Preferential attachment scale-free growth model with random fitness

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We introduce a model which consists in a planar network which grows by adding nodes at a distance \( r \) from the pre-existing barycenter. Each new node position is randomly located through the distribution law \( P(r) \propto 1/r^\gamma \) with \( \gamma > 1 \). The new node \( j \) is linked to only one pre-existing node according to the probability law \( P(i \leftrightarrow j) \propto \eta_i k_i r_{ij}^{-\alpha A} \) (\( 1 \leq i < j; \alpha_A \geq 0 \)); \( k_i \) is the number of links of the \( i \)th node, \( \eta_i \) is its fitness (or quality factor), and \( r_{ij} \) is the distance. We consider in the present paper two models for \( \eta_i \). In one of them, the single fitness model (SFM), we consider \( \eta_i = 1 \forall i \). In the other one, the uniformly distributed fitness model (UDFM), \( \eta_i \) is chosen to be uniformly distributed within the interval (0, 1]. We have determined numerically the degree distribution \( P(k) \). This distribution appears to be well fitted with \( P(k) = P(1) k^{-\lambda} e^{\lambda e_q (k-1)/\kappa} \) with \( q \geq 1 \), where \( e_q^\lambda \equiv [1+(1-q)x]^{1/(1-q)} \) (\( e_1^\lambda = e^x \)) is the q-exponential function naturally emerging within nonextensive statistical mechanics. We determine, for both models, the entropic index \( q \) as a function of \( \alpha_A \) (\( q \) in depends from \( \gamma \)). Additionally, we determine the average topological (or chemical) distance within the network, and the time evolution of the average number of links \( \langle k_i \rangle \). We obtain that, asymptotically, \( \langle k_i \rangle \propto (t/i)^{\beta} \), (\( i \) coincides with the input-time of the \( i \) node) and \( \beta(\alpha_A) \) to both cases.

Scale-free networks are very popular nowadays due to their uncountable applications in different fields of knowledge. These models are typically associated with some physical quantities that are characterized by power-law asymptotic behavior, instead of the usual exponential laws. Most of these models do not take into account the node-to-node Euclidean distance, i.e., the geographical distance. One of them which does take into account this aspect of the problem has been introduced recently and discussed. In this example, as well as in others, strong connection has been revealed with nonextensive statistical mechanics. In the present paper, we follow the lines of which we extend in the sense that we include now a local variable denominated fitness or quality factor, we call this variable \( \eta_i \). The model studied by is herein referred to as the Single Fitness Model (SFM), that is, \( \eta_i = 1 \forall i \), and we
also study the *Uniformly Distributed Fitness Model* (UDFM), in which the local fitness is an independent random variable. We are here interested to focus on the effects of the fitness on the connectivity probability distribution and similar quantities.

![Figure 1: Connectivity distribution $P(k)$ in log-log scale for typical values $\alpha_A$ for UDFM and SFM models. The symbols are numerical results and continuous lines are the best fits in according to equation 2.](image)

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At the plane, we have constructed a complex network which is constituted by nodes and links between nodes. The network grows sequentially by adding a node at each time. We fix the first node ($i = 1$) at the origin. The position of the second node ($i = 2$) is chosen randomly with respect to origin in such way
its distance $r$ obeys the probability law $P(r) \propto 1/r^\gamma$ with $\gamma > 1$. In this paper, we have chosen $\gamma = 5/4$ for all cases studied. Next the second node is linked to first one. Then, the origin is moved into network barycenter formed by the two nodes. The previous procedure is used to include the third site, fourth site, and so on. We link the third node ($j = 3$) to one of the pre-existing nodes in the network. We establish its connection by considering the linking probability given by $P(i \leftrightarrow j = 3) \propto \eta_i k_i/r_i^{\alpha_A}$ ($\alpha_A \geq 0$) and $1 \leq i < j$. The connectivity of $i^{th}$ node is stood for $k_i$ and $\eta_i$ is its fitness and $r_{ij}$ is its distance to new node ($j = 3$). At this early stage we have $k_1 = 1$ and $k_2 = 1$.

The Fitness model is characterized by the presence of factor $\eta_i$ that appear in the linking probability. We define the fitness or quality model in following way: To each node $i < j$ is assigned a uniform random value $\eta_i \in (0,1]$ when the node is born. That value is kept constant while the network grows. In traditional models old nodes are more attractive than new ones. In this model we give a chance for new sites, with high fitness, become competitive.

The network growth process is sequentially repeated up to size desired for network. To avoid node position overlap we consider the lowest distance between them to be unit. If we denote $N$ the total number of nodes one can write linking probability for $i^{th}$ network node to connect the new node $j = N + 1$ as

$$P(i \leftrightarrow j = N + 1) = \frac{\eta_i k_i/r_i^{\alpha_A}}{\sum_{i=1}^{N} \eta_i k_i/r_i^{\alpha_A}}$$  \hspace{1cm} (1)

The dynamics that we have introduced by the above rule does privilege the connection between new nodes and those having many links (high fitness nodes)
since they are not so far from each other. One similar rule was explored [10] for the particular case of uniform distribution of sites within some limited region at the plane, with $n_i = 1 \ \forall i$. The $\alpha_A$ parameter control the influence range of the nodes (or hubs). The case $\alpha_A = 0$ corresponds to the Bianconi fitness model [11] where distance is not relevant. In the present paper we will emphasize the following aspects:

- The network degree distribution in the stationary-state $P(k)$ relative to number of nodes having $k$ links in the $N \to \infty$.
- The temporal dependence of average number of links $\langle k_i \rangle$ and more precisely to know how asymptotically it increase with time $t/i$, $t \geq i$.
- The dependence of average length, $\langle l \rangle$, with $\alpha_A$ parameter.

The numerical results are obtained for networks with 10 000 nodes and 2 000 runs. In this network size the probability distribution $P(k)$ practically enter in the stationary regime. The figure 1 show our numerical results for $P(k)$ (symbols) and fits (continuous lines) for UDFM and SFM models. One can observe in our results that $\alpha_A$ has a strong influence on $P(k)$. The present results are very well fitted by the function

$$P(k) = P(1) k^{-\lambda} e_q^{-(k-1)/\kappa}$$

where q-exponential function is defined by

$$e_q^x = [1 + (1 - q)x]^{1/(1-q)} \quad (e_1^x = e^x).$$

The variable $\kappa$ is the characteristic number of links [1]. In the figure 2 we represent the figure 1 in a $\ln q - \text{linear plot}$. Where $\ln_q(x) = (x^{1-q} - 1)/(1-q)$.

![Figure 4: Temporal dependency of the average connectivity for UDFM model.](image)

Figure 4: Temporal dependency of the average connectivity for UDFM model.

Analizing the figure 1 we have found that the UDFM has raised the number of links per node and it turns out that hubs are more connected. In the figure 2 we present the $(q, \lambda, \kappa)$ evolution curve with $\alpha_A$ for both models studied. We observe that UDFM curve is always above the SFM case. They have the same asymptotic limit, i.e., $q(\alpha_A) = 1$.
Figure 5: Average connectivity exponent for $\alpha_A$ values relative to measures on node $i = 50$.

Figure 6: Average path length for UDFM and SFM models.
The results for the node average connectivity $\langle k_i \rangle$ are indicated in the figure 4 where we show its temporal evolution. Our data for $\langle k_i \rangle$ can be fitted by the function

$$\langle k_i \rangle \propto \left( \frac{t}{i} \right)^{\beta(\alpha_A, n_i)} \quad t \geq i \quad \quad (4)$$

The collapse of the curves for $\alpha_A \geq 2$ indicates an upper limit (natural cutoff) for a node connectivity. As much we increase $\alpha_A$ all hubs reduce its connectivity to a minimal value 2. We explain this result imagining that the linking probability decays strongly for increasing $\alpha_A$ in such way that a node does not see other unless its first neighbours, and therefore they form long node chains. This fact is inherent to the network growing rule, i.e., if we use other rule for example each new site links with more than one node, this limit result would be quite different.

In the figure 5 we show the exponent $\beta(\alpha_A)$ for UDFM and SFM models. The exponent decline down from 0.59 for UDFM and from 0.48 for SFM to a stationary value 0.1. The figure 6 shows the average length $\langle l \rangle$ dependency on $\alpha_A$ for UDFM and SFM models. The average path length, $\langle l \rangle$, of a network is defined as the number of edges in the shortest path between two nodes averaged over all pairs of nodes. We see that, when $\alpha_A$ varies, $\langle l \rangle$ raises rapidly to a saturated value $\sim 12$. The saturation in figure 6 is explained by the $\langle k_i \rangle$ convergence in figures 5 and 4 for $\alpha_A \geq 2$.

Now we will summarize our present paper. Following the lines of the paper [1] we have presented a model in which we add fitness. We study the effect of competition between the relevant variable for connectedness when we include fitness on the network. In UDFM model the popular nodes (many links) do compete with younger nodes when the fitness is an important factor that permits them to get more links. When we raise $\alpha_A$ we favor the linking between first neighbours in the network. When $\alpha_A$ is zero we recover the Bianconi-Barabási model [11]. The average connectivity $\langle k_i \rangle$ and average path length $\langle l \rangle$ are appreciably influenced by factor $\alpha_A$ for UDFM and SFM models. We have shown that degree distribution $P(k)$ is very well fitted by the function proposed that contain $q$-exponential function that emerges naturally from non-extensive statistics. And when we compare our results relative to SFM we have obtained that the number of links per node has raised for all values $\alpha_A$. When the $\alpha_A$ is increased we find that links per hubs has reduced as SFM but it occurs slower. The presence of fitness changes the universality class in the complex network. Then if you want a more complete scenario for this kind of system, fitness should be included.

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