Quantization and the tangent groupoid

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Abstract

This is a survey of the relationship between $C^*$-algebraic deformation quantization and the tangent groupoid in noncommutative geometry, emphasizing the role of index theory. We first explain how $C^*$-algebraic versions of deformation quantization are related to the bivariant $E$-theory of Connes and Higson. With this background, we review how Weyl–Moyal quantization may be described using the tangent groupoid. Subsequently, we explain how the Baum–Connes analytic assembly map in $E$-theory may be seen as an equivariant version of Weyl–Moyal quantization. Finally, we expose Connes’s tangent groupoid proof of the Atiyah–Singer index theorem.

1 Introduction

Quantization theory is concerned with the passage from classical to quantum mechanics (or field theory), and vice versa. Dirac’s famous early insight that the Poisson bracket in classical mechanics is formally analogous to the commutator in quantum mechanics was initially implemented, in a mathematical context, in geometric quantization. This approach is generally felt to be somewhat passé, although certain techniques from it continue to play an important role. What has replaced geometric quantization is the idea of deformation quantization, which emerged in the 1970s independently through the work of Berezin [8, 9] and of Flato and his collaborators [10].

Here quantum mechanics is seen as a deformation of classical mechanics, which should be recovered as $\hbar \to 0$. Hence it is particularly important to study quantum

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theory for a range of values of Planck’s “constant” $\hbar$, and to control the classical limit. This aspect was missing in geometric quantization, as was the idea that one should start from Poisson manifolds, or, even more generally, from Poisson algebras, rather than from symplectic manifolds. Here a Poisson algebra is a commutative algebra $\tilde{A}$ over $\mathbb{C}$ equipped with a Lie bracket $\{ , \}$, such that for each $f \in \tilde{A}$ the map $g \mapsto \{ f, g \}$ is a derivation of $\tilde{A}$ as a commutative algebra. Seen in this light, the quickest definition of a Poisson manifold $P$ is that the space $\tilde{A} = C^\infty(P)$ of smooth functions over it is a Poisson algebra with respect to pointwise multiplication.

The best-known approach to deformation quantization is purely algebraic, and is known as formal deformation quantization or star-product quantization. Here one works with formal power series in $\hbar$; in particular, it is generally impossible to ascribe a numerical value to Planck’s constant. This approach was launched in 1978 [7], and has led to impressive existence and classification results so far. For example, Fedosov proved by an explicit geometric construction that any symplectic manifold can be quantized [22], and Kontsevich, using entirely different methods, extended this result to arbitrary Poisson manifolds [30]. These results belong to the early phase of formal deformation quantization, which has been reviewed by Sternheimer [58].

Recently, the theory has been put on a new footing by Kontsevich and Soibelman, who use a high-powered description of general deformation theory in terms of operads [31, 32]. Their approach uncovers unexpected and fascinating links between deformation quantization, the theory of motives, and the so-called Grothendieck–Teichmüller group in algebraic geometry. This illustrates the phenomenon that despite its original motivation, formal deformation quantization is taking a path that is increasingly remote from physics.

The link between operator algebras and quantum physics has been close ever since von Neumann’s foundational work in both areas. It should, therefore, be no surprise that $C^*$-algebras provide a language for describing deformation quantization that is interesting for both mathematics and physics. The physical interest in the $C^*$-algebraic approach lies partly in the fact that $\hbar$ is now a real number rather than a formal parameter, so that one can study the limit $\hbar \to 0$ in a precise, analytic way, and partly in the possibility of explicitly describing most known examples of quantization as it is used in physics. Mathematically, it turns out that $C^*$-algebraic deformation quantization sheds light on many interesting examples in noncommutative geometry. (In this paper, we always mean “noncommutative geometry” in the sense of Connes [14]. There are constructions involving homotopic algebra and “$\infty$-structures” that go under this name as well, and which are actually closely related to formal deformation quantization; see [59] for a representative paper.)

The $C^*$-algebraic approach to deformation quantization was initiated in 1989 by Rieffel [53], who observed that a number of examples of quantization could be described by continuous fields of $C^*$-algebras in a natural and attractive way. As indicated above, some of his examples involve quantization as physicists know and love it, like Weyl–Moyal quantization and related constructions (see, in particular, [54] for a survey), while others relate to noncommutative geometry. In the latter category, Rieffel’s discovery that the familiar noncommutative tori can be seen as deformation quantizations of ordinary symplectic tori stands out [53, 55]. (Noncommutative tori actually do have potential physical relevance through string theory [15].)
We refer to [34, 54] for surveys of the starting period of $C^*$-algebraic deformation quantization, including references up to 1998. Later work that is relevant to noncommutative geometry includes [35, 36], which will be recalled below, as well as [44]. Very recently, Cadet [12] showed that the Connes–Landi noncommutative four-spheres [17] fall into this context. The general picture of $C^*$-algebraic deformation quantization that emerges from the literature so far is that it is rich in examples and poor in existence and classification theorems; compare this with the formal case!

We now outline the contents of the remainder of this paper; the key concept unifying what follows is Connes’s tangent groupoid [14, 27]. It is clear from its very definition that the bivariant E-theory of Connes and Higson [10, 14, 16] should be closely related to $C^*$-algebraic deformation quantization as formulated by Rieffel [43, 56]. In Section 2 we sketch a direct route from formal deformation quantization to asymptotic morphisms and E-theory, which entices a generalization of Rieffel’s $C^*$-algebraic axioms. In Section 3 we sketch an approach to Weyl–Moyal quantization that is based on a powerful lemma, which in Section 4 we show to underlie the Baum–Connes conjecture [5, 6] in E-theory as formulated in [14]. Since the Baum–Connes conjecture is an issue in index theory, our discussion is intended as a minor contribution to the growing literature on the intimate relationship between deformation quantization, K-theory, and index theory. In the purely algebraic setting, powerful new results have been achieved in this direction [22, 23, 24, 45, 46, 47], whereas $C^*$-algebraic quantization-oriented methods so far have mainly led to new proofs of known results. In the latter spirit, Section 5 contains an exposition of Connes’s tangent groupoid proof of the Atiyah–Singer index theorem [14].

Throughout this paper we use the following convention. $G$ is a Lie groupoid over $G(0)$, with associated convolution $C^*$-algebras $C^*(G)$ and $C^*_r(G)$ [14]. We write $K^*(G)$ for $K_*(C^*(G))$, and similarly $K^*_r(G) = K_*(C^*_r(G))$. This is consistent with the usual identification $K^*(X) = K_*(C_0(X))$, for when a locally compact groupoid $G$ is a space $X$ (in that $G = G(0) = X$ with trivial operations), one has $C^*(X) = C_0(X)$.

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2 From deformation quantization to E-theory

In formal deformation quantization one defines a star-product $\ast$ on a unital Poisson algebra $\hat{A}$ as an associative product on the ring $\hat{A}[[\hbar]]$ of formal power series in one variable with coefficients in $A$ [7]. Such a product is evidently determined by its value on $A$. Writing $f \ast g = \sum_n \hbar^n C_n(f, g)$, where $f, g \in A$, one requires that $C_0(f, g) = fg$ and $C_1(f, g) = C_1(g, f) = i\{f, g\}$. Heuristically, one may think of the restriction of the star-product $\ast$ to $\hat{A}$ as a family of associative products $\ast_\hbar$ on $A$.

Rieffel’s original definition of $C^*$-algebraic deformation quantization [53] was motivated by this interpretation. He defined a “strict” quantization of a given Poisson manifold $P$ as a family $(A_\hbar)_{\hbar \in \mathbb{C}}$ of $C^*$-algebras, equipped with the structure of a continuous field, with the feature that each fiber algebra $A_\hbar$ is the completion of a single
(i.e., $\hbar$-independent) Poisson algebra $\tilde{A}_0$ that is densely contained in the commutative $C^*$-algebra $A_0 = C_0(P)$, equipped with a “deformed” (i.e., $\hbar$-dependent) product $*_{\hbar}$, involution $^*_{\hbar}$, and norm $\| \cdot \|_{\hbar}$. Here one generically takes $\hbar \in I = [0, 1]$, although more general base spaces of the continuous field are occasionally used (as long as the base contains 0 as an accumulation point).

Consequently, one has canonical “quantization” maps $Q_{\hbar} : \tilde{A}_0 \to A_\hbar$ given by $Q_{\hbar}(f) = f$, seen as an element of $A_\hbar$, and for each $f \in A_0$ the map $\hbar \mapsto Q_{\hbar}(f)$ defines a canonical section of the field. By construction, one then has

$$Q_{\hbar}(f) *_{\hbar} Q_{\hbar}(g) = Q_{\hbar}(f *_{\hbar} g)$$

for all $f, g \in \tilde{A}_0$. Hence Rieffel was able to formulate Dirac’s insight mentioned earlier in an asymptotic way by means of the axiom

$$\lim_{\hbar \to 0} \| \frac{i}{\hbar}(Q_{\hbar}(f), Q_{\hbar}(g)) - Q_{\hbar}((f, g)) \|_{\hbar} = 0$$

for all $f, g \in \tilde{A}_0$. Here $[\ , \ ]$ is the commutator with respect to $*_{\hbar}$.

In examples related to Berezin–Toeplitz quantization, however, continuous fields of $C^*$-algebras and quantization maps $Q_{\hbar}$ occur which do not have the feature that $Q_{\hbar}(f)Q_{\hbar}(g)$ is the $Q_{\hbar}$ of something, contra [1]; see [34] and references therein. This called for a more general definition of $C^*$-algebraic deformation quantization [33, 34, 57], whose relationship with formal deformation quantization was rather obscure. We now remove this deficiency.

The algebra $\tilde{A}[[\hbar]]$ used in the formal setting is a $C[[\hbar]]$ algebra, in the sense that there is an injective ring homomorphism from $C[[\hbar]]$ into the center of $\tilde{A}[[\hbar]]$; cf. [38], p. 121. Now the $C^*$-algebraic analogue of such an algebra is a so-called $C(I)$ $C^*$-algebra. Recall that, for a compact Hausdorff space $X$, a $C(X)$ $C^*$-algebra is a $C^*$-algebra $A$ with a unital embedding of $C(X)$ in the center of its multiplier algebra [28]. The structure of $C(X)$ $C^*$-algebras is as follows [48].

A field of $C^*$-algebras is a triple $(X, \{A_x\}_{x \in X}, A)$, where $\{A_x\}_{x \in X}$ is some family of $C^*$-algebras indexed by $X$, and $A$ is a family of sections (that is, maps $f : X \to \prod_{x \in X} A_x$ for which $f(x) \in A_x$) that is i) a $C^*$-algebra under pointwise operations and the natural norm $\|f\| = \sup_{x \in X} \|f(x)\|_{A_x}$, ii) closed under multiplication by $C(X)$, and iii) full, in that for each $x \in X$ one has $\{f(x) \mid f \in A\} = A_x$. The field is said to be continuous when for each $f \in A$ the function $x \mapsto \|f(x)\|$ is in $C(X)$ (this is equivalent to the corresponding definition of Dixmier [59]; cf. [25]). The field is upper semicontinuous when for each $f \in A$ and each $\varepsilon > 0$ the set $\{x \in X \mid \|f(x)\| \geq \varepsilon\}$ is compact.

Thm. 2.3 in [48] now states that a $C(X)$ $C^*$-algebra $A$ defines a unique upper semicontinuous field of $C^*$-algebras $(X, \{A_x = A / C(X, x)A\}_{x \in X}, A)$. Here $C(X, x) = \{f \in C(X) \mid f(x) = 0\}$, and, with slight abuse of notation, $a \in A$ is identified with the section

$$a : x \mapsto \pi_x(a),$$

where $\pi_x : A \to A_x$ is the canonical projection. Moreover, a $C(X)$ $C^*$-algebra $A$ defines a continuous field of $C^*$-algebras whenever the map $x \mapsto \|\pi_x(a)\|$ is lower semicontinuous (and hence continuous) for each $a \in A$ [11].
We return to deformation quantization. In the formal setting, given a Poisson algebra $\tilde{A}$ one could look at general $C[[\hbar]]$ algebras $A$ with the property that $A/\hbar A \cong \tilde{A}$, rather than narrowing the discussion to the free $C[[\hbar]]$ modules $\tilde{A}[[\hbar]]$. This motivates the following definition in the analytic context. As in Rieffel’s discussion, we start from a Poisson manifold instead of a Poisson algebra.

**Definition 1** A $C^*$-algebraic quantization of a Poisson manifold $P$ is a $C(I)$ $C^*$-algebra $A$ such that

1. For each $a \in A$, the function $\hbar \mapsto \|\pi_\hbar(a)\|$ from $I$ to $\mathbb{R}^+$ is lower semicontinuous (and hence continuous);
2. One has $A_0 = A/C(I,0)A \cong C_0(P)$ as $C^*$-algebras;
3. There is a Poisson algebra $\tilde{A}_0$ that is densely contained in $C_0(P)$, and, identifying $A_0$ and $C_0(P)$, there is a cross-section $Q : \tilde{A}_0 \to A$ of $\pi_0$, such that \([\mathbf{3}]\) holds for $Q_\hbar = \pi_\hbar \circ Q$.

This definition (with evident modifications when $I = [0,1]$ is replaced by a more general index set) seems to cover all known examples. It follows from the discussion above that, due to the first condition, $A$ is automatically the section algebra of a continuous field. Let us now assume that this field is trivial away from $\hbar = 0$. This means by definition that $A_\hbar = B$ for all $\hbar \in (0,1]$, and that, under the identification \([\mathbf{3}]\), one has a short exact sequence

$$0 \to CB \to A \to A_0 \to 0. \quad \text{(4)}$$

Here the so-called cone $CB = C_0((0,1],B)$ appears. (Strictly speaking, the fields in our examples are merely isomorphic to those of this form, but there is always a canonical trivialization.)

In this situation, one obtains a homomorphism $Q_*$ from $K_*(A_0)$ to $K_*(B)$, as follows. Since the cone $CB$ is contractible, and therefore has trivial K-theory, the periodic six-term sequence shows that

$$\pi_0 : K_*(A) \to K_*(A_0) \quad \text{(5)}$$

is an isomorphism. (In fact, Bott periodicity is not needed to infer that $\pi_0$ is invertible; the long exact sequence of K-theory with an ad-hoc argument will do.) Here, with abuse of notation, $\pi_0$ stands for the image of the *-homomorphism $\pi_0 : A \to A_0$ under the K-functor. (See \([\mathbf{56}]\) for the analogous result $K_0(A[[\hbar]]) \cong K_0(A)$ in formal deformation quantization.) The K-theory map defined by the continuous field is then simply

$$Q_* = \pi_1 \circ \pi_0^{-1} : K_*(A_0) \to K_*(B). \quad \text{(6)}$$

This map may be described more explicitly, whether or not $A_0$ is commutative, as follows \([\mathbf{20}]\). Denote the unitization of a $C^*$-algebra $C$ without unit by $C^+$, and assume for simplicity that neither $A_0$ nor $B$ (and hence $A$) is unital (this is indeed the case in all our examples). Firstly, for any $n \in \mathbb{N}$, the $C^*$-algebra $M_n(A^+)$ of $n \times n$ matrices over $A^+$ is again a $C(I)$ $C^*$-algebra, and a nontrivial argument shows that it even defines
a continuous field whenever $A$ does \([24]\). The fiber algebras of this field are evidently $M_n(A^+_n)\) at $h = 0$ and $M_n(B^+)$ at $h \in (0, 1]$. Now let $[p] - [q] \in K_0(A_0)$, where $p, q$ are projections in $M_n(A^+_0)$. Extend $p$ and $q$ to continuous sections $h \mapsto p_h$ etc. of the field $M_n(A^+_0)$, and finally put
\[
Q_0([p] - [q]) = [p_1] - [q_1],
\]
which lies in $K_0(B)$ as desired. This is independent of all choices. Of course, the suffix 1 may be replaced by $h$ for any $h \in (0, 1]$. To construct $Q_1$, one works with suspensions as appropriate.

The passage to E-theory is well known \([10, 14, 16, 43, 56]\), as follows. Any cross-section $Q : A_0 \to A$ of $\pi_0$ defines an asymptotic morphism $(Q_h)_{h \in I}$ from $A_0$ to $B$ by $Q_h = \pi_h \circ Q : A_0 \to B$, and all such $Q$ define homotopic asymptotic morphisms. Thus a deformation quantization defines an element of $\tilde{E}(\pi_0, B)$, and therefore a homomorphism from $K_*(A_0)$ to $K_*(B)$. This homomorphism is precisely $Q_*$, which in the context of asymptotic morphisms has an explicit description, too \([26]\): extend the $Q_h$ to maps $Q^*_h : M_n(A^+_n) \to M_n(B^+)$ in the obvious way, and find continuous families of projections $(p_h)_{h \in (0, 1]}$ in $M_n(B^+)$ etc. such that
\[
\lim_{h \to 0} \|Q^*_h(p) - p_h\| = 0.
\]
Then use \([7]\) as above.

In fact, it is sufficient if $Q$ is defined on a dense subspace $\tilde{A}_0$ of $A_0$, as in Definition \([1]\). The corresponding $\ast$-homomorphism from $\tilde{A}_0$ to $C_h((0, 1], B)/CB$ can be extended to $A_0$ by continuity, and this extension may subsequently be lifted to an asymptotic morphism from $A_0$ to $B$, which on $\tilde{A}_0$ is equivalent to the original one.

By the same argument, one may start from a definition of quantization directly in terms of maps $Q_h : A_0 \to B$, as in \([33, 34]\), and arrive at E-theory classes, but in the examples below it will be the $C(I)$ $C^*$-algebras rather than their associated continuous fields or quantization maps that are canonically given. A $C^*$-algebraic quantization has more structure than an asymptotic morphism in E-theory, in that in the latter the maps $Q$ are completely arbitrary, whereas in the former they relate to the Poisson structure on $A_0$, and have to be chosen with care. This is clear from condition 3 in Definition \([1]\) which on the transition from deformation quantization to E-theory does not depend.

### 3 Weyl–Moyal quantization

The first example to consider in any version of quantization theory is the Weyl–Moyal quantization of $T^*(\mathbb{R}^n)$, or more generally, of $T^*(M)$, where $M$ is a Riemannian manifold. In the formal setting this is handled for $\mathbb{R}^n$ in \([7]\) and for general $M$ in \([13, 50]\); for the $C^*$-algebraic formalism we refer to \([54]\) and \([33, 34]\), respectively. In the context of noncommutative geometry and the Baum–Connes conjecture, the “royal path” towards Weyl–Moyal quantization \([13, 54, 55]\) is formulated in terms of Connes’ tangent groupoid (cf. \(\S \ II.5\) in \([13]\)), as follows\footnote{After circulation of this paper as a preprint I heard from Alejandro Rivero that Connes himself suggested this formulation at Les Houches 1995.} An immersion $M \hookrightarrow N$ of manifolds...
defines a manifold with boundary

$$G_{M \to N} = \{0\} \times \nu(M) \cup (0, 1] \times N,$$

(9)

where $\nu(M)$ is the normal bundle of the embedding. The smooth structure on this space was first defined in [27]. If $N = M \times M$ and the embedding is the diagonal map $x \mapsto (x, x)$, the ensuing manifold $G_{M \to M}$, denoted simply by $G_M$ in what follows, is a Lie groupoid over $G_M(0) = I \times M$ in the following way. The fiber at $h = 0$ is $\nu(M) = T(M)$, which is a groupoid over $M$ under the canonical bundle projection and addition in each $T_x(M)$. The fiber at any $h \in (0, 1]$ is the pair groupoid $M \times M$ over $M$. The total space $G_M$, then, is a groupoid with respect to fiberwise operations. This Lie groupoid is the tangent groupoid of $M$. See also [34, 49].

It is quite obvious that $A = C^*(G_M)$ is a $C^*(I)$ $C^*$-algebra, with associated fiber algebras

$$A_0 = C_0(T^*(M));$$
$$A_h = B_0(L^2(M)) \forall h \in (0, 1],$$

(10)

where $B_0(H)$ is the $C^*$-algebra of compact operators on $H$. The continuity of this field may be established in many ways (see [34, 54] and references therein), but in the context of this paper the most appropriate approach is to use the following lemma, due to Blanchard and Skandalis (but apparently first published in [36], which is partly based on Ramazan’s thesis [51]). This lemma generalizes a corresponding result of Rieffel [52], which states that if $H$ is a Lie groupoid fibered over a manifold $X$ by a smooth surjective submersion $\pi : H \to X$ (both $H$ and $X$ may be manifolds with boundary). Suppose that $\pi(x) = \pi(s(x)) = \pi(r(x))$ (where $s$ and $r$ are the source and the range projections in $H$); in that case, each $H_x = \pi^{-1}(x)$ is a Lie subroupoid of $H$, and $H$ is a bundle of Lie groupoids over $X$ with fibers $H_x$ and pointwise operations.

Then $(X, \{C^*(H_x)\}_{x \in X}, C^*(H))$ is a field of $C^*$-algebras, which is continuous at all points $x$ where $H_x$ is amenable. The same statement holds if $C^*(H_x)$ and $C^*(H)$ are replaced by $C^*_r(H_x)$ and $C^*_r(H)$, respectively.

See [1] for the theory of amenable groupoids. Applied to the tangent groupoid $H = G_M$, where $X = I$, this lemma proves continuity of the field (10), since the groupoid $H_0 = T(M)$ is commutative and therefore amenable, and $H_{h \neq 0} = M \times M$ is amenable as well. In fact, equipping the cotangent bundle $T^*(M)$ with the canonical Poisson structure, all of Definition 1 holds [34, 44]; the quantization maps $Q_h$ may be given by Weyl–Moyal quantization with respect to a Riemannian structure on $M$.

4 The Baum–Connes conjecture in E-theory

The Baum–Connes conjecture [5, 6, 14] is an important issue in noncommutative geometry; see [32] for a recent overview focusing on discrete groups, and cf. [61] for a survey of the situation for groupoids. The purpose of this section is to show how
Connes’s E-theoretic description of the analytic assembly map [4, Ch. II] approach fits into the formalism of the previous sections, simultaneously inserting some details omitted in section II.10 of [4]. We will use the notation of [4].

Recall [4, 54] that a (right) $G$ space $P$ is a smooth map $P \xrightarrow{\alpha} G(0)$ along with a map $P \times_{\alpha} G \to P$, $(p, \gamma) \mapsto p\gamma$ (where $\alpha(p) = r(\gamma)$), such that $(p\gamma_1)\gamma_2 = p(\gamma_1\gamma_2)$ whenever defined, $p\alpha(p) = p$ for all $p$, and $\alpha(p\gamma) = s(\gamma)$. The action is called proper when $\alpha$ is a surjective submersion and the map $P \times_{\alpha} G \to P \times P$, $(p, \gamma) \mapsto (p, p\gamma)$ is proper (in that the inverse images of compact sets are compact).

The following construction is crucial for what follows. Let a $G$ space $H$ be a Lie groupoid itself, and suppose the base map $H \xrightarrow{\alpha} G(0)$ is a surjective submersion that satisfies $\alpha \circ s_H = \alpha \circ r_H = \alpha$ as well as the condition that, for each $\gamma \in G$, the map $\alpha^{-1}(r(\gamma)) \to \alpha^{-1}(s(\gamma))$, $h \mapsto h\gamma$, is an isomorphism of Lie groupoids (note that for each $u \in G(0)$, $\alpha^{-1}(u)$ is a Lie groupoid over $\alpha^{-1}(u) \cap H(0)$). In particular, one has $(h_1h_2)\gamma = (h_1\gamma)(h_2\gamma)$ whenever defined.

Under these conditions, one may define a Lie groupoid $H \rtimes G$, called the semidirect product of $H$ and $G$ (see [4] for the locally compact case and [41] (2nd ed.) for the smooth case). The total space of $H \rtimes G$ is $H \times_{\alpha} G$, the base space of units $(H \times G)(0)$ is $H(0)$, the source and range maps are

$$s(h, \gamma) = s_H(h)\gamma; \\
r(h, \gamma) = r_H(h),$$

respectively, the inverse is $(h, \gamma)^{-1} = (h^{-1}, \gamma, \gamma^{-1})$ (note that one automatically has $\alpha(h^{-1}) = \alpha(h)$, so that this element is well defined), and multiplication is given by $(h_1, \gamma_1)(h_2, \gamma_2) = (h_1h_2, \gamma_1\gamma_2)$, defined whenever the product on the right-hand side exists (this follows from the automatic $G$ equivariance of $s_H$ and $r_H$). Well-known special cases of this construction occur when $H$ is a space and $G$ is a groupoid, so that $H \rtimes G$ is a groupoid over $H$, and when $G$ and $H$ are both groups, so that $H \rtimes G$ is the usual semidirect product of groups.

In the context of the Baum–Connes conjecture, the key application of this construction is as follows [4]. Let $P$ be a proper $G$ space. One may define three Lie groupoids, all over $P$.

1. The tangent bundle $T_G(P)$ of $P$ along $\alpha$ (i.e., ker($\alpha_*$), where $\alpha_* : T(P) \to T(G(0))$ is the derivative of $\alpha$) is a $G$ space, with base map $\alpha_0(\xi_p) = \alpha(p)$ (where $\xi_p \in T_G(P)_p$) and with the obvious push-forward action. If $T_G(P)$ is seen as a Lie groupoid over $P$ by inheriting the Lie groupoid structure from $T(P)$ (see Section 3), one may define the semidirect product groupoid $T_G(P) \rtimes G$ over $P$.

2. The fibered product $P \times_{\alpha} P$ is a $G$ space under the base map $\alpha_3(p, q) = \alpha(p) = \alpha(q)$ and the diagonal action $(p, q)\gamma = (p\gamma, q\gamma)$. Moreover, $P \times_{\alpha} P$ inherits a Lie groupoid structure from the pair groupoid $P \times P$ over $P$, becoming a Lie groupoid over $P$. Hence one has the semidirect product groupoid $(P \times_{\alpha} P) \rtimes G$ over $P$.

3. The tangent groupoid $G_P^\prime$ associated to $P$ has a Lie subgroupoid $G_P^\prime$ over $I \times P$ that by definition contains all points $(\xi = 0, \xi_p)$ of $G_P$ whose $\xi_p$ lies in $T_G(P)$,
and all points \((h > 0, p, q)\) for which \(\alpha(p) = \alpha(q)\). It is clear that \(G'_p\) is a bundle of groupoids over \(I\), whose fiber at \(h = 0\) is \(T_G(P)\), and whose fiber at any \(h \in (0, 1]\) is \(P \times _\alpha P\). Combining the \(G\) actions defined in the preceding two items, there is an obvious fiberwise \(G\) action on \(G'_p\) with respect to a base map \(\alpha(h, \cdot) = \alpha_h(\cdot)\), where \(\alpha_h = \alpha_1\) for \(h \in (0, 1]\). This action is smooth, so that one obtains a semidirect Lie groupoid \(G'_p \rtimes G\) over \(I \times P\).

The following two propositions provide the technical underpinning for \(\S II.10.\alpha\) in \([3]\).

**Proposition 1** If \(P\) is a proper \(G\) space, then \(C^* (G'_p \rtimes G)\) is the \(C^*\)-algebra of sections \(A\) of a continuous field of \(C^*\)-algebras over \(I\) with fibers

\[
\begin{align*}
A_0 &= C^*(T_G(P) \rtimes G); \\
A_h &= C^*((P \times _\alpha P) \rtimes G) \forall h \in (0, 1].
\end{align*}
\]

This field is trivial away from \(h = 0\). The same is true if all groupoid \(C^*\)-algebras are replaced by their reduced counterparts.

**Proof.** It is obvious that \(G'_p \rtimes G\) is a bundle of groupoids over \(I\), whose fiber at \(h = 0\) is \(T_G(P) \rtimes G\), and whose fiber at any \(h \in (0, 1]\) is \((P \times _\alpha P) \rtimes G\). Since the corresponding field of \(C^*\)-algebras is obviously trivial away from \(h = 0\), it is continuous at all \(h \in (0, 1]\). If we can show that \(T_G(P) \rtimes G\) is an amenable groupoid, Lemma [3] proves continuity at \(h = 0\) as well.

To do so, we use Cor. 5.2.31 in \([3]\), which states that a (Lie) groupoid \(H\) is amenable iff the associated principal groupoid (that is, the image of the map \(H \to H^{(0)} \times H^{(0)}\), \(h \mapsto (r(h), s(h))\)) is amenable and all stability groups of \(H\) are amenable. As to the first condition, the principal groupoid of \(T_G(P) \rtimes G\) is the equivalence relation on \(P\) defined by \(p \sim q\) when \(q = p\gamma\) for some \(\gamma \in G\). This is indeed amenable, because this equivalence relation is the same time the principal groupoid of \(P \rtimes G\) (over \(P\)), which is proper (hence amenable) because \(P\) is a proper \(G\) space. As to the second condition, the stability group of \(p \in P\) in \(T_G(P) \rtimes G\) is \(T_G(P)_p \rtimes G_p\), where \(G_p\) is the stability group of \(p \in P\) in \(P \rtimes G\). The latter is compact by the properness of the \(G\) action, so that \(T_G(P)_p \rtimes G_p\) is amenable as the semidirect product of two amenable groups.

When \(G\) is trivial, the continuous field of this proposition is, of course, the one defined by the tangent groupoid of \(P\), which coincides with the field defined by the Weyl–Moyal quantization of the cotangent bundle \(T^* (P)\); see Section \([3]\). The general case is a \(G\) equivariant version of quantization, which cannot really be interpreted in terms of quantization, because the fiber algebra at \(h = 0\) is no longer commutative.

**Proposition 2** The \(C^*\)-algebras \(C^*((P \times _\alpha P) \rtimes G)\) and \(C^*(G)\) are (strongly) Morita equivalent, as are the corresponding reduced \(C^*\)-algebras.

**Proof.** It is easily checked that the map \((p, q, \gamma) \mapsto \gamma\) from \((P \times _\alpha P) \rtimes G\) to \(G\) is an equivalence of categories. Since this map is smooth, it follows from Cor. 4.23 in \([3]\) that \((P \times _\alpha P) \rtimes G\) and \(G\) are Morita equivalent as Lie groupoids (and hence as locally
compact groupoids with Haar system). The proposition then follows from Thm. 2.8 in [42].

By [3, 4], the continuous field of Proposition 1 yields a map

\[ Q_* : K^*(T_G(P) \times G) \to K^*((P \times G)). \]  

By Proposition 3 and the fact that the K-theories of Morita equivalent C*-algebras are isomorphic, this map equally well takes values in \( K^*(G) \), and hence, by the K-theory push-forward of the canonical projection \( C^*(G) \to C^*_r(G) \), in \( K^*_r(G) \).

Now suppose that the classifying space \( E_G \) for proper \( G \) actions is a smooth manifold (which is true, for example, when \( G \) is a connected Lie group [4, §II.10,β], or when \( G \) is the tangent groupoid of a manifold). This means that, up to homotopy, there is a unique smooth \( G \)-equivariant map from any proper \( G \) manifold to \( E_G \). In that case, one may put \( P = E_G \) in the above formalism, and, writing

\[ K^*_\text{top}(G) = K^*(T_G(E_G) \times G), \]  

one obtains a map

\[ \mu : K^*_\text{top}(G) \to K^*_r(G). \]  

This is the analytic assembly map in E-theory as defined by Connes. In general, the definition of \( K^*_\text{top}(G) \) is more involved, but the analytic assembly map is constructed using the above construction in a crucial way. The Baum–Connes conjecture (without coefficients) in E-theory states that \( \mu \) be an isomorphism. It remains to be seen how this relates to the Baum–Connes conjecture for groupoids in KK-theory [61], which is a priori stronger even if the assembly maps turn out to be the same. For further comments cf. the end of the next section.

5 The Atiyah–Singer index theorem

We now use the ideas in the preceding sections to sketch two proofs of the Atiyah–Singer index theorem. We refer to [4, 43, 53] for the necessary background. Throughout this section, \( M \) is a compact manifold. Atiyah and Singer [3] define two maps, t-ind and a-ind, from \( K^0(T^*(M)) \) to \( \mathbb{Z} \), and show that they are equal. To define t-ind, let \( M \to \mathbb{R}^k \) be a smooth embedding, defining a normal bundle \( \nu(M) \to M \) and associated pushforwards \( T(M) \to T(\mathbb{R}^k) \) and \( T(\nu(M)) \to T(M) \). Since the latter bundle has a complex structure (or, more generally, is even-dimensional and K-oriented), one has the K-theory Thom isomorphism \( \tau : K^0(T(M)) \to K^0(T(\nu(M))) \). Identifying \( T(\nu(M)) \) with a tubular neighbourhood of \( T(M) \) in \( T(\mathbb{R}^k) \), one has \( T(\nu(M)) \to T(\mathbb{R}^k) \) as an open set, so that one has a natural extension map \( \psi : K^0(T(\nu(M))) \to K^0(T(\mathbb{R}^k)) \). Finally, for \( T(\mathbb{R}^k) = \mathbb{R}^{2k} \) one has the Bott isomorphism \( \beta_k : K^0(\mathbb{R}^{2k}) \to \mathbb{Z} \). Identifying \( T(M) \) with \( T^*(M) \) through some metric, t-ind is the composition

\[ \text{t-ind} = \beta_k \circ \psi \circ \tau : K^0(T^*(M)) \to \mathbb{Z}. \]  

Using some algebraic topology, it is easy to show that

\[ \text{t-ind}(x) = (-1)^{\dim(M)} \int_{T^*(M)} \text{ch}(x) \wedge \pi^*\text{td}(T^*(M) \otimes \mathbb{C}), \]  

where \( \text{ch} \) denotes the Chern character and \( \text{td} \) denotes the Todd class. For further details see [3, §II.10].
where \( \text{ch} : K^0(T^*(M)) \to H^*_c(T^*(M)) \) is the Chern character, \( \pi : T^*(M) \to M \) is the canonical projection, and \( t\text{d}(E) \in H^*(M) \) is the Todd genus of a complex vector bundle \( E \to M \).

The analytic index \( a\text{-ind} : K^0(T^*(M)) \to \mathbb{Z} \) is defined by

\[
a\text{-ind}(\sigma_P) = \text{index}(P). \tag{18}
\]

Here \( P : C^\infty(E) \to C^\infty(F) \) is an elliptic pseudodifferential operator between complex vector bundles \( E \) and \( F \) over \( M \), with principal symbol \( \sigma_P \in K^0(T^*(M)) \), and

\[
\text{index}(P) = \dim \text{ker}(P) - \dim \text{coker}(P). \tag{19}
\]

Atiyah and Singer [3] formulate two axioms which \( t\text{-ind} \) is trivially shown to satisfy, and which uniquely characterize \( t\text{-ind} \) as a map from \( K^0(T^*(M)) \) to \( \mathbb{Z} \). The burden of their proof of the index theorem in K-theory

\[
t\text{-ind} = a\text{-ind} \tag{20}
\]

is to show that \( a\text{-ind} \) satisfies these axioms as well. Combining (17), (18), and (20), one then obtains the usual cohomological form of the index theorem [4], viz.

\[
\text{index}(P) = (-1)^{\dim(M)} \int_{T^*(M)} \text{ch}(\sigma_P) \wedge \pi^* t\text{d}(T^*(M) \otimes \mathbb{C}). \tag{21}
\]

This proof has a number of drawbacks. It is not easy to show that (18) is well defined; one must establish that \( \text{index}(P) \) only depends on the symbol class \( \sigma_P \), and that \( K^0(T^*(M)) \) is exhausted by elements of that form. Furthermore, the definition of \( t\text{-ind} \) looks artificial. All in all, it would seem preferable to have natural map

\[
q\text{-ind} : K^0(T^*(M)) \to \mathbb{Z} \tag{22}
\]

to begin with, and to show that this a priori defined map satisfies both

\[
q\text{-ind}(x) = (-1)^{\dim(M)} \int_{T^*(M)} \text{ch}(x) \wedge \pi^* t\text{d}(T^*(M) \otimes \mathbb{C}) \tag{23}
\]

and

\[
q\text{-ind}(\sigma_P) = \text{index}(P). \tag{24}
\]

This would immediately imply (21).

This program may indeed be realized [14, 21, 26, 60]. We start from the continuous field (10), defining the map (6). Composing this map with the trace \( \text{tr} : K_0(B_0(L^2(M))) \to \mathbb{Z} \), one may put

\[
q\text{-ind} = \text{tr} \circ Q_0. \tag{25}
\]

Connes (cf. Lemma II.5.6 in [14]) claims that this map coincides with \( a\text{-ind} \), which is true, but this equality actually comprises half of the proof of the index theorem! The computations establishing (24) may be found in [21, 26, 40, 60].
One way to prove (23) is to note that the continuous field (11) extends to a continuous field defined by $A_x = A \otimes C(X)$, where $X$ is any compact Hausdorff space (cf. Thm. 2.4 in [20]). Using Cor. 3.2 in [20], the associated maps (6) $Q^X : K^*(T^*(M) \times X) \to K^*(X)$ are easily seen to be natural in $X$, and to be homomorphisms of $K(X)$ modules. Furthermore, a lengthy calculation given in [26] shows that $Q^X_0(\lambda_M) = 1$, where $\lambda_M \in K_0(M \times T^*(M))$ is a generalized Bott element defined in [26]. As shown in [26], by a straightforward topological argument these three properties imply (23). Also see [60] for a different proof.

Another approach to proving (23) is due to Connes; see §II.5 of [14]. First, extend the embedding $M \to \mathbb{R}^k$ to $j : M \to \mathbb{R}^{2k}$ by mapping $x \in \mathbb{R}^k$ to $(x, 0) \in \mathbb{R}^{2k}$. Recall that $G_M$ is the tangent groupoid of $M$, with base $G_M^{(0)} = I \times M$. Now

$$P = G_M^{(0)} \times \mathbb{R}^{2k}$$

is a right $G_M$ space through the obvious map $P \to G_M^{(0)}$, i.e., $\alpha(u, X) = u$, and the action is given by

$$\alpha(v, (r(\gamma), X)\gamma) = (s(\gamma), X + h(\gamma)).$$

Here $h(h = 0, \xi_x) = j_*(\xi_x)$ and $h(h, x, y) = (j(x) - j(y))/\hbar$. This action defines the semidirect product groupoid $P \rtimes G_M$. From (11) and (27) one reads off the source and range projections $s, r : P \to P$ as

$$s(r_G(\gamma), X, \gamma) = (s_G(\gamma), X + h(\gamma));$$
$$r(r_G(\gamma), X, \gamma) = (r_G(\gamma), X).$$

Connes’s first observation (Prop. 7 on p. 104 of [14]) is that

$$K^*(G_M) \cong K^*(P \rtimes G_M).$$

This follows, because $C^*(P \rtimes G_M) \cong C^*(G_M) \rtimes \mathbb{R}^{2k}$ with respect to a suitable action of $\mathbb{R}^{2k}$ on $C^*(G_M)$. This isomorphism is easily established by a Fourier transformation on the $\mathbb{R}^{2k}$ variable, and implies (19) by Connes’s Thom isomorphism [14, 14, 20]. It is interesting to regard (29) as a proof of the Baum–Connes conjecture for $G_M$. Indeed, one may take

$$EG_M = G_M^{(0)} \times \mathbb{R}^{2k};$$

in particular, the $G_M$ action on $P$ is free and proper. As a groupoid and as a $G_M$ space, $T_G(M)(P)$ is just $P \times \mathbb{R}^{2k}$ over $P \times \{0\}$, i.e., the direct product of $P$ as a space with the given $G_M$ action and $\mathbb{R}^{2k}$ as an abelian group with the trivial $G_M$ action. Therefore,

$$C^*(T_G(M)(P) \times G_M) \cong C^*(P \times G_M) \otimes C_0(\mathbb{R}^{2k}),$$

so that, using (14) and Bott periodicity, one has

$$K^*_\text{top}(G_M) \cong K^*(P \rtimes G_M).$$

The analytic assembly map $\mu : K^*_\text{top}(G_M) \to K^*_\text{top}(G_M)$ is precisely the map occurring in Connes’s Thom isomorphism. Note that $K^*_\text{top}(G_M) = K^*(G_M)$, both being isomorphic to $K^*(T^*(M))$. 

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The second main observation \[\text{[14]}\] is that the Lie groupoid \(P \times G_M\) is Morita equivalent to the space

\[\mathcal{B}G_M = \{0\} \times T(\nu(M)) \cup ([0, 1] \times \mathbb{R}^{2k}).\]  \hspace{1cm} (33)

Here \(T(\nu(M)) = \nu(M) \times \mathbb{R}^k\) is actually the normal bundle of \(M \hookrightarrow \mathbb{R}^{2k}\), so this is a special case of \([\text{[4]}]\).

Looking separately at the cases \(\hbar = 0\) and \(\hbar > 0\), it is easily seen that \(\mathcal{B}G_M\) is diffeomorphic to the orbit space \(P/G_M\) of the \(G_M\) action on \(P\) (which also explains the notation, as \(P = E\mathcal{G}_M\)). This coincides with the orbit space \(P/(P \times G_M)\) of the \(P \times G_M\) action on its own base space \(P\), which is free and proper. The orbit space \(\mathcal{B}G_M\) acts trivially on \(P\), and it follows that \(P\) is a \((\mathcal{B}G_M, P \times G_M)\) equivalence \([\text{[42]}]\). Hence \(\mathcal{B}G_M\) and \(P \times G_M\) are Morita equivalent.

It follows that

\[K^*(P \times G_M) \cong K^*(\mathcal{B}G_M),\]  \hspace{1cm} (34)

and hence, by \([\text{[28]}]\),

\[K^*(G_M) \cong K^*(\mathcal{B}G_M).\]  \hspace{1cm} (35)

Now both \(C^*(G_M)\) and \(C^*(\mathcal{B}G_M) = C_0(\mathcal{B}G_M)\) are \(C(I)\) \(C^*\)-algebras, defining continuous fields by Lemma \([\text{[1]}]\). We decorate maps associated to the second field with a hat. For example, the associated maps \([\text{[6]}]\) are \(\mathcal{Q}_\ast : K^*(T^*(M)) \to K_c(B_0(L^2(M)))\) and \(\mathcal{Q}_\ast : K^*(T(\nu(M))) \to K^*(\mathbb{R}^{2k})\), respectively. We have already dealt with \(\mathcal{Q}_0\); it is easily seen that \(\mathcal{Q}_0\) is the extension map \(\psi\). The isomorphism \([\text{[23]}]\), which we call \(\alpha^\ast\), induces isomorphisms \(\alpha^\ast_h : K_c(C^*(G_M)) \to K_c(C_0(\mathcal{B}G_M))\) such that \(\alpha^\ast_h \circ \pi_h = \hat{\pi}_h \circ \alpha^\ast\), for any \(h \in I\). It can be checked that \(\alpha^0_1 : K^0(T^*(M)) \to K^0(T(\nu(M)))\) is the Thom isomorphism \(\tau\), and that \(\alpha^0_1 : K_0(B_0(L^2(M))) \to K^0(\mathbb{R}^{2k})\) is \(\beta^{-1}_k \circ \text{tr}\). It follows from the definition of \(\mathcal{Q}_\ast\) and \(\alpha_h\) that one has \(\alpha^\ast_1 \circ \mathcal{Q}_\ast = \hat{\mathcal{Q}}_\ast \circ \alpha^0_1\). Using \([\text{[14]}]\) and \([\text{[25]}]\), the last equality with \(s = 0\) immediately implies a-ind = t-ind, and hence \([\text{[23]}]\) from \([\text{[17]}]\). This proof of the index theorem has great conceptual beauty.

We close with some comments on the Baum–Connes conjecture in \(E\)-theory in the light of the above considerations. For \(M = \mathbb{R}^k\), the map \(q\)-ind : \(K^0(\mathbb{R}^{2k}) \to \mathbb{Z}\) is the inverse of the Bott map, so that Atiyah’s index theory proof of the Bott periodicity theorem \([\text{[4]}]\) may actually be rewritten in terms of deformation quantization \([\text{[24, 40, 23]}]\). The fact that the “classical algebra” \(C_0(\mathbb{R}^{2k})\) and the “quantum algebra” \(B_0(L^2(\mathbb{R}^k))\) have the same K-theory is peculiar to this special case; for general \(M\) this will, of course, fail. Indeed, the Baum–Connes conjecture may actually be seen as a test of the rigidity of K-theory under deformation quantization. Connes’s interpretation of the Baum–Connes conjecture as a \(G\) equivariant version of Bott periodicity \([\text{[14, 50, II.10.e]}]\) is consistent with this picture, since the field \([\text{[13]}]\) underlying the Baum–Connes conjecture is just a \(G\) equivariant version of the field \([\text{[10]}]\).

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\footnote{Note that the localization to \([0, 1]\) of the continuous field associated to the Heisenberg group used in \([\text{[20]}]\) is the same as the field defined by \(C^*(G_{\mathbb{R}^k})\), cf. §II.2.6 of \([\text{[4]}]\), so that the approach in \([\text{[20]}]\) really is based on deformation quantization.}
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