THE LEFSCHETZ-LUNTS FORMULA FOR DEFORMATION QUANTIZATION MODULES

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Abstract. We adapt to the case of deformation quantization modules a formula of V. Lunts [6] who calculates the trace of a kernel acting on Hochschild homology.

1. Introduction

Inspired by the work of D. Shklyarov (see [9]), V. Lunts has established in [6] a Lefschetz type formula which calculates the trace of a coherent kernel acting on the Hochschild homology of a projective variety (Theorem 4.4). This result has inspired several other works ([2, 7]). In [2], Cisinski and Tabuada recover the result of Lunts via the theory of non-commutative motives. In [7], Polischuk proves similar formulas and applies them to matrix factorization. The aim of this paper is to adapt Lunts formula to the case of deformation quantization modules (DQ-modules) of Kashiwara-Schapira on complex Poisson manifolds. For that purpose, we develop an abstract framework which allows one to obtain Lefschetz-Lunts type formula in symmetric monoidal categories endowed with some additional data.

Our proof relies essentially on two facts. The first one is that the composition operation on the Hochschild homology is compatible in some sense with the symmetric monoidal structures on the categories involved. The second step of the proof relies mainly on the functoriality of the Hochschild class. This suggests that the Lefschetz-Lunts formula is a 2-categorical statement and that it might be possible to build a set-up in the spirit of [1] which would encompass simultaneously the two steps of our proof.

Let us compare briefly the different approaches and settings of [6], [2] and [7] to ours. As already mentioned, we are working in the framework of deformation quantization modules over complex manifolds.

The approach of Lunts is based on a certain list of properties of the Hochschild homology of algebraic varieties (see [6 §3]). These properties mainly concern the behaviour of Hochschild homology with respect to the composition of kernels and the functoriality of the Hochschild
homology. A straightforward consequence of these properties is that the morphism $X \to \text{pt}$ induces a map from the Hochschild Homology of $X$ to the ground field $k$. Such a map does not exist in the theory of DQ-modules. Thus, it is not possible to integrate a single class with value in Hochschild homology and one has to integrate a pair of classes. Then, it seems that the method of V. Lunts cannot be carried out in our context.

In [2], the authors showed that the results of V. Lunts for projective varieties can be derived from a very general statement for additive invariants of smooth and proper differential graded category in the sense of Kontsevich. However, it is not clear that this approach would work for DQ-modules even in the algebraic case. Indeed, the results used to relate non-commutative motives to more classical geometric objects rely on the existence of a compact generator for the derived category of quasi-coherent sheaves. To the best of our knowledge, there are no such results for DQ-modules. Similarly, the approach of [7] does not seem to be applicable to DQ-modules.

The paper is organised as follows. In the first part, we sketch a formal framework in which we can get a formula for the trace of a class acting on a certain homology, starting from a symmetric monoidal category endowed with some specific data. In the second part, we briefly review, following [4], some elements of the theory of DQ-modules. The last part is mainly devoted to the proof of the Lefschetz-Lunts theorems. Then, we briefly explain how to recover some of Lunts results.

2. A general framework for Lefschetz type Theorems.

2.1. A few facts about symmetric monoidal categories and traces. In this subsection we recall a few classical facts concerning dual pairs and traces in symmetric monoidal categories. References for this subsection are [3] §4], [5], [8].

Let $\mathcal{C}$ be a symmetric monoidal category with product $\otimes$, unit object $1_\mathcal{C}$ and symmetry isomorphism $\sigma$. All along the paper, we identify $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$.

Definition 2.1. We say that $X \in \text{Ob}(\mathcal{C})$ is dualizable if there is $Y \in \text{Ob}(\mathcal{C})$ and two morphisms, $\eta : 1_\mathcal{C} \to X \otimes Y$, $\varepsilon : Y \otimes X \to 1_\mathcal{C}$ called coevaluation and evaluation such that the condition (a) and (b) are satisfied:

(a) The composition $X \simeq 1_\mathcal{C} \otimes X \overset{\eta \otimes \text{id}_Y}{\longrightarrow} X \otimes Y \otimes X \overset{\text{id}_X \otimes \varepsilon}{\longrightarrow} X \otimes 1_\mathcal{C} \simeq X$ is the identity of $X$.

(b) The composition $Y \simeq Y \otimes 1_\mathcal{C} \overset{\text{id}_Y \otimes \eta}{\longrightarrow} Y \otimes X \otimes Y \overset{\varepsilon \otimes \text{id}_Y}{\longrightarrow} 1_\mathcal{C} \otimes Y \simeq Y$ is the identity of $Y$. 


We call $Y$ a dual of $X$ and say that $(X,Y)$ is a dual pair.

We shall prove that some diagrams commute. For that purpose recall the useful lemma below communicated to us by Masaki Kashiwara.

**Lemma 2.2.** Let $\mathcal{C}$ be a monoidal category with unit. Let $(X,Y)$ be a dual pair with coevaluation and evaluation morphisms

$$1_\mathcal{C} \xrightarrow{\eta} X \otimes Y, \ Y \otimes X \xrightarrow{\varepsilon} 1_\mathcal{C}.$$  

Let $f : 1_\mathcal{C} \rightarrow X \otimes Y$ be a morphism such that $(\text{id}_X \otimes \varepsilon) \circ (f \otimes \text{id}_X) = \text{id}_X$. Then $f = \eta$.

**Proof.** Consider the diagram:

$$\xymatrix{ 1_\mathcal{C} \ar[r]^f \ar[d]_{\eta} & X \otimes Y \ar[d]^{\text{id}_X \otimes \text{id}_Y \circ f} \\
X \otimes Y \ar[r]_{\eta \otimes \text{id}_X \otimes \text{id}_Y} & X \otimes Y \otimes X \otimes Y \ar[d]_{\text{id}_X \otimes \varepsilon \otimes \text{id}_Y} \\
& X \otimes Y. \ar[u]_{\text{id}_X \otimes \varepsilon \otimes \text{id}_Y} \ar[u]_{\text{id}_X \otimes \varepsilon \otimes \text{id}_Y} \ar[u]_{\text{id}_X \otimes \varepsilon \otimes \text{id}_Y} \ar[u]_{\text{id}_X \otimes \varepsilon \otimes \text{id}_Y}}$$

By the hypothesis, $(\text{id}_X \otimes \varepsilon \otimes \text{id}_Y) \circ (\text{id}_X \otimes \text{id}_Y \circ f) = \text{id}_X \otimes \text{id}_Y$ and $(\text{id}_X \otimes \varepsilon \otimes ; d_Y) \circ (\eta \otimes \text{id}_X \otimes \text{id}_Y) = (\text{id}_X \otimes \text{id}_Y)$. Therefore, $\eta = f$. □

**Proposition 2.3.** If $(X,Y)$ is a dual pair, then for every $Z, W \in \text{Ob}(\mathcal{C})$, there are natural isomorphisms

$$\Phi : \text{Hom}_\mathcal{C}(Z, W \otimes Y) \xrightarrow{\sim} \text{Hom}_\mathcal{C}(Z \otimes X, W)$$

$$\Psi : \text{Hom}_\mathcal{C}(Y \otimes Z, W) \xrightarrow{\sim} \text{Hom}_\mathcal{C}(Z, X \otimes W)$$

where for $f \in \text{Hom}_\mathcal{C}(Z, W \otimes Y)$ and $g \in \text{Hom}_\mathcal{C}(Y \otimes Z, W)$,

$$\Phi(f) = (\text{id}_W \otimes \varepsilon) \circ (f \otimes \text{id}_X)$$

$$\Psi(g) = (\text{id}_X \otimes g) \circ (\eta \otimes \text{id}_Z).$$

**Proof.** see [3, §4]. □

**Remark 2.4.** It follows that $Y$ is a representative of the functor $Z \mapsto \text{Hom}_\mathcal{C}(Z \otimes X, 1_\mathcal{C})$ as well as a representative of the functor $W \mapsto \text{Hom}_\mathcal{C}(1_\mathcal{C}, X \otimes W)$. Therefore, the dual of a dualizable object is unique up to a unique isomorphism.

**Definition 2.5.** For a dualizable object $X$, the trace of $f : X \rightarrow X$ denoted $\text{Tr}(f)$ is the composition

$$1_\mathcal{C} \rightarrow X \otimes Y \xrightarrow{f \otimes \text{id}} X \otimes Y \xrightarrow{\varepsilon} Y \otimes X \xrightarrow{\varepsilon} 1_\mathcal{C}.$$  

Then, $\text{Tr}(f) \in \text{Hom}_\mathcal{C}(1_\mathcal{C}, 1_\mathcal{C})$. 
Remark 2.6. The trace could also by defined as the following composition
\[ 1_C \rightarrow X \otimes Y \xrightarrow{\sigma} Y \otimes X \xrightarrow{id \otimes f} Y \otimes X \xrightarrow{\varepsilon} 1_C. \]
These two definitions of the trace coincide because \((id \otimes f) \sigma = \sigma (f \otimes id)\) since \(\sigma\) is a natural transformation.

Recall the following fact.

Lemma 2.7. With the notation of Definition 2.5, the trace is independent of the choice of a dual for \(X\).

Proof. Let \(Y\) and \(Y'\) two duals of \(X\) with evaluations \(\varepsilon, \varepsilon'\) and co-evaluation \(\eta, \eta'\). By definition of a representative of the functor \(Z \mapsto \text{Hom}_C(Z \otimes X, 1_C)\) there exist a unique isomorphism \(\theta : Y \rightarrow Y'\) such that the diagram
\[
\begin{align*}
\text{Hom}_C(Z, Y') & \xrightarrow{\Phi} \text{Hom}_C(Z \otimes X, 1_C) \\
\theta_0 & \downarrow \phi \\
\text{Hom}_C(Z, Y) & \xrightarrow{\phi} 
\end{align*}
\]
commutes. For \(Z = Y\), the diagram implies \(\varepsilon = \varepsilon' \circ (\theta \otimes \text{id}_X)\). Using the Lemma 2.2 we get that \(\eta = (\text{id}_X \otimes \theta^{-1}) \circ \eta'\). It follows that the diagram
\[
\begin{array}{ccc}
X \otimes Y & \xrightarrow{f \otimes \text{id}} & X \otimes Y \\
\eta^{-1} & \downarrow \id \otimes \theta & \downarrow \id \otimes \theta \\
1_C & \xrightarrow{id \otimes \theta} & Y \otimes X \\
\theta \otimes \text{id} & \downarrow \varepsilon & \varepsilon' \\
X \otimes Y' & \xrightarrow{f \otimes \text{id}} & Y' \otimes X \\
\end{array}
\]
commutes which proves the claim. \(\square\)

Example 2.8. (see [3, §3]) Let \(k\) be a Noetherian commutative ring of finite cohomological dimension. Let \(\text{D}^b(k)\) be the bounded derived category of the category of \(k\)-modules. It is a symmetric monoidal category for \(\otimes^L_k\). We denote by \(\text{D}^b_f(k)\), the full subcategory of \(\text{D}^b(k)\) whose objects are the complexes with finite type cohomology. If \(M \in \text{Ob}(\text{D}^b_f(k))\), its dual is given by \(\text{RHom}_k(M, k)\). The evaluation and the
coevaluation are given by
\[
\text{ev} : \text{RHom}_k(M, k) \overset{L}{\otimes} M \to k
\]
\[
\text{coev} : k \to \text{RHom}_k(M, M) \widetilde{\otimes} M \overset{\sigma}{\otimes} \text{RHom}_k(M, k).
\]
Assume \(k\) is an integral domain, Then \(k\) can be embedded into its field of fraction \(F(k)\). If \(f\) is an endomorphism of \(M\) then the trace of \(f\)
\[
k \overset{\text{coev}}{\to} \text{RHom}_k(M, M) \overset{\sigma}{\sim} \text{RHom}_k(M, k) \overset{\text{ev}}{\to} k
\]
coincides with \(\sum_i (-1)^i \text{Tr}(H^i(\text{id}_{F(k)} \otimes f))\). If \(f = \text{id}_M\), one set
\[
\chi(M) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{F(k)}(H^i(M)).
\]

2.2. The framework. In this section we define a general framework for Lefschetz-Lunts type theorems.

Let \(\mathcal{C}\) be a symmetric monoidal category with product \(\otimes\), unit object \(1_\mathcal{C}\) and symmetric isomorphism \(\sigma\). Let \(k\) be a noetherian commutative ring with finite cohomological dimension.

Assume we are given:
(a) a monoidal functor \((\cdot)^a : \mathcal{C} \to \mathcal{C}\) such that \((\cdot)^a \circ (\cdot)^a = \text{id}_\mathcal{C}\) and \(1_\mathcal{C}^a \simeq 1_\mathcal{C}\)
(b) a symmetric monoidal functor \((L, \mathcal{R}) : \mathcal{C} \to \mathcal{D}^b(k)\) where \(\mathcal{R}\) is
the isomorphism of bifunctor from \(L(\cdot) \otimes L(\cdot)\) to \(L(\cdot \otimes \cdot)\). That is
\[
L(X) \overset{L}{\otimes} L(Y) \overset{\mathcal{R}}{\simeq} L(X \otimes Y)
\]
functorially in \(X\) and \(Y\),
(c) for \(X_i \in \text{Ob}(\mathcal{C})\) \((i = 1, 2, 3)\), a morphism
\[
\vartriangle_2 : L(X_1 \otimes X_2^a) \otimes L(X_2 \otimes X_3^a) \to L(X_1 \otimes X_3^a),
\]
(d) for every \(X \in \text{Ob}(\mathcal{C})\), a morphism
\[
L_{\Delta_X} : k \to L(X \otimes X^a),
\]
these datas verifying the following properties:
(P1) for \(X_1, X_3 \in \text{Ob}(\mathcal{C})\), the diagram
\[
\begin{array}{ccc}
L((X_1 \otimes 1_\mathcal{C}) \otimes L(1_\mathcal{C}^a \otimes X_3)) & \overset{\vartriangle_2}{\longrightarrow} & L(X_1 \otimes X_3) \\
\downarrow & & \downarrow \text{id} \\
L(X_1 \otimes L(X_3)) & \overset{\mathcal{R}}{\longrightarrow} & L(X_1 \otimes X_3)
\end{array}
\]
commutes,
(P2) for $X_1, X_2, X_3, X_4 \in \text{Ob}(\mathcal{C})$, the diagram

\[
\begin{array}{c}
L(X_1 \otimes X_2^a) \otimes L(X_2 \otimes X_3^a) \otimes L(X_3 \otimes X_4^a) \\
\xrightarrow{id \otimes \circ} \\
L(X_1 \otimes X_2^a) \otimes L(X_2 \otimes X_4^a) \\
\xrightarrow{\circ} \\
L(X_1 \otimes X_4^a)
\end{array}
\]

commutes,

(P3) the diagram

\[
\begin{array}{c}
k \xrightarrow{L_{\Delta X}} L(X \otimes X^a) \\
\xrightarrow{L_{\Delta X^a}} L(X^a \otimes X) \xrightarrow{L(\sigma)}
\end{array}
\]

commutes,

(P4) the composition

\[
L(X) \xrightarrow{L_{\Delta X} \otimes \text{id}_{L(X)}} L(X \otimes X^a) \otimes L(X) \xrightarrow{\circ} L(X)
\]

is the identity of $L(X)$ and the composition

\[
L(X^a) \xrightarrow{\text{id}_{L(X^a)} \otimes L_{\Delta X}} L(X^a) \otimes L(X \otimes X^a) \xrightarrow{\circ} L(X^a)
\]

is the identity of $L(X^a)$,

(P5) the diagram

\[
\begin{array}{c}
L(X \otimes X^a) \otimes L(X^a \otimes X) \\
\xrightarrow{\circ \otimes X^a} \\
L(X^a \otimes L(X))
\end{array}
\]

commutes,

(P6) for $X_1$ and $X_2$ belonging to $\text{Ob}(\mathcal{C})$, the diagram

\[
\begin{array}{c}
L((X_1 \otimes X_2)^a) \otimes L((X_1 \otimes X_2)) \\
\xrightarrow{L(\sigma) \otimes L(\sigma)} \\
L((X_2 \otimes X_1)^a) \otimes L(X_2 \otimes X_1)
\end{array}
\]

commutes.
Lemma 2.9. The object $L(X^a)$ is a dual of $L(X)$ with evaluation
\[ \varepsilon := \circ : L(X^a) \otimes L(X) \to k \] and coevaluation \[ \eta := \mathcal{R}^{-1} \circ L_\Delta. \]

Proof. Consider the diagram
\[
\begin{array}{c}
L(X) \xrightarrow{\eta \otimes \text{id}} L(X) \otimes L(X^a) \otimes L(X) \xrightarrow{\text{id} \otimes \varepsilon} L(X) \\
\downarrow \downarrow \downarrow \\
L(X^a) \xrightarrow{\text{id} \otimes \eta} L(X^a) \otimes L(X) \xrightarrow{\varepsilon \otimes \text{id}} L(X) \\
\end{array}
\]
and the diagram
\[
\begin{array}{c}
L(X^a) \xrightarrow{\text{id} \otimes \eta} L(X^a) \otimes L(X) \otimes L(X^a) \xrightarrow{\varepsilon \otimes \text{id}} L(X^a) \\
\downarrow \downarrow \downarrow \\
L(X^a) \xrightarrow{\text{id} \otimes L_\Delta} L(X^a) \otimes L(X^a) \xrightarrow{\varepsilon} L(X^a) \\
\end{array}
\]
These diagrams are made of two squares. The commutations of the left squares are obvious. The squares on the right commute because of the Property (P2). It follows that the two diagrams commute. Property (P4) implies that the bottom line of each diagram is equal to the identity. This proves the proposition.

The preceding proposition shows that $L(X)$ is a dualizable object of $\text{D}^b(k)$. We set $L(X)^* = \text{RHom}_k(L(X), k)$. Recall that $L(X)^* \simeq L(X^a)$.

Let $\lambda : k \to L(X \otimes X^a)$. It defines a morphism
\[
(2.1) \quad \Phi_\lambda : L(X) \xrightarrow{\lambda \otimes \text{id}} L(X \otimes X^a) \otimes L(X) \xrightarrow{\circ} L(X).
\]
Consider the diagram
\[
\begin{array}{c}
L(X) \xrightarrow{L} L(X)^* \xrightarrow{\Phi_\lambda \otimes \text{id}} L(X \otimes L(X)^*) \xrightarrow{L} L(X)^* \otimes L(X) \\
\downarrow \downarrow \downarrow \\
L(X) \xrightarrow{\varepsilon} k \\
\end{array}
\]
and
\[
\begin{array}{c}
L(X) \xrightarrow{\eta} L(X \otimes X^a) \xrightarrow{\Phi_\lambda \otimes \text{id}} L(X) \otimes L(X^a) \xrightarrow{\varepsilon} k \\
\end{array}
\]
Lemma 2.10. The diagram (2.2) commutes.

Proof. By Lemma 2.9, \( L(X^a) \) is a dual of \( L(X) \) with evaluation morphism \( \varepsilon \) and coevaluation morphism \( \eta \). It follows from Lemma 2.7 that the diagram (2.2) commutes. \qed

We identify \( \lambda \) and the image of \( 1_k \) by \( \lambda \).

Theorem 2.11. We have the formula

\[
\text{Tr}(\Phi_\lambda) = L_{X^a} \circ X_{X^a} \lambda.
\]

Proof. By definition of \( \Phi_\lambda \), the diagram (2.3)

\[
\begin{array}{ccc}
L(X) \otimes L(X^a) & \xrightarrow{\Phi_\lambda \otimes \text{id}} & L(X) \otimes L(X^a) \\
\downarrow \eta & & \downarrow \varepsilon \\
L(X \otimes X^a) \otimes L(X) & \xrightarrow{\Delta \otimes \text{id}} & L(X) \otimes L(X^a) \otimes L(X)
\end{array}
\]

commutes.

Thus computing the trace of \( \Phi_\lambda \) is equivalent to compute the lower part of diagram (2.3).

We denote by \( \zeta \) the map

\[
\zeta : L(X^a \otimes X) \simeq k \otimes L(X^a \otimes X) \xrightarrow{L_\Delta \otimes \text{id}} L(X \otimes X^a) \otimes L(X^a \otimes X) \xrightarrow{X^a \otimes X} k.
\]

Consider the diagram (2.4)
This diagram is made of four sub-diagrams numbered from 1 to 4.

1. The sub-diagram 1 commutes by definition of \( \eta \).
2. Notice that \( \mathcal{A} = \circ \) by the Property (P1). Then the sub-diagram
   2 commutes by the Property (P2).
3. The sub-diagram 3 commutes because \( L \) is a symmetric monoidal functor,
4. The sub-diagram 4 commutes by Property (P5).

The right side of the diagram (2.4) is equal to \( L_\Delta \circ L(\sigma) \lambda \). By the
Property (P6), \( L_\Delta \circ L(\sigma) \lambda = L(\sigma) L_\Delta \circ \lambda \) and by the Property
(P3), \( L(\sigma) L_\Delta = L_\Delta a \), the result follows. \( \square \)

3. A short review on DQ-modules

Deformation quantization modules have been introduced and systematically studied in [4]. We shall first recall here the main features of this theory, following the notations of loc. cit.

In all this paper, a manifold means a complex analytic manifold. We denote by \( \mathbb{C} h \) the ring \( \mathbb{C}[[h]] \). A Deformation Quantization algebroid stack (DQ-algebroid for short) on a complex manifold \( X \) with structure sheaf \( \mathcal{O}_X \), is a stack of \( \mathbb{C} h \)-algebras locally isomorphic to a star algebra \( (\mathcal{O}_X[[h]], \ast) \). If \( \mathcal{A}_X \) is a DQ-algebroid on a manifold \( X \) then the opposite DQ-algebroid \( \mathcal{A}_X^{op} \) is denoted by \( \mathcal{A}_X^a \). The diagonal embedding is denoted by \( \delta_X : X \rightarrow X \times X \).

If \( X \) and \( Y \) are two manifolds endowed with DQ-algebroids \( \mathcal{A}_X \) and \( \mathcal{A}_Y \), then \( X \times Y \) is canonically endowed with the algebroid \( \mathcal{A}_{X \times Y} := \mathcal{A}_X \Box \mathcal{A}_Y \) (see [4 §2.3]). We write \( \mathcal{C}_X \) for the \( \mathcal{A}_{X \times X^a} \)-module \( \delta_{X^a} \mathcal{A}_X \) and \( \omega_X \) for the dualizing complex of DQ-modules. We denote by \( D^\prime_{\mathcal{A}_X} \) the duality functor of \( \mathcal{A}_X \)-modules:

\[
D^\prime_{\mathcal{A}_X}(\cdot) := \mathcal{R}\mathcal{H}\mathcal{om}_{\mathcal{A}_X}(\cdot, \mathcal{A}_X).
\]

Consider complex manifolds \( X^i \) endowed with DQ-algebroids \( \mathcal{A}_{X^i} \)
\((i = 1, 2, \ldots)\).

**Notation 3.1.** (i) Consider a product of manifolds \( X \times Y \times Z \). We denote by \( p_i \) the \( i \)-th projection and by \( p_{ij} \) the \((i, j)\)-th projection (e.g., \( p_{13} \) is the projection from \( X_1 \times X_1^a \times X_2 \) to \( X_1 \times X_2 \)). We use similar notations for a product of four manifolds.
(ii) We write \( \mathcal{A}_i \) and \( \mathcal{A}_{ij} \) instead of \( \mathcal{A}_{X^i} \) and \( \mathcal{A}_{X^i \times X^j} \) and similarly with other products. We use the same notations for \( \mathcal{C}_{X^i} \).
(iii) When there is no risk of confusion, we do note write the symbols \( p_{i}^{-1} \) and similarly with \( i \) replaced with \( ij \), etc.
3.1. Hochschild homology. Let $X$ be a complex manifold endowed with a DQ-algebroid $A_X$. Recall that its Hochschild homology is defined by

$$\mathcal{HH}(A_X) := \delta_X^{-1}(C_{X^a} \otimes_{A_{X^a \times X^a}} C_X) \in D^b(C^h_X).$$

We denote by $\mathcal{HH}(X)$ the object $R\Gamma(X, \mathcal{HH}(A_X))$ of $D^b(C^h)$ and by $\mathcal{HH}_0(A_X)$ the $C^h$-module $H^0(\mathcal{HH}(X))$. We also set for $\Lambda$ a closed subset of $X$, $\mathcal{HH}_\Lambda(A_X) := \Gamma_\Lambda \mathcal{HH}(A_X)$ and $\mathcal{HH}_\Lambda^0(A_X) = H^0(R\Gamma_\Lambda(X; \mathcal{HH}(A_X)))$.

**Proposition 3.2.** There is a natural isomorphism

$$\mathcal{HH}(A_X) \simeq R\mathcal{H}om_{A_{X^a \times X^a}}(\omega_X^{-1}, C_X).$$

**Proof.** See [4, §4.1] □

**Proposition 3.3** (Künneth isomorphism). Let $X_i$ ($i = 1, 2$) be complex manifolds endowed with DQ-algebroids $A_i$.

(i) There is a natural morphism

$$R\mathcal{H}om_{A_{12}}(\omega_1^{-1}, C_1) \otimes R\mathcal{H}om_{A_{23}}(\omega_2^{-1}, C_2) \rightarrow R\mathcal{H}om_{A_{12} \times A_{23}}(\omega_1^{-1}, C_2).$$

(ii) If $X_1$ or $X_2$ is compact, this morphism induces a natural isomorphism

$$\mathcal{H}(X_1) \otimes \mathcal{H}(X_2) \sim \mathcal{H}(X_{12}).$$

**Proof.** (i) is clear.

(ii) By [4, §1.5], the modules $\mathcal{H}(X_i)$ for ($i = 1, 2$) and $\mathcal{H}(X_{12})$ are cohomologically complete. If $X_1$ is compact, then the $C^h$-module $\mathcal{H}(A_1)$ belongs to $D^b_+(C^h)$. Thus, $\mathcal{H}(A_1) \otimes_{C^h} \mathcal{H}(A_2)$ is still a cohomologically complete module (see [4, §1.6]).

Applying the functor $gr_h$ to the morphism (3.2), we obtain the usual Künneth isomorphism for Hochschild homology of complex manifolds. Since $gr_h$ is a conservative functor on the category of cohomologically complete modules, the morphism (3.2) is an isomorphism. □

3.2. Composition of Hochschild homology. Let $\Lambda_{ij}$ ($i = 1, 2, j = i + 1$) be a closed subset of $X_{ij}$ and consider the hypothesis

$$p_{i3} \text{ is proper over } \Lambda_{12} \times X_2 \Lambda_{23}.$$ 

We also set $\Lambda_{12} \circ \Lambda_{23} = p_{13}(p_{12}^{-1} \Lambda_{12} \cap p_{23}^{-1} \Lambda_{23}).$

Recall Proposition 4.2.1 of [4].
Theorem 3.4. Let $\Lambda_{ij}$ ($i = 1, 2$ $j = i + 1$) satisfying (3.3). There is a morphism

\[
(3.4) \quad \mathcal{H} \mathcal{H}_{\Lambda_{12}}(A_{12^a}) \circ \mathcal{H} \mathcal{H}_{\Lambda_{23}}(A_{23^a}) \to \mathcal{H} \mathcal{H}_{\Lambda_{12} \circ \Lambda_{23}}(A_{13^a}).
\]

which induces a composition morphism for global sections

\[
(3.5) \quad \circ : \mathcal{H} \mathcal{H}_{\Lambda_{12}}(A_{12^a}) \otimes \mathcal{H} \mathcal{H}_{\Lambda_{23}}(A_{23^a}) \to \mathcal{H} \mathcal{H}_{\Lambda_{12} \circ \Lambda_{23}}(A_{13^a}).
\]

Corollary 3.5. The morphism (3.4) induces a morphism

\[
(3.6) \quad \circ_{\text{pt}} : \mathcal{H} \mathcal{H}(A_1) \otimes \mathcal{H} \mathcal{H}(A_2) \to \mathcal{H} \mathcal{H}(A_1)
\]

which coincides with the morphism (3.1).

Proposition 3.6. (i) Let $\Lambda_{ij}$ ($i = 1, 2, 3$ $j = i + 1$) such that the pair $\Lambda_{12}, \Lambda_{23}$ satisfies (3.3) as well as $\Lambda_{23}, \Lambda_{34}$ with similar notations. The following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{H} \mathcal{H}_{\Lambda_{12}}(A_{12^a}) & \mathcal{H} \mathcal{H}_{\Lambda_{23}}(A_{23^a}) & \mathcal{H} \mathcal{H}_{\Lambda_{12} \circ \Lambda_{23}}(A_{13^a}) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{H} \mathcal{H}_{\Lambda_{12} \circ \Lambda_{23}}(A_{13^a}) & \mathcal{H} \mathcal{H}_{\Lambda_{34}}(A_{34^a}) & \mathcal{H} \mathcal{H}_{\Lambda_{12} \circ \Lambda_{23} \circ \Lambda_{34}}(A_{14^a}).
\end{array}
\]

(ii) Assume that $X_i$ is compact for $i = 1, 2, 3, 4$. Then, the preceding diagram induces a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H} \mathcal{H}(X_{12^a}) & \mathcal{H} \mathcal{H}(X_{23^a}) & \mathcal{H} \mathcal{H}(X_{34^a}) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{H} \mathcal{H}(X_{13^a}) & \mathcal{H} \mathcal{H}(X_{34^a}) & \mathcal{H} \mathcal{H}(X_{14^a}).
\end{array}
\]

Remark 3.7. If $X$ is a complex compact manifold endowed with a DQ-algebroid $\mathcal{A}_X$. Let $\lambda \in \mathcal{H} \mathcal{H}^0(\mathcal{A}_{X \times X^a})$. There is a morphism

\[
(3.7) \quad \Phi_{\lambda : \mathcal{H} \mathcal{H}(X) \to \mathcal{H} \mathcal{H}(X)
\]

given by

\[
\mathcal{H} \mathcal{H}(X) \simeq \mathbb{C}^h \otimes \mathcal{H} \mathcal{H}(X) \xrightarrow{\lambda \otimes \text{id}} \mathcal{H} \mathcal{H}(X \times X^a) \otimes \mathcal{H} \mathcal{H}(X) \rightarrow \mathcal{H} \mathcal{H}(X).
\]
3.3. Hochschild class. Let $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{A}_X)$. We have the chain of morphisms

$$hh_M : \mathcal{RHom}_{\mathcal{A}_X}(\mathcal{M}, \mathcal{M}) \xrightarrow{\sim} D^b_{\mathcal{A}_X}(\mathcal{M}) \overset{\otimes}{\longrightarrow} \mathcal{A}_X \mathcal{M} \xrightarrow{\sim} \mathcal{C}_X.$$

We get a map

$$(3.8) \quad hh_0^M : \text{Hom}_{\mathcal{A}_X}(\mathcal{M}, \mathcal{M}) \rightarrow H^0_{\text{Supp}(\mathcal{M})}(X, \mathcal{HH}(\mathcal{A}_X)).$$

**Definition 3.8.** The image of an endomorphism $f$ of $\mathcal{M}$ by the map (3.8) gives an element $hh_X(\mathcal{M}, f) \in H^0_{\text{Supp}(\mathcal{M})}(X, \mathcal{HH}(\mathcal{A}_X))$ called the Hochschild class of the pair $(\mathcal{M}, f)$. If $f = \text{id}_\mathcal{M}$, we simply write $hh_X(\mathcal{M})$.

**Remark 3.9.** Let $M \in D^b_{\mathcal{C}_h}(\mathcal{C}_h)$ and let $f \in \text{Hom}_{\mathcal{C}_h}(M, M)$. Then the Hochschild class $hh_{\mathcal{C}_h}(M, f)$, of $f$ is obtained by the composition

$$\mathcal{C}_h \rightarrow \mathcal{RHom}_{\mathcal{C}_h}(M, M) \rightarrow M \otimes_{\mathcal{C}_h} \mathcal{RHom}_{\mathcal{C}_h}(M, \mathcal{C}_h) \xrightarrow{f \otimes \text{id}} M \otimes_{\mathcal{C}_h} \mathcal{RHom}_{\mathcal{C}_h}(M, \mathcal{C}_h) \rightarrow \mathcal{RHom}_{\mathcal{C}_h}(M, \mathcal{C}_h) \otimes_{\mathcal{C}_h} M \rightarrow \mathcal{C}_h.$$

Thus it is the trace of $f$ in $D^b_{\mathcal{C}_h}$.

3.4. Actions of Kernels. It is possible to compose kernels in the framework of DQ-modules.

**Definition 3.10 (III).** Let $\mathcal{K}_i \in D^b(\mathcal{A}_{ij})$ ($i = 1, 2, j = i + 1$). One sets

$$\mathcal{K}_1 \circ \mathcal{K}_2 = \mathcal{R}p_{14!}((\mathcal{K}_1 \otimes \mathcal{K}_2) \otimes_{\mathcal{A}_{22a}} \mathcal{C}_{X_2}),$$

$$\mathcal{K}_1 * \mathcal{K}_2 = \mathcal{R}p_{14*}((\mathcal{K}_1 \otimes \mathcal{K}_2) \otimes_{\mathcal{A}_{22a}} \mathcal{C}_{X_2}).$$

We explain how kernels act on Hochschild homology.
Lemma 3.11. Let $\mathcal{K} \in D^b_{coh}(\mathcal{A}_{12})$. There are natural morphisms in $D^b_{coh}(\mathcal{A}_{11})$

\[
\omega^{-1}_{X_1} \to \mathcal{K} \ast \mathcal{D}'(\mathcal{K})
\]

(3.9)

\[
\mathcal{K} \circ \omega_{X_2} \circ \mathcal{D}'(\mathcal{K}) \to \mathcal{C}_{X_1}.
\]

(3.10)

Let $\mathcal{K}$ be an object of $D^b_{coh}(\mathcal{A}_{12\ast})$ with support $\Lambda_{12}$ and let $\Lambda_2$ be a closed subset of $X_2$. Assume that $\Lambda_{12} \times_{X_2} \Lambda_2$ is proper over $X_1$. Then, according to [4] there is a map

\[
\Phi_X : \text{HH}_{\Lambda_2}(\mathcal{A}_{X_2}) \to \text{HH}_{\Lambda_{12} \circ \Lambda_2}(\mathcal{A}_{X_1})
\]

defined as follow. For the sake of brevity, we write $\text{Hom}$ instead of $\text{RHom}$.

\[
\Gamma_{\Lambda_{12}} \text{Hom}_{12\ast}(\mathcal{K}, \mathcal{K}) \circ \Gamma_{\Lambda_2} \text{Hom}_{22\ast}(\omega^{-1}_2, \mathcal{C}_2) \circ \Gamma_{\Lambda_{12}} \text{Hom}_{21\ast}(\omega_2 \circ \mathcal{D}'/\mathcal{K}, \omega_2 \circ \mathcal{D}'/\mathcal{K})
\]
\[
\to \Gamma_{\Lambda_{12} \circ \Lambda_{12}} \text{Hom}_{11\ast}(\mathcal{K} \ast \omega^{-1}_2, \mathcal{K} \circ \mathcal{C}_2) \circ \Gamma_{\Lambda_{12}} \text{Hom}_{21\ast}(\omega_2 \circ \mathcal{D}'/\mathcal{K}, \omega_2 \circ \mathcal{D}'/\mathcal{K})
\]
\[
\to \Gamma_{\Lambda_{12} \circ \Lambda_2} \text{Hom}_{11\ast}(\mathcal{K} \ast \omega^{-1}_2 \ast \omega_2 \ast \mathcal{D}'/\mathcal{K}, \mathcal{K} \circ \omega_2 \circ \mathcal{D}'/\mathcal{K})
\]
\[
\to \Gamma_{\Lambda_{12} \circ \Lambda_2} \text{Hom}(\omega^{-1}_1, \mathcal{C}_1).
\]

There is a canonical section in $\Gamma_{\Lambda_{12}}(X_{12}; \text{Hom}_{12\ast}(\mathcal{K}, \mathcal{K}))$ and a canonical section in $\Gamma_{\Lambda_{12}}(X_{21}; \text{Hom}_{21\ast}(\omega_2 \circ \mathcal{D}'/\mathcal{K}, \omega_2 \circ \mathcal{D}'/\mathcal{K}))$. They are associated with $\text{id}_\mathcal{K} \in \text{Hom}_{12\ast}(\mathcal{K}, \mathcal{K})$ and $\text{id}_{\mathcal{D}'/\mathcal{K}} \in \text{Hom}(\mathcal{D}'/\mathcal{K}, \mathcal{D}'/\mathcal{K})$. This defines the map (3.11).

There is the following result regarding $\Phi_X$.

Proposition 3.12 ([4]). The map $\Phi_X : \text{HH}_{\Lambda_2}(\mathcal{A}_{X_2}) \to \text{HH}_{\Lambda_{12} \circ \Lambda_2}(\mathcal{A}_{X_1})$ in (3.11) is the map $\text{hh}_{X_{12\ast}}(\mathcal{K}) \circ$ given in (3.3).

We denote by $\omega^\text{top}_X$ the dualizing complex associated with the topological manifold $X^\text{top}$ underlying $X$.

Proposition 3.13. Let $X_i$, $(i=1, 2)$ be a complex manifold endowed with a DQ-algebroid $\mathcal{A}_i$. The following diagram commutes.

\[
\begin{array}{ccc}
\text{HH}(\mathcal{A}_{1\ast}) \otimes \text{HH}(\mathcal{A}_{2\ast}) \otimes \text{HH}(\mathcal{A}_2) & \xrightarrow{\cdot \cdot} & \omega^\text{top}_{12} \\
\downarrow & & \\
\text{HH}(\mathcal{A}_{12\ast}) \otimes \text{HH}(\mathcal{A}_{1\ast 2}) & & \\
\end{array}
\]
Proof. It is a consequence of the projection formula and of the associativity of the tensor product. □

Corollary 3.14. The diagram

\[
\begin{array}{ccc}
\text{HH}(X_1^a) \otimes \text{HH}(X_{12}^a) \otimes \text{HH}(X_2) & \overset{\circ \circ}{\rightarrow} & \mathbb{C}^h \\
\text{HH}(X_{12}^a) \otimes \text{HH}(X_{102}^a) & \overset{1_{12}^a}{\rightarrow} & \\
\end{array}
\]

commutes

The composition

\[
\mathbb{C}_X^h \to R\text{Hom}_C(X \times X^a) \overset{\text{hh}(\Delta)}{\rightarrow} \mathcal{H}(X \times X^a)
\]

induces a map

\[
(3.14) \quad \text{hh}(\Delta) : \mathbb{C}^h \to \text{HH}(X \times X^a).
\]

The image of 1_{\mathbb{C}^h} by \text{hh}(\Delta) is \text{hh}(X \times X^a)(\mathcal{C}_X).

Proposition 3.15. The left and right actions of \text{hh}(X \times X^a)(\mathcal{C}_X) on \text{HH}(X) via the morphism (3.5) are the trivial action.

Proof. See [4, §4.3]. □

We define the morphism \(\zeta : \text{HH}(\mathcal{A}_X \times X^a) \to \mathbb{C}^h\) as the composition

\[
\text{HH}(X^a \times X) \simeq \mathbb{C}^h \otimes \text{HH}(X \times X^a) \overset{\text{hh}(\Delta) \otimes \text{id}}{\rightarrow} \text{HH}(X \times X^a) \otimes \text{HH}(X^a \times X) \overset{\text{id} \otimes \text{hh}(\Delta)}{\rightarrow} \mathbb{C}^h.
\]

Corollary 3.16. The diagram below commutes.

\[
\begin{array}{ccc}
\text{HH}(X^a) \otimes \text{HH}(X) & \overset{\gamma}{\rightarrow} & \mathbb{C}^h \\
\text{HH}(X^a \times X) & \overset{\zeta}{\rightarrow} & \\
\end{array}
\]

Proof. It follows from Corollary 3.14 with \(X_1 = X_2 = X\), that the diagram

\[
\begin{array}{ccc}
\text{HH}(X^a) \otimes \mathbb{C}^h \otimes \text{HH}(X) & \overset{\text{id} \otimes \text{hh}(\Delta) \otimes \text{id}}{\rightarrow} & \text{HH}(X^a) \otimes \text{HH}(X \times X^a) \otimes \text{HH}(X) \overset{\circ \circ}{\rightarrow} \mathbb{C}^h \\
\text{HH}(X^a) \otimes \mathbb{C}^h \otimes \text{HH}(X) \overset{\text{id} \otimes \text{hh}(\Delta) \otimes \text{id}}{\rightarrow} & \text{HH}(X \times X^a) \otimes \text{HH}(X^a \times X) & \overset{\circ \circ}{\rightarrow} \mathbb{C}^h \\
\end{array}
\]

commutes which implies the result. □
Finally, an important result of [4] is the following

**Theorem 3.17.** Let \( \Lambda_i \) be a closed subset of \( X_i \times X_{i+1} \) (\( i = 1, 2 \)) and assume that \( \Lambda_1 \times X_2 \Lambda_2 \) is proper over \( X_1 \times X_3 \). Set \( \Lambda = \Lambda_1 \circ \Lambda_2 \). Let \( \mathcal{K}_i \in D_{coh,i}(A_{X_i \times X_{i+1}}^+) \). Then

\[
\text{hh}_{X_13}(\mathcal{K}_1 \circ \mathcal{K}_2) = \text{hh}_{X_12}(\mathcal{K}_1) \circ \text{hh}_{X_23}(\mathcal{K}_2)
\]

as element of \( \text{HH}_0^i(A_{X_1 \times X_3}) \).

**Proof.** See [4, §4.3]. \( \square \)

### 3.5. A monoidal category

In this subsection we collect a few facts concerning the product \( \boxtimes \). Recall that if \( X \) and \( Y \) are two complex manifolds endowed with DQ-algebroids \( A_X \) and \( A_Y \), \( X \times Y \) is canonically endowed with the DQ-algebroid \( A_{X \times Y} := A_X \boxtimes A_Y \). There is a functorial symmetry isomorphism

\[
\sigma_{X,Y} : (X \times Y, A_{X \times Y}) \sim (Y \times X, A_{X \times Y})
\]

and for any triple \((X, A_X), (Y, A_Y)\) and \((Z, A_Z)\) there is a functorial associativity isomorphism

\[
\rho_{X,Y,Z} : (A_X \boxtimes A_Y) \boxtimes A_Z \sim A_X \boxtimes (A_Y \boxtimes A_Z).
\]

We consider the category \( \mathcal{DQ} \) whose objects are the pairs \((X, A_X)\) where \( X \) is a complex manifold and \( A_X \) a DQ-algebroid stack on \( X \) and where the morphisms are obtained by composing and tensoring the identity morphisms, the symmetry morphisms and the associativity morphisms. The category \( \mathcal{DQ} \) endowed with \( \boxtimes \) is a symmetric monoidal category.

We denote by

\[
v : ((X \times Y) \times (X \times Y)^a, A_{(X \times Y) \times (X \times Y)^a}) \rightarrow ((Y \times X) \times (Y \times X)^a, A_{(Y \times X) \times (Y \times X)^a})
\]

the map of defined by \( v := \sigma \times \sigma \).

In this situation, after identifying, \((X \times X)^a \times (Y \times Y)^a\) with \((X \times Y) \times (X \times Y)^a\), there is a natural isomorphism \( C_X \boxtimes C_Y \simeq C_{X \times Y} \) and the morphism \( v \) induces an isomorphism

\[
v^*(C_{X \times Y}) \simeq C_{Y \times X}.
\]

**Proposition 3.18.** The map \( \sigma_{X,Y} \) induce an isomorphism

\[
(3.16) \quad \sigma_* : \mathcal{HH}(A_{X \times Y}) \rightarrow \mathcal{HH}(A_{Y \times X})
\]
Proof. There is the following cartesian square of topological space.

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\sigma} & Y \times X \\
\downarrow{\delta} & & \downarrow{\delta'} \\
(X \times Y) \times (X \times Y) & \xrightarrow{\nu} & (Y \times X) \times (Y \times X).
\end{array}
\]

Then

\[
\sigma_* \mathcal{H} \mathcal{H}(\mathcal{A}_{X \times Y}) \simeq \sigma\delta^{-1}(C_{(X \times Y)^a} \mathcal{L}_{\mathcal{A}_{X \times Y}} \mathcal{C}_{X \times Y})
\]
\[
\simeq \delta'^{-1}\nu(\mathcal{L} \mathcal{C}_{(X \times Y)^a} \mathcal{A}_{X \times Y})
\]
\[
\simeq \delta'^{-1}(\mathcal{C}_{(Y \times X)^a} \mathcal{L}_{\mathcal{A}_{Y \times X}} \mathcal{C}_{Y \times X}).
\]

\[\square\]

The morphism (3.16) induces an isomorphism that we still denote \(\sigma_*\)

\[\sigma_* : \mathcal{H} \mathcal{H}(X \times Y) \to \mathcal{H} \mathcal{H}(Y \times X).\]

The following diagram commutes

(3.17) \[
\begin{array}{ccc}
\mathcal{H} \mathcal{H}(X \times Y) & \xrightarrow{\sigma_*} & \mathcal{H} \mathcal{H}(Y \times X) \\
\uparrow{\mathcal{L}} & & \uparrow{\mathcal{L}} \\
\mathcal{H} \mathcal{H}(X) \mathcal{L} \mathcal{H} \mathcal{H}(Y) & \longrightarrow & \mathcal{H} \mathcal{H}(Y) \mathcal{L} \mathcal{H} \mathcal{H}(X).
\end{array}
\]

**Proposition 3.19.** We have

\[\sigma_* hh_{X \times X^a}(C_X) = hh_{X^a \times X}(C_{X^a}).\]

Proof. Immediate by using Lemma 4.1.4 of [4]. \[\square\]

4. A Lefschetz formula for DQ-modules

Inspired by the Lefschetz formula for Fourier-Mukai functor of V. Lunts (see [3]), we give a similar formula in the framework of DQ-modules.

**Theorem 4.1.** Let \(X\) be a complex compact manifold endowed with a DQ-algebroid \(\mathcal{A}_X\). Let \(\lambda \in \mathcal{H} \mathcal{H}^0(\mathcal{A}_{X \times X^a})\). Let

\[\Phi_\lambda : \mathcal{H} \mathcal{H}(X) \to \mathcal{H} \mathcal{H}(X)\]

be the map (3.7). Then

\[\text{Tr}_{\mathbb{C}^a}(\Phi_\lambda) = hh_{X^a \times X}(C_{X^a})_{X \times X^a} \lambda.\]
Proof. Consider the full subcategory \( \mathcal{C} \) of \( \mathcal{D} \) whose objects are the pair \((X, A_X)\) where \( X \) is a compact manifold. By the results of the Subsection 3.5, the pair \((\text{HH}, \mathfrak{R})\) is a symmetric monoidal functor.

The data are given by
(a) the functor \((\cdot)^a\) which associate to a DQ-algebroid \((X, A_X)\) the opposite DQ-algebroid \((X, A_X^a)\),
(b) the monoidal functor on \( \mathcal{C} \) given by the pair \((\text{HH}, \mathfrak{R})\),
(c) the morphism (3.5),
(d) for each pair \((X, A_X)\) the morphism \(\text{hh}(\Delta_X)\).

We check the properties requested by our formalism:
(i) the Property (P1) is granted by the Corollary 3.5,
(ii) the Property (P2) follows from the Proposition 3.6,
(iii) the Property (P3) follows from the Proposition 3.19,
(iv) the Property (P4) follows from the Proposition 3.15,
(v) the Property (P5) follows from the Proposition 3.16,
(vi) the Property (P6) is clear.

Then the formula follows from Theorem 2.11. □

Corollary 4.2. Let \( X \) be a complex compact manifold endowed with a DQ-algebroid \( A_X \) and let \( \mathcal{K} \in \mathcal{D}_{\text{coh}}(A_X \times X^a) \). Then
\[
\text{Tr}_{\mathcal{C}}(\Phi_{\mathcal{K}}) = \text{hh}_{X^a \times X}(C_{X^a}) \circ \text{hh}_{X \times X}(\mathcal{K}).
\]

Proof. Apply Proposition 3.12 and Theorem 4.1. □

Corollary 4.3. Let \( X \) be a complex compact manifold endowed with a DQ-algebroid \( A_X \) and let \( \mathcal{K} \in \mathcal{D}_{\text{coh}}(A_X \times X^a) \). Then
\[
\text{Tr}_{\mathcal{C}}(\Phi_{\mathcal{K}}) = \chi(R\Gamma(X \times X^a; C_{X^a} \otimes A_X^{X^a} \mathcal{K})).
\]

Proof. Applying the Proposition 3.17 to the right hand side of Theorem 4.2 we get that \(\text{Tr}_{\mathcal{C}}(\Phi_{\mathcal{K}}) = \chi(R\Gamma(X \times X^a; C_{X^a} \otimes A_X^{X^a} \mathcal{K})).\) □

4.1. Applications. We explain how to recover some of the results of the paper [6] of V. Lunts and give a special form of the formula when \( X \) is also symplectic.

Theorem 4.4 ([6]). Let \( X \) be a complex compact manifold and \( \mathcal{K} \in \mathcal{D}_{\text{coh}}(\mathcal{O}_{X \times X}) \). Then,
\[
\sum_i (-1)^i \text{Tr}(H^i(\Phi_{\mathcal{K}})) = \chi(R\Gamma(X \times X; \mathcal{O}_X \otimes \mathcal{K})�).
\]
Proof. We endow $X$ with the trivial deformation. Then, we can apply the Corollary 4.3 and forget $\hbar$. We recover the Theorem 3.9 of [6].

\[ \] 

Corollary 4.5 ([6]). Let $X$ be a complex compact manifold endowed with a DQ-algebroid $\mathcal{A}_X$ and let $K \in \mathbb{D}^b_{\text{coh}}(\mathcal{A}_{X \times X^a})$. Then,

$$
\sum_i (-1)^i \text{Tr}(H^i(\Phi_K)) = \int_X \delta^* \text{ch}(K) \cup \text{td}_X(TX)
$$

where $\text{ch}(K)$ is the Chern class of $K$, $\text{td}_X(TX)$ is the Todd class of the tangent bundle $TX$ and $\delta^*$ is the pullback by the diagonal embedding.

Proof. It is a direct application of Corollary 5.3.5 of [4] and Corollary 4.3.

It is possible to localize $\mathcal{A}_X$ with respect to $\hbar$. We denote by $\mathbb{C}((\hbar))$ the field of formal Laurent series. We set by $\mathcal{A}_X^{\text{loc}} = \mathbb{C}((\hbar)) \otimes_{\mathbb{C}_h} \mathcal{A}_X$. If $\mathcal{M}$ is a $\mathcal{A}_X$-module we denote by $\mathcal{M}^{\text{loc}}$ the $\mathcal{A}_X^{\text{loc}}$-module $\mathbb{C}((\hbar)) \otimes_{\mathbb{C}_h} \mathcal{M}$. We denote by $d_X$ the complex dimension of $X$. In the symplectic case, we have according to [4, §6.3]

**Theorem 4.6.** If $X$ is a symplectic, the complex $\mathcal{H}(\mathcal{A}_X^{\text{loc}})$ is concentrated in degree $-d_X$ and there is a canonical isomorphism

$$
\tau_X : \mathcal{H}(\mathcal{A}_X^{\text{loc}}) \xrightarrow{\sim} \mathbb{C}_X^{\text{h,loc}}[d_X].
$$

We refer the reader to section 6.2 and 6.3 of [4] for a precise description of $\tau_X$.

**Definition 4.7 ([4]).** Let $\mathcal{M} \in \mathbb{D}^b_{\text{coh}}(\mathcal{A}_X^{\text{loc}})$. We set

$$
eu(\mathcal{M}) = \tau_X(\mathcal{H}_X(\mathcal{M})) \in H^d_X(\text{Supp}(\mathcal{M}), X; \mathbb{C}_X)
$$

and call $\neu(\mathcal{M})$ the Euler class of $\mathcal{M}$.

We then have the following

**Proposition 4.8.** Let $X$ be a compact complex symplectic manifold and let $K \in \mathbb{D}^b_{\text{coh}}(\mathcal{A}_{X \times X^a})$. Then,

$$
\sum_i (-1)^i \text{Tr}(H^i(\Phi_K)) = \int_{X \times X} \neu(\mathcal{C}_X^{\text{loc}}) \cup \neu(\mathcal{K}^{\text{loc}})
$$

where $\cup$ is the cup product.

Proof. It is a direct consequences of [4, §6.3] and of Theorem 4.1.
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REFERENCES

[1] A. Căldăraru and S. Willerton. The Mukai pairing. I. A categorical approach. New York J. Math., 16:61–98, 2010.

[2] D.-C. Cisinski and G. Tabuada. Lefschetz and Hirzebruch-Riemann-Roch formulas via noncommutative motives, arXiv:1111.0257. ArXiv e-prints, November 2011.

[3] M. Kashiwara and P. Schapira. Categories and sheaves, volume 332 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 2006.

[4] M. Kashiwara and P. Schapira. Deformation quantization modules, arXiv:1003.3304v2. Astérisque, 2012.

[5] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. Equivariant stable homotopy theory, volume 1213 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[6] V. A. Lunts. Lefschetz fixed point theorems for Fourier-Mukai functors and DG algebras, arXiv:1102.2884. ArXiv e-prints, February 2011.

[7] A. Polishchuk. Lefschetz type formulas for dg-categories, arXiv:1111.0728. ArXiv e-prints, November 2011.

[8] K. Ponto and M. Shulman. Traces in symmetric monoidal categories, arXiv:1107.6032. ArXiv e-prints, July 2011.

[9] D. Shklyarov. Hirzebruch-Riemann-Roch theorem for DG algebras, arXiv:0710.193. ArXiv e-prints, October 2007.

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