Strings in an arbitrary constant magnetic field with arbitrary constant metric and stringy form factors

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Abstract: We quantize the open string in an arbitrary constant magnetic field with a non factorized metric on a torus. We then discuss carefully the vertexes which describe the emission of dipole open strings and closed strings in the non compact limit. Finally we compute various stringy form factors which in the compact case induces a Kähler and complex structure dependence and suppression of some amplitudes with KK states.

Keywords: D-branes, Gauge Theories.
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1. Introduction and conclusions

The construction of “phenomenological” models of particle physics based on D-branes embedded in supersymmetric string compactifications has become a major direction in the development of String Theory (for review, see for instance [1]).

In these D-brane scenarios non-perturbative corrections arise from D-instantons and wrapped Euclidean branes. In principle all instantonic branes which can be consistently included may contribute to the low energy effective action. If the theory contains a gauge sector realized on D(3 + n)-branes wrapped on a cycle \( C \), then Euclidean branes \( E(n - 1) \) wrapped on \( C \) correspond to the instanton sectors of the gauge theory\(^1\). Other Euclidean branes, for instance those wrapped on a cycle \( C' \neq C \), do not possess this interpretation and have been referred to as “exotic” or “stringy” instantons; they have been investigated over the last years in a rapidly growing literature (for a review, see for example [2]).

This interest was sparked by the realization that exotic instantons might provide couplings which are forbidden in perturbation theory but necessary for phenomenological applications; for instance, they have been pointed out as possible sources of neutrino masses or of certain Yukawa couplings in GUT models. The relation of “ordinary” instantonic branes to the field-theoretical description of instantons in supersymmetric gauge theories (as reviewed, for instance, in [3]) has been clarified in detail [4, 5]. In the ordinary cases, the spectrum and interactions of the moduli, i.e. of the physical excitations of open strings with at least one end-point on the instantonic branes, reproduce the ADHM construction of the moduli space of gauge theory instantons. In particular, let us consider the NS sector

\(^1\)The simplest case is represented by the D3/D(–1) system, corresponding to \( n = 0 \).
of the open strings with one end on the $D(3+n)$ and the other end on the $E(n-1)$-brane. The world-sheet condition for physical states has the form

$$L_0 - \frac{1}{2} = N_X + N_\psi + \sum_{i=1}^{3} \frac{\nu_i}{2} = 0,$$  \hspace{1cm} (1.1)$$

where $N_X$ and $N_\psi$ are the occupation numbers for the bosonic and fermionic world-sheet oscillators, while the (positive) angles $2\pi \nu_i$ denote the twist eventually occurring in the three complex internal directions. In writing (1.1) we have taken into account the $1/2$ contribution to the zero-point energy from the four space-time directions, which are of Neumann-Dirichlet type. Ordinary $E(n-1)$-branes impose, in the internal space, the same boundary conditions as the gauge $D(3+n)$-branes do. All twists $\nu^i$ therefore vanish and the ground state $N_X = N_\psi = 0$ is physical; being degenerate in the non-compact directions, it corresponds to the moduli $w_i$ of the ADHM construction. These bosonic mixed moduli enter in an essential way in the ADHM constraints and, once these constraints are solved, they contain in particular the size $\rho$ of the instanton solution. The instanton profile and its dependence on ADHM parameters can be read from the amplitude [5].

$$A_{\mu}(p; w, \bar{w}) = \langle V_{A_{\mu}} \rangle_{\text{mixed disk}} = \langle V_{w}^{(-1)} \rangle A_{\mu}^{(0)}(-p) V_{w}^{(-1)} \rangle$$  \hspace{1cm} (1.2)$$

exactly as the brane gravitational source profile can be read from the amplitude ([6])

$$h_{\mu\nu}(p) = \langle V_{h_{\mu\nu}} \rangle_{\text{disk}} = \langle h_{\mu\nu}(p) | Dp \rangle,$$  \hspace{1cm} (1.3)$$

where $| Dp \rangle$ is the boundary state associated to the given $Dp$ brane.

The main result of this paper is the generating vertexes (3.26, 3.27, 3.30) which allow to compute all amplitudes involving two generic twisted states. An immediate consequence shown in section 4 is that all three points mixed amplitudes computed in literature have missed a momentum dependent normalization factor. This has a certain number of consequences.

- The amplitude (1.2) reads

$$A_{\mu}^{I}(p; w, \bar{w}) = i \langle T^{I} \rangle_{p} A_{\mu}^{(0)}(-p) e^{-i p \cdot x_0} e^{-\frac{1}{2} R^2(\frac{1}{x_0})} A_{\mu}^{\rho} \rho^\nu \rho^\nu \rangle$$  \hspace{1cm} (1.4)$$

where the missed factor is the last one with $R^2(\frac{1}{x_0}) = 4 \ln 2$. As it is clear from its dependence on $\alpha'$ it does not affect much the infrared properties of the instantons and therefore in the non compact limit the result found in literature such as for example ([9],[10], [11],[12]) for the non compact cases are right.

- Would it be only for the previous case it would appear that the consequences are purely of principle, even if interesting. There are nevertheless cases where the additional factor can have a certain impact. As it is clear from the previous expression

\footnote{In order to be explicit, we consider a toroidal orbifold situation, and assume that the branes in the internal space are distinguished by their relative angles or magnetizations.}

\footnote{Nevertheless similar factors have appeared in four points amplitudes, see for example ([7],[8]).}
and from eq. (4.18) the additional factor is sensitive to the transverse momenta. In the compact case this means that the amplitudes involving the creation of a couple of Kaluza-Klein scalar particles are affected: these are the amplitudes analogous to the fermionic ones computed in ([13]).

- Also the action of $D9/D5$ ([14]) acquires higher order corrections which can be useful to shed further light on the non abelian Dirac-Born-Infeld action in the case of very strong magnetic fields.

The paper is organized as follows. In section 2 we consider the quantization of open string on a generic torus with a generic constant gravitational background with a generic constant field strength in the Cartan subalgebra. We do not however discuss the vacuum of the theory in the compact case since it is not so straightforward and its use in the rest of the paper does not affect the main result in any way which cannot be argued about. In section 3 we discuss the construction of the vertexes in the old formalism where we describe the emission of dipole strings from a single “carrier” dicharged one. We start from the simplest case where the magnetic fields are only in two directions and then we derive the generating function of all vertexes (Sciuto - Della Selva - Saito vertex) for the general case. This vertex is a generalization of the old result by Corrigan and Fairlie ([15]). We obtain also the corresponding emission vertex for closed string states. Using this result it would be interesting to compute the boundary state with two twist fields inserted using the technique developed in ([16]) since boundary states have found many applications such as and not only ([17], [6], [18], [19], [20]) (for a review see for example [21]).

Finally in section 4 we use the formalism developed to compute various three points mixed amplitudes which we interpret as form factors. We justify this interpretation and we discuss the difference between amplitudes like $D25/D25'$ and $D25/D23$ as far as zero modes are concerned. Finally we compute the instanton form factor (1.4).

2. Open string in arbitrary constant background

In this section we proceed to quantize the string on a generic torus with non factorized metric and generic constant magnetic field. Nevertheless we do not care of describing the proper vacuum for the zero modes in the compact case since it is not so straightforward to build ([22]) and it is not needed for the rest of the paper. Nevertheless a proper treatment of zero modes and Chan-Paton matrices as in ([16]) for the dipole strings is necessary to solve some apparent inconsistencies ([23], [28]) and correctly explain some phenomena as the gauge rank reduction in presence of discrete $B$ ([24]).

2.1 The action

The action for the spatial compact coordinates, in the following labeled by $i, j = 1, \ldots, d$, of a bosonic string interacting with constant gravitational and 2-form backgrounds is given
by:

\[
S_{\text{bulk}} \equiv - \int d\tau \int_0^\pi d\sigma L = -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left[ G_{ij} \partial_\alpha X^i \partial_\beta X^j \eta^{\alpha\beta} - B_{ij} \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j \right]
\]

(2.1)

where the world-sheet metric is \( \eta_{\alpha\beta} = \text{diag}(-1,1) \) and \( \epsilon^{01} = 1 \). We assume \( B \) and \( G \) to be independent of \( X \). Notice that in flat noncompact space a constant background for \( B \) can be always gauged to \( B = 0 \) by in the case of closed string but not for open string while \( G \) can always be put in a diagonal form by a change of coordinates but we will not do this since we are interested in the compact case.

We consider open strings and therefore add also an open string background consisting of a gauge fields describing an arbitrary field strength \( F_{ij} \) in the Cartan subalgebra. This field strength can be diagonalized in color space. Without a big loss of generality we can assume that \( F_{ij} = F_{0ij} \mathbb{I}_N \oplus F_{\pi ij} \mathbb{I}_N \) where \( \mathbb{I}_N \) is the identity in a \( N \) dimensional color subspace and we have not assumed that \( F_{0ij} \) and \( F_{\pi ij} \) commute. On this background we have different open string sectors: two dipole string and two dicharged ones. In particular of the two dicharged open strings we consider the string with the constant commuting field strength \( F_{0ij} \) on the \( \sigma = 0 \) boundary and \( F_{\pi ij} \) on the \( \sigma = \pi \) ones \(^4\). The action of an open string in a closed toroidal string background interacting with those gauge fields is then given by

\[
S = S_{\text{bulk}} + S_{\text{boundary}}
\]

(2.2)

where \( S_{\text{bulk}} \) is given in Eq. (2.1) and \( S_{\text{boundary}} \) is equal to:

\[
S_{\text{boundary}} = -q_0 \int d\tau A_{0i} \partial_\tau X^i|_{\sigma=0} + q_\pi \int d\tau A_{\pi i} \partial_\tau X^i|_{\sigma=\pi}
\]

(2.3)

where \( q_0 \) and \( q_\pi \) are the charges located at the two end-points.

2.2 Boundary conditions

An open string in the previous metric background given by \( E_{ij} = G_{ij} + B_{ij} \) and in presence of a constant background field \( \hat{F}_{A ij} = 2\pi \alpha' q_A F_{A ij} \) \((A = 0, \pi)\) has boundary conditions

\[
G_{ij} X^{it} + (B - \hat{F}_0)_{ij} \dot{X}_j|_{\sigma=0} = 0, \quad G_{ij} X^{it} + (B - \hat{F}_\pi)_{ij} \dot{X}_j|_{\sigma=\pi} = 0
\]

(2.4)

which can be written either as

\[
\mathcal{E}_{0ij} \partial_+ X^j - \mathcal{E}_{0ij} \partial_- X^j|_{\sigma=0} = 0, \quad \mathcal{E}_{\pi ij} \partial_+ X^j - \mathcal{E}_{\pi ij} \partial_- X^j|_{\sigma=\pi} = 0
\]

(2.5)

or as

\[
\partial_- X^i - R_{0ij}^i \partial_+ X^j|_{\sigma=0} = 0, \quad \partial_- X^i|_{\sigma=\pi} - R_{\pi ij}^i \partial_+ X^j = 0
\]

(2.6)

\(^4\)In compact directions it is also sometimes necessary to add noncommuting Wilson lines, i.e., which do not belong to the Cartan subalgebra. These are included in the transition functions.
when we introduce
\[ \mathcal{E}_{0,\pi} = E^T + \tilde{F}_{0,\pi} = G + (\tilde{F}_{0,\pi} - B) = G + \mathcal{F}_{0,\pi} \tag{2.7} \]
and the orthogonal matrices
\[ R_0 = \mathcal{E}_0^{-1} \mathcal{E}_0^T, \quad R_\pi = \mathcal{E}_\pi^{-1} \mathcal{E}_\pi^T, \tag{2.8} \]
which satisfy the conditions
\[ R_{0,\pi} G^{-1} R_{0,\pi}^T = G^{-1}, \quad R_{0,\pi}^T G R_{0,\pi} = G. \tag{2.9} \]

2.3 String quantization for a generic magnetic constant background on $T^n$.

Given the boundary conditions (2.6) it is immediate to write a solution of the equation of motion which respects the $\sigma = 0$ ones as
\[ X^i(\sigma, \tau) = x^i_0 + \frac{1}{2} \left( X_{Lnzm}^i (\tau + \sigma) + (R_0)_j^i X_{Rnz}^j (\tau - \sigma) \right) \tag{2.10} \]
The $\sigma = \pi$ boundary condition implies\(^5\)
\[ X_{Lnmz}^i (\tau + 2\pi) = R^i_j X_{Lnmz}^j (\tau) \tag{2.11} \]
with\(^\text{\footnotesize Notice that in } X_{Lnzm} \text{ expansion there is also the momentum and that we consider } x^i_0 \text{ only as zero modes.}\)
\[ R = R_\pi^{-1} R_0 \Rightarrow G^{-1} R^T = G^{-1}, \quad R^T G R = G \tag{2.12} \]
To continue it is convenient to introduce the $R$ eigenvectors $v_a$ as
\[ R^i_j v_a^j = e^{-i2\pi \nu a} v_a^i, \quad a = \pm 1, \cdots \pm n. \tag{2.13} \]
where all eigenvectors (but one in odd dimensions) come in couples and are labeled as\(^6\)
\[ v_{-a} = v^a = v_a^\dagger, \quad \nu_{-a} = -\nu_a \tag{2.14} \]
with $a > 0 \to \nu_a \geq 0$\(^7\) and are normalized as\(^\text{\footnotesize If the dimension of the space is odd we take } \nu_{a=0} = 0 \text{ and } v_{a=0} \text{ to be real.}\)
\[ v_a^\dagger G v_b = v_{-a}^\dagger G v_b = \delta_{a,b}. \tag{2.15} \]
For future convenience we write also the spectral decomposition of the $R$ and $G^{-1}$ matrices as
\[ (G^{-1})^{ij} = \sum_a v_a^i v_a^j, \quad (RG^{-1})^{ij} = \sum_a v_a^i e^{-i2\pi \nu a} v_{-a}^j \tag{2.16} \]

\(^5\)Notice that in $X_{Lnzm}$ expansion there is also the momentum and that we consider $x^i_0$ only as zero modes.

\(^6\)If the dimension of the space is odd we take $\nu_{a=0} = 0$ and $v_{a=0}$ to be real.

\(^7\)We use the convention that while indexes $a, b$ label any eigenvalues $c, d$ label those different from zero, i.e. $\nu_c, \nu_d \neq 0$ while $e, f, g$ label those equal to zero, i.e. $\nu_e, \nu_f, \nu_g = 0$
It is then possible to write the expansions for $X_{Lzm}$ and $X_{Rzm}$ as

\[
X_{Lzm}^i(\xi) = \sum_a v^i_a \alpha_n^a e^{-i(n+\nu_a)\xi} \\
X_{Rzm}^i(\xi) = \sum_a (R_0 v^i_a) \beta^a_n e^{-i(n+\nu_a)\xi}
\]

(2.17)

with the understanding that when $n + \nu_a = 0$ then $e^{\pm i(n+\nu_a)\xi} = \xi$. We can therefore write the complete expansion of the string field in a way similar to the usual one as

\[
X^i(\sigma, \tau) = x^i_0 + i\sqrt{2}\alpha' \sum_{a,n/n+\nu_a \neq 0} \frac{\alpha_n^a}{n + \nu_a} \Psi^i_{n,a}(\sigma, \tau) + \sqrt{2}\alpha' \sum_{e/\nu_e = 0} \alpha_0^e \Psi^i_{0,e}(\sigma, \tau)
\]

(2.18)

where the we have defined the eigenfunctions

\[
\Psi^i_{n,a}(\sigma, \tau) = \frac{1}{2} e^{-i(n+\nu_a)\tau} \left( v^i_a e^{-i(n+\nu_a)\sigma} + (R_0 v^i_a) e^{i(n+\nu_a)\sigma} \right), \quad n + \nu_a \neq 0
\]

\[
\Psi^i_{0,e}(\sigma, \tau) = \frac{1}{2} \left( v^i_e (\tau + \sigma) + (R_0 v^i_e) (\tau - \sigma) \right), \quad \nu_e = 0
\]

(2.19)

2.3.1 Completeness of the $\Psi_{n,a}$

We want now show that $\Psi^i_{n,a}(\sigma, \tau)$ form a complete basis for all functions with the same boundary conditions. To this purpose we define the product

\[
\langle X^i(\sigma, \tau), Y^j(\sigma, \tau) \rangle = \int_0^\pi d\sigma X^{i*}(\sigma, \tau) \left[ G \left( \partial_{\tau} - \partial_{\tau} \right) + \mathcal{F}_0 \delta(\sigma) - \mathcal{F}_\pi \delta(\sigma - \pi) \right] Y^j(\sigma, \tau)
\]

(2.20)

It is worth noticing that this product shall also be used for the subspaces where $\mathcal{F}_0 = \mathcal{F}_\pi$ and are not factorisable because of the metric.

Let us now consider a generic field

\[
\Phi^i(\sigma, \tau) = \phi^i + \sum_{a,\alpha} \phi^a_n \Psi^i_{n,a}(\sigma, \tau)
\]

(2.21)

and try to extract the coefficients $\phi^i$. We get immediately that

\[
\langle \Psi_{f,a}, \Phi \rangle = -i\pi(n + \nu_a) \phi^a_f \quad n + \nu_a \neq 0
\]

\[
\langle \mathcal{E}^{-1}_{\sigma} G v_e, \Phi \rangle = \pi \phi^e_0 \quad \nu_e = 0
\]

(2.22)

from which we can easily extract the coefficients $\phi^a_n$. The analogous expressions for $\phi^i$ give

\[
\langle \Psi_{f,e}, \Phi \rangle = -\pi v^e_f \left( \mathcal{E}_\pi \phi + \mathcal{F}_\pi v_f \phi^0_f \right) \quad \nu_e = 0
\]

\[
\langle \mathcal{E}^{-1}_{\sigma} G v_c, \Phi \rangle = -v^c_f \mathcal{E}^T \Delta \mathcal{F} \phi
\]

(2.23)

---

8For the zero modes we could also use a more symmetric formulation with respect to exchanges $\sigma = 0 \leftrightarrow \sigma = \pi$ and $F_0 \leftrightarrow -F_\pi$ using $\Psi^0_{0,a}(\sigma, \tau) = \frac{1}{2} \left( v^0_e (\tau + \sigma) + (R_0 v^0_e) (\tau - \sigma) \right)$ but the computation are less straightforward and the result as complex.

9We take $\int_0 d\sigma \delta(\sigma) = 1$ since taking $\int_0 d\sigma \delta(\sigma) = \frac{1}{2}$ as in [25] would not reproduce the dipole case. Similarly for $\sigma = \pi$. 
with $\Delta F = F_\pi - F_0$. From the previous equations it is possible in principle to recover the $\phi^i$ coefficients since the equations in the last two lines are as many as the components of $\phi^i$ and are independent and are therefore sufficient but the result is rather not instructive. In order to compute the commutation relation we resort therefore to the trick of computing the commutation relations among the previous quantities (2.22, 2.23) when we take $\Phi = X$ and then extract those among the $x^i_0$.

### 2.3.2 Commutation relations

Using the canonical commutation relation

$$[X^i(\sigma, \tau), P_j(\sigma', \tau)] = i\delta^i_j \delta(\sigma - \sigma')$$  \hfill (2.24)

we can deduce the commutation relations (for details see appendix B)

$$[\alpha^a_n, \alpha^b_m] = \delta_{a+b,0} \delta_{m+n,0} (n + \nu_a) \quad n + \nu_a, m + \nu_n \neq 0$$

$$[\alpha^a_0, \alpha^a_n] = 0 \quad n + \nu_a \neq 0$$

$$[x^i, \alpha^a_n] = 0$$

$$[\alpha^a_0, x^i_0] = -i \sqrt{2\alpha} (E^{-1}_\pi G_{-e})^i$$

$$[x^i_0, x^j_0] = +i 2\pi\alpha' \sum_{c/fc \neq 0} w^i_c \frac{1}{f_c} (w^j_c)^j$$

$$- i 2\pi\alpha' \sum_{c/fc \neq 0, \nu_f = 0} \left[ w^i_c \frac{1}{f_c} (w^j_c E^{-T}_\pi v_f)(v_f^j G E^{-T}_\pi) + (E^{-1}_\pi G_{vf})^i(v_f^j E_\pi w_c) \frac{1}{f_c} (w^j_c)^j \right]$$

$$+ i 2\pi\alpha' \sum_{f,g/\nu_f = \nu_g = 0} (E^{-1}_\pi G_{vf})^i \left[ v_f^j F_\pi v_g + \sum_{c/fc \neq 0} (v_f^j E_\pi w_c) \frac{1}{f_c} (w^j_c E^{-T}_\pi v_g) \right] (v_g^j G E^{-T}_\pi)^j$$

(2.25)

The last commutation relations can also be rewritten in a more compact form by using the spectral decomposition of $G^{-1}$ as

$$[x^i_0, x^j_0] = +i 2\pi\alpha' \sum_{c,d/\nu_c = \nu_d \neq 0} (E^{-1}_\pi G_{vc})^i \left[ \sum_{c_1/fc_1 \neq 0} (v_c^i E_\pi w_{c_1}) \frac{1}{f_{c_1}} (w_{c_1}^j E^{-T}_\pi v_d) \right] (v_d^j G E^{-T}_\pi)^j$$

$$+ i 2\pi\alpha' \sum_{f,g/\nu_f = \nu_g = 0} (E^{-1}_\pi G_{vf})^i v_f^j F_\pi v_g (v_g^j G E^{-T}_\pi)^j$$

(2.26)

If we introduce the decomposition\(^{10}\)

$$x^i_0 = \sum_a (E^{-1}_\pi G v_a)^i x^a_0 \quad \Leftrightarrow \quad x^a_0 = v^i_a E_\pi x^i_0$$

(2.27)

\(^{10}\)Notice however that the same decomposition does not simplify the non zero modes commutation relations as better detailed in footnote 12.
then all the previous non vanishing commutation relations involving $x_0$ boil down to
\[
[x^i_0, a^{-f}_0] = i\sqrt{2\alpha'}\delta_{e,f}
\]
\[
[x^e, x^{-d}_0] = i2\pi\alpha' v^f_i F_{\pi} v_f
\]
\[
[x^e, x^{-d}_0] = i2\pi\alpha' \sum_{c_1 \neq 0} (v^1_i E_{\pi} w_{c_1}) \frac{1}{f_{c_1}} (w^1_{c_1} E_{\pi}^T v_d)
\] (2.28)

In the previous commutation relations and in (2.25, 2.26) new vectors $\{w_c\}$ appear. They are related the eigenvectors of $\Delta F$ and are an expression of the fact that $x_0$s feel mostly $\Delta F$ while the other non zero modes feel $\Delta R = R_\pi - R_0$. We define therefore the $\Delta F$ eigenvectors
\[
(\Delta F)_{ij} w^j_a = f_a G_{ij} w^j_a
\] (2.29)
as for $R$, all eigenvectors (but one in odd dimensions) come in couples and are labeled as
\[
w_{-a} = w^*_a, \quad f_{-a} = f^*_a = -f_a
\] (2.30)
and normalized as
\[
w^*_a G w_b = w^T_{-a} G w_b = \delta_{a,b}
\] (2.31)
It is therefore possible to write the spectral decomposition of the $\Delta F$ and $G^{-1}$ matrices as
\[
(G^{-1})^{ij} = \sum_a w^*_a w^j_{-a}, \quad \Delta F_{ij} = \sum_a (Gw_a)_i f_a (Gw_{-a})_j
\] (2.32)

It is then possible to check that $E^{-1}_0 G \ker(\Delta R) = E^{-1}_\pi G \ker(\Delta R) = \ker(\Delta F)$. This happens since
\[
R_\pi - R_0 = 2E^{-1}_0 (F_\pi - F_0) E^{-1}_\pi G = 2E^{-1}_\pi (F_\pi - F_0) E^{-1}_0 G
\] (2.33)
therefore for any $v_f$ such that $\nu_f = 0$, i.e. $R_0 v_f = R_\pi v_f$, the vector $E^{-1}_\pi G v_f$ belongs to $\ker(\Delta F)$ and similarly for $G^{-1}E_{\pi} w_f$ with $f_f = 0$. We can therefore use the $E^{-1}_\pi G v_f$ as basis for $\ker(\Delta F)$ even if they are not orthogonal. This explains why they appear in the $x_0$ commutation relations. Notice finally that generically it is not true that $E^{-1}_0 G [\ker(\Delta R)]^\perp = [\ker(\Delta F)]^\perp$ even if they have the same dimension.

Finally we can check what happens of the $x_0$ commutation relations in special cases. When $\ker(\Delta F) = \emptyset$, i.e. there are no subspaces where $F_0 = F_\pi$ the previous commutation relation for the $x_0$ reduces to
\[
[x^i_0, x^j_0] = i2\pi\alpha' [(\tilde{F}_\pi - \tilde{F}_0)^{-1}]^{ij}
\] (2.34)
while in the case where $[\ker(\Delta F)]^\perp = \emptyset$, i.e. $F_0 = F_\pi$ it reduces to
\[
[x^i_0, x^j_0] = i2\pi\alpha' [E^{-1}_\pi F_\pi E^{-T}_\pi]^{ij} = i2\pi\alpha' G^{ij}
\] (2.35)
upon the use of the spectral decomposition of $G^{-1}$ given in eq. (2.16). Under the same assumption comparing with the usual dipole string expansion we see that $\sqrt{2\alpha'} p^i = (E^{-1}_\pi G v_c)^i a^0_0$ so that the commutation $[a^0_0, x^i]$ becomes $[p^i, x^i] = -iG^{ji}_0$ as expected.
2.3.3 The vacuum and the spectrum

The vacuum of the non zero modes (here we consider only $x^i_0$ as zero modes) can be defined as

$$\alpha^a_n|0_e, 0_c, 0_\alpha\rangle = \alpha^0_n|0_e, 0_c, 0_\alpha\rangle = 0 \quad n + \nu_a > 0$$

(2.36)

Instead for the $x^i_0$ we must find a maximum commuting set among themselves and also with $\alpha^0_n$. The $x^i_0$ commuting with $\alpha^0_n$ are the combinations $x^i_0 = v^i_c \mathcal{E} x_0$ which nevertheless do not commute among themselves. We can form the proper number of commuting linear combinations of $x^i_0$, let us call them $\hat{\pi}^i_0$ ($\hat{c} = 1, \ldots, \frac{1}{2}\dim [\ker(\Delta \mathcal{F})]^\perp$) by diagonalizing at $2 \times 2$ blocks the previous antisymmetric matrix. At the same time we get the corresponding “coordinates” $\hat{\xi}^c_0$ with the usual algebra $[\hat{\xi}^c_0, \hat{\pi}^d_0] = i\delta^c_d$. Finally we can define the vacuum for the zero modes by requiring that

$$\hat{\pi}^\hat{c}_0|0_e, 0_c, 0_\nu\rangle = 0 \quad \hat{c} = 1, \ldots, \frac{1}{2}\dim [\ker(\Delta \mathcal{F})]^\perp$$

(2.37)

This definition of the $x^i_0$ vacuum is the right one only for the non compact case since on the torus $x^i_0$ are not anymore good operators and the vacuum definition is more complex ([22]). The spectrum is then given by

$$\prod_{a} \prod_{n=1}^{\infty} (\alpha_n^a)^{N_{a\nu}}|k^e_c, \kappa^c_0, 0_\alpha\rangle$$

(2.38)

with normalization

$$\langle k^e_c, \kappa^e_0, 0_\alpha|k^e_c, \kappa^c_0, 0_\alpha\rangle = (2\pi)^d \delta^{\dim \ker(\Delta \mathcal{F})} (k^e_c - k^e_c) \delta^{\dim (\ker(\Delta \mathcal{F})^\perp)} (\kappa^c_0 - \kappa^c_0)$$

(2.39)

It is worth stressing that also in half the “twisted” directions there is a “momentum” $\kappa^c_0$ since in the non compact case there is still a translational invariance roughly because it is possible to choose a gauge where $\mathcal{F}_0 - \mathcal{F}_\pi$ is derived from a potential depending on only half the number of coordinates. While this choice is still possible in the compact case the transition functions associated to the gauge bundle break this remaining translational invariance and we are left with a finite Landau levels degeneracy. Notice however that this $\kappa^c_0$ is not a true momentum since it does not appear in the Hamiltonian and it is actually a label for the degeneration. Another way of saying this is that the choice of using $\hat{\pi}^\hat{c}_0$ as annihilators is completely arbitrary and we could and well have chosen a mixture of $\hat{\pi}^\hat{c}_0$ and $\hat{\xi}^\hat{c}_0$ or any other possible combination of commuting operators.

2.3.4 Energy momentum tensor

We can now compute the energy momentum tensor to be

$$T = -\frac{1}{\alpha^\prime} \partial_+ X^T G \partial_+ X = -\frac{1}{4\alpha^\prime} \partial_+ X^T L G \partial_+ X_L = \sum_n L_n e^{-i(n+\sigma)}$$

$$L_n = \sum_a \frac{1}{2} \left( \sum_k : \alpha_n^a \alpha_k^{-a} : + \delta_{n,0} \nu_{a} (1 - \nu_{a}) \right)$$

(2.40)

\[\text{Remember that the extra term is due to the presence of a subleading singularity in the double derivative of the propagators.}\]
It is interesting to notice that the shift can be understood as due to representing the dipole energy momentum tensor in the dicharged string as explained in section 3.3.

2.3.5 OPEs

The basic OPE for the non zero modes is given by \((z = e^{\tau E + i\sigma})\)

\[
X_{L_{nm}}^{i(+)}(z) X_{L_{nm}}^{j(-)}(w) = -2\alpha' \sum_a v^i_a \hat{g}_{-\nu_a} \left(\frac{w}{z}\right) v^j_a \quad |z| > |w|
\] (2.41)

where we have defined the “propagator” as

\[
\hat{g}_{\nu_a}(z) = -\sum_{n-\nu_a>0} \frac{1}{n-\nu_a} z^{n-\nu_a} \quad |z| < 1, \quad -\pi < \text{arg}(z) < \pi
\] (2.42)

From the previous basic OPE we can derive all the others such as for example\(^{12}\)

\[
X_{L_{nm}}^{i(+)}(z) X_{R_{nm}}^{j(-)}(\bar{w}) = -2\alpha' \sum_a v^i_a \hat{g}_{-\nu_a} \left(\frac{\bar{w}}{z}\right) (R_0 v_{-a})^j
\] (2.43)

2.3.6 Ω operator

It is also possible to define the Ω operator which exchanges the string boundaries and therefore maps a string \(X\) with boundary field strength \((F_0, F_\pi)\) into a (generically different) string \(\tilde{X}\) with boundary field strength \((\tilde{F}_0 = -F_\pi, \tilde{F}_\pi = -F_0)\). The minus sign is due to the orientation of the boundaries. Starting from the expected relation

\[
Ω X(\sigma, \tau) Ω^{-1} = \tilde{X}(\pi - \sigma, \tau)
\]

and the consequences of the relation between \(\tilde{F}\) and \(F\)

\[
\tilde{R}_0 = R^{-1}_\pi, \quad \tilde{R}_\pi = R^{-1}_0
\]

\[
\bar{v}_a = R_0 v_a e^{i\pi \nu_a}
\]

(no sum over \(a\) in the last equation) we find that the Ω action on the operators is given by

\[
Ω \alpha_n^a Ω^{-1} = (-)^n \tilde{\alpha}_n^a
\]

\[
Ω x_0 Ω^{-1} = x_0 + \sqrt{2\alpha'\pi} \frac{1 - \tilde{R}_0}{2} \bar{v}_f \tilde{\alpha}_0^f
\]

(2.46)

3. Vertexes and the SDS vertex

In order to explain the derivation in a simple setting we start considering the tachyonic vertexes on \(\mathbb{R}^2\) with coordinates \((x^1, x^2)\). We consider only two magnetized branes with different magnetization, i.e. \(f_0 = (\tilde{F}_0 - B)_{12}\) and \(f_\pi = (\tilde{F}_\pi - B)_{12}\) so the gauge group is \(U(2)\) and the dicharged string describes the matter in the bifundamental.

\(^{12}\) These OPEs and the zero modes commutation relations show why it is difficult to define in a unique way well adapted coordinates: in fact the operators with the simplest relations are \(x_0^a = v_0^a \xi x_0\), \(X_{L_{nm}} a(z) = v_0^a G X_{L_{nm}}(z)\) and \(X_{R_{nm}} a(z) = v_0^a G^T \xi_0 X_{R_{nm}}(z)\).
In this formalism ([26]) we have a carrier string which is the dicharged string coupled to the two different magnetic fields. From this dicharged string we then describe the emission of the dipole strings from the two boundaries. Exactly as in the old superstring formalism where it was natural to describe the emission of NS states from a R string. Differently from the emission of NS states from a R string we have here two non equivalent boundaries and therefore we can emit states of two different dipole strings.

To give the complete picture is also important to describe the coupling of the dicharged string with the closed string: it turns out that it is exactly this coupling which constrains how the splitting in left and right zero modes happens.

We start writing the tachyonic vertexes explicitly, then we give the arguments that lead to their expressions and we construct the generator of all vertexes, i.e. the Scuto-Della Selva-Saito vertex. Finally we discuss all the consistency requests we expect these vertexes must satisfy.

3.1 The tachyonic vertexes for $U(1) \times U(1)$ on $\mathbb{R}^2$

In the following we work with flat complex coordinates but we do not denote this explicitly while in appendix D we are more pedantic and denote the flat vectors explicitly by an underline: for examples the magnetic field in flat coordinates \( f_0 = (F_0 - B)_{12}/\sqrt{\text{det} G} \) will be written as \( f_0 \). The part of the tachyonic vertexes associated with the two twisted directions for the emission of a tachyonic dipole string excitation from the \( \sigma = 0, \sigma = \pi \) boundary and for the closed string tachyon read

\[
\begin{align*}
V_{T_0}(x, k) &\propto e^{-\frac{1}{2} R^2(\nu) \Delta_0(k) x - \Delta_0(k) \cdot e^{i(kZ(x,x) + kZ(x,x))}}; \quad (3.1) \\
V_{T_0}(y, k) &\propto e^{-\frac{1}{2} R^2(\nu) \Delta_\pi(k) y - \Delta_\pi(k) \cdot e^{i(kZ(y,y) + kZ(y,y))}}; \quad (3.2) \\
W_{T_0}(z, \bar{z}, k_L, k_R) &\propto e^{-\frac{1}{2} R^2(\nu) \Delta_\pi(k_L) z - \Delta_\pi(k_L) \cdot e^{i(k_LZ_L(z) + k_LZ_L(z))}}; \quad (3.3)
\end{align*}
\]

where we have introduced the (partial) conformal dimensions

\[
\Delta_0(k) = 2\alpha' \cos^2 \gamma_0 \bar{k} k, \quad \Delta_\pi(k) = 2\alpha' \cos^2 \gamma_\pi \kbar k, \quad \Delta_\nu(k) = 2\alpha' \kbar k \quad (3.4)
\]

with the complex momentum given by \( k = (k_1 + ik_2)/\sqrt{2} \) and with \( e^{i\gamma_0,\pi} = (1 + i\gamma_0,\pi)/\sqrt{1 + \gamma_0^2} \).

We have also defined the positive constant (for \( 0 < \nu < 1 \) the positiveness follows from the \( \psi \) representation given in eq.(C.11))

\[
\frac{1}{2} R^2(\nu) = -\frac{1}{2} \psi(1 - \nu) + \psi(\nu) - 2\psi(1) \quad (3.5)
\]

where \( \nu = |\gamma_0 - \gamma_\pi|/\pi \) and \( \psi(z) = \frac{d \ln \Gamma(z)}{dz} \) is the digamma function. The normal ordering is then better defined splitting zero and non zero modes and it amounts to

\[
\begin{align*}
: e^{i(kZ(x,x) + k\bar{Z}(x,x))} : &= : e^{i\cos \gamma_0 (k e^{i\gamma_0} L_{\zeta,\zeta} (x) + k e^{i\gamma_0} \bar{Z}(x))} : e^{i(\bar{k}_0 + k\bar{z}_0)} \\
: e^{i(kZ(y,y) + k\bar{Z}(y,y))} : &= : e^{i\cos \gamma_\pi (k e^{i\gamma_\pi} L_{\zeta,\zeta} (y) + k e^{i\gamma_\pi} \bar{Z}(y))} : e^{i(\bar{k}_0 + k\bar{z}_0)} \quad (3.6)
\end{align*}
\]
for open string vertexes and to
\[
\begin{align*}
:e^{i(\bar{z}_L + k L \bar{z}_L)} : &= e^{i(\bar{z}_L + k L \bar{z}_L)} \\
:e^{i(\bar{z}_R + k R \bar{z}_L)} : &= e^{i(\bar{z}_R + k R \bar{z}_L)}
\end{align*}
\] (3.7)
for the closed string one. In the previous definition of the normal ordering for the closed
string vertex we introduced the left and right zero modes \(z_{L0}, z_{R0}\). The computation
given in the next subsection is independent on this splitting since in the \(Z, \bar{Z}\) expansion
there are no momentum operators which could be sensitive to this splitting as it happens
also whenever \(k \epsilon \Delta F = 0\). The splitting can nevertheless determined by requiring that
two closed string vertexes (3.3) have the same OPE as the usual ones which is one of
the consistency requirements we write down in section 3.4 and we verify in appendix F.
Therefore here we can simply state the result
\[
z_{L0} = (1 + i\tilde{f}_0)z_0, \quad z_{R0} = (1 - i\tilde{f}_0)z_0.
\] (3.8)

### 3.1.1 Heuristic derivation of the vertexes.

To derive heuristically the previous vertexes we use the same approach used in ([27]). The
idea is to define the vertex for the emission of a dipole state from the dicharged string
starting from a regularized version of the naive vertex, then eliminate the divergence by
dividing it by the normalization required by the emission vertex of the same state from the
corresponding dipole string and finally let the cutoff used to regularize the vertex to zero.

We start by writing the regularized version of the naive emission vertex for the dipole
Tachyon from the \(\sigma = 0\) boundary of the dicharged string. The dipole tachyon is emitted
from the dicharged string and the not normal-ordered but point splitted emission operator
can be written as
\[
[V_{T_0}(x, k)]_{p.s.,} = \exp \left\{ i \gamma_0 \left[ ke^{i\gamma_0}(Z_{L n zm}(-)(x,e^{-\eta}) + Z_{L n zm}(+)(x,x)) + ke^{i\gamma_0}(\bar{Z}_{L n zm}(-)(x,e^{-\eta}) + \bar{Z}_{L n zm}(+)(x,x)) \right] \right\} e^{i(kz_0 + k\bar{z}_0)}
\] (3.9)
Using the boundary conditions this expression can be rewritten as
\[
\begin{align*}
\exp \left\{ i \gamma_0 \left[ ke^{i\gamma_0}(Z_{L n zm}(-)(x,e^{-\eta}) + Z_{L n zm}(+)(x,x)) + ke^{i\gamma_0}(\bar{Z}_{L n zm}(-)(x,e^{-\eta}) + \bar{Z}_{L n zm}(+)(x,x)) \right] \right\} e^{i(kz_0 + k\bar{z}_0)}
\end{align*}
\] (3.10)
where \(\eta > 0\) and, as usual, \(Z_{L n zm}(-)\) is the part of the field \(Z_{L n zm}\) containing the creator
operators with the exclusion of the zero modes \(z_0, \bar{z}_0\). It is then immediate to rewrite the
previous expression as a normal-ordered one
\[
[V_{T_0}(x, k)]_{p.s.,} = \exp \left\{ \cos^2 \gamma_0 \alpha \ k \bar{k} \left( \hat{g}_+(e^{-\eta}) + \hat{g}_-(e^{-\eta}) \right) \right\} \exp \left\{ i \gamma_0 \left[ ke^{i\gamma_0}(Z_{L n zm}(-)(x,e^{-\eta}) + Z_{L n zm}(+)(x,x)) + ke^{i\gamma_0}(\bar{Z}_{L n zm}(-)(x,e^{-\eta}) + \bar{Z}_{L n zm}(+)(x,x)) \right] \right\} e^{i(kz_0 + k\bar{z}_0)}
\] (3.11)
where we have used the “propagator” \(\hat{g}_+(z)\) defined in eq. (2.42).
We want now to eliminate the divergence which appears as $\eta \to 0^+$ and we want to do this by comparing with the analogous procedure we can follow for the usual emission vertex from the dipole “carrier” string in order to subtract as little as possible. Let us therefore perform the same operations as above on the usual vertex operator which describes the emission of the same dipole tachyon from the corresponding dipole string. This dipole string is described by $Z^{(0)}$ and $\bar{Z}^{(0)}$ and has the tachyon itself we are emitting among the excitations. If we start from the analogous expression of (3.10) where we use $Z^{(0)}$ and $\bar{Z}^{(0)}$ in place of $Z$ and $\bar{Z}$ we find

$$[V_{T_{0}}^{(0)}(x,k)]_{\text{p.s.}} = \exp \left\{ \cos^2 \gamma_0 \; \alpha' \; k \bar{k} \left( 2\tilde{g}_0(e^{-\eta}) + 2 \ln x \right) \right\} \exp \left\{ i \cos \gamma_0 \left[ k e^{i\gamma_0} (Z_{L_{nzm}}(0,-)(xe^{-\eta}) + Z_{L_{nzm}}(0,+)(x)) \right. \right.$$ 

$$\left. + k e^{-i\gamma_0} (\bar{Z}_{L_{nzm}}(0,-)(xe^{-\eta}) + \bar{Z}_{L_{nzm}}(0,+)(x)) \right] \right\} : e^{i(kz_{0}^{(0)} + k\bar{z}_{0}^{(0)})} e^{2\alpha'(k\rho^{(0)} - k\bar{\rho}^{(0)})} \ln x.$$  

(3.12)

Dividing this point splitted expression for the non-operatorial factor in the first line

$$N_0(\eta) = \exp \left\{ \cos^2 \gamma_0 \; \alpha' \; k \bar{k} \left( 2\tilde{g}_0(e^{-\eta}) + 2 \ln x \right) \right\}$$  

(3.13)

we get a regularized vertex and letting $\eta \to 0^+$ we recover the usual vertex operator for the emission from the dipole string

$$V_{T_{0}}^{(0)}(x,k) = \lim_{\eta \to 0^+} N_0^{-1}(\eta) [V_{T_{0}}^{(0)}(x,k)]_{\text{p.s.}}.$$  

(3.14)

It is worth stressing that the zero modes $z_{0}^{(0)}$ and $\bar{z}_{0}^{(0)}$ do not give any contribution even if they do commute between themselves because they are not split into different exponentials and that the factor $\cos^2 \gamma_0$ comes from rewriting the vertex using the left moving part only.

If we divide the vertex operator (3.11) for the same factor written in the first line of eq. (3.12) we can still take the $\eta \to 0^+$ since the two dimensional UV divergences are the same and the result is exactly the vertex given in eq. (3.1)

$$V_{T_{0}}(x,k) = \lim_{\eta \to 0^+} N_0^{-1}(\eta) [V_{T_{0}}^{(0)}(x,k)]_{\text{p.s.}}.$$  

(3.15)

For the emission vertex from the $\sigma = \pi$ boundary the same procedure works: the regularized normal-ordered vertex written in term of the untwisted fields $Z^{(\pi)}(y,y)$ and $\bar{Z}^{(\pi)}(y,y)$ must be divided by the product of the non-operatorial factor $e^{\cos^2 \gamma_0 \; \alpha' \; k \bar{k}(2\tilde{g}_0(e^{-\eta})+2\ln |y|)}$ in order to give the usual emission vertex. Another way of getting this result is to use the twist operator $\Omega$.

For the closed string we proceed again as before and we compare with the usual emission vertex expressed through the closed string fields $Z^{(c)}$ and $\bar{Z}^{(c)}$ where the zero modes are treated as they were independent. The normalization factor is easily computed to be

$$\exp \left\{ \alpha' \; k_L \bar{k}_L \left( 2\tilde{g}_0(e^{-\eta}) + 2 \ln z \right) \right\} \exp \left\{ \alpha' \; k_R \bar{k}_R \left( 2\tilde{g}_0(e^{-\eta}) + 2 \ln \bar{z} \right) \right\}$$  

(3.16)

We are however left with the problem of how to split the zero modes $z_0$ and $\bar{z}_0$ into left and right zero modes $z_{0L}$, $z_{0R}$. As stated before and verified in appendix F the splitting can determined by requiring that two closed string vertexes (3.3) have the same OPE as the usual ones.
3.2 The SDS vertex and the generic vertex.

We want now to reproduce the steps done in the previous section for all vertexes at once: this can be done using a generator functional of the vertexes.

Following the spirit of the Sciuto-Della Selva-Saito approach we define a generator functional for the vertexes describing the emission of dipole states from the $\sigma = 0$ boundary of the dicharged string. The idea can be illustrated by an example. Consider the vertex which describes the fluctuations of the gauge vector around the dipole string background expressed in the dipole string fields $X_{(0)}$, we can derive it from a generating functional for the dipole string as

$$\tilde{V}_0(x; \epsilon, k) = \epsilon_i \partial_x X_{(0)}^i(x, x)e^{ik_j X_{(0)}^j(x, x)} = S_0(0) \left| \frac{\epsilon_i \partial_x}{\partial c_1} e^{ik_j \frac{\epsilon_j}{\sigma_0}} \right|_{c=0}$$  \hspace{1cm} (3.17)

where we have introduced the generating vertex

$$S_0(0,c,x) = e^{\sum_{k=0}^{\infty} c_k(x) \partial^k_{\epsilon}(X_{(0)})(x, x)} \hspace{1cm} (3.18)$$

and the derivative $\partial_x$ acts on both the left moving and right moving part. Then it is natural to assume that we can derive the emission vertex for the same state from the dicharged string as

$$V(x; \epsilon, k) = S_0(c, x) \left| \frac{\epsilon_i \partial_x}{\partial c_1} e^{ik_j \frac{\epsilon_j}{\sigma_0}} \right|_{c=0}$$  \hspace{1cm} (3.19)

where $S_0(c,x)$ can be derived by regularizing the analogous for the dicharged string of the generating vertex $S_0(0,c,x)$ as done in the previous section for the tachyonic vertex. We write therefore

$$[S_0(c, x)]_{p.s.} = e^{\sum_{k=0}^{\infty} c_k \partial^k_{x}(X_{(0)})(x, x)}$$

$$= e^{\sum_{k=0}^{\infty} c_k \partial^k_{x}(X_{L_{nm}})(x, x)} + e^{\sum_{k=0}^{\infty} c_k \partial^k_{x}(X_{L_{nm}})(x, x)}$$

$$= e^{\sum_{k=0}^{\infty} c_k \partial^k_{x}(X_{L_{nm}})(x, x)} + e^{\sum_{k=0}^{\infty} c_k \partial^k_{x}(X_{L_{nm}})(x, x)} + \epsilon_0 x_0$$  \hspace{1cm} (3.20)

where to write the last line we used the boundary conditions which allow to write

$$X(x, x) = x_0 + \frac{1}{2} X_{L_{nm}}(x) = x_0 + \epsilon_0^{-1} G X_{L_{nm}}(x)$$  \hspace{1cm} (3.21)

with $x \in \mathbb{R}^+$. We can then deduce the final expression by first normal ordering and then dividing by the regularization factor the previous expression

$$S_0(c, x) = \lim_{\eta \to 0^+} N_0(c, \eta) [S_0(c, x)]_{p.s.}$$

$$= \exp \left\{ - \alpha' \sum_{k,l=0}^{\infty} c_{k} \epsilon \partial^{k}_{u} \epsilon^{-1} G v_{c} v_{c}^{\dagger} G \epsilon^{-T} c_{l} \partial^{l}_{v} \mid_{u=x} \partial^{T}_{v} \mid_{v=x} \right\}$$

$$\exp \left\{ + \alpha' \sum_{l=0}^{\infty} c_{l} \epsilon \epsilon^{-1} G v_{c} v_{c}^{\dagger} G \epsilon^{-T} c_{l} \partial^{l}_{x} \log(x) \right\}$$

$$= e^{\sum_{k=0}^{\infty} c_k \epsilon \partial^k_{x} G \partial^k_{x} X_{L_{nm}}(x) + c_0 x_0 \hspace{1cm} (3.22)}$$
where the last line can also be written as: \( e^{\sum_k c_k^0 \partial^k X(x,x)} \); and in the normal ordering the \( x_0 \) zero modes are not splitted. We have also defined the function

\[
\Delta_a(u/v) = \hat{g} - v_a(u/v) - \hat{g}_0(u/v) = \hat{g} - v_a(u/v) - \ln(1 - u/v)
\]  

(3.23)

which near \( u = v \) can be expanded as

\[
\Delta_a(1 + \eta) = \left\{ \begin{array}{l}
\psi(v_a) - \psi(1) - (1 - v_a)\eta + \frac{(1 - v_a)(1 - v_a)}{4} \eta^2 + O(\eta^3) \quad \nu_a > 0 \\
\psi(1 - |v_a|) - \psi(1) - |v_a|\eta + \frac{|v_a|(1 + |v_a|)}{4} \eta^2 + O(\eta^3) \quad \nu_a \leq 0
\end{array} \right.
\]

(3.24)

The true Sciuto - della Selva-Saito is then obtained by representing the algebra \([c_{ik}, \frac{\partial}{\partial c_{ik}}] = \delta^j_i \delta_{k_1,k_2}\) on the dipole string Fock space. We introduce therefore the dipole string Fock space \( H_\pi \) then we make the identifications\(^{13}\)

\[
1 \rightarrow \langle x(0) = 0 | \langle 0(0) | e^{\frac{i}{\sqrt{2 \alpha'}}} \frac{c_{i0}}{k!} E_0 i \}
\]

\[
\frac{\partial}{\partial c_{ik}} \rightarrow -i \sqrt{2 \alpha'} (k - 1)! (E_0^{-1} G)^i_j \alpha^0_{i0 - k} \sim \partial^k X_0^{-i} x(x) | x = 0
\]  

(3.25)

with \([\alpha^0_{i0}, \alpha^0_{j0}] = n G^{ij} \delta_{m+n,0}\) so that after the substitution we get explicitly

\[
S_0(x) = \langle x(0) = 0 | \langle 0(0) | \exp \left\{ \frac{1}{2} \sum_{k,l=0}^{\infty} \sum_{c/v \neq 0} \alpha^0_{0k} G v_c v_c^\dagger G \alpha^0_{0l} \frac{\partial^{i x}_{|v=x} k!}{l!} \Delta_c(u/v) \right\}
\]

\[
\exp \left\{ -\frac{1}{2} \sum_{l=0}^{\infty} \sum_{c/v \neq 0} \alpha^0_{0l} G v_c v_c^\dagger G \alpha^0_{0l} \frac{\partial^l_{|v=x} |1}{l!} \log(x) \right\}
\]

\[
: \exp \left\{ i \frac{1}{\sqrt{2 \alpha'}} \sum_{k=0}^{\infty} \frac{1}{k!} \alpha^0_{0k} G \alpha^0_{0l} E_0 \partial^k X(x,x) \right\}
\]  

(3.26)

In a similar way we can compute the Sciuto della Selva-Saito vertex for the emission of \( \sigma = \pi \) dipole strings with the help of \( (y < 0) \)

\[
X(y, y) = x_0 + E_\pi^{-1} G X_{L+n,m}(y) - \sqrt{2 \alpha'} \pi R \sum_{f/v \neq 0} v_f \alpha^0_f
\]

as

\[
S_\pi(y) = \langle x(\pi) = 0 | \langle 0(\pi) | \exp \left\{ \frac{1}{2} \sum_{k,l=0}^{\infty} \sum_{c/v \neq 0} \alpha^\pi_{0k} G v_c v_c^\dagger G \alpha^\pi_{0l} \frac{\partial^{i x}_{|v=x} k!}{l!} \Delta_c(u/v) \right\}
\]

\[
\exp \left\{ -\frac{1}{2} \sum_{l=0}^{\infty} \sum_{c/v \neq 0} \alpha^\pi_{0l} G v_c v_c^\dagger G \alpha^\pi_{0l} \frac{\partial^l_{|v=x} |1}{l!} \log(|y|) \right\}
\]

\[
: e^{\frac{i}{\sqrt{2 \alpha'}} \sum_{k} \alpha^\pi_{0k} G \partial^k X_{L+n,m}(y)} e^{\frac{i}{\sqrt{2 \alpha'}} \sum_{k} \alpha^0_{0k} G \partial^k X_{L+n,m}(y)}
\]  

(3.27)

\(^{13}\)In the following we will also use \( \alpha^\pi_n = \sum_{a} v^a_n \alpha^\pi_n \) with \([\alpha^\pi_n, \alpha^\pi_m] = \sum_{a} v^a_n (n + v_a) v^\dagger_n \delta_{m+n,0}\). In the pure dipole string these \( \alpha^\pi_0 \) satisfy \([\alpha^\pi_0, \alpha^\pi_0] = n G^{ij} \delta_{m+n,0}\) and are related to the usual operators \( \hat{\alpha}^\pi_n \), defined in the appendix D, as \( \alpha^\pi_0 = G^{-1} \hat{\alpha}^\pi_0 \). Moreover we find \([x_0^a, \alpha^\pi_0] = i \sqrt{2 \alpha'} (E_\pi^{-1} k)^j \exp(ik_0 x_0) | 0 \rangle = \sqrt{2 \alpha'} (E_\pi^{-1} k)^j \exp(ik_0 x_0) | 0 \rangle \) and similarly \( \alpha^\pi_0 \) \( \exp(ik_0 x_0) | 0 \rangle = \sqrt{2 \alpha'} (E_\pi^{-1})^j \exp(ik_0 x_0) | 0 \rangle \).
where the auxiliary Fock space $\mathcal{H}_\sigma$ is associated with the $\sigma = \pi$ dipole Fock space and hence in principle different from the $\sigma = 0$ auxiliary Fock space. The last line can also be rewritten as: $e^{\frac{1}{\sqrt{2\alpha'}} \sum_k \frac{1}{2} \alpha_{(e,k)}^T \epsilon^{e_1}_{\pi} \theta^k X(y;\sigma)}$; so that we immediately see that $\Omega S_0(x) \Omega^{-1} = \tilde{S}_\pi(y)$ where $\tilde{S}_\pi(y)$ is the SDS vertex associated to the emission from the $\sigma = \pi$ boundary of the discharged string with magnetic field strength $(\tilde{F}_0 = -\tilde{F}_\pi, \tilde{F}_\pi = -F_0)$.

The corresponding closed string emission vertex can be written as generating function as

$$S(c_L, c_R, z, \bar{z}) = \exp \left\{ -\alpha' \sum_{k,l=0}^{\infty} \sum_{c/v_c \neq 0} c_{Lk}^T v_c v_c^{\dagger} c_{Ll} \partial_{|u=z}^k \partial_{|v=z}^l \Delta_c (u/v) \right\}$$

$$\exp \left\{ + \alpha' \sum_{l=0}^{\infty} \sum_{c/v_c \neq 0} c_{L0}^T v_c v_c^{\dagger} c_{Ll} \partial_{z}^l \log(z) \right\}$$

$$\exp \left\{ -\alpha' \sum_{l=0}^{\infty} \sum_{c/v_c \neq 0} c_{R0}^T R_0 v_c v_c^{\dagger} R_0^T c_{RL} \partial_{|u=z}^k \partial_{|v=z}^l \Delta_c (u/v) \right\}$$

$$\exp \left\{ + \alpha' \sum_{l=0}^{\infty} \sum_{c/v_c \neq 0} c_{R0}^T R_0 v_c v_c^{\dagger} R_0 c_{RL} \partial_{\bar{z}}^l \log(\bar{z}) \right\}$$

$$\sum_k c_{Lk}^T \partial^k X_{Lnm}(z) + c_{L0}^T \partial^k X_{Lnm}(z) + c_{R0}^T \partial^k X_{Rnm}(\bar{z}) + c_{Rk}^T \partial^k X_{Rnm}(\bar{z})$$

$$\frac{1}{\sqrt{2\alpha'}} \sum_{k=0}^{\infty} \frac{1}{k!} \alpha_{(e,k)}^T \bar{\epsilon}^k X_{Lnm}(\bar{z}) + i \frac{1}{\sqrt{2\alpha'}} \alpha_{(e,k)}^T \bar{\epsilon}^k X_{Rnm}(\bar{z})$$

$$\langle x_L = 0|0_L| \exp \left\{ \frac{1}{2} \sum_{k,l=0}^{\infty} \sum_{c/v_c \neq 0} \alpha_{Lk}^T G v_c v_c^{\dagger} G \alpha_{Ll} \partial_{|u=z}^k \partial_{|v=z}^l \Delta_c (u/v) \right\}$$

$$\exp \left\{ - \frac{1}{2} \sum_{l=0}^{\infty} \sum_{c/v_c \neq 0} \alpha_{L0}^T G v_c v_c^{\dagger} G \alpha_{Ll} \partial_{\bar{z}}^l \log(\bar{z}) \right\}$$

$$\exp \left\{ i \frac{1}{\sqrt{2\alpha'}} \sum_{k=0}^{\infty} \frac{1}{k!} \alpha_{Lk}^T \partial^k X_{Lnm}(z) + i \frac{1}{\sqrt{2\alpha'}} \alpha_{R0}^T \partial^k X_{Rnm}(\bar{z}) \right\}$$

$$\langle x_R = 0|0_R| \exp \left\{ \frac{1}{2} \sum_{k,l=0}^{\infty} \sum_{c/v_c \neq 0} \alpha_{Rk}^T G R_0 v_c v_c^{\dagger} R_0^T G \alpha_{RL} \partial_{|u=z}^k \partial_{|v=z}^l \Delta_c (u/v) \right\}$$

$$\exp \left\{ - \frac{1}{2} \sum_{l=0}^{\infty} \sum_{c/v_c \neq 0} \alpha_{R0}^T G R_0 v_c v_c^{\dagger} R_0 \partial_{\bar{z}}^l \log(\bar{z}) \right\}$$

$$\exp \left\{ i \frac{1}{\sqrt{2\alpha'}} \sum_{k=0}^{\infty} \frac{1}{k!} \alpha_{Rk}^T \partial^k X_{Rnm}(\bar{z}) + i \frac{1}{\sqrt{2\alpha'}} \alpha_{R0}^T \partial^k X_{Rnm}(\bar{z}) \right\}$$

(3.28)

(3.29)

(3.30)
It is worth stressing that the splitting of zero modes \( x_0 \) in left \( x_{0L} = G^{-1} \mathcal{E}_x x_0 \) and right \( x_{0R} = G^{-1} \mathcal{E}_x^T x_0 \) parts is dictated by the request that the OPE of two of the previous vertexes reproduces the usual OPE (see appendix F).

3.3 Examples of vertexes

We can now put at work the results of the previous section. In this section we write only the part of the vertex in the twisted directions. We start with simplest case, i.e. the tachyonic vertex from the \( \sigma = 0 \) boundary

\[
V_{(0)T_0}(x, k) = \mathcal{S}_0(x) | k_{(0)} \rangle = e^{ik_i X_i(x, x)} : \Rightarrow V_{(0)T_0}(0, k)|0(0)\rangle = | k_{(0)} \rangle \tag{3.31}
\]

so we can compute the emission vertex of the same tachyon from the dicharged string as

\[
V_{T_0}(x, k) = \mathcal{S}_0(x) | k_{(0)} \rangle = e^{i\alpha' k_i (\mathcal{E}_0^{-1} G v_c)^j (v_c^\dagger \mathcal{E}_0^{-T})^j} k_j \Delta_c(1) \nonumber
\]

\[
x^{-\alpha' k_i (\mathcal{E}_0^{-1} G v_c)^j (v_c^\dagger \mathcal{E}_0^{-T})^j} k_j \nonumber
\]

\[
: e^{ik_i X_i(x, x)} : \tag{3.32}
\]

which reduces to eq. (3.1) when we consider the dicharged string on \( \mathbb{R}^2 \). In a similar way we get the tachyonic vertex from the \( \sigma = \pi \) boundary

\[
V_{T_\pi}(y, k) = \mathcal{S}_\pi(y) | k_{(\pi)} \rangle = e^{i\alpha' k_i (\mathcal{E}_\pi^{-1} G v_c)^j (v_c^\dagger \mathcal{E}_\pi^{-T})^j} k_j \Delta_c(1) \nonumber
\]

\[
| y|^{-\alpha' k_i (\mathcal{E}_\pi^{-1} G v_c)^j (v_c^\dagger \mathcal{E}_\pi^{-T})^j} k_j \nonumber
\]

\[
: e^{ik_i X_i(y, y)} : \tag{3.33}
\]

which differs from the \( \sigma = 0 \) one also because it depends on \( \mathcal{E}_\pi \) in stead of \( \mathcal{E}_0 \).

As a second example we consider the fluctuations of the gauge fields. This result will be used later when we compute the instanton form factor. The dipole state associated with the emission of this fluctuation is

\[
V_{(0)A}(x, \epsilon, k) = : \epsilon_j \partial X_j(x, x) e^{ik_i X_i(x, x)} : \Rightarrow V_{(0)A}(0, \epsilon, k)|0(0)\rangle = -i \sqrt{2\alpha'} \epsilon_j (\mathcal{E}_0^{-1} G)^j l_{(0)\epsilon - 1} | k_{(0)} \rangle \tag{3.34}
\]

so we get the emission vertex from the dicharged string as

\[
V_A(x, \epsilon, k) = \mathcal{S}_0(x) \left[ - i \sqrt{2\alpha'} \epsilon_j (\mathcal{E}_0^{-1} G)^j l_{(0)\epsilon - 1} | k_{(0)} \rangle \right] \nonumber
\]

\[
= e^{i\alpha' k_i (\mathcal{E}_0^{-1} G v_c)^j (v_c^\dagger \mathcal{E}_0^{-T})^j} k_j \Delta_c(1) \nonumber
\]

\[
x^{-\alpha' k_i (\mathcal{E}_0^{-1} G v_c)^j (v_c^\dagger \mathcal{E}_0^{-T})^j} k_j \nonumber
\]

\[
: \epsilon_j \partial X_j(x, x) + i \alpha' \left( \frac{2(\theta(-\nu_c) + \nu_c)}{x} k_i [\mathcal{E}_0^{-1} G v_c v_c^\dagger \mathcal{E}_0^{-T})^j] \epsilon_j \right) e^{ik^T X(x, x)} : \tag{3.35}
\]
where $\theta(x) = 1$ for $x > 0$ and 0 otherwise is the Heaviside function. Obviously the same result can be obtained computing (3.19) using the $S(c, x)$ expression given in eq. (3.22). The previous vertex can also be rewritten in a different way by splitting the last term into symmetric and antisymmetric part as

$$V_A(x, \epsilon, k) = e^{\alpha' k^T \theta_{x} G v_c \gamma^0 G \theta^T k} \Delta_c(1)$$

$$x = -\alpha' k^T \theta_{x} G v_c \gamma^0 G \theta^T k$$

$$\left( e^{T \partial X(x, x)} + i \alpha' \frac{1}{x} k^T \theta_{x} G v_c \gamma^0 G \theta^T \epsilon + \alpha' \frac{1}{x} k^T \Theta_{\text{vec}} \epsilon \right) e^{i k T X(x, x)} :$$

(3.36)

where $\Theta_{\text{vec}} = -\Theta_{\text{vec}}^T = i \sum_{c > 0} (1 - 2|\nu_c|) \theta_{x} G (v_c \gamma^0 \gamma^0 - v_c \gamma^0 \gamma^0) G \theta_{x}^T$ and we cannot use the spectral decomposition of $G^{-1}$ in the symmetric part since $\{v_c\}$ do not generically form a basis.

Another interesting example is the energy momentum tensor. It is a descendant of the unity $1_{(0)}$ of the dipole string and therefore it can be computed as

$$T_{(0)1_0}(x) = -\frac{1}{4\alpha'} : \partial_x X_{L(0)}^T G \partial_x X_{L(0)} :$$

$$\Rightarrow T_{(0)1_0}(0)|0\rangle = \frac{1}{2} \alpha_{(0)-1} \alpha_{(0)-1} |0\rangle$$

$$\Rightarrow T_{1_0}(x) = S_0(x) \frac{1}{2} \alpha_{(0)-1} \alpha_{(0)-1} |0\rangle$$

$$= -\frac{1}{4\alpha'} \partial_x X_{L(0)}^T G \partial_x X_{L} + \frac{1}{2} \sum_{c/\nu_c \neq 0} \partial_u |u=x\rangle \partial_v |v=x\rangle \Delta_c(u/v)$$

(3.37)

On the other side the energy momentum tensor of the dicharged string $T(z)$ has exactly the same OPEs with all the dipole string vertexes in dicharged string formalism and therefore we can identify on the boundary $T(x) = T_{(0)1_0}(x)$ and analogously for the $\sigma = \pi$ dipole string $T(y) = T_{(\pi)1_\pi}(y)$ ($y < 0$). Therefore we recover the dicharged string energy momentum (2.40) when we rewrite the second addend as $-\frac{1}{2\pi^2} \sum_{c > 0} (\Delta_c'(1) + \Delta_c''(1) + \Delta_{-c}'(1)) + \Delta_{-c}''(1)) = +\frac{1}{2\pi^2} \sum_{c > 0} \nu_c (1 - \nu_c)$.

### 3.4 Consistency conditions for the vertexes.

We have constructed the SDS vertexes and some explicit examples of vertexes which can be computed from it. We have now ready to state the consistency conditions we expect to be satisfied by the proposed vertexes in order to be identified with the proper emission vertexes of the dipole string states from the dicharged string:

1. open string vertexes have well defined conformal transformations;
2. closed string vertexes has well defined conformal transformations;
3. open string vertexes on the opposite boundaries commute;
4. open string emission vertexes commutes with the closed string vertexes;
5. the OPE of two open string vertexes for the emission of dipole states from the \( \sigma = 0 \) boundary of the dicharged string must have the same coefficients of the OPE of the corresponding open string vertexes for the emission of the same dipole states from the \( \sigma = 0 \) boundary of the dipole string when we properly map the vertexes;

6. the same is true for the emission of dipole states from the \( \sigma = \pi \) boundary;

7. in a similar way the OPE of two closed string vertexes in open string formalism must reproduce the result in the closed string formalism.

Details on how to check the previous constraints are given in appendix F. Here we limit ourselves to some comments.

- Constraints (1-2) are actually a consequence of (5-7) since we have shown that the dicharged energy-momentum tensor can be identified with the dipole energy-momentum tensor. In any case because in the generic OPE

\[
T(z)V(w) = \frac{\Delta V(w)}{(z-w)^2} + \frac{\partial V(w)}{(z-w)} + \text{reg}
\]

there is the derivative of \( V \) from constraints (1-2) we can test the presence of terms like \( x^{-\Delta} \) in the vertex nevertheless the purely numerical normalization factors like \( e^{\delta(\epsilon)\Delta_0(k)} \) cannot be tested.

- The constraints (3-4) check the operatorial structure of vertexes only.

- The constraints (5-7) check both the operatorial and the non-operatorial structure of the vertexes.

4. Stringy form factors

We are now ready to use the previously developed machinery to do some amplitude computations.

4.1 Tachyonic form factor on \( \mathbb{R}^2 \)

As a warming up we consider the tachyonic profile of the \( D25/D25' \) bound state where the first \( D25 \) has a background field strength \( (F_0)_{12} \) switched on and the second \( (F_\pi)_{12} \). The amplitude we consider can be written in the usual CFT formalism as

\[
\mathcal{A} = C_0(\nu)N_0(\nu)^2N_0(F_0) \left( V_{T25'/25}(x_1; q, \lambda) \ V_{T25}(x_2; k_M) V_{T25'/25}(x_3; p, \kappa) \right)
\]

(4.1)

where \( C_0(\nu) \) is the normalization of the mixed amplitude, \( N_0(\nu) \) is the normalization of the dicharged vertex, \( N_0(F_0) \) is the normalization of the dipole vertex and \( \mu = 0, 3, \ldots D - 1 \), \( M = (\mu, i) = 0, \ldots D - 1 \). The dicharged vertex for the tachyon with momentum \( p_\mu \) \( (\mu \neq 1, 2) \), polarizazion \( t(p_\mu, \kappa) \) is given by

\[
V_{T25'/25}(x; p, \kappa) = t(p, \kappa) \ c(x) \ \Delta_{\nu,\kappa}(x) e^{ip_\mu X^\mu(x, x)}
\]

(4.2)
with \( \alpha' p_\mu^2 + \nu (1 - \nu) = 1 \) and where we have introduced the family of twist fields \( \Delta_{\nu,\kappa}(x) \) parametrized by \( \kappa \). The reason of introducing this family is that the vacuum is degenerate and labeled by \( \kappa \) (which we can identify with the “momentum” in direction 2) as in eq. (2.38) but it is not possible to account for this degeneracy introducing a factor \( e^{i\kappa X^2} \) in the vertex because it would change the conformal dimension. The dipole vertex with polarization \( T(k_M) \) and momentum \( k_M = (k_\mu, k_i) \) \( (M = 0, \ldots D - 1, i = 1, 2) \) is given as usual by

\[
V_{T25}(x; k_M) = T(k_M) c(x) e^{i k_M X^M(x, x)} \tag{4.3}
\]

with \( \alpha'(k_M^2 + G_{ij}^0 k_i k_j) = 1 \). The computation is straightforward for all correlators but the three point correlator

\[
\langle \Delta_{-\nu,\lambda}(x_1) e^{ik_1 X^i(x_2, x_2)} \Delta_{\nu,\kappa}(x_3) \rangle = \frac{C}{x_{12}^{\Delta(k_i)} x_{13}^{\nu(1-\nu)-\Delta(k_i)} x_{23}^{\nu(1-\nu)-\Delta(k_i)}} \tag{4.4}
\]

which is completely fixed by conformal invariance up to the constant \( C \). In the previous expression \( \Delta(k_i) = \alpha' G_{ij}^0 k_i k_j \) is the conformal dimension of \( e^{ik_1 X^i(x_2, x_2)} \). To fix the constant \( C \) we take the limit \( x_3 \to 0 \), \( x_1 \to \infty \) and we get

\[
\lim_{x_3 \to 0, x_1 \to \infty} x_1^{\nu(1-\nu)} \langle \Delta_{-\nu,\lambda}(x_1) e^{ik_1 X^i(x_2, x_2)} \Delta_{\nu,\kappa}(x_3) \rangle = \frac{C}{x_2^{\Delta(k_i)}} \tag{4.5}
\]

with \( \frac{1}{2} R^2(\nu) = - (\Delta_{c=1}(1) + \Delta_{c=-1}(1)) \) and where the result of the last line is due to the fact that the correlator boils down to a quantum mechanical computation for the Landau level theory with \( \Theta_{\text{charged}}^{12} = ((\mathcal{F}_\pi - \mathcal{F}_0)^{-1})^{12} \). This result can seem quite strange since it involves \( k_1 \) and \( k_2 \) in an asymmetric way. The reason is that we started with the vacuum annihilated by \( x_1^0 \) but we could as well have started with a vacuum annihilated by any linear combination of \( x_1^0 \) and \( x_2^0 \). In particular choosing the vacuum annihilated by \( x_2^0 \) would reverse the role of the two directions.

Putting all together we get the amplitude

\[
\mathcal{A} = C_0(\nu) N_0(\nu)^2 N_0(F_0) (2\pi)^{D-2} \delta^{D-2}(p_\mu + k_\mu + q_\mu) 2\pi \delta(k_2 + k + \lambda) t(p_\mu, \kappa) t(q_\mu, \lambda) T(k_M) e^{-i\pi\alpha' k_1 k_2} \Theta_{\text{charged}}^{12} e^{ik_1 \kappa} \frac{1}{2} R^2(\nu) e^{ik_1 \kappa} \frac{1}{2} R^2(\nu) e^{ik_1 \kappa} \tag{4.6}
\]

where the last term can be interpreted as form factor of the twisted matter. One could doubt about this interpretation because if we had computed the amplitude with two dipole tachyons with momenta \( k \) and \( l \) each vertex would yield \( e^{-\frac{1}{2} R^2(\nu) \alpha' G_{ij}^0 k_i k_j} \) and \( e^{-\frac{1}{2} R^2(\nu) \alpha' G_{ij}^0 k_i l_j} \) respectively which is not the expected factor \( e^{-\frac{1}{2} R^2(\nu) \alpha' G_{ij}^0 (k+l)_i (k+l)_j} \). Nevertheless this factor is what one gets when factorizing in the dipole string channel because of OPEs.

Let us now discuss the tachyonic profile of a \( D25/D23 \) system. We start considering the \( D25 \) with a vanishing background field strength \((F_0)_{12}\) and with zero Kalb-Ramond
$B_{12} = 0$ so that $\nu = \frac{1}{2}$. Nevertheless this system cannot be obtained taking the $(F_\pi)_{12} \to \infty$ limit rigorously. In fact taking naively this limit we expect the same infinite degeneration of the $D25/D25'$ case and this expectation is generically true unless we both choose $\nu = \frac{1}{2}$ and consider the $D25/D23$ configuration from the beginning in a compact space on which we take the decompactification limit. To understand this point it is necessary to start with a compactified version of the theory on $\mathbb{R}^{D-2} \times T^2$ and either construct explicitly the vacuum or more intuitively take a T-duality and end with a $D24/D24'$ system. In this T-dual version we expect that the two $D24$ meet only once, i.e. that the first Chern class be $c_1 = 1$ but $c_1 = 1$ of more generically $c_1$ finite cannot be true on a space with arbitrary large volume. A naive way of seeing this is to notice that $c_1 \propto (F_0 - F_\pi)_{12} \ vol(T^2)$ so that $c_1$ cannot stay finite when $\ vol(T^2) \to \infty$ with non vanishing $F_0 - F_\pi$. Another way of getting the same result in a slightly more general case with non vanishing $B_{12}$ is to realize that if we require $\nu = \frac{1}{2}$ we must have $(F_0 - B)_{12} (F_\pi - B)_{12} = -\det G$ which allows a solution for $B_{12}$ only when $(F_0 - F_\pi)^2_{12} > 4 \det G$. This last equation can be rewritten as $\frac{c^2_{12}}{(N_0, N_\pi)^2} > 4 \det G$ ($N_0, N_\pi$ arbitrary number of $D24$ and $D24'$ which can be obtained by T-duality) in a compact space so we cannot take the decompactification limit unless we take $c_1 \to \infty$ at the same time. This is exactly the same result we get by taking the $(F_\pi)_{12} \to \infty$ naively. The only way to get the vacuum degeneration $c_1 = 1$ (or finite) in the non compact case is to start already from a $D25/D23$ system. Even if we start from a system $D25/D23$ with $B_{12} = 0$ and a non vanishing background field strength $(F_0)_{12}$ on a compact space and we perform a T-duality to get a $D25/D25'$ system the volume of the $T^2$ on which this latter system lives is bounded as $\det G < \frac{1}{4}$ and so we cannot take the decompactification limit. Explained this point if we assume $c_1 = 1$ and to take the decompactification limit in the proper way then the zero modes correlator gives

$$\langle \lambda = 0, y_0 | e^{ik_1 x_1} \rangle = e^{ik_1 y_0}$$

(4.7)

where $\lambda = \kappa = 0$ is the only possible value of the vacuum degeneration label and $y_0$ is the Wilson line which appears as background only for compact spaces. Notice that the previous state with $c_1 = 1$ (or finite) is the analogous of a momentum eigenstate with discrete momentum while the one entering the spectrum in eq. (2.38) is analogous to one with continuum momentum and so the two are connected as

$$|\lambda\rangle_{\text{continuum}} = \sqrt{2\pi c_1} |\lambda\rangle_{\text{discrete}}$$

(4.8)

where $c_1$ plays the role of the radius. The same kind of issue in defining the lower dimensional brane vacuum is present also for the dipole string ([29],[30]). Finally we can write the amplitude for the $D25/D23$ case as

$$A = C_0(\nu) N_0(\nu)^2 N_0(F_0) (2\pi)^{D-2} \delta^{D-2}(p_\mu + k_\mu + q_\mu) t(p_\mu) t(q_\mu) T(k_M) e^{ik_1 y_0} e^{-\frac{1}{4} R^2(\nu) \delta^{ij} k_i k_j}$$

(4.9)

with $\pi \nu = \frac{1}{2} \pi - \text{arctg} (f_0/\sqrt{\det G_{T^2}})$ and where the discharged tachyon polarizations now depend only on the momenta and not on the parameter labeling the infinite degeneracy of the Landau levels in the non compact case.
4.2 Electromagnetic form factor on $\mathbb{R}^2$

Let us now consider the electromagnetic profile of the $D25/D25'$ matter which can be computed by the correlator

$$A = \mathcal{C}_0(\nu)\mathcal{N}_0(\nu)^2\mathcal{N}_0(F_0) \langle V_{T25'/25} (x_1; q_\mu, \lambda) \ V_{A25}(x_2; k_M, \epsilon_M) V_{T25'/25} (x_3; p_\mu, \kappa) \rangle \quad (4.10)$$

where the dipole vertex is given as usual by

$$V_{A25}(x; k_M, \epsilon) = \epsilon_N(k_M) \ c(x) \partial X^N(x, x) e^{ikMxM(x, x)}$$

with $\alpha'(k_\mu^2 + \mathcal{G}_{ij} k_i k_j) = \epsilon_\mu k_\mu + \epsilon_i \mathcal{G}_{ij} k_j = 0$. The computation is again straightforward for all correlators but the three points correlator

$$\langle \Delta_{-\nu, \lambda}(x_1) \partial X^j(x_2, x_2) e^{ik_i X^i(x_2, x_2)} \Delta_{\nu, \kappa}(x_3) \rangle = \mathcal{C}(x_1, x_2, x_3) \frac{\Delta(k_1)+1}{\Delta(k_1)+1} \frac{\Delta(k_2)-1}{\Delta(k_2)-1} \frac{\Delta(k_3)}{\Delta(k_3)+1}$$

$$= -i \alpha' \left( \frac{x_{23} - x_{13}}{x_{12}} \mathcal{G}_{ij}^j - i \Theta_{\text{vect}}^{ij} \right) k_l e^{-\frac{i}{2}R^2(\nu)\Delta(k_i)} e^{-i\pi \alpha' k_i k_2} \Theta_{\text{charged}}^{ij} e^{ik_1 \kappa} 2\pi \delta(k_2 + \kappa + \lambda)$$

with $\Theta_{\text{vect}} = (1 - 2|\nu|)\mathcal{E}_0^{-1}G \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} G\mathcal{E}_0^{-T}$. The previous correlator is not fixed in the functional dependence on $x$ by conformal invariance since $\partial X^j(x_2, x_2) e^{ik_i X^i(x_2, x_2)}$ is not a good conformal operator because of the cubic pole with the energy-momentum tensor.

We can fix this correlator by first taking the limit

$$\lim_{x_3 \to 0, x_1 \to \infty} x_1^\nu \langle \Delta_{-\nu, \lambda}(x_1) \partial X^j(x_2, x_2) e^{ik_i X^i(x_2, x_2)} \Delta_{\nu, \kappa}(x_3) \rangle =$$

$$= \langle 0_{\nu}, -\lambda | e^{-\frac{i}{2}R^2(\nu)\Delta(k_1)} x_2^\Delta(k_2) e^{ik_i X^i(x_2, x_2)} (\partial X^j(x_2, x_2) + i \alpha' \mathcal{G}_{ij}^j) k_l \rangle 0_{\nu}, \kappa \rangle$$

$$= e^{-\frac{1}{2}R^2(\nu)\Delta(k_i)} x_2^{-\Delta(k_i) - 1} \alpha' \left( i \mathcal{G}_{ij}^j - \Theta_{\text{vect}}^{ij} \right) k_l \langle -\lambda | e^{ik_i x_i} | k \rangle$$

$$= e^{-\frac{1}{2}R^2(\nu)\Delta(k_i)} x_2^{-\Delta(k_i) - 1} e^{-i\pi \alpha' k_1 k_2} \Theta_{\text{charged}}^{ij} \alpha' \left( i \mathcal{G}_{ij}^j - \Theta_{\text{vect}}^{ij} \right) k_l e^{ik_1 \kappa} 2\pi \delta(k_2 + \kappa + \lambda)$$

then computing the amplitude in this limit

$$\lim_{x_1 \to \infty, x_3 \to 0} x_2^2 A = \mathcal{C}_0(\nu)\mathcal{N}_0(\nu)^2\mathcal{N}_0(F_0) (2\pi)^{D-2} \delta^{D-2} (p_\mu + k_\mu + q_\mu) 2\pi \delta(k_2 + \kappa + \lambda)$$

$$= (-i \alpha') t(p_\mu, \kappa) t(q_\mu, \lambda) \left[ \epsilon_\mu (k_M) (p_\mu - q_\mu) - ie_j (k_M) \Theta_{\text{vect}}^{ij} k_l \right] e^{-i\pi \alpha' k_1 k_2} \Theta_{\text{charged}}^{ij} \alpha' e^{-\frac{1}{2}R^2(\nu)\alpha' \mathcal{G}_{ij}^j}$$

and finally reconstructing the correlator such that it reproduces this amplitude without taking the $x_3 \to 0, x_1 \to \infty$ limit.

Again the $D25/D23$ amplitude is slightly different because of zero modes and reads

$$A = \mathcal{C}_0(\nu)\mathcal{N}_0(\nu)^2\mathcal{N}_0(F_0) (2\pi)^{D-2} \delta^{D-2} (p_\mu + k_\mu + q_\mu)$$

$$= (-i \alpha') t(p_\mu) t(q_\mu) \left[ \epsilon_\mu (k_M) (p_\mu - q_\mu) - ie_j (k_M) \Theta_{\text{vect}}^{ij} k_l \right] e^{ik_1 y_0} e^{-\frac{1}{2}R^2(\nu)\alpha' \mathcal{G}_{ij}^j}$$

(4.15)
4.3 Electromagnetic form factor on \( \mathbb{R}^d \)

We want to generalize the previous computation to \( \mathbb{R}^d \) with \( d_K = \dim \ker(\Delta F) \). The first issue is to write the tachyonic discharged vertex. It is not completely trivial since we can immediately write

\[
V_{25/25'}(x; p_\mu, p_f, \kappa) = t(p_\mu, p_f, \kappa) \, c(x) \, \Delta_{\nu, \kappa}(x) \, e^{ip_\mu X^\mu(x) + ip_f X^f(x)}
\]

(4.16)

but then we have to specify what \( X^f(x, x) \) is. There are at least three possibilities \( v^\dagger G X(x, x), v^\dagger \mathcal{E}_0 X(x, x) \) and \( v^\dagger \mathcal{E}_\pi X(x, x) \). By comparing with the state \( | p_f \rangle = e^{i p_f x_0} | 0 \rangle \) and its dimension \( L_0 | p_f \rangle = 2 \alpha' p_f p_{-f} | p_f \rangle \), it turns out that

\[
X^f(x, x) = v^\dagger \mathcal{E}_\pi X(x, x).
\]

(4.17)

In the previous vertex (4.16) \( \nu \) and \( \kappa \) are now \( \frac{1}{2} \dim \ker(\Delta F)^\perp = \frac{1}{2}(d - d_K) \) dimensional vectors which label the Landau levels “frequencies” and their degeneracies. It is then easy to compute

\[
A = C_0(\nu) N_0(\nu)^2 N_0(F_0)(2\pi)^{D-d} \delta^{D-d}(p_\mu + k_\mu + q_\mu)(2\pi)^d \delta^{d}(p_f + k_f + q_f)(-\lambda|e^{i k_c x_0^c}|\mu)
\]

\[
(-ia')t(p) t(q) \left[ \epsilon_\mu(k)(p^\mu - q^\mu) + \epsilon_f(k)(p_{-f} - q_{-f}) - i\epsilon_j(k)\Theta^{j}_{\text{vec}} k_l \right] e^{-\alpha' \sum_{c>0} R^2(\nu_c) \hat{k}_c \hat{k}_{-c}}
\]

(4.18)

where we have defined \( k_f = k^T \mathcal{E}_\pi^{-1} G v_f = \hat{k}_f = k^T \mathcal{E}_0^{-1} G v_c \neq \hat{k}_c = k^T \mathcal{E}_0^{-1} G \bar{v}_c \) which are two possible ways of projecting the momentum \( k_i = (1, \ldots d) \) along the directions with \( \nu_f = 0 \) and \( \nu_c \neq 0 \). We also explicitly have \( -i\epsilon_j \Theta^{j}_{\text{vec}} k_l = \sum_{c>0}(1 - 2\nu_c)(\hat{\epsilon}_c \hat{k}_{-c} - \hat{k}_c \hat{\epsilon}_{-c}) \). Moreover we have left not evaluated the quantum mechanical correlator \( \langle -\lambda|e^{i k_c x_0^c}|\mu \rangle \) because its value depends whether we consider a \( D25/D25' \) system or a \( D25/Dp \) one since for any dimension less than 25 we gain a Wilson line and loose an infinite degeneracy if there is already a zero eigenvalue. Also \( t(p) \) and \( t(q) \) have a different dependence on parameters whether we consider \( D25/D25' \) or \( D25/Dp \) as discussed above.

The previous amplitude can be computed without taking the limit using the correlator

\[
\langle -\nu, -\lambda(x_1) \, \partial X^j(x_2, x_2) e^{ik_c X^c'(x_2, x_2)} \, \Delta_{\nu, \kappa}(x_3) \rangle =
\]

\[
= \frac{-i\alpha \left( 2 \frac{x_{12}}{x_{13}} (\mathcal{E}_0^{-1} C \mathcal{V}_j)^i p_{-f} - 2 \frac{x_{12}}{x_{13}} (\mathcal{E}_0^{-1} C \mathcal{V}_j)^i q_{-f} + \frac{x_{12}}{x_{13}} \mathcal{E}_0^{-1} C \mathcal{V}_j)^i \hat{k}_{-c} - i\Theta^{j}_{\text{vec}} k_l \right)}{x_{12} \alpha'(q_f q_{-f} + p_f p_{-f} - \Delta(k_i)) + \sum_{c>0} \nu_c(1 - \nu_c) + \alpha'(q_f q_{-f} + p_f p_{-f} - \Delta(k_i)) - 1} e^{-\frac{1}{2} \alpha' \sum_{c>0} R^2(\nu_c) \hat{k}_c \hat{k}_{-c} (2\pi)^d \delta^{d}(q_f + k_f + p_f)(-\lambda|e^{i k_c x_0^c}|\mu)}
\]

(4.19)

4.4 The stringy instantonic form factor

It is now immediate to compute the form factor of \( D(-1) \) seen from \( D3 \). Using the notation of ([5]) we get

\[
A^{I}_\mu(p; \bar{w}, w) = i (T^I)_{\alpha}^{\nu} e^\pi \eta_{\nu \mu} \left( w_{\alpha}^{\nu} (\tau e^{i \hat{\omega}_{\alpha \nu}}) \right) e^{-ip \cdot x_0} e^{-2\ln 2 \alpha' p_\mu p^\mu}
\]

(4.20)
where $\mu = 0, 1, 2, 3$ and we have used $\Delta_{\nu} = \frac{1}{2}(x) = -2 \ln(1 + \sqrt{x})$ so that $R^2(\frac{1}{2}) = 4 \ln 2$. We have also used $F_0 = 0$. It is interesting to notice that only for $\nu = \frac{1}{2}$ the $\Theta_{\text{vect}}$ term vanishes. It then follows that an instanton cannot shrink to zero and that its profile is

$$A^I_\mu(x) = \int \frac{d^4 p}{(2\pi)^2} A^I_\mu(p; \bar{w}, w) \frac{1}{p^2} e^{ip \cdot x}$$

$$= (T^I)^u_v \left( w^u_\alpha (\tau_c)^{\alpha \beta} w^\beta_v \right) \bar{\eta}_{\nu \mu} \left\{ -2 \frac{(x-x_0)^\nu}{(x-x_0)^2} (x-x_0)^2 >> \alpha' R^2(\frac{1}{2}) \right. \right.$$  

$$\left. \left. \left. \left. \left. - \frac{2}{3} \frac{(x-x_0)^\nu}{\alpha'^2 R^4} (x-x_0)^2 << \alpha' R^2(\frac{1}{2}) \right) \right(4.21) \right. \right.$$  

Since instantonic amplitudes on non compact space are dominated by the infrared behavior of the instanton it is then clear that such amplitudes essentially are the same as without the form factor. On the contrary the form factor can give relevant contributions when the space is compact and the size is not too large.

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**A. Conventions.**

We define:

- WS metric signature: $\eta_{\alpha \beta} = (-, +); e^{01} = -1$
  
  Space-time metric signature: $G_{\mu \nu} = (-, +, \ldots, +)$;

- Indices: Compact $i, j, \ldots = 1, \ldots d$; non compact $\mu, \nu, \ldots = 0, d + 1 \ldots D$; general $M, N, \ldots = 0, \ldots D$;

- WS coordinates: $\xi_{\pm} = \tau \pm \sigma$ and $\tau_E = i\tau$

$$z = e^{\tau_E + i\sigma}, \quad \bar{z} = e^{\tau_E - i\sigma}, \quad (A.1)$$

the variables $z$ and $\bar{z}$ are defined respectively in the upper-half and in the lower-half complex plane since $\sigma \in [0, \pi]$

- Given a couple of free bosons $X^{1,2} \equiv X^{L2}$ (which we identify with flat coordinates) we make the linear combinations

$$Z = X = \frac{1}{\sqrt{2}} (X^1 + iX^2) \quad \bar{Z} = \bar{X} = Z^\dagger \quad (A.2)$$

Notice that in the main text we do not write the underline explicitly.

- We define the following functions which are used in the definitions of “propagators” for $0 < \epsilon < 1$

$$\hat{g}_\epsilon(z) = g_\epsilon(z) = -\sum_{n=1}^{\infty} \frac{1}{n-\epsilon} z^{n-\epsilon} \quad |z| < 1, |\arg(z)| < \pi$$

$$\hat{g}_{-\epsilon}(z) = g_{1-\epsilon}(z)$$

-
• Background matrices:

\[ E = \| E_{ij} \| = G + B \]
\[ \mathcal{E} = \| \mathcal{E}_{ij} \| = E^T + 2\pi\alpha'q_0F = G + \mathcal{F} \]  
\[ \mathcal{F} = 2\pi\alpha'q_0F - B = \tilde{\mathcal{F}} - B \]  
\[ \mathcal{F} = 2\pi\alpha'q_0F - B = \tilde{\mathcal{F}} - B \]  
\[ \hat{\mathcal{F}} = 2\pi\alpha'q_0F \]  
\[ \mathcal{F} = 2\pi\alpha'q_0F - B = \tilde{\mathcal{F}} - B \]  
\[ \mathcal{F} = 2\pi\alpha'q_0F - B = \tilde{\mathcal{F}} - B \]  

(A.3)

For the dipole strings we can define

\[ \mathcal{E}^{-1} = G^{-1} - \Theta \]  

from which we deduce that

\[ \mathcal{E}G^{-1}\mathcal{E}^T = \mathcal{E}^T G^{-1}\mathcal{E} = G \]
\[ \Theta = \frac{1}{2}(\mathcal{E}^T - \mathcal{E}^{-1}) = -\mathcal{E}^{-1}\mathcal{B}\mathcal{E}^{-T} \]  

(A.6)

B. Details on the dicharged string quantization.

Using the product definition (2.20) and the eigenfunctions (2.19) we find the following basic products

\[ \langle \Psi_{na}, \Psi_{mb} \rangle = -i\pi \delta_{a,b} \delta_{m,n} (n + \nu_a) \quad n, \nu_a, m, \nu_b \neq 0 \]
\[ \langle \Psi_{0e}, \Psi_{ma} \rangle = 0 \quad \nu_e = 0, \quad m + \nu_a \neq 0 \]
\[ \langle w, \Psi_{ma} \rangle = 0 \quad m + \nu_a \neq 0 \]
\[ \langle \Psi_{0e}, \Psi_{0f} \rangle = -\pi^2 \nu_e^2 \mathcal{F}_\pi \nu_f \]
\[ \langle w, \Psi_{0e} \rangle = \pi w^\dagger \mathcal{E}_\pi^T \nu_e \quad \nu_e = 0 \]
\[ \langle w, u \rangle = -w^\dagger \Delta \mathcal{F} u \]  

(B.1)

where we have defined \( \Delta \mathcal{F} = \mathcal{F}_\pi - \mathcal{F}_0 \) and \( w, u \) are arbitrary constant vectors. The computation is almost straightforward when using the right tricks therefore we show a piece of it:

\[ \Psi_{na}^\dagger(\pi, \tau) \mathcal{F}_\pi \Psi_{mb}(\pi, \tau) = \]
\[ = (-)^{m+n} e^{i(n+\nu_a)-(m+\nu_b)} \nu_a^\dagger e^{i\nu_a} \frac{1 + e^{-2i\nu_a R_0^T}}{2} \frac{1 + e^{-2i\nu_a R_0^T}}{2} e^{-i\nu_b} \nu_b \]
\[ = (-)^{m+n} e^{i(n+\nu_a)-(m+\nu_b)} \nu_a^\dagger e^{i\nu_a} \frac{1 + R_\pi^T}{2} \frac{1 + R_\pi^T}{2} e^{-i\nu_b} \nu_b \]
\[ = (-)^{m+n} \frac{1}{4} e^{i(n+\nu_a)-(m+\nu_b)} \nu_a^\dagger e^{i\nu_a} G(R_{\pi}^{-1} - R_{\pi}) e^{-i\nu_b} \nu_b \]
\[ = (-)^{m+n} \frac{1}{4} e^{i(n+\nu_a)-(m+\nu_b)} \left( e^{-i\nu_a \nu_b} \nu_a^\dagger GR_0^{-1} \nu_b - e^{+i\nu_a \nu_b} \nu_a^\dagger GR_0 \nu_b \right) \]  

(B.2)
where we have used a certain number of times $R_\pi = R_0 R^{-1}$, we have rewritten $F_\pi = G(1 - R_\pi)(1 + R_\pi)^{-1}$ and used the definition of the eigenvalues $v_a$ in eq. (2.13) and the orthogonality relation for $R$ in eq. (2.12). We can now define the quantities

$$I_{an} = \langle \Psi_{a,n}, \Phi \rangle = -i\pi(n + \nu_a) \phi_n^a \quad n + \nu_a \neq 0$$

$$I_{w=\varepsilon^{-1} Gv_c} = \langle \varepsilon^{-1} Gv_c, \Phi \rangle = \pi \phi_c^e \quad \nu_e = 0$$

(B.3)

from which we can easily extract the coefficients $\phi_n^a$. The analogous expressions for $\phi_i$ give

$$I_e = \langle \Psi_0 e, \Phi \rangle = -\pi v_e^\dagger (\varepsilon_\pi \phi + \pi F_\pi v_f \phi_0^f) \quad \nu_e = 0$$

$$I_{w=\varepsilon^{-1} Gv_c} = \langle \varepsilon^{-1} G v_c, \Phi \rangle = -v_c^\dagger G E^{-T} \Delta F \phi \quad \nu_e = 0$$

(B.4)

from which it is possible in principle to recover the $\phi^i$ coefficients since the equations in the last two lines are as many as the components of $\phi^i$ and are independent and are therefore sufficient but the result is rather not instructive. In order to compute the commutation relation we resort therefore to the trick of computing the commutation relations among the previous quantities (2.22, 2.23) when we take $\Phi = X$ and then extract those among the $x^i$.

Using the canonical commutation relations (2.24) the commutation relations among the previous defined quantities (B.3, B.4) when $\Phi = X$ are

$$[I_{n,a}, I_{m,b}] = 2\pi^2 \alpha' (n + \nu_a) \delta_{a,b} \delta_{m,n}$$

$$[I_e, I_f] = -i2\pi^2 \alpha' v_e^\dagger F_\pi v_f^\dagger$$

$$[I_e, I_w] = -i2\pi^2 \alpha' v_c^\dagger E_\pi^\dagger \Delta F u^*$$

$$[I_w, I_u] = -i2\pi \alpha' w^\dagger \Delta F u^*$$

(B.5)

from which we can deduce the commutation relations for the string operators written in the main text.

C. Analytic properties of the twisted propagator.

From the normal ordering it turns out natural to define the following function which are used in the definitions of “propagators” for $0 < \epsilon < 1$

$$g_\epsilon(z) = -\sum_{n=1}^{\infty} \frac{1}{n - \epsilon} z^{n-\epsilon} \quad \left| z \right| < 1, -\pi < \text{arg}(z) < \pi$$

(C.1)

We will also use the following more compact notation for $-1 < \epsilon < 1$

$$\hat{g}_\epsilon(z) = \begin{cases} g_\epsilon(z) & 0 < \epsilon < 1 \\ g_{1+\epsilon}(z) & -1 < \epsilon < 0 \end{cases}$$

(C.2)

The reason of the range of $\text{arg}(z)$ is that we put a cut along the negative real axis. The previous function can be defined on all sheets as

$$g_{\epsilon,s}(z) = -\sum_{n=1}^{\infty} \frac{1}{n - \epsilon} z^{n-\epsilon} \quad \left| z \right| < 1, -\pi + 2\pi s < \text{arg}(z) < \pi + 2\pi s$$

(C.3)
Its analytic continuation to the whole complex plane but the negative real axis and $x \geq 1$ part of the real one can be given by the following integral representation which trivially reduces to the previous series in the $|z| < 1$ domain

$$
ge_{\epsilon,s}(z) = - \int_0^{|z|} dx \frac{x^{-\epsilon} e^{i(1-\epsilon)\phi}}{1 - e^{i\phi} x} \quad z = |z| e^{i\phi} \in \mathbb{C} - \{x < 0, x \geq 1\} \tag{C.4}$$

where the path $\gamma$ is the straight line from $t = 0$ to $t = z$. The cut along $x \geq 1$ is because of the logarithmic singularity at $z = 1$ which is due to the integrand pole at $t = 1$.

Given the previous definition it is then easy to see that

$$
ge_{\epsilon,s}(z) = \int_0^{+\infty} dy \frac{y^{-(1-\epsilon)} e^{i(1-\epsilon)\phi}}{e^{i\phi} - y} + g_{1-\epsilon,-s} \left( \frac{1}{z} \right) = C_{\epsilon,s}(arg \, z) + g_{1-\epsilon,-s} \left( \frac{1}{z} \right) \tag{C.6}$$

As it is usual in complex analysis we consider the auxiliary integral $\int dw \frac{w^{1-\epsilon}}{e^{i\phi} + w}$ and then we can easily compute the constant entering the previous relation to be

$$
C_{\epsilon,s}(arg \, z) = \int_0^{+\infty} dy \frac{y^{-(1-\epsilon)}}{e^{i\phi} - y} = \begin{cases} 
\frac{\pi e^{-i\pi s}}{\sin \pi \epsilon} e^{-i2\pi \epsilon s} & 2\pi s < \phi < \pi + 2\pi s \\
\frac{\pi e^{i\pi \epsilon}}{\sin \pi \epsilon} e^{-i2\pi \epsilon s} & -\pi + 2\pi s < \phi < 2\pi s
\end{cases} \tag{C.7}
$$

The previous expression satisfies the following properties

$$C_{1-\epsilon,-s}(arg \, 1/z) = -C_{\epsilon,s}(arg \, z), \quad [C_{\epsilon,s}(arg \, z)]^* = C_{\epsilon,-s}(arg \, z^*), \tag{C.8}$$

which can be derived as consistency conditions from eq. (C.6) and verified directly on the expression (C.7). It is then immediate to write the corresponding property for the $\hat{g}$ function:

$$
\hat{g}_{\epsilon,s}(z) = \hat{C}_{\epsilon,s}(arg \, z) + \hat{g}_{-\epsilon,-s} \left( \frac{1}{z} \right), \quad \hat{C}_{\epsilon,s}(arg \, z) = \begin{cases} 
\frac{\pi e^{i\pi s}}{\sin \pi \epsilon} e^{i2\pi \epsilon s} & 2\pi s < \phi < \pi + 2\pi s \\
\frac{\pi e^{-i\pi \epsilon}}{\sin \pi \epsilon} e^{i2\pi \epsilon s} & -\pi + 2\pi s < \phi < 2\pi s
\end{cases} \tag{C.9}
$$

It is also useful to compare the behavior of $g_{\epsilon}(z)$ with that of the usual untwisted propagator $g_0(z) = \log(1 - z)$, then we get for $\eta > 0$

$$
g_{\epsilon,s}(1-\eta) = \log \eta + \int_0^{1-\eta} dt \frac{1 - t^{-\epsilon}}{1 - t} = \log \eta + \psi(1-\epsilon) - \psi(1) + \epsilon \eta + \frac{\epsilon(\epsilon + 1)}{4} \eta^2 + O(\eta^3) \tag{C.10}
$$

where we used the fact that $\psi = \frac{d \log \Gamma(z)}{dz}$ can be expressed as

$$
\psi(1 + z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n + z)} = -\gamma + \int_0^1 dt \frac{1 - t^z}{1 - t} \tag{C.11}
$$

with $\gamma$ the Euler-Mascheroni constant. We record here a useful $\gamma$ property which we use in the computations

$$
\psi(1 - z) = \psi(z) + \pi \cot \pi z \tag{C.12}
$$

so that taking $\epsilon = \nu_a$ we get

$$\Delta_a(1) = \Delta_{-a}(1) = \pi \cot \pi \nu_a \tag{C.13}$$
D. Dipole open string conventions

D.1 Real formalism

On the magnetic background \( \mathcal{F}_0 = \mathcal{F}_\pi \) the open dipole string field expansion is given by

\[
X^i(z, \bar{z}) = \frac{1}{2} \left( X^i_L(z) + X^i_R(\bar{z}) \right)
\]  

where \( z = e^{rE+i\sigma} \in \mathbb{H} \) (0 \( \leq \sigma \leq \pi \)) and

\[
X^i_L(z) = (G^{-1} \mathcal{E})^i_j \hat{X}^j_{L(0)}(z) + y^i_0
\]
\[
X^i_R(\bar{z}) = (G^{-1} \mathcal{E}^T)^i_j \hat{X}^j_{R(0)}(\bar{z}) - y^i_0
\]

where we have defined the following quantities as in eq. (2.7)

\[
F_{ij} = -B_{ij} + \hat{F}_{ij}
\]
\[
E_{ij} = G_{ij} + F_{ij} = G_{ij} - B_{ij} + \hat{F}_{ij}
\]

and the open string metric given by

\[
G_{ij} = G_{ij} - F_{ik}G_{kh}F_{hj} = E^T_{ik}G_{kh}E_{hj}
\]

along with the non commutativity parameter \( \Theta \) as

\[
(\mathcal{E}^{-1})^{ij} = (G^{-1})^{ij} - \Theta^{ij}.
\]

Moreover we have also defined the fields \( \hat{X}^i_{L(0)} \) and \( \hat{X}^i_{R(0)} \) which have the usual field expansion

\[
\hat{X}^i_{L(0)}(z) = \hat{x}^i - 2\alpha' \hat{p}^i i \ln(z) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\text{sgn}(n)}{\sqrt{|n|}} \hat{a}^i_n z^{-n}
\]
\[
\hat{X}^i_{R(0)}(\bar{z}) = \hat{x}^i - 2\alpha' \hat{p}^i i \ln(\bar{z}) + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\text{sgn}(n)}{\sqrt{|n|}} \hat{a}^i_n \bar{z}^{-n}
\]

but have a different set of commutation relations since the metric \( G \) is replaced by the open string metric \( G \) and the zero modes have a non trivial commutation relation, explicitly

\[
[\hat{x}^i, \hat{x}^j] = i\sqrt{2\alpha'} G^{ij}
\]
\[
[\hat{a}^i_m, \hat{a}^j_n] = \delta_{m+n,0} G_{ij} sgn(m)
\]

The previous expansion (D.7) looks as the usual one because we have used \( \hat{x}^i \) with non vanishing commutator and not \( \hat{x}^i_0 \) defined as \( \hat{x}^i = \hat{x}^i_0 - \pi \alpha' \Theta^i \theta_j G_{lm} \hat{p}^m \) with the usual commutator

\[
[\hat{x}^i_0, \hat{x}^j_0] = 0
\]

Finally \( y^i_0 = \sqrt{\alpha'} G^{ij} \theta_j \) are constants and proportional to the Wilson lines \( \theta \) on the brane at \( \sigma = 0 \) ([31],[32], [29] and [16] ). Notice the \( y^i_0 \) do not enter the expansion of \( X(z, \bar{z}) \) and therefore they do enter the open string vertexes but they do enter the closed string
vertexes in the open string formalism where they are necessary to reproduce from the open string side the phases in the boundary state due to Wilson lines or positions.

On this background the dipole string Hamiltonian is then given by

\[ H_0 - 1 = L_0 = \alpha' G^{\mu \nu} k_\mu k_\nu + \alpha' \tilde{p}^i G_{ij} \tilde{p}^j + \sum_{n=1}^{\infty} nG_{\mu \nu} \alpha_n^i \alpha_n^\nu + nG_{ij} \tilde{a}_n^i \tilde{a}_n^j \]  

(D.10)

In the non compact case the vacuum is defined

\[ a^i_n |0\rangle = p^i |0\rangle = 0, \quad n > 0 \]  

(D.11)

so that the basic OPE reads

\[ X^{(-)i}_L(z) X^{(+)}(w) = -2\alpha' G^{ij} \log(\frac{z - w}{z}) + : \]  

\[ X^{(-)i}_L(z) X^{(+)}(w) : \]  

(D.12)

**D.2 Complex formalism in two dimensions.**

We consider \( \mathbb{R}^2 \) on which we write the background matrices in flat coordinates as

\[ \mathcal{E} = \mathcal{G} - \mathcal{R} + \mathcal{F} = \begin{pmatrix} 1 & \tilde{f} \\ -\tilde{f} & 1 \end{pmatrix}, \quad \tilde{f} = \hat{f} - \hat{b} \]  

(D.13)

from which we derive the open string background

\[ \mathcal{G} = \frac{1}{1 + \tilde{f}^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \cos^2 \gamma \mathbb{I}_2, \quad \Theta = \frac{1}{1 + \tilde{f}^2} \begin{pmatrix} 0 & \tilde{f} \\ -\tilde{f} & 0 \end{pmatrix} = \cos \gamma \sin \gamma \epsilon_2 \]  

(D.14)

where we have defined the angle \( \gamma \) as

\[ e^{i\gamma} = \frac{1 + i\tilde{f}}{\sqrt{1 + \tilde{f}^2}} \Rightarrow \tilde{f} = \tan \gamma, \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2} \]  

(D.15)

If we define the complex fields

\[ Z(z, \bar{z}) = \frac{1}{\sqrt{2}} \left( X^1(z, \bar{z}) + iX^2(z, \bar{z}) \right) = \frac{1}{2}(Z_L(z) + Z_R(\bar{z})) \]  

(D.16)

the boundary conditions become

\[ Z' + i\bar{Z}|_{\sigma=0} = 0, \quad Z' + i\bar{Z}|_{\sigma=\pi} = 0. \]  

(D.17)

or in the worldsheet Wick rotated version

\[ e^{+i\gamma} \partial Z|_x = e^{-i\gamma} \partial Z|_x \quad x \in \mathbb{R}^+ \]  

\[ e^{+i\gamma} \partial Z|_y = e^{-i\gamma} \partial Z|_y \quad y = |y|e^{i\pi} \in \mathbb{R}^- \]
The fields expansions for $Z$ read
\[
Z_L = (1 - i\hat{\alpha}'\hat{Z}_L(0) + w_0 = e^{-i\gamma}Z_L(0) + w_0
= e^{-i\gamma}\left(\hat{z}_0 - 2\alpha'\hat{p} i\ln(z) + i\sqrt{2\alpha'}\sum_{n=1}^{\infty} \frac{\hat{a}_n}{\sqrt{n}} z^{-n} - \frac{\hat{a}^+_n}{\sqrt{n}} z^n\right) + w_0
\]
\[
Z_R = (1 + i\hat{\alpha}'\hat{Z}_R(0) - w_0 = e^{+i\gamma}Z_R(0) - w_0
= e^{+i\gamma}\left(\hat{z}_0 - 2\alpha'\hat{p} i\ln(z) + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty} \frac{\hat{a}_n}{\sqrt{n}} z^{-n} - \frac{\hat{a}^+_n}{\sqrt{n}} z^n\right) - w_0 \tag{D.18}
\]
where we have introduced $Z_L(0) = \sqrt{1 + \hat{\alpha}'^2}Z_L(0)$ and for their complex conjugates $\bar{Z} = \frac{1}{\sqrt{2}}(\bar{X}^1 - i\bar{X}^2)$
\[
\bar{Z}_L = (1 + i\hat{\alpha}'\hat{\bar{Z}}_L(0) + \bar{w}_0 = e^{+i\gamma}\bar{Z}_L(0) + \bar{w}_0
= e^{+i\gamma}\left(\bar{z}_0 - 2\alpha'\hat{p} i\ln(\bar{z}) + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty} \frac{\bar{a}_n}{\sqrt{n}} \bar{z}^{-n} - \frac{\bar{a}^+_n}{\sqrt{n}} \bar{z}^n\right) + \bar{w}_0
\]
\[
\bar{Z}_R = (1 - i\hat{\alpha}'\hat{\bar{Z}}_R(0) - \bar{w}_0 = e^{-i\gamma}\bar{Z}_R(0) - \bar{w}_0
= e^{-i\gamma}\left(\bar{z}_0 - 2\alpha'\hat{p} i\ln(\bar{z}) + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty} \frac{\bar{a}_n}{\sqrt{n}} \bar{z}^{-n} - \frac{\bar{a}^+_n}{\sqrt{n}} \bar{z}^n\right) - \bar{w}_0 \tag{D.19}
\]
with commutation relations
\[
[\bar{z}_0, \bar{z}_0] = 2\pi\alpha'\hat{1}
[\bar{z}_0, \bar{p}_0] = i
[\bar{a}_n, \bar{a}_m] = \delta_{m+n,0} sgn(m) \tag{D.20}
\]
When we define as usual ($n > 0$)
\[
\alpha_n = \sqrt{n}\alpha_n \quad \alpha_{-n} = \sqrt{n}\alpha^+_n \quad \alpha_0 = \sqrt{2\alpha'}\hat{p}
\]
and similarly for the barred quantities, we can write the energy-momentum tensor as
\[
T(z) = -\frac{2}{\alpha'}\partial Z\partial Z = -\frac{1}{2\alpha'}\partial_Z L \partial_Z L = \sum_k L_k \frac{L_k}{z^{k+2}}
\]
with
\[
L_k = \sum_{m=1}^{\infty} \bar{\alpha}^+_m \bar{\alpha}_{m+k} + \sum_{m=0}^{\infty} \alpha^+_m \alpha_{m+k} + \sum_{m=1}^{k} \bar{\alpha}_m \alpha_{k-m} \quad k > 0
\]
\[
L_k = \sum_{m=1}^{\infty} \bar{\alpha}^+_m \alpha_{m+|k|} + \sum_{m=0}^{\infty} \alpha^+_m \alpha_{m+|k|} + \sum_{m=1}^{\infty} \bar{\alpha}_m \alpha_{|k|-m} \quad k < 0
\]
\[
L_0 = \alpha_0\bar{\alpha}_0 + \sum_{m=1}^{\infty} \bar{\alpha}^+_m \bar{\alpha}_m + \sum_{m=1}^{\infty} \alpha^+_m \alpha_m
= \alpha'\hat{p}\bar{\alpha} + \sum_{n=1}^{\infty} n \left(\alpha^+_n \alpha_n + \bar{\alpha}^+_n \bar{\alpha}_n\right) \tag{D.21}
\]
The tachyonic vertex is then
\[ V_T(x, k) = : e^{i k \cdot X(x, x)} : \Lambda \]
\[ = : e^{i \bar{k} \cdot \bar{X}} L(0) (x) : \Lambda \]
\[ = : e^{i \cos \gamma (\bar{k} \cdot \bar{Z} L(0)(x) + \bar{\bar{k}} \cdot \bar{\bar{Z}} L(0)(x))} : \Lambda \]  
(D.22)
where the normal ordering for the zero modes is the trivial one, i.e. \( : e^{i k \cdot \bar{x}} : =: e^{i k \cdot \bar{x}} \) as suggested by the hermiticity of the vertex. The \( \Lambda \) are the hermitian Chan-Paton matrices.

The previous tachyonic vertex has conformal dimension
\[ \Delta(V_T) = \alpha' G_{ij} k_i k_j = 2 \alpha' \cos^2 \gamma \bar{k} \bar{\bar{k}} \]
(D.23)
and OPE
\[ V_T(x, k) V_T(x, l) = e^{\frac{1}{2} 2 \pi \alpha' \cos(2\gamma) (\bar{\bar{k}} - \bar{\bar{l}})(x_1 - x_2) 2 \alpha' \cos^2 \gamma (\bar{ar{k}} - \bar{\bar{l}})} : \Lambda V_T : \]
(D.24)
which must be reproduced by the corresponding emission vertex from the dicharged string.

E. Dicharged string on \( \mathbb{R}^2 \) in complex formalism.

For dicharged strings on \( \mathbb{R}^2 \) it is natural to work in complex formalism which we use in section 3.1. Here we state our normalizations and conventions in this formalism. Using flat coordinates the boundary conditions are
\[ \bar{Z}' + i \dot{\bar{f}}(0) \bar{Z} |_{\sigma = 0} = 0, \quad \bar{Z}' + i \dot{\bar{f}}(\pi) \bar{Z} |_{\sigma = \pi} = 0 \]
(E.1)
or in the worldsheet Wick rotated version
\[ e^{i \gamma_0} \partial Z |_x = e^{-i \gamma_0} \bar{\partial} Z |_x \quad x \in \mathbb{R}^+ \]
\[ e^{i \gamma_\pi} \partial Z |_y = e^{-i \gamma_\pi} \bar{\partial} Z |_y \quad y = |y| e^{i \pi} \in \mathbb{R}^- \]
(E.2)
(E.3)
where we have defined as for the dipole string
\[ e^{i \gamma_0} = \frac{1 + i \dot{f}_0}{\sqrt{1 + \dot{f}_0^2}} \rightarrow \bar{f}_0 = \tan \gamma_0 \]
\[ e^{i \gamma_\pi} = \frac{1 + i \dot{f}_\pi}{\sqrt{1 + \dot{f}_\pi^2}} \rightarrow \bar{f}_\pi = \tan \gamma_\pi \]
(E.4)

It is then immediate to compute the
\[ R_{0,\pi} = \begin{pmatrix} \cos 2 \gamma_{0,\pi} & -\sin 2 \gamma_{0,\pi} \\ \sin 2 \gamma_{0,\pi} & \cos 2 \gamma_{0,\pi} \end{pmatrix} \Rightarrow R = \begin{pmatrix} \cos 2(\gamma_0 - \gamma_\pi) & -\sin 2(\gamma_0 - \gamma_\pi) \\ \sin 2(\gamma_0 - \gamma_\pi) & \cos 2(\gamma_0 - \gamma_\pi) \end{pmatrix} \]
(E.5)
so that we

$$1 > \epsilon = \nu = \left| \gamma_0 - \gamma_\pi \right| > 0$$ \hspace{1cm} (E.6)

For the complex string fields

$$Z(z, \bar{z}) = \frac{1}{2} (Z_L(z) + Z_R(\bar{z})),$$ \hspace{1cm} (E.7)

we have the following expansions\(^{14}\)

$$Z_L(z) = \bar{z}_{L0} + i \sqrt{2\alpha'} \sum_{n=0}^{\infty} \left[ \frac{\bar{a}_{n+1-\epsilon}}{\sqrt{n + 1 - \epsilon}} z^{-n+1-\epsilon} - \frac{\bar{a}_{n+\epsilon}^\dagger}{\sqrt{n + \epsilon}} \bar{z}^{n+\epsilon} \right]$$

$$Z_R(\bar{z}) = \bar{z}_{R0} + i \sqrt{2\alpha'} e^{2i\gamma_0} \sum_{n=0}^{\infty} \left[ \frac{\bar{a}_{n+1-\epsilon}}{\sqrt{n + 1 - \epsilon}} \bar{z}^{n+1-\epsilon} - \frac{\bar{a}_{n+\epsilon}^\dagger}{\sqrt{n + \epsilon}} z^{n+\epsilon} \right]$$ \hspace{1cm} (E.8)

and

$$\bar{Z}_L(z) = \bar{z}_{\bar{L}0} + i \sqrt{2\alpha'} \sum_{n=0}^{\infty} \left[ \frac{\bar{a}_{n+1-\epsilon}^\dagger}{\sqrt{n + 1 - \epsilon}} z^{n+1-\epsilon} + \frac{\bar{a}_{n+\epsilon}}{\sqrt{n + \epsilon}} \bar{z}^{n+\epsilon} \right]$$

$$\bar{Z}_R(\bar{z}) = \bar{z}_{\bar{R}0} + i \sqrt{2\alpha'} e^{-2i\gamma_0} \sum_{n=0}^{\infty} \left[ \frac{\bar{a}_{n+1-\epsilon}^\dagger}{\sqrt{n + 1 - \epsilon}} \bar{z}^{n+1-\epsilon} + \frac{\bar{a}_{n+\epsilon}}{\sqrt{n + \epsilon}} z^{n+\epsilon} \right]$$ \hspace{1cm} (E.9)

with non vanishing commutation relations:

$$[\bar{a}_{n+1-\epsilon}, \bar{a}_{m+1-\epsilon}^\dagger] = \delta_{n,m} n, m \geq 0$$

$$[\bar{a}_{n+\epsilon}, \bar{a}_{m+\epsilon}^\dagger] = \delta_{n,m} n, m \geq 0$$

$$[\bar{z}_0, \bar{z}_0] = \frac{2\pi\alpha'}{\bar{f}_{(0)} - \bar{f}_{(\pi)}}$$ \hspace{1cm} (E.10)

The vacuum is the usual vacuum for the non zero modes and it is defined as

$$a_{n+\epsilon}(0_\epsilon) = \bar{a}_{n+1-\epsilon}(0_\epsilon) = 0 \hspace{1cm} n \geq 0$$ \hspace{1cm} (E.11)

while for zero modes it is necessary to distinguish between the non compact case and the compact one. For the non compact case we define it as

$$x^1|0_\epsilon \rangle = 0 \rightarrow |k, \epsilon \rangle = e^{-ik(\bar{f}_{(0)} - \bar{f}_{(\pi)})x^2} |0_\epsilon \rangle = 0$$ \hspace{1cm} (E.12)

which corresponds choosing \( p = x^1/(2\alpha') \) and \( x = -(\bar{f}_{(0)} - \bar{f}_{(\pi)})x^2 \), as noticed in the main text the choice of \( x^1 \) ad destructor is arbitrary and we could have chosen any linear combination of \( x^1 \) and \( x^2 \) as well.

The compact case is more subtle and it is discussed in a separate paper [22].

\(^{14}\)Notice the redefinitions \( e^{-i\gamma_0} \bar{a}_{n+1-\epsilon} = \bar{a}_{n+1-\epsilon} \) and \( e^{-i\gamma_0} \bar{a}_{n+\epsilon}^\dagger = \bar{a}_{n+\epsilon}^\dagger \) for \( n \geq 0 \) as in [33].
The left-right split of zero modes can be parametrized as

\[ z_{l0} = (1 + \alpha)z_0 + \beta \bar{z}_0 + w_0, \quad z_{r0} = (1 - \alpha)z_0 - \beta \bar{z}_0 - w_0, \]  

(E.13)

and because of the constraints from the closed OPE the parameters can be fixed to \( \alpha = f_0 \) and \( \beta = 0 \).

Energy-momentum tensor is given by

\[ T_\epsilon(z) = -\frac{2}{\alpha'} : \partial \bar{Z} \partial Z : -\frac{\epsilon(\epsilon - 1)}{2z^2} = -\frac{1}{2\alpha'} : \partial Z_L \partial \bar{Z}_L : -\frac{\epsilon(\epsilon - 1)}{2z^2} = \sum_k \frac{L_k}{z^{k+2}} \]  

(E.14)

with

\[
\begin{align*}
L_{k(\epsilon)} &= \sum_{m=1}^{\infty} \bar{\alpha}_{m-\epsilon} \alpha_{m+k-\epsilon} + \sum_{m=0}^{\infty} \bar{\alpha}_{m+\epsilon} \alpha_{m+k+\epsilon} + \sum_{m=1}^{k} \bar{\alpha}_{m-\epsilon} \alpha_{k-m+\epsilon} \quad k > 0 \\
L_{k(\epsilon)} &= \sum_{m=1}^{\infty} \bar{\alpha}_{m+|k|-\epsilon} \alpha_{m-\epsilon} + \sum_{m=0}^{\infty} \bar{\alpha}_{m+|k|+\epsilon} \alpha_{m+\epsilon} + \sum_{m=1}^{|k|} \bar{\alpha}_{m-\epsilon} \alpha_{|k|-m+\epsilon} \quad k < 0 \\
L_{0(\epsilon)} &= \sum_{m=1}^{\infty} \bar{\alpha}_{m-\epsilon} \alpha_{m-\epsilon} + \sum_{m=0}^{\infty} \bar{\alpha}_{m+\epsilon} \alpha_{m+\epsilon} + \frac{1}{2} \epsilon (1 - \epsilon)
\end{align*}
\]  

(E.15)

E.1 OPEs.

Various useful contractions are \(|z| > |w|\):

\[
\begin{align*}
Z_L^{(+, nzm)}(z) \overline{Z}_L^{(-, nzm)}(w) &= -2\alpha' \hat{g}_e(w/z) + \cdots \\
Z_L^{(+, nzm)}(z) \overline{Z}_R^{(-, nzm)}(w) &= -2\alpha' e^{-2i\gamma_0} \hat{g}_e(w/z) + \cdots \\
Z_R^{(+, nzm)}(\bar{z}) \overline{Z}_L^{(-, nzm)}(w) &= -2\alpha' e^{2i\gamma_0} \hat{g}_e(w/\bar{z}) + \cdots \\
Z_R^{(+, nzm)}(\bar{z}) \overline{Z}_R^{(-, nzm)}(w) &= -2\alpha' \hat{g}_e(w/\bar{z}) + \cdots
\end{align*}
\]  

(E.16)

and

\[
\begin{align*}
\overline{Z}_L^{(+, nzm)}(z) Z_L^{(-, nzm)}(w) &= -2\alpha' \hat{g}_{-\epsilon}(w/z) + \cdots \\
\overline{Z}_L^{(+, nzm)}(z) Z_R^{(-, nzm)}(w) &= -2\alpha' e^{2i\gamma_0} \hat{g}_{-\epsilon}(w/z) + \cdots \\
\overline{Z}_R^{(+, nzm)}(\bar{z}) Z_L^{(-, nzm)}(w) &= -2\alpha' e^{-2i\gamma_0} \hat{g}_{-\epsilon}(w/\bar{z}) + \cdots \\
\overline{Z}_R^{(+, nzm)}(\bar{z}) Z_R^{(-, nzm)}(w) &= -2\alpha' \hat{g}_{-\epsilon}(w/\bar{z}) + \cdots
\end{align*}
\]  

(E.17)

F. Details on the vertexes construction on \( \mathbb{R}^n \) and their consistency

In this appendix we give some details of the computations needed to verify that the proposed vertexes and their generators do indeed satisfy all the desired properties. We first consider the problem in its full generality in one case by computing the OPE of two generating vertexes for the emission of dipole states from the \( \sigma = 0 \) boundary of a dicharged string then we examine all the other properties in a simpler setting, i.e. we consider the tachyonic vertexes and \( \text{ker}(\mathcal{F}_\pi - \mathcal{F}_0) = \emptyset \) only.
F.1 OPE of the generating vertexes

The OPE of two vertexes which describe the emission of a dipole state from the dicharged string must have the same coefficients as the OPE of the corresponding emission vertexes from the dipole string therefore we start considering the OPE of two emission vertexes from a dipole string. We consider here the case of the emission from the \( \sigma = 0 \) boundary but we can obviously repeat the same procedure for the \( \sigma = \pi \) vertexes.

More specifically we consider two generating vertexes for the emission of dipole strings from a dipole string and we compute the product of two such vertexes to be \((x,y) > 0\)

\[
S_0(c,x) S_0(d,y) = e^{\pi \alpha' c_0^T \theta dz e^{-2\alpha' \sum_{k,l=0}^{\infty} c_k^T G \sum_{l=0}^{\infty} d_l \partial_c^k \partial_d^l G_0(x,y)} S_0 \{ \{ d_k + \sum_{m=0}^{k} c_m \frac{(x-y)^{k-m}}{(k-m)!} \}, y \}
\]

(F.1)

then we expect that the corresponding OPE for the emission vertexes from a dicharged string to read

\[
S(c,x) S(d,y) = e^{\pi \alpha' c_0^T \theta dz e^{-2\alpha' \sum_{k,l=0}^{\infty} c_k^T G \sum_{l=0}^{\infty} d_l \partial_c^k \partial_d^l G_0(x,y)} S \{ \{ d_k + \sum_{m=0}^{k} c_m \frac{(x-y)^{k-m}}{(k-m)!} \}, y \}
\]

(F.2)

since the OPE coefficients \( e^{\pi \alpha' c_0^T \theta dz e^{-2\alpha' \sum_{k,l=0}^{\infty} c_k^T G \sum_{l=0}^{\infty} d_l \partial_c^k \partial_d^l G_0(x,y)} \) must be the same when we map \( S_0 \) into \( S \). Notice that in the previous expressions we have taken the parameters \( \{ c_k \} \) not to be constants but generic functions and that this does not change the fact that \( S(c,x) \) can be used as generating function for all vertexes. Moreover and more subtly we have expanded \( X_{L,nz,m}(x) \) around \( x = y \) but this expansion is convergent only for a subset of the circle \( |x| = |y| = 1 \) because \( X_{L,nz,m} \) contains both positive and negative frequencies.

Let us verify the previous equation explicitly. The product of two generating vertexes \( S \) reads

\[
S(c,x) S(d,y) = \exp \{-\alpha' \sum_{k,l=0}^{\infty} c_k^T \epsilon_0^{-T} G v_c v_c^\dagger G \epsilon_0^{-1} c_l \partial^k_{u=x} \partial^l_{v=y} \Delta_c(u/v) \}
\]

\[
\exp \{-\alpha' \sum_{k,l=0}^{\infty} d_k^T \epsilon_0^{-T} G v_c v_c^\dagger G \epsilon_0^{-1} d_l \partial^k_{u=y} \partial^l_{v=y} \Delta_c(u/v) \}
\]

\[
\exp \{-2\alpha' \sum_{k,l=0}^{\infty} c_k^T \epsilon_0^{-T} G v_a v_a^\dagger G \epsilon_0^{-1} d_l \partial^k_{v=x} \partial^l_{u=y} \hat{g} - v_a(u/v) \} \exp \{ \frac{1}{2} c_0^T [x,x^T] d_0 \}
\]

\[
: \exp \{ \{ d_k + \sum_{m=0}^{k} c_m \frac{(x-y)^{k-m}}{(k-m)!} \} \partial^k_{y} X(y,y) \}: \quad (F.3)
\]

We then reexpress the last line using \( S(\{ d_k + \sum_{m=0}^{k} c_m \frac{(x-y)^{k-m}}{(k-m)!} \}, y) \) and we use the defi-
nition of $\Delta_a$ so we find

$$S(c, x) S(d, y) = \exp \left\{ - \alpha' \sum_{k,l=0}^{\infty} c_k^T \mathcal{E}_0^{-T} G v_c v_c^\dagger \mathcal{G} v_0^\dagger \mathcal{E}_0^{-1} d_l \partial_v^k |_{v=x} \partial_v^l |_{u=y} [\Delta_c(v, u) - \Delta_{-c}(u, v)] \right\}$$

$$\exp \left( \frac{1}{2} c_0^T \left[ x, x^T \right] d_0 \right)$$

$$e^{-2\alpha' \sum_{k,l=0}^{\infty} c_k^T G^{-1} d_l \partial_v^k \partial_v^l G_0(x,y)} S_0(\{d_k + \sum_{m=0}^{k} c_m \frac{(x-y)k-m}{(k-m)!}\}, y)$$

(F.4)

Using the property of $\hat{g}_{-\nu c}$ we see that we are left with

$$= \exp \left\{ - \alpha' T \mathcal{E}_0^{-T} G v_c v_c^\dagger \mathcal{G} v_0^\dagger \mathcal{E}_0^{-1} d_0 \left[ \hat{C}_{-\nu c} \left( \arg \frac{x}{y} \right) - \hat{C}_0 \left( \arg \frac{x}{y} \right) \right] + \frac{1}{2} c_0^T \left[ x, x^T \right] d_0 \right\}$$

$$e^{-2\alpha' \sum_{k,l=0}^{\infty} c_k^T G^{-1} d_l \partial_v^k \partial_v^l G_0(x,y)} S(\{d_k + \sum_{m=0}^{k} c_m \frac{(x-y)k-m}{(k-m)!}\}, y)$$

(F.5)

which reproduces the desired OPE when we check that the zero modes contribution is the non commutative phase $e^{i\pi \alpha' \epsilon_0^T \Theta d_0}$.

This is exactly what we do now by rewriting the exponential as

$$\frac{1}{\alpha'} \left\{ \frac{1}{2} c_0^T [x_0, x_0^T] d_0 + \alpha' \sum_{c} c_c^T \mathcal{E}_0^T G v_c v_c^\dagger \mathcal{G} v_0^\dagger \mathcal{E}_0^{-T} d_0 \left[ \hat{C}_{-\nu c} \left( \arg \frac{x}{y} \right) - \hat{C}_0 \left( \arg \frac{x}{y} \right) \right] \right\}$$

$$= -i\pi c_0^T \left[ \mathcal{E}_0^{-1} G v_f v_f^\dagger \mathcal{F} v_g v_g^\dagger \mathcal{G} \mathcal{E}_0^{-1} + \mathcal{E}_0^{-1} G v_c v_c^\dagger \mathcal{E} v_c^T \right] \frac{1}{1 - \alpha' \epsilon_0^T v_c^\dagger \mathcal{G} \mathcal{E}_0^{-1} v_c^T} d_0$$

(F.6)

where we used the explicit expressions for $\hat{C}$

$$= -i\pi c_0^T \left[ \mathcal{E}_0^{-1} G v_f v_f^\dagger \mathcal{F} v_g v_g^\dagger \mathcal{G} \mathcal{E}_0^{-1} + \frac{1}{1 - \alpha' \epsilon_0^T v_c^\dagger \mathcal{G} \mathcal{E}_0^{-1} v_c^T} \right] d_0$$

(F.7)

where we used

$$\frac{1}{f_d^T v_c} = -\frac{2}{1 - e^{-i2\pi \nu d}} w_d^\dagger \mathcal{G} \mathcal{E}_0^{-1} G v_c$$

which can be derived from $R$ definition

$$= -i\pi c_0^T \left[ \mathcal{E}_0^{-1} G v_f v_f^\dagger \mathcal{F} v_g v_g^\dagger \mathcal{G} \mathcal{E}_0^{-1} + \frac{1}{1 - e^{-i2\pi \nu d}} \times \left[ +2\mathcal{E}_0^{-1} G v_f v_f^\dagger \mathcal{E}_0^{-1} G v_d v_d^\dagger \mathcal{E} \mathcal{E}_0^{-1} - 2\mathcal{E}_0^{-1} G v_d v_d^\dagger \mathcal{E} \mathcal{E}_0^{-1} + \mathcal{E}_0^{-1} G(1 + R)v_d v_d^\dagger \mathcal{E} \mathcal{E}_0^{-1} \right] \right] d_0$$

(F.8)

where we have used the spectral representation for $G^{-1}$ to write $w_{c_1} w_c^\dagger = G^{-1} - w_{f_1} w_{f_1}^\dagger$, then we used the orthogonality of $G^{-1} \mathcal{E} v_c \in \ker \Delta R$ with $v_c$ and finally we used again
the spectral representation for $G^{-1}$ to write $v_c v_c^\dagger = G^{-1} - v_f v_f^\dagger$

$$= -i\pi c_0^T \left[ \mathcal{E}_\pi^{-1} G v_f v_f^\dagger F_{\pi} v_g v_g^\dagger G \mathcal{E}_\pi^{-T} + \frac{1}{1 - e^{-2\pi v_c}} \times \left[ -\mathcal{E}_\pi^{-1} G v_f v_f^\dagger F_{\pi} (1 - R) v_d v_d^\dagger G \mathcal{E}_\pi^{-T} + \mathcal{E}_\pi^{-1} G v_c v_c^\dagger G (1 - R) \mathcal{E}_0^{-T} - \mathcal{E}_\pi^{-1} G (1 - R) v_c v_c^\dagger G \mathcal{E}_0^{-T} \right] \right] d_0$$

(F.9)

where we used $\mathcal{E}_\pi^{-T} - \mathcal{E}_0^{-T} = -\frac{1}{2} (1 - R) G^{-1} \mathcal{E}_0^{-T}$ in the second term of the second line and the orthogonality of $v_f$ and $v_c$ to substitute $\mathcal{E}_\pi^{-1} \rightarrow \Delta \mathcal{F} \mathcal{E}_0^{-1} = -\frac{1}{2} \mathcal{E}_\pi^{-1} (1 - R)$ in the first term of the second line

$$= -i\pi c_0^T \left[ \mathcal{E}_\pi^{-1} G v_f v_f^\dagger F_{\pi} v_g v_g^\dagger G \mathcal{E}_\pi^{-T} + \mathcal{E}_\pi^{-1} G v_f v_f^\dagger F_{\pi} v_d v_d^\dagger G \mathcal{E}_\pi^{-T} + \mathcal{E}_0^{-1} G v_c v_c^\dagger F_{\pi} \mathcal{E}_0^{-T} \right] d_0$$

$$= -i\pi c_0^T \left[ \mathcal{E}_0^{-1} \mathcal{F} \mathcal{E}_0^{-T} \right] d_0$$

(F.10)

**F.2 Consistency of the tachyonic vertexes**

Given the length of the previous computations we restrict ourselves to show that the desired constraints are satisfied by the tachyonic vertexes in the case $\ker(\mathcal{F}_\pi - \mathcal{F}_0) = \emptyset$.

**F.2.1 Basic building blocks of the tachyonic vertexes and their properties**

We define now some basic build blocks of the tachyonic vertex operators and compute some properties which turn out to be useful in checking the wanted properties needed for defining a good CFT. We define therefore

$$B_0(k) = e^{ik z_0}$$
$$B_L(k, z) = e^{ik X_{\Lambda n zm}^{i}(z)}$$
$$B_R(k, \bar{z}) = e^{ik X_{\Lambda R n zm}^{i}(\bar{z})} = e^{i(R_{\nu}^{T} k)_i X_{\Lambda n zm}^{i}(\bar{z})}$$

(F.11)

where in the last line we used the $X$ boundary conditions. They have the following non trivial products ($|z| > |w|$)

$$B_0(k) B_0(l) = B_0(k + l) e^{-i\pi \alpha'} k_i [(\hat{F}_\pi - \hat{F}_0)^{-1}]_{ij}$$
$$B_L(k, z) B_L(l, w) = e^{ik X_{\Lambda n zm}^{i}(z) + il X_{\Lambda n zm}^{i}(w)} e^{2\alpha' \sum_a k T v_a v_a^\dagger \hat{g}_{-\nu_a,0}(\bar{w})}$$
$$B_R(k, \bar{z}) B_R(l, \bar{w}) = e^{i(R_{\nu}^{T} k)_i X_{\Lambda n zm}^{i}(\bar{z}) + i(R_{\nu}^{T} l)_i X_{\Lambda n zm}^{i}(\bar{w})} e^{2\alpha' \sum_a k T R_{\nu} v_a v_a^\dagger \hat{g}_{-\nu_a,0}(\bar{w})}$$
$$B_L(k, z) B_R(l, \bar{w}) = e^{ik X_{\Lambda n zm}^{i}(z) + i(R_{\nu}^{T} l)_i X_{\Lambda n zm}^{i}(\bar{w})} e^{2\alpha' \sum_a k T v_a v_a^\dagger R_{\nu}^{T} \hat{g}_{-\nu_a,0}(\bar{w})}$$

(F.12)

where we have explicitly written the sheet on which $g_{-\nu_a,0}$ is defined since $-2\pi < arg(\bar{w}/z) < 0$ and we have defined the fundamental sheet $s = 0$ by $\pi < arg(\bar{w}/z) < \pi$ so we can have $s = 0, 1$. 

\[ -37 - \]
Using the previous products it is not difficult to derive the following properties of the commutation / analytic continuation to the range $|z| < |w|$

\[
B_0(k) B_0(l) = B_0(l) B_0(k) e^{-2i\pi\alpha' k_0 \frac{(F_\pi-F_0)^{-1}}{\pi} \nu \arg(w/z)} \exp\{-2\alpha' \sum_{a} k_{a} l_{-a} \frac{\pi}{\sin \pi \nu a} \arg(w/z)\}
\]

\[
[B_L(k, z) B_L(l, w)]_{\text{an.cont.}} = B_L(l, w) B_L(k, z) \exp\{-i4\pi\alpha' k_i \left(\frac{R_0 \beta(-\arg(w/z))}{\frac{1}{1-R} G^{-1}}\right)^{ij} I_j\}
\]

\[
= B_L(l, w) B_L(k, z) \exp\{+i4\pi\alpha' (R_0^T l)_i \left(\frac{\frac{1}{1-R} G^{-1}}{1-R} \right)^{ij} k_j\}
\]

(F.13)

where we have used the spectral decomposition for the matrix $R$ given in eq. (2.16) and some attention to the range of $\arg(\bar{w}/z)$ must be used to get the result.

**F.2.2 The tachyonic vertexes**

We are now in the position of writing the tachyonic vertexes for both the dipole and closed tachyons. The tachyonic emission vertex from the $\sigma = 0$ boundary reads

\[
\mathcal{V}_{T_0}(x, k) = x^{-\alpha' k_0 \gamma_0^{ij} k_j} e^{\alpha' \Delta (1) \hat{k}_c \cdot k} e^{i k_0 x_0^i} : e^{ik_0 x_0^i} X_L^i(x) : e^{ik_0 x_0^i} X_L^j(x) : \Lambda_{(0)}
\]

\[
= x^{-\alpha' k_0 \gamma_0^{ij} k_j} e^{\alpha' \Delta (1) \hat{k}_c \cdot k} e^{ik_0 x_0^i} B_0(k) B_L \left(\frac{1}{2} k, x\right) \Lambda_{(0)}
\]

(F.14)

where $\hat{k}_c = \bar{v}_c \pi^{-T} G k$ and we have written the last line to show why we talked of basic building blocks for the $B$s.

In a similar way we can write the explicit expressions for the emission of a dipole tachyon from the $\sigma = \pi$ boundary of the dicharged string in a compact way as

\[
\mathcal{V}_{T_\pi}(y, l) = |y^{-\alpha' l_0 \gamma_0^{ij} l_j} e^{\alpha' \Delta (1) l, l_\beta} e^{il_0 x_0^i} : e^{il_0 x_0^i} X_L^i(y) : e^{il_0 x_0^i} X_L^j(y) : \Lambda_{(\pi)}\]

\[
= |y^{-\alpha' l_0 \gamma_0^{ij} l_j} e^{\alpha' \Delta (1) l, l_\beta} e^{il_0 x_0^i} B_0(k) B_L \left(\frac{1}{2} \frac{1}{1-R} R L_\pi l, y\right) \Lambda_{(\pi)}
\]

(F.15)

where $l_c = \bar{v}_c \pi^{-T} G l$ and the Chan-Paton matrix is transposed because the color flows in the opposite direction with respect to the $\sigma = 0$ boundary.

Finally we can write the explicit expressions for the emission of a closed string tachyon from the dicharged string including a cocycle $e^{i\pi\alpha' k_0^T S^k L}$ (which was omitted in the main
\[ \mathcal{W}_{T_\mu}(z, \bar{z}; k_L, k_R) = e^{i \pi \alpha' k_L^\alpha} T_{Lk} : e^{\frac{i}{2} k_\mu X_L^\mu(z)} : = e^{\frac{i}{2} k_\mu X_L^\mu(\bar{z})} : \\
\quad - \alpha' k_L G_{ij} k_{Lj} \ e^{\alpha' \Delta_v(1)} k_L v_{Lj}^c e^{ik_L [(G^{-1} \mathcal{E}^\pi)^T g_0 + y_0]} : e^{ik_L i X_{nzm}^i(\bar{z})} ; \\
\quad - \alpha' k_R G_{ij} k_{Rj} \ e^{\alpha' \Delta_v(1)} k_R (R_0 v_{c})^j k_{Rj} (R_0 v_{c})^i e^{ik_R [(G^{-1} \mathcal{E}^T)^T g_0 + y_0]} : e^{ik_R i X_{nzm}^i(\bar{z})} ; \\
\quad = e^{i \pi \alpha' k_L^\alpha} T_{Lk} : e^{\frac{i}{2} k_\mu X_L^\mu(z)} : = e^{\frac{i}{2} k_\mu X_L^\mu(\bar{z})} : \\
\quad - \alpha' k_L G_{ij} k_{Lj} \ e^{\alpha' \Delta_v(1)} k_L v_{Lj}^c k_{Lj} v_{Lc}^c B_0 (\mathcal{E}^\pi)^T G^{-1} k_L B_L (k_L, z) \\
\quad - \alpha' k_R G_{ij} k_{Rj} \ e^{\alpha' \Delta_v(1)} k_R (R_0 v_{c})^j k_{Rj} (R_0 v_{c})^i B_0 (\mathcal{E}^\pi)^T G^{-1} k_R B_R (k_R, \bar{z}) \] (F.16)

where \( k_L = k_R \), we have split the zero modes \( x_0 \) into left \( x_{L0} = G^{-1} \mathcal{E}^\pi x_0 \) and right \( x_{R0} = G^{-1} \mathcal{E}^T_{\pi} x_0 \) ones and explicitly introduced the dependence of the “Wilson lines” (there are actually no Wilson lines in non compact spaces) writing \( y_0 \). The splitting between left and right zero modes can be fixed by imposing that two F.16 have the correct OPE as we verify in section F.2.8, looking in particular to the \( z \) and \( \bar{z} \) dependent c-numbers.

We are now ready to verify the consistency conditions for these vertexes.

### F.2.3 The open string vertexes on the opposite boundaries commute

The product of open string vertexes on the opposite boundaries reads

\[
[V_{T_\mu}(k, x) \ V_{T_\pi}(l, y)]_{\text{an.cont.}} = [V_{T_\mu}(l, y) \ V_{T_\pi}(k, x)] \exp \{-i 2 \pi \alpha' k^T (\hat{F}_\pi - \hat{F}_0)^{-1} l\} \\
\exp \{-i 4 \pi \alpha' k^T \frac{I + R_0}{2} \frac{I}{I - R} G^{-1} \frac{I + R^T}{2}\} \] (F.17)

where the contribution in the second line is from \( B_0 \) and the one in the third from \( B_L \). We have not any contribution from Chan-Paton factors since they acts on two different color spaces.

If we evaluate the different matrices in flat coordinates we get easily

\[
\frac{I + R_0}{2} = \frac{I}{I + \mathcal{E}_0} \\
\frac{I + R^T}{2} = \frac{I}{I - \mathcal{E}_\pi} \\
\frac{I}{I + R} = \frac{1}{2} (I + \mathcal{E}_0)(\mathcal{E}_\pi - \mathcal{E}_\pi)^{-1}(I - \mathcal{E}_\pi) \] (F.18)

and therefore it follows that the request is automatically satisfied.
F.2.4 The open string emission vertex from $\sigma = 0$ commutes with the closed string vertex

The open string emission vertex from $\sigma = 0$ commutes with the closed string vertex:

$$[\mathcal{V}_{T_0}(l, x) \mathcal{W}_{T_{\pi}}(k_L, k_R, z, \bar{z})]_{\text{an.cont.}} = \mathcal{W}_{T_{\pi}}(k_L, k_R, z, \bar{z}) \mathcal{V}_{T_0}(l, x)$$

$$\exp \{ -i 2 \pi \alpha' l^T (\hat{F}_\pi - \hat{F}_0)^{-1} n + \hat{L}^T m \}$$

$$\exp \{ -i 4 \pi \alpha' \frac{l^T + R_0}{2} \frac{R}{R - R_0} G^{-1} k_L \}$$

$$\exp \{ -i 4 \pi \alpha' \frac{l^T + R_0}{2} \frac{R}{R - R_0} G^{-1} R_0 k_R \}$$  \hspace{1cm} (F.19)

where the second line is due to zero modes, the third to the commutation of the open string vertex with the left moving part of the closed one, and the fourth to the commutation with the right moving part. The terms $l^T \ldots n$ automatically cancel while the other terms $l^T \ldots m$

$$\exp \left\{ -i 2 \pi \sqrt{\alpha'} l^T \left[ (\hat{F}_\pi - \hat{F}_0)^{-1} \hat{L}^T - (\hat{F}_\pi - \hat{F}_0)^{-1} (-B) + \frac{\frac{R_0 + R}{2}}{R - R_0} G^{-1} \frac{\frac{R}{2}}{R - R_0} \right] m \right\}$$

$$= \exp \left\{ -i 2 \pi \sqrt{\alpha'} l^T \left[ (\hat{F}_\pi - \hat{F}_0)^{-1} (\hat{L}^T + \hat{F}_\pi) \right] m \right\}$$  \hspace{1cm} (F.20)

are not present in the non compact case.

F.2.5 The open string emission vertex from $\sigma = \pi$ commutes with the closed string vertex

In a similar way the open string emission vertex from $\sigma = \pi$ commutes with the closed string vertex:

$$[\mathcal{V}_{T_\pi}(l, y) \mathcal{W}_{T_{\pi}}(k_L, k_R, z, \bar{z})]_{\text{an.cont.}} = \mathcal{W}_{T_{\pi}}(k_L, k_R, z, \bar{z}) \mathcal{V}_{T_\pi}(l, y)$$

$$\exp \{ -i 2 \pi \alpha' l^T (\hat{F}_\pi - \hat{F}_0)^{-1} n + \hat{L}^T m \}$$

$$\exp \{ -i 4 \pi \alpha' \frac{l^T + R_\pi}{2} \frac{R}{R - R_0} G^{-1} k_L \}$$

$$\exp \{ -i 4 \pi \alpha' \frac{l^T + R_\pi}{2} \frac{R}{R - R_0} G^{-1} R_0 k_R \}$$  \hspace{1cm} (F.21)

where the second line is due to zero modes, the third to the commutation of the open string vertex with the left moving part of the closed one and the fourth to the commutation with the right moving part. The terms $l^T \ldots n$ automatically cancel while the other type of terms $l^T \ldots m$ gives

$$\exp \left\{ -i 2 \pi \sqrt{\alpha'} l^T \left[ (\hat{F}_\pi - \hat{F}_0)^{-1} \hat{L}^T - (\hat{F}_\pi - \hat{F}_0)^{-1} (-B) + \frac{\frac{R_\pi + R}{2}}{R - R_0} \frac{R}{R - R_0} G^{-1} \frac{\frac{R}{2}}{R - R_0} \right] m \right\}$$

$$= \exp \left\{ -i 2 \pi \sqrt{\alpha'} l^T \left[ (\hat{F}_\pi - \hat{F}_0)^{-1} (\hat{L}^T + \hat{F}_\pi) \right] m \right\}$$  \hspace{1cm} (F.22)

which again are not here for non compact case.
The product of two open string vertexes for the emission of dipole states from the $\sigma = 0$ boundary

The product (OPE) of two dipole tachyons must give the same result when computed using the original vertexes or the emission vertexes from the discharged string. We have already verified this property for all vertexes but it is worth to check it directly in a simple case. Therefore we first compute the product and then we consider its OPE at the leading order. The product reads

$$V_T(x_1, k) V_T(x_2, l) = \exp \left\{ -2\alpha' \sum_a \Delta_a(1) \left[ v_a^+ G \frac{1 + R_0^T}{2} k v\_a^+ G \frac{1 + R_0^T}{2} l \right. \\
+ v_a^+ G \frac{1 + R_0^T}{2} l v\_a^+ G \frac{1 + R_0^T}{2} k \right\} \right.$$ 

$$e^{\alpha' \sum_a \delta_a v_a^+ G \frac{1 + R_0^T}{2} (k+l) v\_a^+ G \frac{1 + R_0^T}{2} (k+l)} \times \frac{\alpha' k^T G^{-1} \hat{g}_{(0)}^{-1} k}{x_1} \frac{\alpha' \nu_T G^{-1} \hat{g}_{(0)}^{-1} l}{x_2} \\
\times \exp \left\{ -i\pi \alpha' k^T (\hat{F}_\pi - \hat{F}_\theta)^{-1} l \right\} B_0(k + l) \right.$$ 

$$\times \exp \left\{ -2\alpha' \sum_a \hat{g}_{-\nu_a} \left( \frac{x_2}{x_1} \right) v_a^+ G \frac{1 + R_0^T}{2} k v\_a^+ G \frac{1 + R_0^T}{2} l \right\} \\
: B_L(k, x_1) B_L(l, x_2) : \right.$$ 

$$\times \Lambda(0) \Lambda(0)$$ (F.23)

where the first line is the $x$ independent normalization factor, the terms between the first and the second product symbols are due to the $x$ dependent normalization factor, the terms between the second and the third product symbols are due to the zero modes, the terms between the third and the fourth product symbols are due to the non-zero modes and the last line is due to Chan-Paton factor. This last line is exactly the same as for the product of two vertexes in the dipole string.

In order to check the leading order of the OPE associated to the previous product we expand the “propagator” as

$$\hat{g}_{-\nu_a} \left( \frac{x_2}{x_1} \right) = \ln \left( 1 - \frac{x_2}{x_1} \right) + \Delta_a(1) + O(x_2 - x_1)^2$$ (F.24)

then we use the properties of $\Delta_a$ to write the pure c-number phase associated with the terms $k^T \ldots l$ as

$$\exp \left\{ -i\pi \alpha' k^T \left[ (\hat{F}_\pi - \hat{F}_\theta)^{-1} + \frac{1 + R_0}{2} \frac{1 + R}{G^{-1}} \frac{1 + R_0^T}{2} \right] l \right\}$$ (F.25)

Finally simplifying the previous phase using the $R$s definitions we get the right leading order

$$V_T(x_1, k) V_T(x_2, l) \sim (x_1 - x_2)^{2\alpha' k^T G^{-1} \hat{g}_{(0)}^{-1} l} e^{-i\pi \alpha' k^T (\hat{\Delta} D_0 + \Theta(0)) l} V_T(x_2, k + l)$$ (F.26)
without any constraint.

**F.2.7 The product of two open string vertexes for the emission of untwisted states from the \( \sigma = \pi \) boundary**

It can be verified exactly as for the \( \sigma = 0 \) case done in the previous section.

**F.2.8 The product of two closed string vertexes**

We consider finally the leading order of the product of two closed string tachyons

\[
\mathcal{W}_{T_c}(l, l_R, w, \bar{w}) \sim \exp \left\{ -i2\pi\alpha' k_L^T(\delta_l + k + l) \right\} e^{i\Phi_0(k + l, k + l)} \\
\times \exp \left\{ -i\pi\alpha' k_L^T \frac{1 + R}{1 - R} G^{-1}_L l - i\pi\alpha' R_0' \frac{1 + R}{1 - R} G^{-1}_R l \right\} \\
\times \exp \left\{ -2\alpha' \sum_c \Delta_c(1) k_L^T G v_c l_T^T G v_{-c} - 2\alpha' \sum_c \Delta_c(1) k_R^T R_0 G v_c l_T^R R_0 G v_{-c} \right\} \\
\times e^{i\alpha' \sum_c \Delta_c(1) \left[ (kL + lL)^T G v_c (kL + lL)^T G v_{-c} + (kR + lR)^T R_0 G v_c (kR + lR)^T R_0 G v_{-c} \right]} \\
\times \bar{w}^{2\alpha' k_L^T G^{-1} l - \Delta(kL + lL)} - \bar{w}^{2\alpha' k_R^T G^{-1} l - \Delta(kR + lR)} \\
\times \exp \left\{ -i\pi\alpha' (\mathcal{E}_\pi G k_L)^T (\mathcal{F}_\pi - \mathcal{F}_0)^{-1} (\mathcal{E}_\pi G l_L) \right\} B_0(\mathcal{E}_\pi^T G (kL + lL)) \\
\times \exp \left\{ -i\pi\alpha' (\mathcal{E}_\pi G k_R)^T (\mathcal{F}_\pi - \mathcal{F}_0)^{-1} (\mathcal{E}_\pi G l_R) \right\} B_0(\mathcal{E}_\pi G (kR + lR)) \\
\times \exp \left\{ 2\alpha' k_L^T G^{-1} l \ln \left( 1 - \frac{w}{z} \right) + 2\alpha' \sum_c \Delta_c(1) k_L^T G v_c l_T^T G v_{-c} \right\} B_L(kL + lL, w) \\
\times \exp \left\{ 2\alpha' k_R^T G^{-1} l \ln \left( 1 - \frac{w}{z} \right) + 2\alpha' \sum_c \Delta_c(1) k_R^T R_0 G v_c l_T^R R_0 G v_{-c} \right\} B_R(kR + lR, \bar{w}) \\
\times \exp \left\{ -i2\pi\alpha' (\mathcal{E}_\pi G k_L)^T (\mathcal{F}_\pi - \mathcal{F}_0)^{-1} (\mathcal{E}_\pi G l_L) - i4\pi\alpha' k_L^T R_0 \frac{1}{1 - R} G^{-1} l_L \right\} \right. \\
\left. \tag{F.27} \right.
\]

where the first line is from the \( c \)-number cocycle, the terms between the first and the second product symbols are due to the \( z, \bar{z} \) independent normalization factor, the terms between the second and the third product symbols are due to the \( z, \bar{z} \) dependent normalization factor, the terms between the third and the fourth product symbols are due to the zero modes, the terms between the fourth and the fifth product symbols are due to the non-zero modes and depend on \( w \), the terms between the fifth and the sixth product symbols are due to the non-zero modes with a \( \bar{w} \) dependence. Finally the last line is the due to the commutation of the \( k_R \) terms with the \( l_L \) ones.

The previous equation can then be rewritten in a way which makes clear what con-
straints we get as
\[
\mathcal{W}_T(z, \bar{z}, k_L, k_R, z, \bar{z}) \mathcal{W}_T(w, \bar{w}, l_L, l_R) \sim \\
\exp \left\{ -i2\pi\alpha' k_L^T S L_L \right\} \\
\times \exp \left\{ -i\pi\alpha' \left[ k_L^T \frac{+ R}{- R} G^{-1} l_L + k_R^T R_0 \frac{+ R}{- R} G^{-1} R_0^T l_R + 4k_R^T R_0 \frac{\cancel{\frac{+ R}{- R} G^{-1} l_L}}{\cancel{\frac{+ R}{- R} G^{-1} l_L}} \right] \right\} \\
\times \exp \left\{ -i\pi\alpha' (\mathcal{E}_\pi^T G k_L)^T (\mathcal{F}_\pi - \mathcal{F}_0)^{-1} (\mathcal{E}_\pi^T G l_L) - i\pi\alpha' (\mathcal{E}_\pi G k_R)^T (\mathcal{F}_\pi - \mathcal{F}_0)^{-1} (\mathcal{E}_\pi^T G l_R) \right\} \\
\times (z-w)^{2\alpha' k_L^T G^{-1}l_L} (\bar{z}-\bar{w})^{2\alpha' k_R^T G^{-1}l_R} \mathcal{W}(k_L+l_L, k_R+l_R, w, \bar{w}) \tag{F.28}
\]

We can now compare with the closed string OPEs which in the non compact case with \( k_L = k_R \) and \( l_L = l_R \) reads
\[
\mathcal{W}_T(z, \bar{z}, k_L, k_R) \mathcal{W}_T(w, \bar{w}, l_L, l_R) \sim \\
(z-w)^{2\alpha' k_L^T G^{-1}l_L} (\bar{z}-\bar{w})^{2\alpha' k_R^T G^{-1}l_R} \mathcal{W}(w, \bar{w}, k_L+l_L, k_R+l_R) \tag{F.29}
\]

Simplifying the phase we get the following constraint:
\[
\exp \left\{ -i2\pi\alpha' k_L^T \left[ S - G^{-1} \right] l_L \right\} = 1 \tag{F.30}
\]
from which we fix the matrix \( S \) which enters the cocycle to be \( S = G^{-1} \).

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