On the Schrödinger Flows

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Abstract

We present some recent results on the existence of solutions of the Schrödinger flows, and pose some problems for further research.

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1. Introduction

Recently the research on so-called Schrödinger flow (or Schrödinger map [1]-[4]) has been carried out by several authors. This is an infinite-dimensional Hamiltonian flow defined on the space of mappings from a Riemannian manifold \((M, g)\) into a Kähler manifold \((N, J, h)\), where \(g\) is the Riemannian metric on \(M\), and \(h\) is the Kähler metric on \(N\), with \(J\) being the complex structure on \(N\). This flow is defined by the following equation

\[
 u_t = J(u)\tau(u),
\]

where \(\tau(u)\) is the so-called tension field well-known in the theory of harmonic maps. In local coordinates, \(\tau(u)\) is given by

\[
 \tau(u)^i = \Delta_M u^i - g^{\alpha\beta} \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta}.
\]

Here \(\Delta_M\) is the Laplace-Beltrami operator on \(M\) and \(\Gamma^i_{jk}\) are the Christoffel symbols of the Riemannian connection on \(N\). Obviously, the Schrödinger flows preserves the energy \(E(u)\) of mapping \(u\), i.e. \(E(u(t)) \equiv E(u(0))\), where

\[
 E(u) = \frac{1}{2} \int_M g^{\alpha\beta} h_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} dM.
\]

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Schrödinger flows are related to various theories in mechanics and physics. A well-known and important example is the so-called Heisenberg spin chain system (also called ferromagnetic spin chain system [7]). This is just the Schrödinger flow into $S^2$. Consider $S^2$ as the unit sphere in $\mathbb{R}^3$, then the equation for the system is given by

$$u_t = u \times \Delta u.$$ 

Note that, for a mapping $u$ from $M$ into $S^2$,

$$J(u) = u \times : T_{u}S^2 \to T_{u}S^2$$

is the standard complex structure on $S^2$, and the tension field of the map $u$ into $S^2$ is given by $\tau(u) = \Delta u + |\nabla u|^2 u$. So, we have $u \times \Delta u = J(u)\tau(u)$. Another interesting example of the Schrödinger flow is the anisotropic Heisenberg spin chain system, i.e. the Schrödinger flow into Poincaré disk $H(-1)$.

Comparing to other geometric nonlinear evolutionary systems, such as the heat flow of harmonic maps (parabolic system) and wave maps (hyperbolic system), the study of Schrödinger flows is still at the beginning stage. There are some remarkable results on the existence of solutions for certain specific cases. E.g. for the Heisenberg spin chain system ($N = S^2$), Zhou et. al. [9] proved the global existence for $M = S^1$, and Sulem et. al. [10] proved the local existence for $M = \mathbb{R}^m$. There are some more recent works, see [1], [3] and [11]. For the general case, however, it turns out that even local existence is hard to prove. In this respect, a recent result obtained by Youde Wang and this author ([4]) states

**Theorem** Let $(M, g)$ be a closed Riemannian manifold of dimension $m$, and let $(N, J, h)$ is a closed Kähler manifold. If $m_0$ is the smallest integer greater than $m/2$ (i.e. $m_0 = \lfloor m/2 \rfloor + 1$), and $u_0 \in W^{k,2}(M, N)$ for any $k \geq m_0 + 3$, then the initial value problem for (1.1) with initial value $u_0$ has a unique local solution. Moreover, if $u_0 \in C^\infty(M, N)$, the local solution is $C^\infty$ smooth.

We remark that, the maximal existence time of the local solution in the above result, depends only on the $W^{m_0+1}$-norm of the initial map $u_0$ for any $k$. This is why we can get local existence in the $C^\infty$ case. Also, for the existence part, the regularity of $u_0$ can be lowered to $W^{k,2}$ with $k \geq m_0 + 1$, however we do not know how to get the uniqueness if $k < m_0 + 3$.

In the following, we give a description of the proof of the above Theorem in Section 2 and 3. Then, in Section 4, we pose some important problems for future research of the Schrödinger flows.

### 2. Some inequalities for Sobolev section norms of maps

Let $\pi : E \to M$ be a Riemannian vector bundle over $M$. Then we have the bundle $\wedge^p T^*M \otimes E \to M$ over $M$ which is the tensor product of the bundle $E$ and the induced $p$-form bundle over $M$, where $p = 1, 2, \ldots, \dim(M)$. We define $\Gamma(\wedge^p T^*M \otimes E)$ as the set of all smooth sections of $\wedge^p T^*M \otimes E \to M$. There
exists a induced metric on $\Lambda^p T^*M \otimes E \to M$ from the metric on $T^*M$ and $E$ such that for any $s_1, s_2 \in \Gamma(\Lambda^p T^*M \otimes E)$

$$\langle s_1, s_2 \rangle = \sum_{i_1 < i_2 < \cdots < i_p} \langle s_1(e_{i_1}, \cdots, e_{i_p}), s_2(e_{i_1}, \cdots, e_{i_p}) \rangle,$$

where $\{e_i\}$ is an orthonormal local frame of $TM$. We define the inner product on $\Gamma(\Lambda^p T^*M \otimes E)$ as follows

$$(s_1, s_2) = \int_M \langle s_1, s_2 \rangle(x) dM = \int_M \langle s_1, s_2 \rangle(x) \ast 1.$$

The Sobolev space $L^2(M, \Lambda^p T^*M \otimes E)$ is the completion of $\Gamma(\Lambda^p T^*M \otimes E)$ with respect to the above inner product $\langle \cdot, \cdot \rangle$, we may also define analogously the Sobolev spaces $H^{k,r}(M, \Lambda^p T^*M \otimes E)$ or $H^{k,r}(M, E)$. Let $\nabla$ be the covariant differential induced by the metric on $E$, then we can take the completion of the smooth sections of $E$ in the norm,

$$||s||_{k,r} = ||s||_{H^{k,r}(M,E)} = \left( \sum_{i=0}^{k} \int_M |\nabla^i s|^r dM \right)^{\frac{1}{r}}.$$

We call the above Sobolev spaces as the bundle-valued Sobolev spaces.

In [4] We establish the following interpolation inequality for sections on vector bundles, which was proved for functions on $\mathbb{R}^m$ by Gagliardo and Nirenberg, and for functions on Riemannian manifolds by Aubin ([8]).

**Lemma 2.1** Let $M$ be a compact Riemannian manifold with $\dim(M) = m$ and $E$ be a Riemannian vector bundle over $M$. Let $q, r$ be real numbers $1 \leq q, r \leq \infty$ and $j, m$ integers $0 \leq j \leq n$. Then there exists a constant $C(M)$ depending on $m, n, j, q, r$ and $a$, and on $M$, but not depending on the choice of metrics on $E$, such that for all $s \in C^\infty(E)$:

$$||\nabla^j s||_{L^p} \leq C(M) ||s||_{H^{m,r}}^a ||s||_{L^q}^{1-a}, \quad (2.1)$$

where

$$\frac{1}{p} = \frac{j}{m} + a \left( \frac{1}{r} - \frac{n}{m} \right) + (1-a) \frac{1}{q},$$

for all $a$ in the interval $\frac{j}{n} \leq a \leq 1$, for which $p$ is non-negative. If $r = \frac{m}{n-1} \neq 1$, then the above interpolation inequality is not true for $a = 1$.

The so-called Sobolev section norms of mapping $u \in C^\infty(M, N)$, where $M$ is a closed Riemannian manifold, is defined as the Sobolev section norms of $\nabla u$ where $\nabla u$ is regarded as a section on the bundle $u^*(TN) \otimes T^*M$. Then with $s = \nabla u$, we have by Lemma 2.1,

$$||\nabla^{j+1} u||_{L^p} \leq C ||\nabla u||_{H^{k,r}}^a ||\nabla u||_{L^q}^{1-a}, \quad (2.2)$$

where the constants in (2.1) satisfy the conditions of Lemma 2.1. Obviously, the $H^{k,2}$ norm of maps $u \in C^\infty(M, N)$ is nonlinear with respect to $u$. 
In order to prove Theorem we need to consider the problem of comparing the $W^{k,2}$ norm with $H^{k,2}$ norm of maps $u \in C^\infty(M, N)$ (i.e., Sobolev section norm). We assume that $M$ is a closed Riemannian manifold and $N$ is a compact Riemannian manifold with or without boundary. It will be convenient to imbed $N$ isometrically into some Euclidean space $\mathbb{R}^K$, and consider $N$ as a compact submanifold of $\mathbb{R}^K$. Then the map $u$ can be represented as $u = (u^1, \cdots, u^K)$ with $u^i$ being globally defined functions on $M$. The we have

$$\|u\|_{W^{k,2}}^2 = \sum_{i=0}^k \|D^i u\|^2_{L^2},$$

where

$$\|D^i u\|^2_{L^2} = \sum_{|a|=i} \|D_a u\|^2_{L^2},$$

and $D$ denotes the covariant derivative for functions on $M$. The $H^{k,2}$ norm of $u$ is defined similarly, only we need to replace $D$ by $\nabla$, where $\nabla$ is the covariant derivative for sections of the bundle $u^*(TN)$ over $M$. For simplicity we also write $\nabla u = Du$. In [4] Ding and Wang obtained the following lemma.

**Lemma 2.2** Assume that $k > m/2$. Then there exists a constant $C = C(N,k)$ such that for all $u \in C^\infty(M, N)$,

$$\|Du\|_{W^{k-1,2}} \leq C \sum_{t=1}^k \|\nabla u\|^t_{H^{k-1,2}}, \quad (2.3)$$

and

$$\|\nabla u\|_{H^{k-1,2}} \leq C \sum_{t=1}^k \|Du\|^t_{W^{k-1,2}}. \quad (2.4)$$

### 3. The proof of theorem

In this section we prove the local existence of smooth solutions for the initial value problem of the Schrödinger flow

$$\begin{align*}
\left\{ \begin{array}{l}
 u_t = J(u)\tau(u), \\
 u(\cdot, 0) = u_0 \in C^\infty(M, N)
\end{array} \right. \quad (3.1)
\end{align*}$$

We need to employ an approximate procedure and solve first the following perturbed problem

$$\begin{align*}
\left\{ \begin{array}{l}
 u_t = \epsilon \tau(u) + J(u)\tau(u), \\
 u(\cdot, 0) = u_0 \in C^\infty(M, N),
\end{array} \right. \quad (3.2)
\end{align*}$$

where $\epsilon > 0$ is a small number.

The advantage of (3.2) is that the equation with $\epsilon > 0$ is uniformly parabolic. Hence the initial value problem has a unique smooth solution $u_\epsilon \in C^\infty(M \times [0, T_\epsilon], N)$ for some $T_\epsilon > 0$. The problem is then to obtain a uniform positive
lower bound \( T \) of \( T_\epsilon \), and uniform bounds for various norms of \( u_\epsilon(t) \) in suitable spaces for \( t \) in the time interval \([0, T]\). (Since we shall use \( L^2 \) estimates, the norms are \( W^{k,2}(M, N) \)–norms for all positive integer \( k \).) Once we get these bounds it is clear that the \( u_\epsilon \) subconverge to a smooth solution of (3.1) as \( \epsilon \to 0 \).

Now let \( u = u_\epsilon \) be a solution of (3.2), then it is easy to see that the energy \( E(u(t)) \) is uniformly bounded for \( t \in [0, T_\epsilon] \), i.e.

\[
E(u(t)) \leq E(u_0). \tag{3.3}
\]

In the following we will make estimations on \( L^2 \)–norms of all covariant derivatives \( \nabla^k u \) \( (k = 2, 3, \ldots) \).

**Lemma 3.1** Let \( m_0 = \lfloor m/2 \rfloor + 1 \), where \( \lfloor q \rfloor \) denotes the integral part of a positive number \( q \), and let \( u_0 \in C^\infty(M, N) \). There exists a constant \( T = T(\| u_0 \|_{H^{m_0+1,2}}) > 0 \), independent of \( \epsilon \in [0, 1] \), such that if \( u \in C^\infty(M \times [0, T_\epsilon]) \) is a solution of (3.1) with \( \epsilon \in (0, 1] \) then

\[
T_\epsilon \geq T(\| \nabla u_0 \|_{H^{m_0,2}}) \quad \text{and} \quad \| \nabla u(t) \|_{H^{k,2}} \leq C(k, \| \nabla u_0 \|_{H^{k,2}}) \quad t \in [0, T]
\]

for all \( k \geq m_0 \).

**Proof** Fix a \( k \geq m_0 \), and let \( l \) be any integer with \( 1 \leq l \leq k \). Suppose that \( \alpha \) be a multi-index of length \( l \), i.e. \( \alpha = (a_1, \cdots, a_l) \). Then we have for \( t \leq T_\epsilon \)

\[
\frac{1}{2} \frac{d}{dt} \| \nabla^\alpha \nabla_i u \|_{L^2}^2 = \int_M \langle \nabla^\alpha \nabla_i u, \nabla_i \nabla^\alpha u \rangle. \tag{3.4}
\]

Exchanging the order of covariant differentiation we have (cf. [9])

\[
\nabla_i \nabla^\alpha \nabla_j u = \nabla^\alpha \nabla_i \nabla_j u + \sum \nabla_b R(u)(\nabla_c u, \nabla_d \nabla_j u) \nabla_e \nabla_i u,
\]

where the sum is over all multi-indexes \( b, c, d, e \) with possible zero lengths, except that \( |c| > 0 \) always holds, such that

\[
(b, c, d, e) = \sigma(\alpha)
\]

is a permutation of \( \alpha \). Noting that we may replace \( \nabla_i u \) in the terms of the summation by the right hand side of equation (3.2), the above identity can be rewritten as

\[
\nabla_i \nabla^\alpha \nabla_j u = \nabla^\alpha \nabla_i \nabla_j u + Q \tag{3.5}
\]

with

\[
|Q| \leq C(l, M) \sum |\nabla^{j_1} u| \cdots |\nabla^{j_s} u| \tag{3.6}
\]

where the summation is over all \((j_1, \cdots, j_s)\) satisfying

\[
j_1 \geq j_2 \geq \cdots \geq j_s, \ l + 1 \geq j_i \geq 1, \ j_1 + \cdots + j_s = l + 3, \ s \geq 3. \tag{3.7}
\]
For the first term in the right hand side of (3.5), we may use the equation (3.2) to get
\[ \nabla_a \nabla_i \nabla_t u = \nabla_a \nabla_i (\epsilon \tau(u) + J(u) \tau(u)) - \epsilon \nabla_a \nabla_k \nabla_k u + J(u) \nabla_a \nabla_k \nabla_u \] (3.8)
where we have used the integrability of the complex structure \( J \) of the Kähler manifold \( N \). By exchanging the orders of covariant differentiation as above, we get from (3.5) and (3.8)
\[ \nabla_t \nabla_a \nabla_i u = \epsilon \nabla_k \nabla_k \nabla_a \nabla_i u + J(u) \nabla_k \nabla_k \nabla_a \nabla_i u + Q \]
where \( Q \) satisfies (3.6-3.7). Substituting this into (3.4) and integrating by part we then have
\[ \frac{1}{2} \frac{d}{dt} \| \nabla_a \nabla_i u \| ^2_{L^2} = \int_M \left( -\epsilon |\nabla_a \nabla_i u|^2 - \langle \nabla_k \nabla_a \nabla_i u, J(u) \nabla_k \nabla_i u \rangle + \langle \nabla_a \nabla_i u, Q \rangle \right) \]
Note that the first integrand is non-positive and the second vanishes, so we have by (3.6)
\[ \frac{d}{dt} \| \nabla^l u \| ^2_{L^2} \leq C(l, M) \sum \int_M |\nabla^{l+1} u| |\nabla^{j_1} u| \cdots |\nabla^{j_s} u|, \tag{3.9} \]
and consequently
\[ \frac{d}{dt} \| \nabla^l u \| ^2_{L^2} \leq C(l, M) \sum \int_M |\nabla^{l+1} u| |\nabla^{j_1} u| \cdots |\nabla^{j_s} u|, \tag{3.10} \]
where the summation is over all \((j_1, \cdots, j_s)\) satisfying (3.7).
To treat the integrals in the summation of (3.9), i.e.
\[ I = \int_M |\nabla^{l+1} u| |\nabla^{j_1} u| \cdots |\nabla^{j_s} u|, \tag{3.10} \]
we need the following lemma which can be proved by applying Lemma 2.1, the Hölder inequality and some combination techniques. Especially, the proof of Lemma 3.3 is slightly tricky, for details we refer to [4].

**Lemma 3.2** Let \( I \) be the integral (3.10), where \((j_1, \cdots, j_s)\) satisfy (3.7). If \( 1 \leq l \leq m_0 \), then there exists a constant \( C = C(M, l) \) such that
\[ I \leq C \| \nabla u \| ^A_{H^{m_0, 2}} \| \nabla u \| ^B_{L^2} \| \nabla^{l+1} u \| _{L^2}, \]
where \( A = [l + 3 + (m/2 - 1)s - m/2]/m_0 \) and \( B = s - A \).

**Lemma 3.3** Assume \( l > m_0 \). Then there exists a constant \( C = C(M, l) \) such that
(1) if \( j_1 = l + 1 \),
\[ I \leq C \| \nabla^{l+1} u \| ^2_{L^2} \| \nabla u \| ^{m/m_0}_{H^{m_0, 2}} \| \nabla u \| ^{2-m/m_0}_{L^2}. \]
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(2) if \( j_1 \leq l \),

\[
I \leq C(1 + \|\nabla u\|_{H^{l+2}}^2)(1 + \|\nabla u\|_{H^{l-1}}^4)
\]

where \( A = A(m, l) \).

Now, return to the proof of Lemma 3.1. We first consider the case \( 1 \leq l \leq m_0 \) in (3.9). Then Lemma 3.2 together with (3.3) leads to

\[
\frac{d}{dt}\|\nabla u\|_{H^{m_0+2}} \leq C \sum_{l=1}^{m_0} \sum_{s=3}^{l+3} \|\nabla u\|_{H^{m_0+2}}^{A(s,l)},
\]

where

\[
A(s, l) = [l + 3 + (m/2 - 1)s - m/2]/m_0.
\]

If we let \( f(t) = \|\nabla u(t)\|_{H^{m_0+2}} + 1 \), then we have

\[
f' \leq Cf^A_0, \quad f(0) = \|\nabla u_0\|_{H^{m_0+2}} + 1, \quad (3.11)
\]

where \( A_0 = \max\{A(s, l) : 3 \leq s \leq l + 3, 1 \leq l \leq m_0\} \). The constant \( C \) in (3.11) depends only on \( m_0, M \) and \( N \). It follows from (3.11) that there exists \( T = T(N, \|\nabla u_0\|_{W^{m_0+2}}) > 0 \) and \( K_0 > 0 \) such that

\[
\|\nabla u(t)\|_{H^{m_0+2}} \leq K_0, \quad t \in [0, T]. \quad (3.12)
\]

For any \( k > m_0 \), we need to consider the case \( m_0 < l \leq k \) in (3.9). Lemma 3.3, (3.3) and (3.12) then imply

\[
\frac{d}{dt}\|\nabla u\|_{H^{k+2}}^2 \leq C(1 + \|\nabla u\|_{H^{k+2}}^2)(1 + \|\nabla u\|_{H^{k-1}}^4). \quad (3.13)
\]

For \( k = m_0 + 1 \), we see from (3.12) that the summation in (3.13) is bounded since \( k - 1 = m_0 \). Then, since (3.13) is a linear differential inequality for \( \|\nabla u\|_{H^{k+2}}^2 \), there exists a constant \( K_1 > 0 \) such that

\[
\|\nabla u(t)\|_{H^{m_0+1}} \leq K_1, \quad t \in [0, T]. \quad (3.14)
\]

It now is clear that one can show inductively using (3.13) the existence of \( K_i > 0 \) for any \( i \geq 1 \) such that

\[
\|\nabla u(t)\|_{H^{m_0+i+2}} \leq K_i, \quad t \in [0, T]. \quad (3.15)
\]

Since we assume \( M \) is compact, consequently \( \|u(t)\|_{L^\infty} \) is uniformly bounded for \( t \in [0, T] \).

It is easy to find that the solution to (3.2) with \( \epsilon \in (0, 1) \) must exist on the time interval \([0, T]\). Otherwise, we always extend the time interval of existence to cover \([0, T]\), i.e., we always have \( T_\epsilon \geq T \). Thus, Lemma 3.1 has been proved.

**Proof of Theorem** First, we would like to mention that \( N \) is always regarded as an embedded submanifold of \( \mathbb{R}^K \). If \( u_0 : M \rightarrow N \) is \( C^\infty \), then, Lemma 3.1 claims that the initial value problem (3.2) admits a unique smooth solution \( u_\epsilon \).
which satisfies the estimates in Lemma 3.1. It follows from Proposition 2.2 that, for any \( k > 0 \) and \( \epsilon \in (0, 1] \), there holds
\[
\max_{t \in [0, T]} \| u_\epsilon \|_{W^{k, 2}(M)} \leq C_k(M, u_0),
\]
where \( C_k(M, u_0) \) does not depend on \( \epsilon \). Hence, by sending \( \epsilon \to 0 \) and applying the embedding theorem of Sobolev spaces to \( u \), we have \( u_\epsilon \to u \in C^k(M \times [0, T], N) \) for any \( k \). It is very easy to check that \( u \) is a solution to the initial value problem (3.1). The uniqueness was addressed in Proposition 2.1 in [1].

Finally, if \( u_0 : M \to N \) is not \( C^\infty \), but \( u_0 \in W^{k, 2}(M, N) \), we may always select a sequence of \( C^\infty \) maps from \( M \) into \( N \), denoted by \( u_{i0} \), such that
\[
u_{i0} \to u_0 \quad \text{in} \quad W^{k, 2}, \quad \text{as} \quad i \to \infty.
\]

This together with the definition of covariant differential leads to
\[
\| \nabla u_{i0} \|_{H^{k-1, 2}} \to \| \nabla u_0 \|_{H^{k-1, 2}}, \quad \text{as} \quad i \to \infty.
\]

Thus, there exists a unique, smooth solution \( u_i \), defined on time interval \([0, T_i]\), of the Cauchy problem (3.1) with \( u_0 \) replaced by \( u_{i0} \). Furthermore, it is not difficult to see from the arguments in Lemma 3.1 that if \( i \) is large enough, then there exists a uniform positive lower bound of \( T_i \), denoted by \( T \), such that the following holds uniformly with respect to large enough \( i \):
\[
sup_{t \in [0, T]} \| \nabla u_i(t) \|_{H^{k-1, 2}} \leq C(T, \| \nabla u_0 \|_{H^{k-1, 2}}).
\]

It follows from Lemma 2.2 and the last inequality that
\[
\sup_{t \in [0, T]} \| Du_i(t) \|_{W^{k-1, 2}} \leq C'(T, \| Du_0 \|_{W^{k-1, 2}}),
\]

where \( D \) denotes the covariant derivative for functions on \( M \). Therefore, there exists a \( u \in L^\infty([0, T], W^{k-1, 2}(M, N)) \) such that
\[
u_i \to u \quad \text{[weakly*] in} \quad L^\infty([0, T], W^{k, 2}(M, N))
\]

upon extracting a subsequence and re-indexing if necessary. It is easy to verify that \( u \) is a strong solution to (3.1) (see [4]).

**Remark** For the Schrödinger flow from an Euclidean space into a Kähler manifold, in [4] we obtained similar local existence results.

### 4. Some problems

1. For the one-dimensional case, i.e. \( \dim M = 1 \), we conjecture the Schrödinger flows should exist globally whenever the target \( N \) is a compact Kähler manifold. This is still open, and is supported by the result with \( N \) being Hermitian locally symmetric ([11]).
The result by Terng and Uhlenbeck [2] shows that for some special targets (e.g. complex Grassmannians), the Schrödinger flows are bi-Hamiltonian integrable systems. In their work, they assume that $M = R^1$, and their result can be generalized to compact Hermitian symmetric spaces (cf. [12]). An interesting open problem is, for these special targets, whether or not the Schrödinger flows are bi-Hamiltonian systems if $M = S^1$.

2. For higher dimensional cases, i.e. $\dim M \geq 2$, we believe that the Schrödinger flow may develop finite-time singularities. There are however no such examples known by now.

3. All present results in the study of the Schrödinger flows depend on the global estimates for the solutions. We do not know if one can find some kind of local estimates for the solutions. It has been well known from the research of various geometric flows that local estimates are important for the analysis of singularities. It is therefore desirable to develop some new methods to attack the question before any serious advance can be made for the study of the Schrödinger flows.

References

[1] W. Y. Ding and Y. D. Wang, Schrödinger flows of maps into symplectic manifolds, Science in China A, 41(7)(1998), 746–755.
[2] C. T. Terng and K. Uhlenbeck, Schrödinger flows on Grassmannians, math. DG/9901086.
[3] N. Chang, J. Shatah and K. Uhlenbeck, Schrödinger maps, Comm. Pure Appl. Math., 53(2000), 590–602.
[4] W. Y. Ding and Y. D. Wang, Schrödinger flows into Kähler manifolds, Science in China A, 44(11)(2001), 1446–1464.
[5] L. D. Landau and E. M. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, Phys. Z. Sowj. 8(1935), 153; reproduced in Collected Papers of L. D. Landau, Pergman Press, New York, 1965, 101–114.
[6] T. Aubin, Nonlinear Analysis on Manifolds. Monge-Ampère Equations, Springer-Verlag, Berlin-Heidelberg-New York, 1982.
[7] J. Eells and L. Lemaire, Another report on harmonic maps, Bull. London Math. Soc., 20 (1988), 385–524.
[8] L. Fadeev and L. A. Takhatajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, Berlin-Heidelberg-New York, 1987.
[9] Y. Zhou, B. Guo and S. Tan, Existence and uniqueness of smooth solution for system of ferromagnetic chain, Science in China A, 34(1991), 257–266.
[10] P. Sulem, C. Sulem and C. Bardos, On the continuous limit for a system of classical spins, Commun. Math. Phys., 107 (1986), 431-454.
[11] P. Pang, H. Wang, Y. D. Wang, Schrödinger flow on Hermitian locally symmetric spaces, to appear in Comm. Anal. Geom. .
[12] B. Dai, Ph. D. dissertation of NUS (2002).