The Dehn Function of Richard Thompson’s Group 

$F$ is Quadratic

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March 29, 2022

Abstract

We prove that the Dehn function (that is, the smallest isoperimetric function) of the R. Thompson’s group $F$ is quadratic.

The Richard Thompson group $F$ can be defined by the following presentation

$$\langle x_0, x_1, x_2, \ldots | x_j^{x_i} = x_{j+1} \text{ (} i < j \text{)} \rangle,$$

where $a^b = b^{-1}ab$ by definition. This group was invented by Richard J. Thompson in 1965 during his work in $\lambda$-calculus. In 1977–1979 it was rediscovered by Dydak, Freyd and Heller in their work on homotopy idempotents. Let us mention a few properties of $F$.

1. The group $F$ is finitely presented. Namely, it can be given by

$$\langle x_0, x_1 | x_2^{x_1} = x_3, x_3^{x_1} = x_4 \rangle,$$

where $x_{k+1} = x_1^{x_0}$ for any $k \geq 1$ by definition. So the defining relations have in fact the following form: $x_1^{x_0} = x_1^{x_0 x_1}, x_1^{x_0} = x_1^{x_0^{x_1}}$.

2. Equivalent definition of $F$ can be done in the following way. Let us consider all strictly increasing continuous piecewise linear functions from the closed unit interval onto itself. Take only those of them that are differentiable except at finitely many dyadic rational numbers and such that all slopes are integer powers of 2. Obviously, these functions form a group under composition. This group is exactly $F$.

This research is partially supported by the RFFI grant 99–01–00894 and the INTAS grant 99–1224.
3. The group $F$ has solvable word and conjugacy problems, it does not satisfy any non-trivial group law although it has no free subgroups of rank $> 1$. It is known that $F$ is not elementary amenable. However, the famous problem about amenability of $F$ is still open.

For the history of the group $F$ see [6]; for the proof of these and other results on the group $F$ see also [4, 2, 3, 13].

Let us recall the definition of an isoperimetric function of a group presentation. For our needs it will be sufficient to give this definition for finite presentations only. However, it makes sense for any presentation with finite number of generators. Let $H$ be a group presented by

$$\langle a_1, a_2, \ldots, a_s \mid R_1, R_2, \ldots, R_t \rangle.$$  

(3)

By $F_s$ we denote the free group on $a_1, a_2, \ldots, a_s$. Let $N$ be the normal closure of the defining relators $R_1, R_2, \ldots, R_t$. By definition, $H \cong F_s/N$. For any word $w \in N$ we consider the smallest number $k = k(w)$ such that $w$ is equal in the free group $F_s$ to a product of the form

$$U_1^{-1} R_{i_1}^\pm U_1 U_2^{-1} R_{i_2}^\pm U_2 \cdots U_k^{-1} R_{i_k}^\pm U_k,$$

where $U_i$ are elements of $F_s$. In other words, $k(w)$ is the smallest number of applications of the defining relations to derive $w$ from the empty word. Equivalently, using van Kampen’s Lemma (see [15]), we may say that $k$ is the smallest number of cells in a van Kampen diagram over (3) provided its boundary label is $w$.

A function $p(n)$ is called an isoperimetric function of presentation (3) whenever $k(w)$ does not exceed $p(n)$ for any word $w \in N$ of length $\leq n$. The smallest isoperimetric function of a finite group presentation is called the Dehn function of this presentation. It is easy to see that any Dehn function of a group presentation is non-decreasing.

Usually Dehn functions are compared with respect to the following partial relation $\preceq$ on functions from the set of natural numbers $\mathbb{N}$ to itself. By definition, $p \preceq q$ means that there exists a positive integer constant $C$ such that $p(n) \leq Cq(Cn) + Cn$ for all $n$. This is a pre-order relation on the set of all non-decreasing functions from $\mathbb{N}$ to $\mathbb{N}$. It induces an equivalence relation $\sim$ on the set of these functions. Namely, $p \sim q$ if and only if $p \preceq q$ and $q \preceq p$.

The following result is well known (see [16, 8, 1]). Suppose that $H$ is a finitely presented group and $p, q$ are Dehn functions for two finite presentations of $H$. Then $p$ and $q$ are equivalent, that is, $p \sim q$. Thus for a finitely presented group $H$ there exists a unique (modulo the equivalence relation $\sim$) Dehn function $p(n)$. The group $H$ has solvable word problem if and only if its Dehn function has a recursive upper bound.

Gersten proved in [4] that the Dehn function of $F$ has exponential upper bound. Also he had a conjecture that the Dehn function of $F$ is exponential. However, Guba and Sapir [14] proved that the Dehn function $\Phi(n)$ of $F$ is strictly subexponential, namely, $\Phi(n) \preceq n^{\log n} = 2^{\log^2 n}$. Later Guba [13] improved this result. He proved that the Dehn function of $F$ has a polynomial upper bound $n^5$. Actually it was proved that $\Phi(n) \preceq n^c$ for some constant $c$ between 4 and 5.

In the present article we are going to calculate the Dehn function of $F$ precisely. Here is our main result.
Theorem 1 The Dehn function $\Phi(n)$ of the Richard Thompson group $F$ is quadratic, that is, $\Phi(n) \sim n^2$.

Note that any automatic group has quadratic isoperimetric function \cite{5}. It is still unknown whether $F$ is automatic.

To prove our main result, we do the following two improvements. First of all, we improve the estimate for the area of some “basic” diagrams obtained in \cite{11}. In that paper it was $n^{2.746\ldots}$, here we decrease it to $n^2$. There was a lemma in \cite{14} that the final estimate for the whole Dehn function can be obtained by multiplying the previous estimate by $n^2$. Another important improvement of this paper is to avoid this multiplier.

Our plan is as follows. In Section 1 we recall some known facts about string rewriting systems and normal forms in $F$. Then we define the standard form of each word and reduce the problem to estimate the Dehn function of $F$ to some fact about standard diagrams. Namely, we need to show that each word of length $n$ can be transformed into its standard form in $O(n^2)$ steps. Since the standard form of a word that equals 1 in $F$ freely equals 1, this would imply our main result. The next Section 2 deals with triangular diagrams. Their boundary equations have the form $pq = r$, where $p$, $q$ are positive words with non-decreasing subscripts and $r$ is the standard form of $pq$. In this Section we reduce the problem for standard diagrams to the one for triangular diagrams. Section 3 reduces the problem to rectangular diagrams. Suppose that we conjugate a positive word $q$ by a letter $x_i$. If all subscripts of $q$ are greater than $i$ (we say that the conjugation is successful in this case), then all subscripts increase by 1 after the conjugation. If we repeat this operation several times, and at each step the conjugation is successful, then we get a word obtained from $q$ by increasing all subscripts by $m$, where $m$ is the length of the conjugator $p$. One can draw a diagram of this conjugation. Such diagrams are called rectangular. We find two partial cases of rectangular diagrams and formulate two important statements, Lemma 6 and Lemma 7. Now the problem is reduced to these two cases.

The first of these cases (horizontal diagrams) is studied in Section 4. It contains Lemma 8 which is the crucial point of our proof. Then we go to the other case (vertical diagrams) in Section 5. We describe the process to construct vertical diagrams using auxiliary diagrams with boundary equation $x_1^{x_0} = x_{n+1}$. We need a modification of Lemma 8. This is Lemma 9. The difference between it and Lemma 8 is that here we need to take areas of some subdiagrams with coefficients that are called intensities. After we prove this Lemma, this immediately implies our main result.

As a corollary, we get that $F$ has linear isodiametric function. Let us recall the definition of this concept and formulate the Corollary.

A diameter of a van Kampen diagram is the diameter of its underlying graph, that is, the greatest distance between its points. Suppose that for any word $w$ of length $\leq n$ that is equal to 1 over $\langle 3 \rangle$, one can find a van Kampen diagram (over the same presentation) of diameter $\leq d(n)$ with boundary label $w$. The smallest function $d(n)$ with this property is called the isodiametric function of $\langle 3 \rangle$. As in the case of Dehn functions, there is a similar result that isodiametric functions of different finite presentations of the same group are equivalent.

Corollary 1 The isodiametric function of the Richard Thompson group $F$ is linear.
This follows from the result of Papasoglu [17]. Note that a direct proof of the corollary is relatively easy.

At the end of the paper, we discuss possible applications of our results to some other Thompson-like groups.

The author thanks Mark Sapir for helpful comments.

1 Standard Forms

We will consider some auxiliary string rewriting system. The facts about rewriting systems we are going to use are standard. We just refer to [7].

Let $X$ be the alphabet $\{x_i, x_i^{-1} \ (i \geq 0)\}$. The string rewriting system $\Sigma$ over $X$ is defined by the following rewriting rules:

$$
x_i^{-1}x_i \rightarrow x_i x_i^{-1}, \quad x_i^{-1}x_j \rightarrow x_{j+1} x_i^{-1}, \quad x_j^{-1}x_i \rightarrow x_i x_{j+1}^{-1} \quad (i < j),
$$

$$
x_j x_i \rightarrow x_i x_{j+1}, \quad x_i^{-1}x_j^{-1} \rightarrow x_{j+1} x_i^{-1} \quad (i < j).
$$

Obviously, if $u \rightarrow v$ for some words $u, v$ over $X$, then words $u$ and $v$ are equal in $F$, that is, are equal modulo $[\mathbb{I}]$. (However, $\Sigma$ is not a rewriting system for the group $F$.)

It is clear that $\Sigma$ is Noetherian (terminating) because applying each of its rules decreases words lexicographically, where $x_0 < \cdots < x_n < \cdots < x_n^{-1} < \cdots < x_0^{-1}$. The system $\Sigma$ is also confluent. By Diamond Lemma, it is enough to check that the system is locally confluent. In our case, this means that if rules $ab \rightarrow cd, be \rightarrow fg$ belong to $\Sigma$, where $a, b, c, d, e, f \in X$, then $cde$ and $afg$ can be reduced to the same word. We shall illustrate this for the case of two rules $x_i^{-1}x_j \rightarrow x_{j+1} x_i^{-1}$ and $x_j x_k \rightarrow x_k x_{j+1}$, where $i < j$, $k < j$. We need to show that the words $u = x_{j+1} x_i^{-1} x_k$ and $v = x_i^{-1} x_k x_{j+1}$ have a common descendant. There are three subcases.

a) $i = k$. Here $u \rightarrow x_{j+1} x_i x_i^{-1} \rightarrow x_i x_{j+1} x_i^{-1}, \quad v \rightarrow x_i x_{j+1} x_i^{-1} x_{j+1} \rightarrow x_i x_{j+1} x_i^{-1}.$

b) $i < k$. Here $u \rightarrow x_{j+1} x_k x_i x_i^{-1} \rightarrow x_k x_{j+1} x_i x_i^{-1}, \quad v \rightarrow x_k x_{j+1} x_i x_i^{-1} x_{j+1} \rightarrow x_k x_{j+1} x_i x_i^{-1}.$

c) $i > k$. Here $u \rightarrow x_{j+1} x_k x_i x_i^{-1} \rightarrow x_k x_{j+1} x_i x_i^{-1}, \quad v \rightarrow x_k x_{j+1} x_i x_i^{-1} x_{j+1} \rightarrow x_k x_{j+1} x_i x_i^{-1}.$

All other cases are quite analogous or they are even easier. We left the rest to the reader. Now we can conclude that $\Sigma$ is complete. Therefore, each word $w$ over $X$ has a unique irreducible form over $\Sigma$. We call it the standard form of the word $w$.

By a monotone positive word (an MP-word) over $X$ we mean a word of the form

$$x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $n \geq 0, i_1 \leq i_2 \leq \cdots \leq i_n$.

The following easy statement establishes some elementary properties of the standard form.
Lemma 1 For any word \( w \) over \( \{ x_0^{\pm 1}, x_1^{\pm 1} \} \) of length \( n \), the standard form of \( w \) is graphically equal to \( pq^{-1} \), where \( p, q \) are MP-words, \( |pq^{-1}| = n \) and all subscripts that occur in \( pq^{-1} \) do not exceed \( n \).

Proof. Clearly, all rules of \( \Sigma \) preserve the length. If \( r \) is the standard form of \( w \), then each positive letter in \( r \) is to the left of each negative letter. Otherwise there is an occurrence in \( r \) of the form \( x_i^{-1}x_i, x_i^{-1}x_j \), or \( x_j^{-1}x_i \), where \( i < j \). This contradicts the fact that \( r \) is irreducible. So \( r \) has the form \( pq^{-1} \), where \( p, q \) are positive words. Since \( r \) is irreducible, it has no occurrences of the form \( x_jx_i, x_i^{-1}x_j^{-1} \), where \( i < j \). Thus \( p \) and \( q \) are MP-words.

Let us prove by induction on \( n \) that maximal subscripts in both \( p \) and \( q \) do not exceed \( n \). This is obvious if \( n = 1 \). Now suppose that the fact is true for any word of length \( n \). Take any word \( v \) of length \( n + 1 \). It can be presented as \( wx_n^{\pm 1} \), \( i = 0, 1 \). Let \( pq^{-1} \) be the standard form of \( w \), where \( p, q \) are MP-words with subscripts \( \leq n \). If \( v = wx_0^{-1} \) then \( p(x_0q)^{-1} \) is the standard form of \( v \). Let \( v = wx_0 \). The word \( pq^{-1} \) can be presented as \( x_0^sp_1q_1^{-1}x_0^{-t} \), where \( s, t \geq 0 \) and \( p_1, q_1 \) do not involve \( x_0 \). In this case it is obvious that \( pq^{-1}x_0 \) can be reduced to \( x_0^{s+1}p_2q_2^{-1}x_0^{-t} \), where \( p_2 (q_2) \) is obtained from \( p_1 (q_1) \) by adding 1 to all subscripts. This implies that all subscripts of the result (which will be the standard form of \( v \)) do not exceed \( n + 1 \).

Now let \( v = wx_1^{-1} \). Applying rules of \( \Sigma \) to \( v \), we move the last letter to the left whenever possible. Each move may increase the subscript on this letter by 1 so at the end of the process the subscript will be at most \( n + 1 \) (the equality holds if and only if \( w = x_0^{-n} \)). Also if we move one letter through another, the letter that stays on the right after the move can increase its subscript by 1 but this happens only once. So all subscripts do not exceed \( n + 1 \) when we get the standard form.

The case \( v = wx_1 \) is analogous.

The proof is complete.

Let us recall some well known facts about normal forms in \( F \). Details can be found in [4]. Any element of \( F \) can be expressed uniquely as a word of the form

\[
x_{i_1}^{s_1} \cdots x_{i_m}^{s_m} x_{j_n}^{-t_n} \cdots x_{j_1}^{-t_1},
\]

where \( m, n \geq 0, i_1 \leq \cdots \leq i_m \neq j_n \geq \cdots \geq j_1; s_1, \ldots, s_m, t_1, \ldots, t_n \geq 1 \). Here it is also claimed that if both \( x_i \) and \( x_i^{-1} \) occur in (4) for some \( i \geq 0 \) then either \( x_{i+1} \) or \( x_{i+1}^{-1} \) also occurs. An expression of the form (4) of an element \( g \in F \) is called the normal form of \( g \). (Note that in [4] it is constructed another useful normal form for elements of \( F \).) In particular, each MP-word is a normal form. So the word \( pq^{-1} \), where \( p, q \) are MP-words, may represent the identity of \( F \) if and only if \( p \) and \( q \) are graphically equal.

Let

\[
\mathcal{P}_r = \left\{ x_0, x_1, x_2, \ldots \mid x_j^{x_i} = x_{j+1} \ (0 < j - i \leq r) \right\}
\]

be a group presentation. It is easy to see that for \( r \geq 2 \) the group presented by \( \mathcal{P}_r \) is \( F \). We will usually construct van Kampen diagrams over (5) to estimate the Dehn function of \( F \). Let us compare \( \mathcal{P}_r \) with finite presentation (2).

By \( \psi \) we denote the shift mapping. It takes each letter \( x_i \) to \( x_{i+1} \) (\( i \geq 0 \)). Clearly, \( \psi \) induces an embedding of \( F \) into itself. (This notation will be often used throughout the
Lemma 2 Let $w$ be a word over $\{x_0^{\pm 1}, x_1^{\pm 1}\}$. Suppose that $w$ equals 1 in $F$. Let $\Delta$ be a van Kampen diagram over $P_5$ of area $N$ with boundary label $w$. Then there exists a van Kampen diagram over (2) with boundary label $w$ and area $\leq 13N$.

Proof. We convert each diagram $\Delta$ over $P_5$ into a diagram $\overline{\Delta}$ over (4) in the following way. Each edge labelled by $x_n$, where $n \geq 2$, is replaced by the path labelled by $x_1^{n-1}$. Each cell of $\Delta$ has boundary equation of the form $x_j^{x_i} = x_{j+1}$, where $0 < j - i \leq 5$. If $i = 0$ then the boundary label of its image in $\overline{\Delta}$ will be freely equal to 1. Suppose that $i \geq 1$. Then the boundary equation of the image of this cell in $\overline{\Delta}$ is $x_k^{x_i} = x_{k+1}$ conjugated by $x_0^{-1}$ in the free group, where $k = j - i + 1$, $2 \leq k \leq 6$. It is easy to see that the relations $x_j^{x_i} = x_5$, $x_5^{x_1} = x_6$, $x_6^{x_1} = x_7$ can be derived from the defining relations of (2) in 5, 9, and 13 steps, respectively. Recall how to do this. Clearly, any relation of the form $x_j^{x_i} = x_{j+1}$, where $i \geq 1$, $j \in \{i + 1, i + 2\}$ is obtained from one of the defining relations of (2) by conjugation. So

$$x_6 = x_5^{x_3} = x_4^{x_2x_3} = x_3^{2x_3} x_3$$

(6) after applying three defining relations. The right-hand side of this equality is a word of length 7. If we conjugate (2) by $x_1$ and apply seven more defining relations of (4), then we obtain $x_6^{x_1} = (x_3^{2x_3})^{x_1} = x_4^{x_3x_4}$. It remains to note that conjugation by $x_0$ applied to (4) gives $x_7 = x_4^{x_3x_4}$ in three steps. So we finally have $x_6^{x_1} = x_7$ in 13 steps. (The reader can draw the corresponding van Kampen diagram to see the diagrams for the other two relations as subdiagrams. They have 5 and 9 cells, respectively.)

Now after free cancellations of labels inside cells and inserting diagrams of at most 13 new cells into each old cell, we get the diagram $\overline{\Delta}$ over (2) with at most 13N cells, where $N$ is the area of $\Delta$. The boundary label remains the same since $w$ is a word over $\{x_0^{\pm 1}, x_1^{\pm 1}\}$.

The proof is complete.

Let $w$ be a word over $\{x_0^{\pm 1}, x_1^{\pm 1}\}$ of length $n$. Let $pq^{-1}$ be its standard form. Our aim is to construct a diagram over $P_5$ of area $O(n^2)$ with boundary equation $w = pq^{-1}$. This will immediately imply that the Dehn function $\Phi(n)$ of $F$ is quadratic. Indeed, if $w$ equals 1 in $F$, then $p$ and $q$ must be graphically equal. So this gives a diagram over $P_5$ of area $O(n^2)$ with boundary label $w$. By Lemma 2 there exists a diagram over (2) of area $O(n^2)$ with boundary label $w$. Hence $\Phi(n) \lesssim n^2$. Since $F$ is not hyperbolic, $n^2 \lesssim \Phi(n)$ (this can be easily shown directly). As a result, $\Phi(n) \sim n^2$, that is, the Dehn function of $F$ is quadratic. So we proved the following

Lemma 3 Suppose that there exists a positive constant $C$ such that for any word $w$ over $\{x_0^{\pm 1}, x_1^{\pm 1}\}$, there exists a van Kampen diagram over $P_5$ of area $\leq Cn^2$ with boundary equation $w = pq^{-1}$, where $pq^{-1}$ is the standard form of $w$. Then the Dehn function of $F$ is quadratic.
2 Triangular Diagrams

Lemma 3 says that the problem about the Dehn function of $F$ can be reduced to standard diagrams, that is, to the diagrams with boundary equation $w = pq^{-1}$, where $pq^{-1}$ is the standard form of $w$. Now we are going to define some other class of diagrams and reduce the problem to it. (For any class of diagrams we deal with, the problem is to show that diagrams over $P_5$ from this class satisfy quadratic isoperimetric inequality.)

Let $p$, $q$ be MP-words and let $r$ be the standard form of $pq$. Clearly, $r$ is also an MP-word. We are going to construct a diagram with boundary equation $pq = r$ over $P_5$ and estimate its area. In Lemma 4 we reduce our problem to this kind of diagrams that we call triangular diagrams.

It is useful to describe the structure of triangular diagrams over (1). Let $p$, $q$, $r$ be as above, $q = x_{k_1} \cdots x_{k_m}$. We define a sequence of words $v_0, u_1, v_1, \ldots, u_m, v_m$ by induction. By definition, $v_0 = p$. For any $1 \leq i \leq m$, let $u_i$ be the longest suffix of $v_{i-1}$ that starts from a letter with a subscript greater than $k_i$. (Here $u_i$ is empty if no such a letter exists.) By $v_i$ we mean the word $\psi(u_i)$. Clearly, there exists a rectangular diagram over (1) with $u_i$ at the bottom, $v_i$ at the top, $x_{k_i}$ on the left and on the right, oriented upwards. Gluing all these diagrams in a natural way, we get the diagram with boundary equation $pq = r$, where $p$ is on the bottom, $q$ is on the right, $r$ is the rest of the boundary. The above picture illustrates the process for $p = x_1x_2x_3x_5x_6x_8, q = x_1x_5x_6x_8x_{13}, r = x_1^2x_3x_4x_5x_6x_8x_{10}x_{12}x_{13}$. (We show only subscripts of letters; all edges are oriented either from the left to the right, or upwards.)

Let $w$ be a word over $X$. By $\mu(w)$ we denote the maximal $i$ such that $x_i$ or $x_i^{-1}$ occurs in $w$. If $w$ is empty, then $\mu(w) = 0$ by definition. It is not hard to see from the construction of the triangular diagram over (1) that $\mu(r) = \max(\mu(q), \mu(p) + |q|)$. We will use this formula in the proof of the next lemma, which says that if triangular diagrams satisfy quadratic isoperimetric inequality, then so do standard diagrams.

**Lemma 4** Suppose that there exists a constant $C_0 > 0$ such that for any MP-words $p$, $q$ there exists a van Kampen diagram over $P_5$ of area $\leq C_0n^2$ with boundary equation $pq = r$, where $r$ is the standard form of $pq$, provided $|r| \leq n$ and $\mu(r) \leq n$. Then there exists a positive constant $C$ such that for any word $w$ over $\{x_0^{\pm1}, x_1^{\pm1}\}$, there exists a van Kampen diagram over $P_5$ of area $\leq Cn^2$ with boundary equation $w = pq^{-1}$, where $pq^{-1}$ is the standard form of $w$. 

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Proof. Let $|w| = n$. If $n = 1$, then $w$ coincides with its standard form, so there is nothing to prove. We proceed by induction on $n$. Let $n > 1$. We can decompose $w$ into two factors of almost equal length, namely, $w = w_1w_2$, where $|w_1| = \left\lceil \frac{n}{2} \right\rceil$, $|w_2| = \left\lceil \frac{n+1}{2} \right\rceil$. Clearly, $|w_1| < |w|$ and $|w_2| < |w|$. Thus we can apply the inductive assumption and find diagrams $\Delta_i$ ($i = 1, 2$) over $P_5$ with boundary equations $w_i = p_iq_i^{-1}$, where the area of $\Delta_i$ does not exceed $C|w_i|^2$. It is clear that

$$|w_1|^2 + |w_2|^2 = \left\lceil \frac{n}{2} \right\rceil^2 + \left\lceil \frac{n+1}{2} \right\rceil^2 = \left\lceil \frac{n^2+1}{2} \right\rceil \leq \frac{n^2+1}{2}. $$

Now take the word $q_1^{-1}p_2$ and find its standard form $p_3q_3^{-1}$. Equality $q_1^{-1}p_2 = p_3q_3^{-1}$ holds in $F$ so $q_1p_3 = p_2q_3$ in $F$. It is easy to see that the words $q_1p_3$ and $p_2q_3$ have the same standard form $r$. Indeed, standard forms of both words are positive. They are equal in $F$ so they must be graphically equal. Now let $p$ be the standard form of $p_1p_3$ and let $q$ be the standard form of $q_2q_3$. We can construct 4 triangular diagrams over $P_5$ that have boundary equations $q_1p_3 = r$, $p_2q_3 = r$, $p_1p_3 = p$, $q_2q_3 = q$. All rewriting rules of $\Sigma$ preserve the number of positive and negative letters. So $|p_2| = |p_3|$, $|q_1| = |q_3|$. Thus $|pq^{-1}| = |p_1| + |p_3| + |q_2| + |q_3| = |p_1| + |p_2| + |q_1| + |q_2| = |w_1| + |w_2| = |w| = n$. This also implies $|r| = |q_1| + |p_3| \leq n$. It remains to estimate the maximal subscripts of $p$, $q$, $r$.

From Lemma 7 we know that $\mu(p_1) \leq |w_1|$. Also we know that $|p_3| = |p_2| \leq |w_2|$. Therefore, $\mu(p_1) + |p_3| \leq n$. Recall that $p_3q_3^{-1}$ was obtained as the standard form of $q_1^{-1}p_2$. By induction on $|q_1|$, it is very easy to show that $\mu(p_3)$ does not exceed $\mu(p_2) + |q_1|$. This implies $\mu(p_3) \leq n$. By the formula obtained before this Lemma, we have $\mu(p) \leq n$. Similarly, $\mu(q) \leq n$. The same inequalities also show that $\mu(r) \leq n$.

By the condition of our Lemma, we can assume that each of the 4 triangular diagrams over $P_5$ has area at most $C_0n^2$. So the diagram

over $P_5$ with boundary equation $w = pq^{-1}$ has area at most $C \frac{n^2+1}{2} + 4C_0n^2 \leq Cn^2$ if $C \geq 11C_0$ (recall that $n \geq 2$). It remains to mention that $pq^{-1}$ is the descendant of $w$ with respect to $\Sigma$ so it is the standard form of $w$. This completes the proof.
3 Rectangular Diagrams

Lemmas 4 and 3 imply that if triangular diagrams satisfy quadratic isoperimetric inequality, then the main result follows. Now we are going to introduce so-called rectangular diagrams and reduce the problem to them. Let \( p = x_{i_1} \cdots x_{i_n} \), \( q = x_{j_1} \cdots x_{j_k} \) be two MP-words. We are going to conjugate \( q \) by \( p \). Suppose that \( j_1 > i_1 \). Then the word \( q^{i_1} \) is equal to \( x_{j_1+1} \cdots x_{j_k+1} \) in \( F \). Now suppose that \( j_1 + 1 > i_2 \). Then we can conjugate \( q^{i_1} \) by \( x_{j_2} \) to get the word \( q = x_{j_1+2} \cdots x_{j_k+2} \), and so on. Thus, if for any \( s \) \((0 \leq s < m)\) we have \( j_1 + s > i_{s+1} \), then equality \( q^p = \psi^m(q) \) holds in \( F \). One can easily draw a diagram over (I) for this equality (it will be a rectangle with \( km \) cells). In this case we say that conjugation of \( q \) by \( p \) is successful.

Suppose now that \( p, q \) are MP-words such that conjugation of \( q \) by \( p \) is successful and let \( p = x_{i_1} \cdots x_{i_n} \), \( q = x_{j_1} \cdots x_{j_k} \). Assume that \( k, m, j_k - i_1 \) do not exceed \( n \). We say that rectangular diagrams satisfy quadratic isoperimetric inequality if for any MP-words with the above conditions there exists a diagram over \( \mathcal{P}_5 \) with boundary equation \( q^p = \psi^m(q) \) of area at most \( Ln^2 \), where \( L > 0 \) is a constant. The next Lemma reduces the problem to the case of rectangular diagrams.

**Lemma 5** If rectangular diagrams satisfy quadratic isoperimetric inequality, then so do triangular diagrams.

**Proof.** For any MP-word \( w \) we define its size \( ||w|| \) as the sum of its length and the difference between its greatest and smallest subscript. (If \( r = x_1^3x_3x_4x_5x_6x_7x_8x_9x_{10}x_{12}x_{13} \), then we have \( ||r|| = |r| + 13 - 1 = 23 \).) The size of the empty word is zero by definition.

Note that if an MP-word \( w \) is a product of three factors \( w = w_1w_2w_3 \), then obviously \( ||w_1|| + ||w_2|| \leq ||w|| \).

Let \( p, q \) be MP-words, let \( r \) be the standard form of \( pq \), and let \( |r| \leq n, \mu(r) \leq n \). Obviously, \( ||r|| \leq 2n \). It suffices to show that one can find a van Kampen diagram over \( \mathcal{P}_5 \) of area \( O(||r||^2) \) with boundary equation \( pq = r \).

We can construct a triangular diagram \( T \) with boundary equation \( pq = r \) over (I) as we did before. (It is useful to look at the picture before Lemma 4.) Each letter in \( r \) can be naturally called horizontal or vertical. If \( T \) has at least one cell, then \( r \) has an occurrence of the form \( ab \), where \( a \) is vertical and \( b \) is horizontal. (In the picture, \( r \) has 3 occurrences of this form: \( x_1x_3, x_6x_8, \) and \( x_8x_{10} \).) For each occurrence of this form one can naturally find a rectangular subdiagram in \( T \). (Say, for \( x_6x_8 \) this rectangular subdiagram will have boundary equation \( x_5x_6x_8x_9x_{10}x_11 = x_8x_9x_{11} \).) Denote this rectangular subdiagram in \( T \) by \( R \). Taking \( R \) off \( T \), we get two triangular diagrams, \( T_1 \) and \( T_2 \), where \( T_1 \) is bounded by \( p \), \( r \) and the left side of \( R \) whereas \( T_2 \) is bounded by \( q \), \( r \) and the top side of \( R \). By \( r_1 \) and \( r_2 \) we define the parts of \( r \) that belong to \( T_1, T_2 \), respectively. (In our example, \( r_1 = x_1^2x_3x_4 \), \( r_2 = x_6x_{10}x_{12}x_{13} \).) Note that one of these triangular subdiagrams or both may be empty. If \( T_1 (T_2) \) is not empty, then \( r_1 (r_2) \) ends (starts) with a horizontal (vertical) letter. The subdiagrams \( R, T_1, T_2 \) and the words \( r_1, r_2 \) depend of the choice of the subword \( ab \). We may sometimes write \( r_1(ab), r_1(cd) \) etc if we want to change the subword.

Estimating areas, we may assume without loss of generality that \( r \) starts with a vertical edge and ends with a horizontal edge. (In our example we just take the first \( x_1 \) and \( x_{13} \) off
This may only decrease \(|r|\). Our aim is to find a subword \(ab\) of \(r\) in such a way that \(|r_1|, |r_2| \leq |r|/2\). Let us show this is always possible. Consider all occurrences of the form \(ab\), where \(a\) is vertical, \(b\) is horizontal and \(|r_1| \leq |r|/2\). Subwords with these properties always exist because \(r_1\) is empty for the leftmost one of them. Now let \(ab\) be the rightmost subword of \(r\) with the properties listed above. We claim that \(|r_2| \leq |r|/2\).

Assume the contrary. Obviously, \(r_2\) must have a subword of the form \(cd\), where \(c, d\) are vertical and horizontal letters, respectively. (Otherwise \(r_2\) is empty.) Choose the leftmost of these subwords in \(r_2\). We have a decomposition \(r = r_1r'\), where \(ab\) is contained in \(r'\). Elementary properties of the size imply \(|r_1r'| \leq |r| - |r_2| < |r|/2\). Let us consider the rectangular subdiagram \(R(cd)\). It is clear that \(r_1(cd) = r_1(ab)r'\). This contradicts the fact that \(ab\) was chosen rightmost.

By induction, we may assume that diagrams \(T_1, T_2\) can be filled by cells of \(P_5\) in such a way that the area of each of them will not exceed \(K(||r||/2)^2\), where \(K > 0\) is a constant. The sum of the horizontal side of \(R\) and the vertical side of \(R\) does not exceed \(|r|\). Also the difference between the greatest and the smallest subscript that occur in \(R\) does not exceed the difference between the greatest and the smallest subscript of \(r\). Thus the condition of our Lemma allows to fill \(R\) by cells of \(P_5\) in such a way that the area of the corresponding rectangular diagram will not exceed \(L||r||^2\). So we get a diagram over \(P_5\) with boundary equation \(pq = r\). (It consists of \(T_1, T_2\), and \(R\)) Its area does not exceed

\[
2K\frac{|r|^2}{4} + L||r||^2 \leq K||r||^2
\]

if \(K \geq 2L\). This completes the proof.

Now it remains to show that rectangular diagrams satisfy quadratic isoperimetric inequality. Let us consider two partial cases of this problem.

Case 1 (horizontal diagrams). Let \(q\) be an MP-word and let \(p = x_i\) consist of one letter. Assume that the smallest subscript of \(q\) is greater than \(i\). Then \(q^p = \psi(q)\) modulo (1). Suppose that \(|q| \leq n\) and let \(j - i \leq n\), where \(j\) is the greatest subscript that occurs in \(q\). We need to construct a diagram over \(P_5\) with boundary equation \(q^{x_i} = \psi(q)\) of area \(O(n^2)\). Since all subscripts that occur in \(q\) exceed \(i\), the word \(q\) equals in \(F\) to a word of the form \(v(x_{i+1}, x_{i+2})\). Suppose that we found a word \(v\) such that there exists a van Kampen diagram \(\Delta\) over \(P_5\) of area \(O(n^2)\) with boundary equation \(q = v(x_{i+1}, x_{i+2})\). We can assume that the path labelled by \(v\) is simple (otherwise we can make \(v\) shorter). It is easy to see that \(v\) cannot be very long. Indeed, the perimeter of each cell equals 4. So the length of \(v\) cannot be bigger than the number of cells in \(\Delta\) multiplied by 4. Thus \(|v| = O(n^2)\). (In fact we will find the word \(v\) of linear length.) Conjugating \(x_{i+1}\) or \(x_{i+2}\) by \(x_i\) corresponds to a defining relation of \(P_5\). Thus we can construct a diagram \(\Gamma\) over \(P_5\) of area \(|v|\) with boundary equation \(v(x_{i+1}, x_{i+2})^{x_i} = v(x_{i+2}, x_{i+3})\). Let us consider the diagram \(\psi(\Delta)\). It has boundary equation \(\psi(q) = v(x_{i+2}, x_{i+3})\) and the same area as \(\Delta\). It remains to glue together \(\Delta, \Gamma\) and the mirror copy of \(\psi(\Delta)\) to obtain the desired “horizontal” diagram.

To find a word \(v(x_{i+1}, x_{i+2})\) with the desired properties, one can first decrease all subscripts of \(q\) by \(i + 1\) (this is possible since all of them exceed \(i\)), then express the result as a word of the form \(v(x_0, x_1)\) and then increase all subscripts by \(i + 1\). This leads to the following statement we are going to prove later.
Lemma 6  For any MP-word $q$ such that $|q| \leq n$, $\mu(q) \leq n$, there exists a word $v(x_0, x_1)$ of length $O(n)$ and a van Kampen diagram over $P_5$ of area $O(n^2)$ with boundary equation $q = v(x_0, x_1)$.

Case 2 (vertical diagrams). Now let $q = x_j$ consist of one letter and let $p = x_{i_1}x_{i_2} \cdots x_{i_m}$ be an MP-word. Suppose that conjugation of $q$ by $p$ is successful, that is, $j > i_1$, $j + 1 > i_2$, $j + 2 > i_3$, $\ldots$, $j + m - 1 > i_m$. In this case $x_j^p = x_{j+m}$ in $F$. We need to find a van Kampen diagram $\Gamma$ over $P_5$ of area $O(n^2)$ for this boundary equation provided $j + m \leq n$. In fact we need more. Suppose that we have $l$th power of $x_j$ instead of $x_j$. If we just glue $l$ copies of $\Gamma$ together, the area becomes $l$ times bigger. To save the area, we need to mention that $\Gamma$ should have symmetric structure with respect to a vertical axis. Namely, we want $\Gamma$ to consist of three parts, the left, the central, and the right, such that the central part has $O(n)$ cells and the other two parts are mirror copies of each other. In this case, if we take $l$th power, this will increase the area by $O(n\ell)$ only. To formulate the main fact about vertical diagrams, we need a definition.

A word $t = x_{j_1}^{d_1} \cdots x_{j_h}^{d_h}$ is called a smooth word of rank $j$, where $d_i = \pm 1$ for all $1 \leq i \leq h$, whenever the following conditions hold: a) if $d_i = 1$, then $j + d_1 + \cdots + d_{i-1}$ belongs to $\{j_i + 1, j_i + 2\}$; b) if $d_i = -1$, then $j + d_1 + \cdots + d_{i-1}$ belongs to $\{j_i + 2, j_i + 3\}$. The number $m = d_1 + \cdots + d_h$ is called the height of $t$.

To clarify this definition, let us show how to construct a van Kampen diagram over $P_5$ with boundary equation $x_{j_i}^j = x_{j_i+m}$. First of all, let $s_0 = j$, $s_1 = j + d_1$, $\ldots$, $s_i = j + d_1 + \cdots + d_i$, $\ldots$, $s_h = j + d_1 + \cdots + d_h = j + m$. For each $1 \leq i \leq h$ such that $d_i = 1$ we can take a cell over $P_5$ with boundary equation $x_{s_{i-1}}^x_{j_i} = x_{s_{i-1}+1}$. If $d_i = -1$, then we can take the cell with boundary equation $x_{s_{i-1}}^{x_{j_i}} = x_{s_{i-1}+1}$. In any case, conjugating $x_{s_{i-1}}$ by $x_{j_i}^{d_i}$ gives $x_{s_i}$. So if we glue all these $h$ cells together in a natural way, then we get a diagram over $P_5$ with boundary equation $x_{j_i}^j = x_{j_i+m}$.

Lemma 7  Let $q = x_j$, $p = x_{i_1}x_{i_2} \cdots x_{i_m}$. Suppose that $j + m \leq n$, $i_1 \leq i_2 \leq \cdots \leq i_m$ and for each $0 \leq s < m$ inequalities $j + s > i_{s+1}$ hold, that is, conjugation of $q$ by $p$ is successful. Then there exists a smooth word $t$ of rank $j$, height $m$ and a van Kampen diagram over $P_5$ with boundary equation $p = t$ of area $O(n^2)$. In addition, $|t| = O(n)$.

First of all let us show that Lemmas 6 and 7 imply that rectangular diagrams satisfy quadratic isoperimetric inequality. According to Lemmas 6, 7 and 8, this will imply our main result about the Dehn function of $F$.

Suppose that Lemmas 6 and 7 hold. Take any MP-words $p$ and $q$ such that conjugation of $q$ by $p$ is successful. Let $|p|, |q|$ do not exceed $n$ and let the difference between the greatest subscript of $q$ and the smallest subscript of $p$ also does not exceed $n$. Our aim is to construct a van Kampen diagram over $P_5$ of area $O(n^2)$ with boundary equation $q^p = \psi^m(q)$, where $m = |p|$.

By $j$ we denote the smallest subscript that occurs in $q$, that is, $q$ starts with $x_j$. We can thus apply $\psi^{-j}$ to $q$. Obviously, $|\psi^{-j}(q)| = |q| \leq n$. The greatest subscript of $\psi^{-j}(q)$ equals $\mu(q) - j < \mu(q) - i \leq n$, where $i$ is the smallest subscript of $p$. Then we can apply Lemma 8 and find a word $v$ of length $O(n)$ such that some van Kampen diagram over $P_5$
has boundary equation \( \psi^{-j}(q) = v(x_0, x_1) \) and the area of this diagram is \( O(n^2) \). Adding \( j \) to all subscripts, we get a diagram \( \Delta \) over \( \mathcal{P}_5 \) of the same area with boundary equation \( q = v(x_j, x_{j+1}) \).

By definition, conjugation of \( x_j \) by \( p \) is also successful. By Lemma 3, there exists a smooth word \( t \) of rank \( j \), height \( m \) and a van Kampen diagram \( \Gamma \) over \( \mathcal{P}_5 \) with boundary equation \( p = t \) of area \( O(n^2) \). Besides, \( |t| = O(n) \). By definition of a smooth word, one can form a diagram \( \Xi_0 \) over \( \mathcal{P}_5 \) with boundary equation \( x_j = x_{j+m} \) that has \( |t| \) cells. Let us draw this diagram vertically, that is, \( t \) will be a vertical path. Take all horizontal edges and increase their subscripts by \( 1 \). This gives some diagram \( \Xi_1 \). Note that all defining relations used in \( \Xi_0 \) had difference 1 or 2. (By a difference of a defining relation of the form \( x_j = x_{j+1} \) we mean the number \( j - i \).) Thus in \( \Xi_1 \) these differences will be 2 or 3. Presentation \( \mathcal{P}_5 \) allows any difference from 1 to 5 so \( \Xi_1 \) will be a diagram over \( \mathcal{P}_5 \). The number of its cells is also \( |t| \).

Now for each \( d \in \{0, 1\} \) we have a diagram \( \Xi_d \) of \( |t| \) cells that has boundary equation \( x_j = x_{j+m+d} \). Thus for any word \( v(x_j, x_{j+1}) \) there exists a diagram of \( |t| \cdot |v| \) cells with boundary equation \( v(x_j, x_{j+1}) = v(x_{j+m}, x_{j+m+1}) \). Denote it by \( \Xi \). The number of cells in it is \( O(n^2) \) since both \( |t| \) and \( |v| \) are \( O(n) \).

Let us draw \( \Xi \) in such a way that paths labelled by \( t \) go upwards. Take two copies of \( \Gamma \) and attach them to \( \Xi \) along \( t \) on the left and on the right. The boundary equation of the result will have the form \( v(x_j, x_{j+1}) = v(x_{j+m}, x_{j+m+1}) \). Now let us attach \( \Delta \) on the bottom along the word \( v(x_j, x_{j+1}) \) (recall that its boundary equation was \( q = v(x_j, x_{j+1}) \)) and then attach the mirror copy of \( \psi^m(\Delta) \) on the top along the word \( v(x_{j+m}, x_{j+m+1}) \). The final result of all these operations will have boundary equation \( q^p = \psi^m(q) \), as desired. The area will be \( O(n^2) \) since it consists of 5 parts of area \( O(n^2) \). This shows that rectangular diagrams satisfy quadratic isoperimetric inequality.

So we reduced everything to Lemmas 6 and 7.

## 4 Horizontal Diagrams

We begin to prove Lemma 6. Take an arbitrary MP-word \( q \) such that \( |q| \leq N, \mu(q) \leq N \). We can write \( q = x_0^k x_1^{d_1} \cdots x_m^{d_m} \), where \( k \geq 0, d_1, \ldots, d_m \geq 1, 0 < i_1 < \cdots < i_m \). Recall that \( x_j \) equals \( x_j^{i_j} \) modulo \( 4 \) for any \( j \geq 1 \). If we replace each \( x_j \) \((j \geq 1)\) by \( x_j^{i_j} \) and cancel the result in the free group, then we get the following word \( Q \) over \( \{x_0^{\pm 1}, x_1^{\pm 1}\} \):

\[
Q = x_0^{k-i_1+1} x_1^{d_1} x_0^{-(i_2-i_1)} x_1^{d_2} \cdots x_0^{-(i_m-i_{m-1})} x_1^{d_m} x_0^{i_m-1}.
\]

The length of \( Q \) equals

\[
|Q| = |k - i_1 + 1| + (i_2 - i_1) + \cdots + (i_m - i_{m-1}) + (i_m - 1) + d_1 + \cdots + d_m.
\]

Since \( |k - i_1 + 1| \leq |k| + |i_1 - 1| = k + i_1 - 1 \), the length of \( Q \) does not exceed \( k + 2(i_m - 1) + d_1 + \cdots + d_m = \frac{|q| + 2(i_m - 1)}{3} \leq 3N \) since \( |q| \leq N \). We have \( i_m = \mu(q) \leq N \). Now our aim is to construct a diagram over \( \mathcal{P}_5 \) of area \( O(N^2) \) with boundary equation \( Q \) = \( q \). We can cancel
this equation by \( x_0^k \) on the left. That is, we are going to find the diagram for the following boundary equation:

\[
x_0^{-(i_1-1)}x_1^{d_1}x_0^{-(i_2-1)}x_1^{d_2} \cdots x_0^{-(i_m-1)}x_1^{d_m}x_0^{i_m-1} = x_1^{d_1} \cdots x_1^{d_m}.
\] (7)

The left-hand side of (7) is a product of two factors. The second factor is \( x_0^s \), where \( s = i_m - 1 \) is non-negative. The first factor involves \( x_0 \) in negative powers only. The number of occurrences of \( x_0^{-1} \) into the first factor equals exactly \( s \). This induces the following definition of a certain class of diagrams.

Let \( \Delta \) be a van Kampen diagram over \( \mathcal{P}_5 \) with boundary equation of the form \( y_1^{-1}uy_2 = z \), where \( u, z \) do not have occurrences of \( x_0^\pm \). Let \( y_1, y_2 \) be decomposed as \( y_j = x_0y_jx_0y_jx_0 \cdots y_{j,s-1}x_0y_{js} \), where \( j = 1, 2 \) and no \( x_0^\pm \) occur in any of the \( y_{ji} (1 \leq i \leq s) \). A diagram with these properties (together with the decomposition of its contour) will be called balanced. We will always draw \( u \) on the top, \( z \) on the bottom, \( y_1 (y_2) \) on the left (right).

A very elementary analysis of \( \Delta \) allows to conclude that it contains \( s \) subdiagrams that will be called \( x_0 \)-bands. An \( x_0 \)-band is a diagram with boundary equation of the form \( v^{x_0} = \psi(v) \), where \( v \) contains only letters \( x_0^{\pm h}, h = 1, 2, 3, 4, 5 \). This \( 0 \)-band consists of exactly \( |v| \) cells. The \( i \)th \( x_0 \)-band in \( \Delta \) must connect the \( i \)th occurrence of \( x_0 \) in \( y_1 \) with the \( i \)th occurrence of \( x_0 \) in \( y_2 \). Although these facts about bands are very easy to prove, we prefer not to do this in details because all balanced diagrams we deal with, will already contain the \( x_0 \)-bands with these properties. So we just include these properties of \( x_0 \)-bands into the definition of a balanced diagram. The top of the \( i \)th \( x_0 \)-band \((1 \leq i \leq s)\) will be denoted by \( u_i = u_i(x_1, \ldots, x_5) \). The bottom path of the \( i \)th band will thus be \( \psi(u_i) = u_i(x_2, \ldots, x_6) \).

Let us define a sequence of subdiagrams \( \Theta_0, \Theta_1, \ldots, \Theta_s \) as follows. Take all the \( x_0 \)-bands off \( \Delta \). The complement of the deleted bands will consist of \( s + 1 \) subdiagrams that will be enumerated from top to bottom. So \( \Theta_0 \) has boundary equation \( u = u_1 \). For each \( 1 \leq i < s \) the diagram \( \Theta_i \) is contained between the \( i \)th and the \((i+1)\)th \( x_0 \)-band. Its boundary equation is \( y_{1i}u_{i+1} = \psi(u_i)y_{2i} \). The boundary equation for \( \Theta_s \) is \( y_{1s}z = \psi(u_s)y_{2s} \).

Apply \( \psi^i \) to \( \Theta_{s-i} \) for each \( 0 \leq i \leq s \). Clearly, the bottom label of \( \psi^s(\Theta_0) \) equals \( \psi^s(u_1) = \psi^{s-1}(\psi(u_1)) \), which is the top label of \( \psi^{s-1}(\Theta_1) \). Analogously, the bottom label of \( \psi^{s-i}(\Theta_i) \) coincides with the top label of \( \psi^{s-1}(\Theta_{i+1}) \) for all \( 0 \leq i < s \). So we can glue diagrams \( \psi^s(\Theta_0), \psi^{s-1}(\Theta_1), \ldots, \psi(\Theta_s) \) together along the paths that have equal labels. The boundary equation of the result, which will be denoted by \( \Delta' \), has the form \( (y_1')^{-1}u'y_2' = z \), where \( u' = \psi^s(u), y'_j = \psi^{s-1}(y_{j1})\psi^{s-2}(y_{j2}) \cdots \psi(y_{js-1})y_{js} \) \((j = 1, 2)\) with \( z \) at the bottom. If we glue \( \Delta \) and the mirror image of \( \Delta' \) along the path labelled by \( z \), we get the diagram \( \bar{\Delta} \) with boundary equation of the form \( L = R \), where

\[
L = y_{1s}^{-1}x_0^{-1} \cdots x_0^{-1}y_{11}^{-1}x_0^{-1}ux_0y_{21}x_0 \cdots x_0y_{2s},
R = y_{11}^{-1}\psi(y_{1,s-1})^{-1} \cdots \psi^{s-1}(y_{11})^{-1}\psi(u)\psi^{s-1}(y_{21}) \cdots \psi(y_{2,s-1})y_{2s}.
\] (8)

Recall that applying \( \psi^k \) means increasing all subscripts by \( k \). The word \( R \) in (8) is the word that can be obtained as follows. Take the product \( y_1^{-1}uy_2 \), delete all occurrences of \( x_0^\pm \) and then increase subscript on each letter by some number, which is equal to the number of \( x_0 \)-bands that are contained in \( \Delta \) below the corresponding occurrence of the same letter in \( \Delta \). This rule is quite easy to apply.
If $\Delta$ is a balanced diagram, then we will say that $\bar{\Delta}$ is the result of the collecting process. Obviously, the area of $\bar{\Delta}$ does not exceed the area of $\Delta$ multiplied by 2.

Recall that we wanted to find a diagram with boundary equation (4). Suppose that we found a (balanced) van Kampen diagram $\Delta$ over $\mathcal{P}_5$ of area $O(N^2)$ that has boundary equation of the form

$$x_0^{-(i_1-1)}x_1^{-d_1}x_0^{-(i_2-i_1)}x_1^{-d_2} \cdots x_0^{-(i_m-i_{m-1})}x_1^{-d_m}x_0^{i_{m-1}} = z,$$

where $z$ is any word without occurrences of $x_0^{\pm 1}$. Let us explain how to get (7) from there. We have a balanced diagram $\bar{\Delta}$ with boundary equation $y_1^{-1}uy_2 = z$, where $u = x_1^{d_m}$, $s = i_m - 1$, $y_2 = x_0^n$, $y_1 = x_0^{i_m-i_{m-1}}x_1^{-d_m} \cdots x_1^{-d_1}x_0^{i_1}$. Collecting process applied to $\Delta$ leads to the diagram $\bar{\Delta}$ that has boundary equation $y_1^{-1}uy_2 = (y_j')^{-1}u'y_2$. The word $y_j'$ ($j = 1, 2$) is obtained from $y_j$ by deleting all the $x_0$’s and then increasing subscripts on some letters. Obviously, $y_j'$ is empty whereas each occurrence of $x_1$ into $y_j'$ is replaced by $x_1+d$, where $d$ is the number of the $x_1$’s that come after the given occurrence of $x_1$ in $y_1$. The word $u = x_1^{d_m}$ increases all its subscripts by $s = i_m - 1$ so it becomes $u' = x_1^{d_m}$ (the subscript 1 must be increased here by $i_m - 1 = (i_m - i_{m-1}) + \cdots + (i_1 - 1)$). Analogously, $x_1^{-d_m-1}$ becomes $x_1^{-d_m}$, and so on. As a result, $(y_1')^{-1}u' = x_1^{d_1} \cdots x_1^{d_m}$. So the boundary equation of $\bar{\Delta}$ is exactly (7). The area of $\bar{\Delta}$ is also $O(N^2)$.

The above reasons show that to prove Lemma 8, it just remains to find a suitable diagram with boundary equation (4). This will be done due to the next Lemma, which is a crucial point of our proof.

**Lemma 8** There exist positive integer constants $C_1$, $C_2$, and $D$ such that for any sequence of integers $\alpha_{-n}$, \ldots, $\alpha_{-1}$, $\alpha_0$, $\alpha_1$, \ldots, $\alpha_n$, there exists a word $w = w(x_1, \ldots, x_5)$ and a van Kampen diagram $\Delta$ over $\mathcal{P}_5$ with boundary equation

$$(x_0x_1^{\alpha_{-1}} \cdots x_0x_1^{\alpha_{-n}})^{-1}x_1^{\alpha_0}(x_0x_1^{\alpha_1} \cdots x_0x_1^{\alpha_n}) = w(x_1, \ldots, x_5)$$

and the following conditions hold:

a) $|w| \leq S + Dn$,
b) the area of $\Delta$ does not exceed $(C_1S + C_2n)n$, where $S = \sum_{i=-n}^{n} |\alpha_i|$.

To get (4), it suffices to take $n = i_m - 1$. Clearly, $n < \mu(q) \leq N$. The $\alpha_i$’s will be either zero, or they are equal to $d_m$, $-d_{m-1}$, \ldots, $-d_1$. So $S = d_1 + \cdots + d_m \leq |q| \leq N$. Thus the area of $\Delta$ will be $O(N^2)$, as desired. So Lemma 8 is now finally reduced to Lemma 8.

**Proof of Lemma 8.** We proceed by induction on $n$. First of all let us construct $\Delta$ for the case $0 \leq n \leq 4$. Consider the following diagram:
For each $0 \leq n \leq 4$, we take the corresponding subdiagram with bottom label $w(x_1, \ldots, x_5) = (x_n^{\alpha n-1} \cdots x_1^{\alpha n})^{-1} x_0^{\alpha 0} (x_n^{\alpha n} \cdots x_1^{\alpha n})$. Clearly, $|w| = \sum_{i=-n}^{n} |\alpha_i| = S$. The area of $\Delta$ obviously does not exceed $nS$.

From now suppose that $n \geq 5$. Let us define integers $k, l, m$ in the following way:

\[ k = l = [(n - 4)/3], \quad m = n - k - l - 4. \]

Clearly, $m \geq (n - 4)/3 > 0$. We also have $m \leq n - 2(n - 6)/3 - 4 = n/3$ since $k = l \geq (n - 4)/3$. So the following inequalities hold:

\[ 0 \leq k, l, m \leq \frac{n}{3}, \quad 2k^2 + 4l^2 + 2m^2 \leq \frac{8}{9} n^2. \] (10)

Since $0 \leq k < n$, we can construct a word $u(x_1, \ldots, x_5)$ and a diagram $\Delta_1$ with boundary equation

\[ (x_0x_1^{\alpha_1} \cdots x_0x_1^{\alpha_k})^{-1} x_0x_1^{\alpha_0} (x_0x_1^{\alpha_1} \cdots x_0x_1^{\alpha_k}) = u(x_1, \ldots, x_5). \] (11)

Let $S_1$ denote $\sum_{i=-(k+1)}^{k+1} |\alpha_i|$. (We included the cases $i = \pm (k + 1)$ for convenience.) By the inductive assumption,

\[ |u| \leq S_1 + Dk, \quad \#\Delta_1 \leq (C_1 S_1 + C_2 k)k. \] (12)

Here $\#$ denotes the area of a diagram over $P_5$.

Now let us define 4 new sequences of integers in the following way:

\[ \beta_i = \alpha_{-(k+i+2)}, \quad \gamma_i = \alpha_{k+i+2} \quad (0 \leq i \leq l + 1), \]

and let $\beta_i = \gamma_i = -1$ for all $-l \leq i < 0$. Analogously,

\[ \delta_i = \alpha_{-(k+l+i+4)}, \quad \epsilon_i = \alpha_{k+l+i+4} \quad (0 \leq i \leq m), \]

and let $\delta_i = \epsilon_i = -1$ for all $-m \leq i < 0$.

We will construct 4 diagrams, one for each of these sequences. They will be denoted by $\Delta_2, \Delta_3, \Delta_4, \Delta_5$. Let us describe each of these diagrams precisely, showing its boundary equation and estimating lengths and areas.

\(^1\)During the proof of this Lemma, by a diagram we will always mean a van Kampen diagram over $P_5$. 

15
Let and separately. Let

\[ S \]

respectively. Let

\[ \bar{u} \]

By definition,

\[ (u_0(x_1, x_2) = x_1^{-\alpha-k+1} u_0(x_1, x_2) x_1^{\alpha+1}), \]

\[ \bar{u}'_0(x_1, x_2) = u'_0(x_1, x_2) x_1^{\beta+1}, \]

\[ \bar{u}''_0(x_1, x_2) = u''_0(x_1, x_2) x_1^{\gamma+1}. \]

By definition,

\[ w(x_1, \ldots, x_5) = u'(x_1, \ldots, x_5)^{-1} \bar{u}'_0(x_2, x_3)^{-1} \bar{u}_0(x_4, x_5) \bar{u}''_0(x_2, x_3) w''(x_1, \ldots, x_5). \]

Each of the diagrams \( \Delta_j \) (1 \( \leq \) j \( \leq \) 5), \( \Delta \) will be drawn in such a way that words \( u \), \( w \) (with or without bars and/or dashes) will be on the bottom. The top will be always labelled by \( x_1 \) to a power with zero subscript (directed from the left to the right). The left (right) side of a diagram will correspond to the part of its boundary, which involves exponents with negative (positive) subscript. If \( \Gamma \) is a diagram, then \( \Gamma^{-1} \) is a mirror copy of \( \Gamma \) with respect to a horizontal axis symmetry. For a symmetry with respect to a vertical axis, we will use the notation \( -\Gamma \). Recall also that the operation \( \psi \) applied to a word or a diagram increases all subscripts on the \( x_i \)'s by 1.

The diagram \( \Delta_2 \) has boundary equation

\[ (x_0 x_1^{-1} \cdots x_0 x_1^{-1})^{-1} x_1^{\delta_0} (x_0 x_1^{\delta_1} \cdots x_0 x_1^{\delta_l}) = u'(x_1, \ldots, x_5). \]

Let \( S_2 \) denote \( \sum_{i=l}^{l+1} |\beta_i| = l + \sum_{i=0}^{l+1} |\beta_i| \). (We also included the case \( i = l + 1 \) for convenience.) By the inductive assumption,

\[ |u'| \leq S_2 + Dl, \quad \#\Delta_2 \leq (C_1 S_2 + C_2 l) l. \]

The diagram \( \Delta_3 \) is defined similarly. It has boundary equation

\[ (x_0 x_1^{-1} \cdots x_0 x_1^{-1})^{-1} x_1^{\gamma_0} (x_0 x_1^{\gamma_1} \cdots x_0 x_1^{\gamma_l}) = u''(x_1, \ldots, x_5). \]

If \( S_3 \) denotes \( l + \sum_{i=0}^{l+1} |\gamma_i| \), then inequalities

\[ |u''| \leq S_3 + Dl, \quad \#\Delta_3 \leq (C_1 S_3 + C_2 l) l \]

hold, as above.

Finally, \( \Delta_4 \) and \( \Delta_5 \) have boundary equations

\[ (x_0 x_1^{-1} \cdots x_0 x_1^{-1})^{-1} x_1^{\delta_0} (x_0 x_1^{\delta_1} \cdots x_0 x_1^{\delta_m}) = w'(x_1, \ldots, x_5) \]

and

\[ (x_0 x_1^{-1} \cdots x_0 x_1^{-1})^{-1} x_1^{\gamma_0} (x_0 x_1^{\gamma_1} \cdots x_0 x_1^{\gamma_m}) = w''(x_1, \ldots, x_5), \]

respectively. Let \( S_4 = m + \sum_{i=0}^{m} |\delta_i| \), \( S_5 = m + \sum_{i=0}^{m} |\epsilon_i| \). Then the following estimates hold:

\[ |w'| \leq S_4 + Dm, \quad \#\Delta_4 \leq (C_1 S_4 + C_2 m) m, \]

\[ |w''| \leq S_5 + Dm, \quad \#\Delta_5 \leq (C_1 S_5 + C_2 m) m. \]
Now we shall describe how the diagram $\Gamma$ is constructed. First we draw the diagram $\Delta_1$. Its bottom is labelled by $u(x_1, \ldots, x_5)$. Conjugating this word by $x_0$, gives us an $x_0$-band that consists of $|u|$ cells. Let us attach this band to $\Delta_1$. The bottom path of the result is the word $u(x_2, \ldots, x_6)$. One can then attach $\psi(\Delta_1)^{-1}$ to it. We get the word $u_0(x_1, x_2)$ on the bottom. Multiplying the bottom word from both sides by suitable powers of $x_1$, we get the word $u_0(x_1, x_2)$. Now we conjugate it by $x_0$ and attach the corresponding $x_0$-band. The word $u_0(x_2, x_3)$ will appear on the bottom of the result.

Note that each of the letters $x_2, x_3, x_4, x_5$ commutes with $x_0 x_1^{-1}$. The diagram of this commutativity consists of exactly 2 cells of the presentation $P_5$. So we can conjugate $u_0(x_2, x_3)$ by $(x_0 x_1^{-1})^t$ attaching the diagram of commutativity. Then the same word $u_0(x_2, x_3)$ appears on the bottom. (The area of the diagram we have attached is exactly $2t|u_0|$.) Now the word $(x_0 x_1^{-1})^t$ appears on the left and on the right. We are going to attach to it the diagram
\(-\Delta_2\) on the left and \(\Delta_3\) on the right in a natural way. After that, the word on the bottom becomes equal to \(u'(x_1, \ldots, x_5)^{-1}\bar{u}_0(x_2, x_3)u''(x_1, \ldots, x_5)\). Conjugate this word by \(x_0\) and attach an \(x_0\)-band. The bottom word will be the image of the previous word under \(\psi\), that is, \(u'(x_2, \ldots, x_6)^{-1}\bar{u}_0(x_3, x_4)u''(x_2, \ldots, x_6)\). Now we take \(\psi(\Delta_2)^{-1}\) and \(\psi(\Delta_3)^{-1}\), attaching them to the subwords \(u', u''\), respectively. If we multiply the result on the left and on the right by suitable powers of \(x_1\), then we get exactly the word \(\bar{u}_0'(x_1, x_2)^{-1}\bar{u}_0(x_3, x_4)\bar{u}_0''(x_1, x_2)\). Adding an \(x_0\)-band to the bottom, we increase all subscripts by 1. Then we get a word in \(x_2, \ldots, x_5\). It commutes with \((x_0x_1^{-1})^m\) and we add the corresponding diagram of commutativity. After we attach \(-\Delta_4\) on the left and \(\Delta_5\) on the right (gluing along words labelled by \((x_0x_1^{-1})^m\)), we get the desired diagram \(\Delta\). Its bottom path will be labelled by \(w(x_1, \ldots, x_5)\), which follows from (23). The boundary equation will be exactly as in the statement, which follows from the definition of \(\beta\)'s, \(\gamma\)'s, \(\delta\)'s, and \(\epsilon\)'s.

Let us estimate the length of \(|w|\). From (23) we see that \(|w| = |w'| + |w''| + |\bar{u}_0| + |\bar{u}_0'| + |\bar{u}_0''|\). Using (19), (20), (21), (22), and taking (10) into account, we obtain that \(|w| \leq (S_1 + Dm) + (S_5 + Dm) + (S_1 + 2k) + (S_2 + 2l) + (S_3 + 2l) = 2Dm + (S_1 + \cdots + S_3) + 2k + 4l = 2S_0 + \sum_{n=1}^{4} |\xi| + 2l + 2m = S + 2Dm + (2k + 6l + 2m) \leq S + 2Dm + (2k + 6l + 2m) \leq S + 2Dm + 30n/3 \leq S + Dm \) provided \(D \geq 10\). (We also refer to the definition of the \(S_j\)'s.) So estimate a) holds.

Now we are going to estimate the area of \(\Delta\) in the same way. First of all, we mention that \(\Delta\) consists of two copies of each of the \(\Delta_j, j = 1, 2, 3\) and one copy of each of the \(\Delta_j, j = 4, 5\). Also \(\Delta\) has three \(x_0\)-bands between \(\Delta_j\) and its mirror copy \((j = 1, 2, 3)\). The sum of areas of these bands clearly equals \(|u| + |u'| + |u''|\). The rest of \(\Delta\) can be divided into 3 rectangles. The square of each of these rectangles will be the product of its sides. The first, central, rectangle will have one side of length \(|\bar{u}_0|\) (width) and another side of length \(2l + 2m + 3\) (height). The two other rectangles have width \(|\bar{u}_0|, |\bar{u}_0''|\), respectively and the same height \(2m + 1\). Note that each of the rectangles indeed consists of small squares \(1 \times 1\), where each square represents one defining relation of \(\mathcal{P}_6\). (This explains, by the way, why we prefer to deal with defining relations of the form \(x_j^i = x_{j+i}\), where \(0 < j - i \leq 5\)). Therefore, the area of \(\Delta\) equals

\[2(\#\Delta_1 + \#\Delta_2 + \#\Delta_3) + \#\Delta_4 + \#\Delta_5 + |u| + |u'| + |u''| + |\bar{u}_0|(2l + 2m + 3) + (|u_0'| + |\bar{u}_0''|)(2m + 1).\]

Note that \(|u| \leq S_1 + Dk, |u'| \leq S_2 + Dl, |u''| \leq S_3 + Dm\) by (12), (13), (10). Also recall that \(|\bar{u}_0| = S_1 + 2k, |\bar{u}_0'| = S_2 + 2l, |\bar{u}_0''| = S_3 + 2l\). We will also use the equality \(S_1 + \cdots + S_4 = S + 2l + 2m\), as we did before. Taking into account the estimates of areas of the \(\Delta_j\)'s, we finally get \(\#\Delta \leq C_1(2S_1k + 2S_2l + 2S_3l + S_4m + S_5m) + C_2(2k^2 + 4l^2 + 2m^2) + S_1 + S_2 + S_3 + D(k + 2l) + (S_1 + 2k)(2l + 2m + 3) + (S_2 + S_3 + 4l)(2m + 1)\). This estimate for the area of \(\Delta\) can be rewritten as

\[C_2(2k^2 + 4l^2 + 2m^2) + S_1(2kC_1 + 2l + 2m + 4) + (S_2 + S_3)(2lC_1 + 2m + 2) + (S_4 + S_5)C_1m + P,\]

where \(P = D(k + 2l) + 2k(2l + 2m + 3) + 4l(2m + 1) \leq Dn + 16n^2/9 + 10n/3\). It follows from the definition of \(k, l, m\) that \(l + m + 2 = n - k + 2 \leq n - (n - 6)/3 + 2 \leq 2n/3\). We also know that \(k \leq n/3\) so the coefficient on \(S_1\) does not exceed \(2nC_1/3 + 4n/3 \leq C_1n\) provided \(C_1 \geq 4\). All coefficients on the other \(\Delta_j\)'s satisfy the same inequality. From (10) we also know that the coefficient on \(C_2\) does not exceed \(8n^2/9\). Hence the area of \(\Delta\) does not exceed \(8C_2n^2/9 + (S_1 + \cdots + S_5)C_1n + P = 8C_2n^2/9 + C_1Sn + (2l + 2m)C_1n + P \leq (C_1S + C_2n)n\).
provided \(2C_1(l + m)n + P \leq C_2n^2/9\). So it remains to mention that

\[
9 \frac{2(l + m)C_1n + P}{n^2} \leq 12C_1 + \frac{9D + 30}{n} + 16 \leq 88
\]

if we choose \(D = 10\), \(C_1 = 4\) (recall that \(n \geq 5\)). Thus one can choose \(C_2 = 88\) to make the proof of part b) complete.

**Remark.** We will need a modification of Lemma 8 to prove the Lemma about vertical diagrams. In this case we will need some flexibility to choose \(k, l, m\). Namely, we need to be able later to make a choice of \(k, l, m\) from some interval of length \(\xi n\), where \(\xi\) is a positive constant. Let us show how to modify inequalities (10) for our purposes. Suppose that there exist some (small) positive constants \(\eta, \zeta\) such that inequalities

\[
0 \leq k, l, m \leq \left(\frac{1}{2} - \eta\right)n, \quad 2k^2 + 4l^2 + 2m^2 \leq (1 - \zeta)n^2 \tag{24}
\]

hold. (In (10), we had \(\eta = 1/6\), \(\zeta = 1/9\).) Now for the length of \(w\) we will have \(|w| \leq S + Dn\) if \(2Dm + 2k + 6l + 2m \leq Dn\). This obviously holds if \(D\) is big enough, say, if \(D \geq 5/2\eta\). So a) holds.

It is also easy to find out what happens with b). Indeed, to choose \(C_1\) it just suffices to make each coefficient on the \(S_j\)'s not to exceed \(C_1n\). For the coefficient on \(S_1\) we have \(2kC_1 + 2l + 2m + 4 \leq (1 - 2\eta)nC_1 + 3n \leq nC_1\) if \(C_1 \geq 3/2\eta\). All the other coefficients satisfy the same inequality. Finally, for the choice of \(C_2\) we need \(2C_1(l + m)n + P \leq C_2n^2\zeta\). Since \(P\) can be roughly estimated as \(4n^2 + (3D/2 + 5)n < (3D/10 + 5)n^2\), it suffices to choose some \(C_2\) with \(C_2 \geq \zeta^{-1}(2C_1 + 3D/10 + 5)\).

## 5 Vertical Diagrams

The aim of this Section is to prove Lemma 7. We are going to discuss first how vertical diagrams may look like.

Let \(q = x_j, p = x_{i_1}x_{i_2} \cdots x_{i_m}\), where \(j + m \leq n, i_1 \leq i_2 \leq \cdots \leq i_m\). Suppose that conjugation of \(q\) by \(p\) is successful, that is, for any \(0 \leq s < m\), it holds \(j + s > i_{s+1}\). It is easy to construct a diagram over \(\mathcal{P}_5\) of \(m\) cells with boundary equation \(x_j^p = x_{j+m}\). Indeed, for any \(0 \leq s < m\), we can conjugate \(x_{j+s}\) by \(x_{i_{s+1}}\), which gives \(x_{j+s+1}\). All these cells can be concatenated together. As a result, we get a diagram with boundary equation \(x_j^p = x_{j+m}\).

Consider a diagram \(\Delta\) over \(\mathcal{P}_5\) with the same boundary equation. Let \(x_j\) be written on the top and let \(x_{j+m}\) be written on the bottom. We want \(\Delta\) to have \(m\) subdiagrams \(\Gamma_1, \ldots, \Gamma_m\). For each \(1 \leq s \leq m\), the diagram \(\Gamma_s\) is an \(x_{i_s}\)-band. It “connects” the \(s\)th letter on the left side of \(\Delta\) with the \(s\)th letter on the right side (these letters have the same label \(x_{i_s}\)). The top path of \(\Gamma_s\) must have label of the form \(v_s(x_{i_{s+1}}, \ldots, x_{i_{s+5}})\). The bottom label of \(\Gamma_j\) will thus be the image of this word under \(\psi\), that is, \(v_s(x_{i_{s+2}}, \ldots, x_{i_{s+6}})\). It is convenient to define also diagrams \(\Gamma_0\) and \(\Gamma_{m+1}\). Each of them consists of one edge, the top edge and the bottom edge of \(\Delta\), respectively. We also let \(i_{m+1} = j + m - 1\) by definition.
For each \( 0 \leq s \leq m \), by \( \Theta_s \) we denote the subdiagram of \( \Delta \) that is contained between \( \Gamma_s \) and \( \Gamma_{s+1} \). For any \( i \geq 0 \) we will denote by \( \bar{x}_i \) the vector \( (x_{i}, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}) \). Our aim is to find a way to define the words \( v_s \) and diagrams \( \Theta_s \). The boundary equation for \( \Theta_s \) must be of the form \( v_s(\bar{x}_{i+2}) = v_{s+1}(\bar{x}_{i+1}+1) \) for \( 1 \leq s \leq m \) (the left-hand side always means the top label). As for \( \Theta_0 \), its boundary equation is \( x_j = v_1(\bar{x}_{i+1}) \).

Take any diagram \( \Xi \) with boundary equation \( x_0^n x_1 x_0^n = z \), where \( x_1 \) is written on the top, \( z \) is a word without occurrences of \( x_0^{\pm 1} \) written on the bottom. This diagram must contain \( x_0 \)-bands (recall that this fact can be proved directly but we just assume this because in our situation the diagram will automatically have this feature). The \( i \)th occurrences of \( x_0 \) into \( x_0^n \) written on the left side and on the right side of the diagram. Clearly, the top path of the \( i \)th band can be denoted by \( u_i(\bar{x}_1) \) for some word \( u_i \). The bottom path of the same band will be \( u_i(\bar{x}_2) \). For any \( 1 \leq i < n \), there exists a subdiagram \( T_i \) in \( \Xi \) that is contained between the \( i \)th and the \( (i + 1) \)th \( x_0 \)-band. Its boundary equation is \( u_i(\bar{x}_2) = u_{i+1}(\bar{x}_1) \). We call it (the \( i \)th) transition diagram. One can say that \( T_i \) converts a word in \( x_2, x_3, \ldots \) into a word in \( x_1, x_2, \ldots \). If \( T_i \) and \( T_{i+1} \) (\( 1 \leq i \leq n - 2 \)) are consecutive transition diagrams, then one can concatenate \( \psi(T_i) \) and \( T_{i+1} \) identifying the bottom path of the first diagram and the top path of the second diagram. The paths we identify obviously have the same label \( u_{i+1}(\bar{x}_2) \). The result will be a diagram with boundary equation \( u_i(\bar{x}_3) = u_{i+2}(\bar{x}_1) \). This diagram will be denoted as a product, that is, \( \psi(T_i)T_{i+1} \). One can also form multiple concatenations adjusting them by applying a suitable power of \( \psi \). So we can define products of the form

\[
\psi^k(T_i)\psi^{k-1}(T_{i+1}) \cdots \psi^{k-r}(T_{i+r})
\]

for any \( i, k, r \geq 1 \) such that \( i + r < n, k \geq r \). The boundary equation of the result will be \( u_i(\bar{x}_{k+2}) = u_{i+r+1}(\bar{x}_{k-r+1}) \). This diagram converts a word in \( x_{k+2}, x_{k+3}, \ldots \) into a word in \( x_{k-r+1}, x_{k-r+2}, \ldots \). Thus transition diagrams can be used to define diagrams \( \Theta_s \). We go with the following definitions. Let

\[
\Theta_0 = \psi^{j-2}(T_1) \cdots \psi^{i_1}(T_{j-i_1-1})
\]

(if \( j = 1 \), then \( i_1 = 0 \) so the product is empty, that is, it is a diagram without cells). The boundary equation for \( \Theta_0 \) is

\[
\begin{equation}
\tag{25}
u_1(\bar{x}_j) = u_{j-i_1}(\bar{x}_{i_1+1}),
\end{equation}
\]

where the left-hand side is just \( x_j \).

For any \( 1 \leq s \leq m \) we let \( \Theta_s = \psi^{i_s}(T_{j-i_s+s-1}) \) if \( i_{s+1} = i_s \). We also let \( \Theta_s \) be empty in the case \( i_{s+1} = i_s \). If \( i_{s+1} - i_s \geq 2 \), then we let

\[
\Theta_s = (\psi^{i_{s+1}-1}(T_{j-i_{s+1}+s}) \cdots \psi^{i_{s}+1}(T_{j-i_{s}+s-2}))^{-1}.
\]

For any \( i_{s+1} \geq i_s \) the diagram \( \Theta_s \) will have boundary equation

\[
\begin{equation}
\tag{26}
u_{j-i_s+s-1}(\bar{x}_{i_s+2}) = u_{j-i_{s+1}+s}(\bar{x}_{i_{s+1}+1}).
\end{equation}
\]

So given a diagram \( \Xi \), we construct the diagram \( \Delta \) in the way we have described. (Note that \( \Delta \) depends of \( p, q, \) and \( \Xi \).) To find out what is the area of \( \Delta \), we need to define some
integers $\kappa_1, \ldots, \kappa_{n-1}$. Namely, we let $\kappa_i = k$ whenever $T_i$ or its image under a power of $\psi$ occurs exactly $k$ times in the $\Theta_s$’s ($0 \leq s \leq m$). We say that $\kappa_i$ is the intensity of $T_i$. It is worth noting that the numbers $\kappa_i$ depend of $p$ and $q$ but they are independent on the choice of $\Xi$. Suppose that all words $u_i$ (1 $\leq i \leq n$) in the diagram $\Xi$ have length $O(n)$. Then each $\Gamma_s$ ($1 \leq s \leq m$) has area $O(n)$ since $\#\Gamma_s = |v_s| = O(n)$ (here $v_s$ is some of the $u_i$’s). The sum of areas of all the $\Gamma_s$’s will be $O(n^2)$ since $m \leq n$. When we take the sum of areas of all the $\Theta_s$ (0 $\leq s \leq m$), then it will be the sum of areas of the $T_i$’s, where each $T_i$ occurs exactly $\kappa_i$ times. So

$$\#\Delta = \sum_{i=1}^{n-1} \kappa_i \#T_i + O(n^2).$$  

(27)

Let us estimate $\sum_{i=1}^{n-1} \kappa_i$. It is the total number of factors of the form $T_i$ (1 $\leq i < n$) for all the $\Theta_s$’s. By definition, $\Theta_0$ has $j - i_1 + 1$ factors and each $\Theta_s$ for 1 $\leq s \leq m$ has $|i_{s+1} - i_s - 1|$ factors. So

$$\sum_{i=1}^{n-1} \kappa_i = j - i_1 + 1 + \sum_{i=1}^{n-1} |i_{s+1} - i_s - 1|. \tag{28}$$

Since $|i_{s+1} - i_s - 1| \leq i_{s+1} - i_s + 1$, we can estimate (28) as $j + n + i_{m+1} < 2j + m + n \leq 3n$. So

$$\sum_{i=1}^{n-1} \kappa_i = O(n). \tag{29}$$

The diagram $\Xi$ can be easily constructed by Lemma 8. However, it is clear that diagrams $T_i$ may sometimes have relatively big area. Although the area of the whole diagram $\Xi$ will be $O(n^2)$, for some values of $i$ the area of $T_i$ may exceed $cn^2$ for some constant $c > 0$. Also we have no guarantee that $\kappa_i$ will be small. It may happen that $\kappa_i > dn$ for some constant $d > 0$. So we never get the estimate $O(n^2)$ if we choose $\Xi$ arbitrarily. To avoid this difficulty, we need to construct $\Xi$ in such a way that if $\kappa_i$ is “big” then the area of $T_i$ is “small”. This is always possible by the following Lemma.

Lemma 9. There exist constants $C, C_0, D > 0$ such that for any integers $\alpha_i$, where $|\alpha_i| \leq 1$ for all $-n \leq i \leq n$ and for any non-negative integers $\kappa_i$ (1 $\leq i \leq n$), there exists a van Kampen diagram $\Delta'$ over $P_5$ with boundary equation of the form

$$L^{-1}x_1^{\alpha_1}R = w(\bar{x}_1),$$

where $L = x_0x_1^{\alpha_1} \cdots x_0x_1^{\alpha_n}$, $R = x_0x_1^{\alpha_1} \cdots x_0x_1^{\alpha_n}$ with the following properties:

a) $\Delta'$ contains $x_0$-bands $\Delta_i'$ (1 $\leq i \leq n$), where $\#\Delta_i' \leq Dn$ for all 1 $\leq i \leq n$. The $i$th $x_0$-band connects the $i$th occurrence of $x_0$ into $L$ with the $i$th occurrence of $x_0$ into $R$. Besides, $|w| \leq Dn$.

b) $\#\Delta' \leq C_0 n^2$.

c) Let $\Phi_i$ (1 $\leq i \leq n$) denote the subdiagram in $\Delta'$ that is contained between $\Delta_i'$ and $\Delta_i'+1$, where $\Delta_{n+1}'$ denotes the bottom path of $\Delta'$ labelled by $w(\bar{x}_1)$. Then

$$\sum_{i=1}^{n} \kappa_i \#\Phi_i \leq C \sum_{i=1}^{n} \kappa_i \cdot n.$$
First of all, let us show that Lemma 3 implies what we need. Take \( L = x_0(x_0x_1^{-1})^{n-1} \), \( R = x_0^n \), \( \alpha_0 = 0 \) and consider the diagram \( \Delta' \) from Lemma 3. (The integers \( \kappa_i \) for \( 1 \leq i < n \) are defined by words \( p, q \) from the statement of Lemma 4 for \( \kappa_n \) we can take 0.) The diagram \( \Delta' \) has boundary equation \( Lw = R \). It is easy to see that \( L \) is a smooth word of rank 1. Conjugating \( x_1 \) by \( L \), we get \( x_2 \). There exists a natural van Kampen diagram over \( \mathcal{P}_5 \) with boundary equation \( x_1^2 = x_2 \). It has \( |L| = 2n - 1 \) cells. We can attach two copies of \( \Delta' \) to it. This gives a diagram with boundary equation \( x_1^{2^m} = x_2^m \). Denote this diagram by \( \Xi \). As above, we find \( x_0 \)-bands in \( \Xi \) and define transition diagrams \( T_i \) (1 \( \leq i < n \)). It is now easy to describe how they look like. Let \( w_i(x_i) \) be the top label of \( \Delta'_i \) for each \( 1 \leq i \leq n \). Then \( \Phi_i \) (1 \( \leq i < n \)) has boundary equation \( x_1w_i(x_2) = w_{i+1}(x_1) \). It is easy to see that \( T_1 \) consists of just one edge labelled by \( x_2 \). For each \( 1 \leq i < n \), \( T_i \) has boundary equation \( w_i(x_2) = w_{i+1}(x_1)x_2w_{i+1}(x_1) \). This diagram consists of one cell of the form \( x_3 = x_1^{-1}x_2x_1 \) and two copies of \( \Phi_i \) from both sides:

\[
\begin{array}{c}
\text{\( w_i(x_2, \ldots, x_6) \)} \\
\Phi_i \\
\text{\( x_1 \)} \\
\Phi_i \\
\text{\( w_{i+1}(x_1, \ldots, x_5) \)}
\end{array}
\]

So by (27) the area of \( \Delta \) (constructed by \( p, q, \Xi \)) does not exceed
\[
\sum_{i=1}^{n-1} \kappa_i (2#\Phi_i + 1) + O(n^2) \leq \sum_{i=1}^{n-1} \kappa_i (2Cn + 1) + O(n^2) = O(n^2)
\]

by part c) of Lemma 3 and (29).

To prove Lemma 7, it suffices to define the smooth path \( t \) from the statement and establish some of its properties. Each diagram \( T_i \) can be naturally decomposed into three parts. In our case, \( T_i \) for \( 1 \leq i < n \) consists of the central part which has one cell and two other parts, the left part and the right part, which are mirror copies of each other (and are equal to \( \Phi_i \)). The diagram \( T_i \) consists of its central part only (the left part and the right part of it are empty). When we concatenate some images of the \( T_i \)'s under a power of \( \psi \), the central parts of the factors are concatenated too. So each \( \Theta_s \) (0 \( \leq s \leq m \)) can be naturally decomposed into three parts as well. By \( t_s \) (0 \( \leq s \leq m \)) we denote the word written on the left and the right side of the central part of \( \Theta_s \) (these sides must have the same label by definition).

For any diagram that is decomposed naturally into the three parts, we can define its thickness as the length of the word written on the left and the right side of the central part. All our transition diagrams \( T_i \) (1 \( \leq i < n \)) have thickness at most one. So the thickness of \( \Theta_s \) does not exceed the sum of its factors for any 0 \( \leq s \leq m \). This implies that \( \sum_{s=0}^{m} |t_s| \) does not exceed \( \sum_{i=1}^{n-1} \kappa_i \), which is \( O(n) \) by (29). To decompose \( \Delta \) into three parts, one need to find this decomposition for each of the bands \( \Gamma_s \) (1 \( \leq s \leq m \)). Each word \( v_s \) (1 \( \leq s \leq m \)) is one of the \( u_i \)'s so it has a “central” letter. So the band \( \Gamma_s \) must have a “central” cell. We now can define the word \( t \) as
\[
t = t_0x_{i_1}t_1x_{i_2} \cdots x_{i_m}t_m.
\]

The diagram \( \Delta \) decomposes into three parts, where the central part \( B \) has boundary equation \( t^{-1}x_jt = x_{j+m} \) and the two other parts are mirror copies of each other and they have boundary
equations $q = t$. Since $|t| = \sum_{s=0}^{m} |t_s| + m = O(n) + m = O(n)$, the only thing we need to check is that $t$ is a smooth word of rank $j$, height $m$. (It has been already shown that the area of $\Delta$ is $O(n^2)$ so the same is true for the area of the subdiagram of $\Delta$ with boundary equation $q = t$.)

It is clear that $B$ consists of exactly $|t|$ cells. The boundary equation of each of them has the form $x_k^{t_i} = x_{k+1}$ for some $k > i$. The letters $x_i$ here belong to the left and the right side of $B$. So what we really have to prove is the alternative $k = i + 1$ or $k = i + 2$. Each letter of $t$ is associated with exactly one cell so we need to establish this alternative for each letter of $t$. Each cell that corresponds to a letter from any of the $t_s$ $(0 \leq s \leq m)$ obviously has this property because it is an image of a central cell of some transition diagram under a power of $\psi$. But we know that any defining relation for a central cell of a transition diagram is $x_k^{x_i} = x_{k+1}$, where $k = i + 1$. So we need to check out only the cells of $B$ that correspond to letters $x_i$ ($1 \leq s \leq m$).

Recall that by $u_i(x_1, \ldots, x_5)$ we denoted the top path of the $i$th $x_0$-band of $\Xi$. The central letter of $u_1$ is $x_1$; for all $1 < i \leq n$ the central letter of $u_i(x_1, \ldots, x_5)$ is $x_2$. Now we can look at the boundary equations of the diagrams $\Theta_s$ $(0 \leq s \leq m)$. According to (23), the central letter of the bottom path of $\Theta_0$ is the central letter of the word $u_{j-i_1}(\bar{x}_{i_1+1})$. Thus it is $x_{i_1+1}$ if $j - i_1 = 1$ and it is $x_{i_1+2}$ otherwise. The defining relation that corresponds to the letter $x_{i_1}$ has the form $x^{x_{i_1}} = x_{k+1}$, where $i = i_1$ and $x_k$ is the central letter of the bottom path of $\Theta_0$. This shows that $k = i + 1$ if $j = i_1 + 1$ and $k = i + 2$ if $j - i_1 \geq 2$.

Now consider the boundary equation of $\Theta_s$, where $1 \leq s < m$. According to (26), the bottom path of $\Theta_s$ is $u_{j-i_{s+1}+s}(\bar{x}_{i_{s+1}+1})$. We have $j + s > i_{s+1}$ by the conditions of Lemma 7. If $j - i_{s+1} + s = 1$, then the central letter we deal with is $x_{i_{s+1}+1}$, otherwise it is $x_{i_{s+1}+2}$. So the cell that corresponds to $x_{i_{s+1}}$ has the form $x^{x_{i_{s}}} = x_{k+1}$, where $i = i_{s+1}$, $k = i_{s+1} + 1$ or $k = i_{s+1} + 2$. This shows that our fact is true for letters $x_{i_2}, \ldots, x_{i_m}$.

So $t$ is smooth of rank $j$. Its height is equal to $m$ since $x_j^t = x_{j+m}$. This completes the proof of Lemma 8.

Now we are going to prove Lemma 8. It is clear that for any $n$ there exists at least one diagram with boundary equation of the Lemma. Let $n_0$ be some positive integer. We can assume that for any $n < n_0$ all conditions a) – c) hold. Indeed, the number of our diagrams under the restriction $n < n_0$ is finite because all the $\alpha_i$’s are either 0 or ±1. So if the constants are chosen big enough, then all inequalities in a) – c) hold.

Let $n \geq n_0$. We proceed by induction on $n$. Now we need to define $k, l, m$ such that $k + l + m + 4 = n$ and for some positive $\eta, \zeta$ conditions (24) hold. We need some freedom of choice of $k$ and $m$. In our case it is very important to choose $k$ and $m$ from an interval of length at least $en$ for some positive constant $e$. Let $k, m$ satisfy the following inequalities:

$$n' = \left\lfloor \frac{n}{3} \right\rfloor \leq k, m \leq \left\lfloor \frac{n}{3} \right\rfloor + \left\lceil \frac{n}{30} \right\rceil = n''.$$

Thus each of the $k, m$ can take at least $\lceil n/30 \rceil + 1 > n/30$ integer values. Then we define $l = n - k - m - 4$. We need $l \geq 0$. Clearly, $l \geq n - 2(n/3 + n/30) - 4 = 4(n/15 - 1) \geq 0$ if we take $n_0 = 15$. On the other hand, $l \leq n - 2[n/3] - 4 \leq n - 2(n - 2)/3 - 4 < n/3$. So each
of the \(k, l, m\) does not exceed \(11n/30\) and we can let \(\eta = 2/15\). Finally, \(2k^2 + 4l^2 + 2m^2 < 4n^2((11/30)^2 + 1/9) = 221n^2/225\) so let \(\zeta = 4/225\).

The remark after the proof of Lemma \(8\) shows that for any \(n \geq n_0\) we can choose any \(k\) and \(m\) such that \([n/3] \leq k, m \leq [n/3] + [n/30]\). Thus all estimates of Lemma \(8\) hold. In particular, we have that the area of \(\Delta'\) will be \(O(n^2)\). (Note that \(S = \sum_{i=-n}^{n} |\alpha_i| \leq 2n + 1\) in our case.) Therefore, part b) holds.

It is easy to prove part a). Take any \(x_0\)-band \(\Delta'_i\). If \(i \leq k\), then it is contained in \(\Delta_1\). By the inductive assumption, its area does not exceed \(Dk < Dn\). The \((k + 1)\)th band has area \(|u| \leq Dk < Dn\). The \((k + 2)\)nd band has area \(|\bar{u}_0| \leq 4k + 3 < Dn\) if \(D\) is big enough. The next \(l\) bands consist of two bands that are contained in \(\Delta_2, \Delta_3\), respectively and the third part that has area \(|\bar{u}_0|\). By the assumption, the total area will be at most \(2Dl + 4k + 3 < 2Dn/3 + 2n \leq Dn\) if \(D \geq 6\). The \((k + l + 3)\)rd band satisfies the same condition since words \(u', w'\) also have length at most \(Dl\) each. The next band has area \(|\bar{u}_0'| + |\bar{u}_0''| + |\bar{u}_0| \leq 4k + 8l + 7 < 5n \leq Dn\) if \(D \geq 5\). The area of each of the last \(m\) bands and the length of \(w\) can be estimated from above as \(2Dm + 4k + 8l + 7 < 2Dm + 5n \leq (11D/15 + 5)n < Dn\) if \(D \geq 20\).

It remains to prove the most difficult part c). Note that we have at least two values of \(i\) such that \(\Phi_i\) may have big area. For \(i = k + 1\) the area of \(\Phi_i\) equals \(#\Delta_1 \leq C_0 k^2\). Also for \(i = k + l + 3\) the area of \(\Phi_i\) equals \(#\Delta_2 + #\Delta_3 \leq 2C_0 l^2\). So \(k, l, m\) should be chosen in such a way that \(\kappa_i\) would be small for these values of \(i\). We did not specify yet how \(k, m\) must be chosen. Now we do that. Let

\[
\kappa_{k+1} = \min \left\{ \kappa_i \mid n' \leq i \leq n'' \right\},
\]

\[
\kappa_{n-m-1} = \min \left\{ \kappa_{n-i-1} \mid n' \leq i \leq n'' \right\}.
\]

Since each of the \(k, m\) may have at least \(n/30\) values, it follows from these definitions that

\[
\kappa_{k+1} \leq 30 \cdot \sum_{i=n'}^{n''} \frac{\kappa_i}{n} \leq \frac{30\kappa}{n},
\]

where \(\kappa = \sum_{i=1}^{n} \kappa_i\). Analogously,

\[
\kappa_{k+l+3} = \kappa_{n-m-1} \leq \frac{30\kappa}{n}.
\]

So the most "dangerous" part of the sum \(\sum_{i=1}^{n} \kappa_i \#\Phi_i\) can be estimated as follows:

\[
\kappa_{k+1} \#\Phi_{k+1} + \kappa_{k+l+3} \#\Phi_{k+l+3} \leq 30C_0 \kappa \cdot \frac{k^2 + 2l^2}{n} < 15C_0 \kappa n.
\]

By the inductive assumption, \(\sum_{i=1}^{k} \kappa_i \#\Phi_i \leq Ck \sum_{i=1}^{k} \kappa_i\). Note that \(\Phi_i\) has zero area for \(i = k + 2\) and \(i = k + l + 4\). For each \(k + 3 \leq i \leq k + l + 2\) the diagram \(\Phi_i\) consists of three parts: \(\Phi'_i\), which is a subdiagram of \(\Delta_2\), \(\Phi''_i\), which is a subdiagram of \(\Delta_3\), and the third part that has area \(|\bar{u}_0|\). By the inductive assumption applied to \(\Delta_2\) and \(\Delta_3\), one can
assume that \[ \sum_{i=k+3}^{k+l+2} \kappa_i \# \Phi_i \leq Cl \sum_{i=k+3}^{k+l+2} \kappa_i \] and \[ \sum_{i=k+3}^{k+l+2} \kappa_i \# \Phi_i'' \leq Cl \sum_{i=k+3}^{k+l+2} \kappa_i. \] Since \( |\bar{u}_0| \leq Dn \), we have \[ \sum_{i=k+3}^{k+l+2} \kappa_i \# \Phi_i \leq (2Cl + Dn) \sum_{i=k+3}^{k+l+2} \kappa_i \leq (C(1-2\eta) + D)n \sum_{i=k+3}^{k+l+2} \kappa_i. \] Analogously, for each \( k+l+5 \leq i \leq n \) we can naturally subdivide \( \Phi_i \) into three parts. The third part will have area \( |\bar{u}_0'| + |\bar{u}_0''| + |\bar{u}_0| \leq Dn \). So we get the inequality \[ \sum_{i=k+l+5}^{n} \kappa_i \# \Phi_i \leq (2Cm + Dn) \sum_{i=k+l+5}^{n} \kappa_i \leq (C(1-2\eta) + D)n \sum_{i=k+l+5}^{n} \kappa_i. \] Taking the sum of the estimates for each of the cases: \( 1 \leq i \leq k, \ k+3 \leq i \leq k+l+2, \ k+l+5 \leq i \leq n, \) and \( i \in \{k+1, k+l+3\} \), we obtain

\[ \sum_{i=1}^{n} \kappa_i \# \Phi_i \leq (C(1-2\eta) + D + 15C_0)\kappa n \leq C\kappa n \]

if \( C \geq (D + 15C_0)/2\eta \). This proves part c).

Lemma 9 is proved. Now the main result follows.

Note that we also got polynomial isoperimetric inequalities for the two other R. Thomp-
son groups, \( T \), and \( V \) in [12] (both groups are finitely presented and simple). Results of
the present paper imply some improvements to the estimates from [12]. Details will appear
elsewhere.

Note that the techniques of this paper may probably be applied to other Thompson-like
groups. It looks like similar results can be obtained without any essential changes for groups
\( F_r \), where \( r \geq 3 \) (see [3]). It is also interesting to look from this point of view to some

diagram groups (see [13]), especially for the diagram groups over semigroup presentation
\( \langle x \mid x^3 = x^2 \rangle \).

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