On pointwise Malliavin differentiability of solutions to semilinear parabolic SPDEs

Carlo Marinelli

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Abstract

We obtain estimates on the first-order Malliavin derivative of mild solutions, evaluated at fixed points in time and space, to a class of parabolic dissipative stochastic PDEs on bounded domain of $\mathbb{R}^d$. In particular, such equations are driven by multiplicative Wiener noise and the nonlinear drift term is the superposition operator associated to a locally Lipschitz continuous function satisfying suitable polynomial growth bounds. The main arguments rely on the well-posedness theory in the mild sense for stochastic evolution equations in Banach spaces, monotonicity, and a comparison principle.

1 Introduction

Consider the stochastic evolution equation

$$du + Au \, dt = f(u) \, dt + \sigma(u) B \, dW(t), \quad u(0) = u_0,$$

(1.1)

where $A$ is the negative generator of an analytic semigroup of contractions $S$ on $L^q(G)$, with $q \geq 2$ and $G \subset \mathbb{R}^d$ a smooth bounded domain; $f: \mathbb{R} \to \mathbb{R}$ is a decreasing locally Lipschitz continuous function such that $|f(x)| \lesssim 1 + |x|^m$; $\sigma: \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function; $W$ is a cylindrical Wiener process on a separable Hilbert space $U$, and $B$ is $\gamma$-Radonifying from $U$ to $L^q(G)$. Precise assumptions are given in §2.1 below.

In [8] we proved that the unique mild solution to (1.1), which is continuous and time and space under mild extra assumptions, is such that the law of the random variable $u(t, x)$ is absolutely continuous with respect to Lebesgue measure for every $(t, x) \in \mathbb{R} \times G$. This was achieved considering first the case where $f$ is Lipschitz continuous, then using a localization argument implying that $u(t, x) \in D^{1,p}_{loc}$, hence applying the well-known Bouleau-Hirsch criterion. This reasoning, by its very nature, does not allow to show that $u(t, x)$ belongs to any Malliavin space. On the other hand, for equations with additive noise, i.e. for which $\sigma$ does not depend on $u$, it was proved in [7] that $u(t, x)$ belongs to $D^{1,p}$ for all $p \geq 1$, and even to $D^{k,p}$ is the coefficients $f$ and $\sigma$ are of class $C^k$ with all derivatives satisfying polynomial bounds. Equations driven by additive noise are in fact much easier to treat because the corresponding equations for Malliavin derivatives are deterministic PDEs with random coefficients, for which a large number of analytic tools can be applied pathwise. In the case of equations driven by multiplicative noise it is unfortunately impossible to follow this route, as the Malliavin derivatives of solutions satisfy linear stochastic PDEs of rather unfriendly character: for instance, the initial condition contains a Dirac measure in time.

Our goal is to fill at least in part the gap between the results of [8] and those of [7], showing that $u(t, x)$ belongs to $D^{1,p}$ for all $p \geq 1$. To this purpose, we develop two different approaches: one uses stochastic calculus in vector-valued $L^q$ spaces and monotonicity, and another one a comparison principle for mild solutions to stochastic evolution equations. In the former approach we need a kind of smoothness (or boundedness) assumption on the noise, while in the latter the covariance of the noise is assumed to be a positivity-preserving operator (such an assumption was used in [7] as well).
We refer to the introduction of [7] for references to the (not very extensive) literature on the pointwise Malliavin differentiability of solutions to equations with coefficients growing faster than linearly, as well as for a short discussion on potential applications.

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2 Preliminaries

2.1 Assumptions and notation

The linear unbounded operator $A$ appearing in (1.1) is supposed to be the (negative) generator of a strongly continuous analytic semigroup $S$ on $L^2(G)$, with $G$ a smooth bounded domain of $\mathbb{R}^d$. Furthermore, we assume that $S$ is self-adjoint and Markovian. Then $S$ can be restricted in a consistent manner to analytic semigroups, necessarily of contractions, to all $L^q := L^q(G)$ spaces, $q \geq 2$. Moreover, $S$ can be extended in a unique way from $L^2$ to an analytic contraction semigroup on $L^q(G; H)$ (see [10]). Finally, we assume that $S$ admits a kernel, i.e. that there exists a measurable function $K : \mathbb{R}_+ \times G^2 \rightarrow \mathbb{R}_+$ such that

$$[S(t)\phi](x) = \int_G K_t(x, y)\phi(y) \, dy$$

for every $\phi \in L^2$.

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuously differentiable, decreasing, and such that

$$|f(x)| + |f'(x)| \lesssim 1 + |x|^m \quad \forall x \in \mathbb{R}$$

for some $m \in \mathbb{R}_+$. More generally, it would be enough to assume that $x \rightarrow f(x) - ax$ is decreasing, for some $a \in \mathbb{R}_+$, and that $f$ is just almost everywhere differentiable. This little extra generality causes too much notational and technical nuisance to be justified.

The function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is of class $C^1$ with bounded derivative. The more general case of $\sigma$ being just local Lipschitz continuous with linear growth could be treated as well, again at the cost of some nuisance.

$W$ is a cylindrical Wiener process on a separable Hilbert space $U$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, with $T \in \mathbb{R}_+$, where $(\mathcal{F}_t)_{t \in [0, T]}$ is the completion of the filtration generated by $W$. All random quantities are assumed to be supported on this stochastic basis.

The operator $B$ is assumed to belong to $\gamma(U; L^q)$, the space of $\gamma$-Radonifying operators from $U$ to $L^q$. The precise value of $q$ will be in the range needed for Proposition 2.1 below to hold. Additional assumptions on $B$ will be stated when needed.

The initial datum $u_0$ is assumed to belong to $C_G$, although less is needed for just existence of a mild solution to (1.1).

The Hilbert space on which the Malliavin calculus will be based is $L^2(0, T; L^2_G)$. Here $Q := BB^*$ is a symmetric trace-class operator on $L^2$, in particular $Q^{1/2}$ exists and is itself a bounded operator, for which assume that $\ker Q^{1/2} = \{0\}$. Then $L^2_Q$ is the completion of $L^2$ with respect to the norm induced by the scalar product

$$\langle f, g \rangle_Q := \langle Q^{1/2}f, Q^{1/2}g \rangle_{L^2(G)} = \langle Q^{1/2}f, Q^{1/2}g \rangle_{L^2(G)}.$$

The space $L^2(0, T; L^2_Q)$ will be denoted simply by $H$. We refer to, e.g., [9] for notation and terminology pertinent to Malliavin calculus, as well as to [11] for an exposition geared towards SPDEs.
We shall write $a \lesssim b$ if there exists a constant $C$ such that $a \leq Cb$. If the constant depends on some parameters of interest, we indicate this as subscripts to the symbol. If $a \lesssim b$ and $b \lesssim a$, we shall write $a \simeq b$. To shorten the notation of functional spaces, we set $\mathbb{P} := L^p(\Omega)$, $L^p_r := L^p(G)$, $L^p_rH := L^p(G;H)$, and $L^p_r := L^p(0,T)$, for any $p$ for which they make sense. Moreover, instead of writing, for instance, $L^p(\Omega;L^q(0,T))$ we shall sometimes simply write $\mathbb{P}L^q$, and similarly for other spaces. We shall abbreviate deterministic and stochastic convolutions writing
\[ S \ast g := \int_0^t S(t-s)g(s)\,ds \quad \text{and} \quad S \diamond G := \int_0^t S(t-s)G(s)\,dW(s). \]

### 2.2 Well-posedness

The following well-posedness result for (1.1), that follows from [4, Theorem 4.9] (cf. [8, Proposition 2.5]), allows to consider the Malliavin derivative of the mild solution pointwise, i.e. for each $(t,x) \in [0,T] \times G$.

**Proposition 2.1.** Assume that
\[ \frac{d}{2q} < \frac{1}{2} - \frac{1}{p} \tag{2.1} \]
and $\sigma : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with linear growth. If $u_0 \in L^p(\Omega;C(\overline{G}))$, then (1.1) admits a unique $C(\overline{G})$-valued mild solution $u$, which satisfies the estimate
\[ \mathbb{E}\sup_{t \in \mathbb{T}} \|u(t)\|_{C(\overline{G})}^p \lesssim 1 + \mathbb{E}\|u_0\|_{C(\overline{G})}^p. \]

Condition (2.1) will be in force throughout the paper.

### 2.3 Maximal estimates for convolutions

We shall use two maximal estimates for deterministic and stochastic convolutions. The first one is elementary, and the second one is a special case of [12, Proposition 4.2] (the proof of which is an extension of the factorization method introduced in [3]).

Let $S$ be a strongly continuous analytic semigroup on a Banach space $E$ with generator $-A$. We shall denote by $E_\alpha$, $\alpha \in \mathbb{R}$, the usual domains of powers of (suitable shifts of) $A$, if $\alpha \geq 0$, and the corresponding extrapolation spaces if $\alpha < 0$. If $E = L^q$ or $L^q(G;H)$, with $H$ a Hilbert space, we shall write $E_\alpha^q$ or $E_\alpha^q(H)$, respectively.

**Lemma 2.2.** Let $r \in ]1,\infty]$ and $\eta \in [0,1/r'[, \, [0,1-1/r]$. Then there exists $\varepsilon \in \mathbb{R}_+$ such that
\[ \|S \ast g\|_{C([0,T];E_\eta)} \lesssim T^{\varepsilon}\|g\|_{L^{r'}(0,T;E)}. \]

**Proof.** By the analyticity of $S$,\[
\left\| \int_0^t S(t-s)g(s)\,ds \right\|_{E_\eta} \leq \int_0^t (t-s)^{-\eta}\|g(s)\|_E\,ds \\
\leq \left( \int_0^t s^{-\eta r'}\,ds \right)^{1/r'}\|g\|_{L^{r'}(0,T;E)},
\]
where $\eta r' \in [0,1[$. \hfill \qed

\[ ^1\text{The semigroup and its generator do not need to satisfy the assumptions of 2.1.} \]
Lemma 2.3. Let $E$ be a UMD Banach space, and the positive constants $\alpha$, $p$, and $\eta$ be such that
\[
\alpha \in \left(0, \frac{1}{2}\right], \quad p \in \left[2, \infty\right[, \quad \theta < \alpha - \frac{1}{p}.
\]
Then, for any $G : \Omega \times [0,T] \to \mathcal{L}(U;E)$ such that $Gu$ is measurable and adapted for every $u \in U$, there exists $\varepsilon \in \mathbb{R}_+$ such that
\[
\left\| S \circ G \right\|_{L^p(\Omega;C([0,T];E_\alpha))} \lesssim T^\varepsilon \left\| (t - \cdot)^{-\alpha} G \right\|_{L^p(\Omega;L^p(0,T;\gamma(L^2(0,t;U);E)))}.
\]

2.4 Itô’s formula

Let $E$ be a UMD Banach space. If $\Phi \in L^2(E)$, i.e. $\Phi$ is a continuous bilinear form on $E$, and $T \in \gamma(U;E)$, we set
\[
\text{Tr}_T \Phi := \sum_{n \in \mathbb{N}} \Phi(Th_n, Th_n),
\]
for which it is easily seen that
\[
|\text{Tr}_T \Phi| \leq \left\| \Phi \right\|_{L^2(E)} \left\| T \right\|_{\gamma(U;E)}^2. \tag{2.2}
\]
We shall repeatedly use the following Itô formula, proved in \cite{2}.

Lemma 2.4. Consider the $E$-valued process
\[
u(t) = u_0 + \int_0^t b(s) \, ds + \int_0^t G(s) \, dW(s),
\]
where

(a) $u_0 : \Omega \to E$ is $\mathcal{F}_0$-measurable;

(b) $b : \Omega \times [0,T] \to E$ is measurable, adapted and with paths in $L^1(0,T;E)$;

(c) $G : \Omega \times [0,T] \to \mathcal{L}(U;E)$ is $U$-measurable, adapted, stochastically integrable with respect to $W$, and with paths in $L^2(0,T;\gamma(U;E))$.

For any $\varphi \in C^2(E)$, one has
\[
\varphi(u(t)) = \varphi(u_0) + \int_0^t \varphi'(u(s))b(s) \, ds + \int_0^t \varphi''(u(s))G(s) \, dW(s) + \frac{1}{2} \int_0^t \text{Tr}_G \varphi''(u(s)) \, ds.
\]
The Itô formula will be applied to functions of the type $\left\| \cdot \right\|_{L^q(G;H)}$, with $H$ a Hilbert space. The differentiability of such functions is considered next.

2.5 Differentiability of the $q$-th power of the norm in $L^q$ spaces of Hilbert-space-valued functions

Let $H$ be a Hilbert space, $(X, \mathcal{A}, \mu)$ a $\sigma$-finite measure space, $q \in [2, \infty]$, and denote the Bochner space of $H$-valued functions $\phi$ such that $\left\| \phi \right\|_H \in L^q(\mu)$ by $L^q(\mu;H)$. The duality map of $L^q(\mu;H)$, defined as the function $J : L^q(\mu;H) \to L^{q'}(\mu;H)$ such that
\[
\langle \phi, J(\phi) \rangle = \left\| \phi \right\|^2_{L^q(\mu;H)},
\]
where \( \langle \cdot, \cdot \rangle \) stands for the duality pairing between \( L^q(\mu; H) \) and \( L^{q'}(\mu; H) \), is easily seen to be

\[
J : \phi \mapsto \| \phi \|^q_{L^q(\mu; H)} \| \phi \|^{-q'}_{H} \phi(\cdot).
\]

We shall need differentiability properties of a related functional, namely of

\[
\Phi_q : L^q(\mu; H) \to \mathbb{R}
\]

\[
\phi \mapsto \| \phi \|^q_{L^q(\mu; H)}.
\]

**Proposition 2.5.** One has \( \Phi_q \in C^2(L^q(\mu; H)) \), with

\[
\Phi_q'(u) : v \mapsto q(\| u \|^q_H^{-2} u, v) = q \int \| u(x) \|^q_H^{-2} \langle u(x), v(x) \rangle_H \mu(dx),
\]

\[
\Phi_q''(u) : (v, w) \mapsto q(\| u \|^q_H^{-2} v, w) + q(q - 2) \int \| u(x) \|^q_H^{-4} \langle u(x), v(x) \rangle_H \langle u(x), w(x) \rangle_H \mu(dx).
\]

In particular, for any \( u \in L^q(\mu; H) \),

\[
\Phi_q'(u) = q\| u \|^q_H^{-2} L^q(\mu; H) J(u), \quad \| \Phi_q'(u) \|_{L^{q'}(\mu; H)} = q\| u \|^{q-1}_{L^q(\mu; H)},
\]

and, by Hölder’s inequality,

\[
\| \Phi_q''(u) \|_{L^{q/2}(\mu, H)} \leq q(q - 1)\| u \|^{q-2}_{L^q(\mu; H)}.
\]

The statement about the first-order derivative follows by general properties of the duality mapping, while the statement about the second-order derivative can be obtained either by a direct computation, or showing that \( \| \cdot \|^2_H : L^q(\mu; H) \to L^{q/2}(\mu) \) is of class \( C^1 \), and then applying calculus rules for the composition of differentiable functions between Banach spaces (see, e.g., \( H \)).

### 3 The formal equation for Malliavin derivatives

Let us write the mild formulation of \( (1.1) \), in the equivalent form

\[
u(t, x) = u_0(x) + \int_0^t \int_G K_{t-s}(x, y) f(u(s, y)) dy ds + \int_0^t \int_G K_{t-s}(x, y) \sigma(u(s, y)) W(dy, ds),
\]

where \( W \) is a Brownian sheet on \( [0, T] \times G \) with covariance (in space) equal to \( Q \) (see \( S \) and references therein).

If \( Du(t, x) \) exists for all \( (t, x) \in [0, T] \times G \), taking the Malliavin derivative of both sides yields

\[
Du(t, x) = v_0(t, x) + \int_0^t \int_G K_{t-s}(x, y) f'(u(s, y)) Du(s, y) dy + \int_0^t \int_G K_{t-s}(x, y) \sigma'(u(s, y)) Du(s, y) \tilde{W}(dy, ds),
\]

where \( v_0 : \Omega \times [0, T] \times G \to H \) is defined by

\[
v_0(t, x) := (\tau, z) \mapsto K_{t-\tau}(x, z) \sigma(u(\tau, z)) 1_{[0, T]}(\tau).
\]
Let us define the process $\chi$ that, thanks to the boundedness of $F/C^4$ it was shown in \cite{8} that $\|u(t)\|_{C^4}$ reaches $n$, and $u_n$ is the unique mild solution in $L^p(\Omega; C([0,T]; C^4))$ to the equation

$$
    du_n + Au_n dt = f_n(u_n) dt + \sigma(u_n) B dW, \quad u_n(0) = u_0,
$$

with $f_n : x \mapsto f(x) 1_{[-n,n]}(x) + f(nx/x) 1_{[-n,n]}$. Moreover, $u_n = u$ on $[0,T_n]$ for every $n \in \mathbb{N}$. In particular, since $f_n$ is Lipschitz continuous, it follows by \cite{8} Theorem 3.1 that $Du_n \in L^\infty(0,T; L^p(\Omega; L^\infty(G,H)))$, hence, by construction of $Du$, one has at least $Du \in L^0(\Omega \times [0,T]; L^\infty(G,H))$.

Looking at the above identity for $Du$ as an equation for $L^q(G,H)$-valued processes, writing $v(t) = Du(t,x)$, one is lead to considering the equation

$$
    v(t) = v_0(t) + \int_0^t S(t-s)f'(u(s))v(s) ds + \int_0^t S(t-s)\sigma'(u(s))v(s)B dW(s),
\quad (3.1)
$$

where, for any $\eta \in (d/(2q), 1/2 - 1/p)$,

$$
    \|v_0(t)\|_{L^\infty(G,H)} \lesssim \|S(t-\cdot)\sigma(u_\lambda)\|_{L^2((0,t); H)}\gamma_{(U,L^q)},
\quad \|v_0\|_{L^\infty([0,T] \times G,H)} \lesssim (1 + \|\sigma(u)\|_{C([0,T]; C^4)})\|B\|_{(U,L^q)} T^{(1-2\eta)/2}.
$$

This implies

$$
    \|v_0\|_{L^p((0,T) \times G,H)} \lesssim T \left(1 + \|u\|_{L^p(\Omega; C([0,T]; C^4)))}\right)\|B\|_{(U,L^q)} \lesssim (1 + \|u_0\|_{C^1}) \|B\|_{(U,L^q)}
\quad (3.2)
$$

A mild solution to \ref{5.1} can be constructed by localization arguments. As a first step, let us replace $f_n$, without loss of generality, with the smoother version defined by

$$
    f_n(x) := f(0) + \int_0^x f'(y) \chi_n(y) dy,
$$

with $\chi_n : \mathbb{R} \to [0,1]$ of class $C^1$ such that $|\chi_n(x)| \leq 1$ and

$$
    \text{supp } \chi_n = [-n-1, n+1], \quad \chi_n([-n,n]) = 1.
$$

Let us define the process $F_n := f_n(u)$, and consider the equation

$$
    v_n(t) = v_0(t) + \int_0^t S(t-s)F_n(s)v_n(s) ds + \int_0^t S(t-s)\sigma'(u(s))v_n(s)B dW(s),
\quad (3.1)
$$

that, thanks to the boundedness of $F_n$ and $\sigma'$, admits a (global) mild solution $v_n$ belonging to $\mathcal{P}C^1 L^\infty_x H$, which is unique also in the larger space $L^\infty_x \mathcal{P}L^q_x H$. Defining $v$ to be the process equal to $v_n$ on $[0,T_n]$ for every $n \in \mathbb{N}$, recalling that $T_n$ converges monotonically to $+\infty$, we obtain a mild solution to \ref{3.1} that is necessarily unique in the set of processes that are locally in $L^\infty \mathcal{P}L^q_x H$. Therefore $v = Du$ in $L^0(\Omega \times [0,T]; L^\infty(G,H))$ and, outside a set $N = \Omega \times [0,T]$ of $\mathcal{P}$-null measure zero, $Du1_{[0,T_n]} \in \mathcal{P}C^1 L^\infty_x H$.

Even though $v = Du$ is locally (i.e. on increasing stochastic intervals) in $\mathcal{P}C^1 L^\infty_x H$, it seems not possible to obtain uniform bounds in this space using only the mild form of the equation.
We interpret this as an equation for the \( L \), namely
\[
\text{where } u \text{ is non-random. Therefore, as proved in [8],}
\]

\[
\text{without uniform bounds, even in weaker norms, it is not clear how to show that } u(t, x) \text{ belongs}
\]

to any space \( \mathbb{D}^{1,p} \). The main obstacle to obtaining such uniform bounds is the deterministic

\[
\text{convolution term in (3.1), essentially because any estimate of the } L^p(\Omega) \text{ norm of } f'(u)v, \text{ or of}
\]

\[
f''_n(u)v, \text{ will necessarily involve the } L^s(\Omega) \text{ norm of } v, \text{ for some } s > p, \text{ as } f'(u) \text{ is not in } L^\infty(\Omega),
\]
or, analogously, the norm of \( f''_n(u) \) in \( L^\infty(\Omega) \) may explode as \( n \) tends to infinity. Note, however,

\[
\text{that such (admittedly crude) estimates do not take advantage in any way of the dissipativity of } f. \text{ Estimates that do exploit the dissipative character of the equation will be obtained in the next section, at the cost, however, of a kind of smoothness assumption on the noise.}
\]

4 Estimates with smooth noise

Throughout this section we assume that \( B \) is very regular, i.e. that \( B \in \gamma(U; X) \), with \( X \) a Banach space continuously embedded in \( L^\infty(\Omega) \), which could be, for instance, \( E^\alpha_\delta \) with \( \alpha > d/(2q) \). Note that if \( B_0 \in \gamma(U; L^p) \), the ideal property of \( \gamma \)-Radonifying operators implies that

\[
B_\varepsilon \coloneqq (I + \varepsilon A)^{-\alpha} B_0 \text{ belongs to } \gamma(U; E^\alpha_\delta) \text{ for every } \varepsilon > 0 \text{ and } \lim_{\varepsilon \to 0} B_\varepsilon = B \text{ in } \gamma(U; L^p).
\]

Let \( (f_\lambda)_{\lambda>0} \subset C^1(\mathbb{R}) \) be a collection of decreasing Lipschitz-continuous functions such that \( f_\lambda \) and \( f'_\lambda \) converge pointwise as \( \lambda \to 0 \) to \( f \) and \( f' \), respectively. For instance, one may take (as in the previous section)

\[
f_\lambda(x) := f(0) + \int_0^x f'(y)\chi_\lambda(y) \, dy,
\]

with \( \chi_\lambda : \mathbb{R} \to [0, 1] \) of class \( C^1 \) such that \( |\chi'_\lambda| \leq 1 \) and

\[
\text{supp } \chi_\lambda = [-1/\lambda - 1, 1/\lambda + 1], \quad \chi_\lambda|_{[-1/\lambda, 1/\lambda]} = 1,
\]
or the Yosida approximation

\[
f_\lambda = \frac{1}{\lambda} \left( I - (I - \lambda f)^{-1} \right),
\]

where \( I : \mathbb{R} \to \mathbb{R} \) is the identity function.

Recall that the equation

\[
du_\lambda + Au_\lambda \, dt = f_\lambda(u_\lambda) \, dt + \sigma(u_\lambda)B \, dW, \quad u_\lambda(0) = u_0.
\]

admits a unique mild solution \( u_\lambda \in L^p(\Omega; C([0, T]; C(\overline{G}))) \) for every \( p > 0 \), because \( u_0 \in C(\overline{G}) \) is non-random. Therefore, as proved in [8], \( u_\lambda(t, x) \in \mathbb{D}^{1,p} \) for every \( (t, x) \in [0, T] \times G \) and every \( p > 0 \), with \( Du_\lambda \in L^\infty(0, T; L^p(\Omega; L^\infty(G; H))) \), and

\[
Du_\lambda(t, x) = v_{0,\lambda}(t) + \int_0^t \int_G K_{t-s}(x, y)f'_\lambda(u_\lambda(s, y))Du_\lambda(s, y) \, dy \\
\quad + \int_0^t \int_G K_{t-s}(x, y)\sigma'_\lambda(u_\lambda(s, y))Du_\lambda(s, y) \, d\gamma(dy, ds),
\]

where

\[
v_{0,\lambda}(t, x) := (\tau, z) \mapsto K_{t-\tau}(x, z) \sigma(u_\lambda(\tau, z)) 1_{[0, t]}(\tau).
\]

We interpret this as an equation for the \( L^q(G; H) \)-valued process \( v_\lambda \), with \( v_\lambda(t) : x \mapsto Du_\lambda(t, x) \), namely

\[
v_\lambda(t) = v_{0,\lambda}(t) + \int_0^t S(t-s)f'_\lambda(u_\lambda(s))v_\lambda(s) \, ds + \int_0^t S(t-s)\sigma'(u_\lambda(s))v_\lambda(s)B \, dW(s).
\]
In complete similarity to the previous section one has, for any \( \eta \in |d/(2q)|, 1/2 - 1/p] \),
\[
\left\| \nu_{0, \lambda}(t) \right\|_{L^\infty(G; H)} \lesssim \left\| S(t - \cdot)\sigma(u_\lambda)B \right\|_{L^2(0, t; \gamma(U; E^\gamma))},
\]

hence
\[
\left\| \nu_{0, \lambda} \right\|_{L^p L^{\infty, H}} \lesssim \left( 1 + \|a_0\|_{C([0, T])} \right) \|B\|_{s(U; L^s)} T^{(1 - 2\eta)/2}.
\]
Moreover, since \( f_\lambda' \) is bounded, \( L^\infty \) admits a unique solution \( \nu_\lambda \in \mathbb{F}C_t L^\infty T_x H \).

**Proposition 4.1.** The family of processes \( (\nu_\lambda)_{\lambda>0} \) is bounded in \( L^p(\Omega; C([0, T]; L^q(G; H))) \).

**Proof.** Setting \( \bar{\nu}_\lambda \equiv v_\lambda - \nu_{0, \lambda} \), in view of the boundedness of \( \nu_{0, \lambda} \) it is enough to show that \( (\bar{\nu}_\lambda) \) is bounded in \( L^q(\Omega; L^\infty(0, T; L^q(G; H))) \). To this purpose, let us write
\[
\bar{\nu}_\lambda(t) = \int_0^t S(t - s) f_\lambda'(u_\lambda(s)) \left( \bar{\nu}_\lambda(s) + \nu_{0, \lambda}(s) \right) ds
+ \int_0^t S(t - s) \sigma'(u_\lambda(s)) \left( \bar{\nu}_\lambda(s) + \nu_{0, \lambda}(s) \right) B dW(s),
\]
that is the mild form of the differential equation
\[
d\bar{\nu}_\lambda + A\bar{\nu}_\lambda dt = f_\lambda'(u_\lambda)(\bar{\nu}_\lambda + \nu_{0, \lambda}) dt + \sigma'(u_\lambda)(\bar{\nu}_\lambda + \nu_{0, \lambda}) B dW,
\]
with initial condition \( \bar{\nu}_\lambda(0) = 0 \). Itô’s formula for the \( q \)-th power of the \( L^q(G; H) \) norm applied to a suitable semimartingale approximation of \( \bar{\nu}_\lambda \) (see, e.g., [6] for details) yields
\[
\left\| \bar{\nu}_\lambda(t) \right\|_{L^q H}^q + q \int_0^t \| \bar{\nu}_\lambda(t) \|_{L^q H}^{q-2} \left\langle A\bar{\nu}_\lambda, J(\bar{\nu}_\lambda) \right\rangle ds
+ q \int_0^t \| \bar{\nu}_\lambda \|_{L^q H}^{q-2} J(\bar{\nu}_\lambda) \sigma'(u_\lambda)(\bar{\nu}_\lambda + \nu_{0, \lambda}) B dW(s)
+ \frac{1}{2} q(q - 1) \int_0^t \| \sigma'(u_\lambda)(\bar{\nu}_\lambda + \nu_{0, \lambda}) B \|_{L^q(U; L^q H)}^2 \left\| \bar{\nu}_\lambda \right\|_{L^q H}^{q-2} ds.
\]
Recalling that \( f_\lambda \) is decreasing, one has
\[
\left\langle f_\lambda'(u_\lambda)(\bar{\nu}_\lambda + \nu_{0, \lambda}), J(\bar{\nu}_\lambda) \right\rangle \leq \left\langle f_\lambda'(u_\lambda)\nu_{0, \lambda}, J(\bar{\nu}_\lambda) \right\rangle
\leq \frac{1}{2} \| \bar{\nu}_\lambda \|_{L^q H}^2 + \frac{1}{2} \| f_\lambda'(u_\lambda) \|_{L^q H}^2 \left\| \nu_{0, \lambda} \right\|_{L^q H}^2,
\]
hence
\[
\left\| \bar{\nu}_\lambda \right\|_{L^q H}^{q-2} f_\lambda'(u_\lambda)(\bar{\nu}_\lambda + \nu_{0, \lambda}), J(\bar{\nu}_\lambda) \nu_{0, \lambda}
\leq \frac{1}{2} \| \bar{\nu}_\lambda \|_{L^q H}^q + \frac{1}{2} \| \nu_{0, \lambda} \|_{L^q H}^q \| \nu_{0, \lambda} \|_{L^q H}^2,
\]
where, by Young’s inequality with conjugate exponents \( q/(q - 2) \) and \( q/2 \),
\[
\left\| \bar{\nu}_\lambda \right\|_{L^q H}^{q-2} f_\lambda'(u_\lambda) \| \nu_{0, \lambda} \|_{L^q H}^2 \left\| \nu_{0, \lambda} \right\|_{L^q H}^2
\leq \frac{q - 2}{q} \| \bar{\nu}_\lambda \|_{L^q H}^q + \frac{2}{q} \left\| f_\lambda'(u_\lambda) \right\|_{L^q H}^q \left\| \nu_{0, \lambda} \right\|_{L^q H}^q.
\]
The first term on the right-hand side of (4.2) is thus estimated by
\[
\frac{q-2}{2} \int_0^t \|\bar{v}_\lambda\|^q_{L^q_t H} \, ds + \int_0^t \|f'\lambda(u\lambda)\|^q_{L^q_t} \|v_{0,\lambda}\|^q_{L^q_t H} \, ds
= \frac{q-2}{2} \|\bar{v}_\lambda\|^q_{L^q(0,t;L^q_t H)} + \left\|\|f'\lambda(u\lambda)\|_{L^q_t} \|v_{0,\lambda}\|_{L^q_t H}\right\|^q_{L^q(0,t)}.
\]

Analogously, using the ideal property of \(\gamma\)-Radonifying operators on the diagram
\[
U \xrightarrow{B} X \xleftarrow{\sigma'(u)} L^\infty_x \xrightarrow{\tilde{v}_\lambda + v_{0,\lambda}} L^q L^q H,
\]
denoting the Lipschitz constant of \(\sigma\) by \(L_\sigma\), one has
\[
\left\|\sigma'(u\lambda)(\tilde{v}_\lambda + v_{0,\lambda})B\right\|^2_{\gamma(U;L^q_t H)} \|\bar{v}_\lambda\|^q_{L^q_t H}
\leq L^2_\sigma \|\bar{v}_\lambda + v_{0,\lambda}\|^2_{L^q_t H} \|B\|^2_{\gamma(U;X)} \|\bar{v}_\lambda\|^q_{L^q_t H}
\leq L^2_\sigma \|B\|^2_{\gamma(U;X)} \left(\frac{q-2}{q} \|\bar{v}_\lambda\|^q_{L^q_t H} + \frac{2}{q} \|\bar{v}_\lambda + v_{0,\lambda}\|^q_{L^q_t H}\right).
\]

The third term on the right-hand side of (4.2) is hence estimated by
\[
L^2_\sigma \|B\|^2_{\gamma(U;X)} \left(\frac{1}{2}(q-1)(q-2) \int_0^t \|\bar{v}_\lambda\|^q_{L^q_t H} \, ds + (q-1) \int_0^t \|\bar{v}_\lambda + v_{0,\lambda}\|^q_{L^q_t H} \, ds\right)
= L^2_\sigma \|B\|^2_{\gamma(U;X)} \left(\frac{1}{2}(q-1)(q-2) \|\bar{v}_\lambda\|^q_{L^q(0,t;L^q_t H)} + (q-1) \|\bar{v}_\lambda + v_{0,\lambda}\|^q_{L^q(0,t;L^q_t H)}\right).
\]

Let us denote the stochastic integral on the right-hand side of (4.2), which is a real local martingale, by \(M\), and set
\[
C_1 = C_1(q, B) := \left(\frac{1}{2}(q-2) + \frac{1}{2} q(q-1) L^2_\sigma \|B\|^2_{\gamma(U;X)}\right)^{1/q},
C_2 = C_2(q, B) := (q-1)^{1/2} (L_\sigma \|B\|_{\gamma(U;E_n)})^{2/q}.
\]

Then
\[
\|\bar{v}_\lambda\|_{C_t L^q H} \leq C_1 \|\bar{v}_\lambda(t)\|_{L^q_t L^q H} + C_2 \|v_{0,\lambda}(t)\|_{L^q_t L^q H}
+ \left\|\|f'\lambda(u\lambda)\|_{L^q_t} \|v_{0,\lambda}\|_{L^q_t H}\right\|_{L^q_t} + q^{1/q} (M^*_T)^{1/q},
\]
thus also
\[
\|\bar{v}_\lambda\|_{\mathbb{L}^p C_t L^q H} \leq C_1 \|\bar{v}_\lambda(t)\|_{\mathbb{L}^p L^q_t L^q H} + C_2 \|v_{0,\lambda}(t)\|_{\mathbb{L}^p L^q_t L^q H}
+ \left\|\|f'\lambda(u\lambda)\|_{L^q_t} \|v_{0,\lambda}\|_{L^q_t H}\right\|_{\mathbb{L}^p L^q_t} + q^{1/q} (M^*_T)^{1/q}_{\mathbb{L}^p q}.
\]

The Burkholder-Davis-Gundy inequality yields
\[
\|M^*_T\|_{\mathbb{L}^p q}^{1/q} \approx \|M\|^{1/2}_{T}\|_{\mathbb{L}^p q}^{1/2} = \|\|M\|_T^{1/(2q)}\|_{\mathbb{L}^p q}.
\]
where

\[\left[ M, M \right]_T^{1/2} \leq L_\sigma \| B \|_{\gamma(U,X)} \left( \int_0^T \| \tilde{v}_\lambda \|_{L^q H}^{2(q-1)} \| v_0, \lambda \|_{L^q H}^2 \, ds \right)^{1/2} \]

\[= L_\sigma \| B \|_{\gamma(U,X)} \left( \| \tilde{v}_\lambda \|_{L^q H} + v_0, \lambda \|_{L^q H} \right) \]

and, all norms being on \( L^q(G; H) \),

\[\| \tilde{v}_\lambda \|_{L^q H} + v_0, \lambda \|_{L^q H} \leq \| \tilde{v}_\lambda \|_{L^q H} + \| \tilde{v}_\lambda \|_{L^q H} \]

so that

\[\left[ M, M \right]_T^{1/2} \leq L_\sigma \| B \|_{\gamma(U,X)} \left( (2 - 1/q)^{1/q} \| \tilde{v}_\lambda \|_{L^{q^2} L^q H} + (1/q)^{1/q} \| v_0, \lambda \|_{L^{q^2} L^q H} \right) \]

and

\[\| [M, M]_T^{1/(2q)} \|_{L^q} \leq L_\sigma \| B \|_{\gamma(U,X)} \left( (2 - 1/q)^{1/q} \| \tilde{v}_\lambda \|_{L^{q^2} L^q H} + (1/q)^{1/q} \| v_0, \lambda \|_{L^{q^2} L^q H} \right) \]

Setting

\[C_3 = C_3(q, B) := (2q - 1)^{1/q} L_\sigma \| B \|_{\gamma(U,X)} \]

\[C_4 = C_4(q, B) := L_\sigma \| B \|_{\gamma(U,X)} \]

we are left with

\[\| \tilde{v}_\lambda \|_{L^p C_q L^q H} \leq C_1 \| \tilde{v}_\lambda \|_{L^p L^q H} + C_2 \| v_0, \lambda \|_{L^p L^q H} + C_4 \| v_0, \lambda \|_{L^p L^q H} \]

where

\[\| f'_A (u, \lambda) \|_{L^p L^q} \leq \| f'(u, \lambda) \|_{L^p L^q} \]

and recalling that \( f'_A (x) \leq 1 + |x|^m \) uniformly with respect to \( \lambda \) (see, e.g., [17, p. 295]) and \( u, \lambda \to u \)

in \( L^p(\Omega; C(0, T); C(\Omega)) \) for every \( p > 2 \),

\[\| f'_A (u, \lambda) \|_{L^p L^q, e} \leq 1 + \| u, \lambda \|^m_{L^p L^q, e} \leq 1 + \| u \|^m_{L^p C_q, e} \]

hence

\[\| f'_A (u, \lambda) \|_{L^p L^q} \leq 1 + \| u \|^m_{C_q} \]

with an implicit constant that depends, among others, on \( p \). Since the norm of the continuous embedding \( L^\infty(0, T) \to L^r(0, T) \) is \( T^{1/r} \) for any \( r \geq 1 \), we conclude that, for \( T_0 \) sufficiently small, \( (\tilde{v}_\lambda)_{\lambda \geq 0} \) is bounded in \( L^p(\Omega; L^\infty(0, T_0; L^q(G; H))) \). By an iteration argument, the same statement is true with \( T_0 \) replaced by \( T \).
Proof. Since implies that there exists a sequence \((\tilde{v}^n)_{n \in \mathbb{N}}\) such that \(\tilde{v}^n \to v\) weakly in \(L^\infty(\Omega; L^q(G; H))\) \(n \to \infty\).}

\[ v_n \to v \quad \text{weakly in } L^\infty(\Omega; L^q(G; H))) \]

\[ v_n \to v \quad \text{weakly in } L^p(\Omega \times [0, T]; L^q(G; H)). \]

Passing to the limit as \(\lambda \to 0\) in \(E\), recalling that the linear operators \(\phi \mapsto S \ast \phi\) and \(\Phi \mapsto S \circ \Phi\) are continuous, hence also continuous with respect to the corresponding weak topologies, it follows that \(\zeta\) coincides with the (unique) solution \(v\) to the equation for formal derivatives obtained in the previous section. This also shows that, under the smoothness assumption on \(\zeta\), follows that that \(\zeta = 0\). We shall see later that better regularity of \(Du\) can be obtained.

We are now going to show how the above compactness results imply estimates on the first-order Malliavin derivative of \(u\). We recall that in \(\mathbb{R}\) the basic result \(u(t, x) \in \mathbb{D}_{loc}^{1,p}\) for every \(p \geq 1\) and \((t, x) \in [0, T] \times G\) was proved.

**Theorem 4.4.** Let \(r = p \wedge q\). Then \(u(t, x) \in \mathbb{D}_{loc}^{1,r}\) for almost every \((t, x) \in [0, T] \times G\).

**Proof.** Since \(v_n\) converges to \(v\) weakly in \(L^p(\Omega \times [0, T]; L^q(G; H))\) as \(\lambda \to 0\), Mazur’s lemma implies that there exists a sequence \((z_n)\) defined by

\[ z_n := \sum_{k=1}^{N(n)} \alpha_{n,k} u_{\lambda_k}, \quad \text{with } \alpha_{n,k} \in \mathbb{R}_+, \quad \sum_{k=1}^{N(n)} \alpha_{n,k} = 1, \]

such that \(z_n \to v\) strongly in \(L^p(\Omega \times [0, T]; L^q(G; H))\). Let \((\tilde{u}_n)\) be the sequence defined by

\[ \tilde{u}_n := \sum_{k=1}^{N(n)} \alpha_{n,k} u_{\lambda_k}. \]

Then \(\tilde{u}_n\) converges to \(u\) in \(L^p(\Omega; C([0, T]; C(G)))\) as \(n \to \infty\), and \(z_n(t, x) = D\tilde{u}_n(t, x)\) for every \((t, x) \in [0, T] \times G\) by linearity of \(D\). Let \(r := p \wedge q\). Then \(\tilde{u}_n(t, x)\) converges to \(u(t, x)\) in \(L^r(\Omega)\) for every \((t, x) \in [0, T] \times G\) and \(z_n\) converges to \(v\) in \(L^r([0, T] \times G; L^r(\Omega; H))\), i.e.

\[ \lim_{n \to \infty} \int_0^T \int_G \|z_n(t, x) - v(t, x)\|^r_H dx dt = 0, \]

hence, passing to a subsequence is necessary,

\[ \lim_{n \to \infty} z_n(t, x) = v(t, x) \quad \text{in } L^r(\Omega; H) \]

for almost all \((t, x) \in G_T\). By the closability of \(D\) it follows that \(u(t, x) \in \mathbb{D}_{loc}^{1,r}\) and \(Du(t, x) = v(t, x)\) for almost all \((t, x) \in [0, T] \times G\). \(\square\)
In fact both \( p \) and \( q \) can be taken as large as needed, hence we actually have that, for almost every \((t, x) \in [0, T] \times G, u(t, x) \in \mathbb{D}^{1,r}\) for every \( r \geq 0\).

As a further step, we are going to show that \( v \) is the limit in a stronger topology of solutions to approximating equations. To this purpose, however, we are not going to use (4.1), but another approximation of (3.1). Let us set, for every \( t \),
\[
\mu(t) = \frac{f'(u(t))}{1 - \lambda f'(u(t))} = \frac{F(t)}{1 - \lambda F(t)}, \quad \lambda > 0,
\]
and consider the equation
\[
\overline{v}_\lambda(t) = v_0(t) + \int_0^t S(t - s)F_\lambda(s)\overline{v}_\lambda(s) \, ds + \int_0^t S(t - s)\sigma'(u(s))\overline{v}_\lambda(s)B \, dW(s), \quad \tag{4.3}
\]
which is readily seen to admit a unique solution \( \overline{v}_\lambda \) in \( L^p(\Omega; C([0, T]; L^\infty(G, H))) \), as it follows by boundedness of \( F_\lambda \).

**Proposition 4.5.** The family of processes \((\overline{v}_\lambda)_{\lambda > 0}\) is a Cauchy net in \( L^p(\Omega; C([0, T]; L^q(\overline{G}))) \).

**Proof.** It clearly holds
\[
\overline{v}_\lambda(t) - \overline{v}_\mu(t) = \int_0^t S(t - s)(F_\lambda(s)\overline{v}_\lambda(s) - F_\mu(s)\overline{v}_\mu(s)) \, ds
+ \int_0^t S(t - s)\sigma'(u(s))(\overline{v}_\lambda(s) - \overline{v}_\mu(s))B \, dW(s),
\]
i.e. \( \overline{v}_\lambda - \overline{v}_\mu \) is the unique mild solution to the differential equation
\[
d(\overline{v}_\lambda - \overline{v}_\mu) + A(\overline{v}_\lambda - \overline{v}_\mu) \, dt = (F_\lambda \overline{v}_\lambda - F_\mu \overline{v}_\mu) \, dt + \sigma'(u)(\overline{v}_\lambda - \overline{v}_\mu)B \, dW
\]
with zero initial condition. We are going to obtain estimates on the difference \( \overline{v}_\lambda - \overline{v}_\mu \) applying Itô’s formula, which is formal but harmless, as already mentioned. In the following, \( \langle \cdot, \cdot \rangle \), without subscripts, stands for duality pairing \( \langle \cdot, \cdot \rangle \) induced by the scalar product of \( L^2(G; H) \). One has
\[
\|\overline{v}_\lambda(t) - \overline{v}_\mu(t)\|_{L^2(H)}^2 + \int_0^t \langle A(\overline{v}_\lambda - \overline{v}_\mu), \Phi_q' (\overline{v}_\lambda - \overline{v}_\mu) \rangle \, ds
\]
\[
\leq \int_0^t \langle F_\lambda \overline{v}_\lambda - F_\mu \overline{v}_\mu, \Phi_q' (\overline{v}_\lambda - \overline{v}_\mu) \rangle \, ds
+ \int_0^t \Phi_q' (\overline{v}_\lambda - \overline{v}_\mu)\sigma'(u)(\overline{v}_\lambda - \overline{v}_\mu)B \, dW(s)
+ \frac{1}{2} q(q - 1) \int_0^t \|\sigma'(u)(\overline{v}_\lambda - \overline{v}_\mu)B\|_{\gamma(U; L^2(H))}^2 \|\overline{v}_\lambda - \overline{v}_\mu\|^2 \, ds. \quad \tag{4.4}
\]
To estimate the first term on the right-hand side, let us set
\[
J_\lambda^F := \frac{1}{1 - \lambda F},
\]
so that \( F_\lambda = FJ_\lambda \) and
\[
F_\lambda \overline{v}_\lambda - F_\mu \overline{v}_\mu = FJ_\lambda^F \overline{v}_\lambda - FJ_\mu^F \overline{v}_\mu,
\]
\[
\overline{v}_\lambda - \overline{v}_\mu = J_\lambda^F \overline{v}_\lambda - J_\mu^F \overline{v}_\mu + \overline{v}_\lambda - J_\lambda^F \overline{v}_\lambda - (\overline{v}_\mu - J_\mu^F \overline{v}_\mu),
\]
\[
= J_\lambda^F \overline{v}_\lambda - J_\mu^F \overline{v}_\mu + \lambda F_\lambda \overline{v}_\lambda - \mu F_\mu \overline{v}_\mu.
\]
Then, recalling that \( \Phi_q'(\overline{x}_\lambda - \overline{x}_\mu) = q\|\overline{x}_\lambda - \overline{x}_\mu\|_H^2(\overline{x}_\lambda - \overline{x}_\mu) \),

\[
\langle F_\lambda \overline{x}_\lambda - F_\mu \overline{x}_\mu, \Phi_q'(\overline{x}_\lambda - \overline{x}_\mu) \rangle
= q \langle F_\lambda J^F_\lambda \overline{x}_\lambda - F_\mu J^F_\mu \overline{x}_\mu, \|\overline{x}_\lambda - \overline{x}_\mu\|_H^2 (J^F_\lambda \overline{x}_\lambda - J^F_\mu \overline{x}_\mu) \rangle
+ q \langle F_\lambda \overline{x}_\lambda - F_\mu \overline{x}_\mu, \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2}\lambda F_\lambda \overline{x}_\lambda - \mu F_\mu \overline{x}_\mu \rangle,
\]

where, as \( F \leq 0 \),

\[
\langle F J^F_\lambda \overline{x}_\lambda - F J^F_\mu \overline{x}_\mu, \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2}(J^F_\lambda \overline{x}_\lambda - J^F_\mu \overline{x}_\mu) \rangle
= \int_G \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2} \langle F J^F_\lambda \overline{x}_\lambda - F J^F_\mu \overline{x}_\mu, J^F_\lambda \overline{x}_\lambda - J^F_\mu \overline{x}_\mu \rangle_H
\]

\[
= \int_G \| J^F_\lambda \overline{x}_\lambda - J^F_\mu \overline{x}_\mu \|_H^2 \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2} \leq 0,
\]

and

\[
\langle F_\lambda \overline{x}_\lambda - F_\mu \overline{x}_\mu, \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2}\lambda F_\lambda \overline{x}_\lambda - \mu F_\mu \overline{x}_\mu \rangle
= \int_G \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2} \langle F_\lambda \overline{x}_\lambda - F_\mu \overline{x}_\mu, \lambda F_\lambda \overline{x}_\lambda - \mu F_\mu \overline{x}_\mu \rangle_H
\]

\[
\lesssim (\lambda + \mu) \int_G \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2} (\|F_\lambda\|\|\overline{x}_\lambda\|_H^q + |F_\mu|\|\overline{x}_\mu\|_H^q)
\]

\[
\leq (\lambda + \mu) \|F\|_L^q (\|\overline{x}_\lambda\|_{L^qH} + \|\overline{x}_\mu\|_{L^qH})^q
\]

Therefore

\[
\int_0^t \langle F_\lambda \overline{x}_\lambda - F_\mu \overline{x}_\mu, \Phi_q'(\overline{x}_\lambda - \overline{x}_\mu) \rangle \, ds \lesssim (\lambda + \mu) q \int_0^t \|F\|_{L^q}(\|\overline{x}_\lambda\|_{L^qH} + \|\overline{x}_\mu\|_{L^qH})^q \, ds
\]

and

\[
\left|\int_0^t \langle F_\lambda \overline{x}_\lambda - F_\mu \overline{x}_\mu, \Phi_q'(\overline{x}_\lambda - \overline{x}_\mu) \rangle \, ds\right|^{1/q}
\]

\[
\lesssim (\lambda + \mu)^{1/q} q^{1/q} \left(\|F\|_{L^q}^{1/q} (\|\overline{x}_\lambda\|_{L^qH} + \|\overline{x}_\mu\|_{L^qH})\right)_{L^q(0,t)}.
\]

The remaining terms on the right-hand side of (4.3) can be estimated similarly to the corresponding estimates in the proof of Proposition 4.1. In particular, one has

\[
\frac{1}{2} q(q-1) \int_0^t \|s^u(\overline{x}_\lambda - \overline{x}_\mu)B\|_{\gamma(U,L^qH)}^2 \|\overline{x}_\lambda - \overline{x}_\mu\|_H^{q-2} \, ds
\]

\[
\leq \frac{1}{2} q(q-1) L^2_{\gamma(U,X)} \int_0^t \|\overline{x}_\lambda - \overline{x}_\mu\|_{L^qH}^q \, ds,
\]

hence the third term on the right-hand side of (4.3) raised to power \(1/\gamma\) is dominated by

\[
(q(q-1)/2)^{1/q} L^2_{\gamma(U,X)} |B|_{\gamma(U,X)}^{2/q} \|\overline{x}_\lambda - \overline{x}_\mu\|_{L^q(0,t,L^qH)}^{2/q}.
\]
Denoting by $M$ the stochastic integral on the right-hand side of (4.3), one has
\[
\left\| (M_T^s)^{1/q} \right\|_{L^p(\Omega)} = \left\| (M_T^s)^{1/q} \right\|_{L^p(\Omega)} \lesssim q^{1/q} \left\| \lambda - \mu \right\|_{L^p(\Omega;L^\infty(0,t;L^2_\mu))},
\]
with an implicit constant depending on the norm of $B$ in $\gamma(U;X)$ and $p$, among others. We are left with
\[
\left\| \lambda - \mu \right\|_{L^p(\Omega;C([0,t];L^2_\mu))} \lesssim (\lambda + \mu)^{p/q} \left\| F \right\|_{L^p_\infty(\Omega)} \left\| \lambda + \mu \right\|_{L^p_\infty(0,t;L^2_H)} + \left\| \lambda - \mu \right\|_{L^p(\Omega;L^\infty(0,t;L^2_H))},
\]
so that, for $T_0$ sufficiently small,
\[
\left\| \lambda - \mu \right\|_{L^p(\Omega;C([0,T_0];L^2_\mu))} \lesssim (\lambda + \mu)^{p/q} \left\| F \right\|_{L^p_\infty(\Omega)} \left\| \lambda + \mu \right\|_{L^p_\infty(0,t;L^2_H)} + \left\| \lambda - \mu \right\|_{L^p(\Omega;L^\infty(0,t;L^2_H))}.
\]
Iterating this reasoning over intervals of length $T_0$ covering $[0,T]$, we reach the conclusion that $(\lambda)$ is a Cauchy net in $L^p(\Omega;C([0,T];L^\infty(G;H)))$ if
\[
\left\| \lambda \right\|_{L^p(\Omega;L^\infty(0,T))}^q \left\| \lambda \right\|_{L^p(\Omega;L^\infty(0,T))}^{1-q}
\]
is bounded uniformly with respect to $\lambda$. From
\[
\left\| \lambda \right\|_{L^p_\infty(0,T)}^q \left\| \lambda \right\|_{L^p_\infty(0,T)}^{1-q}
\]
it follows that, for any $r, s > p$ such that $1/p = 1/r + 1/s$,
\[
\left\| \lambda \right\|_{L^p_{r,s}} \lesssim \left\| \lambda \right\|_{L^p_\infty(0,T)}^q \left\| \lambda \right\|_{L^p_{r,s}}^r.
\]
As already seen, $F = f(u)$ belongs to $L^p(\Omega;C([0,T];C(\Omega)))$ for every $p > 0$, hence one only has to show that $(\lambda)$ is bounded in $L^r(\Omega;L^s(0,T;L^2(G;H)))$ for some $r > p$. But this can be obtained exactly as in the proof of Proposition 4.1.

The Cauchy property just proved, coupled with the regularizing properties of the semigroup $S$, allow to obtain strong regularity properties of the process $Du$.

**Theorem 4.6.** The process $Du$ belongs to $L^p(\Omega;C([0,T];L^\infty(G;H)))$ for every $p > 0$, after modification on a subset of $\Omega \times [0,T]$ of measure zero.

**Proof.** Let $\zeta$ be the (strong) limit in $L^p(\Omega;C([0,T];L^\infty(G;H)))$ of the Cauchy sequence $(\lambda)$. Passing to the limit as $\lambda \to 0$ in (4.3), one easily sees that $\zeta$ solves (3.1), hence $\zeta = v = Du$. We are going to improve the regularity of $v$ using the regularizing properties of $S$. In particular, Lemma 2.2 yields
\[
\left\| S * Fv \right\|_{L^p(\Omega;C([0,T];L^\infty(H)))} \lesssim T^\varepsilon \left\| Fv \right\|_{L^p(\Omega;L^\infty(H))}
\]
for every $r > 1$ and $0 \leq \eta < 1 - 1/r$, with $\varepsilon$ a positive constant. Therefore, taking $r = 2p/(p+2)$, so that $1/r = 1/2 - 1/p$, and $\eta > d/(2q)$, one has
\[
\left\| S * Fv \right\|_{L^p(\Omega;L^\infty(H))} \lesssim T^\varepsilon \left\| Fv \right\|_{L^p(\Omega;L^\infty(0,T;L^s(G;H)))},
\]
where
\[
\left\| Fv \right\|_{L^p(\Omega;L^\infty(H))} \leq \left\| F \right\|_{L^\infty_{r,s}} \left\| v \right\|_{L^s_{r,s}}.
\]

14
Let $s, s' \in \mathbb{R}_+$ be such that $1/p = 1/s + 1/s'$. Then

$$
\|S \ast (F \nu)\|_{L^p C_0 L^q_\infty H} \lesssim_T \|F\|_{L^{p'} L^{q'} C_0} \|v\|_{L^{p'} L^{q'}_\infty H},
$$

where, as already mentioned several times, both norms on the right-hand side are finite. The analogous estimate for the stochastic convolution term is similar (actually a bit simpler, as $\sigma'(u)$ is bounded): by Lemma 2.3, taking $\alpha < 1/2$ such that $\eta < \alpha - 1/p$, one has

$$
\|S \circ \sigma'(u)vB\|_{L^p C_0 L^q_\infty(H)} \lesssim_{p,T} \left( \int_0^T \mathbb{E}\|(t - \cdot)^{-\alpha} \sigma'(u)vB\|_{\gamma L^2(0,t;U;L^q_\infty H)}^p \, dt \right)^{1/p},
$$

where

$$
\|(t - \cdot)^{-\alpha} \sigma'(u)vB\|_{\gamma L^2(0,t;U;L^q_\infty H)} \leq \|(t - \cdot)^{-\alpha} \sigma'(u)vB\|_{L^2(0,t;\gamma(U;L^q_\infty H))} \leq L_{\sigma} \|B\|_{\gamma(U;X)} \left( \int_0^t (t - s)^{-2\alpha} \|v(s)\|_{L^q_\infty H}^2 \, ds \right)^{1/2} \lesssim_T L_{\sigma} \|B\|_{\gamma(U;X)} \|v\|_{L^p C_0 L^q_\infty H},
$$

hence

$$
\|S \circ \sigma'(u)vB\|_{L^p C_0 L^q_\infty H} \lesssim_{p,T} L_{\sigma} \|B\|_{\gamma(U;X)} \|v\|_{L^p C_0 L^q_\infty H},
$$

where the right-hand side is certainly finite.

\[\square\]

## 5 Estimates with positivity-preserving covariance

Recall that $L^2_\sigma$ is the completion of $L^2(G)$ with respect to the norm $\|\cdot\|_Q := \|Q^{1/2}f\|_{L^2}$. Throughout this section we shall assume that the bounded operator $Q$ on $L^2$ is positivity preserving, i.e. that $Qg \geq 0$ if $g \in L^2$, $g \geq 0$. Moreover, without loss of generality, we assume that $\sigma \geq 0$.

Let $f_\lambda$ be the Yosida approximation of $f$. As already seen, the Lipschitz continuity of $f_\lambda$ implies that

$$
du_\lambda + Au_\lambda \, dt = f_\lambda(u_\lambda) \, dt + \sigma(u_\lambda)B \, dW, \quad u_\lambda(0) = u_0 \in C(G)
$$

admits a unique mild solution $u_\lambda \in L^p(\Omega; C([0,T];C(G)))$ for all $p \geq 1$, and $v_\lambda := Du_\lambda$ satisfies \[\text{\textbf{4.4}}\].

Let us introduce the auxiliary equation

$$
y_\lambda(t) = v_{0,\lambda}(t) + \int_0^t S(t - s)\sigma'(u_\lambda(s))y_\lambda(s)B \, dW(s).
$$

Both this equation and \[\text{\textbf{4.4}}\] are well-posed in $L^p(\Omega; C([0,T];L^\infty(G; H)))$.

The following comparison result is the main tool to achieve boundedness of the collection $(v_\lambda)$.

**Proposition 5.1.** One has $\|v_\lambda(t,x)\|_H \leq \|y_\lambda(t,x)\|_H$ for almost all $(t,x) \in [0,T] \times G$.

We proceed in several steps.

**Lemma 5.2.** Let $h \in H$ be such that $Qh \geq 0$. Then $\langle v_{0,\lambda}, h \rangle \geq 0$. 

15
The boundedness of \( Qh \in L^2 \) for a.a. \( t \in [0,T] \), hence \( Qh(t) \in L^2 \) for a.a. \( t \in [0,T] \) and \( Qh \in L^2(0,T;L^2) \approx L^2(G_T) \), so the statement \( Qh \geq 0 \) is meaningful. One has

\[
\langle v_{0\lambda}(t,x),h \rangle_H = \int_0^T \int_G K_{t-r}(x,z) \sigma(u_{\lambda}(\tau,z)) 1_{[0,l]}(\tau) [Qh](\tau,z) \, d\tau \, dz \\
= \int_0^T S(t-\tau) \sigma(u_{\lambda}(\tau))[Qh](\tau) \, d\tau,
\]

where \( \sigma \geq 0 \) and \( S \) is a positivity-preserving semigroup. The result then follows immediately. \( \square \)

Let us set, for any \( h \in H \), \( v_{0\lambda}^h := \langle v_{0\lambda}, h \rangle_H \). Then \( v_{0\lambda}^h \) satisfies

\[
v_{0\lambda}^h(t) = \langle v_{0\lambda}, h \rangle_H + \int_0^t S(t-s) f'_\lambda(u_{\lambda}(s)) v_{0\lambda}^h(s) + \int_0^t S(t-s) \sigma'(u_{\lambda}(s)) v_{0\lambda}^h(s) B \, dW,
\]

i.e., by the previous lemma, it is the mild solution to

\[
dv_{0\lambda}^h + Av_{0\lambda}^h = \langle \sigma(u_{\lambda})Qh + f'_\lambda(u_{\lambda})v_{0\lambda}^h \rangle dt + \sigma'(u_{\lambda})v_{0\lambda}^h B \, dW, \quad v_{0\lambda}^h(0) = 0. \tag{5.1}
\]

Completely analogously, \( y_{0\lambda}^h := \langle y_{0\lambda}, h \rangle_H \) is the mild solution to

\[
dy_{0\lambda}^h + Ay_{0\lambda}^h = \langle \sigma(u_{\lambda})Qh + \sigma'(u_{\lambda})y_{0\lambda}^h \rangle B \, dW, \quad y_{0\lambda}^h(0) = 0.
\]

We are going to compare \( v_{0\lambda}^h \) and \( y_{0\lambda}^h \) pointwise for a certain class of vectors \( h \). To do so, we need to impose a regularity assumption on the noise that will be removed later.

**Lemma 5.3.** Assume that \( B \in \gamma(U,X) \), with \( X \) a Banach space continuously embedded in \( L^\infty \). If \( h \in H \) is such that \( Qh \geq 0 \), then

\[
0 \leq \langle v_{\lambda}, h \rangle_H \leq \langle y_{\lambda}, h \rangle_H. \tag{5.2}
\]

**Proof.** The boundedness of \( f'_\lambda \) and the hypothesis on \( B \) imply that the equation for \( v_{0\lambda}^h \) is well-posed in \( L^p(\Omega;C([0,T];L^2)) \), hence we can apply the maximum principle in \( [3] \). This says that if

\[
-\langle \sigma(u_{\lambda})Qh + f'_\lambda(u_{\lambda})\phi, \phi^- \rangle_{L^2} + \frac{1}{2} \| 1_{\{\phi \leq 0\}} \sigma'(u_{\lambda})\phi B \|_{L^2}^2 \leq \| \phi^- \|_{L^2}^2
\]

for every \( \phi \in L^2 \), then \( v_{\lambda}^h \geq 0 \). Since \( \sigma \geq 0 \), \( Qh \geq 0 \), and \( f'_\lambda \) is bounded by \( 1/\lambda \), one obtains

\[
-\langle \sigma(u_{\lambda})Qh + f'_\lambda(u_{\lambda})\phi, \phi^- \rangle_{L^2} \leq \frac{1}{\lambda} \| \phi^- \|_{L^2}^2.
\]

Moreover, thanks to the hypothesis on \( B \) and the boundedness of \( \sigma', \phi \mapsto \sigma'(u_{\lambda})\phi B \) is Lipschitz continuous with values in the space of Hilbert-Schmidt operators, and

\[
\| 1_{\{\phi \leq 0\}} \sigma'(u_{\lambda})\phi B \|_{L^2}^2 \lesssim \| \phi^- \|_{L^2}^2.
\]

The proof that \( v_{\lambda}^h \geq 0 \) is thus completed. The difference \( y_{\lambda}^h - v_{\lambda}^h \) satisfies

\[
y_{\lambda}^h(t) - v_{\lambda}^h(t) = \int_0^t S(t-s)(-f'_\lambda(u_{\lambda}))v_{\lambda}^h \, ds + \int_0^t S(t-s)\sigma'(u_{\lambda})(y_{\lambda}^h(t) - v_{\lambda}^h(t)) B \, dW(s),
\]

hence, again applying the above-mentioned comparison principle, \( y_{\lambda}^h - v_{\lambda}^h \geq 0 \) if

\[
\langle f'_\lambda(u_{\lambda})v_{\lambda}^h, \phi^- \rangle_{L^2} + \frac{1}{2} \| 1_{\{\phi \leq 0\}} \sigma'(u_{\lambda})\phi B \|_{L^2}^2 \lesssim \| \phi^- \|_{L^2}^2,
\]

which is the case because \( f'_\lambda(u_{\lambda}) \leq 0 \) and \( v_{\lambda}^h \geq 0 \). The proof is thus concluded. \( \square \)
In order to remove the assumption on $B$ of the lemma, consider a linear equation of the type

$$w(t) = w_0(t) + \int_0^t S(t-s)F(s)w(s)\,ds + \int_0^t S(t-s)\Sigma(s)w(s)C\,dW(s)$$

for $L^q$-valued processes, where $F$ and $\Sigma$ are bounded random fields. Then results on continuous dependence of solutions on coefficients (cf. [4]), or a direct computation using the maximal estimate of Lemma 2.3, shows that the map

$$\gamma(U; L^q) \to L^p(\Omega; C([0, T]; E_\eta))$$

is continuous.

Now we can prove the crucial estimate.

**Proposition 5.4.** If $h \in H$ is such that $Qh \geq 0$, then

$$0 \leq \langle v_\lambda, h \rangle_H \leq \langle y_\lambda, h \rangle_H. \tag{5.3}$$

**Proof.** Let $\alpha > 0$ be such that $(I + A)^{-\alpha} L^q \to L^\infty$ and set $B_\varepsilon := (I + \varepsilon A)^{-\alpha} B$, $\varepsilon > 0$. Then $B_\varepsilon$ satisfies the hypothesis of the lemma and $B_\varepsilon \to B$ in $\gamma(U; L^q)$ as $\varepsilon \to 0$. Denote the solutions to the equations for $v_\lambda$ and $y_\lambda$ with $B_\varepsilon$ replaced by $B$ by $v_{\lambda, \varepsilon}$ and $y_{\lambda, \varepsilon}$, respectively. Then $0 \leq v_{\lambda, \varepsilon} \leq y_{\lambda, \varepsilon}$ by the lemma, hence $0 \leq v_\lambda \leq y_\lambda$ taking the limit as $\varepsilon \to 0$. \qed

If $h \in L^2([0, T] \times G)$, $h \geq 0$, since we have assumed that $Q$ is positivity preserving, then $Qh \geq 0$ and

$$0 \leq \langle v_\lambda, h \rangle_H = \langle Q^{1/2}v_\lambda, Q^{1/2}h \rangle = \langle Qv_\lambda, h \rangle_H.$$

Since this holds for an arbitrary such $h$, we infer that

$$Qy_\lambda(t, x) \geq Qv_\lambda(t, x) \geq 0 \quad \text{for a.a. } (t, x) \in [0, T] \times G.$$

In view of (5.3), we can thus proceed as follows: $Qv_\lambda \geq 0$ yields

$$\|v_\lambda\|_H^2 = \langle v_\lambda, v_\lambda \rangle_H \leq \langle y_\lambda, v_\lambda \rangle_H$$

and $Qy_\lambda \geq 0$ yields

$$\langle v_\lambda, y_\lambda \rangle_H \leq \langle y_\lambda, y_\lambda \rangle_H = \|y_\lambda\|_H^2.$$

We have thus shown that

$$\|v_\lambda\|_H^2 \leq \|y_\lambda\|_H^2,$$

i.e. the proof of Proposition 5.4 is completed.

Since estimates in $L^p(\Omega; C([0, T]; L^\infty))$ for $\|y_\lambda(t, x)\|_H$ uniform with respect to $\lambda$ can be easily obtained, as $y_\lambda$ satisfies an equation with Lipschitz coefficients, we arrive at the following result.

**Theorem 5.5.** Assume that $Q$ is positivity preserving. Then

$$\sup_{(t, x)} E\|Du(t, x)\|_H^p < \infty$$

for every $p > 0$.

This is the same result obtained in [7], as far as first-order Malliavin derivatives goes, in the much simpler case of additive noise, under the same conditions on $f$, $\sigma$, and $Q$. Here we needed to assume slightly more on the semigroup $S$ in order to extend it from $L^q$ to $L^q(H)$.
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Carlo Marinelli
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
United Kingdom
URL: http://goo.gl/4GKJP