Out-of-Time-Order Correlation in Marginal Many-Body Localized Systems

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We show that the out-of-time-order correlation (OTOC) ⟨W(t)†V(0)†W(t)V(0)⟩ in many-body localized (MBL) and marginal MBL systems can be efficiently calculated by the spectrum bifurcation renormalization group (SBRG). We find that in marginal MBL systems, the scrambling time \( t_{\text{scr}} \) follows a stretched exponential scaling with the distance \( d_{WV} \) between the operators \( W \) and \( V \):
\[
t_{\text{scr}} \sim \exp(\sqrt{d_{WV}/l_0}),
\]
which demonstrates Sinai diffusion of quantum information and the enhanced scrambling by the quantum criticality in non-chaotic systems.

The out-of-time-order correlation (OTOC)\(^{[1–7]}\) was recently proposed to quantify the scrambling and the butterfly effect in quantum many-body dynamics, and has attracted great research interests in quantum gravity\(^{[8–11]}\), quantum information\(^{[12]}\) and condensed matter\(^{[13–17]}\) communities. Consider two local unitary operators \( W \) and \( V \), along with the many-body Hamiltonian \( H \) of the system; the OTOC is defined as
\[
F(t) = \langle W(t)†V(0)†W(t)V(0)⟩,
\]
where \( W(t) = e^{iHt}We^{-iHt} \) and \( V(0) = V \) are the operators at time \( t \) and time 0 respectively. The notation \( \langle \cdots \rangle \) stands for either the expectation value on a pure state of interest (typically a short-range entangled eigenstate at a distance \( d_{WV} \)) or the ensemble average over a mixed state density matrix. The OTOC is closely related to the squared commutator\(^{[2, 3, 11]}\) of the operators:
\[
C(t) = \langle[|W(t), V]|^2⟩ = 2(1 - \text{Re}F(t)).
\]
If \( W \) and \( V \) are far apart local operators, then their squared commutator \( C(t) \) should vanish initially (for \( t = 0 \)). As time evolves, the operator \( W(t) \) will grow in size and complexity, and eventually spread to the location of the operator \( V \), at which point \( C(t) \) develops a finite value. So the growth of the squared commutator \( C(t) \), or the decay of the OTOC \( F(t) \), characterizes the growth of local operators and the spreading of quantum information, which is a phenomenon known as scrambling\(^{[18–21]}\). Typically, the OTOC will remain large until the scrambling time\(^{[20]}\) \( t_{\text{scr}} \) and decay rapidly once \( t > t_{\text{scr}} \). The scrambling time \( t_{\text{scr}} \) generally depends on the distance \( d_{WV} \) between \( W \) and \( V \) operators.

Although we used the spectrum bifurcation renormalization group (SBRG)\(^{[42, 43]}\) approach to calculate the OTOC in MBL and marginal MBL systems. SBRG is an efficient numerical approach to construct the MBL effective Hamiltonian from a given disordered quantum many-body Hamiltonian. The idea of SBRG is similar to the real space renormalization group for excited states (RSRG-X)\(^{[44–49]}\). At each RG step, the leading energy scale term \( H_0 \) in the Hamiltonian is identified and the whole Hamiltonian is rotated to the (block) diagonal basis of \( H_0 \); then the terms in the off-diagonal blocks are reduced by the 2nd order perturbation. SBRG uses Clifford gates to boost the calculation efficiency for qubit models, such that the full spectrum is obtained in one run of RG (in contrast to RSRG-X which targets a single eigenstate at a time). With SBRG we can push the calculation of OTOC to much larger system size (e.g. 256
spins in this work) than exact diagonalization, hence verifying the scaling behaviors of the butterfly light-cones over a much larger scale.

We start by deriving the formula for the OTOC that can be used in the SBRG calculations. The output of SBRG[42] is the MBL effective Hamiltonian,[38, 45, 50–54] which can be written in terms of the stabilizers (1-bits) \( \tau^z_a \) \( (a = 1, 2, \ldots, L \) labels the stabilizers) as,

\[
H_{MBL} = \sum_a \epsilon_a \tau^z_a + \sum_{a,b} \epsilon_{ab} \tau^z_a \tau^z_b + \sum_{a,b,c} \epsilon_{abc} \tau^z_a \tau^z_b \tau^z_c \ldots, \tag{2}
\]

which contains single-body terms \( \epsilon_a \tau^z_a \), two-body terms \( \epsilon_{ab} \tau^z_a \tau^z_b \) and higher-body terms. The key difference between Anderson and MBL insulators is that the two-body and higher-body terms are absent in the former while present in the later. \( H_{MBL} \) can also describe the marginal MBL system, where the major modification is that the stabilizers \( \tau^z_a \) will be quasi-long-ranged. The chance of finding a stabilizer decays as a power-law with its length, instead of exponentially localized in the MBL system.

The stabilizers all commute with each other and also commute with the Hamiltonian, i.e. \( [\tau^z_a, \tau^z_b] = [\tau^z_a, H_{MBL}] = 0 \). To simplify the notation, we denote each product of stabilizers as \( \tau^z_{a_1 a_2 \ldots} = \tau^z_{a_1} \tau^z_{a_2} \tau^z_{a_3} \ldots \). We may further bundle the subscripts indices together and write

\[
H_{MBL} = \sum_A \epsilon_A \tau^z_A, \tag{3}
\]

where \( A = a_1 a_2 \ldots \) stands for a sequence of stabilizer indices and \( \tau^z_A = \prod_{a \in A} \tau^z_a \). Since the Hamiltonian \( H_{MBL} \) is a sum of commuting terms \( \epsilon_A \tau^z_A \), the time-evolution operator \( U(t) \) can be factorized to the product of unitary operators generated by every \( \tau^z_A \) operator independently,

\[
U(t) = e^{-i t H_{MBL}} = \prod_{\tau^z_A} e^{-i \epsilon_A t \tau^z_A}. \tag{4}
\]

The product runs over all \( \tau^z_A \) operators in the Hamiltonian \( H_{MBL} \). The time-dependent operator \( W(t) \) is then given by \( W(t) = U(t)^\dagger U(t) \).

We begin by transforming the \( W \) and \( V \) operators to the \( \tau^z \) basis, which involves a unitary transformation composed of Clifford rotations with perturbative Schrieffer-Wolff corrections [42]. In this basis, \( W \) and \( V \) are then linear combinations of products of Pauli operators. For simplicity, we will neglect the perturbative Schrieffer-Wolff corrections, which are small in the limit of large disorder. In Ref. 42 we tested this approximation by restoring the many-body wave function from the Clifford rotation only, and benchmarking the result with exact diagonalization. Good wave function fidelity is achieved as long as the disorder is strong. We will only consider \( W \) and \( V \) which are products of Pauli operators (called Pauli strings) in the physical basis, which implies that they will remain Pauli strings in the \( \tau^z \) basis (since Clifford rotations map Pauli strings to Pauli strings). Therefore their algebraic relations with \( \tau^z_A \) is rather simple: \( W \) and \( V \) can either commute or anticommute with \( \tau^z_A \) [71]. Let \( C_W \) (or \( A_W \)) be the set of \( \tau^z_A \) that commute (or anticommute) with \( W \). Any \( \tau^z_A \) in \( H_{MBL} \) either belongs to \( C_W \) or \( A_W \) for any given \( W \). With this setup, we can calculate the time-evolution of \( W \) as follows

\[
W(t) = \prod_{\tau^z_A \in C_W} e^{i t \epsilon_A \tau^z_A} W \prod_{\tau^z_A \in A_W} e^{-i t \epsilon_A \tau^z_A} = W \prod_{\tau^z_A \in A_W} e^{-2 i t \epsilon_A \tau^z_A}. \tag{5}
\]

The unitary operators generated by \( \tau^z_A \in C_W \) will annihilate each other by commuting through \( W \), so only those generated by \( \tau^z_A \in A_W \) will survive. Suppose \( W \) is a local operator (e.g. an on-site Pauli operator); then Eq. 5 indicates that the support (or the size) of \( W(t) \) will grow in time. \( W(t) \) starts out with \( W(0) = W \) initially, and as time evolves, \( W \) will expand via a product of non-local operators \( e^{-2 i t \epsilon_A \tau^z_A} \). Each of them gradually evolves from 1 to \( e^{i \epsilon_A t} \tau^z_A \) in the time scale \( \sim \epsilon_A^{-1} \). \( \tau^z_A \) terms that are more non-local typically have smaller energy scales \( \epsilon_A \) in the local Hamiltonian \( H_{MBL} \), and thus take longer time to contribute to \( W(t) \). So the operator \( W(t) \) will grow gradually. Accordingly, as \( W(t) \) becomes non-local, the quantum information associated with \( W \) will be spread throughout the system and can not be retrieved by local measurements, which illustrates the idea of quantum chaos[7, 12, 55, 56] and scrambling[18–20].

The OTOC was proposed to quantify the growth of the operator and the scrambling effect. Here let us discuss the OTOC at “infinite temperature” where the density matrix of the system is simply identity, so that

\[
F(t) = \text{Tr} W(t) V(0) W(0) V(0), \tag{6}
\]

(the daggers are omitted as we assume both \( W \) and \( V \) are Hermitian Pauli operators). Following the similar calculation in Eq. 5, we find

\[
F(t) = \text{Tr} W V W V \prod_{\tau^z_A \in A_W \cap A_V} e^{4 i t \epsilon_A \tau^z_A}, \tag{7}
\]

where \( \tau^z_A \) are the terms in \( H_{MBL} \) that anticommute with both \( W \) and \( V \). As both \( W \) and \( V \) are Pauli strings, regardless of whether they commute or anticommute, this just amounts to an overall sign in \( W V W V = \pm 1 \), which is not important. So we might as well assume \( [W, V] = 0 \) (which is the case for far apart local operators), then the OTOC simply reads \( F(t) = \text{Tr} \prod_{\tau^z_A \in A_W \cap A_V} e^{4 i t \epsilon_A \tau^z_A} \).

The unitary operators can be expanded, i.e.

\[
F(t) = \text{Tr} \sum_{\tau^z_A \in A_W \cap A_V} (\cos(4 \epsilon_A t) + i \tau_A^z \sin(4 \epsilon_A t)). \tag{8}
\]
We take an approximation by dropping all the \( \sin(4\epsilon_A t) \) terms in the expansion (to be justified shortly),\(^{24, 26}\) and arrive at a simple formula for the OTOC of MBL and marginal MBL systems,

\[
F(t) \simeq \prod_{\tau^i_A \in A_W \cap A_V} \cos(4\epsilon_A t). \tag{9}
\]

In numerics, we first run the SBRG on a given quantum many-body Hamiltonian to generate the MBL effective Hamiltonian \( H_{\text{MBL}} \). From \( H_{\text{MBL}} \) we filter out all terms \( \epsilon_A \tau^i_A \) that anticommute with both \( W \) and \( V \) (recall that \( W \) and \( V \) are Pauli strings in the \( \tau^i_A \) basis since we dropped the perturbative Schrieffer-Wolff corrections) and collect their energy coefficients \( \epsilon_A \). Then the OTOC can be evaluated very efficiently according to Eq. 9. We must bear in mind that Eq. 9 does not apply to the thermalized system, because our starting point, the MBL effective Hamiltonian \( H_{\text{MBL}} \), breaks down in the thermalized phase.

The approximation we made in Eq. 9 is to drop all terms in the expansion that contain the product of \( \sin(4\epsilon_A t) \). Such terms will only arise when several different \( \tau^i_A \) operators product to identity so as to survive the trace. However note that all \( \tau^i_A \) in Eq. 9 are taken from the set \( A_W \cap A_V \), within which one must have at least four \( \tau^i_A \) product together to reach the identity (as long as \( W \) and \( V \) commute). Thus the \( \sin(4\epsilon_A t) \) factors appear as products of four or more, whose short-time behavior is suppressed by \( \sim t^4 \) (as \( t \to 0 \)). In conclusion, such terms will never dominate the expansion until after \( t_{\text{scr}} \).

To calculate the OTOC of more general operators \( W \) and \( V \), or to include the perturbative Schrieffer-Wolff corrections, one can expand the operators as a sum of Pauli strings in the \( \tau^i_A \) basis, and express the OTOC as a sum of these operators. In the following, we will only focus on the OTOC of Pauli strings without Schrieffer-Wolff corrections.

The OTOC starts out at 1 and decays to 0. The time for the onset of the decay is defined as the scrambling time \( t_{\text{scr}} \).\(^{15, 20}\) One can estimate the scrambling time based on Eq. 9. At short-time, \( \cos(4\epsilon_A t) \) can be Taylor expanded to \( 1 - \frac{1}{2}(4\epsilon_A t)^2 + \cdots \), so the OTOC behaves as

\[
F(t) = 1 - \frac{1}{2}(4\epsilon_A W_V t)^2 + \cdots, \tag{10}
\]

where the energy scale \( \| \epsilon_A \|_{W,V} \) is defined via

\[
\| \epsilon_A \|_{W,V}^2 = \sum_{\tau^i_A \in A_W \cap A_V} \epsilon_A^2. \tag{11}
\]

So the scrambling time \( t_{\text{scr}} \) is set by this energy scale as \( t_{\text{scr}} = \| \epsilon_A \|_{W,V}^{-1} \). The energy scale \( \| \epsilon_A \|_{W,V} \) is not just a Frobenius norm of the energy coefficients in \( H_{\text{MBL}} \), it also sensitively depends on the operators \( W \) and \( V \).

For Anderson insulators, if the spacial separation between \( W \) and \( V \) is much greater than the localization length, then \( A_W \cap A_V \) is usually an empty set, i.e. there is almost no stabilizer that can anticommute with both \( W \) and \( V \) because all stabilizers are exponentially localized within the localization length. (Recall that all terms in \( H_{\text{MBL}} \) (Eq. 2) are stabilizers for Anderson insulators, and that we’re approximating the stabilizers as a product of Pauli operators by neglecting the perturbative corrections.) In this case \( \| \epsilon_A \|_{W,V} \to 0 \) and hence \( t_{\text{scr}} \to \infty \). So the OTOC will remain finite and not decay in time for far apart \( W \) and \( V \), meaning that there is no scrambling in Anderson insulators.

The situation is different if we add interactions. For MBL systems, far apart \( W \) and \( V \) operators can be connected by many-body interaction terms in \( H_{\text{MBL}} \). A typical contribution comes from the two-body terms \( \epsilon_{ab} \tau^i_a \tau^j_b \) localized around \( W \) and \( \tau^j_b \) localized around \( V \). Then \( \| \epsilon_A \|_{W,V} \approx \| \epsilon_{ab} \| \sim e^{-x_{ab}/\xi} \), where \( x_{ab} \) is the distance between \( \tau^i_a \) and \( \tau^j_b \), which is also roughly the distance \( d_{W,V} \) between \( W \) and \( V \). So the scrambling time \( t_{\text{scr}} \) follows \( \ln t_{\text{scr}} \sim d_{W,V}/\xi \), leading to a logarithmic butterfly light-cone in the MBL system.\(^{23–27}\)

Another direction out of Anderson insulators is to consider quantum critical systems, i.e. marginal MBL systems. In these systems, each stabilizer \( \tau^i_a \) itself becomes power-law quasi localized, and can connect spatially far separated \( W \) and \( V \). Then the energy scale can be dominated by the single-body energy \( \| \epsilon_A \|_{W,V} \approx \| \epsilon_a \| \sim e^{-\sqrt{l_0/\xi}} \), which follows the “stretched exponential” scaling with respect to the length \( l \) of the stabilizer (where \( l_0 \) is a length scale depending on the initial disorder strength). This scaling is an exact result in the free limit by RSRG and has been shown to apply to interacting cases in Fig. 2 as well in Ref. 42. \( l \) is also roughly the distance \( d_{W,V} \) between \( W \) and \( V \). Therefore the scrambling time \( t_{\text{scr}} \) follows \( \ln t_{\text{scr}} \sim \sqrt{d_{W,V}} \), leading to a squared logarithmic butterfly light-cone in the marginal MBL system. Because the scrambling in the marginal MBL system is determined by the single-body energy scale, the butterfly light-cone is not much affected by the absence or presence of the interaction.

To verify the above theoretical proposals, we numer-

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Anderson} & \text{MBL} & \text{Marginal MBL} & \text{Ergodic} \\
\hline
\ln t_{\text{scr}} & \infty & \sim d_{W,V} & \sim d_{W,V}^{1/2} & \text{ln } d_{W,V} \\
\hline
\end{array}
\]

For Anderson insulators, if the spacial separation between \( W \) and \( V \) is much greater than the localization length, then \( A_W \cap A_V \) is usually an empty set, i.e. there is almost no stabilizer that can anticommute with both \( W \) and \( V \) because all stabilizers are exponentially localized within the localization length. (Recall that all terms in \( H_{\text{MBL}} \) (Eq. 2) are stabilizers for Anderson insulators, and that we’re approximating the stabilizers as a product of Pauli operators by neglecting the perturbative corrections.) In this case \( \| \epsilon_A \|_{W,V} \to 0 \) and hence \( t_{\text{scr}} \to \infty \). So the OTOC will remain finite and not decay in time for far apart \( W \) and \( V \), meaning that there is no scrambling in Anderson insulators.

The situation is different if we add interactions. For MBL systems, far apart \( W \) and \( V \) operators can be connected by many-body interaction terms in \( H_{\text{MBL}} \). A typical contribution comes from the two-body terms \( \epsilon_{ab} \tau^i_a \tau^j_b \) with \( \tau^i_a \) localized around \( W \) and \( \tau^j_b \) localized around \( V \). Then \( \| \epsilon_A \|_{W,V} \approx \| \epsilon_{ab} \| \sim e^{-x_{ab}/\xi} \), where \( x_{ab} \) is the distance between \( \tau^i_a \) and \( \tau^j_b \), which is also roughly the distance \( d_{W,V} \) between \( W \) and \( V \). So the scrambling time \( t_{\text{scr}} \) follows \( \ln t_{\text{scr}} \sim d_{W,V}/\xi \), leading to a logarithmic butterfly light-cone in the MBL system.\(^{23–27}\)

Another direction out of Anderson insulators is to consider quantum critical systems, i.e. marginal MBL systems. In these systems, each stabilizer \( \tau^i_a \) itself becomes power-law quasi localized, and can connect spatially far separated \( W \) and \( V \). Then the energy scale can be dominated by the single-body energy \( \| \epsilon_A \|_{W,V} \approx \| \epsilon_a \| \sim e^{-\sqrt{l_0/\xi}} \), which follows the “stretched exponential” scaling with respect to the length \( l \) of the stabilizer (where \( l_0 \) is a length scale depending on the initial disorder strength). This scaling is an exact result in the free limit by RSRG and has been shown to apply to interacting cases in Fig. 2 as well in Ref. 42. \( l \) is also roughly the distance \( d_{W,V} \) between \( W \) and \( V \). Therefore the scrambling time \( t_{\text{scr}} \) follows \( \ln t_{\text{scr}} \sim \sqrt{d_{W,V}} \), leading to a squared logarithmic butterfly light-cone in the marginal MBL system. Because the scrambling in the marginal MBL system is determined by the single-body energy scale, the butterfly light-cone is not much affected by the absence or presence of the interaction.

To verify the above theoretical proposals, we numer-
FIG. 1: A ternary plot (copied from [43]) of the disorder and energy averaged Edwards-Anderson correlator vs coupling constants ($0 < J_{x,y,z} < 1$) for the XYZ spin chain of length $L = 256$. We use this plot to sketch the phase diagram. When $J_z > \max(J_x, J_y)$, the system is in an MBL $Z_2$ spin glass state. When $J_z < J_x = J_y$, the system is in a marginal MBL phase. (The other phases are given by permutations of $x, y, z$.) The white dots correspond to the points in the phase diagram that are shown in Fig. 3.

FIG. 2: Disorder average of the log of the stabilizer energy $\ln \epsilon$ vs stabilizer length $\ell$ on a periodic lattice of 256 spins in the marginal MBL phase of the XYZ spin chain. This figure verifies the scaling $-\ln \epsilon \sim \sqrt{\ell}$. The two rows (of plots) correspond to different points (B and C) in the phase diagram Fig. 1 while the two columns correspond to different horizontal axes: $|i - j|$ vs $\sqrt{|i - j|}$. The stabilizer length is calculated by writing a stabilizer $\tau_z^a$ in the physical basis and dropping all perturbative Schrieffer-Wolff corrections. The result is a product of Pauli operators at different sites. The stabilizer length is then the length of the shortest continuous sequence of sites (on the periodic lattice) that contains all of the Pauli operators. (Due to a slight even-odd effect, only odd stabilizer lengths are shown. 2^{16} disorder samples are used.)

FIG. 3: Disorder average (using a geometric mean (Eq. 14)) of the out-of-time-order correlation (OTOC) (Eq. 6) $\exp \text{avg} \ln |F(t)|$ of $W = \sigma_x^i$ and $V = \sigma^y_j$ showing how the light cone of the (geometric mean) OTOC depends on the time $t$ and distance $d_{WV} = |i - j|$ separation of $W$ and $V$ (on a lattice with 256 spins). Specifically, the light cone grows like $t_{\text{sc}} \sim \exp(\sqrt{d_{WV}/l_0})$. As in Fig. 2, the rows (of plots) correspond to different points in the phase diagram Fig. 1 while the two columns correspond to different horizontal axes: $|i - j|$ vs $\sqrt{|i - j|}$. Fits are shown for the cases when the scrambling time scaling agrees with the choice of horizontal axes. As can be seen from Fig. 4, the statistical errors and finite system size do not significantly affect the linear fit. (2^9 disorder samples are used.)

We numerically measure the OTOC in MBL and marginal MBL systems by SBRG. The model we study is the XYZ spin chain with strong disorder on a periodic 1D lattice. The Hamiltonian is given by [43]

$$H = \sum_{i=1}^L (J_{i,x} \sigma_i^x \sigma_{i+1}^x + J_{i,y} \sigma_i^y \sigma_{i+1}^y + J_{i,z} \sigma_i^z \sigma_{i+1}^z),$$  

where $\sigma_i^\mu$ ($\mu = x, y, z$) are the spin operators on $i$th site.
L = 256) to have converged for all only four times has essentially completely converged by |i − j| ≤ 32 converges very quickly with system size and has essentially completely converged by L = 128, which is only four times |i − j|. Thus, we expect Fig. 3 (for which L = 256) to have converged for all |i − j| ≤ 256/4 = 64 (or \(\sqrt{|i − j|} \leq 8\)). Even when L = 64, which is only twice |i − j|, the OTOC has already mostly converged. Error bars are statistical errors which are calculated using the bootstrap method [61, 62] and are small enough to not have a significant effect on our light cone measurements.

of a 1D lattice of length L = 256. The random couplings \(J_{i,\mu} \in [0, J_{\mu}]\) are independently drawn from the power-law distribution PDF\(P(J_{i,\mu}) = 1/(\Gamma J_{i,\mu})(J_{i,\mu}/J_{\mu})^{1/\Gamma}\), where 0 < \(\Gamma < \infty\) controls the disorder strength. We define

\[ \tilde{J}_{\mu} = J_{\mu}^{1/\Gamma}, \tag{13} \]

and take \(\tilde{J} = (\tilde{J}_{x}, \tilde{J}_{y}, \tilde{J}_{z})\) as the tuning parameters. In this work, a large disorder strength of \(\Gamma = 4\) was usedRef. 43. The model has three spin glass MBL phases corresponding to the large \(\tilde{J}_{x}, \tilde{J}_{y}\) or \(\tilde{J}_{z}\) limits respectively, as shown in Fig. 1, where the spin flip \(\mathbb{Z}_2 \times \mathbb{Z}_2\) symmetry is broken in every many-body eigenstate of the Hamiltonian. The spin glass phases are separated by three phase boundaries, where all the eigenstates become quantum critical, and the system is at the marginal MBL point.

We will focus along the line of \(\tilde{J}_{x} = \tilde{J}_{y}\) and study the behavior of OTOC by SBRG. Details of the SBRG algorithm are given in Ref. 42, 43. In short, SBRG can accurately simulate phases with spectrum bifurcation in the limit of large disorder where the Hamiltonian is written as a sum of products of Pauli operators where each product of Pauli operators has an independently random coefficient. By large disorder, we mean that for every coefficient \(h_i\), the standard deviation of \(\log(h_i)\) is large. SBRG performs well in both the fully MBL and the marginal MBL phases. SBRG does not perform well in or near thermal phases. In this work we keep the largest 1024 additional terms during each RG step. [72] Keeping more terms in \(\Sigma^2\) was beneficial for this work because it allows SBRG to capture more of the small terms in \(H_{\text{MBL}}\) (Eq. 2), which allows us to more accurately calculate the OTOC at larger distances in Fig. 3.

In Fig. 3 we show the color plots of the OTOC \(F(t) = \text{Tr} W(t)V(0)W(t)V(0)\) for local operators \(W = \sigma_i^x\) and \(V = \sigma_j^y\) at sites \(i\) and \(j\) respectively. The choice of the operators is quite generic. The primary consideration is to avoid the operators that commute with most of the local integral of motions (LIOMs) in the MBL system, else it is difficult to observe the decay of the OTOC within reasonable time scale.[23] As the LIOMs in the large-\(J_z\) spin glass phase are mainly \(\sigma_i^z\sigma_{i+1}^z\), we will not choose \(W\) or \(V\) to be \(\sigma^z\) operators. Other than that, we have tried several different choices of \(W\) and \(V\), and the resulting OTOC is similar to what is shown in Fig. 3.

In our calculation, the disorder averaging is done using a geometric mean (which measures the typical value of the OTOC). More specifically, we calculated

\[ \exp \text{avg} \ln |F(t)| = \exp \left( \frac{1}{N_d} \sum_{\delta} \ln |F(t)| \right) \tag{14} \]

where \(\sum_{\delta}\) denotes the summation over \(N_d\) disorder samples. The typical correlation function in a marginal MBL phase, and its crucial difference from the arithmetic mean value (often dominated by rare events) was discussed in many previous studies [63–66] In our case, the geometric mean was used because it was not possible to accurately calculate the ordinary mean at large time and distance separation using our SBRG methods. For a given disorder sample, sometimes SBRG does not manage to find enough terms in Eq. 9, which results in an \(F(t)\) that is too large at large time \(t\). This error can substantially affect the arithmetic mean of \(|F(t)|\), but is negligible in the geometric mean. Therefore we use the
typical OTOC (i.e. geometric mean) to reduce the rare-event effect.

We see in Fig. 3 that in general the OTOC starts out from 1 and decays to 0. The time-scale for the onset of the decay, i.e. the scrambling time \( t_{\text{scr}} \), grows monotonically with the distance \( d_{WV} = |i - j| \) between the operators \( W \) and \( V \). The top row of Fig. 3 is deep in the MBL spin glass phase with \( J_z/J_{x,y} = 8 \), while the middle and bottom row are at the marginal MBL critical points with \( J_x = J_y = J_z = 2J_z \) (see Fig. 1 for a phase diagram). The left column is plotted with \(|i - j|\) as the horizontal axis, while the right column uses \( \sqrt{|i - j|} \). The side-by-side comparison shows that in the MBL phase, the OTOC light cone is logarithmic \( \ln t_{\text{scr}} \sim d_{WV}^{1/2} \), while the marginal MBL light cone obeys \( \ln t_{\text{scr}} \sim d_{WV}^{1/2} \), as expected. On the other hand, if we treat \( d_{WV} \) as a function of time:

\[
d_{WV} \sim (\ln t)^2,
\]

then \( d_{WV} \) can be viewed as the size of the operator \( W(t) \). So Eq. 15 also describes the slow spreading of the quantum information of operator \( W \) in the system. Its transport universality class is known as the Sinai diffusion,[67] which governs the transport in critical Anderson localized system.[68] Our calculation demonstrates that the spreading of quantum information in marginal MBL systems also follows the Sinai diffusion rule. Interaction does not seem to affect the diffusion behavior, probably because the operator growth in the marginal MBL system is dominated by the single-body terms \( \sum_a e_a \tau^z_a \) of the MBL Hamiltonian. The Sinai diffusion of quantum information is also seen in the entanglement growth \( S(t) \sim (\ln t)^2 \) for Ising-like marginal MBL systems, as studied in Ref. 44, 48, 69.

In summary, we demonstrated how the OTOC in MBL and marginal MBL systems can be efficiently calculated using the SBRG approach. The system size can be pushed to several hundred sites, much larger than the previous exact diagonalization studies. We confirmed the logarithmic butterfly light cone \( \ln t_{\text{scr}} \sim d_{WV}^{1/2} \) in the MBL system. We found the marginal MBL system is a faster scrambler due to quantum criticality. Its scrambling is dominated by single-body terms in the MBL effective Hamiltonian, which is different from the MBL cases. Therefore marginal MBL systems have a different butterfly light cone scaling \( \ln t_{\text{scr}} \sim d_{WV}^{1/2} \). In this paper, we focused on the case where \( W \) and \( V \) are both local operators. Our calculation can be generalized to generic operators over regions of finite lengths.

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[29] R. Berkovits and Y. Avishai, Journal of Physics: Condensed Matter 8, 389 (1996).
[30] R. Berkovits and Y. Avishai, arXiv preprint cond-mat/9707066 (1997).
[31] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Physical Review Letters 95, 206603 (2005), cond-mat/0506411.
[32] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006), cond-mat/0506617.
[33] J. Z. Imbrie, ArXiv e-prints (2014), 1403.7837.
[34] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006), cond-mat/0506617.
[35] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Physical Review Letters 95, 206603 (2005), cond-mat/0506411.
[36] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006), cond-mat/0506617.
[37] J. Z. Imbrie, ArXiv e-prints (2014), 1403.7837.
[38] R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics 6, 15 (2015), 1407.0838.
[39] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006), cond-mat/0506617.
[40] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Physical Review Letters 95, 206603 (2005), cond-mat/0506411.
[41] R. Nandkishore and A. C. Potter, Phys. Rev. B 90, 195115 (2014), 1406.0847.
[42] Y.-Z. You, X.-L. Qi, and C. Xu, ArXiv e-prints (2015), 1508.03635.
[43] K. Slagle, Y.-Z. You, and C. Xu, Phys. Rev. B 94, 014205 (2016), 1604.02428.
[44] R. Vosk and E. Altman, Physical Review Letters 110, 067204 (2013), 1205.0026.
[45] B. Swingle, ArXiv e-prints (2013), 1307.0507.
[46] G. Refael and E. Altman, Comptes Rendus Physique 14, 725 (2013), 1402.6008.
[47] D. Pekker, G. Refael, E. Altman, E. Demler, and V. Oganesyan, Physical Review X 4, 011052 (2014), 1307.3253.
[48] R. Vosk and E. Altman, Physical Review Letters 112, 217204 (2014), 1307.3256.
[49] G. Refael, E. Altman, E. Demler, and V. Oganesyan, Physical Review X 4, 011052 (2014), 1307.3253.
[50] M. Serbyn, Z. Papić, and D. A. Abanin, Physical Review Letters 111, 127201 (2013), 1305.5554.
[51] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B 90, 174202 (2014), 1305.4915.
[52] I. H. Kim, A. Chandran, and D. A. Abanin, ArXiv e-prints (2014), 1412.3073.
[53] A. Chandran, I. H. Kim, G. Vidal, and D. A. Abanin, Phys. Rev. B 91, 085425 (2015), 1407.8480.
[54] L. Rademaker, ArXiv e-prints (2015), 1507.07276.
[55] M. Srednicki, Phys. Rev. E 50, 888 (1994).
[56] L. F. Santos and M. Rigol, Phys. Rev. E 81, 036206 (2010).
[57] M. Blake, ArXiv e-prints (2015), 1504.3651.
[58] M. Blake, ArXiv e-prints (2015), 1603.08510.
[59] Y. Gu, X.-L. Qi, and D. Stanford, ArXiv e-prints (2016), 1609.07832.
[60] S. X. Chen, W. Hrdle, and M. Li, Journal of the Royal Statistical Society: Series B (Statistical Methodology) 65, 663 (2003), ISSN 1467-9868, URL http://dx.doi.org/10.1111/1467-9868.00408.
[61] B. Efron, Ann. Statist. 7, 1 (1979), URL http://dx.doi.org/10.1214/aos/1176344552.
[62] D. S. Fisher, Phys. Rev. Lett. 69, 534 (1992).
[63] D. S. Fisher, Phys. Rev. B 50, 3799 (1994).
[64] D. S. Fisher, Phys. Rev. B 51, 6411 (1995).
[65] O. Motrunich, K. Damle, and D. A. Huse, Phys. Rev. B 63, 134424 (2001), cond-mat/0005543.
[66] Y. G. Sinai, Theory Probab. Appl. 27 (1982).
[67] D. Bagrets, A. Altland, and A. Kamenev, ArXiv e-prints (2016), 1605.01657.
[68] R. Vosk and E. Altman, Physical Review Letters 112, 217204 (2014), 1307.3256.
[69] S. Gopalakrishnan, M. Mueller, V. Khemani, M. Knap, E. Demler, and D. A. Huse, ArXiv e-prints (2015), 1502.07712.
[70] In MBL phase or marginal MBL phase, $W$ and $V$ should be an infinite sum of Pauli strings in the $\tau$ basis, with coefficients decaying exponentially with both the disorder and the range [51, 70], see for instance Fig.3 of Ref. 43. Neglecting the Schrieffer-Wolff correction keeps the leading Pauli string in this expansion.
[71] That is, in this work we keep $1024$ (instead of only $256$ as in Ref. 43) additional terms to $\Sigma^2$; see Appendix B in Ref. 43.