DIFFERENCES OF COMPOSITION OPERATORS BETWEEN DIFFERENT HARDY SPACES

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ABSTRACT. In this paper, we give some estimates for the norm and essential norm of the differences of two composition operators between different Hardy spaces.

Keywords: Hardy space, composition operator, difference, norm, essential norm.

1. INTRODUCTION

Let \( \mathbb{D} \) denote the open unit disk of the complex plane \( \mathbb{C} \). We denote the closure and the unit circle of \( \mathbb{D} \) by \( \overline{\mathbb{D}} \) and \( \partial \mathbb{D} \), respectively. Let \( H(\mathbb{D}) \) be the class of functions analytic in \( \mathbb{D} \). Let \( dm = \frac{d\theta}{2\pi} \) denote the normalized Lebesgue measure on \( \partial \mathbb{D} \). The Lebesgue space \( L^p(m) \) will also be denoted by \( L^p(\partial \mathbb{D}) \), \( 0 < p < \infty \). For \( 0 < p < \infty \), let \( H^p \) denote the Hardy space of all \( f \in H(\mathbb{D}) \) such that

\[
\|f\|_p^p = \sup_{0 < r < 1} \int_{\partial \mathbb{D}} |f(r\xi)|^p dm(\xi) < \infty.
\]

Recall that if \( f \in H^p(\mathbb{D}) \), then the radial limits \( \lim_{r \to 1} f(re^{i\theta}) \) exist almost everywhere on \( \partial \mathbb{D} \) and will be denoted also by \( f \), which belongs to \( L^p(\partial \mathbb{D}) \) and

\[
\|f\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta.
\]

The space \( H^\infty(\mathbb{D}) \) consists of all bounded analytic functions on \( \mathbb{D} \), and its norm is given by the supremum norm on \( \mathbb{D} \).

For \( a \in \mathbb{D} \), let \( \sigma_a(z) := \frac{a - z}{1 - \overline{a}z} \) be the disc automorphism that exchanging 0 for \( a \). Let \( \triangle(a, r) := \{ z \in \mathbb{D} : |\sigma_a(z)| < r \} \) denote the pseudohyperbolic disk centered at \( a \) with radius \( r \). For two points \( z, w \in \mathbb{D} \), the pseudohyperbolic distance is given by

\[
\rho(z, w) = |\sigma_w(z)| = \left| \frac{z - w}{1 - \overline{w}z} \right|.
\]

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Let \( \varphi \) and \( \psi \) be two analytic self-maps of \( \mathbb{D} \). We write that 
\[ \sigma(z) = \rho(\varphi(z), \psi(z)) \]
Note that \( \sigma \) also has a radial extension \( \sigma^* \) almost everywhere 
on \( \partial \mathbb{D} \). Indeed, if \( \varphi \neq \psi \), then the radial limits of \( \varphi \) and \( \psi \) can coincide only 
on a set of measure zero. We will use the same notation for the function \( \sigma \) and its radial extension.

Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \). The composition operator \( C_{\varphi} \) on \( H(\mathbb{D}) \) is defined by 
\[ C_{\varphi}f = f \circ \varphi. \]
Furthermore, if \( u \) is a Borel measurable function, a weighted composition operator \( uC_{\varphi} \) on \( H(\mathbb{D}) \) is defined by 
\[ (uC_{\varphi}f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}. \]

The study of the differences of composition operators was started on the Hardy space \( H^2 \). The main purpose for this study is to understand the topological structure of the set of composition operators \( C(H^2) \), see [1, 4, 21]. After that, such related problems have been studied on several spaces of analytic functions by many authors, see for example [7, 10, 11, 13, 14, 15, 16, 17, 18, 22]. Motivated by [14, 17, 18], in this paper, we study the differences of composition operators between different Hardy spaces.

In the Hardy spaces setting, Goebeler [9] showed that for \( 0 < q < p < \infty \), \( C_{\varphi} - C_{\psi} : H^p \to H^q \) is compact if and only if the composition operators \( C_{\varphi} \) and \( C_{\psi} \) between these spaces are both compact, that is, \( |\varphi| < 1 \) and \( |\psi| < 1 \) a.e. on \( \partial \mathbb{D} \). Nieminen and Saksman [16] proved that \( C_{\varphi} - C_{\psi} \) is compact on \( H^p \) for all \( p \in [1, \infty) \) if and only if \( C_{\varphi} - C_{\psi} \) is compact for some \( p \in [1, \infty) \). But the complete characterization of the compactness of \( C_{\varphi} - C_{\psi} : H^p \to H^p \) is still open.

Recently, Saukko in [17] asked the following question: is boundedness and compactness of the difference operator in Hardy spaces enough to guarantee the boundedness and compactness of corresponding weighted composition operators? In this paper, we give partly positive answers to this question. The main results of this paper are the following theorems.

**Theorem 1.1.** Let \( 1 < p < q < \infty \). Suppose \( \varphi \) and \( \psi \) are analytic self-maps of \( \mathbb{D} \). Then \( C_{\varphi} - C_{\psi} : H^p \to H^q \) is bounded if and only if both weighted composition operators \( \sigma C_{\varphi} \) and \( \sigma C_{\psi} \) map \( H^p \) into \( H^q \). Furthermore,

\[
\|C_{\varphi} - C_{\psi}\|_{H^p \to H^q} \approx \sup_{a \in \mathbb{D}} \int_{\partial \mathbb{D}} \left( \frac{1 - |a|^2}{(1 - \bar{a}\varphi(\xi))^2} \right)^{\frac{1}{p}} \left( \frac{1 - |a|^2}{(1 - \bar{a}\psi(\xi))^2} \right)^{\frac{1}{q}} dm(\xi)
\]

\[
\approx \sup_{a \in \mathbb{D}} \int_{\partial \mathbb{D}} \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\xi)|^2} + \frac{1 - |a|^2}{|1 - \bar{a}\psi(\xi)|^2} \right)^2 \rho(\varphi(\xi), \psi(\xi))^q dm(\xi).
\]
Theorem 1.2. Let $1 < p < q < \infty$. Suppose $\varphi$ and $\psi$ are analytic self-maps of $\mathbb{D}$ such that $C_\varphi - C_\psi : H^p \to H^q$ is bounded. Then

$$\|C_\varphi - C_\psi\|_{e,H^p \to H^q}^q \approx \limsup_{|a| \to 1} \int_{\partial \mathbb{D}} \left( \frac{1 - |a|^2}{|1 - \overline{a}\varphi(\xi)|^2} + \frac{1 - |a|^2}{|1 - \overline{a}\psi(\xi)|^2} \right)^{\frac{q}{p}} \rho(\varphi(\xi), \psi(\xi))^q dm(\xi).$$

Recall that the essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators $K$ mapping $X$ into $Y$, that is,

$$\|T\|_{e,X \to Y} = \inf \{ \|T - K\|_{X \to Y} : K \text{ is compact} \},$$

where $X, Y$ are Banach spaces and $\| \cdot \|_{X \to Y}$ is the operator norm.

The present paper is organized as follows. In Section 2, we study weighted composition operators between Hardy spaces. In Section 3, we state some lemmas and give the proofs of Theorems 1.1 and 1.2.

For two quantities $A$ and $B$, we use the abbreviation $A \lesssim B$ whenever there is a positive constant $c$ (independent of the associated variables) such that $A \leq cB$. We write $A \approx B$, if $A \lesssim B \lesssim A$.

2. WEIGHTED COMPOSITION OPERATORS FROM $H^p$ TO $H^q$

In this section, we collect some characterizations of weighted composition operators between different Hardy spaces. Given any measure $\mu$ on $\mathbb{D}$, we denote by $\mu|_{\overline{\mathbb{D}}}$ and $\mu|_{\partial \mathbb{D}}$ its restrictions to the Borel subsets of $\mathbb{D}$ and $\partial \mathbb{D}$, respectively. By Lemma 2.1 of [2], the $s$-Carleson measure on $\overline{\mathbb{D}}$ is defined as follows.

Definition 2.1. Let $0 < p, q < \infty$, $\mu$ be a Borel measure on $\overline{\mathbb{D}}$. Then the measure $\mu$ is called a $\frac{q}{p}$-Carleson measure on $\overline{\mathbb{D}}$ if the inclusion map $I_\mu : H^p \to L^q(\mu, \overline{\mathbb{D}})$ is bounded, i.e., there exists a constant $C$ such that

$$\left( \int_{\overline{\mathbb{D}}} |f(z)|^q d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_p$$

for every $f \in H^p$. Furthermore, $\mu$ is a vanishing $\frac{q}{p}$-Carleson measure on $\mathbb{D}$ if the inclusion map $I_\mu : H^p \to L^q(\mu, \mathbb{D})$ is compact.

For an interval $I \subset \partial \mathbb{D}$, the Carleson square is defined by

$$S(I) = \{ re^{it} \in \mathbb{D} : 1 - |I| < r < 1, e^{it} \in I \},$$
where \( |E| \) denotes the Lebesgue measure of the measurable set \( E \subset \partial \mathbb{D} \). If \( a \in \mathbb{D} \setminus \{0\} \), let \( I_a = \{ e^{i\theta} : \arg(ae^{-i\theta}) \leq \frac{1-|a|}{2} \} \), and denote \( S(a) = S(I_a) \). For convenience, we put \( I_0 = \partial \mathbb{D} \) and \( S(0) = \mathbb{D} \).

Suppose \( u : \partial \mathbb{D} \rightarrow \mathbb{C} \) is a measurable function and \( \varphi \) is an analytic self-map of \( \mathbb{D} \). Define the measure \( \mu_{u,\varphi} \) in \( \overline{\mathbb{D}} \) by

\[
\mu_{u,\varphi}(E) = \int_{\varphi^{-1}(E) \cap \partial \mathbb{D}} |u(z)|^q dm(z)
\]

for all Borel set \( E \subset \overline{\mathbb{D}} \).

We need the following results about weighted composition operators on Hardy spaces from [2] and [5].

**Theorem 2.1.** Suppose \( 0 < p < q < \infty \) and \( 0 < r < 1 \). Let \( u : \partial \mathbb{D} \rightarrow \mathbb{C} \) be a measurable function and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Then the following statements are equivalent:

(i) The weighted composition operator \( uC_\varphi : H^p \rightarrow L^q(\partial \mathbb{D}) \) is bounded.

(ii) \( \mu_{u,\varphi}|_{\partial \mathbb{D}} = 0 \) and \( \|\mu_{u,\varphi}||_{p,q} := \sup_{a \in \mathbb{D}} \|\mu(S(a))\| (1-|a|) \frac{q}{p} < \infty \).

(iii) \( \mu_{u,\varphi}|_{\partial \mathbb{D}} = 0 \) and \( \|\mu_{u,\varphi}||_{p,q,r} := \sup_{a \in \mathbb{D}} \frac{\mu(S(a))}{(1-|a|)^{r/2}} < \infty \).

(iv) \( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1-|a|^2)^{q/2} \|\mu_{u,\varphi}(z)\| \text{d}m(z) < \infty \).

(v) \( \sup_{a \in \mathbb{D}} \|uC_\varphi k_a\|_{L^q(\partial \mathbb{D})} < \infty \), where \( k_a(z) = \left( \frac{1-|a|^2}{(1-|z|^2)^2} \right)^{1/2} \).

Furthermore, \( \|uC_\varphi\|_{H^p \rightarrow L^q(\partial \mathbb{D})} \) and the quantities in (ii), (iii), (iv) and (v) are all comparable with comparability constants depending only on \( p, q \) and \( r \).

**Proof.** The equivalence between (ii) and (iii) can be found in [12]. By the change of variables, for all \( f \in H(\mathbb{D}) \) (see [3] Lemma 2.1, also see [17]),

\[
\|uC_\varphi f\|_{L^q(\partial \mathbb{D})} = \|f\|_{L^q(\mu_{u,\varphi}, \mathbb{D})}.
\]

Therefore, \( uC_\varphi : H^p \rightarrow L^q(\partial \mathbb{D}) \) is bounded if and only if the inclusion map \( I_{\mu_{u,\varphi}} : H^p \rightarrow L^q(\mu_{u,\varphi}, \mathbb{D}) \) is bounded, and \( \|uC_\varphi\| = \|I_{\mu_{u,\varphi}}\|. \) Taking \( f = k_a \), we obtain that (iv) and (v) are equivalent. The equivalence of (i), (ii) and (iv) follows from [2] Theorem 2.5 and the proof of [2] Proposition 2.3. The comparability of the quantities are clear. The proof is complete.

Let \( n \in \mathbb{N} \). Define the partial sum operator \( S_n : H(\mathbb{D}) \rightarrow H(\mathbb{D}) \) by

\[
S_n \left( \sum_{k=0}^{\infty} a_k z^k \right) = \sum_{k=0}^{n} a_k z^k.
\]

Denote \( R_n = I - S_n \). For \( 0 < s < 1 \), we denote \( \mathbb{D}_s = \{ z \in \mathbb{D} : |z| < s \} \).
Theorem 2.2. Suppose $1 < p < q < \infty$ and $0 < r < 1$. Suppose that $u : \partial \mathbb{D} \to \mathbb{C}$ is a measurable function and $\varphi$ is an analytic self-map of $\mathbb{D}$ such that the operator $u \varphi : H^p \to L^q(\partial \mathbb{D})$ is bounded. Then

(i) 
\[ \| u \varphi \|_{e, H^p \to L^q(\partial \mathbb{D})}^q \approx \lim_{s \to 1} \| \mu_{u, \varphi} \|_{\overline{D} \setminus \overline{D}_s}^q \approx \lim_{s \to 1} \| (u \varphi) R_n \|_{H^p \to L^q(\partial \mathbb{D})}^q \approx \lim_{n \to \infty} \| (u \varphi) R_n \|_{H^p \to L^q(\partial \mathbb{D})}^q \approx \sup_{|a| \to 1} \| (u \varphi) k_a \|_{L^q(\partial \mathbb{D})}^q \approx \lim_{n \to \infty} \| (u \varphi) R_n \|_{H^p \to L^q(\partial \mathbb{D})}^q \approx \lim_{n \to \infty} \| (u \varphi) R_n \|_{L^q(\partial \mathbb{D})}^q \approx \lim_{|a| \to 1} \mu_{u, \varphi}(\Delta(a, r)) \cdot (1 - |a|^2)^{\frac{2}{p}}. \]

(ii) For every $0 < \eta < 1$,
\[ \lim_{n \to \infty} \sup_{\| f \|_{p} \leq 1} \int_{\varphi^{-1}(\mathbb{D}_n)} \left| (C \varphi \circ R_n f)(\xi) \right|^q \, dm(\xi) = 0. \]

Proof. (i) First, we prove that
\[ \lim_{s \to 1} \| \mu_{u, \varphi} \|_{\overline{D} \setminus \overline{D}_s}^q \approx \lim_{|a| \to 1} \mu_{u, \varphi}(\Delta(a, r)) \cdot (1 - |a|^2)^{\frac{2}{p}}. \]

Let
\[ t_r(s) = \frac{s - r}{1 - sr}. \]

After a calculation, we get that $\Delta(a, r) \cap (\mathbb{D} \setminus \mathbb{D}_s) \neq 0$ if and only if $|a| \geq t_r(s)$. It is easy to see that $t_r(s)$ is continuous and increasing on $[r, 1)$, and $\lim_{s \to 1} t_r(s) = 1$. Thus,
\[ \lim_{|a| \to 1} \mu_{u, \varphi}(\Delta(a, r)) \cdot (1 - |a|^2)^{\frac{2}{p}} = \lim_{s \to 1} \sup_{|a| \geq t_r(s)} \frac{\mu_{u, \varphi}(\Delta(a, r))}{(1 - |a|^2)^{\frac{2}{p}}} \geq \lim_{s \to 1} \sup_{|a| \geq t_r(s)} \frac{\mu_{u, \varphi}(\Delta(a, r) \cap (\mathbb{D} \setminus \mathbb{D}_s))}{(1 - |a|^2)^{\frac{2}{p}}} = \lim_{s \to 1} \sup_{a \in \mathbb{D}} \frac{\mu_{u, \varphi}(\Delta(a, r) \cap (\mathbb{D} \setminus \mathbb{D}_s))}{(1 - |a|^2)^{\frac{2}{p}}} = \lim_{s \to 1} \| \mu_{u, \varphi} \|_{\overline{D} \setminus \overline{D}_s}^q. \]

Denote $A = \lim_{s \to 1} \| \mu_{u, \varphi} \|_{\overline{D} \setminus \overline{D}_s}^q$. For any $\epsilon > 0$, there exists $0 < t < 1$, such that if $t \leq s < 1$, we have
\[ \| \mu_{u, \varphi} \|_{\overline{D} \setminus \overline{D}_s}^q < A + \epsilon. \]
For any fixed $s \ (0 < s < 1)$, we know that $\triangle(a, r) \subset \mathbb{D} \setminus \mathbb{D}_s$, as $|a|$ close enough to 1. Therefore, there exists a $l, 0 < l < 1$, such that

$$\|\mu_{u, \varphi}|_{\mathbb{D} \setminus \mathbb{D}_s}\|_{p, q, r}^q = \sup_{a \in \mathbb{D}} \frac{\mu_{u, \varphi}(\triangle(a, r) \cap (\mathbb{D} \setminus \mathbb{D}_s))}{(1 - |a|^2)^{\frac{2}{p}}},$$

Hence,

$$A + \epsilon \geq \lim_{|a| \to 1} \sup \frac{\mu_{u, \varphi}(\triangle(a, r))}{(1 - |a|^2)^{\frac{2}{p}}}.$$

Since $\epsilon$ is arbitrary, we obtain

$$\lim_{s \to 1} \|\mu_{u, \varphi}|_{\mathbb{D} \setminus \mathbb{D}_s}\|_{p, q, r}^q \geq \lim_{|a| \to 1} \sup \frac{\mu_{u, \varphi}(\triangle(a, r))}{(1 - |a|^2)^{\frac{2}{p}}}.$$

Now we consider the remainder of the proof. Since $u C_\varphi : H^p \to L^q(\partial \mathbb{D})$ is bounded, by Theorem 2.1, we have $\mu_{u, \varphi}|_{\partial \mathbb{D}} = 0$. See [5, Theorem 5] and the proof of [5, Theorem 2] for the rest of the proof. Although in the proof it is assumed that the function $u$ is analytic, the proof also work if it is only measurable. The comparability of the quantities follows from the proofs.

(ii) Let $\eta \in (0, 1)$ be fixed. For $w \in \mathbb{D}$, let $K_w(z) = \frac{1}{1 - wz}$, $z \in \mathbb{D}$. Then $K_w \in H^\infty \subset H^p'$, where $1/p + 1/p' = 1$. For $f \in H^p$ and $g \in H^p'$, we denote

$$\langle f, g \rangle = \int_{\partial \mathbb{D}} f(\xi)\overline{g(\xi)} dm(\xi).$$

It is easy to see that for every $f \in H^p$,

$$f(w) = \langle f, K_w \rangle \text{ and } \langle R_n f, K_w \rangle = \langle f, R_n K_w \rangle.$$

Thus,

$$|R_n f(w)| = |\langle R_n f, K_w \rangle| = |\langle f, R_n K_w \rangle| \leq \|f\|_p \|R_n K_w\|_\infty.$$  

For all $\xi \in \varphi^{-1}(\mathbb{D}_\eta)$, let $w = \varphi(\xi)$. Then $|w| < \eta$. Since

$$R_n K_w(z) = R_n \left(\sum_{k=0}^{\infty} \overline{w}^k z^k \right) = \sum_{k=n+1}^{\infty} \overline{w}^k z^k,$$

one has

$$\|R_n K_w\|_\infty \leq \frac{\eta^{n+1}}{1 - \eta}.$$

Therefore,

$$\lim_{n \to \infty} \sup_{\|f\|_p \leq 1} \int_{\varphi^{-1}(\mathbb{D}_\eta)} |(C_{\varphi} \circ R_n f)(\xi)|^q dm(\xi) \leq \lim_{n \to \infty} \frac{\eta^{n+1}}{1 - \eta} = 0.$$
3. PROOFS OF MAIN RESULTS

To prove the main results in this paper, we need the following three lemmas.

**Lemma 3.1.** Let $0 < r < 1$. Then there exists a constant $C = C(r) > 0$ such that whenever $a \in \mathbb{D}$ and $z \in \triangle(a, r)$,

$$\left| \frac{a}{C} \rho(z, w) \right| \leq \left| 1 - \frac{1 - \overline{a}z}{1 - \overline{a}w} \right| \leq C|a|\rho(z, w)$$

for every $w \in \overline{\mathbb{D}}$.

**Proof.** The proof is similarly with [17, Lemma 4.3]. We only notice that $|1 - \overline{z}w| \approx |1 - \overline{a}w|$, whenever $a \in \mathbb{D}$, $z \in \triangle(a, r)$ and $w \in \overline{\mathbb{D}}$.

**Lemma 3.2.** Let $0 < r < 1, \gamma > 0$. Then there exist constants $C_1 = C_1(r, \gamma), C_2 = C_2(r, \gamma) > 0$ such that whenever $a \in \mathbb{D}$ and $z \in \triangle(a, r)$,

$$C_1|a|\rho(z, w) \leq \left( \frac{1 - |a|^2}{(1 - \overline{a}z)^2} \right)^\gamma - \left( \frac{1 - |a|^2}{(1 - \overline{a}w)^2} \right)^\gamma \leq C_2|a|\rho(z, w)(1 - |a|^2)^\gamma$$

for every $w \in \overline{\mathbb{D}}$.

**Proof.** Let $a \in \mathbb{D}, w \in \overline{\mathbb{D}}$ and $z \in \triangle(a, r)$. By the proof of [17, Lemma 4.4], we have

$$\left| 1 - \frac{1 - \overline{a}z}{1 - \overline{a}w} \right| \approx \left| 1 - \frac{1 - \overline{a}z}{1 - \overline{a}w} \right|.$$

Applying this, $|1 - \overline{a}z| \approx 1 - |a|^2$ and Lemma 2.1 we get the desired result.

**Lemma 3.3.** [17] Let $f \in H^1$ and $0 < r < 1$. Then there exist a constant $C = C(r)$ such that

$$|f(z) - f(a)| \leq C\rho(z, a)P|f|(a)$$

for every $z \in \triangle(a, r)$. Here $Pf$ is the Poisson transformation of $f$, i.e.,

$$Pf(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{i\theta}|^2} f(e^{i\theta})d\theta, \; f \in L^1(\partial\mathbb{D}).$$

Now we are in a position to prove our main results in this paper.

**Proof of Theorem 1.1.** First, we consider the lower bound. Suppose that $C^\varphi - C^\psi : H^p \rightarrow H^q$ is bounded. Since $H^q$ is compact embedding into $H^p$, we have $C^\varphi - C^\psi : H^p \rightarrow H^p$ is compact. Therefore, by [16, Theorem 1 and Lemma 4], $|\varphi| < 1$ and $|\psi| < 1$ a.e. on $\partial\mathbb{D}$. Thus, $\mu_{\sigma, \varphi}(\partial\mathbb{D}) = 0$. 

Let \( k_a(z) = \left( \frac{1 - |a|^2}{(1 - |a|^2)^2} \right)^{\frac{1}{p}} \). By Lemma 3.2, we get
\[
\|C_\varphi - C_\psi\|_{H^p \to H^q} ^q \\
\geq \sup_{a \in \partial \mathbb{D}} \| (C_\varphi - C_\psi) k_a \|_q ^q \\
= \sup_{a \in \partial \mathbb{D}} \left\| \int_{\partial \mathbb{D}} \left( \frac{1 - |a|^2}{(1 - |a|^2)^2} \right)^{\frac{1}{p}} - \left( \frac{1 - |a|^2}{(1 - \varphi(a))^2} \right)^{\frac{1}{p}} \right\| \, dm(\xi) \\
\geq \sup_{|a| \geq 2^{-3}} \left\| \int_{\varphi^{-1}(\Delta(a, \frac{1}{2})) \cap \partial \mathbb{D}} \frac{|\sigma(\xi)|^q}{(1 - |a|^2)^{\frac{2}{p}}} \, dm(\xi) \right\| \\
= \sup_{|a| \geq 2^{-3}} \frac{\mu_{\sigma, \varphi}(\Delta(a, \frac{1}{2}))}{(1 - |a|^2)^{\frac{2}{p}}}.
\]

Noting that \( \Delta(a, 2^{-3}) \subset \Delta(2^{-2}, 2^{-1}) \) for every \( a \in \Delta(0, 2^{-3}) \), we get
\[
\sup_{|a| < 2^{-3}} \frac{\mu_{\sigma, \varphi}(\Delta(a, 2^{-3}))}{(1 - |a|^2)^{\frac{2}{p}}} \leq \sup_{|a| \geq 2^{-3}} \frac{\mu_{\sigma, \varphi}(\Delta(2^{-2}, 2^{-1}))}{(1 - (2^{-2})^2)^{\frac{2}{p}}}.
\]

Hence
\[
\sup_{a \in \partial \mathbb{D}} \frac{\mu_{\sigma, \varphi}(\Delta(a, 2^{-3}))}{(1 - |a|^2)^{\frac{2}{p}}} \lesssim \sup_{a \in \partial \mathbb{D}} \| (C_\varphi - C_\psi) k_a \|_q ^q.
\]

Thus, by Theorem 2.1, we obtain that \( \sigma C_\varphi : H^p \to L^q(\partial \mathbb{D}) \) is bounded and
\[
\|C_\varphi - C_\psi\|_{H^p \to H^q} ^q \gtrsim \sup_{a \in \partial \mathbb{D}} \| (C_\varphi - C_\psi) k_a \|_q ^q \gtrsim \| \sigma C_\varphi\|_{H^p \to L^q(\partial \mathbb{D})} ^q.
\]

Similarly,
\[
\|C_\varphi - C_\psi\|_{H^p \to H^q} ^q \geq \sup_{a \in \partial \mathbb{D}} \| (C_\varphi - C_\psi) k_a \|_q ^q \gtrsim \| \sigma C_\varphi\|_{H^p \to L^q(\partial \mathbb{D})} ^q,
\]

and hence
\[
\|C_\varphi - C_\psi\|_{H^p \to H^q} ^q \gtrsim \| \sigma C_\varphi\|_{H^p \to L^q(\partial \mathbb{D})} ^q + \| \sigma C_\varphi\|_{H^p \to L^q(\partial \mathbb{D})} ^q. \tag{2}
\]

Next we consider the upper bound.
\[
\|C_\varphi - C_\psi\|_{H^p \to H^q} ^q \\
= \sup_{\|f\|_p \leq 1} \| (C_\varphi - C_\psi) f \|_q ^q \\
= \sup_{\|f\|_p \leq 1} \left( \int_{|\sigma(\xi)| \geq \frac{1}{2}} + \int_{|\sigma(\xi)| < \frac{1}{2}} \right) |f \circ \varphi(\xi) - f \circ \psi(\xi)|^q \, dm(\xi) \\
:= I_1 + I_2.
\]

It is easy to see that
\[
I_1 \lesssim \| \sigma C_\varphi\|_{H^p \to L^q(\partial \mathbb{D})} ^q + \| \sigma C_\varphi\|_{H^p \to L^q(\partial \mathbb{D})} ^q.
\]
By Lemma 2.3,
\[ I_2 = \sup_{\|f\|_p \leq 1} \int |f \circ \varphi(x) - f \circ \psi(x)|^q dm(x) \]
\[ \leq \sup_{\|f\|_p \leq 1} \int |(P|f| \circ \varphi(x)|)^q dm(x) \]
\[ \leq \sup_{\|f\|_p \leq 1} \int (P|f|(z))^q d\mu_{\sigma, \varphi}(z). \]

Let \( \tilde{g}(z) \) denote the harmonic conjugate function of \( g(z) := P|f|(z) \), normalized so that \( \tilde{g}(0) = 0 \), and let \( v = g + \tilde{g} \). Then both \( g \) and \( \tilde{g} \) are belong to the harmonic Hardy space \( h^p \) and hence \( v \in H^p \). By M. Riesz theorem (see [8, Theorem 2.3 of Chapter 3] or [6, Theorem 4.1]) and Minkowski inequality, we get
\[ \|v\|_p \approx \|g\|_p + \|\tilde{g}\|_p \lesssim \|g\|_p \leq \|f\|_p. \]

By Theorem 2.1,
\[ \int \int (P|f|(z))^q d\mu_{\sigma, \varphi}(z) \leq \int |v(z)|^q d\mu_{\sigma, \varphi}(z) \]
\[ \lesssim \|\sigma C_{\varphi}\|_p^{q} \|f\|_p^{q}, \]
Thus,
\[ I_2 \lesssim \|\sigma C_{\varphi}\|_p^{q} \|f\|_p^{q}. \]

Hence
\[ \|C_{\varphi} - C_{\psi}\|_p \lesssim \|\sigma C_{\varphi}\|_p^{q} \|f\|_p^{q}, \]
(3)
Therefore, by (1), (2) and (3), we get
\[ \|C_{\varphi} - C_{\psi}\|_p \approx \sup_{a \in \mathbb{D}} \|\sigma C_{\varphi}(a) - \sigma C_{\psi}(a)\|_q \]
\[ \approx \|\sigma C_{\varphi}\|_p^{q} \|f\|_p^{q}. \]

The result follows by Theorem 2.1.

**Proof of Theorem 1.2.** We can assume that \( \varphi \neq \psi \). Let \( \{a_n\} \) be any sequence in \( \mathbb{D} \) such that \( a_n \to 1 \) as \( n \to \infty \). Let \( k_{a_n}(z) = (1 - |a_n|^2)^{-\frac{1}{q}} \). Then \( \|k_{a_n}\|_{H^p} = 1 \) and \( k_{a_n} \to 0 \) weakly in \( H^p \) as \( n \to \infty \). Let \( S \) be a compact operator from \( H^p \) into \( H^q \). Then \( \lim_{n \to \infty} \|S k_{a_n}\|_{H^q} = 0 \). Hence,
\[ \|C_{\varphi} - C_{\psi} - S\|_{H^p \to H^q} \geq \lim_{n \to \infty} \sup \|\sigma C_{\varphi}(a) - \sigma C_{\psi}(a)\|_q \]
\[ \geq \lim_{n \to \infty} \sup \|\sigma C_{\varphi}(a) - \sigma C_{\psi}(a)\|_q. \]
Thus,

\[ \|C_{\varphi} - C_{\psi}\|_{e,H^p \to H^q} \geq \limsup_{|a| \to 1} \|(C_{\varphi} - C_{\psi})k_a\|_q. \]  

(4)

Now, we prove that

\[ \limsup_{|a| \to 1} \|(C_{\varphi} - C_{\psi})k_a\|_q^q \gtrsim \|\sigma C_{\varphi}\|_{e,H^p \to L^q(\partial D)}^q + \|\sigma C_{\psi}\|_{e,H^p \to L^q(\partial D)}^q. \]  

(5)

Since \(C_{\varphi} - C_{\psi} : H^p \to H^q\) is bounded, by Theorem 1.1, we have \(\sigma C_{\varphi} : H^p \to L^q(\partial D)\) is bounded. Let \(0 < r < 1\). By Lemma 3.2 and Theorem 2.2, we obtain

\begin{align*}
\limsup_{|a| \to 1} \|(C_{\varphi} - C_{\psi})k_a\|_q^q &= \limsup_{|a| \to 1} \int_{\partial D} \left| \left( \frac{1 - |a|^2}{(1 - \overline{a}\varphi(\xi))^2} \right)^{\frac{1}{p'}} - \left( \frac{1 - |a|^2}{(1 - \overline{a}\psi(\xi))^2} \right)^{\frac{1}{p'}} \right|^q dm(\xi) \\
&\gtrsim \limsup_{|a| \to 1} \int_{\varphi^{-1}(\Delta(a,r)) \cap \partial D} \frac{|\sigma(\xi)|^q}{(1 - |a|^2)^{\frac{3}{r}}} dm(\xi) \\
&= \limsup_{|a| \to 1} \frac{\mu_{\sigma,\varphi}(\Delta(a,r))}{(1 - |a|^2)^{\frac{3}{r}}} \\
&\approx \|\sigma C_{\varphi}\|_{e,H^p \to L^q(\partial D)}^q.
\end{align*}

Similarly, we get

\[ \limsup_{|a| \to 1} \|(C_{\varphi} - C_{\psi})k_a\|_q^q \gtrsim \|\sigma C_{\psi}\|_{e,H^p \to L^q(\partial D)}^q. \]

Finally, we prove that

\[ \|C_{\varphi} - C_{\psi}\|_{e,H^p \to H^q} \lesssim \|\sigma C_{\varphi}\|_{e,H^p \to L^q(\partial D)} + \|\sigma C_{\psi}\|_{e,H^p \to L^q(\partial D)}. \]

Since the partial sum operator \(S_n\) is compact, we get

\[ \|C_{\varphi} - C_{\psi}\|_{e,H^p \to H^q} \leq \limsup_{n \to \infty} \|(C_{\varphi} - C_{\psi})R_n\|_{H^p \to L^q(\partial D)}. \]

Denote \(E = \{\xi \in \partial D : |\sigma(\xi)| \geq 1/2\}\) and \(E' = \partial D \setminus E\). Then

\[ I_n(f) := \int_E \|(C_{\varphi} - C_{\psi})R_n f(\xi)\|^q dm(\xi) \]

\[ \leq 2^q \left( \int_E \|\sigma C_{\varphi} R_n f(\xi)\|^q dm(\xi) + \int_E \|\sigma C_{\psi} R_n f(\xi)\|^q dm(\xi) \right) \]

\[ \leq 2^q \left( \|\sigma C_{\varphi} R_n\|_{H^p \to L^q(\partial D)}^q + \|\sigma C_{\psi} R_n\|_{H^p \to L^q(\partial D)}^q \right), \]

whenever \(\|f\|_p \leq 1\) and \(n \in \mathbb{N}\). Thus by Theorem 2.2 (i),

\[ \limsup_{n \to \infty} \sup_{\|f\|_p \leq 1} I_n(f) \lesssim \|\sigma C_{\varphi}\|_{e,H^p \to L^q(\partial D)} + \|\sigma C_{\psi}\|_{e,H^p \to L^q(\partial D)}. \]
Denote

\[ J_n(f) := \int_{E'} |(C\varphi - C\psi) R_n f(\xi)|^q \, dm(\xi). \]

For all \( a, z, w \in \mathbb{D} \), from [8, Lemma 1.4 of Chapter 1] or [20] we see that

\[ \rho(z, w) \leq \frac{\rho(z, a) + \rho(a, w)}{1 + \rho(z, a)\rho(a, w)}. \]

Let \( s \in (0, 1) \) be arbitrary. Suppose \( z \in E' \cap \varphi^{-1}(\mathbb{D}_s) \). By the last inequality we can find \( s' = \frac{1+r}{1+r} \in (0, 1) \) such that \( E' \cap \varphi^{-1}(\mathbb{D}_s) \subset \psi^{-1}(\mathbb{D}_{s'}) \). Thus by Theorem 2.2 (ii),

\[ \lim_{n \to \infty} \sup_{\|f\|_p \leq 1} \int_{E' \cap \varphi^{-1}(\mathbb{D}_s)} |(C\varphi \circ R_n f)(\xi)|^q \, dm(\xi) = 0 \]

and

\[ \lim_{n \to \infty} \sup_{\|f\|_p \leq 1} \int_{E' \cap \varphi^{-1}(\mathbb{D}_s)} |(C\psi \circ R_n f)(\xi)|^q \, dm(\xi) = 0. \]

Hence

\[ \limsup_{n \to \infty} \sup_{\|f\|_p \leq 1} J_n(f) \lesssim \limsup_{n \to \infty} \sup_{\|f\|_p \leq 1} \int_{F} |(C\varphi - C\psi) \circ R_n f(\xi)|^q \, dm(\xi) \]

\[ \lesssim \sup_{\|f\|_p \leq 1} \int_{F} |(C\varphi - C\psi) f(\xi)|^q \, dm(\xi), \]

where \( F = E' \cap \varphi^{-1}(\mathbb{D} \setminus \mathbb{D}_s) \) and we used the fact that the operators \( S_n \) are uniformly bounded (see [23, Proposition 1]), so does \( R_n \).

Using Lemma 3.3, we get

\[ \int_{F} |(C\varphi - C\psi) f(\xi)|^q \, dm(\xi) \lesssim \int_{F} |\sigma(\xi)|^q (P|f| \circ \varphi(\xi))^q \, dm(\xi) \]

\[ = \int_{\varphi(F)} (P|f|(z))^q \, d\mu_{\sigma,\varphi}(z) \]

\[ \lesssim \int_{\mathbb{D} \setminus \mathbb{D}_s} (P|f|(z))^q \, d\mu_{\sigma,\varphi}(z). \]

Let \( \tilde{g}(z) \) denote the harmonic conjugate function of \( g(z) := P|f|(z) \) with \( \tilde{g}(0) = 0 \) and let \( v = g + i\tilde{g} \). Then \( v \in H^p, |g(z)| \leq |v(z)|, \) and

\[ \|v\|_p \approx \|g\|_p \leq \|f\|_p \leq 1. \]

Therefore,

\[ \int_{\mathbb{D} \setminus \mathbb{D}_s} (P|f|(z))^q \, d\mu_{\sigma,\varphi}(z) \lesssim \int_{\mathbb{D} \setminus \mathbb{D}_s} |v(z)|^q \, d\mu_{\sigma,\varphi}(z). \]
Since \( C_\varphi - C_\psi : H^p \to H^q \) is bounded, we have \( \sigma C_\varphi : H^p \to L^q(\partial \mathbb{D}) \) is bounded. Thus, \( \mu_{\sigma, \varphi}\big|_{\partial \mathbb{D}} = 0 \) and
\[
\left\| \left( \mu_{\sigma, \varphi}\big|_{\partial \mathbb{D}} \right) \right\| \leq \|\mu_{\sigma, \varphi}\|_{p, q, r} < \infty.
\]
This shows that \( \mu_{\sigma, \varphi}\big|_{\partial \mathbb{D}} \) is a \( \frac{q}{p} \)-Carleson measure on \( \partial \mathbb{D} \). Then,
\[
\left| \int_F (C_\varphi - C_\psi)f(\xi)\,|q|\,dm(\xi) \right| \leq \left\| \left( C_\varphi - C_\psi \right)f \right\|_{p, q, r}
\]
\[
= \int_D |v(z)|^q \,d\mu_{\sigma, \varphi}(z)
\]
\[
\leq \|\mu_{\sigma, \varphi}\|_{p, q, r}^q.
\]
Letting \( s \to 1 \), we get
\[
\limsup_{n \to \infty} \sup_{\|f\|_p \leq 1} J_n(f) \leq \|\sigma C_\varphi\|_{e, \Sigma^p \to L^q(\partial \mathbb{D})}^q.
\]
Therefore,
\[
\|C_\varphi - C_\psi\|_{e, H^p \to H^q} \leq \limsup_{n \to \infty} \sup_{\|f\|_p \leq 1} \|C_\varphi - C_\psi\|_{H^p \to L^q(\partial \mathbb{D})}^q
\]
\[
= \limsup_{n \to \infty} \sup_{\|f\|_p \leq 1} I_n(f) + \limsup_{n \to \infty} \sup_{\|f\|_p \leq 1} J_n(f)
\]
\[
\leq \|\sigma C_\varphi\|_{e, H^p \to L^q(\partial \mathbb{D})}^q + \|\sigma C_\psi\|_{e, H^p \to L^q(\partial \mathbb{D})}^q.
\]
By (4), (5), (6) and Theorem 2.2, we get the desired result.

**Remark 3.1.** Theorem 1.2 motivate us to study a possible connection between the differences of composition operators and the corresponding weighted composition operators on \( H^p \). We conjecture that
\[
\|C_\varphi - C_\psi\|_{e, H^p \to H^p}^p
\]
\[
\approx \limsup_{|\alpha| \to 1} \int_{\partial \mathbb{D}} \left( \frac{1 - |\alpha|^2}{|1 - \overline{\alpha}\varphi(\xi)|^2} \right) \left( \frac{1 - |\alpha|^2}{|1 - \overline{\alpha}\psi(\xi)|^2} \right) \rho(\varphi(\xi), \psi(\xi))^p \,dm(\xi).
\]

**Remark 3.2.** In [16], Nieminen and Saksman showed that the compactness of \( C_\varphi - C_\psi \) on \( H^p \) is independent of \( p \in [1, \infty) \). Therefore, we also conjecture that \( C_\varphi - C_\psi : H^p \to H^p(1 \leq p < \infty) \) is compact if and only if
\[
\lim_{|\alpha| \to 1} \int_{\partial \mathbb{D}} \left( \frac{1 - |\alpha|^2}{|1 - \overline{\alpha}\varphi(\xi)|^2} + \frac{1 - |\alpha|^2}{|1 - \overline{\alpha}\psi(\xi)|^2} \right) \rho(\varphi(\xi), \psi(\xi)) \,dm(\xi) = 0.
\]
Remark 3.3. In [19], Shapiro showed that the operator \( C_{\varphi} \) is compact on \( H^p \) if and only if
\[
\lim_{|z| \to 1} \frac{N_{\varphi}(z)}{\log \frac{1}{|z|}} = 0,
\]
where \( N_{\varphi} \), called the Nevanlinna counting function, is defined by
\[
N_{\varphi}(z) = \sum_{a \in \varphi^{-1}(z)} \log \frac{1}{|a|}, \quad z \in \mathbb{D} \setminus \varphi(0).
\]

We ask whether \( C_{\varphi} - C_{\psi} : H^p \to H^p \) is compact if and only if
\[
\lim_{|z| \to 1} \rho(\varphi(z), \psi(z)) \left( \frac{N_{\varphi}(z)}{\log \frac{1}{|z|}} + \frac{N_{\psi}(z)}{\log \frac{1}{|z|}} \right) = 0?
\]

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REFERENCES

[1] E. Berkson, Composition operators isolated in the uniform operator topology, *Proc. Amer. Math. Soc.* **81** (2) (1981), 230–232.

[2] O. Blasco and H. Jarchow, A note on Carleson measures for Hardy spaces, *Acta Sci. Math. (Szeged)* **71** (2005), 371–389.

[3] M. Contreras and A. Hernández-Díaz, Weighted composition operators on Hardy spaces, *J. Math. Anal. Appl.* **263** (2001), 224–233.

[4] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[5] Ž. Ćučković and R. Zhao, Weighted composition operators between different weighted Bergman spaces and different Hardy spaces, *Illinois J. Math.* **51** (2007), 479–498.

[6] P. Duren, *Theory of \( H^p \) Spaces*, Academic Press, 1970.

[7] E. Gallardo-Gutiérrez, M. González, P. Nieminen and E. Saksman, On the connected component of compact composition operators on the Hardy space, *Adv. Math.* **219** (2008), 986–1001.

[8] J. Garnett, *Bounded Analytic Functions*, Springer Science & Business Media, 2007.

[9] T. Goebeler, Composition operators acting between Hardy spaces, *Integr. Equ. Oper. Theory* **41** (4) (2001), 389–395.

[10] S. Li, Differences of generalized composition operators on the Bloch space, *J. Math. Anal. Appl.* **394** (2012), 706–711.

[11] M. Lindström and E. Saukko, Essential norm of weighted composition operators and difference of composition operators between standard weighted Bergman spaces, *Complex Anal. Oper. Theory* **9** (6) (2015), 1411–1432.

[12] D. Luecking, Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, *Amer. J. Math.* **107** (1) (1985), 85–111.

[13] B. D. MacCluer, S. Ohno and R. Zhao, Topological structure of the space of composition operators on \( \mathcal{H}^\infty \), *Integr. Equ. Oper. Theory* **40** (2001), 481–494.
[14] J. Moorhouse, Compact differences of composition operators, *J. Funct. Anal.* **219** (2005), 70–92.
[15] P. Nieminen, Compact differences of composition operators on Bloch and Lipschitz spaces, *Comput. Method Funct. Theory* **7** (2007), 325–344.
[16] P. Nieminen and E. Saksman, On compactness of the difference of composition operators, *J. Math. Anal. Appl.* **298** (2) (2004), 501–522
[17] E. Saukko, Difference of composition operators between standard weighted Bergman spaces, *J. Math. Anal. Appl.* **381** (2) (2011), 789–798.
[18] E. Saukko, An application of atomic decomposition in Bergman spaces to the study of differences of composition operators, *J. Funct. Anal.* **262** (9) (2012), 3872–3890.
[19] J. Shapiro, The essential norm of a composition operator, *Ann. Math.* **125** (2) (1987), 375–404.
[20] J. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.
[21] J. Shapiro and C. Sundberg, Isolation amongst the composition operators, *Pacific J. Math.* **145** (1990), 117–152.
[22] Y. Shi and S. Li, Essential norm of the differences of composition operators on the Bloch space, *Math. Ineq. Appl.* **20** (2) (2017), 543–555.
[23] K. Zhu, Duality of Bloch spaces and norm convergence of Taylor series, *Michigan Math. J.* **38** (1) (1991), 89–101.

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