Torsional Topological Invariants

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Making use of the SO(3,1) Lorentz algebra, we derive in this paper two series of Gauss-Bonnet type identities involving torsion, one being of the Pontryagin type and the other of the Euler type. Two of the six identities involve only torsional tensorial entities and establish

\[
\sqrt{-g} g^{\mu \nu \lambda \rho} \left( C_{\alpha \beta}^{\mu \nu} + C_{\mu \nu}^{\alpha \beta} C_{\sigma}^{\alpha \beta} \right) C_{\chi \alpha}^\eta C_{\rho \beta \eta} \\
\sqrt{-g} g^{\mu \nu \lambda \rho} \epsilon_{\alpha \beta \gamma \delta} \left( C_{\alpha \beta}^{\mu \nu} + C_{\mu \nu}^{\alpha \beta} C_{\sigma}^{\alpha \beta} \right) C_{\chi \gamma}^\eta C_{\rho \delta \eta}
\]

as purely torsional topological invariants.

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I. PRELIMINARIES

The underlying geometry of the Einstein-Cartan-Kibble-Sciama gravitational theory \[1\] is the Riemann-Cartan geometry, and the vierbein field \(e^a_\mu\) and the Lorentz-spin connection field \(\omega_{ab}^\mu\) are the basic field variables, where the Latin \(a, b\) are the anholonomic Lorentz indices and the Greek \(\mu\) the holonomic coordinate index. The Lorentz indices are raised and lowered by

\[\eta^{ab} = \eta_{ab} = (1, -1, -1, -1),\]

while the coordinate indices are lowered and raised by

\[g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu,\]

and its inverse \(g^{\mu \nu}\). The affine connection \(\Gamma^\lambda_{\mu \nu}\) is defined in terms of the vierbein field \(e^a_\mu\) and spin-connection field \(\omega_{ab}^\mu\) by

\[\Gamma^\lambda_{\mu \nu} = e^a_\lambda (e^a_\mu,_{\nu} + \omega^a_{b \nu} e^b_\mu),\]

where \(e^a_\lambda\) is the inverse of \(e^{a \lambda}\). The covariant derivatives, denoted with semi-column \(;\) subscripts, with respect to both local Lorentz transformations and general coordinate transformations are defined for generic \(\chi^a_\lambda\) and \(\chi^a_\nu\) according to

\[\chi^a_{\lambda ;\mu} = \chi^a_{\lambda ;\mu} - \omega_{a \mu} \chi^b_\lambda + \Gamma^a_{\nu \mu} \chi^a_\nu,\]

\[\chi^a_{\nu ;\mu} = \chi^a_{\nu ;\mu} + \omega_{b \mu} \chi^b_\lambda - \Gamma^a_{\nu \mu} \chi^a_\lambda.\]

It can be easily verified that

\[e^a_{\mu ;\lambda} = 0,\]

\[e^a_\mu ;\lambda = 0,\]

and, consequently,

\[g^{\mu \nu ;\lambda} = 0,\]

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The affine connection $\Gamma^\lambda_{\mu\nu}$ as defined by (3) is in general not symmetric,

$$\Gamma^\lambda_{\mu\nu} \neq \Gamma^\lambda_{\nu\mu},$$

(9)

giving rise to the torsion tensor $C^\lambda_{\mu\nu}$, which is defined by

$$C^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}.$$  

(10)

Define

$$\varpi_\mu \equiv \frac{1}{4} \sigma_{ab} \omega^{ab \mu},$$

(11)

where

$$\sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b],$$

(12)

with $\{\gamma_a, \gamma_b\} = 2 \eta_{ab}$. The set of matrices $\frac{i}{2} \sigma_{ab}$ satisfy the SO(3,1) Lorentz algebra

$$\frac{i}{2} [\sigma_{ab}, \sigma_{cd}] = \eta_{ac} \sigma_{bd} - \eta_{ad} \sigma_{bc} + \eta_{bd} \sigma_{ac} - \eta_{bc} \sigma_{ad}.$$  

(13)

The Lorentz curvature $R^{ab}_{\mu\nu}$ is defined through

$$\frac{1}{4} \sigma_{ab} R^{ab}_{\mu\nu} \equiv \varpi_{\mu,\nu} - \varpi_{\nu,\mu} + i[\varpi_\mu, \varpi_\nu],$$

(14)

and, as a result of the Lorentz algebra (13), is given by

$$R^{ab}_{\mu\nu} = \omega^{ab \mu,\nu} - \omega^{ab \nu,\mu} - \omega^{ac \mu} \omega^{ab}_{\nu} + \omega^{ac \nu} \omega^{ab}_{\mu},$$

(15)

which has the property

$$R^{\lambda\rho}_{\mu\nu} = \epsilon^\alpha_\lambda \epsilon^\beta_\rho R^{ab}_{\mu\nu} = g^{\alpha\beta} (\Gamma^\lambda_{\sigma\mu,\nu} - \Gamma^\lambda_{\sigma\nu,\mu} - \Gamma^\lambda_{\alpha\mu} \Gamma^\alpha_{\sigma\nu} + \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\sigma\mu}),$$

(16)

where $\Gamma^\lambda_{\mu\nu}$ is defined by (3).

II. GAUSS-BONNET TYPE IDENTITIES

Gauss-Bonnet type identities in Riemann-Cartan curved space-time

$$\sqrt{-g} \epsilon^{\mu\nu\lambda\rho} R^{\alpha\beta}_{\mu\nu} R_{\alpha\beta\lambda\rho} = \text{total derivative},$$

(17)

$$\sqrt{-g} \epsilon^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R_{\gamma\delta\lambda\rho} = \text{total derivative}$$

(18)

can be simply derived on the basis of the SO(3,1) Lorentz algebra and properties of the Dirac matrices. These identities establish

$$\sqrt{-g} \epsilon^{\mu\nu\lambda\rho} R^{\alpha\beta}_{\mu\nu} R_{\alpha\beta\lambda\rho},$$

(19)

$$\sqrt{-g} \epsilon^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R_{\gamma\delta\lambda\rho},$$

(20)

as topological invariants. They are, respectively, the Pontryagin and Euler topological invariants. Based on the SO(4,1) de Sitter algebra, another Gauss-Bonnet type identity

$$\sqrt{-g} \epsilon^{\mu\nu\lambda\rho} R^{AB}_{\mu\nu} R_{AB\lambda\rho} = \text{total derivative},$$

(21)
where indices A and B take on five values (0,1,2,3,5), can be derived, establishing
\[ g^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho}^A R_{\mu\nu\lambda\rho} B \]
(22)
as a Pontryagin type topological invariant for the de Sitter group. It is the difference of the SO(4,1) and SO(3,1) Pontryagin invariants, namely (21) and (17), that led to the identity [6]
\[ g^{\mu\nu\lambda\rho} (R_{\mu\nu\lambda\rho} + \frac{1}{2} C_{\mu\nu}^\alpha C_{\alpha\lambda\rho}) = \partial_\mu (g^{\mu\nu\lambda\rho} C_{\nu\lambda\rho}), \]
(23)
establishing
\[ g^{\mu\nu\lambda\rho} (R_{\mu\nu\lambda\rho} + \frac{1}{2} C_{\mu\nu}^\alpha C_{\alpha\lambda\rho}) \]
as a torsional topological invariant [7, 8].

We now use the same method, based on the SO(3,1) algebra and properties of the Dirac matrices, to derive two more series of topological invariants involving torsion, one being of the Euler type and the other of the Pontryagin type. Define \( \omega_{\mu}^{\prime ab} \)
by
\[ \omega_{\mu}^{\prime ab} \equiv \omega_{\mu}^{ab} + \xi C_{\mu}^{ab}, \]
(25)
where \( \xi \) is an arbitrary parameter, and
\[ C_{\mu}^{ab} = \epsilon^{\alpha\beta\mu\nu} C_{\mu}^{ab \alpha\beta}, \]
which is antisymmetric in a and b. It is convenient to introduce the group algebraic notations
\[ \omega_{\mu}^{\prime \prime} \equiv \omega_{\mu} + \xi C_{\mu}^{\prime}, \]
\[ C_{\mu}^{\prime} = \frac{1}{4} \omega_{\mu}^{\prime \prime}, \]
(26)
The curvature tensor \( R_{\mu\nu}^{\prime ab} \) corresponding to the connection \( \omega_{\mu}^{\prime ab} \) is defined by
\[ \frac{1}{4} \epsilon_{\mu\nu} R_{\mu\nu}^{\prime ab} = \bar{R}_{\mu\nu}^{\prime}, \]
(27)
where
\[ \bar{R}_{\mu\nu}^{\prime} \equiv \omega_{\mu,\nu}^{\prime} - \omega_{\nu,\mu}^{\prime} + i[\omega_{\mu}^{\prime}, \omega_{\nu}^{\prime}], \]
(28)
On account of the SO(3,1) Lorentz algebra, \( R_{\mu\nu}^{\prime ab} \) is explicitly given by
\[ R_{\mu\nu}^{\prime ab} = \omega_{\mu,\nu}^{ab} - \omega_{\nu,\mu}^{ab} - \omega_{\mu,\nu}^{ac} \omega_{\nu}^{b} + \omega_{\nu,\mu}^{ac} \omega_{\nu}^{b}, \]
(29)
which can be expressed in terms of the curvature tensor \( R_{\mu\nu}^{ab} \) and the torsion tensor \( C_{\mu}^{ab} \) as
\[ R_{\mu\nu}^{\prime ab} = R_{\mu\nu}^{ab} + \xi (C_{\mu}^{ab} + C_{\lambda}^{ab} C_{\mu}^{\lambda \nu}) + \xi \omega_{\nu,\mu}^{ac} (C_{\nu}^{ac} C_{\mu}^{b} + C_{\nu}^{ac} C_{\mu}^{b} + C_{\nu}^{ac} C_{\mu}^{b}), \]
(30)
where \( C_{\mu}^{ab} \) denotes
\[ C_{\mu}^{ab} = C_{\mu}^{ab \colon \nu} - C_{\nu}^{ab \colon \mu}, \]
(31)
with \( \colon \) representing covariant derivative in accordance with the definitions (4) and (5), e.g.,
\[ C_{\mu}^{ab \colon \nu} = C_{\mu}^{ab \colon \nu} - \Gamma_{\mu \nu}^{\lambda} C_{\lambda}^{ab} + \omega_{\nu}^{c} C_{\mu}^{cb} + \omega_{\nu}^{c} C_{\mu}^{ac}. \]
(32)
III. Pontryagin Type Identities

We can express the Pontryagin type invariant for $R'^{\alpha \beta}_{\mu \nu}$ in the following form

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} R'^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} = \sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} R^{ab}_{\mu \nu \alpha \beta \lambda \rho} = 2\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr} [\bar{R}_{\mu \nu} R_{\lambda \rho}],$$

where use has been made of

$$\text{Tr}[\sigma_{ab} \sigma_{cd}] = 4(\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}).$$

With $R'_{\mu \nu}$ given by (28), we can express (33) in the form

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} R'^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} = \partial_\mu \left\{ 8\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr} \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\},$$

where use has been made of the fact that $\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho}$ is a constant. With $R'^{ab}_{\mu \nu}$ given by (30), we can expand the left-hand side of (35) as a power series of the parameter $\xi$,

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} R'^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} = \sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \left\{ R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} + \xi R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} + \frac{\xi^2}{2} R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} \left[ R_{\alpha \beta \lambda \rho} + 3R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} \right] + \frac{\xi^3}{6} R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} \left[ R_{\alpha \beta \lambda \rho} + 3R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} \right] \right\} + ...,$$

As a power series in $\xi$, and with $\bar{w}_\rho$ given by (26), the right-hand side of (35) is given by

$$\partial_\mu \left\{ 8\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr} \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\} + \xi \left\{ 2 \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\} + \frac{\xi^2}{2} \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\},$$

We recall that $\bar{w}_\mu$ and $C_\mu$ are given in (26). Since the parameter $\xi$ is arbitrary, we equate terms of equal power in $\xi$ in (37) and obtain a set of four identities. The identity corresponding to the zeroth power in $\xi$ is the original Gauss-Bonnet identity (16) for the Pontryagin invariant. The other three identities are

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} (C_{\alpha \beta \lambda \rho} + C_{\sigma \alpha \beta \lambda \rho}) = \partial_\mu \left\{ 4\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr} \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\},$$

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} [4R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} C_{\lambda \beta \rho \sigma} = \partial_\mu \left\{ 8\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr} \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\} + \frac{\xi^2}{2} \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\},$$

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} (C_{\alpha \beta \lambda \rho} + C_{\alpha \beta \lambda \rho}) = \partial_\mu \left\{ 4\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr} \left[ \bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho \right] \right\} \right\},$$

IV. Euler Type Identities

Let us denote by $\eta_{abcd}$ the totally antisymmetric Minkowski tensor, with

$$\eta_{0123} = -1.$$

Because of the relation

$$\text{Tr}[\gamma_5 \sigma_{ab} \sigma_{cd}] = -4i \eta_{abcd},$$

where

$$\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3,$$

we can express the Euler type invariant in the form

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \epsilon_{\alpha \beta \gamma \delta} R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} = \sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \eta_{abcd} R^{ab}_{\mu \nu \alpha \beta \lambda \rho} = 4i \sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr}[\gamma_5 R'_{\mu \nu} R'_{\lambda \rho}],$$

where $R'_{\mu \nu}$ is defined by (28). Substituting (28) into (44), we obtain

$$\sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \epsilon_{\alpha \beta \gamma \delta} R^{\alpha \beta}_{\mu \nu \alpha \beta \lambda \rho} = \partial_\mu \left\{ 16i \sqrt{-\bar{g}}e^{\mu \nu \lambda \rho} \text{Tr} [\gamma_5 (\bar{w}_\nu \partial_\lambda \bar{w}_\rho + \frac{2}{3} \bar{w}_\nu \bar{w}_\rho \bar{w}_\rho)] \right\},$$

(45)
where use has been made of the relation,

$$[\gamma_5, \varpi'_\mu] = 0. \quad (46)$$

We expand both sides of (45) as power series of $\xi$. The left-hand side is

$$\sqrt{-g}e^{\mu\nu\lambda\rho}\epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\lambda\rho} = \sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} \{ R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\lambda\rho} + \xi^2 R^{\alpha\beta}_{\mu\nu} (C^{\gamma\delta}_{\lambda\rho} + C^{\gamma\delta}_{\rho\lambda}) + \xi^4 (C^{\alpha\beta}_{\mu\nu} + C^{\alpha\beta}_{\nu\mu}) C^{\gamma\eta}_{\lambda\rho} C^{\gamma\eta}_{\rho\lambda}\} + \xi^2 [4R^{\alpha\beta}_{\mu\nu} \lambda^\gamma_\sigma C^{\delta}_{\rho\sigma} + (C^{\alpha\beta}_{\mu\nu} + C^{\alpha\beta}_{\nu\mu}) (C^{\gamma\delta}_{\lambda\rho} + C^{\gamma\delta}_{\rho\lambda})] + \xi^4 [4(C^{\alpha\beta}_{\mu\nu} + C^{\alpha\beta}_{\nu\mu}) C^{\gamma\eta}_{\lambda\rho} C^{\gamma\eta}_{\rho\lambda}] + i \xi g^{\gamma\delta} C^{\gamma\eta}_{\lambda\rho} C^{\gamma\eta}_{\rho\lambda} \delta_{\lambda\rho}. \quad (47)$$

The right-hand side of (45), as a power series of $\xi$, is given by

$$\partial_\mu \{ 16i \sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} \} = \partial_\mu \{ 8i \sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} \} , \quad (49)$$

$$\partial_\mu \{ 16i \sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} \} = \partial_\mu \{ 8i \sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} \} , \quad (50)$$

$$\partial_\mu \{ 16i \sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} \} = \partial_\mu \{ 8i \sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} \} , \quad (51)$$

V. Purely Torsional Topological Invariants

In addition to the original Gauss-Bonnet identities (17) and (18) for the Pontryagin and Euler topological invariants, respectively, we have obtained in this paper six additional identities, which do not exist when torsion vanishes. Three of them, namely (38), (39) and (40), are of the Pontryagin type, and the other three, namely (49), (50) and (51), are of the Euler type. Of the six, two of the identities, namely (40) and (51), are special in that they contain only torsion tensorial entities, just like the previously known torsional identity (23). The right-hand side of the two identities (40) and (51) can be easily evaluated. We have the following results for the traces:

$$e^{\mu\nu\lambda\rho} \epsilon^{\alpha\beta\gamma\delta} C_{\mu\nu} C_{\lambda\rho} = -\frac{i}{2} e^{\mu\nu\lambda\rho} C_{\alpha\beta\gamma\delta} \delta_{\alpha\beta} \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta}. \quad (52)$$

$$e^{\mu\nu\lambda\rho} \epsilon^{\alpha\beta\gamma\delta} C_{\mu\nu} C_{\lambda\rho} = \frac{1}{4} e^{\mu\nu\lambda\rho} C_{\alpha\beta\gamma\delta} \delta_{\alpha\beta} \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta}. \quad (53)$$

where

$$\delta_{\alpha\beta} \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta} \equiv \eta^{\epsilon\ad\beta} \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta}. \quad (54)$$

The identities (40) and (51) then become, respectively,

$$\sqrt{-g}e^{\mu\nu\lambda\rho} (C^{\alpha\beta}_{\mu\nu} + C^{\alpha\beta}_{\nu\mu}) C_{\lambda\rho} \eta \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta} = \partial_\mu \{ 16i \sqrt{-g}e^{\mu\nu\lambda\rho} C_{\alpha\beta\gamma\delta} \delta_{\alpha\beta} \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta} \} , \quad (55)$$

and

$$\sqrt{-g}e^{\mu\nu\lambda\rho} \epsilon^{\alpha\beta\gamma\delta} (C^{\alpha\beta}_{\mu\nu} + C^{\alpha\beta}_{\nu\mu}) C_{\lambda\rho} \eta \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta} = \partial_\mu \{ 16i \sqrt{-g}e^{\mu\nu\lambda\rho} C_{\alpha\beta\gamma\delta} \delta_{\alpha\beta} \epsilon_{\gamma\delta} \epsilon_{\epsilon\eta} \} . \quad (56)$$
These two identities together with the previously known identity

\[ \sqrt{-g} \epsilon^{\mu\nu\lambda\rho}(R_{\mu\nu\lambda\rho} + \frac{1}{2} C^\gamma_{\mu\nu} C_{\alpha\lambda\rho}) = \partial_\mu (\sqrt{-g} \epsilon^{\mu\nu\lambda\rho} C_{\nu\lambda\rho}) \]

are the three identities containing only torsional tensorial entities, establishing

\[ \sqrt{-g} \epsilon^{\mu\nu\lambda\rho}(R_{\mu\nu\lambda\rho} + \frac{1}{2} C^\alpha_{\mu\nu} C_{\alpha\lambda\rho}) = \partial_\mu (\sqrt{-g} \epsilon^{\mu\nu\lambda\rho} C_{\nu\lambda\rho}) \]  

\[ \sqrt{-g} \epsilon^{\mu\nu\lambda\rho}(C^{\alpha\beta}_{\mu\nu} + C^\sigma_{\mu\nu} C^{\alpha\beta}_{\sigma}) C_{\lambda\eta} C_{\rho\delta} \]  

\[ \sqrt{-g} \epsilon^{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\gamma\delta} (C^{\alpha\beta}_{\mu\nu} + C^\sigma_{\mu\nu} C^{\alpha\beta}_{\sigma}) C_{\gamma\eta} C_{\rho\delta} \]

as the three purely torsional topological invariants. We remind ourself that \( C^{\alpha\beta}_{\mu\nu} \) is defined in (31).

We finally remark that we have applied the same consideration to the case of the SO(4,1) de Sitter algebra by combining \( \omega^{\alpha 5}_{\mu} = \frac{1}{4} t^a_{\mu} \), with the SO(3,1) connection \( \omega^{ab}_{\mu} \), defined by (25), to form the SO(4,1) de Sitter connection \( \omega^{AB}_{\mu} \). No additional identity beyond those already obtained is found.

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