On the Hybrid Mean Value of Generalized Dedekind Sums, Generalized Hardy Sums and Kloosterman Sums *

Qing Tian †
School of Science, Xi’an University of Architecture and Technology
Xi’an, 710055, Shaanxi, P. R. China

Abstract
The main purpose of this paper is to study the hybrid mean value problem involving generalized Dedekind sums, generalized Hardy sums and Kloosterman sums, and give some exact computational formulae for them by using the properties of Gauss sums and the mean value theorem of the Dirichlet L-function.

Keywords: Hybrid mean value, Kloosterman sums, Generalized Dedekind sums, Generalized Hardy sums.

MSC: 11F20, 11L05

1. Introduction

Suppose that $k$ is a positive integer, then for an arbitrary integer $h, m, n$, The generalized Dedekind sums are defined by

$$S(h, m, n, k) = \sum_{j=1}^{k} \overline{B}_m \left( \frac{j}{k} \right) \overline{B}_n \left( \frac{hj}{k} \right),$$

where

$$\overline{B}_m(x) = \begin{cases} B_m(x - [x]), & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

$B_m(x)$ is the Bernoulli polynomial, $\overline{B}_m(x)$ defined on the interval $0 < x \leq 1$ is the $n$-th Bernoulli periodic function. For $m = n = 1$, $S(h, 1, 1, q) = S(h, q)$ are the classical Dedekind sums, which were studied by many experts because of the prominent role they play in the transformation theory of the Dedekind eta-function. Some arithmetical properties of $S(h, q)$ can be found in Apostol [1] and Carlitz [2]. The most famous property of the Dedekind sums may be the reciprocity formula ([3][4])

$$S(h, k) + S(k, h) = \frac{h^2 + k^2 + 1}{12hk} - \frac{1}{4}.$$

In [5], Berndt gave certain sums called Hardy sums which are related to the Dedekind sums, and also obtained some arithmetic properties (see [6]). Sitaramachandrarao [7] and Pettet [8] used elementary methods to express the Hardy sums in terms of the Dedekind sums. H.N.Liu [9] generalize the Hardy

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†E-mail: qingtian@xauat.edu.cn (Qing Tian)
sums as follows

\[ s_1(h, m, k) = \sum_{j=1}^{k} (-1)^j \bar{B}_m \left( \frac{j}{k} \right), \quad s_2(h, m, n, k) = \sum_{j=1}^{k} (-1)^j \bar{B}_m \left( \frac{j}{k} \right), \]

\[ s_3(h, n, k) = \sum_{j=1}^{k} (-1)^j \bar{B}_n \left( \frac{hj}{k} \right), \quad s_5(h, m, k) = \sum_{j=1}^{k} (-1)^j \bar{B}_m \left( \frac{j}{k} \right). \]

For \( m = n = 1 \), the sums \( s_1(h, k) = s_1(h, 1, k), s_2(h, k) = s_2(h, 1, 1, k) \), \( s_3(h, k) = s_3(h, 1, k) \) and \( s_5(h, k) = s_5(h, 1, k) \) are classical Hardy sums defined in [5]. H. N. Liu’s research paper [9] also express the generalized Hardy sums in term of generalized Dedekind sums, that is

**Proposition 1.1** Let \( h, q \) be positive integer with \( (h, q) = 1 \), then

\[
\begin{align*}
 s_1(h, m, q) &= 2 \cdot S(h, m, 1, q) - 4 \cdot S \left( \frac{h}{m}, m, 1, q \right), & \text{if } h \text{ is even number} \\
 s_2(h, m, n, q) &= 2^n \cdot S(2h, m, n, q) - S(h, m, n, q), & \text{if } q \text{ is even number} \\
 s_3(h, n, q) &= 2 \cdot S(h, 1, n, q) - 4 \cdot S(2h, 1, n, q), & \text{if } q, n \text{ are odd number} \\
 s_5(h, m, q) &= 2^{m+1} \cdot S(2h, m, 1, q) + 2^{m+1} \cdot S(h, m, 1, 2q) - (2 + 2^{m+2}) \cdot S(h, m, 1, q), & \text{if } h + q \text{ is even number}
\end{align*}
\]

where \( 2 \cdot 2 \equiv 1 \mod q \). Moreover, each one of

\[
\begin{align*}
 s_1(h, m, q) &\quad (h + m \text{ even}), \quad s_2(h, m, n, q) &\quad (h + m + q \text{ odd}) \\
 s_3(h, n, q) &\quad (h + q \text{ odd}), \quad s_5(h, m, k) &\quad (h + m + q \text{ even})
\end{align*}
\]

is zero.

Recently, some authors studied the hybrid mean value of Dedekind sums or Hardy sums with Kloosterman sums defined by

\[
K(n, q) = \sum_{c=1}^{\varphi(q)} e \left( \frac{mc + \bar{c}}{q} \right),
\]

where \( \sum_{c=1}^{\varphi(q)} \) denotes the summation over all \( c \) such that \( (c, q) = 1 \), \( e(y) = \exp(2\pi iy) \) and \( \bar{c} \cdot c \equiv 1 \mod q \).

And they found there are some close relationships between the functions. Y. N. Liu et al. [10] gave several explicit formulae for

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} K^2(a, p) K^2(b, p) S^{k}(a\bar{b}, p)
\]

under the condition \( q = p \) is a prime. H. Zhang et al. [11] and W. Peng et al. [12] also obtained identities for

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} K(a, p) K(b, p) s_1(2a\bar{b}, p)
\]

and

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} K(a, p) K(b, p) s_5(a\bar{b}, p)
\]

respectively.

Naturally, one might consider whether the hybrid mean value be extended to generalized Dedekind sums \( S(h, m, n, q) \) or certain generalized Hardy sums with Kloosterman sums \( K(n, q) \) under the condition of composite number \( q \)? If yes, then what can be expected? These problems may be interesting. In this paper, we shall study the problems and give some exact computational formulae by using the prosperities
of Gauss sums and the mean value theorem of the Dirichlet L-function. That is, we shall prove the following:

**Theorem 1.** Let \( q \) be a square-full number, \( m \equiv n \equiv 1 \mod 2 \). Then we have

\[
\sum_{a=1}^{q} \sum_{b=1}^{q} K(a,q)K(b,q)S(ab,m,n,q) = q^{2-2m-2n} \sum_{l=0}^{m+n} q^{l} \cdot r_{m,n,l} \cdot \prod_{p | q} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^l} \right) \left(1 - \frac{1}{p^{l-m-n+l}} \right)
\]

where

\[
r_{m,n,l} = B_{m+n-l} \sum_{a=0}^{m} \sum_{b=0}^{n} B_{m-a}B_{n-b} \frac{(m)}{(a)} \frac{(n)}{(b)} \frac{(a+b+1)}{(m+n-l)},
\]

\( B_m \) is Bernoulli number, \( \left( \frac{m}{a} \right) = \frac{(-1)^{a(m-a)}}{a!(m-a)!} \), \( \phi(q) = \prod_{p | q} (1 - \frac{1}{p}) \), \( \prod_{p | q} \) denotes the products of all prime divisors of \( q \) and \( \phi(q) = q\phi_1(q) \).

**Theorem 2.** Let \( q \) be a square-full number, \( m \equiv 1 \mod 2 \). Then we have

\[
\sum_{a=1}^{q} \sum_{b=1}^{q} K(a,q)K(b,q)s_1(ab,m,q) = q^{m-2} \sum_{l=0}^{m+1} q^{-m} \cdot r_{m,1,l} \cdot \frac{-2l - 2}{2m-1+1} \cdot \prod_{p | q} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^l} \right) \left(1 - \frac{1}{p^{l-m}} \right)
\]

**Theorem 3.** Let \( q \) be a square-full even number, \( m \equiv n \equiv 1 \mod 2 \). Then we have

\[
\sum_{a=1}^{q} \sum_{b=1}^{q} K(a,q)K(b,q)s_2(ab,m,n,q) = q^{2-2m} \sum_{l=0}^{m+1} r_{m,1,l} \cdot \left( 2m \frac{2m-2l+1}{2m-1+1} - 1 \right) \cdot q^{-m} \cdot \prod_{p | q} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^l} \right) \left(1 - \frac{1}{p^{l-m}} \right)
\]

**Theorem 4.** Let \( q \) be a square-full odd number, \( n \equiv 1 \mod 2 \). Then we have

\[
\sum_{a=1}^{q} \sum_{b=1}^{q} K(a,q)K(b,q)s_3(ab,n,q) = q^{2-2n} \sum_{l=0}^{n+1} r_{1,1,l} \cdot q^l \left( 2 - 4 \cdot \frac{2l - 2n+1 - 2}{2n+2} \right) \cdot \prod_{p | q} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^l} \right) \left(1 - \frac{1}{p^{l-n}} \right)
\]

**Theorem 5.** Let \( q \) be a square-full odd number, \( m \equiv 1 \mod 2 \). Then we have

\[
\sum_{a=1}^{q} \sum_{b=1}^{q} K(2a-1,q)K(2b-1,q)s_5((2a-1)(2b-1),m,q) = q^{2-2m} \sum_{l=0}^{m+1} r_{m,1,l} \cdot q^l \left( 2 - 4 \cdot \frac{2l - 2m+1 - 2}{2m+2} \right) \cdot \prod_{p | q} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^l} \right) \left(1 - \frac{1}{p^{l-m}} \right)
\]

The present work is a generalization of [11] and [12].

For general number \( q > 2 \), we can only get some asymptotic formulae, whether there exits the identities for the hybrid mean value for these sums are open problems.
2. Several Lemmas

Before starting our proof of the theorems, several lemmas will be useful.

**Lemma 2.1.** Let \( h, q \) be positive integers with \( q \geq 3 \) and \( (h, q) = 1 \), \( m \equiv n \equiv 1 \mod 2 \). Then we have

\[
S(h, m, n, q) = \frac{-4m!n!}{(2\pi i)^{m+n}q^{m+n-1}} \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\chi \mod d \chi(-1)=-1} \chi(h)L(m, \chi)L(n, \bar{\chi}),
\]

where \( \sum_{d|q} \) denotes the sums over all divisors of \( q \) and \( L(m, \chi) \) denotes the Dirichlet L-function corresponding to character \( \chi \mod d \).

**Proof.** See Theorem 2.3 of [9].

**Lemma 2.2.** Let \( q \geq 3 \) be an odd number. Then for odd numbers \( h, q \) with \( (h, q) = 1 \), we have

\[
s_5(h, m, q) = 2 \cdot S(h, m, 1, q) - 4 \cdot S(2h, m, 1, q).
\]

**Proof.** From the Proposition 1.1, we know that if \( h + q \) is even number, generalized Hardy sums \( s_5(h, m, q) \) can be expressed in terms of generalized Dedekind sums, that is

\[
s_5(h, m, q) = 2^{m+1} \cdot S(2h, m, 1, q) + 2^{m+1} \cdot S(h, m, 1, 2q) - (2 + 2^{m+2}) \cdot S(h, m, 1, q).
\]

Now we simplify the formula (1). Considering the second part \( S(h, m, 1, 2q) \) firstly, by using Lemma 1 we have

\[
S(h, m, 1, 2q) = -\frac{4m!}{(2\pi i)^{m+1}(2q)^m} \sum_{d|2q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod 2d \chi(-1)=-1} \chi(h)L(m, \chi)L(1, \bar{\chi})
\]

\[
= -\frac{4m!}{(2\pi i)^{m+1}(2q)^m} \left( \sum_{d|q} \frac{(2d)^{m+1}}{\phi(2d)} \sum_{\chi \mod 2d \chi(-1)=-1} \chi(h)L(m, \chi)L(1, \bar{\chi}) \right)
\]

\[
+ \sum_{d|q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod d \chi(-1)=-1} \chi(h)L(m, \chi)L(1, \bar{\chi})
\]

\[
= -\frac{4m!}{(2\pi i)^{m+1}(2q)^m} \left( 2^{m+1} \sum_{d|q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod d \chi(-1)=-1} \chi_2(h)L(m, \chi_2)L(1, \bar{\chi}_2) \right)
\]

\[
+ \sum_{d|q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod d \chi(-1)=-1} \chi(h)L(m, \chi)L(1, \bar{\chi})
\]

where \( \chi_2^0 \) denotes the principal character modulo 2.
From the Euler infinite product formula (see Theorem 11.6 of [13]), we have

\[ L(m, \chi^0_m) = \prod_{p_1} \left( 1 - \frac{\chi(p_1)\chi_0(p_1)}{p_1^m} \right)^{-1} = \prod_{p_1, p_2, \ldots} \left( 1 - \frac{\chi(2)}{2^m} \right)^{-1} \]

\[ = \left( 1 - \frac{\chi(2)}{2^m} \right) \prod_{p_1} \left( 1 - \frac{\chi(p_1)}{p_1^m} \right)^{-1} = \left( 1 - \frac{\chi(2)}{2^m} \right) L(m, \chi) \]

\[ L(1, \chi^0_m) = \prod_{p_2} \left( 1 - \frac{\chi(p_2)\chi_0(p_2)}{p_2^m} \right)^{-1} = \left( 1 - \frac{\chi(2)}{2^m} \right) L(1, \chi) \]

where \( \prod_p \) denotes the product over all primes \( p \).

That is we have the identity

\[ S(h, m, 1, 2q) = -\frac{4m!}{(2\pi)^{m+1} q^m} \cdot \frac{1}{2^m} \cdot 2^{m+1} \cdot \sum_{d | q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod d} \left( 1 - \frac{\chi(2)}{2^m} \right) \left( 1 - \frac{\chi(2)}{2^m} \right) \chi(h) L(m, \chi) L(1, \chi) \]

\[ + \sum_{d | q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod d} \chi(h) L(m, \chi) L(1, \chi) \]

\[ = -\frac{4m!}{(2\pi)^{m+1} q^m} \sum_{d | q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod d} \left( 2 + \frac{1}{2m-1} - \chi(2) - \frac{\chi(2)}{2m-1} \right) \chi(h) L(m, \chi) L(1, \chi) \]

\[ = (2 + \frac{1}{2m-1}) \cdot S(h, m, 1, q) - S(2h, m, 1, q) - \frac{1}{2m-1} \cdot S(2h, m, 1, q) \]  \quad (2)

Combining (1) with (2), it follows that

\[ s_5(h, m, q) = 2 \cdot S(h, m, 1, q) - 4 \cdot S(2h, m, 1, q). \]

This proves Lemma 2.2.

**Lemma 2.3.** Let \( q \geq 3 \) be an integer and \( \chi \) be a non-principal character mod \( q \). Then we have

\[ \sum_{a=1}^{q'} \chi(a) K(a, q) = \tau^2(\chi) \]

**Proof.** From the properties of reduced residue system, it is known that if \( a \) pass through a reduced residue system mod \( q \), then for any integer \( c \) with \( (c, q) = 1 \), \( ac \) also pass through a reduced residue system mod \( q \), by the definition of Gauss sums, we have

\[ \sum_{a=1}^{q} \chi(a) K(a, q) = \sum_{a=1}^{q'} \chi(a) \sum_{c=1}^{q} e \left( \frac{ac + \bar{e}}{q} \right) \]

\[ = \sum_{c=1}^{q'} e \left( \frac{\bar{e}}{q} \right) \sum_{c=1}^{q} \chi(a) e \left( \frac{ac}{q} \right) \]
= \sum_{c=1}^{q} \chi(c) e\left(\frac{c}{q}\right) \sum_{a=1}^{q} \chi(ac) e\left(\frac{ac}{q}\right)
= \tau^2(\chi).

This proves Lemma 2.3.

**Lemma 2.4.** Let \( q \) be a square-full number. Then for any non-primitive character \( \chi \mod q \), we have the identity
\[
\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right) = 0
\]

**Proof.** It is known that \( \tau^2(\chi) \) is a multiplicative function, so without loss of generality we assume that \( q = p^\alpha \), where \( p \) is a prime and \( \alpha \geq 2 \). If \( \chi \) is a non-primitive character modulo \( q = p^\alpha \), then \( \chi \) must be a character modulo \( p^{\alpha-1} \). Note that the trigonometric identity \( \sum_{a=0}^{p-1} e\left(\frac{a}{p}\right) = 0 \). From the properties of the reduced residue system modulo \( p^{\alpha-1} \), it is easy to get
\[
\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right) = \sum_{b=1}^{p^{\alpha-1}} \chi(b) e\left(\frac{b}{p^\alpha}\right) \sum_{a=1}^{p-1} e\left(\frac{a}{p}\right) = 0.
\]

This proves Lemma 2.4.

**Lemma 2.5.** Let \( q \geq 2 \) be an integer, \( m \equiv n \equiv 1 \). We have
\[
\sum_{\chi \mod q} L(m, \chi)L(n, \chi) = -\frac{(2\pi i)^{m+n} \phi(q)}{4m!n!} \cdot \left( \sum_{l=0}^{m+n} r_{m,n,l} \cdot \phi_l(q) \cdot q^{l-m-n} - \frac{B_mB_n\phi_{m+n-1}(q)}{q} \right).
\]

**Proof.** See Theorem 3 of [9].

**Lemma 2.6.** Let \( q \) be square-full number. Then we have
\[
\sum_{\chi \mod q} L(m, \chi)L(n, \chi) = \frac{(2\pi i)^{m+n} \phi(q)}{4m!n!} \cdot \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^{l-m-n+1} \cdot \prod_{p|q} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^l} \right) \left(1 - \frac{1}{q^{l-m-n+1}} \right),
\]

where \( \sum_{\chi \mod q}^* \) denotes the sums over all odd primitive characters mod \( q \).

**Proof.** Noting that \( q \) is square-full number and
\[
\sum_{\chi \mod q} L(m, \chi)L(n, \chi) = \sum_{d|q} \sum_{\chi \mod d} L(m, \chi^{0}_{q})L(n, \chi^{0}_{q}) = \sum_{\chi \mod q}^* L(m, \chi_{q}^{0})L(n, \chi_{q}^{0}),
\]

by using Möbius inverse formula, we have
\[
\sum_{\chi \mod q}^* L(m, \chi)L(n, \chi) = \sum_{\chi \mod q}^* L(m, \chi_{q}^{0})L(n, \chi_{q}^{0})
\]
According to Lemma 2.5, we get

\[
\sum_{\chi \mod q}^* L(m, \chi) L(n, \chi) = -\frac{(2\pi i)^{m+n}}{4m!n!} \cdot \left[ \sum_{l=0}^{m+n} r_{m,n,l} \cdot \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right) \phi_l \left(\frac{q}{d}\right) \left(\frac{q}{d}\right)^{l-m-n} \right. \\
+ \left. B_m B_n \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right) \phi_{m+n-1}\left(\frac{q}{d}\right) \frac{d}{q} \right].
\]

Using the same methods, we get

\[
\sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right) \phi_{m+n-1}\left(\frac{q}{d}\right) \frac{d}{q} = 0.
\]

Due to the discussion above, we obtain

\[
\sum_{\chi \mod q}^* L(m, \chi) L(n, \chi) = -\frac{(2\pi i)^{m+n}}{4m!n!} \cdot \left[ \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^{l-m-n+1} \cdot \prod_{p|q} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^l}\right) \left(1 - \frac{1}{p^{l-m-n+1}}\right) \right].
\]

This proves Lemma 2.6.

**Lemma 2.7.** Let \( q \) be square-full number, then we have the identity

\[
\sum_{\chi \mod q}^* L(m, \chi) L(n, \chi) = -\frac{(2\pi i)^{m+n}}{4m!n!} \cdot \left[ \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^{l-m-n} \cdot q^{l-m-n+1} \cdot \prod_{p|q} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^l}\right) \left(1 - \frac{1}{p^{l-m-n+1}}\right) \right].
\]

**Proof.** Note that

\[
\sum_{\chi \mod 2q} L(m, \chi) L(n, \chi) = \sum_{\chi \mod q} L(m, \chi) L(n, \chi)
\]

\[
= \sum_{d|q} \sum_{\chi \mod d} L(m, \chi) L(n, \chi).
\]
and Möbius inverse formula, we have
\[ \sum_{\chi \mod q \chi(-1) = -1}^* L(m, \chi\chi_2^0)L(n, \overline{\chi\chi_2^0}) = \sum_{\chi \mod q \chi(-1) = -1}^* L(m, \chi\chi_{2q}^0)L(n, \overline{\chi\chi_{2q}^0}) = \sum_{d|q} \mu(d) \sum_{\chi \mod \frac{q}{d} \chi(-1) = -1}^* L(m, \chi\chi_{2q}^0)L(n, \overline{\chi\chi_{2q}^0}) = \sum_{d|q} \mu(d) \sum_{\chi \mod \frac{q}{d} \chi(-1) = -1}^* L(m, \chi\chi_{2q}^0)L(n, \overline{\chi\chi_{2q}^0}) = \sum_{d|q} \mu(d) \sum_{\chi \mod \frac{q}{d} \chi(-1) = -1}^* L(m, \chi)L(n, \overline{\chi}). \]

According to the Lemma 2.6, it follows that
\[ \sum_{\chi \mod q \chi(-1) = -1}^* L(m, \chi\chi_2^0)L(n, \overline{\chi\chi_2^0}) = \frac{(2\pi i)^{m+n}}{4m!n!} \cdot \left[ \sum_{l=0}^{m+n} r_{m,n,l} \cdot \sum_{d|q} \mu(d) \phi \left( \frac{2q}{d} \right) \phi_d \left( \frac{2q}{d} \right) \left( \frac{2q}{d} \right)^{l-m-n+1} \right] + \sum_{d|q} \mu(d) \phi \left( \frac{2q}{d} \right) \cdot \frac{B_mB_n \phi_{m+n-1} \left( \frac{2q}{d} \right)}{\phi_2 \left( \frac{2q}{d} \right)} \]
\[ = \frac{(2\pi i)^{m+n}}{4m!n!} \cdot \sum_{l=0}^{m+n} r_{m,n,l} \cdot \left( 1 - \frac{1}{2l} \right) \cdot 2^{l-m-n} \cdot q^{l-m-n+1} \cdot \prod_{p|q} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p'} \right) \left( 1 - \frac{1}{p'^{-m-n+1}} \right). \]

This proves Lemma 2.7.

**Lemma 2.8.** Let \( q \) be square-full odd number, we have
\[ \sum_{\chi \mod q \chi(-1) = -1}^* \chi(2)L(m, \chi)L(n, \overline{\chi}) \]
\[ = \frac{(2\pi i)^{m+n}}{4m!n!} \cdot \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^{l-m-n+1} \cdot \frac{2^{l-m-n+1} - 2^{m+n} - 2^{l-m-n}}{2^{m+n+2}} \cdot \prod_{p|q} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p'} \right) \left( 1 - \frac{1}{p'^{-m-n+1}} \right). \]

**Proof.** From the proof of Lemma 2.2, we know that
\[ L(m, \chi\chi_2^0)L(n, \overline{\chi\chi_2^0}) = L(m, \chi)L(n, \overline{\chi}) \left( 1 - \frac{\chi(2)}{2m} \right) \left( 1 - \frac{\chi(2)}{2n} \right) \]
\[ = L(m, \chi)L(n, \overline{\chi}) \left[ 1 + \frac{1}{2^{m+n}} - \left( \frac{\chi(2)}{2m} + \frac{\chi(2)}{2n} \right) \right]. \]

Note that \( \sum_{\chi \mod q \chi(-1) = -1}^* \chi(2) = \sum_{\chi \mod q \chi(-1) = -1}^* \overline{\chi}(2) \) we get
\[ \sum_{\chi \mod q \chi(-1) = -1}^* \overline{\chi}(2)L(m, \chi)L(n, \overline{\chi}) \]
From Lemma 2.3 and 2.4 we have if
\[ \text{That is we will prove Theorem 2-5.} \]

\[ \text{and} \]
\[ \text{Note that if} \]
\[ \text{In this section, we shall complete the proof of the theorems.} \]

\[ \text{This proves Lemma 2.8.} \]

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems.

First we give a hybrid mean value formula for generalized Dedekind sums with Kloosterman sums.

Note that if \( \chi \) is primitive character mod \( q \), the Gauss sums \( \tau(\chi) = \sqrt{q} \) and

\[ \left| \sum_{a=1}^{q} \chi(a)K(a, q) \right| = |\tau^2(\chi)| = q. \]

From Lemma 2.3 and 2.4 we have if \( q \) is a square-full number and \( m \equiv n \equiv 1(\text{mod}2) \)

\[ \sum_{a=1}^{q} \sum_{b=1}^{q} K(a, q)K(b, q)S(ab, m, n, q) \]

\[ = -\frac{4m!n!}{(2\pi i)^{m+n}q^{m+n-1}} \sum_{d|q} \phi(d) \sum_{\chi \text{ mod } d \chi(-1)=-1} \sum_{a=1}^{q} \sum_{b=1}^{q} K(a, q)K(b, q) \chi(ab)L(m, \chi)L(n, \bar{\chi}) \]

\[ = -\frac{4m!n!}{(2\pi i)^{m+n}q^{m+n-1}} \sum_{d|q} \phi(d) \sum_{\chi \text{ mod } d \chi(-1)=-1} \left| \sum_{a=1}^{q} \chi(a)K(a, q) \right|^2 \cdot L(m, \chi)L(n, \bar{\chi}) \]

\[ = -\frac{4m!n!}{(2\pi i)^{m+n}q^{m+n-1}} \frac{q^{m+n}}{\phi(q)} \sum_{\chi \text{ mod } q \chi(-1)=-1} q^2 \cdot L(m, \chi)L(n, \bar{\chi}) \]

\[ = q^{4-m-n} \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^l \cdot \prod_{p|q} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^{l-m+n+1}} \right). \]

This completes the proof of Theorem 1.

Now we give some hybrid mean value formulae for generalized Hardy sums and Kloosterman sums.

That is we will prove Theorem 2-5.

From the proposition 1.1, Lemma 2.6 and Lemma 2.8 together, we have if \( q \) is a square-full number and \( m \equiv 1(\text{mod}2) \)

\[ \sum_{a=1}^{q} \sum_{b=1}^{q} K(a, q)K(b, q)s_1(2\bar{a}b, m, q) \]

\[ = 2 \sum_{a=1}^{q} \sum_{b=1}^{q} K(a, q)K(b, q)S(2\bar{a}b, m, 1, q) - 4 \sum_{a=1}^{q} \sum_{b=1}^{q} K(a, q)K(b, q)S(ab, m, 1, q) \]

\[ = -\frac{4m!}{(2\pi i)^{m+1}q^m} \sum_{d|q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \text{ mod } d \chi(-1)=-1} \sum_{a=1}^{q} \left| \chi(a)K(a, q) \right|^2 \cdot (2\bar{\chi}(2) - 4) \cdot L(m, \chi)L(1, \bar{\chi}) \]
This completes the proof of Theorem 2.

In a similar way, we will deduce the identities involving generalized Hardy sums \(s_2(ab, m, n, q), s_3(ab, n, q), s_5((2a - 1)(2b - 1), m, q)\) with Kloosterman sums respectively.

If \(q\) is a square-full even number and \(m \equiv n \equiv 1 \pmod{2}\), we have
\[
\sum_{a=1}^{q} \sum_{b=1}^{q'} K(a, q)K(b, q)s_2(ab, m, n, q) = 2^m \sum_{a=1}^{q} \sum_{b=1}^{q'} K(a, q)K(b, q)S(2ab, m, n, q) - \sum_{a=1}^{q'} \sum_{b=1}^{q} K(a, q)K(b, q)S(ab, m, n, q)
\]
\[
= \frac{4m!}{(2\pi i)^{m+1}q^m} \cdot \frac{q^{m+1}}{\phi(q)} \cdot \sum_{\chi \mod q}^{*} q^2 \cdot (2\chi(2) - 4) \cdot L(m, \chi)L(1, \chi)
\]
\[
= \frac{q^{3-m}}{\phi(q)} \cdot \sum_{l=0}^{m+1} r_{m,n,l} \cdot q^{l-m} \cdot \frac{2^l - 2^{m+2}}{2^{m-1} + 1} \prod_{p|q} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^l}\right) \left(1 - \frac{1}{p^{l-m}}\right).
\]

If \(q\) is a square-full odd number and \(n \equiv 1 \pmod{2}\), we get
\[
\sum_{a=1}^{q} \sum_{b=1}^{q'} K(a, q)K(b, q)s_3(ab, n, q)
\]
\[
= 2 \sum_{a=1}^{q'} \sum_{b=1}^{q} K(a, q)K(b, q)S(ab, 1, n, q) - 4 \sum_{a=1}^{q} \sum_{b=1}^{q'} K(a, q)K(b, q)S(2ab, 1, n, q)
\]
\[
= \frac{4n!}{(2\pi i)^{1+n}q^n} \cdot \frac{q^{n+1}}{\phi(q)} \cdot \sum_{\chi \mod q}^{*} |\chi(a)K(a, q)|^2 \cdot (2 - 4\chi(2)) \cdot L(1, \chi)L(n, \chi)
\]
\[
= \frac{4n!}{(2\pi i)^{1+n}q^n} \cdot \frac{q^{n+1}}{\phi(q)} \cdot \sum_{\chi \mod q}^{*} q^2 \cdot (2 - 4\chi(2)) \cdot L(1, \chi)L(n, \chi)
\]
\[
= \frac{q^{3-n}}{\phi(q)} \cdot \sum_{l=0}^{n+1} r_{1,n,l} \cdot q^{l} \cdot \frac{5 \cdot 2^n - 2^{l+1} + 6}{2^{n-1} + 1} \prod_{p|q} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^l}\right) \left(1 - \frac{1}{p^{l-n}}\right).
\]

If \(q\) is a square-full odd number and \(n \equiv 1 \pmod{2}\), it is clear that if \(a\) pass through a reduced residue system mod \(q\), then \(2a - 1\) also pass through a reduced residue system mod \(q\), that is
\[
\sum_{a=1}^{q'} \chi(2a - 1)K(2a - 1, q) = \sum_{a=1}^{q'} \chi(a)K(a, q) = \tau^2(\chi).
\]
and

\[\sum_{a=1}^{q'} \sum_{b=1}^{q'} K(2a-1, q)K(2b-1, q)s_5((2a-1)(2b-1), m, q)\]

\[= 2 \cdot \sum_{a=1}^{q'} \sum_{b=1}^{q'} K(2a-1, q)K(2b-1, q)S((2a-1)(2b-1), m, q)\]

\[-4 \cdot \sum_{a=1}^{q'} \sum_{b=1}^{q'} K(2a-1, q)K(2b-1, q)S((2a-1)(2b-1), m, q)\]

\[= \frac{4m!}{(2\pi i)^{m+1}q^m} \left[ 2 \cdot \sum_{d|q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod d} \sum_{a=1}^{q'} \chi(2a-1)K(2a-1, q) \right]^2 \cdot L(m, \chi)L(1, \chi)\]

\[-4 \cdot \sum_{d|q} \frac{d^{m+1}}{\phi(d)} \sum_{\chi \mod q} \sum_{a=1}^{q'} \chi(2a-1)K(2a-1, q) \right]^2 \cdot \chi(2)L(m, \chi)L(1, \chi)\]

\[-\frac{4m!}{(2\pi i)^{m+1}q^m} \cdot \frac{q^{m+1}}{\phi(q)} \cdot \sum_{\chi \mod q} \chi(-1)^{-1} \cdot 4 \cdot (2-4\chi(2)) \cdot L(m, \chi)L(1, \chi)\]

\[= \frac{q^{3-m}}{\phi(q)} \cdot \sum_{l=0}^{m+1} r_{m+1, l} \cdot \left( \frac{5 \cdot 2^m - 2^{l+1} + 6}{2^m-1} \right) \cdot q^l \prod_{p|q} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p^l} \right) \left( 1 - \frac{1}{p^{l-m}} \right)\]

This completes the proof of Theorem.

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