Transient entanglement in a spin chain stimulated by phase pulses.

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Dynamics of the one-dimensional open Ising chain under influence of \(\pi\)-pulses is studied. It is shown that the application of a specific sequence of such instant kicks to selective spins stimulates arising of perfect dynamical pairwise entanglement between ends of the spin chain. Analytic formulas for the concurrence dynamics are derived. It is also shown that the time required to perfectly entangle the ends of the chains grows linearly with the number of spins in the chain. The final entangled state of the ending spins is always the same and does not depend on length the chain.

I. INTRODUCTION

Generation and distribution of entanglement between remote particles is a necessary requirement for quantum information transfer \cite{1, 2}. It turns out that spin chains are one of the most promising quantum channels for creation of spin-spin entanglement \cite{3}. It appears, that in spite of existence of correlations in spin chains, the amount of entanglement generated between two particular spins rapidly decays with the distance between them \cite{3, 4} even near the point of quantum phase transition \cite{5}. So that, the implementation of efficient mechanisms for entangling spatially separated subsystems is still a challenging problem, especially for massive particles.

During the last years different types of spin-spin interactions have been extensively studied as possible candidates to entangle remote particles. It has been proved that for certain types of spin-1/2 open Heisenberg chains, it is possible to entangle the ends of the chain by action of stationary Hamiltonians \cite{6} and implementing different mechanisms like mirror symmetry \cite{7}, dimerized models or weak couplings at the ends of the chain \cite{8}, performing a quench \cite{9, 10}, or considering alternating couplings between the spins \cite{11}. Also, several dynamical schemes, such as application of optimized time-dependent magnetic field \cite{12} and periodically time-dependent nearest-neighbor coupling \cite{13} have been proposed to faithfully entangle the ends of an open spin chain. Unfortunately, only a partial (although a quite large) entanglement can be achieved with these schemes. Besides, owing mathematical complexity of the problem, the main tool for studying such systems remains numerical simulations.

The simplest type of interaction between massive particles is the homogeneous Ising chain. Being the simplest type of interaction, it has been a candidate for implementation of quantum information algorithms since the very beginning \cite{14}. It worth noting a recent proposal for measurement-based quantum computation \cite{15} using homogeneous Ising-like interaction. Also, this kind of interaction accompanied by the global \(\pi\) pulses allows a perfect qubit transport, quantum mirrors and universal quantum computation \cite{16}.

On the other hand, the possibility of application of pulses to quantum systems in order to preserve the coherence was discussed several years ago \cite{17}, and recently, the idea of applying optimized \(\pi\) pulses at irregular intervals of time was proposed and studied by Uhrig \cite{18} for efficient control of dephasing in spin systems and later applied for preserving entanglement in dephasing environments \cite{19}.

In this article we propose a simple analytical scheme which allows to create a transient entanglement between the ends of an homogeneous spin chain by means of application of a sequence of \(\pi\) pulses on some particular spins. We apply \(\pi\) pulses not to all qubits (as it was proposed in \cite{16} to control a qubit transport) but only to a part of them, which results in a perfect transient entanglement between the ends the chain.

II. THE MODEL

Let us consider an open chain of \(N\) spins 1/2 with homogeneous Ising-like interaction (the coupling constant is taken to be unity), governed by the Hamiltonian

\[ H = \sum_{j=1}^{N} \hat{s}_{zj} \hat{s}_{zj+1}, \quad (1) \]

and initially prepared in the coherent superposition

\[ |\Psi_0\rangle = \Pi_{j=1}^{N} (+)_j, \quad (+)_j = \frac{1}{\sqrt{2}} \left( |0\rangle_j + |1\rangle_j \right). \]

Since \( |\Psi_0\rangle \) is not an eigenstate of the Hamiltonian Eq. (1), some specific spin-spin correlations arise during the Hamiltonian evolution, \( \hat{U}(t) = e^{-i\hat{H}t} \). The evolution of the density matrix \( \hat{\rho} \) can be found in the following
closed form,
\[
\hat{\rho}(t) = \hat{U}(t) |\Psi_0\rangle \langle \Psi_0| \hat{U}^\dagger(t)
\]
\[
= \left( \frac{1}{2} + \cos \frac{t}{2} \hat{s}_{x1} + 2 \sin \frac{t}{2} \hat{s}_{y1} \hat{s}_{z2} \right)
\]
\[
\times \prod_{j=2}^{N-1} \left[ \frac{1}{2} + \cos \frac{t}{2} \hat{s}_{xz} - 4 \sin^2 \frac{t}{2} \hat{s}_{xz} \hat{s}_{zj} - 2 \hat{s}_{zj} \hat{s}_{zj+1} + 2 \sin \frac{t}{2} \cos \frac{t}{2} \hat{s}_{yz} (\hat{s}_{zj-1} + \hat{s}_{zj+1}) \right]
\]
\[
\left( \frac{1}{2} + \cos \frac{t}{2} \hat{s}_{xN} + 2 \sin \frac{t}{2} \hat{s}_{yN} \hat{s}_{zN-1} \right),
\]
(2)

In the Hamiltonian (1) only nearest neighbors interact, and a dynamical pairwise entanglement is generated between them. As it can easily be seen from (2), the entanglement dynamics between all the pairs in the middle of the chain is the same, the concurrence [20], is given by

\[
C_{jj+1} = \max \left[ 0, \frac{\sin t}{2} - \frac{\sin^2 \frac{t}{2}}{2} \right],
\]
(3)

where \( j = 2, \ldots, N-2 \) and its maximum is \( \sim 0.31 \). Since the spin chain is open, the concurrences for the spins 1 and 2, and the spins \( N-1 \) and \( N \) are different that given in Eq. (3),

\[
C_{12} = C_{N-1 N} = \frac{|\sin t|}{2},
\]
(4)

reaching the maximum of 1/2 at \( t = (2n + 1)\pi/2 \), with \( n = 0, 1, \ldots \).

It is important to stress that non neighboring pairs of spins will never get entangled under the Hamiltonian evolution.

III. ENTAILING THE ENDS OF THE CHAIN WITH \( \pi \)-PULSES

In order to generate entanglement between the ends of the chain let us apply instant \( \pi \) pulses to the first \( N - 1 \) spins at certain times \( t_j \). Such kicks correspond to rotations around the \( y \)-axis, and for the \( j \)-th spin are represented by the operator \( \hat{R}_j = e^{-i\pi \hat{s}_{yj}/2} \). The main idea is as follows: initially all the spins are in eigenstates of \( \hat{s}_{xz} \) so that, in the course of the evolution governed by Eq. (1), some spins become correlated. At the instant \( t_1 = \pi \) we apply a \( \pi \)-rotation to the first \( N - 1 \) spins, producing flip of those spins. Afterwards, the evolution under the Hamiltonian Eq. (1) allowed to continue leading to the recoupling of all the spins. Then, at the instant \( t_2 = 2\pi \) we apply the inverse \( \pi \)-rotation, allowing the system to evolve further and repeat this cycle until \( N - 2 \) spin flips are performed at times \( t_j = j\pi, j = 1, 2, \ldots, N-2 \), so that

\[
|\Psi(t)\rangle = \hat{U}(t - \pi(N-2))\hat{R}^{(-1)^{N-1}}\hat{U}(\pi) \ldots \hat{U}(\pi)\hat{R}^{-1}\hat{U}(\pi)\hat{R}\hat{U}(\pi)|\Psi_0\rangle
\]

where \( \hat{R} = \otimes_{j=1}^{N} \hat{R}_j \). Using (3) we now proceed to compute the reduced density matrix and the (time-dependent) concurrence \( C(t) [20] \).

The three-spin case is special, since the middle spin is coupled to both ends. For a three-spin chain a single \( \pi \) pulse at \( t_1 = \pi \), applied to the first and second spins, produces entanglement between the ends. It is straightforward to obtain the concurrence for the first and third spins:

\[
C_{13}(t) = \left\{ \begin{array}{ll}
\cos^2 \frac{t}{2}, & t \geq \pi \\
0, & t < \pi,
\end{array} \right.
\]
i.e., the first and third spins, which were not entangled before the pulse \( \langle t < \pi \rangle \), become completely entangled at \( t = 2\pi k, k = 1, 2, \ldots \), while \( C_{12}(t) = C_{23}(t) = 0 \) for \( t \geq \pi \).

For longer chains the dynamical behavior of the pairwise entanglement is rather unusual. After the first kick and before the \((N-2)\)-nd kick the pairwise entanglement between any pair of spins disappears. Since the system is closed, the pairwise entanglement transforms into a multipartite entanglement during that time. Moreover, after the \((N-2)\)-nd kick it is possible to obtain the following analytic expression for the concurrence between the ends of the chain:

a) for the even number of spins

\[
C(t) = \left\{ \begin{array}{ll}
\max \left[ 0, \sin \frac{t}{2} \right] - \frac{\cos^2 \frac{t}{2}}{2}, & t \geq (N-2)\pi \\
0, & t < (N-2)\pi,
\end{array} \right.
\]
(5)

and

b) for the odd number of spins

\[
C(t) = \left\{ \begin{array}{ll}
\max \left[ 0, \cos \frac{t}{2} \right] - \frac{\sin^2 \frac{t}{2}}{2}, & t \geq (N-2)\pi \\
0, & t < (N-2)\pi,
\end{array} \right.
\]
(6)

while the nearest neighbors in the middle of the chain recover their original transient entanglement \( C_{jj+1}(t) = \max[0, |\sin t| - \sin^2 t/2]/2 \) for times \( t > (N-2)\pi \).

Thus, the subsystem containing the first and the last spins of the chain stays in an unentangled state until the last kick is applied. As a result the subsystem reaches a maximally entangled state at the moment \( t = (N + 2k - 1)\pi, k = 0, 1, \ldots \), being entangled with the rest of the chain, in accordance with the monogamy property [21]. So that, the time needed to completely entangle the ends of the chain linearly depends on the length of the chain. In particular, at the instant \( t = (N-1)\pi \) the state of the chain is as follows:

\[
|\Psi\rangle = \frac{1}{2^{(N-1)/2}} \prod_{j=2}^{N-1} \left( \begin{array}{l}
|1\rangle_j + (-1)^j |0\rangle_j
\end{array} \right)
\]

\[
\otimes \left[ |00\rangle_{1N} + |11\rangle_{1N} + i(|01\rangle_{1N} + |10\rangle_{1N}) \right].
\]

On the other hand, the concurrence between the spins 1 and 2, and the spins \( N-1 \) and \( N \), remains zero for any \( t > \pi \).
In order to study the behavior of concurrence $C(t)$ when the times $t_j$ of application of the $\pi$-pulses are not commensurable, we have numerically calculated $C(t) = C(t_1, t_2, t_3, \ldots, t_N)$ as a function of $t_1$ and $t_2 > t_1$ at different fixed times. In Fig. 2 we plot the concurrence $C(t)$ for 4 spins at time $t = 3\pi$ (compare with Fig. 1(a)) when $t_1 \in [0.1, 5]$, and $t_2 \in [5, 9]$. We can observe that $C(t_1, t_2)$ has a pronounced maximum, $C = 1$, at $t_1 = \pi$, and $t_2 = 2\pi$ and smoothly decreases when $t_{1,2}$ deviate from these optimal values.

We have studied the dynamics a one-dimensional open Ising chain of $N$ spins assisted by phase kicks at times $t_j = j\pi$, $j = 1, 2, \ldots, N - 2$. It is found that the application of $N - 2$ instant pulses to $N - 1$ spins leads to arising of transient perfect entanglement between the first and the last spins of the chain. One interesting results is that, if the number of pulses is less than required, then every pairwise concurrence would be zero. The periodic behavior of the concurrence between the ends of the chain, $C_{1N}(t)$, reaches the maximum value $C = 1$ at some specific times even when the pulses are stopped being applied and the chain evolves under the Hamiltonian only. This effect substantially enlarges possible applications of the low-dimensional spin chains in quantum information technology.

On the other hand, in this particular model we may observe an important example of a non-trivial effect of local transformations on the entanglement production in non-linear systems.

It is easy to see that combining the present scheme with quench one can entangle any two spins $r$ and $s$, $1 < r < s < N$, in the chain, i.e. to model a quantum router. Really, we just have to “disconnect” the chain from 1 to $r - 1$ and from $s + 1$ to $N$ and then apply the above discussed sequence of pulses.

It worth noting that similar local transformations in the form of instant pulses were also used for another purposes: for atomic squeezing enhancement in the Dicke states, for intensification of the entanglement in continuous-variables systems, and in two-spin systems.

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