Convolution of Roumieu ultradistributions in sequential approach

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November 5, 2019

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Abstract

We consider several general sequential conditions for convolvability of two Roumieu ultradistributions on $\mathbb{R}^d$ in the space $\mathcal{D}'\{M_p\}$ and prove that they are equivalent to the convolvability of these ultradistributions in the sense of Pilipović and Prangoski. The discussed conditions, based on two classes $\mathcal{U}\{M_p\}$ and $\mathcal{U}'\{M_p\}$ of approximate units and corresponding sequential conditions of integrability of Roumieu ultradistributions, are analogous to the known convolvability conditions in the space $\mathcal{D}'$ of distributions and in the space $\mathcal{D}'(M_p)$ of ultradistributions of Beurling type.

Moreover, the following property of the convolution and ultradifferential operator $P(D)$ of class $\{M_p\}$ is proved: if $S, T \in \mathcal{D}'\{M_p\}(\mathbb{R}^d)$ are convolvable, then

$$P(D)(S \ast T) = (P(D)S) \ast T = S \ast (P(D)T).$$

Keywords: Ultradistributions, Convolution of ultradistributions, Approximate unit, Ultradifferential operator.

MSC: 46F05 46F10, 46E10.

1 Introduction

Deep investigations of the convolution of two ultradistributions of Roumieu type (that we call shorter Roumieu ultradistributions) in the non-quasianalytic case were carried out via $\varepsilon$-tensor product by Pilipović and Prangoski in [21] and, with important improvements, by Dimovski,
Pilipović, Prangoski and Vindas in [5]. The authors gave there general functional definitions and proved fundamental results on convolvability and the convolution of Roumieu ultradistributions in a way analogous to the known general approaches of Chevalley and Schwartz in case of distributions. For other aspects of the theory see e. g. [2, 4, 6, 7, 22, 24, 27, 28]. See also the recent article [23] for results concerning the quasianalytic case.

The aim of this paper is to discuss sequential conditions playing a similar role in the study of the convolution of Roumieu ultradistributions to those used in the sequential theories of the convolution of distributions (see [26], [3], [9] and [17]) and ultradistributions of Beurling type (see [11], [12] and [2]). The conditions are based on two types of $\mathcal{R}$-approximate units (Definition 2 and Definition 3), being the counterparts of the approximate units in the sense of Dierolf and Voigt (see [3]). The respective classes $\mathcal{U}^{(M_p)}$ and $\mathcal{U}^{(M_p)}$ of $\mathcal{R}$-approximate units are used in sequential characterization of integrable Roumieu ultradistributions ([18]), analogous to those proved by Pilipović in [20] in case of integrable ultradistributions of Beurling type. As a consequence, we give several sequential definitions of the convolution of Roumieu ultradistributions (Definition 6). We prove in Theorem 4 that all our sequential definitions of the convolution of Roumieu ultradistributions are equivalent to those given in [21] and [5].

Another application of the notion of $\mathcal{R}$-approximate units is presented in the proof of Theorem 5 describing a non-trivial property of the convolution of Roumieu ultradistributions and ultradifferential operators of the class $\{M_p\}$. It is worth to recall that Pilipović in [20] used a different but analogous class of approximate units to prove the same property in case of the convolution of ultradistributions of Beurling type. Our proof of Theorem 5 is based on similar ideas but discussion concerning the class $\mathcal{R}$ plays an essential role in our case.

2 Preliminaries

We consider complex-valued $C^\infty$-functions and Roumieu ultradistributions defined on $\mathbb{R}^d$ (or on an open subset of $\mathbb{R}^d$) using the standard multi-dimensional notation in $\mathbb{R}^d$.

To mark the dimension of $\mathbb{R}^d$, which is essential in some situations, we denote the considered spaces of test functions and the corresponding spaces of Roumieu ultradistributions simply by adding the index $d$ at the end of the respective symbol. Moreover, if necessary, the constant function $1$ on $\mathbb{R}^d$ will be denoted by $1_d$ and the value of $T \in \mathcal{D}_d^{(M_p)}$ on $\varphi \in \mathcal{D}_d^{(M_p)}$ by $\langle T, \varphi \rangle_d$.

The spaces of test functions and Roumieu ultradistributions are defined by a given sequence $(M_p)_{p \in \mathbb{N}_0}$ of positive numbers. Usually some of the following conditions are imposed on the sequence $(M_p)$:

(M.1) (logarithmic convexity) $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{N}$;
(M.2) (stability under ultradifferential operator) $M_p \leq AH^pM_qM_{p-q}$, $p, q \in \mathbb{N}_0$, $q \leq p$;
(M.2') (stability under differential operator) $M_p \leq AH_pM_{p-1}$, $p \in \mathbb{N}$;
(M.3) (strong non-quasi-analyticity)
\[
\sum_{p=q+1}^{\infty} M_{p-1} M_p^{-1} \leq A q M_q M_{q+1}^{-1}, \quad q \in \mathbb{N};
\]
(M.3') (non-quasi-analyticity)
\[
\sum_{p=1}^{\infty} M_{p-1} M_p^{-1} < \infty,
\]
for certain constants \( A > 0 \) and \( H > 0 \). We can and will assume that \( H \geq 1 \).

Clearly, conditions (M.2') and (M.3') are particular cases of conditions (M.2) and (M.3), respectively.

For simplicity, we will assume in the whole paper that the sequence \((M_p)\) satisfies the three conditions (M.1), (M.2) and (M.3), not discussing which of them can be weakened or omitted in the formulations of presented theorems.

It follows by induction from (M.1) that
\[
\frac{M_i}{M_{i-1}} \leq \frac{M_{p+i}}{M_{p+i-1}}, \quad p \in \mathbb{N}_0, \ i \in \mathbb{N},
\]
and thus
\[
\frac{M_q}{M_0} = \prod_{i=1}^{q} \frac{M_i}{M_{i-1}} \leq \prod_{i=1}^{q} \frac{M_{p+i}}{M_{p+i-1}} = \frac{M_{p+q}}{M_p}, \quad p \in \mathbb{N}_0, \ q \in \mathbb{N}.
\]
Consequently, (M.1) implies \( M_p \cdot M_q \leq M_0 M_{p+q} \) for \( p, q \in \mathbb{N}_0 \). For simplicity we assume in the sequel that \( M_0 = 1 \). With this assumption, the last inequality gets the form:

\[
(2.1) \quad M_p \cdot M_q \leq M_{p+q}, \quad p, q \in \mathbb{N}_0.
\]

It will be convenient to extend the sequence \((M_p)_{p \in \mathbb{N}_0}\) to \((M_k)_{k \in \mathbb{N}_d}^0\) by means of the formula:
\[
M_k := M_{k_1+\ldots+k_d}, \quad k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d.
\]
Due to the extended notation we immediately get the extended version of inequality (2.1):

\[
(2.2) \quad M_j \cdot M_k \leq M_{j+k}, \quad j, k \in \mathbb{N}_0^d.
\]

The associated function of the sequence \((M_p)\) is given by
\[
M(\rho) = \sup_{p \in \mathbb{N}_0} \log + \frac{\rho^p}{M_p}, \quad \rho > 0.
\]

For an arbitrary \( k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \) denote by \( D^k \) the differential operator of the form
\[
D^k = D_1^{k_1} \ldots D_d^{k_d} := \left( \frac{1}{i \partial x_1} \right)^{k_1} \ldots \left( \frac{1}{i \partial x_d} \right)^{k_d}.
\]

An essential role in our consideration plays Komatsu’s lemma proved in [16] (see Lemma 3.4 and Proposition 3.5) in which numerical sequences monotonously increasing to infinity are involved. The class of such sequences \((r_p)_{p \in \mathbb{N}_0}\) (with \( r_0 = 1 \)) has been denoted by \( \mathfrak{R} \) in [21] and [5] and we preserve this notation in our paper.
For every \((r_p) \in \mathcal{R}\) we call \((R_p)\) the *product sequence* corresponding to \((r_p)\) if its elements are of the form \(R_p := \prod_{i=0}^{p} r_i\) for \(p \in \mathbb{N}_0\) (i.e. \(R_0 = 1\)).

Let us recall Komatsu’s lemma in the following equivalent form which emphasizes the symmetry of two assertions:

**Lemma 1.** Let \((a_k)_{k \in \mathbb{N}_0}\) be a sequence of nonnegative numbers.

(I) The following two conditions are equivalent:

(A1) \(\exists h > 0 \sup_{k \in \mathbb{N}_0} \frac{a_k}{h^k} < \infty\);

(B1) \(\forall (r_k) \in \mathcal{R} \sup_{k \in \mathbb{N}_0} \frac{a_k}{R_k} < \infty\);

(II) the following two conditions are equivalent:

(A2) \(\forall h > 0 \sup_{k \in \mathbb{N}_0} (h^k a_k) < \infty\);

(B2) \(\exists (r_k) \in \mathcal{R} \sup_{k \in \mathbb{N}_0} (R_k a_k) < \infty\),

where \((R_k)\) is the product sequence corresponding to the sequence \((r_k) \in \mathcal{R}\).

**Remark 1.** The above lemma can be easily extended to the \(d\)-dimensional version concerning sequences \((a_k)_{k \in \mathbb{N}_0^d}\) of nonnegative numbers.

It is worth noticing that Lemma 1 delivers two simple characterizations (dual to each other): 1° of *slowly increasing sequences* (i.e. satisfying \((A_1)\)), 2° of *rapidly decreasing sequences* (i.e. satisfying \((A_2)\)) of nonnegative numbers. They are expressed through respective properties of sequences, described by product sequences corresponding to sequences of the class \(\mathcal{R}\).

It will be convenient to use for \(\lambda > 0\) and \((r_p) \in \mathcal{R}\) the following notation:

\[
\lambda(r_p) = (\overline{r}_p), \quad \text{where} \quad \overline{r}_0 = 1 \quad \text{and} \quad \overline{r}_p = \lambda r_p \quad \text{for} \quad p \in \mathbb{N}.
\]

Clearly, if \((r_p) \in \mathcal{R}\), then \(\lambda(r_p) \in \mathcal{R}\) for \(\lambda \geq 1\). Moreover, \(\lambda(r_p) \in \mathcal{R}\) for \(0 < \lambda < 1\) in case \((r_p) \in \mathcal{R}\) and \(r_1 \geq \lambda^{-1}\).

### 3 Ultradifferentiable Functions

For a given complex-valued function \(\varphi\) on \(\mathbb{R}^d\) and a compact set \(K\) in \(\mathbb{R}^d\) denote

\[
\|
\varphi\|_\infty := \sup_{x \in \mathbb{R}^d} |\varphi(x)|; \quad \|
\varphi\|_K := \sup_{x \in K} |\varphi(x)|.
\]

For a given sequence \((M_p)\), a regular compact set \(K\) in \(\mathbb{R}^d\) and \(h > 0\) the symbol \(C^{(M_p)}_{K,h,d}\) will mean the locally convex space (l.c.s.) of all \(C^\infty\)-functions \(\varphi\) on \(\mathbb{R}^d\) such that

\[
q_{K,h}(\varphi) := \sup_{k \in \mathbb{N}_0^d} \frac{\|D^k \varphi\|_K}{h^{|k|} M_k} < \infty,
\]

(3.1)
with the topology defined by the semi-norm \( q_{K,h} \) given above, while the symbol \( D_{K,h,d}^{(M_p)} \) will mean the Banach space of all \( C^\infty \)-functions \( \varphi \) satisfying \((3.1)\) and having supports contained in \( K \), with the topology of the norm \( q_{K,h} \) in \((3.1)\).

For a fixed sequence \((M_p)\), we consider the following locally convex spaces of ultradifferentiable functions on \( \mathbb{R}^d \):

\[
D_{K,d}^{(M_p)} := \lim_{h \to \infty} D_{K,h,d}^{(M_p)}; \quad D_d^{(M_p)} := \lim_{K \subset \subset \mathbb{R}^d} D_{K,d}^{(M_p)};
\]

\[
E_d^{(M_p)} := \lim_{K \subset \subset \mathbb{R}^d} \lim_{h \to \infty} E_{K,h,d}^{(M_p)},
\]

where the symbol \( K \subset \subset \mathbb{R}^d \) means that compact sets \( K \) grow up to \( \mathbb{R}^d \).

Moreover, for a given \((M_p)\), we define

\[
D_{L^\infty,h,d}^{(M_p)} := \lim_{h \to \infty} D_{L^\infty,h,d}^{(M_p)},
\]

where \( D_{L^\infty,h,d}^{(M_p)} \) is the Banach space of all \( C^\infty \)-functions \( \varphi \) on \( \mathbb{R}^d \) such that

\[
\| \varphi \|_{\infty,h} := \sup_{k \in \mathbb{N}_0^d} \left( (h^k M_k)^{-1} \| D^k \varphi \|_\infty : k \in \mathbb{N}_0^d \right) < \infty,
\]

with the norm \( \| \cdot \|_{\infty,h} \) defined above.

For a given regular compact set \( K \subset \mathbb{R}^d \) and given sequences \((M_p)\) and \((r_p) \in \mathcal{R}\) we denote by \( D_{K,(r_p),d}^{(M_p)} \) the Banach space of all \( C^\infty \)-functions \( \varphi \) on \( \mathbb{R}^d \) having supports contained in \( K \) such that

\[
\| \varphi \|_{K,(r_p)} := \sup_{k \in \mathbb{N}_0^d} \frac{\| D^k \varphi \|_K}{R_{|k|} M_k} < \infty
\]

with the norm \( \| \cdot \|_{K,(r_p)} \) defined above.

The following result is essentially due to Komatsu [16], since it is a consequence of his beautiful Lemma recalled above.

**Proposition 1.** We have the equality

\[
D_{K,d}^{(M_p)} = \lim_{(r_p) \in \mathcal{R}} D_{K,(r_p),d}^{(M_p)},
\]

where the space \( D_{K,d}^{(M_p)} \) is defined in \((3.2)\).

**Proof.** The assertion is an immediate consequence of definitions of \( q_{K,h} \) in \((3.1)\) and \( \| \cdot \|_{K,(r_p)} \) in \((3.5)\) and Part (I) of the \( d \)-dimensional version of Lemma recalled above.
For given \((M_p)\) and \((r_p)\) in \(\mathcal{R}\) we denote by \(D^{(M_p)}_{L^{\infty},(r_p),d}\) the Banach space of all \(C^\infty\)-functions \(\varphi\) on \(\mathbb{R}^d\) such that
\[
\|\varphi\|_{(r_p)} := \sup_{k \in \mathbb{N}_0^d} \frac{\|D^k \varphi\|_\infty}{R_{|k|} M_k} < \infty,
\]
with the norm \(\| \cdot \|_{(r_p)}\) defined in (3.6).

For a given sequence \((M_p)\) the following projective description is shown in [5, 23]:
\[
\tilde{D}^{(M_p)}_{L^{\infty},d} = \lim_{(r_p) \in \mathcal{R}} D^{(M_p)}_{L^{\infty},(r_p),d},
\]
where the equality holds as l.c.s. We denote by \(\tilde{B}^{(M_p)}_{d}\) the completion of \(D^{(M_p)}_{d}\) in \(D^{(M_p)}_{L^{\infty},d}\).

**Remark 2.** Notice that, if necessary, we may assume that a sequence \((r_p)\) in \(\mathcal{R}\) satisfies for a given constant \(c > 0\) the inequality \(r_p > c\) for all \(p \in \mathbb{N}\). In fact, if \(r_p \not\rightarrow \infty\) as \(p \not\rightarrow \infty\), then there is a \(p_0 \in \mathbb{N}\) such that \(r_p > c\) for \(p > p_0\) and we may replace \((r_p)\) by \((\tilde{r}_p)\) in \(\mathcal{R}\) defined by \(\tilde{r}_0 = 1, \tilde{r}_p := r_{p+p_0}\) for all \(p \in \mathbb{N}\). Thus \(\|\varphi\|_{(r_p)} < \infty\) implies \(\|\varphi\|_{(\tilde{r}_p)} < \infty\) for all \(\varphi \in D^{(M_p)}_{d}\).

**Proposition 2.** If \(\varphi_1, \varphi_2 \in D^{(M_p)}_{L^{\infty},d}\), then \(\varphi_1 \cdot \varphi_2 \in D^{(M_p)}_{L^{\infty},d}\). Moreover, for every \((r_p)\) in \(\mathcal{R}\) such that \((r_p)/2 \in \mathcal{R}\), the inequality holds:
\[
\|\varphi_1 \cdot \varphi_2\|_{(r_p)/2} \leq \|\varphi_1\|_{(r_p)/2} \|\varphi_2\|_{(r_p)/2},
\]
where \((r_p)/2\) is meant in the sense of (2.3).

**Proof.** Fix \((r_p)\) in \(\mathcal{R}\) such that \((r_p)/2 = (\tilde{r}_p)\) in \(\mathcal{R}\) (see (2.3) for \(\lambda = \frac{1}{2}\)). Hence \(\tilde{R}_p = 2^{-p} R_p\) for \(p \in \mathbb{N}_0\) and consequently \(\tilde{R}_{|k|} = 2^{-k} R_{|k|}\) for \(k \in \mathbb{N}_0^d\).

If \(\varphi_1, \varphi_2 \in D^{(M_p)}_{L^{\infty},d}\), then \(\varphi_1, \varphi_2 \in D^{(M_p)}_{L^{\infty},(r_p)/2,d}\) by the projective description of the spaces \(D^{(M_p)}_{L^{\infty},d}\). Now it follows from (3.6) that
\[
\|D^j \varphi_i\|_\infty \leq 2^{-j} R_{|j|} M_j \|\varphi_i\|_{(r_p)/2} \quad (i = 1, 2)
\]
for all \(j \in \mathbb{N}_0^d\). Hence, by Leibniz’ formula,
\[
\|D^k (\varphi_1 \varphi_2)\|_\infty \leq \sum_{0 \leq j \leq k} \binom{k}{j} \|D^j \varphi_1\|_\infty \|D^{k-j} \varphi_2\|_\infty
\]
\[
\leq 2^{-k} \|\varphi_1\|_{(r_p)/2} \|\varphi_2\|_{(r_p)/2} \sum_{0 \leq j \leq k} \binom{k}{j} R_{|j|} M_j R_{|k-j|} M_{k-j}
\]
for each \(k \in \mathbb{N}_0^d\).

Recall that condition (M.1) implies inequality (2.2). On the other hand, for every \((r_p)\) in \(\mathcal{R}\) we have
\[
R_p \cdot R_q = \prod_{i=0}^{p} r_i \prod_{i=0}^{q} r_i \leq \prod_{i=0}^{p+q} r_i = R_{p+q}
\]
for $p, q \in \mathbb{N}_0$, due to monotonicity of $(r_p)$, and hence

$$R_{ij} \cdot R_{ik} \leq R_{i[j+k]}, \quad k, j \in \mathbb{N}_0^d.$$  

(3.9)

Therefore, it follows from (3.8) that

$$\frac{\|D^k(\varphi\phi)\|_\infty}{R_{|k|M_k}} \leq \|\varphi\|((r_p)/2)\|\phi\|((r_p)/2) < \infty$$

for all $k \in \mathbb{N}_0^d$, which completes the proof.

**Remark 3.** Notice that the assertion of Proposition 2 is true, in particular, for functions $\varphi_1, \varphi_2 \in D_{d}^{[M_p]}$ and semi-norms of the form (3.3).

### 4 Roumieu Ultradistributions

**Definition 1.** We denote the strong dual of the space $D_{d}^{[M_p]}$ by $D'_{d}^{[M_p]}$ and call it the space of Roumieu ultradistributions.

The strong dual of the space $\hat{B}_{d}^{[M_p]}$, denoted by $D'_{L^{1,d}}^{[M_p]}$ is called the space of Roumieu integrable ultradistributions.

**Remark 4.** The space $D^{[M_p]}$ is dense in $\hat{B}_{d}^{[M_p]}$ and the respective inclusion mapping is continuous. Consequently, the space $D'_{L^{1,d}}^{[M_p]}$ of Roumieu integrable ultradistributions is a subspace of the space $D'_{d}^{[M_p]}$ of Roumieu ultradistributions.

In the sequel, we assume that the sequence $(M_p)$ satisfies conditions (M.1), (M.2') and (M.3'), so we can use the projective description of the considered locally convex spaces of test functions.

**Definition 2.** By an $\mathfrak{R}$-approximate unit we mean a sequence $(\pi_n)$ of ultradifferentiable functions $\pi_n \in D_{d}^{[M_p]}$ converging to 1 in $E_{d}^{[M_p]}$ such that the following property holds for every sequence $(r_p) \in \mathfrak{R}$:

$$\sup_{n \in \mathbb{N}} \|\pi_n\|((r_p)) = \sup_{n \in \mathbb{N}, k \in \mathbb{N}_0^d} (R_{|k|M_k})^{-1}\|D^k\pi_n\|_\infty < \infty,$$

where $(R_p)$ is the product sequence corresponding to $(r_p)$.

**Definition 3.** By a special $\mathfrak{R}$-approximate unit we mean an $\mathfrak{R}$-approximate unit $(\pi_n)$ such that for every compact set $K \subset \mathbb{R}^d$ there exists an index $n_0 \in \mathbb{N}$ such that $\pi_n(x) = 1$ for all $n \geq n_0$ and $x \in K$.

We denote the class of all $\mathfrak{R}$-approximate units on $\mathbb{R}^d$ by $\mathfrak{U}_d^{[M_p]}$ and the class of all special $\mathfrak{R}$-approximate units on $\mathbb{R}^d$ by $\mathfrak{U}_d^{[M_p]}$.

**Remark 5.** By the Denjoy-Carleman theorem, the defined above spaces of ultradifferentiable functions as well as the classes $\mathfrak{U}_d^{[M_p]}$ and $\mathfrak{U}_d^{[M_p]}$ of approximate units contain sufficiently many members.
5 Integrability of Roumieu Ultradistributions

We formulate below a characterization of integrable Roumieu ultradistributions, being an analog of the theorem of Dierolf and Voigt concerning integrable distributions (see [3]) and of the theorem of Pilipović concerning ultradistributions of Beurling type (see [20]). The proof of the theorem is given in [18].

Theorem 1. Let $V \in D^{(M_p)}_d$. The following conditions are equivalent:

(A) $V$ is continuous on $D^{(M_p)}_d$ in the topology induced by $\dot{B}^{(M_p)}_d$, i.e. there are a sequence $(r_p) \in \mathbb{R}$ and a constant $C > 0$ such that the inequality

\[ |\langle V, \varphi \rangle| \leq C \|\varphi\|(r_p) \]

holds for all $\varphi \in D^{(M_p)}_d$;

(B) there is a sequence $(r_p) \in \mathbb{R}$ with the property that for every $\varepsilon > 0$ there exists a regular compact set $K \subset \mathbb{R}^d$ such that the inequality

\[ |\langle V, \varphi \rangle| \leq \varepsilon \|\varphi\|(r_p) \]

holds for $\varphi \in D^{(M_p)}_d$ with supp $\varphi \cap K = \emptyset$;

(C) for every $(\Pi_n) \in \mathbb{U}^{(M_p)}_d$ the sequence $(\langle V, \Pi_n \rangle)$ is Cauchy;

(D) for every $(\Pi_n) \in \mathbb{U}^{(M_p)}_d$ the sequence $(\langle V, \Pi_n \rangle)$ is Cauchy;

(E) there are a sequence $(r_p) \in \mathbb{R}$, a constant $C > 0$ and a regular compact $K \subset \mathbb{R}^d$ such that inequality (5.1) holds for $\varphi \in D^{(M_p)}_d$ with supp $\varphi \cap K = \emptyset$.

6 Convolution of Roumieu Ultradistributions

S. Pilipović and B. Prangoski made in [21] a deep study of the convolution of Roumieu ultradistributions. The study was based on the investigation of the $\varepsilon$ tensor product of the respective spaces of test functions. Let us recall some results proved and observations made in [21].

The authors use the results on the $\varepsilon$ tensor product from [16] to prove that

\[ \dot{B}^{(M_p)}_{d_1} \otimes \varepsilon \dot{B}^{(M_p)}_{d_2} \approx \dot{B}^{(M_p)}_{d_1 \times d_2} \]

in the sense of an isomorphism. They consider, analogously to ideas applied in [19] to the convolution of measures, the following semi-norms in the space $D^{(M_p)}_{L^\infty,d}$:

\[ q_{g,(r_p)}(\varphi) := \sup_{k \in \mathbb{N}_0} \sup_{x \in \mathbb{R}^d} \left| \frac{g(x)D^k\varphi(x)}{R_{|k|}M_k} \right|, \quad \varphi \in D^{(M_p)}_{L^\infty,d} \]

Denote by $D^{(M_p)}_{L^\infty,d}$ the l.c.s. $D^{(M_p)}_{L^\infty,d}$ equipped with the topology defined by the family $\{q_{g,(r_p)} : g \in C_0, (r_p) \in \mathbb{R}\}$ of semi-norms and the strong dual of $D^{(M_p)}_{L^\infty,d}$ by $\left(D^{(M_p)}_{L^\infty,d}\right)'$. The connection
between the strong dual of $\tilde{D}_{L,\infty,d}$ and the space of integrable distributions $D'_{L^1,d}$ was studied in [21]. Such results have recently improved and it is shown that

$$\tilde{D}'_{L,\infty,d} = D'_{L^1,d},$$

as locally convex spaces (cf. [23], Prop 5.3 and Prop 5.4).

These observations allow the authors to give in [21] the following definitions of convolvability and convolution of two Roumieu ultradistributions, analogous to the known definitions of L. Schwartz for distributions (see [25]):

**Definition 4.** Let $S, T \in D'_{d}^{(M_p)}$. If the following condition is satisfied:

$$(s)\quad V_\varphi := (S \otimes T) \varphi^\Delta \in D'_{L^1,2d}$$

for all $\varphi \in D^{(M_p)}_d$, where $\varphi^\Delta$ is the function of the class $\mathcal{E}^{(M_p)}_{2d}$ defined by

$$\varphi^\Delta(x, y) := \varphi(x + y), \quad x, y \in \mathbb{R}^d,$$

then we say that the Roumieu ultradistributions $S, T$ are **convolvable** in the sense of $(s)$. Then the convolution $S \ast T$ of $S$ and $T$ in $D'_{d}^{(M_p)}$ is defined by

$$\langle S \ast T, \varphi \rangle_d := \langle V_\varphi, 1 \rangle_{2d}, \quad \varphi \in D^{(M_p)}_d,$$

where $V_\varphi$ is meant, according to (6.1), as an element of the space $\tilde{D}'_{L,\infty,2d}$ and the constant function 1 is meant as an element of the space $\tilde{D}^{(M_p)}_{L,\infty,2d}$.

For each $a > 0$ consider the subset $\Delta_a := \{(x, y) \in \mathbb{R}^{2d} : |x + y| \leq a\}$ of $\mathbb{R}^{2d}$ and the following subspace of $B_{2d}^{(M_p)}$:

$$B_{2d}^{(M_p)}(\Delta_a) := \{ \varphi \in B_{2d}^{(M_p)} : \text{supp } \varphi \subseteq \Delta_a \}.$$

Denote by $B^{(M_p)}_{\Delta,2d}$ the inductive limit of the spaces defined in (6.4):

$$B^{(M_p)}_{\Delta,2d} := \lim_{a \to \infty} B_{2d}^{(M_p)}(\Delta_a).$$

The following result on equivalence of convolvability conditions for Roumieu ultradistributions was proved in [21]:

**Theorem 2.** Let $S, T \in D'_{d}^{(M_p)}$. The following conditions are equivalent to condition $(s)$ of convolvability for $S$ and $T$:

$$(cs)\quad S \otimes T \in B^{(M_p)}_{\Delta,2d}$$
\[(c_1') \quad S(\bar{T} \ast \varphi) \in \tilde{\mathcal{D}}^{(M_p)}_{L^1,d} \text{ for all } \varphi \in \mathcal{D}^{(M_p)}_d \text{ and for every compact subset } K \text{ in } \mathbb{R}^d, \text{ the mapping} \]
\[
\mathcal{D}^{(M_p)}_d \times \tilde{\mathcal{B}}^{(M_p)}_d \ni (\varphi, \chi) \mapsto \langle S(\bar{T} \ast \varphi), \chi \rangle \in \mathbb{C}
\]
is a continuous bilinear mapping;
\[(c_2') \quad (\varphi \ast \bar{S})T \in \tilde{\mathcal{D}}^{(M_p)}_{L^1,d} \text{ for all } \varphi \in \mathcal{D}^{(M_p)}_d \text{ and for every compact subset } K \text{ in } \mathbb{R}^d, \text{ the mapping} \]
\[
\mathcal{D}^{(M_p)}_d \times \tilde{\mathcal{B}}^{(M_p)}_d \ni (\varphi, \chi) \mapsto \langle (\bar{S} \ast \varphi)T, \chi \rangle \in \mathbb{C}
\]
is a continuous bilinear mapping;
\[(c_3) \quad \bar{S}(\bar{T} \ast \varphi) \in L^1_d \text{ for all } \varphi, \psi \in \mathcal{D}^{(M_p)}_d.\]

Dimovski, Pilipović, Prangoski and Vindas modified conditions \((c_1')\) and \((c_2')\) and proved in [5], under the assumption that the sequence \((M_p)\) satisfies (M.1), (M.2) and (M.3), that they are equivalent to more transparent versions, as given in the following theorem:

**Theorem 3.** Let \(S, T \in \mathcal{D}'^{(M_p)}_d\). The following conditions are equivalent to condition \((c)\) of convolvability for \(S\) and \(T\):

\[(c_1) \quad S(\bar{T} \ast \varphi) \in \mathcal{D}'^{(M_p)}_{L^1,d} \text{ for all } \varphi \in \mathcal{D}^{(M_p)}_d; \]
\[(c_2) \quad (\varphi \ast \bar{S})T \in \mathcal{D}'^{(M_p)}_{L^1,d} \text{ for all } \varphi \in \mathcal{D}^{(M_p)}_d.\]

In the next section we formulate certain sequential conditions of convolvability of Roumieu ultradistributions, connected with Theorem 1 on integrability in \(\mathcal{D}'^{(M_p)}_d\) from section 5.

### 7 Sequential Definitions of Convolution in \(\mathcal{D}'^{(M_p)}_d\)

The notion of \(\mathcal{A}\)-approximate unit makes it possible to consider several sequential definitions of the convolution of Roumieu ultradistributions based on corresponding sequential conditions of convolvability. The conditions require that respective numerical sequences, corresponding to a given pair of Roumieu ultradistributions via certain approximate units, are Cauchy sequence (Cauchy s. in short) for all approximate units from a given class. The first definition of this kind was given for the convolution of distributions by V. S. Vladimirov in [26] and its equivalent versions were discussed in [3] and [9]. Their counterparts for ultradistributions of Beurling type were discussed in [11] (see also [2]).

We will prove in Theorem 4 that all the sequential definitions are equivalent to the definition of the general convolution of Roumieu ultradistributions in the sense of S. Pilipović and B. Prangoski [21]. Our proof of Theorem 4 will be based on the integrability result proved in the previous section.
**Definition 5.** Let \( S, T \in \mathcal{D}_d^{(M_p)} \). We say that \( S, T \) are convolvable in the sense of (v), (ii), (i_1), (i_2), if the corresponding condition below holds for every \( \varphi \in \mathcal{D}_d^{(M_p)} \), respectively:

\[
\begin{align*}
(v) & \quad \langle (S \otimes T, \ i_n \varphi^\Delta)_{2d} \rangle \quad \text{is a Cauchy s. for all } (i_n) \in \mathbb{U}^{(M_p)}_{2d}; \\
(ii) & \quad \langle ((i_n^1 S) \otimes (i_n^2 T), \varphi^\Delta)_{2d} \rangle \quad \text{is a Cauchy s. for all } (i_n^1), (i_n^2) \in \mathbb{U}^{(M_p)}_d; \\
i_1) & \quad \langle ((i_n S) \otimes T, \varphi^\Delta)_{2d} \rangle \quad \text{is a Cauchy s. for all } (i_n) \in \mathbb{U}^{(M_p)}_d; \\
i_2) & \quad \langle (S \otimes (i_n T), \varphi^\Delta)_{2d} \rangle \quad \text{is a Cauchy s. for all } (i_n) \in \mathbb{U}^{(M_p)}_d.
\end{align*}
\]

**Definition 6.** If \( S, T \in \mathcal{D}_d^{(M_p)} \) are convolvable in the sense of (v), (ii), (i_1), (i_2), respectively, then the convolution of \( S \) and \( T \) in \( \mathcal{D}_d^{(M_p)} \) is the corresponding formula below:

\[
\begin{align*}
\langle S \ast^V T, \varphi \rangle_d & := \lim_{n \to \infty} \langle S \otimes T, i_n \varphi^\Delta \rangle_{2d}, \quad \varphi \in \mathcal{D}_d^{(M_p)}, \quad (i_n) \in \mathbb{U}^{(M_p)}_{2d}; \\
\langle S \ast^Π T, \varphi \rangle_d & := \lim_{n \to \infty} \langle (i_n^1 S) \otimes (i_n^2 T), \varphi^\Delta \rangle_{2d}, \quad \varphi \in \mathcal{D}_d^{(M_p)}, \quad (i_n^1), (i_n^2) \in \mathbb{U}^{(M_p)}_d; \\
\langle S \ast^Π T, \varphi \rangle_d & := \lim_{n \to \infty} \langle (i_n S) \otimes T, \varphi^\Delta \rangle_{2d}, \quad \varphi \in \mathcal{D}_d^{(M_p)}, \quad (i_n) \in \mathbb{U}^{(M_p)}_d; \\
\langle S \ast^Π T, \varphi \rangle_d & := \lim_{n \to \infty} \langle S \otimes (i_n T), \varphi^\Delta \rangle_{2d}, \quad \varphi \in \mathcal{D}_d^{(M_p)}, \quad (i_n) \in \mathbb{U}^{(M_p)}_d;
\end{align*}
\]

respectively.

**Remark 6.** Condition (s) of convolvability in Definition 4 guarantees, by the considerations of Pilipović and Prangoski from section 4 in [21], that the convolution \( S \ast T \) in the sense of (6.3) exists in \( \mathcal{D}_d^{(M_p)} \). It follows from Theorem 4 formulated below, that convolvability conditions (v), (ii), (i_1) and (i_2) guarantee that the corresponding sequential convolutions, defined in Definition 6, exist in \( \mathcal{D}_d^{(M_p)} \).

In addition, to the sequential conditions of convolvability in \( \mathcal{D}_d^{(M_p)} \), given in Definition 5, one may consider also the conditions (v), (i_1), (i_2) being the modifications of the above ones consisting in replacing the classes \( \mathbb{U}^{(M_p)}_d \) and \( \mathbb{U}^{(M_p)}_d \) of \( \mathcal{R} \)-approximate units by the classes \( \mathbb{U}^{(M_p)}_{2d} \) and \( \mathbb{U}^{(M_p)}_{2d} \) of special approximate units, respectively. The modified conditions lead to additional sequential definitions of the convolution in \( \mathcal{D}_d^{(M_p)} \) which are counterparts of the known sequential definitions of the convolution of distributions (see [20], [3] and [9]). It follows from Theorem 7 that they are equivalent to all the conditions listed in Definition 6.

**Remark 7.** The convolution of Roumieu ultradistributions in \( \mathcal{D}_d^{(M_p)} \) investigated in [21] and [6] and its sequential versions discussed in this paper is a general notion. It embraces various particular cases, e.g. expressed in terms of supports of given Roumieu ultradistributions. The discussion of such sufficient conditions will be given in forthcoming papers.

Each of the sequential definitions of the convolution of \( S \) and \( T \), under the corresponding conditions, do not depend on the choice of an approximate unit from a given class.
Theorem 4. Let $S, T \in D_d^{(M_p)}$. All the conditions formulated in Definition 7 are equivalent to the condition (s) of convolvability of ultradistributions. If one of these conditions is satisfied, then all corresponding convolutions of Roumieu ultradistributions $S$ and $T$ exist in $D_d^{(M_p)}$ and all the convolutions are equal:

$$S\ast T = S\ast T = S\ast T = S\ast T = S\ast T.$$ 

Proof. We will prove the equivalence of convolvability conditions given in Definitions 7 and 6 and in Theorem 3 according to the following scheme of implications:

$$(\Pi_1) \rightarrow (c_1) \rightarrow (s) \rightarrow (v) \rightarrow (\Pi)$$

$$(\Pi_2) \rightarrow (c_2)$$

Assume condition (\Pi) for a fixed $\varphi \in D_d^{(M_p)}$. This means that the following limit exists for all $(\Pi_1^n), (\Pi_2^n) \in U_d^{(M_p)}$:

$$\lim_{n \to \infty} \langle (\Pi_1^n S) \otimes (\Pi_2^n T), \varphi^\triangle \rangle_{2d} = \alpha$$

for a certain $\alpha \in \mathbb{C}$ which does not depend on the sequences $(\Pi_1^n)$ and $(\Pi_2^n)$. Hence the double limit

$$\lim_{i,j \to \infty} \langle (\Pi_1^i S) \otimes (\Pi_2^j T), \varphi^\triangle \rangle_{2d} = \alpha$$

also exists for all $(\Pi_1^i), (\Pi_2^j) \in U_d^{(M_p)}$. Indeed, if the last equality is not true, then there exist increasing sequences of positive integers $(i_n)$ and $(j_n)$ such that

$$|\langle (\Pi_1^i S) \otimes (\Pi_2^j T), \varphi^\triangle \rangle - \alpha| > \varepsilon,$$

for all $n \in \mathbb{N}$, but this conducts to contradiction with condition (\Pi), because $(\Pi_1^n), (\Pi_2^n) \in U_d^{(M_p)}$. Thus under the existence of the iterated limits and definition of approximate unit we obtain

$$\alpha = \lim_{n \to \infty} \langle (\Pi_1^n S) \otimes T, \varphi^\triangle \rangle_{2d},$$

which ensure condition (\Pi_1).

Next for a fixed $\varphi \in D_d^{(M_p)}$ and arbitrary $(\Pi_n) \in U_d^{(M_p)}$ we may write

$$\langle (\Pi_n S) \otimes T, \varphi^\triangle \rangle_{2d} = \langle (\Pi_n S) \ast T, \varphi \rangle_d = \langle \Pi_n S, \varphi \ast \bar{T} \rangle_d = \langle S(\bar{T} \ast \varphi), \Pi_n \rangle_d, \quad n \in \mathbb{N}$$

for any function $\Pi_n \in D_d^{(M_p)}$. Then according to Theorem 4 and condition (\Pi_1) we have the equality

$$\lim_{n \to \infty} \langle (\Pi_n S) \otimes T, \varphi^\triangle \rangle_{2d} = \langle S(\bar{T} \ast \varphi), 1 \rangle$$

because the definition of family $U_d^{(M_p)}$. Moreover, this implies integrability of ultradistribution $S(\bar{T} \ast \varphi)$ for all $\varphi \in D_d^{(M_p)}$. 

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The proof of implications (ii) ⇒ (iii) ⇒ (iv) is analogous. The equivalence of conditions (c1), (c2) and (s) is given by Theorem 3. From Theorem 1 with $V := (S \otimes T) \varphi^\Delta$ and $(n_n) \in \mathcal{U}_{2d}^{(M_p)}$, we obtain equivalence of conditions (s) and (v).

It remains to note that the implication (v) ⇒ (ii) is obvious, since for every $\varphi \in \mathcal{D}_d^{(M_p)}$ and arbitrary $(n_n^1), (n_n^2) \in \mathcal{U}_{d}^{(M_p)}$ the functions $(n_n^1 \otimes n_n^2) \varphi^\Delta$ belong to $\mathcal{D}_{2d}^{(M_p)}$ for $n \in \mathbb{N}$ and

$$\langle (n_n^1 S \otimes (n_n^2 T), \varphi^\Delta)_{2d} = \langle S \otimes T, \varphi^\Delta (n_n^1 \otimes n_n^2) \rangle_{2d}$$

according to the notion of tensor products in spaces $\mathcal{D}_d^{(M_p)}$ and $\mathcal{U}_{d}^{(M_p)}$ and the respective topological isomorphisms described in Theorems 2.1 and 2.3 in [15]. Condition (v) guarantees that the sequence $\big( (n_n^1 S \otimes (n_n^2 T), \varphi^\Delta)_{2d} \big)$ is Cauchy. Hence $(n_n^1 \otimes n_n^2) \in \mathcal{U}_{2d}^{(M_p)}$.

**Remark 8.** In addition, one may consider the sequential conditions (v), (i), (ii), (iii) of convolvability of $S, T \in \mathcal{D}_d^{(M_p)}$ and the corresponding convolutions $S \ast T$, $\mathcal{D}_d^{(M_p)}$, $S \ast T$, $S \ast T$, $S \ast T$ of $S$ and $T$, replacing the classes $\mathcal{U}_{2d}^{(M_p)}$ and $\mathcal{U}_{d}^{(M_p)}$ of $\mathcal{D}_d^{(M_p)}$ and $\mathcal{U}_{d}^{(M_p)}$ of special $\mathcal{D}_d^{(M_p)}$. The modified conditions of convolvability and the corresponding sequential definitions of the convolution of $S$ and $T$ are equivalent to those considered above.

The equivalence of conditions (s), (v), (ii), (iii) and (iv) of convolvability follows in the same manner as the equivalence of conditions (s), (v), (ii), (iii) and (iv) proved in Theorem 1 above. The equality of the corresponding convolutions is an easy consequence of this result.

**Remark 9.** It is easy to see that the convolution of Roumieu ultradistributions is commutative, i.e. $S \ast T = T \ast S$ for $S, T \in \mathcal{D}_d^{(M_p)}$.

### 8 Ultradifferential Property of the Convolution

Let us consider an ultradifferential operator $P(D)$ defined by Komatsu in [14] as follows:

**Definition 7.** An operator of the form

$$P(D) = \sum_{k \in \mathbb{N}_0^d} c_k D^k, \quad c_k \in \mathbb{C}$$

is called an *ultradifferential operator of class $\{M_p\}$* if for every $L > 0$ there is a constant $C_L$ such that

$$|c_k| \leq C_L \frac{L^k}{M_k}, \quad k \in \mathbb{N}_0^d.$$

The estimation in (8.2) means that for every $L > 0$ we have $\sup_k (L^{-k} M_k |c_k|) < \infty$. Then according to Lemma 1 Part (II) there is a sequence $(u_p) \in \mathcal{D}$ such that $\sup_k (U_{k|L} M_k |c_k|) < \infty$.  

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where \( U_k = \prod_{p \leq k} u_p \). In other words an ultradifferential operator of the form (8.1) is of class \( \{ M_p \} \) if there are \( C > 0 \) and \( (u_p) \in \mathbb{R} \) such that

\[
|c_k| \leq \frac{C}{U_{|k|} M_k}, \quad k \in \mathbb{N}_0^d.
\]

Due to Komatsu \[14\], if \( (M_p) \) satisfies condition (M.2), then \( P(D) \) defines the respective continuous mappings \( \mathcal{D}_d^{(M_p)} \rightarrow \mathcal{D}_d^{(M_p)} \) and \( \mathcal{D}_d^{(M_p)} \rightarrow \mathcal{D}_d^{(M_p)} \). Moreover the series \( P(D)S = \sum_{k \in \mathbb{N}_0^d} c_k D^k S \) converges absolutely in \( \mathcal{D}_d^{(M_p)} \) for every \( S \in \mathcal{D}_d^{(M_p)} \).

In Theorem 5 below we prove an important and non-trivial property of the convolution of Roumieu ultradistributions. In the proof we will need the following very useful result from \[21\]:

**Lemma 2.** For every sequence \((s_p) \in \mathbb{R}\) there exists a sequence \((r_p) \in \mathbb{R}\) such that \( r_p \leq s_p \) for \( p \in \mathbb{N} \) and

\[
R_{p+q} \leq 2^{p+q} R_p R_q \quad \text{for all } p, q \in \mathbb{N}_0.
\]

**Theorem 5.** Let \( S, T \in \mathcal{D}_d^{(M_p)} \) be convolvable and let \( P(D) \) be an ultradifferential operator of class \( \{ M_p \} \). Then \( P(D)S \) and \( T \) and as well as \( S \) and \( P(D)T \) are convolvable and, moreover,

\[
P(D)(S \ast T) = (P(D)S) \ast T = S \ast (P(D)T).
\]

**Proof.** Assume that \( S, T \in \mathcal{D}_d^{(M_p)} \) are convolvable. By the definitions of \( P(D) \) and \( S \ast T \) and by Theorem 4 we have

\[
\langle P(D)(S \ast T), \varphi \rangle = \langle S \ast T, P(-D) \varphi \rangle = \langle (S \otimes T)(P(-D_x) \varphi^\Delta, 1_{2d})
\]

\[
\lim_{n \to \infty} \langle S \otimes T, \pi_n \left[ P(-D_x) \varphi^\Delta \right] \rangle
\]

for all \((\pi_n) \in \mathcal{U}_{2d}^{(M_p)}\) and \( \varphi \in \mathcal{D}_d^{(M_p)} \).

We are going to prove only that the ultradistributions \( P(D)S \) and \( T \) are convolvable in \( \mathcal{D}_d^{(M_p)} \) and the first equality of (8.5) holds, because the remaining part of the assertion follows then directly from Remark 4. To prove the convolvability of \( P(D)S \) and \( T \) we have to show that the sequence \( \left( \langle P(D)S \otimes T, \pi_n \varphi^\Delta \rangle \right)_{n \in \mathbb{N}} \) is convergent.

We have

\[
\langle P(D)S \otimes T, \pi_n \varphi^\Delta \rangle = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \langle D_x^\alpha S \otimes T, \pi_n \varphi^\Delta \rangle = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \langle S \otimes T, -D_x^\alpha \left( \pi_n \varphi^\Delta \right) \rangle
\]

\[
= \langle S \otimes T, P(-D_x) \left( \pi_n \varphi^\Delta \right) \rangle
\]

for all \( n \in \mathbb{N} \), by (8.1) and the absolute convergence of the respective series.

Comparing the last terms in (8.6) and (8.7), we see that the declared assertion will be proved if we show the equalities

\[
P(-D_x) \left( \pi_n \varphi^\Delta \right) = \pi_n P(-D_x) \varphi^\Delta + \nu_n
\]
on $\mathbb{R}^{2d}$ for all $n \in \mathbb{N}$, where $\nu_n$ are certain functions in $\mathcal{D}_d^{(M_p)}$ such that

$$\lim_{n \to \infty} \langle S \otimes T, \nu_n \rangle = 0. \tag{8.9}$$

Applying (8.1) and Leibniz’ rule and then changing the order of summation, we get, for all $n \in \mathbb{N}$, the equalities

$$P(-D_x) \left( \Pi_n \varphi^\Delta \right) = \sum_{\alpha \in \mathbb{N}_0^d} (-1)^{|\alpha|} c_\alpha \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^\beta \varphi)(D^{\alpha-i} \varphi)^\Delta$$

$$= \sum_{i=0}^d \sum_{\beta \in \mathbb{N}_0^d} (-1)^{|\beta+i|} \left( \begin{array}{c} \beta + i \\ i \end{array} \right) c_{\beta+i}(D^{\beta} \varphi)^\Delta$$

$$= \Pi_n P(-D_x) \varphi^\Delta + \nu_n$$

on $\mathbb{R}^{2d}$ with the functions $\nu_n$ defined for $x, y \in \mathbb{R}^d$ by

$$\nu_n(x, y) := \sum_{i \in \mathcal{N}} D^i \Pi_n(x, y) \sum_{\beta \in \mathbb{N}_0^d} (-1)^{|\beta+i|} \left( \begin{array}{c} \beta + i \\ i \end{array} \right) c_{\beta+i} D^\beta \varphi(x + y), \tag{8.10}$$

where $\mathcal{N} := \mathbb{N}_0^d \setminus \{(0, \ldots, 0)\}$. This means equations (8.8) hold for $\nu_n$ defined in (8.10) and for all $n \in \mathbb{N}$. Clearly, $(\nu_n)$ depends on the initial sequence $(\Pi_n)$.

It suffices to show (8.9). Choose $\theta \in \mathcal{D}_d^{(M_p)}$ such that $\theta(x) = 1$ for $x \in \text{supp} \varphi$. By (8.10), we have

$$\langle S \otimes T, \nu_n \rangle = \langle (S \otimes T) \theta^\Delta, \nu_n \rangle, \quad n \in \mathbb{N}.$$ 

Due to assumption that $S$ and $T$ are convolvable in $\mathcal{D}_d^{(M_p)}$, the sequence $\left( \langle (S \otimes T) \theta^\Delta, \Pi_n \rangle \right)_{n \in \mathbb{N}}$ is convergent for every $(\Pi_n) \in \mathcal{U}_d^{(M_p)}$. To prove (8.9) it is enough to show that also $(\Pi_n + \nu_n) \in \mathcal{U}_d^{(M_p)}$. Since $(\nu_n), (\Pi_n) \in \mathcal{U}_d^{(M_p)}$, for each compact set $K$ in $\mathbb{R}^{2d}$ there exists an $n_0 \in \mathbb{N}$ such that $D^n \Pi_n(x, y) = 0$ and $\Pi_n(x, y) = 1$ for $(x, y) \in K, \ i \in \mathcal{N}$ and $n > n_0$. Consequently, in view of (8.10),

$$\Pi_n(x, y) + \nu_n(x, y) = 1 \quad \text{for } (x, y) \in K, \ n > n_0.$$ 

Therefore it remains to prove, for every $(t_p) \in \mathfrak{R}$, that

$$\sup_{n \in \mathbb{N}} \|\nu_n\|_{(t_p)} < \infty, \tag{8.11}$$

since $(\Pi_n) \in \mathcal{U}_d^{(M_p)}$ and (8.11) implies $\sup_{n \in \mathbb{N}} \|\Pi_n + \nu_n\|_{(t_p)} < \infty$.

Fix an arbitrary $(t_p) \in \mathfrak{R}$. The coefficients of the ultradifferential operator $P(D)$ satisfy (8.3) for some $(u_p) \in \mathfrak{R}$. Putting $s_k := \min\{t_k, u_k\}$ for $k \in \mathbb{N}$, we have $(s_p) \in \mathfrak{R}$. By Lemma 2 there exists a sequence $(\tau_p) \in \mathfrak{R}$ such that $\tau_k \leq s_k$ and inequality (8.4) holds. In addition we assume, according to Remark 2 that

$$r_p > 16H^2 \quad \text{for } p \in \mathbb{N}, \tag{8.12}$$
where $H > \frac{1}{4}$ is a constant from condition (M.2).

Let $\alpha \in \mathbb{N}_0^d$ be arbitrarily fixed. For fixed $n \in \mathbb{N}$ and $(r_p) \in \mathfrak{R}$ chosen above, according to the representation (8.10), we have

$$\frac{\|D^\alpha \nu_n\|_\infty}{R_{|\alpha|} M_\alpha} \leq \sum_{i \in N, j \leq \alpha} \left( \sum_{\beta \in \mathbb{N}_0^d} c_{\beta+i} \right) \frac{\|D^{\alpha+i-j} \pi_n\|_\infty}{R_{|\alpha+i-j|} M_{\alpha+i-j}} \sum_{\beta \in \mathbb{N}_0^d} (\beta + i) |c_{\beta+i}|$$

(8.13)

Moreover, we have

$$H \geq 0$$

where

$$\frac{R_{|\alpha+i-j|} R_{|\beta|}}{R_{|\alpha|}} \leq \frac{2^{\beta+j} R_{|\alpha|}}{R_{|\alpha|}} \leq 2^{|\alpha+i|} 2^{\beta+j}$$

and

$$\frac{M_{\alpha+i-j} M_{\beta+j}}{M_\alpha M_\beta M_i} \leq \frac{A H^{\beta+j} M_{\alpha+i}}{M_\alpha M_i} \leq A^2 H^{\alpha+i} H^{\beta+j}. \tag{8.15}$$

Moreover, we have

$$|c_{\beta+i}| \leq \frac{C}{U_{|\beta|} U_{|i|} M_\beta M_i}$$

and

$$R_{|\beta|} R_{|i|} \leq \frac{U_{|\beta|} U_{|i|}}{C} \leq 1,$$}

by (8.3), properties (2.1) of $(M_p)$ and (3.9) of $(u_p) \in \mathfrak{R}$ and because $r_k \leq u_k$ for $k \in \mathbb{N}$. The inequalities in (8.14), (8.15) and (8.16) hold for arbitrary $\alpha, \beta, i, j \in \mathbb{N}_0^d$ such that $j \leq \alpha$ and will be used later together with the following known estimate:

$$\binom{\beta+i}{i} \leq 2^{\beta+i}, \tag{8.17}$$

for a certain $B > 0$ and all $\beta, i \in \mathbb{N}_0^d$.

It follows from (8.13), due to (8.14)–(8.17), that

$$\frac{\|D^\alpha \nu_n\|_\infty}{R_{|\alpha|} M_\alpha} \leq A^2 C \sum_{i \in N, j \leq \alpha} \left( \sum_{\beta \in \mathbb{N}_0^d} c_{\beta+i} \right) \frac{\|D^{\alpha+i-j} \pi_n\|_\infty}{R_{|\alpha+i-j|} M_{\alpha+i-j}} \sum_{\beta \in \mathbb{N}_0^d} (\beta + i) |c_{\beta+i}|$$

(8.18)

According to assumption (8.12), consider the sequences $(\pi_p)$ and $(\overline{\pi}_p)$ of the class $\mathfrak{R}$ defined by $\overline{\pi}_p := r_p/8H$ and $\overline{\pi}_p := r_p/16H^2$, respectively, for $p \in \mathbb{N}$. Clearly,

$$2^{\alpha+i-j}(2H)^{\alpha+i-j} \frac{\|D^{\alpha+i-j} \pi_n\|_\infty}{R_{|\alpha+i-j|} M_{\alpha+i-j}} \leq \|\pi_n\|_{(\overline{\pi}_p)}$$
and
\[ 2^{2|\beta+j|}(2H)^{2|\beta+j|} \left\| \frac{D_{x}^{\beta+j} \varphi^{\Diamond}}{R_{|\beta+j|} M_{\beta+j}} \right\|_{\infty} \leq \| \varphi \|_{(\varphi_{\alpha})} \]
for all \( \alpha, \beta, i, j \in \mathbb{N}_{0}^{d}, j \leq \alpha \). We deduce from (8.18) and the above estimates that
\[ \left\| \frac{D_{x}^{\alpha} \nu_{n}}{R_{|\alpha|} M_{\alpha}} \right\|_{\infty} \leq \frac{A^{2} C}{2^{\alpha}} \| \pi_{n} \|_{(\varphi_{\alpha})} \| \varphi \|_{(\varphi_{\alpha})} \cdot \sum_{i \in \mathcal{N}} \left( \frac{1}{2} \right)^{i} \sum_{\beta \in \mathbb{N}_{0}^{d}} \left( \frac{1}{4H} \right)^{\beta} < \infty. \]
for arbitrary \( \alpha \in \mathbb{N}_{0}^{d} \) and \( n \in \mathbb{N} \). Hence
\[ \sup_{n \in \mathbb{N}} \| \nu_{n} \|_{(r_{p})} = \sup_{n \in \mathbb{N}} \sup_{\alpha \in \mathbb{N}_{0}^{d}} \frac{\| D_{x}^{\alpha} \nu_{n} \|_{\infty}}{R_{|\alpha|} M_{\alpha}} < \infty \]
and, since \( r_{p} \leq t_{p} \) for \( p \in \mathbb{N} \),
\[ \sup_{n \in \mathbb{N}} \| \nu_{n} \|_{(t_{p})} \leq \sup_{n \in \mathbb{N}} \| \nu_{n} \|_{(r_{p})} < \infty, \]
i. e. (8.11) is proved, as required. The assertion of Theorem 5 is proved. \( \square \)

Theorem 5 has also been shown in the quasianalytic case in the forthcoming article [23]. The proof there is via a completely different method (cf. [23], Cor. 5.10).

Acknowledgements

This work was partly supported by the Center for Innovation and Transfer of Natural Sciences and Engineering Knowledge of University of Rzeszów.

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