APPELL-LAURICELLA HYPERGEOMETRIC FUNCTIONS OVER FINITE FIELDS AND ALGEBRAIC VARIETIES

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Abstract. We prove finite field analogues of integral representations of Appell-Lauricella hypergeometric functions in many variables. We consider certain hypersurfaces having a group action and compute the numbers of rational points associated with characters of the group, which will be expressed in terms of Appell-Lauricella functions over finite fields.

1. Introduction

Generalized hypergeometric functions \( n+1F_n(z) \) (the Gauss hypergeometric functions when \( n = 1 \)) over \( \mathbb{C} \) are defined by the power series

\[
n+1F_n \left( \begin{array}{c} a_0, a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{array} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k \cdots (a_n)_k}{(b_1)_k \cdots (b_n)_k} z^k.
\]

Here, \( a_i, b_j \) are complex parameters with \( b_j \not\in \mathbb{Z}_{\leq 0} \), and \( (a)_k = \Gamma(a + k)/\Gamma(a) \) is the Pochhammer symbol. Lauricella’s hypergeometric functions \( F_{D}^{(n)} \), \( F_{A}^{(n)} \), \( F_{B}^{(n)} \) and \( F_{C}^{(n)} \) with \( n \) variables (Appell’s functions \( F_1, F_2, F_3 \) and \( F_4 \) respectively, when \( n = 2 \)) are generalizations of the Gauss hypergeometric functions. For example,

\[
F_{D}^{(n)} \left( \begin{array}{c} a; b_1, \ldots, b_n \\ c \end{array} ; z_1, \ldots, z_n \right) := \sum_{k_i \geq 0} \frac{(a)_{k_1+\cdots+k_n} (b_1)_{k_1} \cdots (b_n)_{k_n} (c)_{k_1+\cdots+k_n} (1)_{k_1} \cdots (1)_{k_n}}{z_1^{k_1} \cdots z_n^{k_n}},
\]

where \( a, b_i, c \in \mathbb{C} \) with \( c \not\in \mathbb{Z}_{\leq 0} \). These functions have integral representations of Euler type, such as

\[
F_{D}^{(n)} \left( \begin{array}{c} a; b_1, \ldots, b_n \\ c \end{array} ; z_1, \ldots, z_n \right) = B(a, c - a)^{-1} \int_0^1 \left( \prod_{i=1}^{n} (1 - z_i u)^{-b_i} \right) u^{a-1}(1 - u)^{c-a-1} du.
\]

Over finite fields, one-variable hypergeometric functions were defined independently by Koblitz [15], Katz [13], Greene [8], McCarthy [19], Fuselier-Long-Ramakrishna-Swisher-Tu [7] and Otsubo [23]. Appell’s functions were defined by Li-Li-Mao [17], He [9], He-Li-Zhang [11] and Ma [18] as generalizations of Greene’s functions, and were defined by Tripathi-Saikia-Barman [29] as generalizations of McCarthy’s functions. For general \( n \), \( F_{D}^{(n)} \) were defined by Frechette-Swisher-Tu [5] and He [10], and \( F_{A}^{(n)} \) were defined by Chetry-Kalita [4] as generalizations of Greene’s functions.

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Otsubo [23] gave a definition of all the Lauricella functions, which will be used in this paper (see subsection 2.1).

In this paper, we prove finite field analogues of integral representations of $F_D^{(n)}$, $F_A^{(n)}$, $F_B^{(n)}$ and $F_C^{(n)}$ (Theorems 3.1, 3.3, 3.4, 3.5 and 3.7). As a corollary, we prove a finite analogue of Karlsson’s formula which relates certain $F_D^{(n)}$ with Gauss hypergeometric functions (Theorem 3.2). Furthermore, we show a finite field analogue (Theorem 3.12) of an integral representation of $F_4(x(1-y), y(1-x))$ due to Burchnall-Chaundy [3].

The reason for the strong analogy between a hypergeometric function over $\mathbb{C}$ and a hypergeometric function over a finite field is that they come from a same algebraic variety. The former is the complex period of the variety and the latter is the trace of Frobenius acting on the $l$-adic étale cohomology. By the Grothendieck-Lefschetz formula, the Frobenius trace is related with the number of rational points on the variety. For example, one-variable hypergeometric functions, over $\mathbb{C}$ and over finite fields, are associated with the variety of the form

$$y^d = (1 - \lambda x_1 \cdots x_n)^{a_0} \prod_{i=1}^n x_i^{a_i} (1 - x_i)^{b_i}.$$  

By computing the number of its rational points over finite fields, Koblitz [15] arrived at his definition of the hypergeometric function.

For the Appell-Lauricella functions, we find naturally corresponding algebraic varieties from the complex integral representations. For example, an algebraic curve $C_{D,\lambda}$ related to $F_D^{(n)}$ is given by

$$y^d = \left( \prod_{i=1}^n (1 - \lambda_i x)^{b_i} \right) x^a (1 - x)^c.$$  

They admit an action of the group $\mu_d$ of $d$th roots of unity, and each of the numbers of $\kappa$-rational points decomposes into $\chi$-components for characters $\chi$ of $\mu_d$, where $\kappa$ is a finite field. By the analogues of integral representations mentioned above, such numbers are expressed in terms of Appell-Lauricella functions over $\kappa$ (Theorems 4.2, 4.4, 4.8, 4.9, 4.11 and 4.13).

According to the decomposition of the numbers, each of the zeta functions decomposes into the Artin $L$-functions. As corollaries of the theorems, we express the Artin $L$-functions in terms of the Appell-Lauricella functions over $\kappa_r$ ($r \geq 1$), where $\kappa_r$ is a degree $r$ extension of $\kappa$ (Corollaries 4.5, 4.10, 4.12 and 4.14).

Furthermore, under some conditions, we will closely look at the curve $X_{D,\lambda}$ which is a smooth projective model of $C_{D,\lambda}$. For each non-trivial character $\chi$ of $\mu_d$, using the result above, the Artin $L$-function $L(X_{D,\lambda}, \chi; t)$ is written in terms of Lauricella functions $F_D^{(n)}(\lambda_1, \ldots, \lambda_n)$ over $\kappa_r$ ($r \geq 1$). By the Grothendieck-Lefschetz formula, the Artin $L$-function $L(X_{D,\lambda}, \chi; t)$ is essentially the characteristic polynomial of the Frobenius acting on the $\chi$-eigenspace $H^1(X_{D,\lambda}, \mathbb{Q}_l)(\chi)$ of the first $l$-adic étale cohomology. By computing its dimension, we will show that the degree of $L(X_{D,\lambda}, \chi; t)$ is $n + 1$ (Theorem 4.7), and hence it follows that $F_D^{(n)}(\lambda_1, \ldots, \lambda_n)$ over $\kappa_r$ ($r \geq 1$) are written as symmetric polynomials of the first $n + 1$ functions.
2. Hypergeometric functions over finite fields

Throughout this paper, let \( \kappa \) be a finite field with \( q \) elements of characteristic \( p \). Let \( \widehat{\kappa} = \text{Hom}(\kappa, \mathbb{Q}^\times) \) denote the group of multiplicative characters of \( \kappa \), and write \( \varepsilon \in \widehat{\kappa} \) for the trivial character. For any \( \eta \in \widehat{\kappa} \), we set \( \eta(0) = 0 \) and write \( \overline{\eta} = \eta^{-1} \). Put, for \( \eta \in \widehat{\kappa} \),

\[
\delta(\eta) = \begin{cases} 1 & (\eta = \varepsilon), \\ 0 & (\eta \neq \varepsilon). \end{cases}
\]

2.1. Definitions. In this subsection, we recall definitions \([23]\) of hypergeometric functions over finite fields.

Fix a non-trivial additive character \( \psi \in \text{Hom}(\kappa, \mathbb{Q}^\times) \). For \( \eta, \eta_1, \ldots, \eta_n \in \widehat{\kappa} \) \((n \geq 2)\), the Gauss sum \( g(\eta) \) and the Jacobi sum \( j(\eta_1, \ldots, \eta_n) \) are defined by

\[
g(\eta) = - \sum_{x \in \kappa^\times} \psi(x) \eta(x) \in \mathbb{Q}(\mu_{p(q-1)}),
\]

\[
j(\eta_1, \ldots, \eta_n) = (-1)^{n-1} \sum_{\eta \in \widehat{\kappa}} \prod_{i=1}^{n} \eta_i(x_i) \in \mathbb{Q}(\mu_{q-1}).
\]

Note that \( g(\varepsilon) = 1 \). Put \( g^\circ(\eta) = g^\delta(\eta) g(\eta) \). Then (cf. \([23, \text{Proposition 2.2 (iii)}]\))

\[
g(\eta) g^\circ(\overline{\eta}) = g(\bar{\eta}) q.
\]

For \( \eta, \ldots, \eta_n \in \widehat{\kappa} \), we have (cf. \([23, \text{Proposition 2.2 (iv)}]\))

\[
j(\eta_1, \ldots, \eta_n) = \begin{cases} 1 - (1 - q)^n & (\eta_1 = \cdots = \eta_n = \varepsilon), \\ \frac{g(\eta_1) \cdots g(\eta_n)}{g^\circ(\eta_1 \cdots \eta_n)} & \text{(otherwise)}. \end{cases}
\]

As an analogue of the Pochhammer symbol \((a)_n = \Gamma(a + n)/\Gamma(a)\), put

\[
(\alpha)_\nu = \frac{g(\alpha \nu)}{g(\alpha)}, \quad (\alpha)^\circ_\nu = \frac{g^\circ(\alpha \nu)}{g^\circ(\alpha)}
\]

for \( \alpha, \nu \in \widehat{\kappa} \). Then, these satisfy

\[
(\alpha)_{\nu \mu} = (\alpha)_{\nu}(\alpha \nu)_{\mu}, \quad (\alpha)^\circ_{\nu \mu} = (\alpha)^\circ_{\nu}(\alpha \nu)^\circ_{\mu},
\]

and

\[
(\alpha)^\circ_{\nu}(\overline{\nu})^\circ_{\mu} = \nu(-1).
\]

Definition 2.1. For \( \alpha_0, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \widehat{\kappa} \), the hypergeometric function over \( \kappa \) is defined by

\[
{}_{n+1}F_n \left( \begin{array}{c} \alpha_0, \alpha_1, \ldots, \alpha_n \\ \beta_1, \ldots, \beta_n \end{array}; \lambda \right) = \frac{1}{1 - q} \sum_{\nu \in \widehat{\kappa}} \frac{\alpha_0(\alpha_1)_\nu \cdots (\alpha_n)_\nu}{(\nu)^\circ(\beta_1)^\circ_\nu \cdots (\beta_n)^\circ_\nu} \nu(\lambda) \quad (\lambda \in \kappa).
\]

Definition 2.2. For \( \alpha, \alpha_1, \ldots, \alpha_n, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \widehat{\kappa} \), Lauricella’s functions over \( \kappa \) are defined as follows. For \( \lambda_1, \ldots, \lambda_n \in \kappa \),

\[
{}_{A}F^{(n)}_A \left( \begin{array}{c} \alpha; \beta_1, \ldots, \beta_n \\ \gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_n \end{array}; \lambda \right)
\]
Properties. Analogues of Appell’s functions are defined by

\[ F_B^{(n)}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \lambda_1, \ldots, \lambda_n) \]

\[ F_C^{(n)}(\alpha; \beta_1, \ldots, \beta_n; \lambda_1, \ldots, \lambda_n) \]

\[ F_D^{(n)}(\alpha; \beta_1, \ldots, \beta_n; \lambda_1, \ldots, \lambda_n) \]

Remark 2.3. A priori, the functions \( F_{n+1}^n \) are \( Q(\mu_{p(q-1)}) \)-valued, but in fact they take values in \( Q(\mu_{q-1}) \) (see [23, Lemma 2.5 (iii)]).

Remark 2.4. By (2.4), (2.3) and (2.1), one shows that, for \( \lambda_i \in \kappa^x \),

\[ F_B^{(n)}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \lambda_1, \ldots, \lambda_n) = (\beta_1 \cdots \beta_n) \left( \prod_{i=1}^{n} (\alpha_i), \gamma \right) \]

\[ F_D^{(n)}(\alpha; \beta_1, \ldots, \beta_n; \lambda_1, \ldots, \lambda_n) = \frac{1}{\lambda_1 \cdots \lambda_n} \left( \frac{1}{\lambda_1} \cdots \frac{1}{\lambda_n} \right). \]

2.2. Properties. We recall some formulas on \( n+1 \) \( F_n \) which will be used in the next section.

Proposition 2.5 ([23, Corollary 3.4 and Corollary 3.6]).

(i) For each \( \alpha \in \kappa^x \) and \( \lambda \in \kappa^x \),

\[ _1F_0 \left( \alpha; \lambda \right) = \left\{ \begin{array}{ll} \overline{\alpha}(1 - \lambda) & (\alpha \neq \varepsilon \text{ or } \lambda \neq 1), \\ 1 - q & (\alpha = \varepsilon \text{ and } \lambda = 1). \end{array} \right. \]
(ii) Suppose that $\beta \neq \gamma$. Then, for $\lambda \neq 0$,

$$-j(\beta, \gamma)F(\alpha, \beta; \gamma; \lambda) = \sum_{u \in F^*} \beta(u)\beta(1-u)\alpha(1-\lambda u) + \delta(\alpha)(1-q)\gamma(\lambda)\beta(\lambda - 1).$$

(The case when $\alpha = \varepsilon$ is not contained in [23, Corollary 3.6], but one shows the case easily by Lemma 2.7.)

**Proposition 2.6 (cf. [23, Theorem 3.2]).** If $n \geq 1$,

$$\binom{\alpha_1, \ldots, \alpha_n, \gamma}{\beta_1, \ldots, \beta_{n-1}, \gamma; \lambda} = q^{\delta(\gamma)}\binom{\alpha_1, \ldots, \alpha_n}{\beta_1, \ldots, \beta_{n-1}; \lambda} + 1 + \frac{\prod_{i=1}^{n}(\alpha_i)\gamma}{(\varepsilon)^{\gamma}\prod_{i=1}^{n}(\beta_i)\gamma} F(\lambda).$$

**Lemma 2.7.** For $\lambda \in F^*$,

$$2F1\left(\frac{\alpha, \varepsilon}{\gamma}; \lambda\right) = \frac{\prod_{i=1}^{n}(\alpha_i)\gamma}{(\varepsilon)^{\gamma}\prod_{i=1}^{n}(\beta_i)\gamma} F(\lambda).$$

**Proof.** By letting $\mu = \gamma \nu$ and using (2.3), we have

$$2F1\left(\frac{\alpha, \varepsilon}{\gamma}; \lambda\right) = \frac{1}{1-q} q^{1-\delta(\nu)} \sum_{\mu} (\alpha_i)_{\mu}^{\gamma}(\lambda) + 1$$

$$= \frac{q}{1-q} \prod_{i=1}^{n}(\alpha_i)\gamma(\lambda) \sum_{\mu} (\alpha_i)^{\gamma}(\lambda)\mu(\lambda) + 1$$

$$= \prod_{i=1}^{n}(\alpha_i)\gamma(\lambda)1F1\left(\frac{\alpha_i}{\gamma}; \lambda\right) + 1.$$

Thus, we obtain the lemma by Proposition 2.5 (i). \qed

The following propositions are slight generalizations of Otsubo’s results [23]. A finite analogue of the Pfaff formula is the following.

**Proposition 2.8 (cf. [23, Theorem 3.13]).** Suppose that $\beta \neq \varepsilon$, $\alpha \neq \gamma$. Then, for $\lambda \neq 1$,

$$\alpha(1-\lambda)2F1\left(\frac{\alpha, \beta}{\gamma}; \lambda\right) = 2F1\left(\frac{\alpha, \varepsilon}{\gamma}; \frac{\lambda}{\lambda-1}\right) + \delta(\beta)(1-q)\frac{\gamma(\lambda)}{g(\gamma)} F(\lambda).$$

**Proof.** By Proposition 2.5 (ii) and letting $v = u(1-\lambda)/(1-\lambda u)$, we have

$$-j(\alpha, \beta, \gamma)2F1\left(\frac{\alpha, \gamma}{\gamma}; \lambda\right) = \sum_{u} \alpha(u)\beta(1-u)\gamma(1-\lambda u)$$

$$= \frac{\alpha(1-\lambda)}{\gamma} \sum_{v} \alpha(v)\beta(1-v)\gamma\left(1-\frac{\lambda u}{\lambda-1}\right).$$
Thus the proposition follows from Proposition 2.5 (ii). □

The following is a finite analogue of the Vandermonde theorem (cf. [26, (1.7.7)]).

**PROPOSITION 2.9** (cf. [23, Theorem 4.3 and Remark 4.4]).

(i) If \( \{\alpha, \mu\} \neq \{\varepsilon, \gamma\} \), then

\[
2F1 \left( \frac{\alpha, \mu}{\gamma}; 1 \right) = q^{-\delta(\mu)} (\gamma)_{\mu} (\alpha)_{\mu}.
\]

(ii) If \( \{\alpha, \mu\} = \{\varepsilon, \gamma\} \) then,

\[
2F1 \left( \frac{\alpha, \mu}{\gamma}; 1 \right) = q^{-\delta(\mu)} (\gamma)_{\mu} (\alpha)_{\mu} - \frac{(1 - q)^2(1 + q)^{\delta(\gamma)}}{q}.
\]

**PROOF.** (i) follows by [23, Theorem 4.3], hence we only have to prove (ii). Suppose that \( \{\alpha, \mu\} = \{\varepsilon, \gamma\} \). By [23, Theorem 4.3] again, it follows that

\[
2F1 \left( \frac{\alpha, \mu}{\gamma}; 1 \right) = 1 + q^{\delta(\gamma)}(1 - q).
\]

On the other hand, if \( \alpha = \varepsilon \) and \( \mu = \gamma \) then,

\[
q^{-\delta(\mu)} (\gamma)_{\mu} = q^{-\delta(\gamma)} (\gamma)_{\mu} = \frac{1}{q},
\]

and if \( \alpha = \gamma \) and \( \mu = \varepsilon \) then,

\[
q^{-\delta(\gamma)} (\gamma)_{\mu} = q^{-1} (\gamma)_{\mu} = \frac{1}{q}.
\]

Thus, we have

\[
q^{-\delta(\mu)} (\gamma)_{\mu} (\alpha)_{\mu} - \frac{(1 - q)^2(1 + q)^{\delta(\gamma)}}{q} = 1 + q^{\delta(\gamma)}(1 - q).
\]

Therefore, we obtain the proposition. □

A finite analogue of the Saalschütz theorem (cf. [26, (2.3.1.3)]) is the following.

**PROPOSITION 2.10** (cf. [23, Theorem 4.11]). Suppose that \( \alpha \neq \varepsilon \), \( \beta \neq \gamma \) and \( \alpha \beta \gamma \neq \varepsilon \). Then,

\[
3F2 \left( \frac{\alpha, \beta, \gamma}{\alpha, \beta, \gamma}; 1 \right) = q^{-(\delta(\gamma))} (\gamma)_{\nu} (\beta)_{\nu} + \frac{g^\gamma(\gamma)g^\varepsilon(\alpha \beta \gamma \varepsilon)}{g(\alpha)g(\beta)g(\gamma)} - (\delta(\gamma)\delta(\nu) + \delta(\beta)\delta(\gamma \nu))(1 - q)^2.
\]

**PROOF.** By [23, Theorem 4.11], we only have to prove for the case when \( \{\alpha, \beta, \mu\} = \{\varepsilon, \gamma, \alpha \beta \gamma \varepsilon\} \) (i.e. \( \alpha \gamma = \nu = \varepsilon \) or \( \beta = \gamma \nu = \varepsilon \)). If \( \alpha \gamma = \nu = \varepsilon \), then the right-hand side of the proposition is equal to \( 3 - q \). On the other hand, by Proposition 2.6 and Lemma 2.7, we have

\[
3F2 \left( \frac{\alpha, \beta, \gamma}{\alpha, \beta, \gamma}; 1 \right) = 3F2 \left( \frac{\alpha, \beta, \varepsilon}{\alpha, \beta, \varepsilon}; 1 \right) = \frac{1}{q} \frac{(\beta)_{\gamma}(\varepsilon)_{\gamma}}{(\varepsilon)_{\gamma}(\beta)_{\gamma}} + 2 - q = 3 - q.
\]

Here, note that \( \alpha \neq \beta \) and \( \beta \neq \varepsilon \) by the assumptions. Similarly, we can prove for \( \beta = \gamma \nu = \varepsilon \). □
3. Finite analogues of integral representations

3.1. The case of $F_D$. For a function $f : (\kappa^n)^n \to \mathbb{C}$, its Fourier transform is a function on $(\kappa^n)^n$ defined by

$$\hat{f}(\nu_1, \ldots, \nu_n) = \sum_{t_i \in \kappa^n} f(t_1, \ldots, t_n) \prod_{i=1}^n \overline{\nu_i(t_i)}.$$ 

Then,

$$f(\lambda_1, \ldots, \lambda_n) = \frac{1}{(q-1)^n} \sum_{\nu_i \in \kappa^n} \hat{f}(\nu_1, \ldots, \nu_n) \prod_{i=1}^n \nu_i(\lambda_i).$$ (3.1)

Over $\mathbb{C}$, Lauricella’s functions $F_D^{(n)}$ have the following integral representations (cf. [16, Theorem 3.4.1]). If $0 < \text{Re}(a) < \text{Re}(c)$,

$$B(a, c-a)F_D^{(n)} \left( \begin{array}{c} a; b_1, \ldots, b_n \\ c \end{array} \right)_{z_1, \ldots, z_n} = \int_0^1 \left( \prod_{i=1}^n (1-z_i u)^{-b_i} \right) u^{a-1} (1-u)^{c-a-1} du.$$ (3.2)

If $0 < \text{Re}(b_i)$ for all $i$ and $\text{Re}(\sum_i b_i) < \text{Re}(c)$, then

$$\left( \prod_{i=1}^n \Gamma(b_i) \right) \Gamma(c - \sum_{i=1}^n b_i) F_D^{(n)} \left( \begin{array}{c} a; b_1, \ldots, b_n \\ c \end{array} \right)_{z_1, \ldots, z_n} = \int_{\Delta} \left( 1 - \sum_{i=1}^n z_i u_i \right)^{-a} \prod_{i=1}^n u_i^{b_i-1} (1 - \sum_{i=1}^n u_i)^{c - \sum_{i=1}^n b_i - 1} du_1 \cdots du_n,$$ (3.3)

where $\Delta := \{(u_1, \ldots, u_n) \in \mathbb{R}^n \mid u_i \geq 0, \sum_i u_i \leq 1\}$. Their finite analogues are as follows.

**Theorem 3.1.**

(i) Suppose that $\alpha \neq \gamma$ and $\beta_i \neq \varepsilon$ for all $i$. Then, for $\lambda_1, \ldots, \lambda_n \in \kappa^n$,

$$-j(\alpha, \overline{\alpha} \gamma) F_D^{(n)} \left( \begin{array}{c} \alpha; \beta_1, \ldots, \beta_n \\ \gamma \end{array} \right)_{\lambda_1, \ldots, \lambda_n} = \sum_{u \in \kappa^n} \left( \prod_{i=1}^n \beta_i(1 - \lambda_i u) \right) \alpha(u) \overline{\alpha}(1-u).$$

(ii) Suppose that $\alpha \neq \varepsilon$ and $\beta_1 \cdots \beta_n \neq \gamma$. Then, for $\lambda_1, \ldots, \lambda_n \in \kappa^n$,

$$(-1)^n \left( \prod_{i=1}^n g(\beta_i) \right) g(\overline{\beta_1} \cdots \overline{\beta_n} \gamma) F_D^{(n)} \left( \begin{array}{c} \alpha; \beta_1, \ldots, \beta_n \\ \gamma \end{array} \right)_{\lambda_1, \ldots, \lambda_n} = \sum_{u_1, \ldots, u_n \in \kappa^n} \pi(1 - \sum_{i=1}^n \lambda_i u_i) \left( \prod_{i=1}^n \beta_i(u_i) \right) \overline{\beta_1} \cdots \overline{\beta_n} \gamma \left( 1 - \sum_{i=1}^n u_i \right).$$

**Proof.** (i) Put

$$f(\lambda_1, \ldots, \lambda_n) = \sum_{u \in \kappa^n} \left( \prod_{i=1}^n \beta_i(1 - \lambda_i u) \right) \alpha(u) \overline{\alpha}(1-u).$$
Letting $s_i = t_i u_i$ for all $i$ and using (2.2) (note that $\alpha \neq \gamma$ and $\beta_i \neq \varepsilon$) and (2.4), we have

$$f(\nu_1, \ldots, \nu_n) = \sum_{t_1, \ldots, t_n \in \mathbb{R}^n} \sum_{u \in \mathbb{R}^n} \alpha(u) \overline{\alpha}(1 - u) \prod_i \beta_i(1 - t_i u) \overline{\beta}(t_i)$$

$$= \sum_{u} \alpha u_1 \cdots u_n (1 - u) \prod_i \beta_i(1 - s_i) \overline{\beta}(s_i)$$

$$= (-1)^n j(\overline{\alpha}, \alpha u_1 \cdots u_n) \prod_i j(\overline{\beta}_i, \beta_i)$$

Thus, (i) follows by (3.1).

(ii) Put

$$g(\lambda_1, \ldots, \lambda_n) = \sum_{u_1, \ldots, u_n \in \mathbb{R}^n} \alpha \overline{\alpha} \left(1 - \sum_i \lambda_i u_i \right) \left(\prod_i \beta_i(u_i)\right) \overline{\beta}_1 \cdots \overline{\beta}_n \gamma \left(1 - \sum_i u_i \right)$$

Letting $s_i = t_i u_i$ for all $i$ and using (2.2) (note that $\alpha \neq \varepsilon$ and $\beta_1 \cdots \beta_n \neq \gamma$) and (2.4), we obtain

$$\hat{g}(\nu_1, \ldots, \nu_n)$$

$$= \sum_{t_1, \ldots, t_n \in \mathbb{R}^n} \sum_{u_1, \ldots, u_n \in \mathbb{R}^n} \alpha \overline{\alpha} \left(1 - \sum_i t_i u_i \right) \left(\prod_i \beta_i(u_i)\right) \overline{\beta}_1 \cdots \overline{\beta}_n \gamma \left(1 - \sum_i u_i \right) \prod_i \overline{\beta}(t_i)$$

$$= \sum_{u_1, \ldots, u_n} \left(\prod_i \beta_i(u_i)\right) \overline{\beta}_1 \cdots \overline{\beta}_n \gamma \left(1 - \sum_i u_i \right) \sum_i \overline{\alpha} \left(1 - \sum_i s_i \right) \prod_i \overline{\beta}(s_i)$$

$$= j(\overline{\beta}_1 \cdots \overline{\beta}_n \gamma, \beta_1 u_1, \ldots, \beta_n u_n) \cdot j(\alpha, \overline{\alpha}_1, \ldots, \overline{\alpha}_n)$$

$$= \frac{\left(\prod_i g(\beta_i)\right) g(\overline{\beta}_1 \cdots \overline{\beta}_n \gamma)}{g^\alpha(\gamma)} \prod_i (\beta_i)_{\nu_i} \left(\frac{\alpha}{\nu_1 \cdots \nu_n} \prod_i (\varepsilon)_{\nu_i}^\alpha \right)$$

Thus, (ii) follows by (3.1).

Let $d \in \mathbb{Z}_{\geq 1}$. Over $\mathbb{C}$, the Gauss hypergeometric functions have the integral representation (cf. [26, (1.6.6)])

$$B(a, c - a)_2 F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; z \right) = \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} \, dt.$$  

If we put $\zeta = \exp(2\pi \sqrt{-1}/d)$, by the change of variable $t = \tau^d$ in (3.4) and using (3.2), we obtain

$$\frac{d}{d \alpha} \left( \frac{\Gamma((d-1)a+c)}{\Gamma((d-1)a+c)} \right)_{2 F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; \zeta \right)}$$

This is a generalization of Karlsson’s formula proved for $d = 2, 3$ [12, (4.10) and (6.1)]. As an application of Theorem 3.1, we obtain a finite analogue of this formula.
Theorem 3.2. Suppose that $d | q - 1$, $\alpha \neq \gamma$ and $\beta \neq \varepsilon$. Let $\varphi_d \in \kappa^\times$ be a character of exact order $d$ and $\xi \in \kappa^\times$ be a primitive $d$th root of unity. Then, for any $\lambda \in \kappa^\times$,

$$F^{(2d-1)}_D \left( \frac{\alpha^d; \alpha^\gamma, \ldots, \alpha^\gamma \beta, \ldots, \beta}{\alpha^{d-1} \gamma}; \xi, \ldots, \xi^{d-1}, \lambda, \xi \lambda, \ldots, \xi^{d-1} \lambda \right)$$

$$= \sum_{i=0}^{d-1} \frac{g(\varphi_d^i \alpha)g^2(\alpha^{d-1} \gamma)}{g(\alpha^d)g^2(\varphi_d^i \gamma)} _2F_1 \left( \frac{\varphi_d^i \alpha, \beta}{\varphi_d^i \gamma}; \lambda^d \right).$$

Proof. For $\lambda = 0$, it is clear. Suppose that $\lambda \neq 0$. By Theorem 3.1 (i), we have

$$-j(\alpha^d, \alpha \gamma)F^{(2d-1)}_D \left( \frac{\alpha^d; \alpha^\gamma, \ldots, \alpha^\gamma \beta, \ldots, \beta}{\alpha^{d-1} \gamma}; \xi, \ldots, \xi^{d-1}, \lambda, \xi \lambda, \ldots, \xi^{d-1} \lambda \right)$$

$$= \sum_{t \in \kappa^\times} \alpha(t^d)\overline{\alpha} \gamma(1 - t^d)\overline{\beta}(1 - \lambda^d t^d)$$

$$= \sum_{i=0}^{d-1} \sum_{t \in \kappa^\times} \varphi_d^i \alpha(t)\overline{\alpha} \gamma(1 - t)\overline{\beta}(1 - \lambda^d t).$$

Here, note that

$$\sum_{i=0}^{d-1} \varphi_d^i(t) = \begin{cases} d & (\varphi_d(t) = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Thus, the theorem follows from Proposition 2.5 (ii). \qed

3.2. The cases of $F_A$ and $F_B$. In the complex case, Lauricella’s functions $F^{(n)}_A$ have the integral representation (cf. [16, Theorem 3.4.1])

$$\left( \prod_{i=1}^{n} B(b_i, c_i - b_i) \right) F^{(n)}_A \left( \frac{a; b_1, \ldots, b_n}{c_1, \ldots, c_n}; z_1, \ldots, z_n \right)$$

$$= \int_0^1 \cdots \int_0^1 \left( 1 - \sum_{i=1}^{n} z_i u_i \right)^{-a} \prod_{i=1}^{n} u_i^{b_i - 1}(1 - u_i)^{c_i - b_i - 1} du_1 \cdots du_n,$$

if $0 < \text{Re}(b_j) < \text{Re}(c_j)$ for all $j$.

Theorem 3.3. Suppose that $\alpha \neq \varepsilon$ and $\beta_i \neq \gamma_i$ for all $i$. Then, for $\lambda_i \in \kappa^\times$,

$$\left( \prod_{i=1}^{n} -j(\beta_i, \overline{\gamma_i}) \right) F^{(n)}_A \left( \frac{\alpha; \beta_1, \ldots, \beta_n}{\gamma_1, \ldots, \gamma_n}; \lambda_1, \ldots, \lambda_n \right)$$

$$= \sum_{\lambda_1, \ldots, \lambda_n \in \kappa^\times} \overline{\alpha} \left( 1 - \sum_{i=1}^{n} \lambda_i u_i \right) \prod_{i=1}^{n} \beta_i(u_i)\overline{\beta_i}(1 - u_i).$$

Proof. Write $f(\lambda_1, \ldots, \lambda_n)$ for the right-hand side of the theorem. Then, putting $s_i = t_i u_i$ and using (2.2) and (2.4), we have

$$\tilde{f}(\nu_1, \ldots, \nu_n) = \sum_{t_1, \ldots, t_n \in \kappa^\times} \sum_{u_1, \ldots, u_n \in \kappa^\times} \overline{\alpha} \left( 1 - \sum_{i=1}^{n} t_i u_i \right) \prod_{i=1}^{n} \beta_i(u_i)\overline{\beta_i}(1 - u_i)\overline{\nu_i}(t_i)$$

.
Thus, we obtain the theorem by (3.1). □

Lauricella’s $F^{(n)}$ have another integral representation ([14], see also [16, Theorem 3.4.1]) as

\[
\left(\prod_{i=1}^{\nu} g(\gamma_i)\right) F_A^{(n)}(a; b_1, \ldots, b_n; c_1, \ldots, c_n) = \int_{\Delta'} \left(\prod_{i=1}^{n} \left(1 - \frac{\lambda_i}{u_i}\right)^{-b_i}\right) \left(\prod_{i=1}^{n} u_i^{-c_i}\right) \Gamma(1 - a) du_1 \cdots du_n,
\]

where $\Delta'$ is a twisted cycle constructed in [14], if $c_1, \ldots, c_n, \sum c_i - a \notin \mathbb{Z}$.

**Theorem 3.4.** Suppose that $\gamma_1, \ldots, \gamma_n, \beta_i \notin \mathbb{Z}$ for all $i$. Then, for $\lambda_i \in \mathbb{K}$,

\[
(-1)^n \left(\prod_{i=1}^{\nu} g(\gamma_i)\right) F_A^{(n)}(\alpha; \beta_1, \ldots, \beta_n; \lambda_1, \ldots, \lambda_n) = \sum_{u_1, \ldots, u_n \in \mathbb{K}} \left(\prod_{i=1}^{n} \beta_i \left(1 - \frac{\lambda_i}{u_i}\right)\right) \left(\prod_{i=1}^{n} \gamma_i(u_i)\right) \alpha \gamma_1 \cdots \gamma_n \left(1 - \sum_{i=1}^{n} u_i\right).
\]

**Proof.** Write $f(\lambda_1, \ldots, \lambda_n)$ for the right-hand side of the theorem. Then, putting $s_i = t_i / u_i$ and similarly as the proof of Theorem 3.3, we have

\[
\hat{f}(\nu_1, \ldots, \nu_n) = \sum_{t_1, \ldots, t_n u_1, \ldots, u_n} \left(\prod_{i=1}^{n} \beta_i \left(1 - \frac{t_i}{u_i}\right)\right) \left(\prod_{i=1}^{n} \gamma_i(u_i)\right) \alpha \gamma_1 \cdots \gamma_n \left(1 - \sum_{i=1}^{n} u_i\right)
\]

\[
= \sum_{u_1, \ldots, u_n \in \mathbb{K}} \left(\prod_{i=1}^{n} \gamma_i(u_i)\right) \alpha \gamma_1 \cdots \gamma_n \left(1 - \sum_{i=1}^{n} u_i\right) \sum_{s_1, \ldots, s_n} \left(\prod_{i=1}^{n} \beta_i \left(1 - s_i\right)\right) \left(\prod_{i=1}^{n} \gamma_i(s_i)\right)
\]

\[
= j(\gamma_1, \ldots, \gamma_n, \alpha \gamma_1 \cdots \gamma_n) \prod_{i=1}^{n} j(\beta_i, \gamma_i)
\]

\[
= \left(\prod_{i=1}^{n} g(\gamma_i)\right) \frac{g^{c}(\alpha)}{g^{c}(\Gamma(1 - c))} \frac{\Gamma(c - \sum_{i=1}^{n} b_i)}{\Gamma(c)} F_B^{(n)}(a_1, \ldots, a_n; b_1, \ldots, b_n; z_1, \ldots, z_n)
\]

Thus, we obtain the theorem by (3.1). □

Lauricella’s functions $F^{(n)}_B$ have the integral representation (cf. [16, Theorem 3.4.1])

\[
\left(\prod_{i=1}^{\nu} \Gamma(b_i)\right) \frac{\Gamma(c - \sum b_i)}{\Gamma(c)} F_B^{(n)}(a_1, \ldots, a_n; b_1, \ldots, b_n; z_1, \ldots, z_n)
\]
where \( \Delta \) is as in (3.3), if \( 0 < \text{Re}(b_i) \) for all \( i \), and \( \text{Re}(\sum b_i) < \text{Re}(c) \).

**Theorem 3.5.** Suppose that \( \alpha_i \neq \varepsilon \) for all \( i \) and \( \beta_1 \cdots \beta_n \neq \gamma \). Then, for \( \lambda_i \in \mathbb{K}^\times \),

\[
(-1)^n \left( \prod_{i=1}^{n} g(\beta_i) \right) \frac{g(\beta_1 \cdots \beta_n \gamma)}{g^c(\gamma)} F_B^{(n)} \left( \alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \lambda_1, \ldots, \lambda_n \right)
\]

\[
= \sum_{u_1, \ldots, u_n \in \mathbb{K}^\times} \left( \prod_{i=1}^{n} \frac{1}{\alpha_i!} (1 - \lambda_i u_i) \right) \left( \prod_{i=1}^{n} \beta_i(u_i) \right) \beta_1 \cdots \beta_n \gamma \left( 1 - \sum_{i=1}^{n} u_i \right).
\]

**Proof.** Write \( f(\lambda_1, \ldots, \lambda_n) \) for the right-hand side of the theorem. Letting \( s_i = t_i u_i \) for all \( h \) and using (2.2) (note that \( \alpha_i \neq \varepsilon \) and \( \beta_1 \cdots \beta_n \neq \gamma \)) and (2.4), we have

\[
\hat{f}(\nu_1, \ldots, \nu_n)
\]

\[
= \sum_{t_1, \ldots, t_n \in \mathbb{K}^\times} \sum_{u_1, \ldots, u_n \in \mathbb{K}^\times} \left( \prod_{i=1}^{n} \beta_i(u_i) \right) \beta_1 \cdots \beta_n \gamma \left( 1 - \sum_{i=1}^{n} u_i \right) \prod_{i=1}^{n} \left( \sum_{t_i(u_i) \in \mathbb{K}^\times} \frac{1}{t_i} \right) \prod_{i=1}^{n} j(t_i, t_i)
\]

\[
= \frac{\left( \prod_{i=1}^{n} g(\beta_i) \right) \frac{g(\beta_1 \cdots \beta_n \gamma)}{g^c(\gamma)}}{\prod_{i=1}^{n} j(t_i, t_i)} \prod_{i=1}^{n} \frac{1}{\alpha_i!} (1 - \lambda_i u_i + \lambda_i u_i) \prod_{i=1}^{n} \frac{1}{\alpha_i!} (1 - \lambda_i u_i). \]

Thus, we obtain the theorem by (3.1). \( \square \)

**Remark 3.6.** Theorem 3.5 is equivalent to Theorem 3.3 via Remark 2.4.

3.3. **The case of \( F_C \).** In the complex case, Lauricella’s functions \( F_C^{(n)} \) have the integral representation (cf. [20, Remark 4.4])

\[
\left( \prod_{i=1}^{n} \frac{\Gamma(1 - c_i)}{\Gamma(1 - a)} \right) \frac{\Gamma(c_1 + \cdots + c_n + 1 - n - a)}{\Gamma(1 - a)} F_C^{(n)} \left( \frac{a; b}{c_1, \ldots, c_n; z_1, \ldots, z_n} \right)
\]

\[
= \int_{\Delta'} \left( 1 - \sum_{i=1}^{n} c_i \right) \left( 1 - \sum_{i=1}^{n} c_i - a \right) \sum_{i=1}^{n} c_i - a - n dt_1 \cdots dt_n,
\]

where \( \Delta' \) is as in (3.5), if \( c_1, \ldots, c_n, \sum_i c_i - a \notin \mathbb{Z} \).

**Theorem 3.7.** Suppose that \( \alpha_1 \cdots \alpha_n, \beta \neq \varepsilon \). Then, for \( \lambda_i \neq \mathbb{K}^\times \),

\[
(-1)^n \left( \prod_{i=1}^{n} g(\alpha_i) \right) \frac{g(\alpha_1 \cdots \alpha_n)}{g^c(\alpha)} F_C^{(n)} \left( \frac{\alpha; \beta}{\gamma_1, \ldots, \gamma_n; \lambda_1, \ldots, \lambda_n} \right)
\]

\[
= \sum_{u_1, \ldots, u_n \in \mathbb{K}^\times} \left( 1 - \sum_{i=1}^{n} u_i \right) \left( \prod_{i=1}^{n} \beta_i(u_i) \right) \beta_1 \cdots \beta_n \gamma \left( 1 - \sum_{i=1}^{n} u_i \right).
\]
Proof. Write $f(\lambda_1,\ldots,\lambda_n)$ for the right-hand side of the theorem. Letting $s_i = t_i/u_i$ and using (2.2) (note that $\nu\gamma_1\cdots\gamma_n, \beta \neq \epsilon$) and (2.4), we have

$$\begin{align*}
\hat{f}(\nu_1,\ldots,\nu_n) &= \sum_{t_1,\ldots,t_n,u_1,\ldots,u_n} \alpha\gamma_1\cdots\gamma_n \left(1 - \sum_i u_i\right) \beta \left(1 - \sum_i t_i\right) \prod_i \nu_i(u_i) \nu_i(t_i) \\
&= \sum_{u_1,\ldots,u_n} \left(\prod_i \nu_i(u_i)\right) \alpha\gamma_1\cdots\gamma_n \left(1 - \sum_i u_i\right) \sum_{s_1,\ldots,s_n} \left(\prod_i \nu_i(s_i)\right) \beta \left(1 - \sum_i s_i\right) \\
&= j(\gamma_1\nu_1,\ldots,\gamma_n\nu_n,\alpha\gamma_1\cdots\gamma_n) j(\nu_1,\ldots,\nu_n,\beta) \\
&= \frac{\prod_i g(\gamma_i) g(\alpha\gamma_1\cdots\gamma_n)}{g^\nu(\alpha\gamma_1\cdots\gamma_n)} \cdot \frac{(\alpha)_{\nu_1}\cdots\nu_n}{\prod_i (\gamma_i)^{\nu_i}} \cdot \frac{(\beta)_{\nu_1}\cdots\nu_n}{\prod_i (\epsilon)^{\nu_i}}.
\end{align*}$$

Thus, we obtain the theorem by (3.1). □

In the complex case, Burchnall-Chaundy [3] proved the expansion formula

$$\begin{align*}
F_4(a; b; c_1, c_2; x(1 - y), y(1 - x)) &= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(1 + a + b - c_1 - c_2)r}{(1)_r(c_1)_r(c_2)_r} x^r y^r \\
&\quad \times 2F_1 \left(\frac{a + r, b + r}{c_1 + r}; x\right) 2F_1 \left(\frac{a + r, b + r}{c_2 + r}; y\right),
\end{align*}$$

(3.6)

(an alternative proof was given by Bailey [2]). From this they deduced, by using (3.4) and $1F_0 \left(\alpha; z\right) = (1 - z)^{-\alpha}$, the integral representation

$$\begin{align*}
B(a, c_1 - a)B(b, c_2 - b)F_4(a; b; c_1, c_2; x(1 - y), y(1 - x)) &= \int_0^1 \int_0^1 u^{a-1} v^{b-1} (1 - u)^{c_1-a-1} (1 - v)^{c_2-b-1} \\
&\quad \times (1 - xu)^{a-c_1-c_2+1} (1 - yv)^{b-c_1-c_2+1} (1 - xu - yv)^{c_1+c_2-a-b-1} dudv,
\end{align*}$$

provided that $0 < \text{Re}(a) < \text{Re}(c_1), 0 < \text{Re}(b) < \text{Re}(c_2)$ and $|x|, |y|$ are small enough to make the double integral convergent. We prove finite analogues of these formulas.

The following lemmas will be used in the proof of Proposition 3.10, from which we will deduce finite analogues of (3.6) and (3.7) (Theorem 3.11 and Theorem 3.12, respectively).

Lemma 3.8 ([27, Theorem 1.1]). For any $x, y \neq 1$,

$$\begin{align*}
\bar{\alpha}(1 - x)\bar{\beta}(1 - y)F_4 \left(\alpha; \beta; \gamma_1, \gamma_2; \frac{-x}{(1 - x)(1 - y)}, \frac{-y}{(1 - x)(1 - y)}\right) &= \frac{1}{(1 - q)^2} \sum_{\mu, \nu} \frac{(\alpha)_{\mu, \beta(\nu)}}{(\bar{\nu})_{\mu, \nu}} (\mu)_{\gamma_1, \nu} (\nu)_{\gamma_1, \mu} 2F_1 \left(\frac{\beta(\nu, \mu)}{\gamma_1}; 1\right) 2F_1 \left(\frac{\gamma_2}{\gamma_1}; 1\right) \mu(x) \nu(y).
\end{align*}$$

Lemma 3.9. Suppose that $\alpha, \beta \notin \{\epsilon, \gamma_1, \gamma_2\}$ and $\alpha\bar{\beta}\gamma_1\gamma_2 \neq \epsilon$. For any $\mu, \nu \in \mathcal{K}^\infty$,

$$2F_1 \left(\frac{\beta(\nu, \mu)}{\gamma_1}; 1\right) 2F_1 \left(\frac{\alpha(\mu, \nu)}{\gamma_2}; 1\right).$$
\[
\begin{align*}
&= \frac{\left(\beta\gamma_1\right)_{\nu} (\gamma_2)_{\nu}}{(\gamma_1)^{\nu}_{\nu} (\gamma_2)^{\nu}_{\nu}} \, \binom{\alpha \beta \gamma_1 \gamma_2, \bar{\mu}, \bar{\nu}}{\beta \gamma_1 \mu, \alpha \gamma_2 \nu} \binom{(\varepsilon)^{\nu}_{\nu}}{(\gamma_1)^{\nu}_{\nu} (\gamma_2)^{\nu}_{\nu}}
&\quad - \frac{(1-q)^2}{q} \left( \delta(\gamma_1 \mu) \delta(\beta \nu) C_1 + \delta(\alpha \mu) \delta(\gamma_2 \nu) C_2 \right),
\end{align*}
\]
where
\[
C_1 := q^{\delta(\gamma_1)} \frac{g(\alpha \beta \gamma_1 \gamma_2) g^{\nu}(\gamma_2)}{g^\nu(\gamma_1) g(\beta \gamma_2)}, \quad C_2 := q^{\delta(\gamma_2)} \frac{g(\alpha \beta \gamma_1 \gamma_2) g^\nu(\gamma_1)}{g(\beta \gamma_1 \gamma_2) g(\gamma_1)}.
\]

**Proof.** Put
\[
L(\mu, \nu) = \binom{\beta \nu}{\gamma_1} 2 F_1 \left( \binom{\alpha \mu, \bar{\nu}}{\gamma_2} ; 1 \right) 2 F_1 \left( \binom{\alpha \mu, \bar{\nu}}{\gamma_2} ; 1 \right),
\]
and
\[
M(\mu, \nu) = q^{-\delta(\gamma_1) - \delta(\alpha \mu)} \frac{(\beta \gamma_1 \gamma_2)_{\nu}}{(\gamma_1)^{\nu}_{\nu} (\gamma_2)^{\nu}_{\nu}}.
\]

First, if \(\{\beta \nu, \bar{\nu}\} \neq \{\varepsilon, \gamma_1\}\) and \(\{\alpha \mu, \bar{\nu}\} \neq \{\varepsilon, \gamma_2\}\), then by Proposition 2.9 (i), we have
\[
L(\mu, \nu) = M(\mu, \nu).
\]
Using Proposition 2.10 (note that \(\{\alpha \beta \gamma_1 \gamma_2, \mu, \bar{\nu}\} \neq \{\varepsilon, \beta \gamma_1 \mu, \alpha \gamma_2 \nu\}\)), we have
\[
M(\mu, \nu) = q^{\delta(\gamma_1) \mu} \frac{(\beta \gamma_1 \gamma_2)_{\nu}}{(\gamma_1)^{\nu}_{\nu} (\gamma_2)^{\nu}_{\nu}} \binom{\alpha \beta \gamma_1 \gamma_2, \bar{\mu}, \bar{\nu}}{\beta \gamma_1 \mu, \alpha \gamma_2 \nu} - \frac{q^{\delta(\gamma_1) \mu} g^\nu(\alpha \gamma_2 \nu)}{g(\beta \gamma_1 \gamma_2) g(\gamma_1) g(\bar{\nu})}
\]
where
\[
N(\mu, \nu) := q^{\delta(\gamma_1) \mu} \frac{g(\alpha \beta \gamma_1 \gamma_2) g^{\nu}(\gamma_2)}{g^\nu(\gamma_1) g(\beta \gamma_2)}.
\]

Therefore, we obtain the formula of the lemma.

Next, if \(\{\beta \nu, \bar{\nu}\} = \{\varepsilon, \gamma_1\}\) (then \(\{\alpha \mu, \bar{\nu}\} \neq \{\varepsilon, \gamma_2\}\)), then by Proposition 2.9 (ii),
\[
L(\mu, \nu) = M(\mu, \nu) - \frac{(1-q)^2}{q} (1 + q)^{\delta(\gamma_1)} \frac{q^{-\delta(\gamma_2) \mu}}{q(\gamma_1)^{\nu}_{\nu} (\gamma_2)^{\nu}_{\nu}}.
\]

By Proposition 2.10, if \(\bar{\nu} = \varepsilon\) and \(\beta \nu = \gamma_1\), then
\[
M(\mu, \nu) = N(\mu, \nu) + \frac{(1-q)^2}{q} \frac{(\bar{\gamma}_2)^{\nu}_{\nu}}{(\gamma_1)^{\nu}_{\nu}},
\]
and if \(\bar{\nu} = \gamma_1\) \(\neq \varepsilon\) and \(\beta \nu = \varepsilon\) (then \(\{\alpha \beta \gamma_1 \gamma_2, \mu, \bar{\nu}\} \neq \{\varepsilon, \beta \gamma_1 \mu, \alpha \gamma_2 \nu\}\)), then
\[
M(\mu, \nu) = N(\mu, \nu).
\]

Consequently, by (3.8), we have
\[
L(\mu, \nu) = N(\mu, \nu) - \delta(\gamma_1 \mu) \delta(\beta \nu) \frac{(1-q)^2}{q} C_1.
\]
Similarly, if \(\{\alpha \mu, \bar{\nu}\} = \{\varepsilon, \gamma_2\}\), then we have
\[
L(\mu, \nu) = N(\mu, \nu) - \delta(\alpha \mu) \delta(\gamma_2 \nu) \frac{(1-q)^2}{q} C_2.
\]

Thus, we obtain the lemma. \(\square\)
For brevity, put \( J := j(\alpha, \overline{\alpha}\gamma_1)j(\beta, \overline{\beta}\gamma_2) \).

**Proposition 3.10.** Suppose that \( \alpha, \beta \notin \{\varepsilon, \gamma_1, \gamma_2\} \) and \( \alpha\beta\gamma_1\gamma_2 \neq \varepsilon \). Then, for any \( x, y \in \kappa^\times \setminus \{1\} \),

\[
J \cdot F_4(\alpha; \beta; \gamma_1, \gamma_2; x(1-y), y(1-x))
\]

\[= \overline{\alpha}(1-x)\overline{\beta}(1-y) \frac{J}{1-q} \sum_{\eta \in \kappa^\times} \frac{(\alpha)_\eta (\beta)_\eta (\alpha\beta\gamma_1\gamma_2)_\eta}{(\varepsilon)_\eta \gamma_1^\eta \gamma_2^\eta \eta} \left( \frac{xy}{(x-1)(y-1)} \right) \times 2F_1 \left( \frac{\alpha, \beta \gamma_1}{\gamma_1 \eta}; \frac{x}{x-1} \right) 2F_1 \left( \frac{\beta, \overline{\beta} \gamma_2^\eta}{\gamma_2 \eta}; \frac{y}{y-1} \right) - S_0(x, y) - S_1(x, y) - S_2(x, y), \]

where

\[
S_0(x, y) := \alpha\beta(-1)j(\alpha\gamma_2, \beta\gamma_1)(x)\gamma_2(y),
\]

\[
S_1(x, y) := j(\alpha\gamma_1\gamma_2, \beta)(x)\alpha\gamma_1(x-1)\beta(y),
\]

\[
S_2(x, y) := j(\alpha\gamma_1\gamma_2, \beta)(y)\beta\gamma_2(x-1)\alpha(y).
\]

**Proof.** By Lemma 3.8 (replace \( x, y \) with \( x/(x-1), y/(y-1) \) respectively),

\[
J \cdot \alpha(1-x)\beta(1-y)F_4(\alpha; \beta; \gamma_1, \gamma_2; x(1-y), y(1-x))
\]

\[= \frac{J}{(1-q)^2} \sum_{\mu, \nu, \eta} \frac{(\alpha)_\mu (\beta)_\nu}{(\varepsilon)_\mu \mu (\varepsilon)_\nu \nu} \left( \frac{\mu, \nu, \eta}{\gamma_1 \gamma_2} ; 1 \right) 2F_1 \left( \frac{\alpha, \beta \gamma_1}{\gamma_1 \eta}; \frac{x}{x-1} \right) 2F_1 \left( \frac{\beta, \overline{\beta} \gamma_2^\eta}{\gamma_2 \eta}; \frac{y}{y-1} \right). \]

Thus, by Lemma 3.9, we obtain that

\[ (3.9) \quad J \cdot \alpha(1-x)\beta(1-y)F_4(\alpha; \beta; \gamma_1, \gamma_2; x(1-y), y(1-x)) = \Phi(x, y) - \alpha(1-x)\beta(1-y)(S_0(x, y) + S_1(x, y) + S_2(x, y)), \]

where

\[
\Phi(x, y) := \frac{J}{(1-q)^2} \sum_{\mu, \nu, \eta} \frac{(\alpha)_\mu (\beta)_\nu}{(\varepsilon)_\mu \mu (\varepsilon)_\nu \nu} \left( \frac{\mu, \nu, \eta}{\gamma_1 \gamma_2} ; 1 \right) 2F_1 \left( \frac{\alpha, \beta \gamma_1}{\gamma_1 \eta}; \frac{x}{x-1} \right) 2F_1 \left( \frac{\beta, \overline{\beta} \gamma_2^\eta}{\gamma_2 \eta}; \frac{y}{y-1} \right). \]

In (3.9), note that, by replacing \( \mu, \nu \) with \( \gamma_1 \mu, \gamma_2 \nu \) respectively and using (2.3) and Proposition 2.5 (i),

\[
\frac{J}{(1-q)^2} j(\alpha\gamma_1, \beta\gamma_2) \sum_{\mu, \nu, \eta} \frac{(\alpha)_\mu (\beta)_\nu}{(\gamma_1)_\mu (\gamma_2)_\nu} \left( \frac{x}{x-1} \right) 2F_1 \left( \frac{\alpha, \beta \gamma_1}{\gamma_1 \eta}; \frac{x}{x-1} \right) 2F_1 \left( \frac{\beta, \overline{\beta} \gamma_2^\eta}{\gamma_2 \eta}; \frac{y}{y-1} \right)
\]

\[= \alpha\beta(-1)j(\alpha\gamma_2, \beta\gamma_1)(x)\gamma_2(y), \]

\[= \alpha(1-x)\beta(1-y)S_0(x, y). \]

By (2.1), for any \( \varphi, \chi \in \kappa^\times \),

\[ (3.10) \quad \frac{\chi_{\phi}}{(\chi^\phi)^{\chi}} = \varphi(-1). \]
Replacing $\mu, \nu$ with $\mu \eta, \nu \eta$ respectively, and using (2.3) and (3.10), we have

\[
(1 - q)^{\frac{3}{J}} \Phi(x, y) = \sum_{\eta, \mu, \nu} \frac{(\alpha)_n(\beta)_n(\alpha \beta \gamma_2)_\eta(\alpha \gamma_2)_\mu(\beta \gamma_2)_\nu}{(\varepsilon \eta)(\gamma_1 \eta)(\gamma_2 \eta)_\eta(\varepsilon \nu)(\gamma_1 \eta)(\gamma_2 \eta)_\nu} \mu \eta \left(\frac{x}{x - 1}\right) \nu \eta \left(\frac{y}{y - 1}\right),
\]
and hence we have

\[
\Phi(x, y) = \frac{J}{1 - q} \sum_{\eta} \frac{(\alpha)_n(\beta)_n(\alpha \beta \gamma_2)_\eta}{(\varepsilon \eta)(\gamma_1 \eta)(\gamma_2 \eta)_\eta} \eta \left(\frac{x y}{(x - 1)(y - 1)}\right) \times \, 2F1 \left(\frac{\alpha \eta, \beta \gamma_1}{\gamma_1 \eta}; \frac{x}{x - 1}; \right) 2F1 \left(\frac{\beta \eta, \gamma_2}{\gamma_2 \eta}; \frac{y}{y - 1}; \right).
\]

Thus, the proposition follows from (3.9). \qed

By Proposition 3.10, we obtain a finite analogue of the Burchnall-Chaundy expansion (3.6), under the assumption $\alpha \beta \gamma_2 \neq \varepsilon$, as follows.

**Theorem 3.11.** Suppose that $\alpha, \beta \notin \{\varepsilon, \gamma_1, \gamma_2\}$ and $\alpha \beta \gamma_2 \neq \varepsilon$. Then, for any $x, y \in \kappa \setminus \{1\}$, we have

\[
J \cdot F_2(\alpha; \beta; \gamma_1, \gamma_2; x(1 - y), y(1 - x)) = \frac{J}{1 - q} \sum_{\eta \in \kappa} \frac{(\alpha)_n(\beta)_n(\alpha \beta \gamma_2)_\eta}{(\varepsilon \eta)(\gamma_1 \eta)(\gamma_2 \eta)_\eta} \eta \left(\frac{x y}{(x - 1)(y - 1)}\right) \times \, 2F1 \left(\frac{\alpha \eta, \beta \gamma_1}{\gamma_1 \eta}; \frac{x}{x - 1}; \right) 2F1 \left(\frac{\beta \eta, \gamma_2}{\gamma_2 \eta}; \frac{y}{y - 1}; \right).
\]

Here, $J$ and $S_0$ are as in Proposition 3.10 and

\[
R_1(x, y) := j(\alpha \beta \gamma_1, \beta j(\alpha \gamma_2, \beta \gamma_2) x(1 - y) \gamma_1(1 - y) \gamma_2, y),
\]

\[
R_2(x, y) := j(\alpha \beta \gamma_1, \beta j(\alpha \gamma_2, \beta \gamma_2) x(1 - y) \gamma_1(\gamma_2 - 1) \gamma_2, y).
\]

**Proof.** Using Proposition 2.8 with $\lambda = x/(x - 1)$ and $\lambda = y/(y - 1)$, we have

\[
2F1 \left(\frac{\alpha \eta, \beta \gamma_1}{\gamma_1 \eta}; \right) 2F1 \left(\frac{\beta \eta, \gamma_2}{\gamma_2 \eta}; \right)
\]

\[
= \left(\alpha \eta(1 - x) \gamma_1(\gamma_1 \eta) + \delta(\beta \eta) \alpha \beta(1 - q)(1 - q) \frac{g(\beta \gamma_1)}{g(\alpha \beta)(\gamma_1 \eta)} \beta \gamma_1 \left(\frac{x}{x - 1}\right) \right)
\]

\[
\times \left(\beta \eta(1 - y) \gamma_2(\gamma_2 \eta) + \delta(\alpha \eta) \alpha \beta(1 - q)(1 - q) \frac{g(\gamma_2 \eta)}{g(\alpha \beta)(\gamma_2 \eta)} \alpha \gamma_2 \left(\frac{y}{y - 1}\right) \right)
\]

\[
\alpha \eta(1 - x) \beta \eta(1 - y) \gamma_1(\gamma_1 \eta) + \delta(\beta \eta) \alpha \beta(1 - q)(1 - q) \frac{g(\beta \gamma_1)}{g(\alpha \beta)(\gamma_1 \eta)} \beta \gamma_1 \left(\frac{x}{x - 1}\right)
\]

\[
\alpha \beta(1 - q)(1 - q) \frac{g(\beta \gamma_2)}{g(\alpha \beta)(\gamma_2 \eta)} \alpha \gamma_2 \left(\frac{y}{y - 1}\right).
\]

Thus, we obtain the theorem by Proposition 3.10, Lemma 2.7, (2.1) and (2.2). \qed
A finite analogue of (3.7) is the following.

**Theorem 3.12.** Suppose that $$\alpha, \beta \not\in \{\varepsilon, \gamma_1, \gamma_2\}$$. Then, for any $$x, y \in \kappa^\times \setminus \{1\}$$,

$$J \cdot F_4(\alpha; \beta; \gamma_1, \gamma_2; x(1 - y), y(1 - x))$$

$$= \sum_{u, v} \alpha(u) \beta(v) \pi(1 - u) \gamma(1 - v)$$

$$\times \frac{\alpha \gamma_1 \gamma_2 (1 - xu) \beta \gamma_2 (1 - yv) \alpha \beta \gamma_1 \gamma_2 (1 - xu - yv)}{S_0(x, y) - S_1(x, y) - S_2(x, y)}.$$

Here, $$J$$ and $$S_i(x, y) (i = 0, 1, 2)$$ are as in Proposition 3.10.

**Proof.** First, suppose that $$\alpha \beta \gamma_1 \gamma_2 = \varepsilon$$. Then, we have a result of Tripathi-Barman [28, Theorem 3.1] (see also [24, Theorem 4.1])

$$J \cdot F_4(\alpha; \beta; \gamma_1, \gamma_2; x(1 - y), y(1 - x))$$

$$= J \cdot 2F_1 \left( \frac{\alpha, \beta}{\gamma_1} ; x \right) 2F_1 \left( \frac{\alpha, \beta}{\gamma_2} ; y \right) - \delta(-x - y) q S_0(x, y),$$

where $$\delta(u) = 0$$ for $$u \in \kappa^\times$$ and $$\delta(0) = 1$$. On the other hand, by using Proposition 2.5 (ii) and letting $$t = ux$$, the first term of the right-hand side of the theorem is

$$\sum_{u, v} \alpha(u) \beta(v) \pi(1 - u) \gamma(1 - v)$$

$$= \sum_u \alpha(u) \pi(1 - u) \gamma(1 - v)$$

$$= J \cdot 2F_1 \left( \frac{\alpha, \beta}{\gamma_1} ; x \right) 2F_1 \left( \frac{\alpha, \beta}{\gamma_2} ; y \right) - \sum_{t \in \kappa} \gamma(1) \left( \frac{x - t}{y - 1 + t} \right).$$

If $$x + y \neq 1$$, then $$(x - t)/(y - 1 + t)$$ runs through $$\kappa \setminus \{-1, x/(y - 1), (x - 1)/y\}$$, and hence we have

$$\sum_{t \in \kappa} \gamma(1) \left( \frac{x - t}{y - 1 + t} \right) = -S_0(x, y) - S_1(x, y) - S_2(x, y).$$

On the other hand, if $$x + y = 1$$, then $$S_0(x, y) = S_1(x, y) = S_2(x, y)$$ and

$$\sum_{t \in \kappa} \gamma(1) \left( \frac{x - t}{y - 1 + t} \right) = (q - 3) S_0(x, y).$$

Therefore, the right-hand side of the theorem is equal to the right-hand side of (3.11), and hence we obtain the theorem.

Secondly, suppose that $$\alpha \beta \gamma_1 \gamma_2 \neq \varepsilon$$, and put

$$\Phi(x, y) = J \cdot \frac{1}{1 - q} \sum_{\eta} \frac{(\alpha)_\eta (\beta)_\eta \alpha \beta \gamma_1 \gamma_2 \eta}{(\gamma_1)_\eta (\gamma_2)_\eta (\varepsilon)_\eta \eta} y \left( \frac{xy}{(x - 1)(y - 1)} \right)$$

$$\times 2F_1 \left( \frac{\alpha \eta, \beta \gamma_1, \gamma \gamma_2}{\gamma_1 \eta} ; x - 1 \right) 2F_1 \left( \frac{\beta \eta, \alpha \gamma_2}{\gamma_2 \eta} ; y - 1 \right).$$
Then, by Proposition 3.10, the left-hand side of the theorem is equal to
\[ \frac{1}{x} \frac{1}{y} \Phi(x, y) - S_0(x, y) - S_1(x, y) - S_2(x, y). \]
By Proposition 2.5 (ii) and letting \( u = s/(sx + 1) \) and \( v = t/(ty + 1) \),
\[
J \left( \frac{\alpha}{(\gamma_1)^2} \right) \left( \frac{\beta}{(\gamma_2)^2} \right) \begin{align*}
&= \sum_{s,t} \alpha(s) \beta(t) \gamma_1(1 - \frac{x}{y}) \beta(t) \gamma_2(1 - t) \gamma_1(1 - y) \\
&= \alpha(1-x) \beta(1-y) \sum_{u,v} \alpha(u) \beta(v) \gamma_1(1-u) \beta(1-v) \gamma_2(1-xu) \gamma(1-yv) \\
&\quad \times \eta(u)(y-1)uv \\
&\times \gamma(u)(1-xu)(1-yv).
\end{align*}
\]
Thus, by Proposition 2.5 (i),
\[
\frac{1}{x} \frac{1}{y} \Phi(x, y) = \sum_{u,v} \alpha(u) \beta(v) \gamma_1(1-u) \beta(1-v) \gamma_2(1-xu) \gamma(1-yv) \\
\times \eta(u)(y-1)uv \\
= \sum_{u,v} \alpha(u) \beta(v) \gamma_1(1-u) \beta(1-v) \gamma_2(1-xu) \gamma(1-yv) \\
\times \eta(u)(y-1)uv \\
= \sum_{u,v} \alpha(u) \beta(v) \gamma_1(1-u) \beta(1-v) \gamma_2(1-xu) \gamma(1-yv) \\
\times \eta(u)(y-1)uv \\
= \sum_{u,v} \alpha(u) \beta(v) \gamma_1(1-u) \beta(1-v) \gamma_2(1-xu) \gamma(1-yv).
\]
Therefore, we obtain the theorem. \( \square \)

4. The number of rational points on some algebraic varieties.

4.1. Rational points and Artin L-functions. In this subsection, we recall the definitions of zeta functions and Artin L-functions of a variety and their properties. For more details, see [25] and [30].

Fix an algebraic closure \( \overline{\kappa} \) of \( \kappa \) and let \( \kappa_r \subset \overline{\kappa} \) be the degree \( r \) extension of \( \kappa \).

Let \( V \) be a variety over \( \kappa \) and put \( N_r(V) = \#V(\kappa_r) \). Then, the zeta function of \( V \) is defined by
\[ Z(V, t) = \exp \left( \sum_{r=1}^{\infty} \frac{N_r(V)}{r} t^r \right) \in \mathbb{Q}[t]. \]

Let \( G \) be a finite abelian group and suppose that \( G \) acts on \( V \) over \( \kappa \). Let \( F \) be the \( q \)-Frobenius acting on \( V(\kappa) \). For \( \chi \in \widehat{G} := \text{Hom}(G, \mathbb{Q}^\times) \) and \( r \in \mathbb{Z}_{\geq 1} \), put
\[ N_r(V; \chi) := \frac{1}{\#G} \sum_{g \in G} \chi(g) \# \{ x \in V(\kappa) \mid F^r(x) = g(x) \} \in \overline{\mathbb{Q}}. \]
The Artin L-function of \( V \) associated to \( \chi \) is defined by
\[ L(V, \chi; t) = \exp \left( \sum_{r=1}^{\infty} \frac{N_r(V; \chi)}{r} t^r \right) \in \overline{\mathbb{Q}}[t]. \]
Since \( N_r(V) = \sum_{\chi \in \hat{G}} N_r(V; \chi) \), we have \( Z(V, t) = \prod_{\chi \in \hat{G}} L(V, \chi; t) \).
Remark 4.1. Let $D_{\lambda}$ be a diagonal hypersurface in $\mathbb{P}^{n-1}$ defined by the equation

$$X_1^d + \cdots + X_n^d = d\lambda X_1^{h_1} \cdots X_n^{h_n},$$

where $\lambda \in \kappa^\times$, $h_i \in \mathbb{Z}_{\geq 1}$, and $\sum_i h_i = d$. A subquotient $G$ of $(\mu_d)^n$ acts on $D_{\lambda}$. The author [22] expresses $N_r(D_{\lambda}; \chi)$ ($\chi \in \hat{G}$) in terms of one-variable hypergeometric functions $dF_{d-1}(\lambda^{d})$ over $\kappa_r$.

4.2. Algebraic varieties related to $F_D$. In this subsection, let $d, a, b_1, \ldots, b_n, c$ be positive integers and let $\lambda_1, \ldots, \lambda_n \in \kappa^\times$. Write $\lambda = (\lambda_1, \ldots, \lambda_n)$. We consider an affine curve $C_{D, \lambda}$ over $\kappa$ defined by the equation

$$(4.1) \quad y^d = \left( \prod_{i=1}^{n} (1 - \lambda_i x)^{b_i} \right) x^a (1 - x)^c.$$ 

Without loss of generality, we assume that $\lambda_1, \ldots, \lambda_n$ are not 1 and distinct. Suppose that $d \not| q - 1$ and let $\mu_d \subset \kappa^\times$ be the subgroup consisting of all the $d$th roots of unity. Then, $\mu_d$ acts on $C_{D, \lambda}$ by $(x, y) \mapsto (x, \xi y)$ ($\xi \in \mu_d$). Fix a generator $\varphi$ of $\kappa^\times$, and put $\varphi_d = \varphi^{(q-1)/d} \in \kappa^\times$ and $\chi = \varphi|_{\mu_d} \in \hat{\mu}_d$. Note that $\hat{\mu}_d = \{ \chi^m \mid m \in \mathbb{Z}/d\mathbb{Z} \}$.

Theorem 4.2. Suppose that $\gcd(d, c) = \gcd(d, b_i) = 1$ for all $i$. Then,

$$N_1(C_{D, \lambda}; \chi^m) = \begin{cases} q & (m = 0), \\ -j(\varphi_d^{ma}, \varphi_d^{mb_1, \ldots, \varphi_d^{mb_n}}) F_D^{(n)} \left( \varphi_d^{ma}, \varphi_d^{mb_1, \ldots, \varphi_d^{mb_n}} ; \lambda_1, \ldots, \lambda_n \right) & (m \neq 0). \end{cases}$$

Proof. Put $f(x) = \left( \prod_{i=1}^{n} (1 - \lambda_i x)^{b_i} \right) x^a (1 - x)^c$. Then, for each $m \in \mathbb{Z}/d\mathbb{Z},$

$$N_1(C_{D, \lambda}; \chi^m) = \frac{1}{d} \sum_{\xi \in \mu_d} \chi^m(\xi) \# \{ (x, y) \in C_{D, \lambda}(\kappa) \mid F(x, y) = (x, \xi y) \} = \frac{1}{d} \sum_{\xi \in \mu_d} \chi^m(\xi) \# \{ x \in \kappa \mid f(x)^{(q-1)/d} = \xi \} + \left( \frac{1}{d} \sum_{\xi \in \mu_d} \chi^m(\xi) \right) \# \{ x \in \kappa \mid f(x) = 0 \}.$$

Therefore, if $m = 0$, then

$$N_1(C_{D, \lambda}; \chi^m) = \sum_{\xi \in \mu_d} \# \{ x \in \kappa \mid f(x)^{(q-1)/d} = \xi \} + \# \{ x \in \kappa \mid f(x) = 0 \} = \# \kappa = q.$$

If $m \neq 0$, then since $\sum_{\xi \in \mu_d} \chi^m(\xi) = 0$, we have

$$N_1(C_{D, \lambda}; \chi^m) = \sum_{\xi \in \mu_d} \chi^m(\xi) \# \{ x \in \kappa \mid f(x)^{(q-1)/d} = \xi \} = \sum_{x \in \kappa} \varphi_d^m(f(x)).$$
Thus, noting that \( \phi_d^{mb_i} \neq \varepsilon \) for all \( i \) by the assumption, the theorem follows from Theorem 3.1 (i).

\[ \square \]

Remark 4.3. Frechette-Swisher-Tu [5, Theorem 5.3] expresses \( N_1(C_{D,\lambda}) \) in terms of a sum of their Appell-Lauricella functions \( F_D^{(n)} \) over \( \kappa \). Since our \( F_D^{(n)} \) coincides with their \( F_D^{(n)} \) under the assumption of Theorem 4.2 by Theorem 3.1 (i), Theorem 4.2 is a refinement of their result.

Next, we consider an affine hypersurface \( S_{D,\lambda} \) of dimension \( n \) over \( \kappa \) defined by the equation

\[ y^d = \left( 1 - \sum_{i=1}^{n} \lambda_i x_i \right)^a \left( \prod_{i=1}^{n} x_i^{b_i} \right) \left( 1 - \sum_{i=1}^{n} x_i \right)^c. \]

The group \( \mu_d \) acts on \( S_{D,\lambda} \) similarly as \( C_{D,\lambda} \).

**Theorem 4.4.** Suppose that \( \gcd(d,a) = \gcd(d,c) = 1 \). Then,

\[ N_1(S_{D,\lambda}; \chi^m) = \begin{cases} q^n & (m = 0), \\ (-1)^n J_D : F_D^{(n)} \left( \phi_d^{m_1} ; \phi_d^{mb_1}, \ldots, \phi_d^{mb_n} ; \phi_d^{m(b_1+\ldots+b_n+c)} ; \lambda_1, \ldots, \lambda_n \right) & (m \neq 0), \end{cases} \]

where \( J_D := j(\phi_d^{mb_1}, \ldots, \phi_d^{mb_n}, \phi_d^{mc}) \).

**Proof.** Similarly as the proof of Theorem 4.2, we have

\[ N_1(S_{D,\lambda}; \chi^0) = \# \kappa^n = q^n, \]

and if \( m \neq 0 \), then

\[ N_1(S_{D,\lambda}; \chi^m) = \sum_{x_1, \ldots, x_n \in \kappa^n} \phi_d^{ma} \left( 1 - \sum_{i=1}^{n} \lambda_i x_i \right) \left( \prod_{i=1}^{n} \phi_d^{mb_i}(x_i) \right) \phi_d^{mc} \left( 1 - \sum_{i=1}^{n} x_i \right). \]

Thus, the theorem follows from Theorem 3.1 (ii). \( \square \)

Fix an integer \( r \geq 1 \). Write \( \phi_{d,r} = \phi_d \circ N_{\kappa_r/\kappa} \in \widehat{\kappa_r} \) where \( N_{\kappa_r/\kappa} \) is the norm map.

**Corollary 4.5.** Put hypergeometric functions over \( \kappa \) as

\[ f_r(\lambda) = F_D^{(n)} \left( \phi_d^{ma}, \phi_d^{m(b_1+\ldots+b_n)} ; \phi_d^{mb_1}, \ldots, \phi_d^{mb_n} ; \phi_{d,r}^{m(c+a)} ; \lambda_1, \ldots, \lambda_n \right), \]

\[ g_r(\lambda) = F_D^{(n)} \left( \phi_d^{ma}, \phi_d^{m(b_1+\ldots+b_n)} ; \phi_d^{mb_1}, \ldots, \phi_d^{mb_n} ; \phi_{d,r}^{m(b_1+\ldots+b_n+c)} ; \lambda_1, \ldots, \lambda_n \right). \]

(i) Suppose that \( \gcd(d,c) = \gcd(d,b_i) = 1 \) for all \( i \). Then,

\[ L(C_{D,\lambda}, \chi^m; t) = \begin{cases} \frac{1}{1 - qt} & (m = 0), \\ \exp \left( \sum_{r=1}^{\infty} j(\phi_d^{ma}, \phi_d^{mc}) f_r(\lambda) t^r \right)^{-1} & (m \neq 0). \end{cases} \]
(ii) Suppose that \( \gcd(d, a) = \gcd(d, c) = 1 \). Then,
\[
L(S_{D, \lambda; \chi^m}; t) = \begin{cases} 
\frac{1}{1 - q^n t} & (m = 0), \\
\exp \left( \sum_{r=1}^{\infty} J_D \cdot g_r(\lambda) \left( \frac{t}{\lambda} \right)^{-1} \right) & (m \neq 0),
\end{cases}
\]
where \( J_D \) is as in Theorem 4.4.

**Proof.** For each \( r \geq 1 \), let \( \varphi' \) be a generator of \( \widehat{\kappa_r^\times} \) such that \( \varphi'|_{\kappa^\times} = \varphi \). By applying Theorems 4.2 and 4.4 for \( \kappa_r \) and \( \varphi' \), we obtain the formulas for \( N_r(C_{D, \lambda}; \chi^m) \) and \( N_r(S_{D, \lambda; \chi^m}) \). Note that \( \varphi_d \) is replaced with \( \varphi' \) and \( \eta_r := \eta \circ N_{\kappa_r / \kappa} \), we have the Davenport-Hasse theorem (cf. [30])
\[
g(\eta_r) = g(\eta)^r.
\]
By this, we have
\[
j(\varphi_d^m, \varphi_{d,r}^m) = j(\varphi_d^m, \varphi_{d,r}^m)^r, \quad j(\varphi_d^m, \varphi_{d,r}^m, \varphi_{d,r}^m) = J_D^r.
\]
Thus, the corollary follows formally. \( \square \)

### 4.3. Smooth compactification of \( C_{D, \lambda} \)

Let \( \overline{C}_{D, \lambda} \) be the projective curve defined by the homogenization of (4.1) with \( x = X/Z, \ y = Y/Z \):
\[
\begin{cases} 
Y^d = Z^e X^a (Z - X)^e \prod_i (Z - \lambda_i X)^{b_i} & (\text{if } d \geq a + \sum_i b_i + c), \\
Z^e Y^d = X^a (Z - X)^e \prod_i (Z - \lambda_i X)^{b_i} & (\text{if } d < a + \sum_i b_i + c),
\end{cases}
\]
where
\[
e := |a + \sum_i b_i + c - d|.
\]
Recall that \( \prod_i \lambda_i (1 - \lambda_i) \prod_{i \neq j} (\lambda_j - \lambda_i) \neq 0 \). The group \( \mu_d \) acts on \( \overline{C}_{D, \lambda} \) by \( \xi \cdot (X : Y : Z) = (X : \xi Y : Z) \) (\( \xi \in \mu_d \)). Suppose that \( e > 0 \). Then, \( \overline{C}_{D, \lambda} \) has the only one point at infinity, denoted by \( \infty \). Since \( \mu_d \) and \( F \) acts on \( \infty \) trivially, we have
\[
N_r(\overline{C}_{D, \lambda}; \chi^m) = N_r(C_{D, \lambda}; \chi^m) = \begin{cases} 
1 & (m = 0), \\
0 & (m \neq 0).
\end{cases}
\]
If \( a > 1 \) (resp. \( b_i > 1, \ c > 1, \ e > 1 \)) then \( \overline{C}_{D, \lambda} \) is singular at \( (0 : 0 : 1) \) (resp. at \( (\lambda_i^{-1} : 0 : 1), \ (1 : 0 : 1), \ \infty \) ). Archinard [1] constructs a desingularization \( \pi : X_{D, \lambda} \to \overline{C}_{D, \lambda} \). Now we suppose
\[
\gcd(d, a) = \gcd(d, b_i) = \gcd(d, c) = \gcd(d, e) = 1.
\]
Then, we have \#\( \pi^{-1}(P) = 1 \) for all \( P \in \{(0 : 0 : 1), (\lambda_i^{-1} : 0 : 1), (1 : 0 : 1), \infty \} \) (see [1, subsection 3.1]), and we obtain, for all \( m \),
\[
N_r(X_{D, \lambda}; \chi^m) = N_r(\overline{C}_{D, \lambda}; \chi^m).
\]
By (4.2), (4.4) and Theorem 4.2, we obtain the following corollary similarly as Corollary 4.5.
Corollary 4.6. Under the assumption (4.3), we have

\[ N_r(X_{D,\lambda}; \chi^m) = \begin{cases} 1 + q^r & (m = 0), \\ -j(\varphi_{d,\lambda}^{\text{mc}}, \varphi_{d,\lambda})^r F_D^{(n)}(\varphi_{d,\lambda}^{\text{mc}}; \lambda_1, \ldots, \lambda_n) & (m \neq 0). \end{cases} \]

Therefore, the Artin \( L \)-function \( L(X_{D,\lambda}, \chi^m; t) \) is expressed in terms of the hypergeometric functions over \( \kappa_r \) \((r \geq 1)\) and the Jacobi sum. In fact, we show that the first \( n + 1 \) functions are sufficient.

Let \( l \neq p \) be a prime number and \( H^i(X_{D,\lambda}, \overline{Q}) (\chi^m) \) be the \( \chi^m \)-eigencomponent of the \( l \)-adic étale cohomology of \( X_{D,\lambda} = X_{D,\lambda} \otimes_{\mathbb{Z}} \mathbb{Z} \), where we fixed an embedding \( \mathbb{Q} \hookrightarrow \overline{Q} \). By the Grothendieck-Lefschetz trace formula (cf. [6, Theorem 2.9])

\[ N_r(X_{D,\lambda}; \chi^m) = \sum_{i=0}^{2} (-1)^i \text{Tr} \left( (F^*)^r \mid H^i(X_{D,\lambda}, \overline{Q}) (\chi^m) \right), \]

we have

\[ L(X_{D,\lambda}, \chi^m; t) = \prod_{i=0}^{2} \det \left( 1 - F^* t | H^i(X_{D,\lambda}, \overline{Q}) (\chi^m) \right)^{-i}. \]

By the following theorem, it follows that the \( F_D^{(n)} \) functions in Corollary 4.6 for \( r = 1, 2, \ldots \) are written as symmetric polynomials of the first \( n + 1 \) functions.

Theorem 4.7. Under the assumption (4.3), if \( m \neq 0 \), then \( L(X_{D,\lambda}, \chi^m; t) \) is a polynomial of degree \( n + 1 \).

Proof. Since \( H^i(X_{D,\lambda}, \overline{Q}) = H^i(X_{D,\lambda}, \overline{Q}) (\chi^0) \) for \( i = 0, 2 \), it suffices to show

\[ d_m := \dim_{\overline{Q}} H^1(X_{D,\lambda}, \overline{Q}) (\chi^m) = n + 1. \]

Since the quotient \( X_{D,\lambda}/\mu_d \) is a rational curve, \( H^1(X_{D,\lambda}, \overline{Q}) (\chi^0) = 0 \) and

\[ \sum_{m=1}^{d-1} d_m = 2 \cdot \text{genus}(X_{D,\lambda}) = (d - 1)(n + 1), \]

by [1, Theorem 4.1] (note that \( d \) and \( n \) are not both even by the assumption (4.3)). Hence, it suffices to show that \( d_m \geq n + 1 \).

By a standard argument using the smooth base change theorem (cf. [6, Theorem 7.3]) and the Artin comparison theorem (cf. [6, Proposition 11.6]), we are reduced to characteristic 0. Regard \( \kappa \) as a residue field of a number field in such a way that the character of \( \mu_d(\mathbb{C}) \cong \mu_d(\kappa) \) induced by \( \chi \) is the inclusion. Put

\[ S = \left\{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid \prod_{i=1}^{n} t_i (1 - t_i) \prod_{j \neq i} (t_j - t_i) \neq 0 \right\}, \]

and let \( f : X_D \to S \) be the relative projective curve over \( \mathbb{C} \) defined by the equation (4.1). Since \( f \) is smooth, the relative algebraic de Rham cohomology \( H^1_{dR}(X_D/S) = R^1 f_* \Omega^1_{X_D/S} \) is a locally free \( \mathcal{O}_S \)-module and \( \text{rank} \mathcal{O}_S H^1_{dR}(X_D/S)(\chi^m) = d_m \).
For $m = 1, \ldots, d - 1$, put a differential 1-form on the fibre $X_{D, \lambda}$ as

$$\omega_m = \frac{y^m}{x(1 - x)} dx.$$ 

We show that it is of the second kind. It may have a pole only at $\infty$. A local parametrization of $X_{D, \lambda}$ at $\infty$ is given by (cf. [1, (7) and (8)])

$$(x, y) = \left(s^{-d}, s^{-(a+c+\sum b_j})(s^d - 1)^{\frac{m_c}{d}} \prod_i (s^d - \lambda_i)^{\frac{m_b}{d^i}} \right),$$

where $s \in \mathbb{C}$ takes values in a neighbourhood of 0 on which $(s^d - 1) \prod (s^d - \lambda_i) \neq 0$. Then, we have

$$\omega_m = -d \cdot s^{-m(a+c+\sum b_j)+d-1}(s^d - 1)^{\frac{m_c}{d}} \prod_i (s^d - \lambda_i)^{\frac{m_b}{d^i}} ds.$$ 

Since $(s^d - 1)^{mc/d} \prod (s^d - \lambda_i)^{mb_i/d}$ is a power series in $s^d$ and $\gcd(d, a+c+\sum b_j) = \gcd(d, e) = 1$ by the assumption (4.3), $\omega_m$ has the trivial residue, thus is of the second kind. Hence, it defines a section of $\mathcal{H}_{d\text{IR}}(X_D/S)(\chi^m)$.

Define a path $\delta : [0, 1] \to X_{D, \lambda}(\mathbb{C})$ by $\delta(t) = (t, \sqrt[d]{(1-t)^e \prod_i (1-\lambda_i t)^{b_i}})$, where the branch of the $d$th root is taken by setting $|\arg(t^e(1-t)^e \prod_i (1-\lambda_i t)^{b_i})| < \pi$ when $\lambda_i$ are close to 0, and continued analytically. Choose a primitive root $\xi \in \mu_d$ and put $\gamma = \delta - \xi \cdot \delta$. Then, we have the period by (3.2),

$$\int_\gamma \omega_m = (1 - \xi^m)B\left(\frac{ma}{d}, \frac{mc}{d}\right) F_D^{(n)}\left(\frac{ma}{d}, \frac{mb_i}{d}, \ldots, \frac{mb_n}{d} : \lambda_1, \ldots, \lambda_n\right).$$

This $F_D^{(n)}$ function satisfies a system of differential equations of rank $n + 1$, which is irreducible by a result of Mimachi-Sasaki [21, Theorem 3.1] and our assumption (4.3). This shows that $\mathcal{H}_{d\text{IR}}(X_D/S)(\chi^m)$ contains an $\mathcal{O}_S$-submodule of rank $n + 1$. Hence $d_n \geq n + 1$ and the theorem is proved. \hfill $\square$

4.4. Algebraic varieties related to $F_A$ and $F_B$. We consider $n$-dimensional affine hypersurfaces $S^1_{A, \lambda}$, $S^2_{A, \lambda}$, and $S_{B, \lambda}$ over $\kappa$ defined by the equations

$$S^1_{A, \lambda} : y^d = \left(1 - \sum_{i=1}^{n} x_i^{a_i}\right)^n \prod_{i=1}^{n} x_i^{b_i}(1 - x_i)^{c_i};$$
$$S^2_{A, \lambda} : y^d = \left(\prod_{i=1}^{n} (x_i - \lambda_i)^{b_i}\right)\left(\prod_{i=1}^{n} x_i^{c_i}\right)\left(1 - \sum_{i=1}^{n} x_i\right)^a;$$
$$S_{B, \lambda} : y^d = \left(\prod_{i=1}^{n} (1 - \lambda_i x_i)^{a_i}\right)\left(\prod_{i=1}^{n} x_i^{b_i}\right)\left(1 - \sum_{i=1}^{n} x_i\right)^c,$$

where $d, a, a_1, \ldots, a_n, b_1, \ldots, b_n, c, c_1, \ldots, c_n \in \mathbb{Z}_{\geq 1}$, and $\lambda_1, \ldots, \lambda_n \in \kappa^\times$. Suppose that $d \mid q - 1$. In the same way as the previous subsection, the group $\mu_q$ acts on these hypersurfaces. Similarly as in the proof of Theorem 4.4, we can show the followings by using Theorems 3.3, 3.4 and 3.5.

Theorem 4.8.

(i) Suppose that $\gcd(d, a) = \gcd(d, c_i) = 1$ for all $i$. Then,

$$N_1(S^1_{A, \lambda}; \chi^m)$$
\begin{align*}
&= \begin{cases} 
q^n & (m = 0), \\
\left( \prod_{i=1}^{n} - j(\varphi_d^{m_{b_i}}; \varphi_d^{m_{c_i}}) \right) \binom{n}{\varphi_d^{m_{(b_1+c_1)}}, \ldots, \varphi_d^{m_{(b_n+c_n)}}; \lambda_1, \ldots, \lambda_n} & (m \neq 0).
\end{cases}
\end{align*}

(ii) Suppose that \( \gcd(d, a_i) = \gcd(d, b_i) = 1 \) for all \( i \).

Then,

\[ N_1(S_{A, \lambda}^2; \chi^m) = \begin{cases} 
q^n & (m = 0), \\
(-1)^n J_A \cdot \binom{n}{\varphi_d^{m_{a_{1}+\sum_{i=1}^{n}(b_i+c_i)}}, \varphi_d^{m_{b_1}}, \ldots, \varphi_d^{m_{b_n+c_n}}; \varphi_d^{m_{(b_1+\cdots+b_n+c)}}, \lambda_1, \ldots, \lambda_n} & (m \neq 0),
\end{cases}
\]

where \( J_A := j(\varphi_d^{m_{b_1+c_1}}, \ldots, \varphi_d^{m_{b_n+c_n}}). \)

**Theorem 4.9.** Suppose that \( \gcd(d, a_i) = \gcd(d, c) = 1 \) for all \( i \).

Then,

\[ N_1(S_{B, \lambda}^2; \chi^m) = \begin{cases} 
q^n & (m = 0), \\
(-1)^n J_B \cdot \binom{n}{\varphi_d^{m_{a_1}}, \ldots, \varphi_d^{m_{a_n}}; \varphi_d^{m_{b_1}}, \ldots, \varphi_d^{m_{b_n+c}}; \lambda_1, \ldots, \lambda_n} & (m \neq 0),
\end{cases}
\]

where \( J_B := j(\varphi_d^{m_{b_1}}, \ldots, \varphi_d^{m_{b_n+c}}). \)

Similarly as Corollary 4.5, we have the following.

**Corollary 4.10.** Put

\[ f_r(\lambda) = \binom{n}{\varphi_d^{d_{a_{1}}}, \varphi_d^{d_{b_1}}, \ldots, \varphi_d^{d_{b_n}}; \varphi_d^{d_{a_{1}+\sum_{i=1}^{n}(b_i+c_i)}}, \varphi_d^{d_{b_1}}, \ldots, \varphi_d^{d_{b_n+c_n}}; \lambda_1, \ldots, \lambda_n}, \]

\[ g_r(\lambda) = \binom{n}{\varphi_d^{d_{r_{a_{1}}}}, \varphi_d^{d_{r_{b_1}}}, \ldots, \varphi_d^{d_{r_{b_n}}}; \varphi_d^{d_{r_{a_{1}+\sum_{i=1}^{n}(b_i+c_i)}}, \varphi_d^{d_{r_{b_1}}}, \ldots, \varphi_d^{d_{r_{b_n+c_n}}}}; \lambda_1, \ldots, \lambda_n}, \]

\[ h_r(\lambda) = \binom{n}{\varphi_d^{d_{r_{a_{1}}}}, \varphi_d^{d_{r_{b_1}}}, \ldots, \varphi_d^{d_{r_{b_n}}}; \varphi_d^{d_{r_{a_{1}+\sum_{i=1}^{n}(b_i+c_i)}}, \varphi_d^{d_{r_{b_1}}}, \ldots, \varphi_d^{d_{r_{b_n+c_n}}}}; \lambda_1, \ldots, \lambda_n}. \]

(i) Suppose that \( \gcd(d, a_i) = \gcd(d, c_1) = 1 \) for all \( i \).

Then,

\[ L(S_{A, \lambda}^1; \chi^m; t) = \begin{cases} 
\frac{1}{1 - q^nt} & (m = 0), \\
\exp \left( \sum_{r=1}^{n} (\prod_{i=1}^{n} - j(\varphi_d^{m_{b_i}}, \varphi_d^{m_{c_i}})^r) \cdot f_r(\lambda) \frac{t^r}{r} \right) & (m \neq 0).
\end{cases}
\]

(ii) Suppose that \( \gcd(d, a_i) = \gcd(d, b_i) = 1 \) for all \( i \).

Then,

\[ L(S_{A, \lambda}^2; \chi^m; t) = \begin{cases} 
\frac{1}{1 - q^nt} & (m = 0), \\
\exp \left( \sum_{r=1}^{\infty} (-1)^n J_A \cdot g_r(\lambda) \frac{t^r}{r} \right) & (m \neq 0),
\end{cases}
\]

where \( J_A \) is as in Theorem 4.8 (ii).
(iii) Suppose that $\gcd(d, a_i) = \gcd(d, c) = 1$ for all $i$. Then,

$$L(S_{B, \lambda}, \chi^m; t) = \begin{cases} 
1 & (m = 0), \\
\frac{1}{1 - q^n t} \exp \left( \sum_{r=1}^{\infty} (-1)^n J_B \cdot h_r(\lambda) \frac{t^r}{r} \right) & (m \neq 0),
\end{cases}$$

where $J_B$ is as in Theorem 4.9.

4.5. Algebraic varieties related to $F_G$. Let $d, a, b, c_1, \ldots, c_n \in \mathbb{Z}_{\geq 1}$ be integers and let $\lambda_1, \ldots, \lambda_n \in \kappa^\times$. Write $S_{C, \lambda}$ for the $n$-dimensional affine hypersurface over $\kappa$ defined by the equation

$$y^d = \left( \prod_{i=1}^{n} x_i^{c_i} \right) \left( 1 - \sum_{i=1}^{n} x_i \right)^a \left( \prod_{i=1}^{n} x_i - \sum_{i \neq i}^{n} \lambda_i \prod_{j \neq i}^{n} x_j \right)^b.$$ 

Similarly as in the previous subsections, suppose that $d \mid q - 1$ and hence, the group $\mu_d$ acts on $S_{C, \lambda}$, and we obtain the following theorem and corollary.

**Theorem 4.11.** Suppose that $\gcd(d, a) = \gcd(d, b) = 1$. Then,

$$N_1(S_{C, \lambda}; \chi^m) = \begin{cases} 
q^n & (m = 0), \\
(-1)^n J_C \cdot F_C^{(n)} \left( \prod_{r=1}^{m} n_{a+b+c_1+\ldots+c_n} ; \prod_{r=1}^{n} \lambda_1, \ldots, \lambda_n \right) & (m \neq 0),
\end{cases}$$

where $J_C = j(\phi_d, n_{a+b+c_1+\ldots+c_n})$.

**Corollary 4.12.** Put

$$f_r(\lambda) = F_C^{(n)} \left( \prod_{d,r} n_{a+b+c_1+\ldots+c_n} ; \prod_{d,r} \lambda_1, \ldots, \lambda_n \right),$$

where $\lambda_1, \ldots, \lambda_n \in \kappa^\times$. Suppose that $\gcd(d, a) = \gcd(d, b) = 1$. Then,

$$L(S_{C, \lambda}, \chi^m; t) = \begin{cases} 
1 & (m = 0), \\
\frac{1}{1 - q^n t} \exp \left( \sum_{r=1}^{\infty} (-1)^n J_C \cdot f_r(\lambda) \frac{t^r}{r} \right) & (m \neq 0),
\end{cases}$$

where $J_C$ is as in Theorem 4.11.

Suppose that $\lambda_1, \lambda_2 \neq 1$. Let $S_{4, \lambda}$ be the affine surface over $\kappa$ defined by the equation

$$y^d = x_1^{(a)} x_2^{(b)} (1 - x_1)^{(c_1-a)} (1 - x_2)^{(c_2-b)} \times (1 - \lambda_1 x_1)^{(a-c_1-c_2)} (1 - \lambda_2 x_2)^{(b-c_1-c_2)} (1 - \lambda_1 x_1 - \lambda_2 x_2)^{(c_1+c_2-a-b)}.$$ 

Here, for $n \in \mathbb{Z}$, $\langle n \rangle \in \{0, \ldots, d - 1\}$ denotes the representative of $n$ mod $d$.

**Theorem 4.13.** Suppose that $\gcd(d, a) = \gcd(d, b) = \gcd(d, c_i-a) = \gcd(d, c_i-b) = 1$ for $i = 1, 2$. Then,

$$N_1(S_{4, \lambda}; \chi^m)$$
Here, $J$ and $S_i$ are as in Theorem 3.12 with $\alpha = \varphi_d^{ma}$, $\beta = \varphi_d^{mb}$, $\gamma_i = \varphi_d^{mc_i}$.

**Proof.** Similarly as in the proof of Theorem 4.4, we have

$$N_1(S_{4,\lambda}; \chi^0) = q^2.$$

For $m \neq 0$,

$$N_1(S_{4,\lambda}; \chi^m) = \sum_{u,v} \varphi_d^{ma}(u)\varphi_d^{mb}(v)\varphi_d^{m(c_1-a)}(1-u)\varphi_d^{m(c_2-b)}(1-v) \times \varphi_d^{m(a-c_1-c_2)}(1-\lambda_1 u)\varphi_d^{m(b-c_1-c_2)}(1-\lambda_2 v)\varphi_d^{m(c_1+c_2-a-b)}(1-\lambda_1 u - \lambda_2 v).$$

Here, note that $\varphi_d^{(n)} = \varphi_d^n$. Thus, the theorem follows by Theorem 3.12. \qed

**Corollary 4.14.** Let the assumptions and notations be as in Theorem 4.13. Put $f_r(\lambda_1, \lambda_2) = F_4(\varphi_d^{ma}, \varphi_d^{mb}, \varphi_d^{mc_i}, \varphi_d^{mc_2}, \lambda_1(1-\lambda_2), \lambda_2(1-\lambda_1)).$

Then,

$$L(S_{4,\lambda}; \chi^m; t) = \begin{cases} \frac{1}{1-q^2t} & (m = 0), \\ \exp \left( \sum_{r=1}^{\infty} J^r \cdot f_r(\lambda_1, \lambda_2) \frac{t^r}{r} \right) \prod_{i=0}^{\infty} (1 - S_i t) & (m \neq 0). \end{cases}$$

**Proof.** Note that, for $\eta \in \kappa'$ and $\eta_r = \eta \circ N_{\kappa'/\kappa}$, if $\lambda \in \kappa$ then $\eta_r(\lambda) = \eta'(\lambda)$. Similarly as Corollary 4.5, we obtain

$$N_r(S_{4,\lambda}; \chi^m; t) = \begin{cases} q^{2r} & (m = 0), \\ J^r f_r(\lambda_1, \lambda_2) + \sum_{i=0}^{2} S_i^r(\lambda_1, \lambda_2) & (m \neq 0), \end{cases}$$

by Theorem 4.13, and hence the corollary follows formally. \qed

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