ON $C^0$ INTERIOR PENALTY METHOD FOR FOURTH ORDER DIRICHLET BOUNDARY CONTROL PROBLEM AND A NEW ERROR ANALYSIS FOR FOURTH ORDER ELLIPTIC EQUATION WITH CAHN-HILLIARD BOUNDARY CONDITION

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Abstract. In this paper, we revisit the $L_2$-norm error estimate for $C^0$-interior penalty analysis of Dirichlet boundary control problem governed by biharmonic operator. In this work, we have relaxed the interior angle condition of the domain from 120 degrees to 180 degrees, therefore this analysis can be carried out for any convex domain. The theoretical findings are illustrated by numerical experiments. Moreover, we propose a new analysis to derive the error estimates for the biharmonic equation with Cahn-Hilliard type boundary condition under minimal regularity assumption.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $n$ denotes the outward unit normal vector to the boundary $\partial \Omega$ of $\Omega$. We assume the boundary $\partial \Omega$ to be the union of line segments $\Gamma_k (1 \leq k \leq l)$ such that their interiors are pairwise disjoint in the induced topology. Consider the following optimal control problem:

$$\min_{p \in Q} J(u, p) := \frac{1}{2} \|u - u_d\|^2 + \frac{\alpha}{2} |p|_{H^2(\Omega)}^2,$$

subject to

$$\Delta^2 u = f \quad \text{in} \quad \Omega,$$

$$u = p \quad \text{on} \quad \partial \Omega,$$

$$\partial u/\partial n = 0 \quad \text{on} \quad \partial \Omega.$$

Here $\alpha > 0$, $f$ and $u_d \in L_2(\Omega)$ denote the regularization parameter, the external force acting on the system and desired observation respectively. The space of admissible controls is given by

$$Q = \{ p \in H^2(\Omega) : \partial p/\partial n = 0 \quad \text{on} \quad \partial \Omega \}.$$  

This article revisits the $L_2$-norm estimate for the optimal control derived in [10]. The analysis therein uses the fact that interior angles of the domain cannot exceed 120 degrees, which is
quite restrictive in applications. This article extends the analysis to any convex polygonal domains. But the extension is non trivial in nature. The main novelties of this article are shortlisted below:

- In Lemma 3.1 we have proven the equality of two bilinear forms over a special class of Sobolev functions. It involves novel functional analytic techniques. This Lemma plays a pivotal role in the analysis.
- In Section 6 we have proposed a novel variational formulation for the problem (6.1) over a new Hilbert space compared to the traditional test space.
- Lemma 6.1 proves a special regularity result for the solution of (6.1) which provides the consistency of the solution (6.6) and hence Galerkin orthogonality (6.8).

Classical non-conforming methods and $C^0$-interior penalty (IP) methods have been two popular schemes to approximate the solutions of higher order equations within the finite element framework. In this connection, we refer to the works of [2, 24, 15, 7, 28, 29, 4, 18, 5, 21, 25, 27, 9, 1] and references therein. These methods are computationally more efficient compared to the one of conforming finite element methods. For the interested readers, we refer to [19] for a discontinuous mixed formulation based analysis of fourth order problems.

In this regard, we would like to remark that mixed schemes are complicated in general and have its restrictions (solution to the discrete scheme may converge to a wrong solution for a fourth order problem if the solution is not $H^3$-regular).

We notice that the literature for the finite element error analysis for higher order optimal control problems is relatively less. In [16], a mixed finite element (Hermann-Miyoshi mixed formulation) analysis is proposed for a fourth order interior control problem. In this work, an optimal order a priori error estimates for the optimal control, state and adjoint state are derived followed by a superconvergence result for the optimal control. For a $C^0$-interior penalty method based analysis of a fourth order interior control problem, we refer to [20]. Therein an optimal order a priori error estimate and a superconvergence result is derived for the optimal control on a general polygonal domain and subsequently a residual based a posteriori error estimates are derived for the construction of an efficient adaptive algorithm.

In [12], abstract frameworks for both a priori and a posteriori error analysis of fourth order interior and Neumann boundary control problems are proposed. The analysis of this paper can be applicable for second and sixth order problems as well.

We continue our discussion on higher order Dirichlet boundary control problems. In this connection, we note that the analysis of Dirichlet boundary control problem is more subtle compared to interior and Neumann boundary control problems. This is due to the fact that the control does not appear naturally in the formulation for Dirichlet boundary control problems. We refer to [10] for the $C^0$-interior penalty analysis of an energy space based fourth order Dirichlet boundary control problem where the control variable is sought from the energy space $H^{3/2}(\partial \Omega)$ (the definition of the space $H^{3/2}(\partial \Omega)$ is given in Section 3). In this work, an optimal order a priori energy norm error estimate is derived and subsequently an optimal order $L_2$-norm error estimate is derived with the help of a dual problem. But the $L_2$-norm error estimate is derived under the assumption that interior angles of the domain should be
less than 120 degrees. This assumption is required to guarantee $H^{5/2+\epsilon}(\Omega)$ regularity for the optimal control \cite{10}.

In this work, we revisit this estimate and extend the angle condition to 180 degrees. Moreover, we propose an alternative error analysis for the solution of bi-harmonic equation with Cahn-Hilliard boundary equation.

The rest of the article is organized as follows. In Section 2 we introduce the $C^0$-interior penalty method and define some general notations and concepts which are important in subsequent discussions. We start Section 3 by showing the equality of two bilinear forms over the space of admissible controls $Q$ which plays a crucial role in establishing the $L_2$-norm error estimate for the optimal control. Subsequently, we derive the optimality system for the model problem (1.1)-(1.2) and its equivalence with the corresponding energy space based Dirichlet boundary control problem. We conclude this section with the discrete optimality system. Section 4 contains the optimal order energy norm estimates for the optimal control, state and adjoint state variables. We derive the optimal order $L_2$-norm estimate for the optimal control variable in Section 5. In Section 6, we propose an alternative approach for the a priori error analysis of $C^0$-interior penalty approximation of stationary Cahn-Hilliard equation under minimal regularity assumption. Section 7 is devoted to the numerical experiment to verify the theoretical findings. We conclude the article with Section 8.

We follow the standard notion of spaces and operators that can be found in \cite{13, 23, 6}. If $S \subset \Omega$ then the space of all square integrable functions defined over $S$ is denoted by $L_2(S)$. When $m > 0$ is an integer, the space of $L_2(S)$ functions whose distributional derivative up to $m$-th order is in $L_2(S)$ is denoted by $H^m(S)$. If $s > 0$ is not an integer then there exists an integer $m > 0$ such that $m - 1 < s < m$. There $H^s(S)$ denotes the space of all $H^{m-1}(S)$ functions which belong to the fractional order Sobolev space $H^{s-m+1}(S)$. When $S = \Omega$, then the $L_2(\Omega)$ inner product is denoted either by $\langle \cdot, \cdot \rangle$ or by its usual integral representation and $L_2(\Omega)$ norm is denoted by $\| \cdot \|$, else it is denoted by $\| \cdot \|_S$. In this context, we mention that $H^{-s}(S)$ denotes the dual of $H^0(S)$ and this duality is denoted by $\langle \cdot, \cdot \rangle_{-s,s,S}$ for positive real $s$.

2. Quadratic $C^0$-Interior Penalty Method

We introduce the $C^0$-interior penalty method in brief for the model problem (1.1)-(1.2). Let $T_h$ be a simplicial, regular triangulation of $\Omega$ \cite{13}. A generic triangle in this triangulation and its diameter are denoted by $T$ and $h_T$ respectively and its maximum over the triangles is called the mesh discretization parameter which is denoted by $h$. The finite element spaces are given by

$$V_h = \{ v_h \in H^1_0(\Omega) : v_h|_T \in P_2(T) \ \forall \ T \in T_h \},$$

$$Q_h = \{ p_h \in H^1(\Omega) : p_h|_T \in P_2(T) \ \forall \ T \in T_h \},$$

where $P_2(T)$ denotes the space of polynomials of degree less than or equal to two on $T$. Sides or edges of a triangle and their lengths are denoted by $e$ and $|e|$ respectively. Set of all edges in a triangulation are denoted by $\mathcal{E}_h$. An edge shared by two triangles is called an interior
edge otherwise boundary edge. The set of all interior and boundary edges are denoted by $E^i$ and $E^b$ respectively. Any $e \in E^i$ can be written as $e = \partial T_+ \cap \partial T_-$ for two adjacent triangles $T_+$ and $T_-$. Let $n_-$ represents the unit normal vector on $e$ pointing from $T_-$ to $T_+$ and set $n_+ = -n_-$. For $s > 0$, define $H^s(\Omega, T_h)$ by

$$H^s(\Omega, T_h) = \{ v \in L_2(\Omega) : v|_T \in H^s(T) \ \forall \ T \in T_h \}.$$ 

For $v \in H^2(\Omega, T_h)$, the jump of normal derivative of $v$ across $e$ is given by

$$[v/\partial n] = \nabla v_+ \cdot n_+ + \nabla v_- \cdot n_-,$$

where $v_\pm = v|_{T_\pm}$. Also, for all $v$ with $\Delta v \in H^1(\Omega, T_h)$, its average and jump across $e$ are given by

$$\{ \Delta v \} = \frac{1}{2} (\Delta v_+ + \Delta v_-),$$

and

$$[\Delta v] = (\Delta v_+ - \Delta v_-),$$

respectively.

For the convenience of notation, we extend the definition of average and jump on the boundary edges also. When $e \in E^b$, there is only one triangle $T$ sharing it. Let $n_e$ denotes the unit outward normal on $e$. For any $v \in H^2(T)$, we set on $e$

$$[v/\partial n] = \nabla v \cdot n_e,$$

and for any $v$ with $\Delta v \in H^1(T)$,

$$\{ \Delta v \} = \Delta v.$$

With the help of the above defined quantities, we define the following mesh dependent bilinear forms, semi-norms and norms which are used in the subsequent analysis.

The discrete bilinear form $a_h(\cdot, \cdot)$ defined on $Q_h \times Q_h$ is given by

$$a_h(p_h, r_h) = \sum_{T \in T_h} \int_T \Delta p_h \Delta r_h \, dx + \sum_{e \in E_h} \int_e \{ \Delta p_h \} [\partial r_h / \partial n_e] \, ds$$

$$+ \sum_{e \in E_h} \int_e \{ \Delta r_h \} [\partial p_h / \partial n_e] \, ds + \sum_{e \in E_h} \sigma |e| \int_e [\partial p_h / \partial n_e] [\partial r_h / \partial n_e] \, ds,$$

where the penalty parameter $\sigma \geq 1$.

Define the discrete energy norm on $V_h$ by

$$\| v_h \|^2_h = \sum_{T \in T_h} \| \Delta v_h \|^2_T + \sum_{e \in E_h} \sigma |e| \| [\partial v_h / \partial n_e] \|^2_e \quad \forall \ v_h \in V_h.$$ \hspace{1cm} (2.1)

Note that (2.1) defines a norm on $V_h$ whereas it is only a semi-norm on $Q_h$ (see [3]). The energy norm $\| \cdot \|_h$ on $Q_h$ is defined by

$$\| p_h \|^2_h = \| p_h \|^2_T + \| p_h \|^2_e \quad \forall \ p_h \in Q_h.$$ \hspace{1cm} (2.2)
In the derivation of the $L^2$-norm error estimate, we need an additional energy norm [resp. semi-norm] $\| \cdot \|_{Q_h}$ on $V_h$ [resp. $Q_h$], [3]. It is defined as follows

$$\|p_h\|_{Q_h}^2 = \sum_{T \in T_h} \|\Delta p_h\|_T^2 + \sum_{e \in E_h} |e| \|\{\Delta p_h\}\|_e^2 + \sum_{e \in E_h} \frac{a}{|e|} \|\{\partial p_h / \partial n_e\}\|_e^2 \quad \forall p_h \in Q_h.$$ 

By the use of trace inequality [6, Section 1.6], we observe that $\| \cdot \|_{Q_h}$ and $\| \cdot \|_h$ are equivalent semi-norms [resp. norms] on $Q_h$ [resp. $V_h$]. Moreover, from the discussions of [7], it follows that $a_h(\cdot, \cdot)$ is coercive and bounded on $Q_h$ with respect to $\| \cdot \|_h$, i.e., there exist positive constants $c, C > 0$ independent of $h$ such that

$$a_h(p_h, p_h) \geq c \|p_h\|_h^2 \quad \forall p_h \in Q_h,$$

$$|a_h(\psi_h, \eta_h)| \leq C \|\psi_h\|_h \|\eta_h\|_h \quad \forall \psi_h, \eta_h \in Q_h.$$ 

For $f \in L^2(\Omega)$, $p_h \in Q_h$, let $v_h(f, p_h) \in V_h$ be the unique solution of the following equation

$$a_h(v_h(f, p_h), w_h) = (f, w_h) - a_h(p_h, w_h) \quad \forall w_h \in V_h.$$ 

From now onwards, we denote by $C$ a generic positive constant that is independent of the mesh parameter $h$.

### 2.1. Enriching Operator

Let $W_h$ be the Hsieh-Clough-Tocher macro finite element space associated with the triangulation $T_h$, [13]. Define $\tilde{Q}_h$ by

$$\tilde{Q}_h = Q \cap W_h.$$ 

We assume there exists a smoothing operator $E_h : Q_h \to \tilde{Q}_h$ which is also known as enriching operator satisfying the following approximation property. We refer to [3] for a detailed discussion on the definition and proof of the following lemma.

**Lemma 2.1.** Let $v \in Q_h$, there hold

$$\sum_{T \in T_h} (h_T^{-4} \| E_h v - v \|_T^2 + h_T^{-2} \| \nabla (E_h v - v) \|_T^2) \leq C \sum_{e \in E_h} \frac{1}{|e|} \left\| \left[ \frac{\partial v}{\partial n} \right] \right\|_e^2$$

and

$$\sum_{T \in T_h} |E_h v - v|_{H^2(T)}^2 \leq C \sum_{e \in E_h} \frac{1}{|e|} \left\| \left[ \frac{\partial v}{\partial n} \right] \right\|_e^2.$$ 

### 3. Auxiliary Results

In this section, we prove the equality of two bilinear forms over the space $Q$ which plays a key role in obtaining the $L^2$-norm estimate on convex domains. Subsequently, we discuss the existence and uniqueness results for the solution to the optimal control problem and derive the corresponding optimality system. At the end of this section, we remark that this
Lemma 3.1. Given \( D \) where \( Z \) implies completeness of \( C \). The following lemma proves the equality of two bilinear forms over \( Q \).

Lemma 2.1, trace inequality for \( H \) is a problem is equivalent to its corresponding Dirichlet control problem, \( [10] \). Define a bilinear form \( a(\cdot, \cdot) : Q \times Q \to \mathbb{R} \) by

\[
a(u, v) = \int_{\Omega} \Delta u \Delta v \, dx.
\]

The following lemma proves the equality of two bilinear forms over \( Q \).

**Lemma 3.1.** Given \( p, q \in Q \) we have

\[
a(q, p) = \int_{\Omega} D^2 q : D^2 p \, dx,
\]

where \( D^2 q : D^2 p = \sum_{i,j=1,2} \frac{\partial^2 q}{\partial x_i \partial x_j} \frac{\partial^2 p}{\partial x_i \partial x_j} \).

**Proof.** Introduce a new function space \( X \) defined by

\[X = \{ \phi \in H^1(\Omega) : \Delta \phi \in L^2(\Omega) \},\]

endowed with the inner product \((\cdot, \cdot)_X\) given by

\[(\phi_1, \phi_2)_X = (\phi_1, \phi_2)_{H^1(\Omega)} + (\Delta \phi_1, \Delta \phi_2),\]

where \((\cdot, \cdot)_{H^1(\Omega)}\) denotes the standard \( H^1(\Omega) \) inner product. It is easy to check that \( X \) is a Hilbert space with respect to \((\cdot, \cdot)_X\) (see [22]). Let \( E_h I_h (p) \) denotes the enrichment of \( I_h (p) \) defined in subsection 2.1 where \( I_h(p) \) is the Lagrange interpolation of \( p \) onto the finite element space \( Q_h \), [6, Chapter 4]. A use of approximation properties of \( I_h \), [6, Lemma 2.1] and triangle inequality yields \( \| E_h I_h (p) \|_X \leq C \| p \|_{H^2(\Omega)} \). Banach Alaoglu theorem asserts the existence of a subsequence of \( \{ E_h I_h (p) \} \) (still denoted by \( \{ E_h I_h (p) \} \) for notational convenience) converging weakly to some \( z \in X \). Continuity of first normal trace operator \( \frac{\partial}{\partial n} : H(div, \Omega) \to H^{-\frac{1}{2}}(\partial \Omega) \), [54] implies the closedness of \( Z \) where \( Z = ker \left( \frac{\partial}{\partial n} \right) \). Therefore, completeness of \( Z \) implies \( z \in Z \). Given \( \phi \in Z \), consider the following problem

\[-\Delta \psi = -\Delta \phi \text{ in } \Omega,\]

\[\partial \psi / \partial n = 0 \text{ on } \partial \Omega.\]

By the elliptic regularity theory, we have \( \| \psi \|_{H^{s+2}(\Omega)} \leq C \| \phi \|_X \) for some \( s > 0 \), depending upon the interior angle of the domain (see [17]) which further implies \( \| \phi \|_{H^{s+2}(\Omega)} \leq C \| \phi \|_X \) \forall \phi \in Z \). It is easy to check that \( Z \) is compactly embedded in \( H^1(\Omega) \). Therefore, \( E_h I_h (p) \) converges strongly to \( z \) in \( H^1(\Omega) \). A combination of Lemma 2.1 trace inequality for \( H^1(\Omega) \) functions [6, Section 1.6] and the \( H^2 \)-regularity of \( p \) implies the strong convergence of \( E_h I_h (p) \) to \( p \) in \( H^1(\Omega) \). The uniqueness of limit implies \( z = p \) and hence \( E_h I_h (p) \) converges weakly to \( p \) in \( Z \).

For any \( \eta \in Z \),

\[a(\eta, E_h I_h (p)) = (\eta, E_h I_h (p))_X - (\eta, E_h I_h (p))_{H^1(\Omega)}.\]

Since, \( E_h I_h (p) \) converges weakly to \( p \) in \( Z \) and \( H^1(\Omega) \), therefore \( (\eta, E_h I_h (p))_X \) converges to \( (\eta, p)_X \) and \( (\eta, E_h I_h (p))_{H^1(\Omega)} \) converges to \( (\eta, p)_{H^1(\Omega)} \). Thus,

\[a(\eta, E_h I_h (p)) \to a(\eta, p).\]
Since \( \| E_h I_h(p) \|_{H^2(\Omega)} \leq C \| p \|_{H^2(\Omega)} \), the subsequence considered in the previous case (i.e. for the space \( X \) which was still denoted by \{ \( E_h I_h(p) \) \}) must have a weakly convergent subsequence denoted by \{ \( E_h I_h(p) \) \} (again for notational convenience!) converges weakly to some \( \tilde{w} \in H^2(\Omega) \). The compact embedding of \( H^2(\Omega) \) in \( H^1(\Omega) \) implies the strong convergence of \( E_h I_h(p) \) to \( \tilde{w} \) in \( H^1(\Omega) \) and hence by the uniqueness of the limit, \( \tilde{w} = p \). Therefore,

\[
\int_{\Omega} D^2 \eta : D^2 E_h I_h(p) \, dx \rightarrow \int_{\Omega} D^2 \eta : D^2 p \, dx \text{ as } h \rightarrow 0.
\]

Next, we aim to show that for \( p, q \in Q \), \( a(q, p) = \int_{\Omega} D^2 q : D^2 p \, dx \). Density of \( C^\infty(\bar{\Omega}) \) in \( H^2(\Omega) \) yields the existence of a sequence \( \{ \phi_m \} \subseteq C^\infty(\bar{\Omega}) \) with \( \phi_m \) converges to \( q \) in \( H^2(\Omega) \). Using Green’s formula, we arrive at

\[
\int_{\Omega} D^2 \phi_m : D^2 E_h I_h(p) \, dx - \int_{\Omega} \Delta \phi_m \Delta E_h I_h(p) \, dx = - \sum_{k=1}^l \int_{\Gamma_k} \frac{\partial^2 \phi_m}{\partial n \partial t} \frac{\partial E_h I_h(p)}{\partial t} \, ds. \tag{3.2}
\]

Applying integration by parts to the right hand side of (3.2) and then taking limit on both sides with respect to \( m \), we find that

\[
\int_{\Omega} D^2 q : D^2 E_h I_h(p) \, dx - a(q, E_h I_h(p)) = \sum_{k=1}^l \int_{\Gamma_k} \frac{\partial q}{\partial n} \frac{\partial^2 E_h I_h(p)}{\partial t^2} \, ds = 0.
\]

Since \( q \in Q \), we conclude that \( a(q, p) = \int_{\Omega} D^2 q : D^2 p \, dx \). \qed

Remark 3.2. We note that the above Lemma helps us to show the equality of the cost functional considered in (1.1) with \( \frac{1}{2} \| u - u_d \|^2 + \frac{\alpha}{2} \| \Delta p \|^2 \) over \( Q \). Therefore if we consider the Optimal Control Problem:

\[
\min_{p \in Q} J_1(u, p) = \frac{1}{2} \| u - u_d \|^2 + \frac{\alpha}{2} \| \Delta p \|^2
\]

subject to

\[
\Delta^2 u = f \text{ in } \Omega, \\
u = p \text{ on } \partial \Omega, \\
\partial u / \partial n = 0 \text{ on } \partial \Omega,
\]

Then this Lemma proves that the solution of this problem is same as the solution of (1.1).

It is easy to check that the bilinear form \( a(\cdot, \cdot) \) defined in (3.1) is elliptic on \( V(= H^2_0(\Omega)) \) and bounded on \( Q \times Q \) (see [6]).

The subsequent results of this section are not new and can be found in [10] but are briefly outlined for the sake of completeness and ease in reading.

For given \( f \in L_2(\Omega) \) and \( p \in Q \), an application of Lax-Milgram lemma [13, 6] gives the existence of an unique \( u_f \in V \) such that

\[
a(u_f, v) = (f, v) - a(p, v) \quad \forall \ v \in V.
\]
Therefore, \( u = u_f + p \) is the weak solution of the following Dirichlet problem:

\[
\begin{align*}
\Delta^2 u &= f \quad \text{in} \quad \Omega \\
u &= p, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

In connection to the above discussion as in [10, 11], the optimal control problem described in (1.1)-(1.2) can be rewritten as

\[
\min_{p \in Q} j(p) := \frac{1}{2} \|u_f + p - u_d\|^2 + \frac{\alpha}{2} \|p\|_{H^2(\Omega)}^2.
\]

The following theorem provides the existence and uniqueness of the solution to the optimal control problem and the corresponding optimality system.

**Theorem 3.3.** The problem (3.3) has a unique solution \((q, u) \in Q \times Q\). Moreover there is an additional variable known as adjoint state \(\phi \in V\) associated to the unique solution and the triplet \((q, u, \phi) \in Q \times Q \times V\) that is (optimal control, optimal state, adjoint state) satisfies the following system, known as the optimality or Karush Kuhn Tucker (KKT) system:

\[
\begin{align*}
u &= u_f + q, \quad u_f \in V, \\
a(u_f, v) &= (f, v) - a(q, v) \quad \forall \ v \in V, \\
a(v, \phi) &= (u - u_d, v) \quad \forall \ v \in V, \\
aa(q, p) &= a(p, \phi) - (u - u_d, p) \quad \forall \ p \in Q.
\end{align*}
\]

**Proof.** From Lemma 3.1, it is clear that for any \(w, v \in Q\), we have \(a(w, v) = \int_\Omega D^2 w : D^2 v \, dx\). The rest of the proof follows from the similar arguments as in [11, Proposition 2.2] and Lemma 3.1. \(\square\)

In the following remark, it is shown that the minimum energy of (1.1)-(1.2) is realized with an equivalent \(H^{3/2}(\partial \Omega)\)-norm of the first trace of the solution of (3.3).

**Remark 3.4.** Since \(\Omega\) is polygonal, we know that the trace of \(Q\) is surjective onto a subspace of \(\prod_{i=1}^m H^{3/2}(\Gamma_i)\) (see [23]) which we refer to as \(H^{3/2}(\partial \Omega)\). The \(H^{3/2}(\partial \Omega)\) semi-norm of any \(p \in H^{3/2}(\partial \Omega)\) can be defined by

\[
|p|_{H^{3/2}(\partial \Omega)} := |u_p|_{H^2(\Omega)} = \min_{w \in Q, w = p \ on \ \partial \Omega} |w|_{H^2(\Omega)}.
\]

Here \(u_p\) is the biharmonic extension of \(p \in H^{3/2}(\partial \Omega)\).

That is \(u_p \in Q\) satisfies

\[
u_p = z + \tilde{p},
\]

such that

\[
\int_\Omega D^2 z : D^2 v \, dx = - \int_\Omega D^2 \tilde{p} : D^2 v \, dx \quad \forall \ v \in V,
\]
where \( \tilde{p} \in Q \) satisfies \( \tilde{p}|_{\partial \Omega} = p \). Hence,
\[
\int_{\Omega} D^2 u_q : D^2 v \, dx = 0 \quad \forall \, v \in V.
\]
In view of Lemma 3.1 we have
\[
a(u_q, v) = 0 \quad \forall \, v \in V.
\]
From (3.6), we have
\[
a(q, v) = 0 \quad \forall \, v \in V,
\]
which implies \( q = u_q \). Therefore, the minimum energy in the minimization problem (1.1)-(1.2) is realized with an equivalent \( H^{3/2}(\partial \Omega) \) norm of the optimal control \( q \).

Now, we define the discrete form of the continuous optimality system.

**Discrete system.** A \( C^0 \)-IP discretization of the continuous optimality system consists of finding \( u_h \in Q_h, \phi_h \in V_h \) and \( q_h \in Q_h \) such that
\[
u_h = u^h_f + q_h, \quad u^h_f \in V_h,
\]
\[
a_h(u^h_f, v_h) = (f, v_h) - a_h(q_h, v_h) \quad \forall \, v_h \in V_h,
\]
\[
a_h(\phi_h, v_h) = (u_h - u_d, v_h) \quad \forall \, v_h \in V_h,
\]
\[
a_h(q_h, p_h) = a_h(\phi_h, p_h) - (u_h - u_d, p_h) \quad \forall \, p_h \in Q_h.
\]

It is easy to check that if \( f = u_d = 0 \) then \( u^h_f = q_h = \phi_h = 0 \) which implies that the discrete system (3.7)-(3.9) is uniquely solvable.

For \( p_h \in Q_h, \ u^h_{ph} \in Q_h \) is defined as follows:
\[
u^h_{ph} = w_h + p_h,
\]
where \( w_h \in V_h \) solves the following equation
\[
a_h(w_h, v_h) = -a_h(p_h, v_h) \quad \forall \, v_h \in V_h.
\]

### 4. Energy Norm Estimate

In this section, we briefly discuss the error estimates for optimal control, state and adjoint variables \( q, u \) and \( \phi \) respectively in the energy norm defined by (2.2). Note that these results can be derived by similar arguments as in [10, Theorem 4.1, Theorem 4.2]. Note that these estimates hold under minimal regularity assumptions.

**Theorem 4.1.** The following optimal order error estimates hold for the optimal control \( q \) in the energy norm:
\[
\|q - q_h\|_h \leq C h^{\min(\gamma_1, 1)} \left( \|q\|_{H^{2+\gamma_1}(\Omega)} + \|\phi\|_{H^{2+\gamma_2}(\Omega)} + \|f\| \right) + \left( \sum_{T \in \mathcal{T}_h} h_T^4 \|u - u_d\|_T^2 \right)^{1/2}.
\]

Here \( \gamma = \min\{\gamma_1, \gamma_2\} \) is the minimum of the regularity index between the adjoint state \( \phi \) and optimal control \( q \). The constant \( C \) depends only upon the shape regularity of the triangulation.
Optimal order error estimates for the optimal state and adjoint state are stated in the following theorem.

**Theorem 4.2.** The optimal state \( u \) and adjoint state \( \phi \), satisfies the following error estimate in energy norm:

\[
\| u - u_h \|_h \leq Ch^{\min(\gamma_1)} \left( \| f \| + \| q \|_{H^{2+\gamma_1}(\Omega)} + \| \phi \|_{H^{2+\gamma_2}(\Omega)} \right),
\]

\[
\| \phi - \phi_h \|_h \leq Ch^{\min(\gamma_1)} \left( \| f \| + \| q \|_{H^{2+\gamma_1}(\Omega)} + \| \phi \|_{H^{2+\gamma_2}(\Omega)} \right),
\]

where \( \gamma_1, \gamma_2 \) and \( \gamma \) are same as in Theorem 4.1.

5. The \( L^2 \)-Norm Estimate

This section is devoted to the \( L^2 \)-norm error estimate for the optimal control. In this section, we assume the domain to be convex unless it is mentioned explicitly otherwise. Note that even with this restriction on the domain, the optimal control \( q \) can be quite rough but it helps the adjoint state \( \phi \) to gain \( H^3 \)-regularity [7], which plays an essential role to derive the desired error estimate.

We begin by taking test functions from \( D(\Omega) \) in (3.5) to obtain

\[
\Delta^2 \phi = u - u_d \quad \text{in} \quad \Omega,
\]

in the sense of distributions. Further, density of \( D(\Omega) \) in \( L^2(\Omega) \) yields

\[
\Delta^2 \phi = u - u_d \quad \text{a.e. in} \quad \Omega, \tag{5.1}
\]

and hence, (5.1) along with the convexity of domain implies \( \nabla (\Delta \phi) \in H(div, \Omega) \). Next, the density of \( C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega}) \) in \( H(div, \Omega) \) space [30] enables us to write

\[
\int_\Omega \nabla (\Delta) \phi . \nabla \psi \, dx + \int_\Omega \Delta^2 \phi \, \psi \, dx = \left\langle \frac{\partial \Delta \phi}{\partial n}, \psi \right\rangle - \frac{1}{2}, \frac{1}{2}, \Omega \quad \forall \psi \in H^1(\Omega). \tag{5.2}
\]

On combining (5.1) and (5.2), we obtain

\[
\int_\Omega \nabla (\Delta) \phi . \nabla \psi \, dx + \int_\Omega (u - u_d) \psi \, dx = \left\langle \frac{\partial \Delta \phi}{\partial n}, \psi \right\rangle - \frac{1}{2}, \frac{1}{2}, \Omega \quad \forall \psi \in H^1(\Omega). \tag{5.3}
\]

Additionally, if \( \psi \in Q \) in (5.3), we find that

\[
a(\phi, \psi) - \int_\Omega (u - u_d) \psi \, dx = -\left\langle \frac{\partial \Delta \phi}{\partial n}, \psi \right\rangle - \frac{1}{2}, \frac{1}{2}, \Omega. \tag{5.4}
\]

We use the following auxiliary result for the subsequent error analysis.

**Lemma 5.1.** For the following variational problem of finding \( w \in H^1(\Omega) \) such that,

\[
(\nabla w, \nabla p) = -\frac{1}{\alpha} \left\langle \frac{\partial (\Delta \phi)}{\partial n}, p \right\rangle \quad \forall \ p \in H^1(\Omega),
\]

there exist a solution \( w \in H^1(\Omega) \) unique up to an additive constant.

**Proof.** The proof follows from the fact that the \( H^1(\Omega) \) semi-norm defines a norm on the quotient space \( H^1(\Omega)/\mathbb{R} \).
The following Lemma provides one of the major difference of this article from [10]. The corresponding Lemma in [10, Lemma 5.2] assumes that the interior angles of the domain should not exceed 120 degrees but in view of Lemma 3.1, we are now able to prove the following Lemma with a more relaxed interior angle condition (all the interior angles are less than 180 degrees). It also helps to establish a more direct relation between the optimal control and adjoint state.

**Lemma 5.2.** The optimal control \( q \) satisfies \( \Delta q \in H^1(\Omega) \) and \( \nabla(\Delta q) \in H(\text{div}, \Omega) \).

**Proof.** Using Lemma 3.1 (3.6) and (5.4), we find that

\[
a(q, p) = -\frac{1}{\alpha} \left\langle \frac{\partial(\Delta \phi)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} \quad \forall \ p \in Q.
\]

A use of integration by parts for \( p \in Q \) in Lemma 5.1 yields

\[
(w, \Delta p) = -\frac{1}{\alpha} \left\langle \frac{\partial(\Delta \phi)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} \quad \forall \ p \in H^1(\Omega).
\]

From (5.5) and (5.6), we find that

\[
(w - \Delta q, \Delta p) = 0 \quad \forall \ p \in Q.
\]

A use of elliptic regularity theory for Poisson equation having Neumann boundary condition on polygonal convex domains along with the fact that \( q \in H^2(\Omega) \) imply that \( w - \Delta q \) belongs to the orthogonal complement of \( L^2_0(\Omega) \), where

\[
L^0_2(\Omega) = \{ \psi \in L_2(\Omega) : \int_{\Omega} \psi = 0 \}.
\]

Therefore, \( \Delta q = w + a \), where \( a \) is some constant function. Hence \( \Delta q \in H^1(\Omega) \). Taking test functions from \( \mathcal{D}(\Omega) \) in (3.6) and using (5.1) together with integration by parts, we obtain

\[
\Delta^2 q = 0 \quad \text{in } \Omega,
\]

in the sense of distributions. The rest of the proof follows from the density of \( \mathcal{D}(\Omega) \) in \( L_2(\Omega) \).

The following Remark and Lemma can be found in [10] but is discussed here for the sake of completeness.

**Remark 5.3.** Since, \( C^\infty(\Omega) \times C^\infty(\bar{\Omega}) \) is dense in \( H(\text{div}, \Omega) \), the following holds

\[
(\Delta^2 q, p) + (\nabla(\Delta q), \nabla p) = \left\langle \frac{\partial(\Delta q)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega},
\]

but \( \Delta^2 q = 0 \) in \( \Omega \), gives

\[
(\nabla(\Delta q), \nabla p) = \left\langle \frac{\partial(\Delta q)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega}.
\]
A use of Green’s identity yields
\[ a(q, p) = \left\langle \frac{\partial(\Delta q)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} - \alpha \left\langle \frac{\partial \Delta \phi}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} \quad \forall \, p \in Q. \tag{5.7} \]

Using (5.7) and (3.6) along with (5.4), we find that
\[ \left\langle \frac{\partial(\Delta \phi)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} = \alpha \left\langle \frac{\partial \Delta q}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} \forall \, p \in Q. \]

The following result is proved in [10] but still we are providing a proof for the convenience of the reader. The following lemma shows that the optimal control \( q \in Q \) and adjoint state \( \phi \in V \) are directly related.

**Lemma 5.4.** For \( p \in H^{\frac{1}{2}}(\partial \Omega) \), we have
\[ \alpha \left\langle \frac{\partial \Delta q}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} = \left\langle \frac{\partial \Delta \phi}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega}. \]

**Proof.** As we know, if \( \Omega \) is a Lipschitz domain then the space \( \left\{ u|_{\partial \Omega} : u \in C^\infty(\mathbb{R}^2) \right\} \) is dense in \( H^{1/2}(\partial \Omega) \) [14, Proposition 3.32]. Therefore, there exists a sequence \( \{ \psi_n \} \subset C^\infty(\partial \Omega) \) such that \( \psi_n \to p \) in \( H^{1/2}(\partial \Omega) \). Let \( u_n \) be the weak solution of the following PDE
\[ \Delta^2 u_n = 0 \text{ in } \Omega, \]
\[ u_n = \psi_n \text{ on } \partial \Omega, \]
\[ \frac{\partial u_n}{\partial n} = 0 \text{ on } \partial \Omega. \]

Clearly, \( u_n \in Q \) and \( u_n|_{\partial \Omega} = \psi_n \) then for any \( \epsilon > 0 \), we have
\[ \left| \left\langle \frac{\partial(\Delta \phi)}{\partial n} - \alpha \frac{\partial(\Delta q)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} \right| = \left| \left\langle \frac{\partial(\Delta \phi)}{\partial n} - \alpha \frac{\partial(\Delta q)}{\partial n}, p - \psi_n \right\rangle_{-\frac{1}{2}, \partial \Omega} \right| \leq \left\| \frac{\partial(\Delta \phi)}{\partial n} - \alpha \frac{\partial(\Delta q)}{\partial n} \right\|_{H^{-1/2}(\partial \Omega)} \left\| p - \psi_n \right\|_{H^{1/2}(\partial \Omega)} \]
\[ \leq \epsilon. \]

This completes the rest of the proof. \( \square \)

A use of Lemma 5.2 along with discrete trace inequality for \( H^1 \) functions and standard interpolation error estimates [6] completes the proof of the following lemma.

**Lemma 5.5.** The optimal control satisfies the following error estimate
\[ \| q - q_h \| + \| q - q_h \|_{Q_h} \leq C h^{\min(\gamma, 1)} \left( \| q \|_{H^{2+\gamma_1}(\Omega)} + \| \nabla(\Delta q) \| + \| \phi \|_{H^{2+\gamma_2}(\Omega)} + \| f \| \right) \]
\[ + \left( \sum_{T \in T_h} h_T^4 \| u - u_d \|_T^2 \right)^{1/2}, \]
where \( \gamma \) and \( C \) are as in Theorem 4.1.
In the following theorem, we derive the optimal order $L_2$-norm estimate for the optimal control $q \in Q$.

**Theorem 5.6.** The optimal control $q$, satisfies the following optimal order error estimate

$$
\|q - q_h\| \leq C h^{2\beta} \left( \|q\|_{H^{2+\beta}(\Omega)} + \|\nabla (\Delta q)\| + \|f\| + \|\phi\|_{H^3(\Omega)} \right),
$$

where $\beta > 0$ is the elliptic regularity for the optimal control $q$.

**Proof.** We deduce the $L_2$-norm error estimate by duality argument. Following the discussion as in [10], the auxiliary optimal control problem is to find $r \in Q$ such that

$$
j(r) = \min_{p \in Q} j(p) := \frac{1}{2} \|u_p - (q - q_h)\|^2 + \frac{\alpha}{2} |p|_{H^2(\Omega)}^2,
$$

where $u_p = w + p$ and $w \in V$ satisfies the following equation

$$
\int_{\Omega} D^2 w : D^2 v \, dx = - \int_{\Omega} D^2 p : D^2 v \, dx \quad \forall \, v \in V.
$$

The standard theory of optimal control problems constrained by partial differential equations provide the existence of a unique solution $r \in Q$ of the above optimal control problem (5.8). For a detailed discussion, we refer to [26, 31]. It is easy to check that $r \in Q$ satisfies the following optimality condition:

$$
\alpha \int_{\Omega} D^2 r : D^2 p \, dx + \int_{\Omega} u_r u_p \, dx = (q - q_h, u_p) \quad \forall \, p \in Q.
$$

From Lemma 3.1, we obtain that

$$
\alpha a(r, p) + (u_r, u_p) = (q - q_h, u_p) \quad \forall \, p \in Q.
$$

This implies,

$$
\alpha a(r, p) + (u_r, p) - a(\xi, p) = (q - q_h, p) \quad \forall \, p \in Q,
$$

with $\xi \in H_0^2(\Omega)$ satisfies the following equation

$$
a(\xi, v) = (u_r - (q - q_h), v) \quad \forall \, v \in H_0^2(\Omega).
$$

Elliptic regularity theory for clamped plate problems on convex domains imply that $\xi \in H^3(\Omega) \cap H_0^2(\Omega)$. From (5.10), we obtain

$$
\Delta^2 \xi = u_r - (q - q_h) \quad \text{in} \quad \Omega,
$$

in the sense of distributions. Since, $D(\Omega)$ is dense in $L_2(\Omega)$, we find that

$$
\Delta^2 \xi = u_r - (q - q_h) \quad \text{a. e. in} \quad \Omega,
$$

$$
\xi = 0; \quad \frac{\partial \xi}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
$$

Therefore $\nabla (\Delta \xi) \in H(div, \Omega)$ which implies $\frac{\partial (\Delta \xi)}{\partial n} \in H^{-1/2}(\partial \Omega)$. Using density of $C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega})$ in $H(div, \Omega)$ (see [30]), we find that

$$
\int_{\Omega} \nabla (\Delta \xi) \cdot \nabla p \, dx + \int_{\Omega} \Delta^2 \xi \, p \, dx = \left( \frac{\partial (\Delta \xi)}{\partial n}, p \right)_{-
\frac{1}{2}, \partial \Omega} \quad \forall \, p \in H^1(\Omega).
$$
Integration by parts along with (5.10) yields
\[
\alpha a(r, p) = -\left\langle \frac{\partial (\Delta \xi)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} \quad \forall \ p \in Q.
\] (5.13)
Choosing test functions from \( D(\Omega) \) in (5.9) and using the density argument, we obtain that
\[
\Delta^2 r = 0 \quad \text{a. e. in } \Omega.
\] (5.14)
Arguments similar to the ones used for proving Lemma 5.2 along with (5.13) yields \( \nabla (\Delta r) \in H(\text{div}, \Omega) \). Therefore \( \frac{\partial (\Delta r)}{\partial n} \in H^{-1/2}(\partial \Omega) \) which further implies
\[
\alpha \left\langle \frac{\partial (\Delta r)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} = \left\langle \frac{\partial (\Delta \xi)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} \quad \forall \ p \in Q.
\] Using Lemma 5.4, we find that
\[
\alpha \left\langle \frac{\partial (\Delta r)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} = \left\langle \frac{\partial (\Delta \xi)}{\partial n}, p \right\rangle_{-\frac{1}{2}, \partial \Omega} \quad \forall \ p \in H^{1/2}(\partial \Omega).
\] (5.15)
To derive the \( L_2 \)-norm error estimate, \( (q) + Q_h \) is used as a test function space. Using the same arguments as in [10, Theorem 5.4], we obtain
\[
\| q - q_h \|^2 = -\int_{\Omega} (q - q_h)(u^h_q - u_q) \, dx - a_h(\xi, u^h_{q-q_h})
+ \alpha a_h(r - r_h, q - q_h) + a_h(\phi - \phi_h, r_h - r) + a_h(\phi - \phi_h, r)
- \int_{\Omega} (u_f - u^h_f) \, r_h \, dx + \int_{\Omega} (u_q - u^h_q)(r - r_h) \, dx + \int_{\Omega} u_r (u^h_r - u_r) \, dx.
\] (5.16)
Now, we estimate each term on the right hand side of (5.16) one by one. The following duality argument is used to find the estimate for the first term.
\[
\| u^h_q - u_q \| = \sup_{w \in L^2(\Omega), w \neq 0} \frac{(u^h_q - u_q, w)}{\| w \|}.
\] (5.17)
Consider the following dual problem
\[
\Delta^2 \phi_w = w \quad \text{in } \Omega,
\phi_w = 0, \quad \frac{\partial \phi_w}{\partial n} = 0 \quad \text{on } \partial \Omega.
\] (5.18)
Let \( P_h(w) \) be the \( C^0 \)-interior penalty approximation of the solution of (5.18). Hence,
\[
(u^h_q - u_q, w) = a_h(\phi_w, u^h_q - u_q)
= a_h(\phi_w - P_h(\phi_w), u^h_q - u_q)
\leq C \| \phi_w - P_h(\phi_w) \|_{Q_h} \| u^h_q - u_q \|_{Q_h}
\leq C h \| w \| \| u^h_q - u_q \|_{Q_h}.
\]
We define \( u^h_{q-q_h} \) as \( u^h_{q-q_h} = v_{0h} + q - q_h \), where \( v_{0h} \in V_h \) solves the following equation
\[
a_h(v_{0h}, v_h) = -a_h(q - q_h, v_h) \quad \forall \ v_h \in V_h.
\]
Using the coercivity of $a_h(\cdot, \cdot)$, we find that
\[
c_1\|v_{0h}\|_{h}^2 \leq a_h(v_{0h}, v_{0h}) = -a_h(q - q_h, v_{0h})
\]
\[
= -\sum_{T \in T_h} \int_T \Delta(q - q_h)\Delta v_{0h} \, dx - \sum_{e \in \mathcal{E}_h} \int_e \{\Delta(q - q_h)\} \{\partial v_{0h}/\partial n\} \, ds
\]
\[
- \sum_{e \in \mathcal{E}_h} \int_e \{\Delta v_{0h}\} \{\partial(q - q_h)/\partial n\} \, ds - \sum_{e \in \mathcal{E}_h} |e| \int_e \{\partial(q - q_h)/\partial n\} \{\partial v_{0h}/\partial n\} \, ds
\]
\[
\leq \left( \sum_{T \in T_h} \|\Delta(q - q_h)\|^2_T + \sum_{e \in \mathcal{E}_h} |e| \|\{\Delta(q - q_h)\}\|^2_e + \sigma \sum_{e \in \mathcal{E}_h} \|\{\partial(q - q_h)/\partial n\}\|^2_e \right)^{1/2}
\]
\[
\left( \sum_{T \in T_h} \|\Delta v_{0h}\|^2_T + (\sigma + 2) \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|\{\partial v_{0h}/\partial n\}\|^2_e + \sum_{e \in \mathcal{E}_h} |e| \|\{\Delta v_{0h}\}\|^2_e \right)^{1/2}
\]
\[
\leq C_2 |e|^\beta \left( \|q\|_{H^{2+\beta}(\Omega)} + \|f\| + \|\nabla(q)\| + |e|^{-\beta} \left( \sum_{T \in T_h} h^4\|u - u_d\|_{T}^2 \right)^{1/2} \right) \|v_{0h}\|_{Q_h}.
\]

Now using the equivalence of $\|.\|_{Q_h}$ and $\|.\|_{h}$ on the finite dimensional space $V_h$, we get the following estimate
\[
\|v_{0h}\|_{Q_h} \leq C_3 |e|^\beta \left( \|q\|_{H^{2+\beta}(\Omega)} + \|\nabla(q)\| + \|f\| + (h^{2-\beta}\|u - u_d\|) \right).
\] (5.19)

A use of triangle inequality along with (5.19) and Theorem 4.1 yields
\[
\|u_{q} - q\|_{Q_h} \leq C_4 |e|^\beta \left( \|q\|_{H^{2+\beta}(\Omega)} + \|f\| + \|\nabla(q)\| + |e|^{-\beta} \left( \sum_{T \in T_h} h^4\|u - u_d\|_{T}^2 \right)^{1/2} \right),
\] (5.20)

and hence, using (5.17) and (5.20), we obtain
\[
\|u_{q} - u\| \leq C_5 h^{1+\beta} \left( \|q\|_{H^{2+\beta}(\Omega)} + \|f\| + \|\nabla(q)\| + h^{2-\beta}\|u - u_d\| \right).
\] (5.21)

The estimate for the second term of the right hand side of (5.16) follows in the same line as in [10] and hence skipped. In order to estimate the third term of the right hand side of (5.16) we note that using similar arguments as in [10] we obtain
\[
\|r\|_{H^{2+\beta}(\Omega)} \leq C\|q - q_h\|.
\] (5.22)

Next
\[
a_h(r - r_h, q - q_h) \leq C\|r - r_h\|_{Q_h}\|q - q_h\|_{Q_h}
\]
\[
\leq C|e|^{2\beta} \left( \|q\|_{H^{2+\beta}(\Omega)} + \|f\| + \|\nabla\|_{H^{3}(\Omega)} + h^{2-\beta}\|u - u_d\| \right) \left( \|r\|_{H^{2+\beta}(\Omega)} + \|\nabla(r)\| \right).
\]
Using the density of $C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega})$ in $H(div, \Omega)$ with respect to the natural norm induced on $H(div, \Omega)$, we find that

$$
\int_\Omega \nabla(\Delta r) \cdot \nabla p \, dx + \int_\Omega \Delta^2 r \, p \, dx = \langle \frac{\partial \Delta r}{\partial n}, p \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega}.
$$

From (5.14) and (5.15), we obtain that

$$
\int_\Omega \nabla(\Delta r) \cdot \nabla p \, dx = \frac{1}{\alpha} \langle \frac{\partial \Delta \xi}{\partial n}, p \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} \quad \forall \, p \in H^1(\Omega).
$$

Note that by taking $p = 1$ in (5.10), we obtain

$$
\int_\Omega (u_r - (q - q_h)) \, dx = 0
$$

and using (5.12), we conclude that (5.23) satisfies the compatibility condition. Taking $p = \Delta r - \frac{1}{|\Omega|} \int_\Omega \Delta r \, dx$ in (5.23) with a use of trace and Poincare-Friedrich’s inequality, we find that

$$
\|\nabla(\Delta r)\| \leq C \|\Delta \xi\|_{H^{-1/2}(\partial \Omega)}.
$$

Using (5.12), we obtain

$$
\left\| \frac{\partial \Delta \xi}{\partial n} \right\|_{H^{-1/2}(\partial \Omega)} \leq C \left( \|\Delta^2 \xi\| + \|\nabla(\Delta \xi)\| \right).
$$

Using the elliptic regularity theory, the solution of (5.11) satisfies

$$
\|\xi\|_{H^3(\Omega)} \leq C \|u_r - (q - q_h)\|
$$

and $\Delta^2 \xi = u_r - (q - q_h)$. Now using (5.24) and (5.25), we find that $\|\nabla(\Delta r)\| \leq C \|q - q_h\|$.

Therefore using (5.22), we have

$$
a_h(r - r_h, q - q_h) \leq C h^{2\beta} \left( \|q\|_{H^{2+\beta}(\Omega)} + \|f\| + \|\phi\|_{H^3(\Omega)} + h^{2-\beta}\|u - u_d\| \right) \|q - q_h\|. \tag{5.26}
$$

Using the same arguments, we obtain the following estimate

$$
a_h(\phi - \phi_h, r_h - r) \leq C h^{1+\beta} \left( \|q\|_{H^{2+\beta}(\Omega)} + \|f\| + \|\phi\|_{H^3(\Omega)} + h^{2-\beta}\|u - u_d\| \right) \|q - q_h\|. \tag{5.27}
$$

Note that as $\phi \in H^1_0(\Omega)$, $\phi_h \in H^1_0(\Omega)$, we obtain

$$
a_h(\phi - \phi_h, r) = 0. \tag{5.28}
$$

The estimates of the remaining terms on the right hand side of (5.10) can be found using the similar arguments as in [10]. Therefore we are skipping them and the result follows. □

6. Alternative Approach of Error Analysis

This section is devoted to the discussion of energy norm error estimate for the solution of fourth order linear elliptic equation with boundary condition of Cahn-Hilliard type [3] under minimal regularity assumption. For the sake of technical simplicity, the following equation is considered

$$
\Delta^2 \psi = g_1 \text{ in } \Omega, \tag{6.1}
$$

$$
\partial \psi / \partial n = 0, \quad \partial(\Delta \psi) / \partial n = g_2 \text{ on } \partial \Omega,
$$
where \( g_1 \in L^2(\Omega) \) and \( g_2 \in L^2(\partial\Omega) \) with the compatibility condition \( \int_{\Omega} g_1 dx = \int_{\partial\Omega} g_2 ds \) being satisfied. We know that the solution of (6.1) is unique up to an additive constant [3].

We refer to [3] for the mediuss error analysis of the solution of (6.1). In this section we propose a new error analysis for the solution in energy norm. We begin by recalling the Hilbert space \( Z \) defined in the proof of Lemma 3.1. The variational formulation of (6.1) is to find \( \psi \in Z \) such that

\[
a(\psi, p) = (g_1, p) - (g_2, p)_{\partial\Omega} \quad \forall \ p \in Z,
\]

(6.2)

If we compare this variational formulation with the corresponding variational formulation (1.1) of [3] then we note that the major difference is in the choice of test or admissible function space. This helps us to establish some special regularity result for the solution of (6.1), and it is one of the main novelties in this section:

**Lemma 6.1.** Let \( \psi \in Z \) be the solution of (6.2). Then \( \Delta \psi \in H^1(\Omega) \) and hence \( \nabla(\Delta \psi) \in H(\text{div}, \Omega) \).

**Proof.** Using Fredholm alternative theory, there exists a weak solution \( w \in H^1(\Omega) \) of the following variational formulation

\[
(\nabla w, \nabla p) = (g_1, p) - (g_2, p)_{\partial\Omega} \quad \forall \ p \in H^1(\Omega).
\]

which is unique up to an additive constant. Applying integration by parts to obtain

\[
(w, \Delta p) = (g_1, p) - (g_2, p)_{\partial\Omega} \quad \forall \ p \in Z.
\]

(6.3)

From (6.2) and (6.3), we find that

\[
(w - \Delta \psi, \Delta p) = 0 \quad \forall \ p \in Z.
\]

Therefore \( w - \Delta \psi \) belongs to the orthogonal complement of \( L^0_2(\Omega) \). Therefore \( \Delta \psi = w + a \), where \( a \) is some constant function. Hence, \( \Delta \psi \in H^1(\Omega) \) and subsequently we obtain the desired result from (6.1). \( \square \)

Note that the solution of (6.1) is unique up to an additive constant. Let \( \psi_1 \in Z \) be a solution of (6.1) and \( c \) be a corner point of \( \Omega \). We define \( \psi_2 \in Z \) as

\[
\psi_2(x) = \psi_1(x) - \psi_{1,c}(x) \quad \forall \ x \in \Omega,
\]

where the constant function \( \psi_{1,c}(x) = \psi_1(c) \ \forall \ x \in \Omega \). Then \( \psi_2(c) = 0 \) and hence \( \psi_2 \) satisfies (6.2). Define

\[
Z^* = \{ p \in Z : p(c) = 0 \}.
\]

In this connection, consider the following variational problem of finding \( \psi'_2 \in Z^* \) such that

\[
a(\psi'_2, p) = (g_1, p) - (g_2, p)_{\partial\Omega} \quad \forall \ p \in Z^*.
\]

(6.4)

Since \( \psi_2 \in Z^* \) satisfies (6.4), \( \psi_2 = \psi'_2 \) holds by the uniqueness of the solution to (6.4) [3]. Therefore, \( \psi'_2 \) possesses the regularity property described in Lemma 6.1.
By an application of Lemma 6.1, we find that $\Delta \psi_2 \in H^1(\Omega)$ and $\nabla(\Delta \psi_2) \in H(div, \Omega)$ which implies

$$\int_{\Omega} \nabla(\Delta) \psi_2 \cdot \nabla \phi \, dx + \int_{\Omega} \Delta^2 \psi_2 \phi \, dx = \left\langle \frac{\partial \Delta \psi_2}{\partial n}, \phi \right\rangle_{\frac{1}{2}, \Omega} \quad \forall \phi \in H^1(\Omega).$$

We consider the finite element space $Z_h$ to be the same as in [3]. Taking $p_h \in Z_h^*$ in (6.5), applying triangle wise integration by parts and (6.1) we obtain

$$a_h(\psi_2, p_h) = (g_1, p_h) - (g_2, p_h)_{\partial \Omega} \quad \forall p_h \in Z_h^*,$$

where $Z_h^*$ is defined by

$$Z_h^* = \{ p_h \in Z_h : p_h(c) = 0 \}.$$

In this direction, we consider the following discrete problem: Find $\psi_h \in Z_h^*$ such that

$$a_h(\psi_h, p_h) = (g_1, p_h) - (g_2, p_h)_{\partial \Omega} \quad \forall p_h \in Z_h^*.$$  

(6.7)

Now we state and prove the main result of this section which gives the error estimate for the solutions of (6.4) and (6.7) in the energy norm.

**Theorem 6.2.** Let $\eta$ and $\eta_h$ be the solutions of (6.4) and (6.7) respectively. Then the following optimal order error estimate holds

$$\| \eta - \eta_h \|_h \leq C \inf_{v_h \in Z_h^*} \| \eta - v_h \|_{Q_h}.$$

**Proof.** By Lax-Milgram lemma, there exists an unique solution of (6.7). From (6.6) and (6.7), we obtain the following Galerkin orthogonality condition

$$a_h(\eta, p_h) = 0 \quad \forall p_h \in Z_h^*.$$  

(6.8)

Let $v_h \in Z_h^*$ be arbitrary. Then using (6.8) we find that

$$\| v_h - \eta_h \|_h = \sup_{\phi_h \in Z_h^*, \phi_h \neq 0} \frac{a_h(v_h - \eta_h, \phi_h)}{\| \phi_h \|_h} = \sup_{\phi_h \in Z_h^*, \phi_h \neq 0} \frac{a_h(v_h - \eta, \phi_h)}{\| \phi_h \|_h} \leq C \| \eta - v_h \|_{Q_h}.$$  

(6.9)

A use of triangle inequality along with (6.9) completes the rest of the proof. $\square$

**Remark 6.3.** The above Theorem provides us the optimal order error estimate for the solution of (6.1) under minimal regularity assumption. Say the solution $\eta$ has $2 + \epsilon$ ($0 \leq \epsilon \leq 1$) regularity, then in the above Theorem if we choose $v_h = I_h \eta$ ($I_h \eta \in Z_h^*$ is the standard piecewise quadratic Lagrange interpolation of $\eta$) then with the help of Lemma 6.1 triangle dependent trace inequality and standard interpolation error estimates we obtain the optimal order error estimate of order $\epsilon$.
7. Numerical Examples

In this section, we verify the theoretical findings by conducting a numerical example. In the example, we validate the \textit{a priori} error estimates derived in the $L_2$-norm and energy norm established in Theorem 4.1, Theorem 4.2 and Theorem 5.6. The MATLAB software has been used for all the computations. To this end, we construct the model problem with known solution. For the ease of constructing numerical example with known solution, we modify the model problem by adding an \textit{a priori} control $p_d \in Q$ in the cost functional $J(\cdot, \cdot)$. The modified optimal control problem reads as:

$$
\min_{p \in Q} \tilde{J}(u, p) := \frac{1}{2} ||u - u_d||^2 + \frac{\alpha}{2} |p - p_d|^2_{H^2(\Omega)},
$$

subject to the condition that $(u, p) \in Q \times Q$ such that $u$ is the weak solution of (1.2). The first order optimality system takes the form:

$$
\begin{align*}
    u &= u_f + p, \\
    a(u_f, v) &= (f, v) - a(p, v) \quad \forall \ v \in V, \\
    a(\phi, v) &= (u - u_d, v) \quad \forall \ v \in V, \\
    \alpha a(p, \tilde{p}) &= a(\phi, \tilde{p}) - (u - u_d, \tilde{p}) + \alpha a(p_d, \tilde{p}) \quad \forall \ \tilde{p} \in Q.
\end{align*}
$$

Similarly, we can write the discrete optimality system as well.

**Example 7.1.** In this example, we consider the domain as $\Omega = (0, 1) \times (0, 1)$ together with the following data:

$$
\begin{align*}
    u(x, y) &= \sin^2 \pi x \sin^2 \pi y + \cos \pi x \cos \pi y, \\
    \phi(x, y) &= \sin^4 \pi x \sin^4 \pi y, \\
    p(x, y) &= \cos \pi x \cos \pi y, \quad u_d(x, y) = u(x, y) - \Delta^2 \phi(x, y), \\
    f(x, y) &= \Delta^2 u(x, y), \quad p_d(x, y) = p(x, y), \quad \alpha = 1.
\end{align*}
$$

The mesh is refined uniformly to confirm \textit{a priori} convergence order. The computed errors and orders of convergence in the energy norm and $L_2$-norm for all the variables are shown in Table 7.1 and Table 7.2, respectively. The example clearly shows the expected rates of convergence. Comparison of the plots of the exact and discrete control, adjoint state and state are shown in the Figures ??, ?? and ?? respectively.
Table 7.1. Errors and orders of convergence in energy norm.

| $h$  | $\|u - u_h\|_h$ | order | $\|\phi - \phi_h\|_h$ | order | $\|q - q_h\|_h$ | order |
|------|----------------|-------|------------------------|-------|----------------|-------|
| 1/4  | 8.33697        | –     | 13.4214                | –     | 5.00562        | –     |
| 1/8  | 4.07916        | 1.0312| 7.05252                | 0.9283| 1.78224        | 1.4899|
| 1/16 | 2.01182        | 1.0198| 3.74453                | 0.9134| 0.88300        | 1.0132|
| 1/32 | 0.98034        | 1.0371| 1.79272                | 1.0626| 0.42386        | 1.0588|
| 1/64 | 0.48293        | 1.0215| 0.85754                | 1.0639| 0.20550        | 1.0445|
| 1/128| 0.23997        | 1.0089| 0.41981                | 1.0305| 0.10162        | 1.0159|

Table 7.2. Errors and orders of convergence in $L_2$-norm.

| $h$  | $\|u - u_h\|_2$ | order | $\|\phi - \phi_h\|_2$ | order | $\|q - q_h\|_2$ | order |
|------|----------------|-------|------------------------|-------|----------------|-------|
| 1/4  | 519.515        | –     | 0.39291                | –     | 519.610        | –     |
| 1/8  | 0.04198        | 13.595| 0.05212                | 2.9144| 0.04101        | 13.629|
| 1/16 | 0.01239        | 1.7604| 0.01805                | 1.5295| 0.01336        | 1.6186|
| 1/32 | 0.00334        | 1.8929| 0.00528                | 1.7732| 0.00379        | 1.8139|
| 1/64 | 0.00086        | 1.9561| 0.00141                | 1.9076| 0.00099        | 1.9299|
| 1/128| 0.00022        | 1.9834| 0.00036                | 1.9665| 0.00025        | 1.9756|

8. Conclusion

In this article, we have derived the $L_2$-norm error estimate for the solution of a Dirichlet boundary control problem on more general domain than the one was studied in [10]. Additionally getting motivated from the technique of deriving an additional regularity result for the optimal control (Lemma 5.2), we have proposed an alternative approach for the error analysis of biharmonic equation of Cahn-Hilliard type boundary condition under minimal regularity assumption. In order to prove these results, we have derived an equality of two well known bilinear forms arising in the context of weak formulation of biharmonic equations and a density result which may be of theoretical interest.

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