A Note on the Variance of the Square Components of a Normal Multivariate within a Euclidean Ball

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Abstract. We present arguments in favor of the inequalities $\text{var}(X^n_2 \mid X \in B_v(\rho)) \leq 2\lambda_n E[X^n_2 \mid X \in B_v(\rho)]$, where $X \sim N_v(0, \Lambda)$ is a normal vector in $v \geq 1$ dimensions, with zero mean and covariance matrix $\Lambda = \text{diag}(\lambda)$, and $B_v(\rho)$ is a centered $v$–dimensional Euclidean ball of square radius $\rho$. Such relations lie at the heart of an iterative algorithm, proposed by Palombi et al. (2012) [6] to perform a reconstruction of $\Lambda$ from the covariance matrix of $X$ conditioned to $B_v(\rho)$. In the regime of strong truncation, i.e. for $\rho \lesssim \lambda_n$, the above inequality is easily proved, whereas it becomes harder for $\rho \gg \lambda_n$. Here, we expand both sides in a function series controlled by powers of $\lambda_n/\rho$ and show that the coefficient functions of the series fulfill the inequality order by order if $\rho$ is sufficiently large. The intermediate region remains at present an open challenge.

1 Introduction

It is intuitively clear that independent random variables develop correlations once constrained within compact multivariate domains. Whenever the mathematical framework rules out closed–form results, a possible approach to studying such correlations is to focus on inequalities among expected values. As a case in point, in this paper we consider a random vector $X \sim N_v(0, \Lambda)$ in $v \geq 1$ dimensions, with $\Lambda = \text{diag}(\lambda)$ and $\lambda = \{\lambda_k\}_{k=1}^v$, whose probability density is cut off sharply outside a Euclidean ball

$$B_v(\rho) = \{x \in \mathbb{R}^v : x^T x < \rho\}.$$  \hfill (1.1)

Owing to the symmetry mismatch between $N_v(0, \Lambda)$ and $B_v(\rho)$, the conditional moments of $X$ admit no exact representation in terms of elementary functions. Our aim is to show that the effect of the spherical truncation on the variance of the square components of $X$ is quantified by the inequalities

$$\Delta_n(\rho; \lambda) \equiv \frac{1}{\rho^2} \{\text{var}(X^n_2 \mid X \in B_v(\rho)) - 2\lambda_n E[X^n_2 \mid X \in B_v(\rho)]\} \leq 0, \quad n = 1, \ldots v.$$  \hfill (1.2)

The interest we have in eq. (1.2) originates from ref. [6], where we have proposed a fixed–point algorithm for the reconstruction of $\Lambda$, in case the only available information amounts to the covariance matrix $S_B$ of $X$ conditioned to $B_v(\rho)$. In particular, in that paper we showed that eq. (1.2) and the correlation inequalities

$$\text{cov}(X^n_2, X^m_2 \mid X \in B_v(\rho)) \leq 0, \quad n \neq m,$$  \hfill (1.3)

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are necessary and sufficient for the convergence of the algorithm. Eq. (1.3) expresses a property of negative association among the square components of \( X \) (see ref. [4]). A proof of it goes beyond the scope of the present paper.

If we denote Gaussian integrals over Euclidean balls by

\[
\alpha_{k\ell m...}(ρ; λ) = \int_{B_v(ρ)} d^v x \frac{x_k^2 x_\ell^2 x_m^2 \ldots}{λ_k λ_\ell λ_m \ldots} \prod_{j=1}^v δ(x_j, λ_j), \quad \delta(y, η) = \frac{e^{-y^2/(2η)}}{(2πη)^{1/2}},
\]

and define \( δ_n ≡ \partial/(∂λ_n) \), we see that \( Δ_n = 2(λ_n^2/ρ^2) [λ_n δ_n(α_n/α)] \). Thus, the inequality \( Δ_n ≤ 0 \) holds true iff \( α_n/α \) is monotonic decreasing in \( λ_n \) with \( λ(n) ≡ \{λ_i\}_{i=n}^\infty \) kept fixed. Since such monotonic behavior is held by both \( α_n \) and \( α \) separately, eq. (1.2) simply means that \( α_n \) is more rapidly decreasing than \( α \). An illustrative example is shown in Fig. 1, where contour plots of \( Δ_n \) at \( v = 2 \) are reported. An analysis of the monotonic properties of averages of monotonic observables of Pólya distributions under linear constraints has been originally proposed by Efron [2]. Unfortunately, the techniques there presented do not carry over to our set–up.

Apart from the covariance reconstruction algorithm, a different motivation to care about eqs. (1.2) and (1.3) has to do with non–linear optimization issues. Thanks to Prékopa’s Theorem [8], \( α(ρ; λ) \) is easily shown to be logarithmic concave in \( ρ \). In sect. 2 we discuss how log–concavity relates to the correlation inequalities. The outcome is that, independently of Prékopa’s Theorem, eqs. (1.2) and (1.3) are alone sufficient to induce log–concavity, while they cannot be deduced from it.

Now, despite the seemingly candid aspect of eq. (1.2), it is difficult to find a unique rigorous proof of it, which works across the whole parameter space \( (ρ; λ) ∈ \mathbb{R}_+ × \mathbb{R}_+^v \). In this paper we propose three different partial arguments. The first two are discussed in sect. 3. They are both straightforward and apply in the regime of strong truncation, i.e. for \( 0 < ρ < λ_n \) resp. \( 0 < ρ < 2λ_n \), independently of \( λ(n) \). In particular, the first one is based on Hölder’s inequality, while the second one makes use of the integral representation of \( Δ_n \). The relative ease of proving eq. (1.2) at strong truncation is certainly due to the large negative values \( Δ_n \) assumes in this regime and its weak dependence upon \( λ(n) \), which in a sense makes the problem nearly 1–dimensional.

The third argument, presented in sects. 4 and 5, applies instead in the regime of weak truncation, i.e. for \( ρ ≫ λ_n \), where proving eq. (1.2) is definitely harder. As \( ρ → ∞ \), we have indeed \( \text{var}(X_n^2 | X ∈ B_v(ρ)) → 2λ_n^2 \) and \( E[X_n^2 | X ∈ B_v(ρ)] → λ_n \), thus \( Δ_n → 0 \). Hence, if eq. (1.2) is correct, it must follow from a cancellation of two positive terms resulting in an increasingly small negative balance. Motivated by the observation that the volume constraint weakens as \( ρ → ∞ \) (and consequently \( α_{k\ell m...} \) becomes well approximated by a

\[1\] Numerical techniques for the computation of \( α_{k\ell m...} \) are discussed in ref. [6].
product of 1-dimensional Gaussian integrals), we expand $\Delta_n$ in a non-elementary-function series around the factorization point. Each term of the expansion factorizes into a 1-dimensional integral along the $n$th direction plus a residual $(v-1)$-dimensional integral in the orthogonal subspace. We prove that such factors get opposite signs as $\rho \to \infty$, thus resulting in negative contributions.

We finally draw our conclusions in sect. 6.

Distributional truncations find application in several frameworks. The specific one, considered in the present paper, turns out to be useful for the compositional analysis of multivariate log-normal data affected by outlying contaminations, where the spherical truncation corresponds to keeping only data with a square Aitchison distance from the mean below a given threshold. This idea is discussed for instance in ref. [7]. In that context, the iterative algorithm of ref. [6] allows for an estimate of the complete covariance matrix from its truncated counterpart.

2 Basic properties of $\alpha_{k\ell m...}$

It is worthwhile starting our discussion by reviewing some trivial properties of the Gaussian integrals, which are used in the sequel. As an alternative notation for $\alpha_{k\ell m...}$ we sometimes adopt the symbol $\alpha_{1:k_1,...:v}$, where $k_j$ counts the multiplicity of the index $j = 1, \ldots, v$. Whenever a directional index has zero multiplicity, we simply drop it. For instance, we write $\alpha_{j:k_j}$ in place of the more pedantic $\alpha_{1:0...j:1...v:0}$. When needed, we declare the integral dimension of $\alpha_{k\ell m...}$ explicitly by writing the latter as $\alpha_{(v)} k\ell m...$. With this in mind, we proceed to a first set of statements.

**Proposition 2.1.** Gaussian integrals fulfill the following properties:

$p_1$) $\alpha_{k\ell m...}(\rho; \lambda)$ is increasing in $\rho$;

$p_2$) $\alpha_{k\ell m...}(\rho; \lambda)$ is separately decreasing in $\lambda_1, \ldots, \lambda_v$;

$p_3$) $\alpha_{k\ell m...}(\rho; \lambda)$ fulfills the scaling equation

$$\rho \partial_\rho + \sum_{r=1}^v \lambda_r \partial_r \alpha_{k\ell m...}(\rho; \lambda) = 0;$$

(2.1)

$p_4$) one-index integrals $\alpha_{k:n}$ follow the hierarchy

$$\alpha_{k:n} \leq (2n-1)\alpha_{k:(n-1)} \leq (2n-1)(2n-3)\alpha_{k:(n-2)} \leq \ldots \leq (2n-1)!! \alpha \leq (2n-1)!!;$$

(2.2)

$p_5$) differentiating $\alpha_{k\ell m...}(\rho; \lambda)$ with respect to $\rho$ yields

$$\rho \partial_\rho \alpha_{k_1...k_n}(\rho; \lambda) = \frac{1}{2} (v + 2n) \alpha_{k_1...k_n}(\rho; \lambda) - \frac{1}{2} \sum_{k=1}^v \alpha_{k_1...k_n k}(\rho; \lambda), \quad n = 0, 1, 2, \ldots$$

(2.3)

$p_6$) $\alpha(\rho; \lambda)$ is logarithmic concave in $\rho$, i.e. it fulfills

$$\alpha(s \rho_1 + (1-s) \rho_2; \lambda) \geq [\alpha(\rho_1; \lambda)]^s [\alpha(\rho_2; \lambda)]^{1-s}, \quad 0 \leq s \leq 1, \quad \rho_1, \rho_2 \in \mathbb{R}_+.$$  

(2.4)

**Proof.** Property $p_1$ follows from the positiveness of the integrand of $\alpha_{k\ell m...}$ and the observation that if $\rho_1 < \rho_2$ then $B_v(\rho_1) \subset B_v(\rho_2)$. $\Box$ Property $p_2$ follows from the observation that $\alpha_{k\ell m...}$ depends on $\rho$ and
\( \lambda \) only via adimensional ratios, as seen by rescaling the integration variable \( x \to x/\sqrt{\rho} \) in eq. (1.4), i.e.

\[
\alpha_{k\ell m...}(\rho; \lambda) = \frac{\rho^{v/2}}{(2\pi)^{v/2}|\Lambda|^{1/2}} \int_{B_0(1)} \mathrm{d}^v v \frac{\rho x_1^2 \rho x_2^2 \rho x_m^2}{\lambda_1 \lambda_2 \lambda_m} \cdots \exp \left\{ - \frac{\rho}{2} \sum_{m=1}^{v} \frac{x_m^2}{\lambda_m} \right\} \\
= \alpha_{k\ell m...}(1; \left\{ \frac{\lambda_1}{\rho}, \ldots, \frac{\lambda_v}{\rho} \right\}) \ .
\]  

(2.5)

When a single variance is downscaled, e.g. \( \lambda \to \lambda' = \{\lambda_1, \ldots, a \lambda_1, \ldots, \lambda_v\} \) with \( 0 < a < 1 \), the change in \( \alpha_{k\ell m...} \) is entirely transferred to the integration region, i.e.

\[
\alpha_{k\ell m...}(\rho; \lambda') = \frac{\rho^{v/2}}{(2\pi)^{v/2}|\Lambda|^{1/2}} \int_{E_v(1;a)} \mathrm{d}^v v \frac{\rho x_1^2 \rho x_2^2 \rho x_m^2}{\lambda_1 \lambda_2 \lambda_m} \cdots \exp \left\{ - \frac{\rho}{2} \sum_{m=1}^{v} \frac{x_m^2}{\lambda_m} \right\},
\]

(2.6)

with

\[
E_v(1; a) = \left\{ x \in \mathbb{R}^v : x_1^2 + \cdots + a x_v^2 + \cdots + x_v^2 < 1 \right\}. 
\]

(2.7)

Since \( B_0(1) \subset E_v(1; a) \), it follows \( \alpha_{k\ell m...}(\rho; \lambda') > \alpha_{k\ell m...}(\rho; \lambda) \). \( \Box \) Property \( p_4 \) follows from the application of the chain rule of differentiation to eq. (2.5). The meaning of the scaling equation is that \( \alpha_{k\ell m...} \) keeps invariant under a change of the units adopted to measure both \( \rho \) and \( \lambda \). \( \Box \) To get convinced about property \( p_4 \), we first notice that

\[
\lambda_k \partial_k \alpha_{1n_1...km_k...vn_v} = \frac{1}{2} \left[ \alpha_{1n_1...km_k...vn_v} - (2n_k + 1)\alpha_{1n_1...km_k...vn_v} \right] ,
\]

(2.8)

as proved by evaluating the derivative on the l.h.s. under the integral sign. From property \( p_2 \), the r.h.s. of eq. (2.8) is recognized to be negative. The proof is completed by taking \( n_j = 0 \) for \( j \neq k \). Note that eq. (2.2) entails the inequalities

\[
\mathbb{E}[X_k^{2n} | X \in B_v(\rho)] \leq (2n - 1)!! \lambda_k^n , \quad n = 1, 2, \ldots ,
\]

(2.9)

with the quantity on the r.h.s. representing the value of the unconditioned \( (2n)\)th univariate moment of \( X_k \). We shall use the lowest order inequalities \( \mathbb{E}[X_k^2 | X \in B_v(\rho)] \leq \lambda_k \) and \( \mathbb{E}[X_k^4 | X \in B_v(\rho)] \leq 3 \lambda_k^2 \) time and again in the sequel. Note also that the larger \( n \), the slower \( \alpha_{k\ell m...} \) saturates to its infinite volume limit. Indeed, if we denote by \( d_n \equiv [(2n - 1)!! - \alpha_{k\ell m...}] / \alpha_{k\ell m...} \) the fractional distance of \( \alpha_{k\ell m...} \) from its infinite volume limit, then eq. (2.2) is equivalent to the inequality chain

\[
d_0(\rho; \lambda) \leq d_1(\rho; \lambda) \leq d_2(\rho; \lambda) \leq \ldots .
\]

(2.10)

As we shall see, this property lies at the heart of most of the difficulties related to proving eq. (1.2). \( \Box \) Property \( p_5 \) follows from eqs. (2.1) and (2.8). \( \Box \) Finally, in order to prove property \( p_6 \), we recall [8]

**Theorem 2.1** (Prékopa). Let \( Q(x) \) be a convex function defined on the entire \( v \)-dimensional space \( \mathbb{R}^v \).

Suppose that \( Q(x) \geq a \), where \( a \) is some real number. Let \( \psi(z) \) be a function defined on the infinite interval \([a, \infty)\). Suppose that \( \psi(z) \) is non-negative, non-increasing, differentiable, and \(-\psi(z) \) is logarithmic concave. Consider the function \( f(x) = \psi(Q(x)) \) \((x \in \mathbb{R}^v)\) and suppose that it is a probability density, i.e.

\[
\int_{\mathbb{R}^v} \mathrm{d}^v x \ f(x) = 1
\]

(2.11)

Denote by \( P\{C\} \) the integral of \( f(x) \) over the measurable subset \( C \) of \( \mathbb{R}^v \). If \( A \) and \( B \) are any two convex sets in \( \mathbb{R}^v \), then the following inequality holds:

\[
(P\{A\})^s \ (P\{B\})^{1-s} \leq P\{s A + (1-s) B\} , \quad 0 \leq s \leq 1 ,
\]

(2.12)

where the linear combination on the l.h.s. denotes the Minkowski sum

\[
s A + (1-s) B \equiv \{ s x + (1-s) y : x \in A, y \in B \}.
\]

(2.13)
Obviously, theorem \[\text{2.1}\] applies if \( f(x) \) is a product of univariate Gaussian densities, as is the case with \( \alpha(\rho; \lambda) \). In addition, if \( x \in B_v(\rho_1) \) and \( y \in B_v(\rho_2) \), from the convexity of the square function \( x \mapsto x^2 \) it follows that

\[
\sum_{k=1}^{v}[sx_k + (1 - s)y_k]^2 \leq s \sum_{k=1}^{v}x_k^2 + (1 - s) \sum_{k=1}^{v}y_k^2 \leq s \rho_1 + (1 - s) \rho_2, \tag{2.14}
\]

i.e. \( sB_v(\rho_1) + (1 - s)B_v(\rho_2) \subseteq B_v(\rho_1 + (1 - s) \rho_2) \). Accordingly, we conclude that

\[
[\alpha(\rho_1; \lambda)]^s [\alpha(\rho_2; \lambda)]^{1-s} \leq \int_{sB_v(\rho_1) + (1 - s)B_v(\rho_2)} d^v x \prod_{m=1}^{v} \delta(x_m, \lambda_m)
\]

\[
\leq \int_{B_v(\rho_1 + (1 - s) \rho_2)} d^v x \prod_{m=1}^{v} \delta(x_m, \lambda_m) = \alpha(s \rho_1 + (1 - s) \rho_2; \lambda). \tag{2.15}
\]

Now, log–concavity is a local property of \( \alpha(\rho; \lambda) \), yet it brings global information about the conditional moments of \( X \). To see this, we observe that since \( \alpha(\rho; \lambda) \) is twice differentiable with respect to \( \rho \), eq. (2.4) is equivalent to

\[
\alpha \partial^2 \rho \alpha - (\partial \rho \alpha)^2 \leq 0. \tag{2.16}
\]

We iterate eq. (2.3) to express the above derivatives in terms of conditional expectations. In first place, evaluating that equation at \( n = 0 \) yields

\[
\partial \rho \alpha = \frac{\nu}{2 \rho} \alpha - \frac{\alpha}{2 \rho} \sum_{k=1}^{v} \frac{\mathbb{E}[X_k^2 \mid X \in B_v(\rho)]}{\lambda_k}. \tag{2.17}
\]

Property \( p_1 \) then implies

\[
\sum_{k=1}^{v} \frac{\mathbb{E}[X_k^2 \mid X \in B_v(\rho)]}{\lambda_k} \leq \nu. \tag{2.18}
\]

Though trivial, eq. (2.18) calls for two remarks. The first one is that a sufficient (but not necessary) condition for it to hold true is \( \mathbb{E}[X_k^2 \mid X \in B_v(\rho)] \leq \lambda_k \) \( \forall k \), which has already been established. In second place, differentiating \( \alpha \) in \( \rho \) an arbitrary number of times generates always symmetric expressions with respect to the directional indices, since \( \rho \) is not tied to any specific direction. In particular, this is the case with the second derivative,

\[
\partial^2 \rho \alpha = \frac{\alpha}{\rho^2} \left\{ \frac{v(v - 2)}{4} - \frac{v}{2} \sum_{k=1}^{v} \frac{\mathbb{E}[X_k^2 \mid X \in B_v(\rho)]}{\lambda_k} + \frac{1}{4} \sum_{k,j=1}^{v} \frac{\mathbb{E}[X_k^2 X_j^2 \mid X \in B_v(\rho)]}{\lambda_k \lambda_j} \right\}. \tag{2.19}
\]

We see that all directional indices are again summed over. We shall come back in sect. 4 to the rational coefficients multiplying the expectation values on the \( r.h.s. \) of eqs. (2.17) and (2.19). For the time being, we finalize our argument by inserting these expressions into eq. (2.16). A little algebra yields

\[
\alpha^2 \left\{ \sum_{k=1}^{v} \frac{\text{var}(X_k^2 \mid X \in B_v(\rho))}{\lambda_k^2} - 2v + 2 \sum_{j \neq k} \frac{\text{cov}(X_j^2, X_k^2 \mid X \in B_v(\rho))}{\lambda_k \lambda_j} \right\} \leq 0. \tag{2.20}
\]

Eq. (2.20) describes the log–concavity of \( \alpha \) in terms of conditional expectations. Did we not know about Prékopa’s Theorem, we could regard it as a result of eqs. (1.2) and (1.3). Unfortunately, the converse does not
For the sake of conciseness, throughout this section we denote conditional expectations by \( \mathbb{E} [\cdot | X \in B_v(\rho)] \) instead of \( \mathbb{E}[\cdot | X \in B_v(\rho)] \). Our starting point consists in regarding eq. (1.2) as an upper bound to \( \mathbb{E}[X_n^4] \). This suggests to consider the wider inequality chain
\[
\mathbb{E}[X_n^4] \leq \mathbb{E}[X_n^2] (2\lambda_n + \mathbb{E}[X_n^2]) \leq \lambda_n (2\lambda_n + \mathbb{E}[X_n^2]) \leq 3\lambda_n^2 . \tag{3.1}
\]
The leftmost bound is in fact a recast of eq. (1.2). If for a moment we give it for granted, the second and third ones follow as an immediate consequence of eq. (1.2). If for a moment we give it for granted, the second and third ones follow as an immediate consequence of eq. (1.2). If for a moment we give it for granted, the second and third ones follow as an immediate consequence of eq. (1.2).

The leftmost bound is in fact a recast of eq. (1.2). If for a moment we give it for granted, the second and third ones follow as an immediate consequence of eq. (1.2). Although our final target is just represented by eq. (1.2), it makes sense to first consider the two rightmost bounds: if they turn out to be violated, eq. (1.2) cannot be correct. The loosest one is once more the trivial inequality \( \mathbb{E}[X_n^4] \leq 3\lambda_n^2 \), which we have already established. By contrast, the inequality
\[
\mathbb{E}[X_n^4] \leq \lambda_n (2\lambda_n + \mathbb{E}[X_n^2]) \tag{3.2}
\]
is less obvious. In sect. 3.1 we prove it. Our argument is based on straightforward algebraic manipulations of the Gaussian integrals over \( B_v(\rho) \). We include it in the present note for a twofold reason: on the one hand it gives a feeling of the optimality of eq. (1.2), on the other it represents the only general result we have, valid across the whole parameter space.

### 3.1 A loose yet general bound to \( \mathbb{E}[X_n^4 | X \in B_v(\rho)] \)

In order to prove eq. (3.2), we use a standard trick, consisting in a rescaling of \( \lambda_n \) by an external parameter \( \tau \), so as to obtain the moments of \( X_n \) by differentiation of \( \alpha \) in \( \tau \). More precisely, we introduce the function
\[
\mathcal{H}(\tau) = \frac{1}{\sqrt{\tau}} \alpha \left( \rho; \left\{ \frac{\lambda_1}{\tau}, \ldots, \frac{\lambda_n}{\tau}, \ldots, \lambda_v \right\} \right) = \frac{1}{\sqrt{\tau}} \int_{B_v(\rho)} d^v x \delta(x_n; \lambda_n/\tau) \prod_{m \neq n} \delta(x_m, \lambda_m) , \tag{3.3}
\]
whose dependence upon \( \rho \) and \( \lambda \) we leave implicit. Differentiating \( \mathcal{H}(\tau) \) under the integral sign yields
\[
\mathbb{E}[X_n^{2k}] = (-1)^k \frac{2^k \lambda_n^k}{\alpha} \frac{\partial^k \mathcal{H}}{\partial \tau^k} \bigg|_{\tau=1} , \quad k = 0, 1, 2, \ldots \tag{3.4}
\]
At the same time, derivatives of \( \mathcal{H}(\tau) \) can be taken via the chain rule of differentiation, which allows us to express them as algebraic combinations of \( \alpha \) and its derivatives in \( \lambda_n \). For instance, with regard to the second and fourth moments, we find
\[
\frac{\partial \mathcal{H}}{\partial \tau} = -\frac{1}{2\tau^{3/2}} \left( \alpha + 2\lambda_n \partial_\alpha \alpha \right) , \tag{3.5}
\]
\[
\frac{\partial^2 \mathcal{H}}{\partial \tau^2} = \frac{1}{4\tau^{5/2}} \left( 3\alpha + 8\lambda_n \partial_\alpha \alpha + 4\lambda_n^2 \partial_\alpha^2 \alpha \right) . \tag{3.6}
\]
Consider first the lowest order derivative. By inserting eq. (3.5) into eq. (3.4) evaluated at \( k = 1 \), we obtain
\[
\mathbb{E}[X_n^2] = \lambda_n [1 - (\alpha_n/\alpha) \partial_\alpha \alpha] . \tag{3.7}
\]
Eq. (3.7) coincides with eq. (2.8) evaluated at \( n_1 = \ldots = n_v = 0 \). Owing to property \( p_2 \) of sect. 2, we infer \( \alpha_n \leq \alpha \) and thus we find again \( \mathbb{E}[X_n^2] = \lambda_n(\alpha_n/\alpha) \leq \lambda_n \). Consider then the fourth moment. If we insert eq. (3.6) into eq. (3.4) evaluated at \( k = 2 \), and then make use of eq. (3.7), we easily arrive at

\[
\mathbb{E}[X_n^4] = 4\lambda_n\mathbb{E}[X_n^2] - \lambda_n^2 + 4\lambda_n^4 \frac{\partial_n^2 \alpha}{\alpha}.
\]  
(3.8)

In order to estimate \( \partial_n^2 \alpha \), we differentiate both sides of eq. (3.7) with respect to \( \lambda_n \). We then invoke again property \( p_2 \) of sect. 2, thus obtaining

\[
\partial_n^2 \alpha = \frac{1}{2\lambda_n} (\partial_n \alpha_n - 3\partial_n \alpha) \leq -\frac{3}{2\lambda_n} \partial_n \alpha = -\frac{3}{4\lambda_n^2} (\alpha_n - \alpha) = -\frac{3\alpha}{4\lambda_n^2} \{\mathbb{E}[X_n^2] - \lambda_n\}.
\]  
(3.9)

This estimate is sufficient to prove eq. (3.2).

### 3.2 First argument in favor of eq. (1.2)

In the regime of strong truncation, eq. (1.2) can be inferred from Hölder’s inequality. We recall that if \( p, q > 1 \) are two numbers satisfying \( 1/p + 1/q = 1 \) and \( X, Y \) are stochastic variables on a given probability space, then \( \mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{1/p}(\mathbb{E}[|Y|^q])^{1/q} \). In our case, we have

\[
\text{var}(X_n^2) = \mathbb{E}\left[(X_n^2 - \mathbb{E}[X_n^2])^2\right] = \mathbb{E}[X_n^4] - \mathbb{E}[X_n^2]^2 = \mathbb{E}[X_n^4] - \mathbb{E}[X_n^2]^2 \\
= \mathbb{E}\left[(X_n^2 - \mathbb{E}[X_n^2]) (X_n^2 + \mathbb{E}[X_n^2])\right] \leq \left[\mathbb{E}[X_n^4] - \mathbb{E}[X_n^2]^2\right] \left[\mathbb{E}[X_n^4] + \mathbb{E}[X_n^2]^2\right] \\
\leq \left\{\mathbb{E}\left[|X_n^2 - \mathbb{E}[X_n^2]|^p\right]\right\}^{1/p} \left\{\mathbb{E}\left[|X_n^2 + \mathbb{E}[X_n^2]|^q\right]\right\}^{1/q}.
\]  
(3.10)

The latter inequality holds true for any finite choice of \( p, q \), provided their reciprocals sum to one. Accordingly, it holds as well in the joint limit \( q \to 1^+ \), \( p = q/(q - 1) \to \infty \), where it amounts to

\[
\text{var}(X_n^2) \leq 2h \mathbb{E}[X_n^2],
\]  
(3.11)

with

\[
h = \lim_{p \to \infty} \left\{\mathbb{E}\left[|X_n^2 - \mathbb{E}[X_n^2]|^p\right]\right\}^{1/p} = \text{ess sup} \left(|X_n^2 - \mathbb{E}[X_n^2]|\right).
\]  
(3.12)

Recall that the essential supremum of a real-valued function \( f \) is defined by \( \text{ess sup} f \equiv \inf\{a \in \mathbb{R} : \mu(\{x : f(x) > a\}) = 0\} \). In particular, the measure \( \mu \) which is understood in eq. (3.12) is the marginal probability measure of \( X_n \), i.e.

\[
d\mu(x_n) = \frac{\alpha^{(\nu-1)}(\rho - x_n^2; \lambda_n)}{\alpha^{(\nu)}(\rho; \lambda)} \delta(x_n, \lambda_n) \, dx_n.
\]  
(3.13)

Owing to the modulating factor \( \alpha^{(\nu-1)}(\rho - x_n^2; \lambda_n) \), \( \mu \) is neither Gaussian nor log–concave (in \( x_n \)). In sect. 4 we shall say more about eq. (3.13) and the factorization of its numerator into functions of resp. \( \lambda_n \) and \( \lambda_n \). For the time being, we observe that \( \mu \) has support in the interval \((-\sqrt{\rho}, \sqrt{\rho})\). Depending on how \( \mathbb{E}[X_n^2] \) compares with \( \rho/2 \), \( h \) might assume one of the values \( h_1 = \rho - \mathbb{E}[X_n^2] \) or \( h_2 = \mathbb{E}[X_n^2] \), as qualitatively represented in Fig. 2. As far as we are concerned, we do not need to establish which among Fig. 2a and Fig. 2b provides the correct qualitative behavior for \( x_n \mapsto |x_n^2 - \mathbb{E}[X_n^2]| \); numerical computations suggest that Fig. 2b is not realized for any choice of \( n, \rho \) and \( \lambda \), yet this information is irrelevant for what follows. More precisely, we distinguish three cases:
Indeed, since $x_n^2 \sim \lambda_n$ (strong truncation): in this case $h \leq \lambda_n$. Indeed, since $\mathbb{E}[X_n^2] \leq \lambda_n$, both $h_1$ and $h_2$ lie below $\lambda_n$. In this region, we have no analytic argument in favour of Fig. 2a or Fig. 2b.

\begin{itemize}
  \item[i)] $\rho \leq \lambda_n$ (strong truncation): in this case $h \leq \lambda_n$. Indeed, since $\mathbb{E}[X_n^2] \leq \lambda_n$, both $h_1$ and $h_2$ lie below $\lambda_n$. In this region, we have no analytic argument in favour of Fig. 2a or Fig. 2b.
  \item[ii)] $\rho > 2\lambda_n$ (weak truncation): in this case $h > \lambda_n$. Indeed, again from $\mathbb{E}[X_n^2] \leq \lambda_n$, we deduce $\rho - \mathbb{E}[X_n^2] > \rho - \lambda_n > \lambda_n \geq 2\mathbb{E}[X_n^2]$. Here, the correct profile of $|x_n^2 - \mathbb{E}[X_n^2]|$ is certainly the one depicted in Fig. 2a.
  \item[iii)] $\lambda_n < \rho < 2\lambda_n$: in this case it is difficult to conclude anything about $h$, except that by continuity there exists a value $\lambda_n < \rho_s(\lambda_n) < 2\lambda_n$, possibly depending on $\lambda_n$, such that $h \leq \lambda_n \Rightarrow \rho \leq \rho_s$.
\end{itemize}

To conclude, the estimate obtained from Hölder’s inequality is certainly as strict as needed for eq. (1.2) to hold true only in case of strong truncation, i.e. for $\rho \leq \lambda_n$. In addition, there is a crossover region where the same estimate might be sufficiently strict, while it becomes definitely too loose in the region of weak truncation.

### 3.3 Second argument in favour of eq. (1.2)

In order to extend the above proof to the region $0 < \rho \leq 2\lambda_n$, we work on the integral representations of $\mathbb{E}[X_n^2] = \lambda_n^2(\alpha_n/\alpha)$ and $\mathbb{E}[X_n^4] = \lambda_n(\alpha_n/\alpha)$. In terms of these, $\Delta_n$ reads

$$
\Delta_n = \frac{\lambda_n^2}{\rho^2} \left[ \frac{\alpha_{n\alpha}}{\alpha} - \left(\frac{\alpha_n}{\alpha}\right)^2 - 2\frac{\alpha_n}{\alpha} \right].
$$

(3.14)

Now, we observe that independently of $v$, $\alpha_{n,k}$ is bounded from above by $(\rho/\lambda_n)^{k-p}\alpha_{n,p}$ for any $p < k$. Indeed, since $x \in B_n(\rho) \Rightarrow -\sqrt{\rho} < x_n < \sqrt{\rho}$, we have

$$
\alpha_{n,k} = \frac{\rho^k}{\lambda_n^k} \int_{B_n(\rho)} d^v x \frac{x_n^{2k}}{\rho^k} \prod_{m=1}^v \delta(x_m, \lambda_m) \leq \frac{\rho^k}{\lambda_n^k} \int_{B_n(\rho)} d^v x \frac{x_n^{2p}}{\rho^p} \prod_{m=1}^v \delta(x_m, \lambda_m) = \frac{\rho^{k-p}}{\lambda_n^{k-p}} \alpha_{n,p}.
$$

(3.15)

Thus, we immediately obtain

$$
\Delta_n \leq \frac{\lambda_n^2}{\rho^2} \left[ (\rho/\lambda_n - 2) \frac{\alpha_{n\alpha}}{\alpha} - \left(\frac{\alpha_n}{\alpha}\right)^2 \right] \leq 0, \quad \text{if } \rho \leq 2\lambda_n.
$$

(3.16)

This conclusion is somewhat conservative, as indeed $\Delta_n \leq 0 \Leftrightarrow \rho \leq \rho_s$, being $\rho_s$ implicitly defined by the non–linear equation $\rho_s = 2\{\lambda_n + \mathbb{E}[X_n^2] | X \in B_n(\rho_s)\}$. By continuity, the latter is certainly fulfilled by some $2\lambda_n < \rho_s \leq 4\lambda_n$. The argument presented here does not apply for $\rho > 4\lambda_n$.  

8
4 Weak truncation expansion

In order to study the variance of the square components of $X$ in the regime of weak truncation, we need to develop an appropriate formalism. To start with, we observe that the constraint $X \in B_v(\rho)$ becomes increasingly unrestrictive as $\rho \to \infty$. As a consequence, we have the asymptotic factorization

$$
\alpha_{1:k_1 \ldots v:k_v}(\rho; \lambda) \left|_{\rho \gg \text{max}_j \{\lambda_j\}} \right. \prod_{j=1}^v \alpha_{j:k_j}(\rho; \lambda_j). \quad (4.1)
$$

The larger is $\rho$, the less is the error made in approximating $\alpha_{1:k_1 \ldots v:k_v}(\rho; \lambda)$ by its factorized counterpart. We aim at characterizing the corrections to eq. (4.1) when $\rho$ is large yet finite. Actually, we are not interested in a complete factorization of the Gaussian integrals: if $\rho \gg \lambda_n$ just for some $1 \leq n \leq v$, we look at the partial factorization occurring along the $n$th direction. Note that: i) in the regime of weak truncation, every rational combination of Gaussian integrals — such as $\Delta_n$ — is led by its factorized counterpart; as we shall see, the latter is subject to relevant simplifications in case of ratios of integrals; ii) 1–dimensional integrals cannot be further simplified, as they amount to lower incomplete gamma functions,

$$
\alpha_{n:k}(\rho; \lambda_n) = \frac{2^k}{\sqrt{\pi}} \gamma \left( k + 1, \frac{\rho}{2\lambda_n} \right), \quad \gamma(s, x) = \int_0^x \frac{e^{-t}}{t^{s-1}} \, dt. \quad (4.2)
$$

4.1 Expansion of Gaussian integrals

In order to present the idea, we first focus on $\alpha$. If $\rho \gg \lambda_n$ for some $1 \leq n \leq v$, we slice the integration domain orthogonally to the $n$th direction, as depicted in Fig. 3. From a geometrical point of view, this corresponds to representing $B_v(\rho)$ as an uncountable union of $(v-1)$–dimensional Euclidean balls, i.e.

$$
B_v(\rho) = \bigcup_{x_n \in (-\sqrt{\rho}, +\sqrt{\rho})} \{ y \in \mathbb{R}^v : y_n = x_n, \ y_{(n)} \in B_{v-1}(\rho - x_n^2) \}. \quad (4.3)
$$

Such technique has been first considered by Ruben [9] with the aim of obtaining an integral recurrence relationship on the dimensionality of $\alpha$. Interpreting the integration domain in terms of eq. (4.3) indeed yields

$$
\alpha^{(v)}(\rho; \lambda) = \int_{-\sqrt{\rho}}^{+\sqrt{\rho}} dx_n \, \delta(x_n, \lambda_n) \, \alpha^{(v-1)}(\rho - x_n^2, \lambda_{(n)}). \quad (4.4)
$$
Since \( \alpha(\rho - x_n^2; \lambda(n)) \) is a smooth function of its first argument \( \rho - x_n^2 \), we propose to expand it in Taylor series around \( x_n^2 = 0 \),

\[
\alpha^{(v-1)}(\rho - x_n^2; \lambda(n)) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lambda_n^k \left( \frac{x_n^2}{\lambda_n} \right)^k \partial^k_\rho \alpha^{(v-1)}(\rho; \lambda(n))
\]

\[
= \alpha^{(v-1)}(\rho; \lambda(n)) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{\lambda_n}{\rho} \right)^k \left( \frac{x_n^2}{\lambda_n} \right)^k \eta_k^{(v-1)}(\rho; \lambda(k)) ,
\]

(4.5)

with the functions \( \eta_k \) defined by

\[
\eta_k^{(v)}(\rho; \lambda) = \begin{cases} 
1, & k = 0, \\
[\alpha^{(v)}(\rho; \lambda)]^{-1} \frac{\lambda}{\rho} k^k \partial^k_\rho \alpha^{(v)}(\rho; \lambda), & k \geq 1.
\end{cases}
\]

(4.6)

When inserted into eq. [4.4], the Taylor series turns into a weak truncation expansion of \( \alpha \), namely

\[
\alpha^{(v)}(\rho; \lambda) = \alpha^{(v-1)}(\rho; \lambda(n)) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{\lambda_n}{\rho} \right)^k \alpha^{(1)}_{n,k}(\rho; \lambda(n)) \eta_k^{(v-1)}(\rho; \lambda(n))
\]

\[
= \alpha^{(1)}(\rho; \lambda(n)) \alpha^{(v-1)}(\rho; \lambda(n)) - \frac{\lambda_n}{\rho} \alpha^{(1)}_{n,1}(\rho; \lambda(n)) \alpha^{(v-1)}(\rho; \lambda(n)) \eta_1^{(v-1)}(\rho; \lambda(n))
\]

\[
+ \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha^{(1)}_{n,1}(\rho; \lambda(n)) \alpha^{(v-1)}(\rho; \lambda(n)) \eta_2^{(v-1)}(\rho; \lambda(n)) + O\left( \frac{\lambda_n^3}{\rho^3} \right) .
\]

(4.7)

Although a complete factorization into functions of \( \lambda_n \) and \( \lambda(n) \) is not exactly realized at finite \( \rho \), we see that it occurs at each order of the expansion. We warn that eq. (4.7) has been obtained upon bringing an infinite sum under an integral sign. Such exchange of limits is delicate, so it is not a priori evident whether the resulting expansion converges or approximates its target just as an asymptotic series. We shall come back to this point later on. We also stress that, while power counting is performed by factors of \( (\lambda_n/\rho)^k \), additional powers and exponentially small terms in \( \rho \) are still hidden within the coefficient functions \( \alpha^{(1)}_{n,k}(\rho; \lambda(n)) \) and \( \eta_k^{(v-1)} \).

To simplify the notation, in the sequel we drop all function arguments, whenever this does not generate confusion. Thus, we shorten eq. (4.7) to

\[
\alpha^{(v)} = \alpha^{(1)} \alpha^{(v-1)} - \frac{\lambda_n}{\rho} \alpha^{(1)}_{n,1}(\rho) \eta_1^{(v-1)} + \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha^{(1)}_{n,1}(\rho) \alpha^{(v-1)} \eta_2^{(v-1)} + O\left( \frac{\lambda_n^3}{\rho^3} \right) .
\]

(4.8)

The same technique can be straightforwardly applied to \( \alpha_{n,p} \). For instance, we have for \( p = 1, 2, \ldots \)

\[
\alpha^{(v)}_{n,1} = \alpha^{(1)}_{n,1} \alpha^{(v-1)} - \frac{\lambda_n}{\rho} \alpha^{(1)}_{n,1}(\rho) \alpha^{(v-1)} \eta_1^{(v-1)} + \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha^{(1)}_{n,1}(\rho) \alpha^{(v-1)} \eta_2^{(v-1)} + O\left( \frac{\lambda_n^3}{\rho^3} \right) ,
\]

(4.9)

\[
\alpha^{(v)}_{n,2} = \alpha^{(1)}_{n,2} \alpha^{(v-1)} - \frac{\lambda_n}{\rho} \alpha^{(1)}_{n,2}(\rho) \alpha^{(v-1)} \eta_1^{(v-1)} + \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha^{(1)}_{n,2}(\rho) \alpha^{(v-1)} \eta_2^{(v-1)} + O\left( \frac{\lambda_n^3}{\rho^3} \right) ,
\]

(4.10)

\[
\vdots
\]

2 The reader will observe that the coefficient function \( \alpha^{(v-1)} \) showing up in each term of the expansion is totally useless, as it simplifies with the one attached to \( \eta_k^{(v-1)} \). Such redundancy is real, yet it turns useful when ratios of Gaussian integrals are considered, as we shall see in sects. 4.2 and 5.
Since the above expansions are all based on eq. (4.5), the coefficient functions \( \eta_k \) are the same independently of \( \rho \). By contrast, the multiplicity of the index \( n \) of the 1-dimensional integrals contributing to each order is shifted forward as \( \rho \) increases.

Now, when it comes to expanding Gaussian integrals with more than one index, the above procedure is carried out in a slightly different way. For instance, in order to expand \( \alpha_{nm} \) we need to take into account factors of \( x_n^2 \) and \( x_m^2 \) under the integral sign. Accordingly, we slice \( B_v(\rho) \) subsequently along the \( n^{th} \) and \( m^{th} \) directions under the assumption \( \rho \gg \max\{\lambda_n, \lambda_m\} \). The analogous of eq. (4.4) reads

\[
\alpha_{nm}^{(v)}(\rho; \lambda) = \int_{-\sqrt{\sigma}}^{+\sqrt{\sigma}} dx_n \frac{x_n^2}{\lambda_n} \int_{-\sqrt{\sigma}}^{+\sqrt{\sigma}} dx_m \frac{x_m^2}{\lambda_m} \delta(x_n, \lambda_n) \int_{-\sqrt{\sigma}}^{+\sqrt{\sigma}} \frac{x_m^2}{\lambda_m} \delta(x_m, \lambda_m) \alpha^{(v-2)}(\rho - x_n^2 - x_m^2; \lambda_{nm}) ,
\]

with \( \lambda_{nm} \equiv \{\lambda_k\}_{k \neq n,m} \). Again, we expand \( \alpha(\rho - x_n^2 - x_m^2; \lambda_{nm}) \) in Taylor series around the point \( x_n^2 + x_m^2 = 0^+ \), thus obtaining

\[
\alpha^{(v-2)}(\rho - x_n^2 - x_m^2; \lambda_{nm}) = \sum_{j=0}^{\infty} \left( \frac{-1}{j!} \right)^j \sum_{k=0}^{j} \left( \frac{j}{k} \right) x_n^{2k} x_m^{2(j-k)} \partial^j_\rho \alpha^{(v-2)}(\rho; \lambda_{nm})
\]

\[
= \alpha^{(v-2)}(\rho; \lambda_{nm}) \sum_{j=0}^{\infty} \left( \frac{-1}{j!} \right)^j \sum_{k=0}^{j} \left( \frac{j}{k} \right) \left( \frac{\lambda_n}{\rho} \right)^k \left( \frac{\lambda_m}{\rho} \right)^{j-k} \left( \frac{x_n^2}{\lambda_n} \right)^k \left( \frac{x_m^2}{\lambda_m} \right)^{j-k} \eta_j^{(v-2)}(\rho; \lambda_{nm}).
\]

Here we have also used Newton’s binomial formula to express each term of the series in products of powers of \( x_n^2 \) and \( x_m^2 \). Inserting this expression back into eq. (4.11) yields

\[
\alpha_{nm}^{(v)} = \alpha_n^{(1)} \alpha_m^{(1)} \alpha^{(v-2)} - \frac{\lambda_n}{\rho} \alpha_n^{(1)} \alpha_m^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} - \frac{\lambda_m}{\rho} \alpha_n^{(1)} \alpha_m^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} + O\left( \frac{\lambda^2}{\rho^2} \right).
\]

Eq. (4.12) can be also used to obtain alternative expansions of \( \alpha_{np} \) if \( \rho \gg \max\{\lambda_n, \lambda_m\} \) for some \( m \), e.g.

\[
\alpha^{(v)} = \alpha^{(1)} \alpha^{(1)} \alpha^{(v-2)} - \frac{\lambda_n}{\rho} \alpha_n^{(1)} \alpha_m^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} - \frac{\lambda_m}{\rho} \alpha_n^{(1)} \alpha_{m,2} \alpha^{(v-2)} \eta_1^{(v-2)} + O\left( \frac{\lambda^2}{\rho^2} \right),
\]

\[
\alpha_n^{(v)} = \alpha_n^{(1)} \alpha^{(v-2)} - \frac{\lambda_n}{\rho} \alpha_{n,2} \alpha^{(v-2)} \eta_2^{(v-2)} - \frac{\lambda_m}{\rho} \alpha_{n,1} \alpha_{n,2} \alpha^{(v-2)} \eta_2^{(v-2)} + O\left( \frac{\lambda^2}{\rho^2} \right),
\]

\[
\vdots
\]

### 4.2 Exercise: relative amplitude of variances and covariances

We make use of the above expansions to qualitatively compare the correlations \( X_n^2 \) has with itself and the other square components of \( X \) in the regime of weak truncation. This is rather instructive, because it shows how analytic cancellations occur in the proposed formalism. In addition, the exercise inspires the following unproved

**Conjecture 4.1.** If \( X \sim N_v(0, \Lambda) \) with \( \Lambda = \text{diag}(\lambda) \), the covariance matrix of the vector \( \{X_n^2\}_{n=1}^v \) conditioned to \( B_v(\rho) \) is diagonally dominant, i.e.

\[
\text{var}(X_n^2 | X \in B_v(\rho)) \geq \sum_{m \neq n} |\text{cov}(X_n^2, X_m^2 | X \in B_v(\rho))| , \quad \rho \in \mathbb{R}_+. \]  

(4.16)
\[\Gamma_{nn}^{(v)} = \frac{1}{\rho^2} \text{var}(X_n^2) = \frac{\lambda_n^2}{\rho^2} \left[ \frac{\alpha_{nn}^{(v)}}{\alpha_{11}^{(v)}} \right]^2, \quad (4.17)\]

\[\Gamma_{nm}^{(v)} = \frac{1}{\rho^2} \text{cov}(X_n^2, X_m^2) = \frac{\lambda_n \lambda_m}{\rho^2} \left[ \frac{\alpha_{nm}^{(v)}}{\alpha_{11}^{(v)}} - \frac{\alpha_{n1}^{(v)}}{\alpha_{11}^{(v)}} \right]^2, \quad n \neq m. \quad (4.18)\]

For illustrative purposes, we show in Fig. 4 a plot of \(|\Gamma_{nn}^{(v)}|\) vs. \(\rho/\lambda_3\) at \(v = 3\) corresponding to the choice \(\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 2, 3\}\). Both \(\Gamma_{nn}\) and \(\Gamma_{nm}\) vanish as \(\rho \to \infty\), yet the former vanishes as \(1/\rho^2\) due to the chosen normalization, whereas the latter is exponentially damped.

We use eqs. (4.18) and (4.19) to work out the expansion of \(\Gamma_{nn}\) and eqs. (4.13) and (4.15) for \(\Gamma_{nm}\). In both cases, in order to expand \(\alpha_{11}^{(1)}\) we rely on the Taylor formula \((1 - x)^{-1} = 1 + x + x^2 + O(x^3)\). Thus, with regard to \(\Gamma_{nn}\) we have

\[\frac{\alpha_{nn}^{(v)}}{\alpha_{11}^{(v)}} = \frac{\alpha_{nn}^{(1)}}{\alpha_{11}^{(1)}} - \frac{\lambda_n}{\rho} \left[ \frac{\alpha_{n3}^{(1)}}{\alpha_{13}^{(1)}} - \frac{\alpha_{n2}^{(1)}}{\alpha_{12}^{(1)}} \right] \eta_1^{(v-1)} + O \left( \frac{\lambda_n^2}{\rho^2} \right), \quad (4.19)\]

\[\left( \frac{\alpha_n^{(v)}}{\alpha_{11}^{(1)}} \right)^2 = \left( \frac{\alpha_n^{(1)}}{\alpha_{11}^{(1)}} \right)^2 - 2 \frac{\lambda_n}{\rho} \left[ \frac{\alpha_{n3}^{(1)}}{\alpha_{13}^{(1)}} - \frac{\alpha_{n2}^{(1)}}{\alpha_{12}^{(1)}} \right] \eta_1^{(v-1)} + O \left( \frac{\lambda_n^2}{\rho^2} \right), \quad (4.20)\]

whence we obtain

\[\Gamma_{nn}^{(v)} = \Gamma_{nn}^{(1)} - \frac{\lambda_n^3}{\rho^3} \left[ \frac{\alpha_{n3}^{(1)}}{\alpha_{13}^{(1)}} - 3 \frac{\alpha_{n2}^{(1)}}{\alpha_{12}^{(1)}} \right] \eta_1^{(v-1)} + O \left( \frac{\lambda_n^2}{\rho^2} \right). \quad (4.21)\]
We see that the leading term of $\Gamma_{nn}$ coincides with its 1–dimensional counterpart. In particular, in Fig. 5 we show the rate at which $\Gamma^{(v)}_{nn}$ approaches $\Gamma^{(1)}_{nn}$ at $v = 3$ and $\lambda = \{1, 2, 3\}$. Since $\lim_{\rho \to \infty} \alpha^{(1)}_{n,k} = (2k - 1)!$, we have $\lim_{\rho \to \infty} (\rho^2/\lambda_n^2) \Gamma^{(1)}_{nn} = 2$, and thus we find again
\[
\Gamma^{(v)}_{nn} \rho \gg \lambda_n \frac{\lambda_n^2}{\rho^2} \left\{ 2 + O \left( \frac{\lambda_n}{\rho} \right) \right\}, \tag{4.22}
\]
apart from exponentially small terms in $\rho$. Analogously, we have
\[
\frac{\alpha^{(v)}_{nm}}{\alpha^{(v)}} = \frac{\alpha^{(1)}_{n} \alpha^{(1)}_{m}}{\alpha^{(1)}_{n} \alpha^{(1)}_{m}} - \frac{\lambda_n}{\rho} \left[ \frac{\alpha^{(1)}_{n,2}}{\alpha^{(1)}_{n}} - \frac{\alpha^{(1)}_{m}}{\alpha^{(1)}_{m}} \right]^2 \eta_1^{(v-2)}
- \frac{\lambda_m}{\rho} \left[ \frac{\alpha^{(1)}_{m,2}}{\alpha^{(1)}_{m}} - \frac{\alpha^{(1)}_{n}}{\alpha^{(1)}_{n}} \right]^2 \eta_1^{(v-2)} + O \left( \frac{\lambda^2}{\rho^2} \right), \tag{4.23}
\]
and
\[
\frac{\alpha^{(v)}_{n,m}}{\alpha^{(v)}} = \frac{\alpha^{(1)}_{n} \alpha^{(1)}_{m}}{\alpha^{(1)}_{n} \alpha^{(1)}_{m}} - \frac{\lambda_n}{\rho} \left[ \frac{\alpha^{(1)}_{n,2}}{\alpha^{(1)}_{n}} - \frac{\alpha^{(1)}_{m}}{\alpha^{(1)}_{m}} \right]^2 \eta_1^{(v-2)}
- \frac{\lambda_m}{\rho} \left[ \frac{\alpha^{(1)}_{m,2}}{\alpha^{(1)}_{m}} - \frac{\alpha^{(1)}_{n}}{\alpha^{(1)}_{n}} \right]^2 \eta_1^{(v-2)} + O \left( \frac{\lambda^2}{\rho^2} \right). \tag{4.24}
\]
When eqs. (4.23)–(4.24) are put into eq. (4.18), an exact cancellation occurs separately among the $O(1)$– and $O(\lambda/\rho)$–terms, so we are left with
\[
\Gamma^{(v)}_{nm} \rho \gg \max \{\lambda_n, \lambda_m\} \frac{\lambda_n \lambda_m}{\rho} O \left( \frac{\lambda^2}{\rho^2} \right). \tag{4.25}
\]
Eqs. (4.22) and (4.25) reflect the behavior observed in Fig. 4.
4.3 Asymptotic vanishing of $\eta_k$ from combinatorial arguments

In order to make the weak truncation expansion effective, we need to characterize the coefficient functions $\eta_k$ and provide an algorithmic recipe for their computation. A trivial property, i.e. the vanishing of $\eta_k$ as $\rho \to \infty$, can be proved from purely combinatorial arguments based on eq. (2.3). This kind of proof nicely follows as a simple application of the scaling eq. (2.1), yet it gives no clue to the vanishing rate of $\eta_k$. We begin with the following.

**Lemma 4.1.** Given a set of $m \geq 0$ distinct indices $\{n_1, \ldots, n_m\}$ and a corresponding set of strictly positive multiplicities $\{k_1, \ldots, k_m\}$, we have

$$\lim_{\rho \to \infty} \rho \partial_\rho \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m (\rho; \lambda) = 0.$$  \hspace{1cm} (4.26)

**Proof.** From eq. (2.3), it follows

$$\lim_{\rho \to \infty} \rho \partial_\rho \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m = \frac{1}{2} \sum_{j=1}^{n} \lim_{\rho \to \infty} \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m + \frac{1}{2} (v + 2k) \lim_{\rho \to \infty} \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m,$$  \hspace{1cm} (4.27)

with $k = \sum_{j=1}^{m} k_j$. The sum index $n$ in eq. (4.27) can either match one of the indices $n_1, \ldots, n_m$ or none of them. Moreover, we have

$$\lim_{\rho \to \infty} \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m = \left\{ \begin{array}{l l}
\lim_{\rho \to \infty} \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m & \text{if } n \neq n_j \forall j = 1, \ldots, m; \\
(2k_j + 1) \lim_{\rho \to \infty} \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m & \text{if } n = n_j \text{ for some } j;
\end{array} \right.$$  \hspace{1cm} (4.28)

as a consequence of the exact factorization of the Gaussian integrals as $\rho \to \infty$ and the standard formula $E[z^{2n}] = (2n - 1)!! (n \geq 0)$, valid for $z \sim \mathcal{N}(0,1)$. Therefore,

$$\lim_{\rho \to \infty} \rho \partial_\rho \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m = \lim_{\rho \to \infty} \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m \cdot \left[ \frac{m - 2k}{2} - \frac{1}{2} \sum_{j=1}^{m} (2k_j + 1) + \frac{1}{2} (v + 2k) \right]$$

$$= \lim_{\rho \to \infty} \alpha^{(v)}_{n_1; k_1} \cdots n_m; k_m \cdot \left[ \frac{m + 2k}{2} - \frac{m + 2k}{2} \right] = 0.$$  \hspace{1cm} (4.29)

From Lemma 4.1 we easily derive the following.

**Proposition 4.1.** As $\rho \to \infty$ all the coefficient functions $\eta_k$ vanish.

**Proof.** Let us define $f_k = (\rho \partial_\rho)^k \alpha^{(v)}$ and $x_k = \sum_{n_1 \cdots n_k} \alpha^{(v)}_{n_1 \cdots n_k}$. We first prove by induction that $f_k$ is a homogeneous linear function of $x_0, \ldots, x_k$. From eq. (2.3), evaluated at $k_1 = \ldots = k_n = 0$, we have indeed

$$\rho \partial_\rho \alpha^{(v)} = \frac{v}{2} \alpha^{(v)} - \frac{1}{2} \sum_{k=1}^{m} \alpha^{(v)}_k = \frac{v}{2} x_0 - \frac{1}{2} x_1 = f_1 (x_0, x_1).$$  \hspace{1cm} (4.30)

Now, suppose that $f_{k-1}$ is a homogeneous linear function of $x_0, \ldots, x_{k-1}$. Then,

$$f_k = (\rho \partial_\rho)^k \alpha^{(v)} = (\rho \partial_\rho)(\rho \partial_\rho)^{k-1} \alpha^{(v)} = \rho \partial_\rho f_{k-1}(x_0, \ldots, x_{k-1})$$

$$= f_{k-1}(\rho \partial_\rho x_0, \ldots, \rho \partial_\rho x_{k-1}) = (\rho \partial_\rho \alpha^{(v)}), \ldots, \sum_{n_1 \cdots n_{k-1} = 1}^{v} \rho \partial_\rho \alpha^{(v)}_{n_1 \cdots n_{k-1}} \right) .$$  \hspace{1cm} (4.31)
The inductive step follows from eq. (2.3) and the assumed linearity of $f_{k-1}$. Hence, we have
\[
\lim_{\rho \to \infty} (\rho \partial_{\rho})^k \alpha^{(v)} = \lim_{\rho \to \infty} f_k(x_0, \ldots, x_k)
= f_{k-1} \left( \lim_{\rho \to \infty} \rho \partial_{\rho} \alpha, \ldots, \sum_{n_1, \ldots, n_{k-1}} \lim_{\rho \to \infty} \rho \partial_{\rho} \alpha^{(v)}_{n_1, \ldots, n_k} \right) = f_{k-1}(0, \ldots, 0) = 0,
\]
where the last equality is again a consequence of the homogeneous linearity of $f_{k-1}$ and the second-to-last one follows from Lemma 4.1. In addition, we know that (see for instance exercise 13, chap. 6 of ref. [3])
\[
\rho^k \partial_{\rho}^k = \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \rho \partial_{\rho}^j \alpha^{(v)} = \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \lim_{\rho \to \infty} \rho \partial_{\rho}^j \alpha^{(v)} = 0,
\]
with the symbols $\binom{k}{j}$ denoting unsigned Stirling numbers of the first kind. Hence, we conclude
\[
\lim_{\rho \to \infty} \eta^{(v)}_k = \lim_{\rho \to \infty} \frac{1}{\alpha^{(v)}} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} (\rho \partial_{\rho})^j \alpha^{(v)} = \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \lim_{\rho \to \infty} (\rho \partial_{\rho})^j \alpha^{(v)} = 0.
\]

4.4 Gaussian representation of $\eta_k$

The above discussion suggests a convenient way to compute the coefficient functions. We have just seen that $\eta_k$ is a linear combination of $f_1, \ldots, f_k$. Moreover, $\forall j \geq 0 f_j$ is a linear combination of $x_0, \ldots, x_j$. We conclude that $\eta_k$ itself can be represented as a linear combination of $x_0, \ldots, x_k$. Since we know how to compute Gaussian integrals with controlled uncertainty, we have a complete recipe for $\eta_k$, provided we determine the coefficients of such linear combinations. To this aim, we concentrate first on the $f_k$’s. Eq. (4.30) gives the analytic expression of $f_1$. By direct calculation we can also derive the expressions
\[
f_2(x_0, x_1, x_2) = \frac{v^2}{4} x_0 - \frac{v + 1}{2} x_1 + \frac{1}{4} x_2,
\]
\[
f_3(x_0, x_1, x_2, x_3) = \frac{v^3}{8} x_0 - \frac{3v^2 + 6v + 4}{8} x_1 + \frac{3v + 6}{8} x_2 - \frac{1}{8} x_3,
\]
\[
\ldots
\]

A generalization is provided by the following

Proposition 4.2. For $k \geq 1$, we have
\[
f_k(x_0, \ldots, x_k) = \sum_{\ell=0}^k d_{k\ell}(v)x_\ell,
\]
where the coefficients $d_{k\ell}(v)$ are defined by
\[
d_{k\ell}(v) = \begin{cases} \sum_{t=0}^k \frac{(-1)^{\ell}}{2^t} \phi_{1-t} \binom{k}{t} \binom{t}{\ell}, & k = 0, \ldots, \ell \\ 0, & \text{otherwise}, \end{cases}
\]
\[
\phi_k = \frac{v!!}{(v-2k)!!} = \begin{cases} 1 & k = 0 \\ (v-2k+2) \phi_{k-1} & k \geq 1, \end{cases}
\]
and the symbols $\binom{k}{\ell}$ denote Stirling numbers of the second kind.
The argument is complete provided we are able to show that $d_{k\ell}(v)$ fulfills the recurrence

$$d_{(k+1)\ell}(v) = \left(\frac{v}{2} + \ell\right) d_{k\ell}(v) - \frac{1}{2} d_{k(\ell-1)}(v).$$

(4.41)

To this aim, it is sufficient to make use of the basic recursive formulae $\{n+1\} = m \{n\} + \{n\}$ and $(\binom{n+1}{m}) = \binom{n}{m} + \binom{n}{m-1}$. We detail the algebra for the sake of completeness:

$$d_{(k+1)\ell}(v) = \sum_{t=0}^{k+1} \frac{(-1)^t}{2^t} \phi_{t-\ell} \left\{ \begin{array}{c} k+1 \\ t \end{array} \right\} (t)$$

$$= \sum_{t=0}^{k+1} \frac{(-1)^t}{2^t} \phi_{t-\ell} \left\{ \begin{array}{c} k \\ t \end{array} \right\} (t) + \sum_{t=1}^{k+1} \frac{(-1)^t}{2^t} \phi_{t-\ell} \left\{ \begin{array}{c} k \\ t-1 \end{array} \right\} (t)$$

$$= \sum_{t=0}^{k+1} \frac{(-1)^t}{2^t} \phi_{t-\ell} \left\{ \begin{array}{c} k \\ t \end{array} \right\} (t) + \sum_{t=1}^{k+1} \frac{(-1)^t}{2^t} \phi_{t-\ell+1} \left\{ \begin{array}{c} k \\ t \end{array} \right\} (t+1)$$

$$= \sum_{t=0}^{k+1} \frac{(-1)^t}{2^t} \phi_{t-\ell} \left\{ \begin{array}{c} k \\ t \end{array} \right\} (t) + \sum_{t=0}^{k} \frac{(-1)^t}{2^{t+1}} \phi_{t-\ell+1} \left\{ \begin{array}{c} k \\ t \end{array} \right\} (t)$$

$$+ \sum_{t=1}^{k+1} \frac{(-1)^t}{2^t} \phi_{t-\ell} \left\{ \begin{array}{c} k \\ t \end{array} \right\} (t) + \frac{1}{2} d_{k(\ell-1)}(v) = \left(\frac{v}{2} + \ell\right) d_{k\ell}(v) - \frac{1}{2} d_{k(\ell-1)}(v).$$

(4.42)

In view of eq. (4.33), it is no surprise that the coefficients $d_{k\ell}(v)$ embody Stirling numbers of the second kind. Recall indeed that Stirling numbers of the first and the second kind are related to each other by the
inversion identity
\[
\max_{j} \sum_{t=0}^{\max(j,k)} (-1)^{t-k} \binom{t}{j} \binom{t}{j} = \delta_{jk}.
\]  
(4.43)

From Proposition 4.2 and eq. (4.43), it follows the following proposition.

**Proposition 4.3.** For \( k \geq 1 \), we have
\[
\eta_{k}(v) = \frac{1}{\alpha(v)} \sum_{\ell=0}^{k} \left( -1 \right)^{k-\ell} \left[ \begin{array}{c} k \\ \ell \end{array} \right] \left( \rho \partial_{\rho} \right)^{\ell} \alpha(v) = \frac{1}{\alpha(v)} \sum_{\ell=1}^{k} \left( -1 \right)^{k-\ell} \left[ \begin{array}{c} k \\ \ell \end{array} \right] \sum_{m=0}^{\ell} \alpha(\ell, m) x_{m}
\]  
(4.44)

with the coefficients \( c_{\ell}(v) \) defined as
\[
c_{\ell}(v) = \left\{ \begin{array}{ll}
\frac{(-1)^{\ell} v!!}{2^{\ell} [v-2(k-\ell)]!!} \left( \begin{array}{c} k \\ \ell \end{array} \right), & 0 \leq \ell \leq k, \\
0, & \text{otherwise}.
\end{array} \right.
\]  
(4.45)

**Proof.** We have all the necessary ingredients to carry out the proof. Again, we detail the algebra for the reader’s convenience:
\[
\eta_{k}(v) = \frac{1}{\alpha(v)} \sum_{\ell=1}^{k} \left( -1 \right)^{k-\ell} \left[ \begin{array}{c} k \\ \ell \end{array} \right] \sum_{m=0}^{\ell} \alpha(\ell, m) x_{m}
\]
\[
= \frac{1}{\alpha(v)} \sum_{m=0}^{\ell} \sum_{m=0}^{k} \left( -1 \right)^{m} \frac{1}{2^{t}} \phi_{t-m} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \begin{array}{c} \ell \\ t \end{array} \right) x_{m}
\]
\[
= \frac{1}{\alpha(v)} \sum_{m=0}^{\ell} \sum_{m=0}^{k} \left( -1 \right)^{m} \frac{1}{2^{t}} \phi_{t-m} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \begin{array}{c} \ell \\ t \end{array} \right) x_{m}
\]
\[
= \frac{1}{\alpha(v)} \sum_{m=0}^{\ell} \sum_{m=0}^{k} \left( -1 \right)^{m} \frac{1}{2^{t}} \phi_{t-m} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \begin{array}{c} \ell \\ t \end{array} \right) x_{m}
\]
\[
= \frac{1}{\alpha(v)} \sum_{m=0}^{\ell} \sum_{m=0}^{k} \left( -1 \right)^{m} \frac{1}{2^{t}} \phi_{t-m} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \begin{array}{c} \ell \\ t \end{array} \right) x_{m}
\]
\[
= \frac{1}{\alpha(v)} \sum_{m=0}^{\ell} \sum_{m=0}^{k} \left( -1 \right)^{m} \frac{1}{2^{t}} \phi_{t-m} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \begin{array}{c} \ell \\ t \end{array} \right) x_{m}.
\]  
(4.46)

Note that on the second line above we could extend the upper bound of the sums over \( m \) and \( t \) from \( \ell \) to \( \infty \) thanks to the property \( \{ a \} \{ b \} = 0 \) if \( b > a \). This in turn allowed us to perform the sum exchange on the third line. By a similar argument, on the last line we could reduce the upper bound of the sum over \( m \) from \( \infty \) to \( k \).

Obviously, computing eq. (4.44) becomes increasingly demanding for larger values of \( k \). Nonetheless, many contributions to the sum on the r.h.s. coincide. In particular, all Gaussian integrals with the same index multiplicities contribute equally, thus we can recast eq. (4.44) to the computationally cheaper expression
\[
\eta_{k}(v) = \sum_{\ell=0}^{k} \sum_{m_{1}=0}^{\ell} \binom{\ell}{m_{1} \ldots m_{v}} \delta_{\ell, m_{1} + \ldots + m_{v}} \frac{\alpha(\ell, m_{1} \ldots m_{v})}{\alpha(v)}.
\]  
(4.47)
### 4.5 Asymptotic sign of \( \eta_k \)

In this and next subsection we put the combinatorial approach on hold and work on the integral representation of the coefficient functions. A first property which turns out to be essential to the last part of the paper concerns the sign assumed by \( \eta_k \) as \( \rho \to \infty \). In regard to this, we state the following

**Proposition 4.4.** As \( \rho \to \infty \) the sign of \( \eta_k \) becomes independent of \( v \) and \( \lambda \). In particular, we have

\[
\lim_{\rho \to \infty} \text{sign } \eta_k^{(v)}(\rho; \lambda) = (-1)^{k-1}. \tag{4.48}
\]

**Proof.** We first express \( \alpha \) in spherical coordinates, i.e. we perform the change of integration variable \( x = ru \) in eq. (1.4), with \( r = ||x|| \), \( u \in \partial \mathcal{B}_v(1) \) and \( \partial \mathcal{B}_v(1) = \{ z \in \mathbb{R}^v : ||z|| = 1 \} \) (in the sequel we write \( dv = r^{v-1} dr du \); here \( du \) embodies the angular part of the spherical Jacobian and the differentials of \( (v-1) \) angles). Thus, we have

\[
\alpha^{(v)}(\rho; \lambda) = \frac{1}{2^v \Gamma(v/2)|A|^{1/2}} \mathbb{M} \left[ \int_0^\sqrt{\rho} dr \, r^{v-1} \exp \left\{ -\frac{r^2 \mathcal{P}(u)}{2} \right\} \right], \quad \mathcal{P}(u) = u^T \Lambda^{-1} u, \tag{4.49}
\]

with \( \mathbb{M} \) representing the uniform average operator on \( \partial \mathcal{B}_v(1) \), namely

\[
\mathbb{M}[g] = \frac{\Gamma(v/2)}{2\pi^{v/2}} \int_{\partial \mathcal{B}_v(1)} du \, g(u). \tag{4.50}
\]

In order to compute \( \eta_k \), we differentiate \( \alpha \) under the integral sign. The first derivative evaluates the radial integral at its upper limit, while the remaining \( k-1 \) ones distribute according to the chain rule of differentiation. Explicitly, we have

\[
\rho^k \partial_{\rho}^{(v)} \alpha^{(v)}(\rho; \lambda) = \frac{\rho^k}{2^v \Gamma(v/2)|A|^{1/2}} \mathbb{M} \left[ \partial_{\rho}^{k-1} \left( \rho^{v/2-1} \exp \left\{ -\rho \mathcal{P}(u) \right\} \right) \right]
\]

\[
= \frac{(-1)^{k-1} \rho^{v/2}}{2^v \Gamma(v/2)|A|^{1/2}} \mathbb{M} \left[ \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (-\phi)^\ell \left( \frac{\rho \mathcal{P}(u)}{2} \right)^{k-1-\ell} \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right]_{\phi=v/2-1}
\]

\[
= \frac{(-1)^{k-1} \rho^{v/2}}{2^v \Gamma(v/2)|A|^{1/2}} \mathbb{M} \left[ Q_{k-1} \left( \frac{\rho \mathcal{P}(u)}{2} \right)^{k-1-\ell} \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right]_{\phi=v/2-1}. \tag{4.51}
\]

Here \( x^\pi \equiv x(x+1)\ldots(x+n-1) \) denotes the \( n^{\text{th}} \) raising factorial of \( x \) and

\[
Q_k(x, a) = \sum_{\ell=0}^{k} \binom{k}{\ell} a^{\ell} x^{k-\ell} \tag{4.52}
\]

is a polynomial in \( x \) of \( k^{\text{th}} \) degree, differing from a Newton polynomial for the presence of \( a^\ell \) in place of \( a^\ell \). We note that the coefficient of the leading term of \( Q_k(x, a) \) is \( \binom{k}{0} a^k = 1 \). It follows that

\[
\rho^k \partial_{\rho}^{(v)} \alpha^{(v)}(\rho; \lambda) \xrightarrow{\rho \to \infty} \frac{(-1)^{k-1}}{\Gamma(\phi+1)|A|^{1/2}} \left( \frac{\rho^{v/2}}{2} \right)^{\phi+k} \mathbb{M} \left[ \mathcal{P}(u)^{k-1} \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right]_{\phi=v/2-1} \tag{4.53}
\]

Eq. (4.48) follows from the positiveness of \( \mathcal{P}(u) \). \( \square \)

Since \( \mathcal{P}(u) > \lambda_{1\text{max}}^{-1} \), being \( \lambda_{\text{max}} = \max_k \{ \lambda_k \} \), we obtain as a by–product an estimate of the exponential damping of \( \eta_k \), namely

\[
\left| \rho^k \partial_{\rho}^{(v)} \alpha^{(v)}(\rho; \lambda) \right| < \frac{\rho^{v/2}}{2^v \Gamma(v/2)|A|^{1/2}} \mathbb{M} \left[ Q_{k-1} \left( \frac{\rho \mathcal{P}(u)}{2} \right)^{k-1-\ell} \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right]_{\phi=v/2-1} e^{-\rho/2\lambda_{\text{max}}}. \tag{4.54}
\]
4.6 A convergence estimate for the expansion

We come back to the issue raised at the beginning of this section: is the weak truncation expansion convergent? It is not difficult to see that the answer lies specifically on the behavior of $\eta_k$ as a function of $k$. Eq. (4.54) shows that the relevant information is brought by $Q_{k-1}(\rho P(u)/2, -\phi)$, particularly by its relative minima/maxima. As $k$ increases, the position of the latter shifts towards larger and larger values of $\rho$, while their absolute size increases. In other words, however we choose a reference scale $\tilde{\rho} > 0$ we always find $k$ such that $\arg \max \{|\eta_k(\rho; \lambda)|\} > \tilde{\rho}$ and $\max \{|\eta_k(\rho; \lambda)|\} > \max \{|\eta_k(\rho; \lambda)|\}$ for $k > \tilde{k}$. For this reason, the convergence issue reduces to quantify the increase rate of $\eta_k$ as a function of $k$.

More quantitatively, we first notice the inequality $\gamma(a, x) < x^a/a$. In order to prove this, we observe that a convenient representation of the lower incomplete gamma function is provided by (see for instance sect. 6 of ref. [1])

$$\gamma(a, x) = \frac{1}{a} x^a e^{-x} M(1, 1 + a, x), \quad (4.55)$$

where

$$M(a, b, x) = 1 F_1(a; b; x) = \sum_{n=0}^{\infty} \frac{a^n x^n}{b^n n!} \quad (4.56)$$

is the confluent hypergeometric function originally introduced by Kummer. Since $1^\pi = n!$ and $(1 + a)^\pi = (1 + a)(2 + a) \ldots (n + a) > n!$ if $a > 0$, it follows

$$M(1, 1 + a, x) = \sum_{n=0}^{\infty} \frac{1^n}{(1 + a)^n n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} < \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \quad (4.57)$$

We can use eq. (4.2) and the above inequality to establish an upper bound to the 1–dimensional Gaussian integrals, namely

$$\alpha^{(1)}_{n,k}(\rho; \lambda_k) < \frac{1}{\sqrt{2\pi k}} \left( \frac{\rho}{\lambda_n} \right)^{k+1/2}. \quad (4.58)$$

Now, let us denote by $X_{n,k}$ the weak truncation expansion of $\alpha_{n,k}$, i.e.

$$X^{(v)}_{n,k}(\rho; \lambda) = \alpha^{(v-1)}(\rho; \lambda_n) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \frac{\lambda_n}{\rho} \right)^p \alpha^{(1)}_{n(k+p)}(\rho; \lambda_n) \eta_p^{(v-1)}(\rho; \lambda_n). \quad (4.59)$$

In view of eq. (4.58), an absolute estimate to $X_{n,k}$ is given by

$$|X^{(v)}_{n,k}(\rho; \lambda)| < \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{\lambda_n}{\rho} \right)^p \alpha^{(1)}_{n(k+p)}(\rho; \lambda_n) \alpha^{(v-1)}(\rho; \lambda_n) \eta_p^{(v-1)}(\rho; \lambda_n)$$

$$< \frac{1}{\sqrt{2\pi}} \left( \frac{\rho}{\lambda_n} \right)^{k+1/2} \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{1}{p + k} \right) \rho^p B_p \alpha^{(v-1)}(\rho; \lambda_k). \quad (4.60)$$

From the above inequality we see that a less than factorial growth of $\eta_p$ with $p$ would make the r.h.s. of eq. (4.60) convergent. In order to estimate $|\rho^p B_p \alpha|$ we make use of eq. (4.51). Since $B_p \alpha(1)$ is a compact domain, we can get rid of the angular average by defining

$$u^* = \arg \max_{u \in \partial B_{\alpha}(1)} \left\{ Q_{p-1} \left( \frac{\rho P(u)}{2}, -\phi \right) \exp \left\{ -\frac{\rho P(u)}{2} \right\} \right\}, \quad (4.61)$$
whence it follows
\[
\left| \rho \partial_\rho \mathcal{A}(\rho; \lambda) \right| \leq \frac{\rho^{v/2}}{2^{v/2} \Gamma(v/2)|\Lambda|^{1/2}} \left| Q_{p-1} \left( \frac{\rho P(u^*)}{2}, -\phi \right) \exp \left\{ -\frac{\rho P(u^*)}{2} \right\} \right.
\]
(4.62)

We have already noticed that \( P(u) > \lambda_{\text{max}}^{-1} \) for all \( u \in \partial B_v(1) \). If we also consider that \( |\lambda|^{1/2} > \lambda_{\text{min}}^{-1} \) being \( \lambda_{\text{min}} = \min_k \{ \lambda_k \} \), then we have \( [P(u)]^{v/2} \langle \lambda^{1/2} \rangle^{-1} < (\lambda_{\text{max}}/\lambda_{\text{min}})^{1/2} \). Multiplying and dividing the r.h.s. of eq. (4.62) by \( P(u^*)^{v/2} \) leads to
\[
\left| \rho \partial_\rho \mathcal{A}(\rho; \lambda) \right| < \frac{1}{\Gamma(v/2)} \left( \frac{\rho_{\text{max}}}{\rho_{\text{min}}} \right)^{v/2} \left( \frac{\rho P(u^*)}{2} \right)^{v/2} \left| Q_{p-1} \left( \frac{\rho P(u^*)}{2}, -\phi \right) \exp \left\{ -\frac{\rho P(u^*)}{2} \right\} \right.
\]
(4.63)

Accordingly, we obtain the estimate
\[
|\lambda_n^{(v)}(\rho; \lambda)| < \frac{1}{\sqrt{2^{v/2} \Gamma((v-1)/2)}} \left( \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \right)^{(v-1)/2} \left( \frac{\rho}{\lambda_n} \right)^{k_1+1/2} \sum_{p=1}^{\infty} \frac{C(v)(\rho)}{p}.
\]
(4.64)

where we have set \( \lambda_{\text{min}} = \min_j \{ \lambda_j \} \), \( \lambda_{\text{max}} = \max_j \{ \lambda_j \} \) and
\[
C(v)(\rho) = \frac{1}{\max_{p \in \mathbb{N}^+}} \left\{ \sum_{\ell=0}^{p-1} \left( -\phi^\ast \right)^\ell x^{p-\ell} \exp \left\{ -x \right\} \right\}
\]
(4.65)

It will be observed that in order to arrive at eq. (4.64), we have gone through a rather long inequality chain, so it is not clear whether the resulting upper bound is finite. For the sum on the r.h.s. to be convergent, it suffices that \( \exists \epsilon > 0, A > 0 \) and \( \rho_0 \) such that \( C(p) < Ap^{-\epsilon} \equiv m(p; A, \epsilon) \) for \( p > \rho_0 \). If this holds true, then we have \( \sum_{p=1}^{\infty} C(p)/p < A\zeta(1 + \epsilon) + \text{const.} \), where \( \zeta \) denotes the Riemann zeta function. In order to evaluate \( C(p) \) analytically, we need to solve a polynomial equation of degree \( (p - \phi^\ast) \) for \( p \gg 1 \). An alternative approach is to compute \( C(p) \) numerically for a set of sufficiently large values of \( p \) and then try to fit data to a model such as \( m(p; A, \epsilon) \), with the parameters \( A \) and \( \epsilon \) depending in general on \( v \). In Fig. 6 (right) we plot numerical determinations of \( C(p) \) for \( v = 2, \ldots, 6 \). We observe that \( C(p) \) is monotonic decreasing for \( v < 5 \) and monotonic increasing for \( v \geq 6 \). In Fig. 6 (left) we report the values of the fitted parameters \( A \) and \( \epsilon \) together with the corresponding \( \chi^2 \)-values (the range chosen for all fits is \( p \in [50, 100] \)). Our numerical experiments suggest that the weak truncation expansion converges uniformly in \( \rho \) at least for \( v \leq 5 \). Nevertheless, the argument is not conclusive, since it assumes that we can extrapolate the fitting model to \( p \to \infty \), which is not mathematically rigorous...

5 Variance reduction in the regime of weak truncation

If we look at eq. (4.21), we see that the next–to–leading contribution to the rescaled variance \( \Gamma_{nn} \) is the sum of a few ratios of 1–dimensional Gaussian integrals, all proportional to \( \eta_1 \). The subsequent terms of the expansion have an increasingly complex structure. Each power of \( \lambda_n/\rho \) couples to several products of coefficient functions \( \eta_k \), always combined so as to give the correct power counting. If \( f(\rho; \lambda) \) is a generic observable, its weak truncation expansion reads
\[
f(\rho; \lambda) = \sum_{q=0}^{\infty} (-1)^q \left( \frac{\lambda_n}{\rho} \right)^q \sum_{\mathcal{Z} \in S_q^q} \mathcal{Z}(\rho; \lambda) \eta_0^{(v-1)}(\rho; \lambda_1) \eta_1^{(v-1)}(\rho; \lambda_2) \cdots \eta_q^{(v-1)}(\rho; \lambda_q) e^q.
\]
(5.1)
We denote by \( \mathcal{E} = \{ e_0, \ldots, e_q \} \) the exponents of \( \eta_0, \ldots, \eta_q \) and by \( S^m_q \) the set of all possible \( \mathcal{E} \)'s corresponding to an overall power counting \( m \), namely

\[
S^m_q = \left\{ \mathcal{E} \in \mathbb{N}_0^{q+1} : \mathcal{P}_q(\mathcal{E}) = m \right\},
\]

with the power counting function \( \mathcal{P}_q(\mathcal{E}) \) defined as

\[
\mathcal{P}_q(\mathcal{E}) = \sum_{k=1}^q k e_k.
\]

Note that if \( m < q \) and \( \mathcal{E} \in S^m_q \), then \( e_{m+1} = \ldots = e_q = 0 \). In this case, we interpret \( \Xi_f^{(m;\mathcal{E})} \) as \( \Xi_f^{(m;\{e_0, \ldots, e_m\})} \).

Recall also that since \( \eta_0 = 1 \), \( e_0 \) never contributes to the power counting. For later convenience, we define \( \mathbf{e} = \{e_1, \ldots, e_q\} \). If \( \mathcal{E} \in S^m_q \), with abuse of notation we also write \( \mathbf{e} \in S^m_q \). Strictly speaking, the presence of \( e_0 \) in eq. (5.1) is necessary in order to properly take into account the leading order of the expansion, namely

\[
\lim_{\rho \to \infty} f(\rho; \lambda) = \lim_{\rho \to \infty} \sum_{e_0=0}^{\infty} \Xi_f^{(0;\{e_0\})}(\rho; \lambda).
\]

The sum over \( e_0 \) in eq. (5.1) extends in principle from 0 to \( \infty \). We use \( e_0 \) to enumerate all contributions proportional to \( \eta_1^{e_1} \cdots \eta_q^{e_q} \). Accordingly, the information concerning the maximum value taken by \( e_0 \) is hidden within \( \Xi_f^{(q;\mathcal{E})} \). We finally stress that \( \Xi_f^{(q;\mathcal{E})} \) depends in general upon all the components of \( \lambda \), yet in the specific cases of \( \Gamma_{nn} \) and \( \Delta_n \) (which are the ones we are interested in) it depends only upon \( \lambda_n \).

Suppose now that \( f(\rho; \lambda) \) and \( g(\rho; \lambda) \) are two observables, which we expand according to eq. (5.1). It is not difficult to prove that the convolution rules needed to obtain the expansion of the algebraic combinations \( f + g \) and \( f \cdot g \) are similar to the Fourier ones. Specifically, we have

\[
\Xi_f^{(q;\mathcal{E})} = \Xi_f^{(q;\mathcal{E})} + \Xi_g^{(q;\mathcal{E})},
\]

and

\[
\Xi_f^{(q;\mathcal{E})} = \sum_{\ell,m=0}^q \delta_{q,\ell+m} \sum_{\mathcal{E} \in S^m_q} \sum_{\mathbf{d} \in S^m_q} \delta_{\mathcal{E},\mathbf{d}} \Xi_f^{(\mathcal{E};\mathbf{d})} \Xi_g^{(m;\mathbf{d})},
\]
Hence, we deduce

\[ \text{This allows us to derive the expansion of the product } \alpha \text{, sufficient to derive the weak truncation expansion of } \Delta_n \text{ to all orders and to consequently prove our main result, represented by the following} \]

\[ \textbf{Theorem 5.1.} \text{ As } \rho \to \infty, \text{ the sign of the coefficient function } \Xi_{\Delta_n}^{(q,v)}(\rho; \lambda_n) \text{ is given by } \]

\[ \lim_{\rho \to \infty} \sum_{e_0} \Xi_{\Delta_n}^{(q,v)}(\rho; \lambda_n) = (-1)^{\sum_{k=1}^{q} e_k - 1}. \quad (5.7) \]

This, in conjunction with eq. (5.48), implies that all terms of the weak truncation expansion of \( \Delta_n \) become negative at sufficiently large \( \rho \).

\[ \textbf{Proof.} \text{ We proceed in subsequent steps. First of all, we observe that } \Delta_n \text{ can be written as} \]

\[ \Delta_n(\rho; \lambda) = \frac{\lambda^2}{\rho^2} \mathcal{D}_n \cdot \mathcal{D}_d, \quad \begin{cases} \mathcal{D}_n = \alpha_n^{(v)} \alpha_n^{(v)} - \alpha_n^{(v)} \alpha_n^{(v)} - 2 \alpha_n^{(v)} \alpha_n^{(v)}, \\ \mathcal{D}_d = [\alpha^{(v)}]^{-2}. \end{cases} \quad (5.8) \]

To work out \( \mathcal{D}_n \), we first review the expansion of \( \alpha_{n,k} \). Explicitly, we have

\[ \alpha_{n,k}^{(v)} = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \frac{(\lambda_n)}{(\rho)} q \alpha_{n,(k+q)}^{(v)} \alpha_{n,(v-1),q}^{(v-1)} \]

\[ = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \frac{(\lambda_n)}{(\rho)} q \alpha_{n,(k+q)}^{(v)} \alpha_{n,(v-1)} \sum_{\xi \in \mathcal{S}_n^q} \left( \prod_{i=0}^{q-1} \delta_{c_{i,0}} \right) \delta_{c_{1,0}} \left[ \rho_0^{(v-1)} \rho_1 \ldots \rho_q^{(v-1)} \right]. \quad (5.9) \]

Hence, we deduce

\[ \Xi_{\alpha_{n,k}}^{(q,v)} = \frac{1}{q!} \alpha_{n,(k+q)}^{(v)} \alpha_{n,(v-1)} \left( \prod_{i=0}^{q-1} \delta_{c_{i,0}} \right) \delta_{c_{1,0}}. \quad (5.10) \]

This allows us to derive the expansion of the product \( \alpha_{n,r} \alpha_{n,s} \). From eq. (5.6) it follows

\[ \Xi_{\alpha_{n,r} \alpha_{n,s}}^{(q,v)} = \sum_{\ell,m=0}^{q} \delta_{q,\ell+m} \frac{1}{\ell! m!} \alpha_{n,(r+\ell)}^{(1)} \alpha_{n,(s+m)}^{(1)} \left[ \alpha^{(v-1)} \right]^2 \]

\[ \cdot \sum_{\xi \in \mathcal{S}_n^q} \sum_{d \in \mathcal{S}_n^q} \delta_{\xi \in \mathcal{C} + d} \left( \prod_{i=0}^{q-1} \delta_{c_{i,0}} \right) \left( \prod_{k=0}^{m-1} \delta_{d_{k,0}} \right) \delta_{c_{1,0}}. \quad (5.11) \]

The inner sums can be performed exactly. Indeed, thanks to the Kronecker symbols \( \delta_{c_{1,0}} \) and \( \delta_{d_{0,0}} \), non–vanishing contributions group according to whether \( \ell = m \) or \( \ell \neq m \), namely

\[ \Xi_{\alpha_{n,r} \alpha_{n,s}}^{(q,v)} = \left[ \alpha^{(v-1)} \right]^2 \sum_{\ell,m=0}^{q} \delta_{q,\ell+m} \frac{1}{\ell! m!} \alpha_{n,(r+\ell)}^{(1)} \alpha_{n,(s+m)}^{(1)} \]

\[ \cdot \left[ \delta_{\ell,m} \delta_{c_{1,0}} \prod_{i \neq \ell}^{q} \delta_{c_{i,0}} + (1 - \delta_{\ell,m}) \delta_{c_{1,0}} \delta_{d_{0,0}} \prod_{i \neq \ell}^{q} \delta_{c_{i,0}} \right]. \quad (5.12) \]
We see that the Kronecker symbol $\delta_{q,\ell+m}$ makes both groups of terms vanish unless $P_q(\ell) = q$, as intuitively understood. Conversely, the only elements $\ell \in S_q^p$ which result in a non-vanishing coefficient function $\Xi(q;\alpha)$ are either those where two different exponents $e_\ell$, $e_m$ equal one (with $\ell + m = q$) while the others vanish, or those where $e_\ell/q = 2$ and $e_i = 0$ for $i \neq q/2$ (of course the latter contribute only when $q$ is even).

From eq. (5.12), we immediately obtain

$$
\Xi(q;\alpha) = [\alpha^{(v-1)}] \sum_{\ell,m=0}^{q} \frac{\delta_{\ell,m}}{\ell! m!} \left( \alpha_{n,(\ell+2)}^{(1)} \alpha_{n,m}^{(1)} - \alpha_{n,(\ell+1)}^{(1)} \alpha_{n,(m+1)}^{(1)} - 2\alpha_{n,(\ell+1)}^{(1)} \alpha_{n,m}^{(1)} \right) 
$$

$$
\cdot \left[ \delta_{\ell,m} \delta_{e_\ell,2} \prod_{i \neq \ell} \delta_{e_i,0} + \left( 1 - \delta_{\ell,m} \delta_{e_\ell,1} \delta_{e_m,1} \right) \prod_{i \neq \ell} \delta_{e_i,0} \right].
$$

The above expression depends upon $\rho$ essentially via the integrals in parentheses (the overall factor $[\alpha^{(v-1)}]^2$ is irrelevant to our aims). Since $\alpha_{n;r} \rightarrow (2r-1)!!$ as $\rho \rightarrow \infty$, we have $(\alpha_{n,(\ell+2)} - \alpha_{n,(\ell+1)}\alpha_{n,(m+1)} - 2\alpha_{n,(\ell+1)}\alpha_{n,m}) \rightarrow (\ell - m)(2\ell + 1)(2\ell - 1)!!(2m - 1)!!$. In particular, this quantity vanishes for $\ell = m$, thus making the first term in square brackets never contribute as $\rho \rightarrow \infty$. A little additional algebra yields

$$
\lim_{\rho \rightarrow \infty} \Xi(q;\alpha)(\rho;\lambda) = 4 \sum_{\ell=0}^{q-1} \sum_{m=0}^{q-1} \frac{\delta_{\ell,m}}{\ell! (m-\ell)!} \frac{2(2\ell-1)!!}{m!} \delta_{e_\ell,1} \delta_{e_m,1} \prod_{i \neq \ell,m} \delta_{e_i,0}.
$$

We notice that the r.h.s. of eq. (5.14) vanishes always for $e_0 \geq 2$, but not necessarily for $e_0 = 0$ or $e_0 = 1$.

As a second step, we work out $Q_d$. To this aim, we first need to evaluate $\Xi(q;\alpha)$. As already done in sect. 4, we make use of the Taylor series $(1 + x)^{-1} = \sum_{p=0}^{\infty} (-1)^p x^p$. From

$$
\alpha^{(v)} = \alpha^{(1)} \alpha^{(v-1)} \left[ 1 + \sum_{q=1}^{\infty} \frac{(-1)^q}{q!} \left( \frac{\lambda_n}{\rho} \right)^q \alpha_{n,q}^{(1)} \eta_q^{(v-1)} \right],
$$

it follows

$$
[\alpha^{(v)}]^{-1} = [\alpha^{(1)} \alpha^{(v-1)}]^{-1} \sum_{p=0}^{\infty} (-1)^p \sum_{\ell_1 \ldots \ell_p = 1} (\lambda_n/\rho)^{\ell_1 + \ldots + \ell_p} \alpha_{n,\ell_1}^{(1)} \ldots \alpha_{n,\ell_p}^{(1)} \eta_{\ell_1} \ldots \eta_{\ell_p}.
$$

On the second line we have reduced the upper limit of the sum over $p$ from $\infty$ to $q$. The reason is that $\delta_{q,\ell_1 + \ldots + \ell_p} = 0$ for $q < p$, owing to $\ell_1 + \ldots + \ell_p \geq p$. On expanding the sums over $\ell_1, \ldots, \ell_p$, we see that all terms proportional to $\eta_1 \ldots \eta_q$ for some $\pi$ coincide. Since permutations of $\ell_1, \ldots, \ell_p$ corresponding to the same $\pi$ give all the same contribution, the latter turns out to be multiplied by an overall numerical factor which is at most $p!$. Of course, permutations of equal indices contribute only once. Therefore, a correct counting of that factor amounts to the multinomial coefficient $p!/([e_1]! \ldots [e_q]!)$, with the constraint $\sum e_j = p$.

In other words, we have

$$
[\alpha^{(v)}]^{-1} = [\alpha^{(1)} \alpha^{(v-1)}]^{-1} \sum_{q=0}^{\infty} (-1)^q \left( \frac{\lambda_n}{\rho} \right)^q \sum_{\ell_1 \ldots \ell_p = 1} (\lambda_n/\rho)^{\ell_1 + \ldots + \ell_p} \alpha_{n,\ell_1}^{(1)} \ldots \alpha_{n,\ell_p}^{(1)} \eta_1 \ldots \eta_q
$$

$$
\left( \sum_{e_k = 0}^{\infty} \frac{1}{k!} \frac{\alpha^{(1)}}{\alpha^{(1)}} \right)^{e_j}
$$

and consequently

$$
\Xi(q;\alpha) = (-1)^q \sum_{j=1}^{p} \alpha_{n,j}^{(1)} \left( \sum_{e_k = 0}^{\infty} \frac{1}{k!} \frac{\alpha^{(1)}}{\alpha^{(1)}} \right)^{e_j}
$$

$$
[\alpha^{(1)} \alpha^{(v-1)}]^{-1}.
$$

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Now, we obtain $\Xi^{(q,\omega)}_{\Delta_n}$ by convolving eq. (5.18) with itself. This yields

$$
\Xi^{(q,\omega)}_{\Delta_n} = \left( -1 \right)^{\sum_{k=1}^{q} e_k} \delta_{e_0,0} \prod_{j=1}^{q} \left( \frac{1}{\alpha^{(1)}_j} \right)^{e_j} \Psi(q,\pi) \right) \left[ \alpha^{(1)(e-1)} - 2 \right],
$$

(5.19)

with the coefficient $\Psi(p,\pi)$ defined by

$$
\Psi(p,\pi) = \sum_{\ell,m=0}^{p} \delta_{p,\ell+m} \sum_{\tau \in S^\ell \pi \in S^m} \delta_{e_\tau + \pi} \left( \sum_{k=1}^{\ell} c_k \right) \left( \sum_{k=1}^{m} d_k \right), \quad |\tau| = q.
$$

(5.20)

From eq. (5.19), it follows

$$
\lim_{\rho \to \infty} \Xi^{(q,\omega)}_{\Delta_n} = \left( -1 \right)^{\sum_{k=1}^{q} e_k} \delta_{e_0,0} \prod_{j=1}^{q} \left( \frac{(2j-1)!!}{j!} \right)^{e_j} \Psi(q,\pi).
$$

(5.21)

As a third step, we convolve eqs. (5.14) and (5.21). In this way we obtain $\Xi^{(q,\omega)}_{\Delta_n}$ directly in the limit $\rho \to \infty$. The algebra is just a little bit intricate, so we detail it. First of all,

$$
\lim_{\rho \to \infty} \Xi^{(q,\omega)}_{\Delta_n} = 4 \sum_{\ell,m=0}^{q} \delta_{q,\ell+m} \sum_{\tau \in S^\ell \pi \in S^m} \delta_{e_\tau + \pi} \left( \sum_{k=1}^{r} \sum_{l=0}^{s} \delta_{n,\tau+s}(r-s)^2 \frac{(2r-1)!!}{r!} \frac{(2s-1)!!}{s!} \right) \cdot \delta_{e_r,1} \delta_{e_s,1} \prod_{i \neq r,s} \delta_{e_i,0} \cdot \left( -1 \right)^{\sum_{k=1}^{q} k} \prod_{k=1}^{m} \left( \frac{(2k-1)!!}{k!} \right)^{d_k} \Psi(m,\pi) \right).
$$

(5.22)

Owing to the Kronecker symbols, we pay no price if we introduce a factor of $(-1)^{\sum_{k=0}^{q} c_k} = 1$ within square brackets. For the same reason, we can also insert additional factors of $[(2k-1)!!/k!] c_k = 1$ for $k \neq r, s$ without pay. Hence,

$$
\lim_{\rho \to \infty} \Xi^{(q,\omega)}_{\Delta_n} = 4\left(-1\right)^{\sum_{k=1}^{q} e_k} \prod_{k=1}^{q} \left( \frac{(2k-1)!!}{k!} \right)^{e_k} \sum_{\ell,m=0}^{q} \delta_{q,\ell+m} \sum_{\tau \in S^\ell \pi \in S^m} \left( -1 \right)^{c_0} \delta_{e_0,0} \delta_{e_\tau + \pi} \sum_{r=0}^{\ell} \sum_{s=0}^{m} \delta_{\ell,\tau+s}(r-s)^2 \delta_{e_r,1} \delta_{e_s,1} \prod_{i \neq r,s} \delta_{e_i,0} \Psi(m,\pi) \theta_{e,\pi}
$$

(5.23)

with $\theta_{e,\pi} = \prod_{i=1}^{q} \theta_{e_i,0}$ being a vector generalization of the Heaviside function

$$
\theta_{a,b} = \begin{cases} 1 & \text{if } a \geq b, \\ 0 & \text{otherwise}. \end{cases}
$$

(5.24)

From eq. (5.23) it follows

$$
\lim_{\rho \to \infty} \sum_{\omega=0}^{\infty} \Xi^{(q,\omega)}_{\Delta_n} = 4\left(-1\right)^{\sum_{k=1}^{q} e_k} \delta_{P_q(\omega),q} \prod_{k=1}^{q} \left( \frac{(2k-1)!!}{k!} \right)^{e_k} \left\{ \Omega^{(q,\pi)} - \Omega^{(q,\pi)} \right\},
$$

(5.25)
In order to complete the proof, we need to show that \( \Omega^{(r,s)}_0 < \Omega^{(r,s)}_1 \) for all \( r, s \). To this aim, we find it convenient to use a slightly different representation of \( \Psi^{(r,s)} \), viz.

\[
\Psi^{(r,s)} = \delta_{P_s(r)} \sum_{m=0}^{p} \sum_{n=0}^{q} \left( \sum_{k=1}^{q} c_k \right) \left( \sum_{k=1}^{q} (e_k - c_k) \right) \theta_{e_n, e_m+1} \prod_{s \neq n}^{1} \theta_{e_s, e_n}.
\]

Suppose that \( c_\ell \geq 1 \) for some \( 1 \leq \ell \leq q \). Then,

\[
\theta_{e_\ell, 1} \Psi^{(r,s)} = \delta_{P_s(r)} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{\ell=1}^{m} \sum_{\ell=1}^{n} \left( \sum_{k=1}^{q} c_k \right) \left( \sum_{k=1}^{q} (e_k - c_k) \right) \theta_{e_n, e_m+1} \prod_{s \neq n}^{1} \theta_{e_s, e_n}.
\]

On performing the change of variable \( e_\ell \rightarrow e_\ell + 1 \), we easily arrive at

\[
\theta_{e_\ell, 1} \Psi^{(r,s)} = \delta_{P_s(r)} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{\ell=1}^{m} \sum_{\ell=1}^{n} \left( \sum_{k=1}^{q} c_k \right) \left( \sum_{k=1}^{q} (e_k - c_k) \right) \theta_{e_n, e_m+1} \prod_{s \neq n}^{1} \theta_{e_s, e_n}.
\]

Analogously, for \( e_\ell \geq 1 \), \( e_r \geq 1 \) and \( 1 \leq s < r < q \), we have

\[
\theta_{e_s, 1} \theta_{e_r, 1} \Psi^{(r,s)} = \delta_{P_s(r)} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{\ell=1}^{m} \sum_{\ell=1}^{n} \left( \sum_{k=1}^{q} c_k \right) \left( \sum_{k=1}^{q} (e_k - c_k) \right) \theta_{e_n, e_m+1} \prod_{s \neq n}^{1} \theta_{e_s, e_n}.
\]

Note that the lower limit of the sum over \( t \) in eq. (5.30) is not fixed at \( \ell \), since \( c_\ell = 0 \) if \( \ell \in S_q \) and \( t < \ell \). The same cannot be done in eq. (5.31) without increasing the resulting sum. Therefore,

\[
\Omega^{(r,s)}_1 = \delta_{P_s(r)} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{\ell=1}^{m} \sum_{\ell=1}^{n} \left( \sum_{k=1}^{q} c_k \right) \left( \sum_{k=1}^{q} (e_k - c_k) \right) \theta_{e_n, e_m+1} \prod_{s \neq n}^{1} \theta_{e_s, e_n},
\]

and

\[
\Omega^{(r,s)}_0 \leq \delta_{P_s(r)} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{\ell=1}^{m} \sum_{\ell=1}^{n} \sum_{s=1}^{r-1} \sum_{s=1}^{r-1} \delta_{e_s, e_r} (r-s)^2 c_s c_r \left( \sum_{k=1}^{q} c_k \right) \left( \sum_{k=1}^{q} (e_k - c_k) \right) \theta_{e_n, e_m+1} \prod_{s \neq n}^{1} \theta_{e_s, e_n}.
\]

Now, it is immediate to prove that

\[
\sum_{\ell=1}^{q} \sum_{r=1}^{q} \sum_{s=1}^{q} \delta_{e_s, e_r} (r-s)^2 c_s c_r < \sum_{\ell=1}^{q} \sum_{k=1}^{q} c_k \left( \sum_{k=1}^{q} (e_k - c_k) \right), \quad \forall \ell, r, s \in S_q.
\]

Indeed, given \( \ell, r, s \in S_q \) each non-vanishing contribution \( c_s c_r > 0 \) with \( s < r \) is weighted by \( (r-s)^2 \) on the l.h.s. and by \( (r^2 + s^2) \) on the r.h.s. The remaining terms on the r.h.s. are \( \sum_{\ell=1}^{q} \sum_{k=1}^{q} c_k (c_k - 1) \geq 0 \) and \( q^2 \sum_{k=1}^{q} c_k (c_k - 1) \geq 0 \). This concludes the proof. \( \square \)
6 Concluding remarks

Conditioning a vector $X \sim N_v(0, \Lambda)$ with $\Lambda = \text{diag}(\lambda)$ to a centered Euclidean ball $B_v(\rho)$ of square radius $\rho$ affects non-trivially the covariance matrix of its square components. Since the conditional moments of $X$ cannot be calculated in closed-form, the only viable approach (besides numerical computation) to characterizing the truncational effects consists in establishing analytic bounds to the conditional correlations (variances and covariances) of the square components of $X$. Such estimates are also referred to in the literature as square correlation inequalities.

In this paper, we specifically focused on the conditional variances. In particular, our aim was proving eq. (1.2). The analyses presented in the previous sections go in this direction, yet they do not solve the problem in a conclusive way. The arguments proposed apply in the opposite regimes of strong and weak truncations. For $0 < \rho < 2\lambda_n$, eq. (1.2) is easily proved. A bigger effort is required for $\rho \gg \lambda_n$. Nothing is said regarding the intermediate region. We conclude with two major criticisms, representing at the same time an outlook of future research:

- the weak truncation region is not sharply defined: the asymptotic property stated by Theorem 5.1 is certainly sufficient to prove that the $p^{th}$ order of the weak truncation expansion of $\Delta_n$ is negative at $\rho > \rho^*_p$, but the theorem does not provide any estimate of $\rho^*_p$. A better characterization of the coefficient functions $\eta_k$ and $\Xi_{\Delta_n}$ far from the asymptotic regime would help identify precise conditions to extend the proof of eq. (1.2) to large yet finite values of $\rho$ along the same lines of Theorem 5.1;

- we also lack a general proof of convergence of the weak truncation expansion. The argument presented in sect. 4 suggests uniform convergence in $v \leq 5$ dimensions, but it is based on a numerical estimate of the vanishing rate of the $p^{th}$ term of the expansion, which cannot be legitimately extrapolated to $p \to \infty$.

The weak truncation expansion of a given observable $f$ (built from Gaussian integrals $\alpha_{k\ell m...}$) is to all extents a perturbative expansion around the factorized value $f$ takes as $\rho \to \infty$. As such, it is affected by the usual problems encountered with perturbative expansions. Having proved a property of $\Delta_n$ to all orders represents the main (non-trivial) contribution of the present paper.

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