Bounding the optimal rate of the ICSI and ICCSI problem

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Abstract—In this work we study both the index coding with side information (ICSI) problem introduced by Birk and Kol in 1998 and the more general problem of index coding with coded side information (ICCSI), described by Shum et al in 2012. We estimate the optimal rate of an instance of the index coding problem. In the ICSI problem case, we characterize those digraphs having min-rank one less than their order and we give an upper bound on the min-rank of a hypergraph whose incidence matrix can be associated with that of a 2-design. Security aspects are discussed in the particular case when the design is a projective plane. For the coded side information case, we extend the graph theoretic upper bounds given by Shanmugam et al in 2014 on the optimal rate of index code.

Index Terms—Index coding, network coding, coded side information, broadcast with side information, min-rank.

I. INTRODUCTION

Since its introduction in [6], the problem of index coding has been generalized in a number of directions [1], [3], [8], [13], [14], [16]. It is a problem that has aroused much interest in recent years; from the theoretical perspective, its equivalence to network coding has established it as an important area of network information theory [18], [17]. In the classical case, a central broadcaster has a data file $x \in \mathbb{F}_q^n$. There are $n$ users each of whom already possesses some subset of components of $x$ as its side-information and each of whom requests some component $x_i$ of the file. The index coding problem is to determine the minimum number of transmissions required so that the demands of all users can be met, given that data may be encoded prior to broadcast. This problem can be associated with a directed graph, or a hypergraph if the case is extended to consider a scenario of $m > n$ users. Several authors have given various bounds on the length of an index code, which refers to the number of transmissions used to meet clients’ demands for a given instance of the problem. It is well known that for the case of linear index coding, the min-rank of the associated side-information graph is the minimal number of broadcasts required. In [24], the authors give several graph theoretic upper bounds based on linear programming. In [16] the authors describe the scenario of linear index coding with coded side information. In this model, users may request a linear combination of the data held by the sender and are assumed to each have some set of linear combinations of the data packets. One motivation for this more general model is that it may serve a larger number of applications than the case for uncoded side-information, such as broadcast relay networks and wireless distributed storage systems. The set-up in [16] does not have an obvious representation in the form of a side-information hypergraph. However, as we show here, practically all the results of [24] can be extended to this case.

In this paper we present new bounds on the optimal rate for different instances of the index coding problem. For the case of uncoded side information the problem will be referred to as an index coding with side information (ICSI) problem. For the case of encoded side information we will describe this as an ICCSI instance. In the first part we give bounds on the minimum number of transmissions required for particular instances of the ICSI problem where the corresponding side-information hypergraph can be associated with the incidence matrix of a design. This comprises Sections II-V. The remainder of the paper is concerned with upper bounds on the total transmission time for the ICCSI problem and extends the results of [24] for this more general case. In Section II we give relevant definitions and results on incidence structures such as designs. In Section III the ICSI problem is described. In Section IV, extending results of [15], we characterize those digraphs having min-rank one less than their order. In Section V we give an upper bound on the min-rank of a hypergraph whose incidence matrix can be associated with that of a 2-design and discuss a security aspect for such special instances of the ICSI problem. In Section VI we describe the ICCSI problem before finally giving several upper bounds on the transmission time of an ICCSI instance based on linear programming.

II. PRELIMINARIES

We establish some notation to be used throughout the paper. We will assume that $q$ is a power of a prime $p$, say $q = p^t$. For any positive integer $n$, we let $[n] := \{1, \ldots, n\}$. We write $\mathbb{F}_q$ to denote the finite field of order $q$ and use $\mathbb{F}_q^{n \times t}$ to denote the vector space of all $n \times t$ matrices over $\mathbb{F}_q$.

Given a matrix $X \in \mathbb{F}_q^{n \times t}$ we write $X_i$ and $X^j$ to denote the $i$th row and $j$th column of $X$, respectively. More generally, for subsets $S \subset [n]$ and $T \subset [t]$ we write $X_S$ and $X_T$ to denote the $|S| \times t$ and $n \times |T|$ submatrices of $X$ comprised of the rows of $X$ indexed by $S$ and the columns of $X$ indexed by $T$ respectively. We write $\langle X \rangle$ to denote the row space of $X$.

A finite incidence structure $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, consists of a pair of finite sets $\mathcal{P}$ (its points) and $\mathcal{B}$ (its blocks), and an incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{B}$. We say that $p$ is contained in or is incident with $B$ if $(p, B) \in \mathcal{I}$.
Definition II.1. Let \( t, v, k \) and \( \lambda \) be positive integers. An incidence structure \( D = (P, B, \mathcal{I}) \) is called a \( t-(v, k, \lambda) \) block design if
1. \( |P| = v; \)
2. \( |B| = k \) for all \( B \in \mathcal{B}; \)
3. every \( t \)-set of points of \( P \) are contained in precisely \( \lambda \) blocks of \( B \).

Often a \( t-(v, k, \lambda) \) block design is simply referred to as a \( t \)-design. Designs are well-studied objects in combinatorics with many applications. The interested reader is referred to \([17], [11], [10]\) for further information, but we present sufficient detail here to meet our purposes. The number of blocks \( b \) of a \( t-(v, k, \lambda) \) design is
\[
b = \binom{v}{t}/\binom{k}{t-1}\lambda
\]
and the number of blocks containing any given point of \( P \) is
\[
r = \binom{v-1}{t-1}/\binom{k-1}{t-1}\lambda,
\]
which is its replication number. In the case of a 2-design we have
\[
r = \lambda(v-1)/(k-1).
\]
An important parameter of a \( t \)-design is its order, defined to be \( n = r - \lambda \).

Definition II.2. Let \( S = (P, B, \mathcal{I}) \) be an incidence structure with \( |P| = v \) and \( |B| = b \). Let the points be labelled \( \{p_1, \ldots, p_v\} \) and the blocks be labelled \( \{B_1, \ldots, B_b\} \). An incidence matrix for \( S \) is a \( b \times v \) matrix \( A = (a_{i,j}) \) with entries in \( \{0, 1\} \) such that
\[
a_{i,j} = \begin{cases} 
1 & \text{if } (p_j, B_i) \in \mathcal{I} \\
0 & \text{if } (p_j, B_i) \notin \mathcal{I}
\end{cases}
\]
The code of \( S \) over \( \mathbb{F}_q \) is the subspace \( C_q(S) \) of \( \mathbb{F}_q^{|P|} \) spanned by the rows of \( A \).

Definition II.3. Let \( S \) be an incidence structure and let \( q \) be a prime power, the \( q \)-rank of \( S \) is the dimension of the code \( C_q(S) \) and is written
\[
\text{rank}_q(S) = \text{dim}(C_q(S)).
\]

The following result was proved by Klemm \([19]\). We will see in Section V that this gives an immediate upper bound on the min-rank of a class of instances of the index coding problem.

Theorem II.4. Let \( D = (P, B) \) be a \( 2-(v, k, \lambda) \) design of order \( n \) and let \( p \) be a prime dividing \( n \). Then
\[
\text{rank}_p(D) \leq \frac{|B| + 1}{2}.
\]
Moreover, if \( p \) does not divide \( \lambda \) and \( p^2 \) does not divide \( n \), then
\[
C_p(D) \perp \subseteq C_p(D)
\]
and \( \text{rank}_p(D) \geq v/2 \).

A \( 2-(n^2 + n + 1, n + 1, 1) \) design, for \( n \geq 2 \), is called a projective plane of order \( n \). A projective plane of order \( n \) is an example of a symmetric design, that is, it has the same number of points as blocks, so \( |P| = |B| \).

The following can be read in \([2] \text{ Theorem 6.3.1}\).

Theorem II.5. Let \( \Pi \) be a projective plane of order \( n \) and \( p \) be a prime such that \( p \mid n \). Then the \( p \)-ary code of \( \Pi \), \( C_p(\Pi) \), has minimum distance \( n + 1 \). Moreover the codewords of minimal weight in \( C_p(\Pi) \) are the scalar multiples of the rows of the incidence matrix of \( \Pi \).

Chouinard, in \([9]\), proved that:

Theorem II.6. Let \( C_p(\Pi) \) be a code arising from a projective plane of prime order \( p \). Then no codeword has weight in the interval \([p + 2, 2p - 1]\).

Definition II.7. A digraph is a pair \( \mathcal{G} = (V, E) \) where:
- \( V \) is the set of vertices of \( \mathcal{G} \),
- \( E \subset V \times V \) is the set of arcs (or directed edges) of \( \mathcal{G} \).

An arc of \( \mathcal{G} \) is an ordered pair \( e = (u, v) \in E(\mathcal{G}) \) for some \( u, v \in V \). In the case that \( u \neq v \), the vertex \( u \) is called the tail of \( e \) and \( v \) the head of \( e \). The arc \( e \) is called an out-going arc of \( u \) and an in-coming arc of \( v \). The out-degree of a vertex \( u \), \( \deg_+(u) \), is the number of out-going arcs, and the in-degree of a vertex \( u \), \( \deg_-(u) \), is the number of in-coming arcs. \( \mathcal{G} \) is called an undirected graph, or a graph, if \( (u, v) \in E(\mathcal{G}) \) whenever \( (v, u) \in E(\mathcal{G}) \). If \( \mathcal{G} \) is a graph then each pair of arcs \( (u, v) \) and \( (v, u) \) are represented by the unordered pair \( \{u, v\} \), which is called an edge. The number of vertices of a digraph is called its order.

We assume that all digraphs have finite order.

Definition II.8. A path in a graph \( \mathcal{G} \) (respectively in a digraph), is a sequence of distinct vertices \( (u_1, u_2, \ldots, u_k) \), such that \( \{u_i, u_{i+1}\} \in E(\mathcal{G}) \) (\( \{u_i, u_{i+1}\} \in E(\mathcal{G}) \), respectively) for all \( i \in [k-1] \). If a path is closed, i.e. \( \{u_k, u_1\} \in E(\mathcal{G}) \) (\( \{u_k, u_1\} \in E(\mathcal{G}) \), respectively), then it is called a circuit. A digraph that is not a graph is called acyclic if it contains no circuits. A graph is acyclic if it has no circuits with at least 3 vertices.

Let \( \nu(\mathcal{G}) \) be the circuit packing number of \( \mathcal{G} \), namely, the maximum number of vertex-disjoint circuits in \( \mathcal{G} \). A feedback vertex set of \( \mathcal{G} \) is a set of vertices whose removal destroys all circuits in \( \mathcal{G} \). Let \( \tau(\mathcal{G}) \) denote the minimum size of a feedback vertex set of \( \mathcal{G} \). We denote by \( \alpha(\mathcal{G}) \) the maximum size of vertex subset such that induced subgraph in \( \mathcal{G} \) is acyclic. Since such a subset of vertices is the complement of a feedback vertex set, we have \( \alpha(\mathcal{G}) = |\mathcal{G}| - \tau(\mathcal{G}) \). In the case that \( \mathcal{G} \) is a graph, \( \alpha(\mathcal{G}) \) is the maximum size of an independent (pairwise non-adjacent) set of vertices.

Definition II.9. A clique of a digraph is a set of vertices that induces a complete subgraph of that digraph. A clique cover of a digraph is a set of cliques that partition its vertex set. A minimum clique cover of a digraph is a clique cover having minimum number of cliques. The number of cliques in such a minimum clique cover of a digraph is called the clique cover number of that digraph. We denote by \( \text{cc}(\mathcal{G}) \) the clique cover number of a digraph \( \mathcal{G} \).

Definition II.10. Let \( \mathcal{G} = (V, E) \) be a digraph of order \( n \). A matrix \( M = (m_{i,j}) \in \mathbb{F}_q^{n \times n} \) is said to fit \( \mathcal{G} \) if
\[
m_{i,j} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } (i, j) \notin E
\end{cases}
\]
The min-rank of \( \mathcal{G} \) over \( \mathbb{F}_q \) is defined to be
\[
\minrk_q(\mathcal{G}) = \min \{ \text{rank}_q(M) : M \text{ fits } \mathcal{G} \}
\]
We also have analogous definitions for a graph.

**Definition II.11.** A (directed) hypergraph \( \mathcal{H} \) is a pair \((V, E)\), where \( V \) is a set of vertices and \( E \) is a set of hyperarcs. A hyperarc \( e \) itself is an ordered pair \((v, H)\), where \( v \in V \) and \( H \subseteq V \), they respectively represent the tail and the head of the hyperarc \( e \).

**Definition II.12.** Let \(|V| = n\) and \(|E| = m\). Let the hyperarcs be labelled \(\{e_1, \ldots, e_m\}\), a matrix \(M = (m_{i,j}) \in \mathbb{F}_q^{m \times n}\) fits the hypergraph if
\[
m_{i,j} = \begin{cases} 
1 & \text{if } j \text{ is the tail of } e_i \\
0 & \text{if } j \text{ does not lie in the head of } e_i
\end{cases}
\]
The min-rank of \( \mathcal{H} \) over \( \mathbb{F}_q \) is defined to be
\[
\text{minrk}_q(\mathcal{H}) = \min\{\text{rank}_q(M) : M \text{ fits } \mathcal{H}\}
\]

### III. INDEX CODING WITH SIDE INFORMATION

The Index Coding with Side Information (ICSI) problem is described as follows. There is a unique sender \( S \), who has a data matrix \( X \in \mathbb{F}_q^{n \times k} \). There are also \( m \) receivers, each with a request for a data packet \( X_i \), and it is assumed that each receiver has some side-information, that is, a client \( i \) has a subset of messages \( X_i \), where \( X_i \subseteq [n] \) for each \( i \in [m] \).

The packet requested by \( i \) in denoted by \( X_f(i) \), where \( f : [m] \to [n] \) is a (surjective) demand function. Here we assume that \( f(i) \not\in X_i \) for all \( i \in [m] \). We may assume that each \( i \)th receiver requests only the message \( X_f(i) \), since a receiver requesting more than one message can be split into multiple receivers, each of whom requests only one message and has the same side information set as the original \( \mathcal{H} \).

For the remainder, let us fix \( t, m, n \) to denote those parameters as described above. Then for any \( \mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_n) \), \( \mathcal{X}_i \subseteq [n] \) and map \( f : [m] \to [n] \), the corresponding instance of the ICSI problem (or the ICSI instance) is denoted by \( \mathcal{I} = (\mathcal{X}, f) \).

It can also be conveniently described by a side-information (directed) hypergraph \([1]\).

**Definition III.1.** Let \( \mathcal{I} = (\mathcal{X}, f) \) be an ICSI instance. The corresponding *side information hypergraph* \( \mathcal{H} = \mathcal{H}(\mathcal{X}, f) \) has vertex set \( V = [n] \) and hyperarc set \( E \), defined by
\[E = \{(f(i), X_i) : i \in [m]\}.
\]

**Remark III.2.** If we have \( m = n \) and \( f(i) = i \) for all \( i \in [n] \), the corresponding side information hypergraph has precisely \( n \) hyperarcs, each with a different origin vertex. It is simpler to describe such an ICSI instance as a digraph \( \mathcal{G} = ([n], E) \), the so-called side information digraph \([3]\). For each hyperarc \((i, X_i) \) of \( \mathcal{H} \), there are \( |X_i| \) arcs \((i, j)\) of \( \mathcal{G} \), for \( j \in X_i \). Equivalently, \( E = \{(i, j) : i, j \in [n], j \in X_i\} \).

**Definition III.3.** Let \( N \) be a positive integer. We say that the map
\[
E : \mathbb{F}_q^{n \times t} \to \mathbb{F}_q^N,
\]
is an \( \mathbb{F}_q \)-code of length \( N \) for the instance \( \mathcal{I} = (\mathcal{X}, f) \) if for each \( i \in [m] \) there exists a decoding map
\[
D_i : \mathbb{F}_q^N \times \mathbb{F}_q^{|X_i|} \to \mathbb{F}_q,
\]
satisfying
\[
\forall X \in \mathbb{F}_q^{n \times t} : D_i(E(X), X_i) = X_f(i),
\]
in which case we say that \( E \) is an \( \mathcal{I} \)-IC. \( E \) is called an \( \mathbb{F}_q \)-linear \( \mathcal{I} \)-IC if \( E(X) = LX \) for some \( L \in \mathbb{F}_q^{N \times n} \), in which case we say that \( L \) represents the code \( E \). If \( t = 1 \), \( E \) is called scalar linear.

The following well-known results quantify the minimal length of a linear index code in respect of its side-information hypergraph (cf. \([13]\)).

**Lemma III.4.** An \( \mathcal{I}(\mathcal{X}, f) \)-IC of length \( n \) over \( \mathbb{F}_q \) has a linear encoding map if and only if there exists a matrix \( L \in \mathbb{F}_q^{N \times n} \) such that for each \( i \in [m] \), there exists a vector \( u(i) \in \mathbb{F}_q^m \) satisfying
\[
\text{Supp}(u(i)) \subseteq X_i \quad (1)
\]
\[
u(i) + e_{f(i)} \in (L). \quad (2)
\]

**Theorem III.5.** Let \( \mathcal{I} = (\mathcal{X}, f) \) be an instance of the ICSI problem, and \( \mathcal{H} \) its hypergraph. Then the optimal length of a \( q \)-ary linear \( \mathcal{I} \)-IC is \( \text{minrk}_q(\mathcal{H}) \). Achievable schemes based on graph-theoretic models for constructing index codes (i.e. upper bounds for index coding) were largely studied \([1, 3, 8, 24]\).

One of these methods comes from the well-known fact that all the users forming a clique in the side information digraph can be simultaneously satisfied by transmitting the sum of their packets \([6]\). This idea shows that the number of cliques required to cover all the vertices of the graph (the clique cover number) is an achievable upper bound.

A lower bound on the min-rank of a digraph was given in \([3]\). An acyclic digraph has min-rank equal to its order (see for instance \([3]\)) and for any subgraph \( \mathcal{G}' \) of a graph \( \mathcal{G} \) we have
\[
\text{minrk}_q(\mathcal{G}') \leq \text{minrk}_q(\mathcal{G}).
\]
Let \( M \) be a matrix that fits \( \mathcal{G} \), the sub-matrix \( M' \) of \( M \) restricted on the rows and columns indexed by the vertices in \( \mathcal{V}(\mathcal{G}') \) is a matrix that fits \( \mathcal{G}' \). These two results are summarized in the following theorem.

**Theorem III.6.** Let \( \mathcal{G} \) be a digraph. Then
\[
\alpha(\mathcal{G}) \leq \text{minrk}_q(\mathcal{G}) \leq \text{cc}(\mathcal{G}).
\]

Instead of covering with cliques, one can cover the vertices with circuits. In \([8]\) the circuit-packing bound was implicitly introduced by the authors. Indeed, Chaudhry and Sprintson construct a linear index code partitioning the graph of the ICSI instance in disjoint circuits. The same bound was explicitly given in the work of Dau et al. \([15]\). It is based on the observation that the existence of a circuit of length \( k \) in the side-information digraph \( \mathcal{G} \) requires at most \( k - 1 \) transmissions to satisfy the demands of the corresponding \( k \) users. Therefore a collection of \( \nu \) vertex disjoint circuits corresponds to a ‘saving’ of at least \( \nu \) transmissions. The bound is stated as follows: Let \( \nu(\mathcal{G}) \) be the circuit-packing number of a graph \( \mathcal{G} \) of order \( n \). Then
\[
\text{minrk}_q(\mathcal{G}) \leq n - \nu(\mathcal{G}).
\]
In [26] the following result is given, leading the authors to introduce the \textit{partition multicast scheme}, which outperforms the circuit-packing number.

**Proposition III.7.** Let $G$ be a graph of order $n$. Then
\[ \min \text{rk}_q(G) \leq n - \min_{v \in V} \text{deg}_G(v), \]
for any $q > n$.

The broadcast rate of an IC-instance $\mathcal{I}$ [11] is defined as follows, with respect to a prime $p$.

**Definition III.8.** Let $\mathcal{I} = (\mathcal{V}, \mathcal{E})$ be an IC instance. We denote by $\beta_i(\mathcal{I})$ the minimal number of symbols required to broadcast the information to all receivers, when the block length is $i$, over all possible extensions of $\mathbb{F}_p$, i.e.
\[ \beta_i(\mathcal{I}) = \inf_q \{ N \mid \exists \text{ a } q\text{-ary index code of length } N \text{ for } \mathcal{I} \}. \]
Moreover we denote by $\beta(\mathcal{I})$ the limit
\[ \beta(\mathcal{I}) = \lim_{i \to \infty} \frac{\beta_i(\mathcal{I})}{i} = \inf \frac{\beta_i(\mathcal{I})}{i}. \]

In the following, we will also use the notation $\beta(G)$ to indicate the broadcast rate of any instance that has $G$ as side-information graph.

The graph parameter $\min \text{rk}_q(G)$ completely characterizes the length of an optimal linear index code. Bar-Yossef et al. [3, 4] showed that in various cases linear codes attain the optimal word length, and they conjectured that the minimum broadcast rate of a graph $G$ was $\min \text{rk}_2(G)$ also for non-linear codes. Lubetzky and Stav in [20] disproved this conjecture.

In the works of Alon et al. [11] and Shanmugam et al. [23], it was shown that results based on partitioning the vertices of a graph $G$ in cliques lead to a family of stronger bounds on $\beta(G)$, starting with an LP relaxation called \textit{fractional chromatic number} [11] and the stronger \textit{fractional local chromatic number} [23]. In [24] the authors extended all these schemes to the case of hypergraphs.

**IV. ON DIRECTED GRAPHS WITH MIN-RANK ONE LESS THAN THE ORDER**

In the work of Dau et al. [15] the authors characterize the undirected graphs of order $n$ having min-rank $n - 1$. Here we extend this result to include directed graphs over a sufficiently large field. Our result relies in part on the following lemma, which is a construction of a digraph $G'$ of min-rank one less that a digraph $G$, obtained from $G$ by contracting an arc.

**Lemma IV.1.** Let $G = (\mathcal{V}, \mathcal{E})$ be a directed graph of order $n$ such that there exist $i_1, i_2 \in \mathcal{V}$ with
\begin{itemize}
  \item[(1)] $(i_1, i_2) \in \mathcal{E}$ and $(i_2, i_1) \notin \mathcal{E}$
  \item[(2)] $\text{deg}_G(i_1) = 1.$
\end{itemize}
Let $G' = (\mathcal{V}', \mathcal{E}')$ with $\mathcal{V}' = \mathcal{V} \setminus \{i_1\}$ and
\[ \mathcal{E}' = (\mathcal{E} \cup \{(i, j_2) \mid (i, j_1) \in \mathcal{E}\}) \setminus \{(i_1, i_2)\} \cup \{(j, i_1) \mid (j, i_1) \in \mathcal{E}\}. \]
Then
\[ \min \text{rk}_q(G) = \min \text{rk}_q(G') + 1 \]
for any $q$.

**Proof.** Let $M = (m_{i,j})$ be a matrix that fits $G$ of minimum rank. We may assume that $i_1 = 1$ and $i_2 = 2$ so that the first two rows of $M$ are
\[ M_1 = (1, \alpha, 0, \ldots, 0) \]
and
\[ M_2 = (0, 1, m_{2,3}, \ldots, m_{2,n}). \]
If $\alpha = 0$ then it is easy to check that deleting the first row and the first column of $M$ we obtain $M'$ of rank $\text{rank}(M) - 1$ that fits $G'$.

Now suppose that $\alpha \neq 0$. We may assume that the rows $M_1, M_2, \ldots, M_{\min \text{rk}_q(G)}$ are linearly independent.

For each vertex $i \in \mathcal{V}' \setminus \{1\}$, label the corresponding vertex in $\mathcal{V}'$ by $i - 1$. Then construct the $(n - 1) \times (n - 1)$ matrix $M'$ whose $i$-th row is obtained from the $i + 1$-th row of $M$ in the following way: for $i = 1, \ldots, \min \text{rk}_q(G) - 1$ let
\[ M'_i = (m_{i+1,1} + m_{i+1,2}, m_{i+1,3}, \ldots, m_{i+1,n}), \]
and for $i = \min \text{rk}_q(G), \ldots, n - 1$ we define
\[ M'_i = (m_{i+1,1} + m_{i+1,2} - \lambda_1(1 + \alpha), m_{i+1,3}, \ldots, m_{i+1,n}) \]
where $\lambda_1 \in \mathbb{F}_q$ satisfies $M_{i+1} = \sum_{r=1}^{\min \text{rk}_q(G)} \lambda_r M_r$ for some $\lambda_r$. The matrix $M'$ fits $G'$, so
\[ \min \text{rk}_q(G') \leq \text{rank}(M') \leq \min \text{rk}_q(G) - 1. \]

Conversely, let $M' = (m'_{i,j})$ be a matrix that fits $G'$ having rank $\min \text{rk}_q(G')$ and suppose the rows $M'_1, M'_2, \ldots, M'_{\min \text{rk}_q(G')}$. The matrix $M'$ is linearly independent. Let $I = \{j \mid (j, i) \in \mathcal{E}\}$ be the set of vertices of $G$ with outgoing arcs directed to $1$. We construct the matrix $M$ such that
\[ M_1 = (1, -1, 0, \ldots, 0), \]
\[ M_i = (m'_{i-1,1}, 0, m'_{i-1,2}, \ldots, m'_{i-1,n-1}), \]
for $i \in I \cap \{2, \ldots, \min \text{rk}_q(G') + 1\}$ and
\[ M_i = (0, m'_{i-1,1}, m'_{i-1,2}, \ldots, m'_{i-1,n-1}), \]
for $i \in \{(n \setminus I) \cap \{2, \ldots, \min \text{rk}_q(G') + 1\}$.

For $i > \min \text{rk}_q(G') + 1$ we have that the $i - 1$-th row of $M'$ is given by
\[ M'_{i-1} = \sum_{r=1}^{\min \text{rk}_q(G')} \lambda_r M'_r, \]
for some $\lambda_r \in \mathbb{F}_q$. If $i \in I$, we put
\[ M_i = (m'_{i-1,1}, 0, m'_{i-1,2}, \ldots, m'_{i-1,n-1}) \]
and hence obtain
\[ M_i = \lambda M_1 + \sum_{r=2}^{\min \text{rk}_q(G')+1} \lambda_{r-1} M_r, \]
where the \( \lambda_r \) are the coefficients in the linear combination of \( M'_{i-1} \), with respect to the first \( \minrk_q(G') \) rows of \( M' \), and
\[
\lambda = \sum_{r \notin I} \lambda_r - \sum_{r=2}^{\minrk_q(G') + 1} \lambda_r - 1 M_r
\]
and we have
\[
M_i = \lambda M_1 + \sum_{r=2}^{\minrk_q(G') + 1} \lambda_r - 1 M_r
\]
and
\[
\minrk_q(G') = \begin{vmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1
\end{vmatrix},
\]
which fits \( \mathcal{G} \). We have \( M_3 = M_4 = M_1 + M_2 \), constructing \( M' \) as in the lemma above we obtain
\[
M' = \begin{vmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{vmatrix},
\]
\( M' \) fits \( G' \). Conversely, from \( M' \) we obtain \( M \), and
\[
\minrk_q(G') = \minrk_q(G) + k \leq n - 2.
\]

**Example IV.2.** Let \( \mathcal{G} \) and \( G' \) be the two digraphs shown in Figure IV.2. The nodes 1 and 2 of \( G \) satisfy the conditions of Lemma IV.1 so we can reduce \( \mathcal{G} \) to \( G' \). Consider the matrix
\[
\begin{vmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1
\end{vmatrix},
\]
which fits \( \mathcal{G} \). We have \( M_3 = M_4 = M_1 + M_2 \), constructing \( M' \) as in the lemma above we obtain
\[
M' = \begin{vmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{vmatrix},
\]
\( M' \) fits \( G' \). Conversely, from \( M' \) we obtain \( M \), and
\[
\minrk_q(G') = \minrk_q(G) + k \leq n - 2.
\]

**Lemma IV.3.** Let \( \mathcal{G} \) be a directed graph of order \( n \) such that \( \tau(\mathcal{G}) = 2 \). Then \( \minrk_q(\mathcal{G}) = n - 2 \), for any \( q > n \).

**Proof.** As observed in Theorem III.6, \( n - \tau(\mathcal{G}) \leq \minrk_q(\mathcal{G}) \), so we need only to prove that \( \minrk_q(\mathcal{G}) \leq n - 2 \).

We may suppose without loss of generality that there does not exist \( i \in V \) with out-degree less than 1, otherwise, from Lemma IV.1 we can delete the node \( i \) and consider the induced subgraph \( G'' \), which satisfies \( \minrk_q(G'') = \minrk_q(G) - 1 \).

Since \( \tau(\mathcal{G}) = 2 \), we have \( \nu(\mathcal{G}) \in \{1, 2\} \). Since \( \minrk_q(\mathcal{G}) \leq n - \nu(\mathcal{G}) \), if \( \nu(\mathcal{G}) = 2 \) then we have our claim immediately. Assume then that \( \nu(\mathcal{G}) = 1 \). We apply Lemma IV.1 iteratively. Note that each time we reduce a graph \( \mathcal{G} \) by an appropriate arc contraction, we obtain \( G' \) with \( \tau(G') = 2 \) and \( \nu(G') = 1 \). Moreover, for each contraction of an arc of the graph, we only shorten the circuits that pass through the node that we delete, and we do not create any new circuit from the fact that the out-degree of the node is 1.

At the point that Lemma IV.1 is no longer applicable, there are two possible cases:

1) the out-degree of each node of the reduced graph \( G' \) is at least 2;
2) there exists \( i_1 \) with out-degree 1 and \( (i_1, i_2), (i_2, i_1) \in E' \).

This last case is not possible, in fact if we consider the circuit \( C = (i_1, i_2) \), from \( \tau(G') = 2 \) we have that there exists a circuit \( C' \) which remains after deleting \( i_2 \). Then, \( C' \) does not pass through \( i_1 \) otherwise it has to pass through \( i_2 \). Then \( C \) and \( C' \) are disjoint, but this is not possible because \( \nu(G') = 1 \).

Therefore, reducing \( \mathcal{G} \) we obtain \( G' \) with \( k \) fewer nodes and all nodes have out-degree at least 2. Then from Proposition III.7 and Lemma IV.1 it follows that
\[
\minrk_q(G) = \minrk_q(G') + k \leq n - 2.
\]

**Corollary IV.4.** Let \( \mathcal{G} \) be a directed graph of order \( n \) such that \( \tau(\mathcal{G}) = 2 \). Then for any \( q > n \), \( \minrk_q(G) = \beta(G) \).

We have now our main result of this section.

**Corollary IV.5.** Let \( \mathcal{G} \) be a graph of order \( n \) and let \( q > n \). Then \( \minrk_q(G) = n - 1 \) if and only if \( \tau(G) = 1 \). Moreover in that case we have \( \beta(G) = n - 1 \) if and only if \( \tau(G) = 1 \).

**Proof.** If \( \tau(G) = 1 \) then \( \nu(G) = 1 \) and we have \( \minrk_q(G) = n - 1 \).

Conversely towards a contradiction assume that \( \tau(G) = 2 \). Then consider a subgraph \( G' \) of \( G \) with \( \tau(G') = 2 \). From Lemma IV.3 we have our claim.

This last theorem implies that the problem of deciding whether or not a digraph has min-rank \( n - 1 \), over a sufficiently large field, can be solved in polynomial time, using a depth-first search algorithm (see for instance [12]) that verifies in a polynomial time whether or not a graph is acyclic.

**Corollary IV.6.** Let \( \mathcal{G} \) be a digraph of order \( n \) and \( q > n \). Then deciding whether \( \minrk_q(G) = n - 1 \) can be done in polynomial time \( (O(n^5)) \).

**Remark IV.7.** In the final stages of the writing of this paper we learned of Ong’s result [21]. In fact Lemma IV.3 (although obtained independently) and its immediate corollary follows from [21, Theorem 1], which is a stronger result, since it holds without any restrictions on \( q \). That is,

**Theorem IV.8** ([21]). Let \( \mathcal{G} \) be a directed graph of order \( n \) satisfying \( \tau(G) \leq 2 \). Then
\[
\minrk_q(G) = \beta(G) = n - \tau(G).
\]

The proof of Theorem IV.8 relies on showing that \( \mathcal{G} \) contains a particular subgraph \( \mathcal{G}_{sub} \) and then devising a coding scheme.
for \( G \) based on the existence of \( G_{2,n} \). The proof given in \([21]\) is a non-trivial graph-theoretic proof and goes through a careful case-by-case analysis. The proof of Lemma \([IV.2]\) given here is rather more straightforward, being based on the construction of a new graph \( G' \) obtained by iterative contractions of the original graph \( G \), following from Lemma \([IV.1]\). Such a result could be helpful also to decrease the size of a graph and thus to optimize the computation of the min-rank of the graph. The hypothesis that \( q > n \) follows since we invoke the partition multicast solution (Proposition \([III.7]\)), therefore requiring the existence of a maximum distance separable code.

In the following table we report the values of the min-rank for graphs and directed graphs with near-extreme min-rank (i.e., \( 1, 2, n - 2, n - 1 \) and \( n \)).

![Figure 2. Forbidden subgraph](image)

| Minrank | Graph \( G \)                                           | Digraph \( D \)         |
|---------|----------------------------------------------------------|-------------------------|
| 1       | \( G \) is complete (trivial)                            | \( D \) is complete (trivial) |
| 2       | \( G \) is 2-colorable [22]                             | for \( q = 2 \), if \( D \) is 3-fair colorable [15] |
| \( n - 2 \) | \( G \) has maximum matching 2 and does not contain the graph in Figure 2 [15] | unknown |
| \( n - 1 \) | \( G \) is a star graph [15]                             | for \( q > n \), \( \tau(D) = 1 \) Corollary \([IV.5]\) for any \( q \), \( \tau(D) = 1 \) Theorem \([V.3]\) |
| \( n \)  | \( G \) has no edges (trivial)                           | \( D \) is acyclic (trivial) [3] |

V. A BOUND FROM T-DESIGNS

In this section we study the case for which an incidence structure, in particular a 2-(\( r^2 + r + 1, r + 1, 1 \)) or projective plane, arises from the side information. This yields an immediate upper bound on the min-rank of the hypergraph, based on known results on the ranks of incidence matrices. Furthermore, we show that secrecy and privacy are attainable for such configurations. Towards secrecy, we show that if an instance fits a projective plane, then a receiver may recover only its requested data, and no more. On the matter of privacy, we identify a constraint on the side information of an adversary hearing the broadcast such that it cannot access the receivers’ requested data. We may assume without loss of generality that \( t = 1 \).

Definition V.1. We said that an instance, \( I = (X, f) \), of the ICSI problem contains an incidence structure \( S = (\mathcal{P}, B) \) if

1) \( \mathcal{P} = [n] \) and \( |B| \leq m \);

2) for each \( i \in [m] \) there exists \( B \in B \) such that \( f(i) \in B \) and \( B \setminus \{f(i)\} \subseteq X_i \).

Moreover we said that the instance coincides with the incidence structure \( \mathcal{S} \) if the following condition is satisfied.

\[ 2' \) for each \( i \in [m] \) there exists \( B \in B \) such that \( f(i) \in B \) and \( B \setminus \{f(i)\} = X_i \).

We immediately obtain the following proposition.

**Proposition V.2.** Let \( I = (X, f) \) be an instance of ICSI problem and \( \mathcal{H} \) be the corresponding hypergraph. If the instance contains a \( 2-(n, k, \lambda) \) design \( D = (\mathcal{P}, B) \) then for all \( q \) a power of a prime \( p \) such that \( p \) divides the order of \( D \) it holds that

\[ \text{minrk}_q(\mathcal{H}) \leq \frac{m + 1}{2}. \]

**Proof.** Let \( D \) be the incidence matrix of \( D \). Then for the Theorem \([III.4]\) we have that the \( p \)-rank of \( D \) is less or equal to \( \frac{m + 1}{2} \).

Now, it is easy to check that \( D \) fits \( \mathcal{H} \), so

\[ \text{minrk}_q(\mathcal{H}) \leq \text{rank}_q(D) \leq \text{rank}_p(D) \]

and that concludes the proof.

**Remark V.3.** To compute the min-rank of a hypergraph is an NP-hard problem [22], however, if there exists a 2-design as in Proposition \([V.2]\) it is possible to have a bound on this value and we can use the linearly independent rows of its incidence matrix to decrease the number of transmissions. We remark further that this result does not require \( q \) to be large, and shows the existence of a class of instances with transmission rate much less than predicted by other bounds.

For example, it is known that if an instance fits the incidence matrix of a projective plane of order \( r \) and \( q > r^2 + r + 1 \) then \( \text{minrk}_q(\mathcal{H}) \leq r^2 + r + 1 - r = r^2 + 1 \) (see, for example [5]), which is significantly greater than the bound \( \text{minrk}_q(\mathcal{H}) \leq (r^2 + r + 2)/2 \), given by Proposition \([V.2]\).

**Example V.4.** Consider the instance of the ICSI problem \( I \) given by \( n = m = 7 \), and \( f(i) = i \) for \( i = 1, \ldots, 7 \). Let the side information be

\[ X_1 = \{2, 3\}, X_2 = \{6, 7\}, X_3 = \{5, 7\}, X_4 = \{2, 5\}, \]

\[ X_5 = \{1, 6\}, X_6 = \{3, 4\}, X_7 = \{1, 4\}. \]

Consider the blocks

\[ B_1 = \{1, 2, 3\}, B_2 = \{2, 6, 7\}, B_3 = \{3, 5, 7\}, B_4 = \{2, 4, 5\}, \]

\[ B_5 = \{1, 5, 6\}, B_6 = \{3, 4, 6\}, B_7 = \{1, 4, 7\}. \]

These blocks form the Fano plane as in Figure 3. This is a 2-(\( 7, 3, 1 \)) design of order 2 and the design is contained in the side information. The 2-rank of the design is 4. Then we can consider 4 linearly independent rows of the incidence matrix of the Fano plane, and encode the message using those reducing the number of transmissions from 7 to 4.

It can be checked that distribution of the ranks of the matrices that fit this incidence is given by

\[ (4,1), (5, 238), (6, 6575), (7, 9570), \]
thus the bound is sharply met in this instance. Moreover, an
optimal encoding matrix \( L \) for this instance must have row
space spanned by the rows of this incidence matrix; there
is a unique optimal solution, up to left multiplication by an
invertible matrix.

Consider now an instance \( \mathcal{I} = (\mathcal{X}, f) \) of the ICSI problem
containing a \( 2-(p^2 + p + 1, p + 1, 1) \) design, where \( p \)
is a prime number. Suppose the matrix \( L \) as above is used as an encoding
matrix. Then we obtain the following result.

**Theorem VI.6.** If \( |\mathcal{X}_A| \leq 2p - 2 \) and for each block \( B \) of the
design \( |\mathcal{X}_A \cap B| \leq p - 1 \), then \( A \) is not able to recover \( X_j \)
for any \( j \notin \mathcal{X}_A \).

**Proof.** If \( p \) is even, then the result follows from the fact that
\( |\mathcal{X}_A| \leq 1 = d - 2 \). Let \( p \) be odd. We know from Theorem
**II.6** that in the code generated by the incidence matrix of a
\( 2-(p^2 + p + 1, p + 1, 1) \) design there are no codewords with
weights in \([p + 2, 2p - 1]\). To recover the message \( X_j \), \( A \)
needs a codeword of weight \( p + 1 \). Such codewords are those
corresponding to some block \( B \), that is a vector of the form

\[
\sum_{i \in B} e_i
\]

and its scalar multiples.

So \( A \) recovers \( X_j \) if and only if there exists \( u + e_j \in C_p \)
with \( \text{Supp}(u) \subseteq \mathcal{X}_A \) and \( |\text{Supp}(u)| = p \). Here \( C \)
means the code of the projective space. Then \( \text{Supp}(u + e_j) = B \)
for some block \( B \), and so \(|(\mathcal{X}_A \cup \{j\}) \cap B| \geq p + 1 \). \( \square \)

VI. INDEX CODING WITH CODED SIDE INFORMATION

In [25] the authors generalized the index coding problem
so that coded packets of a data matrix \( X \) may be broadcast
or part of a user’s cache. This finds applications, for example,
in broadcast channels with helper relay nodes. We present the
model with coded side information in the following section.

A. Preliminaries on the ICCSI Problem

As before there is a data matrix \( X \in \mathbb{F}_q^{n \times t} \) and a set of
\( m \) receivers or users. For each \( i \in [m] \), the \( i \)th user seeks
some linear combination of \( X \), say \( R_iX \) for some \( R_i \in \mathbb{F}_q^n \).
A user’s cache comprises a pair of matrices

\[
V(i) \in \mathbb{F}_q^{d_i \times n} \quad \text{and} \quad \Lambda(i) \in \mathbb{F}_q^{d_i \times t}
\]

related by the equation

\[
\Lambda(i) = V(i)X.
\]

While \( X \) is unknown to user \( i \), it is assumed that any vector \( v \)
in the row spaces of \( V(i) \) and the respective \( \lambda = vX \) can
be generated at the \( i \)th receiver. We denote these respective
row spaces by \( \mathcal{X}(i) := (V(i)) \) and \( \mathcal{L}(i) := \{v \cdot X \mid v \in \mathcal{X}(i)\} \)
for each \( i \). The side information of the \( i \)th user is \( (\mathcal{X}(i), \mathcal{L}(i)) \).
Similarly, the sender has the pair of row spaces \( (\mathcal{X}(S), \mathcal{L}(S)) \)
for matrices

\[
V(S) \in \mathbb{F}_q^{d_S \times n} \quad \text{and} \quad \Lambda(S) = V(S)X \in \mathbb{F}_q^{d_S}
\]

and does not necessarily possess \( X \) itself.

The \( i \)th user requests a coded packet \( R_iX \in \mathcal{L}(S) \) with
\( R_i \in \mathcal{X}(S) \setminus \mathcal{X}(i) \). We denote by \( R \) the \( m \times n \) matrix over \( \mathbb{F}_q \)
with each \( i \)th row equal to \( R_i \). The matrix \( R \) thus represents
the requests of all \( m \) users. We denote by

\[
\mathcal{X} := \{A \in \mathbb{F}_q^{m \times n} : A_i \in \mathcal{X}(i), i \in [m]\},
\]

Figure 3. Fano plane
so that \( \mathcal{X} = \bigoplus_{i \in [m]} \mathcal{X}^{(i)} \) is the direct sum of the \( \mathcal{X}^{(i)} \) as a vector space over \( \mathbb{F}_q \). Similarly, we write \( \bigoplus \mathcal{X}(S) := \bigoplus_{i \in [m]} \mathcal{X}^{(i)} = \{ Z \in \mathbb{F}_q^{m \times n} : \forall i \in [m] : Z_i \in \mathcal{X}^{(i)} \}. \)

For the remainder, we let \( \mathcal{X}, \mathcal{X}(S), \bigoplus \mathcal{X}(S), R \) be as defined above and write \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \) to denote an instance of the ICCSI problem for these parameters. As before, for the ICCSI instance \( \beta_i(\mathcal{I}) \) denotes the minimum broadcast rate for block-length \( t \) where the encoding is over all possible extensions of \( \mathbb{F}_p \). That is, for \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \)

\[
\beta_i(\mathcal{I}) = \inf_{q} \{ N \mid \exists a \text{-q-ary index code of length } N \text{ for } \mathcal{I} \}.
\]

The optimal broadcast rate is given by the limit

\[
\beta(\mathcal{I}) = \lim_{t \to \infty} \frac{\beta_i(\mathcal{I})}{t} = \inf_{t} \frac{\beta_i(\mathcal{I})}{t}.
\]

**Definition VI.1.** Let \( N \) be a positive integer. We say that the map

\[
E : \mathbb{F}_q^{n \times t} \to \mathbb{F}_q^{N},
\]

is an \( \mathbb{F}_q \)-code for \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \) of length \( N \) if for each \( i \)th receiver, \( i \in [m] \) there exists a decoding map

\[
D_i : \mathbb{F}_q^{N} \times \mathcal{X}^{(i)} \to \mathbb{F}_q^{t},
\]

satisfying

\[
\forall X \in \mathbb{F}_q^{n \times t}: D_i(E(X), A) = R_i X,
\]

for some vector \( A \in \mathcal{X}^{(i)} \), in which case we say that \( E \) is an \( \mathcal{I} \)-IC. \( E \) is called an \( \mathbb{F}_q \)-linear \( \mathcal{I} \)-IC if \( E(X) = LV(S) X \) for some \( L \in \mathbb{F}_q^{N \times ds} \) in which case we say that \( L \) represents the code \( E \).

Given an instance \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \) and a matrix \( L \in \mathbb{F}_q^{N \times ds} \) that represents an \( \mathcal{I} \)-IC, we write \( L \) to denote the space \( \langle LV(S) \rangle \).

We have the following (see [5], [25]).

**Lemma VI.2.** Let \( L \in \mathbb{F}_q^{N \times ds} \). Then \( L \) represents a \( \mathbb{F}_q \)-linear \( \mathcal{I} \)-IC index code of length \( N \) if and only if for each \( i \in [m] \), \( R_i \in L + \mathcal{X}^{(i)} \).

**Remark VI.3.** If the equivalent conditions of the above lemma hold we have that for each \( i \in [m] \), \( R_i = b^{(i)} L V(S) + a^{(i)} V^{(i)} \) for some vectors \( a^{(i)}, b^{(i)} \). So User \( i \) decodes its request by computing

\[
R_i X = b^{(i)} L V(S) X + a^{(i)} V^{(i)} X = b^{(i)} Y + a^{(i)} \Lambda^{(i)},
\]

where \( Y \) is the received message.

**Remark VI.4.** The ICCSI problem as introduced before is indeed a special case of the ICCSI problem. Setting \( V(S) \) to be the \( n \times n \) identity matrix, \( R_i = e_i^{(i)} \) and \( V^{(i)} \) to be the \( d_i \times n \) matrix with rows \( V_j^{(i)} = e_i \) for each \( j \in \mathcal{X}_i \) yields \( \mathcal{X}^{(i)} = \{ e_j : j \in \mathcal{X}_i \} \), so that \( \text{Supp}(v) \subset \mathcal{X}_i \) if and only if \( v \in \mathcal{X}^{(i)} \).

The analogue of the min-rank is as follows:

**Definition VI.5 (5).** The min-rank of the instance \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \) of the ICCSI problem over \( \mathbb{F}_q \) is

\[
\kappa(\mathcal{I}) = \min \left\{ \text{rank}(A + R) : \begin{array}{l}
\text{A } \in \mathbb{F}_q^{m \times n}, \\
A_i \in \mathcal{X}^{(i)} \cap \mathcal{X}(S), \forall i \in [m]
\end{array} \right\}.
\]

Note that \( \kappa(\mathcal{I}) \) measures the rank distance of the \( m \times n \) matrix \( R \) to the \( \mathbb{F}_q \)-linear matrix code \( \mathcal{X} \cap (\bigoplus \mathcal{X}(S)) \).

As in the ICSI case, the length of an optimal \( \mathbb{F}_q \)-linear ICCSI index code is characterized by the min-rank of the instance.

**Lemma VI.6 (5).** The length of an optimal \( \mathbb{F}_q \)-linear index code for \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \) is \( \kappa(\mathcal{I}) \).

**B. Approaches from Integer and Linear Programming**

In this section we generalize all the bounds given in [24] (which themselves are generalizations of [26]) to the case of the ICCSI problem. We start with the following definition, introduced in [25] as a coding group, wherein a procedure to detect such a subset is given. It is easy to see that this definition generalizes the definition of a hyperclique for the ICSI case given in [24].

**Definition VI.7.** Let \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \) be an instance of the ICCSI problem. A subset of receivers \( C \subset [m] \) is called generalized clique if and only if either of the following equivalent conditions hold:

1) there exists \( v \in \mathcal{X}(S) \) such that \( \langle v \rangle \subset \langle R_i \rangle + \mathcal{X}^{(i)} \) for all \( i \in C \),

2) \( \text{rank}(R_C + A_C) = 1 \) for some \( m \times n \) matrix \( A \in \mathcal{X} \cap (\bigoplus \mathcal{X}(S)) \).

For simplicity in the following we refer to a generalized clique just as a clique.

The demand \( R_i X \) of each user \( i \) of a clique can be met by sending the message \( v X \) and hence a set of \( \ell \) cliques that partitions the set \([m]\) ensures that all requests can be delivered in at most \( \ell \) transmissions. Minimizing this number for a specific instance can be found via integer programming (see [7], [26], [24]). Recall that the optimal solution of the LP-relaxation of an IP problem returns rational values.

**Definition VI.9.** We denote by \( C \) the set of all cliques of \( \mathcal{I} = (\mathcal{X}, \mathcal{X}(S), R) \). For each clique \( C \in \mathcal{C} \) define the set

\[
\mathcal{R}(C) := \{ v \in \mathbb{F}_q^m : R_i \in \langle v \rangle + \mathcal{X}^{(i)} \ \forall i \in C \}.
\]

**Definition VI.10.** We define the generalized clique cover number of \( \mathcal{I} \), denoted by \( \varphi(\mathcal{I}) \), to be the optimal solution of the following integer programme:

\[
\min \sum_{C \in \mathcal{C}} y_C
\]
where

Consider the instance bound on the transmission rate of the instance over all possible \( \phi \) is called the fractional local generalized clique cover number \( \phi_f(I) \).

**Definition VI.11.** For each \( C \in \mathcal{C} \) fix a vector \( v_C \in \mathcal{R}(C) \). We define the following integer programme with respect to the vectors \( v_C \).

\[
\min k \\
\text{s.t.} \quad \sum_{C: v_C \notin X^{(j)}} y_C \leq k \text{ for all } j \in [m] \\
\quad \sum_{C: j \in C} y_C = 1 \text{ for all } j \in [m] \\
\quad y_C \in \{0, 1\} \text{ for all } C \in \mathcal{C}.
\]

(3)

The LP relaxation of (3) (so with the relaxed constraint \( 0 \leq y_C \leq 1 \) for all \( C \)) is the fractional generalized clique cover number \( \phi_f(I) \).

We denote by \( \phi_I(I, (v_C \in \mathcal{R}(C) : C \in \mathcal{C})) \) the optimal solution of (4), depending on the fixed \( v_C \)'s. The minimum over all possible \( v_C \)'s is called the local generalized clique cover number

\[ \phi_I(I) = \min_{(v_C \in \mathcal{R}(C) : C \in \mathcal{C})} \phi_I(I, (v_C : C \in \mathcal{C})). \]

This is an extension of the local hyperclique cover: for a set of fixed \( v_C \), given user \( j \in [m] \) and some feasible solution to (3), count number of cliques \( C \) in that generalized clique cover such that \( v_C \) is not contained in the side-information \( X^{(j)} \) and let \( k \) be the maximum number of such cliques for each \( j \). The optimal solution of (4) is the minimum value of \( k \) over all possible solutions of (3) and all choices of \( v_C \). The minimum of the LP relaxation of (4) over all possible \( v_C \)'s is called the fractional local generalized clique cover number \( \phi_{fI}(I) \). Both \( \phi_{fI}(I) \) and \( \phi_I(I) \) will be shown to give upper bound on the transmission rate of the instance \( I \).

**Remark VI.12.** Consider the instance \( I \) of the ICCSI problem with \( m = n = 6 \), \( F_q = F_4 = \{0, 1, \alpha, \alpha^2\} \) and \( X^{(S)} = F_5^6 \), where \( \alpha \) is such that \( \alpha^2 = \alpha + 1 \).

\[
\begin{align*}
V^{(1)} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \\
V^{(2)} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \\
V^{(3)} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \\
V^{(4)} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
V^{(5)} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \\
V^{(6)} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},
\end{align*}
\]

and \( R_1 = 100000, R_2 = 010000, R_3 = 001000, R_4 = 000100, R_5 = 000010, R_6 = 000001 \).

Now if we consider the partition \( C_1 = \{1, 2\}, C_2 = \{3, 4\}, C_3 = \{5, 6\} \), and we use \( v_{C_1} = 110000, v_{C_2} = 001100, v_{C_3} = 00001\alpha \), to encode \( X \), then we obtain \( k = 3 \). But using \( v_{C_3} = 00001\alpha \) we have that \( k = 2 \). Clearly the optimal solution of (4) depends on the choice of vectors \( v_C \).

Another approach is based on partition multicast, as described in [24].

**Definition VI.13.** We define the partition generalized multicast number, \( \phi^P(I) \) to be the optimal solution of the following integer program

\[
\min \sum_{M \subset [m]} a_M d_M \\
\text{s.t.} \quad \sum_{M \subset [m]} a_M = 1 \text{ for all } j \in [m] \\
a_M \in \{0, 1\} \text{ for all } M \subset [m], M \neq \emptyset.
\]

(5)

The LP relaxation of (5) is called the fractional partition generalized multicast number, \( \phi^P_f(I) \).

We remark that \( d_M = \dim((R_M)) - \min_{j \in M} \dim((R_M) \cap X^{(j)}) \).

We briefly justify the above: each user is assigned to exactly one multicast group \( M \), so the selected groups \( M \) form a partition of \([m]\). Each member \( j \) of a multicast group \( M \subset [m] \) already has access to at least \( \dim((R_M) \cap X^{(j)}) \) independent vectors in \((R_M) \). As we’ll show in Theorem VI.14 a coding scheme can be applied to ensure delivery of all remaining requests within a group using at most \( d_M \) transmissions. The total number of transmissions required by this scheme is the sum of the \( d_M \)s, over all selected multicast groups \( M \).

The final approach combined partition multicast and local clique covering [24, Definition 10]. The users \([m]\) are partitioned into multicast groups and independently covered by generalized cliques. Each multicast group offers a reduced ICCSI problem, to which a restricted local clique cover is applied.

**Definition VI.14.** Define the following integer programme

\[
\min \sum_{M \subset [m]} a_M t_M \\
\text{s.t.} \quad \sum_{C: v_C \notin X^{(j)}} y_C \leq t_M \text{ for all } j \in M \\
\sum_{M \subset [m]} a_M = 1, \sum_{C: j \in C} y_C = 1 \text{ for all } j \in [m] \\
a_M, y_C \in \{0, 1\} \text{ for all } C \in \mathcal{C}, M \subset [m] \text{ and } t_M \in \mathbb{N}.
\]

(6)

We denote by \( \phi^P_I(I, (v_C \in \mathcal{R}(C) : C \in \mathcal{C})) \) the optimal solution of (6) with respect to \((v_C \in \mathcal{R}(C) : C \in \mathcal{C})\) fixed.
The minimum over all possible choices of $v_C$ is called the partitioned local generalized clique cover number

$$
\varphi_d^0(\mathcal{I}) = \min_{(v_C \in \mathcal{R}(C)) \cap C \subseteq C} \beta_d^0(\mathcal{I}, (v_C \in \mathcal{R}(C) : C \subseteq C)).
$$

The minimum of the LP relaxation of (6) over all possible choices of $v_C$ is called the fractional partitioned local generalized clique cover number $\varphi_d^f(\mathcal{I})$.

Now, we will show that achievable schemes exist for all parameters and hence obtain upper bounds on $\beta(\mathcal{I})$. The basic technique is to use MDS codes. It will be notionally convenient to express $X$ as a column vector of length $n$ over $\mathbb{F}_{q^r}$. We will assume in all cases that $q^r$ is large enough to assure the existence of an $\mathbb{F}_{q^r}$-MDS code of the required length.

**Theorem VI.15.** Let $\mathcal{I} = (\mathcal{X}, \mathcal{X}^{(S)}, R)$. There exist achievable $\mathbb{F}_{q^r}$-linear index codes corresponding to $\varphi(\mathcal{I})$ and $\varphi_f(\mathcal{I})$. In particular, we have

$$
\beta(\mathcal{I}) \leq \varphi_f(\mathcal{I}) \leq \varphi(\mathcal{I}).
$$

**Proof.** For each $C \subseteq \mathcal{C}$ fix a vector $v_C \in \mathcal{R}(C)$. Then given a clique cover $C^{\text{opt}} = \{C \subseteq \mathcal{C} : y_C = 1\}$, corresponding to an optimal solution of (3), and a data vector $X$, we broadcast $\{v_C \cdot X : C \in C^{\text{opt}}\}$. The demands $R_j X$ of each receiver $j \in [m]$ can be met in $|C^{\text{opt}}| = \varphi(\mathcal{I})$ transmissions since $R_j X$ can recover the vector $\hat{v}_C X G^{[r]}$. Now consider the LP relaxation of (3) and let an optimal solution be given by $\{y_C : C \in \mathcal{C}\} \subseteq \mathbb{Q}$. Let $r$ be the least common denominator of the $y_C$ and for each $C$ define the integral weight $\hat{y}_C = r y_C \in [r]$. Denoting the resulting multi-set of cliques by $C^{\text{opt}} = \{y_C, C : C \in \mathcal{C}\}$, and let $y$ be contained in $r$ (not necessarily distinct) cliques of $C^{\text{opt}}$, with each distinct clique $C$ appearing with multiplicity $\hat{y}_C$. Now split each packet $X_i \in \mathbb{F}_{q^r}$ into $r$ packets of equal size, so consider now $X$ as the data matrix

$$
X = \begin{bmatrix}
X_1 & \ldots & X_k
& \vdots & \ddots & \vdots \\
X_1 & \ldots & X_r
\end{bmatrix},
$$

with coefficients in a subfield $\mathbb{F}_{q^r}$ of $\mathbb{F}_q$ where $r$ is the least divisor of $t$ satisfying $r \leq t$. If $q^r > s = \sum_C \hat{y}_C$ then there exists an $\mathbb{F}_{q^r}$-MDS code, so suppose this is the case and let $G$ be a generator matrix of such a code. Now list the elements of $C^{\text{opt}}$ as $C_1, \ldots, C_s$ and assign to each column $G_i$ of $G$ the clique $C_i$. For each clique $C_i$ in $C^{\text{opt}}$, the packet $v_{C_i} X G_i \in \mathbb{F}_{q^r}$ is transmitted. Each transmission corresponds to an $\mathbb{F}_{q^r}$-linear combination of blocks of length $\ell \leq t/r$ over $\mathbb{F}_q$ and there are $s = r \varphi_f(\mathcal{I})$ transmissions in total.

Now consider the receiver $j \in [m]$, which has demanded the vector $R_j X$. We may assume that $j$ is contained in the first $r$ cliques $C_1, \ldots, C_r$ of the list of $s$ cliques. Then all users, including $j$, has received $(v_{C_1} X G_1, \ldots, v_{C_s} X G_s) \in \mathbb{F}_{q^r}^s$. From Remark VI.13 we have $R_j = \alpha_j v_{C_i} + a_j V(j)$ for some $\alpha_j, a_j$ for each $i \in [r]$. Thus $j$ can recover the vector

$$(R_j X G_1, \ldots, R_j X G_s) = (\alpha_j v_{C_1} X G_1 + a_j V(j) X G_1, \ldots, a_j v_{C_s} X G_s + a_j V(j) X G_s).$$

Now

$$(R_j X G_1^r, \ldots, R_j X G^r) = R_j X G^r$$

where $G^r = [G^1, \ldots, G^r]$ is an invertible $r \times r$ matrix, by the MDS property of the code generated by $G$. Then $j$ can decode $R_j X$. Every user receives the $r$ packets it requires and the total number of transmissions is $s$.

**Theorem VI.16.** Let $\mathcal{I} = (\mathcal{X}, \mathcal{X}^{(S)}, R)$. There are achievable linear index codes corresponding to $\varphi(\mathcal{I})$ and $\varphi_f(\mathcal{I})$ implying $\beta(\mathcal{I}) \leq \varphi_f(\mathcal{I}) \leq \varphi(\mathcal{I})$.

**Proof.** Let $C^{\text{opt}} = \{C_1, \ldots, C_s\}$ the set of cliques for which $y_C = 1$ in the optimal solution $(k, (y_C : C \in \mathcal{C}))$ of (4) for some fixed choice of vectors $v_C \in \mathbb{F}_{q^r}^n$. Let $s = \sum_C y_C = |C^{\text{opt}}|$ and let $G$ the generator matrix of an $\mathbb{F}_{q^r}[s, k]$ MDS code. As before we associate a column of $G$ to each clique in $C^{\text{opt}}$, and the sender transmits an encoding of the data vector $X \in \mathbb{F}_{q^r}^n$ as

$$
Y = \sum_{C \in C^{\text{opt}}} v_C X G^C = G(v_C X)_{C \in C^{\text{opt}}, v_C \in \mathbb{F}_{q^r}^n},
$$

which corresponds to $s$ transmissions over $\mathbb{F}_{q^r}$. For any $j \in [m]$, the constraints in the integer programme of (4) require that there are at most $k$ cliques of $C^{\text{opt}}$ with $v_C \notin \mathcal{X}^j$. This means that for any choice of $j$, there are at most $k$ vectors in $\{v_C : C \in \mathcal{C}\}$ not contained in $\mathcal{X}^j$. We have

$$
Y = \sum_{C \in C^{\text{opt}}, v_C \in \mathcal{X}^j} v_C X G^C + \sum_{C \in C^{\text{opt}}, v_C \notin \mathcal{X}^j} v_C X G^C.
$$

Therefore, Receiver $j$, given its side information $\mathcal{X}^j$, can recover

$$
\sum_{C \in C^{\text{opt}}, v_C \notin \mathcal{X}^j} v_C X G^C = \tilde{G}(v_C X)_{C \in C^{\text{opt}}, v_C \notin \mathcal{X}^j},
$$

where

$$
\tilde{G} = [G^C]_{C \in C^{\text{opt}}, v_C \notin \mathcal{X}^j}
$$

is a $k \times k$ submatrix of $G$. Since $\tilde{G}$ is invertible by the MDS property, the user $j$ can retrieve the vector $(v_C X)_{C \in C^{\text{opt}}, v_C \notin \mathcal{X}^j}$. For a clique $C$ containing $j$, using $v_C X$ it is possible to retrieve $R_j X$.

Now consider the LP relaxation of (4) and let $(k, (y_C : C \in \mathcal{C}))$ be an optimal solution for some rationals $0 \leq y_C \leq 1$. This time, let $r$ be the least common denominator of the $y_C$ and for each $C$ define $\hat{y}_C = r y_C / k = r k \in \mathbb{Z}$. As before, every distinct clique $C$ is assigned an integer weight in $[r]$, and we denote the corresponding multi-set of cliques by $C^{\text{opt}}$. Each user is contained in $r$ cliques. Let $s = \sum_C y_C$, let $G$ and $H$ be respective generator matrices of $[s, k]$ and $[s, r]$ MDS codes over $\mathbb{F}_{q^r}$. Again we represent $X$ as an $n \times r$ matrix with each packet $X_i$ in the form of a vector of length $r$ over a subfield of $\mathbb{F}_{q^r}$. Associating the $j$th columns of $G$ and $H$ to the $j$th clique $C_j$ with respect to a fixed listing of the multi-set $C^{\text{opt}}$, the following is transmitted.

$$
Y = \sum_{i=1}^s (v_C X H^i) G^i.
$$
For any $j \in [m]$, the $j$th receiver uses its side information as before to obtain

$$\sum_{i=1}^{k} (v_{ci}, XH^i)G^i,$$

where without loss of generality, $C_1, \ldots, C_k$ are the cliques for which $v_{ci} \notin X^{(j)}$. Moreover, $j$ is in $r$ of these cliques, which we may suppose to be $C_1, \ldots, C_r$. So as before from the MDS property of $G$, $j$ can recover the vector $(v_{c1}, XH^1, \ldots, v_{cr}, XH^r)$, and in particular $(v_{c1}, XH^1, \ldots, v_{cr}, XH^r)$.

Since for each $i \in [r]$, $R_j = c_i v_{ci} + a_i V^{(j)}$ for some $c_i$ and $a_i$, the user $j$ can obtain

$$(R_j, XH^1, \ldots, R_j, XH^r),$$

and therefore obtain $R_j X$ by the MDS property of $H$. Every user receives its required $r$ packets and the total number of transmissions is $k$.

Given an instance $\mathcal{I} = (X, \mathcal{X}^{(S)}, R)$, let $\tilde{n}$ denote the number of distinct equivalence classes of $[m]$ under the relation $i \sim j$ if $\mathcal{X}^{(i)} = \mathcal{X}^{(j)}$. We will use the following result of [5], which generalizes Proposition III.7

**Proposition VI.17.** Let $\mathcal{I} = (X, \mathcal{X}^{(S)}, R)$. If $q > \tilde{n}$ then $\kappa(\mathcal{I}) \leq \max\{n - d_i : i \in [m]\}$. For any $q$, $\kappa(\mathcal{I}) \leq \text{rank}(R)$.

**Proof.** That $\kappa(\mathcal{I}) \leq \text{rank}(R)$ is trivial: $\kappa(\mathcal{I})$ is by definition the minimum rank of an element of the coset $R + \mathcal{X} \cap (\oplus \mathcal{X}^{(S)})$. Indeed, an $F_q$-linear code of length $N = \text{rank}(R)$ exists simply by sending a basis of the rowspace of $R$, in which case no user requires its side-information in order to retrieve its request $R_i X$. That $\kappa(\mathcal{I}) \leq \max\{n - d_i : i \in [m]\}$ is shown in [5].

The essential content of the proof of Proposition VI.17 is that there exists an $N \times n$ matrix $L$ realizing $\mathcal{I}$ for $N \leq \max\{n - d_i : i \in [m]\}$, which corresponds to a multicast solution, so every user can retrieve any linear combination of the $X_i$. In this case the matrix $L$ is such that $\langle L \rangle + \mathcal{X}^{(i)} = F_q^n$ for each $i$.

**Theorem VI.18.** Let $\mathcal{I} = (X, \mathcal{X}^{(S)}, R)$. There are achievable linear index codes of lengths $\varphi^p(\mathcal{I})$ and $\varphi^f_p(\mathcal{I})$, which implies that $\beta(\mathcal{I}) \leq \varphi^p(\mathcal{I})$.

**Proof.** Let $\mathcal{M}$ be a collection of multicast groups $M \subset [m]$ yielding an optimal solution to (5). Let $M \in \mathcal{M}$ and consider the ICCSI instance $\mathcal{I}_M = (\oplus_{J \in M} \mathcal{X}^{(J)}, (R_M), M)$. From Proposition VI.17 for sufficiently large $q$, there exists $L_M \in F_q^{d_M \times n}$ such that each user in $M$ can decode $R_M X$, which uses $d_M$ transmissions. Applying this approach to each $M \in \mathcal{M}$, we find that all users’ requests can be retrieved using at most $\varphi^p(\mathcal{I}) = \sum_{M \in \mathcal{M}} d_M$ transmissions.

Let us consider now the LP relaxation of (3) and let $\{a_M : M \subset [m]\} \subset \mathbb{Q}$ be an optimal solution. Let $r$ denote the least common denominator of the $a_M$ and define $a_M = ra_M \in \mathbb{Z}$. Every multicast group $M$ is assigned an integer weight in $[r]$ and the multi-set of multicast groups is denoted by $\mathcal{M}^{opt}$. Every user is contained in $r$ multicast groups of $\mathcal{M}^{opt}$. As before, we represent the data vector $X \in \mathbb{F}_q^n$, as an $n \times r$ matrix over a subfield of $\mathbb{F}_q$. Let $L_M$ be an $d_M \times n$ matrix satisfying $\langle R_M \rangle \subset \langle L_M \rangle + \mathcal{X}^{(j)}$ for $j \in M$, i.e. such that each user assigned to $M$ can retrieve its requested data $R_j X$. Let $s = \sum_{M} a_M$ and, as before, let $G$ be a generator matrix of an $[s, r]$ MDS code over $\mathbb{F}_q$ with $tr \leq t$ and associate a column $G^i$ of $G$ to each multicast group $M_i$ in $\mathcal{M}$. The sender transmits the $s \mathbb{F}_q^r$-vectors of lengths $d_M$:

$$L_M X G^1, \ldots, L_M X G^s.$$

Let $j \in M_i$ for some $i \in [r]$. User $j$ considers only $r$ vectors, say these are:

$$L_M X G^1, \ldots, L_M X G^r,$$

and by assumption can solve for some vectors $a_i, c_i$

$$R_j = c_i L_M + a_i V^{(j)}.$$

Thus $j$ can recover

$$R_j X G^i = c_i L_M X G^i + a_i V^{(j)} X G^i$$

as User $j$ knows $L_M X G^i$, $V^{(j)} X$ and $G^i$. So, we can compute

$$R_j X[G^1, \ldots, G^r]$$

and from the MDS property it is possible to obtain $R_j X$. 

**Remark VI.19.** Theorem VI.18 generalizes the statement of [24, Theorem 2]. However, the scheme given in the proof of [24, Theorem 2] to establish the upper bound, is incorrect. We assert that the statement of the theorem is still valid since it is special case of Theorem VI.18 and the parameters $\varphi^p$ and $\varphi^f_p$ generalize those given in [18]. We provide an example below to show that the scheme proposed in the proof of [24, Theorem 2] does not work.

Consider the instance of the ICSI problem with $m = n = 4$, $f(i) = i$ for all $i$ and side information $X_1 = \{2\}$, $X_2 = \{3, 4\}$, $X_3 = \{1, 4\}$ and $X_4 = \{1, 3\}$. The graph $G$ associated with this instance is given in Figure 4. It can be checked that $\varphi^p(G) = 3$ and from the LP relaxation we obtain $\varphi^f_p(G) = 5/2$. Consider for example the set $\mathcal{M}^{opt} = \{M_1 = \{1, 2, 3\}, M_2 = \{1, 2, 4\}, M_3 = \{3, 4\}\}$ arising from an optimal solution of the LP problem. Then $r = 2$ and our data matrix is

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$
groups (not necessarily different). Every packet $X_i$ consists of $r$ sub-packets, then we transmit each sub-packet using the scheme corresponding to one of the $r$ multicast groups.

For all $i$, denote by $L_i$ the matrix associated to the scheme used to encode the message for the users contained in the set $M_i$. In particular, we can consider the following matrices

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

Note that only the receivers contained in $M_i$ are able to decode when $L_i$ is used to encode.

Following the scheme given in [27], we do not need to combine the sub-packets $X^1$ and $X^2$, using an MDS code, as in the proof of Theorem VI.18. Therefore, the message transmitted using this scheme will be of type

$$Y = (L_1X^{i_1}, L_2X^{i_2}, L_3X^{i_3})$$

where $i_j \in \{1, 2\}$ for each $j$. Thus it should be possible to find a choice of the $i_j$'s such that all the receivers are able to retrieve the requested packet.

Suppose we choose $i_1 = 1$, $i_2 = 2$ and $i_3 = 1$. Note that in this case the receivers 1, 2 and 4 can retrieve their requested packets, but receiver 3 obtains only the first sub-packet. It can be checked that for all possible choice of $i_j$, there is at least one receiver that obtains only one of its two requested sub-packets. On the other hand, using an $\mathbb{F}_2[3, 2, 2]$ MDS code to combine the sub-packets, we can satisfy all the requests by sending:

$$Y = (L_1X^1, L_2X^2, L_3(X^1 + X^2)).$$

Theorem VI.20. There are achievable linear index codes corresponding to $\varphi_p^p(I)$ and $\varphi_p^p(I)$ implying $\beta(I) \leq \varphi_p^p(I)$.

Proof. Fix a set of coding vectors $\{v_C \in \mathcal{R}(C)\}$ for each $C \in \mathcal{C}$. Let $C_{\text{opt}} = \{C_1, \ldots, C_r\}$ be the set of cliques for which $y_C = 1$ in the optimal solution $\{\{t_M : M \in [m]\}, \{y_C : C \in \mathcal{C}\}\}$ of (6). Fix a multicast group $M$ and let $G$ be a generator matrix of an $[s, t]$ MDS code. Associate each $i$th column of $G$ to the clique $C_i$ in $C_{\text{opt}}$. For this multicast group, the sender transmits

$$Y = \sum_{C \cap M \neq \emptyset} v_C XG^i.$$ 

Given the side-information of User $j \in M$ this sum reduces to one involving only $t_M$ cliques, which we may assume to be $C_1, \ldots, C_{t_M}$, yielding

$$\sum_{i=1}^{t_M} v_C XG^i = (v_{C_1}X, \ldots, v_{C_{t_M}}X)[G^1, \ldots, G^{t_M}],$$ 

and inverting the matrix $[G^1, \ldots, G^{t_M}]$ we can recover $(v_{C_1}X, \ldots, v_{C_{t_M}}X)$. As $j$ is contained in one of these cliques it can decode $R_j X$.

Let us consider, now, the LP relaxation of (6). Let $\{\{t_M, a_M : M \in [m]\}, \{y_C : C \in \mathcal{C}\}\}$ be an optimal solution.

Let $r_1$ denote the least common denominator of the $y_C$ and the $t_M$ and let $r_2$ denote the least common denominator of the $a_M$. Define $\hat{y}_C = r_1y_C$, $\hat{t}_M = r_1t_M$ and $\hat{a}_M = r_2a_M$.

Every clique $C$ is assigned an integral weight in $[r_1]$ and every multicast group $M$ is assigned an integral weight in $[r_2]$.

Denote as before the multi-set of cliques by $\hat{C}$. Given the side-information of User $j$, we can compute

$$\sum_{C \in C_{\text{opt}}} v_C XG^i_j,$$

and inverting the matrix $[G^1, \ldots, G^{t_M}]$ we can recover $(v_{C_1}X, \ldots, v_{C_{t_M}}X)$. As $j$ is contained in one of these cliques it can decode $R_j X$.

Remark VI.21. The parameters $\varphi_p^p$ and $\varphi_p^p$ are not comparable. From the parameters given in [24] we have that there exist instances of the ICSI problem for which $\varphi_p^p(I) \geq \varphi_p^p(I)$.

Now consider the ICCSI instance with $m = n = 3$, $q = 2$, $\mathcal{X}^2 = \mathbb{F}_2$.

$$V^{(1)} = [1 \ 0 \ 1] \quad V^{(2)} = [1 \ 1 \ 1] \quad V^{(3)} = [1 \ 1 \ 1],$$

and $R_1 = 100$, $R_2 = 010$, $R_3 = 001$.

In order to satisfy the requests of a receiver using only one vector then the coding vectors should be

- $v_1 = 100$ or $v_1 = 111$ for User 1;
- $v_2 = 010$ or $v_2 = 101$ for User 2;
- $v_3 = 001$ or $v_3 = 110$ for User 3.
Remark VI.22. The parameters $\varphi^p$ and $\varphi$ are not comparable. From the parameters given in [24], there exist instances of the ICSI problem for which $\varphi(I) \geq \varphi^p(I)$. Now consider the ICCSI instance with $m = n = 2$, $q = 2$, $X(S) = F_2^2$, $V^1(1) = [1 \ 1]$ $V^2(0) = [0 \ 0]$, and $R_1 = 10$, $R_2 = 01$. It is easy to check that using the multicast group partition we need two transmissions, but it can be seen that $\{1, 2\}$ is a clique and that $V_{1,2} = 01 \in R(\{1, 2\})$, yielding $1 = \varphi(I) \leq \varphi^p(I) = 2$.

Remark VI.23. We have $\varphi^p(I) \leq \varphi(I) \leq \varphi^p(I)$. It is easy to check that $\varphi(I) \leq \varphi^p(I)$ as $t$ is at most equal to the number of cliques that form a partition of $[m]$. Then we have also $\varphi^p(I) \leq \varphi^p(I)$. In fact, among the possible optimal solution to obtain we have those where $M = [m]$ and in that case we obtain exactly $\varphi(I)$.

Remark VI.24. It is possible to introduce a weak definition of clique, $C \subseteq [m]$ is called weak clique if for all $i, j \in C$ we have $R_i \in X^{(i)}$ or $\langle R_j \rangle = \langle R_i \rangle$. Using this definition, it is possible to introduce the notion of a weak clique cover, a local weak clique cover and a partitioned local weak clique cover with respective corresponding parameters $w\varphi(I), w\varphi^p(I)$ and $w\varphi(I)$ along with their fractional counterparts.

Remark VI.25. If $C$ is a weak clique then it is also a generalized clique. We can encode the message using the sum of distinct requests as vector $\nu_C$. Moreover from the definition of weak clique, if we consider a clique as a multicast group $M$ then it results $d_M = 1$. Therefore $\varphi^p(I) \leq w\varphi(I)$ and the same holds for the fractional parameters. However also in this case the partitioned local weak clique cover and the partitioned multicast cover are not comparable (see example in Remark VI.24).

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