Quantum Discord for Generalized Bloch Sphere States

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Abstract

In this study for particular states of bipartite quantum system in $2^n \times 2^m$ dimensional Hilbert space state, similar to $m$ or $n$-qubit density matrices represented in Bloch sphere we call them generalized Bloch sphere states (GBSS), we give an efficient optimization procedure so that analytic evaluation of quantum discord can be performed. Using this optimization procedure, we find an exact analytical formula for the optimum positive operator valued measure (POVM) that maximize the measure of the classical correlation for these states. The presented optimization procedure also is used to show that for any concave entropy function the same POVMs are sufficient for quantum discord of mentioned states. Furthermore, We show that such optimization procedure can be used to calculate the geometric measure of quantum discord (GMQD) and then an explicit formula for GMQD is given. Finally, a complete geometric view is presented for quantum discord of GBSS.

Keywords: Quantum Discord, Generalized Bloch Sphere States, Dirac $\gamma$ matrices, Bipartite Quantum System.

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1 Introduction

Quantum correlation and its characterization is an important and one of the most popular research topics but is challenging in quantum information theory. Recently, it was found that quantum correlation is a key resource in broadcasting of quantum states \[1, 2\], quantum state merging \[3, 4\], assisted optimal state discrimination \[5\], remote quantum state preparation \[6\], and so on.

Recently, much effort has been devoted to studying various measures of non-classical correlation\[7, 8, 9, 10, 11, 12, 13, 14\]. As a typical kind of quantum correlation, quantum entanglement \[15\] has been well understood in many aspects and widely applied to quantum communication and quantum computation\[16\]. It is quite well established that entanglement is essential for certain kinds of quantum-information tasks like quantum cryptography and super-dense coding. However, besides the widely studied feature of entanglement, quantum theory exhibits also another form of non-classical correlations which is quantified by the quantum discord (QD). In an other word, entanglement only represents a special kind, but not all, of the quantum correlation.

Quantum discord captures the nonclassical correlations, including but not limited to entanglement. The quantum discord as a measure of quantum correlations, initially introduced by Ollivier and Zurek \[17\] and by Henderson and Vedral \[18\], is a measure of the discrepancy between quantum versions of two classically equivalent expressions for mutual information. Many remarkable applications of QD have been proposed such as the characterization of quantum phase transitions \[19\] and the dynamics of quantum systems under decoherence \[20\]. Besides, quantum discord could be used to improve the efficiency of the quantum Carnot engine \[21\] and to better understand the quantum phase transition and the process of Grover search \[22, 23\]. Because evaluation of quantum discord involves optimization procedure, it is difficult to calculate and it was analytically computed only for some special cases \[24, 25, 26, 27, 28\].
To avoid this difficulty and obtain an analytic analysis, Dakić et al. [29] introduced the GMQD which defined as the nearest distance between the given state and the set of zero-discord states. Luo and Fu [30] introduced another equivalent form for GMQD.

In this paper, we devote to investigate the quantum discord for GBSS in a bipartite system. Our results show that despite of functionality, which is chosen in definition of the entropy for the quantum discord, the same POVM satisfies the optimization procedure. Also our analytical calculation of the GMQD, denote that the optimization procedure does not change and is the same as we have for original quantum discord.

The organization of the paper is as follows. First, a brief review of the concept of the quantum discord is given. Then, we explain the quantum discord for GBSS states, leading to the obtaining the optimum POVMs and analytical solution of quantum discord. Next, an explicit formula for GMQD is presented. Finally, we present a geometric viewpoint, from which GBSS quantum discord can be described clearly. The paper ends with a brief conclusion and one appendix.

2 The Concept of The Quantum Discord

Quantum discord [17, 18] is measured by the difference between the mutual information and the maximal conditional mutual information obtained by local measurement. "Right” Quantum discord of a bipartite state $\rho_{AB}$ in a Hilbert space $H^A \otimes H^B$ is given by [17, 18]

$$D_B(\rho^{AB}) = I(\rho^{AB}) - C_B(\rho^{AB}), \quad (2.1)$$

where

$$I(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}). \quad (2.2)$$

Here, $I(\rho^{AB})$ represents the quantum mutual information (the total amount of correlations) of the two-subsystem state $\rho^{AB}$. $\rho^{A(B)} = Tr_{B(A)}(\rho^{AB})$ is the reduced density matrix for the
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subsystem \( A(B) \). \( S(\rho) = -Tr(\rho \log_2 \rho) \) is the von Neumann entropy of the system in the state \( \rho \). The other quantity, \( C_B(\rho^{AB}) \), is interpreted as a measure of the classical correlation of the two subsystems \( AB \) in the state \( \rho^{AB} \) and it is defined as the maximal information that one can obtain, for example, about \( B \) by performing the complete measurement \( E^B_k \) on \( H^B \),

\[
C_B(\rho^{AB}) = \max_{\{E^B_k\}} \left[ S(\rho^A) - \sum_k p_A|k S(\rho_A|k) \right],
\]

(2.3)

where \( \rho_{A|k} = \frac{1}{p_A|k}\text{Tr}_B(I_A \otimes E^B_k \rho^{AB}) \) is the postmeasurement state of \( A \) after obtaining the outcome \( k \) on \( B \) with the probability \( p_A|k = Tr[I_A \otimes E^B_k \rho^{AB}] \); maximum is taken over all the von Neumann measurement sets \( E^B_k \) on system \( B \).

Similarly, "left" quantum discord is given by

\[
D_A(\rho^{AB}) = I(\rho^{AB}) - C_A(\rho^{AB}),
\]

(2.4)

\[
C_A(\rho^{AB}) = \max_{\{E^A_k\}} \left[ S(\rho^B) - \sum_k p_B|k S(\rho_B|k) \right].
\]

(2.5)

where \( \rho_{B|k} = \frac{1}{p_B|k}\text{Tr}_A(E^A_k \otimes I_B \rho^{AB}) \) is the postmeasurement state of \( B \) after obtaining the outcome \( k \) on \( A \) with the probability \( p_B|k = Tr[E^A_k \otimes I_B \rho^{AB}] \). Note that the difference between those two discords is that the measurement is performed on party \( A \) or on party \( B \), respectively. It is thus expected that this definition of quantum discord is not symmetric with respect to \( A \) and \( B \).

A set of operators \( \{E^A_k(\rho^{AB})\} \) is named POVM if and only if the following two conditions are met: (1) each operator \( E^A_k(\rho^{AB}) \) is positive positive \( \Leftrightarrow \langle \psi | E^A_k(\rho^{AB}) | \psi \rangle > 0, \forall |\psi\rangle \) and (2) the completeness relation is satisfied, i.e.,

\[
\sum_k E^A_k(\rho^{AB}) = 1.
\]

(2.6)
The elements of \( \{ E_k^{A(B)} \} \) are called effects or POVM elements. On its own, a given POVM \( \{ E_k^{A(B)} \} \) is enough to give complete knowledge about the probabilities of all possible outcomes; measurement statistics is the only item of interest.

### 3 Analytical solution for GBSS

Let \( \gamma_\mu, \mu = 1, \ldots, d \), be \( d \) Dirac \( \gamma \) matrices satisfying the anticommuting relations:

\[
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} I. \quad \mu, \nu = 1, \ldots, d.
\]

(3.7)

It follows from relation (3.7) that the \( \gamma \) matrices \( \gamma_\mu \) generate an algebra which, as a vector space, has a dimension \( 2^d \) (For a brief review about Dirac matrices and an explicit construction of \( \gamma \), we refer the reader to [31] or see the Appendix). We consider hermitian matrices \( \lambda_i, i = 1, 2, \ldots, 2^d \) as all possible multiplications of \( \gamma_\mu, \mu = 1, \ldots, d \) up to multiplicative factors \( 1, \pm i \).

Let a basis for the Lie algebra of \( SU(N) \) be given by \( \{ \lambda_i \}_{i=1}^{N^2} \). We will use the following normalization condition for the elements of the Lie algebra of \( SU(N) \)

\[
Tr(\lambda_i \lambda_j) = 2\delta_{ij}.
\]

(3.8)

We will also choose the following relations for commutation and anticommutation relations:

\[
[\lambda_i, \lambda_j] = 2if_{ijk} \lambda_k,
\]

\[
\{\lambda_i, \lambda_j\} = \frac{4}{N} \delta_{ij} I + 2d_{ijk} \lambda_k,
\]

(3.9)

where the \( f_{ijk} \) are the structure constants and the \( d_{ijk} \) are the components of the totally symmetric \( d \)-tensor. These two equations may be combined more succinctly as

\[
\lambda_i \lambda_j = \frac{2}{N} \delta_{ij} I + if_{ijk} \lambda_k + d_{ijk} \lambda_k.
\]

(3.10)

Using these conventions, we may express the POVM elements as

\[
E^{B}_{e^k} = \frac{1}{M} (I_{M \times M} + \sqrt{\frac{M(M-1)}{2}} e^k \lambda^B) = \frac{1}{M} (I_{M \times M} + \sqrt{\frac{M(M-1)}{2}} \Sigma_{i=1}^{M^2-1} e_i^k \lambda^B_i),
\]

(3.11)
\[ E^{A}_{e^k} = \frac{1}{N} (I_{N \times N} + \sqrt{\frac{N(N-1)}{2}} e^k \lambda^A) = \frac{1}{N} (I_{N \times N} + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{N^2-1} e_i^k \lambda^A_i). \] (3.11)

This representation is called a coherence vector representation with \( e^k \) the coherence vector. The constant is a convenient one such that for pure states
\[ e^k \cdot e^k = 1, \quad e^k \star e^k = e^k, \] (3.12)
where the star product is defined by
\[ (a \star b)_l = \sqrt{\frac{N(N-1)}{2}} \frac{1}{N-2} d_{ijl} a_i b_j. \] (3.13)

We consider particular states acting on a bipartite system \( H^A \otimes H^B \) with \( \dim(H^A) = N = 2^n \) and \( \dim(H^B) = M = 2^m \) which possess properties similar to \( m \) or \( n \)-qubit density matrices represented in Bloch sphere, and so we call them generalized Bloch sphere states.

In the case of even dimension \( d \), we denote \( \gamma_s = i^{-\frac{d}{2}} \gamma_1 \gamma_2 \ldots \gamma_d \) by \( \gamma_{d+1} \), then the matrices \( \vec{w} = \{ \gamma_1, \gamma_2, \ldots, \gamma_d, \gamma_{d+1} \} = \{ \gamma_1, \gamma_2, \ldots, \gamma_{2n}, \gamma_{2n+1} = i^{-n} \gamma_1 \gamma_2 \ldots \gamma_{2n} \} \) form a maximally anticommuting set in the algebra of \( \gamma \) matrices (in the case of odd \( d \), the set of matrices \( \gamma_i, i = 1, \ldots, d \) is maximally anticommuting set). By choosing maximally anticommuting sets \( \vec{w} \) the decomposition of density matrices of \( A \) into a Bloch vector has, in general, the following form:
\[ \rho = \frac{1}{N} (I + \vec{c} \vec{w}) = \frac{1}{N} (I + \sum_{j=1}^{2n+1} c_j \gamma_j), \] (3.14)
That is, \( \gamma_j \) are maximally anticommuting set which satisfy
\[ \{ \gamma_i, \gamma_j \} = 2\delta_{ij} I, \] (3.15)
where \( I \) stands for the identity operator and \( \gamma_j \) for \( j = 1, 2, \ldots, 2n+1 \), known as Dirac matrices, are generators of special orthogonal group \( SO(2n+1) \), and represented as traceless Hermitian matrices in a \( 2n \)-dimensional Hilbert space.

Let \( N < M \), thus we can represent the density operators acting on a bipartite system \( H^A \times H^B \) as:
\[
\rho^{AB} = \frac{1}{NM}(I_N \otimes I_M + \sum_{i,j=1}^{2n+1} t_{ij} \gamma_i^A \otimes \gamma_j^B),
\]
(3.16)

where, \( T = [t_{ij}] \) is the correlation matrix. Since quantum correlations are invariant under local unitary transformation, i.e. under transformations of the form \((U_1 \otimes U_2)\rho^{AB}(U_1^\dagger \otimes U_2^\dagger)\) with \( U_1, U_2 \in SU(N) \), we can, without loss of generality, restrict our considerations to some representative class such that \( T \) is diagonal, namely \( T = \text{diag}\{t_1, t_2, ..., t_{2n+1}\} \) [32]. Concerning this representative class, density operators acting on a bipartite system \( H^A \times H^B \) can be written as:

\[
\rho^{AB} = \frac{1}{NM}(I_N \otimes I_M + \sum_{j=1}^{2n+1} t_j \gamma_j^A \otimes \gamma_j^B),
\]
(3.17)

with eigenvalues

\[
\lambda_{i_1, i_2, ..., i_{2n}} = \frac{1}{NM}[1 + (-1)^{i_1}t_1 + (-1)^{i_2}t_2 + ... + (-1)^{i_{2n}}t_{2n} + (-1)^n(-1)^{i_1+i_2+...+i_{2n}}t_{2n+1}],
\]
(3.18)

where \( i_1, i_2, ..., i_{2n} \in \{0, 1\} \). Then the postmeasurement state of \( A(B) \) after obtaining the outcome \( k \) on \( B(A) \), is given by

\[
\rho_k^A = \frac{1}{p_{A|k}}Tr_B(I_A \otimes E_{ek}^B \rho^{AB}) = \frac{1}{N}(I_{N \times N} + \sqrt{\frac{2(M-1)}{M}} \sum_j t_j e_j^k \lambda_j^A)
\]

\[
\rho_k^B = \frac{1}{p_{B|k}}Tr_A(E_{ek}^A \otimes I_B \rho^{AB}) = \frac{1}{M}(I_{M \times M} + \sqrt{\frac{2(N-1)}{N}} \sum_j t_j e_j^k \lambda_j^B).
\]
(3.19)

In the rest of this section we evaluate analytically the quantum discord for any concave quantum entropy functions. To begin, consider the following quantum entropy functions [33]

\[
S(\rho) = -Tr(\rho \log_2 \rho),
\]
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\[ S_R(\rho) = \frac{1}{1-q} \log \text{Tr} \rho^q, \ (0 \leq q \leq 1), \]
\[ S_T(\rho) = \frac{1}{1-q} [\text{Tr} \rho^q - 1], \ (0 < q), \] (3.20)

respectively the von Neumann, Renyi and Tsallis, where \( S_R(\rho) \) and \( S_T(\rho) \) are equal to von Neumann entropy in the limit \( q = 1 \).

Let us introduce

\[ \mu_k = \sqrt{\frac{2(N-1)}{N} \sum (t_j e^k_j)^2}. \] (3.21)

Assume, with no loss of generality, the "left" quantum discord (for the GBSS quantum discord being symmetric between A and B if \( N = M \)). Then using the Eqs. (2.5) and (3.20) the measure of the classical correlation of the two subsystems \( AB \) in the state \( \rho^{AB} \) for above quantum entropy functions is given by

\[ C_A(\rho^{AB}) = \max_{\{E_{Ak}^A\}} \left[ \log M + \frac{1}{N} \sum_k \left[ \frac{1-\mu_k}{2} \log \left( \frac{1-\mu_k}{M} \right) + \frac{1+\mu_k}{2} \log \left( \frac{1+\mu_k}{M} \right) \right] \right], \] (3.22)

\[ C_R(\rho^{AB}) = \max_{\{E_{Ak}^A\}} \left[ \frac{1}{1-q} \log (M^{-(q-1)}) - \frac{1}{M(1-q)} \sum_k \log \frac{M^{-(q-1)}}{2} [(1+\mu_k)^q + (1-\mu_k)^q] \right], \]
\[ C_T(\rho^{AB}) = \max_{\{E_{Ak}^A\}} \left[ \frac{M^{-(q-1)}}{1-q} - \frac{M^{-q}}{2(1-q)} \sum_k [(1+\mu_k)^q + (1-\mu_k)^q] \right]. \] (3.23)

The above equations represents that the maximum value of the measure of the classical correlation for any kind of entropy only depends on \( \mu_k \). Thus we turn our attention to the finding maximum value of the \( \mu_k \). Hence, the problem of finding the maximum value of the measure of the classical correlation is reduced to the problem

\[ \text{maximize } \mu_k = \sqrt{\frac{2(N-1)}{N} \sum (t_j e^k_j)^2}, \]
\[ \text{subject to } e^k e^* = 1, e^k * e^k = e^k. \] (3.24)
Optimization procedure:

Here we present an analytical procedure for optimization of the measure of the classical correlation for GBSS. This procedure also allows us to obtain the optimum POVMs.

Suppose $t_{\text{max}} := \max\{|t_1|, |t_2|, ..., |t_{2n+1}|\}$ and its corresponding coefficient in Eq.(3.19) is $e^k_i \lambda^B_i$. Let the set $\{\lambda^B_j\}$ anticommute with $\lambda_i$ and their related coefficients are $\{e^k_j\}$. Choosing these coefficients and making them zero yield a set of optimum POVMs for quantum discord provided that other coefficients chosen such that, the coefficient $e^k_i$ is maximized. In this case, one can show that the maximum value of $\mu_k$ occurs when the all nonzero coefficients are

$$e^k_j = \pm \sqrt{\frac{1}{N-1}}.$$  \hspace{1cm} (3.25)

To show this, Let a basis for the Lie algebra of SU(N) be given by

$$\{\lambda_j\}_{j=0}^{N^2-1} = \{\lambda_{\alpha_1,\alpha_2,...,\alpha_n}\} = \{\sqrt{\frac{2}{N}} \sigma^1_{\alpha_1} \otimes \sigma^2_{\alpha_2} \otimes \cdots \otimes \sigma^n_{\alpha_n}\},$$ \hspace{1cm} (3.26)

where $\alpha_1, \alpha_2, ..., \alpha_n \in \{0, 1, 2, 3\}$. For the sake of convenience of analysis, we denote

$$t_j \equiv t_{\alpha_1,\alpha_2,...,\alpha_n},$$

$$e^k_j \equiv e^k_{\alpha_1,\alpha_2,...,\alpha_n},$$ \hspace{1cm} (3.27)

so Eq.(3.24) can be written as

$$\text{maximize} \quad \mu_k = \sqrt{\frac{2(N-1)}{N}} (\sum t_{\alpha_1,\alpha_2,...,\alpha_n} e^k_{\alpha_1,\alpha_2,...,\alpha_n})^2$$

subject to $e^k \cdot e^k = 1, e^k \ast e^k = e^k.$ \hspace{1cm} (3.28)

Using Eq.(3.9) we have

$$d_{ijk} = 2Tr(\lambda_i \lambda_j \lambda_k),$$ \hspace{1cm} (3.29)
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or

\[ d(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_n) = 2Tr(\lambda_{\alpha_1, \alpha_2, \ldots, \alpha_n} \lambda_{\beta_1, \beta_2, \ldots, \beta_n} \lambda_{\gamma_1, \gamma_2, \ldots, \gamma_n}), \tag{3.30} \]

where \( \alpha_i, \beta_i \) and \( \gamma_i \in \{0, 1, 2, 3\} \).

Considering the optimization procedure and using Eqs. (3.28) and (3.30) one can show that the maximum value of \( \mu_k \) achieve when Eq. (3.25) is satisfied.

Thus, using Eqs. (3.25) and (3.28) the maximum value of the \( \mu_k \) is given by

\[ \mu_{k,max} = \sqrt{\frac{2}{N} t_{max}}. \tag{3.31} \]

Hence, the measure of the classical correlation due to the von Neumann version entropy is given by

\[ C(\rho^{AB}) = 1 - \frac{\mu_{k,max}}{2} \log(1 + \mu_{k,max}) + \frac{1 + \mu_{k,max}}{2} \log(1 + \mu_{k,max}) \tag{3.32} \]

Finally, from Eqs. (2.2), (2.4) and (3.31), we obtain the quantum discord such as:

\[ D(\rho^{AB}) = \sum_{j=0}^{N^2-1} \lambda_j \log\lambda_j - \frac{1 - \mu_{k,max}}{2} \log(1 - \mu_{k,max}) - \frac{1 + \mu_{k,max}}{2} \log(1 + \mu_{k,max}) + \log(NM) \tag{3.33} \]

This is in agreement with the result obtained by Luo in [35] for \( N = M = 2 \). M.D. Lang and C.M. Caves [36] showed that for the Bell-diagonal states for two qubits discord is zero for classical states, which lie on the Cartesian axes and origin. While Eq. (3.33) shows that for the GBSS, discord is zero only when \( t_1 = t_2 = \ldots = t_{2n+1} = 0 \) (\( \rho^{AB} = \frac{1}{NM} I \)), which lies on the origin.

For the Renyi and Tsallis entropy we get

\[ C_R(\rho^{AB}) = \frac{1}{1-q} \log(M^{-(q-1)}) - \frac{1}{1-q} \log\left(\frac{M^{-(q-1)}}{2}[(1 + \mu_{k,max})^q + (1 - \mu_{k,max})^q] \right) \]
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\[ C_T(\rho^{AB}) = \frac{M^{-(q-1)}}{1-q} \frac{M^{-(q-1)}}{2(1-q)}[(1 + \mu_{k,max})^q + (1 - \mu_{k,max})^q], \]  

(3.34)

and using Eqs. (2.2), (2.4) and (3.31), we obtain the quantum discord such as:

\[ D_R(\rho^{AB}) = \frac{1}{1-q} \log(M^{-(q-1)}) - \frac{1}{1-q} \log Tr(\rho^{AB})^q + \frac{1}{1-q} \log \frac{M^{-(q-1)}}{2}[(1 + \mu_{k,max})^q + (1 - \mu_{k,max})^q], \]

\[ D_T(\rho^{AB}) = \frac{1}{q-1} - \frac{1}{q-1} [Tr(\rho^{AB})^q] + \frac{M^{-(q-1)}}{2(1-q)}[(1 + \mu_{k,max})^q - (1 - \mu_{k,max})^q]. \]

(3.35)

Now, we give an exact analytical formula for the POVM. To do this, without loss of generality, we assume that \( \alpha_1, \alpha_2, ..., \alpha_n \in \{0, 1\} \) and \( \beta_1, \beta_2, ..., \beta_n \in \{0, 1\} \). Then using Eq.(3.11) the POVM elements are given by

\[ E_{e^k}^\alpha \equiv E_{\alpha_1,\alpha_2,..,\alpha_n} = \frac{1}{N}(I - \sqrt{N/2} \sum_{\beta_1,\beta_2,..,\beta_n} (-1)^{\sum_{i=1}^{n}(\alpha_i\beta_i)} \lambda_{\beta_1,\beta_2,..,\beta_n}) \]

\[ = \frac{1}{N}(I - \sum_{\beta_1,\beta_2,..,\beta_n} (-1)^{\sum_{i=1}^{n}(\alpha_i\beta_i)} \sigma_{\beta_1}^1 \otimes \sigma_{\beta_2}^2 \otimes ... \otimes \sigma_{\beta_n}^n) \]

\[ = P_{1}^\pm \otimes P_{2}^\pm \otimes ... \otimes P_{n}^\pm, \]

(3.36)

where \( P_k^\pm = \frac{1}{2}(I \pm n^k.\sigma) \).

4 Geometric Measure of Quantum Discord

The geometric measure of quantum discord \( [29] \) is given by:

\[ D = \min_\chi ||\rho - \chi||^2 \]

(4.37)

where the minimum is over the set of zero-discord states [i.e., \( D(\chi) = 0 \)] and the geometric quantity \( ||\rho - \chi||^2 = Tr(\rho - \chi)^2 \) is the square of Hilbert-Schmidt norm of Hermitian operators.
\( \chi \) can be represented as:

\[
\chi = \sum_k p_k |k\rangle \langle k | \otimes \rho_k, \tag{4.38}
\]

where \( p_k \) is a probability distribution, \( \{ |k\rangle \} \) is an arbitrary orthonormal basis for \( H^A \) and \( \rho_k \) is a set of arbitrary states (density operators) on \( H^B \).

Consider a bipartite system \( H^A \otimes H^B \) with \( \text{dim} H^A = N \) and \( \text{dim} H^B = M \). Let \( L(H^A) \) be the space consisting of all linear operators on \( H^A \). This is a Hilbert space with the Hilbert-Schmidt inner product

\[
\langle X | Y \rangle = Tr(X^\dagger Y)
\]

The Hilbert spaces \( L(H^B) \) and \( L(H^A \otimes H^B) \) are defined similarly. Let \( \{ X_i : i = 1, 2, ..., N^2 \} \) and \( \{ Y_j : j = 1, 2, ..., M^2 \} \) be sets of Hermitian operators which constitute orthonormal bases for \( L(H^A) \) and \( L(H^B) \) respectively. Then

\[
Tr(X_i X_{i'}) = \delta_{ii'}, \quad Tr(Y_j Y_{j'}) = \delta_{jj'}.
\]

\( \{ X_i \otimes Y_j \} \) constitutes an orthonormal (product) basis for \( L(H^A \otimes H^B) \) (linear operators on \( H^A \otimes H^B \)). In particular, any bipartite state \( \rho \) on \( H^A \otimes H^B \) can be expanded as

\[
\rho = \sum_{ij} c_{ij} X_i \otimes Y_j, \tag{4.39}
\]

with \( c_{ij} = Tr(\rho X_i \otimes Y_j) \).

S. Luo and S. Fu introduced the following form of geometric measure of quantum discord \[30\]

\[
D(\rho) = Tr(CC^t) - \max_A Tr(ACC^t A^t), \tag{4.40}
\]

where \( C = [c_{ij}] \) is an \( N^2 \times M^2 \) matrix and the maximum is taken over all \( N \times N^2 \)-dimensional isometric matrices \( A = [a_{ki}] \) such that

\[
a_{ki} = tr(|k\rangle \langle k | X_i) = \langle k | X_i | k \rangle,
\]
where \(\{|k\rangle\}_{k=1}^{N}\) is any orthonormal base for \(H^A\). We can expand the operator \(|k\rangle\langle k|\) in the basis of \(\{X_i\}\) as:

\[
|k\rangle\langle k| = \sum_i a_{ki}X_i, \quad k = 1, 2, ..., N.
\]

A general bipartite state \(\rho\) on \(H^A \otimes H^B\) can be written in this basis as

\[
\rho = \frac{1}{MN} [I_N \otimes I_M + \bar{x}X^A \otimes I_M + I_N \otimes \bar{y}Y^B + \sum_{i=1}^{N^2-1} \sum_{j=1}^{M^2-1} t_{ij}X_i^A \otimes Y_j^B]
\]

(4.41)

where \(\bar{x} \in \mathbb{R}^{N^2-1}\) and \(\bar{y} \in \mathbb{R}^{M^2-1}\) are the coherence vectors of the subsystems A and B. These are given by

\[
x_i = \frac{N}{2} Tr(\rho \lambda_i \otimes I_M), \quad y_j = \frac{M}{2} Tr(\rho I_N \otimes \lambda_j).
\]

The correlation matrix \(T = [t_{ij}]\) is given by

\[
T = [t_{ij}] = \frac{MN}{4} Tr(\rho \lambda_i \otimes \lambda_j).
\]

Based on above definitions, Rana et al. and Hassan et al. have shown that the geometric discord of \(\rho\) is lower bounded. By choosing the orthonormal bases \(\{X_i\}\) and \(\{Y_j\}\) in Eq.(4.39) as the generators of \(SU(N)\) and \(SU(M)\) respectively, that is,

\[
X_1 = \frac{1}{\sqrt{N}} I_N, \quad Y_1 = \frac{1}{\sqrt{M}} I_M,
\]

\[
X_i = \frac{1}{\sqrt{2}} \lambda_{i-1}, \quad i = 2, 3, ..., N^2; \quad Y_j = \frac{1}{\sqrt{2}} \lambda_{j-1}, \quad i = 2, 3, ..., M^2,
\]

the author have shown that

\[
Tr(CC^t) = \frac{1}{NM} + \frac{2}{NM^2} ||\bar{y}||^2 + \frac{2}{NM^2} ||\bar{x}||^2 + \frac{4}{M^2 N^2} ||\vec{T}||^2,
\]

\[
Tr(ACC^tA^t) = \frac{1}{N} \left\{ \frac{1}{M} + \frac{2}{M^2} ||\bar{y}||^2 + \frac{2(N-1)}{N^2 M} \left[ \sum_{j=1}^{N-1} e^{iG(e^j)^t} + \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} e^{iG(e^j)^t} \right] \right\},
\]

(4.42)
where $G = (\vec{x}x^T + \frac{2TT'}{M})$ and

$$e^k = \sqrt{\frac{N}{N-1}}(a_{k2}, a_{k3}, \ldots, a_{kN^2}), \quad k = 1, 2, \ldots, N-1, \quad e^N = -\sum_{k=1}^{N-1} e^k.$$  

Suppose, for a given pure state $|i\rangle\langle i|$ and an unitary operator $U$ acting on $H^A$, there exists an orthogonal operator $O = [O_{\alpha\beta}]$ acting on $\mathbb{R}^{N^2-1}$ such that

$$e^j = (n^j)^tO,$$

where $n^j$ is the coherence (column) vector of the state $|i\rangle\langle i|$ and $e^j$ is the coherence (row) vector of the state $U|i\rangle\langle i|U^\dagger$. By writing $G = \sum_{q=1}^{N^2-1} \eta_q |\hat{f}_q\rangle\langle \hat{f}_q|$, with its eigenvalues arranged in nondecreasing order the author have shown that

$$Tr(ACC^tA^t) = \frac{1}{N^2} \frac{1}{M} + \frac{2}{M^2} \|\vec{y}\|^2 + \frac{2}{NM} \sum_{q=1}^{N^2-1} \eta_q \left[\sum_{l=1}^{N-1} |\langle l|O|\hat{f}_q\rangle|^2\right], \quad (4.43)$$

where $\{1\}$ is the standard (computational) basis states in $H^A$. The desired maximum is obtained by choosing $O$ in in Eq. (4.43) to be that orthogonal matrix which takes the eigenbasis of $G$ matrix to the standard basis in $\mathbb{R}^{N^2-1}$. In general, if we consider all of the generators of $SU(N)$ in Eq. (4.41) the unitary operator $U$ corresponding to the orthogonal matrix $O$ does not exist. Hence, there exists no an explicit formula for GMQD in general, but here we show that in the case of the maximally anticommuting set of $\lambda_j$ there exists an explicit formula for GMQD. Consider the maximally anticommuting set of $\lambda_j$, that is (for $N < M$)

$$\rho^{AB} = \frac{1}{NM}(I_N \otimes I_M + \sum_{i=1}^{2n+1} x_i \lambda_i^A \otimes I_M + I_N \otimes \sum_{i=1}^{2n+1} y_i \lambda_i^B + \sum_{i,j=1}^{2n+1} t_{ij} \gamma_i^A \otimes \gamma_j^B). \quad (4.44)$$

In this case, the unitary operator $U$ exists and it is spinor representation of special orthogonal group $SO(2n + 1)$. Here, the spinor representations of the group $SO(2n + 1)$ are given by $U = e^{\sum_{i<j} \theta_{ij} \lambda_i \lambda_j}$ and we can write the transformation of $\gamma_i$ more explicitly as

$$U \gamma_i U^\dagger = \sum_{j=1}^{2n+1} [SO(2n + 1)]_{ij} \gamma_j.$$
Let $a = \sqrt{\frac{N(N-1)}{2}}$ and

$$E_{e^k}^A = \frac{1}{N}(I_{N\times N} + ae^kA^t) = \frac{1}{N}[W_0 + a(W_1 + W_2 + \ldots + W_{N-1})], \quad (4.45)$$

where

$$W_0 = I_{N\times N}, \quad W_1 = e^{\eta\cdot\vec{w}} = \sum_{i=1}^{2n+1} e_i^{\eta}\gamma_i, \quad W_2 = \sum_{i_1, i_2} e_{i_1i_2}^{k}\gamma_{i_1}\gamma_{i_2} \quad \text{and}$$

$$W_{N-1} = \sum_{i_1 < i_2 < \ldots < i_{N-1}} e_{i_1i_2\ldots i_{N-1}}^{k}\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_{N-1}}. \quad (4.46)$$

The eigenvectors of $G$ matrix have projection only in the subspace $W_1$. Then, to get the maximum value of $Tr(ACC^tA^t)$, we can choose the orthogonal group $SO(2n+1)$ such that it takes the Bloch vector components $e^k$ in the subspace of maximally anticommuting set $W_1$ to the eigenvector of $G$ matrix with the largest eigenvalue. That is, the orthogonal group $SO(2n+1)$ takes the vector $e^k$ to the eigenvector of $G$ matrix with the largest eigenvalue and the transformation of the Bloch vector components $e^k$ are given by

$$e_{i_1}^{k} \rightarrow \sum_{j_1} [SO(2n+1)]_{i_1j_1} e_{j_1}^{k}, \quad e_{i_1i_2}^{k} \rightarrow \sum_{j_1, j_2} [SO(2n+1)]_{i_1j_1} [SO(2n+1)]_{i_2j_2} e_{j_1j_2}^{k},$$

$$e_{i_1i_2\ldots i_{N-1}}^{k} \rightarrow \sum_{j_1, \ldots, j_{N-1}} [SO(2n+1)]_{i_1j_1} \ldots [SO(2n+1)]_{i_{N-1}j_{N-1}} e_{j_1j_2\ldots j_{N-1}}^{k}.$$

Then using the optimization procedure of section \[\text{3}\] one can show that

$$\max Tr(ACC^tA^t) = \frac{1}{N}\left\{\frac{1}{M} + \frac{2}{M^2} ||\vec{y}||^2 + \frac{2}{NM} \eta_{max}\right\}, \quad (4.47)$$

where $\eta_{max}$ is the largest eigenvalue of the matrix $G$. Using Eqs (4.40,4.42,4.47) we get

$$D(\rho) = \frac{2}{N^2M} [||\vec{x}||^2 + \frac{2}{M} ||\vec{T}||^2 - \frac{\eta_{max}}{N-1}], \quad (4.48)$$

This is in agreement with the lower bounded obtained in [38] for the geometric discord of the bipartite system.
Here, for example, we consider the representation of Gamma matrices in \( N = 4 \) dimensions. There are different representations for the Gamma matrices, depending on the basis in which they are written. In the Weyl representation (or chiral representation), for any \( k \), we have \([4.1]\)

\[
E_k^A = \frac{1}{N} \left[ I \pm \sum_{i=1}^{5} e_i^k \gamma_i \pm e_6^k (\sigma_z \otimes \sigma_x) \pm i e_7^k (\sigma_z \otimes \sigma_y) \pm i e_8^k (\sigma_z \otimes \sigma_z) \pm e_9^k (I \otimes \sigma_z) \pm e_{10}^k (I \otimes \sigma_y) \pm e_{11}^k (I \otimes \sigma_x) \pm e_{12}^k (\sigma_y \otimes I) \pm e_{13}^k (\sigma_x \otimes I) \pm e_{14}^k (\sigma_y \otimes \sigma_y) \pm e_{15}^k (\sigma_y \otimes \sigma_z) \right],
\]

where

\[
\begin{align*}
\gamma_1 &= \sigma_x \otimes I, \quad \gamma_2 = i\sigma_y \otimes \sigma_x, \quad \gamma_3 = i\sigma_y \otimes \sigma_y, \quad \gamma_4 = i\sigma_y \otimes \sigma_z, \quad \gamma_5 = i\gamma_1 \gamma_2 \gamma_3 \gamma_4 = -\sigma_z \otimes I.
\end{align*}
\]

Suppose \( e_5^k \neq 0 \), then using the optimization procedure of section [3] we have

\[
e^k = (0, 0, 0, 0, \pm \frac{1}{\sqrt{3}}, 0, 0, \pm \frac{1}{\sqrt{3}}, 0, 0, 0, 0, 0, 0),
\]

then, we choose the the orthogonal matrix \( O \) such that it takes the eigenvector of \( G \) matrix with the largest eigenvalue to the Bloch vector \((0, 0, 0, 0, \pm \frac{1}{\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\).

5 Geometric interpretation

Here, we consider \( \vec{x} = \vec{y} = 0 \) and restrict our considerations to some representative class such that \( T \) is diagonal. In this case, using Eq.(4.48) we get

\[
D(\rho) = \frac{4}{N^2 M^2} \left[ ||\vec{T}||^2 - \frac{t_{max}^2}{(N - 1)} \right],
\]

where \( ||\vec{T}||^2 = t_1^2 + t_2^2 + \ldots + t_{2n+1}^2 \). Now, a complete geometric view is presented for quantum discord and GMQD of GBSS. To do this we determine the feasible region of GBSS. Then we investigate the level surfaces of GMQD.
5.1 Level surfaces of quantum discord

M.D. Lang and C.M. Caves [36] considered the level surfaces of quantum discord for Bell-diagonal states. They showed that the set of Bell-diagonal states for two qubits can be depicted as a tetrahedron in three dimensions.

Here, we point out that the geometric interpretation holds also for generalized Bloch sphere states. A GBSS are specified by the correlation coefficients \( \{t_1, ..., t_{2n+1}\} \). The positivity of \( \rho^{AB} \) implies that

\[
\lambda_{i_1,i_2,...,i_{2n}} = \frac{1}{NM} \left[ 1 + (-1)^{i_1}t_1 + (-1)^{i_2}t_2 + \ldots + (-1)^{i_{2n}}t_{2n} + (-1)^{n}(-1)^{i_1+i_2+\ldots+i_{2n}}t_{2n+1} \right] \geq 0.
\]

(5.52)

The above equation gives \( N \) inequalities and these inequalities determine the region of GBSS. Moreover, by imposing the positivity of partial transposition (PPT) of \( \rho^{AB} \), we obtain \( N \) other inequalities. Partial transposition changes the sign of \( t_{2n+1} \). The region defined by intersection of these \( 2N \) halfspaces is a convex polytope which is the region of separable GBSS. In fact, the set of separable GBSS are specified by \( |t_1| + |t_2| + \ldots + |t_{2n+1}| \leq 1 \). On the other hand, the intersection of halfspaces form a convex polytope, where the intersection of its complement and the region of PPT density matrices, is the region of detectable PPT entangled states[41].

5.2 Level surfaces of geometric quantum discord

In order to quantify level surfaces of quantum discord for GBSS, without loss of generality, we assume that

\[
|t_1| > |t_2| > \ldots > |t_{2n+1}|,
\]

(5.53)

that is \( t_{\text{max}} = t_1 \), then using Eq. (5.53) we have
Quantum Discord

\[ D = \frac{4}{N^2 M^2} \left[ \sum_{i \neq 1} t_i^2 + t_1^2 \left( 1 - \frac{1}{N - 1} \right) \right]. \]  \hspace{1cm} (5.54)

In the above equation the \( D = \text{constant} \) represents an ellipsoidal region in \( 2n + 1 \)-dimensional space and Eq. (5.53) describes a class of planes.

Physical GBSS states belong to the intersection of these planes and ellipsoid.

In fact the level surfaces of geometric discord of GBSS are composed of \( 2n + 1 \) identical intersecting ellipsoids. For the set of Bell-diagonal states for two qubits the level surfaces of geometric discord are composed of three identical intersecting cylinders instead of ellipsoids [42].

When the \( D \) decreases ellipsoids shrink towards the origin and when the \( D \) increases the ellipsoids increase outward from the origin such that in the region of GBSS the maximum value of \( D \) occurs when \( |t_1| = \ldots = |t_{2n+1}| = 1 \). So we have

\[ D_{\text{max}} = \frac{4}{N^2 M^2} \left( 2n + 1 - \frac{1}{N - 1} \right). \]  \hspace{1cm} (5.55)

The separable states is actually bounded by \( |t_1| + |t_2| + \ldots + |t_{2n+1}| \leq 1 \). Since in the region of separable GBSS \( ||T||^2 \leq 1 \), then the maximum value of the \( D \) occurs, when \( ||T||^2 = 1 \) and \( t_{\text{max}} \) is as small as possible. This leads

\[ |t_1| = |t_2| = \ldots = |t_{2n+1}| = \frac{1}{\sqrt{2n + 1}}, \]

and the maximum value of the geometric measure of quantum discord for GBSS is given by

\[ D_{\text{max}} = \frac{4}{N^2 M^2} \left( 1 - \frac{1}{(2n + 1)(N - 1)} \right). \]  \hspace{1cm} (5.56)

This is in agreement with the result obtained in [29] for the set of Bell-diagonal states for two qubits.
6 Conclusion

In summary, we have developed a complete and intuitive analytic picture of the quantum discord problem for GBSS that lends itself to a straightforward analytic algorithm to finding the optimal POVM. In addition, we have shown that the result does not depend on the entropy function. That is the same POVM is optimum for measuring quantum discord of GBSS for any concave entropy function in the definition of quantum discord. Also, we have pointed out that the presented procedure of optimization is also used to find the geometric measure of quantum discord and we provide an analytical expression for the geometric quantum discord. Finally, we have developed the geometric interpretation of original quantum discord and geometric measure of quantum discord for GBSS.
Appendix:

Throughout the paper, we have used the formalism of Dirac $\gamma$ matrices. Therefore, in this appendix we define the algebra of Dirac $\gamma$ matrices and exhibit matrices which realize the algebra in the Euclidean representation and explain our notations and conventions.

To do this, let $\gamma_{\mu}, \mu = 1, \ldots, d$, be a set of $d$ matrices satisfying the anticommuting relations:

$$\gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2\delta_{\mu\nu} I,$$

in which $I$ is the identity matrix. These matrices are the generators of a Clifford algebra similar to the algebra of operators acting on Grassmann algebras. It follows from relations (I-1) that the $\gamma$ matrices generate an algebra which, as a vector space, has a dimension $2^d$.

In the following, we will give an inductive construction ($d \rightarrow d + 2$) of hermitian matrices satisfying (I-1). In the algebra one element plays a special role, the product of all $\gamma$ matrices. The matrix $\gamma_s$:

$$\gamma_s = i^{-\frac{d}{2}} \gamma_1 \gamma_2 \cdots \gamma_d,$$

anticommutes, because $d$ is even, with all other $\gamma$ matrices and $\gamma_s^2 = I$.

In calculations involving $\gamma$ matrices, it is not always necessary to distinguish $\gamma_s$ from other $\gamma$ matrices. Identifying thus $\gamma_s$ with $\gamma_{d+1}$, we have:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I, i, j = 1, \ldots, d, d + 1.$$

The Greek letters $\mu \nu \ldots$ are usually used to indicate that the value $d + 1$ for the index has been excluded.

**An explicit construction of $\gamma_i^{(d)}$**

It is sometimes useful to have an explicit realization of the algebra of $\gamma$ matrices. For $d = 2$,
the standard Pauli matrices realize the algebra:

\[
\gamma^{(d=2)}_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{(d=2)}_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\gamma^{(d=2)}_3 = \gamma^{(d=2)}_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The three matrices are hermitian, i.e., \( \gamma_i = \gamma^\dagger_i \). The matrices \( \gamma_1 \) and \( \gamma_3 \) are symmetric and \( \gamma_2 \) is antisymmetric, i.e., \( \gamma_1 = \gamma_1^\dagger \), \( \gamma_3 = \gamma_3^\dagger \) and \( \gamma_2 = -\gamma_2^\dagger \). To construct the matrices for higher even dimensions, we then proceed by induction, setting:

\[
\gamma^{(d+2)}_i = \sigma_1 \otimes \gamma^{(d)}_i = \begin{pmatrix} 0 & \gamma^{(d)}_i \\ \gamma^{(d)}_i & 0 \end{pmatrix}, \quad i = 1, 2, ..., d + 1,
\]

\[
\gamma_{d+2} = \sigma_2 \otimes I^{(d)} = \begin{pmatrix} 0 & -iI_d \\ iI_d & 0 \end{pmatrix},
\]

where, \( I_d \) is the unit matrix in \( 2^d \) dimensions. As a consequence \( \gamma^{(d+2)}_s \) has the form:

\[
\gamma^{(d+2)}_s = \gamma^{(d+2)}_{d+3} = \sigma_3 \otimes I_d = \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix}.
\]

A straightforward calculation shows that if the matrices \( \gamma^{(d)}_i \) satisfy relations (I-3), the \( \gamma^{(d+2)}_i \) matrices satisfy the same relations. By induction we see that the \( \gamma \) matrices are all hermitian.

from (I-5), it is seen that, if \( \gamma^{(d)}_i \) is symmetric or antisymmetric, \( \gamma^{(d+2)}_i \) has the same property. The matrix \( \gamma^{(d+2)}_{d+2} \) is antisymmetric and \( \gamma^{(d+2)}_s \) S which is also \( \gamma^{d+2}_{d+3} \) is symmetric. It follows immediately that, in this representation, all \( \gamma \) matrices with odd index are symmetric and all matrices with even index are antisymmetric, i.e.,

\[
\gamma_i^t = (-1)^{i+1}\gamma_i.
\]
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