Learning to Reformulate for Linear Programming

Xijun Li\textsuperscript{1,2}, Qingyu Qu\textsuperscript{2}, Fangzhou Zhu\textsuperscript{2}, Jia Zeng\textsuperscript{2}, Mingxuan Yuan\textsuperscript{2}, Kun Mao\textsuperscript{3} and Jie Wang\textsuperscript{1}

\textsuperscript{1}MIRA Lab, USTC  
\textsuperscript{2}Huawei Noah’s Ark Lab  
\textsuperscript{3}Huawei Cloud Co.

\{xijun.li, quqingyu, zhufangzhou, zeng.jia, yuan.mingxuan, maokun\}@huawei.com, jiewangx@ustc.edu.cn

Abstract

It has been verified that the linear programming (LP) is able to formulate many real-life optimization problems, which can obtain the optimum by resorting to corresponding solvers such as OptVerse, Gurobi and CPLEX. In the past decades, a serial of traditional operation research algorithms have been proposed to obtain the optimum of a given LP in a fewer solving time. Recently, there is a trend of using machine learning (ML) techniques to improve the performance of above solvers. However, almost no previous work takes advantage of ML techniques to improve the performance of solver from the front end, i.e., the modeling (or formulation). In this paper, we are the first to propose a reinforcement learning-based reformulation method for LP to improve the performance of solving process. Using an open-source solver COIN-OR LP (CLP) as an environment, we implement the proposed method over two public research LP datasets and one large-scale LP dataset collected from practical production planning scenario. The evaluation results suggest that the proposed method can effectively reduce both the solving iteration number (25\%) and the solving time (15\%) over above datasets in average, compared to directly solving the original LP instances.

1 Introduction

Through many years of practices, it has been verified that the mathematical programming (LP) [Ge et al., 2021] is capable of formulating real-life optimization problems such as planning, scheduling, resource allocation, etc. The LP can obtain the optimum by resorting to corresponding solvers, such as OptVerse [Huawei, 2021], Gurobi [Gurobi, 2021], CPLEX [IBM, 2021], SCIP [ZIB, 2021], etc, which provides the optimal solutions to those real-life optimization problems. Government and business corporation benefit a lot from the practice of mathematical programmings, which thus constantly draws interests from both academics and industry. There are many kinds of mathematical programmings, including linear programming (LP), mixed integer programming (MIP), quadratic programming (QP). In past decades, a collection of classical algorithms (such as simplex [Dantzig, 1987], barrier [Ander- sen et al., 2003], branch and bound [Wolsey, 2007], etc.) have been proposed to solve above mathematical programmings and meanwhile they have been implemented and been integrated in above well-established solvers. Amongst these mathematical programmings, LP is the foundation. Thus many performance improvement of solver can be gained from the research of LP.

Recently, there is a trend of using machine learning (ML) techniques to improve traditional combinatorial optimization solvers [NeurIPS 2021 Competition, 2021] on specific problem distributions. Because in real-life scenarios, a practitioner repeatedly solves problem instances from a specific distribution, with redundant patterns and characteristics. For example, managing a large-scale energy distribution network requires solving very similar optimization problems on a daily basis, with a fixed power grid structure while only the demand changes over time. This change of demand is hard to capture by hand-engineered expert rules, and ML-enhanced approaches offer a possible solution to detect typical patterns in the demand history. A serial of machine learning-based approaches have been proposed to improve the performance of above solvers [Khalil et al., 2016; Gasse et al., 2019; Gupta et al., 2020; Nair et al., 2020].

It should be noted that most of above ML-enhanced approaches focus on using ML techniques to replace some key components within solver. Almost no previous work has thought of accelerating the solving of solver from the most front end, i.e., the modeling from a real-life optimization problem to a mathematical programming. Because we unconsciously think that the human experts are totally responsible for modeling and formulation for real-life optimization problems. The expert-designed formulation is deemed as the ‘perfect’ mathematical programming model and sent to the solver to get the optimal solution, which never thinks of how formulation could affect the performance of corresponding solver. But from some optimization theories and empirical studies [Ge et al., 2021], the formulation (such as the ordering of variables and constraints) is highly related to both the accuracy and solving speed of solver, which leaves the huge space for improving the performance of solvers through reformulating the mathematical programming.

In this paper, from the perspective of reformulation, we propose a machine learning-based automatic reformulation method for LP, in order to accelerate the solving, where a graph
The variables (columns) \( x \).

With above concepts, the basis can be defined as follows:

\[
A \begin{bmatrix} \mathbf{c} \\
\mathbf{x}
\end{bmatrix}
\]

where \( A \) is a linear program is an optimization problem of the form

The contributions of this work are summarized as follows:

• To our best knowledge, we are the first to propose a machine learning-based reformulation method for linear programming and implement the method using an open source software COIN-OR LP (CLP) as back-end solver.

• Extensive experiments have been performed over two public research LP datasets and one large-scale LP dataset collected from practical production planning scenario. The results suggest that the proposed method can effectively reduce both the solving iteration number (25%) and the solving time (15%) in average, compared to directly solving the original LP instances.

• Our proposed method does not restrict the type of solver, which can be implemented over any solver as long as it has corresponding interfaces. Thus the proposed method is a general way to improve the performance of solver.

2 Background

2.1 Linear programming and its initial basis

A linear program is an optimization problem of the form

\[
\begin{align*}
\min \quad & c^T x, \\
\text{s.t.} \quad & Ax \leq b, x \geq 0
\end{align*}
\]

where \( c \in \mathbb{R}^n \) is the objective coefficient vector; \( A \in \mathbb{R}^{m \times n} \) is the constraint coefficient matrix; and \( b \in \mathbb{R}^m \) is the constraint right-hand-side vector. \( x \in \mathbb{R}^n \) is the variable vector. Usually \( A \) is with full row rank. If the condition cannot be met, appropriate unit columns can be added. Note that formulation (1) is the standard form of linear programming. Other form of linear programming can convert to the standard form.

Suppose that \( S \) is a subset of columns of \( A \). We use \( A_S \) denote the \( m \times |S| \) submatrix that contains the columns of \( S \). If \( S \) is an order, then the columns of \( A_S \) are taken to appear in that order. Similarly, if \( d \) is a vector and \( S \) is a subset of row indices, then \( d_S \) is the corresponding subvector. Again, if \( S \) is ordered, the rows of \( d_S \) are also subject to that order.

With above concepts, the basis can be defined as follows:

Definition 1. A basis is a pair \((B, N)\) which splits the variables (columns): \( B = \{B_1, ..., B_m\} \subseteq \{1, ..., n\} \) is an ordered subset of column indices such that \( B = A_B \) is nonsingular. \( B \) is called the basis indices and \( B \) is the basis matrix. The variables (columns) \( x_j (j \in B) \) are called the basic variables. The remaining variables are nonbasic variables. We use \( N = \{j | j \notin B\} \) denote the set of indices of nonbasic variables.

Corresponding to each basis, there exists a basic solution \( x \) given by

\[
x_N = 0
\]

The relation between variables and constraints of LP can be represented by a bipartite graph, where a set of \( n \) nodes in the graph represents the \( n \) variables contained in the LP and the other set of \( m \) nodes correspond to the \( m \) constraints of the LP. The edge between a variable node and a constraint node represents the corresponding variable shows in the constraint. The number of edges indicates the number of non-zeros (NNZ) in the constraint matrix \( A \). An example of the bipartite graph is given in Figure 1. Other information such as the objective coefficients and constraint bounds, etc. can also be added into the bipartite graph. In this way, the lossless representation of the LP can be sent as an input to graph neural networks.

2.2 Bipartite graph representation of linear programming

The relation between variables and constraints of LP can be represented by a bipartite graph, where a set of \( n \) nodes in the graph represents the \( n \) variables contained in the LP and the other set of \( m \) nodes correspond to the \( m \) constraints of the LP. The edge between a variable node and a constraint node represents the corresponding variable shows in the constraint. The number of edges indicates the number of non-zeros (NNZ) in the constraint matrix \( A \). An example of the bipartite graph is given in Figure 1. Other information such as the objective coefficients and constraint bounds, etc. can also be added into the bipartite graph. In this way, the lossless representation of the LP can be sent as an input to graph neural networks.

2.3 Pointer network for combinatorial optimization problem

Combinatorial optimization problems such as Traveling Salesman Problem (TSP), Convex Hull problem, Set Cover problem,
etc. play a fundamental role in the development of computer science, which have many applications in manufacturing, planning, genetic engineering, etc. Many kinds of algorithms have been proposed to solve above combinatorial optimization problem, including dynamic programming [Sumita et al., 2017; Chauhan et al., 2012], cutting plane [Applegate et al., 2003], local search [Zhang and Looks, 2005] and neural network-based search method [Vinyals et al., 2015; Bello et al., 2016]. In recent years, the application of neural networks on combinatorial optimization problem has drawn much more attention than other methods [Vinyals et al., 2015; Bello et al., 2016], especially after the proposal of Pointer Network (PN). The pointer network is a sequence-to-sequence model [Vinyals et al., 2015] originating from the domain of natural language processing. It can learn the conditional probability of an output sequence of elements that are discrete symbols corresponding to positions in an input sequence, which dedicates to dealing with variable size of output dictionary. Specifically, the pointer network is comprised of two recurrent neural networks (RNN), encoder and decoder. The entire sequence-to-sequence output process is divided into two phases, encoding and decoding. In the encoding phase, the encoder reads initial representation (raw data or linear transformation of raw data) of $s_i$ of input sequence $S = \{s_1, s_2, ..., s_n\}$, one at a time step, and transform them into a sequence of latent memory states $\{\text{enc}_i\}_{i=1}^n$. In the decoding phase, the decoder network also maintains a latent memory states $\{\text{dec}_i\}_{i=1}^n$. Then it utilizes the attention mechanism [Vinyals et al., 2015] over $\{\text{enc}_i\}_{i=1}^n$ and $\{\text{dec}_i\}_{i=1}^n$ to produce a probability distribution over $s_i (i = 1, ..., n)$. Then one $s_i$ is pointed and output according to the probability distribution and its corresponding decoding embedding $\text{dec}_i$ is sent as input to the next decoder step. On the TSP, [Vinyals et al., 2015] trained above neural network in a supervised manner to predict the sequence of visited cities. [Bello et al., 2016] trained the network with reinforcement learning method, using the negative tour length as the reward signal.

3 Proposed Solution

3.1 Overview

In this section, our reformulation method is presented. We firstly introduce how we represent a given linear programming and send it as input into a graph neural network. Then the embedding output by the graph neural network is aggregated with a given variable splittings and passed into a pointer network to get a permutation of variables. The permutation is utilized to reformulate the original linear programming, in order to accelerate the solving process of corresponding solver. The entire process is summarized in Figure 2.

3.2 Representation

We adopt the same method as done in [Gasse et al., 2019] to represent a given linear programming as a bipartite graph $(G, C, E, V)$. Specifically, in the bipartite graph, $C \in \mathbb{R}^{n \times c}$ corresponds to the features of the constraints in the LP; $V \in \mathbb{R}^{n \times d}$ denotes the features of the variables in the LP; and an edge $e_{ij} \in E$ between a constraint node $i$ and a variable node $j$ if the corresponding coefficient $A_{ij} \neq 0$. For simplicity, we just attach the value of $A_{ij}$ to the corresponding edge $e_{ij}$.

Next, the bipartite graph representation of LP is sent as input into a two-interleaved graph convolutional neural network (GCNN) [Gasse et al., 2019]. In detail, the graph convolution is broken into two successive passes, one from variable side to constraint side, and one from constraint side to variable side, which can be formulated as follows:

$$
c^{(l+1)}_i \leftarrow f_C \left( c^{(l)}_i, \sum_j g_C(c^{(l)}_j, v^{(l)}_j, e_{ij}) \right), \quad (4)
$$

$$
v^{(l+1)}_j \leftarrow f_V \left( v^{(l)}_j, \sum_i g_V(c^{(l)}_i, v^{(l)}_i, e_{ij}) \right), \quad (5)
$$

where $f_C, g_C, f_V$ and $g_V$ are 2-layer perceptrons with prenorm layer. We adopt the ReLU as the activation function. And $l$ represents the number of times that we perform the convolution. In our implementation, we set $l = 2$. The parameters involved in above GCNN are denoted by $\theta_G$.

3.3 Aggregation

The embedding information of variables can be obtained using the GCNN. However, we perform an aggregation operation over the embedding rather than directly sending the them to pointer networks. There are several reasons why we need to perform aggregation. First, the learning capability of pointer network is limited. According to evaluation report of previous work [Vinyals et al., 2015; Bello et al., 2016], the pointer network can achieve closely optimal results with up to 100 nodes. It performs significantly worse when the number of nodes exceeds 1000. Second, considering all possible permutations of variables of a given LP is intractable. The number of variables of LP that comes from practical scenario can easily exceeds 100. Third, many LPs have its own special structure, which can be exploited to split the variables into several clusters in advance. Many optimization methods such as Benders decomposition [Mavrotas and Makryvelios, 2021; Gharaei et al., 2020] have exploited the structure of LP model to accelerate the searching process. Considering all above, we perform the aggregation using the following steps:

**Splitting.** For a given LP as shown in (1), the variables $x_i (i = 1, ..., n)$ are split up into $k$ disjoint clusters $C_j = \{x_{j1}, x_{j2}, ..., x_{jk}\} (j = 1, ..., k)$. Note that the ordering of variables within one cluster is subject to the ordering of variables in the original LP. The clustering method is not restricted here. It could be specified by human experts or using hyper-graph decomposition method [Manieri et al., 2021].

**Pooling function.** With above splitting clusters $C_j (j = 1, ..., k)$ and variable embedding $v_i (i = 1, ..., n)$, we perform the aggregation for each cluster via:

$$
\Sigma_j = \mathcal{P} \{v_i | x_i \in C_j\} \quad (6)
$$

where $\mathcal{P}$ is a pooling function which could be maximum, minimum, average or other appropriate functions. We still do not restrict the kind of the pooling function here. $\Sigma_j \in \mathbb{R}^d$ can be understood the embedding of splitting cluster.
3.4 Permutation

Given an LP \( lp \), we aim to reformulate the LP by reordering the variables of the original LP, in order to improve the solving performance of the corresponding solver. More specifically, given a sequence of splitting clusters \( \{C^j\}_{j=1}^k \) of \( lp \), we would like to find a permutation \( \pi \) of these clusters to reformulate the original LP. The reformulation is achieved by that 1) between splitting clusters, the variables will be reordered with its cluster according to the permutation \( \pi \); and 2) within each cluster, the order of variables remains the same with the original LP. In this way, the coefficients matrix \( A \) and cost coefficients vector \( c \) will correspondingly be altered. We hope the reformulation of the original LP can improve the solving performance of the solver, which depends on the preference of performance optimization. We formally define the improvement \( R \) of solving performance gained from reformulation as:

\[
R(\pi|lp) = SM(lp) - SM(lp|\pi) \tag{7}
\]

where \( SM(lp) \) denotes that with respect to a solving performance metric \( M \), calling a solver to solve \( lp \); and \( SM(lp|\pi) \) refers to that with respect to the same solving metric \( M \), calling the same solver to solve \( lp \) reformulated using the variable permutation \( \pi \). Using Eq.(7), we can measure how a permutation \( \pi \) can improve the solving performance \( M \) of solver, compared to the original LP without reformulation.

Our aim is to learn a probability distribution \( p(\pi|\{C^j\}_{j=1}^k) \) that given a sequence of splitting clusters \( \{C^j\}_{j=1}^k \) of \( lp \) and corresponding embedding \( \{\Sigma^j\}_{j=1}^k \), can assign higher probabilities to "better" permutations and lower probabilities to "worse" ones. The "better" and "worse" are measured using Eq.(7). Similar to [Vinyals et al., 2015; Bello et al., 2016], the probability distribution \( p(\pi|\{C^j\}_{j=1}^k) \) utilizes the chain rule to factorize the probability of a permutation as:

\[
p(\pi|\{C^j\}_{j=1}^k) = \prod_{j=1}^k p(\pi(j)|\pi(<j), \{C^j\}_{j=1}^k) \tag{8}
\]

We parameterize \( p(\pi|\{C^j\}_{j=1}^k) \) by a pointer network whose parameter is denoted by \( \theta_P \).

3.5 Training method

In our proposed method, there are two main classes of parameters, \( \theta_G \) and \( \theta_P \), to be learned. Theoretically, the parameters could be trained using supervised learning (SL) as done in [Vinyals et al., 2015] or reinforcement learning (RL) as done in [Bello et al., 2016]. However, we adopt the reinforcement learning method instead of supervised learning method. First, getting high-quality labelled data (getting improvement \( R \) gained from reformulation in our context) is expensive especially when the size of LP is large. Because we need to call a solver to solve two LP instances (i.e., original LP and its reformulated LP) each time we calculate the improvement \( R \). Besides, RL is deemed as an effective way to generate better supervised signals, which could help find a more competitive solution than purely supervised learning. Thus we propose to use model-free policy-based RL to learn \( \theta_G \) and \( \theta_P \).
The training objective is to maximize the expected improvement $R$ over a given $L P$ $l_p$, which is formally defined as:

$$J(\theta_G, \theta_P | l_p) = \mathbb{E}_{\pi \sim p_{\theta_G}, \pi_P(|l_p)}[R(\pi | l_p)] \quad (9)$$

In our training phase, the LPs usually come from the same practical scenario $S$ such as production planning, bin packing, etc. Thus the total training objective is defined as:

$$J(\theta_G, \theta_P) = \mathbb{E}_{l_p \sim S}[J(\theta_G, \theta_P | l_p)] \quad (10)$$

We utilize stochastic gradient ascent method to optimize Eq.(10). According to the REINFORCE algorithm, the gradient of Eq.(10) is given as:

$$\nabla_{\theta_G, \theta_P} J(\theta_G, \theta_P | l_p) = \mathbb{E}_{\pi \sim p_{\theta_G}, \pi_P(|l_p)}[(R(\pi | l_p) - b(l_p)) \nabla_{\theta_G, \theta_P} \log p_{\theta_G, \theta_P}(\pi | l_p)] \quad (11)$$

where $b(\cdot)$ is a baseline function independent of $\pi$ and estimates the expected improvement $R$ to reduce the learning variances. To enhance the estimate accuracy of baseline function, we additionally introduce a critic network parameterized by $\theta_c$, which is trained with stochastic gradient descent method over a mean squared error (MSE) between the true improvement $R$ and its prediction $b_c(l_p)$. The MSE loss is defined as:

$$\mathcal{L}(\theta_c) = \mathbb{E}_{l_p \sim S, \pi \sim p_{\theta_G}, \pi_P(|l_p)}[b_c(l_p) - R(\pi | l_p)]^2 \quad (12)$$

Note that in our implementation, all mentioned-above gradients are approximated using Monte Carlo sampling. We give a snippet of pseudo codes in Algorithm 1 (see Appendix).

4 Experimental Evaluation

4.1 Setting up

Implementation detail

In our implementation, we use the open-source LP solver COIN-OR LP (CLP) [COIN-OR Foundation, 2021a] as the "environment" in the reinforcement learning setting, with which the proposed method interact. To ease the interaction, we first adopt the well-developed CLP python interface library, CyLP [COIN-OR Foundation, 2021b] as the interface between the proposed reformulation method and CLP solver. Besides, we develop a new interface that can 1) take as input a given permutation; 2) reformulate a LP instance according to the given permutation; and 3) solve the reformulated LP instance and return the solving metric of interest.

With regard to the GCNN and PN involved in the proposed method, we implemented them using PyTorch [Paszke et al., 2019]. The corresponding hyperparameter is summarized in Table 3 (see Appendix). Note that we use the default parameter of CLP solver when we call the solver to solve a given LP. All experiments are conducted on a computing server, which is equipped with Intel(R) Xeon(R) Platinum 8180M CPU@2.50GHz, a V100 GPU card with 32GB graphic memory and 1TB main memory.

Dataset

The entire evaluation was performed over three sets of Mixed Integer Linear Programming (MILP) problems. All dataset are scenario specific, i.e., they contain problem instances from only a single scenario. Two of them, Balanced Item Placement (BIP) and Workload Apportionment (WA), are from [NeurIPS 2021 Competition, 2021]. The third set is obtained from Huawei Production Planning (HPP). The detailed description about above dataset is summarized in Appendix. Note that we relax the integer constraint of variables in above MILPs to get the LP instance. Within the dataset of BIP and WA, there are respectively in total 10000 LP instances, which are different in the value of coefficients, the size of constraints and variables. And for the dataset of HPP, there are in total 1000 LP instances. The statistical description of above datasets is summarized in Table 1, where $m, n$ and $NNZ$ represent the number of constraints, variables and non-zero coefficients of a linear programming, respectively. For each dataset, 80%, 10% and 10% of instances are used as the training, validation and testing set respectively. Besides, for each dataset, we train the neural networks involved in the proposed method separately. In other words, we have three sets of $(\theta_G, \theta_P, \theta_c)$ to train.

4.2 Evaluation result

Learning convergence

We first give the learning curves of neural network parameters (i.e., $\theta_P$ and $\theta_G$) in Figure 4 (see Appendix), which shows that the parameter learning of neural networks involved in our method can converge over the three datasets. However, compared with small-scale LP instances from BIP, it is relatively harder to learn neural network parameter over the large-scale and complex LP instances from WA and HPP, since the loss variance of former is much smaller than that of latter two.

Reduction of iteration number

We measure how our proposed reformulation method reduces the iteration number of solving process compared to directly solving the original LP. Specifically, we first call the CLP solver to directly solve LP instances from the testing set of above three datasets and record the iteration number of solving process (here refers to the iteration number of simplex method). Then we reformulate these LP instances using the learned neural networks. The CLP solver is called again to solve the reformulated LP instances and the iteration number is recorded subsequently. We compare the iteration number of solving reformulated LP instances against that of solving original ones. Due to the inference randomnes of neural network, here we adopt a $k$-shot inference mechanism, which means that the neural network will infer $k$ times and the best
We continue to measure how our proposed method reduces the solving time. The same procedure as described in Section 4.2 is adopted in this experiment. Different from Section 4.2, we here compare the solving time between original LP instances and its reformulation obtained from our method, instead of the solving iteration number. Note that for each LP instance reformulated by our method, we keep the best result among 3 times of inferences of neural networks. The results are presented in Figure 3. In these figures, the horizontal axis, Ratio of reformulate/original, denotes the ratio of the iteration number of solving reformulated LP instances to that of solving original ones. The vertical axis denotes the cumulative probability distribution of the ratio over the dataset. Several findings can be pointed out: 1) our learning-based reformulation method is effective in reducing the solving iteration number. When $k \geq 3$, almost all original LP instance can be solved with fewer solving iteration number via reformulating by our method; 2) our method can be generalized to unseen data since it still performs well over the testing dataset; and 3) On the LP intances from WA and HPP, our method performs slightly worse than on the ones from BIP, which is in accordance with what we observe in the learning convergence.

**Reduction of solving time**

We continue to measure how our proposed method reduces the solving time. The same procedure as described in Section 4.2 is adopted in this experiment. Different from Section 4.2, we here compare the solving time between original LP instances and its reformulation obtained from our method, instead of the solving iteration number. Note that for each LP instance reformulated by our method, we keep the best result among 3 times of inferences of neural networks. The results are presented in Table 2. From observing the results, several claims can be made: 1) the proposed method can indeed reduce the solving time of given LP instances by reformulating them, which inerably benefits from the reduction of solving iteration number; 2) our method performs slightly worse over the complex and large-scale LP instances but still can reduce at least 8.35% the solving time over the complex LP instances from WA and HPP. Besides, we give a visualization for the reformulation process of our method, in order to figure out what the neural networks have learned (see visual analysis in Appendix A).

| Dataset | Average reduction | Standard error |
|---------|-------------------|---------------|
| BIP | 25.18% | ±2.39% |
| WA | 10.78% | ±0.59% |
| HPP | 15.62% | ±2.47% |

Table 2: Reduction of solving time by our proposed method
References

[Andersen et al., 2003] E. D. Andersen, C. Roos, and T. Terlaky. On implementing a primal-dual interior-point method for conic quadratic optimization. Mathematical Programming, 95(2):249–277, 2003.

[Applegate et al., 2003] David Applegate, Robert Bixby, Vášek Chvátal, and William Cook. Implementing the dantzig-fulkerson-johnson algorithm for large traveling salesman problems. Mathematical Programming, 97(1):91–153, 2003.

[Bello et al., 2016] Irwan Bello, Hieu Pham, Quoc V Le, Mohammad Norouzi, and Samy Bengio. Neural combinatorial optimization with reinforcement learning. arXiv preprint arXiv:1611.09940, 2016.

[Bixby, 1992] Robert E Bixby. Implementing the simplex method: The initial basis. ORSA Journal on Computing, 4(3):267–284, 1992.

[Chauhan et al., 2012] Chetan Chauhan, Ravindra Gupta, and Kshitij Pathak. Survey of methods of solving tsp along with its implementation using dynamic programming approach. International journal of computer applications, 52(4), 2012.

[COIN-OR Foundation, 2021a] COIN-OR Foundation. Coin-or linear programming, https://github.com/coin-or/clp., 2021.

[COIN-OR Foundation, 2021b] COIN-OR Foundation. Cylp, https://github.com/coin-or/cylp., 2021.

[Dantzig, 1987] G. B. Dantzig. Origins of the simplex method. 1987.

[Gasse et al., 2019] Maxime Gasse, Didier Chételat, Nicola Ferroni, Laurent Charlin, and Andrea Lodi. Exact combinatorial optimization with graph convolutional neural networks. arXiv preprint arXiv:1906.01629, 2019.

[Ge et al., 2021] D. Ge, C. Wang, Z. Xiong, and Y. Ye. From an interior point to a corner point: Smart crossover. 2021.

[Gharaei et al., 2020] Abolfazl Gharaei, Mostafa Karimi, and Seyed Ashkan Hoseini Shekarabi. Joint economic lot-sizing in multi-product multi-level integrated supply chains: Generalized benders decomposition. International Journal of Systems Science: Operations & Logistics, 7(4):309–325, 2020.

[Gupta et al., 2020] Prateek Gupta, Maxime Gasse, Elias B Khalil, M Pawan Kumar, Andrea Lodi, and Yoshua Bengio. Hybrid models for learning to branch. arXiv preprint arXiv:2006.15212, 2020.

[Gurobi, 2021] Gurobi. Gurobi solver, https://www.gurobi.com/, 2021.

[Huawei, 2021] Huawei. Optverse solver, https://www.huaweicloud.com/product/modelarts/optverse.html, 2021.

[IBM, 2021] IBM. Cplex, https://www.ibm.com/hk-en/analytics/cplex-optimizer, 2021.

[Khalil et al., 2016] Elias Khalil, Pierre Le Bodic, Le Song, George Nemhauser, and Bistra Dilkina. Learning to branch in mixed integer programming. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 30, 2016.

[Manieri et al., 2021] Lucrezia Manieri, Alessandro Falsone, and Maria Prandini. Hyper-graph partitioning for a multi-agent reformulation of large-scale milps. IEEE Control Systems Letters, 6:1346–1351, 2021.

[Maros, 2002] István Maros. Computational techniques of the simplex method, volume 61. Springer Science & Business Media, 2002.

[Mavrotas and Makryvelios, 2021] George Mavrotas and Evangelos Makryvelios. Combining multiple criteria analysis, mathematical programming and monte carlo simulation to tackle uncertainty in research and development project portfolio selection: A case study from greece. European Journal of Operational Research, 291(2):794–806, 2021.

[Nair et al., 2020] Vinod Nair, Sergey Bartunov, Felix Gjeno, Ingrid van Glehn, Pawel Lichocki, Ivan Lobov, Brendan O’Donoghue, Nicolas Sonnerat, Christian Tjandraatmadja, Pengming Wang, et al. Solving mixed integer programs using neural networks. arXiv preprint arXiv:2012.13349, 2020.

[NeurIPS 2021 Competition, 2021] NeurIPS 2021 Competition. Machine learning for combinatorial optimization, https://www.ecole.ai/2021/ml4co-competition/, 2021.

[Paszke et al., 2019] Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, et al. Pytorch: An imperative style, high-performance deep learning library. Advances in neural information processing systems, 32:8026–8037, 2019.

[Selsam et al., 2018] Daniel Selsam, Matthew Lamm, Benedikt Bünz, Percy Liang, Leonardo de Moura, and David L. Dill. Learning a sat solver from single-bit supervision. arXiv preprint arXiv:1802.03685, 2018.

[Sumita et al., 2017] Hanna Sumita, Yuma Yonebayashi, Naonori Kakimura, and Ken-ichi Kawarabayashi. An improved approximation algorithm for the subpath planning problem and its generalization. In IJCAI, volume 2017, pages 4412–4418, 2017.

[Vinyals et al., 2015] Oriol Vinyals, Meire Fortunato, and Navdeep Jaitly. Pointer networks. arXiv preprint arXiv:1506.03134, 2015.

[Wolsey, 2007] Laurence A Wolsey. Mixed integer programming. Wiley Encyclopedia of Computer Science and Engineering, pages 1–10, 2007.

[Zhang and Looks, 2005] Weixiong Zhang and Moshe Looks. A novel local search algorithm for the traveling salesman problem that exploits backbones. In IJCAI, volume 5, pages 343–384. Citeseer, 2005.

[ZIB, 2021] ZIB. Scip solver, https://www.scipopt.org/, 2021.
Appendix A
Feature used in constructing bipartite graph
Features of constraint:
- rhs: the right-hand side coefficients of LP, i.e. b, normalized with constraint coefficients.
- ub_cons: upper bound of constraint, normalized with all constraints upper bound.
- lb_cons: lower bound of constraint, normalized with all constraints lower bound.
Features of variable:
- lb_var: upper bound of variable, normalized with all variables lower bound.
- ub_var: lower bound of variable, normalized with all variables lower bound.
- lb_var: lower bound of variable, normalized with all variables upper bound.
Features of edge:
- coef: the constraint coefficients of variables, i.e. A, normalized per constraint

Pseudo code of training method

Algorithm 1 Training procedure of the proposed method

Inputs
- \( S \): set of linear programming problems
- \( T \): number of training steps
- \( B \): batch size

Initializes
- \( \theta_G \): parameters for graph convolutional neural network
- \( \theta_P \): parameters for pointer network
- \( \theta_c \): parameters for critic network

for steps \( t = 1 \) to \( T \) do
- \( l_{p_i} \sim S \) for \( i \in \{1, ..., B\} \)
- \( \pi_i \sim p_{\theta_C, \theta_P}(|l_{p_i}|) \) for \( i \in \{1, ..., B\} \)
- Calculate \( R_i \) using Eq. (7) for \( i \in \{1, ..., B\} \)
- \( b_i = l_{\theta_G}(l_{p_i}) \) for \( i \in \{1, ..., B\} \)
- \( g \leftarrow \frac{1}{B} \sum_{i=1}^{B} (R_i - b_i) \nabla_{\theta_G, \theta_P} \log p_{\theta_C, \theta_P}(\pi_i | l_{p_i}) \)
- \( \mathcal{L}_c \leftarrow \frac{1}{T} \sum_{i=1}^{B} (b_i - R_i)^2 \)
- Perform a gradient ascent step to update \( \theta_G \) and \( \theta_P \) using \( g \) respectively
- Perform a gradient descent step to update \( \theta_c \) using \( \nabla_{\theta_c} \mathcal{L}_c \)
end for

return \( \theta_G \), \( \theta_P \) and \( \theta_c \)

Hyperparameters setting

Part of important hyperparameters involved in our method is list in Table 3.

Description of dataset
Balanced Item Placement. This problem deals with spreading items (e.g., files or processes) across containers (e.g., disks or machines) utilizing them evenly. Items can have multiple copies, but at most, one copy can be placed in a single bin. The number of items that can be moved is constrained, modeling the real-life situation of a live system for which some placement already exists. Each problem instance is modeled as a MILP, using a multi-dimensional multi-knapsack formulation.

Workload Apportionment. This problem deals with apportioning workloads (e.g., data streams) across as few workers (e.g., servers) as possible. The apportionment is required to be robust to any one worker’s failure. Each instance problem is modeled as a MILP, using a bin-packing with apportionment formulation.

Huawei Production Planning. The planning and scheduling optimization problems solved in the Huawei production planning engine.

Visual analysis

In order to figure out what the neural network have learned, we give the visualization for the reformulation process of our method, over the three datasets of LP instances. Specifically, from each dataset, we select two LP instances. We reformulate them by the learned neural network of our proposed method. Then we draw the coefficient matrix of original LP instances and reformulated ones respectively, which are shown in Figure 5 to Figure 7. Observing these figures, we can find that 1) our proposed reformulation method indeed capture the characteristics of LP instances originated from different scenarios. Because the pattern of corresponding reformulated LP instances are greatly different across different datasets but are similar between LP instances within the same dataset; 2) The reformulation is relatively stable when the original LP instances are highly similar. All the original LP instances of BIP are with the same number of constraints and variables, which is only different in the value of coefficients. Thus the pattern of the corresponding reformulated LP instances are almost the same (see Figure 5). However, the pattern of reformulated LP instances of WA and HPP is quiet different (see Figure 6 and Figure 7) because the corresponding original LP instances differs in not only the value of coefficients but also the number of constraints and variables.

| Name | Used value |
|------|------------|
| Optimizer | ADAM |
| # epochs (T) | 40000 |
| # splitting cluster | 20 |
| Batch size (B) | 8 |
| Train size | 640 |
| Validation size | 320 |
| Learning rate | 10^{-4} |
| Decay ratio of learning rate | 0.96 |
| Gradient clip normalizer | \( l_2 \) norms |
| Dimension of input embedding in PN | 128 |
| Dimension of hidden layers in PN | 128 |
| Dimension of input embedding in GCNN | 64 |
| # of times that performs convolution (l) | 2 |

Table 3: Hyperparameters setting
Figure 4: Learning convergence result of our proposed method over above three datasets

Figure 5: Visualization of reformulation process over BIP dataset

Figure 6: Visualization of reformulation process over WA dataset

Figure 7: Visualization of reformulation process over HPP dataset