A Note on Covering Young Diagrams with Applications to Local Dimension of Posets

Stefan Felsner, Torsten Ueckerdt

February 25, 2019

Abstract

We prove that in every cover of a Young diagram with \( \binom{2k}{k} \) steps with generalized rectangles there is a row or a column in the diagram that is used by at least \( k+1 \) rectangles. We show that this is best-possible by partitioning any Young diagram with \( \binom{2k}{k} - 1 \) steps into actual rectangles, each row and each column used by at most \( k \) rectangles. This answers two questions by Kim et al. [5].

Our results can be rephrased in terms of local covering numbers of difference graphs with complete bipartite graphs, which has applications in the recent notion of local dimension of partially ordered sets.

1 Introduction

Let \( \mathbb{N} \) denote the set of all natural numbers (i.e., positive integers). For \( x \in \mathbb{N} \) we denote \( [x] = \{1, \ldots, x\} \) to be the set of the first \( x \) natural numbers. A Young diagram with \( r \) rows and \( c \) columns is a subset \( Y \subseteq [r] \times [c] \) such that whenever \( (i, j) \in Y \), then \( (i-1, j) \in Y \) provided \( i \geq 2 \), as well as \( (i, j-1) \in Y \) provided \( j \geq 2 \). A Young diagram\(^1\) is visualized as a set of axis-aligned unit squares that are arranged consecutively in rows and columns, each row starting in the first column, and with every row (except the first) being at most as long as the row above. The number of steps of a Young diagram \( Y \) is the number of different row lengths in \( Y \), i.e., the cardinality of

\[
Z = \{(s, t) \in Y \mid (s+1, t) \notin Y \text{ and } (s, t+1) \notin Y\},
\]

where elements of \( Z \) are called steps of \( Y \). Young diagrams with \( n \) elements, \( r \) rows, \( c \) columns, and \( z \) steps, visualize partitions of the natural number \( n \) into \( r \) unlabeled positive integer summands (summand \( s \) being the length of row \( s \)) with summands on \( z \) different values and largest summand being \( c \).

A generalized rectangle in a Young diagram \( Y \subseteq [r] \times [c] \) is a set \( R \) of the form \( R = S \times T \) with \( S \subseteq [r] \) and \( T \subseteq [c] \) and \( R \subseteq Y \). Note that (unless \( Y = [r] \times [c] \)) not every set of the form \( R = S \times T \) with \( S \subseteq [r] \) and \( T \subseteq [c] \) satisfies \( R \subseteq Y \). A generalized rectangle \( R = S \times T \) with \( S \) being a set of consecutive numbers in \([r]\) and \( T \) being a set of consecutive numbers in \([c]\) is an actual rectangle. A

\(^1\)In the literature our Young diagrams are more frequently called Ferrers diagrams. We stick to Young diagram to be consistent with [5].
Figure 1: Left: A Young diagram $Y$ with $r = 8$ rows, $c = 7$ columns, and $z = 5$ steps. Highlighted are the set $Z$ of steps (gray), the element $(i, j) = (6, 2) \in Y$ (bold boundary), the generalized rectangle $\{2, 4, 5\} \times \{1, 3\}$ (green), and the actual rectangle $\{1, 2\} \times \{4, 5, 6\}$ (orange). Right: The Young diagram $Y_9$ with 9 steps and a $(2, 3)$-local partition of $Y$ with actual rectangles.

A generalized rectangle $R = S \times T$ uses the rows in $S$ and the columns in $T$. See the left of Figure 1 for an illustrative example.

Motivated by applications for the local dimension of partially ordered sets, we investigate covering a Young diagram $Y$ with generalized rectangles such that every row and every column of $Y$ is used by as few generalized rectangles in the cover as possible. We say that $Y$ is covered by a set $C$ of generalized rectangles if $Y = \bigcup_{R \in C} R$, i.e., $Y$ is the union of all rectangles in $C$. In this case we also say that $C$ is a cover of $Y$. If additionally the rectangles in $C$ are pairwise disjoint, we call $C$ a partition of $Y$. For example, the right of Figure 1 shows a Young diagram with a partition into actual rectangles.

**Theorem 1.** For any $k \in \mathbb{N}$, any Young diagram $Y$ can be covered by a set $C$ of generalized rectangles such that each row and each column of $Y$ used by at most $k$ rectangles in $C$ if and only if $Y$ has strictly less than $\left(\binom{2k}{k}\right)$ steps.

We prove Theorem 1 in Section 2, answer the questions raised by Kim et al. in Section 3, and describe the application to local dimension of posets in Section 4.

## 2 Proof of Theorem 1

Throughout we shall simply use the term rectangle for generalized rectangles, and rely on the term actual rectangle when specifically meaning rectangles that are contiguous. For a Young diagram $Y$ and $i, j \in \mathbb{N}$, let us define a cover $C$ of $Y$ to be $(i, j)$-local if each row of $Y$ is used by at most $i$ rectangles in $C$ and each column of $Y$ is used by at most $j$ rectangles in $C$. For $z \in \mathbb{N}$, let $Y_z = \{(s, t) \in [z] \times [z] \mid s + t \leq z + 1\}$ be the (unique) Young diagram with $z$ rows, $z$ columns, and $z$ steps. See the right of Figure 1.

We start with a lemma stating that instead of considering any Young diagram with $z$ steps, we may restrict our attention to just $Y_z$.

**Lemma 2.** Let $i, j, z \in \mathbb{N}$ and $Y$ be any Young diagram with $z$ steps. Then $Y$ admits an $(i, j)$-local cover if and only if $Y_z$ admits an $(i, j)$-local cover with exactly $z$ rectangles.

**Proof.** First assume that $Y$ admits an $(i, j)$-local cover $C$. If $C$ consists of strictly more than $z$ rectangles, then there are $R_1, R_2 \in C$, $R_1 \neq R_2$, such that $R_1, R_2 \subseteq
Figure 2: Transforming a cover of any Young diagram $Y$ with 5 steps into a cover of $Y_5$ (left) and vice versa (right).

$[s] \times [t]$ for some step $(s, t) \in Z$. However, in this case $C - \{R_1, R_2\} + \{R_1 \cup R_2\}$ is also an $(i, j)$-local cover of $Y$ with one rectangle less. Thus, by repeating this argument, we may assume that $|C| = z$.

If $Y \neq Y_z$, there is a row $s$ or a column $t$ that is not used by any step in $Z$. Apply the mapping $\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with $(x, y) \mapsto \begin{cases} (x, y) & \text{if } x < s \\ (x - 1, y) & \text{if } x \geq s \end{cases}$ respectively $(x, y) \mapsto \begin{cases} (x, y) & \text{if } y < t \\ (x, y - 1) & \text{if } y \geq t \end{cases}$

Intuitively, we cut out row $s$ (respectively column $t$), moving all rows below one step up (respectively all columns to the right one step left). This gives an $(i, j)$-local cover of a smaller Young diagram with $z$ steps, and eventually leads to an $(i, j)$-local cover of $Y_z$, as desired. See the left of Figure 2.

On the other hand, if $Y_z$ admits an $(i, j)$-local cover $C = \{R_1, \ldots, R_z\}$, this defines an $(i, j)$-local cover of $Y$ as follows. Index the rows used by the steps $Z$ of $Y$ by $s_1 < \cdots < s_z$ and the columns used by the steps $Z$ of $Y$ by $t_1 < \cdots < t_z$ and let $s_0 = t_0 = 0$. Defining

$$R'_a = \{(s, t) \in Y \mid s_{x-1} < s \leq s_x \text{ and } t_{y-1} < t \leq t_y \text{ for some } (x, y) \in R_a\}$$

for $a = 1, \ldots, z$ gives an $(i, j)$-local cover $\{R'_1, \ldots, R'_z\}$ of $Y$. See the right of Figure 2.

Observe that the construction maps an actual rectangle $R_a$ of $Y_z$ to an actual rectangle $R'_a$ of $Y$. Also, if $\{R_1, \ldots, R_z\}$ is a partition of $Y_z$, then $\{R'_1, \ldots, R'_z\}$ is a partition of $Y$. This will be used in the proof of Item (i) of Theorem 3.

Let us now turn to our main result. In fact, we shall prove the following strengthening of Theorem 1.

**Theorem 3.** For any $i, j, z \in \mathbb{N}$ and any Young diagram $Y$ with $z$ steps, the following hold.

(i) If $z < \binom{i+j}{i}$, then there exists an $(i, j)$-local partition of $Y$ with actual rectangles.

(ii) If $z \geq \binom{i+j}{i}$, then there exists no $(i, j)$-local cover of $Y$ with generalized rectangles.

**Proof.** First, let us prove Item (i). For shorthand notation, we define $f(i, j) := \binom{i+j}{i} - 1$. It will be crucial for us that the numbers $\{f(i, j)\}_{i,j \geq 1}$ solve the

$$...$$
Then by (1). Consider the actual rectangle $(i, j)$-local partition of $Y_z$ with $1 \leq i \leq j$. This follows directly from Pascal’s rule $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$ for any $a, b \in \mathbb{N}$ with $1 \leq b \leq a - 1$.

Due to Lemma 2 it suffices to show that for any $i, j \in \mathbb{N}$ and $z = f(i, j) = \binom{i+j}{i} - 1$, there is an $(i, j)$-local partition of $Y_z$ with actual rectangles.

We define the $(i, j)$-local partition $C$ by induction on $i$ and $j$. For illustrations refer to Figure 3.

If $i = 1$, respectively $j = 1$, then $C$ is the set of rows of $Y_z$, respectively the set of columns of $Y_z$. If $i \geq 2$ and $j \geq 2$, then $z = f(i, j) = f(i-1, j) + f(i, j-1) + 1$ by (1). Consider the actual rectangle $R = [a] \times [z+1-a]$ for $a = f(i-1, j) + 1$. Then $Y_z - R$ splits into a right-shifted copy $Y'$ of $Y_{a-1}$ and a down-shifted copy $Y''$ of $Y_{z-a}$. Note that $a - 1 = f(i-1, j)$ and $z - a = f(i, j-1)$.

By induction we have an $(i-1, j)$-local cover $C'$ of $Y'$ and an $(i, j-1)$-local cover $C''$ of $Y''$, each consisting of pairwise disjoint actual rectangles. Define

$$C = \{R\} \cup C' \cup C''$$

this is a cover of $Y_z$ consisting of pairwise disjoint actual rectangles. Rows 1 to $a$ are used by $R$ and at most $i - 1$ rectangles in $C'$, and rows $a + 1$ to $z$ are used by at most $i$ rectangles in $C''$. Hence each row of $Y_z$ is used by at most $i$ rectangles in $C$. Similarly each column of $Y_z$ is used by at most $j$ rectangles in $C$. Thus $C$ is an $(i, j)$-local partition of $Y_z$ by actual rectangles, as desired.

For $z' < z = f(i, j)$ we obtain an $(i, j)$-local partition of $Y_{z'}$ by restricting the rectangles of the cover $C$ of $Y_z$ to the rows from $z - z'$ to $z$. This yields an $(i, j)$-local partition of a down-shifted copy $Y'$ of $Y_{z'}$.

Now, let us prove Item (ii). Due to Lemma 2 it is sufficient to show that for $i, j \in \mathbb{N}$ the Young diagram $Y_{z'}$ with $z' \geq \binom{i+j}{i}$ admits no $(i, j)$-local cover. If $Y_{z'}$ with $z' > z = \binom{i+j}{i}$ has an $(i, j)$-local cover, then by restricting the rectangles of the cover to the rows from $z' - z$ to $z'$ we obtain an $(i, j)$-local cover of a down-shifted copy of $Y_z$. Therefore, we only have to consider $Y_z$. 

Figure 3: **Left:** The Young diagram $Y_z$ with $z = f(1, 7) = \binom{1+7}{1} - 1 = 7$ steps and a $(1, 7)$-local partition of $Y_z$ into actual rectangles. **Right:** The Young diagram $Y_z$ with $z = f(3, 2) = \binom{3+2}{3} - 1 = 9$ steps, the rectangle $R = [a] \times [z+1-a] = [6] \times [6]$ with $a = f(2, 2) + 1 = 6$, and the Young diagrams $Y'$ and $Y''$ with $f(2, 2) = 5$ and $f(3, 1) = 3$ steps, respectively.
Let $C$ be a cover of $Y_z$. We shall prove that $C$ is not $(i,j)$-local. Again, we proceed by induction on $i$ and $j$, where illustrations are given in Figure 4.

If $i = 1$, then each row is only used by a single rectangle in $C$, otherwise, $C$ would not be $(1,j)$-local. Hence, each row of $Y_z$ is a rectangle in $C$. Thus column 1 of $Y_z$ is used by $z = j + 1$ rectangles, proving that $C$ is not $(i,j)$-local.

The case $j = 1$ is symmetric to the previous by exchanging rows and columns.

Now let $i \geq 2$ and $j \geq 2$. We have $z = \binom{i+j}{i} = \binom{i-1+j}{i-1} + \binom{i+j-1}{j-1}$.

Consider the rectangle $M = [a] \times [z-a]$ for $a = \binom{i-1+j}{i-1}$. Then $Y_z - M$ splits into a right-shifted $Y'$ copy of $Y_a$ and a down-shifted copy $Y''$ of $Y_{z-a}$. Note that $z - a = \binom{i+j-1}{i-1}$.

Let $C'$, respectively $C''$, be the subset of rectangles in $C$ using at least one of the rows $1, \ldots, a$ in $Y_z$, respectively at least one of the columns $1, \ldots, z-a$ in $Y_z$. Note that $C' \cap C'' = \emptyset$ as each generalized rectangle is contained in $Y_z$.

Prune each rectangle in $C'$ to the columns $z - a + 1, \ldots, z$ and each rectangle in $C''$ to the rows $a + 1, \ldots, z$. This yields covers of $Y'$ and $Y''$.

The Young diagram $Y'$ is a copy of $Y_a$ and $a = \binom{i-1+j}{i-1}$. Hence, by induction the pruned cover $C'$ is not $(i-1,j)$-local. If some column $t$ of $Y'$ is used by at least $j+1$ rectangles in $C'$, this column of $Y_z$ is used by at least $j+1$ rectangles in $C$, proving that $C$ is not $(i,j)$-local, as desired. So we may assume that some row $s$ of $Y'$ is used by at least $i$ rectangles in $C'$.

Symmetrically, $Y''$ is a copy of $Y_{z-a}$ and $z - a = \binom{i+j-1}{i-1}$. Hence, the pruned $C''$ is a cover of $Y''$, which by induction is not $(i,j-1)$-local, and we may assume that some column $t$ of $Y''$ is used by at least $j$ rectangles in $C''$. Hence row $s$ in $Y_z$ is used by at least $i$ rectangles in $C'$ and column $t$ in $Y_z$ is used by at least $j$ rectangles in $C''$. As $C' \cap C'' = \emptyset$ and element $(s,t)$ is contained in some rectangle of $C$, either row $s$ of $Y_z$ is used by at least $i+1$ rectangles or column $t$ of $Y_z$ is used by at least $j+1$ rectangles (or both), proving that $C$ is not $(i,j)$-local.

Finally, Theorem 1 follows from Theorem 3 by setting $i = j = k$.

3 Local covering numbers

In [5], Kim et al. introduced the concept of covering a Young diagram with generalized rectangles subject to minimizing the maximum number of rectangles in any row or column. Their motivation was to investigate the relations between
local difference cover numbers and local complete bipartite cover numbers, which are defined as follows\(^2\).

A difference graph is a bipartite graph in which the vertices of one partite set can be ordered \(a_1,\ldots,a_r\) in such a way that \(N(a_i) \subseteq N(a_{i-1})\) for \(i = 2,\ldots,r\), i.e., the neighborhoods of these vertices along this ordering are weakly nested. Equivalently, a bipartite graph \(H = (V,E)\) with bipartition \(V = A \cup B\), \(|A| = r, |B| = c\), is a difference graph if \(H\) admits a bipartite adjacency matrix \(M = (m_{s,t})_{s \in A, t \in B}\) whose support is a Young diagram \(Y \subseteq [r] \times [c]\):

\[
\forall s \in A, t \in B : \quad \{s,t\} \in E \iff (s,t) \in Y \iff m_{s,t} = 1
\]

Then complete bipartite subgraphs \(G\) of \(H\) correspond precisely to generalized rectangles \(R\) in \(Y\). Rows and columns of \(M\) correspond to vertices of \(H\) in \(A\) and \(B\), respectively.

Following the notation in [6], local covering numbers are defined as follows. For a graph class \(\mathcal{F}\) and a graph \(H\), an injective \(\mathcal{F}\)-covering of \(H\) is a set of graphs \(G_1,\ldots,G_t \in \mathcal{F}\) with \(H = G_1 \cup \cdots \cup G_t\). An injective \(\mathcal{F}\)-covering of \(H\) is \(k\)-local if every vertex of \(H\) is contained in at most \(k\) of the graphs \(G_1,\ldots,G_t\), and the local \(\mathcal{F}\)-covering number of \(H\), denoted by \(c^\mathcal{F}_{\mathcal{F}}(H)\), is the smallest \(k\) for which a \(k\)-local injective \(\mathcal{F}\)-cover of \(H\) exists.

Let \(\mathcal{D}\) denote the class of all difference graphs, and \(\mathcal{CB} \subset \mathcal{D}\) the class of all complete bipartite graphs. Clearly, we have \(c^\mathcal{D}_{\mathcal{D}}(H) \leq c^\mathcal{CB}_{\mathcal{CB}}(H)\) for all graphs \(H\). Kim et al. [5] asked whether there is a sequence of graphs \((H_i : i \in \mathbb{N})\) for which \(c^\mathcal{CB}_{\mathcal{CB}}(H_i)\) is constant while \(c^\mathcal{D}_{\mathcal{D}}(H_i)\) is unbounded. They prove that for all graphs \(H\) on \(n\) vertices,

\[
c^\mathcal{CB}_{\mathcal{CB}}(H) \leq c^\mathcal{D}_{\mathcal{D}}(H) \cdot \lceil \log_2(n/2 + 1) \rceil,
\]

by showing that \(c^\mathcal{CB}_{\mathcal{CB}}(H) \leq \lceil \log_2(r + 1) \rceil\) whenever \(H \in \mathcal{D}\) is a difference graph with one partite set of size \(r\). However, no lower bound on \(c^\mathcal{CB}_{\mathcal{CB}}(H)\) for \(H \in \mathcal{D}\) is established in [5]. Specifically, Kim et al. ask for the exact value of \(c^\mathcal{CB}_{\mathcal{CB}}(H_i)\) for the difference graph \(H_i\) corresponding to the Young diagram \(Y_i\). For the case that \(i+1\) is a power of \(2\) they prove the upper bound \(c^\mathcal{CB}_{\mathcal{CB}}(H_i) \leq \log_2(i+1) - 1\).

Using Theorem 1 and \((2k^2)_k = (1 + o(1))\frac{1}{\sqrt{k\pi}}2^{2k}\), we see that

- for every difference graph \(H\) the exact value of \(c^\mathcal{CB}_{\mathcal{CB}}(H)\) is the smallest \(k \in \mathbb{N}\) such that for the number \(z\) of steps\(^3\) of \(H\) it holds \(z < (2k^2)_k\),
- the difference graphs \(H_i, i \in \mathbb{N}\), defined by Kim et al. satisfy

\[
c^\mathcal{CB}_{\mathcal{CB}}(H_i) = (1 + o(1))\frac{1}{2} \log_2 i,
\]

- for this sequence \((H_i : i \in \mathbb{N})\) of difference graphs \(c^\mathcal{D}_{\mathcal{D}}(H_i)\) is constant 1, while \(c^\mathcal{CB}_{\mathcal{CB}}(H_i)\) is unbounded, and
- for all graphs \(H\) on \(n\) vertices,

\[
c^\mathcal{CB}_{\mathcal{CB}}(H) \leq c^\mathcal{D}_{\mathcal{D}}(H) \cdot (1 + o(1))\frac{1}{2} \log_2(n/2).
\]

\(^2\)Deviating from [5], we follow here the terminology and notation of local covering numbers introduced in [6].

\(^3\)In terms of graphs, this is the number of different sizes of neighborhoods in one partite set.
4 Local dimension of posets

The motivation for Kim et al. [5] to study local difference cover numbers comes from the local dimension of posets, a notion recently introduced by Ueckerdt [9].

For a partially ordered set (short poset) \( P = (P, \leq) \), define a partial linear extension of \( P \) to be a linear extension \( L \) of an induced subposet of \( P \). A local realizer of \( P \) is a non-empty set \( L \) of partial linear extensions such that (1) if \( x < y \) in \( P \), then \( x < y \) in some \( L \in L \), and (2) if \( x \) and \( y \) are incomparable (denoted \( x \parallel y \)), then \( x < y \) in some \( L \in L \) and \( y < x \) in some \( L' \in L \). The local dimension of \( P \), denoted \( ldim(P) \), is then the smallest \( k \) for which there exists a local realizer \( L \) of \( P \) with each \( x \in P \) appearing in at most \( k \) partial linear extensions \( L \in L \).

For an arbitrary height-two poset \( P = (P, \leq) \), Kim et al. consider the bipartite graph \( G_P = (P, E) \) with partite sets \( A = \text{min}(P) \) and \( B = P - \text{min}(P) \subseteq \text{max}(P) \) whose edges correspond to the so-called critical pairs:

\[
\forall x \in A, y \in B: \quad \{x, y\} \in E \iff x \parallel y \quad \text{in } P
\]

They prove that

\[
c^D_P(G_P) - 2 \leq ldim(P) \leq c^\mathcal{EB}_P(G_P) + 2,
\]

which also gives good bounds for \( ldim(P) \) when \( P \) has larger height, since we have

\[
ldim(Q) - 2 \leq ldim(P) \leq 2ldim(Q) - 1
\]

for the associated height-two poset \( Q \) known as the split of \( P \) (see [1], Lemma 5.5). Using these results and the ones from the previous section, we can conclude the following for the local dimension of any poset.

**Corollary 4.** For any poset \( P \) on \( n \) elements with split \( Q \) we have

\[
c^D_P(G_Q) - 4 \leq ldim(P) \leq c^D_P(G_Q) \cdot (1 + o(1)) \log_2 n.
\]

5 Ferrers Dimension

The aim of this section is to provide some links to research where related things have been investigated with a different terminology.

A Ferrers diagram is a Young diagram. Typically Ferrers diagrams are defined as graphical visualizations of integer partitions.

Riguet [8] defined a Ferrers relation\(^4\) as a relation \( R \subset X \times Y \) such that

\[
(x, y) \in R \text{ and } (x', y') \in R \implies (x, y') \in R \text{ or } (x', y) \in R.
\]

A relation \( R \subset X \times Y \) can be viewed as a digraph \( D \) with \( V_D = X \cup Y \) and \( E_D = R \). A digraph thus corresponding to a Ferrers relation is a Ferrers digraph. Riguet characterized Ferrers digraphs as those in which the sets \( N^+(v) \) of out-neighbors are linearly ordered by inclusion. Hence, bipartite Ferrers digraphs are exactly the difference graphs.

\(^4\) According to [4] Ferrers relations have also been studied under the names of biorders, Guttman scales, and bi-quasi-series.
By playing with \( x = x' \) and/or \( y = y' \) in the definition of a Ferrers relation it can be shown that Ferrers digraphs without loops are 2+2-free and transitive, i.e., they are interval orders. In general, however, Ferrers digraphs are allowed to have loops.

In the spirit of order dimension the Ferrers dimension of a digraph \( D \) (\( \text{fdim}(D) \)) is the minimum number of Ferrers digraphs whose intersection is \( D \). If \( \mathcal{P} = (P, \leq) \) is poset and \( D_{\mathcal{P}} \) the digraph associated with the order relation (reflexivity implies that \( D_{\mathcal{P}} \) has loops at all vertices), then \( \text{dim}(\mathcal{P}) = \text{fdim}(D_{\mathcal{P}}) \). This was shown by Bouchet [2] and Cogis [3], it implies that Ferrers dimension is a generalization of order dimension. Since Ferrers digraphs are characterized by having a staircase shaped adjacency matrix the complement of a Ferrers digraph is again a Ferrers digraph. Therefore, instead of representing a digraph as intersection of Ferrers digraphs containing \( (D = \bigcap F_i \text{ with } D \subseteq F_i) \). We can as well represent its complement as union of Ferrers digraphs contained in it \( (\overline{D} = \bigcap F_i \text{ with } F_i \subseteq \overline{D}) \). This simple observation is sometimes useful.

The Ferrers dimension of a relation \( R \) (\( \text{fdim}(R) \)) is the minimum number of Ferrers relations whose intersection is \( R \). Note that if \( D \) is the digraph corresponding to a relation \( R \), then \( \text{fdim}(D) = \text{fdim}(R) \). Hence, the result of Bouchet can be expressed as \( \text{dim}(\mathcal{P}) = \text{fdim}(P, P, \leq) \), here we use the notation \( (P, P, \leq) \) to emphasize that we interpret the order as a relation. The interval dimension \( \text{idim}(\mathcal{P}) \) of a poset \( \mathcal{P} \) is the minimum number of interval orders extending \( \mathcal{P} \) whose intersection is \( \mathcal{P} \). Interestingly interval dimension is also nicely expressed as a special case of Ferrers dimension: \( \text{idim}(\mathcal{P}) = \text{fdim}(P, P, <) \). For this and far reaching generalizations see Mitas [7].

Relations \( R \subseteq X \times Y \) with \( X \cap Y = \emptyset \) can be viewed as bipartite graphs. In this setting \( \text{fdim}(R) \) is the global \( D \)-covering number of \( \overline{R} \), i.e., minimum number of difference graphs whose union is the bipartite complement of \( R \).

We believe that it is worthwhile to study local variants of Ferrers dimension.

Acknowledgments

This research has been mostly conducted during the Graph Drawing Symposium 2018 in Barcelona. Special thanks go to Peter Stumpf for helpful comments and discussions.

References

[1] Fidel Barrera-Cruz, Thomas Prag, Heather C. Smith, Libby Taylor, and William T. Trotter. Comparing Dushnik-Miller Dimension, Boolean Dimension and Local Dimension. arXiv preprint 1710.09467, 2017.

[2] André Bouchet. Étude combinatoire des ensembles ordonnés finis. These de Doctorat D’Etat, Universite de Grenoble, 1971.

[3] Olivier Cogis. On the ferrers dimension of a digraph. Discrete Math., 38:47–52, 1982.

[4] David Eppstein, Jean-Claude Falmagne, and Sergei Ovchinnikov. Media theory. Springer, 2008.
[5] Jinha Kim, Ryan R. Martin, Tomáš Masařík, Warren Shull, Heather C. Smith, Andrew Uzzell, and Zhiyu Wang. On difference graphs and the local dimension of posets. arXiv preprint 1803.08641, 2018.

[6] Kolja Knauer and Torsten Ueckerdt. Three ways to cover a graph. Discrete Math., 339(2):745–758, 2016.

[7] Jutta Mitas. Interval orders based on arbitrary ordered sets. Discrete Math., 144:75–95, 1995.

[8] Jacques Riguet. Les relations de Ferrers. C. R. Acad. Sci., Paris, 232:1729–1730, 1951.

[9] Torsten Ueckerdt. Order & Geometry Workshop, 2016.