GALOIS THEORY FOR A CLASS OF COMPLETE MODULAR LATTICES

ALEXANDRE A. PANIN

Department of Mathematics and Mechanics
St.Petersburg State University
2 Bibliotechnaya square,
St.Petersburg 198904, Russia

ABSTRACT. We construct Galois theory for sublattices of certain complete modular lattices and their automorphism groups. A well-known description of the intermediate subgroups of the general linear group over a semilocal ring containing the group of diagonal matrices, due to Z.I.Borewicz and N.A.Vavilov, can be obtained as a consequence of this theory. Bibliography: 3 titles.

INTRODUCTION

We generalize here the results of [PY], [S]. Namely, Galois theory for a class of complete modular lattices is constructed.

By an automorphism of a complete lattice we mean hereafter a bijective mapping of the lattice onto itself which commutes with the supremum and infimum of every subset of the lattice. Other notions and definitions are introduced in [PY].

FORMULATION OF THE MAIN RESULTS

Let \( L \) be a complete modular lattice, \( L_0 \) its finite sublattice, which is a Boolean algebra, \( G \) a subgroup of the group of all automorphisms of the lattice \( L \), \( H = G(L_0) \).

Let \( e_1, e_2, \ldots, e_n \) be the atoms of \( L_0 \). We consider a number of additional conditions (it is supposed that, unless otherwise stated, the indices are changing from 1 to \( n \)).

1991 Mathematics Subject Classification. 11H56, 20E15, 20G15, 20G35, 03G10, 20E07.

Key words and phrases. Modular lattices, automorphism groups of lattices, distribution of subgroups, linear algebraic groups.
Theorem 1. Assuming that the conditions 1° – 12° are fulfilled, for every subgroup \( F \geq H \) of group \( G \):

(i) \( \sigma = \sigma(F) \) is a net collection in \( L'_0 \);
(ii) \( G(K_\sigma) \leq F \);
(iii) if \( M \) is a sublattice of \( L'_0 \) such that \( G(M) \leq F \), then \( G(M) = G(K_\sigma) \).
Theorem 2. Let $\tau = (\tau_{ij})$ be a net collection in $L'_0$, $g \in G$. Provided that the conditions $1^0 - 11^0$, $13^0 - 16^0$ are fulfilled, we have:

(i) if $[g(e_i)]_j \leq \tau_{ij}$ for every $i, j$, then $g \in G(K_\tau)$;

(ii) the index of $G(K_\tau)$ in its normalizer is finite.

Proof of the main results

Proof of the parts (i)–(iii) of Theorem 1 is analogous to the proof of the corresponding assertions of [PY]. We note that instead of properties of the dimension function on $L$ used in [PY], one must apply the modularity law and the following trivial statement:

If $A \subseteq G$, where $A^{-1} = A$ and $a(x) \leq x$ for every $a \in A$ and some $x \in L$, then $a(x) = x$ for every $a \in A$.

Proof of Theorem 2 will be presented below.

Lemma 1. For every $f \in G$ $f(w) = w$.

Proof. It is sufficient to check $[f(w_i)]_j \leq w_j$ for every $i, j$.

It is clear that $w \in L'_0$, therefore for $i = j$ it is just the condition $6^0$.

Let $i \neq j$, $g \in H_{ij}(e_j)$. By the condition $9^0$ there exists $t \in H_{ji}(e_i)$ such that $tg(e_i) = e_j$. Consider an arbitrary coatom $y$ in $e_j$. Then $(tg)^{-1}(y)$ is a coatom in $e_i$, therefore $(tg(w_i))_j \leq y$, whence $[g(w_i)]_j \leq w_j$. Now it remains to apply the condition $6^0$.

Suppose $\tau = (\tau_{ij})$ is a net collection in $L'_0$.

Lemma 2. Let $\rho_{ij} = \tau_{ij} + w_j$. Then:

(i) $\rho = (\rho_{ij})$ is a net collection in $L'_0$;

(ii) $G(K_\rho) = G(K_\tau) \cdot G^w$.

Proof. Let $i, j, k$ be pairwise distinct, and let $g \in H_{ij}(x)$, $x \leq \tau_{ij} + w_j$.

By the conditions $7^0$ and $10^0$ $g \in \langle H, H_{ij}(y), H_{ij}(z) : y \leq \tau_{ij}, z \leq w_j \rangle$.

If $f \in H_{ij}(y)$, where $y \leq \tau_{ij}$, then $[f(\tau_{ki} + w_i)]_j \leq \tau_{kj} + w_j$ by Lemma 1. If $f \in H_{ij}(z)$, where $z \leq w_j$, then $[f(\tau_{ki} + w_i)]_j \leq w_j$.

(i) Apply the condition $8^0$.

(ii) The inclusion $\supseteq$ is trivial. Further, by the condition $15^0$ and by Theorem 7.2 [PY] $G(K_\rho) \subseteq \langle G(K_\tau), G^w \rangle$. It remains to note that $G^w \unlhd G$.

Proof of Theorem 2(i).

A. First, suppose that $\tau_{ij} \leq w_j$ for every $i \neq j$. Since $1 = \sum_{i=1}^{n} g(e_i)$, we have $e_1 = [g(e_1)]_1 + w_1$. By the condition $13^0$ $[g(e_1)]_1 = e_1$. Repeating the proof of Theorem 7.2 [PY], we obtain $g \in \langle H, H_{ij}(x) : x \leq \tau_{ij} \rangle \subseteq G(K_\tau)$.

B. General case. We put $x_i = \sum_{j=1}^{n} \tau_{ij}$. By the definition of a net collection we have $g(x_i) \leq x_i$. Further, it follows from Lemma 1 that $g(x_i + w) \leq x_i + w$, where
and since the restriction of \( g \) to \( L^w \) is an automorphism of this lattice, we have \( g(x_i + w) = x_i + w \) by the condition 140. Thus \( g \in G(K_\rho) \).

It follows from Lemma 2 that \( g = g_1g_2 \), where \( g_1 \in G(K_\tau) \), \( g_2 \in G^w \). Since \( g_2(x_i) \leq x_i \), then \( [g_2(e_i)]_j \leq \tau_{ij} \cdot w_j \) for every \( i \neq j \). We have already proved in the part A that \( [g_2^{-1}(e_i)]_j \leq \tau_{ij} \cdot w_j \) for every \( i \neq j \), therefore \( g_2 \in G(K_\tau) \).

Proof of the part (ii) is conceptually identical with the constructions of §7 of the article [BV].

As in [PY], a complete description of subgroups of the general linear group over a semilocal ring (whose fields of residues have at least seven elements, see [BV]), containing the group of diagonal matrices, can be deduced from Theorems 1 and 2.

ACKNOWLEDGEMENTS

The author is grateful to Professor Anatoly V. Yakovlev for the formulation of the problem and useful discussions.

REFERENCES

[BV] Borewicz Z.I., Vavilov N.A., Subgroups of the full linear group over a semilocal ring containing the group of diagonal matrices, Proc. Steklov Inst. Math. (1980), no. 4, 41–54.

[PY] Panin A.A., Yakovlev A.V., Galois theory for a class of modular lattices, Zap. Nauchn. Semin. POMI 236 (1997), 133–148 (In Russian, English transl. to appear in J. Math. Sci.).

[S] Simonian A.Z., Galois theory for modular lattices, Ph.D. thesis, St.Petersburg State University, 1992, pp. 1–73 (In Russian).

E-MAIL ADDRESS: alex@ap2707.spb.edu