On the Nonlinear Neutral Conformable Fractional Integral-Differential Equation

Rui Li, Wei Jiang, Jiale Sheng, Sen Wang
School of Mathematical Sciences, Anhui University, Hefei, China
Email: jiangwei@ahu.edu.cn

Abstract
In this paper, we investigate the nonlinear neutral fractional integral-differential equation involving conformable fractional derivative and integral. First of all, we give the form of the solution by lemma. Furthermore, existence results for the solution and sufficient conditions for uniqueness solution are given by the Leray-Schauder nonlinear alternative and Banach contraction mapping principle. Finally, an example is provided to show the application of results.

Keywords
Conformable Fractional Derivative, Delay, Existence and Uniqueness, Functional Differential Equation

1. Introduction
The theory of fractional calculus has played a major role in control theory, fluid dynamics, biological systems, economics and other fields [1] [2] [3]. It serves as a valuable tool for the description of memory and hereditary properties of various materials and processes. In recent years, plenty of interesting results have been observed for the Riemann-Liouville and Caputo type fractional derivatives. The definition of the conformable fractional derivative and integral was introduced in 2014 by Khalil et al. [4]. Compared with Riemann-Liouville and Caputo type fractional derivatives, the conformable fractional derivative satisfies the Leibniz rule and chain rule, and can be converted to classical derivative [5]. This is of great help to study fractional differential equations. In the past few years, the conformable fractional derivative has been used in the field of fractional newtonian mechanics, heat equation, biology and so on, and the results are abundant. The conformable fractional optimal control problems with time-delay were studied, which proved that the embedding method, embedding the admissible set into a subset of measures, can be successfully applied to nonlinear prob-
lems [6]. Alharbi et al. utilized the homotopy perturbation method to solve a model of Ambartsumian equation with the conformable derivative and gave the approximate solution of equation [7]. The conformable fractional derivative was utilized to solve the time-fractional Burgers type equations approximately [8].

Over these years, there has been a significant development in fractional functional differential equations. Among them, Li, Liu and Jiang gave sufficient conditions of the existence of positive solutions for a class of nonlinear fractional differential equations with Caputo derivative [9]. Guo et al. studied fractional functional differential equation with impulsive, then they obtained existence, uniqueness, and data-dependent results of solutions to the equation [10]. The existence of positive periodic solutions was given by Zhang and Jiang, for n-dimensional impulsive periodic functional differential equations [11]. In addition, the fractional stochastic functional system driven by Rosenblatt process was investigated by Shen et al., and they obtained controllability and stability results [12].

Time-delay is part of the theoretical fields investigated by many authors, including unbounded time-delay, bounded time-delay, state-dependent time-delay and others. In 2009, the existence and uniqueness of solutions of the Caputo fractional neutral differential equations with unbounded delays were discussed [13]. In 2011, Li and Zhang considered the Caputo fractional neutral integral-differential equations with unbounded delay, which used the fixed point theorem to study the existence of mild solutions of equations [14]. The Caputo fractional neutral integral-differential equations with unbounded delay were discussed in 2013 [15]. The paper used Monch’s fixed point theorem via measures of non-compactness to study the existence of solutions of equations.

Based on the above research background and relevant discussions, we found that few people used conformable derivative to study fractional differential equations with time-delay. In 2019, Mohamed I. Abbas gave the existence of solutions and uniqueness of solution for fractional neutral integro-differential equations by the Hadamard fractional derivative of order \( \alpha \in (0,1) \) and the Riemann-Liouville integral [16]. In this paper, we will discuss the nonlinear neutral fractional integral-differential equation in the frame of the conformable derivative of order \( \alpha \in (1,2) \) and the conformable integral. Then we make the condition 3 weaker to improve feasibility. Considering the following equation:

\[
\begin{align*}
T^\alpha \left[ w(t) - \sum_{i=1}^{p} I^\beta_i u_i(t, w_i) \right] &= I(t, w), \quad t \in [0, \rho], \\
w(t) &= \psi(t), \quad t \in [-\nu, 0],
\end{align*}
\]

(1.1)

where \( T^\alpha \) denotes the conformable fractional derivative of order \( \alpha \), \( 1 < \alpha < 2 \), \( I^\beta \) denotes the conformable fractional integral with order \( \beta \), \( \beta \in (0,1) \), \( i = 1, 2, 3, \ldots, p \), \( p \in \mathbb{N} \), \( \rho, \nu > 0 \) are constants. And for any \( t \in [0, \rho] \), we denote by \( w_i \) the element of \( C([-\nu, 0], R) \) and is defined by \( w_i(\theta) = w(t+\theta) \), \( \theta \in [-\nu, 0] \). Here \( w_i(\cdot) \) represents the history of the state from time \( t-\nu \) up to the present time \( t \). \( l, u : [0, \rho] \times C([-\nu, 0], R) \to R \) are continuous functions that satisfy some hypotheses given later, \( \psi \in C([-\nu, 0], R) \).
The rest of this paper is organized as follows: In Section 2, we introduce the concepts and basic properties of conformable fractional integral and derivative. In Section 3, we give existence results for the solution and sufficient conditions for uniqueness solution by Leray-Schauder nonlinear alternative and Banach contraction mapping principle. In Section 4, the numerical simulation is showed to illustrate the results.

Notations: \( C([0, \rho], R) \) denotes all continuous functions that mapped from \([0, \rho] \) to \( R \) and \( R \) denotes all real numbers. \( R^+ \) denotes all positive real numbers.

2. Preliminaries

In this section, we present some necessary definitions and lemmas to establish our main results.

**Definition 2.1.** ([5]) For a function \( w: [a, +\infty) \to R \), the conformable fractional integral of order \( \alpha (n < \alpha \leq n+1, n \in N) \) of the function \( w \) is defined as follows:

\[
I^\alpha_a w(t) = \frac{1}{n!} \int_a^t (t-x)^{n-\alpha} w(x) \, dx.
\]

If \( a = 0 \), \( I^\alpha_a w(t) \) can be written as \( I^\alpha w(t) \).

**Definition 2.2.** ([5]) For a function \( w: [a, +\infty) \to R \), the conformable fractional derivative of order \( \alpha (n < \alpha \leq n+1, n \in N) \) of the function \( w \) is defined as follows:

\[
T^\alpha_a w(t) = \lim_{\varepsilon \to 0} \frac{w^{[\alpha]-1}(t+\varepsilon(t-a)^{[\alpha]-\alpha}) - w^{[\alpha]-1}(t)}{\varepsilon},
\]

where \( [\alpha] \) denotes the smallest integer greater than or equal to \( \alpha \). If \( a = 0 \), \( T^\alpha_a w(t) \) can be written as \( T^\alpha w(t) \).

**Lemma 2.3** ([4]) If function \( l \) is \( \alpha \) times differential at a point \( t > 0 \) for \( n < \alpha \leq n+1, n \in N \), then

1) \( T^\alpha l(t) = 0 \), for all constant functions \( l(t) = \lambda \);
2) \( l \) is \( n+1 \) times differential simultaneously, then \( T^\alpha l(t) = t^{[\alpha]-\alpha} l^{[\alpha]}(t) \).

**Lemma 2.4.** ([5])

1) For a function \( w: [a, +\infty) \to R \), if \( w^{(\alpha)}(t) \) is continuous, then for any \( t > a \), we have

\[
T^\alpha_a I^\alpha_a w(t) = w(t), \alpha \in (n, n+1];
\]

2) For a function \( w: [a, +\infty) \to R \), if \( w \) is \( n+1 \) times differentiable, then for any \( t > a \), we have

\[
I^\alpha_a T^\alpha_a w(t) = w(t) - \sum_{i=0}^\infty \frac{w^{(i)}(a)(t-a)^i}{i!}, \alpha \in (n, n+1].
\]

**Lemma 2.5.** ([4]) If \( \alpha \in (0, 1] \), functions \( w_1, w_2 \) are \( \alpha \) times differentiable at a point \( t > 0 \), then

1) \( T^\alpha (a_1 w_1 + a_2 w_2) = a_1 T^\alpha (w_1) + a_2 T^\alpha (w_2) \);
\[ T^\alpha \left( t^\rho \right) = p t^{\rho - \alpha}; \]
\[ T^\alpha \left( w_1, w_2 \right) = w_1 T^\alpha \left( w_2 \right) + w_2 T^\alpha \left( w_1 \right); \]
\[ T^\alpha \left( \frac{w_1}{w_2} \right) = \frac{w_1}{w_2} \left[ w_2 T^\alpha \left( w_1 \right) - w_1 T^\alpha \left( w_2 \right) \right]. \]

**Lemma 2.6.** ([17]) (The nonlinear alternative of Leray-Schauder type) Let \( C \) be a Banach space, \( C_1 \) be a closed, convex subset of \( C \) and \( P \) be an open subset of \( C_1 \), \( 0 \in P \). If \( A : P \rightarrow C_1 \) is a continuous, compact map. Then

1) \( A \) has a fixed point in \( P \), or
2) There is a \( w \in \partial P \) (the boundary of \( P \) in \( C_1 \)) and \( \lambda \in (0,1) \), such that \( w = \lambda A(w) \).

**Lemma 2.7.** Let \( I(t) \) be a continuous function, then the fractional differential equation

\[
\begin{cases}
T^\alpha \left[ w(t) - \eta(t) \right] = I(t), & t \in [0, \rho], 1 < \alpha < 2, \\
w(0) = \psi_0, w'(0) = \psi'_0,
\end{cases}
\]

is equivalent to the integral-differential equation

\[
w(t) = \psi_0 - \eta(0) + \left[ \psi'_0 - \eta'(0) \right] t + \eta(t) + \int_0^t \left( t - s \right) s^{\alpha - 2} I(s) \, ds,
\]

where \( \psi_0 = w(0), \ \psi'_0 = w'(0) \).

**Proof:** Consider Equation (2.1), for any \( t \in [0, \rho] \),

\[
T^\alpha \left[ w(t) - \eta(t) \right] = I(t).
\]

Transforming \( \alpha \) times conformable fractional integral on both sides of the equation and using Lemma 2.4, since \( 1 < \alpha < 2 \), we have

\[
w(t) = \psi_0 - \eta(0) + \left[ \psi'_0 - \eta'(0) \right] t + \eta(t) + \int_0^t \left( t - s \right) s^{\alpha - 2} I(s) \, ds,
\]

where \( \psi_0 = w(0), \ \psi'_0 = w'(0) \). The proof is completed. \( \square \)

### 3. Main Results

In this section, we give several results about the fractional integral-differential Equation (1.1).

If \( X = \{ w \mid w \in C([-\upsilon, \rho], \mathbb{R}) \} \) is Banach Space, the norm is defined as \( \|w\| = \sup \left\{ |w(t)|, t \in [-\upsilon, \rho] \right\} \).

Let \( \eta(t) = \sum_{i=1}^{n} I^{\beta_i} u_i \left( t, w_i \right) \). Giving the definition of the operator \( A : X \rightarrow X \),

\[
A w(t) = \begin{cases} \psi_0 + \left[ \psi'_0 - \eta'(0) \right] t + \sum_{i=1}^{n} I^{\beta_i} u_i \left( t, w_i \right) + \int_0^t \left( t - s \right) s^{\alpha - 2} I(s, w_i) \, ds, & t \in [0, \rho], \\
\psi(t), & t \in [-\upsilon, 0].
\end{cases}
\]

where \( \psi_0 = w(0), \ \psi'_0 = w'(0), \ w_0 = w(0 + \theta) = \psi(\theta), \ \theta \in [-\upsilon, 0] \),

\[
\eta(0) = \sum_{i=1}^{n} I^{\beta_i} u_i \left( t, w_i \right) \bigg|_{t=0} = 0.
\]

It should be noticed that Equation (1.1) has solutions if and only if the operator \( A \) has fixed points. So as to achieve the desired goals, we impose the follow-
ing assumptions for the Equation (1.1).

(H1) There exist functions \( \gamma(t), \delta(t) : [0, \rho] \to \mathbb{R} \) and continuous non decreasing functions \( \zeta(t), \phi(t) : [0, \infty] \to [0, \infty] \), such that, for any \((t, w) \in [0, \rho] \times C([-\nu, 0], \mathbb{R})\),
\[
|f(t, w)| \leq \gamma(t) \zeta(w), \\
|\dot{u}_i(t, w)| \leq \delta_i(t) \phi(w).
\]

(H2) Functions \( l_i, u_i : [0, \rho] \times C([-\nu, 0], \mathbb{R}) \to \mathbb{R} \) are continuous. There exist positive functions \( \lambda_i, \mu \) with bounds \( \lambda_i, \mu < \infty \), respectively such that
\[
|u_i(t, x) - u_i(t, y)| \leq \lambda_i \|x - y\|, \\
|f(t, x) - f(t, y)| \leq \mu \|x - y\|.
\]

(H3) There exists constant \( M > 0 \), \( \nu_1 < \nu \), such that
\[
M > \|\varphi\| + \|\varphi_0 - \eta'(0)\| + \sum_{i=1}^{p} \|\phi_i(u_1)\| \rho^\beta_i + \|\varphi_0\| \rho^\alpha \frac{1}{\alpha(\alpha - 1)}.
\]

(H4)
\[
\sum_{i=1}^{p} \|\phi_i(u_1)\| \rho^\beta_i + \|\mu\| \rho^\alpha \frac{1}{\alpha(\alpha - 1)} < 1.
\]

We give an existence result based on the nonlinear alternative of Leray-Schauder type applied to a completely continuous operator.

**Theorem 3.1.** Suppose that the assumptions (H1)-(H3) are satisfied, then the Equation (1.1) has at least one solution.

**Proof.** The operator \( A \) is defined as (3.1). Define
\[
D = \{ \varphi \in C([-\nu, \rho], \mathbb{R}) : \|\varphi\| \leq \nu_1 \}.
\]

Firstly, we prove that operator \( A \) is uniformly bounded. For any \((t, w) \in [0, \rho] \times C([-\nu, 0], \mathbb{R})\), \( i = 1, 2, \ldots, p \), \( p \in \mathbb{N}_+ \), and \( w \in D \), by (H3), we have

\[
|A\varphi(t)| = \left| \varphi_0 + \left[ \varphi_0 - \eta'(0) \right] t + \sum_{i=1}^{p} \int_{0}^{t} \dot{\varphi}_i(s, w) + \int_{0}^{t} \left( t - s \right) s^{\alpha-2} \varphi(s, w) \, ds \right|
\]

\[
\leq \|\varphi_0\| + \|\varphi_0 - \eta'(0)\| t + \sum_{i=1}^{p} \int_{0}^{t} \|\dot{\varphi}_i(s, w)\| + \int_{0}^{t} \left( t - s \right) s^{\alpha-2} \|\varphi(s, w)\| \, ds
\]

\[
\leq \|\varphi_0\| + \|\varphi_0 - \eta'(0)\| t + \sum_{i=1}^{p} \int_{0}^{t} \|\dot{\varphi}_i(t, \|w\|)\| + \int_{0}^{t} \left( t - s \right) s^{\alpha-2} \varphi(t, \|w\|) \, ds
\]

\[
\leq \|\varphi_0\| + \|\varphi_0 - \eta'(0)\| t + \sum_{i=1}^{p} \|\phi_i(u_1)\| \int_{0}^{t} \left( t - s \right) s^{\alpha-2} \, ds + \|\varphi_0\| \int_{0}^{t} \left( t - s \right) s^{\alpha-2} \, ds
\]

\[
\leq \|\varphi_0\| + \|\varphi_0 - \eta'(0)\| \rho + \sum_{i=1}^{p} \|\phi_i(u_1)\| \frac{\rho^\beta_i}{\beta_i} + \|\varphi_0\| \rho^\alpha \frac{1}{\alpha(\alpha - 1)} t
\]

\[
\leq \|\varphi_0\| + \|\varphi_0 - \eta'(0)\| \rho + \sum_{i=1}^{p} \|\phi_i(u_1)\| \frac{\rho^\beta_i}{\beta_i} + \|\varphi_0\| \rho^\alpha \frac{1}{\alpha(\alpha - 1)} t < M.
\]

For any \( t \in [-\nu, 0] \) and \( w \in D \), we have
\[ |Aw(t)| = |\psi'(t)| \leq \|\psi\|. \]

Denote \( M_1 = \max \{ M, \|\psi\| \} \), then
\[ \|Aw(t)\| \leq M_1, \ t \in [-\upsilon, \rho], \ w \in D. \]

This implies that the operator \( A \) is uniformly bounded in \( D \).

Besides, we need to prove that \( AD \) is an equicontinuous set. Let \( w^n \) be a sequence such that \( \lim_{n \to \infty} w^n = w \) in \( D \). Then, for any \( t \in [0, \rho] \), we have
\[
\lim_{n \to \infty} Aw^n(t) = \psi_0 + \left[ \psi'_0 - \eta'(0) \right] t + \lim_{n \to \infty} \sum_{i=1}^{p} s^{\beta_i - 1} u_i(s, w^n) \int_0^t (t-s) s^{\alpha_2 - 2} l(s, w^n) \, ds.
\]

By \( (H_2) \), functions \( l, u_i \) are uniformly continuous, thus, we have
\[
\lim_{n \to \infty} Aw^n(t) = \psi_0 + \left[ \psi'_0 - \eta'(0) \right] t + \sum_{i=1}^{p} s^{\beta_i - 1} u_i(s, w) \int_0^t (t-s) s^{\alpha_2 - 2} l(s, w) \, ds.
\]

For any \( t \in [-\upsilon, 0] \), it is obvious that
\[
\lim_{n \to \infty} Aw^n(t) = \psi(t) = Aw(t).
\]

Therefore, \( Aw(t) \) is continuous and uniformly continuous for \( t \in [-\upsilon, \rho] \), which implies that \( Aw(t) \) is equicontinuous for \( t \in [-\upsilon, \rho] \), \( A \) is continuous in \( C([-\upsilon, \rho], R) \).

Furthermore, we consider \( |Aw(t_2) - Aw(t_1)| \), for any \( t_1, t_2 \in [-\upsilon, \rho] \), \( t_1 < t_2 \).

Case 1. If \( 0 \leq t_1 < t_2 \leq \rho \), for any \( (t, w) \in [0, \rho] \times C([-\upsilon, 0], R) \) and \( w \in D \), \( i = 1, 2, \ldots, p \), we have
\[
|Aw(t_2) - Aw(t_1)| = \left| \psi_0 + \left[ \psi'_0 - \eta'(0) \right] (t_2 - t_1) + \sum_{i=1}^{p} s^{\beta_i - 1} u_i(s, w) \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} l(s, w) \, ds \right|
\]
\[
\leq |\psi'_0 - \eta'(0)|(t_2 - t_1) + \sum_{i=1}^{p} s^{\beta_i - 1} |u_i(s, w)| \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} |l(s, w)| \, ds
\]
\[
+ |\sum_{i=1}^{p} s^{\beta_i - 1} u_i(s, w)| \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} \left( \|w\| \right) \, ds
\]
\[
\leq |\psi'_0 - \eta'(0)|(t_2 - t_1) + \sum_{i=1}^{p} s^{\beta_i - 1} \left( \|u_i\| \right) \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} \left( \|w\| \right) \, ds
\]
\[
+ |\sum_{i=1}^{p} s^{\beta_i - 1} u_i(s, w)| \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} \left( \|w\| \right) \, ds
\]
\[
+ \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} \gamma(t) \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} \left( \|w\| \right) \, ds
\]
\[
+ \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} \gamma(t) \int_0^{t_2} (t_2 - s) s^{\alpha_2 - 2} \left( \|w\| \right) \, ds
\]
by assumption (H1), we have

\begin{align*}
\leq & \left| \nu'_0 - \eta'(0) \right| (t_2 - t_i) + \sum_{i=1}^p |\delta_i| \int_0^{t_2} s^{\beta_i} ds \\
& + \|\phi\| \int_0^{t_2} (t_2 - t_i) s^{\alpha_i} ds + \|\phi\| \int_0^{t_2} (t_2 - s) s^{\alpha_i} ds \\
\leq & \left| \nu'_0 - \eta'(0) \right| (t_2 - t_i) + \sum_{i=1}^p |\delta_i| \frac{t_2^{\beta_i} - t_i^{\beta_i}}{\beta_i} + \|\phi\| \frac{t_2^{\alpha_i} - t_i^{\alpha_i}}{\alpha_i (\alpha_i - 1)}.
\end{align*}

If \( t_2 - t_i \to 0 \), then \( \left| Aw(t_2) - Aw(t_i) \right| \to 0 \).

Case 2. If \( -\nu < t_i < 0 \), for any \((t, w_i) \in [0, \rho] \times C([-\nu, 0], R)\) and \( w \in D \), \( i = 1, 2, \ldots, p \), \( Aw(t_i) = \psi(t_i) \) holds for any \( -\nu < t_i < 0 \), we have

\begin{align*}
\left| Aw(t_2) - Aw(t_i) \right| & = \left| \nu_0 + \left[ \nu'_0 - \eta'(0) \right] t_2 + \sum_{i=1}^p \int_0^{t_2} s^{\beta_i} u_i(s, w_i) ds \\
& + \int_0^{t_2} (t_2 - s) s^{\alpha_i} l(s, w_i) ds - \psi(t_i) \right| \\
& \leq \left| \nu'_0 - \eta'(0) \right| t_2 + |\nu_0 - \psi(t_i)| + \sum_{i=1}^p \int_0^{t_2} s^{\beta_i} |u_i(s, w_i)| ds \\
& + \int_0^{t_2} (t_2 - s) s^{\alpha_i} |l(s, w_i)| ds \\
& \leq \left| \nu'_0 - \eta'(0) \right| t_2 + |\nu_0 - \psi(t_i)| + \sum_{i=1}^p |\delta_i| \int_0^{t_2} s^{\beta_i} ds \\
& + \|\phi\| \int_0^{t_2} (t_2 - s) s^{\alpha_i} ds \\
& \leq \left| \nu'_0 - \eta'(0) \right| t_2 + |\nu_0 - \psi(t_i)| + \sum_{i=1}^p |\delta_i| \frac{t_2^{\beta_i} - t_i^{\beta_i}}{\beta_i} \\
& + \|\phi\| \frac{t_2^{\alpha_i} - t_i^{\alpha_i}}{\alpha_i (\alpha_i - 1)}.
\end{align*}

Since \( \psi(t) \) is a continuous function, if \( t_i \to 0 \), then we have \( \psi(t_i) \to \psi_0 \). If \( t_2 - t_i \to 0 \), then \( \left| Aw(t_2) - Aw(t_i) \right| \to 0 \).

Case 3. If \( -\nu \leq t_i < t_2 < 0 \), for any \( -\nu \leq t_i < t_2 < 0 \) and \( w \in D \),

\[ \left| Aw(t_2) - Aw(t_i) \right| = \left| \psi(t_2) - \psi(t_i) \right|. \]

Since \( \psi(t) \) is a continuous function, if \( t_2 - t_i \to 0 \), then \( \left| Aw(t_2) - Aw(t_i) \right| \to 0 \).

From what has been discussed above, \( AD \) is equicontinuous. By Arzelá-Ascoli theorem, \( AN \) is compact, then \( A \) is completely continuous on \( X \).

For any \((t, w_i) \in [0, \rho] \times C([-\nu, 0], R)\) and \( t \in [0, \rho] \), \( i = 1, 2, \ldots, p \), we have

\[ w(t) = \nu_0 + \left[ \nu'_0 - \eta'(0) \right] t + \sum_{i=1}^p \int_0^{t_2} s^{\beta_i} u_i(s, w_i) ds + \int_0^{t_2} (t_2 - s) s^{\alpha_i} l(s, w_i) ds, \]

For any \( t \in [0, \rho] \), by assumption \((H_d)\), we have

\begin{align*}
|w(t)| & \leq |\nu_0| + |\nu'_0 - \eta'(0)| t + \sum_{i=1}^p t^{\beta_i} u_i(s, w_i) + \int_0^{t_2} (t_2 - s) s^{\alpha_i} l(s, w_i) ds \\
& \leq |\nu_0| + |\nu'_0 - \eta'(0)| t + \sum_{i=1}^p t^{\beta_i} \delta_i(t) \phi(\|w_i\|) + \int_0^{t_2} (t_2 - s) s^{\alpha_i} \gamma(t) \zeta(\|w_i\|) ds \\
& \leq |\nu_0| + |\nu'_0 - \eta'(0)| t + \sum_{i=1}^p t^{\beta_i} \phi(\|w_i\|) \int_0^{t_2} s^{\beta_i} ds + \|\phi\| \int_0^{t_2} (t_2 - s) s^{\alpha_i} ds \\
& \leq |\nu_0| + |\nu'_0 - \eta'(0)| t + \sum_{i=1}^p t^{\beta_i} \phi(\|w_i\|) \int_0^{t_2} s^{\beta_i} ds + \|\phi\| \frac{t_2^{\alpha_i} - t_i^{\alpha_i}}{\alpha_i (\alpha_i - 1)}.
\end{align*}
Consider the assumption (H3), there exists $M \neq \|w(t)\|$. Define set 
\[ P = \{ w \in C([-\nu, \rho], R) : \|w(t)\| < M \} \]. We can show that $A : \overline{P} \to D$ is continuous and completely continuous. Assuming there exists $w \in \partial P$ and $\lambda \in (0,1)$, such that $w = \lambda A(w)$. Then we have 
\[ \|\lambda\| = \frac{\|w\|}{\|A(w)\|} \geq 1. \]

By Lemma 2.6, $A$ has a fixed point in $P$, which implies that there exists at least one solution to the Equation (1.1). The proof is completed.

We will give the uniqueness result of solutions of Equation (1.1):

**Theorem 3.2.** Suppose that the assumptions (H2) and (H4) are satisfied, then the Equation (1.1) has a unique solution.

**Proof.** The operator $A$ is defined as (3.1). For any $t \in [0, \rho]$ and $w^1, w^2 \in C([-\nu, \rho], R)$, by (H2), we have
\[
|Aw^1(t) - Aw^2(t)|
\]
\[
\leq \|\psi_0 + [\psi_0' - \eta'(0)]t + \sum_{i=1}^{p} \|\phi_i(u_i)\| \beta_i t_i \|\frac{t_i^{\alpha}}{\alpha} - \psi_0^{\alpha} (\alpha - 1)\| \psi_0^{\alpha} (\alpha - 1)\|
\]
\[
\leq \|\psi_0 + [\psi_0' - \eta'(0)]t + \sum_{i=1}^{p} \|\phi_i(u_i)\| \beta_i t_i \|\frac{t_i^{\alpha}}{\alpha} - \psi_0^{\alpha} (\alpha - 1)\|
\]
\[
\leq \sum_{i=1}^{p} \|\phi_i(u_i) - \phi_i(u_i)\| \beta_i t_i \|\frac{t_i^{\alpha}}{\alpha} - \psi_0^{\alpha} (\alpha - 1)\|
\]

For any $t \in [-\nu, 0]$, 
\[ |Aw^1(t) - Aw^2(t)| = |\psi(t) - \psi(t)| = 0. \]

By (H2), $A$ is a contraction mapping, then Equation (1.1) has a unique solution. The proof is completed.

**4. An Illustrative Example**

This section presents an example where we apply Theorems 3.1 and 3.2 to some particular cases.
Example 4.1. Consider the fractional integral-differential equation
\[
\begin{align*}
\left[T^\frac{3}{2}\right] \left[w(t) - \sum_{i=1}^3 I^\frac{i}{4} \left(\frac{|w_i|}{(20i + 6t)(1 + |w_i|)}\right)\right] &= \frac{1}{16 - t^2} \left(\frac{|w_i|}{3(1 + |w_i|)} + \frac{1}{5}\right), \\
\psi(t) &= \psi(t) \quad t \in [-\frac{1}{2}, 0].
\end{align*}
\]
where \( T^\frac{3}{2} \) denotes the conformable fractional derivative of order \( \frac{3}{2} \), \( I^\frac{i}{4} \) denotes the conformable fractional integral of the order \( \frac{i}{4} \), \( i = 1, 2, 3 \). If \( w(t) \colon [-\frac{1}{2}, 2] \rightarrow R \), then for any \( t \in [0, 2] \), we define \( w_i(t) = w(t + \theta) \), \( \theta \in [-\frac{1}{2}, 0] \). Functions
\[
u_i(t, w_i) = \frac{|w_i|}{(20i + 6t)(1 + |w_i|)}, \\
l(t, w_i) = \frac{1}{16 - t^2} \left(\frac{|w_i|}{3(1 + |w_i|)} + \frac{1}{5}\right).
\]
The continuous function \( \psi(t) \) satisfies the condition that \( \psi(0) = \psi'(0) = 0 \).

For any \( (t, w_i) \in [0, 2] \times C\left(\left[-\frac{1}{2}, 0\right], R\right), i = 1, 2, 3 \), we have
\[
|\nu_i(t, w_i)| = \frac{|w_i|}{(20i + 6t)(1 + |w_i|)} \leq \frac{1}{20}.
\]
For any \( (t, w_i) \in [0, 2] \times C\left(\left[-\frac{1}{2}, 0\right], R\right), \) we have
\[
|l(t, w_i)| = \frac{1}{16 - t^2} \left(\frac{|w_i|}{3(1 + |w_i|)} + \frac{1}{5}\right) \leq \frac{2}{45}.
\]
For any \( (t, w_i) \in [0, 2] \times C\left(\left[-\frac{1}{2}, 0\right], R\right), \psi(0) = \psi'(0) = \eta(0) = \eta'(0) = 0 \), if \( M > 0.659 \), we have
\[
M > 1, i = 1, 2, 3.,
\]
Consequently, by Theorem 3.1, the Equation (4.1) has at least one solution.

For continuous functions column \( u_i(t, w_i), u_i(t, w_i^2) : [0, 2] \times C\left(\left[-\frac{1}{2}, 0\right], R\right), i = 1, 2, 3 \), we have
\[
|u_i(t, w_i) - u_i(t, w_i^2)| = \left|\frac{|w_i|}{(20i + 6t)(1 + |w_i|)} - \frac{|w_i|}{(20i + 6t)(1 + |w_i^2|)}\right|,
\]
For continuous functions \( l(t, w_i) \), \( l(t, w_i) : [0, 2] \times C\left(\left[-\frac{1}{2}, 0\right], R\right) \), we have

\[
\left\| l(t, w_i) - l(t, w_j) \right\| 
\leq \frac{1}{16 - t^2} \left( \frac{\left| w_i \right|}{3(1 + \left| w_i \right|)} + \frac{1}{5} \right) - \frac{1}{16 - t^2} \left( \frac{\left| w_j \right|}{3(1 + \left| w_j \right|)} + \frac{1}{5} \right)
\]

\[
\leq \frac{\left\| w_i - w_j \right\|}{3(16 - t^2) \left(1 + \left| w_i \right|\right) \left(1 + \left| w_j \right|\right)}
\]

\[
\leq \frac{\left\| w_i - w_j \right\|}{3(16 - t^2)}
\]

For any \( (t, w_i) \in [0, 2] \times C\left(\left[-\frac{1}{2}, 0\right], R\right) \), we have

\[
\sum_{i=1}^{3} \frac{1}{20i^2} \frac{2^i}{4} + \frac{1}{36} \frac{2^3}{3} \frac{3^2}{2} \frac{3}{2} < 0.596 < 1, i = 1, 2, 3.
\]

Thus, by Theorem 3.2, the Equation (4.1) has a unique solution.

### 5. Conclusion

The conformable fractional derivative brings great convenience to the study of fractional functional differential equations due to its unique properties. This paper uses conformable derivative to study the fractional neutral integro-differential equations, and obtains the results of the existence of the solution and the sufficient conditions for the uniqueness of the solution.

### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

### References

[1] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006) Theory and Applications of Fractional Differential Equations, Vol. 204. North Holland Mathematics. Elsevier, Amsterdam.

[2] Miller, K.S. and Ross, B. (1993) An Introduction to the Fractional Calculus and Differential Equations. John Wiley, Hoboken.

[3] Podlubny, I. (1999) Fractional Differential Equations. Academic Press, Cambridge.

[4] Khalil, R., Al Horani, M., Yousef, A. and Sababheh, M. (2014) A New Definition of Fractional Derivative. *Computational and Applied Mathematics*, 264, 65-70.  
https://doi.org/10.1016/j.cam.2014.01.002
[5] Abdeljawad, T. (2015) On Conformable Fractional Calculus. *Computational and Applied Mathematics, 279*, 57-66. https://doi.org/10.1016/j.cam.2014.10.016

[6] Ziaei, E. and Farahi, M.H. (2019) The Approximate Solution of Non-Linear Time-Delay Fractional Optimal Control Problems by Embedding Process. *Mathematical and Information, 36*, 713-727. https://doi.org/10.1093/imamci/dnx063

[7] Alharbi, F.M., Baleanu, D. and Ebaid, A. (2019) Physical Properties of the Projectile Motion Using the Conformable Derivative. *Chinese Journal of Physics, 58*, 18-28. https://doi.org/10.1016/j.cjiph.2018.12.010

[8] Senol, M., Tasbozan, O. and Kurt, A. (2019) Numerical Solutions of Fractional Burgers' Type Equations with Conformable Derivative. *Chinese Journal of Physics, 58*, 75-84. https://doi.org/10.1016/j.cjiph.2019.01.001

[9] Li, X., Liu, S. and Jiang, W. (2011) Positive Solutions for Boundary Value Problem of Nonlinear Fractional Functional Differential Equations. *Applied Mathematics and Computation, 217*, 9278-9285. https://doi.org/10.1016/j.amc.2011.04.006

[10] Guo, T. and Jiang, W. (2012) Impulsive Fractional Functional Differential Equations. *Computers and Mathematics with Applications, 64*, 3414-3424. https://doi.org/10.1016/j.camwa.2011.12.054

[11] Zhang, S., Jiang, W. and Huang, Y. (2013) Existence of Positive Periodic Solutions for a Class of Higher-Dimension Functional Differential Equations with Impulses. *Abstract and Applied Analysis, 2013*, Article ID: 396509. https://doi.org/10.1155/2013/396509

[12] Shen, G., Sakthivel, R. and Ren, Y. (2020) Controllability and Stability of Fractional Stochastic Functional Systems Driven by Rosenblatt Process. *Collectanea Mathematica, 71*, 63-82. https://doi.org/10.1007/s13348-019-00248-3

[13] Zhou, Y., Jiao, F. and Li, J. (2009) Existence and Uniqueness for Fractional Neutral Differential Equations with Infinite Delay. *Nonlinear Analysis: Theory, Methods and Applications, 71*, 3249-3256. https://doi.org/10.1016/j.na.2009.01.202

[14] Li, F. and Zhang, J. (2011) Existence of Mild Solutions to Fractional Integrodifferential Equations of Neutral Type with Infinite Delay. *Advances in Difference Equations, 2011*, Article ID: 963463. https://doi.org/10.1155/2011/963463

[15] Ravichandran, C. and Baleanu, D. (2013) Existence Results for Fractional Neutral Functional Integro-Differential Evolution Equations with Infinite Delay in Banach Spaces. *Advances in Difference Equations, 2013*, 1-12. https://doi.org/10.1186/1687-1847-2013-215

[16] Abbas, M.I. (2019) On the Hadamard and Riemann-Liouville Fractional Neutral Functional Integrodifferential Equations with Finite Delay. *Journal of Pseudo-Differential Operators and Applications, 10*, 505-514. https://doi.org/10.1007/s11868-018-0244-4

[17] Granas, A. and Dugundji, J. (2003) Fixed Point Theory. Springer, New York. https://doi.org/10.1007/978-0-387-21593-8