NP-hardness of computing PL geometric category in dimension 2*

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Abstract

The PL geometric category of a polyhedron $P$, denoted $\text{plgcat}(P)$, is a combinatorial notion which provides a natural upper bound for the Lusternik–Schnirelmann category and it is defined as the minimum number of PL collapsible subpolyhedra of $P$ that cover $P$. In dimension 2 the PL geometric category is at most 3.

It is easy to characterize/recognize 2-polyhedra $P$ with $\text{plgcat}(P) = 1$. Borghini provided a partial characterization of 2-polyhedra with $\text{plgcat}(P) = 2$. We complement his result by showing that it is NP-hard to decide whether $\text{plgcat}(P) \leq 2$. Therefore, we should not expect much more than a partial characterization, at least in algorithmic sense. Our reduction is based on the observation that 2-dimensional polyhedra $P$ admitting a shellable subdivision satisfy $\text{plgcat}(P) \leq 2$ and a (nontrivial) modification of the reduction of Goaoc, Paták, Patáková, Tancer and Wagner showing that shellability of 2-complexes is NP-hard.

1 Introduction

An important notion in homotopy theory is the Lusternik–Schnirelmann category (LS category) of a topological space. This notion is important not only as a purely mathematical object (see, e.g., the book [CLOT03]) but also in computer science as it is closely related to the topological complexity of motion planning; see, e.g, [Far03, Far04, FM20].

The LS category, $\text{cat}(X)$, of a topological space $X$ is the smallest $n$ (if it exists) such that $X$ can be covered by $n$ open sets so that the inclusion of each of the open sets is nullhomotopic in $X$. One difficulty when working with the LS category is that it is often hard to determine. For example, determining whether $\text{cat}(X) = 1$ is equivalent to contractibility of $X$. This is known to be undecidable if $X$ is a simplicial complex of dimension at least 4; see [VKF74, §10] and [Tan16, Appendix] while it is an open problem whether this is decidable for simplicial complexes of dimension 2.

In order to bound the LS category from above we can use some closely related notions. One of them is the geometric category, $\text{gcat}(X)$, which requires that the open sets covering $X$ are already contractible. (For more details see again [CLOT03].) If $X$ is a polyhedron, this is equivalent to finding the minimum number of subpolyhedra covering $X$ each of which is contractible. This may make estimating $\text{gcat}(X)$ sometimes easier. However, determining whether $\text{gcat}(X) = 1$ is still equivalent to contractibility of $X$.

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1Via tools in [HAMS93] (using the exercise on page 8) decidability of this problem is equivalent to determining whether a given balanced presentation of a group presents a trivial group. In this form, the problem is mentioned for example in [BMS02].
Next step in this direction has been done by Borghini [Bor20] who introduced PL geometric category $\text{plgcat}(P)$ of a compact (connected) polyhedron $P$. It is the minimum number of PL collapsible subpolyhedra of $P$ that cover $P$. (See Section 2 for the precise definition of PL collapsibility.) In this case determining whether $\text{plgcat}(P) = 1$ is equivalent to asking whether $P$ is PL collapsible. At least for 2-complexes this is a significant improvement as PL collapsibility of 2-complexes is a purely combinatorial notion which is easy to check. Indeed, it is not hard to derive from known results that this is a polynomially checkable criterion (by performing the collapses greedily on an arbitrary triangulation).

**Proposition 1.** Given a 2-dimensional triangulated polyhedron $P$, it can be checked in polynomial time whether $\text{plgcat}(P) = 1$.

Borghini further proved [Bor20] that a connected $d$-dimensional polyhedron has PL geometric category at most $d + 1$. For connected 2-polyhedra, this means that the only options are 1, 2, or 3. One of the main aims in [Bor20] is to provide a partial characterization of polyhedra $P$ with $\text{plgcat}(P) \leq 2$ (which we do not reproduce here). All these positive results suggest that determining $\text{plgcat}(P)$ could be easy for 2-polyhedra. In particular, one should be curious whether it is possible to extend Borghini’s results to a full characterization that would distinguish 2-polyhedra with PL geometric category equal to 2 from those for which it equals 3.

We will show that this is essentially impossible, at least for an efficiently algorithmically checkable characterization. In technical terms, we show that determining whether $\text{plgcat}(P) \leq 2$ is NP-hard. Let us point out that NP-hard problems are believed not to be solvable in polynomial time. (This is equivalent to the standard conjecture $P \neq NP$ in theory of computation. NP-hardness is discussed in a bit more detail in Section 2.)

**Theorem 2.** Given a 2-dimensional triangulated polyhedron $P$, it is NP-hard to decide whether $\text{plgcat}(P) \leq 2$.

We should also point out that we actually do not know whether recognition of triangulated polyhedra with $\text{plgcat}(P) \leq 2$ belongs to the class NP (not even whether it is decidable). This could be certified by two subpolyhedra witnessing $\text{plgcat}(P) \leq 2$ but we do not know whether we can bound their sizes.

A useful step towards our proof is that we observe a relation between $\text{plgcat}(P) \leq 2$ and shellability (of some triangulation) of $P$. (Shellability will be discussed in detail in Section 2.)

**Proposition 3.** If a 2-dimensional polyhedron $P$ admits a (pure) shellable triangulation, then $\text{plgcat}(P) \leq 2$.

It has been shown by Goaoc, Paták, Patáková, Tancer and Wagner [GPP+19] that shellability is NP-hard already for 2-dimensional simplicial complexes. In addition, the reduction in [GPP+19] is quite resistant with respect to subdivisions. Thus, we could hope to prove Theorem 2 in the following way: Consider a complex $K$ that appears in the reduction in [GPP+19]. If $K$ is shellable, then $\text{plgcat}(|K|) \leq 2$ by Proposition 3 ($|K|$ stands for the polyhedron of $K$). If we were able to show the other implication: ‘if $K$ is not shellable, then $\text{plgcat}(|K|) = 3$’, we would immediately get a proof of Theorem 2. Unfortunately, the other implication, stated this way, is not true: with some more effort (which we do not do here), it could be shown that every complex $K$ from the reduction in [GPP+19] satisfies $\text{plgcat}(|K|) = 2$. However, this problem can be circumvented. We construct certain enriched complex $K^+$ (by attaching a torus in a suitable way to every triangle of $K$—it may be slightly surprising that this indeed helps). It turns out that $\text{plgcat}(|K^+|)$ stays 2 for shellable $K$ but it grows to 3 for non-shellable $K$ (coming from [GPP+19]). This will prove Theorem 2.
We point out that Proposition 3 as stated is not really necessary in the proof of Theorem 2. But we state it here as it provides the motivation for our approach as well as it can be seen as a complementary result to the results of Borghini [Bor20] providing some sufficient (or necessary) conditions for $\text{plgcat}(P) \leq 2$.

We also point out that instead of [GPP+19], it would be in principle possible to use also a modification of reduction by Santamaria-Galvis and Woodroofe [SGW21], where some of the gadgets are slightly simplified. However, some intermediate steps in [GPP+19] are done via collapsibility thus for purposes of this paper it is easier to adapt to the setting in [GPP+19].

**Organization.** Proposition 1 is proved in Section 3. Theorem 2 is proved in Section 4 and Proposition 3 is proved in Section 2.

## 2 Preliminaries

**Simplicial complexes, polyhedra and subdivisions.** Although we assume that the reader is familiar with simplicial complexes (abstract or geometric), we briefly recall these notions up to the level we need in this paper. Because we also want to work with polyhedra, we will be using geometric simplicial complexes (with a single exception that the input for any computational problem we consider is the corresponding abstract simplicial complex). That is, a simplicial complex is for us a collection of geometric simplices embedded in some $\mathbb{R}^m$ such that two simplices intersect in a face of both of them; and a face of any simplex in the complex belongs again to the complex. The dimension of a simplex is the number of its vertices minus one; the dimension of a simplicial complex is the maximum of the dimensions of simplices appearing in the complex. When considering a simplicial complex as input of an algorithmic problem, we switch to the corresponding abstract simplicial complex. Roughly speaking, it records only the combinatorial information which vertices form a simplex. For more details on simplicial complexes, we refer to textbooks such as [RS82, Mat03].

We work with polyhedra as defined in [RS82]. When we say ‘polyhedron’ we always mean a compact polyhedron. Because every compact polyhedron can be triangulated, an equivalent definition is that a polyhedron is the underlying space $|K| := \bigcup_{\sigma \in K} \sigma$ of some finite simplicial complex $K$ (a.k.a. the polyhedron of $K$).

A simplicial complex $K'$ is a subdivision of a complex $K$ if $|K'| = |K|$ and every $\sigma' \in K'$ is a subset of some $\sigma \in K$. Given a subcomplex $L$ of $K$, then the subcomplex $L'$ of $K'$ corresponding to $L$ is the complex $L' := \{\sigma' \in K': \sigma \subseteq |L|\}$.

**Collapsibility and PL collapsibility.** Given a simplicial complex $K$, a face $\sigma \in K$ is free, if it is contained in a unique maximal face. A complex $K$ collapses to a subcomplex $K'$ by an elementary collapse, if $K'$ is obtained from $K$ by removing a pair of faces $\{\sigma, \tau\}$ where $\dim \tau = \dim \sigma + 1$, $\sigma$ is free and $\tau$ is the maximal face containing $\sigma$. We also say in this case that $K$ collapses to $K'$ through $\sigma$. A simplicial complex $K$ collapses to a subcomplex $L$, if there is a sequence of elementary collapses starting collapsing $K$ gradually to $L$. A complex $K$ is collapsible, if $K$ collapses to a point. Because collapsibility preserves the homotopy type, collapsibility of a simplicial complex is often understood as a combinatorial counterpart of the notion of contractibility of a topological space. (In particular, collapsibility of a complex implies contractibility of the corresponding polyhedron.)

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2Some authors allow more general elementary collapse removing a face $\sigma$ and all faces containing it provided that $\sigma$ is contained in a unique maximal face. This is only a cosmetic change in the resulting notion of collapsible complex because this more general elementary collapse can be emulated by a sequence of elementary collapses according to our definition.
A polyhedron $P$ is **PL collapsible** if some triangulation of $P$ is a collapsible simplicial complex. Similarly, a simplicial complex $K$ is **PL collapsible** if $|K|$ is a PL collapsible polyhedron. Here, we should point out a certain subtlety in the definition of PL collapsible simplicial complex: If $K$ is PL collapsible, then there is some triangulation $K'$ of $|K|$ which is collapsible (in the simplicial sense). This triangulation $K'$ needn’t be a priori a subdivision of $K$. However, by [Hud69, Theorem 2.4] we may assume that $K'$ actually is a subdivision of $K$. This also affects our earlier definition of plgcat($|K|$). We get plgcat($|K|$) $\leq k$ if and only if some subdivision of $K$ can be covered by $k$ collapsible subcomplexes while it does not matter with which triangulation of $|K|$ we start.

In general, collapsibility and PL collapsibility of a simplicial complex differ because PL collapsibility allows an arbitrarily fine subdivision before starting the collapses. In this paper, we need both and we carefully distinguish these two notions.

**Shellability.** A simplicial complex $K$ is **pure** if all its (inclusion-wise) maximal faces have the same dimension. A **shelling** of a pure complex $K$ is an ordering of all its maximal faces into a sequence $\vartheta_1, \ldots, \vartheta_m$ such that for every $k \in \{2, \ldots, m\}$ the subcomplex of $K$ with the underlying space $(\bigcup_{i=1}^{k-1} \vartheta_i) \cap \vartheta_k$ is pure and $(\dim \vartheta_k - 1)$-dimensional. (Here we use the notation for geometric simplicial complexes, thus $\vartheta_1, \ldots, \vartheta_m$ are actual geometric simplices.) A complex $K$ is **shellable** if it admits a shelling.

There are some similarities between collapsible and shellable simplicial complexes. However, in general, these two notions differ. For example, on the one hand a collapsible complex is always contractible as an elementary collapse keeps the homotopy type but shellable complexes need not be contractible. On the other hand, the union of two triangles meeting in a single vertex is a complex which is collapsible but not shellable. The following description of 2-complexes admitting a shellable subdivision has been given by Hachimori [Hac08].

**Theorem 4 ([Hac08]).** Let $K$ be a 2-dimensional simplicial complex. Then the following statements are equivalent:

(i) The complex $K$ has a shellable subdivision.

(ii) The second barycentric subdivision $sd^2 K$ is shellable.

(iii) The link of each vertex of $K$ is connected and $K$ becomes collapsible after removing $\tilde{\chi}(K)$ triangles where $\tilde{\chi}$ denotes the reduced Euler characteristic.

Hachimori’s theorem easily implies Proposition 3:

**Proof of Proposition 3.** Let $K$ be a pure shellable triangulation of $P$. By Theorem 4 there is a list of triangles $\tau_1, \ldots, \tau_\ell$ such that the resulting complex $K'$ is collapsible after removing these triangles. Now we build an auxiliary complex $L$ from $K$ by subdividing each of the triangles $\tau_1, \ldots, \tau_\ell$ as in Figure 1. We also build a complex $L'$ by removing the middle triangle $\tau'_i$ from each subdivided $\tau_i$ in $L$. The complex $K'$ is a subcomplex of $L'$ and it is not hard to see that $L'$ collapses to $K'$. Hence $L'$ is collapsible as well. Then $|L'|$ is one of the two collapsible polyhedra covering $P$. The second polyhedron is obtained by taking the union of $\tau'_i$ and connecting them along the 1-skeleton of $L$ so that the resulting complex is collapsible (the connection along the 1-skeleton of $L$ can be, for example, obtained so that we pick two edges in each triangle and then we extend this forest to a spanning tree).
Homology. In our auxiliary computations, we will often need homology groups, including the exact sequence for pairs, the Mayer-Vietoris exact sequence and the Lefschetz duality. In general, we refer to the literature such as [Hat02, Mun84] for details (in case of Lefschetz duality, we will recall its statement when used).

In all our computations, we work with homology with $\mathbb{Z}_2$-coefficients. When working with simplicial complexes, we use simplicial homology. In particular, when we speak of $k$-chains, then we can identify a $k$-chain with a collection of $k$-simplices (in its support). (Similarly, a $k$-cycle is such a collection with trivial boundary; i.e., each $(k-1)$-simplex is in an even number of $k$-simplices of the cycle.) In case of polyhedra, we use singular homology. However, we of course implicitly use that the simplicial and singular homology groups are (naturally) isomorphic (for a simplicial complex and its polyhedron).

NP-hardness and satisfiability. Here we briefly overview a few notions from computational complexity we need in this paper. For more details see, e.g., [AB09, Chapter 2].

A decision problem belongs to the class NP if an affirmative answer to it can be verified in polynomial time using a certificate of polynomial size. A decision problem $X$ is NP-hard if for each problem $Y$ from the class NP there is a polynomial time reduction from $Y$ to $X$. More precisely, given an instance $q$ of the problem $Y$ one can construct in polynomial time in the size of $q$ an instance $p$ of the problem $X$ such that the answer to $q$ is yes if and only if the answer to $p$ is yes.

An important NP-hard problem is the so called 3-satisfiability problem (it also belongs to NP). An input for the 3-satisfiability problem is a 3-CNF formula $\phi$, that is, a boolean formula in conjunctive normal form where every clause contains exactly three literals. A literal is some variable $x$ or its negation $\neg x$; a clause with 3 literals is a (sub)formula of form $(\ell_1 \lor \ell_2 \lor \ell_3)$ where $\ell_i$ are literals. A formula $\phi$ is in conjunctive normal form if it can be written as $\phi = c_1 \land c_2 \land \cdots \land c_m$ where $c_j$ are clauses. An example of a 3-CNF formula is $(x \lor \neg y \lor z) \land (\neg x \lor \neg y \lor t)$.

3 PL collapsibility of 2-complexes

It is a folklore result going back at least to Lickorish (according to [HAMS93]) that simplicial 2-complexes can be collapsed greedily:

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3A literal is some variable $x$ or its negation $\neg x$; a clause with 3 literals is a (sub)formula of form $(\ell_1 \lor \ell_2 \lor \ell_3)$ where $\ell_i$ are literals. A formula $\phi$ is in conjunctive normal form if it can be written as $\phi = c_1 \land c_2 \land \cdots \land c_m$ where $c_j$ are clauses. An example of a 3-CNF formula is $(x \lor \neg y \lor z) \land (\neg x \lor \neg y \lor t)$.
Proposition 5 (see [HAMS93, page 20] or [MF08, Lemma 1 + Corollary 1]). Let $K$ be a collapsible 2-complex. Assume that $K$ collapses to a subcomplex $L$. Then $L$ is collapsible as well. In particular, it can be checked in polynomial time whether a simplicial 2-complex is collapsible.

For PL collapsibility we can essentially deduce the same conclusion as for collapsibility as soon as we observe that PL collapsibility of a 2-complex does not depend on the choice of the subdivision, which also might be a folklore result.

Lemma 6. Let $K$ be a simplicial complex of dimension at most 2 and $K'$ be a subdivision of $K$. Then $K$ is collapsible if and only if $K'$ is collapsible.

In the proof of the lemma we use the following observation.

Observation 7. Let $\tau$ be a triangle with vertices $a, b, c$. Let $K'$ be an arbitrary subdivision of $\tau$. Then $K'$ collapses to the subcomplex $V'$ formed by the subdivision of the edges $ab$ and $bc$.

Proof. We greedily perform collapses through free edges of $K'$ which are not in $V'$. Let $L'$ be the resulting complex. We observe that $L'$ contains no triangle. Indeed, every edge contained in some triangle of $L'$ is either an edge of $V'$ or it has to be contained in both neighboring triangles (otherwise we could continue with collapses). This means, because the dual graph of $K'$ is connected, that once there is a single triangle of $K'$ in $L'$, then $L'$ contains all triangles of $K'$ which is a contradiction.

Thus, $L'$ contains no triangles and it has the same homotopy type as $K'$. That means that $L'$ is a tree. Now we greedily perform collapses of edges not in $V'$ through vertices of degree 1. By essentially the same argument as above, only the edges of $V'$ remain (otherwise, we would find a cycle in $L'$).

Proof of Proposition 5. Let $K$ be the input triangulation of $P$. By definition, $\text{plgcat}(P) = 1$ if and only if $P$ is PL collapsible which occurs if and only if some subdivision $K'$ of $K$ is collapsible. By Lemma 6, it is sufficient to check whether $K$ is collapsible. This can be done in polynomial time due to Proposition 5.

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In this section we prove Theorem 2. As we have sketched in the introduction, in our construction we need to attach a torus to every triangle of a certain intermediate complex. We start with the details regarding this attachment.
Figure 2: Left: The torus $T_r$ with longitude $\lambda_r$. Opposite edges are identified as usual. Right: Splitting $T_r$ to two annuli.

4.1 Attaching tori

First, let us consider the standard torus $T = S^1 \times S^1$. An important curve in $T$ is the longitude $\lambda = S^1 \times \{\cdot\}$ where ‘$\cdot$’ stands for some fixed point in $S^1$.

**Definition 8** (Enriched complex $K^+$). Given a simplicial complex $K$, we define the enriched complex $K^+$ as follows. For each triangle $\tau \in K$ we consider a copy $T_\tau$ of the standard torus with longitude $\lambda_\tau$ triangulated as in Figure 2. We get $K^+$ as a result of gluing all tori $T_\tau$ to $K$ so that we identify $\lambda_\tau$ with $\partial \tau$. In the sequel, we consider $K$ as well as all the tori $T_\tau$ as subcomplexes of $K^+$.

Note that the enriched complex $K^+$ can be constructed in polynomial time in the size of $K$.

**Observation 9.** If $K$ admits a covering by two collapsible subcomplexes $K_1, K_2$ such that both $K_1$ and $K_2$ contain the whole 1-skeleton of $K$ then $K^+$ can be also covered by two collapsible subcomplexes.

**Proof.** Split each $T_\tau$ to two annuli $A_{\tau,1}$ and $A_{\tau,2}$ as in Figure 2. (Both of them are subcomplexes of $T_\tau$ and they share $\lambda_\tau$ on one of their boundaries.) Take $K^+_{\tau,i}$ as the union of $K_i$ and all annuli $A_{\tau,i}$ for $i \in \{1, 2\}$. Then $K^+_{\tau,1}$ and $K^+_{\tau,2}$ cover $K^+$. In addition, they are both collapsible because $K^+_{\tau,i}$ collapses to $K_i$ as each $A_{\tau,i}$ collapses to $\lambda_\tau$. \hfill $\square$

We continue with the main technical lemma for our reduction.

**Lemma 10.** Let $P$ be a polyhedron which is a union of two subpolyhedra $R$ and $T$. Assume that $T = S^1 \times S^1$ is the torus and assume that $R$ and $T$ intersect exactly in the longitude $\lambda = S^1 \times \{\cdot\}$ of $T$. Assume that $P$ can be covered by two contractible subpolyhedra $Q_1, Q_2$. Then $\lambda \subset Q_1, Q_2$ and $\lambda$ is nullhomologous in $R \cap Q_1$ as well as in $R \cap Q_2$.

**Proof.** Let $A_i := T \cap Q_i$ for $i = 1, 2$. The lemma is implied by the following two claims where all the homology is considered with $\mathbb{Z}_2$ coefficients.

**Claim 10.1.**

(i) If $H_1(A_1) = 0$, then $\dim H_1(A_2) \geq 2$.

(ii) If $H_1(A_2) = 0$, then $\dim H_1(A_1) \geq 2$. 

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Claim 10.2.

(i) If $H_1(A_1) \neq 0$, then $\dim H_1(A_1) = 1$, $\lambda$ belongs to $Q_1$ and $\lambda$ is nullhomologous in $R \cap Q_1$.

(ii) If $H_1(A_2) \neq 0$, then $\dim H_1(A_2) = 1$, $\lambda$ belongs to $Q_2$ and $\lambda$ is nullhomologous in $R \cap Q_2$.

Indeed, the conjunction of the claims implies that only option is that $\dim H_1(A_1) = \dim H_1(A_2) = 1$ and thus we can use the conclusions of Claim 10.2. Therefore, it remains to prove the claims.

In each of the claims, we only prove the first item as the other one is symmetric.

Proof of Claim 10.1(i).

Let $N_1$ be the regular neighborhood$^4$ of $A_1$ inside $T$, which is homotopy equivalent to $A_1$; see Figure 3. Then $N_1$ is a surface with boundary. Thus we may apply the Lefschetz duality$^5$ obtaining

$$H_1(N_1, \partial N_1) \cong H^1(N_1) \cong H_1(N_1) \cong H_1(A_1) = 0$$

where the second isomorphism follows from the fact that the homology and the cohomology groups are isomorphic over a field.

Now let $C$ be the closure of the complement of $N_1$ in $T$, that is, $C := T \setminus N_1$. By the excision property of homology, and then by (1)

$$H_1(T, C) \cong H_1(N_1, \partial N_1) = 0.$$  \tag{2}

Finally, we consider the long exact sequence of the pair:

$$\cdots \rightarrow H_1(C) \xrightarrow{i_*} H_1(T) \rightarrow H_1(T, C) \rightarrow \cdots$$

The map $i_*$ is induced by the inclusion $i: C \rightarrow T$. Because of (2), the map $i_*$ is surjective. The inclusion $i$ can be decomposed into inclusions $j: C \rightarrow A_2$ and $k: A_2 \rightarrow T$. (Note that $C \subseteq A_2$ as $A_1$ and $A_2$ cover $T$.) By functoriality of homology, $k_*: H_1(A_2) \rightarrow H_1(T)$ must be surjective as well. Therefore, $\dim H_1(A_2) \geq \dim H_1(T) = 2$.

Proof of Claim 10.2(i). Let $R_1 := R \cap Q_1$. Consider the Mayer-Vietoris exact sequence:

$$\cdots \rightarrow H_1(A_1 \cap R_1) \xrightarrow{f} H_1(A_1) \oplus H_1(R_1) \xrightarrow{g} H_1(Q_1) \rightarrow \cdots$$

$^4$In this case $N_1$ is a 2-manifold with boundary inside $T$ which collapses to $A_1$. For a general definition of regular neighborhood see [RS82, Chapter 3].

$^5$Lefschetz duality (see e.g. Theorem 3.43 in [Hat02] over $\mathbb{Z}_2$). Let $M$ be an $n$-dimensional compact manifold with boundary $N$. Then $H_i(M, N; \mathbb{Z}_2) \cong H^{n-i}(M; \mathbb{Z}_2)$ for every $i$. 

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As we assume that $Q_1$ is contractible, we get $H_1(Q_1) = 0$. Therefore, $f$ is surjective (from exactness). As we also assume that $H_1(A_1) \neq 0$, there is a nonzero vector $v = (z, 0) \in H_1(A_1) \oplus H_1(R_1)$. We know $v \in \text{im} f$ as $f$ is surjective. In particular, $H_1(A_1 \cap R_1) \neq 0$. On the other hand, $A_1 \cap R_1 \subseteq T \cap R = \emptyset$. Therefore, $A_1 \cap R_1 = \emptyset$. This gives $\lambda \subseteq Q_1$ as we need. Using that $f$ is surjective again, we get $\dim H_1(A_1) + \dim H_1(R_1) \leq \dim H_1(A_1 \cap R_1) = 1$. Because $H_1(A_1) \neq 0$ we actually get $\dim H_1(A_1) = 1$ and $\dim H_1(R_1) = 0$. This gives that $\lambda$ is nullhomologous in $R_1 = R \cap Q_1$.

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4.2 Construction form \textbf{[GPP+19]}

As we sketched in the introduction, we use the construction from \textbf{[GPP+19]} as an intermediate step. Given that this construction is somewhat elaborated, we prefer to state it as a blackbox only mentioning the properties that we need in our reduction.

The NP-hardness in \textbf{[GPP+19]} is proved by a reduction from the classical 3-satisfiability problem which is defined in Section 2.

\textbf{Proposition 11} (\textit{[GPP+19]}). There is a polynomial time algorithm that produces from a given 3-CNF formula $\phi$ (with $n$ variables) a pure 2-dimensional complex $K_\phi$ with the following properties.

(i) $K_\phi$ contains pairwise disjoint triangulated 2-spheres $S_1, \ldots, S_n$, one for each variable.

(ii) The second homology group, $H_2(K_\phi)$, is generated by the spheres $S_1, \ldots, S_n$. In particular, $H_2(K_\phi) \cong \mathbb{Z}_2^n$ and no triangle outside the spheres $S_1, \ldots, S_n$ is contained in a 2-cycle.

(iii) If $\phi$ is satisfiable, then there are triangles $\tau_i$ in $S_i$ for every $i \in [n]$ such that $K_\phi$ becomes collapsible after removing these triangles. In addition, for every $i \in [n]$, there are at least two options how to pick $\tau_i$ in $S_i$. (Such a choice can be done independently in each $S_i$ yielding at least $2^n$ collapsible subcomplexes.)

(iv) If an arbitrary subdivision of $K_\phi$ becomes collapsible after removing some $n$ triangles, then $\phi$ is satisfiable.

\textit{Proof.} The proof of the proposition consists mostly of references to \textbf{[GPP+19]}. However, a few items are not as explicitly stated in \textbf{[GPP+19]} as we need them here, thus we explain in detail how all the items of the proposition can be deduced from the text in \textbf{[GPP+19]}.

The construction of $K_\phi$ is given in Section 4 of \textbf{[GPP+19]}. The spheres $S_1, \ldots, S_n$ of item (i) are the spheres $S(u)$ introduced in §4.3 of \textbf{[GPP+19]}. For checking the other items, we first point out that \textbf{[GPP+19]} Proposition 12] states that the number of variables, $n$, is equal to the reduced Euler characteristic $\tilde{\chi}(K_\phi)$.

It is stated in Remark 13 in \textbf{[GPP+19]} that $K_\phi$ is homotopy equivalent to the wedge of $n$ 2-spheres; in particular, $\dim H_2(K_\phi) = n$. Then item (ii) immediately follows as the disjoint spheres $S_1, \ldots, S_n$ generate a subspace of dimension $n$ in $H_2(K_\phi)$. Unfortunately, Remark 13 is only a side remark in \textbf{[GPP+19]} and it is not proved there. Therefore, we explain in Appendix A Proposition 12 how Remark 13 of \textbf{[GPP+19]} follows from their tools.

Item (iii), using $n = \tilde{\chi}(K_\phi)$, is the content of Proposition 8(ii) in \textbf{[GPP+19]} with the addendum that it is also necessary to check the proof: In the beginning of Section 7 of \textbf{[GPP+19]}, it is specified that the triangles are removed in certain regions $D[\ell(u)]$. By checking the construction of $D[\ell(u)]$ in §4.3 of \textbf{[GPP+19]}, these regions are in the correct spheres $(S_i$ in our notation; $S(u)$
in the notation of [GPP+19] and in addition there are at least two choices of the removed triangle for every $i$ (actually exactly three choices).

Item (iv), using $n = \chi(K_\phi)$, is exactly the content of Proposition 8(iii).

\[\square\]

4.3 The final reduction

Proof of Theorem 2. Given a 3-CNF formula $\phi$ and its corresponding complex $K_\phi$ we construct its enriched complex $K_\phi^+$. (See Definition 8.) Theorem 2 is proved by showing that $\phi$ is satisfiable if and only if $\text{plgcat}(K_\phi^+) \leq 2$ as 3-satisfiability is an NP-hard problem.

(a) $\phi$ is satisfiable $\implies$ $K_\phi^+$ can be covered by two collapsible subcomplexes.

Suppose that the formula $\phi$ is satisfiable. Then by Proposition 11(iii) $K_\phi$ is collapsible after removal of $n$ triangles, one from each sphere $S_i$, and for each $S_i$ there are at least two options, say $\tau_i(1), \tau_i(2)$, how to pick such a triangle. Therefore, the subcomplexes

$$K_1 := K_\phi \setminus \{\tau_1^{(1)}, \ldots, \tau_n^{(1)}\}, \quad K_2 := K_\phi \setminus \{\tau_1^{(2)}, \ldots, \tau_n^{(2)}\}$$

are collapsible subcomplexes of $K_\phi$ and they cover it.

Moreover, each of $K_1$ and $K_2$ contains the whole 1-skeleton of $K_\phi$. Indeed, the complex $K_\phi$ is pure thus every edge of $K_\phi$ is contained in at least one triangle and in addition in at least two triangles if it is an edge in some of the spheres $S_i$. In order to get $K_1$ or $K_2$, at most one triangle is removed from each $S_i$. Therefore, each edge of $K_\phi$ is still contained in at least one triangle of $K_1$ and in at least one triangle of $K_2$. Then Observation 9 implies that $K_\phi^+$ can be covered by two collapsible subcomplexes.

(b) A subdivision $\left(K_\phi^+\right)'$ of $K_\phi^+$ can be covered by two collapsible subcomplexes $\implies$ $\phi$ is satisfiable.

First, we sketch the idea: Let $(K_1)^'$ and $(K_2)^'$ be the two collapsible subcomplexes of $(K_\phi^+)'$ covering it. (We point out that $(K_1)^'$ is just a notation not implying that $(K_1)^'$ is a subdivision of some complex $K_\phi^+$. We want to verify the assumption in Proposition 11(iv) in order to deduce that $\phi$ is satisfiable. For this, we need a subdivision of $K_\phi$ such that removing $n$ triangles from this subdivision yields a collapsible complex. In fact, our subdivision will be trivial, thus we need to find $n$ triangles in $K_\phi$ such that their removal yields a collapsible complex. We will take $(K_1^+)'$, say, and we will (essentially) deduce that in each $S_i$ there must be $\tau_i$ such that $(K_1^+)'$ must miss at least one triangle in the subdivided $\tau_i$. These triangles $\tau_i$ are the triangles we want to remove from $K_\phi$. However, we need several intermediate claims to deduce that the resulting complex is indeed collapsible. (We will use the second complex $(K_2^+)'$ only very sparingly in order to verify the assumptions of Lemma 10.)

Let $K_\phi'$ be the subcomplex of $(K_\phi^+)'$ corresponding to $K_\phi$ in this subdivision. (Let us recall that this means that $K_\phi'$ is formed by simplices $\sigma \in (K_\phi^+)'$ such that $\sigma \subseteq |K_\phi|$.)

Claim 11.1. The complex $K_1'$ is a collapsible subcomplex of $K_\phi'$.

Proof. Our aim is to show that $(K_1^+)'$ collapses to $K_\phi'$. Then it follows from Proposition 5 that $K_1'$ is collapsible.
We pick an arbitrary triangle \( \tau \) of \( K_\phi \). Recall that \( T_\tau \) is the torus attached to \( \tau \). (See Definition 8.) Let \( T'_\tau \) be the subcomplex of \((K_\phi^+)\)' corresponding to \( T_\tau \). Note that (the subdivision of) \( \partial \tau \) belongs to \((K_\phi^+)\)' by Lemma 10. We also observe that \( T'_\tau \) is not a subcomplex of \((K_\phi^+)\)' otherwise \((K_\phi^+)\)' would contain a nontrivial 2-cycle which is not possible if it is collapsible.

Now we proceed similarly as in the proof of Observation 7. We greedily perform collapses in \((K_\phi^+)\)' on simplices of \( T'_\tau \) with the exception that we are not allowed to remove the simplices belonging to (the subdivision of) \( \partial \tau \). (See Figure 4 for a realistic example of the intersection of \((K_\phi^+)\)' and \( T'_\tau \).) Let \( L' \) be the resulting complex. We first observe that \( L' \) contains no triangles of \( T'_\tau \) as at least one triangle is missing and the dual graph to our triangulation of \( T'_\tau \) is connected even after removing the dual edges crossing \( \partial \tau \). Therefore, \( L' \cap T'_\tau \) is a graph. Due to our restriction on collapses, subdivided \( \partial \tau \) is inside this graph. We observe that no other (graph theoretic) cycle may belong to this graph. Indeed, another cycle would contain an edge which is not in \( \partial \tau \), thus not contained in any triangle of \( L' \). Therefore, such a cycle could not be filled with a 2-chain, and thus it would be necessarily homologically nontrivial in \( L' \) which is a contradiction with the fact that \( L' \) is contractible (obtained by collapses from a collapsible complex). Thus, we may conclude that \( L' \cap T'_\tau \) is the subdivided \( \partial \tau \) with a collection of pendant trees. However, these pendant trees have to be actually trivial as they get collapsed during the greedy collapses.

 Altogether we have collapsed \((K_\phi^+)\)' to a complex \( L' \) which agrees with \( K'_1 \) on \( K'_\phi \) while we have removed all simplices of \( T'_\tau \) except those that belong to \( K'_\phi \). Now we pick another triangle \( \sigma \) of \( K_\phi \) and we remove (via collapses) the simplices of \( T'_\sigma \) except those belonging to \( K'_1 \) by an analogous approach. After passing through every triangle of \( K_\phi \), we get exactly \( K'_1 \) as required.

**Claim 11.2.** For every triangle \( \tau \in K_\phi \), \( \partial \tau \) is contained in \( |K'_1| \) and it is nullhomologous in \( |K'_1| \).

**Proof.** Let \( P := |K'_\phi| = |(K_\phi^+)'|. \) Let \( R \) be the polyhedron of \( K_\phi \) and all tori of \( K_\phi^+ \) except \( T_\tau \). Let \( Q_1 := |(K_\phi^+)'| \) and \( Q_2 := |(K_2^+)'|. \) Then \( R, |T'_\tau|, Q_1 \) and \( Q_2 \) satisfy the assumptions of Lemma 10. Then we deduce that \( \partial \tau \) is nullhomologous in \( R \cap Q_1 \). Assume that \( \tau \) is such that \( T'_\tau \) is the first torus to be removed in the proof of Claim 11.1 we can choose so. Then
\[ R \cap Q_1 \text{ is exactly the polyhedron of } L' \text{ in the proof of Claim } \text{[11.1]. In particular, } L' \text{ collapses to } K'_1. \text{ As collapses provide a homotopy equivalence, we deduce that } \partial \tau \text{ is nullhomologous in } [K'_1] \text{ as well.} \]

Now, for any triangle \( \tau \in K_1 \) let \( \tau' \) be the subcomplex of \( K'_1 \) corresponding to this triangle.

**Claim 11.3.**

(i) If \( \tau \in K_1 \) is a triangle which does not belong to any of the spheres \( S_1, \ldots, S_n \), then \( \tau' \) is a subcomplex of \( K'_1 \).

(ii) For every \( i \in [n] \), all triangles \( \tau \) in \( S_i \) except exactly one satisfy that \( \tau' \) is a subcomplex of \( K'_1 \).

**Proof.** Let \( \tau \in K_1 \). Due to Claim 11.2 it has to be possible to fill the subdivision of \( \partial \tau \) by some 2-chain \( c = c(\tau) \) in \( K'_1 \).

If \( \tau \) does not belong to any of the spheres \( S_1, \ldots, S_n \), then the only option for \( c \) is to contain all simplices of \( \tau' \). Indeed, if there is another such \( c' \), then considering \( \tau' \) as a 2-chain, we get a nontrivial 2-cycle \( \tau' + c' \) with support at least partially outside the spheres \( S_1, \ldots, S_n \) which contradicts Proposition 11(ii). Therefore, \( \tau' \) must be a subcomplex of \( K'_1 \) which concludes (i).

Now for (ii), take \( i \in [n] \). Then \( K'_1 \) has to miss at least one triangle in \( |S_i| \) otherwise subdivided \( S_i \) forms a non-trivial 2-cycle in \( K'_1 \) which is a contraction with Claim 11.1. Assume that \( \tau \) in \( S_i \) was chosen so that this missing triangle belongs to \( \tau' \). Then \( \partial \tau \) splits (subdivided) \( S_i \) to two hemispheres; one of them is formed by \( \tau' \) and another is formed by the union of subcomplexes \( \sigma' \) taken over all triangles \( \sigma \) in \( S_i \) different from \( \tau \). By using Proposition 11(ii) again, the only options are that \( c = c(\tau) \) contains all the simplices of one or the other (subdivided) hemispheres. But the hemisphere of \( \tau' \) is ruled out as \( \tau' \) misses a triangle of \( K'_1 \). Thus \( c \) has to be filled by the other hemisphere. Then we conclude (ii) for all simplices \( \sigma \) in \( S_i \) except exactly \( \tau \) as required. \( \square \)

In the light of Claim 11.3(ii), let \( \tau_i \) be the unique triangle of \( S_i \) such that \( \tau'_i \) is not a subcomplex of \( K'_1 \). Let \( K'_0 \) be the subcomplex of \( K_1 \) obtained by removing all triangles \( \tau_1, \ldots, \tau_n \) and let \( (K'_0)' \) be the subcomplex of \( K'_1 \) corresponding to \( K'_0 \). Note that Claim 11.3 implies that \( (K'_0)' \) is a subcomplex of \( K'_1 \). See Figure 5 for comparison of \( K'_0, K'_1 \) and \( K'_0 \) after using Claim 11.3.

**Claim 11.4.** \( K'_1 \) collapses to \( (K'_0)' \).

**Proof.** The complexes \( K'_1 \) and \( (K'_0)' \) differ only so that \( K'_1 \) may contain some simplices of \( \tau'_i \) for some \( i \) (except those that subdivide \( \partial \tau_i \)) which are not in \( (K'_0)' \).

Now, we continue analogously as in the proof of Observation 7 or Claim 11.1. We greedily collapse all simplices of \( K'_1 \) in \( \tau'_i \) except those that subdivide \( \partial \tau_i \). We first deduce that the resulting complex contains no triangles of \( \tau'_i \) as at least one triangle was missing in the beginning. Then we deduce that there is no graph-theoretic cycle among simplices of \( \tau'_i \) except the one corresponding to \( \partial \tau_i \) by the same argument as in the proof of Claim 11.1 (using that \( K'_1 \) is collapsible). Then, we deduce that among the simplices of \( \tau'_i \) only the simplices subdividing \( \partial \tau_i \) remain in the complex. After repeating this approach for every \( i \in [n] \) we obtain \( (K'_0)' \). \( \square \)
Figure 5: A schematic drawing of $K'_\phi$, $K'_1$ and $K'_-\phi$. We emphasize that this not really a realistic drawing of $K'_\phi$ (with the same polyhedron as $K_\phi$) as constructed in [GPP+19]. We only attempt to draw as simple complex as possible satisfying conclusions (i) and (ii) of Proposition 11 and so that $K'_1$ is collapsible. (The space inside the spheres is completely hollow.)

Now, we have acquired enough tools to conclude the case (b) and therefore to conclude the proof of the theorem. From Claims 11.1 and 11.4 and Proposition 11 we deduce that $(K'_-\phi)'$ is collapsible. By Lemma 6 we deduce that $K'_-\phi$ is collapsible. Finally, by Proposition 11(iv) we deduce that $\phi$ is satisfiable.

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A Appendix

Our aim in the appendix is to verify Remark 13 in [GPP+19] which is stated but not proved in [GPP+19]. The exact statement we need is given by the following proposition. We will provide all the necessary detail in order to verify correctness of Remark 13 of [GPP+19]. On the other hand, we warn the reader that our proof is not self-contained but it relies on the construction of \( K_\phi \) and partially the notation in [GPP+19]; thus it is necessary to consult the contents of [GPP+19].

**Proposition 12.** The complex \( K_\phi \) from [GPP+19] is homotopy equivalent to the wedge of \( n \) 2-spheres (where \( n \) is the number of variables).

In the proof, we need the following simple lemma.

**Lemma 13.** Let \( K_1, K_2 \) be simplicial complexes. Assume that \( K_1 \cap K_2 \) and \( K_2 \) are contractible, then \( K_1 \) and \( K_1 \cup K_2 \) are homotopy equivalent.

**Proof.** It is well known that contracting a contractible subcomplex is a homotopy equivalence [Mat03, Proposition 4.1.5]. Therefore, we get

\[
|K_1 \cup K_2| \simeq |K_1 \cup K_2|/|K_2| = |K_1|/|K_1 \cap K_2| \simeq |K_1|
\]

as required. \( \square \)

**Proof of Proposition 12** We follow essentially in verbatim the proof of Proposition 12 in [GPP+19]. The only difference is that we use Lemma 13 instead of the weaker statement in [GPP+19]: If \( K_1 \cap K_2 \) and \( K_2 \) are contractible, then \( \tilde{\chi}(K_1 \cup K_2) = \tilde{\chi}(K_1) \) where \( \tilde{\chi} \) stands for the reduced Euler characteristic.

As described in the proof of Proposition 12 in [GPP+19], the complex \( K_\phi \) can be transformed into certain complex \( K' \) by a series of steps when we decompose some intermediate complex as \( K_1 \cup K_2 \) where \( K_2 \) and \( K_1 \cap K_2 \) are contractible, and then we replace the intermediate complex with \( K_1 \). Therefore, using Lemma 13 we get that the resulting complex \( K' \), after performing all these steps is homotopy equivalent to \( K_\phi \).

By a further homotopy equivalence Goaoc et al., [GPP+19], obtain another complex \( K'' \) which is already (obviously) homotopy equivalent to the wedge of \( n \) 2-spheres. Therefore, \( K_\phi \) is homotopy equivalent to the wedge of \( n \) 2-spheres. \( \square \)