MATHEMATICAL ANALYSIS OF THE ACOUSTIC IMAGING MODALITY USING BUBBLES AS CONTRAST AGENTS AT NEARLY RESONATING FREQUENCIES

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Abstract. We analyze mathematically the acoustic imaging modality using bubbles as contrast agents. These bubbles are modeled by mass densities and bulk moduli enjoying contrasting scales. These contrasting scales allow them to resonate at certain incident frequencies. We consider two types of such contrasts. In the first one, the bubbles are light with small bulk modulus, as compared to the ones of the background, so that they generate the Minnaert resonance (corresponding to a local surface wave). In the second one, the bubbles have moderate mass density but still with small bulk modulus so that they generate a sequence of resonances (corresponding to local body waves).

We propose to use as measurements the far-fields collected before and after injecting a bubble, set at a given location point in the target domain, generated at a band of incident frequencies and at a fixed single backscattering direction. Then, we scan the target domain with such bubbles and collect the corresponding far-fields. The goal is to reconstruct both the variable, mass density and bulk modulus of the background in the target region.

1. We show that, for each fixed used bubble, the contrasted far-fields reach their maximum value at, incident, frequencies close to the Minnaert resonance (or the body-wave resonances depending on the types of bubbles we use). Hence, we can reconstruct this resonance from our data. The explicit dependence of these resonances in terms of the background mass density of the background allows us to recover it, i.e. the mass density, in a straightforward way.

2. In addition, this measured contrasted far-fields allow us to recover the total field at the location points of the bubbles (i.e. the total field in the absence of the bubbles). A numerical differentiation argument, for instance, allows us to recover the bulk modulus of the targeted region as well.

1. Introduction and statement of the results. Diffusion by highly contrasted small particles is of fundamental importance in several branches of applied sciences, as for example in material sciences and imaging. In this work, we focus on the acoustic imaging modality using microscaled bubbles as contrast agents, see [10, 19, 20, 17] for more details on related theoretical and experimental studies. We describe

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a modality using the contrasted scattered fields, by the targeted anomaly, measured before and after injecting microscaled bubbles. These bubbles are modeled by mass densities and bulk moduli enjoying contrasting scales. These contrasting scales allow them to resonate at certain incident frequencies. The main goal of this work is to analyze mathematically this contrasted scattered fields in terms of these scales with incident frequencies close to these resonances and derive explicit formulas linking the values of the unknown mass density and bulk modulus of the targeted region to the measured scattered fields.

To describe properly the mathematical model we are dealing with in this work, let us denote by $D$ a small particle in $\mathbb{R}^3$ of the form $D := \varepsilon B + z$, where $B$ is an open, bounded, simply connected set in $\mathbb{R}^3$ with Lipschitz boundary, containing the origin, and $z$ specifies the location of the particle. The parameter $\varepsilon > 0$ characterizes the smallness assumption on the particle. Let us consider a mass density (respectively, bulk modulus) that we note by $\rho_\varepsilon(\cdot)$ (respectively, $k_\varepsilon(\cdot)$) of the form

$$
\rho_\varepsilon(x) := \begin{cases} 
\rho_0(x), & x \in \mathbb{R}^3 \setminus D, \\
\rho_1, & x \in D,
\end{cases}
$$

and

$$
k_\varepsilon(x) := \begin{cases} 
k_0(x), & x \in \mathbb{R}^3 \setminus D, \\
k_1, & x \in D,
\end{cases}
$$

where $\rho_1$ and $k_1$ are positive constants, while $\rho_0$ and $k_0$ are smooth enough functions which are constant outside of a bounded and smooth domain $\Omega$. We denote respectively $\bar{\rho}_0$ and $\bar{k}_0$ to be the values of $\rho_0$ and $k_0$ outside $\Omega$. Thus $\rho_0$ and $k_0$ denote the density and bulk modulus of the background medium, and $\rho_1$ and $k_1$ denote the density and bulk modulus of the bubble respectively.

We are interested in the following problem describing the acoustic scattering by a bubble, see [11] and [12], given by the system

$$
\begin{cases}
\nabla \cdot \left[ \frac{1}{\rho_0} \nabla u \right] + \frac{\omega^2}{k_0} u = 0 \text{ in } \mathbb{R}^3 \setminus D,
\nabla \cdot \left[ \frac{1}{\rho_1} \nabla u \right] + \frac{\omega^2}{k_1} u = 0 \text{ in } D,
\n\left. u_+ - u_- \right|_{\partial D} = 0, \text{ on } \partial D,
\n\frac{1}{\rho_1} \left| \frac{\partial u}{\partial \nu} \right|_+ - \frac{1}{\rho_0} \left| \frac{\partial u}{\partial \nu} \right|_- = 0 \text{ on } \partial D,
\end{cases}
$$

where $\omega > 0$ is a given frequency and $\nu$ denotes the external unit normal to $\partial D$. Here the total field is $u := u^i + u^s$, where $u^i$ denotes the incident field (we restrict to plane incident waves) and $u^s$ denotes the scattered waves which satisfy the following condition

$$
\frac{\partial u^s}{\partial |x|} - i\kappa_0 u^s = o \left( \frac{1}{|x|} \right), \quad |x| \to \infty, \quad \text{(S.R.C)}.
$$

We introduce the notation $\kappa_0^2 := \omega^2 \rho_0 / k_0$ and $\kappa_1^2 := \omega^2 \rho_1 / k_1$. The problem (1) is well posed, see [3, 4] and [9]. In addition, the scattered field $u^s$ can be expanded as

$$
u^s(x, \theta) = e^{i\kappa_0 |x|} \frac{|x|}{|x|} u^\infty(\hat{x}, \theta) + O \left( \frac{1}{|x|^2} \right), \quad |x| \to +\infty,$$

where $\hat{x} := x/|x|$ and $u^\infty(\hat{x}, \theta)$ denotes the far-field pattern corresponding to the unit vectors $\hat{x}, \theta$, i.e. the incident and propagation directions respectively. We are
interested in the regimes where the coefficients satisfy the conditions:
\[ \rho_1 = C_\rho \varepsilon^s, \quad s \geq 0 \] and \[ k_1 = C_k \varepsilon^t, \quad t \geq 0, \]
with positive constants \( C_\rho \) and \( C_k \) which are independent from \( \varepsilon \), and real numbers \( s, t \) assumed to be non negative. The scattering problem described above models the acoustic wave diffracted in the presence of small bubbles. In this case, the parameters \( s \) and \( t \) fix the kind of medium we are considering, see \([12, 11, 18]\) and \([9]\). Recall that the speed of propagation of the sound is \( c_0 := \sqrt{\frac{k_0}{\rho_0}} \) in the background and \( c_1 := \sqrt{\frac{k_1}{\rho_1}} \) in the bubble. We are interested in the following two regimes on the relative speed of propagation inside the bubble as compared to the one in the background, i.e the ratio \( \frac{c_1}{c_0} \).

1. **Moderate relative speed of propagation.** In this case, we assume that \( s = t \), then the relative speed of propagation is uniformly bounded from below and above or
\[
\frac{c_1^2}{c_0^2} = \frac{\rho_0 k_1}{k_0 \rho_1} = \frac{\kappa_0^2}{\kappa_1^2} \sim 1, \quad \text{as} \ \varepsilon \ll 1.
\]

2. **Small relative speed of propagation.** In this case, we assume that \( s < t \), then the relative speed of propagation is small or
\[
\frac{c_1^2}{c_0^2} = \frac{\kappa_0^2}{\kappa_1^2} \sim \varepsilon^{t-s}, \quad \text{as} \ \varepsilon \ll 1.
\]

There is a major difference between these two regimes. In the first one, the contrast between the bubble and the background comes only from the transmission coefficients \( \rho_1/\rho_0 \) across the interface of the bubble. Hence, there might be surface waves created by this contrast if it is pronounce enough. For certain scales of \( \rho_1/\rho_0 \), this is indeed the case, as we will see it later. In the second regime, as the speed of propagation is very small inside the bubble, then the sound wave (i.e. the fluctuation) slows down inside it and might create local spots, body waves, even though the bubble has a small size. For certain scales of \( \frac{c_1^2}{c_0^2} \) this is indeed the case.

To highlight these differences, let us for the moment assume that \( \rho_0 \) is constant everywhere in \( \mathbb{R}^3 \). In this case, the above problem can be equivalently formulated as
\[
\begin{cases}
\Delta u + \kappa_0^2 u = 0 \text{ in } \mathbb{R}^3 \setminus D, \\
\Delta u + \kappa_1^2 u = 0 \text{ in } D, \\
\left. u \right|_+ - \left. u \right|_- = 0, \quad \text{on } \partial D, \\
\left. \frac{1}{\rho_1} \frac{\partial u}{\partial \nu} \right|_- - \left. \frac{1}{\rho_0} \frac{\partial u}{\partial \nu} \right|_+ = 0 \text{ on } \partial D, \\
u - u \text{ satisfies the SRC.}
\end{cases}
\]

As we can see, the contrasts of the medium appear in the transmission conditions through the coefficient \( 1/\rho_1 \) (or equivalently \( \rho_0/\rho_1 \)), and through the speed of propagation, namely \( \rho_0/k_0 \) and \( \rho_1/k_1 \). Based on the Lippmann-Schwinger equation representation of the total fields, the second contrast appears on the (volumetric) Newtonian potential \( \mathcal{N}_D^\omega \) defined, from \( L^2(D) \) to \( H^2(D) \), as
\[
\mathcal{N}_D^\omega[f](x) := \int_D G_\omega(x, y) f(y) dy,
\]
while the first one appears on the (surface) Neumann-Poincaré operator \((K_\nu^\omega)^*\) defined, from \(L^2(\partial D)\) to \(L^2(\partial D)\), by
\[
(K_\nu^\omega)^* [f](x) := p.v. \int_{\partial D} \frac{\partial G_\omega(x, y)}{\partial \nu(x)} f(y) d\sigma(y).
\]
Precisely, the values of the field \(u\) outside the bubble \(D\) is fully computable from the knowledge of \(u(x), x \in D\) and \(\partial \nu u(x), x \in \partial D\). These last quantities are solutions of the following system of integral equations
\[
\begin{align*}
(u(x) - \gamma \omega^2 N_\nu^\omega [u](x) + \alpha S_D^\omega [\partial \nu u](x) = u^i(x), \quad \text{on } D
\end{align*}
\]
and
\[
\alpha \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K_\nu^\omega)^* \right) [\partial \nu u] - \gamma \omega^2 \partial \nu N_\nu^\omega [u](x) = \partial \nu u^i \quad \text{on } \partial D,
\]
where \(S_D^\omega\) is the single layer operator defined, from \(L^2(\partial D)\) to \(H^{\frac{3}{2}}(D)\), as
\[
S_D^\omega[f](x) := \int_{\partial D} G_\omega(x, y) f(y) d\sigma(y),
\]
and \(u^i\) is the incident field such that
\[
\nabla \cdot \left[ \frac{1}{\rho_0} \nabla u^i \right](x) + \frac{\omega^2}{k_0} u^i(x) = 0, \quad x \in D.
\]
and we have adapted the succeeding notations \(\gamma = \beta - \alpha \rho_1/k_1\) and \(\alpha := 1/\rho_1 - 1/\rho_0\) with \(\beta := 1/k_1 - 1/k_0\).
Here, \(G_\omega\) stands for the Green’s functions related to (5). More precisely, \(G_\omega\) is solution, in the distributional sense, of
\[
\nabla \cdot \left[ \rho_0^{-1}(x) \nabla G_\omega(x, z) \right] + \frac{\omega^2}{k_0} G_\omega(x, z) = -\delta(x) \quad \text{for any } x, z \in \mathbb{R}^3.
\]
with the radiation conditions at infinity. For \(\hat{x}\) in the unit sphere, we note by
\[
G_\omega^{\infty}(\hat{x}, \cdot)
\]
the far field associated to \(G_\omega(\cdot, \cdot)\). Unless specified, we use the same notation \(G_\omega(\cdot, \cdot)\) for the general setting when \(\rho_0\) and \(k_0\) are locally variable.

Depending on the scales of the contrasts, we make the following observations.

1. In the first regime, i.e \(s = t\), we have \(\gamma \sim 1\), as \(\varepsilon \ll 1\), and the Newtonian potential is negligible as it scales as \(\varepsilon^2\) as \(\varepsilon \ll 1\). However, if \(s = t = 2\) then the contrasts on the mass densities, i.e. \(1/\alpha\), can approximate the spectrum of the Neumann-Poincaré operator \(K_\nu^0\). For smooth domain \(D\), this operator defined on \(L^2(\partial D)\) has a sequence of real eigenvalues accumulating at 0 in addition to the value \(-\omega_0^2/2\). As the contrast is real, then we can only approximate the highest eigenvalue, which is \(-\omega_0^2/2\). This can be done in this regime as \(\alpha \sim \varepsilon^{-2}\). The frequency \(\omega\) for which this is possible is the Minnaert resonance (corresponding to a surface wave type).

2. In the second regime, if \(s < t\), the high contrasts of the speed of propagation allow the Newtonian operator to dominate the Neumann-Poincaré operator. In addition, if we take \(t - s = 2\), then the contrast of the speed of propagation, \(\gamma \sim \varepsilon^{-2}\), will balance the scale of the Newtonian operator and we might excite its eigenvalues. There is a discrete sequence of such eigenvalues (corresponding to local body waves).
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Microbubbles with scales fitting into the first regime are well known to exist in the nature. However, those related to the second regime, with small speed of propagation, are less known. Nevertheless, there are possibilities to artificially produce them, see the discussion in [23] and also in [22].

A first key observation in our analysis, which happens to be useful for the imaging later on, is that the Minnaert resonance and the sequence of body wave resonances are characterized by the bulk modulus of the bubble and the surrounding local mass density of the background. In addition, we show that the contrasted scattered fields reach their maximum values at, incident, frequencies close to the Minnaert resonance (or the body-wave resonances depending on the types of bubbles we use). This allows us to recover these resonances by measuring the contrasted scattered waves at a band of incident frequencies but at a fixed single backscattering direction.

A second key observation is that this measured contrasted scattered waves allows us to recover the total field at the location point of the bubble. Scanning the targeted region with such bubbles, we can recover the total field there up to a sign (i.e. the total field in the absence of the bubbles).

Based on these observations, we can reconstruct the density and the bulk modulus of the targeted region from the contrasted scattered waves (before and after injecting the bubbles) at a band of incident frequencies but at a fixed single backscattering direction. More details are given in section 2. Nevertheless, let us say it in short here that these contrasted scattered waves encodes the Minnaert and the body-waves resonance in its denominator and the total field in its numerator. From the first one, we extract the mass density while from the second one we derive the bulk modulus of the targeted region.

The following theorems translate these observations with more clear statements. We state the following conditions which are common to both the two results.

**Conditions:** Let Ω be a bounded domain of diameter diam(Ω) of order 1. Let also ρ0 and k0 be two functions of class C1 and are constant outside Ω. They are assumed to be positive functions. Let D := z + εB be a small and Lipschitz smooth domain where z ∈ Ω away from its boundary. The relative diameter of D is small as compared to the diameter of Ω, i.e. \( \frac{ε}{diam(Ω)} << 1 \). The functions ρ0 and k0 are assumed to be independent on the parameter ε.

**Theorem 1.1.** Let the above Conditions be satisfied. In addition, let \( \rho_1 \) and \( k_1 \) be constants enjoying the following scales

\[
\rho_1 = \rho_1 ε^2, \quad k_1 = k_1 ε^2 \quad \text{and} \quad \frac{k_1}{\rho_1} \sim 1, \quad \text{as} \quad ε << 1
\]

and \( \rho_1 \) is large enough such that \( \max_{x ∈ Ω} ρ_0(x) < \rho_1 \).

The solution of the corresponding problem (1), has the following expansions.

1. The scattered field is approximated as

\[
u^s(·, ϑ, ω) = v^s(x, ϑ, ω) - \frac{ω^2 ω^2_M}{k_1(ω^2 - ω^2_M)} |B| ε G_ω(x - z) v(·, ϑ, ω)
\]

\[
+ O \left( \frac{ε^2}{(ω^2 - ω^2_M)^2} \right)
\]
where $\Gamma_f$ with the corresponding eigenfunctions, where obviously compact, selfadjoint and by $\omega$ with radiations conditions at infinity. For $\omega\ll 1$, these expansions are valid under the condition that $\varepsilon \ll 1$. Here, we have

$$\omega_M := \omega_M(z) := \sqrt{\frac{2F_1}{\rho(z)\mu_{\partial B}}}, \quad \text{with} \quad \mu_{\partial B} := \frac{1}{|\partial B|} \int_{\partial B} \int_{\partial B} \frac{(x - y) \cdot \nu(x)}{4 \pi |x - y|} d\sigma(x)d\sigma(y),$$

called the Minnaert frequency\(^1\). In both the expansions $\nu := v(x, \theta, \omega) = v^\infty(\hat{x}, \theta, \omega) + w(x, \theta)$ and $v^\infty := v^\infty(\hat{x}, \theta, \omega)$ is the total field, and is the far field associated to the scattered field, solution of the problem (1) in the absence of the bubble $D$.

Note, from (9), that the Minnaert frequency $\omega_M$ is such that $\omega_M \sim 1$, in terms of $\varepsilon$, and since $\omega$ approaches $\omega_M$, we deduce that $\omega \sim 1$, in terms of $\varepsilon$.

The first mathematical study of the Minnaert resonance was shown in [6] where it was estimated for bubbles injected in a homogeneous background. Later on, a series of works were devoted to its implications in different areas, see [9, 3, 4, 7, 8].

To state the results related to the second regime, we first introduce, with some details, the Newtonian operator $\mathcal{N}_B$ defined from $L^2(B)$ to $L^2(B)$ as\(^2\)

$$\mathcal{N}_B[f](x) := \int_B \Gamma(x, y) f(y) dy = \int_B \rho_0(z) e^{i \omega \sqrt{\frac{|y-x|}{\rho_0(z)}}} \Gamma(x, y) f(y) dy,$$

where $\Gamma(x, \cdot)$ is the fundamental solution of

$$\Delta \Gamma(x, z) + \frac{\omega^2 \rho_0(z)}{k_0(z)} \Gamma(x, z) = -\rho_0(z) \delta(x), \quad x, z \in \mathbb{R}^3,$$

with radiations conditions at infinity. For $\omega = 0$, the operator $\mathcal{N}_B[\cdot]$ is positive, compact, selfadjoint and by $(\rho_0(z) \lambda^B_n, \psi^B_n)_{n \in \mathbb{N}}$ we design its sequence of eigenvalues with the corresponding eigenfunctions, where obviously $(\lambda^B_n, \psi^B_n)_{n \in \mathbb{N}}$ are related to the operator defined through $f(\cdot) \rightarrow \int_B \frac{1}{4 \pi |x - y|} f(y) dy$.

---

\(^1\)Remark that $\mu_{\partial B}$ depends only on the shape of the domain $B$.

\(^2\)For convenience, we’ll omit the dependency notation of $\mathcal{N}_B[z]$ with respect to $z$. 
Theorem 1.2. Let the above Conditions be satisfied. In addition, let \( \rho_1 \) and \( k_1 \) be constants enjoying the following scales

\[
\rho_1 = \rho_0(z) + \mathcal{O}(\varepsilon^j), \quad j > 0, \quad \text{and} \quad k_1 = \kappa_1 \varepsilon^2 \quad \text{as} \quad \varepsilon << 1.
\]

In this regime, the solution of the problem (1), has the following expansions.

1. The scattered field has the approximation

\[
u^s(x, \theta, \omega) = \nu^s(x, \theta, \omega) - \frac{1}{k_1} \frac{\omega^2 \omega_0^2}{(\omega^2 - \omega_0^2)} \left( \int_B e_{\lambda_{\omega_0}}^B \right)^2 \varepsilon G_\omega(x; z) \nu(z, \theta, \omega)
\]

\[+ \mathcal{O} \left( \varepsilon + \frac{\varepsilon^{1+\min\{1;j\}}}{(\omega^2 - \omega_0^2)^2} \right),
\]

uniformly for \( x \) in a bounded domain away from \( D \) and \( \theta \) in the unit sphere.

2. The farfield has the approximation

\[
u^\infty(x, \theta, \omega) = \nu^\infty(x, \theta, \omega) - \frac{\rho_0}{4 \pi k_1} \frac{\omega^2 \omega_0^2}{(\omega^2 - \omega_0^2)} \left( \int_B e_{\lambda_{\omega_0}}^B \right)^2 \varepsilon \nu(z, -\hat{x}, \omega) \nu(z, \theta, \omega)
\]

\[+ \mathcal{O} \left( \varepsilon + \frac{\varepsilon^{1+\min\{1;j\}}}{(\omega^2 - \omega_0^2)^2} \right),
\]

uniformly for \( \theta \) and \( \hat{x} \) in the unit sphere.

These expansions are valid as soon as \( \frac{\varepsilon^h}{\omega^2 - \omega_0^2} = \mathcal{O}(1) \), with \( h < \min\{1, j\} \), as \( \varepsilon << 1 \), where

\[
\omega_{\lambda_{\omega_0}} := \sqrt{\frac{k_1}{\rho_0(z) \lambda_{\omega_0}^B}}.
\]

Observe that \( \left( \int_B e_{\lambda_{\omega_0}}^B(x) \, dx \right)^2 \) means \( \sum_l \left( \int_B e_{\lambda_{\omega_0}}^B(x) \, dx \right)^2 \), where \( l \) is such that

\[
N^\omega_B [e_{\lambda_{\omega_0}}^B (\cdot)] = \lambda_{\omega_0}^B e_{\lambda_{\omega_0}}^B (\cdot).
\]

Here again \( \nu := \nu(x, \theta, \omega) \) and \( \nu^\infty := \nu^\infty(\hat{x}, \theta, \omega) \) is the total field, and is the far field associated to the scattered field, solution of the problem (1) in the absence of the bubble \( D \).

The body-wave resonances have been characterized already in [5, 21] in the framework of dielectric nanoparticles, in the scalar model related to the TM regime of the electromagnetic scattering, with a homogeneous background. There, the contrast comes from the dielectric nanoparticles with high permittivity and moderate permeability. In our context, the contrast comes from the fact that the density of the bubble is moderate while its bulk is still small. At the mathematical level, our formulas in (11) extend those in [5] to the case of the acoustic model, i.e. a divergence form model, with heterogeneous background. As we have said above, such bubble’s contrasts might not be available in nature but can be artificially designed, see [23].

We finish this section with the following observations.

\[^3\text{This can be seen in (102).}\]
1. The approximations in Theorem 1.1 are similar to the ones in Theorem 1.2 up to the multiplicative factor appearing in the dominating term. The additional term $O(\varepsilon)$ appearing in the error of the approximations (11) and (12) can be removed as follows:

(a) The scattered fields are approximated as

$$u^s(x, \theta, \omega) = v^s(x, \theta, \omega) + \frac{\omega^2}{k_1} G_\omega(x, z) v(z, \theta, \omega) \int_D W(x) dx + O\left(\frac{\varepsilon^{1+\min(1, j)}}{(\omega^2 - \omega^2_{n_0})^2}\right)$$

(b) The farfields are approximated as

$$u^\infty(\hat{x}, \theta, \omega) = v^\infty(\hat{x}, \theta, \omega) + \frac{\rho_0}{4\pi k_1} \frac{\omega^2}{\omega^2_{n_0}} v(z, -\hat{x}, \omega) \int_D W(x) dx + O\left(\frac{\varepsilon^{1+\min(1, j)}}{(\omega^2 - \omega^2_{n_0})^2}\right),$$

where $W := (I - \gamma \omega^2 A_{B}^{0})^{-1}$. The term $O(\varepsilon)$ appearing in (11) and (12), is due to the fact that, see (102),

$$\int_D W(x) dx = -\omega^2_{n_0} \frac{\int_D e_{n_0}(x) dx}{(\omega^2 - \omega^2_{n_0})} + O(\varepsilon^3).$$

2. In both Theorem 1.1 and Theorem 1.2, the error terms contain the term $\varepsilon^2 (\omega^2 - \omega^2_{n_0})^2$ or $\varepsilon^2 (\omega^2 - \omega^2_{n_0})^2$, respectively. This is not optimal and we believe it can improved to allow us to take $\omega^2 - \omega^2_{n_0}$ of the order of $\varepsilon$ or less. This can be done at the expense of improving the dominating term of the corresponding approximation. This can be seen from the proofs when the background is homogeneous. In the inhomogeneous background case, its justification makes the computations rather more involved and we prefer to skip this and stick to the results as stated above for clarity.

3. Finally, we do believe that the condition $\max_{x \in \Omega} \rho_0(x) < \rho_1$ used in Theorem 1.1 and the condition (10) appearing in Theorem 1.2 might be removed.

2. An application to the acoustic imaging using resonating bubbles. Based on the expansions given in Theorem 1.1 and Theorem 1.2, in particular (8) and (12), we design the following imaging procedure to reconstruct the mass density $\rho_0$ and bulk modulus $k_0$ inside the bounded domain $\Omega$ where they are variable. This procedure is based on the following measured data.

Let $[\omega_{\min}, \omega_{\max}]$ be an interval of possible incident frequencies under the following conditions

$$\omega_{\min} \leq \sqrt{\frac{2\bar{k}_1}{\max_{z \in \Omega} \rho_0(z) \mu_{\partial B}}} \leq \sqrt{\frac{2\bar{k}_1}{\min_{z \in \Omega} \rho_0(z) \mu_{\partial B}}} \leq \omega_{\max}.$$ 

This condition makes sense as soon as we know a priori a lower bound and an upper bound of the unknown mass density $\rho_0$. Inverse Problems and Imaging Volume 15, No. 3 (2021), 555–597
1. Collect the farfields before injecting the bubble \( D \), i.e. measure the backscattered farfield at a single incident wave \( \theta \) and a band of frequencies \( \omega \in [\omega_{\text{min}}, \omega_{\text{max}}] : v^\infty(\theta, \theta, \omega) \).

2. Collect the farfield after injecting the bubble \( D \), centered at the point \( z \in \Omega \), i.e. measure the backscattered farfield at a single incident wave \( \theta \) and a band of frequencies \( \omega \in [\omega_{\text{min}}, \omega_{\text{max}}] : u^\infty(\theta, \theta, \omega, z) \).

The imaging procedure goes as follows. We set

\[ I(\omega, z) := u^\infty(\theta, \theta, \omega, z) - v^\infty(\theta, \theta, \omega) \]

as the imaging functional, remembering that the incident angle \( \theta \) is fixed. We have the following properties from (8)

\[ I(\omega, z) \sim -\rho_0 \frac{\omega_0^2}{4 \pi k_1 (\omega^2 - \omega_0^2(z)) |B|} \in [v(z, \theta, \omega)]^2. \]

We divide this procedure into two steps:

1. Step 1. From this expansion, we recover \( \omega_0^2(z) \) as the frequency for which the imaging function \( \omega \to I(\omega, z) \) gets its largest value. From the estimation of this resonance \( \omega_0^2(z) \), we reconstruct the mass density at the center of the injected bubble \( z \), based on (9), as follows:

\[ \rho_0(z) = \frac{2k_1}{\omega_0^2(z) \mu_B}. \]

Scanning the domain \( \Omega \) by such bubbles, we can estimate the mass density there.

2. Step 2. To estimate now the bulk modulus, we go back to (15) or (8), and derive the values of the total field \( [v(z, \theta, \omega)]^2 \). This field corresponds to the model without the bubble. Hence, we have at hand \( v(z, \theta, \omega) \) for \( z \in \Omega \) up to a sign (i.e. we know the modulus and the phase up to a multiple of \( \pi \)).

Use the equation \( \nabla \cdot \rho_0^{-1} \nabla v + \omega_0^2 k_0^{-1} v = 0 \) to recover the values of \( k_0 \) in the regions where \( v \) does not change sign. This can be done by numerical differentiation for instance. Other ways are of course possible to achieve this second step. In addition, we have at hand multiple frequency internal data.

The procedure described above uses the Minnaert resonance. The key point to recover the mass density is the explicit dependance of this resonance on the value of the mass density on it’s ‘center’, see (9). We can do the same work using the sequence of resonances coming from the second regime, see (13). Therefore, we may recover \( \rho_0 \) and \( k_0 \) for this regime as well. However, for technical reasons, we need to know the mass density as we use the condition (10). But as we have said earlier, we believe that this condition might be removed.

3. **Proof of Theorem 1.1.** We divide the proof into two steps. In the first step, we provide the expansions in the case when the background is homogeneous. This allows to show the key parts in localizing the resonance and computing the scattered fields from incident frequencies close to these resonances. In the second step, we deal with the case when the background is heterogeneous and show how this perturbation influences the derivation of the expansions and the resonances as well.
3.1. **Constant coefficients.** We assume here that both \( \rho_0 \) and \( k_0 \) are constants everywhere in \( \mathbb{R}^3 \). We recall that \( \rho_1 = \tilde{\rho}_1 \epsilon^2 \), \( k_1 = \tilde{k}_1 \epsilon^2 \), where \( \tilde{\rho}_1 \), \( \tilde{k}_1 \) do not depend on \( \epsilon \). In this case, it is immediate to show that

\[
G_\omega(x, y) = \rho_0 \frac{e^{\kappa_0 |x-y|}}{4\pi |x-y|}, \quad x \neq y, \quad \text{where} \quad \kappa_0 := \omega \sqrt{\rho_0 / k_0}.
\]

Let \( u \) be the solution of (1). From the Lippman-Schwinger representation we have

\[
\tag{17}
\int D G_\omega(x, y) \nabla u(y) dy - \beta \omega^2 \int D G_\omega(x, y) u(y) dy = u'(x),
\]

where \( \alpha := 1/\rho_1 - 1/\rho_0 \) and \( \beta := 1/k_1 - 1/k_0 \).

Since \( \nabla G_\omega(x, y) = -\nabla G_\omega(x, y) \), by integration by parts and (6) we have

\[
\nabla \cdot \int D G_\omega(x, y) \nabla u(y) dy = -\int D G_\omega(x, y) u(y) dy - \int_{\partial D} G_\omega(x, y) \partial_\nu u(y) d\sigma(y),
\]

so (17) becomes

\[
\tag{18}
\int D G_\omega(x, y) u(y) dy + \alpha \int_{\partial D} G_\omega(x, y) \partial_\nu u(y) d\sigma(y) = u'(x),
\]

where \( \gamma = \beta - \alpha \rho_1 / k_1 \). Taking the normal derivative as \( x \to \partial D \) from inside \( D \), from the jump relations of the derivative of the single layer potential we obtain

\[
\left( 1 + \frac{\alpha \rho_0}{2} \right) \partial_\nu u(x) - \gamma \omega^2 \partial_\nu - \int D G_\omega(x, y) u(y) dy + \alpha (K_D^\gamma)^* [\partial_\nu u](x) = \partial_\nu u'(x).
\]

Notice that due to the scaling of \( \rho_1 \) and \( k_1 \), we have \( \gamma = \mathcal{O}(1) \) as \( \epsilon \to 0 \). Expanding in \( \epsilon \) the fundamental solution, we obtain for \( x \) away from \( D \),

\[
\int D G_\omega(x, y) u(y) dy = G_\omega(x, z) \int D u(y) dy + \mathcal{O} \left( \epsilon^2 \| u \|_{L^2(D)} \right),
\]

as by the Cauchy-Schwartz inequality and the fact that \( |y - z| = \mathcal{O}(\epsilon) \) we have

\[
\left| \int D (y - z) u(y) dy \right| \leq \| y - z \|_{L^2(D)} \| u \|_{L^2(D)} = \mathcal{O} \left( \epsilon^2 \| u \|_{L^2(D)} \right).
\]

In the same way, we have

\[
\int_{\partial D} |y - z| \partial_\nu u(y) d\sigma(y) \lesssim \epsilon^2 \| \partial_\nu u \|_{L^2(\partial D)},
\]

so that

\[
\int_{\partial D} G_\omega(x, y) \partial_\nu u(y) d\sigma(y) = G_\omega(x, z) \int_{\partial D} \partial_\nu u(y) d\sigma(y) + \mathcal{O} \left( \epsilon^2 \| \partial_\nu u \|_{L^2(\partial D)} \right)
\]

Therefore, we can rewrite (18) as

\[
u^*(x) = \gamma \omega^2 G_\omega(x, z) \int D u(y) dy - \alpha G_\omega(x, z) \int_{\partial D} \partial_\nu u(y) d\sigma(y) + \mathcal{O} \left( \epsilon^2 \| u \|_{L^2(D)} + \| \partial_\nu u \|_{L^2(\partial D)} \right).
\]

From the equation satisfied by \( u \), see for instance (1), and the divergence theorem, we have

\[
\int D \nabla \cdot \left( \frac{1}{\rho_1} \nabla u \right)(y) dy = -\frac{k_1}{\omega^2 \rho_1} \int_{\partial D} \partial_\nu u(y) d\sigma(y),
\]

\[
\int D u(y) dy = \frac{k_1}{\omega^2} \int D \nabla \cdot \left( \frac{1}{\rho_1} \nabla u \right)(y) dy = \frac{k_1}{\omega^2 \rho_1} \int_{\partial D} \partial_\nu u(y) d\sigma(y).
\]
then
\begin{equation}
(21) \quad u^\varepsilon(x) = -\left(\alpha + \frac{\gamma k_1}{\rho_1}\right) G_\omega(x, z) \int_{\partial D} \partial_\nu u(y) d\sigma(y) + O\left(\varepsilon^2 \|u\|_{L^2(D)} + \|\partial_\nu u\|_{L^2(\partial D)}\right).
\end{equation}

Now, we derive the dominating term of \( \int_{\partial D} \partial_\nu u d\sigma \) and estimate \( \|u\|_{L^2(D)} \) and \( \|\partial_\nu u\|_{L^2(\partial D)} \) in terms of \( \varepsilon \). Let us consider first the case when \( \gamma = 0 \). In this case, the equation \( (19) \) becomes
\begin{equation}
(22) \quad \left(\frac{1}{\alpha} + \frac{\rho_0}{2}\right) I + (K_D^0)^* \left[ \partial_\nu u \right] = \alpha^{-1} \partial_\nu u^i,
\end{equation}
and we can rewrite it as
\begin{equation}
(23) \quad \left(\frac{1}{\alpha} + \frac{\rho_0}{2}\right) I + (K_D^0)^* \left[ \partial_\nu u \right] + \left((K_D^0)^* - (K_D^0)^*\right) \left[ \partial_\nu u \right] = \alpha^{-1} \partial_\nu u^i.
\end{equation}

Let
\begin{align}
A_{\partial D} := \frac{\rho_0^2}{k_0} \mu_{\partial D}, \quad \mu_{\partial D} := \frac{1}{|\partial D|} \int_{\partial D} \int_{\partial D} \frac{(x - y) \cdot \nu(x)}{4\pi |x - y|} d\sigma(x) d\sigma(y),
\end{align}
and
\begin{align}
A(y) := \frac{\rho_0^2}{k_0} \int_{\partial D} \frac{(x - y) \cdot \nu(x)}{4\pi |x - y|} d\sigma(x).
\end{align}

By the divergence theorem we have \( A_{\partial D} > 0 \), and it is immediate that \( A - A_{\partial D} \) has average zero along \( \partial D \). Expanding \( G_\omega(x, y) \) in terms of \( |x - y| \), we obtain
\begin{align}
(K_D^0)^* \left[ \partial_\nu u \right] &:= \int_{\partial D} \frac{\partial G_\omega(x, y)}{\partial \nu(x)} \partial_\nu u(y) d\sigma(y) \\
&= -\int_{\partial D} \frac{\rho_0}{4\pi} \frac{(x - y) \cdot \nu(x)}{|x - y|^3} \sum_{n=0}^{\infty} \frac{(i\kappa_0 |x - y|)^n}{n!} \partial_\nu u(y) d\sigma(y) \\
&\quad + \int_{\partial D} \frac{\rho_0}{4\pi} \frac{(x - y) \cdot \nu(x)}{|x - y|^3} \sum_{n=0}^{\infty} \frac{(i\kappa_0 |x - y|)^{n+1}}{n!} \partial_\nu u(y) d\sigma(y) \\
&= (K_D^0)^* \left[ \partial_\nu u \right] - \frac{\kappa_0^2 \rho_0}{8\pi} \int_{\partial D} \frac{(x - y) \cdot \nu(x)}{|x - y|} \partial_\nu u(y) d\sigma(y) \\
&\quad - \frac{i\kappa_0^3 \rho_0}{12\pi} \int_{\partial D} (x - y) \cdot \nu(x) \partial_\nu u(y) d\sigma(y) \\
&\quad + O(\varepsilon^3 \|\partial_\nu u\|_{L^2(\partial D)}),
\end{align}
and integrating \( (23) \) on \( \partial D \), as \( K_D^0[1] = -\rho_0/2 \), see Appendix 109, we obtain
\begin{align}
\left(\frac{1}{\alpha} - \frac{\kappa_0^2 k_0}{2 \rho_0} A_{\partial D}\right) \int_{\partial D} \partial_\nu u(x) d\sigma(x) &= \frac{1}{\alpha} \int_{\partial D} \partial_\nu u^i(x) dx \\
&\quad + \frac{i\kappa_0^3 \rho_0}{12\pi} \int_{\partial D} \int_{\partial D} (x - y) \cdot \nu(x) \partial_\nu u(y) dy dx \\
&\quad + \frac{\kappa_0^2 k_0}{2 \rho_0} \int_{\partial D} (A(y) - A_{\partial D}) \partial_\nu u(y) dy \\
&\quad + O(\varepsilon^3 \|\partial_\nu u\|_{L^2(\partial D)}).
\end{align}
We can estimate the integral which contains $A(\cdot) - A_{\partial D}$ by rewriting
\[ \int_{\partial D} (A(y) - A_{\partial D}) \partial_\nu u(y) d\sigma(y) = \alpha^{-1} \int_{\partial D} A(y) - A_{\partial D} \left( (\rho_0/2 + 1/\alpha + (K_\omega^*)^{-1}[\partial_\nu u^i])\right) (y) d\sigma(y) \]
\[ \leq \alpha^{-1} \left\| \left( (\rho_0/2 + 1/\alpha + K_\omega^*)^{-1}[A(\cdot) - A_{\partial D}] \right) \partial_\nu u^i \right\|_{L^2(\partial D)} \]
\leq \alpha^{-1} \left\| (\rho_0/2 + 1/\alpha + K_\omega^*)^{-1}[A(\cdot) - A_{\partial D}] \right\|_{L^2(\partial D)} \left\| \partial_\nu u^i \right\|_{L^2(\partial D)}
(27) = \mathcal{O}(\varepsilon^6),

the last equality being a consequence of the fact that $(\rho_0/2 + 1/\alpha + K_\omega^*)^{-1}$ does not scale on $L^2_0(\partial D) := \{ f \in L^2(\partial D) : \int_{\partial D} f d\sigma = 0 \}$, and $A$ and $A_{\partial D}$ scale both as $\varepsilon^2$. Then (26) becomes
\[ \left( 1 - \frac{i\kappa_0^2|D|\rho_0}{4\pi} - \frac{\kappa_0^2 k_0}{2\rho_0} A_{\partial D} \right) \int_{\partial D} \partial_\nu \partial_\omega u \, d\sigma = \frac{1}{\alpha} \int_{\partial D} \partial_\nu u^i \, d\sigma + \mathcal{O}(\varepsilon^6 \left\| \partial_\nu \partial_\omega u \right\|_{L^2(\partial D)} + \varepsilon^6),
where we have used the fact that $\int_{\partial D} (x - y) \cdot \nu(x) d\sigma(x) = \int_D \nabla \cdot [x - y] dx = 3|D|$. Then, multiplying by $\alpha$ (which scales like $\varepsilon^{-2}$), we obtain the expression of the following dominating term of $\int_{\partial D} \partial_\nu u \, d\sigma$,
\[ \left( 1 - \frac{i\kappa_0^2|D|\rho_0}{4\pi} - \frac{\kappa_0^2 k_0}{2\rho_0} A_{\partial D} \right) \int_{\partial D} \partial_\nu u \, d\sigma = \frac{1}{\alpha} \int_{\partial D} \partial_\nu u^i \, d\sigma + \mathcal{O}(\varepsilon^2 \left\| \partial_\nu \partial_\omega u \right\|_{L^2(\partial D)} + \varepsilon^4).

In the general case of $\gamma \neq 0$, instead of identity (22), we have
\[ \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K_\omega^*)^* \right) \partial_\nu |u|(x) - \frac{\omega^2 \gamma}{\alpha} \partial_\nu - \int_D G_\omega(x,y) u(y) dy = \alpha^{-1} \partial_\nu u^i(x).
\]
Integrating on $\partial D$, and integrating by parts the last integral, we obtain
\[ \int_{\partial D} \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K_\omega^*)^* \right) \partial_\nu |u|(x) d\sigma(x) + \frac{\omega^2 \gamma \rho_0}{\alpha} \int_D \int_D G_\omega(x,y) u(y) dy dx + \int D u(x) dx = \alpha^{-1} \int_{\partial D} \partial_\nu u^i(x) d\sigma(x).
\]
Then, with the same estimates as in (26), we obtain
\[ \left( \frac{1}{\alpha} - \frac{i\kappa_0^2|D|\rho_0}{4\pi} - \frac{\kappa_0^2 k_0}{2\rho_0} A_{\partial D} \right) \int_{\partial D} \partial_\nu u(x) d\sigma(x) + \frac{\omega^2 \gamma \rho_0}{\alpha} \int_D u(x) dx
\]
\[ (28) = \alpha^{-1} \int_{\partial D} \partial_\nu u^i(x) d\sigma(x) - \frac{\omega^2 \gamma \kappa_0^2}{\alpha} \int_D \int_D G_\omega(x,y) u(y) dy dx + \text{error},
\]
where
\[ \text{error} := \mathcal{O}(\varepsilon^5 \left\| \partial_\nu u \right\|_{L^2(\partial D)} + \varepsilon^6).
\]
Next, with help of the Cauchy-Schwartz inequality, we estimate the double volume integral as
\[ \left| \frac{\omega^2 \gamma \kappa_0^2}{\alpha} \int_D \int_D G_\omega(x,y) u(y) dy dx \right| \leq \varepsilon \frac{1}{\alpha} \left\| u \right\|_{L^2(D)},
\]
then, the equation (28) takes the following form

\[
\left( \frac{1}{\alpha} - \frac{i k_{0}^{3} |D| \rho_{0}}{4 \pi} - \frac{k_{0}^{2}}{2 \rho_{0}} A_{\beta D} \right) \int_{\partial D} \partial_{\nu} u(x) \, d\sigma(x) + \frac{\omega^{2} \gamma \rho_{0}}{\alpha} \int_{D} u(x) \, dx = \alpha^{-1} \int_{\partial D} \partial_{\nu} u'(x) d\sigma(x) + r,
\]

where

\[
r := \mathcal{O} \left( \varepsilon^{\frac{11}{2}} \|u\|_{L^{2}(\partial D)} + \varepsilon^{5} \|\partial_{\nu} u\|_{L^{2}(\partial D)} + \varepsilon^{6} \right).
\]

We use (20) and the fact that \( \Delta^{i} u = -\kappa_{0}^{2} u^{i} \) to obtain

\[
\left( \frac{1}{\alpha} - \frac{i k_{0}^{3} |D| \rho_{0}}{4 \pi} - \frac{\omega^{2}}{2} A_{\beta D} - \frac{\gamma k_{1} \rho_{0}}{\alpha \rho_{1}} \right) \int_{\partial D} \partial_{\nu} u = -\frac{\omega^{2} \rho_{0}}{\alpha k_{0}} \int_{D} u + r
\]

\[
(30)
\]

From the definition of \( \kappa_{0} \), recall that \( \kappa_{0} := \omega \sqrt{\rho_{0}/k_{0}} \), we can see that the term between parenthesis on the left hand side of the previous equation is cubic polynomial function on \( \omega \). Now, we define the Minnaert frequency \( \omega_{M} \) to be the dominating part of the zero of this cubic polynomial function. To find the dominant part of the zero is equivalent to solve

\[
\omega^{2} = \frac{2}{A_{\beta D}} \left( \frac{1}{\alpha} - \frac{i k_{0}^{3} |D| \rho_{0}}{4 \pi} - \frac{\gamma k_{1} \rho_{0}}{\alpha \rho_{1}} \right) = \frac{2}{A_{\beta D}} \left( \frac{1}{\alpha} \frac{k_{1} \rho_{0}}{\rho_{1}} \right) + \mathcal{O} (\varepsilon) = \frac{2 \bar{K}_{1}}{\mu_{\beta B} \rho_{0}} + \mathcal{O} (\varepsilon),
\]

where the last two equalities are established using the definition of \( \alpha, \beta, \gamma \) and the scales of \( |D| \) and \( A_{\beta D} \). Setting,

\[
\omega_{M}^{2} := \frac{2 \bar{K}_{1}}{\mu_{\beta B} \rho_{0}},
\]

we have

\[
(32) \quad \left( \frac{1}{\alpha} - \frac{i k_{0}^{3} |D| \rho_{0}}{4 \pi} - \frac{\omega^{2}}{2} A_{\beta D} - \frac{\gamma k_{1} \rho_{0}}{\alpha \rho_{1}} \right) = \frac{\varepsilon^{2} \rho_{0} \bar{K}_{1}}{k_{0}} \left( \frac{\omega_{M}^{2} - \omega^{2}}{\omega_{M}^{2}} \right) + \mathcal{O} (\varepsilon^{3}).
\]

The equation (30), using (32), takes the following form

\[
\frac{\varepsilon^{2} \rho_{0} \bar{K}_{1}}{k_{0}} \left( \frac{\omega_{M}^{2} - \omega^{2}}{\omega_{M}^{2}} \right) \int_{\partial D} \partial_{\nu} u(x) \, d\sigma(x) + \mathcal{O} (\varepsilon^{3}) \int_{\partial D} \partial_{\nu} u(x) \, d\sigma(x) = \frac{\omega^{2} \rho_{0}}{\alpha k_{0}} |D| u'(z) + r.
\]

By Cauchy-Schwartz inequality we estimate the second term on the left hand side equation as \( \mathcal{O} \left( \varepsilon^{4} \|\partial_{\nu} u\|_{L^{2}(\partial D)} \right) \). Then, we obtain the following formula

\[
\int_{\partial D} \partial_{\nu} u \, d\sigma = \frac{\kappa_{1}^{2} |D| u'(z)}{(\omega^{2} - \omega_{M}^{2})} + \mathcal{O} \left( \frac{\varepsilon^{7} + r + \varepsilon^{4} \|\partial_{\nu} u\|_{L^{2}(\partial D)}}{\varepsilon^{2} (\omega^{2} - \omega_{M}^{2})} \right)
\]

\[
(33) \quad = \frac{\omega_{M}^{2} k_{0}^{2} |D| u'(z)}{k_{0}} + \mathcal{O} \left( \frac{\varepsilon^{4} + \varepsilon^{7} \|u\|_{L^{2}(\partial D)} + \varepsilon^{2} \|\partial_{\nu} u\|_{L^{2}(\partial D)}}{\left( \omega^{2} - \omega_{M}^{2} \right)} \right)
\]

\[
(34) \quad = \mathcal{O} \left( \frac{\varepsilon^{3}}{\omega^{2} - \omega_{M}^{2}} \right) + \mathcal{O} \left( \frac{\varepsilon^{4} + \varepsilon^{7} \|u\|_{L^{2}(\partial D)} + \varepsilon^{2} \|\partial_{\nu} u\|_{L^{2}(\partial D)}}{\left( \omega^{2} - \omega_{M}^{2} \right)} \right).
\]
To estimate the error term, on the last expression, we need the following a priori estimates.

**Proposition 1.** For \( u = u^i + u^s \), solution of (1), it holds

\[
\| \partial_\nu u \|_{L^2(\partial D)} = \mathcal{O} \left( \frac{\varepsilon^2}{\omega^2 - \omega_M^2} \right),
\]

and

\[
\| u \|_{L^2(D)} = \mathcal{O} \left( \frac{\varepsilon^2}{\omega^2 - \omega_M^2} \right),
\]

under the condition that \( \varepsilon / (\omega^2 - \omega_M^2) \) is small enough.

**Proof.** Let us indicate as \( C \) a generic constant independent of \( \varepsilon \). From (3) we have

\[
(I - \gamma \omega^2 N_B^\omega) [u] + \alpha S_D^\omega[\partial_\nu u] = u^i.
\]

Since \( \gamma = \mathcal{O}(1) \) and thus \( \| N_B^\omega \|_C \xrightarrow{\varepsilon \to 0} 0 \), for \( \varepsilon \) small enough we have that \( I - \gamma \omega^2 N_B^\omega \) is invertible, so (37) takes the following form

\[
u = -\alpha(I - \gamma \omega^2 N_B^\omega)^{-1}[S_D^\omega[\partial_\nu u]] + (I - \gamma \omega^2 N_B^\omega)^{-1}[u^i].
\]

Taking the \( L^2(D) \)-norm in both side of the last equation and using the fact that

\[
\left\| (I - \gamma \omega^2 N_B^\omega)^{-1} \right\|_{L(L^2(D))} \leq C,
\]

to obtain

\[
\| u \|_{L^2(D)} \leq \alpha C \| S_D^\omega[\partial_\nu u] \|_{L^2(D)} + C \| u^i \|_{L^2(D)}.
\]

In order to finish the last estimation we need to show precisely how the single layer potential scales. For this, by definition, we have

\[
\left\| S_D^\omega[f] \right\|_{L^2(D)}^2 := \int_D \int_{\partial D} G_\omega(x, y) f(y) dy dx, \quad \forall f \in L^2(\partial D)
\]

\[
= \varepsilon^5 \int_B \int_{\partial B} G_{\varepsilon \omega}(\eta - \xi) \tilde{f}(\xi) d\xi d\eta := \varepsilon^5 \left\| S_B^\omega[f] \right\|_{L^2(B)}^2
\]

and from the continuity of \( S_B^\omega \) from \( L^2(\partial B) \) to \( H^2(B) \) we have that

\[
\left\| S_D^\omega[f] \right\|_{L^2(D)}^2 = \varepsilon^5 \left\| S_B^\omega[f] \right\|_{L^2(B)}^2 \leq \varepsilon^5 C \left\| f \right\|_{L^2(\partial B)}^2 = \varepsilon^3 C \left\| f \right\|_{L^2(\partial D)}^2,
\]

in particular

\[
\left\| S_D^\omega[\partial_\nu u] \right\|_{L^2(D)} \leq \varepsilon^2 C \left\| \partial_\nu u \right\|_{L^2(\partial D)}.
\]

Combining (38) and (40), we obtain

\[
\| u \|_{L^2(D)} \leq \varepsilon^{-\frac{1}{2}} C \| \partial_\nu u \|_{L^2(\partial D)} + C \| u^i \|_{L^2(D)}.
\]

To manage the term \( \| \partial_\nu u \|_{L^2(\partial D)} \) we use the boundary integral equation given by (4), to write on the boundary

\[
\partial_\nu u = \alpha^{-1} \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K_B^\omega)^* \right)^{-1} \left[ \partial_\nu u^i \right] + \omega^2 \gamma \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K_B^\omega)^* \right)^{-1} \left[ \partial_\nu N_B^\omega \right] [u].
\]
Next, for convenience, we set
\[ T := \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K^*_D) \right)^{-1} \]
and we rewrite (42) as
\[ \frac{\partial u}{\partial \nu} = \frac{1}{\alpha} T \left[ \frac{\partial u^i}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial u^i}{\partial \nu} \right] + \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial u^i}{\partial \nu} \frac{1}{\alpha} T \quad [1] \]
\[ + \frac{\omega^2 \gamma}{\alpha} T \left[ \frac{\partial N^*_D[u]}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial N^*_D[u]}{\partial \nu} \right] \]
\[ + \frac{\omega^2 \gamma}{\alpha} \int_{\partial D} \frac{\partial N^*_D[u]}{\partial \nu} T \quad [1]. \]
\[ \text{(43)} \]

Since \(-\frac{\omega}{2}\) is an eigenvalue of \(K^*_D\) with associated eigenspace consisting of constant functions, we have the estimates
\[ \|T\|_{L(L^2(\partial D))} = \left\| \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K^*_D) \right)^{-1} \right\|_{L(L^2(\partial D))} \leq C\alpha, \]
and on the space of functions with zero average we have
\[ \|T\|_{L(L^2(\partial D))} = \left\| \left( \frac{1}{\alpha} + \frac{\rho_0}{2} + (K^*_D) \right)^{-1} \right\|_{L(L^2(\partial D))} \leq C. \]
\[ \text{(44)} \]
\[ \text{(45)} \]

Now, take the \(L^2(\partial D)\)-norm in both side of (43), with the help of (44) and (45) we obtain
\[ \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D)} \leq \alpha^{-1} \left\| \frac{\partial u^i}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial u^i}{\partial \nu} \right\|_{L^2(\partial D)} + \frac{1}{|\partial D|} \left\| \int_{\partial D} \frac{\partial u^i}{\partial \nu} \right\|_{L^2(\partial D)} \]
\[ + \alpha^{-1} \left\| \frac{\partial N^*_D[u]}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial N^*_D[u]}{\partial \nu} \right\|_{L^2(\partial D)} \]
\[ + \frac{1}{|\partial D|} \left\| \int_{\partial D} \frac{\partial N^*_D[u]}{\partial \nu} \right\|_{L^2(\partial D)} \] \]
\[ \text{(46)} \]

Obviously, we have
\[ \left| \int_{\partial D} \frac{\partial u^i}{\partial \nu} \right| = \left| \int_D \Delta u^i \right| = |k_0^2| \left| \int_D u^i \right| = O(\varepsilon^3) \]
and, by the triangular inequality and the smoothness of \(\partial_\nu u^i\), we obtain
\[ \left\| \frac{\partial u^i}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial u^i}{\partial \nu} \right\|_{L^2(\partial D)} \leq \left\| \frac{\partial u^i}{\partial \nu} \right\|_{L^2(\partial D)} = O(\varepsilon). \]
\[ \text{(47)} \]
\[ \text{(48)} \]

We also have, recalling the definition of the Green function,
\[ \int_{\partial D} \frac{\partial N^*_D[u]}{\partial \nu} \,(x) \,dx = -\rho_0 \int_D u(x) \,dx - \omega^2 \rho_0 \int_D \int_D G(x, y) u(y) \,dy \,dx \]
\[ = -\rho_0 \int_D u(x) \,dx + O\left( \varepsilon^2 \|u\|_{L^2(D)} \right) \]
\[ \text{(20)} \]
\[ \frac{k_1 \rho_0}{\omega^2 \rho_1} \int_{\partial D} \partial_\nu u(x) \,d\sigma(x) + O\left( \varepsilon^2 \|u\|_{L^2(D)} \right) \]
\[ = \frac{\rho_0 \omega^2 |D|}{(\omega^2 - \omega_M^2)} u'(x) + O\left( \varepsilon^4 + \varepsilon^2 \|u\|_{L^2(D)} + \varepsilon^2 \|\partial_\nu u\|_{L^2(\partial D)} \right). \]
\[ \text{(19)} \]
Finally, we obtain
\[
\int_{\partial D} \frac{\partial N_D^\alpha [u]}{\partial \nu}(x) dx = \mathcal{O} \left( \frac{\varepsilon^3}{\omega^2 - \omega_M^2} \right) + \mathcal{O} \left( \frac{\varepsilon^2 \| u \|_{L^2(D)} + \varepsilon \| \partial_{\nu} u \|_{L^2(\partial D)} + \varepsilon^4}{(\omega^2 - \omega_M^2)} \right).
\]

Now, we estimate the last term in the right hand side of (46). For this, we simply write
\[
\left\| \frac{\partial N_D^\alpha [u]}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial N_D^\alpha [u]}{\partial \nu} \right\|_{L^2(\partial D)} \leq \left\| \frac{\partial N_{\tilde{D}}^\alpha [\tilde{u}]}{\partial \nu} \right\|_{L^2(\partial D)} + \frac{1}{|\partial D|} \int_{\partial D} \left| \frac{\partial N_{\tilde{D}}^\alpha [\tilde{u}]}{\partial \nu} \right| dx.
\]
and deal only with the first term since the second one is estimated by (49). For this, by definition and scale, we have
\[
\left\| \frac{\partial N_{\tilde{D}}^\alpha [u]}{\partial \nu} \right\|^2_{L^2(\partial D)} \leq \int_{\partial D} \left| \frac{\partial \nu}{\partial \nu} \right|^2 \int_{\tilde{D}} G_\omega(x - y) u(y) dy d\sigma(x) = \varepsilon^4 \int_{\partial B} \left| \frac{\partial \nu}{\partial \nu} \right|^2 \int_{\tilde{B}} G_\omega(\eta - \xi) \tilde{u}(\xi) d\sigma(\eta)
\]
\[
\leq \varepsilon^4 \left\| \frac{\partial N_{\tilde{D}}^\alpha [\tilde{u}]}{\partial \nu} \right\|^2_{L^2(\partial B)}.
\]
From the continuity of \( N_{\tilde{D}}^\alpha : L^2(\tilde{B}) \to H^2(B) \), we deduce that
\[
\left\| \frac{\partial N_D^\alpha [u]}{\partial \nu} \right\|_{L^2(\partial D)} = \varepsilon^3 \left\| \frac{\partial N_{\tilde{D}}^\alpha [\tilde{u}]}{\partial \nu} \right\|_{L^2(\partial B)} \leq \varepsilon^2 C \| \tilde{u} \|_{L^2(B)} = \varepsilon^2 C \| u \|_{L^2(D)},
\]
and plugging (52) in (50) we obtain
\[
\left\| \frac{\partial N_D^\alpha [u]}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial N_D^\alpha [u]}{\partial \nu} \right\|_{L^2(\partial D)} \leq \varepsilon^2 \| u \|_{L^2(D)} + \varepsilon^{-1} \left| \int_{\partial D} \frac{\partial N_D^\alpha [u]}{\partial \nu} \right|.
\]
Then, by (49), we have
\[
\left\| \frac{\partial N_D^\alpha [u]}{\partial \nu} - \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial N_D^\alpha [u]}{\partial \nu} \right\|_{L^2(\partial D)} \leq \varepsilon^2 \| u \|_{L^2(D)} + \mathcal{O} \left( \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)} \right)
\]
\[
+ \mathcal{O} \left( \frac{\varepsilon^2 \| \partial_{\nu} u \|_{L^2(\partial D)} + \varepsilon^2}{(\omega^2 - \omega_M^2)} \right).
\]
Therefore, by (47), (48), (49) and (53), the formula (46) becomes
\[
\| \partial_{\nu} u \|_{L^2(\partial D)} \leq \mathcal{O} \left( \frac{\varepsilon^2 \| u \|_{L^2(D)} + \varepsilon \| \partial_{\nu} u \|_{L^2(\partial D)} + \varepsilon^2}{(\omega^2 - \omega_M^2)} \right)
\]
and if \( \varepsilon/(\omega^2 - \omega_M^2) \) is small enough we obtain
\[
\| \partial_{\nu} u \|_{L^2(\partial D)} \leq \mathcal{O} \left( \frac{\varepsilon^2 \| u \|_{L^2(D)} + \varepsilon^2}{(\omega^2 - \omega_M^2)} \right).
\]
Substituting this estimate for $\|\partial_\nu u\|_{L^2(\partial D)}$ in (41), to obtain
\[
\|u\|_{L^2(D)} \lesssim \varepsilon^{-1/2} \|\partial_\nu u\|_{L^2(\partial D)} + \|u\|_{L^2(D)} = \frac{\varepsilon^2 \|u\|_{L^2(D)}}{(\omega^2 - \omega_M^2)} + \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)} + \varepsilon^2,
\]
again under the condition $\varepsilon / (\omega^2 - \omega_M^2)$ small enough, we obtain
\[
\|u\|_{L^2(D)} \lesssim \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)}.
\]
This justify (36). Now, use (36) into (55) to get (35).

Recall (33) and rewrite it, using the a priori estimate given by (36) and (35), as
\[
\int_{\partial D} \partial_\nu u(x) \, d\sigma(x) = \frac{\kappa_1^2 \omega_M^2}{(\omega^2 - \omega_M^2)} |D| u^i(z) + O \left( \frac{\varepsilon^4}{(\omega^2 - \omega_M^2)^2} \right).
\]
The formula (21) together with the a priori estimate given in Proposition 1 allow to write
\[
u^a(x) = -\left( \alpha + \frac{\gamma k_1}{\rho_1} \right) G_\omega(x, z) \int_{\partial D} \frac{\partial u}{\partial \nu}(y) \, d\sigma(y) + O \left( \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)} \right)
\]
\[
\overset{(56)}{=} -\left( \alpha + \frac{\gamma k_1}{\rho_1} \right) G_\omega(x, z) \left[ \frac{\kappa_1^2 \omega_M^2}{(\omega^2 - \omega_M^2)} |D| u^i(z) + O \left( \frac{\varepsilon^4}{(\omega^2 - \omega_M^2)^2} \right) \right]
\]
\[+ O \left( \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)} \right)
\]
and the fact that $\alpha + \gamma k_1 / \rho_1 = \rho_1^{-1} + O(1)$, and $\rho_1 = \rho_1 \varepsilon^2$, we rewrite the last formula as
\[
u^a(x) = -\left( \frac{1}{\rho_1} + O(1) \right) G_\omega(x, z) \left[ \frac{\kappa_1^2 \omega_M^2}{(\omega^2 - \omega_M^2)} |D| u^i(z) + O \left( \frac{\varepsilon^4}{(\omega^2 - \omega_M^2)^2} \right) \right]
\]
\[+ O \left( \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)} \right)
\]
\[
\overset{(57)}{=} -\frac{\omega^2 \omega_M^2}{\kappa_1 (\omega^2 - \omega_M^2)} |B| \varepsilon G_\omega(x, z) u^i(z) + O \left( \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)^2} \right),
\]
for $x$ away from $D$ and $\varepsilon / (\omega^2 - \omega_M^2)$ small enough.

**Remark 1.** Recall that, in the constant coefficients case, we have $v(\cdot, \theta, \omega) = u^i(\cdot, \theta, \omega)$, and $v^a = 0$ as there is no scattering. Hence the equation (7) reduces to (57).

Now that (7) is proved, we deduce the corresponding far field
\[
u^\infty(\hat{x}, \theta, \omega) = -\frac{\omega^2 \omega_M^2}{\kappa_1 (\omega^2 - \omega_M^2)} |B| \varepsilon G_\omega^\infty(\hat{x}, z) v(z, \theta, \omega) + O \left( \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)^2} \right).
\]
and using the mixed reciprocity relation \( G^\infty_\omega (\hat{x}, z) = \frac{\tau_0}{4\pi} v(z, -\hat{x}, \omega), \) see (115) in Appendix 5.3, we obtain

\[
\begin{align*}
  u^\infty (\hat{x}, \theta, \omega) &= v^\infty (\hat{x}, \theta, \omega) - \frac{\tau_0}{4\pi k_1} \frac{\omega^2 \omega_M^2}{(\omega^2 - \omega_M^2) B} \varepsilon (z, -\hat{x}, \omega) v(z, \theta, \omega) \\
  &+ \mathcal{O} \left( \frac{\varepsilon^2}{(\omega^2 - \omega_M^2)^2} \right).
\end{align*}
\]

This proves (8) and we finish the proof of Theorem 1.1.

3.2. Variable coefficients. Let us suppose now that the coefficients \( \rho_0, k_0 \) vary smoothly depending on the position while in a bounded domain \( \Omega \), and that they are constant outside \( \Omega \). We warn the reader that we keep the same notations as in the case of constant coefficients and we shall denote by \( G_\omega \) the fundamental solution satisfying (6) with these variable coefficients. In this case, the Lippmann-Schwinger equation writes as

\[
\begin{align*}
  u(x) &= -\nabla \cdot \left[ \int_D \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) G_\omega(x, y) \nabla u(y) dy \right] \\
  &- \omega^2 \int_D \left( \frac{1}{k_1} - \frac{1}{k_0(y)} \right) G_\omega(x, y) u(y) dy = v(x).
\end{align*}
\]

We denote by

\[
\begin{align*}
  I := \nabla \cdot \left[ \int_D \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) G_\omega(x - y) \nabla u(y) dy \right] \\
  = -\int_D \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) \nabla G_\omega(x, y) \cdot \nabla u(y) dy
\end{align*}
\]

moreover we can write it, using integration by parts identities, as

\[
\begin{align*}
  I &= \int_D G_\omega(x, y) \nabla \cdot \left[ \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) \nabla u(y) \right] dy \\
  &- \int_{\partial D} G_\omega(x, y) \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) \partial_n u(y) dy \\
  &= \frac{1}{\rho_1} \int_D G_\omega(x, y) \Delta u(y) dy - \int_D G_\omega(x, y) \nabla \cdot \left[ \frac{1}{\rho_0(y)} \nabla u(y) \right] dy \\
  &- \int_{\partial D} G_\omega(x, y) \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) \partial_n u(y) dy.
\end{align*}
\]

Recall that, on \( D \), we have \( \Delta u + \kappa^2 u = 0 \) and use this to write the previous equation as

\[
\begin{align*}
  I &= \omega^2 \int_D \left( \frac{\rho_1}{k_1(\rho_0(y))} - \frac{1}{k_1} \right) G_\omega(x, y) u(y) dy - \int_D G_\omega(x, y) \nabla \frac{1}{\rho_0(y)} \cdot \nabla u(y) dy \\
  &- \int_{\partial D} G_\omega(x, y) \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) \partial_n u(y) dy.
\end{align*}
\]
Plugging the new expression of $I$ onto the Lippmann-Schwinger equation (58), we obtain

$$u(x) = -\omega^2 \int_D \gamma(y) G_\omega(x, y) u(y) dy + \int_D G_\omega(x, y) \nabla \frac{1}{\rho_0(y)} \cdot \nabla u(y) dy$$

$$+ \int_{\partial D} G_\omega(x, y) \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) \partial_\nu u(y) dy = v(x),$$

(59)

where

$$\gamma(y) := \frac{-1}{k_0(y)} + \frac{\rho_1}{k_1 \rho_0(y)}.$$

By taking the normal derivative from inside we deduce the corresponding integral equation on the boundary. More precisely, for $x \in \partial D$, we have

$$\left[ 1 + \frac{\rho_0(x)}{2} \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(x)} \right) \right] \partial_\nu u(x) - \omega^2 \partial_\nu \int_D \gamma(y) G_\omega(x, y) u(y) dy$$

$$+ \partial_\nu \int_D G_\omega(x, y) \nabla \frac{1}{\rho_0(y)} \cdot \nabla u(y) dy$$

$$+ (K_\omega^*)^\dag \left[ \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(z)} \right) \partial_\nu u(z) \right](x)$$

$$= \partial_\nu v(x).$$

(60)

We use the Lippmann-Schwinger equation to derive an expression for the scattered field. To do this, for $x$ away from $D$ and $y$ such that $|y - z| \sim \varepsilon$, we expand near $z$ the equation (59) to obtain

$$u^s(x) = u^s(x) + \omega^2 G_\omega(x, z) \gamma(z) \int_D u(y) dy - G_\omega(x, z) \int_D \nabla \frac{1}{\rho_0(y)} \cdot \nabla u(y) dy$$

$$- G_\omega(x, z) \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(z)} \right) \int_{\partial D} \partial_\nu u(y) d\sigma(y)$$

$$+ O \left( \varepsilon^2 \|u\|_{H^1(D)} + \|\partial_\nu u\|_{L^2(\partial D)} \right).$$

(61)

We know that

$$\int_D \nabla \frac{1}{\rho_0} \cdot \nabla u dy = -\int_D \frac{1}{\rho_0} \Delta u dy + \int_{\partial D} \frac{1}{\rho_0} \partial_\nu u dy$$

$$= -\omega^2 \rho_1 \int_D \frac{1}{\rho_0} u dy + \int_{\partial D} \frac{1}{\rho_0} \partial_\nu u dy$$

$$= \omega^2 \rho_1 \int_D u dy + \frac{\omega^2 \rho_1}{k_1} \nabla \frac{1}{\rho_0(z)} \cdot \int_D (y - z) u(y) dy$$

$$+ \frac{1}{\rho_0(z)} \int_{\partial D} \partial_\nu u dy + O \left( \varepsilon^2 \|u\|_{L^2(D)} + \varepsilon^2 \|\partial_\nu u\|_{L^2(\partial D)} \right).$$

(62)
Lemma 3.1. Set \( (65) \)

Proposition 2. For \( u^t \) appearing above.

In the next proposition, similarly to Proposition 1, we estimate the error terms \( (64) \) and plug these estimates in \( (62) \) to obtain

\[
\int_D \frac{\partial u}{\partial \nu} \cdot \nabla u \, dy = \frac{\omega^2}{\rho_0} \int_D u \, dy + \frac{1}{\rho_0} \int_{\partial D} \partial u \, d\sigma(y) + O \left( \varepsilon^2 \|u\|_{H^1(D)} + \|\partial u\|_{L^2(\partial D)} + \varepsilon^4 \right)
\]

(63)

Then, the equation \( (61) \) takes the following form

\[
u^s(x) - v^s(x) = -G_\omega(x, z) \left[ -\omega^2 \gamma(z) \int_D u \, dy + \frac{1}{\rho_1} \int_{\partial D} \frac{\partial u}{\partial \nu} \, d\sigma \right]
\]

\[
+ O \left( \varepsilon^2 \|u\|_{H^1(D)} + \|\partial u\|_{L^2(\partial D)} + \varepsilon^4 \right) \quad \text{to} \quad (20) \]

\[
- \frac{1}{\rho_1} G_\omega(x, z) \int_{\partial D} \partial u \, d\sigma(y)
\]

(64)

In the next proposition, similarly to Proposition 1, we estimate the error terms appearing above.

**Proposition 2.** For \( u = u^t + u^s \), the solution of \( (1) \), it holds

\[
\|\partial u\|_{L^2(\partial D)} = O \left( \frac{\varepsilon^2}{\omega^2 - \omega_M^2} \right)
\]

and

\[
\|u\|_{H^1(D)} = O \left( \frac{\varepsilon^{1/2}}{\omega^2 - \omega_M^2} \right)
\]

under the condition that \( \varepsilon \omega^2 / \omega_M^2 \) is small enough.

The next two lemmas are needed to prove the previous proposition.

**Lemma 3.1.** Set \( J_D^w \) the operator defined as

\[
J_D^w := L^2(\partial D) \to L^2(\partial D)
\]

\[
f \to J_D^w[f](x) := \int_{\partial D} \rho_0^{-1}(y) \frac{\partial G_\omega}{\partial \nu}(y, x) f(y) \, d\sigma(y)
\]

then \( (J_D^w)^* [f](x) = \rho_0^{-1}(x) (K_D^w)^* (f)(x) \), \( J_D^0[1] = -1/2 \) and in addition we have

\[
(J_D^w - J_D^0) [1](x) = -\kappa_0 \frac{1}{2} \int_{\partial D} \frac{w(y) \cdot (y - x)}{4 \pi |x - y|} \, d\sigma(y) + O \left( \varepsilon^3 \right)
\]

(67)
Proof. See the appendix.

We need also the following result.

**Lemma 3.2.** Set $B_D$ the operator defined from $L^2(\partial D)$ to $L^2(\partial D)$ as

$$B_D[f](x) := \left[ (\rho_0^{-1}(x)\alpha^{-1}(z) + \frac{1}{2}) I + (J_D^\omega)^* \right] f(x)$$

then $B_D$ is invertible and in addition we have

$$\|B_D^{-1}\|_{\mathcal{L}(L^2(\partial D))} \lesssim \alpha(z) \quad \text{and} \quad \|B_D^{-1}\|_{\mathcal{L}(L^2_0(\partial D))} = \mathcal{O}(1), \quad \text{as } \varepsilon << 1$$

if $\bar{p}_1$ is such that $\rho_0(z) < \bar{p}_1$.

Proof. See the appendix.

Now, we are ready to prove Proposition 2.

**Proof.** Let the two volumetric operators $N_D^\omega : L^2(D) \to L^2(D)$ and $M_D^\omega : (L^2(D))^3 \to L^2(D)$ defined by

$$N_D^\omega[\varphi](x) := \int_D G_\omega(x-y)\gamma(y)\varphi(y)dy \quad \text{and} \quad M_D^\omega[F](x) := \int_D G_\omega(x-y)\nabla_\rho \frac{1}{\rho_0}(y) \cdot F(y)dy.$$ 

Notice that, for $\varepsilon$ small, the operator norms of $N_D^\omega$ and $M_D^\omega$ and their derivatives all go to zero, thus for any fixed coefficient $\lambda$, the operators $I + \lambda N_D^\omega$, $I + \lambda M_D^\omega$, $I + \lambda \nabla N_D^\omega$, and $I + \lambda \nabla M_D^\omega$ are invertible, and their inverse have norm bounded by a constant $C$ independent from $\varepsilon$. Also, we define a boundary integral operators $S_D^\omega$ from $L^2(\partial D)$ to $L^2(D)$ as

$$S_D^\omega[\varphi](x) := \int_{\partial D} G_\omega(x,y) \left( \frac{1}{\rho_1} - \frac{1}{\rho_0(y)} \right) \varphi(y) d\sigma(y).$$

Next, we estimate $u$ with $H^1(D)$-norm. First, we use (59) to write the integral equation satisfied by $u$ as follows

$$u = (I - \omega^2 N_D^\omega)^{-1} [-M_D^\omega[\nabla u] - S_D^\omega[\partial_\nu u] + v]$$

and taking the gradient of (59), we get

$$\nabla u = (I + \nabla M_D^\omega)^{-1} [\omega^2 \nabla N_D^\omega[u] - \nabla S_D^\omega[\partial_\nu u] + \nabla v].$$

We take the $L^2(D)$-norm to obtain

$$\|\nabla u\|_{L^2(D)} = \left\| (I + \nabla M_D^\omega)^{-1} [\omega^2 \nabla N_D^\omega[u] - \nabla S_D^\omega[\partial_\nu u] + \nabla v] \right\|_{L^2(D)}$$

$$\leq C \left[ \omega^2 \|\nabla N_D^\omega[u]\|_{L^2(D)} + \|\nabla S_D^\omega[\partial_\nu u]\|_{L^2(D)} + \|\nabla v\|_{L^2(D)} \right]$$

$$\leq C \left( \varepsilon \|u\|_{L^2(D)} + \varepsilon^{-\frac{1}{2}} \|\partial_\nu u\|_{L^2(\partial D)} + \|\nabla v\|_{L^2(D)} \right),$$

where we used the change of variable techniques as in (51) and (39) and remarking that the leading term\(^4\) of $S_D^\omega[\varphi](\cdot)$ is given by $\rho_1^{-1} \int_{\partial D} G_\omega(\cdot - y)\varphi(y)d\sigma(y)$.

---

\(^4\)To exhibit the leading term of $S_D^\omega$ we should take into account the fact that $\rho_1$ (respectively, $\rho_0$) are constant function (respectively, smooth function) on the spatial variable and behaving as $\varepsilon^{-2}$ (respectively, independent on $\varepsilon$).
From (69), we also have
\[
\|u\|_{L^2(D)} \leq \|(I - \omega^2 N_D^\omega)^{-1}\|_E \left(\|M_D^\omega [\nabla u]\|_{L^2(D)} + \|S_D^\omega [\partial_D u]\|_{L^2(D)} + \|v\|_{L^2(D)}\right)
\]
\[
\leq C \left(\varepsilon^2 \|\nabla u\|_{L^2(D)} + \varepsilon^{-\frac{1}{2}} \|\partial_D u\|_{L^2(\partial D)} + \|v\|_{L^2(D)}\right).
\]
Substituting (70), this becomes
\[
\|u\|_{L^2(D)} \leq C \left(\varepsilon^{-\frac{1}{2}} \|\partial_D u\|_{L^2(\partial D)} + \varepsilon^2 \|\nabla v\|_{L^2(D)} + \|v\|_{L^2(D)}\right)
\]
\[
= C \varepsilon^{-\frac{1}{2}} \|\partial_D u\|_{L^2(\partial D)} + O \left(\varepsilon^\frac{1}{2}\right).
\]
Putting together (70) and (71), we obtain
\[
\|u\|_{L^2(D)} \leq C \left(\varepsilon^{-3} \|\partial_D u\|_{L^2(\partial D)} + \varepsilon^3\right).
\]
To estimate \(\|\partial_D u\|_{L^2(\partial D)}\), we rewrite (60), where we denote by \(\alpha(z) := \rho_0^{-1} - \rho_0^{-1}(z)\), as follows
\[
\left[\left(\rho_0^{-1}(x) \alpha^{-1}(z) + \frac{1}{2}\right) I + \rho_0^{-1}(x) (K_D^\omega)^*\right] [\partial_D u](x) =
\]
\[+ \rho_0^{-1}(x) \alpha^{-1}(z) \partial_D v(x) + \omega^2 \rho_0^{-1}(x) \alpha^{-1}(z) \partial_D N_D^\omega[u](x)
\]
\[+ \rho_0^{-1}(x) \alpha^{-1}(z) \partial_D M_D^\omega [\nabla u](x)
\]
\[+ \alpha^{-1}(z) \int_0^1 (x - z) \cdot \nabla \frac{1}{\rho_0} (z + t(x - z)) \, dt \partial_D u(x)
\]
\[+ \rho_0^{-1}(x) \alpha^{-1}(z) (K_D^\omega)^* \left[\int_0^1 (-z) \cdot \nabla \frac{1}{\rho_0} (z + t(-z)) \, dt \partial_D u(\cdot)\right](x).
\]
We keep the same notation as in lemma 3.2, we invert the operator \(B_D := \left[\left(\rho_0^{-1}(\cdot) \alpha^{-1}(z) + \frac{1}{2}\right) I + \rho_0^{-1}(\cdot) (K_D^\omega)^*\right]\) and after taking the \(L^2(\partial D)\)-norm, the equation (73) becomes
\[
\|\partial_D u\|_{L^2(\partial D)}
\]
\[= \alpha^{-1}(z) \left\|B_D^{-1} \left[\rho_0^{-1} \partial_D v\right] + \omega^2 B_D^{-1} \left[\rho_0^{-1} \partial_D N_D^\omega [u]\right]
\]
\[+ B_D^{-1} \left[\rho_0^{-1} \partial_D M_D^\omega [\nabla u]\right]
\]
\[+ \frac{1}{2} B_D^{-1} \left[\int_0^1 (-z) \cdot \nabla \frac{1}{\rho_0} (z + t(-z)) \, dt \partial_D u\right]
\]
\[+ B_D^{-1} \left[\rho_0^{-1} (K_D^\omega)^* \left[\int_0^1 (-z) \cdot \nabla \frac{1}{\rho_0} (z + t(-z)) \, dt \partial_D u(\cdot)\right]\right]\|_{L^2(\partial D)}
\]
From (68), we have
\[
\left\|B_D^{-1} \left[\int_0^1 (-z) \cdot \nabla \frac{1}{\rho_0} (z + t(-z)) \, dt \partial_D u\right]\right\|_{L^2(\partial D)} \leq \alpha(z) \left\|\int_0^1 (-z) \cdot \nabla \frac{1}{\rho_0} (z + t(-z)) \, dt \right\| \|\partial_D u\|_{L^2(\partial D)} \lesssim \alpha(z) \varepsilon \|\partial_D v\|_{L^2(\partial D)}
\]
and similarly, using the continuity of \((K^\alpha_D)^*\), we obtain
\[
\left\| B_D^{-1} \left[ \rho_0^{-1} (K^\alpha_D)^* \left[ \int_0^1 (\cdot - z) \cdot \nabla \frac{1}{\rho_0} (z + t(\cdot - z)) dt \partial v u(\cdot) \right] \right] \right\|_{L^2(\partial D)} \lesssim \alpha(z) \left\| \partial v u \right\|_{L^2(\partial D)} \lesssim \alpha(z) \varepsilon \left\| \partial v u \right\|_{L^2(\partial D)}.
\]
The equation (74) can be rewritten as
\[
\left\| \partial v u \right\|_{L^2(\partial D)} \lesssim \alpha^{-1}(z) \left\| B_D^{-1} \left[ \rho_0^{-1} \partial v \right] \right\|_{L^2(\partial D)} + B_D^{-1} \left[ \rho_0^{-1} \partial v N_D^\omega[u] \right] + B_D^{-1} \left[ \rho_0^{-1} \partial v M_D^\omega[\nabla u] \right]_{L^2(\partial D)}.
\]
Next, to estimate the right hand side of (75), we proceed in three steps.

1. Step 1: Obviously, we have
\[
\left\| B_D^{-1} \left[ \rho_0^{-1} \partial v \right] \right\|_{L^2(\partial D)} \leq \left\| B_D^{-1} \left[ \rho_0^{-1} \partial v - \frac{1}{|\partial D|} \int_{\partial D} \rho_0^{-1} \partial \nu v \, d\sigma \right] \right\|_{L^2(\partial D)} + \frac{1}{|\partial D|} \left\| \int_{\partial D} \rho_0^{-1} \partial \nu v \, d\sigma \right\| \left\| B_D^{-1} \left[ \rho_0^{-1} \partial v \right] \right\|_{L^2(\partial D)}
\]
and with help of (48), (47) and (68), we have
\[
\left\| B_D^{-1} \left[ \rho_0^{-1} \partial v \right] \right\|_{L^2(\partial D)} = O(1).
\]

2. Step 2: By a change of variables, as in (51), and from the continuity of \(N_D^\omega, M_D^\omega : L^2(B) \rightarrow H^2(B)\), we have
\[
\left\| \rho_0^{-1} \partial \nu \cdot N_D^\omega[u] \right\|_{L^2(\partial D)} \leq \varepsilon C \left\| u \right\|_{L^2(D)}
\]
and
\[
\left\| \rho_0^{-1} \partial \nu \cdot M_D^\omega[\nabla u] \right\|_{L^2(\partial D)} \leq \varepsilon C \left\| \nabla u \right\|_{L^2(D)}.
\]
Now, we use the same approach as previously by considering separately \(\rho_0^{-1} F - |\partial D|^{-1} \int_{\partial D} \rho_0^{-1} F\) and \(|\partial D|^{-1} \int_{\partial D} \rho_0^{-1} F\) for \(F = \partial \nu M_D^\omega[\nabla u]\) and \(F = \partial \nu M_D^\omega[\nabla u]\), to obtain
\[
\left\| B_D^{-1} \left[ \rho_0^{-1} \partial \nu \cdot N_D^\omega[u] \right] - \frac{1}{|\partial D|} \int_{\partial D} \rho_0^{-1} \partial \nu \cdot N_D^\omega[u] \, d\sigma \right\|_{L^2(\partial D)} \leq \sqrt{\varepsilon} C \left\| u \right\|_{L^2(D)}
\]
and
\[
\left\| B_D^{-1} \left[ \rho_0^{-1} \partial \nu \cdot M_D^\omega[\nabla u] \right] - \frac{1}{|\partial D|} \int_{\partial D} \rho_0^{-1} \partial \nu \cdot M_D^\omega[\nabla u] \, d\sigma \right\|_{L^2(\partial D)} \leq \sqrt{\varepsilon} C \left\| \nabla u \right\|_{L^2(D)}.
\]
3. Step 3: We deal with the term
\[
\int_{\partial D} \rho_0^{-1} (x) \partial \nu \cdot M_D^\omega[\nabla u](x) \, d\sigma(x).
\]
Interchanging the integration and using the divergence theorem we get
\[
\int_{\partial D} \frac{\partial_{\nu} M_D^{\omega}[\nabla u](x)}{\rho_0(x)} \, d\sigma(x) = \int_D \nabla \left( \frac{1}{\rho_0(x)} \cdot \nabla u(y) \int_D \nabla \left[ \rho_0^{-1}(x) \nabla G_\omega(x, y) \right] \, dx \right) \, dy
\]
\[
= - \int_D \nabla \left( \frac{1}{\rho_0(x)} \cdot \nabla u(x) \right) dx
\]
\[
- \omega^2 \int_D \nabla \left( \frac{1}{\rho_0(y)} \cdot \nabla u(y) \right) \int_D k_0^{-1}(x)G_\omega(x, y) \, dx \, dy
\]
\[
= \int_D \nabla \left( \frac{1}{\rho_0(x)} \cdot \nabla u(x) \right) dx + O \left( \varepsilon^2 \| \nabla u \|_{L^2(D)} \right)
\]
(76)
\[
= O \left( \varepsilon^2 \| u \|_{H^\nu(D)} + \varepsilon^2 \| \partial_{\nu} u \|_{L^2(\partial D)} + \varepsilon^4 \right).
\]

The analysis of the term
\[
\int_{\partial D} \rho_0^{-1}(x) \partial_{\nu} N_D^{\omega}[u](x) \, d\sigma(x)
\]
is more delicate and needs more efforts. We start by repeating the same steps of the proof of Proposition 1 to obtain
\[
\int_{\partial D} \frac{\partial_{\nu} N_D^{\omega}[u](x)}{\rho_0(x)} \, d\sigma(x) = \int_D \gamma(y) \, u(y) \int_{\partial D} \rho_0^{-1}(x) \partial_{\nu}(x)G_\omega(x, y) \, d\sigma(x) \, dy
\]
\[
= \int_D \gamma(y) \, u(y) \int_D \nabla \left[ \rho_0^{-1}(x) \nabla G_\omega(x, y) \right] \, dx \, dy
\]
(6)
\[
= - \int_D \gamma(x) \, u(x) \, dx - \omega^2 \int_D \gamma(y) \, u(y) \int_D \frac{G_\omega(x, y)}{k_0(x)} \, dx \, dy
\]
\[
= - \gamma(z) \int_D u(x) \, dx - \nabla \gamma(z) \cdot \int_D (x - z) \, u(x) \, dx
\]
\[
+ O \left( \varepsilon^2 \| u \|_{L^2(D)} \right).
\]
(77)

To estimate the second term in the right hand side of the last equation we use (59) to get
\[
\int_D (x - z)u(x) \, dx = -\omega \int_D (x - z)N_D^{\omega}[u](x) \, dx + \int_D (x - z)M_D^{\omega}[\nabla u](x) \, dx
\]
\[
+ \int_D (x - z)S_D^{\omega}[\partial_{\nu} u](x) \, dx + \int_D (x - z)v(x) \, dx
\]
\[
\left| \int_D (x - z)u(x) \, dx \right| \leq \varepsilon^2 \| u \|_{L^2(D)} + \varepsilon^2 \| \nabla u \|_{L^2(D)} + \varepsilon^2 \| \partial_{\nu} u \|_{L^2(\partial D)} + O \left( \varepsilon^4 \right).\]

Finally, the equation (77), with the help of (20), takes the following form
\[
\int_{\partial D} \frac{\partial_{\nu} N_D^{\omega}[u]}{\rho_0} \, d\sigma = \frac{\gamma(z)k_1}{\omega^2 \rho_1} \int_{\partial D} \partial_{\nu} u \, d\sigma
\]
\[
+ O \left( \varepsilon^2 \| u \|_{L^2(D)} + \varepsilon^2 \| \nabla u \|_{L^2(D)} + \varepsilon^2 \| \partial_{\nu} u \|_{L^2(\partial D)} + \varepsilon^4 \right).
\]
(78)
To finish with the estimation of (78), we need to estimate the integral of $\partial_v u$ over $\partial D$. For this, the identity (60) can be rewritten as

$$\frac{1}{2} \left( \rho_1^{-1} + \rho_0^{-1}(x) \right) \partial_v u(x) + (J_D^\omega)^* \left[ (\rho_1^{-1} - \rho_0^{-1}(x)) \partial_v u(x) \right] - \omega^2 \rho_0^{-1}(x) \partial_v N_D^\omega [u](x) + \rho_0^{-1}(x) \partial_v M_D^\omega [\nabla u](x) = \rho_0^{-1}(x) \partial_v v(x).$$

We integrate this equation over $\partial D$ and use (78), (76) and $J_D^\omega(1) = -1/2$ to obtain

$$k_1 \int_{\partial D} \partial_v u(x) d\sigma(x) + \int_{\partial D} \left( J_D^\omega - J_D^0 \right) [1](x) \left[ (\rho_1^{-1} - \rho_0^{-1}(x)) \partial_v u(x) \right] d\sigma(x) = \int_{\partial D} \rho_0^{-1}(x) \partial_v v(x) d\sigma(x) + O \left( \varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2 \|\partial_v u\|_{L^2(\partial D)} + \varepsilon^4 \right).$$

and from (67) we obtain

$$k_1 \int_{\partial D} \partial_v u(x) d\sigma(x) = \frac{k_1^2 \alpha(z)}{2 \rho_0} \int_{\partial D} \left( \nu(y) \cdot \frac{y - x}{|y - x|^2} - \nu(z) \partial_v u(x) \right) d\sigma(x) = \int_{\partial D} \rho_0^{-1}(x) \partial_v v(x) d\sigma(x) + O \left( \varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2 \|\partial_v u\|_{L^2(\partial D)} + \varepsilon^4 \right).$$

Using the same notations as in (24) and the fact that $v$ satisfy the equation (5) we write

$$\left( \frac{k_1}{\rho_1 k_0} - \frac{\omega^2 \alpha(z)}{2 \rho_0} A_{\partial D} \right) \int_{\partial D} \partial_v u(x) d\sigma(x) = -\frac{\omega^2}{k_0(z)} \int_D v(x) dx + O \left( \varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2 \|\partial_v u\|_{L^2(\partial D)} + \varepsilon^4 \right) + \frac{\omega^2 \alpha(z)}{2 \rho_0} \int_{\partial D} (A(x) - A_{\partial D}) \partial_v u(x) d\sigma$$

and using the definition of $A_{\partial D}$ and the estimation as the one given in (27) we obtain

$$\left( \frac{k_1}{\rho_1 k_0} - \frac{\omega^2 \alpha(z) \rho_0}{2 \rho_0} \mu_{\partial D} \right) \int_{\partial D} \partial_v u(x) d\sigma(x) = -\frac{\omega^2}{k_0(z)} \int_D v(x) dx + O \left( \varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2 \|\partial_v u\|_{L^2(\partial D)} + \varepsilon^4 \right).$$

(79)

As in the previous section, we can write

$$\left( \frac{k_1}{\rho_1 k_0} - \frac{\omega^2 \alpha(z) \rho_0}{2 \rho_0} \mu_{\partial D} \right) = \frac{k_1}{\rho_1 k_0} \left( \frac{\omega_1^2 - \omega_0^2}{\omega_M^2} \right) \left( 1 + O \left( \frac{\varepsilon^2}{\omega_M^2 - \omega_M^2} \right) \right),$$

(80)

where now

$$\omega_M^2 := \omega_M^2(z) := \frac{2 k_1}{\rho_0(z) \mu_{\partial B}}.$$
Then
\[
\int_{\partial D} \partial_{\nu} u \, d\sigma = \frac{\mathcal{P}_1 \omega^2 \omega_M^2}{k_1 (\omega^2 - \omega_M^2)} \int_D v(x) \, dx \left[ 1 + \mathcal{O} \left( \frac{\varepsilon^2}{\omega^2 - \omega_M^2} + \varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^{-1} \|\partial_{\nu} u\|_{L^2(\partial D)} + \varepsilon \right) \right]
\]
\[
= \frac{\mathcal{P}_1 \omega^2 \omega_M^2}{k_1 (\omega^2 - \omega_M^2)} \int_D v(x) \, dx + \mathcal{O} \left( \frac{\varepsilon^5}{(\omega^2 - \omega_M^2)^2} + \frac{\varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2 \|\partial_{\nu} u\|_{L^2(\partial D)} + \varepsilon^4}{(\omega^2 - \omega_M^2)^2} \right)
\]

(81)

We derive the corresponding estimate as in (34):
\[
\int_{\partial D} \partial_{\nu} u(x) d\sigma(x) = \mathcal{O} \left( \frac{\varepsilon^3}{\omega^2 - \omega_M^2} \right)
\]
\[
+ \mathcal{O} \left( \frac{\varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2 \|\partial_{\nu} u\|_{L^2(\partial D)} + \varepsilon^4}{\omega^2 - \omega_M^2} \right)
\]

Back substituting these results into (78) we obtain
\[
\int_{\partial D} \rho^{-1}_0(x) \partial_{\nu} \mathcal{N}^D \|u\|(x) \, d\sigma(x) = \frac{\gamma(z) \omega_M^2}{(\omega^2 - \omega_M^2)} \int_D v(x) \, dx + \mathcal{O} \left( \frac{\varepsilon^5}{(\omega^2 - \omega_M^2)^2} \right)
\]
\[
+ \mathcal{O} \left( \frac{\varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2 \|\partial_{\nu} u\|_{L^2(\partial D)} + \varepsilon^4}{(\omega^2 - \omega_M^2)^2} \right)
\]

With the same calculations as for (54), we derive the estimate
\[
\|\partial_{\nu} u\|_{L^2(\partial D)} = \mathcal{O} \left( \frac{\varepsilon^2 \|u\|_{H^1(D)} + \varepsilon \|\partial_{\nu} u\|_{L^2(\partial D)} + \varepsilon^2}{\omega^2 - \omega_M^2} \right).
\]

Hence if \(\varepsilon/(\omega^2 - \omega_M^2)\) is small enough, we have
\[
\|\partial_{\nu} u\|_{L^2(\partial D)} = \mathcal{O} \left( \frac{\varepsilon^2 \|u\|_{H^1(D)} + \varepsilon^2}{\omega^2 - \omega_M^2} \right).
\]

Finally from (72), we obtain (65) and (66).

\[\Box\]

**Remark 2.** A straightforward calculation, from (70), (71) and the estimation of \(\|\partial_{\nu} u\|_{L^2(\partial D)}\), allow to deduce that
\[
\|u\|_{L^2(D)} = \mathcal{O} \left( \frac{\varepsilon^3}{\omega^2 - \omega_M^2} \right) \quad \text{and} \quad \|\nabla u\|_{L^2(D)} = \mathcal{O} \left( \frac{\varepsilon^2}{\omega^2 - \omega_M^2} \right).
\]

Now, we use the smallness of \(\varepsilon/(\omega^2 - \omega_M^2)\), Proposition 2 and developing near \(z\) the function \(v\), we obtain from (81) the following formula
\[
\int_{\partial D} \partial_{\nu} u(x) \, d\sigma(x) = \frac{\omega^2 \omega_M^2 \mathcal{P}_1}{k_1 (\omega^2 - \omega_M^2)} v(z) |D| + \mathcal{O} \left( \frac{\varepsilon^4}{(\omega^2 - \omega_M^2)^2} \right).
\]
Plugging this estimation in (64), and with help of Proposition 2, we obtain, for \( x \) away from \( D \),

\[
(82) \quad u^s(x) = v^s(x) - G_{\omega}(x, z) \frac{\omega_2^2}{k_1 (\omega^2 - \omega_2^2)} \cdot v(z) |B| \varepsilon + O \left( \frac{\varepsilon^2}{(\omega^2 - \omega_2^2)^2} \right),
\]

as it was done for (57).

Using the mixed reciprocity relation, we derive the expansion of the associated far fields as it was mentioned in (8).

4. **Proof of Theorem 1.2.** To avoid additional lengthy computations, we provide the detailed proof in the case the background is constant inside \( \Omega \), i.e \( \rho_0(\cdot) = \rho_0(z) \) and \( k_0(\cdot) = k_0(z) \) in \( \Omega \). The case of variable background in \( \Omega \) can be handled following the steps described in the previous section.

The starting point is, again, the system of the integral equations:

\[
(83) \quad u(x) - \gamma \omega^2 \int_D G_\omega(x, y)u(y)dy + \alpha \int_{\partial D} G_\omega(x, y) \partial_n u(y) d\sigma(y) = v(x), \text{ in } D,
\]

\[
(84) \quad \left[ \left( 1 + \frac{\alpha \rho_0}{2} \right) I + \alpha (K_D^\omega)^* \right] [\partial_n u](x) - \gamma \omega^2 \partial_n - \int_D G_\omega(x, y)u(y)dy = \partial_n v(x), \text{ on } \partial D.
\]

Notice that due to the scaling of \( \rho_1 \) and \( k_1 \), in this regime, we have

\[
(85) \quad \gamma \sim \varepsilon^{-2} \text{ while } \alpha \sim \varepsilon^j, \quad j > 0 \quad \text{as } \varepsilon \to 0
\]

where we recall that \( \gamma := \beta - \alpha \rho_1 k_1^{-1}, \quad \alpha := \rho_1^{-1} - \rho_0^{-1} \) and \( \beta := k_1^{-1} - k_0^{-1} \).

The strategy of the proof is quite similar to the previous section. Indeed, we first provide the a priori estimation of both \( u \) and \( \partial_n u \) and then derive the dominating term of the expansion of the scattered fields. The main difference is the fact that the regimes, fixed by the contrasts of the mass densities and bulk moduli, are different and consequently we have a sequence of different resonances \( \omega_n \)'s instead of the Minnaert one.

4.1. **A priori estimation.** We start with the equation (84), i.e

\[
(86) \quad \left[ \left( 1 + \frac{\alpha \rho_0}{2} \right) I + \alpha (K_D^\omega)^* \right] [\partial_n u](x) = \gamma \omega^2 \partial_n N_D^\omega[u](x) + \partial_n v(x).
\]

As \( \alpha \) is small, see (85), then \( \left[ \left( 1 + \frac{\alpha \rho_0}{2} \right) I + \alpha (K_D^\omega)^* \right]^{-1} \) exists. Taking the inverse in both sides of (86) we obtain

\[
\partial_n u = \gamma \omega^2 \left[ \left( 1 + \frac{\alpha \rho_0}{2} \right) I + \alpha (K_D^\omega)^* \right]^{-1} [\partial_n N_D^\omega[u]]
\]

and then by taking the \( L^2(\partial D) \)-norm, we obtain

\[
\frac{\|\partial_n u\|_{L^2(\partial D)}}{\|\partial_n v\|_{L^2(\partial D)}} \leq \frac{\gamma \omega^2}{\alpha} \left\| \left( \frac{1}{\alpha} + \frac{\rho_0}{2} \right) I + (K_D^\omega)^* \right\|_{L^2(\partial D)} \left\| \partial_n N_D^\omega[u] \right\|_{L^2(\partial D)}
\]

\[
+ \frac{1}{\alpha} \left\| \left( \frac{1}{\alpha} + \frac{\rho_0}{2} \right) I + (K_D^\omega)^* \right\|_{L^2(\partial D)} \left\| \partial_n v \right\|_{L^2(\partial D)}.
\]
Since
\[
\left\| \left[ \left( \frac{1}{\alpha} + \frac{\rho_0}{2} \right) I + (K_D^\omega)^* \right]^{-1} \right\|_{L(L^2(\partial D))} \lesssim \frac{1}{\text{dist} \left( \frac{1}{\alpha} + \frac{\rho_0}{2} ; \sigma ((K_D^\omega)^*) \right)}
\]
\[
= \frac{1}{\left| \frac{1}{\alpha} + \frac{\rho_0}{2} - \frac{1}{2} \right|} \simeq \alpha
\]
and, as in (52), we have
\[
\left\| \partial_\nu N_D^\omega[u] \right\|_{L^2(\partial D)} \leq C \varepsilon^{\frac{1}{2}} \left\| u \right\|_{L^2(D)}
\]
then we obtain
\[
(87) \quad \left\| \partial_\nu u \right\|_{L^2(\partial D)} \lesssim \varepsilon^{-\frac{1}{2}} \left\| u \right\|_{L^2(D)} + \left\| \partial_\nu v \right\|_{L^2(\partial D)}.
\]
Next, recall that \( \Gamma_\omega(\cdot, \cdot) \) is given by
\[
\Gamma_\omega(x, y) = \rho_0(z) \frac{e^{i \kappa_0(z) |x-y|}}{4 \pi |x-y|}, \quad x \neq y,
\]
and consider the equation (83), i.e
\[
(88) \quad u(x) - \gamma \omega^2 N_D^\omega[u](x) = -\alpha S_D^\omega [\partial_\nu u](x) + v(x),
\]
as
\[
N_D^\omega[u](x) := \int_D G_\omega(x, y) u(y) \, dy
\]
\[
= \int_D \Gamma_\omega(x, y) u(y) \, dy + \int_D (G_\omega - \Gamma_\omega)(x, y) u(y) \, dy
\]
\[
= \int_D \Gamma_0(x, y) e^{i \kappa_0(z) |x-y|} u(y) \, dy + \int_D (G_\omega - \Gamma_\omega)(x, y) u(y) \, dy
\]
\[
= \int_D \Gamma_0(x, y) \left[ 1 + \sum_{n \geq 1} \frac{(i \kappa_0(z) |x-y|)^n}{n!} \right] u(y) \, dy
\]
\[
+ \int_D (G_\omega - \Gamma_\omega)(x, y) u(y) \, dy
\]
\[
= N_D^\omega[u](x) + \sum_{n \geq 1} \int_D \Gamma_0(x, y) \frac{(i \kappa_0(z) |x-y|)^n}{n!} u(y) \, dy
\]
\[
+ \int_D (G_\omega - \Gamma_\omega)(x, y) u(y) \, dy.
\]
The equation (88) takes the following form
\[
(89) \quad u(x) - \gamma \omega^2 N_D^\omega[u](x) = -\alpha S_D^\omega [\partial_\nu u](x) + v(x)
\]
\[
+ \gamma \omega^2 \int_D (G_\omega - \Gamma_\omega)(x, y) u(y) \, dy
\]
\[
+ \gamma \omega^2 \rho_0(z) \sum_{n \geq 1} (i \kappa_0(z))^n \int_D \frac{|x-y|^{n-1}}{n!} u(y) \, dy.
\]
We take the $L^2(D)$-norm in both sides of the last equation to obtain
\[
\|u - \gamma \omega^2 \mathcal{N}^D u\|_{L^2(D)} \leq \alpha \|S_D^\gamma [\partial_\nu u]\|_{L^2(D)} + \|v\|_{L^2(D)}
+ |\gamma \omega^2| \|u\|_{L^2(D)} \left[ \int_D \int_D |G_\omega - \Gamma_\omega|^2 (x, y) \, dx \, dy \right]^{\frac{1}{2}}
+ |\gamma \omega^2 \rho_0(z)| \sum_{n \geq 1} |\kappa_0(z)| n \left\| \int_D \frac{1}{n!} - y^{n-1} u(y) \, dy \right\|_{L^2(D)}.
\]
(90)

Recall, see for instance (40), that
\[
\|S_D^\gamma [\partial_\nu u]\|_{L^2(D)} \leq C \varepsilon^2 \|\partial_\nu u\|_{L^2(\partial D)}
\]
and
\[
\sum_{n \geq 1} |\kappa_0(z)| n \left\| \int_D \frac{1}{n!} - y^{n-1} u(y) \, dy \right\|_{L^2(D)} \leq \|u\|_{L^2(D)} \varepsilon^3 \sum_{n \geq 1} |\kappa_0(z)| \geq \varepsilon^{n-1} \frac{n!}{n!}
= O\left(\|u\|_{L^2(D)} \varepsilon^3\right).
\]
In the appendix (5) we prove that the function $(G_\omega - \Gamma_\omega)$ are bounded, then
\[
\left[ \int_D \int_D |G_\omega - \Gamma_\omega|^2 (x, y) \, dx \, dy \right]^{\frac{1}{2}} = O\left(\varepsilon^3\right).
\]
Then (90) becomes
\[
\|u - \gamma \omega^2 \mathcal{N}^D u\|_{L^2(D)} \leq \alpha \varepsilon^2 \|\partial_\nu u\|_{L^2(\partial D)} + \|v\|_{L^2(D)} + \varepsilon \|u\|_{L^2(D)}.
\]
(92)

Plugging (87) in (92), we obtain
\[
\|u - \gamma \omega^2 \mathcal{N}^D u\|_{L^2(D)} \leq \varepsilon^3 + \frac{1}{2} \|\partial_\nu u\|_{L^2(\partial D)} + \|v\|_{L^2(D)} + (\alpha + \varepsilon) \|u\|_{L^2(D)}.
\]
(93)

Let $(\rho_0(z) \lambda^D_n, e^D_n)_{n \in \mathbb{N}}$ the eigensystem of the Newtonian operator $\mathcal{N}^D$ which is positive, compact and selfadjoint on $L^2(D)$. Using this basis, the left hand side of (93) can be computed as\(^5\)
\[
\|u - \gamma \omega^2 \mathcal{N}^D u\|_{L^2(D)}^2 = \sum_n \left| \langle u - \gamma \omega^2 \mathcal{N}^D u; e^D_n \rangle \right|^2
= \sum_n \left| \langle u; e^D_n \rangle \right|^2 \left| 1 - \gamma \omega^2 \rho_0(z) \lambda^D_n \right|^2
+ \sum_{n \neq n_0} \left| \langle u; e^D_n \rangle \right|^2 \left| 1 - \gamma \omega^2 \rho_0(z) \lambda^D_n \right|^2.
\]
(94)

Next, we choose $\omega^2$ such that $\left| 1 - \gamma \omega^2 \rho_0(z) \lambda^D_{n_0} \right| \sim \varepsilon^h$, which implies that
\[
\omega^2 := \frac{1 \pm \varepsilon^h}{\gamma \rho_0(z) \lambda^D_{n_0}}, \text{ or } (\omega^2 - \omega^2_{n_0}) \sim \varepsilon^h \text{ where } \omega^2_{n_0} := \frac{K_1}{\rho_0(z) \lambda^D_{n_0}} \sim 1.
\]
(95)

Since we choose $\omega^2$ close to $\omega^2_{n_0}$ we deduce that, for $n \neq n_0$, the sequence $\left| 1 - \gamma \omega^2 \rho_0(z) \lambda^D_n \right|^2$ is bounded from below. Then, if we set
\[
\sigma := \inf_{n \neq n_0} \left| 1 - \gamma \omega^2 \rho_0(z) \lambda^D_n \right|^2,
\]
\[^5\text{Where } \langle \cdot; \cdot \rangle \text{ stands for the } L^2(D) \text{ inner product.}\]
the equation (94) becomes
\[ \| u - \gamma \omega^2 N_0[u] \|_{L^2(D)}^2 \geq | \langle u; e_n^D \rangle |^2 | 1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D \|^2 + \sigma \sum_{n \neq n_0} | \langle u; e_n^D \rangle |^2. \]

From the previous equation, with the help of (93), we deduce that
\[ \langle u; e_n^D \rangle ^2 \lesssim \left( \sigma^{-1} + \| \partial_u v \|_{L^2(\partial D)}^2 + \| u \|_{L^2(D)}^2 \right)^{-1} \left[ \epsilon^{3+2j} \| \partial_u v \|_{L^2(\partial D)}^2 + \| u \|_{L^2(D)}^2 \right]. \]

and
\[ \sum_{n \neq n_0} | \langle u; e_n^D \rangle |^2 \lesssim \sigma^{-1} \left( \epsilon^{3+2j} \| \partial_u v \|_{L^2(\partial D)}^2 + \| u \|_{L^2(D)}^2 \right). \]

Now, if we sum (96) and (97), we obtain
\[ \| u \|_{L^2(D)}^2 := | \langle u; e_n^D \rangle |^2 + \sum_{n \neq n_0} | \langle u; e_n^D \rangle |^2 \]
\[ \lesssim \left( \sigma^{-1} + \| \partial_u v \|_{L^2(\partial D)}^2 + \| u \|_{L^2(D)}^2 \right)^{-1} \left[ \epsilon^{3+2j} \| \partial_u v \|_{L^2(\partial D)}^2 + \| u \|_{L^2(D)}^2 \right]. \]

then
\[ \| u \|_{L^2(D)}^2 \left( 1 - \frac{(\alpha + \varepsilon)^2}{| 1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D |^2} \right) \lesssim \left[ \epsilon^{3+2j} \| \partial_u v \|_{L^2(\partial D)}^2 + \| u \|_{L^2(D)}^2 \right]. \]

We need \( \alpha \) such that \( 1 - (\alpha + \varepsilon)^2 | 1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D |^{-2} \) is uniformly bounded from below. For this, we see that
\[ 1 - (\alpha + \varepsilon)^2 | 1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D |^{-2} \simeq 1 - \varepsilon^{-2h} (\alpha + \varepsilon)^2. \]

As \( \alpha \sim \varepsilon^j \), where \( j > 0 \), then
\[ 1 - (\alpha + \varepsilon)^2 | 1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D |^{-2} \simeq 1 - \varepsilon^{2(\min(1,j)-h)}. \]

This implies, if \( h < \min(j, 1) \), the boundedness from below of
\[ 1 - (\alpha + \varepsilon)^2 | 1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D |^{-2}. \]

Now, under the condition
\[ h < \min(j, 1) \]
we get an estimation of \( \| u \|_{L^2(D)} \) with respect to \( \| v \|_{L^2(D)} \) and \( \| \partial_u v \|_{L^2(\partial D)} \) as follows
\[ \| u \|_{L^2(D)}^2 \lesssim \frac{1}{| 1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D |^2} \left[ \epsilon^{3+2j} \| \partial_u v \|_{L^2(\partial D)}^2 + \| u \|_{L^2(D)}^2 \right]. \]

We plug (99) into (87) to obtain an estimation of \( \| \partial_u u \|_{L^2(\partial D)} \) with respect to \( \| v \|_{L^2(D)} \) and \( \| \partial_u v \|_{L^2(\partial D)} \). Precisely, we obtain
\[ \| \partial_u u \|_{L^2(\partial D)} \lesssim \| \partial_u v \|_{L^2(\partial D)} + \varepsilon^{-3-2h} \| v \|_{L^2(D)}^2. \]
4.2. Estimation of the scattered field.

We write the integral equation (89) and we develop the incident field \(v\) near the center \(z\) to obtain

\[
(I - \gamma \omega^2 \mathcal{N}^0_D) \left[ u(x) \right] = - \alpha S^2_D \left[ \partial_n u \right](x) + v(z) + \int_0^1 (x - z) \cdot \nabla v(z + t(x - z)) \, dt + \gamma \omega^2 \int_D (G_\omega - \Gamma_\omega)(x, y) u(y) \, dy + \gamma \omega^2 \rho_0(z) i \kappa_0(z) \int_D u(y) \, dy + \gamma \omega^2 \rho_0(z) \sum_{n \geq 2} (i \kappa_0(z))^n \int_D \frac{|x - y|^{n-1}}{n!} u(y) \, dy.
\]

Next, successively, we set \(W\) to be \(W := (I - \gamma \omega^2 \mathcal{N}^0_D)^{-1}[1]\), apply the self adjoint operator \((I - \gamma \omega^2 \mathcal{N}^0_D)^{-1}\) in both sides of the previous equation and integrate over the domain \(D\) to obtain

\[
\int_D u(x) \, dx = v(z) \int_D W(x) \, dx + \int_D W(x) \, dx \int_0^1 (x - z) \cdot \nabla v(z + t(x - z)) \, dt \, dx
- \alpha \int_D W(x) S_D^w \left[ \partial_n u \right](x) \, dx + \gamma \omega^2 \rho_0(z) i \kappa_0(z) \int_D W(x) \, dx \int_D u(y) \, dy
+ \gamma \omega^2 \int_D W(x) \int_D (G_\omega - \Gamma_\omega)(x, y) u(y) \, dy \, dx
+ \gamma \omega^2 \rho_0(z) \sum_{n \geq 2} (i \kappa_0(z))^n \int_D W(x) \, dx \int_D \frac{|x - y|^{n-1}}{n!} u(y) \, dy.
\]

We keep only, on the right hand side, the first term and we estimate the others as an error. For this, we need first an a priori estimation of \(\int_D W(x) \, dx\) and \(\|W\|_{L^2(D)}\).

To do this, we have

\[
\int_D e_n^D(x) \, dx = \int_D 1 e_n^D(x) \, dx = \int_D (I - \gamma \omega^2 \mathcal{N}^0_D) \left[ W \right](x) e_n^D(x) \, dx
= \langle W; e_n^D \rangle \left( 1 - \gamma \omega^2 \rho_0(z) \lambda_n^D \right)
\]

which implies that \(\langle W; e_n^D \rangle = \langle 1; e_n^D \rangle \left( 1 - \gamma \omega^2 \rho_0(z) \lambda_n^D \right)^{-1}\) and then

\[
\int_D W(x) \, dx = \sum_n \langle W; e_n^D \rangle \int_D e_n^D(x) \, dx = \sum_n \frac{(\int_D e_n^D(x) \, dx)^2}{(1 - \gamma \omega^2 \rho_0(z) \lambda_n^D)}
= \frac{(\int_D e_n^D(x) \, dx)^2}{(1 - \gamma \omega^2 \rho_0(z) \lambda_n^D)} + \sum_{n \neq n_0} \frac{(\int_D e_n^D(x) \, dx)^2}{(1 - \gamma \omega^2 \rho_0(z) \lambda_n^D)}.
\]

Obviously, we have

\[
\sum_{n \neq n_0} \frac{(\int_D e_n^D(x) \, dx)^2}{(1 - \gamma \omega^2 \rho_0(z) \lambda_n^D)} = O(\varepsilon^3).
\]

Then

\[
\int_D W(x) \, dx \sim O(\varepsilon^{3-h}).
\]
Similarly,
\[ \|W\|_{L^2(D)}^2 = \sum_n |\langle W, e_n^D \rangle|^2 = \frac{|\langle 1; e_n^D \rangle|^2}{|1 - \gamma \omega^2 \rho_0(z) \lambda_n^D|^2} + \sum_{n \neq n_0} |\langle 1; e_n^D \rangle|^2 \frac{|\langle 1; e_{n_0}^D \rangle|^2}{|1 - \gamma \omega^2 \rho_0(z) \lambda_{n_0}^D|^2}. \]

Then
\[ (104) \quad \|W\|_{L^2(D)} \sim \mathcal{O}(\varepsilon^{3-h}). \]

Now, we are ready to estimate the error parts of (101). To achieve this, we split it as follows

* Estimation of \( I_1 := \int_D W(x) \int_0^1 (x - z) \cdot \nabla v(z + t(x - z)) \, dt \, dx. \)

By the Cauchy-Schwartz inequality, we have
\[ |I_1| \leq \|W\|_{L^2(D)} \left\| \int_0^1 (\cdot - z) \cdot \nabla v(z + t(\cdot - z)) \, dt \right\|_{L^2(D)} = \mathcal{O}(\varepsilon^{4-h}). \]

* Estimation of \( I_2 := \alpha \int_D W(x) S_D \partial_v u(x) \, dx. \)

By applying the Cauchy-Schwartz inequality and the continuity of the single layer, see for instance the inequality (40), we obtain
\[ |I_2| \leq \alpha \|W\|_{L^2(D)} \|S_D \partial_v u\|_{L^2(D)} \lesssim \varepsilon^{3-h+j} \|\partial_v u\|_{L^2(\partial D)} \]

and recall that, see (100), we have
\[ \|\partial_v u\|_{L^2(\partial D)}^2 \lesssim \|\partial_v v\|_{L^2(\partial D)}^2 + \varepsilon^{-3-2h} \|v\|_{L^2(D)}^2 \]
then
\[ |I_2|^2 \lesssim \varepsilon^{6+2j-2h} \left[ \|\partial_v v\|_{L^2(\partial D)}^2 + \varepsilon^{-3-2h} \|v\|_{L^2(D)}^2 \right]. \]

Since the incident field is smooth we have
\[ \|\partial_v v\|_{L^2(\partial D)} = \mathcal{O}(\varepsilon^{2}) \quad \text{and} \quad \|v\|_{L^2(D)}^2 = \mathcal{O}(\varepsilon^{3}). \]

With this \( I_2 = \mathcal{O}(\varepsilon^{3+j-2h}). \)

* Estimation of \( I_3 := \gamma \omega^2 \rho_0(z) i \kappa_0(z) \int_D W(x) \, dx \int_D u(y) \, dy. \)

A straightforward application of (103) and the Cauchy-Schwartz inequality allows to deduce
\[ |I_3| = \left| \gamma \omega^2 \rho_0(z) \kappa_0(z) \int_D W(x) \int_D u(\gamma) \, dy \right| \lesssim \varepsilon^{-2+3-h} \|W\|_{L^2(D)} \|u\|_{L^2(D)} = \varepsilon^{2-h} \|u\|_{L^2(D)} \]
then, with help of (99), we obtain
\[ |I_3|^2 \lesssim \varepsilon^{5-2h} \|u\|_{L^2(D)}^2 \]
\[ \lesssim \varepsilon^{5-2h} \left[ 1 - \gamma \omega^2 \rho_0 \lambda_{n_0}^D \right]^{-2} \left[ \varepsilon^{3+2j} \|\partial_v v\|_{L^2(\partial D)}^2 + \|v\|_{L^2(D)}^2 \right]. \]

Recall again that \( \|\partial_v v\|_{L^2(\partial D)}^2 = \mathcal{O}(\varepsilon^{2}) \) and \( \|v\|_{L^2(D)}^2 = \mathcal{O}(\varepsilon^{3}) \), then we deduce that \( I_3 = \mathcal{O}(\varepsilon^{4-2h}). \)
We know, from (83), that the scattered field is given by
\[ u^s(x) = v^s(x) + \gamma \omega^2 \int_D G_\omega(x,y) u(y) \, dy - \alpha \int_{\partial D} G_\omega(x,y) \partial_n u(y) \, d\sigma(y). \]

For \( x \) away from \( D \), we expand \( G_\omega(\cdot, \cdot) \) near \( z \) to obtain
\[
\begin{align*}
u^s(x) &= v^s(x) + \gamma \omega^2 \int_D u(y) \, dy - \alpha \int_{\partial D} G_\omega(x,z) \partial_n u(y) \, d\sigma(y) \\
&\quad + \gamma \omega^2 \int_D \int_0^1 (y-z) \cdot \nabla G_\omega(x,z+t(y-z)) \, dt \, u(y) \, dy \\
&\quad - \alpha \int_{\partial D} \int_0^1 (y-z) \cdot \nabla G_\omega(x,z+t(y-z)) \, dt \, \partial_n u(y) \, d\sigma(y).
\end{align*}
\] (106)

We need to estimate the two last terms of the previous equation.
* Estimation of $B_1 := \gamma \omega^2 \int_D \int_0^1 (y - z) \cdot \nabla_y G_\omega(x, z + t(y - z)) \, dt \, u(y) \, dy$.

We have
\[
|B_1| \leq \varepsilon^{-2} \| u \|_{L^2(D)} \left[ \int_D \left( \int_0^1 (y - z) \cdot \nabla_y G_\omega(x, z + t(y - z)) \, dt \right)^2 \, dy \right]^{\frac{1}{2}} \\
\leq \varepsilon^{-2} \| u \|_{L^2(D)} \left[ \int_D |y - z|^2 \, dy \right]^{\frac{1}{2}} \leq \varepsilon^{-2} \varepsilon^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} = O \left( \varepsilon^{2-h} \right).
\]

* Estimation of $B_2 := \alpha \int_{\partial D} \int_0^1 (y - z) \cdot \nabla_y G_\omega(x, z + t(y - z)) \, dt \, \partial_s u(y) \, d\sigma(y)$.

We have
\[
|B_2| \leq \varepsilon^j \| \partial_s u \|_{L^2(\partial D)} \left[ \int_{\partial D} \left( \int_0^1 (y - z) \cdot \nabla_y G_\omega(x, z + t(y - z)) \, dt \right)^2 \, d\sigma(y) \right]^{\frac{1}{2}} \\
\leq \varepsilon^j \| \partial_s u \|_{L^2(\partial D)} \left[ \int_{\partial D} |y - z|^2 \, d\sigma(y) \right]^{\frac{1}{2}} \leq \varepsilon^j \varepsilon^{-h} \varepsilon^2 = O \left( \varepsilon^{2+j-h} \right).
\]

Taking into account the estimation of $B_1$ and $B_2$ we rewrite the formula (106) as
\[
u^*(x) = v^*(x) + \gamma \omega^2 G_\omega(x, z) \int_D u(y) \, dy - \alpha G_\omega(x, z) \int_{\partial D} \partial_s u(y) \, d\sigma(y) \\
+ \mathcal{O} \left( \varepsilon^{2-h} \right) \\
\overset{(20)}{=} v^*(x) + \beta \omega^2 G_\omega(x, z) \int_D u(y) \, dy + \mathcal{O} \left( \varepsilon^{2-h} \right).
\]

Now, we use the expression of $\int_D u(x) \, dx$ given in the formula (105) to deduce that
\[
u^*(x) = v^*(x) + \beta \omega^2 G_\omega(x, z) \left[ v(z) \int_D W(x) \, dx + \mathcal{O} \left( \varepsilon^{3-2h+\min(1;j)} \right) \right] \\
+ \mathcal{O} \left( \varepsilon^{2-h} \right) \\
= v^*(x) + \beta \omega^2 G_\omega(x, z) v(z) \int_D W(x) \, dx + \mathcal{O} \left( \varepsilon^{1-2h+\min(1;j)} \right).
\]

Plugging the estimation of $\int_D W(x) \, dx$, given in formula (102), we obtain
\[
u^*(x) = v^*(x) + \beta \omega^2 G_\omega(x, z) v(z) \left[ \frac{\left( \int_D e_{\nu_0}^D(x) \, dx \right)^2}{1 - \gamma \omega^2 \rho_0 \lambda^{D}_{\nu_0}} + \mathcal{O} \left( \varepsilon^3 \right) \right] \\
+ \mathcal{O} \left( \varepsilon^{1-2h+\min(1;j)} \right).
\]

Finally, recalling the value of $\beta := k_1^{-1} - k_0^{-1}$, we have
\[
u^*(x) = v^*(x) + \frac{1}{k_1} \omega^2 G_\omega(x, z) v(z) \left( \int_D e_{\nu_0}^D(x) \, dx \right)^2 \left( 1 - \gamma \omega^2 \rho_0 \lambda^{D}_{\nu_0} \right) + \mathcal{O} \left( \varepsilon^{1-2h+\min(1;j)} \right) \\
\overset{(95)}{=} v^*(x) - \frac{1}{k_1} G_\omega(x, z) v(z) \frac{\omega^2 \rho_0}{\omega^2 - \omega^2_{\nu_0}} \left( \int_D e_{\nu_0}^D(x) \, dx \right)^2 \\
+ \mathcal{O} \left( \varepsilon + \frac{\varepsilon^{1+\min(1;j)}}{\omega^2 - \omega^2_{\nu_0}} \right).
\]
We recall that $k_1 := \frac{k_1}{k_1} \varepsilon^2$, $\int_D e_n^B(x) \, dx = \varepsilon^2 \int_B e_n^B(x) \, dx$ and rewrite the last equation as

$$u^e(x) = u^s(x) - \frac{1}{k_1} G_\omega(x, z)v(z)\varepsilon \frac{\omega^2_0 \omega^2}{(\omega^2 - \omega^2_0)^2} \left(\int_B e_n^B(x) \, dx\right)^2 + O\left(\varepsilon + \varepsilon^{1+\min(1; j)}\right).$$

This justify the equations (11), (12) and Theorem 1.2.

5. Appendix. This section is devoted to justify Lemma 3.1, Lemma 3.2 and the mixed reciprocity condition.

Recall that $\Gamma_\omega(\cdot, \cdot)$ given by $\Gamma_\omega(x, y) = \rho_0(z) \frac{e^{ik_0(z)|x-y|}}{4\pi|x-y|}$ is the fundamental solution of the equation (6) with constant coefficients $\rho_0(z)$ and $k_0(z)$ satisfying the radiation conditions at infinity. By expanding in $|x-y|$ we have

$$\nabla \Gamma_\omega(x, y) = \nabla \Gamma_0(x, y) - \rho_0(z) (y-x) \left(\frac{\kappa^2_0}{8\pi|x-y|} + \frac{i\kappa^3_0}{12\pi} + O(|x-y|)\right).$$

Then, from (6) and since $\Delta \Gamma_\omega(x, y) + \kappa^2_0(z) \Gamma_\omega(x, y) = -\rho_0(z) \delta_y(x)$, integrating $G_\omega(x, y) - \Gamma_\omega(x, y)$ against the Dirac delta $-\delta_y(x)$, we have for any ball $B_R$ of large radius

$$(G_\omega - \Gamma_\omega)(x, y) = -\int_{B_R} \left(\frac{1}{\rho_0(t)} - \frac{1}{\rho_0(z)}\right) \nabla G_\omega(t, x) \cdot \nabla \Gamma_\omega(t, y) \, dt + \omega^2 \int_{B_R} \left(\frac{1}{k_0(t)} - \frac{1}{k_0(z)}\right) G_\omega(x, t) \Gamma_\omega(t, y) \, dt + \int_{\partial B_R} \left(\frac{1}{\rho_0(t)} - \frac{1}{\rho_0(z)}\right) \partial_t \Gamma_\omega(x, t) \, dt.$$

The first integral in (107) is $O(1)$ because

$$\int_{B_R} \left(\frac{1}{\rho_0(t)} - \frac{1}{\rho_0(z)}\right) \nabla G_\omega(t, x) \cdot \nabla \Gamma_\omega(t, y) \, dt = \langle \nabla \rho_0^{-1}(x) \cdot (t-y) \nabla G_\omega(t, x) \cdot \nabla \Gamma_\omega(t, y) \rangle dt + r(t, y),$$

where the term $r$ is a bounded function. We can show that the term $\int_{B_R} (t-y) \nabla G_\omega(t, x) \cdot \nabla \Gamma_\omega(t, y) \, dt$ is bounded by anti-symmetry. The second integral in (107) is bounded due to the smoothness of $k$; and the last integral is also bounded if we take $R$ large enough. Thus by the divergence theorem and the properties of the fundamental solutions $G_\omega$ and $\Gamma_\omega$ we have

$$\int_D \nabla \cdot \left[\rho_0(y)^{-1} \nabla G_\omega(x, y) - \rho_0(z)^{-1} \nabla \Gamma_\omega(x, y)\right] \, dy = \omega^2 \int_D (k_0(y)^{-1} G_\omega(x, y) - k_0(z)^{-1} \Gamma_\omega(x, y)) \, dy$$

and

$$= \omega^2 \int_D (k_0(y)^{-1} - k_0(z)^{-1}) G_\omega(x, y) \, dy$$

(108)

$$+ \omega^2 \int_D k_0(z)^{-1} (G_\omega - \Gamma_\omega)(x, y) \, dy = O(\varepsilon^3).$$
5.1. Proof of Lemma 3.1. To prove Lemma 3.1, we start by recalling the definition of the operator $J_ω^*$.

$$J_ω^* : = L^2(\partial D) \to L^2(\partial D)$$

$$f \to J_ω^*[f](x) := \int_{\partial D} \rho_0^{-1}(y) \frac{\partial G_ω}{\partial y}(x, y) f(y) d\sigma(y).$$

Let $\langle \cdot, \cdot \rangle$ be the $L^2(\partial D)$ inner product and $f$ and $g$ be two functions in $L^2(\partial D)$. We have

$$\langle J_ω^*[f], g \rangle := \int_{\partial D} J_ω^*[f](x) g(x) d\sigma(x)$$

$$= \int_{\partial D} g(x) \int_{\partial D} \rho_0^{-1}(y) \frac{\partial G_ω}{\partial y}(x, y) f(y) d\sigma(y) d\sigma(x)$$

$$= \int_{\partial D} f(x) \rho_0^{-1}(x) \int_{\partial D} g(y) \frac{\partial G_ω}{\partial y}(x, y) d\sigma(y) d\sigma(x)$$

$$= \int_{\partial D} f(x) \rho_0^{-1}(x) \left( K_ω^\ast \right)^* [g](x) d\sigma(x) = \langle f, \rho_0^{-1}(K_ω^\ast)[g] \rangle.$$

This proves that $(J_ω^*)^*[f](\cdot) = \rho_0^{-1}(\cdot) (K_ω^\ast)^*[f](\cdot)$.

Next, by definition, we have

$$J_ω^*[1](x) := \int_{\partial D \cap B(x, \varsigma)} \frac{\partial \omega}{\partial D}(x, y) \rho_0^{-1}(y) d\sigma(y)$$

$$= \int_{\partial D \cap B(x, \varsigma)} \frac{\partial \omega}{\partial D}(x, y) \rho_0^{-1}(y) d\sigma(y)$$

$$+ \int_{\partial D \setminus \partial D \cap B(x, \varsigma)} \frac{\partial \omega}{\partial D}(x, y) \rho_0^{-1}(y) d\sigma(y)$$

where $\varsigma$ is such that $\varsigma << \varepsilon$. From (6), for $\omega = 0$, we have

$$\int_{\partial D \setminus \partial D \cap B(x, \varsigma)} \frac{\partial \omega}{\partial D}(x, y) \rho_0^{-1}(y) d\sigma(y) = \int_{\partial D \setminus B(x, \varsigma)} \nabla \cdot \left[ \rho_0^{-1}(y) \nabla G_0(x, y) \right] dy$$

$$= \int_{\partial D \setminus B(x, \varsigma)} -\delta(y) dy = 0.$$

Then

$$J_ω^*[1](x) \sim \int_{\partial D \cap B(x, \varsigma)} \frac{\partial \omega}{\partial D}(x, y) \rho_0^{-1}(y) d\sigma(y)$$

$$+ \int_{\partial D \cap B(x, \varsigma)} \nu(y) \cdot \nabla (G_0 - \Gamma_0)(x, y) \rho_0^{-1}(y) d\sigma(y).$$

By the same arguments as in (107), we can prove that $\left| \nabla (G_0 - \Gamma_0) \right|(x, y) \sim \frac{1}{|x - y|}$.

This implies,

$$\left| \int_{\partial D \cap B(x, \varsigma)} \nu(y) \cdot \nabla (G_0 - \Gamma_0)(x, y) \rho_0^{-1}(y) d\sigma(y) \right| \lesssim \int_{\partial D \cap B(x, \varsigma)} \frac{1}{|x - y|} d\sigma(y) + \cdots = O(\varsigma).$$
Now, by developing $\rho_0^{-1}(\cdot)$ near $x$, we obtain
\[
J^0_D[1](x) = \int_{\partial D \cap B(x, \varsigma)} \nu(y) \cdot \nabla \left( \frac{1}{|x - y|} \right) d\sigma(y) \\
+ \rho_0(x) \int_{\partial D \cap B(x, \varsigma)} \nu(y) \cdot \nabla \left( \frac{1}{|x - y|} \right) \int_0^1 (y - x) \cdot \nabla \rho_0^{-1}(x + t(y - x)) dt d\sigma(y) \\
+ \mathcal{O}(\varsigma).
\]

An upper bound of the second integral is given by
\[
|\cdots| \lesssim \int_{\partial D \cap B(x, \varsigma)} \frac{|y - x|}{|x - y|} d\sigma(y) = \mathcal{O}(\varsigma^2).
\]

Finally,
\[
J^0_D[1](x) = \int_{\partial D \cap B(x, \varsigma)} \nu(y) \cdot \nabla \left( \frac{1}{|x - y|} \right) d\sigma(y) + \mathcal{O}(\varsigma).
\]

By taking the limit when $\varsigma \to 0$, the error term goes to zero and the first term goes to $-1/2$, see ([14], page 40-41). We deduce that $J^0_D[1](\cdot) = -1/2$. Similar proof works for an arbitrary density $f$, in particular for $f := \rho_0$, and we have
\[
(109) \quad K^0_D[1](\cdot) := J^0_D[\rho_0](\cdot) = -\frac{\rho_0(\cdot)}{2}.
\]

Let us now estimate the variation $\left( J^\omega_D - J^0_D \right)[1](x)$. We have
\[
\left( J^\omega_D - J^0_D \right)[1](x) = \int_D \nabla \cdot \left[ \rho_0^{-1}(y) \nabla (G_\omega - G_0)(x, y) \right] dy \\
= -\frac{\kappa_0^2}{\rho_0(z)} \int_D \Gamma_0(x, y) dy - \frac{\kappa_0^2}{\rho_0(z)} \int_D (\Gamma_\omega - \Gamma_0)(x, y) dy \\
+ \int_D \nabla \cdot \left[ \rho_0^{-1}(y) \nabla G_0(x, y) - \rho_0^{-1}(z) \nabla \Gamma_0(x, y) \right] dy \\
+ \int_D \nabla \cdot \left[ \rho_0^{-1}(y) \nabla G_\omega(x, y) - \rho_0^{-1}(z) \nabla \Gamma_\omega(x, y) \right] dy.
\]

Since $|x - y|$ is small, clearly, we have $\rho_0^{-1}(z) \kappa_0^2 \int_D (\Gamma_\omega - \Gamma_0)(x, y) dy = \mathcal{O}(\varepsilon^3)$ and from (108), we deduce that $\int_D \nabla_y \cdot \left[ \rho_0^{-1}(y) \nabla G_\omega(x, y) - \rho_0^{-1}(z) \nabla \Gamma_\omega(x, y) \right] dy$ and $\int_D \nabla_y \cdot \left[ \rho_0^{-1}(y) \nabla G_0(x, y) - \rho_0^{-1}(z) \nabla \Gamma_0(x, y) \right] dy$ behave as $\varepsilon^3$. Then
\[
\left( J^\omega_D - J^0_D \right)[1](x) = -\rho_0^{-1}(z) \kappa_0^2 \int_D \Gamma_0(x, y) dy + \mathcal{O}(\varepsilon^3)
\]
moreover
\[
\rho_0^{-1}(z) \Gamma_0(x, y) = \frac{1}{4\pi|x - y|} = -\frac{1}{2} \nabla \cdot \left[ \frac{(x - y)}{4\pi|x - y|} \right]
\]
then a simple integration ends the proof of Lemma 3.1.
5.2. Proof of Lemma 3.2. First, for an arbitrary \( \lambda \in \mathbb{R} \), we investigate the invertibility of \( (\lambda + \frac{1}{2}) I + J_D^0 \). For this, let \( f \in L^2(\partial\Omega) \) such that \( f \neq 0 \) and \( (\lambda + \frac{1}{2}) I + (J_D^0)^* \) \( [f] = 0 \), then we have

\[
0 = \int_{\partial\Omega} \left((\lambda + \frac{1}{2}) I + (J_D^0)^*\right) [f] 1 \, d\sigma = \int_{\partial\Omega} f \left((\lambda + \frac{1}{2}) I + J_D^0\right) [1] \, d\sigma = \lambda \int_{\partial\Omega} f \, d\sigma.
\]

With standard argument, see for instance [2], we show that \((\lambda + \frac{1}{2}) I + J_D^0 \) is invertible in \( L^2(\partial\Omega) \) with

\[
\left\| \left((\lambda + \frac{1}{2}) I + J_D^0\right)^{-1} \right\|_{\mathcal{L}(L^2(\partial\Omega))} = O(1/\lambda),
\]

and

\[
\left\| \left((\lambda + \frac{1}{2}) I + J_D^0\right)^{-1} \right\|_{\mathcal{L}(L^2(\partial\Omega))} = O(1), \text{ uniformly on } \lambda.
\]

Next, we investigate the invertibility of \((\lambda + \frac{1}{2}) I + J_D^\omega \) in \( L^2(\partial\Omega) \). For this, we need to compute \( \|J_D^\omega - J_D^0\|_L \).

We have

\[
\|J_D^\omega - J_D^0\|_{\mathcal{L}(L^2(\partial\Omega))} = \left\| \left(J_D^\omega - J_D^0\right)^* \right\|_{\mathcal{L}(L^2(\partial\Omega))}
\]

\[
:= \sup_{\|f\|_{L^2(\partial\Omega)} = 1} \left\| \left(J_D^\omega - J_D^0\right)^* [f] \right\|_{L^2(\partial\Omega)}
\]

\[
= \sup_{\|f\|_{L^2(\partial\Omega)} = 1} \left\| \rho_0^{-1} (K_D^\omega - K_D^0)^* [f] \right\|_{L^2(\partial\Omega)}
\]

\[
\leq \sup_{\|f\|_{L^2(\partial\Omega)} = 1} \left\| (K_D^\omega - K_D^0)^* [f] \right\|_{L^2(\partial\Omega)}
\]

\[
\overset{(25)}{\approx} \sup_{\|f\|_{L^2(\partial\Omega)} = 1} \left\| \int_{\partial\Omega} \frac{\nu(\cdot \cdot \cdot - y)}{|\cdot - y|} f(y) \, d\sigma(y) \right\|_{L^2(\partial\Omega)}
\]

\[
= O(\varepsilon^2).
\]

Then

\[
(\lambda + \frac{1}{2}) I + J_D^\omega = \left((\lambda + \frac{1}{2}) I + J_D^0 - J_D^0 + J_D^\omega\right)
\]

\[
(\lambda + \frac{1}{2}) I + J_D^\omega = \left((\lambda + \frac{1}{2}) I + J_D^0\right) \left[I - \left((\lambda + \frac{1}{2}) I + J_D^0\right)^{-1} (J_D^0 - J_D^\omega)\right],
\]

and, by taking the inverse, we obtain

\[
\left((\lambda + \frac{1}{2}) I + J_D^0\right)^{-1} = \left[I - \left((\lambda + \frac{1}{2}) I + J_D^0\right)^{-1} (J_D^0 - J_D^\omega)\right]^{-1} \left((\lambda + \frac{1}{2}) I + J_D^0\right)^{-1}
\]
we know that \((\lambda + \frac{1}{2}) I + J_D^0)\) exists, then it suffices to prove that the first operator on the right hand side exists also. Using (110) and assuming that \(\lambda^{-1} \varepsilon^2 < 1\), we have

\[
\left\| \left( \lambda + \frac{1}{2} \right) I + J_D^0 \right\|_{L(L^2(\partial D))} \lesssim \frac{\varepsilon^2}{\text{dist} \left( \lambda + \frac{1}{2}; \sigma \left( J_D^0 \right) \right)} \leq \frac{\varepsilon^2}{\lambda} < 1,
\]

then by the Neumann series representation for the inverse operator we deduce that the first operator on the right hand side of (111) exists and consequently \((\lambda + \frac{1}{2}) I + J_D^\omega)^{-1}\) is well defined. Again, by (111) we deduce that

\[
(112) \quad \left\| \left( \lambda + \frac{1}{2} \right) I + J_D^\omega \right\|_{L(L^2(\partial D))} \leq \lambda^{-1}.
\]

Similar arguments allow to obtain the following estimation as well

\[
\left\| \left( \lambda + \frac{1}{2} \right) I + J_D^\omega \right\|_{L(L^2(\partial D))} = \mathcal{O}(1), \text{ uniformly on } \lambda.
\]

From \(J_D^\omega\) we construct the operator \(B_D\) defined from \(L^2(\partial D)\) to \(L^2(\partial D)\) as \(\forall f \in L^2(\partial D), \ x \in \partial D,\)

\[
B_D[f](x) := \left[ \left( \rho_0^{-1}(x) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right] [f](x)
\]

where we recall that \(\alpha(z) := \rho_1^{-1} - \rho_0(z)\) with \(\rho_1 = \overline{\rho}_1 \varepsilon^2\). We have

\[
B_D = \left[ \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right] + \int_0^1 (\cdot - z) \cdot \nabla \rho_0^{-1}(z + t(\cdot - z)) dt I
\]

\[
= \left[ \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right] I
\]

\[
+ \left[ \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right]^{-1} \int_0^1 (\cdot - z) \cdot \nabla \rho_0^{-1}(z + t(\cdot - z)) dt I.
\]

Then

\[
B_D^{-1} = \left[ I + \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right]^{-1} \int_0^1 (\cdot - z) \cdot \nabla \rho_0^{-1}(z + t(\cdot - z)) dt I
\]

\[
(113) \quad \left[ \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right]^{-1}.
\]

We know that \(\left[ \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right]^{-1}\) exists if \(\left( \rho_0^{-1}(z) \alpha^{-1}(z) \right)^{-1} \varepsilon^2 < 1\) or equivalently if \(\rho_0(z) < (1 + \varepsilon^2) \overline{\rho}_1\), but recall that we have assumed \(\overline{\rho}_1\) large enough such that the previous condition is satisfied. Next, it is sufficient to prove that

\[
\zeta := \left\| \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J_D^\omega)^* \right\|_{L^2(\partial D)} \lesssim \frac{\varepsilon^2}{\lambda} < 1.
\]
is less than 1. For this, thanks to (112), we can prove that \( \zeta = \mathcal{O}(\varepsilon) \). Finally, \( B_D \)

is invertible.

From (113), we have

\[
\| B_D^{-1} \|_{L^2(\partial D)} \lesssim \frac{1}{1 - \zeta} \left\| \left( \rho_0^{-1}(z) \alpha^{-1}(z) + \frac{1}{2} \right) I + (J \rho_0^* \alpha^*)^{-1} \right\|_{L^2(\partial D)} \approx \alpha(z).
\]

With the same arguments we can prove that \( \| B_D^{-1} \|_{L^2(\partial D)} = \mathcal{O}(1) \).

5.3. Proof of the mixed reciprocity relation. Set \( \Phi_\omega(x, \cdot) \) to be the fundamental solution of

\[
\nabla_y \cdot \left[ \frac{1}{\rho_0} \nabla \Phi_\omega(x, y) \right] + \frac{\omega^2}{k_0} \Phi_\omega(x, y) = -\delta(y), \quad x, y \in \mathbb{R}^3.
\]

Since \( \rho_0 \) is constant, the previous equation will be reduced to Helmholtz one and \( \Phi_\omega(x, \cdot) \) takes the following form

\[
\Phi_\omega(x, y) := \rho_0 \frac{e^{i \omega \sqrt{\rho_0/k_0} |x-y|}}{4 \pi |x-y|}, \quad x \neq y.
\]

From (6), we can see that both \( G_\omega(x, \cdot) \) and \( \Phi_\omega(x, \cdot) \) satisfy the same equation outside the domain \( \Omega \). Then, for \( x, z \in \mathbb{R}^3 \setminus \overline{\Omega} \), we have the following Green’s formula

\[
G_\omega(x, z) - \Phi_\omega(x, z) = \int_{\partial \Omega} \frac{(G_\omega - \Phi_\omega)(y, z) \partial \Phi_\omega(y, x)}{\partial \nu(y)} \, ds(y)
\]

and we have the associated far field

\[
G_\omega(x, z) - \rho_0 \frac{e^{-i \omega \sqrt{\rho_0/k_0} |x-z|}}{4 \pi} = \rho_0 \frac{e^{-i \omega \sqrt{\rho_0/k_0} |x-z|}}{4 \pi} \int_{\partial \Omega} \frac{(G_\omega - \Phi_\omega)(y, z) \partial \Phi_\omega(y, x)}{\partial \nu(y)} \, ds(y)
\]

\[
- \frac{\partial}{\partial \nu(y)} \left[ (G_\omega - \Phi_\omega)(y, z) e^{-i \omega \sqrt{\rho_0/k_0} |x-y|} \right] \, ds(y)
\]

\[
- \frac{\rho_0^* v^*(\cdot, \cdot) \frac{e^{-i \omega \sqrt{\rho_0/k_0} |x-y|}}{4 \pi}}{\partial \nu(y)} \, ds(y) = 0 \quad \text{for } x, z \in \mathbb{R}^3 \setminus \overline{\Omega}.
\]

But for \( z \in \mathbb{R}^3 \setminus \overline{\Omega} \), we have

\[
\rho_0^* v^*(z, \cdot) = -\int_{\partial \Omega} G_\omega(y, z) \frac{\partial v^*(y, x)}{\partial \nu(y)} - \frac{\partial G_\omega(y, z)}{\partial \nu(y)} v^*(y, x) \, ds(y)
\]

\[
= -\int_{\partial \Omega} G_\omega(y, z) \frac{\partial v^*(y, x)}{\partial \nu(y)} - \frac{\partial G_\omega(y, z)}{\partial \nu(y)} v(y, x) \, ds(y)
\]

\[
+ \int_{\partial \Omega} G_\omega(y, z) \frac{\partial v^i(y, x)}{\partial \nu(y)} - \frac{\partial G_\omega(y, z)}{\partial \nu(y)} v^i(y, x) \, ds(y).
\]
where \( v'(\cdot, \hat{x}) \) is plane wave given, for any \( y \in \mathbb{R}^3 \), by \( v'(y, \hat{x}) = e^{i \omega \sqrt{\rho_0/\kappa_0} \cdot \hat{x} \cdot y} \) and \( v^s(\cdot, \hat{x}) \) is the scattered field associated to the total field \( v(\cdot, \hat{x}) \),

\[
v'(\cdot, \hat{x}) := v^s(\cdot, \hat{x}) + v'(\cdot, \hat{x}),
\]

solution of

\[
\nabla_y \left[ \rho_0^{-1}(y) \nabla v(y, \hat{x}) \right] + \frac{\omega^2}{k_0} v(y, \hat{x}) = 0.
\]

Then, for \( z \in \mathbb{R}^3 \setminus \Omega \), we have

\[
G^\infty_\omega(-\hat{x}, z) - \frac{\rho_0}{4 \pi} e^{i \omega \sqrt{\rho_0/\kappa_0} \cdot \hat{x} \cdot z} = \frac{\rho_0}{4 \pi} \left[ \int_{\partial \Omega} G_\omega(y, z) \frac{\partial v(y, \hat{x})}{\partial \nu(y)} - \frac{\partial G_\omega(y, z)}{\partial \nu(y)} v(y, \hat{x}) \, d\sigma(y) + v^s(z, \hat{x}) \right],
\]

and, since \( z \notin \Omega \), we have

\[
\int_{\partial \Omega} G_\omega(y, z) \frac{\partial v(y, \hat{x})}{\partial \nu(y)} - \frac{\partial G_\omega(y, z)}{\partial \nu(y)} v(y, \hat{x}) \, d\sigma(y) = 0,
\]

then we get

\[
G^\infty_\omega(-\hat{x}, z) = \frac{\rho_0}{4 \pi} v(z, \hat{x}), \quad z \in \mathbb{R}^3 \setminus \overline{\Omega}.
\]

The identity (114) is also true for \( z \) approaching \( \partial \Omega \) as both \( G^\infty_\omega(-\hat{x}, z) \) and \( v(z, \hat{x}) \) have limits as \( z \) approaches \( \partial \Omega \). In addition, repeating the same arguments as above, we deduce that \( \partial_\nu G^\infty_\omega(-\hat{x}, z) = \frac{\rho_0}{4 \pi} \partial_\nu v(z, \hat{x}) \) on \( \partial \Omega \) as well.

Now, for \( z \in \Omega \) and \( x \in \mathbb{R}^3 \setminus \Omega \), we have

\[
G_\omega(x, z) = \int_{\partial \Omega} G_\omega(y, z) \frac{\partial G_\omega}{\partial \nu(y)}(x, y) - \frac{\partial G_\omega}{\partial \nu(y)}(y, z) G_\omega(x, y) \, d\sigma(y).
\]

Then,

\[
G^\infty_\omega(\hat{x}, z) = \int_{\partial \Omega} G_\omega(y, z) \frac{\partial G^\infty_\omega}{\partial \nu(y)}(\hat{x}, y) - \frac{\partial G_\omega}{\partial \nu(y)}(y, z) G^\infty_\omega(\hat{x}, y) \, d\sigma(y),
\]

or

\[
G^\infty_\omega(-\hat{x}, z) = \frac{\rho_0}{4 \pi} \int_{\partial \Omega} G_\omega(y, z) \frac{\partial v}{\partial \nu(y)}(\hat{x}, y) - \frac{\partial G_\omega}{\partial \nu(y)}(y, z) v(\hat{x}, y) \, d\sigma(y)
\]

\[
= \frac{\rho_0}{4 \pi} v(z, \hat{x}).
\]

This proves that

\[
G^\infty_\omega(-\hat{x}, z) = \frac{\rho_0}{4 \pi} v(z, \hat{x}), \quad \text{for } z \in \mathbb{R}^3.
\]
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