MACROSCOPIC AND EDGE BEHAVIOR OF A PLANAR JELLIUM

DJALIL CHAFAI, DAVID GARCÍA-ZELADA, AND PAUL JUNG

Abstract. We consider a planar Coulomb gas in which the external potential is generated by a smeared uniform background of opposite-sign charge on a disc. This model can be seen as a two-dimensional Wigner jellium, not necessarily charge-neutral, and with particles allowed to exist beyond the support of the smeared charge. The full space integrability condition requires low enough temperature or high enough total smeared charge. This condition does not allow at the same time, total charge-neutrality and determinantal structure. The model shares similarities with both the complex Ginibre ensemble and the Forrester–Krishnapur spherical ensemble of random matrix theory. In particular, for a certain regime of temperature and total charge, the equilibrium measure is uniform on a disc as in the Ginibre ensemble, while the modulus of the farthest particle has heavy-tailed fluctuations as in the Forrester–Krishnapur spherical ensemble. We also touch on a higher temperature regime producing a crossover equilibrium measure, as well as a transition to Gumbel edge fluctuations. More results in the same spirit on edge fluctuations are explored by the second author together with Raphael Butez.

1. Introduction

Coulomb gases are systems of charged particles, all of the same sign, where the pair potential between particles is of Coulomb type. If the space in which the particles live is not compact, then an external potential is required to confine the particles from ‘going off to infinity’, since they are all of the same sign. Wigner jelliums are Coulomb gases for which this external potential is precisely the Coulomb potential generated by a charged background of opposite sign. Typically, for the jellium, one imposes an additional constraint that all particles live in some compact region which is equivalent to having an infinite external potential on the complement of this region. We study a simple planar jellium obtained using a uniform background on a centered disc, but where the particles are not confined to live on this disc. Our analysis reveals that this model, seen as a Coulomb gas, cannot be both charge-neutral and determinantal. Moreover, this model shares similarities with both the Ginibre and the spherical model, see Figure 1. In particular, it has a uniform equilibrium on the disc and has heavy-tailed fluctuations at the edge.

The rest of the introduction is devoted to, firstly, a more precise description of the Wigner jellium and its link to the Coulomb gas, and secondly, the main results concerning the particular unconfined jellium model that we study. Section 2 is devoted to general notions and facts on planar potential theory including planar Coulomb gases. It gathers the main known examples and results needed to understand our main results. Section 3 is devoted to the proofs of our main results. Finally Section 4 provides some historical remarks on the jellium and Coulomb gases.

1.1. Wigner jelliums as Coulomb gases. Following Wigner [66], let us consider \( n \) unit negatively charged particles (electrons) at positions \( x_1, \ldots, x_n \) in \( \mathbb{C} = \mathbb{R}^2 \), lying in a positive background of total charge \( \alpha > 0 \) smeared according to a probability measure \( \rho \) on \( \mathbb{C} \) with finite Coulomb energy \( c = \mathcal{E}(\rho) \), see [3] for a general definition of \( \mathcal{E}(\cdot) \). We could alternatively suppose that the particles are positively charged (ions) and the background is negatively charged (electrons), this reversed choice would not affect the analysis of the model. The total energy of the system, counting each pair a single time, is given by

\[
\sum_{i<j} g(x_i - x_j) - \alpha \sum_{i=1}^{n} U_{\rho}(x_i) + \alpha^2 c,
\]
Figure 1. Plot of the external potential $V = -\frac{\alpha}{n}U_\rho$ used later in Lemma 1.1 when the radius of the disc is one. Here, $n$ is the number of particles and $\alpha$ is total charge of the opposite-signed background. In the neighborhood of the origin, the behavior is quadratic just like the potential of the Ginibre ensemble, while outside this neighborhood, the behavior is logarithmic just like the potential of the spherical ensemble.

where $U_\rho = -\log|\cdot| * \rho$ is the logarithmic potential of $\rho$. This matches the energy formula (7) of a Coulomb gas with $V = -\frac{\alpha}{n}U_\rho$. This observation leads us to define the jellium model on $S \subset \mathbb{C}$ with total background charge $\alpha > 0$ and background distribution $\rho$ with $\text{supp}\rho \subset S$ as being the Coulomb gas of the full space $\mathbb{C}$, with potential $V$ given by

$$V = \begin{cases} -\frac{\alpha}{n}U_\rho & \text{on } S \\ +\infty & \text{on } S^c. \end{cases}$$

We say that the system is (charge) neutral when $\alpha = n$. We say that it is uniform when $\rho$ is the uniform distribution on some compact subset of $\mathbb{C}$. The great majority of jellium models studied in the literature are charge-neutral and satisfy $S = \text{supp}\rho$.

Conversely, from the energy formula above and the inversion formula for the logarithmic potential (2), a Coulomb gas with sub-harmonic potential $V$ (meaning $\Delta V \geq 0$) could be seen as a jellium as above with $\alpha \rho = \frac{\Delta V}{4\pi} d\ell_C$ on $S = \mathbb{C}$ where $\ell_C$ stands for the Lebesgue measure on $\mathbb{C}$, but such a $\rho$ is not necessarily a probability measure. When $V$ is not sub-harmonic then $\rho$ is additionally no longer a positive measure but we can still interpret it as a background with opposite charge on $\{|\Delta V| < 0\}$.

The famous example of the complex Ginibre ensemble is a Coulomb gas with potential $V = |\cdot|^2$, for which $\Delta V$ is constant, leading to an interpretation of this Coulomb gas as a degenerate jellium on the full space $\mathbb{C}$ with Lebesgue background. The beautiful example of the Forrester–Krishnapur spherical ensemble is a Coulomb gas with potential $V = (1 + 1/n) \log(1 + |\cdot|^2)$, for which $\Delta V = 4/(1 + 1/n) / (1 + |\cdot|^2)^2$, leading to an interpretation of this Coulomb gas as a jellium on the full space with a heavy tailed background. We can also consider such a background–potential inverse problem for the one-dimensional log-gases of random matrix theory, which can be seen as two-dimensional Coulomb gases confined to the real line, such as the Gaussian Unitary Ensemble.

For instance it follows from the discussion in [29, Section 1.4] that the logarithmic potential of the background measure of total charge $n$ with Lebesgue density

$$x \mapsto \frac{\sqrt{n}}{\pi} \sqrt{1 - \frac{x^2}{2n}} 1_{|x| < \sqrt{2n}}$$

is given on the interval $S = [-\sqrt{2n}, \sqrt{2n}]$ by

$$x \mapsto \frac{x^2}{2} + \frac{n}{2} \left( \log \frac{n}{2} - 1 \right).$$

1.2. Model and main results. In this note, we focus on a very simple planar jellium on the full space $S = \mathbb{C}$, seen as a Coulomb gas $P_n$ defined by (6) with $V = -\frac{\alpha}{n}U_\rho$, $\alpha > 0$, where $\rho$ is the uniform probability distribution on the closed centered disc

$$D_R = \{ z \in \mathbb{C} : |z| \leq R \}$$
of radius $R > 0$. We study the macroscopics and the edge asymptotics of this planar Coulomb gas. We let $\alpha$ and $\beta$ depend on $n$ and we proceed in an asymptotic analysis as $n \to \infty$. The potential $V$ depends on $n$. Our analysis reveals that this special model shares similarities with the complex Ginibre and the Forrester–Krishnapur spherical ensembles.

We look at (a) when this system is well-defined, and in the case it is well-defined, (b) global asymptotics at the level of the equilibrium measure, and (c) edge behavior in the sense of asymptotic analysis of the particle farthest from the origin.

We first need requirements under which the Boltzmann–Gibbs measures exist. The following lemma says that when the total charge of the background is high enough, the confinement effect on the gas is strong enough to define the Boltzmann–Gibbs measure. The condition is natural for Coulomb gases, see for instance [21]. Note that the condition does not allow, at the same time, both charge-neutrality and determinantal structure.

We use the notation $Z_n$ and $P_n$ for the partition function and Gibbs measure of the system of $n$ particles, respectively (see (8) and (6) below for a precise mathematical formulation).

**Lemma 1.1** (Confinement or integrability condition). We have

$$Z_n < \infty \quad \text{if and only if} \quad \alpha - n > \frac{2}{\beta} - 1.$$ 

Moreover if this condition holds then $P_n$ is a Coulomb gas with an external potential

$$V(x) = \frac{\alpha}{2n} \left( \frac{|x|^2}{R^2} - 1 + 2 \log R \right) 1_{|x| \leq R} + \frac{\alpha}{n} \log |x| 1_{|x| > R}.$$ 

In particular, in the determinantal case $\beta = 2$, the condition on $\alpha$ reads $\alpha > n$.

On the other hand, in the neutral case $\alpha = n$, the condition on $\beta$ reads $\beta > 2$.

The proof of Lemma 1.1 is given in Section 3.1 and a plot of $V$ is provided in Figure 1.

Note that the condition does not depend on $R$. Also note that the potential matches the one of the Ginibre ensemble when restricted to the disc of radius $R$, while it is similar to the one of the spherical ensemble when restricted to the region outside of the disc of radius $R$.

We use the notation $X_n = (X_{n,1}, \ldots, X_{n,n})$ and $\mu_{X_n}$ for the random vector of particle locations and the corresponding empirical distribution, respectively (see (9) below).

**Theorem 1.2** (First order global asymptotics: low temperature regime). Suppose that both $\alpha = \alpha_n$ and $\beta = \beta_n$ may depend on $n$ in such a way that

$$\lim_{n \to \infty} n \beta_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\alpha_n}{n} = \lambda \geq 1,$$

and, if $\lambda = 1$, that $\alpha_n - n > \frac{2}{\lambda(n-1)} - 1$ for $n$ large enough. Then for $n$ large enough $Z_n < \infty$, and $P_n$ is well-defined. Moreover, regardless of the way we define the sequence of probability measures $(P_n)_n$ on the same probability space, we have that almost surely,

$$\lim_{n \to \infty} d_{BL}(\mu_{X_n}, \mu_*) = 0,$$

where $\mu_*$ is the uniform distribution on $D_{R/\sqrt{n}}$.

The proof of Theorem 1.2 is given in Section 3.2.

Note that the low temperature regime contains the determinantal case $\beta = 2$. Also note that the case $\lambda < 1$ is useless since Lemma 1.1 tells us in this case that $Z_n = \infty$.

**Theorem 1.3** (First order global asymptotics: high temperature regime). Suppose that both $\alpha = \alpha_n$ and $\beta = \beta_n$ may depend on $n$ in such a way that

$$\lim_{n \to \infty} n \beta_n = \kappa > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\alpha_n}{n} = \lambda \quad \text{with} \quad \kappa(\lambda - 1) > 2.$$ 

Then for $n$ large enough, $Z_n < \infty$, and $P_n$ is well-defined. Moreover, regardless of the way we define the sequence of probability measures $(P_n)_n$ on the same probability space, we have, almost surely,

$$\lim_{n \to \infty} d_{BL}(\mu_{X_n}, \mu_*) = 0$$

where $\mu_*$ has a density $\varphi$ that satisfies the following equation on its support

$$\Delta \log \varphi = 2\pi \kappa \left( \varphi - \frac{\lambda}{\pi R^2} \right).$$
The proof of Theorem 1.3 is given in Section 3.3.

Our last results concern the fluctuation of the edge, in other words the modulus of the farthest particle, in the determinantal case $\beta = 2$. We reveal a phase transition with respect to $\lambda$: the fluctuations are heavy tailed if $\lambda = 1$ and light tailed (Gumbel) if $\lambda > 1$.

When $\lambda = 1$, we know from Theorem 1.2 that the equilibrium $\mu_*$ is supported in $D_R$. The farthest particle will then “feel” $V$ outside $D_R$, which is, according to Lemma 1.1, in this region, logarithmic, and resembles that of the Forrester–Krishnapur spherical ensemble. We can then expect that the fluctuations of the modulus of the farthest particle will be then heavy tailed, however the fluctuation law may differ from that of the spherical ensemble. This intuition is entirely confirmed by the following theorem.

**Theorem 1.4** (Heavy-tailed edge). Suppose that $\beta = 2$ and $\alpha = \alpha_n = n + \kappa_n$ with $\kappa_n > 0$ and $\lim_{n \to \infty} \kappa_n = \kappa > 0$, in such a way that in particular $\lim_{n \to \infty} \alpha_n/n = \lambda = 1$. Then $Z_n < \infty$ by Lemma 1.7 and $P_n$ is well-defined. Moreover

$$\max_{1 \leq k \leq n} |X_{n,k}| \xrightarrow[n \to \infty]{\text{law}} L$$

where $L$ is the law with cumulative distribution function given by

$$t \in \mathbb{R} \mapsto L((-\infty, t]) = \prod_{k=0}^{\infty} \left(1 - \frac{2^{2(k+\kappa)}}{n} \right) 1_{t \geq R}.$$

The proof of Theorem 1.4 is given in Section 1.4.

Note that the law $L$ in Theorem 1.4 has a heavy (right) tail.

Beyond the edge fluctuation, and following [13, proof of Theorem 2.4], it is actually possible to show that the whole (determinantal) point process converges as $n \to \infty$, in such a way that in particular $\lim_{n \to \infty} \alpha_n/n = \lambda = 1$. Moreover, if we define

$$a_n = \frac{\sqrt{n\alpha_n}}{C_n} \quad \text{and} \quad b_n = C_n \left(1 + \frac{1}{2} \sqrt{\frac{C_n}{n}}\right)$$

where $C_n = \log(n) - 2\log\log(n) - \log(2\pi)$ and $C_n = \sqrt{\frac{n}{\alpha_n} R}$, then

$$\max_{1 \leq k \leq n} |X_{n,k}| \xrightarrow[n \to \infty]{\frac{R}{\sqrt[n]{\lambda}}} \frac{R}{\sqrt[2]{\lambda}}$$

and

$$a_n \left(\max_{1 \leq k \leq n} |X_{n,k}| - b_n\right) \xrightarrow[n \to \infty]{\text{law}} G$$

where $G$ is the Gumbel law with cumulative distribution function

$$t \in \mathbb{R} \mapsto G((-\infty, t]) = e^{-e^{-t}}.$$

The proof of Theorem 1.5 is given in Section 3.5.

It is worth mentioning that following [31], one can pass from the heavy tailed law $L$ of Theorem 1.4 to the Gumbel light tailed law $G$ of Theorem 1.5. Namely, if for each $\kappa > 0$, $\xi_\kappa$ is a random variable taking values in $[1, +\infty)$ with cumulative distribution function

$$t \in [1, +\infty) \mapsto \mathbb{P}(\xi_\kappa \leq t) = \prod_{k=0}^{\infty} (1 - t^{-2(k+\kappa)}),$$

The proof of Theorem 1.4 is given in Section 1.4.
and if $\varepsilon_\kappa > 0$ is the unique solution of $\varepsilon_\kappa \exp(\kappa \varepsilon_\kappa) = 1$, then
\[ 2\kappa \left( \xi_\kappa - 1 - \frac{\varepsilon_\kappa}{2} \right) \xrightarrow{\kappa \to \infty} \text{Gumbel}. \]

Furthermore, following [31], taking $\kappa = +\infty$ in Theorem 1.3 leads to Gumbel fluctuations!

A simulation study can be done using the algorithm in [18], see for instance Figure 2.

Note that the edge of one-dimensional models is considered in [23, 25, 23].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{The top graphic is a plot of a simulation of $X_n \sim P_n$, $n = 8$, illustrating Theorem 1.2 and Theorem 1.5 in the case $R = 2$ and $\lambda = 4$. We used the algorithm from [35] with $dt = .5$ and $T = 100000$. About 10 independent copies were simulated and merged and we retained only the last 10% of the trajectories. The bottom graphic shows a histogram of the radii of the same data together with the non-asymptotic radial density for the complex Ginibre ensemble (dashed line, exact formula from determinantal structure) and radial density of equilibrium measure (solid line).}
\end{figure}

Let us end this introduction with two open problems. We believe there should be a version of Theorem 1.3 in the critical case $\kappa(\lambda - 1) = 2$, but it is unclear for us that the functional used in our proof of Theorem 1.3 is well-defined in this case. For criticality, the proof of the large deviation
principle may require a special proof. We should have $-\int \Delta \log \varphi \, d\ell_{\mathbb{C}} = 4\pi$. With the Gauss–Bonnet formula in mind, seeing $-\Delta \log \varphi$ as a curvature suggests a space of Euler characteristic one, which could be thought as the unit disc. A second open problem is to investigate the density $\varphi$ of Theorem 1.3 as $\kappa \to \infty$, in other words, as the high temperature regime approaches the low temperature regime. The question here is to elucidate how or in what sense the density approaches the uniform distribution on $D_{R/\sqrt{\kappa}}$.

2. Essential facts on planar potential theory

This section gathers useful elements of two-dimensional potential theory. We refer, for instance, to [55, 63, 65, 11] for more details on the basic aspects of potential theory used in this note. The Coulomb kernel $g$ in dimension 2 is given on $x \in \mathbb{C} = \mathbb{R}^2 \setminus \{0\}$, by

$$g = -\log |x|.$$ 

It belongs to $L^1_{\text{loc}}(\mathbb{C})$ and constitutes the fundamental solution of the Laplace or Poisson equation, namely $\Delta g = -2\pi \delta_0$ in the sense of Schwartz distributions on $\mathbb{R}^2$. In particular $g$ is super-harmonic, and harmonic on $\mathbb{R} \setminus \{0\}$. The Coulomb potential at point $x \in \mathbb{C}$ generated by a distribution of charges (say electrons) modeled by a probability measure $\mu$ on $\mathbb{C}$ such that $g1_{K^c} \in L^1(\mu)$ for some large enough compact set $K$ is defined by

$$U_\mu(x) = (g * \mu)(x) = \int g(x-y)\,d\mu(y) \in (-\infty, +\infty].$$

We have $U_\mu \in L^1_{\text{loc}}(\mathbb{C})$ and the identity $\Delta g = -2\pi \delta_0$ gives the inversion formula

$$\Delta U_\mu = -2\pi \mu.$$ 

In particular $-U_\mu$ is sub-harmonic in the sense that $\Delta U_\mu \geq 0$. The Coulomb (self-interaction) energy of the (distribution of charges) $\mu$ is defined when it makes sense by

$$E(\mu) = \frac{1}{2} \iint g(x-y)\,d\mu(x)\,d\mu(y) = \frac{1}{2} \int U_\mu(x)\,d\mu(x).$$

A subset of $\mathbb{C}$ has finite capacity when its carries a probability measure with finite Coulomb energy. When this is not the case, we say that the set has zero capacity.

Let us denote by $\mathcal{P}(\mathbb{C})$ the set of probability measures on $\mathbb{C}$, equipped with the topology of weak convergence with respect to continuous and bounded test functions, and its associated Borel $\sigma$-field. This topology is metrized by the bounded-Lipschitz metric

$$d_{\text{HL}}(\mu, \nu) = \sup \left\{ \int f(d\mu - d\nu) : \|f\|_{\infty} \leq 1, \|f\|_{\text{Lip}} \leq 1 \right\}$$

where $f : \mathbb{C} \to \mathbb{R}$ is measurable, $\|f\|_{\infty} = \sup_{x \in \mathbb{C}} |f(x)|$, $\|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|}$.

Let $V : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function playing the role of an external potential, producing an external electric field $-\nabla V$. If $V$ grows faster than $g$ at infinity, the Coulomb energy $\mathcal{E}_V$ with external field is defined by

$$\mu \in \mathcal{P}(\mathbb{C}) \mapsto \mathcal{E}_V(\mu) = E(\mu) + \int V\,d\mu.$$ 

It is lower semi-continuous with compact level sets, strictly convex, and it admits a unique minimizer called the equilibrium measure or Frostman measure denoted

$$\mu_* = \arg \min_{\mathcal{P}(\mathbb{C})} \mathcal{E}_V.$$ 

From this variational formula, there exists a constant $c$ such that except on a set of zero capacity,

$$\begin{cases} U_{\mu_*} + V = c & \text{on the support of } \mu_*, \\ U_{\mu_*} + V \geq c & \text{outside}. \end{cases}$$

In particular $V$ is sub-harmonic on the support of $\mu_*$. Combined with (2), we get, when $V$ has Lipschitz weak first derivative, that

$$d\mu_* = \frac{\Delta V}{2\pi} \, d\ell_{\mathbb{C}} \quad \text{on the support of } \mu_*.$$
In particular the probability measure \( \mu_* \) is supported in \( \{ \Delta V \geq 0 \} \). Furthermore \( \mu_* \) is compactly supported when \( V \) goes to \( +\infty \) at \( \infty \) fast enough.

2.1. Planar Coulomb gases. A planar Coulomb gas with \( n \) particles, potential \( V \), and inverse temperature \( \beta \geq 0 \) is the exchangeable Boltzmann–Gibbs probability measure \( P_n \) on \( \mathbb{C}^n \) given by

\[
\text{d}P_n(x_1, \ldots, x_n) = \frac{e^{-\beta E_n(x_1, \ldots, x_n)}}{Z_n} \text{d}\ell_C(x_1) \cdots \text{d}\ell_C(x_n)
\]

where

\[
E_n(x_1, \ldots, x_n) = \sum_{i<j} g(x_i - x_j) + n \sum_{i=1}^n V(x_i)
\]

and

\[
Z_n = \int e^{-\beta E_n(x_1, \ldots, x_n)} \text{d}\ell_C(x_1) \cdots \text{d}\ell_C(x_n),
\]

The Coulomb gas is well defined when \( Z_n < \infty \). It models a gas of unit charged particles, or more precisely a random configuration of unit charged particles. We should keep in mind that we play here with electrostatics rather than with electrodynamics (no magnetic field). For all \( n \), we define the random empirical measure

\[
\mu X_n = \frac{1}{n} \sum_{k=1}^n \delta_{X_{n,k}} \quad \text{where} \quad X_n = (X_{n,1}, \ldots, X_{n,n}) \sim P_n.
\]

The notation \( \sim \) means that the random variable \( X_n \) has law \( P_n \). We have \( \mathcal{P}(\mu X_n \in A) = P_n\left( \frac{1}{n} \sum_{k=1}^n \delta_{x_k} \in A \right) \) for any Borel subset \( A \subset \mathcal{P}(\mathbb{C}) \). In the low temperature regime \( \beta = \beta_n \) with \( \lim_{n \to \infty} n \beta_n = \infty \), \( (\mu X_n)_n \) satisfies the following large deviation principle: for any Borel subset \( A \subset \mathcal{P}(\mathbb{C}) \) with interior \( \text{int}(A) \) and closure \( \text{clo}(A) \),

\[
- \inf_{\text{int}(A)} \mathcal{E}_V + \mathcal{E}_V(\mu_*) \leq \lim_{n \to \infty} \frac{\log \mathbb{P}(\mu X_n \in A)}{n^2 \beta_n} \leq \lim_{n \to \infty} \frac{\log \mathbb{P}(\mu X_n \in A)}{n^2 \beta_n} \leq - \inf_{\text{clo}(A)} \mathcal{E}_V + \mathcal{E}_V(\mu_*).
\]

We refer to \([39, 61, 6, 36, 19, 32]\) for these large deviation principles for Coulomb gases.

For any Borel measures \( \mu \) and \( \nu \), we define the Kullback–Leibler divergence or relative entropy of \( \nu \) with respect to \( \mu \) by \( D(\nu \mid \mu) = \int \log \frac{d\nu}{d\mu} d\nu \) if \( \nu \) is absolutely continuous with respect to \( \mu \), and \( D(\nu \mid \mu) = +\infty \) otherwise.

In the high temperature regime \( \beta = \beta_n \) with \( \lim_{n \to \infty} n \beta_n = \kappa \in (0, +\infty) \), the same holds true with \( \mathcal{E}_V \) formally replaced by

\[
\mathcal{E}_V + \frac{1}{\kappa} D(\cdot \mid \ell_C)
\]

or, equivalently, formally replaced by

\[
\mathcal{E} + \frac{1}{\kappa} D(\cdot \mid \nu_V)
\]

where \( \nu_V \) has a density proportional to \( e^{-\kappa V} \). We also have to replace \( \mu_* \) by the minimizer of \( \mathcal{E}_V + \frac{1}{\kappa} D(\cdot \mid \ell_C) \). This is known as the crossover regime, which interpolates between \( \nu_V \) and the minimizer of \( \mathcal{E}_V \). The classical Sanov theorem corresponds formally to this regime when we turn off the pair interaction by taking \( g = 0 \). This regime is considered in particular in \([14, 9, 19, 26, 32, 2]\).

From \([10]\), for all \( \varepsilon > 0 \), we get by taking \( A = \{ \mu \in \mathcal{P}(\mathbb{C}) : d_{BL}(\mu X_n, \mu_*) > \varepsilon \} \) that

\[
\sum_n \mathbb{P}(d_{BL}(\mu X_n, \mu_*) \geq \varepsilon) < \infty,
\]

which is a summable convergence in probability. By the Borel–Cantelli lemma, we obtain something known in the probabilistic literature as complete convergence, see for instance \([67]\). In particular, regardless of the way we define the random vectors \( X_n \) on the same probability space, we have that almost surely,

\[
\lim_{n \to \infty} d_{BL}(\mu X_n, \mu_*) = 0.
\]
2.2. Determinantal exact solvability, Ginibre and spherical models. It is useful to rewrite the density of the Coulomb gas $P_n$ defined in (3), provided that $Z_n < \infty$, as
\begin{equation}
\frac{e^{-\beta n \sum_{i=1}^{n} V(x_i)}}{Z_n} \prod_{i<j}^{n} |x_i - x_j|^\beta.
\end{equation}

This includes plenty of famous models from random matrix theory including the following couple of models, and we refer to [33, 46, 40, 30, 29, 64, 65] for more information:

- **Complex Ginibre ensemble.** This corresponds to taking $\beta = 2$ and $V = \frac{1}{2} |\cdot|^2$.

The equilibrium measure is uniform on the unit disc, namely
\[ d\mu = \frac{1}{\pi} \frac{|\cdot|}{1 + |\cdot|^2} d\ell_C, \]
in accordance with (5). This Coulomb gas describes the eigenvalues of a Gaussian random complex $n \times n$ matrix $A$ with density proportional to $e^{-\text{Trace}(AA^\ast)}$ where $A^\ast = \bar{A}^\top$ is the conjugate-transpose of $A$. Equivalently, the entries of $A$ are independent and identically distributed with independent real and imaginary parts having a Gaussian law of mean 0 and variance $1/(2n)$. This gas also appears in various other places in the mathematical physics literature, for instance as the modulus of the wave function in Laughlin’s model of the fractional quantum Hall effect [21], in the description of the vortices in the Ginzburg–Landau model of superconductivity [64], and in a model of rotating trapped fermions [49].

- **Forrester–Krishnapur spherical ensemble.** This corresponds to taking $\beta = 2$ and $V = \frac{n+1}{2n} \log(1 + |x|^2)$.

The equilibrium measure is heavy tailed and given by
\[ d\mu = \frac{1}{\pi (1 + |\cdot|^2)^2} d\ell_C. \]

The name of this gas comes from the fact that it is the image by the stereographical projection of the Coulomb gas on the sphere, with constant potential, onto the complex plane. This Coulomb gas describes the eigenvalues of $AB^{-1}$ where $A$ and $B$ are two independent copies of complex Ginibre random matrices. We can loosely interpret $AB^{-1}$ as a sort of matrix analogue of the Cauchy distribution since when $A$ and $B$ are $1 \times 1$ matrices, this is precisely a Cauchy distribution.

The case $\beta = 2$ has a remarkable integrable structure, called a **determinantal structure**, which provides exact solvability, see for instance [9 [58, 5 [21]. More precisely, if $\beta = 2$ then for all $1 \leq k \leq n$, the $k$-th dimensional marginal distribution of the exchangeable probability measure $P_n$, denoted $P_{n,k}$, has density proportional to
\[ (x_1, \ldots, x_k) \in \mathbb{C}^k \mapsto \det[K_n(x_i, x_j)]_{1 \leq i, j \leq k} \]
where $K_n$ is an explicit kernel which depends on $n$ and $V$. Since $E\mu_{X,1} = P_{n,1}$, it follows in particular that the density of $E\mu_{X,n}$ is proportional to $x \in \mathbb{C} \mapsto K_n(x, x)$. Following [45, 21], if $\beta = 2$ and if $V$ is radially symmetric, say $V = Q(|\cdot|)$, then the point process of radii or moduli (this should be interpreted as a random multi-set)
\[ \{|X_{n,1}|, \ldots, |X_{n,n}|\} \]
has the same law as the point process $\{Y_{n,1}, \ldots, Y_{n,n}\}$ where $R_{n,1}, \ldots, R_{n,n}$ are independent (and not identically distributed) random variables with $R_k$ of density proportional to
\begin{equation}
t \in [0, +\infty) \mapsto \frac{t^{2k-1} e^{-2nQ(t)}}{\pi} \frac{1}{1 + t^2}, \quad 1 \leq k \leq n.
\end{equation}

Following [62, 21, 43, 31], this allows for the asymptotic analysis of the modulus of the farthest particle of the Coulomb gas as $n \to \infty$. In particular, one can analyze:
• Complex Ginibre ensemble. For this gas, the equilibrium measure has an edge and the modulus of the farthest particle tends to this edge, and the fluctuation is described by a Gumbel law. Namely, following [22, 24], if we define
\[ a_n = 2 \sqrt{nc_n} \quad \text{and} \quad b_n = 1 + \frac{1}{2} \sqrt{nc_n} \]
where \( c_n = \log(n) - 2 \log \log(n) - \log(2\pi) \) then
\[ \max_{1 \leq k \leq n} |X_{n,k}| \xrightarrow{p} 1 \quad \text{and} \quad a_n (\max_{1 \leq k \leq n} |X_{n,k}| - b_n) \xrightarrow{\text{law}} G \]
where \( G \) is the Gumbel law with cumulative probability distribution
\[ t \in \mathbb{R} \mapsto G((\infty, t]) = e^{-t}. \]

• Forrester–Krishnapur spherical ensemble. For this gas, the modulus of the farthest particle tends to infinity and has a heavy tail. Namely, following [21, 41, 48].
\[ \max_{1 \leq k \leq n} |X_{n,k}| \xrightarrow{p} +\infty \quad \text{and} \quad \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} |X_{n,k}| \xrightarrow{\text{law}} F \]
where \( F \) is the probability distribution with cumulative distribution function
\[ t \in \mathbb{R} \mapsto F((\infty, t]) = \lim_{k \to \infty} e^{-t^2} \sum_{j=0}^{k-1} \frac{1}{j!} t^j, \]
moreover this law is heavy tailed in the sense that
\[ 1 - F((\infty, t]) = F((t, +\infty)) \sim t^{-2}. \]

The notation \( \xrightarrow{\text{law}} \) and \( \xrightarrow{p} \) stand for convergence in law and in probability respectively.

In random matrix theory and statistical physics, it is customary to speak about macroscopic behavior for \( \mu_X \) and about edge behavior for \( \max_{1 \leq k \leq n} |X_{n,k}| \).

3. Proofs

3.1. Proof of Lemma 1.1

Proof of Lemma 1.1. Following for instance [63, Ch. 0, Example 5.7], we have
\[ U_\rho(x) = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \frac{1}{|x - re^{i\theta}|^2} \, r \, dr \, d\theta \]
\[ = \frac{1}{2} \left( 1 - \frac{|x|^2}{R^2} \right) 1_{|x| < R} - \log \frac{|x|}{R} 1_{|x| > R} + \log R, \]
which is harmonic outside the support of \( \rho \), in accordance with (2). Now we define
\[ W_\rho = -U_\rho \quad \text{and} \quad G(x,y) = g(x-y) + W_\rho(x) + W_\rho(y) \]
then
\[ \sum_{i,j} G(x_i, x_j) = \sum_{i,j} g(x_i - x_j) + (n-1) \sum_{i=1}^n W_\rho(x_i) \]
and
\[ e^{-\beta E_n(x_1, \ldots, x_n)} = e^{-\beta \left( \sum_{i,j} g(x_i - x_j) + \alpha \sum_{i=1}^n W_\rho(x_i) \right)} = e^{-\beta \sum_{i,j} G(x_i, x_j) \prod_{i=1}^n e^{-\beta (\alpha - n+1) W_\rho(x_i)}}. \]

The idea now is to show that the first exponential in the last display is bounded whereas the product of exponentials is integrable. Indeed, we shall use the following properties:
(a) The function \( x \mapsto |W_\rho(x) - \log|x|| \) is bounded for \( |x| \geq 1 \);
(b) The function \( G \) is bounded from below (see Remark 3.1).
(c) For all closed set $F$ and every compact set $K$ such that $F \cap K = \emptyset$, we have
\[
\sup_{(x,y) \in F \times K} |G(x, y)| < \infty.
\]
By (i) and since $W_\rho$ is bounded from below, we have
\[
x \in \mathbb{C} \implies e^{-\beta(\alpha - n + 1)W_\rho(x)} \text{ is integrable } \iff \beta(\alpha - n + 1) > 2.
\]
Thus, using additionally (b), we obtain
\[
\beta(\alpha - n + 1) > 2 \implies Z_n < \infty.
\]
For the converse implication choose $n - 1$ pairwise disjoint compact sets $K_1, \ldots, K_{n-1}$ in a neighborhood of the origin, say the open unit disc. Then, (c) implies that $-\beta \sum_{i < j} G(x_i, x_j)$ is bounded from below whenever $(x_1, \ldots, x_{n-1}) \in K_1 \times \cdots \times K_{n-1}$ and $|x_n| \geq 1$. As $W_\rho$ is bounded from above in the unit disc there exists a constant $C$ such that the integrand is bounded from below by
\[
\frac{C}{|x_n|^{\beta(\alpha - n + 1)}}
\]
whenever $(x_1, \ldots, x_{n-1}) \in K_1 \times \cdots \times K_{n-1}$ and $|x_n| \geq 1$. We conclude that
\[
\beta(\alpha - n + 1) \leq 2 \implies Z_n = \infty.
\]
\[\square\]

**Remark 3.1** (Confinement or integrability condition). The integrability condition in Lemma 1.2 can also be derived using the elementary inequality $|a - b| \leq (1 + |a|)(1 + |b|)$ valid for all $a, b \in \mathbb{C}$, as in [21]. Namely, it gives $\prod_{i < j} |x_i - x_j| \leq (\prod_{i=1}^{n-1} (1 + |x_i|))^{n-1}$ since each $i$ appears exactly in $n - 1$ elements of $\{(i, j) : i < j\}$. Hence, for all $x_1, \ldots, x_n \in \mathbb{C}$ such that $|x_1| > R, \ldots, |x_n| > R$, we have, for some constants $c', c''$,
\[
E_n(x_1, \ldots, x_n) \geq c' - \sum_{i < j} \log |x_i - x_j| + \alpha \sum_{i=1}^{n} \log |x_i| \geq c'' - (\alpha - (n - 1)) \sum_{i=1}^{n} \log |x_i|.
\]
Therefore $Z_n < \infty$ if $x \mapsto 1/|x|^{\beta(\alpha - (n - 1))}$ is integrable at infinity with respect to the Lebesgue measure on $\mathbb{C}$, which holds when $\beta(\alpha - (n - 1)) > 2$.

3.2. Proof of Theorem 1.2 We use the method used in [32] for the proof of (10). This method takes its roots in [26]. In this approach, the relative entropy plays an essential role for tightness, even when it does not appear in the final statement. Beware also that we have here to take into account the fact that the potential $V = -\frac{4}{n} U_\rho$ depends on $n$.

**Proof of Theorem 1.2** Suppose first that $\lambda > 1$. Define $G : \mathbb{C} \times \mathbb{C} \to (-\infty, \infty]$ by
\[
G(x, y) = g(x - y) + W_\rho(x) + W_\rho(y)
\]
and
\[
A_n = \frac{\alpha_n - n + 1}{n} - \frac{4}{n \beta_n}.
\]
Then,
\[
\lim_{n \to \infty} A_n = \lambda - 1 > 0
\]
and $e^{-\beta_n E_n(x_1, \ldots, x_n)}$ writes
\[
\exp \left[ -\beta_n \left( \sum_{i < j} g(x_i - x_j) + \alpha_n \sum_{i=1}^{n} W_\rho(x_i) \right) \right] = \exp \left[ -\beta_n \left( \sum_{i < j} G(x_i, x_j) + A_n \sum_{i=1}^{n} W_\rho(x_i) \right) \right] \prod_{i=1}^{n} \exp \left[ -4W_\rho(x_i) \right],
\]
where $W_\rho = -U_\rho$. Let us define $H_n : \mathbb{C}^n \to (-\infty, +\infty]$ by
\[
H_n(x_1, \ldots, x_n) = \frac{1}{n^4} \sum_{i < j} G(x_i, x_j) + A_n \frac{1}{n} \sum_{i=1}^{n} W_\rho(x_i)
\]
As before, we notice that
\[ e^{-\beta_n E_n} d\sigma_n(x) \] so that the Coulomb gas law \( e^{-\beta_n H_n} d\sigma_n(x) \) is proportional to
\[ e^{-\beta_n H_n} d\sigma_n(x_1, \ldots, x_n). \]

Take any bounded continuous \( f : \mathbb{C} \to \mathbb{R} \). Then following for instance [32], we get that
\[
\frac{1}{n^2 \beta_n} \log \int_{\mathbb{C}^n} e^{-n^2 \beta_n (f \circ i_n + H_n)} d\sigma_n(x_1, \ldots, x_n)
\]
\[
= - \inf_{\mu \in \mathcal{P}(\mathbb{C}^n)} \left\{ \mathbb{E}_\mu [f \circ i_n + H_n] + \frac{1}{n^2 \beta_n} D(\mu \mid \sigma_n) \right\},
\]
where \( \mathcal{P}(\mathbb{C}^n) \) is the set of probability measures on \( \mathbb{C}^n \) and where
\[ i_n(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}, \]
so that it is natural to expect that
\[
- \inf_{\mu \in \mathcal{P}(\mathbb{C}^n)} \left\{ \mathbb{E}_\mu [f \circ i_n + H_n] + \frac{1}{n^2 \beta_n} D(\mu \mid \sigma) \right\}
\]
converges to
\[
- \inf_{\mu \in \mathcal{P}(\mathbb{C})} \left\{ f(\mu) + \frac{1}{2} \int_{\mathbb{C} \times \mathbb{C}} G(x, y) d\mu(x) d\mu(y) + (\lambda - 1) \int_{\mathbb{C}} W_\rho(x) d\sigma(x) \right\}
\]
which is what exactly happens. We refer to [32] for the details.

Suppose now that \( \lambda = 1 \). Define \( H_n : \mathbb{C}^n \to (-\infty, \infty] \) by
\[ H_n(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i<j} G(x_i, x_j) \]
where
\[ G(x, y) = g(x - y) + W_\rho(x) + W_\rho(y) \]
and define
\[ \gamma_n = \beta_n (\alpha_n - n + 1). \]
and
\[ d\sigma_n(x) = \frac{e^{-\gamma_n W_\rho}}{Z_{\sigma_n}} d\ell_C(x). \]

The Coulomb gas law is proportional to
\[ e^{-\beta_n H_n} d\sigma_n(x_1, \ldots, x_n). \]

As before, we notice that
\[
\frac{1}{n^2 \beta_n} \log \int_{\mathbb{C}^n} e^{-n^2 \beta_n (f \circ i_n + H_n)} d\sigma_n(x_1, \ldots, x_n)
\]
\[
= - \inf_{\mu \in \mathcal{P}(\mathbb{C}^n)} \left\{ \mathbb{E}_\mu [f \circ i_n + H_n] + \frac{1}{n^2 \beta_n} D(\mu \mid \sigma_n) \right\},
\]
but now we remark that if \( d\mu = \rho d\ell_C \) we have
\[
\frac{1}{n^2 \beta_n} D(\mu \mid \sigma_n) = \frac{1}{n \beta_n} \int_{\mathbb{C}} \rho \log \rho d\ell_C + \frac{\gamma_n}{n \beta_n} \int_{\mathbb{C}} W_\rho d\mu + \frac{\gamma_n}{n \beta_n} \log \int_{\mathbb{C}} e^{-\gamma_n W_\rho} d\ell_C
\]
if all these terms make sense (holds if \( \rho \) is bounded and compactly supported) and, thus,
\[
\lim_{n \to \infty} \frac{1}{n \beta_n} D(\mu \mid \sigma_n) = 0.
\]
By the same arguments as before we may conclude.
3.3. Proof of Theorem 1.3 As in the proof of Theorem 1.2 we proceed as in [2] for the proof of (10), taking into account the fact that the potential $V = -a U_\rho$ depends on $n$.

Proof of Theorem 1.3. Choose any $\delta > 0$ such that $2 < \delta < \beta(\lambda - 1)$ and define

$$A_n = \frac{\alpha_n - n + 1}{n} - \frac{\delta}{n \beta_n}$$

Then,

$$\lim_{n \to \infty} A_n = \lambda - 1 - \frac{\delta}{\kappa} > 0$$

and we write

$$\exp \left[ -\beta_n \left( \sum_{i<j} g(x_i - x_j) + \alpha_n \sum_{i=1}^n W_\rho(x_i) \right) \right] = \exp \left[ -\beta_n \left( \sum_{i<j} G(x_i, x_j) + A_n \sum_{i=1}^n W_\rho(x_i) \right) \right] \prod_{i=1}^n \exp[-\delta W_\rho(x_i)]$$

where $W_\rho = -U_\rho$ and $G(x, y) = g(x - y) + W_\rho(x) + W_\rho(y)$ are as in the proof of Theorem 1.2. Furthermore, by following the same ideas as for the case $\lambda > 1$ in the proof of Theorem 1.2 we conclude that $\mu_n \to \mu_*$ as $n \to \infty$, where $\mu_*$ is the unique minimizer of

$$\mu \mapsto \int \sigma(x) \, d\mu(x) + \int \lambda |x| \, d\mu(x).$$

We refer to [2] for the details. Then, still following [2], we can get from (1) that

$$\Delta \log \varphi = 2\pi \kappa \varphi - 2\pi \kappa \lambda \frac{1}{\pi R^2}.$$ 

\[\square\]

3.4. Proof of Theorem 1.4.

Proof of Theorem 1.4. It is enough to prove the fluctuation result since it implies the convergence in probability to the edge (just like the central limit theorem implies the weak law of large numbers).

Now, since $\beta = 2$, $P_n$ is determinantal and we can use [12] which we rephrase as follows: the point process of the radii $\{X_{n,1}, \ldots, X_{n,n}\}$ has the same law as the point process $\{Y_{n,0}, \ldots, Y_{n,n-1}\}$ where $Y_{n,0}, \ldots, Y_{n,n-1}$ are independent (not identically distributed) complex random variables such that

$$Y_{n,k} \sim a_{n,k} |z|^{2k} e^{-2(n+\kappa_n)W_\rho(z)} \, d\ell_C(z)$$

with $a_{n,k} = \left( \int C |z|^{2k} e^{-2(n+\kappa_n)W_\rho(z)} \, d\ell_C(z) \right)^{-1}$

is a normalization constant, and $W_\rho = -U_\rho$. In particular,

$$\mathbb{P}(\max_{1 \leq k \leq n} |X_{n,k}| \leq x) = \prod_{k=0}^{n-1} \left( 1 - a_{n,k} \int_{D_{2k}} |z|^{2k} e^{-2(n+\kappa_n)W_\rho(z)} \, d\ell_C(z) \right).$$

By adding a constant, we shall suppose that $W_\rho(z) = \log R$ if $|z| = R$. In that case

$$W_\rho(z) = \log |z| \quad \text{if} \quad |z| \geq R \quad \text{while} \quad W_\rho(z) > \log |z| \quad \text{if} \quad |z| < R.$$

Suppose that $x > R$. Then, by (18) and (19),

$$\mathbb{P}(\max_{1 \leq k \leq n} |X_{n,k}| \leq x) = \prod_{k=0}^{n-1} \left( 1 - a_{n,k} \int_{D_{2k}} |z|^{2k} e^{-2(n+\kappa_n)W_\rho(z)} \, d\ell_C(z) \right).$$

Using the change of indices $k \to n - k - 1$ we obtain

$$\mathbb{P}(\max_{1 \leq k \leq n} |X_{n,k}| \leq x) = \prod_{k=0}^{n-1} \left( 1 - a_{n,n-k-1} \int_{D_{2k}} |z|^{-2(k+\kappa_n+1)} \, d\ell_C(z) \right).$$

The limit can be calculated by using Lebesgue’s dominated convergence theorem.
which is a (scaled) complex Ginibre ensemble. It follows that for any event

\[ \Pr \left( \sum_{k=0}^{n-1} |z_k|^2 \leq 1 \right) \]

But the left-hand side of (20) can be calculated explicitly as

\[ 1 - b_{n,n-k-1} \int_{D_R^2} |z|^{-2(k+\kappa_n+1)} d\ell_C(z) = 1 - a_{n,n-k-1} \int_{D_R^2} |z|^{-2(k+\kappa_n+1)} d\ell_C(z) \]

where

\[ b_{n,n-k-1} = \left( \int_{D_R} |z|^{-2(k+\kappa_n+1)} d\ell_C(z) \right)^{-1}. \]

But the left-hand side of (20) can be calculated explicitly as

\[ 1 - b_{n,n-k-1} \int_{D_R^2} |z|^{-2(k+\kappa_n+1)} d\ell_C(z) = 1 - \left( \frac{R}{x} \right)^{2(k+\kappa_n)} \]

and, if we choose \( \varepsilon \in (0,\kappa) \), then, for \( n \) large enough we have \( \kappa - \varepsilon < \kappa_n \) so that

\[ 1 - \left( \frac{R}{x} \right)^{2(k+\kappa_n-\varepsilon)} \leq 1 - \left( \frac{R}{x} \right)^{2(k+\kappa_n)} . \]

Since

\[ \prod_{k=0}^{\infty} \left( 1 - \left( \frac{R}{x} \right)^{2(k+\kappa_n-\varepsilon)} \right) > 0 \]

we have a domination from below of our product.

**Pointwise convergence.** Now, let us see the convergence of the terms. The coefficient

\[ a_{n,n-k-1}^{-1} = \int_\mathbb{C} |z|^{2(n-k-1)} e^{-2(n+\kappa_n)W_\rho(z)} d\ell_C(z) \]

has an integrand which converges to

\[ |z|^{-2(k+1)} e^{-2\kappa_n W_\rho(z)} \]

and that is dominated by, for instance, \( |z|^{-2(k+1)} e^{-(\kappa+\varepsilon)W_\rho(0)} e^{-2(\kappa-\varepsilon)W_\rho(z)} \) which is integrable. So,

\[ \lim_{n \to \infty} a_{n,n-k-1}^{-1} = \int_{D_R^2} |z|^{-2(k+\kappa_n+1)} d\ell_C(z) \]

and, by evaluating the integrals,

\[ \lim_{n \to \infty} \left( 1 - a_{n,n-k-1}^{-1} \int_{D_R^2} |z|^{-2(k+\kappa_n+1)} d\ell_C(z) \right) = 1 - \left( \frac{R}{x} \right)^{2(k+\kappa)} . \]

**Lebesgue’s dominated convergence theorem.** Having dominated each term from below and having proved the convergence of each term we apply Lebesgue’s dominated convergence theorem and obtain the desired result.

3.5. **Proof of Theorem 1.5.**

**Proof of Theorem 1.5.** It is enough to prove the fluctuation result since it implies the convergence in probability to the edge. Now, let us write \( \alpha_n = n\lambda_n \) with \( \lambda_n \to \lambda > 1 \). From the formula for \( V = -\frac{\partial}{\partial n} U_\rho \) of Lemma 1.4 we get that \( V = V^{\text{Gin}} \) on \( D_R \), where

\[ V^{\text{Gin}} = \frac{\lambda_n}{2R^2} |\cdot|^2. \]

Let \( P_n^{\text{Gin}} \) be the Boltzmann–Gibbs probability measure on \( \mathbb{C}^n \) defined by (6) with potential \( V^{\text{Gin}} \), which is a (scaled) complex Ginibre ensemble. It follows that for any event

\[ A \subset D_R^n = D_R \times \cdots \times D_R = \{ x \in \mathbb{C}^n : \max(|x_1|, \ldots, |x_n|) \leq R \} \]

we have

\[ P_n(A) = \frac{Z_n^{\text{Gin}}}{Z_n} P_n^{\text{Gin}}(A). \]
From Theorem 1.3, the limiting distribution $\mu_+$ under $P_n$ is supported in $D_{R/\sqrt{A}}$. Now $\lambda > 1$ implies $D_{R/\sqrt{A}} \subseteq D_R$. For an arbitrary $\varepsilon \in (0, R(1 - 1/\sqrt{A}))$, let us define the event

$$A_n = D^n_{R/\sqrt{A} + \varepsilon} = \left\{ x \in \mathbb{C}^n : \max(|x_1|, \ldots, |x_n|) \leq R/(\sqrt{A} + \varepsilon) \right\} \subseteq D^n_R.$$

Let $a_n, b_n$, and $G$ be as in Theorem 1.3 and let $\xi \sim G$. Let us define

$$M_n = a_n(\max(|x_1|, \ldots, |x_n|) - b_n).$$

Then it is known, see [8, 21], that

$$\lim_{n \to \infty} P_n^G(A_n^c) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}_P(e^{i\theta M_n}) = \mathbb{E}(e^{i\theta \xi}), \quad \theta \in \mathbb{R}.

\textbf{Lemma 3.2} (Partition functions).

$$\lim_{n \to \infty} \frac{Z_n^G}{Z_n} = 1.

\textbf{Proof of Lemma 3.2.} \text{Since } Z_n P_n(A_n) = Z_n^G P_n^G(A_n) \text{ and since } \lim_{n \to \infty} P_n^G(A_n) = 1 \text{ from } (22), \text{ the desired statement is actually equivalent to}

$$\lim_{n \to \infty} P_n(A_n) = 1.

But from (12), we obtain, denoting $V = Q(|\cdot|)$ and $r = R/\sqrt{A} + \varepsilon$,

$$P_n(A_n) = \prod_{k=1}^n \left( 1 - c_{n,k} \int_r^\infty t^{2k-1} e^{-2nQ(t)} dt \right) \text{ where } c_{n,k}^{-1} = \int_0^\infty t^{2k-1} e^{-2nQ(t)} dt.

\text{It is possible to follow this elementary route and to evaluate the limit of this product by evaluating the integrals. Actually (23) is a weak consequence of } [\text{1}], \text{ which is itself a refinement of } [\text{20}] \text{ Theorem 1.12]. Note that [\text{4}], Theorem 1] deals with potentials not necessarily rotationally invariant, and provides a quantitative exponential upper bound on the probability. Note also that condition (vi) of [\text{3}] translates in our context to } \int_U e^{2nU(r)} d\mu < \infty \text{ which holds precisely when } \lambda > 1. \text{ When } \lambda = 1, \text{ Theorem 1.1 shows that } (23) \text{ does not hold, so that } \lambda > 1 \text{ is the optimal condition on } \lambda \text{ under which (24) can hold.}

Now, using (21), (23), and (22), we obtain

$$|\mathbb{E}_{P_n}(e^{i\theta M_n}) - \mathbb{E}_{P_n^G}(e^{i\theta M_n} 1_{A_n^c})| = |\mathbb{E}_{P_n}(e^{i\theta M_n} 1_{A_n^c})| \leq P_n(A_n^c) = \frac{Z_n^G}{Z_n} P_n^G(A_n^c) \xrightarrow{n \to \infty} 0.

Next, and similarly, using (21),

$$\mathbb{E}_{P_n}(e^{i\theta M_n} 1_{A_n}) = \frac{Z_n^G}{Z_n} \mathbb{E}_{P_n^G}(e^{i\theta M_n} 1_{A_n^c}) = \frac{Z_n^G}{Z_n} \mathbb{E}_{P_n^G}(e^{i\theta M_n}) - \frac{Z_n^G}{Z_n} \mathbb{E}_{P_n^G}(e^{i\theta M_n} 1_{A_n}),$$

and using (23) and (22) we obtain

$$\lim_{n \to \infty} |\mathbb{E}_{P_n}(e^{i\theta M_n} 1_{A_n}) - \mathbb{E}(e^{i\theta \xi})| = 0.

Finally we have obtained as expected that

$$\lim_{n \to \infty} \mathbb{E}_{P_n}(e^{i\theta M_n}) = \lim_{n \to \infty} \mathbb{E}_{P_n^G}(e^{i\theta M_n}) = \mathbb{E}(e^{i\theta \xi}), \quad \theta \in \mathbb{R}.

\textbf{Remark 3.3. An alternative and purely equivalent way to formulate the proof of Theorem 1.5 is to note first that (21) is indeed equivalent to stating that}

$$P_n(\cdot \mid D^n_R) = P_n^G(\cdot \mid D^n_R).

\text{Next, we may deduce from (21), (23), and (22) that}

$$\lim_{n \to \infty} P_n(D^n_R) = \lim_{n \to \infty} P_n^G(D^n_R) = 1.

\text{which implies using (25) and Lebesgue’s dominated convergence theorem that}

$$\lim_{n \to \infty} \left( P_n(\cdot \mid D^n_R) - P_n^G(\cdot \mid D^n_R) \right) = 0.
weakly in $\mathcal{P}(\mathbb{C})$, which implies in turn using again \cite{22} that
\[
\lim_{n \to \infty} \mathbb{E}_{P_n}(e^{i\theta M_n}) = \lim_{n \to \infty} \mathbb{E}_{P_{\mathrm{Gin}}}(e^{i\theta M_n}) = \mathbb{E}(e^{i\theta \xi}), \quad \theta \in \mathbb{R}.
\]

4. Historical Comments

The jellium model was used around 1938 by Eugene P. Wigner in \cite{66} for the modeling of electrons in metals, more than ten years before his renowned works on random matrices. This model was inspired from the Hartree–Fock model of quantum mechanics, see \cite{35, 56, 57, 64}, and \cite{51, 53}. The term jellium was apparently coined by Conyers Herring since the smeared charge could be viewed as a positive “jelly”, see \cite{41}. The model is also known as a one-component plasma with background. As already mentioned, usually charge-neutral jellium models are studied, and this is done typically after restricting the electrons to live on some compact support of positive background. The restriction ensures integrability of the energy and the interest is usually focused on the distribution/behavior of electrons in the “bulk” of the limiting system when the volume of the compact set goes to infinity (thermodynamic limit). There are some exceptions where the edge has been considered, for instance in \cite{15, 34}. Also, the edge of Laughlin states has been considered in \cite{16}. The case \(d = 3\) is considered in \cite{55}, and quoting \cite{56}: “It is also possible to consider the one- and two-dimensional versions of this problem, where the Coulomb potential \(|x|^{-1}\) is replaced by \(−|x|\) and \(−\log|x|\), respectively. In the one-dimensional, classical case, Baxter \cite{7} calculated the partition function exactly. For that case, Kunz \cite{17} showed that the one-particle distribution function exists and that it has crystalline ordering, i.e., the Wigner lattice exists for all temperatures. Brascamp and Lieb \cite{12} showed the same to be true in the quantum mechanical case for one-component fermions when \(\beta\) is large enough. Although we do not deal with the one-dimensional problem here, our methods would apply in that case. In two dimensions there are difficulties connected with the long-range nature of the \(−\log|x|\) potential, and we shall not discuss this here.” For more background literature on the jellium, see also \cite{3, 42, 28, 1, 43, 22}. See in particular \cite{52}, for the fluctuations of non-neutral jelliums.

Historically, Coulomb gas models appeared naturally in statistics around 1920-1930 in the study of the spectrum of empirical covariance matrices of Gaussian samples. Nowadays we speak about the Laguerre ensemble and Wishart random matrices. This was almost ten years before the introduction of the jellium model by Wigner. In the 1950’s, Wigner rediscovered, by accident, these works by reading a statistics textbook, and this motivated him to use random matrices for the modeling of energy levels of heavy nuclei in atomic physics, see \cite{17}. We refer to \cite{10} for these historical aspects. The work of Wigner was amazingly successful, and he received in 1963 a Nobel prize in Physics “for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles.”. The term Coulomb gas is explicitly used by Dyson in his first seminal 1962 paper \cite{27} and by Ginibre \cite{33}, while the term Fermi-gas was used earlier by Mehta & Gaudin \cite{60} and also later by Dyson & Mehta \cite{59}.

Acknowledgments

We would like to thank an anonymous reviewer for his useful remarks on the presentation, and also Sabine Jansen, Mathieu Lewin, and Grégory Schehr for their feedback.

References

[1] M. Aizenman, S. Jansen & P. Jung – “Symmetry breaking in quasi-1D Coulomb systems”, Annales Henri Poincaré 11 (2010), no. 8, p. 1453-1485. [2] G. Akemann & S.-S. Byun – “The high temperature crossover for general 2D Coulomb gases”, preprint arXiv:1808.00319v1, 2018. [3] Alastuey, A. & Jancovici, B. – “On the classical two-dimensional one-component Coulomb plasma”, J. Phys. France 42 (1981), no. 1, p. 1–12. [4] Y. Ameur – “A localization theorem for the planar Coulomb gas in an external field”, arXiv preprint arXiv:1907.00923v1, 2019. [5] Y. Ameur, H. Hedenmalm & N. Makarov – “Random normal matrices and Ward identities”, Ann. Probab. 43 (2015), no. 3, p. 1157–1201. [6] G. W. Anderson, A. Guionnet & O. Zeitouni – An introduction to random matrices, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge University Press, Cambridge, 2010.
