O.Y. Kushel

SPECTRAL PROPERTIES OF ONE CLASS OF SIGN-SYMMETRIC MATRICES

Abstract.

A \( n \times n \) matrix \( A \), which has a certain sign-symmetric structure (\( J \)-sign-symmetric), is studied in this paper. It’s shown, that such a matrix is similar to a nonnegative matrix. The existence of the second in modulus positive eigenvalue \( \lambda_2 \) of a \( J \)-sign-symmetric matrix \( A \), or an odd number \( k \) of simple eigenvalues, which coincide with the \( k \)th roots of \( \rho(A)^k \), is proved under the additional condition, that its second compound matrix is also \( J \)-sign-symmetric. The conditions, when a \( J \)-sign-symmetric matrix with a \( J \)-sign-symmetric second compound matrix has complex eigenvalues, which are equal in modulus to \( \rho(A) \), are given.

**Keywords:** Totally positive matrices, Sign-symmetric matrices, Nonnegative matrices, Compound matrices, Exterior powers, Gantmacher–Krein theorem, eigenvalues.

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1 Introduction

We first state a classical result of F.R. Gantmacher and M.G. Krein (see, for example, [1]).

**Theorem A (Gantmacher–Krein).** If the matrix \( A \) of a linear operator \( A : \mathbb{R}^n \to \mathbb{R}^n \) is positive together with its \( j \)th compound matrices \( A^{(j)} \) \( (1 < j \leq k) \) up to the order \( k \), then the operator \( A \) has \( k \) positive simple eigenvalues \( 0 < \lambda_k < \ldots < \lambda_2 < \lambda_1 \), with the positive eigenvector \( x_1 \) corresponding to the maximal eigenvalue \( \lambda_1 \), and the eigenvector \( x_j \), which has exactly \( j - 1 \) changes of sign, corresponding to the \( j \)-th eigenvalue \( \lambda_j \).

Let us remember, that a matrix, which satisfies the conditions of theorem A, is called strictly \( k \)-totally positive (\( STP_k \)). If a matrix \( A \) is nonnegative together with its \( j \)th compound matrices \( A^{(j)} \) \( (1 < j \leq k) \) up to the order \( k \), then \( A \) is called \( k \)-totally positive (\( TP_k \)). The following statement (see [1], p. 317, theorem 13) easily comes out from the reasons of continuity: *if the matrix \( A \) of a linear operator \( A : \mathbb{R}^n \to \mathbb{R}^n \) is \( k \)-totally positive, then the operator \( A \) has \( k \) nonnegative eigenvalues \( 0 \leq \lambda_k \leq \ldots \leq \lambda_2 \leq \lambda_1 \).*

We next formulate the statement, which follows from the infinite-dimensional statements, proved in [2]. This statement generalizes the concept of total positivity.
Theorem B. Let the matrix $A$ of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be similar to a nonnegative irreducible matrix. Let its second compound matrix $A^{(2)}$ be also similar to a nonnegative irreducible matrix. Then one of the following two cases takes place:

1. The operator $A$ has two positive simple eigenvalues $\lambda_1, \lambda_2$:

$$\rho(A) = \lambda_1 > \lambda_2 > 0.$$ 

Moreover, if the operator $A$ has $l > 1$ eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_{l+1}$, equal in modulus to $\lambda_2$, then each of them is simple, and they coincide with the $l$th roots from $\lambda_2$.

2. There is just three eigenvalues on the spectral circle $|\lambda| = \rho(A)$. Each of them is simple, and they coincide with the 3th roots of $(\rho(A))^3$.

The proof of theorem B is based on the well-known Perron-Frobenius statement, that a nonnegative irreducible matrix has the maximal in modulus simple positive eigenvalue, and very simple reasoning, that similarity transformations do not change the spectrum of a matrix. Let us note, that the peripheral spectrum of the operator $A$ not always will be real. This is the principal difference between the statement of theorem B and the statements, proved in [1]. The necessary and sufficient condition of the existence of complex eigenvalues on the spectral circle $|\lambda| = \rho(A)$ in the case, when both the matrices $A$ and $A^{(2)}$ are similar to nonnegative irreducible matrices with diagonal matrices of similarity transformations, was obtained in [3].

The class of matrices, which are similar to nonnegative matrices together with their second compound matrices is studied in this paper (later on we shall not assume the additional condition of irreducibility). The problem of the detailed description of the spectrum of such matrices is examined. The obtained results are based on the Frobenius theorems of the structure of reducible matrices and combinatorial reasons. The main question, raised in this paper, is: when will such matrices have complex eigenvalues on the largest spectral circle?

2 Tensor and exterior square of the space $\mathbb{R}^n$

Let us briefly remind basic definitions and statements, related with the tensor and exterior squares of the space $\mathbb{R}^n$ (for more detailed information see [4], [5], [3]).

Later on, as the author thinks, it will be more convenient to consider the space $\mathbb{R}^n$ as the space of real-valued functions $x : \{1, \ldots, n\} \to \mathbb{R}$, defined on the finite set of indices $\{1, \ldots, n\}$. Denote such a space by $X$. The basis in the space $X$ consists of the functions $e_i$, for which $e_i(j) = \delta_{ij}$.

The tensor square $X \otimes X$ of the space $X$ is the space of all functions, defined on the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$. The space $X \otimes X$ is isomorphic to the space $\mathbb{R}^{n^2}$.

The exterior square $X \wedge X$ of the space $X$ is the space of all antisymmetric functions (i.e. functions $f(i, j)$, for which the equality $f(i, j) = -f(j, i)$ is true),
defined on the set \{1, \ldots, n\} \times \{1, \ldots, n\}. It is known, that \(X \wedge X\) coincides with the linear span of all exterior products \(x \wedge y\) \((x, y \in X)\), which acts according to the rule:

\[(x \wedge y)(i, j) = x(i)y(j) - x(j)y(i).\]

The space \(X \wedge X\) is isomorphic to the space \(X(W \setminus \Delta)\) of real-valued functions, defined on the set \(W \setminus \Delta\), where \(W\) is a subset of \(\{1, \ldots, n\} \times \{1, \ldots, n\}\), which satisfies the following conditions:

\[W \cup \tilde{W} = (\{1, \ldots, n\} \times \{1, \ldots, n\});\]
\[W \cap \tilde{W} = \Delta.\]  

(Here \(\tilde{W}\) = \{(j, i) : (i, j) \in W\}; \(\Delta = \{(i, i) : i = 1, \ldots, n\}\).)

The following equality is true for the power of the set \(W \setminus \Delta\):

\[N(W \setminus \Delta) = \frac{n^2 - n}{2} = C_n^2.\]

It comes out from the above reasoning, that the space \(X(W \setminus \Delta)\) is isomorphic to the space \(\mathbb{R}^{C_n^2}\).

Every subset \(W \subset \{1, \ldots, n\} \times \{1, \ldots, n\}\) defines a binary relation on the set \(\{1, \ldots, n\}\) (see, for example, [6]). If the set \(W\) satisfies conditions (1) and (2), and, in addition, the inclusion \((i, k) \in W\) follows from the inclusions \((i, j) \in W\) and \((j, k) \in W\) for any \(i, j, k \in \{1, \ldots, n\}\) (i.e. the set \(W\) possesses the property of transitivity), then the relation, defined by the set \(W\), is a linear order relation.

Every subset \(W \subset \{1, \ldots, n\} \times \{1, \ldots, n\}\), which satisfies conditions (1) and (2), uniquely defines a basis \(\{e_i \wedge e_j\}_{(i,j) \in W \setminus \Delta}\) in \(\mathbb{R}^n \wedge \mathbb{R}^n\), which consists of the exterior products of the initial basic vectors. Such a basis in \(X \wedge X\), constructed with respect to the set \(W\), is called a \(W\)-basis.

3 The second compound matrix and the exterior square of a linear operator

The exterior square \(A \wedge A\) of the operator \(A : X \to X\) acts in the space \(X \wedge X\) according to the rule:

\[(A \wedge A)(x \wedge y) = Ax \wedge Ay.\]

Let the operator \(A\) be defined by the matrix \(A = \{a_{ij}\}_{i,j=1}^n\) in the basis \(\{e_i\}_{i=1}^n\). Then the matrix of the operator \(A \wedge A\) in the \(W\)-basis \(\{e_i \wedge e_j\}_{(i,j) \in W \setminus \Delta}\) coincides with the \(W\)-matrix \(A_W^{(2)}\). (Here the matrix \(A_W^{(2)}\) consists of minors \(A \left( \begin{array}{cc} i & j \\ k & l \end{array} \right)\) of the initial matrix \(A\), formed of the rows with numbers \(i\) and \(j\) and the columns with numbers \(k\) and \(l\), where \((i, j), (k, l) \in (W \setminus \Delta)\). The minors \(A \left( \begin{array}{cc} i & j \\ k & l \end{array} \right)\) are numerated in the lexicographic order.)
Note, that if the set $W$ coincides with the set $M = \{(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, n\} : i \leq j\}$, then the corresponding $W$–basis is $\{e_i \wedge e_j\}_{1 \leq i < j \leq n}$, i.e. the canonical basis in the space $\mathbb{R}^n \wedge \mathbb{R}^n$, and the corresponding $W$–matrix is a matrix, which consists of minors $A\left(\begin{array}{cc}i & j \\ k & l \end{array}\right)$, where $1 \leq i < j \leq n$, $1 \leq k < l \leq n$, i.e. the second compound matrix of the matrix $A$.

The following theorem about the eigenvalues of a $W$–matrix is true (see [3]).

**Theorem 1.** Let $W$ be a subset of the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$, which satisfies conditions (1) and (2). Let $\{\lambda_i\}_{i=1}^n$ be the set of all eigenvalues of the matrix $A$, repeated according to multiplicity. Then all the possible products of the type $\{\lambda_i \lambda_j\}$, where $1 \leq i < j \leq n$, forms the set of all the possible eigenvalues of the corresponding $W$–matrix $A^{(2)}_W$, repeated according to multiplicity.

In the case $W = M$ theorem 1 turns into the Kronecker theorem (see [1], p. 80, theorem 23) about the eigenvalues of the second compound matrix.

### 4 Operators, which leave invariant a cone in $\mathbb{R}^n$, and their matrices

Let us remember some widely used definitions. An operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is called **nonnegative**, if it leaves invariant the cone of nonnegative vectors in $\mathbb{R}^n$. It’s well known, that the operator $A$ is nonnegative if and only if its matrix $A$ is nonnegative, i.e. all the elements $a_{ij} (i, j = 1, \ldots, n)$ of the matrix $A$ are nonnegative.

The following theorem is perhaps the best known part of the theory of nonnegative operators (see, for example, [7], p. 14, theorem 4.2).

**Theorem 2 (Perron).** Let the matrix $A$ of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be nonnegative. Then the spectral radius $\rho(A) \geq 0$ is a nonnegative eigenvalue of the operator $A$, with the nonnegative eigenvector $x_1$ corresponding to it.

However, this result is also correct for any matrix, similar to a nonnegative matrix (since a similarity transformation preserves the spectrum of a matrix). The question, how can we see if an arbitrary matrix is similar to a nonnegative matrix, was raised in [3]. The answer was given for the special case, when the matrix of the similarity transformation is diagonal. The following definition was first introduced in [3].

A matrix $A$ of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is called **$J$–sign-symmetric**, if there exists such a subset $J \subseteq \{1, \ldots, n\}$, that both the conditions (a) and (b) are true:

(a) the inequality $a_{ij} \leq 0$ follows from the inclusions $i \in J$, $j \in \{1, \ldots, n\} \setminus J$ and from the inclusions $j \in J$, $i \in \{1, \ldots, n\} \setminus J$ for any two numbers $i, j$;

(b) one of the inclusions $i \in J$, $j \in \{1, \ldots, n\} \setminus J$ or $j \in J$, $i \in \{1, \ldots, n\} \setminus J$ follows from the strict inequality $a_{ij} < 0$.

Note, that the subset $J$ in the definition of $J$–sign-symmetricity in the case of an arbitrary matrix $A$ is not uniquely defined.
The following statement of the similarity of a $\mathcal{J}$–sign-symmetric matrix to a nonnegative matrix was proved in [3] (see [3], theorem 4).

**Theorem 3.** Let $A$ be a $\mathcal{J}$–sign-symmetric matrix. Then it can be represented in the following form:

$$A = D\tilde{A}D^{-1},$$

where $\tilde{A}$ is a nonnegative matrix, $D$ is a diagonal matrix, which diagonal elements are equal to $\pm 1$.

**Corollary.** Let the matrix $A$ of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathcal{J}$–sign-symmetric. Then the spectral radius $\rho(A) \geq 0$ is a nonnegative eigenvalue of the operator $A$.

This statement has the following geometric meaning: if the matrix $A$ of a linear operator $A$ is $\mathcal{J}$–sign-symmetric in some basis $\{e_i\}_{i=1}^n$, then the operator $A$ leaves invariant one of the cones, spanned on the vectors $\{\pm e_i\}_{i=1}^n$.

Later on operators with positive spectral radius will play an important role. Let us formulate a sufficient criteria of the positivity of $\rho(A)$.

**Lemma 1.** Let the matrix $A$ of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathcal{J}$–sign-symmetric. Let at least one element $a_{ii}$, situated on the principal diagonal of $A$, be not equal to zero. Then the spectral radius $\rho(A) > 0$ is a positive eigenvalue of the operator $A$.

□ It follows from the definition of $\mathcal{J}$–sign-symmetricity, that all the elements $a_{ii}$ ($i = 1, \ldots, n$) are nonnegative. Since at least one of $a_{ii}$ is non-zero, then the sum $\sum_{i=1}^n a_{ii}$ is positive. It’s clear (see, for example, [5]), that:

$$\text{tr}(A) = \sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_{ii}.$$  

(Here $\{\lambda_i\}_{i=1}^n$ is the set of all eigenvalues of the operator $A$, repeated according to multiplicity.) So we have, that

$$\rho(A) \geq \frac{1}{n} \sum_{i=1}^n \lambda_i = \frac{1}{n} \sum_{i=1}^n a_{ii} > 0.$$

□

This criteria is sufficient, but, as it’ll be shown later, not necessary for the positivity of $\rho(A)$.

Now we’ll require some notations. Given the set $\alpha$ of $k$ ($1 \leq k \leq n$) indices $i_1, \ldots, i_k$ ($1 \leq i_1 < \ldots < i_k \leq n$). Then

$$A(\alpha) = A \left[ \begin{array}{cccc} i_1 & \ldots & i_k \\ i_1 & \ldots & i_k \end{array} \right]$$

denotes a $k \times k$ submatrix of the matrix $A$, formed of the rows with numbers $i_1, \ldots, i_k$ and the columns with the same numbers. The submatrix $A(\alpha)$ is called
a principal submatrix of the matrix $A$ and it represents the matrix of the restriction of the operator $A$ to a $k$-dimensional coordinate subspace, spanned on the basic vectors $e_{i_1}, \ldots, e_{i_k}$.

Any submatrix of a nonnegative matrix is obviously nonnegative. Let us prove the corresponding property of principal submatrices of a $J$–sign-symmetric matrix.

**Lemma 2.** Let the matrix $A$ of a linear operator $A$ be $J$–sign-symmetric. Then every principal submatrix of the matrix $A$ is also $J$–sign-symmetric.

\[ \square \]

Let $A(\alpha)$ is an arbitrary principal submatrix of the matrix $A$. It’s obvious, that the set $J \cap \alpha$ satisfy both the conditions (a) and (b) in the definition of $J$–sign-symmetricity of the matrix $A(\alpha)$. \[ \blacksquare \]

### 5 Irreducible operators and their matrices

A linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is called irreducible, if it has no invariant $k$-dimensional coordinate subspaces with $0 < k < n$. Otherwise it’s called reducible. An operator $A$ is irreducible (reducible) if and only if its matrix $A$ is irreducible (respectively reducible). Remember, that a matrix $A$ is called reducible, if there exists a permutation of coordinates such that:

\[
P^{-1}AP = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix},
\]

where $P$ is an $n \times n$ permutation matrix (each row and each column have exactly one 1 entry and all others 0), $A_1, A_2$ are square matrices. Otherwise the matrix $A$ is called irreducible.

We next formulate some important properties of irreducible operators (see, for example, [7], [8]).

The following properties of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ are equivalent:

(i) Its matrix $A$ is irreducible;

(ii) Its matrix $A$ has a "path of irreducibility i.e. such a set of indices $\{j_0, j_1, \ldots, j_s\}$, that $j_0 \neq j_1$, $j_1 \neq j_2, \ldots, j_{s-1} \neq j_s$, every index of $\{1, \ldots, n\}$ coincides with one of the indices $\{j_0, j_1, \ldots, j_s\}$ and

\[
a_{j_0j_1}, a_{j_1j_2}, \ldots, a_{j_{s-1}j_s}, a_{j_sj_0} > 0.
\]

(iii) Every nonnegative eigenvector of the operator $A$ is positive.

An irreducible matrix $A$ is called imprimitive, if there exists a permutation of coordinates such that:

\[
P^{-1}AP = \begin{pmatrix} 0 & A_{12} & 0 & \ldots & 0 \\ 0 & 0 & A_{23} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_{h-1h} \\ A_{h1} & 0 & 0 & \ldots & 0 \end{pmatrix},
\]

where $A_{h1}, A_{23}, \ldots, A_{12}$ are square matrices.
where $P$ is an $n \times n$ permutation matrix, $A_{i\ i+1}$ ($i = 1, \ldots, h-1$) and $A_{h\ 1}$ are square matrices. Otherwise the matrix $A$ is called primitive. An operator $A$ is imprimitive (primitive) if and only if its matrix is imprimitive (primitive).

The following sufficient criteria of primitivity was proved in [7] (see [7], p. 49, corollary 1.1): if the matrix $A$ of the operator $A$ is irreducible, and $tr(A) > 0$, then $A$ is primitive.

Let us remember the widely known Frobenius theorem on the spectrum of irreducible operators.

**Theorem 4 (Frobenius).** Let the matrix $A$ of a linear operator $A$ be nonnegative and irreducible. Then the spectral radius $\rho(A) > 0$ is a simple positive eigenvalue of the operator $A$, with the corresponding positive eigenvector $x_1$. If $h$ is a number of the eigenvalues of the operator $A$, which are equal in modulus to $\rho(A)$, then all of them are simple and they coincide with the $h$-th roots of $(\rho(A))^h$. Moreover, the spectrum of the operator $A$ is invariant under rotations by $\frac{2\pi}{h}$ about the origin.

The number of the eigenvalues, which are equal in modulus to $\rho(A)$ is called the index of imprimitivity of an irreducible operator $A$ and denoted $h(A)$. If $h(A) > 1$, then it is equal to the number $h$ of the square blocs in the form (5) of the matrix $A$. If $A$ is primitive, then $h(A) = 1$.

It’s easy to see, that the similarity transformation (3) with a diagonal matrix $D$ preserves the property of irreducibility and the index of imprimitivity of the operator $A$ as well, as other spectral properties (see [3]). So we have:

**Theorem 5.** Let the matrix $A$ of a linear operator $A$ be $J$-sign-symmetric and irreducible. Then the spectral radius $\rho(A) > 0$ is a simple positive eigenvalue of the operator $A$. If $h$ is a number of the eigenvalues of the operator $A$, which are equal in modulus to $\rho(A)$, then all of them are simple and they coincide with the $h$-th roots of $(\rho(A))^h$. Moreover, the spectrum of the operator $A$ is invariant under rotations by $\frac{2\pi}{h}$ about the origin.

And the following criteria is true.

**Lemma 3.** Let the matrix $A$ of the operator $A$ be $J$-sign-symmetric and irreducible. If at least one element $a_{ii}$, situated on the principal diagonal of $A$, is not equal to zero, then $A$ is primitive. In turn, if $A$ is imprimitive, then its principal diagonal contains only zeroes.

□ The proof of lemma 3 comes out from the given above sufficient criteria of the primitivity of nonnegative matrices.

Note, that if the matrix $A$ is $J$-sign-symmetric and irreducible, then the set $J$ in the definition of $J$-sign-symmetricity is uniquely defined (up to the set $\{1, \ldots, n\} \setminus J$).

### 6 Reducible operators and their matrices

The statement, which helps to bring the study of reducible operators in the finite-dimensional space $\mathbb{R}^n$ to the study of irreducible operators is well-known (see, for
Theorem 6 (Frobenius). Let \( A \) be a nonnegative reducible matrix. Then there exists a permutation of coordinates such that:

\[
P^{-1}AP = \hat{A},
\]

where \( P \) is an \( n \times n \) permutation matrix, \( \hat{A} \) is a block-triangular form, where the finite number \( l \leq n \) of square irreducible blocks \( A_j \) (\( j = 1, \ldots, l \)) are situated on the principal diagonal, and zero elements are situated above the principal diagonal:

\[
A = \begin{pmatrix}
A_1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_r & 0 & 0 & \ldots & 0 \\
B_{r+1} & B_{r+2} & \ldots & B_{r+1r} & A_{r+1} & 0 & \ldots & 0 \\
B_{r+21} & B_{r+22} & \ldots & B_{r+2r} & B_{r+2r+1} & A_{r+2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
B_{l1} & B_{l2} & \ldots & B_{lr} & B_{lr+1} & B_{lr+2} & \ldots & A_l
\end{pmatrix}
\]

(6)

Such a representation is uniquely defined (up to the numeration of the blocs).

The spectral radius \( \rho(A) \) is an eigenvalue of the operator \( A \) with the nonnegative eigenvector \( x_1 \) corresponding to it. Moreover, the following equalities are true:

\[
\sigma_p(A) = \bigcup_{j=1}^{l} \sigma_p(A_j), \quad \rho(A) = \max_{j=1,\ldots,l} \{ \rho(A_j) \},
\]

where \( \sigma_p(A_j) \) are the point spectra (i.e. the sets of all eigenvalues), and \( \rho(A_j) \) are the spectral radii of the irreducible blocs \( A_j \) (\( j = 1, \ldots, l \)).

It’s easy to see, that if the matrix \( A \) is \( J \)-sign-symmetric and reducible, then there is \( 2^l \) possible ways of constructing the set \( J \) in the definition of \( J \)-sign-symmetry.

7 The connection between the sets \( J \) and \( W \).

Later on we’ll impose the condition of \( J \)-sign-symmetricity to the second compound matrices \( A^{(2)} \). So we are interested in some special properties of \( A^{(2)} \). The following statement about the link between the structure of the second compound matrices \( A^{(2)} \) and the structure of the matrix \( A^{(2)}_W \), constructed with respect to a set \( W \in \{1, \ldots, n\} \times \{1, \ldots, n\} \), which satisfies conditions (1) and (2), is true.

Let the second compound matrix \( A^{(2)} \) of a matrix \( A \) be \( J \)-sign-symmetric. Then there exists such a set \( W \in \{1, \ldots, n\} \times \{1, \ldots, n\} \), which satisfies conditions (1) and (2), that the corresponding \( W \)-matrix \( A^{(2)}_W \) is nonnegative. Moreover, if \( A^{(2)} \) is irreducible, then \( A^{(2)}_W \) is also irreducible (see [3], theorem 6).
This statement easily follows from the theorem 3 and the given above fact, that the matrix of the exterior square $A \wedge A$ of the operator $A$ in the $W$–basis 
$\{e_i \wedge e_j\}_{(i,j)\in W\wedge \Delta}$ coincides with the $W$–matrix $A^{(2)}_W$.

The method of constructing the set $W$, for which the corresponding $W$–matrix $A^{(2)}_W$ is positive, by the set $J$ in the definition of $J$–sign-symmetricity of $A^{(2)}$ is given in [3] (see the proof of theorems 5 and 6):

The pair $(i, j) \in W$ if and only if one of the following two cases takes place:

(a) $i < j$, and the number $\alpha$ of the pair $(i, j)$ (in the lexicographic numeration), belongs to the set $J$;

(b) $i > j$, and the number $\tilde{\alpha}$ of the pair $(j, i)$ belongs to the set $\{1, \ldots, C^2_n\} \setminus J$.

It’s easy to see, that the number of different ways of the constructing of the set $W$ is equal to the number of different ways of the constructing of the set $J$ in the definition of $J$–sign-symmetricity of the second compound matrix.

Let us generalize the given above method of constructing the set $W$ to the case of $J$–sign-symmetric matrix. Let $A$ be a $J$–sign-symmetric matrix, and let $J$ be a subset of the set $\{1, \ldots, n\}$ in the definition of $J$–sign-symmetricity (i.e. such a subset, that the conditions (a) and (b) are true). Let $A^{(2)}$ be a $J$–sign-symmetric matrix. Let $\tilde{J}$ be a subset of $\{1, \ldots, C^2_n\}$ in the definition of $J$–sign-symmetricity for the matrix $A^{(2)}$. Let us construct a set $\hat{W}(J, \tilde{J}) \subseteq (\{1, \ldots, n\} \times \{1, \ldots, n\})$ with respect to the sets $J$ and $\tilde{J}$ by the following way.

A pair $(i, j)$ belongs to the set $\hat{W}(J, \tilde{J})$ if and only if one of the following four cases takes place:

(a) $i < j$, both the numbers $i, j$ belong either to the set $J$, or to the set $\{1, \ldots, n\} \setminus J$, and the number $\alpha$, corresponding to the pair $(i, j)$ in the lexicographic numeration, belongs to the set $\tilde{J}$;

(b) $i < j$, one of the numbers $i, j$ belongs to the set $J$, and the other belongs to the set $\{1, \ldots, n\} \setminus J$, and the number $\alpha$, corresponding to the pair $(i, j)$ in the lexicographic numeration, belongs to the set $\{1, \ldots, C^2_n\} \setminus \tilde{J}$;

(c) $i > j$, both the numbers $i, j$ belong either to the set $J$, or to the set $\{1, \ldots, n\} \setminus J$, and the number $\alpha$, corresponding to the pair $(j, i)$ in the lexicographic numeration, belongs to the set $\{1, \ldots, C^2_n\} \setminus \tilde{J}$;

(d) $i > j$, one of the numbers $i, j$ belongs to the set $J$, the other belongs to the set $\{1, \ldots, n\} \setminus J$, and the number $\alpha$, corresponding to the pair $(j, i)$ in the lexicographic numeration, belongs to the set $\tilde{J}$.

As it was noticed above, the set $W(J, \tilde{J})$ is called transitive, if the inclusion $(i, k) \in W(J, \tilde{J})$ follows from the inclusions $(i, j) \in W(J, \tilde{J})$ and $(j, k) \in W(J, \tilde{J})$ for any $i, j, k \in \{1, \ldots, n\}$

The set $W(J, \tilde{J})$ is obviously not uniquely defined, but there is a finite number of the possible ways of its constructing.
Generalization of the Gantmacher–Krein theorems to the case of an irreducible $2$–totally $\mathcal{J}$–sign-symmetric matrix.

Now we'll prove the theorem of the spectral properties of an irreducible $\mathcal{J}$–sign-symmetric matrix with a $\mathcal{J}$–sign-symmetric second compound matrix, using the following statement, proved in [3] (see [3], theorem 12).

**Theorem 7.** Let $A$ be a $\mathcal{J}$–sign-symmetric matrix. Let its second compound matrix $A^{(2)}$ be also $\mathcal{J}$–sign-symmetric. Let one of the sets $\hat{W}(\mathcal{J}, \tilde{\mathcal{J}})$ be transitive. Then the two largest in modulus eigenvalues of the operator $A$ are nonnegative.

The following statement about the spectrum of a $\mathcal{J}$–sign-symmetric irreducible matrix with a $\mathcal{J}$–sign-symmetric second compound matrix is true.

**Theorem 8.** Let the matrix $A$ of a linear operator $A$ be $\mathcal{J}$–sign-symmetric and irreducible. Let its second compound matrix $A^{(2)}$ be $\mathcal{J}$–sign-symmetric. Then the operator $A$ has a simple positive eigenvalue $\lambda_1 = \rho(A)$. Moreover, one of the following two cases takes place:

1. If at least one of the possible sets $W(\mathcal{J}, \tilde{\mathcal{J}})$ is transitive, then $h(A) = 1$, the second in modulus eigenvalue $\lambda_2$ of the operator $A$ is nonnegative and different in modulus from the first eigenvalue:

   $0 \leq \lambda_2 < \lambda_1$.

2. If all the possible sets $W(\mathcal{J}, \tilde{\mathcal{J}})$ are not transitive, then there is an odd number $k \geq 1$ of eigenvalues on the spectral circle $|\lambda| = \rho(A)$. All of them are simple and coincide with the $k$th roots of $(\rho(A))^k$.

□ Suppose a transitive set $W(\mathcal{J}, \tilde{\mathcal{J}})$ exists. Enumerate the eigenvalues of the matrix $A$ (repeated according to multiplicity) in order of decrease of their modules:

   $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$.

Then, applying theorem 7, we get, that the two largest in modulus eigenvalues of the operator $A$ are nonnegative: $0 \leq \lambda_2 \leq \lambda_1 = \rho(A)$. It means, that either $\lambda_1$ is a multiple eigenvalue (this contradicts the irreducibility of the operator $A$) or there is only one eigenvalue on the spectral circle $|\lambda| = \rho(A)$. I.e. the equality $h(A) = 1$ is true and the inequalities $0 \leq \lambda_2 < \lambda_1$ are also true.

Let all the possible sets $W(\mathcal{J}, \tilde{\mathcal{J}})$ be not transitive. Let $h(A) = k$ be the index of imprimitivity of the operator $A$. Let us apply the Frobenius theorem to the irreducible operator $A$. We get, that all the eigenvalues of the operator $A$, equal in modulus to $\rho(A)$, are simple and they coincide with the $k$th roots of $\rho(A)^k$.

Show, that the number $k$ is odd in this case. Suppose the opposite: let $k$ be even. Examine the exterior square $A \wedge A$ of the operator $A$ and its matrix $A^{(2)}$. It follows
from theorem 1, that all the eigenvalues \( A \& A \), equal in modulus to \( \rho(A \& A) \), form all the possible products of the type \( \lambda_j \lambda_m \), where \( 1 \leq j < m \leq k \), and \( \lambda_1, \ldots, \lambda_k \) are all the eigenvalues of the operator \( A \), equal in modulus to \( \rho(A) \). If \( k = 2 \), then there is only one negative eigenvalue equal to \( -\rho(A)^2 \) on the spectral circle \( |\lambda| = \rho(A \& A) \). This fact contradicts the Perron theorem (applying the Perron theorem to the operator \( A \& A \), we get, that the spectral radius \( \rho(A \& A) \) is a nonnegative eigenvalue of \( A \& A \)). Now let \( k = 4, 6, 8, \ldots \). Examine \( \lambda_j = \rho(A)e^{2\pi(i-1)/k} \) (\( j = 1, \ldots, k \)) \( - \) all the eigenvalues of the operator \( A \), equal in modulus to \( \rho(A) \). It follows from theorem 1, that all the eigenvalues, equal in modulus to \( \rho(A \& A) \), can be written in the form \( \lambda_j \lambda_m = \rho(A)^2e^{2\pi(i-1)/k} e^{2\pi(m-1)/k} \), where \( 1 \leq j < m \leq k \) (the general number of such eigenvalues is \( C_k^2 = \frac{k(k-1)}{2} \)). It’s easy to see, that there are multiple eigenvalues among them. That is why the operator \( A \& A \) is reducible. It follows from the Frobenius theorem of nonnegative matrices, that the number \( s \) of irreducible blocs \( A_j \) with the property \( \rho(A_j) = \rho(A \& A) \) on the principal diagonal in representation (6) of the reducible operator \( A \& A \) is equal to the multiplicity of the nonnegative eigenvalue \( \rho(A \& A) = \rho(A)^2 \). Let us calculate this multiplicity. For this we write down all the products of the form \( \lambda_j \lambda_m \), where \( 1 \leq j < m \leq k \), equal to \( \rho(A)^2 \):

\[
\begin{align*}
\rho(A)^2 &= \rho(A)^2 e^{2\pi i/k} e^{2\pi(k-1)/k} \\
\rho(A)^2 &= \rho(A)^2 e^{4\pi i/k} e^{2\pi(k-2)/k} \\
\rho(A)^2 &= \rho(A)^2 e^{6\pi i/k} e^{2\pi(k-3)/k} \\
&\quad \ldots \quad \ldots \\
\rho(A)^2 &= \rho(A)^2 e^{2\pi(i(k-1))/k} e^{2\pi(k-1)/k}
\end{align*}
\]

It’s easy to see, that the number of such products is equal to \( \frac{k}{2} - 1 \). As it follows, the number \( s \) of the irreducible blocs \( A_j \) with \( \rho(A_j) = \rho(A \& A) \) is equal to \( \frac{k}{2} - 1 \). Since all the eigenvalues, equal in modulus to \( \rho(A \& A) \), are products of the different \( k \)th roots of \( \rho(A)^k \), they are \( k \)th roots of \( \rho(A)^{2k} \). As it follows, the index of imprimitivity \( h(A_j) \) can not be greater than \( k \) for any \( j = 1, \ldots, s \). The general number of the eigenvalues (taking into account their multiplicities), equal in modulus to \( \rho(A \& A) \), is not greater, than \( (\frac{k}{2} - 1)k \), i.e. the product of the number of the block and the maximal index of imprimitivity of any of them. We came to the contradiction, because there is \( C_k^2 = \frac{k(k-1)}{2} > (\frac{k}{2} - 1)k \) eigenvalues, equal in modulus to \( \rho(A \& A) \).

**Example 1.** Let the operator \( A : \mathbb{R}^5 \to \mathbb{R}^5 \) be defined by the matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

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This matrix is obviously nonnegative and irreducible.

In this case the second compound matrix is the following:

$$A^{(2)} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

It's easy to see, that the matrix $A^{(2)}$ is reducible and $J$–sign-symmetric. In this case we have $2^2 = 4$ ways of constructing the set $J$: $J = \{1, 5, 8, 10, 2, 6, 9\}$, $J = \{4, 7, 3\}$, $J = \{1, 5, 8, 10, 7, 3\}$, $J = \{4, 2, 6, 9\}$. Examine the sets $W$, corresponding to this sets of indices $J$. Such sets $W$ defines non-transitive binary relations on the set of the indices $\{1, 2, 3, 4, 5\}$. The operator $A$ satisfies the conditions of theorem 8, case (2). It's easy to see, that $A$ has the first positive simple eigenvalue $\lambda = \rho(A) = 1$, and there is five (an odd number) eigenvalues $1$, $\frac{2\pi i}{8}$, $\frac{4\pi i}{8}$, $\frac{6\pi i}{8}$ and $\frac{8\pi i}{8}$ on the spectral circle $|\lambda| = 1$, all of them are simple and coincide with 5th roots of 1.

Let us notice, that if we assume the irreducibility of the second compound matrix $A^{(2)}$, as well, as of the initial matrix $A$, then the theorem 8 can be refined.

**Theorem 9.** Let the matrix $A$ of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be $J$–sign-symmetric and irreducible. Let its second compound matrix $A^{(2)}$ be also $J$–sign-symmetric and irreducible. Then one of the following two cases takes place:

1. The set $\hat{W}(J, \tilde{J})$ is transitive. Then $h(A) = 1$, $h(A \wedge A)$ is an arbitrary, and the operator $A$ has two positive simple eigenvalues $\lambda_1$, $\lambda_2$:

$$\rho(A) = \lambda_1 > \lambda_2 \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$  

   If $h(A) = h(A \wedge A) = 1$, then $\lambda_2$ is different in modulus from the other eigenvalues. If $h(A) = 1$, and $h(A \wedge A) > 1$, then the operator $A$ has $h(A \wedge A)$ eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_{h(A \wedge A) + 1}$, equal in modulus to $\lambda_2$, each of them is simple, and they coincide with the $h(A \wedge A)$th roots from $\lambda_2^{h(A \wedge A)}$.

2. the set $\hat{W}(J, \tilde{J})$ is not transitive. Then $h(A) = h(A \wedge A) = 3$, and there is just three eigenvalues on the spectral circle $|\lambda| = \rho(A)$. Each of them is simple, and they coincide with the $3$th roots of $(\rho(A))^3$.

The proof of the theorem 5 can be found in [3] (see [3], theorem 15).

Using lemma 1 and lemma 3, we can prove one else theorem.
Theorem 10. Let the matrix $A$ of a linear operator $A$ be $\mathcal{J}$–sign-symmetric and irreducible. Let its second compound matrix $A^{(2)}$ be $\mathcal{J}$–sign-symmetric. Then the operator $A$ has a simple positive eigenvalue $\lambda_1 = \rho(A)$. Moreover, if at least one element $a_{ii}$, situated on the principal diagonal of $A$, is not equal to zero, then $h(A) = 1$, and the second in modulus eigenvalue $\lambda_2$ of the operator $A$ is nonnegative and different in modulus from the first eigenvalue:

$$0 \leq \lambda_2 < \lambda_1.$$

If $A$ has at least one positive principal minor of the second order, then $\lambda_2 > 0$.

Note, that the conditions for the diagonal elements, given in theorem 10, are only sufficient for the existence of the second in modulus nonnegative eigenvalue of a 2–totally $\mathcal{J}$–sign-symmetric matrix, unlike the conditions for the set $W$, given in theorem 9, which are necessary and sufficient.

9 Generalization of the Gantmacher–Krein theorems to the case of a reducible 2–totally $\mathcal{J}$–sign-symmetric matrix.

Let us describe the spectrum of a reducible $\mathcal{J}$–sign-symmetric matrix with a $\mathcal{J}$–sign-symmetric second compound matrix, using the theorems proved above.

Theorem 11. Let the matrix $A$ of a linear operator $A$ be $\mathcal{J}$–sign-symmetric. Let its second compound matrix $A^{(2)}$ be also $\mathcal{J}$–sign-symmetric. Let, in addition, $\rho(A) > 0$. Then the operator $A$ has a positive eigenvalue $\lambda_1 = \rho(A)$. Let $m \geq 1$ be the multiplicity of the eigenvalue $\rho(A)$. Then there is $m$ sets of eigenvalues on the largest spectral circle $|\lambda| = \rho(A)$, with an odd number $k_j \geq 1$ ($j = 1, \ldots, m$) of eigenvalues in the $j$-th set. The eigenvalues of the $r$-th set coincide with the $k_j$-th roots of $(\rho(A))^{k_j}$.

\[ \square \] Apply theorem 6 to the matrix $A$. We get, that there are $m \geq 1$ of irreducible blocks $A_j$ ($j = 1, \ldots, m$) with $\rho(A_j) = \rho(A)$ on the principal diagonal of form (6) of the matrix $A$. Apply theorem 8 to every submatrix $A_j$, which are, as it follows from lemma 2, irreducible and $\mathcal{J}$–sign-symmetric, with $\mathcal{J}$–sign-symmetric second compound matrices $A_j^{(2)}$. We get, that there is an odd number $k_j \geq 1$ of eigenvalues on the spectral circle $|\lambda| = \rho(A_j)$. All of them are simple and coincide with the $k_j$-th roots of $(\rho(A))^{k_j}$. Applying theorem 6 once again, we get, that all this eigenvalues are the eigenvalues of the operator $A$. \[ \blacksquare \]

10 Concluding remarks

The results of this article can be easily generalized to the case of $k$-totally $\mathcal{J}$-sign-symmetric matrices with $k = 3, 4, 5, \ldots$. 

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O.Y. Kushel
Belorussian State University
address: 220050, Republic of Belarus, Minsk, Nezavisimosti sq., 4.
e-mail: kushel@mail.ru