On the principal eigenvectors of uniform hypergraphs✩

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Abstract: Let $A(H)$ be the adjacency tensor of $r$-uniform hypergraph $H$. If $H$ is connected, the unique positive eigenvector $x = (x_1, x_2, \ldots, x_n)^T$ with $\|x\|_r = 1$ corresponding to spectral radius $\rho(H)$ is called the principal eigenvector of $H$. The maximum and minimum entries of $x$ are denoted by $x_{\text{max}}$ and $x_{\text{min}}$, respectively. In this paper, we investigate the bounds of $x_{\text{max}}$ and $x_{\text{min}}$ in the principal eigenvector of $H$. Meanwhile, we also obtain some bounds of the ratio $x_i/x_j$ for $i, j \in [n]$ as well as the principal ratio $\gamma(H) = x_{\text{max}}/x_{\text{min}}$ of $H$. As an application of these results we finally give an estimate of the gap of spectral radii between $H$ and its proper sub-hypergraph $H'$.

Keywords: Uniform hypergraph, Adjacency tensor, Principal eigenvector, Principal ratio, Weighted incidence matrix

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1. Introduction

Let $G$ be a simple connected graph, and $A(G)$ be the adjacency matrix of $G$. Perron-Frobenius theorem implies that $A(G)$ has a unique unit positive eigenvector corresponding to spectral radius $\rho(G)$, which is usually called the principal eigenvector of $G$. The principal eigenvector plays an important role in spectral graph theory, and there exist some literatures concerning that. In 2000, Papendieck and Recht [16] posed an upper bound for the maximum entry of the principal eigenvector of a graph. Later, Zhao and Hong [23] further investigated bounds for maximum entry of the principal eigenvector of a symmetric nonnegative irreducible matrix with zero trace. In 2007, Ciobă and Gregory [4] improved the bound of Papendieck and Recht [16] in terms of the vertex degree. Das [6, 7] obtained bounds for the maximum entry of the principal eigenvector of the signless Laplacian matrix and distance matrix in 2009 and 2011, respectively. Recently, Das et al. [8] determine upper and lower bounds of the maximum entry of the principal eigenvector of the distance signless Laplacian matrix.

Hypergraph is a natural generalization of ordinary graph (see [1]). A hypergraph $H = (V, E)$ on $n$ vertices is a set of vertices, say $V = \{1, 2, \ldots, n\}$ and a set of edges, say

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which one can get the definition of the eigenvalues of the adjacency tensor of an
hypergraph (see more in Section 2). Obviously, adjacency tensor is a symmetric nonnegative
tensor. The Perron-Frobenius theorem for nonnegative tensors has been established (see
[3], [21] and the references in them). Based on the Perron-Frobenius theorem there exists a
unique positive eigenvector \( x \) with \( \| x \|_r = 1 \) for \( A(H) \) of a connected \( r \)-uniform hypergraph
\( H \). This vector will be called the principal eigenvector of \( H \). Denote by \( x_{\text{max}} \) the maximum
entry of \( x \) and by \( x_{\text{min}} \) the minimum entry of \( x \). The principal ratio, \( \gamma(H) \), of \( H \) is defined
as \( x_{\text{max}}/x_{\text{min}} \).

Recently, Nikiforov [15] presented some bounds on the entry of the principal eigenvector
of an \( r \)-uniform hypergraph (see Section 7 of [15]). Li et al. [9] posed some lower bounds
for principal eigenvector of connected uniform hypergraphs in terms of vertex degrees and
the number of vertices. In this paper, we generalize some classical bounds on the principal
eigenvector to an \( r \)-uniform hypergraph. In Section 2, we introduce some notations and
necessary lemmas. In Section 3, we present some upper bounds on the maximum and
minimum entries in the principal eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \) of a connected \( r \)-uniform hypergraph
\( H \). Meanwhile, we also investigate bounds of the ratio \( x_i/x_j \) for \( i, j \in [n] \) as well as the principal ratio \( \gamma(H) = x_{\text{max}}/x_{\text{min}} \) of \( H \). Based on these results, in Section 4 we
finally give an estimate of the gap of spectral radii between \( H \) and its proper sub-hypergraph
\( H' \).

2. Preliminaries

Let \( H = (V(H), E(H)) \) and \( H' = (V(H'), E(H')) \) be two \( r \)-uniform hypergraphs. If
\( V(H') \subseteq V(H) \) and \( E(H') \subseteq E(H) \), then \( H' \) is called a sub-hypergraph of \( H \). If \( H' \) is a sub-hypergraph of \( H \), and \( H' \neq H \), then \( H' \) is called a proper sub-hypergraph of \( H \). A
hypergraph \( H \) is called a linear hypergraph provided that each pair of the edges of \( H \) has
at most one common vertex (see [2]).

For a vertex \( v \in V(H) \), the degree \( d_H(v) \) is defined as the number of edges containing
\( v \). We denote by \( \delta(H) \) and \( \Delta(H) \) the minimum and maximum degrees of the vertices of \( H \).
In a hypergraph \( H \), two vertices \( u \) and \( v \) are adjacent if there is an edge \( e \) of \( H \) such that
\( \{u, v\} \subseteq e \). A vertex \( v \) is said to be incident to an edge \( e \) if \( v \in e \). A walk of hypergraph \( H \)
is defined to be an alternating sequence of vertices and edges \( v_1 e_1 v_2 e_2 \cdots v_\ell e_\ell v_{\ell+1} \) satisfying
that both \( v_i \) and \( v_{i+1} \) are incident to \( e_i \) for \( 1 \leq i \leq \ell \). A walk is called a path if all vertices
\( E = \{e_1, e_2, \ldots, e_m\} \), where \( e_j = \{i_1, i_2, \ldots, i_\ell\} \), \( i_j \in [n] := \{1, 2, \ldots, n\} \), \( j \in [\ell] \). A
hypergraph is called \( r \)-uniform if every edge contains precisely \( r \) vertices. Let \( H = (V, E) \)
be an \( r \)-uniform hypergraph on \( n \) vertices. The adjacency tensor (see [5]) of \( H \) is defined
as the order \( r \) dimension \( n \) tensor \( A(H) \) whose \((i_1 i_2 \cdots i_r)\)-entry is

\[
(A(H))_{i_1 i_2 \cdots i_r} = \begin{cases} 
\frac{1}{(r-1)!} & \text{if } \{i_1, i_2, \ldots, i_r\} \in E(H), \\
0 & \text{otherwise.}
\end{cases}
\]

Qi [17] and Lim [11] independently introduced the concept of eigenvalues of tensors, from
which one can get the definition of the eigenvalues of the adjacency tensor of an \( r \)-uniform
hypergraph (see more in Section 2).
and edges in the walk are distinct. The length of a path is the number of edges in it. A hypergraph \( H \) is called connected if for any vertices \( u, v \), there is a path connecting \( u \) and \( v \). The distance between two vertices is the length of the shortest path connecting them. The diameter of a connected \( r \)-uniform hypergraph \( H \) is the maximum distance among all vertices of \( H \).

The following definition was introduced by Qi [17].

**Definition 2.1 ([17])** Let \( A \) be an order \( r \) dimension \( n \) tensor, \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \) be a column vector of dimension \( n \). Then \( Ax^{r-1} \) is defined to be a vector in \( \mathbb{C}^n \) whose \( i \)-th component is the following

\[
(Ax^{r-1})_i = \sum_{i_2, \ldots, i_r = 1}^n a_{i_1i_2 \cdots i_r}x_{i_2} \cdots x_{i_r}, \quad i = 1, 2, \ldots, n. \tag{2.1}
\]

If there exists a number \( \lambda \in \mathbb{C} \) and a nonzero vector \( x \in \mathbb{C}^n \) such that

\[
Ax^{r-1} = \lambda x^{r-1}, \tag{2.2}
\]

then \( \lambda \) is called an eigenvalue of \( A \), \( x \) is called an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \), where \( x^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \ldots, x_n^{r-1})^T \). The spectral radius of \( A \) is the maximum modulus of the eigenvalues of \( A \), i.e., \( \rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} \).

For an \( r \)-uniform hypergraph \( H \), the spectrum, eigenvalues and spectral radius of \( H \) are defined to be those of its adjacency tensor \( A(H) \). By using the general product of tensors defined by Shao in [19], \( Ax^{r-1} \) can be simply written as \( Ax \). In the remaining part of this paper, we will use \( Ax \) to denote \( Ax^{r-1} \).

**Theorem 2.1 ([18])** Let \( A \) be a nonnegative symmetric tensor of order \( r \) and dimension \( n \), denote \( \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\} \). Then we have

\[
\rho(A) = \max \left\{ x^T(Ax) : x \in \mathbb{R}^n_+, \|x\|_r = 1 \right\}. \tag{2.3}
\]

Furthermore, \( x \in \mathbb{R}^n_+ \) with \( \|x\|_r = 1 \) is an optimal solution of the above optimization problem if and only if it is an eigenvector of \( A \) corresponding to the eigenvalue \( \rho(A) \).

A novel method, weighted incidence matrix method is introduced by Lu and Man for computing the spectral radii of hypergraphs in [12].

**Definition 2.2 ([12])** A weighted incidence matrix \( B \) of a hypergraph \( H = (V(H), E(H)) \) is a \( |V(H)| \times |E(H)| \) matrix such that for any vertex \( v \) and any edge \( e \), the entry \( B(v, e) > 0 \) if \( v \in e \) and \( B(v, e) = 0 \) if \( v \notin e \).

**Definition 2.3 ([12])** A hypergraph \( H \) is called \( \alpha \)-normal if there exists a weighted incidence matrix \( B \) satisfying

1. \( \sum_{v \in e \in E} B(v, e) = 1 \), for any \( v \in V(H) \);
2. \( \prod_{v \in e \in E} B(v, e) = \alpha \), for any \( e \in E(H) \).
Moreover, the weighted incidence matrix $B$ is called consistent if for any cycle $v_0e_1v_1 \cdots e_\ell v_\ell$ ($v_\ell = v_0$)
\[
\prod_{i=1}^\ell \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.
\]
In this case, $H$ is called consistently $\alpha$-normal.

**Lemma 2.1 ([12])** Let $H$ be a connected $r$-uniform hypergraph. Then $H$ is consistently $\alpha$-normal if and only if $\rho(H) = \alpha^{-\frac{1}{r}}$.

### 3. The principal eigenvectors of uniform hypergraphs

#### 3.1. The extreme components of principal eigenvector

In this subsection we shall present some bounds on $x_{\text{max}}$ and $x_{\text{min}}$ in the principal eigenvector of a connected $r$-uniform linear hypergraph $H$.

The following lemma is very useful for us, so we reproduce the proof of Lu and Man [12].

**Lemma 3.1 ([12])** Suppose that $H$ is a connected $r$-uniform hypergraph. Let $H$ be consistently $\alpha$-normal with weighted incidence matrix $B$. If $x = (x_1, x_2, \ldots, x_n)^T$ is the principal eigenvector of $H$, then for any edge $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} \in E(H)$, we have
\[
B(v_{i_1}, e)^{\frac{1}{r}} \cdot x_{v_{i_1}} = B(v_{i_2}, e)^{\frac{1}{r}} \cdot x_{v_{i_2}} = \cdots = B(v_{i_r}, e)^{\frac{1}{r}} \cdot x_{v_{i_r}}.
\]

**Proof.** Let $\rho$ be the spectral radius of $H$. From Lemma 2.1, $H$ is consistently $\alpha$-normal with $\alpha = \rho^{-r}$. According to Theorem 2.1 and Definition 2.3, we have
\[
\rho = x^T (A(H)x) = r \sum_{e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} \in E(H)} x_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_r}}
\]
\[
= \frac{r}{\alpha^\frac{1}{r}} \sum_{e \in E(H)} \prod_{v_i : v_i \in e} \left( B(v_i, e)^{\frac{1}{r}} x_{v_i} \right)
\]
\[
\leq \frac{r}{\alpha^\frac{1}{r}} \sum_{e \in E(H)} \sum_{v_i : v_i \in e} B(v_i, e) x_{v_i}^r
\]
\[
= \alpha^{-\frac{1}{r}} = \rho,
\]
which implies that in (3.2) equality holds. Therefore the desired equations (3.1) follows. □

**Theorem 3.1** Let $H$ be a connected $r$-uniform linear hypergraph on $n$ vertices with spectral radius $\rho$ and principal eigenvector $x$. For any $v \in V(H)$, if $d$ is the degree of vertex $v$, then
\[
x_v \leq \frac{1}{\sqrt{1 + (r-1) \left(\frac{d}{r}\right)^{\frac{1}{r-1}}}}.
\]
Proof. Let \( e_1, e_2, \ldots, e_d \) be all distinct edges containing \( v \). Denote 
\[
  e_i = \{v, v_{i_1}, v_{i_2}, \ldots, v_{i_{r-1}}\}, \quad i = 1, 2, \ldots, d.
\]
Suppose that \( H \) is consistently \( \alpha \)-normal with weighted incidence matrix \( B \), where \( \alpha = \rho^{-r} \).
For each edge \( e_i \), by (3.1) and AM-GM inequality we have
\[
  \frac{x_{r_{i_1}} + x_{r_{i_2}} + \cdots + x_{r_{i_{r-1}}}}{x_v^r} = \frac{B(v, e_i) + B(v, e_i) + \cdots + B(v, e_i)}{(r-1)B(v, e_i)}
\]
It follows from Hölder inequality that
\[
  \sum_{i=1}^{d} \sum_{j=1}^{r-1} x_{v_{ij}}^r \geq (r-1)\rho^{-r} x_v^r \sum_{i=1}^{d} B(v, e_i)^{\frac{r}{r-1}}
\]
Notice that \( \sum_{j=1}^{n} x_j^r = 1 \) and \( H \) is linear, we obtain
\[
  1 = \sum_{j=1}^{n} x_j^r \geq x_v^r + \sum_{i=1}^{d} \sum_{j=1}^{r-1} x_{v_{ij}}^r
\]
Therefore we have
\[
x_v \leq \frac{1}{\sqrt{1 + (r-1)\left(\frac{\rho^r}{d}\right)^{\frac{1}{r-1}}}}
\]
The proof is completed. \(\square\)
It is known that \( \rho(H) \geq \sqrt{\Delta(H)} \) from [15, Proposition 7.13], so we get the following corollary immediately.

**Corollary 3.1** Let \( H \) be a connected \( r \)-uniform linear hypergraph on \( n \) vertices with principal eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \). Then

\[
x_i \leq \frac{1}{\sqrt{r}}.
\]

When \( r = 2 \), we can obtain the following classical result, which was also proved in [4].

**Corollary 3.2 ([4])** Suppose that \( G \) is a connected graph on \( n \) vertices with spectral radius \( \rho \) and principal eigenvector \( x \). For any \( v \in V(G) \), if \( d \) is the degree of vertex \( v \), then

\[
x_v \leq \frac{1}{\sqrt{1 + \frac{\rho^2}{d}}}.
\]

**Remark 3.1** In Theorem 3.1, we give an upper bound on \( x_v \) for a connected \( r \)-uniform linear hypergraph. With the same symbols as that of Theorem 3.1, Nikiforov presented a bound for general uniform hypergraph as follows [15]. For \( r \)-uniform hypergraph it is proved that

\[
x_v \leq \frac{r}{\sqrt{[\rho^r(r-1)!]^{\frac{1}{r-1}}}}.
\]

(3.3)

Notice that

\[
1 + (r-1) \left( \frac{\rho^r}{d} \right)^{\frac{1}{r-1}} > (r-1) \left( \frac{\rho^r}{d} \right)^{\frac{1}{r-1}} \geq \frac{[\rho^r(r-1)!]^{\frac{1}{r-1}}}{d}.
\]

Therefore when \( H \) is a connected \( r \)-uniform linear hypergraph, Theorem 3.1 has a better bound than (3.3).

In the following we give an upper bound on \( x_{\min} \) for a connected \( r \)-uniform hypergraph, which extends the result of Nikiforov [14] to linear hypergraphs.

**Theorem 3.2** Suppose that \( H \) is a connected \( r \)-uniform linear hypergraph on \( n \) vertices with spectral radius \( \rho \) and the principal eigenvector \( x \). Let \( \delta \) be the minimum vertex degree of \( H \), then

\[
x_{\min} \leq \frac{1}{\sqrt{(r-1) \left( \frac{\rho^r}{\delta} \right)^{\frac{1}{r-1}} + n - \delta(r-1)}}
\]

**Proof.** Let \( u \in V(H) \) be a vertex of minimum degree \( d(u) = \delta \). Let \( e_1, e_2, \ldots, e_\delta \) be all distinct edges containing \( u \). Denote

\[
e_i = \{u, u_{i1}, u_{i2}, \ldots, u_{i\delta-1}\}, \quad i = 1, 2, \ldots, \delta.
\]
Similarly to the proof in Theorem 3.1, we have

\[ \frac{\delta}{\sum_{i=1}^{r-1} x_{uij}} \geq (r - 1) \left( \frac{\rho^r}{\delta} \right) \frac{1}{r - 1} x_u^r. \]

It follows from \( \sum_{i=1}^{n} x_i^r = 1 \) that

\[ x_{\min}^r \leq x_u^r \leq \frac{\delta}{(r - 1) \left( \frac{\rho^r}{\delta} \right) \frac{1}{r - 1}} \leq \frac{1 - [n - \delta(r - 1)] x_{\min}^r}{(r - 1) \left( \frac{\rho^r}{\delta} \right) \frac{1}{r - 1}}, \]

which implies that

\[ \left[ (r - 1) \left( \frac{\rho^r}{\delta} \right) \frac{1}{r - 1} + n - \delta(r - 1) \right] x_{\min}^r \leq 1. \]

This completes the proof of this theorem. \( \square \)

If we take \( r = 2 \) in the above theorem, we obtain the following classical result.

**Corollary 3.3 ([14])** Suppose that \( G \) is a connected graph on \( n \) vertices with spectral radius \( \rho \) and the principal eigenvector \( x \). Let \( \delta \) be the minimum vertex degree of \( G \), then

\[ x_{\min} \leq \sqrt{\frac{\delta}{\rho^2 + \delta(n - \delta)}}. \]

### 3.2. The ratio of entries in principal eigenvector and the principal ratio \( \gamma \)

In this subsection we shall consider the bounds of the ratio \( x_i/x_j \) for \( i, j \in [n] \) as well as the principal ratio \( \gamma(H) \) of a connected \( r \)-uniform hypergraph \( H \).

**Theorem 3.3** Suppose that \( H \) is a connected \( r \)-uniform hypergraph on \( n \) vertices with spectral radius \( \rho \) and the principal eigenvector \( x \). For \( u, v \in V(H) \), if \( d(u, v) = \ell \), then

\[ \frac{1}{\rho^\ell} \leq \frac{x_u}{x_v} \leq \rho^\ell. \]  

\hspace{1cm} (3.4)

**Proof.** It is sufficient to prove the left-hand side of (3.4). From Lemma 2.1, we assume that \( H \) is consistently \( \alpha \)-normal with weighted incidence matrix \( B \). Let \( u = u_0 e_1 u_1 \cdots e_{\ell} u_{\ell} = v \) be a shortest path in \( H \) from \( u \) to \( v \). For edge \( e_i, i = 1, 2, \ldots, \ell \), by (3.1), we have

\[ x_{u_{i-1}} \cdot B(u_{i-1}, e_i)^\frac{1}{\ell} = x_{u_i} \cdot B(u_i, e_i)^\frac{1}{\ell}. \]  

\hspace{1cm} (3.5)

According to Definition 2.3, we see

\[ \prod_{w : w \in e_i} B(w, e_i) = \alpha \text{ and } 0 < B(w, e_i) \leq 1, \ i = 1, 2, \ldots, \ell. \]

Therefore we deduce that

\[ B(u_{i-1}, e_i) B(u_i, e_i) \geq \alpha. \]

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It follows from (3.5) that
\[
x_{u_{i-1}}^2 \geq x_{u_{i-1}}^2 \cdot B(u_{i-1}, e_i)^2 = x_{u_{i-1}} x_u \cdot [B(u_{i-1}, e_i) B(u_i, e_i)]^{1/2} \geq \alpha^{r/2} x_{u_{i-1}} x_u \rho
\]
which yields that
\[
\frac{x_{u_{i-1}}}{x_u} \geq \frac{1}{\rho}, \quad i = 1, 2, \ldots, \ell.
\]
Therefore we deduce that
\[
\frac{x_u}{x_v} = \frac{x_{u_0}}{x_{u_1}} \cdot \frac{x_{u_1}}{x_{u_2}} \cdot \ldots \cdot \frac{x_{u_{\ell-1}}}{x_v} \geq \frac{1}{\rho^r}.
\]
The proof is completed. \qed

**Corollary 3.4** Let $H$ be a connected $r$-uniform hypergraph with spectral radius $\rho$ and the principal eigenvector $x$. If $u, v \in V(H)$ are adjacent, then
\[
\frac{1}{\rho} \leq \frac{x_u}{x_v} \leq \rho.
\]
If spectral radius $\rho(H) \geq \sqrt[4]{\rho}$, then we have the following stronger result.

**Theorem 3.4** Let $H$ be a connected $r$-uniform hypergraph with spectral radius $\rho$ and the principal eigenvector $x$. Let $u, v \in V(H)$, $d(u, v) = \ell$. Then the following statements hold.
(1) If $\rho > \sqrt[4]{\rho}$, then
\[
\left(\frac{\sigma - \sigma^{-1}}{\sigma^{\ell+1} - \sigma^{-(\ell+1)}}\right)^{\frac{1}{r}} \leq \frac{x_u}{x_v} \leq \left(\frac{\sigma^{\ell+1} - \sigma^{-(\ell+1)}}{\sigma - \sigma^{-1}}\right)^{\frac{1}{r}},
\]
where
\[
\sigma = \frac{1}{2} \left(\sqrt{\rho^r + \sqrt{\rho^r - 4}}\right).
\]
(2) If $\rho = \sqrt[4]{\rho}$, then
\[
\frac{1}{\sqrt{(\ell + 1)^2}} \leq \frac{x_u}{x_v} \leq \sqrt{(\ell + 1)^2}.
\]
**Proof.** By Lemma 2.1, we may assume that $H$ is consistently $\alpha$-normal with weighted incidence matrix $B$, where $\alpha = \rho^{-r}$. Let $u = u_0 e_1 u_1 \cdots e_{\ell} u_{\ell} = v$ be a shortest path in $H$ from $u$ to $v$. For any $i = 2, 3, \ldots, \ell$, from (3.1) we have
\[
\begin{aligned}
x_{u_{i-1}}^r \cdot B(u_{i-1}, e_{i-1}) &= x_{u_{i-2}}^r \cdot B(u_{i-2}, e_{i-1}), \\
x_{u_{i-1}}^r \cdot B(u_{i-1}, e_i) &= x_{u_i}^r \cdot B(u_i, e_i).
\end{aligned}
\]
Therefore we conclude that
\[
\sqrt{x_{u_{i-2}}} + \sqrt{x_{u_i}} = \left( \frac{B(u_{i-1}, e_{i-1})}{B(u_{i-2}, e_{i-1})} + \frac{B(u_{i-1}, e_i)}{B(u_i, e_i)} \right) \cdot \sqrt{x_{u_{i-1}}}
\]
\[
= \left( \frac{B(u_{i-1}, e_{i-1})}{\sqrt{B(u_{i-2}, e_{i-1})B(u_{i-1}, e_{i-1})}} + \frac{B(u_{i-1}, e_i)}{\sqrt{B(u_i, e_i)B(u_{i-1}, e_i)}} \right) \cdot \sqrt{x_{u_{i-1}}}
\]
\[
\leq \alpha^{-\frac{1}{2}} [B(u_{i-1}, e_{i-1}) + B(u_{i-1}, e_i)] \cdot \sqrt{x_{u_{i-1}}}
\]
\[
\leq \alpha^{-\frac{1}{2}} \cdot \sqrt{x_{u_{i-1}}} = \sqrt{\rho^r x_{u_{i-1}}^r}.
\]

It follows that
\[
\sqrt{x_{u_i}} \leq \sqrt{\rho^r x_{u_{i-1}}^r} - \sqrt{x_{u_{i-2}}}^r, \; i = 2, 3, \ldots, \ell.
\]

From Corollary 3.4 we have \(x_{u_1} \leq \rho x_{u_0} = \rho x_u\). Denote \(x_{u_{-1}} = 0\), we have
\[
\sqrt{x_{u_i}}^r \leq \sqrt{\rho^r} \sqrt{x_{u_{i-1}}^r} - \sqrt{x_{u_{i-2}}}^r, \; i = 1, 2, \ldots, \ell. \tag{3.6}
\]

(1). It is sufficient to prove the left-hand side. We may rewrite (3.6) as
\[
\left( \begin{array}{c}
\sqrt{x_{u_1}}^r \\
\sqrt{x_{u_{i-1}}^r}
\end{array} \right) \leq \left( \begin{array}{cc}
\sqrt{\rho^r} & -1 \\
1 & 0
\end{array} \right) \left( \begin{array}{c}
\sqrt{x_{u_1}}^r \\
\sqrt{x_{u_{i-2}}}^r
\end{array} \right), \; i = 1, 2, \ldots, \ell.
\]

If \(\rho > \sqrt{4}\) we have
\[
\left( \begin{array}{c}
\sqrt{x_{u_i}}^r \\
\sqrt{x_{u_{i-1}}^r}
\end{array} \right) \leq \left( \begin{array}{cc}
\sqrt{\rho^r} & -1 \\
1 & 0
\end{array} \right)^i \left( \begin{array}{c}
\sqrt{x_{u_1}}^r \\
\sqrt{x_{u_0}}^r
\end{array} \right), \; i = 0, 1, \ldots, \ell. \tag{3.7}
\]

Observe that the fact
\[
\left( \begin{array}{cc}
\sqrt{\rho^r} & -1 \\
1 & 0
\end{array} \right) = P \left( \begin{array}{cc}
\sigma & 0 \\
0 & \sigma^{-1}
\end{array} \right) P^{-1},
\]
where
\[
P = \left( \begin{array}{cc}
1 & 1 \\
\sigma^{-1} & \sigma
\end{array} \right), \; \sigma = \frac{1}{2} (\sqrt{\rho^r} + \sqrt{\rho^r - 4}).
\]

Hence (3.7) now becomes
\[
\left( \begin{array}{c}
\sqrt{x_{u_i}}^r \\
\sqrt{x_{u_{i-1}}^r}
\end{array} \right) \leq P \left( \begin{array}{cc}
\sigma^i & 0 \\
0 & \sigma^{-i}
\end{array} \right) P^{-1} \left( \begin{array}{c}
\sqrt{x_{u_1}}^r \\
\sqrt{x_{u_0}}^r
\end{array} \right), \; i = 0, 1, \ldots, \ell.
\]

Recall that \(x_{u_{-1}} = 0\) and \(x_{u_0} = x_u\), we obtain
\[
\left( \begin{array}{c}
\sqrt{x_{u_i}}^r \\
\sqrt{x_{u_{i-1}}^r}
\end{array} \right) \leq \left( \begin{array}{cc}
1 & 1 \\
\sigma^{-1} & \sigma
\end{array} \right) \left( \begin{array}{cc}
\sigma^i & 0 \\
0 & \sigma^{-i}
\end{array} \right) \left( \begin{array}{cc}
\sigma & 1 \\
\sigma^{-1} & \sigma
\end{array} \right) \left( \begin{array}{c}
\sqrt{x_{u_i}}^r \\
0
\end{array} \right)
\]
Therefore we get
\[
\sqrt{x^r_u} = \sqrt{x^r_u} \leq \frac{\sigma^{\ell+1} - \sigma^{-(\ell+1)}}{\sigma - \sigma^{-1}} \sqrt{x^r_u},
\]
which implies that
\[
\frac{x_u}{x_v} \geq \left( \frac{\sigma - \sigma^{-1}}{\sigma^{\ell+1} - \sigma^{-(\ell+1)}} \right)^2.
\]

(2). If \( \rho = \sqrt{4} \), then (3.6) can be written as
\[
\sqrt{x^r_{u_i}} - \sqrt{x^r_{u_{i-1}}} \leq \sqrt{x^r_{u_{i-1}}} - \sqrt{x^r_{u_{i-2}}}, \quad i = 1, 2, \ldots, \ell.
\]
Therefore we have
\[
\sqrt{x^r_{u_i}} - \sqrt{x^r_{u_{i-1}}} \leq \sqrt{x^r_{u_0}} - \sqrt{x^r_{u_{-1}}} = \sqrt{x^r_u}, \quad i = 0, 1, \ldots, \ell. \tag{3.8}
\]
Summing over all \( i \) of (3.8), we obtain
\[
\sqrt{x^r_v} = \sqrt{x^r_{u_\ell}} - \sqrt{x^r_{u_{-1}}} \leq (\ell + 1) \sqrt{x^r_u},
\]
which yields that
\[
\frac{x_u}{x_v} \geq \frac{1}{\sqrt{(\ell + 1)^2}}.
\]

The proof is completed. \( \square \)

**Remark 3.2** If \( \rho \geq \sqrt{4} \), Theorem 3.4 is much stronger than Theorem 3.3. Indeed, for \( \rho > \sqrt{4} \) we have
\[
\rho^{\ell/2} = (\sigma + \sigma^{-1})^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} \sigma^{-\ell+2i} \geq \sum_{i=0}^{\ell} \sigma^{-\ell+2i} = \frac{\sigma^{\ell+1} - \sigma^{-(\ell+1)}}{\sigma - \sigma^{-1}},
\]
and \((\sqrt{4})^\ell = (2^\ell)^{\frac{\ell}{2}} \geq (\ell + 1)^{\frac{\ell}{2}} = \sqrt{(\ell + 1)^2}\) for \( \rho = \sqrt{4} \).

**Corollary 3.5** Let \( H \) be a connected \( r \)-uniform hypergraph on \( n \) vertices with spectral radius \( \rho > \sqrt{4} \) and principal eigenvector \( x \). Let \( \ell \) be the shortest distance between a vertex having \( x_{\text{min}} \) and a vertex having \( x_{\text{max}} \) as their \( x \) components. Then
\[
\gamma(H) \leq \left( \frac{\sigma^{\ell+1} - \sigma^{-(\ell+1)}}{\sigma - \sigma^{-1}} \right)^{\frac{\sqrt{r}}{r}}. \tag{3.9}
\]

Taking \( r = 2 \) in (3.9), we get the following result which was proved by Cioabă and Gregory [4].
Corollary 3.6 ([4]) Let $G$ be a connected graph of order $n$ with spectral radius $\rho > 2$ and principal eigenvector $x$. Let $\ell$ be the shortest distance from a vertex on which $x$ is maximum to a vertex on which it is minimum. Then
\[
\gamma(G) \leq \frac{\tau^{\ell+1} - \tau^{-(\ell+1)}}{\tau - \tau^{-1}},
\]
where $\tau = (\rho + \sqrt{\rho^2 - 4})/2$.

4. Spectral radius of sub-hypergraph

It is known from Perron-Frobenius theorem that if $H'$ is a proper sub-hypergraph of a connected $r$-uniform hypergraph $H$, then $\rho(H') < \rho(H)$. In this section, we will refine quantitatively the gap between $\rho(H)$ and $\rho(H')$. The main result of this section is inspired by that of [13], and the arguments of Theorem 4.2 have been used in [13]. For ordinary graphs, Nikiforov [13] proved the following result.

Theorem 4.1 ([13]) If $G'$ is a proper subgraph of a connected graph $G$ on $n$ vertices with diameter $D$, then
\[
\rho(G) - \rho(G') > \frac{1}{n\rho(G)^2D}.
\]

The following lemmas will be used in our proof.

Lemma 4.1 ([20]) Let $G$ be a connected graph with diameter $D$, and $G'$ be any connected subgraph obtained by deleting $t$ edges from $G$. Then $\text{diam}(G') \leq (t + 1)D$.

For an $r$-uniform hypergraph $H$, we define a multiple graph $\tilde{H}$ as follows: $\tilde{H}$ has vertex set $V(H)$, and vertices $u$ and $v$ are adjacent in $\tilde{H}$ if and only if $\{u, v\} \subseteq e \in E(H)$.

Lemma 4.2 Let $H$ be an $r$-uniform hypergraph with diameter $D$, and $H'$ be a connected sub-hypergraph obtained by deleting an edge of $H$. Then $\text{diam}(H') \leq r(D + 1)$.

Proof. We first prove that $d_H(u, v) = d_{\tilde{H}}(u, v)$ for any $u$, $v \in V(H)$. Suppose that $u = u_0e_1u_1e_2 \cdots u_{\ell-1}e_\ell u_\ell = v$ is a path in $H$ from $u$ to $v$. Let $\{u_{i-1}, u_i\} = f_i$, $i = 1, 2, \ldots, \ell$, then $u_0f_1u_1f_2 \cdots f_\ell u_\ell = v$ is a walk in $\tilde{H}$. Thus $d_H(u, v) \geq d_{\tilde{H}}(u, v)$. Similarly, we have $d_{\tilde{H}}(u, v) \geq d_H(u, v)$. Therefore $d_H(u, v) = d_{\tilde{H}}(u, v)$. In particular, $\text{diam}(H) = \text{diam}(\tilde{H})$.

Let $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ and $H' = H - e$. Clearly,
\[
\tilde{H}' = \tilde{H} - \bigcup_{1 \leq p < q \leq r} \{v_{i_p}, v_{i_q}\},
\]
and $v_{i_1}$, $v_{i_2}$, $\ldots$, $v_{i_r}$ induces a clique $K_r$ in $\tilde{H}$. Let
\[
H^* = \tilde{H} - \bigcup_{2 \leq p < q \leq r} \{v_{i_p}, v_{i_q}\}.
\]

Clearly, $\text{diam}(H^*) \leq D + 1$, and $H' = H^* - \bigcup_{q=2}^{r} \{v_{i_1}, v_{i_q}\}$. It follows from Lemma 4.1 that $\text{diam}(H') = \text{diam}(\tilde{H}') \leq r(D + 1)$. □
The following result can be found in [13].

**Lemma 4.3 ([13])** Let \( G \) be a connected graph with diameter \( D \), and \( G' \) be any connected subgraph of \( G \) obtained by deleting an edge \( e = uv \). Then for any \( w \in V(G) \) we have
\[
d_{G'}(w, u) + d_{G'}(w, v) \leq 2D.
\]

**Lemma 4.4** Let \( H \) be a connected \( r \)-uniform hypergraph on \( n \) vertices with diameter \( D \), and \( H' \) be any connected sub-hypergraph of \( H \) obtained by deleting an edge \( e \in E(H) \). Then for any \( w \in V(H) \) we have
\[
\sum_{w \in e} d_{H'}(w, v) \leq r(r - 1)(D + 1).
\]

**Proof.** Let \( H' = H - e \), where \( e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \in E(H) \) is an edge of \( H \). For any \( 1 \leq p, q \leq r, p \neq q \), we let
\[
H(v_{i_p}, v_{i_q}) = H' + \{v_{i_p}, v_{i_q}\}.
\]

It is clear that
\[
H(v_{i_p}, v_{i_q}) = H' + \bigcup_{1 \leq s \leq r} \bigcup_{s \neq p} \{v_{i_s}, v_{i_s}\} - \bigcup_{1 \leq t \leq r} \bigcup_{t \neq p, t \neq q} \{v_{i_t}, v_{i_t}\}.
\]

Notice that
\[
\text{diam} \left( H' + \bigcup_{1 \leq s \leq r, s \neq p} \{v_{i_s}, v_{i_s}\} \right) \leq D + 1.
\]

It follows from Lemma 4.1 that \( \text{diam} \left( H(v_{i_p}, v_{i_q}) \right) \leq (r - 1)(D + 1) \). Therefore for any \( w \in V(H) \), from Lemma 4.3 we have
\[
d_{H'}(w, v_{i_p}) + d_{H'}(w, v_{i_q}) \leq 2 \cdot \text{diam} \left( H(v_{i_p}, v_{i_q}) \right) \leq 2(r - 1)(D + 1).
\]

In particular, we have
\[
\begin{align*}
d_{H'}(w, v_{i_1}) + d_{H'}(w, v_{i_r}) &\leq 2(r - 1)(D + 1), \\
d_{H'}(w, v_{i_2}) + d_{H'}(w, v_{i_{r-1}}) &\leq 2(r - 1)(D + 1), \\
\vdots &
\end{align*}
\]

(4.1)

Summing the both sides of (4.1) we obtain
\[
d_{H'}(w, v_{i_1}) + d_{H'}(w, v_{i_2}) + \cdots + d_{H'}(w, v_{i_r}) \leq r(r - 1)(D + 1).
\]

Observe that for any \( u, v \in V(H) \), \( d_{H'}(u, v) = d_{H'}(u, v) \), we have
\[
d_{H'}(w, v_{i_1}) + d_{H'}(w, v_{i_2}) + \cdots + d_{H'}(w, v_{i_r}) \leq r(r - 1)(D + 1).
\]

The proof is completed. \( \square \)
The following theorem give an estimate of spectral radii between $H$ and any proper sub-hypergraph $H'$ of $H$.

**Theorem 4.2** Suppose that $H$ is a connected $r$-uniform hypergraph on $n$ vertices with diameter $D$ and spectral radius $\rho$. If $H'$ is a proper sub-hypergraph of $H$, then

$$\rho(H) - \rho(H') \geq \min \left\{ \frac{r}{n\rho^r(r-1)(D+1)}, \frac{1}{n\rho^rD(\rho^r + r - 1)} \right\}.$$  

**Proof.** From Perron-Frobenius theorem for nonnegative tensors, $\rho(H') \leq \rho(H)$ whenever $H' \subseteq H$. Therefore we may assume that $H'$ is a maximal proper sub-hypergraph of $H$, i.e., $V(H') = V(H)$ and $H'$ differs from $H$ in a single edge $e_0 = \{v_1, v_2, \ldots, v_r\}$. We distinguish the following two cases.

**Case (i).** $H'$ is connected. Let $x$ be the principal eigenvector of $H'$ and $x_w$ be the maximum entry of $x$. For an edge $e$ and vector $x$, we adopt the symbol $x^e := \prod_{v \in e} x_v$ from [10]. Therefore we have

$$\rho(H) - \rho(H') \geq x^T(A(H)x) - x^T(A(H')x)$$

$$= r \sum_{e \in E(H)} x^e - r \sum_{e \in E(H')} x^e$$

$$= rx_0^e.$$  

From Theorem 3.3 and Lemma 4.4, we have

$$x_0^e = \frac{x_{v_1}x_{v_2} \cdots x_{v_r}}{x_w} \cdot \frac{x_w^r}{x_w^r}$$

$$= \frac{x_{v_1}x_{v_2} \cdots x_{v_r}}{x_w} \cdot \frac{x_w^r}{x_w^r}$$

$$\geq \frac{x_w^r}{\rho^{d_{H'}(v_1, w) + d_{H'}(v_2, w) + \cdots + d_{H'}(v_r, w)}}$$

$$\geq \frac{x_w^r}{\rho^{r-1}(D+1)} \geq \frac{1}{n\rho^r(D+1)}.$$  

Therefore we get

$$\rho(H) - \rho(H') \geq \frac{r}{n\rho^r(D+1)}.$$  

**Case (ii).** $H'$ is disconnected. Suppose that $H'$ has $t$ connected components $H_1$, $H_2$, $\ldots$, $H_t$ ($t \geq 2$). It is known that $\rho(H') = \max_{1 \leq i \leq t} \{\rho(H_i)\}$ (see Lemma 9 of [22]). Without loss of generality, we may assume that $\rho(H') = \rho(H_1)$. For convenience, we denote by $|H_1| = m$, $\rho_1 = \rho(H')$, and let $x = (x_1, x_2, \ldots, x_m)^T$ be the principal eigenvector of $H_1$ corresponding to $\rho_1$. Since $H$ is connected, we know that $e_0 \cap H_i \neq \emptyset$, $i = 1, 2, \ldots, t$. Let $e_0 \cap V(H_1) = \{v_1, v_2, \ldots, v_s\}$, $1 \leq s \leq r - 1$. In the light of Theorem 3.3 and $\text{diam}(H_1) \leq D$ we have

$$x_{v_j} \geq \frac{x_{\max}}{\rho_1^{D}}, \quad j = 1, 2, \ldots, s,$$  

(4.2)
where \( x_{\text{max}} = \max\{x_1, x_2, \ldots, x_m\} \). Let \( x_0 = \min\{x_{v_1}, x_{v_2}, \ldots, x_{v_s}\} \). Denote \( y = (y_1, y_2, \ldots, y_m, y_{m+1}, \ldots, y_{m+r-s})^T = (x_1, x_2, \ldots, x_m, \underbrace{x_0, \ldots, x_0}_{r-s})^T \).

It follows that
\[
\|y\|_r^r = 1 + (r-s)\frac{x_0^r}{\rho_1^r}.
\]

Therefore we have
\[
\rho(H) \geq \rho(H + e_0) \geq \frac{y^T(A(H + e_0)y)}{\|y\|_r^r} \geq \frac{\rho_1^r}{\rho_1^r + (r-s)x_0^r} \left( r \sum_{e' \in E(H_1)} y_{e'}^r + ry_0^r \right) \geq \frac{\rho_1^r}{\rho_1^r + (r-1)x_0^r} \left( \rho_1 + \frac{rx_0^r}{\rho_1^{r-1}} \right) = \rho_1 + \frac{\rho_1 x_0^r}{\rho_1^r + (r-1)x_0^r}.
\]

Notice that \( x_0^r \leq 1 \leq \rho_1 \). Hence we obtain that
\[
\rho(H) - \rho_1 \geq \frac{\rho_1 x_0^r}{\rho_1^r + (r-1)x_0^r} \geq \frac{\rho_1 x_0^r}{\rho_1^r + (r-1)\rho_1} = \frac{x_0^r}{\rho_1^{r-1} + r - 1}.
\]

From (4.2) we have
\[
\rho(H) - \rho_1 \geq \frac{x_{\text{max}}}{\rho_1^{rD}} \frac{1}{\rho_1^{r-1} + r - 1} \geq \frac{1}{m\rho_1^{rD}} \frac{1}{\rho_1^{r-1} + r - 1} \geq \frac{1}{n\rho_1^{rD}(\rho_1^{r-1} + r - 1)}.
\]

The proof is completed. □

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