ON THE TOPOLOGY OF A SMALL COVER ASSOCIATED TO A SHELLABLE COMPLEX

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ABSTRACT. From a given shelling of a simplicial complex $K$, we get a cell decomposition for a small cover $Y$ associated to $K$, which is a generalization of the one given in Davis and Januszkiewicz [DJ91] by a Morse function. Then we study its relation with the well-known algebraic decomposition of the mod 2 Stanley-Reisner ring $\mathbb{Z}_2[K]$ from the shelling. In particular we calculate the cohomology groups of $Y$ with integer coefficients.

1. Introduction

Let $\mathbb{Z}_2$ be the field with two elements 0 and 1. In the ring $\mathbb{Z}_2[x_1, \ldots, x_m]$ of polynomials, an ideal $I$ generated by square-free monomials uniquely corresponds to an abstract simplicial complex $K$, which is a collection of subsets of $[m] = \{1, \ldots, m\}$. The quotient $\mathbb{Z}_2[K] = \mathbb{Z}_2[x_1, \ldots, x_m]/I$ is called the Stanley-Reisner ring of $K$.

Suppose the Krull dimension of $\mathbb{Z}_2[K]$ is $n$ and $\deg x_i = 1, i = 1, \ldots, m$. Consider the case when $\mathbb{Z}_2[K]$ is

(a) Cohen-Macaulay, and
(b) admits a homogeneous system of parameters by polynomials of degree one.\(^1\)

That is to say, there exist homogeneous and algebraically independent polynomials $\theta_1, \ldots, \theta_n$ of degree one so that $\mathbb{Z}_2[K]$ is a finitely generated free $\mathbb{Z}_2[\theta_1, \ldots, \theta_n]$-module (see Section 2 for more details).

Let $G = \mathbb{Z}_2^n$ be the direct sum of $n$ copies of $\mathbb{Z}_2$. The objects above have the following topological interpretations: in [DJ91] Davis and Januszkiewicz

\(^1\)We treat $\mathbb{Z}_2$ as a group when considering its actions on spaces.

\(^2\)Since $\mathbb{Z}_2$ is finite, (b) is not a consequence of the Noether Normalization Lemma (cf. [Sta95, Theorem 5.2, p. 34]).
showed that under (a) and (b), there is a $G$-space $Y$, called a small cover,\footnote{For convenience we still use the name “small cover” and the notation $Y$, as in [DJ91, Section 5, pp. 439–444]. Real topological toric manifolds, as generalizations of real quasi-toric manifolds (see below), were introduced and studied in [IFM13]: in the language here they are also small covers.} such that

1. the mod 2 cohomology ring of the Borel construction $EG \times_G Y$ is isomorphic to $\mathbb{Z}_2[K]$;
2. the inclusion $Y \to EG \times_G Y$ induces a surjection in mod 2 cohomology, and $H^*(Y; \mathbb{Z}_2)$ is isomorphic to the quotient $\mathbb{Z}_2[K]/(\theta_1, \ldots, \theta_n)$.

Moreover, with a further assumption that (a1) $K$ is the boundary complex of a convex polytope, $Y$ admits a canonical smooth structure to be a smooth $G$-manifold (referred to as a real quasi-toric manifold), on which one can choose a Morse function to give a cell structure so that a cell is a mod 2 cycle as a cellular chain, and the total number of cells is precisely $\dim H^*(Y; \mathbb{Z}_2)$. However, their construction does not give an explicit basis for $H^*(Y; \mathbb{Z}_2)$.

Go back to algebra, earlier works of Kind and Kleinschmidt [KK79] and Garsia [Gar80] showed that the ring $\mathbb{Z}_2[K]/(\theta_1, \ldots, \theta_n)$ does admit a simple and explicit basis (see Proposition 3.1), provided that $K$ is shellable.\footnote{Actually this basis also works, with $\mathbb{Z}_2$ replaced by $\mathbb{Z}$, for a quasi-toric manifold; see Remark 3.2.} Shellability is an important concept arising from combinatorics; the assumption of shellability is well-known to be (strictly) stronger than (a) but weaker than (a1) (cf. [Sta95, pp. 82–84]).

In this paper we focus on the topological meaning of this basis to a small cover $Y$. We show that:

1. a shelling of $K$ gives a cell structure on $Y$, such that a dual cell (as a cellular cochain) is a mod 2 cocycle, and all dual cells give the basis above; under (a1), this cell structure specializes to the one given in [DJ91] by a Morse function.\footnote{In this case a cell is the core of a piecewise linear handle, which can be smoothed canonically; see Remark 3.5.}
2. With integer coefficients, the corresponding cellular cochain complex of $Y$ is cochain-homotopy equivalent to\footnote{As is well-known, a (co)chain-homotopy equivalence between two (co)chain complexes implies the existence of a (co)chain map between them which induces an isomorphism in (co)homology, and vice versa (cf. [Mun84, Theorem 46.2, pp. 279–280]).} a cochain complex $(\mathcal{C}^\lambda_\lambda, 2\mathcal{D})$,
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where \((\overline{C}_\lambda', \overline{d}')\) is a direct sum of the cellular cochain complexes of a family of full subcomplexes of \(K\) (see Theorem 5.2).\(^7\)

In this way we get all information of the cohomology groups \(H^*(Y; \mathbb{Z})\) (see Corollary 5.3). In particular, we calculate explicitly the (higher) mod 2 Bockstein homomorphisms on \(H^*(Y; \mathbb{Z}_2)\) (see Corollary 5.4).

Buchstaber and Ray [BR98] showed that every unoriented cobordism class contains a real quasi-toric manifold; it is well-known that the real part of a projective toric variety (over \(\mathbb{C}\)) is a real quasi-toric manifold. Partial information of \(H^*(Y; \mathbb{Z})\) was obtained by Trevisan [Tre12], Suciu and Trevisan [ST12], and later by Choi and Park [CP13] (see Corollary 5.3).\(^8\)

The paper is organized as follows. Section 2 is a review of notations and previous results on small covers. The concept of shellability, and the induced filtrations on associated objects are given Section 3. Section 4 is a review of Bockstein homomorphisms. The main theorem, Theorem 5.2 (without proof), and its corollaries are given in Section 5. Sections 6 and 7 are preparations to prove Theorem 5.2; the proof is given in Section 8, which is essentially (again) an induction based on a shelling. In Section 9 we give examples to illustrate Theorem 5.2.

2. SMALL COVERS OVER SIMPLE POLYHEDRAL COMPLEXES

Throughout this paper, let \(K \subset 2^{[m]}\) be an abstract simplicial complex with its geometric realization \(|K|\): \(K\) is a collection of subsets of \([m] = \{1, 2, \ldots, m\}\) (including the empty set), with an element \(\sigma \in K\) called a simplex or a face, so that \(\sigma \in K\) implies \(\tau \in K\), for all \(\tau \subset \sigma\). The geometric realization of the face \(\sigma\) will be denoted by \(|\sigma|\), and its dimension \(\dim(\sigma)\) is one less than the cardinality \(\text{card}(\sigma)\). The dimension of \(|K|\) is the number \(\max\{\dim(\sigma) \mid \sigma \in K\}\).

Denote by \(\text{Vert}(K) \subset [m]\) the vertex set \(\{i \mid \{i\} \in K\}\). Suppose \(\dim(|K|) = n - 1\) and \(\text{Vert}(K) = [m]\). Let \(P\) be the simple polyhedral complex dual to \(|K|\) (cf. [DJ91, p. 428]): as a polyhedron, \(P\) is the cone over the barycentric subdivision \(|K'|\) of \(|K|\); \(P\) has a mirror structure \(F = \{F_1, \ldots, F_m\}\), in which \(F_i\) is the star of the \(i\)-th vertex of \(|K|\) in \(|K'|\).

\(^7\)Here \(2d\) means that the coefficients from coboundary operators are doubled: this explains why a dual cell of \(Y\) gives a mod 2 cocycle. This family of full subcomplexes is due to Suciu and Trevisan [ST12] in the description of the rational cohomology of a real quasi-toric manifold.

\(^8\)When \(Y\) is a pullback of the linear model, \(H^*(Y; \mathbb{Z})\) was given in [DJ91, Corollary 6.11]; see also Yu [Yu14] and [CKT16] on stable decompositions of small covers.
Let $e_i$ be the $i$-th generator of $\mathbb{Z}_2^m$. Suppose there is a characteristic function $\lambda: \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, such that

$$\sigma \in K \implies \text{rank}(\lambda|_\sigma) = \text{card}(\sigma),$$

where $\lambda|_\sigma$ denotes the restriction of $\lambda$ to the subspace spanned by $\{e_i\}_{i \in \sigma}$. For $x \in P$, define $\sigma_x = \{i \mid x \in F_i\}$, and let $G_x \subseteq \mathbb{Z}_2^n$ be the image of $\lambda|_{\sigma_x}$. $G_x$ consists of only the identity if $\sigma_x = \emptyset$.

Denote by $G$ the group $\mathbb{Z}_2^n$. With respect to $\lambda$ and $P$, there is a $G$-space given by the Basic Construction (cf. [DJ91]):

$$Y = \mathbb{Z}_2^n \times P / \sim$$

where $(g', x') \sim (g, x)$ in $Y$ if and only if $x = x'$ and $g^{-1}g' \in G_x$. $G$ acts on $Y$ via $(h, [g, x]) \mapsto [hg, x]$.

Let $BP = E \times_G Y$ be the Borel construction. We have a fiber bundle

$$Y \longrightarrow BP \longrightarrow BG.$$ 

Let $(\mathbb{R}P^\infty, *)$ be the infinite real projective space with a base point.

**Proposition 2.1** (Cf. [DJ91]). Up to homotopy,

$$BP = \bigcup_{\sigma \in K} B'_\sigma; \quad B'_\sigma = \prod_{i=1}^m X_i; \quad X_i = \begin{cases} \mathbb{R}P^\infty & i \in \sigma \\ * & i \notin \sigma \end{cases}$$

moreover, the mod 2 cohomology of $BP$ is isomorphic to the Stanley-Reisner ring of $K$, namely

$$H^i(BP; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \ldots, x_m]/I_K \quad \text{(deg}(x_i) = 1, \ i = 1, \ldots, m),$$

where the Stanley-Reisner ideal $I_K$ is generated by square-free monomials of the form $x_\tau = \prod_{i \in \tau} x_i$ for all $\tau \not\subseteq K$.

Denote by $\mathbb{Z}_2[K]$ the (mod 2) Stanley-Reisner ring of $K$. It can be shown that the Krull dimension of $\mathbb{Z}_2[K]$ is $n$ (cf. [Sta95, pp. 53–54]); by Noether Normalization Lemma, there exist homogeneous polynomials $\theta_1, \ldots, \theta_n$ that are algebraically independent, such that $\mathbb{Z}_2[K]$ is a finitely generated $\mathbb{Z}_2[\theta_1, \ldots, \theta_n]$-module. Such a sequence $\theta_1, \ldots, \theta_n$ is called a homogeneous system of parameters (h.s.o.p. for short).

If additionally $\mathbb{Z}_2[K]$ is a free $\mathbb{Z}_2[\theta_1, \ldots, \theta_n]$-module, then $\mathbb{Z}_2[K]$ is Cohen-Macaulay, and we say that $K$ is a Cohen-Macaulay complex. By a result of Reisner [Rei76], $\mathbb{Z}_2[K]$ is Cohen-Macaulay if and only if $\tilde{H}_i([K]; \mathbb{Z}_2) = 0$ for all $i < n - 1$, and $\tilde{H}_i(\text{Link}(\sigma, K); \mathbb{Z}_2) = 0$ for all $i < \dim \text{Link}(\sigma, K)$, where $\text{Link}(\sigma, K)$ denotes the link of $\sigma$ in $K$. 
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Definition 2.2. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an $(n \times m)$-matrix with $\lambda_i \in \mathbb{Z}_2^m$. We identify $\lambda_i$ with a subset of $[m]$ by its non-zero entries. A linear system of parameters (l.s.o.p. for short) of $\mathbb{Z}_2[K]$ associated to $\lambda$ is an h.s.o.p. of the form

$$l_{\lambda_i} = \sum_{k \in \lambda_i} x_k, \quad i = 1, \ldots, n.$$ 

The criterion below is useful.

Proposition 2.3 (Cf. [Sta95, pp. 81–82]). The following statements are equivalent:

1. $\lambda: \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2^n$ is a characteristic function which satisfies (1);
2. $l_{\lambda_1}, \ldots, l_{\lambda_n}$ is an l.s.o.p. of $\mathbb{Z}_2[K]$ associated to $\lambda$.

The quotient ring $\mathbb{Z}_2[K]/(l_{\lambda_1}, \ldots, l_{\lambda_n})$ has the following topological interpretation.

Proposition 2.4 (Cf. [DJ91, Theorem 5.12]). When $K$ is a Cohen-Macaulay complex admitting a characteristic function $\lambda$, the inclusion $Y \rightarrow BP$ in the fibration (3) induces a surjection in mod 2 cohomology, and we have an isomorphism

$$H^*(Y; \mathbb{Z}_2) \cong \mathbb{Z}_2[K]/(l_{\lambda_1}, \ldots, l_{\lambda_n})$$

of graded rings.

3. Filtrations from shellability

A face of $K$ is called a facet if it is not contained in any larger face. Recall that $K$ is shellable, if there is an ordering of its facets, $\sigma_1, \ldots, \sigma_s$, say, all of the same dimension $n - 1 = \dim |K|$ (i.e., $K$ is pure), together with a filtration $\sigma_1 = K_1 \subset K_2 \subset \cdots \subset K_s = K$, $|K_{j+1}| = |K_j| \cup |\sigma_{j+1}|$, such that for all $j = 1, 2, \ldots, s - 1$, the intersection $|\sigma_{j+1}| \cap |K_j|$ is a union of $(n - 2)$-faces of $|\sigma_{j+1}|$. Equivalently, we have a minimal face $r(\sigma_{j+1})$ among all faces of $K_{j+1}$ that are not contained in $K_j$, called the restriction of $\sigma_{j+1}$, which is unique. Formally we set

$$r(\sigma_1) = \emptyset, \quad x_{r(\sigma_1)} = 1$$

where $x_\tau = \prod_{i \in \tau} x_i$ for $\tau \subset [m]$.

The following basis for $H^*(Y; \mathbb{Z}_2)$ is convenient.

Proposition 3.1 (Cf. [Sta95, pp. 82–83]). Let $\sigma_1, \ldots, \sigma_s$ be a shelling of $K$ with characteristic function $\lambda = (\lambda_i)_{i=1}^n$. Then $\mathbb{Z}_2[K]$ is a free $\mathbb{Z}_2[\lambda_1, \ldots, \lambda_n]$-module with basis $\{x_{r(\sigma_j)}\}_{j=1}^s$. 

Remark 3.2. Garsia [Gar80, Theorem 4.2] showed that Proposition 3.1 holds with $\mathbb{Z}_2$ replaced by $\mathbb{Z}$, where $\lambda: \mathbb{Z}^m \to \mathbb{Z}^n$ satisfies that $\lambda|_{\sigma_j}: \mathbb{Z}^n \to \mathbb{Z}^n$ is an isomorphism of abelian groups, $j = 1, \ldots, s$ (compare (1)).

Throughout this section, let $\lambda$ be a characteristic function with respect to $K$, and suppose $\sigma_1, \ldots, \sigma_s$ is a shelling with filtration $\sigma_1 = K_1 \subset \ldots \subset K_s = K$.

Consider the real moment-angle complex associated to $K$, which is given by

\[
\mathbb{R}Z = \bigcup_{\sigma \in K} B_{\sigma}; \quad B_{\sigma} = \prod_{i=1}^{m} X_i, \quad X_i = \begin{cases} I & \text{if } i \in \sigma \\ \partial I = \{-1, 1\} & \text{if } i \notin \sigma. \end{cases}
\]

It is naturally a $\mathbb{Z}_2^m$-space with the action generated by changing the sign of each coordinate.

Denote by $\text{Ker}\lambda$ the kernel of $\lambda: \mathbb{Z}_2^m \to \mathbb{Z}_2^n$. We see that by (1), $\text{Ker}\lambda$ is isomorphic to $\mathbb{Z}_2^{n-m}$. Suppose $\text{Vert}(K) = [m]$ and $\text{Vert}(K_j) = [m_j]$, $m_j \leq m$, $j = 1, \ldots, s$. Naturally $\lambda$ can be treated as a characteristic function with respect to $K_j$, and we shall denote it by $\lambda|_{K_j}: \mathbb{Z}_2^{m_j} \to \mathbb{Z}_2^n$; let $Y_j$ be the corresponding small cover given by (2), and let $\mathbb{R}Z_j \subset I^m$ be the real moment-angle complex given by (6), with $K$ replaced by $K_j$.

Lemma 3.3. We have the following properties.

(a) As a subgroup of $\mathbb{Z}_2^m$, the action of $\text{Ker}\lambda$ on $\mathbb{R}Z_j$ is free, $j = 1, \ldots, s$.

(b) Endowed with the induced action of $\mathbb{Z}_2^m \cong \mathbb{Z}_2^m/\text{Ker}\lambda$, the orbit space $\mathbb{R}Z_j/\text{Ker}\lambda$ (as a polyhedron) is $\mathbb{Z}_2^m$-equivariantly and piecewise linearly homeomorphic to the small cover $Y_j$.

Proof. For (a), it suffices to prove the $\text{Ker}\lambda$-action is free on $\mathbb{R}Z$. We follow the argument of [Pan08, Theorem 3.12]: an isotropy subgroup of the $\mathbb{Z}_2^m$-action on $\mathbb{R}Z$ is of the form

$$
\mathbb{Z}_2^\sigma = \{(g_1, \ldots, g_m) \in \mathbb{Z}_2^m \mid g_i = 0 \text{ if } i \notin \sigma\},
$$

for some $\sigma \in K$. Since $\lambda|_{\sigma}$ is non-degenerate, $\mathbb{Z}_2^\sigma \cap \text{Ker}\lambda$ is trivial, thus the action is free.

Denote by $\mathbb{R}^m_+$ the first orthant $\{x = (x_i)_{i=1}^m \in \mathbb{R}^m \mid x_i \geq 0, i = 1, \ldots, m\}$, and let $H_i$ be the $i$-th coordinate hyperplane $\{x = (x_i)_{i=1}^m \in \mathbb{R}^m \mid x_i = 0\}$, $i = 1, \ldots, m$. Let $P'_j$ be the intersection $\mathbb{R}Z_j \cap \mathbb{R}^m_+$, with the mirror structure $F^i_j = \{F^i_1, \ldots, F^i_{m_j}\}$, where $F^i_j = P'_j \cap H_i$. (b) can be proved by the following steps:

1. there is a piecewise linear homeomorphism $\iota: P'_j \to P_j$ preserving the mirror structures, where $P_j$ is the simple polyhedron complex dual to $|K_j|$ (cf. [BP02, Chapter 4, pp. 53–55]).
(2) define
\[ \mathbb{R}Z'_j = \mathbb{Z}^m_2 \times P'_j / \sim, \]
where \((g', x') \sim (g, x)\) if and only if \(x = x'\) and \(g^{-1}g' \in \tilde{G}_x\), \(\tilde{G}_x \subset \mathbb{Z}^m_2\)
the subgroup generated by \(\{e_i \mid x \in P'_j\}\) (\(e_i\) is the \(i\)-th generator of \(\mathbb{Z}^m_2\)), then the map \(\mathbb{R}Z'_j \to \mathbb{R}Z_j\) with \([g, x] \mapsto \tilde{g}x\) is a \(\mathbb{Z}^m_2\)-equivariant and piecewise linear homeomorphism;
(3) the map \(\mathbb{R}Z'_j \to Y_j\) with \([g, x] \mapsto [\lambda(g), \iota(x)]\) induces a desired homeomorphism \(\mathbb{R}Z_j / \text{Ker}\lambda = \mathbb{R}Z'_j / \text{Ker}\lambda \to Y_j\).

\[ \square \]

Recall that for \(0 \leq k \leq n\), an \(n\)-dimensional \(k\)-handle \(W\) on a piecewise linear (PL for short) manifold \(X\) is a copy of \(I^k \times I^{n-k}\), attached to the boundary \(\partial X\) via a PL embedding \(f \colon \partial I^k \times I^{n-k} \to \partial X\). The number \(k\) is called the index of the handle. It turns out that the union \(X \cup_f W\) is again a PL \(n\)-manifold (cf. [RS72, Chapter 6]).

For \(\sigma \subset [m]\), let \(I_\sigma = \prod_{i=1}^{m} X_i \subset I^m\) be the cube with \(X_i = I\) if \(i \in \sigma\), or \(X_i = \{1\}\) otherwise. Clearly \(\dim I_\sigma = \text{card}(\sigma)\). By definition (6), \(\mathbb{R}Z_1 = B_{\sigma_1}\); since \(\lambda|_{\sigma_1}\) is non-degenerate, \(\mathbb{R}Z_1 = B_{\sigma_1}\) coincides with the orbit of \(I_{\sigma_1}\) under the action of \(\text{Ker}\lambda\), and the orbit map \(\pi\) maps \(I_{\sigma_1}\) homeomorphically onto \(Y_1\).

For \(j = 1, \ldots, s-1\), \(\mathbb{R}Z_{j+1}\) is obtained from \(\mathbb{R}Z_j\) by attaching \(B_{\sigma_{j+1}}\), a disjoint union of \(2^{m-n}\) cubes of dimension \(n\); consider the cube
\[ I_{\sigma_{j+1}} = I_{r(\sigma_{j+1})} \times I_{\sigma_{j+1} \setminus r(\sigma_{j+1})} \]
in \(\mathbb{R}Z_{j+1}\): the non-degeneracy of \(\lambda|_{\sigma_{j+1}}\) implies that \(B_{\sigma_{j+1}}\) is the orbit of \(I_{\sigma_{j+1}}\) under \(\text{Ker}\lambda\), which are attached along the orbit of \(\partial I_{r(\sigma_{j+1})} \times I_{\sigma_{j+1} \setminus r(\sigma_{j+1})}\).

Moreover, this attaching is along the (topological) boundary \(\partial \mathbb{R}Z_j\) if and only if \(|\sigma_{j+1}|\) is attached along the boundary \(\partial |K_j|\). The orbit map \(\pi\) maps \(I_{\sigma_{j+1}}\) homeomorphically onto its image in \(M_{j+1}\), which is a PL embedding into \(M_j\) when restricted to \(\partial I_{r(\sigma_{j+1})} \times I_{\sigma_{j+1} \setminus r(\sigma_{j+1})}\).

Suppose the cardinality of \(r(\sigma_j)\) is \(k_j\). The proposition below follows from an induction on \(j\) (see Example 6.5 for an illustration).

**Proposition 3.4.** The filtration \(Y_1 \subset \ldots \subset Y_s\) gives a PL handle decomposition of \(M\), where \(Y_1 = I^n\), \(Y_s = Y\) and \(Y_{j+1}\) is obtained from \(Y_j\) by attaching a handle of index \(k_{j+1}\), provided that \(|K_{j+1}|\) is obtained by attaching \(|\sigma_{j+1}|\) along \(\partial |K_j|\), \(j = 1, \ldots, s-1\). In general \(Y_{j+1}\) is obtained from \(Y_j\) by attaching a cell of dimension \(k_{j+1}\) up to homotopy.

**Remark 3.5.** When \(|K|\) is the boundary complex of a convex polytope \(P_0\), where \(P_0\) is the dual of a simple polytope \(P \subset \mathbb{R}^n\), one can choose a generic
linear function $f: \mathbb{R}^n \to \mathbb{R}$, such that the gradient of $f$ gives an ordering $v_1, \ldots, v_s$ of vertices of $P$, and gives an orientation for each edge of $P$. Since a vertex $v$ of $P$ corresponds to a facet $\sigma$ of $K$, and the $n$ edges around $v$ correspond to the $n$ facets of $\sigma$, $f$ gives a shelling $\sigma_1, \ldots, \sigma_s$ of $K$.

We can use the smooth structure of $P \subset \mathbb{R}^n$ to smooth the PL handles of $Y$, which coincides with the smooth handles given in [DJ91, pp. 431–432]: let $p: Y \to P$ be the orbit map associated to the $\mathbb{Z}_2$-action on $Y$, it can be checked directly that $\pi: \mathbb{R}Z \to Y$ maps the interior of $I_{\sigma_i}$ homeomorphically onto a neighborhood $p^{-1}(U_i)$, where $U_i$ is a neighborhood of $v_i \in P$, such that the closure of $p^{-1}(U_i)$ is the smooth handle from the Morse function.

**Remark 3.6.** It is a standard fact in PL topology that, if each attaching of $|\sigma_{j+1}|$ is along $\partial|K_j|$ (in the shelling), then $|K|$ is either a PL sphere or a PL disk of dimension $n - 1$, depending on whether the attaching of the last facet $|\sigma_s|$ is along its whole boundary or not. Therefore when $K$ is shellable (and admits a characteristic function), $\mathbb{R}Z$ (resp. $Y$) is a closed PL $n$-manifold if and only if $|K|$ is a PL $(n - 1)$-sphere.\(^9\)

Let $l_{\lambda_1}, \ldots, l_{\lambda_n}$ be the l.s.o.p. associated to the characteristic function $\lambda$ (see Proposition 2.3), and $l_{\lambda_1}|_j, \ldots, l_{\lambda_n}|_j$ be its image under $\mathbb{Z}_2[K] \to \mathbb{Z}_2[K_j]$ which is induced by the inclusion $K_j \to K$.

**Proposition 3.7.** The ring $\mathbb{Z}_2[K_j]$ is a free $\mathbb{Z}_2[l_{\lambda_1}|_j, \ldots, l_{\lambda_n}|_j]$-module with basis $\{x_r(\sigma_1)\}_{1 \leq j}$, and $H^*(Y_j; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[K]/(l_{\lambda_1}, \ldots, l_{\lambda_n}, x_r(\sigma_{j+1}), \ldots, x_r(\sigma_s))$ as graded rings.

**Proof.** Notice that $K_j$ is shellable, thus is Cohen-Macaulay. The first statement follows from Proposition 3.1. Let $P_j$ be the simple polyhedral complex dual to $|K_j|$. The second one follows from Proposition 2.4, together with the observation that $\mathbb{Z}_2[K] \to \mathbb{Z}_2[K_j]$ coincides with $H^*(BP; \mathbb{Z}_2) \to H^*(BP_j; \mathbb{Z}_2)$, which is induced from the inclusion $BP_j \to BP$ (see Proposition 2.1). □

### 4. Bockstein homomorphisms

For a given space $X$,\(^{10}\) to recover the $2^k$-torsion elements from $H^*(X; \mathbb{Z}_2)$, it suffices to understand—which is a classical method—their behaviors under

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\(^9\)In fact the assumption of shellability is not essential; compare [Cai15, Theorem 2.3].

\(^{10}\)As usual, suppose $H_i(X; \mathbb{Z})$ is finitely generated for all $i$. 

(higher) mod 2 Bockstein homomorphisms. More precisely, we have an exact couple

\[
\begin{array}{c}
H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}) \\
\downarrow \kappa \\
H^*(X; \mathbb{Z}_2),
\end{array}
\]

(8)

where \(\kappa\) is the connecting homomorphism, which is induced from the exact sequence

\[
0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\mod 2} \mathbb{Z}_2 \to 0
\]
on coefficients. The following fact is well-known (cf. [McC01, Chapter 10]).

**Proposition 4.1.** The single-graded spectral sequence \(E^*_k(X)\) associated to (8) satisfies the following properties.

1. \(E^*_1 = H^*(X; \mathbb{Z}_2)\), and the first differential

\[
d_1 : H^n(X; \mathbb{Z}_2) \to H^{n+1}(X; \mathbb{Z}_2)
\]

coincides with the Steenrod square \(Sq^1\). More explicitly, if the mod 2 reduction \([c]\) of an integral singular cochain \(c\) represents a class in \(H^n(X; \mathbb{Z}_2)\), then

\[
d_1[c] = \left[\frac{1}{2} dc\right],
\]

where \(d\) is the coboundary operator on cochains.

2. A \(\mathbb{Z}_2\)-summand in \(H^{n+1}(X; \mathbb{Z})\) induces a \(\mathbb{Z}_2\)-pair in \(H^n(X; \mathbb{Z}_2)\) and \(H^{n+1}(X; \mathbb{Z}_2)\), respectively, which survives to \(E^*_k(X)\), and is connected by the \(k\)-th differential \(d_k\).

3. \(E^*_\infty(X)\) is the mod 2 reduction of the free part of \(H^*(Y; \mathbb{Z})\).

Now we apply the Bockstein spectral sequence to a small cover \(Y\). First by (4), we see that the mod 2 cohomology groups of \(BP\) is generated by monomials of the form

\[
x_{i_1}^{n_1}x_{i_2}^{n_2} \cdots x_{i_l}^{n_l}, \quad \{i_1, \ldots, i_l\} \in K,
\]
on which we have, by Cartan formula,

\[
Sq^1(x_{i_1}^{n_1}x_{i_2}^{n_2} \cdots x_{i_l}^{n_l}) = \sum_{j=1}^{l} n_j x_{i_1}^{n_1} \cdots x_{i_{j-1}}^{n_{j-1}} x_{i_j}^{n_j+1} x_{i_{j+1}}^{n_{j+1}} \cdots x_{i_l}^{n_l}.
\]

(10)

On passage to \(Y\) via the inclusion \(Y \to BP\), by (5), an element from \(\mathbb{Z}_2[K]\) being a cocycle under \(Sq^1\) means that its image lies in the ideal \((l_{\lambda_1}, \ldots, l_{\lambda_n})\).
Example 4.2. Let \(|K|\) be the boundary of a square with \(\text{Vert}(K) = [4]\): \(K\) admits a shelling \(\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\) with their restrictions \(\emptyset, \{3\}, \{4\}, \{1,4\}\), respectively. One checks easily that

\[
\lambda = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}
\]

is a characteristic function with respect to \(K\). By Proposition 2.4, we have

\[
H^*(Y;\mathbb{Z}_2) \cong \mathbb{Z}_2[K]/(l_{\lambda_1}, l_{\lambda_2}) = \mathbb{Z}_2[x_1, \ldots, x_4]/(x_{1,3}, x_{2,4}, l_{\lambda_1}, l_{\lambda_2})
\]

where \(l_{\lambda_1} = x_1 + x_3 + x_4\) and \(l_{\lambda_2} = x_2 + x_4\). By Proposition 3.1, \(H^*(Y;\mathbb{Z}_2)\) has an additive basis \(\{[1], [x_3], [x_4], [x_{1,4}]\}\). We see that in \(\mathbb{Z}_2[K]\), being a free \(\mathbb{Z}_2[l_{\lambda_1}, l_{\lambda_2}]\)-module,

\[
S_q^1(x_3) = x_3^2 = l_{\lambda_1}x_3 + (l_{\lambda_1} + l_{\lambda_2})x_4 + x_{1,4};
\]

\[
S_q^1(x_4) = x_4^2 = l_{\lambda_2}x_4;
\]

\[
S_q^1(x_{1,4}) = x_4^2 = l_{\lambda_1}x_4 + x_1x_4^2 = l_{\lambda_1}x_{1,4},
\]

since \(x_{1,3}\) and \(x_{2,4}\) vanish in \(\mathbb{Z}_2[K]\). Thus \([x_{1,4}]\) and \([x_3]\) are connected by \(S_q^1\), while \([x_4]\) and \([1]\) survive to \(E^*_q(Y)\). Therefore \(Y\) is the Klein bottle.

5. The main theorem

In what follows, let \(\sigma_1, \ldots, \sigma_s\) be a shelling of \(K\) (cf. Section 3), \(\lambda\) a characteristic function and \(Y\) the small cover.

Definition 5.1. Suppose \(\lambda = (\lambda_1, \ldots, \lambda_n), \lambda_i \in \mathbb{Z}_2^m\). Let \(G_\lambda \subset \mathbb{Z}_2^m\) be the subgroup generated by \(\{\lambda_i\}_{i=1}^n\). Via non-zero entries, an element of \(G_\lambda\) corresponds uniquely to a subset of \([m]\).

For \(\omega \subset [m]\), the full subcomplex \(K_\omega = \{\sigma \subset \omega \mid \sigma \in K\}\).

Theorem 5.2. There exist cochain complexes \((\mathcal{C}^*_\lambda, \overline{d})\) and \((C^*(Y), d)\), together with a cochain map\(^\dagger\)

\[
\phi: (\mathcal{C}^*_\lambda, 2\overline{d}) \rightarrow (C^{*+1}(Y), d)
\]

increasing the degrees by one, with the following properties.

(a) \(\phi\) induces an isomorphism of cohomology groups.

(b) \((\mathcal{C}^*_\lambda, \overline{d})\) is a direct sum \(\oplus_{\omega \subset G_\lambda}(\mathcal{C}^*(K_\omega), \overline{d})\), in which \((\mathcal{C}^*(K_\omega), \overline{d})\) is cochain-homotopy equivalent to the reduced singular cochain complex of \(|K_\omega|\), for each \(\omega \in G_\lambda\).

\(^\dagger\)The coboundary operator \(2\overline{d}\) means \((2\overline{d})c = 2(\overline{d}c), c \in \mathcal{C}^*_\lambda\).
(c) $(C^*(Y), d)$ is cochain-homotopy equivalent to the singular cochain complex of $Y$.
(d) $\overline{C}_\lambda^*$ is generated by $\{r(\sigma_j) \in K_{\omega_j} \}_{j=1}^s$ ($r(\sigma_j)$ is the restriction of $\sigma_j$ in the shelling), where $\omega_j \in G_\lambda$ is the unique element which satisfies $\omega_j \cap \sigma_j = r(\sigma_i)$; we have

$$\phi(r(\sigma_j)) = [x_{r(\sigma_j)}]$$

in $H^*(Y; \mathbb{Z}_2)$, where $[\phi(r(\sigma_j))]$ means the mod 2 reduction of $\phi(r(\sigma_j))$.

Since $\phi$ is a cochain map with respect to $2\overline{d}'$, by (9) and (12),

$$[\phi(\overline{d}'(r(\sigma_j))) = \left[\frac{1}{2}\phi(2\overline{d}'(r(\sigma_j))) = Sq^1([x_{r(\sigma_j)})].$$

To understand the cohomology of $(\overline{C}_\lambda^*, 2\overline{d}')$ and $(\overline{C}_\lambda^*, \overline{d}')$, we use the normal form of a morphism between two finitely generated abelian groups: for each $r \geq 0$, we can find two bases for cocycles and coboundaries in $(\overline{C}_\lambda^*, \overline{d}')$, respectively, such that a coboundary is certain integral times of exactly one cocycle (cf. [Mun84, Theorem 11.3, pp. 55–56]). The same bases still work for $(\overline{C}_\lambda^*, 2\overline{d}')$, while those coefficients are doubled. It follows that:

**Corollary 5.3.** Let $\overline{G}_i^\omega$ be the group $\oplus_{\omega \in G_\lambda} \overline{H}^i(|K_\omega|; \mathbb{Z})$, $i \geq -1$,\footnote{We adopt the usual convention that, $\overline{H}^{-1}(|K_\omega|; \mathbb{Z})$ is non-trivial only when $\omega = \emptyset$, which is infinite cyclic.} and let $p$ be an odd prime. Then,

1. the $\mathbb{Z}$-summands (resp. $\mathbb{Z}_{p^k}$-summands, $k \geq 1$) of $\overline{G}_i^\omega$ are in one-to-one correspondence with that of $H^{i+1}(Y; \mathbb{Z})$;\footnote{This result is due to Choi and Park [CP13], under the assumption that $K$ is star-shaped; the isomorphism between the free parts was first given in [ST12].}
2. for $k \geq 1$, the $\mathbb{Z}_{p^k}$-summands of $\overline{G}_i^\omega$ are in one-to-one correspondence with the $\mathbb{Z}_{2p^k+1}$-summands of $H^{i+1}(Y; \mathbb{Z})$.

The remaining 2-torsion elements in $H^*(Y; \mathbb{Z})$ can be understood as follows. Denote by $h_i$ the cardinality of the set $\{r(\sigma_j) \mid \text{card}(r(\sigma_j)) = i, j = 1, \ldots, s\}$, and let $\beta_i(k)$ be the number of $\mathbb{Z}_{2p^k}$-summands of $\overline{G}_i^{\omega-1}$, $k \geq 0$ ($\beta_i(0) = \text{rank} \overline{G}_i^{\omega-1}$). By Propositions 2.4 and 3.1, $h_i = \text{rank} H^i(Y; \mathbb{Z}_2)$, and we define

$$h'_i = h_i - \sum_{k \geq 0} \beta_i(k) - \sum_{k \geq 1} \beta_{i+1}(k), \quad i \geq 0,$$

then the number of $\mathbb{Z}_2$-summands in $H^i(Y; \mathbb{Z})$ is equal to

$$h'_i - h'_{i+1} + h'_{i+2} - \ldots + (-1)^{n-i} h'_n.$$
As an illustration, in Example 4.2, we have $G_\lambda = \{\emptyset, \{1, 3, 4\}, \{2, 4\}, \{1, 2, 3\}\}$ as a set; since $|K_{1,3,4}|$ and $|K_{1,2,3}|$ are contractible, $\bar{G}^*$ is contributed by $\tilde{H}^{-1}(|K_\emptyset|; \mathbb{Z})$ and $\tilde{H}^0(|K_{2,4}|; \mathbb{Z})$, giving the two $\mathbb{Z}$-summands of $H^*(Y; \mathbb{Z})$ in dimensions 0 and 1, respectively.

Together with Proposition 4.1 and the second part of Corollary 5.3, we have:

**Corollary 5.4.** In the corresponding Bockstein spectral sequences, $E^r_{i,k+1}(Y)$ is isomorphic to $\oplus_{\omega \in G_\lambda} \tilde{E}^r_{i,k}(|K_\omega|)$ (i.e., using the reduced mod 2 cohomology), for all $i,k \geq 1$. In particular,

$$E^0_2(Y) \cong \bigoplus_{\omega \in G_\lambda} \tilde{H}^{i-1}(|K_\omega|; \mathbb{Z}_2), \quad i \geq 1.$$ 

The corollary below also follows from Corollary 5.3.

**Corollary 5.5.** The following statements are equivalent.

1. Each element in $H^*(Y; \mathbb{Z})$ is either torsion-free or a two-torsion.
2. $H^*(|K_\omega|; \mathbb{Z})$ is torsion-free, for all $\omega \in G_\lambda$.

**Remark 5.6.** By Corollary 5.3, together with the construction given in [CP13, Theorem 5.10], for any given integer $q > 0$, a $q$-torsion element can exist in the homology $H^*(Y; \mathbb{Z})$ of a real quasi-toric manifold $Y$.

## 6. The cochain complex associated to a small cover

This section is devoted to the construction of the cochain complex $(C^*(Y), d)$ in Theorem 5.2.

**6.1. Oriented cellular chain complex.** An orientation of a $k$-cell $e$ in a CW complex $X$ is a chosen generator in $H_k(e, \partial e)$, where $\partial e$ is the topological boundary of $e$. An oriented cell $e$ will be denoted by $[e]$. Let $(C_*(X), \partial)$ be the associated cellular chain complex, in which the boundary $\partial[e]$ of an oriented $k$-cell is a $\mathbb{Z}$-linear sum of oriented $(k-1)$-cells.

We omit the proof of the standard lemma below:

**Lemma 6.1.** Suppose $X$ is a CW complex with an action of a finite group $G$, such that

(*) for every $g \in G$ and every cell $e \subset X$, $ge$ is again a cell.

Then the cell structure of $X$ descends to that of the orbit space $X/G$. 

The condition (\(\ast\)) implies that each transformation \(g: X \to X\) is cellular, thus \(g_*: C_*(X) \to C_*(X)\) induces a chain map, namely if \(g_*[e] = \varepsilon[ge]\), then \(g_*\partial[e] = \varepsilon\partial[ge]\), \(\varepsilon = \pm 1\), where \(g_*: H_* \to H_*\) is induced by \(g\). Let \(\pi: X \to X/G\) be the orbit map. Since \(\pi\) is also cellular, it induces a chain map \(\pi_*: (C_*(X), \partial) \to (C_*(X/G), \partial)\), i.e.,

\[
\partial\pi_*[e] = \pi_*\partial[e].
\]

The interval \(I = [-1,1]\) admits a decomposition with a single 1-cell \(I\) and two bounding 0-cells \(1_+ = \{1\}\) and \(1_- = \{-1\}\), respectively, such that \(\partial[I] = [1_+] - [1_-]\) (\(\partial[1_+]\) and \(\partial[1_-]\) vanish).

**Definition 6.2.** We call a face \(e\) of the cube \(I^m\), as a convex polytope, a **cubical cell**. More precisely, \(e = \prod_{i=1}^m X_i\), where \(X_i\) is either \(I\), \(1_+\) or \(1_-\).

1. For a cubical cell \(e\), define
   \[
   \sigma_e = \{i \mid X_i = I\}, \quad \tau^+_e = \{i \mid X_i = 1_+\} \quad \text{and} \quad \tau^-_e = \{i \mid X_i = 1_-\},
   \]
   which are disjoint subsets with their union \([m]\).
2. Given \(i \in \sigma_e\), denote by \(\partial^+_i e\) (resp. \(\partial^-_i e\)) the \(i\)-th **front face** (resp. **back face**) of \(e\): it is a cubical cell with the \(i\)-th component \(1_+\) (resp. \(1_-\)) and other components coinciding with \(e\).

By the Eilenberg-Zilber theorem, the oriented cellular chain \([e] = \otimes_{i=1}^m [X_i]\) is endowed with the boundary operator

\[
\partial[e] = \sum_{i=1}^m (-1)^{(\sigma_e,i)}[X_1] \otimes \ldots \otimes [X_{i-1}] \otimes \partial[X_i] \otimes [X_{i+1}] \otimes \ldots \otimes [X_m]
\]

\[
= \sum_{i \in \sigma_e} (-1)^{(\sigma_e,i)}([\partial^+_i e] - [\partial^-_i e]),
\]

where \((\sigma_e,i) = \text{card} (\{j \in \sigma_e \mid j < i\})\). In what follows, the notation \([e]\) means that the cubical cell \(e\) is endowed with the orientation from the Eilenberg-Zilber theorem.

We see that the real moment-angle complex \(\mathbb{R}Z\) has a cell decomposition with cubical cells \(e\) so that \(\sigma_e \in K\) (see (6)). Let \((C_*(\mathbb{R}Z), \partial)\) be the oriented cellular chain complex above, where \(\partial\) follows (16). It turns out that the homology of \((C_*(\mathbb{R}Z), \partial)\) gives the cellular homology of \(\mathbb{R}Z\) (cf. [Cai15, Theorem 3.1]).

With this decomposition, the action of \(\text{Ker} \lambda\) on \(\mathbb{R}Z\) satisfies Lemma 6.1, thus it gives a cell structure on \(Y\). More explicitly, write \(g = (g_i)_{i=1}^m \in \text{Ker} \lambda\), we have

\[
g_*[e] = \otimes_{i=1}^m (g_i)_*[X_i] = (-1)^{(\sigma_e,g)}[ge],
\]
where \((\sigma, g) = \text{card}\{i \in \sigma_e \mid g_i = 1\}\) and \(ge \) is again cubical; if \(g_i = 1\), then \((gi)_*[I] = \{I\}\) reversing the orientation, \((gi)_*[1+] = [1]_+\) and \((gi)_*[1_] = [1]_-\).

Given a shelling \(\sigma_1, \ldots, \sigma_s\) of \(K\), let

\[
\pi_e
\]

be the facet function so that \(f(\sigma)\) is the first facet in the sequence that contains \(\sigma\). For \(g = (gi)_{i=1}^m \in \mathbb{Z}_2^n\), let \(\sigma_g \subset [m]\) be the non-zero entries of \(g\).

**Lemma 6.3.** Given \(\tau \subset [m] \setminus f(\sigma)\), there exists a unique element \(g \in \text{Ker}\lambda\) which satisfies \(\sigma_g \setminus f(\sigma) = \tau\).

**Proof.** Without loss of generality, suppose \(f(\sigma) = \{1, 2, \ldots, n\}\). The non-degeneracy of \(\lambda|_{f(\sigma)}\) implies that \(\text{Ker}\lambda\), as an \((m-1) \times m\)-matrix, has the form

\[
\begin{pmatrix}
1 & 2 & \cdots & n & n+1 & n+2 & \cdots & m \\
1 & 2 & \cdots & * & 1 & 0 & \cdots & 0 \\
2 & * & \cdots & * & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
m & * & \cdots & * & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

Let \(v_i\) be the \(i\)-th row vector of \(\text{Ker}\lambda\). The element \(g = \sum_{i \in \tau} v_{i-n}\) satisfies \(\sigma_g \setminus f(\sigma) = \tau\), which is unique.

Let \(I_{\sigma} = \prod_{i=1}^m X_i\); the cubical cell with \(X_i = \{1\}\) if \(i \notin \sigma\), and \(X_i = I\) otherwise. A cubical cell \(e\) is called canonical, if \(e \subset I_{f(\sigma)}\); equivalently, \(\tau_e^- \subset f(\sigma)\). For instance, \(I_{\sigma}\) is canonical.

**Proposition 6.4.** Let \(\pi : \mathbb{R}Z \to Y\) be the orbit map. For each cubical cell \(e\) of \(\mathbb{R}Z\), there exists a unique \(g_e \in \text{Ker}\lambda\), such that \(g_e e\) is canonical.

Let the cell \(\pi e\), where \(e\) is canonical, be endowed with the orientation \(\pi_*[e]\):

\[
[\pi e] = \pi_*[e].
\]

Then

\[
E_Y = \{[\pi e] \mid e \text{ is canonical}\}
\]

is a basis for \(C_* (Y)\); the boundary operator in \((C_* (Y), \partial)\) follows

\[
\partial[\pi e] = \pi_* \partial[e] = \sum_{i \in \sigma_e} (-1)^{|\sigma_e : i|} (\pi_* [\partial_i^+ e] - \pi_* [\partial_i^- e]).
\]

**Proof.** Consider \(\tau = \tau_e^- \setminus f(\sigma)\). By Lemma 6.3, we have a unique \(g_e \in \text{Ker}\lambda\), such that \(\sigma_g \setminus f(\sigma) = \tau\). Clearly \(g_e e \subset I_{f(\sigma)}\) is canonical. The uniqueness of \(g_e\) implies that \(E_Y\) is a basis. (19) follows from (14) and (16).

Notice that a back face \(\partial_i^- e\) may not be canonical, even if \(e\) is.
Example 6.5. Suppose \( \{1, 2\}, \{2, 3\}, \{1, 3\} \) is a shelling of \( K, \) \( \text{Vert}(K) = [3], \) with their restrictions \( \emptyset, \{3\}, \{1, 3\}, \) respectively. Consider

\[
\lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{with} \quad \text{Ker} \lambda = (1 \ 1) ;
\]

Figures 1 and 2 illustrate the filtrations of \( \mathbb{R} \mathbb{Z} \) and \( Y \), respectively, where the 2-cells of \( Y \) are equipped with canonical orientations.
Lemma 6.7

(Cf. [Cai15, Section 3.2])

For \( g^2 \in \pi \partial^-_3 I_{23} \) that \( \sigma \) where \( j \in \) \[23 \] \( d \) that \( \tau \) under \( \pi \) of the canonical cells, \(-[\pi \partial^-_1 I_{12}]\) and \(-[\pi \partial^-_2 I_{23}]\), respectively, as illustrated.

6.2. The transfer and the cochain complex.

Definition 6.6. Let \([I]^*, [1_+]^*\) and \([1_-]^*\) be the oriented dual of \([1]\), \([1_+]\) and \([1_-]\), respectively. Let \((C^* (\mathbb{R}, \mathbb{Z}), d)\) be the cochain complex dual to \((C_*(\mathbb{R}, \mathbb{Z}), 0)\) (with respect to the Hom-functor): we write an oriented dual cell \([e]^*\) in the form \([e]^* = \otimes_{i=1}^{m} [X_i]^*\), if \([e] = \otimes_{i=1}^{m} [X_i]\). For disjoint subsets \(\sigma, \tau\) of \([m]\), let \(u_{\sigma, \tau} \in C^*(\mathbb{R}, \mathbb{Z})\) be the cochain

\[
(20) \quad u_{\sigma, \tau} = \otimes_{i=1}^{m} \alpha_i, \quad \alpha_i = \begin{cases} \ [I]^* & i \in \sigma \\ [1_+]^* & i \in \tau \\ [1_+]^* + [1_-]^* & \text{otherwise.} \end{cases}
\]

When \(\sigma = \tau = \emptyset\), we denote the cochain by the void word \(\emptyset\).

Recall that \(\tau^+_e\) (resp. \(\tau^-_e\)) is the set of \(1_+\)-components (resp. \(1_-\)-components) of \(e\). We see that \(u_{\sigma, \tau}\) is a sum of \(2^j\) dual cells, namely

\[
(21) \quad u_{\sigma, \tau} = \sum_{\sigma = \sigma_e, \tau \subset \tau^+_e} [e]^*,
\]

where \(j = m - \text{card}(\sigma) - \text{card}(\tau)\). In particular, the void word \(\emptyset\) is a sum of \(2^m\) dual vertices.

For \(g = (g_i)_{i=1}^{m} \in \mathbb{Z}_2^m\), consider \(g^*: C^*(\mathbb{R}, \mathbb{Z}) \to C^*(\mathbb{R}, \mathbb{Z})\): by dualizing (17), we have

\[
(22) \quad g^*([e]^*) = \otimes_{i=1}^{m} g_i^*([X_i]^*) = (-1)^{[\sigma \cap g]} [ge]^*,
\]

where \(g_i^*([I]^*) = -[I]^*, \ g_i^*([1_+]^*) = [1_-]^*\) and \(g_i^*([1_-]^*) = [1_+]^*\) if \(g_i = 1\).

Lemma 6.7 (Cf. [Cai15, Section 3.2]). As an abelian group, \((C^*(\mathbb{R}, \mathbb{Z}), d)\) is generated by \(\{u_{\sigma, \tau}\}_{\sigma, \tau}\), with \(\sigma, \tau\) running through disjoint subsets of \([m]\) so that \(\sigma \in K\). The coboundary operator follows

\[
(23) \quad d(u_{\sigma, \tau}) = \sum_{\sigma \cup \{i\} \in K} (-1)^{[\sigma, i]} u_{\sigma \cup \{i\}, \tau \setminus \{i\}},
\]

where \((\sigma, i) = \text{card}(\{j \in \sigma \mid j < i\})\).
Proof. For the first statement, it suffices to show that each dual cell \([e]^*\) is expressible as a \(\mathbb{Z}\)-linear sum by cochains of the form \(u_{\sigma_\tau} t_{r_\tau}^*\).

We use an induction on \(k = \text{card}(\tau_e^-)\). This is clear when \(k = 0\): in this case \(\sigma_e \cup \tau_e^+ = [m]\), thus \([e]^* = u_{\sigma_e} t_{r_e^+}^*\). Assume that this is true for all \(k < i\), and suppose \(\text{card}(\tau_e^-) = i\). Expanding (20) as a sum of dual cells, we see that each summand in \(u_{\sigma_e} t_{r_e^+}^* - [e]^*\) is in the form \([e]'^*\) so that \(\text{card}(\tau_{e'}^-) < i\). By assumption, \(u_{\sigma_e} t_{r_e^+}^* - [e]^*\) is expressible as a \(\mathbb{Z}\)-linear sum as desired, so is \([e]^*\).

The first statement follows by induction.

Suppose \([e]^* = \otimes_{i=1}^m [X_i]^*\) is the dual of the cubical cell \([e] = \otimes_{i=1}^m [X_i]\). By dualizing (15),

\[
d[e]^* = \sum_{i=1}^m (-1)^{(\sigma_e,i)} [X_1]^* \otimes \ldots \otimes [X_{i-1}]^* \otimes d[X_i]^* \otimes [X_{i+1}]^* \otimes \ldots \otimes [X_m]^*,
\]

where on each component we have \(d[I]^* = 0\), \(d[1_+]^* = [I]^*\) and \(d[1^-]^* = -[I]^*\). Therefore by definition (20),

\[
d(u_{\sigma} t_{r^+}) = \sum_{i=1}^m (-1)^{(\sigma,i)} \alpha_1 \otimes \ldots \otimes \alpha_{i-1} \otimes d\alpha_i \otimes \alpha_{i+1} \otimes \ldots \otimes \alpha_m,
\]

which gives (23). \(\square\)

Let \(G_\lambda\) be the group given in Definition 5.1.

Lemma 6.8. For each \(\sigma \in K\), there exists a unique element \(\omega \in G_\lambda\), such that

\[(24) \quad \omega \cap f(\sigma) = \sigma,\]

where \(f(\sigma)\) is the first facet in the shelling that contains \(\sigma\).

Proof. Without loss of generality, suppose \(f(\sigma) = \{1,2,\ldots,n\}\) and \(\lambda|_{\sigma_1} : \mathbb{Z}_2^n \to \mathbb{Z}_2^m\) is the identity. Suppose \((\lambda_1,\ldots,\lambda_n)\) are the row vectors of \(\lambda\), \(\lambda_i \in \mathbb{Z}_2^m\). Let \(g \in \mathbb{Z}_2^m\) be the element \(\sum_{i \in \sigma} \lambda_i\), and let \(\omega\) be the set of non-zero entries of \(g\). Then \(\omega\) satisfies (24), which is unique. \(\square\)

Recall that the transfer homomorphism \(T_* : C_*(Y) \to C_*(RZ)\) is a chain map defined by

\[(25) \quad T_*(\pi_* [e]) = \sum g'_*[e],\]

where \(e\) is a cubical cell of \(RZ\). It can be checked that the definition is independent of the choice of the pre-image \([e]\), and \(T_*\) is invariant in the sense that

\[(26) \quad g_* \circ T_* = T_*\]
where \( g_\ast : C_\ast(\mathbb{R}Z) \to C_\ast(\mathbb{R}Z) \) is induced by \( g \in \text{Ker}\lambda \). Let \( T^\ast : C^\ast(\mathbb{R}Z) \to C^\ast(Y) \) be the dual of \( T_\ast \). By Lemma 6.7, the dual chain complex \((C^\ast(Y), d)\) of \((C_\ast(Y), \partial)\) is generated by \( \{T^\ast(u_\sigma t_\tau)\}_{\sigma, \tau} \).

Recall that a cubical cell \( e \) is canonical, if \( \tau_e^- \subset f(\sigma_e) \). By Proposition 6.4, \( C^\ast(Y) \) has a basis \( \{T^\ast([e]^-)\}_e \) with \( e \) running through canonical cells of \( \mathbb{R}Z \).

We would like to express each cochain \( T^\ast(u_\sigma t_\tau) \) as a \( \mathbb{Z}\)-linear sum of these basis elements.

We say that a cochain \( c \) is divisible by an integer \( r \), if there exists a cochain \( c' \) so that \( c = rc' \); \( c \) is called primitive if it is not divisible by any integer greater than 1.

**Proposition 6.9.** Given \( \omega \in G_\lambda \) and \( \sigma \in K_\omega \) with \( \text{card}(\sigma) = k \), the cochain \( T^\ast(u_\sigma t_\omega \setminus \sigma) \in C^k(Y) \) is divisible by \( 2^{\mu_k(\omega)} \), where

\[
(27) \quad \mu_k(\omega) = \max\{m - n + k - \text{card}(\omega), 0\}.
\]

In particular, if \( \omega \cap f(\sigma) = \sigma \), then \( \mu_k(\omega) = m - n + k - \text{card}(\omega) \geq 0 \), and

\[
(28) \quad T^\ast(u_\sigma t_\omega \setminus \sigma) = 2^{\mu_k(\omega)} T^\ast(u_\sigma t_{[m] \setminus f(\sigma)}),
\]

where \( T^\ast(u_\sigma t_{[m] \setminus f(\sigma)}) \) is primitive in \( C^\ast(Y) \).

**Proof.** Let \( \tau_1 = (\omega \cap f(\sigma)) \setminus \sigma \) and \( \tau_2 = [m] \setminus (f(\sigma) \cup \omega) \). We see that \( \omega \cap f(\sigma) = \tau_1 \cup \sigma \), and \( \omega \setminus f(\sigma) = [m] \setminus (f(\sigma) \cup \tau_2) \). Since \( \tau_1 \cap \sigma = \emptyset \) and \( f(\sigma) \cap \tau_2 = \emptyset \),

\[
\text{card}(\omega) = \text{card}(\omega \cap f(\sigma)) + \text{card}(\omega \setminus f(\sigma)) = \text{card}(\tau_1) + k + m - n - \text{card}(\tau_2),
\]

namely

\[
\text{card}(\tau_2) - \text{card}(\tau_1) = m - n + k - \text{card}(\omega) = \mu_k(\omega).
\]

Denote \( k_1 = \text{card}(\tau_1) \) and \( k_2 = \text{card}(\tau_2) \). For \( l \in \tau_2 \), let \( g_l \in \text{Ker}\lambda \) be the unique element so that \( \sigma_{g_l} \setminus f(\sigma) = \{l\} \), where \( \sigma_{g_l} \subset [m] \) is the set of non-zero entries of \( g_l \) (see Lemma 6.3). Let \( G_{\tau_2} \) be the subgroup generated by \( \{g_l \mid l \in \tau_2\} \), whose order is clearly \( 2^{k_2} \).

For \( S \subset [m] \), define \( (S, g) = \text{card}(S \cap \sigma_g) \). We see that \( g \in \text{Ker}\lambda \), \( \omega \in G_\lambda \) implies that \((\omega, g)\) is an even number (see Definition 5.1). For \( g \in G_{\tau_2} \),

\[
(29) \quad 0 \equiv (\omega, g) = (\omega \cap f(\sigma), g) + (\omega \setminus f(\sigma), g) = (\sigma, g) + (\tau_1, g) \quad \text{mod} \ 2.
\]
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We expand $u_\sigma t_{\omega \setminus \sigma}$ with respect to $\tau_2$, as a sum of $2^{k_2}$ terms: by definition (20),

$$u_\sigma t_{\omega \setminus \sigma} = \sum_{g \in G_{\tau_2}} \alpha_g, \quad \alpha_g = \bigotimes_{i=1}^m \alpha_i, \quad \alpha_i = \begin{cases} [I]^* & i \in \sigma \\ [1-_i]^* & i \in \sigma_g \cap \tau_2 \\ [1^+_i]^* + [1-_i]^* & i \in f(\sigma) \setminus \omega \\ [1^+_i]^* & \text{otherwise.} \end{cases}$$

Let $p: G_{\tau_2} \to \mathbb{Z}_2^{k_1}$ be the projection sending $g = (g_i)_{i=1}^m$ to $p(g) = (g_i)_{i \in \tau_1}$. For $g \in G_{\tau_2}$, by definition, $g^*$ turns the $[1-_i]^*$-components from $\tau_2$ of $\alpha_g$ into $[1^+_i]^*$-ones, and turns all $[1^+_i]^*$-components from $\sigma_{p(g)} \subset \tau_1$ into $[1-_i]^*$-ones, leaving all $([1^+_i]^* + [1-_i]^*)$-components unchanged; hence by (22), we have

$$g^*(\alpha_g) = (-1)^{\langle \sigma, g \rangle} \alpha'_g, \quad \alpha'_g = \bigotimes_{i=1}^m \alpha'_i, \quad \alpha'_i = \begin{cases} [I]^* & i \in \sigma \\ [1-_i]^* & i \in \sigma_{p(g)} \\ [1^+_i]^* + [1-_i]^* & i \in f(\sigma) \setminus \omega \\ [1^+_i]^* & \text{otherwise.} \end{cases}$$

Notice that by (29), $(-1)^{\langle \sigma, g \rangle} \alpha'_g = (-1)^{\text{card}(\sigma_{p(g)})} \alpha'_g$, which means that $g^*(\alpha_g)$ is only determined by the image $p(g)$, thus we denote

$$\beta_{p(g)} = g^*(\alpha_g) = (-1)^{\text{card}(\sigma_{p(g)})} \alpha'_g.$$  

Together with (26), we see that

$$T^*(u_\sigma t_{\omega \setminus \sigma}) = \sum_{g \in G_{\tau_2}} T^*(\alpha_g) = \sum_{g \in G_{\tau_2}} T^*(g^*(\alpha_g))$$
$$= \sum_{g \in G_{\tau_2}} T^*(\beta_{p(g)}) = NT^*(\sum_{g' \in \text{Im} p} \beta_{g'})$$

(30)

where $N$ is the order of the kernel of $p$, which is clearly divisible by $2^{k_2-k_1} = 2^{\mu_k(\omega)}$ if $k_2 \geq k_1$.

In the special case $\tau_1 = \emptyset$, we have $\mu_k(\omega) = \text{card}(\tau_2)$, $\sigma_{p(g)} = \emptyset$, and

$$\beta_{p(g)} = \alpha'_g = u_\sigma t_{[m] \setminus f(\sigma)}$$

for all $g \in G_{\tau_2}$; now (28) follows from (30), since $N = 2^{k_2}$. Each dual cell appearing in the sum $u_\sigma t_{[m] \setminus f(\sigma)}$ is the dual of a canonical one, hence $T^*(u_\sigma t_{[m] \setminus f(\sigma)})$ is primitive. \qed
7. The simplicial (co)chain complexes

With respect to \( K \subset 2^{[m]} \), we denote by \((\widetilde{C}_*(K), \partial')\) the augmented (ordered) simplicial chain complex of \( K \): \( \widetilde{C}_*(K) = \bigoplus_{q \geq -1} C_q(K) \), where \( C_q(K) \) is generated by oriented simplices \( [\sigma] = [i_0, \ldots, i_q] \) (\( \sigma \in K \), \( i_0 < \ldots < i_q \) (formally \( C_{-1}(K) \) is generated by \([\emptyset]\)) \(^{14}\) together with the boundary operator

\[
\partial' [\sigma] = \begin{cases} 
\sum_{k=0}^q (-1)^k [\sigma \setminus \{i_k\}] & q > 0 \\
[\emptyset] & q = 0 \\
0 & q = -1.
\end{cases}
\]

It can be shown that \((\widetilde{C}_*(K), \partial')\) is chain-homotopy equivalent to the reduced singular chain complex of \( |K| \) (cf. [Mun84]).

**Definition 7.1.** Let \( K \subset K' \subset 2^{[m]} \) be two abstract simplicial complexes. \( K' \) is called a regular (simplicial) expansion of \( K \), if \( |K'| \) is the union of \( |K| \) and a simplex \( |\sigma| \), say, such that \( \dim |\sigma| \geq 0 \) and the set \( \{ \tau \subset \sigma \mid \tau \notin K \} \) has a unique, minimal element, which is denoted by \( r(\sigma) \). \(^{15}\)

We still call \( r(\sigma) \) the restriction of \( \sigma \); \( \sigma \) can be a singleton, in this case \( K' \) is a disjoint union of \( |K| \) and \( |\sigma| \), and \( r(\sigma) = \sigma \).

Geometrically, \( K' \) is a regular expansion of \( K \) if and only if the intersection \( |\sigma| \cap |K| \) is a union of codimension-one faces of \( |\sigma| \).

It is easy to see that, unless \( r(\sigma) = \sigma \), i.e., the whole boundary of \( |\sigma| \) is contained in \( |K| \), there is a strong deformation retraction \( |K'| \to |K| \) along \( |\sigma| \).

**Lemma 7.2.** Suppose \( K' \) is a regular expansion of \( K \) along \( \sigma \), \( r(\sigma) \neq \sigma \). Then there is a simplicial chain map \( \rho: \widetilde{C}_*(K') \to \widetilde{C}_*(K) \) whose restriction to \( \widetilde{C}_*(K) \) is the identity.

**Proof.** We can choose a (any) simplicial approximation of the retraction \( |K'| \to |K| \), which is identity on \( |K'| \).\(^ {16}\) This can be done since \( \sigma \) is a simplex. Then the induced chain map is a desired one. \( \square \)

\(^{14}\)The orientation follows the rule that, a permutation of \( i_k, i_j \in \sigma \) gives a \((-1)\)-sign.

\(^{15}\)\( r(\sigma) \neq \emptyset \) since \( \emptyset \in K \).

\(^{16}\)To preserve the degrees, if in the approximation a simplex is mapped (geometrically) to another one whose dimension is strictly lower, then we set its image to be zero in the target chain group. It can be checked that in this way we get a chain map (cf. [Mun84, pp. 62–63]).
Definition 7.3. We say that $\sigma_1, \sigma_2, \ldots, \sigma_l$ is a regular expanding of $K$, if $K$ admits a filtration $2^{\sigma_1} = K_1 \subset K_2 \subset \cdots \subset K_l = K$, where $2^{\sigma_1}$ denotes all subsets of $\sigma_1 \neq \emptyset$, such that $K_{j+1}$ is a regular expansion of $K_j$ along $\sigma_{j+1}$, $j = 1, \ldots, s - 1$.

In the sequence above, a simplex $\sigma_i$ is called critical, if $r(\sigma_j) = \sigma_j$. We shall denote by $\text{Cri}(K)$ the set of critical faces. Formally we set $r(\sigma_1) = \emptyset$.

We see that, up to homotopy, $|K|$ has a cell decomposition by critical simplices. If $\text{Cri}(K) = \emptyset$, $|K|$ deformation retracts onto a vertex.

Proposition 7.4. Suppose $\sigma_1, \ldots, \sigma_l$ is a regular expanding of $K$ with the filtration $K_1 \subset \cdots \subset K_l$.

Let $\tilde{G}_*(K)$ be the free abelian group generated by $\text{Cri}(K)$,\footnote{If trivially $|K| = \emptyset$, then we define $\tilde{G}_*(K)$ to be generated by a single element $[\emptyset]$ in dimension $-1$.} with each critical simplex suitably oriented (which is trivial if $\text{Cri}(K) = \emptyset$), and let $(\tilde{C}_*(K), \partial')$ be the simplicial chain complex. Then $\tilde{G}_*(K)$ admits a boundary operator $\partial'$ to be a chain complex, and there is a (graded) chain map $\rho: \tilde{C}_*(K) \to \tilde{G}_*(K)$ with the following properties.

(a) $\rho$ is an identity when restricted to $\text{Cri}(K)$:

$$\rho([\sigma]) = [\sigma], \quad \sigma \in \text{Cri}(K).$$

(b) $\rho$ induces an isomorphism in homology.

(c) Let $\rho^*$ be the dual of $\rho$ (by the Hom-functor). If $\sigma_{j+1}$ is critical, then

$$\rho^*([\sigma_{j+1}]^*) = [\sigma_{j+1}]^* + e_{j+1}^+,$$

where $[\sigma_{j+1}]^*$ the oriented dual of $[\sigma_{j+1}]$, and $e_{j+1}^+$ is (if not vanishing) a $\mathbb{Z}$-linear sum with dual simplices of the form $[\sigma_t]^*$ involved, $t > j + 1$.

Proof. We construct $\rho$ inductively from $K_j$ to $K_{j+1}$, $j = 1, \ldots, l - 1$. First consider $K_1 = 2^{\sigma_1}$. Since $\text{Cri}(K_1)$ is empty, by definition, we define $\rho([\sigma]) = 0$ for all $\sigma \subset \sigma_1$ (including the case $\sigma = \emptyset$). Suppose the chain map $\rho_j: \tilde{C}_*(K_j) \to \tilde{G}_*(K_j)$ has been defined as desired.

(I) If $\sigma_{j+1}$ is not critical, let $f_{j+1}: \tilde{C}_*(K_{j+1}) \to \tilde{C}_*(K_j)$ be the map given in Lemma 7.2, and define $\rho_{j+1}$ as the composition $\rho_j \circ f_{j+1}$;

(II) Otherwise if $\sigma_{j+1}$ is critical, we define

$$\mathcal{J}' = \rho_j(\partial'[\sigma_{j+1}])$$

and $\rho_{j+1}([\sigma_{j+1}]) = [\sigma_{j+1}]$ preserving the orientation; $\rho_{j+1}$ coincides with $\rho_j$ on $\tilde{C}_*(K_j)$.
By induction, it can be checked that the chain map \( \rho: \tilde{C}_*(K) \to \tilde{G}_*(K) \) is well defined and satisfies both (a) and (b). (c) follows from (I).

Note that in a regular expanding of \( K \), compared with a shelling (cf. Section 3), each \( \sigma_j \) need not to be a facet, and \( K \) need not to be pure. We have the following relation between them.

**Proposition 7.5.** Suppose \( \sigma_1, \ldots, \sigma_s \) is a shelling of \( K \subset 2^{|m|} \). Given \( \omega \subset [m] \), let \( \sigma_{j_1} \cap \omega, \ldots, \sigma_{j_l} \cap \omega \) be the (non-repeating) list of non-empty simplices in \( \{ \sigma_j \cap \omega \}_{j=1}^s \). Then \( \sigma_{j_1} \cap \omega, \ldots, \sigma_{j_l} \cap \omega \) is a regular expanding of the full subcomplex \( K_\omega \), with their restrictions \( r(\sigma_{j_1}), \ldots, r(\sigma_{j_l}) \), respectively, where \( r(\sigma_{j_k}) \) is the restriction of \( \sigma_{j_k} \) in the given shelling, \( k = 1, \ldots, l \).

**Proof.** Recall that by definition, the restriction \( r(\sigma_{j+1}) \) of \( \sigma_{j+1} \) in the shelling is the minimal element of the set \( \{ \sigma \subset \sigma_{j+1} \mid \sigma \notin K_j \} \). Suppose the intersection \( \sigma_{j+1} \cap \omega \), as a simplex in the full subcomplex \( K_{j+1}|_\omega \) of \( K_{j+1} \), is not empty. If \( r(\sigma_{j+1}) \subset \omega \), then by definition, \( K_{j+1}|_\omega \) is a regular expansion of \( K_j|_\omega \) along \( \sigma_{j+1} \cap \omega \), whose restriction is again \( r(\sigma_{j+1}) \); otherwise \( r(\sigma_{j+1}) \cap \omega \) is a proper subset of \( r(\sigma_{j+1}) \), which means that any simplex \( \sigma \in K_{j+1}|_\omega \) cannot contain \( r(\sigma_{j+1}) \), thus by minimality, \( \sigma \in K_j \), hence \( \sigma \in K_j|_\omega \). The statement follows by induction.

8. Proof of Theorem 5.2

For an abstract simplicial complex \( K \subset 2^{|m|} \), let \((\tilde{C}^*(K), d^\prime)\) (resp. \((\tilde{C}^*(K), d^\prime)\)) be the dual of the chain complex \((\tilde{G}_*(K), \partial^\prime)\) (resp. the reduced simplicial chain complex \((\tilde{C}_*(K), d^\prime)\)) given in Proposition 7.4, and let \( \rho^* : C^*(K) \to \tilde{C}^*(K) \) be the dual of \( \rho \).

Let \( [\sigma]^* \in \tilde{C}^*(K) \) be the oriented dual of \([\sigma] \in \tilde{C}_*(K) \) (since \( K \) is finite, \( C^*(K) \) is generated by dual simplices). It can be checked that, by dualizing (31),

\[
(33) \quad d'[\sigma]^* = \sum_{(\sigma \cup \{i\}) \in K} (-1)^{(\sigma, i)} [\sigma \cup \{i\}]^* \quad (d'[\emptyset]^* = \sum_{\{i\} \in K} [i]^*),
\]

where \((\sigma, i) = \text{card}\{j \in \sigma \mid j < i\}\).

Fix \( \omega \subset [m] \) and consider the map

\[
(34) \quad \varphi : \tilde{C}^*(K_\omega) \to C^*/1(\mathbb{RZ})
\]

which sends \([\sigma]^* \) to \( u_\sigma t_{\omega \backslash \sigma} \) (see Definition 6.6). A comparison of (33) and (23) shows that \( \varphi \) is a (degree-increasing) cochain map with respect to differentials \( d^\prime \) and \( d \).
Now given $\omega \in G_{\lambda}$ (see Definition 5.1), suppose $\sigma_1, \ldots, \sigma_s$ is a shelling of $K$ with restrictions $r(\sigma_1), \ldots, r(\sigma_s)$, respectively, and $\sigma_j \cap \omega, \ldots, \sigma_j \cap \omega$ is a regular expanding of $K_\omega$ (see Proposition 7.5). The lemma below is straightforward:

**Lemma 8.1.** The simplex $\sigma_j \cap \omega, k = 1, \ldots, l$, is critical if and only if $\sigma_j \cap \omega = r(\sigma_j)$. That is to say, the cochain complex $(\overline{C}^*(K_\omega), \overline{d}')$ is generated by $\{\sigma_j | \sigma_j \cap \omega = r(\sigma_j)\}$.

Let $\phi_k|_\omega, k \geq 0$, be the map given by

$$\phi_k|_\omega = \frac{1}{2\mu_k(\omega)} T^* \circ \varphi \circ \rho^* : \overline{C}^{k-1}(K_\omega) \rightarrow C^k(Y),$$

where $T^*$ is the dual transfer homomorphism, and the coefficient $1/\mu_k(\omega)$ is multiplied so that its image is primitive, $\mu_k(\omega) = m - n + k - \text{card}(\omega)$ (see Proposition 6.9).

**Lemma 8.2.** Let $\phi|_\omega: \overline{C}^*(K_\omega) \rightarrow C^{*+1}(Y)$ be the homomorphism whose restriction to $\overline{C}^{k-1}(K_\omega)$ is $\phi_k|_\omega, k = 0, 1, \ldots, n$. If we endow $\overline{C}^*(K_\omega)$ with the coboundary operator $2\overline{d}'$ rather than $\overline{d}'$, then it is a cochain map

$$(\overline{C}^*(K_\omega), 2\overline{d}') \rightarrow (C^{*+1}(Y), d).$$

**Proof.** As a composition of cochain maps, $T^* \circ \varphi \circ \rho^*$ is a cochain map with respect to $\overline{d}'$ and $d$. Choose $c \in \overline{C}^{k-1}(K_\omega)$, we see that

$$\phi(2\overline{d}' c) = \frac{1}{2\mu_{k+1}(\omega)} T^* \circ \varphi \circ \rho^* (2\overline{d}' c)$$

$$= 2\mu_{k+1}(\omega) d \circ T^* \circ \varphi \circ \rho^* (c)$$

$$= d \left( \frac{1}{2\mu(\omega)} T^* \circ \varphi \circ \rho^* (c) \right) = d\phi(c).$$

$\square$

Denote $\overline{C}_\lambda^* = \oplus_{\omega \in G_{\lambda}} \overline{C}^*(K_\omega)$, and let $\phi: (\overline{C}_\lambda^*, 2\overline{d}') \rightarrow (C^{*+1}(Y), d)$ be the cochain map whose restriction to $\overline{C}^*(K_\omega)$ is $\phi|_\omega$.

Now (b) and (c) in Theorem 5.2 follows from Proposition 7.4, and the standard isomorphism between cellular and singular (co)homology, respectively. The first statement in (d) follows from Lemmas 6.8 and 8.1.

It remains to prove (a) and (12). Suppose $K_1 \subset \ldots \subset K_s$ is the filtration associated to the shelling $\sigma_1, \ldots, \sigma_s$ of $K$, and let $Y_1 \subset \ldots \subset Y_s$ be the filtration of $Y$ given in Proposition 3.4. The proof is an induction. Suppose $\omega_j \cap \sigma_j = r(\sigma_j)$,
and denote by $\mathcal{C}_\lambda^*(K_j) = \oplus_{\omega \in G_\lambda} \mathcal{C}_\lambda^*(K_j|_\omega)$ the cochain complex associated to $K_j$, with $K_j|_\omega$ the full subcomplex.

First consider $j = 1$. By definition $r(\sigma_1) = \emptyset$ thus $\omega_1 = \emptyset$, and $\mathcal{C}_\lambda^*(K_1) = \tilde{C}^*(K_{\omega_1})$ concentrates in dimension $-1$ and is generated by $[\emptyset]^*$; by (34) $\phi$ maps $[\emptyset]^*$ to the void word $\emptyset$, and the class with representative $\frac{1}{2^n-n}T^*(\emptyset)$ generates $H^0(Y_1)$ (actually it is the sum of all $2^n$ dual vertices of $Y_1$, an $n$-cube). Therefore (a) and (12) hold for $K_1$.

Now suppose they hold for $K_j$. We treat $\mathcal{C}_\lambda^*(K_j)$ and $\mathcal{C}_\lambda^*(K_{j+1})$ as abelian groups and denote by $D^*_j$ their difference $\mathcal{C}_\lambda^*(K_{j+1}) \setminus \mathcal{C}_\lambda^*(K_j)$. We see that $D^*_j$ is closed under $2d$ and, denote by $(\mathcal{D}^*_j, 2d)$ the corresponding cochain complex. Observe that the relative cochain complex $((\mathcal{C}_\lambda^*(K_{j+1}), D^*_j))$ is canonically isomorphic to $(\mathcal{C}_\lambda^*(K_j), 2d)$. By definition,

$$D^*_j = \oplus_{\omega \in G_\lambda} \mathcal{C}_\lambda^*(K_{j+1}|_\omega) \setminus \mathcal{C}_\lambda^*(K_j|_\omega)$$

is generated by dual simplices among $\{[\sigma_{j+1} \cap \omega]^*\}_{\omega \in G_\lambda}$ so that $\sigma_{j+1} \cap \omega$ is critical in $K_{j+1}|_\omega$ (see Proposition 7.4); by Lemma 6.8, $\omega_{j+1}$ is unique, thus $D^*_j$ is generated by a single element $[r(\sigma_{j+1})]^* \in K_{j+1}|_{\omega_{j+1}}$, which is clearly a cocycle in dimension $k_{j+1} - 1$, $k_{j+1} = \text{card}(r(\sigma_{j+1}))$.

Likewise consider the difference $C^*(Y_{j+1}) \setminus C^*(Y_j)$ of dual cells, which we shall denote by $D^*_{j+1}$; as a subgroup of $(C^*(Y_{j+1}), d)$, $D^*_{j+1}$ is also closed under $d$, and the relative cochain complex $((C^*(Y_{j+1}), D^*_{j+1}), d)$ is canonically isomorphic to $(C^*(Y_j), d)$. More precisely, $D^*_{j+1}$ is a collection of dual cells

$$\{[\pi e]^* | e \text{ is canonical with } r(\sigma_{j+1}) \subset \sigma_e \subset \sigma_{j+1}\},$$

where $\pi : \mathbb{R}Z_{j+1} \to Y_{j+1}$ the orbit map (see Proposition 6.4). Geometrically, up to homotopy, $Y_{j+1}$ is obtained from $Y_j$ by attaching a cell of dimension $k_{j+1}$ (see Proposition 3.4); thus the cohomology of $(D^*_{j+1}, d)$ is infinite cyclic which concentrates in dimension $k_{j+1}$, and we can check directly that it is generated by the class with representative

$$(35) \quad \frac{1}{2^{n_{k_{j+1}}(\omega_{j+1})}}T^*(u_{r(\sigma_{j+1})}t_{\omega_{j+1}}(\sigma_{j+1}),$$

which is primitive by Proposition 6.9 (by (23) it is a cocycle since in $K_{j+1}$, $\sigma_{j+1}$ is the unique facet which contains $r(\sigma_{j+1})$).

We see that the cochain map $\phi$ induces a homomorphism between the long exact sequences associated to pairs $(\mathcal{C}_\lambda^*(K_{j+1}), D^*_j)$ and $(C^*(Y_{j+1}), D^*_{j+1})$. 
respectively:

\begin{equation}
\cdots \leftrightarrow H^t(\overline{C}^*_{\lambda}(K_{j+1}), \overline{D}^*_{j+1}) \leftrightarrow H^t(\overline{C}^*_{\lambda}(K_{j+1})) \leftrightarrow H^t(\overline{D}^*_{j+1}) \leftrightarrow \cdots
\end{equation}

\begin{equation}
\cdots \leftrightarrow H^{t+1}(C^*(Y_{j+1}), D^*_{j+1}) \leftrightarrow H^{t+1}(C^*(Y_{j+1})) \leftrightarrow H^{t+1}(D^*_{j+1}) \leftrightarrow \cdots
\end{equation}

where by induction hypothesis, for any \( t \geq -1 \), the first column is an isomorphism; in (c) of Proposition 7.4, \( c^+_{j+1} \) of (32) vanishes since \([r(\sigma_{j+1})]^* \) is the last dual simplex in \( \overline{C}^*(K_{j+1}|\omega_{j+1}) \), hence the image of \([r(\sigma_{j+1})]^* \) under \( \phi \) is exactly (35), which means that the third column is also an isomorphism (which is non-trivial only when \( t = k_{j+1} - 1 \)). Therefore, the middle column is an isomorphism, from which (a) holds. By Proposition 3.7, as vector spaces, \( H^*(Y_{j+1}; \mathbb{Z}_2) \) is obtained from \( H^*(Y_j; \mathbb{Z}_2) \) by adding a basis element \([x_{r(\sigma_{j+1})}]^* \), which has to be the mod 2 reduction of (35). We see that (12) holds, and the whole proof is completed by induction.

9. Examples

In this section, a simplex \( \{i_0, i_2, \ldots, i_q\} \) will be denoted by \( i_0i_2 \ldots i_q \) for short. First consider Example 4.2 again. We have the shelling

\[ 12, 23, 34, 14 \]

with their restrictions marked. Since \( G_{\lambda} = \{\emptyset, 134, 24, 123\} \) as a set, under the rule \( \omega \cap \sigma = r(\sigma) \), we have \( \emptyset \in K_\emptyset, 4 \in K_{24} \) and \( 3, 14 \in K_{1,3,4} \) (with no restrictions in \( K_{1,2,3} \)).

The shelling above gives a regular expanding of \( K_{1,3,4} \), i.e., 1, 3, 34, 14 (see Proposition 7.5), in which 3 and 14 are critical (see Figure 3). The simplicial retraction maps 34 to 3, therefore \( \rho([3, 4]) = 0 \) and \( \rho([4]) = [3] \) (see Lemma 7.2 and Proposition 7.4); after dualization, we have

\[ d'[3]^* = d'[4]^* = [1, 4]^*, \]

where \( d' \) is the dual operator of \( d \) in Proposition 7.4 (see (33) on \( d' \)), which coincides with the previous calculation (11), by (13).

Example 9.1. Let \( K \) be the boundary of the 3-polytope shown in Figure 4, with characteristic function

\[ \lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \]
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We give $K$ the shelling below with restrictions marked:

$137, 136, 357, 157, 235, 236, 145, 245, 146, 246$.

With the 10 basis elements of $H^*(Y; \mathbb{Z}_2)$ associated to the restrictions above, we calculate their images under $Sq^1$ in $\mathbb{Z}_2[K]$: we have $x_6^2 = l_{\lambda_2}x_6, x_2^2 = l_{\lambda_1}x_2$,

$x_5^2 = (l_{\lambda_1} + l_{\lambda_3})x_5 + x_{1,5} + l_{\lambda_3}x_2 + x_{2,6}, \quad x_4^2 = (l_{\lambda_2} + l_{\lambda_3})x_4 + x_{1,6}$,

together with $Sq^1(x_{1,5}) = (l_{\lambda_1} + l_{\lambda_3})x_{1,5}, Sq^1(x_{2,6}) = (l_{\lambda_1} + l_{\lambda_3})x_{2,6},$

$Sq^1(x_{2,4}) = (l_{\lambda_1} + l_{\lambda_2} + l_{\lambda_3})x_{2,4} + x_{2,4,6},$

$Sq^1(x_{4,6}) = (l_{\lambda_1} + l_{\lambda_3})x_{4,6}$ and $Sq^1(x_{2,4,6}) = (l_{\lambda_1} + l_{\lambda_2} + l_{\lambda_3})x_{2,4,6}$, where $l_{\lambda_1} = x_1 + x_2 + x_7, l_{\lambda_2} = x_3 + x_4 + x_5$ and $l_{\lambda_3} = x_5 + x_6 + x_7$. 

---

**Figure 3.** The regular expanding and the retraction

**Figure 4.**
It is interesting to compare the coboundary $d'$ between critical simplices in each full subcomplexes $K_\omega$, $\omega \in G_\lambda$, and the Bockstein homomorphism $Sq^1$, by (13). We illustrate this in Figure 5, where $\emptyset \in K_\emptyset$ is ignored; notice that no restrictions are contained in full subcomplexes $K_{3,4,5}$ and $K_{1,2,3,4,5,7}$.

By Theorem 5.2 and Corollary 5.4, we can choose $[1]$, $[x_2]$, $[x_6]$ and $[x_{2,6}]$ as a basis for $E^*(Y) = E^*_\infty(Y)$ (since $\tilde{H}^*(|K_\omega|)$ is torsion-free, for all $\omega \in G_\lambda$); other six basis elements in $H^*(Y; \mathbb{Z}_2)$ are connected in pairs by $Sq^1$. As a conclusion,

$$H^k(Y; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k = 2 \\ \mathbb{Z}_2 & k = 3. \end{cases}$$

**Remark 9.2.** When $K$ is shellable in the sense of Björner and Wachs [BW96] (i.e., $K$ need not to be pure), the algebraic basis from shellability (cf. Proposition 3.1) still holds (see [BW97, Theorem 12.3]), and our proof of Theorem 5.2 works as well. Shellable complexes (in the usual sense) are considered in this paper since we are interested in the case when the small cover is a piecewise linear manifold (see Proposition 3.4).

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