Relational Approach to Spin Networks *

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Abstract
Individual spinors in a SU(2) spin network are described by their relations to the background spin network. A ‘covariant’ formulation of these relations yields the de Sitter group SO(3,2) as the fundamental symmetry group. Locally this symmetry group is approximated by the Poincaré group, which leaves invariant (certain) clusters of spinors. The calculated masses of these clusters reproduce the lepton spectrum. Corrections to the approximate Poincaré group, based on the exact SO(3,2) symmetry, deliver interaction terms, identical to those of the standard model. In addition, gravitation is obtained. The calculation of the fine-structure constant reproduces Wyler’s formula.

1 Introduction

The literature on spin networks is dominated by their graph theoretical aspects. In this article the notion of spin networks will be used in an elementary sense. It will denote a set of two-component spinor states, that can be linked together (and unlinked again) according to the quantum mechanical rules for coupling angular momentum.

The word “Relational” in the title refers to the concept of Relational Quantum Mechanics, which was developed by C. Rovelli[2] and others. The approach to spin networks, presented here, will essentially study relations between individual spinors and a background network.

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The approach will also illustrate the role of reduction and emergence in particle physics. It may, therefore, be considered as a contribution to the dispute about reductionism versus emergence in physics research that recently has been brought up again [3].

There is a well-known dictum of Bohr’s:

“Physics is not about reality, but about what we can say about reality.” [4]

To those, who have some background of information theory, this dictum suggests that the object of physics is information, information about reality. Particle physics, in particular, is then the part of physics that is in search for the most elementary building blocks, not of reality, but of information about reality.

The question, whether there may or may not exist a one-to-one relation between information and “reality”, is not relevant for the following. This question cannot be answered by physical means. Therefore, it is not a subject of particle physics, but rather, if at all, of metaphysics. For the following it will be sufficient to understand “reality” simply as the collection of all available information about reality.

The relevance of information theory for understanding quantum mechanics has been accentuated especially by John Wheeler [6], who coined the phrase “It from Bit”. In fact, the decomposition of information into smaller and smaller pieces finally results in a set of yes-no decisions, or binary elements, or, simply, bits. Hence, the question for the most elementary building blocks of information about reality is very easily answered: They are neither strings nor branes, but simply binary elements. Therefore, the ultimate goal of every “reductionist” should be the formulation of physics in terms of binary elements.

Many of us expect to find the most elementary building blocks of reality at scales of the Planck length. At these scales it would hardly be possible to experimentally identify binary elements. But, assuming that there are basic binary elements at accessible scales, are we really prepared to observe such elements? Would we even realize that it is a binary element, when we encounter it in, say, a scattering experiment? Can binary elements at all turn up as isolated physical entities? In case they can, what would binary elements look like? To answer these and other questions we must, obviously, develop a theoretical understanding of how single binary elements would behave, if they were observed in a suitably designed experimental setup.
When we accept binary elements as the ultimate building blocks of reality, we are faced with the challenge to make understandable, how such elements of particle physics, as space-time, masses, charges and interaction can be derived from binary elements. In face of the tremendous difficulties of string theory, to reach its goals on the basis of strings and branes, the intention to base physics on even more elementary building blocks, may appear highly adventurous. Nevertheless, in the past, several serious authors have attempted to establish connections between sets of binary elements and fundamental structures of physics.

The quantum mechanical generalisation of binary elements are spinors. These are described by states in a SU(2)-symmetric Hilbert space. A set of spinors then forms a spin network in the elementary sense defined above. We will see below that, in principle, it makes no difference, whether we start from a set of binary elements or from a set of spinors.

Spin networks were introduced by Roger Penrose [7] almost 40 years ago, in an attempt to describe the geometry of space-time in a purely combinatorial way. Penrose studied systems of two-component spinors, which represent the simplest quantum mechanical objects. He was able to show that large systems of such spinors, generate properties of angular directions in three-dimensional space. Despite this success, the concept of SU(2) based spin networks was considered not rich enough to describe distances [8].

Another attempt to derive space-time from “binary alternatives” was undertaken by C. F. von Weizsäcker [9, 10] and co-workers. In this concept a physical particle is considered as an aggregate of $10^{40}$ binary alternatives, representing the knowledge about the particle. Although indications for a Poincaré symmetric space-time structure were obtained, the work remained at an experimental stage without having established a definite link to empirical particle physics.

Spin networks were re-discovered by Rovelli and Smolin as a basis for studies on quantum gravity [11, 12]. In quantum gravity spin networks are employed to build models of space-time at Planck scales. In contrast to these models, the scales encountered in the present article will turn out to be experimentally accessible.

The relational approach to spin networks is essentially guided by Bohr’s statement[13]

“Physical phenomena are observed relative to different experimental setups” (accentuation by the author).

Bohr’s dictum in a concise way states that we do physics in a “relational” way.
In the context of spin networks, Bohr’s statement can be understood as an invitation, to establish a mathematical description of spinors in a spin network, by their relations to other subsets of the spin network. These subsets then stand for Bohr’s experimental setups. We will find that such relations, in fact, provide individual spinors with “physical” properties that can be compared to well-known elements of the standard model. Therefore, this paper can be understood as an attempt to base the standard model on spin networks.

The outline of the paper is as follows: In section 2 we will construct from spinors a quasi-classical reference object, which can be used as a simple model of an experimental setup. We will obtain the de Sitter group SO(3,2) as symmetry group of the state space of this reference object. A local approximation to this symmetry delivers the Poincaré group as the local kinematical group of the reference. In section 3, four-component Dirac spinors are introduced as means to describe the coupling of two-component spinors to a reference. In section 4, a covariant formulation of this coupling results in a description of Dirac spinors in space-time, by solutions of Dirac’s equation. Sections 5 and 6 describe, how mass relations for different types of linking spinors to references reproduce the leptonic mass spectrum. In Section 7 the interaction term of quantum electrodynamics is derived as a SO(3,2) correction to the approximate Poincaré group. In section 8, the electromagnetic coupling constant is calculated. The result is Wyler’s formula for the fine-structure constant. Sections 9 and 10 shortly discuss how the interaction terms of quantum chromodynamics and weak interaction result from SO(3,2) corrections. Section 11 shows that gravitation is a natural result of the basic SO(3,2) symmetry. Section 12 identifies spin networks are the relational form of sets of binary elements. Section 13 concludes with the statement that binary elements neither need nor allow for a deeper explanatory base.

2 Quasi-classical description of large subsets

Following Bohr’s dictum we will develop a mathematical theory of individual spinors relative to certain reference subsets of the spin network. We would be well advised to choose the references in a most general, but nevertheless realistic way. A suitable candidate for a reference object is the counterpart to a classical rigid body, capable of moving in space-time and rotating with respect to three spatial axes. Such an object may carry a coordinate system. Then the kinematical states of the reference object can be characterised by all possible
coordinate transformations. Of course, we want the theory to be valid for all states of the reference object. This means, the theory shall be covariant with respect to a group of coordinate transformations that act transitive on the state space of the reference object. Therefore, we have two things to do: Firstly, construct a reference object from spinors that comes close to a classical body. Secondly, determine the transformation group that describes the kinematics of this object.

We construct a reference object from a set of spinors by linking them together according to the quantum mechanical rules for coupling angular momenta. Assume that the spinors are linked to total states with well-defined SU(2) quantum numbers. If the reference is sufficiently large, it can be treated by the limit of large quantum numbers, which means quasi-classically. Using the three infinitesimal generators $j_{12}, j_{23}, j_{31}$ of SU(2), as defined below, we can rotate the reference with respect to three orthogonal axes in a quasi-continuous way. These rotations form the group SO(3). For a given total angular momentum, however, the axis of rotation is fixed, as a result of conservation of angular momentum. Therefore the (quasi-classical) motion of the reference will essentially be restricted to a 2-dimensional plane. This means that the reference as a whole performs an “orbital” rotational movement with respect to a (distant) centre in a quasi-continuous way. Hence, a first contribution to the kinematical group is the group SO(2).

Assume that we divide the reference object into several parts. This allows us to take differences of individual angular momenta. To reduce the number of degrees-of-freedom, let us assume that the parts are fixed relative to each other, so that they form the equivalent of a rigid body. Then the operators of differences of angular momentum can be used as infinitesimal generators to rotate the object as a whole relative to its (local) centre. These rotations determine the “orientation” of the object relative to three orthogonal axes. The rotations again form a group SO(3), which extends the kinematical group to the product group SO(3) × SO(2).

Assume now that a naive observer is moving together with the reference object. Then he will not immediately take notice of the orbital motion. However, he may observe that other objects are moving relative to his own coordinate system. He will inevitably develop a perception of “time”, which “passes” although his own coordinate system does not seem to move. Probably he will then decide to measure the elapse of time by comparing it with some suitable periodic
motions of other objects. Therefore, the SO(2)-part of the kinematical group shows properties of “time”, whereas the SO(3)-part describes the orientation of the rigid body in 3-dimensional “space”.

In a next step the observer may develop techniques to move his position relative to the 3-dimensional coordinate system. Alternatively, he may move the reference object together with its coordinate system relative to his own position. Such actions are described by boost operations, which are coordinate transformations that let the SO(3)-coordinate system move in the direction of a coordinate of the SO(2)-system. The boost operations extend, as will be shown below, the kinematical group to SO(3,2). This group contains all coordinate transformations that our hypothetical observer is able to carry out, or, alternatively, the reference object may be subjected to. A description of a single spinor that remains valid after one of these transformations, must therefore be covariant with respect to the operations of SO(3,2). In this way, SO(3,2) becomes the basic symmetry group of a theory of spin networks that will be developed in the following.

To examine the kinematical group in more detail, let us represent the infinitesimal generators of the group operations of the SO(3)-part by differential operators. These will act on functions in an auxiliary 3-dimensional parameter space spanned by the coordinates $x_1, x_2, x_3$. Later it will become clear that we can form states that are “localized” at any given “point” of this parameter space, thus making it a “physical space” in the usual sense.

Rotations of the coordinate system of the reference are then generated by three well-known differential operators

$$ j_{ij} = i(x_i \partial_j - x_j \partial_i), \quad i, j = 1, 2, 3. \quad (1) $$

The SO(2)-part of SO(3,2) can be represented by rotations in a 2-dimensional parameter space spanned by two additional parameters $x_0$ and $x_4$. Then the infinitesimal generator of SO(2) rotations is given by

$$ j_{04} = i(x_0 \partial_4 - x_4 \partial_0). \quad (2) $$

Both parameter spaces, spanned by coordinates $(x_1, x_2, x_3)$ and $(x_0, x_4)$, respectively, are Euclidean. We can combine them to a 5-dimensional pseudo-Euclidean space with the metric

$$ g_{ab} = \text{diag}(+1, -1, -1, -1, +1), \quad a, b = 0, ..., 4. \quad (3) $$
This allows us to express the boost operations by pseudo-rotations in the \((0, i)\)-plane, \(i = 1, 2, 3\), as known from Lorentz transformations. The boost operations are then generated by

\[ j_{0i} = i(x_0 \partial_i - x_i \partial_0), \quad i = 1, 2, 3. \tag{4} \]

The proof is by verifying the commutation relations of the homogeneous Lorentz group. There is a second set of pseudo-rotations, which are generated by

\[ j_{\mu 4} = i(x_\mu \partial_4 - x_4 \partial_\mu), \quad \mu = 0, 1, 2, 3. \tag{5} \]

It can easily be verified that the differential operators \((1, 2, 4, 5)\) satisfy the commutation relations of \(\text{SO}(3,2)\):

\[ [j_{\mu\nu}, j_{\rho\sigma}] = -i[g_{\mu\rho} j_{\nu\sigma} - g_{\mu\sigma} j_{\nu\rho} + g_{\nu\sigma} j_{\mu\rho} - g_{\nu\rho} j_{\mu\sigma}], \tag{6} \]

\[ [j_{\mu 4}, j_{\nu 4}] = -ij_{\mu\nu}, \tag{7} \]

\[ [j_{\mu\nu}, j_{\rho 4}] = i[g_{\nu\rho} j_{\mu 4} - g_{\nu\mu} j_{\rho 4}]. \tag{8} \]

Now consider operators \((5)\) in the neighbourhood of the point \(O = (x_\mu = 0, x_4 = R)\). For \(R\) large compared to the components \(x_\mu\) the operator \(j_{\mu 4}/R\) can be approximated by

\[ p_\mu = -i \partial_\mu, \quad \mu = 0, 1, 2, 3. \tag{9} \]

The operators \(p_\mu\) commute and act as generators of translations in the 4-dimensional subspace defined by \((x_0, ..., x_3)\). Together with \((1)\) and \((4)\) they generate the inhomogeneous Lorentz group (Poincaré group) \(\text{P}(3,1)\). This approximation, valid in the neighbourhood of a given point of the parameter space, is known as group contraction\([14, 15]\).

From eigenstates of the three spatial translation operators we can, by superposition, construct states that are localized at any position \(x_i, i = 1, 2, 3\), in the neighbourhood of \(O\). The time-like translation operator then describes the evolution of this state in time. The translation operators therefore give the abstract parameters \((x_0, ..., x_3)\) the familiar meaning of space-time.

Obviously \(x_i\) and \(p_k\) satisfy the well-known commutation relations

\[ [x_i, p_k] = i\delta_{ik}. \tag{10} \]

Without any indication of a “fourth dimension”, the time coordinate is derived from the parameterisation of the orbital angular momentum, which is subject to a \(\text{SO}(3)\) symmetry.
That ends the construction of a reference object from a subset of the spin network. Its kinematics can be expressed by coordinate transformations within the parameter space. The transformations form the group SO(3,2). The reference object can be treated quasi-classically, but nevertheless it is still a quantum mechanical object.

3 Spinors within spin networks

In this section a quantum mechanical description of a single spinor relative to a quasi-classical reference object will be given.

Spinor states form a complex vector space built from two-component base vectors

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (11)$$

It is well-known that the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (12)$$

are the generators of SU(2) transformations within this vector space.

Now consider a reference object with well-defined quantum numbers \((S, S_3)\) with respect to SU(2). By adding another single spinor to this state, we can obtain either a state

$$(S + \frac{1}{2}, S_3 + \frac{1}{2}) \quad (13)$$

or

$$(S - \frac{1}{2}, S_3 - \frac{1}{2}) \quad (14)$$

or any linear combination of these states. Which of these states is obtained, depends on the orientation of the single spinor relative to the reference state. The relative orientation is neither a property of the single spinor nor of the reference state, but rather a property of the combined system. However, if we keep the orientation of the reference fixed, we are able to formally express the two ways of coupling as a property of the spinor itself. We can do this by simply doubling the spinor components in the following way:

$$|u_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |u_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \quad (15)$$
\[
|v_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
|v_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\] (16)

The first group (15) of base states describes a spinor coupled to the reference object in the sense of (13). The second group (16) describes a spinor coupled to the reference object in the sense of (14). The matrix
\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\] (17)
where \(I\) is the 2 \(\times\) 2 unit matrix, then delivers an eigenvalue of +1, if applied to the first group of spinors (15), and −1, if applied to the second (16).

This formulation is not yet covariant with respect to \(SO(3,2)\). To make it covariant, we have to find 4 \(\times\) 4 matrices that transform together with \(\gamma^0\) in the same way as the state of the reference is transformed by the operations of \(SO(3,2)\). In other words, we have to find a representation of \(SO(3,2)\) by 4 \(\times\) 4 matrices.

The representation of rotations is straightforward. Their generators are obtained from Pauli matrices in the following form
\[
\sigma_{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad i,j,k = 1,2,3.
\] (18)

Boost operations are generated by the 4 \(\times\) 4-matrix
\[
\sigma^{0k} = -\sigma^{k0} = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}.
\] (19)

When we close the algebra of the matrices defined so far, with respect to their commutation product, we find additional matrices
\[
\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}.
\] (20)

We can combine the indices 0 and \(k\) to an index \(\mu = 0,\ldots,3\), and use \(g_{\mu\nu} = \text{diag} \ (+1,-1,-1,-1)\) in the usual way to raise and lower indices.

The matrices (17) and (20) are Dirac’s \(\gamma\)-matrices in the so-called standard or Dirac representation. \(\gamma\)-matrices satisfy the well-known anti-commutation
relations
\[ \{ \gamma_\mu, \gamma_\nu \} = 2 g_{\mu\nu} . \]  

and the commutation relations
\[ \frac{i}{2} [\gamma_\mu, \gamma_\nu] = \sigma_{\mu\nu} . \]  

The 4 × 4 matrices \( s_{\mu\nu} \) and \( s_\mu \), built from Dirac matrices,
\[ s_{\mu\nu} := \frac{1}{2} \sigma_{\mu\nu} \quad \text{and} \quad s_\mu := \frac{1}{2} \gamma_\mu \]
form a representation of SO(3,2). The proof is by verifying the commutation relations \((6,7,8)\) of SO(3,2).

In the following, 4-component spinor states, obtained from the base states \((15,16)\) by applying transformations generated by the matrices \((23)\), will be called Dirac spinors.

To make the vector space of Dirac spinors a Hilbert space, we can add a scalar product, as known from the treatment of the Dirac equation,
\[ \langle \bar{a} | b \rangle \quad \text{with} \quad \langle \bar{a} | = \langle a | \gamma^0 . \]

The introduction of Dirac spinors allows to keep track of the coupling of a single two-component spinor to a reference object. Their introduction does not mean any modification of the basic two-component spin network. It only defines an alternative view on the basic spin network.

4 Spinors in space-time

In this section we will talk about space-time properties of spinors in the neighbourhood of the point of contraction \( O \). Therefore, whenever advantageous, we will replace SO(3,2) by the approximate \( P(3,1) \).

Let \( | P \rangle \) be the state of a quasi-classical reference object with momentum \( P \). By adding a spinor the momentum is changed by an increment \( p \) to
\[ | P' \rangle = | P + p \rangle . \]

In the rest frame, we have \( P' = (P_0 + p_0, 0, 0, 0) \). Since \( P_0 \) is the approximation to \( J_{10} \), the addition of a spinor increments or decrements the value of \( P_0 \) by 1/2. Therefore \( p_0 \) is positive or negative, depending on the coupling of the spinor. This dependency can be expressed by the simple relation
\[ \langle \gamma^0 p_0 - m | \psi \rangle = 0, \quad \text{with} \quad m = \text{const.} \]
Taking into account the transformation properties of $\gamma^\mu$ with respect to $\text{SO}(3,2)$ and $p_\mu$ with respect to $\text{P}(3,1)$, we can write (26) in a covariant form

$$ (\gamma^\mu p_\mu - m) |\psi(p)\rangle = 0 .$$

(27)

This is the Dirac equation with mass $m = 1/2$. The well-defined value $1/2$ sets a numerical mass scale, which is the precondition for the existence of a discrete mass spectrum.

Since the momentum $p_\mu$ in (27) has been “borrowed” from the reference, the transformation properties of $p$ with respect to the Poincaré group are identical to that of $P$. But the apparent space-time properties of the spinor do not reflect degrees of freedom of the spinor itself, but stand for the relation of the spinor to a reference object. There are still only two independent states in the Hilbert space of a spinor. The parallel or anti-parallel way of coupling to a reference is covered by doubling the spinor components to form a Dirac spinor. The momentum information is coded into the coefficients of the Dirac spinor state and reflects the combined state of spinor and reference object.

Consider a momentum eigenstate of the spinor attached to the reference, expressed by a plane wave

$$ e^{i(P_\mu + p_\mu) x_\mu} = e^{iP_\mu x_\mu} e^{i p_\mu x_\mu} .$$

(28)

When we apply a finite translation by a displacement vector $a$ to the spinor but not to the reference, this results in a phase factor

$$ e^{i p_\mu a_\mu} .$$

(29)

If at position $a$ relative to the reference there is another reference object, then this displacement acts as a recoupling of the spinor from the first reference to the second. Invariance with respect to $\text{P}(3,1)$ ensures conservation of energy-momentum for the process of recoupling. Hence, a spinor is able to “carry” momentum from one reference to the other.

In this sense the spinor part of (28) can be regarded as the description of a particle, capable of moving from one reference to the other. Or as we may say: from one experimental setup to another.

Thus we have arrived at a description of a spinor in space-time, where the space-time continuum is defined by the kinematical degrees-of-freedom of macroscopic reference objects.
The “position” of the spinor has been defined relative to the position of the reference by an offset vector \( a \). Of course, there are other ways to determine the position of an object in space. For example, a position can be defined by the intersection of three perpendicular planes. Each plane can be determined by its distance from one of three reference objects. In such a configuration three reference objects contribute to the position of the spinor. The determination of the effective mass of a spinor within such a configuration is not as trivial as in the case of only one reference. In the following section we will derive a “mass formula”, which relates the mass of the one-reference case to effective masses in configurations with two and three reference objects.

Adding a fourth reference to define a spinor position would lead to an over-determined system. Therefore, in the following we consider only up to three references.

5 Mass relations

A few years ago G. González-Martín [16, 17] (G-M in the following) obtained mass relations, based on an universal structure group \( \text{SL}(4, \mathbb{R}) \). G-M’s idea was that the structure group describes a “substrate”, from which particles are generated as “excitations” with certain symmetric and topological properties, which are associated with subgroups of the structure group. G-M found a mass formula for the three massive leptons

\[
m_n = 4\pi \left( \frac{16\pi}{3} \right)^n m_e \quad n = 1, 2 ,
\]

(30)

where \( m_e \) is the electron mass and \( m_1 \) stands for the myon mass, \( m_2 \) to the tauon mass. With the experimental electron mass of 0.5109989 MeV, G-M obtained \( m_\mu = 107.5916 \) MeV and \( m_\tau = 1770.3 \) MeV. (The experimental values are 105, 658 and 1776.99.)

We will see that our approach to space-time properties of spinors leads to an explanation of G-M’s mass formula on the basis of spin networks.

Consider three reference objects with entangled wave functions and a single spinor linked to their wave function. Then, instead of (28), we have a state built from entangled contributions of

\[
e^{i(P'_\mu + P''_\mu + P'''_\mu + p_\mu) x^\mu}.
\]

(31)

The spinor is now added to three reference objects at the same time. What is
then the effective spinor mass in such a configuration compared to the single
reference situation?

The states of the reference objects belong to a product representation of
SO(3,2). To obtain mass relations, the product representation must be made
comparable to the single reference situation treated in the last section. A suit-
able way of achieving this is to convert the product representation into a direct
sum (or rather integral) of irreducible representations of the Poincaré group.
Each representation will then contribute the same value of 1/2 to the mass.
The effective spinor mass is subsequently determined by a sum over all con-
tributing representations.

In a first step we decompose a quasi-classical representation of SO(3,2) into
a set of SO(3,1) representations. Let \( S \) denote the group of SO(3,2) transfor-
mations. Let \( L \) denote the subgroup of Lorentz transformations SO(3,1) con-
tained in SO(3,2), and let \( P \) denote the transformations of the (approximate) Poincaré
group P(3,1).

Consider a reference object with state \( |\Phi\rangle \) in a SO(3,2)-symmetric Hilbert
space \( H_S \). Assume that in the neighbourhood \( \mathcal{N} \) of the origin \( O \) this state is
approximated by a momentum eigenstate. When all Lorentz transformations \( L \)
are applied to this state, a Hilbert space \( H_L \), as a subspace of \( H_S \) is obtained.
This Hilbert space is associated with the point \( O \).

If a (finite) transformation \( s \in S, s \notin L \) is applied to a state of \( H_L \), a new
state is generated, which is not in \( H_L \). Therefore, by applying transformations
of the coset \( Ls \), a non-equivalent Hilbert space \( H_s^L \) is obtained. This Hilbert
space is associated with the point \( sO \). There is a one-to-one relation between
cosets \( Ls \) and Hilbert spaces \( H_s^L \). The set of all cosets \( Ls \) generates the total
Hilbert space \( H_S \).

The set of cosets forms a homogeneous space \( S/L \), where \( S \) acts transitive on
this space and \( L \) is the isotropy group of the origin \( O \); the projection \( \pi : S \rightarrow S/L \) makes \( S \) a principle bundle on \( S/L \) with fiber \( L \).

Adding up all non-equivalent \( H_s^L \) means an integration over the homogeneous
space \( S/L \). The integral delivers a decomposition of \( H_S \) into a sum of non-
equivalent \( H_s^L \)

\[
H_S = \int d\Omega \ H_s^L = \int ds \frac{d\Omega}{ds} \ H_s^L , \tag{32}
\]

where \( d\Omega \) is the infinitesimal volume element in \( S/L \). The Jacobian \( d\Omega/ds \) is
a measure of the number of non-equivalent Hilbert spaces \( H_s^L \) obtained by an
infinitesimal transformation \( ds \). With a properly chosen parameterisation, such
that \( \int ds = 1 \), the Jacobian becomes identical to the volume \( V(S/L) \) of \( S/L \).

The volume of \( S/L \) is determined in [16] as

\[
V(S/L) = \frac{16\pi}{3}.
\]

Volumes of homogeneous spaces were calculated by L. K. Hua [18, 19]. They have been used with some success in semi-empirical mass formulae for more than three decades [20, 21].

In the case of two or three reference objects, the Hilbert space \( H_S \) is obtained from product representations of individual Hilbert spaces \( H_S^{(1)}, H_S^{(2)}, \) and eventually \( H_S^{(3)} \). With the decomposition (32) of each Hilbert space, the integrals contain products of volume factors (Jacobians) \( V(S/L) \). The following factors correspond to one, two and three reference objects, respectively,

\[
\left(\frac{16\pi}{3}\right), \left(\frac{16\pi}{3}\right)^2 \text{ and } \left(\frac{16\pi}{3}\right)^3.
\]

Restricting the resulting Hilbert spaces to such Hilbert spaces that are associated with the point \( O \), means dividing the factors in (34) by \( V(S/L) \). This results in

\[
1, \left(\frac{16\pi}{3}\right) \text{ and } \left(\frac{16\pi}{3}\right)^2.
\]

Next remember that spinors are attached to states of representations of \( P(3,1) \) in the neighbourhood \( \mathcal{N} \) of \( O \), rather than of \( SO(3,1) \) at \( O \). Representations of \( P(3,1) \) in \( \mathcal{N} \) are obtained from representations of \( SO(3,1) \) at \( O \) by adding infinitesimal transformations \( t \in S \). By applying all \( t \) to \( L \), cosets \( Lt \) are obtained. They form a homogeneous space \( P/L \) with a volume [17] of

\[
V(P/L) = V(U(1)) = 4\pi.
\]

This corresponds to an additional integration over the homogeneous space \( P/L \) with the Jacobian \( V(P/L) \).

The Hilbert space \( H_L \) (associated with the point \( O \)) requires a special treatment. It is spanned by momentum eigenstates. In \( \mathcal{N} \), an infinitesimal \( t \) is equivalent to a translation. Since momentum states are eigenstates of the generators of translations, the effect of \( t \) is a mapping of a momentum state onto the same state with an additional phase factor. Since phase factors applied to basic states do not change the Hilbert space, \( H_L \) is not changed by translations and the factor \( V(P/L) \) does not apply.
This argument does not apply to other $H^L_L$ for the following reason. $t$ does not commute with a finite $s$, except when $s$ is in the direction of $t$. Therefore, the action of $t$ can in general not be described by phase factors applied to the base states of $H^L_L$. The special case, when $s$ is in the direction of $t$, delivers a contribution to the integral of measure zero, which can be neglected.

Multiplying the terms in (35) by the appropriate factors results in the volume factors

$$1, \quad 4\pi \left( \frac{16\pi}{3} \right) \quad \text{and} \quad 4\pi \left( \frac{16\pi}{3} \right)^2.$$  

This means, the involvement of two and three reference objects gives rise to a direct sum of non-equivalent Hilbert spaces $H_P$, determined by an integral over a parameter space and a Jacobian given by (37). We use here and in the following the term “direct sum”, although it is in fact a “direct integral”.

The direct sum has formally the same structure as the Hilbert space of a multi-particle system, where the number of particles is determined by the multiplicity (37). Since the spinor mass has the same value $m_e$ in each $H_P$, the resulting effective spinor mass is $m_e$ multiplied by one of the volume factors (37). This reproduces G-M’s mass relations (30).

There is a certain similarity to the Higgs mass generating mechanism: Masses are determined by the coupling of spinors to the background spin network, which in some sense can be regarded as a background field. However, this background field describes quasi-classical objects, which means that it is a classical field. Hence there are no “Higgs particles”.

The fact that the mass relations agree with experimental data, suggests an identification of the three configurations with massive leptons.

6 Massless spinor states

Assume that a spinor is coupled to a reference in such a way that for the resulting spinor momentum the relation $|p_0| = |p|$ holds. This relation defines a massless spinor state. There are principally two configurations, a massless “particle” with $p_0 > 0$ and a massless “anti-particle” with $p_0 < 0$. As long as the flat-space approximation is valid, all states can be generated from a single particle or anti-particle state by a transformation of SO(3,1). Remember that SO(3,1) does not include reflections.

A covariant description of these states is given by the Weyl equation, which
is known not to be invariant under space reflections,

\[(\partial_0 + \sigma \cdot \partial) \psi(x) = 0\]  \hspace{1cm} (38)

For positive \(p_0\) the helicity \(\sigma \cdot p / |p|\) is \(-1\). For negative \(p_0\) (anti-particle) the helicity is \(+1\). This is consistent with the experimental fact that massless leptons exist only with these helicities. Within our approach, the chirality of massless spinor states is a simple consequence of the fact that SO(3,1) does not include reflections.

The spinor part of the solutions of the Weyl equation can also be written as Dirac spinors (refer to standard text books)

\[
\begin{align*}
|u\rangle &= \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \\
|v\rangle &= \sqrt{E} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]  \hspace{1cm} (39)

In this form, massless leptons appear as a superposition of equally weighted particle and anti-particle states of massive leptons. Therefore, massless leptons do not really represent a new phenomenon. Hence, the considerations of the preceding section, regarding the linking to one, two, or three reference objects, apply also to the solutions of the Weyl equation. Each massive lepton is then accompanied by a massless, neutrino-like configuration. Consequently, we can identify three families of leptons, caused by three types of referencing to macro-objects.

The question of how possibly non-zero neutrino masses can be obtained from corrections based on the exact SO(3,2) symmetry is beyond the scope of this paper.

7 Electromagnetic interaction

Consider a spinor state in an SO(3,2) symmetric Hilbert space. SO(3,2) transformations are then generated from infinitesimal generators that consist of an “orbital” and a spin part

\[l_{\mu \nu} + s_{\mu \nu}\]  \hspace{1cm} (40)

and

\[l_{\mu 4} + s_\mu .\]  \hspace{1cm} (41)
The operators (40) generate the transformations of the homogeneous Lorentz group. In the neighbourhood of the point of contraction $O$, the operators (41) are approximated by generators of “translations”

$$p_\mu + s_\mu.$$  

(42)

Since there is no other spin matrix, which near $O$ can be used as an approximation to $s_\mu$, the original matrix is used. The momentum operator $p_\mu$ in (42) stands for the effect of the spinor on the reference object. The term $s_\mu$ in (42) has to be considered as an inheritance from the global SO(3,2) symmetry.

In studying the contraction limit, the older literature, e.g. [15, 22], concentrated on the case $|p| \gg |s|$. Therefore, the contributions of $s_\mu$ were considered negligible compared to those of $p_\mu$. Here the occurrence of $s_\mu$ in (42) demands for a more differentiated examination. Although $p_\mu$ is obtained from the large momentum $P_\mu$ of the macroscopic system, the contributions of $p_\mu$ can not be considered large compared to those of $s_\mu$. In fact, since $p_\mu$ stands for the effect of $s_\mu$ on the macrosystem, both operators have the same magnitude. Therefore, whenever a spinor operator occurs together with the corresponding momentum operator, it is advisable to have a careful look at the involved magnitudes.

In the neighbourhood of the point of contraction $O$, operators (42) are used for infinitesimal translations of magnitude $\epsilon$ only. From

$$[\epsilon s_\mu, \epsilon s_\nu] = -i \epsilon^2 \sigma_{\mu\nu}$$  

(43)

with $\epsilon \to 0$, follows, that in this context not only $p_\mu$ but also $s_\mu$ can be treated like commuting operators. Also in the first terms of an expansion of an infinitesimal transformation

$$I + \epsilon \mu s_\mu + \ldots$$  

(44)

all higher terms may be neglected compared to the unit matrix $I$. Therefore, it is correct to say that translations applied to the solutions of the Dirac equation are generated by $p_\mu$ only. But ignoring $s_\mu$ in the general expression (42), without any justification, would be definitely a serious fault.

Now consider an isolated SO(3,2) based multi-particle system in flat-space approximation. In a Poincaré invariant theory the word “isolated”, applied to a non-interacting multi-particle system, usually means that the modulus of the total 4-momentum is a constant

$$\left(\sum_i p_i^\mu\right)\left(\sum_j p_{j\mu}\right) = \text{const}.$$  

(45)
If the system interacts, we would in addition require that the interactions are confined to members of the system.

In generalizing these remarks to the flat-space approximation of our SO(3,2) invariant theory, it seems reasonable to study configurations, where the modulus of the sum of individual generators (42) is a constant

$$\left[ \sum_i (p_i^\mu + s_i^\mu) \right] \left[ \sum_j (p_{j\mu} + s_{j\mu}) \right] = \text{const}. \quad (46)$$

Based on the translation invariance of the flat space approximation, in addition to (46), relation (45) must still be valid. This allows to write down another constant expression by eliminating contributions of (45) to (46)

$$\sum_{ij} p_j^\mu s_{ji}^\mu + \sum_{ij} s_j^\mu s_{ji}^\mu = \text{const}. \quad (47)$$

The terms $i = j$ in the first sum of (47) are elements of Dirac’s equations for individual spinors. The terms $i \neq j$ define correlations between momentum and spin of different spinors. The terms $i \neq j$ in the second sum define correlations between different spinors. For $i = j$ these terms simply deliver a constant unit matrix. Equation (47) can be understood as a generalization of Dirac’s equation to a multi-particle system.

It will become clear later that the constancy of the left hand side of (47) is equivalent to the clause that interactions within the system are confined to members of the system.

To study the implications of (47), a perturbation approach seems appropriate. This is suggested by those terms of (47) that belong to Dirac’s equation of individual spinors, similar to a non-interacting Poincaré invariant theory. When we start from a system of “free” Dirac particles as a zeroth approximation, we have to treat the other elements of (47) as perturbations. It is fairly evident from the structure of (47) that such an approach will introduce correlations between the originally “free” particles. In fact, we will find that these correlations take on the structure of well-known interaction terms.

Let us start with the first term of (47), and postpone the treatment of the second term to section 10.

To treat the multi-particle system we make use of standard Fock space methods. The ‘free’ part of the system is easily converted into a Fock space formulation for solutions of the Dirac equation. We skip this step and refer to standard textbooks (see e.g. [23]). The field operator of the Dirac field (taken from this
reference) has the form
\[ \psi(x) = (2\pi)^{-3/2} \int d^3p \left( b_s(p)u_s(p)e^{-ipx} + d_s(p)^\dagger v_s(p)e^{ipx} \right). \] (48)

A similar expression defines the adjoint operator \( \psi^\dagger(x) \). \( b_s^\dagger(p), b_s(p) \) are electron emission and absorption operators, \( d_s^\dagger(p), d_s(p) \) are the corresponding operators for positrons. They satisfy the usual anticommutation relations of the Dirac field.

Now let us try to find a Fock space representation for the correlation terms
\[ s_\mu p^\mu \] (49)
in (47). The \( s_\mu \) part is easily converted into a Fock space operator
\[ \int d^3x \bar{\psi}(x) \gamma_\mu \psi(x), \] (50)
where \( \bar{\psi} = \psi^\dagger\gamma_0 \) is the Dirac adjoint field operator. When \( \gamma_\mu \) belongs to a first particle, then the momentum \( p^\mu \) belongs to a second one. Adding a similar Fock space representation for \( p^\mu \), would lead to a perturbation term, that is highly non-linear in \( \psi(x) \):
\[ \int d^3x \bar{\psi}(x) \gamma_\mu \psi(x) \bar{\psi}(x) p^\mu \psi(x). \] (51)

A well-established method to handle non-linear many-body problems is the introduction of a potential that is generated by the second particle and acts on the first, and vice versa. In our case, the second part of (51), which is related to the momentum of the second particle, offers itself to be replaced by a place-holder operator \( A^\mu(x) \). This operator will be defined as an operator in an auxiliary Fock space. The states of this Fock space will formally be treated like independent degrees-of-freedom. However, the perturbation algorithm will be set up in such a way that conservation of 4-momentum is guaranteed at each “vertex”. This condition will ensure that, within the algorithm, \( A^\mu(x) \) will behave in the same way as the original operator (51) would do at each vertex, as far as the balance of 4-momentum is concerned.

So let us try to write the correlation operator in the following form
\[ \int d^3x \bar{\psi}(x) \gamma_\mu \psi(x) A^\mu(x), \] (52)
where the Fock space operator $A^\mu(x)$ has the effect of a quantized vector potential. This potential shall describe the action of the second particle within the correlation term.

We make an ansatz for $A^\mu(x)$ in a general form, which is well-known from quantum electrodynamics (see e.g. [23]),

$$A^\mu(x) = \left( \frac{1}{2\pi} \right)^{-3/2} \int \frac{d^3 k}{k^0 \sqrt{2}} \left( a^j(k) e^{-ikx} + a^j(k)^\dagger e^{ikx} \right),$$

$$j = 1, 2, 3,$$  \hspace{1cm} (53)

and

$$A^0(x) = \left( \frac{1}{2\pi} \right)^{-3/2} \int \frac{d^3 k}{k^0 \sqrt{2}} i \left( a^0(k) e^{-ikx} + a^0(k)^\dagger e^{ikx} \right).$$  \hspace{1cm} (54)

The operators $a^\mu(k)$ and their counterparts $a^\mu(\dagger)$ act as absorption and emission operators for quanta with momentum $k$. As such they satisfy the commutation relations

$$[a^\mu(k), a^\nu(\dagger)] = \delta^{\mu\nu} \delta(k - k').$$  \hspace{1cm} (55)

In (53) and (54) $k^0$ shall be determined by the requirement of energy-momentum conservation at the vertex, when these operators are evaluated within a matrix element. In QED, where $A^\mu(x)$ is interpreted as the massless photon field, this requirement leads to an ‘off-shell’ behaviour of the photon field. In our context, ‘off-shell’ is not defined, since we have not assigned any mass to the place-holder field.

The auxiliary Fock space can be understood as a bookkeeping mechanism for state transitions of the second particle. The momentum operator $p^\mu$ in (57) indicates that $a^\mu(k)^\dagger$ and $a^\mu(k)$ not only create and annihilate states that carry momentum, but themselves contribute to the balance of 4-momentum, when evaluated between states of the auxiliary Fock space.

Using the decomposition of the field operators (48), (53) and (54) into their counterparts in momentum space, the contributions to (52) take on the form

$$\ldots \bar{b}(p + k) \gamma_\mu b(p) a^\mu(k) \ldots$$  \hspace{1cm} (56)

(For a while, we omit the factors $u_s$, $v_s$ and the time dependencies.) An analogous consideration is valid for positron and mixed terms. Here $a^\mu(k)$ has to simulate terms belonging to the second particle of the form

$$\ldots \bar{b}(p' + k) p^\mu b(p') \ldots$$  \hspace{1cm} (57)
Obviously we can integrate over $p'$ and obtain an expression for $a^\mu(k)$ in terms of pairs of annihilation and creation operators of the second particle

$$a^\mu(k) = \int_V d^3(p') j_V(p') \ldots \bar{b}(p' + k) \ldots b(p') \ldots . \tag{58}$$

Here $V$ denotes the integration volume and $j_V$ is the Jacobian belonging to this volume. The integration volume will be evaluated in the next section.

Coming back to the expression (56), we now add space-time dependencies to the emission and absorption operators

$$\ldots \bar{b}(p + k) e^{i(p + k)x} \gamma_\mu b(p) e^{-ipx} a^\mu(k) e^{-ikx} \ldots . \tag{59}$$

After inserting the spin functions $u_s(p)$ and $v_s(p)$, these terms and the corresponding positron and mixed terms add up to a Fock operator in the form

$$\int d^3x : \bar{\psi}(x) \gamma_\mu \psi(x) : A^\mu(x) , \tag{60}$$

where the dots :: stand for normal ordering of emission and absorption operators. This operator has the form of the interaction term of quantum electrodynamics (QED). Therefore, we can conclude that (massive) spin-1/2 particles, based on SO(3,2) symmetry, carry a charge and interact according to the rules of QED.

The coupling constant is hidden in the Jacobian $j_V$, which, obviously, relates transitions $p' \rightarrow p' + k$ to the generation or absorption of a “photon” with momentum $k$. Within a perturbation calculation another Jacobian belongs to the integration over $k$. Our task is therefore, to adjust the Euclidean volume element $dk dp$ to the geometry of the integration volume by a combined Jacobian. This volume has, as we shall see below, an essentially spherical structure. The Jacobian can be understood as the density of states that contribute to an Euclidean volume element. Therefore the evaluation of the Jacobian in the following section will essentially consist on an evaluation of this density.

8 Calculation of the coupling constant

The determination of the Jacobian is, by no means, a trivial task. Nevertheless, it was (unwittingly) attacked 40 years ago by A. Wyler [20], who discovered that the fine-structure constant $\alpha$ can be expressed by volumes of certain symmetric spaces. Unfortunately, Wyler was not able to put his observation into a convincing physical context. Even worse, his mathematical reasoning did not
withstand a closer inspection. Therefore, his work was criticized as fruitless numerology [24]. We will see in the following that Wyler’s approach has a very down-to-earth mathematical basis. But neither Wyler nor his critics were aware of this.

As a matter-of-fact Wyler broke down the problem of determining the Jacobian into several steps, by skilfully making use of some symmetric structures. These structures have well-known volume and surface elements, which Wyler then put together to obtain his formula.

To retrieve these symmetric structures, we take a close look at the parameters that describe particle states in momentum representation. The state of a single particle depends on three independent parameters $p_1, p_2, p_3$ with
\[ p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2. \] (61)

They span a 3-dimensional parameter space. Two particle momenta then span a 6-dimensional parameter space. When the momenta of both particles are given, we can use (61) to calculate the effective mass of the two-particle state. The value of the mass characterizes an irreducible representation of $P(3,1)$. When we hold this mass fixed, the number of independent parameters is reduced to five. Let us denote these parameters by $z = (z_1, z_2, z_3, z_4, z_5), z \in R^5$.

Next consider the Dirichlet problem for Laplace’s equation in $R^5$. The Dirichlet problem consists in finding a solution $\phi(z)$ on some domain $D$, such that $\phi$ on the boundary of $D$ is equal to some given function. Following Wyler, we extend the parameter space into the complex space $C^5$ and make use of the Poisson kernel $P(z, \xi)$, which is defined on a polydisk $D^5$ in $C^5$. We choose the radius of $D^5$ large enough, so that all momenta, that for a given two-particle state may contribute to (58), are enclosed. Then we scale the momentum parameter, so that the largest momentum has the value of 1. Therefore, without loss of generality, we can set the radius of the polydisk to 1, as Wyler did, and write
\[ D^5 = \{ z \in C^5; 1 + |zz'|^2 - 2\bar{z}z' > 0, |zz'| < 1 \} . \] (62)

The Poisson kernel has the well-known reproducing property that analytic functions $\phi(z)$ in $D^5$ are determined by their values on the boundary $Q^5$ of $D^5$
\[ \phi(z) = \int_{Q^5} P(z, \xi) \phi(\xi) d\xi , \] (63)
where
\[ Q^5 = \{ \xi = xe^{i\theta}; x \in R^5, xx' = 1 \} . \] (64)
This integral will be used as a vehicle to study the density of states that contribute to the Euclidean volume element $d\xi$.

The Poisson kernel is given in [19] by

$$P(z,\xi) = \frac{1}{V(Q^5)} \frac{(1 + |zz'|^2 - 2\bar{z}z')^{\frac{5}{2}}}{|(z - \xi)(z - \xi')|^5}.$$  

(65)

For $z = 0$ the Poisson kernel takes on a constant value

$$P(0,\xi) = \frac{1}{V(Q^5)},$$  

(66)

and, therefore, from (63)

$$\phi(0) = \int_{Q^5} P(0,\xi) \phi(\xi) d\xi = \int_{Q^5} \phi(\xi) \frac{d\xi}{V(Q^5)}.$$  

(67)

Each volume element on $Q^5$ then contributes with a factor of $V(Q^5)^{-1}$ to the integral.

The five dimensions of $Q^5$ may appear to indicate that there is a symmetry with respect to $SO(5)$. This is not the case. A rotation from the momentum subspace of the first particle to the second is not a valid symmetry operation and hence cannot contribute to the states in a volume element. Excluding one axis of rotation from the symmetry operations of $Q^5$ means a reduction of the symmetry volume from five to four dimensions. We therefore have to correct the integration volume by a factor of $1/V(S^4)$, where $S^4 = SO(5)/SO(4)$ is the unit sphere in 4 dimensions. We account for this correction by multiplying the volume element of the Poisson integral by this factor. This delivers a new value of $1/(V(Q^5)V(S^4))$ for the density of states on $Q^5$.

Up to now the integral covers states, whose parameters are located on the boundary $Q^5$. To include states, whose parameters are located in the interior of $D^5$, we have to extend the integral by an integration in a radial direction, e.g. in the direction of $z_1$. It is sufficient to integrate over one radial direction, because the other directions are already covered by the integration over $Q^5$. (Remember, that the integration over $Q^5$ has indeed been performed, as if there were a $SO(5)$ symmetry. The error made in this connection has been compensated for only after the integration.)

In preparation for this integration, let us consider the integral

$$\int_0^1 dr \int_{Q^5} \frac{d\xi(r)}{V(Q^5)},$$  

(68)
where \( r \) is the radius of \( Q^5 \). This integral delivers the volume \( V(D^5) \) of \( D^5 \).

Instead of integrating over \( r \), we can multiply the volume element of \( Q^5 \) by \( V(D^5) \) and evaluate the following integral

\[
\int_{Q^5} \frac{V(D^5)}{V(Q^5)} d\xi ,
\]

which delivers the same result, since

\[
\int_{Q^5} \frac{1}{V(Q^5)} d\xi = 1 .
\]

Since the volume element \( d\xi(r) \) is 4-dimensional, the integration in (69) is performed in four orthogonal directions. Therefore, an integration over only one direction can be replaced by a multiplication of the volume element by \( V(D^5)^{\frac{1}{4}} \).

Hence, the integration over a radial direction in \( D^5 \) contributes an additional factor \( V(D^5)^{\frac{1}{4}} \) to the volume element \( d\xi \).

When we perform the integration over the phase factor in (64), which can be done for “reasonable” \( \phi(z) \) (cf. [19]), we obtain another factor of \( 2\pi \).

Remembering that \( 2 \times 2 \) spin components contribute to a two-particle state, we add a factor of 4.

Collecting all factors results in an effective volume factor of

\[
8\pi V(D^5)^{\frac{1}{4}} / (V(S^4) V(Q^5)) .
\]

This is Wyler’s formula.

The volumes \( V(D^5) \) and \( V(Q^5) \) have been calculated by L. K. Hua [19]. \( V(S^4) \) is the volume of the unit sphere \( S^4 \) in 4 dimensions. With

\[
V(Q^5) = \frac{8\pi^3}{3} ,
\]

\[
V(D^5) = \frac{\pi^5}{2^4 5!} ,
\]

\[
V(S^4) = \frac{8\pi^2}{3}
\]

we obtain

\[
\frac{9}{8\pi^4} \left( \frac{\pi^5}{2^4 5!} \right)^{\frac{1}{4}} = \frac{9}{16\pi^3} \left( \frac{\pi}{120} \right)^{\frac{1}{4}} = \frac{1}{137.03608245} .
\]

The best experimental value for \( \alpha \) currently is \( 1/137.035 999 070(98) \).
Wyler’s description of the factor $V(D^5)^\frac{1}{2}$ is not very clear, but his formulations indicate that he regarded this additional volume factor as the Jacobian of the boost operation $z \rightarrow g(z)$. From the explicit form of the Poisson kernel (65) we can easily read off that this interpretation cannot be correct. Nevertheless, Wyler can take the credit for having found a precious key to a basic aspect of particle physics. It was his personal fate that he did not find the matching lock.

The correct value for the fine-structure constant delivers strong evidence that the interaction, found in the last section, is in fact the electromagnetic interaction.

9 Strong interaction

After having successfully reproduced the electromagnetic interaction, we should feel encouraged to look for other interactions that may result from SO(3,2) corrections. This and the next two sections will briefly sketch how these interactions are obtained. We start with an interaction term that is very similar to the electromagnetic term, but is obtained from a different configuration of spinors.

So far we have treated the linkage of a single spinor to up to three reference objects. Now we will try the opposite situation, where three spinors are linked to a single quasi-classical reference object.

The action of the cluster of three spinors on the reference is again a change of its 4-momentum, which will be interpreted as the 4-momentum of the cluster. Let us assume that the cluster of spinors represents a stable configuration with respect to transformations of the Poincaré group. Then there are three independent momentum components of the cluster that have to be distributed among three spinors. To solve this task, we give the operators of the Dirac field an additional index $\psi_k(x)$, where $k = 1, 2, 3$ reference one of the three momentum components. With the help of Gell-Mann’s matrices $\lambda_{a}, a = 1, \ldots, 8$, we can define eight basic configurations

$$\sum_{k=1}^{3} \lambda_{a,ik} \psi_k(x), \quad a = 1, \ldots, 8.$$  \hspace{1cm} (76)

From the octet of Gell-Mann’s matrices any other $3 \times 3$ matrix with trace 0 can be obtained by linear combination. In this sense the configurations (76) form a complete basis. When we insert this ansatz into the interaction term (52),
and perform the same steps as in section 7, we obtain the interaction term of quantum chromodynamics

\[ \int d^3x \bar{\psi}(x) \gamma_\mu \lambda_a \psi(x) G^\mu_a(x). \]  

(77)

Here the Gluon field \( G^\mu_a(x) \) takes the place of the electromagnetic four-potential \( A^\mu(x) \).

A quark appears as a kind of uncompleted particle in the sense that it does not possess a complete set of momentum components relative to a reference. A single quark may contribute just one component, which may be either \( p_1 \), \( p_2 \) or \( p_3 \). Therefore, it is not possible to give a single quark a position relative to a reference. When we associate the colours red, green or blue to each of the momentum components, then only a “white” combination provides a complete momentum vector. This then allows to treat the compound object as a free particle. Since a quark can carry any of the three momentum components, there is a SU(3) symmetry with respect to exchange of the momentum components.

10 Weak interaction

When we write the second term of (47) as a Fock space operator and specialize this operator to the transition \( \mu \rightarrow e + \nu_\mu + \bar{\nu}_e \), we obtain the basic structure of weak interaction

\[ \int d^3x \bar{\psi}_\nu(x) \gamma_\mu \psi_\mu(x) \bar{\psi}_e(x) \gamma^\mu_L \psi_e(x). \]  

(78)

The matrices \( \gamma_\mu L \) contain the usual projection on left-handed components

\[ \gamma_{\mu L} := \gamma_\mu \frac{1 - \gamma_5}{2}, \]  

(79)

since, as we have seen in section 6, there are only left-handed neutrinos.

11 Gravitation

In a SO(3,2) symmetric system, curved space-time obviously is an inherent property. Flat space-time is obtained only as an approximation. Up to now we have tacitly assumed that this approximation is valid. In the following we will try to find configurations, where this assumption is not valid any more.
Consider the SO(3,2) operator (5) from section 2

\[ j_{\mu 4} = i(x_\mu \partial_4 - x_4 \partial_\mu), \quad \mu = 0, 1, 2, 3. \] (80)

In tangential space-time at the point \( O = (0, 0, 0, 0, R) \) the translation operator is proportional to the second part \(-ix_4 \partial_\mu\) of the SO(3,2) operator. The contribution of the first term will, in general, be very small, so that it can be neglected. But, when we examine SO(3,2) correction to the flat-space approximation, we must take care of the first part \( ix_\mu \partial_4 \) in a suitable way.

When we add the first part as a “small” correction to the translation operator, in the same way as we have added correction terms before, we encounter product terms of the form

\[ ix_\mu \partial_4 (-ix_4' \partial'_\mu - ix_4'' \partial''_\mu - \ldots), \] (81)

which result from an evaluation of (46) with \( p_\mu \) replaced by \( j_{\mu 4} \). Here the non-primed term on the left refers to a test particle under examination and the primed terms in brackets shall refer to a large collection of other particles in the neighbourhood of the test particle. We assume that the primed particles are concentrated near \( O \), so that \( x_4', x_4'', \ldots \) are nearly equal to \( R \). When, furthermore, the wave functions of the primed momenta are in phase, the expectation values of the momentum of the primed particles can add up to a large value. The largeness of this value may then eventually over-compensate the smallness of the term \( x_\mu \partial_4 \). Thus, in a perturbation calculation the contribution of (81) may become dominant.

To understand the effect of (81) in a perturbation expansion, we replace \( x_4 \) in (80) by the de Sitter radius \( R \), divide by \( R \) and obtain

\[ i \frac{x_\mu}{R} \partial_4 - i \partial_\mu . \] (82)

This operator stands for a de Sitter displacement operation near \( O \). For \( R \to \infty \) this operation becomes the translation operator. For finite \( R \) it describes the corresponding operation in curved space-time with a radius of curvature \( R \).

Now compare (82) with the term

\[ ix^\mu \partial_4 (-i \partial'_\mu + \ldots) - i \gamma^\mu \partial_\mu , \] (83)

obtained by inserting (81) into a perturbation expansion for a test particle near the point \( O \). It is obvious that (83) has the characteristics of the de Sitter
displacement operator (82) in curved space-time. But now the curvature $1/R$ is replaced by the 4-momentum of primed particles. Therefore, the curvature of space-time, experienced by the test particle, becomes proportional to the sum of 4-momentum of primed particles. Taking into account a distribution of the primed particles over a finite area, we can (in a classical approximation) replace their 4-momentum by an energy-momentum tensor.

It may be of interest that, even in the absence of any other matter, space-time of the test particle is curved as indicated by (82), due to the basic SO(3,2) symmetry.

When Albert Einstein set up the field equations of general relativity, he used the fact that the relation between curvature, expressed by the metric tensor $g_{\mu\nu}$, and the energy-momentum tensor is uniquely determined by the following conditions [25]:

1. There are no higher than second order differential quotients of $g_{\mu\nu}$.
2. The relation is linear in the second order differential quotient.
3. The divergence of the curvature expression is zero.

Although further work will be needed to definitely show that these conditions are satisfied for the quantum mechanical interaction term (81), we can state the following: The curvature, experienced by the test particle, is proportional to the energy-momentum tensor of the particles in its neighbourhood. This proportionality has the following consequences: Since the curvature tensor is linear in the second order differential quotient of $g_{\mu\nu}$, condition 2 is obviously satisfied. The curvature tensor does not contain differential quotients of greater then second order, therefore also condition 1 is satisfied. Condition 3 expresses energy-momentum conservation. Since energy-momentum of the primed particles is conserved, at least in a neighbourhood of $O$, condition 3 also holds.

The aforesaid strongly suggests that the term (83), within a perturbation calculation, defines a theory of quantum gravity that, in the classical (low-energy) limit, reproduces the field equations of general relativity.

The reader will have noticed that the term (83) does not contain any “gravitational field”. It only contains well-defined differential operators acting on wave functions of the involved particles. Therefore, the notorious problems of “quantizing” the “gravitational field” are avoided.
12 Spin networks versus binary elements

The approach has started from a spin network, although its motivation came from information at its most elementary level, represented by a set of binary elements. So the author owes the reader a justification of this proceeding.

Consider a set of binary elements, representing information. The information may be complete, but, in general, it is not. Therefore, we have to provide the possibility of representing incomplete information. A simple method to do this is to form linear combinations with complex coefficients, which interpolate between the states of the binary element. This leads to a vector space, which can easily be extended to a Hilbert space. By this action a binary element is extended to a spinor. The spinor form can be considered as the quantum mechanical representation of a binary element.

As before we can use these spinors to construct a quasi-classical reference. After having done this, we are in a position to look for a covariant description of a single binary element relative to the reference. Since a rotation of the reference, together with its coordinate system, induces a similar rotation of the spinor, the task of finding a covariant description is settled by the spinor form of the binary element. Then spin networks are “relational equivalent” to sets of binary elements.

This consideration shows that Bohr’s statement not only can be applied to the most elementary level of information, but actually can be regarded as the root of quantum mechanics at this basic level.

13 Conclusion

Bohr’s statement on the relational nature of observation has guided us from a most general binary structure, to concepts that in the end, allow us to describe observations relative to experimental setups in the common language of particle physics.

Bohr’s statement has directed us to a description of microscopic phenomena relative to macroscopic setups by interpolating Hilbert spaces. Therefore, quantum mechanics can be understood as a necessary and logical result of the application of Bohr’s statement to the microscopic domain.

Another consequence of this approach is the emergence of Minkowskian space-time. Space-time is obtained as the parameter space of local symmetry transformations of a quasi-classical reference object. It is then employed to de-
scribe binary elements relative to quasi-classical reference objects in a covariant way. Since SO(3,2) contains the translation group only as a local approximation, we observe corrections to translation invariance, that manifest themselves as interactions between certain configurations of spinors.

Note that we have obtained space-time as a property of more or less macroscopic objects. This is polar opposite to attempts to derive physical space-time from discrete structures at Planck scales. Nevertheless, our approach is basically also a discrete one, but discrete at microscopic scales rather than at Planck scales.

“Internal” symmetries, considered as fundamental in the standard model, like SU(3), SU(2) and U(1) have been found to emerge as symmetries of special spinor configurations.

In contrast to the standard model there are no adjustable constants. This means that, when spin-networks are taken as a basis for the standard model, its constants must be calculable from some characteristic configurations of spinors. That this is possible in principle, has been shown for the lepton mass spectrum and the fine-structure constant.

Our approach exhibits a general binary structure as a basis, from which fundamental aspects of particle physics can be derived. Does this imply that reality has the structure of a binary system? Is this structure some kind of a prephysical aspect of reality, or is it rather a logical consequence of our continuous search for more and more elementary structures? In the author’s opinion the latter is true. The binary structure reflects our way of doing science along the concept of reductionism. Reductionism applied to information must, inevitably, lead to the most elementary, which means binary, elements of information. Since binary information cannot be divided further, there cannot be any knowledge beyond what can be coded into the binary structure. Binary structures, therefore, do not require or even allow for a deeper explanatory base. The binary structure serves as a general means to represent knowledge at the most elementary level, but it is not part of the knowledge. It is merely an unspecific logical scheme, comparable to the binary system in computer technology, which is used only to represent information.

After having divided knowledge into their most elementary parts, it should not be a surprise that we can recombine the parts and obtain space-time, particles and interaction. What could come as a surprise is that we can do this without the help of any “law of nature”. In fact, the only principle that we
have applied is formulated in Bohr’s dictum about the relativity of observation. This means that quantum mechanics, space-time, particles and interactions are inevitable consequences, not of fundamental laws of nature, but of how we observe physical phenomena. The transcription of our observation techniques into a mathematical form does not reveal any deep mystery, but only employs the rules of mathematical logic.

Bohr may have had something near to it in mind, when he said: “It is the task of science to reduce deep truths to trivialities”.

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