Possible management of near shore nonlinear surging waves through bottom boundary conditions

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Abstract

We propose an alternative way for managing near shore surging waves, including extreme waves like tsunamis, going beyond the conventional passive measures like the warning system. We study theoretically the possibility of influencing the nonlinear surface waves through a leakage boundary effect at the bottom. It has been found through analytic result, that the controlled leakage at the bottom might regulate the amplitude of the surface solitary waves. This could lead to a possible decay of the surging waves to reduce its hazardous effects near the shore. Our theoretical results are estimated by applying it to a real coastal bathymetry of the Bay of Bengal in India.

Supplementary material for this article is available online

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(Some figures may appear in colour only in the online journal)

1. Introduction

Near shore coastal regions often witness surging of the approaching waves including extreme events like tsunamis [1]. Such a natural phenomenon has also been observed, though in a miniature scale in few rivers around the world as bore waves [2–4]. Famous examples are the river Seine in France and the river Hoogli in India [1, 2]. Such surging waves are suspected to be caused by nonlinear gravity waves, propagating over a decreasing depth bathymetry towards the shore or along upstream river. Such events which can often trigger extremely hazardous effects have attracted intense attention over the centuries and have been studied extensively from both theoretical and practical points of view [1]. The main emphasis of the investigations was to work towards the development of an early warning systems for minimizing the loss of human lives. The present development of the tsunami warning system reached a satisfactory level [5–10], though there is a scope for further research on warning system.

However, such instruments by nature remain as a passive device with not much scope for protecting against the onslaught of surging waves.

There are few situations where the installation of a passive warning system is not enough, while the demand is for more active intervention. This is particularly true for example, in protecting nuclear reactors and related installations, which are located usually at the vicinity of the sea shore due to logistical reasons, against the tsunami threat. As we know, the tsunami of 2004 which brought devastating effects over many countries was a potential threat to the nuclear reactor at Karpakkam in India. The tsunami of 2010 inflicted real calamities in the Fukushima nuclear reactors in Japan [11, 12].

In a relatively smaller scale, the near shore waves and bore waves have caused many destructions to the coastal habitats and in-land rivers throughout the centuries. Therefore along with the traditional warning systems, it is desirable to find ways and means geared towards possible invasive procedures for taming of such hazardous wave phenomena. There are a few suggestions for effective interventions, like planting Mangrove trees along the coastal lines [13], installation of breakwaters at
strategic positions [14–17], and stoppage of erosion by concrete boulders etc.

However, these are mostly indirect ways to counter the surging waves, while we lack proposals on directly attacking the problem, perhaps with the exception of the proposed bubble method, aiming to stop the incoming waves by a stream of fast and strong counter-waves, mixed with air bubbles [18]. Though the last method was proposed more than fifty years back, its feasibility and effectiveness has not been established yet. The attenuation of incident water waves by a curved vane like structure positioned beneath or at the surface of a body of water is described in a patent [19] where the detailed design of the structure is given. An attempt to investigate a way to reduce the devastating effects of a tsunami waves by single and double submerged barrier was carried out in Tel Aviv University [20].

Our aim here is to theoretically explore the effect of a leakage boundary control at the bottom on the surging surface wave amplitude. Though water leakage at the bottom is intuitively a convincing way for regulating a surging surface wave, no such theoretical or experimental proposal seems to exist in the available literature.

In our investigation, the governing nonlinear equations describing unidirectional gravity waves are derived from the basic hydrodynamic equations at the shallow water regime. The key factor responsible for surging of the nonlinear waves approaching to the shore (or in upstream rivers) is the decreasing bathymetry, which triggers the amplitude surge of the surface waves inversely proportional to the water depth, which diminishes continuously along the wave propagation towards the shore.

We intend to study first the effect of the leakage boundary condition (BC) on the nonlinear solitary surface waves of the well-known perturbed Korteweg–de Vries (KdV) equation, propagating in shallow water of constant depth. For analyzing the leakage condition at the bottom we consider the vertical fluid velocity as a function of surface wave profile. By choosing different leakage functions we identify subsequently, through theoretical study, the optimal case inducing maximum amplitude damping to the surface waves.

Our finding shows that the simplest linear dependence of the leakage velocity function on the surface wave profile turns out to be the most favorable one. This knowledge is applied subsequently to a slowly varying bathymetry, which without the leakage condition, as we know would result in a solitary wave solution with increasing amplitude with the water depth decreasing along its propagation. However, when the controlled bottom leakage condition with optimal leakage velocity function, calculated through the proposed analytic method which includes also the near shore bathymetry profile is imposed, the surging amplitude of the wave would meet the counter damping effect. This would result in a managed wave propagating towards the shore with reduced hazardous effect due to the effective damping induced by the changed boundary condition.

Note that, the tsunami-like near shore nonlinear surging waves are really a complex phenomena depending on various aspects of near shore topography, wave interactions with coastal barriers, reflection of waves etc requiring in general a two-dimensional (2D) treatment. However, since our motivation is to stress on the major cause for the nonlinear surfing of the near shore waves, due to the decreasing depth topography and to carry out the analysis at the analytical level, we have to concentrate on some theoretical abstractions for the model, focusing on the primary cause and ignoring the other effects. Since the surging effects of such waves is due to the decreasing depth profile towards the shore, the 1D characteristics of the wave described by KdV equation is grossly sufficient for estimating the hazardous effects of near shore surging waves. Moreover, since such a regime can be analytically handled with the implementation of our controlled boundary leakage mechanism, our concern here is mostly to investigate 1D KdV equation with its different deformations due to variable depth and leakage. Moreover, the tsunami like waves that approaches from deep sea towards the shore, its direction from sea to shore is the main relevant direction which is manifestly one dimensional. 2D effects could necessarily bring extra corrections which could be handled by (2+1)-dimensional wave equations like KP equation which however goes beyond the scope of the present study.

We would like to emphasize that there could be various natural bottom boundary effects inducing damping of the surface wave amplitudes, like porosity [21–30], irregularities, uneven heights, periodic topography, friction [31, 32] apart from the fluid viscosity [33] etc. While the long obstacle can also induce fission of the solitary waves [31]. However our aim here is to study theoretically the possibility of inducing damping effect on the surface wave amplitude artificially created through a controlled leakage mechanism installed at the bottom. This is achieved by changing the bottom BC using the information of the bathymetry profile and the wave solution obtained from our analytic treatment.

An important result of our work is to achieve exact nature of the solutions, in spite of a variable depth bathymetry, which allows finer details and precise predictions of the surface waves. For extracting possible estimates of our theoretical results we have applied it to a real sea shore bathymetry at a high risk coastal zone of the Bay of Bengal near the city of Chennai in south India as presented in a recent in depth study of the subject [34].

Since this is a theoretical work only the idea of implementation of the leakage mechanism based on our analytical result, changing appropriately the bottom BC, could be provided. The boundary leakage device could be implemented at a reasonable distance from the shoreline, where at the initial moment the surface wave profile of the surging waves (for the decreasing bathymetry) should be captured to extract the solution for the propagating wave assuming the changed boundary condition using our analytic algorithm. This nonlinear wave solution together with the information about the near shore bathymetry should be fed to activate the leakage device installed at the bottom. Such a leakage device is supposed to manage the propagating surface wave profile through bottom boundary condition, by reducing the wave amplitude as well as the speed and the width of the
propagating wave, until the surface wave has reached to a tolerable height. The leakage mechanism therefore expected to consume less and less energy by inducing lesser leakage, as the amplitude and width of the surface wave are gradually reduced.

However, for practical implementation of the present theoretical idea, many experimental as well as field work is needed for optimal implementation and practical use of the device.

The paper is organized as follows. In section 2, the shallow water problem in constant depth with a specific class of bottom BC with controlled leakage is considered theoretically. Corresponding surface wave evolution equation is derived for optimal leakage condition and the analysis for other leakage functions is given as supplementary material. In the next section the variable depth problem is taken up, for the optimally controlled leakage and the related wave equation is derived extracting exact solution for a tuned balance between the effects of leakage and the variable bathymetry. For an estimation, we have applied our theoretical result to a near shore bathymetry data in section 4. Section 5 is the concluding section followed by the bibliography. In the supplementary material we have presented our findings to cover different cases of the controlled bottom leakage conditions, ranging from space dependent to time dependent, from vanishing of effective leakage velocity to a desirable leakage conditions etc. The supplementary material also contains our analysis for a possible management of near shore surging waves at another southern coastal point of the Bay of Bengal using real bathymetry data.

2. Bottom boundary condition and its effect on nonlinear surface wave in constant depth

The purpose of this section is to study the effect of BC at the bottom, designed with a initial feedback from the surface wave, where the leakage velocity function would depend on the wave profile and its spatial derivatives calculated analytically through the present scheme. The leakage velocity independent of the wave profile has been investigated earlier [44]. As is well known that, the nonlinear free surface gravity waves propagating in a shallow water in constant depth with the traditional hard bed boundary condition in the form of solitary waves retain their constant amplitude profile with a high degree of stability. However, when the boundary condition is changed to a leakage function dependent on the wave profile itself, as we find here, the solitary waves propagating on the surface suffer an amplitude damping along its propagation. Different forms of the leakage velocity function at the bottom induce different types of damping. Such a controlled leakage at the bottom might be arranged by tuning the device to sustain a consistent leakage BC according to the wave profile derived analytically from the governing equation. Our motivation for this study is to analyze different damping effects corresponding to different leakage functions and identify the case when the damping would be maximum, which is the most desirable feature in the present context.

In the following subsections we derive the corresponding free surface wave equation and investigate the nature of the solitary wave solution with damping caused by different cases of the leakage condition at the bottom.

2.1. Surface wave evolution equation with leakage boundary condition

We consider here the shallow water nonlinear surface-gravity wave, propagating along the positive $x$-direction in a constant water depth with the viscosity and the surface tension of the fluid, which is assumed to be incompressible, are neglected in what follows. We start from the dimensionless basic hydrodynamic equations [35]:

\[
\begin{align*}
\frac{u_t + \epsilon (uu_x + uw_z)}{} &= -p_x, \\
\delta^2 [w_t + \epsilon (uw_x + w w_z)] &= -p_z, \\
\end{align*}
\]

along the $x$ and the $z$ axis, respectively, which are reducible from the Euler equation in the present case.

Here $u, w, p, \eta$ are horizontal and vertical fluid velocity components, pressure and the surface wave profile, respectively, with the subscripts denoting partial derivatives. $\epsilon$ is the amplitude parameter defined by $\epsilon = \frac{a}{l}$ and $\delta = \frac{h}{l}$ is the shallowness parameter, expressed through the maximum amplitude $a$, the water depth $h$ and the wavelength $l$ (see figure 1). $\epsilon$ and $\delta$ are natural parameters supposed to be small, which is consistent with the long wave and the shallow water limit. The continuity equation of the fluid yields

\[ u_x + w_z = 0. \]

Nonlinear variable boundary conditions, valid at the free boundary $z = 1 + \epsilon \eta$, on the other hand, gives

\[ p = \eta, \quad w = \eta_t + \epsilon w\eta_z, \]

while we take the boundary condition for the vertical component of the water velocity at the bottom: $z = 0$ as

\[ w = -\epsilon \delta \alpha G (\eta, \eta_x, \ldots), \]

where $G(\eta, \eta_x, \ldots)$ is assumed, in general, to be an arbitrary function of $\eta$ and its spatial derivatives and $b\delta \alpha$ is a positive constant with $\epsilon$ being a small parameter as defined above. It is important to note here, that usual hard bed scenario with no
leakage one would have \( w = 0 \) at the bottom whereas in our choice the nontrivial leakage velocity function \( G \) may depend functionally on the surface wave profile derived analytically using the surface wave profile as initial condition.

The leakage is considered here to be in the \( \epsilon \) order. Note that the negative sign in equation (4) appears because the leakage velocity occurs along the negative \( z \)-direction, i.e., vertically downward. In order to model shallow water solitary waves, there must be an appropriate balance between non-linearity and dispersion, i.e., \( \delta^2 = O(\epsilon) \) as \( \epsilon \) tends to zero. Thus for any \( \delta \), there exists a region in \((x, t)\)—plane with \( \epsilon \) tending to zero, where this balance remains valid. This region of our interest may be defined by a scaling of independent variables as \( x \to \frac{\delta}{\sqrt{\epsilon}} x, t \to \frac{\delta}{\sqrt{\epsilon}} t \) and \( w \to \frac{\delta^2}{\epsilon} w \) for any values of \( \epsilon \) and \( \delta \). The set of equations (1)-(4) thus becomes,

\[
\begin{align*}
\eta_t + \epsilon (u \eta_x + w \eta_z) &= -p_x, \quad \epsilon [w_t + \epsilon (u \eta_x + w \eta_z)] \\
&= -p_x, \quad u_x + w_z = 0,
\end{align*}
\]

valid at the free surface and at the bottom, respectively, where \( \alpha = \frac{\delta}{\sqrt{\epsilon}} \), with a net outcome of the transformation is to replace \( \delta^2 \) by \( \epsilon \) in equations (1)-(4). Introducing a new frame of reference with stretched time \( \xi = x - t, \tau = \epsilon t \), and boundary conditions in the form

\[
q(\xi, \tau, z; \epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n q_n(\xi, \tau, z),
\]

\[
\eta(\xi, \tau; \epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n \eta_n(\xi, \tau),
\]

(8)

where \( q \) (and related \( q_n \)) represents each of the functions \( u, w \) and \( p \) for the corresponding expansion.

Now to deduce the final evolution equation from the set of complicated nonlinear equations (5)-(7) involving several variables, we have to make the asymptotic multi-scale expansions as explained above. Below, we carry out an explicit order by order calculation to demonstrate the process.

2.1.1. Result at \( \epsilon^0 \) order. At \( \epsilon^0 \) order the above set of equations (5)-(7) is reduced respectively to the following set

\[
\begin{align*}
u_{0t} &= p_{0\xi}, \quad p_{0\xi} = 0, \quad u_{0\xi} + w_{0z} = 0 \quad (9) \\
p_0 &= \eta_0, \quad w_0 = -\eta_0z, \quad (10) \\
w_0 &= 0, \quad (11)
\end{align*}
\]

with equation (10) valid at \( z = 1 \) and (11) at \( z = 0 \). These equations lead to the solutions expressed through \( \eta \) as \( p_0 = \eta_0, \quad u_0 = \eta_0, \quad \eta_0z \), with the appearance of \( \eta \) caused only by the passage of the wave has been imposed, i.e., \( u_0 = 0 \), whenever \( \eta_0 = 0 \).

2.1.2. Result at \( \epsilon \) order. In this order of approximation, two free boundary conditions at \( z = 1 + \epsilon \eta \) are evaluated by performing Taylor expansions of the functions \( u, w, p \) around the point \( z = 1 \). Consequently the following set of equations are obtained from (5)-(7):

\[
\begin{align*}
-\eta_{1\xi} + u_{0\tau} + w_{0\xi} + w_{0\eta} u_{0\xi} &= 0 \\
-\eta_{1\xi} + p_{1\xi} &= w_{0\xi}, \quad u_{1\xi} + w_{1z} = 0 \quad (12)
\end{align*}
\]

with the boundary conditions:

\[
\begin{align*}
p_1 + \eta_0 p_0 &= \eta_1, \\
w_1 + \eta_0 w_0 &= -\eta_{1\xi} + \eta_0z + \eta_0 \eta_{0\xi}, \quad (13)
\end{align*}
\]

valid at \( z = 1 \). We also get from the BC at the bottom: \( z = 0 \), the relation

\[
w_1 = -\alpha G_0(\eta_0, \eta_0z, \ldots) \quad (14)
\]

where \( G_0 \) is the contribution of the leakage function at \( \epsilon^0 \) order. Integrating the third equation of (12) and using its first equation together with the relation \( p_0 = \eta_0 \) obtained earlier. \( w_1 \) can be expressed now as

\[
w_1 = -\left( \eta_{1\xi} + \eta_{0\tau} + \eta_0 \eta_{0\xi} + \frac{1}{2} \eta_{0\xi}z \right) \eta_{0\xi} + \frac{1}{6} \eta_{0\xi}^2 = \alpha G_0, \quad (15)
\]
giving thus all other functions expressed through the fields \( n_0 \) and \( \eta \), only, in this order of approximation. Note, that the expression of \( w_1 \) in equation (13) is obtained independently as surface boundary condition.

Finally comparing \( w_1 \) from equation (15) and from the surface BC in (12) along with eliminating \( \eta \) we obtain the free surface wave equation as

\[
2\eta_{0r} + \frac{1}{3} \eta_{0xx} + 3\eta_0 \eta_{0} + \alpha G_0 = 0, \tag{16}
\]

with an additional term due to the wave profile dependent bottom leakage velocity appearing in the well known integrable KdV equation [36], which however spoils the integrability of the system, in general. With a scaling of the variables as \( U = 9\eta_0, T = \tau/6 \) equation (16) takes a normalized form

\[
U_T + UU_T + U_{xx} + \beta G_0 = 0, \tag{17}
\]

where \( \alpha \) is scaled to \( \beta \) and \( G_0(U, U_T, \ldots) \) is an arbitrary smooth function, originating from the wave profile dependent leakage velocity. It is fascinating to note, that the condition, we impose for the fluid velocity at the bottom through a boundary condition with wave profile dependence makes it way to the nonlinear evolution equation at the surface.

Note that equation (17) is an extension of the KdV equations with arbitrary higher nonlinearity, which in general represents a non integrable system. However an approximate method due to Bogoliubov and Mitropolsky [33, 37, 38] could be applied here for extracting analytic solutions for the wave equation (17), in general, in an implicit form. For explicit analytic solution, one needs to make suitable choices for function \( G_0 \). We focus below on some of such choices with lower order nonlinearities, e.g. \( G_0 = U, U^2, U^3, U^4 \) though this set, in principle, can be extended further. We do not put emphasis on the physical meaning for the individual forms of the leakage function, since our main motivation is to compare theoretically the result of the corresponding wave solutions, to identify the case that would induce maximum damping of the wave amplitude. It is intriguing to note, that similar equations for some of the cases considered by us were obtained earlier [31, 33], though in completely different physical set-ups.

In order that this approximation scheme to be consistent with the condition for the validity of (17), it is required that the leakage coefficient \( \beta \) should be a small parameter of order higher than \( \epsilon \) as \( 1 \gg \beta \gg \epsilon \).

Introducing a phase coordinate \( \phi(T, \beta) = \sqrt{N(T, \beta)/12} \left( \xi - \frac{1}{3} t N(T, \beta) dT \right) \), through a time-dependent function \( N(T, \beta) \), assumed to vary slowly with time, with two different time scales \( t_0 = T, t_1 = \beta T \), we seek a solution of the wave equation following [38]. By expanding \( U(\phi, \beta, T) \) in small parameter \( \beta \) as

\[
U(\phi, \beta, T) = U_0(\phi, t_0, t_1) + \beta U_1(\phi, t_0) + O(\beta^2), \tag{18}
\]

valid for long times, (as large as \( T \sim O(1/\beta) \)), we obtain using (17) an equation containing different powers of \( \beta \). Since, we are interested only to the solution in the first order of \( \beta \), the variable \( t_1 \) is not included in the solution \( U_1 \) since it would contribute only to the higher order. Equating coefficients of the same powers of \( \beta \), equations at different orders are derived, which need to be solved at each order.

### 2.2. \( G_0 = U \)

Our investigation with a different form of leakage velocity functions \( G_0 \) shows interestingly that the simplest linear choice \( G_0 = U \) induces maximum damping of the surface wave which is the optimal condition we are seeking at. Therefore, keeping only the details for the optimal case below we shift the other calculations to the supplementary section listing here the final result for the different cases considered.

The analytic solution of this optimal case governed by the perturbed KdV equation can be carried out by the application of Bogoliubov Mitropolsky approximate scheme. We may note by passing that the equation obtained in this case is mathematically equivalent to the dissipation induced evolution considered in the context of ion-sound waves damped by ion-neutral collisions [33].

Integrating equation (17) for \( G_0 = U \), over the whole range of \( \xi \) we can solve for the total wave amplitude

\[
I(T) = \int_{-\infty}^{\infty} U d\xi \text{ and the total intensity of the wave } P(T) = \int_{-\infty}^{\infty} U^2 d\xi \text{ to get the explicit expressions as } I(T) = I(0) e^{-\beta T} \text{ and } P(T) = P(0) e^{-2\beta T}, \tag{19}
\]

respectively, where \( U(\xi, T) \) and its higher order \( \xi \) derivatives are assumed to vanish at infinity. It is also evident from the exponentially decaying nature, that the wave intensity is not conserved in time, confirming that the integrability of the perturbed KdV equation (17) in this case is lost due to the leakage we have considered here.

Since estimating the damping of the solitary water waves is the main concern of our problem, we take the following relations as the required initial and boundary conditions:

\[
U(\phi, 0, \beta) \sim N_0 \text{sech}^2(\phi), \quad U(\pm \infty, T, \beta) = 0. \tag{20}
\]

Inserting expansion (18) on the perturbed KdV equation (17), we obtain in the zeroth order of \( \beta \),

\[
\rho \frac{\partial U_0}{\partial t_0} + \frac{\partial^3 U_0}{\partial \phi^3} - \frac{4}{3} \frac{\partial U_0}{\partial \phi} + \frac{12}{N} \frac{\partial U_0}{\partial \phi} = 0, \tag{19}
\]

where \( \rho = \frac{12 \sqrt{12}}{N D_{N}} \), with \( N(t_1) \) as an arbitrary function of \( t_1 \), except for the initial condition \( N(0) = N_0 \). The equation in the \( \beta \) order on the other hand takes the form

\[
\frac{\partial U_1}{\partial t_0} + L[U_1] = M[U_0], \tag{20}
\]

where,

\[
M[U_0] = -\frac{\partial U_0}{\partial t_0} - \frac{\partial}{\partial \phi} \frac{\partial U_0}{\partial \phi} \frac{dN}{\partial t_0} - U_0,
\]

\[
L[U_1] = \frac{1}{\rho} \frac{\partial^3 U_1}{\partial \phi^3} - \frac{4}{2N} \frac{\partial U_1}{\partial \phi} + \frac{12}{N} \frac{\partial (U_0 U_1)}{\partial \phi}. \tag{21}
\]
Note that here the contributions from both the terms $U_0$ and $U_1$ enter in the equation. Our aim is now to solve equation (19) for $U_0$ and use equation (20) to study the secular behavior of $U_1$ following Bogoliubov—Mitropolsky scheme, which in turn would help to obtain the final solution $U_0$. The solution scheme is formulated in paper [37]. To clarify the steps we present the the solution for $U_0$ through an ansatz as $U_0 = g(t)f(\phi, t_0)$ to justify the slowly varying amplitude of the wave through function $g$. Since our main concern is to estimate the possible damping of the amplitude of the waves, the required initial condition when the leakage implementation has not yet started and the evolution is governed by the pure KdV equation only is the standard soliton profile $U(\phi, 0, \beta) = N_0 \text{sech}^2(\phi)$. The natural boundary conditions on the other hand for the localized waves is $U(\pm \infty, T, \beta) = 0$. Now putting the ansatz in (19), if we assume $g(t_0) = N(t_0)$, we are left with a deformed KdV equation for $f$. However with a further assumption of a KdV soliton solution for $f = \text{sech}^2(\phi + c_1 t_0)$, which is consistent with the initial and boundary condition the equation is finally solved resulting to the condition $c_1 = 0$, showing independence of $U_0$ on $t_0$. Therefore the form of the solution is obtained as

$$U_0(\phi, t_0, t) = N(t_0) \text{sech}^2(\phi),$$  \hspace{1cm} (22)

where $N(t_0)$ will be determined from the secularity condition considered in the next order. Note that, the boundary and initial conditions for perturbative contribution $U_1$ are $U_1(\pm \infty, t_0) = 0$, $U_1(\phi, 0) = 0$. In order that (18) to be valid for large times while it is a natural requirement that, $U_1(\phi, t_0)$ should not behave secularly with $t_0$ for long time. To eliminate secular behavior of $U_1$ it is necessary that $M[U_0]$ be orthogonal to all solutions $g(\phi)$ of $L^+ [g] = 0$, where the function $g(\phi)$ should satisfy $g(\pm \infty) = 0$. Here $L^+$ is the operator adjoint to $L$ given by

$$L^+ = \frac{1}{\rho} \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} \right) + \frac{4}{\rho} \frac{\partial}{\partial \phi} - \frac{12}{\rho} \text{sech}^2(\phi) \frac{\partial}{\partial \phi},$$  \hspace{1cm} (23)

One can show, that the only possible solution of $L^+ [g] = 0$, with $g(\pm \infty) = 0$, is in the solitonic form $g(\phi) = \text{sech}^2(\phi)$. Thus from the orthogonality requirement we get

$$\int_{-\infty}^{\infty} \text{sech}^2(\phi) M[U_0] d\phi = 0,$$  \hspace{1cm} (24)

which yields a simple first order differential equation for $N(t_0)$, the solution of which is

$$N(t_0) = N(0) \exp \left( -\frac{4 t_0}{3} \right), \quad t_0 = \beta T$$  \hspace{1cm} (25)

for positive small leakage parameter $\beta$ at large time $T$. Therefore using the explicit form of (25) we obtain the final result as

$$U = N(t_0) \text{sech}^2 \phi(\xi, t_0) + O(\beta),$$

$$\phi(\xi, t_0) = \sqrt{\frac{N(t_0)}{12}} \left( \xi + \frac{1}{4 \beta} N(t_0) \right).$$  \hspace{1cm} (26)

The wave solution of equation (17) thus obtained for $G_0 = U$, shows that the amplitude of the solitary wave would decrease with time following (25).

Recall that similar dissipative soliton solution was derived earlier in many different physical situations [31, 33]. Similar analysis for different forms for the leakage velocity functions $G_0$ leads to the following result.

II. $G_0 = U^2 : \quad N(T) = \frac{N(0)}{[1 + \frac{2G_0}{3}] T}$

III. $G_0 = U^3 : \quad N(T) = \frac{N(0)}{\sqrt{[1 + \frac{6G_0}{5}] T}}$

IV. $G_0 = U^2 : \quad N(T) = \frac{N(0)}{\sqrt{[1 + \frac{5G_0}{6}] T}}$

The damping of the wave amplitude suffered by nonlinear leakage functions, is reproduced above while relegating the detailed calculation for obtaining such solutions to the supplementary section. Though we have not directly used this result for nonlinear leakage function, they have influenced our investigation by providing a series of examples for picking up the optimal one with maximum damping, which is our question of relevance here. We discover that the optimal case fortunately coincides with the simplest linear function which is explained in 2.2.

Comparing the above result we may conclude, that the increase of nonlinearity as well as derivatives, of the wave profile in the leakage velocity function weakens the decay rate of the solitonic amplitude and the linear dependence $G_0 = U$ gives the maximum damping effect. Therefore in what follows concerning the variable depth situation we stick only to the linear case.

3. Effect of leakage BC on nonlinear shallow water surface wave in variable depth bathymetry

Propagation of nonlinear shallow water unidirectional waves over variable depth topography has been studied intensively with rich results [1, 31, 32, 35] (see figure 2).

It is known that the slowly variable depth in comparison to the evolution scale of the wave, can lead to the upsurging wave amplitude, for decreasing depth, which occurs when the wave approaches to the shore. In this section we intend to focus on such a situation due to its potentially hazardous consequences and look for its possible intervention through bottom leakage BC. Since in the previous section we have identified the maximum damping effect of surface waves for leakage velocity function linearly dependent on the wave profile, we will apply this particular leakage condition to achieve maximal damping effect. Therefore we take up the problem of nonlinear wave propagation over shallow water of slowly varying depth in the framework of KdV equation, together with a nontrivial leakage condition at the bottom with a leakage function proportional to the surface wave profile, derived analytically using the initial surface wave through a feedback mechanism. It is worth mentioning that this important problem targetted towards managing directly the surging...
waves caused by the decreasing depth bathymetry, has not received the needed attention.

### 3.1. Nonlinear surface wave evolution with variable depth under bottom boundary condition with leakage

Under this physical situation one has to start with the same basic dimensionless hydrodynamic equations considered in the previous section as:

\[
u_t + \epsilon (uu_x + wu_z) = -p_x, \quad \epsilon [w_t + \epsilon (uw_x + ww_z)] = -p_z, \quad u_x + w_z = 0,
\]

(27)

together with the surface boundary conditions

\[p = \eta, \quad w = \eta_x + \epsilon \eta_{xx} \quad \text{valid at } z = 1 + \epsilon \gamma.\]

However, the effect of variable depth and the leakage condition enter through a more general boundary condition at the bottom, varying as \(z = b(x)\):

\[w = \frac{db}{dx} - \epsilon g(x)G(\eta, \eta_x, \ldots).\]

(28)

Note that in comparison with the previous case (4) together with the variable depth function an additional leakage function \(g(x)\) independent of the wave profile \(\eta\) appears with the \(\eta\) dependent leakage function \(G\). We see below that the function \(g\) should depend on the small parameter \(\epsilon\), such that \(b(x) = B(\epsilon x)\). As we have identified in previous section, we assume \(G = \eta\) to get the maximum benefit of damping due to leakage. For detailed investigation we introduce a new set of variables

\[\xi = \frac{1}{\epsilon} \chi(X) - t, \quad X = \epsilon x,\]

(29)

where \(\chi(X)\) will be determined later in equation (32). For solving the above set of equations we would represent the asymptotic solutions as we have used earlier.

We stress again that the hydrodynamic equations involved here are the same as those used in the previous section in dealing with the constant depth problem, except the crucial BC at the bottom.

#### 3.1.1. Result at \(\epsilon^0\) order

At \(\epsilon^0\) order, the above equations are reduced to

\[u_{0\xi} = \chi' p_{0\xi}, \quad p_{0\xi} = 0, \quad \chi' u_{0\xi} + w_{0z} = 0, \]

(30)

together with the boundary conditions \(p_0 = \eta_0, \quad w_0 = -\eta_{0\xi}\), valid at the surface and \(w_0 = 0\), at the variable bottom \(z = B(X)\).

Using the above bulk equations and the boundary conditions we obtain

\[p_0 = \eta_0, \quad u_0 = \chi' \eta_0, \quad w_0 = \chi'^2 \eta_{0\xi} (B - z), \quad \chi'^2 = \frac{1}{D(X)},\]

(31)

where \(D(X) = 1 - B(X)\) and \(\chi'\) is the derivative of \(\chi\) with respect to \(X\). \(\chi\) can be solved explicitly through the bathymetry function for the right moving wave as

\[\chi(X) = \int_0^X \frac{dX'}{\sqrt{D(X')}}.\]

(32)

#### 3.1.2. \(\epsilon\) order approximation

In next order approximation we obtain the set of equations

\[-u_{\xi\xi} + \chi'' u_{0\xi\xi} + w_0 u_{0\xi} = -\chi' p_{1\xi} - p_{0\xi\xi}, \]

\[p_{1\xi} = w_{0\xi\xi}, \quad \chi' u_{1\xi} + u_{0\xi} + w_{0z} = 0\]

(33)

together with the surface boundary conditions

\[p_1 = \eta_1, \quad w_1 + \eta_0 w_{0\xi} = -\eta_{1\xi} + u_0 \chi'' \eta_{0\xi},\]

(34)

and the condition

\[w_1 = u_0 B'(X) - g(X) \eta_0.\]

(35)

valid at the variable bottom with \(B'(X)\) denoting derivative in \(X\). Our aim is to express other field variables only through the wave functions \(\eta_0\) and \(\eta_1\) as

\[p_1 = \eta_1 + \frac{1}{D} \eta_{0\xi\xi} \left[ \frac{1}{2} (1 - z^2) + B(z - 1) \right] \]

(36)

and

\[w_1 = \frac{B'}{D} \eta_0 - \eta_{0\xi} \left( \frac{B - z}{\sqrt{D}} \eta_{0\xi\xi} + \frac{B - z}{\sqrt{D}} \eta_0 \right) \]

\[+ \frac{(B - z)}{D} \frac{\eta_{0\xi}}{D} + \frac{(B - z)}{D^2} \eta_0 \eta_{0\xi} - \frac{\eta_{0\xi\xi}}{D^2} \left[ B \left( \frac{z^2}{2} - z \right) + \frac{(z^2 - 1)}{2} - \frac{B^3}{3} + B - \frac{B}{2} \right].\]

(37)

Using the above expressions we can finally derive the surface wave evolution equation

\[2 \sqrt{D} \eta_{0\xi} + \frac{3}{D} D'' \eta_0 \eta_{0\xi} + \left( \frac{D'}{2 \sqrt{D}} + g \right) \eta_0 + \frac{D}{3} D \eta_{0\xi\xi} = 0.\]

(38)

Note that this variable coefficient KdV equation contains explicitly the bathymetry function \(D(X)\) linked to the variable depth as well as the function \(g(X)\) related to the leakage at the bottom. This variable coefficient KdV equation containing the combined effect of variable depth and the leakage is an important result we have derived here. Different types of variable coefficient KdV like equations were studied earlier for analyzing the possible solutions both in one [39–42] and two dimensions [43, 45].

#### 3.2. Nature of the solitary wave solution

It is evident that in the absence of the leakage \((g = 0)\), our equation (38) would reduce to the KdV equation with variable depth \([1, 31, 35]\):

\[2 \sqrt{D} \eta_{0\xi} + \frac{3}{D} D'' \eta_0 \eta_{0\xi} + \left( \frac{D'}{2 \sqrt{D}} \right) \eta_0 + \frac{D}{3} D \eta_{0\xi\xi} = 0.\]

(39)
When the depth variation occurs in a scale slower than the evolution scale of the wave, the solitary wave solution of equation (39), as is well known, can be expressed as an approximate solution

$$\eta_0 = \frac{A_0}{D} \text{sech}^2 \left[ \sqrt{\frac{3A_0}{4D^3}} \left( \xi - \frac{D^{-1/2}A_0}{2} \right) \right], \quad (40)$$

as given in [35]. Here $A_0$ is the amplitude of the wave for constant depth ($D = 1$). It is clearly seen that the amplitude of the solitary wave increases as $D$ decreases i.e. the water becomes shallower, showing that such waves would approach the shore with surging amplitude. At the same time, at a closer look shows that along such propagation, the velocity of the wave decreases as also the width of the solitonic wave profile. These characteristics of the near shore surging waves are consistent even with the tsunami waves.

It is intriguing to note that for variable bathymetry with uneven depth, irregular depth or periodic topography in place of growing amplitude one gets a damping wave amplitude as explained in [32]. We will be concerned however with the surging waves caused by a smoothly decreasing depth due to their hazardous effects.

Now we will analyze the solution of equation (38) with nontrivial boundary leakage, rewriting it in a more general form

$$a(X)\eta_0X + b(X)\eta_0\eta_0\xi + c(X)\eta_0 + d(X)\eta_{0\xi\xi\xi} = 0, \quad (41)$$

where we have denoted $a(X) = 2\sqrt{D}$, $b(X) = \frac{3}{D}$, $c(X) = \left( \frac{D'}{2\sqrt{D}} + g \right)$ and $d(X) = \frac{b}{D}$. Dividing (41) by $d(X)$ and defining $\eta_0 = \frac{V}{b}$ where $a_1 = \frac{a}{d}$, $b_1 = \frac{b}{d}$ and $c_1 = \frac{c}{d}$ respectively, the equation (41) can be transformed to

$$a_1U_X + UU_k + U_{\xi\xi\xi} + \left( c_1 - a_1 \frac{b_{1\xi}}{b_1} \right) U = 0, \quad (42)$$

which in general cannot be solved exactly. However, we may notice, that for a finer balance tuned between the variable depth bathymetry and the controlled leakage velocity function giving the condition

$$g = -\frac{9D'}{2\sqrt{D}}, \quad (43)$$

i.e., when the leakage function $g$ depends on the near shore variable depth bathymetry in a specific way, the last term of (42) vanishes reducing the equation to a more simple form of variable coefficient KdV equation

$$a_1U_X + UU_k + U_{\xi\xi\xi} = 0, \quad (44)$$

where $a_1 = \frac{a(X)}{d(X)}$. It is interesting to note, that the tuning condition (43) relating the leakage velocity function with the bathymetry function is exactly same as the solvability condition used in [40] for obtaining analytic solutions of a general variable coefficient KdV equation, considered in a formal mathematical setting.

Defining a new coordinate $T = \int \sqrt{D(X)} \, dX$ equation (44) can be transformed into the standard constant coefficient KdV equation

$$U_T + UU_k + U_{\xi\xi\xi} = 0, \quad (45)$$

admitting the well known solitary wave solution

$$U = N_0 \text{sech}^2 \left[ \frac{N_0}{\sqrt{12}} \left( \xi - \frac{V}{3} \int \sqrt{D(X)} \, dX \right) \right].$$

Expressing in terms of the original field variable we get finally the wave solution

$$\eta_0 = A(X) \text{sech}^2 \left[ \frac{N_0}{\sqrt{12}} \left( \xi - V(X) \right) \right].$$

$$A(X) = \left( \frac{D(X)^2}{9} N_0 \right)^{1/3} V(X) = \frac{N_0}{3} \int \sqrt{D(X)} \, dX \quad (46)$$

with the depth function $D(X)$ and leakage velocity function $g(X)$ are tuned as (43). Note that for decreasing depth $D$, which without leakage would make the wave amplitude to surges as in (40), due to the controlled tuning of the leakage the resultant solitonic wave function would suffer a damping of its amplitude as evident from (46) (see figure 3). However a detailed analysis of solitonic solution (46) shows that the speed of the wave would continue to decrease as it propagates towards the shore together with its width. Thus we have achieved control over a surging wave approaching to the shore by inducing combination of feedback and a controlled tuning of the leakage at the bottom. Summarising the scheme of our solution, let us note here that in the first step we have considered the leakage functions (linear as well as nonlinear) in nonlinear KdV type equation for a constant depth.

This nonlinear equation could be solved for all leakage function only approximately using Bogoliubov–Mitropolsky scheme. Comparing these results one may conclude that, the linear leakage function would induce maximum damping, the main relevant point of our concern. Therefore, in our next step applicable to KdV like equation with decreasing depth bathymetry, we have used only the linear leakage function, assuming that the other cases will similarly give less damping effect, though in a nonlinear cases the explicit formula is not possible to derive analytically.

We are fortunate that the linear leakage function which gives us the optimal regime for maximum amplitude decay for surface wave at the same time, solves also the proposed nonlinear KdV type equation with variable bathymetry exactly which gave us the final result (46).

We have studied also various possible extensions of the leakage boundary conditions and their corresponding effects in modifying the nature of the surging solitary waves which might be of practical importance in different other situations. (This material is included as the supplementary material.)

### 4. Estimation of our exact result on real bathymetry

Our theoretical results are of an exact nature with an intention to manage near shore surging waves, by creating an artificial bottom boundary condition, is applied to a real sea shore bathymetry in order to see the effectiveness of our theoretical
findings in the coastal region of Chennai district of the Tamilnadu state. This southeast coast of India was one of the worst affected areas during the 2004 Indian Ocean tsunami. A Coastal Vulnerability Index was developed for this region using eight relative risk variables including near shore bathymetry to know the high and low vulnerable areas. According to one of those risk variables, bathymetry at about 29.11 km of coastline in that area has a high risk rating having high vulnerability, while about 18.55 km of coastline has medium risk rating and about 10.54 km shows low risk rating, which are displayed in Figure 4.

The depth contour of the Chennai coastline, which is constructed from the Naval Hydrographic Charts for 2002, is also given in [34] and is displayed in figure 5. Therefore, we aim to apply our proposed wave management scheme to one of the high risk point in the above coastal zone (N 13° 10.5’ - E 80° 18.75’), which is denoted by the red line in figure 4.

We have drawn the near shore bathymetry using the depth contour data (figure 5) of this shoreline point along the latitude which is given as figure 6. Note also that, the variation along X is in km whereas variation along D(X) is in meter. Hence the depth function is varying very slowly, which is consistent with our theoretical assumptions.

In the absence of the leakage at the bottom, the solitary wave amplitude would increase following (40) with the amplitude as $A_t = \frac{A_0}{D(X)}$. As demonstrated in figure 6, as the depth function flattens out near the shore region, the soliton amplitude $A_1$ develops rapidly to give surging effects. It may also be concluded from the figure that if the artificial leakage device at the bottom is implemented from a certain point away from the shore, as our theoretical findings (46) shows, it could be possible to manage the wave amplitude to the form $A_2 = \frac{N_0 D(X)}{9}$, where $N_0$ is a free constant chosen to match the initial condition consistent with the wave solution in the absence of the leakage condition. It is important to note that, the amplitude decay of the surging solitary waves would be stronger, as may be seen by comparing the wave amplitudes A, B, C in figure 7, when the implementation of the leakage device starts at a longer distance away from the shore. We may see from the result shown in figure 7, that when a solitary surface wave of amplitude of about 1 meter starts approaching towards the shore from around 10.5 km, it would grow to a surging wave of amplitude $\sim 30$ meter at the coast, if no damping leakage device is installed. It is obvious that such a huge wave will produce devastating effects on coastal habitation and costly installations. However we see from the same figure that if we implement a controlled leakage device at the bottom, as proposed here based on our theoretical result, the surging amplitude would gradually decrease according to our
imposes a damping effect on the surface wave to manage its surging amplitude to a reasonable level as it reaches the shore, with possible minimization of its damaging effects.

Note that, though such an idea is consistent with our intuition, no earlier attempts were made to study or implement such an important idea to intervene directly with a hazardous near shore surging waves. Though our first attempt in this direction is theoretical one, we could achieve an analytical result and apply it to a real near shore bathymetry in a tsunami prone coastal region in the Bay of Bengal. However, for practical application of the feedback leakage system proposed here for the possible management of near shore nonlinear surging waves, thorough and intensive laboratory as well as field work must be undertaken.

The majority of the earlier studies, concentrated on the damping of the waves occurring due to natural effects like viscosity, bottom roughness, sand porosity etc in which we have no control. In contrast, our main motivation here is to analyze the impact of artificially created bottom boundary condition created on the swelling wave approaching the shore, with an aim to manage the intensity of the hazardous effect of such near shore wave phenomenon.

Our crucial observation is, that the surging of approaching waves caused by decreasing water depth bathymetry might be thought of as being triggered by effective vertical fluid flow proportional to the gradient of the depth profile, acting as a virtual source emerging from the bottom. Our key idea for managing the growing amplitude of the surface wave is to counter this source by an effective sink through such leakage mechanism creating a downward fluid velocity.

We have considered the propagation of an unidirectional, shallow water, nonlinear free surface gravity wave based on the basic hydrodynamic equations at the shallow water regime and identified first, that a leakage velocity function at the bottom, dependent linearly on the surface wave profile, could induce the maximum desirable damping effect on the amplitude of the surface wave. This knowledge is applied further for managing the surging solitary waves propagating towards the shore, in the slowly decreasing depth. The corresponding evolution equation for the combined effect of leakage and the variable bathymetry turns out to be in the form of a variable depth KdV equation, different from the variable coefficient KdV equation obtained earlier. Though in general this is a non-integrable system, we have found, that for a controlled tuning between the topography and leakage velocity function, the equation becomes exactly solvable, allowing solitary wave solutions with damping amplitude.

A strong point of our result is its exact nature, which allows one to access precise and finer effects and make more accurate predictions. To have an estimation of the range and prediction of our theoretical result, we have applied it to real data from the bathymetry map of the tsunami prone near shore regions on the Bay of Bengal in India. As shown by the real bathymetry, the more extensive installations starting from a further distance into the sea would result in a more effective management of the incoming surging waves. However, since our theoretical result is a preliminary one proposed for the

5. Concluding remarks

The focus of our investigation is a theoretical possibility of managing the intensity of near shore surging waves, including extreme waves like tsunamis and bore waves, by inducing a damping effect through a specially designed leakage mechanism installed at the water bed, starting at a convenient distance from the shore. The idea is to capture the surging surface profile approaching the shore at a suitable time and using this wave profile as an initial condition extract the solitonic solution analytically from an exactly solvable KdV like equation obtained here under variable bathymetry and nontrivial bottom BC. The near shore bathymetry information together with the wave solution derived is fed into the leakage device as a controlled boundary condition. This in turn exact result (46) as the wave propagates towards the shore. In particular when the leakage device is implemented in a region of 0.9 km from the shore (Q₁ in figure 7), the wave amplitude of 1 meter which would otherwise increase to 30 meter without any leakage (D in figure 7), would decrease to an amplitude of ~1.23 meter (denoted by point A in figure 7).

For optimal estimation however, the cost effectiveness and the concrete requirements should be taken into account in deciding the range of such proposed installations. The main Emphasis should possibly be on the protection of sensitive installations like nuclear reactors at the sea coast against the danger of tsunami like waves.

For a further confirmation of our theoretical proposal the same methodology has been applied to another high risk point (N 13° 0’ - E 80° 16.2’) at the same Chennai coastal region. This investigation is detailed in the supplementary material.
first time, where concrete energy estimate and economical viability go beyond its scope. The cost effectiveness and the concrete requirements should be taken into account carefully in deciding the range of such proposed installations. The main emphasis should possibly be on the protection of sensitive installations like nuclear reactors at the sea coast against the danger of tsunami like waves. Therefore the control mechanism for the possible management of the potentially hazardous near shore waves, proposed here, may be considered only in limited strategic areas surrounding costly installations, in order to reduce the intensity of the approaching wave to a safer limit.

Our analysis shows that a significant upsurge could have been experienced by a future surging wave approaching towards these coastal points. For example at the identified northern coastal point (N 13° 10.5' - E 80° 18.75') a wave of nearly 1 meter built at a distance of 10.5 km from the shore would have been developed to a killing height of 30 meter at the shore without any control, as estimated from the near shore coastal bathymetry. However, the surging waves could be managed at the height of 1.23 meters if the proposed leakage installation could start from a distance of 900 meters from the shore. A smaller distance could result to a higher amplitude though significantly lesser than that without control.

Similarly at a southern point (N 13° 0' - E 80° 16.2') the bathymetry would induce equally devastating upsurge for a wave of 1 m created at 11.1 km away, to develop into a 30 meter killer wave at the shore However, it could be controlled to a wave amplitude of 0.4 meter, if the leakage installation starts at a distance of 900 meters (see supplementary material).

Note, that in situations when no analytic solution is possible to achieve and our exact method fails, one could perhaps still use the proposed feedback leakage mechanism. However in such cases, since there would be no exact predictability of the future profile, the surging surface wave profile should be captured at each instant of time to regulate the leakage device at every instance through a direct feedback, until we could reach to a reasonable damping of surface wave amplitude.

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