Lower Bounds of Optimal Exponentials of Thickness in Geometry Rigidity Inequality for Shells

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Abstract The optimal exponentials of the thickness in the geometry rigidity inequality of shells represent the geometry rigidity of the shells. We obtain that the lower bounds of the optimal exponentials are $4/3$, $3/2$, and $1$, for the hyperbolic shell, the parabolic shell, and the elliptic shell, respectively, through the construction of the Ansätze.

Keywords geometry rigidity inequality, shell, nonlinear elasticity, Riemannian geometry

Mathematics Subject Classifications (2010) 74K20(primary), 74B20(secondary).

1 Introduction and Main Results

The geometry rigidity inequality, namely the Friesecke-James-Muller estimate [10, 11], plays a central role in models in nonlinear elasticity. In their basic form, these estimates assert that for a deformation $u \in H^1(\Omega, \mathbb{R}^n)$ the distance of $\nabla u$ to a suitably chosen proper rotation $Q \in \text{SO}(n)$ is dominated in $L^2$ by the distance function of $\nabla u$ to $\text{SO}(n)$. The proof [10] is based on the fact that the nonlinear estimate can be related to the linear one since the tangent space to the smooth manifold $\text{SO}(n)$ at the identity matrix is given by the linear space of all skew-symmetric matrices. In fact, geometric rigidity results are

This work is supported by the National Science Foundation of China, grants no. 61473126 and no. 61573342, and Key Research Program of Frontier Sciences, CAS, no. QYZDJ-SSW-SYS011.
the cornerstone of rigorous derivations of two dimensional plate and shell theories from three-dimensional models in the framework of nonlinear elasticity theory. The $L^2$ version by Friesecke et al. [10] generalized previous work [20, 21, 22, 38, 39] and allowed for the first time the derivation of limiting theories as the thickness of the three-dimensional structure tends to zero without a priori assumptions on the deformations in various scaling regimes [9, 10, 11, 19, 27, 28, 29, 46] and many others.

It is known that the rigidity of a shell is closely related to the optimal constant of thickness in the geometric rigidity estimate [6, 7, 11, 14] and the optimal constant is crucial to shell theories being derived from 3-dimensional elasticity by $\Gamma$-convergence like [9, 10, 19, 27, 28, 29, 46]. As a linear version of the geometric rigidity estimate, the the optimal constants of thickness in Korn’s inequalities have been calculated subject to the Gaussian curvature of the middle surface of a shell under the assumption that the middle surface is given by a single principal curvature coordinate [13, 15, 18]. This assumption that the middle surface being a single principal curvature coordinate is generalized in [47, 48]. In Korn’s inequalities the optimal constant for the plate calculated in [11] scales like $h^2$, for cylindrical shells in [13], $h^{3/2}$, for the positive curvature in [18], $h$, and for the negative curvature in [18], $h^{4/3}$, respectively. It is expected that the analogous nonlinear estimates will have the same scaling of the constant in terms of the shell thickness.

Here we calculate some lower bounds of the optimal exponentials of the thickness in the nonlinear geometry rigidity inequality subject to the curvature of the middle surface.

Let $M \subset \mathbb{R}^3$ be a $C^3$ surface with the induce metric $g$ and a normal field $\vec{n}$. Let $S \subset M$ be an open, simply connected, bounded set with a regular boundary $\partial S$. We consider a shell with thickness $h > 0$

\[ \Omega = \{ p + t\vec{n}(p) \mid p \in S, -h/2 < t < h/2 \}. \]

Let $\kappa$ be the Gaussian curvature of $M$. We say that $\Omega$ is parabolic if

\[ \kappa(p) = 0, \quad |\Pi(p)| > 0 \quad \text{for} \quad p \in \overline{S}, \]  \hspace{1cm} (1.1)

where $\Pi = \nabla \vec{n}$ is the second fundamental form of $M$. If

\[ \kappa(p) > 0 \quad \text{for} \quad p \in \overline{S}, \]  \hspace{1cm} (1.2)

then $\Omega$ is said to be elliptic. In addition, $\Omega$ is said to be hyperbolic if

\[ \kappa(p) < 0 \quad \text{for} \quad p \in \overline{S}. \]

For $A \in \mathbb{R}^{3 \times 3}$, we denote the Euclidean norm by $|A| = \sqrt{\text{tr} AA^T}$. The distance from $A$ to $\text{SO}(3)$ is denoted $\text{dist} (A, \text{SO}(3))$. Let $\mu > 0$ be such that estimate (1.3) below holds.
true. There is a constant $C > 0$, independent of $h > 0$, such that for every $u \in H^1(\Omega, \mathbb{R}^3)$ there exists a constant rotation $Q \in \text{SO}(3)$, such that

$$
\|\nabla u - Q\|_{L^2(\Omega)}^2 \leq \frac{C}{h\mu} \int_{\Omega} \text{dist}^2(\nabla u(z), \text{SO}(3))dz.
$$

(1.3)

Set

$$
\mu(\Omega) = \inf\{ \mu \mid \mu > 0 \text{ such that } (1.3) \text{ holds} \}.
$$

It follows from [11] that

$$
\mu(\Omega) = 2
$$

if $\Omega$ is a plate, and from [27] that

$$
\mu(\Omega) \leq 2
$$

for a shell, respectively.

We have the following.

**Theorem 1.1** If $\Omega$ is hyperbolic, then

$$
\mu(\Omega) \geq 4/3
$$

(1.4)

In the case of $\Omega$ being elliptic,

$$
\mu(\Omega) \geq 1
$$

(1.5)

Next, we consider the case of the parabolic. We need

**Proposition 1.1** Let $M \subset \mathbb{R}^3$ be a parabolic surface without boundary. Then for given $p \in M$, there exists a unique regular geodesic $\gamma(t, p)$ on $M$ such that

$$
\gamma(0, p) = p, \quad |\dot{\gamma}(t, p)| = 1, \quad \nabla_{\dot{\gamma}(t, p)} \vec{n} = 0 \quad \text{for } t \in \mathbb{R}.
$$

(1.6)

Moreover, $\gamma(t, p)$ is a straight line in $\mathbb{R}^3$.

**Theorem 1.2** Let $M \subset \mathbb{R}^3$ be a parabolic surface and let $S \subset M$ be a bounded open set. We suppose that there is a point $p_0 \in S$ such that the following assumption holds: Let $t_- < 0 < t_+$ be such that

$$
\gamma(t_\pm, p_0) \in \partial S, \quad (\nabla_{\tau_\pm}(\gamma(t_\pm, p_0))) \neq 0, \quad \gamma(t, p_0) \in S \text{ for } t \in (t_-, t_+),
$$

(1.7)

where $\gamma$ is given in (1.6) and $\tau_\pm \in T_{\gamma(t_\pm, p_0)}\partial S$ with $|\tau_\pm| = 1$. Then

$$
\mu(\Omega) \geq 3/2.
$$

(1.8)

**Remark 1.1** It is conjectured that all the equal signs in (1.4), (1.5), and (1.8) hold.
The estimates (1.4), (1.5), and (1.8) will be obtained by constructing the Ansätze. The main observation is that such Ansätze may come from the improvement in the ones for the Korn inequality. In the case of the Korn inequality the Ansätze are constructed in [13], [15], and [18], which are based on the main assumption that the middle surface is given by a single principal curvature coordinate, i.e.,

\[ S = \{ \mathbf{r}(z, \theta) \mid (z, \theta) \in [1, 1 + l] \times [0, \theta_0] \}, \]  

where the properties

\[ \nabla_{\partial z} \mathbf{n} = \kappa_z \partial z, \quad \nabla_{\partial \theta} \mathbf{n} = \kappa_\theta \partial \theta \quad \text{for} \quad p \in S \]

hold. In the case of the parabolic or hyperbolic shell, a principal coordinate only exists locally ([48]). There is even no such a local existence for the elliptic shell.

Here we will construct the Ansätze for the Korn inequality without assumption (1.9) and then improve them to obtain the ones for the geometric rigidity estimate.

## 2 Proof of the Main Results

Let \( \nabla \) and \( D \) denote the connection of \( \mathbb{R}^3 \) in the Euclidean metric and the one of \( M \) in the reduced metric, respectively. We have to treat the relationship between \( \nabla \) and \( D \) carefully.

We need a linear operator \( Q \) as follows. Let \( M \) be oriented and \( \mathcal{E} \) be the volume element of \( M \) with the positive orientation. Let \( p \in M \) be given and let \( e_1, e_2 \) be an orthonormal basis of \( T_p M \) with positive orientation, that is,

\[ \det \left( e_1, e_2, \mathbf{n}(p) \right) = 1. \]

We define \( Q : T_p M \to T_p M \) by

\[ Q \alpha = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2 \quad \text{for all} \quad \alpha \in T_p M. \]  

\[ (2.1) \]

\( Q \) is well defined in the following sense: Let \( \hat{e}_1, \hat{e}_2 \) be a different orthonormal basis of \( T_p M \) with positive orientation,

\[ \det \left( \hat{e}_1, \hat{e}_2, \mathbf{n}(p) \right) = 1. \]

Let

\[ \hat{e}_i = \sum_{j=1}^{2} \alpha_{ij} e_j \quad \text{for} \quad i = 1, 2. \]

Then

\[ 1 = \mathcal{E}(\hat{e}_1, \hat{e}_2) = \alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}. \]
Using the above formula, a simple computation yields
\[
(\alpha, \hat{e}_2)\hat{e}_1 - (\alpha, \hat{e}_1)\hat{e}_2 = (\alpha, e_2)e_1 - (\alpha, e_1)e_2.
\]
Clearly, \( Q : T_pM \to T_pM \) is an isometry and
\[
Q^T = -Q, \quad Q^2 = -\text{Id}.
\]
The operator \( Q \) plays an important role in the case of the hyperbolic surface [46]. Let \( S \) be hyperbolic and \( p_0 \in S \) be given. Let \( \psi(p) = (x_1, x_2) : B(p_0, 3\delta) \to \mathbb{R}^2 \) be an asymptotic coordinate system with \( \psi(p_0) = 0 \)
\[
\Pi(\partial x_1, \partial x_1) = \Pi(\partial x_2, \partial x_2) = 0 \quad \text{for} \quad p \in B(p_0, 3\delta),
\]
where \( B(p_0, 3\delta) \subset S \) is the geodesic ball centered at \( p_0 \) with radius \( 3\delta \) where \( \delta > 0 \) is small. It is further assumed that \( \psi(p) = x \) is positively orientated, i.e.,
\[
\det \left( \partial x_1, \partial x_2, \vec{n}(p) \right) > 0 \quad \text{for} \quad p \in B(p_0, 3\delta).
\]
Let \( \varphi \in C^1_0(B(p_0, 3\delta)) \) be given such that
\[
\varphi(p) = 1 \quad \text{for} \quad p \in B(p_0, 2\delta).
\]
We define
\[
f(p) = \varphi(p)x_1(p) \quad \text{for} \quad p \in S,
\]
where \( x_1 \) is the first component of the \( \psi(p) = (x_1, x_2) \) for \( p \in B(p_0, 3\delta) \). Then
\[
f(p) = 0 \quad \text{for} \quad p \in S/ B(p_0, 3\delta).
\]

**Lemma 2.1** Let \( S \) be hyperbolic and \( f \) be given in (2.4). Then
\[
\Pi(QDf, QDf)(p) = 0 \quad \text{for} \quad p \in B(p_0, 2\delta),
\]
where \( D \) is the connection of \( S \) in the induced metric \( g \).

**Proof** Consider the asymptotic coordinate system (2.2). Set
\[
E_1 = a_1\partial x_1, \quad E_2 = b_1\partial x_1 + b_2\partial x_2,
\]
where
\[
a_1 = \frac{1}{\sqrt{g_{11}}}, \quad b_1 = -\frac{g_{12}}{\sqrt{g_{11}\det G}}, \quad b_2 = \sqrt{\frac{g_{11}}{\det G}},
\]
where
\[
G = \left( g_{ij} \right), \quad g_{ij} = \langle \partial x_i, \partial x_j \rangle.
\]
Then $E_1, E_2$ forms an orthonormal frame with positive orientation on $B(p_0, \delta)$ since it follows from (2.3) that
\[
\det \left( E_1, E_2, \vec{n}(p) \right) = \det \left( \partial x_1, \partial x_2, \vec{n}(p) \right) = a_1 b_2 \det \left( \partial x_1, \partial x_2, \vec{n}(p) \right) = 1
\]
for $p \in B(p_0, \delta)$. Using (2.1) and (2.6), we have
\[
\begin{align*}
Q \partial x_1 &= -\frac{1}{a_1} (b_1 \partial x_1 + b_2 \partial x_2), \\
Q \partial x_2 &= \frac{1}{b_2} \left( a_1 + \frac{b_2}{a_1} \right) \partial x_1 + \frac{b_1}{a_1} \partial x_2.
\end{align*}
\]
From (2.4),
\[
f_{x_1} = 1, \quad f_{x_2} = 0 \quad \text{for} \quad p \in B(p_0, 2\delta).
\]
We thus obtain
\[
Df = g^{11} \partial x_1 + g^{12} \partial x_2,
\]
where \(g^{ij} = (g_{ij})^{-1}\), and
\[
(det G)QDf = \det G \frac{g^{12}}{a_1 b_2} (a_1^2 + b_1^2) - \frac{g^{11} b_1}{a_1} \partial x_1 + \frac{\det G}{a_1} (g^{12} b_1 - g^{11} b_2) \partial x_2
\]
\[
= -\sqrt{\det G} \partial x_2 \quad \text{for} \quad p \in B(p_0, 2\delta).
\]
(2.5) follows from (2.2) and (2.8).

**Lemma 2.2** Let $S$ be hyperbolic. Then the shape operator $\nabla \vec{n} : T_pS \to T_pS$ is reversible. Let $\kappa$ be the Gaussian curvature and let $f$ be given in (2.4). Set
\[
Z = (\nabla \vec{n})^{-1}[(\nabla \vec{n})^2]^{1/2} Df, \quad v = \frac{|Df|^2}{\sqrt{|\Pi|^2 - 2\kappa}} \quad \text{for} \quad p \in S.
\]
Then
\[
sym Z \otimes Df = v \Pi \quad \text{for} \quad p \in B(p_0, 2\delta).
\]

**Proof** Let $p \in B(p_0, \delta)$ be given. Let $e_1, e_2$ be an orthonormal basis of $T_pS$ with positive orientation such that
\[
\nabla_{e_1} \vec{n} = \lambda_1 e_2, \quad \nabla_{e_2} \vec{n} = \lambda_2 e_2, \quad \lambda_1 > 0 > \lambda_2.
\]
Then
\[
Z = e_1(f)e_1 - e_2(f)e_2.
\]
Moreover, it follows from Lemma 2.1 and (2.11) that
\[
0 = \Pi \left( e_2(f)e_1 - e_1(f)e_2, e_2(f)e_1 - e_1(f)e_2 \right) = \lambda_1 [e_2(f)]^2 + \lambda_2 [e_1(f)]^2.
\]
Using (2.11) and (2.13), we have
\[
v = \frac{|e_1(f)|^2 + |e_2(f)|^2}{\lambda_1 - \lambda_2} = \frac{|e_1(f)|^2}{\lambda_1} - \frac{|e_2(f)|^2}{\lambda_2}.
\] (2.14)

It follows from (2.11)-(2.14) that
\[
Z \otimes Df(e_1, e_1) = |e_1(f)|^2 = v\lambda_1 = v\Pi(e_1, e_1),
\]
\[
Z \otimes Df(e_2, e_2) = -|e_2(f)|^2 = v\Pi(e_2, e_2),
\]
\[
Z \otimes Df(e_1, e_2) = e_1(f)e_2(f), \quad Z \otimes Df(e_2, e_1) = -e_2(f)e_1(f).
\]

Thus (2.10) follows from the above formulas. □

Let \((M, g)\) be a Riemannian manifold. Let \(T\) be a 2-order tensor field on \((M, g)\) and let \(X\) be a vector field on \((M, g)\). We define the inner multiplication of \(T\) with \(X\) to be another vector field, denoted by \(i(X)T\), given by
\[
\langle i(X)T, Y \rangle = T(X, Y) \quad \text{for} \quad Y \in T_pM, \quad p \in M, \quad g = \langle \cdot, \cdot \rangle.
\]

For any \(y \in H^1(\Omega, \mathbb{R}^3)\), we decompose \(y\) into
\[
y(z) = W(p, t) + w(p, t)\vec{n}(p) \quad \text{for} \quad z = p + t\vec{n}(p) \in \Omega, \quad p \in S, \quad |t| < h/2,
\] (2.15)

where \(w = \langle y, \vec{n} \rangle\) and \(W(\cdot, t)\) is a vector field on \(S\) for \(|t| < h/2\). It follows from (2.15) that
\[
\nabla_{\alpha + t\vec{n}} y = D_\alpha W + w\nabla_\alpha \vec{n} + [\alpha(w) - \Pi(W, \alpha)]\vec{n} \quad \text{for} \quad \alpha \in T_pS,
\] (2.16)
\[
\nabla \vec{n} y = W_t(p, t) + w_t(p, t)\vec{n}(p) \quad \text{for} \quad p \in S, \quad |t| < h/2,
\] (2.17)

where \(\nabla\) and \(D\) are the covariant differentials of the dot metric in \(\mathbb{R}^3\) and of the induced metric in \(S\), respectively, and \(W_t = \partial_t W\) and \(w_t = \partial_t w\).

By defining \(\nabla \vec{n} \vec{n} = 0\), we introduce an 2-order tensor \(P(y)\) on \(\mathbb{R}^3_p\) by
\[
P(y)(\tilde{\alpha}, \tilde{\beta}) = \langle \nabla \nabla \vec{n} \vec{n} y, \tilde{\beta} \rangle \quad \text{for} \quad \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^3.
\] (2.18)

We have

**Lemma 2.3** ([47]) Let \(y \in H^1(\Omega, \mathbb{R}^3)\) be given in (2.15). Then
\[
|\nabla y + tP(y)|^2 = |DW + w\Pi|^2 + |Dw - i(W)\Pi|^2 + |W_t|^2 + w_t^2,
\] (2.19)
\[
|\text{sym } \nabla y + t\text{sym } P(y)|^2 = |\Upsilon(y)|^2 + |\frac{1}{2}X(y)|^2 + w_t^2,
\] (2.20)

where
\[
\Upsilon(y) = \text{sym } DW + w\Pi, \quad X(y) = Dw - i(W)\Pi + W_t.
\] (2.21)
Moreover, the following estimates hold

\[
\sigma|\nabla y|^2 \leq |\nabla y + tP(y)|^2 \leq C|\nabla y|^2,
\]

\[
\sigma|\text{sym } \nabla y|^2 - Ch^2|\nabla y|^2 \leq |\text{sym } \nabla y + t \text{ sym } P(y)|^2 \leq C(|\text{sym } \nabla y|^2 + h^2|\nabla y|^2),
\]

for \( h > 0 \) small, where \(|t| \leq h/2\).

For \( A, B \in \mathbb{R}^{3 \times 3} \), let

\[
T(A) = (A^T A)^{1/2} - I, \quad \Phi(B) = \text{sym } B + \frac{1}{2} B^T B,
\]

where \( I \) is the \( 3 \times 3 \) unit matrix. Then

\[
|T(A)| = \text{dist } (A, \text{SO}(3)) \quad \text{for } A \in \mathbb{R}^{3 \times 3} \quad \text{with } \det A > 0.
\]

**Lemma 2.4**

\[
\frac{|T(A)|}{2 \sqrt{3}} \leq |\Phi(B)| \leq \frac{\sqrt{3} + |A|}{2} |T(A)| \quad \text{for } \det A > 0,
\]

where \( A = I + B \) for \( B \in \mathbb{R}^{3 \times 3} \).

**Proof** (2.24) follows from the identities

\[
T(A)((A^T A)^{1/2} + I) = B + B^T + B^T B = 2\Phi(B), \quad T(A) = 2\Phi(B)((A^T A)^{1/2} + I)^{-1}.
\]

\[\square\]

**Proof of Theorem 1.1** (a) Let the middle surface \( S \) be hyperbolic. Let \( \delta > 0 \) be given in (2.2). Let \( \check{\varphi} \in C^2_0(\mathcal{B}(p_0, 3\delta)) \) be such that

\[
\check{\varphi}(p) = 1 \quad \text{for } p \in \mathcal{B}(p_0, \delta), \quad \check{\varphi}(p) = 0 \quad \text{for } S/\mathcal{B}(p_0, 2\delta).
\]

Let \( f \) and \( Z, v \) be given in (2.4) and (2.9), respectively. First, we look for the ansatze for the Korn inequality in the form

\[
y(z) = W(z) + w(p)n(p),
\]

where

\[
W(z) = \frac{\check{\varphi}(p)\cos(\phi f(p))}{v(p)}Z(p) - tDw(p), \quad w(p) = \check{\varphi}(p)\phi \sin(\phi f(p)), \quad \phi = \frac{1}{h^{1/3}}.
\]

We have

\[
Dw = \phi^2 \check{\varphi} \cos(\phi f)Df + \phi \sin(\phi f)D\check{\varphi},
\]

\[
D^2 w = -\phi^3 \check{\varphi} \sin(\phi f)Df \otimes Df + \phi \sin(\phi f)D^2 \check{\varphi}
+ \phi^2 (\cos(\phi f)Df \otimes D \check{\varphi} + \cos(\phi f)D \check{\varphi} \otimes Df),
\]

8
and
\[ DW = -\frac{w}{v} Z \otimes Df + \frac{\cos(\phi f)}{v} Z \otimes D\hat{\phi} - \frac{\hat{\phi} \cos(\phi f)}{v^2} Z \otimes Dv - tD^2w. \] (2.29)

It follows from (2.29) and (2.10) that
\[ \Upsilon(y) = \frac{\cos(\phi f)}{v} \text{sym} Z \otimes D\hat{\phi} - \frac{\hat{\phi} \cos(\phi f)}{v^2} \text{sym} Z \otimes Dv - tD^2w, \] (2.30)

where \( \Upsilon(y) \) is given in (2.21). In addition, from (2.27) and (2.28), we have
\[ h^{2/3}DW|_{S} \leq C, \quad |t||D^2w| \leq C \quad \text{for} \quad p \in S, \quad |t| \leq h/2, \quad h > 0. \] (2.31)

Let \( \psi(p) = x \) be the asymptotic coordinate system with \( \psi(p_0) = 0 \), given in (2.2). Let \( \delta_0 > 0 \) be given small such that
\[ (0, \delta_0)^2 \subset \psi(B(p_0, \delta)). \]

From (2.28) and (2.7), we obtain
\[
C \geq h^2 \|D^2w\|_{L^2(S)}^2 \geq h^2 \int_{B(p_0, \delta)} |D^2w|^2 dg = \int_{B(p_0, \delta)} \sin^2(\phi f)|Df|^2 dg \\
\geq \sigma \int_{(0, \delta_0)^2} \sin^2(\phi x_1) dx_1 dx_2 = \sigma \int_0^{\delta_0} \sin^2(\phi x_1) dx_1 \\
= \frac{\sigma \delta_0}{2} \int_0^{\delta_0} [1 - \cos(2\phi x_1)] dx_1 \\
\geq \sigma \sum_{k=1}^{m} \frac{h^{1/3}(\frac{1}{4} + k\pi)}{h^{1/3}(\frac{1}{4} + k\pi)} dx_1 = \frac{\sigma h^{1/3}m}{2} \\
\geq \sigma (\delta_0 - \frac{5h^{1/3}\pi}{4}) \geq \sigma \quad \text{for} \quad h > 0 \quad \text{small}, \]
(2.32)

where
\[ m = \left\lfloor \frac{\delta_0}{h^{1/3} \pi} - \frac{1}{4} \right\rfloor. \]

A similar argument as above yields
\[ h^{4/3} \|DW\|_{L^2(S)}^2 \geq \int_{B(p_0, \delta)} \cos^2(\phi f)|Df|^2 dg \geq \sigma \quad \text{for} \quad h > 0 \quad \text{small}. \] (2.33)

Noting that \( |t| \leq h/2 \), from (2.30) and (2.32), we have
\[ \|\Upsilon(y)\|_{L^2(S)}^2 \leq C \quad \text{for} \quad h > 0 \quad \text{small}. \] (2.34)

Using the formulas (2.19), (2.29), and (2.31), we obtain
\[ (1 - Ch^2)\|\nabla y\|^2 \leq C(|DW|^2 + h^2|D^2w|^2 + |w|^2 + |Dw|^2 + |Z|^2) \leq \frac{C}{h^{4/3}} \quad \text{for} \quad z = p + t\bar{n} \in \Omega, \quad |t| \leq h/2, \quad h > 0, \]
(2.35)
which gives
\[ \| \nabla y \|_{L^2(S)}^2 \leq \frac{C}{h^{4/3}}. \]
Moreover, it follows from (2.19) and (2.33) that
\[ \| \nabla y \|_{L^2(S)}^2 \geq \sigma \| \nabla y + tP(y) \|_{L^2(S)}^2 \geq \sigma \| Dw \|_{L^2(S)}^2 \geq \frac{\sigma}{h^{4/3}}, \]
that is,
\[ \frac{\sigma}{h^{4/3}} \leq \| \nabla y \|_{L^2(S)}^2 \leq \frac{C}{h^{4/3}} \quad \text{for } h > 0 \text{ small}, \quad |t| \leq h/2. \quad (2.36) \]
Let \( X(y) \) be given in (2.21). It follows from (2.26) and (2.31) that
\[ \| X(y) \|_{L^2(S)}^2 = \| i (W) \Pi \|_{L^2(S)}^2 = \| \hat{\varphi} \cos(\phi f) \|_{L^2(S)}^2 - \sigma \| Dw \|_{L^2(S)}^2 \geq \sigma \frac{h^{4/3}}{3}, \quad \text{for } h > 0 \text{ small}. \quad (2.37) \]
In addition, by an argument as for (2.32), we have
\[ \| X(y) \|_{L^2(S)}^2 \geq \sigma \int_{B(p_0, \delta)} \cos^2(\phi f) \| (\nabla \bar{n})^2 \|_{L^2(S)}^2 \| W \|_{L^2(S)}^2 \| \tau^2 - C h^{1/3} \|_{L^2(S)}^2 \geq \sigma \quad \text{for } h > 0 \text{ small}. \quad (2.38) \]
From (2.20), (2.34), (2.36), and (2.37), we obtain
\[ \| \text{sym} \nabla y \|_{L^2(S)}^2 \leq \| \text{sym} \nabla y + t \text{sym} P(y) \|_{L^2(S)}^2 + C h^2 \| \nabla y \|_{L^2(S)}^2 \leq C, \quad (2.39) \]
and, by (2.36) and (2.38),
\[ 2 \| \text{sym} \nabla y \|_{L^2(S)}^2 \geq \| \text{sym} \nabla y + t \text{sym} P(y) \|_{L^2(S)}^2 - C h^2 \| \nabla y \|_{L^2(S)}^2 \geq \frac{1}{2} \| X(y) \|_{L^2(S)}^2 - C h^{2/3} \geq \sigma, \quad (2.40) \]
respectively.

Now we consider the ansatze for the geometry rigidity inequality, given by
\[ u(z) = z + \tau y(z), \quad \tau > 4/3, \quad (2.41) \]
where \( y \) is given in (2.25). Then
\[ \nabla u - I = \tau \nabla y. \]
Let \( \Phi(B) \) be given in (2.22). From (2.35) and (2.39), we have
\[ \| \Phi(h^\tau \nabla y) \|_{L^2(S)}^2 = h^{2\tau} \int_S | \text{sym} \nabla y + \frac{h^\tau}{2} \nabla^T y \nabla y |^2 dg \leq h^{2\tau} (2 \| \text{sym} \nabla y \|_{L^2(S)}^2 + h^{2\tau} \int_S | \nabla y |^4 dg) \leq C(1 + h^{2(\tau-4/3)}) h^{2\tau} \leq C h^{2\tau}, \quad (2.42) \]
and from (2.40)
\[ \| \Phi(h^\tau \nabla y) \|_{L^2(S)}^2 \geq h^{2\tau} (\sigma - Ch^{2(\tau-4/3)}) \geq \sigma h^{2\tau}, \]
respectively.

We set \( A = \nabla u \) and \( B = h^\tau \nabla y \) in Lemma 2.4 and use (2.23), (2.24), and (2.42) to obtain
\[ \| \text{dist} (\nabla u, \text{SO}(3)) \|_{L^2(\Omega)}^2 \leq 2\sqrt{3} \| \Phi(h^\tau \nabla y) \|_{L^2(\Omega)}^2 \leq Ch^{1+2\tau}. \]
Similarly, it follows from (2.23), (2.24), (2.35) and (2.43) that
\[ \| \text{dist} (\nabla u, \text{SO}(3)) \|_{L^2(\Omega)}^2 \geq \sigma \| \Phi(h^\tau \nabla y) \|_{L^2(\Omega)}^2 \geq \sigma h^{1+2\tau}. \]
Finally, using (2.44), (2.45), and (2.36), we obtain
\[ \frac{\sigma}{h^{4/3}} \leq \frac{\| \nabla u - I \|_{L^2(\Omega)}^2}{\| \text{dist} (\nabla u, \text{SO}(3)) \|_{L^2(\Omega)}^2} = \frac{h^{2\tau} \| \nabla y \|_{L^2(\Omega)}^2}{\| \text{dist} (\nabla u, \text{SO}(3)) \|_{L^2(\Omega)}^2} \leq \frac{C}{h^{4/3}}. \]
The estimate (1.4) follows.

(b) Let \( S \) be elliptic. We look for the ansatz in the form
\[ u(z) = z + h^\tau y(z) \quad \text{for} \quad z = p + tn(p) \in \Omega, \quad \tau > 1, \]
where
\[ y = -tDw + w\bar{n} \]
is the ansatz, given in the proof [47, Theorem 1.4] for the optimal constant of the Korn inequality. From [47], we have
\[ \| \nabla y \|_{L^\infty(\Omega)} \leq \frac{C}{h^{1/2}}, \quad \frac{\sigma}{h^{1/2}} \leq \frac{\| \nabla y \|_{L^2(\Omega)}}{\| \text{sym} \nabla y \|_{L^2(\Omega)}} \leq \frac{C}{h^{1/2}}. \]
We have
\[ \nabla u - I = h^\tau \nabla y, \quad \Phi(\nabla u - I) = h^\tau (\text{sym} \nabla y + \frac{h^\tau}{2} \nabla^T y \nabla y). \]
It follows from (2.48) that
\[ \frac{\sigma}{h^{1/2}(1 + h^{-1})} \leq \frac{\| \nabla u - I \|_{L^2(\Omega)}^2}{\| \Phi(\nabla u - I) \|_{L^2(\Omega)}^2} \leq \frac{C}{h^{1/2}(1 - Ch^\tau - 1)}. \]
Thus the proof is complete by Lemma 2.4. \( \Box \)

**Proof of Proposition 1.1** Let \( p \in M \) be given and let \( e \in T_p M \) be such that
\[ \nabla_p \bar{n} = 0, \quad |e| = 1. \]
Consider the geodesic
\[ \gamma(t,p) = \exp_p te \quad \text{for} \quad t \in \mathbb{R} \]  \hspace{1cm} (2.49)
on \( M \) in the induced metric \( g \), where \( \exp_q : T_p M \to M \) is the exponential map. We will show that \( \gamma(t,p) \) satisfies (1.6).

From [47, Lemma 2.7], there are a neighbourhood \( N \) of \( p \) and a vector field \( X \) such that \( X(p) = e, |X| = 1, \nabla_X \vec{n} = 0 \) for \( q \in N \). \hspace{1cm} (2.50)

Set \( Y = QX \) where \( Q \) is given in (2.1). Then
\[ \nabla_Y \vec{n} = \lambda Y, \quad \langle X, Y \rangle = 0, \quad |Y| = 1 \quad \text{for} \quad q \in N, \]
where \( \lambda \neq 0 \) is the nonzero principal curvature.

We have
\[
\nabla_X \nabla_Y \vec{n} = \nabla_X (\lambda Y) = X(\lambda)Y + \lambda \nabla_X Y = X(\lambda)Y + \lambda(\nabla_X Y, X) + \lambda(\nabla_X Y, \vec{n}) \vec{n} = X(\lambda)Y + \lambda(D_X Y, X)X.
\]
Thus
\[ X(\lambda)Y + \lambda(D_X Y, X)X = \nabla_Y \nabla_X \vec{n} + \nabla_{[X,Y]} \vec{n} = \lambda(\nabla_Y [X,Y])Y, \]
which yields, by \( \lambda \neq 0 \),
\[ X(\lambda) = \lambda \nabla_Y [X,Y], \quad \langle D_X Y, X \rangle = 0 \quad \text{for} \quad q \in N. \]
We obtain
\[
D_X X = \langle D_X X, Y \rangle Y = -\langle X, D_X Y \rangle Y = 0 \quad \text{for} \quad q \in N, \quad \hspace{1cm} (2.51)
D_X Y = \langle D_X Y, X \rangle X = 0 \quad \text{for} \quad q \in N. \quad \hspace{1cm} (2.52)
\]

Consider the flow by \( X \):
\[ \alpha'(t,q) = X(\alpha(t,q)), \quad \alpha(0,q) = q \quad \text{for} \quad q \in N. \]
Set
\[ \gamma(t,q) = \exp_q tX(q) \quad \text{for} \quad q \in N. \]  \hspace{1cm} (2.53)
The formula (2.51) shows that
\[ \gamma(t,q) = \alpha(t,q), \quad \dot{\gamma}(t,q) = X(\gamma(t,q)) \quad \text{when} \quad \alpha(t,q) \in N. \]  \hspace{1cm} (2.54)

Next, we prove that
\[ \nabla_{\dot{\gamma}(t,q)} \vec{n} = 0 \quad \text{for} \quad (t,q) \in \mathbb{R} \times N. \]  \hspace{1cm} (2.55)
Let \( q \in N \) be given. From (2.54) and (2.50), there is a largest number \( \varepsilon_0 > 0 \) such that (2.55) hold for all \( t \in [0, \varepsilon_0) \). Let \( \varepsilon_0 < \infty \). Clearly

\[
\nabla \dot{\gamma}_{(\varepsilon_0, q)} \vec{n} = 0.
\]

By [47, Lemma 2.7] again, there is a vector field \( Z \) on a neighbourhood of \( \gamma(\varepsilon_0, q) \) such that

\[
Z(\gamma(\varepsilon_0, q)) = \dot{\gamma}(\varepsilon_0, q), \quad |Z| = 1, \quad \nabla_Z \vec{n} = 0.
\]

From the uniqueness of a geodesic, (2.55) would hold true for all \( t \in [0, \varepsilon_1) \) for some \( \varepsilon_1 > \varepsilon_0 \). This contradiction shows that (2.55) hold for all \( t \in (\varepsilon_0, \infty) \).

We extend the vector field \( X \) from \( N \) to \( \hat{N} \), still denoted by \( X \), by

\[
X(\gamma(t, q)) = \dot{\gamma}(t, q) \quad \text{for} \quad (t, q) \in \mathbb{R} \times N,
\]

where

\[
\hat{N} = \{ \gamma(t, q) \mid t \in \mathbb{R}, q \in N \}.
\]

Moreover, we extend \( Y \) from \( N \) to \( \hat{N} \) by paralleling translation \( Y(q) \) to \( Y(\gamma(t, q)) \) along the geodesic \( \gamma(t, q) \) in the induced metric of \( M \). Then \( X \) and \( Y \) forms an orthonormal frame on \( \hat{N} \) with

\[
\nabla_X \vec{n} = 0, \quad \nabla_Y \vec{n} = \lambda Y, \quad \lambda \neq 0, \quad D_X X = D_X Y = 0 \quad \text{for} \quad q \in \hat{N}.
\]

Let

\[
\gamma(t, q) = \gamma_1(t)X + \gamma_2(t)Y + \gamma_3(t)\vec{n}.
\]

Then

\[
\dot{\gamma}(t, q) = \gamma_1'(t)X + \gamma_2'(t)Y + \gamma_3'(t)\vec{n} + \gamma_1(t)\nabla_X X + \gamma_2(t)\nabla_X Y + \gamma_3(t)\nabla_X \vec{n}
\]

\[
= \gamma_1'(t)X + \gamma_2'(t)Y + \gamma_3'(t)\vec{n} \quad \text{for} \quad t \in \mathbb{R}.
\]

Thus the formulas \( \gamma'(t, q) = X(\gamma'(t, q)) \) imply that

\[
\gamma_1(t) = \gamma_1(0) + t, \quad \gamma_2(t) = \gamma_2(0), \quad \gamma_3(t) = \gamma_3(0) \quad \text{for} \quad t \in \mathbb{R}.
\]

Thus \( \gamma(t, q) \) is a straight line in \( \mathbb{R}^3 \) for given \( q \in N \). The proof is complete. \( \square \)

**Lemma 2.5** Let \( p_0 \in S \) be such that (1.7) holds. For any \( a > 0 \), there exists a principal coordinate system \( \psi^{-1} : \subset (-a, a) \times (-\varepsilon, \varepsilon) \to M \) such that

\[
\psi(\gamma(t, p_0)) = (t, 0) \quad \text{for} \quad t \in (-a, a)
\]

where \( \varepsilon > 0 \) is a number small.
**Proof** Let $\gamma(t,q), X,$ and $Y$ be given in the proof of Proposition 1.1. Define

$$
\eta(\gamma(t,q)) = e^{\int_0^t \langle D_Y X, Y \rangle \circ \gamma(s,q) \, ds} \text{ for } \gamma(t,q) \in \hat{N}.
$$

Then

$$
[X, \eta Y] = D_X (\eta Y) - \eta D_Y X = [X(\eta) - \eta (D_Y X, Y)] Y = 0 \text{ for } q \in \hat{N}.
$$

(2.58)

Let $\beta(s,q)$ be the flow by the vector $\eta Y$, i.e., for each $q \in \hat{N}$, there is $\varepsilon(q) > 0$ such that

$$
\dot{\beta}(s,q) = \eta Y(\beta(s,q)), \quad \beta(0,q) = q \text{ for } s \in (-\varepsilon(q), \varepsilon(q)).
$$

Since the interval $[-a, a]$ is compact, there is a constant $\varepsilon > 0$ small such that

$$
\dot{\beta}(s, \gamma(t, p_0)) = \eta Y(\beta(s, \gamma(t, p_0))) \quad \text{for all } (t,s) \in (-a, a) \times (-\varepsilon, \varepsilon).
$$

(2.59)

From [26, p.233, Theorem 9.44], (2.58) implies that

$$
\gamma(t, \beta(s, p_0)) = \beta(s, \gamma(t, p_0)) \quad \text{for all } (t,s) \in (-a, a) \times (-\varepsilon, \varepsilon).
$$

(2.59)

We define $\psi^{-1} : (-a, a) \times (-\varepsilon, \varepsilon) \rightarrow M$ by

$$
\psi^{-1}(x_1, x_2) = \gamma(x_1, \beta(x_2, p_0)).
$$

From Proposition 1.1, there is a $\varepsilon > 0$ such that $\psi(q) = x$ defines a coordinate satisfying (2.56). Furthermore, (2.59) implies that $\partial x_1 = X$ and $\partial x_2 = \eta Y$. □

We make some further preparations for the proof of Theorem 1.2. Let assumption (1.7) hold. Let $a > \max\{|t_-,|, |t_+|\}$ be given. Let $\psi(q) = (x_1, x_2)$ be the principal coordinate given in Lemma 2.5. Let $X$ and $\eta Y$ be the vector field given in Lemma 2.5 such that

$$
\partial x_1 = X, \quad \partial x_2 = \eta Y.
$$

(2.60)

Then

$$
D_X X = D_X Y = 0, \quad D_Y X = \varrho Y, \quad D_Y Y = -\varrho X,
$$

(2.61)

where $\varrho = \langle D_Y X, Y \rangle$. For $x_2 \in (-\varepsilon, \varepsilon)$, let $t_+(x_2) > 0 > t_-(x_2)$ be such that

$$
\gamma(t_{\pm}(x_2), \beta(x_2, p_0)) \in \partial S, \quad \gamma(x_1, \beta(s, p_0)) \in S \quad \text{for } x_1 \in (t_-(x_2), t_+(x_2)).
$$

Set

$$
S_0 = \{ \gamma(x_1, \beta(x_2, p_0)) \mid (x_1, x_2) \in (t_-(x_2), t_+(x_2)) \times (-\varepsilon, \varepsilon) \}.
$$

Then $S_0 \subset S$. 14
We will construct the ansatz with its values supported on $S_0$. Let the functions $\eta$ and $\varphi$ be given in (2.57) and (2.61), respectively, on $S_0$. Set
\[
\varpi(p) = e^{\int_{x_0}^{x} \varphi(s, x_2) ds} \quad \text{for } p \in S_0.
\]

We further define
\[
v(p) = \varpi \varphi'(x_2), \quad b(p) = -\frac{1}{\lambda} Y(v), \quad w(p) = \lambda v(p) - Y(b) \quad \text{for } p \in S_0.
\]

Then
\[
v_{x_1} = v \varphi, \quad Y(v) + b \lambda = 0, \quad Y(b) - \lambda v + w = 0 \quad \text{for } p \in S_0.
\]

Consider the ansatz
\[
y(z) = \begin{cases} 
V(z) + t W(z) + b \vec{n} & \text{for } z = p + t \vec{n}, \quad p \in S_0, \quad |t| < h/2, \\
0 & \text{for } z = p + t \vec{n}, \quad p \in S/S_0, \quad |t| < h/2,
\end{cases}
\]

where
\[
V = vY, \quad W = -b_{x_1} X + wY \quad p \in S_0.
\]

It follows from (2.61) and (2.60) that
\[
D_X V = v_{x_1} Y, \quad D_Y V = -v \varphi X + Y(v)Y,
\]
\[
D_X W = -b_{x_1 x_1} X + w_{x_1} Y, \quad D_Y W = -[Y(b_{x_1}) + \varphi w] X + [Y(w) - b_{x_1} \varphi] Y.
\]

Using the above formulas and from Lemma 2.3, we obtain
\[
|\nabla y + t P(y)|^2 = |D V + t D W + b \Pi|^2 + |D b - i (V + t W) \Pi + |W|^2
\]
\[
\quad = \left( \langle D_X V, X \rangle + t \langle D_X W, X \rangle \right)^2 + \left( \langle D_Y V, Y \rangle + t \langle D_Y W, Y \rangle \right)^2
\]
\[
\quad + \left( \langle D_Y V, X \rangle + t \langle D_Y W, X \rangle \right)^2 + \left( V(v) + t[Y(w) - b_{x_1} \varphi] + b \lambda \right)^2
\]
\[
\quad + b_{x_1}^2 |Y(b) - (v + tw) \lambda|^2 + b_{x_1}^2 + w^2
\]
\[
\quad = t^2 b_{x_1 x_1}^2 + (v_{x_1} + tw_{x_1})^2 + \left\{ (v \varphi + t[Y(b_{x_1}) + \varphi w]) \right\}^2
\]
\[
\quad + [Y(v) + t[Y(w) - b_{x_1} \varphi] + b \lambda]^2 + 2b_{x_1}^2
\]
\[
\quad + [Y(b) - (v + tw) \lambda]^2 + w^2 \quad \text{for } p \in S_0,
\]

where the following formulas have been used
\[
\nabla X \vec{n} = 0, \quad \nabla Y \vec{n} = \lambda Y \quad \text{for } p \in S_0.
\]

Noting (2.63), by a similar computation, we have
\[
|Y(y)|^2 = t^2 \left\{ b_{x_1 x_1}^2 + \frac{1}{2} [w_{x_1} - Y(b_{x_1}) - \varphi w]^2 + [Y(w) - b_{x_1} \varphi]^2 \right\} \quad \text{for } p \in S_0,
\]
\[
|D b - i (V + t W) \Pi + W|^2 = [Y(b) - (v + tw) \lambda + w]^2 = t^2 \lambda^2 w^2 \quad \text{for } p \in S_0.
\]
Proof of Theorem 1.2 Let \( \varphi_0 \in C^4_0(-\varepsilon, \varepsilon) \) be given such that

\[
\varphi_0(x_2) = 1 \quad \text{for} \quad x_2 \in (-\varepsilon/2, \varepsilon/2).
\]

Set

\[
\varphi(x_2) = \varphi_0(x_2) \cos(\phi x_2), \quad \phi = \frac{1}{h^{1/4}}, \quad \text{for} \quad x_2 \in (-\varepsilon, \varepsilon),
\]
in (2.62).

Then it follows from (2.62) that

\[
v(p) = \varphi'(x_2), \quad b(p) = -\frac{1}{\lambda} Y(v), \quad w(p) = \lambda v(p) - Y(b),
\]

Noting \(|t| \leq h/2\) and from (2.65)-(2.67) and (2.20), we obtain

\[
\sigma h^{2r-\frac{1}{2}} \leq \|\nabla u - I\|_{L^2(\Omega)}^2 \leq C h^{2r-\frac{1}{2}}, \quad \sigma h^{2r}(h - h^{2r-2}) \leq \|\Phi(\nabla u - I)\|_{L^2(\Omega)}^2 \leq C h^{2r}(h + h^{2r-2}).
\]
Since \( \det \nabla u = \det (I + h^2 \nabla y) > 0 \) when \( h > 0 \) is small enough, from Lemma 2.4 and the above estimates, we obtain

\[
\frac{\sigma}{h^{3/2}(1 + h^{2(\tau - 3/2)})} \leq \left\| \nabla u - I \right\|^2_{L^2(\Omega)} \leq \frac{C}{h^{3/2}(1 - h^{2(\tau - 3/2)})} \frac{\left\| \text{dist} (\nabla u, \text{SO}(3)) \right\|^2_{L^2(\Omega)}}{L^2(\Omega)}.
\]

The proof is complete. \( \square \)

**Conflict of interest statement**

There is no conflict of interests.

Ethical approval: This article does not contain any studies with human participants or animals performed by the author.

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