THE DEPENDENCE OF THE HELICITY BOUND OF FORCE-FREE MAGNETIC FIELDS ON BOUNDARY CONDITIONS

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ABSTRACT

This paper follows up on a previous study showing that in an open atmosphere such as the solar corona, the total magnetic helicity of a force-free field must be bounded, and the accumulation of magnetic helicity in excess of its upper bound would initiate a nonequilibrium situation resulting in an explosion such as a coronal mass ejection (CME). In the current paper, we investigate the dependence of the helicity bound on the boundary condition for several families of nonlinear force-free fields. Our calculation shows that the magnitude of the helicity upper bound of force-free fields is nontrivially dependent on the boundary condition. Fields with a multipolar boundary condition can have a helicity upper bound 10 times smaller than those with a dipolar boundary condition when helicity values are normalized by the square of their respective surface poloidal fluxes. This suggests that a coronal magnetic field may erupt into a CME when the applicable helicity bound falls below the already accumulated helicity as the result of a slowly changing boundary condition. Our calculation also shows that a monotonic accumulation of magnetic helicity can lead to the formation of a magnetic flux rope applicable to kink instability. This suggests that CME initiations by exceeding helicity bound and by kink instability can both be the consequences of helicity accumulation in the corona. Our study gives insights into the observed associations of CMEs with the magnetic features at their solar surface origins.

Subject headings: MHD — Sun: corona — Sun: coronal mass ejections (CMEs) — Sun: magnetic fields

1. INTRODUCTION

Magnetic helicity is a physical quantity that measures the topological complexity of a magnetic field, such as the degree of linkage and/or twistedness in the field (Moffatt 1985; Berger & Field 1984). In a previous paper (Zhang et al. 2006, hereafter ZFL06) we proposed that in an open atmosphere such as the solar corona, there is an upper bound on the total magnetic helicity that a force-free field can contain. The accumulation of magnetic helicity in excess of this upper bound would initiate a nonequilibrium situation, resulting in a coronal mass ejection (CME). In the current paper, we investigate the dependence of the helicity bound on the boundary condition for several families of nonlinear force-free fields. Our calculation shows that the magnitude of the helicity upper bound of force-free fields is nontrivially dependent on the boundary condition. Fields with a multipolar boundary condition can have a helicity upper bound 10 times smaller than those with a dipolar boundary condition when helicity values are normalized by the square of their respective surface poloidal fluxes. This suggests that a coronal magnetic field may erupt into a CME when the applicable helicity bound falls below the already accumulated helicity as the result of a slowly changing boundary condition. Our calculation also shows that a monotonic accumulation of magnetic helicity can lead to the formation of a magnetic flux rope applicable to kink instability. This suggests that CME initiations by exceeding helicity bound and by kink instability can both be the consequences of helicity accumulation in the corona. Our study gives insights into the observed associations of CMEs with the magnetic features at their solar surface origins.

2. THE MODEL

2.1. The Governing Equation

Following Flyer et al. (2004) and ZFL06, we use the families of power-law axisymmetric force-free fields to understand the basic physical properties of interest.

With axisymmetry, the solenoidal magnetic field $B$ in $r > 1$ can be written in the form

$$B = \frac{1}{r \sin \theta} \left[ \frac{1}{r} \frac{\partial A}{\partial \theta} - \frac{\partial A}{\partial r}, Q(A) \right],$$

where the flux function $A$ defines the poloidal magnetic field and the function $Q$ defines the toroidal (or azimuthal) field. $Q$ is defined as a strict power law in $A$ with the form

$$Q^2(A) = \frac{2\gamma}{n+1}A^{n+1},$$

where $n$ is an odd constant index that must be not less than 5 in order for the field to possess finite magnetic energy in $r > 1$, and $\gamma$ is a free parameter that we choose to be positive without loss of generality. This form of $Q$ reduces the force-free condition to the following governing equation for the flux function $A$:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \mu^2} + \gamma A^n = 0.$$\hspace{1cm}(3)

This governing equation was solved numerically as a boundary value problem within the domain $r > 1$ in Flyer et al. (2004) and ZFL06, subject to the prescribed boundary flux distribution of

$$A|_{r=1} = \sin^2 \theta.$$\hspace{1cm}(4)

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The first new boundary condition has the flux distribution of
\[ A|_{r=1} = \sin^2 \theta. \] (5)

This flux distribution and its associated normal field distribution are plotted in the middle panels of Figure 1. We see that this new boundary condition has its flux concentrated nearer to the equator than that of the dipolar field. This makes it more like solar active regions, with its normal field strength much higher at equatorial regions than that near the poles. Below we will refer this family of power-law axisymmetric force-free fields as bipolar force-free fields, or bipolar fields.

The second new family of solutions are those that we will refer to as multipolar force-free fields, or multipolar fields. They are also the solutions to equation (3), but subject to the boundary condition
\[ A|_{r=1} = \sin^2 \theta \times (1 - 5 \cos^2 \theta). \] (6)

Its flux distribution and associated normal field distribution are plotted in the bottom panels of Figure 1. We see that fields with this boundary condition have both positive and negative magnetic fields in one hemisphere, distinctively different from those in dipolar force-free fields and bipolar force-free fields, where fields in one hemisphere have the same magnetic polarity.

Table 1 presents a comparison of the three boundary conditions. We see that their common feature is that they all have \( A|_{r=1} = 0 \) at solar poles and \( A|_{r=1} = 1 \) at the equator. The dipolar boundary condition and bipolar boundary differ by the contrast of their respective maximum \( B_r|_{r=1} \) values to their \( B_r|_{r=1} \) values at the northern pole. The multipolar boundary condition differs from the dipolar and bipolar boundary conditions by the existence of both magnetic polarities in each hemisphere, indicated by its negative minimum \( A \) value.

In Table 1 we also present the total surface poloidal flux \( (F_p) \) for the three boundary conditions, where
\[ F_p = \int_{r=1} B_r (> 0) \, ds = 2\pi \int_0^{\pi/2} |B_r| \sin \theta \, d\theta. \] (7)

We see that the dipolar and bipolar fields have the same total surface poloidal flux \( (2\pi) \), and the multipolar fields have a larger total surface poloidal flux \( (5.2\pi) \). In the following sections we normalize the calculated magnetic helicity by \( F_p^2 \) in order to make fields with different boundary conditions comparable.

### 3. RESULTS AND ANALYSIS

As in ZFL06, for each new boundary condition, we numerically solve equation (3) for three cases: \( n = 5, 7, \) and 9. In each case, the numerical method, that is, the Newton’s iteration combined with a pseudo-arc-length continuation scheme, guarantees the completeness of each solution branch generated by the \( \gamma \) values.

| TABLE 1 |
| --- |
| **Comparison of the Three Boundary Conditions** |
| **Boundary Condition** | \( A \) (at \( r = 1 \)) | \( B_r \) (at \( r = 1 \)) |
| **Min.** | **Max.** | **Min.** | **Max.** | **Min.** | **F_p** |
| Dipolar | 0.8 | 1.0 | 0 | 2 | 0 | 2 | 2π |
| Bipolar | 0.8 | 1.0 | 0 | 2.25 | 0 | 8 | 2π |
| Multipolar | 0.8 | 1.0 | 0 | 2.25 | 0 | 8 | 5.2π |
Also as in ZFL06, for each solution obtained, we calculate three physical quantities of the field: the total magnetic energy
\[ E = \int_{r>1} \frac{B_r^2}{8\pi} dV = \frac{1}{4} \int_{r=1} (B_r^2 - B_\theta^2 - B_\varphi^2) \sin \theta \, d\theta, \tag{8} \]

total azimuthal flux
\[ F_\varphi = \int_{r>1} B_\varphi r \, dr \, d\theta = \sqrt{\frac{\gamma}{m+1}} \int_{r>1} [A_n^{m+1} \, dr \, d\theta / \sin \theta], \tag{9} \]

and total relative magnetic helicity
\[ H_R = 4\pi \int_{r>1} AB_\varphi r \, dr \, d\theta = 4\pi \sqrt{\frac{\gamma}{m+1}} \int_{r>1} A_{n+1}^{m+2} \, dr \, d\theta. \tag{10} \]

The derivation of these formulae can be found in ZFL06. The only difference is that since we are also considering multipolar fields, the absolute value of \( B_\varphi \) (that is, \( |B_\varphi| \)) is introduced when calculating the total azimuthal flux.

In the geometric simplicity of these force-free fields, the equilibrium in each case is due to the magnetic tension force of the poloidal flux confining the magnetic pressure of the azimuthal equilibrium in each case is due to the magnetic tension force of the fields, the absolute value of only difference is that since we are also considering multipolar fields, the values of total magnetic helicity (but not yet enough for an eruption), then a change of the boundary condition could lower the helicity upper bound, resulting in a nonequilibrium situation and hence a CME eruption under the new boundary condition. For example, if a CME-type eruption can be triggered by various surface field variations (e.g., Chen & Shibata 2000; Amari et al. 2000, 2003a, 2003b).

For example, if \( H_R / F_\varphi^2 \) were 0.3 for a dipolar boundary condition, then an evolutionary change to a bipolar boundary condition would result in a CME eruption, because the applicable upper bound on the

![Figure 2](image-url)
conserved total helicity has been reduced as suggested by our numerical experiments.

A note to address here is that although in this paper we emphasize the role of boundary condition variations, this does not mean that the role of magnetic helicity accumulation becomes less important. A change of the boundary condition may bring in an eruption only when the field has accumulated enough helicity for an eruption under the new boundary condition. If not, the field does not erupt even when the boundary flux distribution is changing. This is consistent with the observation (Zhang et al. 2007) that although flux emergences are indeed found to be associated with CME eruptions, the same rate of flux emergence can also be found when there is no CME or solar activities. This means that flux emergence may be a trigger of a CME eruption, but flux emergence alone does not guarantee an eruption.

Another interesting result of our calculations is that these normalized $H_R/F_p^2$ helicities, estimated from simple axisymmetric power-law force-free fields, lie close to those estimated from observations. Van Driel-Gesztelyi et al. (2003) and Demoulin (2007) pointed out that the $H_R/F_p^2$ numbers, estimated from the extrapolated magnetic fields based on observed photospheric magnetograms, are between 0.02 to 0.2. Our helicity upper bound numbers for dipolar and bipolar fields are just a little higher. Note that the numbers estimated from the observations may be somehow underestimated, because of the limited spatial resolution of the observed magnetograms. So there seems to be consistency between the theoretical $H_R/F_p^2$ helicities and those estimated from observations.

Figure 4 presents four field configurations selected from the $n = 9$ solution curve, positions of which along the curve are illustrated in Figure 2. We see that starting from the potential field (panel A) the curve first reaches the maximum-$\gamma$ field (panel B), then the field with maximum total magnetic energy (panel C), and finally the one with maximum total azimuthal flux (panel D). A clear bubble (representing a flux rope) is presented in the field of panel C but not in the field of panel D, the latter actually possessing more total azimuthal flux than the former. This tells us that although the existence of a flux rope (or

![Figure 3](image_url)

**Fig. 3.**—Variation of the total magnetic helicity ($H_R$) vs. azimuthal flux ($F_p$) along the solution curve for fields with the dipolar (top) and bipolar (bottom) boundary conditions. Here the total magnetic helicity ($H_R$) of each field has been normalized by the square of their corresponding poloidal flux ($F_p^2$).
flux ropes) in the low corona does represent storage of a certain amount of magnetic helicity (see discussions in Zhang & Low 2003), it is not necessary that they are present in the field with a maximum helicity storage.

3.2. Helicity Upper Bound of Multipolar Fields

As in Figure 2, in Figure 5 we present the variations of the total magnetic energy, total azimuthal flux, and total magnetic helicity versus $C_1$ along the solution curve with $n = 5, 7, 9$, but for the multipolar fields. We see that these curves also suggest the existence of upper bounds on the total magnetic energy, total azimuthal flux, and total magnetic helicity, similar to those for dipolar and bipolar fields.

The figure also shows that the magnitude of the helicity upper bound for multipolar fields is smaller than that for dipolar fields. Such a severe reduction of the $H_R/F_p^2$ upper bound not only further confirms our previous result that the helicity upper bound is dependent on the boundary condition, but also brings our theoretical $H_R/F_p^2$ value even closer to those estimated from observations (Regnier et al. 2005). Furthermore, the severe reduction of the helicity upper bound in terms of $H_R/F_p^2$ values may also explain why solar eruptions such as CMEs are more likely to happen in complicated active regions where the multipolar field, by the above property, will take less time to reach its helicity bound, producing an eruption.

As in Figure 4, Figure 7 presents four field configurations selected from the $n = 7$ solution curve of the multipolar fields. We see again a clear bubble in the field of panel C, but not in panel D. Therefore, as with bipolar fields, multipolar fields with maximum helicity storage need not contain a flux rope.

3.3. Kink Instability

A rope of highly helical field is susceptible to an instability that causes the rope to kink (Friedberg 1987). From elementary
calculations, this kink instability sets in if a critical twist is exceeded ($T > T_c$). The exact value of $T_c$ depends on the detailed field models, and could increase from the traditional Kruskal-Shafranov limit $T_c = 2\pi$ to $2.5\pi$ (Hood & Priest 1981) or $4.8\pi$ (Mikic et al. 1990).

Since we have helical flux tubes (or flux ropes) in our solutions, it is interesting to investigate whether these flux ropes have exceeded the kink instability. Figure 8 presents the variation of the average twist ($T$) versus $\gamma$ for two fields. One of the fields is the bipolar $n = 9$ maximum-energy field, presented in panel C of Figure 4. The other is the multipolar $n = 7$ maximum-energy field, presented in panel C of Figure 7. Here $\theta_0$ is the angle from the equator. The average twist ($T$) is obtained from $H_r^0/F_0^p$, where $H_r^0$ is the total relative magnetic helicity in the domain $\Omega'$, enclosed by the $r = 1$ surface and the flux surface with $A = A|_{r=1,\theta=\theta_0}$, and $F_0^p$ is the total surface ($r = 1$) poloidal flux of the domain $\Omega'$.

We see that in both fields the average twist of the central part of the field (that is, where the flux rope is located) has exceeded the kink instability criteria, $T_c = 2.5\pi$ of Hood & Priest (1981) or $T_c = 4.8\pi$ of Mikic et al. (1990). This tells us that with the accumulation of a certain amount of magnetic helicity, the flux...
Fig. 6.—Variation of the total magnetic helicity ($H_B$), normalized by the square of poloidal flux ($F_p^2$), vs. azimuthal flux ($F_\phi$) for fields with the multipolar boundary condition.

Fig. 7.—Same as Fig. 4, but for the $n = 7$ multipolar case.
rope formed in the field can possess a twist number that is larger than the kink instability criteria. If other necessary conditions are favorable, for example, if the field has accumulated enough free magnetic energy, an eruption may happen even before the field has reached its helicity upper bound state. In that sense, reaching the helicity upper bound state may not be a necessary condition for eruption, but the helicity upper bound is a sufficient condition for an eruption to become unavoidable. This also shows that CME eruptions that are initiated by the kink instability (e.g., Torok & Kliem 2005; Fan & Gibson 2007) or by the existence of a helicity upper bound could both be viewed as the consequences of magnetic helicity accumulation, and that they are not mutually exclusive.

4. CONCLUSION

In this paper, we continue our study on the hydromagnetic origin of CMEs in terms of magnetic helicity accumulation. As in a previous paper (ZFL06), we numerically solve equation (3) to get families of axisymmetric power-law force-free fields, but subject to two new boundary conditions.

By analyzing and comparing obtained solutions for three different boundary conditions, we conclude the following:

1. The suggestion that there may be an upper bound on the total magnetic helicity for force-free fields is also found for the two new boundary conditions.

2. The magnitude of the helicity upper bound of force-free fields is nontrivially dependent on the boundary condition. In our examples, the fields with a surface flux distribution more like a simple active region (bipolar fields) have helicity upper bounds that are smaller than those of fields with dipolar boundary conditions. For multipolar fields, the helicity upper bound \( \mathcal{H}_H/F^2 \) can be 10 times smaller than that of a dipolar field. These results provide some insights into the observed association of CMEs with flux emergence and surface field variation. These results also suggest a physical reason why eruptions are more likely to happen in complicated active regions.

3. CME initiations by kink instability and by the existence of a helicity upper bound can both be the result of magnetic helicity accumulation in the corona. They do not exclude each other.

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