The general QCD parametrization and the hierarchy of its parameters.
(Why some simple models of hadrons work so well)

G.Dillon and G.Morpurgo

Università di Genova and Istituto Nazionale di Fisica Nucleare, Sezione di Genova.

1.Introduction.
2.The derivation of the general QCD parametrization.
3.The GP baryon mass formulas.
4.The magnetic moments of the octet baryons to 1st order in flavor breaking.
5.The \( \Delta \rightarrow p + \gamma \) decay, the magnetic moments of the \( \Delta \)'s and related results.
6.Double counting on inserting explicit pion fields in the QCD Lagrangian.
7.The hierarchy of the parameters.
8.The parametrization of the masses of the lowest Ps and V meson nonets.
9.The radiative \( V \rightarrow P + \gamma \) meson decays.
10.The baryons electromagnetic mass differences and the Coleman-Glashow equation.
11.Two relations: a) Between the charge radii of \( p, n, \Delta^+ \); b) Between the radii of \( \pi^+, K^+, K^0 \).
12.Parametrization of the \( \rho \gamma, \omega \gamma \) and \( \phi \gamma \) couplings: Why \( f_{\rho \gamma} : f_{\omega \gamma} \) differs from 3:1 only by \( \sim 10\% \) in spite of flavor breaking.
13.The GP and chiral theories: A few remarks.
14.Comparing the GP with the \( 1/N_c \) method; some comments.
15.Appendix I-A field theoretical derivation of the GP.
16.Appendix II.A summary of the main steps in the derivation of the spin-flavor dependence of the GP terms.

We dedicate this article to the memory of Giuseppe Franco Bassani, past president of the SIF.

1-e-mail: dillon@ge.infn.it ; morpurgo@ge.infn.it
1. Introduction.

This survey summarizes the “General QCD Parametrization” (GP), a procedure derived exactly from QCD, to understand and predict many hadron properties; it applies to the low energy region where, obviously, the perturbative method is out of question.

The GP was formulated [1] to explain why a naive description -such as the simple non relativistic quark model (NRQM)[2]- can reproduce semi-quantitatively many properties of hadrons. A classical example is that of fig.1, where a two parameter NRQM description of the magnetic moments of the lowest octet baryons is compared with the experimental data.

Indeed the reason why the NRQM was so successful remained an open question till the development of the GP.2 A similar problem (why does it work?) existed for other simple models (e.g. the MIT bag model [5] -more limited, but still of interest), also providing a simple description of several facts. The GP solves these problems. Short summaries of the GP were given previously [16]. This report amplifies them.

The starting point of the GP is this: Many hadron properties (e.g. magnetic moments, masses, electromagnetic (e.m.) mass differences, semileptonic decays, strange quark contribution to the proton e.m. form factor etc.) can be parametrized exactly in the spin-flavor space exploiting only a few general properties of QCD. These are:

a) Flavor breaking is due only to the mass term in the Lagrangian,
b) Only quarks carry electric charge,
c) Exact QCD eigenstates can be put in correspondence (for baryons) to a set of three quark-no gluon states and (for mesons) to a set of quark-antiquark-no gluon states,
d) The flavor matrices in the electromagnetic (e.m.) and in the flavor breaking term of the QCD Lagrangian commute.

However, although the above properties are essential in what follows, they would not lead too far without exploiting the "hierarchy" of the parameters [21], mentioned below (see point 3) and illustrated later (Sect.7).

As we will see, the “General QCD Parametrization” has three important features:

1) The spin-flavor structure of the hadron properties, expressed in terms of 2x2 Pauli matrices, is similar to that of the NRQM; yet the GP is fully relativistic, although the procedure to derive it from QCD is non covariant.

2) In spite of the complexity of the QCD interaction terms, the number of important additive terms appearing in the parametrization of several hadronic properties is often rather small, much smaller than one might have expected a priori. One reason for this will be stated in the next point (3).

3) When all the values of the coefficients of each term in the parametrization (the parameters) can be extracted from the experimental data, it turns out that the terms of increasing complexity (for the meaning of “complexity” compare Sect.7) are multiplied by decreasing parameters; we call this the hierarchy of the parameters. The hierarchy reduces the number of significant terms needed to reproduce the data, thus explaining the success of simple models like the NRQM.

4) The hierarchy allows also to define properly the notion of constituent quarks (first introduced in [2] as dressed quarks with their cloud of $q\bar{q}$ and gluons).

The fact that the general QCD parametrization is derived exactly from QCD using only the above properties (a to d) has the following implication: Any specific approximation (good or bad) to the exact QCD Lagrangian sharing the simple properties a)
Figure 1: The measured magnetic moments of the baryons compared with the values (solid lines) calculated with the simple two-parameter formula (Eq.1 in Ref.[1]) of the NRQM with $\mu = 2.793$ and $A = 0.96$ (that is $a = 0.65$) having used as input only the proton (+2.793) and $\Lambda$ (= $-0.613 \pm 0.004$) magnetic moments. [Calculated and (measured) values are neutron = $-1.86$ ($-1.913$); $\Sigma^- = -1.04$ ($-1.160 \pm 0.025$); $\Sigma^+ = +2.68$ (+2.458 $\pm 0.040$); $\Xi^- = -0.05$ ($-0.6507 \pm 0.0025$); $\Xi^0 = -1.43$ ($-1.250 \pm 0.014$); $\mu(\Sigma \rightarrow \Lambda = -1.61$) ($|^\text{exp}| = 1.61 \pm 0.08$).]
to d), listed above (think e.g. to some forms of chiral QCD Lagrangian) also may lead to results with the same general structure as those of the GP (though the values of the parameters so obtained are not necessarily correct).

By this we mean that if one performs a complete calculation with a specific approximate Lagrangian (say a chiral one) and/or with lattice QCD, one may obtain specific numerical values for the parameters mentioned above. But it is not evident that the values so obtained are better than those resulting from the GP, by fitting the largest parameters empirically and using the hierarchy mentioned above to estimate the order of magnitude of the remaining ones. Whether this is so or not depends on the chiral theory used, on the approximations performed in the chiral calculation etc.

To anticipate an example of application of the GP we write below a formula (to be derived in Sect. 3) obtained with the GP [8] for the lowest octet + decuplet baryon masses. This formula - Eq. (1) - improves the Gell Mann-Okubo formula; the particle symbols stay for their masses and $T$ is a definition of the quantity appearing in the equation on the left. We will derive the Eq. (1) at the end of Sect. 3. The standard Gell Mann-Okubo formula is the same as Eq. (1), without the $T$ term.

$$
\frac{1}{2} (p + \Xi^0) + T = \frac{1}{4} (3\Lambda + 2\Sigma^+ - \Sigma^0) \quad T \equiv \Xi^{*-} - \frac{1}{2} (\Omega + \Sigma^{*-}) \quad (1)
$$

The fit to the data depends on whether we use the conventional or the pole values for the masses of the resonances appearing in $T$:

$$
l.h.s. = 1132.36 \pm 0.7 \, \text{MeV}, \quad r.h.s. = 1133.93 \pm 0.04 \, \text{MeV} \quad (\text{conventional})
$$
$$
l.h.s. = 1133.86 \pm 1.25 \, \text{MeV}, \quad r.h.s. = 1133.93 \pm 0.04 \, \text{MeV} \quad (\text{pole}) \quad (2)
$$

(The above numbers need no corrections from the e.m. contributions to the masses). A similar formula (except for the e.m. corrections) was re-obtained only later - by a series of chiral calculations (compare: Ref. [9], its Erratum Ref. [22], and Ref. [23]).

Two other points should be mentioned:

a) The GP does not need, for its derivation, the use of unproven, though widely adopted, assumptions. For instance it is often implied that chiral dynamics is needed to relate QCD to constituent quark models. This is incorrect: the relationship between QCD and constituent quark models depends on the hierarchy mentioned above and is much more general (See Sect. 7).

Neither chiral dynamics nor the notion of pions as quasi-Goldstone bosons play any role in deriving the general parametrization that relates the quark descriptions (both current and constituents) to QCD.

b) We will consider only problems of hadrons with light quarks ($u, d, s$). The fact that the NRQM works quantitatively has always been considered miraculous for the light quarks (although to understand its working raises problems also for the hadrons with non zero charm or beauty). It appears from the GP that the reason why the NRQM works has nothing to do with the low velocity of quarks inside the hadrons, that was implied in [2]: $u, d, s$ quarks can move as fast as one likes.

Finally, besides clarifying why some models work, the general parametrization (with its hierarchy of parameters) leads to a variety of new results, as we shall see.

2. The derivation of the general QCD parametrization.

In the past section we exemplified in fig. 1 (magnetic moments) how the NRQM describes semi-quantitatively the data. Here we show how the general parametrization
is derived from QCD. It is simpler to discuss the masses of the lowest $8+10$ baryons, instead of the magnetic moments. Thus we illustrate the GP first with the masses and consider later the magnetic moments (and other problems). However most of this section, even if referring to the masses, is intended to be a general introduction; it is closely related to Ref.[1] and, in part, to Ref.[26].

The mass $M_B$ of a baryon $B$ is the expectation value of the exact QCD Hamiltonian $H_{QCD}$ in its lowest exact QCD eigenstate $|\Psi_B\rangle$ in the rest system:

$$M(B) = \langle \Psi_B | H_{QCD} | \Psi_B \rangle$$

Of course $|\Psi_B\rangle$, being the exact state of a strongly interacting system of quarks and gluons, is a superposition of an infinite number of Fock states, starting with three quarks. The superposition includes four quarks and one antiquark, three quarks plus one gluon and so on. Schematically:

$$|\Psi_B\rangle = |qqq\rangle + |qqqq\rangle + |qqq, \text{Gluons}\rangle + \ldots$$

where the ellipsis stands for the sum of an infinite number of additional states; the amplitudes that multiply each state (depending on the momenta, spins, flavors, colors of the intervening quarks, antiquarks and gluons) have been left understood.

We now introduce an auxiliary Hamiltonian $\mathcal{H}$, non-relativistic, operating only in the $3q$ sector; the operator $\mathcal{H}$ (we call it the “model Hamiltonian”) has the only purpose of providing a set of baryon states - to be called the model (or auxiliary) states $|\Phi_B\rangle$. $|\Phi_B\rangle$ is constructed (as in the NRQM) with just three quarks and no gluon.

We now write the exact state $|\Psi_B\rangle$ in (4) as:

$$|\Psi_B\rangle = V |\Phi_B\rangle$$

where $V$ is some (very complicated) unitary transformation. In principle $V$ can be expressed in terms of $H$ and $\mathcal{H}$ using the adiabatic construction of the bound states. (See the Appendix I, where the construction of $V$ is related to the Gell Mann-Low adiabatic procedure and to the $U(0,-\infty)$ Dyson operator). Using $V$, the Equation (3) giving the mass $M_B$ of $B$, can be rewritten:

$$M_B = \langle \Phi_B | V^\dagger H_{QCD} V | \Phi_B \rangle$$

The difference between Eq.(6) and Eq.(3) is the following: In Eq.(6) the states are simple; the complexity of the states (4) is transferred to $V$. This has an advantage which is basic in the procedure: $V^\dagger H_{QCD} V$ is indeed a complicated operator, but since it has to act only on the coordinates (space, spin, flavor, color) of the three quarks present in the state $|\Phi_B\rangle$, it must be (after contraction of all the field operators) necessarily a function of these coordinates only. In what follows the three quarks in $|\Phi_B\rangle$ will be numbered 1, 2, 3. Thus, after the elimination of all the creation and destruction operators, $V^\dagger H_{QCD} V$ behaves as a color singlet three body operator acting on 1, 2, 3.

In Eq.(3) $H_{QCD}$ transforms -under space rotations- as a scalar and the same is true for $V^\dagger H_{QCD} V$ because $V$ is invariant under rotations (it is expressed in terms of $H_{QCD}$ and $\mathcal{H}$, both rotationally invariant). As to $\Phi_B$ in Eq.(6), it depends on how we select the model Hamiltonian $\mathcal{H}$. By choosing $\mathcal{H}$ as the simplest, most naive, most unrefined N.R. quark model Hamiltonian, the parametrization of $\langle \Phi_B | V^\dagger H_{QCD} V | \Phi_B \rangle$ is considerably simplified. We select $\mathcal{H}$ so that, for the lowest octet and decuplet baryons, the wave functions $\Phi_B$ have a non relativistic space-spin structure with the following properties:
(1) $\Phi_B$ is the product of a space-spin structure $X(r_1, r_2, r_3)$ symmetrical in $r_1, r_2, r_3$, times a spin-flavor part $W_B(1, 2, 3)$, times a color singlet factor $C(1, 2, 3)$; (2) The space part has orbital angular momentum $L = 0$; thus $X \equiv X_{L=0}(r_1, r_2, r_3)$.

Altogether we have for the baryon $B$:

$$\Phi_B = X_{L=0}(r_1, r_2, r_3) \cdot W_B(1, 2, 3) \cdot C(1, 2, 3) \quad (7)$$

where $C(1, 2, 3)$ is the color factor.

The assumption $L = 0$ and the symmetry of the space wave function imply automatically that the spin factors $W_B(1, 2, 3)$ have the $SU_3$ structure. Omitting the color factor $C(1, 2, 3)$ one has, for instance ($S$ in Eq.(8) below means symmetric in 1, 2, 3):

$$W^+_p = (18)^{-1/2}S[\alpha_1(\alpha_2\beta_3 - \alpha_3\beta_2)u_1u_2d_3]; \quad W^+_{\Delta++} = \alpha_1\alpha_2\alpha_3u_1u_2u_3 \quad (8)$$

The $W_B$'s of the other 8 and 10 states are constructed similarly.

Two remarks must be added:

The first is on the spin functions: Because the operator $V$ must be written in terms of creation and destruction operators of Dirac particles, the spinors appearing in $W_B(1, 2, 3)$ must be four component spinors; otherwise the operation $V$ would not be defined. This is achieved by completing the Pauli spinors of the model wave function with two zeros in the lower components. This is compatible with our non-relativistic definition. This is achieved by completing the Pauli spinors of the model wave function with two zeros in the lower components. This is compatible with our non-relativistic choice of the model Hamiltonian $\hat{H}$. In fact $\hat{H}$ operates on two-component Pauli spinors, but it is formally possible to extend the space of such spinors to that of four component spinors, provided that $\hat{H}$ is extended without connecting the space of the upper and lower components and giving zero when operating on the latter (compare [1]).

The second remark is related to the $X_{L=0}(r_1, r_2, r_3)$ in the wave function. Why $X_{L=0}$ does not carry a baryon index $B$, specifying the baryon we are considering? The answer is that we selected a flavor independent $\hat{H}$, with the masses of the quarks $u, d, s$ in $\hat{H}$ chosen equal, at some intermediate value between those of the $u, d$ and $s$ masses. The whole flavor breaking is assigned to the $V$ operator; recall that $V$ depends on the exact QCD Hamiltonian $H_{QCD}$; this contains the flavor breaking due to the mass differences between $u, d, s$. (We might have made a different choice, constructing the model Hamiltonian $\hat{H}$ with a flavor dependence, but we preferred to assign also this task to $V$).

Consider now briefly the flavor dependence of $H_{QCD}$. The mass term in the QCD Hamiltonian is for the $u, d, s$ quarks:

$$\int d^3r \ m[\tilde{u}_R(x)u_R(x) + \tilde{d}_R(x)d_R(x)] + (m + \Delta m)\tilde{s}_R(x)s_R(x) \quad (9)$$

where the index $R$ refers to the mass renormalization point.

In [1] we never used explicitly the values of the quark masses $m$ (the quarks were then indicated as $\mathcal{P}, \mathcal{N}, \lambda$) nor a value of $\Delta m$, but we had in mind $m$ values (corresponding to a renormalization point around $\Lambda_{QCD}$ of the order $300 - 400 \text{ MeV}$ for $\mathcal{P}, \mathcal{N}$ and $450 - 550 \text{ MeV}$ for $\lambda$, such that $\Delta m/m \cong 1/3$. Subsequently [26] we re-analyzed the results of [1] renormalizing the quark masses at the conventional $q = 1 \text{ GeV}$ (we used then $u, d, s$ symbols for quarks, with $m_u, m_d = \text{a few MeV}$). As it appeared (and as it will emerge also here) the GP procedure is independent of the choice of the mass renormalization point. In fact in [26] we confirmed and extended significantly the

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3Of course $\Delta$ has an appreciable width and, therefore, is not an exact eigenstate of $H_{QCD}$ but this is irrelevant here and will be considered later.
results of Ref.([1]).

Coming back to the description of the model state of a baryon in momentum space, we obtain (performing in Eq.(7) a Fourier transformation of $X_{L=0}$):

$$| \Phi_B \rangle = \sum_{p,w} C^B_{w_1,w_2,w_3} (p_1,p_2,p_3) a_{p_1,w_1}^\dagger a_{p_2,w_2}^\dagger a_{p_3,w_3}^\dagger | 0 \rangle$$

(10)

where $\sum_{p,w}$ stays, as indicated, for the sum over all the $p$'s and $w$'s. In Eq.(10) the $a_{p,w}^\dagger$ are creation operators of quarks of momentum $p$ and spin-flavor-color index $w$, $| 0 \rangle$ is the vacuum state in a Fock space of quarks and gluons. $| \Phi_B \rangle$ belongs to a given representation of $SU_3$ (flavor); in it the masses of the three quarks $u,d,s$ are taken equal and their values identified with their QCD values (in the unbroken flavor limit) at any convenient mass renormalization point.

Because we are in the baryon rest frame, $C^B_{w_1,w_2,w_3} (p_1,p_2,p_3)$ in Eq.(10) contains a factor $\delta(p_1 + p_2 + p_3)$.

We now come back to the unitary correspondence $V$ between the model state $| \Phi_B \rangle$ and the exact QCD state $| \Psi_B \rangle$. Recall that, for the lowest octet and decuplet baryons, $| \Phi_B \rangle$ is a simple 3 quark S state, while, of course, $| \Psi_B \rangle$ (4) contains in addition all kinds of $q\bar{q}$ and gluon states and, moreover, it certainly has terms with a non vanishing orbital angular momentum $L$. For convenience we rewrite the Eqs.(4),(5) as:

$$V| \Phi_B \rangle = | qqq \rangle + | qqq\bar{q} \rangle + | qqq, \text{Gluons} \rangle + ......$$

(11)

Obviously each state in the sum (11) has the same conserved quantum numbers as those of $| \Psi_B \rangle$. The transformation $V$ has the tasks:

a) Of dressing the simple three quarks state $| \Phi_B \rangle$, that is of transforming $| \Phi_B \rangle$ into an infinite sum of states of $q,\bar{q}$ and gluons, the only restriction being that such states have the correct quantum numbers;

b) Of introducing configuration mixing in $L$; this is not present in $| \Phi_B \rangle$ which, as stressed, is a pure $L = 0$ state;

c) Of transforming the Pauli spinor quark states in $| \Phi_B \rangle$ into Dirac 4-spinors.

As already mentioned, in the Appendix I we will show how the existence of $V$ can be established in field theory. Now we proceed directly to the parametrized baryon mass formula (and to the other properties of hadrons). Calling $M_B$ the mass of a baryon of the lowest $8$ or $10$, we start from Eqs.(3),(6) that we recall below:

$$M_B = \langle \Psi_B | H_{QCD} | \Psi_B \rangle = \langle \Phi_B | V^\dagger H_{QCD} V | \Phi_B \rangle$$

(12)

The general form of the parametrization of any physical property (e.g. the baryon $8+10$ masses to be discussed now, or the magnetic moments or any other property) is independent of the selection of a specific $| \Phi_B \rangle$. That is, independently of this selection, the calculated $M_B$ is, in all cases:

$$M_B = \langle \Phi_B | V^\dagger H_{QCD} V | \Phi_B \rangle \equiv \langle W_B | \text{parametrized mass} | W_B \rangle$$

(13)

\footnote{In Ref.[1] the ratio $\Delta m/m$ (associated to the magnitude of flavor breaking) determined from the baryon masses and the baryon octet magnetic moments - was found in both cases $\approx 0.3$. In Ref.[26] (compare its Sect.II) $\Delta m/m \approx m_s/m$ is, of course, totally different (say, in the range $8-25$), but the flavor breaking parameter turned out to be $\approx m_s/(3\Lambda_{QCD}) \approx 0.3$, the same value obtained in Ref.[1]. In the present paper, for simplicity, the quark symbols will be always $u,d,s$ (and, only if necessary, we will specify the mass renormalization point in the QCD Lagrangian).}
In Eq. (13) the last term defines what we call the “parametrized mass” of \( B \); that is, it displays the general parametrization of the mass of \( B \).

We now outline the calculation of the expectation value of \( V^\dagger H_{QCD} V \) in the state \( | \Phi_B \rangle \); this amounts to construct explicitly the last term of Eq. (13), thus expressing the masses as the expectation values of a spin-flavor operator in the 3q spin-flavor functions, \( W_B \). Because \( \Phi_B \) in (13) is a 3 quark state, the only part of \( V^\dagger H_{QCD} V \) that contributes in Eq. (13) is its projection \( \tilde{H} \) in the \( | 3q \rangle \) Fock sector:

\[
\tilde{H} = \sum_{3q} \sum_{3q'} | 3q \rangle \langle 3q | V^\dagger H_{QCD} V | 3q' \rangle | 3q' \rangle,
\]

(14)

where the sums in (14) are now on all possible 3-quark, no gluon Fock states. After normal ordering of all the creation and destruction operators in \( \tilde{H} \) and their contraction with those arising from \( | \Phi_B \rangle \) and \( \langle \Phi_B | \) (see Eq. (13)), the operator \( \tilde{H} \) becomes a function of only the spin-flavor-space variables of the three quarks in \( | \Phi_B \rangle \); thus parametrizing \( \tilde{H} \) means to construct the most general scalar operator in the space \( r_i \), spin \( \sigma_i \)'s, flavor \( f_i \)'s and color operators of the three quarks \( (i = 1, 2, 3) \), keeping obviously only the terms with a non-vanishing expectation value in \( | \Phi_B \rangle \). (We identified already the quarks in the model states \( | \Phi_B \rangle \) as those appearing in the QCD Lagrangian at the selected renormalization point in the no flavor breaking limit). Thus it is straightforward to contract the creation and destruction operators in \( V^\dagger H_{QCD} V \) with those in the auxiliary states \( | \Phi_B \rangle \). This leads to the last term of Eq. (13). After this contraction the projection \( \tilde{H} \) of the field operator \( V^\dagger H_{QCD} V \) in the 3-body sector becomes a scalar (i.e. a rotation invariant) function of the space \( r_i \), spin \( \sigma_i \), flavor \( f_i \) and color operators of the three quarks. One must write the most general expression of that operator. We call it \( \tilde{H}' \) (a different symbol must of course be used because \( \tilde{H} \) (Eq. (14)) operates in Fock space, whereas \( \tilde{H}' \), obtained after contraction of the field operators, is just a 3-body quantum mechanical operator).

The number of independent scalar operators in the spin-flavor space of 3 quarks is of course finite. We use for them the symbol \( Y_\mu(\sigma, f) \), where the index \( \mu \) specifies the operator to which we refer. Thus the most general operator of the space and spin-flavor variables is necessarily

\[
\tilde{H}' = \sum_\mu R_\mu(r, r') Y_\mu(\sigma, f)
\]

(15)

where \( R_\mu(r, r') \) are operators in the coordinate space of the three quarks and: \( r \equiv (r_1, r_2, r_3), \ r' \equiv (r'_1, r'_2, r'_3) \). In the baryon rest system the combination \( \sum_i r_i \) does not intervene; only \( (r_i - r_k) \) and \( (r_j - (1/2)(r_i + r_k)) \) appear \( [i \neq j \neq k] \).

To calculate the masses we must calculate the expectation value of \( \tilde{H}' \), Eq. (15), on \( | \Phi_B \rangle \), given by Eq. (10)\(^5\) (Recall that \( \Phi_B \) is the product of a space part \( X_{L=0}(r_1, r_2, r_3) \) with orbital angular momentum zero and a spin-flavor-(color) factor \( W_B(1, 2, 3) \) carrying the whole \( \mathbf{J} \)). Then the mass \( M_B \) results:

\[
M_B = \sum_\mu \langle X_{L=0}(r) \mid R_\mu(r, r') \mid X_{L=0}(r') \rangle \langle W_B \mid Y_\mu(\sigma, f) \mid W_B \rangle
\]

(16)

\(^5\)One might ask why we do not calculate the (masses\(^2\)) operating similarly with the (Hamiltonian\(^2\)); one might do so, except that the (Hamiltonian\(^2\)) is most probably non renormalizable. (Think, in ordinary quantum mechanics, of using the square of the Hamiltonian of the hydrogen atoms to calculate its levels!)
that is

\[ M_B = \sum_\mu k_\mu \langle W_B \mid Y_\mu(\mathbf{r}, f) \mid W_B \rangle \equiv \langle W_B \mid "\text{parametrized mass}" \mid W_B \rangle \]  

(17)

where the coefficients \( k_\mu \) are:

\[ k_\mu = \langle X_{L=0}(\mathbf{r}) \mid R_\mu(\mathbf{r}, \mathbf{r}') \mid X_{L=0}(\mathbf{r}') \rangle \]

(18)

Because the \( X \)'s in Eq.(18) have \( L = 0 \), the operators \( R_\mu(\mathbf{r}, \mathbf{r}') \) must be rotation invariant; also, because \( H' \) is a scalar, the \( Y_\mu(\mathbf{r}, f) \)'s in Eq.(17) must be flavor-dependent scalar operators constructed with the spins \( \mathbf{s} \)'s of the three quarks. Eqs.(17)(18) give the most general form of the “parametrized masses” of the lowest baryons \( 8 \) and \( 10 \).

To conclude, we summarize the contents of this Section: The two main QCD steps leading to the Eq.(19) of next Sect.3, that will give explicitly the “parametrized mass” of the \( 8 + 10 \) baryons are: 1) The masses are calculated as the expectation values of the exact \( H_{QCD} \) Hamiltonian in the exact states \( \mid \Psi_i \rangle \); 2) Such states \( \mid \Psi_i \rangle \) are related by \( \mid \Psi_B \rangle = V \mid \Phi_B \rangle \), [Eq.(5)], to a set of auxiliary (model) states \( \mid \Phi_i \rangle \).

3. The GP baryon mass formulas.

We now write the general expression for the “parametrized mass” of the octet and decuplet baryons. After integrating on the space coordinates, the “parametrized mass” is a function only of the spin and flavor operators of the three quarks in the model state. Indicating by \( P_i^s \) the projector in the flavor space of the \( s \) quark (\( P_i^s \mathbf{s}_i = \mathbf{s}_i, P_i^s \mathbf{u}_i = P_i^s \mathbf{d}_i = 0 \)), that is \( P_i^s \equiv \text{diag}(0,0,1) \), one has:

\[
"\text{parametrized mass}" = M_0 + B \sum_i P_i^s + C \sum_{i > k} (\mathbf{s}_i \cdot \mathbf{s}_k) + D \sum_{i > k} (\mathbf{s}_i \cdot \mathbf{s}_k)(P_i^s + P_k^s) \\
+ E \sum_{i > k \neq j} (\mathbf{s}_i \cdot \mathbf{s}_k)P_j^s + a \sum_i P_i^s P_k^s + b \sum_i (\mathbf{s}_i \cdot \mathbf{s}_k)P_i^s P_k^s \\
+ c \sum_{i > k \neq j} (\mathbf{s}_i \cdot \mathbf{s}_k)P_j^s(P_i^s + P_k^s) + dP_1^s P_2^s P_3^s 
\]

(19)

where \( M_0, B, C, D, E, a, b, c, d \) are coefficients (or, as we call them, parameters).

A few comments: Barring -at the moment- the e.m. and isospin corrections, the number of masses of the lowest octet and decuplet baryons is \( 8 \) and the parameters \( (M_0, B, C, D, E, a, b, c, d) \) in Eq.(19) are \( 9 \); thus Eq.(19) can certainly be satisfied [in the expressions of the masses only \((a + b)\) intervenes]. Also, spin scalars \( (\mathbf{s}_1 \times \mathbf{s}_2) \cdot \mathbf{s}_3 \) cannot contribute in Eq.(19) as shown in Eq.(26) of Ref.[1]; in fact they are absent in Eq.(19). Note also that \( (P_i^s)^n = P_i^s \) for any \( n \) (this implies that the GP records automatically, to all orders in \( n \), the flavor breaking contributions additive in the quarks).6

The parametrization (19) holds for any magnitude of the flavor breaking term \( \Delta m \) (which is included to all orders), and is independent of the choice of the quark mass

6Explicit formulas relating the \( 8 \) and \( 10 \) baryon masses to the coefficients in Eq.(19) are given in Sect.XII of [1] and in the Appendix B of [26]. We will transcribe them in part in the Appendix II here.
renormalization point. Finally we stress the interest of the following fact. We will see by fitting the masses that in the Eq.(19) the magnitude of the parameters decreases strongly moving to terms with increasing number of indices; this is what we call the hierarchy of the parameters.

The parameters, in MeV, of Eq.(19) fitting the baryon masses, using the pole -1st line- (for discussions of the pole parameters see [24, 25, 26]) or the conventional -2nd line- values for the resonances, are:

\[
\begin{array}{cccccccc}
M_0 & B & C & D & (a + b) & E & c & d \\
1076 & 192, & 45.6 & \pm 0.3, & -13.8 & \pm 0.3, & -16 & \pm 1.4, & 5.1 & \pm 0.3, & -1.1 & \pm 0.7, & 4 & \pm 3 \\
1086 & 194, & 49.2 & \pm 0.3, & -16.4 & \pm 0.2, & -7.5 & \pm 0.8, & 2.5 & \pm 0.2, & 3.1 & \pm 0.4, & -5.7 & \pm 2
\end{array}
\]

The hierarchy of the parameters is evident; the values (20) decrease rather strongly with increasing complexity of the accompanying spin-flavor structure, so that, neglecting c and d in Eq.(19), one obtains -see below- the mass formula Eq.(1) of Sect.1. Its agreement with the data (to 1/1000, Eq.(2)) confirms the smallness of c, d.

Although the subject of the hierarchy (and the question of the correct choice of normalization for the coefficients in an expression like Eq.(19)) will be discussed in Sect.7, a few remarks may be useful also here.

Both C and D multiply terms of type \(\sigma_i \cdot \sigma_k\) but the value of D should be depressed with respect to that of C because the D terms includes a flavor breaking factor while the C term does not. In fact it is \(|D/C| \approx 1/3\). The E term \((E \sum_{i\neq k\neq j}^{i>k}(\sigma_i \cdot \sigma_k)P^s_j)\) has a spin structure similar to that of the D term but its flavor factor has a different index, so that three quarks are involved; this can be interpreted as the exchange of an additional gluon and produces a further reduction. Depending on the use of the conventional or pole values of the masses for the resonances, the additional gluon reduction factor for E is \(\approx 0.22\) (conventional) or \(\approx 0.37\) (pole).

Another remark: Limiting to first order flavor breaking, the Eq.(19) simplifies and contains only 5 parameters instead of 8; thus it reduces to:

\[
\text{"parametrized mass" } \equiv M(1) = M_0 + B \sum_i P_i^s + C \sum_{i>k}(\sigma_i \cdot \sigma_k) + D \sum_{i>k}(\sigma_i \cdot \sigma_k)(P_i^s + P_k^s) + E \sum_{i\neq k\neq j}(\sigma_i \cdot \sigma_k)P^s_j
\]

With five parameters and eight masses, we get in this approximation, three mass relations: 1) The Gell Mann-Okubo formula for the octet; 2) The two equal spacing formulas of Gell Mann for the decuplet: \(\Omega - \Sigma^* = \Xi^* - \Sigma^* = \Sigma^* - \Delta\).

To obtain the generalized Gell Mann-Okubo formula (1) (Sect.1)[8] we consider the 2nd order flavor breaking terms present in the general parametrization [Eq.(19)]; we neglect the 3-quark term with coefficient c, which is very small (see Eq.(20)); this smallness is due to the hierarchy; in other words we keep only the 2nd order term \((a + b)\). Because it is easy to check that the combination of masses \(T\) appearing in Eq.(1) is \(T = -(a + b)/2\), we obtain the Eq.(1) and establish that it is correct to 2nd order in flavor breaking (except for the neglect of c). For seeing that the result is free from electromagnetic corrections compare the footnote 9 of Ref.[8].

4. The magnetic moments of the octet baryons to 1st order in flavor breaking.
To display other results of the GP we consider now the magnetic moments of the lowest octet baryons. In fact the first motivation of this work was to understand why the NRQM two parameters formula (µ is the proton magnetic moment= 2.793 µN):

$$\mathcal{M} = \mu \sum \left( \frac{2}{3} \sigma^p - \frac{1}{3} \sigma^N - \frac{1}{3} \sigma^\lambda \right) + (A/3) \sum \sigma^\lambda$$  \hspace{1cm} (22)

worked so well, as shown in fig.1 (Sect.1). It is convenient, as done originally [1], to perform this calculation of the magnetic moment \( \mathcal{M} \) in the rest frame of the baryon; we will use the standard formula (non covariant, but correct in the rest frame!):

$$\mathcal{M} = (1/2) \int d^3 \mathbf{r} \ (\mathbf{r} \times \mathbf{j}(\mathbf{r}))$$  \hspace{1cm} (23)

where \( \mathbf{j}(\mathbf{r}) \) is the space part of the electromagnetic current \( j_\mu(x) \) at time \( t = 0 \):

$$j_\mu(x) = (ie) \left[ \frac{2}{3} \bar{u}(x) \gamma_\mu u(x) - \frac{1}{3} \bar{d}(x) \gamma_\mu d(x) - \frac{1}{3} \bar{s}(x) \gamma_\mu s(x) \right] \equiv (ie) \left[ \bar{\Psi}(x)(\lambda_3 + \frac{1}{3} \lambda_8) \gamma_\mu \Psi(x) \right]$$  \hspace{1cm} (24)

Of course \( e(1/2)(\lambda_3 + \frac{1}{3} \lambda_8) \) is the charge \( eQ \equiv e[\frac{2}{3}P^u - \frac{1}{3}P^d - \frac{1}{3}P^s] \); the \( P^q \)'s are projectors on \( u, d, s \). [Frequently \( eQ \), will replace \( eP_1 \), the notation used in [1]].

The magnetic moments of the octet baryons \( B \) are:

$$M_z(B) = \mu \sum \langle X_{L=0}(\mathbf{r}) \ | \ R_\mu(\mathbf{r}, \mathbf{r}') \ | \ X_{L=0}(\mathbf{r}') \rangle \langle W_B \ | \ G_\nu(\sigma, f) \ | \ W_B \rangle$$  \hspace{1cm} (25)

From Eq.(25) we obtain the “parametrized magnetic moments” calling:

$$g_\nu = \mu \langle X_{L=0}(\mathbf{r}) \ | \ R_\nu(\mathbf{r}, \mathbf{r}') \ | \ X_{L=0}(\mathbf{r}') \rangle$$  \hspace{1cm} (26)

and writing:

$$M_z(B) = \sum_\nu g_\nu \langle W_B \ | \ G_\nu(\sigma, f) \ | \ W_B \rangle \equiv \langle W_B \ | \ “parametr.magn.mom.” \ | \ W_B \rangle$$  \hspace{1cm} (27)

Here we write the GP of the magnetic moments only to first order in flavor breaking. (For the baryon masses the parametrization was exact to all orders in \( P^s \)). Keeping only terms linear in \( P^s \) the GP of the magnetic moments of the baryon octet, is for each baryon \( B \), a linear combination of 7 terms. It is:

$$M_z(B) = \langle W_B | \sum_{\nu=0}^{7} g_\nu (G_\nu_z) | W_B \rangle \equiv \langle W_B | \sum_{\nu=1}^{7} \bar{g}_\nu (\bar{G}_\nu_z) | W_B \rangle$$  \hspace{1cm} (28)

We will explain in a moment why the two expressions in Eq.(28) - the first containing eight terms and the second seven - are identical. In Eq.(28) the \( W_B \)'s (\( \equiv W_B(1, 2, 3) \)) are, of course, the same defined in Sect.3. As to the \( G_\nu \)'s, they are:

$$G_0 = Tr[Q P^s] \sum_i \sigma_i \quad G_1 = \sum_i Q_i \sigma_i \quad G_2 = \sum_i Q_i P^s_i \sigma_i \quad G_3 = \sum_{i \neq k} Q_i \sigma_k \quad G_4 = \sum_{i \neq j} Q_i Q_j P^s_i \sigma_k \quad G_5 = \sum_{i \neq k} Q_i P^s_i \sigma_k \quad G_6 = \sum_{i \neq k} Q_i P^s_k \sigma_i \quad G_7 = \sum_{i \neq j \neq k} Q_i Q_j P^s_k \sigma_k$$  \hspace{1cm} (29)

\(^7\)We use now for the magnetic moments the notation \( M_z \) instead of \( \mathcal{M} \) - there should be no confusion with the masses.
As we will see, the coefficient \( g_0 \) of \( G_0 \) is expected to be significantly smaller than those of the other terms; therefore here the effect of the term \( G_0 \) is unimportant.

Forgetting \( g_0 \) one might then obtain the seven coefficients \( (g_1 \text{ to } g_7) \) from the seven magnetic moments; but it is unnecessary to discuss now how small is \( g_0 \), because the relation between the eight terms, displayed below in Eq.(30), allows to express all the magnetic moments of the octet baryons in terms of only 7 parameters \( \tilde{g}_i \), with \( i=1 \) to 7. The sums over \( i,j,k \) in Eq.(29) extend from 1 to 3 and \( Q_i \) is the quark charge. Although we wrote \( M_z(B) \) (28) in two forms, the two coincide, due to the relation (30), easily verifiable (compare the proof at the end of this section).

\[
G_0 = -\frac{1}{3} G_1 + \frac{2}{3} G_2 - \frac{5}{6} G_3 + \frac{5}{3} G_4 + \frac{1}{6} G_5 + \frac{1}{6} G_6 + \frac{2}{3} G_7
\]  

(30)

The Eq.(30) leads to the following relations between the \( \tilde{g}_\nu \)'s in the right hand side of Eq.(28) and the \( g_\nu \)'s in the middle expression:

\[
\begin{align*}
\tilde{g}_1 &= g_1 - (1/3)g_0 ; & \tilde{g}_2 &= g_2 + (2/3)g_0 ; & \tilde{g}_3 &= g_3 - (5/6)g_0 \\
\tilde{g}_4 &= g_4 + (5/3)g_0 ; & \tilde{g}_5 &= g_5 + (1/6)g_0 ; & \tilde{g}_6 &= g_6 + (1/6)g_0 \\
\tilde{g}_7 &= g_7 + (2/3)g_0
\end{align*}
\]  

(31)

The magnetic moments of the octet baryons expressed via Eqs.(28,29) in terms of the \( \tilde{g}_\nu \)'s are given by the following expressions (now the baryon symbol indicates its magnetic moment).

\[
P = \tilde{g}_1 \\
\begin{align*}
n &= -(2/3)(\tilde{g}_1 - \tilde{g}_3) \\
\Lambda &= -(1/3)(\tilde{g}_1 - \tilde{g}_3 + \tilde{g}_2 - \tilde{g}_5) \\
\Sigma^+ &= \tilde{g}_1 + (1/9)(\tilde{g}_2 - 4\tilde{g}_4 - 4\tilde{g}_5 + 8\tilde{g}_6 + 8\tilde{g}_7) \\
\Sigma^0 &= -(1/3)(\tilde{g}_1 + 2\tilde{g}_3) + (1/9)(\tilde{g}_2 - 4\tilde{g}_4 + 2\tilde{g}_5 - 4\tilde{g}_6 - 4\tilde{g}_7) \\
\Xi^0 &= -(2/3)(\tilde{g}_1 - \tilde{g}_3) + (1/9)(-4\tilde{g}_2 - 2\tilde{g}_4 + 4\tilde{g}_5 - 8\tilde{g}_6 + 10\tilde{g}_7) \\
\Xi^- &= -(1/3)(\tilde{g}_1 + 2\tilde{g}_3) + (1/9)(-4\tilde{g}_2 - 2\tilde{g}_4 + 8\tilde{g}_5 - 2\tilde{g}_6 - 2\tilde{g}_7) \\
(\Sigma \Lambda) &= -(1/\sqrt{3})(\tilde{g}_1 - \tilde{g}_3 + \tilde{g}_6 - \tilde{g}_7)
\end{align*}
\]  

(32)

From the first seven Eqs. one obtains

\[
\begin{align*}
\tilde{g}_1 &= 2.793 ; & \tilde{g}_2 &= -0.934 ; & \tilde{g}_3 &= -0.076 ; & \tilde{g}_4 &= 0.438 \\
\tilde{g}_5 &= 0.097 ; & \tilde{g}_6 &= -0.147 ; & \tilde{g}_7 &= 0.154
\end{align*}
\]  

(33)

If written with the \( g_\nu \)'s rather than with the \( \tilde{g}_\nu \)'s, using (31), the formulas (32) are changed all in the same way, by the addition of \((-1/3)g_0\) to the r.h.s. of all expressions (e.g. \( n = -(1/3)g_0 - (2/3)(g_1 - g_3) \), etc.). The last Eq. \((\Sigma \Lambda)\) stays unchanged. From it one gets \( \mu(\Sigma \Lambda) = -1.48 \pm 0.04; \) the experimental value is \(-1.61 \pm 0.08; \) errors are still large.

We underline that the structure of the \( G_\nu \)'s is similar to that of a NRQM, but the expression (28) for \( M_z(B) \) is an exact consequence of full QCD (to first order in flavor breaking). We emphasized this often [1],[26], but we repeat it to avoid misinterpreting the Eqs.(28,29) as a sort of generalized NRQM. In fact the results of the GP are exact consequences of QCD; they include all effects of virtual \( q \bar{q} \) pairs and gluons, as well as those of configuration mixing and the relativistic ones.
Note the relative dominance of \( \tilde{g}_1 \) and \( \tilde{g}_2 \) (related to the \( p \) and \( \Lambda \)) in the sum in Eq.(28) (see the numbers in the Eq.(30)); this explains why the fit of the naive NRQM:

\[
M_z(B) = \langle W_B | \tilde{g}_1(G_1)_z + \tilde{g}_2(G_2)_z | W_B \rangle
\]  

(34)

is fairly good.\(^8\) From Eq.(33) we can now explore the details of the hierarchy (the main parameters, related to \( p \) and \( \Lambda \) have been already mentioned): The average value of the reduction factor resulting from the presence of two different indices in a sum (we call it sometimes also the one gluon exchange reduction factor) derived from the values of \( |\tilde{g}_6|, |\tilde{g}_5|, |\tilde{g}_4| \) [the parameters multiplying terms with two different indices, except \( G_3 \) and \( G_7 \) -see below] is 0.25, having adopted 0.33 for the flavor reduction factor derived from \( |\tilde{g}_2| \) and \( |\tilde{g}_1| \). We will proceed using 0.33 for both the flavor and the “one gluon exchange” reduction factors. The maximum discrepancy between the estimated and empirical values is 2.5 for the \( |\tilde{g}_\nu|'s \) with \( \nu = 4, 5, 6 \). Above we did not consider \( \tilde{g}_7 \) for the reasons explained in Ref.[21].

The above values seem consistent, but a serious exception is \( |\tilde{g}_3| \approx 0.08 \). This is much too small: One expects from the hierarchy \( 2.79 \cdot 0.33 \approx 0.92 \), an order of magnitude larger. In Sect.5 we will indicate a possible solution to this question.

In extracting the hierarchy from the orders of magnitude of the \( \tilde{g}_\nu \)’s, we neglected, in the equations above, the presence of \( g_0 \); the reason, we repeat, is that its effect on the order of magnitude of the parameters discussed above is expected to be rather small. We will come back on this in Sect.7 but here we give the order of magnitude of \( g_0 \).

A first estimate of \( g_0 \) is obtained from the recent measurements of the \( \bar{s}s \) contribution to the magnetic moment of the proton. Essentially the question is: How much do the \( \bar{s}s \) pairs inside the proton contribute to its magnetic moment? One can answer to this question with the GP, calculating [35] the magnetic moment of the proton to all orders in flavor breaking. One can thus show that the contribution of the \( s\bar{s} \) to the proton magnetic moment is related to the value of \( g_0 \). One obtains \(|g_0 Tr[QP^s]| \) in the interval \( 1/10 \div 1/30 \). (Another determination of a Trace term -similar order of magnitude- will be obtained, using data from the ratio between the \( \rho - \gamma \) and \( \omega - \gamma \) couplings in Sect.12). Incidentally, these values are one order of magnitude larger than our previous estimates; but they are still small enough to confirm the fact that the Trace terms were indeed negligible when we neglected them. We add that the \( g_0 \) given above leads to a very small (non measurable) contribution of the \( s\bar{s} \) pairs to the proton magnetic moment at small values of \( q^2 \). This agrees with the experiments [79] and also with a lattice plus chiral evaluation [80].

**Proof of Eq.(30).** Define:

\[
\sigma_{i,z} = (1/3) \sum_i P_i \sigma_{i,z}
\]

and consider the expectation values of the above \( z \) components for all octet baryons. One can check that the equation:

\[
(1 + S) \sum_q |\bar{s}s - 3Q| \sum_q = (-2/3) + (5/3)Q - (2/3)S + (4/3)Q S
\]

(where \( Q \) is the baryon charge and \( S \) the baryon strangeness) holds for all the octet baryons (compare the table I in Ref.[1],p.3004). Observe that the \( z \) component of \( G_7 \) given in Eq.(29) can be written: \( G_{7,z} = -S(Q+(1/3)) - [3Q + 2] \sum_s + S \sum_q \). Simplifying in a similar way all the other \( G_{\nu,z} \)’s (with \( \nu \) from 0 to 6) one obtains the Eq.(30). To

---

\(^8\)Interrupting briefly this comparison, we recall that, as well known, [27] the \( SU_3 \) expression of the e.m current, to 1st order in flavor breaking, has just seven parameters. How does this compare with the results here? The point is that, as the derivation of the GP shows, each term in it has a definite dynamical meaning (corresponding to a certain class of Feynman diagrams) that produces a hierarchy of the parameters, with some parameter expected to be larger than others. Instead the dynamics plays no role in the \( SU_3 \) treatment, that only displays (group theoretically) a list of the possible terms.
check this we list the expressions of the \( G_{i,z} \) with \( i = 1 \) to 6: \( G_{1,z} = \sum_q \); \( G_{2,z} = - \sum_s \); \( G_{3,z} = Q - \sum_q \); \( G_{4,z} = -S + 3 \sum_s \); \( G_{5,z} = -(3Q + 1) \sum_s \); \( G_{6,z} = -S \sum_q + \sum_s \).

5. The \( \Delta \rightarrow p + \gamma \) decay, the magnetic moments of the \( \Delta \)'s and related results.

We now apply the GP to the subjects indicated in the title. As a byproduct we will obtain a possible explanation of why the ratio between the proton and neutron magnetic moments is so near to \(-3/2\) (that is \( \tilde{g}_2 \) is so small), as noted in Sect.4. The contents of this section is again related to the hierarchy. Possibly it would have been better to discuss the hierarchy in general at this stage. We preferred to continue here with problems related to the magnetic moments; but one can switch now to Sect.7 on the hierarchy and come back.

For the \( \Delta \rightarrow p + \gamma \) decay \( M1 \) transition see Ref.[33] to which we refer also for details on the transition form factor. (For the \( E2 \) contribution to the transition see Ref.[34]). The GP treatment of the \( M1 \) transition is developed in Ref.[1]; the formula of [1] to calculate both the diagonal magnetic moments of the \( \Delta \)'s and the transition matrix elements \( \Delta \rightarrow \text{Nucleon} + \gamma \), is Eq.(62), in Sect.9.\(^9\) Note that Eq.(62) is obtained omitting the \( \eta \) term in Eq.(61); this is possible because the \( \eta \) term just takes into account (Fermi-Watson theorem-compared, for this application, Ref.[28]) the effect of the final state interaction in the matrix element \( \Delta \rightarrow n\pi \). The omission of \( \eta \) amounts to analyze the data after the extraction of such effects. Moreover Eq.(62) does not include a term \( \approx \mathbf{J} \cdot \text{Tr}(Q P^3) \), negligible and not contributing to the \( \Delta \rightarrow p + \gamma \) decay; thus we continue to ignore below the difference between the \( \tilde{g} \)'s and the \( g \)'s.

Using our present notation, the Eq.(62) of [1] can be rewritten:

\[
\mathbf{M} = \sum_{\text{perm}} \left[ \alpha Q_1 + \delta (Q_2 + Q_3) \sigma_1 \right] + \left[ \beta Q_1 + \gamma (Q_2 + Q_3) \right] \sigma_1 (\sigma_2 \cdot \sigma_3) \tag{35}
\]

The sum over the perm(utations) in Eq.(35) means that to the term (123) displayed one has to add the terms (321) and (231); in Eq.(35) \( \alpha, \beta, \delta, \gamma \) are four real parameters (the same as those of the Eq.(62)); an estimate of their magnitude can now be obtained from the hierarchy:

\[
|\delta/\alpha| \simeq 0.33; \quad |\beta/\alpha| \simeq |\gamma/\alpha| \simeq (0.33)^2 \approx 0.11 \tag{36}
\]

As shown in [1] the Eq.(62) can be rewritten (recall that \( \mathbf{G}_1 = \sum_i Q_i \sigma_i \)):

\[
\mathbf{M} = (\alpha - \delta) \mathbf{G}_1 + (\beta - \gamma) \left[ (1/4)(4|\mathbf{J}|^2 - 7) \mathbf{G}_1 + (1/4) \mathbf{G}_1 (4|\mathbf{J}|^2 - 7) \right] \\
+ \left[ \delta - \beta - 2\gamma + (1/2)\gamma (4|\mathbf{J}|^2 - 7) \right] Q_B \cdot (2\mathbf{J}) \tag{37}
\]

The Eq.(37) can also be used to calculate to higher orders in the hierarchy [23] the magnetic moments \( p, n \) of proton and neutron; it gives (taking the expectation value):

\[
p = \alpha - 3\beta - 2\gamma, \quad n = -(2/3)(\alpha - \delta - 2\beta + 2\gamma) \tag{38}
\]

\(^9\)To avoid confusion we will call that Equation: (62). Also we correct here again some misprints in the Eqs.(63, 64, 66). In Eq.(63i)-third line- replace, in the square brackets, \( \delta - \beta - 2\gamma \) with \( \delta - \beta \); in Eq.(64i) \( \delta - \beta + 2\gamma \) should be \( \delta - \beta - 4\gamma \); in Eq.(66i) write \( F = \delta - \beta - 4\gamma \).
that is:
\[ g_1 = \alpha - 3\beta - 2\gamma, \quad g_3 = \delta - \beta - 4\gamma \] (39)

One can obtain \( p/n \simeq -3/2 \) if \( \delta \simeq \beta + 4\gamma \); it is interesting to note that the hierarchy (does not -of course- prescribe, but) allows this. Indeed \( \delta \) multiplies a term with two indices while \( \beta \) and \( \gamma \) both multiply terms with three indices. If, indeed, \( \delta \simeq \beta + 4\gamma \) (it appears the only possible explanation), the fact that \( p/n \) deviates so little from \(-3/2\) would be due to sheer chance. This is a case of historical interest because in 1965 the successful prediction \([3],[2]\) \( \frac{\mu(p)}{\mu(n)} = -3/2 \) was most important for the acceptance of the quark description. That prediction could not have been made if this chance cancellation (due to \( \delta \approx \beta + 4\gamma \)) had not occurred.\(^{10}\)

We add now a few remarks on the \( \Delta \to p + \gamma \) transition and on the magnetic moments of the \( \Delta \)'s. From Eq.(37) the matrix element \( \mu(\Delta \to p\gamma) \) is:
\[
\mu(\Delta \to p\gamma) = (2/3)\sqrt{2}(\alpha - \delta + \beta - \gamma) \] (40)

The approximate equation just discussed \( \delta \simeq \beta + 4\gamma \) gives:
\[
\mu(\Delta \to p\gamma) = (2/3)\sqrt{2}\mu(p)[1 + 3(\beta - \gamma)/\mu(p)] \] (41)

If \( \beta \) and \( \gamma \) have equal values and opposite signs (with \( \gamma < 0 \) as suggested by the previous relation \( \delta \simeq \beta + 4\gamma \) and by the hierarchy), using \(|\gamma/\alpha| \approx 0.11\) one gets:
\[
\mu(\Delta \to p\gamma) \approx 1.7 \cdot 2/3\sqrt{2}\mu(p)[1 + 3(\beta - \gamma)/\mu(p)] \] (42)

One more point. From the Eq.(37) one can also express in terms of \( \alpha, \beta, \gamma \) the magnetic moments \( \mu(\Delta) \) of the \( \Delta \)'s. It is:
\[
\mu(\Delta^Q) = (\alpha + 2\delta + \beta + 2\gamma)Q_{\Delta} = [\mu(p) + 2\delta + 4\beta + 4\gamma]Q_{\Delta} \] (43)

The Eq.(43) (with the estimate of \( \delta, \beta, \gamma \) given in Ref.[23], Sect.VI) shows that the magnetic moment \( \mu_{\Delta^+} \) of the singly charged \( \Delta^+ \) is expected to be appreciably smaller than \( \mu_p \), but the error in the estimate is still large. Note that it is quite generally \( \mu(\Delta^Q) = kQ + \xi \) with \( \xi \) very small (Ref.[14]) because \( \xi \) is the coefficient of a Trace term (ignored in this section), depressed due to the need of exchanging several gluons. Although the estimate of \( \xi \) in Ref.[14] was much too small, the \( \xi \) term remains negligible with respect to \( kQ \) (for \( Q \neq 0! \)). In practice the magnetic moments of the \( \Delta \)'s are expected to be quasi-proportional to their charges (thus \( \mu_{\Delta^0} \approx 0 \)). It is most difficult (perhaps impossible), but it would be most interesting to have a measurement of \( \mu_{\Delta^0} \); its deviation from zero would give directly the order of magnitude (for this case) of the coefficient of \( Tr(QP^*) \).

6. Double counting on inserting explicit pion fields in the QCD Lagrangian.

We now digress to a problem, marginal with respect to our main line, but important to clarify some treatments appearing frequently in the literature. In fact any calculation

\(^{10}\)Also Leinweber et al. [31] attribute to chance the smallness of the deviation of \( \mu(p)/\mu(n) \) from \( = -3/2 \), as we had (previously) noted [21]. However -though this is not relevant in our treatment- we do not share their claim that this conclusion is not possible in a constituent quark description. See Ref.[32].
based on a quark-gluon Lagrangian to which pion fields are added as explicit degrees of freedom leads to terms duplicating those coming from the standard QCD Lagrangian. This can be shown by the GP; the doubly counted terms can be displayed explicitly.

We do this for the baryon magnetic moments but the argument holds generally. It follows that, if some pion exchange contribution is assumed, it is impossible in any way to extract univocally its amount.

In fact, one often introduces, in addition to the QCD Lagrangian, an explicit $\bar{q}gq\pi$ coupling, say $\pi_i(q(x))\gamma_5\lambda_i\bar{q}(x)$, where $q(x)$ is the quark field. That is, in such treatments (e.g. Manohar and Georgi [38], Krivoruchenko [39]) the pions appear in the Hamiltonian in addition to the quark and gluon fields of QCD; the same occurs phenomenologically in various bag models [Ref.[37]]. In such treatments $\lambda_1, \lambda_2$ (or, in $SU(2), \tau_x, \tau_y$) should enter in the calculation of some hadronic properties, in particular of the magnetic moments. For instance, in [39] the magnetic moments of proton and neutron contain, due to pion exchange, the spin flavor term:

$$\sum_{i \neq k}(\sigma_i \times \sigma_k)(\tau_i \times \tau_k) \tag{44}$$

But the QCD Lagrangian (on which, of course, the GP is based) contains - also including electromagnetism and the flavor breaking mass term - only the flavor matrices $\lambda_3$ and $\lambda_8$. They commute and have a closed algebra. Thus from a QCD calculation, where pions intervene indirectly as $q\bar{q}$ aggregates, one should not get in the final result any $\lambda_1$ or $\lambda_2$; this seems to be in contrast with Eq.(44).

Imagine, as an example, to calculate in pure QCD the magnetic moments, using Feynman diagrams. No matter how complicated is the calculation, no flavor $\lambda_1, \lambda_2$ should appear in the final result, because no $\lambda_1$ or $\lambda_2$ can arise from $\lambda_3$ and $\lambda_8$. Then how can the result (this is the apparent "paradox") contain $\tau_x, \tau_y$'s, that is $\lambda_1, \lambda_2$?

Now the answer: As we will show, in contradiction with the above argument, the term (44) is not logically in contrast with a pure QCD calculation [the detailed treatment is presented in Ref.[36]]. We will show that the term (44) can indeed be present because it can be identically rewritten as a sum of terms not containing at all $\tau_x, \tau_y$. In short, as we will see, adding a term like (44) to a QCD calculation, is not forbidden. But it gives rise to a disturbing case of double counting. We will show in what follows that this double counting is unavoidable in Lagrangians containing explicitly pions in addition to quarks and gluons, unless a rule (that, however, would amount to have solved QCD) is given to subtract from QCD some definite quark-gluon diagrams.

Let us first examine how the fact that Eq.(44) contains $\lambda_1$ and $\lambda_2$ (while only $\lambda_3$ and $\lambda_8$ appear in QCD) is not, in principle, in contrast with QCD.

This is fairly simple (compare, for more details, the Sect.3 of [36]). Call $P_{x}^{ik}$ the Majorana exchange operator of the space coordinates of quarks $i, k$. Let us, for example, consider the magnetic moments of $p, n$ (compare Sect.4) and refer to their calculation, described by the Eq.(25)(Sect.4).

In a QCD calculation it is possible that the Majorana $P_{x}^{ik}$ appears in the $R_\mu(\mathbf{r}, \mathbf{r}')$ factor of Eq.(25), producing terms of the type $\sum_{i \neq k} Q_i \sigma_i P_{x}^{ik}$. We may calculate these terms in Eq.(25). First, let $P_{x}^{ik}$ act on $X_{L=0}$. Because $X_{L=0}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is symmetric in all pairs $i, k$, the operation $P_{x}^{ik}$ does not alter the result of the calculation of the parameters $g_\mu$.

Second, always in $SU(2)$ [we will extend to $SU(3)$ in a moment], consider the operation of $P_{x}^{ik}$ on the $W_B(1, 2, 3)$ factor. Write $P_{x}^{ik} = (1 + \sigma_i \cdot \sigma_k)(1 + \tau_i \cdot \tau_k)/4$, and,
in Eq.(25), let $P_{ik}^x$ operate on the $W_B$ factor using the relation:

$$
\sum_{i \neq k} Q_i \sigma_i \sigma^k = \sum_{i \neq k} Q_i \sigma_i (1 + \sigma_i \cdot \sigma_k) (1 + \tau_i \cdot \tau_k)/4
$$

(45)

with $Q_i = (1/2) \tau_{zi} + 1/6$. Using the identities $\tau_{zi}(\tau_i \cdot \tau_k) = \tau_{zk} - i(\tau_i \times \tau_k)_z$ and $\sigma_{zi}(\sigma_i \cdot \sigma_k) = \sigma_{zk} - i(\sigma_i \times \sigma_k)_z$, Eq.(45) becomes:

$$
4 \sum_{i \neq k} Q_i \sigma_i \sigma^k = \sum_{i \neq k} \left[(Q_i + Q_k) + (1/6)(\tau_i \cdot \tau_k) - 1\right] \left(\sigma_i + \sigma_k\right)_z - (1/2)(\tau_i \times \tau_k)_z (\sigma_i \times \sigma_k)_z
$$

(46)

In the steps leading to (46) we omitted the terms not contributing to the expectation values on real function $W_i$ (see Ref.[1],sect.V).

Because it is $\sum_{i \neq k} [(\tau_i \cdot \tau_k) - 1](\sigma_i + \sigma_k)_z = 0$ for $p, n$ states, using, on the left hand side $P^x X_{L=0} = X_{L=0}$, we obtain the identity, for $p, n$:

$$
\sum_{i \neq k} (\tau_i \times \tau_k)_z (\sigma_i \times \sigma_k)_z = -8 \sum_i Q_i \sigma_i + 4 \sum_{i \neq k} Q_i \sigma_k = -8 G_1 + 4 G_3
$$

(47)

The above equation (47) shows that the expression $\sum_{i \neq k} (\sigma_i \times \sigma_k)_z (\tau_i \times \tau_k)_z$ is already contained in the terms with coefficients $g_1$ and $g_3$ in the GP of the magnetic moments; that is the contribution from the pion term is indistinguishable and cannot be separated from the contributions of type $G_1$ and $G_3$ that already come from the quark gluon dynamics of QCD independently of the pion degrees of freedom. Adding (44) to $G_1$ and $G_3$ amounts to write twice the same terms (double counting). This argument is easily extended from $SU(2)$ to $SU(3)$, that is from terms of type $\sum_{i \neq k} (\sigma_i \times \sigma_k)(\tau_i \times \tau_k)$ to those of type $\sum_{i \neq k} (\sigma_i \times \sigma_k)(\lambda_i \times \lambda_k)$; obviously, if one keeps pion exchange, but excludes kaon exchange (as implied in Ref.[38]), one should insert the appropriate projectors, that is one should consider the term $\sum_{i \neq k} (\sigma_i \times \sigma_k)(\lambda_i \times \lambda_k)(1 - P^s_i)(1 - P^s_k)$. For the detailed calculation consult the Ref.[36]. The result is:

$$
\sum_{i \neq k} (\sigma_i \times \sigma_k)(\lambda_i \times \lambda_k)_3(1 - P^s_i)(1 - P^s_k) = -8 G_1 + 8 G_2 + 4 G_6 + 4 G_3 - 4 G_5 - 4 G_4
$$

(48)

Once more we see that pion exchange is already included in the GP of QCD; in other words it is conceptually impossible to disentangle the pion term from the quark-gluon contribution. Theories that obtain agreement with experiment using this trick are useless. That is, if we do not know, a priori, the contribution of the pion exchange, we cannot separate it from terms due only to gluons. If we somehow assume to know the contribution of pion exchange, then we should be able to solve QCD so as to subtract from gluon exchange that part already accounted by pion exchange. But, then the introduction of pion degrees of freedom, even if quasi Goldstone [38], looks unnecessary.

Note finally the following: The whole argument is based on the equivalence of the Majorana exchange operator, that can occur in a pure quark-gluon QCD, to terms typical of pion exchange. This it is not limited to the magnetic moments; we used them here only as a convenient example.

Although the above calculations, using explicitly the Majorana exchange operator, show directly that it is meaningless to insert in a QCD Hamiltonian a pion-quark interaction, this result is, in fact, already contained in the simple argument at the beginning of this Section: The fact that only the flavor matrices $\lambda_8$ and $\lambda_3$ appear in the strong plus electromagnetic QCD Hamiltonian implies, since $\lambda_8, \lambda_3$ form a closed algebra, that
in the exact expression of any quantity derived from the QCD Hamiltonian, no other flavor operator can be present. That is any effective Lagrangian depending explicitly on the pion field and thus on the $\lambda_1, \lambda_2$ flavor matrices cannot reproduce the results of the original Lagrangian. This remark applies also to a statement on the Skyrmion model (considered by Gross -see [40]- as a step that contributed to clarify the success of the NRQM) and applies also, more importantly, to the effective Lagrangian written by Manohar and Georgi [38] to explain the NRQM with chiral quarks, treating the pseudoscalar lowest nonet as quasi-Goldstone bosons and the other mesons as $\bar{q}q$ aggregates. A few words should be added on Ref.[38], the title of which is “Chiral quarks and the non relativistic quark model”. In writing a quasi-chiral Lagrangian of quarks and gluons plus pions, the latter seen as quasi-Goldstone bosons, the authors of Ref.[38] implicitly assumed, of course, that their Lagrangian was equivalent, for the description of hadrons, to that of QCD. In fact their conclusion is that their effective Lagrangian explains the good results of the NRQM. We find this conclusion unwarranted for more than one reason. First the Authors should have been be able to prove the mathematical equivalence of their Lagrangian to that of QCD, at least for a set of low energy problems. This equivalence is often taken for granted but this is unjustified. The simultaneous presence of quarks and pions in their Lagrangian, leading to an unavoidable double counting in their theory, as shown above, duplicates in an uncontrolled way contributions that are obtained with quarks and gluons alone, creating serious consistency problems. We mentioned specifically this point because one often finds, as already stated, that the way to understand the success of the NRQM is to start from a quasi chiral effective Lagrangian. Of course by leaving free in the quasi chiral effective Lagrangian a sufficient number of appropriately chosen parameters (as done in Ref.[39]) it is possible that some combination of them can be related to the parameters appearing in the GP. But, quite generally, our conclusion [36] is that a Lagrangian with explicit pions cannot reproduce QCD.

7. The hierarchy of the parameters.

In Sections 3,4,5 we exemplified (for the baryon masses and magnetic moments) how the data indicate a “hierarchy” of the GP parameters. At this stage, before examining other cases, we should give explicitly the rules for writing the expressions of the various quantities (masses, magnetic moments, etc.) so that the parameters are defined uniquely. To exemplify, if some term in the parametrization (symmetric in the indices $i, k$) is written as $\sum_{i,k} F_{i,k}$, or if the same term is rewritten as $\sum_{i<k} F_{i,k}$, the number of addenda differs by a factor 2 and the value of the parameter multiplying this term changes accordingly.

The masses of $8+10$ provide again an example to establish the rules. We show this here, rewriting, for convenience, the Eq.(19) of Sect.3 for the $8+10$ baryon masses:

\[
\text{"parametrized mass" } = M_0 + B \sum_i P_i^s + C \sum_{i<k} (\sigma_i \cdot \sigma_k) + D \sum_{i<k} (\sigma_i \cdot \sigma_k) (P_i^s + P_k^s) \\
+ E \sum_{i<k} (\sigma_i \cdot \sigma_k) P_j^s + a \sum_{i<k} P_i^s P_k^s + b \sum_{i<k} (\sigma_i \cdot \sigma_k) P_i^s P_k^s \\
+ c \sum_{i\neq k\neq j} (\sigma_i \cdot \sigma_k) P_j^s (P_i^s + P_k^s) + dP_1^s P_2^s P_3^s
\]
In Eq.(19) -reproduced above- all sums are written so that each different addendum in a sum appears once and only once. For instance the \( E \) term is explicitly

\[
E[(\sigma_1 \cdot \sigma_2)P^s_3 + (\sigma_3 \cdot \sigma_1)P^s_2 + (\sigma_3 \cdot \sigma_2)P^s_1]
\]  

(49)

The hierarchy becomes apparent if one looks at the experimental values of the coefficients in Eq.(19); these were given (in MeV) in Eq.(20) using, for the (wide) decouplet resonances, both the pole and the conventional mass values. We saw that the magnitude of the coefficients, that is of the parameters, decreases with increasing number of indices in a term (we often say “with increasing complexity”).

Quantitatively things go as follows: Each flavor breaking \( P^s \) factor in a sum (representing a term in the parametrization) introduces a reduction factor \( \approx 0.3 \) (compare e.g. the ratio \( D/C \)). In addition to this factor -related to flavor breaking- another reduction factor arises from the number of the different indices in the same sum. Each different pair of indices or, as we call it, each “gluon exchange” pair of indices, introduces a reduction factor \( \approx 0.37 \). The total reduction factor associated to a term is the product of the flavor and gluon exchange reduction factors. The numerical values of the reduction factors given above are those \([26]\) obtained using for the masses of the resonances their pole values. Using the “conventional” masses gives somewhat different reduction factors: the above 0.3 (and 0.37) become (in the order) 0.33 and 0.22.

As to the parameters multiplying the terms proportional to a \( \text{Trace} \) (these \( \text{Trace} \) terms are absent in the mass formulas and - to 1st order flavor breaking - they are unimportant for the magnetic moments), their order of magnitude is generally much smaller than the previous ones; we will consider it in the last part of this Section.

In what follows it is useful to describe the hierarchy in field theoretical terms, that is to establish the correspondence between the General Parametrization and a Feynman diagrams description (compare the Appendix I for a detailed treatment).

Consider the field theoretical calculation of the expectation value of some field operator, call it \( \Omega \) (expressed of course in terms of quark and gluon fields) in some exact state \( |\Psi\rangle \) of the exact QCD Hamiltonian. In the GP this calculation of \( \Omega_{av} \) amounts to that of the expectation value of an effective operator \( \tilde{\Omega} \) on the model state \( \Phi \):

\[
\Omega_{av} \equiv \langle \Psi | \Omega | \Psi \rangle = \langle \Phi | V^\dagger \Omega V | \Phi \rangle \equiv \langle \Phi | \tilde{\Omega} | \Phi \rangle
\]  

(50)

Because \( V \) can be related to the \( U \) operator of Dyson, it can be shown that \( \Omega_{av} \) is finally expressed in terms of Feynman diagrams (Appendix I, Eq.158 -note, in the example of Eq.158 \( \Omega_{av} \) is in fact \( \mathcal{M}_{av} \)):

\[
\Omega_{av} = \langle \Phi | T \{ [\Omega(0)U(+\infty | -\infty)] | \Phi \rangle_C
\]  

(51)

where \( C \) means that only “connected” Feynman diagrams are involved. Clearly, because \( |\Phi\rangle \) is a 3-quark state, all diagrams intervening in the calculation of \( \Omega_{av} \) (Eq.(50)) have three quark lines entering and three outgoing. For instance for the lowest \( 8 + 10 \) baryons \( \Phi \) has been chosen as the product of a space factor \( L = 0 \) state times a spin-flavor factor; thus \( \Omega_{av} \), after integration on the space coordinates, becomes, as exemplified on parametrizing the baryon masses and magnetic moments, a combination of spin-flavor structures. The situation is illustrated graphically in Fig.2. In Fig.2 - that refers to the baryon masses - the part \( \alpha \) corresponds to terms of type \( \sum_i \), the part \( \beta \) to terms of type \( \sum_{i\neq k} \), and the part \( \gamma \) implies \( \sum_{i\neq k\neq j} \). The coefficients that multiply terms \( \sum_{i\neq k} \) imply at least the exchange of one gluon; those multiplying terms \( \sum_{i\neq k\neq j} \) need the exchange of at least two gluons. The coefficients of the part \( \alpha \) of Fig.2 include all diagrams without gluon exchange, but may receive a contribution also from
Figure 2: (α) The class of connected Feynman diagrams giving rise to zero or one-index terms in the spin-flavor space. The diagrams in this class include all diagrams without gluon exchange between different boxes. However in (α) diagrams may be present in which gluon exchange is present between boxes without producing factors $(\sigma_i \cdot \sigma_k)$ or $P_i^sP_k^s$. (β) The class of connected Feynman diagrams corresponding to two-index terms in the spin-flavor space. The diagrams in this class imply the exchange of at least one gluon, but may include diagrams with more than one-gluon exchange not producing spin-flavor factors with three indices. (γ) The class of connected Feynman diagrams corresponding to terms with three indices in the spin-flavor space. These diagrams imply the exchange of at least two gluons. The boxes in (α), (β), (γ) describe the effect of the transformation $V$ on the three quarks in $|\Phi_B\rangle$. 
diagrams exchanging gluons without producing factors $\sigma_i \cdot \sigma_k$ or $P^s_i P^s_k$. On this basis one can list how the diagrams of the portions $\alpha$ or $\beta$ or $\gamma$ of Fig.2 contribute to the the various parameters in the baryon mass formula:

\[
\begin{align*}
M_0 &= M_{0\alpha} + M_{0\beta} + M_{0\gamma}, & C &= C_\beta + C_\gamma & (\text{no } P^s) \\
B &= B_\alpha + B_\beta + B_\gamma, & D &= D_\beta + D_\gamma, & E &= E_\gamma & (1 \text{ } P^s) \\
a &= a_\beta + a_\gamma, & b &= b_\beta + b_\gamma, & c &= c_\gamma, & (2 \text{ } P^s) \\
d &= d_\gamma & (3 \text{ } P^s) (52)
\end{align*}
\]

Note, for illustration, that $d, c$ and $E$ appear only with the suffix $\gamma$; this means that they imply at least the exchange of 2 gluons; $C, D, a, b$ are related to at least one gluon exchange; $M_0, B$ do not show signs of gluon exchange (but may contain, of course, contributions from one or more gluon exchanges in a spin independent way). Finally, because $M_0$ is the spin-flavor independent part of the mass, it can be interpreted roughly as the sum of the masses of the constituent non strange quarks in a $p$ or $n$ and leads (with some assumptions) to an estimate for the value of the masses of constituent quarks.

However, before discussing constituent quarks, we recall that (see Sect.2), we analyzed in Ref.[26] the dependence of the GP from the quark mass renormalization point; there the GP was expressed in terms of light “current quarks” $u, d, s$ with the conventional choice of their mass renormalization point ($g \approx 1 GeV$); to have shown, as in [26], that the GP can be formulated in this conventional QCD frame, is necessary for its consistency.

However, although the conventional choice of the quark masses is necessary for the high energy perturbative QCD, the results of the GP on the low energy properties of hadrons can be obtained both with this conventional choice or with different selections, say with masses of the order $\equiv M(P, N) \approx 300 MeV$ and $M_s \equiv M(\lambda), \approx 500 MeV$- corresponding to a different mass renormalization point. (Such quarks may be called “constituents”; their usefulness in the treatment of the low energy hadronic properties results from the fig.2 plus the hierarchy)\textsuperscript{11}

Indeed the hierarchy favors terms with few indices; therefore diagrams of type $\alpha$ in Fig.2 play a significant role. This role should be experimentally detectable, in the sense that the cases in which, say a proton, is in the form of three constituent quarks may be a measurable fraction of the total. Experiments to see this may be difficult, but, as discussed in [47] or [48] should be interesting.

It is clear that each “box” in Fig.2, when explored on a fine grain basis (that is, analyzed in terms of the conventional “light” $u, d, s$ quarks), contains all possible light quark-antiquark pairs -that is, quark loops- as well as gluons. In other words the Fig.2 describes the effect of the transformation $V$ on the three quarks (for a baryon) in $|\Phi_B\rangle$. Inside each box a quark line may zigzag as much as it likes - compatibly with QCD!- due to emission and reabsorption of gluons; also each box contains quark loops and gluon lines connected in all possible ways consistent the rules of QCD.

We now consider the determination of the parameters in the Eq.(19) giving the parametrized mass formula. We will do this in the constituent quark description, recalling that such description started with the “naive” quark model (Ref.[2]), where the analogy of a hadron to a nucleus composed of dressed nucleons was explicitly

\textsuperscript{11}In 1974 a paper by Melosh appeared [29] with the title:“Quarks: current and constituents” This paper, written at a time when current algebra was the basic description, can be confusing on the notion of constituent quarks. I mention it here only for this reason and refer to [30] for more comments.
introduced; this description was later resurrected in Ref.[13], where a Fermi-Breit approximation of QCD was introduced.

Here, to exemplify how the values (Eq.(20)) of the GP baryon mass parameters can be determined, we will use the constituent treatment as in Ref.[21].

Consider first the ratio of the coefficients $D$ and $C$ in the mass parametrization Eq.(19); this ratio gives the magnitude of the reduction factor related to $1P^s$ flavor breaking. It is:

$$|D/C| = \frac{\langle \geq 1 \text{ gluon exch.} | 1P^s \rangle}{\langle \geq 1 \text{ gluon exch.} | 0P^s \rangle} \approx 0.3$$  (53)

To obtain the reduction factor due to flavor, we start considering the ratio $|D/C|$; a similar procedure can be used for the parameter $c$. The treatment of $D/C$ is given in Ref.[21] after Eq.(6). It starts from the question: Why does the above ratio (53), which is a measure of $\Delta m/m$, where $m$ is, here, the mass of a “constituent” quark in a nucleon and $(m + \Delta m)$ the same quantity for the “strange constituent quark”, give the correct value of the reduction factor due to flavor? We might have written equally well the term $D \sum_{i>k}(\sigma_i \cdot \sigma_k)(P^s_i + P^s_k)$ in Eq.(19) as $2D \sum_{i>k}(\sigma_i \cdot \sigma_k)P^s_i$ because only its expectation value on a symmetric wave function is relevant. Why then we used $D/C$ rather than $2D/C$? The rule adopted in writing the sums in Eq.(19) implies the first choice (and of course a uniform criterion must be adopted). Still, in view of the importance of this point, a direct check is appropriate. For this we compared our GP result with the explicit results of De Rujula, Georgi and Glashow [13]. Note that besides the comparison with Ref.[13], the analysis in [26] confirms the above treatment. In Ref.[21] we also evaluated (using the conventional values of the baryon masses) the ratio $|E/D|$ that gives the reduction factor due to one gluon exchange, obtaining $|E/D| = 0.22$. The same quantity calculated using the pole values of the masses is:

$$|E/D| = \frac{\langle \geq 2 \text{ gluon exch.} | 1P^s \rangle}{\langle \geq 1 \text{ gluon exch.} | 0P^s \rangle} \approx 0.37$$  (54)

We now come back to the hierarchy for the magnetic moments (Sect.4). Using the symbols of Eqs.(33) we can write:

$$|\tilde{g}_2/\tilde{g}_1| = \frac{\langle \geq 0 \text{ gluon exch.} | 1P^s \rangle}{\langle \geq 0 \text{ gluon exch.} | 1P^s \rangle} = 0.934/2.793 \approx 0.33$$  (55)

Thus the reduction factors for flavor breaking from the analysis of the masses and from that of the magnetic moments are the same (0.3 to 0.33).

The Trace terms. Finally we consider the terms of type $g_0 Tr(QP^s)$; these terms -absent in the baryon mass parametrization- play a role in the GP of many quantities, e.g., in the magnetic moments of the octet and the decuplet baryons. In Ref.[1] these Trace terms were ignored -incorrectly- for the magnetic moments of the octet baryons; but then (as shown in detail in Sect.4) they can be combined with the other terms so that, in fact, no error was made to first order in flavor breaking, which was the case considered in Ref.[1]. What is the meaning and order of magnitude of the Trace terms?

A) Meaning - Looking at the GP in terms of Feynman diagrams a Trace term corresponds to a closed quark loop. There may be, as already stated, a large number of quark loops inside each box in fig.2; but it is not to such loops -in which only gluons attach to the loop- that we refer here. We refer to quark loops where an external photon or some other external particle (say a Z meson for weak interactions) attaches
to a vertex in the loop quark circuit. For the magnetic moments one has to do with a photon (the external magnetic field). These Trace terms, that imply the presence of an external field, are not present in Fig. 2 that refers to the baryon masses and should be completed in the presence of external fields (for instance, just having a photon attached to one of the boxes). A case of loop with a photon at a vertex will be considered in Sect. 12 (compare there the Fig. 3). These quark loops are represented by Trace terms.

If a photon vertex is on the loop we meet, in circling the loop, the photon vertex plus, at least, three gluons vertices. The reason for the above “three” is that one gluon is forbidden by color (“forbidden” means that the loop gives zero) and two gluons (+ a photon) is forbidden by the Furry theorem, so that three gluons are the minimum number that can be present.

B) Order of magnitude - A general estimate is not easy, but the circumstance that such a loop is connected to the rest of the diagram by at least three gluons suggests, due to the hierarchy, that its contribution should be comparatively small. One may expect a reduction factor of the order \((0.33)^3 \cdot (1/3) \cdot 3 \approx 3.6 \cdot 10^{-2}\), where the \((1/3)\) appearing in the product is the value of \(\text{Tr}(QP^0)\) and the 3 stays for \(N_c\). (This rough estimate disregards the permutations between the gluons from the loop). An order of magnitude (consistent with the above estimate) of the loop reduction factor derived from specific cases was given near to the end of Sect. 3.8. The parametrization of the masses of the lowest Ps and V meson nonets.

We apply now the GP to the lowest nonets of Pseudoscalar and Vector mesons (compare Ref. [17]). As we will see, the GP leads to formulas that look very similar to those of the NRQM, although now the description is fully relativistic. We will proceed as for baryons, beginning with the model states for mesons. Because the model Hamiltonian for the mesons, \(\mathcal{H}(\text{mesons})\) (written simply \(\mathcal{H}\) below in this Section), is assumed to conserve the orbital angular momentum and to be spin and flavor independent, the model states of the Ps and V nonets are degenerate states of \(\mathcal{H}\) (\(M_0^0\) is the same for all \(i\) in Eq.(56)):

\[
\mathcal{H} | \Phi_i \rangle | \text{no gluons} \rangle = M_0^0 | \Phi_i \rangle | \text{no gluons} \rangle (i = \pi, \eta, \eta', K; \rho, \omega, \Phi, K^*) \quad (56)
\]

In Eq.(56) \( | \Phi_i \rangle \) is a state of a meson at rest with \(L = 0\) and \(M_0^0\) is the common value of the masses of the 18 model states of the \(0^-\) and \(1^-\) lowest mesons. From now on we omit the factor \( | \text{no gluons} \rangle \) in (56). Because the model Hamiltonian \(\mathcal{H}\) is spin and flavor independent, the wave function \(\Phi_i\) of a meson has the form:

\[
\Phi_i = S_c W_i \varphi(r) \quad (57)
\]

where \(S_c\) is the color singlet factor for a meson (that we will not write in what follows, unless necessary), \(W_i\) is the \(q\overline{q}\) spin-flavor wave function of the \(i\)-th meson and \(\varphi(r)\) is a function of the relative distance \(r\) of the quark and antiquark.

Operating as we did for baryons, we define a unitary transformation \(V\) acting on the above meson model states and transforming them into the exact states (to be indicated \( | \psi_i \rangle \)). Recall, once more, that (as for baryons, Sect. 2) the model states are chosen as simple as possible; all the “complications” are hidden in the operator \(V\):

\[
| \psi_i \rangle = V | \Phi_i \rangle \quad (58)
\]

Again our aim will be to parametrize the results. As done for baryons, \(V\) is constructed in terms of the exact QCD Hamiltonian \(H_{\text{QCD}}\); the problem of quark mass renormalization is similar to that considered in Sect. 2 for baryons. Because the QCD Hamiltonian
is the same, the flavor breaking parameter is decently small and can be treated perturbatively (as for baryons).

In the above notation the mass of the \(i\)-th meson is:

\[
M_i = \langle \psi_i \mid H \mid \psi_i \rangle = \langle \Phi_i \mid V^\dagger HV \mid \Phi_i \rangle \tag{59}
\]

Because the \(\Phi_i\)'s are, by construction, two body states (one quark-one antiquark), only the projection \(\tilde{H}\) of the operator \(V^\dagger HV\) in the subspace of these two body states intervenes in the calculation of (59):

\[
\tilde{H} \equiv \sum | 1q, 1\bar{q} \rangle \langle 1q, 1\bar{q} | V^\dagger HV | 1q', 1\bar{q}' \rangle \langle 1q', 1\bar{q}' | \tag{60}
\]

The above statement, that the only part of \(V^\dagger HV\) intervening in the calculation of \(M_i\) (Eq.(59)) is the two body part \(\tilde{H}\), is trivial but essential; in fact calculating the expectation value of the field operator \(V^\dagger HV\) in the state \(| \Phi_i \rangle\) becomes equivalent to calculate the expectation value of a certain quantum mechanical two-body operator \(\tilde{H}\) on the wave function \(\Phi_i\) in ordinary non relativistic two-body quantum mechanics:

\[
M_i = \langle \Phi_i \mid \tilde{H} \mid \Phi_i \rangle \tag{61}
\]

The most general parametrization of the meson masses amounts then to write in ordinary non relativistic two body quantum mechanics the most general operator \(\tilde{H}\) of the relative coordinates and momentum \(r, p\), of the spins \(\sigma_1, \sigma_2\) (1=quark, 2=antiquark) and of the flavor operators \(f\) (see below) invariant with respect to translations and rotations. It follows from Eq.(61) that this most general operator has necessarily the form:

\[
\tilde{H} = \sum_{\nu} R_{\nu}(r, p) G_{\nu}(\sigma, f) \tag{62}
\]

where the \(G_{\nu}\)'s (that of course have nothing to do with the same (boldface) symbols used treating the baryon magnetic moments in Sect.4) is a set of independent operators (specified by the index \(\nu\)) constructed in terms of \(\sigma_1, \sigma_2\) and the flavor operators \(f\) of the quark and the antiquark; \(R_{\nu}(r, p)\) are operators (of which it is not necessary to know the expression) constructed in terms of the relative coordinate and momentum \(r\) and \(p\) of the quark and the antiquark pair in the model state.

Calculating the expectation values of \(\tilde{H}\) (62) on the various mesons \(i\), their masses \(M_i\) can be written:

\[
M_i = \sum_{\nu} g_{\nu}(W_i \mid G_{\nu}(\sigma, f) \mid W_i) \tag{63}
\]

where \(g_{\nu}\) stays for:

\[
g_{\nu} = \langle \varphi(r) \mid R_{\nu}(r, p) \mid \varphi(r) \rangle \tag{64}
\]

\(\varphi(r)\) being the space dependent factor of \(\Phi_i\) in Eq.(57).

Because the model Hamiltonian \(\hat{H}\) was chosen to be independent from the flavor and from \(J\), the space wave function \(\varphi(r)\) in Eq.(64) is independent of the meson index \(i\); the same is true for the coefficients \(g_{\nu}\) in Eqs.(63, 64). Therefore Eq.(63) leads to the mass operator \(M\) (65) (of which, to get the masses, one must take the expectation value on the spin-flavor factor \(W_i\) of the wave function):

\[
M = \sum_{\nu} g_{\nu} G_{\nu}(\sigma, f) \tag{65}
\]
Note that this is a general result; part of its interest (as for the analogous expression for baryons, Eq. (17)) is its simplicity (and the fact that it appears in a notation directly related to the NRQM).

Listing the possible $G_{\nu}$ in Eq. (65) means to find their possible spin and flavor dependence. Ignoring, for the moment, the electromagnetic interactions, the only $\lambda$ matrix that intervenes is $\lambda_s$, that appears in the flavor breaking part of the Hamiltonian. Instead of $\lambda_s$ we consider the projection operator $P^s = \frac{1}{2}(1 - \lambda_s)$ that gives zero when applied to $u$ or $d$ quarks (or antiquarks) and is 1 when applied the $s$ quark (or antiquark). For a system of one quark and one antiquark, and if the isospin $I$ of the state is $\neq 0$ the list of flavor operators is:

$$1, \quad P^s_1, \quad P^s_2$$

(66)

In (66) and in what follows the index 1 will refer to the quark and the index 2 to the antiquark.

For the mesons with $I = 0$ the list above (66) is however incomplete. Indeed the flavor operators in Eq. (66) connect only a quark with an antiquark. For a system of one quark and one antiquark, and if the isospin $I = 0$, also matrix elements of $\hat{H}$ that connect a $u\bar{u}$ state with a $d\bar{d}$; or a $s\bar{s}$ state is not normalized to 1 (differently from the other states); to normalize to 1, we must multiply it by $1/\sqrt{3}$. Here we leave it in the above form, keeping in mind this point.

Taking into account the expressions (67) we must add to the list (66) the operators:

$$\langle z | w \rangle \equiv P^s_1 P^s_2 (68)$$

As to the $\sigma$ dependence of the $G_{\nu}(\sigma, f)$ only the following expressions (69) are possible:

$$1,\quad \sigma_1 \cdot \sigma_2$$

(69)

We can now discuss the parametrization of the meson masses. We begin with the (lowest mass) mesons with $I \neq 0$, ($\pi, K; \rho, K^*$). The flavor breaking parts of $G_{\nu}(\sigma, f)$ can contain only $P^s_1$ and $P^s_2$. The charge conjugation invariance of the Hamiltonian implies that the $G_{\nu}$'s must be invariant for the exchange of 1 and 2; thus the possible $G_{\nu}$'s are $1, \sigma_1 \cdot \sigma_2, P^s_1 + P^s_2, \sigma_1 \cdot \sigma_2(P^s_1 + P^s_2)$. Therefore the most general expression of the mass of a meson with $I \neq 0$ correct to all orders in flavor breaking is:

$$M_{I \neq 0} = A + B\sigma_1 \cdot \sigma_2 + C(P^s_1 + P^s_2) + D\sigma_1 \cdot \sigma_2(P^s_1 + P^s_2)$$

(70)

where $A, B, C, D$ are four real coefficients. Because $\sigma_1 \cdot \sigma_2 = -3$ for $J = 0$ and $\sigma_1 \cdot \sigma_2 = +1$ for $J = 1$, the masses (indicated by the meson symbols) are:

$$\pi = A - 3B (= 138), \quad K = A - 3B + C - 3D (= 495)$$

$$\rho = A + B (= 770), \quad K^* = A + B + C + D (= 894)$$

(71)
where the values given in MeV are approximate values averaged over the charges. The Eqs.(71) imply (in MeV):

\[ A = 612, \quad B = 158, \quad C = 182, \quad D = -58 \quad (72) \]

We now consider the mesons with \( I = 0 \). Their most general mass formula is obtained adding to (70) with \( A, B, C, D \) just determined, another part obtained multiplying the flavor operators (68) with the spin operators (69). We have:

\[
M_{I=0} = A + B\sigma_1 \cdot \sigma_2 + C(P_1^s + P_2^s) + D\sigma_1 \cdot \sigma_2(P_1^s + P_2^s) \\
+ (E + F\sigma_1 \cdot \sigma_2)|z\rangle\langle z| + (H + G\sigma_1 \cdot \sigma_2) (|z\rangle\langle w| + |w\rangle\langle z|) \\
+ (N + T\sigma_1 \cdot \sigma_2)P_1^sP_2^s \quad (73)
\]

The Eq.(73) contains \( A, B, C, D \) \(^{12}\) (determined [Eq.(72)] from \( \pi, K, \rho, K^* \)) and six additional parameters \( E, F, H, G, N, T \). These might be determined from the masses of \( \eta, \eta', \omega, \phi \) + two mixing angles (the vector \( \theta_V \), and pseudoscalar \( \theta_P \)), if these angles were known. Alternatively one can limit to the first-order flavor-breaking approximation, that is disregard the last term in Eq.(73)(this means that the \( N, T \) terms, of 2nd-order in flavor breaking -proportional to \( P_1^sP_2^s \) are disregarded; that is we set: \( N = 0, T = 0 \)). In this 1st order flavor breaking approximation the masses of \( \eta, \eta', \omega, \phi \) fix \( E, F, G, H \); then one can determine, to 1st order in flavor-breaking, the mixing angles.

Here we will first \(^{17}\) consider the Pseudoscalar \( P \) mesons \( \eta, \eta' \); for them in Eq.(73) \( \sigma_1 \cdot \sigma_2 = -3 \). Introducing the abbreviations:

\[ b = A - 3B (= 138 MeV), \quad d = C - 3D (= 357 MeV), \quad f = E - 3F, \quad g = H - 3G \quad (74) \]

where \( b \) and \( d \) are known from (72), the Eq.(73) for the \( P \) mesons with \( I = 0 \) takes the form:

\[ M_{I=0}(P) = b + d(P_1^s + P_2^s) + f|z\rangle\langle z| + g(|w\rangle\langle w| + |w\rangle\langle z|) + O(\Delta m/m)^2 \quad (75) \]

To determine the masses of \( \pi^0, \eta, \eta' \) correct to 1st-order in flavor-breaking we must diagonalize the Equation \(^{13}\)

\[
\begin{pmatrix} b + f - M & f & f + g \\ f & b + f - M & f + g \\ f + g & f + g & b + 2d + f + 2g - M \end{pmatrix} = 0 \quad (76)
\]

One solution of Eq.(76) is the \( \pi^0 \) mass; we get, of course for it: \( \pi^0 = b = A - 3B \); the masses of \( \eta \) and \( \eta' \) are the two roots of:

\[ (b + 2f - M)(b + 2d + f + 2g - M) - 2(f + g)^2 = 0 \quad (77) \]

The solutions of Eq.(77) are \( M = b + k \pm (k^2 + 2g^2 - 4fd)^{1/2} \) where \( 2k \equiv (3f + 2d + 2g) \). Equating the two solutions to the masses of \( \eta(547) \) and \( \eta'(958) \) we obtain \( \eta' + \eta = \]

\(^{12}\) We use here the same symbols of Ref.[17]; in Ref.[26], Eq.(28) \( C \) and \( B \) were -unfortunately-interchanged: \( C \) there is our \( (B) \) here and viceversa.

\(^{13}\) In an arXiv paper [19] L.Durand, while stating that the matrix (76) below was the most general form, added that it did not contain the physical identification of the various contributions. At a question of G.M. on the meaning of this, he answered that the last statement certainly needed a clarification- if that arXiv paper was published, which was uncertain.
\[ 2b + 3f + 2d + 3g \text{ and } (\eta' - \eta)/2 = [2g^2 - 4fd + \frac{1}{4}(3f + 2d + 2g)^2]^{1/2} \text{ where } b \text{ and } d \]

have the value given in Eq.(74). Solving for \( g \) and \( f \) we obtain two possible solutions \((g; f)_1 \) and \((g; f)_2 \). Expressed in MeV, they are:

\[
(g; f)_1 = (-133, +261), \quad (g; f)_2 = (-343, +400) \quad (78)
\]

The ratio \((g/f\sqrt{3})\) between the coefficient \( g \sqrt{3} \) of the (normalized) flavor-breaking term and the coefficient \( 3f \) of the (normalized) unitary singlet term is substantially smaller for the solution \( N.1 \) than for the solution \( N.2 \). Thus we must choose the solution \( N.1 \) (in order to be consistent with the assumption of neglecting terms of order higher than the first in \((\Delta m/m)\)). On choosing \((g; f)_1\) it is straightforward to check that the diagonalization of (77) implies [in addition to \( \pi^0 = (1/\sqrt{2})(u\pi - d\bar{d}) \)]:

\[
\eta = 0.603(u\pi + d\bar{d}) - 0.522 s\bar{s}, \quad \eta' = 0.367(u\pi + d\bar{d}) + 0.854 s\bar{s} \quad (79)
\]

With the usual definition \( \eta = \eta_1 \sin \theta_P + \eta_8 \cos \theta_P \) and \( \eta' = \eta_1 \cos \theta_P - \eta_8 \sin \theta_P \) where \( \eta_1 = (1/\sqrt{3})(u\pi + d\bar{d} + s\bar{s}) \) and \( \eta_8 = (1/\sqrt{6})(u\pi + d\bar{d} - 2s\bar{s}) \), we obtain from Eqs.(79):

\[
\sin \theta_P \simeq -0.39 \quad \text{(that is, } \theta_P \simeq -23^\circ) \quad (80)
\]

This value can be compared with \( \theta_P \simeq -20^\circ \) and \( \theta_P \simeq \pm 24^\circ \) obtained respectively in Refs.[65],[66].

We refer here to the part B of Sect.V of [17] for a discussion (in our opinion now totally obsolete) on the values of the above angle obtained using, instead of the Hamiltonian, its square or other powers. The discussion is obsolete for the reason already stated (Sect.2,footnote 4), that the square of the QCD Hamiltonian is, almost certainly, a non renormalizable operator. It is obvious, from the previous derivation of \( \theta_P \), that the value given above \((-23^\circ)\) is that obtained from QCD neglecting terms of second order in flavour breaking.

Other work comparable to our results described above is that in Ref.[13] and in Ref.[18], the latter closely related to the NRQM. As to Ref.[13], if we identify our \( f \) with the \( \beta \) in the mass-matrix of [13] and include the 1st-order flavor breaking correction \( g \) (that should have been included in the work of [13]-compare the end of Sect.V in Ref.[17], our mass matrix and that of Ref.[13] coincide. Also our results coincide with those of [18]. Here, aside from other minor points, there seems to be a numerical mistake in the Eqs.(5) and (5a) of [18] for the \( \eta \) and \( \eta' \), but, after its correction, the result coincides (again compare for the details the footnote 8 of [17]).

More interesting, is the following remark. We have seen that the pion mass is:

\[
\pi = A - 3B \quad (81)
\]

with \( A = 612, B = 158 \). These values -that refer to \( I \neq 0 \) mesons- are exact QCD values (there is no higher order correction omitted). The smallness of the pion mass, \( \pi = A - 3B \) might be just an accident depending critically on the values of the parameters \( A \) and \( B \) in Eq.(72), with \( B \) multiplied by \((-3)\) in Eq.(71). Assume, for instance, that \( B \) had a value 100 MeV instead of 158 MeV (Eq.(72)). Then, if \( A \) maintains its value 612 MeV, the pion would have a mass of 312 MeV, no longer so small (the \( \rho \) would then have mass 712 MeV).

In QCD a reduction of \( B \) by the above amount \( \approx 40\% \) could arise from an even smaller percentage change of \( \alpha_s \), the quark-gluon coupling. Thus we do not attribute a profound meaning to the smallness of the pion mass. We differ substantially on
this from the standard point of view in chiral QCD, where the pion is seen as a quasi-Goldstone boson, getting its mass from explicit breaking of chiral symmetry due to the small u, d masses. This remark does not question at all, obviously, the phenomenological treatments and results related to PCAC (partial conservation of axial-vector current), insofar as they simply take note of the empirical value of the pion mass.

We now come back to the Vector mesons $V$ with $I = 0$; their discussion is similar to that of the $Ps$ mesons; one has to put in Eq. (70) $\sigma_1 \cdot \sigma_2 = +1$ (instead of $-3$) and therefore one replaces in the previous treatment of the $Ps$ mesons $A - 3B$ with $A + B$; $C - 3D$ with $C + D$; $E - 3F$ with $E + F$; $H - 3G$ with $H + G$. As is well known, the parametrization with $A, B, C, D$ alone is almost sufficient for the $I = 0$ Vector mesons; this means that in the Equation that replaces (74) $E + F \approx 0$ and $H + G \approx 0$. (From this follows that, for instance, the value of the rate $\Phi \to \pi^0\gamma$ is very small; we will come back to this in the next Section.)

A few comments on the relation of the above results with those of the NRQM. To conclude this section we come back to the question raised at the start: Why does the NRQM work so well? The following remarks, that conclude Ref.[17], provide a summary of the answer; of course, for more details, the whole Ref.[17] should be consulted.

(1) Almost all the features of the mass formulas for the lowest meson nonets currently used in a NRQM description are general consequences of the GP; they are similar to those of the NRQM, but they do not depend on it; the only exception is the $(m_1 m_2)^{-1}$ multiplying factor of the $\sigma_1 \cdot \sigma_2$ term in the De Rujula et al. expression [13] for the meson masses, a result that depends specifically on the one gluon exchange potential. Stated differently, the meson mass formulas used in the NRQM are more general than one might have thought. In this respect the situation for the meson masses is quite different, for instance, from that found for the baryon magnetic moments. We recall that the GP expression for the magnetic moments, correct to first order in flavor breaking, had seven parameters whereas the fairly successful parametrization of the NRQM had only two.

(2) The procedure developed above to derive the meson mass formulas clarifies the old question of the mixing angle for the $I = 0$ meson nonets. The mixing angle (say $\theta_P$) can be determined from the GP (that is in a model independent way) knowing the masses of the $P$ mesons provided terms of order higher than the first in the flavor breaking expansion parameter are negligible and are neglected. According to this procedure the linear angle is much more natural than the quadratic or square-root ones; it is the angle resulting from a QCD calculation on expanding the exact result in series of $(\Delta m/m)$ and neglecting terms of order higher than the first.

9. The radiative $V \to P + \gamma$ meson decays.

An early (1965) test of the NRQM consisted in comparing with the data the calculated [4] radiative decays of Vector mesons; the evaluation of these ($M1$) $\gamma$ transitions was possible in 1965 because the magnetic moments of the $P$ and $N$ quarks had been deduced [2] from those of the proton and neutron. We list below many $M1$ transitions of interest, though only a few were measured in those years and the magnetic moment of the $\lambda$ quark was still unknown:

(1) $\omega \to \pi^0 + \gamma$, (2) $\omega \to \eta + \gamma$, (3) $\rho \to \pi + \gamma$, (4) $\rho^0 \to \eta + \gamma$, (5) $K^{*+} \to K^+ + \gamma$, (6) $K^{*0} \to K^0 + \gamma$, (7) $\rho^0 \to \eta' + \gamma$, (8) $\omega \to \eta' + \gamma$, (9) $\phi \to \eta + \gamma$, (10) $\phi \to \eta' + \gamma$, (11) $\phi \to \pi^0 + \gamma$

The $\omega \to \pi^0\gamma$ transition had been measured reasonably well. The NRQM cal-
culation reproduced it and gave the orders of magnitude of the others not too far from reality. Ref.[4] (compare also [6, 7]) contains a list of the transitions calculated at that time.

Because these transitions need in fact a full QCD calculation, one can apply the GP also to this problem. The formalism is developed in Ref.[20](General parametrization of the $V \to P \gamma$ meson decays), that contains a full description of the procedure. Here we only summarize some of the main results. It will appear that the hierarchy is useful also here; and, once more, this explains why the NRQM works.

To clarify this statement, before giving the details, consider the ratio $\Gamma(\omega \to \pi^0 \gamma)/\Gamma(\rho \to \pi^0 \gamma)$; the GP predicts this to be 9 (to all orders in flavor breaking) plus the contributions from processes where the initial $q\bar{q}$ state $\omega$ transforms into 3 gluons and then gives rise to the final state. The hierarchy, of course, implies that such contribution should be a minor one and, in fact, it is not visible (at the level $\pm 15\%$, the experimental error). Other processes and, in particular, $\phi \to \pi^0 \gamma$, lead to the same conclusion. In general, if the 3-gluon diagrams are negligible, the important terms in the GP calculation of the $V \to \pi \gamma$ decays (that being an exact QCD calculation, includes automatically the configuration mixing and all the complexities of the Fock $q\bar{q}$, gluon expansion of the hadron states) reproduce in practice the result of the NRQM.

We now give a short summary of the procedure; that is we will give the main steps of the GP for a transition of type $A_i \to B_j + \gamma$ where $A_i$ is a vector meson (e.g. $\rho, \omega, \phi, K^{*0}, \bar{K}^{*0}, K^{*\pm}$) and $B_j$ is a pseudoscalar meson (e.g $\pi, \eta, \eta', K^0, \bar{K}^0, K^\pm$).

The matrix element for the transition $A_i \to B_j + \gamma$ in the rest frame of $A_i$ is:

$$M_{ji} = \frac{1}{\sqrt{2k \cdot 2E_j(P)}} \int dt \exp(-ikt) \langle B_j(P) | \int d^3r \exp(i\mathbf{k} \cdot \mathbf{r}) \mathbf{j}(\mathbf{r}, t) | A_i(0) \rangle \cdot \epsilon$$ ...

where $\mathbf{j}(\mathbf{r}, t)$ is the quark e.m. current and $\epsilon, \mathbf{k}$ and $k$ are the photon polarization, momentum and energy; $|B_j(P)\rangle$ are respectively the true states of the pseudoscalar meson with momentum $P$ and of the vector meson at rest (the index $i$ will always refer to the $V$ mesons and $j$ to the $P$ mesons); $E_j(P)$ is the energy of the $P$ meson (until further notice $V$ is assumed to be heavier than $P$); $[2E_j(P)]^{-1/2}$ is required by Lorentz invariance, if, as we do, we normalize both $|B_j(P)\rangle$ and $|A_i(0)\rangle$ to one meson per unit volume in the rest system of $A_i$.

On expressing the exact states $|B_j(P)\rangle$ and $|A_i(0)\rangle$ (each of which, being an exact state, is a superposition of an infinite number of Fock states with $q, q'$ and gluons) as $|B_j(P)\rangle = V|\Phi_{B_j}(P)\rangle$ and $|A_i(0)\rangle = V|\Phi_{A_i}(0)\rangle$ where $V$ is the unitary transformation already introduced repeatedly (transforming in this case the $(1q, 1\bar{q})$ model states $\Phi$ into the exact states), the Eq.(82) becomes:

$$M_{ji} = \frac{1}{\sqrt{2k \cdot 2E_j(P)}} \int dt \exp(-ikt) \langle \Phi_{B_j}(P) | V^\dagger | \int d^3r \exp(i\mathbf{k} \cdot \mathbf{r}) \mathbf{j}(\mathbf{r}, t) V|\Phi_{A_i}(0)\rangle \cdot \epsilon$$ ...

In Eq.(83) it is:

$$\mathbf{j}(\mathbf{r}, t) = \exp[i(Ht - \mathbf{G} \cdot \mathbf{r})] \mathbf{j}(0) \exp[-i(Ht - \mathbf{G} \cdot \mathbf{r})]$$ ...

where $\mathbf{G}$ is the momentum operator.

Recall that $\mathbf{G}$ commutes with $V$, whereas, of course, $H$ does not; this corresponds to the fact that the momentum (but not the energy) of the model state must be equal to the momentum of the exact state. For the exact Hamiltonian $H$ one has:

$$H|B_j(P)\rangle = [P^2 + M^2(B_j)]^{1/2}|B_j(P)\rangle, \quad H|A_i(0)\rangle = M(A_i)|A_i(0)\rangle$$ ...

(85)
We repeat that the index $i$ will always refer to a Vector meson (in this case $A_i$ (at rest) -mass $M_i$- and the index $j$ to a $P$ meson ($B_j$), in this case with energy $\sqrt{P_j^2 + M_j^2}$). Inserting Eqs.(83)(84) in Eq.(82) we get:

$$M_{ji} = \frac{1}{\sqrt{2k \cdot 2E_j(P)}} (2\pi^2) \delta^{(3)}(\mathbf{P} + \mathbf{k}) \delta(M_i - k - E_j(P)) \langle \Phi_{B_j}(\mathbf{P}) | V^\dagger j(0) V | \Phi_{A_i}(0) \rangle \cdot \epsilon$$

(86)

The model states of the $P_j, V_i$ mesons are written in the usual way, that is the simplest one compatible with the good quantum numbers. They are (ignoring the color factor):

$$|\Phi_{A_i}(0)\rangle = |\chi(A_i)\varphi(r)\rangle = \sum_{\mathbf{p}} \sum_{\rho_1, \rho_2} \varphi(p) \chi_{\rho_1, \rho_2}(A_i) a_{\mathbf{p}, \rho_1}^\dagger b_{-\mathbf{p}, \rho_2}^\dagger |0\rangle$$

$$|\Phi_{B_j}(\mathbf{P})\rangle = |\chi(B_j)\varphi(r) \exp(i\mathbf{P} \cdot \mathbf{R})\rangle = \sum_{\mathbf{p}} \sum_{\rho_1, \rho_2} \varphi(p) \chi_{\rho_1, \rho_2}(B_j) a_{\mathbf{p}+(\mathbf{P}/2), \rho_1}^\dagger b_{-\mathbf{p}+\mathbf{P}/2, \rho_2}^\dagger |0\rangle$$

(87)

$|0\rangle$ in the above formulas (87) is the bare vacuum of quarks, antiquarks and gluons; $a_{\mathbf{p}, \rho}^\dagger$ and $b_{-\mathbf{p}, \rho}^\dagger$ are creation operators of a quark and, respectively, an antiquark (the question of the renormalization of their masses has been discussed previously); $p$ is their momentum and the index $\rho$ characterizes the spin-flavor status; $\mathbf{P}$ is the momentum of the Ps meson. For a transition $A_i \rightarrow B_j + \gamma$ it is (writing $P = |\mathbf{P}|$):

$$P \equiv P_{ij} = (M_i^2 - M_j^2)/(2M_j)$$

(88)

In the Eqs.(87) $\rho_1$ and $\rho_2$ are indices referring to the spin-flavor state of the quark (1) and antiquark (2) (1 will always refer to the quark $q$ and 2 to the antiquark $\bar{q}$); $\chi_{\rho_1, \rho_2}$ are the spin-flavor functions (of course with spin 1 for the Vector mesons and 0 for the $P_s$ mesons; in spite of the fact that the GP is fully relativistic, the model functions are constructed with Pauli spinors, as already discussed). We recall (Sect.2) that the unitary transformation $V$ [here this $V$ is not the symbol of the vector meson!] operating on the model states, has the property of leading from Pauli to Dirac spinors; $\varphi(r)$ and its Fourier transform $\varphi(p)$, equal for all the states (that is independent from the indices $i$ and $j$), is the (rotation invariant ($L = 0$)) space or momentum part of the model wave function. For more details compare the Ref.[20], Sect.III.14.

The most general vertex for a $V \rightarrow P \gamma$ decay is:

$$G_{ijj} \partial_\alpha A_\beta \partial_\mu V_\nu P \epsilon_{\alpha\beta\mu\nu}$$

(89)

where $A_\alpha$, $V_\nu$, and $P$ are the electromagnetic, vector and pseudoscalar fields, $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita symbol and $G_{ijj}$ is a real constant with the dimensions of a magnetic moment depending on the $i, j$ pair; $G_{ijj} = G_{ij}(p_1^2, p_2^2, p_3^2)$ is a Lorentz invariant that can depend only on invariants constructed with the four momenta $p_1, p_2, p_3$ of the three external “legs” of the $V \leftrightarrow P \gamma$ diagram; because it is $p_1 + p_2 = p_3$ only two such invariants exist, the masses of the $V$ and $P$ mesons, so that:

$$G_{ijj} \equiv G_{ij}(M_i^2, M_j^2)$$

(90)

\[\text{In Ref.[20] the notation (N, P, \lambda) and the expression “constituent quarks” were used for objects that might have been called simply “quarks” (as we do here, after the analysis of [26]). We remarked this already; we note it again here for the readers of Ref.[20].}\]
Once one has \( G_{ij} \) expressed as the parametrized expression for the decay \( i \to j \) under consideration, the rate of the decay \( V_i \to P_j + \gamma \) is given by:

\[
\Gamma(A_i \to B_j \gamma) = G^2_{ij} k^3/(12\pi) \quad (V \to P\gamma)
\]

(91)

If \( P \) (index \( j \)) is heavier than \( A \) (index \( i \)) the same formula holds (except for a factor 3) (See the remarks on this in the Appendix of Ref.[20], after Eq.A5):

\[
\Gamma(B_j \to A_i \gamma) = G^2_{ij} k^3/(4\pi) \quad (P \to V\gamma)
\]

(92)

Now we should “summarize” the contents of Ref.[20]. As a matter of fact this would be too long here and we can just give the final result for the parametrization of \( G_{ij} \) appearing in Eq.(92):

\[
G_{ij}(M_i^2, M_j^2) = 2\langle M_V/M_i \rangle^{1/2} \sum_{\nu=1}^7 \mu_{\nu} f_{\nu}(P) \Gamma_{\nu}(B_j A_i)
\]

(93)

On the r.h.s. the notation is as follows:

1) \( M_V \) is an average mass of the Vector mesons (irrelevant if we consider the ratio between the decay rates of two \( V \) mesons of the same family),

2) \( M_i \) is the mass of the decaying \( V \) meson,

3) The \( \mu_{\nu} \)'s have the dimensions of a magnetic moment of the decaying \( V \) meson; except for \( \mu_{\nu_1} \) all the \( \mu_{\nu} \)'s multiply \( \Gamma_{\nu} \)'s that -as we will see- are either reduced by flavor breaking, or reduced by gluon exchange. This will play a role when we will consider the ratio between two different decay modes (see below);

4) The seven \( \Gamma_{\nu}(B_j A_i) \) will be listed and discussed in a moment (compare the Eqs.(59,60) of Ref.[20], where the notation \( \Pi^P \) etc. was used for what we now call \( P^\mu \) etc.). The values of the \( \Gamma_{\nu} \)'s depend (as the notation indicates) on the transition considered (recall that 1 always refers to the quark and 2 to the antiquark; note also that with the definition \( Q = (2/3)P^u - (1/3)P^d - (1/3)P^s \) the charge of an antiquark is \(-Q\)). These values of the \( \Gamma_{\nu} \)'s appear in the Table at the end of this Section for each transition \( V \to P\gamma \) or \( P \to V\gamma \).

\[
\begin{align*}
\Gamma_1 &= (Q_1 + Q_2) \\
\Gamma_2 &= (Q_1 + Q_2)(P^u_1 + P^s_2) \\
\Gamma_3 &= (Q_1 - Q_2)(P^u_1 - P^s_2) \equiv QS \\
\Gamma_4 &= (Q_1 + Q_2)P^u_1 \cdot P^s_2 \\
\Gamma_5 &= [(Q_1 + Q_2)|z'\rangle\langle z'| + |z'\rangle z'|(Q_1 + Q_2)] \\
\Gamma_6 &= [z'\rangle\langle w|(Q_1 + Q_2) + (Q_1 + Q_2)|w\rangle\langle z'|] \\
\Gamma_7 &= [w\rangle\langle z'|(|Q_1 + Q_2) + (Q_1 + Q_2)|z'\rangle\langle w|] \\
\end{align*}
\]

(94)

In the last form of \( \Gamma_3 \), \( Q \) and \( S \) stay for the total charge and strangeness. The symbols \(|z'\rangle\) and \(|w\rangle\) stay respectively for:

\[
|z'\rangle = (1/\sqrt{3})|u_1\bar{u}_2 + d_1\bar{d}_2 + s_1\bar{s}_2\rangle, \quad |w\rangle = |s_1\bar{s}_2\rangle
\]

(95)

and represent flavor structures corresponding to transitions taking place through an intermediate gluon (indicated in what follows by \( g \))\textsuperscript{15} that, for instance, can occur in

\textsuperscript{15}The accent on \( z' \) is to remind that \( \langle z'|z'\rangle = 1 \), while the \(|z\rangle\) in Sect.8 was normalized to 3.
processes like:

\[
\phi \rightarrow 3g \rightarrow \pi^0 + \gamma, \quad \phi \rightarrow 2g + \gamma \rightarrow \eta + \gamma, \quad \omega \rightarrow 3g \rightarrow \pi^0 + \gamma, \quad \omega \rightarrow 2g + \gamma \rightarrow \eta + \gamma
\]  

It should be remarked that the \( \Gamma \)'s in Eq.(94) are all the flavor expressions contained in the exact transition operator \( V^\dagger j(0)V \); \( \Gamma_5, \Gamma_6, \Gamma_7 \) (the gluon exchange terms) add to the flavor terms \( (Q_i; Q_i P_k; \cdots) \) contained in \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \). Forgetting gluon exchange, these would be the only ones present.

Here we will limit the discussion to some processes that can depend in a clear way from the contribution of gluon exchange -compare for this the Table I.

Let us start with the ratio \((\omega^0 \rightarrow \pi\gamma)/(\rho^0 \rightarrow \pi\gamma)\) discussed at the beginning of this Section as one of the first applications of the NRQM. One has from the Eqs.(94) that only \( \Gamma_1 \) and \( \Gamma_5 \) intervene in \( \omega \rightarrow \pi\gamma \) and only \( \Gamma_1 \) in \( \rho \rightarrow \pi\gamma \):

\[
\frac{\Gamma(\omega^0 \rightarrow \pi\gamma)}{\Gamma(\rho^0 \rightarrow \pi\gamma)} = \left( 3 + \frac{2\mu_5 f_5(P)}{\mu_1 f_1(P)} \right)^2 \times 1.06
\]  

In the formula (97) above, the factor 1.06 is due to the different momenta in the two cases (it comes from the ratio of the third powers of the momenta); when the data will improve, this factor should be recalculated because of the large width of the \( \rho \).

Any -at present non appreciable- deviation from 9.5 of the r.h.s.of (97) (of course after the errors are duly taken into account) is a measure of the contribution to \( \omega \rightarrow \pi\gamma \) via gluons represented by the term \( \mu_5 f_5(P) \). Note that Eq.(97) is an exact consequence of any relativistic field theory that satisfies the assumptions stated in Sect.1. (In particular it is correct to all orders in flavor breaking). Of course QCD belongs to such theories and, in a sense, the Eq.(97) provides a confirmation of the smallness of these 3-gluons contributions in QCD.

A similar example comes from the ratio of the \( \eta'\rho\gamma \) and \( \eta'\omega\gamma \) decays; these depend only on \( \Gamma_1 \) and on \( \Gamma_5, \Gamma_6, \Gamma_7 \), all implying the intervention of 3 gluons, the last two processes with first order flavor breaking. Also here the experimental error is large, but the indications are for a negligible contribution from the gluonic diagrams. Finally a strong confirmation of the smallness of the gluon diagrams contribution comes from smallness of the \( \phi \rightarrow \pi^0\gamma \) decay. The order of magnitude of this decay can be reproduced by the small deviation \( (\approx 1^0) \) of the Vector meson mixing angle \( \theta_V \) in the \( \phi \) meson from its ideal value \( (\theta_V = 35.3^0) \). The theoretical uncertainty, particularly from the form factor, is significant; still, because a sensible estimate of the order of magnitude of the rate \( \phi \rightarrow \pi^0\gamma \) is obtained without invoking the gluon annihilation diagrams appearing in the ninth row of table I, there is no evidence, inside the errors, for the relevance of such diagrams. Even assuming that the deviation of \( \theta_V \) from its ideal value is \( 2^0 \), the gluon contributions, if present, cannot certainly be larger (indeed should be smaller) than the tiny value calculated with this \( 2^0 \) deviation.

**TABLE I.** The values of \( \Gamma_\nu(B_i; A_j) \) in Eq.(94). The abbreviations are indicated at the bottom of the table; the flavor wave function of each meson is assumed to be normalized to one; \( \theta_V \) is taken to have its ideal value, except in the calculation of \( \Gamma_1 \) for \( \phi \rightarrow \pi^0\gamma \).
Consider now the Coleman-Glashow (CG) equation. As mentioned, in their derivation Coleman and Glashow neglected entirely the flavor breaking of the strong interactions. But it was shown in Ref. [44] that the CG formula holds also taking into account

\[ \rho \pi \gamma, \omega \pi \gamma, \rho \eta \gamma, \omega \eta \gamma, \rho \pi' \gamma, \omega \pi' \gamma, \rho \pi^0 \gamma, \sin(\theta_V' - \theta_V), K^{*0} K^0 \gamma, K^{*+} K^{+} \gamma \]

Before concluding this section we note that an impressive amount of work, both experimental and theoretical, has been done in the last decade by the groups of Achasov et al. and Benayoun et al. and by others; unfortunately it is impossible to discuss in detail all this; even limiting to the \( \gamma \) decays of Vector mesons, we had to be necessarily most synthetic in the presentation of our work [20]). However we must cite some papers to refer at least to part of the work mentioned above. For the contributions of Benayoun et al. we refer to: [55, 56, 57, 59]. For Achasov et al. [60, 61]. Compare also [62] and other contributions by the same group.

### 10. The baryons electromagnetic mass differences and the Coleman-Glashow equation.

Another test of the hierarchy [42] is provided by the Coleman-Glashow formula [41] for the baryon octet mass differences. (Below we indicate again the masses with the baryon symbols). The Coleman-Glashow formula is:

\[ p - n = \Sigma^+ - \Sigma^- + \Xi^0 - \Xi^0 \]  

(98)

The present data (after a comparatively recent measurement of the \( \Xi^0 \) mass [43]) give:

\[ l.h.s. = -1.29 \, MeV \]

(99)

\[ r.h.s. = -1.58 \pm 0.25 \, MeV \]

Because the mass difference \( (\Sigma^- - \Sigma^+) \) in (98) is \( \approx 8 \, MeV \), the agreement is amazing [before the measurement in Ref. [43] it was already good (1.29 to be compared with 1.67 \pm 0.6)]. Note that the Coleman-Glashow formula was derived (Ref. [41]) assuming unbroken flavor; but the \( SU(3) \) violation in the baryon octet masses is significant \( (M_\Xi - M_\rho)/(M_\Xi + M_\rho) \approx 17\% \). Here we will examine how, due to the hierarchy, this violation has a rather small effect.

Note again that it was once more the hierarchy, to produce the improved Gell Mann-Okubo mass formula [8] of Sect.1 (Eq.1), with an excellent agreement with the data.

In this Section we shall consider also some relations due to Gal and Scheck (though there the experimental errors are much larger).

Consider now the Coleman-Glashow (CG) equation. As mentioned, in their derivation Coleman and Glashow neglected entirely the flavor breaking of the strong interactions. But it was shown in Ref. [44] that the CG formula holds also taking into account
all the flavor breaking terms, except those with three quark indices (negligible because of the hierarchy).

Here we summarize the derivation of the CG equation taking into account flavor breaking, without performing all the calculations, (see [42]).

It must be underlined that the \( m_u - m_d \) terms and the so called Trace terms were discussed in Sect.4 of Ref.[42]; they do not affect the CG equation. (Incidentally we remark that the \( m_u - m_d \) terms for the generalized Gell Mann-Okubo mass formula (Eq.(1)) were duly taken into account in Ref.[8]).

To derive the CG equation using the GP, call \( \Omega \) the exact QCD operator -to 2nd order in the charge-expressed in terms of the quark fields; \( \Omega \) will represent the e.m. contribution of interest to the baryon mass, that is the e.m. current-current interaction of the quarks in a baryon (proportional to their charge-charge interactions); with respect to the original Coleman-Glashow paper, the new thing is that we take into account the flavor breaking contributions.

Call, as usual, \( |\Psi_B\rangle \) and \( |\Phi_B\rangle \) the exact and model states of the baryon \( B \), writing, once more, in the usual notation: \( |\Phi_B\rangle = |X_{L=0},W_B,S_c\rangle \) (and omitting from now on the color factor \( S_c \)). In Ref.[44] (Eqs.13-16) the e.m. interactions were written explicitly and discussed.

They lead to the terms listed below (because the \( \Omega^- \) has no role in the CG equation, the last line is useless here, but we included it for completeness):

\[
\begin{align*}
Q_i^2, & \quad \tilde{Q}_i Q_k \quad \text{(no flavor breaking)} \\
Q_i^2 P_i^s, & \quad Q_i^2 P_k^s, \quad Q_i Q_k P_i^s, \quad Q_i Q_k P_j^s, \quad (1\text{st order flavor breaking}) \\
Q_i^2 P_i^s P_k^s, & \quad Q_i^2 P_k^s P_j^s, \quad Q_i Q_k P_i^s P_j^s, \quad (2\text{nd order flavor breaking}) \\
Q_i^2 P_i^s P_j^s P_k^s, & \quad Q_i Q_k P_i^s P_j^s P_k^s, \quad (3\text{rd order flavor breaking})
\end{align*}
\]

As to the spin dependence, we proved in [1] that only the following scalars exist:

\[
|\sigma_i \cdot \sigma_k\rangle \quad (i, k = 1, 2, 3) \quad (100)
\]

We recall that the scalar \((\sigma_1 \times \sigma_2) \cdot \sigma_3\) has vanishing expectation value on a spin flavor state having a real wave function \(W_B(1,2,3)\). [The \(W_B\)'s are the usual spin-unitary spin functions of the baryons \( B \)].

After this list of charge, flavor and spin functions, we display the results of the \( V \) transformation:

\[
\langle \Psi_B | \Omega | \Psi_B \rangle = \langle \Phi_B | V^\dagger \Omega V | \Phi_B \rangle \equiv \langle W_B | \widetilde{\Omega} | W_B \rangle \quad (101)
\]

where :

\[
\widetilde{\Omega} = \sum_{\nu} t_{\nu} \Gamma_{\nu}(s,f) \quad \text{with:} \quad t_{\nu} \equiv \langle X_{L=0} | G_{\nu}(r) | X_{L=0} \rangle \quad (102)
\]

The \( \Gamma_{\nu} \)'s depend only on the spin and flavor variables of the quarks in \( \Phi_B \); the \( t_{\nu} \)'s are a set of parameters coming from the integration of the space factors \( G_{\nu}(r) \) of \( \widetilde{\Omega} = \sum_{\nu} G_{\nu}(r) \Gamma_{\nu}(s,f) \) on the space part \( X_{L=0} \) of the baryon model factor. The Eqs.(101),(102) reproduce the usual procedure of the GP. As mentioned, the \( m_u - m_d \) and Trace terms do not affect the CG relation; thus below we transcribe simply (in the Eqs.(103)) the \( \Gamma_{\nu} \) at zero order (call them \( \delta_0 B \)) and at first order (\( \delta_1 B \)) in flavor breaking; Note that the three quark terms with no flavor breaking are already included -that is, they are taken into account- in the original CG (no flavor breaking) formula.

In this case (no flavor breaking) define:

\[
\begin{align*}
\Gamma_1 &= \sum [Q_i^2], \quad \Gamma_2 = \sum [Q_i^2 (\sigma_i \cdot \sigma_k)], \quad \Gamma_3 = \sum [Q_i^2 (\sigma_k \cdot \sigma_j)], \quad \Gamma_4 = \sum [Q_i Q_k], \\
\Gamma_5 &= \sum [Q_i Q_k (\sigma_i \cdot \sigma_k)], \quad \Gamma_6 = \sum [Q_i Q_k (\sigma_i + \sigma_k) \cdot \sigma_j] \quad (103)
\end{align*}
\]

34
In Eq.(103) the sum symbols referring to terms with 1, 2, 3 indices are defined respectively as:

\[
\sum [i] = \sum_{i=1}^{3}, \quad \sum [i, k] = (1/2) \sum_{i,k=1 \atop i \neq k}^{3}, \quad \sum [i, k, j] = (1/6) \sum_{i,k,j=1 \atop i \neq k \neq j}^{3}
\]

We display also the F’s at first order in flavor breaking. (The Ref.([44] contains the list of \(\Gamma\)’s at 2nd order flavor breaking). Of course the 1st order \(\Gamma\)’s are present only in \(\Lambda, \Sigma, \Sigma^*, \Xi, \Xi^*, \Omega\).

\[
\begin{align*}
\Gamma_7 &= \sum \{Q_i^2 P_i^s\}; & \Gamma_8 &= \sum \{Q_i^2 P_i^s (\sigma_i \cdot \sigma_k)\}; & \Gamma_9 &= \sum \{Q_i^2 P_i^s (\sigma_j \cdot \sigma_k)\}; \\
\Gamma_{10} &= \sum \{Q_i^2 P_i^s\}; & \Gamma_{11} &= \sum \{Q_i^2 P_i^s (\sigma_i \cdot \sigma_k)\}; & \Gamma_{12} &= \sum \{Q_i^2 P_i^s (\sigma_i + \sigma_k) \cdot \sigma_i\} \\
\Gamma_{13} &= \sum \{Q_i Q_k P_i^s\}; & \Gamma_{14} &= \sum \{Q_i Q_k P_i^s (\sigma_i \cdot \sigma_k)\}; & \Gamma_{15} &= \sum \{Q_i Q_k P_i^s (\sigma_i + \sigma_k) \cdot \sigma_i\} \\
\Gamma_{16} &= \sum \{Q_i Q_k P_j^s\}; & \Gamma_{17} &= \sum \{Q_i Q_k P_j^s (\sigma_i \cdot \sigma_k)\}; & \Gamma_{18} &= \sum \{Q_i Q_k P_j^s (\sigma_i + \sigma_k) \cdot \sigma_j\}
\end{align*}
\]

In the above list of 1st order flavor breaking terms, the three quark terms \(\Gamma_9, \Gamma_{12}, \Gamma_{15}\) do not contribute to the left and right hand side of the CG formula; the terms \(\Gamma_{16}, \Gamma_{17}, \Gamma_{18}\), do no contribute to \(n\) and \(p\) while the correction for \(\Sigma^+, \Sigma^-, \Xi^-, \Xi^0\) are in each case of the order “something”/3 where the magnitude of “something” is estimated by the hierarchy as 1/9: this is due, in all cases, to the product of a reduction factor \(\approx 1/3\) -presence of a \(P^s\) - and a factor \(\approx 1/3\), due to one more gluon exchange. Thus for each of the above three terms \(\Gamma_{16}, \Gamma_{17}, \Gamma_{18}\) we have a reduction of the order \((1/3)^3 \approx 4.10^{-2}\) with respect to the dominant no-flavor breaking contribution. (Note: The \(\Gamma_{18}\) here is 1/2 that listed in Ref.[44]; this was wrong by a factor 2 -but produced no error because, in Ref.[44] it was not used).

Because experimentally \(\Sigma^- - \Sigma^+ \simeq 8 MeV\) and \(\Xi^- - \Xi^0 \simeq 6.4 MeV\) the reduction given above \((1/3)^3 \approx 4.10^{-2}\) implies an expected difference between the left and right hand sides of the C.G. formula (due to 1st-order flavor breaking terms), of \(\approx 0.2 \pm 0.3 MeV\) that does not disagree with the data.

The \(\Gamma\)’s necessary to calculate the 2nd order flavor breaking correction are given in [44], Eq.(19). The order of magnitude of the terms of interest are estimated using the hierarchy.

Though we do not list here the 2nd order flavor breaking terms, we mention that the estimate of their contribution to the difference between the left and right hand sides of the CG formula is \(\approx 0.02 - 0.1 MeV\).

We finally note the Gal-Scheck relations, derived long ago using the NRQM; again the hierarchy plays an important role (for more details see the Ref.[44]).

The Gal and Scheck relations considered here are those for the baryon masses; they were derived from the NRQM assuming that 3-quark terms were negligible. Three of these relations deal with the masses of wide resonances (the \(\Delta\)’s); thus they are not easily verifiable. Two, the only ones to be displayed below, imply resonances not so wide (\(\Sigma^{*\pm}\) and \(\Xi^{*-0}\)) and can be checked more easily. They are:

\[
\begin{align*}
(\Sigma^{*+} - \Sigma^-) + (\Xi^{*+} - \Xi^0) &= p - n & (-1.2 \pm 0.9 = -1, 29) \\
(1/2)(\Sigma^{*+} + \Sigma^-) - \Sigma^0 &= (1/2)(\Sigma^+ + \Sigma^-) - \Sigma^0 & (1.3 \pm 1.2 = 0.85 \pm 0.12)
\end{align*}
\]

The above relations hold to all orders in flavor breaking.

Finally (independently of the Gal Scheck relations), to stimulate more precise measurements, we write a relation between the baryon electromagnetic masses [42], which
is the analogous for the decuplet of the Coleman Glashow equation for the octet. It can be easily verified using the Eqs. (27)(28) of Ref. [44]; it holds to all orders in flavor breaking:

\[ \delta \Delta^+ - \delta \Delta^0 = \delta \Sigma^{*+} - \delta \Sigma^{*-} + \delta \Xi^{*+} - \delta \Xi^{*-} \]  

(106)

The above Eq. (106) (as well as the one below) might have been written -more simply- suppressing all the \( \delta \) symbols, since they are independent of the \( u,d \) quark mass differences. We kept however the notation used in Ref. [42].

The Eq. (106), plus the Equation: \( \delta \Delta^{++} - \delta \Delta^- = 3(\delta \Delta^+ - \delta \Delta^0) \) (also true to all orders in flavor breaking) might be useful for determining the mass differences between the \( \Delta \)'s.

11. Two relations: a) Between the charge radii of \( p, n, \Delta^+ \); b) Between the radii of \( \pi^+, K^+, K^0 \).

1. On the charge radii of \( p, n, \Delta^+ \).

Buchmann, Hernandez and Faessler [49] derived, using an elaborate quark model including two body gluon and pion exchange, the following relation between the electric charge radii of proton, neutron and \( \Delta \):

\[ r^2(p) - r^2(n) = r^2(\Delta^+) \]  

(107)

We will show that the relation (107) can be reproduced using the GP if one neglects terms with three indices and a closed loop contribution (a Trace term); these are indeed absent in the model of Buchmann et al. and are expected to be small in the GP due to the hierarchy.

The quantities \( r^2(p), r^2(n), r^2(\Delta^+) \) calculated in the rest frame of the baryon considered are scalars under space rotations. Indeed the charge square radius of the baryon \( B \) is defined as:

\[ r^2(B) = \langle W_B | [\text{parametrized } r^2] | W_B \rangle \]  

(108)

where the \( W_B \) are the standard spin-flavour functions, used previously (Sects. 2, 3).

The most general parametrized \( r^2 \) for \( p, n, \Delta^+ \) is linear [45] in the quark charges \( Q \), and it can only contain rotation invariant spin expressions of the type \( (\sigma_i \cdot \sigma_j) \). As to the projection operator \( P^s \) this can only be contained, for \( p, n, \Delta^+ \), in the term \( Tr[QP^s] \) arising from closed internal loops. Thus the most general form of [parametrized \( r^2 \)] is:

\[ [\text{parametrized } r^2] = A \sum_i Q_i + B \sum_{i \neq k} Q_i(\sigma_i \cdot \sigma_k) + C \sum_{i \neq j \neq k} Q_i(\sigma_j \cdot \sigma_k) + D Tr[QP^s] \]  

(109)

where \( A, B, C, D \) are four parameters. (As noted repeatedly, the scalar \( (\sigma_i \times \sigma_j \cdot \sigma_k) \) cannot be present in Eq. (109)). Note that in principle there should be two different \( D \)'s in Eq. (108) multiplying \( Tr[QP^s] \) for \( n, p \) and for \( \Delta \); but because the coefficient \( D \) is negligible in all cases, we wrote [45] the Eq. (109) introducing just one \( D \) (there is of course no difficulty in writing the correct expression).

With a few steps one finally obtains, calculating the expectation values of the appropriate expressions on \( p, n, \Delta \):

\[ r^2(p) = A - 3C - D/3; \quad r^2(n) = -2B + 2C - D/3; \quad r^2(\Delta^+) = A + 2B + C - D/3; \]  

(110)
Note the following: If we only take into account in Eq.(110) the A (additive) and B (two index) terms we get:

\[ r^2(p) = A; \quad r^2(n) = -2B; \quad r^2(\Delta^+) = A + 2B; \quad (111) \]

This means that if the NRQM result \( \left( r^2(p) = A; \quad r^2(n) = 0; \quad r^2(\Delta^+) = A \right) \) is corrected by the two index terms we get the Buchmann et al. result \( r^2(p) - r^2(n) = r^2(\Delta^+) \).

If now we keep also the 3-index terms having C as coefficients as well as the D terms \( r^2(p) - r^2(n) = r^2(\Delta^+) \) is replaced by:

\[ r^2(p) - r^2(n) = r^2(\Delta^+) - 6C + D/3 \quad (112) \]

The coefficient D characterizes the contributions of internal quark closed loops with the probe photon line (the photon exchanged in order to measure the radii) ending on the loop. The reduction factor is around 30 (see the end of Sect.7) so that the D term is negligible.

As to the value of C, we have \( |C/B| \approx 1/3 \). Because, from the experimental knowledge of the radii of \( n, p \) we get, \( |B/A| \approx 0.08 \), and, from the hierarchy we expect \( 6|C/A| \approx 0.16 \) we obtain:

\[ r^2(p) - r^2(n) \cong r^2(\Delta^+) (1 \pm 0.16) \quad (113) \]

We conclude that the main result found in the model of [49] discussed above is correct in QCD except for terms possibly of order 10% to 20%.

As to the derivation from the same model of the quadrupole moment of \( \Delta \) and of the \( \Delta \to p\gamma \) quadrupole transition, they do not appear to follow from QCD, as far as we can see. (see the remarks in Sect.1 and 4 of Ref.[45]).

Following the results described above, Buchmann (with Henley, in some papers) used the GP to treat several problems - mainly on the e.m. properties of hadrons. A partial list of contributions is: Refs.[51, 50, 53, 54]; we do not share however the point of view of Buchmann on the quasi-equivalence between the GP and the large \( N_c \) method underlying some of these papers (compare Sect.13).

2.-On the charge radii of \( \pi^+, K^+, K^0 \).

For \( \pi^+ \), \( K^+ \) \( K^0 \) the calculation [63] is even simpler than the above one because we are dealing with zero-spin mesons. As to the experimental values of the radii listed above their errors are still comparatively large, except (recently) for the \( K^0 \).

The square radius \( r^2(M) \) of a meson \( M \) with e.m. form factor \( F(q^2) \) is:

\[ r^2(M) = -6\frac{dF(q^2)}{dq^2} \bigg|_{q^2=0}; \quad F(q^2) = \langle M(q/2)|\rho(0)|M(-q/2) \rangle \quad (114) \]

Here \( |M(p)\rangle \) is the exact eigenstate of the QCD Hamiltonian for the meson \( M \) with total momentum \( p \), \( \rho(0) = i\overline{\psi}(0)Q\gamma_4\psi(0) \), with \( \psi(x) \) the quark field and \( Q = (1/2)(\lambda_3 + (1/3)\lambda_8) \) is the charge operator. We recall, incidentally, that the \( q^2 \) dependence of the e.m. form factors was given for \( p, n \) in Ref.[46]. The exact \( r^2(M) \) derived from QCD for a meson with an \( L = 0 \) auxiliary state (as the lowest pseudoscalar mesons) is, in the GP:

\[ r^2(M) = \langle W_M|^{parametrized \ r^2}|W_M \rangle \quad (115) \]

where \( W_M \) are the standard spin-flavor functions of the \( \pi \) or \( K \) mesons. Due to the linearity of \( r^2 \) in \( \rho(0) \) the most general “parametrized \( r^2 \)” for \( \pi^+, K^+, K^0 \) is a scalar.
linear in the quark charges $Q_i (i=1,2)$ ($1=\text{quark}, 2=\text{antiquark}$). As to the spins, the “parametrized $r^{2n}$” can contain only $\sigma_1 \cdot \sigma_2$ which, applied to the spin singlet factor in $W_M$ is just $-3$. Therefore the most general “parametrized $r^{2n}$” (a scalar under rotations) is:

$$\text{"parametrized } r^{2n} = A \sum_i Q_i + B \sum_i Q_i P_i^s + C \sum_{i \neq k} Q_i P_k^s + D \text{Tr}[QP^s]$$ \hspace{1cm} (116)$$

In Eq.(116) $A, B, C, D$ are four real parameters; the sums (on $i, k = 1, 2$) are exemplified by the following case for $K^+$: $\sum_{i \neq k} Q_i P_k^s =$ (for $K^+$) $\langle W_{K^+} | Q_1 P_2^s + Q_2 P_1^s | W_{K^+} \rangle = 2/3$. Neglecting the Trace term in Eq.(116) for the reasons already stated (depressed by the Furry theorem at least 30 times with respect to the dominant term $A$), the equation (116) contains three parameters $A, B, C$ and must fit three quantities. $|B/A|$ = the flavor reduction factor factor (from 0.22 to 0.37) for which we also take 1

$$|C| \approx (1/3)|B| \cong (1/9)|A|$$ \hspace{1cm} (117)$$

We thus obtain:

$$r^2(\pi^+) = A \hspace{1cm} (= 0.44 \pm 0.02)$$

$$r^2(K^+) = A + (1/3)B + (2/3)C \hspace{1cm} (= 0.34 \pm 0.05)$$

$$r^2(K^0) = (1/3)B - (1/3)C$$

In general the Eqs.(118) lead to the relation:

$$r^2(\pi^+) - r^2(K^+) = -r^2(K^0) - C \hspace{1cm} (0.10 \pm 0.05 \pm 0.07 \pm 0.007 \pm 0.05)$$ \hspace{1cm} (119)$$

Experimentally $r^2(\pi^+) - r^2(K^+) = 0.10 \pm 0.05$; on the left hand side $r^2(K^0) = -0.077 \pm 0.007$ and $C$ (calculated from the hierarchy, Eq.(117)) is $\pm 0.05$; more precise measurements for $r^2(\pi^+), r^2(K^+)$ would be useful and one would like to have a confirmation of the very precise value of $r^2(K^0)$. The interest in obtaining precise values is due also to the following circumstance. From the Eqs.(118) one has:

$$|B/A| = |r^2(K^+) - r^2(\pi^+) + 2r^2(K^0)/r^2(\pi^+)| \cong | -0.10 \pm 0.05 - 0.154 \pm 0.014 | / (0.44 \pm 0.01)$$ \hspace{1cm} (120)$$

It is important to note that the hierarchy leads to $|B/A| \cong 1/3$; this prediction should be rather solid because based only on flavor breaking (it is also independent of $C$). However with the values that appear in Eq.(120) this expectation is fulfilled only using the $\pm 0.05$ error at two standard deviations or/and if the error in $r^2(K^0)$ has been underestimated in Ref.[52].

12. Parametrization of the $\rho \gamma$, $\omega \gamma$ and $\phi \gamma$ couplings: Why $f_{\rho \gamma}:f_{\omega \gamma}$ differs from 3:1 only by $\sim 10\%$ in spite of flavor breaking?

This Section is based on Ref.[15]. Some numbers appearing in the original treatment are modified but the conclusion for the $\rho \gamma$, $\omega \gamma$ couplings summarized by the title above,
is practically the same as that of [15]. [However the conclusion of [15] on a negative \( \phi - \omega \) mixing angle was incorrect: the angle is positive, \( \approx +1^\circ \).]

To determine the \( \rho \gamma, \omega \gamma \) and \( \phi \gamma \) couplings, we analyze below, using the GP, the \( V - \gamma \) couplings \( f_{V \gamma} \) in the decays \( \rho^0,\omega,\phi \rightarrow e^+e^- \). The above \( V - \gamma \) couplings are, in fact, presented sometimes as an illustration of how well the NRQM + \( SU_3 \) predicts the ratios of the \( f_{V \gamma} \) for \( V = \rho^0,\omega \) and, in part \( \phi \). But \( |f_{\rho \gamma}/f_{\omega \gamma}| \approx 3 \) needs an explanation, because the value 3 is the perfect \( SU_3 \) prediction with no flavor breaking, whereas flavor is broken. Yet we will see, using the GP, that the statement is almost correct (see the title of this section) although this is not obvious. (From now on in this section the \( \phi = 3 \) needs an explanation).

Yet we will see, using the GP, that the statement is almost correct (see the title of this section) although this is not obvious. (From now on in this section the \( \phi = 3 \) needs an explanation).

The coupling \( f_{v\gamma} \) of a neutral vector meson \( v \) to the photon is proportional to the matrix element \( \langle v | j^\alpha(x) | 0 \rangle \), where \( j^\alpha(x) \) is the quark current:

\[
j^\alpha(x) = \frac{ie}{2} \bar{\psi}(x)(\lambda_3 + \frac{1}{3}\lambda_8)\gamma^\alpha\psi(x)
\]  
(121)

in the usual notation, with the color index omitted and \( \psi(x) = u(x), d(x), s(x) \).

Introducing the charge \( Q = \frac{2}{3}P^u - \frac{1}{3}P^d - \frac{1}{3}P^s \) the Eq.(121) becomes

\[
j^\alpha(x) = \frac{ie}{2} \bar{\psi}(x)Q\gamma^\alpha\psi(x)
\]  
(122)

The decays of \( \rho,\omega,\phi \) into \( e^+e^- \) are expressed in terms of the quantities \( f_{v\gamma} \) defined in the rest system of the decaying \( v \) meson, by

\[
f_{v\gamma}\delta_{ik} = \langle v^i | j^k(0) | 0 \rangle, \quad i, k = 1, 2, 3
\]  
(123)

where \( |v^i\rangle \) is the exact state of the QCD Hamiltonian representing the vector meson \( v \) with polarization \( i \) and \( j^k \) is the \( k \)-th component of the quark current (122); similarly \( |0\rangle \) is the exact vacuum of QCD, that is the exact vacuum of quarks, antiquarks and gluons. Of course the General Parametrization connects the exact states to the model states by the unitary transformation \( V \), introduced in Sect.2; the structure of \( V \) depends on the problem considered, and we have to construct \( V \) for our problem here. First we must clarify the meaning of “exact state” introduced above. The states \( |v\rangle \) and \( |0\rangle \) on the r.h.s. of Eq.(123), when written in terms of Fock states in the space of quarks \( q \), antiquarks \( \bar{q} \) and gluons \( G \), are a super-position of an infinite number of Fock states (compare Sect.2). Schematically,

\[
|v\rangle \equiv |v\rangle_{\text{exact}} = |q\bar{q}\rangle + |q\bar{q}q\bar{q}\rangle + |q\bar{q},G\rangle + ... \\
|0\rangle \equiv |0\rangle_{\text{exact}} = |0_{\text{bare}}\rangle + |q\bar{q},G\rangle + ... 
\]  
(124)

where \( |0_{\text{bare}}\rangle \) [perhaps the name \( |0_{\text{model}}\rangle \) instead of \( |0_{\text{bare}}\rangle \) would have been better, but we preferred not to change on this the notation adopted in Ref.(15)], is a zero energy state of the model Hamiltonian with no quarks, antiquarks or gluons. As we exemplified abundantly, the GP starts with the introduction of a model Hamiltonian and of a unitary transformation \( V \) that transforms the simple model states of the model Hamiltonian into the complicated exact states of the exact QCD Hamiltonian. A large freedom exists in the selection of the model Hamiltonian in a given problem, as seen
for baryons and mesons in the Refs.\cite{1, 17, 26} and in the past sections of this survey. In the present case the model Hamiltonian is chosen as a function of $1q - 1\overline{q}$; one of its eigenstates is a state of zero energy without quarks, antiquarks and gluons that in the Eqs.\cite{124} above we called $|0_{bare}\rangle$ and in the following we will call $|0_b\rangle$.

The model states of mesons for the lowest nonet are pure $|q\overline{q}\rangle$ states with $L = 0$, all degenerate in mass. The wave function $F(v^i)$ of the $|q\overline{q}\rangle$ Fock state $|1q, 1\overline{q}, model\rangle$ is factorizable:

$$F(v^i) = \varphi_{L=0}(r) \cdot \chi^i \cdot C$$

(125)
a product of a space factor $\varphi_{L=0}(r)$, a spin-flavor factor $\chi^i$ and a color factor $C$. (In what follows, to simplify the notation, we will omit writing the factor $C$ and the color structure of the e.m. current; they lead to a multiplicative factor $\sqrt{3}$ in the matrix element $\langle v^i|j^k(0)|0\rangle$ that will be reinserted in Eq.\cite{132} below). The exact state, obtained from it via the unitary transformation $V$, is:

$$|v\rangle = V|1q - 1\overline{q}, model\rangle$$

(126)
The transformation $V$ has in this case also the task of transforming the model vacuum state $|0_b\rangle$ into the exact vacuum $|0\rangle$

$$|0\rangle = V|0_b\rangle$$

(127)

Thus, in the rest system of the $v$, it is:

$$V = V_0|0_b\rangle\langle 0_b| + V_1\sum|1q, 1\overline{q}\rangle\langle 1q, 1\overline{q}| + ....$$

(128)

In the Eq.\cite{128} above $|0_b\rangle\langle 0_b|$ is the projector on the $|0_b\rangle$ vacuum Fock state and $\sum|1q, 1\overline{q}\rangle\langle 1q, 1\overline{q}|$ is the projector on the states of the $1q, 1\overline{q}$ no-gluon Fock sector. The dots refer to the projectors of $V$ on the other Fock states, not relevant here.

The matrix element $\langle v^i|j^k(0)|0\rangle$ in Eq.\cite{123} is

$$\langle v^i|j^k(0)|0\rangle = \langle model \ 1q, 1\overline{q}|V^\dagger j^k(0)V|0_b\rangle = \langle F(v^i)|V^\dagger j^k(0)V|0_b\rangle$$

(129)

On the r.h.s. of \cite{129} we inserted the state $|F(v^i)\rangle$ corresponding to the wave function $F(v^i)$ \cite{125} of the model state of the $v$ meson:

$$|F(v^i)\rangle = \sum_p \sum_{\rho_1,\rho_2} g(p)\chi^i_{\rho_1,\rho_2} a_{p,\rho_1}^+ b_{-p,\rho_2}^+|0_b\rangle$$

(130)

where $p = (1/2)(p_1 - p_2)$ is the relative momentum of the quark 1 and antiquark 2 in the model state. In Eq.\cite{130} $a_{p,\rho}^+$, $b_{p,\rho}^+$ are creation operators of a quark and, respectively, antiquark with momentum $p$ and in the spin-flavor state $\rho$; $\rho_1$ and $\rho_2$ refer to the spin-flavor state of the quark 1 and antiquark 2 $[\rho = s, f; s = \text{spin}, f = \text{flavor}]$; here and in the following, 1 always refers to the quark and 2 to the antiquark. As to $\varphi_{L=0}(r)$ (the same for all nonet states) and its Fourier transform $g(p)$, they are the space or momentum factors of the $L = 0$ model wave functions.

Inserting \cite{130} in the last member of \cite{129} we obtain:

$$\langle v^i|j^k(0)|0\rangle = \sum_p \sum_{\rho_1,\rho_2} g^*(p)\chi^{*i}_{\rho_1,\rho_2} \langle 0_b|a_{p,\rho_1} b_{-p,\rho_2}(V^\dagger j^k(0)V)|0_b\rangle$$

(131)
The only part of \((V^\dagger j_k(0)V)\) contributing to the r.h.s. of (131) has, clearly, the form 
\[ G^{k}_{\rho_1,\rho_2}(p)\sigma^+_{p_1}b^+_{p_2}, \]
where \(G^{k}_{\rho_1,\rho_2}(p)\) is some function of \(p\). Using this expression of 
\((V^\dagger j_k(0)V)\) we obtain:
\[
\langle v^i|j^k(0)|0\rangle = \sqrt{3}\sum_p \sum_{\rho_1,\rho_2} g^*(p)\chi^{i^*}_{\rho_1,\rho_2}G^{k}_{\rho_1,\rho_2}(p) \tag{132}
\]
where we reintroduced the factor \(\sqrt{3}\) mentioned above (after Eq.(125)).

At this stage our aim is to display the most general flavor dependence of the right 
hand side of Eq.(132). To achieve this we must first eliminate the \(p\)'s summing over \(p\). Calling:
\[
\xi^k_{\rho_1,\rho_2} = \sum_p g^*(p)G^k_{\rho_1,\rho_2}(p) \tag{133}
\]
we have
\[
\langle v^i|j^k(0)|0\rangle = \sqrt{3}\sum_{\rho_1,\rho_2} \chi^{i^*}_{\rho_1,\rho_2}\xi^k_{\rho_1,\rho_2} \tag{134}
\]
The indices \(\rho \equiv s, f\), as already stated, specify both spin \(s\) and flavor \(f\). To obtain the 
flavor dependence it is convenient to use the fact that both \(\chi\) and \(\xi\) in Eq.(134) can be 
written as products of a spin and a flavor factor. We can then perform the sum over 
the spins obtaining:
\[
\sqrt{3}\sum_{s_1,s_2} \chi^{i^*}_{\rho_1,\rho_2}\xi^k_{\rho_1,\rho_2} = K\delta_{ik}\mathcal{F}_{f_1,f_2} \tag{135}
\]
where \(K\) is a constant and we have put
\[
\mathcal{F}_{f_1,f_2} = \langle v_{f_1,f_2}|\mathcal{F}|0\rangle \tag{136}
\]
Here \(v_{f_1,f_2}\) is the flavor factor of the model wave function of the meson \(v\) and \(\mathcal{F}\) is the 
most general flavor structure corresponding to the operator \((V^\dagger j(0)V)\) (\(j(0)\) here is any 
component of the current). We write
\[
\mathcal{F} = \sum_{f_1',f_2'} |f_1',f_2'|\langle f_1',f_2'| (V^\dagger j(0)V)_{\text{flavor}}|0\rangle \langle 0| \tag{137}
\]
where the projector \(\sum_{f_1',f_2'} |f_1',f_2'|\langle f_1',f_2'|\) refers only to the flavor space of \(1q, 1\bar{q}\) and 
\((V^\dagger j(0)V)_{\text{flavor}}\) is the most general operator that can result in QCD when \((V^\dagger j(0)V)\) 
is calculated and integrated over all variables except flavor. In (137) it is of course 
irrelevant to write \(|0\rangle\) or \(|0_b\rangle\) because the bare and exact vacuum have the same flavor 
properties.

The most general expression of \(\mathcal{F}\) in the calculation of the \(v\gamma\) couplings is easily 
found from the Eqs. given above. It is (as usual, \(Q = (2/3)P^u - (1/3)P^d - (1/3)P^s\)):
\[
\mathcal{F} = A(Q_1+Q_2)(|u\bar{u}|+|d\bar{d}|)\langle 0| + B(Q_1+Q_2)|s\bar{s}\rangle\langle 0| + CTr(2QP^s)\cdot|u\bar{u}+d\bar{d}+s\bar{s}\rangle\langle 0| + h.c. \tag{138}
\]
Here \(A, B, C\) are three coefficients; a fourth parameter in front of \(s\bar{s}\) in the last term 
of the expression is unnecessary because it can be absorbed in the coefficient \(B\). Note 
that there are no terms proportional to \((Q_1 - Q_2)\) because this quantity -not \((Q_1 + Q_2)\)-
is the total charge of the meson, and therefore vanishes for neutral states. Also we 
introduced a factor 2 in the Trace term (not present in [15]) to normalize all terms
Figure 3: The closed loop three gluons contribution to the $\rho^0\gamma$, $\omega\gamma$, $\phi\gamma$ couplings. The loop may be circled by a quark of flavor $u$, $d$ or $s$. The net result is nonvanishing because of flavor breaking, as indicated by the operator $P^s$, and corresponds to the term $CTr(2QP^s)$ in Eq. (138).

properly. Clearly $Tr(2QP^s) \equiv -2/3$, but we left, so far, this operator in this form to keep track of its origin due to a closed loop of type shown in Fig. 3.

On circling the closed loop we must meet, for a $\nu$ meson, in addition to the photon vertex, at least three gluon-quark vertices. Three is due to color and to spin 1 (or to Furry theorem). We stress (this is in fact the basic point of the present discussion) that (138) is the most general parametrization that can emerge from an exact complete QCD calculation. Nothing more complicated than this can be present.

At this point one natural question is: So what? There are three parameters and three mesons ($\rho$, $\omega$, $\phi$) and, without further information, nothing useful can be predicted. However we expect that, due to the hierarchy (Sect. 6), the Trace term in (138) contributes much less than the others; therefore, it may be a fair approximation to neglect it in a first approximation. We will see that the data on the $(f_{\rho\gamma}/f_{\omega\gamma})$ ratio confirm this point and allow to explain why, in spite of flavor breaking, the above ratio is near to 3.\footnote{In Ref. [15] we estimated the ratio $C/A \approx 10^{-2}$; as we will see it is instead $\approx 5 \cdot 10^{-2}$, but this difference does not change the conclusion just stated.} Note that for $C = 0$ (and in the absence of $\rho - \omega$ mixing) we would obtain:

$$f_{\omega\gamma} = KA \frac{2}{3\sqrt{2}}; \quad f_{\rho\gamma} = KA \frac{6}{3\sqrt{2}}$$  \hspace{1cm} (139)

which produces the old NRQM or $SU_3$ result:

$$f_{\rho\gamma} : f_{\omega\gamma} = 3 : 1$$  \hspace{1cm} (140)

What is important, however, is that we have shown that this is an almost exact QCD result obtained neglecting only the (small) contribution due to the Trace term in Eq. (138).

We now discuss the experimental value of $f_{\rho\gamma}/f_{\omega\gamma}$. We have:

$$(f_{\rho\gamma}/f_{\omega\gamma})^2 = (M_\rho/M_\omega)^2 \Gamma(\rho \rightarrow e^+e^-)/\Gamma(\omega \rightarrow e^+e^-)$$  \hspace{1cm} (141)

Here the $(M_\rho/M_\omega)^2$ is due to the square of the photon propagator in the decay $\nu \rightarrow e^+e^-$, proportional to $M_\nu^{-4}$, and to the phase space factor proportional to $M_\nu^2$. Note that in constructing the $\nu$ meson states with the general parametrization method in the rest frame of the decaying meson, one starts from a set of normalized model states which are mass-degenerate for all mesons of the nonet. The differences in mass of the
exact states are incorporated and automatically produced by the flavor breaking part of the $V$ operator which, operating on the model states, transforms them into the exact states. Therefore we must not insert explicitly the normalization factor $(2M_v)^{-1/2}$ of the state of the decaying $v$ meson. For this reason the r.h.s. of (141) contains $(M_\rho/M_\omega)^2$ instead of $(M_\rho/M_\omega)$ that often appears in this formula. We adopt the values $\Gamma(\rho \rightarrow e^+e^-) = 6.77 \pm 0.32$ KeV and $\Gamma(\omega \rightarrow e^+e^-) = 0.60 \pm 0.02$ KeV. Thus, according to Eq.(141), one gets:

$$|f_{\rho\gamma}/f_{\omega\gamma}|_{\text{exp}} = 3.35 \pm 0.07$$  (142)

The deviation from the “perfect” (3:1) result Eq.(140) can be due to the Trace term and to the $\rho - \omega$ mixing (for this compare e.g. Ref. [57],[58]). Taking into account only the Trace term, the Eq.(139) is replaced by:

$$f_{\omega\gamma} = K(A - 2C) \frac{2}{3\sqrt{2}}, \quad f_{\rho\gamma} = KA \frac{6}{3\sqrt{2}}$$  (143)

that is:

$$f_{\rho\gamma} : f_{\omega\gamma} = \frac{3}{1 - 2C/A}$$  (144)

From the experimental value (142)-and having omitted, as stated, the $\rho - \omega$ mixing-one gets: $C/A \approx 0.05$.

Considering now the $\phi$ decay one has:

$$f_{\phi\gamma} : f_{\rho\gamma} = \frac{\sqrt{2}}{3} \frac{B}{A}$$  (145)

and, from the experimental value $\Gamma(\phi \rightarrow e^+e^-) = 1.32 \pm 0.05$ KeV, one obtains:

$$|B/A| = 1.24 \pm 0.09,$$  (146)

a value consistent with that due to the flavor breaking. Note that a positive flavor breaking correction (i.e. $B/A > 1$) in this case is expected from a “quarkonium” model, since the quarkonium wave function at the origin $\varphi_Q(0)$ is larger for a heavier $q\bar{q}$ pair increasing the annihilation probability amplitude.

To summarize: The general parametrization (with neglect of the Trace term) predicts $f_{\rho\gamma} : f_{\omega\gamma} = 3$ in spite of flavor breaking and, at the same time, accounts for the deviation of $|f_{\phi\gamma} : f_{\rho\gamma}|$ from $\sqrt{2}/3$. It is misleading to say, as done sometimes, that the experimental values of the ratios $|f_{\rho\gamma} : f_{\omega\gamma} : f_{\phi\gamma}|$ are 3 : 1 : $\sqrt{2}$, as if flavor breaking were absent. It is true that $|f_{\rho\gamma} : f_{\omega\gamma}|$ is not far from 3:1, but, as expected, $|f_{\rho\gamma} : f_{\phi\gamma}|$ differs appreciably from $\sqrt{2}/3$. That is, flavor breaking in the expected amount is necessary to account for the last ratio.

A last remark is in order. The above analysis refers to $\omega$ and $\phi$ particles with the vector mixing angle at the ideal value $\theta_v = 35.3^\circ$. Allowing for a small deviation $\delta\theta_v$, one has:

$$f_{\omega\gamma}(35.3^\circ + \delta\theta_v)/f_{\omega\gamma}(35.3^\circ) = (\cos \delta\theta_v + \sqrt{2}(B/A) \sin \delta\theta_v) \approx (1 + 1.75\delta\theta_v)$$  (147)

In Ref.[15] we did use erroneously the Eq.(147) and got a negative mixing angle $\delta\theta_v$. We now see that Eq.(147) is compatible with a small positive $\delta\theta_v \approx 1^\circ$. 

43
13. The GP and chiral theories: A few remarks.

In a comparatively recent paper [9] Durand, Ha and Jaczko re-derived, using heavy-baryon chiral perturbation theory, the generalized Gell-Mann-Okubo mass formula (Eq.(1), Sect.1) obtained in 1992 by one of us (Ref.[8]) using the GP. We found their result interesting, especially because it shows that some kind of hierarchy plays a role in chiral QCD.

In an Erratum, Ref.[22], Durand and his collaborators acknowledged the coincidence of their result with [8]. Before illustrating an interesting aspect that we find in the derivation of Durand et al., we wish to reproduce a few words of clarification to [22] already contained in Ref.[23](footnote[3]).

1) The $T$ in the Eq.(1) -of the present survey- is called in [22] the “parameter” $T$. However, as we saw, $T$ is not a parameter; it is a well defined quantity: $T = \Xi^{*-} - (1/2) (\Omega + \Sigma^{*-})$ [the symbols are the baryon masses].

2) Also the statement in [22] from “so is not to be used” to “Our approaches differ in that respect” is not too clear because our $\hat{\alpha}_{MM'}$ -except that the Eq.(1) (Sect.1) includes the e.m. corrections, essential, in this case, to arrive to the level of precision noted in Sect.1.

Of course the fact that some of our results can be obtained from chiral theories is not unexpected, because the only properties of QCD that the GP exploits are the obvious ones indicated in the points a) to d) of Sect.1 of this survey. Thus any relativistic theory (chiral or non chiral) compatible with the general quark-gluon description of QCD and satisfying the above points a)-d) could, if used properly, produce our results.

But the main reason of our interest in the derivation of Durand et al. is the following: The re-derivation of the generalized Gell-Mann-Okubo mass formula by Durand et al. using chiral QCD implies that also their chiral description predicts that certain terms of second order in flavor breaking are very small. (Precisely those terms corresponding to our coefficients $c, d$ in Eq.(19) of Sect.3.) The smallness of $c$ due to the hierarchy (and established directly - see the values of the parameters in Eq.(20)) is all that was used in the General Parametrization to derive the new mass formula: compare Sect.3.

In fact the result of Durand et al. seems to be a case where a chiral procedure, even if after a very long calculation, leads to a prediction in low energy QCD depending only from the hierarchy of the parameters. As we stated repeatedly, the hierarchy appears naturally in the General QCD Parametrization and, we feel, should have appeared earlier and more generally and “spontaneously” in all theories that (as the chiral ones claim) intend to be a good approximation to QCD or at least to provide its main results. Our interest in the result of Durand et al. centered basically on this aspect.

The above point (Why in the usual treatments of chiral QCD the hierarchy remains so hidden?) was the main one that we intended to raise in this Section on the relation between the GP and chiral theories (and this question, after all, remains open). We have mentioned other problems related to chiral theories both in this survey -Sects.6 and 8- and in Ref.[26], but we will not come back to them here.

14. Comparing the GP with the $1/N_c$ method; some comments.

We inserted this section only for completeness; its contents is a summary of two arXiv reports [67, 68] to which we refer for all details. Here we limit to a few comments.

The basis of the “Large $N_c$ method” was a paper of ’t Hooft [69] on an hypothetical QCD with an increasingly large $N_c$ (number of colors). It should be added that ’t
Hooft’s aim was to try to understand quark confinement; he did not use the results of his paper for the dynamical QCD problems, treated later by others.

In the large $N_c$ method, $N_c$ is considered a parameter; one assumes that in the limit $N_c \to \infty$, the QCD strong coupling constant $g$ decreases proportionally to $1/\sqrt{N_c}$. The question if this expansion is meaningful at $N_c = 3$ does not certainly have an obvious answer (see [76]); no one knows the behaviour of QCD for $N_c \to \infty$. However, this expansion in $1/N_c$ (and also in flavour breaking) was widely used in the past years (see e.g. [78]). In Ref.[74] this expansion and also the GP were discussed. This fact (plus the popularity of the large $N_c$ method) have been the reason for inserting here a few remarks on the method.

Let us compare the GP parametrized baryon mass with the same quantity obtained in the $1/N_c$ method. Also there [70], the parametrization of the baryon masses is expressed in terms of 8 parameters (from $c_{(0)}$ to $c_{(3)}$), but these parameters multiply collective rather than individual quark variables. Again, setting to zero the smaller coefficients, one finds a relation between octet and decuplet baryon masses (Eq.(4.6) in [70]). This coincides with the result obtained several years before in [8] and reproduced here as Eq.(1), the generalized Gell Mann Okubo mass formula. But this result was ignored both in [70] (and in [78], [74]).

The re-derivation of the generalized Gell Mann-Okubo mass formula implies, of course, that the $1/N_c$ method is characterized by some hierarchy, at least for the masses, similar to that of the GP; but note the following: Whereas in the large $N_c$ description the reduction factor in the hierarchy is precisely $1/N_c = 1/3$, in the GP 1/3 is only an order of magnitude for the reduction factor. The above feature (“precisely 1/3”) is clearly very restrictive, perhaps too much, in the large $N_c$ procedure. (One can, of course, always find some way out from such problems, but, in so doing, one loses the basic feature of the theory). In this situation, the statement (see Ref.[74]) that the GP gives a reasonably good reproduction of the QCD results, but imposes “mild physical constraints” is, to say the least, unclear [for the meaning of the “mild” above see in [74] the remarks between Eqs.(3.5) and (3.6)]. The same lack of clarity applies to the expression “very general quark model” that Lebed used for the GP in a previous paper [75].

A question arises, of course: Does the large $N_c$ method lead always to the same results of the GP method? The answer is no: below, we will illustrate why the $1/N_c$ method can be incorrect in (at least) some cases.

We comment first on the Coleman-Glashow (CG) relation that we treated (sec.10) by the GP in Ref.[42]. There we showed that neither the $u - d$ mass difference, nor the Trace terms, modify the conclusion, reached in [44], that only a few, small, three index terms violate the CG relation. This explains the “miraculous” precision of the CG relation, originally derived by CG in exact $SU(3)_f$; a precision confirmed by a new measurement of the $\Xi^0$ mass [43].

After the appearance of [42] “On the miracle of the Coleman-Glashow and other baryon mass formulas”, a preprint by Jenkins and Lebed [77] implied that according to the large $N_c$ description it is “natural” (not “miraculous”) that the CG relation is so beautifully verified. It is implied in [77] that the terms neglected are “naturally” expected to be small.

This confidence has no basis. For the CG relation the terms in the GP are many [44, 42]. It is totally unjustified to estimate their global contribution only through the order in $1/N_c$ of a typical term, as done in the $1/N_c$ method. Thus the predictions of the $1/N_c$ expansion do not have a real QCD foundation.

Another simple case where the results of the large $N_c$ method clearly differs from the GP analysis (in the wrong sense) is that of the magnetic moments of $p, n$ and $\Delta$ ’s. The
$1/N_c$ results do not account for the facts, contrary to the statements in Refs. [71, 72, 73]. Due to the omission of effects of order $1/N_c^2$, the $1/N_c$ expansion cannot explain for instance, the $\mu(p)/\mu(n)$ ratio and the $\Delta \to p\gamma$ transition (Sect.5 in this survey).

15. Appendix I - A field theoretical derivation of the GP.

We now relate the General QCD Parametrization to a conventional Feynman diagrams description. Essentially we will show how the unitary transformation $V$ connecting the exact state, say of a baryon, $|\Psi_B\rangle$, to its model state $|\Phi_B\rangle$ can be constructed in a field theoretical frame. We did the same in the Appendix to Ref.[1], except that there we assumed to identify $m$ (the renormalized mass of a quark) with the (not well defined) mass of a constituent quark. As shown in Ref.[26], this is unnecessary; here we will not keep this limitation. In principle the renormalization point for the quark masses can be selected arbitrarily; although it is fixed at some value, we do not need at this stage to specify it explicitly. For the rest we proceed as in the Appendix of Ref.[1]. Here we will deal only with the relation between the $V$ transformation and the conventional field theoretical description. We will illustrate the construction of $V$ only for baryonic 3-quark states, but clearly the same arguments hold in all cases. Call $|qqq\rangle$ a state of three quarks and no gluon and let $\eta = \sum |qqq\rangle\langle qqq|$ be the projection operator into any state (of 3 quarks and no gluons):

$$\eta |qqq\rangle = |qqq\rangle; \quad \eta | \neq qqq\rangle = 0$$

We rewrite $H$ identically as:

$$H = \eta H \eta + (1 - \eta)H(1 - \eta) + \eta H(1 - \eta) + (1 - \eta)H \eta$$

Introduce now the model Hamiltonian $\eta \mathcal{H} \eta$ which is a typical non-relativistic quark model Hamiltonian acting only in the Fock space of the states of three quarks and no gluons. We decompose $H$ as:

$$H = K_0 + K_1$$

with:

$$K_0 = (1 - \eta)H(1 - \eta) + \eta \mathcal{H} \eta; \quad K_1 = \eta H(1 - \eta) + (1 - \eta)H \eta + \eta H \eta - \eta \mathcal{H} \eta$$

having added to $K_0$ and subtracted from $K_1$ the model Hamiltonian $\eta \mathcal{H} \eta$. Referring to the baryons we assume that $\eta \mathcal{H} \eta$ has degenerate eigenvalues $M^B_0$ for all the octet and decuplet baryon states:

$$\eta \mathcal{H} \eta |\Phi_B\rangle = M^B_0|\Phi_B\rangle \quad (B = N, \Lambda, \Sigma, \Xi, \Delta, \Sigma^*, \Xi^*, \Omega)$$

where $|\Phi_B\rangle|0\text{gluons}\rangle$ are the $L = 0$ model states. Because in the three-quark sector $K_0$ and $\mathcal{H}$ coincide, $|\Phi_B\rangle$ are the degenerate eigenstates of $K_0$:

$$K_0|\Phi_B\rangle = M^B_0|\Phi_B\rangle$$

In the part $\eta \mathcal{H} \eta$ of $K_0$ the masses of the $u, d, s$ quarks are assumed to be equal [as implied by Eq.(152)]; the flavor breaking mass term (Eq.(9) of Sect.2) appears in the term $(1 - \eta)H(1 - \eta)$ of $K_0$ and in the term $\eta H \eta$ of $K_1$. The term $(1 - \eta)H(1 - \eta)$ of $K_0$ includes in particular the Hamiltonian of the non interacting gluons; $K_1$ contains
the interaction terms $\eta H(1 - \eta)$ and $1 - \eta H\eta$ of the quark-gluon Hamiltonian.

We now treat $K_0$ as the unperturbed Hamiltonian and $K_1$ as the perturbation; imagine inserting $K_1$ adiabatically and construct the true states $|\Psi\rangle$ with the procedure of Gell Mann and Low (Ref.[64]) (this procedure is not compulsory, but it shows that at least one method of construction exists). Writing $K_1(t) = \exp (+iK_0t)K_1 \exp (-iK_0t)$ the adiabatic $U(t, t_0)$ satisfies:

$$i\dot{U}_\alpha(t, t_0) = \exp (-\alpha |t|) \cdot K_1(t)U_\alpha(t, t_0) \quad (\text{with } \alpha > 0, \ U_\alpha(t_0, t_0) = 1) \quad (154)$$

and the $|\Psi\rangle$’s for the lowest bound states corresponding to the lowest $|\Phi_B\rangle$’s are:

$$|\Psi_B\rangle = \lim_{\alpha \to 0} \exp (-w_B/\alpha) \cdot U_\alpha(0, -\infty)|\Phi_B\rangle \quad (155)$$

where $w_B$ is purely imaginary ($w_B + w_B^* = 0$) [so that the factor $\exp(-w_B/\alpha)$ in front of Eq.(155) is a pure phase factor that eliminates the singularity coming from the $\lim_{\alpha \to 0} U_\alpha(0, -\infty)$; $w_B$ in Eq.(155) is related to the $S = U(+\infty, -\infty)$ matrix element of the $\Phi_B \rightarrow \Phi_B$ transition by $\lim_{\alpha \to 0} \langle \Phi_B | S | \Phi_B \rangle = \exp(2w_B/\alpha)$]. The basic operator $V$ introduced in the text can be therefore written explicitly as:

$$V = \lim_{\alpha \to 0} \exp (-w_B/\alpha) \cdot U_\alpha(0, -\infty) \quad (156)$$

Thus, for instance, the formula for the magnetic moments (omitting Trace terms) has the form used in the text, namely:

$$M = \langle \Phi | V^\dagger \mathcal{M} V | \Phi \rangle \quad (157)$$

It can be seen easily that the above formula (157) is the same as that used frequently for practical calculations:

$$M = \frac{\langle \Phi | T(M(0)S) | \Phi \rangle}{\langle \Phi | S | \Phi \rangle} \equiv \langle \Phi | T(M(0)S) | \Phi \rangle_C \quad (158)$$

where the index $C$ means “connected”. However the formula (156) for $V$ is not that written more frequently:

$$|\Psi\rangle = \lim_{\alpha \to 0} \frac{U_\alpha(0, -\infty)|\Phi\rangle}{\langle \Phi | U_\alpha(0, -\infty)|\Phi\rangle} \quad (159)$$

Although the final formulas for the physical quantities are always the same, in the Eq.(159) the denominator is not a pure phase factor, as it is the factor multiplying $U_\alpha(0, -\infty)$ in Eq.(156). For the GP this might create the problem considered at the end of Sect.8 of [1], that is, the need for an additional parameter in the parametrization of the magnetic moments. But with the $V$ in Eq.(156) this problem does not arise, as discussed in some detail also at the end of the Appendix of Ref.[1].

16. Appendix II - A summary of the main steps in the derivation of the spin-flavor dependence of the GP terms.

A) The spin algebra
How can one obtain the spin-flavor structures that appear, e.g., in the parametrized
expression of the baryon masses (Eq.19) or in that (Eqs.28,29) of the baryon octet magnetic moments?  

In this Appendix we will illustrate, as an example, the derivation of the spin-flavor terms appearing in the magnetic moments, at 1st order in flavor breaking (recall that, as we stated in Sect.2, the model state \( |\Phi_B\rangle \) has been selected with orbital angular momentum \( L = 0 \); this is the reason why, after the \( V \) transformation, only the spins of the quarks appear in the construction of the expressions for the magnetic moment of the baryons). Indeed, all three body space axial vectors vanish on evaluating their expectation value on \( |\Phi_{B,L=0}\rangle \); therefore in evaluating the magnetic moments one has to do with the expectation value of something of the form \( \sum_{\nu} \sum_{p,p'} R_{\nu}(p,p') G_{\nu}(\sigma, f) \), where \( p, p' \) are the two independent momenta of three bodies in the rest system. Because \( R_{\nu}(p,p') \) is a scalar under space rotations, the whole axial vector contributing to the magnetic moment of \( B \) is due to the spins.

Setting \( \langle X_{L=0}|R_{\nu}(p,p')|X_{L=0}\rangle \equiv g_{\nu} \), one obtains for the magnetic moments the expression \( \sum_{\nu} g_{\nu} G_{\nu}(\sigma, f) \). (Compare Sect.IV of [1]).

We will start examining the spin dependence of the most general axial vector operator formed with the spins \( \sigma_1, \sigma_2, \sigma_3 \) of three spin \( 1/2 \) particles (the quarks) and also its dependence on the flavor operators \( f \). This is given in Eq.(160), to which one has to add all terms, possibly with different coefficients, obtained performing any permutation on \( 1,2,3 \).

\[
\sigma_1 [a(f) + b(f) (\sigma_2 \cdot \sigma_3)] + c(f)(\sigma_1 \times \sigma_2) + d(f)(\sigma_1 \times \sigma_3)
\]  

In (160) \( a(f), b(f), c(f), d(f) \) are Hermitian operators constructed with the flavor variables and having real matrix elements between real functions (we call such operators “real”).

One can show [for the proof compare Ref.[1], page 3001] that when calculating the expectation value of the operator (160) on a spin-flavor state with a real wave function and a given value of the total angular momentum \( J \), (a) the cross products terms in Eq.(160) give no contribution, and (b) the term \( \sigma_1 (\sigma_2 \cdot \sigma_3) \) can be rewritten purely in terms of \( \sigma_1 \) and of the \( c \) number \( J \). It follows that the most general Hermitian axial-vector operator \( G(\sigma, f) \) constructed in terms of the \( \sigma_i \) of the three quarks (and of the flavor operators \( f \) ) is, when used for evaluating an expectation value as specified above, a combination of \( \sigma_1 \Gamma^f_1(f), \sigma_2 \Gamma^f_2(f), \sigma_3 \Gamma^f_3(f) \), where \( \Gamma^f_1(f) \), are three operators depending, for a given \( J \), only on \( f \). With the same type of proof one also finds that the scalar:

\[
(\sigma_1 \times \sigma_2) \cdot \sigma_3 : F(f)
\]  

where \( F(f) \) is any Hermitian real flavor-dependent operator has a vanishing expectation value on any real spin-flavor state of three particles. (compare the Eq.(26) in [1] and the proof after it).

We now list a set of relations involving \( \sigma_i \)'s that are useful to develop the GP.

Consider the term \( \sigma_1 (\sigma_2 \cdot \sigma_3) \) that appears in Eq.(160) (multiplied with a real flavor operator \( b(f) \) ) and take its \( z \) component. It is: \( \sigma_1 z (\sigma_2 \cdot \sigma_3) = (1/2) \sigma_1 z [(\sigma_2 + \sigma_3)^2 - 6] \).

Writing: \( 2J = \sigma_1 + \sigma_2 + \sigma_3 \) we obtain:

\[
\sigma_1 z (\sigma_2 \cdot \sigma_3) = \frac{1}{4} \sigma_1 z [(2J - \sigma_1)^2 - 6] + \frac{1}{4} [(2J - \sigma_1)^2 - 6] \sigma_1 z
\]  

\(^{18}\text{The reader should consult [1] for more details on the points treated in this Appendix; in fact this summary cannot replace such a consultation, but we felt it necessary to list here some important points, to give an idea of the methods used.}\)
and, with some algebra:

\[
\sigma_{1z}(\sigma_2 \cdot \sigma_3) = \frac{1}{4}[(4|J|^2 - 7)\sigma_{1z} + \sigma_{1z}(4|J|^2 - 7)] - 2J_z \tag{163}
\]

In calculating the expectation value of \(\sigma_{1z}(\sigma_2 \cdot \sigma_3)\) on a state with a given \(J\) we can write \(|J|^2 = J(J + 1)\), a \(c\) number.

In conclusion the most general axial-vector operator (as far as its expectation value on a real spin-flavor state with a given \(J\) is concerned) is:

\[
G(\sigma, f) = \sigma_1 \Gamma_1^f(f) \quad \text{or} \quad \sigma_2 \Gamma_2^f(f) \quad \text{or} \quad \sigma_3 \Gamma_3^f(f) \tag{164}
\]

where the \(\Gamma_1^f(f)\) are real flavor operators. We did prove, essentially, that the most general axial vector formed with three spin \(\frac{1}{2}\) particles (under the italicized condition above) is a combination of \(\sigma_1, \sigma_2, \sigma_3\) and nothing else. It might seem strange that the only axial vectors are those listed in (164), since we can for instance, multiply \(\sigma_1\) by \((\sigma_1 \cdot \sigma_3)\) or by any other scalar product of the spin matrices and remain with an axial vector. The answer appears from the Eqs. (165) and (166) below where we have limited to the \(z\) components (the \(x\) and \(y\) behave similarly).

\[
\sigma_{1z}(\sigma_1 \cdot \sigma_2) = \sigma_{2z} + i(\sigma_1 \times \sigma_2)_z, \tag{165}
\]

\[
\sigma_{1z}[\sigma_1 \cdot (\sigma_2 \times \sigma_3)] = (\sigma_2 \times \sigma_3)_z + i[\sigma_1 \times (\sigma_2 \times \sigma_3)]_z = (\sigma_2 \times \sigma_3)_z + i(\sigma_1 \cdot \sigma_2)\sigma_{3z} - i(\sigma_1 \cdot \sigma_3)\sigma_{2z} \tag{166}
\]

because the expectation value of \((\sigma_i \cdot \sigma_k)\) is zero as stated above, Eqs. (165) and (166)\([\text{in conjunction with (163)}]\) exemplify how the multiplication of \(\sigma_i\) by scalar products of \(\sigma\)'s does not create new axial vectors in addition to (164) with a non zero expectation value.

B) The flavor algebra

To write all \(G_{\nu}(\sigma, f)\)'s that appear, to all orders in flavor breaking, in the construction of the magnetic moments (and were listed in Eq. (29) of Sect. 4 to first order in flavor breaking), we still need the expression of the most general \(\Gamma^j(f)\)'s appearing in Eq. (164). The \(\Gamma^j(f)\) have been discussed in Sect. VI of Ref. [1]. That presentation is simple and remains valid for the octet magnetic moments to first order flavor breaking. Here we will indicate some changes in notation of this survey with respect to [1] and also some corrections to the list of \(\Gamma^j(f)\)’s; these changes do not affect the 1st order flavor breaking baryon octet magnetic moments given in Ref. [1] and here in Sect. 4.

The changes in notation are the following: 1) The projection operators on the \(P, N, \lambda\) quarks \((P^u, P^d, P^s, 2)\) the projection operator \(P^t\) in the e.m. current is now rewritten as \([(2/3)P^u - (1/3)P^d - (1/3)P^s] \tag{36}\). 3) We usually write \(Q_i\) instead of \(P_i^Q\).

As to the corrections (mentioned above), they consist in the following additions to the \(\Gamma(f)\)'s with respect to those listed in Ref. [1]: In addition to the \(\Gamma(f)\)'s listed in the Eq. (36) of that paper [and, for the \(\Omega^-\), in the line below that of Eq. (36)], a new class of \(\Gamma(f)\) (1st order - or more - flavor breaking) must be considered of the form \(Tr(QP^s), \ P_s^tTr(QP^s), P_s^tP_s^tTr(QP^s)\), and, for the \(\Omega^-\), \(P_s^tP_s^tP_s^tTr(QP^s)\). Note that the treatment of the magnetic moments to 1st order flavor breaking given in [1] remains correct, because the 1st order Trace term, as shown in Sect. 4, Eq. (30), can be expressed in
A last remark on the meaning of “order in flavor breaking”. When we refer to first order flavor breaking, this means to keep only flavor breaking terms linear in the $P^s$. But, because it is $(P^s_i)^n = P^s_i$ for any (integer positive) $n$, the GP includes in fact all orders of the flavor breaking contributions additive in the quarks; flavor-breaking effects of 2nd order or more coming from terms carrying the indices of two different quarks are not included.

C) The coefficients in the baryon mass parametrization

This subject is treated in Sect.XII of Ref.[1] and in the Appendix B of Ref.[26]. Both should be consulted (and, in particular, the first order flavor breaking mass formulas (82)-(83)of Ref.[1] can be useful), but here we limit to list the formulas (given in [26]) leading to the values of the coefficients $M_0, B, C, D, E, (a + b), c, d$ displayed in Sect.3 of this survey.

Because in Sect.3 the treatment of the general parametrization (and therefore of its parameters) refers to the strong interactions only [that is the masses in Eq.(19),Sect.3 are the eigenvalues of $H_{QCD}$, without the e.m. interaction], it is necessary, especially for the smaller coefficients, to extract from the experimental mass values their strong part. Stated otherwise, to determine the coefficients of the parametrization (19), one must use mass values independent of the e.m. and isospin breaking ($m_u \neq m_d$), at least to first order. We did this already when writing the generalized Gell Mann-Okubo mass formula (Eq.(1)).

For the large coefficients $M_0, B, C$, this precision is usually not necessary and we will just use an average mass on the different baryons of a multiplet, e.g. take $N \equiv (n + p)/2$ for the mass of a nucleon or $(\Delta \equiv \Delta^{++} + \Delta^{+} + \Delta^{0} + \Delta^{-})/4$ for the average mass of a $\Delta$. Thus we will determine $M_0, B, C$ as:

$$M_0 = (\bar{N} + \bar{\Delta})/2, \quad B = \Lambda - \bar{N} + 3E, \quad C = (\bar{\Delta} - \bar{N})/6 \quad (167)$$

where the parameter $E$ in (167) (the same appearing in Eq.(168)) will be expressed in terms of the masses in the second equation below: $D, E, (a + b), c, d$ are determined from the following equations (168) that are Coulomb and isospin independent to first order:

$$D = (1/6)[(\Sigma^*^- - \Sigma^-) + (\Sigma^+ - p)] \quad (168)$$

$$E = (1/6)(\Sigma^*^- - \Sigma^-) + (1/12)(\Sigma^0 - 3\Lambda + 2n)$$

$$c = (1/3)[(\Xi^*^- + \Xi^-) - (\Sigma^*^- + \Sigma^-)] - 2E$$

$$a + b = \Xi^- - \Sigma^- + (1/2)(\Sigma^0 - 3\Lambda + 2n) + 2c$$

$$d = \Omega^- - \Delta^{++} + 3(\Xi^0 - \Sigma^*)$$

where in the above formulas $\Delta^-$ stays for:

$$\Delta^- = \Delta^{++} + 3(n - p) \quad (169)$$

---

In the list of Eq.(39) of Ref.[1] the term $G_{7b} = \sum_{i \neq k} P_i^q P_k^q P_k^\lambda \sigma_i$ should also have appeared; it is classified however among the 2nd order flavor breaking terms that were not considered there.
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