A NOTE ON WEYL GROUPS AND CRYSTALLOGRAPHIC ROOT LATTICES

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Abstract. We follow the dual approach to Coxeter systems and show for Weyl groups that a set of reflections generates the group if and only if the related sets of roots and coroots generate the root and the coroot lattices, respectively. Previously, we have proven if \((W, S)\) is a Coxeter system of finite rank \(n\) with set of reflections \(T\) and if \(t_1, \ldots, t_n \in T\) are reflections in \(W\) that generate \(W\) then \(P := \langle t_1, \ldots, t_n^{-1} \rangle\) is a maximal parabolic subgroup of \((W, S)\) [BGRW17, Theorem 1.5]. Here we show if \((W, S)\) is crystallographic as well then all the reflections \(t \in T\) such that \(\langle P, t \rangle = W\) form a single orbit under conjugation by \(P\).

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1. Introduction

If a group \(W\), \(|W| > 2\), contains a set of involutions \(S\) such that \((W, S)\) is a Coxeter system then the simple system \(S\) is in general not uniquely determined by \(W\). The problem to determine all the Coxeter systems for which the possible simple systems \(S\) are conjugate in \(W\) is intensely studied under the name isomorphism problem for Coxeter groups and is still open, see [Müh05] and [MN16]. Let \(T := \{ws^{-1} | w \in W, s \in S\}\) be the set of reflections of \((W, S)\). A special case of the isomorphism problem is the question which groups are strongly reflection rigid, that is for which Coxeter groups all the simple systems that are contained in \(T\) are conjugate in \(W\). Recently there has been made substantial progress (e.g. see [CP10] or [HP17]).

In the dual approach, started independently by Brady and Watt [BW02] and by Bessis [Bes03], Coxeter systems are studied via their set of reflections \(T\). It is also of interest not only to study simple systems in \(T\) but more generally minimal generating sets \(X\) in \(T\). Clearly, each subset \(X\) of \(T\) corresponds to a subset \(R = R(X)\) of the set of roots \(\Phi\) that is related to \((W, S)\).

In this note we study finite Weyl groups, i.e. spherical crystallographic Coxeter systems \((W, S)\), and the related dual Coxeter systems \((W, T)\). In terms of the root and the coroot lattice we give a criterion which tells us when a set of reflections in \(T\) is a minimal generating set of \(W\) (for the notation see the next section). More precisely we show the following.
Theorem 1.1. Let $\Phi$ be a crystallographic root system of rank $n$, $R := \{\alpha_1, \ldots, \alpha_k\} \subseteq \Phi$ non-empty and $\Phi'$ a root subsystem of $\Phi$ containing $R$. Then the following statements are equivalent:

(a) $\Phi' = W_R(R)$
(b) $W_{\Phi'} = W_R$
(c) $L(\Phi') = \text{span}_\mathbb{Z}(\alpha_1, \ldots, \alpha_k)$ and $L((\Phi')^\vee) = \text{span}_\mathbb{Z}(\alpha_1^\vee, \ldots, \alpha_k^\vee)$.

Remark 1.2. Note that for a non-crystallographic root system $\Phi$, the $\mathbb{Z}$-span of $\Phi$ is a lattice that is not invariant under the action of $W = W_\Phi$, the reflection group generated by the reflections given by $\Phi$. Then the equivalence of (b) and (c) does not hold anymore: for instance in the reflection group $I_2(5)$ every two reflections generate the group, but two long roots do not generate the root lattice.

Theorem 1.1 is a generalization of $\text{[BGRW17, Proposition 5.10 and Lemma 5.12]}$. There we characterized quasi-Coxeter elements in terms of the related root lattice in simply laced root systems. Theorem 1.1 should also help to clarify the notation of a quasi-Coxeter element in the literature. Here and in $\text{[BGRW17]}$ it denotes an element in $W$ that has a reduced factorization into reflections such that these reflections generate $W$; while in $\text{[BH16]}$ it is an element such that the roots related to a reduced $T$-factorization of that element generate the root lattice. It is a consequence of Theorem 1.1 that quasi-Coxeter elements in our sense are precisely those elements of $W$ that have a reduced $T$-factorization such that the related roots and the related coroots form a basis of the root and the coroot lattices, respectively.

In $\text{[BGRW17]}$ Corollary 6.10 we characterized the maximal parabolic subgroups of a spherical dual Coxeter system $(W, T)$ of rank $n$ to be precisely those rank $n - 1$ reflection subgroups which generate together with one further reflection the whole group $W$.

Theorem 1.3. $\text{[BGRW17 Corollary 6.10]}$ Let $(W, T)$ be a spherical dual Coxeter system of rank $n$ and $W'$ a reflection subgroup of rank $n - 1$. Then $W'$ is a parabolic subgroup if and only if there exists $t \in T$ such that $\langle W', t \rangle = W$.

Here we show that in the Weyl group case the reflection $t \in T$ so that $t$ and the maximal parabolic subgroup generate $W$ is unique up to conjugacy with elements of the parabolic subgroup.

Theorem 1.4. Let $(W, T)$ be a spherical dual crystallographic Coxeter system and $P$ a maximal parabolic subgroup of $W$. All the reflections $t \in T$ such that $W = \langle P, t \rangle$ form a single orbit under conjugation by $P$.

The following statement is equivalent to Theorem 1.4

Theorem 1.5. Let $(W, T)$ be a spherical dual crystallographic Coxeter system. If $\{\alpha_1, \ldots, \alpha_n\}$ is a simple system for $\Phi$, $P = \langle s_{\alpha_1}, \ldots, s_{\alpha_{n-1}} \rangle$ a parabolic subgroup and $W = \langle P, s_\beta \rangle$, then there is $w \in P$ such that $w(\beta) = \alpha_n$.

Remark 1.6. (a) Our first proof of Theorem 1.4 was case-by-case. Here we follow an idea by Vinberg $\text{[Vin17]}$ of a uniform proof of Theorem 1.4.

(b) Observe that this theorem does not hold in general if we remove the condition that $(W, T)$ is crystallographic, see the next example.

Example 1.7. Consider a Coxeter system $(W, S)$ of type $H_3$. Let $S = \{s_1, s_2, s_3\}$ be such that $s_1$ and $s_3$ commute and such that $s_1 s_2$ and $s_2 s_3$ are of order 5 and 3, respectively. Then
for the parabolic subgroup $W' := \langle s_1, s_3 \rangle$ we obtain

$$\langle W', s_2 \rangle = W = \langle W', s_2s_1s_2 \rangle.$$ 

But $s_2$ and $s_2s_1s_2$ are not conjugated under $W'$ as $s_2$ and $s_1$ are related to roots of different length.

Notice that the results presented here are used by the second named author to show that the Hurwitz action on the set of reduced factorizations of a quasi–Coxeter element is transitive in dual affine Coxeter systems [We17].

The organization of this note is as follows. In Section 2 we give all the necessary definitions. In Section 3 we prove Theorem 1.1 and in Section 4 we provide a criterion for a set of roots to be a simple system and prove Theorem 1.4.

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2. Spherical crystallographic Coxeter systems

Let $(W, S)$ be a spherical Coxeter system of rank $n$, that is $W$ is a finite group and $S \subseteq W$ a set of involutions such that for $s, t \in S$ there exists $m(s, t) \in \mathbb{N}$ with $m(s, s) = 1$, $m(s, t) = m(t, s)$ and $m(s, t) \geq 2$ if $s \neq t$ so that $W$ has the following presentation

$$W = \langle S \mid (st)^{m(s, t)} = 1, \ s, t \in S \rangle.$$ 

Let $V$ be an $\mathbb{R}$-vector space with euclidean product $(\cdot \mid \cdot)$ and let $\alpha \in V$, $\alpha \neq 0$. A reflection $r = s_\alpha$ is a linear transformation that fixes a hyperplane $H_\alpha := \alpha^\perp$ pointwise and sends $\alpha$ to $-\alpha$, i.e.

$$s_\alpha(v) = v - \frac{2(\alpha \mid v)}{(\alpha \mid \alpha)} \alpha$$

for all $v \in V$, see [Hum90] or [Bou02]. The geometric representation $\varphi : W \to \text{GL}(V)$ of the spherical Coxeter system is the faithful representation of $W$ such that $\varphi(s)$ is a reflection $s_\alpha$ for each $s \in S$. We identify $W$ with its image.

Then $W$ leaves the form $(\cdot \mid \cdot)$ invariant and $\Phi := \{w(\alpha) \mid s \in S\} \subset V$ is the root system related to $(W, S)$, i.e. $\Phi$ is a finite set of nonzero vectors in $V$ such that

1. $\text{span}_R(\Phi) = V$,
2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$ and
3. $\Phi \cap \mathbb{R} \alpha = \{\pm \alpha\}$ for all $\alpha \in \Phi$.

Then $W = W_\Phi := \langle s_\alpha \mid \alpha \in \Phi \rangle$.

In this note we always assume that $\Phi$ is crystallographic, i.e. $\frac{2(\alpha \mid \beta)}{(\alpha \mid \alpha)}$ is an integer for all $\alpha, \beta \in \Phi$. In this case the Coxeter system is also said to be crystallographic.

Each root system $\Phi$ contains a so-called simple system $\Delta \subset \Phi$. That is a basis of $V$ such that the expression of each element in $\Phi$ in that basis has either all coefficients non-negative or non-positive integers. In each simple system every two different roots have an obtuse dihedral angle [Hum90, Corollary 1.3]. Coxeter observed that this characterizes the simple systems in irreducible crystallographic root systems, see Lemma 4.1.

The simple systems have a second geometric description. The connected components of $V \setminus \cup_{\alpha \in \Phi} H_\alpha$ are open cones and are called fundamental cones. The hyperplanes bounding a fundamental cone are called walls of the fundamental cone. If $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ is a simple system, then the hyperplanes $H_{\alpha_i}$ are the walls of a fundamental cone $C$ and the
fundamental cone is $C = \{ x \in V \mid (\alpha_i \mid x) > 0 \text{ for } 1 \leq i \leq n \}$. In this way we get another characterization of the simple systems of $\Phi$. They are precisely the sets of roots for which there exists a fundamental cone such that they are orthogonal to its walls and point inwards the fundamental cone.

In this note we follow the dual approach to Coxeter systems. A pair $(W, T)$ consisting of a group $W$ and a generating subset $T$ of $W$ is called dual Coxeter system of finite rank $n$ if there is a subset $S \subseteq T$ with $|S| = n$ such that $(W, S)$ is a Coxeter system, and $T = \{ w s w^{-1} \mid w \in W, s \in S \}$ is the set of reflections for the Coxeter system $(W, S)$. We then call $(W, S)$ a simple system for $(W, T)$. If $S' \subseteq T$ is such that $(W, S')$ is a Coxeter system, then $\{ w s w^{-1} \mid w \in W, s \in S' \} = T$, see [BMMN02] Lemma 3.7. Hence a set $S' \subseteq T$ is a simple system for $(W, T)$ if and only if $(W, S')$ is a Coxeter system. The rank of $(W, T)$ is defined as $|S|$ for a simple system $S \subseteq T$. This is well-defined by [BMMN02] Theorem 3.8.

Let $W'$ be a reflection subgroup of $W$. By [Dye90] $(W', W' \cap T)$ is again a dual Coxeter system. The reflection subgroup generated by $\{ s_1, \ldots, s_m \} \subseteq T$ is called a parabolic subgroup for $(W, T)$ if there is a simple system $S = \{ s_1, \ldots, s_n \}$ for $(W, T)$ with $m \leq n$. This definition differs from the usual notion of a parabolic subgroup generated by a conjugate of a subset of a fixed simple system $S$ (see [Hum90] Section 1.10). However, in [BGRW17] Corollary 4.4 it is shown that both definitions coincide for finite Coxeter groups.

An element $c \in W$ is called a Coxeter element if there exists a simple system $S = \{ s_1, \ldots, s_n \}$ for $(W, T)$ such that $c = s_1 \cdots s_n$. An element $w \in W$ is called quasi-Coxeter element for $(W, T)$ if there exists a $T$-reduced factorization $w = t_1 \cdots t_n$ such that $(t_1, \ldots, t_n) = W$. In particular, every Coxeter element is a quasi-Coxeter element.

Let as usual $\Phi^\vee$ be the set of coroots

$$\alpha^\vee := \frac{2\alpha}{(\alpha \mid \alpha)}, \text{ where } \alpha \in \Phi.$$ 

For a set of roots $R \subseteq \Phi$ we define $L(R) := \text{span}_\mathbb{Z}(R)$ resp. $L(R^\vee) := \text{span}_\mathbb{Z}(R^\vee)$. We call $L(\Phi)$ resp. $L(\Phi^\vee)$ the root lattice resp. the coroot lattice.

Later we will need the following two facts.

**Lemma 2.1.** [Hum90] 2.9] Let $\Phi$ be an irreducible and crystallographic root system of rank $n \geq 2$ and $\Delta := \{ \alpha_1, \ldots, \alpha_n \}$ be a simple system for $\Phi$. Then $\Delta^\vee = \{ \alpha_1^\vee, \ldots, \alpha_n^\vee \}$ is a simple system for $\Phi^\vee$ (and in particular a basis for $L(\Phi^\vee)$).

**Lemma 2.2.** [Bou02] Ch. VI, 3, Cor. to Theorem 1] Let $\Phi$ be a crystallographic root system and $\alpha, \beta \in \Phi$ with $(\alpha \mid \beta) < 0$ and $\alpha \neq -\beta$. Then $\alpha + \beta \in \Phi$.

3. **Characterization of quasi-Coxeter elements: Proof of Theorem 1.1**

**Proof of Theorem 1.1.** The equivalence of (a) and (b) is already part of [BGRW17] Proposition 5.10. Therefore it remains to prove the equivalence of (b) and (c).

We first show that (b) implies (c). So assume (b) and let $t_i := s_{\alpha_i}, 1 \leq i \leq k$. Let $T_\Phi$ be the set of reflections in $W_\Phi$. By [Dye90] Corollary 3.11 (ii), we have $T_\Phi = \{ w t_i w^{-1} \mid 1 \leq i \leq k, w \in W_\Phi \}$. In particular every root in $\Phi'$ has the form $w(\alpha_i)$ for some $w \in W_\Phi$, $1 \leq i \leq k$. Since $W_\Phi = \langle t_1, \ldots, t_k \rangle$, we can write $w = t_{i_1} \cdots t_{i_m} \alpha_{i_j}$ with $1 \leq i_j \leq k$ for each $1 \leq j \leq m$. Since $\Phi'$ is crystallographic it follows that $w(\alpha_i) = t_{i_1} \cdots t_{i_m} (\alpha_{i_j})$ is an integral linear combination of the $\alpha_{i_j}$'s, hence that $\Phi' \subseteq L(R)$. Since $R \subseteq \Phi'$ we get that $L(R) = L(\Phi')$. 
By [Bou02, Ch. VI, Paragraph 1] there is an isomorphism \( \varphi : W_{\Phi'} \xrightarrow{\sim} W_{(\Phi')^\vee} \) with \( \varphi(s_\alpha) = s_{\alpha^\vee} \). Thus
\[
W_{R^\vee} = \langle s_{\alpha_1^\vee}, \ldots, s_{\alpha_k^\vee} \rangle = \varphi(\langle s_{\alpha_1}, \ldots, s_{\alpha_k} \rangle) = \varphi(W_R) = \varphi(W_{\Phi'}) = W_{(\Phi')^\vee}.
\]

Using the same argumentation as in the last paragraph (now for \( (\Phi')^\vee \) and \( R^\vee \) instead of \( \Phi' \) and \( R \)) we obtain \( L((\Phi')^\vee) = L(R^\vee) \), which shows (c).

Now assume (c), that is \( L(R) = L(\Phi') \) and \( L(R^\vee) = L((\Phi')^\vee) \). By [BGRW17, Proposition 5.10] it remains to show that if \( \Phi'' \) denotes the smallest root subsystem of \( \Phi \) containing \( R \), then \( \Phi'' = \Phi' \). Since \( R \subseteq \Phi'' \subseteq \Phi' \) and \( L(R) = L(\Phi') \), we have \( L(\Phi'') = L(R) \). Let \( \gamma \in \Phi' \). It remains to show that \( \gamma \in \Phi'' \). Since \( L(\Phi') = L(R) = L(\Phi'') \), we have
\[
\gamma = \sum_{i=1}^m \mu_i \beta_i
\]
with \( \mu_i \in \mathbb{Z} \) and \( \beta_i \in \Phi'' \). As \( \beta_i \in \Phi'' \) implies \( -\beta_i \in \Phi'' \), we may assume \( \mu_i \in \mathbb{Z}_{>0} \). Therefore we can write
\[
\gamma = \sum_{i=1}^m \beta_i
\]
with \( \beta_i \in \Phi'' \) and we may assume that \( m \) is minimal with that property (note that \( \beta_i = \beta_j \) for \( i \neq j \) is possible and that \( m \) might have changed). We obtain
\[
(\gamma \mid \gamma) = \sum_{i=1}^m (\beta_i \mid \beta_i) + \sum_{i \neq j} (\beta_i \mid \beta_j),
\]
thus
\[
1 = \sum_{i=1}^m \frac{(\beta_i \mid \beta_i)}{(\gamma \mid \gamma)} + \sum_{i \neq j} \frac{(\beta_i \mid \beta_j)}{(\gamma \mid \gamma)}.
\]

Assume \( \gamma \not\in \Phi'' \). This implies \( m \geq 2 \). If the root \( \gamma \) is short, then \( \sum_{i=1}^m \frac{(\beta_i \mid \beta_i)}{(\gamma \mid \gamma)} \geq 2 \), hence \( (\beta_i \mid \beta_j) < 0 \) for some \( i \neq j \). By the minimality of \( m \) we have \( \beta_i \neq -\beta_j \). Therefore \( \beta_i + \beta_j \in \Phi'' \) by Lemma 222, contradicting the minimality of \( m \). Thus \( \gamma \in \Phi'' \).

If the root \( \gamma \) is long, then \( \gamma^\vee \) is short. Since \( L((\Phi')^\vee) = L(R^\vee) \), we can argue as before and obtain \( \gamma^\vee \in (\Phi'')^\vee \). Thus \( \gamma \in \Phi'' \).

\[ \square \]

4. A special generation property of \( W \): Proof of Theorem 1.4

In this section we prove Theorem 1.4. Throughout the section we assume as in the last section that \( \Phi \) is a crystallographic root system. We like to recall that in the crystallographic case simple systems are characterized by their pairwise obtuse angles as has been observed by Coxeter.

**Lemma 4.1.** Let \( R := \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi \) be a set of roots such that
\begin{enumerate}[(i)]
  
  \item \( (\alpha_i \mid \alpha_j) \leq 0 \) for \( i \neq j \) and
  
  \item \( W = \langle s_{\alpha_i} \mid 1 \leq i \leq n \rangle \).
\end{enumerate}

Then \( R \) is a simple system for \( \Phi \).
Proof. We follow the ideas of the proof given in [Doe93, Proposition 4.3]. By Theorem 1.1 and (ii) we obtain that $R$ is a basis of $V$. Put

$$C := \{ x \in V \mid (x \mid \alpha_i) > 0 \text{ for all } i \in \{1, \ldots, n\} \}.$$ 

We claim that $C$ is a fundamental cone of $(W, S)$. Therefore we need to show that for every $\alpha \in \Phi$ for all $x \in C$ either $(x \mid \alpha) > 0$ or $(x \mid \alpha) < 0$. Assume there are $\alpha \in \Phi$, $x_1, x_2 \in C$ such that $(x_1 \mid \alpha) < 0$ and $(x_2 \mid \alpha) > 0$. As $C$ is convex and $(\cdot \mid \alpha)$ continuous, we get $H_\alpha \cap C \neq \emptyset$.

Next we derive a contradiction to that fact. As the roots in $R$ are pairwise obtuse and as $\Phi$ is crystallographic we have

$$(\alpha_i \mid \alpha_j) = -\cos \left(\frac{\pi}{m_{ij}}\right),$$

where $m_{ij}$ is the order of $s_{\alpha_i}s_{\alpha_j}$ for $1 \leq i < j \leq n$. Thus, $M = (m_{ij})$ is a Coxeter matrix and $W = \langle s_{\alpha_1}, \ldots, s_{\alpha_n} \rangle$ is the geometric representation of the Coxeter system $(W(M), R)$ with Coxeter matrix $M$ by [Bou02, V, 4.3]. As $H_\alpha \cap C \neq \emptyset$ implies $s_\alpha(C) \cap C \neq \emptyset$, we get $s_\alpha = \text{id}$ by [Bou02, V, 4.4, Theorem 1], which is not possible.

This shows that $C$ is a fundamental cone and $R$ a simple system, see [Bou02, VI, 1.5, Theorem 2].

Notice that Lemma 1.1 implies if $w$ is a quasi-Coxeter element that is not a Coxeter element, then for each factorization of $w$ into reflections the roots related to that reflections are not pairwise obtuse. And this is precisely the case when for each factorization of $w$ the diagram Carter [Car72] associates to the roots related to this factorization has a circle, see [Car72, Lemma 19].

Proof of Theorem 1.4. Let $\Delta := \{\alpha_1, \ldots, \alpha_n\}$ be a simple system for $\Phi$ such that $P = \langle s_{\alpha_1}, \ldots, s_{\alpha_{n-1}} \rangle$ and let $t \in T$ such that $W = \langle P, t \rangle$. As $\Delta_P := \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a simple system for a subsystem of $\Phi$, the dihedral angles between these roots are pairwise obtuse.

Let $\beta \in \Phi$ such that $s_\beta = t$. Further let $E$ be the cone of $V$ that is cut out by the hyperplanes $H_\alpha$ with $\alpha \in \Delta_P \cup \{\beta\}$. Then

$$E = \{ x \in V \mid (x \mid \alpha) > 0 \text{ for all } \alpha \in \Delta_P \cup \{\beta\} \}$$

contains a fundamental cone.

Assume that $(\alpha_j \mid \beta) > 0$ for some $1 \leq j \leq n - 1$. Then

$$(\alpha_j \mid s_{\alpha_j}(\beta)) = (s_{\alpha_j}(\alpha_j) \mid \beta) = -(\alpha_j \mid \beta) = -(\alpha_j \mid \beta) < 0$$

and we replace $\beta$ by $s_{\alpha_j}(\beta)$.

We claim that the cone $F$ that is cut out by the new hyperplanes is contained in $E$. Let $x \in F$, that is $(x \mid \alpha_i) > 0$ for $1 \leq i \leq n - 1$ and $(x \mid s_{\alpha_j}(\beta)) > 0$. Then

$$(x \mid \beta) = (x \mid s_{\alpha_j}(s_{\alpha_j}(\beta))) = (x \mid s_{\alpha_j}(\beta)) - \frac{2(s_{\alpha_j}(\beta) \mid \alpha_j)(x \mid \alpha_j)}{(\alpha_j \mid \alpha_j)} > 0$$

as $(x \mid s_{\alpha_j}(\beta)) > 0$ and $(s_{\alpha_j}(\beta) \mid \alpha_j) < 0$, but $(x \mid \alpha_j) > 0$. Thus $F$ is contained in $E$.

Next we show that this containment is proper. As $\Delta_P \cup \{\beta\}$ is linear independent, it follows that $M := \{ x \in V \mid (x \mid \alpha_i) > 0 \text{ for } 1 \leq i \leq n - 1 \} \cap H_\beta \neq \emptyset$. Let $y \in M$. Then

$$(y \mid s_{\alpha_j}(\beta)) = (s_{\alpha_j}(y) \mid \beta) = (y \mid \beta) - \frac{2(y \mid \alpha_j)(\alpha_j \mid \beta)}{(\alpha_j \mid \alpha_j)} < 0$$
as \((y \mid \beta) = 0\) and \((y \mid \alpha_j), (\alpha_j \mid \beta) > 0\). Thus \(y\) is in the closure of \(E\), but not in the closure of \(F\), which implies that \(E \neq F\) and that therefore \(F\) is a proper subset of \(E\).

As \(E\) only contains finitely many such subsets, we will get after finitely many steps of this process a set of roots having pairwise obtuse angles. Then Lemma \([4.1]\) implies that the related cone is a fundamental cone and that the obtained set of roots \(\{\alpha_1, \ldots, \alpha_{n-1}, \gamma\}\) is a simple system for \(\Phi\). We got \(\gamma\) from \(\beta\) by conjugating it with elements of \(P\). This shows our claim. \(\square\)

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