**ORIGINAL ARTICLE**

**L^p** bound for the Hilbert transform along variable non-flat curves

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**Funding information**
NSF of China, Grant/Award Number: 11901301; NSF of Jiangsu Province, Grant/Award Number: BK20180721

**Abstract**
We prove the **L**^p bound for the Hilbert transform along variable non-flat curves \((t, u(x) [t]^\alpha + v(x) [t]^\beta)\), where \(\alpha\) and \(\beta\) satisfy \(\alpha \neq \beta, \alpha \neq 1, \beta \neq 1\). Compared with the associated theorem in the work (Guo et al. Proc. Lond. Math. Soc. 2017) investigating the case \(\alpha = \beta \neq 1\), our result is more general while the proof is more involved. To achieve our goal, we divide the frequency of the objective function into three cases and take different strategies to control these cases. Furthermore, we need to introduce a “short” shift maximal function \(M^n\) to establish some pointwise estimates.

**KEYWORDS**
Hilbert transform, shifted maximal function, variable non-flat curves

**MSC (2020)**
42B10, 42B20

**1 | INTRODUCTION**

Let \(u : R \times R \to R\) be a measurable function, the Hilbert transform along variable curves \((t, u(x, y) \cdot [t]^\alpha)\) is defined by

\[
H_\alpha f(x, y) := \text{p.v.} \int_R f(x - t, y - u(x, y) [t]^\alpha) \frac{dt}{t}, \quad \alpha > 0,
\]

where \([t]^\alpha\) represents either \(|t|^\alpha\) or \(\text{sgn}(t)|t|^\alpha\) for \(\alpha \neq 1\) and \([t]^1\) stands for \(t\). The research of \(H_\alpha\) is considered as a complement of the works on the \(L^p\) boundedness of the maximal operator along variable curves \((t, u(x, y) \cdot [t]^\alpha)\) given by

\[
M_\alpha f(x, y) := \sup_{\epsilon > 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x - t, y - u(x, y) [t]^\alpha)| dt.
\]

We refer [3, 5], and reference therein for the investigation of the above maximal function.

Stein and Street [16] proved that \(H_\alpha\) is bounded on \(L^p\) for \(p > 1\) under the assumption that \(\alpha \in \mathcal{N}\) and \(u\) is analytic. Indeed, the objects in that work are all polynomials with analytic coefficients. Note that this analytic condition for \(u\) plays a crucial role in their proof. Based on time-frequency techniques, Lacey and Li [11] removed this type of regularity assumption. More precisely, they achieved the following two single annulus estimates for an arbitrary measurable function \(u\):

\[
\|H_k P_k^{(2)} f\|_{L^\infty} \leq \|P_k^{(2)} f\|_L^2.
\]
where \( \| f \|_{2,\infty} \) means the weak \( L^2 \) norm of \( f \), and

\[
\| H_1 P_k^{(2)} f \|_p \leq \| P_k^{(2)} f \|_p, \quad p > 2.
\]

We refer (2.2) for the definition of \( P_k^{(2)} \). For some convenience, in what follows, we call single annulus estimate a weak estimate. In the case of \( \alpha \neq 1 \), by using the local smoothing estimate which is derived from a decoupling inequality (see [18]) and the Sobolev embedding inequality, Guo et al. [7] established

\[
\| H_\alpha P_k^{(2)} f \|_p \leq \| P_k^{(2)} f \|_p, \quad p > 2.
\]

However, all the above works only devote to the weak estimate. Lately, in two papers [8, 9], Guo et al. gave a detailed discussion of the \( L^p \) boundedness of \( H_\alpha \) for \( \alpha > 1 \). More precisely, the \( L^p \) operator norm of \( H_\alpha \) depends on the choice of \( u(x, y) \).

There are many works that focus on some special \( u \) satisfying

\[
u(x, y) = u(x, 0).
\]

It is corresponding to the operator \( H_\alpha \) defined by

\[
H_\alpha f(x, y) := p.v. \int_R f(x - t, y - u(x)[t]^\alpha) dt, \quad \alpha > 0,
\]

where \( u : R \rightarrow R \) is a measurable function. This operator is closely related to the Carleson’s maximal operator (see, e.g., [4, 13]) in the sense that their \( L^2 \) bounds are equivalent, which can be obtained via the Plancherel theorem and the linearization process. By developing the methods applied in [11] and [12] further, Bateman [1], Bateman and Thiele [2] considered the operator \( H_1 \) and proved

\[
\| H_1 P_k^{(2)} f \|_p \leq \| P_k^{(2)} f \|_p, \quad 1 < p < \infty
\]

and

\[
\| H_1 f \|_p \leq \| f \|_p, \quad \frac{3}{2} < p < \infty.
\]

For \( \alpha \neq 1 \), Guo et al. [7] proved

\[
\| H_\alpha f \|_p \leq \| f \|_p, \quad 1 < p < \infty
\]

by Littlewood–Paley theory and the shifted maximal estimate (see Lemma 2.3). In addition, they also proved

\[
\| H_{1,\alpha} P_k^{(2)} f \|_p \leq C_p \| P_k^{(2)} f \|_p, \quad 1 < p < \infty,
\]

where \( H_{1,\alpha} \) is given by

\[
H_{1,\alpha} f(x, y) := p.v. \int_R f(x - t, y - u(x)t - v(x)[t]^\alpha) dt, \quad \alpha > 0, \quad \alpha \neq 2.
\]

Their proof relies on almost-orthogonality, stationary-phase, and \( TT^* \) methods. However, it is not easy to prove the \( L^p \) bound for \( H_{1,\alpha} \). It is worth mentioning that the method in [7] seems not useful to the case \( \alpha = 1 \) since it strongly depends on the curvature condition coming from \( \alpha \neq 1 \). In turn, one can scarcely obtain the full range of \( p \) when applying the method in [1, 2] to the case \( \alpha \neq 1 \).

Motivated by the above works, here we consider the operator \( H_{\alpha,\beta} \) defined by

\[
H_{\alpha,\beta} f(x, y) := p.v. \int_R f(x - t, y - u(x)[t]^\alpha - v(x)[t]^\beta) dt, \quad \alpha, \beta > 0.
\]
where \( u : \mathcal{R} \to \mathcal{R} \) and \( v : \mathcal{R} \to \mathcal{R} \) are two measurable functions. The natural goal is to prove the \( L^p \) boundedness of this operator. Using the notation in (1.1), we collect the previous related works as follows:

| \( \alpha = \beta = 1 \) | [1, 2] | \( \frac{3}{2} < p < \infty \) and weak estimate \( 1 < p < \infty \) |
| \( \alpha = \beta \neq 1 \) | [7] | \( 1 < p < \infty \) |
| \( \alpha \neq \beta, \alpha = 1, \beta \neq 2 \) | [7] | weak estimate \( 1 < p < \infty \) |

Now, we state our main result.

**Theorem 1.1.** Let \( (\alpha, \beta) \in S_1 \) defined by

\[
S_1 := \{ (\alpha, \beta) \in \mathcal{R}^+ \times \mathcal{R}^+ : \alpha \neq \beta, \alpha \neq 1, \beta \neq 1 \},
\]

then we have

\[
\|H_{\alpha, \beta}f\|_p \preceq_{\alpha, \beta, p} \|f\|_p
\]

(1.2)

holds for \( 1 < p < \infty \).

**Remark 1.2.** As far as we know, it is unsolved for the following two cases:

\[
S_2 := \{ (\alpha, \beta) \in \mathcal{R}^+ \times \mathcal{R}^+ : \alpha \neq \beta, \alpha = 1, \beta \neq 2 \},
\]

\[
S_3 := \{ (1, 2) \}.
\]

We give the following remarks on the case \( p \neq 2 \) since the special case \( p = 2 \) is a direct result of the Carleson maximal estimates.

(1) In the case of \( (\alpha, \beta) \in S_2 \), it seems hard to directly use our approach, since \( u(x)t \) does not have any curvature. We speculate that its proof needs a combination of the methods in previous works and our approach in the current paper. For all this, it remains open.

(2) For \( (\alpha, \beta) \in S_3 \), as the Carleson’s maximal operator, it is the most natural challenge. We are far from knowing how to bound this case.

Next, let us make some comments on our proof.

We first show the new gap in the proof and then give our strategy. Since there are two fractional variable monomial \( u(x)[t]^{\alpha} \) and \( v(x)[t]^{\beta} \), we need a different shifted maximal function \( M^{[n]} \) (see Section 2 for the definition) to control some pointwise estimates, where \( n \) depends on the frequency of the objective function (This fact makes our proof and the corresponding proof in [7] different). More badly, this type of shifted maximal function prevents us from using the vector-valued shifted maximal estimate. To break this barrier, we first divide the frequency of the objective function into three cases, and use different measures to deal with each case. Then we introduce a “short” shift maximal function \( M^{[n]} \) to obtain some pointwise estimates. At last, we can prove the desired estimate by applying both the vector-valued and the scalar shifted maximal estimates.

The following weak estimate is a direct result of Theorem 1.1 by taking \( f \to P_k^{(2)}f \).

**Corollary 1.3** (weak estimate). Let \( (\alpha, \beta) \in S_1 \), then for all \( k \in \mathcal{Z} \)

\[
\|H_{\alpha, \beta}P_k^{(2)}f\|_p \preceq_{\alpha, \beta, p} \|P_k^{(2)}f\|_p
\]

holds for \( 1 < p < \infty \).

The present paper is structured as follows:

In Section 2, we give the identity decomposition, the Littlewood–Paley projection, and several useful estimates such as the shifted maximal estimate. The third section proves Theorem 1.1, while the fourth section provides the proof of Lemma 3.1. Finally, we give the proof of Lemma 3.2 in the last section.
Let us complete this section by describing the notation we shall use in this paper. 

**Notation.** We hereinafter use \( x \leq y \) to stand for there exists a constant \( C \) (which may only depend on fixed parameters such as \( \alpha, \beta \), and \( p \)) such that \( x \leq C y \). We write \( C_\gamma \) to mean that the constant \( C \) depends on \( \gamma \). \( F^\gamma(f) \) is the Fourier transform in the \( y \)-variable of a function \( f \). We use \( \| \cdot \|_p \) to stand for \( \| \cdot \|_{L^p} \).

### 2 SOME PREPARATIONS

Let \( \phi(t) \) be a radial, smooth, and decreasing function that is supported on \( \{ |t| \leq 2 \} \) and equals 1 on \( \{ |t| \leq 1 \} \). Define \( \psi(t) \) by \( \psi(t) = \phi(t) - \phi(2t) \), which is a non-negative smooth function supported on the set \( \{ t : 1/2 \leq |t| \leq 2 \} \). Denote \( \psi_l(t) := \psi(2^{-l} t) \), \( \phi_l(t) := \phi(2^{-l} t) \) (Note that \( \phi_0(t) = \phi(t) \)), then for all \( l_0 \in \mathbb{Z} \),

\[
\phi_{l_0}(t) + \sum_{l \geq l_0} \psi_l(t) = 1, \quad \forall t \in \mathcal{R}
\]  

(2.1)

and

\[
\sum_{l \in \mathbb{Z}} \psi_l(t) = 1, \quad \forall t \in \mathcal{R} \setminus \{0\}.
\]

Define the corresponding Littlewood–Paley projection in the \( y \)-variable of a function \( f \) on \( \mathcal{R} \) by

\[
P_k^{(2)} f(x, y) := \int_{\mathcal{R}} f(x, y - z) \tilde{\psi}_k(z) dz,
\]

(2.2)

where \( \tilde{\psi}_k \) denotes the inverse Fourier transform of the function \( \psi_k \).

We need the following decay estimate, which is a modification of Lemma 2.1 in [10].

**Lemma 2.1.** Let \( \alpha, \beta \) be two positive numbers satisfying \( \alpha \neq \beta \), \( \alpha, \beta \neq 1 \), and \( \psi \) be smooth and supported on \( [1/2, 2] \cup [-2, -1/2] \) and \( \theta \) be smooth and supported on \( [-2, 2] \). For \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \), \( b \in \mathcal{R} \) and \( a > 0 \), let

\[
\Phi_{\alpha, b}(t) = e^{|\lambda_1| |t|^\alpha + |\lambda_2| |t|^\beta} \frac{\theta(bt)\psi(t)}{t}, \quad \Phi_{\alpha, b}(t) = \frac{1}{a} \Phi_{\alpha, b} \left( \frac{t}{a} \right).
\]

(2.3)

Then there exists \( \gamma_0 > 0 \) such that for all \( r \geq 1 \) and \( \omega \in L^2(\mathcal{R}) \),

\[
\left\| \sup_{a > 0, b \in \mathcal{R}} |(\omega \ast \Phi_{\alpha, b})(x)| \right\|_{L^2(\mathcal{R})} \leq \alpha, \beta \ r^{-\gamma_0} \| \omega \|_{L^2(\mathcal{R})}.
\]

**Proof.** Due to the supports of \( \theta \) and \( \psi \), the effective range for \( b \) is \( \{ |b| \leq 4 \} \). Then we can replace \( b \in \mathcal{R} \) by \( |b| \leq 4 \). Following the proof of Lemma 2.1 in [10] line by line can lead to the desired estimate. \( \square \)

Suppose \( \sigma \geq 0 \), we define the shifted maximal operator \( M^{[\sigma]} \) and “short” shifted maximal operator \( M^{[\sigma]} \) as follows:

\[
M^{[\sigma]} f(z) := \sup_{z \in \mathcal{R}, |l| = 1} \frac{1}{|l|} \int_{I(l)} |f(\xi)| d\xi
\]

and

\[
M^{[\sigma]} f(z) := \sup_{z \in \mathcal{R}, |l| = 1} \int_{I(l)} |f(\xi)| d\xi,
\]
where \( I(\sigma) \) denotes a shift of the bounded interval \( I = [a, b] \) given by
\[
I(\sigma) := [a - \sigma |I|, b - \sigma |I|] \cup [a + \sigma |I|, b + \sigma |I|].
\]

Obviously, \( M^{|\sigma|} f(z) \leq M^{|\sigma|} f(z) \).

Our proof needs the following estimates.

**Lemma 2.2** ([14, 15]). Let \( 1 < p < \infty \), we have
\[
\|M[n]f\|_p \leq \log(2 + |n|)\|f\|_p.
\] (2.4)

Here the constant hidden in \( \leq \) is independent of \( |n| \) and \( f \).

**Lemma 2.3** ([7]). Let \( 1 < p < \infty \), \( 1 < q \leq \infty \), we have
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |M[n]f_k|^q \right)^{\frac{1}{q}} \right\|_p \leq \log^2(2 + |n|)\left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^q \right)^{\frac{1}{q}} \right\|_p.
\] (2.5)

Here the constant hidden in \( \leq \) is independent of \( |n| \) and \( f \).

We also require the following lemma to prove a pointwise estimate.

**Lemma 2.4.** Let \( n = n_1 + n_2 \), \( n_1 \geq 0 \) and \( 0 \leq n_2 \leq C_1 \) for some \( C_1 > 0 \), then the following pointwise estimate of \( M^{[n]} \) holds:
\[
M^{[n]}f \leq \sum_{l=0}^{C_1} M^{[n_1+l]}f.
\]

**Proof.** Let \( I = [a, a+1] \), then
\[
\int_{I(n)} |f(\xi)| d\xi = \int_{a+n}^{a+1+n} |f(\xi)| d\xi + \int_{a-n}^{a+1-n} |f(\xi)| d\xi
\]
\[
= \int_{a+n_1+n_2}^{a+1+n_1+n_2} |f(\xi)| d\xi + \int_{a-n_1-n_2}^{a+1-n_1-n_2} |f(\xi)| d\xi.
\]

Since \( 0 \leq n_2 \leq C_1 \), we have
\[
\int_{a+n_1+n_2}^{a+1+n_1+n_2} |f(\xi)| d\xi \leq \sum_{l=0}^{C_1} \int_{a+n_1+l}^{a+1+n_1+l} |f(\xi)| d\xi
\]
and
\[
\int_{a-n_1-n_2}^{a+1-n_1-n_2} |f(\xi)| d\xi \leq \sum_{l=0}^{C_1} \int_{a-n_1-l}^{a+1-n_1-l} |f(\xi)| d\xi.
\]

Combining with these inequalities yields
\[
\int_{I(n)} |f(\xi)| d\xi \leq \sum_{l=0}^{C_1} \int_{I(n_1+l)} |f(\xi)| d\xi,
\]
which completes the proof by taking the supremum over \( a \in \mathcal{R} \) on both sides. \( \Box \)
In this section, we prove Theorem 1.1. Without loss of generality, we assume \( u(x) > 0, v(x) > 0, \) and \( 0 < \alpha < \beta. \) For convenience of notation, we denote \( u(x)[t]^{\alpha} + v(x)[t]^{\beta} \) by \( \Gamma(x, t). \)

Thanks to
\[
P^{(2)}_k H_{\alpha, \beta} f = H_{\alpha, \beta} P^{(2)}_k f
\]
and the Littlewood–Paley theory, it is enough to show
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |H_{\alpha, \beta} P^{(2)}_k f|^2 \right)^{\frac{1}{2}} \right\|_p \leq \| f \|_p. \tag{3.1}
\]

We beforehand give some motivations of the strategy. We first use the identity decomposition like (2.1) to quantify \( \Gamma(x, t) \) since \( H_{\alpha, \beta} \) is similar to the classical Hilbert transform for “small” \( |\Gamma(x, t)|. \) After this process, we establish a useful decay estimate like \( 2^{-\eta n} (\eta > 0) \) for \( |\Gamma(x, t)| \approx 2^n, \) and then obtain the desired estimate for not “small” \( |\Gamma(x, t)| \) by summing over \( n \geq 0. \) The above approach is analogous to that in [7], however, due to the appearance of \( u(x)[t]^{\alpha} \) and \( v(x)[t]^{\beta}, \) our proof is more involved. On the one hand, on account of \( \alpha \neq \beta, \) we need to employ
\[
\phi_{-k/\alpha} \left( u(x) \right)^{\frac{1}{\alpha} t} + \sum_{l > -k/\alpha} \psi_l \left( u(x) \right)^{\frac{1}{\alpha} t} = 1 \tag{3.2}
\]
and
\[
\phi_{-k/\beta} \left( v(x) \right)^{\frac{1}{\beta} t} + \sum_{l > -k/\beta} \psi_l \left( v(x) \right)^{\frac{1}{\beta} t} = 1 \tag{3.3}
\]
to quantify \( u(x)[t]^{\alpha} \) and \( v(x)[t]^{\beta}, \) respectively. On the other hand, to bound some operators by shifted maximal operator, we have to classify the range of \( k \) in (3.1) into several small ranges, and use several different measures to control them.

We return to the proof. Applying (3.2) and (3.3) to \( H_{\alpha, \beta} P^{(2)}_k f \) gives
\[
H_{\alpha, \beta} P^{(2)}_k f(x, y) = H_{\alpha, \beta}^{ll} P^{(2)}_k f(x, y) + H_{\alpha, \beta}^{lh} P^{(2)}_k f(x, y) + H_{\alpha, \beta}^{hl} P^{(2)}_k f(x, y) + H_{\alpha, \beta}^{hh} P^{(2)}_k f(x, y),
\]
where
\[
H_{\alpha, \beta}^{ll} P^{(2)}_k f(x, y) = \text{p.v.} \int_R (P^{(2)}_k f)(x-t, y-\Gamma(x, t)) \phi_{-k/\alpha} \left( u(x) \right)^{\frac{1}{\alpha} t} \phi_{-k/\beta} \left( v(x) \right)^{\frac{1}{\beta} t} \frac{dt}{t},
\]
\[
H_{\alpha, \beta}^{lh} P^{(2)}_k f(x, y) = \sum_{l > -k/\beta} \int_R (P^{(2)}_k f)(x-t, y-\Gamma(x, t)) \phi_{-k/\alpha} \left( u(x) \right)^{\frac{1}{\alpha} t} \psi_l \left( v(x) \right)^{\frac{1}{\beta} t} \frac{dt}{t},
\]
\[
H_{\alpha, \beta}^{hl} P^{(2)}_k f(x, y) = \sum_{l > -k/\alpha} \int_R (P^{(2)}_k f)(x-t, y-\Gamma(x, t)) \psi_l \left( u(x) \right)^{\frac{1}{\alpha} t} \phi_{-k/\beta} \left( v(x) \right)^{\frac{1}{\beta} t} \frac{dt}{t},
\]
\[
H_{\alpha, \beta}^{hh} P^{(2)}_k f(x, y) = \sum_{l > -k/\beta, m > -k/\alpha} \int_R (P^{(2)}_k f)(x-t, y-\Gamma(x, t)) \psi_m \left( u(x) \right)^{\frac{1}{\alpha} t} \psi_l \left( v(x) \right)^{\frac{1}{\beta} t} \frac{dt}{t}.
\]

In order to prove (3.1), it suffices to show (3.1) with \( H_{\alpha, \beta} \) replaced by \( H_{\alpha, \beta}^{ll}, H_{\alpha, \beta}^{lh}, H_{\alpha, \beta}^{hl} \) and \( H_{\alpha, \beta}^{hh}, \) respectively. We stress that the new ingredient in our proof is the estimate of \( H_{\alpha, \beta}^{hh} P^{(2)}_k f. \)
3.1 | The estimate of $H^p_{\alpha,\beta} P_k f$

We need the following lemma, the proof of which is given in Section 4.

Let $\mathcal{M}(x,y) (i = 1, 2)$ be the strong maximal function applied in the $i$-th variable, $H^*_{\kappa}$ be the maximally truncated Hilbert transform applied in the first variable.

**Lemma 3.1** (The first lemma). The following pointwise estimate holds

$$H^p_{\alpha,\beta} P_k f(x,y) \leq \sum_{q \in \mathbb{Z}} \frac{1}{(1 + |q|)^2} \int_q^{q+1} \mathcal{M}(x,y-z) dE \leq \sum_{q \in \mathbb{Z}} \left( \mathcal{M}(x,y-z) \right)^{1/p}$$

where the constant is independent of $u(x), v(x), k, \kappa$ and $f$.

Thanks to Lemma 3.1, the estimate of $H^p_{\alpha,\beta} P_k f$ is a direct consequence. In fact, by Minkowski's inequality, we bound the corresponding norm of the first term on the right side of (3.4) by

$$\sum_{q \in \mathbb{Z}} \frac{1}{(1 + |q|)^2} \int_q^{q+1} \left( \sum_{k \in \mathbb{Z}} |\mathcal{M}(x,y-z)|^2 \right)^{1/2} dE \leq \left( \sum_{k \in \mathbb{Z}} |\mathcal{M}(x,y-z)|^2 \right)^{1/2} \leq \left( \sum_{k \in \mathbb{Z}} |\mathcal{M}(x,y-z)|^2 \right)^{1/2}.$$ 

As a result, we achieve (3.1) with $H_{\alpha,\beta}$ replaced by $H^p_{\alpha,\beta}$ via (3.4) and the vector estimates of $\mathcal{M}(1)$ and $H^*_{\kappa}$.

3.2 | The estimate of $H^p_{\alpha,\beta} P_k f$

In fact, the estimate of $H^p_{\alpha,\beta} P_k f$ is the core content in this article. Changing variables $m \to m - \frac{k}{\alpha}$ and $l \to j - \frac{k}{\beta}$, we have

$$H^p_{\alpha,\beta} P_k f = \sum_{j,m \geq 0} T_{j,m,j}(P_k f)(x, y),$$

where $T_{j,m,j}$ is defined by

$$T_{j,m,j}(x) = \int \psi_{m-k} \psi_{j-k} \frac{1}{|t|} dE.$$ 

To get the desired estimate of $H^p_{\alpha,\beta} P_k f$, it suffices to show there exists $\tau_1 > 0$ such that

$$\left( \sum_{k \in \mathbb{Z}} \left| T_{j,m,j}(P_k f) \right|^2 \right)^{1/2} \leq 2^{-\max\{j,\tau_1\} \tau_1} \left\Vert f \right\Vert_p.$$ 

By Stein–Wainger’s method [17], we first prove the special case $p = 2$, which is described by the following lemma. We postpone the proof in Section 5.

**Lemma 3.2** (The second lemma). (3.5) holds for $p = 2$. 

Thanks to interpolation theorem, it is enough to prove

$$\left\| \left( \sum_{k \in \mathbb{Z}} |T_{k,m,j}(P_k^{(2)} f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq (j^2 \beta^2 + m^2 \alpha^2 + C_{\alpha,\beta})^2 \|f\|_p,$$  \hspace{1cm} (3.6)

where $C_{\alpha,\beta}$ is a constant that depends only on $\alpha$ and $\beta$. As the statement at the beginning of this section, we split the sum of $k$ into three parts

$$\sum_{k \in \mathbb{Z}} = \sum_{k < k_1^1} + \sum_{k_1^1 \leq k < k_2^2} + \sum_{k \geq k_2^2},$$

where

$$k_1^1_{\alpha,\beta,x} = \frac{\log_2 \left( v(x)^{\alpha} u(x)^{-\beta} \right) - \alpha j}{\beta - \alpha}, \quad k_2^2_{\alpha,\beta,x} = \frac{\alpha j m + \log_2 \left( v(x)^{\alpha} u(x)^{-\beta} \right)}{\beta - \alpha}.$$

Note that

$$k_2^2_{\alpha,\beta,x} - k_1^1_{\alpha,\beta,x} = \frac{\alpha \beta (m + j)}{\beta - \alpha} \geq 0.$$  \hspace{1cm} (3.7)

To prove (3.6), it suffices to prove

$$\left\| \left( \sum_{k < k_1^1} |T_{k,m,j}(P_k^{(2)} f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq (2 + j^2 \beta^2 + m^2 \alpha^2) \|f\|_p$$  \hspace{1cm} (3.8)

$$\left\| \left( \sum_{k \geq k_2^2} |T_{k,m,j}(P_k^{(2)} f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq (2 + j^2 \beta^2 + m^2 \alpha^2) \|f\|_p$$  \hspace{1cm} (3.9)

$$\left\| \left( \sum_{k_1^1 \leq k < k_2^2} |T_{k,m,j}(P_k^{(2)} f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq (j^2 \beta^2 + m^2 \alpha^2 + C_{\alpha,\beta})^2 \|f\|_p$$  \hspace{1cm} (3.10)

Denote

$$F_k(x, y) := f(x, 2^{-k} y), \quad (u_k(x), v_k(x)) = 2^k (u(x), v(x)).$$

Changing variables $z \to 2^{-k} z$ gives

$$(P_k^{(2)} f)(x - t, y - u(t)^{\alpha} - v(t)^{\beta}) = \int 2^k \psi(2^k z) f(x - t, y - z - u(t)^{\alpha} - v(t)^{\beta}) dz$$

$$= \int \psi(z) f(x - t, y - 2^{-k} z - u(t)^{\alpha} - v(t)^{\beta}) dz$$

$$= \int \psi(z) f(x - t, 2^{-k} y - z - u_k(x)[t]^{\alpha} - v_k(x)[t]^{\beta}) dz$$

$$= (P_0^{(2)} F_k)(x - t, 2^k y - u_k(x)[t]^{\alpha} - v_k(x)[t]^{\beta}).$$
Thus, we rewrite $T_{k,m,j} \left( P^{(2)}_k f \right)(x, y)$ as

$$
\int_R \left( P^{(2)}_0 F_k \right)(x - t, 2^k y - u_k(x)[t]^\alpha - v_k(x)[t]^\beta) \psi_m \left( u_k(x) \frac{1}{2} t \right) \psi_j \left( v_k(x) \frac{1}{2} t \right) \frac{dt}{t}.
$$

Next, we focus on the pointwise estimate of the following integral:

$$
TP^{(2)}_0 f(x, y) = \int_R \left( P^{(2)}_0 f \right)(x - t, y - u_k(x)[t]^\alpha - v_k(x)[t]^\beta) \psi_m \left( u_k(x) \frac{1}{2} t \right) \psi_j \left( v_k(x) \frac{1}{2} t \right) \frac{dt}{t},
$$

(3.11)

which can yield the associated estimate of $T_{k,m,j} \left( P^{(2)}_k f \right)$ by the scaling arguments.

**The proof of (3.8)** Due to the support of $\psi$, 

$$
\lambda_{x,j} := 2^j v_k(x)^{-\frac{1}{\beta}} \in \left[ 2^{m-1} u_k(x)^{-\frac{1}{\alpha}}, 2^{m+1} u_k(x)^{-\frac{1}{\alpha}} \right]
$$

(3.12)

and the range of $t$ in (3.11) is 

$$
I := \left\{ t : 2^j v_k(x)^{-\frac{1}{\beta}} \leq |t| \leq 2^{j+1} v_k(x)^{-\frac{1}{\beta}} \right\} \cap \left\{ t : 2^{m-1} u_k(x)^{-\frac{1}{\alpha}} \leq |t| \leq 2^{m+1} u_k(x)^{-\frac{1}{\alpha}} \right\}.
$$

Because of

$$
P^{(2)}_0 f(x, y) = \int \check{\phi}(z) f(x, y - z) dz
$$

yielding

$$
|P^{(2)}_0 f(x, y)| \leq \int \frac{1}{1 + |z|^8}|f(x, y - z)| dz
$$

$$
\leq \int \sum_{r \in \mathbb{Z}} \chi_{[r, r+1)}(z) \frac{1}{1 + |z|^8}|f(x, y - z)| dz
$$

$$
\leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |\tau|^8} \int_{\tau}^{\tau+1} |f(x, y - z)| dz,
$$

we bound $|TP^{(2)}_0 f(x, y)|$ by

$$
\sum_{\tau \in \mathbb{Z}} \frac{1}{1 + |\tau|^8} \int_{\tau}^{\tau+1} |f(x - t, y - z - u_k(x)[t]^\alpha - v_k(x)[t]^\beta)| \psi_m \left( u_k(x) \frac{1}{2} t \right) \psi_j \left( v_k(x) \frac{1}{2} t \right) \frac{dt}{t} dz dt.
$$

The following procedure is splitting $I$ into small intervals $\{I_r\}$, which makes $u(x)[t]^\alpha \in I^1_{x,t}$ and $v(x)[t]^\beta \in I^2_{x,t}$ (here $|I^i_{x,t}| \leq 1 (i = 1, 2)$), and estimating the above integral by the shifted maximal function.

Choosing $\delta_{x,j,m} = \lambda_{x,j} \min\{2^{-m\alpha}, 2^{-j\beta}\}$ (this choice of $\delta_{x,j}$ is to get (3.14)). We define

$$
I_r := \left\{ t : \frac{1}{2} \lambda_{x,j} + r \delta_{x,j,m} \leq |t| \leq \frac{1}{2} \lambda_{x,j} + (r + 1) \delta_{x,j,m} \right\},
$$

and then we see $|I_r| = 2 \delta_{x,j,m}$ and

$$
I \subset \left\{ \frac{1}{2} \lambda_{x,j} \leq |t| \leq 2 \lambda_{x,j} \right\} \subset \bigcup_{r=0}^{N_{j,m}-1} I_r.
$$
where \( \frac{3}{2} \lambda_{x,j} \leq N_{j,m} \delta_{x,j} \leq 2 \lambda_{x,j} \) (which yields \( \frac{3}{2} \max\{2^j \beta, 2^{m \alpha}\} \leq N_{j,m} \leq 2 \max\{2^j \beta, 2^{m \alpha}\} \)). So

\[
|TP_0^{(2)} f(x, y)| \leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |r|^8} N_{j,m}^{-1} \sum_{r=0}^{N_{j,m}^{-1}} \int_{I_r} \int_0^{\tau+1} f(x-t, y-z - u_k(x)[t] \alpha - v_k(x)[t] \beta) dt dz.
\]

Since \((t, z) \in I_r \times [\tau, \tau+1]\), we have \(z + u_k(x)[t] \alpha + v_k(x)[t] \beta \in J_r\), which is defined by

\[
J_r := \left[ \tau + u_k(x) \left( \frac{1}{2} \lambda_{x,j} + r \delta_{x,j,m} \right)^\alpha + v_k(x) \left( \frac{1}{2} \lambda_{x,j} + r \delta_{x,j,m} \right)^\beta , \right.
\]

\[
\left. \tau + 1 + u_k(x) \left( \frac{1}{2} \lambda_{x,j} + (r+1) \delta_{x,j,m} \right)^\alpha + v_k(x) \left( \frac{1}{2} \lambda_{x,j} + (r+1) \delta_{x,j,m} \right)^\beta \right].
\]

We also have

\[
|TP_0^{(2)} f(x, y)| \leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |r|^8} N_{j,m}^{-1} \sum_{r=0}^{N_{j,m}^{-1}} \int_{I_r} \int_{J_r} f(x-t, y-z)| dt dz. \tag{3.13}
\]

By the mean value theorem, we see

\[
1 \leq |J_r| \leq 1 + u_k(x) \lambda_{x,j}^{\alpha-1} \delta_{x,j} + v_k(x) \lambda_{x,j}^{\beta-1} \delta_{x,j}.
\]

Recall \( \delta_{x,j,m} = \lambda_{x,j} \min\{2^{-m \alpha}, 2^{-j \beta}\} \). It follows by (3.12) that

\[
u_k(x) \lambda_{x,j}^{\alpha-1} \delta_{x,j,m} + v_k(x) \lambda_{x,j}^{\beta-1} \delta_{x,j,m} \leq 1, \tag{3.14}
\]

which yields \(1 \leq |J_r| \leq 1\). Hence, there exists a positive integer \(C_{\alpha,\beta}(\leq 1)\) such that

\[
J_r \subset \bigcup_{i=0}^{C_{\alpha,\beta}} [\tau + a_{j,m,k}(x)+i, \tau + a_{j,m,k}(x)+i+1], \tag{3.15}
\]

where

\[
a_{j,m,k}(x) := u_k(x) \left( \frac{1}{2} \lambda_{x,j} + r \delta_{x,j,m} \right)^\alpha + v_k(x) \left( \frac{1}{2} \lambda_{x,j} + r \delta_{x,j,m} \right)^\beta.
\]

We first have by (3.12) that

\[
a_{j,m,k}(x) \leq 2^{m \alpha} + 2^j \beta. \tag{3.16}
\]

Furthermore, the definitions of \(\lambda_{x,j}, \delta_{x,j,m}\), and \((u_k, v_k)\) give

\[
a_{j,m,k}(x) = 2^j \beta \left( \frac{1}{2} + \frac{r \min\{2^{-m \alpha}, 2^{-j \beta}\}}{2^j} \right)^\beta + 2^{k(1-\frac{\beta}{\alpha})} u(x) u(x)^{\frac{\beta}{\alpha}} \left( \frac{1}{2} + \frac{r \min\{2^{-m \alpha}, 2^{-j \beta}\}}{2^j} \right)^\alpha.
\]

Combining with (3.13) and (3.15) leads to

\[
|T(P_0^{(2)} f)(x, y)| \leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |r|^8} N_{j,m}^{-1} C_{\alpha,\beta} \sum_{r=0}^{N_{j,m}^{-1}} \sum_{i=0}^{C_{\alpha,\beta}} \int_{I_r} \int_{J_r} f(x-t, y-z)| dt dz.
\]
By (3.16), we observe

\[
\frac{1}{|\Omega|} \int_{\Omega} \int_{r+a_{j,m,k}(x)+i}^{r+a_{j,m,k}(x)+i+1} |f(x-t,y-z)| \, dt \, dz \leq M_1^{[r]}(M_2^{[\tau r(x,k,i)]} f)(x,y),
\]

where

\[
\sigma_1^r := r + \max\{2^{m\alpha-1}, 2^{j\beta-1}\} \leq \max\{2^{j\beta}, 2^{m\alpha}\},
\]

\[
\sigma_i^r(x,k,i) := \tau + i + a_{j,m,k}(x) \leq \tau + C_{\alpha,\beta} + \max\{2^{j\beta}, 2^{m\alpha}\}.
\]

Here $M_1^{[r]}$ and $M_i^{[r]}$ $(i = 1, 2)$ are the shifted maximal operators (defined in Section 2) applied in the $i$-th variable. Notice that $\sigma_i^r(x,k,i)$ depends on $k$, which suggests that our procedure is very different from [7], and more involved. We now have

\[
|T(P_0^{(2)} f)(x,y)| \leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |r|^8} \frac{1}{N_{j,m}} \sum_{r=0}^{N_{j,m}-1} \sum_{l=0}^{N_{j,m-1}} M_1^{[r]}(M_2^{[\sigma^r_l(x,k,i)]} f)(x,y).
\]

**Remark 3.3.** We point out that we do not use $k < k_1^{1,\alpha,\beta}x$ in the proof of (3.19). In fact, (3.19) will be used in the proof of (3.10).

We cannot use the estimate of the shifted maximal operator given in Lemma 2.3 directly. However, we have assumed

\[
\alpha < \beta \quad \text{and} \quad k \leq k_1^{1,\alpha,\beta,x} = \frac{\log_2 (v(x)^{\alpha} v(x)^{-\beta}) - a_{\alpha,\beta}}{\beta - \alpha} \quad \text{in (3.8)}
\]

so that

\[
2^{k(1-\beta/\alpha)} u(x) v(x)^{-\alpha/\beta} 2^{k^{\alpha}} \leq 1.
\]

Thus, by Lemma 2.4, there exists a positive constant $C'_{\alpha,\beta}$ independent of $k$ such that

\[
M_1^{[r]}(M_2^{[\sigma^r_l(x,k,i)]} f)(x,y) \leq \sum_{l=0}^{C'_{\alpha,\beta}} M_1^{[r]}(M_2^{[\sigma^r_l(x,k,i)+l]} f)(x,y),
\]

where

\[
\sigma_3^r(i) := \tau + i + 2^{j\beta} \left(\frac{1}{2} + r \min\{2^{-m\alpha}, 2^{-j\beta}\}\right)^{\beta}.
\]

The scaling arguments give

\[
\left| T_{k,m,j}(P_k^{(2)} f)(x,y) \right| = \left| T(P_0^{(2)} F_k)(x,2^k y) \right|
\]

\[
\leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |r|^8} \frac{1}{N_{j,m}} \sum_{r=0}^{N_{j,m}-1} \sum_{l=0}^{C'_{\alpha,\beta}} M_1^{[r]}(M_2^{[\sigma^r_l(x,k,i)+l]} F_k)(x,2^k y)
\]

\[
\leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |r|^8} \frac{1}{N_{j,m}} \sum_{r=0}^{N_{j,m}-1} \sum_{l=0}^{C'_{\alpha,\beta}} M_1^{[r]}(M_2^{[\sigma^r_l(x,k,i)+l]} F_k)(x,2^k y)
\]

\[
\leq \sum_{r \in \mathbb{Z}} \frac{1}{1 + |r|^8} \frac{1}{N_{j,m}} \sum_{r=0}^{N_{j,m}-1} \sum_{l=0}^{C'_{\alpha,\beta}} M_1^{[r]}(M_2^{[\sigma^r_l(x,k,i)+l]} f)(x,y).
\]
Because of $P_k^{(2)} = P_{k-1}^{(2)} + P_k^{(2)} + P_{k+1}^{(2)}$, we can obtain by the same way that

$$|T_{k,m,j}(P_k^{(2)} f)(x,y)| \leq \sum_{\tau \in \mathbb{Z}} \frac{1}{1 + |\tau|^8} \sum_{j=0}^{N_j(m-1)} \sum_{l=0}^{C_{\alpha,\beta}} \sum_{k=1}^{M_1^{[\frac{3}{2}]}(\tau + j + l)} \left( M_2^{[\frac{3}{2}](\tau + j + l)} P_k^{(2)} f \right)(x,y).$$

So, we can accomplish (3.8) by Minkowski's inequality if we can show that

$$\left\| \left( \sum_{k \in \mathbb{Z}} |M_1^{[\frac{1}{2}]}(M_2^{[\frac{3}{2}](\tau + j + l)} P_k^{(2)} f)(x,y)|^2 \right)^{\frac{1}{2}} \right\|_p \leq (1 + \tau^2)(2 + j^2 \beta^2 + m^2 \alpha^2)^2 \|f\|_p$$

(3.20)

holds for all $0 \leq i \leq C\alpha,\beta$ and $0 \leq l \leq C'\alpha,\beta$, where the constant only depends on $\alpha$ and $\beta$. Thanks to Lemma 2.3 and (3.18), the left-hand side of (3.20) is controlled by

$$\log^2(2 + \sigma_i^2) \left\| \left( \sum_{k \in \mathbb{Z}} |M_2^{[\frac{3}{2}]}(\tau + j + l) P_k^{(2)} f)(x,y)|^2 \right)^{\frac{1}{2}} \right\|_p \leq (2 + j^2 \beta^2 + m^2 \alpha^2) \left\| \left( \sum_{k \in \mathbb{Z}} |P_k^{(2)} f)(x,y)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq \log^4(2 + |\tau|)(2 + j^2 \beta^2 + m^2 \alpha^2)^2 \|f\|_p,$$

which is bounded by the right-hand side of (3.20).

**The proof of (3.9)** The proof is similar to the proof of (3.8). Indeed, taking

$$\lambda_{x,m} = 2^m u_k(x)^{\frac{1}{\alpha}}, \quad \delta_{x,j,m} = \lambda_{x,m} \min\{2^{-m\alpha}, 2^{-j\beta}\},$$

where $\lambda_{x,m}$ plays the same role as $\lambda_{x,j}$ (see (3.12)), and repeating the previous arguments yields

$$|T(\mathcal{P}_0^{(2)} f)(x,y)| \leq \sum_{\tau \in \mathbb{Z}} \frac{1}{1 + |\tau|^s} \sum_{j=0}^{N_j(m-1)} \sum_{l=0}^{C_{\alpha,\beta}} \int_{\tau + a_{j,m,k} + l} \int_{\tau + a_{j,m,k} + l} |f(x - t, y - z)| \, dt \, dz,$$

where $N_{j,m}, I_r$ is defined as before and

$$a_{j,m,k}(x) := u_k(x) \left( \frac{1}{2} \lambda_{x,m} + r \delta_{x,j,m} \right)^{\alpha} + v_k(x) \left( \frac{1}{2} \lambda_{x,m} + \frac{r}{\alpha} \delta_{x,j,m} \right)^{\beta}$$

satisfying

$$a_{j,m,k}(x) = 2^m \left( \frac{1}{2} + r \min\{2^{-m\alpha}, 2^{-j\beta}\} \right)^{\alpha} + 2^k \left( \frac{1}{\alpha} \right)^{\beta} v_k(x) u_k(x)^{-\frac{\beta}{\alpha}} 2^m \left( 1 + r \min\{2^{-m\alpha}, 2^{-j\beta}\} \right)^{\beta}$$

and

$$a_{j,m,k}(x) \leq 2^{m\alpha} + 2^{j \beta}. $$
Since we have assumed that $\alpha < \beta$ and $k \geq k^2_{\alpha, \beta, x} = \frac{\alpha \beta n + \log_{\frac{\beta}{\alpha}} \left( (v(x)^{\alpha} u(x)^{-\beta}) \right)}{\beta - \alpha}$, we have

$$2^{k(1 - \frac{\beta}{\alpha})} v(x) u(x)^{-\frac{\beta}{\alpha}} 2^{m\beta} \leq 1.$$ 

Then along the previous way leads to (3.9).

**The proof of (3.10)** Thanks to (3.7),

$$\# \{ k \in \mathbb{Z} : k^1_{\alpha, \beta, A} \leq k < k^2_{\alpha, \beta, A} \} \leq_{\alpha, \beta} (j \beta + m \alpha).$$

(3.21)

By Minkowski’s inequality and (3.21), the left-hand side of (3.10) is bounded by

$$\left\| \sum_{k^1_{\alpha, \beta, A} \leq k \leq k^2_{\alpha, \beta, A}} \left| T_{k, m, j}(P^{(2)}_k f) \right| \right\|_p \leq \# \{ k \in \mathbb{Z} : k^1_{\alpha, \beta, A} \leq k < k^2_{\alpha, \beta, A} \} \sup_{k \in \mathbb{Z}} \left\| T_{k, m, j}(P^{(2)}_k f)(x, y) \right\|_p \leq (j \beta + m \alpha) \sup_{k \in \mathbb{Z}} \left\| T_{k, m, j}(P^{(2)}_k f)(x, y) \right\|_p.$$ 

Recall $F_k(x, y) = f(x, 2^{-k} y)$. As the statement in Remark 3.3, thanks to (3.19), we deduce by the scaling arguments that

$$\left| T_{k, m, j}(P^{(2)}_k f)(x, y) \right| = \left| T(P^{(2)}_0 F_k)(x, 2^k y) \right| \leq \sum_{r=0}^{N_{j, m} - 1} \frac{1}{N_{j, m}} \sum_{l=0}^{M_{j, k} \left( M_{j, k} \left( x, k, i \right) \right) F_k} \left( x, 2^k y \right) \leq \sum_{r=0}^{N_{j, m} - 1} \frac{1}{N_{j, m}} \sum_{l=0}^{M_{j, k} \left( M_{j, k} \left( x, k, i \right) \right) F_k} \left( x, 2^k y \right).$$

We complete the proof of (3.10) by (3.18) and the scalar shifted maximal estimate (see (2.4) in Lemma 2.2).

### 3.3 The estimates of $H_{\alpha, \beta}^{1h} P^{(2)}_k f$ and $H_{\alpha, \beta}^{hl} P^{(2)}_k f$

Due to the property of the support of $\phi$, $\Gamma(x, t)$ essentially plays the same role as the single fractional monomial. So we expect that the proof is similar to the previous work [7], and easier than the estimate of $H_{\alpha, \beta}^{hl} P^{(2)}_k f$.

We only give a sketch of the estimate of $H_{\alpha, \beta}^{1h} P^{(2)}_k f$. By the variable substitution $l \rightarrow -k/\beta + j$, we get

$$H_{\alpha, \beta}^{1h} P^{(2)}_k f(x, y) = \sum_{j \geq 0} T_{k, j}(P^{(2)}_k f)(x, y),$$

where $T_{k, j}$ is defined by

$$T_{k, j} f(x, y) = \int_R f(x - t, y - u(x)^{\beta} v(x)^{1 - \beta} \phi \left( \frac{u(x)^{1 - \beta}}{\beta} - \frac{v(x)^{\beta}}{\beta} \right) \frac{dt}{t}.$$ 

It suffices to show there exists $\iota > 0$ such that

$$\left\| \sum_{k \in \mathbb{Z}} \left| T_{k, j}(P^{(2)}_k f) \right|^2 \right\|_p \leq 2^{-\beta \iota} \| f \|_p.$$
This, with the $L^2$ bound (which can be obtained by the same way leading to Lemma 3.2), yields that it is enough to show
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |T_k,j(P^{(2)}_k f)|^2 \right)^{1/2} \right\|_p \leq (1 + j)^q \|f\|_p.
\]

Since $t$ is in the support of $\phi_{\frac{x}{t}}(u(x)\frac{1}{t})$, after using the scaling arguments, $u(x)[t]^\alpha + v(x)[t]^\beta$ is approximated by $v(x)[t]^\beta$.

Indeed, along the same way yielding (3.19), if we use the following notation:
\[
\lambda_{x,j} := 2^j v_k(x)^{1/\beta}, \quad \delta_{x,j} = \lambda_{x,j} 2^{-j} \beta, \quad N_{j,m} \delta_{x,j} \in \left[\frac{3}{2} \lambda_{x,j}, 2 \lambda_{x,j}\right],
\]

one can obtain
\[
T_{k,j}(P^{(2)}_k f) \leq \sum_{\tau \in \mathbb{Z}} \frac{1}{1 + |\tau|^\beta} \frac{1}{N_{j,m}} \sum_{r=0}^{N_{j,m}-1} \sum_{i=0}^{C_{\lambda,j}} M_1^{[n_1(r)]} \left( M_2^{[n_2(r)+i+\tau]} P^{(2)}_k f \right)(x,y),
\]

where
\[
n_1(r) = r + 2^{\beta - 1} \leq 2^{\beta}, \quad n_2(r) + i + \tau \leq 2^{\beta} + \tau.
\]

We complete the proof by applying Lemma 2.3 directly.

## 4 | PROOF OF THE FIRST LEMMA

This section devotes to the proof of Lemma 3.1. Since this estimate is independent of $u(x)$, $v(x)$, and $k$, by the scaling arguments, it suffices to show the case $k = 0$:

\[
H_{\alpha,\beta}^{II} P^{(2)}_0 f(x,y) \leq \sum_{q \in \mathbb{Z}} \frac{1}{1 + |q|^\beta} \int_q^{q+1} \mathcal{M}^{(1)} P^{(2)}_0 f(x,y-z)dz + \mathcal{M}^{(1)} P^{(2)}_0 f(x,y) + H^* P^{(2)}_0 f(x,y),
\]

where
\[
H_{\alpha,\beta}^{II} P^{(2)}_0 f(x,y) := p.v. \int_R (P^{(2)}_0 f)(x-t, y-\Gamma(x,t))\phi \left( u(x)\frac{1}{t} \right) \phi \left( v(x)\frac{1}{t} \right) dt.
\]

Due to the support of $\phi$ yielding
\[
|\Gamma(x,t)| \leq u(x)[t]^\alpha + v(x)[t]^\beta \leq 1,
\]

$H_{\alpha,\beta}^{II}$ can be approximated by $T_{\alpha,\beta}^{II}$, which is defined by
\[
T_{\alpha,\beta}^{II} f(x,y) := p.v. \int_R f(x-t, y)\phi \left( u(x)\frac{1}{t} \right) \phi \left( v(x)\frac{1}{t} \right) dt.
\]

To obtain (4.1), it suffices to show
\[
|H_{\alpha,\beta}^{II} P^{(2)}_0 f - T_{\alpha,\beta}^{II} P^{(2)}_0 f| \leq \sum_{q \in \mathbb{Z}} \frac{1}{1 + |q|^\beta} \int_q^{q+1} \mathcal{M}^{(1)} P^{(2)}_0 f(x,y-z)dz
\]

and
\[
|T_{\alpha,\beta}^{II} P^{(2)}_0 f| \leq \mathcal{M}^{(1)} P^{(2)}_0 f(x,y) + H^* P^{(2)}_0 f(x,y).
\]
We first prove (4.3). Denote the Schwartz function $\Phi(z)$ by

$$
\sum_{i=-1}^{1} P_{i}^{(2)} f(x,y) = \int f(x,y-z) \Phi(z) dz,
$$

which, together with $P_{0}^{(2)} = \sum_{i=-1}^{1} P_{i}^{(2)} P_{0}^{(2)}$, yields

$$
P_{0}^{(2)} f(x,y) = \sum_{i=-1}^{1} P_{i}^{(2)} P_{0}^{(2)} f = \int_{R} P_{0}^{(2)} f(x,y-z) \Phi(z) dz.
$$

Then we rewrite $H_{\alpha,\beta}^{(2)} P_{0}^{(2)} f(x,y) - T_{\alpha,\beta}^{(2)} P_{0}^{(2)} f(x,y)$ as

$$
\int \int (P_{0}^{(2)} f)(x-t,y-z) \{ \Phi(z-\Gamma(x,t)) - \Phi(z) \} \phi \left( u(x) \left\lvert \frac{1}{\alpha} \right\rvert t \right) \phi \left( v(x) \left\lvert \frac{1}{\beta} \right\rvert t \right) dt dz. \tag{4.5}
$$

By the mean value theorem and (4.2), we deduce

$$
|\Phi(z-\Gamma(x,t)) - \Phi(z)| = \left| \int_{0}^{1} \frac{d}{ds}(\Phi(z-s\Gamma(x,t))) ds \right| 
\leq | \int_{0}^{1} \Phi'(z-s\Gamma(x,t)) ds | |\Gamma(x,t)| 
\leq | \int_{0}^{1} \frac{ds}{1+|z-s\Gamma(x,t)|^2} |\Gamma(x,t)| 
\leq \sum_{q \in \mathbb{Z}} \frac{1}{(1+|q|^{2})^{2}} X_{[q,q+1]}(z) \left( u(x) |t|^\alpha + v(x) |t|^\beta \right).
$$

With this, we bound (4.5) by

$$
\sum_{q \in \mathbb{Z}} \frac{1}{(1+|q|^{2})^{2}} \int_{q}^{q+1} \int (P_{0}^{(2)} f)(x-t,y-z) \right| \times \left( u(x) |t|^\alpha \phi \left( u(x) \left\lvert \frac{1}{\alpha} \right\rvert t \right) + v(x) |t|^\beta \phi \left( v(x) \left\lvert \frac{1}{\beta} \right\rvert t \right) \right) dt dz.
$$

Here $|\phi| \leq 1$ is applied. Thus, to prove (4.3), it suffices to show

$$
\int \left| (P_{0}^{(2)} f)(x-t,y) \right| \left( u(x) |t|^\alpha \phi \left( u(x) \left\lvert \frac{1}{\alpha} \right\rvert t \right) + v(x) |t|^\beta \phi \left( v(x) \left\lvert \frac{1}{\beta} \right\rvert t \right) \right) dt \leq \mathcal{M}^{(1)}(P_{0}^{(2)} f)(x,y), \tag{4.6}
$$

the left side of which is the sum of

$$
\int \left| (P_{0}^{(2)} f)(x-t,y) \right| u(x) |t|^\alpha \phi \left( u(x) \left\lvert \frac{1}{\alpha} \right\rvert t \right) dt
$$

and

$$
\int \left| (P_{0}^{(2)} f)(x-t,y) \right| v(x) |t|^\beta \phi \left( v(x) \left\lvert \frac{1}{\beta} \right\rvert t \right) dt.
$$
We only show the estimate of the former since the latter can be bounded by the same way. If $\alpha > 1$, we apply $u(x)|t|^\alpha \lesssim 2^\alpha$ derived from the support of $\phi(u(x)^\frac{1}{2}t)$ to obtain

$$
\int \left| \left( P_0^{(2)} f \right) (x - t, y) \right| u(x)|t|^{\alpha - 1} \phi \left( u(x)^{\frac{1}{2} t} \right) dt 
\lesssim\alpha u(x)^{\frac{1}{2}} \int_{|t| \leq 2u(x)^{\frac{1}{2}}} \left| \left( P_0^{(2)} f \right) (x - t, y) \right| dt,
$$

which is bounded by a constant multiple of $\mathcal{M}^{(1)}(P_0^{(2)} f)(x, y)$. If $0 < \alpha < 1$, we have

$$
\int \left| \left( P_0^{(2)} f \right) (x - t, y) \right| u(x)|t|^{\alpha - 1} \phi \left( u(x)^{\frac{1}{2} t} \right) dt \leq \sup_{\varepsilon > 0} \left\{ \int |P_0^{(2)} f(x - t, y)| \varepsilon^{-1} K(\varepsilon^{-1} t) dt \right\}
$$

(4.7)

where $K(t) = |t|^{\alpha - 1}\phi(t)$. It follows from $0 < \alpha < 1$ and the property of $\phi$ that

$$
\|K(t)\|_1 \leq\alpha 1, \ K(t) = K(|t|), \ K(t_1) \leq K(t_2)
$$

whenever $|t_1| \geq |t_2|$. By Theorem 2.1.10 in [6], the right side of (4.7) is bounded by a constant multiple of $\mathcal{M}^{(1)}P_0^{(2)} f(x, y)$, which completes the proof of (4.3).

Next, we show (4.4). Because of the support of $\phi$, the region of integration is $I := \{|t| \leq \sigma(x)\}$, where $\sigma(x) = \min\{2u(x)^{\frac{1}{2}}, 2v(x)^{\frac{1}{2}}\}$. A simple computation gives

$$
|T^{|l|}_{\alpha, \beta} P_0^{(2)} f(x, y)| \leq \left| p.v. \int_I P_0^{(2)} f(x - t, y) \frac{dt}{t} \right|
$$

$$
+ \left| p.v. \int_I P_0^{(2)} f(x - t, y) \frac{\phi \left( u(x)^{\frac{1}{2} t} \right) - 1}{t} \phi \left( u(x)^{\frac{1}{2} t} \right) dt \right|
$$

$$
+ \left| p.v. \int_I P_0^{(2)} f(x - t, y) \frac{\phi \left( v(x)^{\frac{1}{2} t} \right) - 1}{t} dt \right|.
$$

We bound the first term by $H^* P_0^{(2)} f(x, y)$, and the remainder by $C \mathcal{M}^{(1)} P_0^{(2)} f(x, y)$. As a consequence, $|T^{|l|}_{\alpha, \beta} P_0^{(2)} f(x, y)|$ is bounded by a constant multiple of

$$
H^* P_0^{(2)} f(x, y) + \mathcal{M}^{(1)} P_0^{(2)} f(x, y).
$$

This concludes the proof of (4.4).

\section{Proof of the Second Lemma}

We prove Lemma 3.2 in Section 3. The advantage for the case $p = 2$ is that Minkowski’s inequality plays a positive role. By Littlewood–Paley theory, (3.5) for $p = 2$ is a direct result of

$$
\|T_{k,m}(P_k^{(2)} f)\|_2 \leq 2^{-\max\{j, m\} 2} \|P_k^{(2)} f\|_2
$$

(5.1)
for some \( \tau_2 > 0 \). By the scaling arguments, (5.1) is equivalent to the case \( k = 0 \), that is,

\[
\left\| \int_R (P_0^{(2)} f)(x - t, y - \Gamma(x, t)) \psi_l \left( u(x) \frac{1}{\tau_2} t \right) \psi_m \left( v(x) \frac{1}{\tau_2} t \right) \frac{dt}{t} \right\|_2 \\
\leq 2^{-\gamma'_2 \max\{\beta m, \alpha l\}} \left\| P_0^{(2)} f \right\|_2
\]

holds for some \( \gamma'_2 > 0 \). Applying Plancherel’s theorem in the \( y \) variable again, it is enough to show

\[
\left\| \int g(x - t, \eta) e^{-i \left( u(x)[t]^{\alpha} + v(x)[t]^{\beta} \right) \eta} \psi_l \left( u(x) \frac{1}{\tau_2} t \right) \psi_m \left( v(x) \frac{1}{\tau_2} t \right) \frac{dt}{t} \right\|_{L_2^2(\mathbb{R}^2)} \\
\leq 2^{-\gamma'_2 \max\{\beta m, \alpha l\}} \| g \|_2.
\]

We bound the integral on the left side by \((\omega \ast \Phi_{\alpha, \beta}^{a, b})(x, \eta)\) with

\[
\omega = g(x, \eta) = P_0^{(2)} (P_0^{(2)} f)(x, \eta), \quad a = 2^{l} u(x) \frac{1}{\tau_2}, \quad b = 2^{-m} v(x) \frac{1}{\tau_2}, \quad \lambda_1 = 2^{a} \eta, \quad \lambda_2 = \frac{u(x)}{2^{-l} u(x) \frac{1}{\tau_2}} \eta.
\]

Then Lemma 3.2 gives (5.3) with \( 2^{-\gamma'_2 \max\{\beta m, \alpha l\}} \) replaced by \( 2^{-\gamma_0 a l} \). Similarly, the integral on the left side can also be bounded by \((\omega \ast \Phi_{\alpha, \beta}^{a, b})(x, \eta)\) with

\[
\omega = g(x, \eta) = P_0^{(2)} (P_0^{(2)} f)(x, \eta), \quad a = 2^{m} v(x) \frac{1}{\tau_2}, \quad b = 2^{-l} u(x) \frac{1}{\tau_2}, \quad \lambda_1 = 2^{m} \beta \eta, \quad \lambda_2 = \frac{u(x)}{2^{-m} v(x) \frac{1}{\tau_2}} \eta.
\]

By Lemma 3.2, we obtain (5.3) with \( 2^{-\gamma'_2 \max\{\beta m, \alpha l\}} \) replaced by \( 2^{-\gamma_0 \beta m} \). Combing with the above two bound yields (5.3) for \( \gamma'_2 = \gamma_0 \).

**Acknowledgements**

The author would like to thank Prof. Jiecheng Chen for his encouragement and the referee for providing useful comments. This work was supported by the NSF of China (11901301) and the NSF of Jiangsu Province (BK20180721).

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**How to cite this article:** R. Wan, *Lp bound for the Hilbert transform along variable non-flat curves*, Math. Nachr. 296 (2023), 1669–1686. https://doi.org/10.1002/mana.202000490