ON THE MODULI SPACE OF $A_\infty$-STRUCTURES

JOHN R. KLEIN AND SEAN TILSON

To Tom Goodwillie on his sixtieth birthday.

Abstract. We study the moduli space of $A_\infty$ structures on a topological space as well as the moduli space of $A_\infty$-ring structures on a fixed module spectrum. In each case we show that the moduli space sits in a homotopy fiber sequence in which the other terms are representing spaces for Hochschild cohomology.

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1. Introduction

There is a long history in algebraic topology of studying homotopy invariant versions of classical algebraic structures. In the 1960s, a theory of $A_\infty$-spaces, that is, spaces equipped with a coherently homotopy associative multiplication, was developed by Stasheff [St1], [St2] and extended by Boardman and Vogt [BV]. One of the main results of this theory is that such a space has the homotopy type of a loop space provided that its monoid of path components forms a group. There was a hint in Stasheff’s work of an obstruction theory for deciding when a space admits an $A_\infty$-structure. Later work by Robinson in the context of $A_\infty$-ring spectra developed such an obstruction theory, in which
the obstructions lie in Hochschild cohomology [R1], [R2] (see also [A]). Robinson’s theory is directed to the question as to whether the moduli space is non-empty and if so, the problem of enumerating its path components. Furthermore, Robinson’s theory depends on an explicit model for the Stasheff associahedron and is therefore not in any obvious way “coordinate free.” The purpose of this paper is to explain from a different perspective why Hochschild cohomology arises when studying moduli problems associated with $A_{\infty}$-spaces and $A_{\infty}$-rings.

The main problem addressed in this paper is identifying the homotopy type of the moduli space of $A_{\infty}$-ring structures on a fixed spectrum. As a warm up, we first investigate the related problem of identifying the homotopy type of the moduli space of $A_{\infty}$-structures on a fixed topological space.

$A_{\infty}$-structures on a space. An $A_{\infty}$-space is a based space which admits a multiplication that is associative up to higher homotopy coherence. According to Boardman and Vogt [BV, th. 1.27], it is always possible to rigidify an $A_{\infty}$-space to a topological monoid in a functorial way. Moreover, there is an appropriate sense in which the homotopy category of $A_{\infty}$-spaces is equivalent to the homotopy category of topological monoids (even more is true: Proposition 3.2 below says that in the derived sense, function complexes of topological monoids are weak equivalent to the corresponding function complexes of $A_{\infty}$-maps; this appears to be well-known [Lu]). Hence, rather than working with $A_{\infty}$-spaces, we can use topological monoids to define the moduli space of $A_{\infty}$-structures.

**Definition 1.1.** If $X$ is a connected based space, Let $\mathcal{C}_X$ be the category whose objects are pairs

$$(M, h)$$

in which $M$ is a topological monoid and $h: X \to M$ is a weak homotopy equivalence of based spaces. A morphism $(M, h) \to (M', h')$ is a homomorphism $f: M \to M'$ such that $h' = f \circ h$.

We define the moduli space

$$\mathcal{M}_X = |\mathcal{C}_X|$$

to be the geometric realization of the nerve of $\mathcal{C}_X$.

For a map $f: Z \to Y$ of (unbased) spaces we define unstable Hochschild “cohomology”

$$\mathcal{H}(Z; Y)$$
to be the space of factorizations

\[ Z \to LY \to Y \]

of \( f \) in which \( LY = \text{map}(S^1, Y) \) is the free loop space of \( Y \) and \( LY \to Y \) is the fibration given by evaluating a free loop at the base point of \( S^1 \). Notice that \( \mathcal{H}(Z; Y) \) has a preferred basepoint given by the factorization \( Z \to LY \to Y \) in which \( Z \to LY \) is the map given by sending \( z \in Z \) to the constant loop with value \( f(z) \).

If we fix \( Y \), and let \( \text{Top}_{/Y} \) be the category of spaces over \( Y \), then the assignment \( Z \mapsto \mathcal{H}(Z; Y) \) defines a contravariant functor \( \text{Top}_{/Y} \to \text{Top}^* \).

**Remark 1.2.** To justify the terminology, let \( G \) be a topological group. Consider the case when \( Z = BG = Y \) is the classifying space of \( G \). Then the fibration \( LBG \to BG \) is fiber homotopy equivalent to the Borel construction

\[ EG \times_G G_{\text{ad}} \to BG . \]

in which \( G_{\text{ad}} \) denotes a copy of \( G \) considered as a left \( G \)-space via the adjoint action \( g \cdot x := gxg^{-1} \) (see e.g., [KSS, §9]).

If we make the latter fibration into a fiberwise spectrum by applying the suspension spectrum functor to each fiber, then the associated spectrum of global sections is identified with the topological Hochschild cohomology spectrum

\[ \mathbb{HH}^*(S[G]; S[G]), \]

where \( S[G] := \Sigma^\infty(G_\leq) \) is the suspension spectrum of \( G \) (also known as the group ring of \( G \) over the sphere spectrum).

Recall for an \( A_\infty \)-space \( X \), there is an inclusion \( \Sigma X \to BX \) which we may regard as a morphism of \( \text{Top}_{/BX} \). In particular, there is a restriction map

\[ (1) \quad \mathcal{H}(BX; BX) \to \mathcal{H}(\Sigma X; BX). \]

**Theorem A.** Assume \( X \) has a preferred \( A_\infty \)-structure. Then the map \( (1) \) sits in a homotopy fiber sequence

\[ \Omega^2 \mathcal{M}_X \to \mathcal{H}(BX, BX) \to \mathcal{H}(\Sigma X, BX) \]

in which the double loop space \( \Omega^2 \mathcal{M}_X \) is identified with the homotopy fiber at the basepoint.

In fact, after some minor modification, the homotopy fiber sequence of Theorem A admits a double delooping, enabling one to recover the moduli space \( \mathcal{M}_X \). If the underlying space of an object \( Z \in \text{Top}_{/Y} \) is equipped with a basepoint \( \ast_Z \in Z \), then \( \ast_Z \) becomes an object in its
own right and we have a morphism \(*_Z \to Z\). This in turn induces a fibration of based spaces
\[
\mathcal{H}(Z; Y) \to \mathcal{H}(*_Z; Y).
\]
We take
\[
\tilde{\mathcal{H}}(Z, Y) = \text{fiber}(\mathcal{H}(Z; Y) \to \mathcal{H}(*_Z; Y))
\]
to be the fiber at the basepoint. Concretely, this is the space of factorizations \(Z \to LY \to Y\) in which \(Z \to LY\) is a based map (where \(LY\) is given the basepoint consisting of the constant loop defined by the image of \(*_Z\) in \(Y\)).

We can regard \(\tilde{\mathcal{H}}(Z, Y)\) as a kind of reduced unstable cohomology. It defines a contravariant functor on \(\text{Top}^*_{/Y}\), the category of based spaces over \(Y\). In particular, we have a map
\[
(2) \quad \tilde{\mathcal{H}}(BX; BX) \to \tilde{\mathcal{H}}(\Sigma X; BX).
\]

Remark 1.3. The if \(Z' \to Z\) is a map of based spaces over a space \(Y\), then it is easy to see that the associated commutative diagram of based spaces
\[
\begin{array}{ccc}
\tilde{\mathcal{H}}(Z; Y) & \longrightarrow & \tilde{\mathcal{H}}(Z'; Y) \\
\downarrow & & \downarrow \\
\mathcal{H}(Z; Y) & \longrightarrow & \mathcal{H}(Z'; Y)
\end{array}
\]
is homotopy cartesian. In particular since \(\Sigma X \to BX\) is a based map, we can replace \(\mathcal{H}\) by \(\tilde{\mathcal{H}}\) in Theorem A and the statement of the theorem remains valid.

The advantage of using the reduced version is made clear by the following companion to Theorem A.

Addendum B. The map \((2)\) has a preferred non-connective double delooping \(\mathcal{B}^2\tilde{\mathcal{H}}(\Sigma X; X) \to \mathcal{B}^2\tilde{\mathcal{H}}(BX; X)\) which sits in a homotopy fiber sequence
\[
\mathcal{M}_X \to \mathcal{B}^2\tilde{\mathcal{H}}(BX, BX) \to \mathcal{B}^2\tilde{\mathcal{H}}(\Sigma X, BX).
\]

Remark 1.4. The main idea of the proof of Addendum B involves showing that the (moduli) space of \(A_\infty\)-spaces \(Y\) which are weakly equivalent to \(X\) (in an unspecified way) gives a double delooping of \(\tilde{\mathcal{H}}(BX, BX)\), and similarly the space of based spaces \(Y\) which are weakly equivalent to \(X\) is a double delooping of \(\tilde{\mathcal{H}}(\Sigma X, BX)\).
$A_\infty$-ring structures. In recent years the subject of algebraic structures on spectra has been profoundly transformed by the existence of improved models for the category of spectra that admit a strictly associative and commutative smash product. What was once called an $A_\infty$-ring spectrum can now be regarded as simply a monoid object in one of the new models.

Henceforth, we work in one of the good symmetric monoidal categories of spectra. To fix our context we work in symmetric spectra. We will fix a connective commutative monoid object $k$ in symmetric spectra. This will function as our ground ring. The category of (left) $k$-modules will be denoted by $k$-mod.

By virtue of [SS] (cf. [EKMM, chap. II]), the category of $A_\infty$-ring spectra is modeled by the category of monoid objects in $k$-mod and their homomorphisms. This category is denoted by $k$-alg. An object of $k$-alg is called a $k$-algebra. There is a forgetful functor $k$-alg $\to k$-mod.

A morphism of either $k$-mod or $k$-alg is a fibration or a weak equivalence if it is one when considered in the underlying category of symmetric spectra. A morphism is a cofibration if it satisfies the left lifting property with respect to the acyclic fibrations. According to [SS, thm 4.1], these notions underly a model structure on $k$-mod and $k$-alg.

**Definition 1.5.** For a $k$-module $E$, the moduli space

$$\mathcal{M}_E$$

is the classifying space of the category $\mathcal{C}_E$ whose objects are pairs

$$(R, h)$$

in which $R$ is a $k$-algebra and $h: E \to R$ is a weak equivalence of $k$-modules. A morphism $(R, h) \to (R', h')$ is a map of $k$-algebras $f: R \to R'$ such that $h' = f \circ h$.

**Remark 1.6.** Using the same notation for the moduli space in each of the settings, i.e., $k$-modules and spaces, should not give rise to any confusion, since the subscript clarifies the context.

Fix a $k$-algebra $R$ and consider the $k$-algebra $R^e := R \wedge_k R^{op}$. A (left) $R^e$-module is also known as an $R$-bimodule. In particular, $R$ is an $R$-bimodule.

**Definition 1.7.** The topological Hochschild cohomology of $R$ with coefficients in an $R$-bimodule $M$ is the homotopy function complex

$$\text{HH}^\bullet(R; M) := R^e\text{-mod}(R, M),$$
given by the hammock localization of $R$-bimodule maps from $R$ to $M$ appearing in [DK1, 3.1].

Suppose now that $1: R \to A$ is a homomorphism of $k$-algebras. Then $A$ is an $R$-bimodule, and $1: R \to A$ is a map of $R$-bimodules. In this way $HH^\bullet(R; A)$ inherits a distinguished basepoint. Furthermore, if $f: R' \to R$ is a $k$-algebra map, then the induced map $f^*: HH^\bullet(R; A) \to HH^\bullet(R'; A)$ is a map of based spaces. In particular, the homomorphism $R^e \to R$ induces a map $HH^\bullet(R; A) \to HH^\bullet(R^e; A) \simeq \Omega^\infty A$. Let $m: R^e \to R$ be the multiplication. Then the composite $1 \circ m: R^e \to A$ is an $R^e$-module map so there is a distinguished basepoint of $HH^\bullet(R^e; A)$ which corresponds to the unit of $\Omega^\infty A$.

**Definition 1.8.** For a $k$-algebra homomorphism $1: R \to A$, the **reduced** topological Hochschild cohomology is defined to be the homotopy fiber

$$D^\bullet(R; A) := \text{hofiber}(HH^\bullet(R; A) \to HH^\bullet(R^e; A))$$

taken at the distinguished basepoint.

Let

$$T: k\text{-mod} \to k\text{-alg}$$

be the **free functor** (i.e., the tensor algebra); this is left adjoint to the forgetful functor $k\text{-alg} \to k\text{-mod}$, so if $R \in k\text{-alg}$ is an object, we have a $k$-algebra map $TR \to R$. In particular, $R$ has the structure of a $TR$-bimodule.

**Theorem C.** Assume $R \in k\text{-mod}$ is equipped with the structure of an $k$-algebra. Then the map $D^\bullet(R; R) \to D^\bullet(TR; R)$ has a preferred (non-connective) double delooping $B^2D^\bullet(R; R) \to B^2D^\bullet(TR; R)$ which sits in a homotopy fiber sequence of based spaces

$$\mathcal{M}_R \to B^2D^\bullet(R; R) \to B^2D^\bullet(TR; R).$$

**Remark 1.9.** As unbased spaces, $B^2D^\bullet(R; R)$ and $B^2D^\bullet(TR; R)$ are constructed so as to depend only on the underlying $k$-module structure of $R$. However, the $k$-algebra structure induces a preferred basepoint making $B^2D^\bullet(R; R) \to B^2D^\bullet(TR; R)$ into a based map.

The double delooping of $D^\bullet(R; R)$ is formally rigged so that its set of path components gives us the correct value of $\pi_0(\mathcal{M}_R)$. Hence, one cannot use Theorem C to compute $\pi_0(\mathcal{M}_R)$. It seems that the only sufficiently general approach to computing path components is

\[1\text{Alternatively, one can define } HH^\bullet(R; M) \text{ as derived simplicial hom, that is the space of maps } R^e \to M^f \text{ in which } R^e \text{ is a cofibrant approximation of } R \text{ and } M^f \text{ is a fibrant approximation of } M \text{ in } R^e\text{-modules; here we are using a simplicial model structure on } R^e\text{-modules to define mapping spaces.} \]
the obstruction theory of [R1], [R2], [A], which is tailored to making such computations.

By taking the two-fold loop spaces and noticing that the reduced cohomology spaces are obtained by fibering the corresponding unreduced cohomology over $\Omega^\infty R$ in each case, we infer

**Corollary D.** There is a homotopy fiber sequence

$$\Omega^2 M_R \to \text{HH}^\bullet (R; R) \to \text{HH}^\bullet (TR; R),$$

in which $\Omega^2 M_R$ is identified with homotopy fiber at the basepoint of $\text{HH}^\bullet (TR; R)$ that is associated with the $TR$-bimodule map $TR \to R$.

**Remark 1.10.** Corollary D is reminiscent of Lazarev’s [La3, thm. 9.2]. However, the moduli space appearing there is different from ours: the points of Lazarev’s moduli space are $k$-algebras $R'$ whose weak homotopy type as a $k$-module is $R$, but $R'$ does not come equipped with a choice of $k$-module equivalence to $R$.

The proof of Theorem C uses various identifications of $D^\bullet (R; A)$ in the case when $A$ is the $R$-bimodule arising from a $k$-algebra homomorphism $1: R \to A$. One of the key identifications interprets $D^\bullet (R; A)$ as the loop space of the space of derived algebra maps $R \to A$:

**Theorem E.** There is a natural weak equivalence

$$D^\bullet (R; A) \simeq \Omega_1 k\text{-alg}(R, A),$$

where the right side is the based loop space at 1 of the homotopy function complex of $k$-algebra homomorphisms from $R$ to $A$.

**Outline.** Section 2 is mostly language. In section 3 we give the proof of Theorem A and Addendum B. This section is independent of the rest of the paper, and we view it as motivation for the $A_\infty$-ring case. Section 4 is the meat of the paper. The hardest part is to establish an augmented equivalence of $A_\infty$-rings between the trivial square zero extension $S \vee S^{-1}$ and the Spanier-Whitehead dual of $S^1_+$ (this appears in Proposition 4.7). In section 5 we deduce Theorem C. Section 6 develops an augmented version of Theorem C. In section 7 we study the homotopy type of the moduli space in two examples: the trivial square zero extension $S \vee S^{-1}$ and the commutative group ring case $S[G]$.

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2. Preliminaries

**Categories.** Let \( \mathcal{C} \) be a category. If \( X, Y \in \mathcal{C} \) are objects, we let \( \text{hom}_\mathcal{C}(X, Y) \) denote the set of arrows from \( X \) to \( Y \).

We let \(|\mathcal{C}|\) be the (geometric) realization of (the nerve of) \( \mathcal{C} \), i.e., the classifying space. Many of the categories \( \mathcal{C} \) considered in this paper are not small. As usual, in order to avoid set theoretic difficulties and have a well-defined homotopy type \(|\mathcal{C}|\), one has to make certain modifications, such as working in a Grothendieck universe. We will implicitly assume this has been done.

We say that a functor \( f: \mathcal{C} \to \mathcal{D} \) is a weak equivalence if it induces a homotopy equivalence on realizations. We say that a composition of functors
\[
\mathcal{C} \xrightarrow{f} \mathcal{D} \to \mathcal{E}
\]
forms a homotopy fiber sequence if after realization if there is a preferred choice of based, contractible space \( U \) together with a commutative diagram
\[
\begin{array}{ccc}
|\mathcal{C}| & \xrightarrow{|f|} & |\mathcal{D}| \\
\downarrow & & \downarrow |g| \\
U & \longrightarrow & |\mathcal{E}|
\end{array}
\]
which is homotopy cartesian. In this paper, \( U = |\mathcal{U}| \) for a suitable pointed category \( \mathcal{U} \), and the diagram arises from a commutative diagram of functors. We call \( \mathcal{U} \) the contracting data.

**Model categories.** The language of model categories will be used throughout the paper. If \( \mathcal{C} \) is a model category, we let
\[
\mathcal{C}(X,Y)
\]
denote the homotopy function complex from \( X \) to \( Y \), where we use the specific model given by the hammock localization of Dwyer and Kan [DK1, 3.1]. In particular, any zig-zag of the form
\[
X = X_0 \leftarrow X_1 \rightarrow X_2 \leftarrow \cdots \rightarrow X_{n-1} \leftarrow X_n = Y
\]
represents a point of \( \mathcal{C}(X,Y) \). If \( \mathcal{C} \) is a simplicial model category (in the sense of Quillen [Q, chap. 2,§1]), \( X \) is cofibrant and \( Y \) is fibrant, then \( \mathcal{C}(X,Y) \) has the homotopy type of the simplicial function space \( F_{\mathcal{C}}(X,Y) \).

Let \( \mathcal{C} \) be a model category. If \( X \in \mathcal{C} \) is an object, let \( h\mathcal{C}(X) \) denote the category consisting of all objects weakly equivalent to \( X \), in which a morphism is a weak equivalence of \( \mathcal{C} \). An important result used in this
paper, due to Dwyer and Kan \cite[prop. 2.3]{DK2} is the weak equivalence of based spaces

\[ |h\mathcal{C}(X)| \simeq Bhaut(X), \]

where $haut(X)$ is the simplicial monoid of homotopy automorphisms of $X$, which is the union of those components of $h\mathcal{C}(X, X)$ that are invertible in the monoid $\pi_0(h\mathcal{C}(X, X))$.

As mentioned above, there are simplicial model category structures on $k$-mod and $k$-alg (see Schwede and Shipley \cite{SS}). In each case the fibrations and weak equivalences are determined by the forgetful functor to symmetric spectra, and the cofibrations are defined by the left lifting property with respect to the acyclic fibrations.

**Spaces.** Let $\text{Top}$ be the category of compactly generated weak Hausdorff spaces. When taking products, we always mean in the compactly generated sense. Function spaces are to be given the compactly generated, compact-open topology. There is a well-known simplicial model category structure on $\text{Top}$ in which a fibration is a Serre fibration, a weak equivalence is a weak homotopy equivalence and a cofibration is defined by the left lifting property with respect to the acyclic fibrations. Similarly, $\text{Top}_*$, the category of based spaces, is a model category where the weak equivalences, fibrations and cofibrations are given by applying the forgetful functor to $\text{Top}$.

**3. Proof of Theorem A and Addendum B**

**Interpretation of the moduli space.** Let $X$ be a cofibrant based space. Let $JX$ be the free monoid on the points of $X$. Then $JX$ is the topological monoid object of $\text{Top}$ given by

\[ X \cup X^{\times 2} \cup \cdots \cup X^{\times n} \cup \cdots \]

where a point in $X^{\times n}$ represents a word of length $n$. Multiplication is defined by word amalgamation and the identifications are given by reducing words, i.e., dropping the basepoint of $X$ whenever it appears.

The moduli space $\mathcal{C}_X$ has an alternative definition as the category whose objects are pairs

\[ (M, h) \]

in which $h: JX \to M$ is a (topological) monoid homomorphism which restricts to a weak homotopy equivalence $X \to M$. A map $(M, h) \to (M', h')$ is a homomorphism $f: M \to M'$ such that $h' = f \circ h$. There is then a decomposition

\[ \mathcal{C}_X = \coprod_{[N, h]} C_X(N, h) \]
where \([N, h]\) runs through the components of \(C_X\), and \(C_X(N, h)\) denotes the component of \(C_X\) given by \([N, h]\).

Let Mon be the category of topological monoid objects of \(\text{Top}_\ast\). Then Mon inherits the structure of a simplicial model category in which the weak equivalences and the fibrations are defined by the forgetful functor \(\text{Mon} \to \text{Top}_\ast\), and cofibrations are defined by the left lifting property with respect to the acyclic fibrations (this follows from [SS], as well as [SV]). Cofibrant objects are retracts of those objects which are built up from the trivial monoid by sequentially attaching free objects, where a free object is of the form \(JY\). Every object is fibrant.

If \(M \in \text{Mon}\) is an object, then we can form the under category \(M \backslash \text{Mon}\) whose objects are pairs \((N, h)\) in which \(h : M \to N\) is a monoid map. A morphism \((N, h) \to (N', h')\) is a monoid map \(f : N \to N'\) such that \(h' = fh\). By Quillen [Q, chap. 2, prop. 6] \(M \backslash \text{Mon}\) forms a simplicial model category in which a fibration, cofibration and weak equivalence are defined by the forgetful functor. When there is no confusion we simplify the notation and drop the structure map when referring to an object: \(N\) henceforth refers to \((N, h)\).

Let \(h\text{Mon} \subset \text{Mon}\) be the category of weak equivalences and let \(N \in M \backslash \text{Mon}\) be a cofibrant object, where \(M \backslash \text{Mon}\) is the (comma) category of objects of \(\text{Mon}\) equipped map from \(M\). By the Dwyer-Kan equivalence (3), there is a homotopy equivalence

\[
| M \backslash h\text{Mon}(N) | \simeq \text{Bhaut}(N \text{ rel } M),
\]

where the right side is the simplicial monoid of homotopy automorphisms of \(N\) relative to \(M\).

We now specialize to the case where \(M = JX\), \(N\) is cofibrant, and the composite \(X \to JX \to N\) is a weak equivalence of based spaces. We claim that there is a homotopy fiber sequence of based spaces

\[
\text{Bhaut}(N) \text{ rel } JX \to \text{Bhaut}_\text{Mon}(N) \to \text{Bhaut}_{\text{Top}_\ast}(N),
\]

where each map is a forgetful map. The claim can be established as follows: consider the forgetful maps of function spaces

\[
F_{\text{Mon}}(N, N; \text{ rel } JX) \to F_{\text{Mon}}(N, N) \to F_{\text{Top}_\ast}(N, N),
\]

These maps form a homotopy fiber sequence of topological monoids (here we use the fact that \(F_{\text{Top}_\ast}(N, N) \simeq F_{\text{Mon}}(JX, N)\)). Taking homotopy invertible components yields a homotopy fiber sequence of homotopy automorphisms. The fiber sequence (4) is then obtained by taking classifying spaces.
We may enhance the homotopy fiber sequence (4) to another one as follows:

\[ \prod_{[N,h]} B\text{haut}(N_{rel} JX) \to \prod_{[N]} B\text{haut}_{\text{Mon}}(N) \to B\text{haut}_{\text{Top}*}(X). \]

Here, the middle term is a disjoint union over the components of \( h\text{Mon} \) whose objects have underlying space weakly equivalent to \( X \), and the first term is the disjoint union over the components of \( C_X \). Furthermore, the Dwyer-Kan equivalence (3) gives an identification

\[ |C_X(N, h)| \simeq B\text{haut}(N_{rel} JX). \]

Consequently, the homotopy fiber sequence (5) arises from the homotopy fiber sequence of categories

\[ C_X \to \prod_{[N]} h\text{Mon}(N) \to h\text{Top}*_{\text{rel}}(X), \]

where the middle term is a coproduct indexed over the components of \( \text{Mon} \) which have the weak homotopy type of \( X \). The functors appearing in (6) are the forgetful functors. In (6), we can take the contracting data \( U \) to be the category whose objects are pairs \((Y, h)\) in which \( Y \) is a based space and \( h: X \to Y \) is a weak equivalence, where a morphism \((Y, h) \to (Y', h')\) is a map \( f: Y \to Y' \) such that \( h' = f \circ h \). Clearly, \((X, \text{id})\) is an initial object so \( U \) is contractible. The functor \( U \to h\text{Top}*_{\text{rel}}(X) \) is the forgetful functor defined by \((Y, h) \mapsto Y\), and the functor \( C_X \to U \) is the forgetful functor defined by the inclusion.

**Lemma 3.1.** Assume \( X \) is a connected cofibrant based space which is equipped the structure of a topological monoid. Then there are weak equivalences

\[ \Omega^2|h\text{Top}(X)| \simeq \tilde{\mathcal{H}}(\Sigma X, BX). \]

and

\[ \Omega^2|h\text{Mon}(X)| \simeq \tilde{\mathcal{H}}(BX, BX). \]

**Proof.** Since \( |h\text{Top}*_{\text{rel}}(X)| \simeq B\text{haut}*_{\text{rel}}(X) \) it suffices to identify \( \Omega^2 B\text{haut}*_{\text{rel}}(X) \) with \( \tilde{\mathcal{H}}(\Sigma X; BX) \). Since \( h\text{aut}*_{\text{rel}}(X) \) is group-like, we have

\[ \Omega^2 B\text{haut}*_{\text{rel}}(X) \simeq \Omega_1 h\text{aut}*_{\text{rel}}(X) = \Omega_1 F*(X, X), \]

where \( \Omega_1 \) denotes loops taken at the identity, and \( F(X, X) \) is the function space of based maps self-maps of \( X \). Hence,

\[ \Omega_1 F*(X, X) \simeq \Omega_1 F*(X, \Omega BX), \]

\[ \simeq \Omega_1 F*(\Sigma X, BX), \]

\[ \simeq \tilde{\mathcal{H}}(\Sigma X, BX). \]
Claim: For group-like topological monoids $X$ and $Y$, the classifying space functor induces a weak equivalence of homotopy function complexes

$$\text{Mon}(X, Y) \simeq \text{Top}_*(BX, BY).$$

The claim can be proved using model category ideas, using the Moore loop functor. For the sake of completeness, we sketch an alternative low-tech argument here. To prove the claim, it is enough to check the statement when $X$ is cofibrant (in this instance, $\text{Mon}(X, Y) \simeq F_{\text{Mon}}(X, Y)$). It is not difficult to show that such an $X$ is a retract of an object built up from a point by attaching free objects, where a free object is of the form $JU$, in which $U$ is a based space and $JU$ is the free monoid on the points of $U$. By naturality, it is enough to check the statement when $X$ itself is inductively built up by attaching free objects. One can now argue by induction. The basis step is for the zero object $X = \ast$. In this case the claim is trivial.

An auxiliary step is to check the claim for a free object $X = JU$. Since $J$ is a left adjoint to the forgetful functor $\text{Mon} \to \text{Top}_*$, the function space of monoid maps $F_{\text{Mon}}(X, Y)$ coincides with the function space of based maps $F_*(U, Y)$. Since $Y$ is group-like, we have

$$F_*(U, Y) \simeq F_*(U, \Omega BY) \simeq F_*(\Sigma U, Y) \simeq F_*(BJU, Y),$$

where we have used a theorem of James to identify $BJU$ with $\Sigma U$.

This concludes the auxiliary step.

For the inductive step, suppose $(D, S)$ is a cofibration pair of based spaces and the claim is true for $X_0$ and let $JS \to X_0$ be a monoid map. Let $X_1 = \text{colim}(X_0 \leftarrow JS \to JD)$. Then, by the fact that (i) function spaces out of pushouts give rise to pullbacks, and (ii) the classifying space functor preserves homotopy pushouts, we infer that the claim is true for $X_1$. This completes the proof sketch of the claim.

The claim implies $|h\text{Mon}(X)| \simeq B\text{haut}_*(BX, BX)$, and we infer

$$\Omega^2|h\text{Mon}(X)| \simeq \Omega^2B\text{haut}_*(BX, BX),$$

$$\simeq \Omega_1\text{haut}_*(BX, BX),$$

$$\simeq \tilde{H}(BX, BX).$$

Proof of Theorem A and Addendum B. Using Lemma 3.1 and (6), there is a homotopy fiber sequence

$$\Omega^2M_X \to \tilde{H}(BX; BX) \to \tilde{H}(\Sigma X; BX).$$
In the second and third terms, we can replace $\tilde{\mathcal{H}}$ by $\mathcal{H}$, since in each case we are fibering over the same space $\mathcal{H}(\ast, BX) \simeq X$.

Lemma 3.1 also shows that the realization of $\coprod_{[N]} h\text{Mon}(N)$ defines a non-connective double delooping of $\tilde{\mathcal{H}}(BX; BX)$ and $h\text{Top}_*(X)$ is a (connective) double delooping of $\mathcal{H}(\Sigma X; BX)$. The homotopy fiber sequence (6) completes the proof.

We end this section with a result about the relation between $A_\infty$-spaces and topological monoids which gives further justification as to why our moduli space $\mathcal{M}_X$ really is a description of the moduli space of $A_\infty$-structures on $X$. According to [SV], the category of $A_\infty$-spaces, denoted here by $\text{Mon}_{A_\infty}$, forms a simplicial model category where a weak equivalence and a fibration are defined by the forgetful functor to $\text{Top}_*$, and cofibrations are defined by the left lifting property with respect to the acyclic fibrations.

**Proposition 3.2.** Let $X$ and $Y$ be topological monoids, where $Y$ is group-like. Then the inclusion of topological monoids into $A_\infty$ spaces induces a weak equivalence of homotopy function complexes

$$\text{Mon}(X, Y) \simeq \text{Mon}_{A_\infty}(X, Y).$$

**Proof.** Consider the composition

$$\text{Mon}(X, Y) \to \text{Mon}_{A_\infty}(X, Y) \to \text{Top}_*(BX, BY).$$

According to [F, 7.7], [BV, prop. 1.6] the second map is a weak equivalence. By the claim appearing in the proof of Lemma 3.1, the composition is a weak equivalence. It follows that the first map is a weak equivalence. □

**Remark 3.3.** Dylan Wilson pointed out to us that Proposition 3.2 is still true without the group-like condition on $Y$. See [Lu, thm. 4.1.4.4, prop. 4.1.2.6].

4. **Proof of Theorem E**

**Universal differentials and Derivations.** Following Lazarev, we define the $R$-bimodule of *universal differentials*

$$\Omega_{k \to R}$$

to be the homotopy fiber of the multiplication map

$$R \wedge_S R^{\text{op}} \to R$$

in the model category $R^e$-mod. From now on we will assume that $R$ is both fibrant and cofibrant.
For an $R$-bimodule $M$, we consider the trivial square zero extension $R ∨ M$, which is a $k$-algebra. Projection onto the first summand is a morphism of $k$-algebras $R ∨ M → R$. The category of $k$-algebra’s over $R$, denoted $k$-alg/$R$, is a model category and $R ∨ M$ is then an object of it.

The homotopy function complex
\[
\text{Der}(R, M) := k\text{-alg}/R(R, R ∨ M)
\]
is then defined. We make this into a based space using the inclusion $R → R ∨ M$.

Remark 4.1. Suppose that $1: R → A$ is a homomorphism of $k$-algebras and $M$ is an $A$-bimodule. Then the diagram
\[
\begin{array}{ccc}
R ∨ M & → & A ∨ M \\
\downarrow & & \downarrow \\
R & → & A
\end{array}
\]
is homotopy cartesian. We infer that there is a weak equivalence
\[
\text{Der}(R; M) \simeq k\text{-alg}/A(R, A ∨ M).
\]

Lemma 4.2. There is a weak equivalence
\[
\text{Der}(R; Σ^{-1}M) \simeq Ω\text{Der}(R; M).
\]

Proof. The functor $M \mapsto \text{Der}(R; Σ^{-1}M)$ preserves homotopy cartesian squares of bimodules. There is a homotopy cartesian square
\[
\begin{array}{ccc}
Σ^{-1}M & → & * \\
\downarrow & & \downarrow \\
* & → & M
\end{array}
\]
Furthermore, it is evident that $\text{Der}(R; *) = k\text{-alg}/R(R, R)$ is contractible. Hence, the map
\[
\text{Der}(R, Σ^{-1}M) → \text{holim}(* \mapsto \text{Der}(R, M) ← *) \simeq Ω\text{Der}(R, M)
\]
is a weak equivalence.

Proposition 4.3 (Lazarev [La1], Dugger-Shipley [DS]). For any $R$-bimodule $M$, there is a weak equivalence
\[
\text{Der}(R, M) \simeq R^e\text{-mod}(Ω_{k→R}, M).
\]

Remark 4.4. Lazarev’s proof of this statement contains serious gaps. The proof was corrected by Dugger and Shipley, based on an unpublished result of Mandell (cf. [DS, rem. 8.7]).
Given a $k$-algebra homomorphism $1: R \to A$, we can regard $A$ as an $R$-bimodule in the evident way.

**Corollary 4.5.** There is a weak equivalence

$$\text{Der}(R, A) \simeq R^e\text{-mod}(\Omega_{k \to R}, A).$$

Consider the fibration sequence

$$\Omega_{k \to R} \to R \wedge S^R \to R.$$

Apply $R^e\text{-mod}(-, A)$ to this sequence to get a homotopy fiber sequence

$$\text{HH}^\bullet(R; A) \to \Omega^\infty A \to R^e\text{-mod}(\Omega_{k \to R}, A)$$

where we have identified the middle term with $R^e\text{-mod}(R^e, A)$. By shifting the homotopy fiber sequence once over to the left (using the unit component of $\Omega^\infty A \simeq \text{HH}^\bullet(R^e; A)$ as basepoint), we see that there’s a weak equivalence

$$\Omega R^e\text{-mod}(\Omega_{k \to R}, A) \simeq D^\bullet(R; A).$$

If we combine this with Corollary 4.5 and Lemma 4.2, we infer

**Corollary 4.6.** There is a weak equivalence

$$D^\bullet(R; A) \simeq \Omega \text{Der}(R, A).$$

Let $A_1$ be the $k$-algebra

$$A \vee \Sigma^{-1} A$$

in which $\Sigma^{-1} A$ is given the evident $A$-bimodule structure making $A_1$ into a trivial square zero extension of $A$. Also, let $\mathcal{L}A$ denote the $k$-algebra given by the function spectrum

$$F(S^1_+, A),$$

taken in the category of symmetric spectra. The multiplicative structure on $\mathcal{L}A$ arises from the multiplication on $A$ and the diagonal map $S^1 \to S^1 \times S^1$. Then we have

**Proposition 4.7.** There is a weak equivalence of augmented $k$-algebras

$$\mathcal{L}A \simeq A_1.$$

**Proof.** This is claimed by Lazarev [La2, th. 4.1], but we were unable to understand his argument. Fortunately, we were helped out by Mike Mandell, who explained a different proof to us. We sketch Mandel’s argument below when $k = S$ and leave the general case as an exercise for the reader.
The $S$-algebra $\mathcal{L}S := F(S^1_+, S)$ is just the Spanier-Whitehead dual of $S^1_+$ in the category of symmetric spectra; it has the structure of a (commutative) $S$-algebra (cf. [C]).

The evident pairing

$$A \wedge_S \mathcal{L}S^1 \rightarrow \mathcal{L}A$$

is a weak equivalence of $S$-algebras. It is therefore enough to show that $\mathcal{L}S^1 \simeq S \vee S^{-1}$ as $S$-algebras, since then

$$\mathcal{L}A \simeq A \wedge_S \mathcal{L}S^1 \simeq A \wedge_S (S \vee S^{-1}) \cong A_1.$$

Let $R$ be an augmented $k$-algebra. The idea of the remainder of the proof is to study the forgetful map

$$S\text{-alg}/S(R, S \vee S^{-1}) \rightarrow S\text{-mod}/S(R, S \vee S^{-1}),$$

which is just the map

(7)

$$S\text{-alg}/S(R, S \vee S^{-1}) \rightarrow S\text{-alg}/S(TR, S \vee S^{-1}),$$

induced by the algebra homomorphism $TR \rightarrow R$.

Using Proposition 4.3, Remark 4.1 and Lemma 4.2, there is a homotopy fiber sequence

(8)

$$S\text{-alg}/S(R, S \vee S^{-1}) \rightarrow R^e\text{-mod}(R, S) \rightarrow \Omega^\infty S,$$

where the displayed homotopy fiber is taken at the basepoint of $\Omega^\infty S$ given by the unit. Note that the $R^e$-module structure on $S$ arises by augmentation, so an extension by scalars argument shows

(9)

$$R^e\text{-mod}(R, S) \simeq R\text{-mod}(S, S).$$

Combining the fiber sequence (8) with this last identification yields a homotopy fiber sequence

(10)

$$S\text{-alg}/S(R, S \vee S^{-1}) \rightarrow R\text{-mod}(S, S) \rightarrow \Omega^\infty S.$$

With respect to the homomorphism $TR \rightarrow R$, we obtain a diagram

(11) $$\begin{array}{ccc}
S\text{-alg}/S(R, S \vee S^{-1}) & \rightarrow & R\text{-mod}(S, S) \\
\downarrow & & \downarrow \\
S\text{-mod}/S(R, S \vee S^{-1}) & \rightarrow & TR\text{-mod}(S, S)
\end{array}$$

which is homotopy cartesian by (10) (where $R$ replaced by $TR$ in (10) for the bottom horizontal map of (11)). Henceforth, we specialize to the case $R = \mathcal{L}S$ (but the argument below works equally well for any $k$-algebra which is weakly equivalent to $S \vee S^{-1}$ as an augmented $S$-module; cf. Remark 4.8 below).
Note that
\[ \pi_0(S\text{-mod}/S)(R, S \vee S^{-1}) \cong \pi_0(S\text{-mod}(R, S^{-1})) \cong \mathbb{Z}, \]
since, as augmented \( S \)-modules, \( R \simeq S \vee S^{-1} \). Furthermore, up to homotopy, such a weak equivalence corresponds to one of the two possible generators of \( \mathbb{Z} \). To lift either of these weak equivalences to an algebra map, it suffices to show that the right vertical map of the diagram (11) is surjective on \( \pi_0 \). In fact, we will show that the right vertical map is a retraction up homotopy.

It is reasonably well-known that \( R\text{-mod}(S, S) \), considered as an \( S \)-module, coincides up to homotopy with \( S\text{-mod}(B_{\text{alg}} R, S) \) where \( B_{\text{alg}} R \) is the bar construction on \( R \) in the category of augmented \( S \)-algebras. Similarly, one can show that \( TR\text{-mod}(S, S) \) coincides up to homotopy with \( S\text{-mod}(B_{\text{alg}} TR, S) \). We need to understand the map
\[ (12) \quad B_{\text{alg}} TR \rightarrow B_{\text{alg}} R. \]
The bar construction \( B_{\text{alg}} R \) is not hard to identify as an \( S \)-module. The homotopy spectral sequence defined by the skeletal filtration has \( E^2 \)-term \( E^2_{p,q} = \pi_q(S^{-p}) \). It is a spectral sequence of \( \pi_* (S) \)-modules and it evidently degenerates at the \( E^2 \)-page. So we obtain a weak equivalence of \( S \)-modules
\[ B_{\text{alg}} R \simeq \bigvee_{j \geq 0} S. \]
This computation shows \( B_{\text{alg}} R \) coincides with the associated graded of the filtration defined by skeleta.

As for \( B_{\text{alg}} TR \), it coincides with \( B_{\text{mod}} R \), the bar construction of \( R \) considered as an augmented \( S \)-module with respect to the monoidal structure given by the coproduct of augmented modules. Furthermore, \( B_{\text{mod}} R \) is easily identified with \( \Sigma S R \), the (fiberwise) suspension of \( R \) considered as an augmented \( S \)-module. As \( R \simeq S \vee S^{-1} \) as augmented \( S \)-modules, we have \( \Sigma S R \simeq S \vee S \). Therefore (12) amounts to the map
\[ \bigvee_{0 \leq j \leq 1} S \rightarrow \bigvee_{j \geq 0} S \]
given by the inclusion of the 1-skeleton into \( B_{\text{alg}} R \). It is clear that this inclusion is a split summand, so the restriction map \( S\text{-mod}(B_{\text{alg}} R, S) \rightarrow S\text{-mod}(B_{\text{alg}} TR, S) \) is a retraction up to homotopy. In particular, the right vertical map of (11) is a surjection on \( \pi_0 \).

Remark 4.8. The above proof actually shows that any augmented \( S \)-algebra \( R \) equipped with a weak equivalence to \( S \vee S^{-1} \) as an augmented \( S \)-module has a lifting to a weak equivalence as an augmented
S-algebra. Furthermore, the proof gives a homotopy fiber sequence
\[ \prod_{j \geq 2} \Omega^\infty S \to S\text{-alg}/S(R, S \vee S^{-1}) \to S\text{-mod}/S(R, S \vee S^{-1}). \]
A version of this sequence also holds in the unaugmented case.

We apply Proposition 4.7 in the following instance. By the adjunction property, we have
\[ (13) \quad \Omega_1 k\text{-alg}(R, A) \cong k\text{-alg}/A(R, \mathcal{L}A). \]
We are now in a position to deduce Theorem E:

**Corollary 4.9.** Let \( 1: R \to A \) be a k-algebra homomorphism. Then there is a natural weak equivalence
\[ \Omega_1 k\text{-alg}(R, A) \simeq D^\bullet(R; A). \]

**Proof.** This uses the chain of weak equivalences
\[ \Omega_1 k\text{-alg}(R, A) \simeq k\text{-alg}/A(R, A \vee \Sigma^{-1}A), \text{ by (13) and Prop. 4.7} \]
\[ = \text{Der}(R, \Sigma^{-1}A), \quad \text{by definition} \]
\[ \simeq \Omega \text{Der}(R; A), \quad \text{by Lem. 4.2} \]
\[ \simeq D^\bullet(R; A), \quad \text{by Cor. 4.6}. \]

\[ \square \]

### 5. Proof of Theorem C

Let \( R \) be a k-algebra. By essentially the same argument appearing in §3, there is a homotopy fiber sequence of categories
\[ (14) \quad \mathcal{C}_R \to \prod_{[R']} hk\text{-alg}(R') \to hk\text{-mod}(R), \]
in which the decomposition appearing in the middle is indexed over those components of \( hk\text{-alg} \) (= the category of k-algebra weak equivalences), which have the property that \( R' \) is weak equivalent to \( R \) as a k-module. In other words, the middle category is the full subcategory of \( hk\text{-alg} \) whose objects are weak equivalent to \( R \) as a k-module. Similarly, \( hk\text{-mod}(R) \) denotes component the category of weak equivalences which contains \( R \). The categories appearing in (14) have a preferred basepoint determined by the k-algebra \( R \) and \( \mathcal{C}_R \) corresponds to the homotopy fiber at the basepoint.

The contracting data \( \mathcal{U} \) for (14) is given by the category whose objects are pairs \( (N, h) \) where \( N \) is an \( R \)-module and \( h: R \to N \) is a weak equivalence of \( R \)-modules. A morphism \( (N, h) \to (N', h') \) is a
map \( f : N \to N' \) such that \( h' = f \circ h \). The functor \( \mathcal{C}_R \to \mathcal{U} \) is the forgetful functor, as is the functor \( \mathcal{U} \to \mathcal{h}_k\text{-mod}(R) \). Moreover, \( \mathcal{U} \) is contractible, since \( (R, \text{id}) \) is an initial object.

As in §3, the strategy will be to identify the middle and last terms of (14) as double deloopings of the Hochschild cohomology spaces.

**Lemma 5.1.** There are weak equivalences of based spaces

\[
D^\bullet(R; R) \simeq \Omega^2|\mathcal{h}_k\text{-alg}(R)|
\]

and

\[
D^\bullet(TR; R) \simeq \Omega^2|\mathcal{h}_k\text{-mod}(R)|.
\]

**Proof.** Using the Dwyer-Kan equivalence, double loop space of \(|\mathcal{h}_k\text{-alg}(R)|\) taken at the point defined by \( R \) is identified with

\[
\Omega^2\mathcal{B}\text{haut}_{k\text{-alg}}(R) \simeq \Omega_1k\text{-alg}(R, R).
\]

Using Corollary 4.9 applied to the identity map \( R \to R \), we obtain a weak equivalence

\[
\Omega_1k\text{-alg}(R, R) \simeq D^\bullet(R; R).
\]

This establishes the first part of the lemma.

For the second part, we use the chain of identifications,

\[
\Omega^2|\mathcal{h}_k\text{-mod}(R)| \simeq \Omega^2\mathcal{B}\text{haut}_{k\text{-mod}}(R),
\]

\[
\simeq \Omega_1\text{haut}_{k\text{-mod}}(R),
\]

\[
\simeq \Omega_1\text{k-alg}(TR, R),
\]

\[
\simeq D^\bullet(TR; R).
\]

where the last weak equivalence is obtained from Corollary 4.9 applied to the \( k \)-algebra map \( TR \to R \). \( \square \)

**Proof of Theorem C.** Use the homotopy fiber sequence (14) together with Lemma 5.1. Note that with the deloopings, the map

\[
D^\bullet(R; R) \to D^\bullet(TR; R)
\]

has a preferred double delooping given by realizing the forgetful functor

\[
\prod_{[R']}\mathcal{h}_k\text{-alg}(R') \to \mathcal{h}_k\text{-mod}(R).
\]

\( \square \)
6. The augmented case

**Definition 6.1.** For an augmented $k$-algebra $R$ we define the *moduli space of augmented $k$-algebra* structures on $R$,

$$\mathcal{M}_{R/k},$$

to be the classifying space of the category whose objects are pairs $(E, h)$ in which $E$ is an augmented $k$-module and $h: E \to R$ is a weak equivalence of augmented $k$-modules. A morphism $(E, h) \to (E', h')$ is an augmentation preserving map $f: E \to E'$ such that $h' \circ f = h$.

It is a consequence of the definition that:

- There is an evident forgetful map
  $$\mathcal{M}_{R/k} \to \mathcal{M}_R.$$
- There is a homotopy fiber sequence
  $$\Omega^2 \mathcal{M}_{R/k} \to \Omega_1 k\text{-alg}/k(R, R) \to \Omega_1 k\text{-mod}/k(R, R).$$

**Definition 6.2.** Let $M$ be an $R$-bimodule which is augmented over $k$. We set

$$\text{HH}^*(R/k; M) := R^\text{e}-\text{mod}/k(R, M),$$

i.e., the homotopy function complex associated to the augmented bimodule maps $R \to M$.

Given an augmented $k$-algebra map $1: R \to A$, we may regard $A$ as an augmented $R$-bimodule. Then restriction defines a map of based spaces

$$\text{HH}^*(R/k; A) \to \text{HH}^*(R^\text{e}/k; A).$$

and we let $D^\bullet(R/k; A)$ be the homotopy fiber taken at the point defined by the bimodule map $R^\text{e} \to R \to A$.

The following is the augmented version of Theorem C. As the proof is similar, we omit it.

**Theorem 6.3.** There is a homotopy fiber sequence

$$\mathcal{M}_{R/k} \to \mathcal{B}^2 D^\bullet(R/k; A) \to \mathcal{B}^2 D^\bullet(TR/k; A),$$

where the deloopings in each case are defined as in the proof of Theorem C.

7. Examples

Computations are somewhat easier to make in the augmented case, since in the unaugmented setting one needs to understand $k\text{-alg}(R, k)$. 
The square-zero case. Let $R = S \vee S^{-1}$ be the trivial square zero extension of $S$. We wish to study the homotopy type of $\mathcal{M}_R$ in this case. The proof of Proposition 4.7 shows that the augmented version $\mathcal{M}_{R/S}$ is connected. Moreover, inspection of the proof shows that it amounts to a computation of $\Omega \mathcal{M}_{R/S}$. The result is that one gets a weak equivalence

$$\Omega \mathcal{M}_{R/S} \simeq \prod_{j \geq 2} \Omega^\infty S.$$  

(cf. Remark 4.8). In the square-zero case, the relationship between $\Omega^2 \mathcal{M}_{R/S}$ and $\Omega^2 \mathcal{M}_R$ is easy to describe.

**Lemma 7.1.** When $R = S \vee S^{-1}$ is the trivial square zero extension, there is a weak equivalence of based spaces

$$\Omega^2 \mathcal{M}_R \simeq L \Omega^2 \mathcal{M}_{R/S}.$$

*Proof. (Sketch).* Using the homotopy cartesian diagram of $S$-algebras

$$
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
S & \longrightarrow & S \times S
\end{array}
$$

in which each map $S \to S \times S$ is the diagonal, we can apply the homotopy function complex out of $R$ to obtain a homotopy cartesian square

$$
\begin{array}{ccc}
S\text{-alg}(R, R) & \longrightarrow & S\text{-alg}(R, S) \\
\downarrow & & \downarrow \\
S\text{-alg}(R, S) & \longrightarrow & S\text{-alg}(R, S) \times S\text{-alg}(R, S)
\end{array}
$$

which shows

$$S\text{-alg}(R, R) \simeq L S\text{-alg}(R, S).$$

On the other hand, there is a homotopy fiber sequence

$$S\text{-alg}/S(R, R) \to S\text{-alg}(R, R) \to S\text{-alg}(R, S)$$

and Proposition 4.7 gives a weak equivalence $S\text{-alg}/S(R, R) \simeq \Omega_1 S\text{-alg}(R, S)$. A careful check of how the identification is made, which we omit, enables us to deduce

$$S\text{-alg}(R, R) \simeq L S\text{-alg}(R, S).$$
A similar argument in the module case shows $S\text{-mod}(R, R) \simeq L S\text{-mod}(R, S)$. Hence there are weak equivalences

$$\Omega^2 \mathcal{M}_R \simeq \text{Lh'}(\Omega_1 S\text{-alg}(R, S) \to \Omega_1 S\text{-mod}(R, S))$$

$$\simeq \text{Lh'}(S\text{-alg}/S(R, R) \to S\text{-mod}/S(R, R))$$

$$\simeq L \Omega^2 \mathcal{M}_{R/S}.$$ 

□

**Corollary 7.2.** When $R = S \vee S^{-1}$ is the trivial square-zero extension, there is a weak equivalence

$$\Omega^2 \mathcal{M}_R \simeq \prod_{j \geq 2} \Omega^\infty S^{-1} \times \Omega^\infty S^{-2}.$$ 

**Proof.** It was noted already that Remark 4.8 gives the computation $\Omega \mathcal{M}_{R/S} \simeq \prod_{j \geq 2} \Omega^\infty S$. Take the based loops of both sides, then apply free loops and use Lemma 7.1. □

**Commutative group rings.** Suppose $R = S[G] := S \wedge (G_+)$ is the group ring on a topological abelian group $G$ (for technical reasons, we assume that the underlying space of $G$ is cofibrant). Then the adjoint action of $G$ acting on $R$ is trivial, and it is not difficult exhibit a weak equivalence

$$\text{HH}^\bullet(R; R) \simeq F((BG)_+, \Omega^\infty R),$$

where the space on the right is the function space of unbased maps $BG \to \Omega^\infty R$. Similarly,

$$\text{HH}^\bullet(TR; R) \simeq F((\Sigma G)_+, \Omega^\infty R).$$

The map $\text{HH}^\bullet(R; R) \to \text{HH}^\bullet(TR; R)$ is induced in this case by the inclusion $\Sigma G \to BG$ given by the 1-skeleton of $BG$. Let $X_G = BG/\Sigma G$. Using Corollary D, we obtain a weak equivalence

$$\Omega^2 \mathcal{M}_R \simeq F(X_G, \Omega^\infty R).$$

In particular, $\pi_*(\mathcal{M}_R) = R^{2-*}(X_G)$ is the shifted $R$-cohomology of $X_G$ for $k \geq 2$. The space $X_G$ comes equipped with a filtration, so one gets an Atiyah-Hirzebruch spectral sequence converging to its $R$-cohomology.

Let us specialize to case when $G = Z$ is the integers. Then we have $R = S[Z] = S[t, t^{-1}]$ is the Laurent ring over $S$ in one generator. In this instance

$$X_Z \simeq \bigvee_j S^2.$$
is a countable infinite wedge of 2-spheres and we infer
\[ \Omega^2 \mathcal{M}_{S}[\mathbb{Z}] \simeq \Omega^2 \prod_j \Omega^\infty S[\mathbb{Z}]. \]

The right side admits a further decomposition into an countable infinite product of copies of \( \Omega^\infty S \), using the \( S \)-module identification \( S[\mathbb{Z}] \simeq \bigvee_j S \).

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