Hylomorphic solitons for the generalized KdV equation.

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Abstract

In this paper we prove the existence of hylomorphic solitons in the generalized KdV equation. Following [2], a soliton is called hylomorphic if it is a solitary wave whose stability is due to a particular relation between energy and another integral of motion which we call hylenic charge.

Dedicated to our friend Djairo De Figueiredo on the occasion of his 80-th birthday

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1
1 Introduction

The Korteweg–de Vries equation (KdV) was first introduced by Boussinesq (1877) and rediscovered by Diederik Korteweg and Gustav de Vries (1895). It is a model of waves on shallow water surfaces.

Many different variations of the KdV equation have been studied. The most common is the following one which is known as the generalized KdV equation (gKdV):

\[ \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x} W'(u) = 0 \]  
(1)

where \( u = u(t,x) \), and \( W \in C^2(\mathbb{R}) \). If \( W(s) = -s^3 \), then (1) reduces to the usual KdV equation

\[ \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x} = 0 \]  
(2)

If \( W(s) = -\frac{s^{k+2}}{(k+2)(k+1)} \), then (1) reduces to the equation

\[ \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u^k \frac{\partial u}{\partial x} = 0 \]  
(3)

known as the modified KdV equation (mKdV).

In this paper we are interested to the existence of solitary waves and solitons for the gKdV equation. Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A soliton is a solitary wave which exhibits some form of stability so that it has a particle-like behavior.

Using the inverse scattering transform, it is possible to prove that KdV admits soliton solutions and to have an extremely powerful and precise information on them. However the inverse scattering techniques cannot be applied to the generalized KdV equation. In this paper, we shall use the method developed in [5] to prove that equation (1) admits solitons and we will show that they are hylomorphic. Following [2], a soliton is called hylomorphic if its stability is due to a particular interplay between the energy \( E \) and the hylenic charge \( C := \int u^2 dx \) which is another integral of motion. More precisely, a soliton \( u_0 \) is hylomorphic if

\[ E(u_0) = \min \left\{ E(u) \mid \int u^2 dx = C(u_0) \right\}. \]

We will show that eq. (1) admits solitons provided that \( W \) satisfies suitable assumptions (see Theorem 17). In particular, if \( W(s) = -\frac{s^{k+2}}{(k+2)(k+1)} \), hylomorphic solitons exist for \( k = 1, 2, 3 \). So, in this case, we get a different proof of well
known results (see [13]) and we show that the "usual" solitons of mKdV can be considered "hylomorphic".

In Th. [17] we obtain the existence of hylomorphic solitons under a very general set of assumptions on $W$; moreover, in contrast to other results on this topic, these assumptions are easy to verify.

2 Solitary waves and solitons

In this section we construct a functional abstract framework which allows to define solitary waves, solitons and hylomorphic solitons.

2.1 Solitary waves

Solitary waves and solitons are particular states of a dynamical system described by one or more partial differential equations. Thus, we assume that the states of this system are described by one or more fields which mathematically are represented by functions

$$ u : \mathbb{R}^N \to V $$

where $V$ is a vector space with norm $| \cdot |_V$ and which is called the internal parameters space. We assume the system to be deterministic; this means that it can be described as a dynamical system $(X, \gamma)$ where $X$ is the set of the states and $\gamma : \mathbb{R} \times X \to X$ is the time evolution map. If $u_0(x) \in X$, the evolution of the system will be described by the function

$$ u(t, x) := \gamma_t u_0(x). \quad (4) $$

We assume that the states of $X$ have "finite energy" so that they decay at $\infty$ sufficiently fast and that

$$ X \subset L^1_{\text{loc}}(\mathbb{R}^N, V). \quad (5) $$

Thus we are lead to give the following definition:

**Definition 1** A dynamical system $(X, \gamma)$ is called of FT type (field-theory-type) if $X$ is a Hilbert space of functions of type [5].

Let $\mathcal{T}$ be the group of translations in $\mathbb{R}^N$ and $U(V)$ the group of unitary transformations on $V$: set

$$ G = \mathcal{T} \times U(V) $$

Given $(\tau, h) \in G$ we will consider the representation of $G$ on $X \subset L^1_{\text{loc}}(\mathbb{R}^N, V)$ given by

$$ [T_{(\tau, h)} u](x) = h u(x - \tau) $$

For example take $X = L^2(\mathbb{R}^N, \mathbb{C})$, $h = e^{i\theta} \in U(1)$, then

$$ [T_{(\tau, h)} u](x) = e^{i\theta} u(x - \tau) $$

A solitary wave is a state of finite energy which evolves without changing his shape. This informal description can be formalized by the following definition:
Definition 2. A state $u_0 \in X \setminus \{0\}$, is called solitary wave if there is a continuous trajectory 

$t \mapsto (\tau(t), h(t)) \in G$

such that 

$$\gamma_t u_0(x) = h(t)u_0(x - \tau(t))$$

For example, consider a solution of a field equation having the following form: 

$$u(t,x) = u_0(x - vt - x_0)e^{i(v \cdot x - \omega t)}; \quad u_0 \in L^2(\mathbb{R}^N); \quad (6)$$

$x_0, v \in \mathbb{R}^N, \omega \in \mathbb{R}$. Clearly $u(t,x)$ is a solitary wave for every $t \in \mathbb{R}$. The evolution of a solitary wave is a translation plus a unitary change of the internal parameters (in this case the phase).

2.2 Orbitally stable states and solitons

The solitons are solitary waves characterized by some form of stability. To define them at this level of abstractness, we need to recall some well known notions in the theory of dynamical systems.

Definition 3. A set $\Gamma \subset X$ is called invariant if $\forall u \in \Gamma, \forall t \in \mathbb{R}, \gamma_t u \in \Gamma$.

Definition 4. Let $(X, \gamma)$ be a dynamical system and let $X$ be equipped with a metric $d$ (it is not necessary to assume that $d(u, v) = \|u - v\|_X$). An invariant set $\Gamma \subset X$ is called stable (with respect to $d$), if $\forall \varepsilon > 0, \exists \delta > 0, \forall u \in X,$ 

$$d(u, \Gamma) \leq \delta,$$

implies that 

$$\forall t \geq 0, \ d(\gamma_t u, \Gamma) \leq \varepsilon.$$  

Definition 5. Let $(X, d)$ be a metric space and let $T$ be the group of translations. A set $\Gamma \subset X$ is called $T$-compact if for any sequence $u_n(x) \in \Gamma$ there is a subsequence $u_{n_k}$ and a sequence $\tau_k \in T$ such that $u_{n_k}(x - \tau_k)$ is convergent with respect to the metric $d$.

Now we give the definition of orbitally stable state:

Definition 6. Let $(X, \gamma)$ be a dynamical system with $X$ equipped with a metric $d$. A state $u \in X$ is called orbitally stable (with respect to $d$) if $u \in \Gamma \subset X$ where

- (i) $\Gamma$ is an invariant stable set with respect to $d$,
- (ii) $\Gamma$ is $T$-compact (with respect to $d$).
This definition is usually present in the literature relative to the dynamics of PDE’s (see e.g. [7], [13] etc.).

Now we are able to give the definition of soliton:

**Definition 7** Let \((X, \gamma)\) be a dynamical system with \(X\) equipped with a metric \(d\). A soliton is an orbitally stable solitary wave (with respect to \(d\)).

**Remark 8** In our definition, since \((X, \gamma)\) is a dynamical system, the map 
\[ t \mapsto \gamma_t u \]
is continuous with respect to \(\|\cdot\|_X\). In the above definitions, we have introduced a distance \(d\), however we have not supposed that 
\[ d(u, v) = \|u - v\|_X \]
In fact, in some applications this is not true. As we will see in section 4 this is the case for equation (1) where we have 
\[ d(u, v) \leq M \|u - v\|_X \]
for a suitable constant \(M > 0\).

### 2.3 Hylomorphic solitons

We now assume that the dynamical system \((X, \gamma)\) has two constants of motion: the energy \(E\) and the hylenic charge \(C\). At this level of abstraction, of course, the name energy and hylenic charge are conventional.

**Definition 9** Let \((X, \gamma)\) be a dynamical system where \(X\) is equipped with a metric \(d\). A soliton \(u_0 \in X\) is called **hylomorphic** if the set \(\Gamma\) (given by Def. 6) has the following structure
\[ \Gamma = \Gamma(e_0, c_0) = \{u \in X \mid E(u) = e_0, |C(u)| = c_0\} \quad (7) \]
where
\[ e_0 = \min \{E(u) \mid |C(u)| = c_0\}. \quad (8) \]

Notice that, by (8), we have that a hylomorphic soliton \(u_0\) minimizes the energy on 
\[ M_{c_0} = \{u \in X \mid |C(u)| = c_0\}. \quad (9) \]
If \(M_{c_0}\) is a manifold and \(E\) and \(C\) are differentiable, then \(u_0\) satisfies the following nonlinear eigenvalue problem:
\[ E'(u_0) = \lambda C'(u_0). \]
3 Hylomorphic solitons for the nonlinear Schrödinger equation

The solitons for eq. (11), as we will see, are related to the solitons of the nonlinear Schrödinger equation. We recall that the orbital stability for the nonlinear Schrödinger equation has been proved in [7] (see also [1] for the general case and [11] with its references).

Here we shall use a method to prove the existence of hylomorphic solitons for (1) similar to the one presented in [5] (see also [4] and [6]). In this section we will resume this method.

The nonlinear Schrödinger equation is given by

\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + \frac{1}{2} W'(\psi) \]  

where \( \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) and where \( W : \mathbb{C} \to \mathbb{R} \) and

\[ W'(\psi) = \frac{\partial W}{\partial \psi_1} + i \frac{\partial W}{\partial \psi_2}. \]  

We assume that \( W \) depends only on \( |\psi| \), namely

\[ W(\psi) = F(|\psi|) \]  

and so \( W'(\psi) = F'(|\psi|) \frac{\psi}{|\psi|} \).

for some smooth function \( F : [0, \infty) \to \mathbb{R} \). In the following we shall identify, with some abuse of notation, \( W \) with \( F \).

The energy is given by.

\[ E = \int \left( \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right) dx \]  

Moreover the Schrödinger equation has another important integral of motion

\[ C = \int |\psi|^2 dx \]  

to which we will refer as charge.

We make the following assumptions on the function \( W \):

\[ W(0) = W'(0) = 0 \]  

\[ W''(0) = 2E_0 > 0 \]  

if we set

\[ W(s) = E_0 s^2 + N(s), \]  

then,

\[ \exists s_0 \in \mathbb{R}^+ \text{ such that } N(s_0) < 0 \]  

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there exist q, r in $(2, +\infty)$, s. t.

\[ |N'(s)| \leq c_1 s^{r-1} + c_2 s^{q-1} \]  \hspace{1cm} (18)

\[ N(s) \geq -cs^p, \quad c \geq 0, \quad 2 < p < 6 \text{ for } s \text{ large} \]  \hspace{1cm} (19)

We can apply the abstract theory of section 2 setting:

- $X = H^1(\mathbb{R}, \mathbb{C})$, $u = \psi$;
- $d(\psi, \varphi) = \|\psi - \varphi\|_{H^1}$.

**Theorem 10** Let $W$ satisfy (14),..., (19). Then there exists $\delta_\infty > 0$ such that for every $\delta \in (0, \delta_\infty)$ there exist $c_\delta > 0$ and an orbitally stable state $\psi_\delta \in H^1(\mathbb{R}, \mathbb{C})$, such that $\psi_\delta$ minimizes the energy on the manifold

\[ \mathcal{M}_{c_\delta} = \{ u \in X \mid \int |\psi|^2 \, dx = c_\delta \} . \]

Moreover if $\delta_1 < \delta_2$ we have that $c_{\delta_1} > c_{\delta_2}$.

**Proof:** The proof is an immediate consequence of Th. 52 in [5] (see also [4]).

By the above theorem we have that every $\psi_\delta$ is an orbitally stable state; the following theorem shows that it is a soliton.

**Theorem 11** Let $u_\delta \in H^1(\mathbb{R}, \mathbb{C})$ be a orbitally stable state as in Th. 10. Then $u_\delta$ is a solution of the equation

\[ -\frac{1}{2} \Delta u + \frac{1}{2} W'(u) = \omega u \]  \hspace{1cm} (20)

and

\[ \psi_\delta(t, x) := u_\delta(x)e^{-i\omega t} \]  \hspace{1cm} (21)

solves (10). Namely $u_\delta$ is a (hylomorphic) soliton.

**Proof.** See Proposition 59 of [5] (see also [4]).

### 4 Hylomorphic solitons for the generalized KdV equation

First of all let us show that the "good" solutions of equation (1) have two constants of motion.
Proposition 12 Let $W$ be a $C^2$ function and $u$ be a smooth solution of equation (1) and assume that $u(t, \cdot) \in H^1(\mathbb{R})$, $\frac{\partial u}{\partial t}(t, \cdot) \in L^2(\mathbb{R})$. Then $u$ has two integrals of motion: the energy

$$E = \int \left( \frac{1}{2} \left[ \frac{\partial u}{\partial x} \right]^2 + W(u) \right) dx$$

(22)

and the charge

$$C = \frac{1}{2} \int u^2 dx$$

(23)

Proof. Since $\frac{\partial u}{\partial t}(t, \cdot) \in L^2(\mathbb{R})$ and

$$-\frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x} W'(u) = \frac{\partial u}{\partial t},$$

the integral

$$\int \left( -\frac{\partial^2 u}{\partial x^2} + W'(u) \right) \frac{\partial u}{\partial t} dx$$

is well defined and it equals the time derivative of $E(u(t))$:

$$\frac{d}{dt} E(u(t)) = \int \left( -\frac{\partial^2 u}{\partial x^2} + W'(u) \right) \frac{\partial u}{\partial t} dx$$

(24)

Moreover

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x} (W'(u)) = -\frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} - W'(u) \right)$$

(25)

Substituting (25) in (24), we get

$$\frac{d}{dt} E(u) = \int \left( \frac{\partial^2 u}{\partial x^2} - W'(u) \right) \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} - W'(u) \right) dx$$

$$= \frac{1}{2} \int \frac{\partial}{\partial x} \left[ \left( \frac{\partial^2 u}{\partial x^2} - W'(u) \right)^2 \right] dx = 0$$

Then $E$ is constant along the solution $u$.

Let us now show that also $C$ is constant along $u$. By (23) we have

$$\frac{d}{dt} C(u) = \int u \frac{\partial u}{\partial t} dx = \int u \left( -\frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x} W'(u) \right) dx$$

(26)

Let us compute each piece separately:

$$\int u \left( -\frac{\partial^3 u}{\partial x^3} \right) dx = \int \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} dx = \frac{1}{2} \int \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)^2 = 0$$

(27)
Moreover
\[ \int u \frac{\partial}{\partial x} (W'(u)) dx = - \int W'(u) \frac{\partial u}{\partial x} dx = - \int \frac{\partial}{\partial x} W(u) dx = 0 \quad (28) \]

Substituting (28) and (27) in (26) we get
\[ \frac{d}{dt} C(u) = 0 \]
□

We will apply the abstract theory of section 2 setting:

- \( X = H^2(\mathbb{R}) \);
- \( d(u, v) = \| \psi - \varphi \|_{H^1} \).

To this end, we need the following assumption which guarantees that also the weak solutions in \( H^2(\mathbb{R}) \) have the properties required by the theory.

**Assumption 13** We assume that the equation (1) defines a dynamical system on \( X = H^2(\mathbb{R}) \), namely, for any initial data \( u_0 \in H^2(\mathbb{R}) \) there is a unique (weak) solution in \( C(\mathbb{R}, H^2(\mathbb{R})) \) of the Cauchy problem. Moreover we assume that the energy (22) and the charge (23) are conserved integrals.

**Remark 14** Clearly, assumption 13 depends on \( W \). By the existence theory of Kato [9], the assumption
\[ \limsup_{s \to \pm \infty} \frac{-W''(s)}{s^4} \leq 0 \quad (29) \]
implies the existence of a unique global solution of eq. (1) in \( C(\mathbb{R}, H^2(\mathbb{R})) \) and the conservation of (22) and (23). In particular, if \( W = -|u|^{k+2} \) we need \( k < 4 \).

(for the well posedeness of equation (1) see also [12], [10] and their references).

**Theorem 15** Let \( W \) satisfy the assumptions (14),...,(19). Then there exists \( \delta_\infty > 0 \) such that for every \( \delta \in (0, \delta_\infty) \) there exist \( c_\delta > 0 \) and \( u_\delta \in H^2(\mathbb{R}) \) which minimizes the energy \( E \) on the manifold
\[ \mathfrak{M}_{c_\delta} = \left\{ u \in X \mid \int u^2 dx = c_\delta \right\} . \]

If \( \delta_1 < \delta_2 \) we have that \( c_{\delta_1} > c_{\delta_2} \). Moreover, if also assumption 13 holds, then \( u_\delta \) is an orbitally stable state.
Proof: The proof of this theorem is essentially the same than the proof of th. 10 which can be found in [5], Th. 52 (see also [4]). The reason for this relies on the fact that the energy and the charge for eq. (10) given by (12) and (13) are formally the same than the energy and the charge of eq. (11) given by (22) and (23). The fact that in the first case ψ is complex while in the second case u is real-valued does not affect the estimates.

Another difference concerns the space X which is $H^1(\mathbb{R}, \mathbb{C})$ for eq. (10) and $H^2(\mathbb{R})$ for eq. (11).

The proof of the theorem consists in minimizing a suitable functional $K_\delta$ on $X$ (see (2.43) in [5]). In the case of eq. (11), we minimize first the functional $K_\delta$ on $H^1(\mathbb{R})$ and then we can prove that the set $\Gamma_\delta$ of these minimizers is contained in $H^2(\mathbb{R})$. In fact the minimizers satisfy the following eigenvalue equation

$$-\frac{\partial^2 u}{\partial x^2} + W'(u) = \lambda_\delta u$$

and hence, since $W \in C^2$, by the standard elliptic regularization, we have that $\Gamma_\delta \subset H^2(\mathbb{R})$.

□

Remark 16 Assumption (15) is not necessary. It is not restrictive to assume that $E_0 > 0$

Proof. In fact consider the following equations

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x} W_0'(u) = 0$$

where $W_0''(0) = -2E_0 < 0$. In this case it is convenient to consider the equation

$$\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} - \frac{\partial}{\partial x} W'(v) = 0$$

where $W(s) = W_0(s) + (E_0 + 1)s^2$. We have that

$W''(0) = 2 > 0$

and to every solution $v$ of eq. (31) corresponds a solution

$u(t, x) = v(t, x + ct)$ with $c = 2(E_0 + 1)$

of eq. (30). In fact

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x} W_0'(u) = \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - \frac{\partial}{\partial x} W_0'(v)$$

$$= \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - \frac{\partial}{\partial x} [W'(v) - 2(E_0 + 1)v]$$

$$= \frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} - \frac{\partial}{\partial x} W'(v) + c\frac{\partial v}{\partial x} - 2(E_0 + 1)\frac{\partial v}{\partial x}$$

$$= 0$$

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We shall prove that the minimizer $u_\delta$ in Theorem 15 is a soliton (def. 7).

**Theorem 17** Under the assumptions and the notations of Th. 15, the minimizer $u_\delta$ is a (hylomorphic) soliton. Moreover it is a solution of the equation

$$\frac{\partial^2 u_\delta}{\partial x^2} - \frac{\partial}{\partial x} W'(u_\delta) = c_\delta \frac{\partial u_\delta}{\partial x}$$

and

$$U_\delta(t,x) := u_\delta(x - c_\delta t)$$
solves (1).

**Proof of Theorem 17** By th. 15 the minimizer $u_\delta$ is an orbitally stable state. So, in order to show that it is a soliton (def. 7), we need to prove that $u_\delta$ is a solitary wave (def. 2).

Since $u_\delta$ is a minimizer of the energy $E$ on the manifold $\mathcal{M}_{c_\delta}$, there exists a Lagrange multiplier $c_\delta$ s.t.

$$E'(u_\delta) = -c_\delta C'(u_\delta).$$

The above equality can be written as follows

$$-\frac{\partial^2 u_\delta}{\partial x^2} + W'(u_\delta) = -c_\delta u_\delta$$

So, if we take the derivative $\frac{\partial}{\partial x}$ on both side, we get (32). Finally (32) implies that the travelling wave $u(t,x) = u_\delta(x - c_\delta t)$ solves (1) and consequently $u_\delta$ is a solitary wave.

**Corollary 18** Equation (3) admits hylomorphic solitons for $k = 1, 2, 3$.

**Proof:** Take

$$W(s) = -\frac{s^{k+2}}{(k+2)(k+1)}.$$  

(33)

For $k = 1, 2, 3$ the function $W$ satisfies (14), (19) and (29). So, by Remark 14 also the assumption 13 is satisfied. Then, by Theorem 17 equation (3), for $k = 1, 2, 3$, admits hylomorphic solitons.

**Corollary 19** If $W(s) = -|s|^{k+2}$, then equation (11) admits hylomorphic solitons for $k \in (0, 4)$.

**Proof:** The proof is the same as for the above corollary.
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