Polynomial-time Sparse Deconvolution

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Abstract
How can a probability measure be recovered with sparse support from its generalized moments? This problem, called sparse deconvolution, has been the focus of research in mathematics, theoretical computer science, and neural computing. However, there is no polynomial-time algorithm for the recovery. The best algorithm requires $O\left(\text{dimension}^{\text{poly}(1/\epsilon)}\right)$ for $\epsilon$-accurate recovery. We propose the first poly-time recovery method from carefully designed moments that requires $O\left(\text{dimension}^{4 \log(1/\epsilon)/\epsilon^2}\right)$ computations for an $\epsilon$-accurate recovery. This method relies on learning a planted two-layer neural network with two-dimensional inputs, a finite width, and zero-one activation. For learning such networks, we establish the first poly-time complexity, and demonstrate its application in sparse deconvolution.

1 Introduction
Sparse deconvolution. Consider a probability distribution $\mu$ with a finite support:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} \delta_{w_i},$$

where $w_i \in \mathbb{R}^d$ lies on the unit sphere denoted by $S_{d-1}$, and $\delta_{w_i}$ is the Dirac measure at $w_i$. Given a feature map $\Phi : S_{d-1} \rightarrow \mathcal{H}$ whose range lies in a Hilbert space $\mathcal{H}$, the generalized moments of $\mu$ are defined as [7]:

$$\Phi_\mu = \int \Phi(w)d\mu(w) \in \mathcal{H}.$$ (generalized moments)

Sparse deconvolution aims at recovering $\mu$ from generalized moments $\Phi_\mu$ [7, 9]. Inspired by $\ell_1$-regularization in compressed sensing [8], [9] proposes a sparse deconvolution program based on regularization with the variation norm [9] as

$$\min_{\nu} \|\Phi_\mu - \Phi_\nu\|_H^2 + \lambda \|\nu\|_{TV}$$

for a probability measure $\nu$. Solving the above convex program guarantees an exact recovery of $\mu$ even in the presence of noise [9]. However, minimizing with respect to an infinite dimensional measure is difficult to implement. This paper proposes the first implementable polynomial time deconvolution from carefully designed moments. The proposed deconvolution is based on particle programming.

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Particle programming. To implement the deconvolution, one may decompose $\nu$ as an average of Dirac measures at particles $v_1,\ldots,v_n \in \mathcal{S}_{d-1}$ and optimize with respect to $v_1,\ldots,v_n$:

$$\min_{\nu} L(\nu) = \|\Phi \nu - \Phi \mu\|_H^2,$$

such that $\nu = \frac{1}{n} \sum_{i=1}^{n} \delta_{v_i}$. (moment matching)

Contrary to the variation norm regularization, the above particle programming is not convex. Yet, gradient descent on particles converges to the Wasserstein gradient flow recovering the minimized form of $L$ as $n \to \infty$ (for homogeneous $\varphi$) [3, 27]. Deconvolution with finitely many particles has been the focus of research in theoretical computer science and neural computing. Theoretical computer scientists have extensively studied the deconvolution with polynomial moments in the form $\Phi(w) := \text{vector}(w^\otimes k) \in \mathbb{R}^{d^k}$ for tensor decomposition, while neural computing relies on non-polynomial moments. The choice of generalized moments determines deconvolution complexity.

Tensor decomposition. The focus of $k$-tensor decomposition is the recovery from $\Phi(w) := \text{vector}(w^\otimes k) \in \mathbb{R}^{d^k}$. For these moments, moment matching does not necessarily recover $\mu$: The recovery is (information-theoretically) possible if $n < 2^{k-1}$ [13]. As $n$ grows, $k$ is required to increase, which raises the recovery cost. For example, [17] proposes a recovery method with sum-of-squares that requires $k = O(1/\epsilon)$ to achieve an $\epsilon$-accurate solution in $O(d^{\text{poly}(1/\epsilon)})$ [17], which increases with $1/\epsilon$ at an exponential rate. This exponential complexity is, to the best of our knowledge, the best computational complexity established for the sparse measure recovery. Here, we research moments $\Phi \mu$ yielding a polynomial complexity in $1/\epsilon$.

Learning planted neural networks. The community of neural computing uses the feature map $\Phi_p(w) : \mathbb{R}^d \to \mathbb{R}$ lies in the Hilbert space $H = L_2$ (space of squared integrable functions):

$$\Phi_p(w)(x) = \varphi(x^\top w)\sqrt{p(x)},$$

(neural features)

where $p$ (a parameter of the feature map) denotes the input distribution, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-polynomial function called the activation function. Neural generalized moments are defined as $\Phi_p(\mu) = \int \Phi_p(w)d\mu(w)$. The recovery from neural moment matching reduces to optimization of the following function:

$$L(\nu) := \int (\Phi_p(\nu)(x) - \Phi_p(\mu)(x))^2 dx.$$

(neural moment matching)

Minimizing $L(\mu)$ with respect to $\frac{1}{n} \sum_{i=1}^{n} \delta_{v_i}$ is equivalent to a recovery of a planted neural network [4]. The support of $\mu$, i.e., $\{w_1,\ldots,w_n\}$, contains the weights of a planted network with inputs $x$. The support of $\nu$, i.e., $\{v_1,\ldots,v_n\}$, constitutes the weights optimized to recover $\mu$. Although optimizing a single neuron is known to be NP-hard [10], the complexity of learning a planted network is still an open problem. We establish poly-time computational complexity for a particular two-dimensional network. Then, we demonstrate its applications in sparse deconvolution.

2 Main results

We propose a poly-time deconvolution from particles as stated in the next theorem.
Theorem 1. Suppose that the support of $\mu$ is composed of distinct $\{w_i \in S_{d-1}\}_{i=1}^n$ (see $\text{A}_2$). There exists a feature map $\Phi_\mu \in \mathbb{R}^m$, with a finite $m$, that allows an $\epsilon$-recovery of $\mu$. In particular, the randomized Algorithm 3 obtains an $\epsilon$-recovery of $\mu$ from $m = O(dn\epsilon^{-1})$ moments with probability at least $1 - \kappa^{-1}$ in $O(n^3d^2\kappa^2\epsilon^{-2}\log(\epsilon^{-1}))$ time.

This is the first poly-time complexity established for sparse deconvolution. The best known recovery algorithm, which is developed for tensor decomposition, has the exponential complexity $O(d^{\text{poly}(1/\epsilon)})$ [17]. The proposed recovery algorithm is based on gradient descent on neural moment matching with

$$\varphi(a) = \begin{cases} 
0 & a \leq 0 \\
1 & a > 0.
\end{cases}$$ (zero-one activation)

The next theorem establishes a poly-time iteration complexity for gradient descent on neural moment matching.

Theorem 2. Suppose that $\{w_1, \ldots, w_n \in \mathbb{R}^2\}$ belongs to the upper half-circle, and $p$ is the uniform measure on the unit sphere; then, gradient descent in polar coordinates on neural moment matching with zero-one activations obtains an $\epsilon$-recovery (defined in Section 4) with $O(n^2/\epsilon)$ iterations.

Previous global convergence results for particle gradient descent rely on an infinite number of particles, i.e., $n \to \infty$ [5]. For a finite $n$, [13] establishes the convergence of stochastic gradient descent to an $(n^{-\alpha})$-optimal solution for an $\alpha > 0$ under displacement convexity of $L$. The last theorem establishes the first polynomial time convergence for an implementable particle gradient descent with a finite number of particles, thereby demonstrating the power of the particle gradient descent method. Remarkably, the last theorem extends to a broad class of input distribution introduced in Section 8.

The recovery with moment matching is a non-convex program. Polynomial-time convergence of gradient descent is established for only a few non-convex functions including Rayleigh quotient [21, 15], Polyak-Lojasiewicz functions [14], star-convex functions [20], and transformations of convex functions [19]. Neural moment matching does not belong to any of these non-convex programs, for which we establish a novel convergence analysis for gradient descent.

Although the gradient descent algorithm in Theorem 2 enjoys a poly-time iteration complexity, its implementation requires infinite moments. We propose an approximate gradient descent in Algorithm 4 that uses $O(n/\epsilon)$ moments for an $\epsilon$-recovery with the total computational complexity $O(n^4\log(1/\epsilon)/\epsilon^2)$. This recovery is established in Theorem 7 and is experimentally validated in Section 7. Using this algorithm, we propose a poly-time algorithm for sparse deconvolution in Section 6.

3 Background

Minimum mean discrepancy formulation. We define the kernel associated with moments $\Phi$ as $k(w, w') = \langle \Phi(w), \Phi(w') \rangle$, and write $L$ alternatively as the minimum mean discrepancy (MMD):

$$L(\nu) = \text{MMD}_k(\nu, \mu) := \int k(w, w')d\nu(w)d\nu(w') - 2 \int k(w, w')d\nu(w)d\mu(w') + \int k(w, w')d\mu(w)d\mu(w').$$ (MMD)
\(L\) is not convex since the level sets in the contour plot are not convex.

\[\text{[1]}\] has studied optimization of MMD with Wasserstein gradient flows. According to this study, MMD does not obey displacement convexity \([18]\), hence optimization of MMD is computationally feasible only under strong assumptions. Here, we demonstrate the important role of kernels in optimization. When \(x\) is uniformly drawn from the unit circle, the associated kernel with the neural moments has the following closed-form \([6]\):

\[
k(w, w') := \langle \Phi_p(w), \Phi_p(w') \rangle = 1 - \frac{\phi(w, w')}{\pi},
\]

(neural kernel)

where \(\cos(\phi(w, w')) = \langle w, w' \rangle / (\|w\|\|w'\|)\). We introduce notation \(s(\theta) = (\sin(\theta), \cos(\theta)) \in S_1\). For \(d = 2\), we use polar representations as \(w_i = s(\omega_i), v_i = s(\theta_i)\). Assumptions in Theorem \([2]\) ensure \(\theta_i, \omega_i \in [0, \pi)\). Replacing the neural kernel into MMD yield:

\[
(\pi n^2)L(\theta) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} |\theta_i - \omega_j| - \sum_{i=1}^{n} \sum_{j=1}^{n} |\theta_i - \theta_j| - \sum_{i=1}^{n} \sum_{j=1}^{n} |\omega_i - \omega_j|.
\]

(1)

For infinitely many particles, \([3]\) establishes the global convergence of Wasserstein gradient flow. Here, we want to extend this result to a finite number of particles.

**Landscape analysis.** \(L\) is not convex as Figure 1 illustrates. Despite non-convexity, the next lemma proves that \(L\) admits a unique critical point if particles \(\{w_i\}\) are distinct. This result builds upon the landscape analysis in \([22]\), which proves local minima of MMD \(k\) are limited to \(\{\pm w_1, \ldots, \pm w_n\}\).

**Lemma 3.** \(L(\theta)\) admits a unique critical point with distinct \(\{\theta_i \in [0, \pi)\}\) for distinct \(\{\omega_i \in [0, \pi)\}\).

**Proof.** The partial (sub)derivative of \(L\) with respect to \(\theta_i\) reads as

\[
(\pi n^2) \frac{dL}{d\theta_i} = \sum_{j=1}^{n} \text{sign}(\theta_i - \omega_j) - \sum_{j \neq i, j=1}^{n} \text{sign}(\theta_i - \theta_j).
\]

(2)

\(^1\)A detailed derivation is provided in Section \([8]\)
Assuming that all \( \{\theta_i\} \) and \( \{\omega_i\} \) are distinct and not equal, the sum of \( n - 1 \) signs in the first derivative term cannot be equal to the sum of \( n \) signs in the second derivative term. Therefore, \( 0 \not\in dL/d\theta_i \) unless \( \theta_i = \omega_j \) holds for some \( i, j \in \{1, \ldots, n\} \).

**Non-smoothness challenge.** The uniqueness of the critical point does not necessarily guarantee the recovery of the sparse measure in polynomial-time, since \( L \) is not a smooth function. When optimizing smooth functions, gradient descent converges to a critical point in polynomial time \([19]\). However, this is not the case for non-smooth functions \([28]\). We leverage a particular structure of \( L \), which is stated later in Lemma \([5]\) to establish a convergence rate for gradient descent.

## 4 Neural deconvolution

Throughout this section, we established the poly-time convergence of gradient descent for two-dimensional neural moment matching stated in Theorem \([2]\) More precisely, we prove that the distribution of particles contracts to the optimal distribution in Wasserstein distance with gradient descent. The proof idea is inspired by Wasserstein gradient flow view towards gradient descent proposed by \([11]\). Detailed proofs are included in the Appendix (Section \([B]\)).

**Wasserstein distance.** Let \( \nu = \frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_i} \), and \( \mu = \frac{1}{n} \sum_{i=1}^{n} \delta_{\omega_i} \). Assuming \( \omega_1 \leq \cdots \leq \omega_n \), we introduce the following distance

\[
W(\nu, \mu) = \min_{\sigma \in \Lambda} \max_{i} |\theta_{\sigma(i)} - \omega_i|,
\]

where \( \Lambda \) is the set of all permutations of indices \( \{1, \ldots, n\} \). Indeed, \( W \) is Wasserstein-\( \infty \) distance and the permutation \( \sigma \) is a transport map from \( \mu \) to \( \nu \) \([25]\). The next lemma proves that a permutation \( \sigma \) sorting \( \theta_{\sigma(i)} \) is the optimal transport map.

**Lemma 4** (Optimal transport). *Without loss of generality, assume \( \omega_1 \leq \cdots \leq \omega_n \). For \( \theta_{\sigma(1)} \leq \cdots \leq \theta_{\sigma(n)} \), we have \( W(\nu, \mu) = \max_{i \in \{1, \ldots, n\}} |\theta_{\sigma(i)} - \omega_i| \).*

The above property of the optimal transport plays an important role in the proof of Theorem \([2]\).

**The gradient direction.** The gradient of \( L \) can be expressed by cumulative densities as

\[
(\pi n) \frac{dL}{d\theta_i} = 2 \left( \int_{-\pi}^{\theta_i} \mu(\theta) d\theta - \int_{-\pi}^{\theta_i} \nu(\theta) d\theta \right), \quad \nu = \frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_i}.
\]

Thus, moving \( \theta_i \) along the negative gradient compensates the difference between the cumulative densities of \( \mu \), and \( \nu \). The next lemma formalizes this observation.

**Lemma 5** (Gradient direction). *For \( \theta_1 < \cdots < \theta_n \), the following bound holds

\[
-\text{sign}(\theta_i - \omega_i) \left( \frac{dL}{d\theta_i} \right) \leq -\frac{1}{\pi n^2}.
\]
Gradient descent. Gradient descent optimizes $L$ by
\[ \theta_i^{(q+1)} = \theta_i^{(q)} - \gamma \frac{dL}{d\theta_i}(\nu_q), \quad \nu_q := \frac{1}{n} \sum_{i=1}^{n} \delta_{q(i)}, \]
for $i = 1, \ldots, n$, and stepsize $\gamma \in \mathbb{R}_+$. According to Lemma 6, gradient descent contracts $\theta_i$ to $\omega_i$ for sorted $\theta_i$s and $\omega_i$s, which is the optimal transport map. Combining these two observations, the next lemma establishes the convergence $\nu_q \to \mu$.

**Lemma 6.** Suppose that $\omega_1, \ldots, \omega_n \in (-\pi, \pi)$. Starting from distinct $\theta_1^{(0)}, \ldots, \theta_n^{(0)} \in (-\pi, \pi)$, gradient descent obeys
\[ W(\nu_q, \mu) \leq \epsilon \]
for $q \geq \lceil \frac{n^2 \pi^2}{\epsilon} \rceil + 1$, and $\gamma = \epsilon$.

The last lemma concludes the proof of Theorem 2. Note that the result of the last lemma is stronger than those of Theorem 2: The last lemma requires $\omega_i \in (-\pi, \pi)$ while $\omega_i \in [0, \pi)$ is sufficient for Theorem 2.

### 5 Deconvolution with finite moments

In this section, we establish a poly-time deconvolution from a finite number of moments. Note that $\nabla_{\omega} L$ implicitly exploits infinitely many moments. Inspired by random Fourier features [24], we propose Fourier moments as
\[ \Phi_{\mu} = [\Phi_1(\mu), \ldots, \Phi_m(\mu)], \quad \Phi'_{\mu} = [\Phi'_1(\mu), \ldots, \Phi'_m(\mu)], \]
where
\[ \Phi_m(\mu) = \left( \frac{2}{\sqrt{(2m+1)\pi}} \right) \sum_{i=1}^{n} \sin ((2m+1)\omega_i), \]
\[ \Phi'_m(\mu) = \left( \frac{2}{\sqrt{(2m+1)\pi}} \right) \sum_{i=1}^{n} \cos ((2m+1)\omega_i). \]

The above moments are obtained by a truncated Fourier series expansion, which allows us to approximate $\nabla_{\omega} L$. Algorithm 1 is developed based on this gradient approximation. Increasing the number of moments provides a more accurate gradient approximation, thereby allowing deconvolution. The next theorem establishes the required number of moments and computations for sparse deconvolution with Algorithm 1.

**Theorem 7.** Suppose that $\min_{i \neq j} |\omega_i - \omega_j| = \beta > 0$ and $\{\omega_i \in (-\pi, \pi)\}_{i=1}^{n}$. Algorithm 1 with stepsize $\gamma = \min \left\{ \frac{\epsilon}{2}, \frac{\beta}{10} \right\}$, and starting from distinct $\{\theta_i^{(0)}\}_{i=1}^{n}$ obtains $W(\nu_K, \mu) \leq \epsilon$ from $m = 800n(\epsilon^{-1} + \beta^{-1})$ moments with total computational complexity
\[ O \left( n^4 \log \left( \frac{1}{\epsilon} \right) \left( \frac{1}{\epsilon^2} \right) + n^4 \log(\beta) \left( \frac{1}{\beta^2} \right) \right). \]

The last theorem relies on a technical assumption: particles $\{\omega_1, \ldots, \omega_n\}$ are distinct. Remarkably, this assumption was not required for the convergence of gradient descent established in Lemma 6. Random $\{w_1, \ldots, w_n\}$ stratifies this assumption with high probability. Although our theoretical analysis relies on distinctness, we conjecture that deconvolution is possible without this assumption.
Algorithm 1  Deconvolution with finite moments

Require: $\Phi_{\mu}$ defined in Eq.(1), stepsize $\gamma \in \mathbb{R}_+$, 
Let $K = \lfloor 200\pi^2 n^2 \log(4\pi \gamma^{-1}) \gamma^{-1} \rfloor + 1$.
for $i = 1, \ldots, n$ do
  for $q < K$ do
    $\theta^{(q+1)}_i = \theta^{(q)}_i - \frac{\gamma}{n^2 \pi} \left( \langle \Phi(\mu), \Phi'(\nu_q) \rangle - \langle \Phi'(\mu), \Phi(\nu_q) \rangle - \sum_{j \neq i}^{n} \text{sign}(\theta^{(q)}_i - \theta^{(q)}_j) \right)$.
  end for
end for
Return $\nu_t = \sum_{i=1}^{n} \theta^{(K)}_i / n$

Algorithm 2  Deterministic sparse deconvolution

Require: $\Phi_{\mu_q}^{\text{(coordinates)}}, \Phi_{\mu_q}^{\text{(pairs)}}$ in Eq. (1), $\epsilon, \beta$ in $A_3$.
for $q = 1, \ldots, d$ do
  Run Algorithm 1($\Phi_{\mu_q}^{\text{(coordinates)}}, \gamma = \min\{\frac{1}{2} \epsilon, \frac{1}{10} \beta\}$) to get $\{[v_i]_q\}_{i=1}^{n}$
end for
for $q = 1, \ldots, d$ do
  Run Algorithm 1($\Phi_{\mu_q}^{\text{(pairs)}}, \gamma = \min\{\frac{1}{2} \epsilon, \frac{1}{10} \beta\}$) to get $\{[y_i]_q\}_{i=1}^{n}$
end for
for $i = 1, \ldots, n$ do
  $[v'_i]_1 \leftarrow [v_i]_1$
end for
for $i, j, r = 1, \ldots, n, q = 2, \ldots, d$ do
  $\triangledown$ Post-processing to glue the coordinates
  if $|[v_i]_1 + [v_j]_q - [y_r]_q| < \beta/5$ then
    $[v'_i]_q \leftarrow [v_j]_q$
  end if
end for
Return $\{v'_1, \ldots, v'_n\}$

6  Polynomial-time sparse deconvolution

To complete the proof of Theorem 1, we extend the deconvolution to measures with sparse supports in $S_{d-1}$ for $d > 2$. The proof is a reduction from $d$-dimensional to a collection of one-dimensional deconvolutions. Such a reduction allows us to leverage the poly-time deconvolution with Algorithm 7 analyzed in the last section. Leveraging this reduction, we propose a deterministic deconvolution for particles with distinct coordinates. Then, we use this deterministic method to design a randomized algorithm that recovers distinct particles with high probability (even if their coordinates are not distinct).

6.1  Deterministic deconvolution

According to Theorem 7 Algorithm 1 recovers individual coordinates $\{[w_i]_q \in [-\pi, \pi]\}$ from moments of $\mu_q^{\text{(coordinates)}} := \sum_{i=1}^{n} \delta_{[w_i]_q} / n$, which is obtained from Diracs at coordinates of particles.
To reconstruct the vectors from individual coordinates, we also recover

\[ \mu_i^{\text{(pairs)}}(w) = \frac{1}{n} \sum_{i=1}^{n} \delta_{\text{pair}(i, r)}, \quad \text{pair}(i, r) := [w_i]_1 + [w_i]_r, \]

whose particles are sum of first and rth coordinates of \( \{w_1, \ldots, w_n\} \). The recovery of \( \{\text{pair}(i, r)\} \) allows us to reconstruct vectors \( w_1, \ldots, w_n \) under the following condition.

**Assumption \( A_1(\beta) \).** We assume that there exists a positive constant \( \beta > 0 \) such that

\[ \beta \leq \min\{\min_{i \neq j, q} |[w_i]_q - [w_j]_q|, \min_{i \neq j, r > 1} |\text{pair}(i, r) - \text{pair}(j, r)|\}. \]

Under the above assumption, Algorithm 2 recovers \( d \)-dimensional particles \( \{w_1, \ldots, w_n\} \) from moments of \( \{\mu_q^{\text{(coordinates)}}, \mu_r^{\text{(pairs)}}\} \), which uses Algorithm 1 and also a post processing to glue recovered coordinates \( \{[w_i]_q\} \) together and construct \( \{w_i\} \).

**Lemma 8.** Suppose that \( A_1(\beta) \) holds; then, Algorithm 2 returns \( \{v'_i \in \mathbb{R}^d\}_{i=1}^n \) such that

\[ \min_{\sigma \in \Lambda} \max_i \|v'_i - w_{\sigma(i)}\|_\infty \leq \epsilon \]

holds with total computational complexity \( O(n^4d(\log(1/\epsilon)/\epsilon^2 + \log(1/\beta)/\beta^2 + 1)) \).

The recovery complexity linearly scales with dimension, but it is \( O(n^4) \). In that regards, increasing the number of particles is more challenging than increasing the dimension. This challenge is also observed for recovery from tensorial moments: the polynomial recovery is achievable for \( n < d \), while the recovery for \( n \gg d \) is \( O(d^{\text{poly}(1/\epsilon)}) \) with sum-of-squares [17].

### 6.2 Randomized deconvolution

Recall that Theorem 1 ensures the deconvolution of distinct \( \{w_i \in \mathcal{S}_{d-1}\} \), while the recovery established in the last lemma relies on a rather stronger assumption. Here, we established the recovery for distinct particles, thereby completing the proof of Theorem 1.

**Assumption \( A_2(\ell) \).** Assuming that particles \( \{w_i \in \mathbb{R}^d\}_{i=1}^n \) are distinct, there exists a positive constant \( \ell \) such that \( \|w_i - w_j\| \geq \ell \) holds for \( i \neq j \in \{1, \ldots, n\} \).

Suppose that \( Z \in \mathbb{R}^{d \times d} \) is a matrix whose elements are i.i.d. Gaussian random variables. We define distribution \( Z\mu = \frac{1}{n} \sum_{i=1}^{n} \delta_{Zw_i/\|Z\|} \), which is obtained by random projections of particles \( \{w_i\} \). Such random projects ensure \( A_2(\ell) \) with high probability. Algorithm 3 exploits this property for deconvolution.

**Algorithm 3** Randomized deconvolution

| Require: | The random matrix \( Z \in \mathbb{R}^{d \times d} \), constant \( \ell \) in \( A_2(\ell) \) and \( \epsilon > 0 \). |
|----------|---------------------------------------------------------------|
| Run Algorithm 2 \( \Phi(Z\mu^{\text{(coordinates)}}, \Phi(Z\mu^{\text{(pairs)}}}, \beta = \Omega\left(\frac{\ell}{\epsilon^2d/\|Z\|^2}\right), \epsilon = \frac{\epsilon'}{d} \) to get \( \{v'_i\}_{i=1}^n \). |
| Return | \( \|Z\|Z^{-1}v'_1, \ldots, \|Z\|Z^{-1}v'_n \) |
Lemma 9. Suppose \( \{ w_i \in S_{d-1} \} \) and obey \( A_2(\ell) \). With probability at least \( 1 - 1/\kappa \), Algorithm 3 returns \( \{ v_1, \ldots, v_n \in \mathbb{R}^d \} \) for which

\[
\min_{\sigma \in \Lambda} \max_i \| v_i - w_{\sigma(i)} \| = O(\epsilon)
\]

holds with the total computational complexity

\[
O \left( n^8 d^4 \kappa^4 \log \left( \frac{nd}{\ell} \right) \left( \frac{1}{\ell^2} \right) + n^4 d^3 \left( 1 + \log \left( \frac{d}{\epsilon} \right) \frac{1}{\epsilon^2} \right) \right).
\]

7 Experiments

We experimentally validate properties of Algorithm 1 and Algorithm 2 for sparse deconvolution with \( d = 1 \) and \( d = 10 \). Our experiments demonstrate deconvolution of four random particles with Algorithm 1 and also validate the influence of the number of particles on its performance. These experiments are implemented in python and run on Google Colaboratory.

A warm-up experiment. As a warm up example, we validate the recovery of four particles:

\[
\mu = \frac{1}{4} \sum_{i=1}^4 \delta_{\omega_i}
\]

where \( \omega_i \) is drawn i.i.d. from uniform \([0, \pi)\). Figure 2.a) shows particles \( \{ \theta^{(q)}_i \}_{i=1}^4 \) generated by Algorithm 1 and confirms their convergence to \( \omega_1, \ldots, \omega_4 \) established by Theorem 7.

![Figure 2a](image1)

Figure 2: Simulations for Algorithm 1. Figure a) vertical axis: \( \theta^{(q)}_i \) where \( i \) is marked by colors; horizontal axis: \( q \); red lines: \( \omega_1, \ldots, \omega_4 \). We used \( n = 4, \gamma = 0.005, m = 500, \) and \( t = 2000 \). Figure b) Recovery of many particles from \( m = 1000 \) moments with \( \gamma = 0.005 \). Mean and 90% confidence intervals of 6 independent runs.

Deconvolution of many particles. Now, we experimentally validate the recovery for \( \mu \) with larger supports containing 10 and 100 particles uniformly drawn from \([0, \pi)\). Figure 2b) plots \( W(\nu_K, \mu) \) for different \( K \) that controls the number of gradient descent steps in Algorithm 1. The decay of \( W(\nu_K, \mu) \) in this plot confirms the convergence of \( \nu_K \) towards \( \mu \) as \( K \to \infty \). For more particles, a larger \( K \) is required, which is proven by the complexity bound in Theorem 7.
High dimensional deconvolution. Algorithm 2 is run to deconvolute \( w_1, \ldots, w_{20} \in \mathbb{R}^{10} \) whose coordinates are drawn i.i.d. from uniform \([0, 1]\). This random scheme ensures \( A_1(\beta) \) for a small \( \beta \), which is required for the deconvolution. Figure 3 confirms the recovery of \( w_1, \ldots, w_{20} \) with Algorithm 2.

8 Beyond a uniform measure

The convergence of gradient descent in Theorem 2 relies on \( x \) drawn uniformly from unit circle (recall \( x \) generates neural moment matching). Let \( x_\omega = (\sin(\omega), \cos(\omega)) \) where \( \omega \in [0, 2\pi) \) is random with density \( p \) and cumulative density \( P \). We extend the convergence of gradient descent to \( p \) obeying the next assumption.

**Assumption \( A_3(b, B) \).** We assume that \( p \) obeys

- \( p(\omega + \pi/2) = p(\omega + \pi) = p(\omega + 3\pi/2) \) for \( \omega \in [0, \pi/2) \),
- \( p(\omega) > b \) for all \( \omega \in [0, 2\pi) \),
- \( p(\omega) \leq B \) for all \( \omega \in [0, 2\pi) \).

The next proposition characterizes the kernel induced by neural features

**Proposition 10.** Consider neural features \( \Phi_p \) with \( p \) obeying \( A_3 \); then, for \( \theta, \omega \in [0, \pi) \), we have:

\[
k(w_\theta, v_\omega) = 1 - 2 \left| \int_0^\theta p(\alpha) d\alpha - \int_0^\omega p(\alpha) d\alpha \right|_{P(\theta)}.
\]
Replacing the above result into the MMD leads to the following recovery objective:

\[
\left( \frac{n^2}{2} \right) L(\theta) = 2 \sum_{i,j=1}^{n} |P(\theta_i) - P(\omega_j)| - \sum_{i,j=1}^{n} |P(\theta_i) - P(\theta_j)| - \sum_{i,j=1}^{n} |P(\omega_i) - P(\omega_j)|. 
\]

As a special case, replacing a uniform measure \( p \) in the above equation obtains Eq. (1). Comparing the above with Eq. (1) reveals that a change of variables converts the above objective to those presented in Eq. (1). Leveraging such a change of variables, the next Lemma establishes the convergence of gradient descent to input distributions obeying Assumption \( A_3 \).

**Lemma 11.** Under \( A_3(b, B) \), Gradient descent on \( L \) starting from distinct \( \{\theta_i^{(0)} \in [0, \pi)\} \)s after

\[ q \geq \left\lceil \frac{\pi/\gamma}{b^2} \right\rceil + 1 \] iterations obtains \( W(\nu_q, \mu) \leq \gamma B. \)

The absolute continuity of \( p \) (i.e., \( b > 0 \)) is necessary to be able to compute the gradient and also ensure the number of required iterations to be finite in the last Lemma.

## 9 Discussion

This paper establishes the first polynomial time complexity for sparse deconvolution with particle programming. Yet, we believe that particle gradient descent optimizes a broader family of functionals with applications beyond sparse deconvolution.

**Beyond the zero-one activation.** The result of Theorem 2 relies on zero-one activation in neural moment matching. However, this activation function is not continuous. As a result, absolute continuity of input distribution is necessary to get a differentiable objective function. Nonetheless, the input distribution often is supported on i.i.d. samples \( \mathbb{2} \), hence non-continuous. To remedy this issue, one may approximate the zero-one activation with a continuous function. Such an approximation may allow for recovery from a finite number of input samples.

There are different ways to approximate the zero-one activation with a continuous function. For example, sigmoid \( \alpha := (1 + \exp(C\alpha))^{-1} \) approximates the zero-one activation for a large positive \( C \). Remarkably, sigmoid is a classical choice for activation functions in neural networks. In that regard, the poly-time convergence of gradient descent (in Theorem 2) may be extended to learning standard neural networks with sigmoid activations. Another approach for approximation of the zero-one activation is the Fourier series expansion proposed in Appendix \( C \). Remarkably, this expansion allows us to establish recovery of a finite number of moments with Algorithm 1.

**Noisy deconvolution and function approximation.** Consider \( f : \mathbb{R}^d \rightarrow \mathbb{R} \):

\[
f(x) = \int \varphi(w^\top x) d\mu(w),
\]

where \( \mu \) is a probability measure. Given \( w_1, \ldots, w_n \overset{i.i.d}{\sim} \mu \), \( f \) alternatively reads as

\[
f(x) = \frac{1}{n} \sum_{i=1}^{n} \varphi(w_i^\top x) + \epsilon_n,
\]
where we called $\epsilon_n$ an error term. Empirical process theory \cite{23} establishes concentration bounds for $\epsilon_n$ decreasing with $n$. Therefore, the recovery of $w_1, \ldots, w_n$ provides an approximation for $f$. This recovery requires extending our analysis to sparse deconvolution from noisy moments. Such an extension enable us to propose poly-time algorithms for function approximation.

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Appendix

A The necessary assumption

Theorem 2 and Lemmas 6 and 11 assume \( \{w_i\} \) are on the upper-half of unit circle. The next Lemma proves that this assumption is necessary for the recovery.

**Lemma 12.** For the general case \( \{w_i \in S_1\} \), neural moment matching admits global minima not equivalent to \( \mu \) if \( \varphi \) is a step function.

**Proof.** Suppose that for all \( i \in \{1, \ldots, n\} \) there exists an \( i* \in \{1, \ldots, n\} \) such that \( w_i = -w_{i*} \). Consider \( \nu = \frac{1}{n} \sum_{i=1}^{n} \delta_{v_i} \) where \( v_i = -v_{i*} \) also holds for \( i, i* \in \{1, \ldots, n\} \). This symmetric property leads to

\[
\Phi_x \nu = \Phi_x \mu = 0 = L(\nu).
\]

(5)

B Convergence analysis for gradient descent

**Lemma 4** [Optimal transport] Without loss of generality, assume \( \omega_1 \leq \cdots \leq \omega_n \). For \( \theta_{\sigma(1)} \leq \cdots \leq \theta_{\sigma(n)} \), we have \( W(\nu, \mu) = \max_{i \in \{1, \ldots, n\}} |\theta_{\sigma(i)} - \omega_i| \).

**Proof of Lemma 4.** Let \( \sigma^* \) denote the optimal transport obeying

\[
\sigma^* = \arg \min_{\sigma} \max_i |\theta_{\sigma(i)} - \omega_i|.
\]

(6)

The proof idea is simple: if there exists \( i < j \) such that \( \theta_{\sigma^*(i)} > \theta_{\sigma^*(j)} \), then swapping \( \sigma^*(i) \) with \( \sigma^*(j) \) will not increase \( f \). To formally establish this idea, we define transport \( \sigma' \) obtained by the swap as

\[
\sigma'(q) = \begin{cases} 
\sigma^*(q) & q \neq i \\
\sigma^*(j) & q = i \\
\sigma^*(i) & q = j.
\end{cases}
\]

(7)

We prove that \( f(\sigma') \leq f(\sigma) \). Let define the following compact notations:

\[
\Delta_{ij} = \max\{|\theta_{\sigma^*(i)} - \omega_i|, |\theta_{\sigma^*(j)} - \omega_j|\}
\]

\[
\Delta'_{ij} = \max\{|\theta_{\sigma^*(i)} - \omega_i|, |\theta_{\sigma^*(j)} - \omega_j|\} = \max\{|\theta_{\sigma^*(j)} - \omega_i|, |\theta_{\sigma^*(i)} - \omega_j|\}.
\]

(8)

(9)

According to the definition,

\[
f(\sigma') = \max\{\Delta'_{ij}, \max_{q \neq i, j} |\theta_{\sigma^*(q)} - \omega_q|\} \leq \max\{\Delta_{ij}, \max_{q \neq i, j} |\theta_{\sigma^*(q)} - \omega_q|\} = f(\sigma^*)
\]

(10)

holds. Since \( \theta_{\sigma^*(i)} > \theta_{\sigma^*(j)} \), \( \Delta'_{ij} < \Delta_{ij} \) holds as it is illustrated in the following figure. Therefore, \( f(\sigma') \leq W(\nu, \mu) \) holds, in that \( \sigma' \) is also an optimal transport. Replacing \( \sigma^* \) by \( \sigma' \) and repeating the same argument ultimately concludes the proof.
Lemma 5. [Gradient direction] For $\theta_1 < \cdots < \theta_n$, the following bound holds

$$-\text{sign} (\theta_i - \omega_i) \left( \frac{dL}{d\theta_i} \right) \leq -\frac{1}{\pi n^2}. \quad (11)$$

Proof of Lemma 5. The partial derivative $dL/d\theta_i$ consists of two additive components:

$$\left(\pi n^2\right) \frac{dL}{d\theta_i} = \sum_j \text{sign}(\theta_i - \omega_j) - \sum_{j \neq i} \sum_{2i-1}^{2i-n-1} \text{sign}(\theta_i - \theta_j),$$

where

$$\Delta = |\{\omega_m < \theta_i\} - |\{\omega_m > \theta_i\}| \quad (12)$$

$$= 2|\{\omega_m < \theta_i\}| - n \quad (13)$$

$$= n - 2|\{\omega_m > \theta_i\}|. \quad (14)$$

Consider the following two cases:

i. $\theta_i > \omega_i$: In this case, $|\{\omega_m < \theta_i\}| \geq i$ (see Fig. 4). Plugging this into Eq. (13), we get $\Delta \geq 2i - n$ that yields $dL/d\theta_i \geq 1/(\pi n^2)$.

ii. $\theta_i < \omega_i$: In this case, $|\{\omega_m \geq \theta_i\}| \geq n - i + 1$ demonstrated in Fig. 4 Using Eq. (14), we get $\Delta \leq 2i - n - 2$ that leads to $dL/d\theta_i \leq -1/(\pi n^2)$.

Combining the above two results concludes the proof.

Figure 4: The cardinality bound. Left: $\theta_i > \omega_i$. Right: $\theta_i < \omega_i$. 

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Lemma 6. [GD convergence] Suppose that \( \omega_1, \ldots, \omega_n \in (-\pi, \pi) \). Starting from distinct \( \theta^{(0)}_1, \ldots, \theta^{(0)}_n \in (-\pi, \pi) \), gradient descent obeys

\[
W(\nu_q, \mu) \leq \epsilon
\]

for \( q \geq \left\lceil \frac{n^2\pi^2}{\epsilon} \right\rceil + 1 \), and \( \gamma = \epsilon \).

Proof of Lemma. Without loss of generality, we assume that \( \theta^{(q)}_1 \leq \cdots \leq \theta^{(q)}_n \); then, the gradient descent obeys

\[
\left| \theta^{(q+1)}_i - \omega_i \right| = \left| \theta^{(q)}_i - \omega_i - \gamma \frac{dL}{d\theta_i}(\nu_q) \right|. \tag{15}
\]

Invoking Lemma 5 yields

\[
\left| \theta^{(q+1)}_i - \omega_i \right| = \left| \theta^{(q)}_i - \omega_i - \gamma \frac{dL}{d\theta_i} \right|. \tag{16}
\]

Using Lemma 4, we get

\[
W(\nu_{q+1}, \mu) \leq \max_i \left| \theta^{(q+1)}_i - \omega_i \right| \tag{17}
\]

\[
\leq \max_i \left| \theta^{(q)}_i - \omega_i \right| - \gamma \left| \frac{dL}{d\theta_i} \right| \tag{18}
\]

\[
\leq \begin{cases} 
\max \{W(\nu_q, \mu) - \frac{\epsilon}{n^2\pi}, \epsilon\} & W(\nu_q, \mu) \geq \epsilon \\
\epsilon & \text{otherwise} 
\end{cases} \tag{19}
\]

We complete the proof by contradiction. Suppose that \( W(\nu_m, \mu) \geq \epsilon \) holds for \( m = 1, \ldots, q' \) where \( q' = \left\lceil (\pi n)^2 / \epsilon \right\rceil + 1 \); then, the induction over \( k \) yields

\[
W(\nu_{q'}, \mu) < W(\nu_0, \mu) - \pi < 0. \tag{20}
\]

The above inequality contradicts to \( W \geq 0 \). Therefore, there exists \( q_* < q' \) such that \( W(\nu_{q_*}, \mu) \leq \epsilon \).

According to Eq. (73), \( W(\nu_q, \mu) \leq \epsilon \) holds for all \( q \geq q_* \). \qed

C Deconvolution with finite moments

In this section, we establish a polytime complexity for Algorithm 1. Recall this Algorithm exploits the following iterations for deconvolution:

\[
\theta^{(q+1)}_i = \theta^{(q)}_i - \frac{\gamma}{n^2\pi} \left( \langle \Phi(\mu), \Phi'(\nu_q) \rangle - \langle \Phi'(\mu), \Phi(\nu_q) \rangle - \sum_{j \neq i} \text{sign}(\theta^{(q)}_i - \theta^{(q)}_j) \right). 
\]

The next theorem represents the deconvolution complexity with the above iterative scheme.
**Theorem 7.** Suppose that \( \min_{i \neq j} |\omega_i - \omega_j| = \beta > 0 \) and \( \{\omega_i \in (-\pi, \pi)\}_{i=1}^n \). Algorithm 1 with stepsize \( \gamma = \min\{\frac{\epsilon}{2}, \frac{\beta}{10}\} \), and starting from distinct \( \{\theta_i^{(0)}\}_{i=1}^n \) obtains \( W(\nu_K, \mu) \leq \epsilon \) from \( m = 800n(\epsilon^{-1} + \beta^{-1}) \) moments with total computational complexity

\[
O \left( n^4 \log \left( \frac{1}{\epsilon} \right) \left( \frac{1}{\epsilon^2} \right) + n^4 \log(\beta) \left( \frac{1}{\beta^2} \right) \right). 
\]

**Proof of Theorem 7.** The moments \( \Phi_k \) are designed based on Fourier series expansion of \( \text{sign} \), namely the following equation:

\[
\Delta \neq 0 \text{ and } \Delta \in (-\pi, \pi) : \text{sign}(\Delta) = \sum_{r=0}^{\infty} \left( \frac{4}{\pi(2r+1)} \right) \sin((2r+1)\Delta). \quad (21)
\]

Fourier series expansion of order \( m \) approximate \( \text{sign}(\Delta) \) by the following function

\[
g(\Delta) = \sum_{r=0}^{m} \left( \frac{4}{\pi(2r+1)} \right) \sin((2r+1)\Delta). \quad (22)
\]

For \( \Delta = \theta - \omega \), we get

\[
g(\theta - \omega) = \sum_{r=0}^{m} \frac{4}{\pi(2r+1)} \sin((2r+1)\theta) \cos((2r+1)\omega) - \cos((2r+1)\theta) \sin((2r+1)\omega)). \quad (23)
\]

The next lemma establishes two important approximation bounds for \( g \).

**Lemma 13.** For all \( \Delta \in \mathbb{R} - \{0\} \), we have

\[
|g(\Delta) - \text{sign}(\Delta)| \leq 4 \left( \frac{1}{m|\Delta|} + \frac{1}{m} \right). \quad (24)
\]

Furthermore, \( |g(\Delta)| \leq 1.9 \) for \( |\Delta| \leq \frac{\pi}{4} \) and \( m > 12 \).

**Proof.** We prove for \( \Delta > 0 \), and the proof easily extends to \( \Delta < 0 \). According to the definition of \( g \), we have

\[
|g(\Delta) - \text{sign}(\Delta)| = \left| \left( \frac{4}{\pi} \right) \sum_{r=m+1}^{\infty} \frac{\sin((2r+1)\Delta)}{2r+1} \right| \sum_{r=m+1}^{\infty} \frac{\sin((2r+1)\Delta)}{2r+1} \quad (25)
\]

Let define \( F_m \) as

\[
F_m(\Delta) = \sum_{r=1}^{m} \frac{\sin(r\Delta)}{r}. \quad (26)
\]

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Given $F_m(\Delta), S_{m+1}(\Delta)$ reads as:

$$S_{m+1}(\Delta) = F_\infty(\Delta) - F_{2m}(\Delta) - \frac{1}{2} (F_\infty(2\Delta) - F_m(2\Delta)).$$  \hspace{1cm} (27)

Using $\frac{1}{r} \sin(r\Delta) = \int_0^\Delta \cos(rt)dt$, it is easy to show that (see [12] for a detailed derivation):

$$F_m(\Delta) = -\frac{\Delta}{2} + \int_0^\Delta \left( \frac{1}{2 \sin(t/2)} - \frac{1}{t} \right) \sin \left( \left( m + \frac{1}{2} \right) t \right) dt + \int_0^\Delta \sin \left( \frac{t}{t} \right) dt$$  \hspace{1cm} (28)

holds for $\Delta > 0$. Integration by parts yields

$$|\epsilon_m(\Delta)| \leq \left| \int_0^\Delta \left( \frac{1}{2 \sin(t/2)} - \frac{1}{t} \right) I e^{(m+\frac{1}{2})t} dt \right|$$  \hspace{1cm} (29)

$$= \frac{1}{m + \frac{1}{2}} \left( h(\Delta) e^{(m+\frac{1}{2})\Delta} - \lim_{\epsilon \to 0} h(\epsilon) e^{\epsilon} + \int_0^\Delta h'(t) e^{(m+\frac{1}{2})t} dt \right)$$  \hspace{1cm} (30)

$$\leq \frac{1}{m} \left( \int_0^\Delta |h'(t)| dt + |h(\pi)| \right)$$  \hspace{1cm} (31)

$$\leq \frac{3}{2m}.$$  \hspace{1cm} (32)

Replacing the above bound into Eq.(27) yields

$$|S_{m+1}(\Delta)| \leq \left| \int_0^\infty \frac{\sin(t)}{t} dt \right| + \frac{1}{2} \left| \int_0^\Delta \frac{\sin(t)}{t} dt \right| + \frac{3}{m}$$  \hspace{1cm} (33)

An application of the Laplace transform obtains (see [11]):

$$\left| \int_x^\infty \frac{\sin(t)}{t} dt \right| = \left| \int_0^\infty \frac{\sin(x) \cos(t) + \cos(x) \sin(t)}{t} dt \right|$$  \hspace{1cm} (34)

$$= \left| \int_0^\infty \frac{s \sin(x) + \cos(x)}{1 + s^2} \exp(-xs) ds \right|$$  \hspace{1cm} (35)

$$\leq \left| \int_0^\infty \exp(-xs) ds \right| \leq \frac{1}{|x|}.$$  \hspace{1cm} (36)

Replacing this into Eq. 33 concludes the proof for the first inequality.

For the second inequality, we use Eq. (27) together with (32):

$$|g(\Delta)| = \left| \frac{4}{\pi} \left( F_{2(m+1)}(\Delta) - \frac{1}{2} F_{m+1}(2\Delta) \right) \right|$$  \hspace{1cm} (37)

$$\leq \frac{2}{\pi} \left| \int_0^{(2m+\frac{3}{2})\Delta} \frac{\sin(t)}{t} dt \right| + \frac{4}{\pi} \left| \int_0^{(2m+\frac{3}{2})\Delta} \frac{\sin(t)}{t} dt \right| + \frac{3}{m}.$$  \hspace{1cm} (38)
Since the maximum of the sin integral is met for \( \pi \), we have
\[
\left| \int_0^x \frac{\sin(t)}{t} \, dt \right| \leq \left| \int_0^\pi \frac{\sin(t)}{t} \, dt \right| = \frac{\pi}{2}.
\] (39)

For \( 0 < |\Delta| < \frac{\pi}{4} \), we get
\[
\left| \int_{(2m+\frac{3}{2})\Delta}^{(2m+3)\Delta} \frac{\sin(t)}{t} \, dt \right| \leq \int_{(2m+\frac{3}{2})\Delta}^{(2m+3)\Delta} \frac{|\sin(t)|}{t} \, dt \leq \frac{\Delta}{2} \leq \frac{\pi}{8}.
\] (40)

Replacing the last two bounds into Eq. (38) concludes:
\[
|g(\Delta)| \leq 1.9.
\] (41)

In Algorithm 1, the sign is replaced by \( g \) which leads to
\[
\theta_i^{(q+1)} = \theta_i^{(q)} - \frac{\gamma}{n^2\pi} \left( g(\theta_i^{(q)} - \omega_j) - \sum_j \text{sign}(\theta_i^{(q)} - \theta_j^{(q)}) \right)
\] (42)

Without loss of generality, we assume \( \theta_1 \leq \cdots \leq \theta_n \). For the ease of notations, we use the compact notation \( \Delta_j = \theta_j^{(q)} - \omega_j \). The approximation error for the gradient is bounded as
\[
\left| \frac{dL}{d\theta_i}(\nu_q) - \tilde{g}_i \right| \leq \frac{1}{\pi n^2} \sum_r |g(\Delta_r) - \text{sign}(\Delta_r)| \leq \begin{cases} \frac{\pi}{4nm} \left( 1 + \frac{1}{\gamma} \right) & \forall j : |\Delta_j| \geq \gamma, \\
\frac{4}{\pi n m} \left( 1 + \frac{1}{\gamma} \right) \frac{1.91}{n^2 \pi} & \text{otherwise}, \end{cases}
\] (43)

where we use the last Lemma to get the second inequality. Since \( m \geq 800n/\gamma \), we get
\[
\left| \frac{dL}{d\theta_i}(\nu_q) - \tilde{g}_i \right| \leq \begin{cases} 0.01/(n^2 \pi) & |\Delta_j| \geq \gamma, \\
1.91/(n^2 \pi) & \text{otherwise}. \end{cases}
\] (44)

If \( |\Delta_j| \leq \gamma \) for \( j \neq i \), then the cardinality argument in Fig. 4 leads to the following inequality:
\[
n^2\pi \left( \frac{dL}{d\theta_i} \right) \text{sign}(\theta_i - \omega_i) \geq 2.
\] (45)

Combining the above result with Eq. (43) yields
\[
|\theta_i^{(q+1)} - \omega_i| \leq |\theta_i^{(q)} - \omega_i| - \frac{0.08 \gamma}{\pi n^2}, \ |\Delta_i| \geq \gamma.
\] (46)

Since \( |\theta_i^{(q)} - \omega_i| \leq 2\pi \) holds, we get
\[
|\theta_i^{(q+1)} - \omega_i| \leq \left( 1 - \frac{0.02 \gamma}{\pi n^2} \right) |\theta_i^{(q)} - \omega_i|, \ |\Delta_i| \geq \gamma.
\] (47)
Assumption \( A \)

Therefore, yields

Combining the last two inequalities concludes the proof.

Furthermore for all \( \gamma \), holds for all \( \omega \).

Proof of Lemma 8.

\[ \text{Invoking Thm. 7 yields: There exists a permutation } \hat{A} \text{ such that } \]

\[ \text{Algorithm 2 returns } \{v_i^* \in \mathbb{R}^d\}_{i=1}^n \text{ such that } \]

\[ \min \max_{\sigma \in \Lambda} \|v_i^* - w_{\sigma(i)}\|_\infty \leq \epsilon \]

holds with total computational complexity \( O(n^4 d (\log(1/\epsilon)/\epsilon^2 + \log(1/\beta)/\beta^2 + 1)) \).

Proof of Lemma 8

Invoking Thm. 7 yields: There exists a permutation \( \sigma \) of indices \( \{1, \ldots, n\} \) such that

\[ \|[v_i]_q - [w_{\sigma(i)}]_q \leq \min\{\epsilon, \frac{1}{5} \beta\} \] (49)

holds for all \( i = 1, \ldots, n \). Similarly, there exits a permutation \( \sigma' \) such that

\[ \|[y_{\sigma'(i)}]_q - ([w_{\sigma(i)}]_q + [w_{\sigma(i)}]_q) | \leq \epsilon \]

holds for all \( i = 1, \ldots, n \). \( \text{[A] } \beta \) yields

\[ \|[v_i]_q - [v_j]_q \geq |[w_{\sigma(i)}]_q - [w_{\sigma(j)}]_q - 2\epsilon \geq \beta/2 \] (\( \epsilon \leq \beta/4 \)). (51)

Therefore

\[ \|[v_i]_1 + [v_i]_q - [y_{\sigma'(i)}]_q | \leq |[v_i]_1 + [v_i]_q - [w_{\sigma(i)}]_1 - [w_{\sigma(i)}]_q | + \epsilon \leq 3\epsilon. \] (52)

Furthermore for all \( j \neq i \), we get

\[ |[v_i]_1 + [v_i]_q - [y_{\sigma(j)}]_q | \geq |[v_i]_1 + [v_i]_q - [w_{\sigma(j)}]_1 - [w_{\sigma(j)}]_q | - \epsilon \]

\[ \geq |[w_{\sigma(i)}]_1 + [w_{\sigma(i)}]_q - [w_{\sigma(j)}]_1 - [w_{\sigma(j)}]_q | - 3\epsilon \] (53)

\[ \geq \beta - 3\epsilon \geq \beta/5. \] (55)

Combining the last two inequalities concludes the proof.
Lemma 9. Suppose \( \{w_i \in S_{d-1}\} \) and obey \( A_2(\ell) \). With probability at least \( 1 - 1/\kappa \), Algorithm 3 returns \( \{v_1, \ldots, v_n \in \mathbb{R}^d\} \) for which
\[
\min_{\sigma \in \Lambda} \max_i \|v_i - w_{\sigma(i)}\| = O(\epsilon)
\]
holds with the total computational complexity
\[
O\left(n^8 d^4 \kappa^4 \log \left(\frac{nd}{\ell^2}\right) + n^4 d^3 \left(1 + \log \left(\frac{d}{\epsilon}\right) \frac{1}{\epsilon^2}\right)\right).
\]

Proof of Lemma 9. We will repeatedly use the following concentration bound for the condition number of the random matrix \( Z \).

Lemma 14 (\cite{26}). There are universal constants \( c_1 \) and \( c_2 \) such that
\[
P(\|Z\| \leq \alpha \sqrt{d}) \leq (c_1 \alpha)^d \tag{56}
\]
\[
P(\|Z^{-1}\| \leq \alpha^{-1} \sqrt{d}) \leq \exp(-c_2 \alpha^2), \tag{57}
\]
holds for \( \alpha > 0 \).

Let \( Z' = Z/\|Z\| \). The proof-idea is simple: Projected particles \( \{Z'w_i\} \) obeys \( A_3(\beta) \) for a small \( \beta \) with high probability. Therefore, we can leverage Algorithm 2 for the recovery whose theoretical guarantees are established by Lemma 8.

Lemma 15. Under \( A_2(\ell) \), \( \{Z'w_i\}_{i=1}^n \) obeys \( A_3 \) for \( \beta = \Omega(\ell/(\kappa^2 d^3/2n^2)) \) with probability at least \( 1 - 1/\kappa \).

Proof. Let \( v_{ij} := w_i - w_j \), and \( z_q \in \mathbb{R}^d \) denote the rows of \( Z \). Then, \( \zeta_q(ij) = (z_q^\top v_{ij})^2/\|v_{ij}\|^2 \) is a \( \chi^2 \)-square random variable for which the following bound holds
\[
P(\|v_{ij}\|^2 \zeta_q(ij) \leq \kappa^2 \|v_{ij}\|^2) \leq \kappa. \tag{58}
\]
Setting \( \kappa = 1/(10dn^2) \) and union bound over all \( i, j, \) and \( q \) concludes
\[
P\left(\exists i, j, q : |z_q^\top (w_i - w_j)| \leq \frac{\ell}{2\kappa d n^2}\right) \leq \frac{1}{2\kappa}. \tag{59}
\]
Therefore,
\[
|(z_1 + z_r)^\top (w_i - w_j)| \leq |z_1^\top (w_i - w_j)| + |z_r^\top (w_i - w_j)| \leq \ell/(\kappa d n^2) \tag{60}
\]
holds with probability \( 1 - (2\kappa)^{-1} \). According to Lemma 14, the spectral norm of the random matrix \( Z \) is \( O(\kappa \sqrt{d}) \) with probability \( 1 - 1/(2\kappa) \). Combining the spectral bound and the last inequality completes the proof. \qed
Let define \( \hat{w}_i := Zw_i/\|Z\| \). According to Lemma 8, Algorithm 2 returns \( \{v'_i\}_{i=1}^n \) for which
\[
\max_i \min_{\sigma \in \Lambda} \|v'_i - \hat{w}_{\sigma(i)}\|_\infty \leq \frac{\epsilon}{d}
\]
holds with probability \(1 - 1/\kappa\) and a permutation \(\sigma\). According to Lemma 14, there exists a universal constant \(c\) such that
\[
P(\|Z^{-1}\| \geq k\sqrt{d}) \leq \frac{1}{\kappa}.
\]
Recall the output of Algorithm 3:
\[
v_i = \|Z\|Z^{-1}v'_i.
\]
Using the above bound, we complete the proof:
\[
\|v_i - w_{\sigma(i)}\| = \|\|Z\|Z^{-1}v'_i - \|Z\|Z^{-1}(Zw_{\sigma(i)}/\|Z\|)\| \\
\leq \|Z\|\|Z^{-1}\|\|v'_i - \hat{w}_{\sigma(i)}\| \\
\leq O(d)\|v'_i - \hat{w}_{\sigma(i)}\|_\infty \\
\leq O(\epsilon).
\]

E. Beyond a uniform measure

The next proposition characterizes the kernel induced by neural features.

**Proposition 16.** Consider neural features \(\Phi_p\) with \(p\) obeying \(A_3\); then,
\[
k(w_\theta, v_\omega) = 1 - 2\left|\int_0^\theta p(\alpha)d\alpha - \int_0^\omega p(\alpha)d\alpha\right|.
\]
holds for \(\theta, \omega \in [0, \pi)\).

Replacing the above result into \(\text{MMD}\) leads to the following recovery objective:
\[
\left(\frac{n^2}{2}\right) L(\theta) = 2 \sum_{i,j=1}^n |P(\theta_i) - P(\omega_j)| - \sum_{i,j=1}^n |P(\theta_i) - P(\theta_j)| - \sum_{i,j=1}^n |P(\omega_i) - P(\omega_j)|.
\]
As a special case, replacing a uniform measure \(p\) in the above equation obtains Eq. (1).

**Proof of Proposition 10.** According to the definition, we get
\[
k(w, v) = \mathbb{E} [\varphi(\cos(\theta - \alpha))\varphi(\cos(\omega - \alpha))] \\
= \left(2 \left(\int_a^\theta + \int_0^{2\pi} \int_b^{2\pi} p(\alpha)d\alpha\right)\right),
\]
where \(a = \pi/2 + \min\{\theta, \omega\}\) and \(b = 3\pi/2 + \max\{\theta, \omega\}\). \(A_3\) concludes
\[
k(w, v)/2 = P(a) + P(2\pi) - P(b) = \frac{1}{2} - |P(\theta) - P(\omega)|.
\]
Lemma 11. Under $A_3(b, B)$, Gradient descent on $L$ starting from distinct $\{\theta_i^{(0)} \in [0, \pi)\}$s after $q \geq \lceil \frac{\pi n}{2b\gamma} \rceil + 1$ iterations obtains $W(\nu_q, \mu) \leq \gamma B$.

Proof of Lemma 11. The proof is based on incorporating the change of variable as $\Theta_i^{(q)} = P(\theta_i^{(q)})$ into the proof of Lemma 6. Without loss of generality, we assume that $\theta_1^{(q)} \leq \cdots \leq \theta_n^{(q)}$; then, gradient descent obeys

$$\left| \theta_i^{(q+1)} - \omega_i \right| = \left| \theta_i^{(q)} - \omega_i - \gamma \left( \frac{dL}{d\Theta_i} \right) p(\theta_i^{(q)}) \right|. \tag{70}$$

Since $P$ is monotonically increasing, $\text{sign}(\theta_i^{(q)} - \omega_i) = \text{sign}(\Theta_i^{(q)} - \omega_i)$. Using the above result together with Lemma 5, we get

$$\left| \theta_i^{(q+1)} - \omega_i \right| = \left| \theta_i^{(q)} - \omega_i \right| - \gamma p(\theta_i^{(q)}) \left| \frac{dL}{d\Theta_i} \right|. \tag{71}$$

Then, invoking Lemma 4 yields

$$W(\nu_{q+1}, \mu) \leq \max_i \left| \theta_i^{(q)} - \omega_i \right| - \gamma p(\theta_i^{(q)}) \left| \frac{dL}{d\Theta_i} \right| \tag{72}$$

$$\leq \begin{cases} \max \{W(\nu_q, \mu) - \frac{\gamma b}{\pi n^2}, \gamma B\} & W(\nu_q, \mu) \geq B\gamma \\ \gamma B & \text{otherwise} \end{cases} \tag{73}$$

The rest of the proof is the same as the proof of lemma 6. \qed