Bounds for Entanglement via an Extension of Strong Subadditivity of Entropy

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We dedicate this paper to Walter Thirring on the occasion of his 85th birthday

Abstract
Let $\rho_{12}$ be a bipartite density matrix. We prove lower bounds for the entanglement of formation $E_f(\rho_{12})$ and the squashed entanglement $E_{sq}(\rho_{12})$ in terms of the conditional entropy $S_{12} - S_1$, and prove that these bounds are sharp by constructing a new class of states whose entanglements can be computed, and for which the bounds are saturated.

Key Words: entropy, entanglement, strong subadditivity.

1 Introduction
It is well known that in classical probability theory the entropy of a bipartite density, $\rho_{12}$, and its marginal densities $\rho_1$ and $\rho_2$, satisfy the positivity of the conditional entropies:

$$S_{12} - S_1 \geq 0 \quad \text{and} \quad S_{12} - S_2 \geq 0.$$ (1.1)

In quantum mechanics one has a density matrix $\rho_{12}$ and its reduced density matrices $\rho_1$ and $\rho_2$, for which one can define von Neumann entropies, but the analog of (1.1) need not hold. The closest one can come to (1.1) for the von Neumann entropy is the triangle inequality [2], $S_{12} - |S_1 - S_2| \geq 0$.

The main thrust of our paper is to show that the failure of either one of the inequalities (1.1), which can occur only in quantum mechanics, necessarily implies quantum entanglement, which is a peculiar correlation found only in quantum mechanics. More precisely, our main result will be the sharp inequality

$$E \geq \max\{S_1 - S_{12}, S_2 - S_{12}, 0\}$$

where $E$ denotes either one of two measures of entanglement, $E_f$ and $E_{sq}$, which we define now.

Let $\rho_{12}$ be a density matrix on a bipartite system; i.e., on a tensor product of two Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\rho_{12}$ is finitely separable in case it can be decomposed as a convex combination of tensor products:

$$\rho_{12} = \sum_{k=1}^{n} v_k \rho_1^k \otimes \rho_2^k$$ (1.2)

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where the $\nu_k$ are positive and sum to 1, and each $\rho_k^j$ is a density matrix on $\mathcal{H}_a$. A bipartite state is separable if it is in the closure of the set of finitely separable states. A bipartite state that is not separable is entangled.

The first measure of entanglement that we study is the *entanglement of formation* $E_f$, introduced by Bennett et al. [3, 4], which is defined in terms of the von Neumann entropy $S(\rho) = -\Tr(\rho \log \rho)$ by

$$E_f(\rho_{12}) = \inf \left\{ \sum_{j=1}^n \lambda_j S(\Tr_2 \omega_j) : \rho_{12} = \sum_{k=1}^n \lambda_k \omega^k \right\} \quad (1.3)$$

where $\Tr$ and $\Tr_2$, respectively, are the trace over the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the partial trace over $\mathcal{H}_2$ alone. The coefficients $\lambda_k$ in the expansion are required to be positive and sum to 1, and each $\omega^k$ is a state on $\mathcal{H}_1 \otimes \mathcal{H}_2$, which, by the concavity of $S$, may be taken to be a pure state without affecting the value of the infimum. Since the two partial traces of a pure state have the same spectrum and hence the same entropies [2], $E_f(\rho_{12})$ is symmetric in 1 and 2. It is known that $E_f(\rho_{12}) = 0$ if and only if $\rho_{12}$ is separable; see [5] for a discussion of this result in relation to other measures of entanglement. A variational problem for quantum entropy in a general von Neumann algebra setting that is related to (1.3) was introduced by Narnhofer and Thirring [12].

The second measure of entanglement that we study is a smaller quantity, the *squashed entanglement*, which was first defined by Tucci [16]. It was rediscovered by Christandl and Winter [6] who showed that it has many important properties, such as additivity. In several ways it provides a better estimate of the purely quantum mechanical entanglement than entanglement of formation. For a review of the subject, see [8].

The definition of squashed entanglement involves a relaxation of the variational problem defining $E_f$; i.e., it extends the domain over which the minimization is to be taken, in the following way:

Given a decomposition of a bipartite state $\rho_{12} = \sum_{k=1}^n \lambda_k \omega^k$, we may associate a tripartite state $\rho_{123}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ where $\mathcal{H}_3$ is *any* Hilbert space of dimension at least $n$ by letting $\{\phi_1, \ldots, \phi_n\}$ be orthonormal in $\mathcal{H}_3$, and defining

$$\rho_{123} = \sum_{k=1}^n \lambda_k \omega^k \otimes |\phi_k\rangle \langle \phi_k| \quad (1.4)$$

Using only the fact that each $\phi_j$ is a unit vector, and not using the orthogonality, one has that $\rho_{12} = \Tr_3(\rho_{123})$, so that $\rho_{123}$ is an *extension* of $\rho_{12}$, meaning that $\Tr_3 \rho_{123} = \rho_{12}$. (This is only one of many ways one could extend $\rho_{12}$ to a tripartite state $\rho_{123}$. Another way is through *purification* which we shall use in Section 2. The main point to note at present is that there are infinitely many extensions.)

Let $\rho_{23}$ denote $\Tr_1(\rho_{123})$, let $\rho_3$ denote $\Tr_{12}(\rho_{123})$, let $S_{23}$ denote $S(\rho_{23})$, and let $S_3$ denote $S(\rho_3)$, etc. following the notational scheme in [3]. A simple computation shows that for the extension $\rho_{123}$ given in (1.4), now using the fact that $\{\phi_1, \ldots, \phi_n\}$ is orthonormal,

$$S_{13} + S_{23} - S_{123} - S_3 = 2 \sum_{j=1}^n \lambda_j S(\Tr_2 \omega_j) \quad (1.5)$$

The right side is (twice) the quantity appearing in the definition of $E_f$, (1.3).

We recall a standard definition: For any density matrix $\rho_{123}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, the *conditional mutual information* of 1 and 2 given 3 is the quantity $I(1, 2|3)$ defined by

$$I(1, 2|3) := S_{13} + S_{23} - S_{123} - S_3 \geq 0 \quad (1.6)$$

The *strong subadditivity of the quantum entropy* theorem of Lieb and Ruskai [11], is the statement that for all tripartite states $\rho_{123}$,

$$I(1, 2|3) \geq 0 \quad (1.7)$$

By these results and (1.6), the quantity $E_{sq}(\rho_{12})$ defined by

$$E_{sq}(\rho_{12}) = \frac{1}{2} \inf \{ I(1, 2|3) : \rho_{123} \text{ is any tripartite extension of } \rho_{12} \} \quad (1.8)$$
defines a non-negative minorant to $E_t(\rho_{12})$ known as the squashed entanglement of a bipartite state $\rho_{12}$.

Evidently, for all $\rho_{12}$, $E_t(\rho_{12}) \geq E_{sq}(\rho_{12})$. Moreover, as proved in [7], there is equality in (1.7) if and only if $H_3$ has the form

$$H_3 = \bigoplus_{j=1}^m H_{3j}^1 \otimes H_{3j}^2$$

and $\rho_{123}$ has the form

$$\rho_{123} = \bigoplus_{j=1}^m \rho_{1,3j}^1 \otimes \rho_{3j,2}^1 \cdot (1.9)$$

Evidently, for any $\rho_{123}$ of the form (1.9), $\rho_{12} := \text{Tr}_2(\rho_{123})$ is separable. Moreover, if $\rho_{12}$ is separable, and has the decomposition (1.2), and if we take $\rho_{1,3j}^1$ to be an arbitrary purification of $\rho_1$ onto $H_1 \otimes H_{3j}^2$, we can take $H_{3j}$ to be one-dimensional, $\rho_{3j,2}^1 = \rho_2^j$, and with these definitions, $\rho_{123}$ is an extension of the given separable bipartite state $\rho_{12}$ for which equality holds in (1.7). Thus, whenever $\rho_{12}$ is finitely separable, then $E_{sq}(\rho_{12}) = 0$. It is elementary to see that whenever $\rho_{12}$ is finitely separable, then $E_t(\rho_{12}) = 0$. A continuity argument [1] then shows that $E_t(\rho_{12}) = 0$, and hence $E_{sq}(\rho_{12}) = 0$, whenever $\rho_{12}$ is separable. The converse is not obvious, but it has recently been proved in [5] that whenever $S_{13} + S_{23} = S_{123} - S_3$ is sufficiently small, then $\rho_{123}$ is well-approximated by a density $\rho_{123}$ that has the form (1.9). Thus, $E_{sq}(\rho_{12}) > 0$, and hence $E_t(\rho_{12}) > 0$, whenever $\rho_{12}$ is not separable. Hence both $E_t$ and $E_{sq}$ are faithful measures of entanglement.

In general, it is not a simple matter to evaluate the infima that define $E_t$ and $E_{sq}$. In particular, even if the bipartite state $\rho_{12}$ operates on a finite dimensional Hilbert space, there is no known a-priori bound on the dimension of the Hilbert spaces $H_3$ that must be used to nearly minimize $I(1,2|3)$. This makes $E_{sq}$ difficult to evaluate in general, so that sharp lower bounds are of interest.

The non-negativity of $E_{sq}$, as we have explained, is a direct consequence of the strong subadditivity inequality (1.7). We shall prove an extension of (1.7), and show that it provides sharp lower bounds on $E_t$ and $E_{sq}$.

1.1 THEOREM (Extended strong subadditivity). For all tripartite states $\rho_{123}$,

$$I(1,2|3) \geq 2 \max\{S_1 - S_{12}, S_2 - S_{12}, 0\} \cdot (1.10)$$

The inequality $I(1,2|3) \geq \lambda \max\{S_1 - S_{12}, S_2 - S_{12}, 0\}$ can be violated for all $\lambda > 2$.

The inequality (1.10) extends the inequality (1.7) in an obvious way, but as we shall see, it is actually a consequence of the seemingly weaker inequality (1.7). We prove Theorem 1.1 in Section 2.

As a direct consequence of Theorem 1.1 we obtain lower bounds for $E_t$ and $E_{sq}$ that we shall show to be sharp:

1.2 THEOREM (Lower bounds for $E_{sq}$ and $E_t$). For all bipartite states $\rho_{12}$,

$$E_{sq}(\rho_{12}) \geq \max\{S_1 - S_{12}, S_2 - S_{12}, 0\} \cdot (1.11)$$

and

$$E_t(\rho_{12}) \geq \max\{S_1 - S_{12}, S_2 - S_{12}, 0\} \cdot (1.12)$$

Both of these inequalities are sharp in that there exists a class of bipartite states $\rho_{12}$ for which

$$E_t(\rho_{12}) = E_{sq}(\rho_{12}) = S_1 - S_{12} > 0 \cdot (1.13)$$

and for which $S_1$ and $S_{12}$ may take arbitrary non-negative values.

Note that because $E_t \geq E_{sq}$, the inequality (1.12) is implied by (1.11), whereas the fact that (1.12) is sharp implies that (1.11) is sharp.

A weaker form of the inequality (1.11) was given by Christandl and Winter [6]. Their lower bound involves the averaged quantity

$$\frac{1}{2}(S_1 + S_2) - S_{12} \cdot (1.14)$$
in place of $\max\{S_1 - S_{12}, S_2 - S_{12}\}$. The difference can be significant: As we explain in Remark 1.4 below, there exist states $\rho_{12}$ for which $E_{\text{sq}}(\rho_{12}) = S_1 - S_{12}$ is arbitrarily large, but the quantity in (1.14) is negative. Moreover, the argument in [6] relied on a lower bound for the the one-way distillable entanglement in terms of the conditional entropy. This inequality, known as the hashing inequality had been a long-standing conjecture, and its proofs remain complicated. Our contribution is to show how this stronger lower bound follows in a relatively simple manner from strong subadditivity, and to provide the examples that prove the sharpness of these bounds.

The class of bipartite states referred to in the final part of Theorem 1.2 are states that saturate the Araki-Lieb triangle inequality

$$S_{12} \geq |S_1 - S_2|.$$

(1.15)

A characterization of cases of equality has been known for some time, and discussed as an exercise in [14, Ex. 11.16]. Recently, Zhang and Wu [17] proved this characterization by using the conditions for equality in the more difficult strong subadditivity theorem proved by Hayden et al. [7]. We give a short and elementary proof which provides a more detailed characterization of the cases of equality.

1.3 THEOREM (Bipartite states with $S_{12} = S_1 - S_2$). A bipartite state $\rho_{12}$ satisfies

$$S_{12} = S_1 - S_2$$

if and only if

$$\text{rank}(\rho_1) = \text{rank}(\rho_2) \text{rank}(\rho_{12})$$

(1.16)

and $\rho_{12}$ has a spectral decomposition of the form

$$\rho_{12} = \sum_{j=1}^{\text{rank}(\rho_{12})} \kappa_j |\phi_j\rangle \langle \phi_j|,$$

(1.17)

where for each $i, j$,

$$\text{Tr}_1 |\phi_i\rangle \langle \phi_j| = \delta_{i,j} \rho_2.$$

(1.18)

Examples of such states may be constructed by choosing any set $n$ of non-negative numbers $\kappa_j$ with $\sum_{j=1}^n \kappa_j = 1$, and any state $\rho_2$ on $\mathcal{H}_2$, and taking $\sigma_j$ to be a purification of $\rho_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that the ranges of the $\sigma_j$ are mutually orthogonal (which is possible under condition (1.17)). By defining

$$\rho_{12} = \sum_{j=1}^{m} \kappa_j \sigma_j$$

one has

$$S_{12} = -\sum_{j=1}^n \kappa_j \log \kappa_j \quad S_2 = S(\rho_2) \quad \text{and} \quad S_1 = S_1 + S_{12}$$

so that the construction yields examples in which $S_2$ and $S_{12}$ take arbitrary non-negative values.

Moreover, for any $\rho_{12}$ yielding equality in the triangle inequality,

$$E_{\text{sq}}(\rho_{12}) = E_1(\rho_{12}) = S_1 - S_{12}.$$ 

(1.19)

1.4 Remark. Theorem 1.3 yields bipartite states $\rho_{12}$ for which $S_1 - S_{12} = S_2$, and for which $S_1$ and $S_{12}$, and can have arbitrary non-negative values. Therefore, we can arrange that $S_1 - S_{12} = S_2$ is positive, and even arbitrarily large, but the quantity in (1.14) is negative.

For our immediate purpose here, we do not require the full strength of Theorem 1.3 which characterizes all bipartite states $\rho_{12}$ with equality in the triangle inequality $S_{12} \geq |S_1 - S_2|$. To show that our bounds in Theorems 1.1 and 1.2 are sharp, it suffices to observe that the construction described in Theorem 1.3 yields examples of bipartite states $\rho_{12}$ for which $S_1 - S_{12} = S_2$, and for which $S_1$ and $S_{12}$, and can have arbitrary non-negative values, and this is an easy calculation.
An upper bound for $E_t$, and hence for $E_{sq}$ as well, is known under the rubric entanglement never exceeds local entropy. If one inserts the trivial decomposition $\rho_{12} = \rho_{12}$ into the basic definition (1.3), and symmetrizing the bound, one obtains the upper bound:
\[
E_t(\rho_{12}) \leq \min\{S_1, S_2\}.
\] (1.20)

For the states used in (1.19), we see that this upper bound is sharp for both $E_t$ and $E_{sq}$. We are grateful to Matthias Christandl for pointing this out to us.

\section{Proofs}

We shall use purification arguments, which appears to have been first used in [2] to prove the Araki-Lieb triangle inequality. In Lemmas 3 of [2] it is shown that for any pure bipartite state $\rho_{12}$, $\rho_1$ and $\rho_2$ have the same non-zero spectrum, and hence the same entropy. Lemma 4 of [2] shows that given any density matrix $\rho$ on $H$, there is a pure state $\rho_{12}$ on $H \otimes H$ such that $\text{Tr}_2(\rho_{12}) = \rho$.

Using these lemmas, the triangle inequality may then be deduced from the subadditivity of the entropy; i.e., the inequality $S_{23} \leq S_2 + S_3$. Considering any purification $\rho_{123}$ of $\rho_{23}$, $S_{23} = S_1$ and $S_3 = S_{12}$ and hence $S_{12} \geq S_1 - S_2$. By symmetry, one then has (1.15). (Note that one can use essentially the same purification to recover the subadditivity inequality from the triangle inequality; in this sense the inequalities are equivalent.)

\textbf{Proof of Theorem 1.11.} The starting point of our proof is inequality (2.1) below, which appears in [11, Theorem 2], and whose simple proof we recall: Consider any purification $\rho_{1234}$ of $\rho_{123}$. Then since $\rho_{1234}$ is pure, $S_{23} = S_{14}$ and $S_{123} = S_4$. Then
\[
S_{12} + S_{23} - S_1 - S_3 = S_{12} + S_{14} - S_{12} - S_1,
\]
and the right hand side is non-negative by (1.7). This proves
\[
S_{12} + S_{23} \geq S_1 + S_3.
\] (2.1)

Next, adding $S_{12} + S_{23} \geq S_1 + S_3$ and $S_{13} + S_{23} \geq S_1 + S_2$, we obtain
\[
S_{12} + S_{13} + 2S_{23} \geq 2S_1 + S_2 + S_3.
\] (2.2)

Consider any purification $\rho_{1234}$ of $\rho_{123}$. Then we obtain, using $S_{12} = S_{34}$, $S_{23} = S_{14}$, and $S_2 = S_{134}$,
\[
S_{13} + S_{34} - S_{134} - S_3 \geq 2(S_1 - S_{14}),
\]
which is (1.10) with different indices. As a consequence of Theorem 1.3 (1.10) is sharp. \hfill $\Box$

\textbf{2.1 Remark.} The inequality (2.2) is the crux of the matter. In a similar way, one may deduce the related inequalities
\[
2S_{12} + S_{13} + 2S_{23} \geq 2S_1 + S_2 + 2S_3 \quad \text{and} \quad S_{12} + S_{13} + S_{23} \geq S_1 + S_2 + S_3.
\] (2.3)

The second inequality in (2.3) is obtained by averaging (2.1) over permutations of the indices. One may then add (2.1) to this to obtain the first inequality. It is worth noting that the purification argument in leading from strong subadditivity to (2.1) reverses, so that (2.1) is equivalent to strong subadditivity.

\textbf{Proof of Theorem 1.2.} The inequality (1.12) is an immediate consequence of (1.10) and the definition of $E_{sq}$. Then (1.12) follows since $E_t \geq E_{sq}$. Once more, the statement about sharpness is a consequence of Theorem 1.3. \hfill $\Box$

\textbf{2.2 Remark.} It is worth pointing out that the inequality (1.12), but not (1.11), has a direct proof using the concavity of the conditional entropy [11, Theorem 1], which may be seen as a consequence of the joint convexity of the relative entropy theorem: The relative entropy of two states $\rho$ and $\sigma$ is defined to be $S(\rho|\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)]$, \quad \text{for} \quad \rho, \sigma \geq 0 \quad \text{and} \quad \text{Tr} \rho = 1.
which is jointly convex as a limiting case of the concavity theorem [9] Theorem 1]. Then defining \( \sigma_2 \) to be the normalized identity on \( \mathcal{H}_2 \),

\[
S(\rho_{12}|\rho_1 \otimes \sigma_2) = \text{Tr}_{\mathcal{H}_2} \log \rho_{12} - \text{Tr}_1 \log \rho_1 - \log(\dim(\mathcal{H}_2)) .
\]

By the concavity of \( \rho_{12} \mapsto S_{12} - S_2 \), for any decomposition of \( \rho_{12} \) of the form \( \sum_{k=1}^n \lambda_k \omega^k \), in which the \( \omega_k \) are pure,

\[
S_{12} - S_1 \geq \sum_{k=1}^n \lambda_k \left[ S(\omega^k) - S(\text{Tr}_{\mathcal{H}_2} \omega^k) \right] = - \sum_{k=1}^n \lambda_k S(\text{Tr}_{\mathcal{H}_2} \omega^k)
\]

since \( S(\omega^k) = 0 \).

**Proof of Theorem 1.3**. As we have explained, the inequality \( S_{12} \geq |S_1 - S_2| \) is deduced from the subadditivity inequality \( S_{23} \leq S_2 + S_3 \). Consequently, if \( \rho_{12} \) is any bipartite state for which \( S_{12} = S_1 - S_2 \), and \( \rho_{123} \) is any purification of it, \( S_{23} = S_2 + S_3 \) and hence \( \rho_{23} = \rho_2 \otimes \rho_3 \) (2.4)

since only product state saturate the subadditivity inequality.

Next, define \( d_j := \text{rank}(\rho_j) \) for \( j = 1, 2, 3 \). Likewise, define

\[
d_{12} := \text{rank}(\rho_{12}) , \quad d_{23} := \text{rank}(\rho_{23}) \text{ etc.}
\]

Since \( \rho_{123} \) is pure, \( d_{23} = d_3 \) and \( d_{12} = d_3 \). But by (2.4), \( d_{23} = d_2 d_3 \), and therefore,

\[
d_1 = d_2 d_3 ,
\]

which proves the necessity of (1.16).

We now show that if \( S_{12} = S_1 - S_2 \), then \( \rho_{12} \) has a spectral decomposition of the from (1.17) where, for each \( i, j \), (1.18) is satisfied. This will prove that every bipartite state \( \rho_{12} \) for which \( S_{12} = S_1 - S_2 \) has the structure asserted in Theorem 1.3.

Conversely, when \( \rho_{12} \) has the spectral projection (1.17), \( S_2 = -\kappa_j \log \kappa_j \). Moreover, when (1.18) is satisfied, the \( \text{Tr}_2[\phi_j]\langle\phi_j| \) have mutually orthogonal ranges, and each has entropy \( S_2 \), and so

\[
S_1 = - \sum_{j=1}^{\text{rank}(\rho_{12})} \kappa_j \log \kappa_j + \sum_{j=1}^{\text{rank}(\rho_{12})} \kappa_j S_2 = S_{12} + S_2 .
\]

Thus, every every bipartite state with the structure described in (1.17) and (1.18) satisfies \( S_{12} = S_1 - S_2 \).

To prove that the spectral decomposition (1.17) of \( \rho_{12} \) satisfies (1.18), let \( \varphi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \) be such that \( \text{Tr}_3[\varphi] \langle \varphi| = \rho_{12} \). Note that without loss of generality, we may assume that \( \dim(\mathcal{H}_j) = d_j \) for \( j = 1, 2, 3 \). Let us choose bases for \( \mathcal{H}_2 \) and \( \mathcal{H}_3 \) in which \( \rho_2 \) and \( \rho_3 \) are diagonal, and pick any orthonormal basis for \( \mathcal{H}_1 \). For \( j = 1, 2, 3 \), let \( X_j \) be a set of cardinality \( d_j = \dim(\mathcal{H}_j) \). Using the orthonormal bases selected above, we may view the vector \( \varphi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \) with a function \( \varphi(x_1, x_2, x_3) \) on \( X_1 \times X_2 \times X_3 \).

Then the pure state density matrix \( \rho_{123} \) has the matrix elements

\[
\varphi(x_1, x_2, x_3) \varphi^*(x'_1, x'_2, x'_3) .
\]

Taking partial traces, and using the fact that our bases diagonalize \( \rho_2 \) and \( \rho_3 \)

\[
\rho_{23}(x_2, x_3; x'_2, x'_3) = \sum_{x_1} \varphi(x_1, x_2, x_3) \varphi^*(x_1, x'_2, x'_3) ,
\]

\[
\rho_2(x_2; x'_2) = \sum_{x_1, x_3} \varphi(x_1, x_2, x_3) \varphi^*(x_1, x'_2, x_3) = \lambda_{x_2} \delta_{x_2, x'_2} ,
\]
and
\[ \rho_3(x_3; x_3') = \sum_{x_1, x_2} \varphi(x_1, x_2, x_3) \varphi^*(x_1, x_2, x_3') = \mu_{x_3} \delta_{x_3, x_3'} . \]

Then by (2.4),
\[ \sum_{x_1} \varphi(x_1, x_2, x_3) \varphi^*(x_1, x_2', x_3') = \lambda_{x_2} \mu_{x_3} \delta_{x_2, x_2'} \delta_{x_3, x_3'} . \]

Now for each \( x_2 \in X_2, x_3 \in X_3 \), define \( \psi_{x_2, x_3}(x_1) \) by \( \psi_{x_2, x_3}(x_1) = \varphi(x_1, x_2, x_3) \). It follows from (2.7) that
\[ \{ \psi_{x_2, x_3} : x_2 \in X_2, x_3 \in X_3 \} \]
is an orthogonal set of vectors. Moreover, since by assumption each \( \lambda_{x_2} \) and each \( \mu_{x_3} \) is non-zero, none of these vectors is zero: The set of vectors in (2.8) is an orthogonal basis for \( H_1 \).

In the same way, defining \( \chi_{x_3}(x_1, x_2) := \varphi(x_1, x_2, x_3) \),
\[ \{ \chi_{x_3} : x_3 \in X_3 \} \]
is a set of \( d_3 \) orthogonal vectors in \( H_1 \otimes H_2 \), and by (2.6), \( \| \chi_{x_3} \|^2 = \mu_{x_3} \). Now define
\[ \eta_{x_3} := \frac{1}{\sqrt{\mu_{x_3}}} \chi_{x_3} . \]

Then since \( \rho_{12}(x_1, x_2; x_1', x_2') = \sum_{x_3} \varphi(x_1, x_2, x_3) \varphi^*(x_1', x_2', x_3) \)
\[ \rho_{12} = \sum_{x_3} \mu_{x_3} |\eta_{x_3}\rangle \langle \eta_{x_3}| . \]

Now we may rewrite
\[ \sum_{x_1} \varphi(x_1, x_2, x_3) \varphi^*(x_1, x_2', x_3') = \sum_{x_1} \sqrt{\mu_{x_3}} \sqrt{\mu_{x_3'}} \eta_{x_3}(x_1, x_2) \eta_{x_3'}^*(x_1, x_2') \]
Comparing with (2.7), we see that
\[ \sum_{x_1} \sqrt{\mu_{x_3}} \sqrt{\mu_{x_3'}} \eta_{x_3}(x_1, x_2) \eta_{x_3'}^*(x_1, x_2') = \lambda_{x_2} \mu_{x_3} \delta_{x_2, x_2'} \delta_{x_3, x_3'} , \]
and this proves (1.18).

We now prove the final statement. Let \( \rho_{12} \) be such that \( S_{12} = S_1 - S_2 \). Then \( \rho_{12} = \sum_{j=1}^{m} \lambda_j \sigma^j \) where each \( \sigma^j \) is a purification of \( \rho_2 \). Hence
\[ E_i(\rho_{12}) \leq \sum_{j=1}^{m} \lambda_j S(T_1(\sigma^j)) = \sum_{j=1}^{m} \lambda_j S_2 = S_2 = S_1 - S_{12} . \]
By the lower bound in Theorem 1.2 we must have \( E_i(\rho_{12}) = E_{sq}(\rho_{12}) = S_1 - S_{12} . \)

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