Assume that an $N$-bit sequence $S$ of $k$ self-delimiting numbers is given as input. We present space-efficient algorithms for sorting, dense ranking and (competitive) ranking $S$ on the word RAM model with word size $\Omega(\log N)$. Our algorithms run in $O(k + \frac{N}{\log N})$ time and use $O(N)$ bits. The sorting algorithm returns the given numbers in sorted order stored within a bit-vector of $N$ bits whereas our ranking algorithms construct data structures that allows us subsequently to return the (dense) rank of each number $x$ in $S$ in constant time if the position of $x$ in $S$ is given together with $x$.

As an application of our algorithms we give an algorithm for tree isomorphism that runs in $O(n)$ time and uses $O(n)$ bits on $n$-node trees.

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1 Introduction

Collecting and analyzing data became an important factor for economical and strategic decisions of many businesses. The most successful business have figured out, better than their comparatives, how to collect and process data more efficiently. With increasing data not only the running time, but also the space consumption of algorithms becomes important.

The motivation to focus on both space efficiency and time efficiency comes from the observation that growing inputs let algorithms fail the execution because they run out of space, or are slowed down because of cache-faults. A cache-fault occurs when the operating system runs out of fast memory and starts swapping, i.e., it moves part of the data to a slower memory. Therefore we focus on space-efficient algorithms, i.e., algorithms that run (almost) as fast as standard solutions for the problem under consideration, but use less space.

Model of Computation. Our model of computation is the word RAM, where we assume to have the standard operations to read, write as well as arithmetic operations (addition, subtraction, multiplication, modulo, bit shift, AND and OR) take constant time on words of size $w = \Omega(\log N)$ bits where $N$ is the input size in bits. The model has three types of memory. A read-only input memory where the input instances are stored. A read-write working memory that the algorithm uses to compute its result and a write-only output memory that the algorithm uses to output its result. As usual, the space bounds stated for an algorithm or a data structure are the space bounds of their working memory.
Sorting. Sorting is a problem that appears in several applications and has been studied for decades. Usually sorting problems are classified into different categories. In comparison sorting two elements of an input sequence must be compared against each other in order to decide if one comes before another. Pagter and Rauhe [28] gave a comparison-based algorithm that runs on input sequences of \( k \) elements in \( O(k^2/s) \) time by using \( O(s) \) bits for every given \( s \) with \( \log k \leq s \leq k/\log k \). Beame [15] presented a matching lower bound in the so-called branching-program model. Another type of sorting is integer sorting where a sequence of \( k \) integers, each in the range \([0, m) = \{0, \ldots, m-1\}\), has to be sorted. It is known that, for \( m = k^{O(1)} \), integer sorting can be done in linear time: consider the numbers as \( k \)-ary numbers, sort the digits of the numbers in rounds (radix sort) and count the occurrences of a digit by exploiting indirect addressing (counting sort). Pagh and Pagter showed optimal time-space trade-offs for integer sorting. Moreover, Han showed that real sorting (the given sequence consists of real numbers) can be converted in \( O(k\sqrt{\log k}) \) time into integers and then can be sorted in \( O(k\sqrt{\log \log k}) \) time [17].

These algorithms above all assume that the numbers are represented with the same amount of bits. We consider a special case of integer sorting that appears in the field of space-efficient algorithms where numbers are often represented as so-called self-delimiting numbers to lower their total memory usage. A self-delimiting number can be represented in several ways. We use the following easy representation, but our results can easily be adapted to other ones: an integer \( x \in \mathbb{N} \) is denoted by the bit sequence \( 1^\ell \delta(x) \) where \( \ell = \lceil \log x \rceil \) is the length of \( \delta(x) \) and \( \delta(x) \) is the binary representation of \( x \in \mathbb{N} \) without leading zeros. E.g., the self-delimiting numbers of 1, 2, 3, 4 are 101, 11010, 11011, 1110100, respectively.

Assume that \( k \) self-delimiting numbers in the range \([0, m) \) with \( m \leq 2^N \) stored in an \( N \)-bit sequence is given. If the memory is unbounded, then we can simply transform the numbers into integers, use a standard sorting algorithm and transform the sorted numbers back into self-delimiting numbers. However, this approach uses \( \Omega(k \log m) \) bits. For \( k \approx N \approx m \), this is to large to be considered space-efficient. We present a sorting algorithm for self-delimiting numbers that runs in \( O(k + \frac{N}{\log N}) \) time and uses \( O(N) \) bits.

Ranking. A common used rank variation is dense rank where a sequence \( S \) of numbers is given and a mapping \( R \) must be constructed such that, for each \( x \in S \), the dense rank \( R[x] \) is the number of different elements in \( S \) that are smaller than \( x \). E.g., the dense ranks of the numbers of the sequence 6, 9, 2, 2, 0 are 2, 3, 1, 1, 0, respectively. Another variant is competitive rank or just rank, where a sequence \( S \) of numbers is given and a mapping \( R \) must be constructed such that for each \( x \in S \) the rank \( R[x] \) is the number of elements in \( S \) that are smaller than \( x \). E.g., the ranks of the numbers in the sequence 6, 9, 2, 2, 0 are 3, 4, 1, 1, 0.

Without space limitation the rank for a sequence of numbers is easy computable by using a table to count the occurrence of each number in a given sequence and then computing a prefix sum over the entries of the table. However, this approach uses \( \Theta(m \log m) \) bits. We show how to compute rank on a sequence \( S \) of length \( N \) consisting of \( k \) self-delimiting numbers in \( O(k + \frac{N}{\log N}) \) time using \( O(N) \) bits and answer rank queries of a number \( x \) in constant time if the position of \( x \) in \( S \) is given with \( x \).

Graph Algorithms and Tree Isomorphism. In the last five years, several space-efficient algorithms are published. Depth-first search and breadth-first search are the first problems that were considered [2] [3] [13] [11]. Further papers with focus on space-efficient algorithms discuss graph interfaces [3] [10], connectivity problems [5] [9] [14], matching [11] and vertex separators [22]. Several of these results are implemented in an open source GitHub project [24].

As an applications of our new sorting and ranking algorithm, we show a space-efficient
isomorphism algorithm for trees. Graph isomorphism is a problem to decide if there is a bijection between the vertices of two graphs that preserves edges. The problem on general graphs is one of few standard problems in computational complexity theory where it is not known if it belongs to $P$ or if it is $NP$-complete. However, there are a lot of results concerning special graph instances like planar graphs for which isomorphism is known to be in $P$. Weinberg [29] showed that $O(n^2)$ time suffices for 3-connected planar graphs. Hopcroft and Tarjan [19] extended the result for general planar graphs and showed an $O(n \log n)$ time algorithm. That result was improved to $O(n)$ time by Hopcroft and Wong, but with a large constant [20]. Our algorithm for tree isomorphism is based on a tree-isomorphism algorithm described in the textbook of Aho, Hopcroft and Ullman [1], which uses $\Omega(n \log n)$ bits. We improve the space-bound to $O(n)$ bits while maintaining the linear running time. Lindell [26] showed that tree isomorphism is in $L$; however, the algorithm presented has running time of $\mathcal{T}(n) \geq 4\mathcal{T}(n/2)$ because, in one subcase of his algorithm, two subtrees with $n/2$ vertices of each of the given trees have to be cross compared recursively with each other. Hence the running time is in $\Omega(n^2)$.

Outline. We continue our paper by introducing definitions and notations in Section 2. Our results on sorting and (dense) ranking are shown in Section 3. We conclude our paper by presenting a space-efficient tree-isomorphism algorithm. Several proofs are moved into the appendix.

2 Preliminaries

In this paper we use basic graph and tree terminology as given in [10]. We use several concepts of trees in our algorithms. The most basic concept is that a tree is a connected graph without cycles. By choosing any vertex of such a graph as a root the tree becomes a rooted tree. If the nodes of a tree have labels the tree is called labeled tree, otherwise (or if not of interest), the tree is called unlabelled tree. By fixing the order of the children of each node in a rooted tree and assigning consecutive numbers to the nodes by a pre-order traversal the tree becomes an ordinal tree.

We denote by $\deg(v)$ the degree of a node or vertex $v$, i.e., number of neighbors of $v$, and by $\text{desc}(u)$ the number of descendants of a node $u$ in a tree. The height of $u$ is defined as the number of edges between $u$ and the longest path to a descendant leaf. The depth of $u$ is defined as the number of edges between the node and the root. There is a compact representation for ordinal trees consisting of $2n$ bits.

Definition 2.1. (balanced parenthesis representation of an ordinal tree) Every node $u$ is represented by an open parenthesis, followed by the parenthesis representation of every child of $u$, and finally a closed parenthesis.

For this representation Munro and Raman [27] showed a succinct data structure that allows us constant time tree navigation (parent, child, left/right sibling) by supporting the following three operations.

Lemma 2.2. For any balanced parenthesis representation of length $n$ there is an auxiliary structure on top of the representation that consists of $o(n)$ bits and provides, after $O(n)$ initialization time, the following operations in constant time:
(a) $\text{findclose}(i)$ ($i \in \mathbb{N}$): Returns the position of the closing parenthesis matching the open parenthesis at position $i$.
(b) $\text{findopen}(i)$ ($i \in \mathbb{N}$): Returns the position of the open parenthesis matching the closed parenthesis at position $i$. 

(c) \textit{enclose}(i) (i \in \mathbb{N}): Returns the position of the closest open parenthesis part of the parenthesis pair enclosing the open parenthesis at position $i$.

A parenthesis representation is usually stored as a bit sequence where the open parenthesis is represented via a 1 and a closed parenthesis via a 0, or vice-versa. Moreover, using a rank-select structure on that sequence a bidirectional mapping between each node $u$ of an ordinal tree $T'$ and the index of $u$’s open parenthesis can be created.

\textbf{Lemma 2.3. (rank-select \cite{5})} Given access to an $n$ bit sequence $B = (b_1, \ldots, b_n) = \{0, 1\}^n$ ($n \in \mathbb{N}$) there is an $o(n)$-bit rank-select data structure that, after an initialization of $O(n/w)$ time, supports two constant-time operations: \texttt{rank}_B(j) = \sum_{i=1}^{j} b_i (j \in \{1, \ldots, n\}) that returns the number of ones in $(b_1, \ldots, b_j)$ in $O(1)$ time, and \texttt{select}_B(k) = \min\{j \in \{1, \ldots, n\} : \texttt{rank}_B(j) = k\}$ that returns the $k$th position of the $k$th one in $B$.

By combining the techniques above they showed the following Lemma (sketched in Fig. 1).

\textbf{Lemma 2.4.} Given an $n$-node tree $T$ and a root $r$ of $T$ there is an algorithm that computes a data structure representing an ordinal tree $T'$ in $O(n)$ time using $O(n)$ bits such that $T$ and $T'$ are isomorphic. The data structure allows tree navigation in constant time, i.e., given a node $u$ of $T'$ it can return parent, firstChild, leftSibling and rightSibling in constant time.

![Figure 1] The ordinal tree (b) that is isomorphic to the rooted tree (a) can be represented via a balanced parenthesis representation (table below of (a) and (b)) that is computed by executing a DFS in pre-order on (a). The parenthesis pairs of each node are connected by a line and the index of each open parenthesis represent the label of that node in (b).

We often have to operate on sets (of nodes) and also iterate over the elements of these sets. To manage the sets using $O(n)$ bits we use \textit{(uncolored) choice dictionaries} \cite{15,23}.

\textbf{Definition 2.5.} \textit{(uncolored) choice dictionary} Initialized with some parameter $n$ there is a data structure that stores a subset $U'$ out of a universe $U = \{0, \ldots, n-1\}$ and supports the standard dictionary operations add, remove and contains. Moreover, it provides an operation \texttt{choice} that returns an arbitrary element of $U'$. Initialization and all other operation run in $O(1)$ time.

Note, that the \texttt{choice} operation can be used to iterate over the elements stored in the choice dictionary by removing every element that \texttt{choice} returns.

Throughout the paper we use the following two graph isomorphism definitions.
Definition 2.6. (graph isomorphism) Two given graphs \( G = (V_G, E_G) \) and \( H = (V_H, E_H) \) are isomorphic exactly if there exists a bijective mapping \( f : V_G \rightarrow V_H \) such that vertices \( u, v \in V_G \) are adjacent in \( G \) if and only if \( f(u), f(v) \in V_H \) are adjacent in \( H \). Then \( f \) is called an isomorphic mapping.

Buss [7] shows that there is an equivalent definition for the special case that the graphs are rooted trees.

Definition 2.7. (rooted tree isomorphism) By induction two rooted trees \( T \) and \( T' \) are isomorphic if and only if
(a) \( T \) and \( T' \) consist of only one node, or
(b) the roots \( r \) and \( r' \) of \( T \) and \( T' \), respectively, have the same number \( m \) of children, and there is some ordering \( T_1, \ldots, T_m \) of the maximal subtrees below the children of \( r \) and some ordering \( T'_1, \ldots, T'_m \) of the maximal subtrees below the children of \( r' \) such that \( T_i \) and \( T'_i \) are isomorphic for all \( 1 \leq i \leq m \).

3 Sorting and Ranking

In this section we consider sorting and ranking of \( k \) self-delimiting numbers, stored within an \( N \)-bit sequence \( S \). The self-delimiting numbers can be stored one after the other, but there can also be several 0s between them. By using the word-RAM operations, we can scan through \( S \) in \( O(\frac{N}{w}) \) time and identify all self-delimiting numbers.

As a warmup, we start showing an easy stable sorting algorithm for self-delimiting numbers. Our idea is to run first an adaptation of counting sort to presort the numbers in several areas such that each area consists of numbers that require an equal amount of bits. We then sort each area independently by a stable-counting sort algorithm. In our paper \( \log x \) is the binary logarithm \( \log_2 x \).

Lemma 3.1. Given an \( N \)-bit sequence \( S \) consisting of \( k \) self-delimiting numbers, each in the range \([0, m)\) \((m \leq 2^{N/\log N})\), there is an \( O(k + n/w) = O(k + \frac{N}{\log N}) \)-time \( O(N)\)-bit stable-sorting algorithm computing a bit sequence of \( N \) bits that stores the given self-delimiting numbers in sorted order.

Proof. Let \( f(x) = 3 \) for \( x \leq 1 \) and \( f(x) = 2 \lfloor \log x \rfloor + 1 \) for \( x > 1 \) be the number of bits a number \( x \) requires as a self-delimiting number. We start to describe how we presort the numbers into areas where each area contains only numbers that require the same amount of bits to be represented. Create a bit-vector \( C \) with \( f(m) \) entries of \( \lfloor \log N \rfloor \) bits each. We use \( C[i] \) to count how many numbers in \( S \) require \( i \) bits as a self-delimiting number. For that, iterate over the self-delimiting numbers in \( S \) and for each number \( x \in S \) increment \( C[f(x)] \) by one.

We now determine the boundary of each area and store it in an array \( B \), of \( f(m) \) fields of \( \lfloor \log N \rfloor \) bit each. Initialize \( B \) by setting \( B[0] = 0 \) and \( B[i + 1] = B[i] + iC[i] \) with \( i = 0, \ldots, f(x) - 1 \). After computing the boundaries we use them to move the self-delimiting numbers into a new \( N \)-bit sequence \( S' \) as follows. Scan through \( S \) and for each self-delimiting number \( x \), store \( x \) in \( S' \) starting from position \( B[f(x)] \) and increment \( B[f(x)] \) by \( f(x) \). Note that the reordering of the numbers is stable. To sort the numbers inside one area we use a folklore technique. We consider each self-delimiting number of a number \( x \) in an \( 2^w \)-ary system so that it is represented by \( \lfloor f(x)/2^w \rfloor \) digits and use radix sort in combination with a stable counting sort algorithm.
For the following note that \( N > k \) is always true. The bit-vectors \( C \) and \( B \) use \( O(f(m) \log N) = O(N) \) bits, \( S' \) uses \( O(N) \) bits and scanning through \( S \), while computing and filling \( C \) and \( B \) can be done in \( O(k + N/w) \) time. (Roughly speaking, we can scan \( S \) in \( O(N/w) \) time and handle each number separately costs \( O(k) \) time.)

Let \( N_i \) be the number of bits that the \( i \)th area require to store its self-delimiting numbers in \( S' \) and note that \( f(x) = i \) is true for each number \( x \) of the \( i \)th area. The \( i \)th area can contain \( O(N_i/i) \) numbers that can be sorted by radix sort in linear time per number if the numbers are within \([0, \sqrt{N}]\). Larger numbers have to be sorted digit-wise. Hence the running time to sort the \( i \)th area is \( O(N_i/i \log N) = O(N) \) and the total time to sort \( S' \) is \( O(\sum_i N_i/i \log N) = O(N \log N) \). In total, the stable sorting algorithm runs in \( O(N \log N) \) time and uses \( O(N) \) bits.

In our later applications we have numbers in the range \([0, m)\) with \( m \leq 2^N \). Therefore we need to improve our lemma to handle such large numbers. Let \( q = 2^{N/\log N} \). The idea is to put the numbers of \( S \) that are at most \( q \) into a sequence \( S_{\leq q} \) and put the remaining \( O(\frac{N}{\log N}) \) numbers of \( S \) that are greater than \( q \) into a compressed trie \( T_{>q} \). We interpret each number as a string of the alphabet \( \Sigma = \{0, 1\} \). \( S_{\leq q} \) is sorted by Lemma 3.1 and written in an \( N \)-bit sequence \( S' \). Observe that the constructed trie of \( \log N \) entries is a binary tree. Using a DFS on the root of \( T_{>q} \) that prioritizes edges with a 0 over edges with a 1 and append the visited leaves to the sequence \( S' \) computed from Lemma 3.1.

We showed our final result on sorting.

\textbf{Theorem 3.2.} Given an \( N \)-bit sequence \( S \) of \( k \) self-delimiting numbers, there is an \( O(k + N/w) = O(k + \frac{N}{\log N}) \)-time \( O(N) \)-bit stable-sorting algorithm computing a bit sequence of \( N \) bits that stores the given self-delimiting numbers in sorted order.

\textbf{Proof.} It remains to show the running time and space bound of the algorithm above. Recall that \( N > k \) is always true. Since, \( S \) can contain only \( O(\frac{N}{\log q}) = O(\log N) \) numbers that are greater than \( q \). Using the word-RAM operations we can fill \( T_{>q} \) and read it with a DFS in \( O(k + N/w) \) time using \( O(N) \) bits. Sorting \( S_{\leq q} \) costs us \( O(k + N/w) \) time and \( O(N) \) bits. Therefore, the construction of the \( N \)-bit sequence \( S' \) runs in \( O(k + N/w) \) time and uses \( O(N) \) bits.

We now consider ranking of \( k \) self-delimiting numbers. For large numbers, we can use a similar trick as for sorting. Therefore, for the time being, let us assume that the numbers are in the range \([0, m)\) (\( m \leq N \)). A standard approach to compute the (dense) rank is to first sort \( S \) and then to use an array \( P \) of \( \lceil \log n \rceil \) bits, to store a prefix sum over the occurrences of (different) numbers \( x \in S \), i.e., in a first step set \( P[x] = P[x] + 1 \) (for dense rank, set \( P[x] = 1 \) for each \( x \in S \), and in a second step compute the prefix sums on \( P \), i.e., for each \( i = 1, \ldots, m - 2 \), set \( P[i] = P[i - 1] + P[i] \). The rank of a number \( x \) is then \( P[x] \). However, array \( P \) uses \( \Omega(m \log k) \) bits.

To compute the dense rank with less space, we can use a bit-vector \( B \) of \( m \) bits and set \( B[x] = 1 \) for each \( x \in S \). Then, using a rank-select data structure on \( B \), the rank of \( x \) is \( \text{rank}_B(x) \). This approach uses \( O(m) \) bits and takes \( O(m/w) \) time due to the initialization of a rank-select data structure on \( B \).

Since we are looking for an \( O(k + N/r) \)-time solution for some integer \( r \) with \( \log N/2 \leq r \leq w/2 \), we build our own data structure that can return the dense rank of \( x \), restricted to \( x \in S \), in \( O(1) \) time. Our approach is similar to the realization of rank in \([12, 21]\), but we leave several values uninitialized. We construct the rank structure not on a given bit-vector, but on a given sequence consisting of \( n \) numbers that correspond to the ones
in the bit-vector. Furthermore, we partition $B$ into $O(m/r)$ frames of $r$ bits and create an array $P$ that contains the prefix sum of the frames up to the $i$th frame ($i = 0, \ldots, \lceil m/r \rceil$). Subsequently we use a lookup table `POPCNT` that allows to determine the number of ones in the binary representation of each frame. The table `POPCNT` can be easily constructed in time $O(2^r r)$ as follows: for each $z = 0, \ldots, 2^r - 1$ set $\text{POPCNT}[z] = y$, where $y$ is the number of bits for $z$ computed in $O(r)$ time by iterating over the bits of $z$’s binary representation.

A practical alternative to a lookup table is the CPU operation `POPCNT` (available in AMD and Intel processors since 2008) that returns the number of bits set to one within a computer word. For an example, see Fig. 2. For the following lemma we now give up our restriction on $m$.

**Theorem 3.3.** Given a sequence $S$ of $k$ self-delimiting numbers stored in $N$ total bits, an integer $r$ with $\log N/2 \leq r \leq w/2$ and access to the lookup-table `POPCNT` for $r$-long bit-vectors, we can compute a data structure realizing a mapping $R : (N, S) \rightarrow N$ where $R(p, x)$ is the dense rank of a number $x \in S$ and $p$ is the position in $S$ of the first bit used by the self-delimiting number of $x$. The data structure is constructed in $O(k + N/r) = O(k + \frac{N}{\log N})$ time and uses $O(N)$ bits.

**Proof.** We split the sequence $S$ of self-delimiting numbers by putting each number $x \in S$ with $x \leq N$ into a sequence $S_{\leq N}$ and each number $x \in S$ with $x > N$ into a sequence $S_{>N}$. Determine the rank for elements in the sequence $S_{\leq N}$ as described in the paragraphs below. For $S_{>N}$ proceed as follows. Create a bit vector $Q$ of length $O(N)$ with zeros in constant time. Intuitively speaking, if $Q$ is below $S$, the rank of a number $x$ in $S_{>N}$ is written in $Q$ below $x \in S$. In detail our sorting algorithm sorts $S_{>N}$ such that it also moves the positions $p(x)$ with the numbers in $S_{>N}$, i.e., we sort the tuples $(x, p(x))$ after the first component.

Now we determine the rank for each number $x \in S_{>N}$ and use the position $p(x)$ to write the rank in an $N$ bit vector $Q$ at position $p(x)$.

Note that the rank of the number and their positions are at most $N$ so that storing them uses less space than the number itself. In other words, the space for storing several ranks and positions in $Q$ and $I$, respectively, does not overlap. If afterwards a dense rank is queried, check first if it is a number greater than $N$. If so, use $Q$ to answer the query and, else, use the description below.

We start to describe some auxiliary data structures. Let $B$ be a bit-vector of $N$ bits containing the values of $S$, i.e., $B[x] = 1$ for every $x \in S$. Partition $B$ into frames of $r$ bits each. Let $P$ be an array of $\lceil N/r \rceil + 1$ entries, each of $\lceil r \rceil$ bits, that allows us to store a prefix sum over the bits inside the frames of $B$, i.e., $P[0] = 0$ and $P[i + 1] (i = 0, \ldots, \lceil N/r \rceil)$ is the number of ones within $B[0, \ldots, ir]$.

We now show how to construct $B$ and $P$. Apply Theorem 3.2 on $S$ to get a sorted sequence $S' = \{x_0, \ldots, x_{k-1}\}$. Let $j$ (initially $j = 0$) be a counter that we use to count
different numbers in $S'$. Initially set $P[0] = 0$. Now for each $x_i$ with $i = 0, \ldots, k - 1$ do the following: if and only if $B[x_i] = 0$ and increment $j$ by one. Moreover, set $B[x_i] = 1$ and $P[(x_i/r) + 1] = j$.

Having $B$, $P$ and access to POPCNT we can answer the rank of a number $x \in S$ by returning $P[i] + \text{POPCNT}[B[\lceil r \cdot (q+1)r - 1\rceil \& (q \ll (r-p) - 1)]$ with $q = \lfloor x/r \rceil - 1$ and $p = x \mod r$. Initializing the memory of $B$ can be done in constant time by using a succinct data structure from Katoh and Goto [25]. Afterwards, sorting $S$ into $S'$ (Theorem 3.2) and computing $B$ and $P$ can be done in $O(k + N/r)$ time. Both, $B$ and $P$ use $O(N)$ bits and computing $S'$ requires $O(N)$ bits. In total, we use $O(N)$ bits.

To compute the competitive rank we require the information of how many times an element appears in the given sequence. We change our approach of the previous lemma as follows. As before, let $r$ be an integer with $\log N/2 \leq r \leq w/2$. We sort $S$ to get a sorted sequence $S'$. Next, we want to partition $S'$ into regions such that the $i$th region $R_i = S' \cap [ir, \ldots, (i + 1)r]$ for all $i = 0, \ldots, 2w/m - 1$. In detail, we go through $S'$ and store for each non-empty region $R_i$ a pointer $F[i]$ to a sequence $A_i$ of occurrences of each number $x \in S'$ written as self-delimiting numbers. Similar to the usage of $B$ for dense rank (Fig. 2), to solve rank we partition $A_i$ into frames of $r$ bits and we compute an array $P_i$ storing the prefix-sums. More exactly, $P_i[j]$ stores the prefix-sum over all self-delimiting numbers in $S'$ up to the $j$th frame in $A_i$. Fig. 3 sketches an example.

By using static-space allocation we can access each number in $A_i$ in constant time. With access to a lookup table $\text{PREFIXSUM}$ we can compute the prefix sum of the numbers within a frame in constant time. The table $\text{PREFIXSUM}$ can be constructed in time $O(2^r)$ as follows: for each $e = 0, \ldots, 2^r - 1$ read $e$ as a sequence of self-delimiting numbers, sum up their values in a variable $s$, and set $\text{PREFIXSUM}[e] = s$.

We so obtain the next theorem.

\begin{figure}[h]
\centering
\begin{tabular}{c|c|c|c|c|c|c|c|c|c|c}
\hline
$S'$ & (2) & (2) & (2) & (2) & (2) & (2) & (2) & (5) & (5) & (5) & (6) & (8) & (8) & (8) & (256) & (300) & (300) & (300) & (1012) & (1012) \\
\hline
$P$ & 1 & 8 & 8 & 10 & 10 & 10 & & & & & & & & & & & & & & & \\
\hline
$A_1$ & (6) & (3) & (1) & (4) & & & & & & & & & & & & & & & & & & & \\
$A_4$ & (1) & (3) & & & & & & & & & & & & & & & & & & & \\
$A_9$ & (2) & & & & & & & & & & & & & & & & & & & \\
$A_{10}$ & (2) & & & & & & & & & & & & & & & & & & & \\
$P_1$ & 9 & 15 & & & & & & & & & & & & & & & & & & & \\
$P_4$ & 19 & & & & & & & & & & & & & & & & & & & \\
$P_9$ & 21 & & & & & & & & & & & & & & & & & & & \\
\hline
\end{tabular}
\caption{A sketch of our storage schema to realize rank. For each region $R_i$, that contains numbers out of $S'$, a pointer $F[i]$ points to a data structure storing the amount of occurrences for each of the numbers using static space allocation. In addition $P_i$ stores the prefix-sum over the frames up to $A_i$.}
\end{figure}

\begin{theorem}
\label{thm:rank}
Given a sequence $S$ of $k$ self-delimiting numbers stored in $N$ total bits, an integer $r$ with $\log N/2 \leq r \leq w/2$ and access to the lookup-table $\text{PREFIXSUM}$ for $r$-long bit-vectors, we can compute a data structure realizing a mapping $R : (N, S) \rightarrow \mathbb{N}$ where $R(p, x)$ is the rank of a number $x \in S$ and $p$ is the position in $S$ of the first bit used by the self-delimiting number of $x$. The data structure is constructed in time $O(k + N/r) = O(k + \frac{N}{\log N})$ and uses $O(N)$ bits.
\end{theorem}

\begin{proof}
By using exactly the same trick as in the proof of Theorem 3.3 it suffices to consider the case that the given self-delimiting numbers are all in the range $[0, N)$.

We start to describe our storage schema. Partition $[0, N)$ into $[N/r]$ regions where each region $R_i = [ir, (i + 1)r)$ ($i = 0, \ldots, [N/r] - 1$) consists of $r$ different numbers. Create an
array $B$ of $\lceil N/r \rceil$ fields, each of $r$ bits. For each region $R_i$ with $R_i \cap S = \emptyset$, $F[i]$ contains a null pointer, otherwise, $F[i]$ points to a data structure $Q_i$ that contains all information to determine the rank of each number of $R_i \cap S$.

The data structure $Q_i$ consists of multiple auxiliary structures. Let $A_i$ be a sequence of self-delimiting numbers consisting of the number of occurrences of each number $x \in R_i$ in $S$. Note that if a number $x \in R_i$ does not appear in $S$, we store a zero as the number of its occurrences in $S$. Partition $A_i$ into frames of $r$ bits each.

A frame can contain multiple self-delimiting numbers. If this is the case, we want to access the prefix sum over the self-delimiting numbers inside the frame, up to a desired number, quickly. To do that we use a precomputed lookup table $\text{PREFIXSUM}$. However, to do that easily we want that no self-delimiting number is divided between two frames. Therefore, we store each self-delimiting number in $A_i$ such that it either fits into the current not full frame, or it uses the next one. Of course it is not possible to store a large number (of more than $r$ bits) into a single frame. However, in this case a frame contains only one number and we can read it directly without using table lookup. If the number is to large to fit into a single frame, we use the next three frames to store the number in a standard binary representation. In detail, use the next first free frame to store only ones and so mark that the next two frames store the number in a binary representation instead of a self-delimiting number.

To find the start of a $q$th number of $R_i$ inside $A_i$, we use the idea of static space allocation, i.e., we initialize a bit-vector $A_i'$ where the start of each number in $A_i$ is marked and initialize a rank-select data structure on it. Moreover, we use an array $P_i$ of the same size as $A_i$ partitioned into frames of size $r$, where $P_i[f]$ is the prefix-sum up to the $f$th frame of $A_i$, up to the $i$th region.

With the data structures defined above we can answer the rank of a number $x \in S$ as follows: compute the region $i = \lfloor x/r \rfloor$ of $x$. Use the pointer $F[i]$ to jump to a data structure $Q_i$. The number $x$ is the $q$th number inside the region $R_i$ with $q = x \mod r$. Select the starting position $p = \text{select}_{A_i^i}(q)$ of the $q$th number in $A_i$. The position $p$ lies inside the $f$th frame with $f = \lfloor p/r \rfloor$. Read the $f$th frame as a variable $z$ and check if the self-delimiting number $A_i[q]$ occupies the whole frame, i.e., check if $A_i[q]$ encodes the start of a large number. If not, remove all numbers inside $z$ after the number $A_i[q]$ and return the rank of $x$ as $P[f] + \text{PREFIXSUM}[z]$. Otherwise, read the large number as $z$ and return the rank of $x$ as $P[f] + z$.

We now describe the initialization of our data structure. Apply Theorem 3.2 on $S$ to get a sorted sequence $S'$. Initialize $F$ with $\lfloor N/r \rfloor$ entries, each of $r$ bits, and let $j = 0$ be the largest rank so far. Iterate over $S'$ and for each number $x$ check if $F[i]$ with $i = \lfloor x/r \rfloor$ contains a pointer to a data structure $Q_i$. If not, initialize $Q_i$ as follows: run through $S'$ as long as the numbers are part of the same region $R_i$ and count the number $y$ of different numbers in that region. Since $R_i$ contains $y \leq \lceil N/r \rceil$ different numbers out of $S$, we can temporarily effort to use a key value map $D$ of $\lceil y/r \rceil$ entries, each of $2r$ bits, to count the number of occurrences of each number $x \in R_i$ in $S$ and store it in $D$. Let $\#(x, S)$ be the number of occurrences of a number $x$ in $S$.

Run through $S'$ as long as the number in $S$ is part of $R_i$ and store for each unique number $x$, the number of its occurrences $\#(x, S)$ as $D[x] = \#(x, S)$. Afterwards, scan trough $D$ and determine the number $N'$ of bits required to represent all value numbers in $D$ as a self-delimiting numbers. Create $Q_i$ by allocating $3N' + r$ bits for $A_i$, $A_i'$, and $\lceil (3N' + r)/r \rceil$ entries of $r$ bits for $P_i$. We want to fill $A_i$ and initialize $A_i'$ such that it results in a bit pattern where the $q$th 1 marks the start of the number of occurrences of the $q$th number of
We start to describe a folklore algorithm for tree isomorphism that requires \( N \) within \( #(\text{number out of } n) \) bits, the sum of the numbers of region \( \mathcal{R}_i \) that are the same size as \( A_i \) and a rank-select structure uses less than \( (\sum_{x \in \mathcal{R}_i \cap S} O(\log \#(x, S))) + O(r) \) bits. In total, \( Q_i \) uses \( (\sum_{x \in \mathcal{R}_i \cap S} O(\log \#(x, S))) + O(r) \) bits with \( d_i \) being the amount of all numbers in \( S \) that are part of \( \mathcal{R}_i \). We create \( Q_i \) only if \( S \) contains numbers of a region \( \mathcal{R}_i \). Over all \( N/r \) regions, containing all numbers of \( S \), our structure uses \( (\sum_{x=0}^{k} O(\log \#(x, S))) + O(r N/r) = \sum_{x=0}^{k} O(\log \#(x, S)) + O(N) = O(N) \) bits.

For a structure \( Q_i \), the array \( A_i \) and \( A_i' \) can be constructed in \( O(k') \) time, where \( k' \) are the number of self-delimiting numbers in \( \mathcal{R}_i \cap S \). Let the numbers inside the \( i \)th region occupy \( N' \) bits. The rank-select structure can be initialized in \( O(N'/r) \) time. Since our smallest possible self-delimiting number uses \( \Omega(1) \) bits and the largest possible uses \( O(r) \) bits, the sum of the numbers of region \( \mathcal{R}_i \) is within \( N' \) and \( 2N' \) and our amount of numbers within \( N' \) bits are within the bounds \( \Omega(N'/r) \) (if most self-delimiting numbers are large) and \( O(N') \) (if most self-delimiting numbers are small). In worst-case of only \( \Theta(N'/r) \) numbers stored within \( N' \) bits and we end up with a worst-case construction time that is linear to the amount of numbers in \( Q_i \). Therefore, in total our structure can be constructed in \( O(k + N'/r) \) time.

We want to remark that the (dense) rank can be answered in constant time with both data structures above without knowing the position \( p \) if all numbers in the given sequence \( S \) are smaller than \( N \).

### 4 Tree Isomorphism

We start to describe a folklore algorithm for tree isomorphism that requires \( \Theta(n \log n) \) bits on \( n \)-node trees. Let \( T_1 \) and \( T_2 \) be two rooted trees. The algorithm processes the nodes of each tree in rounds. In each round, all nodes of depth \( d = \max, \ldots, 0 \) are processed. Within a round, the goal is to compute a classification number for every node \( u \) of depth \( d \), i.e., a number out of \( \{0, \ldots, n\} \) that represents the structure of the maximal subtree below \( u \). The
correctness of the algorithm is shown in \[1\] and follows from the invariant that two subtrees in the trees \(T_1\) and \(T_2\) gets the same classification number exactly if they are isomorphic.

Since we later want to modify the algorithm, we now describe it in greater detail. In an initial process assign the classification number 0 to every leaf. Then, starting with the maximal depth do the following. First, compute the classification vector of each non-leaf \(v\) of depth \(d\) consisting of the classification numbers of \(v\)'s children. After doing this in both trees, compute the classification number for the non-leafs as follows: sort the components of each classification vector lexicographically. Then assign classification numbers 1, 2, 3, etc. to the vectors such that only equal vectors get the same number. By induction the invariant holds for all new classification numbers. Repeat the whole procedure iteratively for the remaining depths until reaching the root. By the invariant above, both trees are isomorphic exactly if the roots of both trees have the same classification number.

![Figure 4](image)

**Figure 4** An example of the algorithm where the trees \(T_1\) (left) and \(T_2\) (right) are isomorphic. The number inside the nodes and above the vectors are classification numbers. The vectors within the parenthesis are the sorted classification vectors of the maximal subtrees.

The algorithm above traverses the nodes in order of their depth starting from the largest and moving to the smallest until reaching the root. Since no \(O(n)\)-bit data structure is known that has an amortized constant-time operation per returned element for outputting the leaves (or all nodes) of a tree in shrinking depth, we modify the algorithm above to make it space-efficient by traversing the nodes in order of their height starting from height 0 (first round) until reaching the root with largest height (last round).

The difference to the standard approach is that we get classification vectors consisting of classification numbers that were computed in different rounds. To avoid a non-injective mapping of the subtrees, our classification numbers consist of tuples \((h_u, q_u)\) for each node \(u\) where \(h_u\) is the height of \(u\) and \(q_u\) is a number representing the subtree induced by \(u\) and its descendants. Intuitively, \(q_u\) is the old classification number from above.

The same invariant as for the standard algorithm easily shows the correctness of our modified algorithm. The space consumption of our algorithm is determined by

(A) the space for traversing nodes in order of their height,
(B) the space for storing the classification vectors and the classification numbers, and
(C) the space needed by an algorithm to assign new classification number based on the previous computed classification vectors.

We now describe \(O(n)\)-bit solutions for (A) - (C).

**A** Iterator returning vertices in increasing height. The idea of the first iteration round is to determine all nodes of height \(h = 0\) (i.e., all leaves) of the given tree in linear time and to store them in an \(O(n)\)-bit choice dictionary \(C\). While iterating over the nodes of height \(h\) in \(C\), our goal is to determine the nodes of height \(h + 1\) and store them in a choice dictionary \(C'\). The details are described in the next paragraph. If the iteration of the nodes in \(C\) is finished, swap the meaning of \(C\) and \(C'\) and repeat this process iteratively with \(h + 1\) as the new height \(h\) until finally the root is reached.
A node is selected for \( C' \) at the moment when we have processed all of its children. To compute this information, our idea is to give every unprocessed node \( u \) a token that is initially positioned at its leftmost child. The goal is to pass that token over every child of \( u \) from left to right until reaching the rightmost child of \( u \), at which point we mark \( u \) as processed and store it in \( C' \). Initially, no node is marked as processed. Informally speaking, we run a relay race where the children of a node are the runners. Before runners can start their run, they must be marked as processed. The initiative to pass the token is driven by the children of \( u \). Whenever a child \( v \) of \( u \) is processed, we check if either \( v \) is the leftmost child of \( u \) or \( v \)'s left sibling has \( u \)'s token. If so, we move the token to the right sibling \( v' \) of \( v \) and then to the right sibling of \( v' \), etc., as long as the sibling is already marked. If all children of \( u \) are processed, \( u \) becomes marked and part of \( C' \).

Using Lemma 2.2 we can jump from a node to its previous and next sibling in constant time. Thus, all moves of \( u \)'s token can be done in \( O(\text{deg}(u)) \) time until we insert \( u \) in \( C' \). In total, moving all tokens can be done in \( O(n) \) time. Based on the ideas above, we get the following lemma.

\begin{lemma}
Given an unlabelled rooted \( n \)-node tree \( T \) there is an iteration over all nodes of the tree in order of their height that runs in linear time and uses \( O(n) \) bits. The iteration is realized by the following methods:
\begin{itemize}
  \item \texttt{init}(\( T \)): Initializes the iterator and sets height \( h = 0 \).
  \item \texttt{hasNext}: Returns \texttt{true} exactly if nodes are left to output, otherwise, it returns \texttt{false}.
  \item \texttt{next}: Returns a choice dictionary containing nodes of height \( h \) and increments \( h \) by 1.
\end{itemize}
\end{lemma}

\begin{proof}
We first describe the initialization. If not given with the tree, compute the balanced parenthesis representation \( R \) of \( T \) and a tree navigation (Lemma 2.2). Let \( u_l \) and \( u_r \) be the positions of the open and closed parenthesis, respectively, for each node \( u \) in \( R \) and, let \( n' = 0 \) be the number of nodes already returned by the iterator. Initialize \( C \) and \( C' \) as empty choice dictionaries of \( O(n) \) bits. Take \( D \) as a bit-vector of \( 2n \) bits where we mark a node \( u \) as processed by setting \( D[u_l] = 1 \) and take \( P \) as a bit-vector of \( 2n \) bits where we store all tokens for the nodes. In detail the token for node \( u \) is stored in \( P[u_l^1], \ldots, P[u_l^f] \) where \( v^1, \ldots, v^f \) are the children of \( u \). In the following steps we adhere to the following two invariants before and after each round:

1. For a child \( u \) of a node \( v \), \( P[u_l] = 1 \) holds exactly if, for all siblings \( v \) left of \( u \), \( D[v_l] = 1 \) holds and for the next right sibling \( y \) of \( u \), \( D[y_l] = 0 \) holds.
2. If \( D[u_l] = 1 \) for all children of a node \( v \), then \( D[v_r] = 1 \).

For any call of \texttt{hasNext} return \( n' < n \). For a call of \texttt{next} with \( n' = 0 \), do the following. Traverse \( R \) to find all leaves, i.e., for each node \( u \) with \( u_l = u_r + 1 \), store \( u \) into \( C \), set \( D[u_l] = 1 \), and set \( n' \) then to the number of leaves. Afterwards return a copy of \( C \).

For a call of \texttt{next} with \( n' > 0 \), collect the nodes of the next height as follows. Iterate over the nodes \( u \) in \( C \) and do the following. Check if \( u \) is a leftmost child or check for the left sibling \( v \) of \( u \) if \( D[v_l] = 1 \) \& \( P[v_l] = 1 \) holds. It not, continue the iteration over \( C \). Otherwise, start or continue the relay race and move to the right sibling of \( u \) and continue moving further as long as for the next right sibling \( v \), \( D[v_r] = 1 \) holds. Then, set \( P[v_l] = 1 \). If we so reach the rightmost sibling, determine the parent \( x \) of \( u \), then set \( D[x_l] = 1 \), add \( x \) to \( C' \) and increment \( n' \) by one. After iterating over the nodes in \( C \), set \( C := C' \), initialize a new empty choice dictionary \( C' \) and return a copy of \( C \).

Note that our structures all use \( O(n) \) bits. Moreover, we pass the token only over siblings that are already marked in \( D \) and do nothing if we cannot pass a token to the next sibling. Hence, in total we require time linear to the amount of siblings to pass the token over every sibling. Hence, the iteration over all nodes of the given tree requires linear time.
\end{proof}
(B) Storing the classification numbers. We now describe an algorithm to store our classification numbers in an \(O(n)\)-bit storage schema. Recall that a classification vector of a node consists of the classification numbers of its children. Our idea is to use self-delimiting numbers to store the classification numbers and to choose the classification numbers such that their size is bounded by \(O(\log n)\) and by a constant times the number of descendants of \(u\).

We take \((0,0)\) as a classification number for every leaf such that a constant number of bits suffice to represent it. Moreover, after computing the classification number of a node \(u\) the classification vector of \(u\) (i.e., the classification numbers of \(u\)'s children) is not required anymore and thus, they can be overwritten and their space can be used for the classification numbers of \(u\). By the next lemma we can store the classification numbers and vectors.

Lemma 4.2. Given an unlabeled rooted \(n\)-node tree \(T\) and an integer \(c > 0\), there is a data structure using \(O(n)\) bits that initializes in \(O(n)\) time and provides \(\text{read}(u)\) and \(\text{write}(u)\) in constant time and \(\text{vector}(u)\) in \(O(\deg(u))\) time.

- \(\text{read}(u)\) (\(u\) node of \(T\)): Returns the number \(x\) stored for \(u\).
- \(\text{write}(u, x)\) (\(u\) node of \(T\), \(0 \leq x \leq \min(2^{2\deg(u)}, \text{poly}(n))\)): Store number \(x\) for node \(u\) and delete all stored numbers of the descendants of \(u\).
- \(\text{vector}(u)\) (\(u\) node of \(T\)): Returns a bit-vector of length \(\leq 2c\deg(u)\) consisting of the concatenation of the self-delimiting numbers stored for the children of \(u\).

Proof. If not already given, compute the data structures of Lemma 2.2. Recall that \(u_l\) and \(u_r\) are the position of the open and closed parenthesis of a node \(u\), respectively. Allocate a bit-vector \(B\) of \(6cn\) bits. To write a number \(x\) for a node \(u\), transform \(x\) into a self-delimiting number and store it in \(B[6cu_l, 6cu_r]\). Recall that a number \(x\) can be stored as a self-delimiting number with \(f(x) = 3\) bits if \(x \leq 1\) and with \(f(x) = 2\lceil\log x\rceil + 1 \leq 3\lceil\log x\rceil\) bits if \(x > 1\). Since \(x \leq 2c\deg(u)\), \(x\) can be stored with \(3\lceil c\deg(u)\rceil \leq 6c\deg(u)\) bits and since the difference of \(u_l\) and \(u_r\) is \(4c\deg(u)\), we can store \(x\) as a self-delimiting number in \(B[6cu_l, 6cu_r]\). Within this area only the numbers of the descendants of \(u\) are stored and thus, only these are overwritten by writing \(x\). To read a number stored for node \(u\), read the bits inside the memory \(B[6cu_l, 6cu_r]\) as a self-delimiting number.

To return \(\text{vector}(u)\), call \(\text{read}(v)\) for all children of \(u\) and concatenate the returned self-delimiting numbers to one bit-vector.

(C) Computing classification numbers. Let \(D\) be our data structure of Lemma 4.2 where we store all classification vectors. Our next goal is to replace the classification vector \(D.\text{vector}(u)\) of all processed subtrees with root \(u\) and height \(h\) by a classification number \((h, q_u)\) with \(|q_u| = O(\min\{\deg(u), \log n\})\) such that the component-wise-sorted classification vectors are equal exactly if they get the same classification number.

Our idea is to sort \(D.\text{vector}(u)\) by using Theorem 3.2 to obtain a component-wise sorted classification vector and turn this vector into a self-delimiting number for further operation on it. We subsequently compute the dense rank to replace the self-delimiting number in \(D.\text{vector}(u)\) by the tuple (height, dense rank). To make it work we transform each vector into a self-delimiting number by considering the bit-sequence of the vector as a number (i.e., assign the prefix 10 to each vector where \(\ell\) is the length of the vector in bits). We can store all these vectors as self-delimiting numbers in a bit-vector \(Z_{h+1}\) of \(O(n)\) bits. Then we can use Theorem 3.3 applied to \(Z_{h+1}\) to compute the dense-rank, i.e., the classification numbers for all subtrees of height \(h + 1\).

Our Algorithm. We now combine the solutions for (A)-(C).
Sorting and Ranking with Applications to Tree Isomorphism

Lemma 4.3. Given two rooted \( n \)-node trees \( T_1 \) and \( T_2 \) there is an algorithm that recognizes if \( T_1 \) and \( T_2 \) are isomorphic in linear-time using \( O(n) \) bits.

Proof. Computing the leaves and their classification numbers is trivial. So assume we have computed the classification numbers for the subtrees of height \( h \) and we have to compute the classification numbers for the subtrees of height \( h+1 \). We iterate over every subtree with root \( u \) and height \( h+1 \) by using the choice dictionary \( C \) returned by our iterator of Lemma 4.1.

In the following, let \( k_u \) be the number of children of a node \( u \), let \( N_u \) be the number of bits of \( D vect(u) \) and let \( V_i \) be the set of nodes of height \( i \) in both trees. For each node \( u \), we sort the classification numbers of its children stored in \( D vect(u) \) into a new bit-vector \( X \) of \( N_u \) bits by applying Theorem 3.3 (with parameters \( N = N_u \) and \( k = k_u \)) and store it by calling \( D write(u, X) \).

We can now construct the vectors \( Z^T_{h+1} \) and \( Z^T_{h+1} \) for the subtrees of height \( h+1 \) within the trees \( T_1 \) and \( T_2 \), respectively. \( Z^T_{h+1} \) consists of the self-delimiting numbers constructed from the vector \( D vect(u) \) of each node \( u \) in \( T_i \) of height \( h+1 \). We then put \( Z^T_{h+1} \) and \( Z^T_{h+1} \) together into a total sequence \( Z \) and apply Theorem 3.3 (with parameters \( N = O(\sum u \in V_{h+1} N_u) \) and \( k \) being the number of self-delimiting numbers in \( Z \)) to determine the dense rank of the numbers in \( Z \). Then we iterate over the numbers \( q \) in \( Z \) and use their position \( p \) to retrieve their rank \( r_q = R(p, q) \). In parallel, we iterate over the subtrees of height \( h+1 \) in both trees again and replace the classification vector \( q \) by calling \( D write(u, (h+1, r_q)) \).

After computing the classification number for each root of \( T_1 \) and \( T_2 \), we can compare them and decide if \( T_1 \) and \( T_2 \) are isomorphic. If so, we proceed with the next height until reaching the root in both trees. Afterwards, we can easily answer the question if \( T_1 \) and \( T_2 \) are isomorphic or not.

One can easily see that all substeps run with \( O(n) \) bits. Let us next consider the parameters of the calls to Theorem 3.3. Parameter \( k \) summed over all calls is bounded by \( \sum u \deg(u) = O(n) \) and parameter \( N \) over all calls is bounded by \( \sum u \deg(u) \log n = O(n \log n) \) since we made sure that the classification number of each vertex is bounded by \( \text{poly}(n) \). Thus, the total time is bounded by \( O(k + N/w) = O(n) \).

We next consider the calls to Theorem 3.3. Since every vertex is part of one set \( V_h \), \( k \) summed over all calls is bounded as before by \( O(n) \). Moreover, each vertex with its classification number is part of one classification vector. Thus, \( N \) over all calls is bounded as before by \( O(n \log n) \). Choosing \( r = (\log n)/2 \), we obtain a total running time of \( O(n) \). Moreover, the lookup table \( \text{POPCNT} \) can be constructed in \( O(2^{tr}) = O(n) \) time. ▶

We generalize Theorem 4.3 to unrooted trees by determining the center of a tree space efficiently. A center of a tree \( T \) is a set consisting of either one node or two adjacent nodes of \( T \) such that the distance to a leaf is maximal for these nodes compared to all other nodes of \( T \). It is known that every tree has a unique center [18, Theorem 4.2]. If two trees \( T_1 \) and \( T_2 \) are isomorphic, then every isomorphism function maps the center nodes of \( T_1 \) to the center nodes of \( T_2 \).

Lemma 4.4. Given an \( n \)-node tree \( T \) there is a linear-time \( O(n) \)-bit algorithm that determines the center of \( T \).

Proof. Harary [18, Theorem 4.2] determines the center of a tree in rounds where, in each round, all nodes of degree 1 are removed. The algorithm stops if at most two nodes are left, which are selected as the center.

We follow a similar approach, but cannot effort to manipulate the given trees. For our approach we need to be able read and reduce the degree of a node. We also can not use
Lemma 4.1 since it requires a rooted tree. Alternatively, we store the initial degrees of all nodes as self-delimiting numbers by using static space allocation. Note that we can easily increment the numbers in-place in $O(1)$ time. We write the degrees in a bit-vector of $O(n)$ bits.

Similar to Harary’s algorithm, our algorithm also works in rounds. We use a choice dictionary $C$ consisting of the nodes of degree one, which we initially fill by scanning through the whole tree. Then, while we iterate over each node $u \in C$ we delete $u$ from $C$, increment an initial-zero global counter $k$ to be able to determine how many nodes remain in the tree, and decrement the degree of the only node $v$ (neighbor of $u$) with $\deg(v) \geq 2$. Intuitively speaking this simulates the deletion of $u$. If the degree of $v$ is 1 after the decrementation, we add $v$ in a second choice dictionary $C'$. After the iteration over the nodes in $C$ is finished, we swap $C$ with $C'$. If at most two nodes remain (i.e., $k \geq n - 2$), we output the nodes in $C$ as the center of the tree. Otherwise, we start the next round.

Working with both choice dictionaries and storing all degrees can be done with $O(n)$ bits. Moreover, since the total degree in a tree is bounded by $O(n)$, our approach runs in $O(n)$ time.

We can check isomorphism for two given non-rooted trees $T_1$ and $T_2$ by first computing the center $C_{T_1}$ and $C_{T_2}$ of $T_1$ and $T_2$, respectively. Then, we iterate over the 4 possibilities to choose a root $r_{T_1} \in C_{T_1}$ and $r_{T_2} \in C_{T_2}$ and apply Theorem 4.3. $T_1$ and $T_2$ are isomorphic exactly if one possible rooted versions of $T_1$ and $T_2$ are isomorphic. We thus can conclude the next theorem.

**Theorem 4.5.** Given two (unrooted) trees $T_1$ and $T_2$ there is an algorithm that outputs if $T_1$ and $T_2$ are isomorphic in linear-time using $O(n)$ bits.

We next show isomorphism on colored trees. Colored trees are trees where each node has a color assigned, i.e., a number out of $\{0, \ldots, n-1\}$. Two colored trees $T_1 = (V_{T_1}, E_{T_1})$ and $T = (V_{T_2}, E_{T_2})$ are isomorphic if an isomorphic mapping $f : V_{T_1} \to V_{T_2}$ exists such that for each $u \in V_{T_1}$, $u$ and $v = f(u)$ have the same color assigned.

To check isomorphism of uncolored trees we used a classification number whose computation was only influenced by the structure of the tree. To check isomorphism of colored trees we additionally use the color of a node to compute a classification number, i.e., we add to each classification vector of a node $u$ the color of $u$ as the last entry of the vector.

**Theorem 4.6.** Given two (unrooted) colored $n$-node trees $T_1$ and $T_2$, there is an algorithm that outputs if $T_1$ and $T_1$ are isomorphic in linear-time using $O(n + b)$ bits exactly if the colors of all nodes of $T_1$ and $T_2$ can be written within $O(b)$ bits.

**Proof.** Our algorithm for tree isomorphism computes a balanced parenthesis representation of the tree and operates on it. In this representation the nodes of a tree $T$ get a different labeling. The labeling is determined by an traversal over the tree in pre-order. This brings up the problem that we cannot access the color of a node $u$ fast enough if we use the balanced parenthesis representation. Therefore, when computing the representation, i.e., when running a pre-order traversal on the original tree we use a bit-vector $A$ of $O(b)$ bits where we first store the color of the vertex visited first, then the color of the vertex visited second, etc. To access the colors in constant time, we again use static space allocation.

We now describe how to change the proof of Theorem 4.3 for colored tree isomorphism. Recall, when computing a classification number for the nodes of some height $h$, the classification vectors of the children of the nodes of height $h$ are read, transformed into a self-delimiting number and written into a separate bit-vector. Before transforming a vector...
of a node $u$ add $u$’s color, i.e., the $u$th color in $A$, as a last entry of the vector. Then proceed as described in the proof of Theorem 4.3.

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