VALUE DISTRIBUTION OF THE SEQUENCES OF THE DERIVATIVES OF ITERATED POLYNOMIALS

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Abstract. We establish the equidistribution of the sequence of the averaged pullbacks of a Dirac measure at any value in $\mathbb{C} \setminus \{0\}$ under the derivatives of the iterations of a polynomial $f \in \mathbb{C}[z]$ of degree more than one towards the $f$-equilibrium (or canonical) measure $\mu_f$ on $\mathbb{P}^1$. We also show that for every $\mathcal{C}^2$ test function on $\mathbb{P}^1$, the convergence is exponentially fast up to a polar subset of exceptional values in $\mathbb{C}$. A parameter space analog of the latter quantitative result for the monic and centered unicritical polynomials family is also established.

1. Introduction

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d > 1$. Let $\mu_f$ be the $f$-equilibrium (or canonical) measure on $\mathbb{P}^1$, which coincides with the harmonic measure $\mu_K(f)$ on the filled-in Julia set $K(f)$ of $f$ with respect to $\infty$. The exceptional set $E(f) := \{ a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N}} f^{-n}(a) < \infty \}$ of $f$ contains $\infty$ and $\#E(f) \leq 2$. Brolin [2, Theorem 16.1] studied the value distribution of the sequence $(f^n : \mathbb{P}^1 \to \mathbb{P}^1)$ of the iterations of $f$, and established

$$\left\{ a \in \mathbb{P}^1 : \lim_{n \to \infty} \frac{(f^n)_* \delta_a}{dn} = \mu_f \text{ weakly on } \mathbb{P}^1 \right\} = \mathbb{P}^1 \setminus E(f),$$

which is more precise than the classical inclusion $\partial K(f) \subset \bigcup_{n \in \mathbb{N}} f^{-n}(a)$ for every $a \in \mathbb{P}^1 \setminus E(f)$. Here for every $h \in \mathbb{C}(z)$ of degree $> 0$ and every Radon measure $\nu$ on $\mathbb{P}^1$, the pullback $h^*\nu$ of $\nu$ under $h$ is a Radon measure on $\mathbb{P}^1$ so that for every $a \in \mathbb{P}^1$, when $\nu = \delta_a$, $h^*\delta_a = \sum_{w \in h^{-1}(a)} (\deg_w h) \delta_a$ on $\mathbb{P}^1$.

Pursuing the analogy between the roles played by $E(f)$ in (1.1) and by the set of Valiron exceptional values in $\mathbb{P}^1$ of a transcendental meromorphic function on $\mathbb{C}$, Sodin [20], Russakovskii–Sodin [19], and Russakovskii–Shiffman [18] (see also [7, 15]) studied the value distribution of a sequence of rational maps between projective spaces from the viewpoint of Nevanlinna theory, in a quantitative way (cf. [22] Chapter V, §2]). Gauthier and Vigny [10, 1. in Theorem A] studied the value distribution of the sequence $((f^n)_' : \mathbb{P}^1 \to \mathbb{P}^1)$ of the derivatives of iterations of a polynomial $f \in \mathbb{C}[z]$ of degree $> 1$ (cf. [23]) possibly with a polar subset of exceptional values in $\mathbb{C} \setminus \{0\}$, in terms of dynamics of the tangent map $F(z, w) := (f(z), f'(z)w)$ on the tangent bundle $T\mathbb{C}$. The aim of this article is to improve their result in two ways.

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The first improvement of [10] 1. in Theorem A] is qualitative, but with no exceptional values.

**Theorem 1.** Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \). Then for every \( a \in \mathbb{C} \setminus \{0\} \),

\[
\lim_{n \to \infty} \frac{((f^n)')^* \delta_a}{d^n - 1} = \mu_f
\]

weakly on \( \mathbb{P}^1 \).

In Theorem 1 the values \( a = 0, \infty \) are excluded since it is clear that for every \( n \in \mathbb{N} \), \((f^n)' \delta_{\infty}/(d^n - 1) = \delta_{\infty}(\neq \mu_f)\), and it immediately follows from (1.1) and the chain rule that \( \lim_{n \to \infty}((f^n)' \delta_{\infty})/(d^n - 1) = \mu_f \) weakly on \( \mathbb{P}^1 \) if and only if \( E(\mu_f) = \{ \infty \} \). In Gauthier–Vigny [10, 2. and 3. in Theorem A], they also established a result similar to Theorem 1 under the assumption that \( f \) has no Siegel disks (or the assumption that \( f \) is hyperbolic). Our proof of Theorem 1 is independent of their argument even in those cases.

The second improvement of [10] 1. in Theorem A] is quantitative, but with an at most polar subset of exceptional values in \( \mathbb{C} \).

**Theorem 2.** Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \), and suppose that \( E(\mu_f) = \{ \infty \} \). Then for every \( \eta \geq \sup_{z \in \mathbb{C}: \text{superattracting periodic point}} \lim \sup_{n \to \infty}((\deg_z(f^n))^1/n) \), there is a polar subset \( E = E_{\eta \eta} \in \mathbb{C} \) such that for every \( a \in \mathbb{C} \setminus E \) and every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \),

\[
\int_{\mathbb{P}^1} \phi d\left( \frac{((f^n)')^* \delta_a}{d^n - 1} - \mu_f \right) = o(\eta/d^n)
\]

as \( n \to \infty \).

The proof of Theorem 2 is based on Russakovskii–Shiffman [18] mentioned above, and on an improvement of it for the sequence of the iterations of a rational function of degree \( > 1 \) by Drasin and the author [6] (see also [4] and [24] in higher dimensions).

**Remark 1.1.** Under the assumption \( E(\mu_f) = \{ \infty \} \) in Theorem 2, we have \( \sup_{z \in \mathbb{C}: \text{superattracting periodic point}} \lim \sup_{n \to \infty}((\deg_z(f^n))^1/n) \in \{ 1, 2, \ldots, d - 1 \} \), and \( = 1 \) if and only if there is no superattracting cycles of \( f \) in \( \mathbb{C} \). Here we adopt the convention \( \sup_{\emptyset} = 1 \). In the case that \( E(\mu_f) \neq \{ \infty \} \), we point out the following better estimate than that in Theorem 2:

\[
\int_{\mathbb{P}^1} \phi d\left( \frac{((f^n)')^* \delta_a}{d^n - 1} - \mu_f \right) = O(nd^{-n}) \quad \text{as } n \to \infty
\]

for every \( a \in \mathbb{C} \setminus \{0\} \) and every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \), with no exceptional values; indeed, we can assume that \( f(z) = z^d \) without loss of generality (see Remark 3.1), and then \( f^n(z) = z^{dn} \) for every \( n \in \mathbb{N} \) and \( \mu_f \) is the normalized Lebesgue measure \( m_{\partial\mathbb{D}} \) on the unit circle \( \partial\mathbb{D} = \partial K(f) \). For every \( a = re^{i\theta} \quad (r > 0, \theta \in \mathbb{R}) \), every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \), and every \( n \in \mathbb{N} \), we have

\[
\left| \int_{\mathbb{P}^1} \phi d\left( (f^n)'^* \delta_a - \sum_{j=1}^{d^n-1} \delta_{e^{i(\theta+j2\pi)/(dn-1)}}/(d^n - 1) \right) \right| \leq ||\phi||_{C^2} \cdot \left| e^{(\log(nd^{-n}))/dn-1} \right| \leq ||\phi||_{C^2} \cdot Cnd^{-n} \quad \text{for some } C > 0 \text{ independent of both } \phi \text{ and } n, \quad \text{and if } \phi \text{ is } C^2, \quad \text{then by the midpoint method in numerically computing definite integrals, we also have}
\]

\[
\left| \int_{\mathbb{P}^1} \phi d\left( \sum_{j=1}^{d^n-1} \delta_{e^{i(\theta+j2\pi)/(dn-1)}}/(d^n - 1) - m_{\partial\mathbb{D}} \right) \right| \leq ||\phi||_{C^2} \cdot C'd^{-n} \quad \text{for some } C' > 0 \text{ independent of both } \phi \text{ and } n.
Finally, let us focus on the (monic and centered) uncritical polynomials family
\begin{equation}
 f : \mathbb{C} \times \mathbb{P}^1 \ni (\lambda, z) \mapsto z^d + \lambda =: f_\lambda(z) \in \mathbb{P}^1
\end{equation}
of degree \(d > 1\). The parameter space analog of Theorem 1 for the sequence \((f_\lambda^n)'(\lambda)\) in \(\mathbb{C}[\lambda]\) of the derivative of \(f_\lambda^n\) at its unique critical value \(z = \lambda\) in \(\mathbb{C}\) is also obtained by Gauthier–Vigny [10, Theorem 3.7]. We will also establish a parameter space analog of Theorem 2.

**Theorem 3.** Let \(f\) be the monic and centered uncritical polynomials family of degree \(d > 1\) defined as in (1.2). Then for every \(\eta > 1\), there is a polar subset \(E = E_{f, \eta} \subset \mathbb{C}\) such that for every \(a \in \mathbb{C} \setminus E\) and every \(C^2\)-test function \(\phi\) on \(\mathbb{P}^1\),
\[
 \int_{\mathbb{P}^1} \phi(\lambda) d\left(\frac{((f_\lambda^n)'(\lambda))^{\cdot} \delta_a}{d^n - 1} - \mu_{C_d}\right)(\lambda) = O((\eta/d)^n)
\]
as \(n \to \infty\). Here \(C_d\) is the connectedness locus of the family \(f\) in the parameter space \(\mathbb{C}\) and \(\mu_{C_d}\) is the harmonic measure on \(C_d\) with pole \(\infty\).

The proof of Theorem 3 is based on Russakovskii–Shiffman [18] mentioned above, and on a quantitative equidistribution of superattracting parameters by Gauthier–Vigny [9].

In Section 2, we recall a background from complex dynamics. In Sections 3 and 4 we show Theorems 1, 2, and 3, respectively.

**Notation 1.2.** We adopt the convention \(\mathbb{N} = \mathbb{Z}_{>0}\). For every \(a \in \mathbb{C}\) and every \(r > 0\), set \(\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}\). Let \(\delta_z\) be the Dirac measure on \(\mathbb{P}^1\) at each \(z \in \mathbb{P}^1\). Let \([z, w]\) be the chordal metric on \(\mathbb{P}^1\) normalized as \([r, \infty] = 1/\sqrt{1+|r|^2}\) on \(\mathbb{P}^1\) (following the notation in Nevanlinna’s and Tsuji’s books [14, 22]). Let \(\omega\) be the Fubini-Study area element on \(\mathbb{P}^1\) normalized as \(\omega(\mathbb{P}^1) = 1\). The Laplacian \(dd^c\) on \(\mathbb{P}^1\) is normalized as \(dd^c(-\log\cdot, \infty) = \omega - \delta_\infty\) on \(\mathbb{P}^1\).

## 2. Background

### 2.1. Dynamics of rational functions

Let \(f \in \mathbb{C}(z)\) be of degree \(d > 1\). Let \(C(f)\) be the critical set of \(f\). The Julia and Fatou sets of \(f\) are defined by \(J(f) := \{z \in \mathbb{P}^1 : \text{the family } (f^n : \mathbb{P}^1 \to \mathbb{P}^1)_{n \in \mathbb{N}} \text{ is not normal at } z\}\) and \(F(f) := \mathbb{P}^1 \setminus J(f)\), respectively. A component of \(F(f)\) is called a Fatou component of \(f\). A Fatou component \(U\) of \(f\) is mapped by \(f\) properly onto a Fatou component of \(f\). A Fatou component \(U\) of \(f\) is said to be cyclic if there is \(n \in \mathbb{N}\) such that \(f^n(U) = U\). For more details on complex dynamics, see e.g. Milnor’s book [13].

The \(f\)-equilibrium (or canonical) measure \(\mu_f\) on \(\mathbb{P}^1\) is the unique probability Radon measure \(\nu\) on \(\mathbb{P}^1\) such that
\begin{equation}
 f^*\nu = d \cdot \nu \quad \text{on} \quad \mathbb{P}^1
\end{equation}
and that \(\nu(\{a\}) = 0\) for every \(a \in E(f)\); the exceptional set of \(f\) is \(E(f) := \{a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N}} f^{-n}(a) < \infty\}\) = \(\{a \in \mathbb{P}^1 : f^{-2}(a) = \{a\}\}\). Then in fact \(\text{supp} \mu_f = J(f)\), and for every \(n \in \mathbb{N}\), \(\mu_{f^n} = \mu_f\) on \(\mathbb{P}^1\). For more details, see Brolin [2], Lyubich [12], Freire–Lopes–Mañé [8].
Lemma 3.2. explicit which computations would be independent of such a normalization.

3.1. without loss of generality, by replacing \( f \) with \( f^{-1} \circ f \) for such \( c \in \mathbb{C} \setminus \{0\} \) that \( c^{d-1} = a_d^{-1} \) if necessary (for every \( c \in \mathbb{C} \setminus \{0\}, z \mapsto c \cdot z \) is also denoted by \( c \)). In this article, we would not normalize \( f \) as \( |a_d| = 1 \) in order to make it explicit which computations would be independent of such a normalization.

2.2. Dynamics of polynomials. Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \). We note that \( \infty \in E(f) \), \( \#(C(f) \cap \mathbb{C}) \leq d-1 \), and \( (C(f) \cap \mathbb{C}) = (\supp dd^c \log |f'|) \cap \mathbb{C} \).

The filled-in Julia set \( K(f) \) of \( f \) is defined by
\[
K(f) := \{ z \in \mathbb{C} : \limsup_{n \to \infty} |f^n(z)| < \infty \},
\]
whose complement in \( \mathbb{P}^1 \) coincides with the immediate superattractive basin
\[
I_\infty(f) := \{ z \in \mathbb{P}^1 : \lim_{n \to \infty} f^n(z) = \infty \}
\]
of the superattracting fixed point \( \infty \) of \( f \); in particular, \( \lim_{n \to \infty} f^n = \infty \) locally uniformly on \( I_\infty(f) \), and \( K(f) \) is a compact subset in \( \mathbb{C} \). We note that \( F(f) = I_\infty(f) \cup \text{int} K(f) \) and that \( J(f) = \partial K(f) \).

By a standard telescope argument, there exists the locally uniform limit
\[
g_f := \lim_{n \to \infty} -\frac{\log |f^n(\cdot)|}{d^n}
\]
on \( \mathbb{C} \). Setting \( g_f(\infty) := +\infty \), we have \( g_f \circ f = d \cdot g_f \) on \( \mathbb{P}^1 \), and for every \( n \in \mathbb{N} \), we also have \( g_{f^n} = g_f \) on \( \mathbb{P}^1 \). The restriction of \( g_f \) to \( I_\infty(f) \) coincides with the Green function on \( I_\infty(f) \) with pole \( \infty \), and the measure
\[
\mu_K(f) := dd^c g_f + \delta_\infty \quad \text{on} \quad \mathbb{P}^1
\]
coincides with the harmonic measure on \( K(f) \) with pole \( \infty \). In particular, \( \supp \mu_K(f) \subset \partial K(f) \), and in fact \( \mu_K(f) = \mu_f \) on \( \mathbb{P}^1 \). The function \( z \mapsto g_f(z) - \log |z| \) extends harmonically to an open neighborhood of \( \infty \) in \( I_\infty(f) \) so the function \( z \mapsto -\log |z, \infty| - g_f(z) \) extends continuously to \( \mathbb{P}^1 \).

The following is substantially shown in Buff [3, the proof of Theorem 4].

Theorem 2.1 (Buff). Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \), and let \( z_0 \in \mathbb{C} \). If \( g_f(z_0) \geq \max_{c \in (f) \cap \mathbb{C}} g_f(c) \), then \( |f'(z_0)| \leq d^2 \cdot e^{(d-1)g_f(z_0)} \), and the equality never holds if \( (C(f) \cap \mathbb{C}) \cap I_\infty(f) \neq \emptyset \).

For more details on polynomial dynamics and potential theory, see Brolin [2, Chapter III], and also Ransford’s book [17].

3. Proof of Theorem

Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \). For every \( a \in \mathbb{C} \) and every \( n \in \mathbb{N} \), the functions \( (\log |(f^n)'-a|)/(d^n-1) - g_f \) and \( (\log \max\{1,|(f^n)'|\})/(d^n-1) - g_f \) extend continuously to \( \mathbb{P}^1 \). Set \( a_d = a_d(f) := \lim_{n \to \infty} f(z)/z^d \in \mathbb{C} \setminus \{0\} \).

Remark 3.1. Since the question is affine invariant, we could assume \( |a_d| = 1 \) without loss of generality, by replacing \( f \) with \( c^{-1} \circ f \circ c \) for such \( c \in \mathbb{C} \setminus \{0\} \) that \( c^{d-1} = a_d^{-1} \). In this article, we would not normalize \( f \) as \( |a_d| = 1 \) in order to make it explicit which computations would be independent of such a normalization.

Lemma 3.2. On \( I_\infty(f) \setminus \bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(C(f) \cap \mathbb{C}) \),
\[
\lim_{n \to \infty} \left( \frac{\log |(f^n)'|}{d^n - 1} - g_f \right) = 0
\]
locally uniformly.
**Proof.** For every $n \in \mathbb{N}$ and every $z \in \mathbb{C}$, by a direct calculation, we have

\[
(3.1) \quad \frac{\log |(f^n)'(z)|}{d^n - 1} = \frac{\log |d^n \cdot a_d(d^n-1)/(d-1)|}{d^n - 1}
= \frac{1}{d^n - 1} \int_{\mathbb{C}} \log |z - u| (dd^c \log |(f^n)'|(u))
= \frac{1}{d^n - 1} \int_{\mathbb{C}} \sum_{j=0}^{n-1} \left( \int_{\mathbb{C}} \log |z - w| (dd^c \log |f^j|)(w) \right)
= \frac{1}{d^n - 1} \int_{\mathbb{C}} \sum_{j=0}^{n-1} (\log |f^j(z) - w| - \log |a_d| (d^{j-1}(d-1)) (dd^c \log |f'|)(w)
\]
\[
= \frac{1}{d^n - 1} \int_{\mathbb{C}} \sum_{j=0}^{n-1} (\log |f^j(z)| - \log |a_d| (d^{j-1}(d-1)) (dd^c \log |f'|)(w)
- \log |a_d|^{1 - \frac{n}{d^n - 1}}.
\]

Then noting that $g_f \circ f = d \cdot g_f$ on $\mathbb{P}^1$, for every $n \in \mathbb{N}$ and every $z \in \mathbb{P}^1$, we have

\[
(3.2) \quad \frac{\log |(f^n)'(z)|}{d^n - 1} - g_f(z)
= \frac{1}{d^n - 1} \int_{\mathbb{C}} \left( \sum_{j=0}^{n-1} \log |f^j(z)|, w \right) (dd^c \log |f'|)(w)
+ \frac{d - 1}{d^n - 1} \sum_{j=0}^{n-1} (- \log |f^j(z)|, w) - g_f(f^j(z))
+ \left( - \int_{\mathbb{C}} \log |w| (dd^c \log |f'|)(w) + \log d + \log |a_d| \right) \frac{n}{d^n - 1},
\]
which with $\sup_{z \in \mathbb{P}^1} |(f^n)'(z)| - g_f(z) < \infty$ completes the proof. \hfill $\square$

**Lemma 3.3.** There is $C = C_f > 0$ such that for every $n \in \mathbb{N}$ and every $z \in \mathbb{P}^1$,

\[
(3.3) \quad \frac{\log \max \{1, |(f^n)'(z)|\}}{d^n - 1} - g_f(z) \leq \frac{Cn}{d^n - 1}.
\]

**Proof.** Set

\[
(3.4) \quad C = C_f := (d - 1) \cdot \sup_{z \in \mathbb{P}^1} \log |z, w| - g_f(z)
+ (d - 1) \cdot \sup_{w \in C(f) \cap \mathbb{C}} |\log |w, \infty|| + \log d + |\log |a_d|| \in \mathbb{R}_{>0}.
\]

Then for every $n \in \mathbb{N}$ and every $z \in \mathbb{C}$, from (3.2), we have $|(f^n)'(z)| \leq e^{Cn} \cdot e^{(d^n-1)g_f(z)}$, which with $g_f \geq 0$ on $\mathbb{P}^1$ completes the proof. \hfill $\square$

We note that $\max_{c \in \mathbb{C}} f^{-n}(C(f) \cap \mathbb{C}) g_f(c) = \max_{c \in C(f) \cap \mathbb{C}} g_f(c) < \infty$.

by $g_f \circ f = d \cdot g_f$ on $\mathbb{P}^1$. 

\[ \]
Lemma 3.4. For every $a \in \mathbb{C} \setminus \{0\}$,
\[
\lim_{n \to \infty} \int_{\mathcal{P}} \left| \frac{\log |(f^n)' - a|}{d^n - 1} - g_f \right| \, d\omega = 0.
\]

Proof. Fix $a \in \mathbb{C} \setminus \{0\}$. The sequence $((\log |(f^n)' - a|)/(d^n - 1))$ of subharmonic functions on $\mathbb{C}$ is locally uniformly bounded from above on $\mathbb{C}$; indeed, by the chain rule and $\lim \inf_{z \to \infty} |f'(z)| = +\infty$, for every $R > 0$ so large that \{\{z: |z| = R\} $\subseteq I_\infty(f) \setminus \bigcup_{n \in \mathbb{N}, j \in \{0\}} f^{-n}(C(f) \cap \mathbb{C})$, we have $\lim \inf_{n \to \infty} \inf_{|z| = R} |(f^n)'(z)| = +\infty$, which with the maximum modulus principle yields $\sup_{|z| \leq R} |(f^n)'(z) - a| \leq \sup_{|z| = R} 2|g_f(z)|$ for every $n \in \mathbb{N}$ large enough. Then by Lemma 3.3 we have $\lim \sup_{n \to \infty} \sup_{|z| \leq R} (\log |(f^n)' - a|)/(d^n - 1) \leq \sup_{|z| = R} g_f(z) < \infty$. By Lemma 3.2 and $g_f > 0$ on $I_\infty(f)$, for every compact subset $C$ in $I_\infty(f) \setminus \bigcup_{n \in \mathbb{N}, j \in \{0\}} f^{-n}(C(f) \cap \mathbb{C})$, we also have $1/2 \leq |(g_f)' - a|/f^n| \leq 2$ on $C$ for every $n \in \mathbb{N}$ large enough, so in particular
\[
(3.5) \quad \lim_{n \to \infty} \left( \frac{\log |(f^n)' - a|}{d^n - 1} - g_f \right) = \lim_{n \to \infty} \left( \frac{\log |(f^n)'|}{d^n - 1} - g_f \right) = 0
\]
locally uniformly on $I_\infty(f) \setminus \bigcup_{n \in \mathbb{N}, j \in \{0\}} f^{-n}(C(f) \cap \mathbb{C})$.

Let $m_2$ be the Lebesgue measure on $\mathbb{C}$. By a compactness principle for a locally uniformly bounded sequence of subharmonic functions on a domain in $\mathbb{R}^m$ which is not locally uniformly convergent to $-\infty$ (see Azarin [11, Theorem 1.1.1], Hörmander’s book [11, Theorem 4.1.9(a)]), we can choose a sequence $(n_j)$ in $\mathbb{N}$ tending to $+\infty$ as $j \to \infty$ such that the $L_{\infty}(C, m_2)$-limit $\phi := \lim_{j \to \infty}(\log |(f^{n_j})' - a|)/(d^{n_j} - 1)$ exists and is subharmonic on $\mathbb{C}$. Choosing a subsequence of $(n_j)$ if necessary, we have $\phi = \lim_{j \to \infty}(\log |(f^{n_j})' - a|)/(d^{n_j} - 1)$ Lebesgue a.e. on $\mathbb{C}$. Then by (3.5), we have $\phi \equiv g_f$ Lebesgue a.e. on $\mathbb{C} \setminus (K(f) \cup \bigcup_{n \in \mathbb{N}, j \in \{0\}} f^{-n}(C(f) \cap \mathbb{C}))$, and in turn on $\mathbb{C} \setminus K(f)$ by the subharmonicity of $\phi$ and the harmonicity of $g_f$ there. Let us show that $\phi = g_f$ Lebesgue a.e. on the whole $\mathbb{C}$, and then $\lim_{n \to \infty}(\log |(f^n)' - a|)/(d^n - 1) = g_f$ in $L_{\infty}(C, m_2)$, which with the locally uniform convergence (3.5) will complete the proof since $\max_{\mathbb{C} \setminus \bigcup_{n \in \mathbb{N}, j \in \{0\}} f^{-n}(C(f) \cap \mathbb{C})} g_f(c) < \infty$ and the Radon-Nikodim derivative $d\omega/dm_2$ is continuous so locally bounded on $\mathbb{C}$.

By $\log(1/|w, \infty|) - \log \max\{1, |w|\} \leq \log 2$ on $\mathbb{C}$ and Lemma 3.3 for every $n \in \mathbb{N}$, we have
\[
\frac{\log |(f^n)' - a|}{d^n - 1} - g_f \leq \frac{\log |(f^n)' - a|}{d^n - 1} + \left( \frac{\log(1/|(f^n)'|, \infty)}{d^n - 1} - g_f \right) + \frac{\log(1/|a, \infty|)}{d^n - 1}
\]
\[
\leq C_f \cdot n \frac{\log \sqrt{2} + \log(1/|a, \infty|)}{d^n - 1}
\]
on $\mathbb{C}$, so $\phi \leq g_f$ Lebesgue a.e. on $\mathbb{C}$ and in turn on $\mathbb{C}$ by the subharmonicity of $\phi$ and the continuity of $g_f$ on $\mathbb{C}$. Hence $\phi - g_f$ is $\leq 0$ and is upper semicontinuous on $\mathbb{C}$.

Now suppose to the contrary that the open subset $\{z \in \mathbb{C}: \phi(z) < g_f(z)\}$ in $\mathbb{C}$ is non-empty. Then by $\phi \equiv g_f$ on $\mathbb{C} \setminus K(f)$, there is a bounded Fatou
component $U$ of $f$ containing a component $W$ of \{ $z \in \mathbb{C} : \phi(z) < g_f(z)$ \}. Since $\phi \leq g_f = 0$ on $U \subset K(f)$, by the maximum principle for subharmonic functions, we in fact have $U = W$.

Taking a subsequence of $(n_j)$ if necessary, we can assume that $(f^{n_j})(U)$ is locally uniformly convergent to a holomorphic function $g$ on $U$ as $j \to \infty$ without loss of generality. We claim that $g' \equiv a$ on $U$, so we can say $g \in \mathbb{C}[z]$; indeed, fixing a domain $D \subset U = W$, by a version of Hartogs’s lemma on subharmonic functions (see H¨ olmander’s book [11, Theorem 4.1.9(b)]) and the upper semicontinuity of $\phi$, we have\[ \limsup_{n \to \infty} \sup_{D} \log \frac{|(f^{n_j})' - a|}{(d^{n_j} - 1)} \leq \sup_{D} \phi \leq 0. \] Hence $g' = (\lim_{j \to \infty} f^{n_j})' = \lim_{j \to \infty} (f^{n_j})' \equiv a$ on $D$, so $g' \equiv a$ on $U$ by the identity theorem for holomorphic functions.

Hence, under the assumption that $a \neq 0$, the locally uniform limit $g$ on $U$ is non-constant. So by Hurwitz’s theorem and the classification of cyclic Fatou components, there is $N \in \mathbb{N}$ such that $V := f^N(U) = g(U)$ is a Siegel disk of $f$ and, setting $p := \min \{ n \in \mathbb{N} : f^n(V) = V \}$, that $p|n_j - n_N|$ for every $j \geq N$. We can fix a holomorphic injection $h : V \to \mathbb{C}$ such that for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, setting $\lambda := e^{2\pi i \alpha}$, we have $h \circ f^p = \lambda \cdot h$ on $V$, so for every $j \geq N$, $h \circ f^{n_j} = \lambda^{n_j - n_N}/p \cdot (h \circ f^{n_N})$ on $U$. Then taking a subsequence of $(n_j)$ if necessary, there also exists the limit\[ \lambda_0 := \lim_{j \to \infty} \lambda^{(n_j - n_N)/p} \] in $\partial \mathbb{D}$, so that $h \circ g = \lim_{j \to \infty} h \circ f^{n_j} = \lambda_0 \cdot (h \circ f^{n_N})$ on $U$. In particular,\[ (3.6) \quad h \circ f^{n_j} - h \circ g = (\lambda^{(n_j - n_N)/p} - \lambda_0) \cdot (h \circ f^{n_N}) \] on $U$. Set $w_0 := h^{-1}(0) \in V$, so that $f^p(w_0) = w_0$, and fix $z_0 \in f^{-N}(w_0) \cap U$, so that $f^{n_j}(z_0) = w_0$ for every $j \geq N$ and $g(z_0) = \lim_{j \to \infty} f^{n_j}(z_0) = w_0$.

We claim that\[ (3.7) \quad \frac{\log |(f^{n_j})'(z_0) - a|}{d^{n_j} - 1} = \frac{\log |\lambda^{(n_j - n_N)/p} - \lambda_0|}{d^{n_j} - 1} + O(d^{-n_j}) \] as $j \to \infty$; for, by the chain rule applied to both sides in (3.6) and $h'(w_0) \neq 0$ (and $g'(z_0) = a$), we have\[ (3.6) \quad (f^{n_j})'(z_0) - a = (\lambda^{(n_j - n_N)/p} - \lambda_0) \cdot (f^{n_N})'(z_0), \] which also yields $(f^{n_N})'(z_0) \neq 0$ by $(f^{n_j})'(z_0) = (f^{n_j})'(w_0) \cdot (f^{n_N})'(z_0)$ and the assumption $a \neq 0$. We also claim that\[ (3.8) \quad \liminf_{j \to \infty} \frac{1}{d^{n_j}} \log |\lambda^{(n_j - n_N)/p} - \lambda_0| \geq 0 \] (cf. [16] Proof of Theorem 3); indeed, for every domain $D \subset U \setminus f^{-N}(w_0)$, since $h^{-1}$ is Lipschitz continuous on $h(\bigcup_{n \in \mathbb{N}} (f^p)^n(D)) \cup g(D) \subset h(V)$ and $\sup_{D} |h \circ f^{n_N}| > 0$, from (3.6), we observe that\[ (*) \quad \frac{1}{d^{n_j}} \sup_{D} \log |f^{n_j} - g| \leq \frac{1}{d^{n_j}} \log |\lambda^{(n_j - n_N)/p} - \lambda_0| + O(d^{-n_j}) \] as $j \to \infty$. On the other hand, for every domain $\tilde{D}$ intersecting $\partial U$ in \mathbb{C}, fixing $\tilde{z} \in \tilde{D} \cap I_\infty(f) \neq \emptyset$, we observe that\[ (**) \quad \liminf_{j \to \infty} \frac{1}{d^{n_j}} \sup_{\tilde{D}} \log |f^{n_j} - g| \geq g_f(\tilde{z}) > 0. \]
Now fix $z_1 \in U$ and $z' \in \partial U$ such that $\mathbb{D}(z_1, |z' - z_1|) \subset U \setminus f^{-n_N}(w_0)$. Then for every $\epsilon \in (0, |z' - z_1|)$, using Cauchy’s estimate applied to $f^{n_j} - g \in \mathbb{C}[z]$ around $z_1$, we have
\[
|f^{n_j} - g| \leq \sum_{k=0}^{d_{n_j}} \sup_{\mathbb{D}(z_1, |z' - z_1| - \epsilon)} \frac{|f^{n_j} - g|}{(|z' - z_1| - \epsilon)^k} \cdot |z - z_1|^k
\]
\[
\leq \left( \sup_{\mathbb{D}(z_1, |z' - z_1| - \epsilon)} |f^{n_j} - g| \right) \cdot \sum_{k=0}^{d_{n_j}} \left( \frac{|z' - z_1| + \epsilon}{|z' - z_1| - \epsilon} \right)^k
\]
on $\mathbb{D}(z', \epsilon)$, so since $z' \in \mathbb{D}(z', \epsilon) \cap \partial U$ and $\mathbb{D}(z_1, |z' - z_1| - \epsilon) \subset U \setminus f^{-n_N}(w_0)$, by (**) and (**), we have
\[
0 < \left( \liminf_{j \to \infty} \frac{1}{d_{n_j}} \log \sup_{\mathbb{D}(z', \epsilon)} |f^{n_j} - g| \right)
\]
\[
\leq \liminf_{j \to \infty} \frac{1}{d_{n_j}} \log \sup_{\mathbb{D}(z_1, |z' - z_1| - \epsilon)} |f^{n_j} - g| + \log \frac{|z' - z_1| + \epsilon}{|z' - z_1| - \epsilon}
\]
\[
\leq \liminf_{j \to \infty} \frac{1}{d_{n_j}} \log |\lambda^{(n_j-n_N)/p} - \lambda_0| + \log \frac{|z' - z_1| + \epsilon}{|z' - z_1| - \epsilon}
\]
This yields (3.8) as $\epsilon \to 0$.

Once (3.7) and (3.8) are at our disposal, using a version of Hartogs’s lemma on subharmonic functions again, we have
\[
\phi(z_0) \geq \limsup_{j \to \infty} \frac{\log |(f^{n_j})'(z_0) - a|}{d_{n_j} - 1} \geq \liminf_{j \to \infty} \frac{\log |\lambda^{(n_j-n_N)/p} - \lambda_0|}{d_{n_j} - 1} \geq 0,
\]
which contradicts $\phi < g_f = 0$ on $U = W$. \hfill \Box

For every $a \in \mathbb{C} \setminus \{0\}$ and every $C^2$-test function $\phi$ on $\mathbb{P}^1$, by Lemma 3.4, we have
\[
\left| \int_{\mathbb{P}^1} \phi d\left( \frac{(f^{n_j})^*\delta_a}{d^n - 1} - \mu_f \right) \right| = \left| \int_{\mathbb{P}^1} \phi d\left( \frac{\log |(f^{n_j})'(z) - a|}{d^n - 1} - g_f \right) \right|
\]
\[
\leq \left( \sup_{z \in \mathbb{P}^1} \left| \frac{dd^c \phi}{d\omega} \right| \cdot \left| \int_{\mathbb{P}^1} \frac{\log |(f^{n_j})'(z) - a|}{d^n - 1} - g_f \right| d\omega(z) \to 0 \right)
\]
as $n \to \infty$, where the Radon-Nikodim derivative $(dd^c \phi)/d\omega$ on $\mathbb{P}^1$ is bounded on $\mathbb{P}^1$. \hfill \Box

4. PROOF OF THEOREM 2

Let $f \in \mathbb{C}[z]$ be of degree $d > 1$, and suppose that $E(f) = \{\infty\}$. Then
\[
\sup_{z \in \mathbb{C} \setminus \{\text{superattracting periodic point of } f \}} \limsup_{n \to \infty} \left( \sup_{z \in \mathbb{C} \setminus \{\text{superattracting periodic point of } f \}} \frac{\log |(f^{n_j})'(z_0)|^{1/n}}{d^n - 1} - g_f \right)
\]
(recall the convention $\sup_{\emptyset} = 1$). Set $a_d := a_d(f) = \lim_{n \to \infty} f(z)/z^d \in \mathbb{C} \setminus \{0\}$. For every $n \in \mathbb{N}$, the functions $(\log(1/|(f^{n_j})'(\infty)|))/(d^n - 1) - g_f$ and $(\log \max(1, |(f^{n_j})'|))/(d^n - 1) - g_f$ extend continuously to $\mathbb{P}^1$. 


Lemma 4.1. For every $\eta > \sup_{c \in C(f) \cap \mathbb{C} \text{ periodic under } f} \limsup_{n \to \infty} (\deg_c(f^n))^{1/n}$, 
\[
\int_{p^1} \left| \frac{\log(1/[(f^n)'](z))}{d^n - 1} - g_f \right| d\omega(z) = o((\eta/d)^n)
\]
as $n \to \infty$.

Proof. For every $n \in \mathbb{N}$, from (3.2), we have
\[
(4.1) \quad \int_{p^1} \left| \frac{\log |(f^n)'(z)|}{d^n - 1} - g_f(z) \right| d\omega(z) \leq \frac{1}{d^n - 1} \int_{\mathbb{C}} \left( \sum_{j=0}^{n-1} \int_{p^1} \log \frac{1}{|f^j(z), w|} d\omega(z) \right) (dd^c \log |f'|)(w) + C_f \cdot n \frac{1}{d^n - 1},
\]
where $C_f > 0$ is defined in (3.4). By [6, Theorem 2], for every $\eta > \sup_{c \in C(f) \cap \mathbb{C} \text{ periodic under } f} \limsup_{n \to \infty} (\deg_c(f^n))^{1/n}$ and every $w \in \mathbb{C} = \mathbb{P}^1 \setminus E(f)$ under the assumption $E(f) = \{\infty\}$, we have
\[
\int_{p^1} \frac{1}{|f^n(z), w|} d\omega(z) = o(\eta^n)
\]
as $n \to \infty$, which with Lemma 3.3 and $0 \leq \log(1/|w, \infty|) - \log \max\{1, |w|\} \leq \log \sqrt{2}$ on $\mathbb{C}$ completes the proof. \hfill $\square$

Lemma 4.2. For every $\eta > 1$, the Valiron exceptional set 
\[E_V((f^n)'), (\eta^n)) := \left\{ a \in \mathbb{P}^1 : \limsup_{n \to \infty} \frac{1}{\eta^n} \int_{p^1} \log \frac{1}{|(f^n)'(z), a|} d\omega(z) > 0 \right\}
\]of the sequence $((f^n)')$ of the derivatives of the iterations of $f$ with respect to the sequence $(\eta^n)$ in $\mathbb{R}_{>0}$ is a polar subset in $\mathbb{P}^1$.

Proof. This is an application of Russakovskii–Shiffman [18, Proposition 6.2] to the sequence $((f^n)')$ in $\mathbb{C}[z]$ since $\sum_{n \in \mathbb{N}} 1/\eta^n < \infty$ for every $\eta > 1$. \hfill $\square$

For every $\eta > \sup_{c \in C(f) \cap \mathbb{C} \text{ periodic under } f} \limsup_{n \to \infty} (\deg_c(f^n))^{1/n}$, every $a \in \mathbb{C} \setminus E_V(((f^n)'), (\eta^n))$, and every $C^2$-test function $\phi$ on $\mathbb{P}^1$, by Lemmas 4.1 and 4.2 we have
\[
\left| \int_{p^1} \phi \left( \frac{(f^n)' \delta_a}{d^n - 1} - \mu_f \right) \right| = \left| \int_{p^1} \phi dd^c \left( \frac{\log |(f^n)', a|}{d^n - 1} + \frac{\log(1/[(f^n)'(z), \infty])}{d^n - 1} - g_f \right) \right| \leq \left( \sup_{p^1} \left| \frac{dd^c \phi}{d\omega} \right| \right),
\]
\[
\left( \frac{1}{d^n - 1} \int_{p^1} \log \frac{1}{|(f^n)'(z), a|} d\omega(z) + \int_{p^1} \left| \frac{\log(1/[(f^n)'(z), \infty])}{d^n - 1} - g_f \right| d\omega(z) \right) = o((\eta/d)^n)
\]
as $n \to \infty$, where the Radon-Nikodim derivative $(dd^c \phi)/d\omega$ on $\mathbb{P}^1$ is bounded on $\mathbb{P}^1$. \hfill $\square$
5. Proof of Theorem 3

Let \( f : \mathbb{C} \times \mathbb{P}^1 \ni (\lambda, z) \mapsto z^d + \lambda =: f_\lambda(z) \in \mathbb{P}^1 \) be the monic and centered unicritical polynomials family of degree \( d > 1 \). For every \( n \in \mathbb{N} \), \( f_\lambda^n(\lambda), (f_\lambda^n)'(\lambda) \in \mathbb{C}[\lambda] \) are of degree \( d^n, d^n - 1 \), respectively.

5.1. Background on the family \( f \). Recall the definitions in Subsection 2.2. The following constructions are due to Douady–Hubbard [5] and Sibony.

For every \( \lambda \in \mathbb{C} \), \( f_\lambda'(z) = d \cdot z^{d-1} \), so \( C(f_{\lambda}) \cap \mathbb{C} = \{0\} \) and \( f_\lambda(0) = \lambda \).

The connectedness locus \( C_d := \{ \lambda \in \mathbb{C} : \lambda \in K(f_{\lambda}) \} \) of the family \( f \) is a compact subset in \( \mathbb{C} \), and \( H_\infty = H_{d,\infty} := \mathbb{P}^1 \setminus C_d \) is a simply connected domain containing \( \infty \) in \( \mathbb{P}^1 \). Moreover, the locally uniform limit

\[
g_{H_\infty}(\lambda) := g_{f_{\lambda}}(\lambda) = d \cdot g_{f_{\lambda}}(0) = \lim_{n \to \infty} \frac{-\log f_{\lambda}^n(\lambda), \infty}{d^n}
\]
exists on \( \mathbb{C} \). Setting \( g_{H_\infty}(\infty) := +\infty \), the restriction of \( g_{H_\infty} \) to \( H_\infty \) coincides with the Green function on \( H_\infty \) with pole \( \infty \), and the measure

\[
\mu_{C_d} := \omega \cdot g_{H_\infty} = \delta_\infty \quad \text{on} \quad \mathbb{P}^1
\]
coincides with the harmonic measure on \( C_d \) with pole \( \infty \). In particular, \( z \mapsto g_{H_\infty}(z) - \log |z| \) extends harmonically to an open neighborhood of \( \infty \) in \( H_\infty \), and \( \text{supp} \mu_{C_d} \subset \partial C_d \) (in fact, the equality holds).

5.2. Proof of Theorem 3. For every \( n \in \mathbb{N} \), \( \lambda \mapsto (\log |(f_\lambda^n)'(\lambda)|)/(d^n - 1) - g_{H_\infty}(\lambda) \) and \( \lambda \mapsto (\log \max\{1, |(f_\lambda^n)'(\lambda)|\})/(d^n - 1) - g_{H_\infty}(\lambda) \) on \( \mathbb{C} \) extend continuously to \( \mathbb{P}^1 \).

Lemma 5.1. For every \( n \in \mathbb{N} \) and every \( \lambda \in \mathbb{C} \),

\[
\log \max\{1, |(f_\lambda^n)'(\lambda)|\} \leq n \log \left(\frac{d}{d^n - 1}\right) - g_{H_\infty}(\lambda) \leq n \log \left(\frac{d^2}{d^n - 1}\right).
\]

Proof. For every \( n \in \mathbb{N} \) and every \( \lambda \in \mathbb{C} \), by \( g_{f_\lambda^n} = g_{f_\lambda} \) on \( \mathbb{P}^1 \) and \( f_{\lambda} = d \cdot g_{f_{\lambda}} \) on \( \mathbb{P}^1 \), we have \( g_{f_\lambda^n}(\lambda) = g_{f_\lambda}(\lambda) = d \cdot g_{f_\lambda}(0) \geq g_{f_\lambda}(0) \geq \max_{e \in \mathbb{C}(f_{\lambda}) \cap \mathbb{C}} g_{f_\lambda}(e) = \max_{e \in \mathbb{C}(f_{\lambda}) \cap \mathbb{C}} g_{f_\lambda}(e) \), so by Theorem 2.1 we have \( |(f_\lambda^n)'(\lambda)| \leq (d^n)^2 e^{(d^n-1)}g_{f_\lambda}(\lambda) = (d^n)^2 e^{(d^n-1)}g_{f_\lambda}(\lambda) = (d^n)^2 e^{(d^n-1)}g_{H_\infty}(\lambda) \). This with \( g_{H_\infty}(\lambda) \geq 0 \) completes the proof.

Lemma 5.2.

\[
\int_{\mathbb{P}^1} \frac{|\log(1/|(f_\lambda^n)'(\lambda)|, \infty)|}{d^n - 1} - g_{H_\infty}(\lambda) \wedge d\omega(\lambda) = O(n^2 d^{-n})
\]
as \( n \to \infty \).

Proof. For every \( n \in \mathbb{N} \), by the third equality in (3.1) for \( f_\lambda \) evaluated at \( z = \lambda \), we have

\[
\log \left| \frac{|(f_\lambda^n)'(\lambda)|}{d^n - 1} \right| - n \log d = \frac{d - 1}{d^n} \sum_{j=0}^{n-1} \log |f_\lambda^j(\lambda)| = \frac{d - 1}{d^n} \sum_{j=0}^{n-1} \log |f_\lambda^{j+1}(0)|,
\]
so that
\begin{equation}
\int_{\mathbb{P}^1} \left| \log \left[ \frac{(f^n_\lambda)'(\lambda)}{d^n - 1} \right] - g_{H_{\infty}}(\lambda) \right| \omega(\lambda)
\leq \frac{d - 1}{d^n - 1} \sum_{j=0}^{n-1} \int_{\mathbb{P}^1} \left| \log \left[ f^{j+1}_\lambda(0) \right] - d^j \cdot g_{H_{\infty}}(\lambda) \right| \omega(\lambda) + \frac{n \log d}{d^n - 1}
= O(n^2d^{-n}) \quad \text{as } n \to \infty
\end{equation}

since by Gauthier–Vigny [9, §4.3, Proof of Theorem A], we have
\begin{equation}
\int_{\mathbb{P}^1} \left| \log \left[ f^n_\lambda(0) \right] - d^n \cdot g_{H_{\infty}}(\lambda) \right| \omega(\lambda) = O(n)
\end{equation}
as \( n \to \infty \). This with Lemma [5.1] and 0 \leq \log(1/w, \infty) - \log \max\{1, |w|\} \leq \log \sqrt{2} \) on \( \mathbb{C} \) completes the proof. \( \square \)

**Lemma 5.3.** For every \( \eta > 1 \), the Valiron exceptional set
\( E_V((f^n_\lambda)'(\lambda)), (\eta^n)) := \left\{ \lambda \in \mathbb{P}^1 : \limsup_{n \to \infty} \frac{1}{\eta^n} \int_{\mathbb{P}^1} \log \frac{1}{[(f^n_\lambda)'(\lambda), a]} d\omega(\lambda) > 0 \right\} \)
of the sequence \( ((f^n_\lambda)'(\lambda)) \) in \( \mathbb{C}[\lambda] \) with respect to the sequence \( (\eta^n) \) in \( \mathbb{R}_{>0} \) is a polar subset in \( \mathbb{P}^1 \).

**Proof.** This is an application of Russakovskii–Shiffman [18, Proposition 6.2] to the sequence \( ((f^n_\lambda)'(\lambda)) \) in \( \mathbb{C}[\lambda] \) since \( \sum_{n \in \mathbb{N}} 1/\eta^n < \infty \) for every \( \eta > 1 \). \( \square \)

For every \( \eta > 1 \), every \( \lambda \in \mathbb{C} \setminus E_V((f^n_\lambda)'(\lambda)), (\eta^n)) \), and every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \), by Lemmas [5.2] and [5.3] we have
\begin{equation}
\left| \int_{\mathbb{P}^1} \phi(\lambda)d \left( \frac{(f^n_\lambda)'(\lambda))^\ast \delta_\lambda}{d^n - 1} - \mu_{C_3} \right)(\lambda) \right|
= \left| \int_{\mathbb{P}^1} \phi(\lambda)dd^c \left( \frac{\log[(f^n_\lambda)'(\lambda), \lambda]}{d^n - 1} + \frac{\log(1/[(f^n_\lambda)'(\lambda), \infty])}{d^n - 1} - g_{H_{\infty}}(\lambda) \right) \right|
\leq \left( \sup_{\mathbb{P}^1} \left| \frac{dd^c \phi}{d\omega} \right| \right) \cdot \left( \frac{1}{d^n - 1} \int_{\mathbb{P}^1} \log \frac{1}{[(f^n_\lambda)'(\lambda), a]} d\omega(\lambda) + \int_{\mathbb{P}^1} \left| \frac{\log(1/[(f^n_\lambda)'(\lambda), \infty])}{d^n - 1} - g_{H_{\infty}}(\lambda) \right| d\omega(\lambda) \right)
= o((\eta/d)^n) \quad \text{as } n \to \infty,
\end{equation}
where the Radon-Nikodim derivative \( (dd^c \phi)/d\omega \) on \( \mathbb{P}^1 \) is bounded on \( \mathbb{P}^1 \). \( \square \)

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