Measure-valued affine and polynomial diffusions

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Abstract

We introduce a class of measure-valued processes, which – in analogy to their finite dimensional counterparts – will be called measure-valued polynomial diffusions. We show the so-called moment formula, i.e. a representation of the conditional marginal moments via a system of finite dimensional linear PDEs. Furthermore, we characterize the corresponding infinitesimal generators and obtain a representation analogous to polynomial diffusions on $\mathbb{R}_+^m$, in cases where their domain is large enough. In general the infinite dimensional setting allows for richer specifications strictly beyond this representation. As a special case we recover measure-valued affine diffusions, sometimes also called Dawson-Watanabe superprocesses. From a mathematical finance point of view the polynomial framework is especially attractive as it allows to transfer the most famous finite dimensional models, such as the Black-Scholes model, to an infinite dimensional measure-valued setting. We outline in particular the applicability of our approach for term structure modeling in energy markets.

Keywords: measure-valued processes; polynomial and affine diffusions; Dawson-Watanabe type superprocesses; martingale problem; maximum principle; HJM term structure modeling; energy markets

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We introduce a class of (non-negative) measure-valued processes, which we call in analogy to their finite dimensional counterparts measure-valued polynomial diffusions. In spirit of Cuchiero et al. (2019), where the focus was on probability measures, we now transfer the defining property of finite dimensional polynomial processes considered e.g. in Cuchiero et al. (2012); Filipović and Larsson (2016); Larsson and Pulido (2017); Cuchiero et al. (2018) to the state space of (non-negative) measures.

This state space has high relevance for applications since stochastic modeling of infinite dimensional non-negative quantities occurs in many areas. This applies especially to biology and population genetics where measure-valued processes have played a crucial role since the 1970’s, when branching Brownian motion was introduced by Henry McKean, see McKean (1975). Also the polynomial property occurs in this field often naturally,
see, e.g., Blath et al. (2016) for an example of a recently investigated two-dimensional polynomial process.

Similarly to finite dimensional polynomial processes on non-negative state spaces which have been applied in many areas of finance, like credit risk, stochastic volatility, or life insurance liability modeling (see, e.g., Ackerer and Filipović (2020); Ackerer et al. (2018); Arrouy et al. (2020); Biagini and Zhang (2016)), measure-valued polynomial processes are also very well suited for modeling in mathematical finance. Indeed, infinite dimensional non-negative processes are of particular interest in view of term structures as we shall illustrate below by analyzing forward price models in energy markets. Another instance of measure-valued processes concerns Markovian lifts of stochastic Volterra processes (see, e.g., Abi Jaber (2019); Cuchiero and Teichmann (2019, 2020)), which became recently very popular in view of rough volatility modeling initiated by Gatheral et al. (2018). Stochastic portfolio theory, introduced in Fernholz (2002) (see also Fernholz and Karatzas (2009)), is another area of applications where one deals with potentially high or infinite dimensional markets. Hence a measure-valued analog of the finite dimensional setup considered e.g. in Fernholz and Karatzas (2005); Cuchiero (2019) is of interest to study asymptotic relative arbitrages. Measure-valued processes can also be used as state processes for stochastic optimal control problems with applications including model-independent derivatives pricing and two player games with asymmetric information (see, e.g., Cox et al. (2021a)).

The advantage of using measure-valued processes instead of function-valued ones is that many spatial stochastic processes do not fall into the framework of stochastic partial differential equations (SPDEs). Indeed, it is often easier to establish existence of a measure-valued process than of an analogous stochastic partial differential equation, say in some Hilbert space, which would for instance correspond to its Lebesgue density, but which does not necessarily exist. For an introduction, important concepts and results related to measure-valued processes we refer to Etheridge (2000); Dawson (1993) and Li (2010). For recent results in the probability measure valued case on generalized Wasserstein spaces see Larsson and Svaluto-Ferro (2020).

Beyond the general suitability of measure-valued processes for infinite dimensional dynamic modeling, the current polynomial framework allows for high tractability and the possibility to transfer many famous finite dimensional models, such as the Black-Scholes model or the Feller diffusion, to an infinite dimensional measure-valued setting. It offers a unified treatment of a large class of processes that includes these well-known examples but also goes far beyond them as we shall illustrate in Theorem 5.9.

Let us now describe the precise setup in more detail. We denote by $X$ measure-valued polynomial diffusions taking values in the space of (non-negative) measures on a locally compact Polish space $E$. We define them as solutions of martingale problems with continuous trajectories for certain operators $L$ acting on classes of cylindrical functions, i.e. functions $f$ of the form

$$f(\nu) = \phi \left( \int_E g_1(x) \nu(dx), \ldots, \int_E g_m(x) \nu(dx) \right),$$

where $\phi \in C^\infty(\mathbb{R}^m)$, $g_1, \ldots, g_m$ are continuous and bounded, and the argument $\nu$ is a (non-negative) measure. The defining property of a measure-valued polynomial diffusion is that when $L$ is applied to a cylindrical polynomial, i.e. $\phi$ is a polynomial on $\mathbb{R}^m$, then $Lf$ is again a polynomial with the same degree as $f$. Precise definitions are given in
Section 4. For related concepts in general Banach spaces we refer to the inspiring paper Benth et al. (2020).

As a consequence of the defining property of polynomial diffusion we can then prove the following results:

- We show that the conditional marginal moments of measure-valued polynomial diffusions can be represented via a system of finite dimensional linear PDEs, whose (maximal) spatial dimension corresponds to the degree of the moment, see Theorem 5.5. This means that the so-called moment formula holds and that the tractability of finite dimensional polynomial processes can be preserved in this setting. Indeed, for polynomial terminal conditions the infinite dimensional Kolmogorov backward equation can be reduced to finite dimensional system of PDEs.

- We give necessary and sufficient conditions for the existence of measure-valued polynomial diffusions by analyzing the corresponding martingale problem. When the domain of the respective operators is large enough, we obtain a representation of the drift and diffusion part analogous to polynomial diffusions on $\mathbb{R}^m_+$, analyzed in Filipović and Larsson (2016), see Theorem 5.11. Otherwise the infinite dimensional setting allows for much more generality by going strictly beyond this representation, see Theorem 5.9.

- One key ingredient in these proofs is a characterization of the positive maximum principle of the respective operators. To this end we also obtain new optimality conditions which are applicable for all martingale problems on the space of (non-negative) measures, see Section 3.

- By specifying affine type operators in Section 6 we recover Dawson-Watanabe-type superprocesses as the affine subclass of polynomial diffusions. In this case, additionally to the moment formula, the Laplace transform of the process’ marginals is exponentially affine in the initial state. As well-known the corresponding characteristic exponent can then be computed by solving the associated Riccati partial differential equations (see, e.g., Li (2010)). Let us here also mention that infinite dimensional affine processes, however with values in Hilbert spaces, have recently been studied in Schmidt et al. (2020); Cox et al. (2021b).

- We also show under which conditions exponential moment exist, so that uniqueness in law holds and the corresponding martingale problems are well-posed, see Theorem 5.15. This condition is satisfied by the subclass of measure-valued affine diffusions, whose marginal laws are then characterized by the solution of the associated Riccati partial differential equations.

- We provide several concrete specifications in Section 7. Indeed, we consider as underlying space $E$ a finite set of points and recover thus the results of Filipović and Larsson (2016). Moreover, we analyze in detail the case when $E \subseteq \mathbb{R}$, exploiting the characterization of strongly continuous positive groups provided in Arendt et al. (1986), which allows us to specify a concrete and easily applicable form of the infinitesimal generator. We also connect our findings with measure-valued analogs of Pearson diffusions (see, e.g., Forman and Sørensen (2008)) and the Black-Scholes model.
This shows in particular how to construct measure-valued versions of the most famous finite dimensional models in finance and hence why measure-valued affine and polynomial diffusions qualify as tractable models for all high dimensional financial markets potentially involving an infinite number of assets. As already mentioned above, this concerns especially term structures arising from fixed income markets with possibly multiple yield curves (see, e.g., Cuchiero et al. (2016a)), equity derivatives (see Kallsen and Krühner (2015) and the references therein) or forward contracts in energy markets (see Benth et al. (2008); Benth and Krühner (2014)). Time to maturity (or the strike dimension in case of equity derivatives) then takes the role of the spatial structure.

Our results can most notably be used for energy market modeling, in particular for electricity and gas markets, whose essential products are futures and forward contracts as well as options written on these. We concentrate on futures (also called swaps) with delivery over a time interval $[\tau_1, \tau_2]$. Their price at time $t \geq 0$ is denoted by $F(t, \tau_1, \tau_2)$ at $t \leq \tau_1$. Following Benth et al. (2008), $F(t, \tau_1, \tau_2)$ can be written as a weighted integral of instantaneous forward prices $f(t, u)$ with delivery at time $\tau_1 \leq u \leq \tau_2$, i.e.

$$F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2) f(t, u) du,$$

where $w(u, \tau_1, \tau_2)$ denotes some weight function. The crucial reason for using measure-valued processes in electricity, whose essential products are futures and forward contracts as well as options written on these. We concentrate on futures (also called swaps) with delivery over a time interval $[\tau_1, \tau_2]$. Their price at time $t \geq 0$ is denoted by $F(t, \tau_1, \tau_2)$ at $t \leq \tau_1$. Following Benth et al. (2008), $F(t, \tau_1, \tau_2)$ can be written as a weighted integral of instantaneous forward prices $f(t, u)$ with delivery at time $\tau_1 \leq u \leq \tau_2$, i.e.

$$F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} w(u, \tau_1, \tau_2) f(t, u) du,$$

where $w(u, \tau_1, \tau_2)$ denotes some weight function. The crucial reason for using measure-valued processes in electricity and gas modeling now comes from the fact that there is no trading with the instantaneous forwards $f(t, u)$ for obvious reasons. Thus, rather than using $f(t, u) du$ in the expression of the future prices, we can also use a measure, which cannot necessarily be evaluated pointwise. In the companion paper Cuchiero et al. (2021) we establish a Heath-Jarrow-Morton (HJM) approach (see Heath et al. (1992)) with measure-valued processes instead of processes taking values in some Hilbert space of functions. We then obtain a HJM-drift condition that restricts the choice of the process since the drift part is completely determined, but we are free to specify its martingale part as long as we do not leave the state space of (non-negative) measures. Here, the full specification of affine and polynomial diffusions provided in Section 5 comes into play, since it allows us to obtain a rich and flexible class of models while preserving tractability. Indeed, the martingale part given by $Q_1$ and $Q_2$ in Theorem 5.9 depends on continuous functions satisfying certain admissibility conditions. For calibration and pricing purposes these functions can then be parametrized by neural networks. This in turn leads to fast calibration procedures where on the one hand the analytic tractability coming from the affine and polynomial nature and on the other hand (stochastic) gradient descent methods for neural networks can be exploited.

The remainder of the paper is organized as follows. In Section 1.1 we introduce frequently used notation and basic definitions. In Section 2 we define polynomials and cylindrical functions of measure arguments, while in Section 3 we prove optimality conditions for these function, which we need to apply the positive maximum principle later on. In Section 4 we introduce the notion of polynomial operators in the current setting and characterize them as second order differential operator with affine drift and quadratic diffusion part. Section 5 contains the main results, i.e. the moment formula, sufficient conditions for existence and uniqueness and a full characterization when the domain of the infinitesimal generator is rich enough. Section 6 is dedicated to the analysis of the affine subclass and Section 7 concludes with exemplary specifications. In Appendix A
we prove necessary conditions for the positive maximum principle, that in turn implies existence of solutions to the martingale problem, see Appendix B. In Appendix C we recall a characterization of generators of strongly continuous positive groups proved in Arendt et al. (1986).

1.1 Notation and basic definitions

Throughout this article, \( E \) is a locally compact Polish space endowed with its Borel \( \sigma \)-algebra.

- \( M_+(E) \) denotes the set of finite non-negative measures on \( E \), and \( M(E) = M_+(E) - M_+(E) \) the space of signed measures of the form \( \nu_+ - \nu_- \) with \( \nu_+, \nu_- \in M_+(E) \). We usually leave “non-negative” away and just say finite measures for \( M_+(E) \). Both \( M(E) \) and \( M_+(E) \) are equipped with the topology of weak convergence, which turns \( M_+(E) \) into a Polish space. For \( \mu, \nu \in M(E) \) we write \( \mu \leq \nu \) if \( \nu - \mu \in M_+(E) \) and \( |\nu| \) for \( \nu_+ + \nu_- \).

- \( C(E), C_b(E), C_0(E), C_c(E) \) denote, respectively, the spaces of continuous, bounded, vanishing at infinity, and compactly supported real functions on \( E \). We equip the latter three with the topology of uniform convergence, and denote by \( \| \cdot \| \) the supremum norm.

- If \( E \) is non-compact, then \( E^\Delta = E \cup \{ \Delta \} \) denotes the one-point compactification, i.e. a compact Polish space. If \( E \) is compact we write \( E^\Delta = E \). We also define

\[
C_\Delta(E^k) := \{ f_{|E^k} : f \in C((E^\Delta)^k) \},
\]

which is a closed subspace of \( C_b(E^k) \). The spaces \( C_\Delta(E) \) and \( C(E^\Delta) \) can be identified, and we sometimes regard elements of the former as elements of the latter, and vice versa. When \( E \) is compact, \( C(E) = C_b(E) = C_0(E) = C_c(E) = C_\Delta(E) \) holds true and we thus simply write \( C(E) \). Note that the constant function 1 lies in \( C_\Delta(E) \), but of course not in \( C_0(E) \). This is one reason why we introduce the function space \( C_\Delta(E^k) \).

- One important underlying space, taking in the above definition the role of \( E \), will be \( M_+(E) \) if \( E \) is compact, and \( M_+(E^\Delta) \) otherwise. Recall that \( M_+(E) \) is locally compact when \( E \) is compact (see, e.g., (Luther, 1970, Remark 1.2.3)), which however does not hold true when \( E \) is non-compact. Therefore we shall consider \( M_+(E^\Delta) \) in the non-compact case. Identifying \( E^\Delta \) with \( E \), when \( E \) is compact, we just write \( M_+(E^\Delta) \) for both cases. In order to distinguish between the different one-point compactifications, we denote the one-point compactification of \( M_+(E^\Delta) \) by \( M^\Delta_+(E^\Delta) \) and identify it with \( C_\Delta(E) \) and \( M_+(E^\Delta) \) are defined analogously to (1.1) and can be identified with \( C(M^\Delta_+(E^\Delta)) \).

- \( \hat{C}_\Delta(E^k) \) is the closed subspace of \( C_\Delta(E^k) \) consisting of symmetric functions \( f \), i.e., \( f(x_1, \ldots, x_k) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \) for all \( \sigma \in \Sigma_k \), the permutation group of \( k \) elements. \( \hat{C}_0(E^k) \) and \( \hat{C}(E^k) \) are defined similarly. For any \( g \in \hat{C}_\Delta(E^k), h \in \hat{C}_\Delta(E^\ell) \) we denote by \( g \otimes h \in \hat{C}_\Delta(E^{k+\ell}) \) the symmetric tensor product, given by
(g ⊗ h)(x_1, \ldots, x_{k+\ell})
= \frac{1}{(k+\ell)!} \sum_{\sigma \in \Sigma_{k+\ell}} g(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) h(x_{\sigma(k+1)}, \ldots, x_{\sigma(k+\ell)}). \quad (1.2)

We emphasize that only symmetric tensor products are used in this paper.

- Throughout the paper we let
  \[ D \subseteq C_\Delta(E) \]
  be a dense linear subspace containing the constant function 1. We set \( D^\otimes 2 = D \otimes D := \text{span}\{g \otimes g : g \in D\} \). This generalizes of course to \( D^\otimes k \) for \( k > 2 \).

Two key notions that we shall often use are the positive maximum principle, and the positive minimum principle for certain linear operators.

**Definition 1.1.** Fix a Polish space \( \mathcal{X} \) and a subset \( S \subseteq \mathcal{X} \). Moreover, let \( D(\mathcal{A}) \) be a linear subspace of \( C(\mathcal{X}) \).

- An operator \( \mathcal{A} \) with domain \( D(\mathcal{A}) \) is said to satisfy the positive maximum principle on \( S \) if
  \[ f \in D(\mathcal{A}), x \in S, \sup_{S} f = f(x) \geq 0 \quad \text{implies} \quad \mathcal{A}f(x) \leq 0. \]

- An operator \( \mathcal{A} \) with domain \( D(\mathcal{A}) \) is said to satisfy the positive minimum principle on \( S \) if
  \[ 0 \leq g \in D(\mathcal{A}), x \in S, \inf_{S} g = g(x) = 0 \quad \text{implies} \quad \mathcal{A}g(x) \geq 0. \]

**Remark 1.2.** Note that the positive maximum principle implies the positive minimum principle. Indeed, let \( g \geq 0 \) with \( \inf_{S} g = g(x) = 0 \) and set \( f = -g \) (which is possible since \( D \) is a linear space). Then the positive maximum principle yields \( \mathcal{A}f(x) = -\mathcal{A}g(x) \leq 0 \).

Moreover, let \( \mathcal{A} \) be an operator with domain \( D(\mathcal{A}) \) satisfying the positive minimum principle on \( S \) and suppose that \( 1 \in D(\mathcal{A}) \). Then \( \mathcal{A}g = \mathcal{B}g + mg \) for some map \( m \) and some operator \( \mathcal{B} \) satisfying \( \mathcal{B}1 = 0 \) and the positive maximum principle on \( S \). \( \mathcal{B} \) and \( m \) can be explicitly constructed by setting \( \mathcal{B}g(x) = \mathcal{A}(g - g(x))(x) \) and \( m(x) := \mathcal{A}1(x) \).

These notions shall play an important role throughout the paper. Indeed, the positive maximum principle (combined with conservativity) is essentially equivalent to the existence of an \( S \)-valued solution to the martingale problem for \( \mathcal{A} \), see (Ethier and Kurtz, 2009, Theorem 4.5.4). Here it is crucial that \( S \) is locally compact. In our setting the most important case is \( S = M_+(E^\Delta) \) which is locally compact.

The positive minimum principle will be used to characterize certain operators appearing in the generator of measure-valued polynomial diffusions. Note here that for generators of strongly continuous semigroups on \( C(\mathcal{X}) \) the positive minimum principle is equivalent to generating a positive semigroup, if \( \mathcal{X} \) is compact, see (Arendt et al., 1986, Theorem B.II.1.6).

Finally, we indicate bounded pointwise limits by \( \text{bp-lim} \) in the sense of (Ethier and Kurtz, 2009, Appendix 3).
2 Polynomials and cylindrical functions of measure arguments

In this section we recall the notion of polynomials of measure arguments similarly as in Cuchiero et al. (2019).

2.1 Monomials and polynomials

A monomial on $M(E)$ is defined via

$$\langle g, \nu^k \rangle = \int_{E^k} g(x_1, \ldots, x_k) \nu(dx_1) \cdots \nu(dx_k)$$

for some $k \in \mathbb{N}_0$, where $g \in \hat{C}_\Delta(E^k)$ is referred to as the coefficient of the monomial; see, e.g., (Dawson, 1993, Chapter 2). We identify $\hat{C}_\Delta(E^0)$ with $\mathbb{R}$, so that for $k = 0$ we have $\langle g, \nu^k \rangle = g \in \mathbb{R}$. The map $\nu \mapsto \langle g, \nu^k \rangle$ is clearly homogeneous of degree $k$, and $g \mapsto \langle g, \nu^k \rangle$ is linear. Furthermore, the identity $\langle g, \nu^k \rangle \langle h, \nu^\ell \rangle = \langle g \otimes h, \nu^{k+\ell} \rangle$, holds true, where the symmetric tensor product $g \otimes h$ is defined in (1.2).

A polynomial on $M(E)$ is now defined as a (finite) linear combination of monomials,

$$p(\nu) = \sum_{k=0}^{m} \langle g_k, \nu^k \rangle,$$  \hspace{1cm} (2.1)

with coefficients $g_k \in \hat{C}_\Delta(E^k)$, and we shall denote the coefficients vector $(g_0, \ldots, g_m)$ of a polynomial by $\hat{g} = (g_0, \ldots, g_m) \in \bigoplus_{k=0}^{m} \hat{C}_\Delta(E^k)$.

The degree of a polynomial $p$, denoted by $\deg(p)$, is the largest $k$ such that $g_k$ is not the zero function, and $-\infty$ if $p$ is the zero polynomial. The representation (2.1) is unique; see Corollary 2.4 in Cuchiero et al. (2019).

The following example shows that the notion of polynomials coincides with the usual one on $\mathbb{R}^m$ when $E$ is a finite set with $m$ elements.

Example 2.1. Let $E = \{1, \ldots, m\}$ be a finite set. Then every element $\nu \in M(E)$ is of the form

$$\nu = c_1 \delta_1 + \cdots + c_m \delta_m, \quad (c_1, \ldots, c_m) \in \mathbb{R}^m,$$

where $\delta_i$ is the Dirac mass concentrated at $\{i\}$. Monomials take the form

$$\langle g, \nu^k \rangle = \sum_{i_1, \ldots, i_k} g(i_1, \ldots, i_k) c_{i_1} \cdots c_{i_k},$$

where the summation ranges over $E^k = \{1, \ldots, m\}^k$. Therefore, as $g$ can be any symmetric function on $E^k$, we recover all homogeneous polynomials of total degree $k$ in the $m$ variables $c_1, \ldots, c_m$.

In the following we define the function space of polynomials on $M(E)$. 

Definition 2.2. Let
\[ P := \{ \nu \mapsto p(\nu) : p \text{ is a polynomial on } M(E) \} \]
denote the algebra of all polynomials on \( M(E) \) regarded as real-valued maps, equipped with pointwise addition and multiplication.

The subsequent lemma asserts that polynomials on \( M(E) \) can be uniquely extended to polynomials on \( M(E^\Delta) \), which we shall apply regularly. For its proof we refer to Lemma 2.3.1 in Cuchiero et al. (2019)

Lemma 2.3. Each \( p \in P \) is continuous on \( M_+(E) \), sequentially continuous on \( M(E) \), and can be uniquely extended to a polynomial on \( M(E^\Delta) \).

2.2 Directional derivatives

The notion of derivatives of a function on \( M(E) \) that we apply throughout the paper is a directional one. More precisely, a function \( f : M(E) \to \mathbb{R} \) is called differentiable at \( \nu \) in direction \( \delta_x \) for \( x \in E \) if
\[
\partial_x f(\nu) := \lim_{\varepsilon \to 0} \frac{f(\nu + \varepsilon \delta_x) - f(\nu)}{\varepsilon}
\]
exists. We write \( \partial f(\nu) \) for the map \( x \mapsto \partial_x f(\nu) \), and we use the notation
\[
\partial_{x_1x_2\cdots x_k} f(\nu) := \partial_{x_1} \partial_{x_2} \cdots \partial_{x_k} f(\nu)
\]
for iterated derivatives. We write \( \partial^k f(\nu) \) for the corresponding map from \( E^k \) to \( \mathbb{R} \). Observe that for a linear map \( p \in P \) of the form \( p(\nu) = \langle g, \nu \rangle \) we get
\[
\partial_x p(\nu) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int g(y) \varepsilon \delta_x(dy) = g(x)
\]
for each \( x \in E \).

2.3 Cylindrical polynomials with regular coefficients

In order to be able to consider certain differential operators later on, let us introduce subspaces of polynomials with more regular coefficients. Let \( \text{Pol}(\mathbb{R}^m) \) denote the set of polynomials on \( \mathbb{R}^m \) and recall that \( D \subseteq C_\Delta(E) \) is a dense linear subspace containing the constant function 1. We then call cylindrical polynomial an element of the set
\[ P^D := \left\{ \nu \mapsto p(\nu) = \phi(\langle g_1, \nu \rangle, \ldots, \langle g_m, \nu \rangle) : \phi \in \text{Pol}(\mathbb{R}^m), g_k \in D, m \in \mathbb{N}_0 \right\}. \]
Note that \( P^D \) is the subalgebra of \( P \) consisting of all (finite) linear combinations of the constant polynomial and “rank-one” monomials \( \langle g \otimes \cdots \otimes g, \nu^k \rangle = \langle g, \nu \rangle^k \) with \( g \in D \).

\[ ^1 \text{It can be shown that sequential continuity cannot be strengthened to continuity.} \]
Since for \( g_{k_1} \in \hat{C}_\Delta(E^{k_1}) \) and \( g_{k_2} \in \hat{C}_\Delta(E^{k_2}) \), it holds \( \langle g_{k_1}, \nu^{k_1} \rangle \langle g_{k_2}, \nu^{k_2} \rangle = \langle g_{k_1} \otimes g_{k_2}, \nu^{k_1+k_2} \rangle \), we can equivalently write
\[
P^D = \left\{ \nu \mapsto \sum_{k=0}^{m} \langle g_k, \nu^k \rangle : m \in \mathbb{N}_0, \ g_k \in D^{\otimes k} \right\},
\]
where \( D^{\otimes k} \) was introduced right after (1.2). For the sake of completeness, let us restate parts of Lemma 2.7 of Cuchiero et al. (2019), showing that the derivative of \( x \mapsto \partial_x p(\nu) \) has the same regularity as the coefficients of \( p \), the latter being our main motivation to introduce \( P^D \).

**Lemma 2.4.** For any \( p \in P^D \) and \( \nu \in M(E) \), we have \( \partial^k p(\nu) \in D^{\otimes k} \).

### 2.4 Cylindrical functions

Similarly to cylindrical polynomials we also consider cylindrical functions. Define
\[
F^D = \left\{ \nu \mapsto f(\nu) = \phi(\langle g_1, \nu \rangle, \ldots, \langle g_m, \nu \rangle) : \phi \in C^\infty_0(\mathbb{R}^m), g_k \in D, \ m \in \mathbb{N}_0 \right\}.
\]
Since concatenations of continuous functions are continuous, cylindrical functions satisfy the following lemma.

**Lemma 2.5.** Each \( f \in F^D \) is continuous on \( M_+(E) \), sequentially continuous on \( M(E) \), and can be uniquely extended to \( M(E^\Delta) \).

For further use we now define the restriction of \( F^D \) to \( M_+(E^\Delta) \), i.e.
\[
F^D(M_+(E^\Delta)) := \{ f|_{M_+(E^\Delta)} : f \in F^D \}.
\]
Observe that functions in \( F^D \) are first extended to \( M(E^\Delta) \) using Lemma 2.5 and then restricted to \( M_+(E^\Delta) \). Since elements in \( F^D(M_+(E^\Delta)) \) do not need to be compactly supported we also define
\[
F^D_c := F^D(M_+(E^\Delta)) \cap C_c(M_+(E^\Delta)).
\]

**Lemma 2.6.** For any \( f \in F^D \) and \( \nu \in M(E) \), we have \( \partial^k f(\nu) \in D^{\otimes k} \). Moreover \( F^D_c \) is dense in \( C_0(M_+(E^\Delta)) \).

**Proof.** For \( f(\nu) := \langle g, \nu \rangle \) we have \( \partial^k f(\nu) = g^{\otimes k} \in D^{\otimes k} \). The first part follows by the chain rule. Since \( F^D_c \) is a point separating algebra that vanishes nowhere, the locally compact version of the Stone–Weierstrass theorem yields the second part.

Finally, we introduce a set that will be used as domain for linear operators:
\[
\mathcal{D} := \text{span}\{ P^D, F^D \}.
\] (2.2)
Observe that we here do not specify the Banach space containing \( \mathcal{D} \).
3 Optimality conditions

We now develop optimality conditions for cylindrical functions and polynomials of measure arguments, which we need in order to apply the positive maximum principle on $M_+(E^\Delta)$. Our first result, Theorem 3.1, extends the classical first and second order Karush–Kuhn–Tucker conditions for functions on $\mathbb{R}^n_+$ (see, e.g., Bertsekas (1997)).

Throughout the section we let $\mathcal{D}$ be the set defined in (2.2). Note that we use Lemma 2.4 and Lemma 2.5 to extend the polynomials and cylindrical functions from $M_+(E)$ to $M_+(E^\Delta)$.

Theorem 3.1. Let $f \in \mathcal{D}$ and $\nu^*$ satisfy $f(\nu^*) = \max_{\nu \in M_+(E^\Delta)} f(\nu)$. Then the following first and second order optimality conditions are satisfied.

(i) For all $\mu, \nu \in M(E^\Delta)$ with supp$(\mu) \subseteq$ supp$(\nu^*)$, it holds that $\langle \partial f(\nu^*), \mu \rangle = 0$ and $\langle \partial f(\nu^*), \nu \rangle \leq 0$.

In particular, $\partial_x f(\nu^*) \leq 0$ for all $x \in E^\Delta$ and vanishes for all $x \in$ supp$(\nu^*)$.

(ii) For all $\mu \in M(E^\Delta)$ with supp$(\mu) \subseteq$ supp$(\nu^*)$, it holds that $\langle \partial^2 f(\nu^*), \mu^2 \rangle \leq 0$.

In particular, $\partial^2_{xx} f(\nu^*) \leq 0$ for all $x \in$ supp$(\nu^*)$.

Proof. (i): Let $x \in$ supp$(\nu^*)$. We shall first consider the following perturbation $\nu^* - \varepsilon_n \nu_n$ with $(\nu_n)_{n \in \mathbb{N}}$ a sequence of non-negative measures converging to $\delta_x$ and $\varepsilon_n > 0$ converging to 0. To this end we need to construct $(\nu_n)_{n \in \mathbb{N}}$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\nu^* - \varepsilon_n \nu_n$ is a non-negative measure for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $A_n$ be the ball of radius $\frac{1}{n}$ centered at $x$, intersected with supp$(\nu^*)$. Since $A_n \subseteq$ supp$(\nu^*)$, we have $\nu^*(A_n) > 0$ and the measures $\nu_n := \nu^*(\cdot \cap A_n)/\nu^*(A_n)$ converge weakly to $\delta_x$ as $n \to \infty$. Let $\varepsilon_n \in (0, \nu^*(A_n))$ and note that for all $B \in \mathcal{B}(E^\Delta)$

$$\nu^*(B) - \varepsilon_n \frac{\nu^*(B \cap A_n)}{\nu^*(A_n)} \geq \nu^*(B \cap A_n) (\nu^*(A_n) - \varepsilon_n) \geq 0,$$

where the last inequality follows from the fact that $\varepsilon_n \in (0, \nu^*(A_n))$. Hence $\nu^* - \varepsilon_n \nu_n$ is a non-negative measure. Moreover, using that $\nu_n$ is a probability measure, we can compute

$$\|(g_1, \varepsilon_n \nu_n), \ldots, (g_k, \varepsilon_n \nu_n)\|^2 \leq \varepsilon_n^2 \sum_{i=1}^{k} \|g_i\|^2. \quad (3.1)$$

By the maximality of $\nu^*$ and the Taylor expansion of $\phi$ we then have

$$0 \geq f(\nu^* - \varepsilon_n \nu_n) - f(\nu^*)$$

$$= \phi((g_1, \nu^* - \varepsilon_n \nu_n), \ldots, (g_k, \nu^* - \varepsilon_n \nu_n)) - \phi((g_1, \nu^*), \ldots, (g_k, \nu^*))$$

$$= -\varepsilon_n \sum_{i=1}^{k} (g_i, \nu_n) \partial_i \phi((g_1, \nu^*), \ldots, (g_k, \nu^*)) + o(\varepsilon_n)$$

$$= -\varepsilon_n \langle \partial f(\nu^*), \nu_n \rangle + o(\varepsilon_n).$$
Dividing by \( \varepsilon_n \), sending \( n \) to infinity, and using the fact that \( \nu_n \) converges weakly to \( \delta_x \), we obtain \( \partial_x f(\nu^*) \geq 0 \) for \( x \in \text{supp}(\nu^*) \). For general \( x \in E^\Delta \), it holds that \( \nu^* + \varepsilon \delta_x \) is a non-negative measure. By the maximality of \( \nu^* \) we thus have

\[
0 \geq f(\nu^* + \varepsilon \delta_x) - f(\nu^*) = \\
\phi((g_1, \nu^* + \varepsilon \delta_x), \ldots, (g_k, \nu^* + \varepsilon \delta_x)) - \phi((g_1, \nu^*), \ldots, (g_k, \nu^*)) \\
= \varepsilon \sum_{i=1}^k g_i(x) \partial_i \phi((g_1, \nu^*), \ldots, (g_k, \nu^*)) + o(\varepsilon) \\
= \varepsilon \partial_x f(\nu^*) + o(\varepsilon).
\]

Dividing again by \( \varepsilon \) and sending it to 0, we deduce that \( \partial_x f \leq 0 \) for all \( x \in E^\Delta \).

Hence \( \partial_x f = 0 \) for all \( x \in \text{supp}(\nu^*) \) and \( \partial_x f \leq 0 \) for all \( x \in E^\Delta \setminus \text{supp}(\nu^*) \). This proves assertion (i) as for all signed measures \( \mu \in M(E^\Delta) \) with \( \text{supp}(\mu) \subseteq \text{supp}(\nu^*) \)

\[
\langle \partial f(\nu^*), \mu \rangle = \int_{\text{supp}(\mu)} \partial_x f(\nu^*) \mu(dx) = 0.
\]

(ii): Let \( \mu \in M_+(E^\Delta) \) such that \( \text{supp}(\mu) \subseteq \text{supp}(\nu^*) \). Since \( \nu^* \) is a maximum for \( f \), we obtain from the second order Taylor expansion of \( \phi \), the estimate (3.1), and by using (i)

\[
0 \geq f(\nu^* + \varepsilon \mu) - f(\nu^*) = \\
\varepsilon \sum_{i=1}^k \partial_i \phi((g_1, \nu^*), \ldots, (g_k, \nu^*)) \langle g, \mu \rangle \\
+ \frac{1}{2} \varepsilon^2 \sum_{i,j=1}^n \partial_{ij} \phi((g_1, \nu^*), \ldots, (g_k, \nu^*)) \langle g_i \otimes g_j, \mu^2 \rangle + o(\varepsilon^2) \\
= \frac{1}{2} \varepsilon^2 \langle \partial^2 f(\nu^*), \mu^2 \rangle + o(\varepsilon^2).
\]

Therefore, \( \langle \partial^2 f(\nu^*), \mu^2 \rangle \leq 0 \) for \( \mu \in M_+(E^\Delta) \) with \( \text{supp}(\mu) \subseteq \text{supp}(\nu^*) \). In particular, for \( x \in \text{supp}(\nu^*) \) and choosing \( \mu = \delta_x \) we have

\[
\partial^2_{xx} f(\nu^*) = \sum_{i,j=1}^n g_i(x) g_j(x) \partial_{ij} \phi((g_1, \nu^*), \ldots, (g_k, \nu^*)) \leq 0.
\]

For the general signed measure case, define

\[
\mu_n := \sum_{i=1}^m \lambda_i \nu_{n,i},
\]

for \( m \in \mathbb{N}, \lambda_1, \ldots, \lambda_m \in \mathbb{R} \) and \( \nu_{n,i} \) constructed as \( \nu_n \) above with \( x \) replaced by \( x_i \in \text{supp}(\nu^*) \). Letting \( \varepsilon_n \) decrease to 0 sufficiently fast (note that only the negative \( \lambda_i \) have to be taken into account) yields \( \nu^* + \varepsilon_n \mu_n \in M_+(E^\Delta) \). The same reasoning as in (3.2) replacing \( \varepsilon \mu \) by \( \varepsilon_n \mu_n \) and passing to the limit implies that

\[
\langle \partial^2 f(\nu^*), \left( \sum_{i=1}^m \lambda_i \delta_{x_i} \right)^2 \rangle \leq 0.
\]
Passing to the weak closure yields \( \langle \partial^2 f(\nu^*), \mu^2 \rangle \leq 0 \) for all \( \mu \in M(E^\Delta) \) with \( \text{supp}(\mu) \subseteq \text{supp}(\nu^*) \). This implies the assertions of (ii).

**Remark 3.2.** To compare the results of Theorem 3.1 with the classical conditions for functions on \( \mathbb{R}^m_+ \), let \( E = \{1, \ldots, m\} \). Then \( M_+(E) \) can be identified with \( \mathbb{R}^m_+ \). Condition (i) translates to

\[
\partial_i f(\nu^*) = 0, \quad i \in E,
\]

if the \( i \)-th component of \( \nu^* \) is strictly positive and \( \partial_i f(\nu^*) \leq 0 \) otherwise. Condition (ii) corresponds to

\[
y^T \partial^2 f(\nu^*) y \leq 0
\]

for all \( y \in \mathbb{R}^m \) such that \( y_j = 0 \) if \( \nu^*_j = 0 \). In particular

\[
\partial_i^2 f(\nu^*) \leq 0, \quad i \in E,
\]

in the case where the \( i \)-th component of \( \nu^* \) is strictly positive. These come from the classical Karush-Kuhn-Tucker conditions when the inequality constraints describe \( \mathbb{R}^m_+ \).

Our next optimality condition is a slight adaptation of Theorem 3.4 in Cuchiero et al. (2019). In comparison with that result we do not need to work with a group of positive isometries, but just with a positive group. Let us recall here the definition of a positive group (see, e.g., Arendt et al. (1986)).

**Definition 3.3.** A group \( \{T_t\}_{t \in \mathbb{R}} \) on \( C_\Delta(E) \) is positive if for any \( 0 \leq g \in C_\Delta(E) \), we have \( T_t g \geq 0 \).

We shall use the tensor notation \( A \otimes A \) to denote the linear operator from \( D \otimes D \) to \( \hat{C}_\Delta(E^2) \) determined by

\[
(A \otimes A)(g \otimes g) := (Ag) \otimes (Ag)
\]

for a given linear operator \( A : D \to C_\Delta(E) \).

**Theorem 3.4.** Fix \( f \in D \) and \( \nu_* \in M_+(E^\Delta) \) with \( f(\nu_*) = \max_{\nu \in M_+(E^\Delta)} f(\nu) \). Let \( A \) be the generator of a strongly continuous positive group on \( C_\Delta(E) \), and assume the domain of \( A \) contains both \( D \) and \( A(D) \). Then

\[
\langle A^2(\partial f(\nu_*)), \mu \rangle + \langle (A \otimes A)(\partial^2 f(\nu_*)), \mu^2 \rangle \leq 0
\]

for every \( \mu \in M_+(E^\Delta) \) with \( \mu \leq \nu_* \) and \( \text{supp}(\mu) \subseteq \text{supp}(\nu_*) \).

**Proof.** In what follows we adapt the proof of Theorem 3.4 in Cuchiero et al. (2019) to the case \( M_+(E^\Delta) \). Let \( \{T_t\}_{t \in \mathbb{R}} \) be the group generated by \( A \). For any \( \mu \in M_+(E^\Delta) \), the group induces a flow of measures \( \mu_t \in M(E^\Delta) \) via the formula \( \langle g, \mu_t \rangle = \langle T_t g, \mu \rangle \) for \( g \in C_\Delta(E) \). The positivity implies that \( \mu_t \) is non-negative. Indeed, \( \langle T_t g, \mu \rangle \geq 0 \) for all \( g \geq 0 \), whence \( \mu_t \) is a non-negative measure.

Therefore, assuming henceforth that \( \mu \leq \nu_* \), it follows that \( \nu_* + \mu_t - \mu \) is a non-negative measure. Since \( \|T_t g - g\| = O(t) \) for every \( g \in D \), we have \( \langle g, (\mu_t - \mu)^k \rangle = O(t^k) \) for every \( g \in D^\otimes k \). Maximality of \( \nu_* \) and Taylor’s formula then yield

\[
0 \geq f(\nu_* + \mu_t - \mu) - f(\nu_*) = \langle \partial f(\nu_*), \mu_t - \mu \rangle + \frac{1}{2} \langle \partial^2 f(\nu_*), (\mu_t - \mu)^2 \rangle + o(t^2)
\]

\[
= \langle (T_t - \text{id}) \partial f(\nu_*), \mu \rangle + \frac{1}{2} \langle (T_t \otimes T_t - 2T_t \otimes \text{id} + \text{id} \otimes \text{id}) \partial^2 f(\nu_*), \mu^2 \rangle + o(t^2).
\]

(3.3)
We claim that both $A$ and $-A$ satisfy the positive minimum principle (see Definition 1.1) on $E^\Delta$. Indeed, for $0 \leq g \in D$ and $x \in E^\Delta$ with $g(x) = \inf_{E^\Delta} g = 0$, we have

$$Ag(x) = \lim_{t \to 0} \frac{T_t g(x) - g(x)}{t} = \lim_{t \to 0} \frac{T_t g(x)}{t} \geq 0,$$

$$-Ag(x) = \lim_{t \to 0} \frac{T_{-t} g(x) - g(x)}{t} = \lim_{t \to 0} \frac{T_{-t} g(x)}{t} \geq 0;$$

which can just be seen as a consequence of (Arendt et al., 1986, Theorem B.II.16). Since $-\partial f(\nu^*) \geq 0$ and $\partial_x f(\nu^*) = 0$ for all $x \in \text{supp}(\nu^*)$ due to Theorem 3.1, it follows that $A(\partial f(\nu_*))(x) = 0$ for all such $x$. As a result, using that $\text{supp}(\mu) \subseteq \text{supp}(\nu_*)$ and that the domain of $A$ contains $A(D)$, we get

$$\langle (T_t - \text{id})\partial f(\nu_*), \mu \rangle = \langle (T_t - \text{id} - tA)\partial f(\nu_*), \mu \rangle = \frac{1}{2} t^2 \langle A^2(\partial f(\nu_*)), \mu \rangle + o(t^2).$$

Furthermore, using that

$$(T_t \otimes T_t - 2T_t \otimes \text{id} + \text{id} \otimes \text{id})(g \otimes g) = (T_t g - g) \otimes (T_t g - g)$$

for all $g \in D$, we deduce that

$$\langle (T_t \otimes T_t - 2T_t \otimes \text{id} + \text{id} \otimes \text{id})g, \mu^2 \rangle = t^2 \langle (A \otimes A)g, \mu^2 \rangle + o(t^2)$$

for all $g \in D \otimes D$. Inserting (3.5) and (3.6) into (3.3), dividing by $t^2$, and sending $t$ to zero yields

$$0 \geq \frac{1}{2} \langle A^2(\partial f(\nu_*)), \mu \rangle + \frac{1}{2} \langle (A \otimes A)\partial^2 f(\nu_*), \mu^2 \rangle.$$ 

This completes the proof. 

The optimality condition obtained in Theorem 3.4 involves the generator of a strongly continuous positive group and is more subtle than the previous ones. In particular in the finite-dimensional case the generator of a strongly continuous positive group can only be a diagonal matrix, see Lemma 3.6. For the properties of such an operator in the general case, we refer to Remark 3.7 and Appendix C.

Remark 3.5. We claim that for $A$ as in Theorem 3.4, the operator $A^2$ satisfies the positive minimum principle on $E^\Delta$. Indeed, let $g \in D$ and $x \in E^\Delta$ with $g(x) = \inf_{E^\Delta} g = 0$. Then, as $A$ and $-A$ satisfy the positive minimum principle $Ag(x) = 0$. Hence $A^2 g(x) = \lim_{t \to 0} (T_t g(x) - g(x) - A f(x))/t \geq 0$, which proves the claim.

The following lemma illustrates how the conditions of Theorem 3.4 translate to the case where $D = C(E^\Delta)$. This includes in particular the finite dimensional case when $E$ consists of $n \in \mathbb{N}$ points.

Lemma 3.6.  (i) Let $A : C(E^\Delta) \to C(E^\Delta)$ be the generator of a strongly continuous positive group, then

$$Ag(x) = a(x) g(x)$$

for some $a \in C_\Delta(E)$. Moreover $(A \otimes A)g(x,y) = a(x)a(y)g(x,y)$ for $g \in \hat{C}_\Delta(E^2)$. 

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(ii) Let $E$ consist of $n \in \mathbb{N}$ points and let $A$ be the generator of a strongly continuous positive group on $C_\Delta(E) \cong \mathbb{R}^n$. Then $A$ is a diagonal matrix.

Proof. As shown in equation (3.4), $A$ and $-A$ satisfy the positive minimum principle. Moreover, as $D = C(E^\Delta)$, Corollary B.II.1.12 in Arendt et al. (1986) yields that $A$ is bounded and that

$$-\|A\|g \leq Ag \leq \|A\|g,$$

for all $g \geq 0$, (3.7)

where $\|A\|$ denotes the operator norm. Observe that (3.7) yields

$$A((g - g(x))^+(x) = A((g - g(x))^-(x) = 0,$$

showing that $Ag(x)$ can only depend on $g(x)$. The statement follows from the fact that $A$ is a bounded linear operator. The form of $(A \otimes A)g(x, y)$ is then obvious.

The second part directly follows from the first one. Let us underline that a matrix in $\mathbb{R}^{n \times n}$ satisfies the positive minimum principle if and only if the off-diagonal elements are non-negative; see (Arendt et al., 1986, Example B.II.1.4). Since both $A$ and $-A$ satisfy the positive minimum principle on $E$, we conclude that $A$ is a diagonal matrix. \qed

Remark 3.7. Generators of strongly continuous positive groups on $C_\Delta(E)$ have been fully characterized in (Arendt et al., 1986, Theorem B.II.3.14). In Appendix C we recall the main tools behind their characterization by introducing definitions such as flows and cocycles.

4 Polynomial operators

Recall that $E$ is a locally compact Polish space and $D \subseteq C_\Delta(E)$ is a dense linear subspace containing the constant function 1. Set $\mathcal{D}$ as in (2.2). We now define polynomial operators, which constitute a class of possibly unbounded linear operators acting on cylindrical functions and polynomials. For the moment we define these polynomial operators for polynomials on general subsets $S \subseteq M(E)$, similarly as in Section 4 of Cuchiero et al. (2019). To avoid unnecessary loss of generality we work with the set $\mathcal{M}(M(E))$ of measurable real valued maps on $M(E)$.

Remark 4.1. In this section, and also more generally, we often work just with subsets of $M(E)$ despite the fact that the results hold also for subsets of $M(E^\Delta)$. This is due to the fact that the second can be deduced from the first by means of a direct argument. Whenever this is not the case, we will explicitly state it.

Definition 4.2. Fix $S \subseteq M(E)$. A linear operator $L : \mathcal{D} \to \mathcal{M}(M(E))$ is called $S$-polynomial if $L$ maps $P^D$ to $P$ such that for every $p \in P^D$ there is some $q \in P$ with $q|_S = Lp|_S$ and $\deg(q) \leq \deg(p)$.

4.1 Characterization of polynomial operators in the diffusion setting

For a linear operator $L$, the associated carré-du-champ operator (see, e.g., Bakry et al. (2013)) is the symmetric bilinear map $\Gamma : \mathcal{D} \times \mathcal{D} \to \mathcal{M}(M(E))$

$$\Gamma(p, q) = L(pq) - pLq - qLp.$$
Related to the carré-du-champ operator is the notion of a *derivation*, which we recall in the following definition and which can be used to characterize the form of $L$, as a second-order differential operator, typical for diffusion processes.

**Definition 4.3.** Fix $\mathcal{S} \subseteq M(E)$. A symmetric bilinear map $\Gamma: \mathcal{D} \times \mathcal{D} \to \mathcal{M}(M(E))$ is called an $\mathcal{S}$-derivation if for all $p, q, r \in \mathcal{D}$, $\Gamma(pq, r) = p\Gamma(q, r) + q\Gamma(p, r)$ on $\mathcal{S}$.

**Remark 4.4.** As explained in Remark 4.1 we consider only subsets of $M(E)$ and not of $M(E^\Delta)$. Hence for finite measure we shall only speak of $M_+(E)$-derivations (even if $\mathcal{S} = M_+(E^\Delta)$).

**Lemma 4.5.** The carré-du-champs operator $\Gamma$ of a linear operator $L: \mathcal{D} \to \mathcal{M}(M(E))$ is an $M_+(E)$-derivation if and only if $Lf(\nu) = 0$ for each $f$ such that $\partial f(\nu) = 0$ and $\partial^2 f(\nu) = 0$.

**Proof.** Observe first that if $\Gamma$ is an $M_+(E)$-derivation, it holds $\Gamma(1, 1) = 2\Gamma(1, 1)$, hence $L(1) = -\Gamma(1, 1) = 0$. Since for each $p, q, r \in \mathcal{D}$ and $\nu \in M_+(E)$ it holds

$$\Gamma(pq, r)(\nu) = p(\nu)\Gamma(q, r)(\nu) - q(\nu)\Gamma(p, r)(\nu) - p(\nu)q(\nu)r(\nu)L1(\nu),$$

we get that $\Gamma$ is a derivation if and only if $L1 = 0$ and

$$L\left((p - p(\nu))(q - q(\nu))(r - r(\nu))\right)(\nu) = 0,$$

for each $p, q, r \in \mathcal{D}$ and $\nu \in M_+(E)$. Observe now that each $p \in \mathcal{D}$ admits the representation

$$p(\nu) = \phi(\langle g_1, \nu \rangle, \ldots, \langle g_n, \nu \rangle)$$

for some $\phi \in C^\infty(\mathbb{R}^n)$ and some linearly independent $g_1, \ldots, g_n \in \mathcal{D}$. Assuming that $\partial p(\nu) = 0$ and $\partial^2 p(\nu) = 0$ and applying Taylor’s theorem to $\phi$ yields

$$p(\mu) = \sum_{k_1, k_2=1}^n \langle g_{k_1}, \mu - \nu \rangle \langle g_{k_2}, \mu - \nu \rangle \tilde{p}_{k_1, k_2}(\mu)$$

for some $\tilde{p}_{k_1, k_2} \in \mathcal{D}$ such that $\tilde{p}_{k_1, k_2}(\nu) = 0$. The claim then follows by (4.2). \hfill \-box

For a finite-dimensional diffusion it is known that its generator is polynomial if and only if the drift and diffusion coefficients are polynomial of first and second degree, respectively, see Cuchiero et al. (2012) and Filipović and Larsson (2016). The following result is the generalization of this fact to the measure-valued setting.

**Theorem 4.6.** A linear operator $L: \mathcal{D} \to \mathcal{M}(M(E))$ is $M_+(E)$-polynomial and its carré-du-champ operator $\Gamma$ is an $M_+(E)$-derivation if and only if $L$ admits a representation

$$Lf(\nu) = B_0(\partial f(\nu)) + \langle B_1(\partial f(\nu)), \nu \rangle$$

$$+ \frac{1}{2} \left\{ Q_0(\partial^2 f(\nu)) + \langle Q_1(\partial^2 f(\nu)), \nu \rangle + \langle Q_2(\partial^2 f(\nu)), \nu^2 \rangle \right\}, \quad f \in \mathcal{D}, \nu \in M_+(E)$$

for some linear operators $B_0: \mathbb{D} \to \mathbb{R}$, $B_1: \mathbb{D} \to C_\Delta(E)$, $Q_0: \mathbb{D} \otimes \mathbb{D} \to \mathbb{R}$, $Q_1: \mathbb{D} \otimes \mathbb{D} \to C_\Delta(E)$, $Q_2: \mathbb{D} \otimes \mathbb{D} \to \tilde{C}_\Delta(E^2)$. These operators are uniquely determined by $L$. 

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Proof. Theorem A.1 in Cuchiero et al. (2019) yields the result when \( D \) is replaced by \( P^D \). The extension of the first implication from \( P^D \) to \( D \) is then direct. For the converse implication, observe that for each \( f \in D \) and \( \nu \in M_+(E) \) there is a \( p \in P^D \) such that \( \partial(p - f)(\nu) = 0 \) and \( \partial^2(p - f)(\nu) = 0 \). The claim then follows by Lemma 4.5.

\( \square \)

4.2 Dual operators

We would now like to associate to an \( S \)-polynomial operator \( L \) a family of so-called dual operators \((L_k)_{k \in \mathbb{N}}\) which are linear operators mapping the coefficients vector of \( p \) to the coefficients vector of \( Lp \). To this end we recall Definition 2.3 of Cuchiero and Svaluto-Ferro (2021).

Definition 4.7. Fix \( m \in \mathbb{N}_0 \) and let \( L \) be an \( S \)-polynomial operator. An \( m \)-th dual operator corresponding to \( L \) is a linear operator \( L_m : \bigoplus_{k=0}^m D^{\otimes k} \to \bigoplus_{k=0}^m \hat{C}_\Delta(E^k) \) such that \( L_m \vec{g} =: (L_m^0 \vec{g}, \ldots, L_m^m \vec{g}) \) satisfies

\[
Lp(\nu) = \sum_{k=0}^m (L_k \vec{g}, \nu^k) \quad \text{for all } \nu \in S,
\]

where \( p(\nu) = \sum_{k=0}^m \langle g_k, \nu^k \rangle \). Whenever \( L_m \) is a closable operator\(^2\), we still denote its closure by \( L_m : D(L_m) \to \bigoplus_{k=0}^m \hat{C}_\Delta(E^k) \) and its domain by \( D(L_m) \subseteq \bigoplus_{k=0}^m \hat{C}_\Delta(E^k) \).

If \( S = M_+(E) \), then the dual operator is unique. Indeed, in this case the representation (2.1) is unique, which is a consequence of Corollary 2.4 in Cuchiero et al. (2019), and we can thus identify each polynomial with its coefficients vector. We therefore call it the dual operator. It is interesting to note that if \( L \) satisfies (4.3), then

\[
L_m^m(g^{\otimes m}) = mB_1(g) \otimes g^{\otimes(m-1)} + \frac{m(m-1)}{2}Q_2(g \otimes g) \otimes g^{\otimes(m-2)},
\]

\[
L_m^{m-1}(g^{\otimes m}) = mB_0(g)g^{\otimes(m-1)} + \frac{m(m-1)}{2}Q_1(g \otimes g) \otimes g^{\otimes(m-2)},
\]

\[
L_m^{m-2}(g^{\otimes m}) = \frac{m(m-1)}{2}Q_0(g \otimes g)g^{\otimes(m-2)}, \text{ and } L_m^k(g^{\otimes m}) = 0 \text{ for each } k < m - 2.
\]

5 Moment formula and existence of polynomial diffusions on \( M_+(E) \)

Let \( L \) be a linear operator acting on \( D \) for \( D \) specified in (2.2). In this section we study existence and uniqueness of non-negative measure valued polynomial diffusions which we introduce via the martingale problem. In the following let \( S \) stand either for \( M'_+(E^\Delta) \), \( M_+(E^\Delta) \) or \( M_+(E) \).

An \( S \)-valued process \( X \) with càdlàg paths defined on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is called an \( S \)-valued solution to the martingale problem for \( L \) with initial condition \( \nu \in S \) if \( X_0 = \nu \) \( \mathbb{P} \)-a.s. and

\[
N_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds \tag{5.1}
\]

\(^2\)We refer to (Ethier and Kurtz, 2009, Chapter 1) for the precise definition.
defines a local martingale for every $f$ in the domain of $L$. Uniqueness of solutions to the martingale problem is always understood in the sense of law. The martingale problem for $L$ is well–posed if for every initial condition $\nu \in \mathcal{S}$ there exists a unique $\mathcal{S}$-valued solution to the martingale problem for $L$ with initial condition $\nu$.

We are mainly interested in $M_+(E)$-valued solutions with continuous paths (with respect to the topology of weak convergence) corresponding to polynomial operators.

**Definition 5.1.** Let $L$ be $M_+(E)$-polynomial. Any continuous $M_+(E)$-valued solution to the martingale problem for $L$ is called a *measure-valued polynomial diffusion*.

The following lemma relates path continuity of $M_+(E)$- and $M_+(E^\Delta)$-valued solutions to the martingale problem to the carré-du-champ operator being a derivation. This explains why we consider derivations in Theorem 4.6.

**Lemma 5.2.** If the carré-du-champ operator $\Gamma$ associated to $L$ is an $M_+(E)$-derivation, then any $M_+(E)$- or $M_+(E^\Delta)$-valued solution to the martingale problem for $L$ has continuous paths and the corresponding quadratic covariation structure is given by

$$d[N^f, N^g]_t = 2\Gamma(f, g)(X_t)dt,$$

for every $f, g \in \mathcal{D}$.

Conversely, if for every initial condition $\nu \in M_+(E)$ (or $M_+(E^\Delta)$ respectively) there is an $M_+(E)$- (or $M_+(E^\Delta)$ respectively) valued solution to the martingale problem for $L$ with continuous paths, then the carré-du-champ operator $\Gamma$ associated to $L$ is an $M_+(E)$-derivation.

**Proof.** We here state the proof only for $M_+(E)$ since the $M_+(E^\Delta)$-case follows by the same arguments extending $f \in \mathcal{D}$ to $M(E^\Delta)$.

Let $X$ be an $M_+(E)$-valued solution to the martingale problem for $L$. By Proposition 2 in Bakry and Émery (1985), the real-valued process $f(X)$ is continuous for every $f \in \mathcal{D}$, in particular for every linear monomial $f(\nu) = \langle g, \nu \rangle$ with $g \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $C_\Delta(E)$, we can conclude that $X$ is continuous with respect to the topology of weak convergence on $M_+(E)$.

Conversely, if $X$ is a $M_+(E)$-valued solution to the martingale problem for $L$ with continuous paths, then, by Lemma 2.3 and Lemma 2.5, the map $t \mapsto f(X_t)$ is continuous for all $f \in \mathcal{D}$. The result now follows by Proposition 1 in Bakry and Émery (1985). 

### 5.1 Moment formula

Polynomial diffusions are of particular interest in many applications because they satisfy a *moment formula*, which allows to compute the process’ moments in a tractable way. If $E$ is a finite set, the moment formula always holds, but technical conditions, in particular on the dual operators, are needed in the general case. We apply here Theorem 3.4 and Remark 3.20 of Cuchiero and Svaluto-Ferro (2021) (see also Example 3.21 and Lemma 3.2 in Cuchiero and Svaluto-Ferro (2021)) to get Theorem 5.5.

Let $L$ be an $M_+(E)$-polynomial operator, fix $m \in \mathbb{N}_0$, and let $L_m$ be the closable $m$-th dual operator corresponding to $L$ with domain $\mathcal{D}(L_m)$. Before stating the theorem, we...
extend the domain of $L$ to polynomials with coefficients in $\mathcal{D}(L_m)$ by setting
\begin{align*}
p_g &= \sum_{k=0}^{m} \langle g_k, \nu^k \rangle \quad \text{and} \quad Lp_{\bar{g}} = \sum_{k=0}^{m} \langle L^k_m \bar{g}, \nu^k \rangle \quad (5.2)
\end{align*}
for all $\bar{g} \in \mathcal{D}(L_m)$ and $m \in \mathbb{N}_0$. As in the finite dimensional case the moment formula corresponds to a solution of a system of linear ODEs. Note that we shall usually call these differential equations ODEs, even though they correspond in many cases to PDEs or PIDEs depending on the underlying space and the specification of the operators. In the current infinite dimensional setting we also need to make the solution concept precise. The following is an adaptation of Definition 3.3 in Cuchiero and Svaluto-Ferro (2021) to the current measure-valued setting.

**Definition 5.3.** We call a function $t \mapsto \bar{g}_t$ with values in $\mathcal{D}(L_m)$ a solution of the $m+1$ dimensional system of ODEs
\begin{align*}
\partial_t \bar{g}_t &= L_m \bar{g}_t, \quad \bar{g}_0 = g,
\end{align*}
if for every $t > 0$ it holds
\begin{align*}
m \sum_{k=0}^{m} \langle g_{t,k}, \nu^k \rangle = \sum_{k=0}^{m} \langle g_{0,k}, \nu^k \rangle + \int_0^t \sum_{k=0}^{m} \langle L^k_m \bar{g}_s, \nu^k \rangle ds \quad (5.3)
\end{align*}
for all $\nu \in M_+(E^\Delta)$.

**Remark 5.4.** Note that the above solution concept reduces to a more classical one if we take $\nu = \delta_{x_1} + \cdots + \delta_{x_k}$ with $x_i \in E$, $i = 1, \ldots, k$ and $k = 1, \ldots, m$. Indeed, by polarization (5.3) can be transformed into
\begin{align*}
g_{t,0} &= g_{0,0} + \int_0^t L^0_m \bar{g}_s ds \\
g_{t,1}(x_1) &= g_{0,1}(x_1) + \int_0^t L^1_m \bar{g}_s(x_1) ds \\
& \vdots \\
g_{t,m}(x_1, \ldots, x_m) &= g_{0,m}(x_1, \ldots, x_m) + \int_0^t L^m_m \bar{g}_s(x_1, \ldots, x_m) ds
\end{align*}
and thus reduces to a classical (except of the integral form) solution of a multivariate P(I)DE.

**Theorem 5.5** (Dual moment formula). Let $L$ be an $M_+(E)$-polynomial operator, fix $m \in \mathbb{N}_0$, let $L_m$ be the $m$-th dual operator corresponding to $L$, and assume that $L_m$ is closable with domain $\mathcal{D}(L_m)$. Suppose that an $M_+(E^\Delta)$-valued solution $(X_t)_{t \geq 0}$ to the martingale problem for $L$ exists. Fix a coefficients vector $\bar{g} = (g_0, \ldots, g_m) \in \mathcal{D}(L_m)$ and suppose that the following condition holds true.

- There is a solution in the sense of Definition 5.3 of the $m+1$ dimensional system of linear ODEs on $[0, T]$ given by
\begin{align*}
\partial_t \bar{g}_t &= L_m \bar{g}_t, \quad \bar{g}_0 = \bar{g}. \quad (5.4)
\end{align*}
Then, for all $0 \leq t \leq T$ the representation

$$
E \left[ \sum_{k=0}^{m} \langle g_k, X_T^k \rangle \mid \mathcal{F}_t \right] = \sum_{k=0}^{m} \langle g_{T-t,k}, X_t^k \rangle
$$

holds almost surely.

**Proof.** This is simply a consequence of Theorem 3.4 in Cuchiero and Svaluto-Ferro (2021) and Lemma 5.7 below.

**Example 5.6.** Let $L$ be an $M_+(E)$-polynomial operator, fix $m \in \mathbb{N}_0$, and let $L_m$ be the closable $m$-th dual operator corresponding to $L$. Assume that $L_m^0 = 0, \ldots, L_m^{m-1} = 0$ and $L_m^m|_{D^{\otimes m}}$ satisfies the positive minimum principle. If the closure $(Y_t^k)_{t \geq 0}$ satisfies (5.4) for every $g$ in the domain of $\mathcal{T}_m|_{D^{\otimes m}}$. We provide now a simple example to illustrate this mechanism.

Consider the process given by $X_t = S_t \mu$ where $dS_t = \sigma S_t dW_t$, $S_0 = 1$, and $\mu \in M_+(E)$. Since $\langle g, X \rangle$ is a local martingale and $\langle g, X_t \rangle^2 = \langle S_t(g, \mu) \rangle^2 = \frac{1}{2} \int_0^t \langle \sigma^2 g \otimes g, X_s^2 \rangle ds + (\text{local martingale})$, the generator of $X$ is given by $Lp(\nu) = \frac{1}{2} \langle \sigma^2 \partial^2 p(\nu), \nu^2 \rangle$, where $L_n g \otimes n = \sigma^2 \frac{n(n-1)}{2} g \otimes n$ and note that $L_n^n$ is the generator of the semigroup given by $Y_t g \otimes n(z) := E[g \otimes n(Z_{t}^{(n)}) \exp(\int_0^t m(Z_{s}^{(n)}) ds)|Z_0^{(n)} = z]$, where $Z^{(n)}$ is the constant process (hence the process generated by 0) and $m(z) = \sigma^2 \frac{n(n-1)}{2}$. This in particular yields

$$
\langle Y_t g \otimes n, X_t^n \rangle = \langle g, \mu \rangle^n \exp(\sigma^2 t \frac{n(n-1)}{2}) = E[\langle g, \mu \rangle^n S_t^n] = E[\langle g, X_t \rangle^n].
$$

**Lemma 5.7.** Let $L$ be an $M_+(E)$-polynomial operator and suppose that an $M_+(E^\Delta)$-valued solution $(X_t)_{t \geq 0}$ to the martingale problem for $L$ exists.

(i) Then, for every $p \in P^D$,

$$
E[p(X_t)^2] < Ce^{Ct},
$$

for some $C > 0$. Moreover, $N^p$ as defined in (5.1) is a true martingale.

(ii) Moreover, if there is a solution to (5.4) in the sense of Definition 5.3, then Condition (ii) and (iii) of Theorem 3.4 in Cuchiero and Svaluto-Ferro (2021) also hold true.
Proof. Let $\| \cdot \|_{M(E^\Delta)}$ denote the total variation norm on $M(E^\Delta)$. Then for all $\nu \in M_+(E^\Delta)$, it holds $\|\nu\|_{M(E^\Delta)} = \nu(E^\Delta)$ and hence
\[
1 + \|\nu\|_{M(E^\Delta)}^{2m} = 1 + \nu(E^\Delta)^{2m}, \quad \nu \in M_+(E^\Delta).
\]
Thus, for any $p = \sum_{k=0}^m \langle g_k, \nu^k \rangle \in P$, we can estimate
\[
p^2 \leq C(1 + \nu(E^\Delta)^{2m})
\]
and
\[
|L(1 + \nu(E^\Delta)^{2m})| \leq C(1 + \nu(E^\Delta)^{2m})
\]
for some constant $C > 0$. According to Definition 3.18 in Cuchiero and Svaluto-Ferro (2021), this means that every $p$ of degree $m$ is $(C, 1 + \nu(E^\Delta)^{2m})$ bounded. Lemma 3.19 in Cuchiero and Svaluto-Ferro (2021) thus yields that for every $p \in P$,
\[
\mathbb{E}[p(X_t)^2] < Ce^{Ct}
\]
and the local martingale $N^p$ is actually a true (even square integrable) martingale.

Concerning part (ii), Lemma 3.19 in Cuchiero and Svaluto-Ferro (2021) also yields Condition (ii) and (iii) of Theorem 3.4 as long as we have solution to (5.4). \qed

5.2 Existence

Our first main result of this section gives sufficient conditions for the existence of solutions to the martingale problem. Applications of this result are discussed in Section 7. Recall that $E$ is throughout a locally compact Polish space. We start by a definition which will be used to describe the form of $Q_2$ in the representation of $L$ given by (4.3).

**Definition 5.8.** We say that a linear operator $C$ admits a $(\beta, \pi)$-representation if
\[
C(g)(x, y) = \frac{1}{2}(\pi(x, y)g(x, x) + \pi(y, x)g(y, y) + 2\beta(x, y)g(x, y)), \quad g \in D \otimes D \quad (5.5)
\]
where
- $\beta : (E^\Delta)^2 \to \mathbb{R}$ is a symmetric function such that $\beta(x, x) \geq 0$ for all $x \in E^\Delta$;
- $\pi : (E^\Delta)^2 \to \mathbb{R}_+$ is a non-negative function such that $\pi(x, x) = 0$ for all $x \in E^\Delta$;
- for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E^\Delta$, $c_1, \ldots, c_n \in \mathbb{R}_+$, the $n \times n$ matrix
  \[
  A^{(n)} := \beta_n + \left( \begin{array}{ccc}
  \sum_{j=1}^n \frac{c_j}{c_1} \pi(x_1, x_j) \\
  \vdots \\
  \sum_{j=1}^n \frac{c_j}{c_n} \pi(x_n, x_j)
  \end{array} \right) \in \mathbb{S}_+^n, \quad (5.6)
  \]
  where $\beta_n \in \mathbb{S}^n$ with entries $\beta_{n,ij} = \beta(x_i, x_j)$;
- the map $(x, y) \mapsto \frac{1}{2}(\pi(x, y)g(x, x) + \pi(y, x)g(y, y) + 2\beta(x, y)g(x, y))$ lies in $\widehat{C}_\Delta(E^2)$ for all $g \in D \otimes D$. 


Accordingly, our sufficient conditions for the existence of the martingale problem now read as follows.

**Theorem 5.9.** Suppose that for \( i = 1, \ldots, n \), each \( A_i \) is the generator of a strongly continuous positive group on \( C_\Delta(E) \) such that its domain contains both \( D \) and \( A_i(D) \). Let \( L : \mathcal{D} \to C(M_+(E)) \) be a linear operator of form (4.3), where

(i) \( B_0 : \mathcal{D} \to \mathbb{R} \) is given by \( B_0(g) = \langle g, b \rangle \) with \( b \in M_+(E^\Delta) \);

(ii) \( B_1 - \frac{1}{2} \sum_{i=1}^{n} A_i^2 : \mathcal{D} \to C_\Delta(E) \) satisfies the positive minimum principle on \( E^\Delta \);

(iii) \( Q_0 \equiv 0 \);

(iv) \( Q_1 \) is of the form

\[
Q_1(g) = \alpha \text{diag}(g), \quad \text{that is,} \quad Q_1(g)(x) = \alpha(x)g(x, x), \quad g \in D \otimes D,
\]

where \( \alpha \in C_\Delta(E) \) with values in \( \mathbb{R}_+ \);

(v) \( Q_2 \) is of the form

\[
Q_2(g) = C(g) + \sum_{i=1}^{n} (A_i \otimes A_i)(g), \quad g \in D \otimes D,
\]

where \( C \) admits a \((\beta, \pi)\)-representation.

Then \( L \) is \( M_+(E)\)-polynomial and its martingale problem has an \( M_+(E^\Delta)\)-valued solution with continuous paths for every initial condition \( \nu \in M_+(E^\Delta) \).

If additionally, the measure \( b \) and the initial condition \( \nu \) both lie in \( M_+(E) \) and if there exists some function \( m \in C_\Delta(E) \) such that

\[
bp-\lim_{n \to \infty} (B_1(1 - g_n) - m(1 - g_n)) \leq 0,
\]

for some sequence \( (g_n)_{n \in \mathbb{N}} \in D \cap C_\mu(E) \) satisfying \( bp-\lim_{n \to \infty} g_n = 1_E \), then any solution to the martingale problem takes values in \( M_+(E) \).

**Remark 5.10.**

(i) When the state space are probability measures \( M_1(E) \), Theorem 5.6 in Cuchiero et al. (2019) provides sufficient conditions guaranteeing existence of solutions to the corresponding martingale problem. These conditions are stronger and imply the current ones. Indeed, we identify \( B_1 \) and \( Q_2 \) here with \( B \) and \( Q \) there. More specifically, \( C(g) \) is identified with \( \alpha\Psi(g) \) and \( A_i \) with \( A_i \). All other quantities are set to 0. The conditions on \( B \) and \( A_i \) are clearly stronger. In particular, the positive maximum principle implies the positive minimum principle on \( E^\Delta \). Moreover, \( \alpha\Psi \) admits a \((\beta, \pi)\)-representation for \( \pi(x, y) = \pi(y, x) = -\beta(x, y) = \alpha(x, y)1_{\{x \neq y\}} \). Observe indeed that in this case the matrix \( A^{(n)} \) appearing in (5.6) is given by

\[
A_{ij}^{(n)} := -\alpha(x_i, x_j)1_{\{x_i \neq x_j\}} + \sum_{k=1}^{n} \frac{c_k}{c_j} \alpha(x_i, x_k)1_{\{x_i \neq x_k\}}1_{\{i = j\}},
\]

which is positive semidefinite since \( \sum_{i,j=1}^{n} A_{ij}^{(n)} v_i v_j = \sum_{i=1}^{n} \alpha(x_i, x_j) \frac{c_i}{c_j} (v_i - v_j \frac{c_i}{c_j})^2 \geq 0 \) for each \( v_i \in \mathbb{R} \).
(ii) Note that (4.3) imposes the implicit condition that \( C(g) \) has to lie in \( \hat{C}_{\Delta}(E^2) \) for every \( g \in D \otimes D \). If \( D = C_{\Delta}(E) \), then this necessarily yields boundedness conditions on \( \pi \) and \( \beta \), as is seen from Theorem 5.11 below. However, this does not hold for general \( D \subseteq C_{\Delta}(E) \), as one can see by considering \( E = \mathbb{R}, D \subseteq C^1_\Delta(\mathbb{R}) \), and \( \pi(x,y) = \pi(y,x) = -\beta(x,y) = |x-y|^\nu \).

**Proof of Theorem 5.9.** Theorem 4.6 implies that \( L \) is \( M_+(E) \)-polynomial. Lemma B.2(i) yields the existence of an \( M_+(E^A) \)-valued continuous solution to the martingale problem for any initial condition provided that \( L \) satisfies the positive maximum principle on \( M_+(E^A) \), which we shall check below.

Let us now verify that the positive maximum principle on \( M_+(E^A) \) holds true. Let therefore \( \nu^* \in M_+(E^A) \) be a maximizer of \( f \in D \) over \( M_+(E^A) \). Then the optimality conditions in Theorem 3.1 yield

\[
\partial_x f(\nu^*) \leq 0, \quad \langle \partial f(\nu^*), \nu^* \rangle = 0, \quad \partial_{xx} f(\nu^*) \leq 0 \quad \text{and} \quad \langle \partial^2 f(\nu^*), \mu^2 \rangle \leq 0, \tag{5.7}
\]

for each \( x \in E \) and \( \mu \in M(E^A) \) with \( \text{supp}(\mu) \subseteq \text{supp}(\nu^*) \). We now analyze \( Lf(\nu^*) \), which reads due the conditions (i) - (v) as follows:

\[
Lf(\nu^*) = \langle \partial f(\nu^*), b \rangle + \langle B_1(\partial f(\nu^*)), \nu^* \rangle + \frac{1}{2} \langle \alpha \text{diag}(\partial^2 f(\nu^*)), \nu^* \rangle \\
+ \frac{1}{2} \langle C(\partial^2 f(\nu^*)), (\nu^*)^2 \rangle + \frac{1}{2} \sum_{i=1}^n (A_i \otimes A_i)(\partial^2 f(\nu^*)), (\nu^*)^2 \rangle. 
\]

By (5.7) and the positivity of \( b \) and \( \alpha \), we have

\[
\langle \partial f(\nu^*), b \rangle \leq 0 \quad \text{and} \quad \langle \alpha \text{diag}(\partial^2 f(\nu^*)), \nu^* \rangle \leq 0. \tag{5.8}
\]

Next, observe that choosing the signed measure in (5.7) to be \( \sum_{i=1}^n \lambda_i \delta_{x_i} \), for some \( \lambda_i \in \mathbb{R} \) yields \( \sum_{i,j=1}^n \lambda_i \lambda_j \partial^2_{x_i x_j} f(\nu^*) \leq 0 \), which implies that the matrix \( (H_{n,ij})_{ij=1}^n \) given by \( H_{n,ij} := \partial^2_{x_i x_j} f(\nu^*) \) is negative semidefinite. Taking \( \mu \in M_+(E^A) \) such that \( \text{supp}(\mu) \) consists of \( n \) points \( x_1, \ldots, x_n \in \text{supp}(\nu^*) \), we thus have \( \mu = \sum_{i=1}^n c_i \delta_{x_i} \), for some \( c_1, \ldots, c_n > 0 \).

Since \( C(g) \) admits a \( (\beta, \pi) \)-representation, using the notation of Definition 5.8 we can thus write

\[
\langle C(\partial^2 f(\nu^*)), \mu^2 \rangle = \sum_{i,j=1}^n (A^{(n)}) \circ H_n)_{ij} c_i c_j,
\]

where \( \circ \) denotes the Hadamard product, i.e. the componentwise multiplication. As \( A^{(n)} \) is by assumption positive semidefinite and as the Hadamard product between a positive semidefinite and negative semidefinite matrix is negative definite, we can conclude that

\[
\langle C(\partial^2 f(\nu^*)), \mu^2 \rangle \leq 0. \tag{5.9}
\]

Passing to the weak closure yields (5.9).

Finally, observe that from (5.8) and (5.9) we get that

\[
Lf(\nu^*) \leq \langle B_1(\partial f(\nu^*)), \nu^* \rangle + \frac{1}{2} \sum_{i=1}^n (A_i \otimes A_i)(\partial^2 f(\nu^*)), (\nu^*)^2 \rangle.
\]

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As \( B_1 - \frac{1}{2} \sum_{i=1}^{n} A_i^2 \) satisfies the positive minimum principle, we have by (5.7)

\[
\langle B_1(\partial f(\nu^*)) - \frac{1}{2} \sum_{i=1}^{n} A_i^2(\partial f(\nu^*)), \nu^* \rangle \leq 0.
\]

Therefore,

\[
L_f(\nu^*) \leq \frac{1}{2} \sum_{i=1}^{n} A_i^2(\partial f(\nu^*), \nu^*) + \frac{1}{2} \sum_{i=1}^{n} (A_i \otimes A_i)(\partial^2 f(\nu^*))(\nu^*)^2 \leq 0,
\]

where the last inequality follows from Theorem 3.4. This proves the positive maximum principle on \( M_+(E^\Delta) \) and thus the existence of an \( M_+(E^\Delta) \)-valued solution. The remaining statement on \( M_+(E) \)-valued solutions to the martingale problem follows by Lemma B.2(ii).

Even though Theorem 5.9 only gives sufficient conditions for the existence of solutions to the martingale problem for general coefficient domains \( D \), the next result shows that the conditions are sharp. Indeed, we consider the case \( D = C_\Delta(E) \) and get the following characterization.

**Theorem 5.11.** Let \( D = C_\Delta(E) \) and let \( L : D \to C(M_+(E)) \) be a linear operator. Then \( L \) is \( M_+(E) \)-polynomial, there exists an \( M_+(E^\Delta) \)-valued solution to the martingale problem for all initial conditions in \( M_+(E^\Delta) \) and all solutions have continuous paths, if and only if \( L \) satisfies (4.3) with

(i) \( B_0 : C_\Delta(E) \to \mathbb{R} \) is given by \( B_0(g) = \langle g, b \rangle \) with a Radon measure \( b \in M_+(E^\Delta) \);

(ii) For every \( g \in C_\Delta(E) \)

\[
B_1g(x) = \int (g(\xi) - g(x)) \nu_B(x, d\xi) + m(x)g(x)
\]

(5.10)

for some non-negative finite kernel \( \nu_B \) from \( E^\Delta \) to \( E^\Delta \) and some \( m \in C_\Delta(E) \);

(iii) \( Q_0 \equiv 0 \);

(iv) \( Q_1 \) is of the form

\[
Q_1(g)(x) = \alpha(x)g(x, x), \quad g \in C_\Delta(E) \otimes C_\Delta(E)
\]

(5.11)

where \( \alpha \in C_\Delta(E) \) with values in \( \mathbb{R}_+ \);

(v) \( Q_2 \) admits a \((\beta, \pi)\)-representation. Moreover, the parameters \( \pi \) and \( \beta \) are bounded and continuous on \((E^\Delta)^2 \setminus \{x = y\}\) and \( \pi + \pi + 2\beta \in \hat{C}_\Delta(E^2) \), where \( \pi(x, y) = \pi(y, x) \).

For the state space \( M_+(E) \), the following equivalence holds.

A linear operator \( L : D \to C(M_+(E)) \) is \( M_+(E) \)-polynomial, there exists an \( M_+(E) \)-valued solution to the martingale problem for all initial conditions in \( M_+(E) \) and all solutions have continuous paths, if and only if \( L \) satisfies (i)-(v), \( b \in M_+(E) \), and \( \nu_B \) is a kernel from \( E \) to \( E \).
Remark 5.12.  (i) Condition (ii) for the bounded operator $B_1$ is equivalent to the requirement that $B_1g + \|B_1\|g \geq 0$ for all $g \geq 0$. This in turn is also equivalent to $(\exp(tB_1))_{t \geq 0}$ being a positive semigroup on $C_\Delta(E)$ and to $B_1$ satisfying the positive minimum principle; see Theorem B.II.1.3 in Arendt et al. (1986). When $E$ is finite this exactly means that the matrix $B_1$ has non-negative off-diagonal elements. The decomposition as of Condition (ii) then means that such a matrix is decomposed into a transition rate matrix where the diagonal elements are defined such that the rows sum up to 0 and a diagonal matrix (corresponding to $m$) where this procedure is compensated.

(ii) Note that in the case of $D = C_\Delta(E)$, adding to $Q_2$ operators of the form

$$(A \otimes A)g,$$

where $A$ is a generator of a strongly continuous positive group on $C_\Delta(E)$ does not yield more generality, since $A$ is necessarily of the form $Ag(x) = a(x)g(x)$ for some $a \in C_\Delta(E)$ as proved in Lemma 3.6. Therefore, $(A \otimes A)(x, y) = a(x)a(y)g(x, y)$. This however can be absorbed in the function $\beta$.

Proof of Theorem 5.11. Assume that $L$ satisfies (4.3) with conditions (i) to (v). By Lemma A.3 condition (ii) implies that $B_1$ satisfying the positive minimum principle on $E^\Delta$. Hence, all conditions of Theorem 5.9 are satisfied and we get that $L$ is $M_+(E)$-polynomial and that its martingale problem has an $M_+(E^\Delta)$ valued solution with continuous paths for every initial condition $\nu \in M_+(E^\Delta)$. By Theorem 5.9 we also get that any solution with initial value $\nu \in M_+(E)$ takes values in $M_+(E)$ provided that $b \in M_+(E)$, $\nu_B$ is a kernel from $E$ to $E$, and $m \in C_0(E)$. Indeed, it remains to verify that $\lim_{n \to \infty} B_1g_n \geq B_11$ where $g_n \in C_0(E)$ is a sequence such that $\lim_{n \to \infty} g_n = 1_E$. The claim then follows by the dominated convergence theorem.

We now prove the opposite implication. Assume that $L$ is $M_+(E)$-polynomial, then there exists an $M_+(E^\Delta)$-valued solution to its martingale problem for each initial condition in $M_+(E^\Delta)$ and all solutions have continuous paths. Theorem 4.6 and Lemma 5.2 imply that $L$ satisfies (4.3), and due to Lemma B.1 also the positive maximum principle on $M_+(E^\Delta)$. An application of Lemma A.2 then yields the result.

Next, assume that $L$ satisfies (i)-(v), $b \in M_+(E)$, and $\nu_B$ is a kernel from $E$ to $E$. By the first part of the theorem it holds that $L$ is $M_+(E)$-polynomial, there exists an $M_+(E^\Delta)$-valued solution to the martingale problem for all initial conditions in $M_+(E)$, and all solutions have continuous paths. By considering any sequence of functions $g_n \in C_0(E)$ with $0 \leq g_n(x) \uparrow 1$ for all $x \in E$ the claim follows by Lemma B.2(ii).

Finally, assume that $L$ is $M_+(E)$-polynomial, there exists an $M_+(E)$-valued solution to the martingale problem for all initial conditions in $M_+(E)$ and all solutions have continuous paths. By the same argument as above $L$ satisfies (4.3) and due to Lemma B.1 the positive maximum principle on $M_+(E)$. Conditions (i)-(v) are then satisfied by Lemma A.2.

It thus remains to prove that $b \in M_+(E)$, $\nu_B$ is a kernel from $E$ to $E$. As a first step, observe that

$$\mathbb{E}[\langle 1, X_t \rangle] \leq \langle 1, X_0 \rangle + \int_0^t \langle 1, b \rangle + \|m\|\mathbb{E}[\langle 1, X_s \rangle]ds$$

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and Lemma 5.7(i) we can apply the Gronwall inequality to get that by

\[ \mathbb{E}[1, X_t] \leq (X_0(E^\Delta) + b(E^\Delta)t) \exp(||m||t) =: m_1(X_0, t). \]

Next, since \( \tilde{B}_1g := B_1g - ||m||g \) is a bounded dissipative operator on \( C_\Delta(E) \) we know that the range of \( (\ell - \tilde{B}_1) \) is given by \( C_\Delta(E) \) for each \( \ell > 0 \). This implies that the range of \( (K - B_1) \) is given by \( C_\Delta(E) \) for each \( K > ||m|| \). Fix now \( K > ||m|| \), \( (h_n)_{n \in \mathbb{N}} \subseteq C_\Delta(E) \) converging to \( 1_\Delta \) in the bounded pointwise sense, and choose \( (g_n)_{n \in \mathbb{N}} \subseteq C_\Delta(E) \) such that \( (K - B_1)g_n = h_n \). Then we have that

\[
\langle g_n, X_0 \rangle e^0 = -\mathbb{E}[\langle g_n, X_t \rangle] e^{-Kt} - \int_0^t \mathbb{E}[\langle g_n, X_s \rangle e^{-Ks} + \langle g_n, b \rangle + \langle B_1g_n, X_s \rangle e^{-Ks}] ds
\]

\[
= -\mathbb{E}[\langle g_n, X_t \rangle] e^{-Kt} + \frac{1}{K} \langle g_n, b \rangle (e^{-Kt} - 1) - \int_0^t \mathbb{E}[\langle h_n, X_s \rangle e^{-Ks}] ds.
\]

Since \( ||\mathbb{E}[\langle g_n, X_t \rangle]| \leq ||g_n|| ||\mathbb{E}[1, X_t]| \leq ||g_n|| m_1(X_0, t) \) sending \( t \) to infinity we get

\[
\langle g_n, X_0 \rangle = -\frac{1}{K} \langle g_n, b \rangle + \int_0^\infty \mathbb{E}[\langle h_n, X_s \rangle e^{-Ks}] ds,
\]

and hence

\[
\lim_{n \to \infty} \langle g_n, X_0 \rangle = \frac{1}{K} b(\Delta) + \int_0^\infty \mathbb{E}[X_s(\Delta)] e^{-Ks} ds = -\frac{1}{K} b(\Delta),
\]

showing that \( \lim_{n \to \infty} g_n(x) = b(\Delta) = 0 \) for each \( x \in E \). Using that

\[
\langle g_n, X_0 \rangle \leq \sup_{n \in \mathbb{N}} ||h_n|| \int_0^\infty \mathbb{E}[m_1(X_0, s) e^{-Ks}] ds < \infty
\]

we can conclude that \( (g_n)_{n \in \mathbb{N}} \) is a bounded sequence.

Next, observe that \( \lim_{n \to \infty} \langle B_1g_n(x) - Kg_n(x) \rangle = \lim_{n \to \infty} -h_n(x) = -1_{\{x = \Delta\}} \leq 0 \) for each \( x \in E^\Delta \). Inserting the from of \( B_1 \) yields

\[
\left( \lim_{n \to \infty} g_n(x) \right) \left( \nu_B(x, \Delta) - 1_{\{x = \Delta\}}(\nu_B(\Delta, E^\Delta) + K - m(\Delta)) \right) = -1_{\{x = \Delta\}}.
\]

Since inserting \( x = \Delta \) we get that \( (\lim_{n \to \infty} g_n(x)) \neq 0 \), we can conclude that \( \nu_B(x, \Delta) = 0 \) for each \( x \in E \) proving the claim. \( \square \)

**Example 5.13.** In accordance with Example 5.6, suppose that \( L \) satisfies the conditions of Theorem 5.11 for \( B_0 = 0 \) and \( Q_1 = 0 \). Then the dual operator \( L^0_m \) satisfies \( L^0_m = 0, \ldots, L^{m-1}_m = 0 \) and

\[
L^m_m(g^{\otimes m}) = mB_1(g) \otimes g^{\otimes (m-1)} + \frac{m(m-1)}{2} Q_2(g \otimes g) \otimes g^{\otimes (m-2)}.
\]

Observe that \( L^m_m \) satisfies positive minimum principle. Since \( B_1 \) and \( Q_2 \) are bounded the operator \( L^m_m \) generates a strongly continuous positive semigroup and the moment formula can thus be applied.

A quick look at the generator \( L \) of the process \( (S_t(\mu))_{t \geq 0} \) studied in Example 5.6 shows that \( L \) satisfies the conditions of Theorem 5.11 for \( B_0 = 0, B_1 = 0, Q_1 = 0, \) and \( Q_2(g) = \sigma^2 g \) and is thus of the given form for \( C = 0, n = 1, \) and \( A_1 g = \sigma g \).

A similar reasoning can be applied to compute the generator \( L \) of \( (Y_t(\mu))_{t \geq 0} \) for \( dY_t = \sigma Y_t dW_t \). The obtained operator \( L \) does not satisfy the conditions of Example 5.6 but satisfies the conditions of Theorem 5.11 for the parameters \( B_0 = 0, B_1 = 0, Q_1(g)(x) = \sigma^2 g(x, x), \) and \( Q_2 = 0 \).
5.3 Uniqueness in law

Having established existence to the martingale problem, we are now concerned with uniqueness. As in the finite dimensional case, uniqueness in law follows if the (marginal) moments determine the finite-dimensional marginal distributions. This property is known as determinacy of the moment problem. Whenever this holds true, we can then conclude uniqueness in law by relying on the moment formula. The results of this section generalize Theorem 4.2 in Filipović and Larsson (2016) and provide conditions ensuring the existence of exponential moments. These conditions are satisfied by measure-valued affine diffusions treated in Section 6 below.

Lemma 5.14. Let \( L \) be an \( M_+(E) \)-polynomial operator and \((X_t)_{t \geq 0}\) an \( M_+(E^\Delta) \)-valued solution to the corresponding martingale problem with initial value \( X_0 = \nu \). Suppose that the conditions of Theorem 5.5 are satisfied for each \( g \in \bigoplus_{k=0}^m D^{\otimes k} \) and each \( m \in \mathbb{N} \). If for each \( t \geq 0 \) there exists \( \varepsilon > 0 \) with

\[
\mathbb{E}[e^{\varepsilon\langle 1, X_t \rangle}] < \infty, \tag{5.12}
\]

then the law of \( X \) is uniquely determined by \( L \) and \( \nu \).

Proof. Observe that condition (5.12) yields \( \mathbb{E}[e^{\varepsilon\langle g, X_t \rangle}] < \infty \) for each \( g \in D \) with \( |g| \leq 1 \). Following proof of Lemma 4.1 in Filipović and Larsson (2016) we thus get that the law of

\[
((g_1, X_t), \ldots, (g_n, X_t))_{t \geq 0}
\]

is uniquely determined by \( L \) and \( \nu \) for each \( g_1, \ldots, g_n \in D \). Using that \( D \) is a dense subspace of \( C_\Delta(E) \) the claim follows. \qed

If \( L \) admits a representation as described in Theorem 5.9 explicit sufficient conditions for (5.12) can be provided. Indeed, the following theorem assumes a linear growth condition, meaning that \( Q_2 \equiv 0 \), to ensure existence of exponential moments.

Theorem 5.15. Let \( L : D \rightarrow C(M_+(E)) \) be a linear operator satisfying the assumptions of Theorem 5.9 and \( X \) be an \( M_+(E^\Delta) \)-valued solution of the corresponding martingale problem. Assume that \( Q_2 \equiv 0 \) and that \( B_1 : D \rightarrow C_\Delta(E) \) is the generator of a strongly continuous semigroup \((P_t)_{t \geq 0}\). Then for each \( t \geq 0 \) there exists \( \varepsilon > 0 \) such that condition (5.12) holds true and thus the martingale problem for \( X \) is well-posed.

The proof of Theorem 5.15 relies on the following generalization of Theorem 1.3 in Hajek (1985).

Lemma 5.16. Let \((Y_t)_{t \geq 0}\) be a stochastic process satisfying \( dY_t = b^Y_t dt + dM_t \) for some continuous local martingale \( M \) whose quadratic variation is given by \( d[M]_t = (\sigma_t^Y)^2 dt \) and some continuous stochastic processes \( b^Y \) and \( \sigma^Y \). Suppose that \( \sup_{t \geq 0} |b_t^Y| \leq m \) and \( \sup_{t \geq 0} |\sigma_t^Y| \leq \rho \) almost surely for some constants \( m, \rho \). Then for any non-decreasing convex function \( \Phi \) on \( \mathbb{R} \) it holds

\[
\mathbb{E}[\Phi(Y_t)] \leq \mathbb{E}[\Phi(V)],
\]

where \( V \) is a Gaussian random variable with mean \( Y_0 + mt \) and variance \( \rho^2 t \).
**Proof.** Without loss of generality we can assume the existence of an auxiliary Brownian motion $W$ independent of $Y$. Set then

$$Z_t^\pm := Y_0 + mt + M_t \pm \int_0^t (\rho^2 - (\sigma_t)^2)^{1/2} dW_t$$

and observe that $\frac{1}{2}(Z_t^+ + Z_t^-) - Y_t = mt - \int_0^t \sigma_t^2 ds \geq 0$. Moreover, since

$$[Z^\pm]_t = [M]_t + \int_0^t (\rho^2 - (\sigma_t)^2) ds = \rho^2 t,$$

by Lévy characterization theorem we get that $(\frac{1}{2}(Z_t^+ - Y_0 - mt))_{t \geq 0}$ is a standard Brownian motion. This in particular implies that $Z_t^+$ and $Z_t^-$ are both Gaussian random variables with mean $Y_0 + mt$ and variance $\rho^2 t$. We can thus use the properties of $\Phi$ to conclude

$$\mathbb{E}[\Phi(Y_t)] \leq \mathbb{E}[\Phi(\frac{1}{2}(Z_t^+ + Z_t^-))] \leq \frac{1}{2} \mathbb{E}[\Phi(Z_t^+) + \Phi(Z_t^-)] = \mathbb{E}[\Phi(V)].$$

We are now ready to prove Theorem 5.15. The proof is an adaptation of the proof of Lemma C.1 in Filipović and Larsson (2016) to the current setting.

**Proof of Theorem 5.15.** Fix $T \geq 0$ and note that setting $f_s := \langle P_{T-s}1, \cdot \rangle$, by (5.1) we have that

$$N^{f_s}_t = f_s(X_t) - f_s(\nu) - \int_0^t Lf_s(X_r) dr$$

is a local martingale for each $s \in [0, T]$. By Lemma 5.2 we know that

$$[N^{f_1}, N^{f_2}]_t = 2 \int_0^t \Gamma(f_1, f_2)(X_r) dr = \int_0^t \langle \alpha(f_1) \alpha(f_2), X_t \rangle dt.$$

Set then $Y_t := \langle P_{T-t}1, X_t - A_t \rangle$ for $A_t := \langle P_T1, X_0 \rangle + \int_0^t \langle P_{T-s}1, b \rangle ds$. Since

$$d\langle P_{T-t}1, X_t \rangle = -\langle B_t P_{T-t}1, X_t \rangle dt + Lf_t(X_t) dt + \text{(local martingale)}$$

we know that $Y$ is a local martingale whose quadratic variation is given by

$$d[Y]_t = \langle \alpha(P_{T-t}1)^2, X_t \rangle dt =: \sigma_t^2 dt.$$ 

Using that $\sigma_t^2 \leq (\sup_{t \in [0,T]} ||\alpha|| ||P_{T-t}1||) \langle P_{T-t}1, X_t \rangle$ and the fact that $A$ is deterministic and thus bounded on $[0, T]$ we get

$$\sigma_t^2 \leq C(1 + |Y_t|). \quad (5.13)$$

Following now the proof of Lemma C.1 in Filipović and Larsson (2016) we can apply Lemma 5.16 to conclude that

$$\mathbb{E}[e^{\varepsilon|Y_T|}] < \infty$$

for some $\varepsilon > 0$. The claim follows by noting that $\langle 1, X_T \rangle = Y_T$. \qed
6 Affine diffusions on $M_+(E)$

We now define operators of affine type and consider solutions to the associated martingale problems. As we will show these solutions constitute a subclass of polynomial processes, which correspond to classical measure-valued branching Markov diffusions analyzed e.g. in Fitzsimmons (1988); Etheridge (2000); Li (2010) and thus range in the class of measure-valued affine processes. They can be considered as generalizations of the Dawson-Watanabe superprocess (in the terminology of Etheridge (2000)\(^3\)), also called super-Brownian motion, where the constant diffusion coefficient is replaced by a function and where the spatial motion is governed by an operator satisfying the positive minimum principle. As it is well-known for affine processes, additionally to the moment formula, the Laplace transform of the process’ marginals is exponentially affine in the initial state and the characteristic exponent can be computed by solving a Riccati partial differential equation, which we shall introduce in Section 6.2.

6.1 Operators of affine type and their characterization in the diffusion setting

Let us here start by defining operators of affine type, inspired by the classical form of the infinitesimal generator of affine processes on finite dimensional state spaces, see, e.g., Kawazu and Watanabe (1971); Duffie et al. (2003); Cuchiero et al. (2011, 2016b).

Definition 6.1. We say that a linear operator $L : D \to \mathcal{M}(M(E))$ is of affine type on $M_+(E)$ if there exist maps $F : D \to \mathbb{R}$ and $R : D \to C_\Delta(E)$ such that

$$L \exp(\langle g, \cdot \rangle)(\nu) = (F(g) + \langle R(g), \nu \rangle) \exp(\langle g, \nu \rangle)$$

for all $g \in D_-$ and $\nu \in M_+(E)$, where $D_-$ denotes all function in $D$ with values in $\mathbb{R}_-$.

For a finite-dimensional diffusions it is well-known that its generator is of affine type if and only if the drift and diffusion coefficients are affine functions (see, e.g., Kawazu and Watanabe (1971); Duffie et al. (2003)). The following assertion states the same result for the measure-valued setting. We report and prove it here for the reader convenience. It implies in particular that (in the current diffusion setting) operators of affine type constitute a subclass of polynomial operators.

Theorem 6.2. A linear operator $L : D \to \mathcal{M}(M(E))$ is of affine type on $M_+(E)$ and its carré-du-champs operator $\Gamma$ is an $M_+(E)$-derivation if and only if $L$ admits a representation

$$L f(\nu) = B_0(\partial f(\nu)) + \langle B_1(\partial f(\nu)), \nu \rangle$$

$$+ \frac{1}{2} \left( Q_0(\partial^2 f(\nu)) + \langle Q_1(\partial^2 f(\nu)), \nu \rangle \right), \quad f \in D, \nu \in M_+(E) \tag{6.1}$$

for some linear operators $B_0 : D \to \mathbb{R}$, $B_1 : D \to C_\Delta(E)$, $Q_0 : D \otimes D \to \mathbb{R}$, $Q_1 : D \otimes D \to C_\Delta(E)$. The restriction of these operators on $D_-$ and $D_- \otimes D_-$, respectively, are uniquely

\(^3\)Note that in Li (2010) “Dawson-Watanabe superprocess” is used for a class of measure-valued branching processes which can also exhibit jumps. Essentially the class of processes that we obtain here is the subset of processes with continuous trajectories therein.
determined by $L$ and the following relations hold for the functions $F$ and $R$ of Definition 6.1

\begin{align}
F(g) &= B_0(g) + \frac{1}{2}Q_0(g \otimes g), \\
R(g) &= B_1(g) + \frac{1}{2}Q_1(g \otimes g).
\end{align}

(6.2)

(6.3)

**Proof.** Assume first that $L$ satisfies (6.1). Then for $f(\nu) = \exp(\langle g, \nu \rangle)$ with $g \in D_-$ and $\nu \in M_+(E)$ we have

\[
L \exp(\langle g, \cdot \rangle)(\nu) = \exp(\langle g, \nu \rangle)(B_0(g) + \langle B_1(g), \nu \rangle + \frac{1}{2}(Q_0(g \otimes g) + \langle Q_1(g \otimes g), \nu \rangle)).
\]

Hence $L$ is an affine operator with $F(g) = B_0(g) + \frac{1}{2}Q_0(g \otimes g)$ and $R(g) = B_1(g) + \frac{1}{2}Q_1(g \otimes g)$. Hence $L$ is of affine type. Moreover, a direct calculation yields for all $p, q \in \mathcal{D}$

\[
\Gamma(p, q)(\nu) = Q_0\left(\partial p \otimes \partial q(\nu)\right) + \langle Q_1(\partial p \otimes \partial q(\nu)), \nu \rangle \quad \text{for all } \nu \in M_+(E),
\]

which is an $M_+(E)$-derivation.

Conversely, assume that $L$ is of affine type and that its carré-du-champs operator $\Gamma$ is an $M_+(E)$-derivation. We now explain how the affine property and the fact that $\Gamma$ is a derivation imply the form (6.1) on $f(\nu) = \exp(\langle g, \nu \rangle)$ for $g \in D_-$. The extension to all of $\mathcal{D}$ follows then similarly as in the proof of Theorem 4.6. Observe that since $L$ satisfies the affine property it holds that

\[
L \exp(\langle g, \nu \rangle) = (F(g) + \langle R(g), \nu \rangle) \exp(\langle g, \nu \rangle),
\]

for each $g \in D_-$. Define then

\[
B_0(g) := 2F(g) - \frac{1}{2}F(2g), \quad B_1 g := R(g) - \frac{1}{2}R(2g),
\]

\[
Q_0(g \otimes g) := F(2g) - 2F(g), \quad Q_1(g \otimes g) := R(2g) - 2R(g),
\]

and observe that the claim follows by showing that these operators are linear. Set $p_g(\nu) := \exp(\langle g, \nu \rangle)$ and note that

\[
B_0(g) + \langle B_1(g), \nu \rangle = L(2p_g p_g(\nu)^{-1} - \frac{1}{2}p_g^2 p_g(\nu)^{-2})(\nu),
\]

\[
Q_0(g \otimes g) + \langle Q_1(g \otimes g), \nu \rangle = L(p_g^2 p_g(\nu)^{-2} - 2p_g p_g(\nu)^{-1})(\nu),
\]

for each $g \in D_-$. Fix now $g_i, f_i \in D_-$ and $\alpha_i, \beta_i \in \mathbb{R}$ such that $\sum \alpha_i g_i = \sum \beta_i f_i$ and note that

\[
\sum \alpha_i B_0(g_i) - \sum \beta_i B_0(f_i) + \left\langle \sum \alpha_i B_1(g_i) - \sum \beta_i B_1(f_i), \nu \right\rangle = Lp(\nu)
\]

for $p = \alpha_i(2p_{g_i} p_{g_i}(\nu)^{-1} - \frac{1}{2}p_{g_i}^2 p_{g_i}(\nu)^{-2}) - \beta_i(2p_{f_i} p_{f_i}(\nu)^{-1} - \frac{1}{2}p_{f_i}^2 p_{f_i}(\nu)^{-2})$. Since $\partial p(\nu) = 0$ and $\partial p^2(\nu) = 0$ the linearity of $B_0$ and $B_1$ follows by Lemma 4.5. Similarly, fix $g_i, f_i \in D_-$ and $\alpha_i, \beta_i \in \mathbb{R}$ such that $\sum \alpha_i g_i \otimes g_i = \sum \beta_i f_i \otimes f_i$ and note that

\[
\sum \alpha_i Q_0(g_i \otimes g_i) - \sum \beta_i Q_0(f_i \otimes f_i) + \left\langle \sum \alpha_i Q_1(g_i \otimes g_i) - \sum \beta_i Q_1(f_i \otimes f_i), \nu \right\rangle = Lp(\nu)
\]

for $p = \alpha_i(\frac{1}{2} p_{g_i}^2 p_{g_i}(\nu)^{-2} - 2p_{g_i} p_{g_i}(\nu)^{-1}) - \beta_i(\frac{1}{2} p_{f_i}^2 p_{f_i}(\nu)^{-2} - 2p_{f_i} p_{f_i}(\nu)^{-1})$. Since $\partial p(\nu) = 0$ and $\partial p^2(\nu) = 0$ the linearity of $Q_0$ and $Q_1$ follows by Lemma 4.5. \qed
We now introduce the subclass of measure-valued affine diffusions via affine type operators.

**Definition 6.3.** Let $L$ be of affine type on $M_+(E)$. Then any continuous $M_+(E)$-valued solution to the martingale problem for $L$ is called a measure-valued affine diffusion.

### 6.2 Laplace transform and Riccati (partial) differential equations

Due to the additional properties of affine diffusions, not only the moment formula holds true but also the Laplace transform can be computed explicitly via non-linear partial differential equations of Riccati type. We start here by introducing the corresponding solution concept. Recall that we refer to differential equations as ODEs even though they often correspond to PDEs or PIDEs depending on the state space and the involved operators.

**Definition 6.4.** Let $R$ be given by (6.3). Then we call a function $t \mapsto \psi_t$ with values in $D$ a solution to the Riccati ODE

$$\partial_t \psi_t = R(\psi_t), \quad \psi_0 = g \in D,$$

if for every $t > 0$ it holds

$$\langle \psi_t, \nu \rangle = \langle g, \nu \rangle + \int_0^t \langle R(\psi_s), \nu \rangle ds$$

(6.4)

for all $\nu \in M_+(E^\Delta)$.

Note that the reason why (6.4) is called Riccati differential equation is because $g \mapsto R(g)$ is a quadratic function as seen from (6.3).

**Remark 6.5.** Similarly as for the moment formula note that the above solution concept reduces to a more classical solution if we take $\nu = \delta_x$ with $x \in E$. Indeed, (6.4) can then be transformed into

$$\psi_t(x) = g(x) + \int_0^t R(\psi_s)(x) ds$$

and thus reduces to a classical (except of the integral form) solution of a Riccati PDE.

Note that in contrast to strong solutions, which were in the current context for instance considered in (Iscoe, 1986, Appendix) or (Li, 2010, Section 7.1) and where $\psi$ is required to be a continuously (Fréchet) differentiable curve $\psi : \mathbb{R}_+ \to D$, Definition 6.4 corresponds to an (analytically) weak solution concept.

In the following we prove the exponential affine property of the Laplace transform.

**Theorem 6.6.** Let $L$ be an operator of affine type with functions $F$ and $R$. Suppose that an $M_+(E^\Delta)$-valued solution $(X_t)_{t \geq 0}$ to the martingale problem for $L$ exists. Assume furthermore that there is a solution in the sense of Definition 6.4 of the Riccati ODE on $[0, T]$ given by

$$\partial_t \psi_t = R(\psi_t), \quad \psi_0 = g \in D_-,$$

(6.5)
which takes values in $D_-$ where $D_-$ denotes all functions in $D$ with values in $\mathbb{R}_-$. Define furthermore
\[ \phi_t = \int_0^t F(\psi_s)ds \] (6.6)
and suppose it takes values in $\mathbb{R}_-$. Then, for all $0 \leq t \leq T$ the representation
\[ \mathbb{E} \left[ \exp(\langle g, X_T \rangle) \mid \mathcal{F}_t \right] = \exp(\phi_{T-t} + \langle \psi_{T-t}, X_t \rangle) \] (6.7)
holds almost surely.

**Proof.** Due to the fact that $X$ solves the martingale problem for $L$, we can deduce analogously as in the proof of (Li, 2010, Theorem 7.13) that $\exp(\phi_{T-t} + \langle \psi_{T-t}, X_t \rangle)$ is a local martingale. By the assumptions that $\psi$ takes values in $D_-$ and $\phi \in \mathbb{R}_-$, this is actually a true martingale as it is bounded by 1 and we thus have
\[ \exp(\phi_{T-t} + \langle \psi_{T-t}, X_t \rangle) = \mathbb{E}[\exp(\phi_0 + \langle \psi_0, X_T \rangle) \mid \mathcal{F}_t] = \mathbb{E}[\exp(\langle g, X_T \rangle) \mid \mathcal{F}_t], \]
proving the claim. \qed

### 6.3 Existence and uniqueness in law

We can now combine Theorem 6.2, Theorem 5.9 and Theorem 6.6 to get the following existence and uniqueness in law result for measure-valued affine diffusions. Recall that $D \subseteq C(\Delta)(E) \cap C_0(E)$ be a dense linear subspace containing the constant function 1.

**Corollary 6.7.** Let $L : D \to C(M_+(E))$ be a linear operator of form (6.1), where

(i) $B_0 : D \to \mathbb{R}$ is given by $B_0(g) = \langle g, b \rangle$ with $b \in M_+(E^\Delta)$;

(ii) $B_1 : D \to C_\Delta(E)$ satisfies the positive minimum principle on $E^\Delta$;

(iii) $Q_0 \equiv 0$;

(iv) $Q_1$ is of the form
\[ Q_1(g) = \alpha \text{diag}(g), \quad \text{that is,} \quad Q_1(g)(x) = \alpha(x)g(x,x), \quad g \in D \otimes D \]
where $\alpha \in C_\Delta(E)$ with values in $\mathbb{R}_+$.

Then $L$ is of affine type on $M_+(E)$ and its martingale problem has an $M_+(E^\Delta)$-valued solution with continuous paths for every initial condition $\nu \in M_+(E^\Delta)$.

If additionally, the measure $b$ and the initial condition $\nu$ lie in $M_+(E)$ and if there exists some function $m \in C_\Delta(E)$ such that
\[ \liminf_{n \to \infty} (B_1(1 - g_n) - m(1 - g_n)) \leq 0, \]
for some sequence $g_n \in D \cap C_0(E)$ satisfying $\lim_{n \to \infty} g_n = 1_E$, then any solution to the martingale problem takes values in $M_+(E)$.

If, in either of the $M_+(E)$ or $M_+(E^\Delta)$ cases,
\[ \partial_t \psi_t = B_1 \psi_t + \frac{1}{2} \alpha \psi_t^2, \quad \psi_0 = g \in D_- \] (6.8)
has a solution in the sense of Definition 6.4 with values in $D_-$ for all $t \geq 0$, then the corresponding martingale problem is well-posed.
Proof. The existence result is just a consequence of Theorem 6.2 and Theorem 5.9. Concerning the well-posedness of the martingale problem, observe that the function $F$ associated to the operator $L$ of affine type is of the form

$$F(g) = B_0(g) = \langle g, b \rangle$$

for $b \in M_+(E^\Delta)$. Hence, $F$ maps $\psi_t \in D_-$ to $\mathbb{R}_-$ and the assumptions of Theorem 6.6 are satisfied. Hence the Laplace transform of $X_t$ is given by

$$\mathbb{E}[\exp(\langle g, X_t \rangle)] = \exp(\tilde{\phi}_t + \langle \psi_t, X_0 \rangle).$$

Since $g \in D_-$ was arbitrary and $D$ is dense in $C_\Delta(E)$, the law of $X_t$ is uniquely determined for all $t \geq 0$. From (Ethier and Kurtz, 2009, Theorem 4.4.2), we infer that $X$ is a Markov process thus the unique solution to the martingale problem associated to $L$.

Remark 6.8. Let us here explain how the above result relates to the literature on $(\xi, \phi)$- (Dawson-Watanabe) - superprocesses as defined in (Li, 2010, p.42). Indeed, $\xi$ is the spatial motion which is described by its infinitesimal generator $A$ that can be obtained from the current $B_1$ via

$$A g = B_1 g - mg,$$

as explained in Remark 1.2. The branching mechanism $\phi$ (not to be confused with the function defined via (6.6)) then corresponds to

$$\phi(x, g) = -m(x)g(x) + \frac{1}{2} \alpha(x)g(x)^2.$$ 

Hence, in the notation of (Li, 2010, Eq.(2.27)), $b = -m$ and $c = \frac{1}{2} \alpha$. The $B_0$-part can be added as an immigration part as considered in (Li, 2010, Section 9.3).

This identification allows us to apply the results of Li (2010) to our setup. In view of the Riccati equations this means in particular that (6.8) corresponds (up to a change of sign) to (Li, 2010, Eq.(7.4)), which is of the form

$$\partial_t V_t f = AV_t f - \phi(\cdot, V_t f) = AV_t f + mV_t f - \frac{1}{2} \alpha(V_t f)^2 = B_1 V_t f - \frac{1}{2} \alpha(V_t f)^2,$$

$$V_0 f = f (= -g in our notation).$$

Supposing that $A$ generates a Feller semigroup $(P_t)_{t \geq 0}$ and that $m = -b \in C_\Delta(E)$, then by Theorem (Li, 2010, Theorem 7.11) this equation is in turn equivalent to the integral equation (Li, 2010, Eq.(5.32))

$$V_t f = P_t f - \int_0^t P_{t-s} \phi(\cdot, V_s f) ds,$$

which corresponds to a mild solution. As $\psi_t = -V_t f$ with initial value $\psi_0 = g = -f$ and denoting the positive semigroup generated by $B_1$ with $Q$ we thus get a similar equivalence between our solution concept for (6.8) and the following integral equation in weak form (corresponding to a weakly mild solution)

$$\langle \psi_t, \nu \rangle = \langle Q_t g, \nu \rangle + \frac{1}{2} \int_0^t \langle Q_{t-s} \alpha \psi_s^2, \nu \rangle ds,$$

$$= \langle P_t g, \nu \rangle + \int_0^t \langle P_{t-s} (m \psi_s + \frac{1}{2} \alpha \psi_s^2), \nu \rangle ds.$$
By (Li, 2010, Corollary 5.17) which is a consequence of (Li, 2010, Proposition 2.20 and (Eq. 2.33)) it admits a unique negative solution in the domain of $\mathcal{A}$ (in the strong sense as generator of the Feller semigroup $(P_t)_{t \geq 0}$). The latter property is a consequence of (Li, 2010, Theorem 7.11). From this discussion we can now deduce the following corollary.

**Corollary 6.9.** Let the Conditions (i)-(iv) of Corollary 6.7 be satisfied and suppose additionally that $B_1$ is the generator of a strongly continuous positive semigroup. Then (6.8) has a unique solution in the sense of Definition 6.4 with values in $D_-$ for all $t \geq 0$ and the martingale problem is well-posed.

**Proof.** Since $B_1$ is the generator of a strongly continuous positive semigroup, it can be decomposed into $B_1 g = A g + m g$ where $m \in C_\Delta(E)$ and $A$ is the (strong) generator of a Feller semigroup. Hence the assertion follows from Theorem 7.11 and Corollary 5.17 in Li (2010) as explained in Remark 6.8.

**Remark 6.10.** Note that the conditions of Corollary 6.9 are in line with the ones of Theorem 5.15 from which uniqueness in law can be deduced as well.

Similarly as in the polynomial case we get a characterization of affine diffusions when $D = C_\Delta(E)$.

**Corollary 6.11.** Let $D = C_\Delta(E)$ and let $L : D \to C(M_+(E))$ be a linear operator. Then $L$ is of affine type on $M_+(E)$, there exists an $M_+(E^\Delta)$-valued solution to the martingale problem for all initial conditions in $M_+(E^\Delta)$ and all solutions have continuous paths, if and only if $L$ satisfies (4.3) with

(i) $B_0 : C_\Delta(E) \to \mathbb{R}$ is given by $B_0(g) = \langle g, b \rangle$ with a Radon measure $b \in M_+(E^\Delta)$;

(ii) For every $g \in C_\Delta(E)$

$$B_1 g(x) = \int g(\xi) - g(x) \nu_B(x, d\xi) + m(x) g(x)$$  \hspace{1cm} (6.9)

for some non-negative finite kernel $\nu_B$ from $E^\Delta$ to $E^\Delta$ and some $m \in C_\Delta(E)$;

(iii) $Q_0 \equiv 0$;

(iv) $Q_1$ is of the form

$$Q_1(g)(x) = \alpha(x) g(x, x), \quad g \in C_\Delta(E) \otimes C_\Delta(E)$$  \hspace{1cm} (6.10)

where $\alpha \in C_\Delta(E)$ with values in $\mathbb{R}_+$.

For the state space $M_+(E)$, the following equivalence holds.

A linear operator $L : D \to C(M_+(E))$ is of affine type on $M_+(E)$, there exists an $M_+(E^\Delta)$-valued solution to the martingale problem for all initial conditions in $M_+(E^\Delta)$ and all solutions have continuous paths, if and only if $L$ satisfies (i)-(iv), $b \in M_+(E)$, and $\nu_B$ is a kernel from $E$ to $E$.

Moreover, in either case

$$\partial_t \psi_t = B_1 \psi_t + \frac{1}{2} \alpha \psi_t^2, \quad \psi_0 = g \in C_\Delta(E)$$

has a unique solution in the sense of Definition 6.4 with values in $D_-$ for all $t \geq 0$, which implies (6.7) and well-posedness of the martingale problem.
Proof. All assertions except the very last one follow from combining Theorem 6.2 and Theorem 5.11. The last one is a consequence of Corollary 6.9 since $B_1$ is decomposed into $B_1 g = A g + m g$ where $A g(x) = \int g(\xi) - g(x) \nu_B(x, d\xi)$ generates a Feller semigroup corresponding to a pure jump process and $m \in C_\Delta(E)$. \hfill \qed

7 Examples and applications

Having established the general setting for measure-valued affine and polynomial diffusions, we now consider several important examples.

7.1 Finite underlying space

We start with a finite underlying space $E$ consisting of $m \geq 1$ points, i.e. $E = \{1, ..., m \}$. Then $C_\Delta(E) = C(E)$ is finite-dimensional, hence any dense linear subspace is equal to the whole space. In this setting, any $M_+(E)$-valued process is of the form $X_t = \sum_{i=1}^m Z^i_t \delta_i$ for some $\mathbb{R}_+^m$-valued process $Z = (Z^1, ..., Z^m)$. Hence, Theorem 5.11 corresponds to the finite dimensional characterization of polynomial operators with state space $\mathbb{R}_+^m$, as proved in (Filipović and Larsson, 2016, Proposition 6.4). Indeed, comparing with this proposition we recognize the following identifications:

(i) the Radon measure $b \in M_+(E)$ corresponds to the vector $\beta_J \in \mathbb{R}_+^m$;

(ii) the operator $B_1$ corresponds to the submatrix $B_{JJ}$ and \ref{eq:5.10} translates to positive off-diagonal elements;

(iii) the function $\alpha \in C_\Delta(E)$ with values in $\mathbb{R}_+$ corresponds to the vector $\phi \in \mathbb{R}_+^m$;

(iv) the functions $\pi$ and $\beta$ correspond to the matrix $\Pi$ and $\alpha \in \mathbb{S}_+$ while Condition \ref{eq:5.6} translates to $\alpha + \text{Diag}(\Pi^T x) \text{Diag}(x)^{-1} \in \mathbb{S}_+$ for all $x \in \mathbb{R}_+^m$.

7.2 Underlying space $E \subseteq \mathbb{R}$

We consider here the case $E^\Delta = [a, b] \subseteq \mathbb{R}^\Delta$, for some $-\infty \leq a < b \leq \infty$ and set

$$D := \{ f|_E : f \in C^\infty_c(\mathbb{R}) + \mathbb{R} \}.$$ 

Recall that we identify $E^\Delta$ with $E$, when $E$ is compact corresponding to the case $a > -\infty$ and $b < \infty$. Our goal is to analyze Theorem 5.9 in this setting. We therefore start by considering generators of strongly continuous positive groups on $C_\Delta(E)$ such that their domain contains both $D$ and $A(D)$. For results on generators of strongly continuous positive groups we refer to Appendix C, in particular to Definition C.5 and Definition C.7 for the notions of admissibility and lattice isomorphisms. In the following lemma we denote by $C^k_\Delta(E)$ the restriction of $C^k(\mathbb{R})$-functions to $E$, i.e. $C^k_\Delta(E) = \{ f|_E : f \in C^k(\mathbb{R}) \cap C_\Delta(\mathbb{R}) \}$. 

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Lemma 7.1. Let \( V \) be a lattice isomorphism on \( C_\Delta(E) \), which additionally leaves \( C^1_\Delta(E) \) and \( C^2_\Delta(E) \)-functions invariant. Moreover, let \( h \in C^1_\Delta(E) \) and \( \tau \in C^1_\Delta(E) \) be an admissible function in the sense of Definition C.5. Define \( \delta \tau \) via (C.4). Then the operator \( A \) given by

\[
A := V^{-1} \delta \tau V + h,
\] (7.1)

satisfies the conditions of Theorem 3.4. That is, \( A \) is the generator of a strongly continuous positive group of \( C_\Delta(E) \) and its domain contains both \( D \) and \( A(D) \).

Proof. First of all note that the domain of \( A \) is given by \( \{ g \in C_\Delta(E) \mid Vg \in D(\delta) \} \), where \( D(\delta) \) is defined in (C.5). As \( V \) leaves \( C^1_\Delta(E) \) and \( C^2_\Delta(E) \)-functions invariant and since \( \tau \) and \( h \) are in \( C^1_\Delta(E) \), the domain of \( A \) contains \( D \) and \( A(D) \). Moreover, by Theorem C.8 the operator \( A \) generates a strongly continuous positive group on \( C_\Delta(E) \), which yields the result.

Example 7.2. Let \( E = \mathbb{R} \). Set \( Vg = pg \) for some strictly positive function \( p \in C^\infty_\Delta(E) \), \( \tau \equiv 1 \) and \( h \in C^1_\Delta(E) \). Then all the conditions of Lemma 7.1 are satisfied and

\[
Ag = \frac{1}{p}(p'g + pg') + hg = g' + \tilde{h}g, \quad g \in D,
\]

where \( \tilde{h} = \frac{p'}{p} + h \). Hence, the stochastic process associated to \( A \) is a pure drift process killed at rate \( \tilde{h} \).

To consider some bounded interval, let \( E = [0, 1] \). Moreover, let \( \tau \in C^1_\Delta(E) \) with such that \( \tau(0) = \tau(1) = 0 \). Then by Remark C.6 the function \( \tau \) is admissible as it has bounded derivatives. Hence,

\[
Ag = \tau g', \quad g \in D
\]

satisfies the conditions of Lemma 7.1.

We can now reformulate Theorem 5.9 in the current setting as follows.

Corollary 7.3. Let \( L : D \to C(M_+(E)) \) be a linear operator of form (4.3), where \( B_0 \), \( Q_0 \) and \( Q_1 \) satisfy the conditions of Theorem 5.9. Moreover, for \( i = 1, \ldots n \), let \( A_i \) be of form (7.1) such that for \( V, \tau \) and \( h \) the conditions of Lemma 7.1 hold. Suppose that

(i) \( B_1 - \frac{1}{2} \sum_{i=1}^n A_i^2 \) : \( D \to C_\Delta(E) \) satisfies the positive minimum principle on \( E^\Delta \);

(ii) \( Q_2 \) is of the form

\[
Q_2(g) = C(g) + \sum_{i=1}^n (A_i \otimes A_i)(g), \quad g \in D \otimes D,
\]

where \( C \) admits a \((\beta, \pi)\)-representation.

Then conditions (i)-(v) of Theorem 5.9 are satisfied.

Proof. This follows directly from Lemma 7.1. \( \square \)
The rest of the section is devoted to the case $E = \mathbb{R}$. In view of Corollary 7.3 and Remark 3.5, $B_1$ should satisfy the positive minimum principle. Applying Remark 1.2, we thus decompose $B_1$ into an operator $B_1g = Ag + mg$ where $A$ satisfies the positive maximum principle and where $m \in C_\Delta(\mathbb{R})$. It is well-known, that under this condition $A$ is a Lévy type operator, i.e.

$$A g = \gamma g' + \frac{1}{2} \Gamma g'' + \int (g(\cdot + \xi) - g - \chi(\xi)g')F(\cdot, d\xi), \quad g \in D,$$

(7.2) for some continuous functions $\gamma$, $\Gamma$ with $\Gamma \geq 0$, a truncation function $\chi$, and a kernel $F(\cdot, d\xi)$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\int (|\xi|^2 \wedge 1)F(\cdot, d\xi) < \infty$. Every operator of this form satisfies $A1 = 0$ and the positive maximum principle on $\mathbb{R}$. The following result expresses Corollary 7.3 within the current setting.

**Corollary 7.4.** Let $L : D \to C(M_+(E))$ be a linear operator of form (4.3), where $B_0$, $Q_0$ and $Q_1$ satisfy the conditions of Theorem 5.9. Let $B_1$ be given by $B_1g = Ag + mg$ where $m \in C_\Delta(\mathbb{R})$ and $A$ is of form (7.2) with $\Gamma := \sigma^2 + \tau^2$ for some continuous function $\sigma$ and $\tau \in C_\Delta^1(\mathbb{R})$. Moreover, define $Q_2$ via

$$Q_2(g)(x, y) = C(y)(x, y) + \tau(\tau(x)\tau(y)g'(x)g'(y) + h(x)h(y)g(x)g(y),$$

where $C$ admits a $(\beta, \pi)$-representation and $h \in C_\Delta^1(\mathbb{R})$. Then conditions (i)-(v) of Theorem 5.9 hold true.

**Proof.** We apply here Corollary 7.3 with $i = 1$ and the operator by $Ag = \tau g' + h$. Note that $\tau$ is admissible as it is bounded and we work on $E = \mathbb{R}$. Hence, we only have to verify that $B_1 - \frac{1}{2} A^2$ satisfies the positive minimum principle on $\mathbb{R}$. This is the case because of the form of $\Gamma = \sigma^2 + \tau^2$ in $A$ so that we obtain

$$B_1g - \frac{1}{2} A^2 g = (\gamma + \tau h)g' + \frac{1}{2} \sigma^2 g'' + \int (g(\cdot + \xi) - g - \chi(\xi)g')F(\cdot, d\xi) + (m + \tau^2)g, \quad g \in D,$$

which satisfies the positive minimum principle on $\mathbb{R}$ and also $\mathbb{R}^\Delta$ since $D \subset C_\Delta(\mathbb{R})$.

**Remark 7.5.** Note that in Corollary 7.4 a non-zero $\tau$ in the specification of $Q_2$ is coupled with a corresponding diffusive component in the specification (7.2) of $A$ and in turn of $B_1$. This is analogous to (Cuchiero et al., 2019, Corollary 6.3 and Example 6.4) (compare also (Cuchiero et al., 2019, Corollary 6.3 and Example 6.4)).

### 7.3 Measure-valued Pearson diffusions

Certain measure-valued polynomial diffusions resulting from Theorem 5.9 can be considered as generalizations of one-dimensional Pearson diffusions on $\mathbb{R}_+$. We recall here the definition when their state space is some bounded or unbounded interval of $\mathbb{R}$, see, e.g., Forman and Sørensen (2008).

**Definition 7.6.** A real-valued stochastic process $(X_t)_{t \geq 0}$ is called *Pearson diffusion* if it is a weak solution of the following stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$
where $W_t$ is a standard Brownian motion and $x \mapsto b(x)$ and $x \mapsto \sigma^2(x)$ are polynomials respectively of at most first and second order degree, i.e., one has

$$b(x) = b_0 + b_1 x,$$

$$q(x) = \frac{1}{2} \sigma^2(x) = q_0 + q_1 x + q_2 x^2,$$

for some $b_0, b_1, q_0, q_1, q_2 \in \mathbb{R}$. In particular the (extended) generator is defined as

$$Lg(x) = b(x) g'(x) + q(x) g''(x), \quad g \in C^2(\mathbb{R}).$$

**Remark 7.7.** Note that this definition coincides with the notion of one-dimensional polynomial processes with continuous trajectories (see Cuchiero et al. (2012); Filipović and Larsson (2016)).

Following Ascione et al. (2020) we can recognize six different types of Pearson diffusions. The statements concerning the invariant measure hold true if the drift is of the form $b(x) = b_0 + b_1 x = b_1 (x + \frac{b_0}{b_1})$ with $b_1 < 0$ and $b_0 \in \mathbb{R}$.

(i) If $q \equiv q_0 > 0$, then $X_t$ is a Ornstein-Uhlenbeck (OU) process with values in $\mathbb{R}$ and the stationary distribution is a Gaussian distribution.

(ii) If $q(x) = q_1 x$ with $q_1 > 0$, then $X_t$ is a Cox-Ingersoll-Ross (CIR) process with values in $\mathbb{R}_+$ and the stationary is a Gamma distribution.

(iii) If $q(x) = q_0 + q_1 x + q_2 x^2$ with $q_2 < 0$, then $X_t$ is a Jacobi process with values in a compact interval and the stationary distribution is a (scaled) Beta distribution.

(iv) If $q(x) = q_1 x + q_2 x^2$ with $q_1, q_2 > 0$, then $X_t$ is a Fisher-Snedecor process with values in $\mathbb{R}_+$ and the stationary distribution is a Fisher-Snedecor distribution.

(v) If $q(x) = q_2 x^2$ with $q_2 > 0$, then $X_t$ is a reciprocal Gamma (RG) process with values in $\mathbb{R}_+$ and the stationary distribution is a reciprocal Gamma distribution. Note that this corresponds exactly to diffusion characteristic of the Black & Scholes model.

(vi) If $q(x) = q_0 + q_1 x + q_2 x^2$ with $q_2 > 0$ and the discriminant $\Delta_q < 0$, then $X_t$ is a Student process with values in $\mathbb{R}$ and the stationary distribution is a Student distribution.

Since we are here only interested in (non-negative) measure-valued analogs of the cases (ii), (iv) and (v) and not in deriving existence of stationary measures, we only distinguish the following three cases in terms of the characterization given in Theorem 5.11.

(ii): As already discussed in Section 6 the case $Q_2 \equiv 0$ can be seen as a generalization of Cox Ingersoll Ross processes to the measure-valued case.

(iv): Taking the general form of Theorem 5.11 yields a measure-valued analog of the Fisher-Snedecor process.

(v): A measure-valued analog of a multivariate Black-Scholes type model corresponds to the case $Q_1 = 0$ and $\pi \equiv 0$.  

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A Necessary conditions for the positive maximum principle

We here establish the necessary conditions implied by the positive maximum principle for both cases $M_+(E^\Delta)$ and $M_+(E)$. Recall that $D \subseteq C_\Delta(E)$ is a dense linear subspace containing the constant function 1 and let $D$ be as in (2.2).

**Lemma A.1.** Let $L : D \to C(M_+(E))$ be an $M_+(E)$-polynomial operator of form (4.3) satisfying the positive maximum principle on $M_+(E^\Delta)$. Then

(i) $B_0$ is a positive linear functional on $D$, hence there exists a Radon measure $b \in M_+(E^\Delta)$ such that $B_0(g) = \langle g, b \rangle$;

(ii) $B_1$ satisfies the positive minimum principle on $E^\Delta$;

(iii) $Q_0(g \otimes g) = 0$ for all $0 \leq g \in D$;

(iv) $Q_1(g \otimes g)$ and $Q_2(g \otimes g)$ are non-negative functions for all $g \in D$;

(v) $(Q_1(g \otimes g), \nu) = 0$ and $(Q_2(g \otimes g), \nu^2) = 0$ for all $\nu \in M_+(E^\Delta)$ and $0 \leq g \in D$ which are 0 on the support of $\nu$.

Replacing the positive maximum principle on $M_+(E^\Delta)$ by the positive maximum principle on $M_+(E)$ we obtain that conditions (i), (iii), and (iv) still hold true and we also get that

(vi) $B_1$ satisfies the positive minimum principle on $E$;

(vii) $(Q_1(g \otimes g), \nu) = 0$ and $(Q_2(g \otimes g), \nu^2) = 0$ for all $\nu \in M_+(E)$ and $0 \leq g \in D$ which are 0 on the support of $\nu$.

**Proof.** Consider the following function $f(\nu) = -\langle g, \nu \rangle$ with $g \in D$ such that $g \geq 0$. Then $\sup_{M_+(E^\Delta)} f = f(0) = 0$. By the positive maximum principle for $L$, it follows that

$$Lf(0) = -B_0(g) - \langle B_1 g, 0 \rangle \leq 0.$$ 

Hence, $B_0(g) \geq 0$ and since $g$ was arbitrary we conclude that $B_0$ is a positive linear functional on $D$. By assumption $D$ is a linear subspace of $C_\Delta(E)$ that contains an interior point of the cone of non-negative function, namely the constant function 1. In this case (Schaefer, 1999, Theorem and Corollary 2 of Chapter V.5.4) (alternatively (Bourbaki, 2003, Section 2.3.1)) yields that every positive linear map on $D$ can be extended to a positive linear map on $C_\Delta(E)$.

Let now $g$ be as above with the additional property that $g(x) = 0$ for some $x \in E^\Delta$, i.e., $\inf_{E^\Delta} g = g(x) = 0$ and $\sup_{M_+(E^\Delta)} f = f(n\delta_x) = 0$ for every $n \in \mathbb{N}$. By the positive maximum principle for $L$, we have

$$Lf(n\delta_x) = -B_0(g) - \langle B_1 g, n\delta_x \rangle \leq 0.$$ 

Hence

$$-\langle B_1 g, \delta_x \rangle \leq \frac{B_0(g)}{n}$$

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and sending \( n \to \infty \), we conclude that \( B_1g(x) \geq 0 \), which implies the positive minimum principle on \( E^\Delta \).

Consider now the function \( f(\nu) = -((g, \nu))^2 \) for \( g \in D \). As it attains its maximum at 0 with \( \partial f(0) = 0 \) and \( \partial^2 f(0) = -2g \otimes g \), we have again by the positive maximum principle

\[
Lf(0) = -2Q_0(g \otimes g) - 2\langle Q_1(g \otimes g), 0 \rangle - 2\langle Q_2(g \otimes g), 0^2 \rangle \leq 0,
\]

implying that \( Q_0(g \otimes g) \geq 0 \).

On the other hand, consider a function \( \phi \in C_c^\infty(\mathbb{R}) \) restricted to \( \mathbb{R}_+ \) of the form \( \phi(x) = (x - 1)^{2n} \) for \( x \in [0, \frac{1}{2}] \) and \( n \in \mathbb{N} \) such that it attains maximum at 0. Consider \( f(\nu) = \phi((g, \nu)) \) for \( g \in D \) with \( g \geq 0 \). Then \( \sup_{M_+(E^\Delta)} f = f(0) = 1 \), \( \partial f(0) = -2ng \) and \( \partial^2 f(0) = 2(n(2n - 1)(g \otimes g)) \). By the positive maximum principle for \( L \), it thus follows that

\[
Lf(0) = -2nB_0(g) + 2n(2n - 1)Q_0(g \otimes g) \leq 0.
\]

Hence,

\[
Q_0(g \otimes g) \leq \frac{B_0(g)}{2n - 1}
\]

and sending \( n \to \infty \) yields \( Q_0(g \otimes g) \leq 0 \) for \( g \geq 0 \). Hence for all \( g \geq 0 \), \( Q_0(g \otimes g) = 0 \).

Finally, fix \( g \in D \) and \( \nu \in M_+(E^\Delta) \) and consider the function \( f(\mu) = -((g, \nu) - \langle ng, \mu \rangle)^2 \) for \( n \in \mathbb{N} \). Then \( f \leq 0 \), \( \partial f(\nu) = 0 \) and \( \partial^2 f(\nu) = -2n^2g \otimes g \). Hence,

\[
Lf\left(\frac{\nu}{n}\right) = -2n^2\langle Q_1(g \otimes g), \frac{\nu}{n} \rangle - 2n^2\langle Q_2(g \otimes g), \frac{\nu^2}{n^2} \rangle \leq 0.
\]

and thus

\[
\langle Q_1(g \otimes g), \nu \rangle \geq -\frac{\langle Q_2(g \otimes g), \nu^2 \rangle}{n},
\]

implying that \( \langle Q_1(g \otimes g), \nu \rangle \geq 0 \) as \( n \) tends to \( \infty \). Considering the function \( f(\mu) = -((g, \nu) - \langle \nu, \mu \rangle)^2 \), yields by similar arguments that \( \langle Q_2(g \otimes g), \nu^2 \rangle \geq 0 \).

On the other hand, consider again a function \( \phi \in C_c^\infty(\mathbb{R}) \) restricted to \( \mathbb{R}_+ \) of the form \( \phi(x) = (x - 1)^{2n} \) for \( x \in [0, \frac{1}{2}] \) and \( n \in \mathbb{N} \) such that it attains maximum at \( 0 \). Fix \( 0 \leq g \in D \) and \( \nu \in M_+(E^\Delta) \) such that \( g = 0 \) on the support of \( \nu \). Consider \( f(\mu) = \phi((g, \mu)) \). Then \( \sup_{M_+(E^\Delta)} f = f(\nu) = 1 \), \( \partial f(\nu) = -2ng \) and \( \partial^2 f(\nu) = 2n(2n - 1)(g \otimes g) \). By the positive maximum principle for \( L \), it thus follows that

\[
Lf(\nu) = -2nB_0(g) - 2n(B_1(g), \nu) + 2n(2n - 1)\langle Q_1(g \otimes g), \nu \rangle + 2n(2n - 1)\langle Q_2(g \otimes g), \nu^2 \rangle \leq 0.
\]

Hence,

\[
\langle Q_1(g \otimes g), \nu \rangle + \langle Q_2(g \otimes g), \nu^2 \rangle \leq \frac{B_0(g) + \langle B_1(g), \nu \rangle}{2n - 1}
\]

which implies that

\[
\langle Q_1(g \otimes g), \nu \rangle + \langle Q_2(g \otimes g), \nu^2 \rangle = 0.
\]

As both summands are non-negative we obtain

\[
\langle Q_1(g \otimes g), \nu \rangle = 0 \quad \text{and} \quad \langle Q_2(g \otimes g), \nu^2 \rangle = 0
\]

for all \( g \geq 0 \) which are 0 on the support of \( \nu \).

The second part of the lemma can be proved analogously. \( \square \)
Lemma A.2. Suppose that $D = C_\Delta(E)$ and suppose that $L$ satisfies the assumptions of Lemma A.1 with the positive maximum principle on $M_+(E)$ instead of $M_+(E^\Delta)$. Then conditions (i)-(v) of Theorem 5.11 are satisfied.

Proof. We first apply Lemma A.1 to obtain conditions (i), (iii), (iv), (vi), and (vii) stated there. Since its domain is given by $C_\Delta(E)$, the operator $B_0$ is automatically continuous (see (Schaefer, 1999, Theorem V.5.5)) and the Riesz-Markov-Kakutani theorem (e.g. (Rudin, 1987, Theorem 2.14)) yields the representation via a Radon measure $b \in M_+(E^\Delta)$ given in condition (i) of Theorem 5.11.

Condition (ii) of Theorem 5.11 follows by Lemma A.3. Next, in order to establish the form of $Q_0$ observe that by linearity condition (iii) of Lemma A.1 extends to the cone

$$K_+ := \left\{ \sum_{i=1}^{k} \lambda_i (g_i \otimes g_i) \mid k \in \mathbb{N}, \lambda_i \in \mathbb{R}_+, 0 \leq g_i \in D \right\}.$$ 

Since $g = g^+ - g^-$ where $0 \leq g^\pm \in C_\Delta(E)$, we have

$$g \otimes g = \sum_{\lambda \in K_+} 2g^+ \otimes g^+ + 2g^- \otimes g^- - (g^+ + g^-) \otimes (g^+ + g^-).$$

This implies that $D \otimes D = K_+ - K_+$ and $Q_0$ is therefore a continuous linear functional and thus 0 on the whole of $D \otimes D$. This yields condition (iii) of Theorem 5.11.

Next, fix $\nu \in M_+(E)$ and consider $C^\nu := \{ q \in C_\Delta(E) \mid q \equiv 0 \text{ on } \text{supp}(\nu) \}$. Then $C^\nu$ is a linear subspace of $C_\Delta(E)$, which can be represented by $C^\nu = C_+^\nu - C_-^\nu$, where

$$C_+^\nu := \{ 0 \leq q \in C_\Delta(E) \mid q \equiv 0 \text{ on } \text{supp}(\nu) \}.$$ 

Similarly for

$$K_+^{\nu} := \left\{ \sum_{i=1}^{k} \lambda_i (g_i \otimes g_i) \mid k \in \mathbb{N}, \lambda_i \in \mathbb{R}_+, g_i \in C^\nu \right\}$$

we have $K_+^{\nu} = K_+^{\nu} - K_+^{\nu}$, where

$$K_+^{\nu} := \left\{ \sum_{i=1}^{k} \lambda_i (g_i \otimes g_i) \mid k \in \mathbb{N}, \lambda_i \in \mathbb{R}_+, g_i \in C_+^\nu \right\}.$$ 

Therefore the non-negative linear functionals $g \mapsto \langle Q_1(g), \nu \rangle$ and $g \mapsto \langle Q_2(g), \nu^2 \rangle$ are continuous and thus 0 on $K_+^{\nu}$, proving that

$$\langle Q_1(g \otimes g), \nu \rangle = 0 \quad \text{and} \quad \langle Q_2(g \otimes g), \nu^2 \rangle = 0,$$

for all $\nu \in M_+(E)$ and $g \in D$ which are 0 on the support of $\nu$. Lemma A.4 and Lemma A.8 then yield condition (iv) and condition (v) of Theorem 5.11, respectively. 

Lemma A.3. Let $B : C_\Delta(E) \to C_\Delta(E)$ be a linear operator. Then $B$ satisfies the positive minimum principle on $E$ if and only if there is map $m \in C_\Delta(E)$ and a non-negative, finite kernel $\nu_B$ from $E$ to $E^\Delta$ such that (5.10) holds. In this case, $B$ is bounded and satisfies the positive minimum principle on $E^\Delta$. Moreover, there is some non-negative (finite) measure $\nu_B(\Delta, \cdot)$ such that (5.10) holds also for $x = \Delta$. 

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Proof. As explained in Remark 1.2 setting $\overline{B}g(x) := B(g - g(x))(x)$ and $m(x) := B1(x)$ we obtain that $Bg = \overline{B}g + mg$ for a map $m \in C_\Delta(E)$ and an operator $\overline{B}$ satisfying $\overline{B}1 = 0$ and the positive maximum principle on $E$. The claim then follows by Lemma C.2 in Cuchiero et al. (2019).

Lemma A.4. Let $D \subseteq C_\Delta(E)$ be a dense linear subspace containing the constant function 1, and let $Q_1 : D \otimes D \to C_\Delta(E)$ be a linear operator. Then the following conditions are equivalent:

(i) $(Q_1(g \otimes g), \nu) \geq 0$ for all $g \in D$ and $\nu \in M_+(E)$ with equality if $g \equiv 0$ on the support of $\nu$.

(ii) $Q_1(g \otimes g)(x) \geq 0$ for all $g \in D$, $x \in E$ with equality if $g(x) = 0$.

Both imply that $Q_1$ is of form (5.11) for some function $\alpha \in C_\Delta(E)$ with values in $\mathbb{R}_+$. 

Proof. Obviously assertion (ii) implies (i). Conversely take $\nu = \delta_x$ to conclude $Q_1(g \otimes g)(x) \geq 0$ and with equality if $g(x) = 0$ as the support of $\delta_x$ is just $\{x\}$.

For $g \in D \otimes D$ we show now that $Q_1(g)(x)$ depends on $g$ through its values at $(x, x)$. To this end, fix $x \in E$ and note that the map $(g, h) \mapsto Q_1(g \otimes h)(x)$ is bilinear as well as positive semidefinite as $Q_1(g \otimes g)(x) \geq 0$ for all $g \in D$. Hence it satisfies the Cauchy–Schwarz inequality

$$|Q_1(g \otimes h)(x)| \leq \sqrt{Q_1(g \otimes g)(x)} \sqrt{Q_1(h \otimes h)(x)}.$$ 

This together with (ii) implies that $Q_1(g \otimes h)(x)$ depends on $g$ and $h$ only through their values at $x$.

Bilinearity then yields $Q_1(g \otimes h)(x) = g(x)h(x)\alpha(x)$ for some $\alpha : E \to \mathbb{R}$. By non-negativity we also have that $0 \leq Q_1(g \otimes g)(x) = g(x)^2\alpha(x)$ and thus that $\alpha(x) \geq 0$. Since $Q_1(g) \in C_\Delta(E)$ and as $1 \in D$, $x \mapsto \alpha(x)$ lies necessarily in $C_\Delta(E)$. Extending to all of $D \otimes D$ yields (5.11).

Let us now recall the definition of a positive semidefinite kernel.

**Definition A.5.** A symmetric function $K : (E^\Delta)^2 \to \mathbb{R}$ is called positive semidefinite kernel on $E^\Delta$ if

$$\sum_{i,j=1}^n c_ic_jK(x_i, x_j) \geq 0$$

holds for any $x_1, \ldots, x_n \in E^\Delta$, $c_1, \ldots, c_n \in \mathbb{R}$ and $n \in \mathbb{N}$.

For our purposes we shall need the bigger set of copositive kernels, see, e.g., Dobre et al. (2016); Kuryatnikova and Vera (2018). This generalizes the notion of copositive matrices, i.e. symmetric matrices $Q \in \mathbb{R}^{n \times n}$ such that $x^TQx \geq 0$ for all $x \in \mathbb{R}^n$.

**Definition A.6.** A symmetric function $K : E^2 \to \mathbb{R}$ is called copositive kernel on $E$ if

$$\sum_{i,j=1}^n c_ic_jK(x_i, x_j) \geq 0$$

holds for any $x_1, \ldots, x_n \in E$, $c_1, \ldots, c_n \in \mathbb{R}_+$ and $n \in \mathbb{N}$.
Remark A.7. In the following lemma copositive kernels naturally arise from the condition \( \langle Q_2(g \otimes g), \nu^2 \rangle \geq 0 \) for all \( \nu \in M_+(E) \). Note that even in the finite dimensional case a simple characterization of copositive matrices is not available, see, e.g., Hiriart-Urruty and Seeger (2010); Vershik (1987) Indeed, only up to dimension 4, every copositive matrix can be represented as sum of a positive semidefinite one and a matrix with non-negative entries. For dimensions \( n \geq 5 \), this is no longer true and copositive matrices are a strict superset thereof for which no simple characterization is known.

In our case copositivity of \( Q_2(g \otimes g) \) (and some further conditions) translate to the requirements of the \((\beta, \pi)\)-representation, where in particular (5.6) is a rather implicit condition. It can be easily verified when we have the decomposition of \( Q_2(g \otimes g) \) into a positive semidefinite kernel and a non-negative function (see Remark A.9 below).

Lemma A.8. Let \( D = C_\Delta(E) \) be a dense linear subspace containing the constant function 1, and let \( Q_2 : D \otimes D \to \hat{C}_\Delta(E^2) \) be a linear operator. Then the following conditions are equivalent:

(i) \( \langle Q_2(g \otimes g), \nu^2 \rangle \geq 0 \) for all \( g \in D \) and \( \nu \in M_+(E) \) with equality if \( g \equiv 0 \) on the support of \( \nu \).

(ii) For all \( g \in D \) the map \((x, y) \mapsto Q_2(g \otimes g)(x, y)\) is a copositive kernel on \( E \) and \( Q_2(g \otimes g)(x, y) = 0 \) if \( g(x) = g(y) = 0 \).

Both imply that \( Q_2 \) admits a \((\beta, \pi)\)-representation and the corresponding parameters \( \pi \) and \( \beta \) are bounded and continuous on \((E^2)^2 \setminus \{x = y\}\) and \( \pi + \pi + 2\beta \in \hat{C}_\Delta(E^2) \), where \( \pi(x, y) = \pi(y, x) \).

Proof. We start by showing that (i) and (ii) are equivalent and first prove that (ii) implies (i). The copositive kernel property of \( Q_2(g \otimes g) \) in (ii) yields \( \langle Q_2(g \otimes g), \nu^2 \rangle \geq 0 \) for all \( \nu \in M_+(E^2) \) which follows by approximating \( \nu \) weakly via \( \sum_{i=1}^n c_i \delta_{x_i} \). If \( g \equiv 0 \) on the support of \( \nu \), then clearly \( g(x) = g(y) = 0 \) for all \( x, y \in \text{supp}(\nu) \) and by assumption \( Q_2(g \otimes g)(x, y) = 0 \) as well, which yields \( \langle Q_2(g \otimes g), \nu^2 \rangle = 0 \).

Conversely, the fact that \( \langle Q_2(g \otimes g), \nu^2 \rangle \geq 0 \) for all \( \nu \in M_+(E) \) implies that \( Q_2(g \otimes g) \) is a copositive kernel. Next we show that \( Q_2(g \otimes g)(x, y) = 0 \) if \( g(x) = g(y) = 0 \). Indeed, if \( \nu = \delta_x \) and \( g \equiv 0 \) on the support of \( \delta_x \), i.e. \( g(x) = 0 \), we conclude \( Q_2(g \otimes g)(x, x) = 0 \). Similarly, take \( \nu = (\delta_x + \delta_y) \) for \( x \neq y \in E \) to obtain

\[
Q_2(g \otimes g)(x, x) + 2Q_2(g \otimes g)(x, y) + Q_2(g \otimes g)(y, y) = \langle Q_2(g \otimes g), \nu^2 \rangle.
\]

If \( g \equiv 0 \) on the support of \( \nu = (\delta_x + \delta_y) \), then \( g(x) = g(y) = 0 \), which in turn implies together with (i) and \( Q_2(g \otimes g)(x, x) = Q_2(g \otimes g)(y, y) = 0 \) that \( Q_2(g \otimes g)(x, y) = 0 \). This shows the equivalence of (i) and (ii).

To prove form (5.5), let us first note that if \( E \) is a singleton, say \( \{x\} \), (5.5) reduces to

\[
Q_2(g)(x, x) = K(g)(x, x) = \beta(x, x)g(x, x), \quad g \in D \otimes D
\]

with \( \beta(x, x) \geq 0 \), which is obvious by linearity and non-negativity.

For the general case, fix \( x \in E \) and note that the map \((g, h) \mapsto Q_2(g \otimes h)(x, x)\) is bilinear as well as positive semidefinite as \( Q_2(g \otimes g)(x, x) \geq 0 \) for all \( g \in D \). Hence it satisfies the Cauchy–Schwarz inequality

\[
|Q_2(g \otimes h)(x, x)| \leq \sqrt{Q_2(g \otimes g)(x, x)} \sqrt{Q_2(h \otimes h)(x, x)}.
\]
This together with the fact that \( Q_2(g \otimes g)(x, x) = 0 \) if \( g(x) = 0 \) implies that \( Q(g \otimes h)(x, x) \) depends on \( g \) and \( h \) only through their values at \( x \). Proceeding as in the proof of A.4 we can conclude that \( Q_2(g)(x, x) = \beta(x, x)g(x, x) \) for all \( g \in D \otimes D \).

Let us now consider \( Q_2(g \otimes g)(x, y) \) for \( x \neq y \). Recall that copositivity of \((x, y) \mapsto Q_2(g \otimes g)(x, y)\), means that

\[
\sum_{i,j=1}^n c_i c_j Q_2(g \otimes g)(x_i, x_j) = \sum_{i=1}^n c_i^2 \beta(x_i, x_i) g(x_i) g(x_i) + \sum_{i \neq j} c_i c_j Q_2(g \otimes g)(x_i, x_j) \geq 0
\]

for any \( x_1, \ldots, x_n \in E \), \( c_1, \ldots, c_n \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). This together with the fact that \( Q_2(g \otimes g)(x, y) = 0 \) if \( g(x) = g(y) = 0 \) implies that \( Q_2(g \otimes g)(x, y) \) can depend on \( g \) only via \( g(x)^2 \), \( g(y)^2 \) and \( g(x)g(y) \). Indeed, suppose that it depended on \( g(z_i)g(z_2) \) for some \((z_1, z_2) \neq (x, y)\). Choosing \( g \) such that \( g(x) = g(y) = 0 \) and \( g(z_i) \neq 0 \), yields \( Q_2(g \otimes g)(x, y) \neq 0 \), a contradiction. If \( z_1 = x \) and \( z_2 \neq x, y \), we can always choose \( g \) such that the above non-negativity is not satisfied. This proves the claim and we obtain by polarization that the bilinear map \((g, h) \mapsto Q_2(g \otimes h)(x, y)\) depends on \( g \) and \( h \) only through their values at \( x \) and \( y \).

Bilinearity and symmetry then yield \( Q_2(g \otimes h)(x, y) = (g(x), g(y))^\top A(x, y)(h(x), h(y)) \) for some \( A(x, y) \in \mathbb{S}^2 \). Setting \( \pi(x, y) = 2A_{11}(x, y), \pi(y, x) = 2A_{22}(x, y) \) and \( \beta(x, y) = 2A_{12}(x, y) \) yields the following representation

\[
Q_2(g \otimes h)(x, y) = \frac{1}{2}(\pi(x, y)g(x)h(x) + \pi(y, x)g(y)h(y) + 2\beta(x, y)g(x)h(y)).
\]

Extending this to all \( g \in D \otimes D \) gives (5.5).

Let us now prove the remaining properties of \( \pi \) and \( \beta \). By (5.5) we can without loss of generalities set \( \pi(x, x) = 0 \) for each \( x \in E \). Indeed, if this is not the case it suffices to replace \( \beta(x, x) \) with \( \pi(x, x) + \beta(x, x) \).

Concerning the non-negativity of \( \pi(x, y) \) for \( x \neq y \), choose \( g \) such that \( g(x) \neq 0 \) and \( g(y) = 0 \). Then the above form and copositivity of \((x, y) \mapsto Q_2(g \otimes g)(x, y)\) imply

\[
0 \leq c_1^2 Q_2(g \otimes g)(x, x) + c_2^2 Q_2(g \otimes g)(y, y) + 2c_1 c_2 Q_2(g \otimes g)(x, y) \\
= \left(c_1^2 \beta(x, x) + c_1 c_2 \pi(x, y)\right) g(x)^2
\]

for all \( c_1, c_2 \geq 0 \). Hence, \( \pi(x, y) \geq 0 \) and analogously \( \pi(y, x) \geq 0 \). Finally, copositivity yields for all \( n \in \mathbb{N}, x_1, \ldots, x_n \in E \) and \( c_1, \ldots, c_n \in \mathbb{R}_{++} \),

\[
0 \leq \sum_{i=1}^n c_i c_j Q_2(g \otimes g)(x_i, x_j) \\
= \sum_{i=1}^n c_i c_j \left(\frac{1}{2} \pi(x_i, x_j)g(x_i)^2 + \frac{1}{2} \pi(x_j, x_i)g(x_j)^2 + \beta(x_i, x_j)g(x_i)g(x_j)\right) \\
= (c_1 g(x_1), \ldots, c_n g(x_n)) \left(\begin{array}{c} \beta_n + \left(\begin{array}{c} \sum_{j=1}^n c_j^2 \pi(x_1, x_j) \\ \vdots \\ \sum_{j=1}^n c_j^2 \pi(x_n, x_j) \end{array}\right) \end{array}\right) \left(\begin{array}{c} c_1 g(x_1) \\ \vdots \\ c_n g(x_n) \end{array}\right),
\]

where \( \beta_n \in \mathbb{S}^n \) with entries \( \beta_{n,ij} = \beta(x_i, x_j) \). As this holds for all \((g(x_1), \ldots, g(x_n))\) and thus by the density of \( D \) for all vectors in \( \mathbb{R}^n \), Condition (5.6) follows.
The last regularity condition follows from the fact that \((x, y) \mapsto Q_2(g)(x, y) \in \hat{C}_\Delta(E^2)\) for all \(g \in D \otimes D\).

Finally, assume that \(\pi\) is not continuous along the sequence \((x_n, y_n)_{n \in \mathbb{N}}\) converging to \((x, y) \in (E^2)^2\) with \(x \neq y\). Without loss of generalities \(x_n \neq y_m\) for each \(n, m\). Choosing \(g \in C_\Delta(E)\) such that \(g(x_n) = 1\) and \(g(y_n) = 0\) it suffices to use that \(Q_2(g \otimes g) \in C_\Delta(E^2)\) to get a contradiction. Similarly, assuming that \(\pi\) explodes along the sequence \((x_n, y_n)_{n \in \mathbb{N}}\) converging to \((x, y) \in (E^2)^2\) one can without loss of generality construct \(g(x_n)^4 = 1/\pi(x_n, y_n)\) and obtain a contradiction. The properties of \(\beta\) follow form the fact that \(\frac{1}{2}(\pi + \pi) + \beta = Q_2(1 \otimes 1) \in C_\Delta(E^2)\). The continuity of those maps guarantees that the parameters of the \((\beta, \pi)\)-representation satisfy the needed conditions.

**Remark A.9.** Note that (5.6) is clearly implied if \(\beta : (E^2)^2 \to \mathbb{R}\) is a positive semidefinite kernel. If this is the case, we get the following decomposition of the copositive kernel \((x, y) \mapsto Q_2(g \otimes g)(x, y)\)

\[
Q_2(g \otimes g)(x, y) = K(g \otimes g)(x, y) + P(g \otimes g)(x, y), \tag{A.1}
\]

where we set

\[
K(g)(x, y) = \beta(x, y)g(x, y) \quad \text{and} \quad P(g)(x, y) = \frac{1}{2}(\pi(x, y)g(x, x) + \pi(y, x)g(y, y)).
\]

Then \(K\) and \(P\) are linear operators on \(D \otimes D\). Moreover, since \(\beta\) is a positive semidefinite kernel we have

\[
\sum_{i,j} c_i c_j K(g \otimes g)(x_i, x_j) = \sum_{i,j} c_i c_j \beta(x_i, x_j) g(x_i) g(x_j) = \sum_{i,j} \tilde{c}_i \tilde{c}_j \beta(x_i, x_j) \geq 0
\]

where \(\tilde{c}_i = c_i g(x_i)\) for \(i = 1, \ldots, n\), whence \(K(g \otimes g)\) is a positive semidefinite kernel for all \(g \in D\). Moreover, \(P(g \otimes g)\) is non-negative, which follows from the non-negativity of \(\pi\) and since \(\pi(x, x) = 0\), we also have \(P(g \otimes g)(x, x) = 0\).

Expression (A.1) is thus a decomposition a into a positive semidefinite kernel \(K(g \otimes g)\) and a non-negative map \(P(g \otimes g) : (E^2)^2 \to \mathbb{R}_+\).

**B  Existence for martingale problems**

This section is dedicated to establish the (essential) equivalence between the existence of a solution to the martingale problem and the positive maximum principle. Here, \(E\) is a locally compact Polish space, \(D\) a dense linear subspace of \(C_\Delta(E)\) containing the constant function 1, and \(L\) a linear operator with domain \(\mathcal{D}\) (as defined in (2.2)) satisfying (4.3). The first lemma asserts that the positive maximum principle is implied if a solution to the martingale problem exists.

**Lemma B.1.** If there exists an \(M_+(E)\)- (or \(M_+(E^2)\) respectively) valued solution \(X\) to the martingale problem for \(L : \mathcal{D} \to C(M_+(E))\) for each initial condition in \(M_+(E)\) (or \(M_+(E^2)\) respectively), then \(L\) satisfies the positive maximum principle on \(M_+(E)\) (or \(M_+(E^2)\) respectively).

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The proof of Lemma B.1 is well-known and we therefore do not state it here; see for instance the proof of Lemma 2.3 in Filipović and Larsson (2016).

The next lemma is an adaptation of a classical result from Ethier and Kurtz (2009). For the application of this result it is crucial that \( L \) is an operator on the space of \( C_0 \)-functions on a locally compact, separable, metrizable space. Since this is not the case for \( M_+ (E) \) if \( E \) is non-compact, we work on \( M_+ (E^\Delta) \), which is a locally compact Polish space with respect to the topology of weak convergence which follows from (Dawson, 1993, Section 3.1) and (Luther, 1970, Remark 1.2.3, pages 542-543).

**Lemma B.2.** Suppose that \( L \) is of form (4.3) and satisfies the positive maximum principle on \( M_+ (E^\Delta) \).

(i) Then for every initial condition in \( M_+ (E^\Delta) \), there exists a continuous \( M_+ (E^\Delta) \)-valued solution to the martingale problem for \( L : D \to C(M_+ (E)) \).

(ii) Let the linear operator \( B_0 : D \to \mathbb{R} \) be given by \( B_0 (g) = \langle g, b \rangle \) with \( b \in M (E^\Delta) \). If \( X_0 \in M_+ (E) \), \( b (\Delta) \leq 0 \), and \( \text{bp-} \lim_{n \to \infty} (B_1 (1 - g_n) - m (1 - g_n)) \leq 0 \) on \( E^\Delta \), where \( m \in C_\Delta (\mathbb{R}) \) and \( g_n \in D \cap C_0 (E) \) is a sequence such that \( \text{bp-} \lim_{n \to \infty} g_n = 1_E \), then any solution to the martingale problem takes values in \( M_+ (E) \).

**Proof.** We verify the conditions of (Ethier and Kurtz, 2009, Theorem 4.5.4). As already mentioned, \( M_+ (E^\Delta) \) is a locally compact, separable and metrizable space. Moreover, by Lemma 2.6, we have that

\[
F^D_c := F^D (M_+ (E^\Delta)) \cap C_c (M_+ (E^\Delta))
\]

is a dense subset of \( C_0 (M_+ (E^\Delta)) \), to which we restrict the domain of \( L \) for the moment. Moreover, the positive maximum principle yields that \( Lf \big|_{M_+ (E^\Delta)} = Lh \big|_{M_+ (E^\Delta)} \) for all \( f, h \in F^D_c \) (extended to \( M (E^\Delta) \)) such that \( f \big|_{M_+ (E^\Delta)} = h \big|_{M_+ (E^\Delta)} \). Note also that the form of (4.3) implies that \( L(F^D_c) \subseteq C_0 (M_+ (E^\Delta)) \), so that we may regard \( L \big|_{F^D_c} \) as an operator on \( C_0 (M_+ (E^\Delta)) \). This means that all the assumptions of in (Ethier and Kurtz, 2009, Theorem 4.5.4) are satisfied. Define now according to the same theorem the linear operator \( L^0 \) on \( C(M^0_+ (E^\Delta)) \) by

\[
L^0 f \big|_{M_+ (E^\Delta)} = L ((f - f (\emptyset)) \big|_{M_+ (E^\Delta)}), \quad L^0 f (\emptyset) = 0
\]

for all \( f \in C (M^0_+ (E^\Delta)) \) such that \( (f - f (\emptyset)) \big|_{M_+ (E^\Delta)} \in F^D_c \).

Then (Ethier and Kurtz, 2009, Theorem 4.5.4) yields that for every initial condition in \( M^0_+ (E^\Delta) \), there exists a solution \( X_t \) to the martingale problem for \( L^0 \) with càdlàg sample paths taking values in \( M^0_+ (E^\Delta) \). Indeed, we obtain that (5.1) is a bounded local martingale (and thus a true martingale) for each \( f \in F^D_c \) where \( L \) is replaced by \( L^0 \). Moreover, by Proposition 2 in Bakry and Émery (1985) we also know that \( t \mapsto f (X_t) \) is continuous for each \( f \in F^D_c \).

Define now \( \tau_n := \inf \{ t > 0 : \langle 1, X_t \rangle > n \} \) and set \( \tau_\emptyset := \lim_{n \to \infty} \tau_n \). We now aim to show that \( \mathbb{P} (\tau_\emptyset > t) = 1 \), showing that \( \mathbb{P} (X_t \neq \emptyset) = 1 \). Fix \( k \in \mathbb{N} \) and \( \phi (x) := 1 + x^k \). Consider an increasing sequence \( \phi_m \in C^\infty (\mathbb{R}) \) such that \( \phi_m (x) = x \) for \( |x| \leq m \) and
\(\phi_m(x) = 0\) for \(|x| > m + 1\). Set \(f(\nu) := \phi(\langle 1, \nu \rangle)\) and \(f_m := \phi_m \circ f\). By continuity of \(t \mapsto f_m(X_t)\), we already know that
\[
\tau_n < \tau_0 \quad \text{and} \quad \langle 1, X_{\tau_n} \rangle = n
\]
for each \(n < m\). Observe then that \(\partial f_m(\nu) = \partial f(\nu)\) and \(\partial^2 f(\nu) = \partial^2 f_m(\nu)\) for each \(\nu \in M_+(E^\Delta)\) such that \(\langle 1, \nu \rangle \leq m\). This implies that
\[
f(X_{\tau_n \wedge t}) - f(X_0) - \int_0^{\tau_n \wedge t} Lf(X_s)ds
\]
is a bounded local martingale for each \(n, m\) such that \(n, \langle 1, X_0 \rangle < m\), and thus a true martingale.

Now, observe that since \(L\) is given by (4.3) it holds
\[
\|Lf\| \leq Kf
\]
for some \(K > 0\). By Fatou lemma this yields
\[
\mathbb{E}[\lim_{n \to \infty} f(X_{\tau_n \wedge t})] \leq \lim_{n \to \infty} \mathbb{E}[f(X_{\tau_n \wedge t})] \leq \lim_{n \to \infty} \left( f(X_0) + K \int_0^t \mathbb{E}[f(X_{\tau_n \wedge s})]ds \right).
\]
By the Gronwall inequality we can thus conclude that
\[
\mathbb{E}[1 + (\lim_{n \to \infty} n 1_{\{\tau_n \leq t\}} + \lim_{n \to \infty} \langle 1, X_t \rangle 1_{\{\tau_n > t\}})^k] = \mathbb{E}[\lim_{n \to \infty} f(X_{\tau_n \wedge t})] \leq f(X_0) \exp(Kt)
\]
and hence that \(P(\tau > t) = 1\) and \(\mathbb{E}[f(X_t)] \leq f(X_0) \exp(Kt)\).

Finally, we need to prove the local martingale property of (5.1) for \(g \in D \setminus F^D_c\). Since we already know that \((\tau_n)_{n \in \mathbb{N}}\) increases to infinity, the claim follows by noticing that setting \(g_m := \phi_m \circ g\) for \(m\) large enough and proceeding as before we can prove that the process
\[
g(X_{\tau_n \wedge t}) - g(X_0) - \int_0^{\tau_n \wedge t} Lg(X_s)ds
\]
is a true martingale.

For the second part, set \(h_n := 1 - g_n\). We know from Lemma (5.7) that
\[
\langle h_n, X_t \rangle - \langle h_n, X_0 \rangle - \int_0^t B_0 h_n + \langle B_1 h_n, X_s \rangle ds
\]
is a true martingale. We thus get that
\[
\mathbb{E}[\langle h_n, X_t \rangle] = \mathbb{E}[\langle h_n, X_0 \rangle] + \int_0^t \mathbb{E}[B_0 h_n + \langle B_1 h_n, X_s \rangle]ds
\]
\[
\leq \mathbb{E}[\langle h_n, X_0 \rangle] + \mathbb{E}[|B_0 h_n|]t + \int_0^t \langle B_1 h_n - m h_n, X_s \rangle + \|m\| \mathbb{E}[\langle h_n, X_s \rangle]ds,
\]
which by the Gronwall inequality yields
\[
\mathbb{E}[\langle h_n, X_t \rangle] \leq a(t) + \int_0^t a(s) \|m\| e^{\|m\|(t-s)} ds,
\]
47
where $a(t) = \mathbb{E}[\langle h_n, X_0 \rangle] + B_0 h_n t + \int_0^t \langle B_1 h_n - mh_n, X_s \rangle ds$. We then get that

$$\mathbb{E}[\langle h_n, X_t \rangle] \leq e^{\|m\|t} \left( \mathbb{E}[\langle h_n, X_0 \rangle] + |B_0 h_n| t + \int_0^t \langle |B_1 h_n - mh_n|, X_s \rangle ds \right).$$

By the dominated convergence and the assumptions on $X_0$, $B_0$ and $B_1$ we then get

$$\mathbb{E}[X_t(\Delta)] = \mathbb{E}[\lim_{n \to \infty} \langle h_n, X_t \rangle]$$

$$\leq \lim_{n \to \infty} e^{\|m\|t} \left( \mathbb{E}[\langle h_n, X_0 \rangle] + |B_0 h_n| t + \int_0^t \langle |B_1 h_n - mh_n|, X_s \rangle ds \right)$$

$$\leq e^{\|m\|t} (X_0(\Delta) + |b(\Delta)| t)$$

$$\leq 0.$$ 

Non-negativity of $X_t(\Delta)$ implies that $X_t(\Delta) = 0$ a.s., whence $X_t \in M_+(E)$. Finally, note that a càdlàg process $X$ on $M_+(E\Delta)$ such that $X_t(\Delta) = 0$ a.s. is càdlàg also with respect to the topology of weak convergence on $M_+(E)$. \hfill \Box

### C Generators of strongly continuous positive groups

Following Arendt et al. (1986), we recall here the main tools behind the characterization of generators of strongly continuous positive groups. We start by introducing the definitions of $C_\Delta(E)$-derivations, flows and cocycles. The notion of a $C_\Delta(E)$-derivation is similar to Definition 4.3, where we considered however bilinear maps.

**Definition C.1.** An operator $\delta$ on $C_\Delta(E)$ is called $C_\Delta(E)$-derivation if its domain $D(\delta)$ is a subalgebra of $C_\Delta(E)$ containing 1 such that

$$\delta(fg) = (\delta f)g + f(\delta g) \quad \text{for all } f, g \in D(\delta).$$

Note that this implies $\delta 1 = 0$.

**Definition C.2.** A mapping $\Phi : \mathbb{R} \times E^\Delta \to E^\Delta$ is called a flow on $E^\Delta$ if the maps $\Phi_t : E^\Delta \to E^\Delta$ given by $\Phi_t(x) = \Phi(t, x)$ are continuous and satisfy

$$\Phi_0(x) = x, \quad x \in E^\Delta,$$

$$\Phi_s \circ \Phi_t = \Phi_{s+t}, \quad s, t \in \mathbb{R}.$$

It follows from the definition that each $\Phi_t$ is a homeomorphism on $E^\Delta$ and $\Phi_{-t} = \Phi_t^{-1}$. A flow is called continuous if it is continuous with respect to the product topology on $\mathbb{R} \times E^\Delta$.

**Definition C.3.** Given a flow $\Phi$ a family $(k_t)_{t \in \mathbb{R}}$ is called a cocycle of $\Phi$ if

$$k_0 = 1,$$

$$k_{t+s} = k_t \cdot (k_s \circ \Phi_t), \quad s, t \in \mathbb{R}.$$
A cocycle \((k_t)_{t \in \mathbb{R}}\) associated to a flow \(\Phi\) is called continuous if the mapping \((t, x) \mapsto k_t(x)\) from \(\mathbb{R} \times E^\Delta\) into \(\mathbb{R}\) is continuous with respect to the product topology on \(\mathbb{R} \times E^\Delta\).

Let \(\Phi\) be a flow and \((k_t)_{t \in \mathbb{R}}\) a cocycle of \(\Phi\). Then, for every \(t \in \mathbb{R}\),

\[ T_t f = k_t \cdot f \circ \Phi_t \quad (C.1) \]

defines a bounded operator \(T_t\) on \(C_\Delta(E)\), which satisfies the semigroup property \(T_{s+t} = T_s T_t\) for all \(s, t \in \mathbb{R}\). Moreover, by \(\text{(Arendt et al., 1986, Proposition B-II.3.9)}\), if \((T_t)_{t \in \mathbb{R}}\) is a strongly continuous group of positive operators on \(C_\Delta(E)\), then there exist a continuous flow \(\Phi\) on \(E^\Delta\) and a continuous cocycle \((k_t)_{t \in \mathbb{R}}\) of \(\Phi\) such that \((C.1)\) holds. In \(\text{(Arendt et al., 1986, Section B-II.3)}\) the form of the cocycle associated with a positive group is explicitly derived and yields the following characterization, see \(\text{(Arendt et al., 1986, Theorem B-II.3.14)}\). Note that we work here with \(C_\Delta(E)\) instead of \(C_0(E)\).

**Theorem C.4.** An operator \(A\) on \(C_\Delta(E)\) is the generator of a positive group \((T_t)_{t \in \mathbb{R}}\) if and only if there exist a \(C_\Delta(E)\)-derivation which is the generator of a group, a function \(h \in C_\Delta(E)\) and \(p \in C_\Delta(E)\) satisfying \(\inf_{x \in E^\Delta} p(x) > 0\) such that

\[ A = V \delta V^{-1} + h, \quad (C.2) \]

where \(V : C_\Delta(E) \to C_\Delta(E)\) is given by \(V f = pf\). In that case one has the following representation

\[ T_t f(x) = \frac{p(x)}{p(\Phi_t(x))} (\exp \int_0^t h(\Phi(s, x)) ds) f(\Phi_t(x)), \quad (C.3) \]

for all \(f \in C_\Delta(E), t \in \mathbb{R}\) and \(x \in E^\Delta\) and where \(\Phi\) satisfies \((C.1)\).

**C.1 Case \(E \subseteq \mathbb{R}\)**

Fix \(E \subseteq \mathbb{R}\) such that \(E^\Delta = [a, b] \subseteq \mathbb{R}^\Delta\), for some \(-\infty \leq a < b \leq \infty\), where we identify \(E^\Delta\) with \(E\), when \(a > -\infty\) and \(b < \infty\). In this case the form of the derivation can be made more explicit. Indeed, let \(\tau : (a, b) \to \mathbb{R}\) be a continuous function and define \(\delta_\tau\) by

\[ \delta_\tau g := \begin{cases} \tau(x)g'(x) & \text{if } \tau(x) \neq 0, \\ 0 & \text{if } \tau(x) = 0, \end{cases} \quad (C.4) \]

for all \(x \in (a, b)\) and with domain

\[ D(\delta_\tau) = \{ g \in C_\Delta(E) : g \text{ is differentiable in } x \in (a, b) \text{ whenever } \tau(x) \neq 0 \}
\]

and there exists a (necessarily unique) \(f \in C_\Delta(E)\) such that \(\delta_\tau g = f\).

\[ (C.5) \]

Note that \(\delta_\tau\) is a derivation and that the above definition corresponds (up to the modification from \(C[a, b]\) to \(C_\Delta(E)\)) to \(\text{(Arendt et al., 1986, Eq. (3.25))}\) with \(m = \tau\) and \(\tilde{\delta}_m = \delta_\tau\).

For the characterization of generators of positive groups on \(C_\Delta(E)\) for \(E \subseteq \mathbb{R}\) we need two further notions, namely so-called admissibility (see \(\text{(Arendt et al., 1986, Definition 3.16)}\)) of the function \(\tau\) and the notion of a lattice isomorphism.
Definition C.5. A function \( \tau : (a, b) \to \mathbb{R} \) is admissible if it is continuous and the following holds: whenever \( a \leq c < d \leq b \) such that \( \tau(x) \neq 0 \) for \( x \in (c, d) \) and \( \tau(c) = 0 \) or \( c = a = -\infty \) and \( \tau(d) = 0 \) or \( d = b = \infty \), then \( \int_c^1 |\tau(x)| \, dx = \int_d^1 |\tau(x)| \, dx = \infty \) \( z \in (c, d) \). Moreover, if if \( a > -\infty \), then \( m(a) = 0 \) and for \( b < \infty \), \( m(b) = 0 \).

Remark C.6. Note that every Lipschitz continuous function (referring to globally Lipschitz in the case of unbounded intervals) is admissible. Note also that \( \tau \) only needs to be continuous on the open interval \((a, b)\).

Definition C.7. A lattice isomorphism is a one-to-one map \( V : \mathcal{C}_\Delta(E) \to \mathcal{C}_\Delta(E) \) such that \( |Vg| = V|g| \) for all \( g \in \mathcal{C}_\Delta(E) \).

The following theorem is a refinement of Theorem C.4 when \( E^\Delta = [a, b] \) and follows from (Arendt et al., 1986, Theorem B-II.3.28). We only adapt it to the current \( \mathcal{C}_\Delta(E) \)-setting instead of the \( \mathcal{C}([a, b]) \) setting considered in (Arendt et al., 1986, Theorem B-II.3.28). Note that this only differs when \( a = -\infty \) and \( b = \infty \).

Theorem C.8. An operator \( A \) generates a positive group on \( \mathcal{C}_\Delta(E) \) if and only if there exist a lattice isomorphism \( V \) on \( \mathcal{C}_\Delta(E) \), an admissible function \( \tau : (a, b) \to \mathbb{R} \), and \( h \in \mathcal{C}_\Delta(E) \) such that \( A = V^{-1}\delta_\tau V + h \).

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