HOMOLOGY OF PLANAR POLYGON SPACES

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Abstract. In this paper we study topology of the variety of closed planar n-gons with given side lengths $l_1, \ldots, l_n$. The moduli space $M_\ell$ where $\ell = (l_1, \ldots, l_n)$, encodes the shapes of all such n-gons. We describe the Betti numbers of the moduli spaces $M_\ell$ as functions of the length vector $\ell = (l_1, \ldots, l_n)$. We also find sharp upper bounds on the sum of Betti numbers of $M_\ell$ depending only on the number of links $n$. Our method is based on an observation of a remarkable interaction between Morse functions and involutions under the condition that the fixed points of the involution coincide with the critical points of the Morse function.

1. Introduction and statement of the result

Given a string $\ell = (l_1, \ldots, l_n)$ of $n$ positive real numbers $l_i > 0$ one considers the moduli space $M_\ell$ of closed planar polygonal curves having side lengths $l_i$. Points of $M_\ell$ parametrize different shapes of such polygons. Formally $M_\ell$ is defined as the factor space

$$M_\ell = \{(u_1, \ldots, u_n) \in S^1 \times \cdots \times S^1; \sum_{i=1}^n l_i u_i = 0 \in \mathbb{C}\} / SO(2).$$

Here $u_i \in S^1 \subset \mathbb{C}$ denote the unit vectors in the directions of the sides of a polygon; the group of rotations $SO(2)$ acts diagonally on $(u_1, \ldots, u_n)$.

Viewed differently, $M_\ell$ is the configuration space of a planar linkage, a planar mechanism consisting of $n$ bars of length $l_1, \ldots, l_n$ connected by revolving joints. Such mechanisms play an important role in robotics where they describe closed kinematic chains and are used widely as elementary parts of more complicated mechanisms. Knowing the topology of $M_\ell$ (for different vectors $\ell$) can be used in designing control programmes and motion planning algorithms for mechanisms.

The length vector $\ell$ is called generic if $\sum_{i=1}^n l_i \epsilon_i \neq 0$ for any choice $\epsilon_i = \pm 1$.

It is known that for a generic length vector $\ell$ the space $M_\ell$ is a closed smooth manifold of dimension $n - 3$. If the length vector $\ell$ is not generic then $M_\ell$ is a compact $(n - 3)$-dimensional manifold with finitely many singular points.

The moduli spaces $M_\ell$ of planar polygonal linkages were studied extensively by many mathematicians; we will mention W. Thurston and J. Weeks.
Our goal in this paper is to give a general formula for the Betti numbers of the moduli space $M_\ell$ as functions of the length vector $\ell$. Our results cover both generic and non-generic vectors $\ell$. In the case of generic $\ell$ the Betti numbers of $M_\ell$ can easily be extracted from the results of the unpublished thesis of K. Walker [13].

Formulae for Betti numbers of polygon spaces in three-dimensional space are known (see papers of A.A. Klyachko [7] and J.-Cl. Hausmann and A. Knutson [2]). Also, J.-Cl. Hausmann and A. Knutson describe cohomology with $\mathbb{Z}_2$ coefficients of the factor $\bar{M}_\ell$ of $M_\ell$ with respect to the natural involution, see [2], Theorem 9.1.

A. Klyachko in his beautiful work [7] uses a remarkable symplectic structure on the moduli space of linkages in $\mathbb{R}^3$ in an essential way. His technique is based on properties of Hamiltonian circle actions (the perfectness of the Hamiltonian viewed as a Morse function). The subsequent important paper of J.-Cl. Hausmann and A. Knutson employs methods of symplectic topology as well: they apply the method of symplectic reduction. Note that J.-Cl. Hausmann and A. Knutson go one step further and compute the multiplicative structure on cohomology, however their description is not very explicit as it uses the language of generators and relations. Symplectic methods play also a central role in the work of M. Kapovich and J. Millson [5].

The moduli spaces of planar linkages $M_\ell$ do not carry symplectic structures in general. Therefore methods of symplectic topology are not applicable in this problem.

The proof of our main result (see Theorem 1 below) is obtained in a very simple manner, it uses a remarkable interaction between Morse functions and involutions under the condition that fixed points of the involution coincide with the critical points of the Morse function.

To state our main theorem we need the following definitions. A subset $J \subset \{1, \ldots, n\}$ is called short if

$$\sum_{i \in J} l_i < \sum_{i \notin J} l_i.$$ 

The complement of a short subset is called long. A subset $J \subset \{1, \ldots, n\}$ is called median if

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i.$$ 

Clearly, median subsets exist only if the length vector $\ell$ is not generic. Note the following simple observation: any two subsets $J, J' \subset \{1, \ldots, n\}$ have a nonempty intersection $J \cap J' \neq \emptyset$ provided that one of the subsets is long and the other is either long or median.

**Theorem 1.** Fix a link of the maximal length $l_i$, i.e. such that $l_i \geq l_j$ for any $j = 1, 2, \ldots, n$. For every $k = 0, 1, \ldots, n - 3$ denote by $a_k$ and
b_k corresponding to the number of short and median subsets of \( \{1, \ldots, n\} \) of cardinality \( k + 1 \) containing \( i \). Then the homology group \( H_k(M_\ell; \mathbb{Z}) \) is free abelian of rank

\[
a_k + b_k + a_{n-3-k},
\]

for any \( k = 0, 1, \ldots, n - 3 \).

By Theorem 1 the Poincaré polynomial

\[
p(t) = \sum_{k=0}^{n-3} \dim H_k(M_\ell; \mathbb{Q}) \cdot t^k
\]

of \( M_\ell \) can be written in the form

\[
q(t) + t^{n-3}q(t^{-1}) + r(t)
\]

where

\[
q(t) = \sum_{k=0}^{n-3} a_k t^k, \quad r(t) = \sum_{k=0}^{n-3} b_k t^k;
\]

the numbers \( a_k \) and \( b_k \) are described in the statement of Theorem 1.

A proof of Theorem 1 is given below in §5. In the rest of this introduction we illustrate the statement of Theorem 1 by several examples.

**Example 1.** Suppose that \( n = 5 \) and \( l_1 = 3, l_2 = 2, l_3 = 2, l_4 = 1, l_5 = 1 \). Then \( l_1 = 3 \) is the longest link and short subsets of \( \{1, \ldots, 5\} \) containing \( 1 \) are \( \{1\}, \{1, 4\} \) and \( \{1, 5\} \). Hence \( a_0 = 1, a_1 = 2 \) and by Theorem 1 the Poincaré polynomial of \( M_\ell \) equals \( 1 + 4t + t^2 \). We conclude that \( M_\ell \) is a closed orientable surface of genus 2.

**Example 2.** Consider the zero-dimensional Betti number

\[
a_0 + b_0 + a_{n-3}
\]

of \( M_\ell \) as given by Theorem 1. We want to show that this number can take values \( 0, 1, 2 \); the first possibility is clearly equivalent to \( M_\ell = \emptyset \). Without loss of generality we may assume that \( l_1 \leq l_2 \leq \cdots \leq l_n \). If \( \{n\} \) is short then \( a_0 = 1 \) and \( b_0 = 0 \). If \( \{n\} \) is median then \( a_0 = 0 \) and \( b_0 = 1 \); in this case clearly \( M_\ell \) is a single point. If \( \{n\} \) is long then \( a_k = 0 = b_k \) for any \( k \) and hence \( M_\ell = \emptyset \). We obtain that \( M_\ell = \emptyset \) if and only if there are no long one-element subsets of \( \{1, \ldots, n\} \) — a result first established by Kapovich and Millson in [5].

Let us show that the number \( a_{n-3} \) equals \( 0 \) or \( 1 \). Clearly, \( a_{n-3} \) coincides with the number of long two-element subsets \( \{r, s\} \subset \{1, \ldots, n-1\} \). There may exist at most one such pair: if \( \{r', s'\} \) is another long pair with \( r \neq r', r \neq s' \), then \( \{r, n\} \) and \( \{r', s'\} \) would be two disjoint long subsets which is impossible. We obtain that \( a_{n-3} = 1 \) if and only if the pair \( \{n-2, n-1\} \) is long and \( a_{n-3} = 0 \) otherwise.

We see that the moduli space \( M_\ell \) has two connected components if and only if the set \( \{n-2, n-1\} \) is long and \( \{n\} \) is short. In this case the length
vector \( \ell \) must be generic and short subsets \( J \subset \{1, \ldots, n\} \) containing \( n \) are exactly the subsets containing neither \( n-2 \) nor \( n-1 \). We see that the Poincaré polynomial of \( M_\ell \) in this case equals \( 2(1+\ell)^{n-3} \). M. Kapovich and J. Millson \(^5\) showed that if \( M_\ell \) is disconnected then it is diffeomorphic to the disjoint union of two copies of the torus \( T^{n-3} \).

**Example 3.** As another example consider the equilateral case when \( l_j = 1 \) for all \( j \). Assume first that \( n = 2r+1 \) is odd and hence \( \ell \) is generic. The short subsets in this case are subsets of \( \{1, \ldots, n\} \) of cardinality \( \leq r \). We may fix the index \( \{n\} \) as representing the longest link. Hence we find that \( b_k = 0 \) vanishes and \( a_k \) equals

\[
a_k = \begin{cases} 
\binom{n-1}{k} & \text{for } k \leq r-1, \\
0 & \text{for } k \geq r.
\end{cases}
\]

By Theorem \(^1\) the Betti numbers of \( M_\ell \) are given by

\[
b_k(M_\ell) = \begin{cases} 
\binom{n-1}{k} & \text{for } k < r-1, \\
2 \cdot \binom{n-1}{r-1} & \text{for } k = r-1, \\
\binom{n-1}{k+2} & \text{for } k > r-1.
\end{cases}
\]

Note that the sum of Betti numbers in this example equals

\[
\sum_{k=0}^{n-3} b_k(M_\ell) = 2^{n-1} - \binom{n-1}{r}, \quad \text{where } n = 2r+1.
\]

**Example 4.** Consider now the equilateral case \( l_j = 1 \) with \( n \) is even, \( n = 2r+2 \). The length vector is now not generic. The short subsets are all subsets of cardinality \( \leq r \) and the median subsets are all subsets of cardinality \( r+1 \). Hence we find that \( b_k = 0 \) for \( k \neq r \) and

\[
b_r = \binom{2r+1}{r}
\]

and the numbers \( a_k \) are given by formula \(^4\). Applying Theorem \(^1\) we find

\[
b_k(M_\ell) = \begin{cases} 
\binom{n-1}{k} & \text{for } k \leq r-1, \\
\binom{n}{r} & \text{for } k = r, \\
\binom{n-1}{k+2} & \text{for } r+1 \leq k \leq n-3.
\end{cases}
\]
The sum of Betti numbers in this example is
\[ (9) \quad \sum_{k=0}^{n-3} b_k(\mathcal{M}_\ell) = 2^{n-1} - \left( \frac{n-1}{r} \right), \quad \text{where} \quad n = 2r + 2. \]

The results described in Examples 3 and 4 were obtained earlier in [3], [4] by different methods.

In the next section we shall see that Examples 3 and 4 give moduli spaces \( \mathcal{M}_\ell \) with the maximal possible total Betti number for all length vectors \( \ell \) having the given number of links \( n \).

2. Maximum of the total Betti number of \( \mathcal{M}_\ell \)

It is well known that the moduli space of pentagons \( \mathcal{M}_\ell \) with a generic length vector \( \ell = (l_1, \ldots, l_5) \) is a compact orientable surface of genus not exceeding 4, see [8]. In the equilateral case, i.e. if \( \ell = (1,1,1,1,1) \), \( \mathcal{M}_\ell \) is indeed an orientable surface of genus 4 (it is a special case of (6)) and hence the above upper bound for pentagons is sharp. In this section we state a theorem generalizing this result for arbitrary \( n \). Namely, we prove that for any length vector \( \ell = (l_1, \ldots, l_n) \) the sum of the Betti numbers
\[ (10) \quad \sum_{i=0}^{n-3} b_i(\mathcal{M}_\ell) \]
is less or equal than the sum of Betti numbers of the moduli space of the equilateral linkage with the same number of sides \( n \).

**Theorem 2.** Let \( \ell = (l_1, \ldots, l_n) \) be a length vector, \( l_i > 0 \). Denote by \( r \) the number \( \left\lfloor \frac{n-1}{2} \right\rfloor \). Then the sum of Betti numbers of the moduli space \( \mathcal{M}_\ell \) does not exceed
\[ (11) \quad B_n = 2^{n-1} - \left( \frac{n-1}{r} \right). \]
This estimate is sharp: \( B_n \) equals the sum of Betti numbers of the moduli space of planar equilateral \( n \)-gons, see [3], [4].

Note that for \( n \) even the equilateral linkage with \( n \) sides is not generic and hence Theorem 2 does not answer the question about the maximum of the total Betti number on the set of all generic length vectors with \( n \) even.

**Theorem 3.** Assume that \( n \) is even and \( \ell = (l_1, \ldots, l_n) \) is a generic length vector. Then the sum of Betti numbers of \( \mathcal{M}_\ell \) does not exceed
\[ (12) \quad B'_n = 2 \cdot B_{n-1}, \]
where \( B_k \) is defined by (11). This upper bound is achieved on the length vector \( \ell = (1,1,\ldots,1,\epsilon) \) where \( 0 < \epsilon < 1 \) and the number of ones is \( n-1 \).

Note that \( M_{(1,\ldots,1,\epsilon)} \) is diffeomorphic to the product \( M_{(1,\ldots,1)} \times S^1 \) (the number of ones in both cases equals \( 2r+1 \)). Hence the sum of Betti numbers of \( M_{(1,\ldots,1,\epsilon)} \) is twice the sum of Betti numbers of \( M_{(1,\ldots,1)} \).
Proofs of Theorems 2 and 3 are given below in section 6.

The asymptotic behavior of $B_n$ (given by (11)) can be recovered using available information about Catalan numbers

$$C_r = \frac{1}{r+1} \binom{2r}{r} \sim \frac{2^{2r}}{\sqrt{\pi} r^{3/2}},$$

see [12]. One obtains the following asymptotic formula

$$B_n \sim 2^{n-1} \cdot \left(1 - \sqrt[4]{\frac{2}{n\pi}}\right)$$

which is valid for even and odd $n$.

From the discussion of Example 2 we know that in the case when $M_\ell$ is disconnected the sum of Betti numbers of $M_\ell$ equals $2^n - 2$ which is approximately half of $B_n$, see (13).

3. Morse theory on manifolds with involutions

Our main tool in computing the Betti numbers of the moduli space of planar polygons $M_\ell$ is Morse theory of manifolds with involution.

**Theorem 4.** Let $M$ be a smooth compact manifold with boundary. Assume that $M$ is equipped with a Morse function $f : M \to [0, 1]$ and with a smooth involution $\tau : M \to M$ satisfying the following properties:

1. $f$ is $\tau$-invariant, i.e. $f(\tau x) = f(x)$ for any $x \in M$;
2. The critical points of $f$ coincide with the fixed points of the involution;
3. $f^{-1}(1) = \partial M$ and $1 \in [0,1]$ is a regular value of $f$.

Then each homology group $H_i(M; \mathbb{Z})$ is free abelian of rank equal the number of critical points of $f$ having Morse index $i$. Moreover, the induced map $\tau_* : H_i(M; \mathbb{Z}) \to H_i(M; \mathbb{Z})$ coincides with multiplication by $(-1)^i$ for any $i$.

![Figure 1. Surface in $\mathbb{R}^3$](image)

As an illustration for Theorem 4 consider a surface is $\mathbb{R}^3$ (see Figure 1) which is symmetric with respect to the $z$-axis. The function $f$ is the
orthogonal projection onto the z-axis, the involution \( \tau : M \to M \) is given by \( \tau(x, y, z) = (-x, -y, z) \).

The critical points of \( f \) are exactly the intersection points of \( M \) with the z-axis.

**Proof of Theorem 4.** Choose a Riemannian metric on \( M \) which is invariant with respect to \( \tau \).

Let \( p \in M \) be a critical point of \( f \). By our assumption, \( p \) must be a fixed point of \( \tau \), i.e. \( \tau(p) = p \). We claim that the differential of \( \tau \) at \( p \) is multiplication by \(-1\), i.e.

\[
d\tau_p(v) = -v, \quad \text{for any } v \in T_pM.
\]

(14)

Firstly, since \( \tau \) is an involution, \( d\tau_p \) must have eigenvalues \( \pm 1 \). Assume that there exists a vector \( v \in T_pM \) with \( d\tau_p(v) = v \). Then the geodesic curve starting from \( p \) in the direction of \( v \) is invariant with respect to \( \tau \) implying that \( p \) is not isolated in the fixed point set of \( \tau \). This contradicts our assumption and hence \( d\tau_p \) must have eigenvalue \(-1\) only. Note that \( d\tau_p \) is diagonalizable as \( d\tau_p \) preserves the Hessian of \( f \) at \( p \):

\[
H(f)_p : T_p(M) \otimes T_p(M) \to \mathbb{R}
\]

(15)

which is a nondegenerate quadratic form. This proves (14).

Consider the gradient vector field \( v \) of \( f \) with respect to the Riemannian metric. We will assume that \( v \) satisfies the transversality condition, i.e. all stable and unstable manifolds of the critical points intersect transversally. \( v \) is \( \tau \)-invariant which means that

\[
v_{\tau(x)} = d\tau_x(v_x), \quad x \in M.
\]

(16)

The Morse-Smale chain complex \( (C_\ast(f), \partial) \) of \( f \) has the critical points of \( f \) as its basis and the differential is given by

\[
\partial(p) = \sum_q [p : q] q
\]

(17)

where in the sum \( q \) runs over the critical points \( q \) with Morse index \( \text{ind}(q) = \text{ind}(p) - 1 \). The incidence numbers \([p : q] \in \mathbb{Z}\) are defined as follows

\[
[p : q] = \sum_\gamma \epsilon(\gamma), \quad \epsilon(\gamma) = \pm 1,
\]

(18)

where \( \gamma : (-\infty, \infty) \to M \) are trajectories of the negative gradient flow \( \gamma(t) = -v_{\gamma(t)} \) satisfying the boundary conditions \( \gamma(t) \to p \) as \( t \to -\infty \) and \( \gamma(t) \to q \) as \( t \to +\infty \).

Observe that if \( \gamma \) is a trajectory as above then \( \tau \circ \gamma \) is another such trajectory. Indeed, using (16) we find \( (\tau \circ \gamma)' = d\tau(\gamma') = -d\tau(v_{\gamma(t)}) = -v_{\tau(\gamma(t))} \).

Theorem 4 would follow once we show that

\[
\epsilon(\gamma) + \epsilon(\tau \circ \gamma) = 0,
\]

(19)
i.e. the total contribution into \( (18) \) of a pair of symmetric trajectories is zero. Hence all incidence coefficients vanish \([p: q] = 0\) and the differentials of the Morse - Smale complex are trivial.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure2.png}
\caption{Two symmetric trajectories of the negative gradient flow}
\end{figure}

To prove (19) we first recall the definition of the sign \( \epsilon(\gamma) \in \{1,-1\}, \) see [10]. For a critical point \( p \) of \( f \) we denote by \( W^u(p) \) and \( W^s(p) \) the unstable and stable manifolds of \( p \). Recall that \( W^u(p) \) is the union of the trajectories \( \gamma : (-\infty, \infty) \rightarrow M \) satisfying the differential equation \( \gamma'(t) = -v_{\gamma(t)} \) and the boundary condition \( \gamma(t) \rightarrow p \) as \( t \rightarrow -\infty \). The stable manifold \( W^s(p) \) is defined similarly but the boundary condition in this case becomes \( \gamma(t) \rightarrow p \) as \( t \rightarrow +\infty \).

Fix an orientation of the stable manifold \( W^s(p) \) for every critical point \( p \in M \). Since \( W^s(p) \) and \( W^u(p) \) are of complementary dimension and intersect transversally at \( p \), the orientation of \( W^s(p) \) determines a coorientation of the unstable manifold \( W^u(p) \), for every \( p \).

If \( \text{ind}(p) - \text{ind}(q) = 1 \) then \( W^u(p) \) and \( W^s(q) \) intersect transversally along finitely many connecting orbits \( \gamma(t) \) and the structure near each of the connecting orbits looks as shown on Figure 3. Note that the normal

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure3.png}
\caption{The stable and unstable manifolds along \( \gamma(t) \)}
\end{figure}

bundle to \( W^u(p) \) along \( \gamma \) coincides with the normal bundle to \( \gamma \) in \( W^s(q) \).
Hence, the coorientation of \( W^u(p) \) together with the natural orientation of the curve \( \gamma(t) \) determine an orientation of \( W^s(q) \) along \( \gamma \). We set \( \epsilon(\gamma) = 1 \) iff this orientation coincides with the prescribed orientation of \( W^s(q) \); otherwise we set \( \epsilon(\gamma) = -1 \).

To compare \( \epsilon(\gamma) \) with \( \epsilon(\tau \circ \gamma) \) we first observe that the involution \( \tau \) preserves the stable and unstable manifolds \( W^s(p) \) and \( W^u(p) \) and for every critical point \( p \) the degrees of the restriction of \( \tau \) on these submanifolds equal
\[
\deg(\tau|_{W^u(p)}) = (-1)^{\text{ind}(p)}, \quad \deg(\tau|_{W^s(p)}) = (-1)^{n-\text{ind}(p)};
\]
(20) as follows from (14). Hence, applying the involution \( \tau \) to the picture shown on Figure 3, we have to multiply the coorientation of \( W^u(p) \) by \((-1)^{n-\text{i-1}}\) and multiply the orientation of \( W^s(q) \) by \((-1)^{n-\text{i}}\). As the result the total sign will be multiplied by \((-1)^{n-\text{i-1}} \cdot (-1)^{n-\text{i}} = -1\). This proves (19) and completes the proof of the first statement of the theorem. The second statement of the Theorem follows from the first one combined with (20).

Theorem 5. Let \( M \) be a smooth compact connected manifold with boundary. Suppose that \( M \) is equipped with a Morse function \( f : M \to [0, 1] \) and with a smooth involution \( \tau : M \to M \) satisfying the properties of Theorem 4. Assume that for any critical point \( p \in M \) of the function \( f \) we are given a smooth closed connected submanifold
\[
X_p \subset M
\]
with the following properties:

1. \( X_p \) is \( \tau \)-invariant, i.e. \( \tau(X_p) = X_p \);
2. \( p \in X_p \) and for any \( x \in X_p - \{p\} \), one has \( f(x) < f(p) \);
3. the function \( f|_{X_p} \) is Morse and the critical points of the restriction \( f|_{X_p} \) coincide with the fixed points of \( \tau \) lying in \( X_p \). In particular, \( \dim X_p = \text{ind}(p) \).
4. For any fixed point \( q \in X_p \) of \( \tau \) the Morse indexes of \( f \) and of \( f|_{X_p} \) at \( q \) coincide.

Then each submanifold \( X_p \) is orientable and the set of homology classes realized by \( \{X_p\}_{p \in \text{Fix}(\tau)} \) forms a free basis of the integral homology group \( H_*(M; \mathbb{Z}) \). In other words, we claim that the inclusion induces an isomorphism
\[
\bigoplus_{\text{ind}(p) = i} H_i(X_p; \mathbb{Z}) \to H_i(M; \mathbb{Z})
\]
(21)
for any \( i \).

Proof of Theorem 5. First we note that each submanifold \( X_p \) is orientable. Indeed, Theorem 4 applied to the restriction \( f|_{X_p} \) implies that \( f|_{X_p} \) has a unique maximum and unique minimum and the top homology group \( H_i(X_p; \mathbb{Z}) = \mathbb{Z} \) is infinite cyclic where \( i = \dim X_p = \text{ind}(p) \).
For a regular value \( a \in \mathbb{R} \) of \( f \) we denote by \( M^a \subset M \) the preimage \( f^{-1}(-\infty, a] \). It is a compact manifold with boundary. It follows from Theorem 4 that \( f \) has a unique local minimum and therefore \( M^a \) is either empty or connected. For \( a \) slightly above the minimum value \( f(p_0) = \min f(M) \) the manifold \( M^a \) is a disc and the homology of \( M^a \) is obviously realized by the submanifold \( X_{p_0} \subset M^a \).

We proceed by induction on \( a \). Our inductive statement is that the homology of \( M^a \) is freely generated by the homology classes of the submanifolds \( X_p \) where \( p \) runs over all critical points of \( f \) satisfying \( f(p) \leq a \).

Suppose that the statement is true for \( a \) and the interval \([a, b]\) contains a single critical value \( c \). Let \( p_1, \ldots, p_r \) be the critical points of \( f \) lying in \( f^{-1}(c) \). Denote

\[
X = \bigsqcup_{i=1}^r X_{p_i}
\]

(the disjoint union). Then \( f \) induces a Morse function \( \bar{f} : X \to \mathbb{R} \) and we set

\[
X^a = \bar{f}^{-1}(-\infty, a].
\]

Consider the Morse - Smale complexes \( C_*(M^a), C_*(M^b), C_*(X) \) and \( C_*(X^a) \); the first two are constructed using the function \( f \) and the latter two are constructed using the function \( \bar{f} \). We have the following Mayer-Vietoris-type short exact sequence of chain complexes

\[
0 \to C_*(X^a) \to C_*(X) \oplus C_*(M^a) \xrightarrow{\Phi} C_*(M^b) \to 0 \tag{22}
\]

which (by the arguments indicated in the proof of Theorem 4) have trivial differentials and hence the sequence

\[
0 \to H_i(X^a) \to H_i(X) \oplus H_i(M^a) \xrightarrow{\Phi} H_i(M^b) \to 0 \tag{23}
\]

is exact (all homology groups have coefficients \( \mathbb{Z} \)). It follows from Lemma 6 below and the construction of the Morse - Smale complex (compare [10], §7) that the homomorphism \( \Phi \) (which appears in (22) and (23)) coincides with the sum of the chain maps induced by the inclusions \( X \to M^b \) and \( M^a \to M^b \).

For \( i < \dim X \) we have \( H_i(X^a) \to H_i(X) \) is an isomorphism (by Theorem 4 and hence (23)) implies that \( H_i(M^a) \to H_i(M^b) \) is an isomorphism. For \( i \geq \dim X \) we have \( H_i(X^a) = 0 \) and therefore \( \Phi : H_i(X) \oplus H_i(M^a) \to H_i(M^b) \) is an isomorphism. This completes the step of induction. \( \square \)

Here is a minor variation of the Morse lemma which has been used in the proof.

**Lemma 6.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth function having \( 0 \in \mathbb{R}^n \) as a nondegenerate critical point and suppose that for some \( k \leq n \) the restriction \( f|_{\mathbb{R}^k \times \{0\}} : \mathbb{R}^k \times \{0\} \to \mathbb{R} \) also has a nondegenerate critical point at \( 0 \in \mathbb{R}^k \).
Then there exists a neighborhood $U \subset \mathbb{R}^n$ of $0$ and a local coordinate system $x : U \to \mathbb{R}^n$ such that $x(\mathbb{R}^k \times \{0\} \cap U) \subset \mathbb{R}^k \times \{0\}$ and
\[
f(x_1, \ldots, x_n) = \pm x_1^2 + \cdots + \pm x_n^2 + f(0).
\]

Proof. One simply checks that the coordinate changes in the standard proof of the Morse lemma (compare [9], §2) can be chosen so that the subspace $\mathbb{R}^k \times \{0\}$ is mapped to itself.

4. The robot arm distance map

A robot arm is a simple mechanism consisting of $n$ bars (links) of fixed length $(l_1, \ldots, l_n)$ connected by revolving joints, see Figure 4. The initial point of the robot arm is fixed on the plane. The moduli space of a robot arm (i.e. the space of its possible shapes) is
\[
W = \{(u_1, \ldots, u_n) \in S^1 \times \cdots \times S^1\}/SO(2).
\]
Clearly, $W$ is diffeomorphic to a torus $T^{n-1}$ of dimension $n - 1$. A diffeomorphism can be specified, for example, by assigning to a configuration $(u_1, \ldots, u_n)$ the point $(1, u_2u_1^{-1}, u_3u_1^{-1}, \ldots, u_{n-1}u_1^{-1}) \in T^{n-1}$ (measuring angles between the directions of the first and the other links).

Consider the moduli space of polygons $M_\ell$ (where $\ell = (l_1, \ldots, l_n)$) which is naturally embedded into $W$.

We define a function on $W$ as follows:
\[
f_\ell : W \to \mathbb{R}, \quad f_\ell(u_1, \ldots, u_n) = -\left|\sum_{i=1}^n l_iu_i\right|^2.
\]
Geometrically the value of $f_\ell$ equals the negative of the squared distance between the initial point of the robot arm to the end of the arm shown by the dotted line on Figure 4. Note that the maximum of $f_\ell$ is achieved on the moduli space of planar linkages $M_\ell \subset W$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{robot_arm.png}
\caption{Robot arm.}
\end{figure}
An important role play the collinear configurations, i.e. such that \( u_i = \pm u_j \) for all \( i, j \), see Figure 5. We will label such configurations by long and median subsets \( J \subset \{1, \ldots, n\} \) assigning to any such subset \( J \) the configuration \( p_J \in W \) given by
\[
p_J = (u_1, \ldots, u_n)
\]
where \( u_i = 1 \) for \( i \in J \) and \( u_i = -1 \) for \( i \notin J \). Note that \( p_J \) lies in \( M_\ell \subset W \) if and only if the subset \( J \) is median.

**Figure 5.** A collinear configuration \( p_J \) of the robot arm.

**Lemma 7.** The critical points of \( f_\ell : W \to \mathbb{R} \) lying in \( W - M_\ell \) are exactly the collinear configurations \( p_J \) corresponding to long subsets \( J \subset \{1, 2, \ldots, n\} \). Each \( p_J \), viewed as a critical point of \( f_\ell \), is nondegenerate in the sense of Morse and its Morse index equals \( n - |J| \).

This lemma is well-known. It can be found as Proposition 3.3 in [13] and as combination of Theorems 3.1 and 3.2 in [1]; in both these references slightly different notations were used.

5. Proof of Theorem 1

Consider the moduli space \( W \) of the robot arm (defined by (25)) with the function \( f_\ell : W \to \mathbb{R} \) (defined by (26)). There is an involution
\[
\tau : W \to W
\]
given by
\[
\tau(u_1, \ldots, u_n) = (\bar{u}_1, \ldots, \bar{u}_n).
\]
Here the bar denotes complex conjugation, i.e. the reflection with respect to the real axis. It is obvious that formula (28) maps \( \text{SO}(2) \)-orbits into \( \text{SO}(2) \)-orbits and hence defines an involution on \( W \). The fixed points of \( \tau \) are the collinear configurations of the robot arm, i.e. the critical points of \( f_\ell \) in \( W - M_\ell \), see Lemma 7. Our plan it to apply Theorems 4 and 5 to the sublevel sets
\[
W^a = f_\ell^{-1}(-\infty, a]
\]
of \( f_\ell \). Recall that the values of \( f_\ell \) are nonpositive and the maximum is achieved on the submanifold \( M_\ell \subset W \). From Lemma 7 we know that the critical points of \( f_\ell \) are the collinear configurations \( p_J \). The latter are labelled by long subsets \( J \subset \{1, \ldots, n\} \) and \( p_J = (u_1, \ldots, u_n) \) where \( u_i = 1 \) for \( i \in J \) and \( u_i = -1 \) for \( i \notin J \). One has
\[
f_\ell(p_J) = -(L_J)^2.
\]
Here \( L_J = \sum_{i=1}^{n} l_i u_i \) with \( p_J = (u_1, \ldots, u_n) \).

The number \( a \) which appears in (29) will be chosen so that
\[
- (L_J)^2 < a < 0
\]
for any long subset \( J \) such that the manifold \( W^a \) contains all the critical points \( p_J \). The situation is shown schematically on Figure 6.

![Figure 6. Function \( f_\ell : W \to \mathbb{R} \) and the manifold \( W^a \).](image)

For each subset \( J \subset \{1, \ldots, n\} \) we denote by \( \ell_J \) the length vector obtained from \( \ell = (l_1, \ldots, l_n) \) by integrating all links \( l_i \) with \( i \in J \) into one link. For example, if \( J = \{1, 2\} \) then \( \ell_J = (l_1 + l_2, l_3, \ldots, l_n) \). We denote by \( W_J \) the moduli space of the robot arm with the length vector \( \ell_J \). It is obvious that \( W_J \) is diffeomorphic to a torus \( T^{n-|J|} \). We view \( W_J \) as being naturally embedded into \( W \). Note that the submanifold \( W_J \subset W \) is disjoint from \( M_\ell \) (in other words, \( W_J \) contains no closed configurations) if and only if the subset \( J \subset \{1, \ldots, n\} \) is long.

**Lemma 8.** Let \( J \subset \{1, \ldots, n\} \) be a long subset. The submanifold \( W_J \subset W \) has the following properties:

1. \( W_J \) is invariant with respect to the involution \( \tau : W \to W \);
2. the restriction of \( f_\ell \) onto \( W_J \) is a Morse function having as its critical points the collinear configurations \( p_I \) where \( I \) runs over all subsets \( I \subset \{1, \ldots, n\} \) containing \( J \);
3. for any such \( p_I \) the Morse indexes of \( f_\ell \) and of \( f_\ell|_{W_J} \) at \( p_I \) coincide.
4. in particular, \( f_\ell|_{W_J} \) achieves its maximum at \( p_J \in W_J \).

**Proof.** (1) is obvious. Statements (2) and (3) follow from Lemma 7 applied to the restriction of \( f_\ell \) onto \( W_J \). Here we use the assumption that \( J \) is long. Under this assumption the long subset for the integrated length vector \( \ell_J \) are in one-to-one correspondence with the long subsets \( I \subset \{1, \ldots, n\} \) containing \( J \). Statement (4) follows from (3) as the Morse index of \( f_\ell|_{W_J} \) at point \( p_J \) equals \( n - |J| = \dim W_J \). \( \Box \)

Applying Theorems 4 and 5 and taking into account Lemma 8, we obtain:

**Corollary 9.** One has:
(1) If \( \alpha \) satisfies \( (\mathcal{M}) \) then the manifold \( W^\alpha \) (see \( (\mathcal{M}) \)) contains all submanifolds \( W_J \) where \( J \subset \{1, \ldots, n\} \) is an arbitrary long subset.

(2) The homology classes of the submanifolds \( W_J \) form a free basis of the integral homology group \( H_*(W^\alpha; \mathbb{Z}) \).

Next we examine the homomorphism

\[
\phi_* : H_i(W^\alpha; \mathbb{Z}) \to H_i(W; \mathbb{Z})
\]

induced by the inclusion \( \phi : W^\alpha \to W \).

Below we will assume that \( l_1 \geq l_j \) for all \( j \in \{1, \ldots, n\} \), i.e. \( l_1 \) is the longest link. This may always be achieved by relabelling.

We describe a specific basis of the homology \( H_* (W; \mathbb{Z}) \). For any subset \( J \subset \{1, 2, \ldots, n\} \) we denote by \( W_J \) the moduli space of configurations of the robot arm with length vector \( \ell_J \) where all links \( l_i \) with \( i \in J \) are integrated into a single link. Note that \( W_J \) is naturally embedded into \( W \) and

\[
W_J \cap M_\ell = \emptyset
\]

if and only if the set \( J \) is long. Since \( W \) is homeomorphic to the torus \( T^{n-1} \), it is easy to see that a basis of the homology group \( H_* (W; \mathbb{Z}) \) is formed by the homology classes of the submanifolds \( W_J \) where \( J \subset \{1, \ldots, n\} \) runs over all subsets containing 1. We will denote the homology class of \( W_J \) by

\[
[W_J] \in H_{n-|J|} (W; \mathbb{Z}).
\]

Assuming that \( J, J' \subset \{1, \ldots, n\} \) are two subsets with \( |J| + |J'| = n + 1 \) the classes \([W_J]\) and \([W_{J'}]\) have complementary dimensions in \( W \) and their intersection number is given by

\[
[W_J] \cdot [W_{J'}] = \begin{cases} 
+1, & \text{if } |J \cap J'| = 1, \\
0, & \text{if } |J \cap J'| > 1.
\end{cases}
\]

Indeed, if \( J \cap J' = \{i_0\} \) then \( W_J \cap W_{J'} \) consists of a single point \( \{p\} \), the moduli space of a robot arm with all links integrated into one link. Let us show that the intersection \( W_J \cap W_{J'} \) is transversal. A tangent vector to \( W \) at \( p = (u_1, \ldots, u_n) \) can be labelled by a vector \( w = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) (an element of the Lie algebra of the torus \( T^n \)) viewed up to adding vectors of the form \((\lambda, \lambda, \ldots, \lambda)\). Such a tangent vector \( w \) is tangent to the submanifold \( W_J \) iff \( \lambda_i = \lambda_j \) for all \( i, j \in J \). Given \( w \) as above it can be written as

\[
w = w' + w'' + (\lambda_{i_0}, \ldots, \lambda_{i_0})
\]

where \( w' \) has coordinates 0 on places \( i \in J \) and coordinates \( \lambda_i - \lambda_{i_0} \) on places \( i \notin J \); coordinates of \( w'' \) vanish on places \( i \notin J \) and are \( \lambda_i - \lambda_{i_0} \) on places \( i \in J \). Hence every tangent vector to \( W \) is a sum of a tangent vector to \( W_J \) and a tangent vector to \( W_{J'} \).

Now suppose that \( |J \cap J'| > 1 \). We will show that then the submanifold \( W_{J'} \) can be continuously deformed inside \( W \) to a submanifold \( W'_{J'} \) such that
$W_J \cap W_{J'} = \emptyset$. This would prove the second claim in (34). Let us assume that $\{1, 2\} \subset J \cap J'$. Define $g_t : W_{J'} \to W$ by

$$g_t(u_1, \ldots, u_n) = (e^{i\theta}u_1, u_2, \ldots, u_n), \quad t \in [0, 1].$$

Here $\theta$ satisfies $0 < \theta < \pi$. Then $W_{J'} = g_1(W_{J'})$ is clearly disjoint from $W_J$; indeed, the links $l_1$ and $l_2$ are parallel in $W_J$ and make an angle $\theta$ in $W_{J'}$.

It follows that the intersection form in the basis $[W_J], [W_{J'}] \in H_*(W; \mathbb{Z})$, where $J \ni 1, J' \ni 1$, has a very simple form:

$$[W_J] \cdot [W_{J'}] = \begin{cases} 
\pm 1, & \text{if } J \cap J' = \{1\}, \\
0, & \text{if } |J \cap J'| > 1.
\end{cases}$$

(35)

In particular, given $[W_J]$ with $1 \in J$, its dual homology class $\in H_*(W; \mathbb{Z})$ equals $[W_K]$ where $K = CJ \cup \{1\}$; here $CJ$ denotes the complement of $J$ in \{1, \ldots, n\}.

Denote by $A_\ast \subset H_*(W^a; \mathbb{Z})$ (correspondingly, $B_\ast \subset H_*(W^a; \mathbb{Z})$) the subgroup generated by the homology classes $[W_J]$ where $J \subset \{1, \ldots, n\}$ is long and contains 1 (correspondingly, $J$ is long and 1 $\not\in J$). Then

$$H_i(W^a; \mathbb{Z}) = A_i \oplus B_i.$$ 

(36)

Similarly, one has

$$H_i(W; \mathbb{Z}) = A_i \oplus C_i \oplus D_i,$$ 

(37)

where:

- $A_\ast$ is as above;
- $C_\ast \subset H_*(W; \mathbb{Z})$ is the subgroup generated by the homology classes $[W_J]$ with $J \subset \{1, \ldots, n\}$ short and 1 $\in J$;
- $D_\ast$ is the subgroup generated by the classes $[W_J] \in H_*(W; \mathbb{Z})$ where $J$ is median and contains 1.

It is clear that $\phi_\ast$ (see (32)) is identical when restricted to $A_i$, compare (36) and (37). We claim that the image $\phi_\ast(B_i)$ is contained in $A_i$. This would follow once we show that

$$[W_J] : [W_K] = 0$$

(38)

assuming that $[W_J] \in B_i$ and $[W_K]$ is the dual of a class $[W_{J'}] \in C_i$ or $[W_{J'}] \in D_i$, see (35). We have

1. $J$ is long and 1 $\not\in J$,
2. $J'$ is short or median and 1 $\in J'$,
3. $|J| = |J'|$,
4. $K = CJ' \cup \{1\}$.

Here $CJ'$ denotes the complement of $J'$ in \{1, \ldots, n\}. By (34), to prove (38) we have to show that under the above conditions one has $|J \cap K| > 1$. Indeed, suppose that $|J \cap K| = 1$, i.e. $J \cap K = \{j\}$, a single element subset. Then $J'$ is obtained from $J$ by removing the index $j$ and adding the index
1 which leads to a contradiction: indeed, \( J \) is long, \( l_j \leq l_1 \) and \( J' \) is either short or median.

**Corollary 10.** The kernel of the homomorphism 

\[
\phi_i : H_i(W^a; \mathbb{Z}) \to H_i(W; \mathbb{Z})
\]

has rank equal\(^1\) to \( \text{rk} B_i \) and the cokernel has rank \( \text{rk} C_i + \text{rk} D_i \).

Below we skip the coefficient group \( \mathbb{Z} \) from the notations.

One has

\[
H_j(W, W^a) \simeq H_j(N, \partial N) \simeq H^{n-1-j}(N) \simeq H^{n-1-j}(M_\ell).
\]

(39)

Here \( N \) denotes the preimage \( f^{-1}_\ell([a, 0]) \).

Note that \( M_\ell \) is a deformation retract of \( N \). Indeed, consider \( M_\ell \subset N'' \subset N' \subset N \) where \( N' \) is a regular neighborhood of \( M_\ell \) in \( N \) and \( N'' \) is a sublevel set \( N'' = f^{-1}_\ell([a', 0]) \) and \( a < a' < 0 \) is such that \( N'' \) is contained in \( N' \). Since \( M_\ell \subset N' \) and \( N'' \subset N \) are deformation retracts, we have the following diagram

\[
\begin{array}{ccc}
M_\ell & \xleftarrow{i} & N' \\
\downarrow{r} & \nearrow{j} & \downarrow{k} \\
N'' & \xleftarrow{j'} & N
\end{array}
\]

where \( i, j, k \) are inclusions and \( r'ji = 1_{M_\ell} \), \( jir' \simeq 1_{N'} \), \( rkj = 1_{N''} \), \( kjr \simeq 1_N \).

It follows that \( g = r'jr : N \to M_\ell \) is a deformation retraction.

Hence we obtain the following short exact sequence

\[
0 \to \text{coker}(\phi_{n-1-j}) \to H^j(M_\ell) \to \ker \phi_{n-2-j} \to 0
\]

which splits since the kernel of \( \phi_{n-2-j} \) is isomorphic to \( B_{n-2-j} \) (see above) and hence it is free abelian.

This proves that the cohomology \( H^*(M_\ell) \) has no torsion and therefore the homology \( H_*(M_\ell) \) is free as well (by the Universal Coefficient Theorem).

The cokernel of \( \phi_{n-1-j} \) is isomorphic to \( C_{n-1-j} \oplus D_{n-1-j} \) as we established earlier. We find that the rank of \( \text{coker}\phi_{n-1-j} \) equals the number of subsets \( J \subset \{1, \ldots, n\} \) which are short or median and have cardinality \( |J| = j + 1 \). In other words,

\[
\text{rk}(\text{coker}\phi_{n-1-j}) = a_j + b_j,
\]

(41)

where we use the notation introduced in the statement of Theorem \( \mathbf{4} \).

The rank of the kernel of \( \phi_{n-2-j} \) equals the rank of \( B_{n-2-j} \), i.e. the number of long subsets \( J \subset \{2, \ldots, n\} \) of cardinality \( |J| = j + 2 \). Passing to the complements, we find

\[
\text{rk}(\ker \phi_{n-2-j}) = a_{n-3-j}
\]

(42)

i.e. the number of short subsets containing 1 with \( |J| = n - 2 - j \).

---

\(^1\)Note that the kernel of \( \phi_i \) (viewed as a subgroup) is distinct from \( B_i \) in general.
Combining (11), (12) with the exact sequence (10) we finally obtain
\[ \text{rk} H_j(M_\ell) = \text{rk} H^j(M_\ell) = a_j + b_j + a_{n-3-j}. \]
This completes the proof, compare (1).

6. Proofs of Theorems 2 and 3

The proofs are based on Theorem 1 and are purely combinatorial. Let \( \ell = (l_1, \ldots, l_n) \) be a length vector. Without loss of generality we may assume that \( l_1 \leq l_2 \leq \cdots \leq l_n \). By Theorem 1 the sum of Betti numbers of \( M_\ell \) equals twice the number of short subsets plus the number of median subsets of \( \{1, \ldots, n\} \), containing \( n \). We show that the number of such subsets is bounded above by \( B_n \) (given by (11)); moreover, we show that it is bounded above by \( B'_n = 2 \cdot B_{n-1} \) (see (12)) assuming additionally that \( \ell \) is generic and \( n \) is even.

We will treat simultaneously both cases \( n \) even and \( n \) odd. Denote \( r = [(n-1)/2] \) so that \( n = 2r + 2 \) for \( n \) even and \( n = 2r + 1 \) for \( n \) odd.

For \( 1 \leq i \leq n \) we denote by \( S_i(\ell) \) (respectively, \( M_i(\ell) \)) the number of short (respectively, median) subsets of \( \{1, \ldots, n\} \) containing the subset \( \{n-i+1, n-i+2, \ldots, n\} \). Clearly, \( S_{r+1}(\ell) = M_{r+1}(\ell) = 0 \) for \( n \) odd and \( S_{r+1}(\ell) = 0, M_{r+1}(\ell) \leq 1 \) for \( n \) even.

We claim that
\[ 2 \cdot S_i(\ell) + M_i(\ell) \leq 2^{n-i} - \sum_{j=r-i+1}^{r} \binom{n-i}{j} \] 
for all \( 1 \leq i \leq r + 1 \). For \( i = 1 \) inequality (43) gives
\[ 2 \cdot S_1(\ell) + M_1(\ell) \leq 2^{n-1} - \binom{n-1}{r} = B_n \]
which is equivalent to our goal (11). We will prove (43) by induction on \( n \) and by descending induction on \( i \).

For \( n \) odd and \( i = r + 1 \) inequality (43) gives \( 2 \cdot S_{r+1}(\ell) + M_{r+1}(\ell) \leq 0 \), which follows from our remark above. Similarly, for \( n \) even and \( i = r + 1 \) inequality (43) states \( 2 \cdot S_{r+1}(\ell) + M_{r+1}(\ell) \leq 1 \), which is obviously true, see above. These two remarks serve as the initial step of induction.

Assume now that inequality (43) is true (a) for \( i + 1 \) and (b) for all \( i \) and all length vectors \( \ell' = (l'_1, \ldots, l'_m) \) with \( m < n \) odd.

One can write
\[ S_i(\ell) = S_{i+1}(\ell) + S'_i(\ell), \quad M_i(\ell) = M_{i+1}(\ell) + M'_i(\ell) \]
where \( S'_i(\ell) \) and \( M'_i(\ell) \) denote the numbers of short and median subsets of \( \{1, \ldots, n\} \) containing \( \{n-i+1, \ldots, n\} \) and not containing \( n-i \). One observes that
\[ S'_i(\ell) \leq S_{i-1}(\ell), \quad \text{and} \quad S'_i(\ell) + M'_i(\ell) \leq S_{i-1}(\ell) + M_{i-1}(\ell), \]
where
\( \tilde{\ell} = (l_1, l_2, \ldots, l_{n-i-1}, l_{n-i+2}, \ldots, l_n) \).

Hence, using (44) and (45), we obtain
\[
2S_i(\ell) + M_i(\ell) \leq [2S_{i+1}(\ell) + M_{i+1}(\ell)] + [2S_{i-1}(\tilde{\ell}) + M_{i-1}(\tilde{\ell})].
\]

By our inductive hypothesis,
\[
2 \cdot S_i + M_i \leq 2n - i - 2r \sum_{j=r-i+1}^{r-i} \binom{n - i - 1}{j} + \binom{n - i}{r} - \binom{n - i - 1}{j}
\]
and
\[
2S_{i-1}(\tilde{\ell}) + M_{i-1}(\tilde{\ell}) \leq 2n - i - 2r \sum_{j=r-i+1}^{r-i} \binom{n - i - 1}{j}
\]
Adding the last two inequalities and taking into account (47) we obtain
\[
2S_i(\ell) + M_i(\ell) \leq 2n - i - 2r \sum_{j=r-i+1}^{r-i} \binom{n - i - 1}{j} + \binom{n - i}{r} - \binom{n - i - 1}{j}
\]
This completes the proof of Theorem 2.

To prove Theorem 3 we assume that \( n \) is even, \( n = 2r + 2 \), and \( \ell = (l_1, \ldots, l_n) \) is a generic length vector where \( l_1 \leq l_2 \leq \cdots \leq l_n \). We replace the inductive hypothesis (43) by
\[
2 \cdot S_i(\ell) \leq 2n - i - 2r \sum_{j=r-i+1}^{r-i} \binom{n - i - 1}{j}
\]
for \( 1 \leq i \leq r + 1 \). For \( i = 1 \) inequality (48) gives the desired inequality
\[
2 \cdot S_1(\ell) \leq 2n - 2r \binom{2r}{r} = B'_n,
\]
compare (12). For \( i = r + 1 \) inequality (48) states \( S_{r+1}(\ell) \leq 0 \) which is obviously correct; this statement will be the base of induction. To perform the step of induction we use inequalities (44) and (45) which are valid in the case of \( n \) even as well. We find
\[
2 \cdot S_{i+1}(\ell) \leq 2n-i-1 - 2r \sum_{j=r-i}^{r-i} \binom{n - i - 2}{j}
\]
and

\[ 2 \cdot S'_i(\ell) \leq 2^{n-i-1} - 2 \cdot \sum_{j=r-i+1}^{r-1} \binom{n-i-2}{j} \]

(both by the induction hypothesis) and adding the last two inequalities, using (44), and performing transformations similar to the odd case, we obtain (48).

This completes the proof of Theorem 3.

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