SOME REMARKS ON THE SIMPLICIAL VOLUME OF NONPOSITIVELY CURVED MANIFOLDS

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ABSTRACT. We show that any closed manifold with a metric of nonpositive curvature that admits either a single point rank condition or a single point curvature condition has positive simplicial volume. We use this to provide a differential geometric proof of a conjecture of Gromov in dimension three.

1. INTRODUCTION AND RESULTS

The vector space $C_i(X; \mathbb{R})$ of singular $i$-chains of a topological space $X$ comes equipped with a natural choice of basis consisting of the set of all continuous maps from the $i$-dimensional Euclidean simplex into $X$. The $\ell^1$-norm on $C_i(X; \mathbb{R})$ associated to this basis descends to a semi-norm $\|\cdot\|_1$ on the singular homology $H_i(X; \mathbb{R})$ by taking the infimum of the norm within each equivalence class. The simplicial volume, written $\|M\|$, of a closed oriented $n$-manifold $M$ is defined to be $\|[M]\|_1$, where $[M]$ is the fundamental class in $H_n(M; \mathbb{R})$. More concretely,

$$\|M\| = \inf \left\{ \sum |a_i| : \left[ \sum a_i \sigma_i \right] = [M] \in H_n(M, \mathbb{R}) \right\},$$

where the infimum is taken over all singular cycles with real coefficients representing the fundamental class in the top homology group of $M$. This invariant is multiplicative under covers, so the definition can be extended to closed non-orientable manifolds as well.

This invariant was first introduced by Thurston ([Thu77, Chapter 6]) and soon after expanded upon by Gromov ([Gro82]). This topological invariant measures how efficiently the fundamental class of $M$ can be represented by real singular cycles. The simplicial volume relates to other important geometrically defined topological invariants such as the minimal volume invariant defined by,

$$\text{MinVol}(M) = \left\{ \text{Vol}(M, g) : -1 \leq K_g \leq 1 \right\}.$$  

Here $K_g$ denotes the sectional curvatures of the metric $g$. We always have $\text{MinVol}(M) \geq C \|M\|$ for some universal constant $C > 0$ depending only on dimension ([Gro82]).

The minimal volume invariant, and hence simplicial volume, of $M$ vanishes if $M$ admits a nondegenerate circle action, or more generally a polarized $\mathcal{F}$-structure ([Fuk87, CG86, CG90]). On the other hand, positivity of simplicial volume has been established for only a few special classes of manifolds including higher genus surface-by-surface bundles ([HK01]), negatively curved manifolds ([Gro82]), more generally manifolds with hyperbolic fundamental group ([Min01]) including visibility manifolds of nonpositive curvature ([Cao95]), certain generalized graph manifolds ([Kue04, CS17]), and most closed locally symmetric spaces of higher rank and some noncompact finite volume ones as well ([LS06, BK07, KK12, LS09]). Several
other related versions of simplicial volume have been considered (e.g. [FLPS16, BFP15, LPW15]), but we will not concern ourselves with these here.

The relationship between curvature and simplicial volume is delicate. Gromov [Gro82] showed that closed manifolds $M$ with amenable fundamental group have $\|M\| = 0$. This includes manifolds with finite fundamental groups such as those with positive sectional or even Ricci curvatures. Moreover, Cheeger and Gromoll [CG72] showed that any manifold $M$ of nonnegative Ricci curvature has fundamental group which is a finite extension of a crystallographic group. In particular, $\pi_1(M)$ is amenable and $\|M\| = 0$. On the other hand, Lohkamp [Loh94] showed that any manifold of dimension at least three admits a metric of negative Ricci curvature, and hence there can be no relation to simplicial volume. Restricting our attention to nonpositively curved manifolds, Gromov ([Sav82] and see also [Gro82, p.11]) conjectured the following:

**Conjecture 1.1** (Gromov). Any closed manifold of nonpositive sectional curvature and negative Ricci curvature has $\|M\| > 0$.

For the statement of next theorem we need the following definition.

**Definition 1.2.** For any positive (resp. negative) semi-definite linear endomorphism $A : V^n \to V^n$, and for any $k = 1, 2, ..., n$, we define the $k$-th trace of $A$, denoted by $\text{Tr}_k(A)$, to be

$$\text{Tr}_k(A) = \inf_{V_k \subset V^n} \text{Tr}(A|_{V_k}) \quad \left(\text{resp. } \text{Tr}_k(A) = \sup_{V_k \subset V^n} \text{Tr}(A|_{V_k})\right),$$

where $V_k$ is a $k$-dimensional subspace (not necessarily invariant under $A$) of $V^n$, and $A|_{V_k}$ is the restriction of $A$ to $V_k \times V_k$, viewed as a bilinear form. Hence $\text{Tr}(A|_{V_k}) = \sum_{i=1}^{k} A(e_i, e_i)$ for any basis of $V_k$. Equivalently, $\text{Tr}_k(A)$ is the sum of $k$ eigenvalues of $A$ closest to 0.

**Theorem 1.3.** If $M$ is a closed nonpositively curved manifold of dimension $n \geq 3$, then the simplicial volume has a lower bound

$$\|M\| \geq \frac{2(n-1)^n}{n^{n/2} \omega_n} \int_M u^p(x) dV$$

where $\omega_n$ is the volume of the unit round $n$-sphere, and $u(x) = \inf_{v \in T^1_x M} \text{Tr}_2 DdB_v$.

**Remark.** Note that the theorem holds trivially when $u(x) \equiv 0$, and $u(x) > 0$ if and only if every vector in $T^1_x M$ is rank one (see Definition 2.1 and Lemma 2.4 below).

![Figure 1. Gromov's example of a rank one graph manifold.](image)

**Example 1.4.** The example of Figure 1 consists of two hyperbolic surfaces $\Sigma_1$ and $\Sigma_2$ with one puncture each such that the cusps have been truncated and smoothly and symmetrically tapered to a flat metric in a
neighborhood of their round circle boundaries $\partial \Sigma_i$. We form $M$ by gluing $\Sigma_1 \times \partial \Sigma_2$ to $\partial \Sigma_1 \times \Sigma_2$ by the identity isometry along the flat boundary torus $\mathbb{T}^2 = \partial \Sigma_1 \times \partial \Sigma_2$, thus switching the surface and circle factors.

This manifold has an obvious rank one $\mathcal{F}$-structure formed by the local circle bundles, and hence $\|M\| = 0$. In this case, $u(x) = 0$ for all $x$. Indeed, if $v$ is tangent to the circle direction at a point $x$, then $DDB_u(w) = 0$ for all $w \in v^\perp$ and hence $\text{Tr}_2 DDB_v = 0$.

**Corollary 1.5.** Let $M$ be a closed nonpositively curved manifold. If there exists a point $x \in M$ so that every vector in $T^1_x M$ is rank one, then the simplicial volume $|\|M\|| > 0$.

**Corollary 1.6.** Let $M$ be a closed nonpositively curved manifold. If there exists a point $x \in M$ so that every 2-plane in $T^1_x M$ is negatively curved, then the simplicial volume $|\|M\|| > 0$.

**Remark.** The $n \geq 3$ cases of the above corollaries are immediate consequences of Theorem 1.3, and the $n = 2$ case holds since the hypotheses on the metric rule out the torus, Klein-bottle, projective plane and sphere.

In the special case where $M$ is real hyperbolic, $u(x) \equiv 1$ and the simplicial volume $|\|M\|| = \text{Vol}(M)/\sigma_n$ ([Gro82, Thu77]), where $\sigma_n$ is the maximal volume of ideal simplices in $\mathbb{H}^n$.

**Remark.** The above observation immediately yields from Theorem 1.3 the upper bound,

$$\sigma_n \leq \frac{n^{n/2} \omega_n}{2(n - 1)^n}.$$  

While this is inferior to the upper bound of $\frac{\pi}{(n-1)!}$ given by Thurston ([Thu77], see also [HMS81]), we achieve it via a completely indirect computation.

Using Theorem 1.3 we provide a purely differential geometric proof of the following 3-dimensional case of Conjecture 1.1.

**Theorem 1.7.** If $M^3$ admits a nonpositively curved metric with negative Ricci curvature, then the simplicial volume $|\|M\|| > 0$.

For the next theorem we need to introduce the notion of $k$-Ricci curvature.

**Definition 1.8.** For $M$ nonpositively curved and $u, v \in T_x M$, the $k$-Ricci curvature is given by

$$\text{Ric}_k(u, v) = \sup_{V \subset T_x M, \dim V = k} \text{Tr} R(u, \cdot, v, \cdot)|_V = \text{Tr}_k R(u, \cdot, v, \cdot).$$

The next theorem generalizes Theorem 1 of [CW17].

**Theorem 1.9.** Let $M$ be an closed manifold of dimension $n$ admitting a Riemannian metric of nonpositive curvature. If there exists $x \in M$, such that any vector $v_x \in T^1_x M$ satisfies $\text{Ric}_{[\lfloor \frac{n}{2} \rfloor + 1]}(v_x, v_x) < 0$, then the simplicial volume $|\|M\|| > 0$.

**Remark.** Dual to the aforementioned $\ell^1$-norm on $C_*(X; \mathbb{R})$, we also have an $\ell^\infty$-norm $\|\|_{\infty}$ on the real vector spaces $C^*(X; \mathbb{R}) = \text{Hom}(C_*(X; \mathbb{R}), \mathbb{R}) = [C_*(X; \mathbb{R})]^\vee$ appearing in the cochain complex for singular cohomology $H^*(X; \mathbb{R})$. By considering the bounded elements, one obtains a subcomplex of the cochain complex, whose homology yields the *bounded cohomology* $H^b_k(X; \mathbb{R})$. The natural inclusion of cochain complexes
induces a comparison map $c: H^n_b(X; \mathbb{R}) \to H^n(X; \mathbb{R})$ from the bounded cohomology to the ordinary cohomology. Elements in the image are cohomology classes which have bounded representatives, and so admit a well-defined $\ell^\infty$-norm. Hence, the simplicial volume of $M$ vanishes if and only if the comparison map $c: H^n_b(M, \mathbb{R}) \to H^n(M, \mathbb{R})$ in top dimension is the zero map. In the setting of the above theorems, positivity of the simplicial volume implies that the top dimensional comparison map is surjective and in particular, the bounded cohomology $H^n_b(M, \mathbb{R})$ is nontrivial.

Note however, that the pointwise estimate of the above theorem is not necessarily sufficient to give surjectivity of the comparison map for bounded cohomology in lower dimensions analogous to the statement of Theorem 2 in [CW17].

Acknowledgments. The authors would like to thank Jean-François Lafont for helpful discussions.

2. Barycenters and Straightening

2.1. Rank of a manifold. In the context of nonpositively curved manifold, the notion of geometric rank can be viewed as a generalization of the usual rank of a symmetric space. We recall the following definition,

**Definition 2.1.** Let $M$ be a nonpositively curved manifold. For any nonzero vector $v \in TM$, we define the rank of $v$ to be the dimension of the space of all parallel Jacobi fields along the geodesic ray formed by $v$. We say $M$ is rank one if there exists a rank one vector on $TM$, and is higher rank otherwise.

One remarkable result about higher rank manifolds is that they are well understood, and completely classified by the following higher rank rigidity theorem,

**Theorem 2.2.** ([Bal85] [BS87]) If $M$ is simply-connected, nonpositively curved manifold of higher rank, then $M$ is either a product of nonpositively curved manifolds, or a higher rank symmetric space of noncompact type.

For our purpose, we would like to see how the rank is reflected by the conditions of our theorems, in terms of $u(x)$ and $\text{Ric}_k$. In fact, we will see in Lemma 2.4 that if $u(x)$ is not identically zero on $M$, then $M$ is rank one. On the other hand, we have

**Lemma 2.3.** If there exists a point $x \in M$ such that $\text{Ric}_{[\frac{n}{2}]+1}(v_x, v_x) < 0$ for all $v_x \in T^1_xM$, then $M$ is rank one.

**Proof.** We begin to show the universal cover $\tilde{M}$ cannot be a product. If not, $\tilde{M} = M_1^{(n_1)} \times M_2^{(n_2)}$, where $n_1 \leq n/2 \leq n_2$. Any vector $v \in T^1M_1 \oplus 0 \subset T^1\tilde{M}$ has $\text{Ric}_{n_2}(v, v) = 0$ since all vectors in the $M_2$ factor have zero sectional curvature with $v$. This implies $\text{Ric}_{[\frac{n}{2}]+1}(v, v) = 0$ as $[\frac{n}{2}] + 1 \leq n/2 \leq n_2$, contrary to hypothesis.

Secondly, we show it cannot be an irreducible symmetric space either. If not, we define the splitting rank of a symmetric space $\tilde{M}$, denoted by $\text{srk}(\tilde{M})$, to be the maximal dimension of a totally geodesic submanifold $Y \subset \tilde{M}$ which splits off an isometric $\mathbb{R}$-factor. If we take a vector $v$ corresponding to the $\mathbb{R}$-factor of a submanifold $Y$ that attains the splitting rank, all vectors in $T^1Y \subset T^1\tilde{M}$ will have zero sectional curvatures with $v$, hence $\text{Ric}_{\text{srk}(X)}(v, v) = 0$. On the other hand, the splitting rank has been computed explicitly in Table 1 of [Wan16] for all irreducible symmetric spaces of noncompact type, and in particular, we have $[\frac{n}{2}] + 1 \leq \text{srk}(\tilde{M})$. Therefore, we have $\text{Ric}_{[\frac{n}{2}]+1}(v, v) = 0$, giving a contradiction.

If $\tilde{M}$ is neither a product nor symmetric, then $M$ is rank one by Theorem 2.2. \qed
2.2. Busemann functions. Recall that, for any triple \((x, y, \theta) \in \widetilde{M} \times \widetilde{M} \times \partial \widetilde{M}\), the Busemann function \(B\) on \(\widetilde{M}\) is defined by

\[
B(x, y, \theta) = \lim_{t \to \infty} (d_{\tilde{M}}(y, \gamma_\theta(t)) - t)
\]

where \(\gamma_\theta(t)\) is the unique geodesic ray from \(x\) to \(\theta\).

We fix a basepoint \(O\) in \(\widetilde{M}\) and shorten \(B(O, y, \theta)\) to just \(B(y, \theta)\). We note that for fixed \(\theta \in \partial \widetilde{M}\) the Busemann function \(B(x, \theta)\) is convex on \(\widetilde{M}\), and the nullspace of its Hessian \(DdB_{(x, \theta)}\) gives all parallel Jacobi fields along the geodesic ray of \(v_{x, \theta}\). In particular, we have the following lemma:

**Lemma 2.4.** \(v_{x, \theta}\) is rank one if and only if \(|\text{Tr}_2 DdB_{(x, \theta)}| > 0\). As a result, if there exists a point \(p \in M\), so that \(\nu(p) > 0\), then \(M\) is rank one.

**Proof.** For any \(w \in v_{x, \theta}^+ \subset T_xM\), \(DdB_{(x, \theta)}(w) = 0\) if and only if there is a stable Jacobi field \(J\) along the geodesic tangent to \(v_{x, \theta}\) with \(J(0) = w\) and \(J'(0) = 0\). By convexity of the norm of Jacobi fields in nonpositively curved manifolds, \(||J(t)||\) is constant in \(t\) and so \(J(t)\) is simply the parallel translation of \(w\) along \(g^t v_{x, \theta}\). In particular, the existence of such a field implies \(\text{rank}(v_{x, \theta}) \geq 2\). \(\square\)

2.3. Lyapunov exponents. For any \(v \in T^1M\) define the *lower Lyapunov exponent* at \(v\) to be the quantity

\[
\lambda_-^v = \min_{\nu \in \nu^+ \cap T^1M} \lim_{t \to \infty} \frac{1}{t} \log \|\Lambda_{v, t}(u)\|,
\]

where \(\Lambda_{v, t} : v^+ \to (g^t v)^+\) is the unstable Jacobi tensor along the geodesic through \(v\). Under our assumptions \(\lambda_-^v \geq 0\), and \(\text{Tr}_2 DdB_v = 0\) implies \(\lambda_-^v = 0\).

For \(v \in T^1M\) we define the (weak) stable manifold through \(v\) to be

\[
W^s(v) = \{ w \in T^1\widetilde{M} : d_{\tilde{g}}(g^t w, g^t v) \leq C \text{ for all } t \geq 0 \text{ and some } C > 0 \}.
\]

The following is a restatement of Proposition 3.1 of [Con03] in our setting. (In that paper \(\lambda_-^v\) is written \(\lambda_+^v\) since it coincides with the negative of the largest nonpositive stable Lyapunov exponent.)

**Lemma 2.5.** The quantity \(\lambda_-^v\) is constant on the stable manifold \(W^s(v)\), and consequently only depends on the point \(v(\infty) \in \partial \widetilde{M}\).

2.4. Patterson-Sullivan measures and barycenters. Let \(M\) be a compact nonpositively curved rank one manifold, \(\widetilde{M}\) the universal cover of \(M\), and \(\Gamma\) the fundamental group of \(M\). In [Kni98], Knieper showed that there exists a unique family of finite Borel measures \(\{\mu_x\}_{x \in \widetilde{M}}\) fully supported on \(\partial \widetilde{M}\), called the Patterson-Sullivan measures, which satisfies:

(a) \(\mu_x\) is \(\Gamma\)-equivariant, for all \(x \in \widetilde{M}\),

(b) \(\frac{d\mu_x}{d\mu_y}(\theta) = \rho^{hB(x, y, \theta)}\), for all \(x, y \in \widetilde{M}\), and \(\theta \in \partial \widetilde{M}\).

where \(h\) is the volume entropy of \(M\), and \(B(x, y, \theta)\) is the Busemann function of \(\widetilde{M}\).

If \(\nu\) is any finite Borel measure fully supported on \(\partial \widetilde{M}\), by taking the integral of \(B(x, \theta)\) with respect to \(\nu\), we obtain a strictly convex function (see the lemma below)

\[
x \mapsto \mathcal{B}_\nu(x) := \int_{\partial \widetilde{M}} B(x, \theta) d\nu(\theta)
\]

Hence we can define the barycenter \(\text{bar}(\nu)\) of \(\nu\) to be the unique point in \(\widetilde{M}\) where the function attains its minimum. It is clear that this definition is independent of the choice of basepoint \(O\).
Lemma 2.6. If $M$ is rank one and $\nu$ is any finite Borel measure that is fully supported on $\partial \tilde{M}$, then the function

$$x \mapsto \mathcal{B}_\nu(x)$$

is strictly convex.

Proof. It is equivalent to show that the Hessian

$$\int_{\partial M} D^2\mathcal{B}_{(x,0)}(\cdot, \cdot) d\nu(\theta)$$

is positive definite. Suppose to the contrary that there this operator has a nonzero element $w \in T_x M$ in its kernel. Since $\nu$ is fully supported, $D^2\mathcal{B}_{(x,0)}(w) = 0$ for almost all $\theta \in \partial M$, and hence by continuity all $\theta \in \partial M$. With the exception of $v_{x,\theta} = \pm w$, by Lemma 2.4 we obtain that the projection of $w$ to $v_{x,\theta}^\perp$ is the initial vector of a parallel Jacobi field. By continuity of such fields, this extends to some direction in $\pm w$ as well. Hence, by Lemma 2.5, every $\xi \in \partial \tilde{M}$ has $\lambda_\xi^- = 0$.

Since $M$ is rank one, this contradicts Corollary 1.2 of [Kn98] that there are regular vectors in $T^1 M$ and hence points $\xi \in \partial \tilde{M}$ with $\lambda_\xi^+ > 0$. In fact, most points are of this form. \hfill $\square$

2.5. Straightening subordinated to $U$.

Definition 2.7. (Compare [LS06]) Let $\tilde{M}^n$ be the universal cover of an $n$-dimensional manifold $M^n$, and $p : \tilde{M}^n \to M^n$ be the covering map. $U \subset M^n$ is a nonempty open set. We denote by $\Gamma$ the fundamental group of $M^n$, and by $C_*(\tilde{M}^n)$ the real singular chain complex of $\tilde{M}^n$. Equivalently, $C_k(\tilde{M}^n)$ is the free $\mathbb{R}$-module generated by $C^0(\Delta^k, \tilde{M}^n)$, the set of singular $k$-simplices in $\tilde{M}^n$, where $\Delta^k$ is equipped with some fixed Riemannian metric. We say a collection of maps $st_k : C^0(\Delta^k, \tilde{M}^n) \to C^0(\Delta^k, \tilde{M}^n)$ is a straightening subordinated to $U$, if it satisfies the following conditions:

(a) the maps $st_k$ are $\Gamma$-equivariant,
(b) the maps $st_\ast$ induce a chain map $st_\ast : C_*(\tilde{M}^n, \mathbb{R}) \to C_*(\tilde{M}^n, \mathbb{R})$ that is $\Gamma$-equivariantly chain homotopic to the identity,
(c) the image of $st_n$ lies in $C^1(\Delta^n, \tilde{M}^n)$, that is, the top dimensional straightened simplices are $C^1$,
(d) there exists a constant $C$ depending on $\tilde{M}^n$, $U$, and the chosen Riemannian metric on $\Delta^n$, such that for any pair $(f, \delta) \in C^0(\Delta^n, \tilde{M}^n) \times \Delta^n$ satisfying $st_n(f)(\delta) \in p^{-1}(U)$, there is a uniform upper bound on the Jacobian of $st_n(f)$ at $\delta$:

$$|\text{Jac}(st_n(f))(\delta)| \leq C$$

where $st_n(f) : \Delta^n \to \tilde{M}^n$ is the corresponding straightened simplex of $f$.

Theorem 2.8. If $\tilde{M}^n$ admits a straightening subordinated to some nonempty open set $U$, then the simplicial volume of $M$ is positive.

Proof. We choose a nontrivial smooth bump function $\phi(x)$ on $M$, such that $0 \leq \phi \leq 1$, and $\phi(x) = 0$ for all $x \notin U$. Let $\sum_{i=1}^l a_i \sigma_i$ be any homological representative of the fundamental class $[M]$, and $st(\sigma_i)$ be the
straightened simplex of \( \sigma_i \) on \( M \), with lift \( \widetilde{st}(\sigma_i) \) on the universal cover \( \widetilde{M} \). We have

\[
\int_M \phi(x) dV = \int_{\sum a_i \sigma_i} \phi(x) dV = \int_{\sum a_i \widetilde{st}(\sigma_i)} \phi(x) dV
\]

(2.1)

\[
\leq \sum_{i=1}^l |a_i| \cdot \left| \int_{\widetilde{st}(\sigma_i)} \tilde{\phi}(x) d\tilde{V} \right|
\]

(2.2)

\[
= \sum_{i=1}^l |a_i| \cdot \left| \int_{\widetilde{st}(\sigma_i) \cap \rho^{-1}(U)} \tilde{\phi}(x) d\tilde{V} \right|
\]

(2.3)

\[
\leq \sum_{i=1}^l |a_i| \cdot \left| \int_{(\rho \widetilde{st}(\sigma_i))^{-1}(U)} \phi(st(\sigma_i)(\delta)) |Jac(st(\sigma_i))(\delta)| dV_\Delta \right|
\]

(2.4)

\[
\leq \sum_{i=1}^l |a_i| \cdot C Vol(\Delta^n)
\]

(2.5)

where equation (2.1) follows from (b) of Definition 2.7, inequality (2.2) lifts to the universal cover \( \widetilde{M} \), equation (2.3) uses the support of \( \phi \), inequality (2.4) pulls the integral back on \( \Delta^n \), and inequality (2.5) follows from (c)(d) of Definition 2.7.

By taking the infimum over all fundamental class representatives \( \sum a_i \sigma_i \), we have

\[
\| M \| \geq \int_M \phi(x) dV / C Vol(\Delta^n) > 0
\]

\[ \square \]

2.6. **Barycentric straightening.** The barycentric straightening was introduced by Lafont and Schmidt [LS06] (based on the barycenter method originally developed by Besson, Courtois, and Gallot [BCG95]) to show the positivity of simplicial volume of most locally symmetric spaces of noncompact type.

Briefly speaking, any old \( k \)-simplex on a manifold gives \( k + 1 \) vertices, which forms \( k + 1 \) Patterson-Sullivan measures. These measures can be viewed as \( k + 1 \) vertices in the affine space of all measures supported on boundary at infinity of the manifold. Using these vertices, we can fill up a simplex on the space of measures by linear combinations. Finally, applying the barycenter map, it gives a new simplex on the original manifold.

More explicitly, we denote by \( \Delta^k \) the standard spherical \( k \)-simplex (for estimate purpose) in the Euclidean space, that is

\[
\Delta^k = \{(a_1, \ldots, a_{k+1}) \mid a_i \geq 0, \sum_{i=1}^{k+1} a_i^2 = 1 \} \subseteq \mathbb{R}^{k+1},
\]

with the induced Riemannian metric from \( \mathbb{R}^{k+1} \), and with ordered vertices \( (e_1, \ldots, e_{k+1}) \). Given any singular \( k \)-simplex \( f : \Delta^k \to \widetilde{M} \), with ordered vertices \( V = (x_1, \ldots, x_{k+1}) = (f(e_1), \ldots, f(e_{k+1})) \), we define the \( k \)-straightened simplex

\[
st_k(f) : \Delta^k \to \widetilde{M}
\]

\[
st_k(f)(a_1, \ldots, a_{k+1}) := \text{bar} \left( \sum_{i=1}^{k+1} a_i^2 \nu_i \right)
\]

where \( \nu_i = \mu_{x_i} / \| \mu_{x_i} \| \) is the normalized Patterson-Sullivan measure at \( x_i \). We notice that \( st_k(f) \) is determined by the (ordered) vertex set \( V \), and we denote \( st_k(f)(\delta) \) by \( st_V(\delta) \), for \( \delta \in \Delta^k \).
When \( M \) is rank one, the Patterson-Sullivan measures \( \nu_{x_i} \) are fully supported on \( \partial \tilde{M} \), and so are the linear combinations. Hence by Lemma 2.6, the barycenter map is well-defined, and so is the barycentric straightening.

It is easy to see the barycentric straightening satisfies (a)-(c) of Definition 2.7, the proof simply follows verbatim from the higher rank case [LS06, Property (1)-(3)].

To check (d) that the Jacobian of top dimensional straightened simplices are uniformly bounded on \( U \), we estimate as follows (which is also similar to [LS06, Property (4)]). For any \( \delta = \sum_{i=1}^{n+1} a_i e_i \in \Delta^n_\mathfrak{a} \), \( st_n(f)(\delta) \) is defined to be the unique point where the function

\[
x \mapsto \int_{\partial \tilde{M}} B(x, \theta) d \left( \sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right)(\theta)
\]

is minimized. Hence, by differentiating at that point, we get the 1-form equation

\[
\int_{\partial \tilde{M}} dB_{(st_n(\delta), \theta)}(v) d \left( \sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right)(\theta) = 0
\]

which holds identically on the tangent space \( T_{st_n(\delta)} \tilde{M} \). Differentiating in a direction \( u \in T_{\delta}(\Delta^n_\mathfrak{a}) \) in the source, one obtains the 2-form equation

\[
\sum_{i=1}^{n+1} 2a_i(u, e_i) \int_{\partial \tilde{M}} dB_{(st_n(\delta), \theta)}(v) d \nu_{x_i}(\theta)
\]

\[
+ \int_{\partial \tilde{M}} Dd \left( dB_{(st_n(\delta), \theta)}(D_\delta(st_n)(u), v) d \left( \sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right)(\theta) \right) = 0
\]

which holds for every \( u \in T_{\delta}(\Delta^n_\mathfrak{a}) \) and \( v \in T_{st_n(\delta)} \tilde{M} \).

Now we define two positive semidefinite symmetric endomorphisms \( H_\delta \) and \( K_\delta \) on \( T_{st_n(\delta)} \tilde{M} \):

\[
\langle H_\delta(v), v \rangle_{st_n(\delta)} = \int_{\partial \tilde{M}} dB^2_{(st_n(\delta), \theta)}(v) d \left( \sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right)(\theta)
\]

\[
\langle K_\delta(v), v \rangle_{st_n(\delta)} = \int_{\partial \tilde{M}} Dd \left( dB_{(st_n(\delta), \theta)}(v) d \left( \sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right)(\theta) \right)
\]

Note from Lemma 2.6 that \( K_\delta \) is positive definite. From Equation (2.6), we obtain, for \( u \in T_{\delta}(\Delta^n_\mathfrak{a}) \) a unit vector and \( v \in T_{st_n(\delta)} \tilde{M} \) arbitrary, the following

\[
|\langle K_\delta((D_\delta(st_n)(u)), v) \rangle| = \left| -\sum_{i=1}^{n+1} 2a_i(u, e_i) \int_{\partial \tilde{M}} dB_{(st_n(\delta), \theta)}(v) d \nu_{x_i}(\theta) \right|
\]

\[
\leq \left( \sum_{i=1}^{n+1} \langle u, e_i \rangle^2 \right)^{1/2} \left( \sum_{i=1}^{n+1} 4a_i^2 \left( \int_{\partial \tilde{M}} dB_{(st_n(\delta), \theta)}(v) d \nu_{x_i}(\theta) \right) \right)^{1/2}
\]

\[
\leq 2 \left( \sum_{i=1}^{n+1} a_i^2 \int_{\partial \tilde{M}} dB^2_{(st_n(\delta), \theta)}(v) d \nu_{x_i}(\theta) \right)^{1/2}
\]

\[
= 2 \langle H_\delta(v), v \rangle^{1/2}
\]

via two applications of the Cauchy-Schwartz inequality.
For points $\delta \in \Delta^n$ where $st_V$ is nondegenerate, we now pick orthonormal bases $\{u_1, \ldots, u_n\}$ on $T_\delta(\Delta^n)$, and $\{v_1, \ldots, v_n\}$ on $S \subseteq T_{stV(\delta)}(\tilde{M})$. We choose these so that $\{v_i\}_{i=1}^n$ are eigenvectors of $H_\delta$, and $\{u_1, \ldots, u_n\}$ is the resulting basis obtained by applying the orthonormalization process to the collection of pullback vectors $\{(K_\delta \circ D_\delta(st_V))^{-1}(v_i)\}_{i=1}^n$. By the choice of bases, the matrix $\langle (K_\delta \circ D_\delta(st_V)(u_i), v_j) \rangle$ is upper triangular, so we obtain

$$|\det(K_\delta) \cdot \text{Jac}_\delta(st_V)| = |\det((K_\delta \circ D_\delta(st_V)(u_i), v_j))|$$

$$= \prod_{i=1}^n \langle K_\delta \circ D_\delta(st_V)(u_i), v_i \rangle$$

$$\leq \prod_{i=1}^n 2(H_\delta(v_i), v_i)^{1/2}$$

$$= 2^n \det(H_\delta)^{1/2}$$

where the middle inequality is obtained via Equation (2.7). Hence we get the inequality

$$|\text{Jac}_\delta(st_V)| \leq 2^n \cdot \frac{\det(H_\delta)^{1/2}}{\det(K_\delta)} \tag{2.8}$$

3. Proofs of Thereoms

Proof of Theorem 1.3 The theorem holds automatically when $u(x) \equiv 0$, so we can assume $u(p) > 0$ for some $p \in M$. By Lemma 2.4 and 2.6, the barycentric straightening is well defined on the universal cover $\tilde{M}$. We use a similar idea to the proof of Theorem 2.8 except that we pick a bump function $u^\eta(x)$ explicitly.

Let $\sum_{i=1}^l a_i \sigma_i$ be any homological representative of the fundamental class $[M]$, and $st(\sigma_i)$ be the barycentrically straightened simplex of $\sigma_i$. We have the following estimate

$$\int_M u^\eta(x) dV = \int_{[\sum a_i \sigma_i]} u^\eta(x) dV = \int_{[\sum a_i st(\sigma_i)]} u^\eta(x) dV$$

$$\leq \sum_{i=1}^l |a_i| \cdot \left| \int_{st(\sigma_i)} u^\eta(x) dV \right|$$

$$= \sum_{i=1}^l |a_i| \cdot \left| \int_{\Delta^n} u^\eta(st(\sigma_i)(\delta)) \text{Jac}(st(\sigma_i))(\delta) dV_x \right|$$

We now apply the Jacobian estimate from inequality (2.8),

$$|\text{Jac}(st(\sigma_i))(\delta)| \leq 2^n \cdot \frac{\det(H_\delta)^{1/2}}{\det(K_\delta)} \tag{3.4}$$

where

$$\langle H_\delta(v), v \rangle_x = \int_{\partial^n M} dB^2_{x,\theta}(v) dv(\theta)$$

$$\langle K_\delta(v), v \rangle_x = \int_{\partial^n M} Dd B_{x,\theta}(v, v) dv(\theta)$$

with $x$ the image point of $\delta$ under straightened simplex, and $v$ some probability measure fully supported on $\partial \tilde{M}$.\]
At the point $x$ where $u(x) > 0$, we have matrix forms

$$dB^2_{(x,\theta)} = \begin{bmatrix} 1 & 0 \\ 0 & 0^{(n-1)} \end{bmatrix}$$

$$DdB_{(x,\theta)} \geq u(x) \begin{bmatrix} 0 & 0 \\ 0 & I^{(n-1)} \end{bmatrix}$$

under a proper choice of orthonormal basis $e_1, e_2, ..., e_n$ so that $e_1$ is the unit vector at $x$ pointing toward $\theta$. Therefore, we have for any $(x, \theta)$

$$dB^2_{(x,\theta)} + \frac{1}{u(x)} DdB_{(x,\theta)} \geq I^{(n)}$$

Hence after integrating on a probability measure $\nu$, the following holds.

$$H_\delta + \frac{1}{u(x)} K_\delta \geq I^{(n)}$$

We now apply the following lemma from Besson-Courtois-Gallot [BCG95] on $H_\delta$ and $\frac{1}{u(x)} K_\delta$.

**Lemma 3.1.** Let $H$ and $K$ be two $n \times n$ ($n \geq 3$) matrices, where $K$ is positive definite, and $H$ is positive semidefinite. If $H + K \geq I$ and $\text{tr}(H) = 1$, then

$$\frac{\det(H)^{1/2}}{\det(K)} \leq \frac{n^{n/2}}{(n-1)^n}$$

Thus, we obtain from inequality (3.1)-(3.4) the following

$$\int_M u^n(x) dV \leq 2^n \frac{n^{n/2}}{(n-1)^n} \text{vol}(\Delta^n) \sum_{i=1}^{l} |a_i|$$

$$= \frac{n^{n/2} \omega_n}{2(n-1)^n} \sum_{i=1}^{l} |a_i|$$

where $\omega_n$ is the volume of a unit $n$-sphere.

By taking the infimum over all fundamental class representatives $\sum a_i \sigma_i$, we have

$$\|M\| \geq \frac{2(n-1)^n}{n^{n/2} \omega_n} \int_M u^n(x) dV$$

**Proof of Theorem 1.7.** We show $u(x)$ cannot be identically zero under the assumption of negative Ricci curvature, hence by Corollary 1.5 the simplicial volume is positive.

Assume not, $u(x) = 0$ for all $x \in M$, that is, there exists a higher rank unit vector $v_x$ at every point $x$ on $M$. Since $\text{Ric}(v_x, v_x)$ is negative, $v_x$ can only be rank 2, that is, there exists a unique (up to sign) unit vector $u_x \perp v_x$ that is parallel along the geodesic ray formed by $v_x$. In particular, the plane spanned by $u_x$ and $v_x$ has curvature zero. Note that the zero curvature plane has to be unique among the Grassmanian two planes of $T_xM$, because if not, we can take any unit vector $w$ in the intersection of two zero curvature planes, and it is clear that $\text{Ric}(w, w) = 0$ as $M$ is nonpositively curved, this contradicts with the Ricci curvature condition.

We denote on each tangent space $T_xM$ an orthonormal frame $e_1 = v_x$, $e_2 = u_x$, and $e_3$ orthogonal to the zero curvature plane. We may assume the zero curvature plane field is orientable by possibly passing to the double cover of $M$. This gives a smooth vector field $e_3$ after a preferable choice of normal direction of
the plane field, which at each point corresponds to the unique minimal Ricci curvature direction. (Note that $e_1, e_2$ might not be global vector fields.) We now apply the following Bochner’s technique on $e_3$.

Lemma 3.2. (Bochner [Boc46], see also [Pet06] Chapter 7-Exercise 6) If $X$ is a smooth vector field on a closed manifold $M$, then

$$\int_M \text{Ric}(X, X)dV = \int_M (\text{div } X)^2 - \text{Tr}(\nabla X \circ \nabla X)dV$$

If we set $X = e_3$ in the lemma above, we can compute $\text{div } X$ and $\text{Tr}(\nabla X \circ \nabla X)$ at each point using any choice of orthonormal basis. Hence if we fix a point $x \in M$, and pick an orthonormal frame $e_1, e_2$ and $e_3$ on $T^1M$ as above, we have

$$\nabla_{e_1} e_3 = 0 \quad \nabla_{e_2} e_3 = a_1 e_1 + a_2 e_2 \quad \nabla_{e_3} e_3 = a_3 e_1 + a_3 e_2$$

as $e_1, e_2$ are parallel along $e_1$ geodesic ray, so does their orthogonal complement $e_3$. This implies

$$\text{div } X = \langle \nabla_{e_1} e_3, e_1 \rangle + \langle \nabla_{e_2} e_3, e_2 \rangle + \langle \nabla_{e_3} e_3, e_3 \rangle = a_{22}$$

and

$$\text{Tr}(\nabla X \circ \nabla X) = \langle \nabla_{e_1} e_3, e_1 \rangle + \langle \nabla_{e_2} e_3, e_2 \rangle + \langle \nabla_{e_3} e_3, e_3 \rangle = 0 + a_{22}^2 + 0 = a_{22}^2$$

Hence the integrand of the right hand side of the identity in the above lemma is always 0, but the left hand side is strictly negative. This gives a contradiction. $\square$

Proof of Theorem 1.9. We notice that the Ricci condition is an open condition, that is, there exists $\epsilon_0 > 0$ and an open set $U \subset M$, such that $\text{Ric}_{\frac{\epsilon_0}{4} + 1}(v, v) < -\epsilon_0$ for all $v \in T^1M$, and for all $x \in U$. We apply the following Theorem proved in [CW17].

Theorem 3.3. Suppose $M$ is a closed non-positively curved manifold with negative $(\frac{n}{4} + 1)$-Ricci curvature, and $\tilde{M}$ is its Riemannian universal cover. Let $x \in \tilde{M}$, $\theta \in \partial \tilde{M}$, and $v$ be any probability measure that has full support on $\partial \tilde{M}$. Then there exists a universal constant $C$ that only depends on $(M, g)$, so that

$$\frac{\det(\int_{\partial M} dB^2_{\ell, v}(\cdot, \cdot)dv(\theta))^{1/3}}{\det(\int_{\partial M} DdB_{\ell, \theta}(\cdot, \cdot)dv(\theta))} \leq C$$

Moreover, from the proof we see that the above inequality is a pointwise estimate, that is, $C$ only depends on the dimension $n$, the norm of curvature operators $\|R\|, \|\nabla R\|, \|\nabla^2 R\|$, and the corresponding $(\frac{n}{4} + 1)$-Ricci constant at $x$. Therefore, we see the barycentric straightening is a straightening subordinated to $U$ under the curvature assumptions: the Jacobian estimate simply follows from inequality (2.8) and the theorem above. We have the following inequalities:

$$|\text{Jac}(st_n(f))(\delta)| \leq 2^n \frac{\det(\int_{\partial M} dB^2_{\ell, v}(\cdot, \cdot)dv(\theta))^{1/3}}{\det(\int_{\partial M} DdB_{\ell, \theta}(\cdot, \cdot)dv(\theta))} \leq 2^n C$$

whenever $st_n(f)(\delta) \in p^{-1}(U)$. Hence by Theorem 2.8 the simplicial volume of $M$ is positive. $\square$
4. Concluding remarks

In addition to their independent interest, both Corollary 1.5 and Theorem 1.9 represent progress towards Gromov’s conjecture, where the former addresses conditions on the rank and the latter does on the curvature. The two conditions do not have a direct implication between them. Indeed, a rank one vector $v$ may have zero Ricci curvature, hence $\text{Ric}_k$ is also zero at $v$. On the other hand, a higher rank vector $v$ only sees a dimension two zero curvature plane through it, which is strictly weaker than $\text{Ric}_k(v,v) = 0$ when $k > 2$.

To summarize, if we denote by $M^n$ the collection of all $n$-dimensional rank one manifolds that are not covered by our theorems (the higher rank case is clear by Theorem 2.2 and [LS06]), then we have for every $M \in M^n$, $M$ satisfies:

(a) $M$ has nonpositive curvature, negative Ricci curvature, and is rank one.
(b) Every point on $M$ has a higher rank vector.
(c) Every point on $M$ has a vector that has $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} = 0$.

It is worth pointing out that we are not aware of any such examples. Therefore one possible approach to verify Gromov’s conjecture is showing $M^n = \emptyset$. In fact, in Theorem 1.7, we showed $M^3 = \emptyset$ by contradicting (a) with (b). However, our proof relies heavily on the low dimensional constraints that at each point the higher rank vectors give unique zero curvature planes. In higher dimensions, we believe that the dynamical properties of the geodesic flow of rank one manifolds (such as the distribution of regular/singular points at boundary at infinity) hold the key to providing a complete answer to this problem.

We end the discussion by giving an alternate direct proof of Gromov’s Conjecture in dimension three using Perelman’s Geometrization Theorem. More generally, we have,

**Theorem 4.1.** Let $M$ be a closed 3-manifold admitting a metric of non-positive curvature with a point $x \in M$ of negative Ricci-curvature. Then $M$ has positive simplicial volume.

**Proof.** Theorem 0.4 of [CCR04] implies under the hypotheses that $\text{MinVol}(M) > 0$. It is known to experts that $\|M\| = 0$ if and only if $M$ is a graph-manifold in the sense of Waldhausen, and that these have $\text{MinVol}(M) = 0$. For completeness sake, we supply a proof below.

We begin by noting that $M$ must be prime in the sense of Milnor under the connected sum decomposition. Indeed, any incompressible 2-sphere will give a nontrivial element of $\pi_2(M)$, whereas nonpositively curved manifolds are aspherical. For the same reason the resulting prime manifold cannot be $S^1 \times S^2$ or $S^1 \wr S^2$, i.e. $M$ is irreducible.

Now we turn to the JSJ decomposition of $M$. Namely, $M$ is formed by gluing together geometric manifolds $M_i$ (termed “pieces”) with toral quotient boundary components together via diffeomorphisms isotopic to affine maps. Since the connecting $T^2$ (or finite quotients thereof) are amenable, we may apply Gromov’s original cut and paste arguments in [Gro82] (or see Theorem 2 of [CS17] for a direct statement) to show that $\|M\| = \sum \|M, \partial M\|$.

Now we recall the eight geometries: $\mathbb{H}^3, \mathbb{R}^3, S^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times R, \text{Nil, Solv, SL}_2(\mathbb{R})$. Six of the eight geometries are aspherical. Of these, only the hyperbolic manifolds do not admit a positive rank polarized-$F$ structures relative to some boundary $F$-structure on the $T^2$ boundary components. The latter statement follows from the fact that four of the remaining five geometries are Seifert-fibered whose local circle fibrations provide the polarized $F$-structure. Compact manifolds of the last geometry Solv have finite covers which are torus bundles over $S^1$ and therefore have a polarized $F$-structure as well.
In summary, if $M_i$ is a hyperbolic piece then $\|M_i, \partial M_i\| > 0$ and hence $\|M\| > 0$. Conversely, if there are no hyperbolic pieces then $M$ admits a polarized $F$-structure and hence $\minVol(M) = 0$. □

Motivated by these theorems, we offer the strengthened version of Gromov’s conjecture.

**Conjecture 4.2.** Let $M$ be a closed manifold admitting a metric of non-positive curvature with a point $x \in M$ of negative Ricci-curvature. Then $M$ has positive simplicial volume.

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