Solution of coupled nonlinear Schrödinger equations in a focusing-defocusing medium by modified perturbation theory

Jerzy Jaśiński, Mirosław Karpiërzc

Faculty of Physics, Warsaw University of Technology, Koszykowa 75, 00-662 Warszawa

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Abstract—Interaction is considered of bright solitons of different orders and two different wavelengths propagating in a medium focusing for one wavelength and defocusing for the other. The system of nonlinear Schrödinger equations is solved in terms of perturbation theory. The application of the additional postulate to adjust both widths of the solitons and to modify the amplitude by a factor determined by the overlap integral greatly improves the accuracy of the description. Good accuracy of description is confirmed by numerical calculations.

In nonlinear optics, the coupled nonlinear Schrödinger equations have been for many years the main tool for studying the interactions of solitons with each other and with the medium through which they pass [1–6]. In this paper, we consider a nonlinear medium focusing for a wave at one frequency and defocusing for another and the description of interaction between two such waves.

Consider two beams \( U_{\text{Pos}}(x,z) \) and \( U_{\text{Neg}}(x,z) \) interacting with a nonlinear medium of nonlinearity:

\[
\varepsilon_z = \alpha_P |U_{\text{Pos}}|^2 - \alpha_N |U_{\text{Neg}}|^2, \tag{1}
\]

for \( \alpha_P > \alpha_N \geq 0 \). The Nonlinear Schrödinger Equations (NSE) describing the propagation of beams have the form:

\[
i \beta_{\text{Pos}} \frac{\partial U_{\text{Pos}}}{\partial z} + \frac{1}{2} \frac{\partial^2 U_{\text{Pos}}}{\partial x^2} + \left( \alpha_P |U_{\text{Pos}}|^2 - \alpha_N |U_{\text{Neg}}|^2 \right) U_{\text{Pos}} = 0, \tag{2}
\]

\[
i \beta_{\text{Neg}} \frac{\partial U_{\text{Neg}}}{\partial z} + \frac{1}{2} \frac{\partial^2 U_{\text{Neg}}}{\partial x^2} + \left( \alpha_P |U_{\text{Pos}}|^2 - \alpha_N |U_{\text{Neg}}|^2 \right) U_{\text{Neg}} = 0.
\]

This case was considered by many authors [4, 6–10] and discussed using different attitudes – analytical [1, 4, 7, 9], numerical [10] or variational [6, 8].

But the simplest solution of Eqs. (2) describes the case of vanishing field \( U_{\text{Neg}} \). Normalizing the wave function \( U_{\text{Pos}} \) together with the coordinates \((x,z)\) gives NSE in its fundamental form:

\[
i \frac{\partial \Psi}{\partial z} + \frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} + \left| \Psi \right|^2 \Psi = 0. \tag{3}
\]

The series of solutions of Eq. (3) can be obtained via Inverse Scattering Transform (IST) [4]. They represent solitons of different orders \( n = 1, 2, \ldots \). The function \( \Psi_n \) is a quotient of two complex combinations of \( n \) terms of the form \( \exp \left( -2 \eta_l (x-x_{\text{Pos}}) + 2im_l \eta_l (z-z_{\text{Pos}}) \right) \) with arbitrary real \( \eta_l \), \( x_{\text{Pos}} \), and \( z_{\text{Pos}} \) \((l = 1,\ldots,n)\). The coefficients \( \eta_l \) determine the widths and heights of its individual components.

A system of equations like Eqs. (2) for \( \beta_P = \beta_N = 1 \) and focusing nonlinearity for both beams \( \alpha_P = 1, \alpha_N = -1 \) has an analytical solution known as Manakov solitons [4, 10–14]. Based on Manakov’s solution, an analogous solution of the system [Eqs. (2)] for focusing-defocusing nonlinearity has been discussed in [4]. Both fields \( U_{\text{Pos}} \) and \( U_{\text{Neg}} \) are described by solitons of the same order, but unfortunately this solution exists only for the special case where \( \alpha_P = \alpha_N = 1 \) and \( \beta_P = \beta_N = 1 \).

The considered system of Eqs. (2) generalizes this case. To obtain Manakov-type solitons we should introduce additional amplitude factors \( \gamma_{\text{Pos}} \) and \( \gamma_{\text{Neg}} \):

\[
U_{\text{Pos}} = \gamma_{\text{Pos}} \Psi_n, \quad U_{\text{Neg}} = \gamma_{\text{Neg}} \Psi_n. \tag{4}
\]

Assume the real \( \gamma_{\text{Pos}} \) and \( \gamma_{\text{Neg}} \) (their phases are included into initial phase factors of \( \Psi_n \)). Substituting Eqs. (4) into Eqs. (2) we can prove that Eq. (3) is satisfied only for:

\[
\beta_{\text{Pos}} = \beta_{\text{Neg}} = 1, \quad \alpha_P \gamma_{\text{Pos}}^2 - \alpha_N \gamma_{\text{Neg}}^2 = 1. \tag{5}
\]

The requirement (4) implies that solitons of the both fields \( U_{\text{Pos}} \) and \( U_{\text{Neg}} \) have not only the same order \( n \), but the centres \( x_{\text{Pos}} \) of all their corresponding components for \( l_{\text{Pos}} = l_{\text{Neg}} = 1,\ldots,n \) have the same positions. In addition, for any pair of component indices \( l \) and \( k \) the initial phase differences \( \eta_l x_{\text{Pos}} - \eta_k x_{\text{Neg}} \) are the same for both fields.

In this paper we solve the system of Eqs. (2) applying perturbation theory. This attitude is frequently applied to describe various effects connected with interaction between solitons or with the medium [1, 5]. Although the \( U_{\text{Neg}} \) field will not always be small relative to \( U_{\text{Pos}} \), let us treat it initially as a small perturbation. Expressing both fields as series with respect to a small constant quantity \( \alpha \) of order of magnitude \( U_{\text{Neg}} \), we have:

\[
U_{\text{Pos}} = U_{\text{Pos}}^{(0)} + \alpha U_{\text{Pos}}^{(1)} + \alpha^2 U_{\text{Pos}}^{(2)} + \ldots, \quad U_{\text{Neg}} = \alpha U_{\text{Neg}}^{(1)} + \ldots \tag{6}
\]

The unperturbed field \( U_{\text{Pos}}^{(0)} \) satisfies the first equation of the system (2) with vanishing \( U_{\text{Neg}} \). Its solution describes a soliton of arbitrary order \( n \):

\[
U_{\text{Pos}}^{(0)}(x,z) = \gamma_n^{(0)} \Psi_n(x,z), \tag{7}
\]

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with $\gamma_P^{(0)}$ denoting the same symbols as in Eqs. (4) and (5), but written for vanishing $U_{Neg}$. For $n = 1$ this solution gives:

$$U_n^{(1)} = \gamma_n^{(1)}(\gamma)^2 \frac{\partial U_n^{(1)}(x,z)}{\partial x} + \alpha_n [U_n^{(1)}]^2 U_n^{(1)} = 0.$$  \hspace{1cm} (8)

with $\beta = 2n/\beta$ and $\alpha = 2n/\beta_n$. The initial phase $\phi_0$ is arbitrary.

By substituting the expansion (6) into the system (2) we prove that the first-order correction vanishes $U_n^{(1)} = 0$ while $U_n^{(1)}$ satisfies the equation:

$$i\beta_n \frac{U_n^{(1)}}{\partial z} + \frac{1}{2} \frac{\partial^2 U_n^{(1)}}{\partial x^2} + \alpha_n [U_n^{(1)}]^2 U_n^{(1)} = 0.$$  \hspace{1cm} (9)

The equation for the Neg field $U_n^{(1)}$ is linear. Its solution gives a function with a profile like $U_n^{(0)}$, but it may have a different propagating term:

$$U_n^{(1)}(x,z) = A_n \exp{-i\phi} U_n^{(0)}(x,z).$$  \hspace{1cm} (10)

Nevertheless, substituting (10) into Eq. (9), we prove that it is satisfied only for two cases: 1) $\beta_n = 0$ or $\Delta k = 0$ and 2) $U_n^{(0)} = U_1$, and $\Delta k = 2n(1/\beta_n - 1/\beta)$. Of particular interest is the second case, in which two solitons $U_{Pos}$ and $U_{Neg}$ can correspond to two different wavevelenghts.

But one can also consider solutions of Eq. (9), where instead of a true distribution of the refractive index derived from the nonlinearity $\alpha_n U_n^{(0)}^2$ we substitute a distribution that is close, but slightly different. Assume that this slightly different distribution is given by:

$$|U_n^{(0)}(x,z)|^2 = A_n^2 \text{Sech}^2(2\eta_n x).$$  \hspace{1cm} (11)

The parameter $\eta_n$ can slightly differ from one of the numbers $\eta_1, \ldots, \eta_n$, defining $n$-th order soliton (7), but the relation between $A_n$ and $\eta_n$ are assumed the same as the first-order soliton $A = 2n/\sqrt{\alpha_n}$. Moreover, to minimize deviation from the $z$-axis during propagation, assume an approximately symmetrical shape of the soliton $U_n^{(0)}(x,z)$ in the whole range of propagation.

The approximation (11) gives the solution of Eq. (9):

$$U_n^{(1)} = \gamma_n^{(1)}(\gamma)^2 \frac{\partial U_n^{(1)}(x,z)}{\partial x} + \alpha_n [U_n^{(1)}]^2 U_n^{(1)} = 0.$$  \hspace{1cm} (12)

with $\beta = 2n/\beta$ but arbitrary amplitude $\gamma_n^{(1)}$ (or $A_n$) of $U_{Neg}$ field (the multiplier before $U_1$ is for consistency with designations in expressions (4)).

The first non-vanishing correction $U_n^{(2)}$ for $U_{Pos}$ field satisfies a much more complicated equation:

$$i\beta_n \frac{U_n^{(2)}}{\partial z} + \frac{1}{2} \frac{\partial^2 U_n^{(2)}}{\partial x^2} + \alpha_n [U_n^{(1)}]^2 U_n^{(1)} = 0.$$  \hspace{1cm} (13)

Nevertheless, assuming $U_n^{(1)}$ of the form (12) and:

$$U_n^{(2)} = \gamma_n^{(2)} \frac{\partial U_n^{(1)}}{\partial x} U_n^{(1)}$$  \hspace{1cm} (14)

we are able to find the solution of Eq. (13) for real $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$. Substituting (12) and (14) into Eq. (13) gives the condition for the existence of this solution:

$$\gamma_n^{(2)} = \alpha_n \frac{\gamma_n^{(1)}}{2\alpha_n \gamma_n^{(0)}}.$$  \hspace{1cm} (15)

The obtained perturbed solutions of Eqs. (7)–(15) contain at least one case of different propagation constants $\beta_n \neq \beta$, but also close the case of Manakov-type solution, Eqs. (4)–(5). To compare results following from these two attitudes, we should assume equal $\beta_n$ and $\beta$, as in Eq. (5). Now we can see that the amplitudes $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$ as a small quantity we have:

$$\gamma_n^{(1)} = 1 + \frac{1}{\sqrt{\alpha_n \gamma_n^{(0)}}} + \gamma_n^{(2)} = \gamma_n^{(0)} + \gamma_n^{(2)}.$$  \hspace{1cm} (16)

Since the obtained second-order correction in expansion (16) is identical as (15), the conclusions from the exact Manakov-type solution (4)–(5) and the perturbation theory (7)–(15) are the same for the same choice of parameters. But the perturbation theory gives more possibililities to change the parameters of the interacting solitons.

By now the amplitude term $\gamma_n^{(1)}$ is arbitrary, because Eq. (9) is linear. To establish this term let us define the overlap integral $Q$ calculated using the initial field shapes:

$$Q = \frac{\int_{-\infty}^{\infty} [U_s(x,0)]^2 \left| U_{Neg}(x,0) \right| dx}{\int_{-\infty}^{\infty} [U_s(x,0)]^2 dx \int_{-\infty}^{\infty} \left| U_{Neg}(x,0) \right|^2 dx}.$$  \hspace{1cm} (17)

Of course, $0 \leq Q \leq 1$. During propagation the Neg soliton and central peak of the Pos soliton equal their widths. The amplitude of the $U_{Neg}$ field will decrease or increase, depending on the relation between $\eta_n$ and $\eta_p$. But its power always decreases to $Q$ of the initial power ($U_{Neg}$ field is much smaller than $U_{Pos}$, so the power carried by $Pos$ soliton hardly changes). This gives the rule enabling us to determine the amplitude of $U_{Neg}$ field:

$$U_{Neg} = \lim_{z \to \infty} [U_{Neg}(0,z)] = \frac{\eta_n}{\sqrt{\eta_p}} |U_{Neg}(0,0)|.$$  \hspace{1cm} (18)

Of course, both solitons should retain their energy, as proved in [9]. But in the process of adjusting their widths some of the waves quickly escape outside the region of interaction. Thus, the reduction in power applies only to the fields that remain in the system still interacting with each other.

To check the obtained results numerically we assumed $a = 1$, $\alpha = 0.25$, $\beta = 1$ and $\beta = 0.8$. As the input $U_{Pos}$ beam we took a third-order soliton with the central peak corresponding to $\eta_p = 3$ at $z$-axis ($x = 0$) and two side peaks approximately twice higher than the central peak. $U_{Neg}$ beam at the input contained the first-order soliton defined by parameter $\eta_n = 10$ with amplitude $2/3$ height of the central Pos peak. The other initial parameters have

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been adjusted to obtain the side peaks clearly separated from the central maximum. For these values in the input plane on the x-axis we have $\alpha_{\text{u}} |U_{\text{u}}(x, z)|^2 = 0.3 \alpha_{\text{u}} |U_{\text{u}}(x, z)|^2$, which means that at the beginning of propagation the $U_{\text{u}}$ beam reduces the nonlinear susceptibility over 3 times, so the interaction of solitons cannot be considered weak.

In Fig. 1 we can see how the central heights of both solitons change during propagation. Note that the heights of the two fields initially decrease, but after a distance of about 70, fairly regular oscillations of both amplitudes remain in the system with a small relative amplitude of $3.4 \times 10^{-4}$ for $U_{\text{u}}$ and 5 times less for $U_{\text{neg}}$. Moreover, we can see that the amplitude of the Neg soliton changes rapidly at the very initial stage of propagation.

In conclusions, using the perturbation theory we have found solutions in the form of a pair of solitary beams with different orders, propagating in the focusing-defocusing medium. The proposed method gives very good quantitative results and calculated beams are stable at long distances. Similar multi-hump solitons have been reported recently, however, in a nonlocal nonlinear medium [13-15].

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Fig. 1. Central heights of $U_{\text{u}}$ and $U_{\text{neg}}$ fields during propagation over the distance $z_{\text{en}}=500$. $U_{\text{u}}(0,0)=6.03$, $U_{\text{neg}}(0,0)=1.8$.

Fig. 2. The change of power of the propagating Neg field (left) and comparison of the initial and final profiles (right).

Analogous behaviour can be seen for the power $M_{\text{neg}} = \int U_{\text{neg}}(x, z) dx$ of the Neg soliton (left graph of Fig. 2). On the other hand, in the right graph one can observe that the profile of the Neg field and the central peak of the Pos field agree very well (the Pos field is scaled to obtain the same central heights). Adjusting Sech function to the final profile of both beams, we find $\eta_{\text{u}}=2.9686$, which is 1% less than the assumed value $\eta_{\text{u}}=3$ for the initial central peak. For the initial shapes of both considered solitons, we obtain $Q=0.7894$, which gives the estimated value of power with a fairly good 1% accuracy (red line). Using the obtained $\eta_{\text{u}}$, one can calculate $\gamma_{\text{u}}^{(1)} = 0.2874$ and $\gamma_{\text{u}}^{(2)} = 0.0103$, which gives the final height of the Neg soliton drawn by the red line in the right graph of Fig. 1 (accuracy 0.2% nevertheless is a significant change) and the final height of the Pos soliton drawn by the black line in the left graph (accuracy 0.01%).

Of course, the accuracy of the method increases for closer values of the central widths ($\eta_{\text{u}}$ and $\eta_{\text{u}}$). But it is also important to have a clearly marked peak in the centre of the Pos soliton field distribution. In Fig. 3 we show the final fields for the initial central peak height equal 10.3 (the left graph) and 0.25 (the right graph) of the side peaks. All other parameters are the same. In the left graph one can hardly distinguish deformations (however both final fields lost their symmetry), but in the right graph we can see that propagation became unstable – the Neg field is no longer guided by Pos soliton.

Fig. 3. Deformations of the final profiles of Neg solitons.

References
[1] Y. Kivshar, G. P. Agrawal, Optical Solitons, From Fibers to Photonic Crystals, (Amsterdam, Academic Press 2003).
[2] F. Abdullaev, S. Darmanyan, P. Khabibullaev, Optical Solitons, (Springer-Verlag, Berlin, 1993).
[3] G. A. Stegeman, D.N. Christodoulides, M. Segev, IEEE J. Selected Topics Quantum Electron. 6, 1419 (2000).
[4] J. Yang, Nonlinear Waves in Integrable and Nonintegrable Systems, (SIAM, Philadelphia 2010).
[5] Y. Kivshar, B. Malomed, Rev. Mod. Phys. 61, 763 (1989).
[6] P. G. Kevrekidis, D.J. Frantzeskakis, Reviews in Physics 173, 5 (2009).
[7] R. de la Fuente, A. Barthelemy, IEEE J. Quantum Electron. 28, 547 (1992).
[8] H.T. Tran, R. A. Sammut, Phys. Rev. A 52, 3170 (1995).
[9] S. Leble, B. Reichel, Eur. Phys. J. Special Topics 173, 5 (2009).
[10] M. Vijnayajanythi, T.Kanna, M. Lakshmanan, Eur. Phys. J. Special Topics 173, 57 (2009).
[11] S. V. Manakov, Sov. Phys. JETP 38 (1973), 248.
[12] J. Y. Yang, Phys. Rev. E 65, 036606 (2002).
[13] T. Kanna, M. Lakshmanan, Phys. Rev. Lett. 86, 5043 (2001).
[14] M. Jakubowski, K. Steglik, R. Squier, Phys. Rev. E 58, 6752 (1998).
[15] P.S. Jung, W. Krolikowski, U.A. Ladyn, M. Trippenbach, M.A. Karpierz, Phys. Rev. A 95 (2017).
[16] P.S. Jung, M.A. Karpierz, M. Trippenbach, D.N. Christodoulides, W. Krolikowski, Photon. Lett. Poland 10 (2018).
[17] A. Ramanui, M. Trippenbach, P.S. Jung, D.N. Christodoulides, W. Krolikowski, G. Assanto, Opt. Express 29, 8015 (2021).

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