Exponential stability and exact controllability of a system of coupled wave equations by second-order terms (via Laplacian) with only one non-smooth local damping

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The purpose of this work is to investigate the exponential stability of a second-order coupled wave equations by Laplacian with one locally internal viscous damping. The strong stability is established by combining a unique continuation result and a Carleman estimate. Besides, the exponential stability is proved under (PMGC) condition on the damping region without any restriction on wave propagation speed (i.e., whether the two wave equations propagate at the same speed or not). The proof of this result relies on the frequency domain method combined with the multiplier techniques.

KEYWORDS
Carleman estimate, controllability, exponential stability, frequency domain approach, geometric condition, HUM method, viscous damping, wave equation

MSC CLASSIFICATION
35B40, 93C05, 93D15, 93D20, 93D23

1 INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded open set with a Lipschitz boundary $\Gamma$. A coupled wave equations, via Laplacian and with just one non-smooth localized viscous damping, is considered:

\[
\begin{aligned}
 p_t(x, t) - a \Delta p(x, t) + c \Delta q(x, t) + d(x)p_t(x, t) &= 0, \quad \text{in } \Omega \times (0, \infty), \\
 q_t(x, t) - \Delta q(x, t) + c \Delta p(x, t) &= 0, \quad \text{in } \Omega \times (0, \infty), \\
 p &= q = 0, \quad \text{on } \Gamma \times (0, \infty), \\
 p(x, 0) &= p_0(x), \quad q(x, 0) = q_0(x), \quad p_t(x, 0) = p_1(x), \quad q_t(x, 0) = q_1, \quad \text{in } \Omega.
\end{aligned}
\]

(1.1)

where $a > 0$, $c \in \mathbb{R}^+$ such that $a > c^2$ and $d \in L^\infty(\Omega)$ such that

\[
d(x) \geq d_0 \quad \text{in } \omega \subset \Omega \quad \text{and} \quad d(x) = 0 \quad \text{in } \Omega - \omega.
\]

(1.2)

Since $a > c^2$, then a constant $b > 0$ exists and satisfies

\[
a = b + c^2.
\]

(1.3)
In the first part of this manuscript, we shall prove that system (1.1) is exponentially stable assuming that the damping region complies with an appropriated geometric conditions known as Piecewise Multipliers Geometric Conditions and abbreviated (PMGC) (please see [1] and Definition 4.1). In the second part, we will study, under suitable geometric conditions, the exact controllability of the following system:

\[
\begin{align*}
\frac{p_t(x, t) - a \Delta p(x, t) + c \Delta q(x, t)}{p_t(x, t)} &= d(x)v, \quad \text{in } \Omega \times (0, \infty), \\
\frac{q_t(x, t) - \Delta q(x, t) + c \Delta p(x, t)}{q_t(x, t)} &= 0, \quad \text{in } \Omega \times (0, \infty), \\
p &= q = 0, \quad \text{on } \Gamma \times (0, \infty), \\
p(x, 0) &= p_0(x), \quad q(x, 0) = q_0(x), \quad p_t(x, 0) = p_1(x), \quad q_t(x, 0) = q_1, \quad \text{in } \Omega,
\end{align*}
\]

where \( v \) is an appropriate control. The key point used here is that, in a suitable space, the observability of the conservative system related to (1.4) (i.e., when \( v = 0 \)) and the exponential stability of system (1.1) are equivalent (see Proposition 2 in [2] and Theorem 4.1 in [3]). In fact, if the (PMGC) condition is fulfilled by the damping region, then the system (1.1) is exponentially stable. Thus, the observability of the homogeneous system associated to (1.4) takes place in \( \left(H^1_0(\Omega) \times L^2(\Omega)\right)^2 \). By the Hilbert Uniqueness Method (HUM in short) [4], we derive the exact controllability of system (1.4).

Russell [5] first presented the concept of indirect stabilization, which means that just one equation of the coupled system is damped, and since that, it has been an interest of many authors (please see [6–25] and references therein).

In [26] and [27], Alabau considered an abstract system of two second-order evolution equations that are weakly coupled. By proving an indirect observability inequality and thanks to the HUM, she proved that the system is exactly controllable for small damping parameters by means of one boundary control. Later on, Liu and Rao [28] established several weak observability inequalities for a two weakly coupled one-dimensional wave equations, by using the non-harmonic analysis tool. By determining an observability inequality for the linearized problem, Ammar-Khodja et al. [29] proved the null controllability of the reaction diffusion system. Wehbe and Youssef [30, 31] have examined the exact controllability of a weakly coupled wave equations with a localized internal control acting on just one equation. In [32], Alabau and Léauteaud derived, under the condition that the coupling and the control regions both meet the Geometric Control Condition (GCC), internal and boundary controllability results for a symmetric system of two wave-type equation, where only one of them is being controlled. Asymptotic behavior of a system of two wave equations coupled by velocities with just one localized damping was studied by Wehbe et al. (see [33]):

\[
\begin{align*}
\frac{p_t - a \Delta p + c(x)q_t + b(x)p_t}{p_t} &= 0, \quad \text{in } \Omega \times \mathbb{R}_+^*, \\
\frac{q_t - \Delta q - c(x)p_t}{q_t} &= 0, \quad \text{in } \Omega \times \mathbb{R}_+^*, \\
p &= q = 0 \text{ on } \Gamma \times \mathbb{R}_+^*.
\end{align*}
\]

They assumed that the coupling and the damping region intersec. First, they proved that their system is strongly stable without any geometric conditions. Then, two stability results have been obtained. The first one is of type exponential when the speed of the propagation of waves are equal and by assuming that the (PMGC) condition is satisfied. The second result is of type polynomial under the same geometric condition. Recently, in 2021, Gerbi et al. (see [34]) generalized the result obtained in [33] by assuming that the coupling region is a part of the damping region and meets the (GCC).

All works published in the literature studied the stability and the controllability of a system of two wave equations coupled by zero-order terms (\( p \) and \( q \)) or by first-order terms (\( p_t \) and \( q_t \)). An important question that can be asked: What can happen if the wave equations are coupled by a second-order term (via Laplacian)? In fact, there exists a physical interpretation of this type of the system. For example, in 1D, system (1.1) represents the piezoelectric material with magnetic effect (see, for instance, [35–42] and the rich references therein). In this direction, system (1.1) can be considered as a restriction of piezoelectric material in the multidimensional case. The analysis of stability issues in our case is more challenging due to the shape of our domain and the kind of coupling (via Laplacian). This paper seems to be the first that deals with this problem. Motivated by all these works, we intend to establish a two-fold objective:

a. To extend exponential decay results, known for the case of two wave equations coupled by zero-order terms or by first-order terms, to the case coupled by second-order terms.

b. To drop the equality assumption on the wave propagation speed considered in many earlier works.
The outline of this paper is as follows. In Section 2, we shall prove the well-posedness of system (1.1). By the help of a unique continuation result and a Carleman estimate, we derive the strong stability of the system in Section 3. The last section is devoted to the study of both the exponential decay of system (1.1) and the exact controllability of problem (1.4).

2 | WELL-POSEDNESS

We start this section by defining the energy of system (1.1) as follows:

\[ E(t) = \frac{1}{2} \int_{\Omega} (|p_i|^2 + b|\nabla p|^2 + |q_i|^2 + |c\nabla p - \nabla q|^2) \, dx. \]

This later satisfies the following result.

**Lemma 2.1.** Let \( Z = (p, p_t, q, q_t) \) be a regular solution of system (1.1). Then, one has

\[ \frac{d}{dt} E(t) = -\int_{\Omega} d(x)|p_i|^2 dx. \]  

(2.1)

**Proof.** Multiplying (1.1) by \( p \) and (1.1) by \( q \), integrating by parts on \( \Omega \) and then taking the real parts to obtain

\[ \frac{1}{2} \frac{d}{dt} (||p_i||^2 + a||\nabla p||^2) - cRe \left( \int_{\Omega} \nabla q \cdot \nabla p \, dx \right) + \int_{\Omega} d(x)|p_i|^2 dx = 0 \]  

(2.2)

and

\[ \frac{1}{2} \frac{d}{dt} (||q_i||^2 + ||\nabla q||^2) - cRe \left( \int_{\Omega} \nabla p \cdot \nabla q \, dx \right) = 0. \]  

(2.3)

Using the fact that \( a = b + c^2 \) in (2.2), we get

\[ \frac{1}{2} \frac{d}{dt} (||p_i||^2 + b||\nabla p||^2) + c^2 \frac{d}{dt} ||\nabla p||^2 - cRe \left( \int_{\Omega} \nabla q \cdot \nabla p \, dx \right) + \int_{\Omega} d(x)|p_i|^2 dx = 0 \]  

(2.4)

By adding (2.3) and (2.4), (2.1) holds true. \( \square \)

(2.1) means that the system (1.1) is dissipative. The state space is given by

\[ \mathcal{H} = (H^1_0(\Omega) \times L^2(\Omega))^2, \]  

(2.5)

which is a Hilbert space under the inner product

\[ \langle Z, Z_1 \rangle_{\mathcal{H}} = \int_{\Omega} (b\nabla p \cdot \nabla p_1 + r r_1 + (c\nabla p - \nabla q) \cdot (c\nabla p_1 - \nabla q_1) + s s_1) \, dx, \]  

(2.6)

for all \( Z = (p, r, q, s)^T \) and \( Z_1 = (p_1, r_1, q_1, s_1)^T \) in \( \mathcal{H} \). The norm in \( \mathcal{H} \) is denoted by \( || \cdot ||_{\mathcal{H}} \). Define now the linear operator \( A \) in \( \mathcal{H} \) by

\[ A(p, r, q, s)^T = (r, a\Delta p - c\Delta q - dr, s, \Delta q - c\Delta p)^T. \]  

(2.7)

with domain

\[ D(A) := \{ Z := (p, r, q, s) \in \mathcal{H}; r, s \in H^1_0(\Omega) \text{ and } \Delta p, \Delta q \in L^2(\Omega) \}. \]

If we set \( Z = (p, p_t, q, q_t)^T \), then (1.1) may be written as

\[ Z_t = AZ, \quad Z(0) = Z_0, \]  

(2.8)
where \( Z_0 = (p_0, p_1, q_0, q_1)^\top \). It follows that

\[
\Re \langle (AZ, Z)_H \rangle = -\int_{\Omega} |r|^2 \, dx \leq 0, \quad \forall \; Z = (p, r, q, s) \in D(A),
\]

which gives the dissipativity property of \( A \). Now, let \( F = (f, g, h, k) \in H \). The Lax-Milgram theorem guarantees that the equation

\[ -AZ = F, \]

has a unique solution \( Z \in D(A) \). Then, \( A \) is \( m \)-dissipative in \( H \) and so \( 0 \in \rho(A) \). Thanks to the Lumer-Phillips theorem, one concludes that \( A \) generates a \( C_0 \)-semigroup of contractions \( (e^{tA})_{t \geq 0} \). In this case, the solution of (2.8) is given by

\[ Z(t) = e^{tA}Z_0, \quad t \geq 0. \]

Consequently, it holds that:

**Theorem 2.2.** Let \( Z_0 \in H \), then system (2.8) possesses a unique weak solution \( Z \) satisfying

\[ Z \in C^0(\mathbb{R}^+, H). \]

Moreover, if \( Z_0 \in D(A) \), then problem (2.8) has a unique strong solution \( Z \) satisfying

\[ Z \in C^1(\mathbb{R}^+, H) \cap C^0(\mathbb{R}^+, D(A)). \]

### 3 | STRONG STABILITY

We start this part by proving a local unique continuation result for a coupled system of wave equations (see [13, 20]). To do this, let us define the elliptic operator \( B \) by

\[
B : H^2(U) \times H^2(U) \to L^2(U) \times L^2(U)
\]

\[
(p, q) \to (\Delta p, \Delta q)
\]

and the function \( l \) by

\[
l : L^2(U) \times L^2(U) \to L^2(U) \times L^2(U)
\]

\[
(p, q) \to (-\lambda^2 b^{-1}(p + cq), -\lambda^2 b^{-1}(cp + aq)).
\]

In order to prove our result, we need the following Carleman result (see [43] and Theorem 3.5 in [44]).

**Lemma 3.1.** Let \( U \) be a bounded open set in \( \mathbb{R}^N \). Define \( \varphi = e^{\psi \varphi} \) with \( \psi \in C^\infty(\mathbb{R}^N, \mathbb{R}) \) satisfying \( |\nabla \psi| > 0 \), and \( \varphi > 0 \) is large enough. Hence, it holds, for some constants \( \lambda > 0 \) and \( C > 0 \), that

\[
\kappa^3 ||e^{\varphi \psi}p||^2_{L^2(U)} + \kappa ||e^{\varphi \psi} \nabla \psi p||^2_{L^2(U)} \leq C ||e^{\varphi \psi} \Delta p||^2_{L^2(U)}, \quad \forall \; p \in H^2_0(U) \text{ and } \kappa > \kappa_0.
\]

Now, we may present our first result in this section (see Section 4 in [44]).

**Proposition 3.2.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set and \( x_0 \in \Omega \). In a neighborhood \( U \) of \( x_0 \), take a function \( m \) such that \( \nabla m \neq 0 \) in \( U \). Moreover, let \( (p, q) \in H^2(U) \times H^2(U) \) be a solution of \( B(p, q) = l(p, q) \). If \( p = q = 0 \) in \( \{ x \in U ; \; m(x) \geq m(x_0) \} \), then \( p = q = 0 \) in a neighborhood of \( x_0 \).

**Proof.** Set \( W := \{ x \in U ; m(x) \geq m(x_0) \} \). Choose \( U' \) and \( U'' \) neighborhoods of \( x_0 \) such that \( U'' \subseteq U' \subseteq U \). Let \( \chi \in C^\infty(U') \) such that \( \chi = 1 \) in \( U'' \). Set \( (\tilde{p}, \tilde{q}) = (\chi p, \chi q) \). Then, \( (\tilde{p}, \tilde{q}) \in H^2_0(U) \times H^2_0(U) \). Let \( \psi(x) = m(x) - c|x - x_0|^2 \)
and $\phi = e^{\omega \cdot \cdot}$. Then, applying Lemma 3.1 for $\bar{p}$ and next for $\bar{q}$, and adding the resultant inequalities to obtain

$$
\kappa^3 \int_{U'} e^{2\kappa \phi} (|\bar{p}|^2 + |\bar{q}|^2) \, dx + \kappa \int_{U'} e^{2\kappa \phi} (|\nabla \bar{p}|^2 + |\nabla \bar{q}|^2) \, dx \leq C \int_{U'} e^{2\kappa \phi} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx. \tag{3.4}
$$

As $U'' \subseteq U'$ and $\chi \in C_c^\infty (U')$ with $\chi = 1$ in $U''$, we arrive at

$$
\kappa^3 \int_{U''} e^{2\kappa \phi} (|p|^2 + |q|^2) \, dx + \kappa \int_{U''} e^{2\kappa \phi} (|\nabla p|^2 + |\nabla q|^2) \, dx 
\leq C \int_{U''} e^{2\kappa \phi} (|\Delta p|^2 + |\Delta q|^2) \, dx + C \int_{U' \setminus U''} e^{2\kappa \phi} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx.
$$

This leads to

$$
\kappa^3 \int_{U''} e^{2\kappa \phi} (|p|^2 + |q|^2) \, dx \leq C \int_{U''} e^{2\kappa \phi} (|\Delta p|^2 + |\Delta q|^2) \, dx + C \int_{U' \setminus U''} e^{2\kappa \phi} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx.
$$

Since $\Delta p = -\lambda^2 b^{-1}(p + cq)$ and $\Delta q = -\lambda^2 b^{-1}(cp + aq)$, we get

$$
(\kappa^3 - c_1(\lambda, b, c)) \int_{U''} e^{2\kappa \phi} |p|^2 \, dx + (\kappa^3 - c_2(\lambda, b, c, a)) \int_{U''} e^{2\kappa \phi} |q|^2 \, dx \leq C \int_{U''} e^{2\kappa \phi} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx,
$$

where $c_1(\lambda, b, c) := 2\lambda^2 b^{-2}(1 + c^2) > 0$ and $c_2(\lambda, b, c, a) := 2\lambda^2 b^{-2}(a^2 + c^2) > 0$. Hence, we get

$$
(\kappa^3 - \max(c_1(\lambda, b, c), c_2(\lambda, b, c, a))) \int_{U''} e^{2\kappa \phi} (|p|^2 + |q|^2) \, dx \leq C \int_{U''} e^{2\kappa \phi} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx.
$$

Thus, we obtain the existence of $\kappa > 0$ large enough and $C > 0$, such that

$$
\kappa^3 \int_{U''} e^{2\kappa \phi} (|p|^2 + |q|^2) \, dx \leq C \int_{U''} e^{2\kappa \phi} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx. \tag{3.5}
$$

Using the fact that $p = q = 0$ in $W$, we derive that

$$
\kappa^3 \int_{U''} e^{2\kappa \phi} (|p|^2 + |q|^2) \, dx \leq C \int_{S} e^{2\kappa \phi} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx, \tag{3.6}
$$

where $S = U' \setminus U'' \cup W$. Set $\Phi^1_\epsilon = \{x \in U; \phi(x) \leq \phi(x_0) - \epsilon\}$ and $\Phi^2_\epsilon = \{x \in U; \phi(x) \geq \phi(x_0) - \epsilon\}$. There exists $\epsilon > 0$ such that $S \subseteq \Phi^1_\epsilon$. Then choose a ball $B_0$ with center $x_0$ such that $B_0 \subseteq U'' \cap \Phi^2_\epsilon$. Hence, using (3.6), we have

$$
\int_{B_0} (|p|^2 + |q|^2) \, dx \leq \frac{Ce^{-cx}}{\kappa^3} \int_{S} (|\Delta \bar{p}|^2 + |\Delta \bar{q}|^2) \, dx.
$$

Taking $\kappa \to \infty$, we obtain that $p = q = 0$ in $B_0$. The proof has been completed.

**Theorem 3.3** (Calderón theorem). Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set and $\omega \subset \Omega$ with $\omega \neq \emptyset$. If $(p, q) \in H^2(\Omega) \times H^2(\Omega)$ such that $B(p, q) = l(p, q)$ and $p = q = 0$ in $\omega$, then $p = q = 0$ in $\Omega$.

**Proof.** Setting $F = \text{supp } p \cup \text{supp } q$ and following the proof of Theorem 4.2 in [45] by using Proposition 3.2 (stated above) instead of Proposition 4.1 (set out in [45]), we get the desired result.

Now, we are in position to state our stability result.
Theorem 3.4. The semigroup \((e^{tA})_{t \geq 0}\) is strongly stable in the space \(H\), which means that
\[
\lim_{t \to \infty} \|e^{tA}Z_0\|_H = 0, \quad \forall \ Z_0 \in H.
\]

Proof. Since the resolvent of \(A\) is compact in \(H\), then the Arendt-Batty theorem states that system (1.1) is strongly stable if and only if \(A\) does not possess purely imaginary eigenvalues. From Section 2, we have \(0 \in \rho(A)\). It remains to prove that \(\sigma(A) \cap i\mathbb{R}^* = \emptyset\). Let \(\lambda \neq 0\) and \(Z = (p, r, q, s)^T \in D(A) \setminus \{0\}\) such that
\[
AZ = i\lambda Z.
\]
(3.7) may be rewritten as
\[
r = i\lambda p, \tag{3.8}
\]
\[
i\lambda r - a\Delta p + c\Delta q + dr = 0, \tag{3.9}
\]
\[
s = i\lambda q, \tag{3.10}
\]
\[
i\lambda s - \Delta q + c\Delta p = 0. \tag{3.11}
\]
From (2.9) and (3.7), one has
\[
0 = \Re(i\lambda \|Z\|_H) = \Re(\langle AZ, Z \rangle) = -\int_{\Omega} d|\lambda|^2 dx \leq 0. \tag{3.12}
\]
Thus, from (3.8), (3.12), the property of the function \(d\) given in (1.2), and the fact that \(\lambda \neq 0\), we have
\[
dr = 0 \text{ in } \Omega \text{ and consequently } r = p = 0 \text{ in } \omega. \tag{3.13}
\]
Using the fact that \(a = b + c^2\) and (3.13) in (3.9), we get
\[
i\lambda r - b\Delta p - c(c\Delta p - \Delta q) = 0, \text{ in } \Omega. \tag{3.14}
\]
Combining (3.14) and (3.11), one has
\[
i\lambda(r + cs) - b\Delta p = 0, \text{ in } \Omega. \tag{3.15}
\]
Putting (3.13) in (3.15) and using (3.10) and the fact that \(\lambda, c \neq 0\), we infer that
\[
q = s = 0 \text{ in } \omega. \tag{3.16}
\]
Inserting (3.8) and (3.10) in (3.15) and then combining with (3.11) to obtain that
\[
\begin{align*}
\Delta p &= -\lambda^2 b^{-1}(p + cq), \\
\Delta q &= -\lambda^2 b(cp + aq), \\
p &= q = 0 \text{ in } \omega.
\end{align*}
\]
Hence, Theorem 3.3 gives us \(p = q = 0\) in \(\Omega\). Thus, by (3.8) and (3.10), we see that \(r = s = 0\) in \(\Omega\) and so that \(Z = 0\). \(\square\)

4 EXPONENTIAL STABILITY AND EXACT CONTROLLABILITY

4.1 Exponential stability
This subsection examines the exponential stability of system (1.1). First, we introduce the \textbf{(PMGC)} condition, as in [1] (see also [4, 46, 47]), on the subset \(\omega\) where the dissipation is effective.
Definition 4.1 (PMGC). There exist open sets \( \Omega_j \subset \Omega \) with piecewise smooth boundary \( \partial \Omega_j \), and points \( x^j_0, j = 1, 2, \ldots, J \), such that \( \Omega_j \cap \Omega_i = \emptyset \), for any \( 1 \leq i < j \leq J \), and
\[
\Omega \cap \mathcal{N}_\varepsilon \left[ \left( \bigcup_{j=1}^J \gamma_j \right) \cup \left( \Omega \setminus \bigcup_{j=1}^J \Omega_j \right) \right] \subset \omega,
\]
for some \( \varepsilon > 0 \), and for \( S \subset \mathbb{R}^N \),
\[
\mathcal{N}_\varepsilon(S) = \bigcup_{x \in S} \{ y \in \mathbb{R}^N; |x - y| < \varepsilon \}, \quad \gamma_j = \{ x \in \partial \Omega_j; (x - x^j_0) \cdot \nu^j(x) > 0 \} \quad \text{and} \quad \Gamma_j = \partial \Omega_j.
\]
where \( \nu^j \) is the unit normal vector pointing into the exterior of \( \Omega_j \).

For the sequel, we need some additional notations. For each \( j = 1, \ldots, J \), set \( m_j(x) = x - x^j_0 \) and \( R_j = \sup \{ |m_j(x)|, x \in \Omega \} \). Set \( S = \left( \bigcup_{j=1}^J \gamma_j \right) \cup \left( \Omega \setminus \bigcup_{j=1}^J \Omega_j \right) \) and let \( 0 < \varepsilon'_0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon \). Since \( (\overline{\Omega} \setminus \mathcal{N}_{\varepsilon_2}) \cap \overline{\mathcal{N}_{\varepsilon_1}} = \emptyset \) and \( (\overline{\Omega} \setminus \mathcal{N}_{\varepsilon_1}) \cap \overline{\mathcal{N}_{\varepsilon_0}} = \emptyset \), we define the following two cut-off functions \( \varphi_1, \varphi_2 \in W^{1,\infty}(\Omega) \), such that \( 0 \leq \varphi_1, \varphi_2 \leq 1 \) and
\[
\varphi_1 = \begin{cases} 
0 & \text{if } x \in \Omega \setminus \mathcal{N}_{\varepsilon_2}, \\
1 & \text{if } x \in \mathcal{N}_{\varepsilon_1}
\end{cases} \quad \text{and} \quad \varphi_2 = \begin{cases} 
0 & \text{if } x \in \Omega \setminus \mathcal{N}_{\varepsilon_1}, \\
1 & \text{if } x \in \mathcal{N}_{\varepsilon_0}.
\end{cases} \quad (4.1)
\]

Now, as \( (\overline{\Omega} \setminus \mathcal{N}_{\varepsilon'_0}) \cap \overline{\mathcal{N}_{\varepsilon_0}} = \emptyset \), we can define the function \( \psi_j \in W^{1,\infty}(\Omega) \), \( 0 \leq \psi_j \leq 1 \), and we consider the \( C^1 \) vector field \( h_j \) on \( \Omega_j \), such that
\[
\psi_j = \begin{cases} 
1 & \text{if } x \in \overline{\Omega} \setminus \mathcal{N}_{\varepsilon_0} \quad \text{and} \quad h_j = \psi_j m_j.
\end{cases} \quad (4.2)
\]

For simplicity, we denote by \( \Gamma'_j = (\Gamma_j \setminus \gamma_j) \), \( Q_k = \mathcal{N}_{\varepsilon_k}(S) \), for all \( k \in \{0, 1, 2\} \) and \( Q'_0 = \mathcal{N}_{\varepsilon'_0} \) (see Figure 1).

We list two examples of region \( \omega \) that satisfies the geometric conditions (PMGC) ([1]).
Example 4.2.

1. Let $\Omega$ be a rectangle and $S$ be two straight lines that are parallel to adjacent sides of the rectangle, respectively, and that intersect at a point in $\Omega$. Then $\omega = \Omega \cap N_1$, for any $\epsilon > 0$, satisfies (PMGC) with $J = 4$, $\Omega_j$ being subrectangles consisting of $\Gamma$ and $S$ and $x_0^j$ being vertices of the rectangle.

2. Let $x = (y, z)$ with $y \in \mathbb{R}^2$ and $z \in \mathbb{R}$, $Y$ be an open subset in $\mathbb{R}^2$, $f_1(y)$ and $f_2(y)$ be real-valued functions defined on $\bar{Y}$ such that $f_1(y) < f_2(y)$, $\forall y \in Y$, and $f_1(y) = f_2(y)$ for all $y \in \partial Y$. Let $\Omega = \{(y, z); y \in Y, f_1(y) < z < f_2(z)\}$. Defining $\omega = \Omega \cap N_1$, for any $\epsilon > 0$. Then $\omega$ satisfies (PMGC) with $J = 2$. $\Omega_1 = \{(y, z); y \in Y, f_1(y) < z < -\frac{1}{2}\}$, $\Omega_2 = \{(y, z); y \in Y, \frac{1}{2} < z < f_2(y)\}$, and $x_0^j = (y_0, z_j)$, where $y_0 \in \mathbb{R}^2$, $z_1 < 0$ small enough and $z_2 > 0$ large enough.

We are now in position to state our result.

Theorem 4.3. Suppose that $\omega$ meets the (PMGC) condition. Then the semigroup $(\rho^A)_{t \geq 0}$ is exponentially stable; that is, there exists $\mu \geq 1$ and $\sigma > 0$ not depending on $Z_0$, such that

$$\|\rho^A Z_0\|_H \leq \mu e^{-\sigma t}\|Z_0\|_H.$$

(4.3)

According to [48, 49] (see also [50]), a $C_0$-semigroup of contractions $(\rho^A)_{t \geq 0}$ on $H$ satisfies (4.3) if

$$i \mathbb{R} \subset \rho(A),$$

(H1)

$$\sup_{\lambda \in \mathbb{R}} \| (i \lambda I - A)^{-1} \|_{L(H)} = O(1).$$

(H2)

From Theorem 3.4, condition (H1) holds true. Using a contradiction argument, we shall prove condition (H2). For this purpose, suppose that (H2) is false, then there exists $\{\lambda^n, Z^n\}_{n \geq 1} \subset \mathbb{R}^* \times D(A)$ with

$$|\lambda^n| \to \infty \text{ and } \|Z^n\|_H = \|(p^n, r^n, q^n, s^n)\|_H = 1,$$

(4.4)

such that

$$(i \lambda^n - A)Z^n = F^n := (f^n, g^n, h^n, k^n)^\top \to 0 \text{ in } H.$$  

(4.5)

We omit the index $n$ for sake of simplicity. Detailing Equation (4.5), we get

$$\begin{cases}
    i \lambda p - r = f, \\
    i \lambda r - a \Delta p + c \Delta q + dr = g, \\
    i \lambda q - s = h, \\
    i \lambda s - \Delta q + c \Delta p = k.
\end{cases}$$

(4.6)

Inserting (4.6) in (4.6)$_2$ and (4.6)$_3$ in (4.6)$_4$, we get

$$\lambda^2 p + a \Delta p - c \Delta q - i \lambda dp = F_1,$$

(4.7)

$$\lambda^2 q + \Delta q - c \Delta p = F_2,$$

(4.8)

where

$$F_1 := -(g + i \lambda f + df) \text{ and } F_2 := -(k + i \lambda h).$$

(4.9)

Using the fact that $a = b + c^2$ in (4.7), we get

$$\lambda^2 p + b \Delta p + c (c \Delta p - \Delta q) - i \lambda dp = F_1.$$

(4.10)
Now, combining (4.8) and (4.10), we obtain
\[
b\Delta p = -\lambda^2 p - c\lambda^2 q + i\lambda dp + F_1 + cF_2.
\]
(4.11)

Inserting (4.11) in (4.8), we get the following system
\[
\lambda^2 p + b\Delta p + c\lambda^2 q - i\lambda dp = F_3,
\]
(4.12)
\[
a\lambda^2 q + b\Delta q + c\lambda^2 p - i\lambda cdp = F_4.
\]
(4.13)
where
\[
F_3 = F_1 + cF_2 \quad \text{and} \quad F_4 = aF_2 + cF_1.
\]
(4.14)

To prove Theorem (4.3), we need several lemmas.

**Lemma 4.4.** The solution \((p, r, q, s)\) of system (4.6) verifies the following estimations:
\[
\int_\Omega |r|^2\,dx = o(1), \quad \int_\Omega |\lambda p|^2\,dx = o(1) \quad \text{and} \quad \int_\Omega |\lambda p|^2\,dx = o(1).
\]
(4.15)

**Proof.** First, taking the inner product of (4.5) with \(Z\) in \(\mathcal{H}\), using the fact that \(\|Z\|_\mathcal{H} = 1\) and \(\|F\|_\mathcal{H} = o(1)\), one derives
\[
\int_\Omega |r|^2\,dx = -\Re \langle (AZ, Z)_{\mathcal{H}} \rangle = \Re \langle (i\lambda I - A)Z, Z\rangle_{\mathcal{H}} = \Re \langle (F, Z)_{\mathcal{H}} \rangle = o(1).
\]
(4.16)

Now, multiplying (4.6) by \(\sqrt{d}\) and using the first estimation in (4.15) and that \(\|F\|_\mathcal{H} = o(1)\), we get the second estimation in (4.15). Finally, the last estimation in (4.15) is obtained by using the definition of \(d\), given in (1.2), together with the second estimation in (4.15).

**Lemma 4.5.** The solution \((p, r, q, s)\) of system (4.6) verifies
\[
\int_{\Omega \cap Q_1} |\lambda q|^2\,dx = o(1).
\]
(4.17)

**Proof.** Multiply (4.12) by \(\varphi_1\bar{q}\), integrate on \(\Omega\), and apply Green’s formula to find:
\[
\Re \left( \int_\Omega \varphi_1\lambda^2 \bar{p}q\,dx \right) - b\Re \left( \int_\Omega \nabla p \cdot \nabla (\varphi_1\bar{q})\,dx \right) + c\int_\Omega |\lambda q|^2\,dx - \Re \left( i\lambda \int_\Omega dp\varphi_1\bar{q}\,dx \right) = \Re \left( \int_\Omega F_3\varphi_1\bar{q}\,dx \right).
\]
(4.18)

Now, in the same way as previously, by multiplying (4.13) by \(-\varphi_1\bar{p}\), one derives that
\[
-a\Re \left( \int_\Omega \varphi_1\lambda^2 q\bar{p}\,dx \right) + b\Re \left( \int_\Omega \nabla q \cdot \nabla (\varphi_1\bar{p})\,dx \right) - c\int_\Omega |\lambda p|^2\,dx = -\Re \left( \int_\Omega F_4\varphi_1\bar{p}\,dx \right).
\]
(4.19)
Adding (4.18) and (4.19), we get
\[
c\int_\Omega |\varphi_1\lambda q|^2\,dx = (a - 1)\Re \left( \int_\Omega \varphi_1\lambda^2 \bar{p}q\,dx \right) + \Re \left( i\lambda \int_\Omega dp\varphi_1\bar{q}\,dx \right) + c\int_\Omega |\lambda p|^2\,dx
\]
\[
+ b\Re \left( \int_\Omega \bar{q}\nabla p \cdot \nabla \varphi_1\,dx \right) - b\Re \left( \int_\Omega \bar{p}\nabla q \cdot \nabla \varphi_1\,dx \right) + \Re \left( \int_\Omega F_3\varphi_1\bar{q}\,dx \right) - \Re \left( \int_\Omega F_4\varphi_1\bar{p}\,dx \right).
\]
(4.20)
Using Lemma 4.6, the definitions of the functions $\varphi_1$ and $d$, and the fact that $\lambda q$ is uniformly bounded in $L^2(\Omega)$, one gets

$$\left| \Re \left( \int_\Omega \varphi_1 \lambda^2 \overline{p} q \, dx \right) \right| = o(1), \quad \left| \Re \left( i \lambda \int_\Omega dp \varphi_1 q \, dx \right) \right| = o(\lambda^{-1}) \quad \text{and} \quad \left| \int_\Omega \varphi_1 \lambda p^2 \, dx \right| = o(1). \tag{4.21}$$

Using (4.15), (4.17), the definition of the function $\varphi_1$, and the fact that $\nabla p$ and $\nabla q$ are uniformly bounded in $L^2(\Omega)$, we see that

$$\left| \Re \left( \int_\Omega \overline{q} \nabla p \cdot \nabla \varphi_1 \, dx \right) \right| = o(\lambda^{-1}) \quad \text{and} \quad \left| \Re \left( \int_\Omega \nabla q \cdot \nabla \varphi_1 \, dx \right) \right| = o(\lambda^{-1}). \tag{4.22}$$

Using the definition of $F_4$ and $F_3$ given in (4.14), the fact that $\lambda q$ and $\lambda p$ are uniformly bounded in $L^2(\Omega)$ and $\|F\|_H = o(1)$, one has

$$\left| \Re \left( \int_\Omega F_3 \varphi_1 \, dx \right) \right| = o(1) \quad \text{and} \quad \left| \Re \left( \int_\Omega F_4 \varphi_1 \, dx \right) \right| = o(1). \tag{4.23}$$

Inserting (4.21), (4.22), and (4.23) in (4.20), and using the property of $\varphi_1$, we get (4.17).

**Lemma 4.6.** The solution $(p, r, q, s)$ of system (4.6) fulfills

$$\int_{\Omega \cap Q_1} |\nabla p|^2 \, dx = o(1). \tag{4.24}$$

**Proof.** Multiply (4.12) by $-\varphi_1 \overline{\mu}$, integrate on $\Omega$, and then apply Green’s formula to get

$$b \int_\Omega \varphi_1 |\nabla p|^2 \, dx = \int_\Omega \varphi_1 \lambda p^2 \, dx - b \int_\Omega \overline{\mu} \nabla p \cdot \nabla \varphi_1 \, dx + c \int_\Omega \lambda^2 \varphi_1 \overline{q} \, dx - i \lambda \int_\Omega \lambda \varphi_1 |p|^2 \, dx - \int_\Omega F_3 \varphi_1 \, dx. \tag{4.25}$$

Using (4.15) and (4.17), the definition of $\varphi_1$, and the fact that $\|F\|_H = o(1)$ and $\nabla p$ is uniformly bounded in $L^2(\Omega)$, we have

$$\left| \int_\Omega \overline{\mu} \nabla p \cdot \nabla \varphi_1 \, dx \right| = o(\lambda^{-1}), \quad \left| \int_\Omega \lambda^2 \varphi_1 \overline{q} \, dx \right| = o(1), \quad \left| \int_\Omega F_3 \varphi_1 \, dx \right| = o(1). \tag{4.26}$$

Inserting the above estimations in (4.25) and using again Lemma 4.4, we get the desired result (4.24).

**Lemma 4.7.** We have

$$\int_{\Omega \cap Q_0} |\nabla q|^2 \, dx = o(1). \tag{4.27}$$

**Proof.** Multiply (4.13) by $-\varphi_2 \overline{\mu}$, integrate on $\Omega$, then thanks to Green’s formula, we get

$$- a \int_\Omega \varphi_2 \lambda q^2 \, dx + b \int_\Omega \varphi_2 |\nabla q|^2 \, dx + \int_\Omega \overline{\mu} \nabla q \cdot \nabla \varphi_2 \, dx + c \int_\Omega \lambda^2 \varphi_2 \overline{p} \, dx + i \lambda c \int_\Omega \varphi_2 \overline{p} \, dx = - \int_\Omega \varphi_2 F_4 \, dx. \tag{4.28}$$

Using (4.17), (4.15), the definition of $\varphi_2$, and $\|F\|_H = o(1)$, we infer that

$$\left| \int_\Omega \overline{\mu} \nabla q \cdot \nabla \varphi_2 \, dx \right| = o(\lambda^{-1}), \quad \left| \int_\Omega \lambda^2 \varphi_2 \overline{p} \, dx \right| = o(1), \quad \left| \lambda c \int_\Omega \varphi_2 \overline{p} \, dx \right| = o(1), \quad \left| \int_\Omega \varphi_2 F_4 \, dx \right| = o(1).$$

Inserting the above estimations in (4.28), and using (4.17) and the property of $\varphi_2$, one gets (4.27).
Lemma 4.8. The solution \((p, r, q, s)\) of system (4.6) verifies

\[
N \int_{\mathcal{K}_0} (|\lambda p|^2 + |\lambda q|^2) \, dx - (N - 2) b \int_{\mathcal{K}_0} |\nabla p|^2 \, dx - (N - 2) \int_{\mathcal{K}_0} |c\nabla p - \nabla q|^2 \, dx \leq o(1). \tag{4.29}
\]

where \(\mathcal{K}_0 = \Omega \setminus (\Omega \cap Q_0)\).

Proof. The proof consists of two steps.

**Step 1.** In this step, we aim to demonstrate the equation below:

\[
\begin{align*}
\int_{\Gamma_j} \text{div} (h_j) \left( |\lambda p|^2 + |\lambda q|^2 - a|\nabla p|^2 - |\nabla q|^2 \right) \, d\Gamma_j & - 2c(2 - N)\Re \left( \int_{\Gamma_j} \psi_j (\nabla \overline{p} \cdot \nabla q) \, dx \right) \\
+ 2a\Re \left( \sum_{i=1}^N \int_{\Gamma_j} \frac{\partial p}{\partial n_i} \frac{\partial h_j^i}{\partial x_i} \frac{\partial \overline{p}}{\partial x_n} \, dx \right) & + 2\Re \left( \sum_{i=1}^N \int_{\Gamma_j} \frac{\partial q}{\partial n_i} \frac{\partial h_j^i}{\partial x_i} \frac{\partial \overline{q}}{\partial x_n} \, dx \right) \\
- \int_{\Gamma_j} (a|\nabla p|^2 + |\nabla q|^2) (h_j \cdot v_j) \, d\Gamma_j & + 2c \Re \left( \int_{\Gamma_j} \frac{\partial p}{\partial n_j} (h_j \cdot v_j) \, d\Gamma \right) = o(1).
\end{align*}
\tag{4.30}
\]

To do this, multiplying (4.7) by \(2(h_j \cdot \nabla \overline{p})\), integrating on \(\Omega_j\), and then taking the real parts to obtain

\[
\begin{align*}
\Re \left( 2\lambda^2 \int_{\Omega_j} p(h_j \cdot \nabla \overline{p}) \, dx \right) & + \Re \left( 2a \int_{\Omega_j} \Delta p(h_j \cdot \nabla \overline{p}) \, dx \right) - \Re \left( 2c \int_{\Omega_j} \Delta q(h_j \cdot \nabla \overline{p}) \, dx \right) \\
- \Re \left( 2i\lambda \int_{\Omega_j} \frac{\partial p}{\partial n_j} (h_j \cdot \nabla \overline{p}) \, dx \right) & = \Re \left( 2 \int_{\Omega_j} F_1(h_j \cdot \nabla \overline{p}) \, dx \right). \tag{4.31}
\end{align*}
\]

- **Estimation of the second member of** (4.31). First, using the definition of \(F_1\) given in (4.9), we get

\[
2 \int_{\Omega_j} F_1(h_j \cdot \nabla \overline{p}) \, dx = -2 \int_{\Omega_j} (g + df)(h_j \cdot \nabla \overline{p}) \, dx - 2i\lambda \int_{\Omega_j} f(h_j \cdot \nabla \overline{p}) \, dx. \tag{4.32}
\]

Thanks to Green’s formula and the fact that \(p = 0\) on \(\Gamma_j \setminus \gamma_j\) and that \(h_j = 0\) on \(\gamma_j\), one has

\[
-2i\lambda \int_{\Omega_j} f(h_j \cdot \nabla \overline{p}) \, dx = 2i\lambda \int_{\Gamma_j} \overline{p}(h_j \cdot \nabla f) \, dx + 2i\lambda \int_{\Omega_j} \overline{p} f(\text{div} h_j) \, dx. \tag{4.33}
\]

Since \(f \rightarrow 0\) in \(H^1_0(\Omega)\) and \(\lambda \nabla p\) is bounded in \(L^2(\Omega)\), then we find

\[
\left| -2i\lambda \int_{\Omega_j} f(h_j \cdot \nabla \overline{p}) \, dx \right| = o(1). \tag{4.34}
\]

Next, as \(f \rightarrow 0\) in \(H^1_0(\Omega)\), \(g \rightarrow 0\) in \(L^2(\Omega)\), and \(\nabla p\) is bounded in \(L^2(\Omega)\), it holds that

\[
\left| -2 \int_{\Omega_j} (g + df)(h_j \cdot \nabla \overline{p}) \, dx \right| = o(1).
\]

Based on the estimation above and (4.34), one gets that

\[
\left| 2 \int_{\Omega_j} F_1(h_j \cdot \nabla \overline{p}) \, dx \right| = o(1). \tag{4.35}
\]
- **Estimation of the first term in** (4.31). Using Green’s formula, we get

\[
\Re \left( 2\lambda^2 \int_{\Omega_j} p(h_j \cdot \nabla \bar{p}) \, dx \right) = -\int_{\Omega_j} \text{div} (h_j) |\lambda p|^2 \, dx + \int_{\Gamma_j} (h_j \cdot v_j) |\lambda p|^2 \, d\Gamma_j.
\]

Since \( \psi_j = 0 \) on \( \Gamma_j \) and \( p = 0 \) on \( (\Gamma_j \setminus \Gamma_j) \), then we derive that

\[
\Re \left( 2\lambda^2 \int_{\Omega_j} p(h_j \cdot \nabla \bar{p}) \, dx \right) = -\int_{\Omega_j} \text{div} (h_j) |\lambda p|^2 \, dx.
\]  

(4.36)

- **Estimation of the second term in** (4.31). Green’s formula leads to

\[
\Re \left( 2a \int_{\Omega_j} \Delta p(h_j \cdot \nabla \bar{p}) \, dx \right) = a \int_{\Omega_j} \text{div} (h_j) |\nabla p|^2 \, dx - 2a\Re \left( \sum_{i=1}^{N} \int_{\Omega_j} \frac{\partial p}{\partial x_i} \frac{\partial h^n_j}{\partial x_i} \frac{\partial \bar{p}}{\partial x_n} \, dx \right) - a \int_{\Gamma_j} (h_j \cdot v_j) |\nabla p|^2 \, d\Gamma_j + \Re \left( 2a \int_{\Gamma_j} \frac{\partial p}{\partial v_j} (h_j \cdot \nabla \bar{p}) \, d\Gamma_j \right).
\]  

(4.37)

According to the choice of \( \psi_j \), one claims that all the boundary terms are vanishing in (4.37) except those on \( \Gamma_j \). However, on this part of the boundary, \( p = 0 \), and consequently, \( \nabla p = \left( \frac{\partial p}{\partial \nu_j} \right) \cdot \nu = \left( \frac{\partial p}{\partial v_j} \right) v_j \). Thus, one has

\[
-a \int_{\Gamma_j} (h_j \cdot v_j) |\nabla p|^2 \, d\Gamma_j + \Re \left( 2a \int_{\Gamma_j} \frac{\partial p}{\partial v_j} (h_j \cdot \nabla \bar{p}) \, d\Gamma_j \right) = a \int_{\Gamma_j} (h_j \cdot v_j) |\nabla p|^2 \, d\Gamma_j.
\]

Inserting the above equation in (4.37), we have

\[
\Re \left( 2a \int_{\Omega_j} \Delta p(h_j \cdot \nabla \bar{p}) \, dx \right) = a \int_{\Omega_j} \text{div} (h_j) |\nabla p|^2 \, dx - 2a\Re \left( \sum_{i=1}^{N} \int_{\Omega_j} \frac{\partial p}{\partial x_i} \frac{\partial h^n_j}{\partial x_i} \frac{\partial \bar{p}}{\partial x_n} \, dx \right) + a \int_{\Gamma_j} (h_j \cdot v_j) |\nabla p|^2 \, d\Gamma_j.
\]  

(4.38)

- **Estimation of the third term in** (4.31). Thanks to Green’s formula and the definition of the function \( \psi_j \), we obtain

\[
-\Re \left( 2c \int_{\Omega_j} \Delta q(h_j \cdot \nabla \bar{p}) \, dx \right) = 2c \Re \left( \int_{\Omega_j} \nabla q \cdot \nabla (h_j \cdot \nabla \bar{p}) \, dx \right) - 2c \Re \left( \int_{\Gamma_j} \frac{\partial q}{\partial v_j} (h_j \cdot \nabla \bar{p}) \, d\Gamma_j \right).  
\]  

(4.39)

By using the definition of \( h_j \), it holds that

\[
I = 2c \Re \left( \int_{\Omega_j} \psi_j \nabla q \cdot \nabla \bar{p} \, dx \right) + 2c \Re \left( \int_{\Omega_j} (m_j \cdot \nabla \bar{p}) \cdot \nabla \psi_j \, dx \right) + 2c \Re \left( \sum_{i=1}^{N} \int_{\Omega_j} \frac{\partial q}{\partial x_i} h^n_j \frac{\partial^2 \bar{p}}{\partial x_i \partial x_n} \, dx \right).  
\]  

(4.40)
Using the definition of the function \( \psi_j \), (4.24), and (4.27), one derives

\[
|2c \Re \left( \int_{\Omega_j} (m_j \cdot \nabla \bar{p}) (\nabla q \cdot \nabla \psi_j) \, dx \right)| = o(1).
\]

The combination of the last equation and (4.40) yields to

\[
I = 2c \Re \left( \int_{\Omega_j} \psi_j \nabla q \cdot \nabla \bar{p} \, dx \right) + I_1 + o(1). \tag{4.41}
\]

Using Green's formula on \( I_1 \), we obtain

\[
I_1 = -2cN \Re \left( \int_{\Omega_j} \psi_j (\nabla \bar{p} \cdot \nabla q) \, dx \right) - 2c \Re \left( \sum_{i=1}^{N} \int_{\Omega_j} \frac{\partial \bar{p}}{\partial x_i} \left( \nabla \left( \frac{\partial q}{\partial x_i} \right) \cdot h_j \right) \right) - 2c \Re \left( \int_{\Omega_j} (\nabla \psi_j \cdot m_j) (\nabla \bar{p} \cdot \nabla q) \, dx \right) + 2c \Re \int_{\Gamma_j} (h_j \cdot v_j) (\nabla \bar{p} \cdot \nabla q) \, d\Gamma_j. \tag{4.42}
\]

Using the definition of the function \( \psi_j \), (4.24), and (4.27), we get

\[
|2c \Re \left( \int_{\Omega_j} (\nabla \psi_j \cdot m_j) (\nabla \bar{p} \cdot \nabla q) \, dx \right)| = o(1).
\]

This later combined to (4.42) gives us

\[
I_1 = -2cN \Re \left( \int_{\Omega_j} \psi_j (\nabla \bar{p} \cdot \nabla q) \, dx \right) - 2c \Re \left( \sum_{i=1}^{N} \int_{\Omega_j} \frac{\partial \bar{p}}{\partial x_i} \left( \nabla \left( \frac{\partial q}{\partial x_i} \right) \cdot h_j \right) \right) + 2c \Re \int_{\Gamma_j} (h_j \cdot v_j) (\nabla \bar{p} \cdot \nabla q) \, d\Gamma + o(1).
\]

Then (4.41) becomes

\[
I = 2c(1 - N) \Re \left( \int_{\Omega_j} \psi_j \nabla q \cdot \nabla \bar{p} \, dx \right) - 2c \Re \left( \sum_{i=1}^{N} \int_{\Omega_j} \frac{\partial \bar{p}}{\partial x_i} \left( \nabla \left( \frac{\partial q}{\partial x_i} \right) \cdot h_j \right) \right) + 2c \Re \int_{\Gamma_j} (h_j \cdot v_j) (\nabla \bar{p} \cdot \nabla q) \, d\Gamma + o(1).
\]

Inserting the above estimation in (4.39), we derive

\[
- \Re \left( 2c \int_{\Omega_j} \Delta q(h_j \cdot \nabla \bar{p}) \, dx \right) = 2c(1 - N) \Re \left( \int_{\Omega_j} \psi_j \nabla q \cdot \nabla \bar{p} \, dx \right) + 2c \Re \int_{\Gamma_j} (h_j \cdot v_j) (\nabla \bar{p} \cdot \nabla q) \, d\Gamma_j - 2c \Re \left( \sum_{i=1}^{N} \int_{\Omega_j} \frac{\partial \bar{p}}{\partial x_i} \left( \nabla \left( \frac{\partial q}{\partial x_i} \right) \cdot h_j \right) \right) - 2c \Re \left( \int_{\Gamma_j} \frac{\partial q}{\partial v_j} (h_j \cdot \nabla \bar{p}) \, d\Gamma_j \right) + o(1). \tag{4.43}
\]

**Estimation of the fourth term in** (4.31). From the definition of the functions \( d \) and \( h_j \), (4.15), and (4.24), we have

\[
|\Re \left( 2i \lambda \int_{\Omega_j} d p(h_j \cdot \nabla \bar{p}) \, dx \right)| = o(1). \tag{4.44}
\]
Inserting (4.35), (4.36), (4.38), (4.43), and (4.44) in (4.31), we get

\[
- \int_{\Omega_j} \text{div} (h_j)|\lambda p|^2 dx + a \int_{\Omega_j} \text{div} (h_j)|\nabla p|^2 dx - 2a\Re \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\partial q}{\partial x_i} \frac{\partial h_i}{\partial x_n} dx \right) \\
+ 2c(1-N)\Re \left( \int_{\Omega_j} \psi_j \nabla q \cdot \nabla \overline{p} dx \right) - 2c\Re \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\partial h_i}{\partial x_n} \left( \nabla \left( \frac{\partial q}{\partial x_i} \right) \cdot h_j \right) dx \right) \\
+ a \int_{\Gamma_j} (h_j \cdot v_j) |\nabla p|^2 d\Gamma_j + 2c\Re \int_{\Gamma_j} (h_j \cdot v_j) \left( \nabla \overline{p} \cdot \nabla q \right) d\Gamma_j - 2c\Re \left( \int_{\Gamma_j} \frac{\partial q}{\partial v_j} (h_j \cdot \nabla \overline{q}) d\Gamma_j \right) = o(1).
\]

(4.45)

Now, multiplying (4.8) by $2h_j \nabla \cdot \overline{q}$ and repeating the same computations as above yields to

\[
- \int_{\Omega_j} \text{div} (h_j)|\lambda q|^2 dx + \int_{\Omega_j} \text{div} (h_j)|\nabla q|^2 dx - 2a\Re \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\partial q}{\partial x_i} \frac{\partial h_i}{\partial x_n} dx \right) \\
+ 2c\Re \left( \int_{\Omega_j} \psi_j \nabla q \cdot \nabla \overline{p} dx \right) + 2c\Re \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\partial h_i}{\partial x_n} \left( \nabla \left( \frac{\partial q}{\partial x_i} \right) \cdot h_j \right) dx \right) \\
+ \int_{\Gamma_j} (h_j \cdot v_j) |\nabla q|^2 d\Gamma_j - 2c\Re \left( \int_{\Gamma_j} \frac{\partial q}{\partial v_j} (h_j \cdot \nabla \overline{q}) d\Gamma_j \right) = o(1).
\]

(4.46)

Summing (4.45) and (4.46) and multiplying the result by $-1$, we obtain (4.30).

**Step 2.** In this step, we shall prove (4.29). First, using the fact that $a = b + c^2$ and the definition of $h_j$, then summing (4.30) on $j$ from 1 to $J$, we obtain

\[
N \int_{K_0} (|\lambda p|^2 + |\lambda q|^2) dx - (N - 2) b \int_{K_0} |\nabla p|^2 dx - (N - 2) \int_{K_0} |c\nabla p - V q|^2 dx = \\
- \sum_{j=1}^{J} \left( N \int_{\Omega_j \cap \Omega_0} (|\lambda p|^2 + |\lambda q|^2) dx - (N - 2) \int_{\Omega_j \cap \Omega_0} (b|\nabla p|^2 dx + |c\nabla p - V q|^2 dx) \right) \\
- \sum_{j=1}^{J} \left( 2a\Re \left( \sum_{i=1}^{N} \int_{\Omega_j \cap \Omega_0} \frac{\partial q}{\partial x_i} \frac{\partial h_i}{\partial x_n} dx \right) + 2c\Re \left( \sum_{i=1}^{N} \int_{\Omega_j \cap \Omega_0} \frac{\partial q}{\partial x_i} \frac{\partial h_i}{\partial x_n} dx \right) \right) \\
+ \sum_{j=1}^{J} \left( b \int_{\Gamma_j} (h_j \cdot v_j)|\nabla p|^2 d\Gamma_j + \int_{\Gamma_j} (h_j \cdot v_j)|c\nabla p - V q|^2 d\Gamma_j \right) + o(1).
\]

(4.47)

By using (4.15), (4.17), (4.24), and (4.27), it holds that

\[
\begin{aligned}
\sum_{j=1}^{J} \left( N \int_{\Omega_j \cap \Omega_0} (|\lambda p|^2 + |\lambda q|^2) dx - (N - 2) \int_{\Omega_j \cap \Omega_0} (b|\nabla p|^2 dx + |c\nabla p - V q|^2 dx) \right) = o(1), \\
\sum_{j=1}^{J} \left( 2a\Re \left( \sum_{i=1}^{N} \int_{\Omega_j \cap \Omega_0} \frac{\partial q}{\partial x_i} \frac{\partial h_i}{\partial x_n} dx \right) + 2c\Re \left( \sum_{i=1}^{N} \int_{\Omega_j \cap \Omega_0} \frac{\partial q}{\partial x_i} \frac{\partial h_i}{\partial x_n} dx \right) \right) = o(1).
\end{aligned}
\]

(4.48)

Since $(x - x_j^0) \cdot v'(x) > 0$ on $\gamma_j$, then $(x - x_j^0) \cdot v'(x) < 0$ on $\Gamma_j$. Hence, we obtain that

\[
\sum_{j=1}^{J} \left( b \int_{\Gamma_j} (h_j \cdot v_j)|\nabla p|^2 d\Gamma_j + \int_{\Gamma_j} (h_j \cdot v_j)|c\nabla p - V q|^2 d\Gamma_j \right) \leq 0.
\]

(4.49)

Inserting (4.48) and (4.49) in (4.47), we deduce (4.29).
**Observability and exact controllability**

Consider the homogenous system related to (1.4):

\[
\begin{align*}
\psi_t &- a \Delta \psi + c \nabla \phi = 0, \quad \text{in } \Omega \times (0, \infty), \\
\phi_t &- \Delta \phi + c \Delta \psi = 0, \quad \text{in } \Omega \times (0, \infty), \\
\psi |_{\Gamma} &= \phi |_{\Gamma} = 0, \quad \text{on } \Gamma \times (0, \infty), \\
\Phi_0 &= (\psi(x, 0), \phi(x, 0)) = (\psi_0(x), \phi_0(x)) = (\psi_1(x), \phi_1(x)), \quad \text{in } \Omega,
\end{align*}
\]

where \( a > c^2 \) and (1.3) holds. Let \( \Phi = (\psi, \psi_t, \phi, \phi_t) \) be a regular solution of (4.54), its associated total energy is given by

\[
E(t) = \frac{1}{2} \int_{\Omega} (|\psi|^2 + b |\nabla \psi|^2 + c |\phi|^2 + |c\psi - \nabla \phi|^2) \, dx.
\]
We have
\[ E'(t) = 0. \]  \hfill (4.56)

Thus, system (4.54) is conservative in time; that is, its energy \( E(t) \) is constant. System (4.54) is also well-posed and admits a unique solution in the energy space \( \mathcal{H} \). From (4.3), there exists (see [2, 3]) a time \( T_0 > 0 \) such that for all \( T > T_0 \), there exist two constants \( M_1, M_2 > 0 \) such that
\[
M_1\|\Phi_0\|_{\mathcal{H}}^2 \leq \int_0^T \int_\Omega d(x)|\psi_t|^2 \, dx \, dt \leq M_2\|\Phi_0\|_{\mathcal{H}}^2, \forall \Phi_0 = (\psi_0, \psi_1, \varphi_0, \varphi_1) \in \mathcal{H}. \]  \hfill (4.57)

Now, we examine the exact controllability of system (1.4) by using the HUM method. Let \( v_0 \in L^2(0, T; L^2(\omega)) \), we define the control
\[
v(t) = -\frac{d}{dt}v_0(t) \in (H^1(0, T; L^2(\omega)))', \]
where the derivative with respect to \( t \) is taken in the sense of the duality \( H^1(0, T; L^2(\omega)) \) and its dual \( (H^1(0, T; L^2(\omega)))' \), that is,
\[-\int_0^T \frac{d}{dt}v_0(t)\theta(t)dt = \int_0^T v_0(t)\frac{d}{dt}\theta(t)dt, \forall \theta \in H^1(0, T; L^2(\omega)).\]

**Theorem 4.10.** Let \( T > 0 \). Assume that \( \omega \) satisfies the geometric condition (PMGC) and let
\[ Z_0 = (p_0, p_1, q_0, q_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^2, \]
then (1.4) has a unique weak solution
\[ Z = (p, p_i, q, q_i) \in C^0([0, T], (L^2(\Omega) \times H^{-1}(\Omega))^2). \]

**Proof.** Let \((\psi, \psi_t, \phi, \phi_t)\) be the solution of (4.54) associated to \( \Phi_0 = (\psi_0, \psi_1, \varphi_0, \varphi_1) \). Multiplying (1.4)_1 by \( \psi \) and integrating by parts on \((0, T) \times \Omega \), and then summing the resultant equations to get
\[
\int_\Omega p_1(T)\psi(T)dx + \int_\Omega q_1(T)\phi(T)dx - \int_\Omega \psi_1(T)p(T)dx + \int_\Omega \phi_1(T)q(T)dx = \\
\int_\Omega p_1(0)\psi(0)dx + \int_\Omega q_1(0)\phi(0)dx - \int_\Omega \psi_1(0)p(0)dx - \int_\Omega \phi_1(0)q(0)dx + \int_0^T \int_\Omega d(x)v(t)\psi dx \, dt. \]  \hfill (4.59)

Note that \( \mathcal{H}' = (H^{-1}(\Omega) \times L^2(\Omega))^2 \). Consequently, we obtain
\[
\langle (p_1(T, \cdot), -p(T, \cdot), q_1(T, \cdot), -q(T, \cdot)), \Phi(T) \rangle_{\mathcal{H}' \times \mathcal{H}} = \langle (p_1, -p_0, q_1, -q_0), \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}} + \int_0^T \int_\Omega d(x)v(t)\psi dx \, dt = \mathbb{L} (\Phi_0). \]  \hfill (4.60)

Thanks to (4.57), it holds that
\[
\|\mathbb{L}\|_{\mathcal{H}(\mathcal{H}') \to} \leq \|v_0\|_{L^2(0, T; L^2(\omega))} + \|Z_0\|_{\mathcal{H}'} \].  \hfill (4.61)

By the help of the Riesz representation theorem, there exists an element \( V(x, t) \in \mathcal{H}' \) solution of
\[
\mathbb{L}(\Phi_0) = \langle V, \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}}, \forall \Phi_0 \in \mathcal{H}. \]  \hfill (4.62)

Then, \( Z(x, t) = V(x, t) \) is the weak solution of system (1.4). The proof is thus complete.

Now, let us examine the indirect locally internal exact controllability problem which can be formulated as follows: For a given \( T > 0 \) sufficiently large and initial data \( Z_0 \), does there exists a suitable control \( v \) that brings back the solution to equilibrium at time \( T \)? That is, the solution of (1.4) satisfies \( p(T) = p_i(T) = q(T) = q_i(T) = 0 \). Applying the HUM method, one gets the result below:
**Theorem 4.11.** Assume that (1.2) holds and \( \omega \) satisfies the geometric condition \((\text{PMGC})\). For every \( T > M_1 \), where \( M_1 \) is given in (4.57) and for every \( Z_0 \in (H^{-1}(\Omega) \times L^2(\Omega))^2 \), there exists a control \( v(t) \in [H^1(0, T; L^2(\omega))^t] \), such that the solution of (1.4) satisfies \( p(T) = p_1(T) = q(T) = q_1(T) = 0 \).

**Proof.** From (4.57), we may define the seminorm \( \| \Phi_0 \|_H = \int_0^T \int_\omega |\psi_t|^2 dx \), where \( \Phi = (\psi, \psi_t, \varphi, \varphi_t) \) is the solution of (4.54). Take the control \( v = \frac{d}{dt} \psi_t \). Now, we solve the following time reverse problem:

\[
\begin{align*}
X_t - a\Delta X + c\Delta Y &= d \frac{d}{dt} \psi_t, & \text{in } (0, T) \times \Omega, \\
Y_t - \Delta Y + c\Delta X &= 0, & \text{in } (0, T) \times \Omega, \\
X(T, \cdot) &= X_0(T, \cdot) = Y(T, \cdot) &= Y_0(T, \cdot) = 0.
\end{align*}
\]

Using Theorem 4.10, the system (4.63) admits a unique solution \( X_{sol} = (X, X_t, Y, Y_t) \in C^0([0, T], H') \). Define the operator \( \Lambda : H \to (H^{-1}(\Omega) \times L^2(\Omega))^2 \) by \( \Lambda \Phi_0 = (X_0(0), -X(0), Y_0(0), -Y(0)) \), \( \forall \Phi_0 \in H \). Besides, we define the following linear form:

\[
\langle \Lambda \Phi_0, \widetilde{\Phi}_0 \rangle = \int_0^T \int_\omega \psi_t \overline{\psi_t} dx dt = \langle \Phi_0, \widetilde{\Phi}_0 \rangle_H, \quad \forall \Phi_0, \widetilde{\Phi}_0 \in H.
\]

Using Cauchy-Schwarz's inequality in the above inequality, we deduce that

\[
|\langle \Lambda \Phi_0, \Phi_0 \rangle_{H \times H'}| = |\langle \Phi_0, \Phi_0 \rangle_H|, \quad \forall \Phi_0 \in H.
\]

In particular, we have

\[
|\langle \Lambda \Phi_0, \Phi_0 \rangle_{H \times H'}| = \| \Phi_0 \|_H^2, \quad \forall \Phi_0 \in H.
\]

Using (4.57), we observe that the operator \( \Lambda \) is coercive and continuous on \( H \). Thanks to Lax-Milgram theorem, we have \( \Lambda \) is an isomorphism from \( H \) into \( H' \). In particular, for every \( Z_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2 \), there exists a solution \( \Phi_0 \in H \), such that

\[
\Lambda \Phi_0 = -Z_0 = (X_0(0), -X(0), Y_0(0), -Y(0)).
\]

Hence, \( Z = X_{sol} \) is inferred from the fact that the solution of (4.63) is unique. Consequently, we have \( p(T) = p_1(T) = q(T) = q_1(T) = 0 \), which completes the proof. \( \square \)

## 5 | CONCLUSION AND OPEN PROBLEMS

By assuming the \((\text{PMGC})\) condition, the exponential stability of second-order coupled, by Laplacian, wave equations with a single locally internal viscous damping is proved without any restrictions on the speed of wave propagation. Noting that the speed takes an important role in the stability (see table below) in the case of coupled wave equations by displacement \((\text{ecue and eccye})\) or by velocity \((\text{ecue}e \text{ and eccye}e)\).

| Coupling | Wave propagation speed | Stability |
|----------|------------------------|----------|
| Of order 0 (i.e., by displacement) | \( a = 1 \) (resp. \( a \neq 1 \)) | Polynomial of order \( r^{-1} \) (resp. \( r^{-1/2} \)) |
| Of order 1 (i.e., by velocities) \cite{33} | \( a = 1 \) (resp. \( a \neq 1 \)) | Exponential (resp. polynomial of order \( 1/2 \)) |
| Of order 2 | \( a \) | Exponential |

We recall the \((\text{GCC})\) introduced by Rauch and Taylor in \cite{51}.

**Definition 5.1** (see Figure 2). Let \( \omega \subset \Omega \) and \( T > 0 \). The pair \((\omega, T)\) verifies the \((\text{GCC})\) if there exists \( T > 0 \) such that every geodesic traveling at speed one in \( \Omega \) meets \( \omega \) in time \( t < T \).

We state here some open problems:

1. The case when \( \omega \) satisfies the \((\text{GCC})\) condition instead of the \((\text{PMGC})\) condition.
2. The case where the coupling in (1.1) is localized.
3. The case where system (1.1) is damped by, for example, delay term, memory damping, Kelvin-Voigt damping... We recall that, in the case of Kelvin-Voigt damping, the system of two wave equations coupled through a displacement (respectively by velocity) is polynomially stable with order $t^{-\frac{1}{2}}$ (respectively $t^{-1}$) (see [13, 20], respectively). If the viscous damping in system (1.1) is replaced by Kelvin-Voigt damping, can we obtain exponential stability?

AUTHOR CONTRIBUTIONS
Mohammad Akil: Conceptualization; investigation; funding acquisition; writing—original draft; writing—review and editing; visualization; validation; methodology; software; formal analysis; project administration; resources; supervision; data curation. Zayd Hajjej: Resources; data curation; software; formal analysis; project administration; writing—review and editing; visualization; validation; methodology; investigation; conceptualization; funding acquisition; writing—original draft.

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