RIEMANN-ROCH THEOREMS VIA DEFORMATION QUANTIZATION I

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Abstract. We deduce the Riemann-Roch type formula expressing the microlocal Euler class of a perfect complex of $\mathcal{D}$-modules in terms of the Chern character of the associated symbol complex and the Todd class of the manifold from the Riemann-Roch type theorem for periodic cyclic cocycles of a symplectic deformation quantization. The proof of the latter is contained in the sequel to this paper.

1. Introduction

We recall the results of [SS] restricting ourselves to the “absolute” case for the sake of simplicity.

1.1. The index theory for elliptic pairs. Let $X$ be a complex manifold of dimension $\dim \mathbb{C}X = d$. Say that a $\mathcal{D}_X$-module $\mathcal{M}$ is good if for every relatively compact open subset $U$ of $X$ the restriction $\mathcal{M}|_U$ admits a finite filtration $G_\bullet \mathcal{M}|_U$ by $\mathcal{D}_U$-submodules, such that $G_i G_\bullet \mathcal{M}|_U$ is $\mathcal{D}_U$-coherent with good filtration.

An elliptic pair $(\mathcal{M}^\bullet, \mathcal{F}^\bullet)$ on $X$ consists of

• a complex $\mathcal{M}^\bullet$ of $\mathcal{D}_X$-modules with bounded good cohomology,
• a complex $\mathcal{F}^\bullet$ of $\mathbb{C}$-vector spaces with bounded, $\mathbb{R}$-constructible cohomology,

which satisfy

$$\text{char}(\mathcal{M}^\bullet) \cap \text{SS}(\mathcal{F}^\bullet) \subseteq T^*_X X .$$

In the case when $\text{Supp} \mathcal{M}^\bullet \cap \text{Supp} \mathcal{F}^\bullet$ (where the support of a complex of sheaves is understood to be the union of the supports of the cohomologies) is compact, P. Schapira and J.-P. Schneiders proved that

$$\dim H^\bullet(X; \mathcal{F}^\bullet \otimes \mathcal{M}^\bullet \otimes \mathcal{L}_{\mathcal{D}_X} \mathcal{O}_X) < \infty .$$

Thus, the Euler characteristic

$$\chi(X; (\mathcal{M}^\bullet, \mathcal{F}^\bullet)) \overset{\text{def}}{=} \sum (-1)^i \dim H^i(X; \mathcal{F}^\bullet \otimes \mathcal{M}^\bullet \otimes \mathcal{L}_{\mathcal{D}_X} \mathcal{O}_X)$$

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is defined. The index theorem for elliptic pairs ([SS], Theorem 5.1) says that
\[
\chi(X; (\mathcal{M}^\bullet, F^\bullet)) = \int_{T^*X} \mu_{eu}(\mathcal{M}^\bullet) - \mu_{eu}(F^\bullet)
\]
where \(\mu_{eu}(\mathcal{M}^\bullet) \in H^{2d}_{\text{char}(\mathcal{M}^\bullet)}(T^*X; \mathbb{C})\) and \(\mu_{eu}(F^\bullet) \in H^{2d}_{SS(F^\bullet)}(T^*X; \mathbb{C})\) are defined in [SS].

The class \(\mu_{eu}(F^\bullet)\) is the characteristic cycle of the constructible complex \(F^\bullet\) as defined by Kashiwara (see [KS] for more details). For example, if \(Y \subset X\) is a closed real analytic submanifold, one has \(\mu_{eu}(C_Y) = [T^*_Y X]\).

With regard to \(\mu_{eu}(\mathcal{M}^\bullet)\) P. Schapira and J.-P. Schneiders conjectured that it is related to a certain characteristic class of the symbol \(\sigma(\mathcal{M}^\bullet)\) of \(\mathcal{M}^\bullet\) ([SS], Conjecture 8.5, see Conjecture 1.2.2).

1.2. The Riemann-Roch type formula. Let \(\pi: T^*X \to X\) denote the canonical projection.

In what follows we will assume that the complex \(\mathcal{M}^\bullet\) admits a filtration \(F^\bullet\mathcal{M}^\bullet\) by \(O_X\)-submodules which is compatible with the action of \(D_X\) and the filtration \(F^\bullet D_X\) by order and such that the symbol complex of \(\mathcal{M}^\bullet\) defined by
\[
\sigma(\mathcal{M}^\bullet) = \pi^{-1}Gr^F_{\cdot} \mathcal{M}^\bullet \otimes_{\pi^{-1}Gr^F_{\cdot} D_X} O_{T^*X}
\]
has bounded \(O_{T^*X}\)-coherent cohomology. Note that, by definition, \(\text{char}(\mathcal{M}^\bullet) \overset{\text{def}}{=} \text{Supp} \sigma(\mathcal{M}^\bullet)\).

For \(\Lambda\) a closed subvariety of \(T^*X\) let \(K^0_\Lambda(T^*X)\) denote the Grothendieck group of perfect complexes of \(O_{T^*X}\)-modules supported on \(\Lambda\) (i.e. acyclic on the complement of \(\Lambda\) in \(T^*X\)). Let \(K^0_0(T^*X) \overset{\text{def}}{=} K^0_{T^*X}(T^*X)\).

For \(\Lambda\) containing \(\text{char}(\mathcal{M})\) let \(\sigma_\Lambda(\mathcal{M}^\bullet)\) denote the class of \(\sigma(\mathcal{M}^\bullet)\) in \(K^0_\Lambda(T^*X)\).

Remark 1.2.1. Both the characteristic variety and the class of the symbol in the Grothendieck group are independent of the choice of the good filtration. Local existence of good filtration in coherent \(D_X\)-modules is sufficient to define \(\sigma_\Lambda(\mathcal{M}^\bullet)\) for \(\mathcal{M}^\bullet\) with bounded good cohomology.

The Chern character \(ch: K^0_0(T^*X) \to \bigoplus_i H^{2i}(T^*X; \mathbb{C})\) admits a natural extension to the Chern character with supports (Theorem 2.1.1)
\[
ch_\Lambda: K^0_\Lambda(T^*X) \to \bigoplus_i H^{2i}_\Lambda(T^*X; \mathbb{C})
\]
which is functorial with respect to change of support.
In [SS], P. Schapira and J.-P. Schneiders make the following conjecture.

**Conjecture 1.2.2.** Suppose that $\mathcal{M}^\bullet$ is a complex of $\mathcal{D}_X$-modules with bounded good cohomology and $\Lambda$ is a closed conic subvariety of $T^* X$ containing $\text{char}(\mathcal{M}^\bullet)$. Then,

$$\mu \text{eu}(\mathcal{M}^\bullet) = [ch_\Lambda(\sigma(\mathcal{M}^\bullet)) \sim \pi^*Td(TX)]^{2d}.$$  

(For $\alpha$ an element of a graded object we denote by $[\alpha]^p$ the homogeneous component of $\alpha$ of degree $p$.)

We will refer to formulas such as the one in 1.2.2 as Riemann-Roch formulas and refer to the left hand side as “non-commutative” and the right hand side as “commutative”. What follows is an informal discussion of the Riemann-Roch formula and the key ideas and observations which enter into the statement and the proof of the main result of this paper (Theorem 3.3.1) from which Conjecture 1.2.2 follows.

1.2.1. Although Conjecture 1.2.2 is stated in terms of $\mathcal{D}$-modules, it is, in fact of micro-local (i.e. local on $T^*X$) nature and is naturally formulated in terms of modules over the ring $\mathcal{E}_X$ of micro-differential operators. Recall that $\mathcal{E}_X$ is a sheaf of algebras on $T^*X$ equipped with the canonical filtration $F_\bullet \mathcal{E}_X$ by order, the symbol map $\sigma : Gr^F \mathcal{E}_X \to \mathcal{O}_{T^*X}$ and the canonical faithfully flat map $\pi^{-1}\mathcal{D}_X \to \mathcal{E}_X$ of filtered $\pi^{-1}\mathcal{O}_X$-algebras. The characteristic variety of a (coherent) $\mathcal{D}_X$-module $\mathcal{M}$ is the support of the $\mathcal{E}_X$-module $\pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{D}_X} \mathcal{E}_X$.

Let $K^0_\Lambda(\mathcal{D}_X)$ (respectively $K^0_\Lambda(\mathcal{E}_X)$) denote the Grothendieck group of the category of perfect complexes of $\mathcal{D}_X$-modules whose characteristic variety is contained in $\Lambda$ (respectively perfect complexes of $\mathcal{E}_X$-modules supported on $\Lambda$). Extension of scalars gives rise to the map $K^0_\Lambda(\mathcal{D}_X) \to K^0_\Lambda(\mathcal{E}_X)$.

The microlocal Euler class which appears on the non-commutative side of the Riemann-Roch formula is a map $K^0_\Lambda(\mathcal{D}_X) \to H^{2d}_\Lambda(T^*X; \mathbb{C})$, but, in fact, by it’s very definition, is the composition of extension of scalars with the map $\mu \text{eu} : K^0_\Lambda(\mathcal{E}_X) \to H^{2d}_\Lambda(T^*X; \mathbb{C})$.

Similarly, the commutative side of the Riemann-Roch formula depends not on the filtered $\mathcal{D}_X$-module but on the filtered $\mathcal{E}_X$-module obtained by extension of scalars.

1.2.2. The presence of the filtration can be accounted for in ring theoretic terms by the traditional device of the Rees construction. The Rees ring of the filtered ring $(\mathcal{E}_X, F_\bullet)$ is the (graded) $\mathbb{C}[t]$-algebra $\mathcal{R}\mathcal{E}_X = \bigoplus_p F_p \mathcal{E}_X t^p \subset \mathcal{E}_X[t]$. A filtered $\mathcal{E}_X$-module is the same as a
(graded) $t$-torsion-free module over $\mathcal{RE}_X$: $\mathcal{R}$ is an exact functor defined by the formula $\mathcal{R}(\mathcal{M}, F) = \bigoplus_p F_p \mathcal{M}^p$.

The algebra $\mathcal{RE}_X$ comes equipped with the symbol map $\sigma: \mathcal{RE}_X \to \mathcal{O}_{T^*X}$ which annihilates $t$ and the canonical isomorphism $\mathcal{RE}_X[t^{-1}] \cong E_X[t^{-1}, t]$. The symbol complex is given by the formula $\sigma(\mathcal{M}^*) = \mathcal{RM}^* \otimes_{\mathcal{RE}_X} \mathcal{O}_{T^*X}$ and the complex is recovered via the canonical isomorphism $\mathcal{RM}^*[t^{-1}] \cong \mathcal{M}^*[t^{-1}, t]$. The natural input for the Riemann-Roch formula is conveniently summarized as a perfect complex of $\mathcal{RE}_X$-modules.

The use of the Rees construction is not just a neat device to keep track of the filtration. The appearance of $E_X$ and $\operatorname{Gr}^F E_X$ as, respectively, the generic and the special fiber in a one parameter deformation allows application of deformation theoretical methods which play a crucial role in the proof of the Riemann-Roch formula.

1.2.3. It becomes necessary, particularly with the appearance of algebras such as $\mathcal{RE}_X$ to use Hochschild homology, rather than appeal directly to the duality theory for the definition of the micro-local Euler class. The map $\mu_\text{eu} : K^0_\Lambda(\mathcal{E}_X) \to H^{2d}_\Lambda(T^*X; \mathbb{C})$ factors naturally into the composition

$$K^0_\Lambda(\mathcal{E}_X) \xrightarrow{\text{Eu}_X} H^0_\Lambda(T^*X; \mathcal{E}_X \otimes \mathcal{E}_X^{op} \mathcal{E}_X) \xrightarrow{\mu_\text{e}_X} H^{2d}_\Lambda(T^*X; \mathbb{C})$$

where the first map is the Euler class with values in Hochschild homology and is defined in the generality of perfect complexes of modules over sheaves of algebras, and the second map is induced by the morphism

$$\mu_\text{e} : \mathcal{E}_X \otimes \mathcal{E}_X^{op} \mathcal{E}_X \to \mathbb{C}_{T^*X}[2d]$$

in the derived category of sheaves on $T^*X$ called the (canonical) trace density map. The existence of the canonical trace density map is a non-trivial fact which has to do with special properties of the algebra of micro-differential operators.

The above mentioned factorization of the microlocal Euler class essentially reduces the Riemann-Roch formula to the problem of expressing the canonical trace density map in "commutative" terms, i.e., loosely speaking, factoring the composition

$$\mathcal{RE}_X \otimes_{\mathcal{RE}_X} \mathcal{RE}_X^{op} \mathcal{RE}_X \to \mathcal{E}_X \otimes \mathcal{E}_X^{op} \mathcal{E}_X[t^{-1}, t] \xrightarrow{\mu_\text{e}} \mathbb{C}_{T^*X}[t^{-1}, t][2d]$$

through the symbol map. (This reduction is not so surprising, considering the fact that the relevant portion of the Hochschild homology of micro-differential operators is spanned by the Euler classes.)
Before addressing the latter, simpler, problem we make some more observations with regard to the Riemann-Roch formula intended to motivate a gearchange that follows.

1.2.4. The added complication in the Riemann-Roch formula is the fact that commutative side is comprised of a homogeneous component of a manifestly non-homogeneous expression. As much as one would like to remove the brackets in the formula it is not clear how to extend the micro-local Euler class to a manifestly non-homogeneous characteristic class without the reformulation of the problem in terms of Hochschild homology.

Hochschild homology is the natural recipient of the Lefshetz trace map. The Euler class is defined as the Lefshetz trace of the identity and possesses special symmetries not shared by the traces of other endomorphisms. As a result, the Euler class extends to the Chern character

\[ ch^E_X : K^0_A(\mathcal{E}_X) \to H^0_\Lambda(T^*X; CC^\text{per}_* (\mathcal{E}_X)) . \]

On the non-commutative side of the Riemann-Roch formula the transition from Hochschild homology to periodic cyclic homology amounts to switching from the integer grading to the even/odd grading, i.e. allowing not necessarily homogeneous classes which is a welcome change. Due to the same special homological properties of micro-differential operators the canonical trace density map extends in a unique fashion to the morphism (in the derived category of sheaves on \( T^*X \))

\[ \tilde{\mu}_e : CC^\text{per}_* (\mathcal{E}_X) \to \mathbb{C}_{T^*X}[u^{-1}, u] \]

where \( u \) denotes a formal variable of degree 2 (so that, for a complex \( A^\bullet \) multiplication by \( u \) establishes the isomorphism \( A^\bullet \to uA^\bullet[2] \)).

1.2.5. The preceding discussion was intended as a justification of the following reformulation of the Riemann-Roch formula in terms of the (non-commutative) Chern character: for a perfect complex \( \mathcal{M}^\bullet \) of \( \mathcal{R}\mathcal{E}_X \)-modules

\[ \tilde{\mu}_e (ch^E_X (\iota \mathcal{M}^\bullet)) = ch_A (\sigma \mathcal{M}^\bullet) \sim \pi^* Td(TX) , \]

where \( \iota \mathcal{M}^\bullet = \mathcal{M}^\bullet[t^{-1}], \sigma \mathcal{M}^\bullet = \mathcal{M}^\bullet \otimes_{\mathcal{R}\mathcal{E}_X} \mathcal{O}_{T^*X} \), and the equality takes place in \( H^d_\Lambda(T^*X; \mathbb{C}_{T^*X}[t^{-1}, t][u^{-1}, u]) \cong H^\text{even}_\Lambda(T^*X; \mathbb{C}_{T^*X}[t^{-1}, t]) \).

1.2.6. The Chern character which appears on the commutative side of the Riemann-Roch formula may be formulated in terms of periodic cyclic homology of \( \mathcal{O}_{T^*X} \) as the composition

\[ K^0_A(\mathcal{O}_{T^*X}) \xrightarrow{ch^{CC^\text{per}}_{T^*X}} H^0_\Lambda(T^*X; CC^\text{per}_* (\mathcal{O}_{T^*X})) \xrightarrow{\tilde{\mu}_0} H^0_\Lambda(T^*X; \mathbb{C}_{T^*X}[u^{-1}, u]) \]
where the second map is induced by the Hochschild-Kostant-Rosenberg-Connes map $\tilde{\mu}_O : CC^{per}(\mathcal{O}_{T^*X}) \to \Omega_{T^*X}^*[u^{-1}, u]$ and the de Rham isomorphism.

With these notations the Riemann-Roch formula reads

$$\tilde{\mu}_E(ch_X^E(\iota(M^*))) = \tilde{\mu}_O(ch_X^{O_{T^*X}}(\sigma(M^*))) \sim \pi^*Td(TX).$$

Furthermore, the naturality of the Chern character implies that $ch_X^E(\iota(M^*)) = \iota(ch_X^E(M^*))$ (respectively $ch_X^{O_{T^*X}}(\sigma(M^*)) = \sigma(ch_X^{O_{T^*X}}(M^*))$) where $\iota : CC^{per}(\mathcal{R}\mathcal{E}^*_X) \to CC^{per}(\mathcal{E}^*_X)[t^{-1}, t]$ (respectively $\sigma : CC^{per}(\mathcal{R}\mathcal{E}^*_X) \to CC^{per}(\mathcal{O}_{T^*X})$) is induced by $\iota : \mathcal{R}\mathcal{E}^*_X \to \mathcal{E}^*_X[t^{-1}, t]$ (respectively $\sigma : \mathcal{R}\mathcal{E}^*_X \to \mathcal{O}_{T^*X}$). The Riemann-Roch formula now takes the shape

$$\tilde{\mu}_E(\iota(ch_X^E(M^*))) = \mu_O(\sigma(ch_X^{O_{T^*X}}(M^*))) \sim \pi^*Td(TX)$$

and, in the present formulation is valid for any cycle in $CC^{per}(\mathcal{R}\mathcal{E}^*_X)$ (because the image of the canonical trace density map is, in essence, spanned by Chern characters). The above formula is equivalent to the commutativity of the diagram

$$\begin{array}{ccc}
CC^{per}(\mathcal{R}\mathcal{E}^*_X) & \xrightarrow{\sigma} & CC^{per}(\mathcal{O}_{T^*X}) \\
\downarrow\iota & & \downarrow\tilde{\mu}_O \sim \pi^*Td(TX) \\
CC^{per}(\mathcal{E}^*_X)[t^{-1}, t] & \xrightarrow{\tilde{\mu}_E} & \mathbb{C}_{T^*X}[t^{-1}, t][u^{-1}, u]
\end{array}$$

in the derived category of sheaves on $T^*X$.

1.2.7. The non-triviality of the issue of defining the canonical trace density map has to do with the fact that the canonical morphism $\mathbb{C}_{T^*X}[2d] \to \mathcal{E}^*_X \otimes_{\mathcal{E}^*_X \otimes \mathcal{E}^{op}_X} \mathcal{E}^*_X$ furnished by the duality theory is not an isomorphism (in the derived category) and that can be traced to the fact that the canonical map $Gr^*_E \mathcal{E}^*_X \to \mathcal{O}_{T^*X}$ is not an isomorphism.

The trace density map is a retraction of the morphism $\mathbb{C}_{T^*X}[2d] \to \mathcal{E}^*_X \otimes_{\mathcal{E}^*_X \otimes \mathcal{E}^{op}_X} \mathcal{E}^*_X$ and is arranged for by embedding $\mathcal{E}^*_X$ into a larger algebra. The choice more natural in the context of micro-local analysis is the algebra of micro-local operators $\mathcal{E}^{rl}_X$ which bears very little resemblance to $\mathcal{O}_{T^*X}$.

The same may be achieved by embedding $\mathcal{R}\mathcal{E}^*_X$ into a formal one-parameter deformation (with $t$ being the parameter) $A^t_{T^*X}$ of $\mathcal{O}_{T^*X}$ which is uniquely determined by the fact that it contains $\mathcal{R}\mathcal{E}^*_X$. The Riemann-Roch formula in its diagramatic reformulation, follows from...
the analogous statement about $A^l_{T^* X}$:

$$
\begin{align*}
CC^\per_{\bullet}(A^l_{T^* X}) & \xrightarrow{\sigma} CC^\per_{\bullet}(O_{T^* X}) \\
\downarrow & \\
CC^\per_{\bullet}(A^l_{T^* X})[t^{-1}] & \xrightarrow{\tilde{\mu}^A} \mathcal{O}_{T^* X}[t^{-1}, t][u^{-1}, u]
\end{align*}
$$

commutes in the derived category of sheaves on $T^* X$. The latter fact is a particular case of a general Riemann-Roch type theorem concerning symplectic deformations of structure sheaves and whose proof relies on the techniques of formal geometry and invariant theory and constitutes the sequel to the present paper.

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2. Characteristic classes and trace maps

In this section we introduce the ingredients which go into the statement of the main result of this paper (Theorem 4.6.1) and show how Conjecture 1.2.2 follows from it. Proofs are postponed until later sections as indicated.

2.1. Characteristic classes of perfect complexes. The following theorem (Theorem 2.1.1) summarizes the relevant aspects of the requisite characteristic classes for perfect complexes. The details of the construction will appear in a separate publication.

Let $X$ be a topological space, $Z$ a closed subset of $X$. Let $\mathcal{A}$ denote a sheaf of algebras on $X$ such that there is a global section $1 \in \Gamma(X; \mathcal{A})$ which restricts to $1_{\mathcal{A}_x}$.

We denote by $K^i_Z(\mathcal{A})$ the $i$-th $K$-group of the category of perfect complexes of $\mathcal{A}$-modules which are acyclic on the complement of $Z$ and refer the reader to Section 5 for other notations.

**Theorem 2.1.1.** There exists the Chern character $ch^A_{Z,i} : K^i_Z(\mathcal{A}) \to H^{-i}_Z(X; CC^\per_\bullet (\mathcal{A}))$ and the Euler class $\text{Eu}^A_{Z,i} : K^i_Z(\mathcal{A}) \to H^{-i}_Z(X; C^\bullet_{\mathcal{A}})$ natural in $X$, $Z$, and $\mathcal{A}$ such that

- the composition $K^i_Z(\mathcal{A}) \xrightarrow{ch^A_{Z,i}} H^{-i}_Z(X; CC^\per_\bullet (\mathcal{A})) \to H^{-i}_Z(X; C^\bullet_{\mathcal{A}})$ coincides with $\text{Eu}^A_{Z,i}$;
• for a perfect complex $\mathcal{F}^\bullet$ of $\mathcal{A}$-modules supported on $Z$ the class
$\text{Eu}^A_{Z,0}(\mathcal{F}^\bullet) \in H^0_Z(X; C^\bullet(\mathcal{A}))$ coincides with the composition

$k \xrightarrow{1 \mapsto \text{id}} R\text{Hom}_A^\bullet(\mathcal{F}^\bullet, \mathcal{F}^\bullet) \xleftarrow{\sim} (R\text{Hom}_A^\bullet(\mathcal{F}^\bullet, \mathcal{A}) \otimes_k \mathcal{F}^\bullet) \otimes_{A \otimes_k A^{\text{op}}} \mathcal{A} \xrightarrow{\text{ev} \otimes \text{id}} A \otimes_{A \otimes_k A^{\text{op}}} \mathcal{A}.

In what follows we will only consider the components of the Euler class and the Chern character defined on $K^0$ and write $\text{Eu}^A_{Z,0}$ (respectively $\text{ch}^A_{Z,0}$) instead of $\text{Eu}^A_Z$ (respectively $\text{ch}^A_Z$). Nor shall we make notational distinction between $\text{ch}^A_Z$ and its image under the natural map $H^*_Z(X; CC_{-}(\mathcal{A})) \to H^*_Z(X; CC_{\text{per}}(\mathcal{A}))$.

2.2. The microlocal Euler class. For the reader’s convenience we give the definition of the microlocal Euler class.

Suppose that $\Lambda$ is a closed subvariety of $T^*X$ and $\mathcal{N}^\bullet$ is a perfect complex of $\mathcal{E}^\infty_X$-modules acyclic on the complement of $\Lambda$. Consider the morphism in the derived category (the Lefshetz trace map) given by

$\mathcal{E} = \mathcal{E}^\infty_X$, $\pi : T^*X \to X$ is the canonical projection, and the last isomorphism is as in ... .

The microlocal Euler class $\mu \text{eu}(\mathcal{N}^\bullet)$ is defined as the Lefshetz trace of the identity, i.e. as the composite

$\mu \text{eu}(\mathcal{N}^\bullet) : \mathcal{C}_{T^*X} \xrightarrow{1 \mapsto \text{id}} R\text{Hom}_{\mathcal{E}^\infty_X}(\mathcal{N}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} \mathcal{C}_{T^*X}[2d] \to R\text{Hom}_{\mathcal{E}^\infty_X}(\mathcal{N}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} R\text{Hom}_{\mathcal{E}^\infty_X}(\mathcal{N}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} \mathcal{C}_{T^*X}[2d].$

where

$D\mathcal{N}^\bullet \overset{\text{def}}{=} R\text{Hom}_{\mathcal{E}^\infty_X}(\mathcal{N}^\bullet, \mathcal{E}^\infty_X \otimes_{\mathcal{E}^\infty_X} \pi^{-1}\Omega^d_{X}, \pi^{-1}\Omega^d_{X}) \to R\text{Hom}_{\mathcal{E}^\infty_X}(\mathcal{N}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} R\text{Hom}_{\mathcal{E}^\infty_X}(\mathcal{N}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} \mathcal{C}_{T^*X}[2d].$

For a perfect complex $\mathcal{N}^\bullet$ of $\mathcal{E}_X$-modules the microlocal Euler class is defined by

$\mu \text{eu}(\mathcal{N}^\bullet) \overset{\text{def}}{=} \mu \text{eu}(\mathcal{N}^\bullet \otimes_{\mathcal{E}_X} \mathcal{E}^\infty_X) \in H^*_X(T^*X; \mathbb{C}).$

Similarly, the microlocal Euler class of a complex $\mathcal{N}^\bullet$ of $\mathcal{D}_X$-modules with $\text{char}\mathcal{N}^\bullet \subseteq \Lambda$ is defined as $\mu \text{eu}(\mathcal{N}^\bullet \otimes_{\mathcal{D}_X} \mathcal{E}^\infty_X)$.

The microlocal Euler class gives rise to the map

$\mu \text{eu} : K^0_X \to H^*_\Lambda(T^*X; \mathbb{C}).$
2.3. The canonical trace density map. Let $\Delta \subset X \times X$ denote the diagonal. Let $a : T^*X \to T^*X$ denote the antipodal map ($(x, \xi) \mapsto (x, -\xi)$). There is a natural isomorphism of sheaves of algebras $a^{-1}E^{(R)}_X \cong \left( \mathcal{E}^{(R)}_X \right)^{op}$. Put $(\mathcal{E}^{(R)}_X)^a = (\text{id} \times a)^{-1}E^{(R)}_{X \times X}$. Then, there is a natural map $E^R_X \boxtimes (E^R_X)^{op} \to (E^R_{X \times X})^a$, and the canonical structure of an $(E^R_{X \times X})^a$-module on $E^R_X$ which extends the canonical structure of an $E^R_{X \times X}$-$\text{op}$-module.

By a theorem of Kashiwara the canonical map $C_{T^*X} \to R \underline{\text{Hom}}_{E^R_{X \times X}}(\mathcal{C}^R_{\Delta|X \times X}, \mathcal{C}^R_{\Delta|X \times X})$ is an isomorphism in the derived category. There are canonical isomorphisms

\[
\begin{align*}
C_{T^*X}[2d] & \cong R \underline{\text{Hom}}_{E^R_{X \times X}}(\mathcal{C}^R_{\Delta|X \times X}, \mathcal{C}^R_{\Delta|X \times X})[2d] \\
& \cong R \underline{\text{Hom}}_{E^R_{X \times X}}(\mathcal{C}^R_{\Delta|X \times X}, \mathcal{E}^R_{X \times X})[2d] \otimes_{E^R_{X \times X}}^{L} \mathcal{C}^R_{\Delta|X \times X} \\
& \cong \mathcal{C}^{(2d)R}_{\Delta|X \times X} \otimes_{E^R_{X \times X}}^{L} \mathcal{E}^{R}_{X \times X} \\
& \cong E^R_X \otimes_{(E^R_{X \times X})^a}^{L} E^R_X.
\end{align*}
\]

By abuse of notation we will denote by $\mu_\mathcal{E}$ and refer to as the canonical trace density map the composition $E^\otimes_{(E^R_{X \times X})^a} E^R_X \to E^R_X \otimes_{(E^R_{X \times X})^a}^{L} E^R_X \to C_{T^*X}[2d]$ and the composition of the latter with the canonical map $E^\otimes_{E^R_X \boxtimes E^R_X} E^R_X \to E^R_X \boxtimes_{(E^R_{X \times X})^a}^{L} E^R_X$.

2.4. Comparison of the Euler classes. The Euler class of Theorem 2.1 combined with the canonical trace density map gives rise to the map

\[
K^0_\Lambda(E^R_X) \xrightarrow{\text{Eu}^R_X} H^0_\Lambda(T^*X; E^R_X \boxtimes_{E^R_X \boxtimes E^R_X}^{L} E^R_X) \xrightarrow{\mu_\mathcal{E}} H^{2d}_\Lambda(T^*X; \mathbb{C}).
\]

(2.1)

Proposition 2.4.1. The composition (2.1) coincides with the microlocal Euler class. In other words, if $\mathcal{N}^\bullet$ is a perfect complex of $E^R_X$-modules and $\Lambda$ is a closed subvariety of $T^*X$ containing Supp $\mathcal{N}^\bullet$, then $\mu_\text{eu}(\mathcal{N}^\bullet) = \mu_\mathcal{E}(\text{Eu}^R_X(\mathcal{N}^\bullet))$. 

Proof. The Lefshetz trace map
\[ R \text{Hom}_{E_X}(N^\bullet, N^\bullet) \to C^{(2d)}_{\Delta \times X} \otimes L_{E_X \times X} \] is compatible with the Lefshetz trace map
\[ R \text{Hom}_{E_X}(N^\bullet, N^\bullet) \to \mathcal{E}_X \otimes L_{E_X \times X} \] and the isomorphisms
\[ C^{(2d)}_{\Delta \times X} \otimes L_{E_X \times X} \cong \mathcal{E}_X \otimes (E_X \times X)^\ast \mathcal{E}_X \cong \mathbb{C}_{T^\ast X}[2d]. \]

2.5. Complexes of Hochschild chains. With the view onto the passage to cyclic homology we describe the canonical trace density map in terms of the standard complexes of Hochschild chains.

The object \( E_X \otimes L_{E_X \times X} \), regarded as an object of the derived category of sheaves on \( T^\ast X \), is represented by the standard complex of Hochschild chains \( C^\bullet(E_X) \) (see [5.1]). Put
\[ \left( \mathcal{E}_X^{(R)} \right)^\epsilon = \delta_{T^\ast X}(\text{id} \times a)^{-1} \mathcal{E}_X^{(R)} \]
\[ \widehat{C}_p(\mathcal{E}_X^{(R)}) = \delta_{T^\ast X} \mathcal{E}_X^{(R)} \]

The Hochschild differential extends to the map \( b : \widehat{C}_p(\mathcal{E}_X^{(R)}) \to \widehat{C}_{p-1}(\mathcal{E}_X^{(R)}) \). The complex \( \widehat{C}_\bullet(\mathcal{E}_X^{(R)}) \) represents \( \mathcal{E}_X^{(R)} \otimes L_{(E_X \times X)^\ast} \mathcal{E}_X^{(R)} \) in the derived category of sheaves on \( T^\ast X \). There is a canonical map \( C_\bullet(\mathcal{E}_X^{(R)}) \to \widehat{C}_\bullet(\mathcal{E}_X^{(R)}) \).

Let \( \Phi^\epsilon \) denote the global section of \( H^{-2d} \widehat{C}_\bullet(\mathcal{E}_X^{(R)}) \) which corresponds to the global section 1 under the isomorphism induced by \( \mu_\mathcal{E} \).

Lemma 2.5.1. Let \( x_1, \ldots, x_d \) be a local coordinate system on \( X \) and let \( \partial_i = \frac{\partial}{\partial x_i} \). Then, the locally defined Hochschild chain \( \text{Alt}(1 \otimes x_1 \otimes \cdots \otimes x_d \otimes \partial_1 \otimes \cdots \otimes \partial_d) \in C_{2d}(\mathcal{E}_X) \) is a cycle which represents (the restriction to the domain of the coordinate system of) \( \Phi^\epsilon \in \Gamma(T^\ast X; H^{-2d}C_\bullet(\mathcal{E}_X^{(R)})) \).

2.6. Cyclic homology of microdifferential operators. Our immediate goal is to recast the calculation of the micro-local Euler class in terms of the negative (and, ultimately, the periodic) cyclic homology. The passage to cyclic homology is motivated by two developments.

On the one hand, the Euler class (which takes values in Hochschild homology) factors canonically through the Chern character which takes values in negative cyclic homology; thus, the Chern character should be a (significant) refinement of the Euler class. On the other hand,
the periodic cyclic homology is invariant under formal deformations of
algebras and this feature will, ultimately, allow us to evaluate the “non-
commutative” Chern character explicitly in “commutative” terms.

The cyclic differential $B$ extends to the map $B : \widehat{C}_p(E^R_X) \to \widehat{C}_{p+1}(E^R_X)$
and the cyclic complexes $\widehat{CC}^{-}(E^R_X)$ and $\widehat{CC}^{per}(E^R_X)$ are defined as in
starting from $\widehat{C}_n(E^R_X)$. The analogs of (5.3) and (5.4) hold and
there are canonical maps $CC^{-}(E^R_X) \to \widehat{CC}^{-}(E^R_X)$ and $CC^{per}(E^R_X) \to
\widehat{CC}^{per}(E^R_X)$.

The complexes $(\widehat{CC}^{-}(E^R_X), b + uB)$ and $(\widehat{C}_n(E^R_X)[[u]], b)$ have canonical
filtrations by powers of $u$ which will be denoted by $F_*$. Note that the
associated graded complexes $Gr_{\bullet} \widehat{CC}^{-}(E^R_X)$ and $Gr_{\bullet} \widehat{C}_n(E^R_X)[[u]]$
are identical.

**Lemma 2.6.1.** There exists an isomorphism of filtered complexes

$$(\widehat{CC}^{-}(E^R_X), b + uB, F_*) \simeq (\widehat{C}_n(E^R_X)[[u]], b, F_*)$$

which induces the identity map on the associated graded complexes and
any two such are (filtered) homotopic.

**Proof.** Consider the spectral sequence of the filtered complex

$(\text{Hom}^\bullet(\widehat{CC}^{-}(E^R_X), \widehat{C}_n(E^R_X)[[u]]), F_*)$ and use the isomorphism (in the de-

rived category) $\widehat{C}_n(E^R_X) \simeq C_{T^*X}[2d]$. \qed

**Corollary 2.6.2.** There exists an isomorphism of filtered complexes

$$(\widehat{CC}^{per}(E^R_X), b + uB, F_*) \simeq (\widehat{C}_n(E^R_X)[u^{-1}], b, F_*)$$

which induces the identity map on the associated graded objects and
any two such are homotopic.

2.7. The trace density on the periodic cyclic complex. The iso-
morphism (in the derived category) $\widehat{C}_n(E^R_X) \simeq C_{T^*X}[2d]$ is represented,

using the standard truncation functors, by the following diagram of quasi-isomorphisms of complexes:

$$\widehat{C}_n(E^R_X) \leftarrow \tau^{\leq -2d} \widehat{C}_n(E^R_X) \rightarrow
\rightarrow \tau^{\geq -2d, \tau^{\leq -2d}} \widehat{C}_n(E^R_X) \simeq C_{T^*X}[2d] . \ (2.2)$$

The diagram (2.2) gives rise to the diagram of complexes of $\mathbb{C}[u^{-1}, u]$-
modules

$$\widehat{C}_n(E^R_X)[u^{-1}, u] \leftarrow \bigg(\tau^{\leq -2d} \widehat{C}_n(E^R_X)\bigg)[u^{-1}, u] \rightarrow
\rightarrow \bigg(\tau^{\geq -2d, \tau^{\leq -2d}} \widehat{C}_n(E^R_X)\bigg)[u^{-1}, u] \simeq C_{T^*X}[2d][u^{-1}, u]$$
which represents a morphism

\[ \tilde{C}(E^R_X)[u^{-1}, u] \to C_{T^*X}[2d][u^{-1}, u] \]  

in the derived category.

We will denote by \( \tilde{\mu}_E \) and refer to as the canonical trace density map the composition

\[ CC^\text{per} (E_X) \to CC^\text{per} (E_R^X) \to \tilde{C}(E_R^X) \to \tilde{C}(E^R_X)[u^{-1}, u] \to C_{T^*X}[2d][u^{-1}, u] \]  

where the last two maps are furnished, respectively, by Corollary 2.6.2 and (2.3).

Note that, by construction, the map \( \tilde{\mu}_E \) induces the map \( \mu_E \) on the associated (to the filtrations by powers of \( u \)) graded objects.

**Lemma 2.7.1.** For a perfect complex \( \mathcal{N}^* \) of \( E_X \)-modules and \( \Lambda \) a closed subvariety of \( T^*X \) containing \( \text{Supp} \mathcal{N}^* \), \( \mu_E(\text{Eu}^E_X(\mathcal{N}^*)) = [\tilde{\mu}_E(ch^E_X(\mathcal{N}^*))]^{2d} \).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
H^0_A(T^*X; CC^\text{per}(E_X)) & \xrightarrow{\tilde{\mu}_E} & H^0_A(T^*X; C_{T^*X}[2d][u]) \\
\downarrow & & \downarrow u \mapsto 0 \\
H^0_A(T^*X; C^\text{per}(E_X)) & \xrightarrow{\mu_E} & H^{2d}_A(T^*X; \mathbb{C})
\end{array}
\]

\[ \square \]

### 3. The Schapira–Schneiders conjecture

In this section we arrive at the statement of the main result of this paper which implies the Schapira–Schneiders conjecture and explain how it follows from the the analogous theorem for formal symplectic deformations. The proof of the latter is presented in the sequel to this paper.

#### 3.1. The Rees construction.

The sheaf of algebras \( E_X \) carries a natural filtration \( F^*_X \) by order. The Rees ring \( \mathcal{R}E_X \) is the (graded) algebra flat over \( \mathbb{C}[\hbar] \) defined by

\[ \mathcal{R}E_X \overset{\text{def}}{=} \bigoplus_p F_p E_X \cdot t^p \]

The Rees module \( \mathcal{R} \mathcal{N} \) (which is a graded module over \( \mathcal{R}E_X \)) associated to a filtered module \((\mathcal{N}, F_\bullet)\) is defined similarly. Thus defined, the Rees construction extends to an exact functor \( \mathcal{R} \) form the (exact) category of filtered \( E_X \)-modules to the (Abelian) category of graded \( \mathcal{R}E_X \)-modules.

The functor \( \mathcal{R} \) is an embedding with the essential image consisting of
the subcategory of \( t \)-torsion free modules. The induced functor between the respective derived categories is an equivalence.

The filtered complex \((N^\bullet, F^\bullet)\) is *good* if and only if the complex of \( \mathcal{E}_X \)-modules \( \mathcal{R}N^\bullet \) is perfect. The Rees construction restricts to an equivalence between the derived categories of good filtered complexes of \( \mathcal{E}_X \)-modules and perfect complexes of graded \( \mathcal{R}E_X \)-modules. In particular it induces an isomorphism of respective Grothendieck groups.

Setting \( t = 0 \) one obtains the map
\[
\sigma : \mathcal{R}E_X \to Gr^F \mathcal{E}_X \to \mathcal{O}_{T^* X}.
\]
Note that
\[
\sigma(N) \overset{\text{def}}{=} Gr^F N \otimes_{Gr^F \mathcal{E}_X} \mathcal{O}_{T^* X} = \mathcal{R}N \otimes_{\mathcal{R}E_X} \mathcal{O}_{T^* X}.
\]

Consider a filtered complex \((N^\bullet, F^\bullet)\) with \( \mathcal{R}N^\bullet \) perfect over \( \mathcal{R}E_X \) with support contained in \( \Lambda \subset T^* X \). Then,
\[
ch_{\Lambda}^{\mathcal{O}_{T^* X}}(\sigma(N^\bullet)) = \sigma \left( ch_{\Lambda}^{\mathcal{R}E_X}(\mathcal{R}N^\bullet) \right).
\]

The canonical homomorphism of algebras
\[
i : \mathcal{R}E_X \to \mathcal{R}E_X[t^{-1}] \overset{\cong}{\to} \mathcal{E}_X[t^{-1}, t]
\]
induces the natural isomorphism \( \mathcal{R}N^\bullet[t^{-1}] \cong N^\bullet \otimes_{\mathcal{E}_X} \mathcal{E}_X[t^{-1}, t] \). Therefore,
\[
ch^{\mathcal{E}_X}_\Lambda(N^\bullet) = \iota \left( ch_{\Lambda}^{\mathcal{R}E_X}(\mathcal{R}N^\bullet) \right).
\]

In particular the right hand side is independent of \( t \).

### 3.2. Cyclic homology of the structure sheaf

We recall, briefly, the well known calculation of the cyclic homology of the structure sheaf \( \mathcal{O}_M \) of a complex manifold \( M \) of Hochschild-Kostant-Rosenberg-Connes.

Let \( \hat{C}_p(\mathcal{O}_M) \) denote the completion of \( \mathcal{O}_{M \times \mathbb{P}^1} \) along the diagonal. The Hochschild differential extends to the map \( b : \hat{C}_p(\mathcal{O}_M) \to \hat{C}_{p-1}(\mathcal{O}_M) \) and the resulting complex \( \hat{C}_\bullet(\mathcal{O}_M) \) represents \( \mathcal{O}_M \otimes_{\mathcal{O}_{M \times \mathbb{P}^1}} \mathcal{O}_M \) in the derived category. There is a canonical map \( C_\bullet(\mathcal{O}_M) \to \hat{C}_\bullet(\mathcal{O}_M) \).

The assignment \( f_0 \otimes \cdots \otimes f_p \mapsto \frac{1}{p!} f_0 df_1 \wedge \cdots df_p \) extends to a of complexes
\[
\mu_{\mathcal{O}} : \hat{C}_\bullet(\mathcal{O}_M) \to \bigoplus_p \Omega^p_M[p]
\]
which, according to a theorem of Hochschild-Kostant-Rosenberg, is a quasi-isomorphism.
The cyclic differential $B$ extends to the map $B : \hat{C}_p(\mathcal{O}_M) \to \hat{C}_{p+1}(\mathcal{O}_M)$ so that the square
\[
\begin{array}{ccc}
\hat{C}_p(\mathcal{O}_M) & \longrightarrow & \Omega_{M}^p \\
B \downarrow & & \downarrow d \\
\hat{C}_{p+1}(\mathcal{O}_M) & \longrightarrow & \Omega_{M}^{p+1}
\end{array}
\]
commutes. The cyclic complex $\hat{CC}^\text{per}_\bullet(\mathcal{O}_X)$ is defined as in 5.1. Thus, the Hochschild-Kostant-Rosenberg map induces the map of complexes
\[
\tilde{\mu}_\mathcal{O} : \hat{CC}^\text{per}_\bullet(\mathcal{O}_X) \to \Omega_{M}^{\bullet}[u^{-1}, u]
\]
which, according to A. Connes, is a quasi-isomorphism.

The natural inclusion $\mathbb{C}_M[u^{-1}, u] \to CC^\text{per}(\mathbb{C}_M)$ is easily seen to be a quasi-isomorphism and the composition
\[
\mathbb{C}_M[u^{-1}, u] \to CC^\text{per}(\mathbb{C}_M) \to CC^\text{per}_\bullet(\mathcal{O}_X) \to \hat{CC}^\text{per}_\bullet(\mathcal{O}_X) \to \Omega_{M}^{\bullet}[u^{-1}, u]
\]
coincides with the canonical map $\mathbb{C}_M \to \mathbb{O}_M$.

By abuse of notation we will denote by $\tilde{\mu}_\mathcal{O}$ the morphism in the derived category represented by
\[
\tilde{\mu}_\mathcal{O} : CC^\text{per}_\bullet(\mathcal{O}_X) \to \mathbb{C}_M[u^{-1}, u] \xleftarrow{\sim} \mathbb{C}_M[u^{-1}, u] \quad (3.1)
\]
which, by the discussion above, is a one-sided inverse to the map induced by the inclusion $\mathbb{C}_M \to \mathbb{O}_M$.

The Chern character with supports (as in (1.1)) for perfect complexes of $\mathcal{O}_X$-modules supported on a closed subvariety $Z$ of a complex manifold $M$ is defined as the composition
\[
K^0_Z(\mathcal{O}_X) \xrightarrow{\text{ch}_{\mathcal{O}_X}^\text{OX}} H^0_Z(X; CC^\text{per}_\bullet(\mathcal{O}_X)) \xrightarrow{\tilde{\mu}_\mathcal{O}} \bigoplus_p H^p_Z(X; \mathbb{C}).
\]

3.3. The Riemann-Roch formula. In view of Lemma 2.7.1 and Proposition 2.4.1 to prove Conjecture 1.2.2 it is sufficient to show that
\[
\tilde{\mu}_\mathcal{E} \left( \tau \left( ch^\mathcal{R\mathcal{E}_X}_\Lambda(\mathcal{R}\mathcal{N}^\bullet) \right) \right) = \tilde{\mu}_\mathcal{O} \left( \sigma \left( ch^\mathcal{R\mathcal{E}_X}_\Lambda(\mathcal{R}\mathcal{N}^\bullet) \right) \right) \sim \pi^* Td(TX). \quad (3.2)
\]

The above reformulation of Conjecture 1.2.2 in terms of the Chern character with values in periodic cyclic homology is crucial since, as it turns out, the formula (3.2) holds with $ch^\mathcal{R\mathcal{E}_X}_\Lambda(\mathcal{R}\mathcal{N}^\bullet)$ replaced by an arbitrary cycle in the periodic cyclic complex. The latter fact constitutes the main result of the paper and is formulated in the following theorem.
Theorem 3.3.1. The diagram in the derived category of sheaves on $T^*X$

$$
\begin{array}{ccc}
CC^\per_\bullet(\mathcal{R}E_X) & \xrightarrow{\sigma} & CC^\per_\bullet(\mathcal{O}_{T^*X}) \\
\downarrow \iota & & \downarrow \mu_\mathcal{O} \circ \pi^* Td(TX) \\
CC^\per_\bullet(\mathcal{E}_X)[t^{-1}, t] & \xrightarrow{\tilde{\mu}} & \mathbb{C} T^*_X[t^{-1}, t][u^{-1}, u]
\end{array}
$$

is commutative.

Proof. The statement follows from Theorem 4.6.1 applied to the case $M = T^*X$ and $\mathbb{A}_{T^*X}^t$ furnished by Proposition 4.5.1 (with $\theta = \pi^* c_1(TX)/2$), Corollary 4.5.3, and the equality $\tilde{\mathcal{A}}(TM) \sim e^\theta = \pi^* Td(TX)$. □

4. Deformation quantization

4.1. Weyl quantization. Consider the vector space $\mathbb{C}^{2d}$ with coordinates $x_1, \ldots, x_d, \xi_1, \ldots, \xi_d$ as a symplectic manifold equipped with the standard symplectic form $dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_d \wedge d\xi_d$. The sheaf of algebras $\mathbb{A}_{\mathbb{C}^{2d}}^t$ is defined as follows.

For $U$ an open subset of $\mathbb{C}^{2d}$ the $\mathbb{C}[[t]]$-module underlying $\mathbb{A}_{\mathbb{C}^{2d}}^t(U)$ is $\mathcal{O}_{\mathbb{C}^{2d}}(U)[[t]]$ and the product on $\mathbb{A}_{\mathbb{C}^{2d}}^t(U)$ is given by the standard Moyal–Weyl product

$$(f \ast g)(x, \xi) = \exp \left( \frac{t}{2} \sum_{i=1}^{d} \left( \frac{\partial f}{\partial \xi_i} \cdot \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \cdot \frac{\partial g}{\partial \xi_i} \right) \right) f(x, \xi) g(y, \eta) \bigg|_{x=y, \xi=\eta}$$

where $x = (x_1, \ldots, x_d)$, $\xi = (\xi_1, \ldots, \xi_d)$, $y = (y_1, \ldots, y_d)$, $\eta = (\eta_1, \ldots, \eta_d)$.

4.2. Symplectic deformation quantization. Let $(M, \omega)$ denote a symplectic manifold of dimension $\dim_{\mathbb{C}} M = 2d$. For purposes of this paper a (symplectic) deformation quantization of $M$ is a formal one parameter deformation of the structure sheaf $\mathcal{O}_M$, i.e. a sheaf of algebras $\mathbb{A}_M^t$ flat over $\mathbb{C}[[t]]$ equipped with an isomorphism of algebras $\mathbb{A}_M^t \otimes_{\mathbb{C}[[t]]} \mathbb{C} \to \mathcal{O}_M$ and having the following properties:

- it is locally isomorphic to the standard deformation $\mathbb{A}_{\mathbb{C}^{2d}}^t$ of $\mathbb{C}^{2d}$ (see below);
- the Poisson bracket on $\mathcal{O}_M$ defined by $\{f, g\} = \frac{1}{t} [\tilde{f}, \tilde{g}] + t \cdot A_M^t$

where $f$ and $g$ are two local sections of $\mathcal{O}_M$ and $\tilde{f}$, $\tilde{g}$ are their respective lifts $\mathbb{A}_M^t$, coincides with the one induced by the symplectic structure.
To a symplectic deformation quantization $\mathbb{A}_M^t$ one associates a characteristic class $\theta \in H^2(M; 1/t C[[t]])$ (see [NT3] and [E3] below) with the property that the coefficient of $1/t$ is the class of the symplectic form.

In what follows we will use the canonical maps $\sigma : \mathbb{A}_M^t \to \mathcal{O}_M$ and $\iota : \mathbb{A}_M^t \to \mathbb{A}_M^t[t^{-1}]$.

### 4.3. Review of Fedosov connections.

We recall briefly the definition of the characteristic class $\theta$ in terms of the Fedosov connections.

Let $W = \mathbb{C}[[\hat{z}_1, \ldots, \hat{z}_d, \hat{\xi}_1, \ldots, \hat{\xi}_d, t]]$ equipped with the Weyl product denoted $\ast$ while the “commutative” product will not be indicated by any special notations. Thus, equipped with the Weyl product, $W$ is a deformation quantization of the formal completion of $\mathbb{C}^{2d}$ at the origin. The induced symplectic structure is the standard one. Let $\mathfrak{g}$ denote the Lie algebra of continuous derivations of $W$. Let $\tilde{\mathfrak{g}}$ denote the Lie algebra $\mathfrak{g}$ with the bracket $[1/t f, 1/t g] = 1/t (f \ast g - g \ast f)$.

The exact sequence

$$0 \to 1/t C[[t]] \to \tilde{\mathfrak{g}} \xrightarrow{\frac{1}{2}ad} \mathfrak{g} \to 0$$

is a central extension of Lie algebras.

Let $\deg \hat{z}_i = \deg \hat{\xi}_i = 1$, $\deg t = 2$, let $\tilde{\mathfrak{g}}_k$ denote the subspace of homogeneous elements of degree $k$, and let $\mathfrak{g}_k$ denote the image of $\tilde{\mathfrak{g}}_k$. Then $\sum_k \tilde{\mathfrak{g}}_k$ (respectively $\sum_k \mathfrak{g}_k$) is a $\mathbb{Z}$-graded dense subalgebra of $\tilde{\mathfrak{g}}$ (respectively $\mathfrak{g}$).

The Lie algebra $\mathfrak{sp}(2d, \mathbb{C})$ is naturally embedded in both $\tilde{\mathfrak{g}}_0$ and $\mathfrak{g}_0$ as the span of the monomials $\frac{1}{t} \hat{z}_i \hat{z}_j$, $\frac{1}{t} \hat{z}_i \hat{\xi}_j$, and $\frac{1}{t} \hat{\xi}_i \hat{\xi}_j$. The (adjoint) action of $\mathfrak{sp}(2d, \mathbb{C})$ on $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ integrates to an action of the group $Sp(2d, \mathbb{C})$.

Suppose that $M$ is a complex manifold of dimension $\dim_{\mathbb{C}} M = 2d$, $\omega \in \Gamma(M; \Omega^2_M)$ a symplectic form on $M$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a cover of $M$ which trivializes $TM$. Set $U_{\alpha, \beta} = U_\alpha \cap U_\beta$ and let $g_{\alpha, \beta} : U_{\alpha, \beta} \to Sp(2d, \mathbb{C})$ denote the transition functions for $TM$.

Let $\mathbb{W}_M$ denote the sheaf of $\mathcal{O}_M$ algebras which corresponds to the representation $W$ of $Sp(2d, \mathbb{C})$ (“the sheaf of holomorphic section of the associated bundle”). A Fedosov connection is a flat $\mathfrak{g}$-connection $\nabla^\Phi$ on $\mathbb{W}_M \otimes_{\mathcal{O}_M} C^\infty_M$ of a particular kind which we proceed to describe.

Locally (i.e. on a $U_\alpha \in \mathcal{U}$) a Fedosov connection is given by $\nabla^\Phi = d + A_\alpha$, where $A_\alpha = \sum_{k \geq -1} A_\alpha^{(k)}$ with $A_\alpha^{(k)} \in \mathbb{A}_M^1(U_\alpha) \otimes \mathfrak{g}_k$. Moreover, in
Darboux coordinates $z_1, \ldots, z_d, \xi_1, \ldots, \xi_d$, $A^{(-1)} = \frac{1}{t} \sum_i dz_i \otimes \xi_i - d\xi_i \otimes \hat{z}_i$, and $A^{(0)}_\alpha \in A^1_M(U_\alpha) \otimes \mathfrak{sp}(2d, \mathbb{C})$ so that $\nabla^{(0)} = d + A^{(0)}_\alpha$ is a torsion free connection.

A Fedosov connection $\nabla^\Phi$ admits a lifting to a $\tilde{g}$-connection $\tilde{\nabla}^\Phi$ (which is not flat) with curvature $c(\tilde{\nabla}^\Phi) \in \Gamma(M; A^{2, cl}_M \otimes \mathbb{C}[[t]])$ where $A^{2, cl}_M$ is the sheaf of closed 2-forms.

The subsheaf of horizontal sections $(\mathbb{W}_M \otimes \mathcal{O}_M \mathbb{C}^\infty_M)^{\nabla^\Phi}$ is a deformation quantization of $(M, \omega)$ with characteristic class $\theta$ the cohomology class of $c(\tilde{\nabla}^\Phi)$ which is independent of the choice of a lifting.

It is shown in [NT3] that every deformation quantization arises in from a Fedosov connection and that the characteristic classes of isomorphic deformations coincide.

4.4. Homological properties of symplectic deformations. The sheaf of algebras $A^t_M \otimes \mathbb{C}[[t]] (A^t_M)^{op}$ is contained in a unique symplectic deformation quantization $A^t_M \times 2$ of the symplectic manifold $(M, \omega) \times (M, -\omega)$.

Let

$$(A^t_M)^{\epsilon} \overset{def}{=} \delta^{-1} A^t_M \times 2 .$$

Then there is a natural map of sheaves of algebras $A^t_M \otimes \mathbb{C}[[t]] (A^t_M)^{op} \to (A^t_M)^{\epsilon}$.

Let

$$\tilde{\mathcal{C}}_p(A^t_M) \overset{def}{=} \lim_{\leftarrow n} \left( A^t_M \otimes \mathbb{C}[[t]]^{p+1} / I_{(p+1)} \right) ,$$

where (the two-sided ideal) $I_{(p+1)}$ is the kernel of the composition of the multiplication map with the symbol map $(A^t_M \otimes \mathbb{C}[[t]])^{p+1} \to A^t_M \to \mathcal{O}_M$. The Hochschild differential [5.1] extends to this setting and gives rise to the complex $\tilde{\mathcal{C}}_\bullet(A^t_M)$ and the natural map of complexes $C_\bullet(A^t_M) \to \tilde{\mathcal{C}}_\bullet(A^t_M)$. The complex $\tilde{\mathcal{C}}_\bullet(A^t_M)$ represents $A^t_M \otimes \mathbb{C}(A^t_M)^{op}$ in the derived category.

The cyclic complexes $\tilde{\mathcal{C}}_\bullet(C_\bullet(A^t_M))$ and $\tilde{\mathcal{C}}_\bullet^{per}(A^t_M)$ are defined as in [5.1] starting from $\tilde{\mathcal{C}}_p(A^t_M)$. The analogs of [5.3] and [5.4] hold and there are natural maps of complexes $CC_\bullet^{per}(A^t_M) \to \tilde{\mathcal{C}}_\bullet^{per}(A^t_M)$ and $CC_\bullet^{per}(A^t_M) \to \tilde{\mathcal{C}}_\bullet^{per}(A^t_M)$.

Proposition 4.4.1. Suppose that $A^t_M$ is a symplectic deformation quantization of $M$. 
1. Let \( x_1, \ldots, x_d, \xi_1, \ldots, \xi_d \) be a local Darboux coordinate system on \( M \). Then the local section \( \Phi^A \) of \( H^{-2d} \hat{C}_*(A^t_M)[t^{-1}] \) represented by 
\[
\text{Alt}(1 \otimes x_1 \otimes \cdots \otimes x_d \otimes \xi_1 \otimes \cdots \otimes \xi_d / t)
\]
is independent of the choice of the Darboux coordinates.
2. The section \( \Phi^A \) generates \( H^{-2d} \hat{C}_*(A^t_M)[t^{-1}] \) as a \( C_M[t^{-1}, t][2d] \)-module and \( H^p \hat{C}_*(A^t_M)[t^{-1}] = 0 \) for \( p \neq -2d \), thus and there is a unique isomorphism
\[
\mu_A : \hat{C}_*(A^t_M)[t^{-1}] \to C_M[t^{-1}, t][2d] \quad (4.1)
\]
in the derived category of sheaves which maps \( \Phi^A \) to 1.
3. There exists an isomorphism of filtered complexes
\[
(\hat{C}_*(A^t_M)[t^{-1}], b + uB, F^*) \cong (\hat{C}_*(A^t_M)[t^{-1}][u], b, F^*)
\]
which induces the identity map on the associated graded objects and any two such are homotopic.
4. There exists an isomorphism of filtered complexes
\[
(\hat{C}_*^{\text{per}}(A^t_M)[t^{-1}], b + uB, F^*) \cong (\hat{C}_*(A^t_M)[t^{-1}][u^{-1}, u], b, F^*)
\]
which induces the identity map on the associated graded objects and any two such are homotopic.

Proof. Since the statement is local it is sufficient to consider the stalk at the origin of the standard deformation of \( \mathbb{C}^{2d} \). Note that the completion of \( \mathcal{A}^t_{\mathbb{C}^{2d}, 0} \) with respect to the powers of the two-sided ideal which is the kernel of the composition \( \mathcal{A}^t_{\mathbb{C}^{2d}, 0} \to \mathcal{O}_{\mathbb{C}^{2d}, 0} \to \mathbb{C}_0 \) coincides with the Weyl algebra \( W = W(\mathbb{C}^{2d}) \). The induced map \( \hat{C}_*(A^t_{\mathbb{C}^{2d}, 0}) \to \hat{C}_*(W) \) is a quasisomorphism and the calculation of Hochschild homology follows from the standard results for the Weyl algebra. To prove the last two statements one argues as in Lemma 2.6.1 and Corollary 2.6.2. □

By abuse of notation we will denote by \( \mu_A \) the composition
\[
C_*^{\text{per}}(A^t_M) \to \hat{C}_*(A^t_M)[t^{-1}] \overset{(4.1)}{\to} C_M[t^{-1}, t][2d]
\]
as well as \( (4.1) \).

Proceeding as in 2.7 and using Proposition 4.4.1 one defines the morphism
\[
\tilde{\mu}_A : CC_*(A^t_M)[t^{-1}] \to C_M[t^{-1}, t][2d][u^{-1}, u] .
\]
4.5. **Deformation quantization of a cotangent bundle.** For applications of our results to microlocal analysis we will require the particular deformation quantization of a cotangent bundle furnished by the following proposition.

**Proposition 4.5.1.** There exists a deformation quantization $\mathbb{A}^{t}_{T^\ast X}$ of $T^\ast X$ and faithfully flat map $\mathcal{R}E_X \to \mathbb{A}^t_{T^\ast X}$ of algebras over $\mathbb{C}[t]$. The characteristic class $\theta$ of the deformation $\mathbb{A}^t_{T^\ast X}$ is equal to $\frac{1}{2} \pi^* c_1(TX)$ (note that the symplectic form is exact in this case).

**Proof.** Suppose that $x_1, \ldots, x_d$ is a local coordinate system on $X$ and let $\xi_i$ denote the symbol of $\frac{\partial}{\partial x_i}$ viewed as a function on $T^\ast X$. Then $x_1, \ldots, x_d, \xi_1, \ldots, \xi_d$ form a Darboux coordinate system on $T^\ast X$. The map alluded to in Proposition 4.5.1 is given in local coordinates by $x_i \mapsto x_i, t \frac{\partial}{\partial x_i} \mapsto \xi_i$.

The only non-trivial part is the calculation of the characteristic class $\theta$. This will be achieved by explicitly manufacturing a Fedosov connection $\nabla^\theta$ on the sheaf of Weyl algebras $\mathcal{W}_{T^\ast X}$ whose sheaf of horizontal sections is $\mathbb{A}^t_{T^\ast X}$ and computing the Weyl curvature.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ denote a cover of $X$ which trivializes $TX$. Set $U_{\alpha,\beta} = U_\alpha \cap U_\beta$ and let $g_{\alpha,\beta} : U_{\alpha,\beta} \to GL(d, \mathbb{C})$ denote the transition functions for $TX$.

Let $\mathfrak{w}_d$ denote the Lie algebra of continuous derivations of the (topological) $\mathbb{C}$-algebra $\mathbb{C}[[\hat{z}_1, \ldots, \hat{z}_d]]$. Let $\mathfrak{w}_{d,k} = \{\sum_i P_i \frac{\partial}{\partial \hat{z}_i}\}$ where $P_i$ are homogeneous polynomials (in $\hat{z}_1, \ldots, \hat{z}_d$) of degree $\deg P_i = k + 1$. Then $[\mathfrak{w}_{d,k}, \mathfrak{w}_{d,l}] \subseteq \mathfrak{w}_{d,k+l}$ and $\sum_k \mathfrak{w}_{d,k}$ is a dense subalgebra of $\mathfrak{w}_d$.

The Lie algebra $\mathfrak{gl}(d, \mathbb{C})$ is identified with $\mathfrak{w}_{d,0}$ by the map $(a_{i,j}) \mapsto \sum_{i,j} a_{i,j} \hat{z}_i \frac{\partial}{\partial \hat{z}_j}$. The (adjoint) action of $\mathfrak{gl}(d, \mathbb{C})$ on $\mathfrak{w}_d$ integrates to the action of $GL(d, \mathbb{C})$. The action of $\mathfrak{w}_{d,0}$ on $\mathfrak{w}_{d,-1}$ is identified with the standard representation of $\mathfrak{gl}(d, \mathbb{C})$ on $\mathbb{C}^d$.

We begin with a torsion free $\mathfrak{gl}(d, \mathbb{C})$-connection $\nabla^{(0)}$ on $TX$ given on $U_\alpha$ by $d + A^{(0)}_\alpha$, where $A^{(0)}_\alpha \in \mathcal{A}^{1,0}_X \otimes \mathfrak{gl}(d, \mathbb{C})$. The connection $\nabla^{(0)}$ extends to a flat $\mathfrak{w}_d$-connection (called a Kazhdan connection) $\nabla^K$ which is, locally, of the form $d + A_\alpha$, where $A_\alpha \in \mathcal{A}^{1,0}_X \otimes \mathfrak{w}_d$, $A_\alpha = \sum_k A^{(k)}_\alpha$ with $A^{(-1)} = - \sum_i dz_i \otimes \frac{\partial}{\partial \hat{z}_i}$ and $A^{(0)}_\alpha$ as above.

Let $\hat{S} = \mathbb{C}[[\hat{z}_1, \ldots, \hat{z}_d]]$; this is a $(\mathfrak{w}_d, GL(d, \mathbb{C}))$-module. Let $\hat{S}$ denote the corresponding sheaf of $\mathcal{O}_X$-algebras (“holomorphic sections of the associated bundle”). The Kazhdan connection $\nabla^K$ is a connection on
\( \hat{S} \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty \), and the sheaf of horizontal sections \( \left( \hat{S} \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty \right)^{\nabla^K} \) is a sheaf of algebras (since \( w_d \) acts by derivations) isomorphic to \( \mathcal{O}_X \).

Let \( \hat{R} = \mathbb{C}[[\hat{z}_1, \ldots, \hat{z}_d, \hat{\xi}_1, \ldots, \hat{\xi}_d, t]] \). We consider \( \hat{R} \) as an algebra with the Weyl product denoted *. The algebra \( \hat{R} \) is an algebra over \( \hat{S} \) in the obvious way and is isomorphic to the completion of the Rees algebra of the algebra of differential operators on \( \hat{S} \) by the assignment \( t \frac{\partial}{\partial \hat{z}_i} \mapsto \hat{\xi}_i \).

Let \( \text{Der}(\hat{R}) \) denote the Lie algebra of continuous, \( \mathbb{C}[[t]] \)-linear derivations of \( \hat{R} \). The exact sequence
\[
0 \rightarrow \frac{1}{t} \mathbb{C}[[t]] \rightarrow \frac{1}{t} \hat{R} \rightarrow \text{Der}(\hat{R}) \rightarrow 0
\]
is a central extension of Lie algebras. Just as in the case of the Weyl algebra in [13], the Lie algebra \( \mathfrak{sp}(2d, \mathbb{C}) \) is contained in \( \frac{1}{t} \hat{R} \) and \( \text{Der}(\hat{R}) \) is the standard way. The inclusion of \( \mathfrak{gl}(d, \mathbb{C}) \) into \( \mathfrak{sp}(2d, \mathbb{C}) \) is given in terms of the standard embedding by \((a_{i,j}) \mapsto \sum_{i,j} a_{i,j} \hat{z}_i \hat{\xi}_j \). The action of \( \mathfrak{gl}(d, \mathbb{C}) \) integrates to an action of \( GL(d, \mathbb{C}) \).

The identification with the differential operators (above) restricts to the map of Lie algebras \( i : w_d \rightarrow \frac{1}{t} \hat{R} \) whose restriction to \( \mathfrak{gl}(d, \mathbb{C}) \) is
\[
(a_{i,j}) \mapsto \sum_{i,j} a_{i,j} \hat{z}_i \hat{\xi}_j = \sum_{i,j} a_{i,j} \hat{z}_i \hat{\xi}_j - \frac{1}{2} tr(a_{i,j}) .
\] (4.2)

Note that the first summand in the last expression is the standard embedding of \( \mathfrak{gl}(d, \mathbb{C}) \). Since the second summand in the last expression is central the standard embedding and \( i \) coincide in \( \text{Der}(\hat{R}) \).

Let \( \hat{\mathcal{R}} \) denote the sheaf of \( \mathcal{O}_X \)-algebras on \( X \) which corresponds to \( \hat{R} \). The Kazhdan connection \( \nabla^K \) gives rise to a connection, still denoted \( \nabla^K \), on \( \hat{\mathcal{R}} \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty \) via the composition \( \frac{1}{t} \text{ad} \circ i \). The sheaf of horizontal sections \( \left( \hat{\mathcal{R}} \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty \right)^{\nabla^K} \) is isomorphic to the completion of the Rees algebra \( \mathcal{R} \mathcal{D}_X \).

There is a canonical isomorphism of algebras \( \hat{\mathcal{R}} \cong \mathcal{W}_{T^*X} \otimes_{\mathcal{O}_{T^*X}} \mathcal{O}_X \). Since the sheaves of algebras \( \mathcal{W}_{T^*X} \otimes_{\mathcal{O}_{T^*X}} \mathcal{C}_T^\infty \) and \( \pi^* \hat{\mathcal{R}} \otimes_{\mathcal{O}_{T^*X}} \mathcal{C}_T^\infty \) are (non-canonically) isomorphic, it suffices to construct a “Fedosov” \( \text{Der}(\hat{R}) \)-connection on the latter and this is what we will do explicitly, at least in the formal neighborhood of the zero section of \( T^*X \).
Let $z_1, \ldots, z_d$ denote local coordinates on $X$, $\xi_i$ the symbol of $\frac{\partial}{\partial z_i}$, so that $z_1, \ldots, z_d, \xi_1 \ldots \xi_d$ is a local Darboux coordinate system on $T^*X$. The automorphism $\Psi$ of sheaf of algebras $\pi^*R$ given in local coordinates by the formula $\Psi = \exp(ad(-\sum_i \xi_i \hat{z}_i^t))$ is, in fact globally well defined on the formal neighborhood of the zero section of $T^*X$. The $\Psi$-conjugate of the pull-back $\pi^*\nabla K$ of the Kazhdan connection is the desired Fedosov connection: $\nabla^\Phi = (\nabla K)^\Psi$.

Suppose $\tilde{\pi}^*\nabla K$ is a lifting of $\pi^*\nabla K$ to a $\frac{1}{t}\hat{R}$-connection. Then $(\tilde{\pi}^*\nabla K)^\Psi$ is a lifting of $\nabla^\Phi$, and $c(\pi^*\nabla K) = c\left((\pi^*\nabla K)^\Psi\right)$ since the curvature is central and $\Psi$ is an “inner” automorphism.

In order to find an explicit lifting $\pi^*\nabla K$, consider the $\frac{1}{t}\hat{R}$-valued forms $i(A_\alpha)$. Since the $(\mathfrak{m}_d, \mathbb{C})$-valued connection forms $A_\alpha$ satisfy
\[
A_\alpha = dg_{\alpha,\beta}^t(g_{\alpha,\beta})^{-1} + g_{\alpha,\beta}^tA_{\beta}(g_{\alpha,\beta})^{-1},
\]
the forms $iA_\alpha$ satisfy
\[
i(A_\alpha) = i(dg_{\alpha,\beta}^t(g_{\alpha,\beta})^{-1}) + i(g_{\alpha,\beta}^tA_{\beta}(g_{\alpha,\beta})^{-1})
= dg_{\alpha,\beta}^t(g_{\alpha,\beta}^t)^{-1} - \frac{1}{2}tr(dg_{\alpha,\beta}^t(g_{\alpha,\beta}^t)^{-1}) + g_{\alpha,\beta}^ti(A_{\beta})(g_{\alpha,\beta}^t)^{-1},
\]
the last equality obtained using (4.2). Since $A_\alpha^{(0)}$ are $(\mathfrak{gl}(d, \mathbb{C}))$-valued connection forms and trace remains invariant under conjugation the identity
\[
\frac{1}{2}tr(A_\alpha^{(0)}) = \frac{1}{2}tr(dg_{\alpha,\beta}^t(g_{\alpha,\beta}^t)^{-1}) + \frac{1}{2}tr(A_{\beta}^{(0)})
\]
holds. Adding the latter identity to the preceding one one easily concludes that the collection of $\frac{1}{t}\hat{R}$-valued forms $\tilde{A}_\alpha = i(A_\alpha) + \frac{1}{2}tr(A_\alpha^{(0)})$ determine a connection $\tilde{\nabla}^K$ which lifts the Kazhdan connection.

Since the Kazhdan connection is flat and $i$ is a morphism of Lie algebras, it follows that $c(\tilde{\nabla}) = \frac{1}{2}tr((\nabla^{(0)})^2)$ which represents $\frac{1}{2}c_1(TX)$ in de Rham cohomology.

Hence, the cohomology class of the curvature of the connection $\pi^*\nabla K$ is equal to $\frac{1}{2}\pi^*c_1(TX)$ and so is the characteristic class of the deformation.

In what follows $A_\alpha^t_T$ will always refer to the deformation furnished by Proposition 4.5.1.
Lemma 4.5.2. The diagram
\[ C_\bullet(\mathcal{R}\mathcal{E}_X)[t^{-1}] \xrightarrow{\sim} C_\bullet(\mathcal{E}_X)[t^{-1}, t] \]
\[ \downarrow \quad \downarrow \mu_\varepsilon \]
\[ C_\bullet(\mathcal{A}^t_{T^*X})[t^{-1}] \xrightarrow{\mu^t_\lambda} \mathbb{C}_{T^*X}[t^{-1}, t][2d] \]
is commutative.

Proof. The statement is equivalent to the commutativity of the diagram of respective cohomology sheaves in degree \(-2d\). The latter fact follows from the local coordinate representation of \(\Phi^E, \Phi^A\), the map \(\mathcal{R}\mathcal{E}_X \rightarrow \mathcal{A}^t_{T^*X}\), and the definitions of \(\mu_\varepsilon\) and \(\mu^t_\lambda\).

Corollary 4.5.3. The diagram
\[ CC_\bullet(\mathcal{R}\mathcal{E}_X)[t^{-1}] \xrightarrow{\sim} CC_\bullet(\mathcal{E}_X)[t^{-1}, t] \]
\[ \downarrow \quad \downarrow \bar{\mu}_\varepsilon \]
\[ CC_\bullet(\mathcal{A}^t_{T^*X})[t^{-1}] \xrightarrow{\bar{\mu}^t_\lambda} \mathbb{C}_{T^*X}[t^{-1}, t][2d][u^{-1}, u] \]
is commutative.

4.6. The Riemann-Roch theorem for periodic cyclic cocycles.
We are finally ready to state the Riemann-Roch type theorem for symplectic deformations. Consider a complex manifold \(M\) and a symplectic deformation quantization \(\mathcal{A}^t_M\) of \(M\) with the characteristic class \(\theta\).

The deformation quantization \(\mathcal{A}^t_M\) induces the symplectic structure on \(M\). In particular the symplectic structure determines a reduction of the structure group of \(TM\) to \(Sp(2d, \mathbb{C})\). The characteristic classes of \(Sp(2d, \mathbb{C})\)-bundles are given by the invariant functions on the Lie algebra \(sp(2d, \mathbb{C})\). In particular the class \(\hat{A}(TM)\) corresponds to the function
\[ sp(2d, \mathbb{C}) \ni X \mapsto \det \left( \frac{\exp(\text{ad}(X))}{\exp(\text{ad}(\frac{X}{2})) - \exp(-\frac{X}{2})} \right). \]

Theorem 4.6.1. The diagram
\[ CC_\bullet^{per}(\mathcal{A}^t_M) \xrightarrow{\sigma} CC_\bullet^{per}(\mathcal{O}_M) \]
\[ \downarrow \mu^t_\lambda \quad \downarrow \bar{\mu}_\varepsilon \hat{A}(TM) - \epsilon^\theta \]
\[ CC_\bullet^{per}(\mathcal{A}^t_M)[t^{-1}] \xrightarrow{\bar{\mu}^t_\lambda} \mathbb{C}_{M}[t^{-1}, t][u^{-1}, u] \]
is commutative.
Theorem 4.6.1 may be deduced from [NT1] and [NT2]. A complete proof will appear in the sequel to the present paper.

5. Appendix: Review of Hochschild and cyclic homology

In this section we review the basic definitions of the Hochschild and the (negative, periodic) cyclic complex of an algebra, of a sheaf of algebras on a space as well as of the “topological” versions of the above. In particular we establish notational conventions with regard to Hochschild and cyclic complexes which are used in the body of the paper.

In addition we review the results concerning the Hochschild and cyclic homologies of certain examples of (sheaves of) algebras which appear in this paper for the reader’s convenience.

5.1. Hochschild and cyclic complexes of algebras. Let \( k \) denote a commutative algebra over a field of characteristic zero and let \( A \) be a flat \( k \)-algebra with \( 1_A \cdot k \) contained in the center, not necessarily commutative. Let \( \overline{A} = A/k \), and let \( C_p(A) \overset{def}{=} A \otimes_k \overline{A}^{\otimes p} \) and let

\[
b : C_p(A) \rightarrow C_{p-1}(A)
\]

\[
a_0 \otimes \cdots \otimes a_p \mapsto (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p.
\]

Then \( b^2 = 0 \) and the complex \((C_\bullet(A), b)\), called the standard Hochschild complex of \( A \) represents \( A \otimes_k A^{op} \otimes_k A \) in the derived category of \( k \)-modules.

The map

\[
B : C_p(A) \rightarrow C_{p+1}(A)
\]

\[
a_0 \otimes \cdots \otimes a_p \mapsto \sum_{i=0}^p (-1)^{p+i} a_i \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{i-1}.
\]

satisfies \( B^2 = 0 \) and \([B, b] = 0\) and therefore defines a map of complexes

\[
B : C_\bullet(A) \rightarrow C_\bullet(A)[-1].
\]

For \( i, j, p \in \mathbb{Z} \) let

\[
CC^{-}_p(A) = \prod_{i+j = p \mod 2} C_{i+j}(A)
\]

\[
CC^{per}_p(A) = \prod_{i+j = p \mod 2} C_{i+j}(A).
\]
The complex \((CC^{-}_\bullet(A), B + b)\) (respectively \((CC^{\text{per}}_\bullet(A), B + b)\)) is called the negative (respectively periodic) cyclic complex of \(A\).

There are inclusions of complexes
\[
CC^{-}_\bullet(A)[-2] \hookrightarrow CC^{-}_\bullet(A) \hookrightarrow CC^{\text{per}}_\bullet(A)
\]
and the short exact sequence
\[
0 \rightarrow CC^{-}_\bullet(A)[-2] \rightarrow CC^{-}_\bullet(A) \rightarrow C^\bullet(A) \rightarrow 0.
\] (5.3)

In what follows we will use the notation of Getzler and Jones ([GJ]). Let \(u\) denote a variable of degree \(-2\) (with respect to the homological grading. Then the negative and periodic cyclic complexes are described by the following formulas:
\[
CC^{-}_\bullet(A) = (C^\bullet(A)[[u]], b + uB)
\]
\[
CC^{\text{per}}_\bullet(A) = (C^\bullet(A)[[u, u^{-1}], b + uB]).
\] (5.5) (5.6)

5.2. Hochschild and cyclic complexes of sheaves of algebras.
Suppose that \(X\) is a topological space and \(A\) is a flat sheaf of \(k\)-algebras on \(X\) such that there is a global section \(1 \in \Gamma(X; A)\) which restricts to \(1_{A_x}\) and \(1_{A_x} \cdot k\) is contained in the center of \(A_x\) for every point \(x \in X\). Let \(C^\bullet(A)\) (respectively \(CC^{-}_\bullet(A), CC^{\text{per}}_\bullet(A)\)) denote the complex of sheaves of \(k\)-modules associated to the presheaf with value \(C^\bullet(A(U))\) (respectively \(CC^{-}_\bullet(A(U)), CC^{\text{per}}_\bullet(A(U))\)) on an open subset \(U\) of \(X\). Then \(C^\bullet(A)\) represents \(A \otimes_{A \otimes_k A^{\text{op}}} A\) in the derived category of sheaves of \(k\)-modules on \(X\).

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