Research Article

Hermite-Hadamard- and Jensen-Type Inequalities for Interval \((h_1, h_2)\) Nonconvex Function

Hongxin Bai, \(^1\) Muhammad Shoaib Saleem \(^2\), Waqas Nazeer \(^3\), Muhammad Sajid Zahoor, \(^2\) and Taiyin Zhao \(^4\)

\(^1\)School of Data Science and Software Engineering, Baoding University, Baoding, Hebei 071000, China
\(^2\)Department of Mathematics, University of Okara, Okara, Pakistan
\(^3\)Department of Mathematics, Government College University, Lahore 54000, Pakistan
\(^4\)School of Information and Software Engineering, University of Electronic Science and Technology of China, Chengdu 610054, China

Correspondence should be addressed to Waqas Nazeer; nazeer.waqas@gmail.com

Received 13 January 2020; Revised 20 March 2020; Accepted 25 March 2020; Published 21 April 2020

Academic Editor: Viliam Makis

Copyright © 2020 Hongxin Bai et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In the present study, we will introduce the definition of interval \((h_1, h_2)\) nonconvex function. We will investigate some properties of interval \((h_1, h_2)\) nonconvex function. Moreover, we will develop Hermite-Hadamard- and Jensen-type inequalities for interval \((h_1, h_2)\) nonconvex function.

1. Introduction

Since its inception five decades ago, the theory of fuzzy sets has advanced in a variety of ways. Application of the theory of fuzzy sets covered many areas like artificial intelligence, decision theory, computer science, logic operational research, and robotics [1–3]. Initially good books like probability theory by Dubois and Prade in 1988, Behavioral and social science by Smithsons in 1987, and Fuzzy Control by sugeno 1985 and pedrycz 1989 and others have been published. We refer [4, 5] recent developments in this field.

For some other results and application of interval analysis theory, we refer the readers [4, 6–13]. Due to vast application of fuzzy sets, many integral inequalities have been derived by different authors [1–3, 14–17].

Costa [18] there is a new fuzzy version of Jensen-type integral inequality for fuzzy interval valued function. Also in [19], Zhao et al. develop new Harmite-Hadamard-type inequality for h-convex interval valued function.

For more about Hermite-Hadamard inequalities, refer [20–24]. We will introduce the interval \((h_1, h_2)\) non-convex function. The second objective of this article is to develop Hermite-Hadamard- and Jensen-type inequality for above said generalization.

2. Preliminaries

In this section, we define some basic definitions, properties, results, and notations on interval analysis, which are used throughout the paper [17, 25]. Here, \(\mathbb{R}_I\) and \(\mathbb{R}_I^+\) denote the family of all intervals and positive interval and it is equipped with the algebraic operations “+” and “.” given, respectively, by \([x_1, x_2] + [y_1, y_2] = [x_1 + y_1, x_2 + y_2]\) and

\[
\alpha \cdot [x_1, x_2] = \begin{cases} 
[\alpha x_1, \alpha x_2], & \text{if } 0 \leq \alpha, \\
[\alpha x_2, \alpha x_1], & \text{if } \alpha < 0.
\end{cases}
\]  

(1)

A function \(F: V \subset \mathbb{R} \rightarrow \mathbb{R}_I\) with \(F(x) = [f_1(x), f_2(x)]\), where \(f_1, f_2: V \rightarrow \mathbb{R}\) are real functions with \(f_1(x) \leq f_2(x)\) for all \(x \in V\), it is called an interval-valued function.

For intervals \([x, \bar{x}]+[y, \bar{y}]\), the Hausdorff distance is defined by \(d([x, \bar{x}], [y, \bar{y}]) = \max\{|x-y|, |\bar{x} - \bar{y}|\}\).

Then \((\mathbb{R}_I, d)\) is complete.
A set of numbers $f[x_{i-1}, y_i, x_i]_{i=1}^m$ is said to be a tagged partition $P$ of $[x, y]$ if
\[
x = x_0 < x_1 < \cdots < x_m = y,
\]
and if $x_{i-1} \leq y_i \leq x_i$ for all $i = 1, 2, \ldots, m$. Moreover, if we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta x_i \leq \delta$ if for each $i$, then we say that the partition is $\delta$-fine. The family of all $\delta$-fine partitions of $[x, y]$ is denoted by $P(\delta; [x, y])$.

Given $X \in P(\delta, [x, y])$, we define a integral sum of $f$: $[x, y] \rightarrow \mathbb{R}_f$ as follows:
\[
S(f, X, \delta, [x, y]) = \sum_{i=1}^m f(y_i)(x_i - x_{i-1}).
\]

Throughout the paper, $IR$-integrable means interval Riemann integrable. The concept of $IR$-integrable is given in [19], Definition 2.2, is equivalent to $IR$ integral given in [10], Definition 9.1.

\textbf{Definition 1.} Let $f$: $[x, y] \rightarrow \mathbb{R}_f$. $f$ is called $IR$-integrable on $[x, y]$ with $IR$-integral $A = \int_X f(t)dt$, if there exists an $A \in RI$ such that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that
\[
d(S(f, X, \delta, [x, y]), A) < \varepsilon,
\]
for each $X \in P(\delta, [x, y])$. Let $IR([x, y])$ denote the set of all $IR$-integrable functions on $[x, y]$.

\textbf{Definition 2.} (See $p$-convex set, [26]).

The real interval $I$ is known as $p$-convex set if for all $x, y \in I$ and $\alpha \in [0, 1]$, implies that
\[
[ax^p + (1 - \alpha)y^p]^{1/p} \in I,
\]
where $p = 2k + 1$ or $p = (n_1/h_2)$, $n_1 = 2r + 1$, $n_2 = 2t + 1$, and $k, r, t \in N$.

\textbf{Definition 3.} (See $p$-convex function, [26]). For a $p$-convex set $I$, the mapping $f$: $I \rightarrow \mathbb{R}$:
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),
\]
is called $p$-convex function, for all $x, y \in I$, and $\lambda \in [0, 1]$.

\textbf{Definition 4.} (See $h$-convex function, [27]). The nonnegative function $f$: $[a, b] \rightarrow \mathbb{R}$ is $h$-convex if
\[
f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + (1 - \lambda)f(y),
\]
\[
\forall x, y \in [a, b] \text{ and } \lambda \in (0, 1) \text{ and } h \neq 0, h \text{ is nonnegative real-valued function or } f \text{ belongs to the class } SX(h; [a, b]).
\]
If the inequality (7) is reversed, then $f$ is said to be $h$-concave, i.e., $f \in SV(h, I)$.

\textbf{Definition 5.} $(h_1, h_2)$-convex function [28]. Let $h_1, h_2$: $[0, 1] \rightarrow \mathbb{R}^+$ be a nonnegative function, $h_1, h_2 \neq 0$. The nonnegative function $f$: $f$: $[0, 1] \rightarrow \mathbb{R}$ is an $(h_1, h_2)$-convex if
\[
f(\lambda x + (1 - \lambda)y) \leq h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y),
\]
for all $x, y \in I$ and $\lambda \in (0, 1)$ or that $f$ belongs to the class $SX((h_1, h_2); J, \mathbb{R})$. If the inequality (8) is reversed, then $f$ is said to be $(h_1, h_2)$-concave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R})$.

\textbf{Definition 6.} Interval $(h_1, h_2)$-convex function [29]. Let $h_1, h_2$: $[0, 1] \rightarrow \mathbb{R}^+$ be a nonnegative function, $h_1, h_2 \neq 0$. We say that $f$: $J \rightarrow \mathbb{R}$ is an interval $(h_1, h_2)$-convex function or that $f$ belongs to the class $SX((h_1, h_2); J, \mathbb{R})$, if $f$ is nonnegative and for all $x, y \in J$ and $\lambda \in (0, 1)$, we have
\[
h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y) \leq f(\lambda x + (1 - \lambda)y).
\]
If the inequality (9) is reversed, then $f$ is said to be interval $(h_1, h_2)$-concave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R})$.

\textbf{Remark 1}

(1) If $h_2 = 1$, then Definition 6 becomes interval $h$-convex in [19].

(2) If $h_1 = h_2 = 1$, then Definition 6 becomes $P$-function in [30].

(3) If $h_1(t) = t^r$, $h_1 = 1$, then Definition 6 becomes to $s$-convex fuzzy process in [31].

We wind up the current section by introducing the new concept of interval $(h_1, h_2)$ nonconvexity. This idea is inspired by An et al. [29]. Throughout the paper, for interval $[x, \overline{x}]$ and $[y, \overline{y}]$, we have
\[
[x, \overline{x}] \subseteq [y, \overline{y}] \Rightarrow y \leq x \text{ and } \overline{x} \leq \overline{y}.
\]

\textbf{Definition 7.} Interval $(h_1, h_2)$ nonconvex function. Let $h_1, h_2$: $[0, 1] \rightarrow \mathbb{R}^+$ be a nonnegative function, $h_1, h_2 \neq 0$. For a nonconvex function, $f$: $J \rightarrow \mathbb{R}$ is an interval $(h_1, h_2)$ nonconvex function if
\[
h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y) \leq f(\lambda x^p + (1 - \lambda)y^p)^{1/p}.
\]
\[
\forall x, y \in J, \text{ and } \lambda \in (0, 1)
\]
For convenience, in this paper, the class of interval nonconvex function is denoted by $SX((h_1, h_2, p); J, \mathbb{R})$. If the inequality (11) is reversed, then $f$ is said to be interval $(h_1, h_2)$ nonconcave, i.e., $f \in SV((h_1, h_2, p), J, \mathbb{R})$.

Throughout the paper $H(x, y) = h_1(x)h_2(y)$ for $x, y \in [0, 1]$.

\textbf{Theorem 1.} Let $f$: $[r, s] \rightarrow \mathbb{R}^+$ be an interval-valued function such that $f(\lambda) = [f(\lambda), \bar{f}(\lambda)]$. Then $f \in SX((h_1, h_2, p); [r, s], \mathbb{R})$ if and only if $f \in SX((h_1, h_2, p); \bar{f}(\lambda), \mathbb{R})$.

\textbf{Proof.} Let $f$ be a interval $(h_1, h_2)$ nonconvex function and suppose that $x, y \in [r, s]$, $\lambda \in (0, 1)$, then
\[
h_1(\lambda)h_2(1 - \lambda)f(x) + h_1(1 - \lambda)h_2(\lambda)f(y) \leq f(\lambda x^p + (1 - \lambda)y^p)^{1/p},
\]
for all $x, y \in I$ and $\lambda \in (0, 1)$ or that $f$ belongs to the class $SX((h_1, h_2); J, \mathbb{R})$. If the inequality (8) is reversed, then $f$ is said to be $(h_1, h_2)$-concave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R})$. If the inequality (9) is reversed, then $f$ is said to be interval $(h_1, h_2)$-concave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R})$. If the inequality (10) is reversed, then $f$ is said to be interval $(h_1, h_2)$ nonconcave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R})$. If the inequality (11) is reversed, then $f$ is said to be interval $(h_1, h_2)$ nonconcave, i.e., $f \in SV((h_1, h_2), J, \mathbb{R})$.
that is,
\[
\begin{aligned}
\left[ h_1(\lambda)h_2(1-\lambda) f(x) + h_1(1-\lambda)h_2(\lambda) f(y) \\
xh_1(\lambda)h_2(1-\lambda) \bar{f}(x) + h_1(1-\lambda)h_2(\lambda) \bar{f}(y) \right] \\
\leq \left[ \int (\lambda x^p + (1-\lambda)y^p)^{(1/p)} \right] \left( \lambda \bar{f}(x) + (1-\lambda)y^p \right)^{(1/p)}.
\end{aligned}
\]
(13)

It follows that
\[
\begin{aligned}
h_1(\lambda)h_2(1-\lambda) f(x) + h_1(1-\lambda)h_2(\lambda) f(y) \\
\geq \int (\lambda x^p + (1-\lambda)y^p)^{(1/p)} h_1(\lambda)h_2(1-\lambda) \bar{f}(x) \\
+ h_1(1-\lambda)h_2(\lambda) \bar{f}(y) \\
\leq \bar{f}(\lambda x^p + (1-\lambda)y^p)^{(1/p)}.
\end{aligned}
\]
(14)

This shows that \( f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \) and \( \bar{f} \in SV((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \).

Conversely suppose that \( f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \) and \( \bar{f} \in SV((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \), then from above definition and inclusion set (13), it follows that \( f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \).

This completes the proof. \( \square \)

\textbf{Theorem 2.} Let \( f : [r, s] \rightarrow \mathbb{R}_+^t \) be an interval-valued function such that \( f(\lambda) = [f(\lambda), \bar{f}(\lambda)] \).
Then \( f \in SV((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \) if and only if \( \bar{f} \in SX((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \) and \( f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \).

\textbf{Proof.} The proof is similar to that of Theorem 1. \( \square \)

\section{3. Main Result}

\textbf{Theorem 3.} Let \( f : [r, s] \rightarrow \mathbb{R}_+^t \), \( h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_+^t \) and \( H((1/2), (1/2)) \neq 0 \) if \( f \in SX((h_1, h_2, p), [r, s], \mathbb{R}_+^t) \) and \( f \in IR_{[r,s]} \) then
\[ \frac{1}{2H((1/2), (1/2))} \int (\frac{r^p + sp^p}{2})^{(1/p)} \geq \frac{p}{s^p - r^p} \int \lambda^{p-1} f(\lambda) d\lambda \]
\[ \geq [f(r) + f(s)] \int H(x, 1-x) dx. \]
(15)

\textbf{Proof.} By assumption, we have
\[ H((1/2), (1/2)) f(x) + H((1/2), (1/2)) f(y) \]
\[ \geq f((1-x)r^p + xs^p)^{(1/p)} + H((1/2), (1/2)) f((1-x)r^p + xs^p)^{(1/p)} \]
\[ + H((1/2), (1/2)) f((1-x)r^p + xs^p)^{(1/p)} \leq \frac{r^p + sp^p}{2}^{(1/p)} \]
(16)

Integrating above w.r.t. “\( x \)” on [0, 1], we get
\[ \int_0^1 f(xr^p + (1-x)s^p)^{(1/p)} dx \]
\[ + \int_0^1 f((1-x)r^p + xs^p)^{(1/p)} dx \]
\[ \geq \frac{1}{H((1/2), (1/2))} \int \frac{r^p + sp^p}{2}^{(1/p)} \]
(17)
\[ \int_0^1 f((1-x)r^p + xs^p)^{(1/p)} dx \]
\[ + \int_0^1 f((1-x)r^p + xs^p)^{(1/p)} dx \]
\[ \leq \frac{1}{H((1/2), (1/2))} \int \frac{r^p + sp^p}{2}^{(1/p)} \]
(18)

it follows that
\[ \frac{2p}{s^p - r^p} \int_0^1 \lambda^{p-1} f(\lambda) d\lambda \geq \frac{1}{H((1/2), (1/2))} \int \frac{r^p + sp^p}{2}^{(1/p)} \]
\[ \frac{2p}{s^p - r^p} \int_0^1 \lambda^{p-1} f(\lambda) d\lambda \leq \frac{1}{H((1/2), (1/2))} \int \frac{r^p + sp^p}{2}^{(1/p)} \]
(19)

This implies that
\[ \frac{p}{s^p - r^p} \int_0^1 \lambda^{p-1} f(\lambda) d\lambda \leq \frac{1}{2H((1/2), (1/2))} \int \frac{r^p + sp^p}{2}^{(1/p)} \]
(20)

Now by def. of interval nonconvex function, we have
\[ h_1(x)h_2(1-x)f(r) + h_1(1-x)h_2(x)f(s) \]
\[ \leq f(xr^p + (1-x)s^p)^{(1/p)} , \]
integrated with respect to “\( x \)” on [0, 1], we get
\[ f(r) \int_0^1 h_1(x)h_2(1-x)dx + f(s) \int_0^1 h_1(1-x)h_2(x)dx \leq \int_0^1 f(xr^p + (1-x)s^p)^{(1/p)} dx [f(r) + f(s)] \times \int_0^1 h_1(x)h_2(1-x)dx \leq \int_0^1 f(xr^p + (1-x)s^p)^{(1/p)} , \]

it follows that

\[ [f(r) + f(s)] \int_0^1 h_1(x)h_2(1-x)dx \leq \frac{p}{sp - rp} \int_0^1 \lambda^{p-1} f(\lambda) d\lambda. \]

Combining (19) and (22) we get (27).

\[ \Delta_1 = \frac{1}{4H((1/2), (1/2))} \left[ f \left( \frac{3r^p + s^p}{4} \right)^{(1/p)} \right] + f \left( \frac{r^p + 3s^p}{4} \right)^{(1/p)} \]

\[ \Delta_2 = \frac{1}{2} \left[ f(r) + f(s) + 2f \left( \frac{r^p + s^p}{2} \right)^{(1/p)} \right] \int_0^1 H(x, 1-x)dx. \]

**Remark 2**

1. If \( H(x, y) = h_1(x) \) and \( f = \overline{f} \) then Theorem 3 becomes (32), Theorem 5).

2. If \( H(x, y) = h_1(x) \) and \( p = 1 \) then Theorem 3 becomes (19), Theorem 4).

3. If \( h_1(x) = x^p \), \( h_2(x) = x \) and \( p = 1 \) then Theorem 3 becomes (31), Theorem 4).

4. If \( f = \overline{f} \) then Theorem 3 becomes Hermite-Hadamard inequality for \((h_1, h_2)\)-convex function.

**Example 1.** Consider \( h_1(t) = t \), \( h_2(t) = t \) for \( t \in [0, 1] \), \([r, s] = [-1, 1]\) and \( f: [r, s] \to \mathbb{R}^+_1 \) be defined by \( f(\lambda) = [\lambda^p, 4 - e^\lambda] \) and \( p \) be an odd number, then we have

\[ \frac{1}{2H((1/2), (1/2))} f \left( \frac{r^p + s^p}{2} \right)^{(1/p)} = f(0) = [0, 3], \]

and

\[ \frac{p}{sp - rp} \int_0^1 \lambda^{p-1} f(\lambda) d\lambda \]

\[ = \frac{p}{2} \left[ \int_1^1 \lambda^{p-1} f(\lambda) d\lambda \int_1^1 \lambda^{p-1}\left( 4 - e^\lambda \right) d\lambda \right] \]

\[ = \frac{p}{2} \left[ \int_1^1 \lambda^{p-1} \lambda d\lambda - \frac{8}{p} \int_1^1 \lambda^{p-1} e^\lambda d\lambda \right] \]

\[ = \left[ 0, 4 - \frac{e - e^{-1}}{2} \right]. \]

Putting \( z = \lambda^p \) in (24) and simplifying, we get

\[ [f(r) + f(s)] \int_0^1 H(x, 1-x)dx = \left[ 0, 4 - \frac{e + e^{-1}}{2} \right]. \]

Combining (23)–(25), we get

\[ \left[ 0, 3 \right] \supseteq \left[ 0, 4 - \frac{e - e^{-1}}{2} \right] \supseteq \left[ 0, 4 - \frac{e + e^{-1}}{2} \right]. \]

Consequently, verify the Theorem 3.

**Theorem 4.** Let \( f: [r, s] \to \mathbb{R}^+_1 \), \( h_1, h_2: [0, 1] \mathbb{R}^+_1 \), and \( H((1/2), (1/2)) \neq 0 \) if \( f \in SX \left( (h_1, h_2, p), [r, s], \mathbb{R}^+_1 \right) \) and \( f \in IR_{[r,s]} \) then

\[ \frac{1}{4H((1/2), (1/2))} f \left( \frac{r^p + s^p}{4} \right)^{(1/p)} \]

\[ \supseteq \Delta_1 \supseteq \Delta_2 \supseteq [f(r) + f(s)] \left[ 0, 4 - \frac{e + e^{-1}}{2} \right], \]

where

\[ \Delta_1 = \frac{1}{4H((1/2), (1/2))} f \left( \frac{3r^p + s^p}{4} \right)^{(1/p)} + f \left( \frac{r^p + 3s^p}{4} \right)^{(1/p)} \]

\[ \Delta_2 = \frac{1}{2} \left[ \int_0^1 H(x, 1-x)dx \right]. \]

**Proof.** For \([r^p, (r^p + s^p/2)], \) one has

\[ H((1/2), (1/2)) \left( \frac{3r^p + s^p}{4} \right)^{(1/p)} \]

\[ + H((1/2), (1/2)) \left( \frac{r^p + 3s^p}{4} \right)^{(1/p)} \]

\[ \supseteq \int_0^1 f \left( \frac{3r^p + s^p}{4} \right)^{(1/p)} \]

Consequently, we get

\[ \frac{4p}{sp - rp} \int_0^1 \left( \frac{r^p + s^p/2}{1/p} \right)^{(1/p)} \lambda^{p-1} f(\lambda) d\lambda \]

\[ \supseteq \frac{1}{H((1/2), (1/2))} f \left( \frac{3r^p + s^p}{4} \right)^{(1/p)} \]

Similarly for \([r^p + s^p/2, s^p],\)

\[ \frac{4p}{sp - rp} \int_0^1 \left( \frac{r^p + 3s^p}{1/p} \right)^{(1/p)} \lambda^{p-1} f(\lambda) d\lambda \]

Adding (30) and (31), we get
\[
\frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
\leq \frac{1}{4H((1/2), (1/2))^2} \left[ f\left(\frac{3r^p + s^p}{4}\right) + f\left(\frac{r^p + 3s^p}{4}\right) \right] \\
- \frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
\leq \Delta_1,
\]
(32)
and from Theorem 3, we have
\[
\frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{r^p + s^p}{2}\right) \\
= \frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{3r^p + s^p}{4} + \frac{1}{2} f\left(\frac{r^p + 3s^p}{4}\right) \right) \\
\geq \frac{1}{4H((1/2), (1/2))]^2} f\left(\frac{3r^p + s^p}{4} + \frac{r^p + 3s^p}{4}\right) \\
\geq \Delta_1 \\
\geq \frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
\geq \frac{1}{2} \left[ f(r) + f(s) + 2 f\left(\frac{r^p + s^p}{2}\right) \right] \int_0^1 H(x, 1 - x) dx \\
= \Delta_2 \\
\geq \left[ f(r) + f(s) \right] \left[ \frac{1}{2} + H\left(\frac{1}{2}, \frac{1}{2}\right) \right] \int_0^1 H(x, 1 - x) dx,
\]
(33)
and the result follows.

**Theorem 5** (Jensen-type inequality). Let \(w_1, w_2, \ldots, w_n \in \mathbb{R}^+\) with \(n \geq 2\). If \(h_1, h_2\) is nonnegative super multiplicative function and \(f \in SX(h_1, h_2, p)\) and \(x_1, x_2, \ldots, x_n \in \mathbb{I}\). Then, we have following inequality:
\[
f\left(\left[\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right]^{1/p}\right) \geq \left[\frac{1}{W_n} \sum_{i=1}^n w_i \right] \sum_{i=1}^n H\left(\frac{w_i}{W_n}, \frac{W_{n-1}}{W_n}\right) f(x_i),
\]
(34)
where \(W_n = \sum_{i=1}^n w_i\)

**Proof.** When \(n = 2\) inequality (33) is true. Consider the inequality (33) is true for \(n - 1\), then
\[
f\left(\left[\frac{1}{W_{n-1}} \sum_{i=1}^{n-1} w_i x_i\right]^{1/p}\right) \geq \left[\frac{1}{W_{n-1}} \sum_{i=1}^{n-1} w_i \right] \sum_{i=1}^{n-1} H\left(\frac{w_i}{W_{n-1}}, \frac{W_{n-2}}{W_{n-1}}\right) f(x_i),
\]
(35)
and
\[
\frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
\leq \frac{1}{4H((1/2), (1/2))^2} \left[ f\left(\frac{3r^p + s^p}{4}\right) + f\left(\frac{r^p + 3s^p}{4}\right) \right] \\
- \frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
\leq \Delta_1,
\]
(32)
and from Theorem 3, we have
\[
\frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{r^p + s^p}{2}\right) \\
= \frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{3r^p + s^p}{4} + \frac{1}{2} f\left(\frac{r^p + 3s^p}{4}\right) \right) \\
\geq \frac{1}{4[H((1/2), (1/2))]^2} f\left(\frac{3r^p + s^p}{4} + \frac{r^p + 3s^p}{4}\right) \\
\geq \Delta_1 \\
\geq \frac{p}{s^p - r^p} \int_r^s \lambda^{p-1} f(\lambda) d\lambda \\
\geq \frac{1}{2} \left[ f(r) + f(s) + 2 f\left(\frac{r^p + s^p}{2}\right) \right] \int_0^1 H(x, 1 - x) dx \\
= \Delta_2 \\
\geq \left[ f(r) + f(s) \right] \left[ \frac{1}{2} + H\left(\frac{1}{2}, \frac{1}{2}\right) \right] \int_0^1 H(x, 1 - x) dx,
\]
(33)
and the result follows.

4. Conclusions
Hermite-Hadamard- and Jensen-type inequalities are true for this new concept of interval value convexity. Moreover by suitable substitution, the obtained results reduce to the results of [19, 31, 32].

**Data Availability**
All data required for this paper is included within this paper.

**Conflicts of Interest**
The authors do not have any conflicts of interest.

**Authors’ Contributions**
All authors contributed equally in this paper.

**Acknowledgments**
This research was supported by Higher Education Commission, Pakistan.

**References**
[1] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, “Generalized riemann-liouville k-fractional integrals associated with ostrowski type inequalities and error bounds of hadamard inequalities,” IEEE Access, vol. 6, pp. 64946–64953, 2018.
[2] S. M. Kang, G. Farid, W. Nazeer, and B. Tariq, “Hadamard and Fejér–Hadamard inequalities for extended generalized
fractional integrals involving special functions,” *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[3] S. M. Kang, G. Farid, W. Nazeer, and S. Mehmood, “(.hm)-convex functions and associated fractional Hadamard and Fejér–Hadamard inequalities via an extended generalized Mittag-Leffler function,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–10, 2019.

[4] Y. C. Kwan, M. S. Saleem, M. Ghafoor, W. Nazeer, and S. M. Kang, “Hermite Hadamard-type inequalities for functions whose derivatives are η-convex via fractional integrals,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–16, 2019.

[5] Y. C. Kwan, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and S. M. Kang, “Inequalities for a unified interval operator and associated results in fractional calculus,” *IEEE Access*, vol. 7, pp. 126283–126292, 2019.

[6] Y. Chalco-Cano, G. N. Silva, and A. Rufián-Lizana, “On the Newton method for solving fuzzy optimization problems,” *Fuzzy Sets and Systems*, vol. 272, pp. 60–69, 2015.

[7] T. M. Costa, H. Bouwmeester, W. A. Lodwick, and C. Lavor, “Calculating the possible conformations arising from uncertainty in the molecular distance geometry problem using constraint interval analysis,” *Information Sciences*, vol. 415-416, pp. 41–52, 2017.

[8] T. M. Costa, Y. Chalco-Cano, W. A. Lodwick, and G. N. Silva, “Generalized interval vector spaces and interval optimization,” *Information Sciences*, vol. 311, pp. 74–85, 2015.

[9] R. Osuna-Gomez, Y. Chalco-Cano, B. Hernandez-Jimenez, and G. Ruiz-Garzn, “Optimality conditions for generalized differentiable interval-valued functions,” *Information Sciences*, vol. 321, pp. 136–146, 2015.

[10] R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1966.

[11] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, PA, USA, 2009.

[12] H. Roman-Flores, Y. Chalco-Cano, and G. N. Silva, “A note on Gronwall type inequality for interval-valued functions,” in *Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting*, vol. 35, pp. 1455–1458, Edmonton, Canada, June 2013.

[13] Y. Guo, G. Ye, D. Zhao, and W. Liu, “Some integral inequalities for log-h-convex interval-valued functions,” *IEEE Access*, vol. 4, 2016.

[14] M. Matloka, “Convex fuzzy processes,” *Fuzzy Sets and Systems*, vol. 110, pp. 109–114, 2000.

[15] Y.–R. Syau, C.–Y. Low, and T.–H. Wu, “A note on convex fuzzy processes,” *Applied Mathematics Letters*, vol. 15, no. 2, pp. 193–196, 2002.

[16] R. Lowen, “Convex fuzzy sets,” *Fuzzy Sets and Systems*, vol. 3, no. 3, pp. 291–310, 1980.

[17] S. Kang, G. Abbas, G. Farid, and W. Nazeer, “A generalized Fejér-Hadamard inequality for harmonically convex functions via generalized fractional integral operator and related results,” *Mathematics*, vol. 6, no. 7, pp. 122, 2018.

[18] T. M. Costa, “Jensen’s inequality type integral for fuzzy-interval-valued functions,” *Fuzzy Sets and Systems*, vol. 327, pp. 31–47, 2017.

[19] D. F. Zhao, T. Q. An, G. J. Ye, and W. Liu, “New Jensen and Hermite-Hadamard type inequalities for h-convex interval-valued functions,” *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[20] M. Noor, K. Noor, M. Awan, and J. Li, “On Hermite-Hadamard inequalities for h-preinvex functions,” *Filomat*, vol. 28, no. 7, pp. 1463–1474, 2014.

[21] M. A. Noor, K. I. Noor, and M. U. Awan, “A new Hermite-Hadamard type inequality for h-convex functions,” *Creative Mathematics and Informatics*, vol. 24, pp. 191–197, 2015.

[22] B. B. Mohsen, M. U. Awan, M. A. Noor, L. Rahi, K. I. Noor, and B. Almutairi, “New quantum Hermite Hadamard inequalities utilizing harmonic convexity of the functions,” *IEEE Access*, vol. 7, pp. 20479–20483, 2019.

[23] M. Z. Sarikaya, A. Saglam, and H. Yildirim, “On some Hadamard-type inequalities for h-convex functions,” *Journal of Mathematical Inequalities*, vol. 2, no. 3, pp. 335–341, 2008.

[24] S. S. Dragomir, “Inequalities of Hermite-Hadamard type for functions of selfadjoint operators and matrices,” *Journal of Mathematical Inequalities*, vol. 11, no. 1, pp. 241–259, 2017.

[25] G. Farid, W. Nazeer, M. Saleem, S. Mehmood, and S. Kang, “Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications,” *Mathematics*, vol. 6, no. 11, pp. 248, 2018.

[26] K. S. Zhang and J. P. Wan, “p-convex functions and their properties,” *Pure and Applied Mathematics Journal*, vol. 1, no. 23, pp. 130–133, 2007.

[27] S. Varoanecl, “On h-convexity,” *Journal of Mathematical Analysis and Applications*, vol. 326, pp. 303–311, 2007.

[28] M. U. Awan, M. A. Noor, K. I. Noor, and A. G. Khan, “Some new classes of convex functions and inequalities,” *Miskolc Mathematical Notes*, vol. 19, no. 1, pp. 77–94, 2018.

[29] Y. An, G. Ye, D. Zhao, and W. Liu, “Hermite-hadamard type inequalities for interval (h1, h2)-convex functions,” *Mathematics*, vol. 7, no. 5, pp. 436, 2019.

[30] S. S. Dragomir, J. Pecaric, and L. E. Persson, “Some inequalities of Hadamard type,” *Soochow Journal of Mathematics*, vol. 21, pp. 335–341, 1995.

[31] R. Osuna-Gomez, M. D. Jimenez-Gamro, Y. Chalco-Cano, and M. A. Rojas-Medar, “Hadamard and Jensen inequalities for s-convex fuzzy processes,” in *Soft Methodology and Random Information Systems*, pp. 645–652, Springer, Berlin, Germany, 2004.

[32] Z. B. Fang and R. Shi, “On the (p,h)-convex functions and some integral inequalities,” *Journal of Inequalities and Applications*, 2014.