Abstract

Witten recently gave further evidence for the conjectured relationship between the $A$ series of the $N = 2$ minimal models and certain Landau-Ginzburg models by computing the elliptic genus for the latter. The results agree with those of the $N = 2$ minimal models, as can be calculated from the known characters of the discrete series representations of the $N = 2$ superconformal algebra. The $N = 2$ minimal models also have a Lagrangian representation as supersymmetric gauged WZW models. We calculate the elliptic genera, interpreted as a genus one path integral with twisted boundary conditions, for such models and recover the previously known result.

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1. Introduction

The study of $N = 2$ superconformal field theories in two dimensions has played an increasingly important role in string theory in recent years. The main motivation has been to construct vacua of string theory which exhibit space-time supersymmetry, but there are also intriguing connections with topological field theory. Furthermore, the $N = 2$ structure is of great mathematical interest.

Among representations of the $N = 2$ superconformal algebra, those in the discrete series are of particular importance [1] [2]. By assembling left- and right-moving such representations, we may construct exactly solvable examples of $N = 2$ superconformal field theories, the so called minimal models. Modularly invariant such theories obey an $ADE$ classification, much as the modular invariants of affine $SU(2)$ [3] [4].

Several different Lagrangian formulations of $N = 2$ superconformal field theories are known. Important examples include non-linear sigma models with Calabi-Yau target spaces, supersymmetric gauged WZW models and supersymmetric Landau-Ginzburg theories. There is a certain overlap between the different constructions, though.

There are strong reasons to believe that certain supersymmetric Landau-Ginzburg models flow to the $A$ series of the minimal models in the infrared under the renormalization group. To test this conjecture one should find quantities which are effectively computable both for the Landau-Ginzburg models and the minimal models. An interesting example is the elliptic genus [5] [6], which could be viewed as a restriction of the usual partition function. In contrast to the partition function, the elliptic genus has an interpretation as an index of one of the supercharges, and is therefore invariant under a large class of smooth deformations of the theory. This property was recently used by Witten [7] to compute the elliptic genus of the Landau-Ginzburg models by deforming them to free field theories.

Di Francesco and Yankielowicz [8] have shown that the result of Witten’s computation agrees with the elliptic genus of the minimal models. The proof uses the known characters of the discrete series representations of the $N = 2$ superconformal algebra. However, the $A$ series of the $N = 2$ minimal models also have a Lagrangian representation as supersymmetric gauged WZW models. To get a better conceptual understanding of the relationship between the different formulations, it would be valuable to calculate the elliptic genus of the minimal models in a Lagrangian formalism, i.e. as a genus one path integral with twisted boundary conditions. This is the object of the present paper.

The organization of this paper is as follows: In section two we discuss the definition and general characteristics of the elliptic genus and review Witten’s calculation [7] of it for certain Landau-Ginzburg models. In section three we briefly review the $N = 2$ supersymmetric coset models and their formulation as supersymmetric gauged WZW models. In section four we calculate the elliptic genera of the $N = 2$ minimal models in a path integral formalism by deforming the models to a weak coupling limit where a one-loop approximation can be justified.
2. The elliptic genus and Landau-Ginzburg theory

An important quantity characterizing an $N = 2$ superconformal field theory is the partition function

$$Z(q, \gamma_L, \gamma_R) = \text{Tr}(-1)^F q^{J_0} \bar{q}^{\bar{J}_0} \exp(i\gamma_L J_0 + i\gamma_R \bar{J}_0),$$  \hspace{1cm} (1)

where $J_0$ and $\bar{J}_0$ are the global $U(1)$ charges of the left- and right-moving $N = 2$ algebra respectively. Although the partition function of for example the minimal models is known, it is not effectively computable for a general $N = 2$ superconformal field theory.

The situation is much better for the elliptic genus [5][6], which is simply the restriction of the partition function to $\gamma_R = 0$, i.e. $Z(q, \gamma, 0)$. In the case of for example a sigma model, the elliptic genus is a topological invariant of the target space. This is related to the fact that it can be interpreted as an index of the $N = 1$ right-moving supercharge, as we will now explain.

The global right-moving $N = 1$ supersymmetry algebra is $Q^2_R = \bar{L}_0$. The index of $Q_R$, i.e. the difference between the number of bosonic and fermionic states of $\bar{L}_0 = 0$ can be written as

$$\text{Tr}(-1)^F q^{J_0} \bar{q}^{\bar{J}_0},$$  \hspace{1cm} (2)

since states of non-zero $\bar{L}_0$ eigenvalue cancel pairwise. Defined in this way, the index is in general divergent due to the infinite degeneracy of the left-moving degrees of freedom. This could be remedied by the inclusion of a convergence factor, which should commute with $Q_R$. A convenient choice for an $N = 2$ theory is

$$(-1)^F q^{J_0} \exp(i\gamma J_0),$$  \hspace{1cm} (3)

which gives exactly the elliptic genus. States of non-zero $\bar{L}_0$ still cancel, so the elliptic genus is holomorphic in $q$, i.e. $\bar{q}$ independent. For some other general properties, see [5].

Being an index of the right-moving supercharge, the elliptic genus is invariant under smooth deformations of the theory which preserve an $N = 1$ right-moving supersymmetry. This property was recently used by Witten [7] to calculate the elliptic genus of certain Landau-Ginzburg models. These models are given by the action

$$I(\phi, \psi) = \int d^2z \left(-\partial_z \bar{\phi} \partial_z \phi + i\bar{\psi}_- \partial_z \psi_+ + i\bar{\psi}_+ \partial_z \psi_- \right) - (\bar{\phi} \phi)^{k+1} - (k+1) \bar{\phi} \phi \psi_+ \psi_- - (k+1) \bar{\phi} \phi \bar{\psi}_- \bar{\psi}_+,$$  \hspace{1cm} (4)

where $\phi$ and $\psi$ are complex bosonic and fermionic fields respectively.

The global $U(1)$ symmetry with parameter $\gamma$ which is part of the left-moving $N = 2$ algebra acts as [7]

$$\phi \rightarrow \exp\left(\frac{i\gamma}{k+2}\right) \phi,$$

$$\psi_+ \rightarrow \exp\left(\frac{i\gamma}{k+2}\right) \psi_+,$$

$$\psi_- \rightarrow \exp\left(-\frac{i\gamma(k+1)}{k+2}\right) \psi_-.$$

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To calculate the elliptic genus we deform the action by turning off the interactions, which turns the model into a theory of a free complex boson and a free complex fermion. The left-moving $U(1)$ transformations are still given by (5). The calculation of the elliptic genus is now easily performed in an operator formalism. The non-zero modes of $\psi_-, \bar{\psi}_-, \psi_+, \text{and } \bar{\psi}_+$ give rise to the factors
\begin{equation}
\prod_{n=1}^{\infty} \left(1 - q^n \exp\left(-i\gamma \frac{(k+1)}{k+2}\right)\right) \left(1 - q^n \exp\left(i\gamma \frac{(k+1)}{k+2}\right)\right) \left(1 - \bar{q}^n \exp\left(-i\gamma \frac{(k+1)}{k+2}\right)\right) \left(1 - \bar{q}^n \exp\left(i\gamma \frac{(k+1)}{k+2}\right)\right),
\end{equation} 
whereas the contribution of the non-zero modes of $\phi$ and $\bar{\phi}$ is
\begin{equation}
\prod_{n=1}^{\infty} \left[ \left(1 - q^n \exp\left(i\gamma \frac{(k+1)}{k+2}\right)\right) \left(1 - q^n \exp\left(-i\gamma \frac{(k+1)}{k+2}\right)\right) \left(1 - \bar{q}^n \exp\left(i\gamma \frac{(k+1)}{k+2}\right)\right) \left(1 - \bar{q}^n \exp\left(-i\gamma \frac{(k+1)}{k+2}\right)\right) \right]^{-1}.
\end{equation}

The final result, including the zero modes, is
\begin{equation}
Z(q, \gamma, 0) = e^{-\frac{i\gamma k}{k+2}} \frac{1 - \exp\left(i\gamma \frac{(k+1)}{k+2}\right)}{1 - \exp\left(i\gamma \frac{1}{k+2}\right)} \prod_{n=1}^{\infty} \frac{1 - q^n \exp\left(i\gamma \frac{(k+1)}{k+2}\right)}{1 - q^n \exp\left(-i\gamma \frac{(k+1)}{k+2}\right)} \frac{1 - \bar{q}^n \exp\left(i\gamma \frac{(k+1)}{k+2}\right)}{1 - \bar{q}^n \exp\left(-i\gamma \frac{(k+1)}{k+2}\right)}.
\end{equation}

By using the known character formulas for the $N = 2$ discrete series representations, Di Francesco and Yankielowicz [8] have shown that this result agrees with the elliptic genus of the minimal models.

3. The Lagrangian formulation of N=2 coset models

The $N = 1$ coset models [10] constitute an important class of superconformal field theories. Such a model is specified by a Lie group $G$ with a subgroup $H$ and a positive integer $k$ called the level. The central charge of the $N = 1$ superconformal algebra is
\begin{equation}
c = \frac{3}{2} (\dim G - \dim H) - \frac{1}{k + Q_G} (Q_G \dim G - Q_H \dim H),
\end{equation}
where $Q_G$ and $Q_H$ are the dual Coxeter numbers of $G$ and $H$ respectively.

Schmitzer [11] has given a Lagrangian representation of the coset models as supersymmetric, gauged WZW models. The fundamental fields are a $G$ valued bosonic field $g$, a gauge field $A_a$ with values in $\text{Lie } H$, and left- and right-moving fermionic fields $\hat{\psi}_+$ and $\hat{\psi}_-$ respectively with values in $\text{Lie } (G/H)$, i.e. the orthogonal complement of $\text{Lie } H$ in $\text{Lie } G$. The action is
\begin{equation}
I(g, A, \hat{\psi}) = I_B(g, A) + I_F(A, \hat{\psi}),
\end{equation}
where the bosonic part is given by

$$I_B(g, A) = k I_{WZW}(g) + \frac{k}{2\pi} \int d^2z \text{Tr}(A_z g^{-1} \partial_z g - A_z g^{-1} g A_z + A_z g^{-1} A_z g)$$  (11)

and the fermionic part by

$$I_F(A, \hat{\psi}) = \frac{ik}{4\pi} \int d^2z \text{Tr}(\hat{\psi}_+ D_z \hat{\psi}_+ + \hat{\psi}_- D_z \hat{\psi}_-)$$  (12)

Here $I_{WZW}(g)$ is the (level 1) \(G\) Wess-Zumino-Witten action \cite{12} and the covariant derivative $D_\alpha$ is defined as $D_\alpha = \partial_\alpha + [A_\alpha, .]$.

Infinitesimal gauge transformations with parameter $\Lambda \in \text{Lie } H$ act as

$$\delta g = [\Lambda, g]$$
$$\delta \hat{\psi}_+ = [\Lambda, \hat{\psi}_+]$$
$$\delta \hat{\psi}_- = [\Lambda, \hat{\psi}_-]$$
$$\delta A_\alpha = -D_\alpha \Lambda.$$  (13)

The model is invariant under left- and right-moving supersymmetries, with parameters $\epsilon_+$ and $\epsilon_-$ respectively, acting as

$$\delta g = i\epsilon_- g \hat{\psi}_+ + i\epsilon_+ \hat{\psi}_- g$$
$$\delta \hat{\psi}_+ = \epsilon_-(1 - \Pi_H)(g^{-1} D_z g - i\hat{\psi}_+ \hat{\psi}_+)$$
$$\delta \hat{\psi}_- = \epsilon_+(1 - \Pi_H)(D_z g g^{-1} + i\hat{\psi}_- \hat{\psi}_-)$$
$$\delta A_\alpha = 0.$$  (14)

Here $\Pi_H$ is the orthogonal projection of $\text{Lie } G$ on $\text{Lie } H$.

Kazama and Suzuki \cite{13} investigated the conditions under which the $N = 1$ coset models actually possess $N = 2$ supersymmetry. They found that this happens exactly when $G/H$ is a Kähler space. A more algebraic way to formulate this condition is as follows: $G/H$ is a Kähler space exactly when $\text{Lie } (G/H)$ can be decomposed as

$$\text{Lie } (G/H) = T \oplus \bar{T},$$  (15)

where $T$ and $\bar{T}$ are complex conjugate representations of $H$ such that

$$[T, T] \subset T \quad [\bar{T}, \bar{T}] \subset \bar{T}$$  (16)

and

$$\text{Tr}(uv) = 0 \quad \text{for } u, v \in T \quad \text{or } u, v \in \bar{T}.$$  (17)

The simplest example of a Kazama-Suzuki model is $G/H = SU(2)/U(1)$. Note that \cite{9} in this case gives the central charges $c = 3k/(k + 2)$ of the $N = 2$ minimal models.

By scrutinizing the action (10) with (11) and (12), we may in fact see that the model indeed has two left-moving and two right-moving supersymmetries when the
Kazama-Suzuki conditions (15), (16) and (17) are fulfilled [14][15]. Namely, denoting the components of $\hat{\psi}_\pm$ in $T$ and $\bar{T}$ as $\psi_\pm$ and $\bar{\psi}_\pm$ respectively, we may write the fermionic part (12) of the action as

$$I_F(A, \hat{\psi}) = \frac{ik}{2\pi} \int d^2z \text{Tr}(\psi_+ D_z \bar{\psi}_+ + \psi_- D_z \bar{\psi}_-).$$

(18)

We see that there is an $R$-symmetry, i.e. a symmetry which does not commute with the supersymmetries (14), such that $\psi_\pm$ and $\bar{\psi}_\pm$ have charge +1 and $-1$ respectively whereas $g$ and $A$ are uncharged. Furthermore, the condition (16) means that the supersymmetry transformations (14) may be decomposed in two parts that change the charge by +1 and $-1$ respectively. For the right movers this yields an $N = 2$ supersymmetry with parameters $\epsilon_- \bar{\epsilon}_-$ and $\bar{\epsilon}_- \epsilon_-$ acting as

$$\delta g = i\epsilon_- g \bar{\psi}_+ + i\bar{\epsilon}_- g \psi_+$$
$$\delta \psi_+ = \epsilon_- \Pi(g^{-1} D_z g - i\psi_+ \bar{\psi}_+ - i\bar{\psi}_+ \psi_+) - i\epsilon_- \psi_+ \psi_+$$
$$\delta \bar{\psi}_+ = \epsilon_- \bar{\Pi}(g^{-1} D_z g - i\psi_+ \bar{\psi}_+ - i\bar{\psi}_+ \psi_+) - i\epsilon_- \bar{\psi}_+ \bar{\psi}_+$$
$$\delta \psi_- = \delta \bar{\psi}_- = \delta A_\alpha = 0.$$  

(19)

Here $\Pi$ and $\bar{\Pi}$ denote the projections on $T$ and $\bar{T}$ respectively. The analogous decomposition for the left movers is

$$\delta g = i\epsilon_+ \bar{\psi}_+ + i\bar{\epsilon}_+ \psi_+$$
$$\delta \psi_- = \epsilon_+ \Pi(D_z g g^{-1} + i\psi_- \bar{\psi}_- + i\bar{\psi}_- \psi_-) + i\epsilon_+ \psi_- \psi_-$$
$$\delta \bar{\psi}_- = \epsilon_+ \bar{\Pi}(D_z g g^{-1} + i\psi_- \bar{\psi}_- + i\bar{\psi}_- \psi_-) + i\epsilon_+ \bar{\psi}_- \bar{\psi}_-$$
$$\delta \psi_+ = \delta \bar{\psi}_+ = \delta A_\alpha = 0.$$  

(20)

To compute the elliptic genus it is essential to identify the global $U(1)$ symmetry that is part of the left-moving $N = 2$ algebra. The condition that this $U(1)$ transformation commutes with the right-moving supersymmetries (13) means that Lie $G$ may be graded by the $U(1)$ charge of $g^{-1} D_z g$, and that this equals the $U(1)$ charge of $\psi_+$ or $\bar{\psi}_+$ on each subspace of $T$ or $\bar{T}$ respectively. The left-moving supersymmetries (20) parametrized by $\epsilon_+$ and $\bar{\epsilon}_+$, on the other hand, should have $U(1)$ charge +1 and $-1$ respectively. By the same arguments as before we find that Lie $G$ may also be graded by the $U(1)$ charge of $D_z g g^{-1}$. However, this $U(1)$ charge differs from that of $\psi_-$ or $\bar{\psi}_-$ by +1 or $-1$ respectively.

Henceforth we will only consider the case where $G/H = SU(2)/U(1)$, i.e. the $N = 2$ minimal models. A $U(1)$ transformation with parameter $\gamma$ of the left-moving $N = 2$ algebra then acts on the fermionic fields as

$$\delta \psi_- = i\gamma c_- \psi_-$$
$$\delta \psi_+ = i\gamma c_+ \psi_+.$$  

(21)
for some real constants $c_-$ and $c_+$. The fields $\bar{\psi}_\pm$ transform as the complex conjugates of $\psi_\pm$. For the bosonic fields we postulate the transformation laws

\begin{align*}
\delta g &= i\gamma(x_- U g + x_+ g U) \\
\delta A_\alpha &= 0
\end{align*}

(22)

for some real constants $x_-$ and $x_+$. Here $U \in \text{Lie SU}(2)$ is the generator of the gauged $U(1)$ factor in $SU(2)$ normalized so that it has eigenvalues $+1$ and $-1$ when acting on $T$ and $\bar{T}$ respectively in the adjoint representation of $SU(2)$. The requirement that the $U(1)$ transformation (21) and (22) of the left-moving $N=2$ algebra commutes with the right-moving supersymmetries (19) translates into the condition $-x_+ = c_+$. To get the correct charges for the left-moving supersymmetries (20) we must take $x_- = c_- - 1$.

We get an additional relation between $c_-, c_+, x_- \text{ and } x_+$ from the requirement that the action (10) with (11) and (18) be invariant under the $U(1)$ transformation (21) and (22). The fermionic part (18) is invariant at the classical level under the transformations (21), but at the quantum level the symmetry breaks down due to the chiral anomaly. The anomalous variation of the effective action is

\begin{equation}
\delta I_{\text{eff}}^F(A, \hat{\psi}) = i\gamma 2(c_+ - c_-) \frac{1}{2\pi} \int d^2 z \text{Tr}(U F_{\bar{z}z}),
\end{equation}

(23)

where the $U(1)$ gauge field strength is defined as $F_{\bar{z}z} = \partial_{\bar{z}} A_{\bar{z}} - \partial_z A_z$. We see that the integrated anomaly is proportional to the first Chern class of the line bundle on which the $U(1)$ gauge field $A_\alpha$ is a connection. Note that although (18) is proportional to the level $k$, the anomaly (23), being a one-loop quantum effect, is independent of $k$.

The bosonic part (11) of the action, on the other hand, is non-invariant under (22) already at the classical level with

\begin{equation}
\delta I_B(g, A) = i\gamma (-x_- - x_+) \frac{k}{2\pi} \int d^2 z \text{Tr}(U F_{\bar{z}z}).
\end{equation}

(24)

There are no quantum corrections to this result, though, since we take the integration measure $Dg$ in the bosonic path integral to be a product of a Haar measure, invariant under left and right $G$ multiplication, for each point on the world sheet. We thus see that the quantum anomaly (23) from the fermionic action will cancel against the classical anomaly (24) from the bosonic action if we choose $2(c_+ - c_-) = k(x_- + x_+)$. Finally, by means of a gauge transformation (13) we may choose $x_- = x_+$. This gauge choice is convenient in that it allows the $U(1)$ transformation (22) to have a line of fixed points where $\delta g = 0$. These fixed points will be important in the next section. With a suitable normalization, our final transformation laws for the global $U(1)$ symmetry of the left-moving $N=2$ algebra are thus

\begin{align*}
\delta \psi_+ &= \frac{i\gamma}{k+2} \psi_+ \\
\delta \psi_- &= \frac{i\gamma (k+1)}{k+2} \psi_- \quad (25)
\end{align*}
\[ \delta g = -\frac{i\gamma}{k + 2}(Ug + gU) \]
\[ \delta A_\alpha = 0 \]

with \( \bar{\psi}_\pm \) again transforming as the complex conjugates of \( \psi_\pm \).

4. Path integral calculation of the elliptic genus

Our object is now to calculate the elliptic genus of a supersymmetric gauged WZW model based on \( G/H = SU(2)/U(1) \), as discussed in the previous section. In a path integral formulation, this could be interpreted as a genus one vacuum-to-vacuum amplitude with the boundary conditions twisted by a left-moving global \( U(1) \) transformation \((25)\) along one of the cycles of the world sheet. If we take the world-sheet to be a torus with modular parameter \( \tau \), the fields should thus obey the following boundary conditions:

\[
\begin{align*}
g(z + \tau, \bar{z} + \bar{\tau}) &= \exp(-\frac{i\gamma}{k + 2} U) g(z, \bar{z}) \exp(-\frac{i\gamma}{k + 2} U) \\
\psi_+(z + \tau, \bar{z} + \bar{\tau}) &= \exp\left(\frac{i\gamma}{k + 2}\right) \psi_+(z, \bar{z}) \\
\psi_-(z + \tau, \bar{z} + \bar{\tau}) &= \exp\left(\frac{i\gamma(k + 1)}{k + 2}\right) \psi_-(z, \bar{z}) \\
A_\alpha(z + \tau, \bar{z} + \bar{\tau}) &= A_\alpha(z, \bar{z})
\end{align*}
\]

(26)

with corresponding conditions on \( \bar{\psi}_\pm(z, \bar{z}) \). All fields should furthermore be invariant under \( z \to z + 1 \). The elliptic genus is given as the Euclidean path integral

\[
Z(q, \gamma, 0) = \int \mathcal{D}g \mathcal{D}\hat{\psi} \mathcal{D}A e^{-I(g, A, \hat{\psi})},
\]

(27)

where the fields obey the above boundary conditions. As usual \( q = \exp(2\pi i \tau) \).

As previously mentioned, the elliptic genus has an interpretation as an index of the right-moving supercharge, and is therefore invariant under smooth deformations of the theory which preserve this supersymmetry. In a path integral formalism, we describe such a deformation by adding a perturbation of the form \( \lambda I_1(g, A, \hat{\psi}) = \lambda \int d^2z L_1 \) to the action in \((27)\). The perturbation Lagrangian \( L_1 \) should transform into a total derivative under the right-moving supersymmetry. For the perturbed model, we thus get

\[
Z(q, \gamma, 0) = \int \mathcal{D}g \mathcal{D}\hat{\psi} \mathcal{D}A e^{-I(g, A, \hat{\psi}) - \lambda I_1(g, A, \hat{\psi})}.
\]

(28)

To see that the elliptic genus \((28)\) is indeed independent of \( \lambda \), we calculate

\[
\frac{\partial Z(q, \gamma, 0)}{\partial \lambda} = -\int \mathcal{D}g \mathcal{D}\hat{\psi} \mathcal{D}A I_1(g, A, \hat{\psi}) e^{-I(g, A, \hat{\psi}) - \lambda I_1(g, A, \hat{\psi})}.
\]

(29)
A formal argument shows that the expression (29) vanishes. Namely, as long as the right-moving supersymmetry acts freely, we may introduce a collective fermionic coordinate $\theta$ for this symmetry. The path integral measure will thus contain the factor $d\theta$. But the invariance of $I(g, A, \hat{\psi})$ and $I_1(g, A, \hat{\psi})$ under supersymmetry means that the integrand is independent of $\theta$, and the integral thus vanishes by the rules of Grassmann integration. This argument breaks down, though, if the supersymmetry has a fixed point [14], as turns out to be the case in our application. The path integral (28) will then receive contributions from field configurations in a neighbourhood of the fixed point, unless the integrand is zero there. To ensure that the elliptic genus is independent of $\lambda$, we should therefore require $I_1(g, A, \psi)$ to vanish at the fixed point of the right-moving supersymmetry.

Path integrals such as (27) are usually hard to calculate, except at weak coupling where perturbation theory may be used. Our model is certainly strongly coupled, though, so a direct evaluation seems quite difficult. However, by perturbing the model in a suitable way we may take it into the weak coupling regime, and, as argued in the previous paragraph, the elliptic genus is invariant under supersymmetric such perturbations. To construct such a supersymmetric perturbation of the Lagrangian, we note that the global right-moving supersymmetry algebra (19) contains two supercharges $Q_+$ and $\bar{Q}_+$ obeying $\{Q_+, Q_+\} = \{\bar{Q}_+, \bar{Q}_+\} = 0$ and $\{Q_+, \bar{Q}_+\} = 2D_z$. If we take $V$ such that $\{\bar{Q}_+, V\} = 0$, then the perturbation $L_1 = \{Q_+, V\}$ will be invariant under $Q_+$ and transform into a total derivative under $\bar{Q}_+$, i.e. $[Q_+, L_1] = 0$ and $[\bar{Q}_+, L_1] = 2D_zV$.

Since we want $L_1$ to be Grassmann even, uncharged under the right-moving $U(1)$ symmetry and of scaling dimension $(1, 1)$, we see from (19) that $V$ should be Grassmann odd, of right $U(1)$ charge $-1$ and of scaling dimension $(1, 1/2)$. In our case there is indeed a sensible such term, namely $V = \text{Tr}(g^{-1}D_zg\psi_+)$. With this $V$ we get

$$L_1 = \text{Tr}\left(g^{-1}D_zg\Pi(g^{-1}D_zg) + iD_z\bar{\psi}_+\psi_+ + iD_z^2g(\psi_+\bar{\psi}_+ + \bar{\psi}_+\psi_+)\right).$$

Note that at a fixed point of the right-moving supersymmetry (19), $g$ must be a constant and $\psi_+$ and $\bar{\psi}_+$ must vanish. This means that $L_1$ vanishes, so the elliptic genus is indeed invariant under this perturbation according to our previous arguments.

To see that the addition of this term to the action takes us to a weakly coupled theory, it is convenient to change variables from $g \in SU(2)$ to $\phi \in \text{Lie}SU(2)$ defined through

$$g = g_0 \exp(i\phi),$$

where $g_0 \in SU(2)$ is a constant which is invariant under the left-moving $U(1)$ transformations (23), i.e.

$$Ug_0 + g_0U = 0.$$

Expanding $L_1$ we get

$$L_1 = \text{Tr}\left(-D_z\phi\Pi(D_z\phi) + iD_z\bar{\psi}_+\psi_+ - D_z\phi(\psi_+\bar{\psi}_+ + \bar{\psi}_+\psi_+)\right)$$

plus higher order interaction terms.
Under a left-moving $U(1)$ transformation (25) $\phi$ transforms as

$$\delta \phi = -\frac{i\gamma}{k+2}[U,\phi] + \mathcal{O}(\phi^2). \quad (34)$$

The components of $\phi$ which lie in $T$ and $\bar{T}$ thus have the same $U(1)$ charges as $\bar{\psi}_+$ and $\psi_+$ respectively. The remaining component of $\phi$ in Lie $H$, i.e. along the gauged $U(1)$ subgroup of $SU(2)$, has zero charge under the left-moving $U(1)$ transformation, but is in fact pure gauge. To see this, we note that under a gauge transformation (13) $\phi$ transforms as

$$\delta \phi = -i(\Lambda - g_0\Lambda g_0^{-1}) + \mathcal{O}(\phi). \quad (35)$$

Furthermore, for $\Lambda \in \text{Lie } H$,

$$\frac{i\gamma}{k+2}[U,\Lambda - g_0\Lambda g_0^{-1}] = 0, \quad (36)$$

so the component of $\phi$ which is pure gauge indeed has zero $U(1)$ charge, and must therefore belong to Lie $H$.

The Kähler condition (16) means that $\psi_+\bar{\psi}_++\bar{\psi}_+\psi_+ \in \text{Lie } H$. The $\text{Tr}(D_z\phi(\psi_+\bar{\psi}_++\bar{\psi}_+\psi_+))$ term in the action may therefore be gauged away, since it only contains components of $\phi$ that are pure gauge. To lowest non-vanishing order, the Lagrangian $L_1$ thus describes two free bosons and two free fermions.

We may now calculate the elliptic genus in a path integral formalism as a perturbative expansion in the number of loops. This is tantamount to expanding the action of the model around its minima, i.e. around field configurations which solve the classical equations of motion. In the weak coupling limit, i.e. as $\lambda \to \infty$, the integrand in (28) is sharply peaked at the classical field configurations and we may trust the one-loop or Gaussian approximation, where we expand the action to second order in the fields around the minima. In this approximation the calculation amounts to a free-field computation, much as in section two. We see that the essential feature is the charges of the various fields under the left-moving $U(1)$ transformation, and since these agree with those of the Landau-Ginzburg model (confer (5)) in section two we recover the result (8) of [8] for the elliptic genus of the $N = 2$ minimal models.

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