WIDE SHORT GEODESIC LOOPS ON CLOSED RIEMANNIAN MANIFOLDS.

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ABSTRACT. It is not known whether or not the length of the shortest periodic geodesic on a closed Riemannian manifold $M^n$ can be majorized by $c(n)\text{vol}^{\frac{1}{2}}$, or $\tilde{c}(n)d$, where $n$ is the dimension of $M^n$, $\text{vol}$ denotes the volume of $M^n$, and $d$ denotes its diameter. In this paper we will prove that for each $\varepsilon > 0$ one can find such estimates for the length of a geodesic loop with with angle between $\pi - \varepsilon$ and $\pi$ with an explicit constant that depends both on $n$ and $\varepsilon$.

That is, let $\varepsilon > 0$, and let $a = \lceil \frac{1}{\sin(\varepsilon^2/2)} \rceil + 1$. We will prove that there exists a “wide” (i.e. with an angle that is wider than $\pi - \varepsilon$) geodesic loop on $M^n$ of length at most $2n!a^n d$. We will also show that there exists a “wide” geodesic loop of length at most $2(n + 1)!^2 a^{(n+1)^3} \text{FillRad} \leq 2 \cdot n(n + 1)!^2 a^{(n+1)^3} \text{vol}^{\frac{1}{2}}$. Here $\text{FillRad}$ is the Filling Radius of $M^n$.

INTRODUCTION

The two results in this paper are motivated by the following 35 year old question formulated by M. Gromov in [G]. Let $M^n$ be a closed Riemannian manifold of dimension $n$ and volume $\text{vol}$. Gromov asked whether there exists a constant $c(n)$ such that the length of a shortest closed geodesic, $l(M^n)$, on $M^n$ is bounded above by $c(n)\text{vol}^{\frac{1}{2}}$. Similarly, one can ask if there exists a constant $\tilde{c}(n)$, such that $l(M^n) \leq \tilde{c}(n)d$, where $d$ denotes the diameter of $M^n$. The answers to the above questions are known for all closed Riemannian surfaces, (see [BZ], [CK] for surveys of these results), including the most difficult case of a Riemannian 2-sphere, for which the first upper bounds were established by C. B. Croke, (see [C]). Volume upper bounds were also proved by Gromov in the special case of essential manifolds, (see [G]). Finally, note that it easy to see that $l(M^n) \leq 2d$, when $M^n$ is not simply connected. This diameter upper bound does not have to hold in general, as it was shown by F. Balacheff, C. Croke and M. Katz who found an example of a Zoll 2-sphere for which the length of a shortest periodic is greater than twice the diameter, (see [BCK]).

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These are the only known to us curvature-free upper bounds for the length of a shortest periodic geodesic on closed Riemannian manifolds of dimension greater than two. The only estimates for $l(M^n)$ in higher dimensions that involve curvature are (1) our old joint results with A. Nabutovsky, [NR0] where we obtained the upper bound for $l(M^n)$ for the class of manifolds with the sectional curvature $K \geq k$, volume $vol \geq v > 0$ and diameter $d \leq D$, and for the class of manifolds with $K \leq k$, and $vol \leq V$, and (2) a recent theorem of N. Wu and Z. Zhu, (see [WuZ]), in which they established the existence of an upper bound for $l(M^n)$ for the class of 4-manifolds with $vol \geq v$, $d \leq D$ and the Ricci curvature, $|Ric| \leq 3$. The result of Wu and Zhu uses recent theory of J. Cheeger and A. Naber, (see [ChN]) of manifolds with two-sided bound on Ricci curvature.

There is, however, a number of curvature-free estimates for the length of geodesic loops, starting with the result of S. Sabourau, [S] (in which he proved the first curvature-free upper bounds for the length of a shortest geodesic loop on an arbitrary Riemannian manifold) as well as for the length of stationary 1-cycles, geodesic nets, and geodesic loops at each point of a manifold, (see [B], [NR1], [NR2], [R1], [R2], [R3], [R4]). In fact, it is plausible that the curvature-free upper bounds for the length of a shortest periodic geodesic do not exist in higher dimensions. This would make the above results, as well as the result discussed in this paper optimal in some sense.

Let us begin with the following definitions, (see Definitions 0.1, 0.4 in [R4]):

**Definition 0.1.** (a) A minimal (or stationary) geodesic net $\Gamma$ is a multi-graph immersed into a Riemannian manifold $M^n$ satisfying the following two conditions:

1. Each edge of $\Gamma$ is a geodesic segment. Multiple edges between two vertices, integer multiples of the same edge, and edges between a point and itself, i.e. loops are allowed.
2. The sum of unit vectors at each vertex tangent to the edges and directed away from this vertex is equal to zero. These vectors are counted with multiplicities of the respective edges.

(b) If a minimal geodesic net $\Gamma$ has one vertex, we will call it a (minimal, or stationary) geodesic flower, or a geodesic $m$-flower where $m$ is the number of loops that comprise it counted with the multiplicities, (see fig. 2). Individual geodesic loops of the minimal geodesic flower will be called its petals. A minimal geodesic net $\Gamma$ that has at most 2 vertices joined by at most $m$ segments, (counted with multiplicities), will be called a minimal geodesic $m$-cage, (or just a geodesic cage), (see fig. 7). Thus, a geodesic $m$-flower can also be viewed as a geodesic $m$-cage.
Remark. Any immersion of a graph in $M^n$ will be called a net. Nets with a single vertex will be called flowers, while nets with at most two vertices will be called cages.

By Definition 0.1 periodic geodesics are minimal geodesic flowers, while singular geodesic loops are not. The stationarity condition implies that stationary geodesic nets, and, in particular, geodesic flowers are critical points of the (possibly weighted) length functional on the space of (multi-)graphs, (see [NR2]). That is a flower $F$ is a geodesic flower if and only if for any smooth vector field $v$ on a Riemannian manifold $M$, the functional $L(t) = \text{length}(\Phi_v^t(F))$ has a critical point at $t = 0$, where $\Phi_v^t$ is the 1-parameter flow of diffeomorphisms of $v$.

In this paper we will prove the following theorems:

**Theorem 0.2.** Let $M^n$ be a closed Riemannian manifold of dimension $n$ and of diameter $d$. Let $q$ be the smallest integer such that $\pi_q(M^n) \neq \{0\}$. Then for any sufficiently small $\varepsilon > 0$ there exists a geodesic loop $\alpha$ with an angle $\theta$, such that $\pi - \theta < \varepsilon$, and such that its length $l(\alpha) \leq 2q! a^q d \leq 2n!a^nd$, where $a = \lceil \frac{1}{\sin \frac{\varepsilon}{2}} \rceil + 1$. Moreover, this geodesic loop will be petal of a stationary geodesic flower.

**Theorem 0.3.** Let $M^n$ be a closed Riemannian manifold of dimension $n$ and of volume $\text{vol}$. Then for every $\varepsilon > 0$ there exists a geodesic loop $\alpha$ with an angle $\theta$, such that $\pi - \theta < \varepsilon$, and the length $l(\alpha) \leq 2\cdot n(n+1)!a^{(n+1)/3} \text{vol}^{\frac{1}{3}}$, where $a = \lceil \frac{1}{\sin \frac{\varepsilon}{2}} \rceil + 1$. Moreover, this geodesic loop will be petal of a stationary geodesic flower.

The first idea behind the proofs of Theorems 0.2 and 0.3 is a version of pseudo-filling argument pioneered by Gromov in [G], where one obtains a minimal object as an obstruction to filling a homologically non-trivial cycle in $M$. In our case, the wide geodesic loops will be obtained as a subset of a stationary geodesic flower the existence of which will be proven using such a technique. This filling technique is used in conjunction with two additional observations.

(1) It is possible for the net to “degenerate” during the length shortening flow described in [NR0], whereby some of the edges can become smaller and eventually disappear, and the vertices can merge together. In view of this, one can introduce a weighted length shortening flow that will force all of the vertices in the net to merge together.

(2) The vertex stationarity condition implies that endowing one of the loops in the net with a large weight will either force this loop to disappear or to become wide.
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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{A non-degenerate 3-cage}
\end{figure}

\theta-\text{graph is an example of a geodesic net}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{A stationary geodesic flower}
\end{figure}

An example of a geodesic flower

\[\alpha_1, \alpha_2, \alpha_3, \text{ are geodesic loops, stationarity condition is satisfied at } p\]

For example, the main idea of [R4] was to define a weighted length functional on the space of nets such that its gradient-like flow will “force” some edges to shrink to a point, and nets to degenerate into geodesic flowers.
Definition 0.4. (1) Let $\Gamma$ be a net with edges $e_1, \ldots, e_i, \ldots, e_k$. Then $L(\Gamma) = \sum_{i=1}^{k} m_i \text{length}(e_i)$, where $m_i \in \mathbb{Z}_+$ and $\text{length}(e_i)$ is the length of $e_i$. will be called a weighted length functional with weights $m_1, \ldots, m_k$. In particular, given an integer $a \geq 3$, $L_a$ will denote the length functional with the weights $1, a, \ldots, a^{k-1}$.

(2) A net $N$ is stationary (or minimal, or critical) with respect to a weighted length functional $L$ with weights $m_i, i = 1, \ldots, k$, if for any one-parametric smooth flow of diffeomorphisms $\psi_t, t = 0$ is a critical point of $f(t) = L(\psi_t(N))$.

The first variation of the length formula implies that if $N$ is critical with respect to a weighted length functional $L$, then (2) in the above definition implies that each edge of $N$ is a geodesic, and the stationarity condition is satisfied at each vertex of $N$. Note that a critical point $N$ of the weighted length functional $L_a$ is a critical point of the regular length functional, if each edge of $N$ is taken with a multiplicity $m_i$.

**Example.** Let $\varepsilon > 0$ be small enough so that $a = \lceil \frac{1}{\sin \frac{\varepsilon}{2}} \rceil + 1 \geq 3$. Let $\Gamma$ be a 3-cage with vertices $p_1, p_2$ and three edges $e_1, e_2, e_3$. Note that it is possible that $p_1 = p_2$ and that the length of one or more edges of $\Gamma$ is zero. In the latter case, namely when one or more edges of $\Gamma$ is trivial $\Gamma$ will either be a closed curve, of a flower with two petals. Let $L_a(\Gamma) = \text{length}(e_1) + a \cdot \text{length}(e_2) + a^2 \cdot \text{length}(e_3)$ to be a weighted length functional applied to $\Gamma$ with the weights, 1, $a$, $a^2$. Suppose the weighted length shortening process converges to a non-trivial stationary geodesic cage.
Then a non-degenerate 3-cage cannot be stationary. To show that suppose \( \Gamma \) is not degenerate. Let \( v_i^j \) be the unit vector tangent to \( e_i \) at the point \( p_j, j \in \{1, 2\} \), (see fig. 4). Then the stationarity condition implies that for both \( j = 1, 2 \), \( v_1^j + av_2^j + a^2v_3^j = 0 \). These conditions obviously cannot be satisfied as \( a^2 > 1 + a \). Therefore, if \( \Gamma \) is critical then at least one of the \( e_i \)'s has to shrink to a point and the cage degenerates into a flower.

We will next show that one of the “petals” of this flower has to be a geodesic loop with a wide angle, i.e. with an angle \( \theta \), such that \( \pi - \theta < \varepsilon \). There are four possibilities to consider.

**Case 1: Two edges of \( \Gamma \) shrink to a point.** In this case the stationary geodesic flower will be a periodic geodesic, which can be viewed as the geodesic loop with an angle \( \pi \). The length of the periodic geodesic will be at most \( l_1 + al_2 + a^2l_3 \), where \( l_i \) is the length of \( e_i, i \in \{1, 2, 3\} \).

**Case 2: \( e_3 \) shrinks to a point.** In this case the stationary geodesic flower will be the “figure 8”, (see fig. 5). It will have two loops \( \alpha_1 \) obtained from the deformation of \( e_1 \) and \( \alpha_2 \) that is obtained from the deformation of \( e_2 \), and one vertex \( P \). Let \( \theta_i \) be an angle corresponding to \( \alpha_i \), for \( i \in \{1, 2\} \). The stationarity condition implies that the bisectors of \( \theta_1 \) and \( \theta_2 \) are the same. Let \( w_i^j \) be the unit vectors tangent to \( \alpha_i \) at \( P, j \in \{1, 2\} \). Then the lengths of the projections of \( w_i^j \) onto the bisector will be \( \cos \frac{\theta_i}{2} \). The
stationarity condition then implies the following equation $\cos \frac{\theta_1}{2} = a \cos \frac{\theta_2}{2}$.

Thus, $a \sin \left(\pi - \theta_2\right) \leq 1$. It follows that $\sin \left(\pi - \theta_2\right) \leq \frac{1}{a} < \sin \frac{\epsilon}{2}$. Therefore, $\pi - \theta_2 < \epsilon$.

**Case 3:** $e_2$ shrinks to a point. Once again, the stationary geodesic net will be the “figure 8” with loops $\alpha_1, \alpha_3$ corresponding to the edges $e_1, e_3$ with the vertex angles $\theta_1, \theta_3$. The bisectors of the angles will be the same. Let $w_i^j$ be the unit vectors tangent to $\alpha_i, i \in \{1, 3\}$, and $j \in \{1, 2\}$. The lengths of the projections of the vectors onto the bisector will be $\cos \theta_i$, and the stationarity condition will imply that $\cos \frac{\theta_1}{2} = a^2 \cos \frac{\theta_3}{2} = a^2 \sin \left(\frac{\pi - \theta_3}{2}\right)$.

Therefore, $\sin \left(\frac{\pi - \theta_3}{2}\right) \leq \frac{1}{a^2} < (\sin \frac{\epsilon}{2})^2 < \sin \frac{\epsilon}{2}$. Therefore, $\pi - \theta_3 < \epsilon$.

**Case 4:** $e_1$ shrinks to a point. The case is done in a similar fashion.

We then combine this idea with the techniques of [R4] that will be explained in the next section. Let us describe this more formally.

To prove Theorem 0.3 we will use the weighted length functional $L_a$ with the weights $1, a, a^2, \ldots, a^{m-1}$, where $m$ is the number of edges in the $m$-cage. Recall that $a$ is a constant that depends on $\epsilon$, and that increases unboundedly as $\epsilon$ approaches 0. Likewise, in order to prove Theorem 0.4, we will consider nets that correspond to the 1-skeleton of an $m$-simplex, and will use the weighted length functional with weights $1, a, a^2, \ldots, a^k$, where $k = \frac{(m+1)(m+2)}{2}$, i.e. the number of edges in the 1-skeleton.

The non-trivial critical points that correspond to the above weighted length functionals are bouquets of geodesic loops of total length $\leq \tilde{c}(n)d$ and of total length $\leq c(n)vol^\frac{1}{n}$ respectively, where $d$ is the diameter and $vol$ is the volume of $M^n$. Moreover, one of the loops in this net, namely the loop corresponding to the edge with the highest weight, among all of the edges of nonzero length, will have an angle that will approach $\pi$ as $a$ becomes large. We will prove that there always exists such a critical point, using techniques of [R4], modified Gromov's extension technique appearing in [G], and the idea illustrated by the above example.
1. The Proof of Theorem 0.2

First we will need to prove the following lemma.

Lemma 1.1. Let \( \varepsilon > 0 \) be given. Let \( a = \max\{\left\lceil \frac{1}{\sin \frac{\pi}{2}} \right\rceil + 1, 3\} \). Let \( \Gamma \) be a geodesic cage with edges \( e_i, i \in \{0, \ldots, (k - 1)\} \) each taken with the multiplicity \( a^i \). Then \( \Gamma \) is not stationary with respect to \( L_a \), unless it is a geodesic flower. Moreover, let \( F \) be a geodesic flower with the geodesic loops (petals) \( e_i, i = 0, \ldots, k - 1 \), (some of them possibly trivial) with multiplicities \( a^i \). Then if \( F \) is stationary, one of the petals has an angle \( \theta > \pi - \varepsilon \).

Proof. Suppose \( \Gamma \) is stationary with respect to \( L_a \). Let us denote the two vertices of the geodesic cage \( \Gamma \) as \( p_1, p_2 \). Assume that \( p_1 \neq p_2 \). Let \( v_i^j \) be the unit vector tangent to the edge \( e_i \) at the vertex \( p_j \), and diverging from \( p_j \) (see fig. 3). The stationarity condition implies that \( \sum_{i=0}^{k-1} a^i v_i^j = 0 \), which can be restated as \( a^{k-1} v_{k-1}^j = -\sum_{i=0}^{k-2} a^i v_i^j \), for \( j \in \{1, 2\} \). The length of the projection of \( v_i^j \) onto the line passing through \( v_{k-1}^j \) is \( \cos \theta_i^j \), where \( \theta_i^j \) is an angle that \( v_i^j \) is making with \( -v_{k-1}^j \). Then the stationarity conditions becomes \( a^{k-1} = \sum_{i=0}^{k-2} a^i \cos \theta_i^j \leq \sum_{i=0}^{k-2} a^i = a^{k-1} - 1 \), which is a contradiction.

Next let us consider a geodesic flower \( F \). Let \( P \) be its vertex. We will denote the two unit vectors that are tangent to the non-constant loop \( e_i \) and diverging away from \( P \) as \( w_1^1, w_2^2 \). In the case when \( e_i \) is a point, we will let \( w_i^j = 0, j \in \{1, 2\} \). Then the stationarity condition at \( P \) implies that \( \sum_{j=1}^{2} \sum_{i=0}^{k-1} a^i w_i^j = 0 \). Let \( s \) be the maximum index for which the loop \( e_j \) is not trivial. Let \( V = \sum_{j=1}^{2} \sum_{i=s}^{k-1} a^i w_i^j \). Then \( a^s (w_1^1 + w_2^2) = -V \). The length of the projection of \( w_i^j \) onto the line through \( -V, i \in \{1, \ldots, s - 1\} \), where \( \theta_i^j \) is the angle between \( V \) and \( w_i^j \) is \( \cos \theta_i^j \). Also, let \( \cos \frac{\theta_i^j}{2} \) be the projections of \( w_i^j, j \in \{1, 2\} \) onto the line through \( V \). Here \( \frac{\theta_i^j}{2} \) is an angle that \( w_i^j \) makes with \( -V \). Then the stationarity condition implies \( 2a^s \cos \frac{\theta_i^j}{2} = \sum_{j=1}^{2} \sum_{i=0}^{s-1} a^i \cos \theta_i^j \leq \sum_{j=1}^{2} \sum_{i=0}^{s-1} a^i = 2(a^s - 1) \). This implies that \( \sin \frac{\pi - \theta}{2} = \cos \frac{\theta_i^j}{2} < \frac{1}{a-1} \leq \sin \frac{\pi}{2} \). Therefore, \( \pi - \theta < \varepsilon \).

Proof of Theorem 0.2. Let \( \varepsilon > 0 \) be given such that \( \left\lceil \frac{1}{\sin \frac{\pi}{2}} \right\rceil + 1 \geq 3 \). The proof is by contradiction. Let \( f : S^q \longrightarrow M^n \) be a non-contractible map from the round sphere \( S^q \) of dimension \( q \) into \( M^n \). Let us endow \( S^q \) with a sufficiently fine triangulation, so that the the maximal diameter of the simplices on \( M^n \) in the triangulation induced by \( f \) is at most some small positive \( \delta \). Let us consider the disc \( D^{q+1} \), such that \( \partial D^{q+1} = S^q \). We will triangulate it as a cone over \( S^q \).

Now consider the space \( \Gamma M^n \) of the \( m \)-cages on \( M^n \) together with the weighted length functional with weights: \( 1, a, a^2, \ldots, a^{m-1} \). Note that it is
enough to prove that there exists a critical point of “small” (i. e. bounded in terms of the diameter $d$ of $M^n$) but non-zero length on $\Gamma M^n$. Indeed, by Lemma 1.1, this critical point will have to be a stationary geodesic flower, and one of the loops that comprise it will have an angle that is $\varepsilon$-close to $\pi$.

Thus, assume that all of the critical points that correspond to the above weighted length functional are longer than the bound specified in the conclusion of the theorem. We will then show that we can extend the map $f$ to $D^{q+1}$, which would contradict the fact that $f : S^q \to M^n$ is not contractible. Now let us enumerate the edges in the 1-skeleton of $D^{q+1}$. The pseudo-extension procedure will be done by induction on the skeleta of $D^{q+1}$. Note that the map can be naturally extended to the center of the disc by assigning to it an arbitrary point $p$ in $M^n$. Likewise, the extension to the 1-skeleton is trivially accomplished by mapping the edges to minimal geodesic segments that connect this point with the corresponding vertices of the triangulation of the image sphere. The rest of the extension is done via an inductive bootstrap procedure, and amounts to “filling” $m$-cages by $m$-discs for all values of $m \leq q+1$. A similar procedure was used in [R1] and [R4]. One proves the base of induction by contracting 2-cages, i.e. closed curves, by the usual Birkhoff curve shortening process, assuming there are no “short” periodic geodesics, i.e. geodesics of length at most $2d$. These homotopies generate 2-disks that fill the 2-cages.

Now assume that we have extended our map to the $k$-skeleton, we will now describe how to extend $f$ to the $(k+1)$-skeleton of $D^{q+1}$. In order to do that, we will show how to extend $f$ to each $(k+1)$-dimensional simplex of $D^{q+1}$. It will be done by “filling” $(k+1)$-cages by $(k+1)$-dimensional discs. Suppose we want to extend $f$ to simplex $\sigma_{k+1}$. Consider its boundary. It consists of $k+2$ $k$-dimensional simplices. One of the simplices in the boundary is a simplex of $S^q$. Recall that the diameter of its image under $f$ is at most some small number $\delta$, which implies that it can be effectively treated as a point, $q$ (see the remark on page 13 in [R1]). Since the simplex is contractable, one can continuously deform it to a point, while stretching each edge by at most $\delta$. Consider an ordered $(k+1)$-cage $C$ that consists of the points $p, q$ and the geodesic edges that we can arrange into a sequence $e_0, e_1, \ldots, e_k$ starting with an edge with the smallest index, and ending with the edge with the largest simplex. Next we will apply the weighted length shortening process, where the weight corresponding to $e_j$ will be $a^j$. Let us now consider a weighted-length shortening process on $\Gamma M^n$. The construction of this flow is analogous to the Birkhoff Curve Shortening Flow. Likewise, it can be shown that in the absence of non-trivial critical points it is possible to deform $\Gamma M^n$ to the subspace that consists of the constant cages, i. e. the cages, in which both vertices coincide and all of the edges have zero lengths, (see [C] for the detailed description of the Birkhoff curve
shortening process). This flow was formally defined for 1-cycles in a much more general setting in [NR1] and [R1]. The weighted length shortening flow is no different then the length shortening flow applied to the multigraphs, where each edge is allowed to be taken with multiplicities. The deformation of the edge during the length shortening is uniquely defined. Thus, each copy of the multiple edge will be deformed in the same way, and the splitting of the edges is not possible. (See Section 3 of [NR1] for the detailed description of the length shortening flow for nets).

If there are no critical geodesic flowers corresponding to the length functional \(L_{a}\), then any cage \(C\) that corresponds to the 1-skeleton of \((k+1)\)-simplex in \(M^n\) obtained on the prior step of the induction can be contracted to a point that we will denote \(x\) along a 1-parameter family of cages \(C_{k+1}^\tau\), \(\tau \in [0,1]\) of smaller weighted length. We can next construct a 1-parameter family of spheres \(S_k^\tau\) of dimension \(k\) corresponding to \(C_{k+1}^\tau\), where \(S_k^0\) will be the image of the boundary of the given simplex, and \(S_k^1 = \{x\}\). This 1-parameter family of spheres generates a \((k+1)\)-dimensional disc. Spheres are constructed by the procedure of “filling” cages \(C_{k+1}^\tau\) at each \(\tau\) first described in [R1]. For each \(\tau \in [0,1]\), consider \((k+1)\) of \(k\) subcages of \(C_{k+1}^\tau\). That is, we are considering \(k\)-cages obtained from \(C_{k+1}^\tau\) by deleting from it one of the edges, i.e. we are looking at all of the 1-parameter families of \(k\)-tuples \((e_0)_\tau, ..., (e_j)_\tau, ..., (e_k)_\tau\) obtained from the original \((k+1)\)-cage by removing from it one of the curves. By induction assumption for all \(\tau \in [0,1]\), each of these subcages can be “filled” by discs of dimension \(k\), and moreover, the resulting discs will change continuously with respect to \(\tau\).

Next glue these \((k+1)\) \(k\)-dimensional discs as in the boundary of \((k+1)\)-simplex to obtain \(S_k^\tau\). Recall that the last \((k+2)\)nd disk is a point. Moreover, this process is continuous with respect to \(C_{k+1}^\tau\). This one-parameter family of spheres generates the desired \((k+1)\)-dimensional disc that can be used to extend \(f\), (see fig. 6, which depicts how contracting 3-cage generates a 1-parameter family of 2-spheres).

We will next prove the length upper bound. Consider the 1-parameter family \(C_{\tau}^{k+1}\) obtained during the weighted length shortening flow of \(C_{0}^{k+1}\) with the aforementioned weights \(1, a, ..., a^k\). Let us denote the maximal total weighted length of \(C_{\tau}^{k+1}\) over all \(\tau \in [0,1]\) as \(l_{k+1}\). Let us consider an ordered sequence of edges of \(C_{\tau}^{k+1}\). Then the length of \((e_i)_\tau\) is at most \(\frac{l_{k+1}}{a^i}\) for each \(\tau \in [0,1]\) and for all \(i \in \{0, ..., k\}\). For each \(\tau \in [0,1]\) let us consider all of the \(k\)-subcages of \(C_{\tau}^{k+1}\). Their total length, of course, does not exceed the maximal length of the \((k+1)\)-cage, i.e. \(l_{k+1}\). Let us consider the \(k\)-subcage, denoted as \(\bar{C}_k\) of \(C_{\tau}^{k+1}\) that is comprised of \((e_0)_\tau, ..., (e_{k-1})_\tau\). From the estimates of the individual segments, we can see that its total
length can be at most \( l_{k+1} + \frac{l_{k+1}}{a} + \ldots + \frac{l_{k+1}}{a^{k+1}} \). This is the upper bound valid for all of the other subcages of \( C_{\tau}^{k+1} \) for all \( \tau \). As we apply the weighted length shortening with the coefficients \( 1, a, \ldots, a^{k-1} \) to \( C_{\tau}^{k} \), we will obtain a 1-parameter family \( C_{s}^{k}, s \in [0, 1] \), where \( C_{0}^{k} = C_{\tau}^{k} \). Note that the maximal length of the \( k \)-subcages resulting from this flow will be at most \( l_{k+1} + a(\frac{l_{k+1}}{a}) + \ldots + a^{k-1}(\frac{l_{k+1}}{a^{k+1}}) = k \times l_{k+1} \). Let \( l_{k} \) denote the maximal length over all of the (2-parameter) family of the \( k \)-subcages. (One parameter is \( \tau \in [0, 1] \); the second parameter is the discrete parameter describing the choice of \( k \) geodesics out of \( (k + 1) \) segments in the cage). Then we obtain the following recurrent relation between \( l_{k+1} \) and \( l_{k} \), \( l_{k} \leq 2 \cdot l_{k+1} \). This relation will hold for all \( k = q, q - 1, \ldots, 2 \). Now note that originally the length of each edge is bounded by the diameter of the manifold \( d \). Thus, \( l_{q+1} \leq d(1 + a + \ldots a^{q}) = \frac{d(a^{q+1} - 1)}{a - 1} \leq 2da^{q} \). Putting this together we see that \( l_{2} \leq 2q!a^{q}d \).

\[ \square \]

**Example.** Let us separately present the proof of Theorem 0.2 in a simple case of \( q = 2 \). Let \( f : S^{2} \rightarrow M^{n} \) be non-contractive. Let \( a \geq 3 \) be given.

The proof will be by contradiction. We will show that in the absence of stationary “figure 8”, (a flower with two geodesic loops) on \( M^{n} \) of length \( \leq 4a^{3}d \) that contains a loop with an angle that is \( \varepsilon \)-close to \( \pi \), we can extend \( f \) to \( D^{3} \) triangulated as a cone over \( S^{2} \). Recall that \( S^{2} \) is triangulated in such a way that the image of a simplex under \( f \) has diameter at most \( \delta \) for some small \( \delta \) that eventually will approach \( 0 \). The extension will constructed by induction on the skeleta of \( S^{2} \).

**0-skeleton** of \( D^{3} \) consists of the vertices of the triangulation of \( S^{2} \) and of the center of the disc \( \bar{p} \). Thus, we extend to the 0-skeleton by mapping \( \bar{p} \) to an arbitrary point \( p \in M^{n} \).

To extend to the 1-skeleton of \( D^{3} \) assign to an arbitrary 1-edge of the form \([\bar{p}, \bar{v}_{i}]\), where \( \bar{v}_{i} \) is a vertex of the triangulation of \( S^{2} \) a shortest geodesic segment \([p, v_{i}]\), where \( v_{i} = f(\bar{v}_{i}) \) of length at most \( \leq d \) that connects \( p \) with \( v_{i} \).

We extend to the 2-skeleton by assigning to an arbitrary 2-simplex of the form \([\bar{p}, \bar{v}_{i1}, \bar{v}_{i2}]\) the disk generated by the contraction of the image of its boundary to some point. The boundary is mapped to a closed curve of length \( \leq 2d + \delta \). Thus, in the absence of short periodic geodesics it is contractible via the Birkhoff curve shortening process.

Finally, let us extend to the 3-skeleton. Consider an arbitrary 3-simplex \([\bar{p}, \bar{v}_{i1}, \bar{v}_{i2}, \bar{v}_{i3}]\). The image of its boundary was defined on the previous step of the induction. It is a 2-sphere glued from four 2-simplices. One of the four simplices comes from the triangulation of \( S^{2} \), and can, therefore, be made arbitrarily small. One can, thus, contract it to \( q \) over itself. This
allows us to simply treat it as a point $q$. So, one can view the sphere, i.e. the image of the boundary of $[\tilde{p}, \tilde{v}_i, \tilde{v}_{1z}, \tilde{v}_{13}]$ as being formed by connecting two points $p$ and $q$ by geodesic segments $e_1, e_2, e_3$ and then contracting each pair of closed curves to a point. Note also, that this construction provides us with a natural cell decomposition of this sphere into two 0-cells: $p, q$, three 1-cells: $e_1, e_2, e_3$ and three 2-cells. Consider the net that corresponds to the 1-skeleton of this sphere under this decomposition. Let us shorten the weighted length $l(e_1) + a l(e_2) + a^2 l(e_3)$. Under the above weighted length shortening, the net will either contract to a point, or will converge to a critical point, which can either be a stationary figure 8 or a periodic geodesic of length $\leq 4d^2 d$. Thus, assuming there are no critical points, the net can only converge to a point, (see fig. 6). This net shortening generates a 1-parameter family of nets that we will denote $C_\tau$. We can extend $C_\tau$ to one parameter family of 2-spheres, $S^2_\tau$. $S^2_0$ will be the original sphere that corresponds to the image of the boundary of the 3-simplex that we are trying to “fill”, while $S^2_1$ will be a point. We thus will obtain the 3-disk. It now remains to show how to construct $S^2_\tau$. For each $\tau \in [0, 1]$ we consider three pairs of curves and contract them to a point without the length increase. At some point that the length of one or two segments will decrease to zero and segments themselves will degenerate to points. However we can still consider three pairs of curves, where one of the curves in two pairs will be a constant curve. We then fill each of these three pairs of curves by discs as we did above when we were extending to the 2-skeleton, i.e. using the curve shortening process. These 3 2-discs glued together form $S^2_\tau$. Thus, if there is no geodesic stationary figure 8 (or periodic geodesic) of length $\leq 4d$ that is a critical point with respect to $L_\alpha$ then we can extend our map $f$ to the 3-skeleton of $D^3$, reaching a contradiction.

**Figure 6.** Deforming a 2-sphere to a point
In this section we will prove Theorem 0.3. The volume upper bound for the length of a “wide” geodesic loop will follow from the Filling Radius upper bound for the length of this “wide” loop combined with the volume upper bound for the Filling Radius.

In [G] Gromov defines the Filling Radius of $M^n$, $\text{FillRad}M^n$, as the minimal $r$ such that the image of $M^n$ under the Kuratowsky embedding into $L^\infty(M^n)$ bounds in its $r$-neighborhood in $L^\infty(M^n)$. Here Kuratowsky embedding is the map that sends each point $x \in M^n$ to the distance function $d(x, \ast)$.

M. Katz proved that $\text{FillRad}M^n \leq \frac{d}{3}$, where $d$ is the diameter of $M^n$, (see [K]), while M. Gromov had found the first estimate for the filling radius of a closed Riemannian manifold in terms of the volume of $M^n$, (see [G]).

Theorem 2.1. [G] Let $M^n$ be a closed connected Riemannian manifold. Then $\text{FillRad}M^n \leq g(n)(\text{vol}(M^n))^{\frac{1}{n}}$, where $g(n) = (n + 1)n^2(n + 1)!^{\frac{1}{2}}$ and $\text{vol}(M^n)$ denotes the volume of $M^n$.

The constant $g(n)$ was improved to $27^n(n + 1)!$ by S. Wenger in [W]. It was further improved by A. Nabutovsky in [N] to simply $n$.

In this section we will prove the following

Theorem 2.2. Let $M^n$ be a closed Riemannian manifold. For each integer $a \geq 3$ there exists a non-trivial stationary geodesic flower of total length $2(\frac{(n + 1)!}{2})^2 a^{(n + 1)^3} \text{FillRad}M^n$ which is a critical point of the weighted length functional $L_a$.

The last missing ingredient in the proof of Theorem 2.2 is the following straightforward generalization of the Merging Lemma proven in [R4]. It is analogous to the first assertion of Lemma 2.1.

Let $K$ be a net that corresponds to the 1-skeleton of an $m$-simplex. Let $w_0, ..., w_{m+1}$ be its vertices. Consider the set of pairs of vertices $S = \{(w_i, w_j) | 0 \leq i < j \leq m + 1\}$ with an alphabetical order. That is $(w_{i_1}, w_{j_1}) < (w_{i_2}, w_{j_2})$ if and only if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$. Note there is a one-to-one correspondence between the elements of $S$ and the set of edges of $K$. Thus the alphabetical order on $S$ induces an order on the edges of $K$.

Lemma 2.3. (Merging Lemma) Let $K$ be a net corresponding to the 1-skeleton of an $m$-simplex with the vertices $w_0, ..., w_{m+1}$ in a Riemannian manifold $M^n$. Let $e_i, i = 1, ..., \frac{(m+1)(m+2)}{2}$ be the edges, enumerated in the order that corresponds to the alphabetical order on the set $S$ above. Consider the following weighted length functional on $K$: $L(K) =$
\[
\sum_{j=1}^{(m+1)(m+2)} a^{j-1} \text{length}(e_j), \text{ where } a \geq 3. \text{ Then the only non-trivial critical points of this functional are minimal geodesic flowers.}
\]

The proof of the Lemma is presented in [R4] with \( a = 3 \). The proof will not be changed if one substitutes an arbitrary integer \( a \) that is greater than or equal to 3 in lieu of 3, thus it will not be presented in this paper. We will, however, present the proof in the case of \( m = 3 \) for the sake of readability.

**Proof of the Merging Lemma for \( m = 3 \).** Let us begin by demonstrating that a non-degenerate 1-skeleton of a 3-simplex cannot be a critical point of \( L_a = \sum_{i=1}^{6} a^{i-1} \text{length}(e_i) \), (see fig. 7(a)). Let \( v_{ij} \) denote the unit vector tangent to \( e_i \) at \( w_j \). Let us consider in turns the four stationarity conditions at each of the vertex: \( w_0, \ldots, w_3 \).

(1) The stationarity condition at the vertex \( w_0 \) implies that \( v_{10} + a^2 v_{30} + a^3 v_{40} = 0 \);
(2) The stationarity condition at the vertex \( w_1 \) implies that \( v_{11} + a v_{21} + a^4 v_{51} = 0 \);
(3) The stationarity condition at the vertex \( w_2 \) implies that \( a v_{22} + a^2 v_{32} + a^5 v_{62} = 0 \);
(4) Finally, the stationarity condition at the last vertex, \( w_3 \) implies that \( a^3 v_{43} + a^4 v_{53} + a^5 v_{63} = 0 \).

It is obvious that for \( a \geq 3 \) these conditions cannot be satisfied, unless each vertex merges with some other vertex as, for example is depicted in the configurations in fig. 7 (b).

Without loss of generality assume that \( w_0 \) merges with \( w_1 \). This can only happen if the length of edge \( e_1 \) that connects the above vertices decreases to 0. In this case, what can possibly happen with the remaining two vertices \( w_2 \), and \( w_3 \)? Since we know that they have to merge with some other vertices, either \( w_2 \) and \( w_3 \) merge together, i.e. the length of edge \( e_6 \) also decreases to 0, (see fig. 7) or all of the four vertices join together, and the net becomes a flower.

Thus, it remains to show that the net depicted in fig. 7 (b) cannot satisfy the stationarity condition.

In this case the following stationarity condition should be satisfied at the vertex \( w_0 = w_1 = w_2 = w_3 = w_4 = w_5 = w_6 \), (and a similar stationarity condition should be satisfied at \( w_2 = w_3 \). Even if the three edges \( e_2, e_3, e_4 \) which appear with the lower weights all merge together, it would not be enough to compensate for the weight of \( e_5 \). Thus the only way the stationarity condition can be satisfied is if all of the vertices merge together.

\[ \square \]

**Theorem 2.2** combined with **Theorem 2.1** leads to the volume bound for the length of the smallest critical point of \( L_a \). By the Merging lemma, this
Figure 7. The graphs below cannot minimize the weighted length functional $l(\Gamma) = \sum_{j=1}^{6} a^{j-1} e_j$.

Critical point will be a geodesic flower. Moreover, it will follow from the second statement of Lemma 1.1 that one of the geodesic loops will have an angle that is within $\varepsilon$ from $\pi$.

We are now ready to present the proof of Theorem 2.2. Let us outline its proof. Let us fill $M^n$ by a polyhedron $P^{n+1}$ in $L^\infty(M^n)$. Let $\text{id} : M^n \rightarrow M^n$ be the identity map on $M^n$. It is impossible to extend the above map to $P^{n+1}$. Thus, attempting to extend the identity map on $M^n$ to $W^{n+1}$ should fail, and the required geodesic flower that is a critical point of $L_a$ will be an obstruction to this extension. The proof is similar to the proof of the analogous statement of [R4].

The difference between the proofs of Theorems 0.2 and 2.2 is that instead of contracting $m$-cages, we will be contracting 1-skeleta of simplices. The contraction of 1-skeleta will generate the 1-parameter family of (not geodesic) nets. Correspondingly, we can build the spheres and the discs out of these 1-skeleta. The weighted length functionals used to contract these nets will be $\sum_{i=1}^{(n+1)(n+2)/2} a^{i-1} \text{length}(e_i)$, where $e_i$'s are the edges of the 1-skeleton of an $(n+1)$-dimensional simplex, satisfying the condition that the edge $e_1$ is coming out of the vertex $w_0$, edges $e_2, e_3$ are coming out of the vertex $w_1$, edges $e_4, e_5, e_6$ are coming out of the vertex $w_3$, etc. Finally, the last $(n+1)$ edges are coming out of the same vertex $w_{n+1}$. As was stated before, the functional will force the net to degenerate into a flower, and one of the loops of the flower to become wide.

Proof of Theorem 2.2. Suppose there are no “small” critical geodesic flowers satisfying the conditions stipulated in the statement of the theorem.

Let $P^{n+1} \subset L^\infty(M^n)$ be a polyhedron that satisfies:

1. $\partial P^{n+1} = M^n$, when $M^n$ is orientable, and $\partial P^{n+1} = M^n \mod 2$, when $M^n$ is not orientable;

2. $\partial P^{n+1} \subset (\text{FillRad} M^n + \delta)$-neighborhood of $M^n$ for an arbitrarily small $\delta > 0$, (see [G]).

Suppose $P^{n+1}$ is triangulated, and that the diameter of any simplex in this triangulation is smaller than $\delta$. Note that we obtain a triangulation of $M^n$ by restricting the triangulation of $P^{n+1}$.
We will obtain geodesic flowers of “small” length, and with at least one “wide” geodesic loop as an obstruction to the extension of the identity map \( id : M^n \rightarrow M^n \) to \( P^{n+1} \).

This extension is constructed by induction on the dimension of the skeleta of \( P^{n+1} \).

As in the proof of Theorem 0.2, we will first extend to the 0-skeleton. Assign to each \( \tilde{p}_i \in P^{n+1} \), the vertex of the triangulation of \( P^{n+1} \) the vertex \( w_i \in M^n \) which is closest to \( \tilde{p}_i \). In case there is more than one vertex in the triangulation of \( M^n \) that is closest to \( \tilde{p}_i \) we can arbitrarily choose any one. Therefore, \( d(\tilde{p}_i, w_i) \leq \text{FillRad}M^n + \delta \). For the sake of the future reference we can call this step 0 of the extension procedure.

Next, we will extend to the 1-skeleton, (step 1). Let \( [\tilde{p}_i, \tilde{p}_j] \subset P^{n+1} \setminus M^n \) denote a 1-simplex in the triangulation of \( P^{n+1} \). We will assign to it a minimal geodesic segment \( [w_i, w_j] \) that connects \( w_i \) and \( w_j \) of length \( \leq 2\text{FillRad}M^n + 3\delta \). Again, in case there are several minimizing geodesics connecting \( w_i, w_j \), we can choose any one of them.

Next we will extend to the 2-skeleton (step 2). Let \( \tilde{\sigma}_{i_0,i_1,i_2} = [\tilde{p}_{i_0}, \tilde{p}_{i_1}, \tilde{p}_{i_2}] \) be an arbitrary 2-simplex. Its boundary is mapped to a closed curve made out of three geodesic segments that we obtained during step 1 of the extension. The total length of the boundary is at most \( \leq 6\text{FillRad}M^n + 9\delta \). If there is a periodic geodesic of length at most \( 6\text{FillRad}M^n + 9\delta \), the conclusion of Theorem 2.2 is satisfied. In case there are no periodic geodesics on \( M^n \) satisfying the above bound we can contract this curve to a point along the curves of smaller length. Moreover, the absence of “short” periodic geodesics implies that this curve shortening homotopy continuously depends on the initial curve. We will map \( \tilde{\sigma}_{i_0,i_1,i_2} \) to a surface denoted as \( \sigma_{i_0,i_1,i_2} \), that is generated by the above homotopy.

Next let us extend to the 3-skeleton (step 3). Consider an arbitrary 3-simplex \( \tilde{\sigma}_{i_0,i_1,i_2,i_3} = [\tilde{p}_{i_0}, ..., \tilde{p}_{i_3}] \). We know that the boundary of this simplex is mapped to the chain: \( \Sigma_{j=0}^3 (-1)^j \sigma_{i_{j_1},i_{j_2},i_{j_3}} \) by the previous step of the induction. Its one skeleton is a net that we will denote by \( K \) defined in Step 1. Apply the weighted length shortening process for nets. In the absence of the critical geodesic flowers it will be continuously deformed to a point. The weighted length of \( K \) is defined as \( \Sigma_{j=1}^6 a^{j-1} \text{length}(e_j) \), where \( e_j \) is an edge of the 1-skeleton. (We will not explicitly describe this length shortening process, but as we have mentioned before, it can be found in [NR1]).

By the Merging Lemma above only geodesic flowers can be critical points for such a functional.

Next for each time \( \tau \) one can construct a 2-dimensional sphere \( S^2 \) corresponding to each \( K_{\tau} \) obtained by shortening \( K = K_0 \). The sphere is constructed in a way that is analogous to a similar construction in the proof.
of Theorem 0.2. That is for each $\tau \in [0, 1]$ consider $K_\tau$. Consider all triples of the edges that correspond to the boundary of the face of the three simplex. Apply the curve shortening to each of the closed curves formed by these curves. In the absence of short periodic geodesics the process will converge to a point. Thus, we will obtain the 2-discs that these homotopies will generate. Glue those discs together along the common edges to obtain $S_\tau^2$. This 1-parameter family of 2-spheres can be regarded as a 3-disc that we will denote as $\sigma_{i_0, \ldots, i_3}^3$. We will assign it to $\tilde{\sigma}_{i_0, \ldots, i_3}^3$.

If we continue this inductive procedure until we reach the $(n+1)$-skeleton of $P$. We will obtain a singular chain on $M^n$, that has the fundamental class $[M^n]$ as its boundary, and therefore, arrive at a contradiction.

Suppose we have extended the identity map $id : M^n \to M^n$ to the $k$-skeleton of $P^{n+1}$ and now want to extend it to the $(k + 1)$-skeleton. Take an arbitrary $(k + 1)$-simplex of $P^{n+1}$. Let us denote by $N$ the image of its 1-skeleton. To extend the map to this simplex, we will construct a $(k + 1)$-dimensional disc that fills $N$. $N$ consists of $\frac{(k+2)(k+1)}{2}$ edges. Each edge has a weight assigned to it as in the Merging Lemma above. Assuming there is no “small” geodesic flowers, $N$ can be deformed to a point along the 1-parameter family of nets $N_\tau, \tau \in [0, 1]$, where the weighted length of $N_\tau$ decreases with $\tau$. Next for each $\tau$ we construct a sphere $S_k^k$ that fills $N_\tau$. It is constructed by constructing $k$-dimensional discs and gluing them as in the boundary of a $(k + 1)$-dimensional simplex. In order to construct those discs, we consider subnets that are obtained by ignoring a vertex and all the edges that are coming out of this vertex.

The maximal length of each edge in a subnet is $\leq 2\text{FillRad} M^n \frac{(k+2)(k+1)}{2} a^{\frac{(k+2)(k+1)-2}{2}}$, i.e. the maximal number of edges in $N_\tau$ times the maximal weighted length of the edges in $N_\tau$. Thus, the total weighted length of the whole subnet is $\leq 2\text{FillRad} M^n \frac{(k+1)(k+2)(k+1)}{2} a^{\frac{(k+1)(k+2)-2}{2}} a^{\frac{(k+2)(k+1)-2}{2}}$. Proceeding in the manner starting from the $(n + 1)$-skeleton of $P^{n+1}$ we would obtain a bound of $2\text{FillRad} M^n \frac{(n+2)(n+1)}{2} a^{\Sigma_{j=0}^{(n+2)}(k+2)(k+1)-2} \leq 2\text{FillRad} M^n (n+1)^2 a^{(n+1)^3}$. Note also that the maximal number of geodesic loops in the geodesic flower can be estimated by the number of edges in $N$, but taken with multiplicities that correspond to weights, thus it is bounded by the sum $\Sigma_{j=1}^{(n+2)(n+1)} a^{j-1} \leq a^{(n+1)^2}$. □

Proof of Theorem 0.3. We combine the assertions of Theorem 2.2 and Lemma 1.1. The desired assertion follows when we substitute $a = \max \{ \left\lfloor \frac{1}{\sin \frac{\pi}{2}} \right\rfloor + 1, 3 \} + \delta$ in the inequality of Theorem 2.2 and then take the limit as $\delta \to 0$. □
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