A kinetic equation approach to the anomalous Hall effect in a diffusive Rashba two-dimensional electron system with magnetization

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We present a two-band kinetic equation method to investigate the anomalous Hall effect in a Rashba two-dimensional electron system subjected to a homogeneous magnetization. The electron-impurity scattering is taken into account in the self-consistent Born approximation. It is demonstrated that the impurity-density-free anomalous Hall conductivity arises from an intrinsic and a disorder-mediated mechanisms, associated respectively with the electron states under and near the Fermi surface. The intrinsic mechanism relates to a dc-field-induced transition process, or in other words, a linear stationary Rabi process. The disorder-mediated one corresponds to a scattering between impurities and the electrons participating in longitudinal transport. Numerically, the dependencies of the anomalous Hall conductivity on the spin-orbit coupling constants and the strength of the magnetization are demonstrated for both the short- and long-range collisions.

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I. INTRODUCTION

In the presence of a magnetic field, the Hall effect induced by Lorentz force is a powerful tool for measurement of the concentration and the nature of the free carriers. However, in many ferromagnets, a nonvanishing transverse resistivity can also be produced by the spontaneous magnetization, which exists in the absence of external fields. This so-called anomalous Hall effect (AHE) has been extensively studied in ferromagnetic semiconductors in recent years.

AHE has been first predicted about five decades ago. At early stages, to interpret it, two mechanisms, namely skew scattering and side-jump process, have been proposed. These two mechanisms are based on a spin-orbit coupling included into the potential of electron-impurity scattering. Obviously, the corresponding contributions to AHE should rely on the electron-impurity collision.

The recent theoretical interest has been focused on another ”intrinsic” mechanism of AHE. This mechanism has been first discussed by Karplus and Luttinger and clearly rederived by MacDonald et al. and Nagaosa et al. By inclusion of the spin-orbit interaction into Hamiltonian of the free electrons, nontrivial transverse conductivity has been obtained even in the absence of disorder scattering. It makes clear that this AHE is associated with the Berry phase in the momentum space.

More recently, according to the well-known result of Streda in the context of Hall effect in magnetically two-dimensional (2D) systems, Dugaev et al. discussed another mechanism of AHE by means of Kubo formula. This mechanism is related to electron states in the vicinity of Fermi surface, and depends on the electron-impurity scattering but is independent of the impurity density \( N_i \). By considering the short-range disorder collision, it has been found that this collision-related mechanism makes a contribution to anomalous Hall conductivity of the same order of magnitude as the intrinsic one. However, as shown in Ref. \( \text{[20]} \), the Kubo approach becomes questionable when used to investigate the collision-related AHE. Taking the static limit \( \omega \rightarrow 0 \) before or after \( N_i \rightarrow 0 \) leads to completely different results. At the same time, up to now, this AHE has been studied only for short-range electron-impurity scattering. We know that in realistic 2D semiconductors, the electron density is not large enough to screen the charged impurities. The Coulomb interaction between electron and impurity is inevitably long-ranged.

In this paper, we employ a two-band kinetic equation approach to study the impurity-density-independent AHE in 2D Rashba electron systems with magnetization. Within this method, we can consider the time-independent nature of dc transport from the beginning and hence the problem caused by taking the static limit in Kubo formalism can be avoided. At the same time, the long-range electron-impurity collision can be easily handled. The obtained kinetic equations in the helicity basis allow us to interpret the above two mechanisms of AHE in terms of interband processes. The intrinsic AHE arises from a dc-field-induced direct transition between two unperturbed spin-orbit-coupled bands. In another point of view, this AHE can also be understood as a result of a quantum interference between perturbed electrons in different bands in the first order of dc field, i.e. as a linear stationary Rabi process. On the other hand, the electrons joining in the longitudinal transport can be resonantly scattered by the impurities, leading to another interband polarization independent of impurity density. This process corresponds to the collision-related mechanism of AHE. Similar picture has been demonstrated in the previous studies on the spin-Hall effect in 2D systems with spin-orbit coupling. Based on the derived equations, we numerically investigate the dependencies of the AHE on the spin-orbit coupling constant and the strength of exchange field.
is subjected to a homogeneous magnetization \( y \) plane with Rashba spin-orbit interaction. This system is subjected to a homogeneous magnetization \( M \) along the \( z \)-direction. The noninteracting Hamiltonian of the considered system has the form

\[
H_0 = \varepsilon_p + \alpha (\sigma_x p_y - \sigma_y p_x) - M \sigma_z,
\]

where \( p \equiv (p_x, p_y) \equiv (p \cos \phi_p, p \sin \phi_p) \) is the electron momentum, \( \varepsilon_p = p^2/(2m) \), \( \alpha \) is the spin-orbit coupling constant, \( M = g_{\mu_B} M_0 \), and \( \sigma_i \) \( (i = x, y, z) \) are the Pauli matrices. This Hamiltonian can be diagonalized easily, resulting in two eigen wave-functions

\[
u(p) = \frac{1}{\sqrt{2\lambda_p}} \left( \begin{array}{c}
\sqrt{\lambda_p - (1)^{\mu} M} \\
- \sqrt{\lambda_p + (1)^{\mu} M}
\end{array} \right),
\]

and eigenvalues

\[
\varepsilon_p = \frac{p^2}{2m} + (-1)^{\mu} \lambda_p,
\]

with \( \lambda_p = \sqrt{\lambda^2 + \alpha^2 p^2} \) and \( \mu = 1, 2 \). It is useful to introduce a unitary transformation \( U_\rho = (u_1(p), u_2(p)) \), by which the basis of the system is changed from a spin one to a helicity one. At the same time, Hamiltonian (1) becomes a diagonal matrix

\[
H_0 = U_\rho^\dagger H_0 U_\rho = \text{diag}(\varepsilon_1(p), \varepsilon_2(p)).
\]

We consider an electric current flowing along the \( x \) axis when the system is driven by a weak dc field \( E \) along the \( y \)-direction. In the helicity basis, the single-particle operator of this current, \( j_x(p) \equiv U_\rho^\dagger j_x(p) U_\rho \), is given by

\[
\frac{\hbar e}{2m_\rho}(\lambda_p - ma^2) \frac{\hbar e}{2m_\rho}(MP_x + i\lambda_p p_y) \frac{\hbar e}{2m_\rho}(MP_x - i\lambda_p p_y) \frac{\hbar e}{2m_\rho}(\lambda_p + ma^2)
\]

and the corresponding macroscopic quantity is obtained by taking the statistical average over it, \( J_x = \sum_p \text{Tr}[j_x(p) \rho(p)] \), with \( \rho(p) \) being the distribution function. By definition, the anomalous Hall conductivity is determined by \( \sigma_{xy} = J_x/E \).

In the spin basis, the lesser Green’s function, \( \tilde{G}^< \), is a \( 2 \times 2 \) matrix and obeys the Dyson equation

\[
[\tilde{G}^0 - U, \tilde{G}^<] = \Sigma^r \tilde{G}^< + \Sigma^< \tilde{G}^a - \tilde{G}^a \Sigma^< - \tilde{G}^< \Sigma^a,
\]

where \( U \) is the one-body external potential due to the dc field. The self-energies \( \Sigma^<,a \) arise from the electron-impurity interaction. In this paper, we consider a Coulomb interaction between electrons and impurities, which can be described by a potential \( V(p-k) \). This potential corresponds to a scattering of an electron from momentum state \( p \) to state \( k \). Under homogeneous condition, Eq. (5) in the momentum space can be written as

\[
\left\{ \frac{\partial}{\partial T} + eE \cdot \nabla p \right\} \tilde{G}^<(p, T) + i[H_0, \tilde{G}^<(p, T)] = -\frac{\partial \tilde{G}^<}{\partial T}_{\text{scatt}}(6)
\]

with the Wigner distribution function \( \tilde{G}^<(p, T) = -i\tilde{G}^<(p, T) \).

We define a distribution function in helicity basis as \( \rho(p,T) = U^\dagger_\rho \tilde{G}^<\rho(p,T)U_\rho \). To derive the kinetic equation for \( \rho \), we multiply the Eq. (6) from left by \( U^\dagger_\rho \) and from right by \( U_\rho \). Due to the unitarity of the transformation, the right-hand side (rhs) of equation is obtained simply by replacing the Green’s function in spin basis with that in one helicity basis

\[
\frac{\partial \rho}{\partial T} = \int_{-\infty}^{T} dt' \{ \{ \Sigma^>, \tilde{G}^< \} + \{ \Sigma^<, \tilde{G}^> \} \} (T, t')(t, T).
\]

At the same time, using the facts \( U^\dagger_\rho \nabla_p \rho U_\rho = \nabla_p \rho - \nabla_p U^\dagger_\rho \nabla_p \rho - \rho U^\dagger_\rho \nabla_p \rho \) and \( \nabla_p U^\dagger_\rho U_\rho = -U^\dagger_\rho \nabla_p \rho U_\rho \), we find that the left-hand side (lhs) of the kinetic equation for distribution function in the helicity basis takes the form

\[
\left\{ \frac{\partial}{\partial T} + eE \cdot \nabla p \right\} \rho - eE \cdot [\rho, U^\dagger_\rho \nabla_p \rho U_\rho] + i[H_0, \rho].
\]

Further, we assume that the applied dc field is weak enough and only the stationary linear response of the system needs to be considered. Hence, we can divide the distribution function into two parts, \( \rho = \rho_0 + \rho_1 \), with unperturbed distribution function \( \rho_0(p) = \text{diag}(n_F,\varepsilon_1(p), n_F,\varepsilon_2(p)) \) and the function in the first order of the dc field \( \rho_1 \).

To simplify relaxation term (5), it is necessary to express the two-time lesser and greater Green’s functions through one-time Wigner distribution. The powerful tool to perform this is the two-band generalized Kadanoff-Baym ansatz (GBKBA), \( n_F \) which has been successfully applied to investigate the optical properties of semiconductors. \( \text{In the first order of dc-field strength, the GBKBA reads}

\[
G_{1}^{G} = -G_{0}^{G} \rho_{1} + \rho_{G}^{G} - G_{1}^{G} \rho_{0} + \rho_{0} G_{2}^{G},
\]

where \( G_{0}^{G,a} \) and \( G_{1}^{G,a} \) are the Green’s functions in the zero- and first-order of dc field, respectively. Note that \( G_{a}^{G,a} \) are off-diagonal matrices. This fact is obvious from the Dyson equations which they should satisfy. In
Eqs. 9 and 11, for shortness, the time arguments in Green’s functions \((t_1, t_2)\) are not written out. Under the stationary condition, the \(\rho_1\) is independent of the time.

After these simplifications, the scattering term can be formally denoted as a sum of two components: 
\[
\frac{\partial \rho}{\partial T}_{\text{scatt}}^{I} \equiv \frac{\partial \rho}{\partial T}_{\text{scatt}}^{I} + \frac{\partial \rho}{\partial T}_{\text{scatt}}^{II},
\]
where \(\frac{\partial \rho}{\partial T}_{\text{scatt}}^{I}\) is off-diagonal and contains all the elements, associated with the functions \(G^r_{\alpha}\). In this component, the perturbed distribution does not appear because we are only interested in the linear response of the system. \(\frac{\partial \rho}{\partial T}_{\text{scatt}}^{II}\) includes all the remaining part.

It is obvious that the driving force in the lhs of kinetic equation, i.e. the term proportional to \(E\), comprises two components: the first of which, \(\varepsilon E \cdot \nabla p\rho_0\), contains only diagonal elements, while the other one is off-diagonal. According to this fact, we break the kinetic equation into two,
\[
\varepsilon E \cdot \nabla p\rho_0 + i[H_0, \rho^I_1] = -\frac{\partial \rho}{\partial T}_{\text{scatt}}^{I}, \quad (11)
\]
\[
-\varepsilon E \cdot [\rho_0, U_p^+ \nabla p U_p] + i[H_0, \rho^{II}_1] = -\frac{\partial \rho}{\partial T}_{\text{scatt}}^{II}, \quad (12)
\]
which have two solutions \(\rho^I_1\) and \(\rho^{II}_1\), respectively: \(\rho_1 = \rho^I_1 + \rho^{II}_1\).

In this paper, we restrict ourselves to consider the AHE being linear in dc field and to the leading order in the impurity-density expansion. In this case, the Eqs. (11) and (12) are independent of each other. In fact, from the definition of \(\frac{\partial \rho}{\partial T}_{\text{scatt}}^{II}\), it is evident that the perturbed distribution functions \(\rho^I_1\) and \(\rho^{II}_1\) do not enter in the scattering term of off-diagonal Eq. (12). Hence, this equation can be solved independently. On the other hand, \(\rho_1\) and \(\rho^{II}_1\). However, as can be seen below, the contribution to \(\frac{\partial \rho}{\partial T}_{\text{scatt}}^{II}\) from the off-diagonal \(\rho^{II}_1\) is of higher-order of impurity density and can be neglected. Consequently, the Eqs. (11) and (12) become independent and can be resolved separately.

We should note that there is a nonvanishing solution of the Eq. (12) if the collision term on the rhs is ignored. This \(\rho^{II}_1\) leads to a Hall conductivity independent of any electron-impurity collision and results in the well-known intrinsic AHE. We will show below that neglect of the rhs of the Eq. (12) is reasonable in the transport study. However, to resolve Eq. (11), the scattering term can not be disregarded. Hence, the anomalous Hall conductivity produced by the corresponding solution \(\rho^I_1\) becomes collision-related.

**B. Intrinsic anomalous Hall effect**

To carry out the expression of \(\rho^{II}_1\), we start with analyzing the retarded and advanced Green’s functions to the first order of dc field, \(G_{1,0}^{r,a}\), which appear in the scattering term of the rhs of Eq. (12). It is well known that \(G_1^{r,a}\) vanish in a one-band electron gas. From the Dyson equations we can see that in the case of two bands, the diagonal elements of \(G_1^{r,a}\) still vanish but their off-diagonal elements are nonzero and proportional to \((G_{1,0}^{r,a})_{11} - (G_{1,0}^{r,a})_{22}\). \(G_1^{r,a} = \frac{i}{2\pi} \varepsilon E \cdot \sigma_z [G_0^{r,a}, U_p^+ \nabla p U_p]\). Inserting \(G_1^{r,a}\) into the scattering term, we obtain
\[
(\rho^{II}_1)_{\mu\bar{\mu}}(p) = \frac{ie}{\pi} E \cdot \nabla p u^+_{\mu}(p) \bar{u}_{\bar{\mu}}(p) \int d\omega n_F(\omega) \frac{\text{Im}[(G_0^{r,a})_{\mu\bar{\mu}} - (G_0^{r,a})_{\bar{\mu}\mu}]}{\varepsilon_\mu(p) - \varepsilon_{\bar{\mu}}(p)}. \quad (13)
\]

with \(\bar{\mu} = 3 - \mu\). This expression can be further simplified if we consider that the collision broadening of the perturbed Green’s functions \(G^{r,a}_{\alpha}\) play secondary roles in transport and the imaginary parts of these functions reduce to \(\delta\)-functions. In result, the \(\rho^{II}_1\) has a simple form
\[
(\rho^{II}_1)_{\mu\bar{\mu}}(p) = \frac{ie}{\pi} E \cdot \nabla p u^+_{\mu}(p) \bar{u}_{\bar{\mu}}(p) \frac{n_F[\varepsilon_{\bar{\mu}}(p)] - n_F[\varepsilon_\mu(p)]}{\varepsilon_\mu(p) - \varepsilon_{\bar{\mu}}(p)}. \quad (14)
\]

Note that this result can also be derived if we simply neglect the scattering term on the rhs of Eq. (12). Taking the statistical average over the current operator \(j_x(p)\), we find the contribution of \(\rho^{II}_1\) to anomalous Hall conductivity:
\[
\sigma_{xy}^{II} = \frac{M_0 M^2 e^2}{2} \int \frac{dp}{(2\pi)^2} \frac{1}{\lambda^2_p} \left( n_F[\varepsilon_1(p)] - n_F[\varepsilon_3(p)] \right). \quad (15)
\]

This result agrees with the previous study on intrinsic anomalous Hall effect. Obviously, it is independent of any electron-impurity scattering and connects with all the electron states under the Fermi surface. Further, it is clear that this expression of \(\sigma_{xy}^{II}\) is related to the Berry phase and the topology of the energy bands. Here, we will propose another interpretation of this intrinsic AHE by extending the treatment of Zhang and Yang in the study on spin-Hall effect.

In fact, the intrinsic contribution to AHE originates from a quantum interference between two bands per-
turbed by the external field and is associated with the nonvanishing interband dipole moment. The wavefunction up to the first order of the electric field can be written as,

\[ |\varphi_{\mu p}^{(0)}| > + |\varphi_{\mu p}^{(1)}| > \] (16)

with

\[ |\varphi_{\mu p}^{(1)}| > = \sum_{k} \langle \varphi_{\mu k}^{(0)} | eE \cdot r | \varphi_{\mu p}^{(0)} \rangle \| | \varphi_{\mu k}^{(0)} \rangle \] (17)

being the first order perturbation. In order to take into account the statistical characters of the electrons, we will utilize the second quantization formalism. The field operators have the forms, \( \psi_{\mu p} = |\varphi_{\mu p}^{(0)}| > c_{\mu p} + |\varphi_{\mu p}^{(1)}| > c_{\mu p} \) and \( \psi_{\mu p}^{+} = |\varphi_{\mu p}^{(0)}| > c_{\mu p}^{+} + |\varphi_{\mu p}^{(1)}| > c_{\mu p}^{+} \), with the electron creation and annihilation operators, \( c_{\mu p} \) and \( c_{\mu p}^{+} \). By definition, the interband polarization \( \rho_{\mu\bar{\mu}}(p) \) describes the quantum interference of perturbed electrons in different bands, \( \rho_{\mu\bar{\mu}}(p) = | \psi_{\mu p}^{+} \psi_{\mu p} |^2 \) In the first order of dc field, it reads

\[ (\rho_{1I})_{\mu\bar{\mu}}(p) = | \psi_{\mu p}^{+} |^2 \{ n_F[\epsilon_{\mu}(p)] - n_F[\epsilon_{\mu}(p)] \}, \] (18)

where the relations \( | \psi_{\mu p}^{+} |^2 \{ n_F[\epsilon_{\mu}(p)] - n_F[\epsilon_{\mu}(p)] \} \) and \( < c_{\mu p}^{+} | c_{\mu p} > n_F[\epsilon_{\mu}(p)] \) are used. Substituting Eq. (17) into this equation and considering the fact that \( | \psi_{\mu p}^{+} |^2 \{ n_F[\epsilon_{\mu}(p)] - n_F[\epsilon_{\mu}(p)] \} \) finally arrive at expression (14).

It is obvious that this dc-field-induced transition can be interpreted as a linear stationary process. The well-known Rabi oscillation occurs in the presence of ac field and has been widely investigated from the viewpoints of optics, semiconductors, and atomic physics etc. However, the ac field is not the essential factor of occurring the interference, whereas it ensures the time oscillation of the Rabi process. If the applied ac field is replaced by a dc one, the quantum interference also may take place. For that, the necessary condition is the nonvanishing of Rabi frequency, which is proportional to the dipole moment, and the strength of external field.

At the same time, from a perturbative point of view, this component of interband polarization can also be understood as an interband transition between two unpermurbed bands. In this picture, all electrons have finite probability to transit from one band to another. In result, the nonvanishing interband polarization emerges and relates to all the electron states. Note that there is no effect of this transition on diagonal elements of distribution up to the first order of dc field.

C. Disorder-mediated anomalous Hall effect

The distribution functions \( \rho_{1I} \) satisfy the Eq. (11). The lhs of this equation consists of a diagonal driving term \( eE \cdot \nabla_p n_F(\epsilon_{\mu}(p)) \), and an off-diagonal matrix \( i[H_0, \rho_{1I}] \) depending only on off-diagonal elements of distribution function. Before seeking the solution of this equation, we should carry out the self-energy in the self-consistent Born approximation. It takes a complicated form and is presented in the Appendix. Due to the spin-orbit coupling, the wave functions possess an additional momentum dependence, leading to a nonglobal transformation. In result, each element of the self-energy, and hence the scattering term in Eq. (11), becomes a function of all the elements of matrix Green’s function. This fact indicates that the scattering can induce an admixture of the spin-orbit-coupled bands as well as the external field. In result, some novel processes contributing to the distribution functions appear.

To simplify the relaxation term standing on the rhs of Eq. (11), we first analyze the lowest exponent in the impurity-density expansion for the elements of distribution function. Note that the scattering always provides a contribution of order of \( N_i \). Since the diagonal driving term in the lhs of Eq. (11) is independent of impurity density, the diagonal elements of distribution \( \rho_{1I}^{\mu\mu} \) should be of order of \( (N_i)^{-1} \). We substitute these \( \rho_{1I}^{\mu\mu} \) into the off-diagonal parts of the relaxation term. Since the term \( i[H_0, \rho_{1I}] \) in the lhs of Eq. (11) is proportional to the off-diagonal elements of the distribution, the leading order of the off-diagonal \( \rho_{1I}^{\mu\mu} \) in impurity-density expansion should be \( (N_i)^0 \). Hence, all terms containing the off-diagonal \( \rho_{1I}^{\mu\mu} \), as well as \( \rho_{1I}^{\mu\bar{\mu}} \), in \( \frac{\partial}{\partial \tau} \frac{\Delta_{\mu\nu}}{\Delta_{\mu\mu}} \) make contributions of higher-order of impurity density and therefore can be ignored. Under these considerations, we find the coupled equations for diagonal elements of distribution function,
axis. Hence, the lhs of Eq. (19) depends on the angle of momentum through a sine function. From the fundamental triangle relation \( \sin \phi_k = -\sin(\phi_p - \phi_k) \cos \phi_p + \cos(\phi_p - \phi_k) \sin \phi_p \) and the symmetry argument that the terms with \( \sin(\phi_p - \phi_k) \) vanish, it can be followed that the diagonal distribution has a simple angle-dependence \((\rho_1^i)_{\mu\mu} = eE\Phi_{\mu}(p)\sin \phi_p\). Due to the elastic nature of electron-impurity scattering, functions \( \Phi_{\mu}(p) \) can be carried out analytically from an interband transition. It should be noted that the latter interband process entirely arises from the spin-orbit coupling: it results in a collision-related admixture of two bands. In the absence of spin-orbit interaction, as shown in the previous studies, \( \rho_{\mu\nu} \), at any time, does not appear in the equations for \( \rho_{\mu\mu} \), even when the external field is strong.

At the same time, we should note that \( \tau_{\mu\nu} \) is the longitudinal relaxation time defined in the framework of Bloch equations. However, the \( \tau_{2\mu\nu} \) and \( \tau_{3\mu\nu} \) are novel relaxation times and, up to now, have not been studied in the literature.

Further, the off-diagonal elements of distribution are given by

\[
\Phi_{\mu}(p) = \frac{-\partial m \varepsilon_{\mu}(p)}{\partial \varepsilon_{\mu}(p)} \left(\frac{\tau^{-1}_{1\mu\bar{\mu}} + \tau^{-1}_{2\mu\bar{\mu}} \frac{\partial \varepsilon_{\mu}(p)}{\partial p} + \tau^{-1}_{3\mu\bar{\mu}} \frac{\partial \varepsilon_{\mu}(p)}{\partial p}}{\tau^{-1}_{1\mu\bar{\mu}} + \tau^{-1}_{2\mu\bar{\mu}} + \tau^{-1}_{3\mu\bar{\mu}}} \right)^{-1},
\]

where \( \tilde{\rho}_{\mu} \) is determined from equation, \( \varepsilon_{\bar{\mu}}(\tilde{\rho}_{\mu}) = \varepsilon_{\mu}(p) \), and the different relaxation times \( \tau_{i\mu\nu} \) \( (i = 1, 3, \mu, \nu = 1, 2) \) are defined by

\[
\frac{1}{\tau_{i\mu\nu}} = 2\pi n_i \sum |V(p - k)|^2 /A_{i\mu\nu}(p, k),
\]

with \( A_{1\mu\nu}(p, k) = \frac{1}{4}(1 - \cos(p_p - \phi_k))a_1(p, k)\Delta_{\mu\nu}, \)

\( A_{2\mu\nu}(p, k) = \frac{1}{2}a_2(p, k)\Delta_{\mu\nu} \) and \( A_{3\mu\nu}(p - k) = \frac{1}{2}\cos(p_p - \phi_k) a_2(p, k)\Delta_{\mu\nu}. \)

From Eq. (19), we find that the terms in the rhs correspond to two distinct scattering processes: the first two terms, i.e. the terms associated with \( \Delta_{\mu\nu} \), describe the well-known intraband process, while the remaining come

\[
(\rho_1^i)_{12} = (\rho_1^i)_{21} = \frac{\pi}{4\lambda_p} \sum_{k, \mu = 1, 2} |V(p - k)|^2 a_3(p, k)(-1)^\mu \{\Delta_{\mu\bar{\mu}}[(\rho_1^i)_{\mu\mu}(p) - (\rho_1^i)_{\mu\mu}(k)] - \Delta_{\bar{\mu}\mu}[(\rho_1^i)_{\mu\mu}(p) - (\rho_1^i)_{\mu\mu}(k)]\}
\]

with \( a_3(p, k) \equiv [\alpha k \lambda_p \sin(\phi_p - \phi_k) + i\alpha M(p - k \cos(\phi_p - \phi_k))] / \lambda_k \lambda_p. \) It is obvious that this function depends on the momentum angle not only through a sine, but also through a cosine function

\[
(\rho_1^i)_{12}(p) = i\zeta_1(p) \sin \phi_p + \zeta_2(p) \cos \phi_p.
\]

\( \zeta_i(p) \ (i = 1, 2) \) are determined by

\[
\zeta_1(p) = \frac{eE\pi}{4\lambda_p} \sum_{k, \mu = 1, 2} |V(p - k)|^2 a_4(p, k)(-1)^\mu \{\Delta_{\mu\bar{\mu}}\Phi_{\mu}(p)[1 - \cos(\phi_p - \phi_k)] - \Delta_{\bar{\mu}\mu} [\Phi_{\mu}(p) - \Phi_{\mu}(\tilde{\rho}_{\mu}) \cos(\phi_p - \phi_k)]\},
\]

\[
\zeta_2(p) = \frac{eE\pi}{4\lambda_p} \sum_{k, \mu = 1, 2} |V(p - k)|^2 a_5(p, k)(-1)^\mu \{\Delta_{\mu\bar{\mu}}\Phi_{\mu}(p) \sin(\phi_p - \phi_k) - \Delta_{\bar{\mu}\mu} \Phi_{\mu}(\tilde{\rho}_{\mu}) \sin(\phi_p - \phi_k)\}
\]

with \( a_4(p, k) \equiv \alpha M(p - k \cos(\phi_p - \phi_k))] / \lambda_k \lambda_p \) and \( a_5(p, k) \equiv a_3(p, k) - ia_4(p, k). \)

Similarly, from Eq. (22) we can see there are two scattering processes to form the nonvanishing interband polarization. One of which, associated with the terms proportional to \( \Delta_{\mu\bar{\mu}} \), is the conventional interband transition of the perturbed electrons. Due to the scattering admixture of spin-orbit-coupled bands, even an intraband transition, connecting with the first two terms of expressions \( (21) \) and \( (25) \), makes a contribution to the interband polarization.

Based on these arguments about the scattering pro-
cesses in Eqs. (19) and (22), we can generalize our results to the case of arbitrary two-band 2D system. In a two-band system with noninteracting wave function $u_\mu(p)e^{ip\cdot r}$, the interband or intraband processes, induced by impurity scattering, are described through

$$\frac{\partial p}{\partial T}_{\text{scatt,}\mu\nu} = \pi \sum_{k\sigma} |V(p-k)|^2 u_\mu^+(p)u_\sigma(k)u_\sigma^+(k)u_\mu(p)$$

$$\times \left\{ \Delta_{\mu\sigma}[(\rho_1)^{\mu\mu}(p) - (\rho_1)^{\mu\sigma}(k)] + \Delta_{\sigma\nu}[(\rho_1)^{\nu\nu}(p) - (\rho_1)^{\nu\sigma}(k)] \right\}. \quad (26)$$

Note that this equation holds for a general two-band 2D system.

After distribution functions are obtained and the statistical average over $j_x$ is taken, the anomalous Hall current, related to the longitudinal transport, can be determined by

$$J_x^l = 2 \sum_p \left[ \frac{\alpha M_G(p)}{\lambda_p^2} \cos^2 \phi_p + \alpha \zeta_1(p) \sin^2 \phi_p \right]. \quad (27)$$

Since the diagonal elements of the current operator is proportional to $p$, the contribution of diagonal distributions to anomalous Hall current vanishes. It is obvious that such anomalous Hall current has a magnitude of order of $(N_i)^0$.

The procedure for deriving the off-diagonal distribution tells us that this collision-related contribution to anomalous Hall effect originates from a disorder-mediated process. The electrons, influenced by external field, experience the impurity collision and join in longitudinal transport. Such scattering produces diagonal elements of distribution function of order of $(N_i)^{-1}$. At the same time, these electrons scattered by impurities give rise to an interband polarization of order of $(N_i)^0$. The disorder only plays an intermediate role. Note that at each stage of this disorder-mediated process, both interband and intraband scatterings occur.

Note that this disorder-mediated mechanism and the well-studied side-jump mechanism are identical physically. Contributions to Hall conductivity from both the mechanisms are independent of the impurity density but rely on the electron-disorder collision. At the same time, they all are related to the electron states near the Fermi surface. However, formally, these two mechanisms are completely different. It is well known that the side-jump process corresponds to a lateral displacement of the center of the wave-packet during the scattering when the spin-orbit interaction is included into the potential of the electron-impurity scattering. It is associated with the scattering-related term in the current operator. However, in our study for Rashba spin-orbit coupling, the current operator becomes impurity-independent.

We also perform a numerical calculation for a GaAs/AlGaAs heterojunction to investigate the intrinsic and disorder-mediated anomalous Hall effect in a Rashba two-dimensional electron system with magnetization. In calculation, we first consider a long-range scattering between electrons and the remote impurities separated at a distance $s$. The corresponding potential takes the form: $V(p) = U(p)/\kappa(p)$ with $U(p) = \epsilon p^2/2m_0$ for $|p| < 2\pi/\kappa$. The wave-function parameter is given by $b^3 = 3m_0\kappa^2(N_c + \frac{2N_{\text{dep}}}{N_i})$. $N_c$ and $N_{\text{dep}}$ are the electron density and the density of depletion layer charges, respectively. $\kappa(p) = 1 + q_sH(p)/p$ is the factor coming from the Coulomb screening. $q_s = \frac{m_e}{2\pi e n_c}$ and $H(p) = \frac{1}{8}(8b^3 + 9pb^2 + 3bq^2)(b + q)^{-3}$. In calculation, we take $m = 0.067m_e$, $s = 500\, \text{Å}$ and $N_{\text{dep}} = 1 \times 10^{13}\, \text{m}^{-2}$. In our formalism, the intrinsic anomalous Hall effect can be obtained directly by evaluating integral (17). To carry out the disorder-mediated anomalous Hall current by Eq. (27), we should determine first the diagonal elements of the distribution from Eqs. (19) and (20) and then the off-diagonal ones by Eq. (20). For long-range disorders, the results are plotted in Fig. 1 and 2.

In Fig. 1, we plot the disorder-mediated $\sigma_{xy}^{I}$, intrinsic $\sigma_{xy}^{I}$ and total anomalous Hall conductivity $\sigma_{xy}$ as functions of the spin-orbit coupling constants. It can be seen that the values of $\sigma_{xy}^{I}$ and $\sigma_{xy}$ are comparable. For different magnetization $M$, with increasing the SO coupling constant, the intrinsic $\sigma_{xy}^{I}$ always increases, while the $\sigma_{xy}^{I}$ exhibits a complicated behavior: for large $M$ it always increases as well as $\sigma_{xy}^{I}$, but for small $M$ it first increases and then falls, and even becomes negative for large $\alpha$. The similar behaviors can also be seen for total $\sigma_{xy}$. It is clear that $\sigma_{xy}^{I}$ and $\sigma_{xy}$ do not always have the same sign and a compensation occurs for small $M$ and large $\alpha$.

We also calculate the dependence of anomalous Hall effect on the magnetization for different SO coupling constants. The results are shown in Fig. 2. Again, we see a sign change for small $M$ and large $\alpha$. Besides, with increasing the magnetization above a critical value, the $\sigma_{xy}^{I}$
FIG. 1: Dependencies of disorder-mediated $\sigma_{xy}^I$, intrinsic $\sigma_{xy}^{II}$ and total anomalous Hall conductivity $\sigma_{xy}$ on the spin-orbit coupling constant in a Rashba 2D electron system with magnetization. The lattice temperature $T = 0$ K and the electron density $N_e = 1 \times 10^{11}$ cm$^{-2}$.

FIG. 2: Disorder-mediated $\sigma_{xy}^I$, intrinsic $\sigma_{xy}^{II}$ and total anomalous Hall conductivity $\sigma_{xy}$ as functions of the magnetization. The other parameters are the same as that in the Fig. 1.

falls. Taking one with another, in the studied parameter range, the absolute values of disorder-mediated $\sigma_{xy}^I$ are always lesser than the intrinsic $\sigma_{xy}^{II}$.

Further, to demonstrate the collision-related feature of the $\sigma_{xy}^I$, we also compute the Hall conductivity for short-range collision by replacing the potential $V(p - k)$ by a momentum-independent one $u$. Note that the resultant $\sigma_{xy}^I$ is independent of the magnitude of $u$. The calculated results are plotted in Fig. 3. It can be seen that the $\sigma_{xy}^I$ from the short- and long-range collisions exhibit completely different behaviors in the dependencies of the SO coupling constant and magnetization. In the parameter regime: $\alpha < 2 \times 10^{-9}$ eV cm and $M < 3$ meV, where the long-range disorders have strong effect on $\sigma_{xy}$, the Hall conductivity practically can not be affected by the short-range disorder. At the same time, in Fig. 3(b) an abrupt decrease of $\sigma_{xy}^I$ with increasing the $M$ is seen. Both these features can be understood from the fact that for short-range collision, the disorder-mediated mechanism has effect on Hall conductivity approximately when $M$ becomes larger than the chemical potential. Note that this fact agrees with the study in Ref. 20. In Fig. 3(a), the $\sigma_{xy}^I$ gradually descends with ascending $\alpha$ for large $M$ since these $M$ are always larger than the chemical potential in the studied regime of $\alpha$. In contrast to the case of long-range scattering, disorder-mediated contribution induced by the short-range collision always has an opposite sign with respect to the intrinsic one.

We should note that in the discussion above, to exhibit the effect of different mechanisms on AHE, we divide the contribution to Hall conductivity into an intrinsic one and a disorder-mediated one. Actually, in experiment, both can not be distinguished. They together result in a measurable Hall resistivity proportional to the square of the impurity density.
IV. CONCLUSION

We have proposed a two-band kinetic equation approach to the anomalous Hall effect in a 2D electron system with a Rashba spin-orbit interaction and an exchange-field-induced magnetization. The obtained equation has been resolved by considering the electron-impurity collision in the self-consistent Born approximation. It is clear that there exist two contributions to the impurity-density-free AHE. One of which arises from a dc-field-induced transition between two spin-orbit-coupled bands, which can also be understood as a linear stationary Rabi process. Another contribution is related to the collision, but is independent of the impurity density. It comes from a process mediated by disorder: electrons participating in longitudinal transport are scattered again by impurity, yielding a nonvanishing interband polarization. Numerically, we have demonstrated the dependencies of both these contributions on the magnetization and SO coupling-constant: they always exhibit a compensation for short-range disorder, while such compensation occurs in the case of long-range collision only for small magnetization and large SO coupling constants.

Since obtained Eqs. (14) and (26) are valid even for a general two-band 2D system, the interband polarization consisting of such two terms is a universal property of 2D systems with spin-orbit interaction. In result, the quantities, connecting with the interband polarization, such as anomalous Hall current, spin-Hall current etc. should be formed from two distinct processes, which are associated with the electron states under and near the Fermi surface, respectively.

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APPENDIX: SELF-ENERGIES

In the spin basis, electrons experience a spin-independent long-range disorder collision, which can be described by a potential $V(p)$. The self-energies for retarded, advanced and lesser Green’s functions in the self-consistent Born approximation take the forms

$$\Sigma^{r,a,<}(p) = \sum_k |V(p - k)|^2 \bar{G}^{r,a,<}(k). \quad (A.1)$$

When a transformation from spin basis to helicity basis is performed, the forms of the self-energies $\Sigma^{r,a,<} \equiv U_p^\dagger \Sigma^{r,a,<} U_p$ are changed to

$$\Sigma^{r,a,<}(p) = \sum_k |V(p - k)|^2 \bar{U}^\dagger_p U_k \bar{G}^{r,a,<}(k) U^\dagger_k U_p. \quad (A.2)$$

It can be seen that, due to the non-global feature of the transformation, the scalar scattering potential in spin basis becomes a matrix in helicity basis $T(p,k) \equiv U_p^\dagger V(p - k) U_k$. Generally, each element of self-energy matrices contains all the elements of Green’s functions.

It is convenient to express the self-energies through a sum of several matrices. Therefore, we first define several matrices: diagonal matrices $B_i$ ($i = 1..4$)

$$\begin{align*}
(B_1)_{\mu\mu} &= \frac{1}{2} (G + \sigma_z G \sigma_z)_{\mu\mu} = G_{\mu\mu}, \\
(B_2)_{\mu\mu} &= \frac{1}{2} (\sigma_x G \sigma_x + \sigma_y G \sigma_y)_{\mu\mu} = G_{\mu\mu}, \\
(B_3)_{\mu\mu} &= \frac{1}{2} ([\sigma_z, G] + \sigma_z [\sigma_x, G] \sigma_z)_{\mu\mu} = G_{\mu\mu} - G_{\mu\mu},
\end{align*} \quad (A.3)$$

$$\begin{align*}
(B_4)_{\mu\mu} &= \frac{1}{2} (\sigma_z [G, \sigma_z] + \sigma_z [\sigma_x, G] \sigma_z)_{\mu\mu} = G_{\mu\mu} - G_{\mu\mu}, \\
(B_5)_{\mu\mu} &= \frac{1}{2} ([\sigma_z, G] + \sigma_z [\sigma_x, G] \sigma_z)_{\mu\mu} = G_{\mu\mu} - G_{\mu\mu}.
\end{align*} \quad (A.4)$$
(B_4)_{\mu\nu} = \frac{1}{2}([\sigma_y, G] + \sigma_z [\sigma_y, G] \sigma_z)_{\mu\nu} = (-1)^{i} (G_{\mu\nu} + G_{\nu\mu}), 
(A.6)

off-diagonal matrices C_i (i = 1..3)

\begin{align*}
(C_1)_{\mu\nu} &= \frac{1}{2} (G - \sigma_x G \sigma_x)_{\mu\nu} = G_{\mu\nu}, \\
(C_2)_{\mu\nu} &= \frac{1}{2} (\sigma_x G \sigma_x - \sigma_y G \sigma_y)_{\mu\nu} = G_{\mu\nu}, \\
(C_3)_{\mu\nu} &= \frac{1}{2} (\sigma_z G \sigma_z - \sigma_x G \sigma_x)_{\mu\nu} = G_{\mu\nu} - G_{\nu\mu}, \\
(D_i)_{\mu\nu} &= (-1)^{i+1} (C_i)_{\mu\nu}.
\end{align*}

By means of these matrices, we can rewrite the self-energies as

\[ \Sigma(p) = \sum_{k,i=1\ldots4} |V(p-k)|^2 b_i B_i(k) + \sum_{k,i=1\ldots3} |V(p-k)|^2 |c_i C_i(k) + d_i D_i(k)|. \]  
(A.10)

\( b_i \) (i = 1..4), \( c_i \) and \( d_i \) (i = 1..3) are the factors depending on both the momenta \( p \) and \( k \),

\[ b_i = \frac{1}{2\lambda_p \lambda_k} \left(M^2 - (-1)^i \lambda_p \lambda_k + \alpha^2 kp \cos(\phi_p - \phi_k)\right), i = 1, 2 \]  
(A.11)

\[ b_3 = \frac{i\alpha p}{2\lambda_p} \sin(\phi_p - \phi_k), \quad b_4 = \frac{i\alpha M}{2\lambda_p \lambda_k} [k - p \cos(\phi_p - \phi_k)], \]  
(A.12)

\[ c_j = \frac{1}{2\lambda_p \lambda_k} \left(\alpha^2 kp + (M^2 + (-1)^j \lambda_p \lambda_k) \cos(\phi_p - \phi_k)\right), j = 1, 2 \]  
(A.13)

\[ c_3 = \frac{iak}{2\lambda_k} \sin(\phi_p - \phi_k), \quad d_j = \frac{iM}{2\lambda_p \lambda_k} (\lambda_p - (-1)^j \lambda_k), j = 1, 2 \quad d_3 = \frac{1}{2\lambda_p \lambda_k} (\alpha kM \cos(\phi_p - \phi_k) - \alpha pM). \]  
(A.14)

In comparison to the systems without magnetization, the self-energies take more complicated forms due to the additional dependence of wave functions on the module of momentum.

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1 C. M. Hurd, *The Hall Effect in Metals and Alloys* (Plenum Press, New York, 1972); L. Berger and G. Bergmann, in *The Hall Effect and Its Applications* edited by C. L. Chien and C. R. Westgate (Plenum, New York, 1979).
2 For a recent review, see J. Sinova, T. Jungwirth, and J. Černoch, Int. J. Mod. Phys. B 18, 1083 (2004).
3 R. Karplus and J. M. Luttinger, Phys. Rev. 95, 1154 (1954); J. M. Luttinger, Phys. Rev. 112, 739 (1958).
4 J. Smit, Physica 21, 877 (1955); *ibid.* 24, 39 (1958).
5 L. Berger, Phys. Rev. B 2, 4559 (1970); 5, 1862 (1972).
6 For recent works, see A. Crépieux and P. Bruno, Phys. Rev. B 64, 014416 (2001); V. K. Dugaev, A. Crépieux, and P. Bruno, *ibid.* 64, 104411 (2001); A. Crépieux, J. Wunderlich, V. K. Dugaev, and P. Bruno, J. Magn. Magn. Mater. 242-245, 464 (2002).
7 M. C. Chang and Q. Niu, Phys. Rev. B 53, 7010 (1996).
8 G. Sundaram and Q. Niu, Phys. Rev. B 59, 14915 (1999).
9 T. Jungwirth, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. 88, 207208 (2002).
10 D. Culcer, A. H. MacDonald, and Q. Niu, Phys. Rev. B 68, 045327 (2003).
11 M. Onoda and N. Nagaosa, J. Phys. Soc. Jpn. 71, 19 (2002); Phys. Rev. Lett. 90, 206601 (2003).
12 Z. Fang, N. Nagaosa, K. S. Takahashi, A. Asamitsu, R. Mathieu, T. Ogasawara, H. Yamada, M. Kawasaki, Y. Tokura, and K. Terakura, Science 302, 92 (2003).
13 T. Jungwirth, J. Sinova, K. Y. Wang, K. W. Edmonds, R. P. Campion, B. L. Gallagher, C. T. Foxon, Q. Niu, and A. H. MacDonald, Appl. Phys. Lett. 83, 320 (2003).
14 I. V. Solovyev, Phys. Rev. B 67, 174406 (2003).
15 Y. Yao, L. Kleinman, A. H. MacDonald, J. Sinova, T. Jungwirth, D.-S. Wang, E. Wang, and Q. Niu, Phys. Rev. Lett. 92, 037204 (2004).
16 P. Bruno, V. K. Dugaev, and M. Taillefer, Phys. Rev. Lett. 93, 096806 (2004).
17 N. A. Sinitsyn, Q. Niu, J. Sinova, and K. Nomura, Phys. Rev. B 72, 045346 (2005).
18 M. V. Berry, Proc. R. Soc. London, Ser. A 392, 45 (1984).
19 P. Středa, J. Phys. C 15, L717 (1982).
20 V. K. Dugaev, P. Bruno, M. Taillefumier, B. Canals, and C. Lacroix, Phys. Rev. B 71, 224423 (2005).
21 S. Y. Liu and X. L. Lei, cond-mat/0411629; Physics Rev. B 72, 155314 (2005).
22 P. Lipavský, V. Špička, and B. Velicky, Phys. Rev. B 34, 6933 (1986); H. Haug, Phys. Status Solidi (b) 173, 139 (1992).
23 H. Haug and A.-P. Jauho, Quantum Kinetics in Transport and Optics of Semiconductors (Springer, 1996).
24 G. D. Mahan, Many-Particle Physics (Plenum, NY, 1990).
25 S. Zhang and Z. Yang, Phys. Rev. Lett. 94, 066602 (2005).
26 X. L. Lei, J. L. Birman, and C. S. Ting, J. Appl. Phys. 58, 2270 (1985).