Application of the Generalized Bochner Technique to the Study of Conformally Flat Riemannian Manifolds

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Abstract: In this article, we discuss the global aspects of the geometry of locally conformally flat (complete and compact) Riemannian manifolds. In particular, the article reviews and improves some results (e.g., the conditions of compactness and degeneration into spherical or flat space forms) on the geometry “in the large” of locally conformally flat Riemannian manifolds. The results presented here were obtained using the generalized and classical Bochner technique, as well as the Ricci flow.

Keywords: Riemannian manifold; locally conformally flat; curvature operator; scalar and Ricci curvatures; Ricci flow

MSC: 53C20; 53C21; 53C24

1. Introduction and Main Results

One of key analytical methods for differential geometry is the celebrated Bochner technique, which was founded by S. Bochner, K. Yano, A. Lichnerowicz and others in the 1950s and 1960s to study the relationship between topology and curvature of a compact Riemannian manifold (e.g., [1]; ([2] pp. 333–364)). To achieve this, the authors of the method used Hopf’s lemma and Green’s theorem (see also [1]). On the other hand, since the 1970s, complete (noncompact) Riemannian manifolds have been included in the range of research carried out using the Bochner technique. For this, the methods of geometric analysis were developed (e.g., [3–5]). As a result, vanishing theorems of the classical Bochner technique took the form of Liouville type theorems, e.g., [5–7]. In this article, we discuss the global aspects of the geometry of locally conformally flat complete and compact Riemannian manifolds using the generalized and classical Bochner technique, as well as the Ricci flow, e.g., [8,9]. Before that, we summarize and improve some known results (for example, the conditions of compactness and degeneration into spherical or flat space forms) on the geometry in the large of locally conformally Riemannian manifolds. This article continues the series of our works [10–16] on various methods of the classical and generalized Bochner technique.

In this article, $(M,g)$ is a Riemannian manifold, namely, a connected $C^\infty$-manifold $M$ of dimension $n \geq 2$ equipped with a metric tensor $g$ and the Levi-Civita connection $\nabla$. Let $TM$ and $T^*M$ be the tangent and cotangent bundles of $M$. Denote by $\otimes^r T^*M$ the vector bundle of $r$-times covariant tensors on $M$, and by $S' M = S'(T^*M)$ and $\Lambda' M = \Lambda'(T^*M)$—its sub-bundles of symmetric $r$-tensors and differential $r$-forms. Riemannian metrics in the fibres of these bundles are induced by the metric $g$ and will be denoted by the same letter $g$. We will set $|T|^2 = g(T,T)$ for arbitrary $T \in \otimes^r T^*M$.

A locally conformally flat Riemannian manifold $(M,g)$ is determined by the condition (see [17] (p. 60)) that any point $x \in M$ has a neighborhood $\Omega \subset M$ and a $C^\infty$-function $f$ on
\[ \Omega \] such that the Riemannian manifold \((\Omega, e^{\phi g})\) is flat (has zero sectional curvature). For such manifolds, the Weyl conformal curvature tensor \(W\), see (3) in what follows, vanishes. For \(n \geq 4\) this condition is necessary and sufficient (see [17] (p. 60)).

The Riemannian geometry “in the large” studies relations between the curvature and the topology and between local and global characteristics, of a Riemannian manifold. There are many results “in the large” on locally conformally flat Riemannian manifolds (e.g., [3,6,10,11,18,19]). The purpose of this article is to review and improve some known results in this field. The paper presents seven theorems, which provide sufficient conditions for a Riemannian manifold to admit a flat metric or metric of positive constant sectional curvature, as well as analytic compactness conditions for complete locally conformally flat Riemannian manifolds. Our theorems can be viewed as instructive applications (obtained by logical inference and noncomplicated calculations) of deep recent results on the Ricci flow under the strict assumption that the Weyl curvature tensor vanishes, thus, the article can also demonstrate to young (and experienced) researchers a visual connection between various topics of modern Riemannian geometry.

S. Bochner’s theorem (see [1] (pp. 79–80)), according to which a compact locally conformally flat Riemannian manifold of dimension \(n \geq 4\) with a positive definite Ricci tensor is a homological sphere—this is the first best-known result “in the large” on such manifolds. Following this, M. Tani proved (see [20]) that a compact locally conformally flat Riemannian manifold of dimension \(n \geq 3\) with positive constant scalar curvature and positive sectional curvature (or positive definite Ricci tensor) is a manifold of constant positive sectional curvature. Notice that a compact manifold of positive constant sectional curvature for even \(n\) is either a sphere \(S^n\) or a real projective space \(\mathbb{R}P^n\) with canonical metric; and for each odd \(n \geq 3\), there are infinitely many spherical space forms that were completely classified by group-theoretic methods (see [8] for more details). Our first theorem gives a generalization of the results of S. Bochner and M. Tani.

**Theorem 1.** A compact locally conformally flat Riemannian manifold of dimension \(n \geq 4\) with positive sectional curvature admits a metric of positive constant sectional curvature and therefore is diffeomorphic to a spherical space form.

Following M. Tani [20], the study was continued by S. Goldberg, who proved a number of theorems. According to one of them (see [21]), a compact locally conformally flat manifold of dimension \(n \geq 3\) with constant scalar curvature \(s\) and Ricci tensor \(\text{Ric}\) such that \(|\text{Ric}| < s/\sqrt{n-1}\) has constant sectional curvature. Note that here \(s > 0\), and the inequality has an equivalent form \(|\text{Ric}|^2 < s^2/(n-1)\). An analogue of Goldberg’s theorem for a complete locally conformally flat manifold with scalar curvature and Ricci tensor satisfying similar conditions was proved in [22]. We can prove a similar statement without the condition that the scalar curvature is constant.

**Theorem 2.** A compact locally conformally flat Riemannian manifold of dimension \(n \geq 4\) with positive scalar curvature \(s\) satisfying the condition

\[
|\text{Ric}|^2 \leq \begin{cases} 
(4/15) s^2, & n = 4, \\
(83/400) s^2, & n = 5, \\
\frac{n}{n-1} s^2, & n \geq 6,
\end{cases}
\]

admits a metric of positive constant sectional curvature and therefore is diffeomorphic to a spherical space form.

Another S. Goldberg’s theorem (see [21,23]) states that a locally conformally flat manifold of dimension \(n \geq 3\) with positive definite Ricci tensor \(\text{Ric}\) and constant scalar curvature \(s\) such that \(|\text{Ric}|^2 \leq \frac{1}{n-1} s^2\), is a manifold of constant sectional curvature. Note that the inequality \(|\text{Ric}|^2 \geq \frac{1}{n} s^2\) holds for any Riemannian manifold. We formulate a similar statement without the requirement that the scalar curvature is constant.
**Theorem 3.** Let \((M, g)\) be a compact locally conformally flat Riemannian manifold of dimension \(n \geq 4\) with
\[
\text{Ric} > \frac{s}{2(n-1)} g.
\]
Then \(M\) admits a metric of positive constant sectional curvature and therefore is diffeomorphic to a spherical space form.

In particular, when the dimension of \(M\) is equal to three, we have

**Theorem 4.** A three-dimensional compact conformally flat Riemannian manifold with \(\text{Ric} < \frac{1}{2} s \, g\) has constant positive sectional curvature.

We consider a homogeneous Riemannian manifold \((M, g)\), which is determined by the condition that its isometry group is transitive, i.e., for any points \(x, y \in M\) there exists an isometry that translates \(x\) into \(y\) (see [17] (p. 178)). Recall that a homogeneous Riemannian manifold is complete (see [17] (p. 181)). S. Goldberg’s theorem in [23] states that a locally conformally flat homogeneous Riemannian manifold with Ricci tensor \(\text{Ric} \geq 0\) is a locally symmetric space. Since a locally symmetric space has a parallel Ricci tensor, S. Goldberg formulates yet another theorem, according to which a locally conformally flat Riemannian manifold has constant sectional curvature if its Ricci tensor is parallel and is positive or negative definite. Moreover, S. Goldberg does not include the completeness or compactness of the manifold in the conditions of the theorems. We supplement these results by the following.

**Theorem 5.** A simply connected irreducible locally conformally flat homogeneous Riemannian manifold of dimension \(n \geq 3\) with non-negative definite Ricci tensor is the canonical sphere.

Articles [6,7] are devoted to the search for analytic compactness conditions for complete locally conformally flat Riemannian manifolds. The main result of [6] is that an \(n\)-dimensional \((n \geq 3)\) simply connected complete locally conformally flat Riemannian manifold with positive constant scalar curvature is compact if

\[
\int_M \|\text{Ric}\|^p \, d\text{vol}_g < \infty
\]

for all \(p \geq n/2\), where \(\text{Ric} = \text{Ric} - (s/n) \, g\) is the traceless Ricci tensor.

Our theorem and Corollary 1 in what follows complement this result.

**Theorem 6.** A complete locally conformally flat Riemannian manifold of dimension \(n \geq 4\) with positive bounded scalar curvature \(s\) satisfying the condition
\[
\|\text{Ric}\|^2 \leq \begin{cases} 
\frac{(1-\varepsilon)^2}{60} s^2, & n = 4, \\
\frac{3(1-\varepsilon)^2}{400} s^2, & n = 5, \\
\frac{(1-\varepsilon)^2}{n^2(n^2-1)} s^2, & n \geq 6,
\end{cases}
\]

for some constant \(\varepsilon \in (0, 1)\), is compact.

**Remark 1.** For \(n \geq 6\) and \(p = 2\), the above integral inequality from [6] is a consequence of Theorem 6, in particular, this follows from the inequalities with \(\|\text{Ric}\|^2\) presented there and the boundedness requirement for positive scalar curvature.

The main result of [19] states that if a complete locally conformally flat manifold of dimension \(n \geq 3\), whose Ricci tensor satisfies the inequality \(\text{Ric} \geq 0\), belongs to one of the following classes: either flat or locally isometric to the product of a canonical sphere and a line, then the manifold is globally conformally equivalent to either \(\mathbb{R}^n\) or a spherical space form (see [24]). In turn, we will prove a theorem that complements this statement.
Theorem 7. A complete noncompact connected locally conformally flat Riemannian manifold of dimension \( n \geq 4 \) with non-negative sectional curvature and constant scalar curvature is a flat space form, if the following inequality holds for some \( p \geq 1 \):

\[
\int_M \| \text{Ric} \|^p \, d\text{vol}_g < \infty.
\]

2. Proofs of Statements

Recall that using the equality

\[
g(\tilde{R}(X_1 \wedge X_2), X_4 \wedge X_3) = Rm(X_1, X_2, X_3, X_4),
\]

a Riemannian curvature tensor \( Rm \in S^2(\Lambda^2 T^*_x M) \) defines a linear symmetric map \( \tilde{R} : \Lambda^2(T^*M) \rightarrow \Lambda^2(T^*M) \), called the curvature operator (see [21] (p. 83) and [25]). The curvature operator is positive definite (respectively, non-negative definite) if at each point \( x \in M \) all the eigenvalues of the operator \( \tilde{R} \) are positive (respectively non-negative), e.g., [26,27]. Obviously, from the positive definiteness (respectively, non-negative definiteness) of the curvature operator, it follows that the sectional curvature of the manifold is positive (respectively, non-negative). The converse is also true for a locally conformally flat Riemannian manifold.

For any orthogonal unit vector \( X \) and \( Y \), we denote by \( \text{sec}(X \wedge Y) \) the sectional curvature in the direction of the 2-plane \( \sigma = \text{span}(X, Y) \).

**Lemma 1.** For a Riemannian manifold \((M, g)\) of dimension \( n \geq 3 \) with vanishing Weyl tensor \((W = 0)\), the conditions \( \tilde{R} > 0 \) and \( \text{sec} > 0 \) are equivalent.

**Proof.** Recall that a Riemannian curvature tensor is called pure (see [17] (p. 439)) if at every point \( x \in M \) there exists an orthonormal basis \( \{e_1, \ldots, e_n\} \) of the space \( T_xM \), for which the 2-form \( e_i \wedge e_j \) is an eigenform of the curvature operator for all pairwise different \( i, j \). In this case, the equality \( \tilde{R}(e_i \wedge e_j) = \text{sec}(e_i \wedge e_j)e_i \wedge e_j \) holds (see [28] and ([4] pp. 175–176)). Thus, in the case of a pure Riemannian curvature tensor, the conditions \( \tilde{R} > 0 \) and \( \text{sec} > 0 \) are equivalent. To complete the proof, recall (see [17] (pp. 61, 439)) that the condition \( W = 0 \) implies that \((M, g)\) has a pure curvature tensor.

\[\square\]

**Remark 2.** The inequality ‘\( > 0 \)’ in Lemma 1 can be simultaneously (3 times) replaced by ‘\( \geq 0 \)’, as well as by ‘\( < 0 \)’ or ‘\( \leq 0 \)’. In the case of locally conformally flat Riemannian manifolds, the proof of Lemma 1 can also be approached differently. Such manifolds are characterized (see [17] (p. 61) and [29]) by the equality \( Rm = S \odot g \), where

\[
S = \frac{1}{n-2} \left( \text{Ric} - \frac{s}{2(n-1)} g \right)
\]

is the Schouten tensor (see [30,31]) and \( \odot \) is the Kulkarni–Nomizu product of symmetric 2-tensors (see [17] (p. 47)). It was proved in [32] that the curvature operator is diagonalized in the same orthonormal basis \( \{e_1, \ldots, e_n\} \) of the tangent space \( T_xM \) at an arbitrary point \( x \in M \), where the Schouten and Ricci tensors are diagonalized. Namely, in the basis \( \{e_i \wedge e_j\} \), the matrix of the curvature operator is diagonalized so that sectional curvatures stand on its main diagonal. Therefore, for a locally conformally flat Riemannian manifold, the condition \( \tilde{R} > 0 \) and condition \( \text{sec} > 0 \) are equivalent.

**Corollary 1.** A complete locally conformally flat Riemannian manifold of dimension \( n \geq 3 \) with sectional curvature satisfying pointwise pinching conditions of the form \( \|\text{sec}\| \leq B \) and \( \text{sec} \geq \frac{\delta}{n(n-1)} > 0 \) for some real constants \( B \) and \( \delta > 0 \), is compact.
Proof. According to [5], a complete Riemannian manifold of dimension \( n \geq 3 \) is compact if its curvature operator is pointwise bounded, i.e., \( \|\hat{R}(x)\| \leq A \) for any point \( x \in M \) and some constant \( A > 0 \), and satisfies the inequality

\[
\hat{R}(e_i \wedge e_j) \geq \delta \frac{s}{n(n - 1)} e_i \wedge e_j
\]

(2)

for some constant \( \delta > 0 \), positive scalar curvature \( s \) and an arbitrary orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_x M \) for any point \( x \in M \). In the case of a locally conformally flat Riemannian manifold, we can rewrite inequality (2) in the form

\[
\text{sec}(e_i \wedge e_j) e_i \wedge e_j \geq \frac{\delta s}{n(n - 1)} e_i \wedge e_j.
\]

Then, by Lemma 1 and Remark 2, we have \( \text{sec} \geq \frac{\delta s}{n(n - 1)} \). On the other hand, the condition \( \|\hat{R}(x)\| \leq A \) has an equivalent form \( \|\hat{R}(x)\|^2 \leq A^2 \), and, by virtue of Lemma 1, can be represented as follows:

\[
\|\hat{R}(x)\|^2 = \sum_{i<j} \text{sec}^2(e_i \wedge e_j) < A^2.
\]

Therefore, if the sectional curvature is pointwise bounded, i.e., \( \|\text{sec}(x)\| \leq B \) for some constant \( B > 0 \) and arbitrary \( x \in M \), then the curvature operator is pointwise bounded, i.e., \( \|\hat{R}(x)\| \leq A \) for \( A > \sqrt{\frac{1}{2} n(n - 1)} B \).

Remark 3. According to [7], a complete locally conformally flat Riemannian manifold \( (M, g) \) of dimension \( n \geq 3 \) is compact if its Ricci tensor is bounded and satisfies the inequality \( \text{Ric} \geq \epsilon s g \) for some constant \( \epsilon > 0 \), where \( s \) is the scalar curvature. Recall (see [20]) that in an orthonormal basis \( \{e_1, \ldots, e_n\} \) for an arbitrary \( x \in M \), in which the Ricci tensor is diagonalized, \( \text{Ric}(e_i, e_i) = R_{ii} \delta_{ii} \), the sectional curvature of a locally conformally flat manifold can be found from the equality (see [23])

\[
\text{sec}(e_i \wedge e_j) = \frac{1}{n-2} (R_{ii} + R_{jj} - \frac{s}{n-1}).
\]

By this, in the case of \( \text{Ric} \geq \epsilon s g \), we obtain the inequality

\[
\text{sec}(e_i \wedge e_j) \geq \frac{2 \epsilon s}{n-2} \left( \epsilon - \frac{1}{2(n-1)} \right).
\]

Thus, in the case \( \epsilon > \frac{1}{2(n-1)} \), the main result of [7] can be deduced from Corollary 1. Moreover, according to Theorem 1, on such a manifold there exists a metric of positive constant sectional curvature, thus, the manifold is diffeomorphic to a spherical space form.

Proof of Theorem 1. Let \( (M, g) \) be a compact \( n \)-dimensional \( (n \geq 4) \) locally conformally flat Riemannian manifold with positive sectional curvature, then by Lemma 1, its curvature operator is positive definite. On the other hand, in the case of an \( n \)-dimensional \( (n \geq 3) \) compact Riemannian manifold \( (M, g) \) with positive curvature operator, the Ricci flow deforms \( g \) to a metric of constant curvature provided that \( (M, g) \) has the a positive curvature operator \( \hat{R} \), see [8] (p. 15) (and another approach in [33]).

Proof of Theorem 2. Recall that the Riemannian curvature tensor of an arbitrary manifold admits pointwise orthogonal decomposition (see [17] (pp. 47–48)),

\[
Rm = W + V + U,
\]

(3)
where $W$ is the Weyl conformal curvature tensor, and the second and third terms are determined from the equalities $V = \frac{1}{n-2} \text{Ric} \odot g$ and $U = \frac{s}{n(n-1)} g \odot g$ (with the Kulkarni-Nomizu product $\odot$ of tensors). From (3) it follows that

$$\|Rm\|^2 = \|W\|^2 + \|V\|^2 + \|U\|^2.$$  

Thus, a locally conformally flat manifold is characterized by the equality $\|Rm\|^2 = \|V\|^2 + \|U\|^2$. According to [34], a metric on a compact Riemannian manifold of dimension $n \geq 4$ with positive scalar curvature and satisfying the following pinching condition:

$$\|V\|^2 + \|U\|^2 \leq \delta_n \|U\|^2,$$  

(4) where $\delta_4 = 1/5$, $\delta_5 = 1/10$ and $\delta_n = \frac{2}{(n-2)(n+1)}$ for all $n \geq 6$, can be deformed using the Ricci flow to a metric of positive constant sectional curvature. Note that $\|U\|^2 = \frac{2}{n(n-1)} s^2$, therefore, we can rewrite (4) in a visual and more convenient form:

$$\|Rm\|^2 \leq \frac{2(\delta_n + 1)}{n(n-1)} s^2.$$  

(5)

Generally, a Riemannian curvature tensor satisfies the inequality $\|Rm\|^2 \geq \frac{2}{n(n-1)} s^2$. One can prove for a locally conformally flat Riemannian manifold the following equality:

$$\|Rm\|^2 = \frac{4}{n-2} \|\text{Ric}\|^2 - \frac{2}{(n-1)(n-2)} s^2.$$  

(6)

Taking into account (6), the pinching condition (5) in the case of a locally conformally flat manifold takes an equivalent form:

$$\|\text{Ric}\|^2 \leq \frac{(n-2)\delta_n + 2(n-1)}{2n(n-2)} s^2.$$  

(7)

For $n \geq 6$, the above inequality can be rewritten as

$$\|\text{Ric}\|^2 \leq \frac{n}{n^2 - 4} s^2.$$  

For $n = 4$, (7) reads as $\|\text{Ric}\|^2 \leq (4/15) s^2$, and as $\|\text{Ric}\|^2 \leq (83/400) s^2$ for $n = 5$. □

**Proof of Theorem 3.** For an arbitrary point $x \in M$ in the tangent space $T_xM$, we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ that diagonalizes the Schouten tensor (1), then it follows from the equality $Rm = S \odot g$ (see also [23]) that

$$\text{sec}(e_i \wedge e_j) = \frac{1}{n-2} (S_{ii} + S_{jj}).$$  

(8)

By conditions, the Schouten tensor is positive definite, hence, by (8), the sectional curvature of the manifold is everywhere positive. In this case, by Lemma 1, the curvature operator $\bar{R}$ is positive definite. The latter means that the metric $g$ admits the Ricci flow deformation to a metric of positive constant sectional curvature (see [33]). □

**Proof of Theorem 4.** In dimension three, a Riemannian manifold $(M, g)$ is conformally flat if and only if its scalar curvature $s$ is constant and the Ricci tensor $\text{Ric}$ is a Codazzi tensor, i.e., $\nabla \text{Ric}$ is symmetric (see [17] (pp. 62, 435)). On the other hand, any Codazzi tensor with constant trace on a compact Riemannian manifold $(M, g)$ of positive sectional curvature is a constant multiple of $g$ (see [17] (p. 436)). At the same time, in dimension three, a Riemannian manifold has positive sectional curvature if and only if $\text{Ric} < \frac{1}{5} s \circ g$ (see [9] (p. 277)). Since in dimension three the Riemann curvature tensor of $(M, g)$ is determined by the Ricci tensor, we can conclude that Theorem 5 holds. □
Proof of Theorem 5. The S. Goldberg’s theorem states (see [23,35]) that a locally conformally flat homogeneous Riemannian manifold with Ricci tensor $\text{Ric} \geq 0$ is a locally symmetric space (see [17] (p. 297)). Moreover, it is known that a simply connected irreducible locally symmetric space is an Einstein manifold, i.e., $\text{Ric} = (s/n) g$. In this case, from (3) it follows

$$\text{Rm} = \frac{s}{n(n-1)} S \otimes g,$$

therefore, $(M,g)$ has constant sectional curvature. For dimension $n \geq 3$ here, only two cases are possible (see [2] (pp. 386–387)). In the first case, $(M,g)$ has positive constant scalar curvature. Recall that a homogeneous Einstein Riemannian manifold has a complete metric $g$ and is compact when $s > 0$ (see [17] (p. 189)). Since $(M,g)$ is simply connected, according to the classical Hopf result, it is a canonical sphere of certain radius (see [8] for more details). In the second case, $(M,g)$ is a Ricci flat symmetric space (see [2] (p. 387)). Moreover, a homogeneous Ricci flat manifold is flat, thus it is isometric to the product of the flat torus and Euclidean space (see [17] (p. 191)). There cannot be a flat torus factor, since the manifold is assumed to be simply connected. □

Proof of Theorem 6. The main result of [36] states (using the Ricci flow) that a complete Riemannian manifold $(M,g)$ of dimension $n \geq 4$ with positive bounded scalar curvature and satisfying the pinching condition:

$$\|W\|^2 + \|V\|^2 \leq \delta_n (1 - \epsilon)^2 \|U\|^2,$$  \hspace{1cm} (9)

for some $\epsilon \in (0,1)$ and $\delta_4 = 1/5$, $\delta_5 = 1/10$ and $\delta_n = \frac{2}{(n-2)(n+1)}$ for all $n \geq 6$, is compact. Assuming $\text{Rm} = Rm - U$, we rewrite (9) in a more convenient form for further use,

$$\| \text{Rm} \|^2 \leq \delta_n (1 - \epsilon)^2 \frac{2}{n(n-1)} s^2.$$  \hspace{1cm} (10)

One may show the following inequality for a locally conformally flat Riemannian manifold:

$$\| \text{Rm} \|^2 = \frac{4}{n-2} (\| \text{Ric} \|^2 - s^2 / n) = \frac{4}{n-2} \| \text{Ric} \|^2.$$  \hspace{1cm} (11)

By (11), the pinching condition (10) for a locally conformally flat Riemannian manifold takes the form

$$\| \text{Ric} \|^2 \leq \frac{(n-2)\delta_n (1 - \epsilon)^2}{2n(n-1)} s^2.$$  \hspace{1cm} (12)

For $n \geq 6$, from (12) we obtain

$$\| \text{Ric} \|^2 \leq \frac{(1 - \epsilon)^2}{n(n^2 - 1)} s^2.$$

For $n = 4$, (12) reads as $\| \text{Ric} \|^2 \leq \frac{(1 - \epsilon)^2}{60} s^2$, and as $\| \text{Ric} \|^2 \leq \frac{3(1 - \epsilon)^2}{40} s^2$ for $n = 5$. □

Proof of Theorem 7. Recall that a covariant symmetric 2-tensor $h$ is called a Codazzi tensor if $\nabla h$ is a covariant symmetric 3-tensor (see [17] (pp. 436–440); [2] (p. 350)). On the other hand, a covariant symmetric 2-tensor $h$ is harmonic if and only if it is a Codazzi tensor with constant trace (see [2] (p. 350)). It is known that the Ricci tensor on a locally conformal flat Riemannian manifold with constant scalar curvature is a Codazzi tensor, thus the Ricci tensor is harmonic. Side by side, it was proved in [11] that on a connected complete noncompact Riemannian manifold with non-negative sectional curvature there are no nonzero harmonic symmetric two-tensors satisfying the condition

$$\int_M \| h \|^2 \text{ vol}_g < \infty.$$
for some $q \geq 1$. This statement directly implies Theorem 7.

\section{Final Remarks}

Conformally flat manifolds are not classified in full generality, although a conformally flat compact simply connected Riemannian manifold is conformally equivalent to the canonical sphere (of the same dimension), see [17] (p. 62). More precisely, Kuiper’s theorem [37] states the following: If $(M,g)$ is a simply connected locally conformally flat manifold of dimension $n \geq 3$, then there is a conformal immersion $\varphi : M \to S^n$ into a canonical sphere $(S^n, g_0)$. If $M$ is compact then this map is a conformal diffeomorphism of $(M, g)$ with $(S^n, g_0)$. In this case, the metric $\varphi^* g_0$ is pointwise conformal to $g$, that is there exists a smooth scalar function $f$ such that $\varphi^* g_0 = e^{2f} g$. Then from the formula (see [17] (p. 59))

$$s = e^{2f} s_0 - 2(n - 1) \bar{\Delta} f + (n - 1)(n - 2) \| \nabla f \|^2,$$

where $s$ is the scalar curvature of $(M, g)$, it follows that

$$s(M) = s_0 \int_M e^{2f} d \text{vol}_g + (n - 1)(n - 2) \int_M \| \nabla f \|^2 d \text{vol}_g > 0.$$

Here $s(M) = \int_M s \text{dvol}_g$ is the total scalar curvature of $(M, g)$. As a result, from this formula we conclude that the inequality $s(M) \leq 0$ is an analytic obstacle for a simply connected and compact Riemannian manifold to be locally conformally flat. In particular, if the scalar curvature of $(M, g)$ is constant, then it is positive. By the above, the Betti numbers $b_0(M), \ldots, b_n(M)$ of an $n$-dimensional simply connected locally conformally flat Riemannian manifold $(M, g)$ are equal to the Betti numbers of an Euclidian sphere $(S^n, g_0)$, i.e.,

$$b_0(M) = b_n(M) = 1 \quad \text{and} \quad b_i(M) = \cdots = b_{n-1}(M) = 0.$$

This result completes the classical S. Bochner’s theorem (see [1] (pp. 79–80)).

Recall that conformal Killing $p$-forms or, in other words, conformal Killing-Yano $p$-tensors, have been defined on $n$-dimensional Riemannian manifolds ($1 \leq p \leq n - 1$) more than fifty years ago by S. Tachibana and T. Kashihara (see 38,39) as a natural generalization of conformal Killing vector fields (see 40). Since then these forms were extensively studied by many geometers. These considerations were motivated by existence of various applications for these forms (e.g., [41,42]).

The vector space of conformal Killing $p$-forms on an $n$-dimensional compact (without boundary) Riemannian manifold $(M, g)$ has a finite dimension $t_p(M)$ named the Tachibana number (e.g., [12,13,43]). These numbers $t_1(M), \ldots, t_{n-1}(M)$ are conformal scalar invariants of $(M, g)$ and satisfy the duality theorem: $t_p(M) = t_{n-p}(M)$. The theorem is an analog of the well known Poincaré duality theorem for the Betti numbers of compact $(M, g)$. It is known that $t_p(S^n) = \frac{(n+2)!}{(p+1)!(n-p+1)!}$ (see [13]). Therefore, for an $n$-dimensional compact simply connected locally conformal flat Riemannian manifold $(M, g)$ we have $t_p(M) = \frac{(n+2)!}{(p+1)!(n-p+1)!}$. On the other hand, according to [12], a compact locally conformally flat manifold with a negative definite Ricci tensor has all Tachibana numbers equal to zero. This is an analogue of S. Bochner’s theorem, with which we began our article.

\section*{Author Contributions:} methodology, J.M., V.R., S.S. and I.T.; investigation, J.M., V.R., S.S. and I.T.; writing–review and editing, J.M. All authors have read and agreed to the published version of the manuscript.

\section*{Funding:} This research was funded by the grant IGA PrF 2021030 at Palacky University in Olomouc.

\section*{Institutional Review Board Statement:} Not applicable.

\section*{Informed Consent Statement:} Not applicable.

\section*{Data Availability Statement:} Not applicable.

\section*{Conflicts of Interest:} The authors declare no conflict of interest.
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