BOUNDED HANKEL PRODUCTS ON FOCK-SOBOLEV SPACES

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\textbf{Abstract.} Let $F^{2,m}$ denote the Fock-Sobolev space of complex plane. The purpose of this paper is to study the conjecture which was shown to be false for Fock space by Ma-Yan-Zheng-Zhu in 2019. For the certain symbol space, the main result of the paper says that the conjecture is actually true in $F^{2,m}$ if $m$ is a positive integer.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and $dA$ be the area measure. For any fixed non-negative integer $m$, $L^2_m$ is the space of Lebesgue measurable functions $f$ on $\mathbb{C}$ such that the function $f(z)$ is in $L^2(\mathbb{C}, |z|^{2m}e^{-|z|^2}dA(z))$. It is well known that $L^2_m$ is a Hilbert space with the inner product

$$<f, g> = \frac{1}{\pi^m} \int_{\mathbb{C}} f(z) \overline{g(z)} |z|^{2m}e^{-|z|^2}dA(z), f, g \in L^2_m.$$ 

This shows that $\|f\|^2_2 = <f, f>$. The Fock-Sobolev space $F^{2,m}$ consists of all entire functions $f$ in $L^2_m$. Obviously, the Fock-Sobolev space $F^{2,m}$ is a closed subspace of the Hilbert space $L^2_m$. There is an orthogonal projection $P$ from $L^2_m$ onto $F^{2,m}$, which is given by

$$Pf(z) = \frac{1}{\pi^m} \int_{\mathbb{C}} f(w)K_m(z, w)|w|^{2m}e^{-|w|^2}dA(w), \quad f \in F^{2,m},$$

where

$$K_m(z, w) = \sum_{k=0}^{\infty} \frac{m!}{(k + m)!}(z\overline{w})^k$$

is the reproducing kernel of Fock-Sobolev space $F^{2,m}$. Note that the functions

$$e_k(z) = \sqrt{\frac{m!}{(k + m)!}z^k}, \quad k = 0, 1, 2, \cdots,$$

form an orthonormal basis for $F^{2,m}$, see [11].

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Let $D$ be the set of all finite linear combinations of kernel functions. Suppose $\varphi$ is a Lebesgue measurable function on $\mathbb{C}$ that satisfies
\begin{equation}
\int_{\mathbb{C}} |\varphi(w)| |K_m(z, w)||w|^{2m}e^{-|w|^2}dA(w) < \infty
\end{equation}
for $f \in D$. Since $D$ is dense in $F^{2,m}$, so we can densely define the Toeplitz operator with the symbol $\varphi$ on $F^{2,m}$ as follows:
$$T_{\varphi}f(z) = \frac{1}{\pi m!} \int_{\mathbb{C}} \varphi(w)f(w)K_m(z, w)|w|^{2m}e^{-|w|^2}dA(w).$$

The Hankel operator $H_{\varphi}$ with the symbol $\varphi$ is given by $H_{\varphi}f = (I - P)(\varphi f)$, where $I$ is the identity operator on $L^2_{\mathbb{C}}$.

Let $k_m(z, w)$ be the normalization of the reproducing kernel $K_m(z, w)$, i.e., $k_m(z, w) = \frac{K_m(z, w)}{\|K_m(z, w)\|}$. Suppose that $\varphi$ is a Lebesgue measurable function on $\mathbb{C}$ satisfying condition (1), the Berezin transform of $T_{\varphi}$ is
$$\tilde{T}_{\varphi}(z) = \langle T_{\varphi}k_m(w, z), k_m(w, z) \rangle.$$

If $m = 0$, the space is the Fock space $F^2$. For the Fock space $F^2$, there is a Weyl operator $U_z$ on $F^2$ such that $U_zf(w) = f(w - z)k_{F,z}(w)$, where $k_{F,z}(w)$ is normalization of the reproducing kernel $F^2$. By the property of $U_z$, the Berezin of $f$ such that
$$\tilde{f}(z) = \int_{\mathbb{C}} f(z \pm w)e^{-|w|^2}dA(w).$$

Up to our knowledge, the boundedness of a single Toeplitz operator on $F^2$ is still an open problem, see [2, 5, 6]. So many mathematicians consider the products of two operators on Fock space. Sarason’s Toeplitz product problem asks for conditions on $f, g \in F^{2,m}$ such that $T_fT_g$ is bounded. Sarason [18] originally asked this for the Hardy space of unit circle. Unfortunately, Sarason’s conjecture is not true, both for Hardy space and Bergman space, see [1, 15] for counterexamples. However, Sarason’s conjecture is true for Fock space, see [10]. Chen et al. have extended this result to Fock-sobolev space, see [8].

The corresponding question for analogous Hankel operators defined on the $F^2$ was partially resolved by Ma-Yan-Zheng-Zhu in their paper [13], they proved that the following conjecture is false on Fock space.

**Conjecture:** Given analytic functions $f$ and $g$, the Hankel product $H^2_fH^2_g$ is bounded if and only if the function
$$E(f, g)(z) = \left[|f|^2(z) - |f(z)|^2\right] \left[|g|^2(z) - |g(z)|^2\right]$$
is bounded.

The Weyl operator $U_z$ is a key to consider the Operator Theory in $F^2$, the results of [2, 3, 5, 6, 10, 13] are based on the property of $U_z$. However, the translations do not have such good property on Fock-Sobolev space $F^{2,m}$. So the Berezin transform of the function can almost never be computed explicitly. We need some new tools.
For \( f, g \in F^{2,m} \), it is clear that
\[
H_f^* H_g^* = T_f g - T_f T_g = \langle T_f, T_g \rangle.
\]
So the boundedness of the Hankel product is closely related to some open problems. For instance, the commutativity of Toeplitz operators, the fixed point of Berezin transform and so on, see [3]. In this paper, we consider the conjecture on Fock-Sobolev spaces. In section [3] we obtain the boundedness of \( E(f, g) \). We are now prepared to state our first main result.

**Theorem A.** Suppose that \( f \) and \( g \) are functions in the Fock-Sobolev space \( F^{2,m} \). Then \( E(f, g) \) is bounded on \( \mathbb{C} \) if and only if one of the following statements holds.
(a) At least one of \( f \) and \( g \) is constant.
(b) Both \( f \) and \( g \) are linear polynomials.
(c) There are constants \( a, b, c, A, \) and \( B \) such that
\[
f(z) = e^{az+b} + A, \quad g(z) = e^{-az+c} + B.
\]
Recall that
\[
H_f^* H_g^* = T_f g - T_f T_g = \langle T_f, T_g \rangle.
\]
So, \( H_f^* H_g^* = 0 \) means that the semi-commutator \( \langle T_f, T_g \rangle = 0 \). In [3, 13], they obtained \( H_f^* H_g^* = 0 \) on \( F^2(\mathbb{C}) \) if \( f(z) = e^{2n\pi i z} \) and \( g(z) = e^z \) for any integer \( n \). But \( E(f, g) \) is unbounded on \( \mathbb{C} \). In fact, the commutativity of Toeplitz operators on Fock spaces is still an open problem, see [3].

But to our surprise, we have obtained that: for any positive integer \( m \), \( T_{e^a z} T_{e^{-a} z} = T_{e^a z} T_{e^{-a} z} \) on \( F^{2,m} \) if and only if \( ab = 0 \). Moreover, we have showed that there is actually no nontrivial functions \( f \) and \( g \) in \( D \) such that \( H_f^* H_g^* = 0 \), see [16].

**Theorem 1.** Suppose \( m \) is a positive integer, the following conditions are equivalent for any two functions \( f \) and \( g \) in \( D \):
(a) \( H_f^* H_g^* = 0 \) on \( F^{2,m} \).
(b) \( T_f T_g = T_f g \) on \( F^{2,m} \).
(c) \( g \) is a constant.
(d) At least one of \( f \) and \( g \) is a constant.

Define
\[
\mathfrak{D} = D \cup \{ f(z) = ce^{az+b} : a, b, c \in \mathbb{C} \} \cup \{ f(z) = az+b : a, b \in \mathbb{C} \}.
\]
In section [4] we extend the results of the semi-commuting Toeplitz operator to the symbol space \( \mathfrak{D} \). The second main result of this paper is given by the following theorem.

**Theorem B.** Suppose that \( m \) is a positive integer and \( f, g \in \mathfrak{D} \). Then the following statements are equivalent.
(a) \( H_f^* H_g^* = 0 \).
(b) \( T_f T_g = T_f g \).
(c) \( g \) is a constant.
In [13], the authors considered the bounded Hankel product with two special symbol classes. In section 5, we apply Theorem B to show that the conjecture is true if $f, g \in \mathcal{D}$. The result is different from the result of [13].

**Theorem C.** Let $m$ be a positive integer, and suppose that $f, g \in \mathcal{D}$. Then $H_{\mathcal{H}}^*H_{\mathcal{H}}$ is bounded on $F^{2,m}$ if and only if $E(f, g)$ is bounded on $\mathbb{C}$.

Throughout the paper, $m$ will be a fixed positive integer. We say that $a \lesssim b$ if there is a constant $c$ independent of $a$ and $b$ such that $a \leq cb$, where $a$ and $b$ are nonnegative quantities. Similarly, we say $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

### 2. Preliminaries

In this section, we give some preliminary facts. We begin with the estimate of the reproducing kernel, see [11].

**Lemma 2.** Suppose that $f \in F^{2,m}$ and $m$ is a nonnegative integer. Then

$$|K_m(z, \omega)| \lesssim \frac{e^{\frac{1}{2}|z|^2 + \frac{1}{2}|\omega|^2 - \frac{1}{8}|z-\omega|^2}}{(1 + |z||\omega|)^m}.$$ 

The next lemma gives the norm estimate of the Hankel operators, see [17].

**Lemma 3.** Let $f \in F^{2,m}$, then

$$\|H_{\mathcal{H}}^*f_k m(\cdot, z)\|_2^2 = |\overline{f}(z) - \overline{f}(\omega)|^2 = |\overline{f}(z) - \overline{f}(z)|^2.$$

Moreover, there is a positive constant $\varepsilon$ independent of $z$ and $\omega$, such that

$$\|H_{\mathcal{H}}^*f_k m(\cdot, z)\|_2^2 \gtrsim \int_{|z-\omega| < \varepsilon} |f(\omega) - f(z)|^2 d\nu(\omega)$$

for sufficiently large $|z|$.

Easily modifying the proof of Theorem 8.4 in [19], we have the following Lemma.

**Lemma 4.** Suppose $f(z) = az + c$ and $g = bz + d$, where $a, b, c, d \in \mathbb{C}$. Then $H_{\mathcal{H}}^*H_{\mathcal{H}}$ is bounded on $F^{2,m}$.

The following lemma is a key to solve the boundedness of the $D(f, g)$, which can be found in [13].

**Lemma 5.** If $F : C \to X$ is entire and bounded, that is, there is a positive constant $C$ such that $\|F(z)\|_X \leq C$ for all $z \in \mathbb{C}$. Then $F(z) = F(0)$ for all $z \in \mathbb{C}$.

In order to prove the main result of bounded Hankel product, we need the following lemmas.

**Lemma 6.** Let $\{w_i\}_{i=1}^N$ be a finite collection of non-zero complex numbers, then $\{1 - K_m(\cdot, w_i)\}_{i=1}^N$ is linearly independent.

**Lemma 7.** For any nonzero constant $a$ and positive integer $m$, we have $e^{az}$ and $e^{az^2}$ are not in $D$. 
Proof. It is clear that
\[ e^{az} = (az)^m K_m(z, \overline{a}) + q_m(az). \]
Then \( e^{az}, e^{az^2} \notin D. \) \( \square \)

We say that \( f \) is a non-vanishing function if \( f(z) \neq 0 \) for all \( z \in \mathbb{C} \). For the non-vanishing functions \( f \) and \( g \), we have the following result.

Lemma 8. Suppose \( f \) and \( g \) are non-vanishing functions in \( F^{2,m} \). If \( H^*_f H^*_g \) is bounded, then one of the following holds.

(a) \( fg \) is bounded on \( \mathbb{C} \);
(b) There is a constant \( C \) so that\[
\widetilde{fg}(z) = f(z)\overline{g(z)} + C.
\]
Moreover,
\[ T^*_f g = g(0)f + C. \]

Proof. Since \( H^*_f H^*_g \) is bounded, then \( \overline{H^*_f H^*_g} \) is bounded on \( \mathbb{C} \). By definition,\[
| \langle H^*_f H^*_g k_z, k_z \rangle | = | \overline{\widetilde{fg}(z)} - f(z)\overline{g(z)} | = | f(z)\overline{g(z)}| | \overline{\widetilde{fg}(z)} - f(z)\overline{g(z)} | - 1 | < \infty.
\]

If \( | f(z)g(z) | \) is bounded, then \( \widetilde{fg}(z) \) is also bounded.

On the other hand, if \( | f(z)g(z) | \) is unbounded, then \( | f(z)g(z) | \to \infty \) as \( | z | \to \infty \) since \( fg \) is an entire function on \( \mathbb{C} \). There is a constant \( C \) so that\[
\lim_{| z | \to \infty} \frac{\overline{\widetilde{fg}(z)}}{f(z)\overline{g(z)}} = 1 \quad \text{and} \quad \lim_{| z | \to \infty} \frac{\overline{\widetilde{fg}(z)}}{f(z)\overline{g(z)}} - 1 = C \frac{f(z)\overline{g(z)}}{f(z)\overline{g(z)}}.
\]

So there exists a function \( h \) so that\[
\widetilde{fg}(z) = f(z)\overline{g(z)} + h(z),
\]
where \( \frac{h(z)}{f(z)g(z)} \to 0 \) as \( | z | \to \infty \). Using (2), we have \( h(z) \) is bounded on \( \mathbb{C} \). It follows that\[
f(z)\overline{g(z)} + h(z) = \frac{1}{\pi} \int_{\mathbb{C}} f(w)\overline{g(w)}|k_m(w, z)|^2 |w|^2 e^{-|w|^2} dA(w)
= \langle T^*_f T^*_g k_m (\cdot, z), k_m (\cdot, z) \rangle + h(z)k_m (\cdot, z) - h(z)k_m (\cdot, z)
= \langle T^*_f T^*_g k_m (\cdot, z), k_m (\cdot, z) \rangle.
\]

Which shows that\[
\langle (T^*_f - T^*_f T^*_g - h(z))k_m (\cdot, z), k_m (\cdot, z) \rangle = 0.
\]
Since \( T^*_f - T^*_f T^*_g \) is bounded on \( F^{2,m} \), \( h(z) \) is bounded and the reproducing kernels \( span \ F^{2,m} \), we obtain\[
H^*_f H^*_g k_m (\cdot, z) = T^*_f k_m (\cdot, z) - T^*_f T^*_g k_m (\cdot, z)
= h(z)k_m (\cdot, z).
\]
Note that $K_m(\cdot, z) \neq 0$. So,

$$h(z) = \frac{T_jzK_m(\cdot, z) - T_jT_zK_m(\cdot, z)}{K_m(\cdot, z)}.$$  

This shows that $h(z)$ is conjugate holomorphic in $z$. Since $h(z)$ is bounded on $\mathbb{C}$, Liouville’s shows that theorem $h$ is a constant function. This gives $H_T^*H_T = h(0)$ and completes the proof of the lemma.

3. Boundedness of $E(f, g)$

The problem to be studied in this section is to determine the function $f, g$ for which the $E(f, g)$ is bounded on $\mathbb{C}$. It will be assumed that $f$ and $g$ are function in $F^{2, m}$. We now consider the boundedness of $E(f, g)$ when $f$ and $g$ are certain special functions.

Lemma 9. Suppose that $f$ and $g$ are two functions in the Fock-Sobolev space $F^{2, m}$.
(a) If at least one of $f$ and $g$ is constant, then $E(f, g)$ is identically zero;
(b) If both $f$ and $g$ are linear polynomials, then $E(f, g)$ is bounded on $\mathbb{C}$;
(c) Suppose that

$$f(z) = Ce^{q(z)} + C_1, \quad g(z) = e^{-q(z)} + C_2,$$

where $C, C_1, C_2$ are constants and $q$ is a linear polynomial. Then $E(f, g)$ is bounded on $\mathbb{C}$.

Proof. If one of $f$ and $g$ is constant, then we have one of $|\bar{f}|^2(z) - |f(z)|^2$ and $|\bar{g}|^2(z) - |g(z)|^2$ is zero, and hence $E(f, g)$ is identically zero. If $f$ and $g$ are linear polynomials, it follows from Lemma 3 and 4 that $E(f, g)$ is bounded.

Now assume $f(z) = Ce^{q(z)} + C_1$ and $g(z) = e^{-q(z)} + C_2$ where $C, C_1,$ and $C_2$ are constants. Without loss of generality, assume $C = 1$, it follows that

$$E(f, g)(z) = \left| \frac{|f(z)|^2}{f(z)} - \frac{|\bar{f}(z)|^2}{\bar{f}(z)} \right| \left| \frac{|g(z)|^2}{g(z)} - \frac{|\bar{g}(z)|^2}{\bar{g}(z)} \right|$$

$$\approx \int_{\mathbb{C}} |e^{q(\omega)} - e^{q(z)}|^2 |k_m(\omega, z)|^2 |w|^{2m} e^{-|w|^2} dA(w)$$

$$\times \int_{\mathbb{C}} |e^{-q(u)} - e^{-q(z)}|^2 |k_m(u, z)|^2 |u|^{2m} e^{-|u|^2} dA(u).$$

We define

$$I = \int_{\mathbb{C}} |e^{q(\omega)} - e^{q(z)}|^2 |k_m(\omega, z)|^2 |\omega|^{2m} e^{-|\omega|^2} dA(\omega),$$

and

$$J = \int_{\mathbb{C}} |e^{-q(u)} - e^{-q(z)}|^2 |k_m(u, z)|^2 |u|^{2m} e^{-|u|^2} dA(u).$$

Then

$$I = \int_{\mathbb{C}} |e^{q(\omega)} - e^{q(z)}|^2 \frac{|K_m(\omega, z)|^2}{K_m(z, z)} |\omega|^{2m} e^{-|\omega|^2} dA(\omega)$$

$$\lesssim \int_{\mathbb{C}} |e^{q(\omega)} - e^{q(z)}|^2 \frac{|z|^4 |\omega|^2 + |z - \omega|^2}{(1 + |z| |\omega|)^{2m} |\omega|^{2m} e^{-|\omega|^2}} \frac{1 + |z|^{2m}}{e^{|z|^2}} dA(\omega)$$

$$\lesssim |e^{q(z)}|^2 \int_{\mathbb{C}} |e^{-q(\omega)} - 1|^2 |\omega|^{2m} e^{-|\omega|^2} dA(\omega),$$

and
where the second "\( \lesssim \)" comes from Lemma 2. A change of variables gives the last "\( \lesssim \)". It is clear that
\[
\int_{\mathbb{C}^n} |e^{-q(\omega)} - 1|^2 e^{-|\omega|^2} dA(\omega) < \infty.
\]
Similarly,
\[
J \lesssim |e^{-q(z)}|^2 \int_{\mathbb{C}} |e^{q(u)} - 1|^2 e^{-|u|^2} dA(u), \quad \int_{\mathbb{C}} |e^{q(u)} - 1|^2 e^{-|u|^2} dA(u) < \infty.
\]
This means that \( E(f, g)(z) \approx I \times J \) is bounded on \( \mathbb{C}^n \). This finishes the proof. \( \square \)

We now turn to the main result of this section.

**Theorem 10.** Suppose that \( f \) and \( g \) are two functions in the Fock-Sobolev space \( F^{2,m} \). Then \( E(f, g) \) is bounded on \( \mathbb{C} \) if and only if one of the following statements holds:
(a) At least one of \( f \) and \( g \) is constant;
(b) Both \( f \) and \( g \) are linear polynomials.
(c) There are constants \( a, b, c, d, A, \) and \( B \) such that
\[
f(z) = ce^{az+b} + A, \quad g(z) = e^{-az+d} + B.
\]

**Proof.** By Lemma 9 we have shown that each of the conditions (a), (b) and (c) will imply that \( E(f, g) \) is bounded on \( \mathbb{C} \).

To prove the converse. Assume \( E(f, g) \) is bounded on \( \mathbb{C} \). Let \( \varepsilon \) be a positive number, and let \( |z| \to \infty \). Then Lemma 5 shows that
\[
E(f, g)(z) \simeq \int_{\mathbb{C}} |f(\omega) - f(z)|^2 |k_m(\omega, z)|^2 dG(\omega)
\]
\[
\times \int_{\mathbb{C}} |g(u) - g(z)|^2 |k_m(u, z)|^2 dG(u)
\]
\[
\gtrsim \int_{|z-\omega|<\varepsilon} \int_{|z-u|<\varepsilon} |f(\omega) - f(z)|^2 |g(u) - g(z)|^2 dA(u) dA(\omega)
\]
(3)
\[
= \int_{|z|<\varepsilon} \int_{|u|<\varepsilon} |f(z-\omega) - f(z)|^2 |g(z-u) - g(z)|^2 dA(u) dA(\omega).
\]

Let \( F_z(u, \omega) = (f(z-\omega) - f(z))(g(z-u) - g(z)) \), then \( F_z \) is a entire function on \( \mathbb{C} \times \mathbb{C} \). In particular, \( F_z \) is a entire function of \( z \) on \( \mathbb{C} \). Let \( B_\varepsilon = \{ z : |z| < \varepsilon \} \), and let \( X = L^2(B_\varepsilon \times B_\varepsilon, dA \times dA) \). It is an immediate consequence of (3) that
\[
\|F_z\|_X^2 = \int_{B_\varepsilon \times B_\varepsilon} |f(z-\omega) - f(z)|^2 |g(z-u) - g(z)|^2 dA(u) \times dA(\omega) < \infty.
\]
By Lemma 5, \( F_z(u, \omega) = F_0(u, \omega) \) for all \( z \in \mathbb{C}^n \). That is,
(4) \[
(f(z-\omega) - f(z))(g(z-u) - g(z)) = (f(-\omega) - f(0))(g(-u) - g(0))
\]
for all \( z \in \mathbb{C} \), and \( u, \omega \in B_\varepsilon \). Divide with sides by \( u \omega \) and let \( u, \omega \to 0 \). Then above equation becomes
\[
f'(z)g'(z) = f'(0)g'(0).
\]
The following proof is similar to the Theorem 5 in [13]. If \( f'(0)g'(0) = 0 \), then at least one of \( f \) and \( g \) is constant. If \( f'(0)g'(0) \neq 0 \), then each of \( f'(z) \) and \( g'(z) \) is a non-vanishing function. By Lemma 2, Cauchy's integral formula and Weierstrass factorization of functions, there are constants \( a, b, c \) and \( d \) such that
\[
f'(z) = e^{az^2 + bz + c}, \quad g'(z) = e^{-az^2 - bz + d}.
\]
By (4) again,
\[
\int_{\gamma_1} f'(z)dz \times \int_{\gamma_2} g'(z)dz = \int_{\gamma_3} f'(z)dz \times \int_{\gamma_4} g'(z)dz,
\]
where \( \gamma_1 \) is any curve that begins at \( z \) and ends at \( z - \omega \), \( \gamma_2 \) is any curve that begins at \( z \) and ends at \( z - u \), \( \gamma_3 \) is any curve that begins at \( 0 \) and ends at \( \omega \), and \( \gamma_3 \) is any curve that begins at \( 0 \) and ends at \( -u \). By the proof of Theorem 5 in [13], we deduce that \( a \) must to be zero. Thus
\[
f'(z) = e^{bz + c}, \quad g'(z) = e^{-bz + d}.
\]
If \( b = 0 \), then \( f \) and \( g \) are linear polynomials. If \( b \neq 0 \), we can see that
\[
f(z) = \frac{1}{b}e^{bz + c} + C_1, \quad g(z) = -\frac{1}{b}e^{-bz + d} + C_2,
\]
where \( C_1, C_2 \) are constants. The proof is complete. \( \square \)

4. Semi-commuting Toeplitz Operators

In this section, we consider the semi-commuting Toeplitz operators with special symbols on \( F^{2,m} \).

**Proposition 11.** Suppose \( f \in D, m \) is a positive integer and \( a, b \in \mathbb{C} \).

1. If \( T_{f(z)} = T_{f(z)}T_z \), then \( f \) is a constant function.
2. If \( T_{e^{az+bz}} = T_{e^{az+bz}} \), then \( a = 0 \).
3. If \( T_{f(z)}T_{e^{az+bz}} = T_{f(z)}T_{e^{az+bz}} \), then \( f \) is a constant function or \( a = 0 \).

**Proof.** To prove (1). Since \( T_{f(z)}z = T_{f(z)}T_z \) and \( T_{f(z)}T_z 1 = 0 \). This shows that \( T_z f(z) = 0 \). It follow that \( \deg(f) < 1 \). Then \( f \) is a constant function. Similarly, \( a = 0 \) if (2) holds.

To prove (3). Let \( f(z) = \sum_{i=1}^{N} b_i K_m \left( z, \lambda_i \right) \), where \( N < \infty \). Without loss of generality, we assume
\[
a \neq 0, \quad b_i \neq 0, \quad \lambda_i \neq 0, \quad \lambda_i \neq \lambda_\kappa (i \neq \kappa)
\]
for \( 1 \leq i, \kappa \leq N \). By the proof of Theorem 6 in [10], we have
\[
\sum_{i=1}^{N} b_i T_{K_{(z, \lambda_i)}e^{az+bz}} \gamma_i = \sum_{i=1}^{N} b_i \sum_{k_i=0}^{\infty} \sum_{\eta=0}^{k_i+l} \frac{m!^{k_i}}{(k_i + m)! \eta!} \frac{(k_i + l + m)!}{(k_i + l - \eta + m)!} z^{k_i + l - \eta}
\]
and
\[
\sum_{i=1}^{N} b_i T_{K_{(z, \lambda_i)}T_{e^{az+bz}}} \gamma_i = \sum_{i=1}^{N} b_i \sum_{k_i=0}^{\infty} \sum_{\eta=0}^{l} \frac{m!^{k_i}}{(k_i + m)! \eta!} \frac{(l + m)!}{(l - \eta + m)!} z^{k_i + l - \eta}.
\]
Let $j = k_i + l - \eta$, we have
$$
\sum_{i=1}^{N} b_i \sum_{k_i = \max(0, j-l)}^{\infty} \frac{m!}{(k_i + m)!} \frac{(k_i + l + m)!}{(k_i + l - j)!} \lambda_i^{k_i} \bar{a}^{k_i} \bar{a}^{l-j}.
$$

$$
= \sum_{i=1}^{N} b_i \sum_{k_i = \max(0, j-l)}^{\infty} \frac{m!}{(k_i + m)!} \frac{(j + m)!}{(k_i + l - j)!} \frac{(l + m)!}{(j - k_i + m)!} \lambda_i^{k_i} \bar{a}^{k_i} \bar{a}^{l-j}.
$$

Assume $0 = l \leq j$, then
$$
\sum_{i=1}^{N} b_i \sum_{k_i = j}^{\infty} \frac{m!}{(k_i - j)!} \lambda_i^{k_i} \bar{a}^{k_i - j} = \sum_{i=1}^{N} b_i m! \lambda_i^j.
$$

That is,
$$(5) \quad \sum_{i=1}^{N} b_i \lambda_i^j \bar{a}^{m} = \sum_{i=1}^{N} b_i \lambda_i^j$$

for all $j \geq 0$. Define
$$
E_1 = \begin{pmatrix}
\lambda_1^{N-1} & \cdots & \lambda_N^{N-1} \\
\vdots & \ddots & \vdots \\
\lambda_1^{N-1} & \cdots & \lambda_N^{N-1}
\end{pmatrix}, \quad X_1 = \begin{pmatrix}
b_1 \\
\vdots \\
b_N
\end{pmatrix}.
$$

Using (5), we have $E_1 X_1 = 0$, which implies that $|E_1| = 0$. By the property of Vandermonde determinant,
$$
0 = |E_1| = \prod_{i=1}^{N} (e^{\alpha \lambda_i} - 1) \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j).
$$

Since $\{\lambda_i\}$ is a finite collection of distinct non-zero complex numbers, there is a $j$ so that
$$
e^{\alpha \lambda_j} - 1 = 0.
$$

By (5) again, we can obtain
$$
(6) \quad \lambda_i^{N,m} = 1
$$

for all $1 \leq i \leq N$.

If $l = j = 1$, then
$$
\sum_{i=1}^{N} b_i \sum_{k_i = 1}^{\infty} \frac{m!}{k_i!} (k_i + m + 1) \lambda_i^{k_i} \bar{a}^{k_i} = \sum_{i=1}^{N} b_i (m + 1)! \lambda_i \bar{a}.
$$

So we have
$$
\sum_{i=1}^{N} b_i \left( e^{\lambda_i} \lambda_i \bar{a} + (m + 1) \left( e^{\lambda_i} - 1 \right) \right) = \sum_{i=1}^{N} b_i (m + 1)! \lambda_i \bar{a}.
$$

Using (5),
$$
\sum_{i=1}^{N} b_i \lambda_i \bar{a} = \sum_{i=1}^{N} b_i (m + 1)! \lambda_i \bar{a}.$$
Since $m \neq 0$ and $a \neq 0$, the above equation becomes

(7) \[ \sum_{i=1}^{N} b_i \overline{\lambda_i} = 0. \]

Now assume $j \geq l = 2$, it follows that

\[
\sum_{i=1}^{N} b_i \sum_{k_i \geq j-2} \frac{m!}{(k_i + m)! (k_i + 2 - j)!} \overline{\lambda_i} k_i a^{k_i+2-j} = \sum_{i=1}^{N} b_i \frac{j}{k_i \geq j-2} \frac{m!}{(k_i + m)! (k_i + 2 - j)!} \overline{\lambda_i} a^{k_i+2-j}. \]

Then

\[
\sum_{i=1}^{N} \frac{m!}{2} b_i (m+1)(m+2) \overline{\lambda_i} a^2 + \sum_{i=1}^{N} b_i m!(j + m)(2 + m) \overline{\lambda_i}^{-1} a = \sum_{i=1}^{N} b_i \left\{ \sum_{j=1}^{N} \right\} m!(k_i + m + 1) (k_i + m + 2) \overline{\lambda_i}^{-1} a^{k_i+2-j} + m!(j + m)(j + m + 1) \overline{\lambda_i}^{-1} a \]

In fact,

(9) \[ (k + m + 1)(k + m + 2) = (k + 2 - j)(k + 1 - j) + 2(m + j)(k + 2 - j) + (m + j)(m + j - 1). \]

Using (7) and the fact $e^{\lambda_i a} = 1$ for all $1 \leq i \leq N$, one can see that

\[
\sum_{i=1}^{N} b_i \sum_{k_i \geq j+1} \frac{m!(k_i + m + 1)(k_i + m + 2)}{(k_i + 2 - j)!} \overline{\lambda_i}^{-1} a^{k_i+2-j} = \sum_{i=1}^{N} b_i m!(e^{\lambda_i a} - 1) \overline{\lambda_i} a^2 + \sum_{i=1}^{N} 2b_i m!(m + j)(e^{\lambda_i a} - 1 - \overline{\lambda_i} a) \overline{\lambda_i}^{-1} a^2 \] 

\[ + \sum_{i=1}^{N} b_i m!(j + m)(j + m - 1)(e^{\lambda_i a} - 1 - \overline{\lambda_i} a - \overline{\lambda_i}^2 a^2) \overline{\lambda_i}^{-1} a^2 \] 

\[ = \sum_{i=1}^{N} 2b_i m!(m + j) \overline{\lambda_i}^2 a^2 - \sum_{i=1}^{N} b_i m!(m + j)(m + j - 1) \overline{\lambda_i}^{-1} a^2 \] 

\[ - \frac{1}{2} \sum_{i=1}^{N} b_i m!(m + j)(m + j - 1) \overline{\lambda_i} a^2 = - \frac{1}{2} \sum_{i=1}^{N} b_i m!(m + j)(m + j + 3) \overline{\lambda_i} a^2 - \sum_{i=1}^{N} b_i m!(m + j)(m + j - 1) \overline{\lambda_i}^{-1} a^2. \]
Putting this in \( S \), we get
\[
\sum_{i=1}^{N} \frac{m_1}{2} b_i (m + 1)(m + 2) \lambda_i^j \bar{a}^2 + \sum_{i=1}^{N} b_i m! (j + m)(2 + m) \lambda_i^{j-1} \bar{a} \\
= - \frac{1}{2} \sum_{i=1}^{N} b_i m! (j + m + 1)(2 + m) \lambda_i^j \bar{a}^2 - \sum_{i=1}^{N} b_i m! (m + j)(2 + m) \lambda_i^{j-1} \bar{a} \\
+ \sum_{i=1}^{N} b_i m! (j + m + 1) \lambda_i^{j-1} \bar{a} \\
+ \frac{1}{2} \sum_{i=1}^{N} m! b_i (j + m + 1)(2 + m) \lambda_i^j \bar{a}^2 \\
= \sum_{i=1}^{N} b_i m! \lambda_i^j \bar{a}^2 + \sum_{i=1}^{N} 2 b_i m! (m + j) \lambda_i^{j-1} \bar{a}.
\]
So we have
\[
\sum_{i=1}^{N} b_i m! \left( 1 - \frac{(m + 1)(m + 2)}{2} \right) \lambda_i^j \bar{a}^2 \\
= \sum_{i=1}^{N} m! b_i (j + m) \lambda_i^{j-1}.
\]
It is easy to check that
\[
2 - (m + 1)(m + 2) \neq 0
\]
for all \( m > 0 \). This gives
\[
\sum_{i=1}^{N} b_i \lambda_i^j = \frac{m(j + m)}{\bar{a}^2 \left( 1 - \frac{(m + 1)(m + 2)}{2} \right)} \sum_{i=1}^{N} m! b_i (j + m) \lambda_i^{j-1}
\]
Setting \( j = 2 \) in \((11)\), it follows from \((7)\) that
\[
\sum_{i=1}^{N} b_i \lambda_i^2 = 0.
\]
Using iteration method in \((11)\), we can see that
\[
\sum_{i=1}^{N} b_i \lambda_i^j = 0
\]
for all \( j \geq 1 \). Since \( \{ \lambda_i \} \) is a finite collection of distinct non-zero complex numbers, then
\[
\left| \begin{array}{cccc}
\lambda_1 & \cdots & \lambda_N \\
\vdots & \vdots & \vdots \\
\lambda_1 & \cdots & \lambda_N
\end{array} \right| = \prod_{1 \leq i \leq N} \lambda_i \prod_{1 \leq k < i \leq N} (\lambda_i - \lambda_k) \neq 0.
\]
So, \( b_i = 0 \) for all \( 1 \leq i \leq N \). This contradiction shows that \( f \) is a constant or \( a = 0 \). This proof is complete. \( \Box \)
Define
\[ \mathcal{D} = D \cup \{ f(z) = ce^{az} + b : a, b, c \in \mathbb{C} \} \cup \{ f(z) = az + b : a, b \in \mathbb{C} \} \].

Next, we will consider the semi-commuting Toeplitz operator with the \( \mathcal{D} \)-symbol on \( F^{2,m} \).

**Theorem 12.** Suppose that \( m \) is a positive integer and \( f, g \in \mathcal{D} \). Then the following statements are equivalent.

(a) \( H_f^* H_g \neq 0 \).
(b) \( T_f T_g = T_g \).
(c) \( f \neq g \).
(d) At least one of \( f \) and \( g \) is a constant.

**Proof.** Suppose \( f \) and \( g \) are functions in \( \mathcal{D} \). Recall that
\[ H_f^* H_g = T_f - T_f T_g, \quad T_f = T_g. \]
It is easy to check that \( T_z^* T_z \neq T_z T_z \). Using Proposition 11 and the main result of [16] (Theorem 1 in this paper), we have
\[ H_f^* H_g = T_g - T_f T_g = 0, \quad f, g \in \mathcal{D} \]
if and only if one of \( f \) and \( g \) is a constant function if and only if \( f \neq g \). \( \square \)

5. BOUNDEDNESS OF HANKEL PRODUCTS

Now, we consider the boundedness of Hankel products on \( F^{2,m} \). Recall that
\[ \mathcal{D} = D \cup \{ f(z) = ce^{az} + b : a, b, c \in \mathbb{C} \} \cup \{ f(z) = az + b : a, b \in \mathbb{C} \}. \]
Here, \( D \) is the set of all finite linear combinations of kernel functions. Next, we will complete characterize the bounded Hankel product when \( f, g \in \mathcal{D} \).

**Theorem 13.** Suppose \( (f, g) \in \mathcal{D} \) and \( m > 0 \). Then \( H_f^* H_g \) is bounded on \( F^{2,m} \) if and only if at least one of the following statement hold.

(a) One of \( f \) and \( g \) is constant.
(b) There are constant \( a, b, c, d, A \) and \( B \) such that
\[ f(z) = ce^{az} + b + A, \quad g(z) = e^{-az} + d + B. \]
(c) Both \( f \) and \( g \) are linear polynomials.

**Proof.** The sufficiency is clear. We now prove the necessity. Suppose \( H_f^* H_g \) is bounded, then either \( T_f T_g, T_g \) are bounded or \( T_f T_g \) are unbounded. First, if \( H_f^* H_g = 0 \), then at least one of \( f \) and \( g \) is constant by Theorem 12. If \( T_f T_g \) are bounded and \( T_f T_g \neq T_f T_g \), it follows from the the main result of [8] that
\[ f(z) = ce^{az} + b, \quad g(z) = e^{-az} + d + B, \]
where \( a, b, c, d, A, B \in \mathbb{C} \).
Now, assume $T_1, T_2, T_3$ are distinct unbounded operators. To prove the Theorem, we only need to show that the Hankel product $H^* \varphi \psi$ is unbounded if (c) does not hold and $T_1, T_2, T_3$ are distinct unbounded operators. There are three cases for us to consider.

**case 1** Without loss of generality, we assume $f(z) = e^{az}, g(z) = e^{bz}$, $a \neq b$ and $a \neq -b$. Since $a \neq b$, we have $fg$ is unbounded on $\mathbb{C}$. By Lemma 8, there is a constant $C$ such that

(11) $\widetilde{fg}(z) = f(z)g(z) + C.$

Moreover,

(12) $T_\varphi f - g(0)f = C.$

In fact, (11) implies

(13) $T_\varphi f - T_\varphi T_\psi = C.$

Let $f(z) = \sum_{i=0}^{\infty} \frac{a_i}{i!} z^i$ and $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$.

Using (13), it is easy to see that

$$\sum_{i=0}^{\infty} \sum_{n \leq l+i} \frac{a_i}{i!} \frac{a_n}{n!} \frac{(l+i+m)!}{(l+i-n)!} \frac{(l+m)!}{i!} \frac{(l+m)!}{j!} \frac{(l-j)!}{(l-j)!} = \sum_{i=0}^{\infty} \sum_{n \leq l} \frac{a_i}{i!} \frac{a_n}{n!} \frac{(l+i+m)!}{(l-i-n)!} \frac{(l+m)!}{i!} (l-j)! \frac{(l+m)!}{j!} (l-j)!$$

Using the above equation and the proof of Theorem 8 in [16], we have

$$\sum_{i=\max\{0,j\}}^{\infty} \frac{(i+j)!}{i!} \frac{(l+j)!}{(l-j)!} \frac{(l+m)!}{i!} \frac{(l+m)!}{j!} \frac{(l+j)!}{(l-j)!} = \sum_{i=1}^{\infty} \frac{a_i}{i!} \frac{(i+j+m)!}{(i+j)!} \frac{(l+j)!}{(l-j)!} \frac{(l+m)!}{i!} \frac{(l+m)!}{j!} \frac{(l+j)!}{(l-j)!}$$

(14) $\sum_{i=1}^{\infty} \frac{a_{i+j}}{i!} (i+j)!.$

By (14) and the proof of Theorem 8 in [16] (see page 16), we have (14) cannot be true if $ab \neq 0$. This contradiction implies that at least one of $a$ and $b$ is zero. It follows from (12) that $C = 0$. So, $\widetilde{fg}(z) = f(z)g(z)$.

**case 2** Without loss of generality, we assume

$$f(z) = \sum_{i=1}^{N_1} a_i K_m(z, A_i) \quad \text{and} \quad g(z) = \sum_{\lambda=1}^{N_2} b_\lambda K_m(z, B_\lambda).$$
Here $N_1, N_2$ are finite positive integers, and $a_i, A_i, b, B_i \in \mathbb{C} \setminus \{0\}$. Suppose $\{A_i\}$ and $\{B_i\}$ are two finite collection of distinct complex numbers. By hypothesis and Lemma 6, $f$ and $g$ are non-vanishing function.

First, we claim $fg$ is unbounded on $\mathbb{C}$. If not, then by Liouville’s theorem, there is a constant $C$ so that $fg = C$. By Lemma 2 and Weierstrass factorization of functions, there are constants $a, b, c$ and $d$ such that

$$f(z) = e^{a z^2 + b z + c} \quad \text{and} \quad g(z) = e^{-a z^2 - b z + d},$$

where $e^{cd} = C$. Using Lemma 7, we can see that the above $f$ and $g$ are not in $D$. This contradiction shows that $fg$ is unbounded. So, $fg$ is bounded if and only if $f$ and $g$ are both bounded on $\mathbb{C}$. That is, $f$ and $g$ are constant functions. Then $H_f^* H_g = 0$.

Now, assume $fg$ is unbounded. By Lemma 8 again

$$(15) \quad \overline{f} g(z) = f(z) \overline{g(z)} + C \quad \text{and} \quad T_f - \overline{g(0)} = C.\$$

Using $(15)$, we obtain

$$< \overline{g(w)} f(w), K_m(w, z) > = < f(w), g(w) K_m(w, z) >\$$

$$(16) \quad = \sum_{i=1}^{N_1} a_i g(A_i) K_m(z, A_i) = \overline{g(0)} f(z) + C.\$$

So

$$\sum_{i=1}^{N_1} a_i (g(A_i) - \overline{g(0)}) K_m(z, A_i) = C.\$$

By Liouville’s theorem,

$$C = \sum_{i=1}^{N_1} a_i (g(A_i) - \overline{g(0)}).\$$

Using the above two equations, we have

$$\sum_{i=1}^{N_1} a_i (g(A_i) - \overline{g(0)})(K_m(z, A_i) - 1) = 0.\$$

By Lemma 9 $0 = g(0) - g(A_i)$ for all $1 \leq i \leq N_1$, since every $a_i \neq 0$. This implies that $C = 0$. It follows from $(15)$ that $\overline{f} g = f g$. By the main result of [16], at least one of $f$ and $g$ is a constant function.

**case 3** Without loss of generality, assume

$$f(z) = \sum_{i=1}^{N_1} a_i K_m(z, A_i) = \sum_{i=1}^{N_1} \sum_{k_i=0}^{\infty} a_i \frac{m!}{(k_i + m)!} A_i^{k_i} z^{k_i} \quad \text{and} \quad g(z) = e^{b z},$$

where $\{A_i\}$ is a finite collection of distinct non-zero complex numbers, $\{a_i\}$ is a finite collection of non-zero complex numbers, and $b$ is a non-zero constant. If $fg$ is
bounded, then there is a constant \( C \) so that \( f(z) = Ce^{-bz} \). By Lemma \( \ref{lem7} \) \( f(z) \neq e^{bz} \).

This shows that \( fg \) is unbounded.

Since \( fg \) is unbounded, then

\[
\widetilde{fg}(z) = f(z)g(z) + C \quad \text{and} \quad T_{\mathcal{F}}f - g(0)f = C
\]

by Lemma \( \ref{lem8} \) Using (10), we can obtain that there is a constant \( C \) such that

\[
\sum_{i=1}^{N_f} a_i g(A_i) K_m(z, A_i) = g(0) \sum_{i=1}^{N_i} a_i + C.
\]

This gives

\[
\sum_{i=1}^{N_i} a_i (g(A_i) - g(0))(K_m(z, A_i) - 1) = 0.
\]

By Lemma \( \ref{lem9} \) again, \( g(A_i) = g(0) \). Thus, \( C = 0 \). It follows that \( \widetilde{fg} = fg \). By Theorem \( \ref{thm12} \) \( f \) is constant function or \( b = 0 \).

**Case 4** Without loss of generality, assume

\[
f(z) = \sum_{i=1}^{N_i} \sum_{k=0}^{\infty} a_i \frac{m!}{(k_i + m)!} A_i^{k_i} z^{k_i}, \quad g(z) = bz,
\]

where \( \{A_i\} \) is a finite collection of distinct non-zero complex numbers, \( \{a_i\} \) is a finite collection of non-zero complex numbers, and \( b \) is a non-zero constant. For \( l \geq 1 \), we have

\[
(T_{\mathcal{F}}f - T_f T_{\mathcal{F}})e_l
\]

\[
= \sum_{i=1}^{N_i} \sum_{k=0}^{\infty} \frac{m! a_i}{(k_i + m)!} A_i^{k_i} \sqrt{\frac{(l + k_i - 1 + m)!}{(l + m)!}}
\]

\[
\times \left( \frac{(k_i + l + m)!}{(l + k_i - 1 + m)!} - \frac{(l + m)!}{(l + m - 1)!} \right) \epsilon_{k_i + l - 1}.
\]

Let \( L_{l+K-1} \) be the coefficient of \( \epsilon_{l+K-1} \), then

\[
L_{l+K-1} = \sum_{i=1}^{N_i} a_i b \frac{m! A_i^K}{(K + m)!} \sqrt{\frac{(l + K - 1 + m)!}{(l + m)!}}.
\]

If

\[
\sum_{i=1}^{N_i} a_i A_i^K = 0
\]

for all \( K \geq 2 \), then \( f \) is a constant function( see the proof of Theorem 6 in [16]).

So we assume there is an integer \( j \geq 2 \) so that \( \sum_{i=1}^{N_i} a_i A_i^{j - 1} \neq 0 \). It is easy to see that

\[
|L_{l+j-1}| \sim \sqrt{\frac{(l + j - 1 + m)!}{(l + m)!}}.
\]
where the symbol $\sim$ means that the ratio of the two terms involved has a positive finite limit as $l \to \infty$. So, $L_{l+j-1} \to \infty$ as $l \to \infty$. Therefore, $H_f^* H_g$ is unbounded. The other cases can be directly obtained from the above four cases. Thus, $H_f^* H_g$ is unbounded if $T_f g$, $T_f T_g$ are distinct unbounded operators and (c) does not hold. The proof is complete.

Corollary 14. Suppose $(f, g) \in \mathcal{D}$ and $m > 0$. Then $H_f^* H_g$ is bounded on $F^{2,m}$ if and only if $E(f,g)$ is bounded on $\mathbb{C}$. Moreover, $H_f^* H_g$ is compact on $F^{2,m}$ if and only if $E(f,g) = 0$ if and only at least one of $f$ and $g$ is a constant function.

Using Theorem 13 and Corollary 14, we have the following result. Note that the degree of $f, g$ are more than 2 means that $f$ and $g$ are not linear holomorphic polynomials.

Corollary 15. Suppose $(f, g) \in \mathcal{D}$ and $m > 0$. If $T_f T_g$ is unbounded and the degree of $f, g$ are more than 2. Then the following statements are equivalent.
(a) $H_f^* H_g$ is bounded on $F^{2,m}$.
(b) $H_f^* H_g$ is compact on $F^{2,m}$.
(c) $f g = f g$.
(d) $E(f, g) = 0$.
(e) at least one of $f$ and $g$ is a constant function.

Proof. This follows from the proof of Theorem 13.

6. Open problems and remarks

As is well known, the Fock-sobolev space is the Fock space with the radial weight. So, some of them share the same structures, the same properties. For instance, Sarason’s problem, the boundedness and compactness of composition operators, the commuting Toeplitz operators with radial symbols, Ha-plitz product and so on, see [4, 7, 8, 9, 10, 12, 14, 17].

However, Proposition 11 and Theorem 13 show that $H_f^* H_g$ is not only not equal to 0, but also unbounded on the Fock-Sobolev spaces($m > 0$). In fact, the commutativity of Toeplitz operators with general symbols on Fock spaces is still an open problem, so it is difficult to complete characterize the bounded Hankel products on Fock spaces. There are two natural problems, which arise from the main results of [3, 13, 16].

Problem 1. Suppose $f$ and $g$ are function in $F^{2,m}$. Determine the $f$ and $g$ for which the Hankel product $H_f^* H_g = (T_f, T_g) = 0$.

Problem 2. Suppose $f$ and $g$ are function in $F^{2,m}$. Determine the $f$ and $g$ for which the Hankel product $H_f^* H_g = (T_f, T_g)$ is bounded on $F^{2,m}$. 
Remark 16. In [16] and this paper, we obtained some results of semi-commuting Toeplitz operator on $F^2_m$ by direct calculation. This gives the fundamental difference between the geometries of Fock and Fock-Sobolev space. Note that the hypothesis $m \neq 0$ is necessary, since we need the initial value.

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