1. Introduction

In the decades following Shannon’s work, the quest to design codes for the additive white Gaussian noise (AWGN) channel led to the development of a rich theory, revealing a number of beautiful connections between information theory and geometry of numbers. One of the most striking examples is the connection between classical lattice sphere packing and the capacity of the AWGN channel. The main result states that any family of lattice codes with linearly growing Hermite invariant achieves a constant gap to capacity. These classical results and many more can be found in the comprehensive book by Conway and Sloane [5].

The early sphere packing results suggested that lattice codes could achieve the capacity of the AWGN channel and led to a series of works trying to prove this, beginning with [6] and finally completed in [7]. Thus, while there are still plenty of interesting questions to consider, the theory of lattice codes for the single user AWGN channel is now well-established.

However, although the AWGN channel is a good model for deep-space or satellite links, modern wireless communications require to consider more general channel models which include time or frequency varying fading and possibly multiple transmit and receive antennas. Therefore in the last twenty years coding theorists have focused on the design of lattice codes for multiple and single antenna fading channels [3, 23].

However, the question of whether lattice codes can achieve capacity in fading channels has only been addressed recently. The first work that we are aware is due to S. Vituri [25, Section 4.5], and gives a proof of existence of lattice codes achieving a constant gap to capacity for i.i.d SISO channels. It seems that with minor modification this proof is enough to guarantee the existence of capacity achieving lattices. In the single antenna i.i.d fading channels this problem was considered also in [10] and in our paper [24].

In [13] it was shown that polar lattices achieve capacity in single antenna i.i.d fading channels. This is not only an existence result, but does also give an explicit code construction. In [4] the authors prove existence of lattice codes achieving capacity for the compound MIMO channel, where the fading is random during the first $s$ time units, but then gets repeated in blocks of length $s$. This work is most closely related to [21], which was considering a similar question.

In [15, 16] we proved that lattice codes derived from number fields and division algebras do achieve a constant gap to capacity over single and multiple antenna fading channels. As far as we know this was the first result achieving constant gap to MIMO capacity with lattice codes. In [11] the authors corrected and generalized [10] and improved on our gap in the case of Rayleigh fading MIMO channels.

However while in our work [16] the gap to capacity is relatively large, these codes are almost universal in the sense that a single code achieves a constant gap to capacity for...
all stationary ergodic fading channel models satisfying a certain condition for fading (9). With some limitations this gap is also uniform for all such channels (see Remark 1). In this work we are revisiting some of the results in [16] and presenting them from a slightly different and more general perspective. Our approach is based on generalizing the classical sphere packing approach to fading channels. In [16] we introduced the concept of reduced Hermite invariant of a lattice with respect to a linear group of block fading matrices. As a generalization of the classical result for AWGN channels, we proved that if a family of lattices has linearly growing reduced Hermite invariant, it achieves a constant gap to capacity in the block fading MIMO channel. In this work we extend this result and show that given any linearly fading channel model we can define a corresponding notion of reduced Hermite invariant. We also prove that in some cases the reduced Hermite invariant of a lattice is actually a homogeneous minimum with respect to homogeneous form (which depends on the fading model). From this perspective the classical sphere packing result [5, Chapter 3] is just one example of the general connection between linear fading channels and the homogeneous minima of the corresponding forms.

In Section 2 we begin by defining a general linear fading model, which captures several channels of interest for practical applications. In Section 3 we recall how to obtain a finite signal constellation from an infinite lattice under an average power constraint. In Section 4 we review how the classical Hermite invariant can be used as a design criterion to build capacity approaching lattice codes in AWGN channels. In Section 5 we generalize the concept of Hermite invariant to linear fading channels by introducing the general reduced Hermite invariant. We will also show that replacing the Hermite invariant with the reduced one as a code design criterion leads to an analogous capacity result in linear fading channels.

In Section 6 we focus on channels where the fading matrices are diagonal. This brings us to consider ergodic fading single antenna channels. Following [16], we then show how lattice constructions from algebraic number fields can be used to approach capacity in such channels. We begin by considering lattices arising from the canonical embedding of the ring of algebraic integers, then examine the question of improving the gap to capacity using non-principal ideals of number fields [24]. In particular, we show that our information-theoretic problem is actually equivalent to a certain classical problem in algebraic number theory.

Finally in Section 7 we extend the results in [16] and show that in many relevant channel models the reduced Hermite invariant of a lattice is actually a homogeneous minimum of a certain form.

2. General linear fading channel

In this work we consider complex vector-valued channels, where the transmitted (and received) elements are vectors in $\mathbb{C}^k$. A code $C_k$ is a finite set of elements in $\mathbb{C}^k$. We assume that both the receiver and the transmitter know the code.

Given a matrix $H \in M_n(\mathbb{C})$ and a vector $x \in \mathbb{C}^k$, in order to hold on the tradition that a transmitted vector is a row, we introduce the notation

$$H[x] = (H(x^T))^T.$$

More precisely, the ideal lattice construction was considered in the extended version of [24], available at http://arxiv.org/abs/1411.4591v2
Let us assume we have an infinite sequence of random matrices $H_k$, $k = 1, 2, \ldots, \infty$, where for every $k$, $H_k$ is a $k \times k$ matrix. Given such sequence of matrices we can define a corresponding channel. Given an input $x = (x_1, \ldots, x_k)$, we will write the channel output as

$$y = H_k[x] + w,$$

where $w$ is a length $k$ random vector, with i.i.d complex Gaussian entries with variance 1 and zero mean, and $H_k$ is a random matrix representing fading. We assume that the receiver always knows the channel realization of $H_k$ and is trying to guess which was the transmitted codeword $x$ based on $y$ and $H_k$. This set-up defines a linear fading channel (with channel state information at the receiver), where the term “linear” simply refers to the fact that the fading can be represented as the action of a linear transform on the transmitted codeword. This type of channels (but without channel state information) have been considered before in [26].

In the following sections we consider the problem of designing codes for this type of channels. In the remainder of the paper we will assume the extra condition that the determinant of the random matrices $H_k$ is non-zero with probability one. The channel model under consideration captures many communication channels of practical significance. For example when $H_k$ is a deterministic identity matrix, we have the classical additive Gaussian channel. On the other hand if $H_k$ is a diagonal matrix with i.i.d Gaussian random elements with zero mean, we obtain the Rayleigh fast fading channel. Finally, if $H_k$ is a block diagonal matrix we obtain a block fading MIMO channel.

### 3. Lattices and finite codes

As mentioned previously, our finite codes $C$ are simply subsets of elements in $\mathbb{C}^k$. We consider the ambient space $\mathbb{C}^k$ as a metric space with the Euclidean norm.

**Definition 1.** Let $v = (v_1, \ldots, v_k)$ be a vector in $\mathbb{C}^k$. The Euclidean norm of $v$ is $||v|| = \sqrt{\sum_{i=1}^{k} |v_i|^2}$.

Given a transmission power $P$, we require that every codeword $x \in C \subset \mathbb{C}^k$ satisfies the average power constraint

$$\frac{1}{k} \|x\|^2 \leq P.\tag{2}$$

The rate of the code is given by $R = \frac{\log |C|}{k}$. In this work we focus on finite codes $C_k$ that are derived from lattices.

A full lattice $L \subset \mathbb{C}^k$ has the form $L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \cdots \oplus \mathbb{Z}b_{2k}$, where the vectors $b_1, \ldots, b_{2k}$ are linearly independent over $\mathbb{R}$, i.e., form a lattice basis.

Given an average power constraint $P$, the following Lemma suggests that by shifting a lattice and considering its intersection with the $2k$-dimensional ball $B(\sqrt{kP})$ of radius $\sqrt{kP}$, we can have codes having roughly $\text{Vol}(B(\sqrt{kP}))$ elements.

**Lemma 1** (see [8]). Suppose that $L$ is a full lattice in $\mathbb{C}^k$ and $S$ is a Jordan-measurable bounded subset of $\mathbb{C}^k$. Then there exists $x \in \mathbb{C}^k$ such that

$$|(L + x) \cap S| \geq \frac{\text{Vol}(S)}{\text{Vol}(L)}.$$
Let $\alpha$ be an energy normalization constant and $L$ a 2$k$-dimensional lattice in $\mathbb{C}^k$ satisfying $\text{Vol}(L) = 1$. According to Lemma 1 we can choose an element $x_R \in \mathbb{C}^k$ such that for the code
\begin{equation}
C = B(\sqrt{kP}) \cap (x_R + \alpha L)
\end{equation}
we have the cardinality bound
\begin{equation}
|C| \geq \frac{\text{Vol}(B(\sqrt{kP}))}{\text{Vol}(\alpha L)} = \frac{C_k P^k}{\alpha^{2k}},
\end{equation}
where $C_k = \frac{(\pi k)^k}{k!}$. We can now see that given a lattice $L$ with $\text{Vol}(L) = 1$, the number of codewords we are guaranteed to get only depends on the size of $\alpha$.

From now on, given a lattice $L$ and power limit $P$, the finite codes we are considering will always satisfy (4). We note that while the finite codes are not subsets of the scaled lattice $\alpha L$, they inherit many properties from the underlying lattice.

4. Hermite invariant in the AWGN channel

In this section we will present the classical Hermite invariant approach to build capacity approaching codes for the AWGN channel [5, Chapter 3]. We remark that this channel can be seen as an example of our general set-up [1] by assuming that for every $k$ the random matrix $H_k$ is a $k \times k$ identity matrix with probability one. The channel equation can now be written as
\begin{equation}
y = x + w,
\end{equation}
where $x \in C_k \subset \mathbb{C}^k$ is the transmitted codeword and $w$ is the Gaussian noise vector.

After the transmission, the receiver tries to guess which was the transmitted codeword $x$ by performing maximum likelihood (ML) decoding, and outputs
\begin{equation}
\hat{x} = \arg\min_{\tilde{x} \in C_k} \|y - \tilde{x}\| = \arg\min_{\tilde{x} \in C_k} \|x - \tilde{x} + w\|.
\end{equation}

This suggests a simple code design criterion to minimize the error probability. Given a power limit $P$, the codewords of $C_k$ should be as far apart as possible. As the properties of the finite code $C_k$ are inherited from the underlying lattice, we should give a reasonable definition of what it means that lattice points are far apart.

**Definition 2.** The Hermite invariant of a $2k$-dimensional lattice $L_k \subset \mathbb{C}^k$ is defined as
\begin{equation}
h(L_k) = \inf \left\{ \|x\|^2 \mid x \in L_k, x \neq 0 \right\} / \text{Vol}(L_k)^{1/k},
\end{equation}
where $\text{Vol}(L_k)$ is the volume of the fundamental parallelootope of the lattice $L_k$.

**Theorem 1.** Let $L_k \subset \mathbb{C}^k$ be a family of $2k$-dimensional lattice codes satisfying $h(L_k) \geq 2kc$. Then any rate
\begin{equation}
R < \log_2 P - \log_2 \frac{2}{\pi ec}
\end{equation}
is achievable using the lattices $L_k$ with ML decoding.

**Proof.** Given a power limit $P$, recall that the finite codes we are considering are of the form $C = B(\sqrt{kP}) \cap (x_R + \alpha L_k)$. Without loss of generality, we can assume that $\text{Vol}(L_k) = 1$. 

Here \( \alpha \) is a power normalization constant that we will soon solve and which will define the achievable rate. The minimum distance in the received constellation is

\[
d = \min_{x, \bar{x} \in \mathbb{C}, x \neq \bar{x}} \| x - \bar{x} \|
\]

The error probability is upper bounded by

\[
P_e \leq \mathbb{P} \left\{ \| w \|^2 \geq \left( \frac{d}{2} \right)^2 \right\}.
\]

Note that we can lower bound the minimum distance as follows:

\[
d^2 \geq \alpha^2 \min_{x \in L_k \setminus \{0\}} \| x \|^2 \geq \alpha^2 h(L_k) \geq \alpha^2 2^c k.
\]

Therefore we have the upper bound

\[
(5) \quad P_e \leq \mathbb{P} \left\{ \| w \|^2 \geq \frac{\alpha^2 c k}{2} \right\}.
\]

Let \( \epsilon > 0 \). Since \( 2 \| w \|^2 \) is a \( \chi^2 \) random variable with \( 2k \) degrees of freedom, due to the law of large numbers,

\[
\lim_{k \to \infty} \mathbb{P} \left\{ \frac{\| w \|^2}{k} \geq 1 + \epsilon \right\} = \lim_{k \to \infty} \mathbb{P} \left\{ \frac{2 \| w \|^2}{2k} \geq 1 + \epsilon \right\} \to 0
\]

Assuming \( \alpha^2 = \frac{2(1+\epsilon)}{\epsilon} \), we then have that \( P_e \to 0 \) when \( k \to \infty \), and the cardinality bound \( (4) \) implies that

\[
|\mathcal{C}| \geq \frac{C_k P^k}{\alpha^{2k}} = \frac{C_k P^k e^k}{2^k (1 + \epsilon)^k}.
\]

For large \( k \), \( C_k \approx \left( \frac{\pi e}{\sqrt{2\pi k}} \right)^k \) using Stirling’s approximation.

It follows that \( \forall \epsilon > 0 \) we can achieve rate

\[
R = \log_2 P - \log_2 \frac{2(1 + \epsilon)}{\pi e c}.
\]

Since \( \epsilon \) is arbitrary, this concludes the proof. \( \square \)

5. Hermite invariant in general linear fading model

In the previous section we saw how the Hermite invariant can be used as a design criterion to build capacity approaching codes in the AWGN channel. Let us now define a generalization of this invariant for linear fading channels.

Suppose we have an infinite sequence of random matrices \( H_k, k = 1, 2, \ldots, \infty \), where \( H_k \) is a \( k \times k \) matrix. Given an input \( x = (x_1, \ldots, x_k) \), we will write the channel output as

\[
y = H_k[x] + w,
\]

where \( w \) is a length \( k \) random vector, with i.i.d complex Gaussian entries with variance 1 per complex dimension. We assume that the receiver knows the realization of \( H \).

Given a channel realization \( H \), the receiver outputs the ML estimate

\[
\hat{x} = \arg \min_{\hat{x} \in \mathbb{C}} \| H[x] + w - H[\hat{x}] \|.
\]
From the receiver’s perspective this is equivalent to decoding the code
\[ H[C] = \{ H[x] \mid x \in C \} \]
over an AWGN channel.

As we assumed that the finite codes are of the form (3), we have
\[ H[C] \subset \{ H[x] \mid x \in x_R + \alpha L \} = \{ z \mid z \in H[x_R] + \alpha H[L] \}, \]
where
\[ H[L] = \{ H[x] \mid x \in L \}. \]

We can now see that the properties of \( H[C] \) are inherited from the set \( H[L] \).

If we assume that the matrix \( H \) has full rank with probability 1, then the linear mapping \( x \mapsto H[x] \) is a bijection of \( C_k \) onto itself with probability 1.

Assuming that \( L_k \subset C_k \) has basis \( \{ b_1, \ldots, b_{2k} \} \) we have that
\[ H[L_k] = \{ H[x] \mid x \in L_k \} = \mathbb{Z}H[b_1] \oplus \cdots \oplus \mathbb{Z}H[b_{2k}], \]
is a full-rank lattice with basis \( \{ H[b_1], \ldots, H[b_{2k}] \} \). Since it is full-rank, we know that \( h(H[L_k]) > 0 \), but is it possible to choose \( L_k \) in such a way that \( h(H[L_k]) \) would be large irrespective of the channel realization \( H \)? Let us now try to formalize this idea.

We can write the random matrix \( H_k \) in the form
\[ H_k = \det(H_k)^{1/k} H'_k \]
where \( |\det(H'_k)| = 1 \). Clearly, if the term \( |\det(H)|^{1/k} \) happens to be small, it will crush the Euclidean distances of points in \( H[L_k] \). However, we will show that if the random matrices \( H_k \) are “well behaving,” then it is possible to design lattices that are robust against fading.

**Definition 3.** Let \( A \) be a set of invertible matrices such \( \forall A \in A, \ |\det(A)| = 1 \). The *reduced Hermite invariant* [16] of a 2\( k \)-dimensional lattice \( L \subset \mathbb{C}^k \) with respect to \( A \) is defined as
\[ \text{rh}_A(L) = \inf_{A \in A} \{ h(A[L]) \}. \]

It is easy to see that
\[ \inf_{A \in A} \left\{ \inf_{x \in L, x \neq 0} ||A[x]||^2 \right\} = \inf_{x \in L, x \neq 0} \left\{ \inf_{A \in A} ||A[x]||^2 \right\}. \]

This observation suggests the following definition.

**Definition 4.** We call
\[ ||x||_A = \inf_{A \in A} \{ ||A[x]|| \mid A \in A \}, \]
the *reduced norm* of the vector \( x \) with respect to the set \( A \).

With this observation we realize that
\[ \text{rh}_A(L) = \frac{\inf_{A \in A} \{ ||x||_A^2 \mid x \in L, x \neq 0 \}}{\text{Vol}(L)^{1/k}}. \]

If the set \( A \) includes the identity matrix, we obviously have
\[ \text{rh}_A(L) \leq h(L). \]
Suppose that \( \{H_k\}_{k \in \mathbb{N}^+} \) is a fading process such that \( H_k \in M_{k \times k}(\mathbb{C}) \) is full-rank with probability 1, and suppose that the weak law of large numbers holds for the random variables \( \{\log \det(H_k H_k^\dagger)\} \), i.e. \( \exists \mu > 0 \) such that \( \forall \epsilon > 0 \),

\[
\lim_{k \to \infty} \mathbb{P} \left\{ \left| \frac{1}{k} \log \det(H_k H_k^\dagger) - \mu \right| > \epsilon \right\} = 0.
\]

We denote the set of all invertible realizations of \( H_k \) with \( \mathcal{A}_k^* \). Then define

\[
\mathcal{A}_k = \{ |\det(A)|^{-1/k} A \mid A \in \mathcal{A}_k^* \}.
\]

**Theorem 2.** Let \( L_k \subset \mathbb{C}^k \) be a family of \( 2^k \)-dimensional lattice codes satisfying \( r_{h_2}(L_k) \geq 2^{kc} \), and suppose that the channel satisfies (9). Then any rate

\[
R < \log_2 P + \mu - \log_2 \frac{2}{\pi c}
\]

is achievable using the codes \( L_k \) with ML decoding.

**Proof.** Given a power constraint \( P \), recall that we are considering finite codes of the form (3), where \( \alpha \) is a power normalization constant that we will soon solve.

The minimum distance in the received constellation is

\[
d_H = \min_{x, \bar{x} \in \mathbb{C}^n \neq \bar{x}} \| H[x - \bar{x}] \| \geq \min_{x \in L_k, x \neq 0} \| H[\alpha x] \|,
\]

and by the hypothesis on the reduced Hermite invariant,

\[
d_H^2 \geq \alpha^2 \min_{x \in L_k \setminus \{0\}} \| H[x] \|^2 \geq \alpha^2 \text{det}(HH^\dagger)^{1/k} r_{h_2}(L_k) \geq \alpha^2 \text{det}(HH^\dagger)^{1/k} 2^{kc}.
\]

The ML error probability is bounded by

\[
P_e \leq \mathbb{P} \left\{ \| w \|^2 \geq \left( \frac{d_H}{2} \right)^2 \right\}.
\]

Fixing \( \epsilon > 0 \), the law of total probability implies that

\[
P_e \leq \mathbb{P} \left\{ \frac{d_H^2}{4k} \geq 1 + \epsilon \right\} \mathbb{P} \left\{ \| w \|^2 \geq \frac{d_H^2}{4} \mid \frac{d_H^2}{4k} \geq 1 + \epsilon \right\} + \mathbb{P} \left\{ \frac{d_H^2}{4k} < 1 + \epsilon \right\}
\]

\[
\leq \mathbb{P} \left\{ \frac{\| w \|^2}{k} \geq 1 + \epsilon \right\} + \mathbb{P} \left\{ \frac{d_H^2}{4k} < 1 + \epsilon \right\}
\]

\[
\leq \mathbb{P} \left\{ \frac{\| w \|^2}{k} \geq 1 + \epsilon \right\} + \mathbb{P} \left\{ \frac{\alpha^2 c \text{det}(HH^\dagger)^{1/k}}{2} < 1 + \epsilon \right\}
\]

Recall that the first term tends to zero when \( k \to \infty \) due to (6). The second term will tend to zero as well if we choose

\[
\log_2 \left( \frac{2(1 + \epsilon)}{\alpha^2 c} \right) = \mu - \delta
\]

for some \( \delta > 0 \). Equation (4) gives us that

\[
R = \frac{1}{k} \log_2 |C| \leq \log_2 P - \log_2 \frac{\alpha^2}{C_k}
\]
For large \( k \), \( C_k \approx \frac{(\pi e)^{k}}{\sqrt{2\pi k}} \). It follows that we can achieve rate

\[
R = \log_2 P + \mu - \delta - \log_2 \frac{2(1 + \epsilon)}{\pi e c}
\]

Since \( \epsilon \) and \( \delta \) are arbitrary, any rate

\[
R < \log_2 P + \mu - \log_2 \frac{2}{\pi e c}
\]

is achievable. \( \Box \)

6. Code design for diagonal fading channels

Let us now consider a fading channel where for every \( k \) we have \( H_k = \text{diag}[h_1, h_2, \ldots, h_k] \). Assume that each \( h_i \) is non-zero with probability 1 and that \( \{h_i\} \) forms an ergodic stationary random process. In this model, sending a single symbol \( x_i \) during the \( i \)th time unit leads to the channel equation

\[
y_i = h_i \cdot x_i + w_i,
\]

where \( w_i \) is a zero-mean Gaussian complex random variable with variance 1.

The corresponding set of matrices \( A_k \) in (10) is a subset of the set of diagonal matrices in \( M_k(\mathbb{C}) \) having determinant with absolute value 1.

The assumption that the process \( \{h_i\} \) is ergodic and stationary implies that each of the random variables \( h_i \) have equal statistics. Therefore we can simply use \( h \) to refer to the statistics of all \( h_i \). Assuming now also that \( \sum_{i=1}^{k} \frac{1}{k} \log \|h_i\|^2 \) converges in probability to some constant, we have the following.

**Corollary 1.** Suppose that we have a family of lattices \( L_k \subset \mathbb{C}^k \), where \( rh_{A_k}(L_k) \geq 2kc \). Then any rate

\[
R < \mathbb{E}_h \left[ \log_2 P \|h\|^2 \right] - \log_2 \frac{2}{\pi e c}
\]

is achievable with the family \( L_k \) over the fading channel (11).

**Proof.** This statement follows immediately from Theorem 2 where \( \mu = \mathbb{E}_h \left[ \log_2 \|h\|^2 \right] \). \( \Box \)

Given two sets \( \mathcal{A}_k \subseteq \mathcal{A}_k \), we have for any lattice \( L \) that

\[
rh_{\mathcal{A}_k}(L) \geq rh_{\mathcal{A}_k}(L).
\]

From now on, we will fix \( \mathcal{A}_k \) to be the set of all diagonal matrices in \( M_k(\mathbb{C}) \) having determinant with absolute value 1. Note that with this choice, if \( rh_{\mathcal{A}_k}(L_k) \geq 2kc \) then Corollary 1 holds for any channel of the form (11).

Let \((x_1, x_2, \ldots, x_k) \in \mathbb{C}^k \). According to [16, Proposition 8], we have

\[
\|\langle x_1, \ldots, x_k \rangle\|_{\mathcal{A}_k}^2 = k|x_1 \cdots x_k|^{2/k}.
\]

We can now see that a lattice with large reduced Hermite invariant must have the property that the product of the coordinates of any non-zero element of the lattice is large.

**Definition 5.** Given \( x = (x_1, \ldots, x_k) \in \mathbb{C}^k \), we define its *product norm* as \( n(x) = \prod_{i=1}^{k} |x_i| \).

\[2\]More precisely, this result is slightly stronger than the statement of Proposition 8, but it is clear from its proof.
Definition 6. Then the normalized product distance of $L_k$ is

$$\text{Nd}_{p,\text{min}}(L_k) = \inf_{x \in L_k \setminus \{0\}} \frac{n(x)}{\text{Vol}(L_k)^{\frac{1}{2}}}.$$  

Combining (12), (8) and (13) we have that

$$\text{rh}_{A_k}(L_k) = k(\text{Nd}_{p,\text{min}}(L_k))^{2/k}.$$  

This result gives us a more concrete characterization of the reduced Hermite invariant and suggests possible candidates for good lattices.

6.1. Codes from algebraic number fields. The product distance criterion in the previous section had already been derived in [3] by analyzing the pairwise error probability in the special case where the process $\{h_i\}$ is i.i.d Gaussian. The authors also pointed out that lattices that are derived from number fields have large product distance. We will now shortly present this classical construction and then study how close to the capacity we can get using number fields. For the relevant background on algebraic number theory we refer the reader to [18].

Let $K/\mathbb{Q}$ be a totally complex extension of degree $2k$ and $\{\sigma_1, \ldots, \sigma_k\}$ be a set of $\mathbb{Q}$-embeddings, such that we have chosen one from each complex conjugate pair. Then we can define a relative canonical embedding of $K$ into $\mathbb{C}^k$ by

$$\psi(x) = (\sigma_1(x), \ldots, \sigma_k(x)).$$

The following lemma is a basic result from algebraic number theory.

Lemma 2. The ring of algebraic integers $\mathcal{O}_K$ has a $\mathbb{Z}$-basis $W = \{w_1, \ldots, w_{2k}\}$ and $\{\psi(w_1), \ldots, \psi(w_{2k})\}$ is a $\mathbb{Z}$-basis for the full lattice $\psi(\mathcal{O}_K)$ in $\mathbb{C}^k$.

For our purposes the key property of the lattices $\psi(\mathcal{O}_K)$ is that for any non-zero element $\psi(x) = (\sigma_1(x), \ldots, \sigma_k(x)) \in \psi(\mathcal{O}_K)$, we have that

$$\left| \prod_{i=1}^{k} \sigma_i(x) \right|^2 = nr_{K/\mathbb{Q}}(x) \in \mathbb{Z},$$

where $nr_{K/\mathbb{Q}}(x)$ is the algebraic norm of the element $x$. In particular it follows that

$$\left| \prod_{i=1}^{k} \sigma_i(x) \right| \geq 1.$$  

We now know that $\psi(\mathcal{O}_K)$ is a $2k$-dimensional lattice in $\mathbb{C}^k$ with the property that $\text{Nd}_{p,\text{min}}(\psi(\mathcal{O}_K)) \neq 0$ and therefore $\text{rh}_{A_k}(\psi(\mathcal{O}_K)) \neq 0$. This is true for any totally complex number field. Let us now show how the value of $\text{rh}_{A_k}(\psi(\mathcal{O}_K))$ is related to an algebraic invariant of the field $K$.

We will denote the discriminant of a number field $K$ with $d_K$. For every number field it is a non-zero integer.

The following Lemma states some well-known results from algebraic number theory and a translation of these results into our coding-theoretic language.

Lemma 3. Let $K/\mathbb{Q}$ be a totally complex extension of degree $2k$ and let $\psi$ be the relative canonical embedding. Then

$$\text{Vol}(\psi(\mathcal{O}_K)) = 2^{-k} \sqrt{|d_L|}$$

$$\text{Nd}_{p,\text{min}}(\psi(\mathcal{O}_K)) = \frac{2^k}{|d_K|^{\frac{1}{4}}} \quad \text{and} \quad \text{rh}_{A_k}(\psi(\mathcal{O}_K)) = \frac{2k}{|d_K|^{1/2k}}.$$
We have now translated the question of finding algebraic lattices with the largest reduced Hermite invariants into the task of finding the totally complex number fields with the smallest discriminant. Luckily this is a well-known mathematical problem with a tradition of almost a hundred years.

In [17], J. Martinet proved the existence of an infinite tower of totally complex number fields \( \{K_k\} \) of degree \( 2^k \), where \( 2^k = 5 \cdot 2^t \), such that

\[
|d_{K_k}|^{\frac{1}{t}} = G^2,
\]

for \( G \approx 92.368 \). For such fields \( K_k \) we have that

\[
Nd_{p,\min}(\psi(O_{K_k})) = \left(\frac{2}{G}\right)^t \quad \text{and} \quad rh_{A_k}(\psi(O_{K_k})) = \frac{2k}{G}.
\]

Specializing Corollary 1 to the family of lattices \( L_k = \psi(O_{K_k}) \) derived from Martinet’s tower, which satisfy the hypothesis with \( c = 1/G \), we then have the following result:

**Proposition 1.** Finite codes drawn from the lattices \( L_k \) achieve any rate satisfying

\[
R < \mathbb{E}_h \left[ \log_2 P |h|^2 \right] - \log_2 \frac{2G}{\pi e}.
\]

**Remark 1.** We note that given a stationary and ergodic fading process \( \{h_i\} \) the capacity of the corresponding channel is

\[
C = \mathbb{E}_h \left[ \log_2(1 + P|h|^2) \right].
\]

It is easy to prove that the rate achieved in Proposition 1 is a constant gap from the capacity. This gap is also universal in the following sense. Let us consider all ergodic stationary channels with the same first order statistics for \( h \). Then the same sequence of finite codes achieve the same gap to capacity in all the channels simultaneously.

**Remark 2.** We note that the number field towers we used are not the best known possible. It was shown in [9] that one can construct a family of totally complex fields such that \( G < 82.2 \), but this choice would add some notational complications.

**Remark 3.** The families of number fields on which our constructions are based were first brought to coding theory in [12], where the authors pointed out that the corresponding lattices have linearly growing Hermite constant. This directly implies that they are only a constant gap from the AWGN capacity. C. Xing in [27] remarked that these families of number fields provide the best known normalized product distance. Overall number field lattices in fading channels have been well-studied in the literature. However, to the best of our knowledge we were the first to prove that they actually do achieve a constant gap to capacity over fading channels.

6.2. Codes from ideals. As seen in the previous section, lattice codes arising from the rings of algebraic integers of number fields with constant root discriminants will achieve a constant gap to capacity over fading channels. However, known lower bounds for discriminants [19] imply that no matter which number fields we use, the gap cannot be reduced beyond a certain threshold (at least when using our current approach to bound the error probability). It is then natural to ask whether other lattice constructions could lead us closer to capacity. The most obvious generalization is to consider additive subgroups of \( O_K \) and in particular ideals of \( O_K \), which will have non-zero reduced Hermite invariant. Most works concerning lattice codes from number fields focused on either the ring \( O_K \) or a
principal ideal $a\mathcal{O}_K$; a more general setting was considered in [2] and [20], which addressed the question of increasing the normalized product distance using non-principal ideals $I$.

The problem with this approach is that while finding the reduced Hermite invariant of lattices $\psi(\mathcal{O}_K)$ or $\psi(a\mathcal{O}_K)$ is an easy task, the same is not true for $\psi(I)$ when $I$ is non-principal. We will now show how this problem can be reduced to another well-known problem in algebraic number theory and how it can be used to study the performance limits of the lattices $\psi(I)$. Here we will follow the extended arXiv version of [24].

We note that while the concept of reduced Hermite invariant is more general and its information-theoretical meaning is clearer, number theoretic proofs are easier when using the equivalent product distance notation. Therefore we will mostly focus on the product distance in this section.

Let $K$ be a totally complex field of degree $2k$. We will use the notation $N(I) = [\mathcal{O}_K : I]$ for the norm of an ideal $I$. From classical algebraic number theory we have that $N(a\mathcal{O}_K) = |nr_{K/Q}(a)|$ and $N(AB) = N(A)N(B)$.

**Lemma 4.** Suppose that $K$ is a totally complex field of degree $2k$ and that $I$ is an integral ideal in $K$. Then $\psi(I)$ is a $2k$-dimensional lattice in $\mathbb{C}^k$ and

$$\text{Vol}(\psi(I)) = [\mathcal{O}_K : I]2^{-k}\sqrt{|d_K|}.$$ 

This well-known result allows us to compute the volume of an ideal, but computing its normalized product distance is a more complicated issue. In [2, Theorem 3.1] the authors stated the analogue of the following result for the totally real case. It is simply a restatement of the definitions.

**Proposition 2.** Let us suppose that $K$ is a totally complex field of degree $2k$ and that $I$ is an integral ideal of $K$. We then have that

$$Nd_{p,\text{min}}(\psi(I)) = \frac{2^k}{|d_K|^\frac{1}{2}}\min(I),$$

where $\min(I) := \min_{x \in I \setminus \{0\}} \sqrt{\frac{|nr_{K/Q}(x)|}{N(I)}}$.

**Proof.** This result follows from Lemma 4, the definition of the normalized product distance and from noticing that $\sqrt{|nr_{K/Q}(x)|} = |n(\psi(x))|$. \qed

Due to the basic ideal theory of algebraic numbers $\min(I)$ is always larger or equal to $1$. If $I$ is not a principal ideal then we have that $\min(I) \geq \sqrt{2}$. Comparing this to Lemma 5 we find that, given a non-principal ideal domain $O_K$, we should use an ideal $I$, which is not principal, to maximize the product distance. Now there are two obvious questions. Given a non-principal ideal domain $\mathcal{O}_K$, which ideal $I$ should we use and how much can we gain? Before answering these questions we need the following.

**Lemma 5.** [2] For any non-zero element $x \in K$,

$$Nd_{p,\text{min}}(\psi(xI)) = Nd_{p,\text{min}}(\psi(I)).$$

This result proves that every ideal in a given ideal class has the same normalized product distance. It follows that given a ring of integers $\mathcal{O}_K$, it is enough to consider one ideal from every ideal class. Given an ideal $I$ we will denote with $[I]$ the ideal class to which $I$ belongs.
Let us denote with $N_{\text{min}}(K)$ the norm of an ideal $A$ in $K$ with the property that every ideal class of $K$ contains an integral ideal with norm $N(A)$ or smaller. The question of finding the size of $N_{\text{min}}(K)$ is a classical problem in algebraic number theory. We refer the reader to [28] for further reading. The following result is from the extended arXiv version of [24].

**Proposition 3.** Let us suppose that $K$ is a totally complex number field of degree $2k$ and that $I$ is an ideal that maximizes the normalized product distance over all ideals in $K$. We then have that

$$Nd_{p,\text{min}}(\psi(I)) = \frac{2^{k/2}\sqrt{N_{\text{min}}(K)}}{|d_K|^{1/2}}$$

and

$$\text{rh}_{A_k}(\psi(I)) = \frac{2k(N_{\text{min}}(K))^{1/k}}{|d_K|^{1/2}}.$$  

**Proof.** Let $L$ be any ideal in $K$, and suppose that $A$ is an integral ideal in the class $[L]^{-1}$ with the smallest norm. We then have that there exists an element $y \in \mathcal{O}_K$ such that $y\mathcal{O}_K = AL$. As $n(\psi(y)) = \sqrt{N(L)N(A)}$ and $N(A) \leq N_{\text{min}}(K)$ we have that $d_{p,\text{min}}(L) \leq \sqrt{N(L)N_{\text{min}}(K)}$ and $Nd_{p,\text{min}}(\psi(L)) \leq \frac{2^{k/2}\sqrt{N_{\text{min}}(K)}}{|d_K|^{1/2}}$.

Assume that $S$ is an ideal such that $N(S) = N_{\text{min}}(K)$ and choose $I$ as an element from the class $[S]^{-1}$. For any non-zero element $x \in I$, we then have that $x\mathcal{O}_K = IC$, for some ideal $C$ that belongs to the class $[S]$. Therefore we have that $n(\psi(x)) \geq \sqrt{N(I)N(C)}$. □

This result translates the question of finding the product distance of an ideal into a well-known problem in algebraic number theory. It also suggests which ideal class we should use in order to maximize the product distance.

Denote with $\mathcal{K}_{2k}$ the set of totally complex number fields of degree $2k$. Then the optimal normalized product distance over all complex fields of degree $2k$ and all ideals $I$ is

$$\max_{K \in \mathcal{K}_{2k}} \frac{2^{k/2}\sqrt{N_{\text{min}}(K)}}{|d_K|^{1/2}}.$$  

(17)

As far as we know it is an open question whether the maximum in (17) is always achieved when $K$ is principal ideal domain. Some preliminary data can be found in [2]. We point out that Proposition 3 makes this problem computationally much more accessible.

7. **Reduced Hermite invariants as homogeneous forms**

Let us now see how different linear channels define different sets $\mathcal{A}_k$ and how the corresponding reduced norms can be seen as different homogeneous forms. For simplicity we will study the case when we transmit four information symbols $(x_1, x_2, x_3, x_4)$.

In the AWGN channel the receiver sees

$$(x_1, x_2, x_3, x_4) + (w_1, w_2, w_3, w_4),$$

where $w_i$ are Gaussian random variables. Here the set $\mathcal{A}_4^{(1)}$ simply consists of a single element, the $4 \times 4$ identity matrix. Therefore we obviously have

$$|| (x_1, x_2, x_3, x_4) ||^{2}_{\mathcal{A}_4^{(1)}} = |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2.$$  

Let us then consider a channel where the fading stays stable for 2 time units and then changes. Then the received signal will be of the form

$$(h_1 x_1, h_1 x_2, h_2 x_3, h_2 x_4) + (w_1, w_2, w_3, w_4).$$
Following the proof of [16, Proposition 8] we get the following result

Assuming that $h_i$ time unit giving us the following received vector:

Earlier we considered the fast fading channel in which the channel can change during every time unit giving us the following received vector:

In this case we have that

and that

In all the previous examples the channel could be represented as a diagonal action. On the other hand, for a $2 \times 2$ MIMO system, the channel matrix will have block diagonal structure. In this case the received vector can be written as

Here the set $\mathcal{A}_4^{(3)}$ consists of matrices

According to [16, Proposition 8] we have that

We immediately note that all the reduced norms share common characteristics.

**Definition.** A continuous function $F: \mathbb{C}^k \to \mathbb{R}$ is called a homogeneous form of degree $\sigma > 0$ if it satisfies the relation

Given a full lattice $L \in \mathbb{C}^k$ and assuming that Vol($L$) = 1, we can define the homogeneous minimum of the form $F$ as

Setting $|| \cdot ||_{\mathcal{A}_4^{(2)}}^2 = F_{\mathcal{A}_4^{(2)}}$, we can see that each of the squared reduced norms defined previously are homogeneous forms of degree 2.

As we saw in Theorem 2 given a sequence of random matrices $H_k$ of size $k \times k$ and the corresponding sets $\mathcal{A}_k$ in [10], we can use $r_h_{\mathcal{A}_k}$ as a design criterion for building capacity-approaching lattice codes. In many cases of interest, $|| \cdot ||_{\mathcal{A}_k}^2 = F_{\mathcal{A}_k}$ will be a homogeneous
form and \( \text{rh}_k(L) = \lambda(F, L) \). For instance this is the case if we extend the previous examples to general size \( k \) and define

\[
\mathcal{A}_k^{(1)} = I_k,
\]
\[
\mathcal{A}_k^{(2)} = \{ \text{diag}[a_1, a_2, a_3, \ldots, a_k, a_k] \mid |a_1a_2\cdots a_k| = 1, a_i \in \mathbb{C} \},
\]
\[
\mathcal{A}_k^{(3)} = \{ \text{diag}[a_1, a_2, \ldots, a_k] \mid |a_1a_2\cdots a_k| = 1, a_i \in \mathbb{C} \}.
\]

In the case where \( \mathcal{A}_k = \{ I_k \} \), we have recovered the classical connection between sphere packing and AWGN capacity, but we also proved that there exist similar connections between different channel models and the corresponding homogeneous forms.

A natural question is now how close to capacity we can get with these methods by taking the best possible lattice sequences in terms of their homogeneous minimum. We will denote with \( \mathcal{L}_k \) the set of all the lattices \( L \) in \( \mathbb{C}^k \) having \( \text{Vol}(L) = 1 \). This leads us to the concept of absolute homogeneous minimum

\[
\lambda(F) = \sup_{L \in \mathcal{L}_k} \lambda(F, L).
\]

Finding the value of absolute homogeneous minima is one of the central problems in geometry of numbers. As we saw earlier it is a central problem also in the theory of linear fading channels.

In the case \( \mathcal{A}_k = \{ I_k \} \), \( \lambda(F_{\mathcal{A}_k}) \) is the Hermite constant \( \gamma_k \). The value of the Hermite constant for different values of \( k \) has been studied in mathematics for hundreds of years and there exists an extensive literature on the topic. In particular good upper and lower bounds are available and it has been proven that we can get quite close to Gaussian capacity with this approach [5, Chapter 3].

In the case of \( F_{\mathcal{A}_k}^{(3)} \) the problem of finding homogeneous minima has been considered in the context of algebraic number fields and some upper bounds have been provided. Similarly for \( F_{\mathcal{A}_k}^{(2)} \) there exists considerable literature. These and related results can be found in [8]. However for the case of homogeneous forms arising from block diagonal structures there seems to be very little previous research.

**Remark.** While the definition of the reduced Hermite invariant is very natural, we have found very few previous works considering similar concepts. The first reference we have been able to locate is [22]. There the author considered matrices of type \( (19) \) and proved \( (12) \) in this special case. Our results can therefore be seen as a natural generalization of this work. The other relevant reference is [1] where the authors defined the Hermite invariant for generalized ideals in division algebras in the spirit of Arakelov theory. Again their definition is analogous to ours in certain special cases.

**Remark.** We note that the reduced norms in our examples are not only homogeneous forms, but multivariate polynomials and the sets \( \mathcal{A}_k^{(i)} \) are groups. As we obviously have that

\[
||A(x)||_{\mathcal{A}_k^{(i)}}^2 = ||x||_{\mathcal{A}_k^{(i)}}^2,
\]

for any \( A \in \mathcal{A}_k^{(i)} \), we can see that \( ||||_{\mathcal{A}_k^{(i)}} \) is actually a classical polynomial invariant of the group \( \mathcal{A}_k^{(i)} \). At the moment we don’t know what conditions a matrix group \( \mathcal{A}_k \) should satisfy so that the corresponding reduced norm would be a homogenous form. Just as well we don’t know when some power of the reduced norm is a polynomial.
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