Conservative limit of two-dimensional spectral submanifolds

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Abstract. The paper considers two-dimensional spectral submanifolds (SSM) of equilibria of finite dimensional vector fields. SSMs are the smoothest invariant manifolds tangent to an invariant linear subspace of an equilibrium. The paper assumes that the vector field becomes conservative at the zero limit of a parameter. It is known that in the conservative limit there exists a unique sub-centre manifold. It is also known that the non-conservative system has a unique SSM under some conditions. However, it is not clear whether the sub-centre manifold is the limit of the SSM and if this limit is smoothly approached. In this paper, we show that the unique SSM continuously approaches the sub-centre manifold as the system tends to the conservative limit.

1. Introduction. Conservative systems play a significant role in many modelling problems, because most physical laws lead to such idealised systems. In Hamiltonian mechanics, the Hamilton function is a conserved quantity, the Schrödinger equation conserves probability and Maxwell’s equations conserve charge. It is then reasonable to think that under idealised conditions many physical models will be a small perturbation away from conservative systems. Such an assumption is often made in the mechanical vibrations literature, where conservative systems are thought to be good approximations of damped systems displaying slowly decaying vibrations. There are a number of results on the persistence of qualitative behaviour when a conservative system gradually becomes dissipative. The KAM [13] theory guarantees the persistence of quasi-periodic orbits under non-resonance conditions. Fenichel’s result [6] guarantees the persistence of a differentiable manifold, when the manifold is normally hyperbolic. It is however not yet known whether Lyapunov sub-centre manifolds (LSCM) would persist, even though this is frequently assumed in the engineering literature [10].

Lyapunov sub-centre manifolds (LSCM) occur in conservative systems and defined about equilibria. LSCMs are two dimensional invariant manifolds tangent to an invariant linear subspace corresponding to a pair of purely imaginary eigenvalues. Kelley has shown that LSCMs are unique [8, 9] in differentiable and analytic systems under some non-resonance conditions. When a conservative system with an LSCM under a small perturbation becomes dissipative, the same conditions no longer guarantee a unique manifold. Instead, there will be a family of invariant manifolds. To regain uniqueness the manifold has to be differentiable for a specified number of times, depending on the ratio of the eigenvalues at the equilibrium, called the spectral coefficient. If a unique manifold exists, it is the smoothest invariant manifold tangent to an invariant subspace, which is called the spectral submanifold (SSM) [2, 7]. It is unclear whether the conservative limit of an SSM is the LSCM.

One important application of LSCMs are mechanical systems, where the dynamics on the LSCM can serve as a reduced order model taking into account a vibration with a single frequency. Since LSCMs consist of periodic orbits, an LSCM is also a family of nonlinear normal modes (NNM) as defined by Rosenberg [15] and later generalised by Kerschen et
NNMs are thought to provide the skeleton of the dynamics within lightly damped mechanical systems [10]. However, in conservative systems, periodic orbits cease to exist when damping is introduced. Shaw and Pierre [16] have introduced a different definition of NNMs, that is, an invariant manifold tangent to a linear subspace about an equilibrium. They however did not insist on smoothness conditions, which means that under their definition the NNM is not unique. Haller and Ponsioen [7] have recognised the shortcoming of Shaw and Pierre’s definition and introduced the notion of SSMs. SSMs can be calculated as a power series expansion [14, 11] and one can show that each coefficient in such an expansion is a smooth perturbation of the equivalent coefficient in the expansion of the LSCM.

The reason why the connection between LSCMs and SSMs is not yet explored, is that the mathematical techniques dealing with the two types of manifolds are incompatible. The theory of Cabre et al. [2] and de la Llave [3] uses the Banach fixed point theorem in which the operator, whose fixed point is the SSM stops being a contraction or the size of the manifold reduces to a point, when the damping is removed from the system. Kelley’s theorems relies on the fact that the LSCM is a family of periodic orbits, while on an SSM orbits are locally asymptotic to the equilibrium. In this paper we extend the approach taken by de la Llave [3] and utilise a variant of the Riemann-Lebesgue lemma to show that a carefully constructed operator is a contraction and therefore its fixed point, which is the SSM and the LSCM at the same time, uniquely exists and continuously depends on the damping parameter. We note that our modification to [3] is not necessary if the LSCM is normally elliptic, i.e., all orbits are quasi-periodic about the LSCM. This situation may occur in Hamiltonian systems. Another case when our extension is not necessary is when the equilibrium is elliptic, but the LSCM is repelling.

1.1. Notation.

- The operator $D$ denotes differentiation. The subscript $j$ in $D_j$ means a partial derivative with respect to the $j$th argument of the function. A superscript means the order of the derivative, for example $D^k_j$ it the $k$th partial derivative with respect to the $j$th variable.
- Differentiation with respect to a variable named $\tau$ is denoted by $\partial_\tau$.
- The real part and the imaginary part of a complex number $x$ are denoted by $\Re x$ and $\Im x$, respectively.

2. The main result. Consider the differential equation

\[ \dot{x} = A_\varepsilon x + N_\varepsilon(x), \]

where $A_\varepsilon$ is a $2\nu \times 2\nu$ real matrix and $N_\varepsilon : \mathbb{R}^{2\nu} \to \mathbb{R}^{2\nu}$ is a real analytic function [12] for $0 \leq \varepsilon < K$. Without restricting generality, we assume that the origin is an equilibrium of (2.1), that is $N_\varepsilon(0) = 0$ and that $D N_\varepsilon(0) = 0$. We assume that $A_\varepsilon$ leaves $E_\varepsilon$, a two-dimensional linear subspace of $\mathbb{R}^{2\nu}$, invariant, such that $E_\varepsilon = A_\varepsilon E_\varepsilon$. The general problem deals with the existence and uniqueness of an invariant manifold $M_\varepsilon$, that is tangential to $E_\varepsilon$ under the following further assumptions.

1. We assume that there exists a homogeneous linear transformation that brings $A_\varepsilon$ into a diagonal matrix $\Lambda_\varepsilon$. 

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2. We assume that the diagonal elements of $A\varepsilon$ (or eigenvalues of $A\varepsilon$) are complex conjugate pairs with non-zero imaginary parts. These eigenvalues are denoted by $\lambda_k$, $k = 1, 2, \ldots, 2\nu$. We also assume that the real parts of the eigenvalues vanish and have a finite derivate at $\varepsilon = 0$, that is, there exist $\alpha_k$, such that $\Re\lambda_k = \varepsilon\alpha_k$. The right and left eigenvectors of $A\varepsilon$ are denoted by $v_k$ and $v^*_k$, respectively, for $k = 1, 2, \ldots, 2\nu$. We assume that the eigenvectors are normalised such that

$$v^*_k \cdot v_k = 1.$$ 

3. We assume that the complex conjugate pair of eigenvalues of $A\varepsilon$ corresponding to the invariant linear subspace $E\varepsilon$ have indices $\ell$ and $\ell + 1$. Without restricting generality we can also assume that $\Re\lambda_\ell = \Re\lambda_{\ell+1} = -\varepsilon$ and $\Im\lambda_\ell = -\Im\lambda_{\ell+1} = 1$, which can be achieved by re-scaling $\varepsilon$ and time.

4. We assume that there exists an integer $\sigma \geq 2$ such that

$$\mathcal{A} = \max_{1 \leq j \leq 2\nu, j \notin \{\ell, \ell + 1\}} \sup_{0 \leq \varepsilon \leq K} \frac{\Re\lambda_j(\varepsilon)}{\Re\lambda_\ell(\varepsilon)} < \sigma.$$ 

The value $\mathcal{A}$ is called the spectral coefficient. (cf. [7].)

5. For $\varepsilon = 0$ there is an analytic function $c : \mathbb{R}^{2\nu} \to \mathbb{R}^{2\nu}$, such that

$$Dc(x) \cdot (A_0x + N_0(x)) = 0.$$ 

The function $c$ is called the conserved quantity of (2.1) for $\varepsilon = 0$.

6. We define the Hessian $H = D^2c(0)$ and assume that the 2-by-2 submatrix $(\Re v_\ell, \Im v_\ell)^T H (\Re v_\ell, \Im v_\ell)$ is positive definite.

7. The eigenvalues of $A\varepsilon$ are non-resonant, that is,

$$\Im\lambda_j \notin \mathbb{Z}$$

for all $j \neq \ell, \ell + 1$.

**Definition 2.1.** Let us denote the transformation in assumption 1 by $T$, such that $A\varepsilon = T\Lambda\varepsilon T^{-1}$ and define

$$\Delta = \varepsilon^{-1}T\Re(A\varepsilon)T^{-1}, \Omega = Ti\Im(A\varepsilon)T^{-1}.$$ 

**Corollary 2.2.** Due to assumption 2, $\Delta$ is bounded for all $0 \leq \varepsilon < K$. From the definition (2.3) it immediately follows that

$$A\varepsilon = \varepsilon\Delta + \Omega.$$ 

Furthermore, $\Delta$ and $\Omega$ are real matrices that commute, that is, $\Delta\Omega = \Omega\Delta$.

**Proof.** The only statement that does not directly follow from inspecting the definition is that $\Delta$ and $\Omega$ are real matrices. Since the columns of $T$ are the eigenvectors $v_k$ of $A\varepsilon$, the column vectors appear as complex conjugate pairs. Therefore the rows of the inverse $T^{-1}$ are complex conjugate pairs, as well. Evaluating the matrix products in (2.3) shows that both $\Delta$ and $\Omega$ are real matrices. \qed
Remark 2.3. We require an even dimensional phase space of dimension $2\nu$. This is due to the non-resonance condition 7, which does not allow a zero eigenvalue at $\varepsilon = 0$, therefore all eigenvalues must have a non-zero imaginary part at $\varepsilon = 0$. In real systems, such as (2.1) complex eigenvalues come in pairs, therefore an even dimensional phase space is necessary.

Remark 2.4. Assumption 2 requires purely imaginary eigenvalues at $\varepsilon = 0$. This is necessary, because in the proof of Theorem 2.6 we re-scale time by $\varepsilon$, such that $\Delta$ remains bounded as $\varepsilon \to 0$. This assumption can be replaced by the system needing to be Hamiltonian, which satisfies further non-resonance conditions [4]. Without re-scaling time, the contraction in section 3.7 becomes weak, that is, the contraction rate would tend to one as $\varepsilon \to 0$, if assumption 2 is replaced.

Theorem 2.5. Under assumptions 1-7, and for $\varepsilon = 0$ there exists a unique analytic invariant manifold $M_0$ that is tangential to the two-dimensional linear subspace $E_0$ in a sufficiently small neighbourhood of the origin. This invariant manifold consists of periodic orbits and the equilibrium. The manifold $M_0$ is called the Lyapunov sub-centre manifold (LSCM).

Proof. The proof can be found in [9].

By virtue of all orbits on the LSCM being periodic, one can construct an immersion of $M_0$, which is a one-parameter family of $2\pi$-periodic functions, denoted by $W_0 : [0, \gamma) \times [0, 2\pi] \to \mathbb{R}^{2\nu}$ with $W_0(r, 0) = W_0(r, 2\pi) \forall r \in [0, \gamma)$ and $\gamma > 0$. To each periodic orbit as parametrised by $r$ corresponds a period, which is given by $T_0 : [0, \gamma) \to (0, \infty)$. The differential equation that the periodic orbits must satisfy is

$$D_2 W_0(r, \theta) T_0(r) = A_0 W_0(r, \theta) + N_0 (W_0(r, \theta)),$$

which is a result of substituting $W_0$ into equation (2.1) and re-scaling time by $T_0$. The immersion $W_0$ is not uniquely given by (2.4), because of two indeterminacies. Firstly, the parametrisation in $r$ is not fixed. Secondly, a constant phase shift $\theta(\vartheta) = \theta_0 + \vartheta$ with $\theta_0 \in \mathbb{R}$ also leaves equation (2.4) invariant, because (2.1) is autonomous. It is clear that any variations on the parametrisation of $W_0$ gives the same invariant manifold due to Theorem 2.5.

To fix the parametrisation in $r$ we set

$$W_0(0, \theta) = 0$$

and require that

$$\int_0^{2\pi} W^*(\theta) \cdot D_1 W_0(r, \theta) d\theta = 1,$$

where

$$W^*(\theta) = (4\pi)^{-1} \left( v^*_\ell e^{-i\theta} + v^*_{\ell+1} e^{i\theta} \right).$$

Note that $W^*$ is a real valued function. Equation (2.6) is called the growth condition, because it prescribes that the first Fourier components of the periodic orbits grow uniformly with increasing $r$. Integrating (2.6) with respect to $r$ with the initial condition stemming from
(2.5) provides a unique value of \( r \) for each periodic orbit. The uniqueness is because the first order harmonics of \( D_1W_0 \) grow linearly with the distance from the origin while higher order harmonics grow with a higher power of that distance. Hence, in a sufficiently small neighbourhood of the origin the magnitude of the first harmonics uniquely selects a periodic orbit.

The phase shift indeterminacy can be fixed by a so-called phase condition, which has been shown in [1] to determine the phase of a periodic orbit uniquely. Such phase conditions are routinely used to numerically calculate orbits, e.g., in [5]. Our phase condition is given by

\[
(2.8) \quad \int_0^{2\pi} D\dot{W}^* (\theta) \cdot D_1W_0 (r, \theta) \, d\theta = 0.
\]

**Theorem 2.6.** Under assumptions 1-6, for equation (2.1) there exists a unique invariant manifold \( M_\epsilon \) for \( 0 \leq \epsilon < K \), which is

1. tangent to the linear subspace \( E_\epsilon \);
2. analytic in the phase space variables apart from the origin, where it is (at least) \( C^\infty \);
3. continuous in \( \epsilon \) for \( 0 \leq \epsilon < K \) and differentiable at \( \epsilon = 0 \).

Note that point 3 implies convergence of \( M_\epsilon \) to \( M_0 \) as \( \epsilon \to 0 \).

**Proof.** The proof is carried out in the following sections.

3. **The proof of Theorem 2.6.** The technique used by de la Llave [3] and Cabre et al [2] to prove existence and uniqueness of SSMs has two steps. It first approximates the SSM using a power series in the phase space variables and then finds a unique correction using the Banach contraction mapping theorem. It turns out that a straightforward application of this technique is not possible in case of vanishing \( \epsilon \), because the domain where the contraction mapping can be applied shrinks to a point as \( \epsilon \to 0 \). A major difference between SSMs and LSCMs is how nearby trajectories behave. For an SSM of an attracting equilibrium the Lyapunov exponents corresponding to nearby trajectories are real parts of the eigenvalues of the Jacobian of the vector field at the equilibrium. In contrast, the Lyapunov exponents of trajectories near an attracting LSCM vary with initial conditions. Therefore the spectral coefficient defined by (2.2), has no relevance to the dynamics about an LSCM apart from the equilibrium. The proofs in [3, 2] require that the ratio of Lyapunov exponents between the normal and the tangential direction of the manifold remain finite. However as \( \epsilon \to 0 \), the Lyapunov exponent in the tangential direction also tends to zero and therefore about a periodic orbit on the LSCM – not identical to the origin – such a spectral ratio may not be defined. An obvious fix is to require that the LSCM is normally elliptic such that all nearby trajectories have zero Lyapunov exponents. This special case occurs in Hamiltonian systems, which is exploited in [4]. The difference between a normally hyperbolic and a normally elliptic LSCM is illustrated in figure 3.1. In this paper we do not assume a normally elliptic LSCM.

We also note that the period \( T_0 \) of the orbits on the LSCM is generally not polynomial, hence we cannot allow a similar flexibility as in Theorem 1.1 of [2] and describe the dynamics on the SSM by a large class of polynomial vector fields. For more details, see Remark 3.1.

To overcome the issue with non-Hamiltonian limits we take into account the oscillatory nature of the solutions and not just exponential rates. This allows us to better estimate the
Figure 3.1. Two particular cases of the $\varepsilon \to 0$ limit. Blue lines represent orbits on the SSM or the LSCM, red lines denote trajectories near the SSM or LSCM. (a) A typical scenario of a 4 dimensional system with two frequencies for $\varepsilon > 0$, where the equilibrium is asymptotically stable and there exists a unique SSM. (b) as $\varepsilon \to 0$ the SSM turns into an LSCM, and nearby trajectories away from the LSCM converge asymptotically to the periodic orbits of the LSCM away from the equilibrium. The equilibrium is still a centre with purely imaginary eigenvalues. (c) the periodic orbits on the LSCM are of centre type with two $+1$ Floquet multipliers and two elsewhere on the complex unit circle.

contraction rate of the operator whose fixed point is the SSM/LSCM. It turns out that the operator is similar to a one-sided Fourier transform, whose norm is finite with regards to the total variation of the function whose Fourier transform is sought. We then exploit that the total variation of an analytic function can be estimated by its supremum norm (the natural norm of continuous functions) over its domain of analyticity, which leads us to the conclusion that the operator is a contraction independent of parameter $\varepsilon$. The limitation of this approach is that we need a polar coordinate system to uncover the Fourier transform and restrict ourselves to analytic SSMs and LSCMs. The consequence of using a polar coordinate system is that we need a separate argument to show smoothness of SSM/LSCM at the origin, which concludes that the SSM/LSCM is at least $C^\infty$ smooth at the origin. We note that the same analysis is not possible for $C^r$ manifolds, but would perhaps be possible for finitely many times differentiable SSMs with the highest derivatives being of bounded variation or absolutely continuous.

3.1. Approximate solution of the invariance equation. We take two steps to establish the existence and uniqueness of a SSM for $\varepsilon > 0$ and its convergence to the LSCM as $\varepsilon \to 0$. The first step in this process approximates $\mathcal{M}_\varepsilon$ to an order of accuracy, that is greater than the spectral coefficient. The second step finds a unique correction to the approximate solution. In what follows we use growth and phase conditions similar to \((2.6)\) and \((2.8)\) and find a unique solution to the invariance equation. Finally we argue that any solution of the invariance equation can be re-parametrised to satisfy the growth and phase conditions and therefore all
analytic solutions of the invariance equation correspond to a unique $\mathcal{M}_\varepsilon$.

Let us consider the invariance equation of $\mathcal{M}_\varepsilon$ for $\varepsilon \geq 0$, in polar coordinates

$$D_1 W (r, \theta) R (r) + D_2 W (r, \theta) T (r) = A_\varepsilon W (r, \theta) + N_\varepsilon (W (r, \theta)),$$

where $r \in [0, \gamma]$, $0 < \gamma \leq 1$ and $\theta \in [0, 2\pi]$. The invariant manifolds is

$$\mathcal{M}_\varepsilon = \{ W (r, \theta) : r \in [0, \gamma], \theta \in [0, 2\pi] \}.$$

The choice of $\gamma$ will be specified later to make sure that $DN_\varepsilon (W (r, \theta))$ is sufficiently small for $r \in [0, \gamma]$. Note that equation (3.1) is not in the most general form, because $R$ and $T$ are assumed to be independent of $\theta$. Nevertheless, if there is a solution to (3.1), it is an immersion of an invariant manifold. We represent the immersion $W$ as a solution of (3.1) in terms of the LSCM and perturbative terms that are proportional to $\varepsilon$, such that

$$W = W_0 + \varepsilon (W^\leq + W^>) ,$$

where $W_0$ is the immersion of the LSCM and $W_0 + \varepsilon W^\leq$ solves the invariance equation up to order $\sigma - 1$ in $r$. We also assume that

$$R = \varepsilon R^\leq, \quad T = T_0 + \varepsilon T^\leq,$$

where $R^\leq$ and $T^\leq$ are polynomials of order $\sigma - 1$ and $\sigma - 2$ in $r$, respectively. Whether the forms of $R$ and $T$ are appropriate, will be checked in the next section.

Similar to the conservative case we need to apply phase and growth conditions to $W^\leq$ for a unique solution, which are

$$\int_0^{2\pi} W^* (\theta) \cdot D_1 W^\leq (r, \theta) \, d\theta = 0, \quad \int_0^{2\pi} D W^* (\theta) \cdot D_1 W^\leq (r, \theta) \, d\theta = 0.$$

Conditions (2.6) and (2.8) account for near internal resonances on $\mathcal{M}_\varepsilon$ [17]. We note that there is no need to impose conditions on $W^>$, because $W^>$ will be solved for fixed $T$ and $R$, which fixes the parametrisation.

**Remark 3.1.** The function $T$ cannot always be represented by a polynomial, therefore we need that at $\varepsilon = 0$, $T = T_0$. Imagine that the derivatives $D^k T_0(0) = 0$ for $1 \leq k \leq \sigma - 2$, but there is a $l > \sigma - 2$ such that $D^l T_0(0) \neq 0$. In this case $T_0$ is not a constant, yet a polynomial approximation of order $\sigma - 2$ would be constant ($T(r) = T_0(0)$). For such a choice of $T$ there would be no meaningful solution for $W_0$, because the only periodic orbit in the neighbourhood of the equilibrium with period $T_0(0)$ is the equilibrium. Representing the SSM for $\varepsilon > 0$ is however possible with polynomials $R$ and $T$ as long as the polynomials are such that low order resonances are taken into account, as described in Cabre et al [2].
3.2. A power series solution of the invariance equation. In this section we solve the invariance equation (3.1) at \( r = 0 \) up to any order \( n \) in \( r \) and show that the invariant manifold is differentiable up to the same order. Let us take the power series of (3.1) at \( r = 0 \) up to order \( n \), which yields

\[
(3.5) \quad \varepsilon \sum_{k=0}^{n} \binom{n}{k} D^{k+1}_1 W D^{n-k} R^\leq + \sum_{k=0}^{n} \binom{n}{k} D^2 D^{k}_1 W D^{n-k} T = A_\varepsilon D^n_1 W + \frac{\partial^n}{\partial r^n} N_\varepsilon (W).
\]

We use the power series for the derivatives

\[
(3.6) \quad D^n_1 W (0, \theta) = \sum_{k=-n}^{n} a^n_k e^{ik\theta},
\]

to find a solution to (3.5), where \( a^n_k = \overline{a}^{n,-k} \), because \( D^n_1 W \) must be real valued. In (3.6) we assumed a finite order of the expansion, which will turn out to be correct by the end of the following calculation. For now we solve for \( a^n_k \) and not think about the limit in \( k \).

The solution of (3.5) can be carried out in increasing order, starting with \( n = 1 \). For \( n = 1 \) we have

\[
(3.7) \quad \varepsilon D_1 W D R^\leq + D_2 D_1 W T = A_\varepsilon D_1 W,
\]

which is an eigenvector-eigenvalue problem. In terms of the power series expansion (3.7) can be decomposed into Fourier components

\[
(3.8) \quad (\varepsilon D R^\leq + i k T - A_\varepsilon) a^1_k = 0.
\]

Since we are looking for an invariant manifold tangential to the linear subspace \( E_\varepsilon \), the solution is \( D R^\leq (0) = \varepsilon^{-1} \Re \lambda_\ell = -1 \), \( T(0) = \Im \lambda_\ell = 1 \) (see assumption 3) and the eigenvectors are

\[
(3.9) \quad a^1_\ell = \bar{d} \nu_\ell, \quad a^{1,-1}_\ell = \bar{-d} \bar{\nu}_\ell, \quad d \neq 0, \quad d \in \mathbb{C}.
\]

Due to the non-resonance conditions 7, the matrix coefficient in (3.8) cannot be singular for \( k \neq \pm 1 \), therefore \( a^n_0 = 0 \) for \( k \neq \pm 1 \). As a result, we have

\[
(3.10) \quad D_1 W (0, \theta) = d \nu_\ell e^{i\theta} + \bar{d} \bar{\nu}_\ell e^{-i\theta}.
\]

Applying the phase and growth conditions, we find that

\[
\int_0^{2\pi} W^\ast (\theta) \cdot D_1 W (0, \theta) d\theta = \Re d = 1,
\]  
\[
\int_0^{2\pi} D W^\ast (\theta) \cdot D_1 W (0, \theta) d\theta = \Im d = 0,
\]

hence \( d = 1 \). Given that \( D_1 W (0, \cdot) = D_1 W_0 (0, \cdot) \), we find that \( D_1 W^\leq (0, \cdot) = 0 \).

For \( n > 1 \), we can rearrange equation (3.5) at \( r = 0 \) into

\[
(3.11) \quad \varepsilon D_1 W D^n R^\leq + \varepsilon D^n_1 W D R^\leq + D_2 D_1 W D^{n-1} T + D_2 D^n_1 W T = A_\varepsilon D^n_1 W + \eta^n,
\]
where

\[ \eta^n = \frac{\partial^n}{\partial ^n} N_\varepsilon (W) - \varepsilon \sum_{k=1}^{n-2} \binom{n}{k} D_1^{k+1} W D^{n-k} R^\varepsilon - \sum_{k=2}^{n-1} \binom{n}{k} D_2 D_k^1 W D^{n-k} T. \]

We note that \( \eta^n (0, \cdot) \) depends on \( D^k W (0, \cdot) \) with \( k < n \) and therefore (3.11) can be solved recursively. We also series expand

\[ \eta^n (\theta) = \sum_{j=-n}^{n} \eta^n_k e^{ijk} \theta, \text{ with } \eta^n_k = \eta^n_k, \]

which brings (3.11) into

\[
(3.12) \quad (\varepsilon DR^\varepsilon + ikT - A_\varepsilon) a^n_k + \varepsilon a^n_1 D^n R^\varepsilon + ik a^n_1 D^{n-1} T = \eta^n_k.
\]

We know that for \( k = \pm 1 \) the matrix coefficient of \( a^n_k \) in (3.12) is singular, with left singular vectors \( v^*_\ell, v^*_{\ell+1} \) and right singular vectors \( v_\ell, v_{\ell+1} \) as in (3.8). We now dot-multiply (3.12) by the left eigenvectors \( v^*_\ell, v^*_{\ell+1} \) and recall that \( a^n_1 = v_\ell \) and \( a^n_{-1} = v_{\ell+1} \) and that \( v^*_\ell \cdot v_\ell = 1 \), which yields that for \( k = \pm 1 \) (3.12) expands into

\[
(3.13) \quad \varepsilon D^n R^\varepsilon + iD^{n-1} T = v^*_\ell \cdot \eta^n_1,
\]

\[
(3.14) \quad \varepsilon D^n R^\varepsilon - iD^{n-1} T = v^*_{\ell+1} \cdot \eta^n_{-1} = v^*_\ell \cdot \eta^n_1.
\]

The solution of equations (3.13) and (3.14) is

\[
(3.15) \quad D^n R^\varepsilon = \varepsilon^{-1} \Re (v^*_\ell \cdot \eta^n_1), \quad D^{n-1} T = \Im (v^*_\ell \cdot \eta^n_1).
\]

Noticing that \( \eta^n_1 \) is a Fourier coefficient and the growth and phase conditions calculate Fourier coefficients, we can also write

\[
(3.16) \quad D^n R^\varepsilon (0) = \varepsilon^{-1} \int_0^{2\pi} W^* (\theta) \cdot \eta^n (0, \theta) d\theta,
\]

\[ D^{n-1} T (0) = \int_0^{2\pi} DW^* (\theta) \cdot \eta^n (0, \theta) d\theta. \]

We now show that the limit \( \varepsilon \to 0 \) in (3.16) makes sense. Setting \( \varepsilon = 0 \) and applying the growth condition (3.3) to equation (3.11) yields that \( \int_0^{2\pi} W^* (\theta) \cdot \eta^n |_{\varepsilon=0} (0, \theta) d\theta = 0 \). Therefore we can also write that

\[ D^n R^\varepsilon = \int_0^{2\pi} W^* (\theta) \cdot \frac{\eta^n (0, \theta) - \eta^n |_{\varepsilon=0} (0, \theta)}{\varepsilon} d\theta, \]

which tends to a derivative as \( \varepsilon \to 0 \). The coefficients \( a^n_1, a^n_{-1} \) are not determined by (3.12), but the growth and phase conditions (3.3), (3.4), respectively imply that \( a^n_1 = a^n_{-1} = 0 \). For all other \( k \neq \pm 1 \) the solution of (3.12) is given by

\[
(3.17) \quad a^n_k = (\varepsilon D^1 R^\varepsilon + ikT - A_\varepsilon)^{-1} \left( \eta^n_k - \varepsilon a^n_1 D^n R^\varepsilon - ika^n_1 D^{n-1} T \right).
\]
Note that our results $a_k^1 = 0$ for $k \neq \pm 1$ and $a_n^n = a_{n+1}^n = 0$ for $n > 0$ imply that
\begin{equation}
(3.18) \quad a_k^n = (\varepsilon D^1 R^\leq + i k T - A_\varepsilon)^{-1} \eta_k^n, \quad n > 1.
\end{equation}

We now show that the solution to the invariance equation leads to a $n$-times differentiable invariant manifold, in particular at the origin. A $n$-times differentiable immersion, using a Cartesian parametrisation has the form
\begin{equation}
(3.19) \quad \hat{W}(x, y) = \sum_{0 < p + q \leq n} b_{pq} (x + iy)^p (x - iy)^q + O \left( (|x, y|)^{n+1} \right),
\end{equation}
where $b_{pq} = \overline{b}_{pq}$, so that the result is real valued. We show that the coefficients $a_k^n$ uniquely determine $b_{pq}$ and in reverse. If we transform (3.19) to polar coordinates, such that $x + iy = re^{i\theta}$, we then get
\begin{equation}
W(r, \theta) = \hat{W}(r \cos \theta, r \sin \theta) = \sum_{0 < p + q \leq n} b_{pq} r^{p+q} e^{i(p-q)\theta}.
\end{equation}
Calculating the $2k$-th derivative and substituting $p = k + j, q = k - j$ yields
\begin{equation}
D^{2k} \hat{W} (0, \theta) = (2k)! \sum_{j=-k}^{k} b_{(k+j)(k-j)} e^{i2j\theta}
\end{equation}
and calculating the $2k + 1$-th derivative and substituting $p = k + j, q = k + 1 - j$ gives
\begin{equation}
D^{2k} \hat{W} (0, \theta) = (2k+1)! \sum_{j=-k}^{k+1} b_{(k+j)(k+1-j)} e^{i(2j-1)\theta}.
\end{equation}
Therefore we must have
\begin{equation}
(3.20) \quad \begin{cases}
    a_{2j}^{2k} = (2k)! b_{(k+j)(k-j)}, & -k \leq j \leq k \\
    a_{2j+1}^{2k+1} = (2k+1)! b_{(k+j)(k+1-j)}, & -k \leq j \leq k + 1
\end{cases}
\end{equation}
and
\begin{equation}
(3.21) \quad \begin{cases}
    a_{j}^{k} = 0 & |j| > k \\
    a_{2j}^{2k+1} = 0 & j \in \mathbb{Z} \\
    a_{2j+1}^{2k+1} = 0 & j \in \mathbb{Z}
\end{cases}
\end{equation}
In equation (3.20) the terms uniformly map to each other, therefore we only need to show that the claimed zero terms in (3.21) do vanish. From the claim (3.21) and (3.15) it follows that
\begin{equation}
(3.22) \quad D^{2k} R^\leq (0) = D^{2k-1} T (0) = 0, \quad k = 1, 2, \ldots
\end{equation}
We can show that (3.20),(3.21) hold by induction. By the result (3.10), (3.20) holds for the first order terms $a_j^1$. We now show that if (3.20),(3.21) hold for $a_{j}^{n-1}$, it must also hold for $a_{j}^{n}$.
and \( n > 1 \). From equation (3.18) we infer that \( a_n^k = 0 \) if and only if \( \eta_n^k = 0 \). We now expand \( \eta^n \) term-by-term. For the nonlinear term we have

\[
\frac{\partial^n}{\partial r^n} N_\varepsilon (W) \bigg|_{r=0} = \sum_{k=2}^n \sum_{j_1+j_2+\cdots+j_k=n} c_k D^k N_\varepsilon (0) D^{j_1}_1 \cdots D^{j_k}_k W,
\]

which purely depends on \( D^k W (0, \cdot) \), \( k < n \) such that for odd \( n \) only odd and for even \( n \) only even Fourier coefficients appear. Now using (3.15) we infer that the same is true for the remaining terms in \( \eta^n \), therefore

\[
\eta_n^{2k} (0, \theta) = \sum_{j=-k}^k \eta_n^{2k} e^{i2j\theta}, \quad \eta_n^{2k+1} (0, \theta) = \sum_{j=-k}^k \eta_n^{2k+1} e^{i(2j-1)\theta},
\]

which then yields the claimed zero terms in (3.20) through equation (3.17).

The argument in this section has shown that if the manifold exists, it is \( C^\infty \), because it is differentiable any number of times at the origin. Here we have also shown that equation (3.6) stands.

### 3.3. The equation for the correction \( W^> \).

Since we have solved the invariance equation up to order \( \sigma - 1 \) in \( r \), both sides of (3.1) will vanish up to and including order \( \sigma - 1 \) once \( W = W_0 + \varepsilon W^\leq \) is substituted. Therefore we define

\[
(3.23) \quad \varepsilon \hat{F}_\varepsilon = \varepsilon R^\leq \left( D_1 W_0 + \varepsilon D_1 W^\leq \right) + T \left( D_2 W_0 + \varepsilon D_2 W^\leq \right) - A_\varepsilon \left( W_0 + \varepsilon W^\leq \right) - N_\varepsilon \left( W_0 + \varepsilon W^\leq \right),
\]

where \( \hat{F}_\varepsilon = O (r^\sigma) \). We can then remove (3.23) from equation (3.1) and divide by \( \varepsilon \), which yields

\[
(3.24) \quad \varepsilon R^\leq D_1 W^> + T D_2 W^> = A_\varepsilon W^> + \tilde{N} (r, \theta, W^>),
\]

where

\[
(3.25) \quad \tilde{N} (r, \theta, z) = \varepsilon^{-1} \left[ N_\varepsilon (W_0 (r, \theta) + \varepsilon W^\leq (r, \theta) + \varepsilon z) - N_\varepsilon (W_0 (r, \theta) + \varepsilon W^\leq (r, \theta)) \right] - \hat{F}_\varepsilon (r, \theta).
\]

Note that the division by \( \varepsilon \) at the limit \( \varepsilon \to 0 \) represents a derivative, hence equation (3.25) is well defined.

### 3.4. Solving the invariance equation.

Here we formally solve (3.24). We start by writing down the formal solution of the conjugate vector field \( R \) and \( T \), when time is re-scaled by \( \varepsilon^{-1} \), that is

\[
\frac{d}{dt} \rho^l (r) = R^\leq (\rho^l (r)), \quad \rho^0 (r) = r,
\]

\[
\frac{d}{dt} \psi^l (r, \theta) = \varepsilon^{-1} \left( T_0 (\rho^l (r)) + \varepsilon T^\leq (\rho^l (r)) \right), \quad \psi^0 (r, \theta) = \theta.
\]
The solution to the phase $\psi^t$ can be written as

$$\psi^t (r, \theta) = \theta + \varepsilon^{-1} \phi^t (r),$$

where

$$\phi^t (r) = \int_0^t T_0 (\rho^s (r)) + \varepsilon T^\leq (\rho^s (r)) \, ds.$$ 

By assumption 3, we have $R^\leq (0) = 0$, $DR^\leq (0) = -1$, hence there is a $0 < c_\rho = 1 + O (\gamma)$, such that $c_\rho^{-1} e^{-t} \leq \rho^t (r) \leq c_\rho e^{-t}$, therefore all trajectories on the invariant manifold converge to the origin exponentially. We also note that $\psi^t$ is only defined for $\varepsilon > 0$, but we will find that the limit $\varepsilon \to 0$ makes sense when considering the fixed point operator.

To formally solve (3.24) we use the method of characteristics and define

$$U^{r, \theta} (t) = W^> (\rho^t (r), \psi^t (r, \theta)), \tag{3.26}$$

which transforms (3.24) into

$$\varepsilon D U^{r, \theta} (t) = \varepsilon D_1 W^> (\rho^t (r), \psi^t (r, \theta)) R^\leq (\rho^t (r)) + D_2 W^> (\rho^t (r), \psi^t (r, \theta)) (T_0 (\rho^t (r)) + \varepsilon T^\leq (\rho^t (r)))$$

$$= A_t U^{r, \theta} (t) + \tilde{N} (\rho^t (r), \psi^t (r, \theta), U^{r, \theta} (t)). \tag{3.27}$$

Equation (3.27) is an ODE and can be transformed into an integral equation using the variation-of-constants formula:

$$U^{r, \theta} (t) = \exp (\varepsilon^{-1} A_t) U^{r, \theta} (0) + \varepsilon^{-1} \int_0^t \exp (\varepsilon^{-1} A_t (t - \tau)) \tilde{N} (\rho^\tau (r), \psi^\tau (r, \theta), U^{r, \theta} (\tau)) \, d\tau.$$ 

Going back to the definition (3.26) we find that

$$W^> (\rho^t (r), \psi^t (r, \theta)) = \exp (\varepsilon^{-1} A_t) W^> (r, \theta) + \varepsilon^{-1} \int_0^t \exp (\varepsilon^{-1} A_t (t - \tau)) \tilde{N} (\rho^\tau (r), \psi^\tau (r, \theta), W^> (\rho^\tau (r), \psi^\tau (r, \theta))) \, d\tau. \tag{3.28}$$

Rearranging (3.28) yields

$$W^> (r, \theta) = \exp (-\varepsilon^{-1} A_t) W^> (\rho^t (r), \psi^t (r, \theta))$$

$$- \varepsilon^{-1} \int_0^t \exp (-\varepsilon^{-1} A_t \tau) \tilde{N} (\rho^\tau (r), \psi^\tau (r, \theta), W^> (\rho^\tau (r), \psi^\tau (r, \theta))) \, d\tau.$$ 

Since $W^> (r, \theta) = O (r^\sigma)$, we find that $W^> (\rho^t (r), \psi^t (r, \theta)) = O (e^{-\sigma t})$. On the other hand we have $\exp (-\varepsilon^{-1} A_t) = O (e^{\delta t})$, hence $\exp (-\varepsilon^{-1} A_t) W^> (\rho^t (r), \psi^t (r, \theta)) = O (e^{(\delta - \sigma) t})$, which vanishes as $t \to \infty$, because $\delta - \sigma > 0$ as assumed in (2.2). After this reasoning we are left with

$$W^> (r, \theta) = -\varepsilon^{-1} \int_0^\infty \exp (-\varepsilon^{-1} A_t \tau) \tilde{N} (\rho^\tau (r), \psi^\tau (r, \theta), W^> (\rho^\tau (r), \psi^\tau (r, \theta))) \, d\tau.$$ 

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Hence we define the operator

\[(3.29) \quad T \left( W^> \right)(r, \theta) = -\varepsilon^{-1} \int_0^\infty \exp \left( -\varepsilon^{-1} A_\varepsilon \tau \right) N \left( \rho^\tau (r), \psi^\tau (r, \theta), W^> \left( \rho^\tau (r), \psi^\tau (r, \theta) \right) \right) \, d\tau \]

and seek the correction term \( W^> \) as a fixed point of \( T \). The function space, where \( T \) is defined is clarified in the next section.

3.5. Fourier representation. In this section we represent \( W^> \) as a Fourier series, such that

\[ W^> = \sum_{k=-\infty}^{\infty} a_k (r) e^{ik\theta}. \]

The coefficients \( a_k \) form the elements of the vector space

\[ X = \{ a_k : k \in \mathbb{Z}, a_k : C^\omega (\mathbb{R}^2) \} , \]

where \( C^\omega \) denotes real analytic functions. Due to analyticity, \( a_k \) have a unique extension in a neighbourhood of the interval \([0, \gamma]\) within the complex plane, and assume complex values in this neighbourhood. The norm on \( X \) is

\[(3.30) \quad \|a\| = \sup_{k \in \mathbb{Z}} \sup_{s \in \Gamma_\delta} e^{\delta |k|} \left| s^{-\sigma} \gamma^{-1} a_k (\gamma s) \right| , \]

where

\[ \Gamma_\delta = \{ x \in \mathbb{C} : 0 \leq \Re x \leq 1, -\delta \Re x \leq \Im x \leq \delta \Re x \}. \]

The definition is due to the fact that a periodic function is analytic if and only if its Fourier coefficients are exponentially decaying. The norm \( \|-\| \) makes \( X \) a Banach space. The norm (3.30) implies that

\[ |a_k (r)| \leq e^{-\delta |k| \gamma^{-1}} \|a\| , \]

which will be useful later. We also define the unit ball \( B_1 = \{ a \in X : \|a\| \leq 1 \} \) so that the Fourier representation of \( T \) maps from \( B_1 \) into itself.

In order to establish contraction properties of \( T \) we define the linear operator \( U : X \to X \)

\[(Uf) (r, k) = -\varepsilon^{-1} \int_0^\infty \exp \left( ik \varepsilon^{-1} \phi^\tau (r) - i \varepsilon^{-1} A_\varepsilon \tau \right) f_k \left( \rho^\tau (r) \right) \, d\tau \]

\[= -\varepsilon^{-1} \int_0^\infty \exp \left( ik \varepsilon^{-1} \phi^\tau (r) - i \varepsilon^{-1} \Omega \tau \right) \exp \left( -\Delta \tau \right) f_k \left( \rho^\tau (r) \right) \, d\tau , \]

where we used that \( \Delta \) and \( \Omega \) commute (see Corollary 2.2). Operator \( U \) contains the non-trivial part of \( T \), a division by \( \varepsilon \), which appears twice in the following calculations.

Lemma 3.2. The linear operator \( U \) is bounded on \( X \) for \( 0 \leq \varepsilon \ll 1 \) and the bound can be chosen independent of \( \varepsilon \). Moreover, there exists a finite \( C_3 > 0 \), such that

\[ \| (Uf) (r, k) \| \leq \frac{C_3}{1 + |k|} e^{-\delta |k| \gamma^{-1}} \|f\| . \]
The integral kernel of operator $\mathcal{U}$, applied to $f$, can be decomposed into an oscillatory part and an exponential part. We need to reason about the oscillatory part, therefore we define the exponentially scaled function

$$g_k(r, \tau) = \exp(-\Delta \tau) f_k(\rho^\tau(r)).$$

Note that $\lim_{\tau \to \infty} g_k(r, \tau) = 0$, due to the assumption that $\sigma > \Re$. This notation leads us to

$$\mathcal{U}f(r, k) = -\varepsilon^{-1} \int_0^\infty \exp(ik\varepsilon^{-1} \phi^\tau(r) - i\varepsilon^{-1} \Omega \tau) g_k(r, \tau) \, d\tau. \quad (3.31)$$

Noticing that

$$\exp(ik\varepsilon^{-1} \phi^\tau(r) - i\varepsilon^{-1} \Omega \tau) = \partial_\tau \exp(ik\varepsilon^{-1} \phi^\tau(r) - i\varepsilon^{-1} \Omega \tau) \times \exp(ik \varepsilon^{-1} \partial_\tau \phi^\tau(r) - i\varepsilon^{-1} \Omega)^{-1}, \quad (3.32)$$

and substituting into (3.31), we integrate by parts and get

$$\mathcal{U}f(r, k) = (ik\partial_\tau \phi^0(r) - \Omega)^{-1} f_k(r) + \int_0^\infty \exp(-\varepsilon^{-1} \Omega \tau + ik\varepsilon^{-1} \phi^\tau(r)) \partial_\tau \left[(ik\partial_\tau \phi^\tau(r) - \Omega)^{-1} g_k(r, \tau)\right] \, d\tau. \quad (3.33)$$

Due to the non-resonance conditions of assumption 7, there exists $C_1 > 0$ such that

$$\sup_{\tau \in [0, \infty]} \sup_{r \in [0, \gamma]} |(\Omega + ik\partial_\tau \phi^\tau(r))^{-1}| \leq \frac{C_1}{1 + |k|}. \quad (3.34)$$

The non-integral term in (3.33) can be estimated as

$$\left|(ik\partial_\tau \phi^0(r) - \Omega)^{-1} f_k(r)\right| \leq \frac{C_1}{1 + |k|} e^{-\delta |k| \gamma_1 \gamma^1 - \sigma} \|f\|.$$

The integral term in (3.33) is estimated as

$$|I_k| = \left|\int_0^\infty \exp(-\varepsilon^{-1} \Omega \tau + ik\varepsilon^{-1} \phi^\tau(r)) \partial_\tau \left[(ik\partial_\tau \phi^\tau(r) - \Omega)^{-1} g_k(r, \tau)\right] \, d\tau\right| \leq \int_0^\infty \left|\partial_\tau \left[(ik\partial_\tau \phi^\tau(r) - \Omega)^{-1} \exp(-\Delta \tau) f_k(\rho^\tau(r))\right]\right| \, d\tau,$$

where the integral kernel can be expanded as

$$\partial_\tau \left[(ik\partial_\tau \phi^\tau(r) - \Omega)^{-1} \exp(-\Delta \tau) f_k(\rho^\tau(r))\right] = (ik\partial_\tau \phi^\tau(r) - \Omega)^{-2} ik\partial_\tau^2 \phi^\tau(r) \exp(-\Delta \tau) f_k(\rho^\tau(r))$$

$$- (ik\partial_\tau \phi^\tau(r) - \Omega)^{-1} \Delta \exp(-\Delta \tau) f_k(\rho^\tau(r))$$

$$+ (ik\partial_\tau \phi^\tau(r) - \Omega)^{-1} \exp(-\Delta \tau) f_k(\rho^\tau(r)) \rho''(r).$$
First we look at the term \( \partial_\tau^2 \phi^\tau (r) \), which evaluates to

\[
\partial_\tau^2 \phi^\tau (r) = \partial_\tau^2 (T_0 (\rho^\tau (r)) + \varepsilon T^\leq (\rho^\tau (r))) = (DT_0 (\rho^\tau (r)) + \varepsilon DT^\leq (\rho^\tau (r))) \rho^\tau (r)
\]

\[
= (DT_0 (\rho^\tau (r)) + \varepsilon DT^\leq (\rho^\tau (r))) R^\leq (\rho^\tau (r)) = O (\gamma).
\]

Next, using Cauchy’s integral formula, we estimate the derivative \( f'_k \) as

\[
f'_k (r) = \frac{1}{2\pi i} \oint_{|z-r|=\delta} \frac{f_k (z)}{(z-r)^2} dz,
\]

\[
|f'_k (r)| \leq (r\delta)^{-1} \sup_{|z-r| \leq \delta} f_k (z) \leq (r\delta)^{-1} e^{-\delta |k|} \gamma^{1-\sigma} \sup_{|z-r| \leq \delta} r^\sigma \|f\|.
\]

The radius of the contour of the Cauchy integral shrinks to zero at the origin, which allows us to use it to estimate norms. In particular, we can estimate that

\[
|f'_k (r)| \leq \gamma^{1-\sigma} e^{-\delta |k|} r^{\sigma-1} \left( \frac{1 + \delta}{\delta} \right) \|f\|.
\]

Continuing with the estimate we find that there exists \( C_2 > 0 \), such that

\[
(3.35) \quad \left| \partial_\tau \left[ (ik \partial_\tau \phi^\tau (r) - \Omega)^{-1} \exp (-\Delta r) f_k (\rho^\tau (r)) \right] \right| \leq \frac{C_1}{1 + |k|} \gamma^{1-\sigma} e^{-\delta |k|} (\rho^\tau (r))^{\sigma} \|f\|
\]

\[
+ \frac{C_1}{1 + |k|} \mathcal{O}(\gamma) e^{\kappa r} \gamma^{1-\sigma} e^{-\delta |k|} (\rho^\tau (r))^{\sigma} \|f\|
\]

\[
+ \frac{C_1}{1 + |k|} \left( \frac{1 + \delta}{\delta} \right)^{\sigma} e^{\kappa r} \gamma^{1-\sigma} e^{-\delta |k|} (\rho^\tau (r))^{\sigma-1} \rho^\tau (r) \|f\| \leq \frac{C_2}{1 + |k|} \gamma^{1-\sigma} e^{-\delta |k|} r^{\sigma} e^{(N-\sigma)\tau} \|f\|
\]

The results (3.35) is now substituted into \( I_k \), which yields

\[
|I_k| \leq \int_0^\infty \frac{C_2}{1 + |k|} \gamma^{1-\sigma} e^{-\delta |k|} r^{\sigma} e^{(N-\sigma)\tau} \|f\| \, d\tau
\]

\[
= \frac{C_2}{1 + |k|} \gamma^{1-\sigma} e^{-\delta |k|} r^{\sigma} \frac{1}{\sigma - N} \|f\|.
\]

Defining \( C_3 = C_1 + C_2 (\sigma - N)^{-1} \) we find that

\[
|\langle U f \rangle (r, k)| \leq \frac{C_3}{1 + |k|} \gamma^{1-\sigma} e^{-\delta |k|} r^{\sigma} \|f\|.
\]
3.6. Fixed point argument. To show that iterating the operator $T$ yields a unique solution for $W^>$, we use Banach’s fixed point theorem, which requires that $T$ is a contraction on a Banach space. A nonlinear operator is a contraction, if it is Lipschitz continuous with a Lipschitz constant less than one and it maps the unit balls into itself. We will utilise operator $U$ to prove contraction.

We start by formally establishing the equation that defines the Lipschitz constant of $T$, that is,

$$T(W^2_2) - T(W^1_1) = -\varepsilon^{-1} \int_0^\infty \exp(-\varepsilon^{-1}A_\varepsilon \tau) \varepsilon^{-1} [N_\varepsilon(W_0 + \varepsilon W^\leq + \varepsilon W^>_2)(\rho^\tau(r),\psi^\tau(r,\theta))] - N_\varepsilon(W_0 + \varepsilon W^\leq + \varepsilon W^>_1(\rho^\tau(r),\psi^\tau(r,\theta))] d\tau.$$

Using the integral variant of the mean value theorem we note that

$$\varepsilon^{-1} [N_\varepsilon(W_0 + \varepsilon W^\leq + \varepsilon W^>_2) - N_\varepsilon(W_0 + \varepsilon W^\leq + \varepsilon W^>_1)] = \int_0^1 DN_\varepsilon \left(W_0 + \varepsilon W^\leq + \varepsilon W^>_2 + s\varepsilon (W^> - W^>_2)\right) ds (W^>_1 - W^>_2),$$

and to simplify notation, we define

$$B_\varepsilon(r,\theta) = \int_0^1 DN_\varepsilon \left(W_0 + \varepsilon W^\leq + \varepsilon W^>_2 + s\varepsilon (W^> - W^>_2)\right) ds.$$

Using the notation (3.37), equation (3.36) can be re-written into

$$T(W^2_2) - T(W^1_1) = -\varepsilon^{-1} \int_0^\infty \exp(-\varepsilon^{-1}A_\varepsilon \tau) B_\varepsilon(\rho^\tau(r),\psi^\tau(r,\theta)) \times$$

$$\times (W^>_1(\rho^\tau(r),\psi^\tau(r,\theta)) - W^>_2(\rho^\tau(r),\psi^\tau(r,\theta))) d\tau.$$

We now represent both $B_\varepsilon$ and $W^>$ by their Fourier series

$$B_\varepsilon(r,\theta) = \sum_{k=-\infty}^{\infty} B_k(r)e^{ik\theta},$$

$$W^>_1,2 = \sum_{k=-\infty}^{\infty} a^1,2_k(r)e^{ik\theta},$$

which brings (3.38) into

$$T(a^2) - T(a^1) = U \left\{ \sum_{j=-\infty}^{\infty} B_j(a^1_{k-j} - a^2_{k-j}) \right\},$$

where the curly brackets mean the sequence as an element of $X$ for which $k \in \mathbb{Z}$.
3.7. Contraction mapping. In formula (3.39) we can estimate that for a $C_4 > 0$,
\begin{equation}
\tag{3.40}
\sup_{r \in \Gamma_{\gamma, \delta}} |B_k(r)| \leq \gamma C_4 e^{-\delta |k|},
\end{equation}
therefore
\[ \left\| \sum_j B_j \left( a_{k-j}^1 - a_{k-j}^2 \right) \right\| \leq \gamma C_4 \sum_{j=-\infty}^{\infty} e^{-\delta |j|} e^{-\delta |k-j|} \left\| a^1 - a^2 \right\| \]
\[ = \gamma C_4 (\coth \delta + |k|) \left\| a^1 - a^2 \right\|. \]
As a result we find that
\[ \| T(a^2) - T(a^1) \| \leq \max_{k \in \mathbb{Z}} \frac{\gamma C_3 C_4}{1 + |k|} \left( \coth \delta + |k| \right) \left\| a^1 - a^2 \right\| \]
\[ = \gamma C_3 C_4 \coth \delta \left\| a^1 - a^2 \right\|, \]
because for $\delta > 0$, $\coth \delta > 1$. The Lipschitz constant is less than unity when $\gamma < (C_3 C_4 \coth \delta)^{-1}$.

Because $\hat{F}_\varepsilon = O(r^\sigma)$, there exists a $M > 0$, such that $\left\| \hat{F}_\varepsilon \right\| \leq \gamma^{\sigma-1} M$, which implies that
\[ \left\| \hat{U} \hat{F}_\varepsilon \right\| \leq \gamma^{\sigma-1} C_3 M. \]
Now estimating
\[ \| T(a) \| = \| T(a) - T(0) + T(0) \| \leq \| T(a) - T(0) \| + \| T(0) \| \]
\[ \leq \gamma C_3 C_4 \coth \delta \| a \| + \gamma^{\sigma-1} C_3 M, \]
we get that the unit ball $B = \{ a \in X : \| a \| \leq 1 \}$ is mapped into itself if
\[ \gamma C_3 C_4 \coth \delta + \gamma^{\sigma-1} C_3 M \leq 1. \]
Because $\sigma \geq 2$ and $0 < \gamma \leq 1$, we can have the conservative bound
\[ \gamma \leq (C_3 C_4 \coth \delta + C_3 M)^{-1}, \]
which implies a smaller than unity Lipschitz constant, as well.

Because $T$ is continuous in $\varepsilon$, its fixed point must depend continuously on $\varepsilon$. At $\varepsilon = 0$, due to the form of the immersion (3.2), the manifold is also differentiable with respect to $\varepsilon$.

3.8. Uniqueness of the invariant manifold. We have so far shown that given the growth and phase conditions (3.3), (3.4) and a simplified invariance equation (3.1) has a unique solution, which is analytic apart from the origin, where it is $C^\infty$. Here we show that any solution of the most general invariance equation that is tangent to the linear subspace $E_\varepsilon$ and has the same smoothness properties, can be re-parameterised to satisfy (3.3), (3.4) and (3.1), making $\mathcal{M}_\varepsilon$ unique.

The most general invariance equation can be written as
\begin{equation}
\tag{3.41}
DV(z) S(z) = A_\varepsilon V(z) + N_\varepsilon(V(z)),
\end{equation}
where \( z \in \mathbb{R}^2 \) and \( V : \mathbb{R}^2 \to \mathbb{R}^{2\nu} \), \( S : \mathbb{R}^2 \to \mathbb{R}^2 \) are real analytic functions. Let us assume that the pair \( V \) and \( S \) is a solution of (3.41). We are looking for an analytic re-parametrisation \( z = T(r, \theta) \), such that the pair

\[
W = V \circ T \quad \text{and} \quad (R, T) = R = (DT)^{-1} \cdot (S \circ T)
\]

is the unique solution of (3.3), (3.4) and (3.1). The transformation \( T \) is then defined by

\[
(3.42) \quad DT(r, \theta) R(r) = S(T(r, \theta)).
\]

We also need to make sure that \( W \) satisfies the growth and phase conditions, hence we need

\[
(3.43) \quad \int_0^{2\pi} W^* (\theta) \cdot D_1 |V(T(r, \theta))|^{\frac{p}{2}} \, d\theta = 0
\]

\[
\int_0^{2\pi} DW^* (\theta) \cdot D_1 |V(T(r, \theta))|^{\frac{p}{2}} \, d\theta = 0
\]

We already know that (3.42) has a unique analytic solution under the conditions

\[
(3.44) \quad \int_0^{2\pi} W^* (\theta) \cdot D_1 T^{\leq} (r, \theta) \, d\theta = 0
\]

\[
\int_0^{2\pi} DW^* (\theta) \cdot D_1 T^{\leq} (r, \theta) \, d\theta = 0
\]

because this is what is proven in the previous sections. After inspection, we find that a suitable non-singular linear combination of equations (3.43) are only a small nonlinear perturbation of (3.44), because \( DV \) at the origin has a full two-dimensional range identical to \( E_{\varepsilon} \), due to \( M_{\varepsilon} \) being tangent to \( E_{\varepsilon} \). Therefore the proof of uniqueness needs to be slightly modified in section (3.2), such that the algebraic equations become weakly nonlinear with small additional terms.

The smallness of the perturbation can be controlled by the parameter \( \gamma \), hence we conclude that for sufficiently small \( \gamma \) there is a unique analytic solution to (3.42) and (3.43). This shows that there is an analytic re-parametrisation that brings any analytic solution of (3.41) into the unique solution of (3.3), (3.4) and (3.1), which then makes \( M_{\varepsilon} \) unique.

We note that our argument above is not strictly necessary, because it has already been shown, e.g., in Theorem 1.1 of [2], that for \( \varepsilon > 0 \) any analytic solution of (3.41) represents the same invariant manifold and that the difference is only in the parametrisation. For \( \varepsilon = 0 \) the uniqueness is given by Theorem (2.5), hence our solution is a particular parametrisation of the otherwise unique invariant manifold \( M_{\varepsilon} \).

This concludes the proof of Theorem 2.6.

4. Conclusions. In this paper we have shown that under the conditions of Theorem 2.6 SSMs continuously tend to LSCMs, when the system becomes conservative. The result is specific to two-dimensional SSMs. However, in many cases two-dimensions are not sufficient to capture all the important dynamics within a system. It is not clear how the proof could be extended to higher dimensions. The proof also sheds some light on what to expect when the system is finitely many times differentiable. In this case it is expected that a function space containing \( p \geq 1 \) times differentiable functions, whose highest derivative has a finite total variation would be suitable, because in such a space operator \( \mathcal{U} \) would remain bounded.
Acknowledgement. The author would like to thank George Haller, Florian Kogelbauer and Rafael de la Llave for discussions.

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