Classical integrable lattice models through quantum
group related formalism

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Abstract

We translate effectively our earlier quantum constructions to the classical
language and using Yang-Baxterisation of the Faddeev-Reshetikhin-Takhtajan
algebra are able to construct Lax operators and associated $r$-matrices of clas-
sical integrable models. Thus new as well as known lattice systems of different
classes are generated including new types of collective integrable models and
canonical models with nonstandard $r$ matrices.

1 Introduction

The basic aim of present investigation is to show that some effective formalism
developed around quantum algebra and quantum integrable systems can also be
fruitfully applied to the domain of classical models. In particular we are able to
translate most of our results related to the generation of quantum integrable mod-
els [1] to the classical language and present a systematic construction of the Lax
operators $L(\lambda)$ and related classical $r(\lambda,\mu)$-matrices for new as well as existing lat-
tice models with canonical Poisson-bracket structures. It demonstrates that there
exist some fundamental building blocks for both Lax operators and $r$-matrices, out
of which these objects can be easily built up following a classical analog of the
Yang-Baxterisation of the Faddeev-Reshetikhin-Takhtajan algebra [2].

We also show that applying analogous quantum constructions one can recover
cheaply though methodically many classical results, which were possibly discovered
originally using deep intuition. Such examples are the famous Ablowitz-Ladik [AL]
model [3], discrete time Toda chain [DTTC] [4], asymmetric lattice NLS model [5]
etc.. Moreover one can construct now exactly integrable discrete version of derivative
NLS and massive Thirring models, where earlier attempts failed.

Moreover the construction indicates the existence of local as well as global col-
lective integrable models formed by collecting a number of constituent models be-
longing to the same descendant class, a possibility which has been ignored mostly
in lattice model construction. Another new aspect is the formulation of a special
class of integrable systems having nonstandared $r$-matrices with both additive and
difference dependence on spectral parameters.
Hopefully the results presented here would draw attention of classical mathematical physicists and thus serve as a bridge narrowing the existing gap between the two camps: classical and quantum.

2 Building blocks and basic PB relations

Let us consider first simple constant solutions of the classical Yang-Baxter equation:

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \]  (2.1)
given as

\[ r^+ = \alpha \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad r^- = \alpha \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \]

where parameter \( \alpha \) may be taken as the deformation parameter in the classical case. Similarly consider \( L^\pm \) matrices also given in the upper/lower triangular form as

\[ L^+ = \begin{pmatrix} \tau_1^+ & \tau_{21} \\ \tau_2^+ & \tau_2^- \end{pmatrix}, \quad L^- = \begin{pmatrix} \tau_1^- & \tau_{12} \\ \tau_1^- & \tau_2^- \end{pmatrix}, \]

with elements \( \{ \tau \} \) being as yet undefined dependent variables. As we see below the matrices \( r^\pm \) will serve as our building blocks for the construction of spectral parameter dependent classical spectral \( r(\xi, \eta) \)-matrix, while \( L^\pm \) will do the same for the related Lax operators \( L(\xi) \) of the integrable lattice models. To specify now the nature of \( \{ \tau \} \) variables these two sets of matrices are linked through the Poisson bracket (PB) relations

\[ \{ L^\pm \otimes L^\pm \} = [r^\pm, L^\pm \otimes L^\pm], \]  (2.2)
\[ \{ L^+ \otimes L^-, L^- \} = [r^+, L^+ \otimes L^-], \]  (2.3)

which are nothing but the classical analogs of the Faddeev-Reshetikhin-Takhtajan algebra, a well known relation in the subject of quantum group. The relations (2.2-3) in the elementwise form yields the defining PB relations among the \( \tau \) variables as

\[ \{ \tau_{12}, \tau_{21} \} = 2\alpha \left( \tau_1^- \tau_2^- - \tau_1^+ \tau_2^+ \right), \quad \{ \tau_i^\pm, \tau_j^\pm \} = 0, \]  (2.4)
\[ \{ \tau_{1i}, \tau_{1j} \} = \epsilon_i \alpha \tau_{12} \tau_i^\pm, \quad \{ \tau_i^\pm, \tau_{21} \} = \mp \epsilon_i \alpha \tau_{21} \tau_i^\pm, \]  (2.5)

for \( i, j = (1, 2) \) with \( \epsilon_1 = 1, \epsilon_2 = -1 \). Interestingly, when the deforming parameter \( \alpha \to 0 \), a consistent limit of (2.4-5) exists with

\[ (\tau_i^+ + \tau_i^-) \to K_i^0, \quad (\tau_i^+ - \tau_i^-) \to \frac{1}{\alpha} K_i^1, \quad \tau_{ij} \to K_{ij} \]  (2.6)
yielding the PB relations

\[ \{ K_{12}, K_{21} \} = K_1^0 K_2^1 - K_1^1 K_2^0, \quad \{ K_1^0, K_2^0 \} = 0, \]  (2.7)
\[ \{ K_i^0, K_{12} \} = \epsilon_i K_{12} K_i^1, \quad \{ K_i^0, K_{21} \} = -\epsilon_i K_{21} K_i^1, \]  (2.8)

where \( K_i^j, i = 1, 2 \) becomes central elements with trivial PB with all others. The above Poisson algebras (2.4-5) and (2.7-8), as we show below, will play a decisive role in generating two different large classes of integrable models.
3 Lax operators and $r$-matrices through Yang-Baxterisation

Equippt with all the building materials: $L^\pm, r^\pm$ matrices and the relations (2.4-5) or (2.7-8) we can start constructing the Lax operator $L(\xi)$ and the $r(\xi, \eta)$-matrix by stitching together the upper and lower triangular matrices through spectral parameters $\xi, \eta$ as

$$r(\xi, \eta) = f(\xi_{12})[\xi_{12}^{-1}r^+ + \xi_{12}r^-], \quad L(\xi) = \xi^{-1}L^+ + \xi L^-,$$  \hspace{1cm} (3.1)

where $\xi_{12} = \xi / \eta$ and $f(\xi_{12}) = \frac{\alpha}{2(\xi_{12}^{-1} - \xi_{12})}$ is a function with a pole at $\xi_{12} = 1$. To convince ourselves that the $r$ and $L$ matrices thus formed can be considered as the representatives of lattice integrable models, we should check that they satisfy the classical Yang-Baxter equation (CYBE)

$$\{ L_n(\xi) \otimes L_m(\eta) \} = \delta_{mn} [ r(\xi, \eta), L_n(\xi) \otimes L_n(\eta) ].$$  \hspace{1cm} (3.2)

This checking however becomes easy due to the algebra (2.2-3) along with the assumption of vanishing PB at different lattice points (ultralocality condition) and the obvious relation $r^+ + r^- = 2\alpha P$ through the permutation operator $P$ (the classical analog of the Hecke condition). Therefore one may get from (3.1) (after an irrelevant gauge transformation) a genuine Lax operator and the associated $r$-matrix in explicit form as

$$L(\xi) = \begin{pmatrix} \xi \tau_1^- + \frac{1}{\xi} r_1^+ & \tau_1 \tau_2 \xi \tau_2 + \frac{1}{\xi} r^+_2 \\ \tau_{12} & \xi \tau_2^- + \frac{1}{\xi} r^+_2 \end{pmatrix},$$  \hspace{1cm} (3.3)

and

$$r(\xi, \eta) = \begin{pmatrix} a(\xi, \eta) & b(\xi, \eta) \\ b(\xi, \eta) & a(\xi, \eta) \end{pmatrix},$$  \hspace{1cm} (3.4)

where $a(\xi, \eta) = \frac{\alpha^2 + \eta^2}{2(\eta - \xi)}$ and $b(\xi, \eta) = \frac{\alpha \xi}{\eta - \xi}$. Note that expressing spectral parameters in the form $\xi = e^{-\alpha \lambda}, \eta = e^{-\alpha \mu}$ the dependence of the above $r$-matrix (3.4) on trigonometric functions and moreover only on the difference of parameters as $r(\lambda - \mu)$ becomes obvious. Remarkably at deformation parameter $\alpha \to 0$ due to the existing limit (2.6) of the dependent variables and the expansion of spectral parameter $\xi \approx 1 - \alpha \lambda$ the $L(\xi)$ operator (3.3) reduces consistently to

$$L(\lambda) = \begin{pmatrix} K_0^0 + \lambda K_1^1 & K_{12}^1 \\ K_1^0 & K_2^0 + \lambda K_2^1 \end{pmatrix}. $$  \hspace{1cm} (3.5)

At the same time due to $f(\xi_{12}) \to \frac{\alpha}{2(\lambda - \mu)}$, through the use of the Hecke condition trigonometric $r$-matrix (3.4) reduces to its rational limit

$$r(\lambda - \mu) = \frac{P}{\lambda - \mu}.$$  \hspace{1cm} (3.6)

We show in the next section that the Lax operator (3.3) along with the PB relations (2.4-5) between its elements serves as an excellent ancestor lattice model generating a large class of descendants, all sharing the same trigonometric $r(\xi, \eta)$-matrix (3.4). Similarly (3.5) with (2.7-8) is responsible for another descendant class having the rational form (3.6) for the associated $r$-matrix.
4 Classical integrable lattice systems as descendant models

The idea is to insert the Lax operators on a lattice with \( n = [1, N] \) sites and find consistent reductions of the general \( L \) operator (3.3) with PB relations (2.4-5) or of (3.5) with (2.7-8). For such reductions proper change of dependent variables from \( \tau_n \) or \( K_n \) to canonical ones:

\[
\{u_n, p_m\} = \delta_{nm} \quad \text{or} \quad \{\psi_n, \psi_n^\dagger\} = i\delta_{nm}
\]
or to some other physically interesting variables (like in the AL model), would result different classical integrable lattice systems, since by construction these descendant models would be associated with \( r \)-matrix (3.4) or (3.6) and satisfy CYBE (3.2).

It is not difficult to show that symmetric reduction of the form

\[
\tau_1^+ = (\tau_2^+)^{-1} = -\tau_2^- = -(\tau_1^-)^{-1}, \quad \tau_{12} = \tau_{21}^-.
\]

(4.1)

reduces PB relations (2.4-5) to the classical analog of the \( q \)-deformed algebra \( U_q(su(2)) \) [7], while similar symmetric reduction of (2.7-8) as

\[
K_1^1 = K_2^1 = 1, \quad K_1^0 = -K_2^0, \quad K_{12} = K_{21}^0
\]

(4.2)

recovers the \( su(2) \) algebra. The reduction (4.1) in turn, expressed through canonical variables \( (u, p) \), yields the lattice sine-Gordon model [8], while similar symmetric reduction of (2.7-8) as

\[
K_1^1 = K_2^1 = 1, \quad K_1^0 = -K_2^0, \quad K_{12} = K_{21}^0
\]

(4.2)

yields directly the Toda lattice model.

Remarkably, some recently proposed discrete integrable systems [5,10] can also be derived consistently as other possible asymmetric reductions of (3.5). For example,

\[
K_1^1 = 0, \quad K_2^0 = K_1^1 = 1, \quad K_1^0 = \phi_n \psi_n, \quad K_{12} = \psi_n, \quad K_{21} = \phi_n
\]

(4.4)

with \( n = [1, N] \) generates [11] the simple lattice model of [5], while

\[
K_1^1 = 0, \quad K_2^0 = \gamma, \quad K_1^1 = 1, \quad K_1^0 = \psi_i^* \psi_i + \omega_i, \quad K_{12} = \psi_i, \quad K_{21} = \gamma \psi_i^*.
\]

(4.5)
with \( i = 1, 2 \) constructs the Toda-like lattice system considered in [10]. Evidently there exist different other reductions of (3.3) and (3.5) capable of generating other integrable systems. We also get as a bonus simultaneously the Lax operators and the \( r \)-matrices of the constructed models ensuring their integrability.

It is interesting to note that such asymmetries can be made more explicit by considering models with new \( r \)-matrix solution

\[
\tilde{r} = r_0 + 2f,
\]

where \( r_0 \) is the original solution (3.4) or (3.6) and

\[
f = \text{diag}(\eta' - \xi', \eta' + \xi', -(\eta' + \xi'), -(\eta' - \xi'))
\]

with \textit{colour} parameters \( \eta', \xi' \). The expression (4.6-7) as a new solution of (2.1) is a classical statement of the twisting transformation of [12], which can also be checked otherwise by direct insertion. We consider the particular case

\[
\eta' = c\eta + \alpha, \quad \xi' = c\xi + \alpha
\]

with \( c \) being a constant parameter, which through (4.6-8) yields a new type of \( r \)-matrix with sum as well as difference dependence on spectral parameters. Observe that at \( c = 0 \), when \( f = \text{diag}(0, \alpha, -\alpha, 0) \), with (3.4) as \( r_0 \) one recovers from (4.6) exactly the \( r \)-matrix associated with the AL [3] as well as the DTTC [4] models. A natural expectation is therefore that these models somehow must be hidden in our construction. To show that this is indeed the case, we note that for this transformed \( r \)-matrix and with the \( L \) operator taken in the form (3.3) the PB relations are changed from (2.4-5) consistently to

\[
\{\tau_{12}, \tau_{21}\} = 2\alpha \left( (\tau_1^{-}\tau_2^{+} - \tau_1^{+}\tau_2^{-}) + \tau_{12}\tau_{21} \right)
\]

\[
\{\tau_1^{+}, \tau_{12}\} = 2\epsilon_i\alpha\tau_{12}\tau_i^{+}, \quad \{\tau_1^{-}, \tau_{21}\} = -2\epsilon_i\alpha\tau_{21}\tau_i^{-},
\]

with functions \( \tau_i^{-} \) having now trivial PB with all other elements. The explicit asymmetry of (4.9-10) is obvious, which for

\[
\tau_1^{-} = \tau_2^{+} = 0, \quad \tau_1^{+} = \tau_2^{-} = 1, \quad \tau_{12} = b_n, \quad \tau_{12} = b_n^*
\]

with PB \( \{b_m, b_n^*\} = -2\alpha(1 + b_n b_n^*)\delta_{mn} \), yields the same AL model. On the other hand reduction of (4.9-10) as

\[
\tau_1^{-} = -1, \quad \tau_2^{+} = 0, \quad \tau_1^{+} = e^{\alpha p_n}, \quad \tau_{21} = e^{\alpha p_n}, \quad \tau_{12} = -\alpha e^{-\alpha p_n} + \alpha p_n
\]

gives the DTTC, meeting our expectation.

5 **Collective models and models with nonadditive \( r \)-matrices**

An interesting possibility of constructing collective models by joining the constituent integrable elements can be effectively exploited thanks to a symmetry of the CYBE (3.2). In particular, it is easily checked that if \( L(\xi, \tau) \) and \( \tilde{L}(\xi, \tilde{\tau}) \) are Lax operators corresponding to two independent descendant models sharing the same \( r \)-matrix and with \( \{L, \tilde{L}\} = 0 \), then the collective model \( L(\xi, \tau, \tilde{\tau}) = L(\xi, \tau)\tilde{L}(\xi, \tilde{\tau}) \) will also be
integrable with the same \( r \)-matrix. This symmetry allows us to construct integrable models by collecting similar \( L \) operators at each lattice point. A recent construction [10] of Toda-like models by joining several bosonic systems is an example of such collective models. A more exciting example is possibly the construction of lattice massive Thirring model by joining integrable discrete derivative NLS models [13]. The discrete derivative NLS in turn may be given by the \( L \)-operator with reduction of (3.4) as

\[
\tau_{11}^+ = (\tau_2^+)^{-1} = q^{-N_n}, \quad \tau_1^- = \frac{a}{4i} q^{N_n+1}, \quad \tau_2^- = -\frac{a}{4i} q^{-(N_n+1)}, \quad (5.1)
\]

\[
\tau_{12} = \left(\frac{a}{2}\right)^{1/2} A_n = \tau_{21}, \quad q = e^\alpha \quad (5.2)
\]

where \( a \) is the lattice constant, \( N_n = a\psi_n^\dagger \psi_n \), and \( A_n \) represents a \( q \)-boson [14] expressed as \( A_n = \psi_n g(N_n), \quad g^2 = \frac{2N_n}{N_n} \). Using the invariance of (2.4-5) under the exchange of variables

\[
\tau_1^+ \rightleftharpoons \tau_2^-, \quad \tau_2^+ \rightleftharpoons \tau_1^- \]

a similar operator \( L^{(2)} \) with another independent component of \( \psi \) variable is obtained, yielding the collective model \( L_n = L_n^{(1)} L_n^{(2)} \), which is the discrete version of the massive Thirring model.

Extending this idea to more global level we may even insert completely different descendant models at different lattice nodes, preserving only the periodic structure. It seems that such rich possibilities for constructing new classical integrable lattice systems with canonical structures remain mostly unexplored.

Finally we come to another novel construction of a class of lattice models with nonstandard \( r \) matrices. For this we have to choose nontrivial \( c \) in (4.8), while as \( r_0 \) in (4.6) one may take either the trigonometric (3.4) or the rational form (3.6). The function \( f \) naturally brings in new (sum as well as difference) dependence on spectral parameters. The associated \( L \) operator may also be changed accordingly keeping the PB relations (2.4-5) or (2.7-8) unchanged. Or alternatively, the \( L \) operator may again be chosen as (3.3) or (3.5) and get the deformed PB relations from the CYBE. Following the first choice we get [1]

\[
L(\xi) = (\tau_1^- \tau_2^+)^{\lambda+\alpha} L^0(\xi), \quad (5.3)
\]

\( L^0 \) being the original \( L \) operator (3.3) with the trigonometric \( r \) matrix. As has been shown above for \( c = 0 \) this system would contain the AL and the DTTC models. Therefore in the general case it should yield new generalised integrable models with \( r \) matrix having an interesting nonadditive dependence on spectral parameters and at the same time it would naturally recover all the other classes of integrable models discussed here at different particular cases. Similar construction exist equally for models with rational \( r \) matrix.

6 Conclusion

Application of quantum group related constructions to classical integrable discrete systems yields intriguing results in model construction. One gets the Lax operators and associated \( r \) matrices in a systematic way, for which to our knowledge no other simple prescription exists. This also gives the construction of some collective integrable models as well as models with nonstandard \( r \)-matrices. Such possibilities are
worth exploring for generating new type of integrable lattice models.

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