A Combinatorial Game and an Efficiently Computable Shift Rule for the Prefer Max De Bruijn Sequence

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Abstract

We present a two-player combinatorial game over a \( k \)-ary shift-register and analyze it. The game is defined such that, for each of the player, the only way to avoid losing is to play such that the next state of the game is the next window in the well known prefer-max De Bruijn sequence. We then proceed to solve the game, i.e., to propose efficient algorithms that compute the moves for each player. Finally, we show how these algorithms can be combined into an efficiently computable shift-rule for the prefer-max sequence.

1. Introduction

Combinatorial game theory is not only an interesting theory on its own, it links to many other fields of mathematics. For example, Conway and Sloane showed in [1] that the “losing positions” of some combinatorial games give linear error detecting and correcting codes. Using this view of these codes as combinatorial games, Fraenkel and Rahat showed in [2] that the codes can be computed in polynomial time and memory.

In this paper we demonstrate a similar application of combinatorial game theory to coding: we show that the optimal play (when both players apply their only non-losing strategies) of a certain combinatorial game constitutes the well known prefer-max De Bruijn sequence (defined below). Then, parallel to the work of Fraenkel and Rahat in [2], we also show how non-losing strategies can be computed in linear time and memory, yielding an efficiently computable shift-rule (mapping a state of a shift register to its follower) for the sequence.

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We use lower-case Greek letters to denote symbols in the alphabet \( [k] = \{0, \ldots, k - 1\} \) and lower-case Latin letters to denote words in \([k]^*\), except for \(n\) and \(k\) which represent natural numbers and indexes.

2. The pref-max De Bruijn sequence and the corresponding Hamiltonian cycle in the De Bruijn graph

A De Bruijn sequence of order \(n\) on the alphabet \([k] = \{0, \ldots, k - 1\}\) is a cyclic sequence of length \(k^n\) such that every possible string of length \(n\) (member of \([k]^n\)) appears exactly once as a substring [3].

The directed De Bruijn graph of order \(n\) over the alphabet \([k]\) (also described in [3]) is the graph whose vertexes are the strings of length \(n\) over the alphabet \([k]\) (i.e. the set \([k]^n\)) and whose edges are such that each vertex \(v = x\sigma\) is connected with a directed edge to all the vertexes in \(\{\tau x : \tau \in [k]\}\).

There is a one-to-one correspondence between De Bruijn sequences and Hamiltonian cycles in the De Bruijn graph of the same order and alphabet, described in [3] and it is as follows:

1. If, for each \(i\), \(w_i = x_i\sigma_i\), and \((w_1, w_2, \ldots)\) is an Hamiltonian cycle then \((\sigma_1, \sigma_2, \ldots)\) is a De Bruijn sequence.
2. A Hamiltonian cycle can be constructed from a De Bruijn sequence \((\sigma_1, \ldots, \sigma_{k^n})\) by visiting the vertex \(\sigma_1 \cdots \sigma_n\), then \(\sigma_2 \cdots \sigma_{n+1}\) and so on, until we return to where we started.

In this paper we focus on a specific Hamiltonian cycle in the De Bruijn graph called the prefer-max cycle (and the corresponding De Bruijn sequence) defined as follows. For simplicity, we only list the vertexes on the cycle, as the edges are induced.

**Definition 1.** The \((k, n)\)-prefer-max cycle, \((w_i)_{i=0}^{k^n-1}\), is such that \(w_0 = 0^{n-1}(k - 1)\) and if \(w_i = \sigma x\) then \(w_{i+1} = x\tau\) where \(\tau\) is the maximal letter such that \(x\tau \notin \{w_0, \ldots, w_i\}\).

**Example 2.** For example, if we set \(n = 3\) and \(k = 3\), we have: 002 \(\rightarrow\) 022 \(\rightarrow\) 222 \(\rightarrow\) 221 \(\rightarrow\) 212 \(\rightarrow\) 122 \(\rightarrow\) 220 \(\rightarrow\) 202 \(\rightarrow\) 021 \(\rightarrow\) 211 \(\rightarrow\) 112 \(\rightarrow\) 121 \(\rightarrow\) 210 \(\rightarrow\) 102 \(\rightarrow\) 020 \(\rightarrow\) 201 \(\rightarrow\) 012 \(\rightarrow\) 120 \(\rightarrow\) 200 \(\rightarrow\) 001 \(\rightarrow\) 011 \(\rightarrow\) 111 \(\rightarrow\) 110 \(\rightarrow\) 101 \(\rightarrow\) 010 \(\rightarrow\) 100 \(\rightarrow\) 000.
Martin [4] proved that the cycle given in Definition 1 is Hamiltonian, i.e., that for \(k^n - 1\) steps there is always a \(\tau\) such that \(w\tau \notin \{w_0, \ldots, w_i\}\), so \(w_{i+1}\) is well defined, as demonstrated with Example 2.

The Hamiltonian cycle \((w_i)_{i=0}^{k^n-1}\), given in Definition 1, corresponds to the well known prefer-max De Bruijn sequence (defined, e.g., in [5]). One challenge raised in the literature (see, e.g., [6]) for such sequences is finding an efficiently computable function that maps each vertex on the cycle to its follower, called a shift-rule. We will arrive to such a rule at the end of the paper.

3. Two properties of the prefer-max cycle

We first state and prove two properties of the prefer-max cycle. These properties will be used later to prove properties of the game that we are going to propose and for its analysis in the following sections.

For the following propositions, let \(\prec\) be the order induced by the sequence given in Definition 1, i.e., \(w_i \prec w_j\) if \(i < j\) where \((w_i)_{i=0}^{k^n-1}\) is as given in Definition 1. This is a linear order over \([k]^n\) because the sequence represents a Hamiltonian cycle of the De Bruijn graph.

**Proposition 3.** For any \(x, y\) such that \(|x| + |y| = n - 1\), if \(0 < \sigma_1 < \sigma_2\) then \(x\sigma_2y \prec x\sigma_1y\).

**Proof.** By induction on the length of \(y\). If \(y\) is empty, the statement is true from the definition of the prefer-max cycle (Definition 1). For the induction step, assume that the statement is true for all \(y\) of some constant length \(t < n\). We need to show that \(x\sigma_2y\tau \prec x\sigma_1y\tau\) for any \(\tau\) and any \(x\) of length \(n - t - 2\). Since \(x\sigma_1y(k-1) \prec x\sigma_1y(k-2) \prec \cdots \prec x\sigma_1y0\) and because the predecessor on the cycle of each of these vertices is a node in \(\{\varphi x\sigma_1y: 0 < \varphi < k\}\) we have that at least \(k - \tau\) vertices in this set precede \(x\sigma_1y\tau\). The term ‘predecessor’ is well defined here because \(\sigma_1 \neq 0\). It assures us that \(x\sigma_1y(k-1)\) is not the first vertex on the cycle. By the induction hypothesis, we have that \(\varphi x\sigma_2y \prec \varphi x\sigma_1y\), for any \(\varphi\). Therefore, there are at least \(k - \tau\) vertices in \(\{\varphi x\sigma_2y: 0 < \varphi < k\}\) before \(x\sigma_1y\tau\). Since the follower of each of these vertices is in \(\{x\sigma_2y(k-1), x\sigma_2y(k-2), \ldots, x\sigma_2y0\}\) and because \(x\sigma_2y(k-1) \prec x\sigma_2y(k-2) \prec \cdots \prec x\sigma_2y0\), we get that \(x\sigma_2y\tau\) must be before \(x\sigma_1y\tau\).

**Example 4.** Consider again the sequence given in Example 2. If we take, for example, \(x = 0\) and \(y = 2\), we see \(022 \prec 012\). If we take, as another example \(x = 2\) and \(y = 2\), we see that \(222 \prec 212\). More generally we see that \(x2y \prec x1y\)
for any $x$ and $y$ whose combine length is two. Note that the proposition does not say where does $x0y$ fit, it may come before $x2y$, after $x1y$, or between the two.

**Proposition 5.** Considering $(w_i)^{k^n-1}_{i=0}$ from Definition[7] For each $0 \leq i < k^n - 1$, exactly one of the following is true for some $x \in [k]^{n-1}$ and $\sigma \in [k]$: 

- $(w_i, w_{i+1}) = (\sigma x, x\sigma)$ and $\sigma x < 0x$;
- $(w_i, w_{i+1}) = (0x, x\sigma)$ and $(\sigma + 1)x < 0x < \sigma x$;
- $(w_i, w_{i+1}) = ((\sigma + 1)x, x\sigma)$ and $0x < (\sigma + 1)x$.

**Proof.** Let $x$ be a word of length $n - 1$ over $[k]$. From Proposition[3] the subsequence $\{(k-1)x, \ldots, 1x\}$ appears on the cycle in a decreasing order. By Definition[1] the subsequence $\{x(k-1), \ldots, x0\}$ also appears in a decreasing order. Since the predecessor of each node in the second list is either $0x$ or a node from the first list, we get that if $\tau x < 0y$ then the follower of $\tau x$ must be $x\tau$ and that if $0x < \tau x$ then the follower of $\tau x$ must be $x(\tau - 1)$. See an illustration of the proof in Figure[1].

**Example 6.** Consider the sequence given in Example[2] again. If we take, for example, $x = 22$, we see that the predecessor of 222 is 022, the predecessor of 221 is 222, and the predecessor of 220 is 122. We see that, in this case, the leftmost digit of the predecessor is either zero or an increase by one of the rightmost digit of its successor. In this case, the node comes before the beginning of the first list considered in the proof, corresponding to having $d = 0$ in Figure[1].

4. A combinatorial game

Our next step is a proposal of a combinatorial game that will, after some analysis, be used to derive an efficient computation of the predecessor in the prefer-max sequence. In light of Proposition[5] we ask ourselves: Given $x\sigma$, when do we need to increase, keep as is, or take a zero instead of $\sigma$ when computing the predecessor of $x\sigma$. We propose an indirect answer: A combinatorial game whose rules correspond to these options. One player, called Bob, can force an increase step while the other player, called Alice, can force a zero step when Bob does not want to increase. Alice’s goal is to continue as far as possible without repeating a state and Bob’s goal is to get to a repeated state as fast as he can. As we will prove later, the game is constructed such that best option for both player is to step along
Initially, the two lists go together $d$ times for some $0 \leq d \leq k$.

When $0x$ enters, the second list advances and the first does not.

After that, the second list is one step ahead of the first list for the remaining $k - d - 1$ times.

Figure 1: An illustration of the proof of Proposition 5.
the sequence, i.e., that the optimal strategies for both players are equivalent to computing the predecessors of \((w_i)_{i=0}^{k^n-1}\) from Definition 1.

The following definition states the rules and the objectives of the game:

**Definition 7.** The \((n,k)\)-shift-game is a two-player combinatorial game played by Alice and Bob as follows: A play of the game consists of a sequence \(s_0, \ldots, s_m\) such that \(s_0 = 0^n\) and if \(s_t = x\sigma\) then:

\[
s_{t+1} = \begin{cases} 
(\sigma + 1)x & \text{if } B(s_t) = 1, \\
0x & \text{if } B(s_t) = 0 \text{ and } A(s_t) = 1, \\
\sigma x & \text{otherwise};
\end{cases}
\]

where \(A, B : [k]^n \to \{0, 1\}\), are functions, called strategies for Bob and for Alice, respectively, such that \(B(x(k-1)) = A(x0) = 0\) for all \(x \in [k]^{n-1}\). The game ends at state \(s_m\) if \(m > 0\) and \(s_m, s_{m-1} \in \{s_0, \ldots, s_{m-1}\}\). Alice wins if \(m < k^n\) and \(s_m = 0^n\). Bob wins if \(s_m \neq 0^n\). If \(m = k^n\) and \(s_m = 0^n\) its a tie.

Note that this definition introduces the notion of strategies for the players. I.e., that a function \(S : [k]^n \to \{0, 1\}\) is a strategy for Bob if \(S(x(k-1)) = 0\) for all \(x \in [k]^{n-1}\) and it is a strategy for Alice if \(S(x0) = 0\) for all \(x \in [k]^{n-1}\). For future use, we note the following observation:

**Observation 8.** If \(S\) is a strategy for Alice or for Bob and \(S'(w) < S(w)\) for all \(w\), then \(S'\) is also a strategy for the same player.

**Example 9.** A play \((s_0, \ldots, s_{10})\) of the \((2,4)\)-shift-game can be such that \(s_0 = 00, s_1 = 10, s_2 = 01, s_3 = 20, s_4 = 12, s_5 = 21, s_6 = 22, s_7 = 32, s_8 = 23, s_9 = 02, s_{10} = 00\). In this example, Bob plays on moves \(\{00, 20, 12, 22, 32\}\), i.e., \(B(00) = B(20) = B(12) = B(22) = B(32) = 1\), Alice plays on \(\{23, 02\}\), i.e., \(A(23) = A(02) = 1\), and neither play on \(\{10, 12\}\). In this example, Alice won because the play reached 00 in \(m = 10\) moves, which is less than \(4^2 = 16\) moves.

This is a deterministic game with perfect information but it is not impartial (i.e., it is a partisan game), as there are differences, beyond who goes first, between Alice’s and Bob’s goals and moves. See Figure 2 for an illustration of how a round of the game is executed.

The \((n,k)\)-shift-game proposed in Definition 7 is a generalization of a game proposed in [7] for solving a problem in control-theory. There, the game was only
Figure 2: A flowchart depicting a round of the game. Bob has a priority: if $B(s_t) = 1$ then $s_{t+1}$ is $(\sigma + 1)x$ (an increase and shift), no matter how Alice plays. The value $A(s_t)$ is only considered when $B(s_t) = 0$. In this case it determines whether $s_{t+1}$ is $0x$ (a reset and shift) or $s_{t+1}$ is $\sigma x$ (just a shift), corresponding to $A(s_t) = 1$ and to $A(s_t) = 0$, respectively.

for the binary case ($k = 2$) in which case the prefer-max sequence is also called ‘prefer-one’. In terms of Definition\[7\] the binary game was

$$s_{t+1} = \begin{cases} 1x & \text{if } \sigma = 0 \text{ and } B(s_t) = 1, \\ 0x & \text{if } \sigma = 1 \text{ and } A(s_t) = 1, \\ \sigma x & \text{otherwise.} \end{cases}$$

It is easy to verify that it is equivalent to the $(n, 2)$-shift-game. Indeed, the game proposed in \[7\] and the strategies proposed there to solve it can generate a known shift-rule for the prefer-one sequence as shown in \[8\] and in \[9\].

Before we move to solving the $(n, k)$-shift-game, we state and prove a complexity property for it. There has been several criteria in the literature for classifying complexities of De Bruijn sequences, e.g., \[10\]. One such criterion, for binary sequences, is the number of ones in the truth table of the feedback function that generates the sequence. For De Bruijn sequences generated by combinatorial games, this can be translated, e.g., to counting the number of moves that Bob plays, i.e., the number of ones in the truth table of Bob’s strategy. A result that goes along these lines is given in the next proposition:

**Proposition 10.** For every play, $(s_t)_{t=0}^m$, of the $(n, k)$-shift game, let $b = |\{s_t : B(s_t) =$
be the number of times Bob plays and $a = \left| \{ s_t : B(s_t) = 0 \land A(s_t) = 1 \} \right|$ be the number of times Alice plays. Then, $b \leq (n + a)(k - 1)$.

**Proof.** Let $E(\sigma_0, \ldots, \sigma_{n-1}) = \sum_{i=0}^{n-1} \sigma_i$. From Definition[7] we get that $E(s_0) = 0$ and that if $s_t = x\sigma$ then

$$E(s_{t+1}) = \begin{cases} E(s_t) + 1 & \text{if } B(s_t) = 1, \\ E(s_t) - \sigma & \text{if } B(s_t) = 0 \text{ and } A(s_t) = 1, \\ E(s_t) & \text{otherwise.} \end{cases}$$

Since $\sigma \leq k - 1$, then $E(s_m) \geq b - a(k - 1)$, and because $s_i \in [k]^n$ for all $i$, then $E(s_m) \leq (k - 1)n$. So $(k - 1)n + 1 \geq b - a(k - 1)$ and the required result follows from this inequality. \qed

**Example 11.** If we look back at Example[9] we can count and see that $b = 5$ and $a = 2$ and verify that $5 \leq (5 \cdot 2) = 10$.

Next, we turn to establishing a connection between the prefer-max and the shift-game. We will first define a pair of strategies $A^*$ and $B^*$ such that if both Alice and Bob use the respective strategy, the play of the game follows the prefer-max sequence (in reverse). Then, we will show that $A^*$ and $B^*$ are the unique strategies for each of the players, respectively, that guarantee not losing the game. This will give us that the efficient implementations of non-losing strategies for both of the players, that we will provide in Section 5, can serve as an efficient shift-rule for the sequence.

The strategies $A^*$ and $B^*$ use the prefer-max sequence as an internal ‘oracle’, as specified in the following definition:

**Definition 12.** Considering $(w_i)_{i=0}^{k^n-1}$ from Definition[1] let $A^*, B^* : [k]^n \rightarrow \{0, 1\}$ be the strategies for Alice and Bob, respectively, defined by

$$B^*(w_i) = \begin{cases} 1 & \text{if } w_i = x\sigma \text{ and } w_{i-1} = (\sigma + 1)x, \\ 0 & \text{otherwise}; \end{cases}$$

and

$$A^*(w_i) = \begin{cases} 1 & \text{if } (w_i = x\sigma \land w_{i-1} = 0x) \lor B^*(w_i) = 1, \\ 0 & \text{otherwise.} \end{cases}$$
From Definition 1, we can see that $A^*$ is a strategy, i.e., $A^*(x_0) = 0$ for all $x \in [k]^{n-1}$, and the next proposition will show that $B^*$ is also a strategy, i.e., $B^*(x(k-1)) = 0$ for all $x \in [k]^{n-1}$. The next proposition also shows that the computation of $B^*$ can be reduced to a computation of $A^*$ over a slightly alternated input. We will use this fact to simplify the analysis and to allow a succinct implementation of the strategies.

**Proposition 13.** For every $x \in [k]^{n-1}$, $B^*(x(k-1)) = 0$ and, for every $\sigma < k-1$, $B^*(x\sigma) = A^*(x(\sigma + 1))$.

**Proof.** If $w_i = x(k-1)$ then $w_{i+1}$ cannot be $kx$ because this word is not in $[k]^n$, thus $B^*(w_i)$ must be zero. From Proposition 3 and Proposition 5, we can see that if we define $zs: [k]^{n-1} \rightarrow [k]$ by $zs(x) = |\{i : ix < 0x\}| + 1$, then for every $x \in [k]^{n-1}$, and $w_i = x\sigma$, if $\sigma < zs(x)$ then $w_{i+1} = (\sigma + 1)x$, if $\sigma = zs(x)$ then $w_{i+1} = 0x$, and if $\sigma > zs(x)$ then $w_{i+1} = \sigma x$. Therefore, $(B^*(x\sigma) = 1) \iff (\sigma < zs(x))$ and $(A^*(x\sigma) = 1) \iff (\sigma \leq zs(x))$. This means that $B^*(x\sigma) = 1$ if and only if $A^*(x(\sigma + 1)) = 1$. \hfill \Box

**Example 14.** For example, for $n = 3$ and $k = 2$, if both players play according to the strategies above, then the resulting play is $000 \rightarrow 100 \rightarrow 010 \rightarrow 101 \rightarrow 110 \rightarrow 111 \rightarrow 011 \rightarrow 001 \rightarrow 000$ which yields a tie.

We can see that in the example above, the play is exactly the prefer-max cycle in reverse. The following proposition establishes that this is true for all $n$ and $k$:

**Proposition 15.** Let $(s_t)_{t=0}^{k^n-1}$ be the play of the $(n,k)$-shift-game when Alice uses the $A^*$ strategy and Bob uses the $B^*$ strategy. Then, $(s_t)_{t=0}^{k^n-1} = (w_{k^n-t})_{t=1}^{k^n}$ where $(w_i)_{i=0}^{k^n-1}$ is the prefer-max sequence given in Definition 7.

**Proof.** From Proposition 5 we know that if $w_{i+1} = x\sigma$ then either $w_i = \sigma x$, $w_i = 0x$ or $w_i = (\sigma + 1)x$. We also have that $(w_i)_{i=0}^{k^n-1}$ contains all of the elements in $[k]^n$, because of its relation to the prefer-max De Bruijn sequence. Thus, by Definition 7, every step in the play of the game must follow a (reversed) step in $(w_i)_{i=0}^{k^n-1}$ and the proof follows by induction. \hfill \Box

From Proposition 15 we have that if Alice plays according to $A^*$  and if Bob plays according to $B^*$ the game ends with a tie. In the next two propositions, we show that each of them wins against any other strategy. Then, in the two propositions that follow, we show that these strategies are the unique strategies with these properties.
Proposition 16. If Bob applies the strategy $B^*$ he wins against any strategy that Alice may apply which is not $A^*$ and achieves a tie against $A^*$.

Proof. First, from Proposition 15 we know that if the players play by the strategies $A^*$ and $B^*$ respectively we get a tie. Now, assume towards contradiction that there exists a strategy $A$ for Alice which wins against $B^*$. Then, there must be a state $s_t = x\sigma$ where $x \neq 0^{n-1}$ such that $A(s_t) \neq A^*(s_t)$, $B^*(s_t) = 0$ and $\sigma \neq 0$ because otherwise the play is equal to the play with $A^*$ and $B^*$ which is, as said, a tie. Let $s_t$ be the first such state. By Proposition 5 and the Definition 12 of $A^*$, we have that if $A^*(s_t) = 1$ then $\sigma x < x\sigma$, and if $A^*(s_t) = 0$ then $0x < x\sigma$. If $A^*(s_t) = 1$ then $A(s_t) = 0$ and $s_{t+1} = \sigma x$. If $A^*(s_t) = 0$ then $A(s_t) = 0$ and $s_{t+1} = 0x$. In both cases $s_{t+1} \in \{s_0, \ldots, s_t\}$ and $s_{t+1} \neq s_0$ which is in contradiction with the assumption that $A$ is a winning strategy.

Proposition 17. If Alice applies the strategy $A^*$ she wins against any strategy that Bob may apply which is not $B^*$ and achieves a tie against $B^*$.

Proof. Assume towards contradiction that there exists a strategy $B$ for Bob which wins against $A^*$ and let $s_0, \ldots, s_m$ be the play of the game with these two strategies. Consider the function $g : [k]^n \rightarrow \mathbb{N}$ such that $g(w_i) = i$ for all $(w_i)_{i=0}^{k^n-1}$ in the order specified in Definition 1. By the definition of $A^*$ and of $B^*$ (Definition 12), if $B(s_t) = B^*(s_t)$ we have that $g(s_{t+1}) = g(s_t) - 1$. Otherwise, if $B(s_t) = 0$ and $B^*(s_t) = 1$ we have that $A^*(s_t) = 1$ thus $w_{i-1} = (\sigma + 1)x$ and $s_{t+1} = 0x$, so, by Proposition 5 we have that $s_{t+1} \prec s_t$ which gives us that $g(s_{t+1}) < g(s_t)$. If $B(s_t) = 1$ and $B^*(s_t) = 0$ we have to check two cases:

1. If $A^*(s_t) = 1$ then $w_{i-1} = 0x$, $\sigma \neq 0$ and $s_{t+1} = (\sigma + 1)x$ so, by Proposition 5 we have that $s_{t+1} \prec s_t$ which gives us that $g(s_{t+1}) < g(s_t)$.

2. If $A^*(s_t) = 0$ then $w_{i-1} = \sigma x$ and $s_{t+1} = (\sigma + 1)x$ so, by Proposition 5 we have that $s_{t+1} \prec s_t$ which gives us that $g(s_{t+1}) < g(s_t)$.

Thus, by this enumeration of all the three cases, we see that $g$ is a strictly decreasing along the play $s_0, \ldots, s_m$. Since the minimum of $g$ is attained at $s = 0^n$, we get that $s_m = 0^n$ in contradiction to the assumption that Bob wins.

The following two propositions establish the uniqueness of both $B^*$ and $A^*$. Since it is easy to generate the prefer-max sequence from an implementation of these strategies, these results establish a polynomial reduction from computing an efficient shift-rule for the prefer-max sequence to efficient computation of non-losing strategies for both players, a problem that we will solve in the next section.
Proposition 18. $B^*$ is the only non-losing strategy for Bob.

Proof. Assume towards contradiction that there exists a non-losing strategy $B$ for Bob such that $B \neq B^*$. Let $s_0, \ldots, s_m$ be the play of the game where Bob applies $B$ and Alice applies $A^*$. Let $g$ be as in Proposition 17. For $t > 0$, assuming that $s_t = \sigma x$, we have two options:

1. If $B(s_t) = 0$ and $B^*(s_t) = 1$ we have, by the definition of $A^*$, that $A^*(s_t) = 1$. By the definition of $B^*$ then $w_{i-1} = (\sigma + 1)x$. By the definition of the game, we have that $s_{t+1} = 0x$. By Proposition 5, $s_{t+1} \prec s_t$, i.e., $g(s_{t+1}) < g(s_t)$.

2. If $B(s_t) = 1$ and $B^*(s_t) = 0$ we have two cases:
   (a) If $A^*(s_t) = 1$, by the definition of $A^*$, $w_{i-1} = 0x$ and $\sigma \neq 0$. By the definition of the game, $s_{t+1} = (\sigma + 1)x$. By Proposition 5, $s_{t+1} \prec s_t$, i.e., $g(s_{t+1}) < g(s_t)$.
   (b) If $A^*(s_t) = 0$, by the definition of $A^*$, $w_{i-1} = \sigma x$. By the definition of the game, $s_{t+1} = (\sigma + 1)x$. By Proposition 5, $s_{t+1} \prec s_t$, i.e., $g(s_{t+1}) < g(s_t)$.

We get that $g$ is strictly decreasing along the play $s_0, \ldots, s_m$. In particular, $g(s_m) \notin \{g(s_1), \ldots, g(s_{m-1})\}$ which means that $s_m \notin \{s_1, \ldots, s_{m-1}\}$. Since $B$ is assumed to be a non-losing strategy for Bob, therefore, by the definition of the ending condition of the game, we must have $s_m = s_0 = 0^n$.

Since $B \neq B^*$, there must be a $0 \leq t < m$ such that $g(t + 1) < g(t) - 1$. Therefore, the length of the game, $m$, must be smaller than $n^k$. By the definition of the game, Bob loses, in contradiction to the assumption that $B$ is a non-losing strategy for Bob. \hfill \Box

Proposition 19. $A^*$ is the only non-losing strategy for Alice.

Proof. Assume towards contradiction that there exists a non-losing strategy $A$ for Alice such that $A \neq A^*$. Let $s_0, \ldots, s_m$ be the play of game when Alice and Bob use the strategies $A$ and $B^*$, respectively. Let $t$ be the first index in this play such that $A(s_t) \neq A^*(s_t)$. Let $x$ and $\sigma$ be such that $s_t = x\sigma$. Note that $\sigma \neq 0$, as either $A(s_t) = 1$ or $A^*(s_t) = 1$ but not both.

By Proposition 5, since until $s_t$ Alice and Bob applied the strategies $A^*$ and $B^*$, respectively, we have that $(s_i)_{i=0}^t = (w_{k^n-i})_{i=1}^{t+1}$, i.e., the game follows the reversed prefer-max sequence until $s_t$.

We first show that $B^*(s_t)$ must be zero. Otherwise, assume towards contradiction that $B^*(s_t) = 1$. By the definition of $B^*$, $w_{k^n-t-2} = (\sigma + 1)x$. By Proposition 3, since $\sigma \neq 0$, then $(\sigma + 1)x < \sigma x$, i.e., $\sigma x \in \{w_{k^n-i})_{i=1}^{t+1} | s_i \}_{i=0}$. \hfill 11
Contradiction to the assumption that $A$ have

We have that $A$ is a non-losing strategy or the fact (Proposition 17) that $A^*$ is a non-losing strategy. Thus $B^*(s_t) = 0$.

If $A^*(s_t) = 1$ then, by the definition of $A^*$, $w_{k^n - t - 2} = 0x$ which, by Proposition 5 means that $x\sigma < \sigma x$ or $x\sigma = \sigma x$. But, since $A(s_t) \neq A^*(s_t)$, we have that $A(s_t) = 0$ so, by the definition of the game, since $B^*(s_t) = 0$, we have $s_{t+1} = \sigma x$ which is a state that we already visited, i.e., Bob wins. This is in contradiction to the assumption that $A$ is a non-losing strategy for Alice.

If $A^*(s_t) = 0$ then, by the definition of $A^*$ and of $B^*$, $w_{k^n - t - 2} = \sigma x$ which, by Proposition 5 means that $x\sigma < 0x$ or $x\sigma = 0x$. But, since $A(s_t) \neq A^*(s_t)$, we have that $A(s_t) = 1$ so, by the definition of the game, since $B^*(s_t) = 0$, we have $s_{t+1} = 0x$ which is a state that we already visited, i.e., Bob wins. This is in contradiction to the assumption that $A$ is a non-losing strategy for Alice.

5. An efficient computation of the non-losing strategies for both players

In this section we propose algorithms for efficient computation of the strategies for both players in the $(n,k)$-shift-game. As said before, this can also be translated to an efficiently computable shift rule for the prefer max sequence. The main ingredient in this is the function $val : [k]^n \rightarrow \mathbb{N}$ given in the following definition together with two helper functions, head and tail, that we will later use in the algorithms and in their analysis:

**Definition 20.** For a state $s = \sigma_0 \cdots \sigma_{n-1} \in [k]^n$ and $m \in [n]$, let:

\[
val(s, m) = \sum_{i=1}^{n} \sigma_{(m-i) \mod n} k^{i-1}
\]

be the value of $s$ read from position $m$.

\[
val(s) = \max\{\val(s, m) : 0 \leq m < n\}
\]

be the (maximal) value of $s$.

\[
head(s) = \min(\arg \max\{\val(s, m) : 0 \leq m < n\})
\]

be the head of $s$.

\[
tail(s) = (\max\{i < head(s) : \sigma_{i \mod n} \neq 0\}) \mod n
\]

be the tail of $s \neq 0^n$. 


The function $\text{val} : [k]^n \times [n] \to \mathbb{N}$ given in Definition 20 transforms a state $s$ to a number by reading its symbols as a number in base $k$ from the $m$th place to the left in a cyclic order. The function $\text{head} : [k]^n \to [n]$ gives the index $m$ of the place from which reading the value $\text{val}(s,m)$ is maximal. The function $\text{val} : [k]^n \to \mathbb{N}$ gives this maximal value, i.e., $\text{val}(s) = \text{val}(s,\text{head}(s))$. The function $\text{tail} : [k]^n \to [n]$ gives the index of the least significant non-zero digit of $s$ as $\text{val}(s)$ reads it.

**Example 21.** For example, $\text{val}(120, 0) = 0 \cdot 1 + 1 \cdot 3 + 2 \cdot 3^2 = 21$, $\text{val}(120, 1) = 1 \cdot 1 + 0 \cdot 3 + 2 \cdot 3^2 = 19$, and $\text{val}(120, 2) = 2 \cdot 1 + 1 \cdot 3 + 0 \cdot 3^2 = 5$, so $\text{val}(120) = \max\{19, 5, 21\} = 21$ and thus $\text{head}(120) = 0$ and $\text{tail}(120) = 1$.

As the function $\text{tail}$ plays a key role in the algorithm for computing the strategies that we propose in the sequel and, consequently, also in the shift-rule for the prefer-max sequence, we state its computational complexity as follows:

**Proposition 22.** The function $\text{tail}$ can be computed in $O(n)$ time and memory.

**Proof.** We can compute each iteration of the $\text{val}(x,i)$ based on the $\text{val}(x,i-1)$ in $O(1)$ time and memory using the equation $\text{val}(x,i) = (\text{val}(x,i-1) - \tau)/k + \tau \cdot k^n$ where $\tau = \text{val}(x,i-1) \mod k$. Computing $\text{val}(x,1)$ is $O(n)$ and checking which value is the maximal is also $O(1)$ since we only need to check if it is greater than the previous maximum, where the first maximum is $\text{val}(x,1)$. Also, from Definition 20 we can compute $\text{tail}$ from $\text{head}$ in $O(n)$ memory and time. □

We turn now to the analysis of the behavior of the function $\text{val}$ over plays of the shift-game. The following three claims establish bounds on the difference $\text{val}(s_{t+1}) - \text{val}(s_t)$ where $s_t$ and $s_{t+1}$ are consecutive states in a play of the game. If $s_t = x\sigma$ then, by definition, $s_{t+1}$ is either $(\sigma+1)x$, $0x$, or $\sigma x$. Since in the third case $\text{val}(s_t) = \text{val}(s_{t+1})$, we only need to analyze the first two cases.

**Claim 23.** $\text{val}((\sigma+1)x) - \text{val}(x\sigma) \geq k^{\text{head}(x\sigma)}$.

**Proof.** If $h = \text{head}(x\sigma)$ and $x = \sigma_0 \ldots \sigma_{n-2}$ then:

$$\text{val}((\sigma+1)x) \geq \text{val}((\sigma+1)x, (h+1) \mod n)$$

$$= \text{val}((\sigma_{n-1}+1)\sigma_0 \ldots \sigma_{n-2}, (h+1) \mod n)$$

$$= k^h + \sum_{i=1}^{n} \sigma_i ((h+1)-i) \mod n k^{i-1}$$

$$= k^h + \sum_{i=1}^{n} \sigma_i (h-i) \mod n k^{i-1} = k^h + \text{val}(x\sigma).$$ □
Claim 24. If \( x \neq 0^{n-1} \) and \( \text{tail}(x \sigma) = n - 1 \), then \( \text{head}(0x) = (\text{head}(x \sigma) + 1) \mod n \).

Proof. Let \( h = \text{head}(x \sigma) \). Since \( \text{tail}(x \sigma) = n - 1 \), we can write \( x \sigma = 0^{h-1} \sigma_i y \sigma \) and \( 0x = 0^h \sigma_i y \) for some \( y \) and some \( \sigma_i \neq 0 \). Let \( h' = \text{head}(0x) \). From Definition 20, since the symbol at the head cannot be zero, we have that \( h' \geq h \).

Assume towards contradiction that \( h' > h + 1 \). Then \( \text{val}(0x, h') = \text{val}(x \sigma, h' - 1) - \sigma^{h'} < \text{val}(x \sigma, h) - \sigma^{h+1} = \text{val}(0x, h + 1) \) which contradicts that \( h' \) is the head. Thus \( h' = h + 1 \). \[\square\]

Claim 25. If \( x \neq 0^{n-1} \) and \( \text{tail}(x \sigma) = n - 1 \), then \( \text{val}(0x) - \text{val}(x \sigma) \geq -(k - 1) \cdot k^{\text{head}(x \sigma)} \).

Proof. If \( h = \text{head}(x \sigma) \), \( \text{head}(x0) = h' = h + 1 \), \( 0x = 0\sigma_0 \ldots \sigma_{n-2} \), and \( x \sigma = \sigma_0 \ldots \sigma_{n-2} \sigma \) then:

\[
\begin{align*}
\text{val}(0x, h') & = \text{val}(x \sigma, h' - 1) - \sigma^{h'} \\
& = (\sum_{i=1}^{n} \sigma((h+1)-1-i) \mod n^{k^{i-1}}) - \sigma^{h'} \\
& = (\sum_{i=1}^{n} \sigma(h-i) \mod n^{k^{i-1}}) - \sigma^{h} \\
& \geq (\sum_{i=1}^{n} \sigma(h-i) \mod n^{k^{i-1}}) - (k - 1)k^h \\
& = \text{val}(x \sigma, h) - (k - 1)k^h
\end{align*}
\]

The above three propositions give us the tools needed for describing a non-losing strategy for Alice and Bob, as follows. The ‘trick’ for Alice is to force Bob to have \( B(s_t) = 1 \) at least once every \( 2n \) steps of the game if he does not want to lose in these \( 2n \) steps. Then, by Claim 23 and Claim 25, we get that \( \text{val} \) increases more than it may decrease between any two states in which we increase the value, which means that Alice guarantees that the play reaches the state with the maximal value of \( \text{val} \), which is \( (k - 1)^n \), from which Alice can easily win in \( n \) moves.

Definition 26. Let \( A_{\text{tail}} : [k]^n \to \{0, 1\} \) be the strategy for Alice defined by

\[
A_{\text{tail}}(x \sigma) = \begin{cases} 
1 & \text{if } \text{tail}(x \sigma) = n - 1; \\
0 & \text{otherwise}.
\end{cases}
\]

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and let \( B_{\text{tail}} : [k]^n \rightarrow \{0, 1\} \) be the strategy for Bob defined by

\[
B_{\text{tail}}(x\sigma) = \begin{cases} 
1 & \text{if } \sigma < k - 1 \text{ and } \text{tail}(x(\sigma + 1)) = n - 1; \\
0 & \text{otherwise}.
\end{cases}
\]

From Definition 26, we can see that \( A_{\text{tail}} \) and \( B_{\text{tail}} \) are strategies, i.e., \( A_{\text{tail}}(x0) = B_{\text{tail}}(x(k - 1)) = 0 \), since \( \text{tail}(x0) \neq n - 1 \).

Our next goal is to prove that \( A_{\text{tail}} \) is a non-losing strategy for Alice. To this end, we define a variant of the game and establish three claims, as follows.

**Definition 27.** The infinite shift-game goes exactly as the game in Definition 7, the only difference being that a play does not end when \( s_m \in \{s_1, \ldots, s_{m-1}\} \), just when \( s_m = s_0 = 0^n \). In this game, Bob wins when the play goes forever, Alice wins if \( s_m = 0^n \) and there is no option for a tie.

**Claim 28.** In any play \( s_0, s_1, \ldots \) of the infinite shift-game where Alice applies the strategy \( A_{\text{tail}} \) and Bob applies a strategy \( B \), if \( t_0 < t_1 < t_2 \) are such that \( B(s_{t_0}) = B(s_{t_2}) = 1 \), \( A_{\text{tail}}(s_{t_1}) = 1 \), and \( B(s_t) = 0 \) for all \( t_0 < t < t_2 \), then \( s_{t'} \) is of the form \( 0x \) for all \( t_1 < t' \leq t_2 \).

**Proof.** By the definition of \( \text{tail} \) we have that if \( \text{tail}(\sigma_0 \cdots \sigma_{n-1}) = \max\{i : \sigma_i \neq 0\} \) for some \( \sigma_0 \cdots \sigma_{n-1} \) then \( \text{tail}(0\sigma_0 \cdots \sigma_{n-2}) = \max\{i \leq n - 2 : \sigma_i \neq 0\} \).

We use this fact to prove, by induction, that for all \( t_1 < t \leq t_2 \) if \( s_t = \sigma_0 \cdots \sigma_{n-1} \) then \( \text{tail}(s_t) = \max\{i : \sigma_i \neq 0\} \). The base of the induction is by the definition of \( A_{\text{tail}} \) that gives us that \( \text{tail}(s_{t_1}) = n - 1 \). The induction step splits into the following two cases: If \( \sigma_{n-1} \neq 0 \) then \( s_{t+1} = 0\sigma_2 \cdots \sigma_{n-2} \) because \( A_{\text{tail}}(s_t) = 1 \). If \( \sigma_{n-1} = 0 \) then \( s_{t+1} = 0\sigma_2 \cdots \sigma_{n-2} \) because the state shifts. In both cases, the symbol that enters at the left is 0 and the invariant \( \text{tail}(s_t) = \max\{i : \sigma_i \neq 0\} \) is kept. \( \square \)

**Claim 29.** In any play \( s_0, s_1, \ldots \) of the infinite shift-game where Alice applies the strategy \( A_{\text{tail}} \), Bob applies a strategy \( B \), and Alice loses, there must be infinitely many indexes \( t \) such that \( B(s_t) = 1 \).

**Proof.** Assume towards contradiction that there is a \( t \) such \( B(s_t) = B(s_{t+1}) = \cdots = B(s_{t+2n}) = 0 \) and \( s_t = \sigma_0 \cdots \sigma_{n-1} \). Let \( m \geq t \) be the first index that satisfies \( \text{tail}(s_{t+m}) = n - 1 \). By Definitions 27 of the infinite game and by Definition 26 of \( A_{\text{tail}} \), \( s_{t+m} = \sigma_m \cdots \sigma_{n-1} \sigma_0 \cdots \sigma_{m-1} \). By the arguments used in the previous proof, \( s_{t+m+n} = 0^n \). This contradicts the assumption that Alice loses. \( \square \)
Claim 30. In any play $s_0,s_1,\ldots$ of the infinite shift-game where Alice applies the strategy $A_{\text{tail}}$ and Bob applies a strategy $B$, if $t_1 < t_2$ are such that $B(s_{t_1}) = B(s_{t_2}) = 1$ then $\text{val}(s_{t_1+1}) < \text{val}(s_{t_2+1})$.

Proof. Assume without loss of generalization that $B(s_t) = 0$ for all $t_1 < t < t_2$. If $A_{\text{tail}}(s_t) = 0$ for all $t_1 < t < t_2$ then $s_{t_2+1}$ is a rotation of $s_{t_1+1}$ with one of its digits increased by one. By the definition of $\text{val}$, this gives that $\text{val}(s_{t_2+1}) > \text{val}(s_{t_1+1})$. If there is a $t_1 < t < t_2$ such that $A(s_t) = 1$, we are in the case covered by Claim 28. Then:

$$\text{val}(s_{t_1+1}) = \text{val}(s_t)$$

(1) is because the state rotates when $B(s) = A(s) = 0$. (2) is by Claim 28 and by Claim 25. (4) is by Claim 24. (5) is by Claim 23.

Proposition 31. $A_{\text{tail}}$ is a non-losing strategy for Alice both in the infinite game and in the finite game.

Proof. Assume towards contradiction that there is a strategy $B$ for Bob such that the play of $B$ against $A_{\text{tail}}$ of the infinite game is the infinite sequence $s_0,s_1,\ldots$ where $s_t \neq 0^n$ for all $t > 0$. By Claim 29 and Claim 30, value of the states must grow without an upper bound along the game. This is a contradiction since, by definition, $\text{val}(s) < (k-1)^n$ for all $s$. This means that Bob cannot force the game to loop before reaching $0^n$, i.e., that he cannot win the finite game either.

Proposition 32. $A_{\text{tail}} = A^*$.

Proof. From Proposition 31, we have that $A_{\text{tail}}$ is a non losing strategy, and from Proposition 19 we know that $A^*$ is the only non-losing strategy for Alice.

Proposition 33. $B_{\text{tail}} = B^*$

Proof. From Proposition 31, we know that $A_{\text{tail}}$ is a non losing strategy and from Proposition 32 we know that $A_{\text{tail}} = A^*$. From Proposition 13 we know that
\[ B^*(w\sigma) = A^*(w(\sigma + 1)) \] and from Definition 26 we can see that \( A_{tail}(w(\sigma + 1)) = B_{tail}(w\sigma) \), so we get:

\[ B^*(w\sigma) = A^*(w(\sigma + 1)) = A_{tail}(w(\sigma + 1)) = B_{tail}(w\sigma) \]

Thus \( B_{tail} = B^* \).

\[ \square \]

6. An efficiently computable shift rule

By Proposition 32 and Proposition 33 we have that \( A^* = A_{tail} \) and that \( B^* = B_{tail} \). In this section, we elaborate a simple construction that uses this fact, and the fact that both \( A_{tail} \) and \( B_{tail} \) can be computed efficiently (Proposition 22), to provide an efficient construction of the prefer-max sequence:

**Theorem 34.** The function

\[
    \text{shift}(x\sigma) = \begin{cases} 
    (\sigma + 1)x & \text{if } \sigma < k - 1 \text{ and } \text{tail}(x(\sigma + 1)) = n - 1; \\
    0x & \text{else, if } \text{tail}(x\sigma) = n - 1; \\
    \sigma x & \text{otherwise}. 
    \end{cases}
\]

maps each vertex on the prefer-max cycle to its predecessor. It can be computed in \( O(n) \) time and memory.

**Proof.** From Proposition 33 and Proposition 32 we know that \( A^* = A_{tail} \) and that \( B^* = B_{tail} \). Thus, we get that \( (s_t)^{k-1}_{t=0} = (w_{k^n-1-t})^{k-1}_{t=0} \), and from Definition 26 we get that \( s_{t+1} = \text{shift}(s_t) \). By Proposition 22 we can calculate \( \text{tail} \) in \( O(n) \) time and memory and that the only part that requires computation.

\[ \square \]

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8. References

References

[1] J. H. Conway, N. J. A. Sloane, Lexicographic codes: Error-correcting codes from game theory, IEEE Transactions on Information Theory 32 (3) (1986) 337–348.

[2] A. S. Fraenkel, O. Rahat, Complexity of error-correcting codes derived from combinatorial games, in: International Conference on Computers and Games, Springer, 2002, pp. 201–212.

[3] N. G. de Bruijn, A combinatorial problem, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen. Series A 49 (7) (1946) 758.

[4] M. H. Martin, A problem in arrangements, Bull. Amer. Math. Soc. 40 (1934) 859–864.

[5] S. W. Golomb, S. Golomb, Shift Register Sequences, Aegean Park Press, 1981.

[6] J. Sawada, A. Williams, D. Wong, A simple shift rule for k-ary de bruijn sequences, Discrete Mathematics 340 (3) (2017) 524 – 531.

[7] G. Weiss, A combinatorial game approach to state nullification by hybrid feedback, in: Decision and Control, 2007 46th IEEE Conference on, IEEE, 2007, pp. 4643–4647.

[8] J. Sawada, A. Williams, D. Wong, A surprisingly simple de bruijn sequence construction, Discrete Mathematics 339 (1) (2016) 127–131.

[9] H. Fredricksen, Generation of the ford sequence of length 2n, n large, Journal of Combinatorial Theory, Series A 12 (1) (1972) 153–154.

[10] A. H. Chan, R. A. Games, E. L. Key, On the complexities of de bruijn sequences, Journal of Combinatorial Theory, Series A 33 (3) (1982) 233–246.