Jet production from the perturbative QCD pomeron with a running coupling constant

Mikhail Braun \( a,b \)^\*, Gian Paolo Vacca \( b \)

\( a \) Department of Particles, University of Santiago de Compostela
\( b \) Department of Physics, University of Bologna
\( b \) Istituto Nazionale di Fisica Nucleare - Sezione di Bologna.

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Abstract.

An analysis of minijet production from the hard pomeron with a running coupling constant is performed. Two supercritical pomerons found in the numerical study are taken into account. The calculated inclusive jet production rate is finite at small \( k_\perp \) and behaves like \( 1/k_\perp^{-4} \) at high \( k_\perp \) modulo factors coming from logarithmic terms. The average \( k_\perp \) is found to be very large (\( \sim 10 - 13 \) GeV/c) and practically independent of energy. This is interpreted as an indication that at present energies we are still far from the asymptotics and that, apart from supercritical pomerons, other states contribute significantly.
1 Introduction.

In this paper we continue the analysis of a model where a running coupling constant is included into the dynamics of reggeized gluons. This model [1,2] is based on the bootstrap relation [3] between the reggeized gluon trajectory $\omega(q)$ and the interaction kernel for a gluon pair $K_q(q_1, q_1')$, which guarantees that the production amplitudes satisfy unitarity in the one-reggeized-gluon-exchange approximation [4]. To satisfy it both $\omega$ and $K$ are written as functionals of a single function $\eta(q)$ which is proportional to $q^2$ in the BFKL fixed coupling case [5]. For the running coupling case the asymptotic form $\eta(q) \approx q^2/2\alpha_s(q^2)$ at $q \to \infty$ can be derived by comparing the evolution equation with the GLAP equation in this limit [1,2].

We recall the basic equation for the pomeron with a running coupling:

$$(-\omega(q_1) - \omega(q_2))\phi(q_1) + \int \frac{d^2q_1'}{(2\pi)^2} K_q^{(vac)}(q_1, q_1')\phi(q_1') = E(q)\phi(q_1)$$

where

$$\omega(q) = -\alpha_s N_c \eta(q) \int \frac{d^2q_1}{(2\pi)^2} \frac{1}{\eta(q_1)\eta(q_2)} , \quad q = q_1 + q_2$$

$$K_q(q_1, q_1') = -2T_1T_2\alpha_s \left( \frac{\eta(q_1)}{\eta(q_1')} + \frac{\eta(q_2)}{\eta(q_2')} \right) \frac{1}{\eta(q_1 - q_1')} - \frac{\eta(q)}{\eta(q_1)\eta(q_2')}$$

$T_{1(2)}$ are the colour operators for the two interacting gluons and $N_c$ is the number of colours; in the vacuum channel $T_1T_2 = -N_c$. Function $\phi$ is the amputated wave function. We will be interested in its computation for the forward scattering ($q = 0$), so dividing by $\eta$ once and twice we get the semi-amputated and the full wave functions, $\psi$ and $\Phi$ respectively. The bootstrap condition is

$$\int \frac{d^2q_1'}{(2\pi)^2} K_q^{(g_{uon})}(q, q_1, q_1') = \omega(q) - \omega(q_1) - \omega(q_2)$$

where in the gluon channel $T_1T_2 = -N_c/2$.

In the following, as in [1,2], we use a parametrization $\eta(q) = (b/2\pi)(q^2 + m^2)\ln((q^2 + m^2)/\Lambda^2)$ with $b = (1/4)(11 - (2/3)N_F)$ and $m \geq \Lambda$ and we restrict to the physical case $N_c = 3$. We choose $\Lambda = 0.2GeV$ for four acting flavours.

In our work [6] we studied the vacuum channel equation [1] numerically. Its spectrum relates to the pomeron trajectory as $\alpha(q) = 1 - E(q) \approx 1 + \Delta - \alpha' q^2$. It turned out that two supercritical pomeron states appear as normalizable solutions of [1]. Choosing $m = m_1 = 0.82GeV$, we found for them

$$\Delta_0 = 0.384, \quad \alpha'_0 = 0.250 (GeV/c)^{-2}; \quad \Delta_1 = 0.191, \quad \alpha'_1 = 0.124 (GeV/c)^{-2}$$

Their wave functions were also computed for the forward case $q = 0$.

As an application, total cross sections for the $\gamma^*\gamma^*$ scattering, and more qualitatively, for the $\gamma^*p$ and $pp$ scattering, were studied in [6] to see the unitarization effects. It turned out that the contribution from multiple pomeron exchanges became significant only for superhigh energies of the order of 100TeV. This seems to support the idea that we are still very far from the asymptopia at present energies, so that other states from the two-gluon spectrum give a large contribution.

In this paper we apply our model with the two found supercritical pomerons to jet production. This process has been extensively studied in the framework of the standard BFKL approach with
a fixed coupling [7,8]. As well-known, this analysis has lead to some far-reaching conclusions as to
the importance of mini-jet production at high energies and the logarithmic rise of the multiplicity.
However in the fixed coupling approach, various ad hoc modifications of the basic BFKL model had
to be introduced to cut off the spectrum at low \( k_\perp \) and also to correctly reproduce the high \( k_\perp \) tail of
the spectrum. Both these problems are naturally resolved by the introduction of a running coupling
in our model, in which no new parameters appear in contrast to the fixed coupling approach.

In section 2 we derive the basic equations and study the asymptotic behaviour of the jet production
cross section. In section 3 we present our numerical results. Discussion and conclusions follow in
section 4. In the Appendix we study the asymptotic form of the pomeron wave functions.

2 General formalism.

In this section we consider the scattering of two highly virtual photons (or heavy ”onia”) to make the
derivation more rigorous. To obtain the formula for the inclusive jet production let us recall the total
cross section for one pomeron exchange \( \Box \):

\[
\sigma = \frac{1}{4} \int d^2 r d^2 r' \rho_q(r) \rho_p(r') \int \frac{d^2 q_1 d^2 q'_1}{(2\pi)^4} G(\nu, 0, q_1, q'_1) \prod_{i=1,2} (1 - e^{iq_i r})(1 - e^{iq'_i r'})
\]

(4)

Here \( q(p) \) is the momentum of the projectile (target); \( \nu = qp = (1/2)s \); \( G(\nu, q, q_1, q'_1) \) is the non-
amputated pomeron Green function in the momentum space, \( q \) being the total momentum of the
two gluons and \( q_1(q'_1) \) being the initial (final) momentum of the first gluon; \( \rho_{q(p)} \) is the dipole colour
density of the projectile (target). For the virtual photons we take the colour dipole densities from \( \Box \).

In the high energy limit the dominant term of the Green function comes from the two mentioned
supercritical pomeron states:

\[
G_P(\nu, q, q_1, q'_1) = \sum_{i=0,1} \nu^{\alpha_i(q)-1} \Phi_i(q_1, q_2) \Phi_i^*(q'_1, q'_2)
\]

(5)

where \( \alpha_i \) and \( \Phi_i \) are the trajectories and wave functions of the leading (0) and subleading (1) pomerons.

Minijets appear as intermediate gluon states in the Green function \( \Box \). They possess arbitrary \( k_\perp \)
subject to condition \( k_\perp^2 << s \). To calculate the inclusive cross-section for their production we have
only to split the Green function in \( \Box \) as indicated in Fig. 1 and drop the integration over \( k_\perp \), i.e. to substitute

\[
G(\nu, 0, q_1, k'_1) \Rightarrow \int \frac{d^2 k_1 d^2 k'_1}{(2\pi)^4} G(\nu, 0, q_1, k_1)V(k_1, k'_1)G(\nu, 0, k_1, q'_1)\delta^{(2)}(k_1 + k_\perp - k'_1)
\]

(6)

where, for the running coupling case,

\[
V(k_1, k'_1) = 6 \frac{\eta(k_1)\eta(k'_1)}{\eta(k_1 - k'_1)} - 3\eta(0)
\]

(7)

Remembering that \( q_1 + q_2 = q'_1 + q'_2 = q = 0 \) we obtain

\[
I(y, k_\perp) \equiv \frac{d^3 \sigma}{dy d^2 k_\perp} = \int d^2 r d^2 r' \rho_q(r) \rho_p(r') \int \frac{d^2 q_1 d^2 q'_1}{(2\pi)^4} (1 - e^{iq_1 r})(1 - e^{iq'_1 r'}) \int \frac{d^2 k_1 d^2 k'_1}{(2\pi)^4} \delta^{(2)}(k_1 + k_\perp - k'_1)G(\nu, 0, q_1, k_1)G(\nu, 0, k_1, q'_1)\left[6 \frac{\eta(k_1)\eta(k'_1)}{\eta(k_1 - k'_1)} - 3\eta(0)\right]
\]

(8)
The appropriate kinematical variables are the rapidity and transverse momentum of the observed gluon (jet), $y = \frac{1}{2} \ln \frac{k^+}{k^-}$ and $k_\perp$, respectively; we also have $s_i = 2\nu_i$ and $s(2) = k_\perp \sqrt{s} e^{-\nu_j}$. It is convenient to use a mixed (momentum-coordinate) representation for the Green functions:

$$G(\nu, 0, r, k) = \int \frac{d^2q_1}{(2\pi)^2} e^{iq_1r} G(\nu, q_1, k)$$

(9)

Defining $\Delta \Phi_i(r) = \Phi_i(r) - \Phi_i(0)$ we can write

$$G(\nu, 0, r, k) - G(\nu, 0, 0, k) = \sum_{i=0,1} \nu^\Delta_i \Delta \Phi_i(r) \Phi^*_i(k)$$

(10)

We also introduce

$$R_i^{\rho(p)} \equiv \langle \Delta \Phi_i \rangle_{\rho(p)} = \int d^2r \rho_\rho(p) r \Delta \Phi_i(r)$$

(11)

In this notation and also using both the semiamputated and full wave functions we find

$$I(y, k_\perp) = \sum_{i,j=0,1} R_i^{\rho} R_j^{\rho} \int \frac{d^2k_1}{(2\pi)^2} \int \frac{d^2k_2}{(2\pi)^2} \left[ \psi_i(k_1) \psi_j(k_1 + k_\perp) \eta(k_\perp) \right] - 3\eta(0) \Phi_i(k_1) \Phi_j(k_1 + k_\perp)$$

(12)

The found supercritical pomerons are isotropic in the transverse space. As a result the inclusive cross-section $I(y, k_\perp)$ also turns out to be isotropic. Therefore we can integrate over azimuthal angles in (12). Defining for the full and semi-amputated wave functions the integrated quantities

$$\hat{\Phi}(k_1, k_\perp) = \int_0^{2\pi} d\alpha \Phi(k_1 + k_\perp)$$

$$\hat{\psi}(k_1, k_\perp) = \int_0^{2\pi} d\eta \Phi(k_1 + k_\perp)$$

(13)

where $\alpha$ is the angle between $k_1$ and $k_\perp$, we finally obtain the inclusive cross section

$$\frac{d^2\sigma}{dy dk_\perp^2} = \frac{3}{4} \sum_{i,j=0} e^{-y(\Delta_i - \Delta_j)} \left[ \frac{k_\perp^2 s}{4} \hat{\psi}(\Delta + \Delta_i) \hat{\psi}(\Delta + \Delta_j) \right] R_i^{\rho} R_j^{\rho} \int \frac{dk_1^2}{(2\pi)^3} \left[ \psi_i(k_1) \psi_j(k_1 + k_\perp) \eta(k_\perp) - \eta(0) \Phi_i(k_1) \Phi_j(k_1 + k_\perp) \right]$$

(14)

The results of numerical calculations of the cross-section (14) and also its generalization to more interesting cases of hadronic targets or/and projectiles will be discussed in the next section. In the rest of this section we shall study the asymptotical behaviour of the found inclusive jet production cross-section at very small and very large transverse momenta and also its $y$ -dependence.

As to the latter, all $y$-dependence in (14) comes from the factor $\exp(-y(\Delta_i - \Delta_j))$ which has its origin in the existence of two different supercritical pomerons. Evidently in the limit $s \to \infty$ this dependence dies out, since the relative contribution of the subdominant pomeron becomes negligible.

The model thus predicts an asymptotically flat $y$ plateau at very high energies.

At small $k_\perp$ the cross-section (14) evidently goes down as $k_\perp^{\Delta_i}$, since all other factors are finite in this limit. However one should remember that (9) gives the dominant contribution only while $(k_\perp^2 s)$ continues to be large. At too small $k_\perp$, when the above quantity becomes finite, all other states from the spectrum of the two-gluon equation (1), hitherto neglected, begin to give comparable or even dominant contribution, so that the found $k_\perp^{\Delta_i}$ behaviour ceases to be valid.

To find the asymptotic behaviour of the inclusive cross section for $k_\perp \to \infty$ we need to know the behaviour of the pomeron wave functions in the momentum space at $q \to \infty$ and in the ordinary space...
at $r \to 0$. In the Appendix we show that

$$
\psi(q) \sim \frac{1}{q^2} (\ln q^2)^\beta
$$

$$
\psi(r) \sim (\ln \frac{1}{r})^{\beta+1}
$$

(15)

where $\beta = -1 - \frac{3}{\ln \epsilon}$ so that in the forward scattering case $\beta$ depends just on the intercept of the corresponding pomeron state. Let us study now the behaviour of (12) for which is valid for naturally behaved $f$, such that

the standard asymptotical behaviour

$$
\langle n \rangle = \int d^3q \frac{d\sigma}{dydk_\perp} a(y) \frac{k_\perp^2}{k_\perp^4} e^{-\ln^2 k_\perp^2 / a^2(y)}
$$

(21)

the standard asymptotical behaviour $\langle n \rangle = a \ln s + b$ is obtained. Indeed integrating (12) we get

$$
\int dydk_\perp \frac{d\sigma}{dydk_\perp} = B s^\Delta_0 \ln s + C s^\Delta_0
$$

(22)

where $\Delta_0$ is the pomeron energy dependence and a logarithmic factor coming from the pomeron energetic dependence and a logarithmic factor coming from the pomeron energetic dependence

$$
so that in (20) the power factor due the pomeron energetic dependence and a logarithmic factor coming from the pomeron energetic dependence

$$
\text{at large } s \to \infty \text{ and } k_\perp \to 0. \text{ In the Appendix we show that}
$$

$$
\int d^2 k_1 \Phi_i(k_1) \Phi_j(k_1 + k_\perp) \to \Phi_j(k_\perp) \int d^2 k_1 \Phi_i(k_1) \sim \frac{1}{k_\perp^2} (\ln k_\perp^2)^{\beta_j-1}
$$

To analyze the first term we use the identity

$$
\int \frac{d^2 k_1}{(2\pi)^2} \psi_i(k_1) \psi_j(k_1 + k_\perp) = 2\pi \int_0^\infty rdr J_0(k_\perp r) \psi_i(r) \psi_j(r)
$$

(17)

and the relation

$$
\int_0^\infty rdr J_0(k_\perp r) J_0(k_\perp r) f(r) = -\frac{1}{k_\perp^2} \int_0^\infty rdr J_1(k_\perp r) f'(r)
$$

(18)

which is valid for naturally behaved $f$, such that $[r f(r) J_1(k_\perp r)]_{r=0}^{\infty} = 0$.

Putting $f(r) \sim (\ln \frac{1}{r})^{\beta_i + \beta_j + 2}$ we obtain the leading behaviour

$$
\int d^2 k_1 \psi_i(k_1) \psi_j(k_1 + k_\perp) \sim \frac{1}{k_\perp^2} (\ln k_\perp^2)^{\beta_i + \beta_j + 1}
$$

(19)

So at $k_\perp \to \infty$ we find for the inclusive cross-section (12)

$$
I(y, k_\perp) \sim \sum_{i,j=0,1} R_i^p R_j^p \frac{k_\perp^2 s}{4} \frac{\Delta_i + \Delta_j}{\Delta_i - \Delta_j} e^{-y(\Delta_i - \Delta_j)} \frac{1}{k_\perp^2} (\ln k_\perp^2)^{\beta_i + \beta_j}
$$

(20)

This asymptotics corresponds to the standard quark-counting rules behaviour ($\sim 1/k_\perp^4$), modified by a power factor due the pomeron energetic dependence and a logarithmic factor coming from the pomeron wave function, i.e. from the running of the coupling. Note that in (20) the $k_\perp$ and $s$ dependencies are separated. As a result we find that the average $\langle k_\perp \rangle$ is finite and independent of $s$ and $\langle k_\perp^2 \rangle \sim s^\Delta$, since it formally diverges for (20) and one has to restrict $k_\perp^2 < s$.

For the the multiplicity

$$
\langle n \rangle = \frac{1}{\sigma} \int d^3q d\sigma d^2k_\perp
$$

the standard asymptotical behaviour $\langle n \rangle = a \ln s + b$ is obtained. Indeed integrating (12) we get

$$
\int dydk_\perp \frac{d\sigma}{dydk_\perp} = B s^\Delta_0 \ln s + C s^\Delta_0.
$$

Since the total cross-section has the form $\sigma = A s^\Delta_0$ at large $s$, we get the mentioned asymptotical expression for the multiplicity. Of course this result is valid only in the extreme limit $s \to \infty$. At large but finite $s$ the existence of two different pomerons leads to some additional non-trivial $s$-dependence.

It is instructive to compare our asymptotic results with those obtained in the BFKL fixed coupling model. In the latter case the inclusive cross section is of course badly behaved in the $k_\perp \to 0$ limit due to scale invariance. It is also very different in the high $k_\perp$ limit. At very large $k_\perp$ such that

$$
\ln k_\perp \sim \sqrt{\ln s}
$$

one finds

$$
\left( \frac{d^2\sigma}{dydk_\perp^2} \right)_{BFKL} \sim a(y) \frac{(k_\perp^2 s)^\Delta}{k_\perp^4} e^{-\ln^2 k_\perp^2 / a^2(y)} \frac{1}{\ln k_\perp^2 \sqrt{\ln s}}
$$

(22)
where \( a^2 \sim (\ln s - 4y^2/\ln s) \). Thus for large \( k_\perp \) the BFKL cross section goes down faster than any power. Also one obtains that \( \langle \ln k_\perp \rangle \sim \sqrt{\ln s} \) so that both \( k_\perp \) and \( k_\perp^2 \) grow with \( s \).

To bring these predictions in better correspondence with the physical reality, as mentioned, various modifications of this orthodox BFKL approach have been introduced. In particular in [7,8] the fusion of gluons via "fan" diagrams was assumed to take place at high gluonic densities, which was considered as a way to partially restore the s-channel unitarity. Then, under some additional assumptions, a \( 1/k_\perp^4 \) asymptotic behaviour similar to (20) was found. In our model such a behaviour naturally follows without introducing additional assumptions nor imposing the unitarity restrictions.

3 Numerical results.

Taking the wave function evaluated numerically in [3] we have computed the cross section (14).

In Fig.2 we present \( d^2\sigma/dydk_\perp \) for the process \( \gamma^*\gamma^* \) (in units \( c = 1 \)). We have chosen the projectile photon to have virtuality \( Q = 5\text{GeV}/c \) and the target one to have virtuality \( P = 1\text{GeV}/c \). The center of mass energy is \( \sqrt{s} = 540\text{GeV} \).

Of course, processes involving hadronic targets or/and projectiles are much more interesting from the practical point of view. However these require some non-perturbative input for the colour densities of the colliding hadrons. A possible way to introduce it is evidently to convert Eq. (1) into an evolution equation in \( 1/x \) and take initial conditions for it from the existing experimental data. Postponing this complicated procedure for future studies, we use here a simpler approach, taking for the hadron (proton) a Gaussian colour density with a radius corresponding to the observed electromagnetic one. Such an approximation, in all probability, somewhat underestimates the coupling of the hadron to the pomeron, since the coupling of the latter to constituent gluons is neglected. Nevertheless, we hope that it gives a reasonable estimate for the inclusive cross-section. Using this approach we get the inclusive jet production cross-sections for the \( \gamma^*p \) and \( pp \) scattering shown in Fig.3 and 4 respectively.

A common feature of jet production in all processes is that the cross-section reaches a maximum at \( k_\perp \approx 1\text{GeV}/c \) from which it monotonously goes down both for smaller and larger \( k_\perp \).

In Fig. 5 the cross-section \( d\sigma/dy \) integrated over \( k_\perp \) in the interval \( 0.5 \div 20\text{GeV}/c \) is presented for the process \( pp \). The limitations in the numerical calculations of the wave functions do not allow us to study higher values of \( k_\perp \), so that we cannot numerically reach the region where the asymptotical behaviour (20) is strictly valid.

In Fig. 6 we show jet multiplicities as a function of \( s \) for the three studied processes \( \gamma^*\gamma^*, \gamma^*p \) and \( pp \). As one observes, their magnitude and behaviour prove to be quite similar.

We also tried to estimate the average \( \langle k_\perp \rangle \). Unfortunately, although it exists according to (20), its value results very sensitive to the high momentum tail of the pomeron wave function, poorly determined from our numerical calculations. To avoid this difficulty we chose to calculate the average \( \langle \ln k_\perp/\Lambda \rangle \) instead. This average at fixed \( y \) depends on \( s \) and \( y \) only due to the existence of two different pomerons, so that this dependence should die out at large enough \( s \). In fact the found average \( \langle \ln k_\perp/\Lambda \rangle \) turns out to be practically independent of \( y \) and very weakly dependent on \( s \) in the whole studied range of \( s \) and \( y \), rising from 3.97 at \( \sqrt{s} = 20\text{GeV} \) to 4.19 at \( \sqrt{s} = 20\text{TeV} \). These values imply a rather high average \( k_\perp \) rising from 10.6 \( \text{GeV}/c \) at \( \sqrt{s} = 20\text{GeV} \) to 13.2 \( \text{GeV}/c \) at
\( \sqrt{s} = 20 \, TeV \).

4 Discussion

Our calculations show that the introduction of a running coupling constant on the basis of the bootstrap condition cures all the deseases of the orthodox BFKL approach for jet production. At high \( k_\perp \) the cross-section becomes well-behaved and more or less in accordance with the expectations based on the quark counting rules. At small \( k_\perp \) no singularity occurs, although contributions from other states is expected to dominate.

As to \( s \)- and \( y \)-dependence, our predictions are even simpler than in the BFKL approach, since the running of the coupling converts the branch point in the complex angular momentum plane, corresponding to the BFKL pomeron, into poles, of which only two are located to the right of unity and contribute at high energies. As a result at superhigh energies, when only the dominant pomeron survives, the \( y \)-dependence completely disappears and the \( s \)-dependence reduces to the standard \( s^{\Delta_0} \) factor. At smaller \( s \) some \( y \)- and extra \( s \)-dependence appears due to the existence of two supercritical pomerons.

With all these refinements, some basic predictions of the BFKL theory are reproduced. The cross-section for minijet production rises fast and saturates the total cross-section as \( s \rightarrow \infty \). Jet multiplicities rise logarithmically.

However \( \langle k_\perp \rangle \) turns out to depend on \( s \) weakly and its calculated value results pretty high, of the order of 10-12 \( GeV/c \). This should be contrasted to the experimentally observed much lower values of \( \langle k_\perp \rangle \) rising with energy. A natural explanation of this discrepancy follows from the conclusion made in [6] from the study of the structure functions and total cross-sections in our model: at present energies the contribution from the two supercritical pomerons only covers a part of the observed phenomena because we are still rather far from the real asymptotics. The bulk of the contribution comes from other states, which produce a much softer spectrum of particles. The observed rise of the \( \langle k_\perp \rangle \) is then related to the dying out of these subasymptotical states. Our prediction is then that the rise of \( \langle k_\perp \rangle \) should saturate at the level of 10-12 \( GeV/c \).

This circumstance has also to be taken into account when discussing the numerical results presented in Figs. 2-6. Probably they also illustrate predictions for considerably higher energies than the present ones, at which, according to the estimate made in [6], they should account for \( \sim 20\% \) of the observed spectra.

To obtain predictions better suited for present energies one should evidently take into account all states present in the spectrum of the pomeron equation (1) and not only the two supercritical pomerons. To realize this program an evolution equation in \( \nu \) following from (1) seems to be an appropriate tool. As mentioned, it could also effectively take into account the nonperturbative effects related to the pomeron coupling to physical hadrons. A work in this direction is now in progress.

5 Appendix
5.1 The momentum space behaviour

The asymptotic form of the pomeron eigenvalue equation at high momentum is [1,2]

\[ \ln \frac{\ln q^2}{\ln m^2} \psi(q) + \alpha m^2 \ln \frac{m^2}{q^2 \ln q^2} = \tilde{\epsilon} \psi(q) + \frac{1}{\pi} \int \frac{d^2 q' \psi(q')}{[(q - q')^2 + m^2] \ln [(q - q')^2 + m^2]} \]  

(23)

where we have put \( E = (3/b) \tilde{\epsilon} \) (we consider the case \( m = m_1 \) for simplicity). Let us study the last (integral) term, which we denote as \( D \). Changing the variable \( q' = |q| \kappa \) we present it in the form

\[ D = \frac{1}{\pi} \int \frac{d^2 \kappa \psi(|q| \kappa)}{[(n - \kappa)^2 + m^2 / q^2] \ln [q^2 (n - \kappa)^2 + m^2]} \]

\[ = \frac{1}{\pi} \int \frac{d^2 \kappa \psi(|q| (n + \kappa))}{(\kappa^2 + m^2 / q^2) \ln [q^2 \kappa^2 + m^2]} \]  

(24)

with \( n^2 = 1 \). In the last form it is evident that the leading terms at \( q \to \infty \) come from the integration region of small \( \kappa \). Then we split the total \( \kappa \) space into two parts: \( \kappa > \kappa_0 \) and \( \kappa < \kappa_0 \) where \( \kappa_0 \) is a small number

\[ \kappa_0 << 1 \]  

(25)

The contributions from these two parts we denote as \( D_1 \) and \( D_2 \), respectively. Our first task is to show that \( D_2 \) cancels the kinetic term in (23) irrespective of the asymptotics of \( \psi(q) \).

With small enough \( \kappa_0 \) and for a ”good enough” \( \psi \)

\[ \psi(|q|(n + \kappa)) \simeq \psi(|q|n) = \psi(q) \]  

(26)

so that we can take it out of the integral over \( \kappa \) in \( D_2 \). Actually (26) is our definition of a ”good” function, so that we shall have to check if this condition is indeed satisfied for the found asymptotical \( \psi(q) \).

With (26) \( D_2 \) simplifies to

\[ D_2 = \frac{\psi(q)}{\pi} \int_0^{\kappa_0^2 q^2} \frac{d^2 q'}{(q'^2 + m^2) \ln (q'^2 + m^2)} = \psi(q) \ln \frac{\ln \kappa_0^2 q^2 + m^2}{\ln m^2} \]  

(27)

As \( q \to \infty \) we assume that also

\[ \kappa_0 q \to \infty \]  

(28)

Evidently this condition fixes the manner in which \( \kappa_0 \) goes to zero as \( q \) turns large. This should be taken into account when verifying condition (23). Then (27) gives

\[ D_2 = \psi(q) \ln \frac{\ln \kappa_0^2 q^2}{\ln m^2} \]  

(29)

We have also

\[ \ln \ln \kappa_0^2 q^2 = \ln (\ln q^2 + \ln \kappa_0^2) \]  

But according to (28)

\[ \ln q^2 >> |\ln \kappa_0^2| \]  

so that we have

\[ \ln \ln \kappa_0^2 q^2 = \ln \ln q^2 + \frac{1}{\ln q^2} \ln \kappa_0^2 + \cdots \]
Then our final result is

\[ D_2 = \psi(q) \left( \ln \frac{\ln q^2}{\ln m^2} + O(1/\ln q^2) \right) \]  

(30)

The first term exactly cancels the kinetic energy (first) term in the pomeron equation. The correction term in (30) evidently is much smaller than \( \tilde{\epsilon} \psi(q) \), since the factor which multiplies the function \( \psi \) goes to zero in (30). So we can safely neglect it.

As a result, the asymptotic equation becomes

\[ \frac{am^2 \ln m^2}{q^2 \ln q^2} = \tilde{\epsilon} \psi(q) + D_1 \]  

(31)

with the term \( D_1 \) given by

\[ D_1 = \frac{1}{\pi} \int d^2 q' \psi(q') \frac{\theta((q-q')^2 - \kappa_0^2 q'^2)}{(q-q')^2 \ln(q-q')^2} \]  

(32)

where we have omitted the terms \( m^2 \) in the denominator because of (28).

As in [2] we pass to the function \( \chi(q) \) defined by

\[ \chi(q) = \psi(q) q^2 \ln q^2 \]  

(33)

Multiplying (31) by \( q^2 \ln q^2 \) we find an equation for \( \chi \)

\[ am^2 \ln m^2 = \tilde{\epsilon} \chi(q) + \frac{1}{\pi} \int d^2 q' \frac{\chi(q') \theta((q-q')^2 - \kappa_0^2 q'^2)}{q'^2 \ln q'^2} \frac{q^2 \ln q^2}{(q-q')^2 \ln(q-q')^2} \]  

(34)

At this point we make a second assumption. Namely we assume that in the integral term of (34) values \( m << q' << q \) give the dominant contribution (as typical for logarithmic integrals). Of course, this assumption is also to be checked for the final asymptotics. With this assumption, we can forget about the \( \theta \) function and also put the last factor to unity in the integral term of (34). We obtain

\[ am^2 \ln m^2 = \tilde{\epsilon} \chi(q) + \int_0^{q^2} \frac{dq'^2 \chi(q')}{q'^2 \ln q'^2} \]  

(35)

Differentiating with respect to \( q^2 \)

\[ \tilde{\epsilon} \frac{d\chi}{dq^2} = -\frac{\chi}{q^2 \ln q^2} \]  

(36)

or

\[ \tilde{\epsilon} \frac{d\chi}{d \ln q^2} = -\chi \]  

(37)

with a solution

\[ \chi(q) = A \exp \left( \frac{\ln \ln q^2}{\tilde{\epsilon}} \right) = A (\ln q^2)^{-1/\tilde{\epsilon}} \]  

(38)

The initial function \( \psi(q) \) has the asymptotics

\[ \psi(q) = \frac{A}{q^2} (\ln q^2)^\beta \]  

(39)

where

\[ \beta = -1 - \frac{1}{\tilde{\epsilon}} = -1 - \frac{3}{bE} \]  

(40)

Now we have to check that our assumptions are indeed fulfilled for the found asymptotics.
Let us begin with the second assumption that the values \( m << q' \ll q \) give the bulk of the contribution to the integral in (33). Evidently, in order that the integral be dominated by large values of \( q' \), it should diverge as \( q \to \infty \). This leads to the condition

\[
E < 0, \quad \beta > -1
\]

(41)

So our asymptotics can only be valid for negative energies, that is, for bound states.

Now for the second part of this assumption. To prove that values \( q' \ll q \) dominate we shall calculate the contribution from the region \( q' \gg q \) and show that it is smaller. The corresponding integral is

\[
I = A q^2 \ln q^2 \int_{q^2}^{\infty} dq' (\ln q'^2)^{\beta - 1}/q'^4
\]

(42)

In terms of \( x = \ln q'^2 \)

\[
I = q^2 \ln q^2 \int_{\ln q^2}^{\infty} dx x^{\beta - 1} \exp(-x)
\]

(43)

The integral over \( x \) can be developed in an asymptotic series in \( 1/\ln q^2 \):

\[
\int_{\ln q^2}^{\infty} dx x^{\beta - 1} \exp(-x) = -\int_{\ln q^2}^{\infty} x^{\beta - 1} d\exp(-x) = \frac{(\ln q^2)^{\beta - 1}}{q^2} + (\beta - 1) \int_{\ln q^2}^{\infty} dx x^{\beta - 2} \exp(-x) = ...
\]

(44)

Thus, we conclude that the integral \( I \) has the asymptotics

\[
I = A (\ln q^2)^{\beta}
\]

(45)

to be compared to the contribution from the region \( q' \ll q \) which behaves as \((\ln q^2)^{\beta+1}\). We see that we have lost one power of \( \ln q^2 \), so that the region \( q' \gg q \) indeed can be neglected.

Now to the assumption (26). We have explicitly

\[
\psi(|q|(n + \kappa)) = (A/q^2)(n + \kappa)^{-2}(\ln q^2(n + \kappa)^2)^{\beta} = (A/q^2)(1 - 2n\kappa - \kappa^2 + 4(n\kappa)^2)(\ln q^2)^{\beta}(1 + (\beta(2n\kappa + \kappa^2))/\ln q^2)
\]

(46)

and it is evident that (32) is satisfied with the behaviour of \( \kappa \) as indicated in (28). Indeed take \( \kappa \sim q^{-\delta} \) with \( \delta < 1 \). Then \( q\kappa \sim q^{1-\delta} \to \infty \) and all the correcting terms in (10) have the order \( q^{-\delta} \).

Note that this does not mean that (26) is quite obvious. It is not valid for, say, the exponential function. In fact we have

\[
\exp(aq^2(n + \kappa)^2) = \exp(aq^2) \exp(aq^2(2n\kappa + \kappa^2))
\]

and since \( q\kappa \) is large the second factor cannot be neglected.

In conclusion, we have verified that our assumptions are fulfilled and therefore the asymptotics (39) is correct for negative energies.

### 5.2 The coordinate space behaviour

We are also interested in the coordinate space behaviour of the semi-amputated wave function

\[
\psi(r) = \int_{0}^{\infty} \frac{qdq}{2\pi} J_0(qr)\psi(q)
\]

(47)
To estimate $\psi(r)$ for $r \to 0$ we make use of the asymptotic momentum behaviour found previously \(39\), so we can write
\[
\psi(r) \approx \int_0^{q_0} \frac{qdq}{2\pi} J_0(qr)\psi(q) + \int_{q_0}^\infty \frac{qdq}{2\pi} J_0(qr) ln(q^2) \beta q^2 \quad (48)
\]

The first integral for $r \to 0$ is finite; on the other hand the second integral results to be not bounded. Infact using the integration variable $y = qr$ we split the $y$ integration region in two parts, $q_0 r < y < y_0 \ll 1$ and $y \geq y_0$; defining the two contribution $I_1$ and $I_2$ respectively, we have
\[
I_1 \sim \int_{q_0 r}^{y_0} \frac{dy}{y} (ln \frac{y}{r})^\beta \sim (ln \frac{1}{r})^{\beta+1} \quad (49)
\]
and
\[
I_2 \sim \int_{y_0}^{\infty} \frac{dy}{y} J_0(y)(ln y + ln \frac{1}{r})^\beta \sim (ln \frac{1}{r})^\beta \quad (50)
\]
So the asymptotic small $r$ behaviour will be divergent, precisely
\[
\psi(r) \sim (ln \frac{1}{r})^{\beta+1} \quad (51)
\]

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7 Figure captions

Fig.1. The substitution for the Green function necessary to calculate the inclusive jet production cross-section.

Fig.2. Inclusive jet production cross-sections for the process $\gamma^*\gamma^*$ at $\sqrt{s} = 540 GeV$ with $Q^2 = 25(GeV/c)^2$ and $P^2 = 1(GeV/c)^2$.

Fig.3. Inclusive jet production cross-sections for the process $\gamma^*p$ at $\sqrt{s} = 540 GeV$ with $Q^2 = 25(GeV/c)^2$.

Fig.4. Inclusive jet production cross-sections for the process $pp$ at $\sqrt{s} = 540 GeV$.

Fig.5. Cross-sections $d\sigma/dy$ for the $pp$ process at $\sqrt{s} = 540 GeV$.

Fig.6. Multiplicities $\langle n \rangle$ as a function of the center of mass energy $\sqrt{s}$ for the processes $\gamma^*\gamma^*$ (the solid curve), $\gamma^*p$ (the dashed curve) and $pp$ (the dotted curve).
\[ \Rightarrow \]
Photon-Photon jet production.
Photon-Proton jet production.
Fig. 5

Proton-Proton scattering

\[ \frac{\text{dsigma}}{\text{dy}} \]

\[ \text{(mbarn)} \]
Fig. 6