Lower bounds for Rankin–Selberg $L$-functions on the edge of the critical strip

by

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1. Introduction. The value distributions of $L$-functions on the edge of the critical strip are important in number theory. In particular, the lower bounds for the $L$-values along the edge $\Re s = 1$ are closely related to the determination of zero-free regions of the corresponding $L$-functions.

In this paper, we are interested in the lower bounds of Rankin–Selberg $L$-functions along the edge of the critical strip. More precisely, let $F/\mathbb{Q}$ be a number field of degree $n_F$, and let $\pi_1$ and $\pi_2$ be unitary cuspidal automorphic representations of $\text{GL}_{n_1}(\mathbb{A}_F)$ and $\text{GL}_{n_2}(\mathbb{A}_F)$ respectively. Assume that the central characters of $\pi_1$ and $\pi_2$ are unitary and normalized so that they are trivial on the diagonally embedded copies of the positive reals. Note that, with this normalization, the Rankin–Selberg $L$-functions $L(s, \pi_1 \times \tilde{\pi}_1)$ and $L(s, \pi_2 \times \tilde{\pi}_2)$ both have simple poles at $s = 1$. The main subject of this paper is a lower bound for the Rankin–Selberg $L$-function $L(1 + it, \pi_1 \times \tilde{\pi}_2)$, which in turn yields a new zero-free region for $L(s, \pi_1 \times \tilde{\pi}_2)$.

In [Sh80], Shahidi showed that $L(1 + it, \pi_1 \times \tilde{\pi}_2) \neq 0$ for every $t \in \mathbb{R}$. Since then, there has been substantial progress on the determination of lower bounds and zero-free regions for these Rankin–Selberg $L$-functions, especially in the following two scenarios. On the one hand, if either $\pi_1$ or $\pi_2$ is self-dual, then it is known, by the work of Moreno [Mo85, Theorem 3.3] and Sarnak [Sa04], that $L(s, \pi_1 \times \tilde{\pi}_2)$ has the classical zero-free region of the de la Vallée Poussin type. More explicitly, let $\varepsilon(\pi_1)$ and $\varepsilon(\pi_2)$ be the analytic conductors of $\pi_1$ and $\pi_2$ respectively and assume that $\pi_2$ is self-dual; then Brumley [HB19, Theorem A.1] proved an explicit zero-free region

$$\sigma \geq 1 - \frac{c}{(n_1 + n_2)^3 \log(\varepsilon(\pi_1)\varepsilon(\pi_2)(|t| + 2)^{n_1 n_F})}$$
for some absolute constant $c > 0$, with the possible exception of a real simple zero if $\pi_1$ is also self-dual. On the other hand, the special case with $\pi_1 = \pi_2$ has also attracted much attention. Goldfeld and Li [GL18 Theorem 1.3] obtained the lower bound

$$L(1 + it, \pi \times \overline{\pi}) \gg \frac{1}{(\log(|t| + 2))^3},$$

where $\pi$ is an irreducible cuspidal unramified representation of $GL_n(\mathbb{A}_Q)$ with $n \geq 2$, tempered almost everywhere, and this lower bound immediately leads to the zero-free region for $L(s, \pi \times \overline{\pi})$ (see [GL18 Theorem 1.2])

$$\sigma > 1 - \frac{c_\pi}{(\log(|t| + 2))^5}, \quad |t| \geq 1.$$

These results were improved by Humphries and Brumley [HB19, Theorem 1.9] and generalized to an arbitrary number field $F$ and to possibly ramified representations $\pi$ over $GL_n(\mathbb{A}_F)$, with the lower bound

$$L(1 + it, \pi \times \overline{\pi}) \gg \frac{1}{\log(|t| + 3)}$$

and the corresponding zero-free region for $L(s, \pi \times \overline{\pi})$:

$$\sigma > 1 - \frac{c_\pi}{\log(|t| + 2)}, \quad |t| \geq 1. \quad (1.1)$$

Humphries and Thorner [HT22] further dropped the temperedness requirement, and made explicit the dependence of the constant $c_\pi$ upon $\pi$ in (1.1), finding a zero-free region

$$\sigma \geq 1 - \frac{c}{\log(c(\pi)n(|t| + e)^{n_1 n_F})}, \quad |t| \geq 1,$$

for some effective absolute constant $c > 0$.

The study of general Rankin–Selberg $L$-functions is much more difficult, and our knowledge is rather limited. A lower bound of the form

$$L(1 + it, \pi_1 \times \pi_2) \gg \frac{1}{(c(\pi_1)c(\pi_2)(|t| + 2))^{N_{\pi_1, \pi_2}}}$$

for some constant $N_{\pi_1, \pi_2} > 0$ and the corresponding zero-free region

$$\sigma \geq 1 - \frac{c}{(c(\pi_1)c(\pi_2)(|t| + 2))^{N_{\pi_1, \pi_2}}}$$

were obtained by Brumley [Br06 Theorem 5] in connection with the strong multiplicity one theorem, and by Gelbart and Lapid [GL06 Theorem 1] for a broad family of $L$-functions. Brumley [LB13] gave an explicit value for the exponent $N_{\pi_1, \pi_2}$. More precisely, if $\pi_1 \neq \pi_2$, then he showed in [LB13 Theorem A.1] that

$$L(1 + it, \pi_1 \times \pi_2) \gg_{F, n_1, n_2, \varepsilon} c(\Pi_1 \times \Pi_2)^{-\frac{1}{2}}(1-\frac{1}{n_1 + n_2})^{-\varepsilon}, \quad (1.2)$$
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where

\[ c(\Pi_t \times \tilde{\Pi}_t) = c(\pi_1 \times \tilde{\pi}_1)c(\pi_2 \times \tilde{\pi}_2)c(\pi_1 \times \tilde{\pi}_2, 1 + it)c(\tilde{\pi}_1 \times \pi_2, 1 - it). \]

In particular, by the estimate

\[ c(\Pi_t \times \tilde{\Pi}_t) \leq (c(\pi_1)c(\pi_2))^{2n_1+2n_2}(|t| + 2)^{2n_F n_1 n_2}, \]

the lower bound (1.2) can be rewritten as

\[ L(1 + it, \pi_1 \times \tilde{\pi}_2) \gg (c(\pi_1)c(\pi_2))^{-(n_1+n_2-1)-\varepsilon}(|t| + 2)^{-n_F n_1 n_2 \left(1 - \frac{1}{n_1+n_2}\right)^{-\varepsilon}}. \]

Further, Brumley [LB13, Theorem A.1] also derived the corresponding zero-free regions for $L(s, \pi_1 \times \tilde{\pi}_2)$, and showed that the above lower bounds still hold in these zero-free regions.

In this paper, we seek to improve upon the above results of Brumley in the $t$-aspect, in particular the lower bound (1.5), and obtain the following theorem.

**Theorem 1.1.** Let $F/\mathbb{Q}$ be a number field of degree $n_F$, and let $\pi_1$ and $\pi_2$ be unitary cuspidal automorphic representations of $GL_{n_1}(\mathbb{A}_F)$ and $GL_{n_2}(\mathbb{A}_F)$ respectively. Assume that $\pi_1 \neq \pi_2$, and that the central characters of $\pi_1$ and $\pi_2$ are unitary and normalized so that they are trivial on the diagonally embedded copies of the positive reals. Then

\[ L(1 + it, \pi_1 \times \tilde{\pi}_2) \gg (c(\pi_1)c(\pi_2))^{-(n_1+n_2-1)-\varepsilon}(|t| + 2)^{-n_F n_1 n_2 \left(1 - \frac{1}{n_1+n_2}\right)^{-\varepsilon}}, \]

where the $\gg$-constant depends upon $F$, $n_1$, $n_2$ and $\varepsilon$ only.

**Remark 1.2.** Brumley [LB13] used the notation $L(s, \pi_1 \times \tilde{\pi}_2)$ for the complete Rankin–Selberg $L$-function. Our (finite) $L$-function $L(s, \pi_1 \times \tilde{\pi}_2)$ is what he denoted by $L^\infty(s, \pi_1 \times \tilde{\pi}_2)$.

**Remark 1.3.** The work of Brumley [LB13, Theorem A.1] also considered the case $\pi_1 = \pi_2$, and similar results were obtained for the lower bounds and zero-free regions. As is clear from Section 3 below, our arguments can also be applied to derive the corresponding improvements of these results in the $t$-aspect. However, these results have now been superseded by the above-mentioned works of Goldfeld and Li [GL18], of Humphries and Brumley [HB19], and of Humphries and Thorner [HT22]. Hence we skip the relevant discussions in this paper.

**Remark 1.4.** Recently, Harcos and Thorner [HT23] considered similar problems on lower bounds and zero-free regions, not for individual Rankin–Selberg $L$-functions $L(s, \pi_1 \times \tilde{\pi}_2)$ themselves but for their $GL_1$-twists $L(s, \pi_1 \times (\pi_2 \otimes \chi))$ as $\chi$ runs through Dirichlet characters.

Applying the results of Li [Li10], we can also derive from the above lower bound in Theorem 1.1 the corresponding zero-free region for the Rankin–
Selberg $L$-function $L(s, \pi_1 \times \pi_2)$. Since the arguments are essentially the same as the first part of the proof to [LB13, Theorem A.1], we omit the details here and only present our result in the following corollary.

**Corollary 1.5.** Under the assumptions of Theorem 1.1 for every $\varepsilon > 0$ there exists a constant $c = c(F, n_1, n_2, \varepsilon) > 0$ such that

$$L(s, \pi_1 \times \pi_2) \gg_{F, n_1, n_2, \varepsilon} \left( c(\pi_1) c(\pi_2) \right)^{(n_1 + n_2 - 1) - \varepsilon} \left( |t| + 2 \right)^{-\frac{n_1 n_2}{2} \left( 1 - \frac{1}{n_1 + n_2} \right) - \varepsilon}$$

in the region

$$\left\{ s = \sigma + it \mid \sigma \geq 1 - \frac{c}{(c(\pi_1) c(\pi_2))^{(n_1 + n_2 - 1) + \varepsilon} \left( |t| + 2 \right)^{-\frac{n_1 n_2}{2} \left( 1 - \frac{1}{n_1 + n_2} \right) + \varepsilon}. \right\}$$

**2. Preliminaries.** Let $F$ be a number field of degree $n_F$, and let $\pi_1$ and $\pi_2$ be distinct unitary cuspidal automorphic representations of $GL_{n_1}(\mathbb{A}_F)$ and $GL_{n_2}(\mathbb{A}_F)$ respectively. Assume that the central characters of $\pi_1$ and $\pi_2$ are unitary and normalized so that they are trivial on the diagonally embedded copies of the positive reals.

Since the Rankin–Selberg $L$-functions $L(s, \pi_1 \times \pi_1)$ and $L(s, \pi_2 \times \pi_2)$ both have simple poles at $s = 1$, we may write their Laurent series expansions at $s = 1$ as

$$L(s, \pi_1 \times \pi_1) = \sum_{k=-1}^{\infty} A_k (s - 1)^k, \quad L(s, \pi_2 \times \pi_2) = \sum_{k=-1}^{\infty} B_k (s - 1)^k.$$

**Lemma 2.1.** For every $\varepsilon > 0$, there exists a constant $C_1 = C_1(\varepsilon) > 0$ such that

$$|A_{-1}| \leq C_1 c(\pi_1)^{\varepsilon}, \quad |A_0| \leq C_1 c(\pi_1)^{\varepsilon},$$

$$|B_{-1}| \leq C_1 c(\pi_2)^{\varepsilon}, \quad |B_0| \leq C_1 c(\pi_2)^{\varepsilon}.$$

**Proof.** The lemma follows from [Li10, Theorem 2]; see also [LB13, (43)].

Further, for every $t \in \mathbb{R}$ consider the isobaric representation

$$\Pi_t = (\pi_1 \otimes |\det|^{it/2}) \boxplus (\pi_2 \otimes |\det|^{-it/2});$$

then we have the factorization of $L$-functions

$$L(s, \Pi_t \times \tilde{\Pi}_t) = L(s, \pi_1 \times \pi_1) L(s, \pi_2 \times \pi_2) L(s + it, \pi_1 \times \pi_2) L(s - it, \pi_1 \times \pi_2)$$

and the corresponding factorization (1.3) of analytic conductors. In particular, by the bound [Br06, (8)] we have the estimate (1.4).

By the factorization (2.1), since $\pi_1$ and $\pi_2$ are assumed to be distinct, the Rankin–Selberg $L$-function $L(s, \Pi_t \times \tilde{\Pi}_t)$ has a double pole at $s = 1$ and
Hence our lemma follows by combining the above estimates. Accordingly, we write the Laurent series expansion of $L(s, \Pi_t \times \tilde{\Pi}_t)$ at $s = 1$ as

$$L(s, \Pi_t \times \tilde{\Pi}_t) = \sum_{k=-2}^{\infty} r_k(s-1)^k.$$ 

**Lemma 2.2.** For every $\varepsilon > 0$, there exists a constant $C_2 = C_2(\varepsilon) > 0$ such that

$$|r_{-1}| + |r_{-2}| \geq C_2(c(\Pi_t) \cdot c(\tilde{\Pi}_t))^{-\frac{1}{2}}(1 - \frac{1}{n_1 + n_2})^{-\varepsilon} \geq C_2(c(\pi_1) c(\pi_2))^{-(n_1 + n_2 - 1) - \varepsilon} |\varepsilon + 2|^{-n_1 n_2 (1 - \frac{1}{n_1 + n_2})^{-\varepsilon}}.$$ 

**Proof.** This follows immediately from [Br06, Theorem 3].

**Lemma 2.3.** Let $|t| \geq 1$. Then

$$|r_{-1}| + |r_{-2}| \leq 3C_1^2(c(\pi_1) c(\pi_2))^\varepsilon |L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2 + C_1^2(c(\pi_1) c(\pi_2))^\varepsilon \left| \frac{d}{dt}(|L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2) \right|,$n

where $C_1$ is as given in Lemma 2.1.

**Proof.** By the factorization (2.1), we have

$$r_{-2} = A_{-1} B_{-1} |L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2,$n

so by Lemma 2.1 we have

$$|r_{-2}| = |A_{-1}||B_{-1}||L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2 \leq C_1^2 c(\pi_1)^\varepsilon c(\pi_2)^\varepsilon |L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2.$$ 

Also, by (2.1) we have

$$r_{-1} = (A_{-1} B_0 + A_0 B_{-1}) |L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2 + 2A_{-1} B_{-1} \Re(L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2) \cdot \overline{L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)}) = (A_{-1} B_0 + A_0 B_{-1}) |L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2 + A_{-1} B_{-1} \frac{d}{dt}(|L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2),$$ 

so again by Lemma 2.1 we have

$$|r_{-1}| \leq (|A_{-1}||B_0| + |A_0||B_{-1}|) |L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2 + |A_{-1}||B_{-1}| \left| \frac{d}{dt}(|L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2) \right| \leq C_1^2 c(\pi_1)^\varepsilon c(\pi_2)^\varepsilon |L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2 + C_1^2 c(\pi_1)^\varepsilon c(\pi_2)^\varepsilon \left| \frac{d}{dt}(|L(1 + i\varepsilon, \pi_1 \times \tilde{\pi}_2)|^2) \right|.$$ 

Hence our lemma follows by combining the above estimates.
3. Proof of Theorem 1.1. In this proof we will freely use the notations introduced in Section 2. Further, for simplicity we also write

\[ g(t) = |L(1 + it, \pi_1 \times \pi_2)|^2, \]
\[ \theta = \frac{n_F n_1 n_2}{2} \left( 1 - \frac{1}{n_1 + n_2} \right). \]

For illustration, let us begin with a brief sketch of the proof. A crucial observation is that, by Lemmas 2.2 and 2.3,

\[ g(t) + |g'(t)| \gg |r_{-1}| + |r_{-2}| \gg (|t| + 2)^{-n_F n_1 n_2 \left( 1 - \frac{1}{n_1 + n_2} \right)^{-\varepsilon}} = (|t| + 2)^{-2\theta - \varepsilon}, \]

where for simplicity we allow the \( \gg \)-constant to depend upon \( \pi_1 \) and \( \pi_2 \). This estimate implies that \( g(t) \) and \( |g'(t)| \) cannot be small simultaneously. Now assume for the sake of contradiction that \( g(t) \) is “small” over an interval \( [t_1, t_2] \), namely

\[ g(t) \ll (|t| + 2)^{-2\theta - 2\varepsilon} \quad (t_1 \leq t \leq t_2), \]

and consider the Fundamental Theorem of Calculus

\[ g(t_1) - g(t_2) = \int_{t_1}^{t_2} (-g'(t)) \, dt. \]

On the one hand, since both \( g(t_1) \) and \( g(t_2) \) are small, the left-hand side of (3.3) is small. On the other hand, since \( g(t) \) is small over \( [t_1, t_2] \), the absolute value \( |g'(t)| \) has to be large; in fact, it can be shown that over \( [t_1, t_2] \) we have \( g'(t) < 0 \) and

\[ -g'(t) \gg (t + 2)^{-2\theta - \varepsilon} \quad (t_1 \leq t \leq t_2), \]

so the right-hand side of (3.3) cannot be small. With appropriate choices of the implicit constants, we will be able to reach a contradiction to (3.3).

Proof of Theorem 1.1. The theorem is well known if \( n_1 = n_2 = 1 \), so henceforth we assume that either \( n_1 \geq 2 \) or \( n_2 \geq 2 \). Note that this assumption implies that the exponent \( \theta \) defined in (3.2) has a lower bound

\[ \theta \geq 1/2. \]

Let \( \varepsilon > 0 \). By [LB13, Theorem A.1], there exists a constant \( C'_{\varepsilon} > 0 \) with

\[ |L(1 + it, \pi_1 \times \pi_2)| \geq C'_{\varepsilon} c(\Pi_t \times \Pi_t) \left( 1 - \frac{1}{n_1 + n_2} \right)^{-\frac{1}{2}} \left( 1 - \frac{1}{n_1 + n_2} \right)^{-\frac{\varepsilon}{2(n_1 + n_2 + n_F n_1 n_2)}}. \]

Recall the bound (1.4):

\[ c(\Pi_t \times \Pi_t) \leq (c(\pi_1)c(\pi_2))^{2n_2 + 2n_2} (|t| + 2)^{2n_F n_1 n_2}; \]
then over the interval $[-1,1]$ the above lower bound gives

$$\left| L(1 + it, \pi_1 \times \pi_2) \right| \geq C_\varepsilon' \left( c(\pi_1)c(\pi_2) \right)^{-\frac{(n_1+n_2-1)\varepsilon}{2}} (|t| + 2)^{-2\theta - \varepsilon}$$

Further, if $|t| < t_1$, then also

$$g(t) \geq C_\varepsilon'^2 (|t| + 2)^{-2\theta - 2\varepsilon} \quad (t \in \mathbb{R}),$$

where the function $g(t)$ is defined in (3.1),

$$C_\varepsilon = \min \left\{ \frac{C_\varepsilon'}{3\theta}, \left( c(\pi_1)c(\pi_2) \right)^{-\frac{(n_1+n_2-1)\varepsilon}{2}}, \sqrt{\frac{C_2}{6C_1^2(2\theta + 2\varepsilon) \left( c(\pi_1)c(\pi_2) \right)^{-\frac{n_1+n_2-1}{2} - \varepsilon}}} \right\}$$

and $C_1$ and $C_2$ are the constants defined in Lemmas 2.1 and 2.2 respectively. Note in particular that

$$C_\varepsilon \gg \left( c(\pi_1)c(\pi_2) \right)^{-\frac{(n_1+n_2-1)\varepsilon}{2}},$$

where the $\gg$-constant depends upon $F$, $n_1$, $n_2$ and $\varepsilon$ only. Hence Theorem 1.1 follows from the claim (3.6) immediately.

By the construction of $C_\varepsilon$ and the estimate (3.5), the claim (3.7) is obviously valid over $[-1,1]$, so it suffices to prove the claim for $|t| > 1$, and we will argue by contradiction.

Assume that there exists some point $|t_0| > 1$ such that

$$g(t_0) < C_\varepsilon^2 (|t_0| + 2)^{-2\theta - 2\varepsilon}.$$  

Without loss of generality, we may assume that $t_0 > 1$. Define

$$S_1 = \{ 0 \leq t < t_0 \mid g(t) \geq C_\varepsilon^2 (t + 2)^{-2\theta - 2\varepsilon} \}.$$ 

Since $[0,1] \subseteq S_1$, we have $S_1 \neq \emptyset$, so $t_1 = \sup(S_1)$ exists and is finite, and

$$t_1 \geq 1, \quad g(t_1) = C_\varepsilon^2 (t_1 + 2)^{-2\theta - 2\varepsilon}.$$ 

Also, we define

$$S_2 = \{ t > t_0 \mid g(t) \geq C_\varepsilon^2 (t + 2)^{-2\theta - 2\varepsilon} \} \cup \{ \infty \}, \quad t_2 = \inf(S_2),$$

so $t_0 \in (t_1,t_2)$ and hence

$$g(t) < C_\varepsilon^2 (t + 2)^{-2\theta - 2\varepsilon} \quad (t_1 < t < t_2).$$ 

Further, if $t_2 < \infty$, then also

$$g(t_2) = C_\varepsilon^2 (t_2 + 2)^{-2\theta - 2\varepsilon}.$$ 

As indicated at the beginning of this section, combining Lemmas 2.2 and 2.3 we have
\[
C_2 \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-\varepsilon} (t+2)^{-2\theta - \varepsilon} \leq |r_{-1}| + |r_{-2}|
\]
\[
\leq 3C_1^2 \left( c(\pi_1) c(\pi_2) \right)^{\varepsilon} g(t) + C_1^2 \left( c(\pi_1) c(\pi_2) \right)^{\varepsilon} |g'(t)|.
\]
Hence
\[
|g'(t)| \geq \frac{C_2}{C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (t+2)^{-2\theta - 3\varepsilon} - 3g(t).
\]
In particular, over the interval \((t_1, t_2)\), by (3.11) we have
\[
g(t) < C_\varepsilon^2 (t+2)^{-2\theta - 2\varepsilon} \leq \frac{C_2}{6C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (|t| + 2)^{-2\theta - 2\varepsilon},
\]
so the estimate (3.13) gives
\[
|g'(t)| \geq \frac{C_2}{C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (t+2)^{-2\theta - \varepsilon} - 3g(t)
\]
\[
\geq \frac{C_2}{C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (t+2)^{-2\theta - \varepsilon}
\]
\[
- 3 \cdot \frac{C_2}{6C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (t+2)^{-2\theta - 2\varepsilon}
\]
\[
\geq \frac{C_2}{2C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (t+2)^{-2\theta - \varepsilon} \quad (t_1 < t < t_2).
\]
In particular, \(g'(t) \neq 0\) over \((t_1, t_2)\), so \(g'(t)\) must keep the same sign in this interval. If \(g'(t) > 0\) over \((t_1, t_2)\), then \(g(t)\) is a strictly increasing function, and in particular
\[
g(t_0) > g(t_1) = C_\varepsilon^2 (t_1 + 2)^{-2\theta - 2\varepsilon} > C_\varepsilon^2 (t_0 + 2)^{-2\theta - 2\varepsilon},
\]
in contradiction to the assumption (3.9). Hence over the interval \((t_1, t_2)\) we have \(g'(t) < 0\) and so
\[
(3.14) \quad -g'(t) = |g'(t)| \geq \frac{C_2}{2C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (t+2)^{-2\theta - \varepsilon}.
\]
If \(t_2 < \infty\), then by (3.10) and (3.12) we have
\[
\int_{t_1}^{t_2} \frac{C_2}{2C_1^2} \left( c(\pi_1) c(\pi_2) \right)^{(n_1+n_2-1)-2\varepsilon} (t+2)^{-2\theta - \varepsilon} \, dt
\]
\[
\leq \int_{t_1}^{t_2} \left(-g'(t)\right) \, dt = g(t_1) - g(t_2) = C_\varepsilon^2 (t_1 + 2)^{-2\theta - 2\varepsilon} - C_\varepsilon^2 (t_2 + 2)^{-2\theta - 2\varepsilon}
\]
\[
= \int_{t_1}^{t_2} C_\varepsilon^2 (2\theta + 2\varepsilon)(t+2)^{-2\theta - 2\varepsilon - 1} \, dt,
\]
which is simply

\begin{equation}
(3.15) \quad \int_{t_1}^{t_2} C_\varepsilon^2 \frac{2\theta + 2\varepsilon}{(t + 2)^{1+\varepsilon}} (t + 2)^{-2\theta - \varepsilon} \, dt
\end{equation}

\begin{equation}
\geq \int_{t_1}^{t_2} \frac{C_2}{2C_1^2} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon} (t + 2)^{-2\theta - \varepsilon} \, dt.
\end{equation}

On the other hand, over the interval \((t_1, t_2)\), by the construction of the constant \(C_\varepsilon\) we deduce

\begin{equation}
C_\varepsilon^2 \frac{2\theta + 2\varepsilon}{(t + 2)^{1+\varepsilon}} \leq \frac{C_2}{6C_1^2(2\theta + 2\varepsilon)} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon} \frac{2\theta + 2\varepsilon}{1+2}
\end{equation}

\begin{equation}
< \frac{C_2}{2C_1^2} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon},
\end{equation}

so we should have

\begin{equation}
\int_{t_1}^{t_2} C_\varepsilon^2 \frac{2\theta + 2\varepsilon}{(t + 2)^{1+\varepsilon}} (t + 2)^{-2\theta - \varepsilon} \, dt
\end{equation}

\begin{equation}
< \int_{t_1}^{t_2} \frac{C_2}{2C_1^2} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon} (t + 2)^{-2\theta - \varepsilon} \, dt,
\end{equation}

a contradiction to \((3.15)\). Hence we must have \(t_2 = \infty\), and so by \((3.14)\),

\begin{equation}
-g'(t) \geq \frac{C_2}{2C_1^2} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon} (t + 2)^{-2\theta - \varepsilon} \quad (t > t_1).
\end{equation}

Now for every \(t_0 > t_1\), by \((3.10)\) we have

\begin{equation}
C_\varepsilon^2 (t_1 + 2)^{-2\theta - 2\varepsilon} = g(t_1) \geq g(t_1) - g(t_0) = \int_{t_1}^{t_0} (-g'(t)) \, dt
\end{equation}

\begin{equation}
\geq \int_{t_1}^{t_0} \frac{C_2}{2C_1^2} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon} (t + 2)^{-2\theta - \varepsilon} \, dt
\end{equation}

\begin{equation}
= \frac{C_2}{2C_1^2} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon} (t_1 + 2)^{1+2\varepsilon} \frac{1-2\theta-\varepsilon}{1+2\theta+\varepsilon-1-2\theta-\varepsilon}.
\end{equation}

By \((3.4)\) we see that \(\theta \geq 1/2\), so \(1-2\theta-\varepsilon < 0\), and letting \(t_0 \to \infty\) in the above estimate gives

\begin{equation}
C_\varepsilon^2 (t_1 + 2)^{-2\theta - 2\varepsilon} \geq \frac{C_2}{2C_1^2} (\mathcal{c}(\pi_1)\mathcal{c}(\pi_2))^{-(n_1+n_2-1)-2\varepsilon} (t_1 + 2)^{1+2\varepsilon} \frac{1-2\theta-\varepsilon}{2\theta+\varepsilon-1}.
\end{equation}
This implies
\[ C_2^2 \geq \frac{C_2}{2C_1^3} (c(\pi_1)c(\pi_2))^{-\left(n_1+n_2-1\right)-2\varepsilon (t_1+2)^{1+\varepsilon}} \frac{1}{2\theta + \varepsilon - 1}, \]
and so
\[ (t_1 + 2)^{1+\varepsilon} \leq C_2^2 \cdot \frac{2C_2}{C_2} (c(\pi_1)c(\pi_2))^{n_1+n_2-1+2\varepsilon} \cdot (2\theta + \varepsilon - 1) \]
\[ \leq \frac{C_2}{6C_1^2(2\theta + 2\varepsilon)} (c(\pi_1)c(\pi_2))^{-\left(n_1+n_2-1\right)-2\varepsilon} \]
\[ \cdot \frac{2C_1^2}{C_2} (c(\pi_1)c(\pi_2))^{(n_1+n_2-1)+2\varepsilon} \cdot (2\theta + \varepsilon - 1) \]
\[ = \frac{2\theta + \varepsilon - 1}{6(\theta + \varepsilon)} < 1, \]
which is impossible since \( t_1 \geq 1 \) by (3.10). This contradiction shows that there cannot exist a point \(|t_0| > 1\) satisfying (3.9). Hence the claim (3.7), and so (3.6), is valid everywhere, namely
\[ |L(1 + it, \pi_1 \times \pi_2)| \geq C_\varepsilon (|t| + 2)^{-\theta - \varepsilon} \]
\[ \geq \frac{C_\varepsilon'}{3^\theta} (c(\pi_1)c(\pi_2))^{-\left(n_1+n_2-1\right)-\varepsilon} (|t| + 2)^{-\frac{n_1+n_2}{2} \left(1 - \frac{1}{n_1+n_2}\right)} - \varepsilon. \]

This completes our proof of Theorem 1.1. \( \blacksquare \)

Acknowledgements. The author would like to thank Dorian Goldfeld for stimulating this research, and for many insightful discussions. The author is also grateful to the referee for the careful reading of the paper and for many constructive suggestions.

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