Hard Lefschetz properties and distribution of spectra in singularity theory and Ehrhart theory

ANTOINE DOUAI
Université Côte d’Azur, CNRS, LJAD, FRANCE
Email address: antoine.douai@univ-cotedazur.fr

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Abstract

We discuss the distribution of the spectrum at infinity of a convenient and nondegenerate Laurent polynomial (singularity side) and the distribution of the Newton spectrum of a polytope (Ehrhart theory side). To this end, we study a hard Lefschetz property for Laurent polynomials and for polytopes and we give combinatorial criteria for this property to be true. This provides informations about a conjecture by Katzarkov-Kontsevich-Pantev.

1 Introduction

Let $P$ be a lattice polytope in $\mathbb{R}^n$ (the convex hull of a finite set in $N := \mathbb{Z}^n$). Define, for a nonnegative integer $\ell$, $L_P(\ell) := \text{Card}((\ell P) \cap N)$. Then $L_P$ is a polynomial in $\ell$ of degree $n$, the Ehrhart polynomial of $P$ and

$$1 + \sum_{m \geq 1} L_P(m)z^m = \frac{\delta_0 + \delta_1 z + \cdots + \delta_n z^n}{(1 - z)^{n+1}}$$

(1)

where the $\delta_j$’s are nonnegative integers. The vector $\delta_P = (\delta_0, \cdots, \delta_n)$ is the $\delta$-vector of the polytope $P$. The first result in the study of the distribution of the $\delta$-vector is probably the symmetry property $\delta_i = \delta_{n-i}$ for $i = 1, \cdots, n$, which is actually a characterization of reflexive polytopes [13]. The second one concerns the unimodality of the $\delta$-vector of a reflexive polytope: taking into account the previous symmetry property, one could expect $\delta_0 \leq \delta_1 \leq \cdots \leq \delta_{[n/2]}$ and $\delta_{[n/2]} \geq \delta_{[n/2]+1} \geq \cdots \geq \delta_n$. This is indeed what happens in dimension less or equal than five [12], but this unimodality may fail in dimension greater or equal to six [17], [18]. On the other hand, singularity theory meets Ehrhart theory by the means of the $\delta$-vector: the spectrum at infinity of a tame Laurent polynomial determines the $\delta$-vector of its Newton polytope and both coincide if the latter is reflexive [7]. This interplay encourages us also to study the unimodality (and more generally, the distribution) of the spectrum at infinity of a regular function.

Classically, unimodality can be seen as a combinatorial application of the hard Lefschetz theorem, see [22] for instance where it is shown that the Poincaré polynomial of a smooth complex projective variety is unimodal, and we are naturally led to study a hard Lefschetz property for regular functions (singularity side) and for polytopes (Ehrhart theory side). On the singularity
side, the hard Lefschetz property for a Laurent polynomial \( f \) is given by the multiplication by \( f \) on a graded Jacobi ring (this is an old story, see [20] for instance). The hard Lefschetz property for a polytope \( P \) is provided by the hard Lefschetz property for the orbifold cohomology of the orbifold associated with \( P \) by the work of Borisov, Chen and Smith [3]. Both are related by a mirror theorem. This is detailed in Section 3, where we also give a combinatorial criterion for these hard Lefschetz properties to be satisfied. For instance, we get (see Theorem 3.5 and Proposition 3.8):

**Theorem 1.1** Let \( P \) be a full dimensional lattice polytope containing the origin as an interior point and let \( \Sigma_P \) be the fan over the faces of \( P \). For a \( n \)-dimensional cone \( \sigma \in \Sigma_P \), let \( \text{Box}(\sigma) \) be the set of \( v \in \mathbb{N} \) such that \( v = \sum_{\rho_i \subseteq \sigma} \rho_i b_i \) for some \( 0 \leq q_i < 1 \), where \( \rho_i \) denotes the ray generated by the vertex \( b_i \) of \( P \). Then \( P \) satisfies the hard Lefschetz property if and only if

\[
[\nu(v)] = (\text{dim} \sigma(v) - 1)/2 \text{ if } \nu(v) \not\in \mathbb{N}
\]

and

\[
\nu(v) = \text{dim} \sigma(v)/2 \text{ if } \nu(v) \in \mathbb{N}
\]

for all \( v \in \cup \sigma \text{Box}(\sigma) \) (the union is taken over all the \( n \)-dimensional cones \( \sigma \in \Sigma_P \), where \( \sigma(v) \) denotes the smallest cone of \( \Sigma_P \) containing \( v \), \( \nu \) is the Newton function of \( P \) and \([a]\) denotes the integral part of \( a \).

When applied to a simplex \( \Delta \), this criterion reduces to an arithmetic condition on its weight (the *weight* of a simplex \( \Delta := \text{conv}(v_0, \cdots, v_n) \) is the tuple \( Q(\Delta) = (q_0, \cdots, q_n) \), arranged by increasing order, where \( q_i := |\det(v_0, \cdots, \hat{v}_i, \cdots, v_n)| \) for \( i = 0, \cdots, n \), the simplex \( \Delta \) is said to be *reduced* if \( \gcd(q_0, \cdots, q_n) = 1 \) and it is moreover *reflexive* if \( q_i \) divides \( \mu := q_0 + \cdots + q_n \) for \( i = 0, \cdots, n \), see Section 4.1 for details). To give an idea, here is the kind of statement that we can get at the end (it should be emphasized that the simplex \( \Delta \) satisfies the hard Lefschetz property if and only if the Laurent polynomial \( f_{\Delta}(u) = \sum_{i=0}^n u^{q_i} \) satisfies the hard Lefschetz property):

**Proposition 1.2** Assume that the reduced and reflexive simplex \( \Delta \) of weight \( (q_0, \cdots, q_n) \) satisfies the hard Lefschetz property. Then,

\[
\frac{2\mu}{q_n} = n + 1 + m(q_n)
\]

(2)

where \( m(q_n) \) denotes the multiplicity of \( q_n \) in the tuple \( (q_0, \cdots, q_n) \).

See Remark 4.3. For example, it is readily seen that Equation (2) fails for the three dimensional reflexive and reduced simplex \( \Delta \) of weight \( (1, 1, 1, 3) \): this simplex does not satisfy the hard Lefschetz property. Actually, we have a stronger statement (a necessary and sufficient condition, see Proposition 4.1 and Corollary 4.2), and it follows from our computations that the hard Lefschetz properties are not common at all: for instance, we check that the hard Lefschetz property is true for 5 out the 147 four dimensional reduced and reflexive simplices described in [4].

This has an interpretation in Hodge theory: as noticed in [19], a Laurent polynomial \( f \) satisfies the hard Lefschetz property if and only if the mixed Hodge structure produced by the Laplace transform of its Gauss-Manin system is of Hodge-Tate type. As a by-product, we get some informations about a conjecture by Katzarkov-Kontsevich-Pantev [15] Conjecture 3.6 (we use here
the version of the conjecture given in \cite{KP2007} and we will refer to it as the KKP conjecture). For instance, we deduce from the criteria alluded to above a necessary and sufficient condition for a Laurent polynomial $f$ whose Newton polytope is a reflexive simplex to satisfy the KKP conjecture, see Proposition \ref{prop:necessary} We hope that this will be useful in order to understand more clearly which polynomials satisfy this conjecture (this question may arise).

About the distribution of the $\delta$-vector of a polytope (and this was after all our starting point), it turns out that the hard Lefschetz properties studied in this paper give the unimodality of weighted $\delta$-vectors in the sense of \cite{St}. In particular, the $\delta$-vector of a reflexive polytope which satisfies the hard Lefschetz property is unimodal. This is discussed in Section 5.

2 Spectra

In this section, we recall some results from \cite{GP}. Let $N$ be the lattice $\mathbb{Z}^n$ and let $P \subset N_{\mathbb{R}}$ be a full dimensional lattice polytope containing the origin as an interior point. We will denote by $V(P)$ the set of its vertices. Let $\Sigma_P$ be the fan in $N_{\mathbb{R}}$ obtained by taking the cones over the faces of $P$

We will always assume that $\Sigma_P$ is simplicial and we will denote by $X_{\Sigma_P}$ the complete toric variety associated with the fan $\Sigma_P$. The Newton function $\nu : N_{\mathbb{R}} \to \mathbb{R}$ which takes the value 1 at the vertices of $P$ and which is linear on each cone of $\Sigma_P$. The Milnor number of $P$ is $\mu_P := n! \text{vol}(P)$ where the volume $\text{vol}(P)$ is normalized such that the volume of the cube is equal to 1. We define the Newton spectrum of $P$ as

$$\text{Spec}_P(z) := (1 - z)^n \sum_{v \in N} z^{\nu(v)}. \quad (3)$$

Let $f(u) = \sum_{m \in \mathbb{Z}^n} a_m u^m$ be a Laurent polynomial defined on $(\mathbb{C}^*)^n$. The Newton polytope $P$ of $f$ is the convex hull of $\text{supp} f := \{m \in \mathbb{Z}^n, a_m \neq 0\}$ in $\mathbb{R}^n$. We will always assume in this text that $f$ is convenient and nondegenerate in the sense of Kouchnirenko \cite{Kouchnirenko} (all the definitions used in this paper are also detailed in \cite{GP}).

Let $A_f := \mathcal{B}/\mathcal{L}$ where $\mathcal{B} := \mathbb{C}[u_1, u_1^{-1}, \ldots, u_n, u_n^{-1}]$ and $\mathcal{L} := \langle u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n} \rangle$ is the ideal generated by the partial derivative $u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n}$ of $f$. We define an increasing filtration $\mathcal{N}_{\alpha}$ on $\mathcal{B}$, indexed by $\mathbb{Q}$, by setting

$$\mathcal{N}_{\alpha} \mathcal{B} := \{g \in \mathcal{B}, \text{supp}(g) \in \nu^{-1}(]-\infty; \alpha])\}$$

where $\nu$ is the Newton function of the Newton polytope $P$ of $f$ and $\text{supp}(g) = \{m \in \mathbb{N}^n, a_m \neq 0\}$ if $g = \sum_{m \in \mathbb{N}^n} a_m u^m \in \mathcal{B}$. By projection, the Newton filtration $\mathcal{N}_{\alpha}$ on $\mathcal{B}$ induces the Newton filtration $\mathcal{N}_{\alpha}$ on $A_f$ and the spectrum at infinity of $f$ is given by

$$\text{Spec}_f(z) = \sum_{\alpha \in \mathbb{Q}} \text{dim}_{\mathbb{C}}(\text{gr}_\alpha^\mathcal{N} A_f) z^\alpha. \quad (4)$$

Both spectra are related: if $f$ is a convenient and nondegenerate Laurent polynomial with Newton polytope $P$, we have $\text{Spec}_f(z) = \text{Spec}_P(z)$ \cite[Corollary 2.2]{GP}.

In this paper, we are interested in the distribution of $\text{Spec}_f(z)$ and $\text{Spec}_P(z)$ and it will be useful to decide when these spectra are honest polynomials. Recall that a lattice polytope $P$ is reflexive if it contains the origin as an interior point and if its polar polytope $P^\circ := \{y \in M_{\mathbb{R}}, \langle y, x \rangle \leq 1 \text{ for all } x \in P\}$ is a lattice polytope.

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Proposition 2.1 [7, Proposition 5.1] Let $P$ be a full dimensional lattice polytope containing the origin as an interior point. The following are equivalent:

1. $\text{Spec}_P(z)$ is a polynomial,
2. $P$ is reflexive,
3. $\text{Spec}_P(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$ where $(\delta_0, \cdots, \delta_n)$ is the $\delta$-vector of $P$. 

Thanks to the identification alluded to above, we get:

Corollary 2.2 Let $f$ be a convenient and nondegenerate Laurent polynomial. Then its spectrum at infinity $\text{Spec}_f(z)$ is a polynomial if and only if its Newton polytope $P$ is reflexive. 

3 The hard Lefschetz property for polynomials and polytopes

Let us begin with the singularity side. Let $f$ be a convenient and nondegenerate Laurent polynomial defined on $(\mathbb{C}^*)^n$. The multiplication by $f$ induces maps

$$[f] : \text{gr}^N_\alpha A_f \longrightarrow \text{gr}^N_{\alpha+1} A_f$$

for $\alpha \in \mathbb{Q}$. The following definition can be found in [19] (see for instance [20, Section 7] for a motivation):

Definition 3.1 Let $f$ be a convenient and nondegenerate Laurent polynomial on $(\mathbb{C}^*)^n$. We will say that $f$ satisfies the hard Lefschetz property (HL) if the multiplication by $f$ induces isomorphisms

$$[f]^{n-1-2k} : \text{gr}^N_{\alpha+k} A_f \longrightarrow \text{gr}^N_{\alpha+n-1-k} A_f$$

for $0 \leq k \leq [(n-1)/2]$ and $\alpha \in [0,1]$ and

$$[f]^{n-2k} : \text{gr}^N_k A_f \longrightarrow \text{gr}^N_{n-k} A_f$$

for $0 \leq k \leq [n/2]$.

Let now $P$ be a full-dimensional lattice polytope containing the origin as an interior point and let

- $X_P$ be the Deligne-Mumford stack associated by [3, Section 3] with the stacky fan $\Sigma_P := (\mathbb{Z}^n, \Sigma_P, V(P))$ (we will refer to it as the stack of $P$; recall that we assume that the complete toric variety $X_{\Sigma_P}$ is simplicial and that $V(P)$ denotes the set of the vertices of $P$),
- $I_{X_P} = \bigsqcup_{\ell \in F} X_\ell$ be the decomposition into connected components of the inertia orbifold of $X_P$,
- $H_{2a}(X_P, \mathbb{C}) := \oplus_{\ell \in F} H^{2(\alpha - \text{age}(X_\ell))}(X_\ell, \mathbb{C})$ be the orbifold cohomology groups of $X_P$ where $\text{age}(X_\ell)$ the age of the sector $X_\ell$ (see [11, Section 4.1 and Definition 4.8]),
- $f_P$ be the Laurent polynomial $f_P(u) := \sum_{b \in V(P)} u^b$. 

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The following wonderful result is due to [3], with a little help from Kouchnirenko [14] (the orbifold cohomology is equipped with the orbifold cup-product); notice that we will mainly work with simplices, in which case the result can be also found in [10]:

**Proposition 3.2** [3] There is an isomorphism of \( \mathbb{Q} \)-graded rings

\[
\varphi : H_{orb}^2(\mathcal{X}_P, \mathbb{C}) \cong \text{gr}_N^* A_f. 
\]

**Proof.** Notice first that \( f_P \) is convenient (because \( P \) contains the origin as an interior point) and nondegenerate (thanks to the simpliciality assumption) with respect to its Newton polytope \( P \). By [14, Théorème 4.1], the map \( \partial : (\mathbb{C}[u, u^{-1}])^n \to \mathbb{C}[u, u^{-1}] \) defined by \( \partial(b_1, \cdots, b_n) = b_1 u \frac{\partial f}{\partial u_1} + \cdots + b_n u \frac{\partial f}{\partial u_n} \) is strict with respect to the Newton filtration. Hence, and thanks to the properties of the Newton filtration [14], [6], the graded ring \( \text{gr}_N^* A_f \) is nothing but the "Stanley-Reisner presentation" of \( \mathcal{X} \) given by the right hand side of [3, Theorem 1.1] and the result follows from loc. cit. \( \square \)

It should be emphasized that Proposition 3.2 provides an isomorphism of rings, and this depends on the special form of \( f_P \), from which we also get \( \varphi^{-1}([f_P]) \in H^2(\mathcal{X}_0, \mathbb{C}) \) where \( \mathcal{X}_0 \) denotes the untwisted sector.

According to [11, Proposition 3.2], the cohomology \( H^*(\mathcal{X}_\ell, \mathbb{C}) \) of the twisted sector \( \mathcal{X}_\ell \) is a \( H^*(\mathcal{X}_0, \mathbb{C}) \)-module under the orbifold cup-product, and this module structure is related to the standard cup-product on \( H^*(\mathcal{X}_\ell, \mathbb{C}) \). We now give the following counterpart of Definition 3.1:

**Definition 3.3** Let \( P \) be a full dimensional lattice polytope in \( \mathbb{R}^n \) containing the origin as an interior point. We will say that \( P \) satisfies the hard Lefschetz property (HL) if there exists \( \omega \in H^2(\mathcal{X}_0, \mathbb{C}) \) such that the orbifold product by \( \omega \) induces isomorphisms

\[
\omega^{n-2k} : H_{orb}^{2(\alpha+k)}(\mathcal{X}_P, \mathbb{C}) \cong H_{orb}^{2(\alpha+n-1-k)}(\mathcal{X}_P, \mathbb{C}) \quad (7)
\]

for \( 0 \leq k \leq \lfloor (n-1)/2 \rfloor \) and \( \alpha \in ]0, 1[ \) and

\[
\omega^{-2k} : H_{orb}^{2k}(\mathcal{X}_P, \mathbb{C}) \cong H_{orb}^{2(n-k)}(\mathcal{X}_P, \mathbb{C}) \quad (8)
\]

for \( 0 \leq k \leq \lfloor n/2 \rfloor \).

**Remark 3.4** Assume that the spectrum at infinity of \( f \) is a polynomial (see Corollary 2.2). Then \( f \) satisfies (HL) if and only if the multiplication by \( f \) induces isomorphisms

\[
[f]^{n-2k} : \text{gr}_k^* A_f \cong \text{gr}_{n-k}^* A_f \quad (9)
\]

for \( 0 \leq k \leq \lfloor n/2 \rfloor \). On the other hand, a reflexive polytope \( P \) satisfies (HL) if and only if there exists \( \omega \in H^2(\mathcal{X}_0, \mathbb{C}) \) such that the orbifold product by \( \omega^{-2k} \) induces isomorphisms

\[
\omega^{-2k} : H_{orb}^{2k}(\mathcal{X}_P, \mathbb{C}) \cong H_{orb}^{2(n-k)}(\mathcal{X}_P, \mathbb{C}) \quad (10)
\]

for \( 0 \leq k \leq \lfloor n/2 \rfloor \).
We now give criteria for the hard Lefschetz property to be true. Let $P$ be a full dimensional lattice polytope in $\mathbb{R}^n$ containing the origin as an interior point and let $X_P$ be the stack of $P$. For $\ell \in F$, we put $\ell^{-1} := I(\ell)$ where $I$ is the involution on $F$ induced by the involution on the inertia orbifold $I_{X_P} = \prod_{\ell \in F} X^{\ell}_I$ defined in [1] (4.3). In this text, $[x]$ denotes the integral part of $x$.

**Theorem 3.5** The polytope $P$ satisfies (HL) if and only if

$$[\text{age}(X^{\ell}_I)] = [\text{age}(X^{\ell-1}_I)]$$

for all $\ell \in F$.

**Proof.** In what follows, we put $i_{\ell} := \text{age}(X^{\ell}_I)$. We first assume that $P$ satisfies the hard Lefschetz property ([7]). Because the orbifold product by $\omega \in H^2(X_0, \mathbb{C})$ preserves the cohomology of each sector $X^{\ell}_I$, we have the isomorphisms

$$\omega^{n-1-2k} : H^{2(\alpha+k)}(X^{\ell}_I, \mathbb{C}) \cong H^{2(\alpha+n-1-k)}(X^{\ell}_I, \mathbb{C})$$

for $\ell \in F$, $\alpha \in [0,1]$ and $k \leq [(n-1)/2]$. By the definition of the orbifold cohomology, we get in particular the isomorphisms

$$\omega^{n-1-2k} : H^{2(\alpha+k-i_{\ell})}(X^{\ell}_I, \mathbb{C}) \cong H^{2(\alpha+n-1-k-i_{\ell})}(X^{\ell}_I, \mathbb{C}). \tag{12}$$

Since (12) is void if $\alpha - i_{\ell} \notin \mathbb{Z}$, we may assume that $\alpha - i_{\ell} \in \mathbb{Z}$ and therefore that $\alpha = i_{\ell} - [i_{\ell}]$ (recall that $\alpha \in [0,1]$). Because $n_{\ell} := \dim X^{\ell}_I = n - i_{\ell} - i_{\ell-1}$ (see for instance [1] Lemma 4.6)), the isomorphisms (12) are in turn equivalent to

$$\omega^{n-1-2k} : H^{2(k-[i_{\ell}])}(X^{\ell}_I, \mathbb{C}) \cong H^{2(n_{\ell}-1-k+i_{\ell}+i_{\ell-1}-[i_{\ell}])}(X^{\ell}_I, \mathbb{C}).$$

Because $i_{\ell} + i_{\ell-1} \in \mathbb{Z}$ and $i_{\ell} \notin \mathbb{Z}$, we have $i_{\ell} + i_{\ell-1} = [i_{\ell}] + [i_{\ell-1}] + 1$ and we finally get the isomorphisms

$$\omega^{n-1-2k} : H^{2(k-[i_{\ell}])}(X^{\ell}_I, \mathbb{C}) \cong H^{2(n_{\ell}-1-k+[i_{\ell}]+1-[i_{\ell}])}(X^{\ell}_I, \mathbb{C}). \tag{13}$$

Because $\dim X^{\ell}_I = n - i_{\ell} - i_{\ell-1}$, we have $i_{\ell} + i_{\ell-1} \leq n$ and therefore we may assume that $[i_{\ell}] \leq [(n-1)/2]$; thus, we can put $k = [i_{\ell}]$ in (13) in order to get the isomorphism

$$H^0(X^{\ell}_I, \mathbb{C}) \cong H^{2(n_{\ell}-[i_{\ell}]+[i_{\ell-1}])}(X^{\ell}_I, \mathbb{C}).$$

It follows that $[i_{\ell-1}] - [i_{\ell}] \leq 0$. In particular, we have also $[i_{\ell-1}] \leq [(n-1)/2]$ and by symmetry we get $[i_{\ell}] - [i_{\ell-1}] \leq 0$. This shows that $[i_{\ell}] = [i_{\ell-1}]$.

The result is shown similarly if $P$ satisfies the hard Lefschetz property ([8]) and we get the converse going backward, applying the hard Lefschetz theorem for the ordinary cohomology of $X^{\ell}_I$ provided by [24] Theorem 1.13 and using [1] Proposition 3.2. \hfill \Box

**Remark 3.6** Theorem 3.5 has been suggested by [11]. If $P$ is reflexive, the ages are integers and $P$ satisfies (HL) if and only if $\text{age}(X^{\ell}_I) = \text{age}(X^{\ell-1}_I)$. This result is stated in loc. cit.

**Corollary 3.7** Let $P$ be as above. If $f_P$ satisfies (HL) then $[\text{age}(X^{\ell}_I)] = [\text{age}(X^{\ell-1}_I)]$ for all $\ell \in F$. The converse holds true if $\dim_{\mathbb{C}} H^2(X_0, \mathbb{C}) = 1$. 

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Proof. Assume first that $f_P$ satisfies (HL). We use Proposition 3.2 and Theorem 3.5 in order to get the conditions on the ages. Conversely, the equality of the ages shows that the hard Lefschetz property hold for $P$ (again by Theorem 3.5) and the assumption ensures that $\varphi^{-1}([f_P])$ is a non-zero multiple of the cohomology class $\omega$ in Definition 3.3. 

Fortunately, we have a combinatorial description of condition (11). Recall the stacky fan $\Sigma_P = (\mathbb{Z}^n, \Sigma_P, \mathcal{V}(P))$ of $P$. For $\sigma$ a $n$-dimensional cone in the fan $\Sigma_P$, we denote by $\text{Box}(\sigma)$ the set of the elements $v \in \mathbb{N}$ such that $v = \sum_{\rho_i \subseteq \sigma} q_i b_i$ for some $0 \leq q_i < 1$ ($\rho_i$ is the ray generated by the vertex $b_i$ of $P$). Let $\text{Box}(\Sigma_P)$ be the union of $\text{Box}(\sigma)$ for all $n$-dimensional cones $\sigma \in \Sigma_P$. By [3, Proposition 4.7 and Remark 5.4], we have the following facts:

- the sectors $\mathcal{X}_v$ are parametrized by $v \in \text{Box}(\Sigma_P)$,
- $\dim \mathcal{X}_v = n - \dim \sigma(v)$,
- $\text{age}(\mathcal{X}_v) = \nu(v)$

where $\sigma(v)$ the smallest cone of $\Sigma_P$ containing $v$ and $\nu$ is the Newton function of $P$. The following is Theorem 1.1 in the introduction:

**Proposition 3.8** Let $P$ be a full dimensional polytope containing the origin as an interior point. Then (11) holds true if and only if

$$[\nu(v)] = (\dim \sigma(v) - 1)/2 \text{ if } \nu(v) \notin \mathbb{N}$$

and

$$\nu(v) = \dim \sigma(v)/2 \text{ if } \nu(v) \in \mathbb{N}$$

for all $v \in \text{Box}(\Sigma_P)$.

**Proof.** Let $v \in \text{Box}(\Sigma_P)$. Because $\dim \mathcal{X}_v = n - \text{age}(\mathcal{X}_v) - \text{age}(\mathcal{X}_{v-1})$ and $\dim \mathcal{X}_v = n - \dim \sigma(v)$, we have $\text{age}(\mathcal{X}_v) + \text{age}(\mathcal{X}_{v-1}) = \dim \sigma(v)$. Therefore, $[\text{age}(\mathcal{X}_v)] + [\text{age}(\mathcal{X}_{v-1})] = \dim \sigma(v) - 1$ if $\text{age}(\mathcal{X}_v) \notin \mathbb{N}$ and $[\text{age}(\mathcal{X}_v)] + [\text{age}(\mathcal{X}_{v-1})] = \dim \sigma(v)$ if $\text{age}(\mathcal{X}_v) \in \mathbb{N}$. Now, we use the fact that $\text{age}(\mathcal{X}_v) = \nu(v)$. 

**Remark 3.9** Assume that each set of vertices of the same $(n-1)$-dimensional face of $P$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Then the toric variety $X_{\Sigma_P}$ is smooth and $\Sigma_P$ is the canonical (in the sense of [3]) stacky fan associated with $\Sigma_P$. We have $X_P = X_{\Sigma_P}$ and equality (11) holds true since there are no twisted sectors. This matches with [14, Proposition 3.4].

## 4 Application to simplices

We apply the results of Section 3 to simplices and we deduce some consequences about the KKP conjecture.
4.1 Hard Lefschetz property for simplices

In this text, we will say that the polytope $\Delta := \text{conv}(v_0, \cdots, v_n)$ is a simplex if its vertices $v_i$ belong to the lattice $\mathbb{Z}^n$ and if it contains the origin as an interior point. We will denote by $f_\Delta$ the Laurent polynomial

$$f_\Delta(u) = \sum_{i=0}^{n} u^{v_i}$$

(14)
on $(\mathbb{C}^*)^n$ where $u^b := u_1^{b_1} \cdots u_n^{b_n}$ if $b = (b_1, \cdots, b_n) \in \mathbb{N}^n$. The weight of a simplex $\Delta$ is the tuple $Q(\Delta) = (q_0, \cdots, q_n)$, arranged by increasing order, where

$$q_i := |\det(v_0, \cdots, \hat{v_i}, \cdots, v_n)|$$

for $i = 0, \cdots, n$. We have $\mu_{\Delta} = q_0 + \cdots + q_n$ (for short we will denote $\mu_{\Delta}$ by $\mu$). The simplex $\Delta$ is said to be reduced if $\gcd(q_0, \cdots, q_n) = 1$. In this case, we set

$$F := \left\{ \frac{\ell}{q_i} | 0 \leq \ell \leq q_i - 1, \ 0 \leq i \leq n \right\}$$

and we denote by $f_1, \cdots, f_k$ the elements of $F$ arranged by increasing order. We also define $d_i := \text{Card}\{j| q_j f_i \in \mathbb{Z}\}$. Notice that $f_1 = 0$ and $d_1 = n + 1$.

According to [8, Section 3.4], the sectors of $X_{\Delta}$ are labelled by the set $F$, the age of the sector $X_{f_\ell}$ is $\text{age}(X_{f_\ell}) = 0$ and

$$\text{age}(X_{f_\ell}) = d_1 + \cdots + d_{\ell-1} - \mu_{f_\ell}$$

(15)

for $\ell = 2, \cdots, k$ (where it is understood that $\text{age}(X_{f_k}) = d_1 - \mu_{f_2}$).

By Remark [3.9], the hard Lefschetz property is true if $(q_0, \cdots, q_n) = (1, \cdots, 1)$. The following criterion works for the remaining cases:

**Proposition 4.1** Let $\Delta$ be a reduced simplex of weight $(q_0, \cdots, q_n)$ such that $q_n \geq 2$. Then $\Delta$ satisfies (HL) if and only if $f_\Delta$ satisfies (HL). And this happens if and only if

$$[d_1 + \cdots + d_{i-1} - \mu f_i] = \frac{d_1 - d_i - 1}{2}$$

for $i \geq 2$ and $d_1 + \cdots + d_{i-1} - \mu f_i \notin \mathbb{Z},$

$$d_1 + \cdots + d_{i-1} - \mu f_i = \frac{d_1 - d_i}{2}$$

for $i \geq 2$ and $d_1 + \cdots + d_{i-1} - \mu f_i \in \mathbb{Z}$.

**Proof.** We deduce the first assertion from Corollary [3.7]. The remaining ones follow from Proposition [3.8] and equality (15) because $\dim X_{f_\ell} = d_1 - 1$, see [8].

Recall that a reduced simplex $\Delta$ of weight $(q_0, \cdots, q_n)$ is reflexive if and only if $q_i$ divides $\mu = q_0 + \cdots + q_n$ for $i = 0, \cdots, n$, see [4].
Corollary 4.2 A reduced and reflexive simplex $\Delta$ of weight $(q_0, \cdots, q_n)$ with $q_n \geq 2$ satisfies (HL) if and only if

$$2\mu f_i = d_1 + 2(d_2 + \cdots + d_{i-1}) + d_i$$

for $i = 2, \cdots, k$.

Proof. Follows from Proposition 4.1 because the ages are nonnegative integers if $\Delta$ is reflexive. □

Remark 4.3 Assume that the reduced and reflexive simplex $\Delta$ satisfies (HL). Then, if $q_n > 1$, it follows from Corollary 4.2 that we must have

$$\frac{2\mu}{q_n} = n + 1 + m(q_n)$$

(16)

where $m(q_n)$ denotes the multiplicity of $q_n$ in the tuple $(q_0, \cdots, q_n)$ because $f_2 = 1/q_n$ and $d_2 = m(q_n)$: this is Proposition 1.2 in the introduction. Most of the time it will be enough to show that this necessary condition does not hold in order to show that the hard Lefschetz condition (HL) fails. For instance, let us consider the three dimensional reflexive and reduced simplex $\Delta$ of weight $(1, 1, 1, 3)$. We have $\mu = 6$, $q_n = 3$, $m(q_n) = 1$ and Equation (16) is $4 = 5$: $\Delta$ does not satisfy (HL).

Example 4.4 Reduced and reflexive simplices are classified up to dimension four in [4]. Using Corollary 4.2 and/or Remark 4.3 we get the following statements:

- two dimensional reduced and reflexive simplices satisfy the hard Lefschetz property;
- if $n = 3$, there are 14 reduced and reflexive simplices (up to unimodular transformations) and the hard Lefschetz property hold only for the simplices with weights $(1, 1, 1, 1)$ and $(1, 1, 2, 2)$;
- if $n = 4$, there are 147 reduced and reflexive simplices (up to unimodular transformations) and the hard Lefschetz property hold only for the simplices $\Delta$ with weights $(1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 2)$, $(1, 1, 2, 2, 2)$, $(1, 2, 3, 3, 3)$ and $(1, 2, 2, 3, 4)$.

4.2 Application to KKP Conjecture for reflexive simplices

Let $f$ be a convenient and nondegenerate Laurent polynomial on $(\mathbb{C}^*)^n$ and let $P$ be its Newton polytope. We keep in this section the notations of [9] and [19]. It is known that $f$ defines a mixed Hodge structure $MHS_f := (H, F^\bullet H, W^\bullet H)$ where $H = \oplus_{\alpha \in [0,1]} \text{gr}^V G$, $G$ denoting the localized Laplace transform of the Gauss-Manin system of $f$ and $V_\bullet$ being the Kashiwara-Malgrange filtration defined in [9, 2.e]. The mixed Hodge structure $MHS_f$ is said to be of Hodge Tate type if

1. $W_{2i+1}H = W_{2i}H$ for $i \in \mathbb{Z}$,
2. the filtrations $F^\bullet H$ and $W_{2\bullet}$ are opposite, that is $\text{gr}^p F_q H = 0$ for $p \neq q$.

By [5] Proposition 1.2.5, the oppositiveness is equivalent to the decomposition $F^p H \oplus W_{2p-2}H = H$ and also to $H = \oplus_p F^p H \cap W_{2p}H$. 

9
Proposition 4.5 \( {\text{[19, Lemma 2.4 and Corollary 2.6]}} \) Let \( f \) be a convenient and nondegenerate Laurent polynomial on \((\mathbb{C}^*)^n\). The following are equivalent:

1. the mixed Hodge structure \( \text{MHS}_f \) is of Hodge Tate type,
2. \( f \) satisfies the hard Lefschetz property of Definition 2.1,
3. \( \dim \text{gr}^F H = \dim \text{gr}^W 2p H. \)

This is closely related to the "\( f_{p,q} = h_{p,q} \)" part of \( [15, \text{Conjecture 3.6}] \), which amounts to the equality \( \dim \text{gr}^F H = \dim \text{gr}^W 2p H \) if the Newton polytope \( P \) is reflexive \( [19, 3.a] [21] \) (we will refer to it as the KKP conjecture). We keep the notations of Section 4.1.

Proposition 4.6 Let \( \Delta \) be a reduced and reflexive simplex in \( \mathbb{R}^n \) with weight \( Q(\Delta) = (q_0, \ldots, q_n) \), where \( q_n \geq 2 \). The Laurent polynomial \( f_\Delta \) defined by (14) satisfies the KKP conjecture if and only if

\[
2 \mu f_i = d_1 + 2(d_2 + \cdots + d_{i-1}) + d_i
\]

for \( i = 2, \ldots, k \).

Proof. By Proposition 4.5, the KKP conjecture is true for \( f \) if and only if \( f \) satisfies (HL). Thus, the result follows from Proposition 4.1 and Corollary 4.2.

If \( \Delta \) is a reduced and reflexive simplex in \( \mathbb{R}^n \), it follows from Example 4.4 that the KKP conjecture is

- for \( n = 2 \): true for \( f_\Delta \),
- for \( n = 3 \): true for \( f_\Delta \) if and only if \( f_\Delta(u_1, u_2, u_3) = u_1 + u_2 + u_3 + 1/(u_1^a u_2^b u_3^c) \) with \( (a, b, c) = (1, 1, 1), (1, 2, 2), \)
- for \( n = 4 \): true for \( f_\Delta \) if and only if \( f_\Delta(u_1, u_2, u_3, u_4) = u_1 + u_2 + u_3 + u_4 + 1/(u_1^a u_2^b u_3^c u_4^d) \) with \( (a, b, c, d) = (1, 1, 1, 1), (1, 1, 1, 2), (1, 2, 2, 2), (2, 3, 3, 3), (2, 2, 3, 4). \)

Up to unimodular transformations, there exists a unique reduced simplex of weight \( (q_0, q_1, \ldots, q_n) \): an algorithm in order to construct it is given in [4, Theorem 3.6] and provides the Laurent polynomials alluded to above.

5 Application to the distribution of spectral numbers

We apply the previous results to the study of the distribution of the spectrum at infinity of a Laurent polynomial.
5.1 Unimodality (introduction): reflexive case

Recall that a polynomial \( a_0 + a_1 z + \cdots + a_n z^n \) is unimodal if there exists an index \( j \) such that \( a_i \leq a_{i+1} \) for all \( i < j \) and \( a_i \geq a_{i+1} \) for all \( i \geq j \). We first study the unimodality of the spectrum at infinity of a convenient and nondegenerate Laurent polynomial satisfying the assumption of Corollary 2.2. So let us assume that

\[
\text{Spec}_f(z) = 1 + d(1)z + \cdots + d(n-1)z^{n-1} + z^n.
\]

The results in this subsection are known in combinatorics, and we rewrite them in the framework of singularity theory. The first one is due to Hibi:

**Proposition 5.1** [12] Let \( f \) be a Laurent polynomial whose spectrum at infinity is a polynomial. We have \( 1 \leq d(1) \leq d(i) \) for \( i \leq [n/2] \). In particular, \( \text{Spec}_f(z) \) is unimodal if \( n \leq 5 \).

**Proof.** Let \( P \) be the Newton polytope of \( f \) and let \( \delta_P(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n \) be its \( \delta \)-vector. By Proposition 2.1, Corollary 2.2 and [7, Corollary 2.2], \( P \) is reflexive and \( \text{Spec}_f(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n \). By [12], we have \( \delta_0 \leq \delta_1 \leq \delta_j \) for \( 2 \leq j \leq [n/2] \). The inequalities follow and we use then the symmetry \( d(i) = d(n - i) \) in order to get the unimodality for \( n \leq 5 \). \( \square \)

Nevertheless, in this situation the spectrum at infinity needs not to be unimodal if \( n \geq 6 \). The following counter-example is provided by [13]:

**Proposition 5.2** Let \( s \geq 2 \), \( k \geq 2 \) be two integers and let \( n := sk \). Let \( f_{\Delta_{\text{Payne}}} \) be the Laurent polynomial defined by

\[
f_{\Delta_{\text{Payne}}}(u_1, \cdots, u_n) := u_1 + \cdots + u_n + \frac{1}{u_1 \cdots u_{n-1} u_n^s}
\]
on \((\mathbb{C}^*)^n\). Then,

1. \( f_{\Delta_{\text{Payne}}} \) is convenient and nondegenerate,
2. the Milnor number of \( f_{\Delta_{\text{Payne}}} \) is equal to \( s(k+1) \),
3. \( \text{Spec}_{f_{\Delta_{\text{Payne}}}}(z) = 1 + z + \cdots + z^{sk} + z^{(s-1)k} + z^{(s-2)k} + \cdots + z^k \),
4. the spectrum at infinity of \( f_{\Delta_{\text{Payne}}} \) is unimodal if and only if \( s = 2 \),
5. \( f_{\Delta_{\text{Payne}}} \) satisfies the hard Lefschetz property (9) if and only if \( s = 2 \).

**Proof.** Let \( \Delta_{\text{Payne}} := \text{conv}(e_1, \cdots, e_n, -\sum_{i=1}^n q_i e_i) \) where \( n := sk \), \( (e_1, \cdots, e_n) \) is the canonical basis of \( \mathbb{R}^n \) and \( (q_1, \cdots, q_n) := (1, \cdots, 1, s) \) where 1 is counted \( sk \)-times. The simplex \( \Delta_{\text{Payne}} \) is reduced and reflexive and is the Newton polytope of \( f_{\Delta_{\text{Payne}}} \). Its weight is \( (q_0, q_1, \cdots, q_n) = (1, \cdots, 1, s) \) where 1 is counted \( sk \)-times and \( \mu_\Delta = s(k+1) \). The nondegeneracy follows from the fact that the facets of \( \Delta_{\text{Payne}} \) are simplices. The assertion on the Milnor number follows from [14].

We have \( f_1 = 0, f_2 = 1/s, \cdots, f_s = (s-1)/s, d_1 = n + 1, d_2 = \cdots = d_s = 1 \). Define \( \beta_1 := 0 \) and \( \beta_i := d_1 + \cdots + d_{i-1} - \mu f_i = k(s - (i - 1)) \)
for \( i = 2, \ldots, s \). By \([8], [10]\) the spectrum at infinity of \( f_\Delta \) is given by \( \beta_1, \beta_1 + 1, \ldots, \beta_1 + d_1 - 1, \ldots, \beta_k, \beta_k + 1, \ldots, \beta_k + d_k - 1 \). For the last assertion, notice that the necessary and sufficient condition of Corollary 4.2 is \( s = 2(i-1) \) for \( i = 2, \ldots, s \) and is satisfied only for \( s = 2 \).

\[
\text{\textbf{Remark 5.3}} \quad \text{Because } \Delta_{P\text{ayne}} \text{ is reflexive, } \text{Spec}_{f_{\Delta_{P\text{ayne}}}}(z) \text{ is equal to the } \delta\text{-vector of } \Delta_{P\text{ayne}}, \text{ see Proposition 2.7. This formula for the } \delta\text{-vector of } \Delta_{P\text{ayne}} \text{ can already be found in [18].}
\]

Another positive result is given by the following:

\[
\text{\textbf{Proposition 5.4}} \quad \text{Let } f \text{ be a Laurent polynomial whose spectrum at infinity is a polynomial and let } P \text{ be its Newton polytope. Assume that } X_{\Sigma_P} \text{ has a crepant resolution. Then } \text{Spec}_f(z) \text{ is unimodal.}
\]

\[
\text{\textit{Proof.}} \quad \text{Let } \rho : Y \rightarrow X_{\Sigma_P} \text{ be the resolution alluded to and let } \varphi : N_R \rightarrow \mathbb{R} \text{ be the function such that } \varphi \text{ is linear on each cone } \sigma \text{ of } \Sigma_P \text{ and } \varphi(v_i) = 1 \text{ for all primitive generators } v_i \text{ of the rays of } \Sigma_P. \text{ By [2, Theorem 4.3], and because } \varphi \text{ is equal to the Newton function } \nu \text{ (the polytope } P \text{ is reflexive by Corollary 2.2), the Newton spectrum of } P \text{ defined by (3) is equal to the stringy E-function of } X_{\Sigma_P} \text{ [2, Definition 3.1]. By [2, Theorem 3.12], we get } \text{Spec}_P(z) = \sum_{i \geq 0} \dim H^{2i}(Y, \mathbb{C}) z^i. \text{ Now, apply hard Lefschetz to } Y \text{ in order to get the unimodality of } \text{Spec}_P(z), \text{ hence the unimodality of } \text{Spec}_f(z). \]

\[
\text{\textbf{Remark 5.5}} \quad \text{The converse is not true: the simplex } \Delta \text{ in } \mathbb{R}^4 \text{ of weight } (1,1,1,1,2) \text{ is reduced, reflexive, and the variety } X_{\Sigma_{\Delta}} \text{ does not have a crepant resolution (the number of lattice points on the boundary is five). However, its Newton spectrum is } \text{Spec}_{\Delta}(z) = 1 + z + 2z^2 + z^3 + z^4 \text{ and is unimodal.}
\]

Last, unimodality and the hard Lefschetz property are of course related:

\[
\text{\textbf{Proposition 5.6}} \quad \text{Let } f \text{ be a convenient and nondegenerate Laurent polynomial on } (\mathbb{C}^\ast)^n \text{ whose spectrum at infinity is a polynomial. Assume that } f \text{ satisfies the hard Lefschetz property of definition 3.1. Then } \text{Spec}_f(z) \text{ is unimodal.}
\]

\[
\text{\textit{Proof.}} \quad \text{The hard Lefschetz property shows that } [f] : \text{gr}_i \mathcal{A}_f \rightarrow \text{gr}_i \mathcal{A}_f \text{ is injective for } i \leq n/2 \text{ and surjective for } i > n/2. \]

\[
\text{\textbf{5.2 Unimodality: general case}}
\]

We consider in this section the general case, that is when \( f \) does not necessarily satisfy the conditions of Corollary 2.2. We write

\[
\text{Spec}_f(z) = \sum_i d(\alpha_i) z^{\alpha_i}
\]

where \( d(\alpha_i) := \dim \text{gr}_{\alpha_i} \mathcal{A}_f \) and \( \alpha_i \in \mathbb{Q} \), the rational numbers \( \alpha_i \) being arranged by increasing order. The symmetry property \( z^n \text{Spec}_f(z^{-1}) = \text{Spec}_f(z) \) suggests the following definition:
**Definition 5.7** We will say that the spectrum at infinity of a convenient and nondegenerate Laurent polynomial is unimodal if

\[ d(\alpha_1) \leq d(\alpha_2) \leq \cdots \leq d(\alpha_\ell) \]

for all \( \alpha_\ell \leq n/2 \).

Unlike the results of the previous section, this unimodality may fail if \( n \leq 5 \) or if \( f \) satisfies the hard Lefschetz property (HL) (see example 5.9 below). So what gives in this case the hard Lefschetz property? Let us write \( \text{Spec}_f(z) = \sum_{\alpha \in [0,1]} z^\alpha \text{Spec}_f^\alpha(z) \) where \( \text{Spec}_f^\alpha(z) \in \mathbb{Q}[z] \).

**Proposition 5.8** Let \( f \) be a convenient and nondegenerate Laurent polynomial and let \( P \) be its Newton polytope.

1. \( \text{Spec}_f^0(z) \) is a polynomial of degree \( n \), with nonnegative integer coefficients.
2. For \( \alpha \in [0,1] \), \( \text{Spec}_f^\alpha(z) \) is a polynomial of degree at most \( n-1 \), with nonnegative integer coefficients.
3. \( z^n \text{Spec}_f^0(z^{-1}) = \text{Spec}_f(z) \) and \( z^{n-1} \text{Spec}_f^0(z^{-1}) = \text{Spec}_f^{1-\alpha}(z) \) for \( \alpha \in [0,1] \).
4. Assume that \( f \) satisfies the hard Lefschetz property of definition [\ref{3.1}] Then the polynomials \( \text{Spec}_f^\alpha(z) \) are unimodal for \( \alpha \in [0,1] \).

**Proof.** For the two first assertions, see [\ref{7} Corollary 2.2], [\ref{7} Proposition 2.4] and [\ref{7} Proposition 2.6]. The third one follows from the symmetry property \( z^n \text{Spec}_f(z^{-1}) = \text{Spec}_f(z) \). For the last assertion, notice that the hard Lefschetz assumption shows that \([f] : \text{gr}^N_{\alpha+i-1} A_f \to \text{gr}^N_{\alpha+i} A_f\) is injective for \( i \leq (n-1)/2 \) and surjective for \( i > (n-1)/2 \) for \( \alpha \in [0,1] \) (the case \( \alpha = 0 \) has been considered in Proposition [\ref{5.6}]).

**Example 5.9** Let \( f \) be the Laurent polynomial defined by \( f(u_1, u_2, u_3) = u_1 + u_2 + u_3 + 1/2u_1^2u_2^2u_3^3 \) on \((\mathbb{C}^*)^3\). Then \( \text{Spec}_f(z) = 1 + 2z + z^{4/3} + z^{5/3} + 2z^2 + z^3 \) and it is not unimodal in the sense of Definition [\ref{5.7}]. We check however that \( f \) satisfies (HL) using Proposition [\ref{5.4}]. We have \( \text{Spec}_f^0(z) = 1 + 2z + 2z^2 + z^3 \), \( \text{Spec}_f^{1/3}(z) = z \), \( \text{Spec}_f^{2/3}(z) = z \) and this is consistent with Proposition [\ref{5.8}].

The previous results have a combinatorial interpretation. For \( \beta \in [1,0] \), we define, after [\ref{23}], the weighted \( \delta \)-vector \( \delta_\beta^\beta(z) := (1-z)^{n+1} \sum_{m \geq 0} L^\beta_P(m)z^m \) where \( L^\beta_P(m) \) denotes the number of lattice points \( v \) in \( mP \) such that \( \nu(v) - [\nu(v)] = \beta \).

**Corollary 5.10** Let \( f \) be a convenient and nondegenerate Laurent polynomial and let \( P \) be its Newton polytope. Then \( \delta^\beta_P(z) \) is a polynomial of degree at most \( n \). Assume moreover that \( f \) satisfies the hard Lefschetz property of definition [\ref{5.7}]. Then \( \delta^\beta_P(z) \) is unimodal for \( \beta \in [1,0] \).

**Proof.** By [\ref{7} Theorem 4.1], we have \( \text{Spec}_f(z) = \sum_{\beta \in [1,0]} z^\beta \delta^\beta_P(z) \). Thus, \( \delta^0_P(z) = \text{Spec}_f^0(z) \) and \( \delta^\beta_P(z) = z \text{Spec}_f^{\beta+1}(z) \) for \( \beta \in [1,0] \).
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