Random Attractors for the Stochastic Benjamin-Bona-Mahony Equation on Unbounded Domains

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Abstract

We prove the existence of a compact random attractor for the stochastic Benjamin-Bona-Mahony Equation defined on an unbounded domain. This random attractor is invariant and attracts every pulled-back tempered random set under the forward flow. The asymptotic compactness of the random dynamical system is established by a tail-estimates method, which shows that the solutions are uniformly asymptotically small when space and time variables approach infinity.

Key words. Stochastic Benjamin-Bona-Mahony Equation, random attractor, pullback attractor, asymptotic compactness.

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1 Introduction

This paper is concerned with the asymptotic behavior of solutions of the stochastic Benjamin-Bona-Mahony (BBM) equation on an unbounded three-dimensional channel. Let $Q = D \times \mathbb{R}$ where $D$ is a bounded open subset of $\mathbb{R}^2$. Consider the following BBM equation on $Q$:

$$
du - d(\Delta u) - \nu \Delta u dt + \nabla \cdot \overrightarrow{F}(u) dt = gd t + h dw, \quad x \in Q, \quad t > 0,
$$

where $\nu$ is a positive constant, $g = g(x)$ and $h = h(x)$ are given functions defined on $Q$, $\overrightarrow{F}$ is a smooth nonlinear vector function, and $w$ is a two-sided real-valued Wiener process on a probability space which will be specified later.

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Stochastic differential equations arise from many physical systems when random uncertainties are taken into account. The long-term behavior of random systems is captured by a pullback random attractor. This concept was introduced in [21, 22] as an extension to stochastic systems of the attractors theory of deterministic equations in [8, 26, 31, 37, 40]. The existence of random attractors has been studied extensively by many authors, see, e.g., [3, 11, 16, 20, 21, 22] and the references therein. Notice that the partial differential equations (PDEs) studied in these papers are all defined in bounded domains. In the case of unbounded domains, the existence of random attractors for PDEs is not well understood yet. In this case, as far as we know, the existence of random attractors is only established for the Reaction-Diffusion equation on $\mathbb{R}^n$ in [12] recently. In this paper, we will investigate the existence of a compact random attractor for the stochastic BBM equation defined on the unbounded channel $Q$.

Notice that Sobolev embeddings are no longer compact for the unbounded domain $Q$. This introduces a major obstacle for proving the existence of attractors for the Benjamin-Bona-Mahony equation defined on $Q$. For some deterministic equations, the difficulty caused by the unboundedness of domains may be overcome by the energy equation approach or by the tail-estimates approach. The energy equation method was developed by Ball in [9, 10] and used by many authors (see, e.g., [23, 25, 27, 31, 32, 36, 45]). The tail-estimates approach was developed in [11] for deterministic PDEs and used in [11, 2, 1, 30, 33, 35, 38, 39]. In this paper, we will develop a tail-estimates approach for weakly dissipative stochastic PDEs like the Benjamin-Bona-Mahony equation and prove the existence of compact random attractors on unbounded domains. The idea is based on the observation that the solutions of the equation are uniformly small when space and time variables are sufficiently large.

We mention that the Benjamin-Bona-Mahony equation was proposed in [13] as a model for propagation of long waves which incorporates nonlinear dispersive and dissipative effects. In the deterministic case, the existence and uniqueness of solutions were studied in [6, 7, 13, 14, 15, 18, 24, 28, 29], and the global attractors were investigated in [5, 17, 19, 38, 42, 43, 44]. In order to deal with the stochastic Benjamin-Bona-Mahony equation, we need to transform the stochastic equation with a random term into a deterministic one with a random parameter. This transformation will change the structure of the original equation and hence cause some extra difficulties for deriving uniform
estimates on solutions, especially on the tails of solutions for large space and time variables. For instance, if we take the inner product of the deterministic BBM equation with $u$ in $L^2(Q)$, then the nonlinear term disappears, and hence the uniform estimates in this case are not hard to get. However, after transformation, this property is lost and the nonlinear term does not disappear when performing energy estimates. This is the reason why much effort of this paper is devoted to deriving the uniform estimates on the tails of solutions (see Section 4 for details).

This paper is organized as follows. In the next section, we review the pullback random attractors theory for random dynamical systems. In Section 3, we define a continuous random dynamical system for the stochastic Benjamin-Bona-Mahony equation on $Q$. Then we derive the uniform estimates of solutions in Section 4, which include the uniform estimates on the tails of solutions. Finally, in Section 5, we establish the asymptotic compactness of the random dynamical system and prove the existence of a pullback random attractor.

In the sequel, we adopt the following notations. We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and the inner product of $L^2(Q)$, respectively. The norm of a given Banach space $X$ is written as $\| \cdot \|_X$. We also use $\| \cdot \|_p$ to denote the norm of $L^p(Q)$. The letters $c$ and $c_i$ ($i = 1, 2, \ldots$) are generic positive constants which may change their values from line to line or even in the same line.

Throughout this paper, we will frequently use the embedding inequality

$$\| u \|_\infty \leq \beta_0 \| u \|_{H^2(Q)}, \quad \forall \ u \in H^2(Q), \quad (1.2)$$

and the Poincare inequality

$$\| \nabla u \|^2 \geq \lambda \| u \|^2, \quad \forall \ u \in H^1_0(Q), \quad (1.3)$$

where $\beta_0$ and $\lambda$ are positive constants.

2 Preliminaries

In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems. The reader is referred to [3, 11, 20, 22] for more details.

Let $(X, \| \cdot \|_X)$ be a separable Hilbert space with Borel $\sigma$-algebra $B(X)$, and $(\Omega, \mathcal{F}, P)$ be a probability space.
Definition 2.1. \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\) is called a metric dynamical system if \(\theta : \mathbb{R} \times \Omega \to \Omega\) is \((\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})\)-measurable, \(\theta_0\) is the identity on \(\Omega\), \(\theta_{s+t} = \theta_t \circ \theta_s\) for all \(s, t \in \mathbb{R}\) and \(\theta_t P = P\) for all \(t \in \mathbb{R}\).

Definition 2.2. A continuous random dynamical system (RDS) on \(X\) over a metric dynamical system \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\) is a mapping
\[
\Phi : \mathbb{R}^+ \times \Omega \times X \to X \quad (t, \omega, x) \mapsto \Phi(t, \omega, x),
\]
which is \((\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))\)-measurable and satisfies, for \(P\)-a.e. \(\omega \in \Omega\),

(i) \(\Phi(0, \omega, \cdot)\) is the identity on \(X\);

(ii) \(\Phi(t+s, \omega, \cdot) = \Phi(t, \theta_s \omega, \cdot) \circ \Phi(s, \omega, \cdot)\) for all \(t, s \in \mathbb{R}^+\);

(iii) \(\Phi(t, \omega, \cdot) : X \to X\) is continuous for all \(t \in \mathbb{R}^+\).

Hereafter, we always assume that \(\Phi\) is a continuous RDS on \(X\) over \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\).

Definition 2.3. A random bounded set \(\{B(\omega)\}_{\omega \in \Omega}\) of \(X\) is called tempered with respect to \((\theta_t)_{t \in \mathbb{R}}\) if for \(P\)-a.e. \(\omega \in \Omega\),
\[
\lim_{t \to \infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \beta > 0,
\]
where \(d(B) = \sup_{x \in B} \|x\|_X\).

Definition 2.4. Let \(\mathcal{D}\) be a collection of random subsets of \(X\). Then \(\mathcal{D}\) is called inclusion-closed if \(D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\) and \(\tilde{D} = \{\tilde{D}(\omega) \subseteq X : \omega \in \Omega\}\) with \(\tilde{D}(\omega) \subseteq D(\omega)\) for all \(\omega \in \Omega\) imply that \(\tilde{D} \in \mathcal{D}\).

Definition 2.5. Let \(\mathcal{D}\) be a collection of random subsets of \(X\) and \(\{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\). Then \(\{K(\omega)\}_{\omega \in \Omega}\) is called an absorbing set of \(\Phi\) in \(\mathcal{D}\) if for every \(B \in \mathcal{D}\) and \(P\)-a.e. \(\omega \in \Omega\), there exists \(t_B(\omega) > 0\) such that
\[
\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq t_B(\omega).
\]

Definition 2.6. Let \(\mathcal{D}\) be a collection of random subsets of \(X\). Then \(\Phi\) is said to be \(\mathcal{D}\)-pullback asymptotically compact in \(X\) if for \(P\)-a.e. \(\omega \in \Omega\), \(\{\Phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}\) has a convergent subsequence in \(X\) whenever \(t_n \to \infty\), and \(x_n \in B(\theta_{-t_n}\omega)\) with \(\{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\).
Definition 2.7. Let $\mathcal{D}$ be a collection of random subsets of $X$ and $\{A(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then $\{A(\omega)\}_{\omega \in \Omega}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for $\Phi$ if the following conditions are satisfied, for $P$-a.e. $\omega \in \Omega$,

(i) $A(\omega)$ is compact, and $\omega \mapsto d(x, A(\omega))$ is measurable for every $x \in X$;

(ii) $\{A(\omega)\}_{\omega \in \Omega}$ is invariant, that is,

$$\Phi(t, \omega, A(\omega)) = A(\theta_t \omega), \quad \forall t \geq 0;$$

(iii) $\{A(\omega)\}_{\omega \in \Omega}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$,

$$\lim_{t \to \infty} d(\Phi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)), A(\omega)) = 0,$$

where $d$ is the Hausdorff semi-metric given by $d(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} \|y - z\|_X$ for any $Y \subseteq X$ and $Z \subseteq X$.

The following existence result on a random attractor for a continuous RDS can be found in [11, 22].

Proposition 2.8. Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of $X$ and $\Phi$ a continuous RDS on $X$ over $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. Suppose that $\{K(\omega)\}_{\omega \in \mathcal{K}}$ is a closed absorbing set of $\Phi$ in $\mathcal{D}$ and $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $X$. Then $\Phi$ has a unique $\mathcal{D}$-random attractor $\{A(\omega)\}_{\omega \in \Omega}$ which is given by

$$A(\omega) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \Phi(t, \theta_{-t} \omega, K(\theta_{-t} \omega)).$$

In this paper, we will determine a collection of random subsets for the stochastic Benjamin-Bona-Mahony equation on $Q$, and prove the equation has a $\mathcal{D}$-random attractor in $H^1_0(Q)$.

3 Stochastic Benjamin-Bona-Mahony equations

In this section, we discuss the existence of a continuous random dynamical system for the stochastic Benjamin-Bona-Mahony equation defined on an unbounded channel. Let $D$ be a bounded open subset of $\mathbb{R}^2$ and $Q = D \times \mathbb{R}$. Consider the stochastic BBM equation defined on $Q$:

$$du - d(\Delta u) - \nu \Delta u dt + \nabla \cdot \overrightarrow{F}(u) dt = g dt + hdw, \quad x \in Q, \quad t > 0,$$  

where $\overrightarrow{F}(u) = F_x(u) = u - u^3$, $\nu > 0$, $\Delta$ is the Laplacian, $g$ is a bounded function, and $h$ is a white noise. The equation is known as the stochastic BBM equation, and its solutions form a continuous random dynamical system. The existence of a random attractor for this equation can be proven using the existence result from Proposition 2.8.
with the boundary condition
\[ u|_{\partial Q} = 0, \quad (3.2) \]
and the initial condition
\[ u(x, 0) = u_0(x), \quad x \in Q, \quad (3.3) \]
where \( \nu \) is a positive constant, \( g \in L^2(Q) \) and \( h \in H^1_0(Q) \) are given, \( w \) is a two-sided real-valued Wiener process on a probability space which will be specified below, and \( \vec{F} \) is a smooth nonlinear vector function given by \( \vec{F}(s) = (F_1(s), F_2(s), F_3(s)) \) for \( s \in \mathbb{R} \), where \( F_k \) \((k = 1, 2, 3)\) satisfies
\[ F_k(0) = 0, \quad |F_k'(s)| \leq \gamma_1 + \gamma_2|s|, \quad s \in \mathbb{R}, \quad (3.4) \]
where \( \gamma_1 \) and \( \gamma_2 \) are positive constants. Denote by
\[ G_k(s) = \int_0^s F_k(t)dt \quad \text{and} \quad \vec{G}(s) = (G_1(s), G_2(s), G_3(s)), \quad s \in \mathbb{R}. \quad (3.5) \]
Then it follows from (3.4) that, for \( k = 1, 2, 3, \)
\[ |F_k(s)| \leq \gamma_1|s| + \gamma_2|s|^2 \quad \text{and} \quad |G_k(s)| \leq \gamma_1|s|^2 + \gamma_2|s|^3. \quad (3.6) \]

Note that the classical Benjamin-Bona-Mahony equation with \( F_k(s) = s + \frac{1}{2}s^2 \) indeed satisfies condition (3.4). Let \( \beta_0 \) and \( \lambda \) be the positive constants in (1.2) and (1.3), respectively, and denote by
\[ \delta = \min \{ \nu, \frac{1}{4} \nu \lambda \} \quad \text{and} \quad \beta = 4\beta_0 \gamma_2 \|h\|_{H^1}. \quad (3.7) \]
Then choose a sufficiently large number \( \alpha \) such that
\[ \alpha > \frac{128 \beta^2}{\delta^2}. \quad (3.8) \]
As we will see later, these constants \( \alpha, \beta \) and \( \delta \) prove useful when deriving uniform estimates on the solutions.

In the sequel, we consider the probability space \((\Omega, \mathcal{F}, P)\) where
\[ \Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}, \]
\( \mathcal{F} \) is the Borel \( \sigma \)-algebra induced by the compact-open topology of \( \Omega \), and \( P \) the corresponding Wiener measure on \((\Omega, \mathcal{F})\). Define the time shift by
\[ \theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}. \]
Then \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\) is a metric dynamical system. For our purpose, we need to convert the stochastic equation (3.1) with a random term into a deterministic one with a random parameter. To this end, we consider the stationary solutions of the one-dimensional equation:

\[
dy + \alpha y dt = dw(t),
\]

where \(\alpha\) satisfies (3.8). The solution to (3.9) is given by

\[
y(\theta_t \omega) = -\alpha \int_{-\infty}^{0} e^{\alpha \tau} (\theta_t \omega)(\tau) d\tau, \quad t \in \mathbb{R}.
\]

It is known that there exists a \(\theta_t\)-invariant set \(\tilde{\Omega} \subseteq \Omega\) of full \(P\) measure such that \(y(\theta_t \omega)\) is continuous in \(t\) for every \(\omega \in \tilde{\Omega}\), and the random variable \(|y(\omega)|\) is tempered (see, e.g., [3, 11, 20, 21]).

Put \(z(\theta_t \omega) = (I - \Delta)^{-1} hy(\theta_t \omega)\) where \(\Delta\) is the Laplacian with domain \(H^1_0(Q) \cap H^2(Q)\). By (3.9) we find that

\[
dz - d(\Delta z) + \alpha(z - \Delta z) dt = hdw.
\]

Let \(v(t, \omega) = u(t, \omega) - z(\theta_t \omega)\), where \(u(t, \omega)\) satisfies (3.1)-(3.3). Then for \(v(t, \omega)\) we have that

\[
v_t - \Delta v_t - \nu \Delta v = -\nabla \cdot \vec{F}(v + z(\theta_t \omega)) + g + \alpha z(\theta_t \omega) + (\nu - \alpha) \Delta z(\theta_t \omega),
\]

with the boundary condition

\[
v|_{\partial Q} = 0,
\]

and the initial condition

\[
v(0, \omega) = v_0(\omega).
\]

By a Galerkin method as in [28, 29], it can be proved that under the assumption (3.4), for \(P\)-a.e. \(\omega \in \Omega\) and for all \(v_0 \in H^1_0(Q)\), problem (3.10)-(3.12) has a unique solution \(v(\cdot, \omega, v_0) \in C([0, \infty), H^1_0(Q))\) with \(v(0, \omega, v_0) = v_0\). Further, the solution \(v(t, \omega, v_0)\) is continuous with respect to \(v_0\) in \(H^1_0(Q)\) for all \(t \geq 0\). Throughout this paper, we always write

\[
u(t, \omega, u_0) = v(t, \omega, v_0) + z(\theta_t \omega), \quad \text{with} \quad v_0 = u_0 - z(\omega).
\]

Then \(u\) is a solution of problem (3.1)-(3.3) in some sense. We now define a mapping \(\Phi : \mathbb{R}^+ \times \Omega \times H^1_0(Q) \rightarrow H^1_0(Q)\) by

\[
\Phi(t, \omega, u_0) = u(t, \omega, u_0), \quad \forall (t, \omega, u_0) \in \mathbb{R}^+ \times \Omega \times H^1_0(Q).
\]
Note that Φ satisfies conditions (i), (ii) and (iii) in Definition 2.2. Therefore, Φ is a continuous random dynamical system associated with the stochastic Benjamin-Bona-Mahony equation on Q. In what follows, we will prove that Φ has a D-random attractor in \( H^1_0(Q) \), where D is a collection of random subsets of \( H^1_0(Q) \) given by

\[
D = \{ B : B = \{ B(\omega) \}_{\omega \in \Omega}, B(\omega) \subseteq H^1_0(Q) \text{ and } e^{-\frac{1}{8} \delta t} d(B(\theta_t \omega)) \to 0 \text{ as } t \to \infty \},
\]

where \( \delta \) is the positive constant in (3.7) and

\[
d(B(\theta_t \omega)) = \sup_{u \in B(\theta_t \omega)} \| u \|_{H^1_0(Q)}.
\]

Notice that D contains all tempered random sets, especially all bounded deterministic subsets of \( H^1_0(Q) \).

### 4 Uniform estimates

In this section, we derive uniform estimates on the solutions of the stochastic Benjamin-Bona-Mahony equation defined on Q when \( t \to \infty \), which include the uniform estimates on the tails of solutions as both \( x \) and \( t \) approach infinity. These estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the random dynamical system.

From now on, we always assume that D is the collection of random subsets of \( H^1_0(Q) \) given by (3.15). We first derive the following uniform estimates on \( v \) in \( H^1_0(Q) \).

**Lemma 4.1.** Assume that \( g \in L^2(Q) \), \( h \in H^1_0(Q) \) and (3.1) holds. Let \( B = \{ B(\omega) \}_{\omega \in \Omega} \in D \) and \( v_0(\omega) \in B(\omega) \). Then for \( P \)-a.e. \( \omega \in \Omega \), there is \( T = T(B, \omega) > 0 \) such that for all \( t \geq T \),

\[
\| v(t, \theta_t \omega, v_0(\theta_t \omega)) \|_{H^1_0(Q)} \leq r_1(\omega),
\]

where \( r_1(\omega) \) is a positive random function satisfying

\[
e^{-\frac{1}{8} \delta t} r_1(\theta_t \omega) \to 0 \text{ as } t \to \infty.
\]

**Proof.** Taking the inner product of (3.10) with \( v \) in \( L^2(Q) \) we find that

\[
\frac{1}{2} \frac{d}{dt} (\| v \|^2 + \| \nabla v \|^2) + \nu \| \nabla v \|^2 = -\int_Q v \nabla \cdot \overrightarrow{F}(v + z(\theta_t \omega)) dx
\]
By the Young inequality, the second term on the right-hand side of (4.2) is bounded by
\[ (g + \alpha z(\theta t \omega) + (\nu - \alpha) \Delta z(\theta t \omega), v). \] (4.2)

By (3.5) we have \( \nabla \cdot \overrightarrow{G}(u) = \overrightarrow{F}(u) \cdot \nabla u \) and hence, by (1.2) and (3.6), the nonlinear term on the right-hand side of (4.2) satisfies
\[
- \int_Q v \nabla \cdot \overrightarrow{F}(v + z(\theta t \omega)) \, dx = - \int_Q (u - z(\theta t \omega)) \nabla \cdot \overrightarrow{F}(v + z(\theta t \omega)) \, dx \\
= - \int_Q u \nabla \cdot \overrightarrow{F}(u) \, dx + \int_Q z(\theta t \omega) \nabla \cdot \overrightarrow{F}(v + z(\theta t \omega)) \, dx \\
= \int_Q \overrightarrow{F}(u) \cdot \nabla u \, dx + \int_Q z(\theta t \omega) \nabla \cdot \overrightarrow{F}(v + z(\theta t \omega)) \, dx \\
= \int_Q \nabla \cdot \overrightarrow{G}(u) \, dx + \int_Q z(\theta t \omega) \nabla \cdot \overrightarrow{F}(v + z(\theta t \omega)) \, dx \\
\leq \gamma_1 \int_Q |v + z(\theta t \omega)| |\nabla z(\theta t \omega)| \, dx + \gamma_2 \int_Q |v + z(\theta t \omega)|^2 |\nabla z(\theta t \omega)| \, dx \\
\leq \gamma_1 (||v|| + ||z(\theta t \omega)||) ||\nabla z(\theta t \omega)|| + 2\gamma_2 \int_Q |v|^2 |\nabla z(\theta t \omega)| \, dx + 2\gamma_2 \int_Q |z(\theta t \omega)|^2 |\nabla z(\theta t \omega)| \, dx \\
\leq \frac{1}{8} \nu \lambda ||v||^2 + c_1 |y(\theta t \omega)|^2 + 2\gamma_2 ||\nabla z(\theta t \omega)||_\infty ||v||^2 + 2\gamma_2 ||z(\theta t \omega)||_2^2 ||\nabla z(\theta t \omega)|| \\
\leq \frac{1}{8} \nu \lambda ||v||^2 + c_1 |y(\theta t \omega)|^2 + 2\gamma_2 |\delta_0| ||\nabla z(\theta t \omega)||_{H^2} ||v||^2 + c_2 |y(\theta t \omega)|^3 \\
\leq \frac{1}{8} \nu \lambda ||v||^2 + c_1 |y(\theta t \omega)|^2 + 2\gamma_2 |\delta_0| ||y(\theta t \omega)||_{H^1} ||y(\theta t \omega)||_1 ||v||^2 + c_2 |y(\theta t \omega)|^3. \] (4.3)

By the Young inequality, the second term on the right-hand side of (1.2) is bounded by
\[
||(g + \alpha z(\theta t \omega) + (\nu - \alpha) \Delta z(\theta t \omega), v) \leq \frac{1}{8} \nu \lambda ||v||^2 + c_3 (||g||^2 + ||z(\theta t \omega)||^2 + ||\Delta z(\theta t \omega)||^2) \\
\leq \frac{1}{8} \nu \lambda ||v||^2 + c_4 (1 + |y(\theta t \omega)|^2). \] (4.4)

It follows from (4.2)-(4.3) that
\[
\frac{d}{dt} ||v||_{H^1}^2 + 2\nu ||\nabla v||^2 \leq \frac{1}{2} \nu \lambda ||v||^2 + 4\beta_0 \gamma_2 ||y(\theta t \omega)||_{H^1} ||v||^2 + c_1 (1 + |y(\theta t \omega)|^2 + |y(\theta t \omega)|^3). \] (4.5)

By (4.3) we have that
\[
2\nu ||\nabla v||^2 \geq \nu ||\nabla v||^2 + \nu \lambda ||v||^2. \] (4.6)
By (4.5) and (4.6) we get
\[\frac{d}{dt} \|v\|_{H^1}^2 + \nu \|\nabla v\|^2 + \frac{1}{2} \nu \chi \|v\|^2 \leq 4 \beta_0 \gamma_0 \|h\|_{H^1} |y(\theta_t \omega)| \|v\| + c(1 + |y(\theta_t \omega)|^2 + |y(\theta_t \omega)|^3),\]
which along with (3.7) implies that
\[\frac{d}{dt} \|v\|^2_{H^1} + (\delta - \beta |y(\theta_t \omega)|) \|v\|^2_{H^1} \leq c(1 + |y(\theta_t \omega)|^2 + |y(\theta_t \omega)|^3). \tag{4.7}\]

Multiplying (4.7) by \(e^{\int_0^t (\delta - \beta |y(\theta_r \omega)|) \, dr}\) and then integrating over \((0, s)\) with \(s \geq 0\), we obtain that
\[\|v(s, \omega, v_0(\omega))\|^2_{H^1} \leq e^{-\delta s + \beta \int_0^t |y(\theta_r \omega)| \, dr} \|v_0(\omega)\|^2_{H^1} + c \int_0^s (1 + |y(\theta_\sigma \omega)|^2 + |y(\theta_\sigma \omega)|^3) e^{\delta (\sigma - s) + \beta \int_s^\sigma |y(\theta_r \omega)| \, dr} \, d\sigma. \tag{4.8}\]

We now replace \(\omega\) by \(\theta_{-t} \omega\) with \(t \geq 0\) in (4.8) to get that, for any \(s \geq 0\) and \(t \geq 0\),
\[\|v(s, \theta_{-t} \omega, v_0(\theta_{-t} \omega))\|^2_{H^1} \leq e^{-\delta s + \beta \int_0^t |y(\theta_{-r} \omega)| \, dr} \|v_0(\theta_{-t} \omega)\|^2_{H^1} + c \int_0^{s-t} (1 + |y(\theta_\sigma \omega)|^2 + |y(\theta_\sigma \omega)|^3) e^{\delta (\sigma - s + t) + \beta \int_\sigma^{s-t} |y(\theta_r \omega)| \, dr} \, d\sigma. \tag{4.9}\]

By (4.9) we find that, for all \(t \geq 0\),
\[\|v(t, \theta_{-t} \omega, v_0(\theta_{-t} \omega))\|^2_{H^1} \leq e^{-\delta t + \beta \int_{-t}^0 |y(\theta_r \omega)| \, dr} \|v_0(\theta_{-t} \omega)\|^2_{H^1} + c \int_{-t}^0 (1 + |y(\theta_\sigma \omega)|^2 + |y(\theta_\sigma \omega)|^3) e^{\delta (\sigma + t) + \beta \int_{\sigma}^0 |y(\theta_r \omega)| \, dr} \, d\sigma. \tag{4.10}\]

Note that \(|y(\theta_r \omega)|\) is stationary and ergodic (see, e.g. [20]). Then it follows from the ergodic theorem that
\[\lim_{t \to \infty} \frac{1}{t} \int_{-t}^0 |y(\theta_r \omega)| \, d\tau = E(|y(\omega)|).\]

On the other hand, we have
\[E(|y(\omega)|) \leq \left( E(|y(\omega)|^2) \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2\alpha}},\]
which shows that
\[\lim_{t \to \infty} \frac{1}{t} \int_{-t}^0 |y(\theta_r \omega)| \, d\tau < \frac{2}{\sqrt{2\alpha}}. \tag{4.11}\]
By (3.8) and (4.11) we find that there is $T_0(\omega) > 0$ such that for all $t \geq T_0(\omega)$,

$$\beta \int_{-t}^{0} |y(\theta_\tau)\omega)| d\tau < \frac{2\beta t}{\sqrt{2\alpha}} < \frac{1}{8} \delta t.$$  \hspace{2cm} (4.12)

By (4.10) and (4.12) we find that, for all $t \geq T_0(\omega)$,

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{H^1}^2 \leq e^{-\frac{1}{8}\delta t} \|v_0(\theta_{-t}\omega)\|_{H^1}^2 + c \int_{-t}^{0} (1 + |y(\theta_\tau\omega)|^2 + |y(\theta_\tau\omega)|^3)e^{\delta \sigma + \beta \int_{\sigma}^{0} |y(\theta_\tau\omega)| d\tau} d\sigma.$$

Note that $|y(\theta_\sigma\omega)|$ is tempered, and hence by (4.12), the integrand of the second term on the right-hand side of (4.13) is convergent to zero exponentially as $\sigma \to -\infty$. This shows that the following integral is convergent:

$$r_0(\omega) = c \int_{-\infty}^{0} (1 + |y(\theta_\sigma\omega)|^2 + |y(\theta_\sigma\omega)|^3)e^{\delta \sigma + \beta \int_{\sigma}^{0} |y(\theta_\tau\omega)| d\tau} d\sigma.$$

It follows from (4.13)-(4.14) that, for all $t \geq T_0(\omega)$,

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{H^1}^2 \leq e^{-\frac{1}{8}\delta t} \|v_0(\theta_{-t}\omega)\|_{H^1}^2 + r_0(\omega).$$  \hspace{2cm} (4.15)

On the other hand, by assumption, $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and hence we have

$$e^{-\frac{1}{8}\delta t} \|v_0(\theta_{-t}\omega)\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty,$$

from which and (4.15) we find that there is $T = T(B, \omega) > 0$ such that for all $t \geq T$,

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{H^1}^2 \leq 2r_0(\omega).$$

Let $r_1(\omega) = \sqrt{2r_0(\omega)}$. Then we get that, for all $t \geq T$,

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{H^1} \leq r_1(\omega).$$  \hspace{2cm} (4.16)

Next, we prove $r_1(\omega)$ satisfies (4.11). Replacing $\omega$ by $\theta_{-t}\omega$ in (4.13) we obtain that

$$r_0(\theta_{-t}\omega) = c \int_{-\infty}^{0} (1 + |y(\theta_\sigma\omega)|^2 + |y(\theta_\sigma\omega)|^3)e^{\delta \sigma + \beta \int_{\sigma}^{0} |y(\theta_\tau\omega)| d\tau} d\sigma$$

$$= c \int_{-\infty}^{-t} (1 + |y(\theta_\sigma\omega)|^2 + |y(\theta_\sigma\omega)|^3)e^{\delta (\sigma + t) + \beta \int_{\sigma}^{0} |y(\theta_\tau\omega)| d\tau} d\sigma.$$
\[
\leq c \int_{-\infty}^{-t} (1 + |y(\theta_{s}\omega)|^2 + |y(\theta_{s}\omega)|^3)e^{\frac{4}{3}\delta(\sigma+t)+\beta \int_{\sigma}^{t}|y(\theta_{s}\omega)|d\sigma}d\sigma.
\]
\[
\leq ce^{\frac{4}{3}\delta t} \int_{-\infty}^{-t} (1 + |y(\theta_{s}\omega)|^2 + |y(\theta_{s}\omega)|^3)e^{\frac{4}{3}\delta\sigma+\beta \int_{\sigma}^{0}|y(\theta_{s}\omega)|d\sigma}d\sigma.
\]
\[
\leq ce^{\frac{4}{3}\delta t} \int_{0}^{t} (1 + |y(\theta_{s}\omega)|^2 + |y(\theta_{s}\omega)|^3)e^{\frac{4}{3}\delta\sigma+\beta \int_{\sigma}^{0}|y(\theta_{s}\omega)|d\sigma}d\sigma. \tag{4.17}
\]
Note that the last integral in the above is indeed convergent since the integrand converges to zero exponentially by (4.12). Then we have
\[
e^{-\frac{1}{4}\delta t}r_1(\theta_{-t}\omega) = e^{-\frac{1}{4}\delta t}\sqrt{2r_0(\theta_{-t}\omega)}
\]
\[
\leq \sqrt{2}ce^{-\frac{1}{4}\delta t}\left( \int_{-\infty}^{t} (1 + |y(\theta_{s}\omega)|^2 + |y(\theta_{s}\omega)|^3)e^{\frac{4}{3}\delta\sigma+\beta \int_{\sigma}^{0}|y(\theta_{s}\omega)|d\sigma}d\sigma \right)^\frac{1}{2} \to 0, \quad \text{as } t \to \infty,
\]
which along with (4.10) completes the proof. \hfill \Box

**Lemma 4.2.** Assume that \(g \in L^2(Q), \ h \in H^1_0(Q)\) and (3.4) holds. Let \(B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\) and \(v_0(\omega) \in B(\omega)\). Then for \(P\text{-a.e. } \omega \in \Omega\), every \(s \geq 0\) and \(t \geq 0\), we have
\[
\|v_s(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{H^1}^2 \leq c + ce^{-2\delta s+2\beta \int_{s-t}^{t}|y(\theta_{s}\omega)|d\sigma}\|v_0(\theta_{-t}\omega)\|_{H^1}^2
\]
\[
+ c \left( \int_{-t}^{s-t} (1 + |y(\theta_{s}\omega)|^2 + |y(\theta_{s}\omega)|^3)e^{\frac{4}{3}(\delta-t)+\beta \int_{\sigma}^{s-t}|y(\theta_{s}\omega)|d\sigma}d\sigma \right)^2
\]
\[
+ c \left( \|z(\theta_{s-t}\omega)\|_{H^1}^2 + \|z(\theta_{s-t}\omega)\|_{H^2}^2 \right),
\]
where \(c\) is a positive deterministic constant.

**Proof.** Taking the inner product of (3.10) with \(v_t\) in \(L^2(Q)\) we obtain that
\[
\|v_t\|^2 + \|\nabla v_t\|^2 + \nu(\nabla v, \nabla v_t) = \int_Q \bar{F}(v + z(\theta_t\omega)) \cdot \nabla v_t dx + (g + \alpha z(\theta_t\omega) + (\nu - \alpha)\Delta z(\theta_t\omega), v_t). \tag{4.18}
\]
We now estimate every term in the above. First we have
\[
\nu|\nabla v, \nabla v_t)| \leq \nu\|\nabla v\| \|\nabla v_t\| \leq \frac{1}{4}\|\nabla v_t\|^2 + \nu^2\|\nabla v\|^2. \tag{4.19}
\]
By (3.6), the nonlinear term in (4.18) is bounded by
\[
|\int_Q \bar{F}(v + z(\theta_t\omega)) \cdot \nabla v_t dx| \leq \gamma_1 \int_Q |v + z(\theta_t\omega)| \|\nabla v_t\| dx + \gamma_2 \int_Q |v + z(\theta_t\omega)|^2 \|\nabla v_t\| dx
\]
\[ \leq \frac{1}{4} \| \nabla v_t \|^2 + c(\| v \|^2 + \| v \|_4^4 + \| z(\theta_t \omega) \|^2 + \| z(\theta_t \omega) \|_4^4) \]
\[ \leq \frac{1}{4} \| \nabla v_t \|^2 + c(\| v \|^2 + \| v \|_H^4 + \| z(\theta_t \omega) \|^2 + \| z(\theta_t \omega) \|_H^4). \tag{4.20} \]

For the last term on the right-hand side of (4.18) we have
\[ |(g + \alpha z(\theta_t \omega) + (\nu - \alpha) \Delta z(\theta_t \omega), v_t) | \leq \frac{1}{2} \| v_t \|^2 + c(\| g \|^2 + \| z(\theta_t \omega) \|^2 + \| \Delta z(\theta_t \omega) \|^2). \tag{4.21} \]

Then it follows from (4.18)-(4.21) that
\[ \| v_t \|^2 + \| \nabla v_t \|^2 \leq c_1(\| v \|_H^4 + \| v \|_H^4) + c_1(1 + \| z(\theta_t \omega) \|^2 + \| z(\theta_t \omega) \|_H^4 + \| z(\theta_t \omega) \|_H^4)
\[ \leq c\| v \|_H^4 + c(1 + \| z(\theta_t \omega) \|^2 + \| z(\theta_t \omega) \|_H^4 + \| z(\theta_t \omega) \|_H^4), \]
which shows that, for all \( t \geq 0 \),
\[ \| v(t, \omega, v_0(\omega)) \|_H^4 \leq c \| v(t, \omega, v_0(\omega)) \|_H^4 + c(1 + \| z(\theta_t \omega) \|^2 + \| z(\theta_t \omega) \|_H^4 + \| z(\theta_t \omega) \|_H^4). \tag{4.22} \]

First replacing \( t \) by \( s \) and then replacing \( \omega \) by \( \theta_t \omega \) in (4.22), we get that, for all \( s \geq 0 \) and \( t \geq 0 \),
\[ \| v(s, \theta_{-t} \omega, v_0(\theta_{-t} \omega)) \|_H^4 \]
\[ \leq c \| v(s, \theta_{-t} \omega, v_0(\theta_{-t} \omega)) \|_H^4 + c(1 + \| z(\theta_{s-t} \omega) \|^2 + \| z(\theta_{s-t} \omega) \|_H^4 + \| z(\theta_{s-t} \omega) \|_H^4), \]
which along with (4.19) completes the proof. \(

We are now ready to derive the uniform estimates on the tails of solutions when \( x \) and \( t \) approach infinity, which are crucial for proving the asymptotic compactness of the equation. To this end, for every \( x \in Q = \Omega \times \mathbb{R} \), we will write \( x = (x_1, x_2, x_3) \) where \( (x_1, x_2) \in \Omega \) and \( x_3 \in \mathbb{R} \). Given \( k > 0 \), denote by \( Q_k = \{(x_1, x_2, x_3) \in Q : |x_3| < k\} \), and \( Q \setminus Q_k \) the complement of \( Q_k \).

**Lemma 4.3.** Assume that \( g \in L^2(Q), h \in H_0^1(Q) \) and (3.4) holds. Let \( B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D} \) and \( v_0(\omega) \in B(\omega) \). Then for every \( \epsilon > 0 \) and \( P\text{-a.e. } \omega \in \Omega \), there exist \( T = T(B, \omega, \epsilon) > 0 \) and \( k_0 = k_0(\omega, \epsilon) > 0 \) such that for all \( t \geq T \),
\[ \int_{Q \setminus Q_{k_0}} (|v(t, \theta_{-t} \omega, v_0(\theta_{-t} \omega))|^2 + |\nabla v(t, \theta_{-t} \omega, v_0(\theta_{-t} \omega))|^2) \, dx \leq \epsilon. \]
Proof. Take a smooth function \( \phi \) such that \( 0 \leq \phi \leq 1 \) for all \( s \in \mathbb{R} \) and

\[
\phi(s) = \begin{cases} 
0, & \text{if } |s| < 1, \\
1, & \text{if } |s| > 2.
\end{cases}
\] (4.23)

Then there is a positive constant \( c \) such that \( |\phi'(s)| + |\phi''(s)| \leq c \) for all \( s \in \mathbb{R} \). Multiplying (3.10) by \( \phi^2(\frac{x^2}{k^2})v \) and then integrating with respect to \( x \) on \( Q \), we get

\[
\int_Q \phi^2(\frac{x^2}{k^2})v v_t \, dx - \int_Q \phi^2(\frac{x^2}{k^2})v \Delta v_t \, dx - \nu \int_Q \phi^2(\frac{x^2}{k^2})v \Delta v \, dx
= - \int_Q \phi^2(\frac{x^2}{k^2})v \nabla \cdot \overrightarrow{F} (v + z(\theta_t \omega)) \, dx + \int_Q \phi^2(\frac{x^2}{k^2})v (g + \alpha z(\theta_t \omega) + (\nu - \alpha) \Delta z(\theta_t \omega)) \, dx.
\] (4.24)

We now deal with the left-hand side of the above. For the first term on the left-hand side of (4.24) we have

\[
\int_Q \phi^2(\frac{x^2}{k^2})v v_t \, dx = \frac{1}{2} \frac{d}{dt} \int_Q \phi^2(\frac{x^2}{k^2}) |v|^2 \, dx.
\] (4.25)

We also have

\[
- \int_Q \phi^2(\frac{x^2}{k^2})v \Delta v_t \, dx = \int_Q \phi^2(\frac{x^2}{k^2})(\nabla v_t \cdot \nabla v) \, dx + \int_Q v \left( \nabla v_t \cdot \nabla \phi^2(\frac{x^2}{k^2}) \right) \, dx
= \frac{1}{2} \frac{d}{dt} \int_Q \phi^2(\frac{x^2}{k^2}) |v|^2 \, dx + \int_Q v \left( \nabla v_t \cdot \nabla \phi^2(\frac{x^2}{k^2}) \right) \, dx.
\] (4.26)

The last term on the left-hand side of (4.24) satisfies

\[
- \nu \int_Q \phi^2(\frac{x^2}{k^2})v \Delta v \, dx = \nu \int_Q \phi^2(\frac{x^2}{k^2}) |\nabla v|^2 \, dx + \nu \int_Q v \left( \nabla v \cdot \nabla \phi^2(\frac{x^2}{k^2}) \right) \, dx.
\] (4.27)

Then it follows from (4.24)-(4.27) that

\[
\frac{1}{2} \frac{d}{dt} \int_Q \phi^2(\frac{x^2}{k^2}) |v|^2 \, dx + \nu \int_Q \phi^2(\frac{x^2}{k^2}) |\nabla v|^2 \, dx
= - \int_Q v \left( \nabla v_t \cdot \nabla \phi^2(\frac{x^2}{k^2}) \right) \, dx + \nu \int_Q v \left( \nabla v \cdot \nabla \phi^2(\frac{x^2}{k^2}) \right) \, dx
+ \int_Q \phi^2(\frac{x^2}{k^2}) \left( \overrightarrow{F} (v + z(\theta_t \omega)) \cdot \nabla v \right) \, dx + \int_Q v \left( \overrightarrow{F} (v + z(\theta_t \omega)) \cdot \nabla \phi^2(\frac{x^2}{k^2}) \right) \, dx
+ \int_Q \phi^2(\frac{x^2}{k^2}) v (g + \alpha z(\theta_t \omega) + (\nu - \alpha) \Delta z(\theta_t \omega)) \, dx.
\] (4.28)

Next, we estimate the right-hand side of (4.28). For the first term we have

\[
| \int_Q v \left( \nabla v_t \cdot \nabla \phi^2(\frac{x^2}{k^2}) \right) \, dx | \leq \int_Q |v| |\nabla v_t| |2\phi \phi'(\frac{x^2}{k^2})| \frac{2|x_3|}{k^2} \, dx
\]
By (1.2) and (3.6), for the second term of (4.31) we have

\[ \int_{k \leq |x_3| \leq \sqrt{x_2}} |v| |\nabla v_1| |2\phi \phi' \left( \frac{x_3^2}{k^2} \right)| \leq \frac{c}{k} \int_{k \leq |x_3| \leq \sqrt{x_2}} |v| |\nabla v_1| dx \leq \frac{c}{k} \int_{k \leq |x_3| \leq \sqrt{x_2}} |v| |\nabla v_1| dx \leq \frac{c}{k} \|v\| \|\nabla v_1\| \leq \frac{c}{k} \|v\|^2 + \frac{c}{k} \|v\|^2. \] (4.29)

Similarly, the second term on the right-hand side of (4.28) is bounded by

\[ \nu \int_Q \left( \nabla \cdot \nabla \phi^2 \left( \frac{x_3^2}{k^2} \right) \right) dx \leq \frac{c}{k} \|v\|^2 + \frac{c}{k} \|v\|^2. \] (4.30)

For the third term on the right-hand side of (4.28) we have

\[ \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \overline{F}(v + z(\theta_1 \omega)) \cdot \nabla v \right) dx \]

\[ = \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \overline{F}(u) \cdot \nabla u \right) dx - \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \overline{F}(v + z(\theta_1 \omega)) \cdot \nabla z(\theta_1 \omega) \right) dx \]

\[ = \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \overline{G}(u) \cdot \nabla \phi^2 \left( \frac{x_3^2}{k^2} \right) \right) dx - \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \overline{F}(v + z(\theta_1 \omega)) \cdot \nabla z(\theta_1 \omega) \right) dx \]

\[ = - \int_Q \overline{G}(u) \cdot \nabla \phi^2 \left( \frac{x_3^2}{k^2} \right) dx - \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \overline{F}(v + z(\theta_1 \omega)) \cdot \nabla z(\theta_1 \omega) \right) dx. \] (4.31)

By (3.6), the first term of the above is bounded by

\[ |\int_Q \overline{G}(u) \cdot \nabla \phi^2 \left( \frac{x_3^2}{k^2} \right) dx| \leq \int_Q \left| (\gamma_1 |u|^2 + \gamma_2 |u|^3) \right| |2\phi \phi' \left( \frac{x_3^2}{k^2} \right)| \leq \frac{2 |x_3|}{k^2} dx \]

\[ \leq \frac{c}{k} \int_{k \leq |x_3| \leq \sqrt{x_2}} (\gamma_1 |u|^2 + \gamma_2 |u|^3) dx \leq \frac{c}{k} \|u\|^2 + \frac{c}{k} \|u\|^3 \]

\[ \leq \frac{c}{k} \|v\|^2 + \frac{c}{k} \|v\|^3 + \frac{c}{k} \|z(\theta_1 \omega)\|^2 + \frac{c}{k} \|z(\theta_1 \omega)\|^3. \] (4.32)

By (1.2) and (3.6), for the second term of (4.31) we have

\[ |\int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \overline{F}(v + z(\theta_1 \omega)) \cdot \nabla z(\theta_1 \omega) \right) dx| \]

\[ \leq \gamma_1 \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) |v + z(\theta_1 \omega)| |\nabla z(\theta_1 \omega)| dx + \gamma_2 \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) |v + z(\theta_1 \omega)| \|\nabla z(\theta_1 \omega)\| dx \]

\[ \leq \gamma_1 \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( |v| + z(\theta_1 \omega) \right) \|\nabla z(\theta_1 \omega)\| dx + \gamma_2 \int_Q \phi^2 \left( \frac{x_3^2}{k^2} \right) |v + z(\theta_1 \omega)| \|\nabla z(\theta_1 \omega)\| dx \]

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It follows from (4.31)-(4.33) that
\[
\gamma_1 + 16 \nu \lambda \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( |z(\theta t \omega)|^2 + |\nabla z(\theta t \omega)|^2 \right) \, dx \\
+ 2 \gamma_2 \|\nabla z(\theta t \omega)\|_\infty \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |z(\theta t \omega)|^2 |\nabla z(\theta t \omega)| \, dx \\
\leq \frac{1}{16} \nu \lambda \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( |z(\theta t \omega)|^2 + |\nabla z(\theta t \omega)|^2 \right) \, dx \\
+ 2 \gamma_2 \beta_0 \|\nabla z(\theta t \omega)\|_{H^2} \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |z(\theta t \omega)|^2 |\nabla z(\theta t \omega)| \, dx \\
\leq \frac{1}{16} \nu \lambda \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( |z(\theta t \omega)|^2 + |\nabla z(\theta t \omega)|^2 \right) \, dx \\
+ 2 \gamma_2 \beta_0 \|h\|_{H^1} |y(\theta t \omega)| \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |z(\theta t \omega)|^2 |\nabla z(\theta t \omega)| \, dx. \quad (4.33)
\]

It follows from (4.31) and (4.33) that
\[
\left| \int \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( \nabla^2 (v + z(\theta t \omega)) \cdot \nabla v \right) \, dx \right| \\
\leq c \|v\|^2 + c \|v\|^3 + c \|z(\theta t \omega)\|^2 + c \|\nabla z(\theta t \omega)\|^3 \]
\[
+ \frac{1}{16} \nu \lambda \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) \left( |z(\theta t \omega)|^2 + |\nabla z(\theta t \omega)|^2 \right) \, dx \\
+ 2 \gamma_2 \beta_0 \|h\|_{H^1} |y(\theta t \omega)| \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |v|^2 \, dx + c \int \phi^2 \left( \frac{x_3^2}{k^2} \right) |z(\theta t \omega)|^2 |\nabla z(\theta t \omega)| \, dx. \quad (4.34)
\]

By (3.6), the fourth term on the right-hand side of (4.25) is bounded by
\[
\left| \int \phi \left( \frac{x_3^2}{k^2} \right) \cdot \nabla \phi \left( \frac{x_3^2}{k^2} \right) \, dx \right| \\
\leq \int \phi \left( \frac{x_3^2}{k^2} \right) |\nabla (v + z(\theta t \omega))| \left( 2 \phi \phi' \left( \frac{x_3^2}{k^2} \right) \right) \left( 2 |x_3| \right) \, dx \\
\leq \int_{|x_3| \leq \sqrt{2}k} \phi \left( \frac{x_3^2}{k^2} \right) |\nabla (v + z(\theta t \omega))| \left( 2 \phi \phi' \left( \frac{x_3^2}{k^2} \right) \right) \left( 2 |x_3| \right) \, dx \\
\leq \frac{c}{k} \int \left( |v + z(\theta t \omega)| + |v + z(\theta t \omega)|^2 \right) |v| \, dx \\
\leq \frac{c}{k} (\|v\|^2 + \|v\|^3 + \|z(\theta t \omega)\|^2 + \|z(\theta t \omega)\|^3). \quad (4.35)
\]
By the Young inequality, the last term on the right-hand side of (4.28) is bounded by
\[
\left| \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) v(g + \alpha z(\theta t \omega) + (\nu - \alpha) \Delta z(\theta t \omega)) \, dx \right|
\leq \frac{1}{16} \nu \lambda \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 \, dx + c \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) (g^2 + |z(\theta t \omega)|^2 + |\Delta z(\theta t \omega)|^2) \, dx.
\]
Finally, by (4.28)-(4.30) and (4.34)-(4.36), we find that
\[
\frac{d}{dt} \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) (|v|^2 + |\nabla v|^2) \, dx + 2\nu \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx 
\leq \left( \frac{1}{4} \nu \lambda + \beta |y(\theta t \omega)| \right) \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 \, dx + c \nu \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) (|v|^2 + ||\nabla v||^2) + \frac{c}{k} ||v||^3 + \frac{c}{k} ||\nabla v||^2 + c \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) (g^2 + |z(\theta t \omega)|^2 + |\Delta z(\theta t \omega)|^2 + |\nabla z(\theta t \omega)|^2 + |z(\theta t \omega)|^2 |\nabla z(\theta t \omega)|) \, dx 
+ \frac{c}{k} \left( ||z(\theta t \omega)||^2 + ||z(\theta t \omega)||^3 \right) .
\]
We now deal with the second term on the left-hand side of the above. Note that
\[
\int_Q |\nabla \left( \phi \left( \frac{x^2}{k^2} \right) v \right) |^2 \, dx = \int_Q |v \nabla \phi \left( \frac{x^2}{k^2} \right) + \phi \left( \frac{x^2}{k^2} \right) \nabla v |^2 \, dx 
\leq 2 \int_Q |v|^2 |\nabla \phi \left( \frac{x^2}{k^2} \right)|^2 \, dx + 2 \int_Q |\phi \left( \frac{x^2}{k^2} \right)|^2 |\nabla v|^2 \, dx 
\leq 2 \int_{k \leq |x| \leq \sqrt{2}k} |v|^2 |\phi \left( \frac{x^2}{k^2} \right)|^2 \left( \frac{|2|x|^2|}{k^2} \right) \, dx + 2 \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx 
\leq \frac{c}{k} \int_{k \leq |x| \leq \sqrt{2}k} |v|^2 \, dx + 2 \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx 
\leq \frac{c}{k^2} \|v\|^2 + 2 \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx 
\leq \frac{c}{k^2} \|v\|^2 + 2 \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx.
\]
Since \( v \in H^1_0(Q) \) we have \( \phi \left( \frac{x^2}{k^2} \right) v \in H^1_0(Q) \) and hence by (1.3) and (4.38) we get
\[
\int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 \, dx \leq \frac{1}{\lambda} \int_Q |\nabla \left( \phi \left( \frac{x^2}{k^2} \right) v \right) |^2 \, dx 
\leq \frac{c}{k^2} \|v\|^2 + \frac{2}{\lambda} \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx,
\]
and hence we have
\[
\int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx \geq \frac{1}{2} \lambda \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 \, dx - \frac{c}{2k^2} \|v\|^2.
\]
By (4.39) we find that
\[
2\nu \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |\nabla v|^2 \, dx \geq \nu \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 \, dx + \frac{1}{2} \nu \lambda \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 \, dx - \frac{c\nu}{2k^2} \|v\|^2.
\]
On the other hand, we have

\[ \|v\|^3_H \leq c \|v\|^3_{H^1} \leq c + \|v\|^4_{H^1}. \]  

(4.41)

By (4.37) and (4.40)-(4.41) we obtain that, for all \( k \geq 1 \),

\[ \frac{d}{dt} \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) \left( |v|^2 + |
abla v|^2 \right) dx + \nu \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |
abla v|^2 dx + \frac{1}{2} \nu \lambda \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 dx \]

\[ \leq \left( \frac{1}{4} \nu \lambda + \beta |y(\theta t)|. \right) \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) |v|^2 dx + \frac{c}{k} (\|v\|^2 + \|\nabla v\|^2) + \frac{c}{k} (1 + \|v\|^4_{H^1}) + \frac{c}{k} \|\nabla v_t\|^2 \]

\[ + c \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) \left( g^2 + |\Delta z(\theta t)|^2 + \|D z(\theta t)|^2 + |z(\theta t)|^2 \right) \|\nabla z(\theta t)| dx \]

\[ + \frac{c}{k} \left( \|z(\theta t)|^2 + \|z(\theta t)|^3 \right). \]  

(4.42)

By (3.7) and (4.42) we find that

\[ \frac{d}{dt} \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) \left( |v|^2 + |
abla v|^2 \right) dx + \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) (\delta - \beta |y(\theta t)|) \left( |v|^2 + \|\nabla v\|^2 \right) dx \]

\[ \leq \frac{c}{k} (\|v\|^2 + \|\nabla v\|^2) + \frac{c}{k} (1 + \|v\|^4_{H^1}) + \frac{c}{k} \|\nabla v_t\|^2 \]

\[ + c \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) \left( g^2 + |z(\theta t)|^2 + \|\Delta z(\theta t)|^2 + \|D z(\theta t)|^2 + |z(\theta t)|^2 \right) \|\nabla z(\theta t)| dx \]

\[ + \frac{c}{k} \left( \|z(\theta t)|^2 + \|z(\theta t)|^3 \right). \]  

(4.43)

Multiplying (4.43) by \( e^{- \int_0^t (\delta - \beta |y(\theta t)|) \, dt} \) and then integrating over \((0, t)\), we get that, for all \( t \geq 0 \),

\[ \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) (|v(t, \omega, v_0(\omega))|^2 + |\nabla v(t, \omega, v_0(\omega))|^2) \, dx \]

\[ \leq e^{- \int_0^t (\delta - \beta |y(\theta t)|) \, dt} \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) (|v_0(\omega)|^2 + |\nabla v_0(\omega)|^2) \, dx \]

\[ + \frac{c}{k} \int_0^t e^{\int_0^t (\delta - \beta |y(\theta t)|) \, dt} ds + \frac{c}{k} \int_0^t e^{\int_0^t (\delta - \beta |y(\theta t)|) \, dt} \|v(s, \omega, v_0(\omega))\|^2_{H^1} \, ds \]

\[ + \frac{c}{k} \int_0^t e^{\int_0^t (\delta - \beta |y(\theta t)|) \, dt} \|v(s, \omega, v_0(\omega))\|^4_{H^1} \, ds \]

\[ + \frac{c}{k} \int_0^t e^{\int_0^t (\delta - \beta |y(\theta t)|) \, dt} \|\nabla v(s, \omega, v_0(\omega))\|^2 \, ds \]

\[ + c \int_0^t e^{\int_0^t (\delta - \beta |y(\theta t)|) \, dt} \left( \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) g^2 dx \right) ds \]

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Replacing $\omega$ by $\theta_{-t}\omega$ in the above, we find that, for all $t \geq 0$,

$$
\int_{Q} \phi^2 \left( \frac{x^2}{K^2} \right) \left( |v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 + |\nabla v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 \right) dx 
\leq e^{-\int_{0}^{t} (\delta - \beta |y(\theta_{-t}\omega)|) dt} \int_{Q} \phi^2 \left( \frac{x^2}{K^2} \right) \left( |v_0(\theta_{-t}\omega)|^2 + |\nabla v_0(\theta_{-t}\omega)|^2 \right) dx 
\leq e^{-\delta t + \beta \int_{0}^{t} |y(\theta_{-t}\omega)| dt} \left\| v_0(\theta_{-t}\omega) \right\|_{H^1}^2 
\leq e^{-\frac{7}{8} \delta t} \left\| v_0(\theta_{-t}\omega) \right\|_{H^1}^2, \quad \text{for all } t \geq T_0(\omega).
$$

In what follows, we estimate every term on the right-hand side of (4.44). For the first term, by (4.12) we have

$$
e^{-\int_{0}^{t} (\delta - \beta |y(\theta_{-t}\omega)|) dt} \int_{Q} \phi^2 \left( \frac{x^2}{K^2} \right) \left( |v_0(\theta_{-t}\omega)|^2 + |\nabla v_0(\theta_{-t}\omega)|^2 \right) dx
\leq e^{-\delta t + \beta \int_{0}^{t} |y(\theta_{-t}\omega)| dt} \left\| v_0(\theta_{-t}\omega) \right\|_{H^1}^2 \leq e^{-\delta t + \beta \int_{0}^{t} |y(\theta_{-t}\omega)| dt} \left\| v_0(\theta_{-t}\omega) \right\|_{H^1}^2
\leq e^{-\frac{7}{8} \delta t} \left\| v_0(\theta_{-t}\omega) \right\|_{H^1}^2,
$$
Since $v_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and $B = \{B(\omega)\}_{\omega \in \Omega} \subset \mathcal{D}$, the right-hand side of (4.45) tends to zero as $t \to \infty$. Therefore, given $\epsilon > 0$, there is $T_1 = T_1(B, \omega, \epsilon) > 0$ such that for all $t \geq T_1$,

\[
e^{-f_0^t(\delta-\beta|y(\theta_{-t}\omega)|)} \int_Q \phi^2 \left(\frac{3}{2} \frac{\omega}{k^2}\right) \left(|v_0(\theta_{-t}\omega)|^2 + |\nabla v_0(\theta_{-t}\omega)|^2\right) dx \leq \epsilon. \tag{4.46}
\]

Note that the second term on the right-hand side of (4.44) satisfies

\[
\frac{c}{k} \int_0^t e^{f_s^t(\delta-\beta|y(\theta_{-t}\omega)|)} ds = \frac{c}{k} \int_0^t e^{\delta(s-t)-\beta f_s^t |y(\theta_{-t}\omega)|} ds = \frac{c}{k} \int_{-t}^0 e^{\delta s + \beta f_s^0 |y(\theta_{-t}\omega)|} ds = \frac{c}{k} \int_{-t}^0 e^{\delta s + \beta f_s^0 |y(\theta_{-t}\omega)|} ds \tag{4.47}
\]

By (4.12), the integrand in (4.47) converges to zero exponentially as $s \to -\infty$, and hence the following integral is well-defined:

\[
r_1(\omega) = \int_{-\infty}^0 e^{\delta s + \beta f_s^0 |y(\theta_{-t}\omega)|} ds. \tag{4.48}
\]

It follows from (4.47) and (4.48) that, for all $t \geq 0$,

\[
\frac{c}{k} \int_0^t e^{f_s^t(\delta-\beta|y(\theta_{-t}\omega)|)} ds \leq \frac{c}{k} r_1(\omega). \tag{4.49}
\]

By (4.9), the third term on the right-hand side of (4.44) is bounded by

\[
\frac{c}{k} \int_0^t e^{f_s^t(\delta-\beta|y(\theta_{-t}\omega)|)} ||v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))||_{H^1}^2 ds 
\leq \frac{c}{k} \int_0^t e^{f_s^t(\delta-\beta|y(\theta_{-t}\omega)|)} e^{\delta s + \beta f_s^0 |y(\theta_{-t}\omega)|} ds 
\int_0^t \left(1 + |y(\theta_{s}\omega)|^2 + |y(\theta_{s}\omega)|^3 e^{(\delta-\beta|y(\theta_{-t}\omega)|)} ds \right) ds. \tag{4.50}
\]

By (4.12), the first term on the right-hand side of (4.50) is given by

\[
\frac{c}{k} \int_0^t e^{f_s^t(\delta-\beta|y(\theta_{-t}\omega)|)} e^{\delta s + \beta f_s^0 |y(\theta_{-t}\omega)|} ||v_0(\theta_{-t}\omega)||_{H^1}^2 ds 
= \frac{c}{k} \int_0^t e^{-\delta t + \beta f_s^t |y(\theta_{-t}\omega)|} ||v_0(\theta_{-t}\omega)||_{H^1}^2 ds 
= \frac{c}{k} \int_0^t e^{-\delta t + \beta f_s^t |y(\theta_{-t}\omega)|} ||v_0(\theta_{-t}\omega)||_{H^1}^2 ds
\]
By (4.52)-(4.54) we obtain that
for all 
\( t \geq T_0(\omega) \). Since \( te^{-\frac{7}{2}\delta t} \|v_0(\theta_{-t}\omega)\|_{H^1}^2 \) tends to zero as \( t \to \infty \), there is \( T_2 = T_2(B, \omega) > 0 \) such that for all \( t \geq T_2 \),
\[
\frac{c}{k} \int_0^t e^{f_1^s(\delta-\beta)|y(\theta_{-s}\omega)|} ds \leq \frac{c}{k} e^{-\frac{7}{2}\delta t} \|v_0(\theta_{-t}\omega)\|_{H^1}^2, \tag{4.51}
\]
For the second term on the right-hand side of (4.50) we have
\[
\frac{c}{k} \int_0^t e^{f_1^s(\delta-\beta)|y(\theta_{-s}\omega)|} ds \leq \frac{c}{k} e^{-\frac{7}{2}\delta t} \|v_0(\theta_{-t}\omega)\|_{H^1}^2\|v_0(\theta_{-t}\omega)\|_{H^1},
\]
and
\[
r_2(\omega) = \int_{-\infty}^0 e^{\frac{1}{2}\delta s + \beta f_2^s} |y(\theta_{s}\omega)| ds,
\]
By (4.12) we know that the following integrals are convergent:
\[
r_3(\omega) = \int_{-\infty}^0 (1 + |y(\theta_{s}\omega)|^2 + |y(\theta_{s}\omega)|^3) e^{\frac{1}{2}\delta s + \beta f_2^s} |y(\theta_{s}\omega)| ds,
\]
By (4.52)-(4.54) we obtain that
\[
\frac{c}{k} \int_0^t e^{f_1^s(\delta-\beta)|y(\theta_{-s}\omega)|} ds \leq \frac{c}{k} r_2(\omega)r_3(\omega), \tag{4.55}
\]
Then it follows from (4.50) and (4.55) that, for all \( t \geq T_2 \),
\[
\frac{c}{k} \int_0^t e^{f_1^s(\delta-\beta)|y(\theta_{-s}\omega)|} ds \|v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{H^1}^2 ds \leq \frac{c}{k} (1 + r(\omega) r_2(\omega)). \tag{4.56}
\]
For the fourth term on the right-hand side of (4.44), by (4.9), we have
\[
\frac{c}{k} \int_0^t e^{f_1^s(\delta-\beta)|y(\theta_{-s}\omega)|} ds \|v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{H^1}^2 ds
\]
For the second term on the right-hand side of (4.57), we have
\[ T \]
and hence it follows from (4.58) that there is
\[ \int_{-t}^{s-t} (1 + |y(\theta_{\sigma})|^2 + |y(\theta_{\sigma})|^3) e^{\delta(s-s+t) + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\sigma} d\sigma \right)^2 ds. \]

We now deal with the first term on the right-hand side of the above, which is given by
\[
\frac{c}{k} \int_{0}^{t} e^{f_{t}^{s}(\delta-|y(\theta_{r-t})|) d\tau} \left( e^{-\delta s + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\tau} \|v_{0}(\theta_{-t\omega})\|_{H^{1}}^{2} \right) ds
\]

By (4.12), we know that
\[ e^{-\delta s + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\sigma} \|v_{0}(\theta_{-t\omega})\|_{H^{1}}^{2} \to 0 \quad \text{as} \quad t \to \infty, \]
and hence it follows from (4.58) that there is \( T_{3} = T_{3}(B, \omega) \) such that for all \( t \geq T_{3}, \)
\[
\frac{c}{k} \int_{0}^{t} e^{f_{t}^{s}(\delta-|y(\theta_{r-t})|) d\tau} \left( e^{-\delta s + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\tau} \|v_{0}(\theta_{-t\omega})\|_{H^{1}}^{2} \right) ds \leq \frac{c}{k}. \]

For the second term on the right-hand side of (4.57), we have
\[
\frac{c}{k} \int_{0}^{t} e^{f_{t}^{s}(\delta-|y(\theta_{r-t})|) d\tau} \left( \int_{-t}^{s-t} (1 + |y(\theta_{\sigma})|^2 + |y(\theta_{\sigma})|^3) e^{\delta(s-s+t) + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\sigma} d\sigma \right)^2 ds
\]

\[
\leq \frac{c}{k} \int_{0}^{t} e^{f_{t}^{s}(\delta-|y(\theta_{r-t})|) d\tau} \left( \int_{-t}^{s-t} (1 + |y(\theta_{\sigma})|^2 + |y(\theta_{\sigma})|^3) e^{\frac{1}{8}\delta(s-s+t) + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\sigma} d\sigma \right)^2 ds
\]

\[
\leq \frac{c}{k} \int_{0}^{t} e^{f_{t}^{s}(\delta-|y(\theta_{r-t})|) d\tau} ds \left( \int_{-t}^{s-t} (1 + |y(\theta_{\sigma})|^2 + |y(\theta_{\sigma})|^3) e^{\frac{3}{8}\delta(s-s+t) + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\sigma} d\sigma \right)^2 ds
\]

\[
\leq \frac{c}{k} \int_{0}^{s-t} e^{f_{t}^{s}(\delta-|y(\theta_{r-t})|) d\tau} ds \left( \int_{-t}^{s-t} (1 + |y(\theta_{\sigma})|^2 + |y(\theta_{\sigma})|^3) e^{\frac{3}{8}\delta(s-s+t) + \beta \int_{s-t}^{s} |y(\theta_{\sigma})| d\sigma} d\sigma \right)^2 ds. \]
Note that (4.12) implies the convergence of the integrals:

\[
r_4(\omega) = \int_{-\infty}^{0} e^{\frac{1}{4}\delta s + \beta \int_{s}^{0} |y(\theta_s \omega)| \, ds} \, ds,
\]

and

\[
r_5(\omega) = \int_{-\infty}^{0} (1 + |y(\theta_s \omega)|^2 + |y(\theta_s \omega)|^3) e^{\frac{3}{4}\delta s + \beta \int_{s}^{0} |y(\theta_s \omega)| \, ds} \, ds.
\]

Therefore it follows from (4.60)-(4.62) that, for all \( t \geq 0 \),

\[
\frac{c}{k} \int_{0}^{t} e^{\int_{t}^{s} (\delta - \beta |y(\theta_{s-t} \omega)|) \, ds} \left( \int_{-\infty}^{s-t} (1 + |y(\theta_s \omega)|^2 + |y(\theta_s \omega)|^3) e^{\delta (s-t) + \beta \int_{\theta_s \omega}^{\theta_{s-t} \omega} |y(\theta_s \omega)| \, d\sigma} \right)^2 \, ds \leq \frac{c}{k} r_4(\omega) r_5^2(\omega).
\]

By (4.57), (4.59) and (4.63) we find that, for all \( t \geq T_3 \),

\[
\frac{c}{k} \int_{0}^{t} e^{\int_{t}^{s} (\delta - \beta |y(\theta_{s-t} \omega)|) \, ds} \|v(s, \theta_{s-t} \omega, v_0(\theta_{s-t} \omega))\|_{H^1}^4 \, ds \leq \frac{c}{k} (1 + r_4(\omega) r_5^2(\omega)).
\]

Note that \( g \in L^2(Q) \), and hence for given \( \epsilon > 0 \), there is \( k_1 = k_1(\epsilon) > 0 \) such that, for all \( k \geq k_1 \),

\[
\int_{|x| \geq k} g^2(x) \, dx \leq \epsilon,
\]

from which the sixth term on the right-hand side of (4.44) is bounded by

\[
c \int_{0}^{t} e^{\int_{t}^{s} (\delta - \beta |y(\theta_{s-t} \omega)|) \, ds} \left( \int_{Q} \phi^2 \left( \frac{x^2}{k^2} \right) g^2 \, dx \right) \, ds
\leq c \int_{0}^{t} e^{\int_{t}^{s} (\delta - \beta |y(\theta_{s-t} \omega)|) \, ds} \left( \int_{|x| \geq k} \phi^2 \left( \frac{x^2}{k^2} \right) g^2 \, dx \right) \, ds
\leq c \epsilon \int_{0}^{t} e^{\int_{t}^{s} (\delta - \beta |y(\theta_{s-t} \omega)|) \, ds} \, ds \leq c \epsilon r_1(\omega),
\]

where \( r_1(\omega) \) is given by (4.48). Since \( (I - \Delta)^{-1} h \in H^2(Q) \), there is \( k_2 = k_2(\omega) > 0 \) such that for all \( k \geq k_2 \),

\[
\int_{|x| \geq k} (|(I - \Delta)^{-1} h|^2 + |\Delta(I - \Delta)^{-1} h|^2) \, dx \leq \epsilon.
\]

Note that \( z(\theta_t \omega) = (I - \Delta)^{-1} h y(\theta_t \omega) \). By (4.60), the seventh term on the right-hand side of (4.44) satisfies

\[
c \int_{0}^{t} e^{\int_{t}^{s} (\delta - \beta |y(\theta_{s-t} \omega)|) \, ds} \left( \int_{Q} \phi^2 \left( \frac{x^2}{k^2} \right) (|z(\theta_{s-t} \omega)|^2 + |\Delta z(\theta_{s-t} \omega)|^2) \, dx \right) \, ds
\]

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Note that the eighth term on the right-hand side of (4.44) satisfies
\[ \int_{|x| \geq k} \int_{0}^{t} e^{\int_{0}^{s} e^{\int_{t}^{s} |\nabla z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^2} dx} ds \]

where \( r_6(\omega) \) is given by
\[ r_6(\omega) = \int_{-\infty}^{0} e^{\delta_{s+t} f_{s}^{0} |y(\theta_{s}\omega)|} dy \int_{0}^{t} e^{\int_{s}^{t} |y(\theta_{s}\omega)|} ds \leq c r_6(\omega), \]

Note that \( r_6(\omega) \) is well-defined by (4.12). Similarly, we can find a random function \( r_7(\omega) \) such that the eighth term on the right-hand side of (4.44) satisfies
\[ c \int_{0}^{t} e^{\int_{0}^{s} e^{\int_{t}^{s} |\nabla z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^2} dx} ds \leq c r_7(\omega). \]

For the last term on the right-hand side of (4.44) we have
\[ \frac{c}{k} \int_{0}^{t} e^{\int_{0}^{s} e^{\int_{t}^{s} |\nabla z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^2} dx} ds \]

where \( r_8(\omega) \) is given by
\[ r_8(\omega) = \int_{-\infty}^{0} e^{\delta_{s+t} f_{s}^{0} |y(\theta_{s}\omega)|} dy \int_{0}^{t} e^{\int_{s}^{t} |y(\theta_{s}\omega)|} ds \leq \frac{c}{k} r_8(\omega), \]

We now deal with the fifth term on the right-hand side of (4.44). By Lemma 4.2 we have
\[ \frac{c}{k} \int_{0}^{t} e^{\int_{0}^{s} e^{\int_{t}^{s} |\nabla z(\theta_{s-t}\omega)|^2 + |z(\theta_{s-t}\omega)|^2} dx} ds \leq \frac{c}{k} \int_{0}^{t} e^{\int_{0}^{s} e^{\int_{t}^{s} |y(\theta_{s}\omega)|} dx} ds \]

\[ + \frac{c}{k} \int_{0}^{t} e^{\int_{0}^{s} e^{\int_{t}^{s} |y(\theta_{s}\omega)|} dx} e^{-2\delta_{s+2t} f_{s}^{0} |y(\theta_{s}\omega)|} \int_{0}^{t} e^{\int_{s}^{t} |v_0(\theta_{s-t}\omega)|^4} ds \]
Let \( T \) be a positive deterministic constant, independent of \( \omega \).

\[
+ \frac{c}{k} \int_0^t e^{\int_0^t \beta \phi(t_0, \omega) dt_0} \left( \int_{t_0}^{s-t} \left( 1 + |y(\theta_{s-t})|^2 + |y(\theta_{s-t})|^{3/2} \right) e^{\delta(\sigma-s+t) + \beta \int_0^t |y(\theta_{s-t})| dt_0} d\sigma \right)^2 ds + \frac{c}{k} \int_0^t e^{\int_0^t \beta \phi(t_0, \omega) dt_0} \left( \|z(\theta_{s-t})\|^2 + \|z(\theta_{s-t})\|^4_{H^1} + \|z(\theta_{s-t})\|^2_{H^2} \right) ds.
\]

(4.70)

Note that the last term of the above has estimates similar to (4.69), which along with (4.49), (4.59) and (4.63) imply that, there are \( r_9(\omega) \) and \( T_4 = T_4(B, \omega) > 0 \) such that for all \( t \geq T_4 \),

\[
\frac{c}{k} \int_0^t e^{\int_0^t \beta \phi(t_0, \omega) dt_0} \|\nabla v_s(s, \theta_{s-t}, \omega, v_0(\theta_{s-t}))\|^2 ds \leq \frac{c}{k} (1 + r_9(\omega)).
\]

(4.71)

Let \( T_5 = T_5(B, \omega, \epsilon) = \max\{T_1, T_2, T_3, T_4\} \) and \( k_3 = k_3(\epsilon) = \max\{k_1, k_2\} \). Then it follows from (4.44), (4.46), (4.49), (4.56), (4.61), (4.69) and (4.71) that, for all \( t \geq T_5 \) and \( k \geq k_3 \),

\[
\int_Q \phi^2 \left( \frac{x^2}{k^2} \right) \left( |v(t, \theta_{-t}, \omega, v_0(\theta_{-t}))|^2 + |\nabla v(t, \theta_{-t}, \omega, v_0(\theta_{-t}))|^2 \right) dx \leq \epsilon(1 + r_{10}(\omega)) + \frac{c}{k} r_{10}(\omega),
\]

(4.72)

where \( r_{10}(\omega) \) is a positive random function. By (4.72) we find that there is \( k_4 = k_4(\omega, \epsilon) > 0 \) such that for all \( t \geq T_5 \) and \( k \geq k_4 \),

\[
\int_{|x_3| \geq \sqrt{2k}} \left( |v(t, \theta_{-t}, \omega, v_0(\theta_{-t}))|^2 + |\nabla v(t, \theta_{-t}, \omega, v_0(\theta_{-t}))|^2 \right) dx \\
\leq \int_Q \phi^2 \left( \frac{x^2}{k^2} \right) \left( |v(t, \theta_{-t}, \omega, v_0(\theta_{-t}))|^2 + |\nabla v(t, \theta_{-t}, \omega, v_0(\theta_{-t}))|^2 \right) dx \\
\leq \epsilon(2 + r_{10}(\omega)),
\]

which completes the proof.

(4.73)

In the sequel, we derive uniform estimates of the solutions on bounded domains which are necessary for verifying the asymptotic compactness of the stochastic Benjamin-Bona-Mahony equation. To this end, we define \( \psi = 1 - \phi \) where \( \phi \) is the function given in (4.23). Fix \( k \geq 1 \) and let \( \tilde{v}(x, t, \omega) = \psi(\frac{x^2}{k^2}) v(x, t, \omega) \). Then \( \tilde{v}(\cdot, t, \omega) \in H_0^1(Q_{2k}) \) and

\[
\|\tilde{v}(t, \omega)\|_{H_0^1(Q_{2k})} \leq c \|v(t, \omega)\|_{H^1(Q_{2k})}, \quad \forall t \geq 0, \ \omega \in \Omega,
\]

(4.73)

where \( c \) is a positive deterministic constant, independent of \( \omega \in \Omega \) and \( k \geq 1 \). Note that

\[
\tilde{v}_t = \psi v_t,
\]

(4.74)
\[ \Delta \tilde{v} = \Delta \psi v + 2 \nabla \psi \cdot \nabla v + \psi \Delta v, \quad (4.75) \]
\[ \Delta \tilde{v}_t = (\Delta \psi) v_t + 2 \nabla \psi \cdot \nabla v_t + \psi \Delta v_t. \quad (4.76) \]

By (4.75) and (4.76) we have
\[ \psi \Delta v = \Delta \tilde{v} - v \Delta \psi - 2 \nabla \psi \cdot \nabla v, \quad (4.77) \]
and
\[ \psi \Delta v_t = \Delta \tilde{v}_t - v_t \Delta \psi - 2 \nabla \psi \cdot \nabla v_t. \quad (4.78) \]

Multiplying (3.10) by \( \psi \) we get that
\[ \psi v_t - \psi \Delta v_t - \nu \psi \Delta v = -\psi \nabla \cdot \vec{F} (v + z(\theta_t \omega)) + \psi g + \alpha \psi z(\theta_t \omega) + (\nu - \alpha) \psi \Delta z(\theta_t \omega). \quad (4.79) \]

Substituting (4.74) and (4.77)-(4.78) into (4.79) we find that
\[ \tilde{v}_t - \Delta \tilde{v}_t - \nu \Delta \tilde{v} = -\psi \nabla \cdot \vec{F} (v + z(\theta_t \omega)) + \psi g + \alpha \psi z(\theta_t \omega) + (\nu - \alpha) \psi \Delta z(\theta_t \omega) \]
\[ - v_t \Delta \psi - 2 \nabla \psi \cdot \nabla v_t - \nu v \Delta \psi - 2 \nu \nabla \psi \cdot \nabla v. \quad (4.80) \]

Consider the eigenvalue problem:
\[ - \Delta \tilde{v} = \lambda \tilde{v} \quad \text{in} \quad Q_{2k}, \quad \tilde{v} |_{\partial Q_{2k}} = 0. \quad (4.81) \]

Then problem (4.81) has a family of eigenfunctions \( \{e_j\}_{j=1}^{\infty} \) with corresponding eigenvalues \( \{\lambda_j\}_{j=1}^{\infty} \) such that \( \{e_j\}_{j=1}^{\infty} \) is an orthonormal basis of \( L^2(Q_{2k}) \) and
\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty \quad \text{as} \quad j \to \infty. \]

Given \( n \), let \( X_n = \text{span}\{e_1, \cdots, e_n\} \) and \( P_n : L^2(Q_{2k}) \to X_n \) be the projection operator. For \( \tilde{v} \), we have the following estimates in \( H^1_0(Q_{2k}) \).

**Lemma 4.4.** Assume that \( g \in L^2(Q) \), \( h \in H^1_0(Q) \) and (3.4) holds. Let \( B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D} \) and \( v_0(\omega) \in B(\omega) \). Then for every \( \epsilon > 0 \) and \( P \)-a.e. \( \omega \in \Omega \), there exist \( T = T(B, \omega, \epsilon) > 0 \) and \( N = N(\omega, \epsilon) > 0 \) such that for all \( k \geq 1 \), \( t \geq T \) and \( n \geq N \),
\[ \|(I - P_n)\tilde{v}(t, \theta_{-t} \omega, \tilde{v}_0(\theta_{-t} \omega))\|_{H^1_0(Q_{2k})} \leq \epsilon. \]
Proof. Let $\tilde{v}_{n,1} = P_n \tilde{v}$ and $\tilde{v}_{n,2} = \tilde{v} - \tilde{v}_{n,1}$. Then applying $I - P_n$ to (4.80) and taking the inner product of the resulting equation with $\tilde{v}_{n,2}$ in $L^2(Q_{2k})$ we obtain that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \tilde{v}_{n,2} \|^2 + \| \nabla \tilde{v}_{n,2} \|^2 \right) + \nu \| \nabla \tilde{v}_{n,2} \|^2 = - \left( \psi \nabla \cdot \vec{F}(v + z(\theta_t \omega)), \tilde{v}_{n,2} \right) + (\psi g, \tilde{v}_{n,2})
\]
\[
+ (\alpha \psi z(\theta_t \omega) + (\nu - \alpha) \psi \Delta z(\theta_t \omega), \tilde{v}_{n,2}) - (v_t \Delta \psi + 2 \nabla \psi \cdot \nabla v_t + \nu \psi \Delta \psi + 2 \nu \nabla \psi \cdot \nabla v, \tilde{v}_{n,2}) \cdot (4.82)
\]
By (3.4), the nonlinear term in the above is bounded by
\[
| \left( \psi \nabla \cdot \vec{F}(v + z(\theta_t \omega)), \tilde{v}_{n,2} \right) | \leq \int_{Q_{2k}} \psi \left( \frac{v^2}{k^2} \right) | \vec{F}'(v + z(\theta_t \omega))| | \nabla v + \nabla z(\theta_t \omega)| | \tilde{v}_{n,2} | dx
\]
\[
\leq \int_{Q_{2k}} | \gamma_1 + \gamma_2 (v + z(\theta_t \omega))| | \nabla v + \nabla z(\theta_t \omega)| | \tilde{v}_{n,2} | dx
\]
\[
\leq c ( \| \nabla v \|^2 + \| \nabla z(\theta_t \omega) \|^2 ) \| \tilde{v}_{n,2} \| + c ( \| \nabla v \|^2 + \| \nabla z(\theta_t \omega) \|^2 ) \left( \| v \|_6 + \| z(\theta_t \omega) \|_6 \right) \| \tilde{v}_{n,2} \|_3
\]
\[
\leq c ( \| v \|_{H^1} + \| z(\theta_t \omega) \|_{H^1} ) \| \tilde{v}_{n,2} \| + c ( \| v \|^2_{H^1} + \| z(\theta_t \omega) \|^2_{H^1} ) \| \nabla \tilde{v}_{n,2} \|_2 \| \tilde{v}_{n,2} \|^{\frac{1}{2}}
\]
\[
\leq \frac{1}{16} \nu \| \nabla \tilde{v}_{n,2} \|^2 + c \lambda_{n+1}^{-\frac{3}{2}} \| v \|^2_{H^1} + \| z(\theta_t \omega) \|^2_{H^1} + c \lambda_{n+1}^{-\frac{7}{2}} \left( \| v \|^4_{H^1} + \| z(\theta_t \omega) \|^4_{H^1} \right)
\]
\[
\leq \frac{1}{16} \nu \| \nabla \tilde{v}_{n,2} \|^2 + c \lambda_{n+1}^{-1} + c \left( \lambda_{n+1}^{-\frac{1}{2}} + \lambda_{n+1}^{-1} \right) \left( \| v \|^4_{H^1} + \| z(\theta_t \omega) \|^4_{H^1} \right).
\]
(4.83)
Note that the second term on the right-hand side of (4.82) is bounded by
\[
| (\psi g, \tilde{v}_{n,2}) | \leq \| g \| \| \tilde{v}_{n,2} \| \leq \lambda_{n+1}^{-\frac{3}{2}} \| g \| \| \nabla \tilde{v}_{n,2} \| \leq \frac{1}{16} \nu \| \nabla \tilde{v}_{n,2} \|^2 + c \lambda_{n+1}^{-1}.
\]
(4.84)
For the third term on the right-hand side of (4.82) we have
\[
| (\alpha \psi z(\theta_t \omega) + (\nu - \alpha) \psi \Delta z(\theta_t \omega), \tilde{v}_{n,2}) | \leq c ( \| z(\theta_t \omega) \| + \Delta z(\theta_t \omega) ) \| \tilde{v}_{n,2} \|
\]
\[
\leq c ( \| v \|_{H^1} ) \| \tilde{v}_{n,2} \| \leq c \lambda_{n+1}^{-\frac{1}{2}} \| y(\theta_t \omega) \| \| \nabla \tilde{v}_{n,2} \| \leq \frac{1}{16} \nu \| \nabla \tilde{v}_{n,2} \|^2 + c \lambda_{n+1}^{-1} \| y(\theta_t \omega) \|^2.
\]
(4.85)
Similarly, we can check that the last term on the right-hand side of (4.82) is bounded by
\[
| (v_t \Delta \psi + 2 \nabla \psi \cdot \nabla v_t + \nu \psi \Delta \psi + 2 \nu \nabla \psi \cdot \nabla v, \tilde{v}_{n,2}) |
\]
(4.86)
which along with (4.89) shows that, for all $n \geq N$,

$$\lambda_{n+1} \geq \max\{1, \lambda\} \quad \text{and} \quad \lambda^{-\frac{1}{2}}_{n+1} \leq \epsilon,$$

(4.88)

where $\lambda$ is the positive constant in (1.3). By (4.87) and (4.88) we have, for all $n \geq N$ and $t \geq 0$,

$$\frac{d}{dt} \| \tilde{v}_{n,2} \|_{H^1}^2 + \frac{3}{2} \nu \| \nabla \tilde{v}_{n,2} \|_{L^2}^2 \leq c \left( \| v \|_{H^1}^2 + \| \dot{v}_t \|_{H^1}^2 + \| y(\theta t\omega) \|^2 + |y(\theta t\omega)|^4 \right).$$

(4.89)

Note that (4.12) and (4.88) imply

$$\frac{3}{2} \nu \| \nabla \tilde{v}_{n,2} \|_{L^2}^2 \geq \nu \| \nabla \tilde{v}_{n,2} \|_{L^2}^2 + \frac{1}{2} \nu \lambda_{n+1} \| \tilde{v}_{n,2} \|_{L^2}^2 \geq \nu \| \nabla \tilde{v}_{n,2} \|_{L^2}^2 + \frac{1}{2} \nu \lambda \| \tilde{v}_{n,2} \|_{L^2}^2 \geq \delta \| \tilde{v}_{n,2} \|_{H^1}^2,$$

which along with (4.89) shows that, for all $n \geq N$ and $t \geq 0$,

$$\frac{d}{dt} \| \tilde{v}_{n,2} \|_{H^1}^2 + \delta \| \tilde{v}_{n,2} \|_{H^1}^2 \leq c \left( \| v \|_{H^1}^2 + \| \dot{v}_t \|_{H^1}^2 + \| y(\theta t\omega) \|^2 + |y(\theta t\omega)|^4 \right).$$

(4.90)

Integrating (4.90) over $(0, t)$, we find that, for all $n \geq N$ and $t \geq 0$,

$$\| \tilde{v}_{n,2}(t, \omega) \|_{H^1}^2 \leq e^{-\delta t} \| \tilde{v}_{n,2}(0, \omega) \|_{H^1}^2 + c \epsilon \int_0^t e^{\delta(s-t)} \left( 1 + \| v(s, \omega, v_0(\omega)) \|_{H^1}^4 + \| v_s(s, \omega, v_0(\omega)) \|_{H^1}^2 + |y(s\omega)|^2 + |y(s\omega)|^4 \right) ds.$$

Replacing $\omega$ by $\theta_{-t}\omega$ in the above, we get that, for all $n \geq N$ and $t \geq 0$,

$$\| \tilde{v}_{n,2}(t, \theta_{-t}\omega) \|_{H^1}^2 \leq e^{-\delta t} \| \tilde{v}_{n,2}(0, \theta_{-t}\omega) \|_{H^1}^2 + c \epsilon \int_0^t e^{\delta(s-t)} \left( 1 + \| v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) \|_{H^1}^4 + \| v_s(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) \|_{H^1}^2 + |y(s\omega)|^2 + |y(s\omega)|^4 \right) ds.$$
For the first term on the right-hand side of (4.91), by (4.73) we have
\[
e^{-\delta t} \| \tilde{v}_{n,2}(0, \theta - t \omega) \|^2_{H^1} \leq e^{-\delta t} \| \tilde{v}_0(\theta - t \omega) \|^2_{H^1} \leq c e^{-\delta t} \| v_0(\theta - t \omega) \|^2_{H^1},
\]
which converges to zero as \( t \to \infty \), and hence there is \( T_1 = T_1(B, \omega, \epsilon) > 0 \) such that for all \( t \geq T_1 \),
\[
e^{-\delta t} \| \tilde{v}_{n,2}(0, \theta - t \omega) \|^2_{H^1} \leq \epsilon. \tag{4.92}
\]

For the second term on the right-hand side of (4.91), by (4.73), Lemma 4.2 and the proof of Lemma 4.3 we can show that there are \( T_2 = T_2(B, \omega, \epsilon) > 0 \) and \( r(\omega) > 0 \) such that for all \( t \geq T_2 \),
\[
\int_0^t e^{\delta(s-t)} \left( 1 + \| v(s, \theta - t \omega, v_0(\theta - t \omega)) \|^2_{H^1} + \| v(s, \theta - t \omega, v_0(\theta - t \omega)) \|^2_{H^1} + | y(\theta - t \omega) |^2 + | y(\theta - t \omega) |^4 \right) ds \\
\leq c(1 + r(\omega)). \tag{4.93}
\]
The details for the proof of (4.93) are omitted here. Let \( T = \max\{T_1, T_2\} \). Then by (4.91)-(4.93) we get that, for all \( n \geq N \) and \( t \geq T \),
\[
\| \tilde{v}_{n,2}(t, \theta - t \omega) \|^2_{H^1} \leq \epsilon + c \epsilon(1 + r(\omega)),
\]
which completes the proof. \( \square \)

**Lemma 4.5.** Assume that \( g \in L^2(Q) \), \( h \in H^1_0(Q) \) and (3.24) holds. Let \( B = \{ B(\omega) \}_{\omega \in \Omega} \subset D \), \( t_m \to \infty \) and \( v_{0,m} \in B(\theta_{-t_m} \omega) \). Suppose \( v(t, \theta - t \omega, v_{0,m}) \) satisfies (3.10)-(3.11) with initial condition \( v_{0,m} \) and
\[
\tilde{v}_m(x, t, \theta - t \omega) = \psi \left( \frac{x^2}{k^2} \right) v(x, t, \theta - t \omega, v_{0,m}),
\]
where \( k \geq 1 \) is fixed. Then for \( P \)-a.e. \( \omega \in \Omega \), the sequence \( \{ \tilde{v}_m(t_m, \theta_{-t_m} \omega) \}_{m=1}^\infty \) has a convergent subsequence in \( H^1_0(Q) \).

**Proof.** By Lemma 4.1 for \( P \)-a.e. \( \omega \in \Omega \), there is \( T_1 = T_1(B, \omega) \) such that for all \( t \geq T_1 \),
\[
\| v(t, \theta - t \omega, v_0(\theta - t \omega)) \|_{H^1_0(Q)} \leq r(\omega), \tag{4.94}
\]
where \( r(\omega) \) is a positive random function. Since \( t_m \to \infty \), there is \( M_1 = M_1(B, \omega) \) such that for all \( m \geq M_1, t_m \geq T_1 \), and hence by (4.94) we have
\[
\| v(t_m, \theta_{-t_m} \omega, v_0(\theta_{-t_m} \omega)) \|_{H^1_0(Q)} \leq r(\omega), \quad \forall \ m \geq M_1,
\]

which implies that
\[
\|v(t_m, \theta_{-t_m}, v_{0,m})\|_{H_0^1(Q)} \leq r(\omega), \quad \forall \ m \geq M_1. \tag{4.95}
\]
By (4.95) we find that
\[
\|\tilde{v}_m(t_m, \theta_{-t_m}, \omega)\|_{H_0^1(Q)} \leq cr(\omega), \quad \forall \ m \geq M_1. \tag{4.96}
\]
Given \(\epsilon > 0\), it follows from Lemma 4.4 that there are \(T_2 = T_2(B(\omega, \epsilon))\) and \(N = N(\omega, \epsilon)\) such that for all \(t \geq T_2\),
\[
\|(I - P_N)\tilde{v}(t, \theta_{-t}, \omega)\|_{H_0^1(Q_{2k})} \leq \epsilon. \tag{4.97}
\]
Take \(M_2 = M_2(B, \omega, \epsilon)\) large enough such that \(t_m \geq T_2\) for \(m \geq M_2\). Then we get from (4.97) that
\[
\|(I - P_N)\tilde{v}_m(t_m, \theta_{-t_m}, \omega)\|_{H_0^1(Q_{2k})} \leq \epsilon, \quad \forall \ m \geq M_2. \tag{4.98}
\]
On the other hand, (4.96) shows that the sequence \(\{P_N\tilde{v}_m(t_m, \theta_{-t_m}, \omega)\}\) is bounded in the finite-dimensional space \(P_NH_0^1(Q_{2k})\) and hence is precompact in \(P_NH_0^1(Q_{2k})\), which along with (4.98) implies the precompactness of \(\{\tilde{v}_m(t_m, \theta_{-t_m}, \omega)\}\) in \(H_0^1(Q_{2k})\). Note that \(\tilde{v}_m(x, t_m, \theta_{-t_m}, \omega) = 0\) for \(x \notin Q_{2k}\) and hence \(\{\tilde{v}_m(t_m, \theta_{-t_m}, \omega)\}\) is precompact in \(H_0^1(Q)\). \(\square\)

Next we establish the asymptotic compactness of the solutions of problem (3.10)-(3.12).

**Lemma 4.6.** Assume that \(g \in L^2(Q)\), \(h \in H_0^1(Q)\) and (3.4) holds. Let \(B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D},\) \(t_n \to \infty\) and \(v_{0,n} \in B(\theta_{-t_n}, \omega)\). Then for \(P\text{-a.e. } \omega \in \Omega\), the sequence \(\{v(t_n, \theta_{-t_n}, \omega, v_{0,n})\}_{n=1}^\infty\) has a convergent subsequence in \(H_0^1(Q)\).

**Proof.** Given \(\epsilon > 0\), it follows from Lemma 4.3 that, for \(P\text{-a.e. } \omega \in \Omega\), there are \(T_1 = T_1(B, \omega, \epsilon)\) and \(k_0 = k_0(\omega, \epsilon)\) such that for all \(t \geq T_1\),
\[
\int_{Q \setminus Q_{k_0}} (|v(t, \theta_{-t}, \omega, v_0(\theta_{-t}, \omega))|^2 + |\nabla v(t, \theta_{-t}, \omega, v_0(\theta_{-t}, \omega))|^2) \, dx \leq \epsilon. \tag{4.99}
\]
Let \(N_1 = N_1(B, \omega, \epsilon)\) be large enough such that \(t_n \geq T_1\) for \(n \geq N_1\). Then by (4.99) we have
\[
\int_{Q \setminus Q_{k_0}} (|v(t_n, \theta_{-t_n}, \omega, v_0(\theta_{-t_n}, \omega))|^2 + |\nabla v(t_n, \theta_{-t_n}, \omega, v_0(\theta_{-t_n}, \omega))|^2) \, dx \leq \epsilon, \quad \forall \ n \geq N_1,
\]
which implies that
\[
\int_{Q \setminus Q_{k_0}} \left( |v(t_n, \theta_{-t_n} \omega, v_{0,n})|^2 + |\nabla v(t_n, \theta_{-t_n} \omega, v_{0,n})|^2 \right) \, dx \leq \epsilon, \quad \forall \, n \geq N_1. \tag{4.100}
\]
Denote by
\[
\tilde{v}_n(x, t_n, \theta_{-t_n} \omega) = \psi \left( \frac{x^2}{k^2} \right) v(x, t_n, \theta_{-t_n} \omega, v_{0,n}).
\]
Then from Lemma [4.5] we know that, up to a subsequence, \{\tilde{v}_n(t_n, \theta_{-t_n} \omega)\} is convergent in \(H^1_0(Q)\), which shows that \{\tilde{v}_n(t_n, \theta_{-t_n} \omega)\} is a Cauchy sequence in \(H^1_0(Q)\), and hence also a Cauchy sequence in \(H^1(Q_{k_0})\). Note that \(\tilde{v}_n(t_n, \theta_{-t_n} \omega) = v(t_n, \theta_{-t_n} \omega, v_{0,n})\) in \(Q_{k_0}\) and thus \{\tilde{v}_n(\omega)\} is a Cauchy sequence in \(H^1(Q_{k_0})\). This along with (4.100) shows that \{\tilde{v}(t, \omega, v_{0,n})\} is a Cauchy sequence in \(H^1_0(Q)\), as desired. \[QED\]

5 Random attractors

In this section, we prove the existence of a \(\mathcal{D}\)-random attractor for the random dynamical system \(\Phi\) associated with the stochastic Benjamin-Bona-Mahony equation on the unbounded channel \(Q\).

By (5.13) and (5.14), \(\Phi\) satisfies
\[
\Phi(t, \theta_{-t} \omega, u_0(\theta_{-t} \omega)) = u(t, \theta_{-t} \omega, u_0(\theta_{-t} \omega)) = v(t, \theta_{-t} \omega, v_{0}(\theta_{-t} \omega)) + z(\omega), \tag{5.1}
\]
where \(v_0(\theta_{-t} \omega) = u_0(\theta_{-t} \omega) - z(\theta_{-t} \omega)\). Let \(B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\) and define
\[
\tilde{B}(\omega) = \{v \in H^1_0(Q) : \|v\|_{H^1_0} \leq \|u(\omega)\|_{H^1_0} \|z(\omega)\|_{H^1_0}, \ u(\omega) \in B(\omega)\}. \tag{5.2}
\]
We claim that \(\tilde{B} = \{\tilde{B}(\omega)\}_{\omega \in \Omega}\) belongs to \(\mathcal{D}\) provided \(B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\). Note that \(B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\) implies that
\[
\lim_{t \to \infty} e^{-\frac{t}{\delta} d(B(\theta_{-t} \omega))} = 0. \tag{5.3}
\]
Since \(z(\omega)\) is tempered, by (5.2) and (5.3) we have
\[
\lim_{t \to \infty} e^{-\frac{t}{\delta} d(\tilde{B}(\theta_{-t} \omega))} \leq \lim_{t \to \infty} e^{-\frac{t}{\delta} d(B(\theta_{-t} \omega))} + \lim_{t \to \infty} e^{-\frac{t}{\delta} \|z(\theta_{-t} \omega)\|_{H^1_0}} = 0,
\]
which shows \(\tilde{B} = \{\tilde{B}(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\). Then by Lemma [4.1] for \(P\)-a.e. \(\omega \in \Omega\), if \(v_0(\omega) \in \tilde{B}(\omega)\), there is \(T_1 = T_1(\tilde{B}, \omega)\) such that for all \(t \geq T_1\),
\[
\|v(t, \theta_{-t} \omega, v_0(\theta_{-t} \omega))\|_{H^1_0} \leq r(\omega), \tag{5.4}
\]
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where $r(\omega)$ is a positive random function satisfying
\[ e^{-\frac{1}{T}t}r(\theta t\omega) \to 0 \quad \text{as } t \to \infty. \] (5.5)

Denote by
\[ K(\omega) = \{ u \in H^1_0(Q) : \| u \|_{H^1_0} \leq r(\omega) + \| z(\omega) \|_{H^1_0} \}. \] (5.6)

Then by (5.5) we have
\[ \lim_{t \to \infty} e^{-\frac{1}{T}t}d(K(\theta t\omega)) \leq \lim_{t \to \infty} e^{-\frac{1}{T}t}r(\theta t\omega) + \lim_{t \to \infty} e^{-\frac{1}{T}t}\| z(\theta t\omega) \|_{H^1_0} = 0, \]
which implies that $K = \{ K(\omega) \}_{\omega \in \Omega} \in \mathcal{D}$. We now show that $K$ is also an absorbing set of $\Phi$ in $\mathcal{D}$.

Given $B = \{ B(\omega) \}_{\omega \in \Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$, by (5.1) and (5.4) we find that, for all $t \geq T_1$,
\[ \| u(t, \theta t\omega, u_0(\theta t\omega)) \|_{H^1_0} \leq \| v(t, \theta t\omega, v_0(\theta t\omega)) \|_{H^1_0} + \| z(\omega) \|_{H^1_0} \leq r(\omega) + \| z(\omega) \|_{H^1_0}, \]
which along with (5.1) and (5.6) implies that
\[ \Phi(t, \theta t\omega, B(\theta t\omega)) \subseteq K(\omega), \quad \forall \ t \geq T_1, \] (5.7)
and hence $K = \{ K(\omega) \}_{\omega \in \Omega} \in \mathcal{D}$ is a closed absorbing set of $\Phi$ in $\mathcal{D}$. In a word, we have proved the following result.

**Lemma 5.1.** Assume that $g \in L^2(Q)$, $h \in H^1_0(Q)$ and (3.4) holds. Let $K = \{ K(\omega) \}_{\omega \in \Omega}$ be given by (5.6). Then $K = \{ K(\omega) \}_{\omega \in \Omega} \in \mathcal{D}$ is a closed absorbing set of $\Phi$ in $\mathcal{D}$.

As an immediate consequence of Lemma 4.6, we find that $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $H^1_0(Q)$.

**Lemma 5.2.** Assume that $g \in L^2(Q)$, $h \in H^1_0(Q)$ and (3.4) holds. Then the random dynamical system $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $H^1_0(Q)$; that is, for $P$-a.e. $\omega \in \Omega$, the sequence \( \{ \Phi(t_n, \theta t_n\omega, u_{0,n}) \} \) has a convergent subsequence in $H^1_0(Q)$ provided $t_n \to \infty$, $B = \{ B(\omega) \}_{\omega \in \Omega} \in \mathcal{D}$ and $u_{0,n} \in B(\theta t_n\omega)$.

**Proof.** Since $B = \{ B(\omega) \}_{\omega \in \Omega}$ belongs to $\mathcal{D}$, so does $\tilde{B} = \{ \tilde{B}(\omega) \}_{\omega \in \Omega}$ which is given by (5.2). Then it follows from Lemma 4.6 that, for $P$-a.e. $\omega \in \Omega$, up to a subsequence, \( \{ v(t_n, \theta t_n\omega, v_{0,n}) \} \) is convergent in $H^1_0(Q)$, where $v_{0,n} = u_{0,n} - z(\theta t_n\omega) \in \tilde{B}(\theta t_n\omega)$. This along with (5.1) shows that, up to a subsequence, \( \{ \Phi(t_n, \theta t_n\omega, u_{0,n}) \} \) is convergent in $H^1_0(Q)$.

\[ \square \]
We are now in a position to establish the existence of a $\mathcal{D}$-random attractor for $\Phi$.

**Theorem 5.3.** Assume that $g \in L^2(Q)$, $h \in H^1_0(Q)$ and (3.14) holds. Let $\mathcal{D}$ be the collection of random sets given by (3.15). Then the random dynamical system $\Phi$ has a unique $\mathcal{D}$-random attractor in $H^1_0(Q)$.

**Proof.** Notice that $\Phi$ has a closed absorbing set $\{K(\omega)\}_{\omega \in \Omega}$ in $\mathcal{D}$ by Lemma 5.1, and is $\mathcal{D}$-pullback asymptotically compact in $H^1_0(Q)$ by Lemma 5.2. Hence the existence of a unique $\mathcal{D}$-random attractor follows from Proposition 2.8 immediately. \qed

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