HIGHER-ORDER LINKING FORMS FOR 3-MANIFOLDS

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Abstract. Given a closed, oriented, connected 3-manifold, M, we define higher-order linking forms on the higher-order Alexander modules of M. These higher-order linking forms generalize similar linking forms for knots previously studied by the author, which were themselves generalizations of the classical Blanchfield linking form for a knot. We also investigate the effect of the construction known as “infection by a knot” on these linking forms.

1. Introduction

We define linking forms, \( B_{\ell R}(M) \), associated to any closed, oriented, connected 3-manifold, \( M \), and any Ore domain, \( R \), such that \( \mathbb{Z} \Gamma \subset R \subset K \Gamma \) where \( \phi : \pi_1(M) \to \Gamma \) is a coefficient system, such that \( \Gamma \) is poly-torsion-free-abelian. Such linking forms have been used in a number of papers (see [1], [3], [4], [5], and [6]). However, the technical definitions and properties of them (particularly, the effect of infection on them) have not previously appeared in the literature.

Higher-order Alexander modules and higher-order linking forms for knots and for closed 3-manifolds with \( \beta_1(M) = 1 \) were introduced in [7] and further developed in [2] and [11]. Higher-order Alexander modules for 3-manifolds in general were defined and investigated in [9]. In Section 2, we define higher-order linking forms for 3-manifolds which are defined on these higher-order Alexander module.

It should be pointed out that the coefficients that we consider are more general than those used in much of the previous related work. First of all, we allow our coefficients to be unlocalized. In particular, the modules on which our linking forms are defined might not have homological dimension 1 and the forms themselves might be singular. This differs from much of the previous work (for instance, [7] and [8]) where the primary focus of study was over coefficients that were localized in order to obtain a principal ideal domain. Moreover, we allow \( \Gamma \) to be an arbitrary poly-torsion-free-abelian group. Some of the previous work (for instance, [9] and [11]) focused on the case where \( \Gamma = \pi_1(M)/\pi_1(M)_{r}^{(n)} \), the quotient of the fundamental group by the \( n \)th term of the (rational) derived series.

In Section 3 we investigate the effect of the construction known as “infection by a knot” on these higher-order linking forms for 3-manifolds. The construction of infecting a knot by a knot has been used extensively (for example, see [7], [8], and [2]). The effect of this construction on the higher-order Alexander modules of knots was studied in [2]. The effect on the higher-order linking forms for knots was studied in [11]. Infesting a 3-manifold by a knot was defined in [10].

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2. Definition of Higher-Order Linking Forms for 3-manifolds

In order to define our linking forms, we will need a coefficient system that embeds in its right ring of quotients. (A right ring of quotients is the non-commutative analogue of a quotient field.) It was shown in [7] that the group rings of a certain class of groups, namely poly-torsion-free-abelian groups have this property.

**Definition 2.1.** A group $\Gamma$ is poly-torsion-free-abelian (PTFA) if it admits a normal series $1 = G_n \triangleleft G_{n-1} \triangleleft \ldots \triangleleft G_0 = \Gamma$ of subgroups such that the factors $G_i/G_{i+1}$ are torsion-free abelian.

**Proposition 2.2** ([7], Prop. 2.5). If $\Gamma$ is PTFA, it follows that $\mathbb{Z}\Gamma$ is an Ore domain, and therefore it is possible to define the right ring of fractions of $\mathbb{Z}\Gamma$.

Suppose $\phi : \pi_1(M) \to \Gamma$ is a coefficient system, where $\Gamma$ is PTFA. Then $\mathbb{Z}\Gamma$ has a right ring of fractions, which we will denote by $\mathbb{K}\Gamma$. This right ring of fractions, $\mathbb{K}\Gamma$, is always a flat $\mathbb{Z}\Gamma$-module. (See [13], Prop. II.3.5.) If $\mathcal{R}$ is an Ore domain such that $\mathbb{Z}\Gamma \subset \mathcal{R} \subset \mathbb{K}\Gamma$, then $\mathbb{K}\Gamma$ is also the right ring of fractions of $\mathcal{R}$. (Such $\mathcal{R}$ could be $\mathbb{Z}\Gamma$ itself or could result from localizing any Ore set of $\mathbb{Z}\Gamma$.)

**Theorem 2.3.** Suppose $M$ is a closed, connected, oriented 3-manifold and $\phi : \pi_1(M) \to \Gamma$ is a PTFA coefficient system. If $\mathcal{R}$ is an Ore domain such that $\mathbb{Z}\Gamma \subset \mathcal{R} \subset \mathbb{K}\Gamma$, then there is a linking form defined on the torsion submodule of $H_1(M; \mathcal{R})$:

$$\mathcal{B}_{\mathcal{R}} : TH_1(M; \mathcal{R}) \to (TH_1(M; \mathcal{R}))/\#.$$

Here we use $\mathcal{M}$ to denote $\mathrm{Hom}_\mathcal{R}(\mathcal{M}, \mathbb{K}\Gamma/\mathcal{R})$. Also give any left $R$-module $\mathcal{M}$, we use $\overline{\mathcal{M}}$ to denote the usual associated right $R$-module resulting from the involution of $R$. The module $TH_1(M; \mathcal{R})$ on which $\mathcal{B}_{\mathcal{R}}$ is defined is referred to as a higher-order Alexander module of $M$. (Such modules were defined and studied in [9], where the focus was on the case where $\Gamma = \pi_1(M)/\pi_1(M)^{(n)}$.)

**Proof.** The short exact sequence $0 \to \mathcal{R} \to \mathbb{K}\Gamma \to \mathbb{K}\Gamma/\mathcal{R} \to 0$ gives rise to the Bockstein sequence of right $\mathcal{R}$-modules:

$$\xymatrix{ H_2(M; \mathbb{K}\Gamma) \ar[r]^-{\psi} & H_2(M; \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\mathrm{P.D.}} & H_2(M; \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\mathrm{P.D.}} \ar[l]^-{\psi} \ar[r] & H_2(M; \mathbb{K}\Gamma) \ar[d]^-{\kappa} & H_2(M; \mathbb{K}\Gamma) \ar[d]^-{\kappa} \ar[l]^-{\psi} \ar[r] & H_2(M; \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\mathrm{P.D.}} \ar[l]^-{\psi} \ar[r] & \cdots \ar[l]. }$$

Since $\mathbb{K}\Gamma$ is a flat $\mathcal{R}$-module, $TH_1(M; \mathcal{R})$ is the kernel of the map $H_1(M; \mathcal{R}) \to H_1(M; \mathcal{R}) \otimes_{\mathcal{R}} \mathbb{K}\Gamma \cong H_1(M; \mathbb{K}\Gamma)$. Using the Bockstein sequence above, we have $TH_1(M; \mathcal{R}) = \operatorname{im} B \cong \operatorname{coker} \psi$. Hence in order to define $\mathcal{B}_{\mathcal{R}}$ on $TH_1(M; \mathcal{R})$, it suffices to define a map on $H_2(M; \mathbb{K}\Gamma/\mathcal{R})$ such that $\operatorname{im} \psi$ is in the kernel.

Consider the following commutative diagram of right $\mathcal{R}$-modules.

$$\xymatrix{ H_2(M; \mathbb{K}\Gamma) \ar[r]^-{\psi} & H_2(M; \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\operatorname{P.D.}} & H_2(M; \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\operatorname{P.D.}} \ar[l]^-{\psi} \ar[r] & H_2(M; \mathbb{K}\Gamma) \ar[d]^-{\kappa} & H_2(M; \mathbb{K}\Gamma) \ar[d]^-{\kappa} \ar[l]^-{\psi} \ar[r] & H_2(M; \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\operatorname{P.D.}} \ar[l]^-{\psi} \ar[r] & \cdots \ar[l] \ar[d]^-{\operatorname{P.D.}} \ar[l]^-{\psi} \ar[r] & \cdots }$$

$$\xymatrix{ \operatorname{Hom}_\mathcal{R}(H_1(M; \mathcal{R}), \mathbb{K}\Gamma) \ar[r] & \operatorname{Hom}_\mathcal{R}(H_1(M; \mathcal{R}), \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\delta^\#} & \operatorname{Hom}_\mathcal{R}(H_1(M; \mathcal{R}), \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\delta^\#} \ar[l]^-{\delta^\#} \ar[r] & \operatorname{Hom}_\mathcal{R}(H_1(M; \mathcal{R}), \mathbb{K}\Gamma) \ar[d]^-{\delta^\#} & \operatorname{Hom}_\mathcal{R}(H_1(M; \mathcal{R}), \mathbb{K}\Gamma) \ar[d]^-{\delta^\#} \ar[l]^-{\delta^\#} \ar[r] & \operatorname{Hom}_\mathcal{R}(H_1(M; \mathcal{R}), \mathbb{K}\Gamma/\mathcal{R}) \ar[d]^-{\delta^\#} \ar[l]^-{\delta^\#} \ar[r] & \cdots \ar[l]. }$$
Here P.D. is the Poincaré duality isomorphism, $\kappa$ is the Kronecker evaluation map, and $j^#$ is induced by the inclusion map.

Since $K\Gamma$ is a torsion-free $R$-module, it follows that $\text{Hom}_R(TH_1(M; R), K\Gamma) = 0$. In other words, the lower left corner of the above diagram is 0. Therefore the image of $\psi$ is in the kernel of the composition $j^# \circ \kappa \circ \text{P.D.}$. Hence, there is a well-defined map, $B\ell_R$, such that the following diagram is commutative.

$$
\begin{array}{c}
H_2(M; K\Gamma/\mathcal{R}) \xrightarrow{B} TH_1(M; \mathcal{R}) \\
\text{P.D.} \downarrow \\
H^1(M; K\Gamma/\mathcal{R}) \\
\kappa \downarrow \\
(H_1(M; \mathcal{R}))^# \\
j^# \downarrow \\
(TH_1(M; \mathcal{R}))^#
\end{array}
$$

3. THE EFFECT OF INFECTION BY A KNOT ON $B\ell_R$

In this section, we consider the effect of infection by a knot on these higher-order linking forms. Let $M$ be a closed, connected, oriented 3-manifold, and let $\eta$ be an embedded, oriented, nullhomologous circle in $M$. Then $\eta$ has a well-defined meridian, $\mu_\eta$, and longitude, $\ell_\eta$. Delete the interior of a tubular neighborhood of $\eta$. Replace it with the exterior, $E(J)$ of some knot $J$ in $S^3$, identifying $\mu_\eta$ with the reverse of the longitude $\ell_J$ of $J$, and $\ell_\eta$ with the meridian $\mu_J$ of $J$. Denote the result $M(\eta, J)$, the result of infecting $M$ by $J$ along $\eta$.

Let $\phi: \pi_1(M) \to \Gamma$ be a PTFA coefficient system, and $\mathcal{R}$ be an Ore domain such that $\mathbb{Z}\Gamma \subset \mathcal{R} \subset K\Gamma$. Since there is a degree one map (rel boundary) $f: E(J) \to E(\text{unknot})$, there is a degree one map from $M(\eta, J)$ to $M$, which is the identity outside of $E(J)$. Hence the following composition of maps defines coefficient systems on $E(J)$, $M(\eta, J)$, and $M$:

$$
\pi_1(E(J)) \xrightarrow{i^*} \pi_1(M(\eta, J)) \xrightarrow{f_*} \pi_1(M) \xrightarrow{\phi} \Gamma.
$$

First, we investigate the effect of infecting a 3-manifold by a knot on the higher-order Alexander modules, $TH_1(M; \mathcal{R})$, on which the higher-order linking forms, $B\ell_R(M)$, are defined. The effect of infecting a knot by a knot on the higher-order Alexander modules of knots was studied in Section 8 of [2].

**Proposition 3.1.** If $\phi(\eta) = 1$, then $H_1(M(\eta, J); \mathcal{R}) \cong H_1(M; \mathcal{R})$. If $\phi(\eta) \neq 1$, then $H_1(M(\eta, J); \mathcal{R}) \cong H_1(M; \mathcal{R}) \oplus H_1(E(J); \mathcal{R})$.

**Proof.** We begin by stating and proving the following necessary lemma.
Lemma 3.2. If $\phi(\eta) = 1$, then $H_\ast (E(J); R) \cong H_\ast (E(J); Z) \otimes Z R$. If $\phi(\eta) \neq 1$, then $H_\ast (E(J); R) \cong H_\ast (E(J); Z[t, t^{-1}]) \otimes Z[t, t^{-1}] R$, where $R$ is a left $Z[t, t^{-1}]$-module by the homomorphism $t \mapsto \phi(\eta)$.

Proof. Let $M(\eta)$ denote the result of deleting the interior of a tubular neighborhood of $\eta$ from $M$. By the Seifert-Van Kampen Theorem, we have the following presentations of $\pi_1(M(\eta, J))$ and $\pi_1(M)$:

$$\pi_1(M(\eta, J)) = \langle \pi_1(M(\eta)), \pi_1(E(J)) | \mu_\eta = \ell_j^{-1}, \ell_\eta = \mu_j \rangle$$

$$\pi_1(M) = \langle \pi_1(M(\eta)), \mu_\eta = 1, \ell_\eta = 0 \rangle$$

The map $f_*: \pi_1(M(\eta, J)) \to \pi_1(M)$ is the identity map on $\pi_1(M(\eta))$ and is the Hurewicz map on $\pi_1(E(J)) \to \mathbb{Z} \cong \langle t \rangle$ which sends $\ell_j \mapsto 1$ and $\mu_j \mapsto t$. Therefore, the map $\psi \circ f_* \circ i_*: \pi_1(E(J)) \to \Gamma$ that defines the coefficient system on $E(J)$ factors through the Hurewicz map, and thus we have the following commutative diagram:

$$\mathbb{Z} \pi_1(E(J)) \xrightarrow{\psi} R$$

$$\downarrow$$

$$\mathbb{Z}[t, t^{-1}]$$

Here $\psi: t \mapsto \phi(\eta)$.

If $\phi(\eta) \neq 1$, then $\psi$ is a monomorphism. It follows from [12, Lemma 1.3] that $R$ is a free, and therefore flat $\mathbb{Z}[t, t^{-1}]$-module. If $C_\ast (E(J); Z \pi_1)$ denotes the chain complex of the universal cover of $E(J)$ with the action of $Z \pi_1(E(J))$ on it, then we have:

$$H_\ast (E(J); R) = H_\ast (C_\ast (E(J); Z \pi_1) \otimes_{Z \pi_1(E(J))} R)$$

$$\cong H_\ast (C_\ast (E(J); Z \pi_1) \otimes_{Z \pi_1(E(J))} \mathbb{Z}[t, t^{-1}] \otimes_{Z[t, t^{-1}]} R)$$

$$\cong H_\ast (E(J); Z[t, t^{-1}]) \otimes_{Z[t, t^{-1}]} R.$$
Lemma 3.2, \( \psi \) exact sequences:

\[
\begin{array}{cccc}
H_1(\partial E(J)) & H_1(E(J)) \oplus H_1(M(\eta)) & H_1(M(\eta, J)) & 0 \\
\downarrow f_* & \downarrow f_* & \downarrow f_* & \\
H_1(\partial E(U)) & H_1(M(\eta)) & H_1(M) & 0.
\end{array}
\]

Proof. Recall that there is a degree one map \( f \) on the infinite cyclic cover, it follows that \( \psi \) is an epimorphism. Hence \( H_1(M(\eta, J); \mathcal{R}) \cong H_1(M(\eta); \mathcal{R})/\text{im}(\psi_2). \) Similarly, \( H_1(M; \mathcal{R}) \cong H_1(M(\eta); \mathcal{R})/\text{im}(\psi_2). \) Therefore, \( H_1(M(\eta, J); \mathcal{R}) \cong H_1(M; \mathcal{R}). \)

Suppose \( \phi(\eta) \neq 1. \) Since \( H_1(\partial E(J); \mathbb{Z}) \to H_1(E(J); \mathbb{Z}) \) is an epimorphism, by Lemma 3.2 \( \psi_1 \) is an epimorphism. Hence \( H_1(M(\eta, J); \mathcal{R}) \cong H_1(M(\eta); \mathcal{R})/\text{im}(\psi_2). \) Furthermore, since \( H_1(E(U); \mathbb{Z}[t, t^{-1}]) = 0, \) it follows that \( H_1(E(U); \mathcal{R}) = 0. \) Hence \( H_1(M; \mathcal{R}) \cong H_1(M(\eta); \mathcal{R})/\text{im}(\psi_2). \) Therefore, \( H_1(M(\eta, J); \mathcal{R}) \cong H_1(E(J); \mathcal{R}) \oplus H_1(1; \mathcal{R}). \)

Corollary 3.3. If \( \phi(\eta) \neq 1, \) then

\[
TH_1(M(\eta, J); \mathcal{R}) \cong TH_1(M; \mathcal{R}) \oplus H_1(E(J); \mathcal{R}) \cong TH_1(M; \mathcal{R}) \oplus (A_0(J) \otimes \mathbb{Z}[t, t^{-1}]), \]

where \( A_0(J) \) is the classical Alexander module of \( J. \)

Proof. Since \( A_0(J) = H_1(E(J); \mathbb{Z}[t, t^{-1}]) \) is annihilated by the Alexander polynomial, it follows that \( (A_0(J) \otimes \mathbb{Z}[t, t^{-1}]) \) is a torsion module. The result now follows from Proposition 3.1 and Lemma 3.2.

We now consider the effect of infecting a 3-manifold by a knot on the higher-order linking forms for 3-manifolds. The effect of infecting a knot by a knot on the higher-order linking forms for knots was shown in Section 4 of [11].

Proposition 3.4. If \( \phi(\eta) = 1, \) then the linking forms \( \text{B}\ell(\mathcal{R}(M(\eta, J)) : TH_1(M(\eta, J); \mathcal{R}) \to (TH_1(M(\eta, J); \mathcal{R}))^{\#} \) and \( \text{B}\ell(\mathcal{R}(M) : TH_1(M; \mathcal{R}) \to (TH_1(M; \mathcal{R}))^{\#} \) are isomorphic.

Proof. Recall that there is a degree one map \( f : M(\eta, J) \to M. \) By Proposition 3.1, \( f \) induces an isomorphism between \( TH_1(M(\eta, J); \mathcal{R}) \) and \( TH_1(M; \mathcal{R}). \)
We have the following commutative diagram:

\[
\begin{array}{ccc}
H_2(M(\eta, J); K\Gamma/\mathcal{R}) & \xrightarrow{f^*} & H_2(M; K\Gamma/\mathcal{R}) \\
\| \quad \| & & \| \\
H^1(M(\eta, J); K\Gamma/\mathcal{R}) & \xrightarrow{f^*} & H^1(M; K\Gamma/\mathcal{R}) \\
| \quad | & & |
\kappa
| & & |
\| \quad \| & & \| \\
(TH_1(M(\eta, J); \mathcal{R}))^\# & \xrightarrow{f^*} & (TH_1(M; \mathcal{R}))^\# \\
| \quad | & & |
\| \quad \| & & \| \\
B^\ell_R(M(\eta, J)) & \xrightarrow{\kappa} & B^\ell_R(M) \\
| \quad | & & |
\| \quad \| & & \| \\
(\eta, J)^\# & \xrightarrow{\kappa} & (\eta, J)^\# \\
\end{array}
\]

Therefore \( B^\ell_R(M(\eta, J)) = f^* \circ B^\ell_R(M) \circ f_\ast \), and hence \( B^\ell_R(M(\eta, J)) \) and \( B^\ell_R(M) \) are isomorphic.

In the remainder of this section, we show how the linking forms \( B^\ell_R(M) \) and \( B^\ell_R(M(\eta, J)) \) are related when \( \phi(\eta) \neq 1 \). We begin by defining a linking form on \( E(J) \) with coefficients that are compatible with viewing \( J \) as the infecting knot of an infection.

**Proposition 3.5.** If \( \phi(\eta) \neq 1 \), then for any knot \( J \), there is a linking form \( B^\ell_R(J) : H_1(E(J); \mathcal{R}) \to (H_1(E(J); \mathcal{R}))^\# \) where the coefficient system is induced by the composition \( \pi_1(E(J)) \xrightarrow{1} \pi_1(M(\eta, J)) \xrightarrow{\phi} \pi_1(M) \xrightarrow{\zeta} \Gamma \).

**Proof.** We consider the Bockstein sequence:

\[
H_2(E(J); \mathcal{K}\Gamma) \to H_2(E(J); \mathcal{K}\Gamma/\mathcal{R}) \xrightarrow{B} H_1(E(J); \mathcal{R}) \to H_1(E(J); \mathcal{K}\Gamma).
\]

From Lemma \[3.3.2\] we have that \( H_1(E(J); \mathcal{R}) \cong H_1(E(J); \mathbb{Z}[t, t^{-1}] \otimes \mathbb{Z}[t, t^{-1}] \mathcal{R} \). Since \( H_1(E(J); \mathbb{Z}[t, t^{-1}]) = \mathcal{A}_0(J) \) is annihilated by the Alexander polynomial, it follows that \( H_1(E(J); \mathcal{R}) \) is a torsion module. Hence \( H_1(E(J); \mathcal{K}\Gamma) = 0 \), and by Poincaré duality, \( H_2(E(J); \mathcal{K}\Gamma) = 0 \). Therefore the map \( B \) above is an isomorphism. We define the linking form \( B^\ell_R(J) \) to be the composition of the following maps:

\[
H_1(E(J); \mathcal{R}) \xrightarrow{B^{-1}} H_2(E(J); \mathcal{K}\Gamma/\mathcal{R}) \xrightarrow{P.D.} H^1(E(J), \partial E(J); \mathcal{K}\Gamma/\mathcal{R}) \xrightarrow{\pi^*} H^1(E(J); \mathcal{K}\Gamma/\mathcal{R}) \xrightarrow{\zeta} \mathcal{H}_1(E(J); \mathcal{R})^\#,
\]

where \( P.D. \) is the Poincaré duality isomorphism, \( \pi^* \) is the map in the long exact sequence of a pair and \( \zeta \) is the Kronecker evaluation map. \( \square \)
We now show that $B_\ell\mathcal{R}(J)$ is determined by the classical Blanchfield linking form on $J$. In the proof of Lemma 3.2, we considered the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Z}\pi_1(E(J)) & \rightarrow & \mathcal{R} \\
\downarrow & \swarrow_{\psi} & \\
\mathbb{Z}[t, t^{-1}] & & \\
\end{array}
$$

Here $\psi : t \mapsto \phi(\eta)$. If $\phi(\eta) \neq 1$, then $\psi$ and $\overline{\psi} : \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}] \rightarrow \mathcal{K}/\mathcal{R}$ are monomorphisms. Furthermore we have a map $\psi_* : A_0(J) = H_1(E(J); \mathbb{Z}[t, t^{-1}]) \rightarrow H_1(E(J); \mathcal{R})$.

**Proposition 3.6.** If $\phi(\eta) \neq 1$, then for all $x, y \in A_0(J)$,

$$
B_\ell\mathcal{R}(J)(\psi_*(x), \psi_*(y)) = \overline{\psi}(B_\ell_0(J)(x, y)),
$$

where $B_\ell_0(J)$ is the classical Blanchfield linking form on $J$.

**Proof.** The classical Blanchfield linking form on $J$ is the composition of the following maps:

$$
H_1(E(J); \mathbb{Z}[t, t^{-1}]) \xrightarrow{B_\ell^{-1}} H_2(E(J); \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]) \xrightarrow{P.D.} H^1(E(J); \mathcal{K}/\mathcal{R}) \xrightarrow{\pi^*} H^1(E(J); \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]) \xrightarrow{\kappa} \text{Hom}_{\mathbb{Z}[t, t^{-1}]}(H_1(E(J); \mathbb{Z}[t, t^{-1}]), \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]),
$$

where $P.D.$ is the Poincaré duality isomorphism, $\pi^*$ is the map in the long exact sequence of a pair and $\kappa$ is the Kronecker evaluation map.
We have the following commutative diagram:

\[
\begin{array}{ccc}
H_1(E(J); \mathbb{Z}[t,t^{-1}]) & \xrightarrow{\psi_*} & H_1(E(J); \mathcal{R}) \\
B^{-1} & & B^{-1} \\
H_2(E(J); \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]) & \xrightarrow{\bar{\psi}_*} & H_2(E(J); \mathcal{K}/\mathcal{R}) \\
P.D. & & P.D. \\
H^1(E(J), \partial E(J); \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]) & \xrightarrow{\bar{\psi}_*} & H^1(E(J), \partial E(J); \mathcal{K}/\mathcal{R}) \\
\pi^* & & \pi^* \\
\kappa & & \kappa \\
\text{Hom}_{\mathbb{Z}[t,t^{-1}]}(H_1(E(J); \mathbb{Z}[t,t^{-1}]), \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]) & \xrightarrow{\bar{\psi}_#} & \text{Hom}_\mathcal{R}(H_1(E(J); \mathbb{Z}[t,t^{-1}]), \mathcal{K}/\mathcal{R}) \\
& & \xrightarrow{\psi^*} \\
& & \text{Hom}_\mathcal{R}(H_1(E(J); \mathbb{Z}[t,t^{-1}]), \mathcal{K}/\mathcal{R})
\end{array}
\]

The composition of maps in the left column is the classical Blanchfield linking form $\mathcal{B}_0(J)$, and in the right column is $\mathcal{B}_R(J)$.

Since the diagram commutes, $\psi^* \circ \mathcal{B}_R(J) \circ \psi_* = \bar{\psi}_# \circ \mathcal{B}_0(J)$. Evaluating these maps on $x, y \in \mathcal{A}_0(J)$, gives the desired result. \( \square \)

We now show the relationship between the linking forms $\mathcal{B}_R(M)$ and $\mathcal{B}_R(M(\eta, J))$ when $\phi(\eta) \neq 1$. In this case, it follows from Corollary 3.3 that the following is a split short exact sequence:

\[
H_1(E(J); \mathcal{R}) \xrightarrow{i_\ast} TH_1(M(\eta, J); \mathcal{R}) \xrightarrow{\iota_T} TH_1(M; \mathcal{R}).
\]

If we choose a splitting $g$, we have the following theorem that relates $\mathcal{B}_R(M(\eta, J))$, $\mathcal{B}_R(M)$, and $\mathcal{B}_R(J)$.

**Theorem 3.7.** If $\phi(\eta) \neq 1$, then $\mathcal{B}_R(M(\eta, J)) \cong \mathcal{B}_R(M) \oplus \mathcal{B}_R(J)$. That is, for any $x_1, y_1 \in TH_1(M; \mathcal{R})$ and $x_2, y_2 \in H_1(E(J); \mathcal{R})$,

\[
\mathcal{B}_R(M)(x_1, y_1) + \mathcal{B}_R(J)(x_2, y_2) = \mathcal{B}_R(M(\eta, J)) (g(x_1) + i_\ast(x_2), g(y_1) + i_\ast(y_2)).
\]
Before giving the proof, we state a corollary that follows immediately from Proposition 3.6 and Theorem 3.7.

**Corollary 3.8.** If \( \phi(\eta) \neq 1 \), then for any \( x_1, y_1 \in TH_1(M; R) \) and \( x_2, y_2 \in A_0(J) \),

\[
\mathcal{B}_\ell(R)(x_1, y_1) + \psi(\mathcal{B}_\ell(0)(x_2, y_2)) = \mathcal{B}_\ell(R)(\eta, J)(g(x_1) + i_*({\psi}_*(x_2)), g(y_1) + i_*({\psi}_*(y_1)))
\]

From Corollary 3.3, we know that every element in \( TH_1(M(\eta, J); R) \) can be written as \( g(x_1) + i_*({\psi}_*(x_2)) \) for some \( x_1 \in TH_1(M; R) \) and \( x_2 \in A_0(J) \). Hence the corollary above shows that the linking form on \( M(\eta, J) \) is completely determined by the linking form on \( M \) and the classical Blanchfield linking form on \( J \). We now prove Theorem 3.7.

**Proof.** We have the following diagram.

\[
\begin{array}{ccc}
H_1(E(J); R) & \xrightarrow{i_*} & TH_1(M(\eta, J); R) \\
\mathcal{B}_\ell(J) & & \mathcal{B}_\ell(M(\eta, J)) \\
H_1(E(J); R)^\# & \xrightarrow{i^*} & TH_1(M(\eta, J); R)^\# \\
\end{array}
\]

where \( g^\# \) is the dual of \( g \). Notice that since \( f_* \circ g = \text{id} \), it follows that \( g^\# \circ f^* = \text{id} \). The isomorphism in the theorem will be given by \( i_* \oplus g \). Hence the theorem will follow from the following four claims.

1. \( i^* \circ \mathcal{B}_\ell(M(\eta, J)) \circ i_* = \mathcal{B}_\ell(J) \) which establishes:

\[
\mathcal{B}_\ell(M(\eta, J))(i_*(x_1), i_*(y_1)) = \mathcal{B}_\ell(J)(x_1, y_1).
\]

2. \( g^\# \circ \mathcal{B}_\ell(M(\eta, J)) \circ g = \mathcal{B}_\ell(M) \) which establishes:

\[
\mathcal{B}_\ell(M(\eta, J))(g(x_2), g(y_2)) = \mathcal{B}_\ell(M)(x_2, y_2).
\]

3. \( g^\# \circ \mathcal{B}_\ell(M(\eta, J)) \circ i_* = 0 \) which establishes:

\[
\mathcal{B}_\ell(M(\eta, J))(i_*(x_1), g(y_2)) = 0.
\]

4. \( i^* \circ \mathcal{B}_\ell(M(\eta, J)) \circ g = 0 \) which establishes:

\[
\mathcal{B}_\ell(M(\eta, J))(g(x_2), i_*(y_1)) = 0.
\]
The first claim follows immediately from the following commutative diagram.
To prove the second claim, we consider the following commutative diagram.

$$
\begin{align*}
H_2(M(\eta,J);\mathcal{K}/\mathcal{R}) & \xrightarrow{f_*} H_2(M;\mathcal{K}/\mathcal{R}) \\
P.D. & \xrightarrow{B} \xleftarrow{B} P.D. \\
H^1(M(\eta,J);\mathcal{K}/\mathcal{R}) & \xrightarrow{f^*} H^1(M;\mathcal{K}/\mathcal{R}) \\
\kappa & \xrightarrow{\ell} TH_1(M(\eta,J);\mathcal{R}) \\
(H_1(M(\eta,J);\mathcal{R}))^# & \xrightarrow{f^*} (H_1(M;\mathcal{R}))^# \\
j^# & \xrightarrow{\ell} B\ell_{\mathcal{R}}(M(\eta,J)) \\
(TH_1(M(\eta,J);\mathcal{R}))^# & \xrightarrow{f^*} (TH_1(M;\mathcal{R}))^#
\end{align*}
$$

From the diagram above we have $f^* \circ B\ell_{\mathcal{R}}(M) \circ f_* = B\ell_{\mathcal{R}}(M(\eta,J))$. Therefore,

$$
g^# \circ f^* \circ B\ell_{\mathcal{R}}(M) \circ f_* \circ g = g^# \circ B\ell_{\mathcal{R}}(M(\eta,J)) \circ g.
$$

Since $f_* \circ g = \text{id}$ and $g^# \circ f^* = \text{id}$, it follows that $g^# \circ B\ell_{\mathcal{R}}(M(\eta,J)) \circ g = B\ell_{\mathcal{R}}(M)$. Hence the second claim is proved.

We have established that we have the following commutative diagram whose rows are exact.

$$
\begin{align*}
H_1(E(\eta,J);\mathcal{R}) & \xrightarrow{i_*} TH_1(M(\eta,J);\mathcal{R}) \\
\xrightarrow{f_*} TH_1(M;\mathcal{R}) \\
\xrightarrow{f^*} TH_1(M;\mathcal{R})^#
\end{align*}
$$

Since $f^* \circ B\ell_{\mathcal{R}}(M) \circ f_* = B\ell_{\mathcal{R}}(M(\eta,J))$, it follows that

$$
g^# \circ B\ell_{\mathcal{R}}(M(\eta,J)) \circ i_* = g^# \circ f^* \circ B\ell_{\mathcal{R}}(M) \circ f_* \circ i_*
$$

$$
i^* \circ B\ell_{\mathcal{R}}(M(\eta,J)) \circ g = i^* \circ f^* \circ B\ell_{\mathcal{R}}(M) \circ f_* \circ g
$$

But since the rows are exact, $f_* \circ i_* = 0$ and $i^* \circ f^* = 0$. Therefore $g^# \circ B\ell_{\mathcal{R}}(M(\eta,J)) \circ i_* = 0$ and $i^* \circ B\ell_{\mathcal{R}}(M(\eta,J)) \circ g = 0$. \hfill \boxed

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