Automorphic hyperfunctions and period functions

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1 Introduction

Lewis, [6], has given a relation between Maass forms and period functions. This paper investigates that relation by means of the invariant hyperfunctions attached to automorphic forms.

1.1 Maass forms. A cuspidal Maass form is a function on the upper half plane \( \mathbb{H}^+ = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \) that satisfies \( u(-1/z) = u(z) \) and has an expansion

\[
u(z) = \sum_{n \neq 0} a_n W_{0,s-1/2}(4\pi |n| y)e^{2\pi inx}.
\]

We write \( x = \text{Re} \, z \) and \( y = \text{Im} \, z \) for \( z \in \mathbb{H}^+ \), and use the Whittaker function \( W_\nu(\cdot) \), see, e.g., [12], 1.7. One can express \( W_\nu \) in terms of a modified Bessel function:

\[
W_\nu(y) = \sqrt{y/\pi} K_\nu(y/2).
\]

These Maass forms occur as eigenfunctions in the spectral decomposition of the Laplacian in \( L^2(\Gamma_\text{mod} \backslash \mathbb{H}^+, dx dy/y^2) \), with \( \Gamma_\text{mod} := \text{PSL}_2(\mathbb{Z}) \). The eigenvalue is \( s(1-s) \). For any given \( s \) the space of such Maass forms has finite dimension. The dimension is non-zero only for an infinite discrete set of points on the line \( \text{Re} \, s = \frac{1}{2} \). For more information concerning Maass forms see, e.g., [14], §3.5–6.

One calls a Maass form even, respectively odd, if \( u(-\bar{z}) = u(z) \), respectively \( u(-\bar{z}) = -u(z) \). In terms of the Fourier coefficients this amounts to \( a_{-n} = a_n \), respectively \( a_{-n} = -a_n \).

Although spectral theory states that cuspidal Maass forms exist, none of them is explicitly known. There are computational results, see, e.g., [13], [2], and [3].

1.2 Period functions. In [3] and [7], Lewis and Zagier show that there is a bijective linear map from the space of cuspidal Maass forms of weight 0 for a fixed value of \( s \) to the space of holomorphic functions \( \psi : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C} \) that satisfy

\[
\psi(z) - \psi(z+1) = (z+1)^{-2s} \psi \left( \frac{z}{z+1} \right),
\]

and \( \psi(1) = 0 \), \( \lim_{z \to \infty, z \in \mathbb{R}} \psi(z) = 0 \). In [2] this bijection is given in terms of a sequence of integral transforms; the approach in [3] uses the \( L \)-series attached to the Maass form. In [15] Zagier gives indications that the function \( \psi \) generalizes
the period polynomial associated to holomorphic cusp forms. So the name period function is appropriate.

For even, respectively odd Maass forms, Lewis and Zagier show that $\psi(z) = \pm z^{-2s} \psi\left(\frac{1}{z}\right)$. Under this assumption, equation (1.2) is equivalent to

$$w(z) - w(z + 1) = \pm z^{-2s} \psi\left(1 + \frac{1}{z}\right). \quad (1.3)$$

Applying (1.3) to $z, z + 1, z + 2, \ldots$, and using the behavior as $z \to \infty$, Lewis and Zagier find

$$\pm \psi(z) = \sum_{n=0}^{\infty} (z + n)^{-2s} \psi\left(1 + \frac{1}{z + n}\right). \quad (1.4)$$

This means that $\psi$ corresponds to an eigenfunction of the transfer operator of Mayer, [9]. Theorem 2 in [9] shows that the $\pm 1$-eigenvectors of the transfer operator are closely related to the zeros of the Selberg zeta function. Lewis remarks that not only the cuspidal Maass forms, but also some Eisenstein series should yield eigenvectors. Zagier, [15], indicates how to obtain period functions by meromorphic continuation of a partial Eisenstein series. For $\zeta(2s) = 0$ these functions are eigenfunctions of the transfer operator.

1.3 Boundary form. Lewis, [6], §6 (c), gives formal computations with the boundary form associated to a Maass form as a motivation for his method. The present paper arose from the wish to make this precise, and to understand the map $u \mapsto \psi$ from Maass forms to period functions in terms of the boundary form.

The boundary forms that we use are hyperfunctions on the boundary of the upper half plane. These hyperfunctions are $SL_2(\mathbb{Z})$-invariant vectors in a principal series representation. Actually, the hyperfunctions related to the most interesting automorphic forms are distributions; that aspect we do not discuss in this paper.

The hyperfunction point of view turns out to give two interpretations of the period function $\psi$. The first one, in Theorem 5.11, arises naturally when describing any hyperfunction associated to a modular form (even for exponentially increasing modular forms). The second interpretation is more complicated. We shall consider a type of parabolic cohomology with values in the hyperfunctions. In Proposition 9.7 we show that the hyperfunctions associated to a class of modular forms (containing the cuspidal Maass forms and some Eisenstein series) correspond to classes in these cohomology groups. In Section 10 we use a map from hyperfunctions to holomorphic functions on the upper half plane to arrive at cohomology classes with holomorphic functions as values. Such a class is determined by one function that turns out to satisfy (1.2), but with $s$ replaced by $1 - s$.

It should be emphasized that we do not recover all results of Lewis and Zagier. We do not prove that each period function satisfying (1.2) with the prescribed behavior at 1 and $\infty$ comes from a Maass form.
1.4 Geodesic decomposition. To obtain the second interpretation of the period function, we use what we call $\Gamma$-decompositions of hyperfunctions.

Let the boundary of the upper half plane be written as a finite union $\bigcup_{j=1}^{n} I_j$, with closed intervals $I_j$ that intersect each other only in their end points, and where the end points are cusps. Any hyperfunctions $\alpha$ on the boundary of $\mathfrak{H}^+$ can be written as a sum $\sum_{j=1}^{n} \alpha_j$, such that the support of the hyperfunction $\alpha_j$ is contained in $I_j$. There are many possibilities to arrange this. For the hyperfunctions associated to cuspidal Maass forms, holomorphic modular cusp forms, and some Eisenstein series, this can be done in a neat way, which we shall call the geodesic decomposition. We shall show in 7.14 that the cocycles attached to holomorphic cusp forms can be derived from this decomposition.

1.5 Overview. From the representational point of view it is more convenient to work with modular forms on the group $\text{PSL}_2(\mathbb{R})$ than on the upper half plane $\mathfrak{H}$. This step is carried out in Section 2. Actually, we do not restrict ourselves to the modular group, but work with a general cofinite discrete subgroup $\Gamma$, that is required to possess cusps. Section 3 discusses some properties of hyperfunctions. Section 4 recalls facts concerning the principal series of representations of $\text{PSL}_2(\mathbb{R})$. In Section 5 we give the relation between automorphic forms and invariant hyperfunctions.

The subject of Section 7 is the geodesic decomposition of hyperfunctions associated to automorphic forms with polynomial growth. As a preparation we discuss in Section 6 the Fourier expansion at a cusp, which we put at $\infty$. The condition of polynomial growth is only imposed at $\infty$. At other (not $\Gamma$-equivalent) cusps the growth may be arbitrary.

The geodesic decomposition represents the invariant hyperfunction as a finite sum. An infinite sum is considered in Section 8. This is related to the transfer operator, discussed in Section 14.

In Section 8 we reformulate our results on the universal covering group of $\text{PSL}_2(\mathbb{R})$, and give a cohomological interpretation of $\Gamma$-decompositions. In Section 14 we return to the period function $\psi$.

1.6 Thanks. I thank E.P. van den Ban, J.J. Duistermaat, J.B. Lewis and D. Zagier for their interest, help, and useful discussions.

Many of the ideas in this paper are present in the work of Lewis, or have been the subject of our discussions during Lewis’s visits to Utrecht. Zagier has brought the work of Lewis to my attention, and has shown interest in this approach. Van den Ban showed me the argument in 3.4. Over the years Duistermaat has repeatedly told me that invariant boundary forms should give insight into automorphic forms.
2 Automorphic forms

2.1 Examples of modular forms. In Section 1 we have already seen cuspidal Maass forms. For Re $s > 1$ the Eisenstein series is given by

$$G(s; z) = \frac{\Gamma(s)}{\pi^s} \sum_{p,q \in \mathbb{Z}}' \frac{y^s}{|qz + p|^{2s}}.$$  

(2.1)

The prime denotes that $(p, q) = (0, 0)$ is omitted. From this one can derive the following Fourier expansion:

$$G(s; z) = 2\Lambda(2s)y^s + 2\Lambda(2s - 1)y^{1-s} + \sum_{n \neq 0} \frac{2\sigma_{2s-1}(|n|)}{|n|^s} W_{0, s-1/2}(4\pi|n|y)e^{2\pi inz},$$

(2.2)

with $\Lambda(u) = \pi^{-u/2} \Gamma\left(\frac{u}{2}\right) \zeta(u)$, $\zeta$ the zeta function of Riemann, and the divisor sum $\sigma_v(m) = \sum_{d|m} d^v$. The Fourier expansion defines $G(s; z)$ for all $s \in \mathbb{C}$ except $s = 0, 1$. We have $G(s; -1/z) = G(s; z)$.

Holomorphic modular cusp forms occur for even “weights” $2k = 12$ and $2k \geq 16$. They have a Fourier expansion of the form $h(z) = \sum_{n=1}^{\infty} c_ne^{2\pi inz}$ and satisfy $h(-1/z) = z^{2k}h(z)$.

These various types of modular forms can be unified by working on the group $\operatorname{PSL}_2(\mathbb{R})$.

2.2 Notations. Put $G := \operatorname{PSL}_2(\mathbb{R})$. Elements of $G$ are indicated by a representative in $\operatorname{SL}_2(\mathbb{R})$. So $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ denote the same element of $G$.

Notations: $k(\theta) := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, and $p(z) := \begin{pmatrix} \sqrt{\pi x}/\sqrt{\pi y} \\ 0 \end{pmatrix}$ for $z \in \mathfrak{H}^+$, $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.

Conjugation by $\begin{pmatrix} -1 \ 0 \\ 0 \ 1 \end{pmatrix} \in \operatorname{PGL}_2(\mathbb{R})$ gives an outer automorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto j \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $G$; it is an involution.

Elements of $G$ act on the upper half plane $\mathfrak{H}^+$ by fractional linear transformations: $z \mapsto (a \ b) \begin{pmatrix} 1 \ c \ d \\ 0 \ 1 \end{pmatrix} z : = \frac{az + b}{cz + d}$. This action is the restriction of the action of $G$ on the complex projective line $\mathbb{P}_\mathbb{C}^1 \supset \mathfrak{H}^+$ defined by the same formula.

The Lie algebra $\mathfrak{g}_r$ of $G$ is generated by $H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. By $\mathfrak{g}$ we denote its complexification $\mathfrak{g}_r \otimes \mathbb{C}$. A convenient basis of $\mathfrak{g}$ is $\mathfrak{W}$, $\mathfrak{E}^+$, $\mathfrak{E}^-$, with $\mathfrak{W} = X - Y$ and $\mathfrak{E}^\pm := H \pm i(X + Y) \in \mathfrak{g}$. The Casimir operator is $\omega := -\frac{1}{4}\mathfrak{E}^+\mathfrak{E}^- + \frac{1}{4}\mathfrak{W}^2 - \frac{1}{2}\mathfrak{W}$; it determines a bi-invariant differential operator on $G$.

$N := \{ (1 \ x) : x \in \mathbb{R} \}$ is a unipotent subgroup of $G$, and $A := \{ p(iy) : y > 0 \}$ a real torus of dimension 1. $P := NA$ is a parabolic subgroup of $G$. The group $K := \{ k(\theta) : \theta \in \mathbb{R} \mod \pi \mathbb{Z} \}$ is a maximal compact subgroup of $G$. As Haar a measure on $K$ we use $dk = \frac{1}{2} d\theta$, with $k = k(\theta)$.

2.3 Discrete subgroup. We consider a cofinite discrete subgroup $\Gamma$ of $G$ with at least one cuspidal orbit. By conjugation we arrange that $\infty$ is a cusp of $\Gamma$, and
that \( p(i + 1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) generates the subgroup \( \Gamma_\infty \) of elements of \( \Gamma \) that fix \( \infty \). Note that \( \Gamma \) is allowed to have more than one \( \Gamma \)-orbit of cusps.

The fundamental example in this paper is the modular group \( \Gamma_{\text{mod}} := \text{PSL}_2(\mathbb{Z}) \). Here the set of cusps is \( \mathbb{P}_1 \); it consists of one \( \Gamma_{\text{mod}} \)-orbit. The elements \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = k(\pi/2) \) generate \( \Gamma_{\text{mod}} \).

2.4 Automorphic forms. By an automorphic form we mean a function \( u : G \to \mathbb{C} \) that satisfies

i) \( u(\gamma g) = u(g) \) for all \( \gamma \in \Gamma \),

ii) \( u(gk(\theta)) = u(g)e^{ir\theta} \) for all \( k(\theta) \in K \), for some \( r \in 2\mathbb{Z} \), the weight,

iii) \( \omega u = s(1 - s)u \) for some \( s \in \mathbb{C} \), the spectral parameter.

Note that there are no growth conditions. This definition is insensitive to the change \( s \mapsto 1 - s \) in the spectral parameter.

2.5 From upper half plane to group. Let \( u \) be a cuspidal Maass form as in 1.1, and put \( u_0(p(z)k(\theta)) := u(z) \). It is not difficult to check that \( u_0 \) is an automorphic form for \( \Gamma_{\text{mod}} \) with weight 0 and eigenvalue \( s(1 - s) \). The same holds for the Eisenstein series. We use the same notation for \( z \mapsto E(s; z) \) and \( p(z)k(\theta) \mapsto E(z; p(z)k(\theta)) := E(s; z) \).

To a holomorphic cusp form \( H \) of weight 2 we associate the function \( h(p(z)k(\theta)) := y^k H(z)e^{2i k \theta} \). This is an automorphic form of weight 2 k with eigenvalue \( k - k^2 \).

2.6 Each automorphic form is determined by the function \( z \mapsto u(p(z)) \) on \( \mathcal{S}^+ \), and satisfies an elliptic differential equation. So it is a real analytic function.

The Lie algebra acts by differentiation on the right. For an automorphic form \( u \) with weight \( r \) and spectral parameter \( s \) we have \( W u = iru \), and \( E^\pm u \) is an automorphic form of weight \( r \pm 2 \), with the same spectral parameter. \( E^+ E^- u \) is always a multiple of \( u \). If \( u \) is an automorphic form on \( G \) with weight \( 2k \), then the function \( z \mapsto y^{-k} u(p(z)) \) is holomorphic if and only if \( E^- u = 0 \).

Automorphic forms for the group \( \Gamma_{\text{mod}} \) are called modular forms.

2.7 Reflection. We define the involution \( j \) on functions on \( G \) by \( j f : g \mapsto f(j(g)) \). It satisfies \( j \circ W = -W \circ j \), and \( j \circ E^\pm = E^\mp \circ j \).

If the involution \( j \) leaves \( \Gamma \) invariant (as is the case for \( \Gamma_{\text{mod}} \)), then \( j \) preserves \( \Gamma \)-invariance on the left, and maps automorphic forms to automorphic forms with the same spectral parameter and opposite weight. The corresponding eigenspace decomposition in weight 0 gives the decomposition of Maass forms in even and odd ones.

3 Hyperfunctions

We consider the sheaves of hyperfunctions on the real line \( \mathbb{R} \) and on the circle \( T := \mathbb{R} \mod \pi\mathbb{Z} \). For a point of view that works in higher dimension we refer to, e.g., [1].
3.1 Holomorphic and analytic functions. Let $\mathcal{O}$ denote the sheaf of holomorphic functions on the complex projective line $\mathbb{P}_C^1$.

A real analytic function on an open set $U \subset \mathbb{R}$ is the restriction of an element of $\mathcal{O}(W)$, where $W \supset U$ is an open set in $\mathbb{C}$, that may depend on the function. So the sheaf $\mathcal{A}$ of real analytic functions on $\mathbb{R}$ is the restriction $\mathcal{O}|_\mathbb{R}$. In the sequel we use ‘analytic’ as abbreviation of ‘real analytic’, and say ‘holomorphic’ when we mean ‘complex analytic’.

3.2 Hyperfunctions on $\mathbb{R}$. (See [1], §1.1–3 for proofs and further information.) Let $U \subset \mathbb{R}$ be open, and choose some open $W \subset \mathbb{C}$ such that $U \subset W$. Hyperfunctions on $U$ are elements of $\mathcal{O}(W \smallsetminus U)$ mod $\mathcal{O}(W)$. This does not depend on the choice of $W$. We denote the linear space of hyperfunctions on $U$ by $\mathcal{B}(U)$. This defines the sheaf $\mathcal{B}$ of hyperfunctions on $\mathbb{R}$. Intuitively, a hyperfunction represented by $g \in \mathcal{O}(W \smallsetminus U)$ is the jump in $g$ when we cross $U$. A more fancy definition of the sheaf of hyperfunctions is $\mathcal{B} = \mathcal{H}_B^0(\mathbb{C}, \mathcal{O})$ (sheaf cohomology).

Multiplication of representatives makes $\mathcal{B}$ into an $\mathcal{A}$-module. We map $\mathcal{A}(U)$ into $\mathcal{B}(U)$ by sending $g \in \mathcal{O}(V)$, with $V \subset \mathbb{C}$ open, $V \supset U$, to the hyperfunction represented by $\theta \mapsto g(\theta)$ on $V \cap \{\theta > 0\}$ and $\theta \mapsto 0$ on $\{\theta < 0\} = \{z \in \mathbb{C} : \Im z < 0\}$.

3.3 Support. The support $\text{Supp}(\alpha)$ of a hyperfunction $\alpha \in \mathcal{B}(U)$ is the smallest closed subset $C \subset U$ such that the restriction of $\alpha$ to $U \smallsetminus C$ is zero. A representative $g \in \mathcal{O}(W)$ of $\alpha$ extends holomorphically to the points of $U \smallsetminus \text{Supp}(\alpha)$.

3.4 Parting. The sheaf $\mathcal{B}$ is flasque. This means that the restriction maps $\mathcal{B}(V) \to \mathcal{B}(U)$ are surjective for all open $U \subset V \subset \mathbb{R}$.

Any $\alpha \in \mathcal{B}(I)$ can be broken up at each point $a \in I$: We can write $\alpha = \alpha_+ + \alpha_-$ with $\alpha_{\pm} \in \mathcal{B}(I)$, $\text{Supp}(\alpha_-) \subset [a, \infty) \cap I$, $\text{Supp}(\alpha_+) \subset (-\infty, a] \cap I$. Indeed, consider $\beta \in \mathcal{B}(I \smallsetminus \{a\})$ that restricts to $\alpha$ on $I \cap (a, \infty)$ and to 0 on $I \cap (-\infty, a)$. The flasqueness implies that there is an element of $\mathcal{B}(I)$ restricting to $\beta$ on $I \smallsetminus \{a\}$. This element we take as $\alpha_+$, and $\alpha_- := \alpha - \alpha_+$.

We call the decomposition $\alpha = \alpha_+ + \alpha_-$ a parting of $\alpha$ at $a$. It is well defined in the stalk $\mathcal{B}_a$. Al partings of $\alpha$ at $a$ are obtained by replacing $\alpha_{\pm}$ by $\alpha_{\pm} \pm \nu$, where $\nu \in \mathcal{B}(I)$ satisfies $\text{Supp} \nu \subset \{a\}$.

3.5 Duality. Let $\mathcal{B}_b(I) := \{\alpha \in \mathcal{B}(I) : \text{Supp}(\alpha) \text{ is bounded}\}$ be the space of hyperfunction on the open interval $I$ with compact support. A duality between $\mathcal{A}(\mathbb{R})$ and $\mathcal{B}_b(\mathbb{R})$ is given by

$$\langle \varphi, \alpha \rangle := \int_C \varphi(\theta) g(\theta) \frac{d\theta}{\pi}$$

for $\varphi \in \mathcal{A}(\mathbb{R})$, $g \in \mathcal{O}(W \smallsetminus \text{Supp}(\alpha))$ a representative of $\alpha \in \mathcal{B}_b(\mathbb{R})$ and $C$ any contour around $\text{Supp}(\alpha)$ contained in $W$ and in the domain of a holomorphic function extending $\varphi$, see Figure 9.3. The use of the variable $\theta$ on $\mathbb{C}$, and the measure $\frac{d\theta}{\pi}$ will become clear in §9.2.

3.6 Reflection. $j \varphi(\theta) := \varphi(-\theta)$ defines an involution $j$ in $\mathcal{A}(\mathbb{R})$. We define the involution $j$ in $\mathcal{B}(\mathbb{R})$ by the action $g \mapsto -jg$ on representatives. In this way $j$ respects the injection $\mathcal{A}(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$ and satisfies $(j \varphi, j \alpha) = \langle \varphi, \alpha \rangle$.

3.7 The circle $T$. The fact that $\mathcal{B}$ is a sheaf means that the definition of hyperfunctions is local, and can be transferred to any real manifold of dimension 1.
We need hyperfunctions on the circle \( T := \mathbb{R} \mod \pi \mathbb{Z} \). There are many ways to embed \( T \) into \( \mathbb{P}^1_{\mathbb{C}} \), for example, by \( \theta \mapsto e^{2i\theta} \) we view \( T \) as the unit circle in \( \mathbb{P}^1_{\mathbb{C}} \). In the sequel it is convenient to identify \( T \) to the common boundary of the upper half plane \( \mathcal{H}^+ \) and the lower half plane \( \mathcal{H}^- := \{ z \in \mathbb{C} : \text{Im } z < 0 \} \). This we accomplish by the map \( \text{pr} : \mathbb{R} \to \mathbb{P}^1_{\mathbb{C}} : \theta \mapsto \cot \theta \).

We use the standard cyclic ordering on \( T \) induced by \( \mathbb{R} \subset T \). Intervals in \( T \) are formed with respect to this ordering: \([-1, 1]\) is the same as the corresponding interval in \( \mathbb{R} \), but \([1, -1] = [1, \infty) \cup \{ \infty \} \cup (-\infty, -1] \). The map \( \text{pr} \) is strictly decreasing. In \([12]\) we shall explain why we do not choose the increasing map \( \theta \mapsto -\cot \theta \).

3.8 Analytic functions and hyperfunctions. We define the sheaves \( \mathcal{A}_T \) of analytic functions on \( T \), and \( \mathcal{B}_T \) of hyperfunctions on \( T \) in the same way as above: analytic functions on \( U \subset T \) are the restrictions of holomorphic functions on some open set in \( \mathbb{P}^1_{\mathbb{C}} \) containing \( U \), and hyperfunctions on \( U \) are the elements of \( \mathcal{O}(W \setminus U) \mod \mathcal{O}(W) \) for any fixed \( W \supset U \).

3.9 Duality. There is a duality between \( \mathcal{A}_T(T) \) and \( \mathcal{B}_T(T) \) given by

\[
\langle f, \alpha \rangle := \int_{C_+} f(\tau)g(\tau) \frac{d\tau}{\pi(1 + \tau^2)} + \int_{C_-} f(\tau)g(\tau) \frac{d\tau}{\pi(1 + \tau^2)},
\]

for \( \varphi \in \mathcal{A}_T(T) \), \( g \) a representative of \( \alpha \in \mathcal{B}_T(T) \), and \( C_\pm \) contours in the intersections of the domains of representatives; see Figure \( \ref{fig:contours} \), and note the orientation.
3.10 Representative. The hyperfunction $\alpha \in \mathcal{B}_T(T)$ can be recovered from the linear form $\varphi \mapsto \langle \varphi, \alpha \rangle$ on $\mathcal{A}_T(T)$. For each $\tau_0 \in \mathbb{C} \setminus \mathbb{R}$ the function $h_{\tau_0} : \tau \mapsto \frac{1}{2\pi} \frac{\log |\tau - \tau_0|}{\tau - \tau_0}$ determines an element of $\mathcal{A}_T(T)$. One can check that $g(\tau_0) := \langle h_{\tau_0}, \alpha \rangle$ is the unique representative of $\alpha$ that is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies $g(i) + g(-i) = 0$.

3.11 Reflection. The reflection on $\mathbb{R}$ corresponds to the reflection induced by $\tau \mapsto -\tau$ in $\mathbb{P}_C^1$. It defines an involution $\jmath$ in $\mathcal{A}_T(T)$ and $\mathcal{B}_T(T)$ (again use an additional minus sign in the representatives of hyperfunctions). This involution respects the embedding $\mathcal{A}_T \to \mathcal{B}_T$ and leaves the duality invariant.

3.12 Basis. For each $r \in \mathbb{Z}$ we define $\varphi_{2r} : \tau \mapsto \left( \frac{t + i}{t} \right)^r$ in $\mathcal{A}_T(T) \subset \mathcal{B}_T(T)$.

We have $\langle \varphi_{2r}, \varphi_{2n} \rangle = \delta_{r+n}$.

Any $\varphi \in \mathcal{A}_T(T)$ has an expansion $\varphi = \sum c_r \varphi_{2r}$, corresponding to a Laurent expansion converging on an annulus of the form $p^{-1} < \left| \frac{t + i}{t} \right| < p$ for some $p > 1$.

Any $\alpha \in \mathcal{B}_T(T)$ can be represented by a function that is of the form $-\frac{1}{2} d_0 - \sum_{r=-\infty}^{\infty} d_r \left( \frac{t + i}{t} \right)^r$ on $\mathcal{H}^-$, and $\frac{1}{2} d_0 + \sum_{r=-\infty}^{\infty} d_r \left( \frac{t + i}{t} \right)^r$ on $\mathcal{H}^+$. The condition on the coefficients is

$$d_r = O(p^{|r|}) \quad \text{as} \quad |r| \to \infty \quad \text{for each} \quad p > 1. \quad (3.1)$$

This shows that $\mathcal{B}_T(T)$ corresponds to the space of linear forms $L : \mathcal{A}_T(T) \to \mathbb{C}$ that satisfy $L \varphi_{2r} = O \left( p^{(|r|)} \right)$ as $|r| \to \infty$ for each $p > 1$.

4 Principal series of representations

4.1 Induced representation. For each $\nu \in \mathbb{C}$ we denote by $\pi_\nu$ the representation of $G$ by right translation in the space $M^\nu$ of classes of functions $f : G \to \mathbb{C}$ satisfying $f(p \cdot z \cdot y) = y^{(1+\nu)/2}$ and $\int_{K} |f(k)|^2 \, dk < \infty$. This gives the induced representation of $G$ corresponding to a character of the parabolic subgroup $P = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \right\}$.

$M^\nu$ is the Hilbert space $L^2(K, dk)$ with a $G$-action $\pi_\nu$ depending on $\nu$. This representation is bounded. Under the pairing $(f_1, f_2) \mapsto \int_{K} f_1(k)f_2(k) \, dk$ the representations $\pi_\nu, M^\nu$ and $\pi_{-\nu}, M^{-\nu}$ are dual to each other. See, e.g., [7], Chap. III, §2. Any $\nu \in \mathbb{C}$, the letter $H$ is used to indicate these spaces. We employ $M$ to avoid confusion with cohomology groups.

4.2 Realization of the induced representation. The elements of $M^\nu$ are sections of a line bundle over $P \setminus G \cong K$. Here we view $P \setminus G$ as $T \subset \mathbb{P}_{\mathbb{C}}^1$, the boundary of $\mathcal{H}^+$. We identify $Pk(\theta)$ with $\tau = \cot \theta$. The right translation in $P \setminus G$ by $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in G$ corresponds to $\tau \mapsto g^{-1} \cdot \tau = \frac{d \tau - b}{c \tau + a}$.

If we would have chosen $Pk(\theta) \mapsto -\cot \theta$, then the action would correspond to $\tau \mapsto j(g)^{-1} \cdot \tau = \frac{-d \tau - b}{-c \tau + a}$. The presence of $j$ in this formula we dislike so much, that we accept that $\theta \mapsto \cot \theta$ inverts the order.
In terms of the variable \( \tau \) we find:

\[
\pi_\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(\tau) = \left( \frac{1 + \tau^2}{(ct - a)^2 + (dt - b)^2} \right)^{(1+\nu)/2} \varphi \left( \frac{d \tau - b}{-ct + a} \right),
\]

\[
\langle f_1, f_2 \rangle = \frac{1}{\pi} \int_T f_1(\tau)f_2(\tau) \frac{d\tau}{1+\tau^2}.
\]

### 4.3 Reflection

The reflection \( j \) in \( M' \) considered as a space of functions on \( G \) corresponds to the reflection \( j \varphi(\tau) = \varphi(-\tau) \) in the functions on \( T \). It satisfies \( j \pi_\nu(g) = \pi_\nu(j(g)) \).

### 4.4 Analytic functions and hyperfunctions

The formula defining \( \pi_\nu(g) \) in the functions on \( T \) preserves analyticity. Let \( M_\omega' := \mathcal{A}_T(T) \) be the space of analytic functions on \( T \). So \( (\pi_\nu, M_\omega') \) is an algebraic subrepresentation of \( (\pi_\nu, M') \). The factor \( \left( \frac{1 + \tau^2}{(ct - a)^2 + (dt - b)^2} \right)^{(1+\nu)/2} \) is holomorphic on a neighborhood of \( T \) in \( \mathbb{P}_C^1 \).

The formula defining \( \pi_\nu(g) \) also makes sense when applied to representatives of hyperfunctions. Let \( M'_\omega := \mathcal{B}_T(T) \) be the space of hyperfunctions on \( T \). This gives a representation \( (\pi_\nu, M'_\omega) \). We have \( (\pi_\nu(g)\varphi, \alpha) = \langle \varphi, \pi_{-\nu}(g)^{-1}\alpha \rangle \) for \( \varphi \in M'_\omega \) and \( \alpha \in M_\omega' \).

In between \( M_\omega' \) and \( M' \) there is the \( \pi_\nu(G) \)-invariant space \( C^\infty(T) \). Its dual, the space of distributions, sits between \( M' \) and \( M'_\omega \). We do not consider these spaces in this paper.

### 4.5 \( K \)-finite vectors

All elements of \( M_\omega' \) are differentiable vectors of \( (\pi_\nu, M') \). The action of \( g \) satisfies \( d\pi_\nu(W)\varphi_{2r} = 2ir\varphi_{2r} \) and \( d\pi_\nu(E^k)\varphi_{2r} = (1 + \nu \pm 2r)\varphi_{2r\pm 2} \). The reflection satisfies \( j\varphi_{2r} = \varphi_{-2r} \).

Let \( M_k^+ \subset M_\omega' \) be the space of finite linear combinations of \( \varphi_{2r} \). It is invariant under \( d\pi_\nu(g) \). The \( (g, K) \)-modules \( (d\pi_\nu, M_k^+) \) have been classified, see, e.g., \cite{6}, Chap. VI, §5. We note the following facts:

\( d\pi_\nu(M_k^+) \) is irreducible if and only if \( \nu \in \mathbb{C} \setminus (1 + 2\mathbb{Z}) \). In this case \( (\pi_\nu, M_k^+) \) and \( (\pi_{-\nu}, M_{-k}^-) \) are isomorphic. The isomorphism is determined up to a factor. We choose \( \iota(\nu) : M_k^+ \to M_{-k}^- \) given by

\[
\iota(\nu)\varphi_{2r} = \left( \frac{1 - \nu}{2} \right)_{|r|} \left( \frac{1 + \nu}{2} \right)_{|r|}^{-1}.
\]

We have \( \iota(-\nu)\iota(\nu) = 1 \).

If \( \nu > 0 \) is odd, then \( M_k^+ \) has two irreducible subspaces \( D_k^+(\nu + 1) := \bigoplus_{2r \geq \nu + 1} \mathbb{C}\varphi_{2r} \) and \( D_k^-(\nu + 1) := \bigoplus_{2r \leq \nu - 1} \mathbb{C}\varphi_{2r} \). These are the discrete series representations.

If \( \nu < 0 \) is odd, then there is the irreducible finite dimensional subspace \( E(\nu + 1) := \bigoplus_{|2r| < \nu - 1} \mathbb{C}\varphi_{2r} \). For \( k \in \mathbb{N} := \mathbb{Z}_{\geq 1} \) there are the following exact sequences of \( (g, K) \)-modules:

\[
0 \to \begin{array}{c} D_k^+(2k) \oplus D_k^-(2k) \to \end{array} M_{2k}^{+,-} \to \begin{array}{c} E(2k) \to \end{array} 0
\]

\[
0 \to \begin{array}{c} E(2k) \to \end{array} M_{2k}^{+,-} \to \begin{array}{c} D_k^+(2k) \oplus D_k^-(2k) \to \end{array} 0
\]
The homomorphisms are unique up to a factor. A possible choice for the first one is the continuation of \(i(\nu)\) to \(\nu = 2k - 1\), and for the second one \(\text{res}_{\nu=1-2k} i(\nu)\).

**4.6 Extension.** Let \(\nu \in \mathbb{C} \setminus (1 + 2\mathbb{Z})\). The factor \(\left(\frac{1+\nu}{2}\right)|_r \left(\frac{1+\nu}{2}\right)|_{2r}^{-1}\) has polynomial growth as \(|r| \to \infty\). This implies that we have extensions \(i(\nu) : M_{\omega}^\nu \to M_{\omega}^{-\nu}\) and \(i(\nu) : M_{\omega}^\nu \to M_{\omega}^{-\nu}\). These isomorphisms respect the \(G\)-action, and satisfy \(\langle \varphi, i(\nu)\alpha \rangle = \langle i(\nu)\varphi, \alpha \rangle\).

**4.7 The space \(E(2k)\).** Let \(k \in \mathbb{N}\). Multiplication by \((\tau^2 + 1)^{k-1}\) gives a bijection from \(E(2k)\) onto the polynomials in \(\tau\) of degree at most \(2k - 2\). The action \(\pi_{1-2k} \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\) corresponds to \(F \mapsto F|_{2-2kg}^{-1}\), where \(\left(F|_{2r} \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right)(\tau) := (c\tau + d)^{-2r}F(\frac{ac + bd}{c^2 + d^2})\).

**4.8** For \(k \in \mathbb{N}\) we give a map \(\alpha \mapsto \alpha^{(2k)}\) from \(M_{\omega}^{2k-1}\) to the polynomials of degree at most \(2k - 2\), extending the composition \(M_{\omega}^{2k-1} \to E(2k) \to (\text{polynomials})\), by

\[
\alpha^{(2k)}(X) := \langle h_X, \alpha \rangle, \text{ with } h_X(\tau) := -(2i)^{2k-2}(\tau^2 + 1)^{1-k}(\tau - X)^{2k-2}.
\]

A computation shows that \(\alpha \mapsto \alpha^{(2k)}\) respects the \(G\)-action, and vanishes on \(\sum d_r \varphi_{2r} \in M_{\omega}^{2k-1}\) with \(d_r = 0\) for \(|r| \geq k\).

## 5 Automorphic hyperfunctions

**5.1 Poisson integral.** In Theorem 3 of [4], Helgason shows that all eigenfunctions of the Laplacian in the non-Euclidean plane can be described as the Poisson integral of a hyperfunction. In our notation this result states that each eigenfunction \(F \in C^\infty(G/K)\) of the Casimir operator \(\omega\) with eigenvalue \(\frac{1}{4}(1 - \nu^2)\) can be written as

\[
F(p(z)k(\theta)) = \langle \pi_{-\nu}(p(z))\varphi_0, \alpha \rangle
\]

for some \(\alpha \in M_{\omega}^\nu = B_{T}(\Gamma)\). In [4], §5, especially 5.5, we see that this is only a very special case of general results for symmetric spaces.

**5.2 Hyperfunctions and \((g, K)\)-modules.** Let \(\alpha \in M_{\omega}^\nu\). For each \(\varphi \in M_{\omega}^{-\nu}\) we put \(T_\alpha\varphi(g) := \langle \pi_{-\nu}(g)\varphi, \alpha \rangle\). This defines a linear map \(T_\alpha : M_{\omega}^{-\nu} \to C^\infty(G)\) that intertwines \(\pi_{-\nu}\) with the action of \(G\) by right translation.

If we restrict \(T_\alpha\) to \(M_{\omega}^{-\nu}\) we get a \((g, K)\)-module in \(C^\infty(G)\) that is isomorphic to a quotient of \(M_{\omega}^{-\nu}\). Conversely, Helgason’s proof can easily be generalized to show that each such \((g, K)\)-module is described by an unique \(\alpha \in M_{\omega}^\nu\).

Under this correspondence the property \(\pi_{\nu}(\gamma)\alpha = \alpha\) for some \(\gamma \in G\) is equivalent to \((T_\alpha\varphi)(\gamma g) = (T_\alpha\varphi)(g)\) for all \(\varphi \in H_{\omega}^{-\nu}\). Actually, it suffices to let \(\varphi\) run through the \(\varphi_{2r}\).

**5.3 Definition.** Let \(A_{\omega}^\nu(\Gamma)\) be the space of \(\alpha \in M_{\omega}^\nu\) that satisfy \(\pi_{\nu}(\gamma)\alpha = \alpha\) for all \(\gamma \in \Gamma\). The elements of \(A_{\omega}^\nu(\Gamma)\) we call automorphic hyperfunctions for \(\Gamma\).
If $\alpha$ is an automorphic hyperfunction, then $T_\alpha M_K^{-\nu}$ is a $(g, K)$-module consisting of linear combinations of automorphic forms, and all $(g, K)$-modules isomorphic to a quotient of $M_K^{-\nu}$ in which the weight vectors are automorphic forms arise in this way.

If $j(\Gamma) = \Gamma$, then $j$ maps $A_{\omega,\infty}(\Gamma)$ into itself. We have $j(T_\alpha \varphi) = T_{j_1}(j_2 \varphi)$.

5.4 Hyperfunction for holomorphic automorphic forms. As an example we consider a holomorphic automorphic form $H$ for $\Gamma$ with even weight $2k$. So $H \left( \frac{az + b}{cz + d} \right) = (cz + d)^{2k} H(z)$ for $z \in \mathcal{H}$. If $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$, we do not impose any condition at the cusps of $\Gamma$, so $k$ may be negative.

Define $\alpha \in M_{2k-1}^{2k-1}$ to be the hyperfunction represented by the function $g$ equal to $0$ on $\mathcal{H}$ and given by $g(\tau) = (-1)^k 4^{-k} (1 + \tau)^k H(\tau)$ for $\tau \in \mathcal{H}$. The transformation behavior of $H$ under $\Gamma$ implies that $\alpha \in A_{2k-1}^{2k-1}$. A computation shows:

$$\langle \pi_{1-2k}(p(z)) \varphi_{2r}, \alpha \rangle = \frac{(-1)^k 4^{-k}}{\pi} y^{1-k} \int_{C_+} H(\tau)(\tau - \bar{z})^{k+r-1}(\tau - z)^{k-r-1} \, d\tau.$$  

This vanishes if $r < k$, and yields $y^k H(z)$ for $r = k$. So the automorphic form on $G$ corresponding to $H$ is equal to $T_\alpha \varphi_{2k}$. The element $T_\alpha \varphi_{2k}$ is a lowest weight vector in the $(g, K)$-module it generates. If $2k \geq 2$, this $(g, K)$-module is isomorphic to $D^+(2k)$; it is the image of $M_K^{-2k} \to D_K^+(2k) \oplus D_K^-(2k) \to D_K^+(2k)$. If $k \leq 0$, the $(g, K)$-module is not irreducible. It is isomorphic to $M_K^{-2k} \mod D_K^+(2k).

5.5 Maass forms. Any automorphic form of weight zero generates a $(g, K)$-module that is the quotient of some $M_K^{\nu}$. Helgason’s result quoted above shows that these automorphic forms all arise from automorphic hyperfunctions. In Section 8 we shall give an explicit construction of the hyperfunction corresponding to automorphic forms with polynomial growth at the cusp $\infty$.

If the eigenvalue is $s(1-s)$ with $s \notin \mathbb{Z}$, then both $\nu = 2s - 1$ and $\nu = 1 - 2s$ are possible; the resulting automorphic hyperfunctions are unique, and are related by $i(\nu)$. If $s \in \mathbb{Z}$, only one of these choices will work, the hyperfunction need not be unique.

If $j(\Gamma) = \Gamma$, and $\alpha \in A_{\omega,\infty}(\Gamma)$ corresponds to the Maass form $u$, then $j \alpha$ corresponds to the Maass form $z \mapsto u(-\bar{z})$.

5.6 Eisenstein series in the domain of absolute convergence. For $\Re s > 1$ we define

$$h_s(\tau) := \frac{-i}{2} \pi^{-s} \Gamma(s) \sum_{p,q \in \mathbb{Z}}^\prime \left( p^2 + q^2 \right)^{-s} \frac{p\tau - q}{q\tau + p}.$$  

This converges absolutely for all $\tau \in \mathbb{C} \setminus \mathbb{R}$. The convergence is uniform on compact sets in $\mathcal{H}^+ \cup \mathcal{H}^-$. Let $\epsilon_\ast^\ast$ be the hyperfunction on $T$ represented by $h_s$. The integral for $\langle \varphi, \epsilon_\ast^\ast \rangle$ can be evaluated term by term. For each term the integrand has only one pole on $T$, at $\tau = \frac{-q}{p}$. We obtain $\langle \varphi, \epsilon_\ast^\ast \rangle = \pi^{-s} \Gamma(s) \sum_{p,q} \varphi \left( \frac{-q}{p} \right) \left( p^2 + q^2 \right)^{-s}$.

If we take $\varphi(\tau) = \pi_{2s-1}(p(z)) \varphi_0(\tau) = y^s$.
\[
\left( \frac{\tau^2 + 1}{(\tau - z)(\tau - \bar{z})} \right)^s, \text{ then we find } G(s; z). \text{ For } s \not\in \mathbb{Z} \text{ this determines the hyperfunction uniquely. So for } \Re s > 1, s \not\in \mathbb{Z}, \text{ we have } \varepsilon^*_s \in A_{-2s}^1(\Gamma_{\text{mod}}). \text{ The relation giving the equivalence of } \pi_{1-2s}(\gamma)h_s \text{ and } h_s \text{ for } s \not\in \mathbb{Z} \text{ extends to } s \in 1 + 2\mathbb{Z}, s \geq 2. \text{ Hence } \varepsilon^*_s \in A_{-2s}^1(\Gamma_{\text{mod}}) \text{ for all } s \text{ with } \Re s > 1. \text{ It corresponds to the Eisenstein series, and satisfies } \varepsilon^*_s \in A_{1-2s}(\Gamma_{\text{mod}}). \]

5.7 Exponentially growing Poincaré series. In their construction and meromorphic continuation of Poincaré series, Miatello and Wallach, [10], explicitly give the linear form corresponding to an automorphic hyperfunction. Their context is much wider than ours: Lie groups with real rank one. Their Poincaré series have in general exponential growth at a cusp.

5.8 Question. Are there automorphic forms that generate a \((g, K)\)-module which is not the quotient of some \(M_K^\nu\)?

In the sequel we consider automorphic hyperfunctions as the principal objects.

5.9 Modular case. For the modular group, automorphic hyperfunctions are closely related to functions satisfying \([12]\). This is the subject of the remaining part of this section. Theorem 5.11 is the main result. The intermediate result Proposition 5.15 is valid for all cofinite discrete groups \(\Gamma\) with cusps.

5.10 Definition. Let \(\nu \in \mathbb{C} \setminus (1 + 2\mathbb{Z})\). We define \(\Psi_{\text{mod}}(\nu)\) to be the linear space of holomorphic functions \(\psi : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}\) that satisfy

\[
\psi(\tau) = \psi(\tau + 1) + (\tau + 1)^{\nu - 1} \psi\left(\frac{\tau}{\tau + 1}\right), \quad (5.1)
\]

\[
\lim_{\Im \tau \to \infty} \left(\psi(\tau) + \tau^{-\nu} \psi\left(\frac{-1}{\tau}\right)\right) + \lim_{\Im \tau \to -\infty} \left(\psi(\tau) + \tau^{-\nu} \psi\left(\frac{-1}{\tau}\right)\right) = 0. \quad (5.2)
\]

The existence of both limits is part of condition (5.2). Equation (5.1) is equation \([12]\), with \(2s\) replaced by \(\nu + 1\).

5.11 Theorem. For each \(\nu \in \mathbb{C} \setminus (1 + 2\mathbb{Z})\) there is a bijective linear map \(A_{-\nu}^\infty(\Gamma_{\text{mod}}) \to \Psi_{\text{mod}}(\nu) : \alpha \mapsto \psi_{\alpha}\).

Remarks. The proof is given in \([12] [13]\). The map is the composition of the maps described in Lemma 5.13 and Proposition 5.15. For \(\nu \in 1 + 2\mathbb{Z}\) the map \(\alpha \mapsto \psi_{\alpha}\) from automorphic hyperfunctions to holomorphic functions on \(\mathbb{C} \setminus (-\infty, 0]\) that satisfy (5.1) is well defined, but we have no bijectivity.

Equation (5.1) is essential in the definition of \(\Psi_{\text{mod}}(\nu)\). The limit condition (5.2) is a normalization, needed to obtain injectivity.

5.12 Definitions. The space \(\Psi_{\text{mod}}(\nu)\) is contained in the space \(\Psi(\nu)\) of holomorphic functions on the smaller domain \(\mathbb{C} \setminus \mathbb{R}\) that satisfy (5.1) and (5.2).

For \(\nu \in \mathbb{C}\), let \(\mathcal{F}(\nu)\) be the space of holomorphic functions \(f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}\) that satisfy

\[
f(\tau) = f(\tau + 1), \quad (5.3)
\]

\[
f(\tau) = O(1) \text{ as } |\Im \tau| \to \infty, \text{ and } f(i\infty) + f(-i\infty) = 0. \quad (5.4)
\]
If a 1-periodic holomorphic function on $\mathfrak{H}^\pm$ is $O(1)$ as $\pm \Im \tau \to \infty$, it has a power series expansion in $e^{\pm 2\pi i \tau}$. Hence $f(\pm i \infty)$ makes sense; it is the constant term in the expansion.

5.13 Lemma. Let $\nu \in \mathbb{C}$, $\nu \notin 1 + 2\mathbb{Z}$. The relations

$$\psi(\tau) = f(\tau) - \tau^{-1-\nu} f \left( \frac{-1}{\tau} \right)$$

(5.5)

$$f(\tau) = \frac{1}{1 + e^{\pi i \nu}} \left( \psi(\tau) + \tau^{-1-\nu} \psi \left( \frac{-1}{\tau} \right) \right) \quad \text{for } \tau \in \mathfrak{H}^\pm$$

(5.6)

define a bijective linear map $\mathcal{F}(\nu) \to \Psi(\nu) : f \mapsto \psi$.

Remarks. J. Lewis has shown me these transformations. See 10.9 for a cohomological interpretation.

If $\nu \in 1 + 2\mathbb{Z}$, then (5.5) defines a map from $\mathcal{F}(\nu)$ to the holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$ that satisfy (5.1).

Proof. For $f \in \mathcal{F}(\nu)$ define $\psi$ by (5.5). Then (5.6) turns out to give back $f$. The periodicity (5.3) for $f$ is equivalent to (5.1) for $\psi$, and (5.2) is a direct reformulation of (5.4).

5.14 Definition. Let $\nu \in \mathbb{C} \setminus (1 + 2\mathbb{Z})$. We define $\mathcal{F}_{\text{mod}}(\nu)$ to be the subspace of $\mathcal{F}(\nu)$ corresponding to $\Psi_{\text{mod}}(\nu) \subset \Psi(\nu)$ under the map $f \mapsto \psi$ of Lemma 5.13.

The $f \in \mathcal{F}_{\text{mod}}(\nu)$ are characterized by the property that $\tau \mapsto f(\tau) - \tau^{-1-\nu} f \left( \frac{-1}{\tau} \right)$ extends holomorphically across $(0, \infty)$. This is equivalent to $\tau \mapsto f(\tau) - (-\tau)^{-1-\nu} f \left( \frac{-1}{\tau} \right) = -(-\tau)^{-1-\nu} \psi \left( \frac{-1}{\tau} \right)$ having a holomorphic extension across $(-\infty, 0)$.

For $\nu \in 1 + 2\mathbb{Z}$ we define $\mathcal{F}_{\text{mod}}(\nu)$ as the space of $f \in \mathcal{F}(\nu)$ for which $\tau \mapsto f(\tau) - \tau^{-1-\nu} f \left( \frac{-1}{\tau} \right)$ extends holomorphically to $\mathbb{C} \setminus \{0\}$.

5.15 Proposition. Let $\nu \in \mathbb{C}$, and $\Gamma \subset G$ as in 23. There is an injective linear map $A_{\nu,\omega}(\Gamma) \to \mathcal{F}(\nu) : \alpha \mapsto f_\alpha$ such that $\tau \mapsto (1 + \tau^2)^{1/(1+\nu)}/2 \ f_\alpha(\tau)$ represents the restriction of the hyperfunction $\alpha$ to the open subset $T \setminus \{\infty\}$ of $T$.

If $\Gamma = \Gamma_{\text{mod}}$, then the image of $A_{\nu,\omega}(\Gamma) \to \mathcal{F}(\nu) : \alpha \mapsto f_\alpha$ is equal to $\mathcal{F}_{\text{mod}}(\nu)$.

Remarks. Here we do not need to exclude $\nu \in 1 + 2\mathbb{Z}$. See 5.16 for the case corresponding to holomorphic automorphic forms.

Proof. The restriction $\alpha_0$ of $\alpha$ to the open $N$-orbit $T \setminus \{\infty\} \subset T$ is represented by a function $g$ that is holomorphic on at least the strips $0 < |\Im \tau| < \varepsilon$ for some $\varepsilon > 0$. Take $\varepsilon < 1$. Then $F : \tau \mapsto (1 + \tau^2)^{-1/(1+\nu)}/2 \ g(\tau)$ is also holomorphic on these strips. The invariance of $\alpha_0$ under $\pi_\nu \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$ implies that $F(\tau - 1) = F(\tau) + q(\tau)$, with $q$ holomorphic on $|\Im \tau| < \varepsilon$. So $F$ represents a hyperfunction on $\mathbb{R}$ that is invariant under the translations $\tau \mapsto \tau + n$ with $n \in \mathbb{Z}$. It determines a hyperfunction on the circle, and that hyperfunction can be represented by a function holomorphic on the complement of the circle in $\mathbb{P}_1^\infty$. 

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This function is unique up to an additive constant. So $F$ can be replaced by
the unique function $f_{\alpha}$ of the form

$$f_{\alpha}(\tau) = \begin{cases} \frac{1}{2} A_0(\alpha) + \sum_{n=1}^{\infty} A_n(\alpha) e^{2\pi in\tau} & \text{for } \tau \in \mathfrak{H}^+, \\ -\frac{1}{2} A_0(\alpha) - \sum_{n=1}^{\infty} A_{-n}(\alpha) e^{-2\pi in\tau} & \text{for } \tau \in \mathfrak{H}^- \end{cases} \tag{5.7}$$

The function $\tau \mapsto (1 + \tau^2)^{(1+\nu)/2} f_{\alpha}(\tau)$ is holomorphic on $(\mathfrak{H}^+ \setminus i[1, \infty)) \cup (\mathfrak{H}^- \setminus (-i[1, \infty))$. It represents $\alpha_0$. If $f_{\alpha}$ would vanish, then $\alpha_0 = 0$. As $\infty$ cannot be a fixed point of the whole group $\Gamma$, this implies $\alpha = 0$. Hence $\alpha \mapsto f_{\alpha}$ is injective.

It is clear that $f_{\alpha} \in \mathcal{F}(\nu)$. To see that it is in $\mathcal{F}_{\text{mod}}(\nu)$ if $\Gamma = \Gamma_{\text{mod}}$, we note that $\tau \mapsto (1 + \tau^{-2})^{(1+\nu)/2} f_{\alpha} \left( \frac{-1}{\tau} \right)$ represents the restriction of $\alpha = \pi_{\nu} \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \alpha$ to $T \setminus \{0\}$. The gluing conditions on $(0, \infty)$ and $(-\infty, 0)$ between the representatives are just the existence of holomorphic extensions across $(0, \infty)$ and $(-\infty, 0)$ that characterize $\mathcal{F}_{\text{mod}}(\nu)$ inside $\mathcal{F}(\nu)$.

Let $\Gamma = \Gamma_{\text{mod}}$, and start with $f \in \mathcal{F}_{\text{mod}}(\nu)$. Clearly, $\tau \mapsto (1 + \tau^2)^{(1+\nu)/2} f(\tau)$ represents a hyperfunction $\beta_0 \in \mathcal{B}_T (T \setminus \{\infty\})$, that satisfies $\pi_{\nu} \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \beta_0 = \beta_0$. Put $\beta_{\infty} := \pi_{\nu} \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \beta_0 \in \mathcal{B}_T (T \setminus \{0\})$. A representative of $\beta_{\infty}$ is $\tau \mapsto (1 + \tau^{-2})^{(1+\nu)/2} f \left( \frac{-1}{\tau} \right)$. The definition of $\mathcal{F}_{\text{mod}}(\nu)$ implies that $\beta_0$ and $\beta_{\infty}$ coincide on $(0, \infty)$ and $(-\infty, 0)$. So there exists $\beta \in M_{\nu, \omega}$ restricting to $\beta_0$ on $T \setminus \{\infty\}$ and to $\beta_{\infty}$ on $T \setminus \{0\}$. Clearly $\pi_{\nu} \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \beta = \beta$. We have to check the invariance under the other generator $\left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$ of the modular group. The support of $\pi_{\nu} \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \beta - \beta$ is contained in $\{\infty\}$. On $|\tau| > 2$, $\tau \notin \mathbb{R}$, this hyperfunction is represented by

$$\left( \frac{\tau^2 + 1}{1 + (\tau - 1)^2} \right)^{(1+\nu)/2} \left( 1 + (\tau - 1)^{-2} \right)^{(1+\nu)/2} f \left( \frac{-1}{\tau - 1} \right) - (1 + \tau^{-2})^{(1+\nu)/2} f \left( \frac{-1}{\tau} \right) = (1 + \tau^{-2})^{(1+\nu)/2} \left( \frac{\tau}{\tau - 1} \right)^{-1} f \left( \frac{-1}{\tau - 1} \right) - f \left( \frac{-1}{\tau} \right) - (1 + \tau^{-2})^{(1+\nu)/2} f \left( \frac{-1}{\tau} \right) \right).$$

The fact that $f \in \mathcal{F}_{\text{mod}}(\nu)$ implies that the quantity between brackets is holomorphic on a neighborhood of $\frac{-1}{\tau} = 1$. This shows that $\pi_{\nu} \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \beta - \beta$ vanishes on a neighborhood of $\infty$.

**5.16 Holomorphic automorphic forms.** Let $H$ be a holomorphic automorphic form $H$ for $\Gamma$ of even weight $2k$. It has a Fourier expansion

$$H(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$$

converging for $z \in \mathfrak{H}^+$. If it is bounded at the cusp $\infty$, then the sum is over $n \geq 0$. Meromorphy at the cusp corresponds to a sum over $n \geq -N$ for some $N \geq 1$.  

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\[ F_n u_{2r} \text{ is a linear combination of:} \]

| \( n \) | \( \nu \neq 0 \) | \( \nu = 0 \) |
|-------|-------------|-------------|
| 0     | \( y^{(1+\nu)/2} \) and \( y^{(1-\nu)/2} \) | \( y^{1/2} \) and \( y^{1/2} \log y \) |
| 0 < 0 | \( W_{r,\nu/2}(4\pi ny) \) | \( W_{-r,\nu/2}(4\pi |n|y) \) |

Table 1: Basis elements for the spaces of Fourier terms of automorphic forms with polynomial growth.

Suppose that \( \alpha \in A^{2k-1}_\omega(\Gamma) \) corresponds to \( H \) as indicated in 5.4. From the representative \( g \) in 5.4 we subtract the function \( p : \tau \mapsto (-1)^k 4^{-k} \left(1 + \tau^2 \right)^k \)
\( P(\tau) \), with \( P(z) := \frac{1}{2}a_0 + \sum_{n=-\infty}^{\infty} a_n e^{2\pi inz} \). The function \( P \) is holomorphic on \( \mathbb{C} \), hence \( g - p \) represents the restriction of \( \alpha \) to \( T \setminus \{\infty\} \). We see that
\[ f_\alpha(\tau) = (-1)^k 4^{-k} \cdot \begin{cases} \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi in\tau} & \text{for } \tau \in \mathfrak{H}^+, \\ -\frac{1}{2}a_0 - \sum_{n=1}^{\infty} a_{-n} e^{-2\pi in\tau} & \text{for } \tau \in \mathfrak{H}^-. \end{cases} \]

5.17 Other examples. In 6.18 we shall see that for other automorphic hyperfunctions \( \alpha \) as well the function \( f_\alpha \) is closely related to the Fourier expansion of the automorphic forms associated to \( \alpha \).

6 Fourier expansion

In this section we give an explicit description of a representative of an automorphic hyperfunction that has polynomial growth at the cusp \( \infty \). The main result is Lemma 6.13.

Some parts of this section are technical. Reading 6.1–6.2, the notations in 6.12, and Lemma 6.13 suffices, if one is willing to accept later on some results on representatives of hyperfunctions.

6.1 System of automorphic forms. We work with an automorphic hyperfunction \( \alpha \in A^{2k-1}_\omega(\Gamma) \). To it corresponds a system \( (u_{2r})_{r \in \mathbb{Z}} \) of automorphic forms given by \( u_{2r}(g) = \langle \pi_\nu(g) \phi_{2r}, \alpha \rangle \). This system satisfies the differential equations
\[ E_+^r u_{2r} = (1 - \nu \pm 2r)u_{2r \pm 2} \quad \text{for } r \in \mathbb{Z}. \quad (6.1) \]

6.2 Polynomial growth. We say that an automorphic hyperfunction \( \alpha \) has polynomial growth at \( \infty \) if \( u_{2r}(p(z)) = O(y^a) \) as \( y \to \infty \), uniformly in \( x \), for each weight \( 2r \in \mathbb{Z} \).

The hyperfunctions associated to Eisenstein series and to cuspidal Maass forms have polynomial growth. The Poincaré series studied in [10] have in
The cuspidal Maass forms and Eisenstein series discussed in Sections 1 and 2 have been given by their Fourier expansion at $\infty$. In general we have:

$$u_{2r}(p(z)) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} F_n u_{2r}(y),$$

$$F_n u_{2r}(y) := \int_{x \in \mathbb{R} \text{mod} \mathbb{Z}} e^{-2\pi i n x} u_{2r}(p(iy + x)) \, dx.$$ 

The Fourier terms $F_n u_{2r}$ satisfy a differential equation with a two-dimensional solution space; see, e.g., [8], Chap. IV, §2, or [1], §4.1, 4.2, 4.4. For $n \neq 0$ the condition of polynomial growth imposes an additional condition. See Table 1 for the possibilities.

**6.4 Systems of Fourier terms.** The differential equations (6.1) imply the same equations for each of the systems $(F_n u_{2r})_r$ separately. So these systems are linear combinations of systems of solutions of the corresponding differential equation. We restrict ourselves to the case of polynomial growth, so we are lead to systems of the functions in Table 1 that satisfy (6.1). In Table 2 we give all possibilities. For $n \neq 0$ the space of Fourier terms with polynomial growth has dimension 1. For $n = 0$ the dimension equals 2, except in the case of odd negative values of $\nu$. Further remarks on the systems in the table:

- **a.** This is the standard form of $M_K^{-\nu}$ as the induced representation from $P$ to $G$.
- **b.** If $\nu$ is odd, some of the functions in the system of Fourier term vanish. This

| $n = 0$ | $\nu \notin 1 + 2\mathbb{Z} \geq 0, \nu \neq 0$ | $y^{(1-\nu)/2}$, $\Gamma(\frac{1-\nu}{2} + |r|)^{-1} \left( \frac{1+\nu}{2} \right)_{|r|} y^{(1+\nu)/2}$ | a | c |
| $\nu \in 1 + 2\mathbb{Z}, \nu > 0$ | $y^{(1-\nu)/2}$, $\Gamma(\frac{1-\nu}{2} + |r|)^{-1} \left( \frac{1+\nu}{2} \right)_{|r|} y^{(1+\nu)/2}$, $\text{sign}(r) \Gamma(1-\nu) y^{1/2}$ | a | c | d |
| $\nu = 0$ | $y^{1/2}$, $y^{1/2} (\log y + l_r)$ | a | e |
| $n \neq 0, \varepsilon = \text{sign } n$ | $\frac{(-1)^r}{\Gamma(\frac{1-\nu}{2} + \varepsilon r)} W_{r,\nu/2} (4\pi |n| y) e^{2\pi i n x}$ | b |

Table 2: Bases of the spaces of systems of Fourier terms with polynomial growth. For each system $2r \in 2\mathbb{Z}$ denotes the weight. We give the function on $G$ in the point $p(z)$ with $z = x + iy \in \mathbb{H}^+$.
is due to the fact that the quickly decreasing Whittaker functions span a 
\((g, K)\)-module that is not a quotient of either \(M^K_\nu\) or \(M^K_{-\nu}\) if \(\nu \in 1 + 2\mathbb{Z}\).

c. For negative, odd \(\nu\) the functions vanish if \(|2r| \geq 1 - \nu\), due to the factor 
\(\left(\frac{1 + \tau}{2}\right)|r|\). For odd, positive \(\nu\) the Gamma factor produces zeros for \(|2r| \leq \nu - 1\).

d. For odd positive \(\nu\) the equations (6.1) give no coupling between \(2r \geq \nu + 1\) and \(2r \leq -1 - \nu\). This explains the presence of three linearly independent systems.

e. At \(\nu = 0\) the systems a and c are linearly dependent. The \(l_r\) are determined up to an additive constant.

6.5 \(N\)-equivariant hyperfunctions. In 6.6–6.10 we describe each system in Table 2 by means of a hyperfunction that transform according to \(\pi_\nu \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right) \beta = e^{-2\pi in\beta}\). Actually, in Table 2 we have chosen factors that seem natural when working with the functions in Table 6. Other factors turn out to be more natural when working with hyperfunctions. We have summarized the results in the Tables 3 and 4.

6.6 Support in \(\infty\). Any \(\pi_\nu(N)\)-invariant hyperfunction with support \(\{\infty\}\) has a representative \(g\) holomorphic on a neighborhood of \(\infty\), with \(\infty\) itself deleted. The difference \(\pi_\nu \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right) g - g\) has to have a holomorphic extension to \(\tau = \infty\) for each real \(x\). An analysis of the unique representative of the form 
\[g(\tau) = (1 + \tau^{-2})^{(1+\nu)/2} \sum_{k=1}^{\infty} r_k \tau^k\]
leads to a solution that is valid for all \(\nu \in \mathbb{C}\):

\[\mu \in M_{-\nu}, \quad \text{represented by } \tau \mapsto -\frac{i}{2} \tau. \quad (6.2)\]

The corresponding linear form on \(M_{-\nu}\) is \(\varphi \mapsto \varphi(\infty)\). This gives the system a in Table 3.

In general, all solutions are multiples of \(\mu\). But if \(\nu \in \mathbb{Z}_{\geq 1}, \nu \geq 1\), the solution space has dimension 2, and is spanned by \(\mu\) and

\[\mu^*_\nu, \quad \text{represented by } \tau \mapsto -\frac{i}{2} \tau^{\nu+1} \left(1 + \tau^{-2}\right)^{(1+\nu)/2}. \quad (6.3)\]

A computation shows that \(\langle \pi_\nu(p(z))\varphi_{2r}, \mu^*_\nu \rangle\) is equal to

\[\lim_{R \to \infty} \frac{1}{\pi} \int_{|\tau| = R} -\frac{i}{2} \tau^{\nu+1} \left(1 + \tau^{-2}\right)^{(1+\nu)/2} y^{(1-\nu)/2} \left(\left(\frac{1 + \tau^2}{(\tau - z)(\tau - \bar{z})}\right)^{(1-\nu)/2} \left(\frac{\tau - \bar{z}}{\tau - z}\right)^r \frac{d\tau}{1 + \tau^2}\right)
\]

\[= y^{(1-\nu)/2} \cdot \text{coefficient of } u^\nu \text{ in } (1 - uz)^{(\nu-1)/2 + r} (1 - u\bar{z})^{(\nu-1)/2 + r}
\]

\[= y^{(1-\nu)/2} \cdot (2iy \text{sign } r)^\nu \frac{\Gamma\left(\frac{\nu+1}{2} + |r|\right)}{\Gamma(\nu + 1) \Gamma\left(\frac{-\nu}{2} + |r|\right)}\]
Figure 3: The contours $I_+$ and $I_-$. 

\[
\varphi(\tau) t_n(\nu) e^{2\pi in\tau} (1 + \tau^2)^{1/2} \frac{d\tau}{\pi(1 + \tau^2)}.
\]  

For each $n \in \mathbb{Z}$ this defines a linear form on $A_T(T)$, with value bounded by the supremum norm of $\varphi$ on $T$. So its values on the basis elements $\varphi_{2r}$ are estimated by $O(1)$. In 3.12 we see that the linear form is given by a hyperfunction, which we call $\kappa_n(\nu)$.

If $\pm n > 0$ we can deform the path of integration into $I_\pm$, indicated in Figure 3. The contour should be contained in the domain of $\varphi$. The resulting integral converges for all $\nu \in \mathbb{C}$. If we take the contour inside the region $c^{-1} < |\frac{\tau + i}{\tau - 1}| < c$ for some $c > 1$, we find that the value on $\varphi_{2r}$ is $O(c|r|)$. So this integral extends the definition of $\kappa_n(\nu)$ for $n \neq 0$.

In all cases $\nu \mapsto \langle \varphi, \kappa_n(\nu) \rangle$ is holomorphic on the domain of definition of $\kappa_n(\nu)$. If we compute $\langle \pi_{-\nu}(p(\cdot)) \varphi_{2r}, \kappa_n(\nu) \rangle$ by means of (6.4), we find

\[
\begin{align*}
(-1)^r \frac{\Gamma(r\nu)}{\Gamma(1/2 + r) \Gamma(1/2 - r)} y^{(1+\nu)/2} & \quad \text{if } n = 0, \\
(-1)^r (\pi|n|)^{-1/2} y^{(1+\nu)/2} e^{2\pi inx} W_{\pm r,\nu/2} & \quad \text{if } \pm n > 0.
\end{align*}
\]

Thus we have a multiple of system $c$, respectively system $b$. This also shows that $\kappa_n(\nu)$ behaves under $\pi_{\nu}(\begin{pmatrix}1 & x \\ 0 & 1 \end{pmatrix})$ according to the character $\begin{pmatrix}1 & x \\ 0 & 1 \end{pmatrix} \mapsto e^{-2\pi inx}$ of $N$. 

So $\mu^*_{\nu}$ determines a multiple of the system $c$ if $\nu$ is even, and of system $d$ if $\nu$ is odd. 

6.7 Integral over the open $N$-orbit. Let $n \in \mathbb{Z}$. Put $n = \pm|n|$, with $\pm = +$ if $n = 0$. We define $t_0(\nu) := \sqrt{\pi} \Gamma(-\nu/2)^{-1}$, and $t_n(\nu) := 1$ if $n \neq 0$. If $\text{Re}\nu < 0$ the following integral converges for each $\varphi \in A_T(T), n \in \mathbb{Z}$:

\[
\int_{-\infty}^{\infty} \varphi(\tau) t_n(\nu) e^{2\pi in\tau} (1 + \tau^2)^{1/2} \frac{d\tau}{\pi(1 + \tau^2)}.
\]
Representative. Let $\pm n > 0$. As we have seen in 3.10, we obtain a representative of $\kappa_n(\nu)$ by
\[
g(\tau_0) = \prod \int_{I_\pm} \frac{1 + \tau\tau_0}{\tau - \tau_0} e^{2\pi i \nu} (1 + \tau^2)^{(\nu-1)/2} d\tau.
\]
In this integral $\tau_0$ is either in $\mathcal{H}_\pm$, or inside the contour $I_\pm$. We conclude that $g(\tau_0) = O(1)$ as $|\text{Im}\ \tau_0| \to \infty$ uniformly for $\text{Re}\ \tau_0$ in compact sets.

In the case that $\tau_0 \in \mathcal{H}_\pm$, and $|\text{Im}\ \tau_0| < 1$, we deform the contour in such a way that $\tau_0$ is outside it. This gives
\[
g(\tau_0) = \pm e^{2\pi i \nu} (1 + \tau_0^2)^{(1+\nu)/2} + \prod \int_{I_\pm} \frac{1 + \tau\tau_0}{\tau - \tau_0} e^{2\pi i \nu} (1 + \tau^2)^{(\nu-1)/2} d\tau.
\]

Thus we see that the restriction of $\kappa_n(\nu)$ to $T \setminus \{\infty\}$ can be represented by 0 on $\mathcal{H}_\pm$ and $\tau \to \pm e^{2\pi i \nu} (1 + \tau^2)^{(1+\nu)/2}$ on $\mathcal{H}_\pm \setminus (\pm i|1, \infty)$.

For $n = 0$ and $\text{Re}\ \nu < -\frac{1}{2}$ we can draw similar conclusions.

Continuation of $\kappa_0(\nu)$. Let $\nu \in \mathbb{C} \setminus (-2\mathbb{N})$, and take
\[
p(\tau) := \begin{cases} t_0(\nu) (1 + \tau^2)^{(1+\nu)/2} & \text{if } \text{Im} \ \tau \geq 0, \ \tau \not\in i[1, \infty), \\ 0 & \text{if } \tau \in \mathcal{H}_- \end{cases}
\]
\[
q(\tau) := \tau^{1+\nu} (1 + \tau^{-2})^{(1+\nu)/2} \frac{\Gamma(1 + \frac{\nu}{2}) e^{-\pi i \text{sign}(\text{Im}\ \tau)/2}}{2i \sqrt{\pi}} \text{ for } \tau \not\in \mathbb{R} \cup i[-1, 1].
\]
(In 6.7 we have defined $t_0(\nu) = \sqrt{\pi} \Gamma(-\nu/2)^{-1}$.)

The functions $p$ and $q$ define hyperfunctions on $T \setminus \{\infty\}$, respectively $T \setminus 0$. They have been chosen in such a way that their difference is holomorphic on $(0, \infty)$ and on $(-\infty, 0)$. So together they define a hyperfunction on $T$, which we call $\beta$ for the moment. We have the following integral representation for each $\varphi \in \mathcal{A}_T(T)$:
\[
\langle \varphi, \beta \rangle = \int_{A_+} \varphi(\tau)p(\tau) \frac{d\tau}{\pi(1 + \tau^2)} + \sum_\pm \int_{B_\pm} \varphi(\tau)q(\tau) \frac{d\tau}{\pi(1 + \tau^2)},
\]
(6.5)
with contours as indicated in Figure 3. This shows that \( \langle \varphi, \beta \rangle \) is holomorphic in \( \nu \in \mathbb{C} \setminus (-2\mathbb{N}) \).

For Re\( \nu < 0 \) we can move off the contours \( B_\pm \) to infinity, and obtain the integral in (6.4) with \( n = 0 \). Thus we have extended the definition of \( \kappa_0(\nu) \) to \( \mathbb{C} \).

For \( \nu \in 2\mathbb{Z}_{\geq 0} \) the function \( p \) vanishes, and the support of \( \kappa_0(\nu) \) is \( \{ \infty \} \). In fact \( \kappa_0(\nu) = (-1)^{\nu/2} \pi^{-1/2} \Gamma \left( 1 + \frac{\nu}{2} \right) \mu_\nu \) if \( \nu \in 2\mathbb{N} \), and \( \kappa_0(0) = \pi^{-1/2} \mu \) (see 6.4).

The \( \pi_\nu(N) \)-invariance, and the expression for \( \pi_\nu(p(z))\varphi_{2\nu}, \kappa_0(\nu) \) stay valid by holomorphy.

**6.10 Logarithmic case.** Let \( \nu = 0 \). The function \( \tau \mapsto \left( \frac{-i}{\pi} \log \tau - \frac{\text{sign} \text{ Im} \tau}{2} \right) \tau \)

\( (1 + \tau^2)^{1/2} \) on \( \mathbb{C} \setminus (\mathbb{R} \cup i[-1,1]) \) defines a hyperfunction on a neighborhood of \( \infty \) that fits nicely with the hyperfunction on \( T \setminus \{ \infty \} \) represented by \( \tau \mapsto -(1 + \tau^2)^{1/2} \) on \( \mathbb{H}^+ \setminus i[1,\infty) \) and \( \tau \mapsto 0 \) on \( \mathbb{H}^- \). We call it \( \lambda(0) \). It satisfies \( \pi_0(\nu) \lambda(0) = \lambda(0) \) for all \( x \in \mathbb{R} \). It is the continuation to \( \nu = 0 \) of the family \( \lambda : \nu \mapsto \frac{2}{\nu \sqrt{\pi}} (\kappa_0(\nu) - \pi^{-1/2} \mu) \).

A computation shows that \( \langle \pi_0(p(z))\varphi_{2\nu}, \lambda(0) \rangle \) is equal to \( \frac{2}{\sqrt{\pi}} \log y \) plus a well defined but complicated multiple of \( \sqrt{\pi} \).

**6.11 Fourier terms of \( \alpha \).** Let \( n \in \mathbb{Z} \) and consider a representative \( g \) of the automorphic hyperfunction \( \alpha \). Define

\[
g_n(\tau) := \int_{x=0}^{1} e^{2\pi i n x} \pi_\nu \left( \begin{array}{c} x \\ 0 \\ 1 \end{array} \right) g(\tau) \, dx
\]

\[
= \int_{x=0}^{1} e^{2\pi i n x} \left( \frac{\tau - i}{\tau - x - i} \right)^{(1+\nu)/2} \left( \frac{\tau + i}{\tau - x + i} \right)^{(1+\nu)/2} g(\tau - x) \, dx.
\]

This defines \( g_n \) on a set \( U \) contained in the domain of \( g \) such that \( T \cup U \) is a neighborhood of \( T \). Let \( \mathcal{F}_n \alpha \) be the hyperfunction represented by \( g_n \); this

| \( n \) | \( \nu \in \mathbb{C} \) | \( \kappa_0(\nu) \) | \( \mu_\nu \) | \( \lambda(0) \) | \( \kappa_0(\nu) \) |
|---|---|---|---|---|---|
| 0 | \( \mathbb{C} \) | \( \mu \) | \( \tau \mapsto \frac{-i}{\pi} \tau \) | \( a \) | 1 |
| 0 | \( \mathbb{C} \) | \( \kappa_0(\nu) \) | \( \text{see 6.8 and 6.9} \) | \( c \) | 1 |
| 1 \( + 2\mathbb{Z}_{\geq 0} \) | \( \mu_\nu \) | \( \text{see 6.3} \) | \( d \) | \( \frac{(-1)^{\nu/2} \pi^{-1/2} \Gamma \left( 1 + \frac{\nu}{2} \right) \mu_\nu}{(1 + \frac{\nu}{2})} \) |
| \( \{ 0 \} \) | \( \lambda(0) \) | \( \text{see 6.10} \) | \( e \) | \( \frac{2}{\pi} \) |
| \( \neq 0 \) | \( \mathbb{C} \) | \( \kappa_0(\nu) \) | \( \text{see 6.8} \) | \( b \) | \( (\pi |n|)^{-1/2} \) |

Table 3: \( \mathbb{N} \)-equivariant hyperfunctions, satisfying \( \pi_\nu \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \beta = e^{2\pi i n x} \beta \).

Each hyperfunction determines a multiple of a system of eigenfunctions in Table 2.
| support                        | $p$          |
|-------------------------------|--------------|
| $\mu$                         | $\{\infty\}$| 0             |
| $\kappa_\nu(\nu)$            | $T$          | $e^{2\pi i n \nu} (1 + \tau^2)^{1+i\nu} / 2$ $n \neq 0$ |
| $\kappa_0(\nu)$              | $T$          | $\sqrt{\pi} \Gamma(-\nu/2)^{-1} (1 + \tau^2)^{1+i\nu} / 2$ $\nu \not\in 2\mathbb{Z}_{\geq 0}$ |
| $\lambda(0)$                 | $T$          | $- (1 + \tau^2)^{1/2}$                         |
| $\mu^*_\nu$                  | $\{\infty\}$| 0             |

Table 4: Support and restriction to $T_0 := T \setminus \{\infty\}$ of the hyperfunctions in Table 3.

The restriction to $T_0$ is represented by a function $\tau \mapsto \pm p(\tau)$ on $\mathcal{S}_\nu \setminus (\pm i)[1, \infty)$, $\tau \mapsto 0$ on $\mathcal{S}_\nu^\mp$. For $\kappa_\nu(\nu)$ the convention is $\pm n \geq 0$, $\pm = +$ if $n = 0$.

does not depend on the choice of the representative $g$. One can check that $\pi_\nu \left( \frac{1}{1} \right) F_{n}\alpha = e^{-2\pi i \nu} F_{n}\alpha$.

In Proposition 6.13 we have discussed the function $f_\alpha$; see also (5.7). On a neighborhood of $T \setminus \{\infty\}$ we can replace $g$ by $\tau \mapsto (1 + \tau^2)^{1+i\nu} / 2 f_\alpha(\tau)$. Then we obtain the following representative $g_n^*$ of the restriction of $F_{n}\alpha$ to $T \setminus \{\infty\}$:

$$
\begin{array}{c|c|c}
\tau & 0 < \text{Im} \tau < 1 & -1 < \text{Im} \tau < 0 \\
n > 0 & g_n^*(\tau) = A_n(\alpha)p_n(\tau) & g_n^*(\tau) = 0 \\
n = 0 & g_0^*(\tau) = \frac{1}{2} A_0(\alpha)p_0(\tau) & g_0^*(\tau) = -\frac{1}{2} A_0(\alpha)p_0(\tau) \\
n < 0 & g_n^*(\tau) = 0 & g_n^*(\tau) = -A_n(\alpha)p_n(\tau)
\end{array}
$$

Here $p_n(\tau) = (1 + \tau^2)^{1+i\nu} / 2 e^{2\pi i n \tau}$. For $n \neq 0$ we conclude that $F_{n}\alpha = A_n(\alpha)\kappa_\nu(\nu)$.

By interchanging the order of integration we obtain $\langle \pi_\nu(p(z))\varphi_2r, F_{n}\alpha \rangle = \int_0^1 e^{2\pi i n x} \left( \pi_\nu(p(z))\varphi_2r, \pi_\nu \left( \frac{1}{1} \right) \alpha \right) dx' = F_{n}ux_2r(p(z))$.

6.12 Notation. We write $f_\alpha = f_0^0 + f_\nu^\nu$, with $f_0^0(\tau) := \frac{1}{2} A_0(\alpha) \text{sign} \text{(Im } \tau)$.

6.13 Lemma. Let $\alpha \in A_{\nu, \omega}(\Gamma)$, and suppose that it has polynomial growth at $\infty$. There is a decomposition $\alpha = F_{n}\alpha + \alpha^c$, with $\alpha^c \in M_{\nu, \omega}^c$, such that the function $\tau \mapsto (1 + \tau^2)^{1+i\nu} / 2 f_\alpha^c(\tau)$ represents the restriction of $\alpha^c$ to $T \setminus \{\infty\}$, and such that the Fourier term of order 0 is given by

$$
\mathcal{F}_{0}\alpha = \begin{cases}
B_0(\alpha)\mu + \pi^{-1/2} \Gamma(-\nu/2)A_0(\alpha)\kappa_0(\nu) & \text{if } \nu \not\in \mathbb{Z}_{\geq 0}, \\
B_0(\alpha)\mu - A_0(\alpha)\lambda(0) & \text{if } \nu = 0, \\
B_0(\alpha)\mu + C_0(\alpha)\mu^*_\nu + \pi^{-1/2} \Gamma(-\nu/2)A_0(\alpha)\kappa_0(\nu) & \text{if } \nu \geq 1 \text{ is odd}, \\
B_0(\alpha)\mu + C_0(\alpha)\kappa_0(\nu) & \text{if } \nu \geq 2 \text{ is even}.
\end{cases}
$$
for each \( \mu \) the definition of \( \mu \)

**Definition of 6.14**

**Proof.**

The integrals in (6.7) define functions that are holomorphic on a neighborhood \( \tau \). In (6.6) the point \( \tau \) is indicated in Figure 3 on page 138 inside the domain of \( \varphi \). In the same way as in 6.7 we show that this linear form is given by an hyperfunction. We define \( \alpha^\varepsilon \) to be this hyperfunction. The 1-periodicity of \( f_\alpha^\varepsilon \) gives \( \alpha^\varepsilon = \alpha^\varepsilon \).

**Remark.** See 6.7 and 6.9 for the definition of the hyperfunction \( \kappa_0(\nu) \), 6.8 for the definition of \( \mu \) and \( \mu^* \), and 6.10 for \( \lambda(0) \).

**6.14 Definition of \( \alpha^\varepsilon \).** The function \( f_\alpha^\varepsilon \) decreases quickly at \( \pm i \infty \). This implies that for each \( \varphi \in A_T(T) \) the integrals in

\[
\frac{1}{\pi} \sum_{\pm} \int_{I_\pm} \varphi(\tau) (\pm f_\alpha^\varepsilon(\tau)) \left( 1 + \tau^2 \right)^{(\nu-1)/2} d\tau
\]

converge absolutely, if we take contours \( I_+ \) and \( I_- \) as indicated in Figure 3 on page 138. In the same way as in 6.7 we show that this linear form is given by an hyperfunction. We define \( \alpha^\varepsilon \) to be this hyperfunction. The 1-periodicity of \( f_\alpha^\varepsilon \) gives \( \alpha^\varepsilon = \alpha^\varepsilon \).

**6.15 Representative of \( \alpha^\varepsilon \).** Like we did in 6.8, we obtain a representative of \( \alpha^\varepsilon \) by defining

\[
g_\alpha^\varepsilon(\tau_0) := \frac{1}{2\pi i} \sum_{\pm} \int_{I_\pm} \frac{1 + \tau \tau_0}{\tau - \tau_0} (\pm f_\alpha^\varepsilon(\tau)) \left( 1 + \tau^2 \right)^{(\nu-1)/2} d\tau.
\]

In (6.6) the point \( \tau_0 \in \mathcal{H}^\varepsilon \), with \( \varepsilon = + \) or \( - \), is supposed to be inside the contour \( I_\varepsilon \). By taking the contours wide enough, we conclude that \( g_\alpha^\varepsilon(\tau_0) = O(1) \) as \( |\text{Im} \tau_0| \to \infty \), uniformly for \( \text{Re} \tau_0 \) in compact sets.

In (6.7) we suppose that \( \tau_0 \in \mathcal{H}^\varepsilon \) is between the real axis and the contour \( I_\varepsilon \). The integrals in (6.7) define functions that are holomorphic on a neighborhood of \( \mathbb{R} \). The latter of the integrals does not depend on \( \tau_0 \). The function \( \tau \mapsto \left( 1 + \tau^2 \right)^{(\nu+1)/2} f_\alpha^\varepsilon(\tau) \) represents the restriction of \( \alpha^\varepsilon \) to \( T \setminus \{ \infty \} \).

**6.16 Fourier expansion.** Consider \( \varphi = \pi_{\varepsilon^{-1}}(p(z))\varphi_{\varepsilon^{-1}} \), and insert the series expansion of \( f_\alpha^\varepsilon \). We interchange the order of summation and integration, and obtain

\[
(\pi_{\varepsilon^{-1}}(p(z))\varphi_{\varepsilon^{-1}}, \alpha^\varepsilon) = \sum_{n \neq 0} e^{2\pi i n x} \frac{(-1)^c(\pi|n|)^{(1+n)/2} A_n(\alpha)}{1 + r \text{ sign } n} W_r \text{ sign } n, \nu/2(4\pi|n|y).
\]

From 6.11 and the fact that hyperfunctions are determined by their values on the \( \varphi_{\varepsilon^{-1}} \), it follows that \( \alpha - \alpha^\varepsilon = F_\alpha \).

**6.17 The Fourier term of order zero.** In Table 3 we see that \( (F_0\varphi_{2r}) \) is a linear combination of systems of Fourier terms. These are represented by \( N \)-invariant hyperfunctions, as indicated in Table 3. We have seen that the restriction of \( F_0 \alpha \) to \( T \setminus \{ \infty \} \) is represented by \( \tau \mapsto \frac{1}{2} A_0(\alpha) \text{ sign } (\text{Im } \tau) \left( 1 + \tau^2 \right)^{(1+n)/2} \). In
Table 4 we check what the multiples of $\kappa_0(\nu)$, respectively $\lambda(0)$, have to be; we see also that $A_0(\alpha)$ has to vanish if $\nu \geq 2$ is even. Thus we get the expression for $F_{\alpha 0}$ in the lemma; this serves as the definition of $B_0(\alpha)$ and $C_0(\alpha)$.

**6.18 Fourier coefficients.** We have seen in the course of the proof of Lemma 6.13 that for $\pm n > 0$

$$F_{\alpha n}(p(z)) = \frac{(-1)^{r} (\pi |n|)^{-(1+\nu)/2}}{\Gamma\left(\frac{1-\nu}{2} \pm r\right)} A_n(\alpha) e^{2\pi i r x} W_{\pm r, \nu/2} (4\pi |n| y),$$

and that $F_{\alpha n}(p(z))$ is equal to

$$B_0(\alpha) y^{(1-\nu)/2}$$

\[ + \begin{cases} 
A_0(\alpha) \frac{\Gamma\left(-\frac{\nu}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{1-\nu}{2} + r\right)} y^{(1+\nu)/2} & \text{if } \nu \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}, \\
-\frac{1}{\sqrt{\pi}} A_0(\alpha) y^{1/2} (\log y + l_\nu) & \text{if } \nu = 0, \\
(A_0(\alpha) - i \text{sign}(r) C_0(\alpha)) \frac{\Gamma\left(-\frac{\nu}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{1-\nu}{2} + r\right)} y^{(1+\nu)/2} & \text{if } \nu \geq 1 \text{ is odd}, \\
C_0(\alpha) \frac{(-1)^{r}}{1 \pm r + |r|} y^{(1+\nu)/2} & \text{if } \nu \geq 2 \text{ is even.} 
\end{cases} \]

**6.19 Isomorphism.** Let $\nu \in \mathbb{C} \setminus (1 + 2\mathbb{Z})$. The isomorphism $\iota(\nu) : M_{-\nu}^0 \to M_{-\nu}^0$ induces an isomorphism $\iota(\nu) : A_{-\nu}^\nu(\Gamma) \to A_{-\nu}^\nu(\Gamma)$, see (1.1). It preserves polynomial growth at $\infty$. A consideration of Fourier terms leads to

$$A_0(\iota(\nu) \alpha) = (\pi |n|)^{-\nu} \Gamma\left(\frac{1+\nu}{2}\right) \Gamma\left(\frac{1-\nu}{2}\right)^{-1} A_n(\alpha) \quad \text{if } n \neq 0,$$

$$B_0(\iota(\nu) \alpha) = \frac{1}{\sqrt{\pi}} \Gamma\left(-\frac{\nu}{2}\right) \Gamma\left(\frac{1-\nu}{2}\right)^{-1} A_0(\alpha) \quad \text{if } \nu \notin \mathbb{Z}_{\geq 0},$$

$$= \Gamma\left(\frac{1+\nu}{2}\right)^{-1} C_0(\alpha) \quad \text{for } \nu \in 2\mathbb{N},$$

$$A_0(\iota(\nu) \alpha) = \sqrt{\pi} \Gamma\left(\frac{1+\nu}{2}\right) \Gamma\left(\frac{\nu}{2}\right)^{-1} B_0(\alpha) \quad \text{if } \nu \notin \mathbb{Z}_{\leq 0},$$

$$C_0(\iota(\nu) \alpha) = \Gamma\left(\frac{1+\nu}{2}\right) B_0(\alpha) \quad \text{for } \nu \in -2\mathbb{N}. $$

The isomorphism $\iota(0)$ is the identity.

**6.20 Maass forms.** For the hyperfunction $\alpha \in A_{-\nu}^{2s-1}(\Gamma_{\text{mod}})$ associated to the cuspidal Maass form in (1.1) we have $A_0(\alpha) = B_0(\alpha) = 0$ and $A_n(\alpha) = (\pi |n|)^{s} \Gamma(1-s) a_n$ for $n \neq 0$. We have chosen $\nu = 2s - 1$; the choice $\nu = 1 - 2s$ would be as good; it gives $s(2s - 1) \alpha \in A_{-\nu}^{2s}(\Gamma_{\text{mod}})$.

**6.21 Eisenstein series.** In 5.6 we have given $\varepsilon^*_s \in A_{-\nu}^{2s}(\Gamma_{\text{mod}})$ for $\text{Re } s > 1$. Here the choice $\nu = 1 - 2s$ seems the natural one if one tries to get a hyperfunction from the series in (2.1). From the Fourier expansion in (2.2) we obtain, for $s \notin 1 + \frac{1}{4} \mathbb{Z}_{\leq 0}$:
\[
A_0(\varepsilon_s^*) = 2\sqrt{\pi} \frac{\Gamma(s)}{\Gamma(s - \frac{1}{2})} \Lambda(2s - 1)
\]

This suggests to consider the family \(\varepsilon_s := \frac{1}{2} \Gamma(s - 1)^{-1} \varepsilon_s^*\).

6.22 Proposition. The family \(s \mapsto \varepsilon_s\) is holomorphic on \(\mathbb{C}\).

Remark. We call a family \(s \mapsto \varepsilon_s\) of hyperfunctions holomorphic if \(s \mapsto \langle \varphi, \varepsilon_s \rangle\) is holomorphic for each \(\varphi \in A_T(T)\).

Proof. For \(\tau \in \mathcal{H}^\pm\) we have

\[
f_{\varepsilon_s^*}(\tau) = \pm 2\pi^{1-s} (s - 1) \sum_{n=1}^\infty \sigma_{1-2s}(n)e^{\pm 2\pi in\tau}.
\]

This converges uniformly for \(\tau\) in compact sets. Thus we obtain holomorphy of \(s \mapsto \varepsilon_s\).

The Fourier term of order 0 is

\[
F_0\varepsilon_s = \frac{\Lambda(2s)}{\Gamma(s - 1)} \mu + (s - 1)\Lambda(2s - 1)\kappa_0(1 - 2s) \quad \text{for } s \neq \frac{1}{2},
\]

\[
= \left(\frac{\Lambda(2s)}{\Gamma(s - 1)} + \pi^{-1/2}(s - 1)\Lambda(2s - 1)\right) \mu - \sqrt{\pi} \left(s - \frac{1}{2}\right) (s - 1)\Lambda(2s - 1)\lambda(1 - 2s) \quad \text{for } s \text{ near } \frac{1}{2}.
\]

This shows the holomorphy of \(s \mapsto \varepsilon_s\). The \(\Gamma\text{mod}\)-invariance extends by holomorphy.

6.23 Functional equation. We have \(\Gamma(-s)\varepsilon_{1-s} = \Gamma(1-s)\iota(2s-1)\varepsilon_s\) for \(s \notin \mathbb{Z}\).

7 Geodesic decomposition

7.1 Period polynomials. Let \(H\) be a holomorphic modular cusp form of weight \(2k \geq 12\). The period polynomial \(r_H\) associated to \(H\) is

\[
r_H(X) := \int_0^\infty H(\tau)(X - \tau)^{2k-2} d\tau,
\]

see, e.g., [15]. The cusps 0 and \(\infty\) can be replaced by any pair \((\xi, \eta)\) of cusps. The path of integration should approach \(\xi\) and \(\eta\) along a geodesic for the non-Euclidean metric on \(\mathcal{H}^+\). In this way we arrive at a homogeneous 1-cocycle \(R_H\) with values in the polynomials of degree at most \(2k - 2\):

\[
R_H(\xi, \eta; X) := \int_\xi^\eta H(\tau)(X - \tau)^{2k-2} d\tau.
\]
It satisfies \( R_H(\gamma \cdot \xi, \gamma \cdot \eta) = R_H(\xi, \eta)|_{2-2k} \gamma^{-1} \) for \( \xi, \eta, \theta \in \mathbb{P}^1_Q, \gamma \in \Gamma_{\text{mod.}} \).

(See 4.7 for the action \( F \mapsto F_{2-2k}g \).)

7.2 Discussion. We want to generalize this to hyperfunctions associated to cuspidal Maass forms and, as far as possible, to Eisenstein series. We try to integrate a representative \( g \) of the automorphic hyperfunction \( \alpha \) along a path as given in Figure 8 on page 28. If \( g \) stays bounded on geodesics approaching the cusps \( \xi \) and \( \eta \), then this is no problem. This holds for \( \xi, \eta \in \Gamma \cdot \infty \) if \( \alpha = \alpha^c \).

But if we use a principal value interpretation of the integral near \( \xi \) and \( \eta \), we can extend this approach to more automorphic hyperfunctions.

We arrive at quantities \( \alpha[\xi, \eta] \in M_{\nu}^\vee \) that just fail to be cocycles. In the case that \( \alpha \) corresponds to a holomorphic cusp form, the map \( M^{2k-1} \to E(2k) \) in 4.8 sends \( \alpha[\xi, \eta] \) to a multiple of \( R_H(\xi, \eta) \).

7.3 Definition. Let \( X \subset T, X \neq \emptyset \), be invariant under \( \Gamma \). We call a \( \Gamma \)-decomposition \( p \) of \( \alpha \in A_{\nu,\omega}^\vee(\Gamma) \) on \( X \) a map \( \{ (\xi, \eta) \in X^2 : \xi \neq \eta \} \to M_{\nu-}\omega^\vee : (\xi, \eta) \mapsto \alpha[\xi, \eta]_p \) that satisfies

a) \( \text{Supp}(\alpha[\xi, \eta]_p) \subset [\xi, \eta] \) for all \( \xi, \eta \in X, \xi \neq \eta \).

b) \( \alpha = \sum_{j=1}^n \alpha[\xi_{j-1}, \xi_j]_p = \alpha \) whenever \( \xi_1, \ldots, \xi_n \in X \) satisfy \( \xi_0 < \xi_1 < \cdots < \xi_n = \xi_0 \).

c) \( \alpha[\gamma \cdot \xi, \gamma \cdot \eta] = \pi_\nu(\gamma)\alpha[\xi, \eta] \) for all \( \gamma \in \Gamma \) and \( \xi, \eta \in X, \xi \neq \eta \).

In condition a) the closed interval \([\xi, \eta] \subset T\) is understood to refer to the cyclic order on the circle \( T \). In condition b) the \( \xi_j \) are supposed to go around \( T \) only once: the intervals \([\xi_{j-1}, \xi_j] \) intersect each other only in the end points.

We define \( A_{\nu,\omega}^\vee(\Gamma, X) = \{ \alpha \in A_{\nu,\omega}^\vee(\Gamma) : \alpha \) has a \( \Gamma \)-decomposition on \( X \} \).

7.4 Partings and \( \Gamma \)-decompositions. Let \( \xi \in X \), and let \( I \neq T \) an open interval containing \( \xi \) such that \( T \setminus I \) contains a point \( \eta \in X \). The image of the decomposition \( \alpha = \alpha[\eta, \xi]_p + \alpha[\xi, \eta]_p \) in \( B_T(I) \) determines a parting of \( \alpha \) at \( \xi \) (see 3.4). The map \( \mathbb{R} \to T : \theta \mapsto \cot \theta \) that we used to define \( B_T \) is decreasing.

For a parting at \( \xi \) of a hyperfunction on \( T \) the support of \( \alpha_- \) is to the left of \( \xi \), and that of \( \alpha_+ \) to the right.

Conversely, if we have a parting \( \alpha = \alpha_+ + \alpha_- \) in the stalk \( (B_T)_x \) for each \( \xi \in X \), we get a decomposition satisfying a) and b) in 7.3. Take open intervals \( I_x \) and \( I_\eta \) containing \( \xi \), respectively \( \eta \), with empty intersection and determine \( \alpha[\xi, \eta]_p \) by its restrictions:

\[
\alpha[\xi, \eta]_p|_{I_x} = \alpha_+ \quad \text{on} \quad I_x,
\]
\[
\alpha[\xi, \eta]_p|_{(\xi, \eta)} = \alpha \quad \text{on} \quad (\xi, \eta),
\]
\[
\alpha[\xi, \eta]_p|_{I_\eta} = \alpha_- \quad \text{on} \quad I_\eta,
\]
\[
\alpha[\xi, \eta]_p|_{(\eta, \xi)} = 0 \quad \text{on} \quad (\eta, \xi).
\]

Condition c) is equivalent to \( \pi_\nu(\gamma)\alpha_{\xi, \pm} = \alpha_{\gamma \cdot \xi, \pm} \) for all \( \xi \in X, \gamma \in \Gamma \).

Let us write \( X \) as a disjoint union of \( \Gamma \)-orbits: \( X = \bigsqcup_{\xi \in \mathcal{I}} \Gamma \cdot \xi \). Finding a \( \Gamma \)-decomposition of \( \alpha \) on \( X \) is equivalent to finding a parting \( \alpha = \alpha_+ + \alpha_- \) at
each \( \xi \in \Xi \) such that \( \nu_\nu(\delta)\alpha_{\xi,\pm} = \alpha_{\xi,\pm} \) for each \( \delta \in \Gamma_\xi := \{ \gamma \in \Gamma : \gamma \cdot \xi = \xi \} \). Thus we have obtained:

7.5 **Proposition.** Let \( \alpha \in A^\nu_{\omega}(\Gamma) \).

i) Let \( \xi \in T \). Then \( \alpha \in A^\nu_{\omega}(\Gamma, \Gamma : \xi) \) in the following cases:

a) \( \Gamma_\xi = \{ 1 \} \).

b) There is a pairing \( \alpha = \alpha_- + \alpha_+ \) of \( \alpha \) at \( \xi \) that satisfies \( \pi_\nu(\delta)\alpha_{\pm} = \alpha_{\pm} \)
in the stalk \( (\mathcal{B}_T)_\xi \) for all \( \delta \in \Gamma_\xi \).

ii) Let \( \{ X_j \}_{j \in J} \) be a collection of non-empty, \( \Gamma \)-invariant subsets of \( T \), and put \( X := \bigcup_{j \in J} X_j \). Then \( \alpha \in A^\nu_{\omega}(\Gamma, X) \) if and only if \( \alpha \in A^\nu_{\omega}(\Gamma, X_j) \) for all \( j \in J \).

**Remark.** It may very well be true that \( A^\nu_{\omega}(\Gamma, X) = A^\nu_{\omega}(\Gamma) \) for all \( X \). In this paper we direct our attention to \( X = \Gamma \cdot \infty \).

7.6 **Definition.** Let \( \alpha \) be a hyperfunction (not necessarily automorphic), represented by \( g \in \mathcal{O}(U \setminus T) \) for some neighborhood \( U \) of \( T \) in \( \mathbb{P}^1_\mathbb{C} \). We define \( \alpha \) to have **geodesic approach at \( \infty \)** if for each holomorphic function \( \varphi \) on a neighborhood of \( \infty \) the following conditions are satisfied:

a) For each \( \xi \in \mathbb{R} \) and for each sufficiently large \( y > 0 \) the function

\[
t \mapsto \frac{i}{\pi} \left( \frac{\varphi(\xi + it)g(\xi + it)}{1 + (\xi + it)^2} + \frac{\varphi(\xi - it)g(\xi - it)}{1 + (\xi - it)^2} \right)
\]

is integrable on \([y, \infty)\) (with respect to the measure \( dt \)).

b) For all \( \xi, \xi_1 \in \mathbb{R} \):

\[
\lim_{y \to \infty} \int_{x = \xi}^{\xi_1} \frac{1}{\pi} \left( \frac{\varphi(x + iY)g(x + iY) - \varphi(x - iY)g(x - iY)}{1 + (x + iY)^2} \right) \, dx = 0.
\]

Let \( m \in G \). We define \( \alpha \) to have geodesic approach at \( m \cdot \infty \) if \( \pi_\nu(m)\alpha \) has geodesic approach at \( \infty \). This does not depend on the choice of \( \nu \in \mathbb{C} \).

7.7 **Discussion.** The integral of the function in condition a) is a principal value variant of the integral \( \int_{L_\xi} \varphi(\tau)g(\tau) \frac{d\tau}{\pi(1 + \tau^2)} \), where \( L_\xi \) is the path from \( \xi + iy \) vertically upward to \( \infty \) in \( \mathbb{H}^+ \), and then from \( \infty \) vertically upward to \( \xi - iy \). If \( g \) is bounded on a neighborhood of \( \infty \), then this integral exists.

The formulation in condition a) allows some cancellation between both parts of the integral. Similarly, the integral in condition b) is a principal value form of \( \int_{M(\xi, \xi_1; Y)} \varphi(\tau)g(\tau) \frac{d\tau}{\pi(1 + \tau^2)} \), where \( M(\xi, \xi_1; Y) \) consists of a path from \( \xi + iy \) to \( \xi_1 + iY \) and a path from \( \xi_1 - iY \) to \( \xi - iy \). These two conditions allow us to define integrals of \( \varphi(\tau)g(\tau) \frac{d\tau}{\pi(1 + \tau^2)} \) from a point \( z \in \mathbb{H}^+ \) via \( \infty \) to the points \( \bar{z} \in \mathbb{H}^- \), where \( \infty \) is crossed along two conjugate geodesics in \( \mathbb{H}^+ \) and \( \mathbb{H}^- \). The particular choice of the geodesic does not matter. The choice of the representative \( g \) does not influence the conditions.
Let \( p \in G \) be such that \( p \cdot \infty = \infty \). Then \( \pi_p(p) \alpha \) has a representative \( g_1 : \tau \mapsto J(\tau)^{(1+r)/2} g(p^{-1} \cdot \tau) \), with \( J \) holomorphic on a neighborhood of \( \infty \).

As \( p \) has the form \( \left( \begin{smallmatrix} 0 & * \\ 0 & * \end{smallmatrix} \right) \), conditions a) and b) for \( g \) and \( g_1 \) are equivalent. This implies that the definition of geodesic approach at \( m \cdot \infty \) does not depend on \( m \), only on \( m \cdot \infty \).

**7.8 Principal value integrals.** Geodesic approach at \( \xi \in T \) allows us to integrate over paths passing the point \( \xi \) along any pair of conjugate geodesics we like.

If \( L \) is a path that crosses \( T \) at a finite number of points along pairs of conjugate geodesics, we denote by \( \text{pv} \int_L \phi(\tau) g(\tau) d\tau \) the integral over \( L \) in which the contributions along each of the pairs of geodesics has the interpretation given above.

If we deform \( L \) in such a way that the end points and the points at which \( T \) is crossed are kept fixed (the corresponding pairs of conjugate geodesics may change), then the integral does not change.

**7.9 Lemma.** Let the function \( g \) represent \( \alpha \in M'_{\nu} \) on a neighborhood of \( \infty \). Define for \( \xi \in \mathbb{R} \) and \( t \) large:

\[
F_+(\xi, t) := \frac{1}{2} \left( \frac{g(\xi + it)}{1 + (\xi + it)^2} + \frac{g(\xi - it)}{1 + (\xi - it)^2} \right),
\]

\[
F_-(\xi, t) := \frac{1}{2t} \left( \frac{g(\xi + it)}{1 + (\xi + it)^2} - \frac{g(\xi - it)}{1 + (\xi - it)^2} \right).
\]

i) Condition a) in the definition of geodesic approach at \( \infty \) in 7.6 is equivalent to the integrability (for the measure \( dt \)) of all \( F_\pm(\xi, \cdot) \) on each interval \([y, \infty)\) with \( y \) large.

ii) If \( F_+(x, t) = o(t) \) and \( F_-(x, t) = o(1/t) \) as \( t \to \infty \) uniformly for \( x \) between \( \xi \) and \( \xi_1 \), then condition b) in the definition of geodesic approach at \( \infty \) is satisfied.

iii) Let \( L_\xi \) be the path along from \( \xi + iy \) via \( \infty \) to \( \xi - iy \) indicated above, and let \( \varphi \) be holomorphic on a neighborhood of \( \infty \) containing \( L_\xi \). If \( \alpha \) has geodesic approach at \( \infty \), then \( \text{pv} \int_{L_\xi} \varphi(\tau) g(\tau) \frac{d\tau}{\pi(1+\tau^2)} \) is equal to

\[
\frac{i}{\pi} \int_y^\infty F_+(\xi, t) \sum_\pm \varphi(\xi \pm it) \, dt + \frac{i}{\pi} \int_y^\infty F_-(\xi, t) \sum_\pm \pm t \varphi(\xi \pm it) \, dt.
\]

**Proof.** Take \( \varphi(\tau) = 1 \) and \( \varphi(\tau) = \frac{1}{\tau - \xi} \) in condition a) to see that \( F_{\xi, \pm} \) is integrable.

We note that \( \sum_\pm \varphi(\xi \pm it) \) and \( \sum_\pm \pm t \varphi(\xi \pm it) \) are bounded as \( t \to \infty \).

(Use the holomorphy of \( \varphi \) at \( \infty \).) A computation shows that the integral of the function in condition a) equals the sums of the integrals in part iii). This gives part iii) and the converse implication in part i).

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Figure 5: The contour $Q(\xi, \eta)$ used in the definition of the geodesic decomposition is the union of $Q_+(\xi, \eta)$ and $Q_-(\xi, \eta)$. Near $\xi$ and $\eta$ the contours are pieces of conjugate geodesics in $\mathcal{H}^+$ and $\mathcal{H}^-$. 

Part ii) follows from the facts that the integral in condition b) is equal to 

$$\frac{1}{\pi} \int_{\xi}^{\xi+i} \left( F_+(x, Y) \sum_{\pm} \pm \varphi (\xi \pm i Y) + F_-(x, Y) \sum_{\pm} Y \varphi (\xi \pm i Y) \right) dx,$$

and that $\sum_{\pm} \pm \varphi (\xi \pm i Y) = O(1/Y)$ as $Y \to \infty$.

**7.10 Lemma.** The following hyperfunctions have geodesic approach at $\infty$:

i) $\alpha^c$ for each $\alpha \in A_{\nu-2}^c(\Gamma)$ with polynomial growth at $\infty$.

ii) $\mu$ and $\lambda(0)$.

iii) $\kappa_0(\nu)$ for $\Re \nu < 1$.

**Proof.** If a representative $g$ is bounded on vertical lines uniformly for $\Re \tau$ in compact sets, then the functions $F_\pm$ in Lemma 7.9 satisfy $F_\pm(\xi, t) = O(t^{-2})$. This suffices for $\alpha^c$, see 6.17. 

The representative $g(\tau) = \frac{-i}{2} \tau$ of $\mu$ satisfies $F_\pm(\xi, t) = O(t^{-2})$.

For part iii) we use the representative $q$ in 6.9 for $\nu \not\in -2\mathbb{N}$. This gives $F_\pm(\xi, t) = O(t^{\Re \nu - 2})$. For $\nu \in -2\mathbb{N}$ we proceed as for $\alpha^c$.

Finally, we check that $F_\pm(\xi, t) = O(t^{-2} \log t)$ for $\lambda(0)$.

**7.11 Lemma.** Let $\alpha \in A_{\nu-2}^c(\Gamma)$ have polynomial growth at $\infty$. Then $\alpha$ has geodesic approach at all points of the orbit $\Gamma \cdot \infty$ if one of the following conditions is satisfied:

a) $\Re \nu < 1$.

b) $A_0(\alpha) = 0$, and if $\nu \in \mathbb{N}$, then $C_0(\alpha) = 0$.

**Proof.** The geodesic approach at $\infty$ follows directly from Lemma 7.10. Use the $\Gamma$-invariance for the other points of $\Gamma \cdot \infty$.

**7.12 Geodesic decomposition.** Let $X \subset$ be the set of points at which a given hyperfunction $\beta$ has geodesic approach.

Let $\xi, \eta \in X, \xi \neq \eta$. We define for $\varphi \in \mathcal{A}_T(T)$:

$$\langle \varphi, \beta[\xi, \eta] \rangle := \text{pv} \int_{Q(\xi, \eta)} \varphi(\tau) g(\tau) \frac{d\tau}{\pi(1+\tau^2)}.$$
with the contour $Q(\xi, \eta)$ given in Figure 3. It is understood that the region between $Q(\xi, \eta)$ and the interval $[\xi, \eta]$ is contained in the domain of $\varphi$. To see that this defines $\beta[\xi, \eta]_g$ as a hyperfunction, we estimate $(\varphi_{2r}, \beta[\xi, \eta]_g)$ in terms of the supremum norm of $\varphi_{2r}$ and its first derivative (with respect to a local coordinate). (Use part iii) of Lemma 7.4.)

Consider $\langle h_{\tau_0}, \beta[\xi, \eta]_g \rangle$ (see 3.10), with $\tau_0$ outside $Q(\xi, \eta)$, to see that the hyperfunction $\beta[\xi, \eta]_g$ has support contained in $[\xi, \eta]$. If $\theta \in X \cap (\xi, \eta)$, then $\beta[\xi, \eta]_g = \beta[\xi, \theta]_g + \beta[\theta, \eta]_g$.

If $T = \bigcup_{j=1}^m [\xi_j, \xi_{j+1}]$ is a partition of $T$ with $\xi_{m+1} = \xi_1 < \cdots < \xi_m \in X$ such that the intervals $[\xi_j, \xi_{j+1}]$ intersect each other only in their end points, then $\beta = \sum_{j=1}^m \beta[\xi_j, \xi_{j+1}]_g$. This we call the geodesic decomposition on $X$. In the sequel we write $\beta[\xi, \eta]$ instead of $\beta[\xi, \eta]_g$.

**Lemma 7.13** $\Gamma$-behavior. Let $m \in G$. If $\xi, \eta, m \cdot \xi, m \cdot \eta \in X$, then a computation shows that $\tau(\xi, m \cdot \xi) = (\tau_0(m) \cdot \beta)[m \cdot \xi, m \cdot \eta]$. So, if $\alpha$ is an automorphic hyperfunction, the set $X$ of points at which it has geodesic approach is $\Gamma$-invariant, and the geodesic decomposition of $\alpha$ on $X$ is a $\Gamma$-decomposition.

**Theorem 7.14** Reflection. If $\beta$ has geodesic approach at $\xi, \eta, -\xi, -\eta$, then $\langle \beta[\xi, \eta] \rangle = (i \alpha)[-\eta, -\xi]$.

**Theorem 7.15** Let $\alpha \in \mathcal{A}_\infty^c(\Gamma)$ have polynomial growth at $\infty$. Then $\alpha$ has geodesic decomposition on the orbit $\Gamma \cdot \infty$ if one of the following conditions is satisfied:

a) $\text{Re} \nu < 1$.

b) $A_0(\alpha) = 0$, and if $\nu \in \mathbb{N}$, then $C_0(\alpha) = 0$.

**Remarks.** One finds the Fourier coefficients $A_0(\alpha)$ and $C_0(\alpha)$ in Lemma 7.13.

If $\alpha$ corresponds to a system cusp forms, then condition b) holds.

Of course, if $\Gamma$ has more than one cuspidal orbit, we may move $\Gamma$-inequivalent cusps to $\infty$ by conjugation. In particular if at each cusp one of the conditions holds, then we have geodesic decomposition on the set of all cusps. Note that the theorem allows $\alpha$ to have terrible growth at cusps that are not $\Gamma$-equivalent to $\infty$.

**Proof.** Directly from Lemma 7.13.

**Locally holomorphic cusp forms.** Let $\alpha \in H^{2k-1}_c(\Gamma)$ correspond to a holomorphic cusp form $H$ for $\Gamma$ of weight $2k \geq 2$ (see 5.4). Then $\alpha$ has geodesic decomposition on the set $X$ of all $\Gamma$-cusps. We consider the image under the map $\beta \mapsto \beta^{(2k)}$ discussed in 7.4. For $\xi, \eta \in X$, $\xi \neq \eta$, we find the period integral discussed in 7.4.

$$(\alpha[\xi, \eta])^{(2k)} = \int_{Q(\xi, \eta)} h_X(\tau)(-1)^k 4^{-k} (1 + \tau^2)^k H(\tau) \frac{d\tau}{\pi(1 + \tau^2)}$$

$$= \frac{1}{4\pi} \int_{\xi}^{\eta} H(\tau) (\tau - X)^{2k-2} d\tau = \frac{1}{4\pi} R_H(\xi, \eta).$$

**Eisenstein family.** The Eisenstein family $\varepsilon_s$, introduced in 6.21, has geodesic decomposition on $\mathbb{P}^1_\mathbb{Q}$ for $\text{Re} s > 0$. From the explicit expression in 7.6.
for $\langle \varphi, \varepsilon_s \rangle$ converging for $\text{Re } s > 1$, one expects that the geodesic decomposition has the following form

$$\langle \varphi, \varepsilon_s[\xi, \eta] \rangle = \frac{1}{2\pi} (s-1) \sum_{p,q} f_{p,q} \left( p^2 + q^2 \right)^{-s} \varphi \left( \frac{-p}{q} \right)$$

(7.2)

for $\text{Re } s > 1$ and $\xi, \eta \in \mathbb{P}^1 \mathbb{Q}$, with $f_{p,q} := 1$ if $-\frac{p}{q} \in \langle \xi, \eta \rangle$, $f_{p,q} := \frac{1}{2}$ if $-\frac{p}{q} = \xi$ or $\eta$, and $f_{p,q} := 0$ otherwise.

This turns out to be true. In the proof one uses the representative $g_s(\tau) = \frac{-1}{2\pi} \pi^{-s} (s-1) \sum_{p,q} f_{p,q} \left( p^2 + q^2 \right)^{-s} \text{Pr}_{p,q}$ in the definition of $\langle \varphi, \varepsilon_s[\xi, \eta] \rangle$. The main point is to interchange the order of integration and summation. This is no problem on most of $Q(\xi, \eta)$. But near $\xi$, and near $\eta$, we have to treat together terms for which the points $-\frac{p}{q}$ are symmetrical with respect to $\xi$, respectively $\eta$.

One can also show that $s \mapsto \langle \varphi, \varepsilon_s[\xi, \eta] \rangle$ is holomorphic on $\text{Re } s > 0$.

## 8 Infinite sums

The geodesic decomposition allows us to write some automorphic hyperfunctions as a finite sum. Under additional conditions an infinite decomposition is possible.

### 8.1 Weak convergence

Limits and infinite series of hyperfunctions we consider in the weak sense: $\lim_{n \to \infty} \beta_n = \beta$ means $\lim_{n \to \infty} \langle \varphi, \beta_n \rangle = \langle \varphi, \beta \rangle$ for each $\varphi$ that is holomorphic on a neighborhood in $\mathbb{P}^1 \mathbb{Q}$ of $\text{Supp}(\beta) \cup \bigcup_{n \geq N} \text{Supp}(\beta_n)$ for some $N$, and similarly for convergence of series.

### 8.2 Theorem

Let $\xi \in T$, $\xi \neq \infty$. Suppose that $\alpha \in A^{\nu, \omega}(\Gamma)$ has polynomial growth at $\infty$, has geodesic approach at $\xi$, and satisfies the following conditions:

a) $\text{Re } \nu < 1$,

b) if $A_0(\alpha) \neq 0$, then $\text{Re } \nu < 0$.

Then

$$\alpha[\xi, \infty] = \frac{1}{2} B_0(\alpha) \mu + \sum_{n=1}^{\infty} \alpha[\xi - n, \xi + n],$$

$$\alpha[\infty, \xi] = \frac{1}{2} B_0(\alpha) \mu + \sum_{n=1}^{\infty} \alpha[\xi - n, \xi - n + 1].$$

Remarks. The conditions imply that $\alpha$ has geodesic approach at $\infty$. The $\Gamma$-orbit of $\xi$ contains the set $\xi + \mathbb{Z}$. So $\alpha$ has also geodesic approach at all points $\xi + n$, $n \in \mathbb{Z}$.

The theorem applies to hyperfunctions associated to cuspidal Maass forms; then $B_0(\alpha) = 0$. The statement is in general false for hyperfunctions associated to holomorphic cusp forms. The hyperfunction $\varepsilon_s$ satisfies the conditions for
Re \( s > \frac{1}{2} \). The presence of the term \( \frac{1}{2} B_0(\alpha) \mu \) is very clear if one considers (7.2) on page 30 for \( Re s > 1 \).

Proof. See 8.3.1.

8.3 Reformulation. By conjugation of \( \Gamma \) we can arrange that \( \xi = 0 \). Application of the result for \( [0, \infty] \) to \( j (n) \in A'_{\omega}(j(\Gamma)) \) gives the result for \( [\infty, 0] \). The conditions stay valid under both transformations.

Let \( \varphi \) be holomorphic in a neighborhood of \( \infty \). The weak interpretation of the series means that we have to prove \( \lim_{n \to \infty} \langle \varphi, \alpha|n, \infty \rangle \rangle = \frac{1}{2} B_0(\alpha) \). We write \( \varphi(\tau) = \psi(1/\tau) \), with \( \psi \) holomorphic on a compact neighborhood of 0. By \( \| \cdot \| \) we denote the supremum norm on this neighborhood. Let \( C_n \) be the vertical line \( \text{Re} \tau = n \), and let \( g \) denote a representative of \( \alpha \). If \( n \) is large enough (depending on \( \varphi \) and \( g \)) we have \( \langle \varphi, \alpha|n, \infty \rangle \rangle = I_n(g) \), where \( I_n(h) := \frac{1}{\pi} \text{pv} \int_{C_n} \psi(\frac{1}{\tau}) h(\tau) d\tau \). Taking the principal value interpretation into account, we have

\[
I_n(h) = \frac{i}{\pi} \sum_{\pm} \int_0^\infty \psi \left( \frac{1}{n \pm it} \right) h(n \pm it) \frac{dt}{1 + (n \pm it)^2}.
\]

The conditions allow us to write the representative as \( g(\tau) = g_0^\nu(\tau) + \pi^{-1/2} \Gamma(-\frac{\nu}{2}) A_0(\nu) g_\nu + B_0(\alpha) g_\mu \), with \( g_\nu \) and \( g_\mu \) representatives of \( \kappa_0(\nu) \) and \( \mu \). We want to show that \( I_n(h) = o(1) \) as \( n \to \infty \) for \( h = g_0^\nu \), and for \( h = g_\nu \) if \( \text{Re} \nu < 0 \), and that \( \lim_{n \to \infty} I_n(g_\mu) = \frac{1}{\pi} \psi(0) \).

8.4 Contribution of \( \alpha^\nu \). Let \( \text{Re} \nu < 1 \). We use the expression for \( g_0^\nu \) in (6.7) on page 30. So \( I_n(g_0^\nu) \) splits up into three terms.

The third term in (6.7) contributes a constant to \( g_0^\nu \). The corresponding integral is estimated by

\[
\int_0^\infty \frac{dt}{1 + (n + it)^2} \leq \frac{1}{n} \int_0^\infty \frac{dt}{(1 + it)^2 + 1/n^2} = o(1).
\]

The middle term in (6.7) is \( O\left( \frac{1}{n^2} \right) \). The corresponding integral is \( O(n^{-2}) \).

For the first term we use the periodicity of \( f_0^\nu \) and obtain

\[
I_n = \frac{i}{\pi} \sum_{\pm} \int_0^\infty \psi \left( \frac{1}{n \pm it} \right) f_0^\nu(0 \pm it) \left( 1 + (n \pm it)^2 \right)^{(\nu - 1)/2} dt.
\]

The integral over \([1, \infty)\) we estimate by \( \int_1^\infty \| \psi \| |e^{-\epsilon t}| |n + it|^{\text{Re} \nu - 1} dt \). This is \( O\left( n^{\text{Re} \nu - 1} \right) = o(1) \), for some \( \epsilon > 0 \).

The function \( \tau \mapsto f_\nu^\varphi(\tau) (1 + \tau^2)^{(1+\nu)/2} \) represents the restriction of \( \alpha \) to \( T \setminus \{ \infty \} \). As \( \alpha \) has geodesic approach at \( n \), the integral \( \int_0^1 \sum_{\pm} \varphi_1(n \pm it) f_0^\nu(\pm it) \left( n + (1 + it)^2 \right)^{(\nu - 1)/2} dt \) converges for all \( \varphi_1 \) holomorphic on a neighborhood of \( n + i[-1, 1] \). Put \( G_+(t) := f_0^\nu(\pm it) \), and \( G_-(t) := it \sum_{\pm} \varphi_1(\pm it) \). The choice \( \varphi_1(\tau) = (1 + \tau^2)^{(1-\nu)/2} \) shows that \( G_+ \) is integrable on \([0, 1]\). The integrability
of $G_-$ follows from $\varphi_1(\tau) = (\tau - n) (1 + \tau^2)^{(1-\nu)/2}$. We obtain
\[
\int_0^1 \sum_{\pm} \psi \left( \frac{1}{n \pm it} \right) f_n^c(\pm it) \left( 1 + (n \pm it)^2 \right)^{(\nu-1)/2} dt \\
= \int_0^1 \frac{1}{2} G_+(t) \sum_{\pm} \psi \left( \frac{1}{n \pm it} \right) \left( 1 + (n \pm it)^2 \right)^{(\nu-1)/2} dt \\
+ \int_0^1 \frac{1}{2} G_-(t) \frac{1}{t} \sum_{\pm} \psi \left( \frac{1}{n \pm it} \right) \left( 1 + (n \pm it)^2 \right)^{(\nu-1)/2} dt
\]
The integral with $G_+$ is $O(\|\psi\| n^{\text{Re} \nu - 1}) = o(1)$. Next we note that
\[
\frac{1}{t} \sum_{\pm} \psi \left( \frac{1}{n \pm it} \right) \left( 1 + (n \pm it)^2 \right)^{(\nu-1)/2} = O \left( n^{\text{Re} \nu - 3} \|\psi\|' + n^{\text{Re} \nu - 2} \|\psi\| \right)
\]
for $0 < t < 1$. So the other integral is $o(1)$ as well.

8.5 Contribution of $k_0(\nu)$. If $\nu \in -2\mathbb{N}$, then $g_\nu(\tau) = O(1)$ for $\text{Re} \tau \geq 2$. This can be shown by the method of 6.8 (deform the line of integration into a narrow $I_+$). So $I_n(g_\nu) = O(1/n)$.

For $\nu \notin -2\mathbb{N}$ we take $g_\nu(\tau) = \tau^{1+\nu} (1 + \tau^{-2})^{(1+\nu)/2} e^{-\pi i \nu \text{sign}(\text{Im} \tau)/2}$, see 6.3.
\[
I_n(g_\nu) = \frac{i}{\pi} \int_0^\infty \sum_{\pm} \psi \left( \frac{1}{n \pm it} \right) (n \pm it)^{-1} \left( 1 + (n \pm it)^2 \right)^{(1+\nu)/2} e^{\pi i \nu / 2} dt \\
= n^{\nu-1} \frac{i}{\pi} \int_0^\infty \sum_{\pm} O(\|\psi\|) (t \mp i)^{-1} \left( 1 + O(n^{-2}) \right) dt
\]
The integral converges if $\text{Re} \nu < 0$. Under that condition we find $I_n(g_\nu) = O(n^{\text{Re} \nu}) = o(1)$.

Necessity. The integral $I_n(g_\nu)$ with $\psi(\tau) = (1 + \tau^2)^{(1+\nu)/2}$ equal $n^\nu$ times a non-zero function of $\nu$. So the bound $\text{Re} \nu < 0$ is needed if $A_0(\alpha) \neq 0$.

8.6 Contribution of $\mu$. $I_n(g_\mu) = \lim_{T \to \infty} \frac{1}{\pi} \int_{-iT}^{n+iT} \psi \left( \frac{1}{\tau} \right) \frac{-i}{2} \tau \frac{d\tau}{1+\tau^2} = \frac{1}{2} \psi(0)$, for each $n$.

9 Universal covering group

To see that the geodesic decomposition of automorphic hyperfunctions is closely related to 1-cocycles it is better not to work on $\text{PSL}_2(\mathbb{R})$, but on its universal covering group.

9.1 Universal covering group. The universal covering group $\tilde{G}$ is a central extension of $G = \text{PSL}_2(\mathbb{R})$ with center $\tilde{Z} \cong \mathbb{Z}$. As an analytic variety it is isomorphic to $\tilde{\mathfrak{g}}^+ \times \mathbb{R}$. This isomorphism is written as $(z, \theta) \mapsto \tilde{p}(z) \hat{k}(\theta)$ which covers the isomorphism $\tilde{\mathfrak{g}}^+ \times (\mathbb{R} \mod \pi \mathbb{Z}) \to G : (z, \theta) \mapsto p(z)k(\theta)$.
There are injective continuous group homomorphisms $\mathbb{R} \to G : x \mapsto \tilde{n}(x)$, $\mathbb{R}_{>0} \to G : y \mapsto \tilde{a}(y)$, and $\mathbb{R} \to G : \theta \mapsto \tilde{k}(\theta)$, covering respectively $x \mapsto \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$, $y \mapsto \left(\begin{smallmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{\pi} \end{smallmatrix}\right)$, and $\theta \mapsto k(\theta)$. We have $\tilde{p}(z) = \tilde{n}(x)\tilde{a}(y)$. The center of $\tilde{G}$ is $\tilde{Z} := \tilde{k}(\pi \mathbb{Z})$. The group $\tilde{P} := \tilde{p}(\tilde{\theta}^+)$ is isomorphic to $P$; it is the connected component of 1 in the parabolic subgroup $\tilde{Z}\tilde{P}$ of $\tilde{G}$.

The projection $\tilde{G} \to G$ we write as $g \mapsto \tilde{g}$. We define a lifting $\SL_2(\mathbb{R}) \to \tilde{G} : g \mapsto \tilde{g}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tilde{p}(z) = \tilde{p}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) -1 k(-\arg(cz + d))$, with $-\pi < \arg \leq \pi$.

Some properties: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ if $\arg(c i + d) \in (\pi, \pi)$, $\tilde{p}(z) = \tilde{p}(\tilde{k}(\theta))$ for $-\pi < \theta < \pi$.

If $\tilde{g} := \tilde{g}(\hat{\theta})$ is the full original in $\tilde{G}$ of the discrete subgroup $\Gamma$. It contains $\tilde{Z}$.

9.2 $\tilde{T} := \tilde{P}\backslash\tilde{G}$ is a covering of $T = P\backslash G$. We use the coordinate $\theta$ corresponding to $\tilde{r} := \tilde{P}(\theta)$ in $\tilde{T}$ $\theta \mapsto \tau = \cot \theta$. The covering map $\tilde{T} \to T$ corresponds to $pr : \theta \mapsto \tau$. We denote the right action of $\tilde{G}$ on $\tilde{T}$ in terms of the coordinate $\theta$ by $g : \theta \mapsto \theta \cdot g$. We have $\theta \cdot \tilde{k}(\theta_1) = \theta + \theta_1$. If $\arg(c i + d) \in (\pi, \pi)$, then $\theta \mapsto \theta \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the strictly increasing analytic function given by $\theta \mapsto \arg((ia - b) \sin \theta + (d - ic) \cos \theta)$ on a neighborhood of $\theta = 0$. This function satisfies $(\theta + \pi) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \theta \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) + \pi$.

The automorphism $j$ of $G$ is covered by the automorphism $\tilde{p}(z)\tilde{a}(\theta) \mapsto \tilde{p}(\tilde{z})\tilde{a}(\tilde{\theta})$ of $\tilde{G}$, also denoted by $j$. It induces an involution $j$ in the functions on $\tilde{G}$, which corresponds to the reflection $\theta \mapsto -\theta$ in $\tilde{T}$.

9.3 Representations. Let $(\tilde{\pi}_\nu, \tilde{M}_\nu)$ be the representation of $\tilde{G}$ in the functions on $\tilde{G}$ that transform on the left according to $\tilde{p}(z) \mapsto y^{(1+\nu)/2}$. (Note that the $\tilde{Z}$-behavior is not fixed. This is an induced representation from $\tilde{P}$ to $\tilde{G}$.) Usually one would induce from $\tilde{Z}\tilde{P}$ to $\tilde{G}$.) This representation can be realized in the functions on $\tilde{T}$. This leads to the action $\tilde{\pi}_\nu(\tilde{G})$ in the analytic functions $\tilde{M}_\nu \subset \tilde{A}(\mathbb{R})$ on $\tilde{T}$, and in the space of hyperfunctions $\tilde{M}_\nu := \tilde{B}(\mathbb{R})$. We identify $\tilde{T}$ with $\mathbb{R}$ by means of the coordinate $\theta$. Note that $\tilde{\pi}_\nu(\tilde{k}(\zeta))$ acts as the translation over $\zeta$.

Let $\tilde{M}_\nu := \tilde{B}_b(\mathbb{R})$ be the subspace of $\tilde{M}_\nu$ of hyperfunctions with bounded support. It is invariant under $\tilde{\pi}_\nu$, and there is a duality between $(\tilde{\pi}_\nu, \tilde{M}_\nu)$ and $(\tilde{\pi}_{-\nu}, \tilde{M}_{-\nu})$, that can be described with explicit integrals, see [..]

Functions on $G$ correspond to functions on $\tilde{G}$ that are invariant under the center $\tilde{Z}$. In this way we obtain the following identifications:

$$M_\nu = \left(\tilde{M}_\nu\right)^\hat{2}, \quad M_{-\nu} = \left(\tilde{M}_{-\nu}\right)^\hat{2}, \quad \tilde{A}_\nu(\Gamma) = \left(\tilde{M}_{-\nu}\right)^\hat{1}.$$}

These identifications respect the reflection $j$. The action of $\tilde{\pi}_\nu(g)$ in the $\pi_\nu(\tilde{Z})$-invariant spaces on the right corresponds to the action of $\pi_\nu(\tilde{g})$ in the spaces on the left.
9.4 Projection. There is also a natural map \( \sigma : \hat{M}_b^\nu \to M_{\nu - \omega}^p \). If \( \beta \in \hat{M}_b^\nu \) has support inside an interval \( I \) of length smaller than \( \pi \) it corresponds, via composition with \( \pi : \theta \mapsto \cot \theta \), to a hyperfunction on \( T \) with support inside \( \text{pr}(I) \). This hyperfunction we define to be \( \sigma \beta \). We extend \( \sigma \) to \( \hat{M}_b^\nu \) by additivity, using a decomposition based on partings as discussed in \( \text{III} \). This linear map \( \sigma \) respects the reflection \( j \) and satisfies \( \pi_\nu(\hat{g})\sigma = \sigma \pi_\nu(g) \). The kernel of \( \sigma \) consists of the elements of the form \((\pi_\nu(\hat{k}(\pi)) - 1)\beta_1 \) with \( \beta_1 \in \hat{M}_b^\nu \).

9.5 Cohomology. Let \( X \subset \hat{T} \) be invariant under \( \hat{\Gamma} \). Let \( A \) be a \( \hat{\Gamma} \)-module. We consider the complex \( C_X^\nu(\hat{\Gamma}, A) \) defined by

\[
C_X^\nu(\hat{\Gamma}, A) := \{ c : X^{n+1} \to A : c(\xi_0 : \gamma_1, \ldots, \xi_n : \gamma) = \gamma^{-1} c(\xi_0, \ldots, \xi_n) \}
\]

\[
dc(\xi_0, \ldots, \xi_n+1) := \sum_{i=0}^{n+1} (-1)^i c(\xi_0, \ldots, \hat{\gamma}_i, \ldots, \xi_n+1).
\]

\( Z_X^\nu(\hat{\Gamma}, A), B_X^\nu(\hat{\Gamma}, A), \) and \( H_X^\nu(\hat{\Gamma}, A) \) are the corresponding groups of cocycles, coboundaries and cohomology classes.

If \( \Gamma \) has only one cuspidal orbit, and \( X \) is the full original in \( \hat{T} \) of the set of cusps, then \( H_X^1(\hat{\Gamma}, A) \) is the usual parabolic cohomology, in which the inhomogeneous 1-cocycle \( \eta \) have the additional property \( \eta(\pi) \in (\pi - 1)A \) if \( \pi \) fixes an element of \( X \). (To a homogeneous 1-cocycle \( c \) corresponds \( \eta_c : \gamma \mapsto c(0 : \gamma^{-1}, 0) \).

9.6 \( \Gamma \)-decomposition and 1-cocycles. Suppose that \( \alpha \in A_{\nu - \omega}^p(\Gamma) = \left( \hat{M}_{\nu - \omega}^p \right)^\hat{\Gamma} \) has a \( \Gamma \)-decomposition \( p \) on \( X \). Let \( \hat{X} \) be the full original of \( X \) in \( \hat{T} \). We construct a cocycle \( c(\alpha, p) \in Z_X^1(\hat{\Gamma}, \hat{M}_b^\nu) \) in the following way:

a) If \( \xi, \eta \in \hat{X}, \xi < \eta < \xi + \pi \), then \( c(\alpha, p; \eta, \xi) \) is the hyperfunction with support inside \( [\xi, \eta]_p \) that satisfies \( \sigma c(\alpha, p; \eta, \xi) = \alpha(\text{pr}(\eta), \text{pr}(\xi))_p \).

The properties of \( \alpha[.,.]_p \) imply \( c(\alpha, p; \eta, \theta) + c(\alpha, p; \theta, \xi) = c(\alpha, p; \eta, \xi) \) if \( \xi < \theta < \eta < \xi + \pi \), and \( c(\alpha, p; \eta, \gamma, \xi) = \pi_\nu(\gamma)^{-1} c(\alpha, p; \eta, \xi) \) for all \( \gamma \in \hat{\Gamma} \).

b) If \( \eta \geq \xi + \pi \), then we take intermediate points \( \theta = \xi < \theta \_1 < \cdots < \theta_k = \eta \), with \( \theta_j < \theta_{j-1} + \pi \), and define \( c(\alpha, p; \eta, \xi) = \sum_{j=1}^{k-1} c(\alpha, p; \theta_j, \theta_{j-1}) \). This does not depend on the choice of the intermediate points.

c) \( c(\alpha, p; \xi, \eta) := 0, c(\alpha, p; \xi, \eta) := -c(\alpha, p; \eta, \xi) \) if \( \eta < \xi \).

This turns out to define a 1-cocycle. It has the additional properties that \( \text{Supp}(c(\alpha, p; \xi, \eta)) \) is contained in the closed interval in \( \hat{T} \) with end points \( \xi \) and \( \eta \), and that \( c(\alpha, p; \eta, \xi) \) and the \( \hat{\Gamma} \)-invariant hyperfunction on \( \hat{T} \) corresponding to \( \alpha \) have the same restriction to the open interval \( (\xi, \eta) \).

If \( p \) is the geodesic decomposition of \( \alpha \), we write \( c_\alpha \) instead of \( c(\alpha, p) \).

If \( j(\Gamma) = \Gamma \), then \( j(c_\alpha(\xi, \eta)) = c_{\alpha\alpha}(\eta, -\xi) \).

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9.7 Proposition. Let $X$ be a non-empty $\Gamma$-invariant subset of $T$. Denote its full original in $T$ by $\tilde{X}$.

i) Let $\alpha \in A^\nu_\omega(\Gamma, X)$.

a) The class $[\alpha]$ of $c(\alpha, p)$ in $H^1_\chi(\tilde{\Gamma}, \tilde{M}_\nu')$ does not depend on the choice of the $\Gamma$-decomposition $p$.

b) There is a bijective map from the set of $\Gamma$-decompositions of $\alpha$ on $X$ to the set of $h \in C^0_\chi(\tilde{\Gamma}, \tilde{M}_\nu')$ that satisfy $\text{Supp}(h(\xi)) \subset \{\xi\}$ for all $\xi \in \tilde{X}$.

ii) The map $\alpha \mapsto [\alpha]$ is an injection $A^\nu_\omega(\Gamma, X) \to H^1_\chi(\tilde{\Gamma}, \tilde{M}_\nu')$. The image consists of those classes that have a representative $c \in C^1_\chi(\tilde{\Gamma}, \tilde{M}_\nu')$ satisfying $\text{Supp}(c(\xi, \eta)) \subset [\xi, \eta]$ for all $\xi, \eta \in \tilde{X}$, $\xi < \eta$.

Remark. $A^\nu_\omega(\Gamma, X)$ is the subset of $\alpha \in A^\nu_\omega(\Gamma)$ that have a $\Gamma$-decomposition on $X$, see the Lemmas 9.8–9.10.

Proof. See the Lemmas 9.8–9.10.

9.8 Lemma. For each $c \in C^1_\chi(\tilde{\Gamma}, \tilde{M}_\nu')$ that satisfies $\text{Supp}(c(\xi, \eta)) \subset [\xi, \eta]$ for all $\xi, \eta \in \tilde{X}$, $\xi < \eta$, there is a unique $\alpha \in A^\nu_\omega(\Gamma, X)$ and a unique $\Gamma$-decomposition $p$ of $\alpha$ on $X$ such that $c = c(\alpha, p)$.

Proof. Let $c$ be given. For $x, y \in T$, $x \neq y$, we can find $\xi, \eta \in \tilde{X}$ such that $x = \text{pr}(\xi)$, $y = \text{pr}(\eta)$, and $\eta < \xi < \eta + \pi$. We define $A(x, y) := \sigma c(\xi, \eta)$. So $\text{Supp}(A(x, y)) \subset [x, y]$. The freedom in the choice of $\xi$ and $\eta$ is a translation over a multiple of $\pi$, hence the choice of $\xi$ and $\eta$ does not influence the definition of $A(x, y)$. If there are $\alpha$ and $p$ with $c(\alpha, p) = c$, then $\alpha[x, y]_p = A(x, y)$.

The properties $A(x, z) + A(z, y) = A(x, y)$ and $A(\gamma \cdot x, \gamma \cdot y) = \pi_\nu(\gamma)A(x, y)$ are easily checked.

Now consider $x, y, z, u \in T$, $x \neq y, z \neq u$. Choose $\xi, \eta, \zeta, \nu \in \tilde{T}$ above these elements such that $\eta < \xi < \eta + \pi$ and $\nu < \zeta < \nu + \pi$. Put $p_1 := c(\xi, \eta) + c(\eta, \xi - \pi)$, and $p_2 := c(\zeta, \nu) + c(\nu, \zeta - \pi)$. Then $A(x, y) + A(y, x) = \sigma p_1$, and $A(z, u) + A(u, z) = \sigma p_2$. The cocycle properties show that $\sigma(p_2 - p_1) = \sigma(c(\xi, \zeta) + c(\zeta - \pi, \xi - \pi)) = 0$. This implies that $\alpha := A(x, y) + A(y, x)$ does not depend on the choice of $x \neq y$. We have $\alpha \in A^\nu_\omega(\Gamma)$ and $\alpha[x, y]_p := A(x, y)$ defines a $\Gamma$-decomposition of $\alpha$ such that $c(\alpha, p) = c$.

9.9 Lemma. Let $\alpha \in A^\nu_\omega(\Gamma, X)$, and let $p$ be a $\Gamma$-decomposition of $\alpha$ on $X$. Then each $\Gamma$-decomposition $q$ of $\alpha$ on $X$ has the form $q = p(h)$, where

$$\alpha[\text{pr}(\xi), \text{pr}(\eta)]_{p(h)} := \alpha[\text{pr}(\xi), \text{pr}(\eta)]_p + \sigma h(\eta) - \sigma h(\xi),$$

for $\eta < \xi < \eta + \pi$, $\xi, \eta \in \tilde{X}$, where $h$ runs through the elements of $C^0_\chi(\tilde{\Gamma}, \tilde{M}_\nu')$ that satisfy $\text{Supp}(h(\xi)) \subset \{\xi\}$ for $\xi \in \tilde{X}$.

For such $h$ we have $c(\alpha, p(h)) = c(\alpha, p) + dh$. 

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Proof. The correspondence in Lemma 9.8 associates $\alpha$ and $p(h)$ to the cocycle $c(\alpha, p) + dh$. This shows that $p(h)$ is a $\Gamma$-decomposition.

Let $p$ and $q$ be $\Gamma$-decompositions of $\alpha$ on $X$. One can check in 9.6 that the cocycle $c := c(\alpha, q) - c(\alpha, p)$ satisfies $\text{Supp}(c(\xi, \eta)) \subset \{\xi, \eta\}$. So for $\xi \neq \eta$ we have $c(\xi, \eta) = c_r(\xi, \eta)$ with $\text{Supp}(c_r(\xi, \eta)) \subset \{\xi\}$ and $\text{Supp}(c_r(\xi, \eta)) \subset \{\eta\}$. A consideration of the cocycle relation for three different points shows that $c(\xi, \eta) = c(\xi)$, $c_r(\xi, \eta) = c_r(\eta)$, and $c(\xi) = -c_r(\xi)$, the $\Gamma$-behavior is $c_r(\xi \cdot \gamma) = \pi_{\nu}(\gamma)c_r(\gamma)$. Take $h := c_r$. Then $c(\alpha, q) = c(\alpha, p) + dh$, with $h$ satisfying the condition in the lemma.

9.10 Lemma. Let $\alpha \in A^\omega_{-\omega}(\Gamma, X)$ and let $p$ be a $\Gamma$-decomposition of $\alpha$ on $X$. If $c(\alpha, p) \in B^1_X(\Gamma, M^\nu)$, then $\alpha = 0$.

Proof. Suppose $c(\alpha, p) = dh$ for some $h \in C^0_{\tilde{\omega}}(\tilde{\Gamma}, M^\nu)$. Fix some $\theta \in \tilde{X}$. The support of $h(\theta)$ is contained in some bounded closed interval $I$. Take $N \in \mathbb{N}$ large, such that $N \pi + \min(J) > \max(J) + 4\pi$. So there is a closed interval $I$ of length $2\pi$ between $\theta$ and $\theta + N\pi$ that does not intersect $J$ or $J + N\pi$.

\[\text{Supp}(h(\theta)) \quad \text{Supp}(h(\theta + N\pi))\]

The restriction of $c(\alpha, p; \theta, \theta + N\pi)$ to $I$ vanishes. So for any $\xi, \eta \in I$, $\eta < \xi < \eta + \pi$, we have $\text{Supp}(c(\alpha, p; \xi, \eta)) \subset \{\eta, \xi\}$, and hence $\alpha[pr(\xi), pr(\eta)]p$ has support contained in $\{pr(\xi), pr(\eta)\}$. So $\alpha$ has restriction zero on each open interval in $T$ bounded by two different points of $X$. As $X$ is infinite, we have $\alpha = 0$.

10 Image in a fixed weight

To return to Lewis’s period function we make a transition from hyperfunctions to functions. In Proposition 10.4, we associate to automorphic hyperfunctions cocycles with holomorphic functions as values. Proposition 10.10 shows that this leads to solutions of (1.2).

10.1 Functions of complex weight. Let $q \in \mathbb{C}$. We define $C^\infty(\tilde{G})_q$ to be the set of functions in $C^\infty(\tilde{G})$ that satisfy $f(gk(\theta)) = f(q)e^{iq\theta}$. This is the space of functions of weight $q$. These spaces are invariant under left translation by elements of $\tilde{G}$.

Functions in $f \in C^\infty(\tilde{G})_q$ are fully determined by the corresponding functions $z \mapsto y^{-q/2}f(p(z))$ on $\tilde{S}^+$. The left translation $L_{g_1}f(g) = f(g_1g)$ in $C^\infty(\tilde{G})_q$ is a right $\tilde{G}$-action. It corresponds to the right $\tilde{G}$-action $g_1 : F \mapsto F|_{g_1}\tilde{k}\gamma$ in the functions on $\tilde{S}^+$ defined by $F|_{\gamma}\tilde{k}(\pi m) = e^{qim\pi / \gamma}$ for $m \in \mathbb{Z}$, and $F|_{\gamma}\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)(z) = (cz + d)^{-q}F\left(\frac{az + b}{cz + d}\right)$ for $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{SL}_2(\mathbb{R})$, $-\pi < \arg(ci + d) < \pi$. 36
10.2 Map to weight $q$. The space $\tilde{M}^{-\nu}$ contains $\varphi : \theta \mapsto e^{i\varphi \theta}$ for each $q \in \mathbb{C}$. If we view $\tilde{M}^{-\nu}$ as a space of functions on $\tilde{G}$, then $\varphi_q$ spans the intersection $\tilde{M}^{-\nu} \cap C^\infty(\tilde{G})_q$.

If $q = 2r \in 2\mathbb{Z}$, then this function is invariant under translation by $\pi$, and corresponds to the function $\varphi_{2r} \in \mathcal{A}_T(T)$.

By $\rho_q \beta : g \mapsto (\tilde{\pi}_\nu(g)\varphi_q, \beta)$ we define a linear map $\tilde{M}^{-\nu}_b \to C^\infty(\tilde{G})_q$. It satisfies $\rho_q \circ \tilde{\pi}_\nu(g) = L_{g^{-1}} \circ \rho_q$.

The map $\rho_q$ induces a homomorphism $H^1_X(\tilde{\Gamma}, \tilde{M}_b^\nu) \to H^1_X(\tilde{\Gamma}, C^\infty(\tilde{G}))$, with the left action $\gamma \mapsto L_{\gamma^{-1}}$ of $\tilde{\Gamma}$ on $C^\infty(\tilde{G})$. This may be uninteresting for general weights $q$, as the cohomology group with values in $C^\infty(\tilde{G})_q$ may vanish. (This is the case for $\Gamma = \Gamma_{mod}$ and $q \not\in 2\mathbb{Z}$.)

10.3 Image in weight $1 - \nu$. The differentiation relations in 4.5 extend to $\varphi_q$ with arbitrary $q$. This implies that $\mathbf{E}^\pm (\rho_q \beta) = (1 - \nu \pm q) \rho_{q \pm 2} \beta$. For weight $q = 1 - \nu$ we have $\mathbf{E}^- (\rho_{1-\nu} \beta) = 0$. So the corresponding functions on $\tilde{\mathcal{H}}^+$ are holomorphic. We define the linear map $P$ from $\tilde{M}^\nu_b$ to the holomorphic functions on $\tilde{\mathcal{H}}^+$ by

$$ P\beta(z) := y^{(\nu-1)/2} \langle \tilde{\pi}_\nu(\tilde{\beta}(z))\varphi_{1-\nu}, \beta \rangle. $$

$P\beta$ is a holomorphic function on $\tilde{\mathcal{H}}^+$, satisfying

$$ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^\nu \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} \beta(z) = (a - cz)^{-\nu-1} (P\beta) \left( \begin{array}{c} dz - b \\ -cz + a \end{array} \right) \bigg|_{1-\nu} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} (z) $$

for $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R})$ with $\text{arg}(ci + d) \in (-\pi, \pi)$.

10.4 Proposition. Suppose that $\alpha \in A_{\nu,\omega}(\Gamma)$ has geodesic approach on the elements of a $\Gamma$-invariant set $X \subset T$. Let $g$ be a representative of $\alpha$.

Consider $\xi, \eta \in X$, $\xi \neq \eta$, such that $\infty \not\in (\xi, \eta)$. Choose $\xi \succ \eta \in \tilde{T} \cap [-\pi, 0]$ such that $\text{pr} \xi = \xi$ and $\text{pr} \eta = \eta$. Then

$$ \left( P_{c\alpha}(\xi, \eta) \right)(z) = \frac{1}{\pi} \text{pv} \int_{\gamma(\xi, \eta)} (z - \tau)^{\nu-1} \left( 1 + \tau^2 \right)^{-(1+\nu)/2} g(\tau) d\tau, $$

$\text{pv}$ denotes the principal value.
with the contour $Q(\xi, \eta)$ as indicated in Figure 6 on page 28.

The function $z \mapsto \left(Pc_\alpha(\tilde{\xi}, \tilde{\eta})\right)(z)$ has a holomorphic extension from $\mathcal{H}^+$ to $\mathbb{C} \setminus (-\infty, \eta]$ if $\eta \neq \infty$, and an extension to $\mathbb{C} \setminus [\xi, \infty)$ if $\xi \neq \infty$. If $\infty < \xi < \eta < \infty$, then the extension to $\mathcal{H}^-$ across $(-\infty, \xi)$ is $e^{2\pi i \nu}$ times the extension across $(\eta, \infty)$.

Remarks. In the Figures 6 and 7 we have drawn the contours and the lines where arg $(z-\tau)$ and arg $(1+\tau^2)$ jump. The standard choice of the argument does not work if $\eta = \infty$. Then we take arg $(z-\tau) \in (-2\pi,0)$, see Figure 6. If we extend $Pc_\alpha(\tilde{\xi}, \tilde{\eta})$ into $\mathcal{H}^-$, we have to adapt the choice of arg $(z-\tau)$ continuously. In all cases the curve $Q(\xi, \eta)$ should stay inside the domain of the representative $g$ of $\alpha$. The principal value interpretation of the integral has been introduced in Proposition 5.13.

If $\infty < \xi < \eta < \infty$, then we can replace $(1+\tau^2)^{-\nu/2} g(\tau)$ by $f_\alpha(\tau)$, see Proposition 5.13.

Proof. As $\xi \neq \eta$, the length of $[\hat{\eta}, \tilde{\xi}]$ is strictly less than $\pi$. Hence the hyperfunction $c_\alpha(\tilde{\xi}, \tilde{\eta})$ on $\tilde{T}$ corresponds to the hyperfunction $\alpha(\xi, \eta)$ on $T$ with support inside $[\xi, \eta]$. We compute the quantity $\langle \tilde{\pi}_{-\nu}(\tilde{p}(z))\varphi_{1-\nu}, c_\alpha(\tilde{\xi}, \tilde{\eta}) \rangle$ as $\langle h_z, \alpha(\xi, \eta) \rangle$, where $h_z$ is the holomorphic function on a neighborhood of $[\xi, \eta]$ such that $\theta \mapsto h_z(\text{pr}(\theta))$ coincides with $\theta \mapsto \tilde{\pi}_{-\nu}(\tilde{p}(z))\varphi_{1-\nu}(\theta)$ on a neighborhood of $[\tilde{\eta}, \tilde{\xi}]$.

We have $\tilde{\pi}_{-\nu}(\tilde{p}(z))\varphi_{1-\nu}(\theta) = y_{1}^{(1-\nu)/2} e^{(1-\nu)\theta}, \tilde{p}(z) = \tilde{k}(\theta)\tilde{p}(\tilde{\xi})$. For $-\pi < \theta < 0$ we find $\tilde{\pi}_{-\nu}(\tilde{p}(z))\varphi_{1-\nu}(\theta) = y_{1}^{(1-\nu)/2} (\cos \theta - \tilde{z} \sin \theta)^{-\nu/2}$. For $\theta \in (-\pi,0)$ and $\tau = \cot \theta \in T \setminus \{\infty\}$ we have arg $(\cos \theta - \tilde{z} \sin \theta) = \sqrt{1+\tau^2} = \arg (z - \tau) - \frac{1}{2} \arg (1+\tau^2)$. Hence $h_z(\tau) = y_{1}^{(1-\nu)/2} (z - \tau)^{-\nu/2} (1+\tau^2)^{(1-\nu)/2}$, on a neighborhood of $[\xi, \eta]$. This leads to the integral representation in the proposition. The holomorphic extensions into $\mathcal{H}^-$ clearly exist.

Let $\infty < \xi < \eta < \infty$. The continuation of $Pc_\alpha(\tilde{\xi}, \tilde{\eta})$ across $(-\infty, \xi)$ is computed with the jump of arg $(z-\tau)$ on a curve that passes above $Q(\xi, \eta)$, and the other continuation with arg $(z-\tau)$ jumping on a curve below $Q(\xi, \eta)$. So in the former integral the argument of $z - \tau$ is $2\pi$ more than in the other.

10.5 Reflection. If we choose originals $\tilde{\xi}$ and $\tilde{\eta}$ in $[0, \pi]$, then we get a similar
integral for $P_{c_\alpha}(\hat{\xi}, \hat{\eta})$, with $(z - \tau)^{\nu - 1}$ replaced by $(\tau - z)^{\nu - 1}$.

Let $j(\Gamma) = \Gamma$, and $\eta < \infty$. If we use $\tau \mapsto -g(-\tau)$ as the representative of $j\alpha$, we obtain \( \left( P_j(c_\alpha(\hat{\xi}, \hat{\eta})) \right) (z) = \left( P_{c_\alpha}(\hat{\xi}, \hat{\eta}) \right) (z) \), where we use the continuation of $P_{c_\alpha}(\hat{\xi}, \hat{\eta})$ across $(\eta, \infty)$.

10.6 Holomorphic cusp form. If $\alpha \in A^{2k-1}(\Gamma)$ corresponds to a holomorphic cusp form of weight $2k$, then $P_{c_\alpha}(\hat{\xi}, \hat{\eta})(z)$ is a polynomial in $z$ that turns out to be $(-1)^k4^{1-k}$ times the period polynomial $R_H(\xi, \eta)$ introduced in $\S 3$.

10.7 Eisenstein series. Take $\xi = \infty$ and $\eta = 0$. From $\S 7.17$ we find for $\text{Re } s > 1$ and $z \in \mathbb{C} \setminus (-\infty, 0)$:

\[
\left( P_{c_{\alpha}}(0, \frac{\pi}{2}) \right)(z) = \frac{1}{2} \pi^{-s}(s - 1) \sum_{p,q} f_{p,q}(p^2 + q^2)^{-s} \left( (z - \tau)^{-2s} \right) \bigg|_{\tau = -p/q} \nonumber
\]

\[
= \pi^{-s}(s - 1) \left( \frac{1}{2} \zeta(2s)(1 + z^{-2s}) + \sum_{p,q \geq 1} (qz + p)^{-2s} \right),
\]

with $f_{p,q}$ as in $\S 7.17$, and $\arg(qz + p) \in (-\pi, \pi)$.

10.8 Interpretation. Let $\alpha$ be as in Proposition 10.4. This proposition shows that for $\xi, \eta$ projecting to $\Gamma \cdot \infty$ with distance strictly smaller than $\pi$ the image $\hat{\mathcal{P}}_{c_\alpha}(\hat{\xi}, \hat{\eta})$ has an extension into $\hat{\mathcal{S}}^-$ across part of $T$. In this way we can view $\hat{\mathcal{P}}_{c_\alpha}$ as a cocycle with values in the $G$-module of holomorphic functions with domain of the form $P_{\Gamma_1 \setminus \Gamma}$, where $I \subset T$ depends on the function. The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\text{arg}(ci + d) \in (-\pi, \pi)$ is given by the same formula as above, with the standard choice of the argument of $cz + d$. This forces $F|_{q,k} = e^{\pi i q} F$ on $\mathcal{S}^\pm$.

10.9 Modular group. The covering $\check{\Gamma}_{\text{mod}}$ of $\Gamma_{\text{mod}}$ is generated by $w := \hat{k} \left( \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right)$ and $n := \hat{n}(1)$, subject to the relations $w^2 n = nw^2$ and $n w n = w^3$.

Let $X = \mathbb{P}^1 \setminus \check{\Gamma}_{\text{mod}}$, and $\check{\Gamma} = \text{pr}^{-1} X$. Any $c \in C^1(\check{\Gamma}_{\text{mod}}, A)$ for a $\check{\Gamma}_{\text{mod}}$-module $A$ is determined by $p = c \left( 0 \cdot \omega^{-1}, 0 \right) = c \left( -\frac{1}{2}, 0 \right)$, with the relations $(n w n + n) p = (1 + w + w^2) p$ and $(n - 1)(w + 1) p = 0$. (To check this, it is convenient to work with the corresponding inhomogeneous cocycle $\gamma \mapsto c \left( 0 \cdot \gamma^{-1}, 0 \right)$.)

The cocycle is a coboundary if and only if $p = (1 - w) f$ for some $f \in A$ satisfying $n f = f$. If we can solve $f$ from $(1 + w) p = (1 - w^2) f$, then $c \in B^1(\check{\Gamma}_{\text{mod}}, A)$. This equation is always solvable if $A$ is the module of holomorphic functions on $\mathcal{S}^{\pm} \cup \check{\mathcal{S}}$ with the action of weight $q \in \mathbb{C} \setminus 2\mathbb{Z}$ indicated above. This is used in Lemma 5.13.

If $\alpha$ satisfies the assumptions of Proposition 10.4 and $c = c_\alpha$, then $p = P_{c_\alpha}(\hat{\xi}, \hat{\eta})$. The relation $c_\alpha(0, -\frac{1}{2}) = c_\alpha(0, -\frac{1}{2}) + c_\alpha(-\frac{1}{2}, -\frac{1}{2})$ corresponds to $p = n^{-1} p + n w^{-1} p$. This is the functional equation for the $\psi$-function in (5.1), but with weight $1 + \nu$ replaced by $1 - \nu$. In that case $F := (1 - w^2) f =$
(1 + w)p is given by

\[ F(z) = \frac{-1}{\pi} \left( \frac{\text{pv} \int_{Q(\infty, 0)} (z - \tau)^{\nu-1} \left(1 + \tau^2\right)^{-1/2} g(\tau) d\tau}{\text{pv} \int_{Q(0, \infty)} (z - \tau)^{\nu-1} \left(1 + \tau^2\right)^{-1/2} g(\tau) d\tau} \right)\]

with \(I_+\) as in Figure 3 on page 18, and \(\arg(z - \tau) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\) if \(z \in \mathcal{Y}^+\) and in \((-\frac{3\pi}{2}, \frac{3\pi}{2})\) if \(z \in \mathcal{Y}^-\). By \(I_+ - I_-\) we mean the union of both contours, with opposite orientation of \(I_-\). The function \(F\) on \(\mathcal{Y}^+ \cup \mathcal{Y}^-\) is holomorphic and has period 1. It can be defined for other \(\Gamma\) as well.

10.10 Proposition. Suppose that \(\alpha \in A^\nu_\omega(\Gamma)\) has polynomial growth at \(\infty\) and satisfies one of the conditions a) and b) in Theorem 7.13, implying geodesic decomposition on \(\Gamma \cdot \infty\). Define

\[ F_\alpha(z) := \frac{-1}{\pi} \text{pv} \int_{I_+ - I_-} (z - \tau)^{\nu-1} \left(1 + \tau^2\right)^{-1/2} g(\tau) d\tau, \]

where \(g\) represents \(\alpha\), and \(I_+\) and \(I_-\) arg \((z - \tau)\) are as indicated above. Then, for \(z \in \mathcal{Y}^\pm\),

\[ F_\alpha(z) = \pm \frac{2ie^{\pm \pi iu/2}}{(2\pi)^\nu \Gamma(1 - \nu)} \sum_{m=1}^{\infty} m^{-\nu} A_{\pm m}(\alpha) e^{\pm 2\pi i m z} \]

\[ \pm i e^{\pm \pi i u/2} \sin \frac{\pi \nu}{2} B_0(\alpha) \pm (\text{if } \nu = 0) i A_0(\alpha). \]

If \(\nu \notin 1 + 2\mathbb{Z}\), \(\nu \notin 2\mathbb{N}\), then \(F_\alpha(z) = 2i\sqrt{\pi} e^{\pm \pi iu/2} \Gamma(1 + \nu/2) \Gamma(1 - \nu/2) \]

\[ \tilde{f}_{i(\nu)} \text{ on } \mathcal{Y}^\pm, \text{ with } f_{i(\nu)} \text{ the function associated to } i(\nu) \in A^\nu_\omega(\Gamma) \text{ in Proposition 5.13.} \]

If \(\nu \notin 1 + 2\mathbb{Z}, \nu \notin 2\mathbb{N}, \text{ and } \Gamma = \Gamma_{\text{mod}}, \text{ then } P_C(-\frac{\pi}{2}, 0) = \frac{1}{\sqrt{\pi}} \Gamma(1 - \nu) \]
\[ \Gamma(1 - \nu/2)^{-1} \psi_{i(\nu)}, \text{ where } \psi_{i(\nu)} \in \Psi_{\text{mod}}(-\nu) \text{ is the image of } i(\nu) \text{ under the map in Theorem 5.11.} \]

Remark. See (5.7) and Lemma 5.13 on page 18 for the coefficients \(A_n(\alpha)\) and \(B_0(\alpha)\), and (1.3) on page 13 for the definition of the isomorphism \(\iota(\nu)\).

Proof. For any hyperfunction \(\beta\) with representative \(h\) we define

\[ J_\nu(z, h) = \frac{-1}{\pi} \text{pv} \int_{I_+ - I_-} (z - \tau)^{\nu-1} \left(1 + \tau^2\right)^{-1/2} h(\tau) d\tau. \]

This does not depend on the choice of the representative \(h\), so we also write \(J_\nu(z, \beta)\). We have \(F_\alpha(z) = J_\nu(z, \alpha)\).
Take $z \in \mathcal{H}$, with $\zeta \in \{1, -1\}$. By taking the contours $I_{\pm}$ wide, we see that $J_\nu(z - x, h) = J_\nu(z, \pi; \nu(\frac{1 - i}{2}) + h)$. This implies that $F_\nu(z) = J_\nu(z, \alpha)$ has a Fourier expansion $F_\nu(z) = \sum_{m \in \mathbb{Z}} F(m, \zeta) e^{2\pi i m z}$ for $z \in \mathcal{H}$, and $F(m, z)$ is given by $e^{2\pi i m z} J_\nu(z, \mathcal{F}_m \alpha)$, see 6.11.

In [10.11] [10.11] we determine $J_\nu(z, h)$ for representatives $h$ of the various Fourier $N$-equivariant hyperfunctions that can occur in $\mathcal{F}_m \alpha$. These computations give the Fourier expansion of $F_\nu$ indicated in the proposition.

Let $\nu \in \mathbb{C} \setminus (1 + 2\mathbb{Z})$. The relation with $\langle (\nu) \alpha \rangle$ follows from 6.19. In the modular case we have already seen that $F_\nu(z) = p(\tau) + \tau^\nu p(-1/\tau)$, with $p = P_{c_\alpha}(-\frac{1}{2}, 0)$. The relations in Lemma 5.13 allow us to express $\psi_\nu(\nu) \alpha$ in terms of $p$.

10.11 General remarks. Let $N > |\text{Re} \, x|$ and $0 < \varepsilon < |\text{Im} \, z|$. We have $J_\nu = -\delta J_\nu - \sum_{\delta = 1, -1} \delta J_\nu^\delta$, with

$$J_\nu^\delta(z, h) = \sum_{\pm} \frac{1}{\pi} \int_{\xi = -N}^{N} (z - \xi \mp i\varepsilon)^{\nu - 1} \left(1 + (\xi \pm i\varepsilon)^2\right)^{-1/2} h(\xi \pm i\varepsilon) \, d\xi,$$

$$J_\nu^0(z, h) = \frac{i}{\pi} \int_{t = \varepsilon}^{\infty} \sum_{\pm} (z - \delta N \mp it)^{\nu - 1} \left(1 + (\delta N \pm it)^2\right)^{-1/2} h(\delta N \pm it) \, dt$$

$$= \frac{i}{\pi} \int_{t = \varepsilon}^{\infty} \sum_{\pm} f_\delta(\delta N \pm it) \frac{h(\delta N \pm it)}{1 + (\delta N \pm it)^2} \, dt,$$

with $f_\delta(\tau) = -\delta e^{\pi i(1-\delta)\nu/2} (1 - \frac{\tau}{\nu})^{-1} (1 + \tau^{-2})^{(1-\nu)/2}$.

10.12 Case $m = 0$, hyperfunction $\mu$. For the representative $h(\tau) = \frac{2\pi}{\text{Im} \, z} \tau$ representing $\mu$ we can take $\varepsilon = 0$. Then $J_\nu^0$ vanishes, and $J_\nu^1(z, h) = (f_1, \mu|N, \infty|) = \frac{1}{2} f_1(\infty) = -\frac{1}{2} e^{\pi i \nu},$ and $J_\nu^{-1}(z, h) = -\langle f_{-1}, \mu|\infty, -N| \rangle = -\frac{1}{2} f_{-1}(\infty) = -\frac{1}{2}.$ Hence $J_\nu(z, \mu) = \zeta e^{\pi i \nu/2} \sin \frac{\pi \nu}{2}$.

10.13 Case $m > 0$, hyperfunction $\kappa_\nu(\nu)$. We consider $\nu \in \mathbb{C} \setminus (-2\mathbb{N})$, with $\text{Re} \nu < 1$. Take $p$ and $q$ as in 6.9. We find

$$J_\nu^\delta(z, q) = \frac{-\delta e^{\pi i(1+\delta)\nu/2} \Gamma(1 + \nu/2)}{2\pi \sqrt{\pi}} \int_{\varepsilon}^{\infty} \sum_{\pm} \mp i (t \mp i\delta N \pm i\varepsilon)^{\nu - 1} \, dt$$

$$= \frac{-\delta e^{\pi i(1+\delta)\nu/2} \Gamma(1 + \nu/2)}{4\pi \sqrt{\pi}} \sum_{\pm} (\varepsilon \mp i\delta N \pm i\varepsilon)^{\nu},$$

$$J_\nu^0(z, p) = -\frac{1}{2\sqrt{\pi} \Gamma(1 - \nu/2)} \left( e^{\pi i \nu}(-z + N + i\varepsilon)^\nu - (z + N - i\varepsilon)^\nu \right).$$

The limits for $\varepsilon \downarrow 0$ exist, and after taking $\varepsilon = 0$ it does no longer hurt that we use different representatives. As $N \to \infty$, all terms tend to zero, and we obtain $J_\nu(z, \kappa_\nu(\nu)) = 0$ if $\nu \notin -2\mathbb{N}$. (One can also check for finite $N$ that the terms cancel each other.)

10.14 Case $m = 0$, $\nu = 0$, hyperfunction $\lambda(0)$. We find

$$J_0(z, \lambda(0)) = \lim_{\nu \to 0} \frac{2}{\sqrt{\pi}} \left( J_\nu(z, \kappa_0(\nu)) - \frac{1}{\sqrt{\pi}} J_\nu(z, \mu) \right) = -\zeta i.$$
10.15 \( \kappa_m(\nu) \) given by an integral. Let \( \text{Re} \nu < 0 \) if \( m = 0 \). We write a representative of \( \kappa_m(\nu) \) as in [6.8]

\[
g(\tau) = \frac{1}{2\pi i} \int_{I} \frac{1 + \tau \tau_0}{\tau_0 - \tau} e^{2\pi i m \tau_0} \left(1 + \tau_0^2\right)^{\nu-1/2} d\tau_0,
\]

where \( I \) is a path of the form \( I_+ \) if \( m \geq 0 \), and of the form \( I_- \) otherwise. In both cases we take \( I \) between the contours \( I_+ \) and \( I_- \) that we use to compute \( J_\nu(z, \kappa_m(\nu)) \). If we insert this representation of \( g \) into the definition of \( J_\nu(z, \kappa_m(\nu)) \), everything converges absolutely, provided we take the principal value interpretation of the outer integral. Hence we can interchange the order of integration, and find:

\[
J_\nu(z, \kappa_m(\nu)) = \frac{1}{2\pi i} \int_{I} \frac{-1}{\pi} \text{pv} \int_{I_+-I_-} \frac{1 + \tau \tau_0}{\tau_0 - \tau} (z - \tau)^{\nu-1} \left(1 + \tau^2\right)^{-(1+\nu)/2} \frac{1 + \tau \tau_0}{\tau_0 - \tau} d\tau \cdot e^{2\pi i m \tau_0} \left(1 + \tau_0^2\right)^{\nu-1/2} d\tau_0.
\]

Next we enlarge the contours \( I_\pm \) into wider contours \( \hat{I}_\pm \) of the same type, but such that \( \hat{I} \) is contained in one of them. This changes the inner integral into

\[
\text{pv} \int_{I_+-I_-} (z - \tau)^{\nu-1} \left(1 + \tau^2\right)^{-(1+\nu)/2} \frac{1 + \tau \tau_0}{\tau_0 - \tau} d\tau + 2\pi i (z - \tau_0)^{\nu-1} \left(1 + \tau_0^2\right)^{(1-\nu)/2}.
\]

The integrand in the integral over \( \hat{I}_\pm \) is \( O(\tau^{-2}) \) as \( |\tau| \to \infty \). So we do not need a principal value interpretation. The integral can be deformed into integrals over vertical lines at \( \text{Re} \tau = -M \) and \( \text{Re} \tau = M \). The limit as \( M \) tends to infinity yields 0. We are left with

\[
J_\nu(z, \kappa_m(\nu)) = \frac{-1}{\pi} \int_I (z - \tau_0)^{\nu-1} e^{2\pi i m \tau_0} d\tau_0.
\]

If \( m = 0 \), this vanishes by an explicit computation. If \( \text{sign} \text{Im} z \neq \text{sign} m \), then we move off the path \( \hat{I} \) upwards or downwards, and obtain \( J_\nu(z, \kappa_m(\nu)) = 0 \) if \( \zeta m < 0 \) in the case \( \text{Re} \nu > 0 \) we have a holomorphic function of \( \nu \) that we compute for \( \text{Re} \nu \) large, and find \( J_\nu(z, \kappa_m(\nu)) = 2\zeta e^{\pi i \zeta \nu/2} \Gamma(1 - \nu)^{-1} (2\pi|m|)^{-\nu} \). 

10.16 Discussion. For \( \alpha \in A_\nu^v(\Gamma_{\text{mod}}) \) corresponding to a cuspidal Maass form, and for \( \alpha = \varepsilon_s \) with \( \text{Re} s < 1, s \notin \mathbb{Z} \), the period function \( \psi_\alpha \) has turned out to arise in two different ways:

- Theorem 5.11 and its proof show that an analysis of representatives of \( \alpha \) leads to \( \psi_\alpha \). (This works for all \( \alpha \in A_\nu^v(\Gamma) \).)

- The geodesic decomposition of \( \iota(\nu)\alpha \) leads to a cocycle on \( \overline{\Gamma} \) with values in the hyperfunctions on \( \overline{T} \) with bounded support. Testing against the lowest weight vector \( \varphi_{1+\nu} \) in \( M_\nu^v \) leads to a cocycle with values in the holomorphic functions on \( \mathfrak{H}^+ \cup \mathfrak{H}^- \). This cocycle is determined by one value, that turns out to be a multiple of \( \psi_\alpha \).
11 Transfer operator

We conclude this paper with some remarks on the transfer operator. We restrict ourselves to results that are a direct consequence of the previous sections.

11.1 Transfer operator. Let \( \alpha \in \mathcal{A}_{\nu} \) satisfy the assumptions of Theorem 8.2, and \( B_0(\alpha) = 0 \). This means that \( \alpha \) corresponds to a cuspidal Maass form. Then \( \alpha(\infty, 0) = \sum_{n=0}^{\infty} \alpha[-n-1, -n] \) (weak convergence). This implies that

\[
P_{c_\alpha} \left( -\frac{\pi}{2}, 0 \right) (z) = \sum_{n=0}^{\infty} P_{c_\alpha} \left( -\frac{\pi}{2} + \arctan n, -\frac{\pi}{2} + \arctan(n+1) \right) (z)
\]

for each \( z \in \mathbb{C} \setminus (-\infty, 0) \). Put \( p_\alpha = P_{c_\alpha} \left( -\frac{\pi}{2}, 0 \right) \). As \( j \tilde{\pi}_\nu \left( n \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right) (c_{j_1} \alpha \left( -\frac{\pi}{2}, 0 \right)) = c_\alpha \left( -\frac{\pi}{2} + \arctan n, -\frac{\pi}{2} + \arctan(n+1) \right) \), we have

\[
P_{c_\alpha} \left( -\frac{\pi}{2} + \arctan n, -\frac{\pi}{2} + \arctan(n+1) \right) (z) = \left( z + n \right)^{\nu-1} p_{j_1} \left( 1 + \frac{1}{n+z} \right)
\]

(see 9.2 for the action on \( \tilde{T} \), and 10.5 for the reflection). So for \( j \alpha = \pm \alpha \), the function \( p_\alpha \), and its multiple \( \psi_{j(\nu)} \alpha \), are eigenfunctions of the transfer operator

\[
L_{1-\nu} : f \mapsto \sum_{n=0}^{\infty} \left( z + n \right)^{\nu-1} f \left( 1 + \frac{1}{n+z} \right)
\]

of Mayer, see [3]. (We have shifted the functions.) Note that the convergence implies \( f(1) = 0 \) if \( \text{Re} \nu \geq 0 \). This implies that \( \psi_\alpha(1) = 0 \) for \( \alpha \) associated to cuspidal Maass forms.

11.2 Transfer operator for hyperfunctions. We define

\[
\mathcal{L}_\nu := \sum_{n=0}^{\infty} j \tilde{\pi}_\nu \left( n \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right)
\]

as an operator in \( M_{\nu, \omega} \). Its domain consists of those \( \beta \in M_{\nu, \omega} \) with support contained in \( [\infty, 0] \) for which the sum converges weakly.

We have seen that, for \( \alpha \) as above, \( \alpha(0,1) \) is in the domain of \( \mathcal{L}_\nu \), and is an eigenvector with eigenvalue \( \pm 1 \) if \( j \alpha = \pm \alpha \).

11.3 Eisenstein series. Theorem 8.2 can be applied to \( \varepsilon_s \) for \( \text{Re} s > \frac{1}{2} \). For these values of \( s \) the hyperfunction \( \varepsilon_s(\infty, 0) \) is in the domain of \( \mathcal{L}_{1-2s} \), and \( \mathcal{L}_{1-2s} \varepsilon_s(\infty, 0) = \varepsilon_s(\infty, 0) - \frac{1}{2} \pi^{-s} (s-1) \zeta(2s) \mu_s \). (We have used \( j \varepsilon_s = \varepsilon_s \).)

This suggests that \( \varepsilon_s(\infty, 0) \) would be an eigenvector of \( \mathcal{L}_{1-2s} \) if \( \zeta(2s) = 0 \). But we have no continuation of our results to these values of \( s \). After continuation of the expression for \( P_{c_{\varepsilon_s}}(0, -\frac{\pi}{2}) \) in 10.7, we obtain an eigenfunction of \( L_{2s} \).
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