Defining the out-of-limit state of the rocks in the vicinity of the working contour

A I Chanyshev¹-², I M Abdulin¹, O E Belousova¹ and O A Lukyashko¹

¹Chinakal Institute of Mining, Siberian Branch of Russian Academy of Sciences, Krasny Prospekt, 54, 630091, Novosibirsk, Russia
²Novosibirsk State University of Economics and Management, Kamenskaya st., 56, 630099, Novosibirsk, Russia

a.i.chanyshev@gmail.com

Abstract. The problem of defining the stress-strain state of a rock mass near the working contour is solved by the Cauchy stress vector and the displacement vector set on it. To do this, you also need to know the elastic properties and the passport curve of the material "tangent stress – shift" with a section of extreme deformation. The information obtained in solving the problem allows us to judge the condition and remaining reserve of strength of the material both on the contour of the rock mass itself and in its vicinity.

1. Introduction

In the development of minerals by open or mine methods, questions about the state of the rock mass near the outlines of mine workings are relevant. This information is necessary to prevent collapses, rock bursts, sudden emissions of gas and coal. Traditionally, it is obtained from solving boundary value problems of rock mechanics either in the Dirichlet formulation, when a displacement vector is specified on the entire working contour (the 2nd boundary value problem), or in the Neumann formulation, when the Cauchy stress vector (1st boundary value problem) is set, or in the formulation of Robin, when a vector of Cauchy stresses is specified on a part of the boundary, and on the other one - a displacement vector (3rd or mixed problem) [1-3]. When solving these boundary-value classical problems, it is required to know the entire geometry of the computational domain; moreover, it is also required to know the loading conditions of the rock mass at “infinity”. Both are not always achievable. At the same time, obtaining information on the state of the rock mass near the mine working is a priority. In this situation, the essential solution to the problem can be obtained by setting the Cauchy stress vector and simultaneously the displacement vector on the circuit. Let us give a solution to this problem in the case when the rock mass near the working contour is in a state of destruction - at the out-of-limit stage of deformation. If the stress expresses the resistance of the material to deformation, then at this stage, the resistance of the material to deformation decreases with increasing strain, i.e., here comes what is traditionally called destruction. The proposed statement of the problem may seem impossible. But it’s enough to recall the simplest situation: there is a surface of the Earth free of stresses (the Cauchy vector is equal to zero vector); on the other hand, there are 3D measurements of displacements of the Earth’s surface made from satellites (a given vector of displacements of the Earth’s points). A similar picture can be seen in mine conditions - there is a working contour free of stress, there are displacements. So, this statement of the problem is feasible.
2. Experimental data of A. N. Stavrogin and his followers on hard loading of rocks [4]

Figure 1 shows the diagrams of sulfide ore deformation in the form of two dependencies $\sigma_3 = f(\varepsilon_1)$, $\sigma_1 = g(\varepsilon_2)$ where the axes 1, 2, 3 correspond to the axes $z, \varphi, r$ of the cylindrical coordinate system.

The diagrams are obtained with axial and lateral compression of cylindrical samples. The samples were initially subjected to hydrostatic compression, then, at a constant level of lateral pressure, they were brought to destruction by axial compression with a controlled change in the axial displacement (hard loading). The stress and strain tensors in these experiments were as follows:

$$
T_\sigma = \begin{pmatrix}
\sigma_3 + \Delta \sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_2
\end{pmatrix},
T_\varepsilon = \begin{pmatrix}
\varepsilon_3 + \Delta \varepsilon_1 & 0 & 0 \\
0 & \varepsilon_2 + \Delta \varepsilon_2 & 0 \\
0 & 0 & \varepsilon_2 + \Delta \varepsilon_2
\end{pmatrix}.
$$

Figure 1. Diagrams of stress and strain changes for sulfide ore indicating the level of lateral pressure.

If we take the following basis as a tensor basis [5]

$$
T_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
T_2 = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
..., T_6 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
$$

(1)

then the dependencies between the coordinates of the tensors $T_\sigma$, $T_\varepsilon$ will have the form like in Figure 1, i.e., they are not passport, they depend on the level of lateral pressure.

If we take as a basis

$$
T_1 = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
T_2 = \frac{1}{\sqrt{6}} \begin{pmatrix}
-2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
T_3 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
$$

(2)

Figure 2. Diagrams of stress and strain changes for sulfide ore in the basis (1).
then the relations between the coordinates of the tensors $T_\sigma$, $T_\varepsilon$ will look like Figure 2 (here the coordinates of the tensors $T_\sigma$, $T_\varepsilon$, along the third unit vector $S_3 = \Omega = 0$ during the loading of the samples).

It is seen that the basis (2) is again not proper, since the curves in Figure 2 depend on the level of lateral pressure. The next step is to turn the basis (2) around the unit vector $S_3$ by an angle $\varphi$. The new basis is based on the following formulas

$$T_m = T_1 \cos \varphi - T_2 \sin \varphi, \quad T_\varepsilon = T_1 \sin \varphi + T_2 \cos \varphi.$$  \hspace{1cm} (3)

If $\varphi = 40^\circ$, then the initial dependences of Figure 1 in the new basis (3) will take the form of Figure 3. We can see that the behavior of the curves does not depend on the level of lateral pressure. One curve is a straight line (on the left), the other is a "single curve". Linear sections on these curves determine the theory of elasticity of different – modulus materials, sections up to the limits of strength – the theory of plasticity, beyond the limit of strength we have the equations of out-of-limit deformation.

3. Plane strain of out-of-limit deformation

As a simplified version let's consider plane strain. For it, instead of (2), we have a tensor basis of the form

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (4)

Here, 1, 2 are the main axes of the stress and strain tensors (the axes coincide $\varepsilon_3 = 0$). The defining deformation relations for the rotated basis (4) have the form:

in the case of elasticity

$$\Omega_m = \frac{S_m}{2k}, \quad \Omega_i = \frac{S_i}{2\mu},$$  \hspace{1cm} (5)

in the case of out-of-limit deformation
\[
\Omega_m = \frac{S_m}{2k}, \quad \Omega_s = \frac{\varepsilon_i + \varepsilon_s}{\sqrt{2}} \sin \varphi, \quad \varphi = \frac{\varepsilon_i - \varepsilon_s}{\sqrt{2}} \cos \varphi, \quad \Omega_p = \frac{S_i}{2 \mu_s},
\]

where \( \tan \chi = S_i / (\Omega_p - \Omega_s) = 2 \mu_s \) defines the module of the recession \( 2 \mu_s \). [6], \( \varphi = \varphi_s \) - rotation angle of the initial basis (3). Substitution of (5), (6) in the equilibrium equations and in the conditions of compatibility of deformations:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{sx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad \frac{\partial \varepsilon_x}{\partial x} = \frac{\partial \varepsilon_y}{\partial x}, \quad \frac{\partial \varepsilon_y}{\partial y} = \frac{\partial \varepsilon_{xy}}{\partial x} + \frac{\partial \varepsilon_{xy}}{\partial y},
\]

leads the problem of out-of-limit deformation to a characteristic equation of the form

\[
A_4 \eta^4 + A_3 \eta^3 + A_2 \eta^2 + A_1 \eta + A_0 = 0, \quad \eta = dy / dx.
\]

Then we assume that if a system of differential equations has characteristics, they are arranged symmetrically relative to the first main direction for the stress tensor. Substitution \( \eta = \tan(\theta + \beta) \) turns (7) to the biquadrate equation

\[
B_4 \lambda^4 + B_2 \lambda^2 + B_0 = 0,
\]

where

\[
\lambda = \tan \beta, B_4 = a - b \sin 2 \varphi_s, B_2 = -2 \left[ \frac{2T}{T + b \cos 2 \varphi_s} \right], B_0 = a + b \sin 2 \varphi_s, a = \frac{1}{2 \mu_s} - \frac{1}{2k}, b = \frac{1}{2 \mu_s} + \frac{1}{2k}.
\]

Solving (8) we find

\[
\lambda_{1,2,3,4} = \pm \sqrt{2} \sqrt{\frac{2T}{T + b \cos 2 \varphi_s} \pm \sqrt{\frac{a + b \sin 2 \varphi_s}{a - b \sin 2 \varphi_s}}}, \quad \lambda = \sqrt{\frac{2T}{T + b \cos 2 \varphi_s} \pm \sqrt{\frac{a + b \sin 2 \varphi_s}{a - b \sin 2 \varphi_s}}},
\]

and ratios on the characteristics:

\[
\left( \cos 2 \beta + \frac{2BT}{0/(\beta - CT)} \right) d\sigma + dT + \frac{2T}{\sin 2 \beta} \left( \cos 2 \beta + \frac{2BT}{0/(\beta - CT)} \right) d\theta - \frac{T}{0/(\beta - CT) \sin 2 \beta} dw_z = 0,
\]

where

\[
A = \frac{\cos^2 \varphi_s}{2k} - \frac{\sin^2 \varphi_s}{2 \mu_s}, \quad B = \frac{\sin 2 \varphi_s}{4} \left( \frac{1}{\mu_s} + \frac{1}{k} \right), \quad C = \frac{\sin^2 \varphi_s}{2k} - \frac{\cos^2 \varphi_s}{2 \mu_s}.
\]

Thus, we obtain four characteristics of the out-of-limit deformation and four relations on them that connect the maximum tangential stress \( T \), the angle \( \theta \), the average stress \( \sigma \) and the rotation angle \( w_z \). It is shown that the system of equations (9) at each point of the destruction area has the unique solution (its determinant is not zero) provided that both the Cauchy stress vector and the displacement vector are set at the same time as the body boundary.

4. Defining of the stress-strain state in the vicinity of the working contour

We consider a three-dimensional body with a boundary \( S \). Let on this boundary at some point \( M_0 \) the Cauchy stress vector with coordinates \( p_i^0 = \sigma_i n_j \) is set simultaneously with the vector of displacement \( \vec{u}_s \) with coordinates
depending on the coordinates of the surface $z = f(x, y)$. In the formulas (10) $\sigma_{ij}$ are stresses; $n_j$ are the coordinates of the normal vector in the coordinate system $xOyz$. In (10), (11) the set values are $p_i^a$, $p_j^a$, $p_k^a$, $u_x^a$, $u_y^a$, $u_z^a$. By this data it is required to restore at the point $M_0$ the entire stress tensor $T_\sigma = \|\sigma_i\|$, the entire strain tensor $T_\varepsilon = \|\varepsilon_i\|$, the components of the rotation vector $\omega_i$, $\omega_j$, $\omega_k$, a total of 15 values.

We will assume that at this point $M_0$ there is out-of-limit deformation of the material.

The equations of out-of-limit deformation of the material are as follows [7-9]:

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x + \sigma_y + \sigma_z}{K}, \quad K = \frac{E}{(1 - 2\nu)},$$

(12)

where $E$ is Young's modulus, $\nu$ is Poisson's ratio. Denote the difference of tensors $T_\sigma$ and $T_\varepsilon$ as $D_\sigma$, $D_\varepsilon$, then

$$D_\sigma = \sum_{i=2}^{6} S_i T_{i\alpha}, \quad D_\varepsilon = \sum_{i=2}^{6} \Omega_i T_{i\alpha}.$$

(13)

We will assume that deviators $D_\sigma$, $D_\varepsilon$ are parallel [7]:

$$S_2 / \Omega_2 = S_3 / \Omega_3 = ... = S_6 / \Omega_6.$$

(14)

We enter the lengths of the deviators

$$|D_\sigma| = \sqrt{\sum_{i=2}^{6} S_i^2}, \quad |D_\varepsilon| = \sqrt{\sum_{i=2}^{6} \Omega_i^2}.$$

The dependence on the $BC$ section in Figure 4 is considered fair.

Figure 4. Diagrams of changes in the length of the stress deviator versus the length of the strain deviator for elasticity $(OA)$, plasticity $(AB)$, out-of-limit deformation $(BC)$.

We denote

$$\tan \chi = \frac{\tau_p}{\gamma}, \quad \tan \chi = 2\mu, \quad t = \frac{|D_\sigma|}{|D_\varepsilon|} - 2\mu, \quad \tau_p = 2\mu \gamma.$$

(15)

As a result, we obtain the system of equations

$$\begin{align*}
\left[\frac{(K + 2t) \varepsilon_x}{3} + (K - t)(\varepsilon_y + \varepsilon_z) / 3\right] \alpha + t \varepsilon_x \beta + t \varepsilon_y \gamma &= p_x^a, \\
\left[t \varepsilon_y \alpha + \left(\frac{(K + 2t) \varepsilon_y}{3} + (K - t)(\varepsilon_x + \varepsilon_z) / 3\right) \beta + t \varepsilon_z \gamma\right] &= p_y^a, \\
\left[t \varepsilon_z \alpha + t \varepsilon_x \beta + \left(\frac{(K + 2t) \varepsilon_z}{3} + (K - t)(\varepsilon_x + \varepsilon_y) / 3\right) \gamma\right] &= p_z^a,
\end{align*}$$

(16)

where
\[ \alpha = \frac{f'_x}{\sqrt{f''_x^2 + f''_y^2} + 1}, \quad \beta = \frac{f'_y}{\sqrt{f''_x^2 + f''_y^2} + 1}, \quad \gamma = \frac{-1}{\sqrt{f''_x^2 + f''_y^2} + 1}, \quad \vec{n} = \frac{1}{\sqrt{f''_x^2 + f''_y^2} + 1}(f'_x, f'_y, -1). \]

In addition, we have a system of equations

\[
\begin{align*}
(du_x & = [\varepsilon_x + (\varepsilon_{xy} + \omega) f'_x] dx + [\varepsilon_{xy} - \omega_x + (\varepsilon_{xx} + \omega_y) f'_y] dy, \\
(du_y & = [\varepsilon_{yy} + \omega_y + (\varepsilon_{yx} - \omega_x) f'_x] dx + [\varepsilon_{yx} + (\varepsilon_{yy} - \omega_x) f'_y] dy, \\
(du_z & = [\varepsilon_{xz} - \omega_y + \varepsilon_x f'_x] dx + [\varepsilon_{zx} + \omega_x + \varepsilon_x f'_y] dy, \\
\end{align*}
\]

(17)

from which six partial differential equations follow \((z = f(x, y) - \text{surface equation})\).

Eventually, we get a system of 9 equations for determining 9 values: \(\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yx}, \varepsilon_{xx}, \varepsilon_{yy}, \omega_x, \omega_y, \omega_z\). From this system we find the unknown values as functions of the parameter \(t\). Iterations are performed on parameter \(t\) using (15) until \(t\) is stabilized.

Some aspects of this work are considered in [10].

5. Conclusions

1. It is shown that setting the Cauchy stress vector on the surface of the body and the displacement vector is sufficient to find the stress-strain state of the arbitrary working contour both in case of elasticity and in case of out-of-limit deformation.

2. The obtained solution exists, it depends solely and continuously on the input data.

Acknowledgments

The research was carried out within the state assignment of Ministry of Science and Higher Education of the Russian Federation (theme No. AAAA-A17-117122090002-5) and was supported by RFBR grant 18-05-00757.

References

[1] Tikhonov A N and Samarsky A A 1999 The Equations of Mathematical Physics (Moscow: MSU Press) (in Russian)

[2] Vladimirov V S 1981 The Equations of Mathematical Physics (Moscow: Nauka) (in Russian)

[3] Muskhelishvili N I 1977 Some Basic Problems of the Mathematical Theory of Elasticity (Dordrecht: Springer)

[4] Stavrogin A N and Tarasov B G 2001 Experimental Physics and Rock Mechanics (St. Petersburg: Nauka) (in Russian)

[5] Novozhilov V V 1963 J. Appl. Math. Mech. 27 (5) 1219–43

[6] Petukhov I M and Linkov A M 1983 Mechanics of Mountain Impacts and Emissions (Moscow: Nedra) (in Russian)

[7] Ilyushin A A 1963 On the Basics of the General Mathematical Theory of Plasticity (Moscow: Izd-vo AN SSSR) (in Russian)

[8] Zhukov A M 1954 Engineering Review 20 37–41 (in Russian)

[9] Kachanov L M 1956 Fundamentals of the Theory of Plasticity (Moscow: Gostekhizdat) (in Russian)

[10] Chanyshchev A I and Abdulin I M 2019 J. Min. Sci. 55 (4) 538–46