EXTRINSIC PANEITZ OPERATORS AND \(Q\)-CURVATURES FOR HYPERSURFACES

ANDREAS JUHL

Abstract. For any hypersurface \(M\) of a Riemannian manifold \(X\), recent works introduced the notions of extrinsic conformal Laplacians and extrinsic \(Q\)-curvatures. Here we derive explicit formulas for the extrinsic version \(P_4\) of the Paneitz operator and the corresponding extrinsic fourth-order \(Q\)-curvature \(Q_4\) in general dimensions. In the critical dimension \(n = 4\), this result yields a closed formula for the global conformal invariant \(\int_M Q_4\,d\text{vol}\) (for closed \(M\)) and various decompositions of \(Q_4\), which are analogs of the Alexakis/Deser-Schwimmer type decompositions of global conformal invariants. These results involve a series of obvious local conformal invariants of the embedding \(M^4 \hookrightarrow X^5\) (defined in terms of the Weyl tensor and the trace-free second fundamental form) and a non-trivial local conformal invariant \(C\). In turn, we identify \(C\) as a linear combination of two local conformal invariants \(J_1\) and \(J_2\). We also observe that these are special cases of local conformal invariants for hypersurfaces in backgrounds of general dimension. Moreover, in the critical dimension \(n = 4\), a linear combination of \(J_1\) and \(J_2\) can be expressed in terms of obvious local conformal invariants of the embedding \(M^4 \hookrightarrow X^5\). This finally reduces the non-trivial part of the structure of \(Q_4\) to the non-trivial invariant \(J_3\). For totally umbilic \(M\), the invariants \(J_i\) vanish, and the formula for \(P_4\) substantially simplifies. For closed \(M^4 \hookrightarrow \mathbb{R}^5\), we relate the integrals of \(J_i\) to functionals of Guven and Graham-Reichert. Moreover, we establish a Deser-Schwimmer type decomposition of the Graham-Reichert functional of a hypersurface \(M^4 \hookrightarrow X^5\) in general backgrounds. In this context, we find one further local conformal invariant \(J_3\). Finally, we derive an explicit formula for the singular Yamabe energy of a closed \(M\). The resulting explicit formulas show that it is proportional to the total extrinsic fourth-order \(Q\)-curvature. This observation confirms a special case of a general fact and serves as an additional cross-check of our main result. Throughout, we carefully discuss the relation of our formulas to the recent literature.

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1
1. Introduction and formulation of the main results

The significance of the Yamabe operator

\[ P_2 = \Delta - \left( \frac{n}{2} - 1 \right) J \]

and the Paneitz operator

\[ P_4 = \Delta^2 - \delta((n-2)Jh - 4P)d + \left( \frac{n}{2} - 2 \right) \left( \frac{n}{2} \right)^2 - |\nabla|^2 - \Delta(J) \]

in geometric analysis is well-known ([B95] CY95] CGY02] [C05] C18] DGH08] J09] BJ10] and references therein). These differential operators are defined on any Riemannian manifold \((M, g)\) of respective dimension \(n \geq 2\) and \(n \geq 3\). Paneitz [P08] discovered the operator \(P_4\) in general dimensions in 1983. Around the same time, it appeared in dimension 4 in several other contexts in [FTS2] R84 [ESS5]. Here we use the following conventions. The dimension of \(M\) is denoted by \(n\), \(\delta\) is the divergence operator on 1-forms, \(\Delta = \delta d + d \delta\) the non-positive Laplacian, \(2(n-1)J = \text{scal}\) and \((n-2)P = \text{Ric} - Jh\). \(P\) is the Schouten tensor of \(h\). It naturally acts on 1-forms. The operators \(P_2\) and \(P_4\) are the first two elements in the sequence of so-called GJMS-operators \(P_{2N}\) (with \(2N \leq n\) for even \(n\) and \(N \geq 1\) for odd \(n\)) [GJMS92]. These self-adjoint geometric differential operators have as leading term a power of the Laplacian \(\Delta\) and are covariant

\[ e^{(\frac{n}{2}+N)\varphi}P_{2N}(h)(f) = P_{2N}(h)(e^{(\frac{n}{2}-N)\varphi}f) \]

under conformal changes \(h \mapsto \hat{h} = e^{2\varphi}h, \varphi \in C^\infty(M)\), of the metric. The original definition of the GJMS-operators rests on the ambient metric of Fefferman and Graham [FG12].

The quantities \(Q_2 = J\) and \(Q_4 = \frac{n}{2}J^2 - |\nabla|^2 - \Delta(J)\) are known as Branson’s \(Q\)-curvature (of respective order 2 and 4). The general Branson’s \(Q\)-curvatures are defined by

\[ P_{2N}(1) = \left( \frac{n}{2} - N \right) (-1)^N Q_{2N} \]

if \(2N < n\), and by a continuation in dimension argument for \(2N = n\). For even \(n\), the case \(2N = n\) will be referred to as the critical case. A remarkable property of the critical \(Q\)-curvature \(Q_n\) is its transformation law

\[ e^{4\varphi}Q_n(h) = Q_n(h) + (-1)^N \nabla P_n(h)(\varphi), \ \varphi \in C^\infty(M). \]

(1.2)

This property may be derived from the conformal covariance (1.1) in the non-critical case using a continuation in dimension argument [B95]. Combining (1.2) with the self-adjointness of \(P_n\) and \(P_n(h)(1) = 0\) it follows that for a closed manifold \(M\) the integral

\[ \int_M Q_n(h) dvol_h \]

is a (global) conformal invariant. This invariant for the conformal class \([h]\) is related to the conformal anomaly of the renormalized volume of a Poincaré-Einstein metric associated to \(h\) [GZ03]. In dimension \(n = 4\), the integrand of this anomaly is just a constant multiple of \(J^2 - |\nabla|^2\). The complexity of formulas for \(P_{2N}\) and \(Q_{2N}\) dramatically increases with \(N\). However, a recursive structure enables one to derive explicit formulas at least for \(P_6\) and \(P_8\). For more details, we refer to [J13] J16] [FG12].

In the recent works [GW15] GW17] [JO21], generalizations \(P_N\) of the GJMS-operators \(P_N\) and \(Q_N\) of Branson’s \(Q\)-curvatures \(Q_N\) were introduced in the context of the singular Yamabe
problem for hypersurfaces. These are called extrinsic conformal Laplacians and extrinsic \( Q \)-curvatures. In [GW15], the operators \( P_N \) were defined in terms of conformal tractor calculus. The alternative approach in [JO21] rests on an extension of the notion of residue families as developed on [JO9]. The latter method proves the self-adjointness of all extrinsic conformal Laplacians by connecting them to scattering theory (see Theorem 3.1).

We assume that the closed manifold \( M \) is the boundary of a compact manifold \( (X, g) \) with \( h \) being induced by \( g \). Let \( \iota : M \hookrightarrow X \) denote the embedding. The operators \( P_N(g) \) still act on \( C^\infty(M) \) and, for even \( N \), have leading term a power of the Laplacian of \( M \). But their lower-order terms depend on the embedding \( \iota \). For odd \( N \), their leading terms depend on the trace-free part \( \hat{L} = L - H h \) of the second fundamental form \( L \). Here \( H \) is the mean curvature, i.e., \( \text{tr}(L) = n H \). The embedding \( M \hookrightarrow X \) is called totally umbilic if \( \hat{L} = 0 \). Again, the operators \( P_N \) are self-adjoint, and they are covariant

\[
e^{\frac{n-2N}{2}\iota^*\varphi}P_N(g)(f) = P_N(g)(e^{\frac{n-2N}{2}\iota^*\varphi}f) \quad (1.3)
\]

under conformal changes \( g \mapsto \hat{g} = e^{2\varphi}g, \varphi \in C^\infty(X) \). We recall that, in contrast to (1.1), the property (1.3) concerns conformal changes of the metric \( g \) on the ambient space \( X \).

In the following, extrinsic conformal Laplacians and extrinsic \( Q \)-curvatures will be denoted by boldface letters. For simplicity, we often omit their dependence on the metric. The operator \( P_1 \) vanishes, and the first two non-trivial extrinsic conformal Laplacians are given by (see [GW15, Proposition 8.5], [JO21, Sections 13.10-13.11])

**Proposition 1.** It holds

\[
P_2(g) = P_2(h) + \frac{n - 2}{4(n - 1)}|\hat{L}|^2, \quad n \geq 2
\]

and

\[
P_3(g) = 8\delta(\hat{L}d) + \frac{n - 3}{2}\delta\delta(\hat{L}) - (n - 3)(\hat{L}, P) + (n - 1)(\hat{L}, F), \quad n \geq 3.
\]

Here \( F \) is the conformally invariant Fialkow tensor (see (2.7)). The corresponding extrinsic \( Q \)-curvatures are

\[
Q_2(g) = Q_2(h) - \frac{1}{2(n - 1)}|\hat{L}|^2 \quad \text{with} \quad Q_2(h) = J^h
\]

and

\[
Q_3(g) = \frac{4}{n - 2}(\delta\delta(\hat{L}) - (n - 3)(\hat{L}, P) + (n - 1)(\hat{L}, F)).
\]

These satisfy the fundamental transformation laws

\[
e^{2\iota^*\varphi}Q_2(\hat{g}) = Q_2(g) - P_2(g)(\varphi)
\]

if \( n = 2 \) and

\[
e^{2\iota^*\varphi}Q_3(\hat{g}) = Q_3(g) + P_3(g)(\varphi)
\]

if \( n = 3 \).

Here we used the general convention \((-1)^N P_{2N}(1) = (n/2 - N)Q_{2N} \) and \( P_N(1) = n^{-N}Q_N \) for odd \( N \) (see also the comments after Theorem 3.1).

In the critical dimension \( n = 2 \), the extrinsic \( Q \)-curvature \( Q_2 \) is a linear combination of \( J \) and \( |\hat{L}|^2 \). The integrals of both terms are conformally invariant. This implies the conformal invariance of \( \int_M Q_2 d\text{vol}_h \) for a closed surface.

In the critical dimension \( n = 3 \), the extrinsic \( Q \)-curvature \( Q_3 \) is proportional to \( (\hat{L}, F) \), up to a total divergence. This implies the conformal invariance of \( \int_M Q_3 d\text{vol}_h \) for a closed \( M \). In

\footnote{In [JO21], the leading term of \( P_{2N} \) is a constant multiple of \( \Delta^N \).}

\footnote{In [JO21], we used the convention \( P_N(1) = 2^{-N}Q_N \) for all \( N \).}
connection with the study of conformal anomalies for CFT on manifolds with boundaries, it has been argued in [Fu15, Si16] that the boundary terms of anomalies are linear combinations of \( \odot(L, \nabla) \) and \( \text{tr}(L^3) \). The Fialkow equation (2.8) shows that \( \odot(L, \nabla) \) can be written as such a linear combination. Note also the formula or \( Q_3 \) in general dimensions has a simple pole at \( n = 2 \) and its residue at \( n = 2 \) is a multiple of \( 3 \)
\[
\delta\odot(L) + \odot(L, \nabla) + H\odot(L)^2.
\]

The fact that this quantity is a constant multiple of the singular Yamabe obstruction \( B_3 \) is a special case of [JO21, Theorem 11.6]. We shall see another instance of it in connection with \( Q_4 \).

Now we state the main result of this paper. The following theorem displays a formula for the extrinsic Paneitz operator \( P_4 \) for general background metrics in general dimensions. The formulation requires some more notation. In the following, we shall use a bar to distinguish curvature quantities of the background metric \( g \) on \( X \) from curvature quantities of the induced metric \( h \) on \( M \). Accordingly, it will be convenient to denote the background metric \( g \) also by \( \bar{g} \). Let \( \bar{\nabla} \) be the Weyl tensor of the background metric \( \bar{g} \) and let \( \bar{\nabla}_{ij} = \bar{\nabla}_{0ij0} \) be defined by inserting a unit normal vector \( N = \partial_0 \) of \( M \) into the first and the last slot of \( \bar{\nabla} \). Then \( \bar{\nabla} \) is a trace-free conformally invariant symmetric bilinear form on \( M \). It naturally acts on 1-forms on \( M \). The component \( \bar{P}_{00} \) is defined by inserting two unit normal vectors into the Schouten tensor \( \bar{P} \) of \( \bar{g} \). Let \( \nabla \) be the Levi-Civita derivative of the background metric \( \bar{g} \). The symbol \( \delta \) also will be used for the divergence operator on symmetric bilinear forms on \( M \).

**Theorem 1.** Assume that \( n \geq 4 \). Then
\[
P_4 = \Delta^2 - \delta((n-2)Jh - 4P)d + \delta \left( \frac{4(n-5)}{n-2} \hat{L}^2 + \frac{n^2-12n+16}{2(n-1)(n-2)} \hat{L}^2 h + \frac{4(n-1)}{n-2} \bar{\nabla} \right) d + \left( \frac{n}{2} - 2 \right) Q_4.
\]

Here \( Q_4 \) is the sum of

- the intrinsic \( Q_4 \) of \((M, h)\),
- the four divergence terms
  \[
  \frac{2(n-1)}{(n-3)(n-2)} \delta \odot(L) + \frac{2(n-1)}{(n-3)(n-2)} \delta \odot(L^2) + \frac{4}{n-3} \delta(\odot(L)) + \frac{3n-4}{2(n-1)(n-2)} \Delta(\odot(L)^2),
  \]
- the derivative terms
  \[
  2\hat{L}_{ij} \odot(L)_{00} - \frac{4}{n-2} \hat{L}_{ij} \odot(L)_{00} + 2(\hat{L}, \text{Hess}(H)),
  \]
- the four \( \bar{\nabla} \)-terms
  \[
  \frac{2(n-1)}{(n-3)(n-2)} (\bar{\nabla}^2) - \frac{2(n-4)(n-1)}{(n-3)(n-2)} (P, \bar{\nabla}) + \frac{4(3n-5)(n-1)}{(n-3)(n-2)^2} (\hat{L}^2, \bar{\nabla}) - \frac{2(n-1)^2}{(n-3)(n-2)} H(\hat{L}, \bar{\nabla}),
  \]
- the four Schouten tensor terms
  \[
  - \frac{2}{(n-3)(n-2)} (\hat{L}^2, P) - \frac{n^3-5n^2+18n-20}{2(n-3)(n-2)} J\bar{\nabla}^2 + 2H(\hat{L}, P) - 2|\hat{L}|^2 \bar{P}_{00}
  \]
- and the four quartic \( L \)-terms
  \[
  - 3H^2|L|^2 - \frac{2(n-3)}{n-2} H \text{tr}(L^3) + \frac{2(5n^2-14n+9)}{(n-3)(n-2)^2} \text{tr}(L^4) - \frac{15n^4-49n^3+36n^2+24n-32}{8(n-3)(n-2)^2(n-1)^2} |\hat{L}|^4.
  \]

Here all scalar products, norms, traces, and divergences are defined by the metric \( h \).

The following comments are in order.

As noted above, in [JO21] a different normalization of \( P_4 \) has been used. In fact, in [JO21] we set \( P_4(g) = 9P_4(h) \) if \( g \) is the conformal compactification of a Poincaré-Einstein metric with

\[3\text{For a justification of the limit, we refer to [JO21] Section 13.10.}\]
conformal infinity \([h]\). Here we adopt the convention that \(P_4(g) = P_4(h)\) in the Poincaré-Einstein case. This implies that

\[
\int_{M^4} Q_4 \, d\text{vol}_h = 3!4!/9 \int_{M^4} V_4 \, d\text{vol}_h = 16 \int_{M^4} V_3 \, d\text{vol}_h,
\]

where \(V_4\) is the fourth singular Yamabe renormalized volume coefficient. For a discussion of the Yamabe energy \(\int_{M^4} V_3 \, d\text{vol}_h\), we refer to Section 11.

Formula (1.4) makes the self-adjointness of \(P_4\) obvious.

Of course, the four total divergence terms in (1.5) vanish by integration on a closed \(M\). Since \((\tilde L^2, \tilde W), |\tilde W|^2\) and the quartic terms \(\text{tr}(\tilde L^4), |\tilde L|^4\) in (1.9) are conformally invariant, one obtains another conformally covariant operator by removing these terms.

The sum in Theorem 1 describing \(Q_4\) has a simple pole in \(n = 3\). The residue of \(Q_4\) at \(n = 3\) equals the sum of the total divergence terms

\[
4\delta \delta (\tilde W) + 4\delta (\tilde L^2) + 4\delta (\tilde L \delta (\tilde L))
\]

the normal derivative term \(-4\tilde L^{ij} \nabla_0 (\tilde W)_{0ij0}\), the Weyl tensor terms

\[
8|\tilde W|^2 + 4(\tilde P, \tilde W) + 32(\tilde L^2, \tilde W) - 8\tilde H(\tilde L, \tilde W),
\]

the Schouten tensor terms

\[
12(\tilde L^2, \tilde P) - 4J|\tilde L|^2
\]

and

\[
24 \text{tr}(\tilde L^4) - 8|\tilde L|^4.
\]

The sum of these terms actually coincides with \(24\mathcal{B}_3\), where \(\mathcal{B}_3\) is the singular Yamabe obstruction of the embedding \(M^4 \hookrightarrow X^5\) [GHW19 Proposition 1.1], [JO22 Theorem 1]. This relation is another special case of [JO21 Theorem 11.6], and it gives another proof of the conformal invariance of \(\mathcal{B}_3\). In (1.9), the normal derivative of the Weyl tensor has a coefficient that is singular at \(n = 3\). The connection between \(Q_4\) and \(\mathcal{B}_3\) actually explains the appearance of this term in \(Q_4\) by its appearance in \(\mathcal{B}_3\).

The conformal covariance of the operator displayed in Theorem 1 can be confirmed by direct calculations - for an outline of the arguments, we refer to Section 12.4.

The formula in Theorem 1 is written in terms of the tensors \(\tilde W\) and \(\tilde P\) of the background metric, their first-order normal derivatives as well as the intrinsic Schouten tensor \(P\) (and its trace \(J\)) and the second fundamental form \(L\) (and its trace \(nH\)) of the embedding \(M \hookrightarrow X\). Alternative formulas can be obtained using the trace-free part of the Fialkow tensor \(\mathcal{F}\).

In view of the particular significance of the result in the critical dimension \(n = 4\), we separately formulate this special case. For \(n = 4\), Theorem 1 reduces to

**Corollary 1.** In the critical dimension \(n = 4\), the extrinsic Paneitz operator \(P_4\) is given by

\[
P_4 = \Delta^2 - \delta(2Jh - 4P) d + \delta \left(14\tilde L^2 - \frac{4}{3} |\tilde L|^2 h + 6\tilde W\right) d.
\]

(1.10)

Note that the right-hand side of (1.10) is a sum of \(P_4\) and three individually conformally covariant operators.

Next, we consider the extrinsic \(Q\)-curvature \(Q_4\) in the critical dimension \(n = 4\) more closely. Let \(M^4\) be closed. Like the total integral of \(Q_4\), the total integral of \(Q_4\) over \(M\) is a global conformal invariant. In fact, combining the conformal transformation property [JO21 Section 10]

\[
e^{4\tau\varphi}(\varphi) Q_4(\tilde g) = Q_4(g) + P_4(g)(\varphi)
\]

(1.11)

with \(P_4(g)(1) = 0\) and the self-adjointness of \(P_4\), shows that the total integral of \(Q_4\) is an invariant of the conformal class \([g]\). The following result describes this global conformal invariant.
Corollary 2. For a closed 4-manifold $M$, it holds

$$
\int_M Q_4dvol_h = \int_M \left( 2|J|^2 - 2|P|^2 + \frac{9}{2} |\nabla^4|a_1^2 \right) dvol_h
$$

$$+ \int_M \left( 2(\hat{L}, \nabla_0(\hat{P})) - 4\hat{L}^j, \nabla_0(\hat{W})_{0j0} + 2(\hat{L}, \text{Hess}(H)) + 2H(\hat{L}, P) - 9H(\hat{L}, \hat{W}) \right) dvol_h
$$

$$+ \int_M \left( 8(\hat{L}^2, P) - 2\hat{P}_{00}|\hat{L}|^2 - 3J|\hat{L}|^2 - 3H^2|\hat{L}|^2 + 21(\hat{L}^2, \hat{W}) - H \text{tr}(\hat{L}^3) \right) dvol_h
$$

$$+ \int_M \left( \frac{33}{2} \text{tr}(\hat{L}^4) - \frac{14}{3} |\hat{L}|^4 \right) dvol_h. \quad (1.12)
$$

Several comments are in order.

The first integral on the right-hand side of (1.12) does not depend on $L$, the second integral is linear in $\hat{L}$, all terms except the last one in the third integral are quadratic in $\hat{L}$, and the terms in the last integral are quartic in $\hat{L}$.

The right-hand side of (1.12) can be decomposed as a sum of a series of global conformal invariants. More precisely, the first integral is the sum of the global conformal invariant

$$
\int_M Q_4(h)dvol_h = \int_M (2J^2 - 2|P|^2)dvol_h
$$

and the integral of a constant multiple of the local conformal invariant $|\nabla|a_1^2$. The quartic terms $|\hat{L}|^4$, $\text{tr}(\hat{L}^4)$ and $(\hat{L}^2, \hat{W})$ are local conformal invariants. The conformal invariance of $\int_M Q_4dvol_h$ implies that the remaining terms define a global conformal invariant. In fact, this sum is the integral of another local conformal invariant. We set

$$
C \overset{\text{def}}{=} 2(\hat{L}, \nabla_0(\hat{P})) - 4\hat{L}^j, \nabla_0(\hat{W})_{0j0} + 2(\hat{L}, \text{Hess}(H)) + 2H(\hat{L}, P) - 9H(\hat{L}, \hat{W})
$$

$$+ 8(\hat{L}^2, P) - 2\hat{P}_{00}|\hat{L}|^2 - 3J|\hat{L}|^2 - 3H^2|\hat{L}|^2 - H \text{tr}(\hat{L}^3) + 2\delta\hat{L}(\hat{L}^2) + \frac{1}{2} \Delta(|\hat{L}|^2). \quad (1.13)
$$

All terms in the latter sum except the last two divergence terms were taken from (1.12). Note that $C = 0$ if $L = 0$. The advantage of adding these two divergence terms becomes clear in the following result.

Theorem 2. Let $n = 4$. Then $e^{4\varphi(\varphi)}\hat{C} = C$ for all $\varphi \in C^\infty(X)$, i.e., $C$ is a local invariant of the conformal class $[g]$.

The local conformal invariant $C$ contains two terms with normal derivatives of the curvature tensor of the background metric: $(\hat{L}, \nabla_0(\hat{P}))$ and $L^j, \nabla_0(\hat{W})_{0j0}$. It turns out that $C$ is a linear combination

$$
C = -4J_1 + 2J_2 \quad \text{(1.14)}
$$

of two local conformal invariants $J_i$ containing these two normal derivative terms, respectively. These local invariants are given by

$$
J_1 \overset{\text{def}}{=} \hat{L}^j, \nabla_0(\hat{W})_{0j0} + 2H(\hat{L}, \hat{W}) + \frac{2}{9}\delta\hat{L})^2 - 2(\hat{L}^2, P) + J|\hat{L}|^2 - \delta\hat{L}(\hat{L}^2) \quad \text{(1.15)}
$$

and

$$
J_2 \overset{\text{def}}{=} (\hat{L}, \nabla_0(\hat{P})) + H(\hat{L}, P) - \frac{1}{2}H(\hat{L}, \hat{W}) + (\hat{L}, \text{Hess}(H))
$$

$$+ \frac{4}{9}\delta\hat{L})^2 - \hat{P}_{00}|\hat{L}|^2 + \frac{1}{2}J|\hat{L}|^2 - \frac{3}{2}H^2|\hat{L}|^2 - \frac{1}{2}H \text{tr}(\hat{L}^3) - \delta\hat{L}(\hat{L}^2) + \frac{1}{4} \Delta(|\hat{L}|^2). \quad \text{(1.16)}
$$

The integrated invariant $J_1$ also appears in [ASZ21] in the context of anomalies of CFT’s on manifolds with boundary. Generalizations for embeddings $M^4 \hookrightarrow X^n$ with $n \geq 5$ have been found in [CHBRS21].
For a closed hypersurface $M^4 \hookrightarrow X^5$, the integrals of these invariants define generalizations of the conformally invariant Willmore functional of hypersurfaces $M^2 \hookrightarrow X^3$ in the sense that the leading terms of their Euler-Lagrange equations (for variations of the embedding) are constant multiples of $\Delta^2(H)$.

It seems to be of independent interest that both invariants $J_1$ and $J_2$ allow generalizations for general dimensions. In fact, the quantities

$$J_1 \overset{\text{def}}{=} \hat{L}^{ij} \nabla_0(\hat{W})_{0ij0} + 2H(\hat{L}, \hat{W}) + \frac{n-2}{n-1}|\delta(\hat{L})|^2 - \frac{n-2}{n-3}(\hat{L}^2, P) - \frac{n-2}{(n-3)(n-6)}J|\hat{L}|^2 + \frac{n-4}{(n-3)(n-6)}\Delta(|\hat{L}|^2) - \frac{1}{n-3}\delta(\hat{L}^2)$$

and

$$J_2 \overset{\text{def}}{=} (\hat{L}, \nabla_0(\hat{P})) + (\hat{L}, \operatorname{Hess}(H)) + H(\hat{L}, P) - \frac{n-3}{n-2}H(\hat{L}, \hat{W}) + \frac{n}{(n-1)^2}|\delta(\hat{L})|^2 - \frac{1}{(n-3)(n-6)}J|\hat{L}|^2 - \frac{3}{2}H^2|\hat{L}|^2 - \frac{n-3}{n-2}H \operatorname{tr}(\hat{L}^3) + \frac{n-4}{n-3}(\hat{L}^2, P) - \frac{1}{n-3}\delta(\hat{L}^2) + \frac{n-5}{2(n-3)(n-6)}\Delta(|\hat{L}|^2)$$

are local conformal invariants of weight $-4$ of an embedding $M^n \hookrightarrow X^{n+1}$, i.e., it holds

$$e^{4\ast(\varphi)}\hat{J}_j = J_j$$

for $j = 1, 2$ (Proposition 8.1). Both invariants have a simple formal pole at $n = 3$ with residue $-\mathcal{D}((\hat{L}^2)_0)$. Here $\mathcal{D} : b \mapsto \delta(b) + (P, b)$ is a conformally covariant operator $S^3_0(M) \to C^\infty(M)$ on trace-free symmetric 2-tensors on $M^3$ and $(\hat{L}^2)_0$ denotes the trace-free part of $\hat{L}^2$. Note that the term $\mathcal{D}((\hat{L}^2)_0)$ contributes to the singular Yamabe obstruction $B_3$ of $M^3 \hookrightarrow X^4$ [GHHW19, IO22]. We also note that both invariants $J_i$ have a simple formal pole at $n = 6$ with residues being proportional to the local invariant $P_2(|\hat{L}|^2)$ of weight $-4$.

Now we return to the critical dimension $n = 4$. The above results imply the following decomposition of the critical extrinsic $Q$-curvature of order 4 in terms of local conformal invariants.

**Theorem 3.** In the critical dimension $n = 4$, the extrinsic $Q$-curvature $Q_4$ admits the decomposition

$$Q_4 = Q_4 + \frac{9}{2}I_4 + C + 21I_6 + \frac{33}{2}I_2 - \frac{14}{3}I_1 + \delta(\hat{L}^2) + \frac{1}{6}\Delta(|\hat{L}|^2) + 4\delta(\hat{L}\delta(\hat{L})) + 3\delta(\hat{W})$$

$$Q_4 = a \operatorname{Pf}_4 + \sum_j b_j I_j + \text{total divergence}$$

where the local conformal invariants $I_j$ are defined in Section 12.2. In particular, $Q_4(g)$ is a linear combination of the Pfaffian of $(M, h)$, local conformal invariants of the embedding $M \hookrightarrow X$ and a divergence term, i.e., it holds

$$Q_4 = a \operatorname{Pf}_4 + \sum_j b_j I_j + \text{total divergence}$$

with the Pfaffian density $\operatorname{Pf}_4$ and local conformal invariants $I_j$ of the embedding $M \hookrightarrow X$.

Some further comments are in order.

In the first line of (1.17), all terms except the intrinsic $Q_4$ of $(M, h)$ are local conformal invariants. Likewise, all terms in the second line are total divergences. The local conformal invariants in (1.18) are intrinsic and extrinsic.

The transformation law (1.11) shows that the conformal variation of the sum of the total divergences in the second lone of (1.17) is given by the second-order part in (1.10). In fact, a direct calculation confirms that this conformal variation equals

$$14\delta(\hat{L}^2 d\varphi) - \frac{4}{3}B(|\hat{L}|^2 d\varphi) + 6\delta(\hat{W} d\varphi).$$
In other words, it is natural to view the pair \((P_4, Q_4)\) as the sum of the pairs
\[
(P_4, Q_4) \quad \text{and} \quad (P_4^c, Q_4^c)
\]
with
\[
P_4 \overset{\text{def}}{=} \delta \left( 14 \hat{L}^2 - \frac{4}{3} |\hat{L}|^2 h + 6 \hat{W} \right) d\text{ and } Q_4^c \overset{\text{def}}{=} \delta \delta (\hat{L}^2) + \frac{1}{6} \Delta (|\hat{L}|^2) + 4 \delta (\hat{L} \delta (\hat{L})) + 3 \delta \delta (\hat{W}),
\]
and a linear combination of the local conformal invariants \(C, I_1, I_2, I_3, I_6\). Both pairs satisfy the same conformal transformation law. It is also worth noting that for any linear combination \(\hat{Q}_4^c\) of \(\delta \delta (\hat{L}^2), \Delta (|\hat{L}|^2), \delta (\hat{L} \delta (\hat{L}))\) and \(\delta \delta (\hat{W})\) there is a second-order operator \(\hat{P}_4^c\) so that the pair \((\hat{P}_4^c, \hat{Q}_4^c)\) satisfies the same conformal transformation law as \((P_4, Q_4)\). So the most interesting and most complex part of the structure of the pair \((P_4, Q_4)\) is the local conformal invariant \(C\) or, equivalently, the local conformal invariants \(J_1\) and \(J_2\).

Moreover, it turns out that
\[
J_1 - 2J_2 = -\frac{4}{3} |\hat{L}|^4 + 3 \text{tr}(\hat{L}^4) + \hat{L}^{kl} \hat{L}^{ij} W_{kijl} + 3 (\hat{L}^2, \hat{W}) - \frac{1}{2} |\hat{W}|^2,
\]
where the right-hand side is a linear combination of obvious local conformal invariants (Corollary 12.6). This identity leads to the following equivalent decomposition of the critical \(Q_4\).

**Corollary 3.** The critical extrinsic \(Q\)-curvature \(Q_4\) admits the decomposition
\[
Q_4 = Q_4 - 3J_1 + \frac{9}{2} J_4 - 3J_5 + 18J_6 + \frac{1}{2} J_7 - \frac{10}{3} J_1 + \frac{27}{2} J_2
\]

\[
+ \delta \delta (\hat{L}^2) + \frac{1}{6} \Delta (|\hat{L}|^2) + 4 \delta (\hat{L} \delta (\hat{L})) + 3 \delta \delta (\hat{W}).
\]

This result finally describes \(Q_4\) in terms of trivial conformal invariants \(I_j\), the non-trivial conformal invariant \(J_1\), and some divergence terms.

In general dimensions, the extrinsic \(Q_4\) can be written as the sum of \(Q_4\), a linear combination of the local conformal invariants \(I_1, I_2, I_4, I_6, -4J_1 + 2J_2\), the product of \(n - 4\) with a curvature term \(E_4\) and a divergence term (Theorem 8.5). \(E_4\) admits a continuation to the critical dimension \(n = 4\) and the conformal variation of \(\int_M E_4 d\text{vol}\) equals the divergence term in (1.17) or (1.20) (Remark 8.5). This generalizes the observation that in the explicit formula \(H\) for \(Q_3\) the conformal variation of \(\int_M (L, P) d\text{vol}\) is given by \(\delta \delta (L)\).

The analog of the decomposition (1.18) for \(Q_2(g)\) for a surface \(M^2 \hookrightarrow X^3\) is obviously true since it is true for \(Q_2(h)\). In this case and for closed \(M\), the total integral of \(Q_2\) is a linear combination of the Euler characteristic of \(M\) and the total integral of the local conformal invariant \(|\hat{L}|^2\). The latter integral is the Willmore energy of \(M \hookrightarrow X\). Similarly, \(Q_3\) for \(M^3 \hookrightarrow X^4\) is a linear combination of the local conformal invariant \((\hat{L}, \hat{F})\) and a divergence term.

Formulas (1.18) and (1.20) are analogs of the Deser-Schwimmer decomposition of global conformal invariants [DS93] (established by Alexakis in the monograph [A12] and a series of papers).

In [GR20], Graham and Reichert studied the renormalized volume of minimal hypersurfaces in a Poincaré-Einstein background. This led to new conformally invariant energies. Moreover, an explicit formula for such an energy was derived if the boundary of the hypersurface is a four-manifold. In Section 9 we shall examine the Graham-Reichert energy functional of four-dimensional hypersurfaces from the perspective of an analog of the Deser-Schwimmer decomposition. This also reveals one further local conformal invariant of \(M^4 \hookrightarrow X^5\). It is given by
\[
J_3 \overset{\text{def}}{=} \frac{1}{2} \hat{L}^{ij} \nabla_{\hat{g}}(\hat{W})_{0ij0} + H(\hat{L}, \hat{W}) + (P, \hat{W}) - \hat{B}_{00} + \frac{1}{2} \delta \delta (\hat{W}),
\]
where $\bar{B}$ is the Bach tensor of the background metric (Corollary 3.6). Note that $J_3 = 0$ if $W = 0$.

For other results on extrinsic analogs of the Deser-Schwimmer classification, we refer to MN18.

Finally, we return to the operator $P_4$ in general dimensions and take a closer look at its structure for a totally umbilic hypersurface, i.e., if $\bar{L} = 0$. In this case, Theorem 1 reduces to the following result.

**Corollary 4.** Assume that $\bar{L} = 0$ and $n \geq 4$. Then

$$P_4 = P_4(f) + 4 \frac{n - 1}{n - 2} \delta(Wd) + \left(\frac{n}{2} - 2\right) \frac{2(n - 1)}{(n - 2)(n - 3)} \left(\frac{n - 1}{n - 2} |W|^2 - (n - 4)(P, W) + \delta(\bar{W})\right). \tag{1.22}$$

In particular, it holds $P_4 = P_2$ iff $\bar{W} = 0$. As a consequence,

$$Q_4 = Q_4 + \frac{2(n - 1)}{(n - 2)(n - 3)} \left(\frac{n - 1}{n - 2} |W|^2 - (n - 4)(P, W) + \delta(\bar{W})\right). \tag{1.23}$$

Thus, in the critical dimension $n = 4$, it holds

$$P_4(f) = P_4(f) + 6\delta(Wdf) \quad \text{and} \quad Q_4 = Q_4 + \frac{9}{2} |W|^2 + 3\delta(\bar{W}). \tag{1.24}$$

The assumption $\bar{L} = 0$ is a conformally invariant condition. The conformally invariant condition $\bar{W} = 0$ yields another interesting special case. If $\bar{g} = dr^2 + h_r$ so that $g_+ = \bar{r}^{-2}\bar{g}$ satisfies $\text{Ric}(g_+) + n(g_+) = 0$ (we shall refer to this case as the Poincaré-Einstein case), then it holds $\bar{L} = 0$ and $\bar{W} = 0$. Hence, in this case, formula (1.22) shows that $P_4$ reduces to the intrinsic Paneitz operator $P_4$ of $M$. Formula (1.24) implies that the total integral of the critical $Q_4$ equals

$$\int_{M^4} \left(Q_4 + \frac{9}{2} |W|^2\right) dvol_h$$

if $\bar{L} = 0$. Now the Chern-Gauss-Bonnet formula

$$8\pi^2 \chi(M) = \frac{1}{4} \int_{M^4} |W|^2 dvol_h + \int_{M^4} Q_4 dvol_h \tag{1.25}$$

immediately implies that the total integral of the critical $Q_4$ is conformally invariant if $\bar{L} = 0$. In this special case, one easily sees that $f \mapsto P_4(f) + 6\delta(Wdf)$ is a conformally covariant operator: both $P_4$ and $\delta(Wd)$ are conformally covariant. Also one can directly verify the fundamental transformation law (1.11) for the critical $Q_4$-curvature if $\bar{L} = 0$. The residue at $n = 3$ of the right-hand side of (1.23) is a constant multiple of $2|W|^2 + (P, W) + \delta(\bar{W})$ and this quantity is a multiple of the singular Yamabe obstruction $B_3$ (if $\bar{L} = 0$).

We finish this section with a detailed review of the paper.

In Section 2, we fix notation and collect basic identities for later references. Section 3 briefly recalls the relation between extrinsic conformal Laplacians and scattering theory for the singular Yamabe metric. It implies a description of the conformal Laplacians in terms of the asymptotic expansions of eigenfunctions of the singular Yamabe metric (see (3.3)). This is followed up in Section 4 by a discussion of the first few terms in the asymptotic expansion of such eigenfunctions, which suffice to prove Proposition 1. Moreover, Theorem 1.6 gives a preliminary formula for $P_4$. This formula describes $P_4$ in terms of data written in coordinates for which the singular Yamabe metric has a simple normal form. These coordinates have been used in JO21 and recently in

5The authors of [AS21] informed us that in a forthcoming paper they prove that the integrated invariant $J_3$ is a linear combination of the integrals of the invariants $I_5$, $\tilde{I}_6$ and $I_7$ (see (2.13)).

6Here $|W|^2 = W_{ijkl}W^{ijkl}$. 
As a corollary, we rederive a formula for the singular Yamabe obstruction \( P \), get the desired formula for the usual Paneitz operator in terms of a Poincaré-Einstein metric. However, to derive a formula for \( P_4 \), it remains to evaluate it further and perform a conformal change. These delicate tasks are realized in the following sections. The most subtle part concerns the study of two normal derivatives of the scalar curvature of the background metric. In Section 5, we derive a formula for the total integral of \( Q_4 \) in the critical dimension \( n = 4 \) (Corollary 4), and Section 7 contains a proof of the formula for \( P_4 \) in the general case (Theorem 1). As a corollary, we rederive a formula for the singular Yamabe obstruction \( B_3 \) of \( M^3 \to X^4 \) by taking a formal residue at \( n = 3 \) (Corollary 7.7). In the following sections, we look closely at the structure of \( Q_4 \). In particular, we prove that \( J_1 \), \( J_2 \) and \( C \) are local conformal invariants (Theorem 2) and establish the decomposition in Theorem 3. The main technical result is Lemma 8.2. We round off this section with the formulation of an analogous conjectural decomposition of \( Q_n \) in general dimensions. Section 9 contains similar results related to the conformally invariant functional \( E_{GR} \) introduced in [GR20]. In this context, we find the local conformal invariant \( J_3 \) (Corollary 9.6). Corollary expresses \( E_{GR} \) in terms of the Euler characteristic of \( M \) and the integrals of the local invariants \( J_1 \) and \( J_2 \). For a flat background, we relate in Section 10 the energy functionals defined by \( C \) to a functional of Guven. In Section 11, we derive a formula for the singular Yamabe energy of \( M^4 \to X^5 \) (Theorem 11.17). It requires solving the singular Yamabe problem to sufficiently high order. The result is proportional to the total integral of \( Q_4 \). Since this relation is a special case of a general result, the calculation serves as an additional cross-check. In the Appendix, we collect various technical results. The first section outlines a direct proof of the conformal covariance of the operator displayed in Theorem 1. Section 12.2 reviews the known trivial and non-trivial extrinsic conformal invariants of hypersurfaces \( M^4 \to X^5 \). In Section 12.3, we briefly review the roles of Deser-Schwimmer type decompositions for extrinsic conformal invariants of hypersurfaces in other mathematical and physical contexts. Section 12.4 provides a new proof of the conformal invariance of the basic conformal invariant \( W_m \) introduced in [BGW21b]. Section 12.5 contains a proof of the relation (1.19).

Some of the results of this paper were announced in [J21]. The results in the paper [BGW21b] overlap with the current results. Among other things, the authors of [BGW21b] derive an explicit formula for the extrinsic Paneitz operator \( P_4 \) in general dimensions\(^7\). In Section 12.6 we (almost) prove that this formula is equivalent to Theorem 1 (see also Remark 7.1).

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2. Notation and basic identities

All manifolds \( X \) are smooth. For a manifold \( X \), \( C^\infty(X) \) is the space of smooth functions on \( X \). Metrics on \( X \) are denoted by \( g \). \( dvol_g \) is the Riemannian volume element defined by \( g \). Let \( \mathfrak{X}(X) \) be the space of smooth vector fields on \( X \). The Levi-Civita connection of \( g \) is denoted by \( \nabla^g \) or simply \( \nabla \) for \( X \in \mathfrak{X}(X) \) if \( g \) is understood. In these terms, the curvature tensor \( R \) of the Riemannian manifold \( (X, g) \) is defined by \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \) for vector fields \( X, Y, Z \in \mathfrak{X}(X) \). The components of \( R \) are defined by \( R(\partial_i, \partial_j)(\partial_k) = R_{ijk}^l \partial_l \). Ric and scal are the Ricci tensor and the scalar curvature of \( g \). On a manifold \( (X, g) \) of dimension \( n \), we set \( 2(n-1)J = \text{scal} \) and define the Schouten tensor \( P \) of \( g \) by

\[
(n-2)P = \text{Ric} - Jg
\]

(if \( n \geq 3 \). Let \( W \) be the Weyl tensor. Then the curvature tensor admits the decomposition \( R = W - P \otimes g \). We recall that \( W \) vanishes in dimension 3. The Cotton tensor \( C \) is defined by

\[
C_{ijk} = \nabla_k(P)_{ij} - \nabla_j(P)_{ik}.
\]

\(^7\)In contrast to [BGW21b], the arguments in the present paper do not rely on computer calculations.

\(^8\)However, we do not verify the equivalence of the terms which are quartic in \( L \) (see (1.19)).
Then

\[(n - 3)C_{ijk} = \nabla^i(W)_{ijk}.\]

Finally, let

\[B_{ij} = \nabla^k(C)_{ijk} + p^{kl}W_{iklj}\]

be the Bach tensor. These conventions are as in [JO23].

For a metric \(g\) on \(X\) and \(u \in C^\infty(X)\), let \(\text{grad}_g(u)\) be the gradient of \(u\) with respect to \(g\) so that \(g(\text{grad}_g(u), V) = (du, V)\) for all \(V \in \mathfrak{X}(X)\). \(g\) defines pointwise scalar products \((\cdot, \cdot)\) and norms \(|\cdot|\) on \(\mathfrak{X}(X)\), on forms and on general tensors. \(\delta^g\) is the divergence operator on differential forms or symmetric bilinear forms. On forms, it coincides with the negative adjoint \(-d^*\) of the exterior differential \(d\) with respect to the Hodge scalar product defined by \(g\). Let \(\Delta_g = \delta^g d\) be the non-positive Laplacian on \(C^\infty(X)\). On the Euclidean space \(\mathbb{R}^n\), it equals \(\sum_i \partial_i^2\). In addition, \(\Delta\) will also denote the Bochner-Laplacian (when acting on \(L\), say).

A metric \(g\) on a manifold \(X\) with boundary \(M\) induces a metric \(h\) on \(M\). In such a setting, we distinguish the curvature quantities of \(g\) and \(h\) by adding a bar to those of \(g\). The covariant derivative, the curvature tensor and the Weyl tensor of \((X, g)\) are \(\nabla, \bar{R}\) and \(\bar{W}\). Similarly, \(\bar{\text{Ric}}\) and \(\text{scal}\) are the Ricci tensor and the scalar curvature of \(g\).

The following conventions coincide with those in [JO21, JO22].

A hypersurface is given by an embedding \(\iota : M \hookrightarrow X\). Accordingly, tensors on \(X\) are pulled back by \(\iota^*\) to \(M\). In practice, we often omit this pull-back. For a hypersurface \(\iota : M \hookrightarrow X\) with the induced metric \(h = \iota^*(g)\) on \(M\), the second fundamental form \(L\) is defined by \(L(X, Y) = -h(\nabla^X \!_\! Y, N)\) for vector fields \(X, Y \in \mathfrak{X}(M)\) and a unit normal vector field \(\partial_0 = N\). We set \(nH = \text{tr}_L(L)\) if \(M\) has dimension \(n\). Then \(H\) is the mean curvature of \(M\). Let \(\bar{L} = L - Hh\) be the trace-free part of \(L\). Sometimes we identify \(L\) with the shape operator \(S\) being defined by \(h(X, S(Y)) = L(X, Y)\).

We use metrics, as usual, to raise and lower indices. In particular, we set \((L^2)_{ij} = L^k_i L^l_j + h^{kl}L_{il}L_{kj}\) and similarly for higher powers of \(L\). We always apply the Einstein summation convention, i.e., we sum over repeated indices.

The 1-form \(\bar{\text{Ric}}_0 \in \Omega^1(M)\) is defined by \(\bar{\text{Ric}}_0(X) = \text{Ric}(X, \partial_0)\) for \(X \in \mathfrak{X}(M)\). Similarly, we write \(b_0\) for the analogous 1-form defined by a bilinear form \(b\), and we let \(\bar{W}_0\) be the 3-tensor on \(M\) with components \(\bar{W}_{ijk0}\), i.e., we always insert the normal vector \(\partial_0\) into the last slot. Moreover, we set \(\bar{W}_{ij} = \bar{W}_{0ij0}\). We define \(\bar{(L, \bar{W}_0)} \in \Omega^1(M)\) by \(\bar{L}^{ij}\bar{W}_{ij0}\).

The curvatures of the background metric \(g\) on \(X\) and the induced metric \(h\) on the hypersurface \(M\) are connected through the Gauss equations

\[
\iota^* R = R + \frac{1}{2} L \otimes L, \tag{2.1}
\]

\[
\iota^* \text{Ric} = \text{Ric} + L^2 - nH L + \bar{G}, \tag{2.2}
\]

\[
\iota^* J = J + \frac{1}{2(n - 1)} |\bar{L}|^2 - \frac{n}{2} H^2 + \bar{P}_{00} \tag{2.3}
\]

with \(\bar{G}_{ij} \overset{\text{def}}{=} \bar{R}_{0ij0}\) and the Codazzi-Mainardi equation

\[
\nabla_j (L)_{ik} - \nabla_i (L)_{jk} = \bar{R}_{ijk0}. \tag{2.4}
\]

Taking traces gives

\[
\delta(L) - ndH = (n - 1)\bar{P}_0 \quad \text{and} \quad \delta(\bar{L}) - (n - 1)dH = (n - 1)\bar{P}_0. \tag{2.5}
\]

The trace-free part of the Codazzi-Mainardi equation states that

\[
\nabla_j (\bar{L})_{ik} - \nabla_i (\bar{L})_{jk} + \frac{1}{n - 1} \delta(\bar{L})h_{ik} - \frac{1}{n - 1} \delta(\bar{L})h_{jk} = \bar{W}_{ijk0}. \tag{2.6}
\]
For any embedding $\iota : M^n \hookrightarrow X^{n+1}$ ($n \geq 3$), the tensor
\[
\mathcal{F} \equiv \iota^* \bar{P} - P + H \bar{L} + \frac{1}{2} H^2 h
\]
(2.7)
is conformally invariant: $\hat{\mathcal{F}} = \mathcal{F}$. The invariance of $\mathcal{F}$ also follows from the identity
\[
(n - 2) \left( \iota^* \bar{P} - P + H \bar{L} + \frac{1}{2} H^2 h \right) = \hat{\mathcal{L}}^2 - \frac{\lvert \hat{\mathcal{L}} \rvert^2}{2(n-1)} h + \overline{W}
\]
(2.8)
([J09, Lemma 6.23.3]). Following [V13] and [GW15], we refer to $\mathcal{F}$ as the Fialkow tensor and to the equation (2.8) as the Fialkow equation. Taking the trace in (2.8), yields the Gauss equation
\[
(2.3)
The trace-free part $\hat{\mathcal{F}}$ of $\mathcal{F}$ is given by
\[
\hat{\mathcal{F}} = \frac{1}{n-2} \hat{\mathcal{L}}^2 - \frac{1}{n(n-2)} \lvert \hat{\mathcal{L}} \rvert^2 h + \frac{1}{n-2} \overline{W}.
\]
(2.9)
Finally, we show that $\mathcal{F}$ naturally appears in the Gauss equation for the Weyl tensor. We calculate
\[
\iota^* \overline{W} = \iota^* \overline{R} + \iota^* \bar{P} \otimes h
\]
\[
= R + \frac{1}{2} L \otimes L + \iota^* \bar{P} \otimes h
\]
\[
= W - P \otimes h + \frac{1}{2} L \otimes L + \iota^* \bar{P} \otimes h
\]
\[
= W + \frac{1}{2} L \otimes L + (\mathcal{F} - H \bar{L} - \frac{1}{2} H^2 h) \otimes h
\]
\[
= W + \frac{1}{2} L \otimes \bar{L} + \mathcal{F} \otimes h
\]
using the Gauss equation (2.1) and the decompositions of the curvature tensors. This proves the Gauss equation
\[
\iota^* \overline{W} = W + \frac{1}{2} L \otimes \bar{L} + \mathcal{F} \otimes h
\]
(2.10)
for the Weyl tensor.

3. EXTRINSIC CONFORMAL LAPLACIANS AND THE SCATTERING OPERATOR

There are two different methods to define extrinsic conformal Laplacians $P_N$. In [GW15], Gover and Waldron defined these operators in terms of compositions of so-called Laplace-Robin operators. The latter notion has its origin in conformal tractor calculus. But Laplace-Robin operators are also linked to representation theory [JO20] and scattering theory [JO21]. From the point of view developed in [JO21], the extrinsic conformal Laplacians appear in terms of so-called residue families as introduced in [J09] in the setting of Poincaré-Einstein metrics. This also provides a natural definition of extrinsic $Q$-curvatures. Roughly speaking, residue families may be viewed as curved versions of symmetry breaking operators in representation theory [KS15]. One of the main results in [JO21] states that both approaches define the same operators.

Moreover, the following result states that the extrinsic conformal Laplacians can be identified with residues of the geometric scattering operator of the singular metric $\sigma^{-2}\tilde{g}$, where $\sigma \in C^\infty(X)$ satisfies the condition
\[
\text{scal}_{\sigma^{-2}\tilde{g}} = -n(n+1)
\]
(at least asymptotically). In other words, $\sigma$ is a solution of a singular Yamabe problem. For more details on the structure of $\sigma$, we refer to Section [11].
Theorem 3.1 ([JO21, Theorem 4]). If \( \sigma \) and \( N \in \mathbb{N} \) satisfies \( 1 \leq N \leq n \) and \( (n/2)^2 - (N/2)^2 \) avoids the discrete spectrum of \(-\Delta_{\sigma^2g}\), then

\[
P_N \sim \text{Res}_{\lambda=\frac{N-1}{2}}(S(\lambda)).
\]

The operator \( S(\lambda) \) is the scattering operator of the Laplacian of the singular Yamabe metric \( \sigma^{-2}\hat{g} \).

Here a comment on the convention of the normalization of \( P_{2N} \) is in order. In contrast to the normalization \( P_{2N} = (2N-1)!^2 \Delta^N + LOT \) used in [JO21], we normalize \( P_{2N} \) so that its leading term is \( \Delta^N \). Accordingly, we adapt the normalization of the extrinsic \( Q \)-curvatures. Then \( Q_{2N} \) reduces to \((-1)^N Q_{2N}\) in the Poincaré-Einstein case.

Theorem 3.1 extends a result of [GZ03] for GJMS-operators. A different perspective was taken in [CMY21] by defining extrinsic conformal Laplacians through the residues of the scattering operator. However, this paper did not clarify the relation of these residues to the operators defined in the works of Gover and Waldron. Since \( |d\sigma|^2 = 1 \) on \( M \), the singular metric \( \sigma^{-2}g \) is asymptotically hyperbolic, and the proof of the above result again rests on results in scattering theory as developed in [GZ03]. The definition of the scattering operator \( S(\lambda) \) combines the existence of local asymptotic expansions of eigenfunctions of the Laplacian \( \Delta_{\sigma^2g} \) with the meromorphic continuation of the global resolvent. However, the residues of the scattering operator, which are of interest here, can be described only in terms of the local asymptotic expansion of eigenfunctions.

In order to analyze these expansions, one has to choose suitable coordinates. In [JO21], we utilized so-called adapted coordinates (which are best suited for the study of residue families). In these coordinates, the metric \( \sigma^{-2}\hat{g} \) takes the form \( s^{-2}(a(s)ds^2 + h_s) \) with some coefficient \( a \in C^\infty(X) \). On the other hand, one also may use coordinates so that the metric \( \sigma^{-2}\hat{g} \) takes the form \( \tilde{r}^{-2}(d\tilde{r}^2 + h_{\tilde{r}}) \). Then the metric \( \tilde{g} \) is conformally related to the original metric \( g \), i.e., it holds

\[
\tilde{g} = e^{2\omega} \hat{g}
\]

with some \( \omega \in C^\infty(X) \) so that \( \omega^*(\omega) = 0 \). In these terms, assume that \( u \) satisfies

\[
-\Delta_{\tilde{r}^{-2}(d\tilde{r}^2 + h_{\tilde{r}})}u = \lambda(n - \lambda)u
\]

and has \( f \in C^\infty(M) \) as boundary value. Then \( u \) has an asymptotic expansion of the form

\[
u \sim \sum_{j = 0}^n \tilde{r}^{\lambda + j} T_j(\lambda)(f) + \sum_{j = 0}^n \tilde{r}^{n-\lambda + j} T_j(n - \lambda)\lambda S(\lambda)(f)
\]

with meromorphic one-parameter families \( T_j(\lambda) \) of differential operators on \( M \). For more details, we refer to [JO21] Section 7. There are analogous expansions in terms of adapted coordinates. But since \( \omega^*(\omega) = 0 \), the scattering operator does not depend on these coordinates. The meromorphic families \( T_N(\lambda) \) has a simple pole at \( \lambda = \frac{n-1}{2} \), and it holds

\[
P_N \sim \text{Res}_{\lambda=\frac{n-1}{2}}(T_N(\lambda))
\]

(see [JO21, Theorem 9.3]); recall that \( P_N \) is self-adjoint.

4. Solution operators and the proof of Proposition \( \blacksquare \)

We assume that the metric \( g_+ = r^{-2}(dr^2 + h_r) \) has constant scalar curvature \(-n(n+1)\).

In the present section, we derive formulas for the solution operators \( T_j(\lambda) \) for \( j \leq 4 \). We apply the results for the metric \( \hat{g}_+ = \hat{r}^{-2}\hat{g} \) to prove Proposition \( \blacksquare \). Finally, we provide a preliminary formula for \( P_4 \) in terms of the metric \( \hat{g} \) (see (3.2)).

\[\text{These coordinates have been used in [JO21, Section 7] and also in [CMY21]. The latter paper derived formulas for } P_2 \text{ and } P_3 \text{ which are equivalent to those in Proposition \( \blacksquare \).}\]
We expand the Laplacian in the form
\[ \Delta_g' = \sum_{j \geq 0} r^{\lambda+j} T_j(\lambda)(f) = \sum_{j \geq 0} r^{\lambda+j} f_j \]
contributing to the formal asymptotic expansion of an eigenfunction \( u \) so that
\[ \Delta_g'(u) + \lambda(n-\lambda)u = 0. \]
We expand the Laplacian in the form
\[ \Delta_{g+} = r^2 \partial_r^2 + (1-n)r \partial_r + r^2 \frac{1}{2} \text{tr}(h^{-1}_r h'_r) \partial_r + r^2 \Delta_{h_r}, \]
where
\[ v(r) = \frac{1}{2} \text{tr}(h^{-1}_r h'_r) = 1 + rv_1 + r^2 v_2 + \cdots. \quad (4.1) \]
Here the prime \('\) denotes the derivative in \( r \). Let \( J \) be defined for the metric \( \bar{g} = dr^2 + h_r \).

**Lemma 4.1.** Assume that \( g_+ = r^{-2}(dr^2 + h_r) \) is a metric of constant scalar curvature \(-n(n+1)\). Then
\[ J = -\frac{1}{2r} \text{tr}(h^{-1}_r h'_r) \]
or, equivalently,
\[ rJ = -\frac{v'}{v}. \quad (4.2) \]

**Proof.** We recall that the transformation law for scalar curvature under the conformal change \( e^{2\varphi} g \) of the metric \( g \) on a manifold \( M \) of dimension \( n \) reads
\[ e^{2\varphi} J = J - \Delta_g(\varphi) - \frac{n-2}{2} |d\varphi|^2_g. \]
We apply this law for \( \bar{g} = r^2 g_+ \) on \( X \). Then
\[ r^2 J = J_{g+} - \Delta_{g_+}(\log r) - \frac{n-1}{2} |d \log r|^2_{g_+}. \]
Now \( J_{g+} = -\frac{n+1}{2} \),
\[ \Delta_{g_+}(\log r) = -n + \frac{1}{2} r \text{tr}(h^{-1}_r h'_r) \]
and \( |d \log r|^2 = 1 \) imply the assertion. \( \square \)

Lemma 4.1 implies
\[ \Delta_{g_+}(r^\lambda f) = -\lambda(n-\lambda)r^\lambda f - \lambda r^{\lambda+2} J f + r^{\lambda+2} \Delta_{h_r}(f) \]
for \( f \in C^\infty(M) \). Hence
\[ (\Delta_{g_+} + \lambda(n-\lambda))(r^\lambda f) = r^{\lambda+2}(\Delta - \lambda J)(f) + r^{\lambda+3}(\Delta' - \lambda J')(f) + r^{\lambda+4}(\Delta'' - \lambda/2 J'')(f) + \cdots, \]
where we use the expansion \( \Delta_{h_r} = \Delta + r\Delta' + r^2 \Delta'' + \cdots \). It follows that the boundary value \( f \in C^\infty(M) \) of \( u \) is free and \( f_1 = 0 \). Moreover, we find
\[ (\Delta - \lambda J)(f) - (\lambda + 2)(n-\lambda - 2) f_2 + \lambda(n-\lambda) f_2 = 0, \]
i.e.,
\[ T_2(\lambda)(f) = f_2 = \frac{1}{2(n-2\lambda-2)} (\Delta - \lambda J)(f). \quad (4.3) \]
Next, we get
\[ (\Delta' - \lambda J')(f) - (\lambda + 3)(n-\lambda - 3) f_3 + \lambda(n-\lambda) f_3 = 0, \]
i.e.,
\[ T_3(\lambda)(f) = f_3 = \frac{1}{3(n-2\lambda-3)}(\Delta' - \lambda \Delta')(f). \] (4.4)

Similarly, we find
\[ (\Delta'' - \lambda/2 \Delta''')(f) + (\Delta - (\lambda + 2)\Delta)f_2 = 4(n-2\lambda - 4)f_4. \]

Hence
\[ T_4(\lambda)(f) = \frac{1}{32(\frac{n}{2} - \lambda - 1)(\frac{n}{2} - \lambda - 2)} \times ((\Delta - (\lambda + 2)\Delta)(\Delta - \lambda \Delta) + 4(\frac{n}{2} - \lambda - 1)\Delta''(f) - 2\lambda(\frac{n}{2} - \lambda - 1)\Delta'''). \] (4.5)

We summarize these results in

**Lemma 4.2.** If \( r^{-2}(dr^2 + h_r) \) has constant scalar curvature \(-n(n+1)\), then \( T_1 = 0 \) and
\[ T_2(\lambda) = \frac{1}{2(n-2\lambda-2)}(\Delta - \lambda \Delta), \quad T_3(\lambda) = \frac{1}{3(n-2\lambda-3)}(\Delta' - \lambda \Delta') \]

and
\[ T_4(\lambda) = \frac{1}{32(\frac{n}{2} - \lambda - 1)(\frac{n}{2} - \lambda - 2)} \times ((\Delta - (\lambda + 2)\Delta)(\Delta - \lambda \Delta) + 4(\frac{n}{2} - \lambda - 1)\Delta''(f) - 2\lambda(\frac{n}{2} - \lambda - 1)\Delta'''). \]

Thus (3.3) and Lemma 4.2 imply the preliminary formulas
\[ P_2 = -4 \text{Res}_{\lambda = \frac{n}{2} - 1}(T_2) = \Delta - \frac{n-2}{2} \tilde{\Delta} \quad \text{and} \quad P_3 \sim \text{Res}_{\lambda = \frac{n}{2} - 2}(T_3) \sim \Delta' - \frac{n-3}{2} \tilde{\Delta}'. \]

We recall that in these formulas, the right-hand sides are defined with respect to the metric \( \hat{g} \).

In order to further evaluate these results, we apply Lemma 4.1. By expanding the relation (4.2) into power series in \( r \), we obtain identities for the normal derivatives of \( \tilde{\Delta} \) in terms of the volume coefficients \( v_1 \). Comparing these formulas with known formulas for the volume coefficients in terms of the metric will imply formulas for normal derivatives of \( \tilde{\Delta} \) in terms of the metric. First, we note that
\[ \frac{\nu'_{v}}{v} = v_1 + (2v_2 - v_1^2)r + (3v_3 - 3v_1v_2 + v_1^3)r^2 + (4v_4 - 4v_1v_3 - 2v_2^2 + 4v_1^2v_2 - v_1^4)r^3 + \cdots. \]

Next, we have

**Lemma 4.3.** In general dimensions, it holds
\[ v_1 = nH, \]
\[ 2v_2 = -\overline{\text{Ric}}_{00} - |\hat{L}|^2 + n(n-1)H^2 = \overline{\text{Ric}}_{00} + \text{scal} - \text{scal}, \]
\[ 6v_3 = -\nabla_0(\overline{\text{Ric}})_{00} + 2(\hat{L}, \hat{G}) - (3n - 2)H\overline{\text{Ric}}_{00} + 2\text{tr}(\hat{L}^3) - 3(n - 2)H|\hat{L}|^2 + n(n-1)(n-2)H^3, \]

where \( \tilde{G}_{ij} \equiv \tilde{R}_{0ij0} \).

**Proof.** These formulas coincide with the corresponding terms in the expansion of the volume form in [AGV81] Theorem 3.4. Note that this is obvious for \( v_1 \) and \( v_2 \) but requires applying the Gauss equations
\[ \text{scal} - \text{scal} = 2\overline{\text{Ric}}_{00} + |\hat{L}|^2 - n^2H^2 \quad \text{and} \quad \overline{\text{Ric}} - \text{Ric} = \tilde{G} - nHL - L^2 \]

for \( v_3 \). Equivalent formulas can be found in [GG19] Section 2. \( \square \)
Note that (4.2) implies \( v_1 = 0 \). Thus Lemma 4.3 shows that \( H = 0 \). Therefore, we get
\[
2v_2 = -\bar{J}, \quad 3v_3 = -\bar{J}' \quad \text{and} \quad 4v_4 - 2v_2^2 = -\frac{1}{2} \bar{J}'',
\]
or, equivalently,
\[
2v_2 = -\bar{J}, \quad 3v_3 = -\bar{J}' \quad \text{and} \quad 4v_4 = -\frac{1}{2} \bar{J}'' + \frac{1}{2} \bar{J}^2. \tag{4.6}
\]

Now combining the first relation in (4.6) with Lemma 4.3 gives
\[
\text{Ric}_{00} + |L|^2 = \bar{J}.
\tag{4.7}
\]
We compare this relation with the Gauss identity
\[
2\text{Ric}_{00} = \text{scal} - \text{scal} - |L|^2
\]
(recall that \( H = 0 \)). We get
\[
2\bar{J} - 2|L|^2 = 2n\bar{J} - 2(n - 1)\bar{J} - |L|^2
\]
Equivalently, we find
\[
\bar{J} = \bar{J} - \frac{1}{2(n - 1)}|L|^2.
\tag{4.8}
\]
Now (4.8) gives
\[
P_2 = \Delta - \frac{n - 2}{2} \left( \bar{J} - \frac{1}{2(n - 1)}|L|^2 \right).
\]
This is the first part of [CMY21] Theorem 5.5. The formula is also in [GW15] Proposition 8.5 and [JO21] Section 13.11. Hence
\[
P_2 = \bar{P}_2 + \frac{n - 2}{4(n - 1)}|L|^2
\]
with \( P_2 \) being the Yamabe operator (note that in our convention, \( \Delta \) is negative). This proves the first part in Proposition \[1\].

Next, we calculate \( \bar{J}' \). The second relation in (4.6) and Lemma 4.3 imply
\[
6v_3 = -\bar{\nabla}_0(\text{Ric})_{00} + 2(\bar{L}, \bar{g}) + 2 \text{tr}(\bar{L}^3) \equiv -2\bar{J}'.
\]
In order to evaluate this formula, we apply the following result. Let \( \bar{G} \) be the Einstein tensor \( \bar{G} = \text{Ric} - n\bar{g}\bar{J} = \text{Ric} - \frac{1}{\text{scal}}\bar{g} \) of \( \bar{g} \). The following result calculates the normal component of the normal derivative of \( \bar{G} \).

**Lemma 4.4.** In general dimensions, it holds
\[
\bar{\nabla}_0(\bar{G})_{00} = -\delta(\text{Ric}_0) - nH\text{Ric}_{00} + (L, \text{Ric}). \tag{4.9}
\]

**Proof.** We prove a more general relation which will be important later. The metric \( \bar{g} \) takes the form \( d\bar{v}^2 + h_r \) in geodesic normal coordinates. The second Bianchi identity implies \( 2\delta^g(\text{Ric}) = d\text{scal} \). Hence
\[
\bar{\nabla}_0(\bar{Ric})(\partial_0, \partial_0)
= \delta^g(\text{Ric})(\partial_0) - \bar{g}^{ij}\bar{\nabla}\bar{\partial}_i(\text{Ric})(\partial_j, \partial_0)
= \frac{1}{2}(d\text{scal}, \partial_0) - \bar{g}^{ij}\partial_i(\text{Ric}(\partial_j, \partial_0)) + \bar{g}^{ij}\text{Ric}(\bar{\nabla}\bar{\partial}_i(\partial_j), \partial_0) + \bar{g}^{ij}\text{Ric}(\partial_j, \bar{\nabla}\bar{\partial}_i(\partial_0))
= \frac{1}{2}(d\text{scal}, \partial_0) - h_r^{ij}\partial_i(\text{Ric}(\partial_j, \partial_0)) + h_r^{ij}\text{Ric}(\bar{\nabla}\bar{\partial}_i(\partial_j) - (L_r)_{ij}\partial_0, \partial_0) + h_r^{ij}\text{Ric}(\partial_j, \bar{\nabla}\bar{\partial}_i(\partial_0))
= \frac{1}{2}(d\text{scal}, \partial_0) - \delta^{h_r}(\text{Ric}_0) - nH_r\text{Ric}_{00} + h_r^{ij}\text{Ric}(\partial_j, \bar{\nabla}\bar{\partial}_i(\partial_0))
on any level surface of $r$. Here $\delta^{hr}$ denotes the divergence operator for the induced metric on the level surfaces of $r$. Similarly, $L_r$ and $H_r$ are the second fundamental form and the mean curvature of these level surfaces. Therefore, using $\nabla_{\partial_0}(\partial_0) = (L_r)_{id} h^{ak}_{\partial_k}$, we obtain
\[
\nabla_0(\hat{G})_{00} = -\delta^{hr}(\hat{\text{Ric}}_0) - nH_r \hat{\text{Ric}}_{00} + h_{ij}^{\hat{\text{Ric}}} (L_r)_{id} \hat{\text{Ric}}_{jk},
\]
i.e., we have proved the relation
\[
\nabla_0(\hat{G})_{00} = -\delta^{hr}(\hat{\text{Ric}}_0) - nH_r \hat{\text{Ric}}_{00} + (L_r, \hat{\text{Ric}}) h_r,
\]
on any level surface of $r$. The assertion is the case $r = 0$.

Now Lemma 4.4 (using $H = 0$) implies
\[
\nabla_0(\hat{G})_{00} = -\delta(\hat{\text{Ric}}_0) + (\hat{L}, \hat{\text{Ric}}).
\]
Hence
\[
\nabla_0(\hat{\text{Ric}})_{00} = n\hat{J}' - \delta(\hat{\text{Ric}}_0) + (\hat{L}, \hat{\text{Ric}}).
\]
It follows that
\[
-2\hat{J}' = -n\hat{J}' + \delta(\hat{\text{Ric}}_0) - (\hat{L}, \hat{\text{Ric}}) + 2(\hat{L}, \hat{G}) + 2\text{tr}(\hat{L}^3),
\]
i.e.,
\[
(n - 2)\hat{J}' = \delta(\hat{L}) + (\hat{L}, \hat{\text{Ric}}) + 2(\hat{L}, \hat{G} - \hat{\text{Ric}}) + 2\text{tr}(\hat{L}^3)
\]
by (2.25). Thus the Gauss identity $\hat{G} - \hat{\text{Ric}} = -\text{Ric} - \hat{L}^2$ yields the desired formula
\[
(n - 2)\hat{J}' = \delta(\hat{L}) + (\hat{L}, \hat{\text{Ric}}) - 2(\hat{L}, \hat{\text{Ric}}).
\]

We summarize these results in

**Proposition 4.5.** If $r^{-2}(dr^2 + h_r) = r^{-2} \hat{g}$ has constant scalar curvature $-n(n+1)$, then $H = 0$ and it holds
\[
\hat{J} = J - \frac{1}{2(n-1)}|\hat{L}|^2 \quad \text{and} \quad (n - 2)\hat{J}' = \delta(\hat{L}) + (\hat{L}, \hat{\text{Ric}}) - 2(\hat{L}, \hat{\text{Ric}}).
\]
In particular, it holds $\hat{J} = J$ and $\hat{J}' = 0$ if $\hat{L} = 0$.

Now, in order to prove the second part of Proposition 1, we have to calculate $\hat{J}'$, i.e., $\hat{J}'$ for the metric $\hat{g}$. Because of (4.4) it only remains to calculate $\hat{\text{Ric}}$. The conformal transformation law for the Ricci tensor gives
\[
\hat{\text{Ric}}_{ij} = \text{Ric}_{ij} - (n - 1) \text{Hess}_{ij}(\omega) - \bar{\Delta}(\omega) h_{ij} - (n - 1) d\omega^2 h_{ij}
\]
\[
= \text{Ric}_{ij} - (n - 1) L_{ij} \partial_0(\omega) - \bar{\Delta}(\omega) h_{ij} - (n - 1) d\omega^2 h_{ij}
\]
using $\omega = 0$. By $\partial_0(\omega) = -H$, we get
\[
\hat{L}^j \hat{\text{Ric}}_{ij} = \hat{L}^j \text{Ric}_{ij} + (n - 1)H |\hat{L}|^2.
\]
Now we apply the formula
\[
\Delta' = [\delta, H_1]d = \delta(H_1 d) - H_1 \delta d
\]
for the metric variation of the Laplacian. Here
\[
\begin{cases}
H_1 = v_1 = 0 & \text{on } \Omega^0(M), \\
v_1 Id - h_{(1)} = -2 \hat{L} & \text{on } \Omega^1(M).
\end{cases}
\]
We recall that $v_1 = 0$. Hence $\Delta' = -2\delta(\hat{L}d)$. Thus, using (4.4), we get
\[
P_3 \sim -\delta(\hat{L}d) - \frac{n - 3}{4(n - 2)}(\delta\delta(\hat{L}) + (\hat{L}, \hat{\text{Ric}}) - 2(\hat{L}, \hat{\text{Ric}}) + (n - 1)H |\hat{L}|^2).
This is the second part of [CMY21, Theorem 5.5]. This formula can also be found in [GW11, Proposition 8.5] and [JO21, Proposition 13.10.1].

We continue discussing $P_4$. Combining the formula for $T_4(\lambda)$ in Lemma 4.2 for the metric $\hat{g}_+ = \hat{r}^{-2}(dr^2 + h_\hat{r}) = \hat{r}^{-2}\hat{g}$ with

$$P_4 \sim \text{Res}_{\lambda = \frac{n}{2} - 2}(T_4)$$

implies the following preliminary formula for $P_4$. Here we use the notation $\hat{J}$ and $\hat{G}$ for the quantities $J$ and $G$ for the metric $\hat{g}$.

**Theorem 4.6.** The extrinsic Paneitz operator is given by

$$P_4(f) = \left(\Delta - \frac{n}{2}\hat{J}\right) \left(\Delta - \left(\frac{n}{2} - 2\right)\hat{J}\right) f - 2(\hat{d}\hat{J}, df) - 4\delta((\hat{h}_{(2)} - \hat{h}_{(1)}^2)df) - (n - 4)\hat{J}'' f$$

$$= \Delta^2(f) - \delta((n - 2)\hat{J}df + 4\hat{h}_{(2)}df) + 4\delta(\hat{h}_{(1)}^2df) + \left(\frac{n}{2} - 2\right) Q_4 f,$$

where $\hat{h}_{(1)} = 2\hat{L}$, $\hat{h}_{(2)} = \hat{L}^2 - \hat{G}$ and

$$Q_4 = \frac{n}{2}\hat{J}^2 - 2\hat{J}' - \Delta(\hat{J}).$$

In particular, $P_4$ is self-adjoint.

**Proof.** The formula

$$\text{Res}_{\lambda = \frac{n}{2} - 2}(T_4) \sim \left(\Delta - \frac{n}{2}\hat{J}\right) \left(\Delta - \left(\frac{n}{2} - 2\right)\hat{J}\right) + 4\Delta'' - (n - 4)\hat{J}''$$

shows that

$$P_4 = \left(\Delta - \frac{n}{2}\hat{J}\right) \left(\Delta - \left(\frac{n}{2} - 2\right)\hat{J}\right) + 4\Delta'' - (n - 4)\hat{J}''.$$  

Now we apply the general formula

$$\Delta'' = [\delta, \mathcal{H}_2] - \mathcal{H}_1[\delta, \mathcal{H}_1]d$$

for the second metric variation of the Laplacian (see the discussion at the end of the section). Here

$$\begin{align*}
\mathcal{H}_2 &= v_2 \quad \text{on } \Omega^0(M), \\
\mathcal{H}_2 &= v_2 \text{Id} - v_1 h_{(1)} + (h_{(1)}^2 - h_{(2)}) \quad \text{on } \Omega^1(M).
\end{align*}$$

Since $v_1 = 0$, the variation formula simplifies to

$$\Delta'' = [\delta, \mathcal{H}_2]d$$

with $\mathcal{H}_2 = v_2 \text{Id} + (h_{(1)}^2 - h_{(2)})$. But $v_2 = -1/2\hat{J}$. This proves the assertion. \qed

We finish with a discussion of the variation formulas for the Laplacian used in the above proofs. More precisely, let $h_r = h + rh_{(1)} + r^2h_{(2)} + \cdots$ be a family of metrics on $M$. Then $\Delta_{h_r} = \Delta_h + r\Delta'_h + r^2\Delta''_h + \cdots$. We refer to $\Delta'_h$ and $\Delta''_h$ as to the first and second metric variation of the Laplacian at $h$.

The arguments rest on the following observation. Let $h_0 = h$ and let the Hodge star operators for $h_0$ and $h_r$ be denoted by $*_0$ and $*_r$, respectively. Let $\mathcal{H}(r) \overset{\text{def}}{=} *_0^{-1}*$ acting on $\Omega^r(M)$. Then it holds

$$\mathcal{H}(r) = v(r)$$

as multiplication operators acting on $C^\infty(M)$. Next, we establish an analogous relation for $\mathcal{H}(r)$ acting on $\Omega^1(M)$. The relation

$$h_r(X, Y) = h_0(T_rX, Y) \quad \text{for } X, Y \in \mathfrak{X}(M)$$

10Note that in [CMY21, Theorem 5.5] the sign of $H$ is misprinted (they use a different sign convention for $H$).
defines an isomorphism \( T_r \in \text{End}(TM) \), i.e., \( T_r(\partial_i) = (T_r)^i_k \partial_k \) with \((h_r)_{ij} = (T_r)^i_k h_{kj}\). In other words, \( T_r \) arises by regarding \( h_r \) as an endomorphism using \( h_0 \). Let \( T_r^* \in \text{End}(T^*M) \) be its dual. Then
\[
   h_r(\alpha, \beta) = h_0(\alpha, (T_r^{-1})^* \beta) \quad \text{for} \ \alpha, \beta \in \Omega^1(M).
\]
Let \( l(r) \stackrel{\text{def}}{=} (T_r^{-1})^* \) acting on \( \Omega^1(M) \).

**Lemma 4.7.** For any metric \( h \), it holds
\[
   \mathcal{H}(r) = v(r)l(r) : \Omega^1(M) \to \Omega^1(M).
\]

**Proof.** On the one hand, we rewrite the defining relation
\[
   \omega \wedge *_{r} \eta = h_r(\omega, \eta)\text{dvol}_r, \ \omega, \eta \in \Omega^1(M)
\]
for the star-operator \(*_r\) as
\[
   \omega \wedge *_{r} \eta = h_0(\omega, l(r)\eta)v(r)\text{dvol}_0.
\]
On the other hand, the defining relation for the star-operator \(*_0\) implies
\[
   \omega \wedge \eta' = h_0(\omega, *_0^{-1} \eta')\text{dvol}_0, \ \eta' \in \Omega^{n-1}(M).
\]
Hence for \( \eta' = *_{r} \eta \), we obtain
\[
   h_0(\omega, *_0^{-1} *_{r} \eta)\text{dvol}_0 = \omega \wedge *_{r} \eta = h_0(\omega, l(r)\eta)v(r)\text{dvol}_0.
\]
It follows that
\[
   *_0^{-1} *_{r} = v(r)l(r).
\]
The proof is complete. \( \square \)

We expand \( \mathcal{H}(r) = \text{Id} + r\mathcal{H}_1 + r^2\mathcal{H}_2 + \cdots \). Then
\[
   \mathcal{H}(r)^{-1} = \text{Id} - r\mathcal{H}_1 + r^2(-\mathcal{H}_2 + \mathcal{H}_1^2) + \cdots .
\]
Now formula \((4.14)\) for \( \mathcal{H}(r) \) may be used to expand the Laplacian \( \Delta_{h_r} \). We write \( \Delta_{h_r} = \delta_r d \) with \( \delta_r = *_{r}^{-1} d *_{r} \) acting on \( \Omega^1(M) \). Now
\[
   \delta_r = \mathcal{H}(r)^{-1} *_0^{-1} d *_0 \mathcal{H}(r) = \mathcal{H}(r)^{-1} \delta_0 \mathcal{H}(r).
\]
Hence
\[
   \delta_r = \delta_0 + r[\delta_0, \mathcal{H}_1] + r^2([\delta_0, \mathcal{H}_2] - \mathcal{H}_1[\delta_0, \mathcal{H}_1]) + \cdots .
\]
Therefore,
\[
   \Delta_{h_r} = \delta_r d = \Delta_h + r[\delta, \mathcal{H}_1]d + r^2([\delta, \mathcal{H}_2]d - \mathcal{H}_1[\delta, \mathcal{H}_1]d) + \cdots .
\]
In other words, the first variation of \( \Delta \) at \( h \) under the perturbation \( h_r \) is given by the operator
\[
   \Delta'_h = [\delta, \mathcal{H}_1]d.
\]
Now Lemma \[4.7\] implies that \( \mathcal{H}_1 = v_1 \text{Id} - h_{(1)} = \frac{1}{2} \text{tr}_h(h_{(1)}) \text{Id} - h_{(1)} \) on \( \Omega^1(M) \). This proves the variation formula
\[
   (d/dt)_{|0} (\Delta_{h + th_{(1)}})(u) = \delta(\mathcal{H}_1 du) - \mathcal{H}_1 \delta du
\]
\[
   = \frac{1}{2} \delta(\text{tr}_h(h_{(1)}) du) - \delta(h_{(1)} du) - \frac{1}{2} \text{tr}(h_{(1)}) \delta du
\]
\[
   = \frac{1}{2} (d \text{tr}_h(h_{(1)}), du) - (\text{Hess}(u), h_{(1)}) - (\delta(h_{(1)}), du)
\]
(with scalar products, \( \delta \) and Hess defined by \( h \)) which is well-known \[BS7\], (1.185). Similarly, for the second variation, we obtain
\[
   \Delta_{h}'' = [\delta, \mathcal{H}_2]d - \mathcal{H}_1[\delta, \mathcal{H}_1]d.
\]
Lemma \[4.7\] implies that \( \mathcal{H}_2 = v_2 \text{Id} - v_1 h_{(1)} + (h_{(1)}^2 - h_{(2)}) \) on \( \Omega^1(M) \). Note that if \( \mathcal{H}_1 \) on functions vanishes, then the formula for the second variation reduces to \( \Delta_{h}'' = [\delta, \mathcal{H}_2]d \). These arguments establish the formulas used in the proofs of Proposition \[4.5\] and Theorem \[4.6\].
5. Proof of Corollary \[1\]

Theorem \[1\] shows that the further discussion of $P_4$ and $Q_4$ requires a good understanding of the term $J''$. Therefore, we next consider the quantity $J''$ if the metric $r^{-2}(dr^2+h_r)$ has scalar curvature $-n(n+1)$. Then we apply the results to prove Corollary \[2\].

We first describe the volume coefficient $v_4$ in terms of the background metric and $L$.

**Lemma 5.1.** In general dimensions, it holds

\[
24v_4 = -\nabla_0^2(\text{Ric})_{00} + 2L^i\nabla_0(\text{Ric})_{0ij0} - 4nH\nabla_0(\text{Ric})_{00} \\
+ 3(\text{Ric}_{00})^2 - 2|\tilde{\nabla}^0|^2 + 8nH(L, \tilde{\nabla}) - 8(L^2, \tilde{\nabla}) + 6\text{Ric}_{00}(\text{Ric})_{00} + 24\sigma_4(L),
\]

where $\sigma_4(L)$ is the fourth elementary symmetric function in the eigenvalues of $L$. Equivalently, it holds

\[
24v_4 = -\nabla_0^2(\text{Ric})_{00} + 2L^i\nabla_0(\text{Ric})_{0ij0} - (4n-2)H\nabla_0(\text{Ric})_{00} \\
+ 3(\text{Ric}_{00})^2 - 2|\tilde{\nabla}^0|^2 + 8(n-2)H(L, \tilde{\nabla}) - 8(L^2, \tilde{\nabla}) \\
+ 6|L|^2\text{Ric}_{00} - (n-1)(3n-4)H^2\text{Ric}_{00} + 24\sigma_4(L).
\]

**Proof.** This is \[JO22\] Lemma 6.7. Its equivalence to \[AGVS1\] Theorem 3.4] follows by combining the calculation on page 483 of this reference with the Gauss equation. The proofs in these references differ. \(\square\)

Now we use Lemma 5.1 to describe the second normal derivative of $\tilde{J}$ if the metric $r^{-2}(dr^2+h_r)$ has scalar curvature $-n(n+1)$. We recall that this condition implies $H = 0$. Thus Lemma 5.1 gives

\[
24v_4 = -\nabla_0^2(\text{Ric})_{00} + 2L^i\nabla_0(\text{Ric})_{0ij0} + 3(\text{Ric}_{00})^2 - 2|\tilde{\nabla}^0|^2 - 8(L^2, \tilde{\nabla}) + 6|L|^2\text{Ric}_{00} + 24\sigma_4(\tilde{L}).
\]

The following unconditional result generalizes Lemma 4.4. It calculates the normal component of the second normal derivative of the Einstein tensor $\tilde{G} = \text{Ric} - n\tilde{\nabla}\tilde{J}$ of a general metric $\tilde{g}$.

**Lemma 5.2.** In general dimensions, it holds

\[
\nabla^2_0(\tilde{G})_{00} = -(n+1)H\nabla_0(\text{Ric})_{00} + 2nHJ' \\
+ 2(\tilde{L}, \nabla(\text{Ric})) - \delta(\nabla_0(\text{Ric})_0) + (\tilde{L}, \nabla_0(\text{Ric})) \\
+ H\delta(\text{Ric})_0 - (n-1)(dH, \text{Ric}) + 2(\delta(\tilde{L}), \text{Ric}) - \delta((\tilde{L}\text{Ric})_0) \\
+ |L|^2\text{Ric}_{00} - (L^2, \text{Ric}) + (\text{Ric}_{00})^2 - (\tilde{\nabla}, \text{Ric}).
\]

**Proof.** We recall that in normal geodesic coordinates the metric $g$ takes the form $dr^2+h_r$ with $h_r = h + 2rL + \cdots$. We also recall the formulas

\[
nH' = -|L|^2 - \text{Ric}_{00}
\]

and $L' = L^2 - \tilde{\nabla}$

for the variation of $H$ and $L$ under the normal exponential map \[HP99\]. Here $'$ denotes the derivative in the variable $r$. Moreover, let $\delta' \overset{\text{def}}{=} (d/dr)|_0(\delta h_r)$. Then

\[
\delta'(\omega) = -2(\tilde{L}, \nabla(\omega))h - 2(\delta(L), \omega)h + n(dH, \omega)h
\]

for $\omega \in \Omega^1(M^4)$ \[B87\] (1.185). Now differentiating \[1.10\] implies

\[
\nabla^2_0(\tilde{G})_{00} = -\delta'(\text{Ric})_0 - \delta(\partial_r(\text{Ric}))_0 - nH\text{Ric}_{00} - nH\nabla_0(\text{Ric})_{00} \\
+ (L', \text{Ric}) + L^i\partial_r(\text{Ric})_0 - 4(L^2, \text{Ric}),
\]

where $\delta' = \delta((\tilde{L}\text{Ric})_0)$.
where $L' = (d/dr)|_0(L_r)$. Note that the last term is caused by the derivative of $h_r$. Hence implies
\[
\nabla_0^2(G)_{00} = 2(L, \nabla (\text{Ric}_0)) + 2(\delta(L), \text{Ric}_0) - n(dH, \text{Ric}_0)
\]
\[
- \delta(\nabla_0(\text{Ric}_0) - \delta((L \text{Ric}_0) - \delta(H \text{Ric}_0) + |L|^2 \text{Ric}_0 + (\text{Ric}_00)^2 - nH \nabla_0(\text{Ric}_0))
\]
\[
+ (L^2, \text{Ric}) - (\mathcal{G}, \text{Ric}) + (L, \nabla (\text{Ric})) + 2(L^2, \text{Ric}) - 4(L^2, \text{Ric})
\]
at $r = 0$. Here we used the relations
\[
\nabla_0(\text{Ric}_0) = \partial_r(\text{Ric}_0) - (L \text{Ric}_0) \text{ and } \nabla_0(\text{Ric}_0)_{ij} = \partial_r(\text{Ric}_0)_{ij} - (L \text{Ric}_0 + \text{Ric}_0 L)_{ij}.
\]
Now, separating the trace-free part of $L$, we obtain
\[
\nabla_0^2(G)_{00} = 2(\dot{L}, \nabla (\text{Ric}_0)) + 2H \delta(\text{Ric}_0) + 2(\delta(\dot{L}), \text{Ric}_0) - (n - 2)(dH, \text{Ric}_0)
\]
\[
- \delta(\nabla_0(\text{Ric}_0)) - \delta((\dot{L} \text{Ric}_0) + (\text{Ric}_00)^2 - nH \nabla_0(\text{Ric}_00)
\]
\[
+ (L^2, \text{Ric}) - (\mathcal{G}, \text{Ric}) + (\dot{L}, \nabla (\text{Ric})) + H \text{scal}' - H \nabla (\text{Ric}_00).
\]
Simplification leads to the result
\[
\nabla_0^2(G)_{00} = - (n + 1)H \nabla_0(\text{Ric}_0) + H \text{scal}'
\]
\[
+ 2(\dot{L}, \nabla (\text{Ric}_0)) - \delta(\nabla_0(\text{Ric}_0)) + (\dot{L}, \nabla (\text{Ric}))
\]
\[
+ H \delta(\text{Ric}_0) - (n - 1)(dH, \text{Ric}_0) + 2(\delta(\dot{L}), \text{Ric}_0) - (L \text{Ric}_0)
\]
\[
+ |L|^2 \text{Ric}_00 - (L^2, \text{Ric}) + (\text{Ric}_00)^2 - (\mathcal{G}, \text{Ric}).
\]

The proof is complete. \hfill \Box

This result is an extension of [1022, Lemma 6.12]. Note that in the second formula, only the coefficients of $H \nabla_0(\text{Ric}_0), (dH, \text{Ric}_0)$ and $H \dot{\mathcal{Y}}'$ depend on the dimension of $M$.

Now we apply Lemma 5.2 for the background metric $\tilde{g} = dr^2 + h_r$ so that $g_+ = r^{-2}\tilde{g}$ has scalar curvature $-n(n + 1)$. We obtain
\[
\nabla_0^2(G)_{00} = 2(\dot{L}, \nabla (\text{Ric}_0)) - \delta(\nabla_0(\text{Ric}_0)) + (\dot{L}, \nabla (\text{Ric}))
\]
\[
+ 2(\delta(\dot{L}), \text{Ric}_0) - \delta((\dot{L} \text{Ric}_0) + (\text{Ric}_00)^2 - (\mathcal{G}, \text{Ric})
\]
\[
\text{using } H = 0. \text{ Now combining (5.1) and (5.5) gives}
\]
\[
24v_4 = -n \dot{\mathcal{Y}}' - 2(\dot{L}, \nabla (\text{Ric}_0)) + \delta(\nabla_0(\text{Ric}_0)) - (\dot{L}, \nabla (\text{Ric}))
\]
\[
- 2(\delta(\dot{L}), \text{Ric}_0) + \delta((\dot{L} \text{Ric}_0) + (\dot{L}^2, \text{Ric}) - (\text{Ric}_00)^2 + (\mathcal{G}, \text{Ric})
\]
\[
+ 2L^2 \nabla_0(\text{Ric}_00) + 3(\text{Ric}_00)^2 - 2|\mathcal{G}|^2 - 8(L^2, \mathcal{G}) + 6|L|^2 \text{Ric}_00 + 24\sigma_4(\dot{L}).
\]
By $24v_4 = -3\dot{\mathcal{Y}}' + 3\dot{\mathcal{Y}}^2$ (see (4.6)), this result implies the formula
\[
(n - 3)\dot{\mathcal{Y}}' = -3\dot{\mathcal{Y}}^2 - 2(\dot{L}, \nabla (\text{Ric}_0)) + \delta(\nabla_0(\text{Ric}_0)) - (\dot{L}, \nabla (\text{Ric}))
\]
\[
- 2(\delta(\dot{L}), \text{Ric}_0) + \delta((\dot{L} \text{Ric}_0) + (\dot{L}^2, \text{Ric}) - (\text{Ric}_00)^2 + (\mathcal{G}, \text{Ric})
\]
\[
+ 2L^2 \nabla_0(\text{Ric}_00) + 3(\text{Ric}_00)^2 - 2|\mathcal{G}|^2 - 8(L^2, \mathcal{G}) + 5|L|^2 \text{Ric}_00 + 24\sigma_4(\dot{L}).
\]
We summarize these results in

**Proposition 5.3.** If $g_+ = r^{-2}(dr^2 + h_r) = r^{-2}\tilde{g}$ has constant scalar curvature $-n(n + 1)$, then

\[
(n - 3)\dot{\mathcal{Y}}' = -3\dot{\mathcal{Y}}^2 - 2(\dot{L}, \nabla (\text{Ric}_0)) + \delta(\nabla_0(\text{Ric}_0)) - (\dot{L}, \nabla (\text{Ric}))
\]
\[
- 2(\delta(\dot{L}), \text{Ric}_0) + \delta((\dot{L} \text{Ric}_0) + (\dot{L}^2, \text{Ric}) + (\mathcal{G}, \text{Ric}) + 2L^2 \nabla_0(\text{Ric}_00) + 3(\text{Ric}_00)^2 - 2|\mathcal{G}|^2 - 8(L^2, \mathcal{G}) + 5|L|^2 \text{Ric}_00 + 24\sigma_4(\dot{L}).
\]
In particular, it holds
\[(n-3)\bar{J}'' = -3\bar{J}^2 + (\bar{g}, \bar{\text{Ric}}) + 2(\bar{\text{Ric}_{00}})^2 - 2|\bar{\mathcal{G}}|^2 + \delta(\nabla_0(\text{Ric})_0)\]
if \(\bar{L} = 0\).

Now partial integration shows

**Corollary 5.4.** Let \(M\) be closed. Then
\[(n-3)\int_M \bar{J}'' \text{dvol}_h = \int_M (\bar{g}, \bar{\text{Ric}}) + 2(\bar{\text{Ric}_{00}})^2 - 2|\bar{\mathcal{G}}|^2 + \delta(\nabla_0(\text{Ric})_0) + 2\bar{L}^i\bar{V}_{0i} \text{dvol}_h.\] (5.7)

In particular, it holds
\[(n-3)\int_M \bar{J}'' \text{dvol}_h = \int_M (\bar{g}, \bar{\text{Ric}}) + 2(\bar{\text{Ric}_{00}})^2 - 2|\bar{\mathcal{G}}|^2 \text{dvol}_h.\] (5.8)
if \(\bar{L} = 0\).

Corollary 5.4 shows that \(\bar{J}''\) and its integral substantially simplify under the assumption \(\bar{L} = 0\).

Now, to further evaluate the integrals in Corollary 5.4, we will derive formulas for some of the ingredients.

**Lemma 5.5.** If \(r^{-2}(dr^2 + h_r) = r^{-2}\bar{g}\) has constant scalar curvature \(-n(n+1)\), then it holds
\[\bar{\text{Ric}_{00}} = J - \frac{2n-1}{2(n-1)}|\bar{L}|^2\]
\[\bar{\mathcal{G}} = P + \frac{1}{n-2}L^2 - \frac{2n-3}{2(n-1)(n-2)}|\bar{L}|^2 h + \frac{n-1}{n-2}W.\]

In particular, it holds
\[\bar{\text{Ric}_{00}} = J\] and \[\bar{\mathcal{G}} = P + \frac{n-1}{n-2}W\]
if \(\bar{L} = 0\).

**Proof.** By the Gauss equation, it holds
\[2\bar{\text{Ric}_{00}} = 2nJ - 2(n-1)J - |\bar{L}|^2 - n(n-1)H^2.\]
Using (4.8), we get
\[2\bar{\text{Ric}_{00}} = 2nJ - 2(n-1)J - \frac{n}{n-1}|\bar{L}|^2 - |\bar{L}|^2 - n(n-1)H^2.\]
This proves the first relation using \(H = 0\). Next, we evaluate the decomposition \(\bar{\mathcal{G}} = \bar{P} + \bar{P}_{00} h + \bar{W}\) under the assumption \(H = 0\). We apply the Fialkow equation
\[\bar{P} = P + \frac{1}{n-2} \left(\bar{L}^2 - \frac{1}{2(n-1)}|\bar{L}|^2 h + \bar{W}\right)\]
(see (2.8) for \(H = 0\)) and
\[\bar{P}_{00} = \frac{1}{n-1}(\bar{\text{Ric}_{00}} - J) = -\frac{1}{n-1}|\bar{L}|^2\]
We will apply these terms for the metric $\hat{g} = e^{2\omega}\hat{g}$. Hence, by the non-linear contributions of $\omega$ (see (5.10)), we have

$$
\hat{g} = \mathcal{P} + \frac{1}{n-2} \left( \hat{L}^2 - \frac{1}{2(n-1)}|\hat{L}|^2 h + \hat{W} \right) - \frac{1}{n-1}|\hat{L}|^2 h + \hat{W} = \mathcal{P} + \frac{1}{n-2} \hat{L}^2 - \frac{2n-3}{2(n-1)(n-2)}|\hat{L}|^2 h + \frac{n-1}{n-2}\hat{W}.
$$

The proof is complete. $\Box$

Now we can give the

**Proof of Corollary** We calculate the integrals in (5.7) for the metric $\hat{\hat{g}} = e^{2\omega}\hat{g}$. We start with a discussion of the last two terms. First, we note that

$$(\hat{L}, \nabla_0(\hat{\text{Ric}})) = (n-1)(\hat{L}, \nabla_0(\hat{\mathcal{P}})) \quad \text{and} \quad \hat{L}^{ij}\nabla_0(\hat{R})_{0i0j} = \hat{L}^{ij}\nabla_0(\hat{\mathcal{P}})_{ij} + \hat{L}^{ij}\nabla_0(\hat{W})_{ij}.$$ 

Hence

$$
\int_M \left( -(\hat{L}, \nabla_0(\hat{\text{Ric}})) + 2\hat{L}^{ij}\nabla_0(\hat{R})_{0i0j} \right) dvol_h = \int_M \left( -(n-3)(\hat{L}, \nabla_0(\hat{\mathcal{P}})) + 2\hat{L}^{ij}\nabla_0(\hat{W})_{ij} \right) dvol_h.
$$

We will apply these terms for the metric $\hat{\hat{g}} = e^{2\omega}\hat{g}$. First, we note that

$$
\hat{\hat{\nabla}}_0(\hat{W})_{ij} = \nabla_0(\hat{W})_{ij} + 2H\hat{W}_{ij}.
$$

Second, the results in Section 8 on the conformal variation of $(\hat{L}, \nabla_0(\hat{\mathcal{P}}))$ (applied to the conformal factor $e^{2\omega}$) (see (8.13)) show that

$$
\int_M (\hat{L}, \hat{\nabla}_0(\hat{\mathcal{P}})) dvol_h = \int_M (\hat{L}, \nabla_0(\hat{\mathcal{P}})) dvol_h + \int_M \left( (\hat{L}, \text{Hess}(H)) + 2H(\hat{L}, \hat{\mathcal{P}}) - H(\hat{L}, \hat{g}) - 2H^2|\hat{L}|^2 - H\text{tr}(\hat{L}^3) - \omega''|\hat{L}|^2 \right) dvol_h + \int_M 2H^2|\hat{L}|^2 dvol_h.
$$

Here we utilized the properties $\omega = 0$ and $\partial_0(\omega) = -H$ on $M$. The term in the last line is caused by the non-linear contributions of $\omega$ (see Remark 8.4). Hence formula (5.7) for $\hat{\hat{g}}$ reads

$$(n-3)\int_M \hat{\hat{\nabla}}^2 dvol_h = \cdots + \int_M \left( -(\hat{L}, \nabla_0(\hat{\text{Ric}})) + 2\hat{L}^{ij}\nabla_0(\hat{R})_{0i0j} \right) dvol_h = \cdots + \int_M \left( -(n-3)(\hat{L}, \nabla_0(\hat{\mathcal{P}})) + 2\hat{L}^{ij}\nabla_0(\hat{W})_{ij} \right) dvol_h = \cdots + \int_M \left( -(n-3)(\hat{L}, \nabla_0(\hat{\mathcal{P}})) + 2\hat{L}^{ij}\nabla_0(\hat{W})_{ij} + 4H(\hat{L}, \hat{W}) \right) dvol_h - (n-3)\int_M \left( (\hat{L}, \text{Hess}(H)) + 2H(\hat{L}, \hat{\mathcal{P}}) - H(\hat{L}, \hat{g}) - H\text{tr}(\hat{L}^3) - \omega''|\hat{L}|^2 \right) dvol_h. \tag{5.9}
$$

Now we restrict to $n = 4$ and find

$$
\hat{\hat{J}}^2 - \hat{\hat{J}}'' = (\hat{L}, \nabla_0(\hat{\mathcal{P}})) - 2\hat{L}^{ij}\nabla_0(\hat{W})_{ij} - 4H(\hat{L}, \hat{W}) \tag{5.10}
$$

up to a total divergence. By [CMY21, Lemma 5.4], the term $\hat{\omega}''$ equals

$$
\hat{\omega}'' = \frac{n+1}{2}H^2 + \hat{\hat{J}} - J + \frac{1}{2(n-1)}|\hat{L}|^2 = \frac{1}{2}H^2 + \hat{\hat{P}}_{00} + \frac{1}{n-1}|\hat{L}|^2.
$$
using the Gauss equation
\[ \bar{J} - J = \bar{P}_{00} - \frac{n}{2}H^2 + \frac{1}{2(n-1)}|\bar{L}|^2. \]

The terms in the last line of (5.10) do not depend on \( H \) and can be expressed in terms of the original metric using the formulas derived above. Here we apply the formulas
\[ \hat{J} = J - \frac{1}{2(n-1)}|\bar{L}|^2, \quad \hat{\text{Ric}}_{00} = J - \frac{2n-1}{2(n-1)}|\bar{L}|^2 \]
(see (4.8) and Lemma 5.5), the decomposition
\[ \hat{\text{Ric}} = \text{Ric} + \hat{\bar{G}} + \bar{\nabla}^2 \]
(5.11)
and
\[ \hat{\bar{G}} = \bar{P} + \frac{1}{n-2}\bar{L}^2 - \frac{2n-3}{2(n-1)(n-2)}|\bar{L}|^2h + \frac{n-1}{n-2}\bar{W} \]
(5.12)
(see Lemma 5.5). Recall that \( H = 0 \) for the metric \( \bar{\bar{g}} \). The terms in the second line of (5.10) do not depend on \( J \). For their simplification, we apply the decomposition \( \bar{G} = \bar{P} + \bar{P}_{00}h + \bar{W} \) and the Fialkow equation (2.8). Note also that \( 24\sigma_4(L) = 3|\bar{L}|^4 - 6\text{tr}(L^4) \). Then a calculation yields
\[ \hat{J}^2 - \hat{J}'' = J^2 - |P|^2 + \frac{9}{4}|W|^2 \]
\[ + (\bar{L}, \bar{\nabla}_0(P)) - 2\bar{L}^i\bar{\nabla}_0(W)_{ij} + (\bar{L}, \text{Hess}(H)) + H(\bar{L}, P) - \frac{9}{2}H(\bar{L}, \bar{W}) \]
\[ + \frac{21}{2}(\bar{L}^2, \bar{W}) + 4(\bar{L}^2, P) - |\bar{L}|^2\bar{P}_{00} - \frac{3}{2}|\bar{L}|^2 - \frac{3}{2}H^2|\bar{L}|^2 - \frac{1}{2}H\text{tr}(\bar{L}^4) \]
\[ - \frac{7}{3}|\bar{L}|^4 + \left( \frac{9}{4} + 6 \right)\text{tr}(\bar{L}^4), \]
up to a divergence term. This proves Corollary 2.

6. \( P_4 \) and \( Q_4 \) for Totally Umbilic Hypersurfaces

In the present section, we prove Corollary 1. Although this result is an obvious consequence of Theorem 1, the following proof prepares the proof of Theorem 1 in Section 7.

The following result makes the right-hand side of the last relation in Proposition 5.3 explicit.

Lemma 6.1. Assume that \( r^{-2}(dr^2 + h_r) = r^{-2}\bar{g} \) has constant scalar curvature \( -n(n+1) \). If \( \bar{L} = 0 \) then
\[ (n-3)\hat{J}'' = (n-3)|P|^2 - \frac{(n-1)(n-2)}{(n-2)^2}|W|^2 + \frac{(n-4)(n-1)}{(n-2)}(P, \bar{W}) + \delta(\bar{\nabla}_0(\text{Ric})_0). \]
In particular, it holds
\[ \hat{J}'' = |P|^2 - \frac{9}{4}|W|^2 + \delta(\bar{\nabla}_0(\text{Ric})_0) \]
in the critical dimension \( n = 4 \).

Proof. This is a direct calculation using (5.11), (5.12) and \( \text{Ric}_{00} = J \).

Now
\[ Q_4 = \frac{n}{2}\hat{J}^2 - 2\hat{J}'' - \Delta(\hat{J}) \]
(see (4.13)) implies
Corollary 6.2. The extrinsic $Q$-curvature $Q_4$ of a totally umbilic hypersurface is given by

$$Q_4 = \frac{n}{2} J^2 - 2|\mathcal{P}|^2 - \Delta(J) + \frac{2}{n} \left( \frac{n-1}{(n-2)} |\hat{\mathcal{W}}|^2 - \frac{2}{n-2} \frac{(n-4)(n-1)}{(n-3)(n-2)} (\mathcal{P}, \hat{\mathcal{W}}) - \frac{2}{n-3} \delta(\hat{\nabla}_0(\hat{\text{Ric}})_0) \right).$$

(6.1)

Note that the first three terms define the intrinsic $Q$-curvature $Q_4$ of $h$. In particular, in the critical dimension $n = 4$, this formula reads

$$Q_4 = Q_4 + \frac{9}{2} |\hat{\mathcal{W}}|^2 - 2\delta(\hat{\nabla}_0(\hat{\text{Ric}})_0).$$

(6.2)

Hence

$$\int_M Q_4 d\text{vol}_h = \int_M \left( Q_4 + \frac{9}{2} |\hat{\mathcal{W}}|^2 \right) d\text{vol}_h$$

if $M$ is closed.

It remains to evaluate the divergence term $\delta(\hat{\nabla}_0(\hat{\text{Ric}})_0)$ in terms of the original metric instead of $\hat{r}^2\sigma^{-2}\hat{g}$. This will be done below. The general case is much more complicated due to a complicated structure of $\hat{P}''$. It will be discussed in the next section.

In order to calculate the divergence term $\delta(\hat{\nabla}_0(\hat{\text{Ric}})_0)$, we apply the properties

- $\omega = 0$ on $M$,
- $\partial_0(\omega) = -H$ on $M$,
- $\partial_0^2(\omega) = (n+1)/2H^2 + \bar{J} - J$ on $M$

(if $L = 0$) [CMY21] Lemma 5.4.

Lemma 6.3. Assume that $\hat{L} = 0$. Then in any dimensions

$$\delta(\hat{\nabla}_0(\hat{\text{P}})_0) = \delta(\hat{\nabla}_0(\hat{\text{P}})_0) - \Delta(\hat{\text{P}}_{00} + H^2).$$

Proof. In the following, we work in general dimensions. We first note that

$$\hat{\nabla}_0(\hat{\text{P}})_{0a} = \hat{\nabla}_0 \left( \hat{\text{P}} - \overline{\text{Hess}}(\omega) + d\omega \otimes d\omega - \frac{1}{2} |d\omega|^2 \hat{g} \right)_{0a}. $$

Now

$$\hat{\nabla}_0(\hat{\text{P}})_{0a} = \partial_0(\hat{\text{P}}_{0a}) - \hat{\text{P}}(\hat{\nabla}_0(\partial_0), \partial_a) - \hat{\text{P}}(\partial_0, \hat{\nabla}_0(\partial_a))$$

$$= \partial_0(\hat{\text{P}}_{0a}) - \hat{\text{P}}(\hat{\nabla}_0(\partial_0) + 2\partial_0(\omega)\partial_0 - \text{grad}(\omega), \partial_a) - \hat{\text{P}}(\partial_0, \hat{\nabla}_0(\partial_a) + \partial_0(\omega)\partial_a + \partial_a(\omega)\partial_0)$$

$$= \nabla_0(\hat{\text{P}})_{0a} - 2\partial_0(\omega)\hat{\text{P}}_{0a} + \hat{\text{P}}(\text{grad}(\omega), \partial_a) - \partial_0(\omega)\hat{\text{P}}_{0a} - \partial_a(\omega)\hat{\text{P}}_{0a}$$

$$= \hat{\nabla}_0(\hat{\text{P}})_{0a} + 2H\hat{\text{P}}_{0a}. $$

Similarly, we find

$$\hat{\nabla}_0(\text{Hess}(\omega))_{0a} = \nabla_0(\text{Hess}(\omega))_{0a} + 2H\text{Hess}(\omega)_{0a},$$

$$\hat{\nabla}_0(d\omega \otimes d\omega)_{0a} = \nabla_0(d\omega \otimes d\omega)_{0a},$$

$$\hat{\nabla}_0(|d\omega|^2 \hat{g})_{0a} = \nabla_0(|d\omega|^2 \hat{g})_{0a} = 0$$

using $\omega = 0$ on $M$. Hence

$$\hat{\nabla}_0(\hat{\text{P}})_{0a} = \nabla_0(\hat{\text{P}})_{0a} - \nabla_0(\text{Hess}(\omega))_{0a} + \nabla_0(d\omega \otimes d\omega)_{0a} + 2H\hat{\text{P}}_{0a} - 2H\text{Hess}(\omega)_{0a}. $$

Now

$$\nabla_0(d\omega \otimes d\omega)_{0a} = \partial_0(\partial_0(\omega)\partial_a(\omega)) - (d\omega \otimes d\omega)(\partial_0, \hat{\nabla}_0(\partial_a)) = \partial_0(\omega)\partial_a^2(\omega) = H\partial_a(H)$$

and

$$\nabla_0(\text{Hess}(\omega))_{0a} = \partial_0(\text{Hess}(\omega))_{0a} + 2H\text{Hess}(\omega)_{0a}.$$
using \( \partial_0(\omega) = -H \), and
\[
\nabla_0(\text{Hess}(\omega))_{0a} = \partial_0(\text{Hess}(\omega)_{0a}) - \text{Hess}(\omega)(\partial_0, \nabla_0(\partial_a))
\]
\[
= \partial_0(\text{Hess}(\omega)_{0a}) - L^b_0 \text{Hess}(\omega)_{0b}
\]
\[
= \partial_0(\text{Hess}(\omega)_{0a}) + H \partial_a(H)
\]
using
\[
\text{Hess}(\omega)_{0a} = \partial^2_{0a}(\omega) - \bar{\Gamma}^b_{0a} \omega_b = \partial^2_{0a}(\omega) = -\partial_a(H).
\]
But
\[
\partial_0(\text{Hess}(\omega)_{0a}) = \partial_0(\partial^2_{0a}(\omega) - \bar{\Gamma}^b_{0a} \omega_b)
\]
\[
= \partial_a \partial^2_{0a}(\omega) - \bar{\Gamma}^b_{0a} \partial_a(\omega)
\]
\[
= \partial_a \partial^2_{0a}(\omega) + L^b_0 \partial_b(H)
\]
\[
= \partial_a \partial^2_{0a}(\omega) + H \partial_a(H).
\]
We combine these results with the formula \( \partial^2_{0a}(\omega) = \frac{n+1}{2} H^2 + \bar{J} - \bar{J} \). Then
\[
\partial_0(\text{Hess}(\omega)_{0a}) = \partial_a \left( \frac{n+1}{2} H^2 + \bar{J} - \bar{J} \right) + H \partial_a(H),
\]
i.e.,
\[
\partial_0(\text{Hess}(\omega)_{0a}) = d \left( \frac{n+1}{2} H^2 + \bar{J} - \bar{J} \right) + H dH
\]
Hence
\[
\nabla_0(\text{Hess}(\omega))_{0a} = d \left( \frac{n+1}{2} H^2 + \bar{J} - \bar{J} \right) + 2 H dH
\]
Combining these results yields
\[
\nabla_0(\bar{P})_{0a} = \nabla_0(\bar{P})_{0a} - \partial_a \left( \frac{n+1}{2} H^2 + \bar{J} - \bar{J} \right) - 2 H \partial_a(H) + H \partial_a(H) + 2 H \bar{P}_{0a} + 2 H \partial_a(H).
\]
By \( \bar{P}_0 = -dH \) (see (2.5)), we find
\[
\nabla_0(\bar{P})_{0a} = \nabla_0(\bar{P})_{0a} - \partial_a \left( \frac{n+2}{2} H^2 + \bar{J} - \bar{J} \right).
\]
Now the Gauss equation \( \bar{J} = \bar{P}_0 = -\frac{n}{2} H^2 \) completes the proof. \( \square \)

As a consequence of Corollary 6.2 and Lemma 6.3, we obtain the formula
\[
Q_4 = Q_4 + \frac{2}{n-3} \frac{(n-1)^2}{(n-2)^2} |\bar{W}|^2 - 2 \frac{(n-4)(n-1)}{(n-3)(n-2)} (\bar{P}, \bar{W}) - \frac{2(n-1)}{n-3} \left( \delta(\nabla_0(\bar{P})_0) - \Delta(\bar{P}_0 + H^2) \right)
\]
(6.3)
if \( \hat{L} = 0 \). The following result further simplifies this formula.

**Lemma 6.4.** Assume that \( \hat{L} = 0 \). Then
\[
\delta(\nabla_0(\bar{P})_0) - \Delta(\bar{P}_0 + H^2) = -\frac{1}{n-2} \delta(\bar{W}).
\]
**Proof.** Let the Cotton tensor \( C \) of \( \bar{g} \) be defined by \( \bar{C}_{ikj} = \nabla_j(\bar{P})_{ik} - \nabla_k(\bar{P})_{ij} \). Then
\[
\bar{C}_{ikj} = \frac{1}{n-2} \text{div}_1(\bar{W})_{ikj}
\]
and we calculate
\[ \nabla_0(\bar{P})_{0a} = (\nabla_0(\bar{P})_{0a} - \nabla_a(\bar{P})_{00}) + \nabla_a(\bar{P})_{00} \]
\[ = \tilde{C}_{0a0} + \nabla_a(\bar{P})_{00} \]
\[ = \frac{1}{n-2} \text{div}_1(\bar{W})_{0a} + \nabla_a(\bar{P})_{00}. \]

But
\[ \text{div}_1(\bar{W})_{0a0} = \sum_{i=0}^{n} \nabla_i(\bar{W})_{0a0} = \sum_{i=0}^{n} \partial^i(\bar{W})_{0a0} \]
\[ - \bar{W}(\nabla^i(\partial_i), \partial_0, \partial_a, \partial_0) - \bar{W}(\partial_i, \nabla^i(\partial_0), \partial_a, \partial_0) - \bar{W}(\partial_i, \partial_0, \nabla^i(\partial_a), \partial_0) - \bar{W}(\partial_i, \partial_0, \partial_a, \nabla^i(\partial_0)) \]
\[ = \sum_{i=1}^{n} \partial^i(\nabla^i_{\partial_a}) + \bar{W}(\nabla^i(\partial_i), \partial_a) + \bar{W}(\partial_i, \nabla^i(\partial_a)) - \sum_{i,j=1}^{n} H h^{ij} \bar{W}_{ij0} - \sum_{i,j=1}^{n} H h^{ij} \bar{W}_{0aj} \]

using \( \nabla^i(\partial_i) = L^i j \partial_j = H h^{ij} \partial_j \) since \( \hat{L} = 0 \). Hence we get
\[ \text{div}_1(\bar{W})_{0a0} = -\delta(\bar{W}) a \]

since \( \bar{W} \) is trace-free. Therefore,
\[ \nabla_0(\bar{P})_{0a} = -\frac{1}{n-2} \delta(\bar{W}) a + \nabla_a(\bar{P})_{00}. \quad (6.4) \]

Moreover, we find
\[ \nabla_a(\bar{P})_{00} = \partial_a(\bar{P}_{00}) - 2\bar{P}(\nabla_a(\partial_0), \partial_0) \]
\[ = \partial_a(\bar{P}_{00}) - 2\bar{P}(L^b_a \partial_b, \partial_0) \]
\[ = \partial_a(\bar{P}_{00}) - 2\bar{P} \partial_a(\partial_0) \]
\[ = \partial_a(\bar{P}_{00}) + 2\partial_a(H) \]
\[ = \partial_a(\bar{P}_{00}) + \partial_a(H^2) \quad (6.5) \]

using \( \hat{L} = 0 \) and \( \bar{P}_0 = -dH \) (by (2.34)). Now we apply \( \delta \) to the relations (6.4) and (6.5) of 1-forms. Combining the resulting identities proves the assertion.

Combining (6.3) with Lemma 6.4 proves the formula for \( Q_4 \) in Corollary 4.

Finally, we can make the formula for \( P_4 \) in Theorem 4.10 fully explicit in the totally umbilic case. By \( \hat{L} = 0 \), we get \( \hat{h}_{(1)} = 2\hat{L} = 0 \), \( J = J \) (Proposition 5.3) and
\[ \hat{h}_{(2)} = \hat{\gamma} = -\bar{P} - \frac{n-1}{n-2} \bar{W} \]

(see Lemma 5.5) so that
\[ P_4 = \Delta^2 - \delta((n-2)J h + 4\bar{P}d) + 4\frac{n-1}{n-2} \delta(\bar{W}d) + \left( \frac{n}{2} - 2 \right) Q_4. \]

This formula completes the proof of Corollary 4.

Remark 6.5. Assume that \( \hat{L} = 0 \). We combine the conformal transformation law for \( P_4 \) with the formula in (1.22). By taking the residue at \( n = 3 \), it follows that the scalar \( R \) defined by \( R \) satisfies \( e^{4\varphi} \circ R = R \). Lemma 6.6 shows that this relation actually follows from the identity
\[ \delta(\bar{W}d\varphi) - (\delta(\bar{W}), d\varphi) - (\text{Hess}(\varphi), \bar{W}) = 0. \]

\(^{11}\)The components for \( i = 0 \) vanish.
The fundamental transformation property of the critical $Q_4$ also can be seen as a consequence of the conformal covariance of $P_4$ for general $n$. Therefore, we finish this section with a direct proof of the conformal covariance of the operator $P_4$. In fact, we confirm the transformation law

$$e^{(\frac{\alpha}{2}+2)\varphi}\hat{P}_4(f) = P_4(e^{(\frac{\alpha}{2}-2)\varphi}f)$$

using the explicit formula (11.22). By the known covariance of the intrinsic $P_4$ and the invariance of $|\hat{W}|^2$, it suffices to prove the conformal covariance of the operator

$$f \mapsto 2(n-3)\delta(\hat{W}df) + \left(\frac{n}{2} - 2\right)\left(-(n-4)(P,\hat{W}) + \delta(\hat{W})\right)f.$$

**Lemma 6.6.** It holds

$$e^{(\frac{\alpha}{2}+2)\varphi}\delta(\hat{W}df) - \delta(\hat{W}d)(e^{(\frac{\alpha}{2}-2)\varphi}f) = -\left(\frac{n}{2} - 2\right)\delta(\hat{W}d\varphi)e^{(\frac{\alpha}{2}-2)\varphi}f$$

and

$$e^{(\frac{\alpha}{2}+2)\varphi}\delta(\hat{W}) - \delta(\hat{W})e^{(\frac{\alpha}{2}-2)\varphi} = ((n-2)\delta(\hat{W}d\varphi) + (n-4)(\delta(\hat{W}),d\varphi))e^{(\frac{\alpha}{2}-2)\varphi},$$

up to non-linear terms in $\varphi$.

**Proof.** First, we recall that the symmetric bilinear form $\hat{W}$ satisfies $\hat{W} = \hat{W}$ and $\text{tr}(\hat{W}) = 0$. Now the conformal transformation laws (12.11) and (12.13) imply

$$e^{(\frac{\alpha}{2}+2)\varphi}\delta(\hat{W}df) = \delta(e^{(\frac{\alpha}{2}-2)\varphi}d\varphi) + \left(\frac{n}{2} - 2\right)(d\varphi,\hat{W}df)e^{(\frac{\alpha}{2}-2)\varphi}$$

$$= \delta(\hat{W}d(e^{(\frac{\alpha}{2}-2)\varphi}f)) - \left(\frac{n}{2} - 2\right)\delta(\hat{W}d\varphi e^{(\frac{\alpha}{2}-2)\varphi}f) + \left(\frac{n}{2} - 2\right)(df,\hat{W}d\varphi)e^{(\frac{\alpha}{2}-2)\varphi}$$

$$= \delta(\hat{W}d)(e^{(\frac{\alpha}{2}-2)\varphi}f) - \left(\frac{n}{2} - 2\right)\delta(\hat{W}d\varphi)e^{(\frac{\alpha}{2}-2)\varphi}f.$$  

This proves the first assertion. Next, we calculate

$$e^{(\frac{\alpha}{2}+2)\varphi}\delta(\hat{W}) = \delta(e^{(\frac{\alpha}{2}+2)\varphi}\hat{W}) = \left(\frac{n}{2} - 2\right)(d\varphi,e^{(\frac{\alpha}{2}+2)\varphi}\hat{W})$$

$$= \delta(\hat{W}d(e^{(\frac{\alpha}{2}-2)\varphi}f)) + \left(\frac{n}{2} - 2\right)(\text{grad}^{(\hat{W}e^{(\frac{\alpha}{2}-2)\varphi})}) + \left(\frac{n}{2} - 2\right)(d\varphi,\delta(e^{(\frac{\alpha}{2}-2)\varphi}\hat{W}))$$

$$= \delta(\hat{W}d\varphi)(e^{(\frac{\alpha}{2}-2)\varphi}) + \left(\frac{n}{2} - 2\right)(\delta(\hat{W}),d\varphi)e^{(\frac{\alpha}{2}-2)\varphi}.$$  

Now combing this relation with the identity

$$\delta\delta(e^{\lambda\varphi}\hat{W}) = (\delta\delta(\hat{W}) + \lambda(\delta(\hat{W}),d\varphi) + \lambda\delta(\hat{W}d\varphi))e^{\lambda\varphi}$$

for $\lambda \in \mathbb{R}$, we find

$$e^{(\frac{\alpha}{2}+2)\varphi}\delta(\hat{W}) = (\delta\delta(\hat{W})) + (n-2)\delta(\hat{W}d\varphi) + (n-4)(\delta(\hat{W}d\varphi))e^{(\frac{\alpha}{2}-2)\varphi}.$$  

This proves the second relation. \qed

Therefore, it remains to prove that

$$-(n-4)(n-3)\delta(\hat{W}d\varphi) + \left(\frac{n}{2} - 2\right)\left(-(n-4)(\text{Hess}(\varphi),\hat{W}) + (n-2)\delta(\hat{W}d\varphi) + (n-4)(\delta(\hat{W}),d\varphi)\right) = 0.$$

This equation is trivial for $n = 4$. For $n \neq 4$, it is equivalent to

$$-2(n-3)\delta(\hat{W}d\varphi) + (n-4)(\text{Hess}(\varphi),\hat{W}) + (n-2)\delta(\hat{W}d\varphi) + (n-4)(\delta(\hat{W}),d\varphi) = 0$$

or in turn to

$$-\delta(\hat{W}d\varphi) + (\text{Hess}(\varphi),\hat{W}) + (\delta(\hat{W}),d\varphi) = 0.$$  

However, this is obvious.
7. **The general case of Theorem 1 and the singular Yamabe obstruction B_3**

In the present section, we prove Theorem 1 and discuss its consequence for \( n = 3 \). We first determine the second-order part of \( P_4 \). By (4.12), it suffices to calculate the sum

\[-\delta((n-2)\hat{J}d + 4\hat{h}_{(2)}d) + 4\delta(h^2_{(1)}d).\]

In order to express that operator in terms of the given data, we apply the formulas \( \hat{h}_{(1)} = 2\hat{L} \),

\[\hat{J} = J - \frac{1}{2(n-1)}|\hat{L}|^2 \]

(by (4.8)) and

\[\hat{h}_{(2)} = \hat{L}^2 - \hat{G}\]

\[= \hat{L}^2 - \left( P + \frac{1}{n-2} \hat{L}^2 - \frac{2n-3}{2(n-1)(n-2)}|\hat{L}|^2 h + \frac{n-1}{n-2}W \right)\]

\[= \frac{n-3}{n-2} \hat{L}^2 - P + \frac{2n-3}{2(n-1)(n-2)}|\hat{L}|^2 h - \frac{n-1}{n-2}W\]

(see Lemma 5.5). Now a direct calculation yields

\[(n-2)\hat{J}d + 4\hat{h}_{(2)}d = (n-2)Jd - 4P + 4\left( 3n - 5 \right) \hat{L}^2 - \frac{n^2 - 12n + 16}{2(n-1)(n-2)}|\hat{L}|^2 h - \frac{n-1}{n-2}W.\]

This implies the displayed terms in Theorem 1. In particular, this proves Corollary 1.

**Remark 7.1.** [BGW21b, Corollary 1.1] states that the second-order part of \( P_4 \) equals

\[\delta(4P - (d-3)J) \delta + \delta \left( 8\hat{L}^2 + \frac{d^2 - 4d - 1}{2(d-1)(d-2)}|\hat{L}|^2 + 4(d-2)\bar{F} \right) d,\]

where \( d = n + 1 \) and \( (d-3)\bar{F} = (\hat{L}^2 - \frac{1}{n} |\hat{L}|^2 + W) \). In terms of the dimension \( n \) of \( M \), this formula reads

\[\delta(4P - (n-2)J) \delta + \delta \left( 8 + \frac{4}{n-2} \right) \hat{L}^2 \right) d + \delta \left( \frac{n^2 - 12n + 16}{2(n-1)(n-2)}|\hat{L}|^2 h + \frac{n-1}{n-2}W \right) d.\]

This formula matches with the result in (1.4).

We continue with the

**Proof of the formula for Q_4 in Theorem 1.** As in the critical dimension \( n = 4 \), the arguments rest on (4.13). First, we extend the arguments in the proof of Corollary 2 in Section 5 to determine the non-divergence terms. In fact, formula (5.9) implies

\[\hat{J}'' = \frac{1}{n-3} \left( -3\hat{L}^2 + (\hat{G}, \hat{Ric}) + 2\hat{L}(\hat{Ric}_{00})^2 + 2(\hat{G})^2 + (\hat{L}^2, \hat{Ric}) + 5|\hat{L}|^2 \hat{Ric}_{00} - 8(\hat{L}^2, \hat{G}) + 24\sigma_4(\hat{L}) \right)\]

\[+ \left( -\hat{L}, \nabla_0(\hat{P}) \right) + 2 \frac{2}{n-3} \hat{L}^{ij} \nabla_0(\hat{W})_{ij} + \frac{4}{n-3} H(\hat{L}, \hat{W})\]

\[- \left( \hat{L}, \text{Hess}(H) \right) + 2H(\hat{L}, \hat{P}) - H(\hat{L}, \hat{G}) - H \text{tr}(\hat{L}^3) - \omega''|\hat{L}|^2,\]

up to divergence terms. Simplification of these terms by utilizing the relations indicated in the proof of Corollary 2 yields the displayed non-divergence terms.
2. It remains to determine the divergence terms in $Q_A$. These consist of $-\Delta(\hat{J})$ and the divergence terms in $\hat{J}''$. Proposition \ref{prop:delta} shows that the divergence terms in $\hat{J}''$ are given by

$$\frac{1}{n-3} \left( -2\langle 1, \nabla (\hat{Ric}_0) \rangle - 2\langle \delta(1), \hat{Ric}_0 \rangle + \delta(\nabla_0(\hat{Ric}_0)) + \delta(\langle \hat{L}\hat{Ric}_0 \rangle) \right)$$

$$= \frac{1}{n-3} \left( -2\delta(\hat{Ric}_0) + \delta(\nabla_0(\hat{Ric}_0)) + \delta(\langle \hat{L}\hat{Ric}_0 \rangle) \right)$$

$$= \frac{1}{n-3} \left( -\delta(\hat{Ric}_0) + \delta(\nabla_0(\hat{Ric}_0)) \right)$$

$$= -\frac{n-1}{n-3} \delta(\hat{P}_0) + \frac{n-1}{n-3} \delta(\nabla_0(\hat{P}_0))$$

for the metric $\hat{g}$. Now

$$\hat{P}_{0a} = \hat{P}_{0a} - \text{Hess}_{0a}(\omega) = \hat{P}_{0a} + \partial_a(H) = \frac{1}{n-1} \delta(\hat{L}_a)$$

(by \ref{eq:delta}) and the vanishing of $\omega$ on $M$) shows that

$$\delta(\hat{L}_0) = \frac{1}{n-1} \delta(\hat{L}_0).$$

Moreover, the calculation of $\delta(\nabla_0(\hat{P}_0))$ rests on the following generalizations of Lemma \ref{lem:delta} and Lemma \ref{lem:delta0}.

**Lemma 7.2.** In general dimensions, it holds

$$\delta(\nabla_0(\hat{P}_0)) = \delta(\nabla_0(\hat{P}_0)) - \Delta \left( \frac{n+1}{2} H^2 + \frac{1}{2(n-1)} |\hat{L}|^2 \right) - 2\delta(LdH) + \frac{2}{n-1} \delta(H\delta(\hat{L})).$$

**Proof.** An extension of the arguments in the proof of Lemma \ref{lem:delta} shows that

$$\nabla_0(\text{Hess}(\omega))_{0a} = \partial_a \left( \frac{n+1}{2} H^2 + \frac{1}{2(n-1)} |\hat{L}|^2 \right) + 2H\partial_a(H) + 2(LdH)_a.$$

Hence

$$\hat{\nabla}_0(\hat{P})_{0a} = \nabla_0(\hat{P})_{0a} - \partial_a \left( \frac{n+1}{2} H^2 + \frac{1}{2(n-1)} |\hat{L}|^2 \right) + 2H\partial_a(H) + 2(LdH)_a$$

$$= \hat{\nabla}_0(\hat{P})_{0a} - \partial_a \left( \frac{n+1}{2} H^2 + \frac{1}{2(n-1)} |\hat{L}|^2 \right) - 2H\partial_a(H) + 2(LdH)_a + H\partial_a(H)$$

by \ref{eq:delta}. Now the Gauss equation

$$\hat{J} - J = \hat{P}_{00} + \frac{1}{2(n-1)} |\hat{L}|^2 - \frac{n}{2} H^2$$

simplifies the latter result to

$$\hat{\nabla}_0(\hat{P})_{0a} = \nabla_0(\hat{P})_{0a} - \partial_a \left( \frac{n+1}{2} H^2 + \frac{1}{n-1} |\hat{L}|^2 \right) - 2(LdH)_a + \frac{2}{n-1} H\delta(\hat{L}_a).$$

Now an application of the operator $\delta$ proves the assertion. \hfill $\Box$

**Lemma 7.3.** In general dimensions, it holds

$$\delta(\nabla_0(\hat{P}_0)) - \Delta(\hat{P}_{00} + H^2) = -\frac{1}{n-2} \delta(\hat{W}) - \frac{1}{n-2} \delta(\hat{L}^i \hat{W}_{0i}) - \frac{2}{n-1} \delta(L\delta(\hat{L})) + 2\delta(LdH).$$
Proof. We extend the arguments in the proof of Lemma 6.3. The relation
\[ \text{div}_1(W)_{0a} = -\delta(W)_a - \bar{L}^{ij}W_{i0aj} - \bar{L}^{ij}W_{i0aj} = -\delta(W)_a - \bar{L}^{ij}W_{0ija} \]
implies
\[ \nabla_0(\bar{P})_{0a} = \frac{1}{n-2} \text{div}_1(W)_{0a} + \nabla_a(\bar{P})_{00} = -\frac{1}{n-2} \delta(W)_a - \frac{1}{n-2} \bar{L}^{ij}W_{0ija} + \nabla_a(\bar{P})_{00}. \]
But
\[ \nabla_a(\bar{P})_{00} = \partial_a(\bar{P}_{00}) - 2\bar{P}(\nabla_a(\partial_0), \partial_0) = \partial_a(\bar{P}_{00}) - 2\bar{L}^b_a\bar{P}_{0b} = \partial_a(\bar{P}_{00}) + 2\bar{L}^b_a\partial_b(H) - 2 \frac{n}{n-1}(L\delta(\bar{L}))_a = \partial_a(\bar{P}_{00}) + 2H\partial_a(H) + 2\bar{L}^b_a\partial_b(H) - 2 \frac{n}{n-1}(L\delta(\bar{L}))_a \]
by \( \bar{P}_0 = -dH + \frac{1}{n-1}\delta(\bar{L}) \) (see (2.5)). Therefore,
\[ \delta(\nabla_0(\bar{P})_{00}) = -\frac{1}{n-2} \delta(\bar{W}) - \frac{1}{n-2} \delta(\bar{L}^{ij}W_{0ija}) + \Delta(\bar{P}_{00} + H^2) + 2\delta(LdH) - 2 \frac{n}{n-1}(L\delta(\bar{L})). \]
The proof is complete. \( \square \)

Using these results, we simplify the divergence terms in \( \bar{Y}'' \) as
\[
\frac{n-1}{n-3} \left( -\frac{1}{n-1} \delta(L\delta(\bar{L})) + \delta(\nabla_0(\bar{P})_{00}) - \Delta(\bar{P}_{00} + H^2) - \frac{1}{n-1}(\nabla_0(\bar{L}))^2 \right) \\
- 2\delta(LdH) + \frac{2}{n-1} \delta(H\delta(\bar{L})) \right) \\
= \frac{n-1}{n-3} \left( -\frac{1}{n-1} \delta(L\delta(\bar{L})) - \frac{1}{n-2} \delta(\bar{W}) - \frac{1}{n-2} \delta(\bar{L}^{ij}W_{0ija}) - \frac{2}{n-1} \delta(L\delta(\bar{L})) + 2\delta(LdH) \\
- \frac{1}{n-1} \Delta(\nabla_0(\bar{L}))^2 - \frac{2}{n-1} \delta(H\delta(\bar{L})) \right) \\
= \frac{n-1}{n-3} \left( -\frac{1}{n-2} \delta(\bar{W}) - \frac{3}{n-1} \delta(L\delta(\bar{L})) - \frac{1}{n-2} \delta(\bar{L}^{ij}W_{0ija}) - \frac{1}{n-1} \Delta(\nabla_0(\bar{L}))^2 \right). \tag{7.2} \]

Next, the following identity enables us to replace the contribution by the Weyl tensor in the latter sum by contributions in terms of \( \bar{L} \).

Lemma 7.4. In general dimensions, it holds
\[ \delta(\bar{L}^{ij}W_{0ija}) = \delta(\bar{L}^2) - \frac{1}{2} \Delta(\nabla_0(\bar{L}))^2 - \frac{n-2}{n-1} \delta(L\delta(\bar{L})). \]

Proof. First, the trace-free Codazzi-Mainardi equation (see (2.6)) implies
\[
\bar{L}^{ij}W_{0ija} = \bar{L}^{ij} \left( \nabla_i(\bar{L})_{aj} - \nabla_a(\bar{L})_{ij} + \frac{1}{n-1} \delta(\bar{L})_ih_{aj} - \frac{1}{n-1} \delta(\bar{L})_a\delta_{ij} \right) \\
= \bar{L}^{ij} \nabla_i(\bar{L})_{aj} - \bar{L}^{ij} \nabla_a(\bar{L})_{ij} + \frac{1}{n-1} \bar{L}^i_a\delta(\bar{L})_i. \tag{7.3} \]

Second, the relations \( d(|\bar{L}|^2)_a = 2\bar{L}^{ij}\nabla_a(\bar{L})_{ij} \) and
\[ (\delta \bar{L}_a) = \bar{L}^i_a\nabla_i(\bar{L})_{aj} \text{ and } \delta(\bar{L}_a^2) = \nabla_i(\bar{L}_a)^j \bar{L}_{ja} + \bar{L}_a^i\nabla_i(\bar{L})_{ja} \]
imply
\[
\delta(\hat{L}^2)_a - \frac{1}{2}d(|\hat{L}|^2)_a - \frac{1}{2}\hat{L}(\hat{L})_a = \nabla^i\hat{L}^i\hat{L}_{ja} + \hat{L}^i\nabla_a(\hat{L})_{ij} - \frac{1}{2}\hat{L}^i\nabla_a(\hat{L})_{ij}
= \frac{1}{2}\delta(\hat{L}^2)_{ja} + \hat{L}^i\nabla_a(\hat{L})_{ij} - \hat{L}^i\nabla_a(\hat{L})_{ij}.
\]  
\eqnum{7.4}

Taking the difference of (7.3) and (7.4) shows the identity
\[
\hat{L}^i\hat{W}_{\partial ij} - \delta(\hat{L}^2)_a + \frac{1}{2}d(|\hat{L}|^2)_a + \frac{1}{2}(\hat{L}(\hat{L}))_a = \frac{1}{n-1}(\hat{L}(\hat{L}))_a - \frac{1}{2}(\hat{L}(\hat{L}))_a
\]
of 1-forms. Equivalently, we find
\[
\hat{L}^i\hat{W}_{\partial ij} = \delta(\hat{L}^2) - \frac{1}{n-1}\hat{L}(\hat{L}).
\]  
\eqnum{7.5}

Now an application of the operator \(\delta\) to the last relation proves the assertion.

Combining (7.2) with Lemma 7.4 yields

\textbf{Lemma 7.5.} The divergence terms of \(Q_4\) are given by the sum of
\[
\frac{2(n-1)}{(n-3)(n-2)}\delta(\hat{W}) + \frac{2(n-1)}{(n-3)(n-2)}d(\hat{L})^2 + \frac{4}{n-3}\delta(\hat{L}(\hat{L}))
+ \left(\frac{2}{n-3} - \frac{n-1}{(n-2)(n-3)}\right)\Delta(|\hat{L}|^2)
\]

and
\[
-\Delta(J) + \frac{1}{2(n-1)}\Delta(|\hat{L}|^2).
\]

Now Lemma 7.5 implies (1.5). This completes the proof of Theorem 1.\(\Box\)

Finally, we discuss the residue formula \[JO21\]
\[
24B_3 = \text{Res}_{n=3}(Q_4).
\]  
\eqnum{7.6}

We shall use this identity to reproduce the known formula for \(B_3\). Theorem 1 shows that \(\text{Res}_{n=3}(Q_4)\) equals the sum of the divergence terms
\[
4\delta(\hat{W}) + 4\delta(\hat{L}^2) + 4\delta(\hat{L}(\hat{L}))
\]
(see also Lemma 7.5), the terms
\[
-4\hat{L}^i\nabla_0(\hat{W})_{\partial ij} - 8H(\hat{L}, \hat{W})
\]
and
\[
8|\hat{W}|^2 + 4(P, \hat{W}) + 32(\hat{L}^2, \hat{W}) + 12(\hat{L}^2, P) - 4|\hat{L}|^2 - 8|\hat{L}|^4 + 24\text{tr}(\hat{L}^4).
\]

Note also that \(\sigma_4(\hat{L}) = 0\) for \(n = 3\) and that this relation is equivalent to \(2\text{tr}(\hat{L}^4) = |\hat{L}|^4\). Hence it holds \(-8|\hat{L}|^4 + 24\text{tr}(\hat{L}^4) = 4|\hat{L}|^4\).

Now we compare this result with the formula for \(B_3\) in \[JO22\] Theorem 1|\footnote{Alternative formulas for \(B_3\) were given in \[CGHW19\] Proposition 1.1] and its arXiv-version. We refer to \[JO22\] Section 6.5] for discussing the relations between these formulas.} This formula implies
\[
24B_3 = 8|\hat{W}|^2 + 4(P, \hat{W}) + 32(\hat{L}^2, \hat{W}) + 12(\hat{L}^2, P) - 4|\hat{L}|^2 + 4|\hat{L}|^4 - 8H(\hat{L}, \hat{W}) - 4\hat{L}^i\nabla_0(\hat{W})_{\partial ij} - 8\hat{L}^i\nabla_0(\hat{W})_{\partial ij} + 4|\hat{W}|^2 + 12\delta(\hat{L}^2) - 4\Delta(|\hat{L}|^2) + 4\delta(\hat{W}).
\]  
\eqnum{7.7}

Hence
\[
\text{Res}_{n=3}(Q_4) = 24B_3.
\]
equals
\[ -8\delta(\dot{L}^2) + 4\delta(\dot{L}\delta(\dot{L})) + 4\Delta(\vert\dot{L}\vert^2) + 8\dot{L}^{ij}\nabla_k\dot{W}_{kij} = 4\vert\dot{W}_0\vert^2. \] (7.8)

In order to simplify this sum, we apply the following result.

**Lemma 7.6.** In any dimension \( n \geq 2 \), it holds
\[ \vert\dot{W}_0\vert^2 - 2\dot{L}^{ij}\nabla_k\dot{W}_{kij} = -2\delta(\dot{L}^{ij}\dot{W}_{-ij}). \]

**Proof.** First, we note that
\[
\delta(\dot{L}^{ij}\dot{W}_{-ij}) = \nabla^k(\dot{L})^{ij}\dot{W}_{kij} + \dot{L}^{ij}\nabla_k\dot{W}_{kij}.
\]
Hence
\[ \vert\dot{W}_0\vert^2 - 2\dot{L}^{ij}\nabla_k\dot{W}_{kij} = \vert\dot{W}_0\vert^2 + 2\nabla^k(\dot{L})^{ij}\dot{W}_{kij} - 2\delta(\dot{L}^{ij}\dot{W}_{-ij}). \]

Thus, it remains to prove that
\[ \vert\dot{W}_0\vert^2 + 2\nabla^k(\dot{L})^{ij}\dot{W}_{kij} = 0. \] (7.9)

The trace-free Codazzi-Mainardi equation (2.0) implies that \( \vert\dot{W}_0\vert^2 = \dot{W}_{ij0}\dot{W}^{ij0} \) equals the sum of
\[
\frac{1}{(n-1)^2} \left( \delta(\dot{L})_j\delta(\dot{L})^j\dot{h}_{ik}h^{jk} + \delta(\dot{L})_i\delta(\dot{L})^i\dot{h}_{jk}h^{jk} - \delta(\dot{L})_j\delta(\dot{L})^i\dot{h}_{ik}h^{jk} - \delta(\dot{L})_i\delta(\dot{L})^j\dot{h}_{jk}h^{ik} \right)
\]
\[ = \frac{1}{(n-1)^2} \left( 2n(\delta(\dot{L}), \delta(\dot{L})) - 2(\delta(\dot{L}), \delta(\dot{L})) \right) = \frac{2}{n-1}(\delta(\dot{L}), \delta(\dot{L})), \]
\[ = -\frac{2}{n-1} \left( \nabla^j(\dot{L})_{ik}\dot{h}^i + \nabla^j(\dot{L})_{jk}\dot{h}^i \right) = -\frac{4}{n-1}(\delta(\dot{L}), \delta(\dot{L})). \]

and
\[
\nabla_j(\dot{L})_{ik}\dot{h}^{ij} - \nabla_j(\dot{L})_{ik}\dot{h}^{ij} = \nabla_i(\dot{L})_{jk}\dot{h}^{ik} + \nabla_i(\dot{L})_{jk}\dot{h}^{ik} = 2\nabla_j(\dot{L})_{ik}\dot{h}^{ik} - 2\nabla_j(\dot{L})_{ik}\dot{h}^{ik}.
\]

Hence
\[ \vert\dot{W}_0\vert^2 = -\frac{2}{n-1}\delta(\dot{L})^2 + 2\nabla_j(\dot{L})_{ik}\nabla^j(\dot{L})^{ik} - 2\nabla_j(\dot{L})_{ik}\nabla^j(\dot{L})^{ik}. \] (7.10)

On the other hand, (2.0) shows that
\[
\nabla^k(\dot{L})^{ij}\dot{W}_{kij} = \nabla^k(\dot{L})^{ij} \left( \nabla_i(\dot{L})_{kj} - \nabla_k(\dot{L})_{ij} + \frac{1}{n-1}\delta(\dot{L})_{ik}\dot{h}_{kj} - \frac{1}{n-1}\delta(\dot{L})_{kj}\dot{h}_{ik} \right)
\]
\[ = \nabla^k(\dot{L})^{ij}\nabla_i(\dot{L})_{kj} - \nabla^k(\dot{L})^{ij}\nabla_k(\dot{L})_{ij} + \frac{1}{n-1}\delta(\dot{L})_{ij}\delta(\dot{L})^i. \] (7.11)

Combining (7.10) and (7.11) proves (7.9). The proof is complete.

Lemma 7.6 shows that the sum (7.3) equals
\[ 8\delta(\dot{L}^2) - 8\delta(\dot{L}^2) + 4\delta(\dot{L}\delta(\dot{L})) + 4\Delta(\vert\dot{L}\vert^2). \] (7.12)

But Lemma 7.3 for \( n = 3 \) shows that the sum (7.12) vanishes.

**Corollary 7.7.** \( \text{Res}_{n=3}(Q_4) = 24\mathbb{B}_3 \).
8. The invariants \( \mathcal{J}_1, \mathcal{J}_2 \) and \( \mathcal{C} \). Proofs of Theorem 2 and Theorem 3

The first main result of the present section is

**Proposition 8.1.** In general dimensions, the quantities

\[
\mathcal{J}_1 \overset{\text{def}}{=} \hat{L}^i j \hat{\nabla}_0 (\hat{\nabla}^j)_0 + 2H(\hat{L}, \hat{\nabla}) + \frac{n-2}{(n-1)(n-6)} \delta \langle \hat{L} \rangle^2 - \frac{n-2}{(n-3)(n-6)} J \hat{L}^2 + \frac{n-4}{(n-3)(n-6)} \Delta (\hat{L}^2) - \frac{1}{n-3} \delta \langle \hat{L}^2 \rangle \)  \quad (8.1)
\]

and

\[
\mathcal{J}_2 \overset{\text{def}}{=} (\hat{L}, \hat{\nabla}(\hat{P})) + (\hat{L}, \text{Hess}(H)) + H(\hat{L}, \hat{P}) - \frac{n-3}{n-2} H(\hat{L}, \hat{\nabla}) + \frac{n}{(n-1)^2} |\delta \langle \hat{L} \rangle|^2 - \frac{1}{(n-3)(n-6)} J \hat{L}^2 - \frac{3}{2} H^2 \hat{L}^2 - \frac{n-3}{n-2} H \text{tr}(\hat{L}^3) + \frac{n-4}{n-3} (\hat{L}^2, \hat{P}) - \frac{1}{n-3} \delta \langle \hat{L}^2 \rangle - \frac{n-5}{2(n-3)(n-6)} \Delta (\hat{L}^2) \)  \quad (8.2)
\]

are local conformal invariants of weight \(-4\) of an embedding \( M^n \hookrightarrow X^{n+1} \), i.e., it holds

\[ e^{4\alpha} \langle \mathcal{J}_i \rangle = \mathcal{J}_i \]

for \( i = 1, 2 \) and all \( \alpha \in C^\infty(X) \).

Some comments are in order.

Both invariants \( \mathcal{J}_i \) vanish if \( \hat{L} = 0 \) and have a simple formal pole at \( n = 3 \). The formal residue of \( \mathcal{J}_1 \) at \( n = 3 \) equals

\[ -\delta \langle \hat{L}^2 \rangle + \frac{1}{3} \Delta (\hat{L}^2) - (\hat{L}^2, \hat{P}) + \frac{1}{3} J \hat{L}^2 - (\delta \langle \hat{L}^2 \rangle)_o + (\hat{P}, (\hat{L}^2)_o) = -\mathcal{D}(\langle \hat{L}^2 \rangle) , \]

where the operator \( \mathcal{D}(b) = \delta \langle b \rangle + (P, b) \) acts on trace-free symmetric bilinear forms and \( \langle \hat{L}^2 \rangle \) denotes the trace-free part of \( \hat{L}^2 \). It is well-known that \( \mathcal{D} : b \mapsto \delta \langle b \rangle + (P, b) \) is a conformally covariant operator \( S^2_0(M) \to C^\infty(M) \) on trace-free symmetric bilinear forms on \( M^3 \). We recall that the term \( \mathcal{D}(\langle \hat{L}^2 \rangle)_o \) contributes to the singular Yamabe obstruction \( B_3 \) of \( M^3 \hookrightarrow X^4 \) (see (1.11)). The formal residue of \( \mathcal{J}_2 \) at \( n = 3 \) also equals \( -\mathcal{D}(\langle \hat{L}^2 \rangle) \).

We also note that both invariants \( \mathcal{J}_i \) have a simple formal pole at \( n = 6 \) with residues being proportional to the local invariant \( P_2(\langle \hat{L}^2 \rangle) \) of weight \(-4\).

We shall see that, in dimension \( n = 4 \), a linear combination of \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) equals \( \mathcal{C} \) (defined in (1.13)). Thus, Proposition 8.1 proves Theorem 2. Theorem 3 then is an easy consequence.

In order to prove Proposition 8.1 it suffices to prove the vanishing of the respective conformal variations

\[ (\mathcal{J}_i(g))^\ast [\varphi] = (d/dt)|_0(e^{4\alpha} \langle \mathcal{J}_i \rangle) (e^{2t\varphi} g) \]

of \( \mathcal{J}_i \) at the metric \( g \). We shall use the bullet notation also for the conformal variation of other scalar curvature quantities.

**Lemma 8.2.** It holds

\[
(\hat{L}, \text{Hess}(H) + \hat{\nabla}_0(\hat{P}))^\ast [\varphi] = \frac{1}{2} |\hat{L}|^2 \Delta (\varphi) - \frac{2n}{n-1} (\hat{L} \delta (\hat{L}), d \varphi) + H(\hat{L}, \text{Hess}(\varphi))
\]

\[
- |\hat{L}|^2 \delta_0 (\varphi) - (\hat{L}, \hat{P}) \partial_0 (\varphi) + \frac{n-3}{n-2} \text{tr}(\hat{L}^3) \partial_0 (\varphi) + 3H |\hat{L}|^2 \partial_0 (\varphi) + \frac{n-3}{n-2} (\hat{L}, \hat{\nabla}) \partial_0 (\varphi)
\]

\[ - \frac{1}{2} \delta (|\hat{L}|^2 d \varphi) + 2 \delta (\hat{L}^2 d \varphi) \]  \quad (8.3)
and
\[
(\hat{L}^j \nabla_0(W)_{0ij0})^\bullet[\varphi] = -2(\hat{L}, \nabla_0)\partial_0(\varphi) + 2\delta(\hat{L}^2 d\varphi) - 2(\hat{L}^2, \text{Hess}(\varphi)) - 2\frac{n-2}{n-1}(\hat{L}\delta(\hat{L}), d\varphi) + |\hat{L}|^2 \Delta(\varphi) - \delta(|\hat{L}|^2 d\varphi).
\]

(8.4)

**Proof.** In the following calculations, all non-linear terms in \( \varphi \) will be omitted without mentioning. We first calculate the conformal variation of
\[
(\hat{L}, \text{Hess}(\varphi)) + (\hat{L}, \nabla_0(\hat{P})).
\]

We recall that
\[
e^\varphi \hat{H} = H + \partial_0(\varphi) \quad \text{and} \quad \text{Hess}_{ij}(u) = \text{Hess}_{ij}(u) - u_i \varphi_j - u_j \varphi_i + h_{ij}(d\varphi, du).
\]

Hence
\[
\text{Hess}_{ij}(\hat{H}) = \text{Hess}_{ij}(\hat{H}) - (\hat{H}_i \varphi_j + \hat{H}_j \varphi_i) + h_{ij}(d\varphi, d\hat{H})
\]
\[
= \text{Hess}_{ij}(H - \varphi) - e^{-\varphi}(H_i \varphi_j + H_j \varphi_i) + e^{-\varphi}(d\varphi, dH + \partial_0(d\varphi))
\]
\[
= e^{-\varphi} \text{Hess}_{ij}(H) - e^{-\varphi}(H_i \varphi_j + H_j \varphi_i) + H \text{Hess}_{ij}(e^{-\varphi})
\]
\[
e^{-\varphi} \text{Hess}_{ij}(\partial_0(\varphi)) - e^{-\varphi}(H_i \varphi_j + H_j \varphi_i) + h_{ij}(d\varphi, dH)
\]
\[
e^{-\varphi} \text{Hess}_{ij}(H) - e^{-\varphi}2(\hat{H}_i \varphi_j + \hat{H}_j \varphi_i) + e^{-\varphi} \text{Hess}_{ij}(\partial_0(\varphi)) - e^{-\varphi} H \text{Hess}_{ij}(\varphi)
\]
\[
+ h_{ij}e^{-\varphi}(d\varphi, dH).
\]

Therefore, we get
\[
e^{4\varphi} \hat{L}^j \text{Hess}_{ij}(\hat{H}) = \hat{L}^j (\text{Hess}_{ij}(H) - 2(H_i \varphi_j + H_j \varphi_i) + \text{Hess}_{ij}(\partial_0(\varphi)) - H \text{Hess}_{ij}(\varphi)).
\]

Hence
\[
(\hat{L}, \text{Hess}(\varphi))^\bullet[\varphi] = -4(\hat{L}dH, d\varphi) + (\hat{L}, \text{Hess}(\partial_0(\varphi))) - H(\hat{L}, \text{Hess}(\varphi)).
\]

(8.5)

Next, we calculate
\[
\nabla_0(\hat{P})_{ij} = e^{-\varphi} \nabla_0(\hat{P} - \text{Hess}(\varphi))_{ij}
\]
\[
e^{-\varphi}(\nabla_0(\hat{P} - \text{Hess}(\varphi))_{ij} - (\hat{P}_{0ij} \varphi_i + \hat{P}_{i0j} \varphi_j + 2\hat{P}_{ij} \partial_0(\varphi)))
\]
\[
e^{-\varphi} \nabla_0(\hat{P})_{ij} - e^{-\varphi} \nabla_0(\text{Hess}(\varphi))_{ij} - e^{-\varphi}(\hat{P}_{0ij} \varphi_i + \hat{P}_{i0j} \varphi_j) - 2e^{-\varphi} \hat{P}_{ij} \partial_0(\varphi)
\]

using the general transformation law
\[
\nabla_i(\partial_j) = \nabla_i(\partial_j) + \partial_i(\varphi) \partial_j + \partial_j(\varphi) \partial_i - g_{ij} \text{grad}(\varphi).
\]

(8.6)

Now we contract with \( \hat{L} \). Then
\[
(\hat{L}, \nabla_0(\hat{P}))^\bullet[\varphi] = -(\hat{L}, \nabla_0(\text{Hess}(\varphi))) - 2(\hat{L} \partial_0, d\varphi) - 2(\hat{L}, \partial_0)(d\varphi).
\]

(8.7)

We continue discussing the term \( \nabla_0(\text{Hess}(\varphi)) \). We find
\[
\nabla_0(\text{Hess}(\varphi))_{ij} = \partial_0(\text{Hess}_{ij}(\varphi)) - \text{Hess}(\varphi)(\nabla_0(\partial_i), \partial_j) - \text{Hess}(\varphi)(\partial_i, \nabla_0(\partial_j))
\]
\[
= \partial_0(\text{Hess}_{ij}(\varphi)) - L^j_i \text{Hess}_{ij}(\varphi) - L^j_i \text{Hess}_{ij}(\varphi) - 2L^k_i \partial_k(\varphi)
\]

using
\[
\text{Hess}_{ij}(u) = \text{Hess}_{ij}(u) + L_{ij} \partial_0(u) \quad \text{and} \quad \nabla_0(\partial_i) = L^k_i \partial_k.
\]

Here we use \( \Gamma^j_0 = L^j_i \) and \( \tilde{\Gamma}^0_0 = 0 \). Hence
\[
\nabla_0(\text{Hess}(\varphi))_{ij} = \partial_0(\text{Hess}_{ij}(\varphi)) - \hat{L}^j_i \text{Hess}_{ij}(\varphi) - \hat{L}^j_i \text{Hess}_{ij}(\varphi) - 2H \text{Hess}_{ij}(\varphi) - 2L^k_{ij} \partial_k(\varphi).
\]

(8.8)
But
\[
\partial_0(\text{Hess}_{ij}(\varphi)) = \partial_0(\partial^0_{ij}(\varphi) - \Gamma^k_{ij} \varphi_k)
\]
\[
= \partial^0_{ij}(\partial_0(\varphi)) - \Gamma^k_{ij} \partial^0_k(\varphi) - \partial_0(\Gamma^k_{ij}) \varphi_k
\]
\[
= \partial^0_{ij}(\partial_0(\varphi)) - \Gamma^1_{ij} \partial^0_1(\varphi) - \Gamma^0_{ij} \partial^0_2(\varphi) - \partial_0(\Gamma^1_{ij}) \varphi_l - \partial_0(\Gamma^0_{ij}) \varphi_l
\]
\[
= \text{Hess}_{ij}(\partial_0(\varphi)) - \partial_0(\Gamma^1_{ij}) \varphi_l - \Gamma^0_{ij} \partial^0_2(\varphi) - \partial_0(\Gamma^0_{ij}) \varphi_l.
\]

Here \( k \) runs from 0 to \( n \) and the tangential index \( l \) runs from 1 to \( n \) (\( n = 4 \)). In the last equality, we used the fact that for tangential indices, the restriction of the Christoffel symbols \( \bar{\Gamma}^l_{ij} \) to \( M \) coincide with the Christoffel symbols \( \Gamma^l_{ij} \) of the induced metric \( h \) on \( M \). Now the general variation formula
\[
\delta(\Gamma^k_{ij}) = \frac{1}{2} \left( \nabla_i(\delta(g))^k_{ij} + \nabla_j(\delta(g))^k_{ij} - \nabla^k(\delta(g))_{ij} \right)
\]
(with \( \nabla \) for \( g \)) implies
\[
\partial_0(\Gamma^l_{ij}) = \nabla_i(L)^l_j + \nabla_j(L)^l_i - \nabla^l(L)_{ij}
\]
\[
= \nabla_i(\ddot{L})^l_j + \nabla_j(\ddot{L})^l_i - \nabla^l(\dot{L})_{ij} + H_j h^l_i + H_i h^l_j - H^l_{ij}
\]
using \( h_r = h + 2Lr + \cdots \). Moreover, the identity \( \Gamma^0_{ij} = -\frac{1}{2} h^l_{ij} \) and the expansion \( h_r = h + 2Lr + (L^2 - \bar{G})r^2 + \cdots \) imply
\[
\Gamma^0_{ij} = -L_{ij},
\]
\[
\partial_0(\Gamma^0_{ij}) = -L^2_{ij} + \bar{G}_{ij}.
\]

Hence
\[
\partial_0(\text{Hess}_{ij}(\varphi)) = \text{Hess}_{ij}(\partial_0(\varphi))
\]
\[
- (\nabla_i(L)^l_j + \nabla_j(L)^l_i - \nabla^l(L)_{ij}) \varphi_l - H_j \varphi_i - H_i \varphi_j + H^l \varphi h_{ij}
\]
\[
+ L_{ij} \partial^0_0(\varphi) + L^2_{ij} \partial_0(\varphi) - \bar{G}_{ij} \partial_0(\varphi).
\]

Thus (8.8) implies
\[
\nabla_0(\text{Hess}(\varphi))_{ij} = \text{Hess}_{ij}(\partial_0(\varphi))
\]
\[
- (\nabla_i(\ddot{L})^l_j + \nabla_j(\ddot{L})^l_i - \nabla^l(\dot{L})_{ij}) \varphi_l - H_j \varphi_i - H_i \varphi_j + H^l \varphi h_{ij}
\]
\[
- ((L^2)_{ij} + 2H \dot{L}_{ij} + H^2 h_{ij}) \partial_0(\varphi) + L_{ij} \partial^0_0(\varphi) - \bar{G}_{ij} \partial_0(\varphi)
\]
\[
- \dot{L}^l_i \text{Hess}_{ij}(\varphi) - \ddot{L}^l_j \text{Hess}_{il}(\varphi) - 2H \text{Hess}_{ij}(\varphi).
\]

By contraction with \( \dot{L} \), we obtain
\[
(\dot{L}, \nabla_0(\text{Hess}(\varphi))) = (\dot{L}, \text{Hess}(\partial_0(\varphi))
\]
\[
- 2\dot{L}^l_j \nabla_i(\lddot{L})_{ij} \varphi_l + \dot{L}^l_j \nabla^l(\dot{L})_{ij} \varphi_l - 2\dot{L}^l_j H_i \varphi_j
\]
\[
- \text{tr}(\dot{L}^3) \partial_0(\varphi) - 2H |\dot{L}|^2 \partial_0(\varphi) + |\dot{L}|^2 \partial^0_0(\varphi) - (\dot{L}, \bar{G}) \partial_0(\varphi)
\]
\[
- 2(\dot{L}^2, \text{Hess}(\varphi)) - 2H (\dot{L}, \text{Hess}(\varphi)).
\]

Reordering gives
\[
(\dot{L}, \nabla_0(\text{Hess}(\varphi))) = (\dot{L}, \text{Hess}(\partial_0(\varphi))
\]
\[
+ \dot{L}^l_j \nabla^l(\dot{L})_{ij} \varphi_l - 2\dot{L}^l_j \nabla_i(\dot{L})_{ij} \varphi_l - 2L^2 H_i \varphi_j - 2(\dot{L}^2, \text{Hess}(\varphi)) - 2H (\dot{L}, \text{Hess}(\varphi))
\]
\[
+ |\dot{L}|^2 \partial^0_0(\varphi) - \text{tr}(\dot{L}^3) \partial_0(\varphi) - 2H |\dot{L}|^2 \partial_0(\varphi) - (\dot{L}, \bar{G}) \partial_0(\varphi).
\]
Now summarizing the conformal variations (8.5) and (8.13) gives
\[ \delta \hat{\mathcal{L}} = \frac{1}{2} (d(|\hat{L}|^2), d\varphi) = \frac{1}{2} \delta(|L|^2 d\varphi) - \frac{1}{2} |\hat{L}|^2 \Delta(\varphi) \] (8.11)
and
\[ \hat{L}^i \nabla_i (\hat{L})_j \varphi_l = \delta(\hat{L}^2 d\varphi) - (\hat{L}^2, \text{Hess}(\varphi)) - (\hat{L} \delta(\hat{L}), d\varphi). \] (8.12)

These relations show that
\[ (\hat{L}, \nabla_0 \text{Hess}(\varphi)) = (\hat{L}, \text{Hess}(\partial_0(\varphi))) \]
\[ - \frac{1}{2} |\hat{L}|^2 \Delta(\varphi) + 2(\hat{L} \delta(\hat{L}), d\varphi) - 2H(\hat{L}, \text{Hess}(\varphi)) - 2(\hat{L} dH, d\varphi) \]
\[ + |\hat{L}|^2 \partial_0^2(\varphi) - \text{tr}(\hat{L}^3) \partial_0(\varphi) - 2H|\hat{L}|^2 \partial_0(\varphi) - (\hat{L}, \tilde{G}) \partial_0(\varphi) \]
\[ + \frac{1}{2} \delta(|\hat{L}|^2 d\varphi) - 2\delta(\hat{L}^2 d\varphi). \]

Combining this result with (8.7) yields
\[ (\hat{L}, \nabla_0 (\hat{P}))^*[\varphi] = -(\hat{L}, \text{Hess}(\partial_0(\varphi))) \]
\[ + \frac{1}{2} |\hat{L}|^2 \Delta(\varphi) - 2(\hat{L} \delta(\hat{L}), d\varphi) - 2(\hat{L} P_0, d\varphi) - 2(\hat{L}, \hat{P}) \partial_0(\varphi) \]
\[ + 2H(\hat{L}, \text{Hess}(\varphi)) + 2(\hat{L} dH, d\varphi) \]
\[ - |\hat{L}|^2 \partial_0^2(\varphi) + \text{tr}(\hat{L}^3) \partial_0(\varphi) + 2H|\hat{L}|^2 \partial_0(\varphi) + (\hat{L}, \tilde{G}) \partial_0(\varphi) \]
\[ - \frac{1}{2} \delta(|\hat{L}|^2 d\varphi) + 2\delta(\hat{L}^2 d\varphi). \] (8.13)

Now summarizing the conformal variations (8.5) and (8.13) gives
\[ (\hat{L}, \text{Hess}(H) + \nabla_0 (\hat{P}))^*[\varphi] \]
\[ = \frac{1}{2} |\hat{L}|^2 \Delta(\varphi) - 2(\hat{L} \delta(\hat{L}), d\varphi) - 2(\hat{L} dH, d\varphi) - 2(\hat{L} P_0, d\varphi) + H(\hat{L}, \text{Hess}(\varphi)) \]
\[ - |\hat{L}|^2 \partial_0^2(\varphi) - 2(\hat{L}, \hat{P}) \partial_0(\varphi) + (\hat{L}, \tilde{G}) \partial_0(\varphi) + \text{tr}(\hat{L}^3) \partial_0(\varphi) + 2H|\hat{L}|^2 \partial_0(\varphi) \]
\[ - \frac{1}{2} \delta(|\hat{L}|^2 d\varphi) + 2\delta(\hat{L}^2 d\varphi). \]

By \( \delta(\hat{L}) = (n-1)dH + (n-1)\hat{P}_0 \) (Codaacci-Mainardi), we have
\[ -2(\hat{L} \delta(\hat{L}), d\varphi) - 2(\hat{L} dH, d\varphi) - 2(\hat{L} P_0, d\varphi) = \left(-2 - \frac{2}{n-1}\right)(\hat{L} \delta(\hat{L}), d\varphi) = -\frac{2n}{n-1}(\hat{L} \delta(\hat{L}), d\varphi). \]

Therefore, we conclude that
\[ (\hat{L}, \text{Hess}(H) + \nabla_0 (\hat{P}))^*[\varphi] \]
\[ = \frac{1}{2} |\hat{L}|^2 \Delta(\varphi) - \frac{2n}{n-1}(\hat{L} \delta(\hat{L}), d\varphi) + H(\hat{L}, \text{Hess}(\varphi)) \]
\[ - |\hat{L}|^2 \partial_0^2(\varphi) - 2(\hat{L}, \hat{P}) \partial_0(\varphi) + (\hat{L}, \tilde{G}) \partial_0(\varphi) + \text{tr}(\hat{L}^3) \partial_0(\varphi) + 2H|\hat{L}|^2 \partial_0(\varphi) \]
\[ - \frac{1}{2} \delta(|\hat{L}|^2 d\varphi) + 2\delta(\hat{L}^2 d\varphi). \] (8.14)
Now the decomposition $\mathcal{G} = \mathcal{P} + \mathcal{P}_0 h + \nabla W$ implies $(\hat{L}, \mathcal{G}) = (\hat{L}, \mathcal{P}) + (\hat{L}, \nabla W)$. Hence
\[
(\hat{L}, \text{Hess}(H) + \nabla_0(\mathcal{P}))^* \mathcal{F} = \frac{1}{2} |\hat{L}|^2 \Delta(\phi) - \sum_{n=1}^{2n-1} (\hat{L} \phi (\hat{L}, d \phi) + H(\hat{L}, \text{Hess}(\phi))
- |\hat{L}|^2 \partial^2_0(\phi) - (\hat{L}, \mathcal{P}) \phi (\partial_0(\phi) + (\hat{L}, \nabla W) \partial_0(\phi) + \text{tr}(\hat{L}^3) \partial_0(\phi) + 2H |\hat{L}|^2 \partial_0(\phi)
- \frac{1}{2} \delta(\hat{L}^2 d \phi) + 2\delta(\hat{L}^2 d \phi).
\]

Next, we use the Fialkow equation (2.8) to write
\[
(\hat{L}, \mathcal{P}) = \left( \hat{L}, \mathcal{P} - H\hat{L} + \frac{1}{n-2} \hat{L}^2 + \frac{1}{n-2} \nabla W \right)
= (\hat{L}, \mathcal{P}) - H |\hat{L}|^2 + \frac{1}{n-2} \text{tr}(\hat{L}^3) + \frac{1}{n-2} (\hat{L}, \nabla W).
\]

Therefore, we find
\[
(\hat{L}, \text{Hess}(H) + \nabla_0(\mathcal{P}))^* \mathcal{F}
= \frac{1}{2} |\hat{L}|^2 \Delta(\phi) - \sum_{n=1}^{2n-1} (\hat{L} \phi (\hat{L}, d \phi) + H(\hat{L}, \text{Hess}(\phi))
- |\hat{L}|^2 \partial^2_0(\phi) - (\hat{L}, \mathcal{P}) \phi (\partial_0(\phi) + \text{tr}(\hat{L}^3) \partial_0(\phi) + 3H |\hat{L}|^2 \partial_0(\phi) + \frac{1}{n-2} (\hat{L}, \nabla W) \partial_0(\phi)
- \frac{1}{2} \delta(\hat{L}^2 d \phi) + 2\delta(\hat{L}^2 d \phi).
\]

This proves the first variation formula. For the proof of the second variation formula, we first observe that
\[
(\hat{L}^{ij} \nabla_0(\nabla W)_{0ij}) = e^{-\phi} \hat{L}^{ij} \nabla_0(\nabla W)_{0ij} = e^{-\phi} \hat{L}^{ij} \nabla_0(e^{2\phi} \nabla W)_{0ij}
= e^{-\phi} \hat{L}^{ij} \nabla_0(\nabla W)_{0ij0} + 2\partial_0(\phi) e^{2\phi} \nabla W_{0ij0}
= -4\phi \hat{L}^{ij} \nabla_0(\nabla W)_{0ij0} + 2e^{-4\phi} \partial_0(\phi)(\hat{L}, \nabla W).
\]

Now the general transformation law (8.6) implies
\[
\nabla_0(\nabla W)_{0ij0} = \nabla_0(\nabla W)_{0ij0} - 2W(\partial_0(\phi) \partial_0(\phi), \partial_i, \partial_j, \partial_0) + W(\nabla(\phi), \partial_i, \partial_j, \partial_0)
- W(\partial_0(\phi) \partial_0(\phi), \partial_i, \partial_j, \partial_0) - W(\partial_0(\phi) \partial_0(\phi), \partial_i, \partial_0(\phi), \partial_j, \partial_0)
- 2W(\partial_0(\phi), \partial_i, \partial_j, \partial_0(\phi)) + W(\partial_0(\phi), \partial_i, \partial_j, \text{grad}(\phi))
= \nabla_0(\nabla W)_{0ij0} - 6\partial_0(\phi) W_{ij0} + W_{0ij0} \text{grad}(\phi).
\]

Hence the right-hand side of (8.15) equals
\[
e^{-4\phi} (\hat{L}^{ij} \nabla_0(\nabla W)_{0ij0} - 4\partial_0(\phi)(\hat{L}, \nabla W) + 2\hat{L}^{ij} \nabla W_{\text{grad}(\phi)ij0}).
\]

Therefore, we get
\[
(\hat{L}^{ij} \nabla_0(\nabla W)_{0ij0})^* \mathcal{F} = -2(\hat{L}, \nabla W) \partial_0(\phi) + 2\hat{L}^{ij} \nabla W_{\text{grad}(\phi)ij0}.
\]

Now we further simplify the term
\[
\hat{L}^{ij} \nabla W_{\text{grad}(\phi)ij0}
using the trace-free Codazzi-Mainardi equation

\[ \nabla_j (\hat{L})_{ik} - \nabla_i (\hat{L})_{jk} + \frac{1}{n-1} \delta(\hat{L})_{ij} h_{ik} - \frac{1}{n-1} \delta(\hat{L})_{ij} h_{jk} = W_{ijk0} \]

(see (2.6)). It follows that

\[ \hat{L}^{ij} W_{\text{grad}^v(\varphi)ij0} = \hat{L}^{ij} \nabla_i (\hat{L})_{\text{grad}^v(\varphi)j} - \hat{L}^{ij} \nabla_{\text{grad}^v(\varphi)}(\hat{L})_{ij} + \frac{1}{n-1} \hat{L}^{ij} \delta(\hat{L})_{\text{grad}^v(\varphi)ij} \]

\[ = \hat{L}^{ij} \nabla_i (\hat{L})_{\text{grad}^v(\varphi)j} - \frac{1}{2} (d(|\hat{L}|^2), d\varphi) + \frac{1}{n-1} \hat{L}^{ij} \delta(\hat{L})_{\text{grad}^v(\varphi)ij}. \]

Now (8.12) shows that

\[ \hat{L}^{ij} \nabla_i (\hat{L})_{j} \varphi = \delta(\hat{L}^2 d\varphi) - (\hat{L}^2, \text{Hess}(\varphi)) - (\hat{L} \delta(\hat{L}), d\varphi). \]

Hence

\[ \hat{L}^{ij} W_{\text{grad}^v(\varphi)ij0} = \delta(\hat{L}^2 d\varphi) - (\hat{L}^2, \text{Hess}(\varphi)) - \frac{1}{2} (d(|\hat{L}|^2), d\varphi) + \frac{1}{n-1} \hat{L} \delta(\hat{L}), d\varphi) \]

\[ = \delta(\hat{L}^2 d\varphi) - (\hat{L}^2, \text{Hess}(\varphi)) - \frac{n-2}{n-1} \hat{L} \delta(\hat{L}), d\varphi) + \frac{1}{2} |\hat{L}|^2 \Delta(\varphi) - \frac{1}{2} (d(|\hat{L}|^2 d\varphi) \]

(see also (7.5)). Thus, we obtain

\[ (\hat{L}^{ij} \nabla_0(W))_{0ij0} \varphi = -2(\hat{L}, \overline{W}) \partial_0(\varphi) + 2 \delta(\hat{L}^2 d\varphi) - 2(\hat{L}^2, \text{Hess}(\varphi)) - \frac{n-2}{n-1} (\hat{L} \delta(\hat{L}), d\varphi) + |\hat{L}|^2 \Delta(\varphi) - \delta(\hat{L}^2 d\varphi). \]

This proves the second formula. \( \square \)

Remark 8.3. The arguments in the second part of the above proof show that

\[ e^\varphi \nabla_0(W)_{0ij0} = \nabla_0(W)_{0ij0} - 4 \partial_0(\varphi) \overline{W}_{ij0} + W_{\text{grad}(\varphi)ij00} + W_{0ij \text{grad}(\varphi)}. \]

Thus, the conformal transformation law \( e^\varphi \nabla_0(W)_{0ij0} = \nabla_0(W)_{0ij0} - 4 \partial_0(\varphi) \overline{W}_{ij0} + W_{\text{grad}(\varphi)ij00} + W_{0ij \text{grad}(\varphi)}. \) for the Cotton tensor \( \tilde{C} \) implies the conformal invariance \( e^\varphi \nabla_0(W)_{0ij0} = \nabla_0(W)_{0ij0} \) of the trace-free symmetric bilinear form

\[ S_{ij} \overset{\text{def}}{=} \nabla_0(W)_{0ij0} - \nabla_0(W)_{0ij0} - 4 \partial_0(\varphi) \overline{W}_{ij0} + W_{\text{grad}(\varphi)ij00} + W_{0ij \text{grad}(\varphi)} \]

(8.17) on \( M \). As a consequence, the scalar curvature quantity \( J_5 \overset{\text{def}}{=} (\hat{L}, S) \) is a conformal invariant of weight \(-4\). For more details on \( J_5 \), we refer to Section (2.2).

For \( n = 3 \), the conformally invariant symmetric tensor \( S \) recently appeared in [CG19, Lemma 2.1] in connection with the study of the manifold variation of the conformally invariant functional

\[ \int_X |W|^2 dvol_g + 8 \int_{M^3} (\hat{L}, \overline{W}) dvol_h \]

(8.18) on a four-manifold \( X \) with boundary \( M \) (for more details, we also refer to [GZ20]). This functional generalizes the conformally invariant functional

\[ \int_X |W|^2 dvol_g \]

of closed four manifolds \( X \). Critical metrics of the latter functional are Bach-flat. The boundary term in (8.18) may be regarded as an analog of the Gibbons-Hawking-York term leading to a well-defined variational problem for the Einstein-Hilbert functional on a manifold with boundary.\(^{13}\)

\(^{13}\)The fact that the variational problem of the functional (8.18) is well-defined also suggests to expect that the same combination of a bulk and a boundary term contributes to the integrated conformal anomaly of CFT’s on a four-manifold with boundary [S16] (14)].
It was noted in [GZ20] that critical points of the functional \(8.18\) are Bach-flat and satisfy the equation \(S = 0\).

For \(n = 3\), the conformal invariant \(\mathcal{J}_5 = (\hat{L}, S)\) of weight \(-4\) also contributes to the singular Yamabe obstruction \(\mathcal{E}_3\) of \(M^3 \hookrightarrow X^4\). Indeed, we calculate

\[
(\hat{L}, S) = -\hat{L}^4 \nabla_0 (\hat{W})_{0ij0} - 2\hat{L}^2 \nabla^k (\hat{W})_{kij0} + 4H(\hat{L}, \hat{W})
\]

\[
= -\hat{L}^4 \nabla_0 (\hat{W})_{0ij0} - 2\hat{L}^2 \nabla^k (\hat{W})_{kij0} - (\hat{L}^2, \hat{W}) + 3H(\hat{L}, \hat{W}) - \hat{L}^2 \hat{L}^k \hat{W}_{kij0} + 4H(\hat{L}, \hat{W})
\]

\[
= -\hat{L}^4 \nabla_0 (\hat{W})_{0ij0} - 2\hat{L}^2 \nabla^k (\hat{W})_{kij0} - 2H(\hat{L}, \hat{W}) + 2(\hat{L}^2, \hat{W}) + 2\hat{L} \hat{L}^k \hat{W}_{kij0}
\]

(for the second equality see \(12.24\) in the proof of Proposition \(12.5\)). Comparing this formula with

\[
12\mathcal{E}_3 = 6D((\hat{L}^2)_0) + 2D(\hat{W})
\]

\[
- 2\hat{L}^2 \nabla^0 (\hat{W})_{0ij0} - 4\hat{L} \nabla^k (\hat{W})_{kij0} - 4H(\hat{L}, \hat{W}) + 2|\hat{L}|^4 + 16(\hat{L}^2, \hat{W}) + 4|\hat{W}|^2 + 2|\hat{W}_0|^2
\]

(see [JO22] Theorem 1) yields

\[
12\mathcal{E}_3 = 2\mathcal{J}_5 + 6D((\hat{L}^2)_0) + 2|\hat{L}|^4 + 2D(\hat{W}) + 12(\hat{L}^2, \hat{W}) + 4|\hat{W}|^2 + 2|\hat{W}_0|^2. \quad (8.19)
\]

**Remark 8.4.** In the proof of Corollary \(4\) in Section \(3\) we need to know how \((\hat{L}, \hat{\nabla}_0 (\hat{P}))\) transforms under the conformal change from \(\hat{g}\) to \(\hat{g} = e^{2\omega} \hat{g}\). In addition to the terms which are linear in \(\omega\), this also requires determining the non-linear contributions by \(\omega\). By the conformal transformation law

\[
\hat{P} = \hat{P} - \hat{\nabla}_0 (\hat{\omega}) + d\omega \otimes d\omega - \frac{1}{2} |d\omega|^2 \hat{g},
\]

all non-linear contributions by \(\omega\) are caused by \((\hat{L}, \hat{\nabla}_0 (\hat{\nabla}(\hat{\omega})))\) and

\[
(\hat{L}, \hat{\nabla}_0 (d\omega \otimes \omega)) - \frac{1}{2} (\hat{L}, \hat{\nabla}_0((d\omega|^2 \hat{g})).
\]

But \(\omega = 0\) on \(M\) implies \(\hat{\nabla}_0 (d\omega \otimes \omega)_{ij} = 0\), and one easily sees that \(\hat{\nabla}_0 ((d\omega|^2 \hat{g}))_{ij}\) is a multiple of \(h_{ij}\). The latter term vanishes by contraction with \(\hat{L}\). It remains to determine the non-linear contributions which are caused by \(\hat{\nabla}_0 (\hat{\nabla}(\hat{\omega}))_{ij}\). But

\[
\hat{\nabla}_0 (\hat{\nabla}(\hat{\omega}))_{ij} = \hat{\nabla}_0 (\hat{\nabla}(\hat{\omega}))_{ij} - 2\hat{\nabla}(\hat{\omega})_{ij} \partial_i (\omega)
\]

\[
= \hat{\nabla}_0 (\hat{\nabla}(\hat{\omega}))_{ij} - 2\hat{L}_j (\partial_i (\omega))^2
\]

\[
= \hat{\nabla}_0 (\hat{\nabla}(\hat{\omega}))_{ij} - 2\hat{L}_j H^2
\]

using \(\omega = 0\), \(\partial_i (\omega) = -H\) and \(\hat{\nabla}_0 (\hat{\nabla}(\hat{\omega}))_{ij} = \hat{\nabla}_0 (\hat{\nabla}(\hat{\omega}))_{ij} + L_{ij} \partial_\omega (\omega) = L_{ij} \partial_\omega (\omega)\) on \(M\). Therefore, the additional contribution to \((\hat{L}, \hat{\nabla}_0 (\hat{P}))\) is the term \(2H^2 |\hat{L}|^2\).

With the above preparations, we are able to give a

**Proof of Proposition 8.7.** For the proofs of the conformal invariance of \(\mathcal{J}_1\), we combine \(8.3\) and \(8.4\) with the variation formulas

\[
(H(\hat{L}, \hat{W}))^* [\varphi] = (\hat{L}, \hat{W}) \partial_\hat{\omega} (\varphi), \quad (\hat{L}^2, \hat{P})^* [\varphi] = -(\hat{L}^2, \hat{\nabla}(\varphi)), \quad (\hat{J}|\hat{L}|^2)^* [\varphi] = -|\hat{L}|^2 \Delta (\varphi), \quad (12.21), (12.26)
\]

and

\[
(|\delta(\hat{L})|^2)^* [\varphi] = 2(n - 1)(\hat{L} \delta(\hat{L}), d\varphi), \quad (8.20)
\]

The latter relation follows from

\[
e^{4\varphi} (\delta(\hat{L}), \delta(\hat{L})) = e^{4\varphi} (\delta(\hat{L}), e^{4\varphi} \delta(\hat{L})^2) h^{ij} = (\delta(\hat{L}), (n - 1) \hat{L} d\varphi)_i (\delta(\hat{L}), (n - 1) \hat{L} d\varphi)_j h^{ij}
\]

using \(12.3\) (for \(\lambda = -1\)). These variation formulas easily imply that the conformal variations of \(\mathcal{J}_1\) and \(\mathcal{J}_2\) vanish. We omit the details. \(\square\)
As an application of the local invariants \( \mathcal{J}_1, \mathcal{J}_2 \) in general dimensions, we obtain the following decomposition of \( \mathcal{Q}_4 \).

**Theorem 8.5.** In general dimensions, the extrinsic \( Q \)-curvature \( \mathcal{Q}_4 \) admits the decomposition

\[
\mathcal{Q}_4 = Q_4 - \frac{15n^3 + 49n^3 + 36n^3 + 24n^3 + 36n^3 + 24n^3 + 32}{8(n-3)(n-2)^2(n-1)} I_1 + \frac{2(5n^2 - 14n + 9)}{(n-3)(n-2)^2} I_2 + \frac{2(n-1)^2}{(n-3)(n-2)^2} I_3 + \frac{4(3n-5)(n-1)}{(n-3)(n-2)^2} I_2
\]

\[-4 \mathcal{J}_1 + 2 \mathcal{J}_2 + (n - 4) \mathcal{E}_4 + \text{total divergence},
\]
where

\[
\mathcal{E}_4 \overset{\text{def}}{=} \frac{4}{n-3} \dot{L}^{ij} \hat{\nabla}_0 (\hat{W})_{0ij0} - \frac{2(n-1)}{(n-3)(n-2)} (P, \hat{W}) + \frac{8}{n-3} H(\hat{L}, \hat{W})
\]

\[-\frac{2(4n-7)}{(n-3)(n-2)} (\hat{L}^2, P) - \frac{n^3 + n^2 + 8n - 20}{2(n-1)(n-3)(n-6)} J|\hat{L}|^2 + \frac{2}{(n-1)^2} |\delta(\hat{L})|^2.
\]

**Remark 8.6.** In the critical dimension \( n = 4 \), we have

\[
\mathcal{E}_4 = 4 \dot{L}^{ij} \hat{\nabla}_0 (\hat{W})_{0ij0} - 3(P, \hat{W}) + 8H(\hat{L}, \hat{W}) - 9(\hat{L}^2, P) - \frac{5}{2} J|\hat{L}|^2 + \frac{2}{9} |\delta(\hat{L})|^2.
\]

A calculation shows that

\[
\left( \int_M \mathcal{E}_4 d\text{vol}_h \right)^\bullet [\varphi] = \int_M \varphi \left[ 4 \delta(\hat{L}^2) + \frac{1}{6} \Delta(|\hat{L}|^2) + 2 \delta(\hat{L}) + 3 \delta(\hat{W}) \right] d\text{vol}_h.
\]

The integrand on the right-hand side is given by the divergence part of \( \mathcal{Q}_4 \) (see (1.17)). This result also follows using general principles. Let \( n > 4 \). Combining the variation formula

\[
\left( \int_M \mathcal{Q}_4 d\text{vol}_h \right)^\bullet [\varphi] = (n - 4) \int_M \varphi \mathcal{Q}_4 d\text{vol}_h
\]

(and similarly for \( \mathcal{Q}_4 \)) with the decomposition \( \mathcal{Q}_4 = Q_4 + \mathcal{I}_1 + (n - 4) \mathcal{E}_4 \) the local conformal invariant \( \mathcal{I} \) of weight \(-4\) and a total divergence \( \delta \) yields

\[
(n - 4) \int_M \varphi \mathcal{Q}_4 d\text{vol}_h = (n - 4) \int_M \varphi \mathcal{Q}_4 d\text{vol}_h + (n - 4) \int_M \varphi \mathcal{I}_1 d\text{vol}_h + (n - 4) \left( \int_M \mathcal{E}_4 d\text{vol}_h \right)^\bullet [\varphi].
\]

We divide by \( n - 4 \) and conclude that

\[
(n - 4) \int_M \varphi \mathcal{E}_4 d\text{vol}_h + \int_M \varphi \delta d\text{vol}_h = \left( \int_M \mathcal{E}_4 d\text{vol}_h \right)^\bullet [\varphi].
\]

Now continuation in dimension implies

\[
\left( \int_M \mathcal{E}_4 d\text{vol}_h \right)^\bullet [\varphi] = \int_M \varphi \delta d\text{vol}_h
\]

for \( n = 4 \).

The following result is the special case of Proposition 8.1 in the critical dimension \( n = 4 \).

**Lemma 8.7.** In dimension \( n = 4 \), the quantities

\[
\mathcal{J}_1 \overset{\text{def}}{=} \dot{L}^{ij} \hat{\nabla}_0 (\hat{W})_{0ij0} + 2H(\hat{L}, \hat{W}) + \frac{2}{9} |\delta(\hat{L})|^2 - 2(\hat{L}^2, P) + J|\hat{L}|^2 - 2\delta(\hat{L}^2)
\]

and

\[
\mathcal{J}_2 \overset{\text{def}}{=} (\hat{L}, \hat{\nabla}_0 (\hat{P})) + H(\hat{L}, \hat{P}) - \frac{1}{2} H(\hat{L}, \hat{W}) + (\hat{L}, \text{Hess}(H))
\]

\[
+ \frac{4}{9} |\delta(\hat{L})|^2 - \bar{P}_{00} |\hat{L}|^2 + \frac{1}{2} J|\hat{L}|^2 - \frac{3}{2} H^2 |\hat{L}|^2 - \frac{1}{2} H \text{tr}(\hat{L}^3) - \delta(\hat{L}^2) + \frac{1}{4} \Delta(|\hat{L}|^2)
\]

are local conformal invariants of weight \(-4\) of the embedding \( M^4 \hookrightarrow X^5 \), i.e., it holds \( e^{4\varphi(\hat{\mathcal{J}}_i) - \mathcal{J}_i} = \mathcal{J}_i \) for all \( \varphi \in C^\infty(X) \), \( i = 1, 2 \).
The integrated invariant $J_1$ in the critical dimension $n = 4$ was discovered in [AS21] (for more details, we refer to Section 12.3).

We continue to consider the invariants $J_1$ and $J_2$ for $n = 4$. Then the relation $C = -4J_1 + 2J_2$ immediately implies

**Corollary 8.8.** $C^*[\varphi] = 0$.

This also completes the proof of Theorem 2.

We continue with the

**Proof of Theorem 3.** Corollary 2 and Lemma 7.5 show that, in the critical dimension $n = 4$, the extrinsic $Q$-curvature $Q_4$ is given by the sum of

$$2J^2 - 2|P|^2 + C + \frac{9}{2}|W|^2 + 21(\hat{L}^2, W) + \frac{33}{2} \text{tr}(\hat{L}^4) - \frac{14}{3}|\hat{L}|^4$$

and the divergence terms

$$-\Delta(J) + 2\Delta(|\hat{L}|^2) + \frac{1}{6} \Delta(|\hat{L}|^2) + 6\delta(L\delta(\hat{L})) + 3\delta(\tilde{W}) + 3\delta(L^b \tilde{W}_{0ij}) .$$

By Theorem 2, the individual terms in (8.23), except $2J^2 - 2|P|^2$ are local conformal invariants. Since $\text{Pf}_4 = J^2 - |P|^2 + \frac{1}{8}|W|^2$, this proves Theorem 3.

We finish this section with the formulation of a conjectural decomposition of the critical extrinsic $Q$-curvature in higher dimensions. Its role in more general contexts will be discussed in Section 12.3.

**Conjecture 8.9.** Let $n$ be even. Then the critical extrinsic $Q$-curvature $Q_n(g)$ is a linear combination of $Q_n(h)$, local conformal invariants of the embedding $M \hookrightarrow X$ and a total divergence. By the Deser-Schwimmer decomposition of $Q_n(h)$ (see [A12]), this is equivalent to the existence of a decomposition of $Q_n(g)$ as a linear combination of the Pfaffian of $(M, h)$, local conformal invariants of the embedding and a total divergence.

Let $n$ be odd. Then $Q_n(g)$ is a linear combination of local conformal invariants of the embedding $M \hookrightarrow X$ and a total divergence.

9. The Graham-Reichert functional

In [GR20], Graham and Reichert studied the asymptotic expansion of the volume of minimal hypersurfaces $M$ (of arbitrary codimension) in a Poincaré-Einstein background $X$. The coefficient of $\log \varepsilon$ ($\varepsilon$ being a cut-off parameter) in these expansions is a global conformal invariant. We shall refer to it as the Graham-Reichert functional.

In the codimension-one special case, the following result describes the structure of the Graham-Reichert functional $\mathcal{E}_{GR}$ of $M^4 \hookrightarrow X^5$ from the perspective of an analog of Conjecture 8.9 (see also Corollary 9.8).

**Lemma 9.1.** It holds

$$8\mathcal{E}_{GR} = \int_M (J^2 - |P|^2) dvol_h + \int_M \left( -(\hat{L}^2, P) - (P, \tilde{W}) + \frac{1}{2} |\hat{L}|^2 + \frac{1}{9} |\delta(\hat{L})|^2 + B_{00} \right) dvol_h + \int_M \left( \frac{1}{12} |\hat{L}|^4 - \frac{1}{4} \text{tr}(\hat{L}^4) \right) dvol_h - \int_M \left( \frac{1}{4} |\tilde{W}|^2 + \frac{1}{2}(\hat{L}^2, \tilde{W}) \right) dvol_h .$$

(9.1)
or, equivalently,

\[
8\mathcal{E}_{GR} = \int_M (J^2 - |P|^2) \, d\text{vol}_h
\]
\[
+ \frac{1}{2} \int_M \mathcal{J} \, d\text{vol}_h - \int_M \left( \frac{1}{2} \tilde{L}^{ij} \tilde{\nabla}_0 (\tilde{W})_{0j0} + H(\tilde{L}, \tilde{W}) + (P, \tilde{W}) - \tilde{B}_{00} \right) \, d\text{vol}_h
\]
\[
+ \int_M \left( \frac{1}{12} |\tilde{L}|^4 - \frac{1}{4} \text{tr}(\tilde{L}^4) \right) \, d\text{vol}_h - \int_M \left( \frac{1}{4} |\tilde{W}|^2 + \frac{1}{2} (\tilde{L}^2, \tilde{W}) \right) \, d\text{vol}_h. \tag{9.2}
\]

We recall that $J^2 - |P|^2 = Pf_4 - \frac{1}{4} |W|^2$.

All integrals in Lemma 9.1 are conformally invariant (see Remark 9.3).

Proof. In the codimension-one case, [GR20, Proposition 5.1] is equivalent to

\[
8\mathcal{E}_{GR} = \int_M \left( |dH|^2 - H^2 |\tilde{L}|^2 + 3H^4 \right) \, d\text{vol}_h
\]
\[
+ \int_M \left( 2h h^{ij} \tilde{\nabla}_0 (\tilde{P})_{ij} + 4(\tilde{P}_0, dH) + 5H^2 h^{ij} \tilde{P}_{ij} - 8\tilde{P}_{00} H^2 \right) \, d\text{vol}_h
\]
\[
+ \int_M \left( -\tilde{P}^{ij} \tilde{P}_{ij} + (\tilde{P}_0, \tilde{P}_0) + (h^{ij} \tilde{P}_{ij})^2 - h^{ij} \tilde{B}_{ij} \right) \, d\text{vol}_h, \tag{9.3}
\]

where $B$ is the Bach tensor of the background metric. Let $G = \overline{\text{Ric}} - 4\bar{g}$ be the Einstein tensor of $\bar{g}$ on $X^5$. Then

\[
\tilde{\nabla}_0 (\tilde{P})_{00} = \frac{1}{3} \tilde{\nabla}_0 (\overline{\text{Ric}} - \overline{J} \bar{g})_{00}
\]
\[
= \frac{1}{3} \tilde{\nabla}_0 (G)_{00} + \tilde{\nabla}_0 (J \bar{g})_{00}
\]
\[
= \frac{1}{3} (-\delta (\overline{\text{Ric}}) - 4H \overline{\text{Ric}}_{00} + (L, \overline{\text{Ric}})) + \overline{J}.
\]

using Lemma 4.4. Hence

\[
h^{ij} \tilde{\nabla}_0 (\tilde{P})_{ij} = \tilde{\nabla}_0 (\overline{J}) - \tilde{\nabla}_0 (\tilde{P})_{00}
\]
\[
= \frac{1}{3} (\delta (\overline{\text{Ric}}) + 4H \overline{\text{Ric}}_{00} - (L, \overline{\text{Ric}}))
\]
\[
= \delta (\tilde{P}_0) + 4H \tilde{P}_{00} - (L, \tilde{P}).
\]

This yields the relation

\[
\int_M H h^{ij} \tilde{\nabla}_0 (\tilde{P})_{ij} \, d\text{vol}_h = \int_M (4H^2 \tilde{P}_{00} - H(L, \tilde{P}) + H \delta (\tilde{P}_0)) \, d\text{vol}_h.
\]

Now, abbreviating the first integral in (9.3) by $(\cdot)_{\text{Guven}}$, we find

\[
8\mathcal{E}_{GR} = (\cdot)_{\text{Guven}} + \int_M \left(-2H (\dot{L}, \tilde{P}) + 3H^2 h^{ij} \tilde{P}_{ij} + 2(dH, \tilde{P}_0) \right) \, d\text{vol}_h
\]
\[
+ \int_M \left( -\tilde{P}^{ij} \tilde{P}_{ij} + (\tilde{P}_0, \tilde{P}_0) + (h^{ij} \tilde{P}_{ij})^2 - h^{ij} \tilde{B}_{ij} \right) \, d\text{vol}_h
\]

using partial integration. Next, we substitute the Fialkow equation

\[
\tilde{P} = P - H \dot{L} - \frac{1}{2} H^2 h + \frac{1}{2} \left( \dot{L}^2 - \frac{|\dot{L}|^2}{6} h + \overline{W} \right)
\]
(one may verify the relation
Remark 9.2. As a cross-check of the coefficients in the first integral in the last line of (9.2),
with
Finally, the conformal transformation law
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\[ 8E_{GR} = \int_M (J^2 - |P|^2) d\text{vol}_h + \int_M \left( \frac{1}{2} |\bar{\dot{L}}|^2 - (\bar{\dot{L}}^2, P) + \frac{1}{12} |\bar{L}|^4 - \frac{1}{4} \text{tr}(\bar{\dot{L}}^4) \right) d\text{vol}_h 
\]
\[ - \int_M \left( (P, \bar{\nabla}) + \frac{1}{2} (\bar{\dot{L}}^2, \bar{\nabla}) + \frac{1}{4} |\bar{\nabla}|^2 + h^{ij} \bar{B}_{ij} \right) d\text{vol}_h. \]

But the relation $3\bar{P}_0 + 3dH = \delta(\bar{L})$ (Codazzi-Mainardi) shows that
\[ |dH|^2 + 2(dH, \bar{P}_0) + |\bar{P}_0|^2 = \frac{1}{9} |\delta(\bar{L})|^2. \]

Thus, we finally arrive at
\[ 8E_{GR} = \int_M (J^2 - |P|^2) d\text{vol}_h + \int_M \left( \frac{1}{9} |\delta(\bar{L})|^2 + \frac{1}{2} |\bar{\dot{L}}|^2 - (\bar{\dot{L}}^2, P) + \frac{1}{12} |\bar{L}|^4 - \frac{1}{4} \text{tr}(\bar{\dot{L}}^4) \right) d\text{vol}_h 
\]
\[ - \int_M \left( (P, \bar{\nabla}) + \frac{1}{2} (\bar{\dot{L}}^2, \bar{\nabla}) + \frac{1}{4} |\bar{\nabla}|^2 + h^{ij} \bar{B}_{ij} \right) d\text{vol}_h. \]

This implies the assertion using $h^{ij} \bar{B}_{ij} = -\bar{B}_{00}$ since $\bar{B}$ is trace-free. \qed

Remark 9.3. As a cross-check of the coefficients in the first integral in the last line of (9.2),

one may verify the relation
\[ \int_M (|dH|^2 - H^2)|\bar{L}|^2 + 3H^4) d\text{vol}_h 
\]
\[ = \int_M (J^2 - |P|^2) d\text{vol}_h + \int_M \frac{1}{2} J_1 d\text{vol}_h + \int_M \left( \frac{1}{12} |\bar{L}|^4 - \frac{1}{4} \text{tr}(\bar{\dot{L}}^4) \right) d\text{vol}_h 
\]
for the flat background $\mathbb{R}^5$ using (10.1) and $\delta(\bar{L}) = 3dH$.

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First, we note that
\[ \left( \int_M \left( \frac{1}{2} \bar{\dot{L}}^{ij} \nabla_0(\bar{W})_{0ij0} + H(\bar{\dot{L}}, \bar{W}) \right) d\text{vol}_h \right)^* [\varphi] = \int_M \bar{\dot{L}}^{ij} \bar{W}_{kij0} \varphi^k d\text{vol}_h = -\int_M \delta(\bar{W}_{0ij}) \varphi d\text{vol}_h 
\]
(\text{using} \ (8.10) \ \text{and partial integration}). Second, we have
\[ \left( \int_M (P, \bar{\nabla}) d\text{vol}_h \right)^* [\varphi] = -\int_M (\text{Hess}(\varphi), \bar{W}) d\text{vol}_h = -\int_M \delta(\bar{W}) \varphi d\text{vol}_h. \]

Finally, the conformal transformation law
\[ e^{2\varphi} \bar{B}_{ij} = B_{ij} + (n - 4)(C_{ij} + C_{jk})\varphi^k + (n - 4)W_{kij} \varphi^k \varphi^l \]
with $C_{ij} = \nabla_k(P)_{ij} - \nabla_j(P)_{ik}$ implies (for $n = 5$)
\[ \left( \int_M \bar{B}_{00} d\text{vol}_h \right)^* [\varphi] = 2 \int_M \bar{C}_{000} \varphi^k d\text{vol}_h = -2 \int_M \delta(\bar{C}_{000}) \varphi d\text{vol}_h. \]
But, the general formula
\[(n - 3)C_{ijk} = \text{div}_1(W)_{ijk} = \nabla^a(W)_{aijk}\]
implies (for \(n = 5\))
\[2\tilde{C}_{00k} = \overline{\text{div}_1(W)}_{00k} = \nabla^a(W)_{a00k}.\]

Now
\[
\nabla^a(W)_{a00k} = \delta^a(\overline{W}_{a00k}) - \overline{W}(\nabla^a(\partial_a), \partial_0, \partial_0, \partial_0, \nabla^a(\partial_k)) - \overline{\nabla}(\partial_a, \nabla^a(\partial_0), \partial_0, \partial_k) - \nabla(\partial_a, \nabla^a(\partial_0), \partial_0, \nabla^a(\partial_0), \partial_k).
\]

Note that the terms for \(a = 0\) vanish. Thus, it suffices to let the summation run only over the tangential index \(a = 1, \ldots, 4\). But for such \(a\) it holds \(\nabla^a(\partial_k) = \nabla^a(\partial_0) = L_{aj}\partial_j\).

Hence
\[
\nabla^a(W)_{a00k} = \delta(\overline{W})_k - \overline{W}_{aj0k}L^{aj} - \overline{W}_{a0jk}L^{aj} = \delta(\overline{W})_k - \overline{W}_{aj0k}L^{aj} = \delta(\overline{W})_k - \overline{W}_{a0jk}L^{aj}
\]
and we find
\[2\tilde{C}_{00k} = \delta(\overline{W})_k + \overline{W}_{0ajk}L^{aj}.\]

Therefore, we get the variation formula
\[-\left(\int_M B_{00d\text{vol}_h}\right) [\varphi] = \int_M \left(\delta\delta(\overline{W}) + \delta(\overline{W}_{0ij}L^{ij})\right) \varphi d\text{vol}_h.\]

These conformal variation formulas imply that the conformal variation of (9.4) vanishes. \(\square\)

It is also worth emphasizing the conformal invariance of the second integral in the decomposition (9.1) of the functional \(\mathcal{E}_{GR}\).

**Corollary 9.4.** The integral
\[\int_M \left(\hat{L}^2, P\right) - \frac{1}{2} J|\hat{L}|^2 - \frac{1}{9} \delta(\hat{L})^2 + (P, \overline{W}) - B_{00}\right) d\text{vol}_h\]
(9.6)
is conformally invariant.

This result also follows from the following local fact.

**Lemma 9.5.** The curvature quantity
\[J_4 \overset{\text{def}}{=} -\frac{2}{9} \delta(\hat{L})^2 - 2(\hat{L}, P) + J|\hat{L}|^2 - 2(P, \overline{W}) + 2B_{00} - \delta\delta(\hat{L}^2) - \delta\delta(\overline{W})\]
(9.7)
of the embedding \(M^4 \hookrightarrow X^5\) is conformally invariant of weight \(-4\), i.e., it holds
\[e^{4\alpha(\varphi)} J_4 = J_4\]
for all \(\varphi \in C^\infty(X)\).

**Proof.** We prove that the conformal variation of \(J_4\) vanishes. We recall that
\[(\hat{L}^2, P)^{\star} [\varphi] = -\langle \hat{L}^2, \text{Hess}(\varphi) \rangle, \quad (J|\hat{L}|^2)^{\star} [\varphi] = -|\hat{L}|^2 \Delta(\varphi)\]
and
\[(\delta(\hat{L})^2)^{\star} [\varphi] = 6(\delta(\hat{L}), \hat{L} d\varphi)\]
(see (8.20)). Hence the conformal variation of the first three terms in (9.7) equals
\[2(\hat{L}^2, \text{Hess}(\varphi)) - |\hat{L}|^2 \Delta(\varphi) + \frac{4}{3} (\delta(\hat{L}), \hat{L} d\varphi)\]
\[= -2(\delta(\hat{L}), d\varphi) + (d(|\hat{L}|^2), d\varphi) + \frac{4}{3} (\hat{L} \delta(\hat{L}), d\varphi) + 2\delta(\hat{L}^2 d\varphi) - \delta(|\hat{L}|^2 d\varphi)\]
By (7.23) (for \(n = 4\)), this sum equals
\[-2(\hat{L}^2, \overline{W}_{0ij}, d\varphi) + 2\delta(\hat{L}^2 d\varphi) - \delta(|\hat{L}|^2 d\varphi).\]
We also recall that \((\delta \delta \hat{L}^2)\)^\*[\(\varphi\)] = \(2\delta (\hat{L}^2 d\varphi) - \delta (|\hat{L}|^2 d\varphi)\) (see (12.4)). Thus,
\[
(\frac{2}{9} |\delta \hat{L}|^2 - 2|\hat{L}|^2, P) + J|\hat{L}|^2 - \delta |\hat{L}|^2)\]^\*[\(\varphi\)] = -2(\hat{L}^j \overline{W}_{0j}, d\varphi) = 2\delta (\hat{L}^j \overline{W}_{0j}) \varphi - 2\delta (\hat{L}^j \overline{W}_{0j}) \varphi). \tag{9.8}
\]
Next, we calculate
\[
(\overline{B}_{00})^\*[\varphi] = 2\overline{C}_{00k} \varphi^k
= -2\delta (\overline{C}_{00}) \varphi + 2\delta (\overline{C}_{00}) \varphi
= -\delta (\overline{W}) \varphi - \delta (\overline{W}) \varphi + 2\delta (\overline{C}_{00}) \varphi) \tag{9.9}
\]
using arguments in the proof of Remark 9.3 and (9.5). Finally, we find
\[
(P, \overline{W})^\*[\varphi] = -(\text{Hess}(\varphi), \overline{W})
= (d\varphi, \delta (\overline{W})) - \delta (\overline{W} d\varphi)
= -\delta (\overline{W}) - \delta (\overline{W}) - \delta (\overline{W} d\varphi). \tag{9.10}
\]
Summarizing the results (9.8)–(9.10) shows that the conformal variation of (9.7) (up to the last term) equals
\[
4\delta (\overline{C}_{00}) \varphi - 2\delta (\hat{L}^j \overline{W}_{0j}) \varphi - 2\delta (\overline{W}) \varphi + 2\delta (\overline{W} d\varphi) = 2\delta (\overline{W} \varphi) + 2\delta (\hat{L}^j \overline{W}_{0j}) \varphi - 2\delta (\overline{W}) \varphi + 2\delta (\overline{W} d\varphi) \quad \text{(by (9.9))}
= 2\delta (\overline{W} d\varphi)
= (\delta \delta (\overline{W}))^\*[\varphi].
\]
This proves that \((\mathcal{J}_4)^\*[\varphi] = 0\). The proof is complete. \(\square\)

As a corollary of Lemma 9.5 and the conformal invariance of \(\mathcal{J}_1\), we obtain the following improvement of Remark 9.3.

**Corollary 9.6.** The curvature quantity
\[
\mathcal{J}_3 \overset{\text{def}}{=} \mathcal{J}_4 - \mathcal{J}_1 = \frac{1}{2} \hat{L}^j \overline{\nabla}_0 (\overline{W})_{0j0} + H(\hat{L}, \overline{W}) + (P, \overline{W}) - \overline{B}_{00} + \frac{1}{2} \delta \delta (\overline{W}) \tag{9.11}
\]
is conformally invariant of weight \(-4\).

Note that \(\mathcal{J}_3\) vanishes if \(\overline{W} = 0\).

Finally, we rewrite the second relation in Lemma 9.1 in terms of the Pfaffian \(\text{Pf}_4\) and the invariants \(\mathcal{I}_j\), \(\mathcal{J}_1\). In particular, this requires expressing \(|W|^2\) and \(\mathcal{J}_3\) in terms of the invariants \(\mathcal{I}_j\).

**Lemma 9.7.** In dimension \(n = 4\), it holds
\[
|W|^2 = |\overline{W}|^2 + \frac{7}{3} \mathcal{I}_1 - 4 \mathcal{I}_2 - 2 \mathcal{I}_4 + 4 \mathcal{I}_5 - 4 \mathcal{I}_6. \tag{9.12}
\]

**Proof.** In general dimensions, the Gauss equation (2.10) implies
\[
|W|^2 = |\overline{W}|^2 - \frac{1}{2} (\hat{L} \otimes \hat{L} - \mathcal{F} \otimes h|^2
= |\overline{W}|^2 + \frac{1}{4} |\hat{L} \otimes \hat{L}|^2 + |\mathcal{F} \otimes h|^2 - (\hat{L} \otimes \hat{L}, \overline{W}) - 2(\mathcal{F} \otimes h, \overline{W}) + (\hat{L} \otimes \hat{L}, \mathcal{F} \otimes h).\]
Now using
\[ |\hat{L} \odot \hat{L}|^2 = 8|\hat{L}|^4 - 8 \text{tr}(\hat{L}^4), \]
\[ |\mathcal{F} \odot h|^2 = 4(n-2)|\mathcal{F}|^2 + 4 \text{tr}(\mathcal{F})^2, \]
\[ (\hat{L} \odot \hat{L}, \hat{W}) = -4 \hat{L}^i \hat{L}^j \hat{W}_{ijkl}, \]
\[ (\mathcal{F} \odot h, \hat{W}) = 4(\mathcal{F}, \hat{W}), \]
\[ (\hat{L} \odot \hat{L}, \mathcal{F} \odot h) = -8(\mathcal{F}, \hat{L}^2). \]
We apply these results in dimension \( n \). Then
\[ |\hat{W}|^2 = 8|\hat{W}|^2 + 2|\hat{L}|^4 - 2 \text{tr}(\hat{L}^4) + 8|\mathcal{F}|^2 + 4 \text{tr}(\mathcal{F})^2 + 4I_5 - 8(\mathcal{F}, \hat{W}) = 8(\mathcal{F}, \hat{L}^2). \]
In order to make that relation explicit, we recall that
\[ 2\mathcal{F} = \hat{L}^2 - \frac{1}{6}|\hat{L}|^2 h + \hat{W}. \]
Hence
\[ \text{tr}(\mathcal{F}) = \frac{1}{6}|\hat{L}|^2, \]
\[ |\mathcal{F}|^2 = -\frac{1}{18}I_1 + \frac{1}{4}I_2 + \frac{1}{4}I_4 + \frac{1}{2}I_6, \]
\[ (\mathcal{F}, \hat{W}) = \frac{1}{2}I_4 + \frac{1}{2}I_6, \]
\[ (\mathcal{F}, \hat{L}^2) = -\frac{1}{12}I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_6. \]
Summarizing these results completes the proof.

Moreover, \([AS22]\) provides the additional relation
\[ J_3 = \frac{1}{2}I_5 + \frac{1}{2}I_6 - \frac{1}{4}I_7, \]
up to a divergence. Note that this formula implies that the integral of \( J_3 \) vanishes if \( \hat{L} = 0 \) using Lemma \([J3]\).

Now combining \((9.12)\) and \((9.13)\) with the formula
\[ 8\mathcal{E}_{GR} = \int_M \left( \text{Pf}_4 - \frac{1}{8}|\hat{W}|^2 + \frac{1}{2}J_1 - J_3 + \frac{1}{12}I_1 - \frac{1}{4}I_2 - \frac{1}{4}I_4 - \frac{1}{2}I_6 \right) \text{dvol}_h \]
(see \([GR20] 2\)) gives the following result.

**Corollary 9.8.**
\[ 8\mathcal{E}_{GR} = \int_M \left( \text{Pf}_4 + \frac{2}{3}J_1 - \frac{5}{24}I_1 + \frac{1}{4}I_2 - \frac{1}{8}I_3 - I_5 - \frac{1}{2}I_6 + \frac{1}{4}I_7 \right) \text{dvol}_h. \]

**Remark 9.9.** Corollary \([7.8]\) represents the integrand of the functional \( \mathcal{E}_{GR} \) (for a hypersurfaces \( M^4 \hookrightarrow X^5 \)) as a linear combination of the Pfaffian of \( M \), local conformal invariants of the embedding and total divergences. Similarly, the Graham-Reichert functional for a closed surface \( M^2 \hookrightarrow X^{n+1} (n \geq 2) \) is a constant multiple of \( \int_M (|H|^2 + \text{tr}_h(\overline{P})) \text{dvol}_h \) \([GR20]\ Corollary 5.3].

For \( n = 2 \), this integral equals
\[ \int_M (H^2 + J - P_{00}) \text{dvol}_h = \int_M (J + \frac{1}{2}|\hat{L}|^2) \text{dvol}_h \]
(by the Gauss identity). For \( n = 3 \), it equals \( \int_M (|H|^2 + J - P_{00} - P_{11}) \text{dvol}_h \), where \( \{\partial_0, \partial_1\} \) is an orthonormal basis of the normal space of \( M \). This conformal invariant appears in the logarithmic term of the entanglement entropy in \([SUS]\) \((1.1),(A6)\]. For general \( n \), the Gauss equation shows that
\[ n(|H|^2 + \text{tr}_h(\overline{P})) = \text{scal} + \langle \hat{L}, \hat{L} \rangle + \left( -\bar{R}_{ab} + \frac{n-2}{n-1} \text{Ric}_a + (n-2)(H,H) \right) \]
(if \( n \geq 2 \)). Here \( \{ \partial_a \} \) is an orthonormal basis of the normal space of \( M \). The integral of \( J \) is a constant multiple of the Euler characteristic of \( M \) (Gauss-Bonnet), \((\hat{L}, \hat{L}) \) and the last term in brackets are local conformal invariants of \( M^2 \hookrightarrow X^{n+1} \).

Another representation of \( \mathcal{E}_{GR} \) is given in [BGW21b, Section 4]. It is a consequence of the theory developed in [AGW21].

10. Energy functionals for a flat background

Here we take a closer look at the energy functionals

\[
\mathcal{C}_M \overset{\text{def}}{=} \int_M C \text{dvol}_h \quad \text{and} \quad (\mathcal{J}_i)_M \overset{\text{def}}{=} \int_M J_i \text{dvol}_h
\]

for a closed hypersurface \( M^4 \hookrightarrow \mathbb{R}^5 \) in a flat background. By the Gauss equation, all curvature data can be expressed in terms of \( L \). As a consequence of the conformal invariance of \( C \), we find

**Lemma 10.1.** Assume that \( M^4 \hookrightarrow \mathbb{R}^5 \). Then

\[
\mathcal{I}_M \overset{\text{def}}{=} \int_M \left( |dH|^2 - H \text{tr}(\hat{L}^3) + \frac{1}{2} H^2 |\hat{L}|^2 \right) \text{dvol}_h
\]

is Möbius invariant, i.e., it holds \( \mathcal{I}_{\gamma(M)} = \mathcal{I}_M \) for all Möbius transformations \( \gamma \) of \( \mathbb{R}^5 \).

**Proof.** Let \( X = \mathbb{R}^5 \). We evaluate the integral

\[
\int_M C \text{dvol}_h = \int_M \left( 2(\hat{L}, \text{Hess}(H)) + 2H(\hat{L}, P) + 8(\hat{L}^2, P) - 3|\hat{L}|^2 - 3H^2|\hat{L}|^2 - H \text{tr}(\hat{L}^3) \right) \text{dvol}_h.
\]

Partial integration and \( \delta(\hat{L}) = 3dH \) (Codazzi-Mainardi) show that

\[
\int_M (\hat{L}, \text{Hess}(H)) \text{dvol}_h = -\int_M (\delta(\hat{L}), dH) \text{dvol}_h = -\int_M 3(dH, dH) \text{dvol}_h.
\]

The Gauss equation and the Fialkow equation imply

\[
J = 2H^2 - \frac{1}{6} |\hat{L}|^2 \quad \text{and} \quad P = -\frac{1}{2} \hat{L}^2 + \frac{1}{12} |\hat{L}|^2 h + H \hat{L} + \frac{1}{2} H^2 h \tag{10.1}
\]

and \( 2H(\hat{L}, P) = -H \text{tr}(\hat{L}^3) + 2H^2 |\hat{L}|^2 \). Hence

\[
3|\hat{L}|^2 J = 6H^2 |\hat{L}|^2 - \frac{1}{2} |\hat{L}|^4 \quad \text{and} \quad 8(\hat{L}^2, P) = -4 \text{tr}(\hat{L}^4) + 8H \text{tr}(\hat{L}^3) + 4H^2 |\hat{L}|^2 + \frac{2}{3} |\hat{L}|^4.
\]

These results imply

\[
\mathcal{C}_M = \int_M C \text{dvol}_h = \int_M \left( -6|dH|^2 + 6H \text{tr}(\hat{L}^3) - 3H^2 |\hat{L}|^2 + \frac{7}{6} |\hat{L}|^4 - 4 \text{tr}(\hat{L}^4) \right) \text{dvol}_h \tag{10.2}
\]

Now the Möbius invariance of \( \mathcal{C}_M \) implies the assertion. \( \square \)

**Lemma 10.2.** For an embedding \( M^4 \hookrightarrow \mathbb{R}^5 \), it holds

\[
\int_M \mathcal{J}_1 \text{dvol}_h = \int_M \left( 2|dH|^2 - 2H \text{tr}(\hat{L}^3) + H^2 |\hat{L}|^2 - \frac{1}{3} |\hat{L}|^4 + \text{tr}(\hat{L}^4) \right) \text{dvol}_h
\]

and

\[
\int_M \mathcal{J}_2 \text{dvol}_h = \int_M \left( |dH|^2 - H \text{tr}(\hat{L}^3) + \frac{1}{2} H^2 |\hat{L}|^2 - \frac{1}{12} |\hat{L}|^4 \right) \text{dvol}_h.
\]

\(^{14}\) Even for \( \hat{L} = 0 \), this formulas differs from (9.2). In view of \( \text{Win} = 0 \) and \( \mathcal{I}_7 = 0 \) (Lemma 7.6), the formula in [BGW21b] is equivalent to \( 8\mathcal{E}_{GR} = \int_M J^2 - |P|^2 + \frac{1}{4} \mathcal{I}_4 \). On the other hand, (10.2) simplifies to \( 8\mathcal{E}_{GR} = \int_M J^2 - |P|^2 - \frac{1}{4} \mathcal{I}_4 \).
In particular, we find
\[ 2 \int_M J_2 \, d\text{vol}_h - \int_M J_1 \, d\text{vol}_h = \int_M \left( \frac{1}{6} \bar{L}^4 - \text{tr} (\bar{L}^4) \right) \, d\text{vol}_h. \] (10.3)

The relation (10.3) reflects properties of the invariants \( J_i \) for general backgrounds. In fact, this relation follows by combining Proposition 12.5 with Remark 12.4 and \( I_5 = \frac{1}{6} I_1 - 2 I_2 \) (see (12.22)).

Another calculation using (10.1) shows that
\[ J^2 - |P|^2 = 3 H^4 + H \text{tr}(\bar{L}^3) - \frac{3}{2} H^2 |\bar{L}|^2 + \frac{1}{12} |\bar{L}|^4 - \frac{1}{4} \text{tr}(\bar{L}^4). \] (10.4)

Hence the conformal invariance of the integral \( \int_M (J^2 - |P|^2) \, d\text{vol}_h \) implies that the integral
\[ \int_M \left( 3 H^4 + H \text{tr}(\bar{L}^3) - \frac{3}{2} H^2 |\bar{L}|^2 \right) \, d\text{vol}_h \]
is Möbius invariant. Combining this with Lemma 10.1 shows that the energy functional
\[ \mathcal{E}_G \overset{\text{def}}{=} \frac{1}{4} \int_M \left( |dH|^2 + 3 H^4 - H^2 |\bar{L}|^2 \right) \, d\text{vol}_h \] (10.5)
is Möbius invariant. This result is related to \([G05, (61)]\). Graham and Reichert \([GR20]\) proved that the functional \( \mathcal{E}_G \) is a special case of a global conformal invariant
\[ \mathcal{E}_{GR} = \frac{1}{4} \int_M (|dH|^2 + \cdots) \, d\text{vol}_h \]
of \( M \hookrightarrow X \) which appears in the asymptotic expansion of the renormalized volume of a minimal hypersurface with boundary \( M \hookrightarrow X \) in a Poincaré-Einstein background with conformal infinity \((X, [g])\) (see Section 9). Graham and Reichert noticed that the original calculation in \([G05]\) dropped a factor of \(-2\). The corrected result \([GR20, (1.1)]\) for \( n = 5 \) states the Möbius invariance of (10.3) \( ^{15} \). For more results on the functional \( \mathcal{E} \), we refer to \([GR20]\).

The results in \([GR20]\) suggest regarding the functional \( \mathcal{E}_G \) as a natural analog of the Willmore functional.

Combining (10.2), (10.4) and (10.5), we find the relation
\[ C_M = -24 \mathcal{E}_G + 6 \int_M (J^2 - |P|^2) \, d\text{vol}_h + \int_M \left( \frac{2}{3} |\bar{L}|^4 - \frac{5}{2} \text{tr}(\bar{L}^4) \right) \, d\text{vol}_h \] (10.6)
for \( M^4 \hookrightarrow \mathbb{R}^5 \). In particular, this again shows the Möbius invariance of \( \mathcal{E}_G \). Together with the Chern-Gauss-Bonnet formula (12.5), we obtain the relation
\[ C_M = 24 \pi^2 \chi(M) - 24 \mathcal{E}_G - \frac{3}{4} \int_M |W|^2 \, d\text{vol}_h + \int_M \left( \frac{2}{3} |\bar{L}|^4 - \frac{5}{2} \text{tr}(\bar{L}^4) \right) \, d\text{vol}_h. \]
The difference \( 24 \pi^2 \chi(M) - 24 \mathcal{E}_G \) can be expressed in terms of the universal invariant \( W_m \) (introduced in \([BCW21b]\)); the proof of the following formula is given in Section 12.4.

**Lemma 10.3.** \( 24 \pi^2 \chi(M) - 24 \mathcal{E}_G = \int_M (W_m + \frac{11}{6} |\bar{L}|^4 - \frac{5}{2} \text{tr}(\bar{L}^4)) \, d\text{vol}_h. \)

Finally, we observe that the relation (10.9) generalizes to embeddings \( M^4 \hookrightarrow S^5 \) in the round sphere \( S^5 \) if
\[ \mathcal{E}_G \overset{\text{def}}{=} \frac{1}{4} \int_M \left( |dH|^2 + 3 H^4 - H^2 |\bar{L}|^2 + 6 H^2 + 3 \right) \, d\text{vol}_h. \] (10.7)

By \([GR20, (1.2)]\), this energy functional again is a special case of the Graham-Reichert energy functional \( \mathcal{E}_{GR} \). In the present case, it holds \( \bar{P} = \frac{1}{2} \bar{g} \) (with \( \bar{g} \) being the round metric on \( S^5 \)) and \( \bar{J} = \frac{5}{2} \). Then \( \bar{P}_{00} = \frac{1}{2} \). The Gauss identity shows that \( J = J_{\text{lat}} + 2 \), where \( J_{\text{lat}} = 2 H^2 - \frac{1}{3} |\bar{L}|^2 \) is

\[^{15}\text{Note that } (GR20) \text{ uses a different normalization of } H.\]
These results imply (10.6) for flat background. Now we calculate of the energy (10.7).

\[ \int_M C\,d\text{vol}_h = \int_M \left( -6|dH|^2 + 2H(L,P) + 8(L^2, P) - |L|^2 - 3J|L|^2 - 3H^2|L|^2 - H\,\text{tr}(L^3) \right)\,d\text{vol}_h \]

where \( C_{\text{lat}} \) is given by the same formula as in the case of a flat background. Similarly, the Fialkow equation shows that \( \mathcal{P} \) be the expansion of the solution of the Yamabe energy \([G17]\).

Note that Definition 11.1 implies that the expansion of \( v(r) = d\text{vol}_{h_r}/d\text{vol}_h = \sum_{k \geq 0} r^k v_k \) defines the volume coefficients \( v_k \in C^\infty(M) \) (see (4.1)).

We first recall the definition of the coefficient \( V_n \) for a hypersurface \( M^n \hookrightarrow X^{n+1} \). In normal geodesic coordinates, the metric \( g \) takes the form \( g = dr^2 + h_r \) with a one-parameter family \( h_r \). Then the expansion

\[ v(r) = d\text{vol}_{h_r}/d\text{vol}_h = \sum_{k \geq 0} r^k v_k \]

defines the volume coefficients \( v_k \in C^\infty(M) \) (see (4.1)).

We recall from Section 3 that \( \sigma \) solves the singular Yamabe problem for the hypersurface \( M^n \hookrightarrow X^{n+1} \) with the background metric \( g \) if the scalar curvature of \( \sigma^{-2}g \) equals \(-n(n+1)\).

**Definition 11.1.** Let

\[ \sigma(r) = \sum_{k \geq 1} \sigma^{(k)} r^k = r + r^2 \sigma^{(2)} + r^3 \sigma^{(3)} + \cdots \]

be the expansion of the solution \( \sigma(r) \) of the singular Yamabe problem of \( M^n \hookrightarrow X^{n+1} \) in normal geodesic coordinates. Let \( V_n \) be the coefficient of \( r^n \) in the expansion of the function

\[ (1 + r\sigma^{(2)} + r^2 \sigma^{(3)} + \cdots)^{-(n+1)} v(r) \]

Note that Definition 11.1 implies that the expansion of

\[ d\text{vol}_{\sigma^{-2}g} = r^{-(n+1)}(1 + \sigma^{(2)} r + \sigma^{(3)} r^2 + \cdots)^{-(n+1)} v(r)\,dr\,d\text{vol}_h \]

involves a term \( V_n r^{-1} dr \,d\text{vol}_h \). By integration, this shows that the total integral of \( V_n \) defines the coefficient of \( \log(\varepsilon) \) in the expansion of the volume. The integral \( \int_M V_n d\text{vol}_h \) is the singular Yamabe energy \([G17]\).
The following result describes the coefficients $V_k$ for $k \leq 4$ in terms of $\sigma(k)$ and $v_k$ for $k \leq 4$ in the respective critical dimensions. It directly follows from the definition.

**Lemma 11.2.** In the respective critical dimensions, it holds

\[
V_2 = 6\sigma_2^2 - 3\sigma_3 - 3\sigma_2 v_1 + v_2,
\]

\[
V_3 = -20\sigma_2^3 + 20\sigma_2\sigma_3 - 4\sigma_4 + 10\sigma_2^2 v_1 - 4\sigma_3 v_1 - 4\sigma_2 v_2 + v_3
\]

and

\[
V_4 = 70\sigma_2^4 - 105\sigma_2^2\sigma_3 + 15\sigma_3^2 + 30\sigma_2\sigma_4 - 5\sigma_5
\]

\[
- 35\sigma_2^3 v_1 + 30\sigma_2\sigma_3 v_1 - 5\sigma_4 v_1 + 15\sigma_2^2 v_2 - 5\sigma_3 v_2 - 5\sigma_2 v_3 + v_4.
\]

In order to determine the coefficients $V_k$ for $k \leq 4$, we need explicit formulas for the coefficients $\sigma(k)$ for $k \leq 5$ and $v_k$ for $k \leq 4$. We first display such formulas for the coefficients $\sigma(k)$.

**Lemma 11.3 ([JO21]).** In general dimensions, it holds

\[
\sigma_2 = \frac{1}{2n}v_1,
\]

\[
\sigma_3 = \frac{2}{3(n-1)}v_2 - \frac{1}{3n}v_1^2 + \frac{1}{3(n-1)}J.
\]

and

\[
\sigma_4 = \frac{3}{4(n-2)}v_3 - \frac{9n^2 - 20n + 7}{12n(n-1)(n-2)}v_1v_2 + \frac{6n^2 - 11n + 1}{24n^2(n-2)}v_1^3
\]

\[
+ \frac{2n-1}{6n(n-1)(n-2)}v_1J + \frac{1}{4(n-2)}J' + \frac{1}{8n(n-2)}\Delta(v_1).
\]

Note that $3\sigma_3 = 2v_2 + \cdots$ for $n = 2$ and $4\sigma_4 = 3v_3 + \cdots$ for $n = 3$.

Note also that $\sigma_4$ has a simple pole at $n = 2$ with $\text{Res}_{n=2}(\sigma_4) \sim B_2$.

In connection with the discussion of $V_4$, we shall apply the following consequences for $n = 4$.

**Corollary 11.4.** In the critical dimension $n = 4$, it holds

\[
\sigma_2 = \frac{1}{8}v_1,
\]

\[
\sigma_3 = -\frac{1}{12}v_1^2 + \frac{2}{9}v_2 + \frac{1}{9}J
\]

and

\[
\sigma_4 = \frac{53}{768}v_1^3 - \frac{71}{288}v_1v_2 + \frac{3}{8}v_3 + \frac{7}{144}Jv_1 + \frac{1}{8}J' + \frac{1}{8}\Delta(\sigma_2).
\]

Finally, we need the following formula for $\sigma_5$.
Lemma 11.5 ([JO21]). In general dimensions, it holds
\[
\sigma(5) = -\frac{n+1}{10(n-3)} |d\sigma(2)|^2 \\
+ \frac{1}{5(n-3)} \Delta'(\sigma(2)) + \frac{1}{5(n-3)} \Delta(\sigma(3)) + \frac{3n-1}{20(n-3)(n-2)n} \Delta(\sigma(2)) v_1 \\
+ \frac{1}{10(n-3)} J'' + \frac{n-1}{4(n-3)(n-2)n} J' v_1 + \frac{2(3n-5)}{15(n-3)(n-1)^2} J v_2 \\
- \frac{4n-3}{20(n-2)(n-1)n} J v_1^2 + \frac{1}{30(n-1)^2} J^2 \\
+ \frac{48n^4 - 247n^3 + 387n^2 - 179n + 3}{60(n-3)(n-2)(n-1)n^2} v_1^2 v_2^2 - \frac{2(3n^2 - 11n + 10)}{15(n-3)(n-1)^2} v_2^2 \\
- \frac{24n^4 - 110n^3 + 133n^2 - 24n - 3}{120(n-3)(n-2)n^3} v_1^4 - \frac{16n^2 - 53n + 27}{20(n-3)(n-2)n} v_1 v_3 + \frac{4}{5(n-3)} v_4.
\]

Note that \(\sigma(5)\) has a simple pole at \(n = 3\) with \(\text{Res}_{n=3}(\sigma(5)) \sim B_3\).

In particular, we obtain

Corollary 11.6. In the critical dimension \(n = 4\), it holds
\[
24\sigma(5) = -\frac{1133}{640} v_1^4 - \frac{213}{20} v_1 v_3 + \frac{653}{80} v_1^2 v_2 + \frac{224}{45} v_2 + \frac{96}{5} v_4 \\
- \frac{13}{20} J v_1^2 + \frac{112}{45} J v_2 + \frac{9}{4} J' v_1 + \frac{4}{45} J^2 + \frac{12}{5} J'' \\
+ \frac{24}{5} \Delta(\sigma(3)) + \frac{33}{20} v_1 \Delta(\sigma(2)) + \frac{24}{5} \Delta'(\sigma(2)) - 12|d\sigma(2)|^2.
\]

Note that \(5\sigma(5) = 4v_4 + \cdots\) for \(n = 4\).

Next, formulas for the volume coefficients \(v_k\) for \(k \leq 4\) in terms of the curvature of the background metric \(g\) were displayed in Lemma 4.3 and Lemma 5.1.

These results imply

Corollary 11.7. In general dimensions, it holds \(2\sigma(2) = H\) and
\[
3(n-1)\sigma(3) = -\text{Ric}_{00} - |\bar{\nabla}|^2 + J = -(n-1)\bar{P}_{00} - |\bar{L}|^2.
\]

Hence
\[
3\sigma(3) = J - \bar{J} - \frac{1}{2(n-1)} |\bar{L}|^2 - \frac{n}{2} H^2.
\]

Moreover, we have
\[
24\sigma(4) = -3\nabla_0\text{Ric}_{00} + J' + 6(\bar{\nabla}, \bar{\nabla}) + 3\Delta(H) + 6 \text{tr}(\bar{L}^3) + 13H|\bar{L}|^2 - 7H\text{Ric}_{00} + 10H \bar{J}
\]
for \(n = 3\).

Proof. The formula for \(\sigma(2)\) is obvious from Lemma 4.3 and Lemma 11.3. Similarly, these results imply
\[
3(n-1)\sigma(3) = (-\text{Ric}_{00} - |\bar{L}|^2 + n(n-1)H^2) - n(n-1)H^2 + J.
\]

This identity simplifies the second claim. Finally, a direct calculation yields the third formula. \(\square\)

The formulas in Corollary 11.7 are equivalent to [CG19] (2.16), (2.17), (2.19).

Next, we use the above results to find explicit formulas for the coefficients \(V_k\) \((k \leq 4)\). First of all, combining Lemma 11.2 with Lemma 4.3 and Corollary 11.7 gives
Lemma 11.8. Let $n = 2$. Then

$$2V_2 = 12\sigma_2^2 - 6\sigma_3 - 6\sigma_2 v_1 + 2v_2$$

$$= -2v_2 + \frac{1}{4}v_1^2 - 2\bar{J}$$

$$= \text{Ric}_{00} + |\bar{L}|^2 - H^2 - 2\bar{J}.$$

Corollary 11.9. Let $n = 2$. Then

$$V_2 = \frac{1}{4}|\bar{L}|^2 - \frac{1}{2}\bar{J}.$$

Hence

$$\int_M V_2 d\text{vol}_h = \frac{1}{4} \int_M |\bar{L}|^2 d\text{vol}_h - \pi \chi(M).$$

Proof. By the Gauss equation for $\bar{J}$, we have

$$\frac{1}{4}|\bar{L}|^2 - \frac{1}{2}\bar{J} = \frac{1}{4}|\bar{L}|^2 - \frac{1}{2}(\bar{J} - \bar{P}_{00} - \frac{1}{2}|\bar{L}|^2 + H^2).$$

But $\bar{P}_{00} = \text{Ric}_{00} - \bar{J}$. This implies the first assertion. The second claim follows from the Gauss-Bonnet formula. □

In particular, the total integral of $V_2$ is conformally invariant.

Similarly, Lemma 11.2 implies

Lemma 11.10. Let $n = 3$. Then

$$V_3 = -20\sigma_3^2 + 20\sigma_2 \sigma_3 - 4\sigma_4 + 10\sigma_2^2 v_1 - 4\sigma_3 v_1 - 4\sigma_2 v_2 + v_3.$$

Evaluation of this formula using the above results yields

Lemma 11.11. Let $n = 3$. Then

$$6V_3 = -\frac{8}{9}v_1^3 + 4v_1 v_2 - 12v_3 - 4v_1 \bar{J} - 6\bar{J} - \Delta(v_1)$$

$$= 2\nabla_0(\text{Ric}_{00}) - 6\bar{J} - 4(\bar{L}, \bar{G}) + 8H\text{Ric}_{00} - \Delta(v_1) - 12H \bar{J} - 4\text{tr}(\bar{L}^2).$$

(11.2)

Next, we simplify (11.2). The second Bianchi identity enables us to remove the normal derivatives in (11.2). Let $\bar{G} = \text{Ric} - \frac{4}{9}\text{scal}\bar{g}$ be the Einstein tensor of the background metric $\bar{g}$ on $X$. The argument rests on the identity

$$\nabla_0(\text{Ric}_{00}) - 3\bar{J} = \nabla_0(\bar{G})_{00} - \frac{1}{2}\delta(\text{Ric}_{00}) - 3H\text{Ric}_{00} + (L, \text{Ric})$$

(11.3)

(see Lemma 4.4).

Corollary 11.12. Let $n = 3$. Then it holds

$$6V_3 = -2\delta(\bar{L}) - 4(\bar{L}, \mathcal{F}) + \Delta(H).$$

Thus, $V_3$ is a sum of a linear combination of the local conformal invariant $(\bar{L}, \mathcal{F})$ and some divergence terms. In particular,

$$\int_M V_3 d\text{vol}_h = -\frac{2}{3} \int_M (\bar{L}, \mathcal{F}) d\text{vol}_h.$$

Corollary 11.12 implies the conformal invariance of the total integral of $V_3$ using that of the Fialkow tensor $\mathcal{F}$. 
Proposition 11.15. Lemma 4.3 and Lemma 5.1 gives

Therefore, we obtain

On the other hand, (2.5) gives

By Lemma 11.14.

Let Lemma 11.13.

First, we again use the second Bianchi identity to remove the second-order normal derivatives

We continue with a simplification of the formula in Proposition 11.15.

Finally, the evaluation of the latter result using the formulas for the volume coefficients

Now we turn to the discussion of \( V_4 \) in dimension \( n = 4 \). Lemma 11.2 gives

Lemma 11.13. Let \( n = 4 \). Then

The evaluation of this formula using Corollary 11.4 and Corollary 11.6 yields

Lemma 11.14. Let \( n = 4 \). Then

Finally, the evaluation of the latter result using the formulas for the volume coefficients \( v_k \) in Lemma 11.3 and Lemma 5.1 gives

Proposition 11.15. Let \( n = 4 \). Then

First, we again use the second Bianchi identity to remove the second-order normal derivatives of the Ricci tensor. Lemma 5.2 implies that

equals the sum of

\[
-15 H \nabla_0 (\check{\text{Ric}})_{00} + 24 H \check{J}' + 24 H \nabla_0 (\check{\text{Ric}})_{00} - 60 H \check{J}' = 9 H \nabla_0 (\check{\text{Ric}})_{00} - 36 H \check{J}' = 9 H \nabla_0 (\check{\text{Ric}})_{00} - 36 H \check{J}' = -9 H \delta(\check{\text{Ric}}) - 36 H \check{J}' + 9 H (L, \check{\text{Ric}})
\]
and
\[
6(\mathring{L}, \nabla (\overline{\text{Ric}}_0)) - 3\delta(\nabla_0(\overline{\text{Ric}})_0) + 3(\mathring{L}, \nabla_0(\overline{\text{Ric}})) + 3H\delta(\overline{\text{Ric}}_0) - 9(dH, \overline{\text{Ric}}_0) + 6(\delta(\mathring{L}), \overline{\text{Ric}}_0) - 3\delta(\mathring{L}(\overline{\text{Ric}})_0) + 3L^2\overline{\text{Ric}}_{00} - 3(L^2, \overline{\text{Ric}}) + 3(\overline{\text{Ric}}_{00})^2 - 3(\mathring{G}, \overline{\text{Ric}}).
\] (11.5)

Thus, we have proved

**Proposition 11.16.** 24\(\mathcal{V}_4\) equals the sum of (11.4), (11.5),
\[
-6\mathring{L}^i\nabla_0(R)_{0ij0},
\]

and
\[
60|d\sigma(2)|^2 - 48H\Delta(\sigma(2)) - 24\Delta'(\sigma(2)) - 24\Delta(\sigma(3))
\] (11.6)

and
\[
6|\mathring{G}|^2 - 5(\overline{\text{Ric}}_{00})^2 + 4\overline{\text{Ric}}_{00}\mathring{J} + 4J^2 + 24(\mathring{L}^2, \mathring{G}) - 10|\mathring{L}|^2\overline{\text{Ric}}_{00} + 4|\mathring{L}|^2J - 12H(\mathring{L}, \mathring{G}) + 27H^2\overline{\text{Ric}}_{00} - 36H^2J + 18\text{tr}(\mathring{L}^4) + 12H\text{tr}(\mathring{L}^3) - 9H^2|\mathring{L}|^2 - 5|\mathring{L}|^4 + 9H^4.
\]

Since we are only interested in the total integral of \(\mathcal{V}_4\), we may ignore the total divergences in Proposition 11.16. These are the terms \(\delta(\nabla_0(\overline{\text{Ric}})_0), \delta(\mathring{L}(\overline{\text{Ric}})_0)\) in (11.4) and \(\Delta(\sigma(3))\) in (11.6). Furthermore, partial integration shows that
\[
\int_M -9H\delta(\overline{\text{Ric}}_0) - 9(dH, \overline{\text{Ric}}_0) = 0
\]
and
\[
\int_M 6(\mathring{L}, \nabla (\overline{\text{Ric}}_0)) + 6(\delta(\mathring{L}), \overline{\text{Ric}}_0) = 0.
\]

Therefore, we may omit these four terms in (11.4) and (11.5).

Next, we evaluate the terms in (11.6). We recall that we omit the term \(\Delta(\sigma(3))\). The variation formula
\[
\Delta'(u) = -2(L, \text{Hess}(u)) - 2(\delta(L), du) + (d \text{tr}(L), du)
\]
implies
\[
\Delta'(u) = -2(L, \text{Hess}(u)) - 4(dH, du) - 2(\overline{\text{Ric}}_0, du)
\]
using \(\delta(L) = 4dH + 3\overline{\text{P}}_0\) (Codazzi-Mainardi). Hence
\[
60|d\sigma(2)|^2 - 48H\Delta(\sigma(2)) - 24\Delta'(\sigma(2)) = 15|dH|^2 - 24H\Delta(H) + 24((L, \text{Hess}(H)) + 2|dH|^2 + (\overline{\text{Ric}}_0, dH))
\]

By partial integration, the integral of this sum equals
\[
\int_M (87|dH|^2 - 24(\delta(L), dH) + 24(\overline{\text{Ric}}_0, dH))dvol_h = -9\int_M |dH|^2dvol_h
\]
again using \(\delta(L) = 4dH + 3\overline{\text{P}}_0\). On the other hand, the integrals of (11.4) and (11.5) contribute
\[
-3\int_M (dH, \overline{\text{Ric}}_0)dvol_h.
\]
Together with the above terms, this gives
\[
\int_M -3(dH, \overline{\text{Ric}}_0) - 9|dH|^2dvol_h = -3\int_M (dH, \delta(\mathring{L}))dvol_h = 3\int_M (\mathring{L}, \text{Hess}(H))dvol_h.
\]

Now simplification of the remaining terms proves
Theorem 11.17. Let $n = 4$. Then
\[
8 \int_M V_4 \text{dvol}_h = \int_M \left( J^2 - |P|^2 + \frac{9}{4} |W|^2 \right) \text{dvol}_h \\
+ \int_M \left( \langle \hat{L}, \nabla_0(\bar{P}) \rangle - 2 \hat{L}^2 \hat{\nabla}_0(\bar{W})_{0ij0} + \langle \hat{L}, \text{Hess}(H) \rangle + H(\hat{L}, P) - \frac{9}{2} H(\hat{L}, \bar{W}) \right) \text{dvol}_h \\
+ \int_M \left( 4(\hat{L}^2, P) - |\hat{L}|^2 \hat{P}_{00} - \frac{3}{2} |\hat{L}|^2 - \frac{3}{2} H^2 |\hat{L}|^2 - \frac{1}{2} H \text{tr}(\hat{L}^3) \right) \text{dvol}_h \\
+ \int_M \left( \frac{21}{2} (\hat{L}^2, \bar{W}) + \frac{33}{4} \text{tr}(\hat{L}^4) - \frac{7}{3} |\hat{L}|^4 \right) \text{dvol}_h.
\]

Proof. We first verify the first two terms in the second line. The terms in (11.5) and (5.8) yield the contributions
\[
- 2 \hat{L}^2 \hat{\nabla}_0(\bar{R})_{0ij0} + (\hat{L}, \nabla_0(\bar{\text{Ric}})) \\
= -2 \hat{L}^2 \hat{\nabla}_0(\bar{W})_{0ij0} - \frac{2}{3} (\hat{L}, \nabla_0(\bar{\text{Ric}})) + (\hat{L}, \nabla_0(\bar{\text{Ric}})) \\
= -2 \hat{L}^2 \hat{\nabla}_0(\bar{W})_{0ij0} + (\hat{L}, \nabla_0(\bar{P}))
\]
using the relation
\[
3 \hat{L}^2 \hat{\nabla}_0(\bar{R})_{0ij0} = 3 \hat{L}^2 \hat{\nabla}_0(\bar{W})_{0ij0} + (\hat{L}, \nabla_0(\bar{\text{Ric}})).
\]
The remaining terms follow by direct calculation using the following identities. First, we note that $8\bar{J} = \text{scal}$ and $\bar{\text{Ric}} = 3 \bar{P} + \bar{J} h$. Now it holds
\[
\bar{J} = J + \bar{P}_{00} + \frac{1}{6} |\hat{L}|^2 - 2 H^2
\]
(Gauss equation) and
\[
\bar{P} = P - H \hat{L} - \frac{1}{2} H^2 h + \frac{1}{2} \left( \hat{L}^2 - \frac{1}{6} |\hat{L}|^2 h + \bar{W} \right)
\]
(Fialkow equation). Finally, $\bar{\text{Ric}}_{00} = 3 \bar{P}_{00} + J$ and $\hat{G} = \bar{P} + \bar{P}_{00} h + \bar{W}$. Using these identities, we express all terms in terms of $J, P, \bar{P}_{00}, \bar{W}$ and $H, \hat{L}$. We omit the details. □

Comparing Corollary 2 with Theorem 11.17 confirms the relation (11.1). Alternatively, Theorem 11.17 and the relation (11.1) confirm Corollary 2.

12. Appendix

This appendix contains the following additional issues.

- A direct check of the conformal covariance of the operator $P_4$ in Theorem 1.
- A brief discussion of Deser-Schwimmer type decompositions of conformal invariants of hypersurfaces $M^4 \hookrightarrow X^5$ in geometric analysis and physics.
- A proof of a decomposition of the local conformal invariant $W_m$ (introduced in [BGW21b]) in terms of basic local conformal invariants (Proposition 12.2).
- A proof of the relation (12.5).
- A proof of the equivalence of the formula for $Q_4$ in Theorem 1 to a formula in [BGW21b] (at least up to terms which are quartic in $L$).
12.1. The conformal covariance of $P_4$. The formulation of Theorem 1 contains the claim that the displayed formula defines a conformally covariant operator. Here we verify this fact by a direct calculation extending the arguments at the end of Section 8.

First, we recall the conformal transformation laws

$$e^{(\lambda+2)\varphi}\delta(e^{-\lambda\varphi}\omega) = \delta(\omega) + (n-2-\lambda)(d\varphi,\omega)$$

(12.1)

for $\omega \in \Omega^1(M)$ and

$$e^{(\lambda+2)\varphi}\delta(e^{-\lambda\varphi}f) = \Delta(f) - \lambda(\nabla \delta(f)) + (n-2-\lambda)(d\varphi,\omega) = \lambda(n-2-\lambda)|d\varphi|^2 f$$

(12.2)

for $f \in C^\infty(M)$. Moreover, it holds

$$e^{(\lambda+2)\varphi}\delta(e^{-\lambda\varphi}b) = \delta(\omega) + (n-2-\lambda)\iota_{\text{grad}(\varphi)}(b) - \text{tr}(b)d\varphi = \delta(\omega) + (n-2-\lambda)bd\varphi - \text{tr}(b)d\varphi$$

(12.3)

for symmetric bilinear forms $b$ and $\lambda \in \mathbb{R}$. Here we use the same symbol for a bilinear form and the corresponding endomorphism on $\Omega^1(M)$.

Now, by the discussion at the end of Section 8, it suffices to prove that the operator

$$f \mapsto \delta \left( \frac{1}{2} \frac{n^2-12n+16}{(n-3)(n-2)} \tilde{\Delta}^2 f \right)$$

is conformally covariant. We denote this operator by $R_4 = r_4 + c_4$ (with $c_4$ denoting its zeroth-order term) and prove that

$$e^{(\frac{n}{2}+2)\varphi}R_4(e^{2\varphi}g)(f) = R_4(g)(e^{(\frac{n}{2}-2)\varphi}f)$$

for all $f \in C^\infty(M)$, $\varphi \in C^\infty(X)$ and all $g$. We recall that in these formulas we suppress the pull-back operator $\iota^*$. It suffices to prove that the conformal variation operator

$$f \mapsto (d/dt)|_0 \left( e^{(\frac{n}{2}+2)\frac{\varphi}{\tau}}R_4(e^{2\varphi}g)(e^{-(\frac{n}{2}-2)\varphi}f) \right)$$

vanishes for all $\varphi \in C^\infty(X)$ and all $g$. The latter operator is the sum of the conformal variation operator of the second-order operator $r_4$ and the conformal variation

$$(c_4(g))*[\varphi] = (d/dt)|_0(e^{4\varphi}c_4(e^{2\varphi}g))$$

of the zeroth-order term. Only the conformal variation of $c_4$ contains normal derivatives of $\varphi$. Lemma 8.2 implies that these terms are given by

$$-2|\tilde{\Delta}^2 f| - 2(\tilde{\Delta}^2 f)\partial_0(\varphi) + 2\frac{n^2-12n+16}{n-2} \text{tr}(\tilde{\Delta}^2 f)\partial_0(\varphi) + 6H|\tilde{\Delta} f|\partial_0(\varphi) + 2\frac{n^3-4n^2+2n-2}{n-2} (\tilde{\Delta} f, \hat{\nabla})\partial_0(\varphi)$$

$$+ \frac{8}{n-1}(\tilde{\Delta} f)\partial_0(\varphi) - \frac{2(n-1)^2}{n-3}(\tilde{\Delta} f, \hat{\nabla})\partial_0(\varphi)$$

$$+ 2(\tilde{\Delta} f)\partial_0(\varphi) + 2|\tilde{\Delta} f|\partial_0(\varphi) - 6H|\tilde{\Delta} f|\partial_0(\varphi) - \frac{2(n-1)^2}{n-3} \text{tr}(\tilde{\Delta}^3 f)\partial_0(\varphi).$$

However, this sum obviously vanishes. Next, the conformal variation operator of $r_4$ equals $-\lambda(n-2-\lambda)|d\varphi|^2 f$.

In order to determine the tangential terms in $(c_4)^*[\varphi]$, we again apply Lemma 8.2 and the variation formulas in the following result.
Lemma 12.1. In general dimensions, it holds
\[
(\delta(\hat{L}^2)^*[\varphi] = (n-2)\delta(\hat{L}^2 d\varphi) - \delta(|\hat{L}|^2 d\varphi) + (n-4)(\delta(\hat{L}^2), d\varphi), \tag{12.4}
\]
\[
(\delta(\hat{L} \delta(\hat{L}))^*[\varphi] = (n-1)\delta(\hat{L}^2 d\varphi) + (n-4)(\hat{L} \delta(\hat{L}), d\varphi) \tag{12.5}
\]
and
\[
(\Delta(|\hat{L}^2|)^*[\varphi] = (n-6)\delta(|\hat{L}|^2 d\varphi) - (n-4)|\hat{L}|^2 \Delta(\varphi). \tag{12.6}
\]

Proof. We recall that the trace-free part of \( L \) satisfies \( \hat{L} = e^{2\varphi} \hat{L} \). The transformation laws \( \text{(12.1)} \) (for \( \lambda = 2 \)) and \( \text{(12.3)} \) (for \( \lambda = 0 \)) imply
\[
e^{4\varphi} \delta \hat{L}^2 = \delta(e^{2\varphi \delta \hat{L}}) + (n-4) e^{2\varphi}(d\varphi, \delta \hat{L})
\]
\[
= \delta(\hat{L}^2) + (n-2)\delta(\hat{L}^2 d\varphi) - \delta(|\hat{L}|^2 d\varphi) + (n-4)(d\varphi, \delta \hat{L}^2)
\]
(up to non-linear terms) using \( \hat{L}^2 = \hat{L}^2 \). This proves the first relation. Similarly, \( \text{(12.1)} \) (for \( \lambda = 2 \)) and \( \text{(12.3)} \) (for \( \lambda = -1 \)) imply
\[
e^{4\varphi} \delta \hat{L} \delta(\hat{L}) = \delta(e^{2\varphi \delta \hat{L}}) + (n-4) e^{2\varphi}(d\varphi, \hat{L} \delta(\hat{L}))
\]
\[
= \delta(\hat{L} e^{\varphi \delta \hat{L}}) + (n-4)(d\varphi, \hat{L} \delta(\hat{L}))
\]
\[
= \delta(\hat{L} \delta(\hat{L})) + (n-1)\delta(\hat{L}^2 d\varphi) + (n-4)(d\varphi, \hat{L} \delta(\hat{L}))
\]
(up to non-linear terms). This proves the second relation. Finally, the transformation law \( \text{(12.2)} \) (for \( \lambda = 2 \)) implies
\[
e^{4\varphi} \Delta(\hat{L})^2 = \Delta(|\hat{L}|^2) - 2\delta(\hat{L}^2 d\varphi) + (n-4)(d(|\hat{L}|^2), d\varphi)
\]
\[
= \Delta(|\hat{L}|^2) + (n-6)\delta(|\hat{L}|^2 d\varphi) - (n-4)|\hat{L}|^2 \Delta(\varphi)
\]
(up to non-linear terms) using \( |\hat{L}|^2 = e^{-2\varphi}|\hat{L}|^2 \). The proof is complete. \( \square \)

Lemma 12.1 shows that the tangential terms in \( (c_4)^*[\varphi] \) are given by the product of \( \frac{2}{n-2} - 2 \) with
\[
\frac{2(n-1)}{(n-3)(n-2)} \left[ (n-2)\delta(\hat{L}^2 d\varphi) - \delta(|\hat{L}|^2 d\varphi) + (n-4)(\delta(\hat{L}^2 d\varphi) - (\hat{L}^2, \text{Hess}(\varphi))) \right]
\]
\[
+ \frac{4}{n-3} \left[ (n-1)\delta(\hat{L}^2 d\varphi) + (n-4)(\hat{L} \delta(\hat{L}), d\varphi) \right]
\]
\[
+ \frac{2(n-4)}{2(n-1)(n-2)} \left[ (n-6)\delta(|\hat{L}|^2 d\varphi) - (n-4)|\hat{L}|^2 \Delta(\varphi) \right]
\]
\[
+ |\hat{L}|^2 \Delta(\varphi) - \frac{4n}{n-1} (\hat{L} \delta(\hat{L}), d\varphi) + 2H(\hat{L}, \text{Hess}(\varphi)) - \delta(|\hat{L}|^2 d\varphi) + 4\delta(\hat{L}^2 d\varphi)
\]
\[
- \frac{4}{n-3} \left[ 2\delta(\hat{L}^2 d\varphi) - (\hat{L}^2, \text{Hess}(\varphi)) - 2\frac{n-2}{n-1}(\hat{L} \delta(\hat{L}), d\varphi) + |\hat{L}|^2 \Delta(\varphi) - \delta(|\hat{L}|^2 d\varphi) \right]
\]
\[
+ \frac{2(n^2-9n+12)}{(n-3)(n-2)}(\hat{L}^2, \text{Hess}(\varphi)) + \frac{n^3-5n^2+18n-20}{2(n-3)(n-2)(n-1)} |\hat{L}|^2 \Delta(\varphi) - 2H(\hat{L}, \text{Hess}(\varphi)).
\]

This sum vanishes. Summarizing the above results shows that the conformal variation of \( R_4 \) vanishes.

12.2. Extrinsic conformal invariants of hypersurfaces. The scalar invariants
\[
\begin{align*}
I_1 &= |\hat{L}|^2, I_2 = \text{tr}(\hat{L}^2), \\
I_3 &= |\hat{W}|^2, I_4 = |\hat{W}|^2, \\
I_5 &= \hat{L}^i \hat{L}^k \hat{W}^{ikij}, I_6 = (\hat{L}^2, \hat{W}), I_7 = |\hat{W}_0|^2
\end{align*}
\]
of an embedding $M^4 \hookrightarrow X^5$ are obvious local conformal invariants of weight $-4$. Here we set

$$|\mathring{W}|^2 \overset{\text{def}}{=} \mathring{W}_{ij}\mathring{W}^{ij} \quad \text{with} \quad \mathring{W}_{ij} \overset{\text{def}}{=} \mathring{W}_{0ij0}$$

and

$$|\mathring{W}_i|^2 \overset{\text{def}}{=} \mathring{W}_{ijkl}\mathring{W}^{ijkl} \quad \text{and} \quad |\mathring{W}_{0i}|^2 \overset{\text{def}}{=} \mathring{W}_{ijkl0}\mathring{W}^{ijkl0};$$

in these definitions, all indices $i,j,k,l$ are tangential. The above invariants are defined in terms of the trace-free part $\mathring{L}$ of $L$ and the Weyl tensor $\mathring{W}$ of the background metric.

Note that the local conformal invariant $|\mathring{W}|^2 = \mathring{W}_{ijkl}\mathring{W}^{ijkl}$ of weight $-4$ is a linear combination of $\mathcal{I}_3$ and the other invariants (Lemma 9.7). Likewise, the invariant $\mathcal{I}_5 = \mathring{L}^{ij}\mathring{L}^{kl}\mathring{W}_{ijkl}$ is a linear combination of the other invariants (see (12.22)). The above conformal invariants are invariant under a change of the orientation of the normal vector.

In addition, we have the non-trivial local conformal invariants $\mathcal{J}_1$ and $\mathcal{J}_2$ (defined in (1.15) and (1.16)). Note that the definitions of $\mathcal{J}_1$ and $\mathcal{J}_2$ contain the respective normal derivative terms $\mathring{L}^{ij}\nabla_0(\mathring{W})_{0ij0}$ and $(\mathring{L}, \nabla_0(\mathring{P}))$. Corollary 12.6 shows that $\mathcal{J}_1 + 2\mathcal{J}_2$ again is a linear combination of the above invariants. Note also that $\mathcal{J}_1$ and $\mathcal{J}_2$ both contain non-trivial divergence terms - they are conformally invariant only with these divergence terms.

Next, we have the local conformal invariant $\mathcal{J}_3$ (see (1.21)). In a forthcoming paper, Astaneh and Solodukhin will prove that the integral of $\mathcal{J}_3$ is a linear combination of the integrals of $\mathcal{I}_3$, $\mathcal{I}_5$ and $\mathcal{I}_7$ (see (9.13)).

Finally, we note that $\mathring{L}^{ij}\nabla^k(\mathring{W})_{kij0}$ is a local conformal invariant in dimension $n = 4$ (see the proof of [JO22, Lemma 6.27]). Lemma 7.6 shows that its integral reduces to the conformal invariant $\int_M |\mathring{W}^0|^2 d\text{vol}_h$. However, the divergence term $\delta(\mathring{L}^{ij}\mathring{W}^k_{ij0})$ itself is a conformal invariant of weight $-4$ (see the comment after Remark 12.4).

The latter observation is related to the local conformal invariant $\mathcal{J}_5 = (\mathring{L}, S)$ (defined in Remark 8.3). In fact, we find

$$(\mathring{L}, S) = \mathring{L}^{ij}(\nabla_0(\mathring{W})_{0ij0} - \mathring{C}_{ij0} - \mathring{C}_{j0i}) + 4H(\mathring{L}, W)$$

$$= -\mathring{L}^{ij}\nabla^k(\mathring{W})_{kij0} + 4H(\mathring{L}, \mathring{W})$$

using $\mathring{C}_{ijk} = \frac{1}{3}\nabla^l(\mathring{W})_{lijk}$. But

$$\mathring{L}^{ij}\nabla^k(\mathring{W})_{kij0} = \mathring{L}^{ij}\mathring{W}^k_{ij0} - (\mathring{L}^2, \mathring{W}) + 4H(\mathring{L}, \mathring{W}) - \mathring{L}^{ij}\mathring{L}^{kl}\mathring{W}_{ijkl}$$

in dimension $n = 4$ (see (12.24) in the proof of Proposition 12.5). Hence

$$\mathcal{J}_5 = (\mathring{L}^2, \mathring{W}) + \mathring{L}^{ij}\mathring{L}^{kl}\mathring{W}_{ijkl} - \mathring{L}^{ij}\mathring{W}^k_{ij0}. \tag{12.7}$$

Note that this relation again implies the conformal invariance of $\mathring{L}^{ij}\mathring{W}^k_{ij0}$.

It is an open problem to classify all local conformal invariants of weight $-4$ of a hypersurface $M^4 \hookrightarrow X^5$. An easier problem is classifying all global conformal integrals attached to an embedding $M^4 \hookrightarrow X^5$.

### 12.3. Decompositions of conformal anomalies

The local conformal invariants in Section 12.2 are also of interest in other parts of geometric analysis and theoretical physics.

Let $(X^{n+1}, g)$ be a compact odd-dimensional manifold with smooth even-dimensional boundary $M^n$. We consider the boundary value problem for the Yamabe operator $P_2(g)$ on $X$ with Dirichlet or Robin boundary conditions. The constant term $a_{n+1}$ in the small-time asymptotic expansion of the trace of the heat kernel of this boundary value problem is a global conformal invariant.\footnote{\textsuperscript{16}We are grateful to S. Solodukhin for informing us about this result [AS22].} It is given by an integral of curvature invariants of the embedding $M \hookrightarrow X$. This result is a consequence of the conformal index property of the critical heat kernel coefficient.

\textsuperscript{17}The coefficient $a_{n+1}$ is also called the critical heat kernel coefficient.
The heat kernel coefficient \( a_{n+1} \) may be regarded as the integrated conformal anomaly of the functional determinant of the boundary value problem \([\text{BG94} \text{ Section } 2]\).

It is expected that \( a_{n+1} \) is a linear combination of the Euler characteristic \( \chi(M) \) and integrals of local conformal invariants of the embedding \( M \hookrightarrow X \). However, in general, the structure of the local conformal invariants in that decomposition is unknown.

For \( n = 2 \), the heat kernel coefficient \( a_3 \) is a linear combination of the Euler-characteristic \( \chi(M) \) and the integral of \( |L|^2 \) (see \([\text{CQ97} \text{ Section } 2]\)). For \( n = 4 \), the heat kernel coefficient \( a_5 \) has been determined in \([\text{BGKV97}]\), and its decomposition has been studied recently in \([\text{AS21}]\).

It involves the invariants \( I_j \) listed in Section \([12.2]\) and \( J_1 \).

The conformal invariance of the functional \( \int_M J_1 d\text{vol}_h \) is one of the main results of \([\text{AS21}]\) \((29)\). Supported by the decomposition of \( a_5 \), Astaneh and Solodukhin state that the integral of the Pfaffian and the integrals of the conformally invariant curvature quantities \( I_j \), together with the integral of \( J_1 \), form a basis of all conformally invariant integrals associated to \( M^4 \hookrightarrow X^5 \).

We recall that the existence of the local invariants \( J_2 \) (or \( C \)), \( J_3 \), \( J_4 \), and \( J_5 \) does not contradict that completeness statement since their integrals are linear combinations of the other invariants.

For odd \( n \), the situation is different. Then \( a_{n+1} \) is a conformally invariant sum of an integral on \( X \) and a boundary integral on \( M \). Its decomposition is expected to have the form

\[
a\chi(X) + \sum c_j \int_X I_j d\text{vol}_g + \sum b_j \int_M J_j d\text{vol}_h
\]

with local conformal invariants \( I_j \) of \( X \) (only depending on \( \text{W} \)) and local conformal invariants \( J_j \) of the embedding \( M \hookrightarrow X \) (only depending on \( \text{W} \) and \( L \)).

For \( n = 3 \), the heat kernel coefficient \( a_4 \) is a linear combination of the Euler characteristic \( \chi(X) \), \( \int_X |\text{W}|^2 d\text{vol}_g \) and the boundary integrals

\[
\int_M (\text{L}, \text{W})d\text{vol}_h \quad \text{and} \quad \int_M \text{tr}(\text{L}^3)d\text{vol}_h
\]

\([\text{BG94} \text{ Theorem } 3.7]\). In higher dimensions, the situation is much less understood.

Of course, there are similar problems for more general conformally invariant boundary value problems.

The above decompositions may be regarded as analogs of the Deser-Schwimmer classification of global conformal anomalies of CFTs on closed manifolds established by Alexakis (see \([\text{A12}]\) and its references).

In the framework of CFTs on manifolds \( X^{n+1} \) with boundary \( M^n \) (BCFT), it is a key problem to classify the (integrated) conformal anomalies. These quantities are expected to have analogous decompositions. More precisely, one expects that, for even \( n \), they have the form

\[
a\chi(M) + \sum b_j \int_M J_j d\text{vol}_h
\]

with local conformal invariants \( J_j \) of the embedding \( M \hookrightarrow X \) (consisting of extrinsic and intrinsic invariants). Similarly, for odd \( n \), they should decompose as in \([12.8]\).

Moreover, these decompositions should follow from corresponding decompositions of the anomalies themselves. In these decompositions, the Euler form \( E_n \) of \( M \) (for even \( n \)) is responsible for the Euler characteristic of \( M \). Likewise, for odd \( n \), the Euler form \( E_{n+1} \) of \( X \), together with a boundary term \( E_{n+1}^\partial \) on \( M \) - according to the Chern-Gauss-Bonnet theorem on manifolds with boundary (see \([\text{GS1} \text{ Chapter } 4]\)) - are responsible for the Euler characteristic \( \chi(X) \) contributing to \([12.8]\).

For \( n = 2 \) and \( n = 3 \), the boundary terms of the respective anomalies decompose as \( aJ + b|L|^2 \) and \( aE_4^\partial + b_1 \text{tr}(\text{L}^3) + b_2(L, \text{W}) \), up to divergence terms. Note that \( Q_2 \) and \( Q_3 \) (see Proposition\(^{18}\)) is the functional \( I_8 \) in the notation of \([\text{AS21}]\).
have the same structure as the boundary terms in these decompositions. Theorem 3 implies a decomposition of \( \int_Q Q_4 \text{dvol}_h \) which is of the form (12.9). The relevant local invariants are listed in Section 12.2 and include \( J_1 \).

Generalizing [AS21], the authors of [CHBRS21] determined the most general form of the boundary terms in the conformal anomaly of a CFT on a manifold \( X \) of dimension \( d \geq 5 \) with a boundary (or defect) \( M \) of dimension 4. The identity [CHBRS21] \((3.1)\) gives the general form of the anomaly of such a BCFT. Apart from the Pfaffian of \( M \), it contains two non-trivial invariants \( J_1, J_2 \), and a series of functionals of \( \overline{W} \) and \( \tilde{L} \). In the case \( d = 5 \), this confirms the decomposition (12.9) with the local invariants \( I_j \) and \( J_1 \) listed in Section 12.2. In particular, in the codimension-one case, [CHBRS21] \((6.2)\) states that the decomposition of the conformally invariant Graham-Reichert functional reduces to

\[
8 \mathcal{E}_{GR} = \int_M \left( \text{Pf}_4 + \frac{1}{2} J_1 - \frac{3}{8} I_1 + \frac{1}{4} I_2 - \frac{1}{8} I_3 + \frac{1}{2} I_4 - \frac{1}{2} I_5 + \frac{1}{2} I_6 + \frac{1}{4} I_7 \right) \text{dvol}_h.
\]

The Gauss equations imply the relation \( J_1 = J_1 + \frac{1}{2} I_1 - I_2 - I_6 \) (up to a divergence) and we get

\[
8 \mathcal{E}_{GR} = \int_M \left( \text{Pf}_4 + \frac{1}{2} J_1 - \frac{5}{24} I_1 + \frac{1}{4} I_2 - \frac{1}{8} I_3 + \frac{1}{2} I_4 - \frac{1}{2} I_5 + \frac{1}{4} I_7 \right) \text{dvol}_h.
\]

Note that this formula (slightly) differs from (9.14).

The anomalies listed in [CHBRS21] Section 3.3 contain three more local invariants, one of which is an extrinsic analog of the 4-form \( \text{tr}(W \wedge W) \) (first Pontrjagin form) [21]. In contrast to the invariants discussed above, these change signs under a simultaneous change of the orientations on \( X \) and \( M \).

From the perspective of the AdS/CFT duality, the conformal (quantum) anomalies of determinants (conformal index) of conformally covariant operators on a manifold \((X, g)\) (of even dimension) appear as duals of the anomaly of the renormalized volume of an associated Poincaré-Einstein metric with conformal infinity \([g]\) [GZ03]. The latter anomaly is proportional to the total integral \( \int_X Q_n(g) \text{dvol}_g \) of Branson’s critical \( Q \)-curvature [GZ03]. This result extends to a relation between the total integral \( \int_X Q_n(h) \text{dvol}_h \) and the anomaly of the renormalized volume of the singular Yamabe metric \( \sigma^{-2} g \) on \( X \) [G17, GW17, JO21].

There is an analog of the AdS/CFT duality for CFTs, which relates anomalies of BCFTs to geometric anomalies of dual theories. This duality naturally involves the study of conformal invariants of submanifolds (see [RT17] and its references). In particular, on the geometric side, this leads to the study of the conformal anomaly of the renormalized volume of a minimal hypersurface with boundary \( M \) in a Poincaré-Einstein background with conformal infinity \( X \) as initiated in [GW99]. Graham and Reichert [GR20] analyzed these conformal anomalies. In particular, they derived an explicit formula for this global conformal invariant of an embedding of \( M^4 \hookrightarrow X^n (n \geq 5) \). Parallel work [Z21] led to equivalent formulas. Finally, from the perspective of BCFTs, these results were discussed in [CHBRS21].

For more details on anomalies, we refer to [FV11, Fu15, S16, AS21, HH17]. For further generalizations and a unified discussion of geometric anomalies, we refer to [AGW21].

### 12.4. The invariants \( W_m \) and \((D(L), L)\)

In [BGW21b], the authors derived formulas for \( P_4 \) and \( Q_4 \) in general dimensions using conformal tractor calculus. A central role plays the local conformal invariant \( W_m \). In the present section, we discuss this invariant. A full comparison of our results with the corresponding results in [BGW21b] in the critical dimension \( n = 4 \) will be given in the Section 12.6.

---

19 The Weyl tensor in [CHBRS21] has the opposite sign.

20 Here \( W \) is regarded as an \( \text{End}(TM) \)-valued 2-form.
The local conformal invariant \( W^m \) is the sum of the term \( (\Delta(\hat{L}), \hat{L}) \) and some curvature terms. This is an interesting result on its own. In order to better understand the conformal invariant \( W^m \), we next describe this invariant in terms of obviously conformally invariant terms. The original proof of the conformal invariance of \( W^m \) depends on heavy tractor calculus machinery.

From \([BGW21b]\), we recall the definition

\[
W^m \overset{\text{def}}{=} \frac{1}{2}(\hat{L}, \Delta(\hat{L}))+\frac{4}{3}\delta(\hat{L}\delta(\hat{L}))+\frac{3}{2}\Delta(|\hat{L}|^2) - 6(\hat{L}, \hat{C}_0) + 4(\hat{L}^2, P) - \frac{7}{2}J|\hat{L}|^2 + 6H(\hat{L}, \hat{W}),
\]

where \( \hat{C} \) is the Cotton tensor of the background metric and \( (\hat{C}_0)_{ij} = \hat{C}_{ij0} \). This step uses the fact that the formula \( \mathcal{F} = \frac{1}{2}L^2 + \frac{8}{3}|L|^2 h \) (see (2.9)) implies the relation \(-6H \text{ tr}(\hat{L}^3) + 12H(\hat{L}, \mathcal{F}) = 6H(\hat{L}, \hat{W})\).

**Proposition 12.2.** Let \( n = 4 \). Then

\[
W^m = \frac{1}{2}(D(\hat{L}), \hat{L}) - 3\mathcal{J}_1 + 3\mathcal{I}_5 + 3\mathcal{I}_6 - \frac{3}{2}\mathcal{I}_7 - 6\delta(\hat{L}, \hat{W}_0),
\]

where
\[
D(t)_{ij} \overset{\text{def}}{=} \Delta(t)_{ij} - 2(P \circ t + t \circ P)_{ij} - Jt_{ij} - \frac{2}{3}(\nabla_i \delta(t)_{j} + \nabla_j \delta(t)_i) + h_{ij}(P, t) + \frac{1}{3}h_{ij}\delta(t)
\]
is a conformally covariant operator \( S^2_0(M) \to S^2_0(M) \) on trace-free symmetric \( 2 \)-tensors: \( e^{\varphi}\hat{D}(t) = D(e^{-\varphi}t) \).

Some comments are in order.

We recall that the scalar product \( (\hat{L}, \hat{W}_0) \in \Omega^1(M) \) is defined as \( \hat{L}^\flat \hat{W} \). On (trace-free, symmetric) \( 2 \)-tensors, the second-order operator

\[
D(\hat{L}, \hat{W}_0) = 2\hat{L}^\flat \hat{L} + 2H(\hat{L}, \hat{W}) + \frac{2}{9}|\delta(\hat{L})|^2 - 2(\hat{L}^2, P) + 2|\hat{L}|^2 + \delta(\hat{L}^2)
\]

(see (1.15)) satisfies \( e^{\hat{L}^\flat(\varphi)}\mathcal{J}_1 = \mathcal{J}_1 \) (Lemma 8.7). Moreover, the divergence \( \delta(\hat{L}, \hat{W}_0) \) is a conformal invariant of weight \(-4\) (see the comment after Remark 12.3). Therefore, Proposition \([BGW21b]\) Theorem 1.2 (see also \([BGW21a]\) Theorem 1.5).

In general dimensions, the operator
\[
D(t)_{ij} \overset{\text{def}}{=} \Delta(t)_{ij} - 2(P \circ t + t \circ P)_{ij} - Jt_{ij} - \left(\frac{n}{2} - 1\right) Jt_{ij} - \frac{4}{n+2}(\nabla_i \delta(t)_{j} + \nabla_j \delta(t)_i)
\]

\[
+ \frac{4}{n}h_{ij}(P, t) + \frac{8}{n(n+2)}h_{ij}\delta(t)
\]

maps trace-free symmetric \( 2 \)-tensors to trace-free symmetric \( 2 \)-tensors. \( D \) is conformally covariant in the sense that

\[
e^{\frac{n}{2}+1-p}D(t) = D(e^{\frac{n}{2}+1-p}\varphi t).
\]

The operator \( D \) in Proposition 12.2 is its special case in dimension \( n = 4 \).

A conformally covariant generalization of the operator \( D \) to an operator on trace-free symmetric \( p \)-tensors was discovered in \([W86]\). It satisfies

\[
e^{\frac{n}{2}+1-p}D(t) = D(e^{\frac{n}{2}+1-p}\varphi t).
\]

We refer to \([J88]\) Chapter 2 for an ambient metric derivation of it. Matsumoto \([M13]\) used an ambient metric approach to define analogs of the GJMS-operators acting on trace-free symmetric \( 2 \)-tensors. On divergence-free and trace-free symmetric \( 2 \)-tensors, the second-order operator \( P_2 \) in this sequence at an Einstein metric acts like a linear combination of \( D \) in (12.13) and the obviously conformally invariant operator \( t_{ij} \mapsto W_{ijkl} t^{kl} \).
The formula (12.13) and the related action functional
\[ \int_M (D(t), t) d\text{vol}_h \]
also have been derived in [AHR] (correcting [EO]). Since \( D \) acts on trace-free symmetric tensors, it is not possible to read off the full operator \( D \) from the associated action.

In [B96, Section 5], Branson classified all conformally covariant second-order operators acting on irreducible tensor bundles. Moreover, explicit formulas are given for operators on \( p \)-forms and trace-free symmetric \( p \)-tensors. The operator in (12.13) should be a particular case of [B96, (5.7)]\(^2\). Branson also showed that the operator \( D \) is unique, up to a constant multiple of the above action of the Weyl tensor.

**Proof.** The definitions of \( D \) and \( J_1 \) yield
\[ \frac{1}{2}(D(\hat{L}), \hat{L}) = \frac{1}{2}(\Delta(\hat{L}), \hat{L}) - 2(\hat{L}, \hat{L}^2) - \frac{1}{2}J|\hat{L}|^2 - \frac{2}{3}J^{ij} \nabla_i \delta(\hat{L})_j \]
(12.14)
and
\[ 3J_1 = 3L^{ij} \nabla_0(\hat{W})_{0ij0} + 6H(\hat{L}, \hat{W}) + \frac{2}{3}\delta(\hat{L})^2 - 6(\hat{L}^2, P) + 3|\hat{L}|^2 - 3\delta\hat{L}^2. \]
Hence
\[ \frac{1}{2}(D(\hat{L}), \hat{L}) \geq 3J_1 = \frac{1}{2}(\Delta(\hat{L}), \hat{L}) + 4(\hat{L}, \hat{L}^2) - \frac{7}{2}J|\hat{L}|^2 - \frac{2}{3}\hat{L}^{ij} \nabla_i \delta(\hat{L})_j - \frac{2}{3}\delta(\hat{L})^2 + 3\delta\hat{L}^2 \]
\[ - 6H(\hat{L}, \hat{W}) - 3\hat{L}^{ij} \nabla_0(\hat{W})_{0ij0}. \]

It follows that the difference \( W_m - \frac{1}{2}(D(\hat{L}), \hat{L}) + 3J_1 \) equals
\[ \frac{3}{2}\Delta|\hat{L}|^2 + \frac{4}{3}\delta(\hat{L})^2 - 3\delta\hat{L}^2 + \frac{2}{3}\hat{L}^{ij} \nabla_i \delta(\hat{L})_j + \frac{2}{3}|\hat{L}|^2 \]
\[ - 6\hat{L}, \hat{C}_0(0) + 12H(\hat{L}, \hat{W}) + 3\hat{L}^{ij} \nabla_0(\hat{W})_{0ij0}. \]

Now the trace-free Codazzi-Mainardi equation implies the relation
\[ 3\Delta|\hat{L}|^2 - 6\delta\hat{L}^2 + 4\delta(\hat{L})^2 = -6\delta(\hat{L}^{ij}, \hat{W}_{0ij0}) = -6\delta(\hat{L}, \hat{W}_0) \]
(see (7.5)). Thus, the latter sum equals
\[ -\frac{2}{3}\delta(\hat{L})^2 + \frac{2}{3}\hat{L}^{ij} \nabla_i \delta(\hat{L})_j + \frac{2}{3}|\hat{L}|^2 \]
\[ + 12H(\hat{L}, \hat{W}) - 3\delta(\hat{L}, \hat{W}_0) - 6\hat{L}, \hat{C}_0(0) + 3\hat{L}^{ij} \nabla_0(\hat{W})_{0ij0} \]
\[ = 12H(\hat{L}, \hat{W}) - 3\delta(\hat{L}, \hat{W}_0) - 6\hat{L}, \hat{C}_0(0) + 3\hat{L}^{ij} \nabla_0(\hat{W})_{0ij0}. \]

Now we note that
\[ -2\hat{L}, \hat{C}_0(0) + \hat{L}^{ij} \nabla_0(\hat{W})_{0ij0} = -\hat{L}^{ij} \nabla^k(\hat{W})_{kij0} \]
\[ = -\hat{L}^{ij} \nabla^k \hat{W}_{kij0} + (\hat{L}^2, \hat{W}) - 4H(\hat{L}, \hat{W}) + \hat{L}^{ij} \hat{L}^{kl} \hat{W}_{kij0} \]
using \( 2(\hat{C}_0)_{ij} = 2\hat{C}_{ij0} = \nabla^a(\hat{W})_{aij0} \) and (12.22). Hence
\[ W_m - \frac{1}{2}(D(\hat{L}), \hat{L}) + 3J_1 = -3\delta(\hat{L}, \hat{W}_0) - 3\hat{L}^{ij} \nabla^k \hat{W}_{kij0} + 3(\hat{L}^2, \hat{W}) + 3\hat{L}^{ij} \hat{L}^{kl} \hat{W}_{kij0}. \]
Now we apply Lemma 7.6. This completes the proof. \( \square \)

\(^2\)The curvature terms in this formula differ from the above result.
The decomposition \([\mathcal{X}]\) contains the local conformal invariant \((D(\hat{L}), \hat{L})\). We next express the resulting conformally invariant action functional

\[
\int_{M^4} (D(\hat{L}), \hat{L}) \, d\text{vol}_h
\]

in terms of the basic conformal invariants listed in Section \([12.2]\). It seems remarkable that the result does not involve derivatives of \(L\).

**Proposition 12.3.** For a closed four-manifold \(M\), it holds

\[
\int_{M} (D(\hat{L}), \hat{L}) \, d\text{vol}_h = \int_{M} \left( -\frac{7}{6} I_1 + 2 I_2 - \frac{3}{2} T_7 \right) \, d\text{vol}_h = \int_{M} \left( -I_5 - \frac{1}{2} I_7 \right) \, d\text{vol}_h.
\]

**Proof.** An identity of Simons \([HP99]\) states that for any hypersurface \(M^m \hookrightarrow X^{n+1}\) with the second fundamental form \(L\), it holds

\[
\Delta(L)_{ij} = n \text{Hess}_{ij}(H) + nH L^2_{ij} - L_{ij}|L|^2 + L^s_i \bar{R}^k_{ijs}k - L^s_i \bar{R}^k_{ij} + \nabla R_{ij} + \nabla_i \bar{R}_{jk}.
\]

In the following, it will be convenient to restate that identity in terms of covariant derivatives of the hypersurface only. In fact, the Gauss identity for the curvature tensor shows that the above identity is equivalent to

\[
\Delta(L)_{ij} = n \text{Hess}_{ij}(H) + nH L^2_{ij} - L_{ij}|L|^2 + \nabla R_{ij} + \nabla_i \bar{R}_{jk}.
\]  

(12.15)

Now let \(n = 4\). The identity \((\Delta(L), L) = (\Delta(\hat{L}), \hat{L}) + 4H \Delta(H)\) and the relation (12.15) imply

\[
(\Delta(\hat{L}), \hat{L}) = 4(\hat{L}, \text{Hess}(H)) + 4H \text{tr}(L^3) - |L|^4 + (L^2)_{ij} \bar{R}^k_{ijs} - L^s_i \bar{R}^k_{ij} + L^s_i \nabla R_{ij} + L^s_i \nabla (\bar{R}^0)_{ij}.
\]

We integrate and apply partial integration. Hence

\[
\int_{M} (D(\hat{L}), \hat{L}) \, d\text{vol}_h = \int_{M} \left( -\frac{7}{6} I_1 + 2 I_2 - \frac{3}{2} T_7 \right) \, d\text{vol}_h \quad \text{(by definition)}
\]

\[
= \int_{M} \left( -4 \delta(\hat{L}, dH) + \frac{4}{3} \delta(\hat{L}, \delta(\hat{L})) - 4(\hat{L}^2, P) - 4|\hat{L}|^2 + 4H \text{tr}(L^3) - |L|^4 \right) \, d\text{vol}_h
\]

\[
+ \int_{M} \left( (L^2)_{ij} \bar{R}^k_{ijs} - L^s_i \bar{R}^k_{ij} + L^s_i \nabla R_{ij} + L^s_i \nabla (\bar{R}^0)_{ij} \right) \, d\text{vol}_h. \quad (12.16)
\]

The Codazzi-Mainardi equation \(\delta(\hat{L}) = 3dH + 3\bar{P}_0\) shows that the sum of the latter two integrals equals

\[
\int_{M} (4(\delta(\hat{L}, \bar{P}_0) - (\delta(L), \bar{R}^0) - 4(\hat{L}^2, P) - J|\hat{L}|^2 + 4H \text{tr}(L^3) - |L|^4 + \text{curvature tensor terms}).
\]

Thus, using

\[
4H \text{tr}(L^3) - |L|^4 = 4H \text{tr}(\hat{L}^3) + 4H^2 |\hat{L}|^2 - |\hat{L}|^4,
\]

this sum can be written as the sum of the integral

\[
\int_{M} \left( -4(\hat{L}^2, P) - J|\hat{L}|^2 + 4H \text{tr}(\hat{L}^3) + 4H^2 |\hat{L}|^2 - |\hat{L}|^4 \right) \, d\text{vol}_h \quad (12.17)
\]

\footnote{The first relation in Proposition \([12.3]\) corrects a typo in formula (20) in \([AS21]\).}
and the integral
\[ \int_M \left( (\delta(\hat{L}), \hat{P}_0) - (dH, \hat{\text{Ric}}_0) + (L^2)^{ij} \hat{\text{R}}_{ik}^{\ j} - L^{ij} L^{rs} \hat{\text{R}}_{rijs} + L^{ij} \nabla^k \hat{\text{R}}_{ikj0} \right) \, dvol_h \] (12.18)
of curvature terms. Now the Fialkow equation
\[ P = \hat{P} + H \hat{L} + \frac{1}{2} H^2 h - \frac{1}{2} \hat{L}^2 + \frac{1}{12} \hat{L}^2 h - \frac{1}{2} \hat{W} \]
(see 2.8) and the Gauss equation \( J = J - \hat{P}_{00} - \frac{1}{3} |\hat{L}|^2 + 2H^2 \) (see (2.3)) imply that the integral (12.17) equals
\[ \int_M -4(\hat{L}^2, \hat{P}) - |\hat{L}|^2 + \hat{P}_{00} |\hat{L}|^2 + 2(\hat{L}^2, \hat{W}) + 2 \text{tr}(\hat{L}^4) - \frac{7}{6} |\hat{L}|^4. \] (12.19)
It remains to calculate the integral (12.18). First, the decomposition \( \hat{R} = \hat{W} - \hat{P} \odot g \) implies
\[ L^{ij} L^{rs} \hat{R}_{rijs} = L^{ij} L^{rs} \hat{W}_{rijs} - 2(\hat{L}^2, \hat{P}) - 2H(\hat{L}, \hat{W}) + 4H(\hat{L}, \hat{P}) + 6H^2 (J - \hat{P}_{00}). \]
Second, we have
\[ (L^2)^{ij} \hat{R}_{ik}^{\ j} = (L^2, \hat{\text{Ric}} - \bar{g}) = (L^2, 3\hat{P} + \bar{h}) - (L^2, \hat{P} + \hat{P}_{00} h + \hat{W}). \]
We combine these results and simplify. Then
\[ (L^2)^{ij} \hat{R}_{ik}^{\ j} - L^{ij} L^{rs} \hat{R}_{rijs} = -L^{ij} L^{rs} \hat{W}_{rijs} + 4(\hat{L}^2, \hat{P}) + |J| \hat{L}|^2 - \hat{P}_{00} |\hat{L}|^2 - (\hat{L}^2, \hat{W}). \]
Finally, we calculate
\[ L^{ij} \nabla^k \hat{R}_{ki0} = \hat{L}^{ij} \nabla^k \hat{R}_{ki0} + H \nabla^k \hat{\text{Ric}}_{k0} \]
\[ = \hat{L}^{ij} \nabla^k \hat{W}_{kij0} - \hat{L}^{ij} \nabla^k (\hat{P}_0)_{ij} h_{kj} + H \nabla^k \hat{\text{Ric}}_{k0}. \]
Hence
\[ \int_M L^{ij} \nabla^k \hat{R}_{ki0} \, dvol_h = \int_M \left( \hat{L}^{ij} \nabla^k \hat{W}_{kij0} + (\delta(\hat{L}), \hat{P}_0) - (dH, \hat{\text{Ric}}_0) \right) \, dvol_h. \] (12.20)
Now summarizing these results shows that
\[ \int_M (D(\hat{L}), \hat{L}) \, dvol_h = \int_M \left( (\hat{L}^2, \hat{W}) + 2 \text{tr}(\hat{L}^4) - \frac{7}{6} |\hat{L}|^4 - L^{ij} L^{rs} \hat{W}_{rijs} - \hat{L}^{ij} \nabla^k \hat{W}_{ki0} \right) \, dvol_h. \]
But
\[ \int_M \hat{L}^{ij} \nabla^k \hat{W}_{kij0} \, dvol_h = \frac{1}{2} \int_M |\hat{W}|^2 \, dvol_h \]
by Lemma 17.6. This proves the first relation. In general dimensions, the Gauss equation (2.10) for the Weyl tensor implies
\[ \hat{I}_5 = \hat{I}_5 + \text{tr}(\hat{L}^4) - |\hat{L}|^4 + 2(\hat{L}^2, \hat{F}) \]
\[ = \hat{I}_5 + \text{tr}(\hat{L}^4) - |\hat{L}|^4 + \frac{2}{n-2} \left( \text{tr}(\hat{L}^4) - \frac{1}{2(n-1)} |\hat{L}|^4 + (\hat{L}^2, \hat{W}) \right). \] (12.21)
In particular, for \( n = 4 \), we obtain
\[ \hat{I}_5 = \hat{I}_5 - \frac{7}{6} \hat{I}_4 + 2 \hat{I}_2 + \hat{I}_6. \] (12.22)
Combining this identity with the first relation proves the second relation. \( \square \)

The following result is a local version of the second relation in Proposition 12.3.

**Remark 12.4.** Let \( n = 4 \). Then \( (D(\hat{L}), \hat{L}) = -I_5 - \frac{1}{2} I_7 - \delta(\hat{L}, \hat{W}_0). \)
Proof. The arguments in the proof of Proposition \[12.3\] lead to the additional divergence terms

\[
4\delta(\hat{L}dH) - \frac{4}{3}\delta(\hat{L}\delta(\hat{L})) + \delta(L\text{Ric}_0)
\]  

(in \[12.19\])

\[
+ \delta(\hat{L}\bar{P}_0) - \delta(H\text{Ric}_0)
\]  

(by \[12.20\])

\[- \delta(\hat{L}W_{\cdot ij0}).
\]

By the Codazzi-Mainardi equation, all terms except the last one cancel. 

As a byproduct, Remark \[12.4\] implies that \(\delta(\hat{L}, \bar{W}_0)\) is a local conformal invariant of weight 

\(-4\).

We finish this section with the

Proof of Lemma \[10.3\]. Proposition \[12.2\] shows that for a flat background

\[
\int_M Wm\text{vol}_h = \frac{1}{2} \int_M (D(\hat{L}), \hat{L})\text{vol}_h - 3 \int_M J_1\text{vol}_h.
\]

But \(\frac{1}{2} \int_M (D(\hat{L}), \hat{L})\text{vol}_h = \int_M (-\frac{7}{6}J_1 + 2J_2)\text{vol}_h\) by Proposition \[12.3\]. Hence

\[
\int_M Wm\text{vol}_h = \int_M \left( \frac{5}{12}J_1 - 2J_2 - 6|dH|^2 + 6H \text{tr}(\hat{L}^3) - 3H^2|\hat{L}|^2 \right)\text{vol}_h
\]

using the first relation in Lemma \[10.2\]. But Hopf’s formula for \(\chi(M)\) \[G04, Theorem 5.7\] states that

\[
24\pi^2\chi(M) = 18 \int_M \sigma_4(L)\text{vol}_h.
\]

Thus, Newton’s identity for \(\sigma_4(L)\) gives

\[
24\pi^2\chi(M) = \int_M \left( \frac{9}{4}J_1 - \frac{9}{2}J_2 + 6H \text{tr}(\hat{L}^3) - 9H^2|\hat{L}|^2 + 18H^4 \right)\text{vol}_h.
\]

Therefore, we get

\[
\int_M \left( Wm + \frac{11}{6}J_1 - \frac{5}{2}J_2 \right)\text{vol}_h - 24\pi^2\chi(M) = \int_M \left( -6|dH|^2 + 6H^2|\hat{L}|^2 - 18H^4 \right)\text{vol}_h.
\]

The proof is complete. 

\[12.5.\] Proof of the relation \[1.19\]. In the present section, we derive the relation \(1.19\). We first prove the following result.

Proposition \[12.5\]. In the critical dimension \(n = 4\), it holds

\[
J_1 - 2J_2 = (D(\hat{L}), \hat{L}) + 3(\hat{L}^2, \bar{W}) + 2\hat{L}^{kl}\hat{L}^{ij}W_{kijl} - \frac{4}{3}|\hat{L}|^4 + 3\text{tr}(\hat{L}^4) + \delta(\hat{L}, \bar{W}_0). \tag{12.23}
\]

Since all other terms on the right-hand side of \(12.23\) are conformally invariant, this identity confirms the conformal invariance of \(J_1 - 2J_2\).

Proof. In general dimensions, it holds

\[
\nabla^k(\bar{W})_{kij0} = (n - 2)C_{ij0} = (n - 2)\nabla^0(\bar{P})_{ij} - (n - 2)\nabla_j(\bar{P})_{i0}.
\]

On the other hand, for tangential \(\partial_k\), we find

\[
\nabla^k(\bar{W})_{kij0} = \nabla^k\bar{W}_{kij0} - \hat{L}^{kl}\bar{W}_{kijl} + \hat{L}_i^k\bar{W}_{k0j0} + nH\bar{W}_{0ij0}
\]

\[
= \nabla^k\bar{W}_{kij0} - \hat{L}^{kl}\bar{W}_{kijl} - \hat{L}_i^k\bar{W}_{k0j0} + nH\bar{W}_{0ij0}
\]

\[
= \nabla^k\bar{W}_{kij0} - \hat{L}^{kl}\bar{W}_{kijl} + H\bar{W}_{0ij0} - \hat{L}_i^k\bar{W}_{k0j0} - H\bar{W}_{0ij0} + nH\bar{W}_{0ij0}
\]

\[
= \nabla^k\bar{W}_{kij0} - \hat{L}^{kl}\bar{W}_{kijl} - \hat{L}_i^k\bar{W}_{k0j0} + nH\bar{W}_{0ij0} \tag{12.24}
\]
using $\nabla_i(\partial_j) = \nabla_i(\partial_j) - L_{ij} \partial_0$ and $\nabla_k(\partial_0) = L_k^m \partial_m$. Hence
\begin{equation}
\nabla^k(W)_{kij0} = \nabla^0(W)_{0ij0} + \nabla^kW_{kij0} - \hat{L}^k_{ij}W_{0k0j} + nH\bar{W}_{ij}. \tag{12.25}
\end{equation}
Combining both results gives the relation
\begin{equation}
(n - 2)\nabla^0(\bar{P})_{ij} - \nabla^0(W)_{0ij0} = \nabla^kW_{kij0} + (n - 2)\nabla_j(\bar{P})_{0i} - \hat{L}^k_{ij}W_{0k0j} + nH\bar{W}_{ij}.
\end{equation}
Together with the trace-free Codazzi Mainardi equation \cite{12.6}, we obtain
\begin{equation}
(n - 2)\nabla^0(\bar{P})_{ij} - \nabla^0(W)_{0ij0}
= \nabla^k\nabla_i(\hat{L})_{kj} - \nabla^k\nabla_k(\hat{L})_{ij} + \frac{1}{n - 1}\nabla^k\delta(\hat{L})_i h_{kj} - \frac{1}{n - 1}\nabla^k\delta(\hat{L})_k h_{ij}
+ (n - 2)\nabla_j(\bar{P})_{0i} - \hat{L}^k_{ij}W_{kij} - \hat{L}^0_{ij}W_{0k0j} + nH\bar{W}_{ij}.
\end{equation}
Now we commute the covariant derivatives in the first term. Then
\begin{equation}
(n - 2)\nabla^0(\bar{P})_{ij} - \nabla^0(W)_{0ij0}
= \nabla_i\delta(\hat{L})_j - R^k_{ijlt}Q^l_k - \nabla^k\delta(\hat{L})_i + \frac{1}{n - 1}\nabla_j\delta(\hat{L})_i - \frac{1}{n - 1}\Delta\delta(\hat{L})_i
+ (n - 2)\nabla_j(\bar{P})_{0i} - \hat{L}^k_{ij}W_{kij} - \hat{L}^0_{ij}W_{0k0j} + nH\bar{W}_{ij}.
\end{equation}
Next, we note that
\begin{equation}
\nabla_j(\bar{P})_{0i} = \partial_j(\bar{P}_{0i}) - \bar{P}(\nabla_j(\partial_i), \partial_0) - \bar{P}(\partial_i, \nabla_j(\partial_0))
= \partial_j(\bar{P}_{0i}) - \bar{P}(\nabla_j(\partial_i), \partial_0) + L_{ij}\bar{P}_{00} - L^j_i\bar{P}(\partial_i, \partial_0)
= \nabla_j(\bar{P}_{0i}) + L_{ij}\bar{P}_{00} - L^j_i\bar{P}_{il},
\end{equation}
and that the Codazzi-Mainardi equation implies
\begin{equation}
(n - 1)\nabla_j(\bar{P}_{0i}) = \nabla_j\delta(\hat{L})_i - (n - 1)\text{Hess}_{ij}(H).
\end{equation}
These identities yield
\begin{equation}
(n - 2)\nabla^0(\bar{P})_{ij} - \nabla^0(W)_{0ij0}
= -\Delta(\hat{L})_i + \nabla_i\delta(\hat{L})_j + \nabla_j\delta(\hat{L})_i - (n - 2)\text{Hess}_{ij}(H) - \frac{1}{n - 1}\delta\delta(\hat{L})_i
- (n - 2)L^k_{ij}\hat{P}_{il} + (n - 2)L_{ij}\bar{P}_{00} - R^k_{ijlt}Q^l_k - \nabla^k\delta(\hat{L})_i + \frac{1}{n - 1}\nabla_j\delta(\hat{L})_i
+ \hat{L}^k_{ij}W_{kij} - \hat{L}^0_{ij}W_{0k0j} + nH\bar{W}_{ij}. \tag{12.26}
\end{equation}
The identity \cite{12.26} will also be important in Section \ref{12.6}. In the critical dimension $n = 4$, we contract this identity with $\hat{L}$ and obtain
\begin{equation}
2\hat{L}^j_i\nabla^0(\bar{P})_{ij} - \hat{L}^j_i\nabla^0(W)_{0ij0}
= -(\hat{L}, \Delta(\hat{L})) + 2\hat{L}^j_i\nabla_i\delta(\hat{L})_j - 2\hat{L}^2(\bar{P}) - 2H(\bar{L}, \bar{P}) + 2\bar{P}_{00}|\hat{L}|^2
- \hat{L}^j_i\hat{L}^k_{ij}R_{kij} + 2(\hat{L}^2, \bar{P}) + \frac{1}{2}|\hat{L}|^2 - \hat{L}^j_i\hat{L}^k_{ij}W_{kij} - (\hat{L}^2, \bar{W}) + 4H(\bar{L}, \bar{W}). \tag{12.27}
\end{equation}
Now the definitions of $J_1$ and $J_2$ give
\begin{equation}
2\mathcal{J}_2 - \mathcal{J}_1
= 2\hat{L}^j_i\nabla_0(\bar{P})_{ij} - \hat{L}^j_i\nabla_0(W)_{0ij0}
+ 2(\hat{L}^2, \bar{P}) + 2H(\bar{L}, \bar{P}) - 3H(\bar{L}, \bar{W}) + 2(\hat{L}, \text{Hess}(H)) - 2\bar{P}_{00}|\hat{L}|^2 - 3H^2|\hat{L}|^2 - H\text{tr}(\hat{L}^3)
+ \frac{2}{3}\delta(\hat{L})^2 - \delta\delta(\hat{L})^2 + \frac{1}{2}\Delta(|\hat{L}|^2).
\end{equation}
In this formula, we substitute the first line on the right-hand side by the sum displayed in (12.27). This yields
\[
2J_2 - J_1 = -(\hat{L}, \Delta(\hat{L})) + 2\hat{L}^i \nabla_i \delta(\hat{L}) - 2(\hat{L}^2, \hat{P} - \hat{P}) - 2H(\hat{L}, \hat{P} - \hat{P}) + 2(\hat{L}^2, \hat{P}) + J|\hat{L}|^2
- \hat{L}^j \hat{L}^k \nabla_j \delta(\hat{L}) - \hat{L}^j \hat{L}^k \nabla_j \delta(\hat{L}) - (\hat{L}^2, \hat{W}) + H(\hat{L}, \hat{W}) - 3H^2|\hat{L}|^2 - H \operatorname{tr}(\hat{L}^3)
+ \frac{2}{3} \delta(\hat{L})^2 - \delta(\hat{L}^2) + \frac{1}{2} \Delta(|\hat{L}|^2).
\]
In order to simplify that result, we apply the identities
\[
\delta(\hat{L} \delta(\hat{L})) = \delta(\hat{L})^2 + \hat{L}^i \nabla_i \delta(\hat{L}),
\]
\[
\hat{L}^i \hat{L}^j \hat{L}^k \nabla_{ki} = -2(\hat{L}^2, \hat{P}) + \hat{L}^i \hat{L}^j \hat{L}^k \nabla_{kij},
\]
together with the consequences
\[
2(\hat{L}, \hat{P} - \hat{P}) = -2H|\hat{L}|^2 + \operatorname{tr}(\hat{L}^3) + (\hat{L}, \hat{W}),
\]
\[
2(\hat{L}^2, \hat{P} - \hat{P}) = -2H \operatorname{tr}(\hat{L}^3) - H^2|\hat{L}|^2 + \operatorname{tr}(\hat{L}^4) - \frac{1}{6} |\hat{L}|^4 + (\hat{L}^2, \hat{W})
\]
of the Fialkow equation. Then we obtain
\[
2J_2 - J_1 = -(\hat{L}, \Delta(\hat{L})) + 4(\hat{L}^2, \hat{P}) + J|\hat{L}|^2
- \hat{L}^j \hat{L}^k \nabla_{kij} - 2(\hat{L}^2, \hat{W}) - \operatorname{tr}(\hat{L}^4) + \frac{1}{6} |\hat{L}|^4
- \frac{4}{3} \delta(\hat{L})^2 + 2\delta(\hat{L} \delta(\hat{L})) - \delta(\hat{L}^2) + \frac{1}{2} \Delta(|\hat{L}|^2).
\]
Finally, (12.22) shows that
\[
\hat{L}^i \hat{L}^j \hat{L}^k \nabla_{kij} = \hat{L}^i \hat{L}^j \hat{L}^k \nabla_{kij} - \frac{7}{6} |\hat{L}|^4 + 2 \operatorname{tr}(\hat{L}^4) + (\hat{L}^2, \hat{W}).
\]
Therefore, we get the final result
\[
2J_2 - J_1 = -(\hat{L}, \Delta(\hat{L})) + 4(\hat{L}^2, \hat{P}) + J|\hat{L}|^2
- 2\hat{L}^j \hat{L}^k \nabla_{kij} - 3(\hat{L}^2, \hat{W}) + \frac{4}{3} |\hat{L}|^4 - 3 \operatorname{tr}(\hat{L}^4)
- \frac{4}{3} \delta(\hat{L})^2 + 2\delta(\hat{L} \delta(\hat{L})) - \delta(\hat{L}^2) + \frac{1}{2} \Delta(|\hat{L}|^2).
\]
Combining this result with
\[
(D(\hat{L}), \hat{L}) = \Delta(\hat{L}, \hat{L}) - 4(\hat{L}^2, \hat{P}) - J|\hat{L}|^2 + \frac{4}{3} \delta(\hat{L})^2 - \frac{4}{3} \delta(\hat{L} \delta(\hat{L}))
\]
(by (12.12)) finally yields
\[
J_1 - 2J_2 = (D(\hat{L}), \hat{L}) + 3(\hat{L}^2, \hat{W}) + 2\hat{L}^j \hat{L}^k \nabla_{kij} - \frac{4}{3} |\hat{L}|^4 + 3 \operatorname{tr}(\hat{L}^4)
+ \delta(\hat{L}^2) - \frac{1}{2} \Delta(|\hat{L}|^2) - \frac{2}{3} \delta(\hat{L} \delta(\hat{L})).
\]
Now Lemma 7.4 completes the proof.

\[\square\]

**Corollary 12.6.** In the critical dimension \( n = 4 \), it holds
\[
J_1 - 2J_2 = -\frac{4}{3}I_1 + 3I_2 + I_5 + 3I_6 - \frac{1}{2}I_7.
\]

**Proof.** Combine Proposition 12.5 with Remark 12.3. \(\square\)

This proves the relation (1.19).
12.6. Some comments on \[\text{BGW21b}\]. Remark 1.1 shows that the formula for the second-order part of \(P_4\) given in \[\text{BGW21b}\] is equivalent to our formula for this part.

In the present section, we prove that the formula for \(Q_4\) in general dimensions displayed in Theorem 1 is equivalent to the formula

\[
Q_4 = Q_4 + \frac{2}{n}(\hat{L}, \Delta (\hat{L})) + \frac{1}{n-3} \left[ 2(n-1)\delta \delta \hat{F} + \frac{3n^2-3n-2}{2(n-1)} \Delta (|\hat{L}|^2) + 4\delta (\hat{L} \delta (\hat{L})) \right] (12.28)
\]

In order to verify the equivalence of both formulas, we calculate the difference of the sum

\[
(1.5) + (1.6) + (1.7) + (1.8)
\]

and the sum

\[
(12.28) + \frac{2}{n}(\hat{L}, \Delta (\hat{L})) + (12.29) + (12.30) + (12.31) + (12.32)
\]

up to terms which are quartic in \(L\).

We first verify that the divergence terms in (1.5) coincide with the divergence terms in (12.28). Indeed, (2.9) implies

\[
\delta \delta \hat{F} = \frac{1}{n-2} \delta \delta (|\hat{L}|^2) - \frac{1}{n(n-2)} \Delta (|\hat{L}|^2) + \frac{1}{n-2} \delta \delta (|W|^2).
\]

Hence (12.28) reads

\[
\frac{2(n-1)}{n-3} \delta \delta (|W|^2) + \frac{2(n-1)}{n-3} \delta \delta (|\hat{L}|^2) + \frac{3n^2-3n-2}{2(n-1)(n-2)} \Delta (|\hat{L}|^2) + \frac{4}{n-2} \delta (\hat{L} \delta (\hat{L})).
\]

But this sum coincides with (1.5).

In order to proceed, we use the identity (12.29) to replace the normal derivative term \((\hat{L}, \nabla_0 (\hat{P}))\) in (1.6) by \(\hat{L}^j \nabla_0 (\hat{W})_{0ij0}\). We find that the sum (1.6) equals

\[
2(\hat{L}, \text{Hess}(\hat{H})) + 2(\hat{L}, \nabla_0 (\hat{P})) - \frac{2}{n-2} \hat{L}^i \nabla_0 (\hat{W})_{0ij0} - \frac{2(n-1)}{n-3} \hat{L}^i \nabla_0 (\hat{W})_{0ij0}
\]

\[
= \frac{2}{n-2}(\hat{L}, \Delta (\hat{L})) + \frac{1}{n-2} \hat{L}^i \nabla_0 \delta (\hat{L})_j - \frac{2(n-1)}{n-3} \hat{L}^i \nabla_0 (\hat{W})_{0ij0}
\]

\[
- 2\hat{L}^j \hat{L}^i \hat{P}_{il} + 2(\hat{L}^2) \hat{P}_{00} - \frac{2}{n-2} \hat{L}^j \hat{L}^i R^k_{i kl} - \frac{2}{n-2} \hat{L}^j \hat{L}^i R^k_{ikl}
\]

\[
- \frac{2}{n-2} \hat{L}^j \hat{L}^k \nabla W_{kij0} - \frac{2}{n-2} (\hat{L}^2, \hat{W}) + \frac{2n}{n-2} H(\hat{L}, \hat{W}).
\]

The difference of this sum and the sum

\[
\frac{2}{n} (\hat{L}, \Delta (\hat{L})) - \frac{2(n-1)}{n-3} \hat{L}^i \hat{C}_{ij0} + \frac{6(n-1)}{n(n-3)} \hat{L}^i \nabla_0 (\hat{W})_{0ij0}
\]

\[
= \frac{2}{n} (\hat{L}, \Delta (\hat{L})) - \frac{2(n-1)}{n-3} \hat{L}^i \nabla_0 (\hat{W})_{0ij0} + \frac{6(n-1)}{n(n-3)} \hat{L}^i \nabla_0 (\hat{W})_{0ij0}
\]

displayed in \[\text{BGW21b}\] Corollary 1.1. Here we omit the terms which are quartic in \(L\). Accordingly, we shall omit the verification of the coincidence of the respective terms which are quartic in \(L\).
Therefore, the sum (12.33) further reduces to the sum of

\[
\frac{4(n-1)}{n(n-2)} \left[ n(\hat{L}, \text{Hess}(H)) + L^{ij} \nabla_i (\text{Ric}_0)_j + \hat{L}^{ij} \nabla^k \bar{R}_{ikj0} + (L^2)_{ij} \bar{R}_{ik} - L^{ij} L^{rs} \bar{R}_{rijs} \right] + \frac{2(n-1)}{(n-2)(n-3)} \left[ L^{ij} \nabla^k \bar{W}_{kij0} - \frac{6(n-1)}{n(n-3)} \hat{L}^{ij} \nabla^k \bar{W}_{kij0} \right] + \frac{4}{n-2} \hat{L}^{ij} \nabla_i \delta(\hat{L})_j,
\]

and

\[
\frac{4(n-1)}{n(n-2)} (L^2)_{ij} \bar{R}_{ik} - L^{ij} L^{rs} \bar{R}_{rijs}
\]

up to terms which are quartic in \( L \). The decomposition \( \bar{R} = \bar{W} - \bar{P} \otimes g \) yields

\[
L^{ij} \nabla^k \bar{R}_{ikj0} = -L^{ij} \nabla^k \bar{W}_{kij0} + L^{ij} \nabla_j (\bar{P}_0)_l - nH \delta(\bar{P}_0).
\]

Therefore, the sum (12.33) further reduces to the sum of

\[
\frac{4(n-1)}{n(n-2)} \left[ n(\hat{L}, \text{Hess}(H)) + L^{ij} \nabla_i (\text{Ric}_0)_j + \hat{L}^{ij} \nabla_j (\bar{P}_0)_l + H \delta(\bar{P}_0) - nH \delta(\bar{P}_0) \right] + \frac{4}{n-2} \hat{L}^{ij} \nabla_i \delta(\hat{L})_j,
\]

and

\[
\frac{4(n-1)}{n(n-2)} ((L^2)_{ij} \bar{R}_{ik} - L^{ij} L^{rs} \bar{R}_{rijs})
\]

up to terms which are quartic in \( L \). Now \( \delta(\hat{L}) = (n-1)dH + (n-1)\bar{P}_0 \) (Codazzi-Mainardi) implies that the first sum vanishes. In the second sum, we use the Gauss equations for \( \bar{R} \), (12.21) and the Fialkow equation to replace curvature contributions of the background metric \( g \) by curvature contributions of the induced metric \( h \). Simplification gives

\[
\frac{4(n-1)}{n(n-2)} ((L^2, \text{Ric}) - \hat{L}^{ij} L^{rs} \bar{R}_{rijs})
\]

and

\[
\frac{4}{n-2} ((L^2, \text{Ric}) - \hat{L}^{ij} L^{rs} \bar{R}_{rijs})
\]

up to terms which are quartic in \( L \). Now (12.34)
up to terms which are quartic in $L$. In the latter formula, we decompose $R = W - P \otimes h$ and substitute $\text{Ric} = (n - 2)P + Jh$.

Finally, the terms in (12.30) - (12.32) yield

$$- \frac{2(n-1)(n-4)}{(n-2)(n-3)} (P, \nabla^2 W) + \frac{2(n-1)^2}{(n-2)^2(n-3)} (\nabla^4 W, \nabla^2 W) + \left( \frac{4(n-1)^2}{(n-2)^2(n-3)} + \frac{2(n-1)}{(n-2)(n-3)} \right) (\hat{L}^2, \nabla^2 W)$$

$$- \frac{6n^2 - 38n + 48}{(n-2)^2(n-3)} (\hat{L}^2, P) + \frac{2(n-1)(n-4)}{n(n-2)(n-3)} J L^2 + \frac{2(n-1)}{n-3} H (L, \nabla^2 W).$$

(12.35)

Now, we use these results to determine the remaining terms in the difference of both formulas for $Q_4$.

- The term $(P, \nabla^2 W)$. Its coefficient

$$\frac{2(n-1)(n-4)}{(n-2)(n-3)}$$

in (12.35) coincides with the coefficient in (1.7).

- The term $(\nabla^4 W, \nabla^2 W)$. Its coefficient

$$\frac{2(n-1)^2}{(n-2)^2(n-3)}$$

in (12.36) coincides with its coefficient in (1.7).

- The term $H (L, \nabla^2 W)$. It contributes to (1.7) and to (12.34) with the respective coefficients

$$- \frac{2(n-1)^2}{(n-2)(n-3)} \quad \text{and} \quad \frac{4n}{n-3} - \frac{2}{n-2}.

The sum of these coefficients equals

$$\frac{2(n-1)}{n-3}.$$

On the other hand, it contributes to (12.35) with the coefficient

$$\frac{2(n-1)}{n-3}.$$

Thus, the term $H (L, \nabla^2 W)$ has the same coefficient in both formulas for $Q_4$. The fact that, for $n = 4$, this coefficient equals $-6$ is reflected by the contribution $-3J_1$ in the decomposition of $Q_4$ in Corollary 3.

- The term $(\hat{L}^2, \nabla^2 W)$. It contributes to (1.7) and (12.34) by

$$\frac{4(3n-5)(n-1)}{(n-2)^2(n-3)} \quad \text{and} \quad -\frac{4}{n-3} \frac{2}{n-2} - \frac{4}{n-3} \frac{2}{n-2} = -\frac{6(n-1)}{(n-2)(n-3)}.$$

The sum of these coefficients equals

$$\frac{2(3n-4)(n-1)}{(n-2)^2(n-3)}.$$

On the other hand, it contributes to (12.35) by

$$\frac{4(n-1)^2}{(n-2)^2(n-3)} + \frac{2(n-1)}{(n-2)(n-3)} = \frac{2(3n-4)(n-1)}{(n-2)^2(n-3)}.$$

Thus, the contributions of $(\hat{L}^2, \nabla^2 W)$ to both formulas for $Q_4$ coincide.

- The term $(\hat{L}^2, P)$. It contributes to (1.8) and (12.34) with the respective coefficients

$$- \frac{2(n^2 - 9n + 12)}{(n-2)(n-3)} \quad \text{and} \quad -4.$$

The sum of these coefficients equals

$$-\frac{6n^2 - 38n + 48}{(n-2)(n-3)}.$$

This coefficient coincides with the coefficient of $(\hat{L}^2, P)$ in (12.35).
• The term $|\tilde J|^2$. It contributes to $|\bar L|^2$ and with the respective coefficients
$$\frac{4(n-1)}{n-2} + \frac{2}{n-2} = \frac{-2}{n} \quad \text{and} \quad -\frac{n^3 - 5n^2 + 18n - 20}{2(n-3)(n-2)(n-1)}.$$ The sum of these coefficients coincides with the sum
$$\frac{2(n-1)(n-4)}{n-2(n-3)} - \frac{n^3 + 5n^2 -20n +20}{2n(n-1)} = -\frac{n^3 - 6n^2 + 24n -24}{2(n-3)(n-2)(n-1)}$$ of its contributions to (12.35) and (12.32).

• The term $L^{ij} L^{kr} W_{ij}^r$. It contributes to (12.34) with the coefficient
$$\frac{4(n-1)}{n-2} - \frac{2}{n-2} = \frac{-2(n+3)}{n(n-3)}.$$ This coefficient coincides with its coefficient in (12.32).

• The terms $|L|^2 P_{00}$ and $H(L, P)$. Their contributions in (1.8) cancel against their respective contributions in (12.34).

In the critical dimension $n = 4$, we may proceed more directly to compare the formulas for $Q_4$ displayed in Corollary 3 and [BGW21b, Theorem 1.2]. [BGW21b, Theorem 1.2] is the special case of [BGW21b, Corollary 1.1] for $n = 4$. It gives the decomposition
$$Q_4 = Q_4 + Wm + U + \text{divergence term}$$
with $Wm$ as in (12.11), the local conformal invariant
$$U \overset{\text{def}}{=} 18(\tilde F, \tilde F) + 6(|\tilde L|^2, \tilde F) + \frac{49}{24}|\tilde L|^4 + \frac{9}{2} L^{ij} \nabla^k \nabla_{kij0} - \frac{7}{2} \hat L^{ij} \hat L^{kl} W_{ijkl}$$ and the divergence terms
$$\frac{8}{3} \delta(L \delta(\tilde L)) + 6 \delta(\tilde L) - \frac{1}{12} \Delta(|\tilde L|^2) = \frac{8}{3} \delta(L \delta(\tilde L)) + 3 \delta(\tilde L^2) - \frac{5}{6} \Delta(|\tilde L|^2) + 3 \delta(\tilde \nabla).$$

We rewrite the sum in (12.36) in terms of the invariants $I_2, J_1$, and a divergence term. Combining Proposition 12.2 and Remark 12.3 and (12.22) gives
$$Wm = \left( -\frac{1}{2} I_5 - \frac{1}{4} I_7 - \frac{1}{2} \delta(\tilde L, \tilde \nabla_0) \right) + \left( -3 J_1 + 3 I_5 + 3 I_6 - \frac{3}{2} I_7 - 6 \delta(\tilde L, \tilde \nabla_0) \right) = -\frac{7}{2} I_1 + 6 I_2 + \frac{5}{2} I_5 + 6 I_6 - \frac{7}{4} I_7 - 3 J_1 - \frac{13}{2} \delta(\tilde L, \tilde \nabla_0)$$ Moreover, we easily calculate
$$U = \frac{15}{2} \text{tr}((\tilde L^4) + \frac{1}{6} |\tilde L|^4 + \frac{9}{2} |\tilde \nabla|^2 + 12 |\tilde L|^2, \tilde \nabla) + \frac{9}{2} L^{ij} \nabla^k \nabla_{kij0} - \frac{7}{2} \hat L^{ij} \hat L^{kl} W_{ijkl}$$
$$= \frac{1}{6} I_1 + \frac{15}{2} I_2 + \frac{9}{2} I_4 - \frac{7}{2} I_5 + 12 I_6 + \frac{9}{4} I_7 + \frac{9}{2} \delta(\tilde L, \tilde \nabla_0)$$ using Lemma 7.6. Finally, the sum (12.37) coincides with the sum
$$4 \delta(\hat L \delta(\tilde L)) + 3 \delta(\tilde L^2) + \frac{1}{6} \Delta(|\tilde L|^2) + 3 \delta(\tilde \nabla) + 2 \delta(\tilde L, \tilde \nabla_0)$$ using Lemma 7.4. Note that the latter sum coincides with the second line of (1.20), up to the last term (being a local conformal invariant).

Summarizing these result, we find that (12.36) reads
$$Q_4 = Q_4 - \frac{10}{3} I_1 + \frac{27}{2} I_2 + \frac{9}{2} I_4 - I_5 + 18 I_6 + \frac{1}{2} I_7 - 3 J_1$$
$$+ 4 \delta(\hat L \delta(\tilde L)) + 3 \delta(\tilde L^2) + \frac{1}{6} \Delta(|\tilde L|^2) + 3 \delta(\tilde \nabla).$$

This shows that the formula for $Q_4$ in Corollary 3 (or equivalently in Theorem 3) coincides with the formula in [BGW21b, Theorem 1.2].
References

[AGV81] E. Abbena, A. Gray and L. Vanhecke, Stein’s formula for the volume of a parallel hypersurface in a Riemannian manifold, *Annali Sc. Norm. Sup. Pisa* 8, (3), (1981), 473–493.

[AHR] J. B. Achour, E. Huguet and J. Renaud, Conformally invariant wave equation for a symmetric second rank tensor (spin-2) in d-dimensional curved background, *Physical Review D* 89, 064041 (2014).

[AG21] C. Arias, R. Gover and A. Waldron, Conformal geometry of embedded manifolds with boundary from universal holographic formulae, *Advances in Math.* 384 (2021) 107700.

[AS21] A. Astaneh and S. Solodukhin, Boundary conformal invariants and the conformal anomaly in five dimensions, *Physics Letters B* 816, (2021), 136282.

[A12] S. Alexakis, The decomposition of global conformal invariants, *Annals of Mathematics Studies* 182, Princeton University Press, Princeton, NJ, 2012.

[AGW21] C. Arias, R. Gover and A. Waldron, Conformal geometry of embedded manifolds with boundary from universal holographic formulae, *Advances in Math.* 384 (2021) 107700.

[AS22] A. Astaneh and S. Solodukhin, private communication.

[BJ10] H. Baum and A. Juhl, *Conformal Differential Geometry: Q-Curvature and Conformal Holonomy*. Oberwolfach Seminars 40, 2010.

[B87] A. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 10, Springer-Verlag, (1987).

[BGW21a] S. Blitz, R. Gover and A. Waldron, Conformal fundamental forms and the asymptotically Poincaré-Einstein condition. arXiv:2107.10381v1

[BGW21b] S. Blitz, R. Gover and A. Waldron, Generalized Willmore energies, Q-curvatures, extrinsic Paneitz operators, and extrinsic Laplacian powers. arXiv:2111.00179v1

[B95] T. Branson, Sharp inequalities, the functional determinant, and the complementary series, *Trans. Amer. Math. Soc.* 347, (1995), 3671–3742.

[B96] T. Branson, Nonlinear phenomena in the spectral theory of geometric linear differential operators, *Proc. Symp. Pure Math.* 59, (1996), 27-65.

[B05] T. Branson, Q-curvature and spectral invariants. *Rend. Circ. Mat. Palermo* (2) Suppl. 75, (2005), 11–55.

[BG94] T. Branson, The functional determinant of a four-dimensional boundary value problem, *Trans. Amer. Math. Soc.* 344, (2), (1994), 479–531.

[BGKV97] T. Branson, P. Gilkey, K. Kirsten and D. Vassilevich, Heat kernel asymptotics with mixed boundary conditions, *Nucl. Phys. B* 563, (3), (1999), 603–626.

[CHBRS21] A. Chalabi, C. Herzog, A. O’Bannon, R. Robinson and J. Sisti, Weyl anomalies of four-dimensional boundaries and defects, *J. High Energ. Phys.* 166, (2022).

[CY95] S.-Y. Alice Chang and P. Yang, Extremal metrics of zeta function determinant on 4-manifolds, *Ann. of Math.* (2) 142, (1), (1995), 171–212.

[CG19] S.-Y. Alice Chang and Y. Ge, Compactness of conformally compact Einstein manifolds in dimension 4. *Advances in Math.* 340, (2018), 588–652.

[CQ97] S.-Y. Alice Chang and J. Qing, The zeta function determinants on manifolds with boundary, *J. Funct. Anal.* 147, (1997), 327–362.

[CY95] S.-Y. Alice Chang and P. Yang, Extremal metrics of zeta function determinant on 4-manifolds, *Ann. of Math.* (2) 142, (1), (1995), 171–212.

[CGY02] S.-Y. Alice Chang, M. Gursky and P. Yang, An equation of Monge-Ampere type in conformal geometry and 4-manifolds of positive Ricci curvature, *Ann. of Math.* (2) 155, (3), (2002), 709–787.

[C18] S.-Y. Alice Chang, Conformal Geometry on Four Manifolds, *Proc. Int. Cong. of Math.* (2018), 1, 119–146.

[CMY21] S.-Y. Alice Chang, S. McKeown and P. Yang, Scattering on singular Yamabe spaces. arXiv:2109.02014

[CK04] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*. Mathematical Surveys and Monographs 110, AMS (2004).

[DS93] S. Deser and A. Schwimmer, Geometric classification of conformal anomalies in arbitrary dimensions, *Physics Letters B.* 309, 279 (1993).

[DGH08] Z. Djadli, C. Guillarmou and M. Herzlich, *Opérateurs géométriques, invariants conformes et variétés asymptotiquement hyperboliques*, Panoramas et Synthèses 26, Société Mathématique de France, 2008.

[ES85] M. Eastwood and M. Singer, A conformally invariant Maxwell gauge. *Physics Letters 107A*, 2, (1985), 73–74.

[EO] J. Erdmenger and H. Osborn, *Conformally covariant differential operators: symmetric tensor fields*, Class. Quantum Grav. 15, (1998), 273-280.

[FG12] C. Fefferman and C. R. Graham, *The Ambient Metric*. Annals of Math. Studies 178, Princeton University Press, 2012. arXiv:0710.0919
EXTRINSIC PANEITZ OPERATORS AND Q-CURVATURES FOR HYPERSURFACES

S. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary), SIGMA Symmetry Integrability Geom. Methods Appl. 4 (2008), paper 036, 3p.

M. Rangamani and T. Takayanagi, Holographic Entanglement Entropy, Lecture Notes in Physics 931 (2017)

R. Riegert, A nonlocal action for the trace anomaly. Physics Letters B 134, no. 1–2, 56–60 (1984).

S. N. Solodukhin, Entanglement entropy, conformal invariance and extrinsic geometry, Physics Letters B, 665, 305–309 (2008). arXiv:0802.3117

S. N. Solodukhin, Boundary terms of conformal anomaly, Physics Letters B, 752, 131–134 (2016). arXiv:1510.04566v4

Y. Vyatkin, Manufacturing conformal invariants of hypersurfaces, PhD thesis, University of Auckland, 2013.

T. J. Willmore, Riemannian Geometry, Oxford Science Publications, 1993.

V. Wünsch, On conformally invariant differential operators, Math. Nachr. 129, 269–281 (1986).

Y. Zhang, Graham-Witten’s conformal invariant for closed four dimensional submanifolds. J. Math. Study 54, 200–226 (2021). arXiv:1703.08611

Humboldt-Universität, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany
Email address: juhl.andreas@googlemail.com