Scharlemann-Thompson untelescoping of Heegaard splittings is finer than Casson-Gordon’s

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1 Introduction

Let $H_1 \cup_P H_2$ be a Heegaard splitting of a closed 3-manifold $M$, i.e., $H_i$ $(i = 1, 2)$ is a handlebody in $M$ such that $H_1 \cup H_2 = M$, $H_1 \cap H_2 = \partial H_1 = \partial H_2 = P$. In [13], M.Scharlemann, and A.Thompson had introduced a process for spreading $H_1 \cup_P H_2$ into a “thinner” presentation. The idea was polished to show that if the original Heegaard splitting is irreducible, then we can spread it into a series $(A_1 \cup_P B_1) \cup \cdots \cup (A_n \cup_P B_n)$ such that each $A_i \cup_P B_i$ is a strongly irreducible Heegaard splitting. In this paper, we call this series of strongly irreducible Heegaard splittings a Scharlemann-Thompson untelescoping (or S-T untelescoping) of $H_1 \cup_P H_2$. On the other hand, preceding [13], A.Casson, and C.Gordon [2] had proved that if $H_1 \cup_P H_2$ is weakly reducible and not reducible, then there exists an incompressible surface of positive genus in $M$. This result is proved by using the following argument.

Let $\Delta = \Delta_1 \cup \Delta_2$ be a weakly reducing collection of compressing disks for $P$ (for the definitions of the terms, see section 2). Then $P(\Delta)$ denotes the surface obtained from $P$ by compressing along $\Delta$. Let $\hat{P}(\Delta)$ be the surface obtained from $P(\Delta)$ by discarding the components that are contained in either $H_1$ or $H_2$. Suppose that $\Delta$ has minimal complexity (for the definition of the complexity, see section 4). Then we can show that the irreducibility of $H_1 \cup_P H_2$ implies that no component of $\hat{P}(\Delta)$ is a 2-sphere. Then, by using a relative version of Haken’s theorem [3], we can show that $\hat{P}(\Delta)$ is incompressible.

With adopting the above notations, we will see, in section 4, that the closure of each component of $M - \hat{P}(\Delta)$ naturally inherits a Heegaard splitting from $H_1 \cup_P H_2$ if $\hat{P}(\Delta)$ contains no 2-sphere component. Hence we obtain a series of Heegaard splittings, say $(C_1 \cup_{Q_1} D_1) \cup \cdots \cup (C_m \cup_{Q_m} D_m)$. If $\hat{P}(\Delta)$ is incompressible, then this series is called a Casson-Gordon untelescoping (or C-G untelescoping) of $H_1 \cup_P H_2$. Then we will also see that C-G untelescoping of $H_1 \cup_P H_2$ can be regarded as one that appears in a process for obtaining S-T untelescoping from $H_1 \cup_P H_2$ (Remark 4.1).

We note that these two untelescoping are used in many articles and equally useful. For example, C-G untelescoping was used by M.Boileau, and J.-P.Otal [4] for studying Heegaard splittings of the 3-dimensional torus, by J.Schultens [15] for studying Heegaard splittings of $(\text{surface}) \times S^1$, by M.Lustig, and Y.Moriah [8] for studying the exteriors of wide knots and links, and by the author [7] for studying the Heegaard splittings of the exteriors of two bridge knots. S-T untelescoping was used, for example, by Scharlemann-Schultens [12], Schultens [13], Morimoto [1], and Morimoto-Schultens [10] for studying the Heegaard splittings...
of the exteriors of non-prime knots. However, it seems that it is not known that whether these two concepts are the same one or not.

Hence, it is natural to ask:

Question. Are these two untelescopings essentially the same?

Since C-G untelescoping of $H_1 \cup_P H_2$ can be regarded as an untelescoping that appears in a process for obtaining S-T untelescoping from $H_1 \cup_P H_2$, the above question can be strengthened as in the following form.

Question’. Is S-T untelescoping essentially finer than C-G untelescoping?

The purpose of this paper is to show that the answer to Question’ is positive.

Theorem. There exist infinitely many closed, orientable, Heegaard genus 4 3-manifolds such that each 3-manifold $M$ admits a genus 4 Heegaard splitting $V \cup_P W$ with the following properties.

1. There is a S-T untelescoping of $V \cup_P W$ which decomposes $M$ into three pieces, say $(V_1 \cup_{P_1} W_1) \cup (V_2 \cup_{P_2} W_2) \cup (V_3 \cup_{P_3} W_3)$.

2. The Heegaard splitting $V \cup_P W$ is decomposed into exactly two pieces by any C-G untelescoping.

3. There is a C-G untelescoping $(V_1' \cup_{P_1'} W_1') \cup (V_2' \cup_{P_2'} W_2')$ of $V \cup_P W$, such that $V_1 \cup_{P_1} W_1$ is the Heegaard splitting that appeared in the above 1, and that $V_2' \cup_{P_2'} W_2'$ is weakly reducible. Moreover, $(V_2 \cup_{P_2} W_2) \cup (V_3 \cup_{P_3} W_3)$ is a C-G untelescoping of $V_2 \cup_{P_2} W_2$, where $V_2 \cup_{P_2} W_2$ and $V_3 \cup_{P_3} W_3$ are Heegaard splittings that appeared in the above 1.

2 Preliminaries

Throughout this paper, we work in the piecewise linear category. For a submanifold $H$ of a manifold $M$, $N(H,M)$ denotes a regular neighborhood of $H$ in $M$. When $M$ is well understood, we often abbreviate $N(H,M)$ to $N(H)$. Let $N$ be a manifold embedded in a manifold $M$ with $\dim N = \dim M$. Then $\Fr_M N$ denotes the frontier of $N$ in $M$. For the definitions of standard terms in 3-dimensional topology, we refer to [1] or [2].

A 3-manifold $C$ is a compression body if there exists a compact connected closed surface $F$ such that $C$ is obtained from $F \times [0,1]$ by attaching 2-handles along mutually disjoint simple closed curves in $F \times \{1\}$ and capping off the resulting 2-sphere boundary components which are disjoint from $F \times \{0\}$ by 3-handles. The subsurface of $\partial C$ corresponding to $F \times \{0\}$ is denoted by $\partial_+ C$. Then $\partial_- C$ denotes the subsurface $\partial C - \partial_+ C$ of $\partial C$. A compression body $C$ is said to be trivial if $C$ is homeomorphic to $F \times [0,1]$ with $\partial_- C$ corresponding to $F \times \{0\}$. A compression body $C$ is called a handlebody if $\partial_- C = \emptyset$. A compressing disk $D(\subset C)$ of $\partial_+ C$ is called a meridian disk of the compression body $C$.

Remark 2.1. The following properties are known for compression bodies.
1. The compression bodies are irreducible.

2. By extending the cores of the 2-handles in the definition of the compression body $C$ vertically to $F \times [0,1]$, we obtain a union of mutually disjoint meridian disks $D$ of $C$ such that the manifold obtained from $C$ by cutting along $D$ is homeomorphic to a union of $\partial_- C \times [0,1]$ and some (possibly empty) 3-balls. This gives a dual description of compression bodies. That is, a connected 3-manifold $C$ is a compression body if there exists a compact (not necessarily connected) closed surface $F$ without 2-sphere components and a union of (possibly empty) 3-balls $B$ such that $C$ is obtained from $F \times [0,1] \cup B$ by attaching 1-handles to $F \times \{0\} \cup \partial B$. We note that $\partial_- C$ is the surface corresponding to $F \times \{1\}$.

Let $N$ be a cobordism between two closed surfaces $F_1, F_2$ (possibly $F_1 = \emptyset$ or $F_2 = \emptyset$), i.e., $F_1 \cup F_2$ is a partition of the components of $\partial N$.

**Definition 2.1.** We say that $C_1 \cup_P C_2$ (or $C_1 \cup C_2$) is a Heegaard splitting of $(N, F_1, F_2)$ (or simply, $N$) if it satisfies the following conditions.

1. $C_i$ ($i = 1, 2$) is a compression body in $N$ such that $\partial_- C_i = F_i$,
2. $C_1 \cup C_2 = N$, and
3. $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$.

The surface $P$ is called a Heegaard surface of $(N, F_1, F_2)$ (or, $N$). The genus of $P$ is called the genus of the Heegaard splitting.

**Definition 2.2.**

1. A Heegaard splitting $C_1 \cup_P C_2$ is reducible if there exist meridian disks $D_1, D_2$ of the compression bodies $C_1, C_2$ respectively such that $\partial D_1 = \partial D_2$

2. A Heegaard splitting $C_1 \cup_P C_2$ is weakly reducible if there exist meridian disks $D_1, D_2$ of the compression bodies $C_1, C_2$ respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. If $C_1 \cup_P C_2$ is not weakly reducible, then it is called strongly irreducible.

3. A Heegaard splitting $C_1 \cup_P C_2$ is trivial if either $C_1$ or $C_2$ is a trivial compression body.

### 3 Scharlemann-Thompson untelescoping

Let $C_1 \cup_P C_2$ be a Heegaard splitting of $(M, F_1, F_2)$. By 2 of Remark 2.1, we see that $C_1$ is obtained from $F_1 \times [0,1] \cup 0$-handles by attaching 1-handles. Recall that $C_2$ is obtained from $\partial_+ C_2 \times [0,1]$ by attaching 2-handles, and 3-handles. Then, by using an isotopy which pushes $\partial_+ C_2 \times [0,1]$ out of $C_2$, we identify $\partial_+ C_2 \times [0,1]$ with $N(\partial_+ C_1, C_1)$. This identification together with the above handles gives the following handle decomposition of $M$.

$$M = F_1 \times [0,1] \cup (0\text{-handles}) \cup (1\text{-handles}) \cup (2\text{-handles}) \cup (3\text{-handles})$$

We note that there are huge variety of ways for giving handle decompositions for $H_1, H_2$. Suppose that:
there exists a proper subset of the 0-handles $\cup$ 1-handles such that some subset the 2-handles$\cup$3-handles do not intersect the 0-handles and 1-handles at all.

Then we can arrange the order of the handles non-trivially to obtain submanifolds $N_1, \ldots, N_n$ such that

\[N_1 = (F_1^{(1)} \times [0, 1]) \cup (0\text{-handles}) \cup (1\text{-handles}) \cup (2\text{-handles}) \cup (3\text{-handles}),\]

\[N_2 = N_1 \cup (F_1^{(2)} \times [0, 1]) \cup (0\text{-handles}) \cup (1\text{-handles}) \cup (2\text{-handles}) \cup (3\text{-handles}),\]

\[\vdots \]

\[N_n = N_{n-1} \cup (F_1^{(n)} \times [0, 1]) \cup (0\text{-handles}) \cup (1\text{-handles}) \cup (2\text{-handles}) \cup (3\text{-handles}),\]

where $F_1^{(1)} \cup F_1^{(2)} \cup \cdots \cup F_1^{(n)}$ is a partition of the components of $F_1$, and each handle is one from the handle decompositon of $M$. Suppose that the following properties are satisfied.

1. $N_1$ is connected, and $N_n = M$.

2. At each stage $k$ ($2 \leq k \leq n$), let $\partial_k^-$ denotes the union of the components of $\partial N_{k-1}$ to which the 1-handles are attached. Then $\partial_k^- \cup (F_1^{(k)} \times [0, 1]) \cup (0\text{-handles}) \cup (1\text{-handles}) \cup (2\text{-handles}) \cup (3\text{-handles})$ is connected, where these handles are those that appeared in the stage $k$.

3. Each component of $\partial N_1, \partial N_2, \ldots, \partial N_{n-1}$ is a 2-sphere.

Then at each stage $k$ let $I_k = \partial_k^- \times [0, 1]$, and $C_1^{(k)} = I_k \cup (F_1^{(k)} \times [0, 1]) \cup (0\text{-handles}) \cup (1\text{-handles})$, where $\partial_k^- \times \{0\}$ and $\partial_k^-$ are identified, and (0-handles), (1-handles) are those that appeared in the stage $k$. Since each component of $\partial N_k$ is not a 2-sphere (above condition 3), we see that each component of $C_1^{(k)}$ is a compression body by 2 of Remark 2.1. Then we see that $C_1^{(k)}$ is connected (above condition 2), hence $C_1^{(k)}$ is a compression body. Hence $\partial_+ C_1^{(k)}$ is a connected surface, and let $J_k = \partial_+ C_1^{(k)} \times [0, 1]$, and $C_2^{(k)} = J_k \cup (2\text{-handles}) \cup (3\text{-handles})$, where $\partial_+ C_1^{(k)} \times \{1\}$ and $\partial_+ C_1^{(k)}$ are identified, and (2-handles), (3-handles) are those that appeared in the stage $k$. Then we see that $C_2^{(k)}$ is a compression body by the above condition 3. It is clear from the construction that we obtained a submanifold, say $R_k$, of $M$ with the Heegaard splitting $C_1^{(k)} \cup C_2^{(k)}$. Moreover it is clear that $M$ can be regarded as obtained from $R_1, \ldots, R_n$ by identifying their boundaries. We call the decomposition of $M$ into the series of Heegaard splittings $(C_1^{(1)} \cup C_2^{(1)}) \cup \cdots \cup (C_1^{(n)} \cup C_2^{(n)})$ an untelescoping of the Heegaard splitting $C_1 \cup_p C_2$.

**Definition 3.1.** The above untelescoping is called a Scharlemann-Thompson untelescoping (or S-T untelescoping) if each Heegaard splitting $C_1^{(k)} \cup C_2^{(k)}$ is non-trivial, and strongly irreducible.

**Remark 3.1.** It is known that every irreducible Heegaard splitting of 3-manifolds with incompressible boundary admits a S-T untelescoping (see, for example, [1]).
4 Casson-Gordon untelescoping

In [2], A. Casson and C. McA. Gordon proved that if a Heegaard splitting of a closed 3-manifold is weakly reducible, and not reducible, then M contains an incompressible surface of positive genus. In this section, we introduce the arguments in their proof, and give the definition of Casson-Gordon untelescoping.

Let M be a closed, orientable 3-manifold, and $H_1 \cup P H_2$ a Heegaard splitting of M. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a weakly reducing collection of disks for P, i.e., $\Gamma_i (i = 1, 2)$ is a union of mutually disjoint, non-empty meridian disks of $H_i$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then $P(\Gamma)$ denotes the surface obtained from P by compressing P along $\Gamma$. Let $\hat{P}(\Gamma) = P(\Gamma) - (the \ components \ of \ P(\Gamma) \ which \ are \ contained \ in \ H_1 \ or \ H_2)$. In [2], Casson-Gordon proved:

**Proposition 4.1.** Let M be a closed, orientable 3-manifold, and $H_1 \cup P H_2$ a Heegaard splitting of M. Suppose that $H_1 \cup P H_2$ is weakly reducible. Then either
1. $H_1 \cup P H_2$ is reducible, or
2. there exists a weakly reducing collection of disks $\Delta$ for P such that each component of $\hat{P}(\Delta)$ is an incompressible surface in M, which is not a 2-sphere.

This result is proved by using the following argument.

In general, for a closed surface F, we define a complexity $c(F)$ of F as follows.

$$c(F) = \Sigma(1 - \chi(F^i)),$$

where the sum is taken over each component $F^i$ of F that is not a 2-sphere.

Let $\Delta = \Delta_1 \cup \Delta_2$ be a weakly reducing collection of disks for P such that $c(\hat{P}(\Delta))$ is minimal among the weakly reducing collection of disks of P. We can show that if $\hat{P}(\Delta)$ contains a 2-sphere component, then $H_1 \cup P H_2$ is reducible, and this gives conclusion 1. (We note that, for the proof of this assertion, the maximality of $\Delta$ is not necessary.) If no component of $\hat{P}(\Delta)$ is a 2-sphere, then the minimality of $c(\hat{P}(\Delta))$ together with a relative version of Haken’s lemma shows that each component of $\hat{P}(\Delta)$ is incompressible, and this gives conclusion 2, and this completes the proof of the proposition.

Now, we introduce some terminologies. Let $H_1 \cup P H_2$ be a Heegaard splitting of a closed M. $\Delta = \Delta_1 \cup \Delta_2$ a weakly reducing collection of disks for P. Suppose that $H_1 \cup P H_2$ is not reducible, hence no component of $\hat{P}(\Delta)$ is a 2-sphere. Let $M_1, \ldots, M_n$ be the closures of the components of $M - \hat{P}(\Delta)$. Let $M_{j,i} = M_j \cap H_i (j = 1, \ldots, n, i = 1, 2)$.

**Lemma 4.2.** For each j, we have either one of the following.
1. $M_{j,2} \cap P \subset Int(M_{j,1} \cap P)$, and $M_{j,1}$ is connected.
2. $M_{j,1} \cap P \subset Int(M_{j,2} \cap P)$, and $M_{j,2}$ is connected.
Claim 1. Each type (1) $M_i$ is a component of $\text{cl}(H_i - N(\Delta_i, H_i))$ ($i = 1$ or 2).

Claim 2. If we immediately have:

(2) There is a component $E$ of $\text{cl}(H_i - N(\Delta_i, H_i))$ ($i = 1$ or 2) such that $M_i$ is the union of $E$ and the components of $N(\Delta_{3-i}, H_{3-i})$ intersecting $E$.

By the definition of $P(\Delta)$, we have:

Claim 1. Each type (1) $M_i$ is amalgamated to the adjacent component in $H_{3-i}$ for obtaining some $M_k$.

Let $E_i$ ($i = 1$ or 2) be a component of $\text{cl}(H_i - N(\Delta_i, H_i))$. By the definition, we immediately have:

Claim 2. If $E_i \cap \Delta_{3-i} = \emptyset$, then $E_i$ is a type (1) component of $M - P(\Delta)$.

Suppose $E_i \cap N(\Delta_{3-i}, H_i) \neq \emptyset$. Let $\hat{\Delta}_{3-i}$ be the union of the components of $\Delta_{3-i}$ intersecting $E_i$. Let $\hat{E}_{3-i}$ be the union of the components $G$ of $\text{cl}(H_{3-i} - N(\Delta_{3-i}, H_{3-i}))$ such that $G \cap P \subset E_i$. By the definition of $\hat{\Delta}_{3-i}$ and Claim 2, we see that each component of $\hat{E}_{3-i}$ is type (1). Moreover it is clear that $\hat{E}_{3-i}$ is the components that are amalgamated to $E_i$ as in Claim 1. Let $\hat{M} = E_i \cup \hat{E}_{3-i}$. Since $\hat{E}_{3-i}$ is already amalgamated within $\hat{M}$, Claim 2 shows that no component of $\partial \hat{M}$ is contained in $H_1$ or $H_2$. Hence $\hat{M} = M_j$ for some $j$. We easily see, conversely, that each $M_j$ is obtained from some component of $M_i \cap H_i$ ($i = 1$ or 2) as in the above manner. Note that these give Lemma 4.2 with regarding $E_i$ as $M_{j,1}$ (conclusion 1), or $M_{j,2}$ (conclusion 2).

**Definition 4.1.** We call the components $M_{j,i}$ ($i = 1, 2$) satisfying the conclusion $i$ of Lemma 4.2 big. If $M_{j,i}$ is not big, then each component of $M_{j,i}$ is called small.

Note that the closure of each component of $H_i - \hat{P}(\Delta)$ ($i = 1, 2$) is a component of some $M_{j,i}$. Hence it is either big or small.

**Lemma 4.3.** Big components, and small components of $H_i - \hat{P}(\Delta)$ appear alternately in $H_i$, i.e., no pair of big components are adjacent in $H_i$, and no pair of small components are adjacent in $H_i$.

**Proof.** Suppose that $M_{j,i}$ is a big component, and $E_i$ the closure of a component of $H_i - \hat{P}(\Delta)$, which is adjacent to $M_{j,i}$. Let $E_{3-i}$ be the closure of a component of $H_{3-i} - \hat{P}(\Delta)$ such that $E_{3-i} \cap E_i \neq \emptyset$. Since $E_{3-i} \cap M_{j,i} \neq \emptyset$, and $E_{3-i} \cap P$ is not contained in $M_{j,i}$ (see the Proof of Lemma 4.2), we see that $E_{3-i}$ is big. This shows that $E_i$ is small.

Suppose that $G_i$ is a small component, and $E_i$ the closure of a component of $H_i - \hat{P}(\Delta)$, which is adjacent to $G_i$. Let $M_{j,3-i}$ be the big component intersecting $G_i$. Then we have: $(E_i \cap P) \cap M_{j,3-i} \neq \emptyset$, and $E_i \cap P$ is not contained in $M_{j,3-i}$ (see the Proof of Lemma 4.2). These show that $E_i$ is big.

This completes the proof of Lemma 4.3.

Now we show that we can naturally obtain an untelescoping from the decomposition of $M$ by $M_{j,i}$’s. We explain this by giving concrete descriptions for one example. Giving general descriptions will easily follows from this example.
Let $H_1 \cup P H_2$ be a genus 4 Heegaard splitting with a maximal weakly reducing collection of disks $\Delta = \Delta_1 \cup \Delta_2$ as in Figure 4.1.

In this case, $M$ is decomposed into three components, say $M_1$, $M_2$, $M_3$, by $\hat{P}(\Delta)$, where (i) $M_{1,1}$ ($M_{3,1}$ resp.) is a genus two handlebody which is big, and $M_{1,2}$ ($M_{3,2}$ resp.) is a genus 1 handlebody with $M_{1,2} \cap P$ ($M_{3,2} \cap P$ resp.) a torus with one hole, and (ii) $M_{2,2}$ is a genus two handlebody which is big, and $M_{2,1}$ is a 3-ball with $M_{2,1} \cap P$ an annulus.

Then we have the following handle decomposition of $M$.

$$M = (M_{1,1} \cup M_{1,2}) \cup (M_{3,1} \cup M_{3,2}) \cup (M_{2,1} \cup M_{2,2})$$

$$= \{(0\text{-handle}) \cup 2 \times (1\text{-handle})\} \cup \{2 \times (2\text{-handle}) \cup (3\text{-handle})\}$$

$$\cup \{(0\text{-handle}) \cup 2 \times (1\text{-handle})\} \cup \{2 \times (2\text{-handle}) \cup (3\text{-handle})\}$$

$$\cup \{1\text{-handle}\} \cup \{2 \times (2\text{-handle}) \cup (3\text{-handle})\}$$

Hence we can obtain an untelescoping

$$M = (C_1^{(1)} \cup C_2^{(1)}) \cup (C_1^{(2)} \cup C_2^{(2)}) \cup (C_1^{(3)} \cup C_2^{(3)})$$

corresponding to this handle decomposition.
Definition 4.2. Let $\Delta$ be a weakly reducing collection of disks for $P$ such that each component of $P(\Delta)$ is an incompressible surface in $M$, which is not a 2-sphere. We call the untelescoping of $H_1 \cup P H_2$ obtained as above with such $\Delta$ a Casson-Gordon untelescoping (or C-G untelescoping) of $M$.

Remark 4.1. Let $(C_1^{(1)} \cup C_2^{(1)}) \cup \cdots \cup (C_1^{(n)} \cup C_2^{(n)})$ be a C-G untelescoping of $H_1 \cup P H_2$, and $R_k = C_1^{(k)} \cup C_2^{(k)}$ ($k = 1, \ldots, n$). By definition, $\partial R_k$ is incompressible in $R_k$. Hence, by Remark 3.1, we see that $C_1^{(k)} \cup C_2^{(k)}$ admits a S-T untelescoping. This shows that a S-T untelescoping can be regarded as a (possibly trivial) refinement of C-G untelescoping.

5 Heegaard genus two link not admitting unknotting tunnel

The tunnel number $t(L)$ of a link $L$ in the 3-sphere $S^3$ is the minimal number of the components of the union of mutually disjoint arcs, called tunnels, $\tau$ such that $\partial \tau \subset L$, and $\text{cl}(S^3 - N(L \cup \tau))$ is a handlebody. We note that the definiton implies that the exteior $E(L) = \text{cl}(S^3 - N(L))$ admits a Heegaard splitting of genus $(t(L) + 1)$. Hence we see that the Heegaard genus of $E(L)$ is less than or equal to $t(L) + 1$, where Heegaard genus of a 3-manifold $M$ is the minimal genus of the Heegaard splittings of $M$. Note that if we restrict our attention to Heegaard splittings of $E(L) \cup \partial E(L) \cup \emptyset$, then the Heegaard genus of $E(L)$ is exactly $t(L) + 1$. However, if we change the partition of $\partial E(L)$, then they may be different. In this section, we give a concrete example of a link not satisfying the equality.

In the remainder of this section, let $L = L_1 \cup L_2$ be a link as in Figure 5.1, where $L_1$ is a $(4, 3)$ torus knot, and $L_2$ is a push out of a meridian curve of $L_1$, i.e., $L$ is a connected sum of $(4, 3)$ torus knot and a Hopf link.

![Figure 5.1](image-url)

Proposition 5.1. $t(L) = 2$.

Proof. Let $t_1, t_2$ be arcs as in Figure 5.1.

It is easily verified that $\text{cl}(S^3 - N(L \cup t_1 \cup t_2))$ is a genus three handlebody. Hence, we have $t(L) \leq 2$.

By Morimoto [9], it is shown that the set of two component composite tunnel number one links coinsides with the set of links each element of which is a connected sum of a two bridge knot and a Hopf link. Since $(4, 3)$ torus knot is a 3-bridge knot, we see that $t(L) > 1$. 

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Hence we have \( t(L) = 2 \).

**Proposition 5.2.** The Heegaard genus of \( E(L) \) is 2.

**Proof.** It is easy to see that \( E(L) \) does not admit a genus one Heegaard splitting. Hence it is enough to show that \( E(L) \) admits a genus 2 Heegaard splitting.

Let \( t \) be an arc as in Figure 5.2. Let \( E(L_2) = \text{cl}(S^3 - N(L_2)) \). We may suppose that \( N(L_1 \cup t) \subset \text{Int}E(L_2) \). The deformations in Figure 5.2 show that \( \text{cl}(E(L_2) - N(L_1 \cup t)) \) is a genus 2 compression body, say \( C_2 \), with \( \partial C_2 = \partial N(L_2) \). Let \( N_1 = N(L_1, N(L_1 \cup t)) \), and \( C_1 = \text{cl}(N(L_1 \cup t) - N_1) \). We note that \( C_1 \) is a genus 2 compression body with \( \partial C_1 = \partial N_1 \). Hence \( C_1 \cup C_2 \) gives a genus 2 Heegaard splitting of \( \text{cl}(S^3 - (N_1 \cup N(L_2))) \), a exterior of \( L \).

**6 Proof of Theorem**

Let \( D(2) \) be a Seifert fibered manifold with orbit manifold a disk with two exceptional fibers. Let \( L_W \) be a Whitehead link (Figure 6.1), \( W = \text{cl}(S^3 - N(L_W)) \), and \( T_1, T_2 \) the boundary components of \( W \).
Let $A(1)$ be a Seifert fibered manifold with orbit manifold an annulus with one exceptional fiber. Then let $N$ be a 3-manifold obtained from $D(2)$, $W$, and $A(1)$ by identifying $\partial D(2)$ and $T_1$ by a homeomorphism taking a regular fiber of $D(2)$ to a meridian curve, then identifying a component of $\partial A(1)$ and $T_2$ by a homeomorphism taking a regular fiber to a meridian curve. We note that $N$ is what is called a full Haken manifold in [6], and this implies:

**Proposition 6.1.** The Heegaard genus of $N$ is 2.

By Proposition 6.1, we see that $N$ admits a handle decomposition as in the following.

(N1) $(0\text{-handle}) \cup 2 \times (1\text{-handle}) \cup (2\text{-handle}),$

or, dually,

(N2) $(\text{torus} \times [0,1]) \cup (1\text{-handle}) \cup 2 \times (2\text{-handle}) \cup (3\text{-handle}).$

Let $N^i$ ($i = 1, 2$) be a copy of $N$, and $D(2)^i$, $W^i$, $A(1)^i$, $T_1^i$, $T_2^i$ the pieces of $N^i$ corresponding to $D(2)$, $W$, $A(1)$, $T_1$, $T_2$ respectively. Let $E = \operatorname{cl}(S^3 - N(L))$, where $L = L_1 \cup L_2$ is the link in section 5. Note that, by the proof of Proposition 6.1, $E$ admits a handle decomposition as in the following.

(E1) $(\partial N(L_1) \times [0,1]) \cup (1\text{-handle}) \cup (2\text{-handle}),$

or, dually,

(E2) $(\partial N(L_2) \times [0,1]) \cup (1\text{-handle}) \cup (2\text{-handle}).$

Here we note that $E$ admits a decomposition $E = E(4, 3) \cup R$, where $E(4, 3)$ is a exterior of $(4, 3)$ torus knot, and $R$ a Seifert fibered manifold with orbit manifold a disk with two holes and no exceptional fibers (i.e., $R = \text{disk with two holes} \times S^1$), where a regular fiber of $R$ is identified with a meridian curve. Let $M$ be a closed 3-manifold obtained from $N_1 \cup N_2$, and $E$ by identifying their boundaries so that the Seifert fibrations in $A(1)^1$, $A(1)^2$, and $R$ do not meet on each glueing torus.

Let $X$ be a closed Haken manifold. Then, by [3], there is a maximal perfectly embedded Seifert fibered manifold $\Sigma$ which is called a characteristic Seifert pair for $X$. Note that $\partial \Sigma$ consists of tori in $X$. If there is a pair of components of $\partial \Sigma$ which are parallel in $X$, then we eliminate one of them from the system. By repeating this procedure, we finally obtain a system of tori, say $\mathcal{T}$, in $X$, the elements of which are mutually non-parallel in $X$. In this paper, we call the decomposition of $X$ by $\mathcal{T}$ a torus decomposition of $X$. Then let $G_\mathcal{T}$ be the graph such that the vertices of $G_\mathcal{T}$ correspond to the components of $X - \mathcal{T}$, and the edges of $G_\mathcal{T}$ correspond to the components of $\mathcal{T}$. We call $G_\mathcal{T}$ a characteristic graph of $X$.

By the construction we immediately see that the decomposition

$$D(2)^1 \cup_{T_1^1} W^1 \cup_{T_2^1} A(1)^1 \cup (R \cup E(4, 3)) \cup A(1)^2 \cup_{T_1^2} W^2 \cup_{T_2^2} D(2)^2$$

is a torus decomposition of $M$, where the characteristic graph is as follows.
Proposition 6.2. The Heegaard genus of $M$ is 4.

Proof. Recall the decomposition $M = N_1 \cup_{\partial N(L_1)} E \cup_{\partial N(L_2)} N_2$. By taking the handle decompositions (N1) for $N_1$, (E1) for $E$, and (N2) for $N_2$, we see that $M$ admits a handle decomposition with one 0-handle, four 1-handles, four 2-handles, and one 3-handle. This shows that $M$ admits a genus 4 Heegaard splitting. Hence $g(M) \leq 4$.

It is known that any Haken manifolds admitting genus $g$ Heegaard splittings are decomposed into at most $3g - 3$ components by torus decomposition. Hence any Haken manifold with genus 3 Heegaard splitting is decomposed into at most 6 pieces by the torus decomposition. Recall that $M$ is decomposed into 8 pieces by the torus decomposition. Hence $g(M) \geq 4$.

These show that $g(M) = 4$. \qed

Let $V \cup W$ be the genus 4 Heegaard splitting obtained in the proof of Proposition 6.2. Then the handle decompositions used there ( (N1) for $N_1$, (E1) for $E$, and (N2) for $N_2$ ) naturally gives an untelescoping

$$M = (V_1 \cup W_1) \cup (V_2 \cup W_2) \cup (V_3 \cup W_3),$$

where $V_1 \cup W_1$ is a genus 2 Heegaard splitting of $N_1$, $V_2 \cup W_2$ is a genus 2 Heegaard splitting of $E$, and $V_3 \cup W_3$ is a genus 2 Heegaard splitting of $N_2$.

Proposition 6.3. The untelescoping $M = (V_1 \cup W_1) \cup (V_2 \cup W_2) \cup (V_3 \cup W_3)$ is a S-T untelescoping.

Proof. In general, by the arguments in section 4, we easily see that if a genus two Heegaard splitting is weakly reducible, then the ambient 3-manifold is either reducible or admits a genus 1 Heegaard splitting. Each of the manifolds $N_1$, $E$, $N_2$ is not reducible or does not admit genus one Heegaard splitting. Hence we see that each $V_i \cup W_i$ ($i = 1, 2, 3$) is strongly irreducible. \qed

Proposition 6.4. $M$ cannot be decomposed into more than two pieces by any C-G untelescoping on $V \cup W$.

Proof. Assume that $M$ is decomposed into three pieces by a C-G untelescoping on $V \cup W$. Let $\Delta = \Delta_V \cup \Delta_W$ be the system of weakly reducing pair of disks, and

$$M = (V_1 \cup W_1) \cup (V_2 \cup W_2) \cup (V_3 \cup W_3)$$

the C-G untelescoping.

Let $M_i = V_i \cup W_i$ ($i = 1, 2, 3$). Recall, from section 4, that each $M_i$ has exactly one big component. Without loss of generality, we may suppose that $V$ contains two big components. Since $M$ is irreducible, we see that each big component is a handlebody whose genus is at least two. It is easy to see that these together with Lemma 4.3 imply:
\(\Delta_V\) consists of a disk which separates \(V\) into two genus two handlebodies which are big components.

By exchanging subscripts, if necessary, we may suppose that these big components correspond to \(V_1\) and \(V_3\). This implies:

Claim 1. Each component of \(\partial M_1, \partial M_3\) is a torus, hence each component of \(\partial M_2\) is a torus.

We note that the small components of \(M_2\) is just a regular neighborhood of \(\Delta_V\) in \(V\). Hence Claim 1 shows that the big components of \(M_2\) is a genus two handlebody and \(\partial M_2\) consists of two tori such that one is the boundary of \(M_1\), and the other is the boundary of \(M_2\). From these observations, we can show that the configuration of \(\Delta\) must be as in Figure 6.2.

![Figure 6.2](image)

Claim 2. Either \(M_1 = N_1, M_1 = N_2, M_3 = N_1\), or \(M_3 = N_2\).

**Proof.** Note that each piece of the torus decomposition of \(M\) is simple. Hence each incompressible torus is isotopic to a member of the tori giving the torus decomposition of \(M\). Since each \(M_i\) admits genus two Heegaard splitting, it can be decomposed into at most three pieces by the torus decomposition. These imply that either \(M_1\) or \(M_3\) is decomposed into exactly three pieces. Without loss of generality, we may suppose that \(M_1\) is decomposed into three pieces. Then note that \(\partial M_1\) consists of a torus. These together with the examination of the characteristic graph show that \(M_1\) is either \(D(2)^1 \cup W^1 \cup A(1)^1\), or \(D(2)^2 \cup W^2 \cup A(1)^2\), and this proves Claim 2.

Since the argument is symmetric, we may suppose \(M_1 = N_1\) in the remainder of this paper.

Claim 3. \(M_3 = N_2\).

**Proof.** Suppose not. Since \(M_2, M_3\) are decomposed into at most three pieces by the torus decomposition, we see that \(M_2 = E(4,3) \cup R \cup A(1)\). However, this contradicts Theorem of [6], since \(R\) is not a 2-bridge link exterior.
By Claim 3, we see that $M_2 = E(L)$. Then, by Figure 6.2, we have $t(L) = 1$, contradicting Proposition 5.1.

Finally, by the arguments of the proof of Proposition 6.3, we see that $M$ cannot be decomposed into more than three pieces, and this completes the proof of Proposition 6.4.

Proposition 6.3 gives conclusion 1 of Theorem. Proposition 6.4 gives conclusion 2 of Theorem. By the construction, we immediately have conclusion 3 of Theorem.

This completes the proof of Theorem.

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