On proper complex equifocal submanifolds

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Abstract

First we show that a curvature-adapted proper complex equifocal submanifold is a principal orbit of a Hermann type action under certain condition. Next we show that a proper complex equifocal submanifold is curvature-adapted under certain condition.

1 Introduction

C.L. Terng and G. Thorbergsson [TT] introduced the notion of an equifocal submanifold in a (Riemannian) symmetric space, which is defined as a compact submanifold with globally flat and abelian normal bundle such that the focal radii for each parallel normal vector field are constant. U. Christ [Ch] showed that all irreducible equifocal submanifolds of codimension greater than one in a symmetric space of compact type are homogeneous and hence they are principal orbits of hyperpolar actions. Furthermore, according to the classification of a hyperpolar action by A. Kolbross [Kol], they are principal orbits of Hermann actions. For (not necessarily compact) submanifolds in a Riemannian symmetric space of non-compact type, the equifocality is a rather weak property. So, we [Koi1,2] introduced the notion of a complex focal radius as a general notion of a focal radius and defined the notion of a complex equifocal submanifold as a submanifold with globally flat and abelian normal bundle such that the complex focal radii for each parallel normal vector field are constant and that they have constant multiplicities. Here we note that the notion of a complex focal radius defined for submanifolds in a general symmetric space but, in the case where the ambient symmetric space is of non-negative curvature, all complex focal radii are real, that is, they are focal radii and hence the complex equifocality is equivalent to the equifocality. Furthermore, we [Koi3] defined the notion of a proper complex equifocal submanifold as a complex equifocal submanifold whose complex focal structure has certain kind of regularity. Here we note that the notion of a proper complex equifocal submanifold in a general symmetric space can be defined but, in the case where the ambient symmetric space is of compact type, it is easy to show that this notion coincides with the notion of a complex equifocal (i.e., equifocal) submanifold. On the other hand, E. Heintze, X. Liu and C. Olmos [HLO] defined the notion of an isoparametric submanifold with flat section in a (general) Riemannian manifold as a submanifold with globally flat and abelian normal bundle such that the sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction. In the case where the ambient space is a symmetric space of non-compact type, we [Koi2] showed the following fact (see Theorem 15 of [Koi2]):
All isoparametric submanifolds with flat section are complex equifocal and, conversely, all curvature-adapted and complex equifocal submanifolds are isoparametric submanifolds with flat section.

Here the curvature-adaptedness means that, for any normal vector \( v \), the normal Jacobi operator for \( v \) preserves the tangent space invariantly and it commutes with the shape operator for \( v \). Also, we [Koi6] showed that, for a curvature-adapted complex equifocal submanifold \( M \), it is proper complex equifocal if and only if it admits no non-Euclidean type focal point on the ideal boundary of the ambient symmetric space, where the notion of a non-Euclidean type focal point on the ideal boundary was introduced in [Koi6]. Therefore, for a submanifold \( M \) in a symmetric space of non-compact type, the following two statements are equivalent:

(I) \( M \) is curvature-adapted and proper complex equifocal.

(II) \( M \) is a curvature-adapted isoparametric submanifold with flat section admitting no non-Euclidean type focal point on the ideal boundary.

As a counterpart of a Hermann action on a symmetric space of compact type, we [Koi3] introduced the notion of a Hermann type action on a symmetric space of non-compact type and showed that principal orbits of a Hermann type action are curvature-adapted and proper complex equifocal, that is, they are as in the above statement (II). A curvature-adapted submanifold in a symmetric space of non-compact type is a generalized notion of a Hopf hypersurface in a complex hyperbolic space and a curvature-adapted complex equifocal submanifold in the space means a Hopf hypersurface with constant principal curvatures. Hopf hypersurfaces with constant principal curvatures in a complex hyperbolic space is classified by J. Berndt [B1] and curvature-adapted hypersurfaces with constant principal curvatures in a quaternionic hyperbolic space is classified by him [B2]. According to Theorems A and C (also Remark 1.1) in [Koi5], it is shown that any irreducible homogeneous complex equifocal submanifold of codimension greater than one admitting a totally geodesic focal submanifold (or a totally geodesic parallel submanifold) occurs as a principal orbit of a Hermann type action. On the other hand, we [Koi7] showed the following homogeneity theorem:

All irreducible proper complex equifocal \( C^\omega \)-submanifolds of codimension greater than one are homogeneous.

Here "\( C^\omega \)" means the real analyticity.

Assumption. In the sequel, we assume that all submanifolds are of class \( C^\omega \).

In this paper, we first prove the following fact in terms of these facts.

Theorem A. If \( M \) is an irreducible curvature-adapted proper complex equifocal submanifold of codimension greater than one in a symmetric space \( G/K \) of non-compact type, then \( M \) is a principal orbit of a Hermann type action on \( G/K \).

Remark 1.1. In this theorem, we cannot replace "proper complex equifocal" by "complex equifocal". In fact, principal orbits of the \( N \)-action on an irreducible symmetric space
Let an abelian subspace \( b \) of \( p := T_{eK}(G/K) \), \( a \) be a maximal abelian subspace of \( p \) containing \( b \) and \( \triangle \) be the root system with respect to \( a \). Set \( \triangle^\text{abs} := \{ \text{abs} \circ \alpha |_{b} \mid \alpha \in \triangle \} \setminus \{ 0 \} \), where \( \alpha |_{b} \) is the restriction of \( \alpha \) to \( b \) and \( \text{abs} \) is the function over \( \mathbb{R} \) defined by \( \text{abs}(t) := |t| \) \( (t \in \mathbb{R}) \). This set \( \triangle^\text{abs} \) is independent of the choice of \( a \). Next we prove the following fact.

**Theorem B.** Let \( M \) be a proper complex equifocal submanifold in a symmetric space \( G/K \) of non-compact type. If arbitrary two elements of \( \triangle^\text{abs}_{g^{-1}(T_{gK}M)} \) are linearly independent, then \( M \) is curvature-adapted, where \( gK \) is an arbitrary point of \( M \).

In particular, we have the following fact.

**Corollary C.** Let \( M \) be a proper complex equifocal submanifold in a symmetric space \( G/K \) of non-compact type. If \( \text{codim} M = \text{rank}(G/K) \) and the root system of \( G/K \) is reduced, then \( M \) is curvature-adapted.

From Theorems A and B, we have the following fact.

**Corollary D.** Let \( M \) be an irreducible proper complex equifocal submanifold of codimension greater than one in a symmetric space \( G/K \) of non-compact type. If arbitrary two elements of \( \triangle^\text{abs}_{g^{-1}(T_{gK}M)} \) are linearly independent, then \( M \) is a principal orbit of a Hermann type action on \( G/K \).

## 2 Basic notions and facts

In this section, we recall basic notions introduced in [Koi1∼3,6]. We first recall the notion of a complex equifocal submanifold. Let \( M \) be an immersed submanifold with abelian normal bundle in a symmetric space \( N = G/K \) of non-compact type. Denote by \( A \) the shape tensor of \( M \). Let \( v \in T_{x}M \) and \( X \in T_{x}M \) \( (x = gK) \). Denote by \( \gamma_{v} \) the geodesic in \( N \) with \( \dot{\gamma}_{v}(0) = v \). The strongly \( M \)-Jacobi field \( Y \) along \( \gamma_{v} \) with \( Y(0) = X \) (hence \( Y'(0) = -A_{v}X \)) is given by

\[
Y(s) = (P_{\gamma_{v}|_{[0,s]}} \circ (D^{co}_{sv} - sD^{si}_{sv} \circ A_{v}))(X),
\]

where \( Y'(0) = \vec{\gamma}_{v}Y \), \( P_{\gamma_{v}|_{[0,s]}} \) is the parallel translation along \( \gamma_{v}|_{[0,s]} \) and \( D^{co}_{sv} \) (resp. \( D^{si}_{sv} \)) is given by

\[
\begin{align*}
D^{co}_{sv} &= g_{s} \circ \cos(\sqrt{-1}\text{ad}(sg^{-1}_{s}v)) \circ g^{-1}_{s} \\
\text{(resp. } D^{si}_{sv} &= g_{s} \circ \frac{\sin(\sqrt{-1}\text{ad}(sg^{-1}_{s}v))}{\sqrt{-1}\text{ad}(sg^{-1}_{s}v)} \circ g^{-1}_{s} \quad )
\end{align*}
\]
Here \( ad \) is the adjoint representation of the Lie algebra \( \mathfrak{g} \) of \( G \). All focal radii of \( M \) along \( \gamma_v \) are obtained as real numbers \( s_0 \) with \( \ker(D^0_{s_0 v} - s_0 D^s_{0 v} \circ A_v) \neq \{0\} \). So, we call a complex number \( z_0 \) with \( \ker(D^{co}_{z_0 v} - z_0 D^s_{z_0 v} \circ A^c_v) \neq \{0\} \) a complex focal radius of \( M \) along \( \gamma_v \) and call \( \dim \ker(D^{co}_{z_0 v} - z_0 D^s_{z_0 v} \circ A^c_v) \) the multiplicity of the complex focal radius \( z_0 \), where \( A^c_v \) is the complexification of \( A_v \) and \( D^{co}_{z_0 v} \) (resp. \( D^s_{z_0 v} \)) is a \( \mathbb{C} \)-linear transformation of \((T_{x}N)^c\) defined by

\[
D^{co}_{z_0 v} = g^c_x \circ \cos(\sqrt{-1\text{ad}^c(z_0 g^c_x^{-1} v)}) \circ (g^c_x)^{-1},
\]

\[
\text{resp. } D^s_{z_0 v} = g^c_x \circ \sin(\sqrt{-1\text{ad}^c(z_0 g^c_x^{-1} v)}) \circ (g^c_x)^{-1},
\]

where \( g^c_x \) (resp. \( ad^c \)) is the complexification of \( g_x \) (resp. \( ad \)). Here we note that complex focal radii along \( \gamma_v \) indicate the positions of focal points of the extrinsic complexification \( M^c(\hookrightarrow G^c/K^c) \) of \( M \) along the complexified geodesic \( \gamma^c_{v,x} \), where \( G^c/K^c \) is the anti-Kaehler symmetric space associated with \( G/K \) and \( \iota \) is the natural immersion of \( G/K \) into \( G^c/K^c \). See Section 4 of [Koi2] about the definitions of \( G^c/K^c \), \( M^c(\hookrightarrow G^c/K^c) \) and \( 
\gamma^c_{v,x} \). Also, for a complex focal radius \( z_0 \) of \( M \) along \( \gamma_v \), we call \( z_0 v \in (T^+_{x} M)^c \) a complex focal normal vector of \( M \) at \( x \). Furthermore, assume that \( M \) has globally flat normal bundle, that is, the normal holonomy group of \( M \) is trivial. Let \( \tilde{v} \) be a parallel unit normal vector field of \( M \). Assume that the number (which may be 0 and \( \infty \)) of distinct complex focal radii along \( \gamma_{\tilde{v},x} \) is independent of the choice of \( x \in M \). Furthermore assume that the number is not equal to 0. Let \( \{r_{i,x} \mid i = 1, 2, \ldots \} \) be the set of all complex focal radii along \( \gamma_{\tilde{v},x} \), where \( |r_{i,x}| < |r_{i-1,x}| \) or \( |r_{i,x}| = |r_{i+1,x}| \) & \( \text{Re} \, r_{i,x} > \text{Re} \, r_{i+1,x} \) or \( |r_{i,x}| = |r_{i+1,x}| \) & \( \text{Re} \, r_{i,x} = \text{Re} \, r_{i+1,x} \) & \( \text{Im} \, r_{i,x} = -\text{Im} \, r_{i+1,x} < 0 \). Let \( r_i \) \( (i = 1, 2, \ldots) \) be complex valued functions on \( M \) defined by assigning \( r_{i,x} \) to each \( x \in M \). We call these functions \( r_i \) \( (i = 1, 2, \ldots) \) complex focal radius functions for \( \tilde{v} \). We call \( r_i \tilde{v} \) a complex focal normal vector field for \( \tilde{v} \). If, for each parallel unit normal vector field \( \tilde{v} \) of \( M \), the number of distinct complex focal radii along \( \gamma_{\tilde{v},x} \) is independent of the choice of \( x \in M \), each complex focal radius function for \( \tilde{v} \) is constant on \( M \) and it has constant multiplicity, then we call \( M \) a complex equifocal submanifold.

Next we shall recall the notion of a proper complex equifocal submanifold. For its purpose, we first recall the notion of a proper complex isoparametric submanifold in a pseudo-Hilbert space. Let \( M \) be a pseudo-Riemannian Hilbert submanifold in a pseudo-Hilbert space \((V, \langle , \rangle)\) immersed by \( f \). See Section 2 of [Koi1] about the definitions of a pseudo-Hilbert space and a pseudo-Riemannian Hilbert submanifold. Denote by \( A \) the shape tensor of \( M \) and by \( T^\perp M \) the normal bundle of \( M \). Note that, for \( v \in T^\perp M, A_v \) is not necessarily diagonalizable with respect to an orthonormal base. We call \( M \) a Fredholm pseudo-Riemannian Hilbert submanifold (or simply Fredholm submanifold) if the following conditions hold:

\begin{enumerate}
\item[(F-i)] \( M \) is of finite codimension,
\item[(F-ii)] There exists an orthogonal time-space decomposition \( V = V_- \oplus V_+ \) such that \((V, \langle , \rangle) \mid_{V_\pm}\) is a Hilbert space and that, for each \( v \in T^\perp M, A_v \) is a compact operator with respect to \( f^* \langle , \rangle) \mid_{V_\pm}\) and hence the normal exponential map \( \exp^\perp : T^\perp M \to V \) of \( M \) is a Fredholm map with respect to the metric of \( T^\perp M \) naturally defined from
\end{enumerate}
for each parallel normal vector field $\tilde{v}$, the number of distinct complex principal
curvatures of direction $\tilde{v}$ is independent of the choice of $x \in M$ and each complex principal
curvature function of direction $\tilde{v}$ is constant on $M$ and has constant multiplicity.

Furthermore, if, for each $v \in T^\perp M$, there exists a pseudo-orthonormal base of $(T_x M)^c$
($x$ : the base point of $v$) consisting of the eigenvectors of the complexified shape operator
$A_v^c$, then we call $M$ a proper complex isoparametric submanifold. Then, for each $x \in M$,
there exists a pseudo-orthonormal base of $(T_x M)^c$ consisting of the common-eigenvectors
of the complexified shape operators $A_v^c$‘s ($v \in T_x M$) because $A_v^c$’s are commutative.
Let $\{E_i \mid i \in I\}$ ($I \subset \mathbb{N}$) be the family of subbundles of $(TM)^c$ such that, for each $x \in M$,
$\{E_i(x) \mid i \in I\}$ is the set of all common-eigenspaces of $A_v^c$’s ($v \in T_x^\perp M$).
Note that $(T_x M)^c = \bigoplus_{i \in I} E_i(x)$ holds. There exist smooth sections $\lambda_i (i \in I)$ of $((T^\perp M)^c)^*$ such
that $A_v^c = \lambda_i(v) \text{id}$ on $E_i(\pi(v))$ for each $v \in T^\perp M$, where $\pi$ is the bundle projection
of $(T^\perp M)^c$. We call $\lambda_i (i \in I)$ complex principal curvatures of $M$ and call subbundles
$E_i (i \in I)$ of $(T^\perp M)^c$ complex curvature distributions of $M$. Note that $\lambda_i(v)$ is one of the
complex principal curvatures of direction $v$. Set $l_i := \lambda_i^{-1}(1) \subset (T_x M)^c$ and $R_i$
be the complex reflection of order two with respect to $l_i$, where $i \in I$. Denote by $W_M$
the group generated by $R_i$’s ($i \in I$) which is independent of the choice of $x \in M$ up to
isomorphism. We call $l_i$’s complex focal hyperplanes of $(M,x)$. Let $N = G/K$ be a
symmetric space of non-compact type and $\pi$ be the natural projection of $G$ onto $G/K$.
Let $(g,\sigma)$ be the orthogonal symmetric Lie algebra of $G/K$, $f = \{X \in g \mid \sigma(X) = X\}$
and $\mathfrak{p} = \{X \in g \mid (X) = -X\}$, which is identified with the tangent space $T_{\pi K} N$. Let
$\langle , \rangle$ be the Ad($G$)-invariant non-degenerate symmetric bilinear form of $g$ inducing the
Riemannian metric of $N$. Note that $\langle , \rangle|_{\mathfrak{p} \times f}$ (resp. $\langle , \rangle|_{\mathfrak{p} \times \mathfrak{p}}$) is negative (resp. positive)
definite. Denote by the same symbol $\langle , \rangle$ the bi-invariant pseudo-Riemannian metric
of $G$ induced from $\langle , \rangle$ and the Riemannian metric of $N$. Set $\mathfrak{g}^c := \mathfrak{p}$, $\mathfrak{g}^- := f$ and
$\langle , \rangle_{\mathfrak{g}^c} := -\pi^*\langle , \rangle + \pi_\mathbb{C}^*\langle , \rangle$, where $\pi_{\mathbb{C}}$ (resp. $\pi_{\mathbb{R}}$) is the projection of $g$ onto $\mathfrak{g}^c$ (resp.
$\mathfrak{g}_{\mathbb{C}}$). Let $H^0(0,1,\mathfrak{g})$ be the space of all $L^2$-integrable paths $u : [0,1] \rightarrow \mathfrak{g}$ (with respect
to $\langle , \rangle_{\mathfrak{g}^c}$). Let $H^0(0,1,\mathfrak{g}^-)$ (resp. $H^0(0,1,\mathfrak{g}^c)$) be the space of all $L^2$-integrable paths
$u : [0,1] \rightarrow \mathfrak{g}^c$ (resp. $u : [0,1] \rightarrow \mathfrak{g}^c$) with respect to $-\langle , \rangle |_{\mathfrak{g}^c \times \mathfrak{g}^c}$ (resp. $\langle , \rangle |_{\mathfrak{g}^c \times \mathfrak{g}^c}$). It is clear that
$H^0(0,1,\mathfrak{g}) = H^0(0,1,\mathfrak{g}^-) \oplus H^0(0,1,\mathfrak{g}^c)$. Define a non-degenerate symmetric
bilinear form $\langle , \rangle_0$ of $H^0(0,1,\mathfrak{g})$ by $\langle u, v \rangle_0 := \int_0^1 \langle u(t), v(t) \rangle dt$. It is easy to show that the
decomposition $H^0(0,1,\mathfrak{g}) = H^0(0,1,\mathfrak{g}^-) \oplus H^0(0,1,\mathfrak{g}^c)$ is an orthogonal time-space
decomposition with respect to $\langle , \rangle_0$. For simplicity, set $H^0_0 := H^0(0,1,\mathfrak{g}_\pm)$ and $\langle , \rangle_{0,H^0_\pm} :=$
$-\pi^*_{H_0}(\ , \ )_0 + \pi^*_{H_0}(\ , \ )_0$, where $\pi_{H_0}$ (resp. $\pi_{H_0}^*$) is the projection of $H^0([0,1],g)$ onto $H_0^0$ (resp. $H_0^0$). It is clear that $\langle u, v \rangle_{0, H_0^0} = \int_0^1 \langle u(t), v(t) \rangle_{g_\pm} \ dt \ (u, v \in H^0([0,1],g))$. Hence $(H^0([0,1],g), \langle \ , \ , \rangle_{0, H_0^0})$ is a Hilbert space, that is, $(H^0([0,1],g), \langle \ , \ , \rangle_0)$ is a pseudo-Hilbert space. Let $H^1([0,1],G)$ be the Hilbert Lie group of all absolutely continuous paths $g : [0,1] \to G$ such that the weak derivative $g'$ of $g$ is squared integrable (with respect to $\langle \ , \rangle_g$), that is, $g^{-1}_*g' \in H^0([0,1],g)$. Define a map $\phi : H^0([0,1],g) \to G$ by $\phi(u) = g_u(1) (u \in H^0([0,1],g))$, where $g_u$ is the element of $H^1([0,1],G)$ satisfying $g_u(0) = e$ and $g_u^{-1}g'_u = u$. We call this map the parallel transport map (from 0 to 1). This submersion $\phi$ is a pseudo-Riemannian submersion of $(H^0([0,1],g), \langle \ , \ , \rangle_0)$ onto $(G, \langle \ , \ , \rangle)$. Let $\pi : G \to G/K$ be the natural projection. It follows from Theorem A of [Koi1] (resp. Theorem 1 of [Koi2]) that, in the case where $M$ is curvature adapted (resp. of class $C^\infty$), $M$ is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric. In particular, if components of $(\pi \circ \phi)^{-1}(M)$ are proper complex isoparametric, then we call $M$ a proper complex equifocal submanifold.

Next we recall the notion of an isoparametric submanifold in a general Riemannian manifold, which was introduced in [HLO]. Let $M$ be a submanifold in a Riemannian manifold $N$. The submanifold $M$ is called an isoparametric submanifold if it has (globally) flat normal bundle, if it has section and if the sufficiently close parallel submanifolds of $M$ are of constant mean curvature with respect to the radial direction. In the case where $N$ is a symmetric space of non-compact type, all isoparametric submanifolds with flat section in $N$ are complex equifocal and, conversely all curvature-adapted and complex equifocal submanifolds in $N$ are an isoparametric submanifolds with flat section.

Next we recall the notion of a non-Euclidean type focal point on the ideal boundary $N(\infty)$ for submanifold $M$ in a Hadamard manifold $N$, which was introduced in [Koi6]. Denote by $\nabla$ the Levi-Civita connection of $N$ and $A$ the shape tensor of $M$. Let $\gamma_v : [0, \infty) \to N$ be the normal geodesic of $v \in T^*_x\, M$. If there exists an $M$-Jacobi field $Y$ along $\gamma_v$ satisfying $\lim_{t \to \infty} \frac{||Y_t||}{t} = 0$, then we call $\gamma_v(\infty) (\in N(\infty))$ a focal point on the ideal boundary $N(\infty)$ of $M$ along $\gamma_v$, where $\gamma_v(\infty)$ is the asymptotic class of $\gamma_v$. We call $\text{Span}\{Y_0 | Y : a \text{-} M$-Jacobi field along $\gamma_v \text{ s.t. } \lim_{t \to \infty} \frac{||Y_t||}{t} = 0 \}$ the nullity space of the focal point $\gamma_v(\infty)$. Also, if there exists a $M$-Jacobi field $Y$ along $\gamma_v$ satisfying $\lim_{t \to \infty} \frac{||Y_t||}{t} = 0$ and $\text{Sec}(v, Y(0)) < 0$, then we call $\gamma_v(\infty)$ a non-Euclidean type focal point on $N(\infty)$ of $M$ along $\gamma_v$, where $\text{Sec}(v, Y(0))$ is the sectional curvature for the 2-plane spanned by $v$ and $Y(0)$.

### 3 Proofs of Theorems A and B

In this section, we shall prove Theorems A and B. First we prepare a lemma to prove Theorem A. Let $M$ be a proper complex equifocal submanifold in a symmetric space $G/K$ of non-compact type. We may assume that $eK \in M$ ($e$ : the identity element of $G$) by operating an element of $G$ to $M$ if necessary and hence the constant path $\hat{0}$ at the zero element 0 of $\hat{g}$ is contained in $\hat{M} := (\pi \circ \phi)^{-1}(M)$. Denote by $M^\circ$ the component of $\hat{M}$ containing $\hat{0}$. Fix a unit normal vector $v$ of $M$ at $eK$. Set $p := T_{eK}(G/K)$ and $b := T_{eK}^\perp M$. Let $p = a + \sum_{\alpha \in \Delta_+} p_\alpha$ be the root space decomposition with respect to a
maximal abelian subspace $a$ in $p$ containing $b$. Let $\triangle_b := \{\alpha|_b \mid \alpha \in \triangle \text{ s.t. } \alpha|_b \neq 0\}$ and $p = 3_{\beta}(b) + \sum_{\beta \in (\triangle_b)_+} p_{\beta}$ be the root space decomposition with respect to $b$, where $3_{\beta}(b)$ is the centralizer of $b$ in $p$. For convenience, we denote $3_{\beta}(b)$ by $p_0$. Then we have $p_\beta = \sum_{\alpha \in \triangle_+ \text{ s.t. } \alpha|_b = \pm \beta} p_\alpha$ (\(\beta \in (\triangle_b)_+\)) and $p_0 = a + \sum_{\alpha \in \triangle_+ \text{ s.t. } \alpha|_b = 0} p_\alpha$. Denote by $A$ (resp. $\tilde{A}$) the shape tensor of $M$ (resp. $\tilde{M}_0$). Also, denote by $R$ the curvature tensor of $G/K$. Let $m_A := \max_{v \in h(0)} \sharp \text{Spec} A_v$ and $m_R := \max_{v \in h(0)} \sharp \text{Spec} R(\cdot, v)$, where we note that $m_R = \sharp(\triangle_b)_+$. Let $U := \{v \in b \setminus \{0\} \mid \sharp \text{Spec} A_v = m_A, \sharp \text{Spec} R(\cdot, v)^2 = m_R\}$, which is an open dense subset of $b \setminus \{0\}$. Fix $v \in U$. Note that $\text{Spec} R(\cdot, v)^2 = \{-\beta(v)^2 \mid \beta \in (\triangle_b)_+\}$. Since $v \in U$, $\beta(v)^2$'s ($\beta \in (\triangle_b)_+$) are mutually distinct. Let $\text{Spec} A_v = \{\lambda^v_1, \ldots, \lambda^v_{m_A}\}$ ($\lambda^v_1 \geq \cdots \geq \lambda^v_{m_A}$). Set

$$I_v^0 := \{i \mid p_0 \cap \ker(A_v - \lambda^v_i \text{id}) \neq \{0\}\},$$

$$I_v^{\beta} := \{i \mid p_\beta \cap \ker(A_v - \lambda^v_i \text{id}) \neq \{0\}\},$$

$$(I_v^{\beta})^+ := \{i \mid p_\beta \cap \ker(A_v - \lambda^v_i \text{id}) \neq \{0\}, \; |\lambda^v_i| > |\beta(v)|\},$$

$$(I_v^{\beta})^- := \{i \mid p_\beta \cap \ker(A_v - \lambda^v_i \text{id}) \neq \{0\}, \; |\lambda^v_i| < |\beta(v)|\},$$

$$(I_v^{\beta})^0 := \{i \mid p_\beta \cap \ker(A_v - \lambda^v_i \text{id}) \neq \{0\}, \; |\lambda^v_i| = |\beta(v)|\}. $$

Let $F$ be the sum of all complex focal hyperplanes of $(\tilde{M}_0, \tilde{0})$. Denote by $pr_R$ the natural projection of $b^\circ$ onto $b$ and set $F_R := pr_R(F)$. Then we have the following facts.

**Lemma 3.1.** Assume that $M$ is curvature-adapted and that $G/K$ is not a hyperbolic space. Then the set $(I_v^{\beta})^0$ is empty and the spectrum of $A_v|_{\sum_{\beta \in (\triangle_b)_+} P_\beta}$ is equal to

$$\left\{\frac{\beta(v)}{\tanh(\beta(Z))} \mid \beta \in (\triangle_b)_+ \text{ s.t. } (I_v^{\beta})^+ \neq \emptyset\right\}$$

and

$$\cup \left\{\frac{\beta(v)}{\tanh(\beta(Z))} \mid \beta \in (\triangle_b)_+ \text{ s.t. } (I_v^{\beta})^- \neq \emptyset\right\}$$

for some $Z \in b$.

**Proof.** From $v \in U$, we have $\beta(v) \neq 0$ for any $\beta \in (\triangle_b)_+$. Hence, since $M$ is curvature-adapted and proper complex equifocal, it follows from Theorem 1 of [Ko12] that $(I_v^{\beta})^0 = \emptyset$. Set $c_{\beta,i,v}^+ := \frac{\beta(v)}{\lambda_i^v}$ (\(i \in (I_v^{\beta})^+ (\beta \in (\triangle_b)_+)\)) and $c_{\beta,i,v}^- := \frac{\lambda_i^v}{\beta(v)} (i \in (I_v^{\beta})^- (\beta \in (\triangle_b)_+))$. According to the proof of Theorems B and C in [Ko6], we have

$$F = \left(\bigcup_{\beta \in (\triangle_b)_+ \text{ s.t. } (I_v^{\beta})^+} \left(\sum_{(i,j) \in (I_v^{\beta})^+} \mathbb{Z} (\beta^c)^{-1} (\arctanh c_{\beta,i,v}^+ + j\pi\sqrt{-1})\right)\right)$$

and

$$F_R = \left(\bigcup_{\beta \in (\triangle_b)_+ \text{ s.t. } (I_v^{\beta})^+} \left(\sum_{(i,j) \in (I_v^{\beta})^+} \beta^{-1} (\arctanh c_{\beta,i,v}^+)\right)\right)$$

Also, since $G/K$ is not a hyperbolic space, the intersection of all the hyperplanes constructing $F_R$ is non-empty. Here we note that, in the case where $M$ is a totally umbilic
hyperbolic hypersurface in a hyperbolic space, $F_R$ is empty. Take an element $Z$ of the intersection. Then we have

$$\lambda_i^v = \begin{cases} \frac{\beta(v)}{\tanh \beta(Z)} & (i \in (I_{\beta}^+)^+ \\
\beta(v) \tanh \beta(Z) & (i \in (I_{\beta}^+)^-) \end{cases}.$$ 

Hence we have

$$\text{Spec} \left( A_v|_{\sum_{\beta \in (\Delta_b)_+} p_{\beta}} \right) = \left\{ \frac{\beta(v)}{\tanh \beta(Z)} \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I_{\beta}^v)^+ \neq \emptyset \right\}$$

$$\cup \left\{ \beta(v) \tanh \beta(Z) \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I_{\beta}^v)^- \neq \emptyset \right\}.$$ 

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By using this lemma, we prove Theorem A.

**Proof of Theorem A.** Let $M$ be as in the statement of Theorem A. In the case where $G/K$ is a hyperbolic space, $M$ is an isoparametric hypersurface (other than a horosphere) in the space and hence it occurs as a principal orbit of a Hermann type action. So we suffice to show the statement in the case where $G/K$ is not a hyperbolic space. Let $Z$ be as in the proof of Lemma 3.1 and $\tilde{Z}$ be the parallel normal vector field of $M$ with $\tilde{Z}_{eK} = Z$. Denote by $\eta_{t\tilde{Z}}$ $(0 \leq t \leq 1)$ the end-point map for $t\tilde{Z}$ (i.e., $\eta_{t\tilde{Z}}(x) = \exp^+(t\tilde{Z}_x)$ $(x \in M)$), where $\exp^+$ is the normal exponential map of $M$. Set $M_t := \eta_{t\tilde{Z}}(M)$, which is a parallel submanifold or a focal submanifold of $M$. In particular, $M_1$ is a focal submanifold of $M$. Let $\tilde{v}$ be the parallel tangent vector field on the flat section $\Sigma_{eK}$ with $\tilde{v}_{eK} = v(\in U)$. Note that $\tilde{v}_{(\eta_{t\tilde{Z}}(M))}$ is a normal vector of $M_t$. Denote by $A^I$ the shape tensors of $M_t$ $(0 < t \leq 1)$. It is clear that there exists a positive number $\varepsilon$ such that $M_t$'s $(1 - \varepsilon < t < 1)$ are parallel submanifolds of $M$. According to the proof of Theorems B and C of [Ko6], there exists a complex linear function $\phi_i$ on $\mathcal{E}$ with $\phi_i(v) = \lambda_i^v$ and $\phi_i^{-1}(1) \subset F$ for each $i \in I_0^0$ with $\lambda_i^v \neq 0$. Fix $i_0 \in I_0^0$. According to (3.1), $\phi_i^{-1}(1)$ coincides with one of $(\beta^c)^{-1}(\text{arctanh}(\beta_{i,v}^{-1}) + j\pi \sqrt{-1})$'s $(\beta \in (\Delta_b)_+)$, $(i,j) \in (I_{\beta}^v)^+ \times Z$ and $(\beta^c)^{-1}(\text{arctanh}(\beta_{i,v}^{-1}) + (j + \frac{1}{2})\pi \sqrt{-1})$'s $(\beta \in (\Delta_b)_+), (i,j) \in (I_{\beta}^v)^- \times Z$. Since $(\beta^c)^{-1}(\text{arctanh}(\beta_{i,v}^{-1}) + j\pi \sqrt{-1}) = \beta^{-1}(\text{arctanh}(\beta_{i,v}^+)) + (\beta^c|_{\sqrt{-1}})^{-1}(j\pi \sqrt{-1})$ and $(\beta^c)^{-1}(\text{arctanh}(\beta_{i,v}^{-1}) + (j + \frac{1}{2})\pi \sqrt{-1}) = \beta^{-1}(\text{arctanh}(\beta_{i,v}^+)) + (\beta^c|_{\sqrt{-1}})^{-1}(j + \frac{1}{2})\pi \sqrt{-1})$, we have $\phi_i^{-1}(1) = \beta^{-1}(\text{arctanh}(\beta_{i,v}^+)) + (\beta^c|_{\sqrt{-1}})^{-1}(0)$ for some $\beta \in (\Delta_b)_+$. Thus we have $Z \beta^{-1}(\text{arctanh}(\beta_{i,v}^+)) \subset \phi_i^{-1}(1)$, which implies that

$$(3.2) \quad \eta_{Z,*}(p_0 \cap \ker(A_v - \lambda_i^v \text{id})) = \{0\}.$$ 

Easily we can show $\eta_{Z,*}(p_0 \cap \ker(A_v) \subset T_{\eta_{t\tilde{Z}}(M)}(M_1)$ and

$$(3.3) \quad A^I_{\tilde{v}_{(\eta_{t\tilde{Z}}(M))}}|_{(\eta_{t\tilde{Z}}(M))_*(p_0 \cap \ker(A_v))} = 0.$$ 

By delicate discussion in terms of Lemma 3.1, we can show

$$\text{Spec} A^I_{\tilde{v}_{(\eta_{t\tilde{Z}}(M))}}|_{\sum_{\beta \in (\Delta_b)_+} \langle \eta_{t\tilde{Z}} \rangle \cdot (p_{\beta})}$$

$$= \left\{ \frac{\beta(v)}{\tanh((1 - t)\beta(Z))} \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I_{\beta}^v)^+ \neq \emptyset \right\}$$

$$\cup \left\{ \beta(v) \tanh((1 - t)\beta(Z)) \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I_{\beta}^v)^- \neq \emptyset \right\}.$$ 

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for each \( t \in (1 - \varepsilon, 1) \). Denote by \( \text{Gr}_m(G/K) \) the Grassmann bundle of \( G/K \) consisting of \( m \)-dimensional subspaces of the tangent spaces, where \( m := \dim M_1 \). Define \( D_t \) \((1 - \varepsilon < t < 1)\) by

\[
D_t := (\eta \tilde{Z})_*(p_0 \cap \ker A_v) + \sum_{\beta \in (\Delta_b)_+ \text{ s.t. } (I^\beta_\beta)^- \neq \emptyset} (p_\beta \cap \ker \left( A^{t}_{\eta \tilde{Z}(eK)} - \beta(v) \tanh((1 - t)\beta(Z))I - \right)).
\]

From (3.2), (3.3) and (3.4), we can show \( D_t \in \text{Gr}_m(G/K) \), \( \lim_{t \to 1 - 0} D_t = T_{\eta \tilde{Z}(eK)}M_1 \) (in \( \text{Gr}_m(G/K) \)) and

\[
\text{Spec } A^1_{\eta \tilde{Z}(eK)} = \{0\} \cup \left\{ \lim_{t \to 1 - 0} \beta(v) \tanh((1 - t)\beta(Z)) \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I^\beta_\beta)^- \neq \emptyset \right\} = \{0\}.
\]

Thus we have \( A^1_{\eta \tilde{Z}(eK)} = 0 \). Since the relation holds for any \( v \in U \) and \( U \) is open and dense in \( b(= T_{eK}^\perp M) \), \( A^1_{\eta \tilde{Z}(eK)} = 0 \) holds for any \( v \in T_{eK}^\perp M \). Set \( L := \eta \tilde{Z}^{-1}(\eta \tilde{Z}(eK)) \). Take any \( x \in L \). Similarly we can show \( A^1_{\eta \tilde{Z}(x)} = 0 \) for any \( v \in T_x^\perp M \), where \( \eta \) is the parallel tangent vector field on the section \( \Sigma_x \) of \( M \) through \( x \) with \( \eta \tilde{Z}(eK) = \eta \tilde{Z}(x) \) and \( \sum_{x \in L} \{ \eta \tilde{Z}(x) \mid v \in T_x^\perp M \} = T_{\eta \tilde{Z}(eK)}M_1 \). Hence we see that \( A^1 \) vanishes at \( \eta \tilde{Z}(eK) \). Similarly we can show that \( A^1 \) vanishes at any \( y \) for any \( y \in M_1 \) other than \( \eta \tilde{Z}(eK) \). Therefore \( M_1 \) is totally geodesic in \( G/K \). By the way, since \( M \) is irreducible, proper complex equifocal and \( \text{codim } M \geq 2 \), it follows from the homogeneity theorem of [Koi7] that \( M \) is homogeneous. Hence it follows from Theorem A of [Koi5] that \( M \) occurs as a principal orbit of a complex hyperpolar action on \( G/K \). Furthermore, since this action admits a totally geodesic singular orbit \( M_1 \) and it is of cohomogeneity greater than one, it follows from Theorem C and Remark 1.1 of [Koi5] that it is orbit equivalent to a Hermann type action. Therefore we see that \( M \) is a principal orbit of a Hermann type action.

q.e.d.

![Diagram](image-url)
Next we shall prove Theorem B. For its purpose, we prepare the following lemma.

**Lemma 3.2.** Let $\psi_1$ and $\psi_2$ be a skew-symmetric $\mathbb{C}$-linear transformation and a symmetric $\mathbb{C}$-linear transformation of a (finite dimensional) anti-Kaehlerian space $(V, \langle , \rangle)$, respectively. Take $X \in V \setminus \{0\}$. Assume that

$$
\left( \cosh((a + bk)\psi_1) - \frac{\sinh((a + bk)\psi_1)}{\psi_1} \circ \psi_2 \right)(X) = 0
$$

for all $k \in \mathbb{Z}$, where $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0\}$. Let $S$ be the minimal subset of the spectrum of $\psi_1^2$ satisfying $\{X, \psi_2(X)\} \subset \bigoplus_{\mu \in S} \ker(\psi_1^2 - \mu \id)$. Then we have $b\sqrt{\mu} \in \pi\sqrt{-1}\mathbb{Z}$ for any $\mu \in S$.

**Proof.** Let $X = \sum_{\mu \in S} X_\mu$ and $\psi_2(X) = \sum_{\mu \in S} \psi_2(X)_\mu$, where $X_\mu, \psi_2(X)_\mu \in \ker(\psi_1^2 - \mu \id)$. Thus it follows from the assumption that

$$
\cosh((a + bk)\sqrt{\mu})X_\mu - \frac{\sinh((a + bk)\sqrt{\mu})}{\sqrt{\mu}} \psi_2(X)_\mu = 0 \quad (\mu \in S, k \in \mathbb{Z}).
$$

From the minimality of $S$, either $X_\mu$ or $\psi_2(X)_\mu$ is not equal to the zero vector. Hence we have $b\sqrt{\mu} \in \pi\sqrt{-1}\mathbb{Z}$. q.e.d.

By using this lemma, we prove Theorem B.

**Proof of Theorem B.** Let $M$ and $G/K$ be as in the statement of Theorem B. Let $\{\lambda_i \mid i \in I\}$ be the set of all complex principal curvatures of $\tilde{M}_0$ at $\hat{0}$ and set $l_i := \lambda_i^{-1}(1) \ (i \in I)$. Since the group generated by the complex reflections $R_j$’s $(i \in I)$ of order 2 with respect to $l_i$ is discrete, $\mathcal{L} := \{l_i \mid i \in I\}$ is equal to the sum $\bigcup_{j=1}^r \mathcal{L}_j$ of subfamilies $\mathcal{L}_j := \{l_{ik} \mid k \in \mathbb{Z}\}$ ($j = 1, \cdots, r$) of parallel complex hyperplanes equidistant to one another. For each $j \in \{1, \cdots, r\}$, we have $\lambda_j^{-1}(1) = \lambda_{ij}^{-1}(1 + kb_j)$ for some $b_j \in \mathbb{C}$. For simplicity, we denote $\lambda_j$ by $\bar{X}_j \ (j = 1, \cdots, r)$. Let $a$ be a maximal abelian subspace of $\mathfrak{p} := T_{eK}(G/K)$ containing $\mathfrak{b} := T_{eK}M$, $\Delta$ be the root system with respect to $a$ and $\mathfrak{p} = a + \sum_{\alpha \in \Delta^+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to $a$. For simplicity we set $\bar{\alpha} := a|_{\mathfrak{b}} (\alpha \in \Delta)$. Set $U := \{v \in \mathfrak{b} \mid \beta(v)2^r_s (\beta \in \Delta^+_{ab}) \text{ are linearly independent over } Q\}$, which is dense in $\mathfrak{b}$ because arbitrary two elements of $\Delta^+_{ab}$ are linearly independent (over $\mathbb{R}$) by the assumption. Fix $v \in U$. The set $FR_v$ of all complex focal radii of $M$ along $\gamma_v$ is given by

$$
FR_v = \{z \in \mathbb{C} \mid \ker(Dv_0^0 \circ D_v^0 \circ A_v^0) \neq \{0\}\}.
$$

For simplicity, set $Q_v(z) := Dv_0^0 \circ D_v^0 \circ A_v^0$. The constant path $\hat{v}$ at $v$ is the horizontal lift of $v$ to 0. On the other hand, the set $FR_0$ of all complex focal radii of $\tilde{M}_0$ along $\gamma_0$ is given by

$$
FR_0 = \left\{ \frac{1 + kb_j}{\lambda_j(v)} \mid j = 1, \cdots, r, \ k \in \mathbb{Z} \right\}.
$$
Since $FR_v = FR_v$, we have $\text{Ker} Q_v(1 + k b_j) \neq \{0\}$ $(j = 1, \ldots, r, k \in \mathbb{Z})$. Set $\overline{E}_{jk} := \text{Ker} Q_v\left(1 + \frac{k b_j}{\lambda_j(v)}\right)$ and $E_{jk} := \text{Ker} \left(\tilde{A}_v - \frac{1}{1 + kb_j} \text{id}\right)$. It is easy to show that $(\pi \circ \phi)^* (E_{jk}) = \overline{E}_{jk}$. Also, we can show that $\overline{E}_{jk}$'s $(k \in \mathbb{Z})$ coincide with one another. Hence, for each $j \in \{1, \ldots, r\}$, we have

$$Q_v\left(\frac{1 + k b_j}{\lambda_j(v)}\right)|_{\overline{E}_{j0}} = \left(\cosh\left(\frac{1 + k b_j}{\lambda_j(v)}\right) - \frac{\sinh\left(\frac{1 + k b_j}{\lambda_j(v)}\right)}{\lambda_j(v)}\right) \circ A_v|_{\overline{E}_{j0}} = 0$$

$(k \in \mathbb{Z})$. Fix $X(\neq 0) \in \overline{E}_{j0}$. Set $p_\beta := \bigoplus_{\alpha \in \Delta^+ \text{ s.t. } \alpha \circ \beta \vDash \beta \in \Delta^\text{abs}} p_\alpha$ $(\beta \in \Delta^\text{abs})$. Let $S$ be the minimal subset of $\Delta^\text{abs}$ satisfying $\{X, A^c_\beta X\} \subset \bigoplus_{\beta \in S} p^c_\beta$. Then, according to Lemma 3.2, we have $\frac{b_j(v)}{\lambda_j(v)} \in \pi \sqrt{-1} \mathbb{Z}$ for any $\beta \in S$. Suppose $\sharp S \geq 2$, where $\sharp(*)$ is the cardinal number of $*$. Then $\beta(v)^2$'s $(\beta \in S)$ are linearly dependent over $Q$. On the other hand, it follows from $v \in U$ that they are linearly independent over $Q$. Thus a contradiction arises. Therefore we obtain $\sharp S = 1$. That is, we have $X \in p^c_{\beta_0}$ and $A^c_\beta X \in p^c_{\beta_0}$ for some $\beta_0 \in \Delta^\text{abs}$. Hence we have $A^c_\beta X = \frac{b_0(v)}{\text{tanh}(b_0(v)/\lambda_j(v))} X$. This together with the arbitrariness of $X(\in \overline{E}_{j0})$ implies $R^c(\cdot, v)v|_{\overline{E}_{j0}} \subset \overline{E}_{j0}$ and $[A_v, R^c(\cdot, v)v]|_{\overline{E}_{j0}} = 0$. On the other hand, we have $\sum_{j=1}^{r} \overline{E}_{j0} = (T_{eK} M)^c$. Therefore we have $R^c(\cdot, v)v((T_{eK} M)^c) \subset (T_{eK} M)^c$ and $[A_v, R^c(\cdot, v)v] = 0$. Hence we have $R(\cdot, v)v(T_{eK} M) \subset T_{eK} M$ and $[A_v, R(\cdot, v)v] = 0$. Furthermore, since $v$ is an arbitrary element of $U$ and $U$ is dense in $\mathfrak{b}$, $R(\cdot, v)v(T_{eK} M) \subset T_{eK} M$ and $[A_v, R(\cdot, v)v] = 0$ holds for any $v \in \mathfrak{b}$. By the same discussion, we can show that the same fact holds for any point of $M$ other than $eK$. Therefore $M$ is curvature-adapted.

q.e.d.

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