Chern-Simons Splitting of 2+1D Pure Yang-Mills Theory at Large Distances

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Abstract

Holomorphic quantization of 2+1 dimensional pure Yang-Mills theory is studied at large distances. Previously we have shown that (Yildirim, 2013, arXiv:1311.1853) topologically massive Yang-Mills theory exhibits a Chern-Simons splitting behavior at large scales, similar to the topologically massive AdS gravity model. This splitting occurs in the form of a sum of two half Chern-Simons parts in the near Chern-Simons limit of both topologically massive Yang-Mills theory and its gravitational analogue. In the pure Chern-Simons limit, the two split parts add up to give the original Chern-Simons term in the action. However, in the opposite limit of the gravitational analogue, the Einstein-Hilbert term with a negative cosmological constant can be written as the difference between two half Chern-Simons parts. With this motivation, we investigate the gauge theory version of this limit, showing that pure Yang-Mills theory acts like two split half Cherns-Simons theories at large distances. At very large distances these terms cancel, making the Yang-Mills theory trivial as expected due to existence of a mass gap. Just as in the topologically massive case, this splitting behavior can be exploited to incorporate link invariants of knot theory.

1 Introduction

It is known that topologically massive AdS gravity action can be written as two Chern-Simons (CS) actions \cite{1-3}. For a dynamical metric $\gamma_{\mu\nu}$, the action is

\[ S = \int d^3 x \left[ -\sqrt{-\gamma} (R - 2\Lambda) + \frac{1}{2\mu} e^{\alpha\rho\nu} \left( \Gamma^\alpha_{\mu\beta} \partial_\nu \Gamma^\beta_{\rho\alpha} + \frac{2}{3} \Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\nu\beta} \Gamma^\beta_{\rho\alpha} \right) \right]. \] (1.1)

With

\[ A^\pm \mu \ a \ b [e] = \omega^a_{\mu} \ b [e] \pm e^a_{bc} e_\mu \ e^c, \] (1.2)
where \( e_\mu^a \) is the dreibein and \( \omega_\mu^a_b [e] \) is the torsion-free spin connection, the action (1.1) splits into two CS parts as

\[
S[e] = -\frac{1}{2} \left( 1 - \frac{1}{\mu} \right) S_{CS}[A^+ [e]] + \frac{1}{2} \left( 1 + \frac{1}{\mu} \right) S_{CS}[A^- [e]]
\]  

(1.3)

where

\[
S_{CS} = \frac{1}{2} \int \epsilon^{\mu\nu\rho} \left( A_\mu^a \partial_\nu A_\rho^b + \frac{2}{3} A_\mu^a c A_\nu^c b A_\rho^b a \right).
\]

(1.4)

In the near CS limit (small \( \mu \)), the theory acts like a sum of two half CS theories as

\[
S[e] \approx \frac{1}{2\mu} S_{CS}[A^+ [e]] + \frac{1}{2\mu} S_{CS}[A^- [e]].
\]

(1.5)

This is analogous to what we have observed in ref. [4] for topologically massive Yang-Mills (TMYM) theory at large distances. We have shown that the inner product of TMYM can be written as

\[
\langle \psi_0 | \psi_0 \rangle_{TMYM_{k}} = \langle \psi | \psi \rangle_{CS_{k/2}} \langle \psi | \psi \rangle_{CS_{k/2}} + O(1/m^2),
\]

(1.6)

where the labels \( k \) and \( k/2 \) indicate the level of the CS term in the Lagrangian. It can be seen from (1.6) that the splitting occurs at large distances compared to \( 1/m \), by keeping the first order and neglecting all higher order terms. An approximation like this is not necessary for the gravitational analogue, since the model does not have a mass gap. In the pure Einstein-Hilbert limit (\( \mu \to \infty \)), (1.1) becomes

\[
S[e] = \frac{1}{2} S_{CS}[A^- [e]] - \frac{1}{2} S_{CS}[A^+ [e]].
\]

(1.7)

Corresponding limit in the gauge theory version gives a pure Yang-Mills (YM) theory. This provides good motivation to study pure YM theory at large distances to see whether or not the analogy can be extended to this limit and a similar splitting can be observed. This is the main goal of this paper.

## 2 Yang-Mills Theory In 2+1 Dimensions

The YM action is given by

\[
S_{YM} = -\frac{k}{4\pi} \frac{1}{4m} \int_{\Sigma \times [t_i, t_f]} d^3x \ Tr (F_{\mu\nu} F^{\mu\nu}).
\]

(2.1)

where \( \Sigma \) is an orientable two dimensional surface and \( m \) is in mass dimensions. \( A_\mu = -i A_\mu^a t^a \) where \( t^a \) are matrix representatives of the Lie algebra \([t^a, t^b] = i f^{abc} t^c\) and in the fundamental representation they are normalized as \( Tr(t^a t^b) = \frac{1}{2} \delta^{ab} \). The dimensionless factor \( \frac{k}{4\pi} \) is normally not necessary but here it is inserted for future convenience. The equations of motion are given by \( D_\nu F^{\mu\nu} = 0 \), where \( D_\nu = \partial_\nu + [A_\mu, \cdot] \).
In the temporal gauge $A_0 = 0$ with complex coordinates $A_z = \frac{1}{2}(A_1 + iA_2)$ and $A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2)$, the conjugate momenta are given by

$$
\Pi^{az} = \frac{k}{8\pi m} F^{a0z} \quad \text{and} \quad \Pi^{a\bar{z}} = \frac{k}{8\pi m} F^{a0\bar{z}}.
$$

(2.2)

With $E_z = \frac{i}{2m} F^{0z}$ and $E_{\bar{z}} = -\frac{i}{2m} F^{0\bar{z}}$, the momenta can be rewritten as

$$
\Pi^{az} = \frac{ik}{4\pi} E^a_z \quad \text{and} \quad \Pi^{a\bar{z}} = -\frac{ik}{4\pi} E^a_{\bar{z}}.
$$

(2.3)

With these coordinates, the symplectic two form is given by

$$
\Omega = \frac{ik}{4\pi} \int_{\Sigma} (\delta E^a_z \delta A^a_{\bar{z}} + \delta A^a_z \delta E^a_{\bar{z}}).
$$

(2.4)

The generator of infinitesimal gauge transformations is $G^a = \frac{ik}{4\pi} (D_z E^a_{\bar{z}} - D_{\bar{z}} E^a_z)$. Then the Gauss’ law is given by $G^a = 0$, which is also one of the equations of motion.

To show how CS splitting happens, we want to write $\Omega$ in two $CS$-like parts. Here, the term “CS-like” means that the symplectic two-form is written in terms of fields that transform like non-abelian gauge fields. To do this, we introduce new coordinates

$$
\tilde{A}_i = A_i + E_i \quad \text{and} \quad \hat{A}_i = A_i - E_i.
$$

(2.5)

Since $E_i$ is gauge covariant, $\tilde{A}_i$ and $\hat{A}_i$ transform like gauge fields. Now, $\Omega$ can be written as

$$
\Omega = \frac{ik}{4\pi} \int_{\Sigma} (\delta \tilde{A}^a_z \delta A^a_{\bar{z}} - \delta A^a_z \delta \hat{A}^a_{\bar{z}}).
$$

(2.6)

The phase space is Kähler with the potential

$$
K = \frac{k}{4\pi} \int_{\Sigma} (\tilde{A}^a_z A^a_{\bar{z}} - A^a_z \hat{A}^a_{\bar{z}}).
$$

(2.7)

Using $\tilde{A}_i$ and $\hat{A}_i$ as connections, new covariant derivatives $\tilde{D}_i$ and $\hat{D}_i$ can be defined that are going to be useful later.

### 2.1 Parametrization of Gauge Fields

When $\Sigma$ is simply connected, we can define a field $A_i$ such that it satisfies [5]

$$
\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] = 0.
$$

(2.8)

With this flatness condition, we can define an $SL(N, \mathbb{C})$ matrix $U$, given by

$$
U(x, 0, C) = \mathcal{P} \exp \left( - \int_C^{\infty} (A_z d\bar{z} + A_{\bar{z}} dz) \right),
$$

(2.9)
which is invariant under small deformations of the path $C$ and it gauge transforms as $U^g = gU$, where $g \in SU(N)$. From (2.9), it follows that

$$A_z = -\partial_z U U^{-1} \text{ and } A_z = U^\dagger \partial_z U^\dagger$$

and

$$\dot{A}_z = -\partial_z U U^{-1} \text{ and } \dot{A}_z = U^\dagger \partial_z U^\dagger.$$  \hfill (2.10)

Since $\tilde{A}_i$ and $\hat{A}_i$ transform like gauge fields, they can be parametrized like one. Using new $SL(N, \mathbb{C})$ matrices $\tilde{U}$ and $\hat{U}$, defined similar to (2.9), we can write

$$\tilde{A}_z = -\partial_z \tilde{U} U^{-1} \text{ and } \tilde{A}_z = \tilde{U}^\dagger \partial_z \tilde{U}^\dagger,$$

and

$$\hat{A}_z = -\partial_z \hat{U} U^{-1} \text{ and } \hat{A}_z = \hat{U}^\dagger \partial_z \hat{U}^\dagger.$$ \hfill (2.11)

Also, $\tilde{A}$ and $\hat{A}$ can be written similar to (2.11). We can use (2.5) to relate the matrices $\tilde{U}$ and $\hat{U}$ with $U$. Assuming $\tilde{U} = U M$ and $\hat{U} = U N$, we can solve (2.5) with

$$M(z, \bar{z}) = N^{-1}(z, \bar{z}) = \mathcal{P} \exp \left( \frac{i}{2m} \int_0^{\bar{z}} \mathcal{F}^{0\bar{z}} \, d\bar{w} \right), \hfill (2.12)$$

where $\mathcal{F}^{0\bar{z}} = U^{-1} \mathcal{F}^{0\bar{z}} U$. It can be seen that $\mathcal{F}$ is gauge invariant, so is $M$. With these new matrices, we can write the $E$-field components as

$$E_z = U^\dagger N^\dagger \partial_z M^\dagger U^\dagger = -U^\dagger N^\dagger \partial_z N^\dagger U^\dagger$$

$$E_{\bar{z}} = -U \partial_z MM^{-1} U^{-1} = U \partial_z NN^{-1} U^{-1}. \hfill (2.13)$$

We can also write the $E$-fields that are given by $E_i = \tilde{A}_i - A_i = \hat{A}_i - \hat{A}_i$, as

$$E_z = \tilde{E}_z = U^\dagger M^\dagger N^\dagger U^{-1} = U \partial_z NN^{-1} U^{-1}$$

$$E_{\bar{z}} = \tilde{E}_{\bar{z}} = U^\dagger M^\dagger N^\dagger U^{-1} = U^\dagger N^\dagger \partial_z N^\dagger U^\dagger.$$ \hfill (2.14)

\section*{2.2 The Wave-Functional}

At least for the gauge theories that we are interested in, such as CS, YM or TMYM, the wave-functional can be factorized as $\psi = \phi \chi$. Here, $\phi$ is the part that satisfies Gauss’s law constraint $G^a \Psi = 0$ and $\chi$ is the gauge invariant part that is necessary for $\psi$ to satisfy the Schrödinger’s equation $\mathcal{H} \psi = \mathcal{E} \psi$. To find $\phi$, the standard technique is to make an infinitesimal gauge transformation on $\psi$, then force the Gauss’ law to obtain a condition that is solved by the Wess-Zumino-Witten(WZW) action. Once $\phi$ is known, then the Schrödinger’s equation can be tackled to find $\chi$.

At this point we need to choose a polarization. Choosing the holomorphic polarization gives $\Phi[A_z, \bar{A}_z, \hat{A}_z] = e^{-\frac{i}{2}K[\Phi]}[A_z, \hat{A}_z]$, where $K$ is the Kähler potential given in (2.7), $\Phi$ and $\psi$ are
the pre-quantum and quantum wave-functionals. \( \chi \) is a function of both \( A_\tilde{z} \) and \( \tilde{A}_\tilde{z} \) and since it is gauge invariant, it has to be a function of the difference of these variables, which is \( E_\tilde{z} \).

From (2.6), upon quantization we can write
\[
A^a_\tilde{z} \psi = \frac{4\pi}{k} \frac{\delta \psi}{\delta A^a_\tilde{z}} \quad \text{and} \quad \tilde{A}^a_\tilde{z} \psi = -\frac{4\pi}{k} \frac{\delta \psi}{\delta A^a_\tilde{z}}.
\] (2.17)

Now, we make an infinitesimal gauge transformation on \( \psi \),
\[
\delta \epsilon \psi[A_\tilde{z}, \tilde{A}_\tilde{z}] = \int d^2z \left( \delta \epsilon \frac{\delta \psi}{\delta A^a_\tilde{z}} + \delta \epsilon \frac{\delta \tilde{A}^a_\tilde{z}}{\delta A^a_\tilde{z}} \right) \psi.
\] (2.18)

Using (2.17), \( \delta A^a_\tilde{z} = D_\tilde{z} e^a \) and \( \delta \tilde{A}^a_\tilde{z} = \tilde{D}_\tilde{z} e^a \), (2.18) can be rewritten as
\[
\delta \epsilon \psi = \frac{k}{4\pi} \int d^2z e^a \left( \tilde{D}_\tilde{z} A^a_\tilde{z} - D_\tilde{z} \tilde{A}^a_\tilde{z} \right) \psi.
\] (2.19)

Using (2.5), (2.19) becomes
\[
\delta \epsilon \psi = \frac{k}{4\pi} \int d^2z e^a \left( \partial_\tilde{z} \left( A^a_\tilde{z} - \tilde{A}^a_\tilde{z} \right) \right) \psi.
\] (2.20)

After applying the Gauss’ law \( G^a \psi = 0 \), we get
\[
\delta \epsilon \psi = \frac{k}{4\pi} \int d^2z e^a \left( \partial_\tilde{z} A^a_\tilde{z} - \tilde{D}_\tilde{z} \tilde{A}^a_\tilde{z} \right) \psi.
\] (2.21)

This condition is solved by \( \psi = \phi \chi \) with
\[
\phi[A_\tilde{z}, \tilde{A}_\tilde{z}] = \exp \left[ \frac{k}{2} \left( S_{ZW}(\tilde{U}) - S_{ZW}(U) \right) \right],
\] (2.22)

or equally
\[
\phi[A_\tilde{z}, \tilde{A}_\tilde{z}] = \exp \left[ \frac{k}{2} \left( S_{ZW}(U) - S_{ZW}(\tilde{U}) \right) \right].
\] (2.23)

The equivalence of (2.22) and (2.23) can be shown by using the Polyakov-Wiegmann(PW) identity [6, 7] with \( \tilde{U} = UM \) and \( \tilde{U} = UM^{-1} \).

Holomorphic components of gauge fields acting on \( \phi \) gives \( A \) and \( \tilde{A} \) fields, as
\[
A^a_\tilde{z} \phi = \frac{4\pi}{k} \frac{\delta \phi}{\delta A^a_\tilde{z}} = A^a_\tilde{z} \phi \quad \text{and} \quad \tilde{A}^a_\tilde{z} \phi = -\frac{4\pi}{k} \frac{\delta \phi}{\delta A^a_\tilde{z}} = \tilde{A}^a_\tilde{z} \phi.
\] (2.24)
2.3 The Schrödinger’s Equation

The YM Hamiltonian is given by

\[ H = T + V \]

with

\[ T = \frac{m}{\alpha} E^a_z E^a_{\bar{z}} \]

and

\[ V = \frac{\alpha}{m} B^a B^a, \]

(2.25)

where \( \alpha = \frac{4\pi}{k} \), \( B = \frac{i k}{2\pi} F_{z\bar{z}} \). We are interested in finding the vacuum wave-functional, which is given by \( H \psi_0 = 0 \), or with using Euclidean metric,

\[ E^a_z E^a_{\bar{z}} \psi_0 + \frac{1}{64m^2} F^a_{z\bar{z}} F^a_{z\bar{z}} \psi_0 = 0. \]

(2.26)

For both YM and TMYM theories, \( \chi \) is typically in the form of \( e^{-\frac{1}{m^2} \int F^2} \) [4, 8, 9](except for very small distances), since they have the same Hamiltonian in the temporal gauge. Since this decay behavior cannot be polarization dependent and it comes from the existence of a mass gap, we expect it to be valid here as well. To check this assumption we will first focus on the potential energy term. Using (2.24), the B-field acting on \( \psi \) is

\[ F^a_{z\bar{z}} \psi = (\partial_z A^a_{\bar{z}} - D_{\bar{z}} A^a_z) \psi \]

\[ = D_{\bar{z}} (A^a_{\bar{z}} - A^a_z) \psi \]

\[ = D_{\bar{z}} \left( -E^a_{z\bar{z}} \psi - \frac{4\pi}{k} \delta \chi \delta \tilde{A}^a_{\bar{z}} \phi \right), \]

(2.27)

where \( E_{z\bar{z}} \) is given by \( \tilde{A}_{\bar{z}} - A_z \). If we consider only the potential energy term, the gauge invariant part of the vacuum wave-function is given by

\[ \chi = \exp \left( -\frac{k}{4\pi} \int \frac{E^a_{z\bar{z}} \mathcal{E}^a_z}{\Sigma} \right) = \exp \left( -\frac{k}{4\pi} \int \frac{\tilde{A}^a_{\bar{z}} - A^a_z}{\Sigma} \mathcal{E}^a_z \right) = 1 + O(1/m^2). \]

(2.28)

Now, to study the kinetic energy term we look at \( E^a_z \psi \). Using (2.24), we can write

\[ E^a_z \psi = (A^a_z - \tilde{A}^a_{\bar{z}}) \phi \chi \]

\[ = E^a_z \psi + \frac{4\pi}{k} \phi \left( \frac{\delta \chi}{\delta \tilde{A}^a_{\bar{z}}} + \frac{\delta \chi}{\delta A^a_z} \right). \]

(2.29)

Since \( \chi = \chi[\tilde{A}^a_{\bar{z}}, A^a_z] = \chi[A^a_{\bar{z}} - A^a_z] \), we can write

\[ \frac{\delta \chi}{\delta A^a_z} = - \frac{\delta \chi}{\delta \tilde{A}^a_{\bar{z}}} = \frac{\delta \chi}{\delta E^a_z}. \]

(2.30)

This leads to

\[ E^a_z E^a_z \psi = E^a_z E^a_z \psi. \]

(2.31)

Thus, as far as only the kinetic energy term is concerned, a constant \( \chi \) is sufficient [8]. If we neglect the potential energy term, the vacuum condition forces \( \mathcal{E}^a_z \) to be zero. This means that the matrix \( M \) has to be a holomorphic function of \( \bar{z} \).
Our goal is to study the large distance behavior of the theory by neglecting second and higher order terms in $1/m$. Neglecting the potential energy term is standard practice in these type of cases. The reasoning is as follows. Since the energy is low, charges move very slowly and do not create significant magnetic fields. When the potential term is not neglected, we expect the full solution for (2.26) to be an interpolation between the kinetic energy eigenstate and potential energy eigenstate. Since neither of these states has a first order term in $1/m$, even if we do not neglect the potential term, there should not be any first order contribution. This result is consistent with ref. [8]. Thus, for the scales that we study, $\chi = 1$ can be taken safely(at least when the potential energy is neglected), just like in TMYM theory [4].

2.4 The Measure

From (2.7), we can write the metric for the space of gauge fields $A$ as

$$ds^2 = -4 \int Tr(\delta A_z \delta A_z - \delta A_z \delta A_z) = 4 \int Tr[\tilde{D}_z(\delta U \delta U^{-1})D_z(U^{-1} \delta U) - D_z(\delta U^{*} \delta U^{-1})\tilde{D}_z(\tilde{U}^{-1} \delta \tilde{U})].$$

(2.32)

Similar to the analysis in refs. [10] and [4], the gauge invariant measure is

$$d\mu(A) = det(\tilde{D}_z D_z) det(D_z \tilde{D}_z) d\mu(U) d\mu(\hat{U}).$$

(2.33)

For a certain choice of local counter terms ($\int Tr(\hat{A}_z A_z + \hat{A}_z A_z)$),

$$det(\tilde{D}_z D_z) det(D_z \tilde{D}_z) = constant \times e^{2c_A (S_{WZW}(U) + S_{WZW}^{*}(\hat{U}))},$$

where $c_A$ is the quadratic Casimir in the adjoint representation given by $c_A = f_{ammn} f_{bmn}$. To simplify the notation, we will continue with $H_1 = U \hat{U}$ and $H_2 = \hat{U} U$. These two matrices are $SU(N)$ gauge invariant and belong to the coset $SL(N, \mathbb{C})/SU(N)$. Now, the gauge invariant measure can be written as

$$d\mu(A) = e^{2c_A (S_{WZW}(H_1) + S_{WZW}^{*}(H_2))} d\mu(H_1) d\mu(H_2).$$

(2.35)

2.5 Chern-Simons Splitting

For holomorphic polarizations, the inner product is given by

$$\langle \psi | \psi \rangle = \int d\mu(A) e^{-K} \psi^* \psi,$$

(2.36)

where $K$ is the Kähler potential. Using (2.7), (2.22) and (2.23) with PW identity gives

$$e^{-K} \psi^* \psi = e^{\frac{K}{2} (S_{WZW}(H_1) - S_{WZW}(H_2))} \chi^* \chi.$$
As we have seen in subsection 2.3,\
\[ \chi^* \chi = 1 + \mathcal{O}(1/m^2). \]  
(2.38)

Now, using (2.35), (2.37) and (2.38), we can write the inner product as
\[ \langle \psi_0 | \psi_0 \rangle = \int d\mu(H_1)d\mu(H_2)e^{(2c_A+\frac{k}{2})S_{WZW}(H_1)+(2c_A-\frac{k}{2})S_{WZW}(H_2)} + \mathcal{O}(1/m^2). \]  
(2.39)

The inner product for CS theory is given by
\[ \langle \psi | \psi \rangle_{CS} = \int d\mu(H)e^{(2c_A+k)S_{WZW}(H)}, \]  
(2.40)

where \( H = U^\dagger U \). It can be seen that the YM inner product consists of two half-level CS parts that cancel as \( m \to \infty \), since \( H_1 = H_2 = H \) in this limit. The YM inner product can be written as
\[ \langle \psi_0 | \psi_0 \rangle_{YM_k} = \langle \psi | \psi \rangle_{CS_{k/2}} \langle \psi | \psi \rangle_{CS_{-k/2}} + \mathcal{O}(1/m^2). \]  
(2.41)

It appears that YM inner product splits into two CS terms at large distances, just as we expected by studying the gravitational analogue (1.7).

3 Conclusion

We have shown that YM theory acts like a topological theory at certain limited scales. When the distance is large enough but finite, YM theory splits into two CS terms with levels \( k/2 \) and \(-k/2\) very similar to the splitting of topologically massive AdS gravity model. At very large distances, these two CS terms cancel to make YM theory trivial. Together with our previous work on TMYM theory [4], we have shown that both YM and TMYM theories exhibit CS splitting at large scales, just as predicted by their gravitational analogues. For both YM and TMYM theories, this limited topological region can be exploited to incorporate link invariants of knot theory, as shown in ref. [4].

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