RECONSTRUCTIBLE GRAPHS, SIMPLICIAL FLAG COMPLEXES OF HOMOLOGY MANIFOLDS AND ASSOCIATED RIGHT-ANGLED COXETER GROUPS

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Abstract. In this paper, we investigate a relation between finite graphs, simplicial flag complexes and right-angled Coxeter groups, and we provide a class of reconstructible finite graphs. We show that if $\Gamma$ is a finite graph which is the 1-skeleton of some simplicial flag complex $L$ which is a homology manifold of dimension $n \geq 1$, then the graph $\Gamma$ is reconstructible.

1. Introduction

In this paper, we investigate a relation between finite graphs, simplicial flag complexes and right-angled Coxeter groups, and we provide a class of reconstructible finite graphs. We show that if $\Gamma$ is a finite graph which is the 1-skeleton of some simplicial flag complex $L$ which is a homology manifold of dimension $n \geq 1$, then the graph $\Gamma$ is reconstructible.

A graph $\Gamma$ is said to be reconstructible, if any graph $\Gamma'$ with the following property $(\ast)$ is isomorphic to $\Gamma$.

$(\ast)$ Let $S$ and $S'$ be the vertex sets of $\Gamma$ and $\Gamma'$ respectively. Then there exists a bijection $f : S \to S'$ such that the subgraphs $\Gamma_{S-\{s\}}$ and $\Gamma'_{S'-\{f(s)\}}$ are isomorphic for any $s \in S$, where $\Gamma_{S-\{s\}}$ and $\Gamma'_{S'-\{f(s)\}}$ are the full subgraphs of $\Gamma$ and $\Gamma'$ whose vertex sets are $S - \{s\}$ and $S' - \{f(s)\}$ respectively.

The following open problem is well-known as the Reconstruction Conjecture.

Problem (Reconstruction Conjecture). Every finite graph with at least three vertices will be reconstructible?

Some classes of reconstructible graphs are known (cf. [3], [20], [21], [22], [23], [26]) as follows: Let $\Gamma$ be a finite graph with at least three vertices.

(i) If $\Gamma$ is a regular graph, then it is reconstructible.
(ii) If $\Gamma$ is a tree, then it is reconstructible.
(iii) If $\Gamma$ is not connected, then it is reconstructible.
(iv) If $\Gamma$ has at most 11 vertices, then it is reconstructible.

Our motivation to consider graphs of the 1-skeletons of some simplicial flag complexes comes from the following idea on right-angled Coxeter groups and their nerves.
Details of Coxeter groups and Coxeter systems are found in [3], [6] and [19], and details of flag complexes, nerves, Davis complexes and their boundaries are found in [8], [9] and [24].

Let $\Gamma$ be a finite graph and let $S$ be the vertex set of $\Gamma$. Then the graph $\Gamma$ uniquely determines a finite simplicial flag complex $L$ whose 1-skeleton $L^{(1)}$ coincide with $\Gamma$. Here a simplicial complex $L$ is a flag complex, if the following condition holds:

$(**)\quad$ For any vertex set $\{s_0, \ldots, s_n\}$ of $L$, if $\{s_i, s_j\}$ spans 1-simplex in $L$ for any $i, j \in \{0, \ldots, n\}$ with $i \neq j$ then the vertex set $\{s_0, \ldots, s_n\}$ spans $n$-simplex in $L$.

Also every finite simplicial flag complex $L$ uniquely determines a right-angled Coxeter system $(W, S)$ whose nerve $L(W, S)$ coincide with $L$ (cf. [1], [8], [9], [10], [12]). Here for any subset $T$ of $S$, $T$ spans a simplex of $L$ if and only if the parabolic subgroup $W_T$ generated by $T$ is finite (such a subset $T$ is called a spherical subset of $S$).

Moreover it is known that every right-angled Coxeter group $W$ uniquely determines its right-angled Coxeter system $(W, S)$ up to isomorphisms ([28], [18]). By this corresponding, we can identify a finite graph $\Gamma$, a finite simplicial flag complex $L$, a right-angled Coxeter system $(W, S)$ and a right-angled Coxeter group $W$.

Let $\Gamma$ and $\Gamma'$ be finite graphs, let $L$ and $L'$ be the corresponding flag complexes, let $(W, S)$ and $(W', S')$ be the corresponding right-angled Coxeter systems, and let $W$ and $W'$ be the corresponding right-angled Coxeter groups, respectively. Then the following statements are equivalent:

1. $\Gamma$ and $\Gamma'$ are isomorphic as graphs;
2. $L$ and $L'$ are isomorphic as simplicial complexes;
3. $(W, S)$ and $(W', S')$ are isomorphic as Coxeter systems;
4. $W$ and $W'$ are isomorphic as groups.

Also, for any subset $T$ of the vertex set $S$ of the graph $\Gamma$, the full subgraph $\Gamma_T$ of $\Gamma$ with vertex set $T$ corresponds the full subcomplex $L_T$ of $L$ with vertex set $T$, the parabolic Coxeter system $(W_T, T)$ generated by $T$, and the parabolic subgroup $W_T$ of $W$ generated by $T$.

Hence we can consider the reconstruction problem as the problem on simplicial flag complexes and also as the problem on right-angled Coxeter groups.

Moreover, the right-angled Coxeter system $(W, S)$ associated by the graph $\Gamma$ defines the Davis complex $\Sigma$ which is a CAT(0) space and we can consider the ideal boundary $\partial \Sigma$ of the CAT(0) space $\Sigma$ (cf. [1], [2], [5], [8], [9], [10], [12], [15], [16], [24]). Then the topology of the boundary $\partial \Sigma$ is determined by the graph $\Gamma$, and the topology of $\partial \Sigma$ is also a graph invariant.

Based on the observations above, we can obtain the following lemma from results of F. T. Farrell [13, Theorem 3], M. W. Davis [10, Theorem 5.5] and [17, Corollary 4.2] (we introduce details of this argument in Section 3).

**Lemma 1.1.** Let $(W, S)$ be an irreducible Coxeter system where $W$ is infinite and let $L = L(W, S)$ be the nerve of $(W, S)$. Then the following statements are equivalent:

1. $W$ is a virtual Poincaré duality group.
2. $L$ is a generalized homology sphere.
3. $\tilde{H}^i(L_{S-T}) = 0$ for any $i$ and any non-empty spherical subset $T$ of $S$. 


Here a generalized homology \( n \)-sphere is a polyhedral homology \( n \)-manifold with the same homology as an \( n \)-sphere \( S^n \) (cf. [10, Section 5], [11], [25, p.374], [27]). Also detail of (virtual) Poincaré duality groups is found in [7], [10], [11], [13].

In Lemma 1.1 we particularly note that the statement (3) is a local condition of \( L \) which determines a global structure of \( L \) as the statement (2). This implies the following theorem.

**Theorem 1.2.** Let \( \Gamma \) be a finite graph with at least 3 vertices and let \((W, S)\) be the right-angled Coxeter system associated by \( \Gamma \) (i.e. the 1-skeleton of the nerve \( L(W, S) \) of \((W, S)\) is \( \Gamma \)). If the Coxeter group \( W \) is a virtual Poincaré duality group, then the graph \( \Gamma \) is reconstructible. Hence,

(i) if \( \Gamma \) is the 1-skeleton of some simplicial flag complex \( L \) which is a generalized homology sphere, then the graph \( \Gamma \) is reconstructible, and

(ii) in particular, if \( \Gamma \) is the 1-skeleton of some flag triangulation \( L \) of some \( n \)-sphere \( S^n \) \((n \geq 1)\), then the graph \( \Gamma \) is reconstructible.

Here, based on this idea, we investigate a finite graph which is the 1-skeleton of some simplicial flag complex which is a homology manifold as an extension of a generalized homology sphere, and we show the following theorem.

**Theorem 1.3.** Let \( \Gamma \) be a finite graph with at least 3 vertices.

(i) If \( \Gamma \) is the 1-skeleton of some simplicial flag complex \( L \) which is a homology \( n \)-manifold \((n \geq 1)\), then the graph \( \Gamma \) is reconstructible.

(ii) In particular, if \( \Gamma \) is the 1-skeleton of some flag triangulation \( L \) of some \( n \)-manifold \((n \geq 1)\), then the graph \( \Gamma \) is reconstructible.

Here detail of homology manifolds is found in [10, Section 5], [11], [25, p.374], [27].

2. Proof of Theorem 1.3

We prove Theorem 1.3.

**Proof of Theorem 1.3.** Let \( \Gamma \) be a finite graph which is the 1-skeleton of some simplicial flag complex \( L \) which is a homology manifold of dimension \( n \geq 1 \). Then we show that the graph \( \Gamma \) is reconstructible.

Let \( \Gamma' \) be a finite graph and let \( L' \) be the finite simplicial flag complex associated by \( \Gamma' \). Also let \( S \) and \( S' \) be the vertex sets of the graphs \( \Gamma \) and \( \Gamma' \) respectively.

Now we suppose that the condition \((*)\) holds:

\((*)\) There exists a bijection \( f : S \to S' \) such that the subgraphs \( \Gamma_{S-\{s\}} \) and \( \Gamma'_{S'-\{f(s)\}} \) are isomorphic for any \( s \in S \).

To show that the graph \( \Gamma \) is reconstructible, we prove that the two graphs \( \Gamma \) and \( \Gamma' \) are isomorphic, i.e., the two simplicial flag complexes \( L \) and \( L' \) associated by \( \Gamma \) and \( \Gamma' \) respectively are isomorphic.

Let \( s_0 \in S \) and set \( s'_0 = f(s_0) \). Then the two subgraphs \( \Gamma_{S-\{s_0\}} \) and \( \Gamma'_{S'-\{s'_0\}} \) are isomorphic by the assumption \((*)\), and the two subcomplexes \( L_{S-\{s_0\}} \) and \( L'_{S'-\{s'_0\}} \) are isomorphic.

The condition \((*)\) implies that the local conditions of \( L \) and \( L' \) coincide. Hence since \( L \) is a homology \( n \)-manifold, we obtain that \( L' \) is also a homology \( n \)-manifold.
(here we note that $n \geq 1$). Indeed if $L$ is a homology $n$-manifold, then by [25, Lemma 63.1],

$$H_i([L'], |L'| - \sigma') \cong H_i([L], |L| - \sigma) \cong H_i(\mathbb{S}^n),$$

for any simplex $\sigma'$ of $L'$ and some corresponding simplex $\sigma$ of $L$. Hence $L'$ is also a homology $n$-manifold. Here we note that there exists $s_0 \in S - \text{Lk}(\sigma, L)^{(0)}$ and for $s' = f(s_0)$, $L_{S-\{s_0\}}$ and $L'_{S-\{s'_0\}}$ are isomorphic.

By [25, Corollary 63.3], the homology $n$-manifolds $L$ and $L'$ have the structure as the unions of some $n$-simplexes. Also we note that the links $\text{Lk}(s_0, L)$ and $\text{Lk}(s'_0, L')$ are homology $(n-1)$-sphere, and the pairs $(L_{S-\{s_0\}}, \text{Lk}(s_0, L))$ and $(L'_{S-\{s'_0\}}, \text{Lk}(s'_0, L'))$ are relative homology $n$-manifolds.

Let $\phi : S - \{s_0\} \to S' - \{s'_0\}$ be a map which induces an isomorphism of the subgraphs $\Gamma_{S-\{s\}}$ and $\Gamma'_{S-\{f(s)\}}$. Then $\phi$ induces an isomorphism of the two simplicial flag complexes $L_{S-\{s_0\}}$ and $L'_{S-\{s'_0\}}$. Then for any $t \in S - \{s_0\}$, by the above observation on the homology manifolds, $t \in \text{Lk}(s_0, L)$ if and only if $\phi(t) \in \text{Lk}(s'_0, L')$.

Hence the map $\bar{\phi} : S \to S'$ defined by $\bar{\phi}|_{S-\{s_0\}} = \phi$ and $\bar{\phi}(s_0) = s'_0$ induces an isomorphism of the two graphs $\Gamma$ and $\Gamma'$.

Therefore the graph $\Gamma$ is reconstructible. \hfill $\square$

3. Appendix: Virtual Poincaré Duality Coxeter groups and reconstructible graphs

We introduce a relation of virtual Poincaré duality Coxeter groups and reconstructible graphs, which is our motivation of this paper.

**Definition 3.1** (cf. [7], [10], [11], [13]). A torsion-free group $G$ is called an $n$-dimensional Poincaré duality group, if $G$ is of type FP and if

$$H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & (i \neq n) \\ \mathbb{Z} & (i = n). \end{cases}$$

Also a group $G$ is called a virtual Poincaré duality group, if $G$ contains a torsion-free subgroup of finite-index which is a Poincaré duality group.

On Coxeter groups and (virtual) Poincaré duality groups, the following results are known.

**Theorem 3.2** (Farrell [13, Theorem 3]). Suppose that $G$ is a finitely presented group of type FP, and let $n$ be the smallest integer such that $H^n(G; \mathbb{Z}G) \neq 0$. If $H^n(G; \mathbb{Z}G)$ is a finitely generated abelian group, then $G$ is an $n$-dimensional Poincaré duality group.

**Remark.** It is known that every infinite Coxeter group $W$ contains some torsion-free subgroup $G$ of finite-index in $W$ which is a finitely presented group of type FP and $H^*(G; \mathbb{Z}G)$ is isomorphic to $H^*(W; \mathbb{Z}W)$. Hence if $n$ is the smallest integer such that $H^n(W; \mathbb{Z}W) \neq 0$ and if $H^n(W; \mathbb{Z}W)$ is finitely generated (as an abelian group), then $W$ is a virtual Poincaré duality group of dimension $n$.

**Theorem 3.3** (Davis [10, Theorem 5.5]). Let $(W, S)$ be a Coxeter system. Then the following statements are equivalent:

1. $W$ is a virtual Poincaré duality group of dimension $n$. 

(2) $W$ decomposes as a direct product $W = W_{T_0} \times W_{T_1}$ such that $T_1$ is a spherical subset of $S$ and the simplicial complex $L_{T_0} = L(W_{T_0}, T_0)$ associated by $(W_{T_0}, T_0)$ is a generalized homology $(n - 1)$-sphere.

**Theorem 3.4 (17 Corollary 4.2).** Let $(W, S)$ be an irreducible Coxeter system, let $L = L(W, S)$ and let $0 \leq i \in \mathbb{Z}$. Then the following statements are equivalent:

1. $H^i(W; \mathbb{Z}W)$ is finitely generated.
2. $H^i(W; \mathbb{Z}W)$ is isomorphic to $\tilde{H}^{i-1}(L)$.
3. $\tilde{H}^{i-1}(L_{S-T}) = 0$ for any non-empty spherical subset $T$ of $S$.

Here $L_{S-T} = L(W_{S-T}, S - T)$.

We obtain the following lemma from results above.

**Lemma 3.5.** Let $(W, S)$ be an irreducible Coxeter system where $W$ is infinite and let $L = L(W, S)$. Then the following statements are equivalent:

1. $W$ is a virtual Poincaré duality group.
2. $L$ is a generalized homology sphere.
3. $\tilde{H}^i(L_{S-T}) = 0$ for any $i$ and any non-empty spherical subset $T$ of $S$.

**Proof.** (1) $\Leftrightarrow$ (2): We obtain the equivalence of (1) and (2) from Theorem 3.3 since $(W, S)$ is irreducible.

(1) $\Rightarrow$ (3): We obtain this implication from Theorem 3.3 because if $W$ is a virtual Poincaré duality group then $H^i(W; \mathbb{Z}W)$ is finitely generated for any $i$.

(3) $\Rightarrow$ (1): Suppose that $\tilde{H}^i(L_{S-T}) = 0$ for any $i$ and any non-empty spherical subset $T$ of $S$. Then by Theorem 3.3, $H^{i+1}(W; \mathbb{Z}W)$ is finitely generated for any $i$. Since $W$ is infinite, $H^{i_0}(W; \mathbb{Z}W)$ is non-trivial for some $i_0$ (cf. [7], [14]). Hence by Theorem 3.2, $W$ is a virtual Poincaré duality group.

As the observation in Section 1, Lemma 3.5 implies the following.

**Theorem 3.6.** Let $\Gamma$ be a finite graph with at least 3 vertices and let $(W, S)$ be the right-angled Coxeter system associated by $\Gamma$. If the Coxeter group $W$ is a virtual Poincaré duality group, then the graph $\Gamma$ is reconstructible.

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