Asymptotically Unbiased, Efficient, and Consistent Properties of the Bayes estimator in the Binomial Distribution with Prior Beta

E G Simanjuntak¹, Widiarti¹,a, D Kurniasari¹, and Amanto¹

¹Departement of Mathematics, Faculty of Mathematics and Natural Sciences, University of Lampung, Indonesia, Jl. Soemantri Brojonegoro No. 1, Bandar Lampung, Indonesia.

email: widiarti08@gmail.com

Abstract. This study will examine the characteristics of the Bayes estimator in the Binomial distribution with prior Beta theoretically and empirically. The theoretical result shows that the Bayes estimator in this distribution is an asymptotically unbiased and consistent, but inefficient estimator. Meanwhile, empirically, Bayes's estimator is an unbiased estimator, efficient, and consistent.

Keyword: Bayes estimator, unbiased, efficient, consistent, binomial distribution.

1. Introduction

The estimation method is a statistical tool that is useful for estimating the value of population parameters based on sample data. Basically, the estimation is divided into two, namely point estimation and interval estimation [1]. There are two methods to estimate the point, namely the classical method and the Bayes method. In the classical method, conclusions are based on information from a random sample drawn from the population. Meanwhile, the Bayes method uses or combines subjective knowledge about the probability distribution of the unknown parameter with the information obtained from the sample data. Subjective knowledge about the probability distribution of the unknown parameter is an initial distribution that provides information about a parameter called a prior. After the observations are made, the information in the prior distribution is combined with the information with the sample data through the Bayes theorem, and the results are expressed in a distribution called the posterior distribution which then becomes the basis for inference in the Bayes method [2]. In the Bayes method, all parameters in the model are treated as variables while in the classical method the parameters are considered as constants. So that if a case occurs, that is, in different situations and places of observation, the parameters change, then with the Bayes principle this problem can be solved [3]. The Bayes method has been developed to estimate small area parameters such as the Empirical Bayes (EB), Empirical Best Linear Unbiased Prediction (EBLUP), Hierarchical Bayes (HB), and Spatial EBLUP. Several studies related to the application of this method
are Widiarti et al. [4] using Spatial EBLUP method for estimating per capita expenditure in Lampung Province, Zou et al. [5] using EB method to be applied to highway safety, Pusponegoro et al. [6] using Spatial EBLUP method for estimating poverty, Najera [7] using HB method for estimating stunting, Buil-Gil [8] using Spatial EBLUP method for estimating perceived neighbourhood disorder, and Li [9] using EB method for estimate road safety.

This study will examine the characteristics of the Bayes estimator such as unbiased, minimum variance (efficiency), and consistency of the Binomial distribution with the prior Beta. The empirical study was carried out through a simulation using RStudio-1.1.463.

2. Method

2.1. Research Data
The research data were simulation data generated by RStudio-1.1.463. Data generated using the Binomial distribution \((m = 5, \text{ and } m = 10)\) with prior Beta. The sample sizes selected were 50, 100, 500, and 1000. These various sample sizes were used to prove whether the sample size had an effect on the characteristics studied. The simulation was carried out by selecting 3 pairs \((\alpha, \beta)\) which had different variance, namely \((1, 3), (10, 6), \text{ and } (20, 24)\) with each of the variations, namely 0.0375, 0.0138, 0.0054. The pairs \((\alpha, \beta)\) do not have a significant difference in variance because the variance of the beta distribution is not more than 1 and \(0 < x < 1\).

2.2. Bayes Method
In the classical approach, the parameter \(\theta\) is an unknown fixed quantity. Whereas in the Bayesian approach, \(\theta\) is seen as a quantity whose variation is described by the probability distribution (called the prior distribution). It is a subjective distribution, based on a person’s beliefs and formulated before data is retrieved. Then, the sample is taken from the population indexed \(\theta\) and the prior distribution is adjusted according to this sample information. The adjusted prior is called the posterior distribution. This adjustment is made using the Bayes rule [10].

2.3. Estimator Characteristics
An estimator is called best estimator if it has the following characteristics:

1. Unbiased Estimator
An estimator is unbiased if the mean of its sampling distribution is the true parameter value. That is, an estimator \(\hat{p}\) is unbiased if and only if
\[
E(\hat{p}) = \int \hat{p} f(\hat{p}|p) \, d\hat{p} = p
\]
where \(f(\hat{p}|p)\) is the sampling distribution of the estimator \(\hat{p}\) given the parameter \(p\) [11].

2. Efficient Estimator
Let \(\hat{p}\) be an unbiased estimator of a parameter \(p\) in the case of point estimation. \(\hat{p}\) is called an efficient estimator of \(p\) if and only if the variance of \(\hat{p}\) attains the Rao-Cramér lower bound. Let \(X_1, X_2, \ldots, X_n\) be iid with common pdf \(f(x;p)\) for \(p \in \Omega\), and \(\hat{p} = u(X_1, X_2, \ldots, X_n)\) be a statistic with mean \(E(\hat{p}) = E[u(X_1, X_2, \ldots, X_n)] = k(p)\), then
\[
Var(\hat{p}) \geq \frac{[k'(p)]^2}{n \, I(p)}
\]
If \(\hat{p} = u(X_1, X_2, \ldots, X_n)\) is an unbiased estimator of \(p\), so that \(k(p) = p\), then the Rao-Cramér inequality becomes [12]
\[
Var(\hat{p}) \geq \frac{1}{n \, I(p)}
\]
3. **Consistent Estimator**

An estimator is consistent if the sample size \(n\) is enlarged to near infinity, the estimator value will tend to approach the population parameter value \([1]\). An estimator \(\hat{p}\) of \(p\) based on a random sample of size \(n\) is said to be consistent if for any small \(\varepsilon > 0\), then
\[
\lim_{n \to \infty} P(|\hat{p} - p| < \varepsilon) = 1
\]
\[
\lim_{n \to \infty} P(|\hat{p} - p| \geq \varepsilon) = 0
\]

The following two conditions are sufficient to define consistency \([13]\).

1. \(\lim_{n \to \infty} E(\hat{p}) = p\)
2. \(\lim_{n \to \infty} Var(\hat{p}) = 0\)

### 3. Result and Discussion

In the Bayes estimator, the probability function \(f(x, p)\) is expressed by the conditional probability function \(f(x_i|p)\), so that for \(X_1, ..., X_n\) a random sample of a population with a Binomial distribution with parameter \(p\) can be written as follows:

\[
X_i \sim \text{Bin}(m, p) \iff f(x_i|p) = \binom{m}{x_i} p^{x_i}(1 - p)^{m-x_i}, x_i = 0, 1, 2, ..., n, i = 1, 2, ..., n, \quad \text{i lainnya}
\]

The prior distribution for \(X_i \sim \text{Bin}(m, p), i = 1, 2, ..., n\) is \(p \sim \text{Beta}(\alpha, \beta)\). So the probability function of the prior distribution is

\[
\pi(p) = \begin{cases} 
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1 - p)^{\beta-1}, & 0 < p < 1 \\
0 & \text{p lainnya}
\end{cases}
\]

The likelihood function of \(X_i \sim \text{Bin}(m, p)\):

\[
f(x_1, x_2, ..., x_n|p) = \prod_{i=1}^{n} \binom{m}{x_i} p^{x_i}(1 - p)^{m-x_i} = \prod_{i=1}^{n} \binom{m}{x_i} p^{\sum_{i=1}^{n} x_i}(1 - p)^{mn - \sum_{i=1}^{n} x_i}
\]

Joint probability density function of \(X_1, ..., X_n\) and \(p\) is:

\[
f(x_1, x_2, ..., x_n, p) = \pi(p) f(x_1, x_2, ..., x_n|p)
\]

\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \prod_{i=1}^{n} \binom{m}{x_i} p^{\sum_{i=1}^{n} x_i-1}(1 - p)^{mn + \sum_{i=1}^{n} x_i-1} dp
\]

Marginal function of \(X_i\) is:

\[
m(x_i) = \int_{0}^{\infty} f(x_i, p) dp
\]

\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[ \prod_{i=1}^{n} \binom{m}{x_i} \right] p^{\sum_{i=1}^{n} x_i-1}(1 - p)^{mn + \sum_{i=1}^{n} x_i-1} dp
\]

\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[ \prod_{i=1}^{n} \binom{m}{x_i} \right] B \left[ \left( \alpha + \sum_{i=1}^{n} x_i \right), \left( mn + \beta - \sum_{i=1}^{n} x_i \right) \right]
\]

The posterior distribution can be written as follows:

\[
\pi(p|x_i) = \frac{f(x_i, p)}{m(x_i)}
\]
\[
\begin{align*}
\Gamma(\alpha + \beta) & = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \prod_{i=1}^{n} \left(\frac{m}{x_i}\right)^{\alpha + \left(\sum_{i=1}^{n} x_i - 1\right)(1 - p)^{mn + \beta - \sum_{i=1}^{n} x_i}} \\
\Gamma(\alpha + \beta) & = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \prod_{i=1}^{n} \left(\frac{m}{x_i}\right)^{\left(\alpha + \sum_{i=1}^{n} x_i\right) (mn + \beta - \sum_{i=1}^{n} x_i)} \\
& = B\left[\left(\alpha + \sum_{i=1}^{n} x_i\right), (mn + \beta - \sum_{i=1}^{n} x_i)\right]
\end{align*}
\]

So, the Bayes estimator for \( p \) is:
\[
\hat{p} = E(p|x_i) = \int_{0}^{1} p \pi(p|x_i) \, dp
\]
\[
= \int_{0}^{1} p \frac{p^{\alpha + \sum_{i=1}^{n} x_i - 1}(1 - p)^{mn + \beta - \sum_{i=1}^{n} x_i - 1}}{B\left[\left(\alpha + \sum_{i=1}^{n} x_i\right), (mn + \beta - \sum_{i=1}^{n} x_i)\right]} \, dp
\]
\[
= \frac{\alpha + \sum_{i=1}^{n} x_i}{mn + \alpha + \beta}
\]

### 3.1. Estimator Characteristics

The study of the characteristics of the Bayes estimator will be carried out theoretically and empirically. Based on the above calculations, the Bayes estimator for the \( p \) parameter in the Binomial distribution with prior Beta is \( \hat{p} = (\alpha + \sum_{i=1}^{n} x_i)/(mn + \alpha + \beta) \). Furthermore, the characteristics of \( \hat{p} \) will be evaluated, namely unbiased, efficient, and consistent.

1. **Unbiased**
   
   The estimator \( \hat{p} \) is unbiased for \( p \) if \( E(\hat{p}) = p \). The following shows whether \( \hat{p} \) is an unbiased estimator for \( p \).
   
   \[
   E(\hat{p}) = E\left(\frac{\sum_{i=1}^{n} X_i + \alpha}{mn + \alpha + \beta}\right) = \frac{\alpha + mn}{mn + \alpha + \beta}
   \]
   
   Since \( E(\hat{p}) \neq p \), it can be concluded that \( \hat{p} \) is a biased estimator for \( p \). However, asymptotically, \( \hat{p} \) is an unbiased estimator because \( \lim_{n \to \infty} E(\hat{p}) = p \).

   \[
   \lim_{n \to \infty} E(\hat{p}) = \lim_{n \to \infty} \left(\frac{\alpha + mn}{mn + \alpha + \beta}\right) = \lim_{n \to \infty} \left(\frac{1}{mn + \alpha + \beta}\right) \alpha + \lim_{n \to \infty} \left(\frac{mn}{mn + \alpha + \beta}\right)p = 0 + p = p
   \]

2. **Efficient**

   The estimator \( \hat{p} \) is said to be efficient if the variance of \( \hat{p} \) reaches the lower bound Rao-Cramer. Next we will show the variance of Bayes estimator as follows.

   \[
   \text{var}(\hat{p}) = \text{var}\left(\frac{\sum_{i=1}^{n} X_i + \alpha}{mn + \alpha + \beta}\right)
   \]
   
   Since \( \text{var}(\alpha X) = \alpha^2 \text{var}(X) \), so

   \[
   \text{var}(\hat{p}) = \frac{\text{var}(\sum_{i=1}^{n} X_i)}{(mn + \alpha + \beta)^2}
   \]
   
   To solve for \( \text{var}(\sum_{i=1}^{n} X_i) \), it is necessary to first know the distribution of \( \sum_{i=1}^{n} X_i \). Let \( Y = \sum_{i=1}^{n} X_i \), then the probability density function of \( Y \) will be sought.

   \( X_i \sim \text{Binomial}(m,p) \)
   
   Let \( Y = \sum_{i=1}^{n} X_i \)

   \[
   M_Y(t) = E(e^{tY}) = E(e^{t \sum_{i=1}^{n} X_i}) = [M_X(t)]^n, \text{ since } X_i \sim \text{Binomial}(m,p)
   \]
\[
\begin{align*}
Y & \sim \text{Binomial}(mn, p) \\
f(y) &= \binom{mn}{x} p^x (1 - p)^{mn-x} \\
\text{So } \text{var}(\sum_{i=1}^{n} X_i) &= mn p (1-p), \text{ then} \\
\text{var}(\hat{p}) &= \frac{\text{var}(\sum_{i=1}^{n} X_i)}{(mn + \alpha + \beta)^2} = \frac{mn p (1-p)}{(mn + \alpha + \beta)^2}
\end{align*}
\]

It can be seen that the variance of \( \hat{p} \) is \( \frac{mn p (1-p)}{(mn + \alpha + \beta)^2} \), then the Rao-Cramer lower bound will be sought as follows.

\[
\begin{align*}
\ln f(x|p) &= \ln \binom{m}{x} + x \ln p + (m - x) \ln(1-p) \\
\frac{\partial \ln f(x|p)}{\partial p} &= \frac{x}{p} \cdot \frac{m-x}{1-p} \\
\left(\frac{\partial \ln f(x|p)}{\partial p}\right)^2 &= \left(\frac{x}{p} \cdot \frac{m-x}{1-p}\right)^2 \\
&= \frac{(x-mp)^2}{p^2(1-p)^2} \\
\text{var}(X) &= E(X - E(X))^2 = E(X - mp)^2 = mp (1-p)
\end{align*}
\]

\[
\begin{align*}
l(p) &= E \left( \frac{\partial \ln f(x|p)}{\partial p} \right)^2 \\
&= E \left( \frac{(x-mp)^2}{p^2(1-p)^2} \right) \\
&= \frac{\text{var}(X)}{p^2(1-p)^2} \\
&= \frac{mp (1-p)}{p^2(1-p)^2} \\
&= \frac{p(1-p)}{p^2(1-p)^2} \\
HB &= \frac{1}{n} l(p) = \frac{1}{n} \frac{m}{p(1-p)} = \frac{p(1-p)}{mn}
\end{align*}
\]

Since the variance of \( \hat{p} \) doesn’t reach the lower bound Rao-Cramer, it can be concluded that \( \hat{p} \) is an inefficient estimator.

3. Consistent

The estimator \( \hat{p} \) is consistent if \( \lim_{n\to\infty} \text{var}(\hat{p}) = 0 \). The following shows whether \( \hat{p} \) is a consistent estimator.

\[
\begin{align*}
\lim_{n\to\infty} \text{var}(\hat{p}) &= \lim_{n\to\infty} \frac{mn p (1-p)}{(mn + \alpha + \beta)^2} \\
&= \lim_{n\to\infty} \frac{mn - mn p}{(mn + \alpha + \beta)^2} \\
&= \lim_{n\to\infty} \frac{m^2 n^2 + 2mn \alpha + 2mn \beta + 2\alpha \beta + \alpha^2 + \beta^2}{mn - mn p} \\
&= \lim_{n\to\infty} \frac{n^2}{n^2} \frac{m^2 + \frac{2mn \alpha}{n^2} + \frac{2mn \beta}{n^2} + \frac{2\alpha \beta}{n^2} + \frac{\alpha^2}{n^2} + \frac{\beta^2}{n^2}}{rac{mn - mn p}{n^2}}
\end{align*}
\]
\[ \frac{0}{m^2} = 0 \]

Since \( \lim_{n \to \infty} \text{var}(\hat{p}) = 0 \), it can be concluded that \( \hat{p} \) is a consistent estimator.

3.2. Simulation

Based on 3 pairs \((\alpha, \beta)\), sample size \((n)\), and number of experiments \((m)\), it will be seen the effect of \((\alpha, \beta), n, \) and \(m\) on bias, variance and MSE value. Based on Table 1 and Figure 1, it can be seen that when 5 independent trials are carried out with \((\alpha, \beta) = (1,3), (10,6)\) will produce a bias value that is getting smaller / closer to 0 when the sample size is enlarged. Meanwhile, when \((\alpha, \beta) = (20,24)\) will produce in a fluctuating bias value. When conducted 10 independent trials, the three pairs \((\alpha, \beta)\) will produce a fluctuating bias value. So it can be concluded that the Bayesian estimator is asymptotically unbiased when it is carried out 5 independent trials with \((\alpha, \beta) = (1, 3)\) and \((10, 6)\), and is biased for the others.

Table 1. The bias value of the Bayes estimator.

| \(\alpha\) | \(\beta\) | \(n\) | \(m=5\) | \(m=10\) |
|---|---|---|---|---|
| 1 | 3 | 50 | 0.001014 | 0.000138 |
| | | 100 | 0.000942 | 0.000554 |
| | | 500 | 0.00028 | 3.3E-05 |
| | | 1000 | 4.6E-05 | 7.61E-05 |
| | | 50 | 0.001154 | 0.00017 |
| 10 | 6 | 100 | 0.001092 | 0.000172 |
| | | 500 | 0.000309 | 0.000488 |
| | | 1000 | 0.000123 | 0.000224 |
| 20 | 24 | 50 | 2.84E-06 | 0.001516 |
| | | 100 | 0.000318 | 5.46E-06 |
| | | 500 | 0.000618 | 5.92E-05 |
| | | 1000 | 0.000262 | 9.04E-05 |
**Figure 1.** Histogram of the bias value on several sample sizes and parameter values.

Based on Table 2 and Figure 2, for all pairs \((\alpha, \beta)\) will result in a smaller variance value when the sample size and trials is enlarged. When the pairs \((\alpha, \beta)\) fluctuate, the variance values tend not to change significantly. So it can be concluded that the Bayesian estimator is an efficient estimator.

| \(\alpha\) | \(\beta\) | \(n\) | m=5 Variance | m=10 Variance |
|----------|----------|-----|-------------|-------------|
| 1        | 3        | 50  | 0.000774    | 0.000328    |
|          |          | 100 | 0.000356    | 0.000194    |
|          |          | 500 | 7.37E-05    | 3.74E-05    |
|          |          | 1000| 4.11E-05    | 1.9E-05     |
|          |          | 50  | 0.000864    | 0.000429    |
| 10       | 6        | 100 | 0.000473    | 0.000232    |
|          |          | 500 | 9.58E-05    | 4.67E-05    |
|          |          | 1000| 4.66E-05    | 2.45E-05    |
|          |          | 50  | 0.000711    | 0.000426    |
| 20       | 24       | 100 | 0.00041     | 0.000222    |
|          |          | 500 | 9.27E-05    | 4.99E-05    |
|          |          | 1000| 4.82E-05    | 2.39E-05    |
Based on Table 3 and Figure 3, for all pairs \((\alpha, \beta)\) the MSE value will be smaller when the sample size and trials is enlarged. When the pairs \((\alpha, \beta)\) fluctuate, the MSE values tend not to change significantly. So it can be concluded that Bayes estimator is a consistent estimator.

**Figure 2.** Histogram of Bayes estimator’s variance on several sample sizes and parameter values.

**Table 3.** MSE value of Bayes estimator.

| \(\alpha\) | \(\beta\) | \(n\) | \(\text{MSE}_{m=5}\) | \(\text{MSE}_{m=10}\) |
|-----------|-----------|------|----------------|----------------|
| 1         | 3         | 50   | 0.000775       | 0.000328       |
| 1         | 3         | 100  | 0.000357       | 0.000195       |
| 1         | 3         | 500  | 7.38E-05       | 3.74E-05       |
| 1         | 3         | 1000 | 4.11E-05       | 1.9E-05        |
| 1         | 3         | 50   | 0.000865       | 0.000429       |
| 1         | 3         | 100  | 0.000474       | 0.000232       |
| 1         | 3         | 500  | 9.59E-05       | 4.7E-05        |
| 1         | 3         | 1000 | 4.66E-05       | 2.46E-05       |
| 1         | 3         | 50   | 0.000711       | 0.000428       |
| 1         | 3         | 100  | 0.000411       | 0.000222       |
| 1         | 3         | 500  | 9.31E-05       | 4.99E-05       |
| 1         | 3         | 1000 | 4.83E-05       | 2.39E-05       |

| 10        | 6         | 50   | 0.000775       | 0.000328       |
| 10        | 6         | 100  | 0.000357       | 0.000195       |
| 10        | 6         | 500  | 7.38E-05       | 3.74E-05       |
| 10        | 6         | 1000 | 4.11E-05       | 1.9E-05        |
| 10        | 6         | 50   | 0.000865       | 0.000429       |
| 10        | 6         | 100  | 0.000474       | 0.000232       |
| 10        | 6         | 500  | 9.59E-05       | 4.7E-05        |
| 10        | 6         | 1000 | 4.66E-05       | 2.46E-05       |
| 10        | 6         | 50   | 0.000711       | 0.000428       |
| 10        | 6         | 100  | 0.000411       | 0.000222       |
| 10        | 6         | 500  | 9.31E-05       | 4.99E-05       |
| 10        | 6         | 1000 | 4.83E-05       | 2.39E-05       |

| 20        | 24        | 50   | 0.000775       | 0.000328       |
| 20        | 24        | 100  | 0.000357       | 0.000195       |
| 20        | 24        | 500  | 7.38E-05       | 3.74E-05       |
| 20        | 24        | 1000 | 4.11E-05       | 1.9E-05        |
| 20        | 24        | 50   | 0.000865       | 0.000429       |
| 20        | 24        | 100  | 0.000474       | 0.000232       |
| 20        | 24        | 500  | 9.59E-05       | 4.7E-05        |
| 20        | 24        | 1000 | 4.66E-05       | 2.46E-05       |
| 20        | 24        | 50   | 0.000711       | 0.000428       |
| 20        | 24        | 100  | 0.000411       | 0.000222       |
| 20        | 24        | 500  | 9.31E-05       | 4.99E-05       |
| 20        | 24        | 1000 | 4.83E-05       | 2.39E-05       |
Figure 3. Histogram of MSE value Bayes estimator on several sample sizes and parameter values.

4. Conclusion
The result of this study show that the Bayes estimator in the Binomial distribution with prior Beta is
\[
\hat{p} = \frac{\alpha + \sum X_i}{mn + \alpha + \beta}
\]
which theoretically has asymptotically unbiased and consistent, but not efficient. But empirically, Bayes estimator is asymptotically unbiased, efficient, and consistent estimator.

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