Finite Horn Monoids and Near-Semirings

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Describing complex objects as the composition of elementary ones is a common strategy in computer science and science in general. This paper contributes to the foundations of knowledge representation and database theory by introducing and studying the sequential composition of propositional Horn theories. Specifically, we show that the notion of composition gives rise to a family of monoids and near-semirings, which we will call Horn monoids and Horn near-semirings in this paper. Particularly, we show that the combination of sequential composition and union yields the structure of a finite idempotent near-semiring. We also show that the restricted class of proper propositional Krom-Horn theories, which only contain rules with exactly one body atom, yields a finite idempotent semiring. On the semantic side, we show that the immediate consequence or van Emden-Kowalski operator of a theory can be represented via composition, which allows us to compute its least model semantics without any explicit reference to operators. This bridges the conceptual gap between the syntax and semantics of a propositional Horn theory in a mathematically satisfactory way. Moreover, it gives rise to an algebraic meta-calculus for propositional Horn theories. In a broader sense, this paper is a first step towards an algebra of rule-based logical theories and in the future we plan to adapt and generalize the methods of this paper to wider classes of theories, most importantly to first-, and higher-order logic programs, and non-monotonic logic programs under the stable model or answer set semantics and extensions thereof.

Additional Key Words and Phrases: computational logic, knowledge representation and reasoning, algebraic structures in logic

ACM Reference Format:
Christian Antić. 2020. Finite Horn Monoids and Near-Semirings. 1, 1 (September 2020), 20 pages.

1 INTRODUCTION

Propositional Horn theories are rule-based formalisms written in a sublanguage of classical propositional logic with important applications in artificial intelligence and computer science in general (cf. [11, 18]). Its rule-like structure naturally induces the structure of a monoid on the space of all such theories as we are going to demonstrate in this paper.

Semigroups and monoids are fundamental algebraic structures ubiquitous in mathematics and theoretical computer science. In this paper, we contribute to the foundations of knowledge representation and database theory by introducing and studying the sequential composition of propositional Horn theories as an associative binary operation on theories with unit. Specifically, we show that the notion of composition gives rise to a family of finite monoids and near-semirings (Theorem 3.2), which we will call Horn monoids and Horn near-semirings in this paper. We also show that the restricted class of proper propositional Krom-Horn theories, which only contain rules with exactly one body atom, yields an idempotent semiring (Theorem 3.3). On the semantic side, we show that the immediate consequence or
van Emden-Kowalski operator of a theory can be represented via composition (Theorem 6.2), which allows us to compute its least model semantics without any explicit reference to operators (Theorem 6.7). This bridges the conceptual gap between the syntax and semantics of a propositional Horn theory in a mathematically satisfactory way. We then proceed by studying decompositions of theories (Section 5). As the main result in this regard, we show that acyclic theories can be decomposed into a product of single-rule theories (Theorem 5.6). Finally, we provide some results on decompositions of general propositional Horn theories (Section 5.2).

From a logical point of view, we obtain a meta-calculus for reasoning about propositional Horn theories. From an algebraic point of view, this paper establishes a bridge between propositional Horn theories and (finite) semigroup theory, which enables us to transfer algebraic concepts from the rich theory of finite semigroups and monoids to the setting of propositional Horn theories and extensions thereof.

In a broader sense, this paper is a first step towards an algebra of rule-based logical theories and in the future we plan to adapt and generalize the methods of this paper to wider classes of theories, most importantly for first-, and higher-order logic programs [1, 5, 17, 19], and non-monotonic logic programs under the stable model [9] or answer set semantics [15] and extensions thereof (cf. [3, 4, 8, 16]).

2 PRELIMINARIES

In this section, we recall the syntax and semantics of propositional Horn theories, and the algebraic structures occurring in the rest of the paper.

2.1 Algebraic Structures

We recall some basic algebraic notions and notations by mainly following the lines of [12, 13].

Given two sets $A$ and $B$, we write $A \subseteq_k B$ in case $A$ is a subset of $B$ with $k$ elements, for some non-negative integer $k$, and given an object $o$ of size $k$, we will write $A \subseteq_o B$ instead of $A \subseteq_{\text{size}(o)} B$. We denote the identity function on a set $A$ by $1d_A$.

A partially ordered set (or poset) is a set $L$ together with a reflexive, transitive, and anti-symmetric binary relation $\leq$ on $L$.

A semigroup is a set $S$ together with an associative binary operation $\cdot$ on $S$. A monoid is a semigroup containing a unit element $1$ such that $1x = x = 1$ holds for all $x$. A group is a monoid which contains an inverse $x^{-1}$ for every $x$ such that $xx^{-1} = x^{-1}x = 1$. A left (resp., right) zero is an element $0$ such that $0x = 0$ (resp., $x0 = 0$) holds for all $x \in S$. An ordered semigroup is a semigroup $S$ together with a partial order $\preceq$ that is compatible with the semigroup operation, meaning that $x \preceq y$ implies $zx \preceq zy$ and $xz \preceq yz$ for all $x, y, z \in S$. An ordered monoid is defined in the obvious way. A relation $\sim$ on the set $S$ is called left (resp., right) compatible with the operation on $S$ if $x \sim y$ implies $zx \sim zy$ (resp., $xz \sim yz$), for all $x, y, z \in S$. It is called compatible if $x \sim y$ implies $wxz \sim wyz$, for all $w, x, y, z \in S$. A left (resp., right) compatible relation is called a left (resp., right) congruence. A compatible relation is called a congruence. A relation on a semigroup $S$ is a congruence if, and only if, it is both a left and a right congruence. A non-empty subset $I$ of $S$ is called a left (resp., right) ideal if $SI \subseteq I$ (resp., $IS \subseteq I$), and a (two-sided) ideal if it is both a left and right ideal.

A near-semiring is a set $S$ together with two binary operations $+$ and $\cdot$ on $S$, and a constant $0 \in S$, such that $(S, +, 0)$ is a monoid and $(S, \cdot)$ is a semigroup satisfying the following laws:

1. $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in S$ (right-distributivity); and
2. $0 \cdot x = 0$ for all $x \in S$.
We recall the syntax and semantics of propositional Horn theories. We say that a least model with respect to set inclusion called the single-rule theory \( r \) is idempotent if \( x + x = x \) holds for all \( x \in S \). A semiring is a near-semiring \((S, +, \cdot, 0)\) such that \( + \) is commutative and additionally to the laws of a near-semiring the following laws are satisfied:

1. \( x \cdot (y + z) = x \cdot y + x \cdot z \) for all \( x, y, z \in S \) (left-distributivity); and
2. \( x \cdot 0 = 0 \) for all \( x \in S \).

2.2 Propositional Horn Theories

We recall the syntax and semantics of propositional Horn theories.

2.2.1 Syntax. In the rest of the paper, \( A \) denotes a finite alphabet of propositional atoms.

A (propositional Horn) theory over \( A \) is a finite set of rules of the form

\[
a_0 \leftarrow a_1, \ldots, a_k, \quad k \geq 0,
\]

where \( a_0, \ldots, a_k \in A \) are propositional atoms. It will be convenient to define, for a rule \( r \) of the form \( 1, \text{head}(r) = \{a_0\} \) and \( \text{body}(r) = \{a_1, \ldots, a_k\} \) extended to theories by \( \text{head}(P) = \bigcup_{r \in P} \text{head}(r) \) and \( \text{body}(P) = \bigcup_{r \in P} \text{body}(r) \). In this case, the size of \( r \) is \( k \). A fact is a rule with empty body and a proper rule is a rule which is not a fact. We denote the facts and proper rules in \( P \) by \( \text{facts}(P) \) and \( \text{proper}(P) \), respectively. We call a rule \( r \) of the form Krom\(^1\) if it has at most one body atom, and we call \( r \) binary if it contains at most two body atoms. A tautology is any Krom rule of the form \( a \leftarrow a \), for \( a \in A \). We call a theory Krom if it contains only Krom rules, and we call it binary if it consists only of binary rules. A theory is called a single-rule theory if it contains exactly one non-tautological rule.

For two propositional Horn theories \( P \) and \( R \), we say that \( P \) depends on \( R \) if the intersection of \( \text{body}(P) \) and \( \text{head}(R) \) is not empty. We call \( P \) acyclic if there is a mapping \( \ell : A \rightarrow \{0, 1, 2, \ldots\} \) such that for each rule \( r \in P \), we have \( \ell(\text{head}(r)) > \ell(\text{body}(r)) \), and in this case we call \( \ell \) a level mapping for \( P \). Of course, every level mapping \( \ell \) induces an ordering on rules via \( r \preceq s \) if \( \ell(\text{head}(r)) \leq \ell(\text{head}(s)) \). We can transform this ordering into a total ordering by arbitrarily choosing a particular linear ordering within each level, that is, if \( \ell(a) = \ell(b) \) then we can choose between \( a < b \) or \( b < a \).

Define the reverse of \( P \) by

\[
P^{rev} = \text{facts}(P) \cup \{ A \leftarrow \text{head}(r) \mid r \in \text{proper}(P) : A \in \text{body}(r) \}.
\]

Roughly, we obtain the reverse of a theory by reversing all the arrows of its proper rules.

2.2.2 Semantics. An interpretation is any set of atoms from \( A \). We define the entailment relation, for every interpretation \( I \), inductively as follows: (i) for an atom \( a, I \models a \) if \( a \in I \); (ii) for a set of atoms \( B, I \models B \) if \( B \subseteq I \); (iii) for a rule \( r \) of the form \( 1, I \models r \) if \( I \models \text{body}(r) \) implies \( I \models \text{head}(r) \); and, finally, (iv) for a propositional Horn theory \( P, I \models P \) if \( I \models r \) holds for each rule \( r \in P \). In case \( I \models P \), we call \( I \) a model of \( P \). The set of all models of \( P \) has a least element with respect to set inclusion called the least model of \( P \) and denoted by \( \text{LM}(P) \). We say that \( P \) and \( R \) are equivalent if \( \text{LM}(P) = \text{LM}(R) \).

Define the van Emden-Kowalski operator \([20]\) of \( P \), for every interpretation \( I \), by

\[
T_P(I) = \{ \text{head}(r) \mid r \in P : I \models \text{body}(r) \}.
\]

We have the following well-known operational characterization of models in terms of the van Emden-Kowalski operator \([20]\).

\(^1\)Krom rules where first introduced and studied by [14]
Proposition 2.1. An interpretation $I$ is a model of $P$ if, and only if, $I$ is a prefixed point of $T_P$.

We call an interpretation $I$ a supported model of $P$ if $I$ is a prefixed point of $T_P$. We say that $P$ and $R$ are subsumption equivalent if $T_P = T_R$.

The following constructive characterization of least models is due to [20].

Proposition 2.2. The least model of a propositional Horn theory coincides with the least fixed point of its associated van Emden-Kowalski operator, that is, for any theory $P$ we have

$$LM(P) = lfp(T_P).$$

3 COMPOSITION

In the rest of the paper, $P$ and $R$ denote propositional Horn theories over some joint alphabet $A$.

Definition 3.1. We define the (sequential) composition of $P$ and $R$ by

$$P \circ R = \{ head(r) \leftarrow body(S) \mid r \in P, S \subseteq R : head(S) = body(r) \}.$$

We will write $PR$ in case the composition operation is understood.

Roughly, we obtain the composition of $P$ and $R$ by resolving all body atoms in $P$ with the 'matching' rule heads of $R$.

Notice that we can reformulate sequential composition as

$$P \circ R = \bigcup_{r \in P} ((r) \circ R),$$

which directly implies right-distributivity of composition, that is,

$$(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$$

holds for all propositional Horn theories $P, Q, R$. (4)

However, the following counter-example shows that left-distributivity fails in general:

$$\{a \leftarrow b, c\} \circ (\{b\} \cup \{c\}) = \{a\} \quad \text{and} \quad (\{a \leftarrow b, c\} \circ \{b\}) \cup (\{a \leftarrow b, c\} \circ \{c\}) = \emptyset.$$

This shows that, unfortunately, the space of all propositional Horn theories (over $A$) fails to form a semiring with respect to union and sequential composition (but see Theorems 3.2 and 3.3).

Of course, we can write $P$ as the union of its facts and proper rules, that is,

$$P = facts(P) \cup proper(P).$$

So we can rewrite the composition of $P$ and $R$ as

$$PR = (facts(P) \cup proper(P))R = facts(P)R \cup proper(P)R = facts(P) \cup proper(PR),$$

which shows that the facts in $P$ are preserved by composition, that is, we have

$$facts(P) \subseteq facts(PR).$$

Define the unit theory (over $A$) by the propositional Krom-Horn theory

$$1_A = \{a \leftarrow a \mid a \in A\}.$$
In the sequel, we will often omit the reference to $A$.

We are now ready to state the main structural result of the paper.

**Theorem 3.2.** The space of all propositional Horn theories forms a finite monoid with respect to sequential composition ordered by set inclusion with the neutral element given by the unit theory, and it forms a finite idempotent near-semiring with respect to union and composition with the left zero given by the empty theory.

**Proof.** We prove the associativity of composition in Section 6.1 (the proof is given after Theorem 6.1). By definition of composition, we have

$$P \circ 1 = \{ \text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq_r 1 : \text{head}(S) = \text{body}(r) \}.$$  

Now, by definition of 1, we have $\text{head}(S) = \text{body}(S)$ and therefore $\text{body}(S) = \text{body}(r)$. Hence,

$$P \circ 1 = P.$$

Similarly, we have

$$1 \circ P = \{ \text{head}(r) \leftarrow \text{body}(S) \mid r \in 1, S \subseteq_1 P : \text{head}(S) = \text{body}(r) \}.$$  

As $S$ is a 1-subset of $P$, $S$ is a singleton $S = \{s\}$, for some rule $r$ with $\text{head}(s) = \text{body}(r)$ and, since $\text{body}(r) = \text{head}(r)$ holds for every rule in 1, $\text{head}(s) = \text{head}(r)$. Hence,

$$1 \circ P = \{ \text{head}(r) \leftarrow \text{body}(s) \mid r \in 1, s \in P : \text{head}(s) = \text{body}(r) \} = \{ \text{head}(s) \leftarrow \text{body}(s) \mid s \in P \} = P.$$

This shows that 1 is neutral with respect to composition. That composition is compatible with set inclusion is obvious. We now turn our attention to the operation of union. In 4 we argued for right-distributivity of composition. That the empty set is a left zero is obvious. \hfill $\Box$

### 3.1 Propositional Krom-Horn Theories

Recall that we call a propositional Horn theory $Krom$ if it contains only rules with at most one body atom. This includes interpretations, unit theories, and permutations.

For propositional Krom-Horn theories $K$ and $H$, sequential composition simplifies to

$$K \circ H = \text{facts}(K) \cup \{ a \leftarrow b \mid a \leftarrow c \in K, c \leftarrow b \in H, a, b, c \in A \}.$$

We have the following structural result as a specialization of Theorem 3.2.

**Theorem 3.3.** The space of all propositional Krom-Horn theories forms a subnear-semiring with respect to sequential composition and union. Moreover, Krom theories distribute from the left, that is, for any propositional Horn theories $P$ and $R$, we have

$$K(P \cup R) = KP \cup KR.$$

This implies that the space of proper propositional Krom-Horn theories forms a finite idempotent semiring.
Proof. For any proper Krom rule \( r = a \leftarrow b \), we have
\[
\{ a \leftarrow b \} \circ (P \cup R) = \{ a \leftarrow c \mid b \leftarrow c \in P \cup R, c \in A \}
\]
\[
= \{ a \leftarrow c \mid b \leftarrow c \in P, c \in A \} \cup \{ a \leftarrow c \mid b \leftarrow c \in R, c \in A \}
\]
\[
= \{ (a \leftarrow b) \circ P \} \cup \{ (a \leftarrow b) \circ R \}
\]
Hence, we have
\[
K(P \cup R) = f \text{ acts}(K) \cup \text{ proper}(K)(P \cup R)
\]
\[
= f \text{ acts}(K) \cup \bigcup_{r \in \text{ proper}(K)} \{ r \}(P \cup R)
\]
\[
= f \text{ acts}(K) \cup \bigcup_{r \in \text{ proper}(K)} \{ (r) \} \cup \{ r \} R
\]
\[
= f \text{ acts}(K) \cup \text{ proper}(K)P \cup \text{ proper}(K)R
\]
\[
= (f \text{ acts}(K) \cup \text{ proper}(K))P \cup (f \text{ acts}(K) \cup \text{ proper}(K))R
\]
\[
= KP \cup KR
\]
which shows that Krom theories distribute from the left, that his, the space of all Krom theories forms a semiring (cf. Theorem 3.2).
\(\square\)

3.1.1 Interpretations. Formally, interpretations are propositional Krom-Horn theories containing only rules with empty bodies (i.e., facts), which gives interpretations a special compositional meaning.

**Proposition 3.4.** Every interpretation \( I \) is a left zero, that is, for any propositional Horn theory \( P \), we have
\[
IP = I.
\]
(8)
Consequently, the space of interpretations forms a right ideal. \(^2\)

Proof. We compute
\[
IP = \{ \text{head}(P) \} \circ \text{body}(I)P \overset{20,19}{=} I \circ \emptyset P \overset{19}{=} I \circ \emptyset \overset{18}{=} f \text{ acts}(I) = I.
\]
(9)
\(\square\)

**Corollary 3.5.** A propositional Horn theory \( P \) commutes with an interpretation \( I \) if, and only if, \( I \) is a supported model of \( P \), that is,
\[
PI = IP \iff I \in \text{Supp}(P).
\]

Proof. A direct consequence of Proposition 3.4 and the forthcoming Theorem 6.2.
\(\square\)

3.1.2 Permutations. With every permutation \( \pi : A \rightarrow A \), we associate the propositional Krom-Horn theory \( \pi = \{ \pi(a) \leftarrow a \mid a \in A \} \).
\(^2\)See Corollary 6.4.
For instance, we have

\[ \pi(a \ b) = \left\{ \begin{array}{c} a \leftrightarrow b \\ b \leftrightarrow a \end{array} \right\} \quad \text{and} \quad \pi(a \ b \ c) = \left\{ \begin{array}{c} a \leftrightarrow b \\ b \leftrightarrow c \\ c \leftrightarrow a \end{array} \right\}. \]

The inverse \( \pi^{-1} \) of \( \pi \) translates into the language of theories as

\[ \pi^{-1} = \pi^{rev}. \]

Interestingly, we can rename the atoms occurring in a theory via permutations and composition by

\[ \pi \circ P \circ \pi^{rev} = \{ \pi(head(r)) \leftrightarrow \pi(body(r)) \mid r \in P \}. \]

We have the following structural result.

**PROPOSITION 3.6.** The space of all permutation theories forms a group.

### 3.2 Idempotent Theories

Recall that we call \( P \) idempotent if \( P^2 = P \). Idempotents play a key role in the structure theory of finite semigroups [12, 13]. We therefore investigate here the most basic properties of idempotent theories.

**PROPOSITION 3.7.** A propositional Horn theory \( P \) is idempotent if, and only if,

\[ \text{proper}(P) \text{facts}(P) \subseteq \text{facts}(P) \quad \text{and} \quad \text{proper}(P) = \text{proper}(\text{proper}(P)P). \] (10)

**PROOF.** The theory \( P \) is idempotent if, and only if, we have (i) \( \text{facts}(P^2) = \text{facts}(P) \) and (ii) \( \text{proper}(P^2) = \text{proper}(P) \).

For the first condition in (10), we compute

\[ \text{facts}(P^2) = P^20 = P0 = \text{facts}(P) \quad \text{and} \quad \text{proper}(P^2) = \text{proper}(P). \]

For the second condition in (10), we compute

\[ \text{proper}(P^2) = \text{proper}(\text{facts}(P) \cup \text{proper}(P))P = \text{proper}(\text{facts}(P) \cup \text{proper}(P))P = \text{proper}(\text{proper}(P)P). \]

\[ \square \]

**COROLLARY 3.8.** Every interpretation is idempotent.

**PROOF.** A direct consequence of Proposition 3.7 and of (8). \[ \square \]

### 3.3 Proper Theories

Recall from 18 that we can extract the facts of a propositional Horn theory \( P \) by computing the right reduct of \( P \) with respect to the empty set, \( P^0 \), and by sequentially composing \( P \) with the empty set, \( P \circ 0 \). Unfortunately, there is no analogous characterization of the proper rules in terms of composition.
The proper rules operator satisfies the following identities, for any propositional Horn theories \( P \) and \( R \), and interpretation \( I \):

\[
\begin{align*}
\text{proper} \circ \text{proper} &= \text{proper} \quad (11) \\
\text{proper} \circ \text{facts} &= \emptyset \quad (12) \\
\text{proper}(1) &= 1 \quad (13) \\
\text{proper}(I) &= \emptyset \quad (14) \\
\text{proper}(P) \circ \emptyset &= \text{proper}(P^\emptyset) = \emptyset \quad (15) \\
\text{proper}(P \cup R) &= \text{proper}(P) \cup \text{proper}(R). \quad (16)
\end{align*}
\]

Of course, we have \( \text{proper}(P) = P \) if, and only if, \( P \) contains no facts, that is, if \( P^\emptyset = \emptyset \). The last identity \((16)\) says that the proper rules operator is compatible with union; however, the following counter-example shows that it is not compatible with sequential composition:

\[
\text{proper}\left(\{a \leftarrow b, c\} \circ \left\{ \begin{array}{c} b \leftarrow b \\ c \end{array} \right\}\right) = \{a \leftarrow b\}
\]

whereas

\[
\text{proper}\left(\{a \leftarrow b, c\}\right) \circ \text{proper}\left(\left\{ \begin{array}{c} b \leftarrow b \\ c \end{array} \right\}\right) = \emptyset.
\]

**Proposition 3.9.** The space of all proper propositional Horn theories forms a subnear-semiring of the space of all propositional Horn theories with zero given by the empty set.

**Proof.** The space of proper theories is closed under composition. It remains to show that the empty set is a zero, but this follows from the fact that it is a left zero by \((8)\) and from the observation that for any proper theory \( P \), we have \( P \circ \emptyset \equiv \text{facts}(P) = \emptyset \). \( \square \)

### 4 ALGEBRAIC TRANSFORMATIONS

In this section, we study algebraic operations of theories expressible via composition and other operators.

Our first observation is that we can compute the heads and bodies via

\[
\text{head}(P) = PA \quad \text{and} \quad \text{body}(P) = \text{proper}(P)^{rev} A.
\]

Moreover, we have

\[
\text{head}(PR) \subseteq \text{head}(P) \quad \text{and} \quad \text{body}(PR) \subseteq \text{body}(R).
\]

#### 4.1 Reducts

Reducing the rules of a theory to a restricted alphabet is a fundamental operation on theories which can be algebraically computed via composition (Proposition 4.2).

**Definition 4.1.** We define the **left** and **right reduct** of a propositional Horn theory \( P \), with respect to some interpretation \( I \), respectively by

\[
I_P = \{ r \in P \mid I \models \text{head}(r) \} \quad \text{and} \quad P^I = \{ r \in P \mid I \models \text{body}(r) \}.
\]
Our first observation is that we can compute the facts of $P$ via the right reduct with respect to the empty set, that is, we have

$$P^\emptyset = P \circ \emptyset = \text{facts}(P).$$

On the contrary, computing the left reduct with respect to the empty set yields

$$\emptyset P = \emptyset.$$

Moreover, for any interpretations $I$ and $J$, we have

$$JI = I \cap J \quad \text{and} \quad I^J = I.$$

Notice that we obtain the reduction of $P$ to the atoms in $I$, denoted $P|_I$, by

$$P|_I = I(P) = (I P)^I.$$ (21)

As the order of computing left and right reducts is irrelevant, in the sequel we will omit the parentheses in (21). Of course, we have

$$A P = P A = A P A = P.$$

Moreover, we have

$$1^I = I 1 = I = 1 I$$

$$1^I \circ 1^J = 1^{I \cap J} = 1^I \circ 1^J = 1^I \cap 1^J$$

$$1^I \cup 1^J = 1^{I \cup J}.$$ (24)

We now want to relate reducts to composition and union.

**Proposition 4.2.** For any propositional Horn theory $P$ and interpretation $I$, we have

$$I P = 1^I \circ P \quad \text{and} \quad P^I = P \circ 1^I.$$ (25)

Moreover, we have

$$P|_I = 1^I \circ P \circ 1^I.$$ (26)

Consequently, for any propositional Horn theory $R$, we have

$$I(P \cup R) = 1^I \cup 1^R \quad \text{and} \quad (P \circ R)^I = 1^P \circ R^I$$

$$I(P \cup R)^I = 1^P \cup 1^R \quad \text{and} \quad (P \circ R)^I = P \circ R^I.$$ (28)

**Proof.** We compute

$$1^I \circ P = \{\text{head}(r) \leftarrow \text{body}(s) \mid r \in 1^I, s \in P : \text{head}(s) = \text{body}(r)\}$$

$$= \{\text{head}(r) \leftarrow \text{body}(s) \mid r \in 1, s \in P : \text{head}(s) = \text{body}(r), \text{head}(r) = \text{body}(r) \in I\}$$

$$= \{\text{head}(s) \leftarrow \text{body}(s) \mid s \in P : \text{head}(s) \in I\}$$

$$= \{s \in P \mid I \models \text{head}(s)\}$$

$$= IP.$$
Similarly, we compute
\[
P \circ I^1 = \{ \text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq_r I^1 : \text{head}(S) = \text{body}(r) \}
\]
\[
= \{ \text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq_r 1 : \text{head}(S) = \text{body}(r), \text{head}(S) = \text{body}(S) \subseteq I \}.
\]
By definition of 1, we have \(\text{body}(S) = \text{head}(S)\) and therefore \(\text{body}(S) = \text{body}(r)\) and \(\text{body}(r) \subseteq I\). Hence, (29) is equivalent to
\[
\{ \text{head}(r) \leftarrow \text{body}(r) \mid r \in P : I \models \text{body}(r) \} = P^I.
\]
Finally, we have
\[
I(P \cup R)^{25} \cong I^1(P \cup R) \cong I^1P \cup I^1R \cong I_P \cup I_R
\]
and
\[
I(PR)^{25} \cong (I^1PR) = ((I^1P)R)^{25} \cong I_P \cup I_R
\]
with the remaining identities in (28) holding by analogous computations. □

The computation of composition can be simplified as follows.

**Proposition 4.3.** For any propositional Horn theories \(P\) and \(R\), we have
\[
P \circ R = p^{\text{head}(R)} \circ \text{body}(P) R.
\]

**Proof.** By definition of reducts, we have
\[
P \circ R = p^{\text{body}(P)} \circ \text{head}(R) R \cong p^{\text{body}(P)} \text{head}(R) R \cong p^{\text{body}(P)} \text{head}(R) R \cong p^{\text{head}(R)} \circ \text{body}(P) R.
\]

\[\square\]

### 4.2 Adding and Removing Body Atoms

We now want to study algebraic transformations of rule bodies. For this, first notice that we can manipulate rule bodies via composition on the right. For example, we have
\[
\{ a \leftarrow b, c \} \circ \{ b \leftarrow b, c \} = \{ a \leftarrow b \}.
\]
The general construction here is that we add a tautological rule \(b \leftarrow b\) for every body atom \(b\) of \(P\) which we want to preserve, and we add a fact \(c\) in case we want to remove \(c\) from the rule bodies in \(P\). We therefore define, for an interpretation \(I\),
\[
I^\Theta = I^{A-I} \cup I.
\]
Notice that \(I^\Theta\) is computed with respect to some fixed alphabet \(A\). For instance, we have
\[
A^\Theta = A \quad \text{and} \quad \emptyset^\Theta = 1.
\]
The first equation yields another explanation for 17, that is, we can compute the heads in \(P\) by removing all body atoms of \(P\) via
\[
\text{head}(P) = PA^\Theta = PA.
\]
Interestingly enough, we have
\[ I \ominus I = ((A - I) \cup I) \ominus I = I \]
Moreover, in the example above, we have
\[ \{c\} \otimes = \begin{cases} a \leftarrow a \\ b \leftarrow b \\ c \end{cases} \]
and \[ \{a \leftarrow b, c\} \circ \{c\} = \{a \leftarrow b\} \]
as desired. Notice also that the facts of a theory are, of course, not affected by composition on the right, that is, we cannot expect to remove facts via composition on the right (cf. 6).

We have the following general result.

**Proposition 4.4.** For any propositional Horn theory \( P \) and interpretation \( I \), we have
\[ PI \otimes = \{\text{head}(r) \leftarrow (\text{body}(r) \cup I) \mid r \in P\}. \]

In analogy to the above construction, we can add body atoms via composition on the right. For example, we have
\[ \{a \leftarrow b\} \circ \{b \leftarrow b, c\} = \{a \leftarrow b, c\}. \]
Here, the general construction is as follows. For an interpretation \( I \), define
\[ I^\otimes = \{a \leftarrow (\{a\} \cup I) \mid a \in A\}. \]
For instance, we have
\[ A^\otimes = \{a \leftarrow A \mid a \in A\} \quad \text{and} \quad \emptyset^\otimes = 1. \]
Interestingly enough, we have
\[ I^\otimes I^\otimes = I^\otimes \quad \text{and} \quad I^\otimes I = I. \]
Moreover, in the example above, we have
\[ \{c\}^\otimes = \begin{cases} a \leftarrow a, c \\ b \leftarrow b, c \\ c \leftarrow c \end{cases} \]
as desired. As composition on the right does not affect the facts of a theory, we cannot expect to append body atoms to facts via composition on the right. However, we can add arbitrary atoms to *all* rule bodies simultaneously and in analogy to Proposition 4.4, we have the following general result.

**Proposition 4.5.** For any propositional Horn theory \( P \) and interpretation \( I \), we have
\[ PI^\otimes = \text{facts}(P) \cup \{\text{head}(r) \leftarrow (\text{body}(r) \cup I) \mid r \in \text{proper}(P)\}. \]

We now want to illustrate the interplay between the above concepts with an example.
Example 4.6. Let \( A = \{a, b, c\} \). Consider the propositional Horn theories

\[
P = \left\{ \begin{array}{l}
c \\ a \leftarrow b, c \\ b \leftarrow a, c 
\end{array} \right\} \quad \text{and} \quad \pi_{(a,b)} = \left\{ \begin{array}{l}
a \leftarrow b \\ b \leftarrow a 
\end{array} \right\}.
\]

Roughly, we obtain \( P \) from \( \pi_{(a,b)} \) by adding the fact \( c \) to \( \pi_{(a,b)} \) and to each body rule in \( \pi_{(a,b)} \). Conversely, we obtain \( \pi_{(a,b)} \) from \( P \) by removing the fact \( c \) from \( P \) and by removing the body atom \( c \) from each rule in \( P \). This can be formalized by as

\[
P = \{c\}^* \pi_{(a,b)} \{c\} \quad \text{and} \quad \pi_{(a,b)} = 1 \{a,b\} P \{c\} \ominus.
\]

5 DECOMPOSITION

In this section, we study sequential decompositions of theories. Notice that it is not possible to add arbitrary rules to a theory via composition, so the goal is to develop techniques for the algebraic manipulation of theories.

The following construction will be useful.

Definition 5.1. Define the closure of \( P \) with respect to \( A \) by

\[
cl_A(P) = 1_A \cup P.
\]

We first start by observing that every rule \( r = a_0 \leftarrow a_1, \ldots, a_k, k \geq 3, \) can be decomposed into a product of binary theories by

\[
\{r\} = \{a_0 \leftarrow a_1, \ldots, a_k\} = \{a_0 \leftarrow a_1, a_2\} \circ \prod_{i=2}^{k-1} cl\{a_{i-1}, \ldots, a_{i+1}\}(\{a_i \leftarrow a_i, a_{i+1}\}).
\]

In the next two subsections we study decompositions of acyclic and arbitrary theories.

5.1 Acyclic Theories

We first want to study decompositions of acyclic theories which have the characteristic feature that their rules can be linearly ordered via a level mapping, which means that there is an acyclic dependency relation between the rules of the theory.

Lemma 5.2. For any propositional Horn theories \( P \) and \( R \), if \( P \) does not depend on \( R \), we have

\[
P(Q \cup R) = PQ.
\]

Proof. We compute

\[
P(Q \cup R) \equiv p\text{head}(Q \cup R) \circ \text{body}(P)(Q \cup R) \equiv p\text{head}(Q \cup R) \circ (\text{body}(P)Q \cup \text{body}(P)R). \quad (33)
\]

Since \( P \) does not depend on \( R \), we have \( \text{body}(P) \cap \text{head}(R) = \emptyset \) and, hence,

\[
\text{body}(P)R = \{r \in R \mid \text{body}(P) \models \text{head}(r)\} = \{r \in R \mid \text{head}(r) \in \text{body}(P)\} = \emptyset.
\]

\[\text{Here, we have } \{c\}^* = 1 \cup \{c\} \text{ and } \{c\}^* \pi_{(a,b)} = \pi_{(a,b)} \cup \{c\} \text{ by the forthcoming equation } 47.\]

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Moreover, we have

\[ p_{\text{head}}(Q \cup R) = p_{\text{head}}(Q) \cup p_{\text{head}}(R) \]

\[ = \{ r \in P \mid \text{body}(r) \subseteq \text{head}(Q) \cup \text{head}(R) \} \]

\[ = \{ r \in P \mid \text{body}(r) \subseteq \text{head}(Q) \} \]

\[ = p_{\text{head}}(Q). \]

Hence, 33 is equivalent to

\[ p_{\text{head}}(Q) \circ \text{body}(P) = P. \]

\[ \square \]

Roughly, we obtain the closure of a theory by adding all possible tautological rules of the form \( a \leftarrow -a \), for \( a \in A \). Of course, the closure operator preserves semantical equivalence.

**Lemma 5.3.** For any propositional Horn theories \( P \) and \( R \), in case \( P \) does not depend on \( R \), we have

\[ P \cup R = \cl_{\text{head}}(R)(P) \cl_{\text{body}}(P)(R) \quad \text{and} \quad \cl_A(P \cup R) = \cl_A(P) \cl_A(R). \]  

(34)

Moreover, for any alphabets \( A \) and \( B \), we have

\[ \cl_A(\cl_B(P)) = \cl_{A \cup B}(P). \]

**Proof.** We compute

\[ \cl_{\text{head}}(P) \cl_{\text{body}}(P)(R) = (1_{\text{head}}(R) \cup P)(1_{\text{body}}(P) \cup R) \]

\[ = 1_{\text{head}}(R) \cl_{\text{body}}(P)(R) \cup P(1_{\text{body}}(P) \cup R) \]

\[ = 1_{\text{head}}(R) \cup 1_{\text{body}}(P) \cup P(1_{\text{body}}(P) \cup R) \]

\[ = R \cup P(1_{\text{body}}(P) \cup R) \]

\[ = R \cup P1_{\text{body}}(P) \]

\[ = R \cup P \]

where the fourth equality follows from \( 1_{\text{head}}(R) \cap \text{body}(P) = \emptyset \) and \( \text{head}(R) \cup R = R \), the fifth equality holds since \( P \) does not depend on \( R \) (Lemma 5.2), and the last equality follows from \( \text{body}(P) = P \).

For the second equality, we compute

\[ \cl_A(P) \cl_A(R) = (1 \cup P)(1 \cup R) \]

\[ = 1 \cup R \cup P(1 \cup R) \]

\[ = 1 \cup R \cup P (1 \cup R) \]

\[ = \cl_A(P \cup R) \]

where the third identity follows from the fact that \( P \) does not depend on \( R \) (Lemma 5.2).

Lastly, we compute

\[ \cl_A(\cl_B(P)) = 1_A \cup 1_B \cup B \]

\[ = 1_{A \cup B} \cup P = \cl_{A \cup B}(P). \]
Example 5.4. We define a family of acyclic theories over $A$, which we call elevator theories, as follows: given a sequence $(a_1, \ldots, a_n) \in A^n$, $1 \leq n \leq |A|$, of distinct atoms, let $E_{(a_1, \ldots, a_n)}$ be the acyclic theory
\[ E_{(a_1, \ldots, a_n)} = \{a_1\} \cup \{a_i \leftarrow a_{i-1} \mid 2 \leq i \leq n\}. \]
So, for instance, $E_{(a,b,c)}$ is the theory
\[ E_{(a,b,c)} = \begin{cases} a \\ b \leftarrow a \\ c \leftarrow b \end{cases}. \]
The mapping $\ell$ given by $\ell(a) = 1$, $\ell(b) = 2$, and $\ell(c) = 3$ is a level mapping for $E_{(a,b,c)}$ and we can decompose $E_{(a,b,c)}$ into a product of single-rule theories by
\[ E_{(a,b,c)} = (1(b,c) \cup \{a\})(1(c) \cup \{b \leftarrow a\})(1(a) \cup \{c \leftarrow b\}) = \text{cl}_{(b,c)}(\{a\})\text{cl}_{(c)}(\{b \leftarrow a\})\text{cl}_{(c)}(\{c \leftarrow b\}). \]

We now want to generalize the reasoning pattern of Example 5.4 to arbitrary acyclic theories. For this, we will need the following lemma.

Lemma 5.5. For any propositional Horn theories $P$ and $R$ and alphabet $B$, if $P$ and $1_B$ do not depend on $R$, we have
\[ \text{cl}_B(\text{cl}_{\text{head}(R)}(P)\text{cl}_{\text{body}(P)}(R)) = \text{cl}_{B \cup \text{head}(R)}(P)\text{cl}_{B \cup \text{body}(P)}(R). \] (35)

Proof. Since $P$ does not depend on $R$, as a consequence of Lemma 5.3 we have
\[ \text{cl}_{\text{head}(R)}(P)\text{cl}_{\text{body}(P)} = P \cup R. \]

Hence,
\begin{align*}
\text{cl}_B(\text{cl}_{\text{head}(R)}(P)\text{cl}_{\text{body}(P)}(R)) &= \text{cl}_B(P \cup R) \\
&= 1_B \cup P \cup R \\
&\overset{34}{=} \text{cl}_{\text{head}(R)}(1_B \cup P)\text{cl}_{\text{body}(1_B \cup P)}(R) \quad (\text{head}(R) \cap B = \emptyset) \\
&= \text{cl}_{B \cup \text{head}(R)}(P)\text{cl}_{B \cup \text{body}(P)}(R). \\
\end{align*}

\[ \Box \]

We further will need the following auxiliary construction. Define, for any linearly ordered rules $r_1 < \ldots < r_n$, $n \geq 2$,
\[ bh_i(r_1 < \ldots < r_n) = \text{body}(r_1, \ldots, r_{i-1}) \cup \text{head}(\{r_{i+1}, \ldots, r_n\}). \]

We are now ready to prove the following decomposition result for acyclic theories.

Theorem 5.6. We can sequentially decompose any acyclic propositional Horn theory $P = \{r_1 < \ell \ldots < \ell r_n\}$, $n \geq 2$, linearly ordered by a level mapping $\ell$, into single-rule theories as
\[ P = \prod_{i=1}^{n} \text{cl}_{bh_i(P)}(r_i). \]
This decomposition is unique up to reordering of rules within a single level.\footnote{See the construction of the total ordering \(<_t\) in Section 2.2.1.}

**Proof.** The proof is by induction on the number \(n\) of rules in \(P\). For the induction hypothesis \(n = 2\) and \(P = \{r_1 <_t r_2\}\), we proceed as follows. First, we have

\[
\cl_{bh_i}(p) (\{r_1\}) \cl_{bh_i}(p) (\{r_2\}) = (1_{\text{head}}(r_2) \cup \{r_1\}) (1_{\text{body}}(r_1)) \cup \{r_2\}) \quad (36)
\]

\[
= 1_{\text{head}}(r_2) (1_{\text{body}}(r_1)) \cup \{r_2\}) \cup \{r_1\}) (1_{\text{body}}(r_1)) \cup \{r_2\} \). \quad (37)
\]

Since Krom theories distribute from the left (Theorem 3.3), we can simplify 36, by applying 25 and 24, into

\[
1_{\text{head}}(r_2) \cap \text{body}(r_1) = 1_{\text{head}}(r_2) \cap \text{body}(r_1) = 1_{\text{head}}(r_2) \cup \text{body}(r_1) \cup r_1, r_2. \quad (38)
\]

Now, since \(r_1\) does not depend on \(r_2\), that is, \(\text{head}(r_2) \cap \text{body}(r_1) = \emptyset\), the first term in 38 equals \(1_\emptyset = \emptyset\) which implies that 38 is equivalent to \(P\) as desired.

For the induction step \(P = \{r_1 <_t \ldots <_t r_{n+1}\}\), we proceed as follows. First, by definition of \(bh_i\), we have

\[
\prod_{i=1}^{n+1} (\cl_{bh_i}(p) (\{r_i\})) = \prod_{i=1}^{n+1} (1_{bh_i}(p) \cup \{r_i\}) \quad (39)
\]

\[
= \prod_{i=1}^{n} (1_{bh_i}(p) \cup \{r_i\}) \cup 1_{\text{head}}(r_{n+1}) \cup \{r_i\} \cup 1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\}. \quad (40)
\]

Second, by idempotency of union we can extract the term

\[
1_{\text{head}}(r_{n+1}) = 1_{\text{head}}(r_{n+1}) \ldots 1_{\text{head}}(r_{n+1}) \quad (n \text{ times})
\]

occurring in 40 thus obtaining

\[
\left[ 1_{\text{head}}(r_{n+1}) \cup \prod_{i=1}^{n} (1_{bh_i}(p) \cup \{r_i\}) \cup 1_{\text{head}}(r_{n+1}) \cup \{r_i\} \cup 1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\}. \quad (41)
\]

Now, since \(r_1\) and \(1_{bh_i}(r_{1}, \ldots, r_{n})\) do not depend on \(r_{n+1}\), we can simplify 41 further to

\[
\left[ 1_{\text{head}}(r_{n+1}) \cup \prod_{i=1}^{n} (1_{bh_i}(p) \cup \{r_i\}) \cup 1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\}. \quad (42)
\]

By applying the induction hypothesis to 42, we obtain

\[
(1_{\text{head}}(r_{n+1}) \cup \{r_1, \ldots, r_n\}) (1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\})
\]

which, by right-distributivity of composition, is equal to

\[
1_{\text{head}}(r_{n+1}) (1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\}) \cup \{r_1, \ldots, r_n\} (1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\}) \quad (43)
\]

Now again since \(r_1\) does not depend on \(r_{n+1}\), for all \(1 \leq i \leq n\), as a consequence of 7, 25, 24, and 32, the term in 43 equals

\[
1_{\text{head}}(r_{n+1}) (1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\}) \cup \{r_1, \ldots, r_n\} (1_{\text{body}}(r_{n+1}) \cup \{r_{n+1}\}) \quad (44)
\]

Finally, since \(1_{\text{head}}(r_{n+1}) \cap \text{body}(r_{n+1}) = 1_\emptyset = \emptyset\), 44 equals \(\{r_1, \ldots, r_{n+1}\}\), which proves our theorem. \(\square\)
5.2 General Decompositions

We now wish to generalize Lemma 5.3 to arbitrary propositional Horn theories. For this, we define, for every interpretation $I$ and disjoint copy $I' = \{ a' \mid a \in I \}$ of $I$,

$$[I \leftarrow I'] = cl_{A - I}((a \leftarrow a' \mid a \in I)) = \{ a \leftarrow a' \mid a \in I \} \cup \{ a \leftarrow a \mid a \in A - I \}.$$  

We have

$$[A \leftarrow A'] = \{ a \leftarrow a' \mid a \in A \}$$

and therefore

$$P[A \leftarrow A'] = \{ head(r) \leftarrow body(r) \mid r \in P \} \quad \text{and} \quad [A' \leftarrow A]P = \{ head(r)' \leftarrow body(r) \mid r \in P \}.$$  

Moreover, we have

$$[A \leftarrow A'][A' \leftarrow A] = 1_A. \quad (45)$$  

We are now ready to prove the main result of this subsection which shows that we can represent the union of theories via composition.

**Theorem 5.7.** For any propositional Horn theories $P$ and $R$, we have

$$P \cup R = cl_{head(R)}(P[A \leftarrow A'])cl_{body}(P[A \leftarrow A'])(R)cl_{A}([A' \leftarrow A]).$$

**Proof.** Since $P[A \leftarrow A']$ does not depend on $R$, we have, as a consequence of Lemma 5.3,

$$P[A \leftarrow A'] \cup R = cl_{head(R)}(P[A \leftarrow A'])cl_{body}(P[A \leftarrow A'])(R).$$

Finally, we have

$$(P[A \leftarrow A'] \cup R)cl_{A}([A' \leftarrow A]) \supseteq P[A \leftarrow A']cl_{A}([A' \leftarrow A]) \cup R cl_{A}([A' \leftarrow A])$$

$$= P[A \leftarrow A']([1_A \cup [A' \leftarrow A]]) \cup R ([1_A \cup [A' \leftarrow A]])$$

$$= P[A \leftarrow A']([A' \leftarrow A]) \cup R$$

$$\supseteq P \cup R$$

where the third identity follows from the fact that $P[A \leftarrow A']$ does not depend on $1_A$ and $R$ does not depend on $[A' \leftarrow A]$ (apply Lemma 5.2). \hfill \Box

**Example 5.8.** Let $A = \{a, b, c\}$ and consider the theory $P = R \cup \pi_{(b,c)}$ where

$$R = \begin{cases} a \leftarrow b, c \\ a \leftarrow a, b \\ b \leftarrow a \end{cases} \quad \text{and} \quad \pi_{(b,c)} = \begin{cases} b \leftarrow c \\ c \leftarrow b \end{cases}.$$  

We wish to decompose $P$ into a product of $R$ and $\pi_{(b,c)}$ according to Theorem 5.7. For this, we first have to replace the body of $R$ with a distinct copy of its atoms, that is, we compute

$$R[A \leftarrow A'] = \begin{cases} a \leftarrow b', c' \\ a \leftarrow a', b' \\ b \leftarrow a' \end{cases}.$$  

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Notice that \( R[A \leftarrow A'] \) no longer depends on \( \pi_{(b,c)} \). Next, we compute the composition of

\[
\text{cl}_{\text{head}}(\pi_{(b,c)})(R[A \leftarrow A']) = \begin{cases} 
  a \leftarrow b', c' \\
  a \leftarrow a', b' \\
  b \leftarrow a' \\
  b \leftarrow b \\
  c \leftarrow c
\end{cases}
\text{and} \quad \text{cl}_{\text{body}}(R[A \leftarrow A'])(\pi_{(b,c)}) = \begin{cases} 
  b \leftarrow c \\
  c \leftarrow b \\
  a' \leftarrow a' \\
  b' \leftarrow b' \\
  c' \leftarrow c'
\end{cases}
\]

by

\[
R' = \text{cl}_{\text{head}}(\pi_{(b,c)})(R[A \leftarrow A']) \circ \text{cl}_{\text{body}}(R[A \leftarrow A'])(\pi_{(b,c)}) = \begin{cases} 
  a \leftarrow b', c' \\
  a \leftarrow a', b' \\
  b \leftarrow a' \\
  b \leftarrow c \\
  c \leftarrow b
\end{cases}
\]

Finally, we need to replace the atoms from \( A' \) in the body of \( R' \) by atoms from \( A \), that is, we compute

\[
R' \circ \text{cl}_A(A' \leftarrow A) = \begin{cases} 
  a \leftarrow b', c' \\
  a \leftarrow a', b' \\
  b \leftarrow a' \\
  b \leftarrow c \\
  c \leftarrow b
\end{cases} \circ \begin{cases} 
  a \leftarrow a \\
  b \leftarrow b \\
  c \leftarrow c \\
  a' \leftarrow a \\
  b' \leftarrow b \\
  c' \leftarrow c
\end{cases} = p
\]

as expected.

Unfortunately, at the moment we have no decomposition result of arbitrary theories analogous to Theorem 5.6 which remains as future work (cf. Section 7.1).

6 ALGEBRAIC SEMANTICS

We now want to reformulate the fixed point semantics of propositional Horn theories in terms of sequential composition. For this, we first show that we can express the van Emden-Kowalski operators via composition and we then show how propositional Horn theories can be iterated bottom-up to obtain their least models.

6.1 The van Emden-Kowalski Operator

Recall from Proposition 2.2 that the least model of a theory can be computed by a least fixed point iteration of its associated immediate consequence or van Emden-Kowalski operator. The next results show that we can simulate the van Emden-Kowalski operators on a syntactic level without any explicit reference to operators.

**Theorem 6.1.** For any propositional Horn theories \( P \) and \( R \), and interpretation \( I \), we have

\[
T_P \cup T_R = T_{P \cup R} \quad \text{and} \quad T_P \circ T_R = T_{P \circ R} \quad \text{and} \quad T_I = Id_A.
\]

**Proof.** The first and last identities hold trivially. We want to prove the second identity. Let \( I \) be an arbitrary interpretation. By definition of composition, an atom \( a \) is in \( T_{PR}(I) \) if there is a rule \( r \in PR \) such that \( \text{head}(r) = \{a\} \) and \( \text{body}(r) \subseteq I \). In this case, \( r \) is of the form \( r = a \leftarrow \text{body}(S) \), for some \( S \subseteq R \). Since \( \text{body}(r) = \text{body}(S) \) and
body(r) ⊆ I, we have I ⊨ body(S), which implies head(S) ⊆ TR(I). By definition of composition, there must be a rule s = a ← head(S) ∈ P such that \( r = \{s\} S \). Hence, body(s) = head(S) ⊆ TR(I) which implies a ∈ Tp(Tr(I)).

For the other direction, a is in Tp(Tr(I)) if there is a rule r ∈ P such that head(r) = \{a\} and body(r) ⊆ TR(I). This implies that there is a subset S of R such that body(r) = Ts(I) or, in other words, body(r) = head(S) and body(S) ⊆ I. Now let s = head(r) ← body(S) = a ← body(S). By construction, s is in P or R and since body(s) = body(S) ⊆ I, we have a ∈ Tp,R(I).

We are now ready to prove the associativity of sequential composition.

**Proof of Theorem 3.2.** As an immediate consequence of Theorem 6.1, the mapping which assigns to each propositional Horn theory its van Emden-Kowalski operator is a monoid morphism. Since composition of operators is associative and since morphisms preserve associativity, we conclude the associativity of sequential composition.

The next result shows that we can represent the van Emden-Kowalski operator of a theory via composition on a syntactic level.

**Theorem 6.2.** For any propositional Horn theory P and interpretation I, we have

\[ T_P(I) = P \circ I. \]

**Proof.** A direct consequence of Theorem 6.1 and Tp(I) = Tr ∘ TI. Another proof is given by the computation

\[ T_P(I) = head(P^I) \xrightarrow{\text{17}} P^I A \xrightarrow{\text{25}} (P(I^1))A = P((1^1)A) \xrightarrow{\text{25}} P(I^1 A) \xrightarrow{\text{20}} P(I \cap A) = PI \]

where the last equality follows from I being a subset of A. \(\square\)

As a direct consequence of Proposition 2.1, we have the following algebraic characterization of (supported) models.

**Corollary 6.3.** An interpretation I is a model of P if, and only if, PI ⊆ I. Moreover, I is a supported model of P if, and only if, PI = I.

**Corollary 6.4.** The space of all interpretations forms an ideal.

**Proof.** By Proposition 3.4, we know that the space of interpretations forms a right ideal and Theorem 6.2 implies that it is a left ideal—hence, it forms an ideal. \(\square\)

The following result shows that composition is compatible with subsumption equivalence.

**Corollary 6.5.** Two propositional Horn theories P and R are subsumption equivalent if, and only if, for any interpretation I, we have PI = RI, which implies that subsumption equivalence is a congruence with respect to sequential composition.

**Proof.** The first assertion is an immediate consequence of Theorem 6.2. It remains to show that subsumption equivalence is a congruence or, equivalently, that it is compatible with composition. Assume that P is subsumption equivalent to R and let Q and S be some arbitrary propositional Horn Theories. We need to show that QPS is subsumption equivalent to QRS. For this, let I be an arbitrary interpretation, and compute

\[ (QPS)I = QP(SI) = QR(SI) = (QRS)I. \]

\(\square\)
6.2 Least Models

We interpret propositional Horn theories according to their least model semantics and since least models can be constructively computed by bottom-up iterations of the associated van Emden-Kowalski operators (cf. Proposition 2.2), we can finally reformulate the fixed point semantics of propositional Horn theories in terms of sequential composition (Theorem 6.7).

Definition 6.6. Define the unary Kleene star and plus operations by

\[ P^* = \bigcup_{n \geq 0} P^n \quad \text{and} \quad P^+ = P^* P. \]

Moreover, define the omega operation by

\[ P^\omega = P^+ \circ \emptyset \overset{18}{=} \text{facts}(P^+). \]

For instance, for any interpretation \( I \), we have

\[ I^* = 1 \cup I \quad \text{and} \quad I^+ = I \quad \text{and} \quad I^\omega = I. \]

Interestingly enough, we can add the atoms in \( I \) to \( P \) via

\[ P \cup I \overset{8}{=} P \cup IP \overset{47}{=} (1 \cup I)P \overset{47}{=} I^* P. \]

Hence, as a consequence of 5 and 48, we can decompose \( P \) as

\[ P = \text{facts}(P)^* \circ \text{proper}(P) \]

which, roughly, says that we can sequentially separate the facts from the proper rules in \( P \).

We are now ready to characterize the least model of a theory via composition as follows.

Theorem 6.7. For any propositional Horn theory \( P \), we have

\[ LM(P) = P^\omega. \]

Proof. A direct consequence of Proposition 2.2 and Theorem 6.2. \( \square \)

Corollary 6.8. Two propositional Horn theories \( P \) and \( R \) are equivalent if, and only if, \( P^\omega = R^\omega \).

7 CONCLUSION

This paper contributed to the foundations of knowledge representation and reasoning by introducing and studying the (sequential) composition of propositional Horn theories. We showed in our main structural result (Theorem 3.2) that the space of all such theories forms a finite monoid with respect to composition, and a finite idempotent near-semiring when adding union. We called these structures Horn monoids and Horn near-semirings. Moreover, we showed that the restricted class of propositional Krom-Horn theories are distributive and therefore its proper instances form an idempotent semiring (Theorem 3.3). From a logical point of view, we obtained an algebraic meta-calculus for reasoning about propositional Horn theories. Algebraically, we obtained a correspondence between propositional Horn theories and finite semigroups, which enables us to transfer concepts from the rich literature on finite semigroups to the logical setting. In a broader sense, this paper is a first step towards an algebra of logical theories and we expect interesting concepts and results to follow.

Manuscript submitted to ACM
7.1 Future Work

In the future, we plan to extend the constructions and results of this paper to wider classes of logical theories as, for example, first-, and higher-order logic programs [1, 2, 17], disjunctive datalog [7], and non-monotonic logic programs under the stable model [9] or answer set semantics [15] (cf. [3, 4, 8, 16]). The first task is non-trivial as function symbols give rise to infinite semigroups and monoids, whereas the non-monotonic case is more difficult to handle algebraically due to negation as failure [6] occurring in rule bodies (and heads). Even more problematic, disjunctive rules yield non-deterministic behavior which is more difficult to handle algebraically. Nonetheless, we expect interesting results in all of the aforementioned cases to follow.

Another major line of research is to study (sequential) decompositions of various theories. Specifically, we wish to compute decompositions of arbitrary propositional Horn theories (and extensions thereof) into "prime" theories in the vein of Theorem 5.6, where we expect permutation theories (Section 3.1.2) to play a fundamental role in such decompositions. For this, it will be necessary to resolve the issue of "prime" or indecomposable theories. Algebraically, it will be of central importance to study Green’s relations [10] (cf. [12, 13]) in the finite Horn monoids (and near-semirings) introduced in this paper. From a practical point of view, a mathematically satisfactory theory of theory decompositions is relevant to modular knowledge representation and optimization of reasoning.

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