Monodromy Transform Approach to Solution of Some Field Equations in General Relativity and String Theory

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Abstract

A monodromy transform approach, presented in this communication, provides a general base for solution of space-time symmetry reductions of Einstein equations in all known integrable cases, which include vacuum, electrovacuum, massless Weyl spinor field and stiff fluid, as well as some string theory induced gravity models. There were found a special finite set of functional parameters which are defined as the set of monodromy data for the fundamental solution of associated spectral problem. These monodromy data consist of the functions of the spectral parameter only. Similarly to the scattering data in the inverse scattering transform, the monodromy data can be used for characterization of any local solution of the field equations. A "direct" and "inverse" problems of such monodromy data transform admit unambiguous solutions. For the linear singular integral equation with a scalar (i.e. non-matrix) kernel, which solves the inverse problem of this monodromy transform, an equivalent regularization – a Fredholm linear integral equation of the second kind is constructed in several convenient forms. The existence and uniqueness of the local solution for arbitrary choice of the monodromy data can be proved using a simple iterative method. This solution is effectively constructed in terms of homogeneously convergent functional series.

1 Introduction

The fundamental nature of Einstein equations as well as beautiful discovery of the existence of a large class of two-dimensional completely integrable systems made very natural various expectations and conjectures of the integrability of the Einstein equations, at least for the space-times with an Abelian two-dimensional isometry group, when the reduced dynamical equations are effectively two-dimensional. First of all, this concerned the Einstein equations
for gravitational fields in vacuum, which integrability was conjectured and even motivated partially long ago (see the papers of Geroch, Maison). However, the actual discovery of very rich internal structure of these equations and development of effective methods for the construction of infinitely large classes of their solutions actually have been started more then twenty years ago. It is necessary to mention here a variety of more or less general and well known new methods and results, such as Belinski and Zakharov formulation of the inverse scattering method and their construction of vacuum $N$-soliton solutions, the constructions of Bäcklund transformations of Harrison and of Neugebauer, the infinite dimensional algebra of internal symmetries of stationary axisymmetric electrovacuum Einstein - Maxwell equations, found by Kinnersley and Chitre and "exponentiation" of some of these symmetries made by Hoenselaers, Kinnersley and Xanthopoulos. Later it was shown, that besides the vacuum case, the integrability properties are possessed by two-dimensional space - time symmetry reductions of Einstein equations in the presence of the massless matter fields – the electromagnetic fields (Kinnersley and Chitre, Hauser and Ernst, GA), or/and Weyl massless two-component spinor (neutrino) field (GA), or/and minimally coupled scalar field, or/and stiff fluid with $p = \epsilon$ (Belinski), or electromagnetic field with dilaton (Belinski and Ruffini), as well as of some string theory induced gravity models with axion, dilaton and electromagnetic fields (e.g., Bakas, Sen, Gal'tsov and Kechkin).

In this communication we present a sketch of general approach to the analysis and solution of all mentioned above integrable space - time symmetry reductions of Einstein equations (of both, hyperbolical and elliptical types). This approach, called a "monodromy transform approach", is based on and it develops the results of the author’s papers. It leads a) to a definition in the most general context of a convenient set of functional parameters – "monodromy data", which analytical properties on the spectral plane are closely related to various physical and geometrical properties of solutions, and b) to a construction of pure linear integral equations (of Cauchy and then, of Fredholm types) equivalent to the original reduced field equations and admitting a construction of their general local solutions in terms of homogeneously converging functional series. Various applications of this approach to a classification of solutions, exact linearization of various boundary value problems and to explicit construction of new classes of exact solutions are expected to be considered elsewhere.

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1The lack of space urges the author to avoid a detail citation and to refer the reader to the references in a few papers cited below, but mainly – to a large and very useful F.J. Ernst’s collection of related references and abstracts, accessible throw http://pages.slic.com/gravity, as well as to gr-qc and hep-th data bases.

2In many intermediate statements were argued very briefly and for the analytical case only. However, all these considerations are valid also for a larger class of solutions with very low order of differentiability (namely, $C^3$ for the metric components). The corresponding rigorous proof can be found in the recently published paper of Hauser and Ernst (I. Hauser, F.J. Ernst, gr-qc/9903104), where many closely related statements were proved in fullest detail, but in a different context of the analysis of the group structure of the solution space of the Ernst equations and characteristic initial value problem.
2 Generalized Ernst equations

The space-time symmetry ansatz of existence of an Abelian two-dimensional space-time isometry group, provided all field components and potentials also possess this symmetry, provides a reduction of all mentioned above cases or eventually, of Einstein - Maxwell - Weyl equations to generalized form of the Ernst equations, except for axion - dilaton gravity, which leads (as it is already known) to a matrix analog of these equations. In the differential form notation the reduced Einstein - Maxwell - Weyl equations can be written as

\[
\begin{align*}
    d^*d\mathcal{E} + \frac{d(\alpha + i\delta)}{\alpha} \ast d\mathcal{E} - \frac{(d\mathcal{E} + 2\Phi d\Phi)}{\text{Re}\mathcal{E} + \Phi \Phi} \ast d\mathcal{E} &= 0 \\
    d^*d\Phi + \frac{d(\alpha + i\delta)}{\alpha} \ast d\Phi - \frac{(d\mathcal{E} + 2\Phi d\Phi)}{\text{Re}\mathcal{E} + \Phi \Phi} \ast d\Phi &= 0 \\
    d^*d\alpha &= 0, \quad d\beta \equiv -\epsilon^*d\alpha \\
    d^*d\gamma &= 0, \quad d\delta \equiv \ast d\gamma.
\end{align*}
\]

(1)

where \(\mathcal{E}(x^1,x^2)\) and \(\Phi(x^1,x^2)\) are complex scalar Ernst potentials; \("\ast\) is a Hodge star operator, such that \(d^*d\) is the two-dimensional d'Alambert or Laplace operator in the hyperbolical \((\epsilon = 1)\) or elliptical \((\epsilon = -1)\) case respectively, defined on the orbit space \((x^1,x^2)\). The real functions \(\alpha(x^1,x^2)\) – a measure of area on the orbits and \(\gamma(x^1,x^2)\) – a potential for neutrino current vector, are arbitrary "harmonical" functions, provided \(d\alpha \wedge d\alpha \neq 0\). These functions determine two other auxiliary real functions – their "harmonical" conjugates \(\beta(x^1,x^2)\) and \(\delta(x^1,x^2)\).

3 Equivalent "spectral" \(N \times N\) - matrix problem

For each of the integrable reductions of Einstein equations considered above we use similar associated complex \(N \times N\) - matrix problems \((N = 2\) for vacuum fields, \(N = 3\) for the models with electromagnetic and Weyl spinor fields and \(N = 4\) for string theory induced gravity models with axion, dilaton and electromagnetic fields) for the four unknown matrix functions

\[
\begin{align*}
    U(\xi,\eta), \quad V(\xi,\eta), \quad \Psi(\xi,\eta,w), \quad W(\xi,\eta,w)
\end{align*}
\]

which should satisfy two groups of conditions. The first one is a deformation problem for a linear system with given (case dependent) structures of canonical forms of coefficients and normalization at some reference point \((\xi_0,\eta_0)\):

\[
\begin{align*}
    2i(w - \xi)\partial_\xi \Psi &= U(\xi,\eta)\Psi & (U)_{\text{e}n} &= U(0) & \Psi(\xi,\eta_0,w) &= I \\
    2i(w - \eta)\partial_\eta \Psi &= V(\xi,\eta)\Psi & (V)_{\text{e}n} &= V(0)
\end{align*}
\]

(3)

These equations follow immediately from generalized Kinnersley equations derived in [3].
The second group of conditions implies the existence for $\Psi$ of a Hermitian integral of certain structure with case dependent constant matrix $\Omega$:

\[
\begin{align*}
\left\{ \begin{array}{l}
\Psi^\dagger W \Psi = W_0(w) \\
W_0^\dagger(w) = W_0(w)
\end{array} \right. \quad \left\| \quad \frac{\partial W}{\partial w} = 4i\Omega \right. (4)
\end{align*}
\]

where $w$ is complex ("spectral") parameter and $\xi, \eta$ are geometrically defined space-time coordinates: $\xi = \beta + j\alpha, \eta = \beta - j\alpha$ with $j = 1$ for $\epsilon = 1$ and $j = i$ for $\epsilon = -1$. Thus, for the hyperbolic case ($\epsilon = 1$) the coordinates $(\xi, \eta)$ are two real light cone coordinates, while for the elliptical case ($\epsilon = -1$) these coordinates are complex conjugated to each other. The canonical forms of $U$ and $V$ matrices (up to a permutation of diagonal elements) are

\[
U^{N=2}_{(0)} = \text{diag}(i, 0) \quad \quad U^{N=3}_{(0)} = \text{diag}(i + a(\xi), 0, 0) \quad \quad U^{N=4}_{(0)} = \text{diag}(i, i, 0, 0)
\]

\[
V^{N=2}_{(0)} = \text{diag}(i, 0) \quad \quad V^{N=3}_{(0)} = \text{diag}(i + b(\eta), 0, 0) \quad \quad V^{N=4}_{(0)} = \text{diag}(i, i, 0, 0)
\]

where $a(\xi) = 2\partial_\xi \gamma, b(\eta) = 2\partial_\eta \gamma$ and a spinor field potential $\gamma$ is an arbitrary real solution of $\partial_\xi \partial_\eta \gamma = 0$, provided, for $\epsilon = -1$, $|\text{Im} a(\xi)| < 1$ and $|\text{Im} b(\eta)| < 1$ at least for $\xi, \eta$ close enough to $\xi_0, \eta_0$. The matrices $\Omega$ are constant:

\[
\Omega^{N=2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \quad \Omega^{N=3} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \quad \Omega^{N=4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}
\]

For any solution of (1), the matrices (3) can be calculated explicitly and for any solution (2) of (3), (4) the solution of the (generalized) Ernst equations where the components of $G\xi, \eta$ are algebraically related to the components of metric and electromagnetic potential $\Psi^\dagger W \Psi$.

4 Direct problem of the monodromy transform and definition of the monodromy data

Let us consider at first the linear system (3). It can be shown [5], that any its solution $\Psi(\xi, \eta, w)$ is holomorphic on the spectral plane outside a cut $L = L_+ \cup L_-$, which structure is shown on the Figure. Four endpoints of this cut are the branchpoints of $\Psi$. The local behaviour of $\Psi$ at the cuts $L_+$ and $L_-$ is characterized by monodromy matrices $T_+(w)$ and $T_-(w)$, which describe the transformations of $\Psi$ along the paths, surrounding the branchpoints on $L_+$ and $L_-:

\[
\Psi \xrightarrow{T_+} \tilde{\Psi} = \Psi \cdot T_+(w), \quad T_+(w) = I - (1 + e^{-2i|\sigma|_w}) \frac{L_+ (w) \otimes k_+(w)}{(L_+ (w) \cdot k_+(w))} (5)
\]
where $2i[\sigma]_+ = \pi a(w)$ and $2i[\sigma]_- = \pi b(w)$. If a spinor field vanishes, i.e. for $a(w) = b(w) = 0$, the branchpoints at the ends of $L_+$ and $L_-$ are algebraic branchpoints of the orders $\frac{1}{2}$ or $-\frac{1}{2}$ and therefore, we have $T_{\pm}^2(w) \equiv I$. The structure of monodromy matrices $T_{\pm}(w)$ allows to associate with any fundamental solution $\Psi$ four complex vector functions $k_{\pm}(w)$, $l_{\pm}(w)$, defined (due to a homogeneity of these expressions) in a projective sense and depending upon the spectral parameter only:

$$
k_{\pm}(w) = (1, u_{\pm}(w), v_{\pm}(w)), \quad l_{\pm}(w) = (1, p_{\pm}(w), q_{\pm}(w)), \quad (6)$$

Following [4, 5] we can find that (6) are equivalent to some constraint on the monodromy data (6), which unambiguously relates the components of the vectors $l_{\pm}$ and $k_{\pm}$ and hence, the functions $u_{\pm}(w)$ and $v_{\pm}(w)$ can represent a complete set of the monodromy data for the entire problem (3) - (4). In the case of axion - dilaton gravity instead of vectors (6) we have $2 \times 4$ matrices of the structure $k_{\pm}(w) = (I, u_{\pm}(w))$, where $I$ is a $2 \times 2$ unit matrix and $u_{\pm}(w)$ are arbitrary $2 \times 2$ matrix functions. These monodromy data can be associated with any local solution of the reduced field equations and therefore, this construction solves the direct problem of our monodromy transform.

![Figure 1](image-url)

Figure 1: The structures of the cut $L = L_+ \cup L_-$ on the spectral plane $w$ and domains of holomorphicity of the monodromy data functions $u_{\pm}(w)$ and $v_{\pm}(w)$ in the hyperbolic ($\epsilon = 1$) and the elliptic ($\epsilon = -1$) cases.

5 Inverse problem of the monodromy transform and equivalent singular integral equations

Simple arguments, similar to once used in [4, 5], show that certain components of local algebraic structure of $\Psi$ on $L_{\pm}$ - the components of two complex vectors $\varphi_{\pm}(\xi, \eta, w)$ should satisfy the similar sets of linear singular integral equations.
(In the case of gravity with axion, dilaton and electromagnetic fields $\varphi_\pm (\xi, \eta, w)$ are $2 \times 4$ matrices.) Omitting farther the suffices $\pm$ and keeping in mind that, for example, $k(\tau) \equiv k_+(\tau)$ for $\tau \in L_+$ and $k(\tau) \equiv k_-(\tau)$ for $\tau \in L_-$, we can write these integral equations in the form

$$\nu(\xi, \eta, \tau)\varphi(\xi, \eta, \tau) + \frac{1}{\pi i} \int_L \frac{[\lambda e^{i\sigma}]}{\zeta - \tau} \mathcal{H}(\tau, \zeta) \varphi(\xi, \eta, \zeta) d\zeta = -k(\tau) \quad (7)$$

where a Cauchy principal value integral is used, and the coefficients are

$$\nu(\xi, \eta, \tau) = -\{\lambda e^{i\sigma}\}_\tau \mathcal{H}(\tau, \tau), \quad \mathcal{H}(\tau, \zeta) = (k(\tau) \cdot l(\zeta))$$

$[\ldots]_\tau$ and $\{\ldots\}_\tau$ are a "jump" and a "continuous part" of functions at the point $\zeta \in L$. The functions $\lambda(\xi, \eta, w) = \sqrt{(w - \xi)(w - \eta)/(w - \xi_0)(w - \eta_0)}$ and $2\sigma(\xi, \eta, w) = \int_{L_+} a(\zeta)/(w - \zeta) d\zeta + \int_{L_-} b(\zeta)/(w - \zeta) d\zeta$.

Thus, the integral equation (7) is determined completely in terms of functions $[\ldots]_\tau$ and $\{\ldots\}_\tau$. Besides that, the Ernst potentials and all of the field components can be calculated as path integrals, which are also determined in terms of monodromy data and the corresponding solution of (7) [4, 5]. Therefore, the solution of (7) solves the inverse problem of our monodromy transform.

6 Equivalent Fredholm equation: the existence and uniqueness of local solutions for arbitrary chosen monodromy data

In accordance with the well known theory of linear singular integral equations, the equation (7) within the class of solutions regular on the cut, possesses an important property, that the index of its characteristic part is equal to zero for arbitrary chosen monodromy data functions. This means that this equation admits various equivalent regularizations. We present two equivalent forms of the corresponding (quasi-) Fredholm equations which are left and right regularizations of (7) (the dependence upon $\xi, \eta$ is not shown here explicitly):

$$\phi(\tau) + \int_L \mathcal{F}(\zeta, \tau) \phi(\zeta) d\zeta = h(\tau), \quad \omega(\tau) + \int_L \mathcal{G}(\zeta, \tau) \omega(\zeta) d\zeta = k(\tau) \quad (8)$$

where $\phi(\tau) = -\mathcal{H}(\tau, \tau)\varphi(\tau)$ and the following relations take place

$$\phi(\tau) = -\frac{1}{B(\tau)Z(\tau)} \mathcal{R}_+ [B(\tau)\omega(\tau)], \quad h(\tau) = -\frac{1}{B(\tau)Z(\tau)} \mathcal{R}_+ [B(\tau)k(\tau)]$$

The kernels $\mathcal{F}(\zeta, \tau)$ and $\mathcal{G}(\zeta, \tau)$ are determined by the expressions

$$\mathcal{F}(\zeta, \tau) = \frac{B(\zeta)Z(\zeta)}{B(\tau)Z(\tau)} \mathcal{R}_+ [B(\tau)\mathcal{S}(\tau, \zeta)]$$

$$\mathcal{G}(\zeta, \tau) = B(\zeta)\mathcal{R}_- [\mathcal{S}(\tau, \zeta)]$$

$$\mathcal{S}(\tau, \zeta) = \frac{\mathcal{H}(\tau, \zeta) - \mathcal{H}(\zeta, \zeta)}{i\pi \mathcal{H}(\zeta, \zeta)(\zeta - \tau)}$$
where the operators $\mathcal{R}_\tau$, $\tilde{\mathcal{R}}_\tau$ and auxiliary functions possess the expressions

\[
\mathcal{R}_\tau [f(\tau)] = A(\tau)f(\tau) - B(\tau)Z(\tau) \frac{1}{i\pi} \int_\gamma \frac{f(\zeta)}{Z(\zeta)(\zeta - \tau)} d\zeta
\]

\[
\tilde{\mathcal{R}}_\tau [f(\tau)] = A(\tau)f(\tau) + \frac{1}{Z(\tau)} \frac{1}{i\pi} \int_\gamma \frac{B(\zeta)Z(\zeta)}{(\zeta - \tau)} f(\zeta) d\zeta
\]

\[
A(\tau) = \sin[\sigma]\tau,
\]

\[
B(\tau) = i \cos[\sigma]\tau,
\]

\[
Z(\tau) = i[\lambda]_\tau e^{i(\sigma)}.\]

(We note here, that for electrovacuum ($\sigma \equiv 0$) we have $A(\tau) = 0$, $B(\tau) = i$.)

The local solution of each of the equations (8) for any given set of monodromy data can be constructed by the known iterative method. In particular,

\[
\Phi(\tau) = \Phi_0(\tau) + \sum_{n=1}^{\infty} (\Phi_n(\tau) - \Phi_{n-1}(\tau)),
\]

\[
\Phi_0(\tau) = h(\tau), \quad \Phi_n(\tau) = h(\tau) - \int_\gamma F(\tau, \zeta) \Phi_{n-1}(\zeta) d\zeta \quad (9)
\]

For local solutions, when the coordinates $\xi$ and $\eta$ are close enough to their initial values $(\xi_0, \eta_0)$, it is easy to prove a homogeneous convergence of the series (9) and therefore, the existence as well as the uniqueness of the solution.

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