Research Article

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On regular solutions to compressible radiation hydrodynamic equations with far field vacuum

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Abstract: The Cauchy problem for three-dimensional (3D) isentropic compressible radiation hydrodynamic equations is considered. When both shear and bulk viscosity coefficients depend on the mass density $\rho$ in a power law $\rho^{\delta}$ (with $0 < \delta < 1$), based on some elaborate analysis of this system’s intrinsic singular structures, we establish the local-in-time well-posedness of regular solution with arbitrarily large initial data and far field vacuum in some inhomogeneous Sobolev spaces by introducing some new variables and initial compatibility conditions. Note that due to the appearance of the vacuum, the momentum equations are degenerate both in the time evolution and viscous stress tensor, which, along with the strong coupling between the fluid and the radiation field, make the study on corresponding well-posedness challenging. For proving the existence, we first introduce an enlarged reformulated structure by considering some new variables, which can transfer the degeneracies of the radiation hydrodynamic equations to the possible singularities of some special source terms, and then carry out some singularly weighted energy estimates carefully designed for this reformulated system.

Keywords: radiation hydrodynamics, three dimensions, local existence, regular solutions, far field vacuum, degenerate viscosity

MSC 2020: 35A01, 35A09, 35Q35, 35M11

1 Introduction

It is well known that the radiation effects become remarkable in some regime when the temperature is high. Radiation sometimes contributes largely to energy density, momentum density, and pressure, for instance, in astrophysics and inertial confinement fusion. Radiation transfer is usually the most effective mechanism that affects the energy exchange in fluids, so it is necessary to take effects of the radiation field into consideration in the classical hydrodynamic framework. The equations of radiation hydrodynamics result from the balances of particles, momentum, and energy. From a microscopic point of view, the radiation field is composed of photons. We first introduce some basic concepts necessary for describing the radiation field and its interaction with matter. At any time $t$, we need $2d$ variables to specify the state of a photon in phase space, namely, $d$ position variables and $d$ velocity (or momentum) variables. Usually, we can denote by $x$ the $d$ position variables, and replace the $d$ momentum variables equivalently with frequency $\nu$ and the travel direction $\Omega$ of the photon. Via these variables, we then define the phase-space distribution function $f = f(t, x, \nu, \Omega)$ for photons such that

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\[dn = f(t, x, v, \Omega) dx dv d\Omega,\]

where \(n\) is the number of photons; \(dn\) is the number of photons (at time \(t\)) at space point \(x\) in a volume element \(dx\), with local frequency \(v\) in a frequency interval \(dv\), and traveling in a direction \(\Omega\) in the cubic angle element \(d\Omega\). In the radiation transport, we usually use the specific radiation intensity \(I(t, x, v, \Omega)\) to replace the distribution function \(f\). The specific radiation intensity is defined as follows:

\[I(t, x, v, \Omega) = chvf(t, x, v, \Omega),\]

where \(c\) is the vacuum speed of light and \(h\) is Planck constant. The physical interpretation of \(I\) is contained in the relationship

\[\frac{1}{c}I_t + \Omega \cdot \nabla I = A_r,\]

where \(A_r\) is a collision term given by

\[A_r = \sigma_e - \alpha_s I + \int_0^{\infty} \int_{S^2} \left( \frac{v'}{v} \sigma_s I' - \sigma_r I \right) d\Omega' dv',\]

\(I = I(t, x, v, \Omega), I' = I(t, x, v', \Omega'), t \geq 0\) is the time, \(x \in \mathbb{R}^3\) is the Eulerian spatial coordinate, \(v, v' \geq 0\) are the frequency of photons, \(\Omega, \Omega' \in S^2\) are the travel direction of photons, and \(S^2 \subset \mathbb{R}^3\) denotes the unit sphere in \(S^2, \sigma_e(t, x, v, \Omega, \rho) \geq 0\) is the rate of energy emission due to spontaneous process, and \(\sigma_s(t, x, v, \Omega, \rho) \geq 0\) denotes the absorption coefficient that may depend on the mass density \(\rho\).

Similarly to absorption, a photon can undergo scattering interactions with matter, and the scattering interaction serves to change the photon’s characteristics \(v'\) and \(\Omega'\) to a new set of characteristics \(v\) and \(\Omega\). To quantitatively describe the scattering event, one requires a probabilistic statement concerning this change, which leads to the definition of the “differential scattering coefficient” \(\sigma_s(v' \rightarrow v, \Omega' \cdot \Omega, \rho) \equiv \sigma_s(v' \rightarrow v, \Omega' \cdot \Omega, \rho)\) that may depend on \(\rho\) such that the probability of a photon being scattered from \(v'\) to \(v\) contained in \(dv'\), from \(\Omega'\) to \(\Omega\) contained in \(d\Omega\), and traveling a distance \(ds\) is given by \(\sigma_s(v' \rightarrow v, \Omega' \cdot \Omega, \rho) dv' d\Omega ds\). Therefore, the time rates of outscattering and inscattering within a unit volume element are expressed as follows:

\[
\text{outscattering} = \int_0^{\infty} \int_{S^2} \sigma_s(v' \rightarrow v, \Omega' \cdot \Omega, \rho) I d\Omega' dv',
\]

\[
\text{inscattering} = \int_0^{\infty} \int_{S^2} \sigma_s(v' \rightarrow v, \Omega' \cdot \Omega, \rho) I d\Omega' dv',
\]

where \(\sigma_s = O(\rho)\).

Concerning the effect of radiation on the dynamic properties of the fluid is very significant, we introduce the following two quantities to describe this effect:

\[
\begin{align*}
F_r &= \int_0^{\infty} \int_{S^2} I(t, x, v, \Omega) \Omega d\Omega dv, \\
P_r &= \int_0^{\infty} \int_{S^2} I(t, x, v, \Omega) \Omega \cdot \Omega d\Omega dv,
\end{align*}
\]
which are called the radiation flux and the radiation pressure tensor, respectively.

Now we take radiation effect into consideration for viscous (barotropic) fluids to have the following isentropic radiation hydrodynamics equations in 3D space:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{1}{c} \frac{\partial I}{\partial t} + \Omega \cdot \nabla I = A_t, \\
\rho \frac{\partial v}{\partial t} + \text{div}(\rho v) = 0, \\
\left(\rho u + \frac{1}{c^2} F\right)_t + \text{div}(\rho u \otimes u + B) + \nabla P_m = \text{div} \mathbb{T},
\end{array} \right.
\end{aligned}
\]

where $A_t$ is defined by (1.2). The unknown functions $\rho, u = (u^1, u^2, u^3)^T$ represent the density and the velocity, respectively. $P_m$ is the material pressure with the following equation of state for the polytropic fluid:

\[
P_m = A\rho^y, \quad A > 0, \quad y > 1,
\]

where $A$ is a constant and $y$ is the adiabatic exponent. $\mathbb{T}$ denotes the viscous stress tensor with the form

\[
\mathbb{T} = 2\mu(\rho) \nabla u + \lambda(\rho) \text{div} u I_3,
\]

where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the deformation tensor and $I_3$ is the $3 \times 3$ identity matrix,

\[
\mu(\rho) = \alpha \rho^\delta, \quad \lambda(\rho) = \beta \rho^\delta,
\]

for some constant $\delta \geq 0$, $\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho) + \frac{2}{3} \mu(\rho)$ is the bulk viscosity coefficient, and $(\alpha, \beta)$ are both constants satisfying

\[
\alpha > 0, \quad 2\alpha + 3\beta \geq 0.
\]

According to (1.1) and (1.3), system (1.4) can be rewritten as follows:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{1}{c} \frac{\partial I}{\partial t} + \Omega \cdot \nabla I = A_t, \\
\rho \frac{\partial v}{\partial t} + \text{div}(\rho v) = 0, \\
\left(\rho u + \frac{1}{c^2} F\right)_t + \text{div}(\rho u \otimes u + B) + \nabla P_m = \text{div} \mathbb{T} - \frac{1}{c^2} \int_0^\infty \int S^2 A_t \omega d\Omega d\omega.
\end{array} \right.
\end{aligned}
\]

In the current paper, we are concerned with the local-in-time well-posedness of regular solution to the Cauchy problem (1.9) with the following initial data and far field behavior:

\[
(I, \rho, u)_{t=0} = (I_0(x, v, \Omega), \rho_0(x), u_0(x)) \quad \text{for} \quad (x, v, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^* \times S^2,
\]

\[
(I, \rho, u) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0, \quad (v, \Omega) \in \mathbb{R}^* \times S^2.
\]

Throughout this paper, we will adopt the following simplified notations, and most of them are for the homogeneous and inhomogeneous Sobolev spaces:

\[
\begin{aligned}
|f|_p = ||f||_{L^p(\mathbb{R}^3)}, \quad ||f||_{L^r, r} = ||f||_{W^{r, r}(\mathbb{R}^3)}, \quad ||f||_{L^2} = ||f||_{H^s(\mathbb{R}^3)}, \quad D^k = D^{k, 2}, \\
D_1 = \{f \in L^k(\mathbb{R}^3) : ||f||_{L^k} = ||\nabla f||_{L^2} < \infty\}, \quad ||f||_{L^p, r} = ||f||_{D^{k, r}(\mathbb{R}^3)}, \\
||f||_{L^2, r} = ||f||_{L^r, r}, \quad \int f \, dx = \int f \, dx,
\end{aligned}
\]

\[
X([0, T]; Y) = X([0, T]; Y(\mathbb{R}^3)), \quad \|f, g\|_{X} = \|f\|_{X} + \|g\|_{X}.
\]

A detailed study of homogeneous Sobolev space can be found in the study by Galdi [9].

When radiation effects are not considered, there is a lot of literature on the well-posedness of strong/classical solutions for the isentropic compressible Navier-Stokes equations. For the constant viscous flow...
(i.e., $\delta = 0$ in (1.7)), when $\inf \rho_0(x) > 0$, the local well-posedness of classical solutions to the Cauchy problem follows from the standard symmetric hyperbolic-parabolic structure satisfying the well-known Kawashima’s condition, cf. [13,22,27], which has been extended to be a global one by Matsumura-Nishida [21] for initial data close to a nonvacuum equilibrium in some Sobolev space $H^s(\mathbb{R}^3) (s > \frac{5}{2})$. However, these approaches do not work when $\inf \rho_0(x) = 0$, which occurs when some physical requirements are imposed, such as finite total initial mass and energy in the whole space. One of the main issues in the presence of vacuum is the degeneracy of the time evolution operator, which makes it hard to understand the behavior of the velocity field near the vacuum. Via imposing some initial compatibility conditions, Cho et al. [4] established the local well-posedness of strong solutions with vacuum, which, recently, has been shown to be a global one with small energy by Huang et al. [11]. We also refer to Lions [19], and Feireisl et al. [8] to readers and the references therein for the existence theory of global weak solutions with finite energy.

For degenerate viscous flow (i.e., $\delta > 0$ in (1.7)) without considering radiation, when $\inf \rho_0(x) = 0$, instead of the uniform elliptic structure in the constant viscous flow, the viscosity degenerates when density vanishes, which raises the difficulty of the problem to another level. Recently, some significant progress has been made on the well-posedness of smooth solutions in 3D space. In the study by Li et al. [16], via introducing a “quasi-symmetric hyperbolic”–“degenerate elliptic” coupled structure to control the behavior of the velocity $u$ near the vacuum, they showed that the unique 3D regular solution with vacuum exists locally in time, which has been extended to be a global one with small density but possibly large velocity by Xin and Zhu [31]. Recently, for the case $0 < \delta < 1$, by using an elaborate elliptic approach on the operators $L(\rho^{\delta-1}u)$ and some initial compatibility conditions, the existence of 3D local regular solution with far field vacuum has also been obtained by Xin and Zhu [32]. Some other interesting results can also be found in [3,5,10].

When the effect of radiation is taken into consideration, studying the radiation hydrodynamics equations becomes more complicated for both inviscid and viscous fluids because of the complexity and mathematical difficulty. For Euler-Boltzmann equations of inviscid fluids, the local well-posedness and finite time blow up of classical solutions away from the vacuum were studied by Jiang and Zhong [33] in multi-dimensional (M-D) space, and see also the study by Jiang and Wang [12] for a simplified 3D isentropic model. Later, Li and Zhu [18] considered the system discussed in [12] and proved the local existence of a unique regular solution with vacuum by means of the theory of quasi-linear symmetric hyperbolic systems and some technique tools. They also showed that the regular solution will blow up if the initial density vanishes in some local domain. For Navier-Stokes-Boltzmann equations of viscous fluids, Ducomet and Nečasová [7] showed that the global existence theory for the compressible viscous fluids developed in [21] can be generalized to the radiation hydrodynamics. Li and Zhu [17] established the existence of local strong solutions for the isentropic flow in homogeneous Sobolev space for large initial data satisfying the initial compatibility conditions. Ducomet et al. [6] obtained the existence of global weak solutions for some radiation hydrodynamic system, where the velocity $u$ may develop uncontrolled time oscillations on the hypothetical vacuum zones. Some other interesting results for the related radiation hydrodynamics models can also be seen in [2,24,25] and so on.

Our goal in the present paper is to show the local existence and uniqueness of the 3D regular solution with far field vacuum to the Cauchy problem (1.9)--(1.10) for $0 < \delta < 1$. It is worth pointing out that when vacuum appears, one would encounter some difficulties as follows. First, (1.9) is a system of fluid equations coupled with a nonlinear integro-differential hyperbolic equation, which makes the corresponding calculation very complicated. Second, the degeneracy of the time evolution and elliptic operator in momentum equations caused by the vacuum makes it very difficult to determine and control the behavior of velocity $u$ near the vacuum, which is the key point in the vacuum-related problems (see Xin and Zhu [32]). Finally, besides the difficulties mentioned earlier, additional difficulties in the proof lie in dealing with the nonlinear terms and the coupled cross terms between radiation field and fluid field, which prevent us obtaining the uniform a priori estimates. To this end, we shall employ the delicate energy estimates based on the well-designed reformulated structure obtained during the linearization and some physically reasonable assumptions on the radiation quantities.
The rest this paper is organized as follows. In Section 2, we first present some physically reasonable assumptions on the radiation quantities and then conclude this section by stating the main result. Section 3 is devoted to establishing the local-in-time well-posedness of regular solution to the Cauchy problem (1.9)–(1.10). Finally, we give an appendix to list some lemmas that are frequently used in our proof.

# 2 Hypotheses and main result

In this section, we first make some physical assumptions on the radiation quantities and then state the main result.

## 2.1 Hypotheses on radiation quantities

The general form of radiation coefficients is usually not known because it is difficult to evaluate these physical coefficients in quantum mechanics. Physically speaking, the radiation coefficients can be written as $\sigma_\rho \rho_\sigma \sigma_\rho \rho_\sigma \sigma_\rho \rho_\sigma$, and then $A_\sigma \sigma I_\nu \nu_\sigma I_\nu \nu_\sigma d\Omega d'$.

Next we will make some physically reasonable assumptions on the radiation coefficients $\sigma_\rho \rho_\sigma$, $\sigma_\sigma$, and $\sigma_\rho \rho_\sigma$, which are similar as in [15, 17, 30].

**H1** *(Differential scattering coefficient).* Let $\sigma_\sigma = \rho \bar{\sigma}_\sigma(v' \rightarrow v, \Omega'), \Omega = \rho \bar{\sigma}_\sigma^s(v \rightarrow V, \Omega \cdot \Omega') = \rho \bar{\sigma}_\sigma^a$, with the functions $\bar{\sigma}_\sigma \geq 0$ and $\bar{\sigma}_\sigma^a \geq 0$ satisfying

$$
\left(\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{\nu'}{\nu'} |\bar{\sigma}_\sigma^a| d\Omega' d\nu' \right)^{\lambda_1} \leq \Lambda,
$$

and for $s = 1, 2, 3$

$$
\left(\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} |\bar{\sigma}_\sigma^a| d\Omega' d\nu' \right)^{\lambda_1} \leq \Lambda,
$$

where $\lambda_1 = 1$ or $1/2$, $\lambda_2 = 1$ or 2, and $\Lambda$ is a fixed constant.

**H2** *(Emission and absorption coefficient).* Let $\sigma_\sigma = \sigma_\rho \rho_\sigma(t, x, v, \Omega, \rho) = \rho \bar{\sigma}_\rho \rho_\sigma(t, x, v, \Omega, \rho)$ and $\sigma_\rho = \sigma_\rho(t, x, v, \Omega, \rho) = \rho \bar{\sigma}_\rho \rho_\sigma(t, x, v, \Omega, \rho)$, where $\bar{\sigma}_\rho \geq 0$, $\bar{\sigma}_\rho \rho_\sigma \geq 0$. Due to $\phi = \frac{\lambda v}{\gamma-1} \rho^{\gamma-1}$, for $s = 1, 2, 3$, we assume

$$
\| \bar{\sigma}_\rho \|_{L^1_t L^2_s(\mathbb{R}^4; C(0, T); H^s)} \leq M(\| \phi_1 \|_3)(1 + \| \phi_1 \|_3),
$$

$$
\| \bar{\sigma}_\rho \rho_\sigma \|_{L^1_t L^2_s(\mathbb{R}^4; C(0, T); L^2)} \leq M(\| \phi_1 \|_3)(1 + \| \phi_1 \|_3),
$$

$$
\| \bar{\sigma}_\rho \rho_\sigma \|_{L^1_t L^2_s(\mathbb{R}^4; C(0, T); L^2)} \leq M(\| \phi_1 \|_3)(1 + \| \phi_1 \|_3),
$$

$$
\| \bar{\sigma}_\rho \rho_\sigma \|_{L^1_t L^2_s(\mathbb{R}^4; C(0, T); L^2)} \leq M(\| \phi_1 \|_3)(1 + \| \phi_1 \|_3),
$$

and for $s = 0, 1, 2$,
\[
\begin{align*}
\left\| \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) - \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) \right\|_{L^2(\mathbb{R}^4 \times S^2; C([0, T], H^s))} & \leq M(\|\phi^3\|_p) (1 + \|\phi^3 - \phi^3\|_p), \\
\left\| \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) - \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) \right\|_{L^2(\mathbb{R}^4 \times S^2; C([0, T], H^s))} & \leq M(\|\phi^3\|_p) (1 + \|\phi^3 - \phi^3\|_p), \\
\end{align*}
\]  

(2.4)

where \( M = M(\cdot) \) denotes a strictly increasing function from \([0, \infty)\) to \([1, \infty)\).

**Remark 2.1.** It follows from (2.3)–(2.4) that for \( s = 0, 1, 2, 3 \),

\[
\begin{align*}
\left\| \sigma_{\delta} \right\|_{L^2(\mathbb{R}^4 \times S^2; C([0, T], H^s))} & \leq M(\|\phi\|_p) \left( \frac{1}{\|\phi\|_p} (1 + \|\phi\|_p) \right), \\
\left\| \sigma_{\delta} \right\|_{L^2(\mathbb{R}^4 \times S^2; C([0, T], H^s))} & \leq M(\|\phi\|_p) \left( \frac{1}{\|\phi\|_p} (1 + \|\phi\|_p) \right),
\end{align*}
\]

(2.5)

and for \( s = 0, 1, 2, 3 \),

\[
\begin{align*}
\left\| \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) - \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) \right\|_{L^2(\mathbb{R}^4 \times S^2; C([0, T], H^s))} & \leq M(\|\phi^3\|_p) (1 + \|\phi^3 - \phi^3\|_p), \\
\left\| \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) - \sigma_{\delta}(\cdot, \cdot, \cdot, \cdot, \cdot, \rho^3) \right\|_{L^2(\mathbb{R}^4 \times S^2; C([0, T], H^s))} & \leq M(\|\phi^3\|_p) (1 + \|\phi^3 - \phi^3\|_p).
\end{align*}
\]

(2.6)

**Remark 2.2.** A general expression of \( \sigma_{\alpha} \) and \( \sigma_{\delta} \) used for describing Compton Scattering process are given by (see [23])

\[
\sigma_{\alpha}(t, x, v, \Omega, \rho, \theta) = D_0 \rho \theta^{-1} \exp \left( - \frac{D_0}{\theta^2} \left( \frac{\nu - v_0}{v_0} \right)^2 \right), \quad \sigma_{\delta} = \delta_{\alpha}(t, x, v \rightarrow v', \Omega \cdot \Omega') \rho,
\]

(2.7)

where \( v_0 \) is the fixed frequency and \( D_i \) \((i = 1, 2)\) are positive constants. It is not difficult to see that, together with the Boyle and Gay-Lussac laws for polytropic gas, \( \sigma_{\alpha} \) also satisfies assumptions (2.3)–(2.6).

### 2.2 Main result

Before stating the main result, let us first introduce the definition of regular solution to the Cauchy problem (1.9)–(1.10).

**Definition 2.1.** Let \( T > 0 \) be a finite constant. The triple \((I, \rho, u)\) is called a regular solution to the Cauchy problem (1.9)–(1.10), if \((I, \rho, u)\) satisfies this problem in the sense of distributions and

\begin{enumerate}
  \item \( I \in L^2(\mathbb{R}^+ \times S^2; C([0, T], H^s)), \quad I_i \in L^2(\mathbb{R}^+ \times S^2; C([0, T], H^2)); \)
  \item \( \inf_{x \in \mathbb{R}^+} \rho(t, x) = 0 \) for \( 0 \leq t \leq T, \quad 0 < \rho^{-1} \in C([0, T], H^0), \)
  \item \( \nabla \rho^{-1} \in L^2([0, T]; \mathbb{R}^2 \cap D^2), \quad \nabla \rho^{-1} \in L^2([0, T]; D^2), \quad \rho^{-1} \nabla u \in C([0, T]; L^2) \cap L^2([0, T]; D^2), \quad \rho^{-1} \nabla u \in L^2([0, T]; D^2), \quad \rho^{-1} \nabla u \in L^2([0, T]; D^2). \)
\end{enumerate}

Via denoting \( \sigma_{\alpha} = \sigma_{\alpha}(t = 0, x, v, \Omega, \rho_0) \) and
then our main result can be stated as follows.

**Theorem 2.1.** Let parameters \((y, \delta, \alpha, \beta)\) satisfy

\[
1 < y \leq \frac{4}{3}, \quad 0 < \delta < 1, \quad \alpha > 0, \quad 2\alpha + 3\beta \geq 0.
\]

(2.8)

If the initial data \((I_0, \rho_0, u_0)\) satisfies

\[
I_0 \in L^2(\mathbb{R}^* \times S^2; H^2), \quad (0 < \rho_0^{\gamma^{-1}}, u_0) \in H^2, \quad \nabla \rho_0^{\delta^{-1}} \in D^1 \cap D^2, \quad \nabla \rho_0^\alpha \in L^4,
\]

(2.9)

and the initial compatibility conditions

\[
\begin{cases}
\nabla u_0 = \frac{1-\delta}{\rho_0^{-2}} g_1, & L u_0 = \rho_0^{1-\delta} g_2, \\
\n\int_0^{\infty} \int_{S^2} \nabla \rho_0^{\alpha} \Omega d\Omega dv = \rho_0^\alpha g_3,
\end{cases}
\]

(2.10)

for some \(g_i \in L^2 (i = 1, 2, 3)\), then there exist a time \(T > 0\) and a unique regular solution \((I, \rho, u)\) in \([0, T] \times \mathbb{R}^3 \times \mathbb{R}^* \times S^2\) to the Cauchy problem (1.9)–(1.10), satisfying

\[
t^\delta u \in L^\infty([0, T]; D^3), \quad \frac{t}{t} u_t \in L^\infty([0, T]; D^2) \cap L^2([0, T]; D^3),
\]

\[
\rho^{\gamma^{-1}} \nabla u_t \in L^\infty([0, T]; L^2), \quad \rho^{\delta^{-1}} \nabla u_t \in L^2([0, T]; L^2),
\]

\[
u_t \in L^2([0, T]; L^2), \quad \frac{t}{t} u_{tt} \in L^\infty([0, T]; L^2) \cap L^2([0, T]; D^4),
\]

\[
\rho^\delta \in L^\infty([0, T]; L^\infty \cap D^{1,6} \cap D^{2,3} \cap D^4),
\]

\[
\nabla \rho \in C([0, T]; D^1 \cap D^2), \quad \nabla \log \rho \in L^\infty([0, T]; L^\infty \cap L^6 \cap D^{1,3} \cap D^2).
\]

Moreover, \((I, \rho, u)\) is also a classical solution to (1.9)–(1.10) for \(t \in (0, T]\).

**Remark 2.3.** The condition \(y \leq \frac{4}{3}\) in (2.8) is a technical assumption used to derive the high-order uniform \textit{a priori} estimates for the radiation intensity (see the proof of Lemma 3.4).

**Remark 2.4.** One can find the following class of initial data \((I_0, \rho_0, u_0)\) satisfying the conditions (2.9)–(2.10):

\[
I_0 \in L^2(\mathbb{R}^* \times S^2; C^3_0(\mathbb{R}^3)), \quad \rho_0(x) = \frac{1}{1 + |x|^{\frac{4\alpha}{3(1-\delta)}}, \quad u_0 \in C^3_0(\mathbb{R}^3),
\]

where \(\frac{3}{4\alpha(1-\delta)} \leq x \leq \frac{1}{4\alpha(1-\delta)}\).

**Remark 2.5.** As mentioned in the abstract, the compatibility conditions (2.10) are also necessary for the existence of the unique regular solutions obtained in Theorem 2.1. Indeed, the one shown that in the second one of (2.10) (resp. (2.10)2) plays a key role in the derivation of \(u_t \in L^\infty([0, T]; L^2(\mathbb{R}^3))\) (resp. \(\rho^{\delta^{-1}} \nabla u_t \in L^\infty([0, T]; L^2(\mathbb{R}^3))\)), which will be used in the uniform estimates for \(|u|_{D^1}\) (resp. \(|u|_{D^2}\)).

### 3 Local-in-time well-posedness

The purpose of this section is to prove Theorem 2.1. We first reformulate the original problems (1.9)–(1.10) to (3.2)–(3.4) by introducing some new variables and then establish the corresponding local existence and
uniqueness of classical solutions to (3.2)–(3.4). Finally, we show that the existence result of the problem (3.2)–(3.4) implies Theorem 2.1.

### 3.1 Reformulation

Via introducing the following new variables:

\[
\phi = \frac{Ay}{y-1} \rho^{y-1}, \quad \psi = \frac{\delta}{\delta-1} \nabla \rho^{\delta-1} = \frac{\delta}{\delta-1} \left( \frac{Ay}{y-1} \right)^{\frac{1}{\delta}} \nabla \rho^{\frac{1}{\delta-1}} = (\psi^{(1)}, \psi^{(2)}, \psi^{(3)}),
\]

then the Cauchy problem (1.9) and (1.10) can be equivalently rewritten as follows:

\[
\begin{aligned}
\frac{1}{c} I_t + \nabla I &= A_r, \\
\phi_t + u \cdot \nabla \phi + (y-1) \phi \text{div} u &= 0, \\
u_t + u \cdot \nabla u + \nabla \phi + a \phi \nabla^2 L u &= \psi \cdot Q(u) - \frac{1}{c} \int_0^\infty \int_{S^2} \Lambda \omega d\Omega dv, \\
\psi_t + \nabla (u \cdot \psi) + (\delta-1) \psi \text{div} u + \delta a \phi \nabla \psi \text{div} u &= 0, \\
(I, \phi, u, \psi)|_{t=0} &= (I_0, \phi_0, u_0, \psi_0) = \left( I_0(x, \nu, \Omega), \frac{Ay}{y-1} \rho_0^{y-1}(x), u_0(x), \frac{\delta}{\delta-1} \nabla \rho_0^{\delta-1}(x) \right)
\end{aligned}
\]

for \((x, \nu, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2, \quad (I, \phi, u, \psi) \to (0, 0, 0, 0) \quad \text{as} \quad |x| \to +\infty \quad \text{for} \quad t \geq 0,

where \(A_r = \rho \Lambda_r, \quad \Lambda_r \) is defined by (2.1),

\[
a = \left( \frac{Ay}{y-1} \right)^{\frac{1}{\delta}}, \quad e = \frac{\delta - 1}{2(y-1)} < 0,
\]

and \(L(u)\) and \(Q(u)\) are given by

\[
Lu = -a \Delta u - (a + \beta) \nabla \text{div} u, \quad Q(u) = a(\nabla u + (\nabla u)^T) + \beta \text{div} u \mathbb{1}_3.
\]

To solve the Cauchy problem (1.9) and (1.10) locally in time, we first establish the local well-posedness of classical solution to the problem (3.2)–(3.4).

**Theorem 3.1.** Let (2.8) hold. Assume the initial data \((I_0, \phi_0, u_0, \psi_0)\) satisfy

\[
I_0 \in L^2(\mathbb{R}^3 \times S^2; H^3), \quad 0 < \phi_0, u_0 \in H^3, \quad \psi_0 \in D^1 \cap D^2, \quad \nabla \phi_0^e \in L^4,
\]

and the initial compatibility conditions

\[
\begin{aligned}
\nabla u_0 &= \phi_0^e g_1, \\
Lu_0 &= \phi_0^e 2g_2, \\
\nabla (a \phi_0^e Lu_0 + \frac{1}{c} \int_0^\infty \int_{S^2} \Lambda_0 \Omega d\Omega d\nu) &= \phi_0^e g_3,
\end{aligned}
\]

for some \(g_i \in L^2 (i = 1, 2, 3), \) then there exist a time \(T_0 > 0\) and a unique classical solution \((I, \phi, u, \psi = \frac{\alpha}{\gamma-1} \nabla \phi)\) in \([0, T_0] \times \mathbb{R}^3 \times \mathbb{R}^+ \times S^2\) to the Cauchy problem (3.2)–(3.4), satisfying
To prove Theorem 3.1, we begin by considering the following linearized problem:

\[
\begin{aligned}
I_{\epsilon, \eta} + \Omega \cdot \nabla I_{\epsilon, \eta} &= \vec{A}_{\epsilon, \eta}^\prime, \\
\phi_{t, \epsilon, \eta} + \nu \cdot \nabla \phi_{\epsilon, \eta} + (\gamma - 1) \phi_{\epsilon, \eta} \nabla v &= 0, \\
u_{t, \epsilon, \eta} + \nu \cdot \nabla v + \nabla \phi_{\epsilon, \eta} + a \sqrt{(h_{\epsilon, \eta})^2 + \epsilon^2 L u_{\epsilon, \eta}} &= 0,
\end{aligned}
\]

\[
\begin{aligned}
&= \psi_{\epsilon, \eta} \cdot Q(v) - \frac{1}{C} \int_0^\infty \int_{\mathbb{S}^2} \vec{A}_{\epsilon, \eta} \cdot \nabla \Omega \nabla v, \\
&\quad \text{for } (x, v, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \quad \text{as } |x| \to +\infty \quad \text{for } t \geq 0,
\end{aligned}
\]

where both \( \epsilon \) and \( \eta \) are positive constants and

\[
\psi_{\epsilon, \eta} = \frac{a \delta}{\delta - 1} \nabla h_{\epsilon, \eta}, \quad \vec{A}_{\epsilon, \eta}^\prime = \sigma_\epsilon - \sigma_\eta I_{\epsilon, \eta} + \int_0^\infty \int_{\mathbb{S}^2} \frac{v}{\sqrt{v^2 \sigma_\epsilon \sigma_\eta - \sigma_\epsilon^2 I_{\epsilon, \eta}^2}} \nabla \Omega \nabla v',
\]

\( \chi, v = (v^{(1)}, v^{(2)}, v^{(3)}) \) and \( g \) are known functions satisfying \( (x, v, g)(t = 0) = (I_0, u_0, (\phi_0 + \eta)^{2\nu}) \) and

\[
\begin{aligned}
\chi &\in L^2(\mathbb{R}^3 \times \mathbb{S}^2; C([0, T]; H^3)), \quad \chi_t \in L^2(\mathbb{R}^3 \times \mathbb{S}^2; C([0, T]; H^3)), \\
g &\in L^\infty \cap C([0, T] \times \mathbb{R}^3), \quad \forall g \in C([0, T]; H^3), \quad g_t \in C([0, T]; H^3), \\
\forall g_{t, \eta} &\in L^2([0, T]; L^2), \quad v \in C([0, T]; H^\infty) \cap L^2([0, T]; H^\infty), \\
w &\in C([0, T]; H^3) \cap L^2([0, T]; D^3), \quad w_t \in L^2([0, T]; D^3), \quad t^\nu v \in L^\infty([0, T]; D^\nu), \\
t^\nu w &\in L^\infty([0, T]; D^3) \cap L^2([0, T]; D^3), \quad t^\nu w_t \in L^\infty([0, T]; D^3),
\end{aligned}
\]

where \( T > 0 \) is an arbitrary constant.

Next, by using standard argument (4,14,20), the global well-posedness of classical solutions \((I_{\epsilon, \eta}, \phi_{\epsilon, \eta}, u_{\epsilon, \eta}, h_{\epsilon, \eta})\) in \([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2\) to (3.8) can be obtained for \( \epsilon \) and \( \eta \) are both positive.

**Lemma 3.1.** Let (2.8) hold and \( \epsilon > 0, \eta > 0 \). Assume \((I_0, \phi_0, u_0)\) satisfies (3.5) and (3.6). Then for any \( T > 0 \), there exists a unique classical solution \((I_{\epsilon, \eta}, \phi_{\epsilon, \eta}, u_{\epsilon, \eta}, h_{\epsilon, \eta})\) in \([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2\) to (3.8), satisfying

\[
\begin{aligned}
\end{aligned}
\]
3.3 A priori estimates independent of \((\epsilon, \eta)\)

Let \(I_{\epsilon}^{\eta}, \phi_{\epsilon}^{\eta}, u_{\epsilon}^{\eta}, h_{\epsilon}^{\eta}\) be a classical solution obtained in Lemma 3.1 and the initial data satisfy (3.5) and (3.6). Now, we are going to establish the uniform a priori estimates for \(I_{\epsilon}^{\eta}, \phi_{\epsilon}^{\eta}, u_{\epsilon}^{\eta}, h_{\epsilon}^{\eta}\), which are independent of \(\epsilon, \eta\). For this purpose, we first choose a positive constant \(c_0\) independent of \(\eta\) such that

\[
2 + \eta + \|\phi_0^\eta - \eta\|_1 + \|L^\eta u_0^\eta\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} + \|\nabla h_0^\eta\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} + \|u_0^\eta\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} + \|g_0^\eta\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} + \|g_0^\eta\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} \leq c_0,
\]

where

\[
\begin{align*}
g_1^\eta &= (\phi_0^\eta)^\epsilon \nabla u_0, \\
g_2^\eta &= a(\phi_0^\eta)^\epsilon L u_0^\eta,
\end{align*}
\]

\[
\begin{align*}
g_3^\eta &= (\phi_0^\eta)^\epsilon \nabla \left(a(\phi_0^\eta)^\epsilon L u_0^\eta + \frac{1}{\epsilon} \int_0^\infty \mathcal{A}_v^\eta \, d\Omega \, dv\right),
\end{align*}
\]

Remark 3.1. It follows from (3.12) and the definition of Lamé operator that

\[
\begin{align*}
al((\phi_0^\eta)^\epsilon u_0^\eta) &= g_2^\eta - \frac{\delta}{\delta - 1} G(\psi_0^\eta, u_0^\eta), \\
(\phi_0^\eta)^\epsilon u_0^\eta &\rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty,
\end{align*}
\]

where \(\psi_0^\eta = \frac{\alpha \delta}{\delta - 1} \nabla h_0^\eta\), and

\[
G = \alpha \psi_0^\eta \cdot \nabla u_0^\eta + \alpha \text{div}(u_0^\eta \otimes \psi_0^\eta) + (\alpha + \beta)(\psi_0^\eta \text{div} u_0^\eta + \psi_0^\eta \cdot (\nabla u_0^\eta) + u_0^\eta \cdot \nabla \psi_0^\eta).
\]

As a consequence of Lemma A.6 and (3.11) and (3.12), one can deduce that

\[
\begin{align*}
|((\phi_0^\eta)^\epsilon u_0^\eta)|_{L^2} &\leq C(|g_1^\eta|_{L^2} + |G(\psi_0^\eta, u_0^\eta)|_{L^2}) \leq C_1, \\
|((\phi_0^\eta)^\epsilon \nabla u_0^\eta)|_{L^2} &\leq C(|g_1^\eta|_{L^2} + |\nabla \psi_0^\eta|_{L^2} + |u_0^\eta|_{L^2} + |\psi_0^\eta|_{L^2} |\nabla u_0^\eta|_{L^2}) \leq C_1, \\
|((\phi_0^\eta)^\epsilon \nabla \phi_0^\eta)|_{L^2} &\leq C_2,
\end{align*}
\]

where \(C_1\) is a positive constant depending on \(c_0, A, \Lambda, \alpha, \beta, y, \) and \(\delta\), but independent of \(\epsilon, \eta\).

We assume there exist some time \(T^* \in (0, T)\) and positive constants \(c_i\) \((i = 1, ..., 7)\) such that

\[
1 < c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq c_5 \leq c_6 \leq c_7,
\]

and
\[
\begin{align*}
&\sup_{0 \leq t \leq T^*} \|q(t)\|_{L^2(R \times S^2; H')} \leq c_1^2, \\
&\sup_{0 \leq t \leq T^*} \|v(t)\|_{L^2(R \times S^2; H')} \leq c_2^2, \\
&\sup_{0 \leq t \leq T^*} \|v(t)\|_{L^2(R \times S^2; H')} + \int_0^t \|V(t)\|_{L^2}^2 + \|V(t)\|_{H^1}^2 \, dt \leq c_3^2, \\
&\sup_{0 \leq t \leq T^*} \|v(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2 + \|g \nabla v(t)\|_{L^2}^2 \, dt \leq c_4^2, \\
&\sup_{0 \leq t \leq T^*} \|v(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2 + \|g \nabla v(t)\|_{L^2}^2 \, dt \leq c_5^2, \\
&\sup_{0 \leq t \leq T^*} \|v(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2 + \|g \nabla v(t)\|_{L^2}^2 \, dt \leq c_6^2, \\
&\sup_{0 \leq t \leq T^*} \|v(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2 + \|g \nabla v(t)\|_{L^2}^2 \, dt \leq c_7^2,
\end{align*}
\]

(3.16)

where \(T^*\) and \(c_i\) \((i = 1, \ldots, 7)\) will be determined later, and depend only on \(c_0\) and the fixed constants \((A, \alpha, \beta, \gamma, \delta, \Lambda, T)\).

Hereafter, \(M = M(c_0)\) denotes a strictly increasing function from \([0, \infty)\) to \([1, \infty)\), and \(C \geq 1\) denotes a generic positive constant. Both \(M(c_0)\) and \(C\) depend only on \((c_0, A, \alpha, \beta, \gamma, \delta, \Lambda, T)\) and may be different from line to line. Moreover, in the rest of Section 3.3, without causing ambiguity, we simply denote

\[
(I_{t=0}^\eta, \phi_{t=0}^\eta, u_{t=0}^\eta, h_{t=0}^\eta) \text{ as } (I_0, \phi_0, u_0, h_0), \quad (I_{t=\eta}^\eta, \phi_{t=\eta}^\eta, u_{t=\eta}^\eta, h_{t=\eta}^\eta) \text{ as } (I, \phi, u, h), \quad (g_{t=0}^\eta, g_{t=2}^\eta, g_{t=4}^\eta) \text{ as } (g_0, g_2, g_4).
\]

### 3.3.1 The a priori estimates for \(\phi\) and \(\psi\)

The following two lemmas give the estimates for \(\phi\) and \(\psi\).

**Lemma 3.2.**

\[
\|
\begin{align*}
\phi(t) - \eta\|_3 &\leq Cc_0, \quad |\phi(t)|_2 \leq Cc_0c_2, \quad |\phi(t)|_{L^2} \leq Cc_0c_3, \\
\|\phi(t)\|_{L^2} &\leq Cc_0c_2, \quad |\phi(t)|_2 \leq Cc_0c_3, \quad \int_0^t \|\phi(t)\|_{L^2}^2 \, ds \leq Cc_0^2c_2^2,
\end{align*}
\]

for \(0 \leq t \leq T_1 = \min\{T^*, (1 + Cc_0)^{-2}\}\).

**Proof.** First, the standard arguments for the transport equation and (3.16) yield that for \(0 \leq t \leq T_1\),

\[
\|
\begin{align*}
\phi - \eta\|_3 &\leq C\left(\|\phi_0 - \eta\|_3 + \eta \int_0^t \|\nabla \phi\|_3 \, ds\right) \exp\left(\int_0^t C\|\nabla \phi\|_4 \, ds\right) \leq Cc_0.
\end{align*}
\]

Second, according to the equation (3.8)\(_2\), one can obtain that for \(0 \leq t \leq T_1\),
\[ \begin{aligned}
&\|\Phi(t)\|_2 \leq C(\|v\|_3 |\nabla \Phi|_6 + |\Phi|_{\infty} |\nabla v|_2) \leq C_2 c_4, \\
&|\Phi(t)|_{L^2} \leq C(\|v\|_{\infty} + |\nabla \Phi|_6 |v|_1 + |\Phi|_{\infty} |v|_2) \leq C_2 c_5, \\
&|\Phi(t)|_{L^2} \leq C(\|v\|_{\infty} |\nabla \Phi|_6 + |\Phi|_{\infty}) \leq C_2 c_6, \\
&|\Phi(t)|_{L^2} = -v_i \cdot \nabla \Phi - v \cdot \nabla v_i - (\gamma - 1)(\Phi_i \text{div} v + \Phi \text{div} v_i)_{L^2} \\
&\leq C(\|v\|_{\infty} |\nabla \Phi|_2 + |\Phi|_{\infty} |v|_1) \leq C_2 c_6.
\end{aligned} \]

Moreover, one also has that for \(0 \leq t \leq T\),
\[
\int_0^t \| \Phi(t) \|_2^2 \, dt \leq C \int_0^t (\|v_i \cdot \nabla \Phi|_1 + \|v \cdot \nabla \Phi|_1 + \|\Phi_i \text{div} v|_1 + \|\Phi \text{div} v_i|_1)^2 \, dt \leq C_2 c_6^2.
\]

The proof of Lemma 3.2 is complete. \(\square\)

**Lemma 3.3.**
\[
|\psi(t)|_{L^2}^2 + \|\psi(t)\|_{L^2, B^1(D)}^2 \leq C_2^2, \quad |\psi_i(t)|_{L^2}^2 \leq C_2^4, \quad |h_i(t)|_{L^2}^2 \leq C_2^3 c_6, \\
|\psi(t)|_{L^2}^2 + \int_0^t (|\psi_i(s)|_{L^2}^2 + |h_i(s)|_{L^2}^2) \, ds \leq C_2^4, \quad \text{for } 0 \leq t \leq T_i.
\]

**Proof.** According to the equation (3.8) and \(\psi = a\delta \nabla h\), one can deduce that
\[
\psi_i + \sum_{l=1}^3 A_i(v) \partial_l \psi + B^i(v) \psi + a\delta (g \nabla \text{div} v + \nabla g \text{div} v) = 0,
\]
where \(A_i(v) = (a_{ij}^l)_{3 \times 3} (i, j, l = 1, 2, 3)\) are symmetric with
\[
a_{ij}^l = v^l \quad \text{for } i = j; \quad \text{otherwise } a_{ij}^l = 0,
\]
and \(B^i(v) = (\nabla v)^l\).

First, let \(\zeta = (\zeta_1, \zeta_2, \zeta_3)^T (1 \leq |\zeta| \leq 2 \text{ and } \zeta = 0, 1, 2)\). Applying \(\partial_l \zeta \) to (3.17), multiplying by \(2 \partial_l \zeta \psi\) and integrating over \(R^3\), one obtains
\[
\frac{d}{dt} |\partial_l \zeta \psi|_2 \leq C(\|v\|_{\infty} |\nabla \Phi|_2 + |\Theta|_2),
\]
where
\[
\Theta = \partial_l (B^l \Phi) - B^l \partial_l \psi + \sum_{l=1}^3 (\partial_l (A_i \partial_l \psi) - A_i \partial_l \partial_l \psi) + a\delta (g \nabla \text{div} v + \nabla g \text{div} v).
\]

For \(|\Theta|_2\), it can be estimated as follows:
\[
|\Theta|_2 \leq C(\|v \nabla |v|_2 + |g \nabla|_2 + |v \nabla|_1 + |g \nabla|_1) \quad \text{for } |\zeta| = 1,
\]
\[
|\Theta|_2 \leq C(\|v \nabla|_1 + |v \nabla|_2 + |g \nabla|_2 + |v \nabla|_1 + |g \nabla|_1 + |v \nabla|_2 + |g \nabla|_1 + |v \nabla|_2) \quad \text{for } |\zeta| = 2.
\]

Plugging the aforementioned estimates for \(|\Theta|_2\) into (3.18) and using the Gagliardo-Nirenberg inequality, one obtains
\[
\frac{d}{dt} \|\psi(t)\|_{L^2, B^1(D)} \leq C(c_2 \|\psi(t)\|_{L^2, B^1(D)} + |g \nabla \text{div} v|_{B^1(D)} + c_2^2),
\]
which, along with the Gronwall inequality, implies that for \(0 \leq t \leq T_i\),
\[
|\psi(t)|_{L^2(D)} \leq C_0 + C_0^2 t + \int_0^t |g \nabla \text{div}_D v| ds \exp(C_0 t) \leq C_0.
\]

Second, according to equations (3.17), one has that for \(0 \leq t \leq T_1\),
\[
\begin{align*}
|\psi(t)|_{L^2(D)} &\leq C(|v|_{C^2} + |v|_{C^2} + |g|^2|v|_{L^2}) \leq C_2^2, \\
|\nabla \psi(t)|_{L^2(D)} &\leq C(|v|_{C^2} + |v|_{C^2} + |g|^2|v|_{L^2}) \leq C_2^2, \\
\int_0^t |\psi(s)|^2 ds &\leq C \int_0^t (|v|^2|\psi|^2 + |v|^2|\psi|^2 + |vi|^2|v|^2 + \psi^2|v|^2) ds \leq C_4^2,
\end{align*}
\]

On the other hand, Lemma A.1 (Appendix A) and (3.16) yield that
\[
|g \nabla v|_{C^2} \leq C|g \nabla v|_{C^2} \leq C_0^2 C_4^2.
\]

Finally, it follows from (3.19) and the equation (3.8) that for \(0 \leq t \leq T_1\),
\[
|h(t)|_{L^2(D)} \leq C(|v|_{C^2} + |g \nabla v|_{C^2}) \leq C_0^2 C_4^2,
\]
\[
\int_0^t |h(s)|^2 ds \leq C \int_0^t (|v|_{C^2} |\psi|^2 + |v|_{C^2} |\psi|^2 + |g|_{C^2} |v|^2 + |g \nabla v|^2)| ds \leq C_4^2,
\]

where one has used
\[
|g \nabla v|_{C^2} \leq C(|v|_{C^2} |v|_{L^2} + |g \nabla v|_{C^2}) \leq C(|v|_{C^2} |v|_{L^2} + |g \nabla v|_{L^2}) \leq C_4^2.
\]

The proof of Lemma 3.3 is complete. \(\square\)

### 3.3.2 The \textit{a priori} estimates for \(I\)

#### Lemma 3.4.
\[
\begin{align*}
\|I\|_{L^2(\mathbb{R}^3 ; C([0,T_1]; H^3))} &\leq C_0, \\
\|I\|_{L^2(\mathbb{R}^3 ; C([0,T_1]; H^3))} &\leq M(c_0) C^{2 \kappa + 1}, \\
\|I\|_{L^2(\mathbb{R}^3 ; C([0,T_1]; L^2))} &\leq M(c_0) C_4^{2 \kappa + 1},
\end{align*}
\]

for \(0 \leq t \leq T_2 = \min\{T^*, (1 + M(c_0) C^{2 \kappa + 1})\} \) with \(\kappa = 1/(y - 1) \geq 3\).

**Proof.** First, multiplying (3.8) by \(2J\) and integrating over \(\mathbb{R}^3\), one has
\[
\frac{d}{dt} |J|^2 \leq C \left( |\sigma|_L^2 |J|_2 + |J|_2 |\phi|_{C^2} \|\phi\|_{L^2(\mathbb{R}^3 ; 2^k)} \left( \int_0^\infty \int_{S'} \frac{v}{|v'|} |\pi|^2 d\Omega' d\nu' \right) \right)^2
\]
\[
\leq C \left( |\sigma|_L^2 + |J|_2^2 + c_0^2 \int_0^\infty \int_{S'} \frac{v}{|v'|} |\pi|^2 d\Omega' d\nu' \right),
\]

where one has used the fact \(\sigma \geq 0\) and \(c_0^2 \geq 0\).
Second, by applying $\delta_i^j(1 \leq |\zeta| \leq 3)$ to (3.8)$_1$, multiplying the resulting equation by $\partial^\zeta I$, and integrating over $\mathbb{R}^3$, one can obtain
\[
\frac{1}{2} \frac{d}{dt} \int |\partial^\zeta I|^2 dx + \int \left( \sigma_a + \int_0^\infty \sigma_a' d\Omega' d\nu' \right) |\partial^\zeta I|^2 dx = -\int \left( \partial^\zeta_i(\sigma_a) - \sigma_a \partial^\zeta_j I \right) \partial^\zeta_j I dx - \int_0^\infty \int_0^\infty \sigma_a \left( \partial^\zeta_i (\phi^j \sigma^j) - \phi^j \partial^\zeta_j I \right) d\Omega' d\nu' \partial^\zeta I dx
\]
\[+ \int_0^\infty \int_0^\infty \sigma_a \partial^\zeta_j I dx + \int_0^\infty \int_0^\infty \sigma_a \partial^\zeta_j (\phi^i \chi^i) d\Omega' d\nu' \partial^\zeta I dx \equiv \sum_{i=1}^n R_i. \tag{3.21} \]

According to Lemma 3.2, Hölder's inequality, and Gagliardo-Nirenberg inequality, one obtains
\[
R_i \leq C\|\nabla \sigma_{a0}\|_2 \|\nabla I\|_2 \quad \text{for } |\zeta| = 1, \\
R_i \leq C\|\nabla \sigma_{a0}\|_2 \|\nabla I\|_2 \quad \text{for } |\zeta| = 2, \\
R_i \leq C\|\nabla \sigma_{a0}\|_2 \|\nabla I\|_2 \quad \text{for } |\zeta| = 3, \\
R_i \leq C\|\nabla \phi\|_1 \|\nabla I\|_2 \quad \text{for } |\zeta| = 1, \\
R_i \leq C\|\nabla \phi\|_2 \|\nabla I\|_2 \quad \text{for } |\zeta| = 2, \\
R_i \leq C\|\nabla \phi\|_2 \|\nabla I\|_2 \quad \text{for } |\zeta| = 3, \\
R_i \leq C\|\partial^\zeta \sigma_a\|_2 + \|\partial^\zeta I\|_2, \\
R_i \leq C c_{01} \|\partial^\zeta I\|_2 \left( \int_0^\infty \int_0^\infty \frac{v}{\sqrt{v'}} \left| \sigma_a \right|^2 d\Omega' d\nu' \right) \left( \int_0^\infty \int_0^\infty \frac{\sqrt{v'}}{v} \left| \sigma_a \right|^2 d\Omega' d\nu' \right)^{\frac{1}{2}} \\
\leq C\|I\|_1^\frac{1}{2} + C c_{01} c_1^2 \left( \int_0^\infty \int_0^\infty \frac{v}{\sqrt{v'}} \left| \sigma_a \right|^2 d\Omega' d\nu' \right)^{\frac{1}{2}} \quad \text{for } |\zeta| = 1, \\
R_i \leq C\|\nabla I\|_1 \|\nabla I\|_2 + C c_{01} c_1^2 \left( \int_0^\infty \int_0^\infty \frac{v}{\sqrt{v'}} \left| \sigma_a \right|^2 d\Omega' d\nu' \right) \quad \text{for } |\zeta| = 2, \\
R_i \leq C\|\nabla I\|_1 \|\nabla I\|_2 + C c_{01} c_1^2 \left( \int_0^\infty \int_0^\infty \frac{v}{\sqrt{v'}} \left| \sigma_a \right|^2 d\Omega' d\nu' \right) \quad \text{for } |\zeta| = 3. \\
\]

Plugging the aforementioned estimates for $R_i$ ($i = 1, \ldots, 4$) into (3.21), summing up all $\zeta(1 \leq |\zeta| \leq 3)$, and combining (3.20), one arrives at
\[
\frac{d}{dt} \|I\|_1^2 \leq M(c_0) \|c_0^2 t\|_1^2 + C \|\sigma_a\|_2^2 + C c_{01} c_1^2 \int_0^\infty \int_0^\infty \frac{v}{\sqrt{v'}} \left| \sigma_a \right|^2 d\Omega' d\nu'. \tag{3.22} \]

Integrating (3.22) over $\mathbb{R}^+ \times S^2$, using the hypotheses H1–H2 and the Gronwall inequality, one has that for $0 \leq t \leq T_2$,
\[
\|I\|_2^2 \leq C(\|I\|_2^2 + M(c_0) c_{10} c_{11} t) \exp(2M(c_0) c_{10} t) \leq C T_2. \tag{3.23} \]

According to the equation (3.8)$_1$ and (3.23), one can obtain that for $0 \leq t \leq T_2$,
\[
\|I\|_2^2 \leq C(\|I\|_2^2 + M(c_0) c_{10} c_{11} t) \quad \text{for } 0 \leq t \leq T_2. \\
\]
Finally, differentiating (3.8) with respect to $t$, one can similarly obtain
\[
\|I_0\|\mathcal{L}_t(L^2(\mathbb{R}^n \times S^2); L^2_t) \leq C\left(\|\nabla I_0\|L^2(\mathbb{R}^n \times S^2); L^2_t) + \|\sigma_0\|\mathcal{L}_t(L^2(\mathbb{R}^n \times S^2); L^2_t) + \int_0^\infty \int_{S^2} |\sigma| \, d\Omega \, d\nu \right)
\]
\[
+ \left(\|\phi^x_0\|\mathcal{L}_t(L^2(\mathbb{R}^n \times S^2); L^2_t) + \|\phi^x\|_{L^2(\mathbb{R}^n \times S^2); L^2_t} \right) \left(\int_0^\infty \int_{S^2} |\sigma| \, d\Omega \, d\nu \right)^{\frac{1}{2}} \leq M(c_0)\frac{2^{2e+3}}{4^3}.
\]

The proof of Lemma 3.4 is complete. □

### 3.3.3 The a priori estimates for auxiliary variables $\varphi$ and $f$

\[
\varphi = h^{-1} = \varphi - 2e, \quad f = \varphi \varphi = \frac{2ae\delta}{\delta - 1} \nabla \log \phi.
\] (3.24)

**Lemma 3.5.**

\[
\|\varphi(t)\|_{L^{2e+1}(\mathbb{R}^n \times \mathbb{R}^3)} + \|f(t)\|_{L^{2e+1}(\mathbb{R}^n \times \mathbb{R}^3)} \leq Cc_0^6,
\]

\[
h(t, x) > \frac{1}{2c_0}, \quad \frac{2}{3} \eta^{-2e} < \varphi(t, x) < 2|\varphi_0|_\infty \leq 2c_0,
\]

\[
\|\varphi(t)\|_{L^{2e+1}(\mathbb{R}^n \times \mathbb{R}^3)} + \|f(t)\|_{L^{2e+1}(\mathbb{R}^n \times \mathbb{R}^3)} \leq Cc_0^6,
\]

for $0 \leq t \leq T_3 = \min\{T^*, (1 + M(c_0)c_0^{-2e+2})\}$.

**Proof. Step 1:** Estimate on $\varphi$. First, according to (3.8) and the definition of $\varphi$ in (3.24), one deduces that $\varphi$ satisfies the following transport equation:

\[
\varphi_t + \nabla \varphi - (\delta - 1)g\varphi^2 \nabla \nu = 0,
\] (3.25)

which, along with the standard characteristic method, yields that

\[
\frac{2}{3} \eta^{-2e} < \varphi(t, x) < 2|\varphi_0|_\infty \leq 2c_0 \quad \text{for} \ (t, x) \in [0, T_3] \times \mathbb{R}^3.
\] (3.26)

Second, by the standard energy estimates for the equation (3.25), one can obtain

\[
\frac{d}{dt} |\nabla \varphi|_6 \leq C\left(\|\nabla \varphi\|_6 + |\varphi|_6 |g \nabla \nu_6| + |\nabla \nu|_6|\nabla \varphi|_6\right),
\]

\[
\frac{d}{dt} |\nabla^2 \varphi|_3 \leq C\left(\|\nabla \varphi\|_6 + |\varphi|_6 |g \nabla \nu_6| + |\nabla \nu|_6|\nabla \varphi|_6 + |\nabla \varphi_6|_6 |g \nabla \nu|_6\right)
\]

\[
\frac{d}{dt} |\nabla^3 \varphi|_2 \leq C\left(\|\nabla \varphi\|_6 + |\varphi|_6 |g \nabla \nu_6| + |\nabla \varphi_6|_6 |g \nabla \nu|_6\right)
\]

\[
\frac{d}{dt} |\nabla^4 \varphi|_1 \leq C\left(\|\nabla \varphi\|_6 + |\varphi|_6 |g \nabla \nu_6| + |\nabla \varphi_6|_6 |g \nabla \nu|_6\right) + |\nabla \nu_6|_6 |\nabla \varphi_6|_6 |g \nabla \nu|_6 + |\nabla \nu|_6 |\nabla \varphi_6|_6 |g \nabla \nu|_6\right)
\]

\[
+ |\nabla \nu_6|_6 |\nabla \varphi_6|_6 |g \nabla \nu|_6 + |\nabla \nu|_6 |\nabla \varphi_6|_6 |g \nabla \nu|_6\right) + |\nabla \nu_6|_6 |\nabla \varphi_6|_6 |g \nabla \nu|_6 + |\nabla \nu|_6 |\nabla \varphi_6|_6 |g \nabla \nu|_6\right)$. 

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which, along with (3.26) and the Gronwall inequality, yield that
\[ |\varphi(t)|_{D^0} + |\varphi(t)|_{D^1} + |\varphi(t)|_{D^2} \leq C c_0 \quad \text{for} \ 0 \leq t \leq T_s. \quad (3.27) \]

Finally, according to equations (3.25)–(3.27), one can similarly obtain
\[ \|\varphi(t)\|_{L^2; D^0} \leq C c_0 \quad \text{for} \ 0 \leq t \leq T_s. \quad (3.28) \]

**Step 2:** Estimate on \( f \). First, by the definition of \( f \) in (3.24), (3.27)–(3.28), and Lemma 3.3, one has that for \( 0 \leq t \leq T_s \),
\[
\begin{align*}
|f(t)|_{\infty} &\leq C c_0, \quad |f(t)|_{6} \leq |\psi |_{|} |\varphi |_{\infty} \leq C c_0^2, \\
|\nabla f(t)|_{3} &\leq C(|\nabla \varphi |_{6} |\psi |_{6} + |\varphi |_{|} |\nabla \psi |_{3}|) \leq C c_0^2, \\
|\nabla^2 f(t)|_{2} &\leq C(|\varphi |_{|} |\nabla^2 \psi |_{2} + |\psi |_{|} |\nabla \varphi |_{2} + |\nabla \varphi |_{|} |\nabla \psi |_{2}|) \leq C c_0^2, \\
|f_t(t)|_{3} &\leq C(|\psi |_{|} |\varphi_t |_{6} + |\varphi |_{|} |\nabla \psi_t |_{3}|) \leq C c_0^2, \\
|\nabla f_t(t)|_{2} &\leq C(|\varphi |_{|} |\nabla \varphi_t |_{3} + |\varphi |_{|} |\nabla^2 \psi_t |_{2} + |\nabla \psi |_{|} |\varphi_t |_{6} + |\nabla \varphi |_{|} |\psi_t |_{3}|) \leq C c_0^2.
\end{align*}
\]

Second, it follows from the definition that \( f \) satisfies the following equations:
\[
f_t + \sum_{i=1}^{3} A_i(v) \delta f + B_i(v)f + a\delta(|g\varphi\nabla v + \varphi\nabla g - \nabla (\varphi g) - (\delta - 1)g\varphi - \nabla v) = 0.
\]

The proof of Lemma 3.5 is complete. \( \square \)

### 3.3.4 The *a priori* estimates for \( u \)

On the basis of estimates obtained in Lemmas 3.2–3.5, now we are ready to give the lower order estimates for \( u \).

**Lemma 3.6.**
\[
|\sqrt{h} \nabla u(t)|_{D^3}^{2} + \|u(t)|_{D^3}^{2} + \int_{0}^{t} \left( \|\nabla u(s)|_{D^3}^{2} + |u_t(s)|_{D^3}^{2}\right) ds \leq C c_0^4,
\]
\[
(\|u |_{D^3}^{2} + |h\nabla^2 u |_{D^3}^{2} + |u_t |_{D^3}^{2}) + \int_{0}^{t} \left( \|u(s)|_{D^3}^{2} + |u_t(s)|_{D^3}^{2}\right) ds \leq M(c_0)c_0^2c_5,
\]
for \( 0 \leq t \leq T_0 = \min\{T^*, (1 + M(c_0)c_5)^{-28(k+20)}\} \).

**Proof.** **Step 1:** Estimate on \( |u|_{2} \). First, multiplying (3.8) by \( u \) and integrating over \( \mathbb{R}^3 \), according to Hölder’s inequality, Young’s inequality, and Gagliardo-Nirenberg inequality, one has
\[
\frac{1}{2} \frac{d}{dt} |u|_{2} + a((h^2 + \varepsilon^2)^{\frac{1}{2}} |u|_{2} + a(\alpha + \beta)(h^2 + \varepsilon^2)^{\frac{1}{2}} |u|_{2}) \leq Q(u) - \psi \cdot Q(v) + \frac{1}{c} \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla u| \cdot d\Omega \, dv + \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla v| \cdot d\Omega \, dv,
\]
\[
\leq C |u|_{2} \left( |\nabla \varphi |_{2} + |\psi |_{2} + \sqrt{h} |\nabla u|_{2} |\varphi |_{\infty} + |\psi |_{\infty} |\nabla v|_{2} + \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla u| \cdot d\Omega \, dv \right),
\]
\[
\leq C(1 + |\nabla \varphi |_{\infty} + |\nabla v|_{2} + |\psi |_{\infty} |\psi |_{\infty} |u|_{2} + \frac{aa}{2} |\sqrt{h} |u|_{2}^{2} + Cc_0^2 + C |u|_{2} \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla u| \cdot d\Omega \, dv) \triangleq R_5.
\]
Next we estimate the last term $R_5$ on the right-hand side of (3.29). By Hölder’s inequality and the hypotheses $H1$–$H2$, one obtains

$$
R_5 = Cu \left[ \int_0^\infty \int_{S^2} |\sigma_e - \sigma_J| \, d\Omega \, dv \right] ^{\frac{1}{2}} 
\leq Cu \left[ \left( \int_{S^2} |\sigma_e - \sigma_J|^2 \, d\Omega \, dv \right) ^{\frac{1}{2}} \right] 
\leq Cu \left[ \left( \int_{S^2} |\sigma_e|^2 \, d\Omega \, dv \right) ^{\frac{1}{2}} \right] 
\leq M(c_0)(|u|_2^2 + c_0^2).
$$

Substituting (3.30) into (3.29), one arrives at

$$
\frac{d}{dt} |u|_2^2 + |\sqrt{h} \nabla u|_2^2 \leq M(c_0)(c_0^2|u|_2^2 + c_0^2),
$$

which, along with the Gronwall inequality, yields that for $0 \leq t \leq T_a$,

$$
|u(t)|_2^2 + \int_0^t |\sqrt{h} \nabla u(s)|_2^2 \, ds \leq C(|u_0|_2^2 + M(c_0)c_0^2t) \exp(M(c_0)c_0^2t) \leq Cc_0^2.
$$

**Step 2:** Estimate on $|\nabla u|_2$. Multiplying (3.8) by $u_t$ and integrating over $\mathbb{R}^3$, one can obtain

$$
\frac{1}{2} \frac{d}{dt} \left( a(h^2 + \varepsilon^2)^2 |\nabla u|_2^2 + a(\alpha + \beta)(h^2 + \varepsilon^2)^2 |\nabla u|_2^2 \right) + |u_t|_2^2 
\leq - \int \left( \nabla \cdot v + \nabla \phi \cdot \nabla + \nabla \cdot \nabla \phi + \nabla \cdot \nabla \cdot Q(u) - \nabla \cdot Q(v) + \frac{1}{\varepsilon} \int \int_{S^2} \nabla \Omega \, d\Omega \, dv \right) \cdot u_t \, dx
$$

which, along with the Gronwall inequality, yields that for $0 \leq t \leq T_a$,

$$
|\sqrt{h} \nabla u(t)|_2^2 + \int_0^t |u_t(s)|_2^2 \, ds \leq C(c_0^2 + M(c_0)c_0^2t) \exp(M(c_0)c_0^2t) \leq Cc_0^2.
$$

**Step 3:** Estimate on $|u|_{\theta'}$. First, it follows from (3.8) and the definition of Lamé operator that

$$
aL(\sqrt{h^2 + \varepsilon^2}u) = -u_t - \nabla \cdot \nabla \phi + \nabla \cdot Q(u) - \nabla (\sqrt{h^2 + \varepsilon^2}u) - \frac{1}{\varepsilon} \int \int \nabla \Omega \, d\Omega \, dv.
$$
Then according to (3.14) for the definition of $G$, (3.30), and Lemma A.6 (Appendix A), one can obtain that
\[
|\sqrt{h^2 + \varepsilon^2} u|_{L^2} \leq C \left( |u_t + v \cdot \nabla v + \nabla \psi - \psi \cdot Q(v) + G(\sqrt{h^2 + \varepsilon^2}, u)|_{L^2}^{\infty} \int \frac{1}{\sqrt{h^2 + \varepsilon^2}} d\Omega dv \right)
\]
\leq C |u_t|_{L^2} + M(c_0)c_g^2 c_s^2.
\] (3.36)

It thus follows from (3.34) and (3.36) that for $0 \leq t \leq T_s$,
\[
\int_0^t \left( |H^2 u(s)|_{L^2}^2 + |\nabla^2 u(s)|_{L^2}^2 \right) ds \leq Cc_0^2.
\] (3.37)

Second, differentiating the equations (3.8) with respect to $t$, one has
\[
u u_t + a \sqrt{h^2 + \varepsilon^2} L(u_t) = -(v \cdot \nabla v)_t - \frac{\alpha h}{\sqrt{h^2 + \varepsilon^2}} Lu + (\psi \cdot Q(v))_t - \frac{1}{c} \int_0^\infty (\mathcal{A}_t)_{t} \Omega d\Omega dv.
\] (3.38)

Multiplying (3.38) by $u_t$ and integrating over $\mathbb{R}^3$, one arrives at
\[
\frac{1}{2} \frac{d}{dt} |u_t|_{L^2}^2 + aa(h^2 + \varepsilon^2\nabla u_t) + a(a + \beta)(h^2 + \varepsilon^2)\nabla u_t
\]
\[
= \int \left( -(v \cdot \nabla v)_t - \nabla \phi_t - a\nabla \sqrt{h^2 + \varepsilon^2} \cdot Q(u)_t + (\psi \cdot Q(v))_t - \frac{\alpha h}{\sqrt{h^2 + \varepsilon^2}} Lu - \frac{1}{c} \int_0^\infty (\mathcal{A}_t)_{t} \Omega d\Omega dv \right)
\cdot u_t \, dx.
\] (3.39)

which, along with the hypotheses $H1$–$H2$ and Lemmas 3.2–3.5, yields that
\[
\frac{d}{dt} |u_t|_{L^2}^2 + |(h^2 + \varepsilon^2)\nabla u_t|_{L^2}^2 \leq C(c_0^2|u_t|_{L^2}^2 + |\nabla^2 u|_{L^2}^2) + M(c_0)c_g^{2v+6}.
\] (3.40)

Integrating (3.40) over $(\tau, t)(\tau \in (0, t))$ and using (3.37), one has
\[
|u_t(t)|_{L^2}^2 + \int_{\tau}^t |\nabla u(t,s)|_{L^2}^2 ds \leq |u_t(\tau)|_{L^2}^2 + Cc_0^2 \int_{\tau}^t |u(s)|_{L^2}^2 ds + Cc_0^4 + M(c_0)c_g^{2v+6} t.
\] (3.41)

It follows from (3.8) that
\[
|u_t(\tau)|_{L^2} \leq C(|\nabla \phi|_{L^2} + |(h + \varepsilon)\nabla u|_{L^2} + |\nabla v|_{L^2} + ||(\mathcal{A})||_{L^2(\mathbb{R}^{3};L^2)} + ||\nabla^2 u||_{L^2(\mathbb{R}^{3};L^2)} + ||\nabla^2 u||_{L^2(\mathbb{R}^{3};L^2)} + ||\nabla^2 u||_{L^2(\mathbb{R}^{3};L^2)}).
\]
which, together with the time continuity of \((I, \phi, u, \psi)\), (2.3) and (3.11), implies that
\[
\limsup_{\tau \to 0} |u_\tau(t)|_2 \leq M(c_0)c_0^5.
\]
Letting \(\tau \to 0\) in (3.41) and using the Gronwall inequality, one can obtain
\[
|u(t)|_2 + \int_0^t |\sqrt{\gamma} \nabla u(s)|_2^2 \, ds \leq M(c_0)c_0^5 \quad \text{and} \quad \int_0^t |\nabla u(s)|_2^2 \, ds \leq M(c_0)c_0^5,
\]
for \(0 \leq t \leq T_0\).

It thus follows from (3.36) that for \(0 \leq t \leq T_0\),
\[
|h^2 u(t)|_2 + |\sqrt{\gamma} h^2 + \varepsilon^2 u(t)|_2 \leq M(c_0)c_0^5 \quad \text{and} \quad |u(t)|_2 \leq M(c_0)c_0^5.
\]

As a consequence of Lemma A.6 and (3.35), one also has
\[
|\sqrt{\gamma} h^2 + \varepsilon^2 u|_2 \leq C \left( |u_t + v \cdot \nabla v + \nabla \phi - \psi \cdot Q(v)|_2^2 + |G(\sqrt{\gamma} h^2 + \varepsilon^2, u)|_2^2 + \int_0^\infty \int_{\mathbb{R}^3} |A_r|_2^2 \, d\Omega \, dv \right),
\]
which implies that
\[
|\sqrt{\gamma} h^2 + \varepsilon^2 \nabla u|_2 \leq C \left( |\nabla u|_6 + |\nabla^3 u|_2 \right) \leq C |\nabla u|_6 + |\nabla^3 u|_2,
\]

for \(0 \leq t \leq T_0\),
\[
\int_0^t \left( |h^2 u(s)|_2^2 + |\sqrt{\gamma} h^2 u(s)|_2^2 + |u(s)|_2^2 \right) \, ds \leq M(c_0)c_0^7.
\]

The proof of Lemma 3.6 is complete. \( \square \)

The following lemma gives the higher order estimates for \(u\).

**Lemma 3.7.**
\[
(|\sqrt{\gamma} \nabla u|_2^2 + |u_t|_2^2 + |u|_2^2 + |h^2 \nabla u|_2^2)(t) + \int_0^t |u(s)|_2^2 \, ds \leq M(c_0)c_0^{18.5},
\]
\[
\int_0^t \left( |u_t(s)|_2^2 + |u(s)|_2^2 + |h^2 \nabla u(s)|_2^2 + |(h^2 u)_t(s)|_2^2 \right) \, ds \leq M(c_0)c_0^{10},
\]
for \(0 \leq t \leq T_5 \leq \min\{T^*, (1 + M(c_0)c_0) - 2^{10.25}\}\).

**Proof.** First, multiplying (3.38) by \(u_t\) and integrating over \(\mathbb{R}^3\), using Hölder’s inequality, Young’s inequality, and Gagliardo-Nirenberg inequality, one has
\[
\frac{1}{2} \frac{d}{dt} \left( a\left( h^2 + \varepsilon^2 \right) \nabla u_t |_2^2 + a(\alpha + \beta) \left( h^2 + \varepsilon^2 \right) \nabla u_t |_2^2 \right) + |u_t|_2^2
\]

\[
= \int \left( -\nabla \cdot (v \cdot v) - \nabla \phi - \frac{ah}{\sqrt{h^2 + \varepsilon^2}} h L u + \frac{ah}{\sqrt{h^2 + \varepsilon^2}} \nabla h \cdot Q(u_t) + (\psi \cdot Q(v))_t \right)_t
\]

\[
- \frac{1}{c} \int_0^\infty (\mathcal{A}_t) \Omega d \Omega d \nu
\]

\[
+ \int \frac{ah}{\sqrt{h^2 + \varepsilon^2}} h_t (a|\nabla u_t|^2 + (\alpha + \beta)|\nabla u_t|^2) d\nu
\]

\[
\leq C|u_t|_2 \left( \|v\|_2 + \|\nabla \phi\|_2 + [h_t]_\infty \|\nabla^2 u_t\|_2 + [\psi]_\infty [\phi]_\infty [\nabla \nabla u_t]_2 + [\psi_t]_2 \|\nabla v\|_\infty + [\psi]_\infty \|\nabla v_t\|_2 \right.
\]

\[
+ \left. \left. \|\nabla (\mathcal{A})\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n \setminus \mathbb{R}^n \times \mathbb{R}^n \setminus \mathbb{R}^n \times \mathbb{R}^n \setminus \mathbb{R}^n \times \mathbb{R}^n) \left( [\nabla (\mathcal{A})_{\Omega}] d \Omega d \nu \right) \right) \right)_t
\]

\[
\leq \frac{1}{8} |u_t|_2^2 + M(c_0) c_0^3 \sqrt{h} \nabla u_t |_2^2 + M(c_0) c_0^{2+14}. \tag{3.47}
\]

Integrating (3.47) over \((\tau, t)(\tau \in (0, t))\), one can obtain

\[
|\sqrt{h} \nabla u_t(t)|_2 + \int_0^t |u_t(s)|_2^2 ds \leq (h^2 + \varepsilon^2) |\nabla u_t(\tau)|_2^2 + M(c_0) c_0^3 \int_0^t |\nabla \nabla u_t|_2^2 ds + M(c_0) c_0^{2+14}. \tag{3.48}
\]

It follows from the equations (3.8) that

\[
|\sqrt{h} \nabla u_t(\tau)|_2 \leq C \left( |\sqrt{h} \nabla (v \cdot v) + \nabla \phi - \psi \cdot Q(v))|_2 + \left. \left. \sqrt{h} \nabla \left( a\sqrt{h^2 + \varepsilon^2} L u + \frac{1}{c} \int_0^\infty (\mathcal{A}_t) \Omega d \Omega d \nu \right) \right) \right)_t \tag{3.49}
\]

which, along with the hypotheses \(H1–H2\), (3.11)–(3.12) and (3.15), implies that

\[
\limsup_{\tau \to 0} |\sqrt{h} \nabla u_t(\tau)|_2 \leq C |\phi^0 \nabla u_0 \cdot \nabla u_0|_2 + [\phi^0 \nabla^2 \phi_0]_2 + [\phi_0 \nabla (\psi_0 \cdot Q(u_0))]_2 + [g_5]_2 + \varepsilon \|\nabla u_0\|_2 \tag{3.50}
\]

\[
\leq C |\nabla u_0|_\infty [g_5]_2 + [\phi_0^0 \nabla u_0 \cdot \nabla u_0]_2 + [\phi_0 \nabla \nabla u_0]_2 + [\phi_0 \nabla u_0]_2 + [\phi_0^0 \nabla \nabla u_0]_2 + [\psi_0]_\infty + [\psi_0]_\infty + [\psi_0]_\infty + [g_5]_2 + [g_5]_2
\]

\[
+ \|u_0\|_\infty \leq M(c_0) c_0^3. \tag{3.50}
\]

Letting \(\tau \to 0\) in (3.48) and using the Gronwall inequality, one concludes that for \(0 \leq t \leq T_0\),

\[
|\sqrt{h} \nabla u_t(t)|_2 + \int_0^t |u_t(s)|_2^2 ds \leq M(c_0) c_0^6 \quad \text{and} \quad |u_t(t)|_2^2 \leq M(c_0) c_0^7, \tag{3.51}
\]

which, together with (3.44)–(3.45), yields that

\[
|\sqrt{h^2 + \varepsilon^2} u|_2 + |\sqrt{h^2 + \varepsilon^2} \nabla u_t|_2 + |h \nabla^2 u|_2 \leq M(c_0) c_0^{8.25} \quad \text{and} \quad |\nabla^2 u|_2 \leq M(c_0) c_0^{2.25}. \tag{3.51}
\]

Second, it follows from (3.38) and Lemma A.6 that for \(0 \leq t \leq T_5\),
\[
|\sqrt{h^2 + e^2} u_t|_{L^p} \leq C \left( u_{tt} + (v \cdot \nabla v)_t + \nabla \phi_t - (\psi \cdot Q(v))_t + \frac{ah}{\sqrt{h^2 + e^2}} h_t L u \right)_{L^p} + |G(\sqrt{h^2 + e^2}, u)|_{L^2} + \int_0^\infty \int (\mathfrak{A}_r \delta) d\Omega d\nu
\]
\[
\leq M(c_0) |u_{tt}|_{L^2} + c_6^{e^2},
\]
\[
|\sqrt{h^2 + e^2} u_t|_{L^2} \leq C(|\sqrt{h^2 + e^2} u_t|_{L^2} + |\nabla \sqrt{h^2 + e^2} u_t|_{L^2} + |\nabla \sqrt{h^2 + e^2} u_t|_{L^2})
\leq M(c_0) |u_{tt}|_{L^2} + c_6^{e^2}),
\]
\[
|(h^2 u_t)|_{L^2} \leq C(|h^2 u_t|_{L^2} + |h_t|_{L^6} |u|^2 u_{t} \leq M(c_0) |u_{tt}|_{L^2} + c_6^{e^2}),
\]
\[
|u|_{L^p} \leq C \left( (h^2 + e^2)^{\frac{1}{2}} (u_t + v \cdot \nabla v + \nabla \phi - \psi Q(v)) \right)_{L^p} + \left( (h^2 + e^2)^{-\frac{1}{2}} \int_0^\infty \int \mathfrak{A}_r \Omega d\Omega d\nu \right)_{L^p}
\leq M(c_0) |u_{tt}|_{L^2} + c_6^{e^2}).
\]

On the other hand, according to equation (3.8), for multi-index \( \zeta \in \mathbb{R}^n \), one has
\[
aL(\sqrt{h^2 + e^2} u_t) = -\sqrt{h^2 + e^2} \sqrt{v} \left( (h^2 + e^2)^{-\frac{1}{2}} (u_t + v \cdot \nabla v + \nabla \phi - \psi \cdot Q(v)) \right)
\]
\[
- \sqrt{h^2 + e^2} \sqrt{v} \left( (h^2 + e^2)^{-\frac{1}{2}} \int_0^\infty \int \mathfrak{A}_r \Omega d\Omega d\nu \right) - G(\sqrt{h^2 + e^2}, \nabla u),
\]
which, along with Lemma A.6 (Appendix A), yields that
\[
|\sqrt{h^2 + e^2} \nabla^2 u|_{L^p} \leq C \left( \sqrt{h^2 + e^2} \nabla^2 \left( (h^2 + e^2)^{-\frac{1}{2}} (u_t + v \cdot \nabla v + \nabla \phi - \psi \cdot Q(v)) \right) \right)_{L^p}
\]
\[
+ \left( \sqrt{h^2 + e^2} \nabla^2 \left( (h^2 + e^2)^{-\frac{1}{2}} \int_0^\infty \int \mathfrak{A}_r \Omega d\Omega d\nu \right) \right)_{L^p}
\]
\[
+ \left| \nabla^2 u \right|_{L^p} + \left| \nabla^2 u \right|_{L^p} + \left| \nabla^2 u \right|_{L^p} + \left| \nabla^2 u \right|_{L^p}
\leq M(c_0) \left( u_{tt} + c_6^{e^2}). \right)
\]

It thus follows from (3.51)–(3.53) that for \( 0 \leq t \leq T_5 \),
\[
\int_0^t \left( (h^2 u_t(s))_{L^p} + |u_t(s)|_{L^p} + |u(s)|_{L^p} + |h^2 u_t(s))_{L^p} + |(h^2 u_t(s))_{L^p}| \right) ds \leq M(c_0) c_6^{10}.
\]

The proof of Lemma 3.7 is complete.

Finally, we establish the time-weighted energy estimates for \( u \).

**Lemma 3.8.**
\[
t^t \left( u_t + u_{tt} + u_{tt} \right)(t) + \int_0^t \left( |u_t(s)|_{L^p} + |u_t(s)|_{L^p} \right) ds \leq M(c_0) c_6^{21},
\]
for \( 0 \leq t \leq T_6 = \min\{T^*, (1 + M(c_0) c_6)^{-2(12,25)} \}.\)

**Proof.** First, differentiating (3.38) with respect to time \( t \), one has
Second, multiplying (3.54) by $u_t$ and integrating over $\mathbb{R}^3$, one can obtain

$$
\frac{1}{2} \frac{d}{dt} |u_t|^2 + a(\sqrt{h^2 + \epsilon^2} \nabla u_t)^2 + a(a + \beta)(h^2 + \epsilon^2) \text{div} u_t^2
= \int \left( - (v \cdot \nabla) u_t - \nabla \phi_t - \frac{ah}{\sqrt{h^2 + \epsilon^2}} h_t \nabla \psi - \frac{ah}{\sqrt{h^2 + \epsilon^2}} h_t \psi \right) - \frac{2ah}{\sqrt{h^2 + \epsilon^2}} h_t \eta_t
- \frac{ae^2 h^2}{(h^2 + \epsilon^2)^2} u_t - \frac{1}{c} \int_0^\infty \int (\mathcal{A}_t) u_t \Omega d\Omega d\nu.
$$

(3.55)

Multiplying (3.55) by $s$ and integrating over $(r, \tau) \in (0, t)$, one arrives at

$$
t \left| u_t(t)^2 \right| + \int_t^\infty \int_{|\nabla u_t|} \left| \sqrt{h} \nabla u_t \right| d|s| \leq t \left| u_t(\tau)^2 \right| + M(c_0) c_0^{2+18.5}(1 + t).
$$

(3.56)

It follows from Lemmas 3.7 and A.5 (Appendix A) that there exists a sequence $\{s_k\}_{k=1}^\infty$ such that

$$
s_k \to 0 \quad \text{and} \quad s_k |u_t(s_k, x)| \to 0, \quad \text{as} \quad k \to \infty.
$$

Letting $\tau = s_k$ in (3.56) and using the Gronwall inequality, one has that for $0 \leq t \leq T_0$,

$$
t \left| u_t(t)^2 \right| + \int_0^t \int_{|\nabla u_t|} \left| \sqrt{h} \nabla u_t(s) \right| d|s| \leq M(c_0) c_0^{2+18.5},
$$

(3.57)

and

$$
\int_0^t \int_{|\nabla u_t|} \left| \sqrt{h} \nabla u_t(s) \right| d|s| \leq M(c_0) c_0^{2+19.5}.
$$

According to (3.52) and (3.57), one obtains

$$
t \left| \nabla^2 u_t(t) \right| \leq M(c_0) c_0^{6+10.25}, \quad t \left| \nabla^2 u(t) \right| \leq M(c_0) c_0^{6+11.25}.
$$

(3.58)

Finally, according to (3.38) and Lemma A.6, one obtains that for $0 \leq t \leq T_0$,

$$
\left| \sqrt{h^2 + \epsilon^2} u_t \right| \leq C \left( ||u_t||_{L^2} + ||v \cdot \nabla u_t||_{L^2} + ||\nabla \phi_t||_{L^2} + \frac{ah}{\sqrt{h^2 + \epsilon^2}} h_t \nabla \psi + \frac{ah}{\sqrt{h^2 + \epsilon^2}} h_t \psi \right)
+ \left| G(\sqrt{h^2 + \epsilon^2} u_t) \right|_{L^2} \leq C \left( ||u_t||_{L^2} + ||v \cdot \nabla u_t||_{L^2} + ||\nabla \phi_t||_{L^2} + \frac{ah}{\sqrt{h^2 + \epsilon^2}} h_t \nabla \psi + \frac{ah}{\sqrt{h^2 + \epsilon^2}} h_t \psi \right)
+ \left| G(\sqrt{h^2 + \epsilon^2} u_t) \right|_{L^2} \leq M(c_0) (||u_t||_{L^2} + ||v \cdot \nabla u_t||_{L^2} + c_6^{6+11.25}),
$$

(3.59)

which, along with (3.57)–(3.58), yield that...
The proof of Lemma 3.8 is complete. □

From Lemmas 3.2–3.8, for $0 \leq t \leq T^* = \min\{T^*, (1 + M(c_0)c_0^{-2(k+1.25)})\}$, one has

\[
\begin{align*}
&\left(\|\phi - \phi^{\infty}\|_{L^2}^2 + \int_0^t \|\phi_{tt}(s)\|_{L^2}^2\, ds \right) + \int_0^t \|\phi_{tt}(s)\|_{L^2}^2\, ds \leq Cc_0^6, \\
&\left(\int_0^t \|\phi(t,s)\|_{L^2}^2\, ds \right) \leq Cc_0^6,
\end{align*}
\]

\[
\begin{align*}
&\int_0^t \left(\|\phi_{tt}(s)\|_{L^2}^2 + \|\phi_{ttt}(s)\|_{L^2}^2\right)\, ds \leq Cc_0^6, \\
&\int_0^t \|\phi_{tt}(s)\|_{L^2}^2\, ds \leq Cc_0^6,
\end{align*}
\]

\[
\begin{align*}
&\int_0^t \left(\|\phi_{ttt}(s)\|_{L^2}^2 + \|\phi_{tttt}(s)\|_{L^2}^2\right)\, ds \leq Cc_0^6,
\end{align*}
\]

Then, defining the constants $c_i$ ($i = 1, \ldots, 7$)

\[
\begin{align*}
c_1 &= Cc_0, \\
c_2 &= Cc_1 = Cc_0 = M(c_0)c_0^{-2} = M(c_0)(Cc_0)^{-2}, \\
c_3 &= M(c_0)c_4^9 = M(c_0)(Cc_0)^{9(k+1.25)}, \\
c_4 &= M(c_0)c_5^2c_0^{-2.25} = M(c_0)^{2}(Cc_0)^{-3.25(k+1.25)}, \\
c_5 &= M(c_0)c_6^{1.75} = M(c_0)^{1.75}(Cc_0)^{1.75(k+1.25)}, \\
c_6 &= M(c_0)c_7^{2.25} = M(c_0)^{2.25}(Cc_0)^{3.25(k+1.25)},
\end{align*}
\]

and the time $T^*$

\[
T^* = \min\{T^*, (1 + M(c_0)c_0^{-2(k+1.25)})\},
\]

one can conclude that for $0 \leq t \leq T^*$,
\[
(\|\phi - \phi^\infty\|_2^2 + \|\phi_t\|_2^2 + \|\phi_{tt}\|_2^2)(t) + \int_0^t \|\phi_{tt}(s)\|_2^2 \, ds \leq c_2^2,
\]
\[
\|\psi_t\|_{L^2[0,T]} \leq c_2^2, \quad |h_t(t)|_{L^\infty} + |\psi_t(t)|_{L^2} \leq c_0^2,
\]
\[
|\psi_t(t)|_{L^2} + \int_0^t (|\psi_{tt}(s)|_2^2 + |u_t(s)|_2^2) \, ds \leq c_2^2,
\]
\[
\|I(t)\|_{L^2(\mathbb{R}^d \times S^1; H^0)} \leq c_2^2, \quad \|I_t(t)\|_{L^2(\mathbb{R}^d \times S^1; H^0)} \leq c_2^2,
\]
\[
h > \frac{1}{2\varepsilon_0}, \quad \|\phi_t(t)\|_{L^2(\mathbb{R}^d \times S^1; H^0)} + \|\phi_{tt}(t)\|_{L^2(\mathbb{R}^d \times S^1; H^0)} \leq c_2^2,
\]
\[
\varphi > \frac{2}{3} \eta^{-2\varepsilon}, \quad \|\phi_t(t)\|_{L^2(\mathbb{R}^d \times S^1; H^0)} + \|\phi_{tt}(t)\|_{L^2(\mathbb{R}^d \times S^1; H^0)} \leq c_2^2
\]
\[
(|\sqrt{n} u|_2^2 + ||u||_2^2)(t) + \int_0^t (||\nabla u(s)||_2^2 + |u_t(s)|_2^2) \, ds \leq c_4^2,
\]
\[
(|u|_{D^2}^2 + |h\nabla u|_2^2 + |u_t|_2^2 + h^2 |\nabla u|_{D^1}^2)(t) + \int_0^t |u_t(s)|_{D^2}^2 \, ds \leq c_4^2,
\]
\[
(|h_t|_{D^2}^2 + |u_t|_{D^2}^2 + |u|_{D^1}^2 + |h
abla u|_{D^1}^2 + |h\nabla u_t|_{D^1}^2)(t) + \int_0^t |u_t(s)|_{D^2}^2 \, ds \leq c_3^2,
\]
\[
\int_0^t (|u_t(s)|_2^2 + |u(s)|_{D^2}^2 + |h\nabla u(s)|_{D^1}^2 + |h\nabla u_t(s)|_{D^1}^2) \, ds \leq c_3^2,
\]
\[
t(|u|_{D^2}^2 + |u_t|_2^2 + |u|_{D^1}^2)(t) + \int_0^t s |u_t(s)|_{D^2}^2 + |u(s)|_{D^2}^2 \, ds \leq c_2^2.
\]

3.4 Passing to the limit $\varepsilon \to 0$

With the help of $(\varepsilon, \eta)$-independent estimates obtained in (3.62), we will establish the local-in-time existence result for the following linearized problem (3.63) without an artificial viscosity (i.e., $\varepsilon = 0$) under the assumption that $\phi^\infty_0 \geq \eta$.

\[
\begin{aligned}
\frac{1}{c} I^\eta_t + \Omega \cdot \nabla I^\eta = \Lambda^\eta, \\
\phi^\eta_t + v \cdot \nabla \phi^\eta + (\gamma - 1) \phi^\eta \text{div } v = 0, \\
u^\eta_t + v \cdot \nabla v + \nabla \psi^\eta + \eta^\eta L u^\eta = \psi^\eta \cdot Q(v) - \frac{1}{c} \int_0^\infty \int_{S^1} \Lambda^\eta \Omega \, d\Omega \, dv, \\
h^\eta_t + v \cdot \nabla h^\eta + (\delta - 1) g \text{div } v = 0, \\
(I^\eta, \phi^\eta, u^\eta, h^\eta)|_{\varepsilon = 0} = (I_0^\eta, \phi_0^\eta, u_0^\eta, h_0^\eta)
\end{aligned}
\]

\[
\begin{aligned}
\tilde{h}^\eta |_{x = 0} + \eta u_0 + (\phi_0 + \eta)^{\eta(\eta)} \quad \text{for } (x, v, \Omega) \in \mathbb{R}^2 \times \mathbb{R}^d \times S^1, \\
(I^\eta, \phi^\eta, u^\eta, h^\eta) \to (0, \eta, 0, \eta^{\eta(\eta)}) \quad \text{as } |x| \to \infty \quad \text{for } t \geq 0,
\end{aligned}
\]

where $\psi^\eta = \frac{\eta}{\delta - 1} \nabla h^\eta$. 

\[\square\]
Theorem 3.2. Let (2.8) hold. Assume the initial data \((I_0, \phi_0, u_0, h_0)\) satisfies (3.5) and (3.6), then there exist a time \(T^* > 0\) independent of \(\eta\) and a unique classical solution

\[\left( I^\eta, \phi^\eta, u^\eta, h^\eta, \psi^\eta = \frac{a\delta}{\delta - 1}\nabla h^\eta \right)\]

in \([0, T^*] \times \mathbb{R}^3 \times \mathbb{R}^+ \times S^2\) to (3.63) satisfying (3.10) with \(T^*\) replaced by \(T^*\). Moreover, \((I^\eta, \phi^\eta, u^\eta, h^\eta)\) satisfies the estimates in (3.62) independent of \(\eta\).

Proof. Step 1: Existence. First, according to Lemmas 3.1–3.8, one can see that for every fixed \(\epsilon > 0\) and \(\eta > 0\), there exist a time \(T^*\) independent of \((\epsilon, \eta)\) and a unique strong solution \((I^{\epsilon, \eta}, \phi^{\epsilon, \eta}, u^{\epsilon, \eta}, h^{\epsilon, \eta})\) in \([0, T^*] \times \mathbb{R}^3 \times \mathbb{R}^+ \times S^2\) to the linearized problem (3.8) satisfying the estimates in (3.62), which are independent of \((\epsilon, \eta)\).

Second, using the characteristic method and standard energy estimates for the equation (3.8), one can obtain that for \(0 \leq t \leq T^*\),

\[|h^{\epsilon, \eta}(t)|_{\infty} + |\nabla h^{\epsilon, \eta}(t)|_2 + |h^{\epsilon, \eta}(t)|_2 \leq C(\eta, \alpha, \beta, \gamma, \delta, T, \phi_0, u_0). \tag{3.64}\]

Then, based on the uniform estimates in (3.62) independent of \((\epsilon, \eta)\), estimates in (3.64) independent of \(\epsilon\), and Lemma A.3 (Appendix A), one concludes that for any \(R > 0\), there exists a subsequence of solutions (still denoted by \((I^{\epsilon, \eta}, \phi^{\epsilon, \eta}, u^{\epsilon, \eta}, h^{\epsilon, \eta})\)), which converges to a limit \((I^\eta, \phi^\eta, u^\eta, h^\eta)\) in the following strong sense:

\[I^{\epsilon, \eta} \rightarrow I^\eta \quad \text{in} \quad L^2(\mathbb{R}^+ \times S^2; C([0, T^*]; H^2(B_R))), \quad \text{as} \quad \epsilon \rightarrow 0,
\]

\[\phi^{\epsilon, \eta} \rightarrow \phi^\eta \quad \text{in} \quad L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T^*]; H^2)), \quad \text{(3.65)}\]

\[\phi_i^{\epsilon, \eta} \rightarrow \phi_i^\eta \quad \text{in} \quad L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T^*]; H^2)),
\]

where \(B_R\) is a ball centered at origin with radius \(R\).

On the other hand, the uniform estimates in (3.62) independent of \((\epsilon, \eta)\) and estimates in (3.64) independent of \(\epsilon\) imply that there exists a subsequence (of subsequence chosen earlier) of solutions (still denoted by \((I^{\epsilon, \eta}, \phi^{\epsilon, \eta}, u^{\epsilon, \eta}, h^{\epsilon, \eta})\)), which converges to a limit \((I^\eta, \phi^\eta, u^\eta, h^\eta)\) as \(\epsilon \rightarrow 0\) in the following weak or weak* sense:

\[I^{\epsilon, \eta} \rightharpoonup I^\eta \quad \text{in} \quad L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T^*]; H^2)), \]

\[I^{\epsilon, \eta} \rightharpoonup I^\eta \quad \text{in} \quad L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T^*]; H^2)), \]

\[(\phi^{\epsilon, \eta} - \phi^\eta, u^{\epsilon, \eta}) \rightharpoonup (\phi^\eta - \phi^\eta, u^\eta) \quad \text{in} \quad L^\infty([0, T^*]; H^2), \]

\[(\phi_i^{\epsilon, \eta}, \psi^{\epsilon, \eta}, h^{\epsilon, \eta}) \rightharpoonup (\phi_i^\eta, \psi^\eta, h^\eta) \quad \text{in} \quad L^\infty([0, T^*]; H^2), \]

\[u_i^{\epsilon, \eta} \rightharpoonup u_i^\eta \quad \text{in} \quad L^\infty([0, T^*]; H^2), \]

\[(\phi_i^{\epsilon, \eta}, \nabla \phi_i^{\epsilon, \eta}, \nabla^2 \phi_i^{\epsilon, \eta}) \rightharpoonup (\phi_i^\eta, \nabla \phi_i^\eta, \nabla^2 \phi_i^\eta) \quad \text{in} \quad L^\infty([0, T^*]; L^2), \]

\[(\nabla^2 \phi_i^{\epsilon, \eta}, \nabla^2 \phi_i^{\epsilon, \eta}) \rightharpoonup (\nabla^2 \phi_i^\eta, \nabla^2 \phi_i^\eta) \quad \text{in} \quad L^\infty([0, T^*]; L^2), \]

\[\frac{t}{\nabla^2 u_i^{\epsilon, \eta}, u_i^{\epsilon, \eta}, \nabla^2 u_i^{\epsilon, \eta}} \rightharpoonup \frac{t}{\nabla^2 u_i^\eta, u_i^\eta, \nabla^2 u_i^\eta} \quad \text{in} \quad L^\infty([0, T^*]; L^2), \]

\[(\nabla \phi_i^{\epsilon, \eta}, \phi_i^{\epsilon, \eta}, \phi_i^{\epsilon, \eta}, f^{\epsilon, \eta}) \rightharpoonup (\nabla \phi_i^\eta, \phi_i^\eta, \phi_i^\eta, f^\eta) \quad \text{in} \quad L^\infty([0, T^*]; L^2), \]

\[(h^{\epsilon, \eta}, h_i^{\epsilon, \eta}, f^{\epsilon, \eta}, f_i^{\epsilon, \eta}) \rightharpoonup (h^\eta, h_i^\eta, f^\eta, f_i^\eta) \quad \text{in} \quad L^\infty([0, T^*]; L^2), \]

\[(\nabla \phi_i^{\epsilon, \eta}, \nabla \phi_i^{\epsilon, \eta}) \rightharpoonup (\nabla \phi_i^\eta, \nabla \phi_i^\eta) \quad \text{in} \quad L^\infty([0, T^*]; L^2), \]

\[\nabla u_i^{\epsilon, \eta} \rightharpoonup \nabla u_i^\eta \quad \text{in} \quad L^\infty([0, T^*]; H^2), \]

\[u_i^{\epsilon, \eta} \rightarrow u_i^\eta \quad \text{in} \quad L^2([0, T^*]; H^2), \]

\[\phi_i^{\epsilon, \eta} \rightarrow \phi_i^\eta \quad \text{in} \quad L^2([0, T^*]; H^2), \]

\[(\psi_i^{\epsilon, \eta}, h_i^{\epsilon, \eta}, u_i^{\epsilon, \eta}) \rightharpoonup (\psi_i^\eta, h_i^\eta, u_i^\eta) \quad \text{in} \quad L^2([0, T^*]; L^2), \]

\[t^{\epsilon} \nabla u_i^{\epsilon, \eta}, \nabla^3 u_i^{\epsilon, \eta} \rightharpoonup t^{\epsilon} \nabla u_i^\eta, \nabla^3 u_i^\eta \quad \text{in} \quad L^2([0, T^*]; L^2), \]
which, along with the lower semi-continuity of weak or weak∗ convergence, imply that \((I^n, \phi^n, u^n, h^n)\) satisfies the corresponding estimates in (3.62) and (3.64) except those weighted estimates for \(u^n\).

It follows from the strong convergence in (3.65) and the weak or weak∗ convergence in (3.66) that

\[
\sqrt{h^n}(\nabla u^n, \nabla u^n) \xrightarrow{\ast} \sqrt{h^n}(\nabla u^3, \nabla u^3) \quad \text{in} \quad L^\infty([0, T^*]; L^2),
\]

\[
h^{\alpha, \alpha}u^n \xrightarrow{\ast} h^{\alpha, \alpha}u^3 \quad \text{in} \quad L^\infty([0, T^*]; H^1),
\]

\[
(h^{\alpha, \alpha}u^n, h^{\alpha, \alpha}u^n) \xrightarrow{\ast} (h^{\alpha, \alpha}u^3, h^{\alpha, \alpha}u^3) \quad \text{in} \quad L^2([0, T^*]; L^2),
\]

\[
h^{\alpha, \alpha}u^n \xrightarrow{\ast} h^{\alpha, \alpha}u^3 \quad \text{in} \quad L^2([0, T^*]; D_1^1 \cap D^2).
\]

Moreover, combining (3.65)–(3.67), one can easily show that \((I^n, \phi^n, u^n, h^n)\) is a weak solution in the sense of distributions to (3.63), satisfying

\[
\begin{align*}
I^n &\in L^2(\mathbb{R}^+ \times S^2; C([0, T^*]; H^0)), \quad I^n \in L^2(\mathbb{R}^+ \times S^2; C([0, T^*]; H^2)), \\
\phi^n - \eta &\in L^\infty([0, T^*]; H^0), \quad h^n \in L^\infty([0, T^*] \times \mathbb{R}^3), \quad \nabla h^n \in L^\infty([0, T^*]; H^0), \\
h^n &\in L^\infty([0, T^*]; H^0), \quad u^n \in L^\infty([0, T^*]; H^0) \cap L^2([0, T^*]; H^2), \\
u^n &\in L^\infty([0, T^*]; H^2) \cap L^2([0, T^*]; D_1^1), \quad u^n \in L^2([0, T^*]; D_1^1), \\
t^n &\in L^\infty([0, T^*]; D^4), \quad t^n u^n \in L^\infty([0, T^*]; D^4) \cap L^2([0, T^*]; D^3), \\
t^n u^n &\in L^\infty([0, T^*]; D^2) \cap L^2([0, T^*]; D^3),
\end{align*}
\]

which imply that the weak solution \((I^n, \phi^n, u^n, h^n)\) of (3.63) is actually a strong one.

**Step 2:** Uniqueness and time continuity. Since \(h^n > \frac{1}{2\alpha}\), the uniqueness and time continuity of the strong solutions can be obtained by using same arguments as in Lemma 3.1. Finally, Theorem 3.2 is proved.

### 3.5 Local solvability of nonlinear problem away from vacuum

In this section, under the assumption that \(\phi^0 \geq \eta\), we will prove the local well-posedness of classical solutions to the following nonlinear problem:

\[
\begin{align*}
\frac{1}{c}I^n + \Omega \cdot \nabla I^n &= A^n, \\
\frac{1}{c} \frac{\partial I^n}{t} + u^n \cdot \nabla \phi^n + (y - 1)\phi^n \text{div}u^n &= 0, \\
u^n + u^n \cdot \nabla u^n + \nabla \phi^n + ah^n u^n &= \psi^n \cdot Q(u^n) - \frac{1}{c} \int_0^\infty \int_{S^2} A^n \Omega d\Omega d\nu, \\
h^n + u^n \cdot \nabla h^n + (\delta - 1)(\phi^n)^{2\delta} \text{div}u^n &= 0, \\
(I^n, \phi^n, u^n, h^n)|_{t=0} &= (I^0, \phi^0, u^0, h^0) = (I^0, \phi^0, u^0, h^0) \\
(I^n, \phi^n, u^n, h^n) &\to (0, \eta, 0, \eta^{2\delta}) \quad \text{as} \ |x| \to +\infty \quad \text{for} \ t \geq 0,
\end{align*}
\]

where \(\psi^n = \frac{a\delta}{\delta - 1} \nabla h^n\).

**Theorem 3.3.** Let (2.8) hold and \(\eta > 0\). Assume the initial data \((I_0, \phi_0, u_0, h_0)\) satisfies (3.5) and (3.6), then there exist a time \(T_* > 0\) independent of \(\eta\) and a unique classical solution:

\[
(I^n, \phi^n, u^n, h^n = (\phi^n)^{2\delta}, \psi^n = \frac{a\delta}{\delta - 1} \nabla h^n)
\]

in \([0, T_*] \times \mathbb{R}^3 \times \mathbb{R}^+ \times S^2\) to (3.69) with \(T_*\) is independent of \(\eta\). Moreover, \((I^n, \phi^n, u^n, h^n)\) satisfies the estimates in (3.62) with \(T^*\) replaced by \(T_*\).
The proof of Theorem 3.3 is based on the classical iteration scheme and the conclusions obtained in Sections 3.2–3.4. Next, let \((I^0, \phi^0, u^0, h^0)\) be the solution to the following problem:

\[
\begin{aligned}
&\frac{1}{c} W_t + \Omega \cdot \nabla W = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^+ \times S^2, \\
&X_t + u_0 \cdot \nabla X = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
&Y_t - \Delta Y = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
&Z_t + u_0 \cdot \nabla Z = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
&(W, X, Y, Z)_{|t=0} = (I_0^0, \phi_0^0, u_0^0, h_0^0) \\
&= (I_0, \phi_0 + \eta, u_0, (\phi_0 + \eta)^{2\nu}) \quad \text{for } (x, \nu, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2, \\
&(W, X, Y, Z) \to (0, \eta, 0, \eta^{2\nu}) \quad \text{as } |x| \to \infty \quad \text{for } t \geq 0.
\end{aligned}
\]  

(3.70)

Choosing a time \(\hat{T} \in (0, T^*)\) small enough such that

\[
\sup_{0 \leq t \leq \hat{T}} \|I^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 \leq c_2^2, \quad \sup_{0 \leq t \leq \hat{T}} \|I^0(t)\|_{L^2(\mathbb{R}^3 \times S^2; \mu')} \leq c_2^2,
\]

\[
\sup_{0 \leq t \leq \hat{T}} \|\nabla I^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 \leq c_2^2, \quad \sup_{0 \leq t \leq \hat{T}} \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \int_0^t \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) dt \leq c_2^2,
\]

\[
\sup_{0 \leq t \leq \hat{T}} \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) + \int_0^t \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) dt \leq c_2^2,
\]

\[
\sup_{0 \leq t \leq \hat{T}} \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) + \int_0^t \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) dt \leq c_2^2,
\]

\[
\sup_{0 \leq t \leq \hat{T}} \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) + \int_0^t \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) dt \leq c_2^2,
\]

\[
\text{ess sup } \sup_{0 \leq t \leq \hat{T}} \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) + \int_0^t \left(\|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2 + \|u^0(t)\|_{L^2(\mathbb{R}^3 \times S^2)}^2\right) dt \leq c_2^2.
\]

(3.71)

\textbf{Proof.} \textbf{Step 1:} Existence. Let the beginning step of our iteration be \((\chi, \nu, g) = (I^0, u^0, h^0)\), then one can obtain a classical solution \((I^1, \phi^1, u^1, h^1)\) to (3.63). Inductively, one constructs the approximate sequences of solutions \((I^{k+1}, \phi^{k+1}, u^{k+1}, h^{k+1})\) as follows: given \((I^k, \phi^k, u^k, h^k)\) for \(k \geq 1\), define \((I^{k+1}, \phi^{k+1}, u^{k+1}, h^{k+1})\) by solving the following problem:

\[
\begin{aligned}
&\frac{1}{c} I^{k+1} + \Omega \cdot \nabla I^{k+1} = A^k, \\
&\phi^{k+1} + u^k \cdot \nabla \phi^{k+1} + (y - 1)\phi^{k+1} \text{div} u^k = 0, \\
u^{k+1} + u^k \cdot \nabla u^k + \Phi^{k+1} + \Phi^{k+1} u^k + \Delta u^{k+1} \\
= \psi^{k+1}, \quad Q(u^k) - \frac{1}{c} \int_0^\infty \mathcal{A}^k(t) \Omega d\Omega dv, \\
h^{k+1} + u^k \cdot \nabla h^{k+1} + (\delta - 1)h^{k+1} \text{div} u^k = 0, \\
(I^{k+1}, \phi^{k+1}, u^{k+1}, h^{k+1})_{|t=0} = (I_0^k, \phi_0^k, u_0^k, h_0^k) \\
= (I_0, \phi_0 + \eta, u_0, (\phi_0 + \eta)^{2\nu}) \quad \text{for } (x, \nu, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2, \\
(I^{k+1}, \phi^{k+1}, u^{k+1}, h^{k+1}) \to (0, \eta, 0, \eta^{2\nu}) \quad \text{as } |x| \to \infty \quad \text{for } t \geq 0.
\end{aligned}
\]  

(3.72)
where

\[
A^k_t = \sigma^k - \sigma^k \nu I_k + \int_0^\infty \left( \frac{\nu}{\nu'} \sigma^k I_k - (\sigma^k) \nu I_k \right) d\Omega' d\nu',
\]

\[
\bar{A}^k_t = \sigma^k - \sigma^k \nu I_k + \int_0^\infty \left( \frac{\nu}{\nu'} \bar{\sigma}^k I_k - \bar{\sigma}^k I_k \right) d\Omega' d\nu',
\]

\[
\sigma^k = (\phi^k) \sigma^k, \quad \sigma^k = \sigma(I, x, \nu, (\phi^k)\nu),
\]

\[
\sigma^k = (\phi^k) \sigma^k, \quad \sigma^k = \sigma(I, x, \nu, (\phi^k)\nu),
\]

\[
\sigma^k = (\phi^k) \sigma^k, \quad \sigma^k = \sigma(I, x, \nu, (\phi^k)\nu),
\]

\[
\sigma^k = (\phi^k) \sigma^k, \quad \sigma^k = \sigma(I, x, \nu, (\phi^k)\nu),
\]

\[
\sigma^k = (\phi^k) \sigma^k, \quad \sigma^k = \sigma(I, x, \nu, (\phi^k)\nu),
\]

\[
\sigma^k = (\phi^k) \sigma^k, \quad \sigma^k = \sigma(I, x, \nu, (\phi^k)\nu),
\]

\[
\sigma^k = (\phi^k) \sigma^k, \quad \sigma^k = \sigma(I, x, \nu, (\phi^k)\nu),
\]

By replacing \((\chi, v, g) = (I^k, u^k, h^k)\), and \((\phi^k, u^k, h^k) (k = 1, \ldots)\) satisfying the uniform estimates (3.62), one can solve problem (3.72).

Let

\[
\psi^{k+1} = \frac{a\delta}{\delta - 1} \nabla h^{k+1}, \quad \phi^{k+1} = (h^{k+1})^{-1},
\]

\[
f^{k+1} = \psi^{k+1} \phi^{k+1} - \frac{a\delta}{\delta - 1} \nabla \log h^{k+1},
\]

then direct calculation shows that problem (3.72) can be rewritten as follows:

\[
\begin{cases}
\frac{1}{c} f^{k+1} + \Omega \cdot \nu f^{k+1} = A^k_t, \\
\phi^{k+1} + u^k \cdot \nabla \phi^{k+1} + (y - 1) \phi^{k+1} \nu I_k = 0, \\
\phi^{k+1} (u^k + u^k) \cdot \nabla \phi^{k+1} + \nabla \phi^{k+1} + aI f^{k+1} \\
= f^{k+1} \cdot Q(u^k) - \frac{1}{c} \phi^{k+1} \int_0^\infty \int_{S^2} A^k_t d\Omega d\nu, \\
\end{cases}
\]

\[
\begin{cases}
f^{k+1} + \sum_{l=1}^3 \sigma^k(u^k) \Xi f^{k+1} + B^r(u^k) f^{k+1} + a\delta \phi^{k+1} \nu I_k \nabla \phi^{k+1} \\
= - a\delta \phi^{k+1} \nabla h^k \nu I_k + (\delta - 1) (\phi^k) (\phi^{k+1})^2 \nabla u^k = 0, \\
\phi^{k+1} + u^k \cdot \nabla \phi^{k+1} - (\delta - 1) (\phi^k) (\phi^{k+1})^2 \nabla u^k = 0, \\
(I^{k+1}, \phi^{k+1}, u^{k+1}, f^{k+1}, \phi^{k+1})|_{t=0} = (I_0^g, \phi_0^g, u_0^g, f_0^g, \phi_0^g) \\
= \left(I_0, \phi_0 + \eta, u_0, \frac{2e\alpha \delta}{\delta - 1} \phi_0 + \eta, (\phi_0 + \eta)^{2\nu}\right) (x, v, \nu, \Omega) \in R^3 \times R^+ \times S^2, \\
(I^{k+1}, \phi^{k+1}, u^{k+1}, f^{k+1}, \phi^{k+1}) \rightarrow (0, \eta, 0, 0, \eta^{-2\nu}) \quad \text{as} \ |x| \rightarrow +\infty \quad t \geq 0.
\end{cases}
\]

Next we are going to show the sequence of approximate solutions \((I^k, \phi^k, u^k, f^k, \phi^k)\) converges strongly to a limit \((I^\eta, \phi^\eta, u^\eta, f^\eta, \phi^\eta)\). Let

\[
I^{k+1} = I^{k+1} - I^k, \quad \phi^{k+1} = \phi^{k+1} - \phi^k, \quad u^{k+1} = u^{k+1} - u^k, \quad f^{k+1} = f^{k+1} - f^k, \quad \phi^{k+1} = \phi^{k+1} - \phi^k.
\]

Then, it follows from (3.74) that
Now we are ready to give several estimates, which will be used later. To facilitate the discussion, we will adopt the following notations in the rest of this subsection

\[ R^k(t) = (|\nabla u_k|_{L^\infty} + |\varphi^{k-1} h_k|_{L^\infty} + |\nabla u^{k-1}|_{L^\infty}), \quad F^k(t) = |u_k^t|_3 + |u^{k-1}|_{L^\infty}\nabla u^{k-1}|_3, \]
\[ G^k(t) = (|f^k|_{L^\infty} + |f^{k-1}|_{L^\infty} + |h^{k-1}|_{L^\infty} + |\nabla u^{k-1}|_{L^\infty}). \]

First, from Remark 3.2 at the end of this subsection, one has

**Lemma 3.9.**

\[ \varphi^{k+1} \in L^\infty([0, T]; H^0), \quad \tilde{f}^{k+1} \in L^\infty([0, T]; H^0) \quad \text{for } k = 1, 2, \ldots. \]

Then multiplying (3.75) by \( 2\varphi^{k+1} \) and integrating over \( \mathbb{R}^3 \), one arrives at

\[ \frac{d}{dt} |\varphi^{k+1}|^2 \leq C |\varphi^{k+1}|_2 (R^k(t)) |\varphi^{k+1}|_2 + |\sigma^k|_3 |\nabla \varphi^k|_6 + R^{k-1}(t) |\varphi^k|_2 + |\nabla \sigma^k|_2 |\varphi^{k-1}|_{L^\infty} + |\varphi^k|_2 |\nabla u^k|_{L^\infty}. \]

Applying \( \delta^k(|\varphi^k|) = 1 \) to (3.75), multiplying the resulting equation by \( 2\delta^k(\varphi^{k+1}) \), and integrating over \( \mathbb{R}^3 \), one has
which, along with (3.76), yields that
\[
\frac{d}{dt}\|\tilde{\varphi}^{k+1}\|_1^2 \leq C\|\varphi^{k+1}\|_1^2 + C\sigma^{-1}\|\varphi^{k+1}\|_1^2,
\]  
(3.77)

where \(\sigma \in (0, 1)\) is a constant to be determined later.

For \(\tilde{f}^{k+1}\), multiplying (3.75) by \(2\tilde{f}^{k+1}\) and integrating over \(\mathbb{R}^3\), one can obtain
\[
\frac{d}{dt}\|\tilde{f}^{k+1}\|_1^2 \leq C\|\varphi^{k+1}\|_1^2 + C\sigma^{-1}\|\varphi^{k+1}\|_1^2 + \sigma(\|\nabla \tilde{f}^1\|_1 + \|\tilde{f}^{k+1}\|_1),
\]  
(3.78)

where one has used
\[
|\Gamma_{\tilde{f}}^1| \leq C(\|\nabla \tilde{f}^1\|_1 + \|\nabla \varphi\|_1\|f^{k+1}\|_1 + \|f^{k+1}\|_1\|\nabla \varphi\|_1 + \|\nabla \tilde{f}^{k+1}\|_1\|\nabla \varphi\|_1 + \|\tilde{f}^{k+1}\|_1\|\nabla \varphi\|_1),
\]
\[
|\Gamma_{\tilde{f}}^2| \leq C(\|\nabla \nabla \tilde{f}^1\|_1 + \|f^{k+1}\|_1\|\nabla \varphi\|_1 + \|\nabla \tilde{f}^{k+1}\|_1\|\nabla \varphi\|_1 + \|\tilde{f}^{k+1}\|_1\|\nabla \varphi\|_1),
\]
\[
|\Gamma_{\tilde{f}}^3| \leq C(\|\tilde{f}^{k+1}\|_1\|\nabla \varphi\|_1 + \|f^{k+1}\|_1\|\nabla \varphi\|_1 + \|\nabla \tilde{f}^{k+1}\|_1\|\nabla \varphi\|_1 + \|\tilde{f}^{k+1}\|_1\|\nabla \varphi\|_1).
\]

For \(\tilde{g}^{k+1}\), multiplying (3.75) by \(2\tilde{g}^{k+1}\) and integrating over \(\mathbb{R}^3\), one has
\[
\frac{d}{dt}\|\tilde{g}^{k+1}\|_1^2 \leq C\|\varphi^{k+1}\|_1^2 + C\sigma^{-1}\|\varphi^{k+1}\|_1^2 + \sigma(\|\nabla \tilde{g}^1\|_1 + \|\tilde{g}^{k+1}\|_1).
\]  
(3.79)

Applying \(\partial_2(\langle \cdot \rangle) = 1\) to (3.75) and integrating the resulting equation over \(\mathbb{R}^3\), one obtains
\[
\frac{d}{dt}\|\tilde{\varphi}^{k+1}\|_1^2 \leq C\|\nabla \tilde{\varphi}^{k+1}\|_1^2 + \|\nabla \varphi\|_1^2 + \|\nabla \tilde{\varphi}^{k+1}\|_1^2 + \|\varphi\|_1^2.
\]  
(3.80)

For \(\tilde{T}^{k+1}\), multiplying (3.75) by \(\tilde{T}^{k+1}\) and integrating over \(\mathbb{R}^+ \times S^2 \times \mathbb{R}^3\), one has
\[
\frac{d}{dt}\|\tilde{T}^{k+1}\|_{L^2(\mathbb{R}^+, S^2; \mathbb{R}^3)}^2 \leq C\|\nabla \varphi^{k+1}\|_1^2 + \sigma(\|\tilde{T}^{k+1}\|_{L^2(\mathbb{R}^+, S^2; \mathbb{R}^3)}^2) + C\sigma^{-1}\|\tilde{T}^{k+1}\|_{L^2(\mathbb{R}^+, S^2; \mathbb{R}^3)}^2
\]  
(3.81)

where one has used (2.6).

For \(\tilde{u}^{k+1}\), multiplying (3.75) by \(2\tilde{u}^{k+1}\) and integrating over \(\mathbb{R}^3\), one has
\[
\frac{d}{dt} \sqrt{\varphi^{k+1} \vartheta^{k+1} |_2^2} + 2a\alpha \nabla \vartheta^{k+1} |_2^2 + 2(a + \beta) | \text{div} \vartheta^{k+1} |_2^2 \\
= \int \left( \varphi^{k+1} \vartheta^{k+1} - 2\varphi^{k+1}(u^k \cdot \nabla \vartheta^k + \vartheta^k \cdot \nabla u^k) - 2(\varphi^{k+1}(u^k + u^{k-1} \cdot \nabla u^{k-1}) + \varphi^{k+1} \nabla \varphi^{k+1} + \varphi^{k+1} \nabla \varphi^{k-1}) + 2(f^{k+1} \cdot Q(\vartheta^k) + T^{k+1} \cdot Q(u^{k-1})) \right) d\Omega \\
+ 2(f^{k+1} \cdot Q(\vartheta^k) + T^{k+1} \cdot Q(u^{k-1})) - \frac{2}{c} \varphi^{k+1} \int_0^\infty \int_{\mathbb{S}^2} \nabla \vartheta^{k+1} \Omega d\Omega dv + \frac{2}{c} \varphi^k L_2 \cdot \nabla \vartheta^{k+1} dx \leq \sum_{j=1}^5 J_i.
\]  

(3.82)

Using Hölder’s inequality, (3.73), and equation (3.74), we estimate terms \(J_i (i = 1, \ldots, 5)\) as follows:

\(J_1 = \int \left( \varphi^{k+1} \vartheta^{k+1} - 2\varphi^{k+1}(u^k \cdot \nabla \vartheta^k + \vartheta^k \cdot \nabla u^k) \right) \cdot \vartheta^{k+1} dx \leq C \sqrt{\varphi^{k+1} \vartheta^{k+1} |_2 \left( |\nabla \vartheta^{k+1} |_2 |u^k|_{\infty} |\vartheta^k|_2 + 1 + |\sqrt{\varphi^{k+1} \vartheta^{k+1}} |_2 (|\nabla \vartheta^k|_{\infty} + |\vartheta^{k-1}|_{\infty} h^2 |\nabla u^k|_{\infty}) \right) + |\vartheta^{k+1}|_{\infty} h^2 |\nabla u^k|_{\infty}} \right).\)

\(J_2 = -2 \int (\overline{\varphi}^{k+1}(u^k + u^{k-1} \cdot \nabla u^{k-1}) + \varphi^{k+1} \nabla \overline{\varphi}^{k+1} + \overline{\varphi}^{k+1} \nabla \varphi^{k+1}) \cdot \vartheta^{k+1} dx \leq C \left( |\sqrt{\varphi^{k+1} \vartheta^{k+1}} |_2 |\nabla \vartheta^{k+1} |_2 f^k(t) + |\nabla \varphi^{k+1}|_{\infty} + |\varphi^{k+1}|_{\infty} \right) \cdot \vartheta^{k+1} dx \right).\)

\(J_3 = 2 \int (f^{k+1} \cdot Q(\vartheta^k) + T^{k+1} \cdot Q(u^{k-1})) \cdot \vartheta^{k+1} dx \leq C \left( |\sqrt{\varphi^{k+1} \vartheta^{k+1}} |_2 |\varphi^{k+1}|_{\infty} \frac{1}{h^2} |\varphi^{k+1}|_{\infty} |\nabla \vartheta^k |_2 + |\overline{T}^{k+1} |_2 |\nabla \vartheta^{k+1} |_2 \right).\)

\(J_4 = -\frac{2}{c} \int_0^\infty \int_{\mathbb{S}^2} \varphi^{k+1} \overline{A}^{k+1} \Omega \cdot \vartheta^{k+1} d\Omega dv dx \leq C |\sqrt{\varphi^{k+1} \vartheta^{k+1}} |_2 |\overline{A}^{k+1} |_2 \left( |\vartheta^{k+1}|_{\infty} \right),\)

\(J_5 = \frac{2}{c} \int \varphi^k L_2 \cdot \vartheta^{k+1} dx = \frac{2}{c} \int \left( \varphi^{k+1} - \varphi^k \right) L_2 \cdot \vartheta^{k+1} dx \leq C \left( |\sqrt{\varphi^{k+1} \vartheta^{k+1}} |_2 |\vartheta^{k+1}|_{\infty} \frac{1}{h^2} \right).\)

Substituting the estimates for \(J_i (i = 1, \ldots, 5)\) into (3.82) and using Young’s inequality, one obtains

\[
\frac{d}{dt} \sqrt{\varphi^{k+1} \vartheta^{k+1} |_2^2} + 2a\alpha \nabla \vartheta^{k+1} |_2^2 + 2(a + \beta) | \text{div} \vartheta^{k+1} |_2^2 \leq C \left( |\nabla \vartheta^k|_{\infty} + |\vartheta^{k+1}|_{\infty} \frac{1}{h^2} \right) + C |\vartheta^{k+1}|_{\infty} \frac{1}{h^2} \right),
\]  

(3.83)

where one has used

\[
\| \overline{A}^{k+1} \|_{L^p(\mathbb{R}^2 ; \mathbb{L}^2)} \leq C \left( 1 + \|\sigma^{k+1} \|_{L^p(\mathbb{R}^2 ; \mathbb{L}^2)} \right),
\]

\[
\| \nabla \vartheta^{k+1} \|_{L^2(\mathbb{R}^2 ; \mathbb{L}^2)} \leq C \left( 1 + \|\sigma^{k+1} \|_{L^p(\mathbb{R}^2 ; \mathbb{L}^2)} \right),
\]

\[
\| \vartheta^{k+1} \|_{L^2(\mathbb{R}^2 ; \mathbb{L}^2)} \leq C \left( 1 + \|\sigma^{k+1} \|_{L^p(\mathbb{R}^2 ; \mathbb{L}^2)} \right),
\]  

(3.84)

Next multiplying (3.75) by \(2\vartheta^{k+1} \text{div} \vartheta^{k+1} \) and integrating over \(\mathbb{R}^3\), one has

\[
\frac{d}{dt} \left( |\nabla \vartheta^{k+1} |_2^2 + (a + \beta) | \text{div} \vartheta^{k+1} |_2^2 \right) + 2| \varphi^{k+1} \vartheta^{k+1} |_2^2 \\
= 2 \int (-\varphi^{k+1}(u^k \cdot \nabla \vartheta^k + \vartheta^k \cdot \nabla u^k) - \varphi^{k+1}(u^k + u^{k-1} \cdot \nabla u^{k-1}) - \varphi^{k+1} \nabla \varphi^{k+1} + \varphi^{k+1} \nabla \varphi^{k-1} + f^{k+1} \cdot Q(\vartheta^k) + T^{k+1} \cdot Q(u^{k-1})) \\
- \frac{1}{c} \varphi^{k+1} \int_0^\infty \int_{\mathbb{S}^2} \nabla \vartheta^{k+1} \Omega d\Omega dv + \frac{1}{c} \varphi^k L_2 \cdot \nabla \vartheta^{k+1} dx \leq \sum_{j=6}^{11} J_i.
\]  

(3.85)
Then the terms $J_i$ ($i = 6, \ldots, 9$) on the right-hand side of (3.85) can be estimated as follows:

$$J_6 = -2 \int \phi^{k+1}(u^k \cdot \nabla u^k + \tau^k \cdot \nabla u^{k-1}) \cdot \tau^{k+1} \, dx$$

$$\leq Q \phi^{k+1} \left( \frac{1}{|\Omega|} \int \phi^{k+1} \tau^{k+1} \left| \int (u^k \cdot \nabla u^k + |\tau^k| \nabla u^{k-1}) \right| \right) \cdot \tau^{k+1} \, dx$$

$$J_7 = -2 \int \phi^{k+1}(u^k + u^{k-1} \cdot \nabla u^{k-1}) \cdot \tau^{k+1} \, dx$$

$$= -2 \frac{d}{dt} \int \phi^{k+1}(u^k + u^{k-1} \cdot \nabla u^{k-1}) \cdot \tau^{k+1} \, dx + 2 \int (\phi^{k+1}(u^k + u^{k-1} \cdot \nabla u^{k-1}))_t \cdot \tau^{k+1} \, dx$$

$$\leq -2 \frac{d}{dt} \int \phi^{k+1}(u^k + u^{k-1} \cdot \nabla u^{k-1}) \cdot \tau^{k+1} + Q \phi^{k+1} \left( \int (u^k \cdot \nabla u^k + |\tau^k| \nabla u^{k-1}) + |\tau^k| \nabla u^k \left| \left| \nabla \phi^{k+1} \right| \right|_{L^2} + |\tau^k| \nabla u^{k-1} \left| \left| \nabla \phi^{k+1} \right| \right|_{L^2} \right)$$

$$+ |\tau^k| \nabla u^k \left| \left| \nabla \phi^{k+1} \right| \right|_{L^2} + |\tau^k| \nabla u^{k-1} \left| \left| \nabla \phi^{k+1} \right| \right|_{L^2} \left( |\nabla u^k|_{L^2} + |\nabla u^{k-1}|_{L^2} \right) \left( |\nabla \phi^{k+1}|_{L^2} + |\nabla \phi^{k+1}|_{L^2} \right)$$

$$J_8 = -2 \int (\nabla \phi^{k+1} \cdot \tau^{k+1}) \cdot \nabla \phi^{k+1} \cdot \tau^{k+1} \, dx$$

$$= -2 \frac{d}{dt} \int \phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1} \, dx - 2 \int \left( (\nabla \phi^{k+1} \cdot \tau^{k+1}) (\nabla \phi^{k+1} \cdot \tau^{k+1}) \right) \, dx$$

$$\leq -2 \frac{d}{dt} \int \phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1} + Q \phi^{k+1} \left( \frac{1}{|\Omega|} \int \phi^{k+1} \tau^{k+1} \left| \nabla \phi^{k+1} \right| \right) \cdot \tau^{k+1} \, dx$$

$$+ \frac{d}{dt} \int \phi^{k+1} \left( (\nabla \phi^{k+1} \cdot \tau^{k+1}) \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, dx + \frac{d}{dt} \int \phi^{k+1} \left( (\nabla \phi^{k+1} \cdot \tau^{k+1}) \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, dx$$

$$J_9 = 2 \int (f^{k+1}, Q(\tau^k) + \tau^{k+1} \cdot Q(u^k-1)), \tau^{k+1} \, dx$$

$$= 2 \frac{d}{dt} \int \phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1} \, dx + 2 \int \left( (\nabla \phi^{k+1} \cdot \tau^{k+1} \cdot \tau^{k+1} - (\nabla \phi^{k+1} \cdot \tau^{k+1} \cdot \tau^{k+1}) \right) \, dx$$

$$\leq 2 \frac{d}{dt} \int \phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1} + Q \phi^{k+1} \left( \frac{1}{|\Omega|} \int \phi^{k+1} \tau^{k+1} \left| \nabla \phi^{k+1} \right| \cdot \tau^{k+1} \, dx$$

$$+ \frac{d}{dt} \int \phi^{k+1} \left( \nabla \phi^{k+1} \cdot \tau^{k+1} \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, dx + \frac{d}{dt} \int \phi^{k+1} \left( \nabla \phi^{k+1} \cdot \tau^{k+1} \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, dx$$

$$+ \frac{d}{dt} \int \phi^{k+1} \left( (\nabla \phi^{k+1} \cdot \tau^{k+1}) \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, dx$$

$$J_{10} = \frac{2}{|\Omega|} \int \int_{0}^{1} \left( \phi^{k+1} \cdot \left\{ \phi^{k+1} \cdot \nabla \phi^{k+1} \right\} \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, d\Omega \, dx$$

$$= \frac{2}{|\Omega|} \frac{d}{dt} \int \int_{0}^{1} \left( \phi^{k+1} \cdot \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, d\Omega \, dx - \frac{2}{c} \int \int_{0}^{1} \left( \phi^{k+1} \cdot \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, d\Omega \, dx$$

$$\leq \frac{2}{|\Omega|} \frac{d}{dt} \int \int_{0}^{1} \left( \phi^{k+1} \cdot \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, d\Omega \, dx + Q \phi^{k+1} \left( \frac{1}{|\Omega|} \int \phi^{k+1} \tau^{k+1} \left| \nabla \phi^{k+1} \right| \cdot \tau^{k+1} \, dx$$

$$+ \frac{d}{dt} \int \phi^{k+1} \left( \nabla \phi^{k+1} \cdot \tau^{k+1} \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, dx + \frac{d}{dt} \int \phi^{k+1} \left( \nabla \phi^{k+1} \cdot \tau^{k+1} \nabla \phi^{k+1} \right) \cdot \tau^{k+1} \, dx$$

$$J_{11} = \frac{2}{c} \int \phi^{k+1} \cdot \nabla \phi^{k+1} \cdot \tau^{k+1} \, dx = \frac{2}{c} \int \left( -\phi^{k+1} \cdot \nabla \phi^{k+1} \cdot \tau^{k+1} \right) \, dx$$

$$= -2 \frac{d}{dt} \int \phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1} \, dx + 2 \int \left( (\phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1} + \phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1}) \right) \, dx$$

$$\leq -2 \frac{d}{dt} \int \phi^{k+1} \nabla \phi^{k+1} \cdot \tau^{k+1} + Q \phi^{k+1} \left( \frac{1}{|\Omega|} \int \phi^{k+1} \tau^{k+1} \left| \nabla \phi^{k+1} \right| \cdot \tau^{k+1} \, dx$$

For the terms $J_i$ ($i = 10, 11$), according to (3.75) and Hölder’s inequality, one has
From the definition of $L_2$ and $\tilde{A}_k$, one arrives at

\begin{equation}
|\langle L_2 \rangle|_3 \leq C(\|\tilde{A}_k^k - \sigma_h^k\|_{L^2(\mathbb{R}^n; S^2)} + \|\sigma_h^k\|_{L^2(\mathbb{R}^n; S^2)} + \|\tilde{A}_k^k - \sigma_h^k\|_{L^2(\mathbb{R}^n; S^2)}) + \\
+ \|\tilde{A}_k^k\|_{L^2(\mathbb{R}^n; S^2)} + \|\tilde{A}_k^k\|_{L^2(\mathbb{R}^n; S^2)} + \|\tilde{A}_k^k - \sigma_h^k\|_{L^2(\mathbb{R}^n; S^2)} + C,
\end{equation}

\[ (3.86) \]

Substituting the estimates for $J_i$ ($i = 6, \ldots, 11$) into (3.85) and using (3.84) and (3.86), one deduces that

\begin{equation}
\frac{d}{dt}(aa|\nabla u|_2^2 + a(a + \beta)|\nabla \tilde{u}|_2^2 + |\tilde{\varphi}_k^k|^2 + |\tilde{\varphi}_k^k|^2) \leq J_{12} + E_b^k(t)|\nabla u|_2^2 + C|\nabla \tilde{u}|_2^2 + \sigma|\tilde{\varphi}_k^k|^2 + |\tilde{\varphi}_k^k|^2
\end{equation}

\[ (3.87) \]

where

\begin{equation}
J_{12} = -2 \frac{d}{dt} \int \left( \tilde{\varphi}_k^k(u_k^k + u_k^k \cdot \nabla u_k^k + \nabla \varphi_k) - \tilde{f}_k^k \cdot Q(u_k^k) - \frac{1}{c} \int_0^\infty \int \tilde{\varphi}_k^k \tilde{A}_k^k \Omega_0 \Omega_0 \nabla v + \frac{1}{c} \tilde{\varphi}_k^k \tilde{L}_2 \right) \cdot \nabla \tilde{u}^k \nabla x,
\end{equation}

and $E_b^k(t)$ satisfies

\[ \int_0^t E_b^k(t) \nabla x \leq C + C_\sigma^{-1} t \quad \text{for } 0 \leq t \leq \hat{T}. \]

It follows from (3.77)–(3.78), (3.80)–(3.81), (3.83), and (3.87) that

\begin{equation}
\frac{d}{dt}(|\nabla \tilde{\varphi}_k^k|_2^2 + |\tilde{\varphi}_k^k|_2^2) \leq \left( \int (\nabla \varphi_k) \cdot (\nabla \tilde{u}) + (\nabla \varphi_k) \cdot (\nabla \tilde{u}) \right) \left( |\nabla \tilde{\varphi}_k^k|_2^2 + |\tilde{\varphi}_k^k|_2^2 \right)
\end{equation}

\[ (3.88) \]

where $\nu, \sigma \in (0, 1)$ are sufficiently small constants, which will be determined later and $N_b^k(t)$ satisfies

\[ \int_0^t N_b^k(t) \nabla x \leq C(1 + \sigma^{-1} t) \quad \text{for } 0 \leq t \leq \hat{T}. \]

Conversely, for $J_{12}$, one has

\[ \int_0^t J_{12} \nabla x \leq C(\|\nabla \tilde{u}^k\|_2^2 + |\tilde{\varphi}_k^k|_2^2 + |\tilde{\varphi}_k^k|_2^2). \]

\[ (3.90) \]

For $\nabla \tilde{u}^k$, according to (3.75) and Lemma A.6 (Appendix A), one obtains

\begin{equation}
|\nabla \tilde{u}^k|_{L^2} \leq C(\|\nabla \tilde{u}^k\|_2 + |\nabla \tilde{u}^k|_2 + |\tilde{\varphi}_k|_2 + |\tilde{\varphi}_k|_2 + |\tilde{\varphi}_k|_2 + |\tilde{\varphi}_k|_2).
\end{equation}

\[ (3.91) \]

Denote
\[ \Gamma^k(t, v) = \sup_{0 \leq s \leq t} \| \nabla \varphi^k(s) \|_2^2 + \sup_{0 \leq s \leq t} \| \varphi^k(s) \|_2^2 + \sup_{0 \leq s \leq t} \| \varphi^k(s) \|_{L^2(R^+ \times S^2)}^2 + \sup_{0 \leq s \leq t} \| \nabla \varphi^k(s) \|_{L^2(R^+ \times S^2)}^2. \]

According to the aforementioned estimates (3.88)–(3.91) and the Gronwall inequality, one concludes that

\[ \Gamma^k(t, v) + \int_0^t (aa|\nabla \varphi^{k+1} |^2 + \nabla |\varphi^{k+1} \eta^k |^2) \, ds \leq C \left( \int_0^t (aa(\sigma + v)(|\nabla \varphi^k |^2 + |\nabla \varphi^{k-1} |^2) + \sigma |\nabla \varphi^k |^2) \, ds \right)^{\frac{1}{2}} \exp(C + C^2 t) + C a_t \int_0^t \exp(C + C^2 t). \]

Choose \( v_0, a_0 \in (0, 1) \) and \( 0 < T \leq \hat{T} \) small enough such that

\[ C a_0 \exp(C \leq \frac{v_0}{32}, \quad C v_0 \exp(C \leq \frac{1}{32})(T + 1) \exp(C a_0^{-1} T) \leq 4. \]

Then one can easily obtain

\[ \sum_{k=1}^\infty \left( \Gamma^k(T, v_0) + \int_0^T (aa|\nabla \varphi^{k+1} |^2 + v_0|\varphi^{k+1} \eta^k |^2) \, ds \right) \leq C < \infty, \quad (3.92) \]

which, along with the uniform (with respect to \( k \)) estimates (3.62), yields that

\[ \lim_{k \to \infty} \left( \| \varphi^{k+1} \|_{L^2(R^+ \times S^2)} + \| \varphi^{k+1} \|_{L^2(R^+ \times S^2)} \right) = 0. \quad (3.93) \]

Then there exists a subsequence (still denoted by \( (I^k, \phi^k, u^k, \varphi^k, f^k) \)) and limit functions \((I^\infty, \phi^\infty, u^\infty, \varphi^\infty, f^\infty)\) such that for any \( k \in [1, 3) \),

\[ I^k \to I^\infty \quad \text{in} \quad L^2(R^+ \times S^2; \ L^\infty([0, T]; \ H^s(R^3))), \]

\[ \phi^k \to \phi^\infty \quad \text{in} \quad L^\infty([0, T]; \ H^s(R^3)), \]

\[ f^k \to f^\infty \quad \text{in} \quad L^\infty([0, T]; \ L^\infty(R^3)), \]

\[ \varphi^k \to \varphi^\infty \quad \text{in} \quad L^\infty([0, T]; \ L^\infty(R^3)), \]

\[ u^k \to u^\infty \quad \text{in} \quad L^\infty([0, T]; \ L^\infty(R^3) \cap D^s(R^3)). \]

Conversely, by virtue of the uniform estimates in (3.62), there exists a subsequence (still denoted by \( (I^k, \phi^k, u^k, \varphi^k, f^k) \)) converging to the limit \((I^\infty, \phi^\infty, u^\infty, \varphi^\infty, f^\infty)\) in the weak or weak* sense. According to the lower semi-continuity of norms, the corresponding estimates in (3.62) for the limit function \((I^\infty, \phi^\infty, u^\infty, \varphi^\infty, f^\infty)\) still hold except those weighted estimates for \( u^l \). Therefore, one concludes that

\[ (I^\infty, \phi^\infty, u^\infty, \varphi^\infty, f^\infty) \] is a weak solution in the sense of distributions to the following Cauchy problem:

\[
\begin{align*}
\frac{1}{c} I^\infty_t + \Omega \cdot \nabla I^\infty &= A^\infty_t, \\
\phi^\infty_t + u^\infty \cdot \nabla \phi^\infty + (y - 1) \phi^0 \div u^\infty &= 0, \\
\varphi^\infty(u^\infty_t + u^\infty \cdot \nabla u^\infty + \nabla \varphi^\infty) &= -a L u^\infty + f^\infty \cdot Q(u^\infty) - \frac{1}{c} \phi^\infty \int_0^\infty \int_{S^2} \Pi^\infty_t \Omega d \Omega d v, \\
f^\infty_t + \sum_{l=1}^3 A_l(u^\infty) \partial_t f + B^l(u^\infty) f + a \delta \Psi \div u^\infty &= 0, \\
\Phi^\infty_t + u^\infty \cdot \nabla \Phi^\infty - (\delta - 1) \Phi^0 \div u^\infty &= 0, \\
I^\infty(t, \phi^\infty, u^\infty, \varphi^\infty, f^\infty) &= I_0 + (\phi_0 + \eta)^{-2} \phi_0(u_0 + \eta)^{-2} \Psi_0 \in R^3 \times R^+ \times S^2, \\
(I^\infty, \phi^\infty, u^\infty, \varphi^\infty, f^\infty) &\to (0, \eta, 0, 0, (\eta)^{-2} \Psi) \quad \text{as} \quad |x| \to +\infty \quad \text{for} \quad t \geq 0.
\end{align*}
\]
It should be emphasized that the conclusions obtained earlier are not sufficient to show the existence of desired strong solution to the Cauchy problem (3.69).

To this end, one first shows the strong convergence of \( \psi^k \). In fact, from (3.62), one obtains
\[
|\psi^{k+1} - \psi^k|_6 = \left| \frac{\psi^{k+1} - \psi^k}{\varphi^k} \right|_6 \leq C(|\varphi^k|_\infty |\varphi^{k+1} - \psi^{k+1}|_6 + |f^k|_6 |\varphi^{k+1}|_\infty),
\]
which, along with (3.94), yields that
\[
\psi^k \to \psi^\eta \text{ in } L^\infty([0, T]; L^5(\mathbb{R}^3)) \text{ as } k \to \infty.
\]
(3.96)

Then, according to (3.73), (3.94), and (3.96), one has
\[
|f^k - \varphi^\eta \psi^\eta|_6 \leq C(|\varphi^k - \varphi^\eta|_\infty |\psi^k|_6 + |\psi^k - \psi^\eta|_6 |\varphi^\eta|_\infty) \to 0 \text{ as } k \to \infty,
\]
which implies that
\[
f^\eta(t, x) = \psi^\eta(t, x) \text{ a.e. on } [0, T] \times \mathbb{R}^3.
\]
(3.97)

Second, one also needs to show the following relationship between \( \phi^\eta \) and \( f^\eta, \psi^\eta, \psi^\delta \):
\[
f^\eta = \frac{2\eta \delta}{\delta - 1} \nabla \log \phi^\eta, \quad \phi^\eta = (\phi^\eta)^{2\epsilon}, \quad \psi^\eta = \frac{a \delta}{\delta - 1} \nabla (\phi^\eta)^{2\epsilon}.
\]
(3.98)

Indeed, (3.98) can be obtained via the same argument as used in [32].

Moreover, denote \( h^\eta = (\varphi^\eta)^{-1} \), it follows from (3.94), the uniform positivity for \( (\varphi^k, \psi^k) \), and the upper bounds of the norms of \( \phi^\eta, u^\eta, \psi^\eta, f^\eta \) that
\[
\begin{align*}
h^\eta \Delta u^\eta &\to h^\eta \Delta u^\eta \text{ in } L^\infty([0, T]; H^1), \\
\sqrt{h^\eta} (\nabla u^\eta, \nabla u^\eta) &\to \sqrt{h^\eta} (\nabla u^\eta, \nabla u^\eta) \text{ in } L^\infty([0, T]; L^2), \\
h^\eta \Delta^2 u^\eta &\to h^\eta \Delta^2 u^\eta \text{ in } L^2([0, T]; D^1 \cap D^2), \\
(h^\eta \nabla^2 u^\eta) &\to (h^\eta \nabla^2 u^\eta) \text{ in } L^2([0, T]; L^2).
\end{align*}
\]
(3.99)

Thus, the corresponding weighted estimates for \( u^\eta \) shown in (3.62) still hold for the limit functions. Based on the estimates (3.62), the strong/weak convergence shown in (3.94), (3.96), and (3.99) and the relations (3.97) and (3.98), one can easily show that
\[
\left( I^\eta, \phi^\eta, u^\eta, h^\eta = (\phi^\eta)^{2\epsilon}, \psi^\eta = \frac{a \delta}{\delta - 1} (\nabla \phi^\eta)^{2\epsilon} \right)
\]
is a weak solution in the sense of distributions to the Cauchy problem (3.69), satisfying
\[
\begin{align*}
I^\eta &\in L^2(\mathbb{R}^3 \times S^2; C([0, T]; H^4)), \\
I^\eta &\in L^2(\mathbb{R}^3 \times S^2; C([0, T]; H^3)), \\
\phi^\eta - \eta &\in L^\infty([0, T]; H^3), \quad \psi^\eta \in L^\infty([0, T]; D^1 \cap D^2), \\
\frac{2}{3} \eta^{2\epsilon} &< \phi^\eta, \phi^\eta \in L^\infty \cap D^{1,6} \cap D^{2,3} \cap D^{3}, \quad f^\eta \in L^\infty \cap L^6 \cap D^{1,3} \cap D^2, \\
u^\eta &\in L^\infty([0, T]; H^2) \cap L^2([0, T]; H^3), \quad u^\eta \in L^\infty([0, T]; H^4) \cap L^2([0, T]; D^2), \\
u^\eta &\in L^2([0, T]; L^2), \quad t^2 u^\eta \in L^\infty([0, T]; D^0), \\
t^2 u^\eta &\in L^\infty([0, T]; D^2) \cap L^2([0, T]; D^3), \quad t^2 u^\eta \in L^\infty([0, T]; L^2) \cap L^2([0, T]; D^2).
\end{align*}
\]

**Step 2:** Uniqueness and time continuity. The uniqueness and time continuity follows easily from the same procedure as in Lemma 3.1. Finally, Theorem 3.3 is proved.

**Remark 3.2.** We conclude this subsection by giving the proof of Lemma 3.9.

**Proof.** Let \( X(x) \in C_0^\infty(\mathbb{R}^3) \) be a truncation function satisfying
Define, for any \( R > 0 \), \( X_R(x) \equiv X\left(\frac{|x|}{R}\right) \), \( \bar{\varphi}^{k+1,R} = \bar{\varphi}^{k+1}X_R \). Then from (3.75),

\[
\varphi_t^{k+1,R} + u^k \cdot \nabla \varphi^{k+1,R} - \varphi^{k+1,k} \cdot \nabla X_R + \nabla \varphi^k X_R = -(1 - \delta)(\varphi^k \text{div} u^k + \varphi^{k-1} \text{div} \nabla X_R + \Gamma_3)X_R. \tag{3.101}
\]

Multiplying (3.101) by \( 2\varphi^{k+1,R} \) and integrating over \( \mathbb{R}^3 \), one has

\[
\frac{d}{dt} \| \varphi^{k+1,R} \|_2 \leq C([|\nabla u^k|_{L^\infty} + |\varphi^{k+1} h^k|_{L^1}|h^k|_{L^1} |\nabla \varphi^{k+1,R}|_2 + |\varphi^k|_{L^\infty} |\nabla u^{k-1}|_2

+ |\varphi^k|_{L^\infty} |\varphi^k|_{L^1} |\nabla u^{k-1}|_2 + |\nabla \varphi^k|_6 + |\varphi^k|_{L^\infty} |\nabla u^k|_2 + |\varphi^{k-1}|_{L^\infty} |\nabla \nabla X_R|_2)) \tag{3.102}
\]

where \( \tilde{C} > 0 \) is a generic constant depending on \( C \) and \( \eta \) but independent of \( R \). Then applying the Gronwall inequality to (3.102), one obtains

\[
|\varphi^{k+1,R}(t)|_2 \leq \tilde{C} \exp(\tilde{C} t) \quad \text{for } (t, R) \in [0, \hat{T}] \times [0, \infty).
\]

It follows from Fatou’s lemma (Lemma A.2 in Appendix A) that

\[
\varphi^{k+1} \in L^\infty([0, \hat{T}]; L^2), \tag{3.103}
\]

which, along with

\[
\tilde{\varphi}^{k+1} \in L^\infty([0, \hat{T}]; L^\infty \cap D^{1,6} \cap D^{2,3} \cap D^3),
\]

yields that

\[
\varphi^{k+1} \in L^\infty([0, \hat{T}]; H^\infty).
\]

Similarly, one can also show that \( \tau^{k+1} \in L^\infty([0, \hat{T}]; H^\infty). \)

\[ \square \]

### 3.6 Limit from the nonvacuum flow to the flow with far field vacuum

Based on the local-in-time uniform estimates in (3.62), now we are ready to give the proof of Theorem 3.1.

**Proof.** The proof will be divided into four steps.

**Step 1:** The locally uniform positivity of \( \phi \). For \( \eta \in (0, 1) \), one sets

\[
\phi_0^\eta = \phi_0 + \eta, \quad \psi_0^\eta = \frac{a\delta}{\delta - 1} \nabla (\phi_0 + \eta)^2 e, \quad h_3^\eta = (\phi_0 + \eta)^2 e, \quad \bar{\mathcal{X}}_0^\eta = \bar{\mathcal{X}}_0(t = 0, x, v, \phi_0^\eta, l_0^\eta, l_0^\eta, \eta).
\]

Then the initial compatibility conditions of the perturbed initial data can be given as follows:

\[
\begin{cases}
\nabla u_0 = (\phi_0 + \eta)^2 e g_1^\eta, \\
L u_0 = (\phi_0 + \eta)^2 e g_2^\eta,
\end{cases}
\]

\[
\begin{cases}
\nabla \left( a(\phi_0 + \eta)^2 e L u_0 + \frac{1}{c} \int_0^\infty \bar{\mathcal{X}}_0^\eta \Omega d\Omega d\nu \right) = (\phi_0 + \eta)^2 e g_3^\eta,
\end{cases}
\]

where

\[
\begin{cases}
g_1^\eta = \frac{\phi_0^e}{(\phi_0 + \eta)^2 e} g_1, \\
g_2^\eta = \frac{\phi_0^{2e}}{(\phi_0 + \eta)^2 e} g_2,
\end{cases}
\]

\[
g_3^\eta = \frac{\phi_0^{3e}}{(\phi_0 + \eta)^3 e} \left( g_3 - \frac{a\eta \nabla \phi_0^e \phi_0^e L u_0}{(\phi_0 + \eta)} + \frac{1}{c} \int_0^\infty \phi_0^\eta \left( \frac{\phi_0^e}{(\phi_0 + \eta)^2 e} \nabla \bar{\mathcal{X}}_0^\eta - \nabla \bar{\mathcal{X}}_0 \right) \Omega d\Omega d\nu \right).
\]
It follows from (3.5) and (3.6) that there exists a \( \eta_1 > 0 \) such that if \( 0 < \eta < \eta_1 \), then
\[
1 + \eta + \| \phi_0^\eta \|_3 + \| \psi_0^\eta \|_3 + \| u_0 \|_3 + \| u_0 \|_3 + \| g_0^\eta \|_2 + \| g_0^\eta \|_2 + \| g_0^\eta \|_2 + \| g_0^\eta \|_2 + \| g_0^\eta \|_2 \leq \mathcal{C}_0,
\]
where \( \mathcal{C}_0 \) is a positive constant independent of \( \eta \). Thus, for the initial data \( (I_0^\eta, \phi_0^\eta, u_0^\eta, \psi_0^\eta) \), the problem (3.69) admits a unique classical solution \( (I_0^\eta, \phi^\eta, u^\eta, \psi^\eta) \) satisfying the local uniform estimates in (3.62) with \( c_0 \) replaced by \( c_0 \), and the life span \( T^* \) is also independent of \( \eta \).

Moreover, one has the following:

**Lemma 3.10.** For any \( R_0 > 0 \) and \( \eta \in (0, 1) \), there exists a constant \( a_{R_0} > 0 \) such that
\[
\phi^\eta(t, x) \geq a_{R_0} > 0, \quad \forall (t, x) \in [0, T] \times B_{R_0},
\]
where \( a_{R_0} \) is independent of \( \eta \).

**Proof.** One only needs to consider the case when \( R_0 \) is suitable large.

First, due to
\[
\phi_0 \in L^2, \quad \psi_0 = \frac{a\delta}{\delta - 1} \psi_0^2 \in L^2 \cap D^2,
\]
one can see that \( \nabla \psi_0^2 \in L^\infty \), which means that the initial vacuum occurs if and only if in the far field. Then there exists a constant \( \mathcal{C}_R \) independent of \( \eta \) such that for any \( R' > 2 \),
\[
\phi_0^\eta(x) \geq \mathcal{C}_R + \eta > 0, \quad \forall x \in B_{R'}.
\]

Second, let \( x(t; x_0) \) be the solution to the following initial value problem:
\[
\begin{cases}
\frac{\partial}{\partial t} x(t; x_0) = u(t, x(t; x_0)), \\
x(0; x_0) = x_0.
\end{cases}
\]

Let \( B(t, R') = \{x(t; x_0)| x_0 \in B_{R'}\} \) be the closed regions that are the images of \( B_{R'} \) under the flow map (3.108). It follows from (3.2) that
\[
\phi^\eta(t, x) = \phi^\eta_0(x_0) \exp\left(-(y-1) \int_0^t \text{div} u^\eta(s; x(s; x_0)) \, ds\right).
\]

According to (3.62), one has
\[
\int_0^t \| \text{div} u^\eta(t, x(t; x_0)) \|_2 \, ds \leq t \left( \int_0^t \| u^\eta(t, x(t; x_0)) \|_2^2 \, ds \right)^{1/2} \leq c_2 t^{1/2}.
\]

It thus follows from (3.107) and (3.109) that
\[
\phi^\eta(t, x) \geq \mathcal{C}^* (\mathcal{C}_R + \eta) > 0, \quad \forall x \in B(t, R'),
\]
where \( \mathcal{C}^* = \exp\left(-(y-1)c_2 T^{1/2}\right) \).

Moreover, it follows from (3.60)–(3.62) and (3.108) that
\[
|x_0 - x| = |x_0 - x(t; x_0)| \leq \int_0^t |u^\eta(t, x(t; x_0))| \, dt \leq c_3 t \leq 1 \leq R'/2,
\]
for all \( (t, x) \in [0, T] \times B_{R'}, \) which implies that \( B_{R'/2} \subset B(t, R') \). Thus, the desired conclusion can be achieved by taking \( R' = 2R_0 \) and \( a_{R_0} = \mathcal{C}^* \mathcal{C}_R \). \( \square \)
**Step 2:** Existence. First, by virtue of the uniform (with respect to $\eta$) estimates in (3.62), one can see that there exists a subsequence (still denoted by $(I^\eta, \phi^\eta, u^\eta, \psi^\eta)$) converging to a limit $(I, \phi, u, \psi)$ in the following weak or weak$^*$ sense:

\[
\begin{align*}
I^\eta & \rightharpoonup^* I, & \text{in } L^2(\mathbb{R}^+ \times S^2; L^{\infty}(0, T; H^2)), \\
I^\eta & \rightharpoonup^* I, & \text{in } L^2(\mathbb{R}^+ \times S^2; L^{\infty}(0, T; H^2)), \\
(\phi^\eta - \eta, u^\eta) & \rightharpoonup (\phi, u) & \text{in } L^\infty([0, T]; H^2), \\
u^\eta & \rightharpoonup u & \text{in } L^2([0, T]; H^0), \\
\phi^\eta & \rightharpoonup \phi & \text{in } L^\infty([0, T]; L^6 \cap D^{1,6} \cap D^3), \\
f^\eta & \rightharpoonup f & \text{in } L^\infty([0, T]; L^6 \cap D^{1,3} \cap D^2), \\
\psi^\eta & \rightharpoonup \psi & \text{in } L^\infty([0, T]; D^1 \cap D^2), \\
\phi_t^\eta & \rightharpoonup \phi_t & \text{in } L^\infty([0, T]; H^2), \\
(u_t^\eta, \psi_t^\eta) & \rightharpoonup (u_t, \psi_t) & \text{in } L^\infty([0, T]; H^1), \\
\psi_t^\eta & \rightharpoonup \psi_t & \text{in } L^\infty([0, T]; L^6 \cap D^{1,3} \cap D^2), \\
f_t^\eta & \rightharpoonup f_t & \text{in } L^\infty([0, T]; L^3 \cap D^1). 
\end{align*}
\]

According to the lower semi-continuity of norms, then the corresponding estimates (3.62) still hold for the limit functions $(I, \phi, u, \psi)$ except those weighted estimates for $u$.

Second, Lemmas A.3 (Appendix A) and 3.10 guarantee that there exists a subsequence (still denoted by $(I^\eta, \phi^\eta, u^\eta, \psi^\eta)$) satisfying

\[
\begin{align*}
I^\eta & \rightarrow I & \text{in } L^2(\mathbb{R}^+ \times S^2; C([0, T]; H^2(B_0))), \\
(\phi^\eta - \eta, \psi^\eta, \phi^\eta) & \rightarrow (\phi, \psi, \phi) & \text{in } C([0, T]; H^4(B_0)), \\
u^\eta & \rightarrow u & \text{in } C([0, T]; H^2(B_0)).
\end{align*}
\]

Arguing as in the proof of (3.98), one can also verify that

\[
f = \frac{2a\varepsilon\delta}{\delta - 1} \nabla \phi, \quad \varphi = \phi^{2\varepsilon}, \quad \psi = \frac{a\delta}{\delta - 1} \nabla \phi^{2\varepsilon}. \tag{3.114}
\]

Conversely, according to the estimates (3.62) except those weighted estimates for $u$, Lemma 3.10, the weak or weak$^*$ convergence in (3.112) and the strong convergence in (3.113), one can show that for $h = \varphi^{-1}$,

\[
\begin{align*}
\sqrt{h^2}(\nabla u^\eta, \nabla u^\eta) & \rightharpoonup \sqrt{h}(\nabla u, \nabla u) & \text{in } L^\infty([0, T]; L^2), \\
h^2\nabla^2 u^\eta & \rightharpoonup h^2\nabla^2 u & \text{in } L^\infty([0, T]; H^2), \\
h^2\nabla^2 u^\eta & \rightharpoonup h^2\nabla^2 u & \text{in } L^2([0, T]; D^1 \cap D^2), \\
(h^2\nabla^2 u^\eta)_t & \rightharpoonup (h^2\nabla^2 u)_t & \text{in } L^2([0, T]; L^2). 
\end{align*}
\]

Then the corresponding weighted estimates for $u$ shown in (3.62) still hold for the limit functions. Therefore, one can conclude that $(I, \phi, u, \psi)$ is the weak solution in the sense of distributions to the Cauchy problem (3.2)–(3.4).

**Step 3:** Uniqueness. The uniqueness can be easily obtained by the same argument as in Theorem 3.3.

**Step 4:** Time continuity. First, the time continuity of $(I, \phi, u, \psi)$ follows from the similar argument used in Lemma 3.1.

For the velocity $u$, the uniform estimates obtained earlier and the Sobolev embedding theorem imply that

\[
u \in C([0, T]; H^2) \cap C([0, T]; \text{weak } H^3) \quad \text{and} \quad \phi^\eta \nabla u \in C([0, T]; L^2). \tag{3.116}
\]

Then it follows from (3.2) that

\[
\phi u_t \in L^2([0, T]; H^2), \quad (\phi u_t)_t \in L^2([0, T]; L^2),
\]
which, along with the Lemma A.3, yields that \( \varphi u \in C([0, T_1]; H^1) \).

Thus, the continuity of \( u \in C([0, T_1]; H^3) \) can be obtained by using the classical elliptic estimates and

\[
a u = - \varphi \left( u_t + u \cdot \nabla u + \nabla \varphi - \psi \cdot Q(u) - \frac{1}{c} \int_0^\infty \int_{\Omega} \phi d\Omega dv \right).
\]

For \( h \nabla^2 u \), due to

\[
h \nabla^2 u \in L^{\infty}([0, T_1]; H^1) \cap L^2([0, T_1]; D^2), \quad (h \nabla^2 u)_t \in L^2([0, T_1]; L^2),
\]

and the Sobolev embedding theorem, one obtains

\[
h \nabla^2 u \in C([0, T_1]; H^3),
\]

which, along with (3.2)_3, yields that \( u_t \in C([0, T_1]; H^3) \).

The proof of Theorem 3.1 is complete. \( \square \)

### 3.7 Proof of Theorem 2.1

With Theorem 3.1 at hand, now we turn to the proof of Theorem 2.1.

**Step 1:** The local well-posedness of regular solutions. First, Theorem 3.1 guarantees that there exists a time \( T_0 > 0 \) such that the problem (3.2)–(3.4) has a unique regular solution \( (I, \phi, u, \psi) \) satisfying the regularities (3.7), which implies that

\[
(I, I_t) \in L^2(\mathbb{R}^+ \times S^2; C([0, T_1] \times \mathbb{R}^3)), \quad \phi \in C([0, T_1] \times \mathbb{R}^3), \quad (u, \nabla u) \in C([0, T_1] \times \mathbb{R}^3). \quad (3.117)
\]

Second, from the transformation in (3.1), one has

\[
\rho(t, x) = \left( \frac{y-1}{Ay} \right)^{\frac{1}{\gamma-1}}(t, x) \quad \text{and} \quad \frac{\partial \rho}{\partial \phi}(t, x) = \frac{1}{y-1} \left( \frac{y-1}{Ay} \right)^{\frac{1}{\gamma-1}} \phi^{\frac{1}{\gamma-1}}(t, x). \quad (3.118)
\]

Then multiplying (3.2)_2 by \( \frac{\partial \rho}{\partial \phi}(t, x) \) yields the continuity equation (1.9)_2; and multiplying (3.2)_3 by \( \rho \) gives the momentum equations (1.9)_3.

Thus, we have shown that \( (I, \rho, u) \) is a weak solution in the sense of distributions to the Cauchy problem (1.9) and (1.10) and satisfying the regularities in Definition 2.1 and (2.11). Moreover, it follows from the continuity equation that \( \rho(t, x) > 0 \) for \([0, T_1] \times \mathbb{R}^3\). In summary, the Cauchy problem (1.9) and (1.10) has a unique regular solution \( (I, \rho, u) \).

**Step 2:** The local well-posedness of classical solutions. Now we show that the regular solution obtained above is indeed a classical one in positive time.

First, according to (3.117) and (3.118), one concludes that

\[
(I, I_t) \in L^2(\mathbb{R}^+ \times S^2; C([0, T_1] \times \mathbb{R}^3)), \quad (\rho, \nabla \rho, \rho_t, u, \nabla u) \in C([0, T_1] \times \mathbb{R}^3).
\]

Second, it follows from the classical Sobolev embedding:

\[
L^2([0, T_1]; H^1) \cap W^{1,2}([0, T_1]; H^{-1}) \hookrightarrow C([0, T_1]; L^2), \quad (3.119)
\]

and the regularities in (2.11) that

\[
tu_t \in C([0, T_1]; H^2) \quad \text{and} \quad u_t \in C([T, T_1] \times \mathbb{R}^3). \quad (3.120)
\]

It remains to show that \( \nabla^2 u \in C([T, T_1] \times \mathbb{R}^3) \). According to (3.2)_3, one has

\[
a u = - \varphi \left( u_t + u \cdot \nabla u + \nabla \varphi - \psi \cdot Q(u) - \frac{1}{c} \int_0^\infty \int_{\Omega} \phi d\Omega dv \right) = \varphi \phi^{\frac{1}{\gamma-1}}. \quad (3.121)
\]
which, along with (2.11), yields that
\[ t\phi^{-2\mathbb{H}} H \in L^\infty([0, T]; H^2). \quad \text{(3.122)} \]

On the other hand, one has
\[ (t\phi^{-2\mathbb{H}} H) = \phi^{-2\mathbb{H}} H + t(\phi^{-2\mathbb{H}} H) + t\phi^{-2\mathbb{H}} H \in L^2([0, T]; L^2). \quad \text{(3.123)} \]

It thus follows from the classical embedding theorem:
\[ L^\infty([0, T]; H^1) \cap W^{1,2}([0, T]; H^{-1}) \hookrightarrow C([0, T]; L^q) \quad \text{for } q \in [2, 6), \quad \text{(3.124)} \]

and (3.121)–(3.123) that
\[ t\phi^{-2\mathbb{H}} H \in C([0, T]; W^{1,4}) \quad \text{and} \quad t\nabla^2 u \in C([0, T]; W^{1,4}). \]

Again the Sobolev embedding theorem implies that \( \nabla^2 u \in C((0, T] \times \mathbb{R}^2). \)

Finally, the proof of Theorem 2.1 is complete. \( \square \)

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Appendix
A Basic lemmas

This appendix is devoted to listing some useful lemmas that were used frequently in the previous sections. The first one is the well-known Gagliardo-Nirenberg inequality.

Lemma A.1. [9] Let \( 1 \leq q, r \leq \infty \), and \( p \) satisfies

\[
\frac{1}{p} = \frac{j}{d} + \left( \frac{1}{r} - \frac{m}{d} \right) + \left( 1 - \frac{1}{q} \right) \quad \text{and} \quad \frac{j}{m} \leq \zeta \leq 1.
\]

Then there exists a generic constant \( C > 0 \) depending only on \( j, m, d, q, r, \) and \( \zeta \) such that for \( f \in L^q(\mathbb{R}^d) \cap D^{m,r}(\mathbb{R}^d) \),

\[
\|Df\|_{L^p} \leq C\|D^m f\|_{L^{p,\zeta}}^{\frac{1}{m} - \zeta},
\]

with the exceptions that if \( j = 0, rm < d, q = \infty \) we assume that \( f \) vanishes at infinity or \( f \in L^{\hat{q}} \) for some finite \( \hat{q} > 0 \), while if \( 1 < r < \infty \) and \( m - j - d / r \) is a nonnegative integer we take \( j / m \leq \zeta < 1 \).

The second one is the well-known Fatou’s lemma, which can be found in [27].

Lemma A.2. [27] Given a measure space \((V, \mathcal{F}, \nu)\) and a set \( X \in \mathcal{F} \), let \( \{f_n\} \) be a sequence of \((\mathcal{F}, \mathcal{B}_{R_{\infty}})\)-measurable nonnegative functions \( f_n : X \to [0, \infty) \). Define the function \( f : X \to [0, \infty] \) by setting

\[
f(x) = \liminf_{n \to \infty} f_n(x),
\]

for every \( x \in X \). Then \( f \) is \((\mathcal{F}, \mathcal{B}_{R_{\infty}})\)-measurable, and

\[
\int f(x) \, d\nu \leq \liminf_{n \to \infty} \int f_n(x) \, d\nu.
\]

The third one shows some compactness results obtained via the Aubin-Lions Lemma.

Lemma A.3. [29] Let \( X_0 \subseteq X \subseteq X_1 \) be three Banach spaces. Suppose that \( X_0 \) is compactly embedded in \( X \) and \( X \) is continuously embedded in \( X_1 \). Then the following statements hold.

1. If \( J \) is bounded in \( L^p([0, T]; X_0) \) for \( 1 \leq p < \infty \), and \( \frac{\partial J}{\partial t} \) is bounded in \( L^1([0, T]; X_0) \), then \( J \) is relatively compact in \( L^p([0, T]; X) \).

2. If \( J \) is bounded in \( L^\infty([0, T]; X_0) \) and \( \frac{\partial J}{\partial t} \) is bounded in \( L^p([0, T]; X_0) \) for \( p > 1 \), then \( J \) is relatively compact in \( C([0, T]; X) \).

The following calculus inequalities can be found in Majda [21].

Lemma A.4. [21] Let \( r, a^* \) and \( b \) be constants such that

\[
\frac{1}{r} = \frac{1}{a^*} + \frac{1}{b}, \quad \text{and} \quad 1 \leq a^*, b, r \leq \infty.
\]

\( \forall s \geq 1, \) if \( f, g \in W^{s,a^*} \cap W^{s,b} (\mathbb{R}^3) \), then it holds that

\[
|\nabla^s (fg) - f \nabla^s g|_r \leq C_s (|\nabla f|_{a^*} |\nabla^{s-1} g|_b + |\nabla^s f|_a |g|_b),
\]

\[
|\nabla^s (fg) - f \nabla^s g|_r \leq C_s (|\nabla f|_{a^*} |\nabla^{s-1} g|_b + |\nabla^s f|_a |g|_b),
\]

Radiation hydrodynamic equations
where $C_s > 0$ is a constant depending only on $s$, and $\nabla^s \delta f$ ($s > 1$) is the set of all $\delta f$ with $|\zeta| = s$. Here, \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 \) is a multi-index.

The following lemma is used to obtain the time-weighted estimates of the velocity $u$.

**Lemma A.5.** [1] If $f(t, x) \in L^2([0, T]; L^2)$, then there exists a sequence $s_k$ such that

$$s_k \to 0 \quad \text{and} \quad s_k |f(s_k, x)|^2_2 \to 0 \quad \text{as } k \to +\infty.$$

The last one gives the regularity estimates for the Lamé operator in $\mathbb{R}^3$.

**Lemma A.6.** [30] If $u \in D^{1,q}(\mathbb{R}^3)$ with $1 < q < +\infty$ is a weak solution to the problem

\[
\begin{align*}
- \alpha \Delta u - (\alpha + \beta) \nabla \text{div} u &= \mathcal{Z}, \\
u &\to 0 \quad \text{as } |x| \to +\infty,
\end{align*}
\]

then it holds that

$$|u|_{D^{1,2,q}} \leq C |\mathcal{Z}|_{D^{1,q}},$$

where the constant $C > 0$ depends only on $\alpha$, $\beta$, and $q$. 
