Vanishing Mean Oscillation Spaces Associated with Operators Satisfying Davies-Gaffney Estimates

Yiyu Liang, Dachun Yang and Wen Yuan

Abstract Let $(\mathcal{X}, d, \mu)$ be a metric measure space, $L$ a linear operator which has a bounded $H_\infty$ functional calculus and satisfies the Davies-Gaffney estimate, $\Phi$ a concave function on $(0, \infty)$ of critical lower type $p_\Phi \in (0, 1]$ and $\rho(t) \equiv t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0, \infty)$. In this paper, the authors introduce the generalized VMO space $\text{VMO}_{\rho, L}(\mathcal{X})$ associated with $L$, and establish its characterization via the tent space. As applications, the authors show that $(\text{VMO}_{\rho, L}(\mathcal{X}))^* = \text{B}_{\Phi, L}^*(\mathcal{X})$, where $L^*$ denotes the adjoint operator of $L$ in $L^2(\mathcal{X})$ and $\text{B}_{\Phi, L}^*(\mathcal{X})$ the Banach completion of the Orlicz-Hardy space $H_{\Phi, L}^*(\mathcal{X})$.

1 Introduction

John and Nirenberg [24] introduced the space $\text{BMO}(\mathbb{R}^n)$, which is defined to be the space of all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} \equiv \sup_{\text{ball } B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty,$$

where and in what follows, $f_B \equiv \frac{1}{|B|} \int_B f(x) \, dx$. The space $\text{BMO}(\mathbb{R}^n)$ was proved to be the dual of the Hardy space $H^1(\mathbb{R}^n)$ by Fefferman and Stein in [14].

Sarason [28] introduced the space $\text{VMO}(\mathbb{R}^n)$, which is defined to be the space of all $f \in \text{BMO}(\mathbb{R}^n)$ such that

$$\lim_{c \to 0} \sup_{\text{ball } B \subset \mathbb{R}^n, r_B \leq c} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx = 0,$$

where $r_B$ denotes the radius of the ball $B$. In order to represent $H^1(\mathbb{R}^n)$ as a dual space, Coifman and Weiss [8] introduced the space $\text{CMO}(\mathbb{R}^n)$, which is defined to be the closure of all infinitely differentiable functions with compact support in the $\text{BMO}(\mathbb{R}^n)$ norm and

2010 Mathematics Subject Classification. Primary 42B35; Secondary 42B30, 46E30, 30L99.

Key words and phrases. metric measure space, operator, bounded $H_\infty$ functional calculus, Davies-Gaffney estimate, Orlicz function, Orlicz-Hardy space, BMO, VMO, molecule, dual.

The second author is supported by the National Natural Science Foundation (Grant No. 10871025) of China and Program for Changjiang Scholars and Innovative Research Team in University of China.

*Corresponding author
was originally denoted by the symbol VMO (\(\mathbb{R}^n\)) in [8], and proved that \((\text{CMO} (\mathbb{R}^n))^* = H^1(\mathbb{R}^n)\). For more properties of BMO (\(\mathbb{R}^n\)), VMO (\(\mathbb{R}^n\)) and CMO (\(\mathbb{R}^n\)), we refer the reader to Janson [18] and Bourdaud [5].

Let \(L\) be a linear operator in \(L^2(\mathbb{R}^n)\) that generates an analytic semigroup \(\{e^{-tL}\}_{t \geq 0}\) with kernels satisfying an upper bound of Poisson type. The Hardy space \(H^1_L(\mathbb{R}^n)\), the BMO space \(\text{BMO}_L(\mathbb{R}^n)\) and Morrey spaces associated with \(L\) were introduced and studied in [4, 13, 11]. Duong and Yan [12] further proved that \((H^1_L(\mathbb{R}^n))^* = \text{BMO}_L^*(\mathbb{R}^n)\), where and in what follows, \(L^*\) denotes the adjoint operator of \(L\) in \(L^2(\mathbb{R}^n)\). Moreover, recently, Deng et al. [9] introduced the space \(\text{VMO}_L(\mathbb{R}^n)\), the space of vanishing mean oscillation associated with operator \(L\), and proved that \((\text{VMO}_L(\mathbb{R}^n))^* = H^1_{L^*}(\mathbb{R}^n)\) and also

\[ \text{VMO}^\Delta (\mathbb{R}^n) = \text{CMO}(\mathbb{R}^n) = \text{VMO}_{\Delta} (\mathbb{R}^n) \]

with equivalent norms. Let \(\Phi\) on \((0, \infty)\) be a continuous, strictly increasing, subadditive function of upper type 1 and of critical lower type \(p_\Phi^{-} \leq 1\) but near to 1 (see Section 2.4 below for the definition). Let \(\rho(t) = t^{-1}/\Phi^{-1}(t^{-1})\) for all \(t \in (0, \infty)\). A typical example of such Orlicz functions is \(\Phi(t) \equiv t^p\) for all \(t \in (0, \infty)\) and \(p \leq 1\) but near to 1. Jiang and Yang [22] introduced the VMO-type space \(\text{VMO}_{\rho,L}(\mathbb{R}^n)\) and proved that the dual space of \(\text{VMO}_{\rho,L^*}(\mathbb{R}^n)\) is the space \(B_{\Phi,L}(\mathbb{R}^n)\), where \(B_{\Phi,L}(\mathbb{R}^n)\) denotes the Banach completion of the Orlicz-Hardy space \(H_{\Phi,L}(\mathbb{R}^n)\) in [23].

Let \(L\) be a second order divergence form elliptic operator with complex bounded measurable coefficients and \(\Phi\) a continuous, strictly increasing, concave function of critical lower type \(p_\Phi^{-} \in (0, 1]\). Jiang and Yang [20] studied the VMO-type spaces \(\text{VMO}_{\rho,L}(\mathbb{R}^n)\) and proved that the dual space of \(\text{VMO}_{\rho,L^*}(\mathbb{R}^n)\) is the space \(B_{\Phi,L}(\mathbb{R}^n)\), where \(B_{\Phi,L}(\mathbb{R}^n)\) denotes the Banach completion of the Orlicz-Hardy space \(H_{\Phi,L}(\mathbb{R}^n)\) in [19]. (We remark that the assumptions on \(p_\Phi\) in [19, 20] can be relaxed into the same assumptions on \(p_\Phi^{-}\); see Remark 2.2(ii) below.) In particular, when \(\Phi(t) \equiv t\) for all \(t \in (0, \infty)\), then \(\rho(t) \equiv 1\) and \((\text{VMO}_{1,L}(\mathbb{R}^n))^* = H^1_{L^*}(\mathbb{R}^n)\), which was also independently obtained by Song and Xu [29], where \(H^p_{L^*}(\mathbb{R}^n)\) denotes the Hardy space first introduced by Hofmann and Mayboroda [16] (see also [17]).

Let \((\mathcal{X}, d)\) be a metric space endowed with a doubling measure \(\mu\) and \(L\) a non-negative self-adjoint operator satisfying Davies-Gaffney estimates. Hofmann et al. [15] introduced the Hardy space \(H^1_L(\mathcal{X})\) associated to \(L\). Jiang and Yang [21] further introduced the Orlicz-Hardy space \(H_{\Phi,L}(\mathcal{X})\). Anh [1] studied the VMO space \(\text{VMO}_L(\mathcal{X})\) associated to \(L\) and proved that the dual space of \(\text{VMO}_L(\mathcal{X})\) is the Hardy space \(H^1_L(\mathcal{X})\). Recently, Duong and Li [10] observed that the assumption “\(L\) is a non-negative self-adjoint operator” in [15] can be replaced by a weaker assumption that “\(L\) has a bounded \(H_\infty\) functional calculus on \(L^2(\mathcal{X})^*\)” and introduced the Hardy space \(H^p_L(\mathcal{X})\) with \(p \in (0, 1]\), which was further generalized by Anh and Li [2] to the Orlicz-Hardy spaces \(H_{\Phi,L}(\mathcal{X})\).

From now on, we always assume that \(L\) is a linear operator which has a bounded \(H_\infty\) functional calculus and satisfies Davies-Gaffney estimates and that \(\Phi\) is a continuous, strictly increasing, concave function of critical lower type \(p_\Phi^{-} \in (0, 1]\). In this paper, we introduce the generalized VMO space \(\text{VMO}_{\rho,L}(\mathcal{X})\) associated with \(L\), and establish its characterization via the tent space in [21]. Then, we further prove that \((\text{VMO}_{\rho,L}(\mathcal{X}))^* = B_{\Phi,L^*}(\mathcal{X})\), where \(B_{\Phi,L^*}(\mathcal{X})\) denotes the Banach completion of the
Orlicz-Hardy space $H_{\Phi,L^*}(\mathcal{X})$ in [2]. When $\Phi(t) \equiv t$ for all $t \in (0, \infty)$, we denote $\text{VMO}_{\rho,L}(\mathcal{X})$ simply by $\text{VMO}_{L}(\mathcal{X})$. As a special case of the main results in this paper, we show that $(\text{VMO}_{L}(\mathcal{X}))^* = H^1_{L^*}(\mathcal{X})$, which, when $L$ is nonnegative self-adjoint, was already obtained by Anh [1].

Precisely, the paper is organized as follows. In Section 2, we recall some known notions and notation concerning metric measure spaces $\mathcal{X}$, then describe some basic assumptions on the considered operator $L$ and the Orlicz function $\Phi$ and present some properties of the operator $L$ and the Orlicz function $\Phi$ considered in this paper.

In Section 3, we first obtain the $p$-Carleson measure characterization (see Theorem 3.1 below) of the space $\text{BMO}_{\rho,L}(\mathcal{X})$ in [2] via first establishing a Calderón reproducing formula (see Proposition 3.3 below). Differently from the Calderón reproducing formula in [21, Proposition 4.6], the Calderón reproducing formula in Proposition 3.3 below holds for all molecules instead of atoms in [21], which brings us some extra difficulty due to the lack of the support of molecules. Then we introduce the generalized VMO space $\text{VMO}_{\rho,L}(\mathcal{X})$ associated with $L$, and the tent space $T_{\Phi,\infty}^\infty(\mathcal{X})$ and establish some basic properties of these spaces. In particular, we characterize the space $\text{VMO}_{\rho,L}(\mathcal{X})$ via $T_{\Phi,\infty}^\infty(\mathcal{X})$; see Theorem 3.4 below. To this end, we first need make clear the dual relation between $H_{\Phi,L^*}(\mathcal{X})$ and $\text{BMO}_{\rho,L}(\mathcal{X})$ (see Theorem 3.2 below), which is deduced from a technical result on the optimal representation of finite linear combinations of molecules (see Theorem 3.1 below). We remark that variants of Theorems 3.1 and 3.2 below have already been given respectively in [2, Theorems 3.15, 3.13 and 3.16] without a detailed proof of [2, Theorem 3.15]. We give a detailed proof of Theorem 3.1 below which induces more accurate indices appearing in Theorems 3.1 and 3.2 below, comparing with [2, Theorems 3.13 and 3.15] (see Remark 3.2 below). Moreover, the proof of Theorem 3.1 below simplifies the proof of [15, Theorem 5.4] in a subtle way, the proof of [15, Theorem 5.4] strongly depends on the support of atoms; see Remark 3.1 below.

In Section 4, we first obtain, in Theorem 4.1 below, the dual space of the tent space $T_{\Phi,\infty}^\infty(\mathcal{X})$ in Definition 3.4 below, from which, we further deduce that $(\text{VMO}_{\rho,L}(\mathcal{X}))^* = B_{\Phi,L^*}(\mathcal{X})$ in Theorem 4.2 below, where $B_{\Phi,L^*}(\mathcal{X})$ denotes the Banach completion of $H_{\Phi,L^*}(\mathcal{X})$. In particular, we obtain $(\text{VMO}_{L}(\mathcal{X}))^* = H^1_{L^*}(\mathcal{X})$.

Finally we make some conventions on notation. Throughout the whole paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. The constant with subscripts, such as $C_1$, does not change in different occurrences. We also use $C(\gamma, \cdots)$ to denote a positive constant depending on the indicated parameters $\gamma, \cdots$. The symbol $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. We also set $\mathbb{N} \equiv \{1, 2, \cdots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. The symbol $B(x, r)$ denotes the ball $\{y \in \mathcal{X} : d(x, y) < r\}$; moreover, let $CB(x, r) \equiv B(x, Cr)$. For a measurable set $E$, denote by $\chi_E$ the characteristic function of $E$ and by $E^c$ the complement of $E$ in $\mathcal{X}$.

2 Preliminaries

In this section, we first recall some notions and notation on metric measure spaces and then describe some basic assumptions on the considered operator $L$ in this paper and
its functional calculus; finally, we also present some basic assumptions and properties on Orlicz functions.

2.1 Metric measure spaces

Throughout the whole paper, let \( \mathcal{X} \) be a set, \( d \) a metric on \( \mathcal{X} \) and \( \mu \) a nonnegative Borel regular measure on \( \mathcal{X} \). Moreover, assume that there exists a constant \( C_1 \geq 1 \) such that for all \( x \in \mathcal{X} \) and \( r > 0 \),

\[
V(x, 2r) \leq C_1 V(x, r) < \infty,
\]

where \( B(x, r) \equiv \{ y \in \mathcal{X} : d(x, y) < r \} \) and

\[
V(x, r) \equiv \mu(B(x, r)).
\]

Observe that if \( d \) is further assumed to be a quasi-metric, then \( (\mathcal{X}, d, \mu) \) is called a space of homogeneous type in the sense of Coifman and Weiss [7] (see also [8]).

Notice that the doubling property (2.1) implies the following strong homogeneity property: there exist some positive constants \( C \) and \( n \), depending on \( C_1 \), such that

\[
V(x, \lambda r) \leq C \lambda^n V(x, r)
\]

uniformly for all \( \lambda \geq 1 \), \( x \in \mathcal{X} \) and \( r > 0 \). The parameter \( n \) measures the dimension of the space \( \mathcal{X} \) in some sense. Also, there exist constants \( C \in (0, \infty) \) and \( N \in [0, n] \), depending on \( C_1 \), such that

\[
V(x, r) \leq C \left( 1 + \frac{d(x, y)}{r} \right)^N V(y, r)
\]

uniformly for all \( x, y \in \mathcal{X} \) and \( r > 0 \). Indeed, the property (2.4) with \( N = n \) is a simple corollary of the strong homogeneity property (2.3). In the cases of Euclidean spaces, Lie groups of polynomial growth and, more generally, Ahlfors regular spaces, \( N \) can be chosen to be 0.

In what follows, for any ball \( B \subset \mathcal{X} \), we set

\[
U_0(B) \equiv B \quad \text{and} \quad U_j(B) \equiv 2^j B \setminus 2^{j-1} B \quad \text{for} \ j \in \mathbb{N}.
\]

The following covering lemma established in [1, Lemma 2.1] plays a key role in the sequel.

**Lemma 2.1.** For any \( \ell > 0 \), there exists \( N_\ell \in \mathbb{N} \), depending on \( \ell \), such that for all balls \( B(x_B, \ell r) \), with \( x_B \in \mathcal{X} \) and \( r > 0 \), there exists a family \( \{ B(x_B, i, r) \} \) of balls such that

i) \( B(x_B, \ell r) \subset \bigcup_{i=1}^{N_\ell} B(x_B, i, r) \);

ii) \( N_\ell \leq C \ell^n \);

iii) \( \sum_{i=1}^{N_\ell} \chi_{B(x_B, i, r)} \leq C \).

Here \( C \) is a positive constant independent of \( x_B \), \( r \) and \( \ell \).
2.2 Holomorphic functional calculi

We now recall some basic notions of holomorphic functional calculi introduced by McIntosh [25].

Let $0 < \nu < \gamma < \pi$. Define the closed sector $S_\nu$ in the complex plane $\mathbb{C}$ by setting $S_\nu \equiv \{ z \in \mathbb{C} : |\arg z| \leq \nu \} \cup \{ 0 \}$ and denote by $S_\nu^0$ its interior. We employ the following subspaces, $H_\infty(S_\nu^0)$ and $\Psi(S_\nu^0)$, of the space $H(S_\nu^0)$ of all holomorphic functions on $S_\nu^0$: 

$$ H_\infty(S_\nu^0) \equiv \left\{ b \in H(S_\nu^0) : \|b\|_{L_\infty(S_\nu^0)} \equiv \sup_{z \in S_\nu^0} |b(z)| < \infty \right\} $$

and

$$ \Psi(S_\nu^0) \equiv \{ \psi \in H(S_\nu^0) : \text{there exist } s \in (0, \infty) \text{ and } C \in (0, \infty) \text{ such that} \right.$$  

$$ \text{for all } z \in S_\nu^0, |\psi(z)| \leq C|z|^s(1 + |z|^{2s})^{-1} \}.$$

Given $\nu \in (0, \pi)$, a closed operator $L$ in $L^2(\mathbb{R}^n)$ is called to be of type $\nu$ if $\sigma(L) \subset S_\nu$, where $\sigma(L)$ denotes its spectra, and if for all $\gamma > \nu$, there exists a positive constant $C_\gamma$ such that for all $\lambda \notin S_\gamma$, $\|(L - \lambda I)^{-1}\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C_\gamma |\lambda|^{-1}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two linear normed spaces and $T$ be a continuous linear operator from $\mathcal{X}$ to $\mathcal{Y}$. Here and in what follows, $\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}}$ denotes the operator norm of $T$ from $\mathcal{X}$ to $\mathcal{Y}$. Let $\theta \in (\nu, \gamma)$ and $\Gamma$ be the contour $\{ \xi = re^{\pm i\theta} : r \geq 0 \}$ parameterized clockwise around $S_\nu$. Then if $L$ is of type $\nu$ and $\psi \in \Psi(S_\nu^0)$, the operator $\psi(L)$ is defined by

$$\psi(L) \equiv \frac{1}{2\pi i} \int_{\Gamma} (L - \lambda I)^{-1}\psi(\lambda) \, d\lambda,$$

where the integral is absolutely convergent in $\mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ (the class of all bounded linear operators in $L^2(\mathbb{R}^n)$). By the Cauchy theorem, we know that $\psi(L)$ is independent of the choices of $\nu$ and $\gamma$ such that $\theta \in (\nu, \gamma)$. Moreover, if $L$ is one-to-one and has dense range, and $b \in H_\infty(S_\gamma^0)$, then $b(L)$ is defined by setting $b(L) \equiv (\psi(L))^{-1}(b\psi)(L)$, where $\psi(z) \equiv z(1 + z)^{-2}$ for all $z \in S_\gamma^0$. It was proved by McIntosh [25] that $b(L)$ is a well-defined linear operator in $L^2(\mathbb{R}^n)$. Moreover, the operator $L$ is said to have a bounded $H_\infty$-calculus in $L^2(\mathbb{R}^n)$, if for all $\gamma \in (\nu, \pi)$, there exists a positive constant $\tilde{C}_\gamma$ such that for all $b \in H_\infty(S_\gamma^0)$, $b(L) \in \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ and

$$\|b\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq \tilde{C}_\gamma \|b\|_{L_\infty(S_\gamma^0)}.$$

2.3 Assumptions on the operator $L$

Throughout the whole paper, we always suppose that the considered operators $L$ satisfy the following assumptions.

Assumption (L)$_1$. The operator $L$ has a bounded $H_\infty$-calculus in $L^2(\mathcal{X})$.

Assumption (L)$_2$. The semigroup $\{e^{-tL}\}_{t \geq 0}$ generated by $L$ is analytic on $L^2(\mathcal{X})$ and satisfies the Davies-Gaffney estimate, namely, there exist positive constants $C_2$ and $C_3$
such that for all closed sets $E$ and $F$ in $\mathcal{X}$, $t \in (0, \infty)$ and $f \in L^2(E)$,

\begin{equation}
\|e^{-tL}f\|_{L^2(F)} \leq C_2 \exp \left\{ -\frac{\operatorname{dist} (E, F)^2}{C_3 t} \right\} \|f\|_{L^2(E)},
\end{equation}

where and in what follows, $\operatorname{dist} (E, F) \equiv \inf_{x \in E, y \in F} d(x, y)$ and the space $L^2(E)$ denotes the set of all $\mu$-measurable functions on $E$ such that $\|f\|_{L^2(E)} \equiv \left\{ \int_E |f(x)|^2 \, d\mu(x) \right\}^{1/2} < \infty$.

**Remark 2.1.** By the functional calculus of $L$ on $L^2(\mathcal{X})$, it is easy to see that if an operator $L$ satisfies Assumptions $(L)_1$ and $(L)_2$, the adjoint operator $L^*$ also satisfies Assumptions $(L)_1$ and $(L)_2$ and, therefore, the following Lemmas 2.2 and 2.3 also hold for $L^*$.

By Assumptions $(L)_1$ and $(L)_2$, we have the following technical result which was obtained by Anh and Li in [2, Proposition 2.2].

**Lemma 2.2.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$. Then for any fixed $k \in \mathbb{Z}_+$ (resp. $j, k \in \mathbb{Z}_+$ with $j \leq k$), the family $\{(t^2 L)^k e^{-t^2 L}, t > 0\}$ (resp. $\{(t^2 L)^j (I + t^2 L)^{-k}, t > 0\}$) of operators also satisfies the Davies-Gaffney estimate (2.7) with positive constants $C_2, C_3$ depending only on $n$ and $k$ (resp. $n$, $j$ and $k$).

By (2.6), we have the following useful lemma.

**Lemma 2.3.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$. Then for any fixed $k \in \mathbb{N}$, the operator given by setting, for all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$,

\[ S^k_t f(x) \equiv \left( \int_{\Gamma(x)} \left| (t^2 L)^k e^{-t^2 L} f(y) \right|^2 \frac{d\mu(y)}{V(x, t)} \, dt \right)^{1/2}, \]

is bounded on $L^2(\mathcal{X})$.

### 2.4 Orlicz functions

Let $\Phi$ be a positive function on $\mathbb{R}_+ \equiv (0, \infty)$. The function $\Phi$ is called of upper (resp. lower) type $p$ for some $p \in [0, \infty)$, if there exists a positive constant $C$ such that for all $t \in [1, \infty)$ (resp. $t \in (0, 1)$) and $s \in (0, \infty)$,

\begin{equation}
\Phi(st) \leq Ct^p \Phi(s).
\end{equation}

Obviously, if $\Phi$ is of lower type $p$ for some $p \in (0, \infty)$, then $\lim_{t \to 0} \Phi(t) = 0$. So for the sake of convenience, if it is necessary, we may assume that $\Phi(0) = 0$. If $\Phi$ is of both upper type $p_1$ and lower type $p_0$, then $\Phi$ is called of type $(p_0, p_1)$. Let

\begin{equation}
p^+_\Phi \equiv \inf\{p \in (0, \infty) : \text{there exists a positive constant } C \text{ such that (2.8) holds for all } t \in [1, \infty) \text{ and } s \in (0, \infty)\}
\end{equation}

and

\begin{equation}
p^-\Phi \equiv \sup\{p \in (0, \infty) : \text{there exists a positive constant } C
\end{equation}
such that (2.8) holds for all $t \in (0, 1)$ and $s \in (0, \infty)$.

It is easy to see that $p^\Phi_\Phi \leq p^\Phi_\Phi$ for all $\Phi$. In what follows, $p^\Phi_\Phi$ and $p^\Phi_\Phi$ are respectively called the critical lower type index and the critical upper type index of $\Phi$.

Throughout the whole paper, we always assume that $\Phi$ satisfies the following assumption.

**Assumption (\(\Phi\)).** Let $\Phi$ be a positive, continuous, strictly increasing function on $(0, \infty)$ which is of critical lower type $p^\Phi_\Phi \in (0, 1]$. Also assume that $\Phi$ is concave.

**Remark 2.2.** (i) Recall that the function $\Phi$ is called of strictly lower type $p$ if (2.8) holds with $C \equiv 1$ for all $t \in (0, 1)$ and $s \in (0, \infty)$. Then the strictly critical lower type index $p^\Phi_\Phi$ of $\Phi$ is defined by

$$p^\Phi_\Phi \equiv \sup\{p \in (0, \infty) : \Phi(st) \leq t^p\Phi(s) \text{ holds for all } t \in (0, 1) \text{ and } s \in (0, \infty)\}.$$  

Obviously, $p^\Phi_\Phi \leq p^\Phi_\Phi \leq p^\Phi_\Phi$. Moreover, it was proved in [19, Remark 2.1] that $\Phi$ is also of strictly lower type $p^\Phi_\Phi$. In other words, $p^\Phi_\Phi$ is attainable.

However, $p^\Phi_\Phi$ and $p^\Phi_\Phi$ may not be attainable. For example, for $p \in (0, 1]$, if $\Phi(t) \equiv t^p$ for all $t \in (0, \infty)$, then $\Phi$ satisfies Assumption (\(\Phi\)) and $p^\Phi_\Phi = p^\Phi_\Phi = p$; for $p \in [1/2, 1]$, if $\Phi(t) \equiv t^p/\ln(e + t)$ for all $t \in (0, \infty)$, then $\Phi$ satisfies Assumption (\(\Phi\)) and $p^\Phi_\Phi = p = p^\Phi_\Phi$, $p^\Phi_\Phi$ is not attainable, but $p^\Phi_\Phi$ is attainable; for $p \in (0, 1/2]$, if $\Phi(t) \equiv t^p\ln(e + t)$ for all $t \in (0, \infty)$, then $\Phi$ satisfies Assumption (\(\Phi\)) and $p^\Phi_\Phi = p = p^\Phi_\Phi$, $p^\Phi_\Phi$ is attainable, but $p^\Phi_\Phi$ is not attainable.

(ii) We observe that, via the Aoki-Roelwicz theorem in [3, 26], all results in [2, 19, 20, 21] are still true if the assumptions on $p^\Phi_\Phi$ are replaced by the same assumptions on $p^\Phi_\Phi$.

Notice that if $\Phi$ satisfies Assumption (\(\Phi\)), then $\Phi(0) = 0$. For any positive function $\Phi$ of critical lower type $p^\Phi_\Phi$, if we set $\Phi(t) \equiv \int_0^t \frac{\Phi(s)}{s} \, ds$ for $t \in [0, \infty)$, then by [30, Proposition 3.1], $\Phi$ is equivalent to $\Phi$, namely, there exists a positive constant $C$ such that $C^{-1}\Phi(t) \leq \Phi(t) \leq C\Phi(t)$ for all $t \in [0, \infty)$; moreover, $\Phi$ is a positive, strictly increasing, concave and continuous function of critical lower type $p^\Phi_\Phi$. Notice that all our results of this paper are invariant on equivalent Orlicz functions. From this, we deduce that all results with $\Phi$ as in Assumption (\(\Phi\)) also hold for all positive functions $\tilde{\Phi}$ of the same critical lower type $p^\tilde{\Phi}$ as $\Phi$.

Let $\Phi$ satisfy Assumption (\(\Phi\)). A measurable function $f$ on $\mathcal{X}$ is said to be in the space $L^\Phi(\mathcal{X})$ if $\int_{\mathcal{X}} \Phi(|f(x)|) \, d\mu(x) < \infty$. Moreover, for any $f \in L^\Phi(\mathcal{X})$, define

$$\|f\|_{L^\Phi(\mathcal{X})} \equiv \inf \left\{ \lambda \in (0, \infty) : \int_{\mathcal{X}} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.$$  

Since $\Phi$ is strictly increasing, we define the function $\rho(t)$ on $(0, \infty)$ by

$$\rho(t) \equiv \frac{t^{-1}}{\Phi^{-1}(t^{-1})}$$  

for all $t \in (0, \infty)$, where $\Phi^{-1}$ is the inverse function of $\Phi$. Then the types of $\Phi$ and $\rho$ have the following relation. If $0 < p_0 \leq p_1 \leq 1$ and $\Phi$ is an increasing function, then $\Phi$ is of type $(p_0, p_1)$ if and only if $\rho$ is of type $(p_0^{-1}, p_1^{-1})$; see [30] for its proof.
3 The Space $\text{VMO}_{\rho, L}(\mathcal{X})$

In this section, we introduce the generalized vanishing mean oscillation spaces associated with $L$. Throughout this section, we always assume that $L$ satisfies Assumptions (L)$_1$ and (L)$_2$.

We first recall the notion of tent spaces in [27], which when $\mathcal{X} \equiv \mathbb{R}^n$ were first introduced by Coifman, Meyer and Stein [6].

For any $\nu > 0$ and $x \in \mathcal{X}$, let $\Gamma_{\nu}(x) \equiv \{(y, t) \in \mathcal{X} \times (0, \infty) : d(x, y) < \nu t\}$ denote the cone of aperture $\nu$ with vertex $x \in \mathcal{X}$. For any closed set $F$ of $\mathcal{X}$, denote by $\mathcal{R}_\nu F$ the union of all cones with vertices in $F$, namely, $\mathcal{R}_\nu F \equiv \bigcup_{x \in F} \Gamma_{\nu}(x)$; and for any open set $O$ in $\mathcal{X}$, denote the tent over $O$ by $T_\nu(O)$, which is defined as $T_\nu(O) \equiv |\mathcal{R}_\nu(O^c)|^0$. It is easy to see that $T_\nu(O) = \{(x, t) \in \mathcal{X} \times (0, \infty) : d(x, O^c) \geq \nu t\}$. In what follows, we denote $\mathcal{R}_1(F)$, $\Gamma_1(x)$ and $T_1(O)$ simply by $\mathcal{R}(F)$, $\Gamma(x)$ and $O$, respectively.

For all measurable functions $g$ on $\mathcal{X} \times (0, \infty)$ and $x \in \mathcal{X}$, define

$$A_\nu(g)(x) \equiv \left( \int_{\Gamma_{\nu}(x)} |g(y, t)|^2 \frac{d\mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2},$$

and

$$C_\rho(g)(x) \equiv \sup_{B \ni x} \frac{1}{\rho(\mu(B))} \left( \frac{1}{\mu(B)} \int_B |g(y, t)|^2 \frac{d\mu(y) dt}{t} \right)^{1/2},$$

where the supremum is taken over all balls $B$ containing $x$. We denote $A_1(g)$ simply by $A(g)$.

Recall that for $p \in (0, \infty)$, the tent space $T^p_\rho(\mathcal{X})$ is defined to be the space of all measurable functions $g$ on $\mathcal{X} \times (0, \infty)$ such that $\|g\|_{T^p_\rho(\mathcal{X})} \equiv \|A(g)\|_{L^p(\mathcal{X})} < \infty$, which when $\mathcal{X} \equiv \mathbb{R}^n$ was introduced by Coifman, Meyer and Stein [6] and when $\mathcal{X}$ is a space of homogeneous type by Russ in [27]. Let $\Phi$ satisfy Assumption ($\Phi$). In what follows, we denote by $T_\Phi(\mathcal{X})$ the space of all measurable functions $g$ on $\mathcal{X} \times (0, \infty)$ such that $A(g) \in L^\Phi(\mathcal{X})$, and for any $g \in T_\Phi(\mathcal{X})$, define its norm by

$$\|g\|_{T_\Phi(\mathcal{X})} \equiv \|A(g)\|_{L^\Phi(\mathcal{X})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \Phi \left( \frac{A(g)(x)}{\lambda} \right) d\mu(x) \leq 1 \right\};$$

the space $T^{\infty}_\Phi(\mathcal{X})$ is defined to be the space of all measurable functions $g$ on $\mathcal{X} \times (0, \infty)$ satisfying $\|g\|_{T^{\infty}_\Phi(\mathcal{X})} \equiv \|C_\rho(g)\|_{L^{\infty}(\mathcal{X})} < \infty$.

Recall that a function $a$ on $\mathcal{X} \times (0, \infty)$ is called a $T_\Phi(\mathcal{X})$-atom if

(i) there exists a ball $B \subset \mathcal{X}$ such that $\text{supp} \ a \subset \tilde{B}$;

(ii) $\mathcal{F}_B |a(x, t)|^2 \frac{d\mu(x) dt}{t} \leq [\mu(B)]^{-1} [\rho(\mu(B))]^{-2}$.

Since $\Phi$ is concave, from Jensen’s inequality and Hölder’s inequality, we deduce that for all $T_\Phi(\mathcal{X})$-atoms $a$, $\|a\|_{T_\Phi(\mathcal{X})} \leq 1$; see [21] for the details. Moreover, the following atomic decomposition for elements in $T_\Phi(\mathcal{X})$ is just [21, Theorem 3.1].

**Lemma 3.1.** Let $\Phi$ satisfy Assumption ($\Phi$). Then for any $f \in T_\Phi(\mathcal{X})$, there exist $T_\Phi(\mathcal{X})$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that for almost every $(x, t) \in \mathcal{X} \times (0, \infty),

$$f(x, t) = \sum_{j=1}^\infty \lambda_j a_j(x, t),$$

(3.1)
and the series converges in $T_\Phi(\mathcal{X})$. Moreover, there exists a positive constant $C$ such that for all $f \in T_\Phi(\mathcal{X})$,

$$
\Lambda(\{\lambda_j a_j\}_{j=1}^\infty) \equiv \inf \left\{ \lambda > 0 : \sum_{j=1}^{\infty} \mu(B_j) \Phi \left( \frac{|\lambda_j|}{\lambda \mu(B_j) \rho(B_j)} \right) \leq 1 \right\} \leq C\|f\|_{T_\Phi(\mathcal{X})},
$$

where $\hat{B}_j$ appears as the support of $a_j$.

**Definition 3.1.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, $\Phi$ satisfy Assumption $(\Phi)$, $\rho$ be as in (2.11), $M \in \mathbb{N}$, $\epsilon \in (0, \infty)$ and $B$ be a ball. A function $\beta \in L^2(\mathcal{X})$ is called a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball $B$ if there exists a function $b \in \mathcal{D}(L^M)$ such that

(i) $\beta = L^M b$;

(ii) For every $k \in \{0, 1, \ldots, M\}$ and $j \in \mathbb{Z}_+$, there holds

$$
\| (r_B^k L)^j b \|_{L^2(U_j(B))} \leq r_B^{Mk} 2^{-j\epsilon} |\mu(2^j B)|^{1/2} |\rho(\mu(2^j B))|^{-1},
$$

where $U_j(B)$ for $j \in \mathbb{Z}_+$ is as in (2.5).

Let $\phi = L^M \nu$ be a function in $L^2(\mathcal{X})$, where $\nu \in \mathcal{D}(L^M)$. Following [15, 16], for $\epsilon > 0$, $M \in \mathbb{N}$ and a fixed $x_0 \in \mathcal{X}$, we introduce the space

$$
\mathcal{M}^{M,\epsilon}_{\Phi}(L) \equiv \left\{ \phi = L^M \nu \in L^2(\mathcal{X}) : \|\phi\|_{\mathcal{M}^{M,\epsilon}_{\Phi}(L)} < \infty \right\},
$$

where

$$
\|\phi\|_{\mathcal{M}^{M,\epsilon}_{\Phi}(L)} \equiv \sup_{j \in \mathbb{Z}_+} \left\{ 2^{j\epsilon} \|V(x_0, 2^j)\|^{1/2} \rho(V(x_0, 2^j)) \sum_{k=0}^{M} \|L^k \nu\|_{L^2(U_j(B(x_0, 1)))} \right\};
$$

see also [2].

Notice that if $\phi \in \mathcal{M}^{M,\epsilon}_{\Phi}(L)$ for some $\epsilon > 0$ with norm 1, then $\phi$ is a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball $B(x_0, 1)$. Conversely, if $\beta$ is a $(\Phi, M, \epsilon)_L$-molecule adapted to any ball, then $\beta \in \mathcal{M}^{M,\epsilon}_{\Phi}(L)$.

Let $A_t$ denote either $(I + t^2 L)^{-1}$ or $e^{-t^2 L}$ and $A_t^*$ either $(I + t^2 L^*)^{-1}$ or $e^{-t^2 L^*}$. For any $f \in (\mathcal{M}^{M,\epsilon}_{\Phi}(L^*))^*$, the dual space of $\mathcal{M}^{M,\epsilon}_{\Phi}(L^*)$, we claim that $(I - A_t)^M f \in L^1_{\text{loc}}(\mathcal{X})$ in the sense of distributions. Indeed, for any ball $B$, if $\psi \in L^2(B)$, then it follows form the Davies-Gaffney estimate (2.7) and Remark 2.1 that $(I - A_t^*)^M \psi \in \mathcal{M}^{M,\epsilon}_{\Phi}(L^*)$ for every $\epsilon > 0$. Thus, there exists a non-negative constant $C(t, r_B, \text{dist}(B, x_0))$, depending on $t$, $r_B$ and $\text{dist}(B, x_0)$, such that for all $\psi \in L^2(B)$,

$$
|\langle (I - A_t^*)^M \psi, \psi \rangle| \equiv |\langle (I - A_t^*)^M \psi, \psi \rangle| \leq C(t, r_B, \text{dist}(B, x_0))\|f\|_{(\mathcal{M}^{M,\epsilon}_{\Phi}(L^*))^*} \|\psi\|_{L^2(B)},
$$

which implies that $(I - A_t)^M f \in L^2_{\text{loc}}(\mathcal{X})$ in the sense of distributions.

Finally, for any $M \in \mathbb{N}$, define

$$
\mathcal{M}^{M,L}_{\Phi}(\mathcal{X}) \equiv \bigcap_{\epsilon > n(1/p_{\Phi} - 1/p_{\Phi}^*)} (\mathcal{M}^{M,\epsilon}_{\Phi}(L^*))^*,
$$

where $p_{\Phi}^*$ and $p_{\Phi}^+$ are, respectively, as in (2.9) and (2.10).
**Definition 3.2.** Let $L$, $\Phi$ and $\rho$ be as in Definition 3.1 and $M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})$. A function $f \in M^{\mu}_{\rho,L}(X)$ is said to be in the space $\text{BMO}^{\mu}_{\rho,L}(X)$ if

$$\|f\|_{\text{BMO}^{\mu}_{\rho,L}(X)} \equiv \sup_{B \subset X} \frac{1}{\rho(B)} \left[ \frac{1}{\mu(B)} \int_B |(1 - e^{-r^2 L})^M f(x)|^2 \, d\mu(x) \right]^{1/2} < \infty,$$

where the supremum is taken over all balls $B$ of $X$.

Now, let us recall some notions on the Orlicz-Hardy spaces associated with $L$. For all $f \in L^2(X)$ and $x \in X$, define

$$S_L f(x) \equiv \left( \int_{I(x)} \left| \int_{(0,\infty)} t^2 L e^{-t^2 L} f(y) \, dt \right|^2 \frac{d\mu(y)}{t} \right)^{1/2}.$$ 

The **Orlicz-Hardy space** $H_{\Phi,L}(X)$ is defined to be the completion of the set $\{f \in L^2(X) : S_L f \in L^\Phi(X)\}$ with respect to the quasi-norm $\|f\|_{H_{\Phi,L}(X)} \equiv \|S_L f\|_{L^\Phi(X)}$.

The Orlicz-Hardy space $H_{\Phi,L}(X)$ was introduced and studied in [2] (see also [21]). If $\Phi(t) \equiv t^p$ for $p \in (0,1]$ and all $t \in (0,\infty)$, then the space $H_{\Phi,L}(X)$ coincides with the Hardy space $H^p_L(X)$, which was introduced and studied by Duong and Li [10].

Let the space $H_{\Phi,\text{fin},L}^{\text{mol},M}(X)$ denote the spaces of finite linear combinations of $(\Phi, M, \epsilon)_L$-molecules. By [2, Corollary 3.8], we obtain that $H_{\Phi,\text{fin},L}^{\text{mol},M}(X)$ is dense in $H_{\Phi,L}(X)$; see also [21, Corollary 4.2].

In what follows, for $M \in \mathbb{N}$, let $C(M)$ be the **positive constant** such that

$$C(M) \int_0^\infty t^{2(M+1)} e^{-2t^2} \, dt = 1. \quad (3.5)$$

Recall that a variant of the following representation of finite linear combinations of molecules was given by [2, Theorem 3.15] without a detailed proof. The following Theorem 3.1 gives more accurate ranges of $\epsilon$ and $M$, comparing with [2, Theorem 3.15].

**Theorem 3.1.** Let $L$, $\Phi$ and $M$ be as in Definition 3.2, and $\epsilon \in (0, M - \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2}))$.

Assume that $f = \sum_{i=0}^N \lambda_i a_i$, where $N \in \mathbb{N}$, $\{a_i\}_{i=0}^N$ is a family of $(\Phi, 2M, \epsilon)_L$-molecules, $\{\lambda_i\}_{i=0}^N \subset \mathbb{C}$ and $\sum_{i=0}^N |\lambda_i| < \infty$. Then there exists a representation of $f = \sum_{i=0}^{2N} \mu_i m_i$, where $\{m_i\}_{i=1}^{2N}$ are $(\Phi, M, \epsilon)_L$-molecules, $\{\mu_i\}_{i=0}^{2N} \subset \mathbb{C}$ and $\sum_{i=0}^{2N} |\mu_i| \leq C \|f\|_{H_{\Phi,L}(X)}$, where $C$ is a positive constant, depending only on $X, L, M, \epsilon$ and $n$.

**Proof.** Throughout this proof, we choose $\bar{p}_\Phi \in (0, p_\Phi)$ such that $M > \frac{n}{2}(\frac{1}{\bar{p}_\Phi} - \frac{1}{2})$ and $\epsilon \in (0, M - \frac{n}{2}(\frac{1}{\bar{p}_\Phi} - \frac{1}{2}))$. Therefore, $\Phi$ is of lower type $\bar{p}_\Phi$ and hence $\rho$ of upper type $1/ \bar{p}_\Phi - 1$.

Since $\{a_i\}_{i=0}^N$ is a family of $(\Phi, 2M, \epsilon)_L$-molecules, by definition there exist a family $\{b_i\}_{i=0}^N$ of functions and a family $\{B_i\}_{i=0}^N$ of balls such that for every $i \in \{0,1,\cdots,N\}$, $a_i = L^{2M} b_i$ satisfies Definition 3.1(ii). Fix a point $x_0 \in X$. Let $\bar{C}(M) \equiv \frac{2C(M)}{M+1}$, where...
$C(M)$ is as in (3.5). Then $\bar{C}(M) \int_0^\infty t^{2(M+2)} e^{-2t^2} \frac{dt}{t} = 1$. By this and the $L^2$-functional calculus, for $f = \sum_{i=0}^N \lambda_i a_i \in L^2(\mathcal{X})$, we have

$$ f = \bar{C}(M) \int_0^\infty (\ell^2 L)^{M+2} e^{-2\ell^2 L} \frac{dt}{t} = \bar{C}(M) \int_{K_1}^\infty (\ell^2 L)^{M+2} e^{-2\ell^2 L} \frac{dt}{t} + \bar{C}(M) \int_0^{K_1} \cdots \equiv f_1 + f_2, $$

where $K_1$ is a positive constant which is determined later.

Let us start with the term $f_1$. Set $\mu \equiv N^{-1} \|f\|_{H_{\ell, L}(\mathcal{X})}$. Substituting $f = \sum_{i=0}^N \lambda_i a_i$ into $f_1$, we have

$$ f_1 = \bar{C}(M) \sum_{i=0}^N \lambda_i \int_{K_1}^\infty (\ell^2 L)^{M+2} e^{-2\ell^2 L} a_i \frac{dt}{t} = \sum_{i=0}^N \mu_i m_{i, K_1}, $$

where $\mu_i \equiv \bar{C}(M) \mu$, $m_{i, K_1} \equiv L^M f_{i, K_1}$, and

$$ f_{i, K_1} \equiv \mu_i^{-1} \lambda_i \int_{K_1}^\infty t^{2(M+2)} L^2 e^{-2t^2 L} a_i \frac{dt}{t}. $$

Then, obviously, $\sum_{i=0}^N |\mu_i| = \sum_{i=0}^N \mu_i = C(M) \|f\|_{H_{\ell, L}(\mathcal{X})}$. We now claim that for an appropriate choice of $K_1$ and $i \in \{0, 1, \ldots, N\}$, $m_{i, K_1}$ is a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball $B_i$. Observe that $a_i = L^{2M} b_i$, for $i \in \{0, 1, \ldots, N\}$. By Minkowski's inequality, for $k \in \{0, 1, \ldots, M\}$, $i \in \{0, 1, \ldots, N\}$ and $j \in \mathbb{Z}_+$,

$$ \left\| (r_{B_i}^2 L)^k f_{i, K_1} \right\|_{L^2(U_j(B_i))} \leq \mu_i^{-1} \lambda_i \int_{K_1}^\infty t^{-2M} \left\| (t^2 L)^{2(M+1)} e^{-2t^2 L} (r_{B_i}^2 L)^k b_i \right\|_{L^2(U_j(B_i))} \frac{dt}{t} $$

$$ \leq \mu_i^{-1} \lambda_i \sum_{l=0}^\infty \int_{K_1}^\infty t^{-2M} \left\| (t^2 L)^{2(M+1)} e^{-2t^2 L} \left( \chi_{U_j(B_i)} \left[ (r_{B_i}^2 L)^k b_i \right] \right) \right\|_{L^2(U_j(B_i))} \frac{dt}{t} $$

$$ \equiv \mu_i^{-1} \lambda_i \sum_{l=0}^\infty H_l, $$

where $U_l(B_i)$ for $l \in \mathbb{Z}_+$ is as in (2.5). When $l < j - 1$, by Lemma 2.2, $\mu(2^l B_i) \lesssim 2^{n(j-l)} \mu(2^j B_i)$, $\rho(2^j B_i) \lesssim 2^{n(j-l)(1/\ell_\Phi - 1)}$ and Definition 3.1(ii), we conclude that

$$ H_l \lesssim \int_{K_1}^\infty t^{-2M} \left\| (r_{B_i}^2 L)^k b_i \right\|_{L^2(U_j(B_i))} \left( \frac{t}{2^l r_{B_i}} \right)^{\epsilon + n(1/\ell_\Phi - 1/2)} \frac{dt}{t} $$

$$ \lesssim \int_{K_1}^\infty t^{-2M} r_{B_i}^{2M} \left[ \mu(2^l B_i) \right]^{-1/2} \left[ \rho(2^j B_i) \right]^{-1} \left( \frac{t}{2^l r_{B_i}} \right)^{\epsilon + n(1/\ell_\Phi - 1/2)} \frac{dt}{t} $$

$$ \lesssim r_{B_i}^{2M} \left[ \mu(2^j B_i) \right]^{-1/2} \left[ \rho(2^j B_i) \right]^{-1} \left( \frac{r_{B_i}}{K_1} \right)^{2M - 2^{l-j} \frac{2}{\ell_\Phi} - \frac{2}{1-\ell_\Phi} \left( \frac{1}{2} - \frac{1}{2} \right)} \left( \frac{r_{B_i}}{K_1} \right)^{2M - 2^{l-j} \frac{2}{\ell_\Phi} - \frac{2}{1-\ell_\Phi} \left( \frac{1}{2} - \frac{1}{2} \right)}.$$
When $l \in \{j-1, j, j+1\}$, from Lemma 2.2 and Definition 3.1(ii), it follows that

\[
H_l \lesssim \int_{K_1}^{\infty} \left\| (r_{B_i}^2 L)^k b_i \right\|_{L^2(U_j(B_i))} \frac{dt}{t} \\
\lesssim r_{B_i}^{2M} 2^{-j} \mu_0 \lesssim \rho(\mu_0 B_j))^{-1} \left( \frac{r_{B_i}}{K_1} \right)^{2M} .
\]

When $l > j + 1$, by Lemma 2.2, $\mu(2^j B_i) \lesssim \mu(2^l B_i)$, $\rho(\mu_0 B_j)) \lesssim \rho(\mu_0 B_i)$ and Definition 3.1(ii), we obtain

\[
H_l \lesssim \int_{K_1}^{\infty} t^{-2M} \left\| (r_{B_i}^2 L)^k b_i \right\|_{L^2(U_j(B_i))} \left( \frac{t}{2^l r_{B_i}} \right)^{\epsilon} \frac{dt}{t} \\
\lesssim r_{B_i}^{2M} 2^{-j} \mu_0 \lesssim \rho(\mu_0 B_j))^{-1} \left( \frac{r_{B_i}}{K_1} \right)^{2M} .
\]

Combining these estimates, by choosing $K_1 > \max \{r_{B_1}, \cdots, r_{B_N}\}$, we further conclude that there exists a positive constant $\tilde{C}$, independent of $i$, such that

\[
\left\| (r_{B_i}^2 L)^k f_{i,K_1} \right\|_{L^2(U_j(B_i))} \\
\leq \tilde{C} r_{B_i}^{2M} 2^{-j} \mu_0 \lesssim \rho(\mu_0 B_j))^{-1} \left( \frac{r_{B_i}}{K_1} \right)^{2[M - \frac{1}{2} - \frac{1}{2p} - \frac{1}{2}]} .
\]

Then, by choosing

\[
K_1 \equiv \max_{0 \leq i \leq N} \left\{ r_{B_i}\left[ \tilde{C} \mu_0^{-1} \max_{0 \leq i \leq N} |\lambda_i| \right]^{\frac{2M}{M - \frac{1}{2} - \frac{1}{2p} - \frac{1}{2}}} \right\},
\]

we see that for $i \in \{0, 1, \cdots, N\}$, $m_{i,K_1}$ is a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball $B_i$, which shows the claim.

We now consider the term $f_2$. Set $\mu \equiv N^{-1} \|f\|_{H_{\Phi,L}(\mathcal{X})}$. Substituting $f = \sum_{i=0}^{N} \lambda_i a_i$ into $f_2$, we have

\[
f_2 = \tilde{C}(M) \sum_{i=0}^{N} \lambda_i \int_{0}^{K_1} (t^2 L)^{M+1} e^{-t^2 L} (t^2 L e^{-t^2 L} a_i) \frac{dt}{t} = \sum_{i=0}^{N} \mu_i m_{i,K_1},
\]

where $\mu_i \equiv C(M) \mu$, $m_{i,K_1} \equiv L^M f_{i,K_1}$, and

\[
f_{i,K_1} \equiv \mu_i^{-1} \lambda_i \int_{0}^{K_1} t^{2(M+1)} e^{-t^2 L} (t^2 L e^{-t^2 L} a_i) \frac{dt}{t} .
\]

Then, obviously, $\sum_{i=0}^{N} |\mu_i| = \sum_{i=0}^{N} \mu_i = C(M) \|f\|_{H_{\Phi,L}(\mathcal{X})}$. We now claim that for $K_1$ as above and $i \in \{0, 1, \cdots, N\}$, $m_{i,K_1}$ is a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball $2^{K_0} B_i$, where $K_0 \in (0, \infty)$ is determined later. To show the claim, for $i \in \{0, 1, \cdots, N\}$ and $j \in \mathbb{Z}_+$, set $\Omega_{j,K_0} \equiv 2^{j+K_0+2} B_i \setminus 2^{j+K_0-2} B_i$ and write

\[
f_{i,K_1} \equiv \mu_i^{-1} \lambda_i \int_{0}^{K_1} t^{2(M+1)} e^{-t^2 L} \left( [t^2 L e^{-t^2 L} a_i] \chi_{\Omega_{j,K_0}} \right) \frac{dt}{t}.
\]
When \( l < j \), by Minkowski’s inequality, for \( k \in \{0, 1, \ldots, M\} \), \( i \in \{0, 1, \ldots, N\} \) and \( j \in \mathbb{Z}_+ \),

\[
\left\| \left( \frac{2^{2K_0}r_{B_i}^2}{2^{2K_0}r_{B_i}^2} L \right)^k g_{i, K_1, K_0} \right\|_{L^2(U_j(2^{2K_0}B_i))}
\leq \mu^{-1} |\lambda_i| r_{B_i}^{2M} \left\| \int_0^{K_1} \left( \frac{t}{r_{B_i}} \right)^{2M-2k} 2^{2kK_0} \right\|_{L^2(U_j(2^{2K_0}B_i))}
\times \left( t^2 L \right)^{k+1} e^{-t^2 L} \left( \int \mu_{\Omega_j, K_0} - \frac{1}{2} \right)_{L^2(U_j(2^{2K_0}B_i))}
\leq \mu^{-1} |\lambda_i| \sum_{l=0}^{\infty} \int_0^{K_1} \left( \frac{t}{r_{B_i}} \right)^{2M-2k} 2^{2kK_0} \left\| \chi_{U_j(2^{2K_0}B_i)} t^2 L L^2(U_j(2^{2K_0}B_i)) \right\|_{L^2(U_j(2^{2K_0}B_i))}
\equiv \mu^{-1} |\lambda_i| \sum_{l=0}^{\infty} H_l.
\]

When \( l < j - 2 \), from Lemma 2.2, \( \mu(2^{j+K_0}B_i) \lesssim 2^{n(j-l)} \mu(2^{j+K_0}B_i), \rho(\mu(2^{j+K_0}B_i)) \lesssim 2^{n(j-l)(1/\rho_k-1)} \rho(\mu(2^{j+K_0}B_i)) \) and Definition 3.1(ii), it follows that

\[
H_l \lesssim \int_0^{K_1} \left( \frac{t}{r_{B_i}} \right)^{2M-2k} 2^{2kK_0} \| a_i \|_{L^2(U_j(2^{2K_0}B_i))} \left( \frac{t}{2^{j+K_0}r_{B_i}} \right)^{\epsilon + n(1/\rho_k-1/2)} dt \frac{t}{t}
\lesssim \int_0^{K_1} \left( \frac{t}{r_{B_i}} \right)^{2M-2k} 2^{2kK_0} r_{B_i}^{4M} 2^{-l(1+K_0)} \left( \mu(2^{j+K_0}B_i) \right)^{-1/2} \rho(\mu(2^{j+K_0}B_i))^{-1}
\times \left( \frac{t}{2^{j+K_0}r_{B_i}} \right)^{\epsilon + n(1/\rho_k-1/2)} dt \frac{t}{t}
\lesssim (2^{2K_0}r_{B_i})^{2M-2j} \left( (2^{j+K_0}B_i) \right)^{-1/2} \rho(\mu(2^{j+K_0}B_i))^{-1/2} 2^{M-2k \epsilon + n(1/\rho_k-1/2)} 2^{M-2k-2M+2k}.
\]

When \( l \in \{j, j+1, \ldots, j+2\} \), by Lemma 2.2 and Definition 3.1(ii), we see that

\[
H_l \lesssim \int_0^{K_1} \left( \frac{t}{r_{B_i}} \right)^{2M-2k} 2^{2kK_0} \| a_i \|_{L^2(U_j(2^{2K_0}B_i))} dt \frac{t}{t}
\lesssim (2^{2K_0}r_{B_i})^{2M-2j} \epsilon \left( (2^{j+K_0}B_i) \right)^{-1/2} \rho(\mu(2^{j+K_0}B_i))^{-1/2} 2^{-2K_0(M-k+\epsilon/2)} K_1^{2M-2k} r_{B_i}^{2M+2k}.
\]

When \( l > j+2 \), from Lemma 2.2, \( \mu(2^{j}B_i) \lesssim \mu(2^{j}B_i), \rho(\mu(2^{j+K_0}B_i)) \lesssim \rho(\mu(2^{j+K_0}B_i)) \) and Definition 3.1(ii), we infer that

\[
H_l \lesssim \int_0^{K_1} \left( \frac{t}{r_{B_i}} \right)^{2M-2k} 2^{2kK_0} \| a_i \|_{L^2(U_j(2^{2K_0}B_i))} \left( \frac{t}{2^{j+K_0}r_{B_i}} \right)^{\epsilon} dt \frac{t}{t}
\lesssim \int_0^{K_1} \left( \frac{t}{r_{B_i}} \right)^{2M-2k} 2^{2kK_0} r_{B_i}^{4M} 2^{-l(1+K_0)} \left( \mu(2^{j+K_0}B_i) \right)^{-1/2} \rho(\mu(2^{j+K_0}B_i))^{-1}
\times \left( \frac{t}{2^{j+K_0}r_{B_i}} \right)^{\epsilon + n(1/\rho_k-1/2)} dt \frac{t}{t}
\lesssim (2^{2K_0}r_{B_i})^{2M-2j} \epsilon \left( (2^{j+K_0}B_i) \right)^{-1/2} \rho(\mu(2^{j+K_0}B_i))^{-1/2} 2^{M-2k} r_{B_i}^{2M+2k}.
\]
\[
\times \left( \frac{t}{2^j + K_0 r_{B_i}} \right)^\epsilon \frac{dt}{t} \\
\leq \left( 2^{2K_0 r_{B_i}} \right) 2^{M-2^j} \left( \mu(2^{j+K_0 B_i}) \right)^{-1/2} [\rho(\mu(2^{j+K_0 B_i}))]^{-1} 2^{-j \epsilon} \\
\times 2^{-2K_0(M-k+\epsilon)} K_1^{2M-2k+\epsilon} r_{B_i}^{2M+2k-\epsilon} 
\]

Then we estimate \( h_{i,K_1} \). By Minkowski's inequality and Definition 3.1(ii), for \( k \in \{0, 1, \ldots, M\} \), \( i \in \{0, 1, \ldots, N\} \) and \( j \in \mathbb{Z}_+ \), we conclude that

\[
\left\| (2^{2K_0 r_{B_i}} L)^k h_{i,K_1} \right\|_{L^2(U_j(2^{K_0 B_i}))} \\
\leq \mu^{-1} |\lambda_i| r_{B_i}^{2M} \left( t \right)^{2M-2^k} 2^{2kK_0} \\
\times (t^2) + e^{-t^2L} \left( t^2 L e^{-t^2 L a_1} \right) X_{\Omega_i, K_0} \left( t \right) \frac{dt}{t} \left\| L^2(U_j(2^{K_0 B_i})) \right\| \\
\leq \mu^{-1} |\lambda_i| \int_0^{K_1} \left( t \right)^{2M-2^k} 2^{2kK_0} \left( \frac{t}{2^j + K_0 r_{B_i}} \right)^{-n(1/p_\Phi - 1/2)} \left\| t^2 L e^{-t^2 L a_1} \right\|_{L^2(\mathcal{X})} \frac{dt}{t} \\
\leq \left( 2^{2K_0 r_{B_i}} \right) 2^{M-2^j} \left( \mu(2^{j+K_0 B_i}) \right)^{-1/2} [\rho(\mu(2^{j+K_0 B_i}))]^{-1} \\
\times 2^{-2K_0(M-k+\epsilon+\frac{n}{2}(1/p_\Phi - \frac{1}{2}))} K_1^{2M-2k+\epsilon} r_{B_i}^{2M+2k-\epsilon-n(1/p_\Phi-1/2)}.
\]

Combining these estimates, by choosing \( K_1 > \max \{ r_{B_1}, \ldots, r_{B_N} \} \), we further see that

\[
\left\| (2^{2K_0 r_{B_i}} L)^k f_{i,K_1} \right\|_{L^2(U_j(2^{K_0 B_i}))} \leq \left( 2^{2K_0 r_{B_i}} \right) 2^{M-2^j} \left( \mu(2^{j+K_0 B_i}) \right)^{-1/2} [\rho(\mu(2^{j+K_0 B_i}))]^{-1} \\
\times 2^{-2K_0(M-k+\epsilon/2)} K_1^{2M-2k+\epsilon} r_{B_i}^{2M+2k}.
\]

Then, by choosing

\[
K_0 \equiv \max_{0 \leq k \leq M} \left( \frac{\ln \left( K_1^{2M-2k+\epsilon} + \frac{n}{2}(1/p_\Phi - \frac{1}{2}) \right) \max_{0 \leq i \leq N} \left\{ r_{B_i}^{2M+2k} \right\}}{2 \ln (M-k+\epsilon/2)} \right),
\]

we conclude that for \( i \in \{0, 1, \ldots, N\} \), \( m_{i,K_1} \) is a \((\Phi, M, \epsilon)_L\)-molecule adapted to the ball \( 2^{K_0 B_i} \), which shows the claim, and hence completes the proof of Theorem 3.1. \( \square \)

**Remark 3.1.** We point out that the proof of Theorem 3.1 also works for [15, Theorem 5.4]. Moreover, due to the lack of the support of molecules, we show that \( m_{i,K_1} \) for \( i \in \{1, \ldots, N\} \) is a \((\Phi, M, \epsilon)_L\)-molecule adapted to the ball \( 2^{K_0 B_i} \), instead of \( B_i \) as in the proof of [15, Theorem 5.4], which also simplifies the proof of [15, Theorem 5.4].

By Theorem 3.1, the argument same as the proofs of [2, Theorems 3.13 and 3.16], we obtain the following dual theorem. We omit the details.
**Theorem 3.2.** Let \( L, \Phi, \rho \) and \( M \) be as in Definition 3.2. Then for any function \( f \in \text{BMO}^M_{\rho,L}(\mathcal{X}) \), the linear functional \( \ell \), defined by \( \ell(g) \equiv \langle f, g \rangle \) initially on \( H^\text{mol, \Phi, fin, L}^M(\mathcal{X}) \) with \( \tilde{M} > M \) and \( \epsilon \in (0, M - \frac{n}{2} \frac{1}{p_\Phi} - \frac{1}{2}) \), has a unique extension to \( H_{\Phi,L^*}(\mathcal{X}) \) and, moreover, 
\[
\|\ell\|_{(H_{\Phi,L^*}(\mathcal{X}))^*} \leq C\|f\|_{\text{BMO}^M_{\rho,L}(\mathcal{X})} \quad \text{for some nonnegative constant } C \text{ independent of } f.
\]

Conversely, for any \( \ell \in (H_{\Phi,L^*}(\mathcal{X}))^* \), there exists \( f \in \text{BMO}^M_{\rho,L}(\mathcal{X}) \) such that \( \ell(g) \equiv \langle f, g \rangle \) for all \( g \in H^\text{mol, \Phi, fin, L}^M(\mathcal{X}) \) and \( \|f\|_{\text{BMO}^M_{\rho,L}(\mathcal{X})} \leq C\|\ell\|_{(H_{\Phi,L^*}(\mathcal{X}))^*} \), where \( C \) is a nonnegative constant independent of \( \ell \).

**Remark 3.2.** (i) Theorem 3.1 is just [2, Theorems 3.15] but with the ranges of indices \( M \) and \( \epsilon \) replaced, respectively, by \( M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2}) \) and \( \epsilon \in (0, M - \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})) \).

(ii) By Theorem 3.2, we see that for all \( M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2}) \), the spaces \( \text{BMO}^M_{\rho,L}(\mathcal{X}) \) for different \( M \) coincide with equivalent norms; thus, in what follows, we denote \( \text{BMO}^M_{\rho,L}(\mathcal{X}) \) simply by \( \text{BMO}_{\rho,L}(\mathcal{X}) \).

The following two propositions are just [2, Propositions 3.11 and 3.12] (see also [21, Propositions 4.4 and 4.5]).

**Proposition 3.1.** Let \( L, \Phi, \rho \) and \( M \) be as in Definition 3.2. Then \( f \in \text{BMO}_{\rho,L}(\mathcal{X}) \) if and only if \( f \in \mathcal{M}^M_{\Phi,L}(\mathcal{X}) \) and
\[
\sup_{B \subset \mathcal{X}} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B \left| \left[ I - (I + t^2 L)^{-1} \right]^M f(x) \right|^2 \, d\mu(x) \right]^{1/2} < \infty.
\]
Moreover, the quantity appeared in the left-hand side of the above formula is equivalent to \( \|f\|_{\text{BMO}^M_{\rho,L}(\mathcal{X})} \).

**Proposition 3.2.** Let \( L, \Phi, \rho \) and \( M \) be as in Definition 3.2. Then there exists a positive constant \( C \) such that for all \( f \in \text{BMO}_{\rho,L}(\mathcal{X}) \),
\[
\sup_{B \subset \mathcal{X}} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B \left| (t^2 L)^M e^{-t^2 L} f(x) \right|^2 \, d\mu(x) \, dt \right]^{1/2} \leq C\|f\|_{\text{BMO}^M_{\rho,L}(\mathcal{X})}.
\]

The following Proposition 3.3 and Lemma 3.2 are a kind of Calderón reproducing formulae.

**Proposition 3.3.** Let \( L, \Phi, \rho \) and \( M \) be as in Definition 3.2, \( \epsilon, \epsilon_1 \in (0, \infty) \) and \( \tilde{M} > M + \epsilon_1 + \frac{n}{2} + \frac{N}{2} \frac{1}{p_\Phi} - 1 \), where \( N \) is as in (2.4). Fix \( x_0 \in \mathcal{X} \). Assume that \( f \in \mathcal{M}^M_{\Phi,L}(\mathcal{X}) \) satisfies that
\[
(3.6) \quad \int_{\mathcal{X}} \frac{|(I - (I + L)^{-1})^M f(x)|^2}{1 + [d(x,x_0)]^{n+\epsilon_1+2N(1/p_\Phi-1)}} \, d\mu(x) < \infty.
\]
Then for all \( (\Phi, \tilde{M}, \epsilon)_{L^*} \)-molecules \( \alpha \),
\[
\langle f, \alpha \rangle = C(M) \int_{\mathcal{X} \times (0,\infty)} (t^2 L)^M e^{-t^2 L} f(x) t^2 L^* e^{-t^2 L^*} \alpha(x) \frac{d\mu(x) \, dt}{t},
\]
where \( C(M) \) is as in (3.5).
Proof. For $R > \delta > 0$, write

\[
C(M) \int_\delta^R \int \chi (t^2 L)^M e^{-t^2 L^*} f(x) t^{2L^*} e^{-t^2 L^*} \alpha(x) \frac{d\mu(x)}{t} dt
\]

\[
= \left\langle f, C(M) \int_\delta^R (t^2 L^*)^{M+1} e^{-t^2 L^*} \alpha \frac{dt}{t} \right\rangle
\]

\[
= \left\langle f, \alpha - C(M) \int_\delta^R (t^2 L^*)^{M+1} e^{-t^2 L^*} \alpha \frac{dt}{t} \right\rangle.
\]

Since $\alpha$ is a $(\Phi, \tilde{M}, \epsilon)_L$-molecule, by Definition 3.1, there exists $b \in L^2(\mathcal{X})$ such that $\alpha = (L^*)^{\tilde{M}} b$. Notice that

\[
f = [I - (I + L)^{-1} + (I + L)^{-1}]^M f
\]

\[
= \sum_{k=0}^{M} \binom{M}{k} [I - (I + L)^{-1}]^{M-k} (I + L)^{-k} f = \sum_{k=0}^{M} \binom{M}{k} [I - (I + L)^{-1}]^M L^{-k} f,
\]

where $\binom{M}{k}$ denotes the binomial coefficient, which, together with $H_{\infty}$-functional calculus, further implies that

\[
\left\langle f, \alpha - C(M) \int_\delta^R (t^2 L^*)^{M+1} e^{-t^2 L^*} \alpha \frac{dt}{t} \right\rangle
\]

\[
= \sum_{k=0}^{M} \binom{M}{k} \left\langle [I - (I + L)^{-1}]^M f, L^{\tilde{M}-k} b - C(M) \int_\delta^R (t^2 L^*)^{M+1} e^{-t^2 L^*} (L^*)^{\tilde{M}-k} b \frac{dt}{t} \right\rangle
\]

\[
= \sum_{k=0}^{M} \binom{M}{k} \left\langle [I - (I + L)^{-1}]^M f, C(M) \int_0^\delta (t^2 L^*)^{M+1} e^{-t^2 L^*} (L^*)^{\tilde{M}-k} b \frac{dt}{t} \right\rangle
\]

\[
+ \sum_{k=0}^{M} \binom{M}{k} \left\langle [I - (I + L)^{-1}]^M f, C(M) \int_R^\infty (t^2 L^*)^{M+1} e^{-t^2 L^*} (L^*)^{\tilde{M}-k} b \frac{dt}{t} \right\rangle
\]

\[
\equiv \sum_{k=0}^{M} \binom{M}{k} (H + J).
\]

For J, by (3.6) and Hölder’s inequality, we conclude that

\[
|J| \lesssim \left\{ \int_{\mathcal{X}} \frac{|(I - (I + L)^{-1})^M f(x)|^2}{1 + [d(x, x_0)]^{n+\epsilon_1 + 2\lambda(1/p_\phi - 1)}} d\mu(x) \right\}^{1/2}
\]

\[
\times \left\{ \int_{\mathcal{X}} \left[ \int_R^\infty (t^2 L^*)^{M+\tilde{M}-k+1} e^{-t^2 L^*} b(x) \frac{1}{t^{2(M-k)+1}} dt \right]^2 \right\}^{1/2}
\]

\[
\times \left( 1 + [d(x, x_0)]^{n+\epsilon_1 + 2\lambda(1/p_\phi - 1)} \right) d\mu(x) \right\}^{1/2}
\]

\[
= \left\{ \int_{\mathcal{X}} \frac{|(I - (I + L)^{-1})^M f(x)|^2}{1 + [d(x, x_0)]^{n+\epsilon_1 + 2\lambda(1/p_\phi - 1)}} d\mu(x) \right\}^{1/2}
\]

\[
\times \left( 1 + [d(x, x_0)]^{n+\epsilon_1 + 2\lambda(1/p_\phi - 1)} \right) d\mu(x) \right\}^{1/2}
\]

\[
\leq \left\{ \int_{\mathcal{X}} \frac{|(I - (I + L)^{-1})^M f(x)|^2}{1 + [d(x, x_0)]^{n+\epsilon_1 + 2\lambda(1/p_\phi - 1)}} d\mu(x) \right\}^{1/2}
\]

\[
\times \left( 1 + [d(x, x_0)]^{n+\epsilon_1 + 2\lambda(1/p_\phi - 1)} \right) d\mu(x) \right\}^{1/2}
\]
Vanishing Mean Oscillation Spaces Associated with Operators

\[
\lesssim \int_R^\infty \left\| (t^2 L^*)^{M+\tilde{M}-k+1} e^{-2t^2 L^*} b \left( 1 + [d(\cdot, x_0)]^{n+\epsilon_1+2N(1/p_\Phi-1)} \right)^{1/2} \right\|_{L^2(\mathcal{X})} \times \frac{1}{t^{2(M-k)+1}} \, dt.
\]

Let \( B_0 \equiv B(x_0, 1) \). Notice that there exist \( \tilde{N}, d \in \mathbb{N} \) such that for all \( j \in \mathbb{N}, j \geq \tilde{N} \),

\[
U_j(B_0) \subset \bigcup_{i=-d}^d U_{j+i}(B),
\]

where \( B \) is the ball adapted to \( \alpha \) and \( U_j(B) \) for \( j \in \mathbb{Z}_+ \) is as in (2.5). By choosing \( j_0 \geq \tilde{N} \), we conclude that

\[
|J| \lesssim \int_R^\infty \left\| (t^2 L^*)^{M+\tilde{M}-k+1} e^{-2t^2 L^*} b \right\|_{L^2(\mathcal{X})} \left( 1 + [d(\cdot, x_0)]^{n+\epsilon_1+2N(1/p_\Phi-1)} \right)^{1/2} \frac{1}{t^{2(M-k)+1}} \, dt
\]

\[+ \sum_{j=j_0+1}^\infty \int_R^\infty \left\| (t^2 L^*)^{M+\tilde{M}-k+1} e^{-2t^2 L^*} b \right\|_{L^2(\mathcal{X})} \left( 1 + [d(\cdot, x_0)]^{n+\epsilon_1+2N(1/p_\Phi-1)} \right)^{1/2} \frac{1}{t^{2(M-k)+1}} \, dt \equiv J_1 + J_2.
\]

For all \( \tilde{c} > 0 \), let \( C_1 \equiv 2^{k_0(n+\epsilon_1+2N(1/p_\Phi-1))} \| b \|_{L^2(\mathcal{X})} \) and \( R_1 \equiv \left( \frac{C_1}{\tilde{c}} \right)^{2(M-k)} \), then for all \( R > R_1 \), we obtain

\[
J_1 \lesssim 2^{k_0(n+\epsilon_1+2N(1/p_\Phi-1))} \int_R^\infty \frac{dt}{t^{2(M-k)+1}} \left\| b \right\|_{L^2(\mathcal{X})} \lesssim \tilde{c}.
\]

Letting \( C_2 \equiv \frac{2^{k_0(n+\epsilon_1+2N(1/p_\Phi-1))}}{R_1} \) and \( R_1 \equiv \left( \frac{C_2}{\tilde{c}} \right)^{2(M-k)} \), then for all \( R > R_1 \), we know that

\[
J_2 \lesssim \sum_{j=j_0+1}^\infty 2^{\frac{1}{2}(n+\epsilon_1+2N(1/p_\Phi-1))} \]

\[\times \sum_{i=-d}^d \left\{ \int_R^\infty \left\| (t^2 L^*)^{M+\tilde{M}-k+1} e^{-2t^2 L^*} \left( X\tilde{U}_{j+i}(B) b \right) \right\|_{L^2(\mathcal{X})} \frac{1}{t^{2(M-k)+1}} \, dt \right. \]

\[+ \left. \int_R^\infty \left\| (t^2 L^*)^{M+\tilde{M}-k+1} e^{-2t^2 L^*} \left( X\tilde{U}_{j+i}(B) b \right) \right\|_{L^2(\mathcal{X})} \frac{1}{t^{2(M-k)+1}} \, dt \right\},
\]

where \( \tilde{U}_{j+i}(B) \equiv 2^{j+i+1}B \setminus 2^{j+i-1}B \). Then, since

\[
\int_R^\infty \left\| (t^2 L^*)^{M+\tilde{M}-k+1} e^{-2t^2 L^*} \left( X\tilde{U}_{j+i}(B) b \right) \right\|_{L^2(\mathcal{X})} \frac{1}{t^{2(M-k)+1}} \, dt
\]
\[
\sum_{j=0}^{\infty} 2^{j(n+\epsilon_1+2N(1/p_\Phi-1))} \left\| (\delta^2 L^*)^{M-\ell+1} e^{-2\delta^2 L^* \bar{M}-k} \chi_{(\cup_{j=j_0-1}^{j_0})^c (B)} (L^*) b \right\|_{L^2(U_j (B))}.
\]
By the $L^2$-functional calculus, we see that $\lim_{\delta \to 0} (\delta^2 L^*)^{M-\ell+1} e^{-2\delta^2 L^* (L^*)} \bar{M}-k b = 0$ in $L^2(\mathcal{X})$ and, by Lemma 2.2, we know that

\[
\sum_{j=j_0+1}^{\infty} 2^{j(n+\epsilon_1+2N(1/p_\Phi-1))} \left\| (\delta^2 L^*)^{M-\ell+1} e^{-2\delta^2 L^* \bar{M}-k} \chi_{(\cup_{j=j_0-1}^{j_0})^c (B)} (L^*) b \right\|_{L^2(U_j (B))}.
\]
Theorem 3.3. Let $\rho$ for some $\epsilon$ be as in Definition 3.2 and $\epsilon \in (0, \infty)$. If $f \in BMO_{\rho,L}(\mathcal{X})$, then for any $(\Phi, M, \epsilon)_{L^2}$-molecule $\alpha$, there holds

$$\langle f, \alpha \rangle = C(M) \int_{\mathcal{X} \times (0, \infty)} (t^2 L^*)^M e^{-\epsilon t^2 L^*} f(x) t^2 L^* e^{-\epsilon t^2 L^*} \alpha(x) \frac{d \mu(x)}{t} dt$$

Recall that a measure $d \mu$ on $\mathcal{X} \times (0, \infty)$ is called a $\rho$-Carleson measure if

$$\|d \mu\|_{\rho} \equiv \sup_{B \subset \mathcal{X}} \left\{ \frac{1}{\mu(B)[\rho(\mu(B))]^2} \int_B |d \mu| \right\}^{1/2} < \infty,$$

where the supremum is taken over all balls $B$ of $\mathcal{X}$.

Using Theorem 3.2 and Proposition 3.2, similar to the proof of [21, Theorem 4.2], we obtain the following $\rho$-Carleson measure characterization of $BMO_{\rho,L}(\mathcal{X})$.

Theorem 3.3. Let $L, \Phi, \rho$ and $M$ be as in Definition 3.2. Fix $x_0 \in \mathcal{X}$. Then the following are equivalent:

(i) $f \in BMO_{\rho,L}(\mathcal{X})$;

(ii) $f \in \mathcal{M}_{\Phi,L}(\mathcal{X})$ satisfies that

$$\int_{\mathcal{X}} \frac{|(I - (I + L)^{-1})^M f(x)|^2}{1 + [d(x,x_0)]^{n + \epsilon_1 + 2N(1/p_\Phi^{-1})}} d \mu(x) < \infty$$

for some $\epsilon_1 \in (0, \infty)$, and $d \mu_f$ is a $\rho$-Carleson measure, where $d \mu_f$ is defined by $d \mu_f(x,t) \equiv |(t^2 L^*)^M e^{-\epsilon t^2 L^*} f(x)|^2 \frac{d \mu(x)}{t} dt$ for all $(x,t) \in \mathcal{X} \times (0, \infty)$.

Moreover, $\|d \mu_f\|_{\rho}$ is equivalent to $\|f\|_{BMO_{\rho,L}(\mathcal{X})}$. 

From

$$S_1 = \left\langle \bar{f}, \int_0^\delta (t^2 L^*) e^{-\epsilon t^2 L^*} (L^*)^{M-k} b \frac{dt}{t} \right\rangle = \left\langle \bar{f}, \left( e^{-\epsilon t^2 L^*} - I \right) (L^*)^{M-k} b \right\rangle,$$

and

$$\lim_{\delta \to 0} \left\| \left( e^{-\epsilon t^2 L^*} - I \right) (L^*)^{M-k} b \right\|_{L^2(\mathcal{X})} = 0,$$

it follows that $\lim_{\delta \to 0} H = 0$, which completes the proof of Proposition 3.3. \hfill \Box

Instead of [21, Proposition 4.6] by Proposition 3.3 here, repeating the proof of [21, Corollary 4.3], we obtain the following Lemma 3.2. The details are omitted.

Lemma 3.2. Let $L, \Phi, \rho$ and $M$ be as in Definition 3.2 and $\epsilon \in (0, \infty)$. If $f \in BMO_{\rho,L}(\mathcal{X})$, then for any $(\Phi, M, \epsilon)_{L^2}$-molecule $\alpha$, there holds

$$\langle f, \alpha \rangle = C(M) \int_{\mathcal{X} \times (0, \infty)} (t^2 L^*)^M e^{-\epsilon t^2 L^*} f(x) t^2 L^* e^{-\epsilon t^2 L^*} \alpha(x) \frac{d \mu(x)}{t} dt.$$
Proof. It follows from Proposition 3.1 and the proof of Lemma 3.2 that (i) implies (ii).

To show that (ii) implies (i), let \( \widetilde{M} > M + \epsilon_1 + \frac{\lambda}{2} + \frac{\lambda}{2}(\frac{1}{\rho} - 1) \). From Proposition 3.3, we deduce that

\[
\langle f, g \rangle = C(M) \int_{X \times (0, \infty)} (t^2 L^M e^{-t^2 L} f(x) t^2 L^e e^{-t^2 L} g(x)) \frac{d\mu(x) dt}{t},
\]

where \( g \) is any finite combination of \( (\Phi, M, \epsilon)L^* \)-molecules. Then \( t^2 L^e e^{-t^2 L} g \in T_{\Phi}(X) \).

By Lemma 3.1, there exist \( \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \) and \( T_{\Phi}(X) \)-atoms \( \{a_j\}_{j=1}^{\infty} \) supported in \( \{B_j\}_{j=1}^{\infty} \) such that (3.1) and (3.2) hold. This, together with Fatou’s lemma and Hölder’s inequality, implies that

\[
|\langle f, g \rangle| = \left| C(M) \int_{X \times (0, \infty)} (t^2 L^M e^{-t^2 L} f(x) t^2 L^e e^{-t^2 L} g(x)) \frac{d\mu(x) dt}{t} \right| \leq \sum_j |\lambda_j| \int_{X} |(t^2 L^M e^{-t^2 L} f(x) a_j(x, t))| \frac{d\mu(x) dt}{t}
\]
\[
\leq \sum_j |\lambda_j| \|a_j\|_{T^2(X)} \left( \int_{B_j} |(t^2 L^M e^{-t^2 L} f(x))|^2 \frac{d\mu(x) dt}{t} \right)^{1/2}
\]
\[
\leq \sum_j |\lambda_j| \|d\mu_j\|_{\rho} \lesssim \|t^2 L^e e^{-t^2 L} g\|_{T_{\Phi}(X)} \|d\mu_j\|_{\rho} \sim \|g\|_{H_{\Phi, L^*}(X)} \|d\mu_j\|_{\rho}.
\]

By this and Theorem 3.2, we conclude that \( f \in (H_{\Phi, L^*}(X))^* = \text{BMO}_{\rho, L}(X) \), which completes the proof of Theorem 3.3. \( \square \)

Now we introduce the space \( \text{VMO}_{\rho, L}(X) \).

**Definition 3.3.** Let \( L, \Phi, \rho \) and \( M \) be as in Definition 3.2. An element \( f \in \text{BMO}_{\rho, L}(X) \) is said to be in the **space** \( \text{VMO}_{\rho, L}(X) \) if it satisfies the following limiting conditions

\[
\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0, \text{ where } x_0 \in X \text{ is a fixed point, } c \in (0, \infty),
\]

\[
\gamma_1(f) \equiv \lim_{c \to 0} \sup_{B : r_B \leq c} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B |(I - e^{-r_B^2 L})^M f(x)|^2 d\mu(x) \right]^{1/2},
\]

\[
\gamma_2(f) \equiv \lim_{c \to \infty} \sup_{B : r_B \geq c} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B |(I - e^{-r_B^2 L})^M f(x)|^2 d\mu(x) \right]^{1/2},
\]

and

\[
\gamma_3(f) \equiv \lim_{c \to \infty} \sup_{B : B \subset [B(x_0, c)]} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B |(I - e^{-r_B^2 L})^M f(x)|^2 d\mu(x) \right]^{1/2}.
\]

For any \( f \in \text{VMO}_{\rho, L}(X) \), define \( \|f\|_{\text{VMO}_{\rho, L}(X)} \equiv \|f\|_{\text{BMO}_{\rho, L}(X)} \).
Definition 3.4. Let $\Phi$ satisfy Assumption ($\Phi$) and $\rho$ be as in (2.11). The space $T_{\Phi,\psi}^\infty(\mathcal{X})$ is defined to be the space of all $f \in T_{\Phi}^\infty(\mathcal{X})$ satisfying $\eta_1(f) = \eta_2(f) = \eta_3(f) = 0$ with the same norm as the space $T_{\Phi}^\infty(\mathcal{X})$, where $x_0 \in \mathcal{X}$ is a fixed point, $c \in (0, \infty)$,

$$\eta_1(f) = \lim_{c \to 0} \sup_{B : r_B \leq c} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_{B} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right]^{1/2},$$

$$\eta_2(f) = \lim_{c \to \infty} \sup_{B : r_B \geq c} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_{B} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right]^{1/2},$$

and

$$\eta_3(f) = \lim_{c \to \infty} \sup_{B : r_B \leq c} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_{B} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right]^{1/2}.$$ 

It is easy to see that $T_{\Phi,\psi}^\infty(\mathcal{X})$ is a closed linear subspace of $T_{\Phi}^\infty(\mathcal{X})$.

Further, denote by $T_{\Phi,\psi}^\infty(\mathcal{X})$ the space of all $f \in T_{\Phi}^\infty(\mathcal{X})$ with $\eta_1(f) = 0$, and $T_{2,\beta}^0(\mathcal{X})$ the space of all $f \in T_{2,\beta}^0(\mathcal{X})$ with bounded support. Obviously, we have $T_{2,\beta}^0(\mathcal{X}) \subset T_{\Phi,\psi}^\infty(\mathcal{X}) \subset T_{\Phi,1}^\infty(\mathcal{X})$. Finally, denote by $T_{\Phi,0}^\infty(\mathcal{X})$ the closure of $T_{2,\beta}^0(\mathcal{X})$ in the space $T_{\Phi,1}^\infty(\mathcal{X})$.

Lemma 3.3. Let $L$ and $\Phi$ be as in Definition 3.1, and $T_{\Phi,\psi}^\infty(\mathcal{X})$ and $T_{\Phi,0}^\infty(\mathcal{X})$ defined as above. Then $T_{\Phi,\psi}^\infty(\mathcal{X})$ and $T_{\Phi,0}^\infty(\mathcal{X})$ coincide with equivalent norms.

Proof. Since $T_{2,\beta}^0(\mathcal{X}) \subset T_{\Phi,\psi}^\infty(\mathcal{X})$ and $T_{\Phi,\psi}^\infty(\mathcal{X})$ is a closed linear subspace of $T_{\Phi}^\infty(\mathcal{X})$, we conclude that $T_{\Phi,\psi}^\infty(\mathcal{X}) = T_{2,\beta}^0(\mathcal{X}) \subset T_{\Phi,\psi}^\infty(\mathcal{X})$.

Conversely, for any $f \in T_{\Phi,\psi}^\infty(\mathcal{X})$, by the definition of $T_{\Phi,\psi}^\infty(\mathcal{X})$, for any $\epsilon > 0$, there exist positive constants $a_0$, $b_0$ and $c_0$ such that

$$\sup_{B : r_B \leq a_0} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_{B} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right]^{1/2} < \epsilon,$$

$$\sup_{B : r_B \geq b_0} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_{B} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right]^{1/2} < \epsilon,$$

and

$$\sup_{B : B \subset (x_0, c_0)\mathbb{C} \mu(\mu(B))} \left[ \frac{1}{\mu(B)} \int_{B} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right]^{1/2} < \epsilon.$$ 

Let $K_0 \equiv \max\{a_0^{-1}, b_0, c_0\}$ and, for all $(y,t) \in \mathcal{X} \times (0, \infty)$,

$$g(y,t) \equiv f(y,t)\chi_{B(x_0, 2K_0) \times ((2K_0)^{-1}, 2K_0)}(y,t).$$

Obviously, $g \in T_{2,\beta}^0(\mathcal{X})$. To complete the proof of Lemma 3.3, we need show that

$$\|f - g\|_{T_{\Phi}^\infty(\mathcal{X})} \lesssim \epsilon.$$
We consider the following three cases for all balls $B$ in (3.7), (3.8) and (3.9).

Case (i) $r_B < a_0$ or $r_B > b_0$. In this case, from (3.7) and (3.8), we deduce that

$$\|f - g\|_{T^\infty_B(x)}^2 \leq \frac{2}{\mu(B)[\rho(\mu(B))]^2} \int_B |f(y,t)|^2 \frac{d\mu(y)}{t} dt \leq 2\epsilon.$$ 

Case (ii) $a_0 \leq r_B \leq b_0$ and $B \subset [B(x_0,c_0)]^c$. In this case, by (3.9), we conclude that

$$\|f - g\|_{T^\infty_B(x)}^2 \leq \frac{2}{\mu(B)[\rho(\mu(B))]^2} \int_B |f(y,t)|^2 \frac{d\mu(y)}{t} dt \leq 2\epsilon.$$ 

Case (iii) $a_0 \leq r_B \leq b_0$ and $B \cap B(x_0,c_0) \neq \emptyset$. In this case, we have

$$\int_B |f(y,t) - g(y,t)|^2 \frac{d\mu(y)}{t} dt \leq \int_{0}^{(2K_0)^{-1}} \int_B |f(y,t)|^2 \frac{d\mu(y)}{t} dt \leq \int_{0}^{(2K_0)^{-1}} \int_{B(x_B,2^k a_0)} |f(y,t)|^2 \frac{d\mu(y)}{t} dt,$$

where $x_B$ is the center of $B$ and $k$ the smallest integer such that $2^k a_0 > r_B$. Then, by Lemma 2.1, we pick a family of balls with the same radius $a_0$, $\{B(x_B,i,a_0)\}_{i=1}^{N_k}$, such that $B(x_B,2^k a_0) \subset \bigcup_{i=1}^{N_k} B(x_B,i,a_0)$, $N_k \lesssim 2^k a_0$ and $\sum_{i=1}^{N_k} \chi_{B(x_B,i,a_0)} \lesssim 1$. Therefore, combining the fact that $\rho$ is an increasing function, we obtain

$$\int_B |f(y,t) - g(y,t)|^2 \frac{d\mu(y)}{t} dt \leq \sum_{i=1}^{N_k} \int_{B(x_B,i,a_0)} |f(y,t)|^2 \frac{d\mu(y)}{t} dt \lesssim \epsilon \sum_{i=1}^{N_k} \mu(B(x_B,i,a_0)) \lesssim \epsilon \mu(B)[\rho(\mu(B))]^2,$$

which completes the proof of Lemma 3.3.

**Definition 3.5.** Let $L$, $\Phi$, $\rho$ and $M$ be as in Definition 3.2. The space $\tilde{\text{VMO}}^M_{\rho,L}(\mathcal{X})$ is defined to be the space of all elements $f \in \text{BMO}^M_{\rho,L}(\mathcal{X})$ that satisfy the following limiting conditions $\tilde{\gamma}_1(f) = \tilde{\gamma}_2(f) = \tilde{\gamma}_3(f) = 0$, where $c \in (0, \infty)$,

$$\tilde{\gamma}_1(f) \equiv \lim_{\epsilon \to 0} \sup_{B : r_B \leq \epsilon} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B |(I + r_B^2 L)^{-1} M f(x)|^2 d\mu(x) \right]^{1/2},$$

$$\tilde{\gamma}_2(f) \equiv \lim_{\epsilon \to \infty} \sup_{B : r_B \geq \epsilon} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B |(I + r_B^2 L)^{-1} M f(x)|^2 d\mu(x) \right]^{1/2},$$

and

$$\tilde{\gamma}_3(f) \equiv \lim_{\epsilon \to 0} \sup_{B : r_B \leq \epsilon} \frac{1}{\rho(\mu(B))} \left[ \frac{1}{\mu(B)} \int_B |(I + r_B^2 L)^{-1} M f(x)|^2 d\mu(x) \right]^{1/2}.$$
and

\[
\overline{\gamma}_1(f) \equiv \lim_{c \to \infty} \sup_{B, B \subset [B(0, c)]^c} \frac{1}{\rho(\mu(B))}[\frac{1}{\mu(B)}\int_B \left| (I - [I + r_B^2L]^{-1})^M f(x) \right|^2 d\mu(x)]^{1/2}.
\]

**Proposition 3.4.** Let \( L, \Phi, \rho \) and \( M \) be as in Definition 3.2. Then \( f \in \text{VMO}^M_{\rho, L}(\mathcal{X}) \) if and only if \( f \in \overline{\text{VMO}}^M_{\rho, L}(\mathcal{X}) \).

**Proof.** Suppose that \( f \in \overline{\text{VMO}}^M_{\rho, L}(\mathcal{X}) \). To see \( f \in \text{VMO}^M_{\rho, L}(\mathcal{X}) \), it suffices to show that

\[
(3.10) \quad \frac{1}{\rho(\mu(B))}[\frac{1}{\mu(B)}\int_B \left| (I - e^{-r_B^2L})^M f(x) \right|^2 d\mu(x)]^{1/2} \lesssim \sum_{k=0}^{\infty} 2^{-k} \delta_k(f, B),
\]

where

\[
(3.11) \quad \delta_k(f, B) = \sup_{B' \subset 2^{k+1} B : r_B \in [2^{-1} r_B, r_B]} \frac{1}{\rho(\mu(B))}[\frac{1}{\mu(B)}\int_B \left| (I - r_B^2L)^{-1} f(x) \right|^2 d\mu(x)]^{1/2}.
\]

Indeed, since \( f \in \overline{\text{VMO}}^M_{\rho, L}(\mathcal{X}) \), by Definition 3.5 and Proposition 3.1, we conclude that \( \delta_k(f, B) \lesssim \|f\|_{\text{BMO}_{\rho, L}(\mathcal{X})} \) and for all \( k \in \mathbb{Z}_+ \),

\[
\lim_{c \to 0, B : r_B \leq c} \sup_{B' \subset 2^{k+1} B : r_B \in [2^{-1} r_B, r_B]} \delta_k(f, B) = \lim_{c \to \infty, B : r_B \leq c} \sup_{B' \subset 2^{k+1} B : r_B \in [2^{-1} r_B, r_B]} \delta_k(f, B) = 0.
\]

Then by the dominated convergence theorem for series, we have

\[
\gamma_1(f) = \lim_{c \to 0, B : r_B \leq c} \sup \frac{1}{\rho(\mu(B))}[\frac{1}{\mu(B)}\int_B \left| (I - e^{-r_B^2L})^M f(x) \right|^2 d\mu(x)]^{1/2} \lesssim \sum_{k=1}^{\infty} 2^{-k} \lim_{c \to 0, B : r_B \leq c} \delta_k(f, B) = 0.
\]

Similarly we see that \( \gamma_2(f) = \gamma_3(f) = 0 \), and hence \( f \in \text{VMO}^M_{\rho, L}(\mathcal{X}) \).

Let us now prove (3.10). Write

\[
(3.12) \quad f = (I - [I + r_B^2L]^{-1})^M f + \left\{ I - (I - [I + r_B^2L]^{-1})^M \right\} f \equiv f_1 + f_2.
\]

By Lemma 2.2, we have

\[
(3.13) \quad \left\| (I - e^{-r_B^2L})^M f_1 \right\|_{L^2(B)} \leq \sum_{k=0}^{\infty} \left\| (I - e^{-r_B^2L})^M (f_1 \chi_{U_k(B)}) \right\|_{L^2(B)} \lesssim \sum_{k=0}^{\infty} e^{-c2^{2k}} \left\| f_1 \chi_{U_k(B)} \right\|_{L^2(\mathcal{X})}
\]
\[ \lesssim \rho(\mu(B))|\mu(B)|^{1/2} \sum_{k=0}^{\infty} e^{-c2k} 2^{kn} \delta_k(f, B) \]

\[ \lesssim \rho(\mu(B))|\mu(B)|^{1/2} \sum_{k=0}^{\infty} 2^{-k} \delta_k(f, B), \]

where \( U_k(B) \) for all \( k \in \mathbb{Z}_+ \) is as in (2.5), \( c \) is a positive constant and the third inequality follows from Lemma 2.1 that there exists a collection, \( \{B_{k,1}, B_{k,2}, \ldots, B_{k,N_k}\} \), of balls such that each ball \( B_{k,i} \) is of radius \( r_B \), \( B(x_B, 2^{k+1}r_B) \subset \bigcup_{i=1}^{N_k} B_{k,i} \) and \( N_k \lesssim 2^{nk} \).

To estimate the remaining term, by the formula that (which relies on the fact that \( (I - (I + r^2L)^{-1})(r^2L)^{-1} = (I + r^2L)^{-1} \) for all \( r \in (0, \infty) \)), and Minkowski’s inequality, we obtain

\[
\left\| \left( I - e^{-r_B^2L} \right)^{M} f_2 \right\|_{L^2(B)} \\
\lesssim \sum_{j=1}^{M} \left\{ \int_{B} \left| \left( I - e^{-r_B^2L} \right)^{M-j} \left[ - \int_{0}^{r_B} \frac{s}{r_B} e^{-s^2L} ds \right]^{j} f_1(x) \right|^{2} d\mu(x) \right\}^{1/2} \\
\lesssim \sum_{j=1}^{M} \sum_{i=0}^{M-j} \int_{0}^{r_B} \cdots \int_{0}^{r_B} \frac{s_1}{r_B} \cdots \frac{s_i}{r_B} \left\| e^{-(ir_B^2+s_1^2+\cdots+s_i^2)L} f_1 \right\|_{L^2(B)} ds_1 \cdots ds_j \\
\lesssim \sum_{j=1}^{M} \sum_{i=0}^{M-j} \int_{0}^{r_B} \cdots \int_{0}^{r_B} \frac{s_1}{r_B} \cdots \frac{s_i}{r_B} \int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{-c(2^k r_B)^2} \left\| f_1 U_k(B) \right\|_{L^2(\mathcal{X})} ds_1 \cdots ds_j \\
\lesssim \rho(\mu(B))|\mu(B)|^{1/2} \sum_{k=0}^{\infty} e^{-c2k} 2^{kn} \delta_k(f, B) \\
\lesssim \rho(\mu(B))|\mu(B)|^{1/2} \sum_{k=0}^{\infty} 2^{-k} \delta_k(f, B),
\]

where \( c \) is a positive constant and in the penultimate inequality, we used the fact that \( \int_{0}^{r_B} \cdots \int_{0}^{r_B} \frac{s_1}{r_B} \cdots \frac{s_i}{r_B} ds_1 \cdots ds_j \sim 1 \). Combining the estimates (3.13) and (3.15), we obtain (3.10), which further implies that \( \widetilde{\text{VMO}}_{\rho,L}^{M}(\mathcal{X}) \subset \text{VMO}_{\rho,L}^{M}(\mathcal{X}) \).

By borrowing some ideas from the proof of [16, Lemma 8.1], then similar to the proof above, we conclude that \( \text{VMO}_{\rho,L}^{M}(\mathcal{X}) \subset \widetilde{\text{VMO}}_{\rho,L}^{M}(\mathcal{X}) \) and the details are omitted. This finishes the proof of Proposition 3.4. \( \Box \)

We now characterize the space \( \text{VMO}_{\rho,L}^{M}(\mathcal{X}) \) via the tent space.
Theorem 3.4. Let $L$, $\Phi$ and $\rho$ be as in Definition 3.1, $M$, $M_1, M_2 \in \mathbb{N}$ and $M_1 \geq M > \frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{2}\right)$. Then the following are equivalent:

(i) $f \in \text{VMO}_{\rho, L}(X)$;

(ii) $f \in \mathcal{M}_{\Phi, L}(X)$ and $(t^2L)^{M_1}e^{-t^2L}f \in T_{\Phi, V}(X)$.

Moreover, $\|(t^2L)^{M_1}e^{-t^2L}f\|_{T_{\Phi, V}(X)}$ is equivalent to $\|f\|_{\text{BMO}_{\rho, L}(X)}$.

Proof. We first show that (i) implies (ii). Let $f \in \text{VMO}_{\rho, L}(X)$. By Proposition 3.2, we know that $(t^2L)^{M_1}e^{-t^2L}f \in T_{\Phi, V}$. To see that $(t^2L)^{M_1}e^{-t^2L}f \in T_{\Phi, V}(X)$, we claim that it suffices to show that for all balls $B$,

$$
(3.16) \quad \frac{1}{\rho(\mu(B))|\mu(B)|^{1/2}} \left[ \int_B \left| (t^2L)^{M_1}e^{-t^2L}f(x) \right|^2 \frac{d\mu(x)}{t} \right]^{1/2} \leq \sum_{k=0}^{\infty} 2^{-k} \delta_k(f, B),
$$

where $\delta_k(f, B)$ is as in (3.11). Indeed, since $f \in \text{VMO}_{\rho, L}(X) = \text{VMO}_{\rho, L}(X)$, we conclude that for each $k \in \mathbb{N}$, $\delta_k(f, B) \lesssim \|f\|_{\text{BMO}_{\rho, L}(X)}$ and

$$
\lim_{c \to 0} \sup_{B : r_B \leq c} \delta_k(f, B) = \lim_{c \to \infty} \sup_{B : r_B \geq c} \delta_k(f, B) = \lim_{c \to \infty} \sup_{B : B \subset [x_0, c]} \delta_k(f, B) = 0.
$$

Then from the dominated convergence theorem for series, we infer that

$$
\eta_1(f) = \lim_{c \to 0} \sup_{B : r_B \leq c} \frac{1}{\rho(\mu(B))|\mu(B)|^{1/2}} \left[ \int_B \left| (t^2L)^{M_1}e^{-t^2L}f(x) \right|^2 \frac{d\mu(x)}{t} \right]^{1/2} \lesssim \sum_{k=1}^{\infty} 2^{-k} \lim_{c \to 0} \sup_{B : r_B \leq c} \delta_k(f, B) = 0.
$$

Similarly we see that $\eta_2(f) = \eta_3(f) = 0$, and hence $(t^2L)^{M_1}e^{-t^2L}f \in T_{\Phi, V}(X)$.

Let us now prove (3.16). Write $f \equiv f_1 + f_2$ as in (3.12). Then by Lemmas 2.2 and 2.3, similar to the estimate of (3.13), we have

$$
(3.17) \quad \left\{ \int_B \left| (t^2L)^{M_1}e^{-t^2L}f_1(x) \right|^2 \frac{d\mu(x)}{t} \right\}^{1/2} \leq \sum_{k=0}^{\infty} \left\{ \int_B \left| (t^2L)^{M_1}e^{-t^2L}(f_1\chi_{U_k})(x) \right|^2 \frac{d\mu(x)}{t} \right\}^{1/2}
$$

\begin{align*}
\leq & \|f_1\|_{L^2(4B)} + \sum_{k=3}^{\infty} \left[ \int_0^{r_B} \exp \left\{ -\frac{(2^k r_B)^2}{ct^2} \right\} \frac{dt}{t} \right]^{1/2} \|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})} \\
\leq & \|f_1\|_{L^2(4B)} + \sum_{k=3}^{\infty} \left\{ \int_0^{r_B} \left[ \frac{t^2}{(2^k r_B)^2} \right]^{n+1} \frac{dt}{t} \right\}^{1/2} \|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})} \\
\leq & \rho(\mu(B))|\mu(B)|^{1/2} \sum_{k=0}^{\infty} 2^{-k} \delta_k(f, B),
\end{align*}
where $U_k(B)$ for all $k \in \mathbb{Z}_+$ is as in (2.5) and $c$ is a positive constant. Applying (3.14), Lemma 2.2 and $M_1 > M$ to $f_2$, we see that

\begin{equation}
(3.18) \quad \left\{ \int_B \left| (t^2 L)^{M_1} e^{-t^2 L} f_2(x) \right|^2 \frac{d\mu(x)}{t} \right\}^{1/2} \leq \sum_{j=1}^{M} \left\{ \int_B \left| (t^2 L)^{M_1} e^{-t^2 L} r_B L^{-j} f_1(x) \right|^2 \frac{d\mu(x)}{t} \right\}^{1/2} \\
\leq \sum_{j=1}^{M} \sum_{k=0}^{\infty} \left\{ \int_B \left[ \frac{t^2}{r_B} \right]^{2j} \left| (t^2 L)^{M_1-j} e^{-t^2 L} (f_1 U_k)(x) \right|^2 \frac{d\mu(x)}{t} \right\}^{1/2} \\
\leq \sum_{j=1}^{M} \left\{ \frac{2}{\sum_{k=0}^{\infty} \int_0^{r_B} \left( \frac{t^2}{r_B} \right)^{2j} \frac{dt}{t} \right\}^{1/2} \| f_1 \|_{L^2(4B)} \\
+ \sum_{k=3}^{\infty} \int_0^{r_B} \exp \left\{ -\frac{(2^k r_B)^2}{ct^2} \right\} \frac{dt}{t} \right\}^{1/2} \| f_1 U_k(B) \|_{L^2(\mathcal{X})} \right\}^{1/2} \\
\leq \| f_1 \|_{L^2(4B)} + \sum_{k=3}^{\infty} \left( \int_0^{r_B} \left[ \frac{t^2}{(2^k r_B)^2} \right]^{n+1} \frac{dt}{t} \right) \frac{1}{2} \| f_1 U_k(B) \|_{L^2(\mathcal{X})} \right\}^{1/2} \\
\leq \rho(\mu(B)) \| \mu(B) \|^{1/2} \sum_{k=0}^{\infty} 2^{-k} \delta_k(f, B).
\end{equation}

The estimates (3.17) and (3.18) imply (3.16), which completes the proof that (i) implies (ii).

Conversely, let $f \in \mathcal{M}_{\Phi, L}(\mathcal{X})$ and $(t^2 L)^{M_1} e^{-t^2 L} f \in T_{\Phi, v}(\mathcal{X})$. By Proposition 3.2, we conclude that $f \in \text{BMO}_{\rho, L}(\mathcal{X})$. For any ball $B$, write

\begin{equation}
\left( \int_B \left| (I - e^{-r_B^2 L})^M f(x) \right|^2 d\mu(x) \right)^{1/2} = \sup_{\| g \|_{L^2(\mathcal{X})} \leq 1} \left| \int_B \left( I - e^{-r_B^2 L} \right)^M f(x) g(x) \, d\mu(x) \right| \\
= \sup_{\| g \|_{L^2(\mathcal{X})} \leq 1} \left| \int_B f(x) \left( I - e^{-r_B^2 L^*} \right)^M g(x) \, d\mu(x) \right|.
\end{equation}

Notice that for any $g \in L^2(B)$, $(I - e^{-r_B^2 L^*})^M g$ is a multiple of a $(\Phi, M, \epsilon)_{L^*}$-molecule; see [16, p. 43]. Then by Lemma 3.2 and Hölder’s inequality, we obtain

\begin{equation}
\left( \int_B \left| (I - e^{-r_B^2 L})^M f(x) \right|^2 d\mu(x) \right)^{1/2} \approx \sup_{\| g \|_{L^2(\mathcal{X})} \leq 1} \left| \int_B \left( t^2 L)^{M_1} e^{-t^2 L} f(x) t^2 L^* e^{-t^2 L^*} \left( I - e^{-r_B^2 L^*} \right)^M g(x) \, d\mu(x) dt \right| \\
\approx \sum_{k=0}^{\infty} \left\{ \int_{V_k(B)} \left| (t^2 L)^{M_1} e^{-t^2 L} f(x) \right|^2 \frac{d\mu(x)}{t} \right\}^{1/2}
\end{equation}
where \( V_0(B) \equiv \hat{B} \) and \( V_k(B) \equiv (2^k B) \setminus (2^{k-1} B) \) for \( k \in \mathbb{N} \). In what follows, for \( k \geq 2 \), let \( V_{k,1} \equiv (2^k B) \setminus (2^{k-2} B \times (0, \infty)) \) and \( V_{k,2} \equiv V_k(B) \setminus V_{k,1}(B) \).

For \( k \in \{0, 1, 2\} \), by Lemmas 2.2 and 2.3, we conclude that

\[
I_k = \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \int_{V_k(B)} \left| \int_{V_k(B)} t^2 L^* e^{-t^2 L^*} \left( I - e^{-r^2_B L^*} \right)^M g(x) \right|^2 \frac{d\mu(x) \, dt}{t} \right\}^{1/2}
\]

\[
\lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \left\| \left( I - e^{-r^2_B L^*} \right)^M g \right\|_{L^2(X)} \lesssim 1.
\]

Now for \( k \geq 3 \), write

\[
I_k \lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \int_{V_{k,1}(B)} \left| \int_{V_{k,1}(B)} t^2 L^* e^{-t^2 L^*} \left( I - e^{-r^2_B L^*} \right)^M g(x) \right|^2 \frac{d\mu(x) \, dt}{t} \right\}^{1/2} + \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \int_{V_{k,2}(B)} \cdots \right\}^{1/2} \equiv I_{k,1} + I_{k,2}.
\]

Since for any \( (y, t) \in V_{k,2}(B) \), \( t \geq 2^{k-2} r_B \), from Minkowski’s inequality and Lemmas 2.2 and 2.3, it follows that

\[
I_{k,2} = \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \int_{V_{k,2}(B)} \left| \int_{V_{k,2}(B)} t^2 L^* e^{-t^2 L^*} \left( I - e^{-r^2_B L^*} \right)^M g(x) \right|^2 \frac{d\mu(x) \, dt}{t} \right\}^{1/2}
\]

\[
= \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \int_{V_{k,2}(B)} \left| \int_{V_{k,2}(B)} t^2 L^* e^{-t^2 L^*} \left[ - \int_{0}^{r_B^2} L^* e^{-s L^*} \, ds \right]^M g(x) \right|^2 \frac{d\mu(x) \, dt}{t} \right\}^{1/2}
\]

\[
\lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \int_{0}^{r_B^2} \cdots \int_{0}^{r_B^2} \left\{ \int_{V_{k,2}(B)} \left| t^2 (L^*)^{M+1} \right|^M g(x) \right|^2 \frac{d\mu(x) \, dt}{t} \right\}^{1/2} \, ds_1 \cdots ds_M
\]

\[
\lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \int_{0}^{r_B^2} \cdots \int_{0}^{r_B^2} \left\{ \int_{2^{2k-2} r_B}^{t^4} \frac{t^4 \|g\|^2_{L^2(B)}}{(t^2 + s_1 + \cdots + s_M)^{2(M+1)}} \, dt \right\}^{1/2} \, ds_1 \cdots ds_M
\]

\[
\lesssim 2^{-2kM}.
\]
Similarly, we see that $I_{k,1} \lesssim 2^{-2kM}$. Let $\bar{p}_\Phi \in (0, p_\Phi)$ such that $M > \frac{n}{2} \left( \frac{1}{p_\Phi} - \frac{1}{2} \right)$. Combining the above estimates and the fact that $\rho$ is of upper type $1/\bar{p}_\Phi - 1$, we finally conclude that

$$\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \left[ \int_B \left| \left( I - e^{-r_{2L}^B} \right)^M f(x) \right|^2 d\mu(x) \right]^{1/2}$$

$$\lesssim \sum_{k=0}^\infty 2^{-2kM} \frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \sigma_k(f, B)$$

$$\lesssim \sum_{k=0}^\infty 2^{-k[2M-n(1/\bar{p}_\Phi-1/2)]} \frac{\sigma_k(f, B)}{\rho(\mu(2^k B))[\mu(2^k B)]^{1/2}}.$$

Since $(t^2L)^M e^{-t^2L}f \in T^{\infty}_{\Phi,r}(X) \subset T_\Phi(X)$, from $M > \frac{n}{2} \left( \frac{1}{p_\Phi} - \frac{1}{2} \right)$ and the dominated convergence theorem for series, we infer that

$$\gamma_1(f) = \lim_{c \to 0} \sup_{c \leq r \leq c} \frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \left[ \int_B \left| \left( I - e^{-r_{2L}^B} \right)^M f(x) \right|^2 d\mu(x) \right]^{1/2}$$

$$\lesssim \sum_{k=1}^\infty 2^{-k[2M-n(1/\bar{p}_\Phi-1/2)]} \lim_{c \to 0} \sup_{c \leq r \leq c} \frac{\sigma_k(f, B)}{\rho(\mu(2^k B))[\mu(2^k B)]^{1/2}} = 0.$$

Similarly, $\gamma_2(f) = \gamma_3(f) = 0$, which implies that $f \in \text{VMO}_{\rho,L}^M(X)$, and hence completes the proof of Theorem 3.4.

**Remark 3.3.** It follows from Theorem 3.4 that for all $M \in \mathbb{N}$ and $M > \frac{n}{2} \left( \frac{1}{p_\Phi} - \frac{1}{2} \right)$, the spaces $\text{VMO}_{\rho,L}^M(X)$ coincide with equivalent norms. Thus, in what follows, we denote the $\text{VMO}_{\rho,L}^M(X)$ simply by $\text{VMO}_{\rho,L}(X)$.

### 4 The Dual Space of $\text{VMO}_{\rho,L}(X)$

In this section, we show that the dual space of $\text{VMO}_{\rho,L}(X)$ is $B_{\Phi,L}^\ast(X)$, where the space $B_{\Phi,L}^\ast(X)$ denotes the Banach completion of the space $H_{\Phi,L}^\ast(X)$; see Definition 4.3 and Theorem 4.2 below.

The proof of the following proposition is similar to that of [23, Proposition 4.1]; we omit the details here.

**Proposition 4.1.** Let $\Phi$ satisfy Assumption $(\Phi)$. Then the dual space of $T_\Phi(X)$ is $T^{\infty}_\Phi(X)$. Moreover, the pairing

$$\langle f, g \rangle \to \int_{X \times (0, \infty)} f(y, t)g(y, t) \frac{d\mu(y)dt}{t}$$

for all $f \in T_\Phi(X)$ and $g \in T^{\infty}_\Phi(X)$ realizes $T^{\infty}_\Phi(X)$ being equivalent to the dual of $T_\Phi(X)$.

We now introduce a new tent space $\tilde{T}_\Phi(X)$ and present some properties.
Definition 4.1. Let \( p \in (0, 1) \). The space \( \tilde{T}_\Phi(\mathcal{X}) \) is defined to be the space of all \( f = \sum_{j=1}^{\infty} \lambda_j a_j \) in \( (T_\Phi^\infty(\mathcal{X}))^* \), where \( \{a_j\}_{j=1}^\infty \) are \( T_\Phi(\mathcal{X}) \)-atoms and \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) such that \( \sum_{j=1}^{\infty} |\lambda_j| < \infty \). If \( f \in \tilde{T}_\Phi(\mathcal{X}) \), then define \( \|f\|_{\tilde{T}_\Phi(\mathcal{X})} \equiv \inf\{\sum_{j=1}^{\infty} |\lambda_j|\} \), where the infimum is taken over all the possible decompositions of \( f \) as above.

By [16, Lemma 3.1], \( \tilde{T}_\Phi(\mathcal{X}) \) is a Banach space. Moreover, from Definition 4.1, it is easy to deduce that \( T_\Phi(\mathcal{X}) \) is dense in \( \tilde{T}_\Phi(\mathcal{X}) \); in other words, \( \tilde{T}_\Phi(\mathcal{X}) \) is a Banach completion of \( T_\Phi(\mathcal{X}) \).

Lemma 4.1. Let \( \Phi \) satisfy Assumption (\( \Phi \)). Then \( T_\Phi(\mathcal{X}) \) is a dense subspace of \( \tilde{T}_\Phi(\mathcal{X}) \) and there exists a positive constant \( C \) such that for all \( f \in T_\Phi(\mathcal{X}) \), \( \|f\|_{\tilde{T}_\Phi(\mathcal{X})} \leq C\|f\|_{T_\Phi(\mathcal{X})} \).

Proof. Let \( f \in T_\Phi(\mathcal{X}) \). By Theorem 3.1, there exist \( T_\Phi(\mathcal{X}) \)-atoms \( \{a_j\}_{j=1}^\infty \) and \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) such that (3.1) and (3.2) hold.

For any \( L \in \mathbb{N} \), set \( \sigma_L \equiv \sum_{j=1}^{L} |\lambda_j| \). Since \( \Phi \) is of upper type 1, by this together with \( \rho(t) = t^{-1}/\Phi^{-1}(t^{-1}) \) for all \( t \in (0, \infty) \), we obtain

\[
\sum_{j=1}^{\infty} \mu(B_j) \Phi \left( \frac{|\lambda_j|}{\sigma_L \mu(B_j) \rho(\mu(B_j))} \right) \geq \sum_{j=1}^{L} \mu(B_j) \Phi \left( \frac{1}{\sigma_L \mu(B_j) \rho(\mu(B_j))} \right) |\lambda_j| \geq 1,
\]

which implies that

\[
\sum_{j=1}^{L} |\lambda_j| \lesssim \Lambda(\{\lambda_j a_j\}_{j=1}^{\infty}) \lesssim \|f\|_{\tilde{T}_\Phi(\mathcal{X})}.
\]

Letting \( L \to \infty \), we further conclude that \( \sum_{j=1}^{\infty} |\lambda_j| \lesssim \|f\|_{\tilde{T}_\Phi(\mathcal{X})} \).

Since \( f \in T_\Phi(\mathcal{X}) \) and \( (T_\Phi(\mathcal{X}))^* = T_\Phi^\infty(\mathcal{X}) \), we see that

\[
f \in T_\Phi(\mathcal{X}) \subset ((T_\Phi(\mathcal{X}))^*)^* = (T_\Phi^\infty(\mathcal{X}))^*.
\]

Thus, \( f \in (T_\Phi^\infty(\mathcal{X}))^* \) and \( \|f\|_{(T_\Phi^\infty(\mathcal{X}))^*} \lesssim \|f\|_{T_\Phi(\mathcal{X})} \). Recall that for any \( \ell \in (T_\Phi^\infty(\mathcal{X}))^* \), its \( (T_\Phi^\infty(\mathcal{X}))^* \) norm is defined by

\[
\|\ell\|_{(T_\Phi^\infty(\mathcal{X}))^*} = \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})} \leq 1} |\ell(g)|.
\]

Observe also that \( a_j \in (T_\Phi^\infty(\mathcal{X}))^* \) for all \( j \in \mathbb{N} \). Now, from these observations, the monotone convergence theorem and Hölder’s inequality, it follows that

\[
\left\| f - \sum_{j=1}^{L} \lambda_j a_j \right\|_{(T_\Phi^\infty(\mathcal{X}))^*} = \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})} \leq 1} \int_{\mathcal{X} \times (0, \infty)} \left| f(x,t) - \sum_{j=1}^{L} \lambda_j a_j(x,t) \right| g(x,t) \frac{d\mu(x) dt}{t} \leq \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})} \leq 1} \int_{\mathcal{X} \times (0, \infty)} \sum_{j=L+1}^{\infty} |\lambda_j||a_j(x,t)g(x,t)| \frac{d\mu(x) dt}{t} = \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})} \leq 1} \sum_{j=L+1}^{\infty} |\lambda_j| \int_{B_j} |a_j(x,t)g(x,t)| \frac{d\mu(x) dt}{t}.
\]
\[
\lim_{L \to \infty} \sum_{j=L+1}^{\infty} |\lambda_j| \|a_j\|_{T^2_\Phi(\mathcal{X})} \|g\chi_{B_j}\|_{T^2_\Phi(\mathcal{X})} \leq \sum_{j=L+1}^{\infty} |\lambda_j| \to 0,
\]
as \(L \to \infty\). Thus, the series in (3.1) converges in \((T^\infty_\Phi(\mathcal{X}))^*\), which further implies that \(f \in \widetilde{T_\Phi(\mathcal{X})}\) and \(\|f\|_{T_\Phi(\mathcal{X})} \leq \sum_{j=1}^{\infty} |\lambda_j| \lesssim \|f\|_{T_\Phi(\mathcal{X})}\). This finishes the proof of Lemma 4.1.

**Lemma 4.2.** Let \(\Phi\) satisfy Assumption (\(\Phi\)). Then \(T^2_{\Phi,\beta}(\mathcal{X})\) is dense in \(\widetilde{T_\Phi(\mathcal{X})}\).

**Proof.** Since \(T_\Phi(\mathcal{X})\) is dense in \(\widetilde{T_\Phi(\mathcal{X})}\), to prove this lemma, it suffices to prove that \(T^2_{\Phi,\beta}(\mathcal{X})\) is dense in \(T_\Phi(\mathcal{X})\) in the norm \(\|\cdot\|_{T_\Phi(\mathcal{X})}\).

Fix \(x_0 \in \mathcal{X}\). For any \(g \in T_\Phi(\mathcal{X})\) and \(k \in \mathbb{N}\), let \(g_k \equiv g\chi_{O_k}\), where
\[
O_k \equiv \{(x, t) \in \mathcal{X} \times (0, \infty) : \text{dist}(x, x_0) < k, t \in (1/k, k)\}.
\]
By the dominated convergence theorem and the continuity of \(\Phi\), we conclude that for any \(\lambda > 0\),
\[
\lim_{k \to \infty} \int_{\mathcal{X}} \Phi \left( \frac{A(g - g_k)(x)}{\lambda} \right) d\mu(x) = \int_{\mathcal{X}} \Phi \left( \frac{A(g - g_k)(x)}{\lambda} \right) d\mu(x) = 0,
\]
which implies that \(\lim_{k \to \infty} \|g - g_k\|_{T_\Phi(\mathcal{X})} = 0\). Then, by Lemma 4.1, we see that
\[
\|g - g_k\|_{T_\Phi(\mathcal{X})} \lesssim \|g - g_k\|_{T_\Phi(\mathcal{X})} \to 0,
\]
as \(k \to \infty\), which completes the proof of Lemma 4.2.

**Lemma 4.3.** Let \(\Phi\) satisfy Assumption (\(\Phi\)). Then \((\widetilde{T_\Phi(\mathcal{X}))}^* = T^\infty_\Phi(\mathcal{X})\) via the pairing
\[
\langle f, g \rangle \to \int_{\mathcal{X} \times (0, \infty)} f(y, t)g(y, t) \frac{d\mu(y) dt}{t}
\]
for all \(f \in \widetilde{T_\Phi(\mathcal{X})}\) and \(g \in T^\infty_\Phi(\mathcal{X})\).

**Proof.** By Proposition 4.1 and the definition of \(\widetilde{T_\Phi(\mathcal{X})}\), we see that \((T_\Phi(\mathcal{X}))^* = T^\infty_\Phi(\mathcal{X})\) and \(T_\Phi(\mathcal{X}) \subset \widetilde{T_\Phi(\mathcal{X})}\), which further implies that \((\widetilde{T_\Phi(\mathcal{X}))}^* \subset T^\infty_\Phi(\mathcal{X})\).

Conversely, let \(g \in T^\infty_\Phi(\mathcal{X})\). Then for any \(f \in \widetilde{T_\Phi(\mathcal{X})}\), choose a sequence \(T_\Phi(\mathcal{X})\)-atoms \(\{a_j\}_{j=1}^{\infty}\) and \(\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}\) such that \(f = \sum_j \lambda_j a_j\) in \((T^\infty_\Phi(\mathcal{X}))^*\) and \(\sum_j |\lambda_j| \lesssim \|f\|_{T_\Phi(\mathcal{X})}\). Thus, by Hölder’s inequality, we obtain
\[
|\langle f, g \rangle| \leq \sum_j \int_{\mathcal{X} \times (0, \infty)} |a_j(x, t)g(x, t)| \frac{d\mu(x) dt}{t} \leq \|g\|_{T^\infty_\Phi(\mathcal{X})} \sum_j |\lambda_j| \lesssim \|g\|_{T^\infty_\Phi(\mathcal{X})} \|f\|_{T_\Phi(\mathcal{X})},
\]
which implies that \(g \in (\widetilde{T_\Phi(\mathcal{X}))}^*\), and hence completes the proof of Lemma 4.3.
Lemma 4.4. Let $\Phi$ satisfy Assumption (Φ). If $f \in \tilde{T}_\Phi(\mathcal{X})$, then

\begin{equation}
\|f\|_{\tilde{T}_\Phi(\mathcal{X})} = \sup_{g \in T^2_{2,\Phi}(\mathcal{X}), \|g\|_{T^2_{\Phi}(\mathcal{X})} \leq 1} \left| \int_{\mathcal{X} \times (0, \infty)} f(x,t)g(x,t) \frac{d\mu(x)}{t} \right|.
\end{equation}

Proof. Let $f \in \tilde{T}_\Phi(\mathcal{X})$. From Lemma 4.2, we deduce that

\begin{equation}
\|f\|_{\tilde{T}_\Phi(\mathcal{X})} = \sup_{\|g\|_{T^\infty_{\Phi}(\mathcal{X})} \leq 1} \left| \int_{\mathcal{X} \times (0, \infty)} f(x,t)g(x,t) \frac{d\mu(x)}{t} \right|.
\end{equation}

Thus, for any $\beta > 0$, there exists $g \in T^\infty_{\Phi}(\mathcal{X})$ such that $\|g\|_{T^2_{2,\Phi}(\mathcal{X})} \leq 1$ and

\begin{equation}
\left| \int_{\mathcal{X} \times (0, \infty)} f(x,t)g(x,t) \frac{d\mu(x)}{t} \right| \geq \|f\|_{\tilde{T}_\Phi(\mathcal{X})} - \frac{\beta}{2}.
\end{equation}

Observe here that $fg \in L^1(\mathcal{X} \times (0, \infty))$. Fix $x_0 \in \mathcal{X}$. Let

\[ O_k \equiv \{(x,t) \in \mathcal{X} \times (0, \infty) : \text{dist}(x,x_0) < k, 1/k < t < k\}. \]

Then there exists $k \in \mathbb{N}$ such that

\begin{equation}
\left| \int_{\mathcal{X} \times (0, \infty)} f(x,t)g(x,t)\chi_{O_k} \frac{d\mu(x)}{t} \right| \geq \|f\|_{\tilde{T}_\Phi(\mathcal{X})} - \beta.
\end{equation}

Obviously, $g\chi_{O_k} \in T^2_{2,\Phi}(\mathcal{X})$. Thus, (4.1) holds, which completes the proof of Lemma 4.4. \qed

The following lemma is a slight modification of [8, Lemma 4.2]; see also [22]. We omit the details here.

Lemma 4.5. Let $\Phi$ satisfy Assumption (Φ). Suppose that $\{f_k\}_{k=1}^\infty$ is a bounded family of functions in $\tilde{T}_\Phi(\mathcal{X})$. Then there exist $f \in \tilde{T}_\Phi(\mathcal{X})$ and a subsequence $\{f_{k_j}\}_{j=1}^\infty$ of $\{f_k\}_{k=1}^\infty$ such that for all $g \in T^2_{2,\Phi}(\mathcal{X})$,

\[ \lim_{j \to \infty} \int_{\mathcal{X} \times (0, \infty)} f_{k_j}(x,t)g(x,t) \frac{d\mu(x)}{t} = \int_{\mathcal{X} \times (0, \infty)} f(x,t)g(x,t) \frac{d\mu(x)}{t}. \]

Theorem 4.1. Let $\Phi$ satisfy Assumption (Φ). Then $(T^\infty_{\Phi,\chi}(\mathcal{X}))^*$, the dual space of the space $T^\infty_{\Phi,\chi}(\mathcal{X})$, coincides with $\tilde{T}_\Phi(\mathcal{X})$ in the following sense:

For any $g \in \tilde{T}_\Phi(\mathcal{X})$, define the linear function $\ell$ by setting, for all $f \in T^\infty_{\Phi}(\mathcal{X})$,

\begin{equation}
\ell(f) \equiv \int_{\mathcal{X} \times (0, \infty)} f(x,t)g(x,t) \frac{d\mu(x)}{t}.
\end{equation}

Then there exists a positive constant $C$, independent of $g$, such that

\[ \|\ell\|_{(T^\infty_{\Phi}(\mathcal{X}))^*} \leq C \|g\|_{\tilde{T}_\Phi(\mathcal{X})}. \]
Conversely, for any \( \ell \in (T^\infty_{\Phi}(\mathcal{X}))^\ast \), there exists \( g \in \widetilde{T}_\Phi(\mathcal{X}) \) such that (4.2) holds for all \( f \in T^\infty_{\Phi}(\mathcal{X}) \) and \( \|g\|_{\widetilde{T}_\Phi(\mathcal{X})} \leq C\|\ell\|_{(T^\infty_{\Phi}(\mathcal{X}))^\ast} \), where \( C \) is a positive constant independent of \( \ell \).

**Proof.** From Lemma 4.2, we infer that \( T^\infty_{\Phi,V}(\mathcal{X}) \subset T^\infty_{\Phi}(\mathcal{X}) = (\widetilde{T}_\Phi(\mathcal{X}))^\ast \), which further implies that \( \widetilde{T}_\Phi(\mathcal{X}) \subset (\widetilde{T}_\Phi(\mathcal{X}))^\ast \subset (T^\infty_{\Phi,V}(\mathcal{X}))^\ast \).

Conversely, let \( \ell \in (T^\infty_{\Phi,V}(\mathcal{X}))^\ast \). Notice that for any \( f \in T^2_{2,b}(\mathcal{X}) \), without loss of generality, we may assume that \( \text{supp} \ f \subset K \), where \( K \) is a bounded set in \( \mathcal{X} \times (0,\infty) \). Then we have \( \|f\|_{T^\infty_{\Phi,V}(\mathcal{X})} = \|f\|_{T^\infty_{\Phi}(\mathcal{X})} \leq C(K)\|f\|_{T^2_{2,b}(\mathcal{X})} \). Thus, \( \ell \) induces a bounded linear functional on \( T^2_{2,b}(\mathcal{X}) \). Let \( O_k \) be as in the proof of Lemma 4.4. By the Riesz representation theorem, there exists a unique \( g_k \in L^2(O_k) \) such that for all \( f \in L^2(O_k) \),

\[
\ell(f) = \int_{\mathcal{X} \times (0,\infty)} f(x,t)g_k(x,t) \frac{d\mu(x)dt}{t}.
\]

Obviously, \( g_{k+1}O_k = g_k \) for all \( k \in \mathbb{N} \). Let \( g = g_1\chi_{O_1} + \sum_{k=2}^{\infty} g_k\chi_{O_{k-1}} \). Then \( g \in L^2_{\text{loc}}(\mathcal{X} \times (0,\infty)) \) and for any \( f \in T^2_{2,b}(\mathcal{X}) \), we have

\[
\ell(f) = \int_{\mathcal{X} \times (0,\infty)} f(y,t)g(y,t) \frac{d\mu(y)dt}{t}.
\]

Set \( \tilde{g}_k \equiv g\chi_{O_k} \). Then for each \( k \in \mathbb{N} \), obviously, we see that \( \tilde{g}_k \in T^2_{2,b}(\mathcal{X}) \subset T_{\Phi}(\mathcal{X}) \subset \widetilde{T}_\Phi(\mathcal{X}) \). Then from Lemma 4.4, it follows that

\[
\|\tilde{g}_k\|_{\widetilde{T}_\Phi(\mathcal{X})} = \sup_{f \in T^2_{2,b}(\mathcal{X}),\|f\|_{T^\infty_{\Phi}(\mathcal{X})} \leq 1} \left| \int_{\mathcal{X} \times (0,\infty)} f(x,t)g(x,t)\chi_{O_k}(x,t) \frac{d\mu(x)dt}{t} \right| = \sup_{f \in T^2_{2,b}(\mathcal{X}),\|f\|_{T^\infty_{\Phi}(\mathcal{X})} \leq 1} |\ell(f\chi_{O_k})| \leq \sup_{f \in T^2_{2,b}(\mathcal{X}),\|f\|_{T^\infty_{\Phi}(\mathcal{X})} \leq 1} \|\ell\|_{(T^\infty_{\Phi,V}(\mathcal{X}))^\ast} \|f\|_{T^\infty_{\Phi}(\mathcal{X})} \leq \|\ell\|_{(T^\infty_{\Phi,V}(\mathcal{X}))^\ast}.
\]

Thus, by Lemma 4.5, there exist \( \tilde{g} \in \widetilde{T}_\Phi(\mathcal{X}) \) and \( \{\tilde{g}_k\}_{j=1}^{\infty} \subset \{\tilde{g}_k\}_{k=1}^{\infty} \) such that for all \( f \in T^2_{2,b}(\mathcal{X}) \),

\[
\lim_{j \to \infty} \int_{\mathcal{X} \times (0,\infty)} f(x,t)\tilde{g}_k(x,t) \frac{d\mu(x)dt}{t} = \int_{\mathcal{X} \times (0,\infty)} f(x,t)\tilde{g}(x,t) \frac{d\mu(x)dt}{t}.
\]

On the other hand, notice that for sufficient large \( k_j \), we have

\[
\ell(f) = \int_{\mathcal{X} \times (0,\infty)} f(x,t)g(x,t) \frac{d\mu(x)dt}{t} = \int_{\mathcal{X} \times (0,\infty)} f(x,t)\tilde{g}_k(x,t) \frac{d\mu(x)dt}{t} = \int_{\mathcal{X} \times (0,\infty)} f(x,t)\tilde{g}(x,t) \frac{d\mu(x)dt}{t},
\]

which implies that \( g = \tilde{g} \) almost everywhere, and hence \( g \in \widetilde{T}_\Phi(\mathcal{X}) \). By a density argument, we conclude that (4.2) also holds for \( g \) and all \( f \in T^\infty_{\Phi,V}(\mathcal{X}) \), which completes the proof of Theorem 4.1. \( \square \)
**Definition 4.2.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, $\Phi$ satisfy Assumption $(\Phi)$, $M \in \mathbb{N}$, $M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})$ and $\epsilon \in (n(1/p_\Phi - 1/p_\Phi^+), \infty)$. An element $f \in (\text{BMO}_{\rho,L}(\mathcal{X}))^*$ is said to be in the space $H^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ if there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $(\Phi, M, \epsilon)_L$-molecules $\{\alpha_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\text{BMO}_{\rho,L}(\mathcal{X}))^*$ and

$$
\Lambda(\{\lambda_j \alpha_j\}_{j=1}^\infty) \equiv \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty \mu(B_j) \Phi \left( \frac{|\lambda_j|}{\lambda \mu(B_j) p(\mu(B_j))} \right) \leq 1 \right\} < \infty,
$$

where for each $j$, $\alpha_j$ is adapted to the ball $B_j$.

If $f \in H^{M,\epsilon}_{\Phi,L}(\mathcal{X})$, then its norm is defined by $\|f\|_{H^{M,\epsilon}_{\Phi,L}(\mathcal{X})} \equiv \inf \{\Lambda(\{\lambda_j \alpha_j\}_{j=1}^\infty)\}$, where the infimum is taken over all the possible decompositions of $f$ as above.

By [21, Theorem 5.1], we see that for all $M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})$ and $\epsilon \in (n(1/p_\Phi - 1/p_\Phi^+), \infty)$, the spaces $H^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ and $H^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ coincide with equivalent norms.

Let us introduce the Banach completion of the space $H^{M,\epsilon}_{\Phi,L}(\mathcal{X})$.

**Definition 4.3.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, $\Phi$ satisfy Assumption $(\Phi)$, $\epsilon \in (n(1/p_\Phi - 1/p_\Phi^+), \infty)$ and $M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})$. The space $B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ is defined to be the space of all $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\text{BMO}_{\rho,L}(\mathcal{X}))^*$, where $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ with $\sum_{j=1}^\infty |\lambda_j| < \infty$ and $\{\alpha_j\}_{j=1}^\infty$ are $(\Phi, M, \epsilon)_L$-molecules. If $f \in B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$, define $\|f\|_{B^{M,\epsilon}_{\Phi,L}(\mathcal{X})} \equiv \inf \{\sum_{j=1}^\infty |\lambda_j|\}$, where the infimum is taken over all the possible decompositions of $f$ as above.

By [16, Lemma 3.1], we know that $B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ is a Banach space. Moreover, from Definition 4.2, it is easy to deduce that $H^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ is dense in $B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$. More precisely, we have the following lemma.

**Lemma 4.6.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, $\Phi$ satisfy Assumption $(\Phi)$, $\epsilon \in (n(1/p_\Phi - 1/p_\Phi^+), \infty)$ and $M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})$. Then

i) $H^{M,\epsilon}_{\Phi,L}(\mathcal{X}) \subset B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ and the inclusion is continuous.

ii) For any $\epsilon_1 \in (n(1/p_\Phi - 1/p_\Phi^+), \infty)$ and $M_1 > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})$, the spaces $B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ and $B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$ coincide with equivalent norms.

**Proof.** From Definition 4.3 and the molecular characterization of $H^{M,\epsilon}_{\Phi,L}(\mathcal{X})$, it is easy to deduce i).

Let us prove ii). By symmetry, it suffices to show that $B^{M,\epsilon}_{\Phi,L}(\mathcal{X}) \subset B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$. Let $f \in B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$. By Definition 4.3, there exist $(\Phi, M, \epsilon)_L$-molecules $\{\alpha_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\text{BMO}_{\rho,L}(\mathcal{X}))^*$ and $\sum_{j=1}^\infty |\lambda_j| \lesssim \|f\|_{B^{M,\epsilon}_{\Phi,L}(\mathcal{X})}$. By i), for each $j \in \mathbb{N}$, we see that $\alpha_j \in H^{M,\epsilon}_{\Phi,L}(\mathcal{X}) \subset B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$ and $\|\alpha_j\|_{B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})} \lesssim \|\alpha_j\|_{H^{M,\epsilon}_{\Phi,L}(\mathcal{X})} \lesssim 1$. Since $B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$ is a Banach space, we see that $f \in B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$ and $\|f\|_{B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})} \lesssim \sum_{j=1}^\infty |\lambda_j| \|\alpha_j\|_{B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})} \lesssim \|f\|_{B^{M,\epsilon}_{\Phi,L}(\mathcal{X})}$. Thus, $B^{M,\epsilon}_{\Phi,L}(\mathcal{X}) \subset B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$, which completes the proof of Lemma 4.6. \qed
Since the spaces $B_{\Phi,L}^M(\mathcal{X})$ coincide for all $\epsilon \in (n(1/p_\Phi - 1/p_\Phi^*), \infty)$ and $M > \frac{2}{3_n} - \frac{1}{2}$, in what follows, we denote $B_{\Phi,L}^M, B_{\Phi,L}^M(\mathcal{X})$ simple by $B_{\Phi,L}(\mathcal{X})$.

**Lemma 4.7.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, and $\Phi$ satisfy Assumption $(\Phi)$. Then $(B_{\Phi,L}(\mathcal{X}))^* = \text{BMO}_{p,L^*}(\mathcal{X})$.

**Proof.** Since $(H_{\Phi,L}(\mathcal{X}))^* = \text{BMO}_{p,L^*}(\mathcal{X})$ and $H_{\Phi,L}(\mathcal{X}) \subset B_{\Phi,L}(\mathcal{X})$, by duality, we conclude that $(B_{\Phi,L}(\mathcal{X}))^* \subset \text{BMO}_{p,L^*}(\mathcal{X})$.

Conversely, let $\epsilon \in (n(1/p_\Phi - 1/p_\Phi^*), \infty)$, $M > \frac{2}{3_n} - \frac{1}{2}$ and $f \in \text{BMO}_{p,L^*}(\mathcal{X})$. For any $g \in B_{\Phi,L}(\mathcal{X})$, by Definition 4.3, there exist $(\Phi, M_\epsilon$, $\epsilon)$-$L$-molecules $\{\alpha_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $g = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\text{BMO}_{p,L^*}(\mathcal{X}))^*$ and $\sum_{j=1}^\infty |\lambda_j| \lesssim \|g\|_{B_{\Phi,L}(\mathcal{X})}$. Thus,

$$\langle f, g \rangle \lesssim \sum_{j=1}^\infty |\lambda_j| \|f, \alpha_j\| \lesssim \sum_{j=1}^\infty |\lambda_j| \|f\|_{\text{BMO}_{p,L^*}(\mathcal{X})} \|\alpha_j\|_{H_{\Phi,L}(\mathcal{X})} \lesssim \|f\|_{B_{\Phi,L}(\mathcal{X})} \|g\|_{B_{\Phi,L}(\mathcal{X})},$$

which implies that $f \in (B_{\Phi,L}(\mathcal{X}))^*$, and hence completes the proof of Lemma 4.7.

Let $M \in \mathbb{N}$. For all $F \in L^2(\mathcal{X} \times (0, \infty))$ with bounded support, define

$$(4.3) \quad \pi_{L,M} F = C(M) \int_0^\infty (t^2 L)^M e^{-t^2 L} F(\cdot, t) \, \frac{dt}{t},$$

where $C(M)$ is as in (3.5).

**Proposition 4.2.** Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, $\Phi$ satisfy Assumption $(\Phi)$ and $M \in \mathbb{N}$. Then the operator $\pi_{L,M}$, initially defined on $T_{2,b}^2(\mathcal{X})$, extends to a bounded linear operator

i) from $T_{2,b}^2(\mathcal{X})$ to $L^2(\mathcal{X})$;

ii) from $T_{\Phi}(\mathcal{X})$ to $H_{\Phi,L}(\mathcal{X})$, if $M > \frac{2}{3_n} - \frac{1}{2}$;

iii) from $\widetilde{T}_{\Phi}(\mathcal{X})$ to $B_{\Phi,L}(\mathcal{X})$, if $M > \frac{2}{3_n} - \frac{1}{2}$;

iv) from $T_{\Phi}^\infty(\mathcal{X})$ to VMO$_{p,L^*}(\mathcal{X})$.

**Proof.** i) and ii) were established in [2, Proposition 3.6] (see also [21, Lemma 3.1]).

By Lemma 4.2, we know that $T_{2,b}^2(\mathcal{X})$ is dense in $T_{\Phi}(\mathcal{X})$. Let $f \in T_{2,b}^2(\mathcal{X})$. From ii) and Lemma 4.6, we deduce that $\pi_{L,M} f \in H_{\Phi,L}(\mathcal{X}) \subset B_{\Phi,L}(\mathcal{X})$. Moreover, by Definition 4.1, there exist $T_{\Phi}(\mathcal{X})$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $(T_{\Phi}^\infty(\mathcal{X}))^*$ and $\sum_{j} |\lambda_j| \lesssim \|f\|_{T_{\Phi}(\mathcal{X})}$. In addition, for any $g \in \text{BMO}_{p,L^*}(\mathcal{X})$, we have $(t^2 L^*)^M e^{-t^2 L^*} g \in T_{\Phi}^\infty(\mathcal{X})$. Thus, by $(T_{\Phi}(\mathcal{X}))^* = T_{\Phi}^\infty(\mathcal{X})$, we conclude that

$$\langle \pi_{L,M} f, g \rangle = C(M) \int_{\mathcal{X} \times (0, \infty)} f(x, t) \sum_{j}^{\infty} \lambda_j C(M) a_j(x, t) (t^2 L^*)^M e^{-t^2 L^*} g(x) \, \frac{dt}{t} \, \frac{dx}{t}$$

$$= \sum_{j=1}^\infty \lambda_j C(M) \int_{\mathcal{X} \times (0, \infty)} a_j(x, t) (t^2 L^*)^M e^{-t^2 L^*} g(x) \, \frac{dt}{t} \, \frac{dx}{t}$$
Vanishing Mean Oscillation Spaces Associated with Operators

\[
\sum_{j=1}^{\infty} \lambda_j \langle \pi_{L,M}(a_j), g \rangle,
\]

which implies that \( \pi_{L,M}(f) = \sum_{j=1}^{\infty} \lambda_j \pi_{L,M}(a_j) \) in \((\text{BMO}_{\rho,L^*}(\mathcal{X}))^*\). By ii), we further conclude that

\[
\|\pi_{L,M}(f)\|_{B_{\Phi,L}(\mathcal{X})} \leq \sum_{j=1}^{\infty} |\lambda_j| \|\pi_{L,M}(a_j)\|_{B_{\Phi,L}(\mathcal{X})} \lesssim \sum_{j=1}^{\infty} |\lambda_j| \|\pi_{L,M}(a_j)\|_{H_{\Phi,L}(\mathcal{X})} \lesssim \|f\|_{T_\Phi(\mathcal{X})}.
\]

Since \( T_{2,b}^2(\mathcal{X}) \) is dense in \( T_\Phi(\mathcal{X}) \), we see that \( \pi_{L,M} \) extends to a bounded linear operator from \( T_{2,b}^2(\mathcal{X}) \) to \( B_{\Phi,L}(\mathcal{X}) \), which completes the proof of iii).

Let us now prove iv). From Lemma 3.3, we infer that \( f \) and \( L \) maps \( T_{2,b}^2(\mathcal{X}) \) continuously into \( \text{VMO}_{\rho,L}(\mathcal{X}) \).

Let \( f \in T_{2,b}^2(\mathcal{X}) \). By i), we see that \( \pi_{L,M}f \in L^2(\mathcal{X}) \). Notice that (3.3) and (3.4) with \( L \) and \( L^* \) exchanged implies that \( L^2(\mathcal{X}) \subset \mathcal{M}_{\Phi,L}^1(\mathcal{X}) \), when \( M_1 \in \mathbb{N} \) and \( M_1 > \frac{2}{2} (\frac{1}{\rho_\Phi} - \frac{1}{2}) \).

Thus, \( \pi_{L,M}f \in \mathcal{M}_{\Phi,L}^{M_1}(\mathcal{X}) \). To show \( \pi_{L,M}f \in \text{VMO}_{\rho,L}(\mathcal{X}) \), by Theorem 3.4, we still need to prove iv) that \( (t^2 L)^{M_1} e^{-t^2 \pi_{L,M}} f \in T_{2,b}^{\infty}(\mathcal{X}) \).

For any ball \( B \equiv B(x_B, r_B) \), let \( V_0(B) \equiv \bar{B} \) and \( V_k(B) \equiv (2^k B) \setminus (2^{k-1} B) \) for any \( k \in \mathbb{N} \). For all \( k \in \mathbb{Z}_+ \), let \( f_k \equiv f \chi_{V_k(B)} \). Thus, for \( k \in \{0, 1, 2\} \), by Lemma 2.2 and i), we see that

\[
\left[ \int_B \left( (t^2 L)^{M_1} e^{-t^2 \pi_{L,M}} f_k(x) \right)^2 \frac{d\mu(x) \, dt}{t} \right]^{1/2} \lesssim \|\pi_{L,M} f_k \|_{L^2(\mathcal{X})} \lesssim \|f_k\|_{T_2^2(\mathcal{X})}.
\]

For \( k \geq 3 \), let \( V_{k,1}(B) \equiv (2^k B) \setminus (2^{k-2} B \times (0, \infty)) \) and \( V_{k,2}(B) \equiv V_k(B) \setminus V_{k,1}(B) \). We further write \( f_k = f_k \chi_{V_{k,1}(B)} + f_k \chi_{V_{k,2}(B)} \equiv f_{k,1} + f_{k,2} \). From Minkowski’s inequality, Lemma 2.3 and Hölder’s inequality, we deduce that

\[
\left[ \int_B \left( (t^2 L)^{M_1} e^{-t^2 \pi_{L,M}} f_{k,2}(x) \right)^2 \frac{d\mu(x) \, dt}{t} \right]^{1/2} \sum_{j=1}^{\infty} \int_{B_{2^{-j}r_B}} \int_{B_{2^{-j}r_B}} \left[ \int_{2^{k-2} B} \left( (s^2 + t^2)^{M_1} e^{-(s^2 + t^2) L}(f_{k,2}(\cdot, s))(x) \right)^2 \frac{d\mu(x) \, dt}{t} \right]^{1/2} ds \, ds
\]

\[
\lesssim 2^{-2k M_1} \int_{2^{k-2} r_B} \|f_{k,2}(\cdot, s)\|_{L^2(\mathcal{X})}^2 \frac{ds}{s} \lesssim 2^{-2k M_1} \|f_{k,2}\|_{T_2^2(\mathcal{X})}.
\]

Similarly, we have
\[
\left[ \int_B \left| (t^2L)^{M_1} e^{-t^2L} \pi_{L,M} f_k(x) \right|^2 \frac{d\mu(x)}{t} \right]^{1/2} \leq 2^{-2kM_1} ||f_k||_{L^2(X)}.
\]

Let \( \tilde{p}_\Phi \in (0, \tilde{p}_\Phi) \) such that \( M > \frac{\tilde{p}_\Phi}{2} \left( \frac{1}{\tilde{p}_\Phi} - \frac{1}{2} \right) \) and \( M_1 > \frac{\tilde{p}_\Phi}{2} \left( \frac{1}{\tilde{p}_\Phi} - \frac{1}{2} \right) \). Combining the above estimates, since \( \Phi \) is of lower type \( \tilde{p}_\Phi \), we finally conclude that
\[
\frac{1}{\rho(\mu(B))|\mu(B)|^{1/2}} \left[ \int_B \left| (t^2L)^{M_1} e^{-t^2L} \pi_{L,M} f_k(x) \right|^2 \frac{d\mu(x)}{t} \right]^{1/2} \leq 2^{-2kM_1} ||f_k||_{L^2(X)}.
\]

Since \( f \in \mathcal{T}_{\Phi,2}(\mathcal{X}) \subset \mathcal{T}_{\Phi,1}(\mathcal{X}) \), we have
\[
\frac{1}{\rho(\mu(2kB))|\mu(2kB)|^{1/2}} ||f_k||_{L^2(X)} \lesssim ||f||_{\mathcal{T}_{\Phi,1}(\mathcal{X})}
\]
and, for all fixed \( k \in \mathbb{N} \),
\[
\lim_{c \to 0} \sup_{B : r_B \leq c} \frac{||f_k||_{L^2(X)}}{\rho(\mu(2kB))|\mu(2kB)|^{1/2}} = \lim_{c \to \infty} \sup_{B : r_B \geq c} \frac{||f_k||_{L^2(X)}}{\rho(\mu(2kB))|\mu(2kB)|^{1/2}} = \lim_{c \to \infty} \sup_{B : |B| \leq c} \frac{||f_k||_{L^2(X)}}{\rho(\mu(2kB))|\mu(2kB)|^{1/2}} = 0.
\]

Thus, by the dominated convergence theorem for series, we further conclude that
\[
\eta_1((t^2L)^{M_1} e^{-t^2L} \pi_{L,M} f) = \lim_{c \to 0} \sup_{B : r_B \leq c} \frac{1}{\rho(\mu(2kB))|\mu(2kB)|^{1/2}} \left[ \int_B \left| (t^2L)^{M_1} e^{-t^2L} \pi_{L,M} f(x) \right|^2 \frac{d\mu(x)}{t} \right]^{1/2} \leq \sum_{k=0}^{\infty} 2^{-2k[M_1 - \frac{\tilde{p}_\Phi}{2} \left( \frac{1}{\tilde{p}_\Phi} - \frac{1}{2} \right)]} \lim_{c \to \infty} \sup_{B : r_B \leq c} \frac{||f_k||_{L^2(X)}}{\rho(\mu(2kB))|\mu(2kB)|^{1/2}} = 0.
\]

Similarly, we have \( \eta_2((t^2L)^{M_1} e^{-t^2L} \pi_{L,M} f) = \eta_3((t^2L)^{M_1} e^{-t^2L} \pi_{L,M} f) = 0 \), and hence \( (t^2L)^{M_1} e^{-t^2L} \pi_{L,M} f \in \mathcal{T}_{\Phi,1}(\mathcal{X}) \), which completes the proof of Proposition 4.2.
Lemma 4.8. Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, and $\Phi$ satisfy Assumption $(\Phi)$. Then $\text{VMO}_{p,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $\text{VMO}_{p,L}(\mathcal{X})$.

Proof. Let $f \in \text{VMO}_{p,L}(\mathcal{X})$ and $M > \frac{n}{2}(\frac{1}{p_\Phi} - \frac{1}{2})$. Then by Theorem 3.4, we have $h \equiv (i^2L)^M e^{-i^2L} f \in T^\infty_{\Phi,L}(\mathcal{X})$. Similarly to the proof of Proposition 4.2, by Lemma 3.3, there exist $\{h_k\}_{k \in \mathbb{N}} \subset T^2_{\Phi,L}(\mathcal{X}) \subset T^\infty_{\Phi,L}(\mathcal{X})$ such that $\|h - h_k\|_{T^\infty_{\Phi,L}(\mathcal{X})} \to 0$, as $k \to \infty$. Thus, by i) and iv) of Proposition 4.2, we see that $\pi_{L,1} h_k \in L^2(\mathcal{X}) \cap \text{VMO}_{p,L}(\mathcal{X})$ and

\[
\|\pi_{L,1}(h - h_k)\|_{\text{BMO}_{p,L}(\mathcal{X})} \lesssim \|h - h_k\|_{T^\infty_{\Phi,L}(\mathcal{X})} \to 0,
\]
as $k \to \infty$.

Let $\alpha$ be a $(\Phi, M, \epsilon)_L$-molecule. Then by the definition of $H_{\Phi,L}(\mathcal{X})$, we know that $e^{-\epsilon L}\alpha \in T_{\Phi,L}(\mathcal{X})$, which, together with Lemma 3.2, the fact that $(T_{\Phi,L}(\mathcal{X}))^* = T^\infty_{\Phi,L}(\mathcal{X})$ and $(H_{\Phi,L}(\mathcal{X}))^* = \text{BMO}_{p,L}(\mathcal{X})$, further implies that

\[
\langle f, \alpha \rangle = C(M) \int_{\mathcal{X} \times (0, \infty)} (i^2L)^M e^{-i^2L} f(x) t^{i^2L} e^{-i^2L} \alpha(x) \frac{d\mu(x)}{t} dt \\
= \lim_{k \to \infty} C(M) \int_{\mathcal{X} \times (0, \infty)} h_k(x) t^{i^2L} e^{-i^2L} \alpha(x) \frac{d\mu(x)}{t} dt \\
= \frac{C(M)}{C_1} \lim_{k \to \infty} \int_{\mathcal{X}} (\pi_{L,1} h_k(x)) \alpha(x) d\mu(x) = \frac{C(M)}{C_1} \langle \pi_{L,1} h, \alpha \rangle.
\]

Since the set of finite combinations of molecules is dense in $H_{\Phi,L}(\mathcal{X})$, we then see that $f = \frac{C(M)}{C_1} \pi_{L,1} h$ in $\text{BMO}_{p,L}(\mathcal{X})$.

Now, for each $k \in \mathbb{N}$, let $f_k \equiv \frac{C(M)}{C_1} \pi_{L,1} h_k$. Then $f_k \in \text{VMO}_{p,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ and, moreover, by (4.4), we have $\|f - f_k\|_{\text{BMO}_{p,L}(\mathcal{X})} \to 0$, as $k \to \infty$, which completes the proof of Lemma 4.8.

In what follows, the symbol $\langle \cdot, \cdot \rangle$ in the following theorem means the duality between the space $\text{BMO}_{p,L}(\mathcal{X})$ and the space $B_{\Phi,L}^*(\mathcal{X})$ in the sense of Lemma 4.7 with $L$ and $L^*$ exchanged.

Theorem 4.2. Let $L$ satisfy Assumptions $(L)_1$ and $(L)_2$, and $\Phi$ satisfy Assumption $(\Phi)$. Then the dual space of $\text{VMO}_{p,L}(\mathcal{X})$, $(\text{VMO}_{p,L}(\mathcal{X}))^*$, coincides with the space $B_{\Phi,L}^*(\mathcal{X})$ in the following sense:

For any $g \in B_{\Phi,L}^*(\mathcal{X})$, define the linear functional $\ell$ by setting, for all $f \in \text{VMO}_{p,L}(\mathcal{X})$,

\[
\ell(f) \equiv \langle f, g \rangle.
\]

Then there exists a positive constant $C$ independent of $g$ such that

\[
\|\ell\|_{(\text{VMO}_{p,L}(\mathcal{X}))^*} \leq C \|g\|_{B_{\Phi,L}^*(\mathcal{X})}.
\]

Conversely, for any $\ell \in (\text{VMO}_{p,L}(\mathcal{X}))^*$, there exist $g \in B_{\Phi,L}^*(\mathcal{X})$ such that (4.5) holds and a positive constant $C$, independent of $\ell$, such that

\[
\|g\|_{B_{\Phi,L}^*(\mathcal{X})} \leq C \|\ell\|_{(\text{VMO}_{p,L}(\mathcal{X}))^*}.
\]
Proof. By Lemma 4.7, we have $(B_{\Phi,L^*}(\mathcal{X}))^* = \text{BMO}_{\rho,L}(\mathcal{X})$. Definition 3.3 implies that $\text{VMO}_{\rho,L}(\mathcal{X}) \subset \text{BMO}_{\rho,L}(\mathcal{X})$, which further implies that $B_{\Phi,L^*}(\mathcal{X}) \subset (\text{VMO}_{\rho,L}(\mathcal{X}))^*$.

Conversely, let $M > \frac{2}{9}\left(\frac{1}{\rho} - \frac{1}{\ell}\right)$ and $\ell \in (\text{VMO}_{\rho,L}(\mathcal{X}))^*$. By Proposition 4.2, $\pi_{L,1}$ is bounded from $T_{\Phi,v}^{\infty}(\mathcal{X})$ to $\text{VMO}_{\rho,L}(\mathcal{X})$, which implies that $\ell \circ \pi_{L,1}$ is a bounded linear functional on $T_{\Phi,v}^{\infty}(\mathcal{X})$. Thus, by Theorem 4.1, there exists $g \in \widetilde{T}_\Phi(\mathcal{X})$ such that for all $g \in T_{\Phi,v}^{\infty}(\mathcal{X})$, $\ell \circ \pi_{L,1}(f) = \langle f, g \rangle$.

Now, suppose that $f \in \text{VMO}_{\rho,L}(\mathcal{X}) \cap L^2(\mathcal{X})$. By Theorem 3.4, we conclude that $(t^2L)^M e^{-t^2L}f \in T_{\Phi,v}^{\infty}(\mathcal{X})$. Moreover, from the proof of Lemma 4.8, we deduce that $f = \frac{C(M)}{C_1} \pi_{L,1}((t^2L)^M e^{-t^2L}f)$ in $\text{BMO}_{\rho,L}(\mathcal{X})$. Thus

\begin{equation}
\ell(f) = \frac{C(M)}{C_1} \ell \circ \pi_{L,1}((t^2L)^M e^{-t^2L}f) = \frac{C(M)}{C_1} \int_{\mathcal{X} \times (0,\infty)} (t^2L)^M e^{-t^2L}f(x,t) \frac{d\mu(x)dt}{t}.
\end{equation}

By Lemma 4.2, $T_{2,b}^2(\mathcal{X})$ is dense in $\widetilde{T}_\Phi(\mathcal{X})$. Since $g \in \widetilde{T}_\Phi(\mathcal{X})$, we choose $\{g_k\}_{k \in \mathbb{N}} \subset T_{2,b}^2(\mathcal{X})$ such that $g_k \to g$ in $\widetilde{T}_\Phi(\mathcal{X})$. By iii) of Proposition 4.2, we see that $\pi_{L^*,M}(g), \pi_{L^*,M}(g_k) \in B_{\Phi,L^*}(\mathcal{X})$ and

$$\|\pi_{L^*,M}(g - g_k)\|_{B_{\Phi,L^*}(\mathcal{X})} \lesssim \|g - g_k\|_{\widetilde{T}_\Phi(\mathcal{X})} \to 0,$$

as $k \to \infty$. This, together with (4.6), Theorem 4.1, the dominated convergence theorem and Lemma 4.7, implies that

\begin{equation}
\ell(f) = \frac{C(M)}{C_1} \lim_{k \to \infty} \int_{\mathcal{X} \times (0,\infty)} (t^2L)^M e^{-t^2L}f(x,t) \frac{d\mu(x)dt}{t} = \frac{C(M)}{C_1} \lim_{k \to \infty} \int_{\mathcal{X}} f(x) \int_0^{\infty} (t^2L^*)^Me^{-t^2L^*}(g_k(\cdot,t))(x) \frac{dt}{t} d\mu(x) = \frac{1}{C_1} \lim_{k \to \infty} \langle f, \pi_{L^*,M}(g_k) \rangle = \frac{1}{C_1} \langle f, \pi_{L^*,M}(g) \rangle.
\end{equation}

Since $\text{VMO}_{\rho,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $\text{VMO}_{\rho,L}(\mathcal{X})$, we finally conclude that (4.7) holds for all $f \in \text{VMO}_{\rho,L}(\mathcal{X})$, and $\|f\|_{(\text{VMO}_{\rho,L}(\mathcal{X}))^*} = \frac{1}{C_1} \|\pi_{L^*,M}f\|_{B_{\Phi,L^*}(\mathcal{X})}$. In this sense, we have $(\text{VMO}_{\rho,L}(\mathcal{X}))^* \subset B_{\Phi,L^*}(\mathcal{X})$, which completes the proof of Theorem 4.2.

**Acknowledgements.** The authors would like to thank Dr. Bui The Anh and Dr. Renjin Jiang for some helpful discussions on the subject of this paper and, especially, for Dr. Bui The Anh to give us his preprint [1]. The authors would also like to thank the referee for his/her several valuable remarks which made this article more readable.

**References**

[1] B. T. Anh, Functions of vanishing mean oscillation associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, Submitted.
Vanishing Mean Oscillation Spaces Associated with Operators

[2] B. T. Anh and J. Li, Orlicz-Hardy Spaces associated to operators satisfying bounded $H_\infty$ functional calculus and Davies-Gaffney estimates, J. Math. Anal. Appl. 373 (2011), 485-501.

[3] T. Aoki, Locally bounded linear topological space, Proc. Imp. Acad. Tokyo 18 (1942), 588-594.

[4] P. Auscher, X. T. Duong and A. McIntosh, Boundedness of Banach space valued singular integral operators and Hardy spaces, Unpublished manuscript, 2005.

[5] G. Bourdaud, Remarques sur certains sous-espaces de BMO($\mathbb{R}^n$) et de bmo($\mathbb{R}^n$), Ann. Inst. Fourier (Grenoble) 52 (2002), 1187-1218.

[6] R. R. Coifman, Y. Meyer and E. M. Stein, Some new function spaces and their applications to harmonic analysis, J. Funct. Anal. 62 (1985), 304-335.

[7] R. R. Coifman and G. Weiss, Analyse harmonique non-commutative sur certains espaces homogènes, Lecture Notes in Math. 242, Springer, Berlin, 1971.

[8] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977), 569-645.

[9] D. Deng, X. T. Duong, L. Song, C. Tan and L. Yan, Functions of vanishing mean oscillation associated with operators and applications, Michigan Math. J. 56 (2008), 529-550.

[10] X. T. Duong and J. Li, Hardy spaces associated to operators satisfying bounded $H_\infty$ functional calculus and Davies-Gaffney estimates, Preprint.

[11] X. T. Duong, J. Xiao and L. Yan, Old and new Morrey spaces with heat kernel bounds, J. Fourier Anal. Appl. 12 (2007), 87-111.

[12] X. T. Duong and L. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, J. Amer. Math. Soc. 18 (2005), 943-973.

[13] X. T. Duong and L. Yan, New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications, Comm. Pure Appl. Math. 58 (2005), 1375-1420.

[14] C. Fefferman and E. M. Stein, $H^p$ spaces of several variables, Acta Math. 129 (1972), 137-193.

[15] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, Mem. Amer. Math. Soc. (to appear).

[16] S. Hofmann and S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, Math. Ann. 344 (2009), 37-116.

[17] S. Hofmann and S. Mayboroda, Correction to “Hardy and BMO spaces associated to divergence form elliptic operators”, arXiv: 0907.0129.

[18] S. Janson, Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation, Duke Math. J. 47 (1980), 959-982.

[19] R. Jiang and D. Yang, New Orlicz-Hardy spaces associated with divergence form elliptic operators, J. Funct. Anal. 258 (2010), 1167-1224.

[20] R. Jiang and D. Yang, Generalized vanishing mean oscillation spaces associated with divergence form elliptic operators, Integral Equations Operator Theory 67 (2010), 123-149.
[21] R. Jiang and D. Yang, Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates, Commun. Contemp. Math. 13 (2011), 331-373.
[22] R. Jiang and D. Yang, Predual spaces of Banach completions of Orlicz-Hardy spaces associated with operators, J. Fourier Anal. Appl. 17 (2011), 1-35.
[23] R. Jiang, D. Yang and Y. Zhou, Orlicz-Hardy spaces associated with operators, Sci. China Ser. A 52 (2009), 1042-1080.
[24] F. John and L. Nirenberg, On functions of bounded mean oscillation, Commun. Pure Appl. Math. 14 (1961), 415-426.
[25] A. McIntosh, Operators which have an $H_\infty$ functional calculus, Miniconference on operator theory and partial differential equations (Canberra), Proc. Centre Math. Anal. Austral. Nat. Univ., Vol. 14, Austral. Nat. Univ., Canberra, 1986, pp. 210-231.
[26] S. Rolewicz, On a certain class of linear metric spaces, Bull. Acad. Polon. Sci. 5 (1957), 471-473.
[27] E. Russ, The atomic decomposition for tent spaces on spaces of homogeneous type, CMA/AMSI Research Symposium “Asymptotic Geometric Analysis, Harmonic Analysis, and Related Topics”, 125-135, Proc. Centre Math. Appl. Austral. Nat. Univ., 42, Austral. Nat. Univ., Canberra, 2007.
[28] D. Sarason, Functions of vanishing mean oscillation, Trans. Amer. Math. Soc. 207 (1975), 391-405.
[29] L. Song and M. Xu, VMO spaces associated with divergence form elliptic operators, Math. Z. (DOI 10.1007/s00209-010-0774-6).
[30] B. E. Viviani, An atomic decomposition of the predual of BMO($\rho$), Rev. Mat. Iberoamericana 3 (1987), 401-425.

Yiyu Liang, Dachun Yang and Wen Yuan (Corresponding author)

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People’s Republic of China

E-mails: yyliang@mail.bnu.edu.cn (Y. Liang)
dcyang@bnu.edu.cn (D. Yang)
wenyuan@bnu.edu.cn (W. Yuan)