Existence and Smoothness of Navier-Stokes Equations

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Abstract
In this paper we propose new method for proving of existence of global solutions for 3D Navier-Stokes equations. This complies an application to the Clay Institute Millennium Prize Navier Stokes Problem. The proposed method can be applied for investigation of global solutions for other classes of PDEs.

Key words: Navier-Stokes equations, global existence.

AMS subject classification: Primary 35Q30, 76D05, 46E35; Secondary 35B65, 35K55.

1 Introduction

In this article we investigate the following Navier-Stokes Equations (1.1)

\[
\begin{aligned}
   u_t + uu_x + vu_y + wu_z + \frac{1}{\rho} p_x - \sigma u_{xx} - \sigma u_{yy} - \sigma u_{zz} &= 0 \\
   v_t + uv_x + vv_y + wv_z + \frac{1}{\rho} p_y - \sigma v_{xx} - \sigma v_{yy} - \sigma v_{zz} &= 0 \\
   w_t + uw + vw_y + ww_z + \frac{1}{\rho} p_z - \sigma w_{xx} - \sigma w_{yy} - \sigma w_{zz} &= 0
\end{aligned}
\]

\[
\begin{aligned}
   u_x + v_y + w_z &= 0 \text{ in } (0,\infty) \times \mathbb{R}^3 \\
   u(0,x,y,z) &= u_0(x,y,z), \quad v(0,x,y,z) = v_0(x,y,z) \\
   w(0,x,y,z) &= w_0(x,y,z) \text{ in } \mathbb{R}^3.
\end{aligned}
\]
where \( p, u, v, w : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) are unknown, \( u_0, v_0, w_0 \in C^\infty(\mathbb{R}^3) \) are given functions.

This is a system of partial differential equations that governs the flow of a viscous incompressible fluid. \( \rho \) is the density, \( u \) the velocity vector, \( P \) the pressure. The first three equations of (1.1) are Cauchy's momentum equations where the first term is the accelerating time varying term, the second and third are the convective and the hydrostatic terms respectively. The physical example of the convective term can be described as a river that is converging, the case where the term is increasing and the river diverging the case where the term is decreasing. The hydrostatic term describe flow from high pressure to low pressure. The forth term is the viscosity term with the coefficient \( \nu \) the kinematical viscosity. This term is said to describe the ability of the fluid to induce motion of neighboring particles. On the right hand side we have the external forces density term. This term can include: gravity, magneto-hydrodynamic force, and so on. The fourth equation of (1.1) is the nullification of the divergence due to incompressibility condition. Turbulent fluid motions are believed to be well modeled by the Navier-Stokes equations. But, due to the complexity of the equations most of our understanding relies on laboratory experiments. This is the main reason why it is necessary to know basic features of the equations like existence and smoothness of the equations' solutions. In the case of the 3D version of the NS equations the existence problem is an unsolved issue([1]).

We recall that global existence of weak solutions of the Naiver-Stokes equations is known to hold in every space dimension. Uniqueness of weak solutions and global existence of strong solutions is known in dimension two [4]. In dimension three, global existence of strong solutions of the Navier-Stokes equations in thin three-dimensional domains began with the papers [5] and [6], where is used the methods in [2] and [3].

In this paper we propose new method for investigation of equations (1.1). The proposed method gives existence of classical solutions for the problem (1.1).

Without loss of generality we assume that \( \rho = \nu = 1 \). For a set \( A \) such that \( A \subset \mathbb{R} \) or \( A \subset \mathbb{R}^2 \) or \( A \subset \mathbb{R}^3 \), with \( \mu(A) \) we will denote it measure.

Let \( R^3 = \bigcup_{j=1}^{\infty} D_j \), where \( D_j \) be bounded subsets of \( R^3 \) satisfying the following conditions

1. \( D_i \cap D_j = \emptyset \) for \( i \neq j, i,j \in \{1,2,\ldots\} \),
2. \( D_j \) and \( D_{j+1}, j \in \{1,2,\ldots\} \), are adjoining,
3. if \((x_1, y_1, z_1) \in D_j\) for some \(j \in \{1, 2, \ldots\}\), is fixed and

\[
\begin{align*}
D_{x_1} & = \{(y, z) \in \mathbb{R}^2 : (x_1, y, z) \in D_j\}, \\
D_{y_1} & = \{(x, z) \in \mathbb{R}^2 : (x, y_1, z) \in D_j\}, \\
D_{z_1} & = \{(x, y) \in \mathbb{R}^2 : (x, y, z_1) \in D_j\}, \\
D_{x_1y_1} & = \{z \in \mathbb{R} : (x_1, y_1, z) \in D_j\}, \\
D_{x_1z_1} & = \{y \in \mathbb{R} : (x_1, y, z_1) \in D_j\}, \\
D_{y_1z_1} & = \{x \in \mathbb{R} : (x, y_1, z_1) \in D_j\},
\end{align*}
\]

We have

\[
\begin{align*}
\mu(D_{x_1}) & \leq \mu(D_j), \\
\mu(D_{y_1}) & \leq \mu(D_j), \\
\mu(D_{z_1}) & \leq \mu(D_j), \\
\mu(D_{x_1y_1}) & \leq \mu(D_j), \\
\mu(D_{x_1z_1}) & \leq \mu(D_j), \\
\mu(D_{y_1z_1}) & \leq \mu(D_j), \ j \in \{1, 2, \ldots\}.
\end{align*}
\]

For every \(j \in \{1, 2, \ldots\}\) we denote with \(D_{||j||}\) a compact subset of \(D_j\) such that \(D_{||j||} \neq D_j\).

for a set \(B \subset \mathbb{R}^3\) and a function \(f: \mathbb{R}^3 \to \mathbb{R}\) with \(f|_B\) we denote the restriction of \(f\) to \(B\).

Our main result is as follows

**Theorem 1.1.** Let \(u_0, v_0, w_0 \in C^\infty(\mathbb{R}^3)\) be such that
\[
\begin{align*}
u_0|_{D_j}, w_0|_{D_j} & \in C^\infty(D_j), \\
\text{supp}u_0|_{D_j}, \text{supp}v_0|_{D_j}, \text{supp}w_0|_{D_j} & \subseteq D_{||j||} \in \{1, 2, \ldots\},
\end{align*}
\]

and

\[
\begin{align*}
|\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_0 (x, y, z) | & \leq C\alpha_1 \alpha_2 \alpha_3 K (1 + \sqrt{x^2 + y^2 + z^2})^{-k}, \\
|\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} v_0 (x, y, z) | & \leq C\alpha_1 \alpha_2 \alpha_3 K (1 + \sqrt{x^2 + y^2 + z^2})^{-k}, \\
|\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} w_0 (x, y, z) | & \leq C\alpha_1 \alpha_2 \alpha_3 K (1 + \sqrt{x^2 + y^2 + z^2})^{-k}, \quad (1.2)
\end{align*}
\]

on \(\mathbb{R}^3\), for any \(a_1, a_2, a_3 \in \mathbb{N}_0\) and positive constant \(K\).

Then the problem (1.1) has a solution \((u, v, w, p) \in (C^\infty([0, \infty) \times \mathbb{R}^3))^4\) such that

\[
\begin{align*}
\int_{\mathbb{R}^3} |u(t, x, y, z)|^2 \, dx \, dy \, dz & \leq C_1 \int_{\mathbb{R}^3} |v(t, x, y, z)|^2 \, dx \, dy \, dz \leq C_1 \\
\int_{\mathbb{R}^3} |w(t, x, y, z)|^2 \, dx \, dy \, dz & \leq C_1 \int_{\mathbb{R}^3} |p(t, x, y, z)|^2 \, dx \, dy \, dz \leq C_1
\end{align*}
\]

for some positive constant \(C_1\).
Remark 1.2 If \( u_0 \neq 0, v_0 \neq 0, w_0 \neq 0 \), then we obtain a nontrivial solution of the system (1.1).

2 Preliminaries

Definition 2.1 let \((X, d)\) be a metric space and \(M\) be a subset of \(X\). The mapping \(T : M \rightarrow X\) is said to be expansive if there exists a constant \(h > 1\) such that

\[
d(Tx, ty) > hd(x, y)
\]

for any \(x, y \in M\).

Theorem 2.2 ([7], Theorem 2.4). Let \(X\) be a nonempty closed convex subset of a Banach space \(E\). Suppose that \(T\) and \(S\) map \(X\) into \(E\) such that

1. \(S\) is continuous and \(S(X)\) resides in a compact subset of \(E\).
2. \(T : X \mapsto E\) is expansive.
3. \(S(X) \subset (I - T)(E)\) and \([x = Tx + Sy, \ y \in X \rightarrow x \in X\) (or \(S(X) \subset (I - T)(X)\)).

then there exist a point \(x^* \in X\) such that

\[
Sx^* + Tx^* = x^*
\]

Theorem 2.3. Let \(X\) be a nonempty closed convex subset of a Banach space \(E\) and \(Y\) is a nonempty compact subset of \(E\) such that \(X \subset Y\), \(Y \neq X\). Suppose that \(T\) and \(S\) map \(X\) into \(E\) such that

1. \(S\) is continuous and \(S(X)\) resides in \(Y\).
2. \(T : X \mapsto E\) is linear, continuous and expansive, and \(T : X \mapsto Y\) is onto, and \(\{x - z : x \in X; z \in S(X)\} \subset Y\).

Then there exists an \(x_2 \in X\) such that

\[
Sx^* + Tx^* = x^*
\]

Proof. Since \(Y\) is compact and \(S(X)\) resides in \(Y\), we have that the first condition of Theorem 2.2 holds. Because \(T : X \mapsto E\) is expansive, we have
that the second condition of Theorem 2.2 holds. Note that $T^{-1} : Y \to E$
exists, it is linear and contractive with a constant $l \in (0,1)$. Let $z \in S(X)$
be arbitrarily chosen and fixed. Set

$$A = \{y - z : y \in Y\}.$$ 

Take $y_0 \in Y$ arbitrarily. Define the sequence $\{y_n\}_{n \in \mathbb{N}}$ as follows.

$$y_{n+1} = T^{-1}y_n - z, \quad n \in \mathbb{N} \cup \{0\}.$$ 

Then

$$||y_2 - y_1|| = ||T^{-1}y_1 - T^{-1}y_0|| \leq l||y_1 - y_0||,$$

$$||y_3 - y_2|| = ||T^{-1}y_2 - T^{-1}y_1|| \leq l||y_2 - y_1||,$$

$$\leq l^2||y_1 - y_0||.$$ 

Using the principle of the mathematical induction, we get

$$||y_{n+1} - y_n|| \leq l^n||y_1 - y_0||, \quad n \in \mathbb{N}$$

Now, from $m > n, m, n \in \mathbb{N}$, we find

$$||y_m - y_n|| \leq ||y_m - y_{m-1}|| + \ldots + ||y_{n+1} - y_n|| \leq (l^{m-1} + \ldots + l^n)||y_1 - y_0|| \leq l^n \sum_{j=0}^{\infty} l^j||y_1 - y_0|| = \frac{l^n}{1-l}||y_1 - y_0||.$$ 

Therefore $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence of elements of $Y \subset E$. Since $E$
is a Banach space, it follows that the sequence $\{y_n\}_{n \in \mathbb{N}}$ is convergent to an
element $y^* \in E$. Because $\{y_n\}_{n \in \mathbb{N}} \subset Y$ and $Y \subset E$ is compact, we have that

$$y^* = T^{-1}y^* - z$$

Or

$$z^* = Tz^* + z, \quad z^* = T^{-1}y^* \in X$$

Because $z \in S(X)$ was arbitrarily chosen, we conclude that $S(X) \subset (I - T)(X)$, i.e.,
the third condition of Theorem 2.2 holds. Hence Theorem 2.2, it follows that there exists an $x^* \in X$
such that

$$Tx^* + Sx^* = x^*.$$ 

This completes the proof.
3 Proof of the Main Result

From the fourth equation of the system (1.1), we get

\[ u(u_x + v_y + w_z) = 0, \quad v(u_x + v_y + w_z) = 0, \quad w(u_x + v_y + w_z) = 0. \]

Then the system (1.1) we can rewrite in the form

\[
\begin{align*}
    u_t + uu_x + vu_y + wu_z + u(u_x + v_y + w_z) + p_z - u_{xx} - u_{yy} - u_{zz} &= 0 \\
    v_t + uv_x + vv_y + vw_z + v(u_x + v_y + w_z) + p_z - v_{xx} - v_{yy} - v_{zz} &= 0 \\
    w_t + uw_x + vw_y + ww_z + w(u_x + v_y + w_z) + p_z - w_{xx} - w_{yy} - w_{zz} &= 0 \\
    u_x + v_y + w_z &= 0 \quad \text{in} \ (0, \infty) \times \mathbb{R}^3 \end{align*}
\]

whereupon

\[
\begin{align*}
    u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} &= 0 \\
    v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} &= 0 \\
    w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} &= 0 \\
    u_x + v_y + w_z &= 0 \quad \text{in} \ (0, \infty) \times \mathbb{R}^3 \\
    u(0, x, y, z) &= u_0(x, y, z), \quad v(0, x, y, z) = v_0(x, y, z) \quad \text{(3.1)} \\
    w(0, x, y, z) &= w_0(x, y, z) \quad \text{in} \ \mathbb{R}^3.
\end{align*}
\]

Remark 3.1. We note that using the fourth equation of (3.1) we can obtain the system (1.1).

Step 1. Let \( j \in \{1, 2, \ldots\} \) be arbitrarily chosen. Firstly, we consider the Problem

\[
\begin{align*}
    u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} &= 0 \\
    v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} &= 0 \\
    w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} &= 0 \\
    u_x + v_y + w_z &= 0 \quad \text{in} \ (0, 1] \times D_j, \\
    u(0, x, y, z) &= u_0(0, x, y, z), \quad v(0, x, y, z) = v_0(0, x, y, z), \quad \text{(3.2)} \\
    w(0, x, y, z) &= w_0(0, x, y, z) \quad \text{in} \ D_j.
\end{align*}
\]
We will prove that the problem (3.2) has a solution \((u, v, w, p)\) such that \(u, v, w, p \in C^1([0,1], C_0^2(D_j))\).
Let \((x_0, y_0, z_0) \in D_j\) be arbitrarily chosen.

For \(u, v, w, p \in C^1([0,1], C_0^2(D_j))\), we define

\[
I_1^{ij}(u, v, w, p) = \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} (u(t, \alpha, \beta, \gamma) - u_0(\alpha, \beta, \gamma)) dydz_1d\beta dy_1d\alpha dx_1
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} (u^2(s, \alpha, \beta, \gamma)) dydz_1d\beta dy_1d\alpha ds
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} u(s, \alpha, \beta, \gamma)v(s, \alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha dx_1 ds
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} u(s, \alpha, \beta, \gamma)w(s, \alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha dx_1 ds
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} p(s, \alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha dx_1 ds
\]

\[
I_2^{ij}(u, v, w, p) = \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} (v(t, \alpha, \beta, \gamma) - v_0(\alpha, \beta, \gamma)) dydz_1d\beta dy_1d\alpha dx_1
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} u(s, \alpha, \beta, \gamma)v(s, \alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha ds
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} v^2(s, \alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha dx_1 ds
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} v(s, \alpha, \beta, \gamma)w(s, \alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha dx_1 ds
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{\gamma_1}^{\gamma_1} p(s, \alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha dx_1 ds
\]
Lemma 3.2. Every solution \((u,v,w,p) \in (C^1([0,1], C_0^2(D_j)))^4\) of the system

\[ l^{1j}(u,v,w,p) = 0 \]
\[ I_{21}(u,v,w,p) = 0 \]  

(3.3)

\[ I_{31}(u,v,w,p) = 0 \]

\[ I_{41}(u,v,w,p) = 0 \]

is a solution of the problem (3.2).

**Proof.** Consider the equation

\[ I_{11}(u,v,w,p) = 0 \quad \text{for} \quad (t,x,y,z) \in [0,1] \times D_j. \]

We differentiate it with respect to \( t \) and we get

\[
0 = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u_t (t, \alpha, \beta, \gamma) \, dy dz_1 d\beta dy_1 d\alpha + \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u^2 (t, \alpha, \beta, \gamma) \, dy dz_1 d\beta dy_1 \, d\alpha \\
+ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u(t, \alpha, \beta, \gamma) v(t, \alpha, \beta, \gamma) \, dy dz_1 d\beta \, d\alpha \\
+ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u(t, \alpha, \beta, \gamma) w(t, \alpha, \beta, \gamma) \, dy dz_1 d\beta dy_1 \, d\alpha \\
+ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} p(t, \alpha, \beta, \gamma) \, dy dz_1 d\beta dy_1 \, d\alpha \\
- \int_{y_0}^{y_1} \int_{z_0}^{z_1} u(t, x, \beta, \gamma) \, dy dz_1 d\beta dy_1 \\
- \int_{x_0}^{x_1} \int_{z_0}^{z_1} u(t, \alpha, y, \gamma) \, dy dz_1 d\alpha \\
- \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u(t, \alpha, \beta, z) \, dy dz_1 d\beta \, d\alpha.
\]

We differentiate the last equality with respect to \( x \) and we obtain

\[
0 = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u_t (t, \alpha, \beta, \gamma) \, dy dz_1 d\beta dy_1 d\alpha + \int_{y_0}^{y_1} \int_{z_0}^{z_1} u^2 (t, x, \beta, \gamma) \, dy dz_1 d\beta dy_1 \\
+ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u(t, \alpha, \beta, \gamma) v(t, \alpha, \beta, \gamma) \, dy dz_1 d\beta \, d\alpha.
\]
Again we differentiate with respect to \( t,x,y,z \) and we find

\[
0 = \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} u_t(t,x,\beta,\gamma) \, dy \, dz_1 \, d\beta \, dy_1
\]

\[
+ \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} (u^2(t,x,\beta,\gamma))_x \, dy \, dz_1 \, d\beta \, dy_1
\]

\[
+ \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} u(t,x,\beta,\gamma) \, v(t,x,\beta,\gamma) \, dy \, dz_1 \, d\beta
\]

\[
+ \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} p_x(t,x,\beta,\gamma) \, dy \, dz_1 \, d\beta \, dy_1
\]

\[
- \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} u_{xx}(t,x,\beta,\gamma) \, dy \, dz_1 \, d\beta \, dy_1
\]

\[
- \int_{z_0}^{z_1} \int_{z_0}^{z_1} u(t,x,y,\gamma) \, dy \, dz_1 \, d\beta \, dy_1 - \int_{y_0}^{y} \int_{y_0}^{y_1} u(t,x,\beta,z) \, d\beta \, dy_1, \quad (t,x,y,z) \in [0,1] \times D_j
\]

Now we differentiate twice the last equation with respect to \( y \) and we find

\[
0 = \int_{z_0}^{z_1} \int_{z_0}^{z_1} u_t(t,x,y,\gamma) \, dy \, dz_1 + \int_{z_0}^{z_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} (u^2(t,x,y,\gamma))_x \, dy \, dz_1
\]

\[
+ \int_{z_0}^{z_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} \int_{z_0}^{z_1} (u(t,x,y,\gamma)v(t,x,y,\gamma))_y \, dy \, dz_1
\]

\[
+ \int_{z_0}^{z_1} u(t,x,y,\gamma) \, w(t,x,y,\gamma) \, dy + \int_{z_0}^{z_1} \int_{z_0}^{z_1} p_x(t,x,y,\gamma) \, dy \, dz_1
\]
We differentiate twice with respect to \( z \) the last equation and we get

\[
0 = u(t,x,y,z) + (u^2(t,x,y,z))_x + (u(t,x,y,z)v(t,x,y,z))_y \\
+ (u(t,x,y,z)w(t,x,y,z))_z + p_z(t,x,y,z) \\
- u_{xx}(t,x,y,z) - u_{yy}(t,x,y,z) - u_{zz}(t,x,y,z), \quad (t,x,y,z) \in [0,1] \times D_j
\]

which we differentiate twice in \( x, y \) and \( z \) and we find

\[
u(0,x,y,z) = v_0(x,y,z) \quad \text{in} \quad D_j.
\]

After we put \( t=0 \) in \( I_2^{1j} = 0 \) and differentiate twice in \( x, y \) and \( z \) the equation \( I_2^{1j} = 0 \), we obtain

\[
v(0,x,y,z) = v_0(x,y,z) \quad \text{in} \quad D_j.
\]

After we put \( t=0 \) in \( I_3^{1j} = 0 \) and differentiate twice in \( x, y \) and \( z \) the equation \( I_3^{1j} = 0 \), we obtain

\[
w(0,x,y,z) = w_0(x,y,z) \quad \text{in} \quad D_j.
\]

This completes the proof.

The proof of the existence result is based on theorem 2.2.

Let \( \mathcal{X} \) be the set of all equicontinuous families of functions of the space

\[
\{g \in C^1([0,1], C_0^2(D_j)) : supp_{(x,y,z)} g \subset D_{jj} \}
\]
with respect to the norm
\[ ||f|| = \max \{ \max_{t \in [0,1]} |f(t,x,y,z)| \}, \]

\[
\max_{t \in [0,1]} |f(t,x,y,z)|, \\
\max_{t \in [0,1]} |f_x(t,x,y,z)|, \\
\max_{t \in [0,1]} |f_y(t,x,y,z)|, \\
\max_{t \in [0,1]} |f_z(t,x,y,z)|, \\
\max_{t \in [0,1]} |f_\xi(t,x,y,z)|, \\
\max_{t \in [0,1]} |f_\eta(t,x,y,z)|, \\
\max_{t \in [0,1]} |f_\zeta(t,x,y,z)|, \\
\max_{t \in [0,1]} |f_\tau(t,x,y,z)|,
\]

and \( \tilde{x}_1 = \tilde{x}_1 \cup \{ u_0, v_0, w_0 \} \), \( \tilde{x}_1 = \overline{\tilde{x}_1} \), i.e., \( \tilde{x}_1 \) is the completion of \( \tilde{x}_1 \), and

\[ x_1 = \{ f \in \tilde{x}_1 : ||f|| \leq \frac{1}{2} \sqrt{\mu(D_j)} \} \]

where \( \mu(D_j) \) is the measure of the set \( D_j \). Let also,

\[ N_{1j} := \max(\max_{D_j} |u_0|, \max_{D_j} |v_0|, \max_{D_j} |w_0|) \]

We take \( \epsilon > 0 \) so that

\[ \epsilon (3M_{1j}^2 + 6M_{1j} + N_{1j}) (\mu(D_j))^2 \leq M_{1j}. \]

We set

\[ Y^1 = \{ f \in \tilde{x}_1 : ||f|| \leq (1 + \epsilon) M_{1j} \} \]

By the construction of \( X^1 \) and \( Y^1 \), We have that \( X^1 \) is a compact subset of \( Y^1 \) and

\( Y^1 \) is a compact subset of \( C^1((0,1], C_0^2(D_j)). \)

For \( u,v,w,p \in Y^1 \) we define the operators

\[ S_{1j}^1 (u,v,w,p) = -\epsilon u + \epsilon l_{1j}^1, \quad T_{1j}^1 (u,v,w,p) = (1 + \epsilon)u, \]

\[ S_{2j}^1 (u,v,w,p) = -\epsilon v + \epsilon l_{2j}^1, \quad T_{2j}^1 (u,v,w,p) = (1 + \epsilon)v, \]

\[ S_{3j}^1 (u,v,w,p) = -\epsilon w + \epsilon l_{3j}^1, \quad T_{3j}^1 (u,v,w,p) = (1 + \epsilon)w, \]

\[ S_{4j}^1 (u,v,w,p) = -\epsilon p + \epsilon l_{4j}^1, \quad T_{4j}^1 (u,v,w,p) = (1 + \epsilon)p, \]

\[ S_{1j}^1 = (S_{1j}^1, S_{2j}^1, S_{3j}^1, S_{4j}^1), \quad T_{1j}^1 = (T_{1j}^1, T_{2j}^1, T_{3j}^1, T_{4j}^1). \]

For $u,v,w,p \in X^j$ we have that

$$||I_1^{ij}(u,v,w,p)|| \leq (||u|| + ||v||^2 + ||u||.||v|| + ||u||.||w|| + ||p|| + ||u||
+ ||u|| + ||u|| + N_{1j}) (\mu(D_j))^2$$

$$\leq (3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2.$$  

Therefore, using our choice of $\epsilon$

$$||S_1^{1j}|| \leq \epsilon(||u|| + \epsilon(3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2
\leq \epsilon M_{1j} + \epsilon(3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2
\leq (1+\epsilon) M_{1j}.$$  

As in above we have

$$||S_2^{1j}|| \leq \epsilon(||v|| + \epsilon(3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2
\leq \epsilon M_{1j} + \epsilon(3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2
\leq (1+\epsilon) M_{1j}.$$  

$$||S_3^{1j}|| \leq \epsilon(||w|| + \epsilon(3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2
\leq \epsilon M_{1j} + \epsilon(3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2
\leq (1+\epsilon) M_{1j},$$

$$||S_4^{1j}|| \leq \epsilon(||p|| + 3\epsilon M_{1j}) (\mu(D_j))^2
\leq \epsilon M_{1j} + 1 M_{1j}
= (1+\epsilon) M_{1j}.$$  

Therefore, for $(u,v,w,p) \in X^j$ we have that

$$S_1^{1j}(u,v,w,p) \in Y^1, \ i = 1,2,3,4.$$  

Then

$$S_1^{1j}: X^1 \times X^1 \times X^1 \times X^1 \rightarrow Y^1 \times Y^1 \times Y^1 \times Y^1$$

and it is continuous.

The operator
is an expansive operator with constant $1+\varepsilon > 1$ and if $(u,v,w,p) \in Y^1 \times Y^1 \times Y^1 \times Y^1$, then

$$\left( \frac{1}{1+\varepsilon} u, \frac{1}{1+\varepsilon} v, \frac{1}{1+\varepsilon} w, \frac{1}{1+\varepsilon} p \right) \in X^1 \times X^1 \times X^1 \times X^1,$$

and

$$\left( T_{1}^{1j} \left( \frac{1}{1+\varepsilon} u, \frac{1}{1+\varepsilon} v, \frac{1}{1+\varepsilon} w, \frac{1}{1+\varepsilon} p \right), T_{2}^{1j} \left( \frac{1}{1+\varepsilon} u, \frac{1}{1+\varepsilon} v, \frac{1}{1+\varepsilon} w, \frac{1}{1+\varepsilon} p \right), T_{3}^{1j} \left( \frac{1}{1+\varepsilon} u, \frac{1}{1+\varepsilon} v, \frac{1}{1+\varepsilon} w, \frac{1}{1+\varepsilon} p \right), T_{4}^{1j} \left( \frac{1}{1+\varepsilon} u, \frac{1}{1+\varepsilon} v, \frac{1}{1+\varepsilon} w, \frac{1}{1+\varepsilon} p \right) \right) = (u,v,w,p).$$

Consequently $T_{i}^{1j}: X^1 \times X^1 \times X^1 \times X^1 \rightarrow Y^1 \times Y^1 \times Y^1 \times Y^1$ is onto.

From here and from Theorem 2.3, it follows that the operator $T_{i}^{1j} + S_{i}^{1j}$ has a fixed point $(u_1,v_1,w_1,p_1)$ in $X^1 \times X^1 \times X^1$. For it we have

$$\begin{align*}
T_{1}^{1j} (u_1, v_1 w_1, p_1) + S_{1}^{1j} (u_1, v_1 w_1, p_1) &= u_1 \\
T_{2}^{1j} (u_1, v_1 w_1, p_1) + S_{2}^{1j} (u_1, v_1 w_1, p_1) &= v_1 \\
T_{3}^{1j} (u_1, v_1 w_1, p_1) + S_{3}^{1j} (u_1, v_1 w_1, p_1) &= w_1 \\
T_{4}^{1j} (u_1, v_1 w_1, p_1) + S_{4}^{1j} (u_1, v_1 w_1, p_1) &= p_1
\end{align*}$$

or

$$\begin{align*}
(1+\varepsilon) u_1 - \varepsilon u_1 + T_{1}^{1j} (u_1, v_1 w_1, p_1) &= u_1 \\
(1+\varepsilon) v_1 - \varepsilon v_1 + T_{2}^{1j} (u_1, v_1 w_1, p_1) &= v_1 \\
(1+\varepsilon) w_1 - \varepsilon w_1 + T_{3}^{1j} (u_1, v_1 w_1, p_1) &= w_1 \\
(1+\varepsilon) p_1 - \varepsilon p_1 + T_{4}^{1j} (u_1, v_1 w_1, p_1) &= p_1
\end{align*}$$
Disjointed sets of $\mathbb{R}^3$

Figure 1: A sketch of the division of $\mathbb{R}^3$ to disjointed subsets $D_1, D_2, D_3$ etc, that illustrates how the proof of existence steps are done.

Whereupon

\[ I_1^{ij}(u_1, v_1, w_1, p_1) = 0, \quad I_2^{ij}(u_1, v_1, w_1, p_1) = 0 \]

\[ I_3^{ij}(u_1, v_1, w_1, p_1) = 0, \quad I_4^{ij}(u_1, v_1, w_1, p_1) = 0 \]

Hence and Lemma 3.2 we obtain that $(u_1, v_1, w_1, p_1)$ is a solution of the system (3.2) for which $u_1, v_1, w_1, p_1 \in C^1([0,1], C_0^2(D_j))$.

**Remark 3.3.** If we assume that

\[ u_1(x,y,z) = u_0(x,y,z), \]
\[ v_1(x,y,z) = v_0(x,y,z), \]
\[ w_1(x,y,z) = w_0(x,y,z), \quad t \in [0,1], \quad (x,y,z) \in D_j \]

then using $I_i^{ij}(u_1, v_1, w_1, p_1)(t,x,y,z) = 0, \quad t \in [0,1], \quad (x,y,z) \in D_i$

for $l = 1, 2, 3$ we get

\[ t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z_1} (u_0(\alpha, \beta, \gamma))^2 \, dy \, dz_1 \, d\beta \, dy_1 \, d\alpha \, dx_1 \]
\begin{align}
& + t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, \beta, \gamma) v_0(\alpha, \beta, \gamma) dydz_1d\beta d\alpha dx_1 \\
& + t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, \beta, \gamma) w_0(\alpha, \beta, \gamma) dyd\beta dy_1d\alpha dx_1 \\
& + \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} p_1(\alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha ds \\
& - t \int_{y_0}^{y} \int_{z_0}^{z} u_0(x, \beta, \gamma) dydz_1d\beta dy_1 \\
& - t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, \beta, z) dydz_1d\alpha dx_1 \\
& - t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, y, \gamma) dydz_1d\alpha dx_1 = 0, t \in [0, 1], (x, y, z) \in D_i \\
\end{align}

\begin{align}
& t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, \beta, \gamma) v_0(\alpha, \beta, \gamma) dydz_1d\beta dy_1d\alpha dx_1 \\
& + t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} (v_0(\alpha, \beta, \gamma))^2 dydz_1d\beta d\alpha dx_1 \\
& + t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} v_0(\alpha, \beta, \gamma) w_0(\alpha, \beta, \gamma) dyd\beta dy_1d\alpha dx_1 \\
& + \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} p_1(s, \alpha, \beta, \gamma) dydz_1d\beta d\alpha dx_1 ds \\
& - t \int_{y_0}^{y} \int_{z_0}^{z} v_0(x, \beta, \gamma) dydz_1d\beta dy_1 \\
& - t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} v_0(\alpha, \beta, z) dydz_1d\alpha dx_1 \\
& - t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} v_0(\alpha, y, \gamma) dydz_1d\alpha dx_1 = 0, t \in [0, 1], (x, y, z) \in D_i \\
\end{align}
Now we consider the problem

\[
+ t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} w_0(\alpha, \beta, \gamma) \, dydz_1 \, d\beta \, d\alpha \, dx_1
\]

\[
+ t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} (w_0(\alpha, \beta, \gamma))^2 \, dyd\beta \, dy_1 \, dx_1
\]

\[
+ \int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{0}^{\gamma} p_1(s, \alpha, \beta, \gamma) \, dyd\beta \, dy_1 \, dx_1 \, ds
\]

\[
- t \int_{y_0}^{y} \int_{z_0}^{z} \int_{x_0}^{x} \int_{0}^{\gamma} w_0(x, \beta, \gamma) \, dydz_1 \, d\beta \, dy_1
\]

\[
- t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} w_0(\alpha, \beta, z) \, d\beta \, dy_1 \, dx_1
\]

\[
- t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} w_0(\alpha, y, z) \, dydz_1 \, d\alpha \, dx_1 = 0 \quad t \in [0,1], (x,y,z) \in D_j
\]

\[
t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, \beta, \gamma) \, dydz_1 \, d\beta \, dy_1 \, da
\]

\[
+ t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, \beta, \gamma) \, dyd\beta \, dy_1 \, dx_1
\]

\[
+ t \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u_0(\alpha, \gamma, \gamma) \, dydz_1 \, d\beta \, dx_1 = 0, \quad t \in [0,1], (x,y,z) \in D_j
\]

(3.6)

(3.7)

If the initial functions \( u_0, v_0, w_0 \) and \( p_0 \) do not satisfy (3.4), (3.5), (3.6), (3.7), then

\[
( u_0, v_0, w_0 ) \not\equiv (u_1, v_1, w_1 ) \text{ on } [0,1] \times D_j
\]

**Step 2.** Now we consider the problem

\[
\begin{cases}
  u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\
  v_t + (uw)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} = 0 \\
  w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\
  u_x + v_y + w_z = 0 \quad \text{in } (1,2] \times D_j \\
  u(1,x,y,z) = u_1(1,x,y,z), \quad v(1,x,y,z) = v_1(1,x,y,z), \\
  w(1,x,y,z) = w_1(1,x,y,z) \quad \text{in } D_j.
\end{cases}
\]

(3.8)
We will prove that the problem (3.8) has a solution \((u,v,w,p)\) such that
\[ u, v, w, p \in C^1([1,2], C_0^2(D)) \].
Let \((x_0,y_0,z_0)\) be arbitrarily chosen.

For \((u, v, w, p) \in (C^1([1,2], C_0^2(D)))^4\) we define

\[ l_1^{2j}(u,v,w,p) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} (u(t,\alpha,\beta,\gamma) - u_1(1,\alpha,\beta,\gamma)) \, dydz_1d\beta dy_1d\alpha dx_1 \]

\[ + \int_{1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u^2(s,\alpha,\beta,\gamma) \, dydz_1d\beta dy_1d\alpha ds \]

\[ + \int_{1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u(s,\alpha,\beta,\gamma)v(s,\alpha,\beta,\gamma) \, dydz_1d\beta dy_1d\alpha dx_1 ds \]

\[ + \int_{1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u(s,\alpha,\beta,\gamma)w(s,\alpha,\beta,\gamma) \, dydz_1d\beta dy_1d\alpha dx_1 ds \]

\[ + \int_{1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} p(s,\alpha,\beta,\gamma) \, dydz_1d\beta dy_1d\alpha ds \]

\[ - \int_{1}^{t} \int_{y_0}^{y} \int_{z_0}^{z} u(s,\alpha,\beta,\gamma) \, dydz_1d\beta dy_1d\alpha ds \]

\[ - \int_{1}^{t} \int_{x_0}^{x} \int_{z_0}^{z} u(s,\alpha,\gamma) \, dydz_1d\beta dy_1d\alpha dx_1 ds \]

\[ - \int_{1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u(s,\alpha,\beta,\gamma) \, dy\, dy_1\, dy_1\, dz_1 \]

\[ l_2^{2j}(u,v,w,p) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} (v(t,\alpha,\beta,\gamma) - v_1(1,\alpha,\beta,\gamma)) \, dydz_1d\beta dy_1d\gamma dx_1 \]

\[ + \int_{1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} u(s,\alpha,\beta,\gamma)v(s,\alpha,\beta,\gamma) \, dydz_1d\beta dy_1d\alpha ds \]

\[ + \int_{1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} v^2(s,\alpha,\beta,\gamma) \, dydz_1d\beta dy_1d\alpha dx_1 ds \]
+ \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z v(s, \alpha, \beta, \gamma)w(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dy_1 \, d\alpha \, dx_1 \, ds 

+ \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z p(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

- \int_1^t \int_{y_0}^y \int_{z_0}^z v(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

- \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z v(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

- \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z v(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

\frac{l^2}{3}(u, v, w, p) = 
\int_{x_0}^x \int_{y_0}^y \int_{z_0}^z (w(t, \alpha, \beta, \gamma) - w_1(1, \alpha, \beta, \gamma)) \, dy \, d\beta \, dy_1 \, d\gamma \, dx_1 

+ \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z u(s, \alpha, \beta, \gamma)w(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dy_1 \, d\alpha \, ds 

+ \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z v(s, \alpha, \beta, \gamma)w(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

+ \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z w^2(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dy_1 \, d\alpha \, dx_1 \, ds 

+ \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z p(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dy_1 \, d\alpha \, dx_1 \, ds 

- \int_1^t \int_{y_0}^y \int_{z_0}^z w(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

- \int_1^t \int_{x_0}^x \int_{z_0}^z w(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

- \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z w(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

- \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z w(s, \alpha, \beta, \gamma) \, dy \, d\beta \, dz_1 \, d\alpha \, dx_1 \, ds 

19
\[ I^{2j}_4 (u, v, w, p) = \]
\[ \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z u(s, \alpha, \beta, \gamma) dy dz_1 d\beta dy_1 d\alpha ds \]
\[ + \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z v(s, \alpha, \beta, \gamma) dy dz_1 d\beta dx_1 ds \]
\[ + \int_1^t \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z w(s, \alpha, \beta, \gamma) dy dz_1 d\beta dx_1 ds. \]

We note that after we differentiate with respect to \( t \) and twice with respect to \( x, y \) and \( z \) the system

\[ I^{2j}_1 (u, v, w, p) = 0, \quad I^{2j}_2 (u, v, w, p) = 0, \quad (3.9) \]
\[ I^{2j}_3 (u, v, w, p) = 0, \quad I^{2j}_4 (u, v, w, p) = 0, \]

we get the system (3.8). After we put \( t = 1 \) in \( I^{2j}_1 = 0 \) and differentiate twice in \( x, y \) and \( z \) the equation \( I^{2j}_1 = 0 \) we obtain

\[ u(1, x, y, z) = u_1(1, x, y, z) \quad \text{in} \quad D_{ij}. \]

After we put \( t = 1 \) in \( I^{2j}_2 = 0 \) and differentiate twice in \( x, y \) and \( z \) the equation \( I^{2j}_2 = 0 \) we obtain

\[ v(1, x, y, z) = v_1(1, x, y, z) \quad \text{in} \quad D_{ij}. \]

After we put \( t = 1 \) in \( I^{2j}_3 = 0 \) and differentiate twice in \( x, y \) and \( z \) the equation \( I^{2j}_3 = 0 \) we obtain

\[ w(1, x, y, z) = w_1(1, x, y, z) \quad \text{in} \quad D_{ij}. \]

Consequently every solution \( (u, v, w, p) \in (C^1([1,2], C^2_0(D_j)))^4 \) of (3.9) is a solution of the problem (3.8).

Let \( \bar{\mathcal{X}} \) be a equicontinuous family of functions of the space

\[ \{ g \in C^1([1,2], C^2_0(D_j)), \supp_{x,y,z} g \subset D_{ij} \} \]

with respect to the norm

\[ ||f|| = \max \{ \max_{t \in [1,2], (x,y,z) \in D_j} |f(t,x,y,z)| \}. \]
\[
\max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \\
\max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \quad \max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \\
\max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \quad \max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \\
\max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \quad \max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \\
\max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \quad \max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \\
\max_{\mathbf{x} \in [1,2]} |f(t,x,y,z)|, \\
\]
\[f \in \tilde{X}^2,\]
and
\[
\tilde{X}^2 = \tilde{X}^2 \cup \{u_1(t,x,y,z), \ v_1(t,x,y,z), \ w_1(t,x,y,z)\},
\]
\[
\tilde{X}^2 = \tilde{X}^2, \text{ i.e., } \tilde{X}^2 \text{ is the completion of } \tilde{X}^2, \text{ and}
\]
\[X^2 = \{f \in \tilde{X}^2: ||f|| \leq M_j = \frac{1}{2\sqrt{\mu(D_j)}}\}
\]
And
\[Y^2 = \{f \in \tilde{X}^2: ||f|| \leq (1 + \epsilon) M_j\}\]
Note that \(X^2\) is a compact subset of \(Y^2\).

For \(u,v,w,p \in Y^2\) we define the operators
\[
S_{1}^{2j} (u,v,w,p) = -e u + e I_1^{2j}, \quad T_{1}^{2j} (u,v,w,p) = (1 + \epsilon)u,
\]
\[
S_{2}^{2j} (u,v,w,p) = -e v + e I_2^{2j}, \quad T_{2}^{2j} (u,v,w,p) = (1 + \epsilon)v,
\]
\[
S_{3}^{2j} (u,v,w,p) = -e w + e I_3^{2j}, \quad T_{3}^{2j} (u,v,w,p) = (1 + \epsilon)w,
\]
\[
S_{4}^{2j} (u,v,w,p) = -e p + e I_4^{2j}, \quad T_{4}^{2j} (u,v,w,p) = (1 + \epsilon)p,
\]
\[
S_{i}^{2j} = ( S_{1}^{2j}, S_{2}^{2j}, S_{3}^{2j}, S_{4}^{2j} ), \quad T^{2j} = ( T_{1}^{2j}, T_{2}^{2j}, T_{3}^{2j}, T_{4}^{2j} ).
\]
For \(u,v,w,p \in X^2\) we have that \(S_{i}^{2j}(u,v,w,p) \in Y^2\), \(i = 1,2,3,4\), i.e.,
\[
S_{i}^{2j}: \ X^2 \times X^2 \times X^2 \times X^2 \to Y^2 \times Y^2 \times Y^2 \times Y^2
\]
and it is continuous.

The operator
\[
T^{2j} : X^2 \times X^2 \times X^2 \times X^2 \to Y^2 \times Y^2 \times Y^2 \times Y^2
\]

is an expansive operator with constant \(1+ \epsilon > 1\) and if \((u,v,w,p) \in Y^2 \times Y^2 \times Y^2 \times Y^2\), then

\[
\left(\frac{1}{1+ \epsilon} \cdot u, \frac{1}{1+ \epsilon} \cdot v, \frac{1}{1+ \epsilon} \cdot w, \frac{1}{1+ \epsilon} \cdot p\right) \in X^2 \times X^2 \times X^2 \times X^2,
\]

and

\[
\left(\begin{array}{c}
T_1^{2j} \left(\frac{1}{1+ \epsilon} \cdot u, \frac{1}{1+ \epsilon} \cdot v, \frac{1}{1+ \epsilon} \cdot w, \frac{1}{1+ \epsilon} \cdot p\right),
T_2^{2j} \left(\frac{1}{1+ \epsilon} \cdot u, \frac{1}{1+ \epsilon} \cdot v, \frac{1}{1+ \epsilon} \cdot w, \frac{1}{1+ \epsilon} \cdot p\right),
T_3^{2j} \left(\frac{1}{1+ \epsilon} \cdot u, \frac{1}{1+ \epsilon} \cdot v, \frac{1}{1+ \epsilon} \cdot w, \frac{1}{1+ \epsilon} \cdot p\right),
T_4^{2j} \left(\frac{1}{1+ \epsilon} \cdot u, \frac{1}{1+ \epsilon} \cdot v, \frac{1}{1+ \epsilon} \cdot w, \frac{1}{1+ \epsilon} \cdot p\right)
\end{array}\right) = (u,v,w,p).
\]

Consequently \(T^{2j} : X^2 \times X^2 \times X^2 \times X^2 \to Y^2 \times Y^2 \times Y^2 \times Y^2\) is onto.

From here and from Theorem 2.3, it follows that the operator \(T^{2j} + S^{2j}\) has a fixed point \((u_2,v_2,w_2,p_2)\) in \(X^2 \times X^2 \times X^2 \times X^2\). For it we have

\[
\begin{align*}
T_1^{2j} (u_2, v_2, w_2, p_2) + S_1^{2j} (u_2, v_2, w_2, p_2) &= u_2 \\
T_2^{2j} (u_2, v_2, w_2, p_2) + S_2^{2j} (u_2, v_2, w_2, p_2) &= v_2 \\
T_3^{2j} (u_2, v_2, w_2, p_2) + S_3^{2j} (u_2, v_2, w_2, p_2) &= w_2 \\
T_4^{2j} (u_2, v_2, w_2, p_2) + S_4^{2j} (u_2, v_2, w_2, p_2) &= p_2
\end{align*}
\]

Or,

\[
\begin{align*}
(1+ \epsilon) \cdot u_2 - \epsilon & \cdot u_2 + l_1^{2j} (u_2, v_2, w_2, p_2) = u_2 \\
(1+ \epsilon) \cdot v_2 - \epsilon & \cdot v_2 + l_2^{2j} (u_2, v_2, w_2, p_2) = v_2 \\
(1+ \epsilon) \cdot w_2 - \epsilon & \cdot w_2 + l_3^{2j} (u_2, v_2, w_2, p_2) = w_2 \\
(1+ \epsilon) \cdot p_2 - \epsilon & \cdot p_2 + l_4^{2j} (u_2, v_2, w_2, p_2) = p_2
\end{align*}
\]

Whereupon

\[
\begin{align*}
l_1^{2j} (u_2, v_2, w_2, p_2) &= 0, l_2^{2j} (u_2, v_2, w_2, p_2) = 0 \\
l_3^{2j} (u_2, v_2, w_2, p_2) &= 0, l_4^{2j} (u_2, v_2, w_2, p_2) = 0
\end{align*}
\]

Therefore \((u_2,v_2,w_2,p_2)\) is a solution of the system \((3.8)\) for which \(u_2, v_2, w_2, p_2 \in C^1([1,2], C_0^2(D))\).

We note that

\[
u_1(1,x,y,z) = u_2(1,x,y,z),
\]

\[22\]
\[ v_1(1,x,y,z) = v_2(1,x,y,z), \]
\[ w_1(1,x,y,z) = w_2(1,x,y,z), \]
\[ p_1(1,x,y,z) = p_2(1,x,y,z), \]

whereupon

\[ u_{1x}(1,x,y,z) = u_{2x}(1,x,y,z), \]
\[ v_{1x}(1,x,y,z) = v_{2x}(1,x,y,z), \]
\[ w_{1x}(1,x,y,z) = w_{2x}(1,x,y,z), \]
\[ p_{1x}(1,x,y,z) = p_{2x}(1,x,y,z), \]
\[ u_{1y}(1,x,y,z) = u_{2y}(1,x,y,z), \]
\[ v_{1y}(1,x,y,z) = v_{2y}(1,x,y,z), \]
\[ w_{1y}(1,x,y,z) = w_{2y}(1,x,y,z), \]
\[ p_{1y}(1,x,y,z) = p_{2y}(1,x,y,z), \]
\[ u_{1z}(1,x,y,z) = u_{2z}(1,x,y,z), \]
\[ v_{1z}(1,x,y,z) = v_{2z}(1,x,y,z), \]
\[ w_{1z}(1,x,y,z) = w_{2z}(1,x,y,z), \]
\[ p_{1z}(1,x,y,z) = p_{2z}(1,x,y,z), \]
\[ u_{1xx}(1,x,y,z) = u_{2xx}(1,x,y,z), \]
\[ v_{1xx}(1,x,y,z) = v_{2xx}(1,x,y,z), \]
\[ w_{1xx}(1,x,y,z) = w_{2xx}(1,x,y,z), \]
\[ p_{1xx}(1,x,y,z) = p_{2xx}(1,x,y,z), \]
\[ u_{1yy}(1,x,y,z) = u_{2yy}(1,x,y,z), \]
\[ v_{1yy}(1,x,y,z) = v_{2yy}(1,x,y,z), \]
\[ w_{1yy}(1,x,y,z) = w_{2yy}(1,x,y,z), \]
\[ p_{1yy}(1,x,y,z) = p_{2yy}(1,x,y,z), \]
\[ u_{1zz}(1,x,y,z) = u_{2zz}(1,x,y,z), \]
\[ v_{1zz}(1,x,y,z) = v_{2zz}(1,x,y,z), \]

\[ w_{1zz}(1,x,y,z) = w_{2zz}(1,x,y,z), \]

\[ p_{1zz}(1,x,y,z) = p_{2zz}(1,x,y,z). \]

Hence and (2.2), (2.4), we get

\[ u_{1t}(1,x,y,z) = u_{2t}(1,x,y,z), \]

\[ v_{1t}(1,x,y,z) = v_{2t}(1,x,y,z), \]

\[ w_{1t}(1,x,y,z) = w_{2t}(1,x,y,z), \]

\[ p_{1t}(1,x,y,z) = p_{2t}(1,x,y,z). \]

Consequently

\[
\begin{align*}
(u(t,x,y,z), v(t,x,y,z), w(t,x,y,z), p(t,x,y,z)) = \\
(u_1(t,x,y,z), v_1(t,x,y,z), w_1(t,x,y,z), p_1(t,x,y,z)) & \in (C^1([0,1], C_0^2(D)) \times C^1([1,2], C_0^2(D)))^4 \\
(u_2(t,x,y,z), v_2(t,x,y,z), w_2(t,x,y,z), p_2(t,x,y,z)) & \in (C^1([0,2], C_0^2(D)))^4
\end{align*}
\]

belongs to \((C^1([0,2], C_0^2(D))))^4\) and it is a solution to the problem

\[
\begin{align*}
\begin{cases}
 u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\
v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} = 0 \\
w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\
u_x + v_y + w_z = 0 \text{ in } (0,2] \times D_j, \\
u(0,x,y,z) = u_0(x,y,z), \quad v(0,x,y,z) = v_0(x,y,z), \\
w(0,x,y,z) = w_0(x,y,z) \text{ in } D_j.
\end{cases}
\end{align*}
\]

Then we consider the problem
and as above we construct a solution

\[
\begin{align*}
\begin{cases}
  u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\
v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} = 0 \\
w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\
u_x + v_y + w_z = 0 \quad \text{ in } (2,3) \times D_j \\
u(2, x, y, z) = u_2(2, x, y, z), \quad v(2, x, y, z) = v_2(2, x, y, z), \\
w(2, x, y, z) = w_2(2, x, y, z) \quad \text{ in } D_j
\end{cases}
\end{align*}
\]

and so on. Consequently

\[
(u_3, v_3, w_3, p_3) \in (C^1([2,3], C^2_0(D_j)))^4
\]

and so on. Consequently

\[
(u_1(t, x, y, z), v_1(t, x, y, z), w_1(t, x, y, z), p_1(t, x, y, z)) \in (C^2([0,1], C^2_0(D_j)))^4
\]

\[
(u_2(t, x, y, z), v_2(t, x, y, z), w_2(t, x, y, z), p_2(t, x, y, z)) \in (C^2([1,2], C^2_0(D_j)))^4
\]

\[
(u_3(t, x, y, z), v_3(t, x, y, z), w_3(t, x, y, z), p_3(t, x, y, z)) \in (C^2([2,3], C^2_0(D_j)))^4
\]

\[
(u_4(t, x, y, z), v_4(t, x, y, z), w_4(t, x, y, z), p_4(t, x, y, z)) \in (C^2([3,4], C^2_0(D_j)))^4
\]

belongs to \((C^1([0,\infty), C^0(D_j)))^4\) and it is a solution to the problem \((3.2)\).

Note that

\(\text{supp}u^j, \text{supp}v^j, \text{supp}w^j, \text{supp}p^j \subset D_j \subset D_j \quad \text{ for any } \quad j = 1, 2, \ldots\)

and then

\[
\begin{align*}
\left. (D_{txyz}^{\alpha} u^j, D_{txyz}^{\alpha} v^j, D_{txyz}^{\alpha} w^j, D_{txyz}^{\alpha} p^j) \right|_{\partial D_j} \\
= \left. (D_{txyz}^{\alpha} u^{j+1}, D_{txyz}^{\alpha} v^{j+1}, D_{txyz}^{\alpha} w^{j+1}, D_{txyz}^{\alpha} p^{j+1}) \right|_{\partial D_{j+1}} \\
= 0
\end{align*}
\]

for any \(\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3), \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \{0, 1, \ldots\}\). Also,
Remark 3.4. We note that in mth step we have

\[ l_1^{m_j} (u, v, w, p) \]

\[ = \int_{t}^{x} \int_{x_0}^{x_1} \int_{y}^{y_1} \int_{y_0}^{y_1} \int_{z}^{z_1} \int_{z_0}^{z_1} (u(t, \alpha, \beta, \gamma)) dydz_1 d\beta dy_1 d\alpha dx_1 \]

\[ + \int_{t}^{t} \int_{m-1}^{l} \int_{x}^{x_1} \int_{x_0}^{x_1} \int_{y}^{y_1} \int_{y_0}^{y_1} \int_{z}^{z_1} \int_{z_0}^{z_1} u^2(s, \alpha, \beta, \gamma) dydz_1 d\beta dy_1 d\alpha ds \]

\[ + \int_{t}^{t} \int_{m-1}^{l} \int_{x}^{x_1} \int_{x_0}^{x_1} \int_{y}^{y_1} \int_{y_0}^{y_1} \int_{z}^{z_1} \int_{z_0}^{z_1} u(s, \alpha, \beta, \gamma)v(s, \alpha, \beta, \gamma) dydz_1 d\beta d\alpha dx_1 ds \]

\[ + \int_{t}^{t} \int_{m-1}^{l} \int_{x}^{x_1} \int_{x_0}^{x_1} \int_{y}^{y_1} \int_{y_0}^{y_1} \int_{z}^{z_1} \int_{z_0}^{z_1} u(s, \alpha, \beta, \gamma)w(s, \alpha, \beta, \gamma) dyd\beta dy_1 d\alpha dx_1 ds \]

is a solution to the problem (1.1) which belongs to the space \((C^2([0, \infty) \times \mathbb{R}^3))^4\). Using the system (1.1) we have that \((u,v,w,p)\) belongs to the space \((C^2([0, \infty) \times \mathbb{R}^3))^4\).

Therefore

\[ \text{supp}\ u, \text{supp}\ v, \text{supp}\ w, \text{supp}\ p \subset D_1 \cup D_2 \cup \ldots \]

Also,

\[ \int_{R^3} |u(t, x, y, z)|^2 \, dx \, dy \, dz = \sum_{j=1}^{\infty} \int_{D_j} |u(t, x, y, z)|^2 \, dx \, dy \, dz \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty, \]

\[ \int_{R^3} |v(t, x, y, z)|^2 \, dx \, dy \, dz = \sum_{j=1}^{\infty} \int_{D_j} |v(t, x, y, z)|^2 \, dx \, dy \, dz \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty, \]

\[ \int_{R^3} |w(t, x, y, z)|^2 \, dx \, dy \, dz = \sum_{j=1}^{\infty} \int_{D_j} |w(t, x, y, z)|^2 \, dx \, dy \, dz \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty, \]

\[ \int_{R^3} |p(t, x, y, z)|^2 \, dx \, dy \, dz = \sum_{j=1}^{\infty} \int_{D_j} |p(t, x, y, z)|^2 \, dx \, dy \, dz \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty, \]

This completes the proof.
\[ I_{2}^{mj}(u, v, w, p) = \]
\[ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} (v(t, \alpha, \beta, \gamma)) \, dydz_1d\beta dy_1d\alpha dz_1 \]
\[ + \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} u(s, \alpha, \beta, \gamma) v(s, \alpha, \beta, \gamma) \, dydz_1d\beta dy_1d\alpha ds \]
\[ + \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} v^2(s, \alpha, \beta, \gamma) \, dydz_1d\beta dy_1d\alpha dz_1ds \]
\[ + \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} v(s, \alpha, \beta, \gamma) w(s, \alpha, \beta, \gamma) \, dydz_1d\beta dy_1d\alpha dz_1ds \]
\[ + \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} p(s, \alpha, \beta, \gamma) \, dydz_1d\beta dz_1ds \]
\[ - \int_{m-1}^{t} \int_{y_0}^{y_1} \int_{z_0}^{z_1} v(s, x, \beta, \gamma) \, dydz_1d\beta dy_1ds \]
\[ - \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{z_0}^{z_1} v(s, \alpha, y, \gamma) \, dydz_1d\alpha dz_1ds \]
\[ - \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{y_0}^{y_1} v(s, \alpha, \beta, z) \, d\beta dy_1d\alpha dz_1ds \]

\[ I_{3}^{mj}(u, v, w, p) = \]
\[
\begin{align*}
\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} (w(t, \alpha, \beta, \gamma)) \, dydz_1d\beta dy_1dx_1 \\
- \ w_{m-1}(m-1, \alpha, \beta, \gamma)) \, dydz_1d\beta dy_1dx_1 \\
\end{align*}
\]

\[
\begin{align*}
+ \int_{m-1}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{u(s, \alpha, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dy_1d\alpha dx_1ds \\
+ \int_{m-1}^{t} \int_{x_0}^{x} \int_{x_0}^{z} \int_{z_0}^{z_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{v(s, \alpha, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta d\alpha dx_1ds \\
+ \int_{m-1}^{t} \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{x_0}^{y} \int_{y_0}^{y_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{w(s, \alpha, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dy_1d\alpha dx_1ds \\
+ \int_{m-1}^{t} \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{p(s, \alpha, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dy_1d\alpha dx_1ds \\
- \int_{m-1}^{t} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{w(s, x, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dy_1d\alpha dx_1ds \\
- \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{w(s, \alpha, y, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dx_1ds \\
- \int_{m-1}^{t} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{w(s, \alpha, \beta, z)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dy_1d\alpha dx_1ds \\
\end{align*}
\]

\[
I_{4}^{m_{j}} (u, v, w, p) = \\
\int_{0}^{t} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{u(s, \alpha, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dy_1d\alpha dx_1ds \\
+ \int_{m-1}^{t} \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{v(s, \alpha, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta d\alpha dx_1ds \\
+ \int_{m-1}^{t} \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{w(s, \alpha, \beta, \gamma)}{w(s, \alpha, \beta, \gamma)} \, dydz_1d\beta dy_1d\alpha dx_1ds \\
\end{align*}
\]

Here \((x_0, y_0, z_0) \in D, j \in \{1, 2, \ldots\} \).
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