Integrability of \(n\)-dimensional dynamical systems of type \(E_7^{(1)}\) and \(E_8^{(1)}\)

Tomoyuki Takenawa

Faculty of Marine Technology, Tokyo University of Marine Science and Technology
Echujima 2-1-6, Koto-ku, Tokyo 135-8533, Japan
E-mail: takenawa@e.kaiyodai.ac.jp TEL & FAX: 81 3 5245 7457

Abstract

We propose an \(n\)-dimensional analogue of elliptic difference Painlevé equation. Some Weyl group acts on a family of rational varieties obtained by successive blow-ups at \(m\) points in \(\mathbb{P}^n(\mathbb{C})\), and in many cases they include the affine Weyl groups with symmetric Cartan matrices as subgroups. It is shown that the dynamical systems obtained by translations of these affine Weyl groups possess commuting flows and that their degrees grow quadratically. For the \(E_7^{(1)}\) and \(E_8^{(1)}\) cases, existence of preserved quantities is investigated. The elliptic difference case is also studied.

1 Introduction

Since the introduction of the singularity confinement method by Grammaticos et al. [9, 15], the seeking of discrete integrable systems progressed extensively. Particularly, based on the pioneering work by Looijenga [14], Sakai classified the relationship between discrete Painlevé equations and rational surfaces [17]. On the other hand, it has become clear that the property of singularity confinement (which is equivalent to the possibility of lifting maps to the sequence of isomorphisms of rational surfaces) does not guarantee integrability, and the notion of growth order of the algebraic degree was proposed as its reinforcement [11, 19, 22]. While the two-dimensional case has been studied in detail, there are few studies that include the relationship between higher-dimensional integrable systems and higher-dimensional algebraic varieties.

As studied by Coble [5] and Dolgachev-Ortland [6], it is known that some Weyl groups act as pseudo-isomorphisms (isomorphisms except sub-varieties of co-dimension 2 or higher) on a family of rational varieties obtained by successive blow-ups at \(m\) (\(m \geq n+2\)) points in \(\mathbb{P}^n(\mathbb{C})\). Here, the Weyl group is given by the Dynkin diagram in Fig. 1 and denoted by \(W(n, m)\). The Weyl group \(W(n, m)\) increases in size if \(m\) increases (Table 1).

In this paper, we study the dynamical systems defined by translations of the affine Weyl group with the symmetric Cartan matrix included in \(W(n, m)\). If \(m \geq n + 7\), \(W(n, m)\) includes the affine Weyl group of type \(E_8^{(1)}\). However, it is adequate to consider the case of \(m = n + 7\), because the affine Weyl group acts trivially on the part obtained by the blow-up at the \(i\)-th point for \(i > n + 7\). The same argument holds true for the case of \(m = n + 4\).
Figure 1: $W(n, m)$ Dynkin diagram

Table 1: Types of Weyl groups

| $n \setminus m$ | 6   | 7   | 8   | 9  | 10 | 11 | 12 |
|-----------------|-----|-----|-----|----|----|----|----|
| 2               | $E_6$ | $E_7$ | $E_8$ | $E_8^{(1)}$ | ind.* | ind.* | ind.* |
| 3               | $D_6$ | $E_7$ | $E_7^{(1)}$ | ind.† | ind.*† | ind.*† | ind.*† |
| 4               | $A_6$ | $D_7$ | $E_8$ | ind.‡ | ind.†‡ | ind.*‡ | ind.*‡ |
| 5               | -    | $A_7$ | $D_8$ | $E_8^{(1)}$ | ind.¶ | ind.†¶ | ind.*¶ |
| 6               | -    | -    | $A_8$ | $D_9$ | ind.‖ | ind.†‖ | ind.*‖ |
| 7               | -    | -    | -    | $A_9$ | $D_{10}$ | ind. ‖ | ind. ‖ |

1. "ind." implies an indefinite type.
2. *, †, and ‡ imply that $W(E_8^{(1)})$, and $W(E_7^{(1)})$ are included, respectively.

$(n \geq 5)$ or $n + 5$ $(n \geq 3)$. The Weyl group $W(n, m)$ includes the affine Weyl group of type $E_8^{(1)}$ or type $E_7^{(1)}$ in each case. In the case of $n = 2, m = 9$, our dynamical system coincides with the elliptic difference Painlevé equation proposed by Sakai [17], from which discrete and continuous Painlevé equations are obtained by degeneration. The case when $n = 3, m = 8$ was studied in [21], and it was revealed that this system can be reduced to the case of $n = 2, m = 9$.

Our dynamical systems possess commuting flows by construction, while a general element of infinite order in the Weyl group of an indefinite type does not possess such flows [19]. In this paper, we show that the degrees of the systems grow in the quadratic order, and therefore, their algebraic entropy is zero. We employ the theory on the relationships between the algebraic degree and the actions on cohomology groups for birational (or rational) dynamical systems, which has been studied by several authors for two-dimensional autonomous cases (for example, Bellon [2], Cantat [4], and Diller-Favre [7]). In [20], the author studied two-dimensional non-autonomous birational cases and showed that the degrees of discrete Painlevé equations grow at most quadratically. Bedford and Kim [1] recently analyzed these relationships for certain higher-dimensional autonomous dynamical systems proposed by Borkaa et al. [3], which occur as a degenerated case in our systems.

Investigation of preserved quantities of our systems are divided into (i) to solve for parameters (if it is achieved, the parameters are considered to be independent variables), and (ii) to solve for dependent variables. Although our systems preserve some hypersurfaces, it might be impossible to solve them even for their parameters. However, it can be done if all the points of the blow-ups lie on an elliptic curve. We also present a conjecture related to preserved quantities for dependent variables in this case.

This article is organized as follows. In Section 2, we review the relationship between
rational varieties and groups of Cremona transformations, and we introduce \( n \)-dimensional dynamical systems associated with affine Weyl groups. In Section 3, we calculate the algebraic degrees of the systems and show that they grow quadratically. In Section 4, the existence of preserved quantities for \( E_7^{(1)} \) and \( E_8^{(1)} \) cases is investigated. In Section 5, the case where all the points of the blow-ups lie on an elliptic curve is studied. Section 6 is devoted to conclusions.

## 2 Defining manifolds and Weyl group actions

Let \( m \geq n + 2 \). Let \( X_{n,m} \) be the configuration space of ordered \( m \) points in \( \mathbb{P}^n \):

\[
PGL(n+1) \bigg/ \left\{ \begin{pmatrix} a_{01} & \cdots & a_{0m} \\ a_{11} & \cdots & a_{1m} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \right\} \bigg/ \left( \mathbb{C}^\times \right)^m,
\]

where \( PGL(n+1) \) denotes the group of general projective linear transformations of the dimension \( n + 1 \). \( X_{n,m} \) is a quasi-projective variety of the dimension \( n(m - n - 2) \). We also consider \( X_{n,m}^1 \cong X(n,m + 1) \) with a natural projection \( \pi : X_{n,m}^1 \to X_{n,m} \):

\[
\begin{pmatrix} a_{01} & \cdots & a_{0m} & x_0 \\ a_{11} & \cdots & a_{1m} & x_1 \\ \vdots & \cdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & x_n \end{pmatrix} \mapsto \begin{pmatrix} a_{01} & \cdots & a_{0m} \\ a_{11} & \cdots & a_{1m} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix},
\]

where each fiber is \( \mathbb{P}^n \) and \( X_{n,m} \) is referred to as the parameter space.

Let \( A \in X_{n,m} \), and let \( X_{n,m}^1 \) be the rational variety obtained by successive blow-ups at the points \( P_i = (a_{0i} : \cdots : a_{ni}) \) \((i = 1, 2, \cdots, m)\) from \( \mathbb{P}^n \). We denote the family of rational projective varieties \( X_A \) \((A \in X_{n,m})\) as \( \widetilde{X}_{n,m}^1 \), which also has the natural fibration \( \pi : \widetilde{X}_{n,m}^1 \to X_{n,m} \).

Let \( E \) be the divisor class on \( X_A \) of the total transform of a hyper-plane in \( \mathbb{P}^n \), and let \( E_i \) be the exceptional divisor class generated by the blow-up at a point \( P_i \). The group of divisor classes of \( X_A \): \( \text{Pic}(X_A) \simeq H^1(X_A, \mathcal{O}^\times) \simeq H^2(X_A, \mathbb{Z}) \) (the second equivalence arises from the fact that \( X_A \) is a rational projective variety) is described as the lattice

\[
\text{Pic}(X_A) = \mathbb{Z}E \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \cdots \oplus \mathbb{Z}E_m.
\]

It should be noted that this cohomology group is independent of \( A \), while \( X_A \) is not isomorphic to \( X_A' \) for \( A' \neq A \) \( \in X_{n,m} \) in general.

Let \( e \in H_2(X_A, \mathbb{Z}) \) be the class of a generic line in \( \mathbb{P}^n \), and let \( e_i \) be the class of a generic line in the exceptional divisor of the blow-up at a point \( P_i \). Then, \( e, e_1, e_2, \ldots, e_m \) form a basis of \( H_2(X_A, \mathbb{Z}) \simeq (H^2(X_A, \mathbb{Z}))^* \) (the equivalence is guaranteed by the Poincaré duality), and the intersection numbers are given by

\[
\langle E, e \rangle = 1, \langle E, e_j \rangle = 0, \langle E_i, e \rangle = 0, \langle E_i, e_j \rangle = -\delta_{i,j}.
\]
Following Dolgachev-Ortland [6], we adopt the root basis \( \{ \alpha_0, \cdots, \alpha_{m-1} \} \subset H^2(X_A, \mathbb{Z}) \) and the co-root basis \( \{ \alpha^\vee_0, \cdots, \alpha^\vee_{m-1} \} \subset H_2(X_A, \mathbb{Z}) \) as

\[
\begin{align*}
\alpha_0 &= E - E_1 - E_2 - \cdots - E_{n+1}, \\
\alpha^\vee_i &= (n-1)e - e_1 - e_2 - \cdots - e_{n+1},
\end{align*}
\]

then \( \langle \alpha_i, \alpha^\vee_j \rangle = -2 \) holds for any \( i \), and these root bases define the Dynkin diagram of type \( T_{2,n+1,m-n-1} \) by assigning a root \( \alpha_i \) to each vertex \( \alpha_i \) and connecting two distinct vertices \( \alpha_i \) and \( \alpha_j \) if \( \langle \alpha_i, \alpha^\vee_j \rangle = 1 \) (in our case \( \langle \alpha_i, \alpha^\vee_j \rangle = 0 \) or 1 for \( i \neq j \)) (Fig. 1).

Let us define the root lattice \( Q = Q(n, m) \subset H^2(X_A, \mathbb{Z}) \) and the co-root lattice \( Q^\vee = Q^\vee(n, m) \subset H_2(X_A, \mathbb{Z}) \) as \( Q = \mathbb{Z} \alpha_0 \oplus \mathbb{Z} \alpha_1 \oplus \cdots \oplus \mathbb{Z} \alpha_{m-1} \) and \( Q^\vee = \mathbb{Z} \alpha^\vee_0 \oplus \mathbb{Z} \alpha^\vee_1 \oplus \cdots \oplus \mathbb{Z} \alpha^\vee_{m-1} \), respectively. For each \( \alpha_i \), the formulae

\[
\begin{align*}
\tau_{\alpha_i}(D) &= D + \langle D, \alpha^\vee_i \rangle \alpha_i \quad \text{for any } D \in Q \\
\tau_{\alpha_i}(d) &= d + \langle \alpha_i, d \rangle \alpha^\vee_i \quad \text{for any } d \in Q^\vee
\end{align*}
\]

define linear involutions (termed simple reflections) of the bi-lattice \( (Q, Q^\vee) \), and they generate the Weyl group \( W \) of type \( T_{2,n+1,m-n-1} \), which we denote as \( W_s(n, m) \).

These simple reflections correspond to certain birational transformations on the fiber space \( \pi : X^1_{n,m} \rightarrow X_{n,m} \). Let us define birational transformations \( r_{i,j} \) \( (1 \leq i < j \leq m) \) and \( r_{i_0,i_1,\cdots,i_n} \) \( (1 \leq i_0 < \cdots < i_n \leq m) \) on the fiber space as follows:

\[
r_{i,j} : ( \cdots \mid \; a_i \; \mid \; \cdots \mid \; a_j \; \mid \; \cdots \mid \; x \;) \rightarrow ( \cdots \mid \; a_j \; \mid \; \cdots \mid \; a_i \; \mid \; \cdots \mid \; x \;);
\]

and \( r_{i_0,i_1,\cdots,i_n} \) is the standard Cremona transformation with respect to the points \( P_{i_0}, P_{i_1}, \cdots, P_{i_n} \), for example, \( r_{1,2,\cdots,n+1} \) is the composition of a projective transformation and the standard Cremona transformation with respect to the origins \( (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \) as

\[
r_{1,2,\cdots,n+1} : (A \mid x) = ( A_1, \cdots, n+1 \mid A_{n+2}, \cdots, m \mid x \;) \rightarrow A^{-1}_{1,\cdots,n+1}(A \mid x) \]

\[
= \begin{pmatrix}
I_{n+1} & \cdots & a'_{ij} & \cdots & x'_{ij} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \vdots \\
\vdots & & & & \ddots
\end{pmatrix}
\rightarrow \begin{pmatrix}
I_{n+1} & \cdots & a'^{-1}_{ij} & \cdots & x'^{-1}_{ij} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \vdots \\
\vdots & & & & \ddots
\end{pmatrix},
\]

where \( A_{j_1,j_2,\cdots,j_k} \) denotes the \((n+1) \times k\) matrix \( (a_{j_1} \mid \cdots \mid a_{j_k}) \), and \( I_k \) denotes the \( k \times k \) identity matrix.

Remark 2.1. The birational map \( r_{i,j} \) cannot be defined on the fibers of the parameters that satisfy \( P_i = P_j \), in which case we blow-up at \( P_j \) after at \( P_i \). Although the points \( P_i \) and \( P_j \) should be exchanged by \( r_{i,j} \), it is impossible to simultaneously preserve the order of blow-ups. The condition \( P_i = P_j \) is equivalent to the condition that the divisor class \( \alpha_i + \cdots + \alpha_{j-1} = E_i - E_j \in H^2(X_A, \mathbb{Z}) \) is effective. This situation was studied in detail by Saito and Umemura [16] in \( n = 2 \) cases and its relationship with the notion of flop was revealed. The birational map \( r_{1,2,\cdots,n+1} \) cannot be defined for the fibers of parameters satisfying \( A_{1,\cdots,n+1} = 0 \), which is equivalent to the points \( P_1, P_2, \cdots, P_{n+1} \) being on the same hyper-plane in \( \mathbb{P}^n \) and the divisor class \( \alpha_0 = E - E_1 - E_2 - \cdots - E_{n+1} \in H^2(X_A, \mathbb{Z}) \) being effective. Such a hyper-surface is referred nodal one (cf. Prop. 2.3).
Let $w$ denote the reflection $r_{i,j}$ or $r_{i_0,i_1,\ldots,i_n}$. The reflection $w$ acts on the parameter space $X_{n,m}$ and preserves the fibration $\pi : X_{n,m}^1 \to X_{n,m}$. As previously mentioned, $H^2(X_A,\mathbb{Z})$ is independent of $A \in X_{n,m}$. Hence, $w$ defines an action on this cohomology group. Moreover, the induced birational map $w : X_A \to X_{w(A)}$ for generic $A \in X_{n,m}$ is a pseudo-isomorphism, i.e., an isomorphism except sub-manifolds of co-dimension 2 or higher, and the lines corresponding to the classes $e$ and $e_i$ can be chosen in a manner such that they do not meet the excluded part. Since $H_2(X_A,\mathbb{Z})$ is also independent of $A \in X_{n,m}$, $w$ defines an action on this homology group and preserves the intersection form $\langle \cdot, \cdot \rangle : H^2(X_A,\mathbb{Z}) \times H_2(X_A,\mathbb{Z}) \to \mathbb{Z}$.

The birational maps $r_{i,i+1}$ and $r_{1,2,\ldots,n+1}$ correspond to the simple reflections $r_{\alpha_i,\bar{\alpha}}$ $(1 \leq i \leq m-1)$ and $r_{0,\alpha}$, respectively. Indeed, their push-forward actions on $H^2(X_A,\mathbb{Z}) \to H^2(X_{w(A)},\mathbb{Z})$ ($w = r_{i,i+1}$ or $r_{1,2,\ldots,n+1}$) and on $H_2(X_A,\mathbb{Z}) \to H_2(X_{w(A)},\mathbb{Z})$ are given by the formulae:

$$
\begin{align*}
 r_{i,i+1}(D) &= D + \langle D, \alpha_i^\vee \rangle \alpha_i \\
r_{i,i+1}(d) &= d + \langle \alpha_i, d \rangle \alpha_i^\vee \\
r_{1,2,\ldots,n+1}(D) &= D + \langle D, \alpha_0^\vee \rangle \alpha_0 \\
r_{1,2,\ldots,n+1}(d) &= d + \langle \alpha_0, d \rangle \alpha_0^\vee
\end{align*}
$$

(3)

for any $D \in H^2(X_A,\mathbb{Z})$ and any $d \in H_2(X_A,\mathbb{Z})$. The formulae (3) are extensions of (2) onto these (co)-homology groups.

Let $W(n,m)$ denote the group generated by $r_{i,i+1}$ ($i = 1, 2, \ldots, m-1$) and $r_{1,2,\ldots,n+1}$. It should be noted that $W(n,m)$ acts on the $X_{n,m}$ except “the nordal set” (see Prop. 2.4).

**Lemma 2.2.** $W(n,m) \cong W_s(n,m)$ holds during the correspondence $r_{i,i+1} \simeq r_{\alpha_i,\bar{\alpha}}$ $(1 \leq i \leq m-1)$ and $r_{1,2,\ldots,n+1} \simeq r_{0,\alpha}$.

**Proof.** It is sufficient to show that for any $w_s \in W_s(n,m)$, there exists a unique element $w \in W(n,m)$, which coincides with $w_s$ on $Q(n,m)$. From (3), such $w \in W(n,m)$ can be easily constructed. Hence, we show the uniqueness of such $w$.

Let both $w, w' \in W(n,m)$ coincide with $w_s \in W_s(n,m)$ on $Q(n,m)$. It is sufficient to prove that $w^{-1} \circ w'$ is the identity on $\Pic(X_A) \cong H^2(X_A,\mathbb{Z})$.

Since $w^{-1} \circ w'$ induces the identity of $Q(n,m)$ and $Q'(n,m)$, we can assume that $w^{-1} \circ w'$ acts on $H^2(X_A,\mathbb{Z})$ and on $H_2(X_A,\mathbb{Z})$ as $E \mapsto E + (n+1)xK, E_i \mapsto E_i + xK, e_i \mapsto e_i + yK$ for some $x, y \in \mathbb{R}, K \in H^2(X_A,\mathbb{Z})$ and $k \in H_2(X_A,\mathbb{Z})$. By $\langle \alpha_i, d \rangle = \langle \alpha_i, (w^{-1} \circ w')(d) \rangle$ and $\langle D, \alpha_i^\vee \rangle = \langle (w^{-1} \circ w')_*(D), \alpha_i^\vee \rangle$ for $D = E, E_i$ and $d = e, e_i$, we have $K = \frac{n+2}{n+1}E - E_1 - E_2 - \cdots - E_m$ and $k = (n+1)e - e_1 - e_2 - \cdots - e_m$.

By $\langle D, d \rangle = \langle (w^{-1} \circ w')_*(D), (w^{-1} \circ w')(d) \rangle$, we have

$$
\begin{align*}
 x + y + pxy &= 0 \\
 x + \frac{n+1}{n+2}y + \frac{n+1}{n+2}pxy &= 0 \\
 (n+1)x + y + (n+1)pxy &= 0,
\end{align*}
$$

where $p = \frac{(n+1)^2}{n+1} - m$; therefore, $x = y = 0$ holds.

For a real root $\alpha = w_s(\alpha_i)$ with $w \in W(n,m)$, we also define the reflection $r_\alpha$ as $w \circ r_{\alpha_i} \circ w^{-1}$. Then, the formulae

$$
\begin{align*}
r_{\alpha*}(D) &= D + \langle D, \alpha^\vee \rangle \alpha \\
r_{\alpha*}(d) &= d + \langle \alpha, d \rangle \alpha^\vee
\end{align*}
$$

(4)
hold for any $D \in H^2(X_A, \mathbb{Z})$ and any $d \in H_2(X_A, \mathbb{Z})$. It should be noted that both $E - E_{i_0} - E_{i_1} - \cdots - E_{i_n}$ and $E_i - E_j$ are real roots and their dual roots are $(n - 1)e - e_{i_0} - e_{i_1} - \cdots - e_{i_n}$ and $e_i - e_j$, respectively.

We have shown the following proposition.

**Proposition 2.3** (Coble, Dolgachev-Ortland). Let $m \geq n + 2$.

1. The actions $r_{i,i+1}$ (1 ≤ $i$ ≤ $m - 1$) and $r_{1,2,\ldots,n+1}$ generate the Weyl group $W(n, m)$ corresponding to the Dynkin diagram of Fig. 2.

2. Each element $w \in W(n, m)$ defines an action on $H^2(X_A, \mathbb{Z})$ and $H_2(X_A, \mathbb{Z})$, and preserves the intersection form ⟨, ⟩: $H^2(X_A, \mathbb{Z}) \times H_2(X_A, \mathbb{Z}) \to \mathbb{Z}$.

3. The birational maps $r_{i,i+1}$ and $r_{1,2,\ldots,n+1}$ correspond to the simple reflections $r_{\alpha_i}$ (1 ≤ $i$ ≤ $m - 1$) and $r_{\alpha_0}$, respectively. For a real root $\alpha = w(\alpha_i)$ and the reflection $r_\alpha = w \circ r_{\alpha_i} \circ w^{-1}$, the formulae (3) hold for any $D \in H^2(X_A, \mathbb{Z})$ and any $d \in H_2(X_A, \mathbb{Z})$.

4. Each $r_{i,j}$ or $r_{i_0,\ldots,i_n}$ is an element of $W(n, m)$.

**Proposition 2.4.** The pseudo isomorphism $w : X_A \to X_{w(A)}$ is defined for any $w \in W(n, m)$ if and only if the divisor class $w_*(\alpha_0)$ is not effective for any $w \in W(n, m)$. A hyper-surface in $X_A$ is referred to as nodal if its class is $w_*(\alpha_0)$ for some $w \in W(n, m)$.

**Proof.** Suppose $w_*(\alpha_0)$ is effective in $X_A$ for some $w$, then $\alpha_0$ is effective in $X_{w^{-1}(A)}$ and hence $r_{\alpha_0} \circ w^{-1} : X_A \to X_{r_{\alpha_0}w^{-1}(A)}$ is not defined. Conversely, suppose $w_*(\alpha_0)$ is not effective for any $w$. If $w : X_A \to X_{w(A)}$ is defined for some $w$, then $\alpha_0 = w_*(w^{-1}(\alpha_0))$ is not effective in $X_{w(A)}$ and hence $r_{\alpha_i} \circ w : X_A \to X_{r_{\alpha_i}w(A)}$ is also defined. By induction, we can define $w : X_A \to X_{w(A)}$ for any $w$.

Let $N_{n,m}$ denote the set of $A \in X_{n,m}$ such that $X_A$ admits a nodal hyper-surface. From Prop. 2.3, the Weyl group $W(n, m)$ acts on $X_{n,m} \setminus N_{n,m}$ as an automorphism.

In many cases, the Weyl group $W(n, m)$ of an affine or indefinite type may include affine sub-groups of the type $A^{(1)}, D^{(1)}, E^{(1)}$ (symmetric Cartan matrix type). For example, when $m \geq n + 7$, $W(n, m)$ includes the affine Weyl group of type $E_8^{(1)}$, as shown in Fig. 3. However, it is sufficient to consider the case of $m = n + 7$, because the affine Weyl group is generated by $r_{1,2,\ldots,n+1}, r_{n-1,n}, r_{n,n+1}, \ldots, r_{n+6,n+7}$ and hence it acts trivially on the part obtained by the blow-up at the $i$-th point for $i \geq n + 7$. When $n \geq 3, m \geq n + 5$, $W(n, m)$ includes the affine Weyl group of type $E_7^{(1)}$ as shown in Fig. 4.

Next, based on Kac’s book [12] (§ 6.5), we introduce translations on $H^2(X_A, \mathbb{Z})$ and $H_2(X_A, \mathbb{Z})$, which were defined in the vector spaces $\mathfrak{h}$ and $\mathfrak{h}^\vee$ in that book, and we realize
them as birational transformations on $\tilde{X}_{n,m}^1$. These translations can be considered as autonomous dynamical systems on $\tilde{X}_{n,m}^1$ or non-autonomous ones on $\mathbb{P}^3$.

i) Let $\beta_0, \beta_1, \cdots, \beta_l \in Q(n, m)$ generate an affine Weyl group $W(R^{(1)})$ of symmetric type s.t. $\beta_1, \cdots, \beta_l$ generate the finite Weyl group $W(R)$. Let $Q(R^{(1)})$ denote the root lattice generated by $\beta_i$’s and let $\delta$ and $\delta^\vee$ denote the null vector and its dual, respectively. Here, $\langle \delta, \alpha^\vee \rangle = \langle \alpha, \delta^\vee \rangle = 0$ holds.

For example, in the $E_8^{(1)}$ case, we have $\beta_0 = \alpha_{n+6}, \beta_i = \alpha_{i+n-2}$ ($1 \leq i \leq 7$), $\beta_8 = \alpha_0$ and

\[
\begin{align*}
\delta &= 3\beta_8 + 2\beta_1 + 4\beta_2 + 6\beta_3 + 5\beta_4 + 4\beta_5 + 3\beta_6 + 2\beta_7 + \beta_0 \\
&= 3E - 3 \sum_{i=1}^{n-2} E_i - \sum_{i=n-1}^{n+7} E_i \\
\delta^\vee &= 3\beta_8^\vee + 2\beta_1^\vee + 4\beta_2^\vee + 6\beta_3^\vee + 5\beta_4^\vee + 4\beta_5^\vee + 3\beta_6^\vee + 2\beta_7^\vee + \beta_0^\vee \\
&= 3(n-1)e - 3 \sum_{i=1}^{n-2} e_i - \sum_{i=n-1}^{n+7} e_i.
\end{align*}
\]

ii) Let $\theta$ denote the highest root $\delta - \beta_0$ and let $\theta^\vee$ denote $\delta^\vee - \beta_0^\vee$, then $\theta$ is a positive root of $W(R)$ and thus a positive real root of $W(R^{(1)})$. Let $t_{\beta_0}$ denote the birational transformation $r_{\theta} \circ r_{\beta_0}$ on $\tilde{X}_{n,m}^1$. From \textbf{(4)}, for any $D \in H^2(X_A, \mathbb{Z})$ and any $d \in H_2(X_A, \mathbb{Z})$, we have

\[
t_{\beta_0}(D) = r_{\theta_*}(D + \langle D, \beta_0^\vee \rangle \beta_0) = D - \langle D, \delta^\vee \rangle \alpha_{n+6} + \langle D, \delta^\vee + \beta_0^\vee \rangle \delta.
\]

and

\[
t_{\beta_0}(d) = d - \langle \delta, d \rangle \beta_0^\vee + \langle \delta + \beta_0, d \rangle \delta^\vee.
\]

iii) For any element $\beta$ of the root lattice $Q(R^{(1)})$, we define $t_{\beta_*}$ as

\[
t_{\beta_*}(D) = D - \langle D, \delta^\vee \rangle \beta + \langle D, -\frac{1}{2} \langle \beta, \beta^\vee \rangle \delta^\vee + \beta^\vee \rangle \delta.
\]
and
\[ t_{\beta_*}(d) = d - \langle \delta, d \rangle \beta^\vee + \langle -\frac{1}{2}(\beta, \beta^\vee)\delta + \beta, d \rangle \delta^\vee. \]

iv) Additivity of \( t_{\beta_*} \). For any \( \alpha, \beta \in Q(R^{(1)}) \),
\[ t_{\alpha*} \circ t_{\beta_*} = t_{\alpha + \beta_*} \]
holds. In fact, since \( \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle \) for symmetric root systems, we have \( t_{\alpha*} \circ t_{\beta_*}(D) = t_{\alpha*}(D - \langle D, \delta^\vee \rangle \alpha + \langle D, -\frac{1}{2}(\beta, \beta^\vee)\delta^\vee + \beta^\vee \rangle \delta) = D - \langle D, \delta^\vee \rangle (\alpha + \beta) + \langle D, -\frac{1}{2}(\alpha, \alpha^\vee) - \frac{1}{2}(\beta, \beta^\vee) - \langle \beta, \alpha^\vee \rangle \rangle \delta^\vee + \alpha^\vee + \beta^\vee \rangle \delta = t_{\alpha + \beta_*} \).

v) For any \( w \) in \( W(R^{(1)}) \) and \( \beta \in Q(R^{(1)}) \), we have
\[ t_w(\beta)_* = w_* \circ t_{\beta_*} \circ w_*^{-1}. \]
Indeed, \( w_* \circ t_{\beta_*} \circ w_*^{-1}(D) = w_*(w_*^{-1}(D) - \langle \omega^{-1}(D), \delta^\vee \rangle \beta + \langle \omega_*^{-1}(D), \frac{1}{2}(\beta, \beta^\vee)\delta^\vee + \beta^\vee \rangle \delta) \).

Now, since \( w_*(\delta) = \delta \) and \( \langle , , \rangle \) is preserved by \( w \), (8) holds.

vi) Since for any \( \beta \) there exists \( w \in W(R^{(1)}) \) such that \( \beta_i = w_*(\beta_0) \), any real root \( \beta \) of \( W(R^{(1)}) \) can be written as \( \beta = w_*(\beta_0) \) for some \( w \in W(R^{(1)}) \). Thus, for any real root \( \beta \),
\[ t_{\beta_*} = w_* \circ t_{\beta_0} \circ w_*^{-1} \]
holds. From (7), for any element \( \beta = k_0\beta_0 + k_1\beta_{n-1} + \cdots + k_l\beta_l \) of the root lattice \( Q(R^{(1)}) \), \( t_{\beta_*} = t_{\beta_0}^{k_0} \circ t_{\beta_1}^{k_1} \circ \cdots \circ t_{\beta_l}^{k_l} \) is an element of \( W(R^{(1)}) \).

vii) For \( \beta_i \), we define the birational transformation \( t_\beta \in W(n, m) \) on \( \bar{X}_{n,m}^1 \) by
\[ t_{\beta_i} = w \circ t_{\beta_0} \circ w^{-1}, \]
where \( w \in W(R^{(1)}) \) such that \( \beta_i = w(\beta_0) \). Moreover, for any \( \beta = k_0\alpha_0 + k_1\beta_1 + \cdots + k_l\beta_l \in Q(R^{(1)}) \), we define the birational transformation \( t_\beta \in W(n, m) \) on \( \bar{X}_{n,m}^1 \) by \( t_\beta = t_{\beta_0}^{k_0} \circ t_{\beta_1}^{k_1} \circ \cdots \circ t_{\beta_l}^{k_l} \). Now, on the level of birational transformations, we have \( t_{\alpha*} \circ t_{\beta} = t_{\alpha + \beta} \) and \( t_{w_*(\alpha)} = w_\circ t_{\alpha} \circ w^{-1} \) for any \( \alpha, \beta \in Q(R^{(1)}) \) and any \( w \in W(R^{(1)}) \).

These formulae are guaranteed by the fact that the action on \( H^2(X_A, \mathbb{Z}) \) uniquely determines a birational transformation in \( W(n, m) \) if it exists.

Now, we have:

**Proposition 2.5.** Let \( \beta \in Q(R^{(1)}) \), there exists a unique birational transformation \( t_\beta \in W(n, m) \) which acts on \( H^2(X, \mathbb{Z}) \) and \( H_2(X, \mathbb{Z}) \) as (2) and (6).

**Example 2.6.** In \( E_8^{(1)} \) case, \( t_{\beta_8} = t_{\alpha_0} \) is given by
\[ t_{\alpha_0} = r_{1,2,\ldots,n-2,n+2,n+3,n+4} \circ r_{1,2,\ldots,n-2,n+5,n+6,n+7} \circ r_{1,2,\ldots,n-2,n+2,n+3,n+4} \circ r_{1,2,\ldots,n-2,n-1,n+1}. \]

In the case of \( n = 2, m = 9 \), this system coincides with the elliptic difference Painlevé equation proposed by Sakai [17].

From Prop. 2.5 the following theorem holds.
Theorem 2.7. Let \( l \geq 2 \) and let \( \beta_0, \beta_1, \cdots, \beta_l \in Q(n, m) \) generate an affine Weyl group \( W(R^{(l)}) \) with symmetric Cartan matrix. Let \( \beta \in Q(R^{(l)}) \) s.t. \( \beta \notin \mathbb{Z}\delta \). There exist non-trivial flows commuting with \( t_\beta \) on \( \overline{X_{n,m}} \), i.e. 
\[ \exists t' \in W \text{ s.t.} \]
\[ (t')^k \neq (t_\alpha)^l \quad (k, l \in \mathbb{Z}, k \neq 0) \text{ and } t_\alpha \circ t' = t' \circ t_\alpha \]

3 Algebraic degrees of dynamical systems

Let \( X, Y, Z \) be varieties obtained by blow-ups from \( \mathbb{P}^n \), and let \( \varphi : X \twoheadrightarrow Y \) be a dominant rational map, i.e., the closure of \( \varphi(X) \) coincides with \( Y \). Let \( \varphi^* \) denotes the pull-buck of \( \varphi : \text{Pic}(Y) \to \text{Pic}(X) \). We define the algebraic degree of \( \varphi \) by \( \deg \varphi \) as the degree of the deduced rational map \( \varphi : \mathbb{P}^n \twoheadrightarrow \mathbb{P}^n \).

Proposition 3.1. The degree of \( \varphi \) coincides with the coefficient of \( E \) of \( \varphi^*(E) \), where \( E \) denotes the class of a hyper-plane in \( \mathbb{P}^n \).

Proof. Write \( \varphi : \mathbb{P}^n \twoheadrightarrow \mathbb{P}^n \) as \((f_0(x)) : f_1(x) : \cdots : f_n(x))\) by polynomials \( f_i \), where \( f_i \)'s are simplified if possible. The class \( E \in \text{Pic}(Y) \) corresponds to a hyper-plane in \( Y \):
\[ a_0y_0' + a_1y_1' + \cdots + a_ny_n' = 0 \]
and the class \( \varphi^*(E) \) corresponds to the hyper-surface in \( X \):
\[ a_0f_0(x) + a_1f_1(x) + \cdots + a_nf_n(x) \]
Hence the coefficient of \( E \) of \( \varphi^*(E) \) is the degree of the polynomials \( f_i \).

Remark 3.2. Let \( X_i \ (i \in \mathbb{Z}) \) be varieties obtained by blow-ups from \( \mathbb{P}^n \), and let \( \varphi_i : X_i \twoheadrightarrow X_{i+1} \) be a dominant rational map.
\[ \lim \inf_{k \to \infty} \frac{1}{k} \log \deg(\varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_0) \quad \text{(10)} \]
is referred to as the algebraic entropy of the dynamical systems \( \{\varphi_i \} \). This notion is closely related to other entropies. Indeed, by Gromov-Yomdin’s theorem \[10, 23\], for a surjective morphism \( f \) from a Kähler complex manifold \( X \) to itself, the topological entropy of \( f \) equals \( \log \lambda(f) \), where \( \lambda(f) \) is the spectral radius of \( f^* : H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R}) \).

On the other hand, its algebraic degree is given (or defined) by the action on a linear subspace \( H^2(X, \mathbb{R}) \).

Definition. For a morphism \( \psi : X \to Y \), let \( \psi^* : \mathcal{P}(Y) \to \mathcal{P}(X) \) denote the set-correspondence defined by the pull-buck, i.e. for \( V \subset Y \), \( \psi^*(V) \) is defined by
\[ \psi^*(V) = \{ x \in X \ ; \ \psi(x) \in V \} \]
Let \( \varphi : X \twoheadrightarrow Y \) be a dominant rational map. Let \( X' \) be the varieties obtained by successive blow-ups \( \pi : X' \twoheadrightarrow X \), which eliminates the determinacy of \( \varphi \), and let \( \varphi' : X' \to Y \) be the map lifted from \( \varphi \) (Fig. 5). The set correspondence \( \varphi_{c*} : \mathcal{P}(X) \to \mathcal{P}(Y) \) is defined by
\[ \varphi_{c*} := \varphi' \circ \pi_{c*} \]
For any $i,j$,

\[ \varphi = (\varphi^{-1})^*_c \text{ if } \varphi \text{ is birational} \]

and the set correspondence $\varphi^*_c : \mathcal{P}(Y) \to \mathcal{P}(X)$ is defined by

\[ \varphi^* := \pi \circ \varphi^*_c. \]

**Definition.** We define the indeterminate component set $\mathcal{C}(\varphi)$ for a dominant rational map $\varphi : X \dashrightarrow Y$ as

\[
\mathcal{I}(\varphi) := \{ L \in X; \exists S \subset Y: \text{irreducible hyper-surface s.t. } L \text{ is an irreducible component of } \varphi^*_c(S) \text{ and } \dim L < n - 1 \} \\
\mathcal{C}(\varphi) := \{ L \in Y; \exists S \subset X: \text{irreducible hyper-surface s.t. } L \text{ is an irreducible component of } \varphi^*_c(S) \text{ and } \dim L < n - 1 \}.
\]

**Proposition 3.3.** Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be dominant rational maps. Then

\[ (g \circ f)^*(D) = f^* \circ g^*(D) \]

holds if and only if $\mathcal{C}(f) \cap \mathcal{I}(g) = \emptyset$.

**Proof.** Without loss of generality, we can assume that $D$ is an irreducible hyper-surface. There exist irreducible hyper-surfaces $D_1, D_2, \ldots, D_m \subset Y$, subvarieties $L_1, L_2, \ldots, L_l \in \mathcal{I}(g)$, and positive integers $r_1, r_2, \ldots, r_m$ such that $g^*_c(D) = D_1^r + D_2^r + \cdots + D_m^r + L_1 + \cdots + L_l$ (“+” implies set sum) and $g^*(D) = r_1D_1 + r_2D_2 + \cdots + r_mD_m$. Similarly, for each $D_i$, there exist irreducible hyper-surfaces $D_{i,1}, D_{i,2}, \ldots, D_{i,m_i} \subset X$, subvarieties $L_{i,1}, L_{i,2}, \ldots, L_{i,m_i} \in \mathcal{I}(f)$, and positive integers $r_{i,1}, r_{i,2}, \ldots, r_{i,m_i}$ such that $f^*_c(D_i) = D_{i,1}^r + \cdots + D_{i,m_i}^r + L_{i,1} + \cdots + L_{i,m_i}$ and $f^*(D_i) = r_{i,1}D_{i,1} + \cdots + r_{i,m_i}D_{i,m_i}$. Hence, we have

\[
f^* \circ g^*(D) = \sum_{i=1}^m \sum_{j=1}^{m_i} r_{i,j}D_{i,j}.
\]

Now, it can be easily observed that $(g \circ f)^*(D) - f^* \circ g^*(D) \geq 0$ holds, and the equality holds iff $L_i \in \mathcal{C}(f)$ for some $i$. \hfill $\Box$

The following proposition immediately follows from Prop. 3.3.

**Proposition 3.4.** Let $X_i$ ($i \in \mathbb{Z}$) be varieties obtained by blow-ups from $\mathbb{P}^n$, and let $\varphi_i : X_i \dashrightarrow X_{i+1}$ be a dominant rational map. Then,

\[ (\varphi_{i+j} \circ \cdots \circ \varphi_i)^* = \varphi_{i+j}^* \circ \cdots \circ \varphi_i^* \tag{11} \]

for any $i, j$ if and only if $\mathcal{C}(\varphi_{i+j} \circ \cdots \circ \varphi_i) \cap \mathcal{I}(\varphi_{i+j})$ holds for any $i, j$.  

---

**Figure 5:** Lift of $\varphi$
A sequence of dominant rational maps satisfying condition \((11)\) for any \(i, j \geq 0\) is referred to as “analytically stable” in [18] (cf. [7]).

**Corollary 3.5.** A sequence of pseudo-isomorphisms is analytically stable.

**Example 3.6.** Set \(n = 4\) in example 2.6. Further, an element of \(\text{Pic}(X_A)\) is written as \(bE + b_1E_1 + b_2E_2 + \cdots + b_{11}E_{11}\). For the translation \(t_{a_0}\) in example 2.6, the induced map \(t_{a_0}^* : \text{Pic}(X_{t_{a_0}(A)}) \rightarrow \text{Pic}(X_A)\) is described as a linear transformation for the vector \([b, b_1, b_2, \cdots, b_{11}]^t\) as

\[
\begin{bmatrix}
28 & 9 & 9 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 \\
-27 & -8 & -9 & -1 & -1 & -1 & -4 & -4 & -4 & -4 & -4 \\
-15 & -5 & -5 & 0 & -1 & -1 & -2 & -2 & -2 & -2 & -2 \\
-15 & -5 & -5 & 0 & -1 & -1 & -2 & -2 & -2 & -2 & -2 \\
-6 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-6 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-6 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-6 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-6 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-6 & -2 & -2 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
\end{bmatrix}
\]

Since the cells of the Jordan normal form are

\[
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
\]

the elements of \((t_{a_0}^l)^*\) grow quadratically with respect to \(l\).

The following theorem generalizes this example.

**Theorem 3.7.** Let \(\beta_0, \beta_1, \cdots, \beta_l \in Q(n, m)\) generate an affine Weyl group \(W(R^{(1)})\) of the symmetric type. Let \(\beta \in Q(R^{(1)})\) be not in \(\mathbb{Z}\delta\), and let \(t_\beta\) be the translation defined as in the previous section. There exists \(\lim_{l \to \infty} (\deg t_\beta^l)/l^2 > 0\), and therefore, the algebraic entropy of \(t_\beta\): \(\lim_{l \to \infty} \frac{1}{l} \log \deg t_\beta^l\) is 0.

**Remark 3.8.** From (1), if \(\beta \in \mathbb{Z}\delta\), then \(t_\beta\) is the identity.

**Proof.** From Prop. 2.5, we have

\[
(t_\beta^l)^*(E) = t_{\delta \cdot \beta}^l(E) = t_{-\delta \cdot \beta}^l(E)
\]

\[
= E + l\langle E, \delta \rangle \beta - \frac{1}{2} \langle \beta, \beta \rangle l^2 \delta + l \beta \delta
\]

\[
= E + (n - 1)l \beta - \left(\frac{1}{2} \langle \beta, \beta \rangle (n - 1) l^2 + \langle E, \beta \rangle l\right) \delta;
\]

therefore, the coefficient of \(E\) is in the order \(l^2\) except the case where \(\langle \beta, \beta \rangle = 0 \iff \beta \in \mathbb{Z}\delta\). 

---

\(^1\) In [1], Bedford and Kim referred to this notion “(1, 1)-regularity”. Here, the term “(1, 1)” arises from the fact that \(c_1(\text{Pic}(X)) = c_1(H^1(X, \mathcal{O}^*)) \simeq H^{1,1}(X, \mathbb{Z}) := H^{1,1}(X) \cap H^2(X, \mathbb{Z})\), where \(c_1\) is the 1st Chern map.
### 4 Preserved divisor classes

In this section, for the $E_7^{(1)}$ and $E_8^{(1)}$ cases, we investigate the existence of quantities on the fibers $X_A$ preserved by the translation $t_\beta : X_A \to X_{t_\beta(A)}$.

In order to find preserved quantities, we have to chose coordinates of $X_{1,n,m}$ so that the preserved hyper-surfaces are fixed by the time evolution. It should be noted that the class of the proper transform of a hyper-surface in $\mathbb{P}^3$ is $k_0E - \sum_{i=1}^m k_iE_i \in \text{Pic}(X_A)$, where $k_0 \geq 1$ is the degree of the surface and $k_i \geq 0 (i = 1, 2, \cdots, m)$ is its multiplicity at the point $P_i$.

Let $m \geq n + 7$. We set $\beta_0, \beta_1, \cdots, \beta_l \in Q(n,m)$ in a manner such that they generate the affine Weyl group $W(E_8^{(1)})$, as described in Section 2. The result for the $E_7^{(1)}$ case is presented at the end of this section.

we show the following theorem.

**Theorem 4.1.** Let $\beta \in Q(E_8^{(1)})$ be not in $\mathbb{Z}\delta$. For a generic parameter $A \in X(n,m)$, $\delta$ is a unique effective class (up to a constant multiple) preserved by the dynamical system $t_\beta$.

**Remark 4.2.** Unfortunately, uniqueness is still a conjecture.

The parameters $A \in X(n,m)$ can be normalized as

\[
\begin{pmatrix}
I_{n-2} & b_{11} & b_{12} & \cdots & b_{19} & c_{0,n+8} & \cdots & c_{0,m} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\
b_{n-2,1} & b_{n-2,2} & \cdots & b_{n-2,9} & \vdots & \vdots & & \vdots \\
0 & \varphi(u_1) & \varphi(u_2) & \cdots & \varphi(u_9) & \vdots & \vdots & \cdots \vdots \\
0 & \varphi'(u_1) & \varphi'(u_2) & \cdots & \varphi'(u_9) & c_{n,n+8} & \cdots & c_{n,m}
\end{pmatrix},
\]

(12)

where $(b_{ij})_{1 \leq i \leq n-2, 1 \leq j \leq 4}$ can be fixed as

\[
(b_{ij})_{1 \leq i \leq n-2, 1 \leq j \leq 4} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and $\varphi(u)$ is the Wierstrass $\varphi$ function of periods $(1, \tau)$. The linear system of $\delta$ is then given by the hyper-surface

\[x_{n-1}x_{n+1}^2 = x_n^3 - g_2x_{n-1}^2x_n - g_3x_{n-1}^3,
\]

where $g_2$ and $g_3$ are constants determined by the periods.

The action of $W(E_8^{(1)})$ on $(x_{n-2} : x_{n-1} : x_n) \in \mathbb{P}^3$ is the same as that with Sakai’s elliptic difference Painlevé equation [17]; hence, we arrived at the following corollary.

**Corollary 4.3.** The period $\tau$ is preserved by $W(E_8^{(1)})$, and the actions on parameters $u_i$ are linear transformations, which are described as follows:

i) $r_{\beta_8}$ acts on $u_i$ and on $u (\neq u_i)$ as
\[ u_i \mapsto u_i - \frac{2}{3}(u_1 + u_2 + u_3) \text{ (} 1 \leq i \leq 3), \quad u_i \mapsto u_i + \frac{1}{3}(u_1 + u_2 + u_3); \]
\[ u (\neq u_i) \mapsto u + \frac{1}{3}(u_1 + u_2 + u_3); \]

ii) \( r_{\beta_j} (1 \leq j \leq 7) \) acts as
\[ \begin{align*}
& u_i \mapsto u_i \ (i \neq j, j + 1), \quad u_j \mapsto u_{j+1}, \quad u_{j+1} \mapsto u_j \ \text{u (\neq u_i)} \mapsto u; \\
& \text{iii) } r_{\beta_0} \text{ acts as} \\
& u_i \mapsto u_i \ (i \neq 8, 9), \quad u_8 \mapsto u_9, \quad u_9 \mapsto u_8, \quad u(\neq u_i) \mapsto u.
\end{align*} \]

**Proof of the uniqueness of Th. 4.1**

Fix \( \beta \in Q(E^{(1)}_n) \) and suppose \( D \in \text{Pic}(X_A) \) is preserved by \( t_{\beta} \). Since the coefficients of \( \beta \) and \( \delta \) in formula (5) should be zero, \( D \) is in the sub-lattice spanned by \( \alpha_i(1 \leq i \leq n-3, n-1 \leq i \leq n+6), \delta, \) and the canonical divisor class \( K_{X_A} = -(n+1)E + (n-1) \sum_{i=1}^{m} E_i \).

The proof is based on the following elementary conjecture. The author can prove only the case of \( n = 2, 3 \) in an elementary way.

**Conjecture 4.4.** Let the points \( P_i \ (1 \leq i \leq m) \) be in a generic position. The dimension of the linear system \( |k_0E - \sum_{i=1}^{m} k_i E_i| \) is given by (negative dimension implies the empty set in this case):

\[ n+1H_{k_0} - 1 + \sum_{s=1}^{m} (-1)^s \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq m} \max\{ n+1H_{k_{i_1} + k_{i_2} + \cdots + k_{i_s} - (s-1)k_0 - s} \}, (13) \]

where \( uH_v = \binom{u+v-1}{v} \) denotes the repeated combination.

**Lemma 4.5.** Let \( k_i > k_j \geq 0 \). For divisors

\[ D = k_0 E - \sum_{i=1}^{m} k_i E_i \]

and

\[ D' = k_0 E - \left( \sum_{i=1}^{m} k_i E_i \right) - E_i + E_j, \]

the inequality \( \dim(|D|) \leq \dim(|D'|) \) holds, where the equality holds if and only if \( k_i = k_{j+1} \).

**Proof.** Without loss of generality, we can assume \( i = 1, j = 2 \). We define function \( f_d(k_1, k_2, x, y) \ (d = 1, 2, 3, \cdots \) and \( x, y \in \mathbb{R} \) recursively as

\[ f_1(k_1, k_2, x, y) = -\binom{n+k_1 + x + y - 1}{n} \lor 0 \]

\[ f_d(k_1, k_2, x, y) = f_{d-1}(k_1, k_2, x, y) - f_{d-1}(k_1, k_2, x + k_{d+1}, y). \]

It is sufficient to show that \( f_d(k_1, k_2, x, y) \) decreases with respect to \( y \). By induction for \( d \), it is easy to show that

\[ \frac{\partial^{s+t} f_d}{\partial x^s \partial y^t} \leq 0 \text{ for } s = 0, 1, 2, \cdots \text{ and } t = 0, 1. \]

From Conj. 4.4, \( \dim(|z\delta|) = 0 \) and \( \dim(|-zK_{X_A}|) < 0 \) hold for \( z > 0 \in \mathbb{R} \), and thus, by Lemma 4.5, \( \delta \) (up to a constant multiple) is the unique effective class preserved by \( t_{\beta} \).
(Uniqueness of the Th. 4.1)

Results for the $E_7^{(1)}$ case

Let $m \geq n + 5$. We set the root basis $\beta_i$ as $\beta_7 = \alpha_0$, $\beta_i = \alpha_{n-3+i}$ ($1 \leq i \leq 6$), and $\beta_0 = \alpha_{n+4}$. Further, we also set the null root $\delta$ as $\delta = 2\beta_7 + \beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 3\beta_5 + 2\beta_6 + \beta_0$. The translation $t_\beta$ is defined by (5). For generic parameters, $\delta$ is preserved by $t_\beta$ ($\beta \notin \mathbb{Z}\delta$).

The parameters $A \in X(n,m)$ can be normalized as

$$
\begin{pmatrix}
I_{n-2} & b_{11} & b_{12} & \cdots & b_{18} & c_{0,n+6} & \cdots & c_{0,m} \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& b_{n-2,1} & b_{n-2,2} & \cdots & b_{n-2,8} & \vdots & \vdots & \vdots \\
& \phi(u_1) & \phi(u_2) & \cdots & \phi(u_8) & \vdots & \vdots & \vdots \\
& \phi'(u_1) & \phi'(u_2) & \cdots & \phi'(u_8) & c_{n,n+6} & \cdots & c_{n,m}
\end{pmatrix},
$$

(14)

where $(b_{ij})_{1 \leq i \leq n-3, 1 \leq j \leq 4}$ can be fixed as

$$(b_{ij})_{1 \leq i \leq n-3, 1 \leq j \leq 5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  

The corresponding divisor of $\delta$ is given by the pencil of the hyper-surfaces

$$C_1(x^2_{n-1} - x_{n-2}x_n + g_2x_{n-2}x_{n-3} + g_3x_{n-3}^2) + C_2(x_{n-2}^2 - x_{n-3}x_n) = 0,$$

where $C_1$ and $C_2$ are constants s.t. $(C_1, C_2) \in \mathbb{P}^1$.

The action of $W(E_7^{(1)})$ on $(x_{n-3} : x_{n-2} : x_{n-1} : x_n) \in \mathbb{P}^3$ is the same with that on $X(3,8+1)$, which was studied in [21]. From the result of [21], the action of $W(E_7^{(1)})$ on $\mathbb{P}^n$ preserves each hyper-surface, which is independent of the parameters $A$, and hence it also preserves $\tau$ and $(C_1 : C_2)$. Moreover, the action can be reduced to the action on the lower dimensional rational variety $(\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{C}^{n-3}$ through Segré embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3: (x_0 : x_1, y_0 : y_1) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1)$.

5 Elliptic difference case

Since the time evolution of the parameters $A$ may not be solved in general as in the previous section, the systems should not be considered as $n$-dimensional but $n + mn - ((n+1)^2 - 1) = n + (the \ freedom \ of \ m \ points) - (the \ freedom \ of \ PGL(n))$-dimensional. On the other hand, Kajiwara et al. [13] has proposed a birational representation of the Weyl group $W(n,m)$, in which all the points of blow-ups lie on a certain elliptic curve in $\mathbb{P}^m$, and the actions on the parameters can be written as linear transformations on a torus\footnote{This calculation was carried out in a rather heuristic manner. We can recover it in an algebro geometric way for the elliptic curves of degree $n + 1$ [8].}. In this

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section, we show that the time evolution of the parameters can be solved in the case where
the points of the blow ups are parameterized as in Kajiwara et al. Further, we represent
the dynamical systems explicitly.

Suppose that the points $P_i$ ($i = 1, 2, \cdots, m$) are on the curve

$$ \left\{ P(z) = \left( \frac{\lambda + z_1 - z}{|z_1 - z|}; \cdots; \frac{\lambda + z_{n+1} - z}{|z_{n+1} - z|} \right) \mid z \in \mathbb{C} \right\}, $$

(15)

where $[z] = z$, sin $z$ or $\theta_1(z)$: a theta function whose zero points are $\mathbb{Z} + \mathbb{Z} \tau$ with the
order 1, $\lambda = z_0 - z_1 - z_2 - \cdots - z_{n+1} \in \mathbb{C}$, and the point $P_i$ corresponds to $z = z_i \in \mathbb{C}$. The following proposition is due to [13].

**Proposition 5.1** (Kajiwara et al.). The birational maps $r_{i,i+1}$ act on the parameter space
$(z_1, \cdots, z_m, \lambda)$ as

$$ r_{i,i+1} : (z_1, \cdots, z_i, z_{i+1}, \cdots, z_{n+1}; \lambda) \mapsto (z_1, \cdots, z_i, z_i, \cdots, z_{n+1}; \overline{\lambda}), $$

(16)

where

$$ \overline{\lambda} = \left\{ \begin{array}{ll} \lambda & (i \neq n + 1) \\ \lambda + z_{n+1} - z_{n+2} & (i = n + 1) \end{array} \right\} ,$$

and they act on $\mathbb{P}^n$ as

$$ r_{i,i+1} : \mathbf{x} \mapsto \mathbf{x}, $$

(17)

where

$$ \mathbf{x} = (x_0 : \cdots : x_i : x_{i+1} : \cdots : x_n) \quad \text{(if } i = 1, 2, \cdots, n) $$

$$ \mathbf{x} = \text{diag} \left( \begin{array}{ccc} [z_{n+1} - z_{n+2}][\lambda + z_{n+1} - z_{n+2}][z_1 - z_{n+2}] & [z_{n+1} - z_{n+2}][\lambda + z_{n+1} - z_{n+2}][z_2 - z_{n+2}] \\
[|z_1 - z_{n+1}|][|z_{n+1} - z_{n+2}|][\lambda + z_{n+1} - z_{n+2}] & [|z_2 - z_{n+1}|][|z_{n+1} - z_{n+2}|][\lambda + z_{n+1} - z_{n+2}] \\
\vdots & \vdots \\
\vdots & \vdots \end{array} \right) \mathbf{x} \quad \text{(if } i = n + 1) $$

$$ \mathbf{x} = \mathbf{x} \quad \text{(if } i = n + 2, n + 3, \cdots, m - 1) .$$

The birational maps $r_{1,2,\cdots,n+1}$ act on the parameter space as

$$ r_{1,2,\cdots,n+1} : (z_i, \lambda) \mapsto (z_i, -\lambda), \quad \text{where} \quad \overline{z_i} = \left\{ \begin{array}{ll} \lambda + z_i & (1 \leq i \leq n + 1) \\
z_i & (n + 2 \leq i \leq m) \end{array} \right\} $$

(18)

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and they act on \(\mathbb{P}^n\) as
\[
    r_{1,\cdots,n+1} : (x_0 : \cdots : x_n) \mapsto (x_0^{-1} : \cdots : x_n^{-1}).
\]
Moreover, by \(r_{i,i+1}\) and \(r_{1,2,\cdots,n+1}\), the point \(P(z)\) \((z \neq z_i)\) is mapped to the point \(\overline{P}(z)\):
\[
    \overline{P}(z) = \left( \frac{\lambda + \overline{z}_1 - z}{\overline{z}_1 - z} : \cdots : \frac{\lambda + \overline{z}_{n+1} - z}{\overline{z}_{n+1} - z} \right),
\]
where \((\overline{z}_1, \cdots, \overline{z}_m; \overline{\lambda})\) are given by (16) and (15), respectively.

**Proof.** Notice that \(P_1 = (0 : \cdots : 0 : 1 : 0 : \cdots : 0)\) \((1\text{-th component})\). It is easy to prove for \(r_{1,2,\cdots,n+1}\). Indeed, by the standard Cremona transformation with respect to \(P_1, \cdots, P_{n+1}\), the point \(P(z)\) \((z \neq z_i i = 1, 2, \cdots, n + 1)\) is mapped to the point
\[
    \left( \frac{z_1 - z}{\lambda + z_1 - z} : \cdots : \frac{z_{n+1} - z}{\lambda + z_{n+1} - z} \right),
\]
Suppose (18), then (21) coincides with (20), while \(\overline{P}(z_i) = P_i(z_i)\) is also satisfied for \(i = 1, 2, \cdots, n + 1\). The case of \(r_{i,i+1}\) is also shown by the definition of \(r_{i,i+1}\) in Section 2, normalization of \((P_1, \cdots, P_{i+1}, P_i, \cdots, P_{n+2})\) to
\[
    \begin{pmatrix}
        I_{n+1} \\
        \frac{\lambda + \overline{z}_1 - \overline{z}_{n+2}}{\overline{z}_1 - \overline{z}_{n+2}} \\
        \vdots \\
        \frac{\lambda + \overline{z}_{n+1} - \overline{z}_{n+2}}{\overline{z}_{n+1} - \overline{z}_{n+2}}
    \end{pmatrix}
\]
and the Riemann relation
\[
    [a + b][a - b][c + z][c - z] + (abc \text{ cyclic}) = 0.
\]
\(\square\)

**Remark 5.2.** In (16) and (18), \(\lambda \in \mathbb{C}^n\) is an extra-parameter, i.e., \(\lambda\) is independent of the parameter \(A\).

In order to investigate the preserved quantities for dependent variables, we present the following conjecture, which has been verified for \(n \leq 5\) by M. Eguchi (in a private seminar).

**Conjecture 5.3.** Suppose that the blowup points \(P_i \in \mathbb{P}^n\) lie on an elliptic curve of degree \(n + 1\), then \(\dim | -K_X| = 0\) holds if \(n\) is even and \(\dim | -\frac{1}{2}K_X| = 1\) holds if \(n\) is odd.

This conjecture suggests that our system of odd dimension may reduces to \((n - 1)\)-dimensional system (cf. [21]).

**Example 5.4.** The translation \(t_{a_0}\) of example [26] acts on the parameters as
\[
    z_i \mapsto \begin{cases} 
        z_i + 9\lambda + 4w & (1 \leq i \leq n - 2) \\
        z_i + 5\lambda + 2w & (i = n - 1, n, n + 1) \\
        z_i + 2\lambda + w & (n + 2 \leq i \leq n + 7) \\
        z_i & (n + 8 \leq i \leq m)
    \end{cases} \\
    \lambda \mapsto -5\lambda - 2w,
\]
where \( w = 2z_{n-1} + 2z_n + 2z_{n+1} - z_{n+2} - \cdots - z_{n+7} \) and acts trivially on any point \( P(z) \) on the elliptic curve except on \( P_1 \) as \( z \mapsto z \).

Next, we represent the action of \( t_{α_0} \) on \( \mathbb{P}^n \) explicitly. For simplicity, we set \( n = 4 \). One can easily generalize this calculation for \( n \)-dimensional case. In this study, the actions of \( r_{1,2,6,7,8} \) and \( r_{1,2,9,10,11} \) are calculated. Let \( i_1, i_2, i_3 \) be integers such that \( 6 \leq i_1 < i_2 < i_3 \leq m \). By a calculation similar to the one on page 36 of [21] (see also [5]), we have

\[
(r_{1,2,i_1,i_2,i_3}(x) = \begin{pmatrix}
P(z_1) & P(z_2) & P(z_3) & P(z_4) \\
\lambda & \lambda & \lambda & \lambda \\
l_{1,2,i_1,i_2,i_3} & -l_{1,i_1,i_2,i_3} & l_{1,2,i_1,i_3} & -l_{1,2,i_1,i_3}
\end{pmatrix},
\]

where

\[
l_{k_1,k_2,k_3,k_4} = \frac{P(z) \ P(z_{k_1}) \cdots P(z_{k_4}) | \ P(z) \ P(z_{k_1}) \cdots P(z_{k_4})}{| \ x \ P(z_{k_1}) \cdots P(z_{k_4}) |},
\]

and \( z_i \) is given by Prop [5,1].

**Example 5.5.** The translation \( t_{α_0} \) of the \( E_7^{(1)} \) case discussed at the end of the previous section acts on the parameters as

\[
z_i \mapsto \begin{cases}
z_i + 4\lambda + 3w & (1 \leq i \leq n - 3) \\
z_i + 3\lambda + 2w & (n - 2 \leq i \leq n + 1) \\
z_i + \lambda + w & (n + 2 \leq i \leq n + 5) \\
z_i & (n + 6 \leq i \leq m)
\end{cases}
\]

\[
\lambda \mapsto -3\lambda - 2w,
\]

where \( w = z_{n-2} + z_{n-1} + z_n + z_{n+1} - z_{n+2} - \cdots - z_{n+5} \) and acts trivially on any point \( P(z) \) on the elliptic curve except on \( P_1 \).

## 6 Conclusion

In this paper, we have proposed \( n \)-dimensional dynamical systems associated with translations of affine Weyl groups, which are included in a Weyl group of indefinite type. In order to examine their integrability, we computed their algebraic entropy and investigated the existence of preserved divisor classes. We also studied the case where the points of blowing-up are on some elliptic curve. Following is a comparison of our systems with the elliptic discrete Painlevé equation.

- The elliptic discrete Painlevé is an isomorphism of a family of rational surfaces. Its degree grows in the quadratic order. The evolution of the parameters can be written by elliptic functions.
- Our system in general case is a pseudo-isomorphism of a family of rational varieties. Although its degree grows in the quadratic order, the evolution of the parameter may not be able to be solved.
- Our system in the elliptic case is a pseudo-isomorphism of a family of rational varieties. Its degree grows in the quadratic order, and the evolution of the parameter can be written
by elliptic functions. Moreover, we have conjectured that the odd-dimensional system reduces to \((n - 1)\)-dimensional one.

For further comparison, we give some properties of dynamical systems associated with a general element of infinite order of the Weyl group \(W(n,m)\). Such system is a pseudo-isomorphism of a family of rational varieties. Its degree grows exponentially, i.e. the algebraic entropy is positive. The evolution of the parameter may not be able to be solved.

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