Seshadri’s criterion and openness of projectivity

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Abstract. We prove that projectivity is an open condition for deformations of algebraic spaces with rational singularities.

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1. Introduction

Many projective varieties have deformations that are not projective, not even algebraic in any sense; K3 and elliptic surfaces furnish the best known examples. In these cases the non-algebraic deformations are also very far from being algebraic. A typical non-projective K3 surface does not contain any compact curves and is not bimeromorphic to an algebraic surface. The latter property is not an accident.

Let \( g : X \to \mathbb{D} \) be a smooth, proper morphism of a complex manifold to a disc. If the central fiber \( X_0 \) is projective, then it is Kähler, and the \( X_s \) are also Kähler for \( |s| \ll 1 \) [9]. A Kähler variety that is bimeromorphic to an algebraic variety is projective [16]. Thus if \( X_s \) is bimeromorphic to an algebraic variety, then it is projective for \( |s| \ll 1 \). We can summarize this somewhat imprecisely as follows:

- Projectivity is an open condition in the category of Moishezon manifolds.

Note, however, that even if all fibers of \( g : X \to \mathbb{D} \) are projective, \( g \) need not be projective over any smaller disc \( \mathbb{D}_\epsilon \); see [1] or [13, Example 4]. Examples of Hironaka—reproduced in [6, Appendix B]—show that projectivity is also not a closed condition.

The aim of this paper is to prove that projectivity is an open condition for deformations of complex algebraic spaces with rational singularities. (We use ‘proper algebraic space over \( \mathbb{C} \)’ as a synonym of ‘proper, complex analytic space that is bimeromorphic to a projective variety’.)

A result of this type, with rather strong restrictions on the singularities, is in [10, Theorem 12.2.10]; see Paragraph 1 for some comments. I have been trying to remove the restrictions, and a key problem that emerged was that the argument used Kleiman’s criterion for ampleness, which is not known to apply to algebraic spaces.

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By contrast, Seshadri’s criterion does hold for algebraic spaces, and leads to a quite
general openness result.

First, I state the 1-parameter version. This is clearer and its proof uses all the essential
ideas. The general version, Theorem 2, is substantially stronger.

**Theorem 1.** Let \( g : X \to \mathbb{D} \) be a proper, flat morphism of complex analytic spaces. Assume that

1. \( X_0 \) is projective,
2. the fibers \( X_s \) have rational singularities for \( s \neq 0 \), and
3. \( g \) is bimeromorphic to a projective morphism \( g^p : X^p \to \mathbb{D} \).

Then \( g \) is projective over a smaller punctured disc \( \mathbb{D}^o \subset \mathbb{D} \).

**Remark.**

1. Note that we make no assumptions on the singularities of \( X_0 \).
2. Even if \( g \) is smooth, usually \( g \) is not projective over the disc \( \mathbb{D}^o \). This is quite typical
for simultaneous resolutions of deformations of surfaces with Du Val singularities; see [1]
or [13, Example 4].
3. Assumption (1.2) can be relaxed to \( X_s \) having 1-rational singularities (see Definition 11), but a similar openness property does not hold for algebraic spaces with slightly
worse singularities. There are flat, proper families of normal, compact surfaces where all
fibers are bimeromorphic to \( \mathbb{P}^2 \), yet the projective fibers correspond to a countable, dense
set on the base; see Example 6. In this example, each surface has a single singular point,
which is simple elliptic (that is, biholomorphic to a cone over an elliptic curve). These are
among the mildest surface singularities, but they are not rational.
4. Assumption (1.3) is automatic if \( g : X \to \mathbb{D} \) is obtained by base change from a
morphism of algebraic spaces, and it implies that every fiber of \( g \) is an algebraic space. A
converse is conjectured in [14, Conjecture 2], and proved in [19, Theorem 1.4] and [18,
Theorem 1.2] for smooth morphisms.
5. Instead of assumption (1.3), we could assume that all fibers are algebraic spaces and
the irreducible components of the Chow–Barlet space parametrizing 1-cycles on \( X/\mathbb{D} \) are
proper over \( \mathbb{D} \); see [2] or Proposition 13.

**Idea of proof.** After shrinking \( \mathbb{D} \) we may assume that \( X \) retracts to \( X_0 \). Since \( X_0 \) is projective,
it has an ample line bundle \( L \). Let \( \Theta \in H^2(X, \mathbb{Q}) \) be the pull-back of \( c_1(L) \) to \( X \). Note that
\( \Theta \) is a topological cohomology class that is usually not the Chern class of a holomorphic
line bundle.

Fix a very general \( s \in \mathbb{D} \) and let \( p_s \in C_s \subset X_s \) be a pointed curve. We show in
Proposition 13 that it has a specialization to \( p_0 \in C_0 \subset X_0 \) such that \( \text{mult}_{p_0} C_0 \geq
\text{mult}_{p_s} C_s \). Thus

\[
\Theta \cap [C_s] = \Theta \cap [C_0] \geq \epsilon \cdot \text{mult}_{p_0} C_0 \geq \epsilon \cdot \text{mult}_{p_s} C_s,
\]

where the first \( \geq \) is the easy direction of Seshadri’s criterion 7 on \( X_0 \). Therefore \( \Theta_s := \Theta|_{X_s} \in H^2(X_s, \mathbb{Q}) \) satisfies the assumption of Seshadri’s criterion on \( X_s \).

It remains to show that Seshadri’s criterion works for (possibly non-algebraic) cohomology classes. This is done in Proposition 9, using some foundational work comparing
numerical and homological equivalence in Proposition 12. This is where rationality of the singularities of $X_s$ is used.

Going from very general fibers to all fibers over a smaller disc uses Proposition 17.

The general version is the following, whose proof is completed at the end of the paper.

**Theorem 2.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that $g$ is flat over $S^*$. Assume that

1. $X_0$ is projective for some $0 \in S$,
2. the fibers $X_s$ have rational singularities for $s \in S^*$, and
3. $g$ is bimeromorphic to a projective morphism $g^p : X^p \to S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i$$

is projective.

**Remark.**

(2.4) Note that $g$ is not assumed flat and the comments in (1.6) also apply to $X \to S$.

The Lefschetz principle then gives the analogous result for algebraic spaces.

**COROLLARY 3**

Let $k$ be a field of characteristic 0, and $g : X \to S$ a proper morphism of algebraic spaces that are of finite type over $k$. Let $S^* \subset S$ be a dense, Zariski open subset such that $g$ is flat over $S^*$. Assume that

1. $X_0$ is projective for some $0 \in S$, and
2. the fibers $X_s$ have 1-rational singularities for $s \in S^*$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i$$

is projective. □

**Question 4.** Is Corollary 3 also true in positive or mixed characteristic? Our proof uses topological cohomology groups in an essential way; it is not clear what would replace them.

Theorem 1 was developed to complete the proof of the second part of the following; for details, see the original paper [13].

**Theorem 5** [13, Theorem 1]. Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces. Fix a point $0 \in S$ and assume that the fiber $X_0$ is projective, of general type, and with canonical singularities. Then there is an open neighborhood $0 \in U \subset S$ such that

1. the plurigenera $h^0(X_s, \omega_{X_s}^{[r]})$ are independent of $s \in U$ for every $r$, and
The fibers $X_s$ are projective for every $s \in U$. □

Example 6. Let $E \subset \mathbb{P}^2$ be a smooth cubic. Fix $m \geq 10$ and let $X \to S$ be the universal family of surfaces obtained by blowing up $m$ distinct points $p_i \in E$, and then contracting the birational transform of $E$. (So $S$ is an open subset in $E^m$.) It is easy to see that such a surface $S_t$ is

(6.1) non-projective if the $p_i^t$ and $L$ are linearly independent in $\text{Pic}(E)$, and

(6.2) projective if the $p_i^t$ satisfy a unique linear relation $\sum_i n_i [p_i^t] \sim nL|_E$, where $0 < n_i < \frac{1}{3}n$ and $n = \frac{1}{3} \sum_i n_i$.

Thus the projective fibers correspond to a countable union of hypersurfaces $H(n_1, \ldots, n_m) \subset S$. All fibers have simple elliptic singularities and trivial canonical class. (For $m = 12$, the singularities are biholomorphic to cones over smooth plane cubics.)

Correction to [10, Chapter 12]. Statements [10, 12.2.6 and 12.2.10] are incorrect. The applications of these results concern families of 3-folds with $\mathbb{Q}$-factorial, terminal singularities. With these additional assumptions, the proofs given there are correct. However, trying to formulate them with minimal sets of assumptions led to errors.

First, in [10, 12.2.6], one should also assume that the fibers $X_s$ have rational singularities. This is necessary since the proof uses [10, 12.1.5]. No other changes needed.

The bigger problem is with [10, 12.2.10]. During the proof, we claim to apply Kleiman’s ampleness criterion to algebraic spaces. However [8] states and proves the criterion for quasi-divisorial schemes. This class includes all varieties that are either projective, or proper and $\mathbb{Q}$-factorial, but it is not clear that the fibers $X_s$ are quasi-divisorial.

A proof for $\mathbb{Q}$-factorial, 3-dimensional algebraic spaces is explained in [11, 5.1.3], this is enough for the applications in [10]. A higher dimensional Kleiman criterion needed for [10, 12.2.10] was recently established by Villalobos-Paz [21]. With this in place, the rest of the proof of [10, 12.2.10] is correct.

2. Seshadri’s criterion and variants

The criterion—proved by Seshadri but first published in [5, Section I.7]—is the following.

**Theorem 7.** Let $X$ be a proper algebraic space and $L$ a line bundle on $X$. Then $L$ is ample iff there is an $\epsilon > 0$ such that

$$\deg L|_C \geq \epsilon \cdot \text{mult}_p C$$

(7.1)

for every integral curve $C \subset X$ and every $p \in C$.

By linearity, then the same holds for all 1-cycles $Z$ on $X$.

An important observation is that while the criterion is frequently stated for schemes only, it in fact holds for proper algebraic spaces, even if they are reducible and non-reduced. This is in marked contrast with Kleiman’s criterion, which is still not known for algebraic spaces in general, though a recent result of Villalobos-Paz [21] proves Kleiman’s criterion
for \(\mathbb{Q}\)-factorial algebraic spaces with log terminal singularities over a field of characteristic 0.

In algebraic geometry, Seshadri’s criterion is usually used to show that a line bundle \(L\) is ample. Here I focus on a reformulation of Theorem 7.

COROLLARY 8

Let \(X\) be a proper algebraic space. Then \(X\) projective iff there is a line bundle \(L\) and an \(\epsilon > 0\) such that

\[
\deg L|_C \geq \epsilon \cdot \mult p C
\]

(8.1)

for every integral curve \(C \subset X\) and every \(p \in C\).

□

Now we make a slight twist and replace the line bundle \(L\) by a cohomology class \(\Theta \in H^2(X(\mathbb{C}), \mathbb{Q})\). The key point is that in our applications we will be able to find such a cohomology class \(\Theta\), but it usually will not be a \((1, 1)\) class.

PROPOSITION 9

Let \(X\) be a proper algebraic space over \(\mathbb{C}\) with 1-rational singularities (see Definition 11). Then \(X\) is projective iff there is a cohomology class \(\Theta \in H^2(X(\mathbb{C}), \mathbb{Q})\) and an \(\epsilon > 0\) such that

\[
\Theta \cap [C] \geq \epsilon \cdot \mult p C
\]

(9.1)

for every integral curve \(C \subset X\) and every \(p \in C\).

Here \([C] \in H_2(X(\mathbb{C}), \mathbb{Q})\) denotes the homology class of \(C\), and we tacitly use the identification \(H_0(X(\mathbb{C}), \mathbb{Q}) \cong \mathbb{Q}\) to view \(\Theta \cap [C]\) as a number. The proof relies on comparing numerical and homological equivalence for curves; we discuss this next.

DEFINITION 10

Let \(X\) be a proper, complex analytic space. For \(K = \mathbb{Q}\) or \(\mathbb{R}\), let \(N_1(X, K)\) denote the \(K\)-vector space generated by compact complex curves \(C \subset X\), modulo numerical equivalence. That is,

\[
\sum a_i A_i \equiv \sum b_j B_j \quad \text{iff} \quad \sum a_i \deg L|_{A_i} = \sum b_j \deg L|_{B_j}
\]

for every holomorphic line bundle \(L\) on \(X\).

Let \(H_2^{\text{alg}}(X, K) \subset H_2(X, K)\) denote the vector subspace generated by homology classes of compact complex curves. Sending an algebraic homology class to a numerical equivalence class gives a natural surjection

\[
R^{\text{hom}}_{\text{num}}: H_2^{\text{alg}}(X, \mathbb{Q}) \twoheadrightarrow N_1(X, \mathbb{Q}).
\]

(10.1)
We see in Proposition 12 that $R_{\text{num}}^{\text{hom}}$ is an isomorphism if $X$ has 1-rational singularities. In this case the inverse map gives an injection

$$N_1(X, \mathbb{Q}) \hookrightarrow H_2(X, \mathbb{Q}). \quad (10.2)$$

Note, however, that the natural map is the one in (10.1), and (10.2) exists only if $X$ has 1-rational singularities.

**DEFINITION 11**

Let $X$ be a normal, complex analytic space and $g : Y \to X$ a resolution of singularities. $X$ has *rational singularities* iff $R^i g_* \mathcal{O}_Y = 0$ for every $i > 0$. The $R^i g_* \mathcal{O}_Y$ are independent of the choice of $Y$, so this is a property of $X$ only. If $R^1 g_* \mathcal{O}_Y = 0$, then $X$ is said to have 1-rational singularities.

**Proof of Proposition 9.** If $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$, then we can view $C \mapsto \Theta \cap [C]$ as a linear map

$$\Theta \cap : N_1(X, \mathbb{Q}) \to \mathbb{Q}.$$ 

By definition, line bundles span the dual space of $N_1(X, \mathbb{Q})$, so there is a line bundle $L$ on $X$ and an $m > 0$ such that $\deg(L|_C) = m \cdot \Theta \cap [C]$ for every integral curve $C \subset X$. Thus

$$\deg(L|_C) = m \cdot \Theta \cap [C] \geq m \epsilon \cdot \text{mult}_p C$$

for every integral curve $C \subset X$ and every $p \in C$. Then $L$ is ample by Theorem 7, so $X$ is projective.

It remains to show that $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$ if $X$ has 1-rational singularities only. This is proved in Proposition 12. \hfill \square

As a consequence of Lefschetz’s theorems, sending a curve $C$ to its homology class $[C]$ descends to an injection $N_1(X, \mathbb{Q}) \to H_2(X, \mathbb{Q})$ for smooth projective varieties, see [4, p. 161]. Next we show that the same holds if $X$ has 1-rational singularities; the key ingredient is [10, Proposition 12.1.4]. \hfill \square

**PROPOSITION 12**

Let $X$ be a proper, complex analytic space that is bimeromorphic to a projective variety. Assume that $X$ has 1-rational singularities. Then sending a curve to its homology class gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X, \mathbb{Q})$.

**Proof.** Take a projective resolution $g : Y \to X$. Let $Z$ be a numerically trivial 1-cycle. We need to show that $[Z] \in H_2(X, \mathbb{Q})$ is the 0 class.

For every closed, irreducible, analytic curve $C \subset X$ there is a closed, irreducible, analytic curve $C_Y \subset Y$ such that $g(C_Y) = C$. (In general, $C_Y \to C$ may have degree > 1.) Thus every $Z \in N_1(X, \mathbb{Q})$ lifts to $Z_Y \in N_1(Y, \mathbb{Q})$ such that $g_*(Z_Y) = Z$ (as cycles).
Let $N_1(Y/X, \mathbb{Q}) \subset N_1(Y, \mathbb{Q})$ denote the vector subspace generated by compact complex curves $C \subset Y$ that map to a point in $X$. If there is a $Z'_Y \in N_1(Y/X, \mathbb{Q})$ such that $Z_Y \equiv Z'_Y$, then $[Z] = g_*[Z_Y] = g_*[Z'_Y] = 0$ in $H_2(X, \mathbb{Q})$.

Otherwise, there is a line bundle $L_Y$ on $Y$ that is trivial on $N_1(Y/X, \mathbb{Q})$ but $(L_Y \cdot Z_Y) \neq 0$. The key point is that, by [10, Proposition 12.1.4], some (positive) power $L^m_Y$ descends to a line bundle $L$ on $X$, and then $(L \cdot Z) = m(L_Y \cdot Z_Y) \neq 0$ gives a contradiction. (This is where we use that $X$ has 1-rational singularities.)

□

### 3. Chow–Barlet spaces with marked multiplicities

Let $g : X \to S$ be a proper morphism. We are interested in the set of proper 1-cycles $Z \subset X$ with a marked point $p \in Z$ such that $g(Z) \subset S$ is a single point and $\text{mult}_p Z = m$. The corresponding coarse moduli space is usually called a Chow variety in the algebraic case and a Barlet space in the complex analytic setting; see [2] for their theory.

We need only a rough approximation of these by a countable union of projective morphisms. This can be easily derived from the classical theory of Chow varieties for projective spaces.

**PROPOSITION 13**

Let $g : X \to S$ be a proper morphism of complex analytic spaces that is bimeromorphic to a projective morphism. Fix $m \in \mathbb{N}$. Then there are countably many diagrams of complex analytic spaces over $S$,

$$
\begin{array}{ccc}
& C_i & \longrightarrow & W_i \times_S X \\
& w_i & & \downarrow \sigma_i \\
W_i & & \longrightarrow & W_i
\end{array}
$$

(13.1)

indexed by $i \in I$, such that

- (13.2) the $w_i : C_i \to W_i$ are proper, of pure relative dimension 1 and flat over a dense, Zariski open subset $W_i^\circ \subset W_i$,
- (13.3) the fiber of $w_i$ over any $p \in W_i^\circ$ has multiplicity $m$ at $\sigma_i(p)$,
- (13.4) the $W_i$ are irreducible, the structure maps $\pi_i : W_i \to S$ are projective, and
- (13.5) the fibers over all the $W_i^\circ$ give all irreducible curves that have multiplicity $m$ at the marked point.

**Proof.** By assumption, there is a bimeromorphic morphism $r : Y \to X$ such that $Y$ is projective over $S$. The Chow variety of curves on $Y/S$ exists and its irreducible components are projective over $S$ (cf. [12, Section I.5] or [2]). The universal curve over it parametrizes all pointed curves on $Y$. If we have a family of pointed curves

$$
\begin{array}{ccc}
& C_Y & \longrightarrow & W \times_S Y \\
& w_Y & & \downarrow \sigma_Y \\
W & & \longrightarrow & W
\end{array}
$$

(13.6)
taking its image on \( X \) gives

\[
\begin{array}{c}
C \\
\downarrow w \\
\downarrow s \\
W
\end{array} \rightarrow \begin{array}{c}
W \times_X X
\end{array}
\]

(13.7)

Here \( w : C \to W \) is proper, of relative dimension 1 and flat over a dense, Zariski open subset \( W^o \subseteq W \). The multiplicity of a fiber at a section is an upper semicontinuous function on \( W^o \). For each \( m \in \mathbb{N} \), let \( W^m \subseteq W \) denote the closure of the set of points \( p \in W^o \) for which \( \text{mult}_{\sigma_i(p)} C_p = m \). We repeat this for all irreducible components of \( W \setminus W^o \). At the end we get countably many diagrams as in (13.1) that satisfy (13.2)–(13.4) but not yet (13.5).

Let \( X^o \subseteq X \) be the largest open set over which \( r \) is an isomorphism. The above procedure gives all irreducible pointed curves that have nonempty intersection with \( X^o \). Equivalently, all curves that are not contained in \( X \setminus X^o \). (We may also get some curves contained in \( X \setminus X^o \).

We can now use dimension induction to get countably many diagrams that give us all curves on \( g : (X \setminus X^o) \to S \). The union of these families with the previous ones gives countably many diagrams that satisfy (13.2)–(13.5). \( \square \)

4. Projectivity of very general fibers

Here we prove that, under the assumptions of Theorem 2, there are many projective fibers.

PROPOSITION 14

Notation and assumptions as in Theorem 2. Then there is a Euclidean open neighborhood \( 0 \in U \subset S \) and countably many nowhere dense, closed, analytic subsets \( \{ H_j \subset U : j \in J \} \), such that \( X_s \) is projective for every \( s \in U \setminus \bigcup_j H_j \).

Proof. First choose \( 0 \in U \subset S \) such that \( X_U \) retracts to \( X_0 \). Since \( X_0 \) is projective, it has an ample line bundle \( L \). Let \( \Theta \in H^2(X_U, \mathbb{Q}) \) be the pull-back of \( c_1(L) \) to \( X_U \). Note that \( \Theta \) is a topological cohomology class that is usually not the Chern class of a holomorphic line bundle.

Consider now the diagrams (13.1) indexed by the countable set \( I \). Let \( J \subset I \) index those diagrams for which \( H_i := \pi_i(W_i) \subseteq S \) is nowhere dense in \( U \).

We aim to show that the restriction \( \Theta_s := \Theta|_{X_s} \) satisfies (9.1) for \( s \in U \setminus \bigcup_j H_j \).

Indeed, pick a curve \( C_s \subset X_s \) and a point \( p_s \in C_s \). Set \( m = \text{mult}_{p_s} C_s \). By assumption, there is an \( i \in I \setminus J \) and a diagram as in (13.1)

\[
\begin{array}{c}
C_i \\
\downarrow w_i \\
\downarrow \sigma_i \\
W_i
\end{array} \rightarrow \begin{array}{c}
W_i \times_X X
\end{array}
\]

(14.1)

with a dense, Zariski open subset \( W^o_i \subseteq W_i \), such that
(14.2) \((C_s, p_s)\) is one of the fibers of \(w_i\) over \(W_i^\circ\),
(14.3) \(\text{mult}_{\sigma(p)} C_p = m\) for all \(p \in W_i^\circ\), and
(14.4) \(\pi_i : W_i \to S\) is projective and its image contains \(0 \in S\).

Thus there is a disc \(D\) (say of radius \(> 1\)) and a holomorphic map \(\tau : D \to W_i\) such that \(\pi_i(\tau(0)) = 0 \in S\) and \(\pi_i(\tau(1)) = s \in S\). After pulling back and discarding embedded points we get

\[
\begin{array}{cc}
C^0 & \subset \rightarrow D \times_S X \\
\sigma | \uparrow & \\
D & \downarrow \sigma
\end{array}
\]

(14.5)

where \(w\) is flat. Let \(C^D_t\) denote the fiber over \(t \in D\) and \(p_t = \sigma(t)\) the marked point. Thus \(C^D_0\) lies over \(0 \in S\) and \(C^D_1\) is identified with our original curve \(C_S\).

Note that

\[
\text{mult}_{p_t} C^D_t = \text{mult}_{p_1} C^D_1 = \text{mult}_{p_s} C_S \quad \text{for all } t \in D^\circ.
\]

Since multiplicity is an upper semi-continuous function, we conclude that

\[
\text{mult}_{p_0} C^D_0 \geq \text{mult}_{p_t} C^D_t = \text{mult}_{p_s} C_S.
\]

Here \(C^D_0\) is a 1-cycle on the projective scheme \(X_0\), and \(\Theta_0\) is the Chern class of an ample line bundle on \(X_0\). Thus

\[
\Theta \cap [C^D_0] \geq \epsilon \cdot \text{mult}_{p_0} C^D_0,
\]

by the easy direction of Theorem 7, where \(\epsilon\) depends only on \(X_0\) and \(\Theta_0\). Putting these together gives that

\[
\Theta_s \cap [C_s] = \Theta \cap [C^D_1] = \Theta \cap [C^D_0] \geq \epsilon \cdot \text{mult}_{p_0} C^D_0 \geq \epsilon \cdot \text{mult}_{p_s} C_S.
\]

Thus \(X_s\) is projective by Proposition 9.

5. Projectivity of general fibers

We prove a result about the set of projective fibers of proper analytic maps. To formulate it, let \(g : X \to S\) be a proper morphism of complex analytic spaces and set

\[
\text{PR}_S(X) := \{s \in S : X_s \text{ is projective}\}.
\]

We prove in Proposition 17 that \(\text{PR}_S(X)\) either contains a dense, Zariski open subset or it is contained in a countable union of Zariski closed, nowhere dense subsets. Results of this type appear in many places, for example, [7, 15, 18–20]. All the arguments below can be found in them.

The next lemma is frequently stated in the algebraic case as in [3, p. 43], but the proof works for proper analytic maps as well. See, for example, [17, p. 10].

**Lemma 15.** Let \(g : X \to S\) be a proper morphism of complex analytic spaces. Then there is a dense, Zariski open subset \(S^0 \subset S\) such that \(g^\circ : X^\circ \to S^0\) is a topologically locally trivial fiber bundle.
Applying this inductively, we get the following.

**COROLLARY 16**

Let $g : X \to S$ be a proper morphism of complex analytic spaces. Then the sheaves $R^i g_* \mathcal{Z}_X$ are constructible in the analytic Zariski topology. □

**PROPOSITION 17**

Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then there is a dense, Zariski open subset $S^\circ \subset S$ such that

(17.1) either $X$ is locally projective over $S^\circ$,

(17.2) or $\text{PR}_S(X) \cap S^\circ$ is locally contained in a countable union of Zariski closed, nowhere dense subsets.

**Remark 17.3.** The following example clarifies (17.2). Let $L \subset \mathbb{C}^2$ be a general line and $Z$ its image in a complex torus $\mathbb{C}^2/\mathbb{Z}^4$. Then $Z$ is irreducible and everywhere dense in the Euclidean topology. However, if $U \subset \mathbb{C}^2/\mathbb{Z}^4$ is contractible, then $Z \cap U$ is a countable union of Zariski closed, nowhere dense subsets of $U$.

**Proof.** By passing to a Zariski open subset, we may assume that $R^2 g_* \mathcal{O}_X$ is locally free and $R^2 g_* \mathcal{Z}_X$ is locally constant. By passing to the universal cover, we may also assume that $R^2 g_* \mathcal{Z}_X$ is a constant sheaf. The exponential sequence now shows that the holomorphic line bundles on $X$ are given by the kernel of

$$\partial : R^2 g_* \mathcal{Z}_X \to R^2 g_* \mathcal{O}_X,$$

and for $s \in S$, the holomorphic line bundles on $X_s$ are given by the kernel of

$$H^2(X_s, \mathcal{Z}_{X_s}) = R^2 g_* \mathcal{Z}_X|_{\{s\}} \to R^2 g_* \mathcal{O}_X|_{\{s\}} = H^2(X_s, \mathcal{O}_{X_s}).$$

Let $\Theta$ be a global section of $R^2 g_* \mathcal{Z}_X$. If $\partial \Theta \equiv 0$, then there is a global line bundle $L_\Theta$ corresponding to it. If such an $L_\Theta$ is ample on some $X_s$, then it is ample over a Zariski open neighborhood of $s$ and we are in case (17.1). Assume now that this is not the case. If $\partial \Theta$ is not identically 0, then $\partial \Theta = 0$ defines a Zariski closed, nowhere dense subset $H_\Theta \subset S$. We claim that $\text{PR}_S(X)$ is contained in the union of these $H_\Theta$. Indeed, pick $s \in S$ in their complement. Then every line bundle on $X_s$ is the restriction of a line bundle on $X$ (up to numerical equivalence) and these are not ample by assumption. □

**Complement 18.** Notation and assumptions are as in Proposition 17. Assume in addition that $g$ is bimeromorphic to a projective morphism and we are in case (17.1). Then $X$ is projective over $S^\circ$.

**Proof.** Choose a projective modification $\pi : X' \to X$. Set $g' := g \circ \pi$. By passing to a Zariski open subset, we may assume that $R^2 g_* \mathcal{O}_X$ and $R^2 g'_* \mathcal{O}_{X'}$ are locally free, and $R^2 g_* \mathcal{Z}_X$ and $R^2 g'_* \mathcal{Z}_{X'}$ are locally constant. Since $g'$ is projective, the monodromy on the algebraic classes of $R^2 g'_* \mathcal{Z}_{X'}$ is finite. Since $X$ is normal, a holomorphic line bundle is
trivial on \(X\) iff its pull-back to \(X'\) is trivial, hence the natural map \(R^2g_*\mathbb{Z}_X \to R^2g'_*\mathbb{Z}_{X'}\) is injective on algebraic classes. Therefore the monodromy on the algebraic classes of \(R^2g_*\mathbb{Z}_X\) is also finite.

Once the monodromy on the kernel of \(\partial : R^2g_*\mathbb{Z}_X \to R^2g_*\mathcal{O}_X\) is finite, we can trivialize it by a finite cover of \(S\). We thus find a relatively ample line bundle \(L_{\Theta}\) after a finite cover of \(S\), which then gives a relatively ample line bundle on \(X\). □

**Proof of Theorem 2.** By Proposition 14, \(PR_S(X)\) contains the complement of a countable union of Zariski closed, nowhere dense subsets. Therefore, by the Baire category theorem, \(PR_S(X)\) is not contained in a countable union of closed, nowhere dense subsets. Thus we are in case (17.1) and \(g\) is locally projective over a dense, Zariski open subset \(S^\circ \subset S\). Global projectivity over \(S^\circ\) is given by Complement 18.

We finish the proof by induction applied to \(S \setminus S^\circ\). □

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