Null Zig-Zag Wilson Loops in $\mathcal{N} = 4$ SYM

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Abstract

In planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory we have studied supersymmetric Wilson loops composed of a large number of light-like segments, i.e., null zig-zags. These contours oscillate around smooth underlying spacelike paths. At one-loop in perturbation theory we have compared the finite part of the expectation value of null zig-zags to the finite part of the expectation value of non-scalar-coupled Wilson loops whose contours are the underlying smooth spacelike paths. In arXiv:0710.1060 [hep-th] it was argued that these quantities are equal for the case of a rectangular Wilson loop. Here we present a modest extension of this result to zig-zags of circular shape and zig-zags following non-parallel, disconnected line segments and show analytically that the one-loop finite part is indeed that given by the smooth spacelike Wilson loop without coupling to scalars which the zig-zag contour approximates. We make some comments regarding the generalization to arbitrary shapes.
1 Introduction and Summary

Wilson loops play a privileged role in the AdS/CFT correspondence [1]. Their study has branched into a wide variety of topics in the correspondence including most recently scattering amplitudes and integrability. Alday and Maldacena put forward a Wilson loop composed of light-like segments as the strong coupling dual of planar gluon scattering amplitudes [2, 3]. At strong 't Hooft coupling, such a Wilson loop is described by a macroscopic fundamental string whose worldsheet is a minimal area embedding in $\text{AdS}_5$ with boundary given by the Wilson loop contour. Amazingly, when that same Wilson loop was considered at weak coupling in planar $\mathcal{N} = 4$ SYM perturbation theory, it agreed with the gluon scattering amplitudes. The correspondence between the light-like Wilson loop and gluon scattering amplitudes appears in fact to be an all-orders statement in planar $\mathcal{N} = 4$ SYM and its string theory dual, i.e. IIB strings on $\text{AdS}_5 \times S^5$, [4–9].

1.1 Introduction

It is well known that the Maldacena-Wilson loop, given in general by

\[ W = \frac{1}{N} \text{Tr}_R \mathcal{P} \exp \oint_C d\tau \left( i \dot{x}^\mu A_\mu + |\dot{x}| \Theta I \Phi_I \right) \] (1)

may be thought of as the holonomy of an infinitely massive $W$-boson obtained through Higgsing the $SU(N + 1)$ theory to $SU(N)$, or equivalently, geometrically, by separating a single D3-brane from a stack of $N + 1$; the long string stretching back to the stack being the $W$-boson. Although the AdS/CFT correspondence is in general better set in Euclidean space, it is interesting to consider Lorentzian contours. In particular, choosing $x^\mu$ null produces a Wilson loop without coupling to scalar fields

\[ W_0 = \frac{1}{N} \text{Tr}_R \mathcal{P} \exp i \oint_C d\tau \dot{x}^\mu A_\mu \] (2)

Interpreting such an object as an holonomy of an infinitely massive particle appears strange, since we are now requiring its worldline to be null. It turns out however, that another interpretation of $W_0$ in terms of gluon scattering amplitudes is possible, as was shown by Alday and Maldacena in Ref. [3]. There is no sense to the concept of scattering amplitudes in a conformal field theory such as $\mathcal{N} = 4$; conformal invariance precludes the existence of asymptotic states. The reflection of this issue is the appearance of IR divergences corresponding to the exchange of soft particles between the external particle legs. The introduction of an IR regulator breaks conformal invariance and allows the definition and computation of these scattering amplitudes. This can be achieved in the string dual of $\mathcal{N} = 4$ SYM through the introduction of a D3-brane at a small distance $r_0$ from the center of $\text{AdS}_5$, effectively introducing into the spectrum a gap or IR cut-off proportional to $r_0$. Upon such a D3-brane one can consider the scattering of open strings. In fact, due to the
Figure 1: An example of a null zig-zag Wilson loop. Here the smooth underlying curve approximated by the zig-zag is a spacelike rectangle.

warping of the $AdS$ metric these are strings of large proper momentum, and as for the case of flat space, their scattering is captured by the growth of a macroscopic connected string worldsheet joining the incoming and outgoing strings. The picture is completed through a T-duality transformation on the boundary coordinates which inverts the $AdS_5$ space into another $AdS_5$, the IR regulator of the former becoming a UV regulator of the latter - the familiar cut-off associated with the approach to the boundary in AdS/CFT. What one has in the new $AdS_5$ is then a macroscopic worldsheet which ends upon a contour at the boundary. This contour is defined by the momenta $k_i^\mu$ of the incoming and outgoing strings in the original $AdS$. Under the action of the T-duality, they are mapped to points $y_i^\mu$ on the boundary of the new $AdS$ through

$$y_{i+1} - y_i = 2\pi k_i$$ \hspace{1cm} (3)

The shape of the resulting Wilson loop is then evident: the conservation of the $k^\mu$ implies a closed contour, the null-ness of the $k^\mu$ (i.e. $k^2 = 0$) implies light-like segments.

In [3] Alday and Maldacena considered a null zig-zag which approximated a smooth (spacelike) rectangle, see Fig. 1. They argued that in the limit of large $n$, where $n$ counts the number of null segments, the finite part of the vacuum expectation value should match that of a Wilson loop without coupling to scalars, whose contour is the underlying smooth space-like path. For the rectangular Wilson loop, they observed that one can see this explicitly at leading order in perturbation theory. We will show this example presently, as it serves as a warm-up for the circle which we will consider afterwards. Before we do this however we must qualify what is meant by the “finite part” of the expectation value. The divergences in the gluon scattering amplitudes mentioned above manifest themselves in the null zig-zag as cusp divergences. At leading order in perturbation theory one can see these cusp divergences by considering the gluon exchange between two null segments which meet at an angle, see Fig. 2. We will now evaluate this diagram. Using the Feynman gauge and dimensional regularization, the gluon propagator is

$$\langle A^a_n(x) A^b_0(0) \rangle = -\frac{\Gamma(1-\epsilon) \, g^2 \mu^{2\epsilon} \delta^{ab} \eta_{\mu\nu}}{4\pi^{2-\epsilon} \left[ -x^2 + i\epsilon \right]^{1-\epsilon}}$$ \hspace{1cm} (4)
where $\eta_{\mu\nu} = (+, -, -, -)$ and the dimension of spacetime is $4 - 2\epsilon$. Expanding (2) to second order and taking its expectation value, one finds for the diagram shown in figure 2 the following contribution

$$\langle W_0 \rangle = 1 + \frac{\lambda (\pi \mu^2)^\epsilon}{4} \frac{\Gamma(1 - \epsilon)}{4\pi^2} \int_0^1 dt_1 \int_0^1 dt_2 \frac{\dot{x}_1 \cdot \dot{x}_2}{[-(x_1 - x_2)^2]^{1-\epsilon}}$$

$$= 1 - \frac{\lambda (\pi \mu^2)^\epsilon}{16\pi^2} \frac{\Gamma(1 - \epsilon)}{2t_1(1 - t_2)^{1-\epsilon}}$$

$$= 1 - \frac{\lambda (\mu^2 \pi s)^\epsilon}{16\pi^2} \frac{\Gamma(1 - \epsilon)}{2^{1-\epsilon}}$$

where $x_1^\mu = t_1 k_1^\mu$, $x_2^\mu = (1 - t_2) k_2^\mu$, and $s = (k_1 + k_2)^2$; the $k_i$ being null momenta. Clearly by expanding in $\epsilon$ we will also produce finite terms of order $\epsilon^0$. When drawing the correspondence between $\langle W_0 \rangle$ and the non-scalar-coupled Wilson loop, we ignore these finite terms. We are interested in those finite pieces which do not scale with the number of cusps $N$. At one-loop we therefore consider only the non-cusp contributions to the expectation value. These are finite, and in the limit of large $N$ will produce both $N$-dependent and $N$-independent terms

$$\lim_{N \to \infty} [\langle W_0 \rangle^{1\text{-loop}} - \langle W_0 \rangle^{1\text{-loop}\text{cusp}}] = A N^2 + B N + C + O(N^{-1}).$$

The non-scalar-coupled Wilson loop is also divergent for spacelike contours. Indeed, the scalar coupling in Eq. (1) may be viewed as a regularization of the non-scalar-coupled loop: the loop-to-loop propagator in Eq. (1) is proportional to $(|\dot{x}_1| |\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2)/(x_1 - x_2)^2$, for smooth contours, this object has no singularity when $x_1 \to x_2$. We will compute the "finite-part" of the non-scalar-coupled Wilson loop by dimensional regularization. The statement which we shall prove is then

$$C = \langle W_{NSC} \rangle^{1\text{-loop}}_{\text{finite}}$$

where $W_{NSC}$ is the non-scalar-coupled Wilson loop whose smooth, spacelike contour is approximated by the null zig-zag.
1.2 Summary of results

We summarize the results we obtain here, and leave the detailed calculations to the following section.

- Anti-parallel lines
  We find, by explicit calculation, the relation Eq. (7) true for the contour being two anti-parallel lines.

- Circular loop
  We calculate the one-loop expectation value of Wilson loop on null zigzags approximating a circle, again confirming the relation Eq. (7).

- Non-parallel lines
  We consider the contour composed of two separated space-like line segments at an arbitrary angle. We find that the relation Eq. (7) is indeed correct. We argue that this result allows one to extend Alday & Maldacena’s argument [3] to any smooth, space-like shape.

1.3 Outline

The rest of the paper is organized as follows. In section 2, we review the equality, ref Eq. (7) for anti-parallel lines. In section 3, we consider two separated lines at an arbitrary angle. In section 4, we consider a circular contour. With these relations, we argue for the extension of the equality to any smooth, space-like shape in section 5.

2 Anti-parallel lines

As discussed in the previous section the simplest setting in which to prove our result is the anti-parallel lines; indeed this case was already considered in [3]. We present the calculation in some detail as it serves to clarify the other contours considered in subsequent sections.

First we specify coordinates of the anti-parallel lines and their zigzag approximation. One straight line spans the two points \((0,0,0,0)\) and \((0,L,0,0)\). Its zigzag sequence consists of points

\[
\{x(1), y(1), z(1), x(2), y(2), z(2), \ldots, x(N), y(N), z(N)\}
\]  

(8)

where \(N\) is some integer and \(N \gg 1\) and

\[
x_{(j)}^\mu = \frac{L}{N} \left(0, (j - 1), 0, 0\right), \quad y_{(j)}^\mu = \frac{L}{N} \left(1/2, (j - 1/2), 0, 0\right), \quad z_{(j)}^\mu = \frac{L}{N} \left(0, j, 0, 0\right)
\]

(9)

specifies the null zig-zag approximation as \(y_{(j)} - x_{(j)}\) and \(z_{(j)} - y_{(j)}\) are null vectors. We may then specify another straight line spanning \((0,0,T,0)\) and \((0,L,T,0)\) with barred variables \(\bar{x}_{(j)}, \bar{y}_{(j)}, \bar{z}_{(j)}\), defined as \(\bar{x}_{(j)} = x_{(j)} + (0,0,T,0)\) and similarly for \(\bar{y}_{(j)}\) and \(\bar{z}_{(j)}\).
Figure 3: Finite contribution to the null zig-zag rectangle at 1-loop.

We begin by considering those contributions stemming from the diagram pictured in Fig. 3. On the bottom straight line the gluon line is attached to the segment \( x(i) + (y(i) - x(i)) t \), while on top one it is attached to \( \bar{z}(j) + (\bar{y}(j) - \bar{z}(j)) \bar{t} \).

We find the following result\(^1\)

\[
\langle W \rangle_{x(i) \bar{y}(j)} = \frac{\lambda}{16\pi^2} \int_0^1 dt \int_0^1 d\bar{t} \frac{1}{2} \frac{1}{(i - j - t)(i - j - \bar{t}) + T^2 N^2/L^2}
\]

The contribution from \( \langle W \rangle_{y(i) x(j)} \) will yield the same result, once we have summed over \( i \) and \( j \). The contributions \( \langle W \rangle_{x(i) \bar{x}(j)} \) and \( \langle W \rangle_{y(i) \bar{y}(j)} \) are zero, as the null-momenta defining the two null segments are (anti-)parallel in this case and thus have zero inner product. We will show that in the large \( N \) limit the contributions

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \left( \langle W \rangle_{x(i) \bar{y}(j)} + \langle W \rangle_{y(i) \bar{x}(j)} \right) = 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \langle W \rangle_{x(i) \bar{y}(j)}
\]

are equal to the analogous diagram in the computation of \( W_{N \text{SC}} \). In the large \( N \) limit Eq. (11) becomes

\[
\lim_{N \to \infty} \frac{\lambda}{16\pi^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^1 dt \int_0^1 d\bar{t} \frac{1}{(i - j - t)(i - j - \bar{t}) + T^2 N^2/L^2} \left( N - k \atop \frac{(k+t)(k+\bar{t}) + T^2 N^2/L^2}{(k+t)(k+\bar{t}) + T^2 N^2/L^2} \right) \left( t \to -t, \bar{t} \to -\bar{t} \right) + \frac{N}{t\bar{t} + T^2 N^2/L^2}
\]

\[
= \frac{\lambda}{16\pi^2} \int_0^1 \frac{d\bar{t}}{\sqrt{g^2 - 4h}} \left( f(g/2 + \sqrt{g^2/4 - h}) - f(-g/2 - \sqrt{g^2/4 - h}) \right)
\]

\[
+ f(-g/2 + \sqrt{g^2/4 - h}) - f(g/2 - \sqrt{g^2/4 - h}) \right) - \frac{\lambda}{16\pi^2} N Li_2(-\frac{T^2}{N^2 L^2})
\]

\[
= \frac{\lambda}{16\pi^2} \left( 2 \frac{L}{T} \arctan \frac{L}{T} - \log \left( 1 + \frac{L^2}{T^2} \right) \right) + \mathcal{O}(N^{-1})
\]  

\(\lambda, \epsilon \) have been set to zero here since the diagram is finite.
where \( g \equiv (t + \bar{t}) \), \( h \equiv (t\bar{t} + T^2N^2/L^2) \), \( f(x) \equiv (x + N)(\psi(1 + x) - \psi(1 + x + N)) \), \( \psi \) is Digamma function; \( L_2 \) is Dilogarithm function. In third line of Eq. (12) we have performed the summation over \( k \) analytically, then in the final line taken large \( N \) limit.

This expression is precisely the contribution stemming from the diagram in Fig. 3, however with a space-like anti-parallel lines (shown in gray) replacing the null zig-zag, i.e.,

\[
\langle W_{NSC} \rangle \rightarrow \frac{\lambda}{16\pi^2} \int_0^1 dt \int_0^1 d\bar{t} \left\{ \frac{L^2}{L^2 (t - \bar{t})^2 + T^2} \right\} = \frac{\lambda}{16\pi^2} \left( \frac{2L}{T} \arctan \frac{L}{T} - \log \left( 1 + \frac{L^2}{T^2} \right) \right).
\]

(13)

3 Non-parallel lines

In order to generalize Alday & Maldacena’s argument to any smooth, space-like curve, we should consider two separated segments at one angle \( \theta \), see Fig. 4. In this section we will compare the finite parts of the gluon exchange between the two separated segments to that of the null zigzag approximation shown in Fig. 5. The length of the horizontal segment is taken to be 1, while the distance between the two segments along the vertical axis is given by \( a \). We parameterize their zigzag approximations as follows

\[
\begin{align*}
x(i) &= \frac{1}{N} \left( 0, i - 1, 0, 0 \right), \quad y(i) = \frac{1}{N} \left( 1/2, i - 1/2, 0, 0 \right), \quad z(i) = \frac{1}{N} \left( 0, i, 0, 0 \right), \\
\bar{x}(i) &= \frac{1}{N} \left( 0, (i - 1), (i - 1) \tan \theta + aN, 0 \right), \quad \bar{z}(i) = \frac{1}{N} \left( 0, i, i \tan \theta + aN, 0 \right), \\
\bar{y}(i) &= \frac{1}{N} \left( 1/2 \sec \theta, (i - 1/2), (i - 1/2) \tan \theta + aN, 0 \right)
\end{align*}
\]

(14)

and take the index on barred variables to run to \( cN \), where \( c \) is some rational number accounting for the different length of the angled segment. The one-loop expectation value of Wilson loop in the large \( N \) limit is given by

\[
\langle W_\emptyset \rangle = \frac{\lambda}{16\pi^2} \lim_{N \to \infty} \sum_{i=1}^N \sum_{j=1}^N cN \left( \langle W \rangle y(i) \bar{x}(j) + \langle W \rangle x(i) \bar{y}(j) + \langle W \rangle x(i) \bar{y}(j) + \langle W \rangle y(i) \bar{y}(j) \right)
\]

(15)
Next we intend to prove that integrations in Eq. (17) can be simplified to a desired form under summation over $i,j$, for example

$$
\sum_{j=1}^{cN} \sum_{i=1}^{N} \int_{0}^{1} \int_{0}^{1} dt \, ds \, \frac{1}{(2i-2j+2-s-t)^2+(2aN+(2j-2+t) \tan \theta)^2} - \frac{1}{(s-t \sec \theta)^2}
$$

$$
\equiv \alpha + \beta + \delta + \zeta
$$

Note that the last term in each of the denominators in Eq. (16) is small compared to the remaining terms; it can be used to expand these integrands. One can check that the correction terms are order of $N^{-4}$, thus their contributions to $\langle W_0 \rangle$ are negligible in the large $N$ limit. Then we have

$$
\langle W \rangle_{y(i)\bar{y}(j)} = \int_{0}^{1} \int_{0}^{1} dt \, ds \, \frac{1 + \sec \theta}{(2i-2j+2-s-t)^2+(2aN+(2j-2+t) \tan \theta)^2} + O(N^{-4})
$$

$$
\langle W \rangle_{x(i)\bar{y}(j)} = \int_{0}^{1} \int_{0}^{1} dt \, ds \, \frac{1 + \sec \theta}{(2i-2j-2+s+t)^2+(2aN+(2j-2+t) \tan \theta)^2} + O(N^{-4})
$$

$$
\langle W \rangle_{x(i)\bar{x}(j)} = \int_{0}^{1} \int_{0}^{1} dt \, ds \, \frac{1 - \sec \theta}{(2i-2j+s-t)^2+(2aN+(2j-2+t) \tan \theta)^2} + O(N^{-4})
$$

$$
\langle W \rangle_{y(i)\bar{y}(j)} = \int_{0}^{1} \int_{0}^{1} dt \, ds \, \frac{1 - \sec \theta}{(2i-2j-s+t)^2+(2aN+(2j-2+t) \tan \theta)^2} + O(N^{-4})
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$$

$$
= \sum_{j=1}^{cN} \sum_{i=1}^{N} \int_{0}^{1} \int_{0}^{1} dt \, ds \, \frac{1}{(i-j)^2+(aN+j \tan \theta)^2} + O(N^{-1})
$$

Suppose $c > 1$, the left-hand side of Eq. (18) is

$$
\left\{ \sum_{j=1}^{N} \sum_{i=j+1}^{N} + \sum_{j=1}^{N} \sum_{i=1}^{j-1} + \sum_{j=N+1}^{N} \sum_{i=1}^{N} + \sum_{i=1}^{N} \sum_{j=1}^{N-1} \right\} \int_{0}^{1} \int_{0}^{1} dt \, ds \, \frac{1}{(2i-2j+2-s-t)^2+(2aN+(2j-2+t) \tan \theta)^2}
$$

$$
\equiv \alpha + \beta + \delta + \zeta
$$
From the definition of $\alpha$, we find

$$\alpha > \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \frac{1}{(2i - 2j + 2)^2 + (2aN + 2j \tan \theta)^2}$$

$$= \left\{ \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} - \sum_{j=1}^{N-1} (i = j + 1) + \sum_{j=1}^{N-1} (i = N + 1) \right\} \frac{1}{4} \frac{1}{(i - j)^2 + (aN + j \tan \theta)^2}$$

$$= \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \frac{1}{4} \frac{1}{(i - j)^2 + (aN + j \tan \theta)^2} + O(N^{-1})$$

(20)

and

$$\alpha < \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \frac{1}{4} \frac{1}{(i - j)^2 + (aN + (j - 1) \tan \theta)^2}$$

$$= \left\{ \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} - \sum_{j=1}^{N-1} (i = N) + \sum_{j=1}^{N-1} (i = 0) \right\} \frac{1}{4} \frac{1}{(i - j)^2 + (aN + j \tan \theta)^2}$$

$$= \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \frac{1}{4} \frac{1}{(i - j)^2 + (aN + j \tan \theta)^2} + O(N^{-1})$$

(21)

So in the large $N$ limit we have

$$\alpha = \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \frac{1}{4} \frac{1}{(i - j)^2 + (aN + j \tan \theta)^2} + O(N^{-1})$$

(22)

This approach to simplify $\alpha$ as Eq. (22) works well for $\beta$, $\delta$, $\zeta$. Putting these simplified expressions together, we get the right-hand side of Eq. (18). If $c \leq 1$, the double summation over $i, j$ can be grouped in a different way, but one can repeat the process as Eq. (20-22) to verify Eq. (18).

Using the same method as in the other expressions in Eq. (17), we obtain

$$\langle W_0 \rangle = \frac{\lambda}{16\pi^2} \sum_{i=1}^{N} \sum_{j=1}^{cN} \frac{1}{(i - j)^2 + (aN + j \tan \theta)^2} + O(N^{-1})$$

$$- \frac{\lambda}{16\pi^2} \int_{0}^{1} dt \int_{0}^{c} ds \frac{1}{(s - t)^2 + (a + s \tan \theta)^2}$$

(23)

This result is just the expression when one calculate one-loop expectation value of Wilson loop defined on Fig. 4.
Figure 6: Null zig-zag Wilson loop which approximates a spacelike circle.

4 Circle

We approximate a space-like circle of unit radius with a null zig-zag specified by a sequence points as in Eq. (8) with

\[ x_j = \left( 0, 0, -\cos(j + 1/2)\varphi, \sin(j + 1/2)\varphi \right); \]
\[ y_j = \left( \sin(\varphi/2), 0, -\cos(\varphi/2) \cos j\varphi, \cos(\varphi/2) \sin j\varphi \right); \]
\[ z_j = \left( 0, 0, -\cos(j - 1/2)\varphi, \sin(j - 1/2)\varphi \right) \]

where \( \varphi = 2\pi/N \) since the contour is closed, see Fig. 6. The one-loop expectation value of the null zig-zag Wilson loop in the large \( N \) limit is given by

\[ \langle W_\emptyset \rangle_{1\text{-loop}} = \frac{\lambda}{16\pi^2} \lim_{N \to \infty} \sum_{i,j=1}^N \left( \langle W \rangle_{x(i) x(j)} + \langle W \rangle_{y(i) y(j)} + \langle W \rangle_{x(i) y(j)} + \langle W \rangle_{y(i) x(j)} \right) \tag{25} \]

where \( \langle W \rangle_{x(i) x(j)} \) stands for the contribution from \( x(i) + t(y(i) - x(i)) \) and \( y(j) + t(z(j) - y(j)) \), \( \langle W \rangle_{x(i) y(j)} = \langle W \rangle_{y(i) y(j)} \) and \( \langle W \rangle_{x(i) y(j)} = \langle W \rangle_{y(i) x(j)} \).

\[ \langle W \rangle_{x(i) x(j)} = \int_0^1 dt \int_0^1 ds \frac{-\sin((i-j)\varphi/2)\eta^2/2}{(s-t)\eta \cos((i-j)\varphi/2) - (1+s\eta^2) \sin((i-j)\varphi/2)} \tag{26} \]
\[ \langle W \rangle_{y(i) y(j)} = \int_0^1 dt \int_0^1 ds \frac{-\cos((i-j)\varphi/2)\eta^2/2}{\sin((i-j)\varphi/2)^2 + s\eta^2 \cos((i-j)\varphi/2)^2 - \eta(s+t) \sin((i-j)\varphi/2)^2} \tag{27} \]

where \( \eta \equiv -\tan(\varphi/2) \). The integration over \( s, t \) can be performed straightforward, but it isn’t easy to obtain analytic result by summing over \( i, j \) from that. We will try another approach.

Let us first consider the contribution from \( \langle W \rangle_{x(i) x(j)} \), notice that there is no cusp...
arising. From Eq. (26), we have\(^2\)

\[
\sum_{i,j=1}^{N} \langle W \rangle_{x(i)x(j)} = \sum_{k=1}^{N/2-1} \int_{0}^{1} dt \int_{0}^{1} ds \frac{N \tan(\pi/N)^2}{(s-t)b + 1 + st \tan(\pi/N)^2}
\]

\[
= \sum_{k=2}^{N/2-1} \int_{0}^{1} dt \int_{0}^{1} ds \frac{N \tan(\pi/N)^2}{(s-t)b + 1 + st \tan(\pi/N)^2} + O(N^{-1})
\]

\[
= N \tan^2 \frac{\pi}{N} \left( \sum_{k=2}^{N/2-1} \frac{1}{k} \right) + O(N^{-1})
\]

\[
= N \tan^2 \frac{\pi}{N} \sum_{z=1}^{\infty} \left( \frac{2}{2z-1} - \frac{1}{z} \right) \frac{N/2-1}{\sum_{k=2}^{N/2-1} b^{2z-2}}
\]

where \( b \equiv \tan(\pi/N) \cot(k\pi/N) \). When \( k \in [2, N/2 - 1], (s-t)b + 1 \in [1/2, 3/2] \), thus we can use \( st \tan(\pi/N)^2 \) as a parameter to expand the integrand in second line and find the correction terms are vanishing in the large \( N \) limit. In addition, \( b \in [0, 1/2] \) can be used to expand the functions \( \log(1 \pm b) \) as indicated in last equation of Eq. (28).

For any integer \( z \geq 1 \), analytic result of \( \sum_{k=2}^{N/2-1} b^{2z-2} \) can be obtained. We find that only \( z = 1 \) will survive in the large \( N \) limit. Thus we have,

\[
\lim_{N \to \infty} \sum_{i,j=1}^{N} \langle W \rangle_{x(i)x(j)} = \frac{\pi^2}{2} + O(N^{-1})
\]  

(29)

Now consider \( \langle W \rangle_{x(i)y(j)} \), we need take out the contribution from cusp divergence when \( i = j, j - 1 \). From Eq. (27), we get

\[
\sum_{i \neq j, j-1}^{N} \langle W \rangle_{x(i)y(j)} = \frac{N}{2} \left( \sum_{k=1}^{N/2-1} \int_{0}^{1} dt \int_{0}^{1} ds \frac{1}{s-t} \left( \frac{1}{\cot(\pi/N) \tan(k\pi/N) + s} - (s \to t) \right) \right)
\]

\[
= 2N \sum_{k=2}^{N/2-1} \int_{0}^{1} ds \log \frac{1-s}{s} \sum_{z=1}^{\infty} \frac{s^{2z-1}}{(\cot(\pi/N) \tan(k\pi/N))^2} - \frac{N}{2} \log^2 2
\]

\[
= N \sum_{z=1}^{\infty} \frac{1-2z H(2z)}{4z^2} \left\{ \tan^2(\pi/N) \sum_{k=1}^{N-1} \cot^2(\pi/k) - 2 \right\} - \frac{N}{2} \log^2 2
\]

(30)

where \( H(x) \equiv \psi(1 + x) + \gamma, \psi \) is Digamma function and \( \gamma \) is Euler constant.

\(^2\)In second line, we have calculated \( k = 1 \) case individually. The result is \( N \{ \log 2(\log(1+a) - \log(1-a)) + \log(1+a) \log(1-a) + \text{Li}_2(a) + \text{Li}_2(a/(1+a)) - \text{Li}_2(2a/(1+a)) \}, \) here \( a \equiv \tan^2(\pi/N) \). Its contributions is vanishing in the large \( N \) limit.
We find in the large $N$ limit,

$$
\lim_{N \to \infty} \sum_{i \neq j, j-1}^N \langle W \rangle_{x(i) y(j)} = \#N + \frac{\pi^2}{2} + \mathcal{O}(N^{-1})
$$

with $\#$ being a real finite number,

$$
\# \equiv \sum_{z=1}^{\infty} \frac{1 - 2z H(2z)}{4z^2} \left\{ (-1)^{z+1} 2^{2z} \pi^{2z} \frac{B_{2z}}{(2z)!} - 2 \right\} - \frac{1}{2} \log^2 2
$$

where $B_n$ is the $n$th Bernoulli number. After these lengthy algebra, we obtain

$$
\left[ \langle W_0 \rangle^{1-\text{loop}} - \langle W_0 \rangle^{1-\text{loop}}_{\text{cusps}} \right] = \frac{\lambda}{16\pi^2} \frac{2}{\pi^2} (\pi^{2} + \#N) + \mathcal{O}(N^{-1}) \sim \frac{\lambda}{8} (33)
$$

up to some term proportional to $N$. We also can reproduce this finite part for expectation value of Wilson loop defined on a space-like circle. Parameterize a circle with unit radius by $x^\mu(t) = (0, \cos 2\pi t, \sin 2\pi t, 0)$. By definition, we find its one-loop expectation value,

$$
\langle W_C \rangle_{\text{NSC}} = \frac{\lambda(\pi\mu^2)^\epsilon}{4} \frac{\Gamma(1-\epsilon)}{4\pi^2} \int_0^1 dt \int_0^1 ds \frac{\dot{x}(t) \cdot \dot{x}(s)}{[-(x(t) - x(s))^2]^{1-\epsilon}}
$$

$$
= \frac{-\lambda(\pi\mu^2)^\epsilon}{4} \frac{\Gamma(1-\epsilon)}{4\pi^2} \frac{2\pi}{2^{1-\epsilon}} \int_0^{2\pi} d\theta \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} (\cos \theta)^{k+1} \frac{\Gamma(1 + k - \epsilon)}{\Gamma(1-\epsilon)} \right\}
$$

$$
= \frac{-\lambda(\pi\mu^2)^\epsilon}{4} \frac{\Gamma(1-\epsilon)}{4\pi^2} \frac{2^\epsilon \sqrt{\pi} \Gamma(2-\epsilon)\Gamma(-1/2 + \epsilon)}{\Gamma(1-\epsilon)\Gamma(1 + \epsilon)} = 1 + \frac{\lambda}{8} + \mathcal{O}(\epsilon^1) (34)
$$

5 Comment on generalization to arbitrary curves

We have verified the Eq. (7) on two separated lines in section 3. This result should be able to be used to generalize this equation to any smooth, space-like shape as follows. First one can approximate a smooth space-like curve by a spacelike $n$-sided polygon. Building upon that polygon a zig-zag approximation, we know from the results of section 3 that the finite part of the non-scalar-coupled space-like Wilson loop is recovered. Then taking $n$ to scale like the number of zig-zags, any effects associated with the corners of the $n$-gon, or self-interactions of the $n$-gon edges themselves will contribute only to the piece which scales as the number of zig-zags. The finite part should then reproduce the finite part of the smooth, non-scalar-coupled, space-like curve being approximated. Indeed, we have verified that this proposal for generalizing to general curves works for the case of a circle, by using a regular $n$-gon as an approximation and applying the results of section 3.

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