Projected images of the Sierpinski tetrahedron and other fractal imaginary cubes

Hideki Tsuiki
tsuiki@i.h.kyoto-u.ac.jp

Abstract

A projected image of a Sierpinski tetrahedron has a positive measure if and only if the images O, P, Q, R of the four vertices satisfy $p\vec{OP} + q\vec{OQ} + r\vec{OR} = 0$ for odd integers $p, q, r$. This fact was essentially obtained by Kenyon (Symbolic dynamics and its applications 135, 1992). We reformulate his proof through the notion of projection of differenced radix expansion sets and obtain similar results for other fractal objects. Specifically, we study projections of H-fractal and T-fractal as well as Sierpinski tetrahedron, which are fractal imaginary cubes (i.e., fractal objects that are projected to squares along three orthogonal directions just as a cube) of the first two degrees, and characterize the directions from which these fractals are projected to sets with positive measures.

1 Introduction

A Sierpinski tetrahedron (i.e., three-dimensional Sierpinski gasket) is a three-dimensional fractal object defined as the attractor of the iterated function system (IFS) \( \{ f_d(x) = \frac{x + d}{2} \mid d \in D_S \} \) where $D_S \subset \mathbb{R}^3$ is the set of four vertices of a regular tetrahedron (see Fig. 1(a)). That is, it is the unique non-empty compact set $S_\infty$ satisfying $S_\infty = \bigcup_{d \in D_S} f_d(S_\infty)$ that exists due to the theory of self-similar fractals by Hutchinson [5].

![Figure 1: Projected images of the Sierpinski tetrahedron](image)
A Sierpinski tetrahedron has a remarkable property that it is projected to squares along the three orthogonal directions connecting the middle points of opposite edges (Fig. 1(a)). An object that is projected to squares along three orthogonal directions just as a cube is called an imaginary cube [15, 16] (See Fig. 2 and 3 for examples of imaginary cubes and Definition 1 for the precise definition of an imaginary cube), and Sierpinski tetrahedron is a fractal imaginary cube. Moreover, as Fig. 1(b) shows, a Sierpinski tetrahedron is projected to figures with positive Lebesgue measures along many other directions, and they always form shapes that tile $\mathbb{R}^2$. In this paper, we characterize directions from which Sierpinski tetrahedron and other similar fractal objects are projected to sets with positive measures.

Because the IFS of a Sierpinski tetrahedron does not perform rotations, projected images of a Sierpinski tetrahedron are two-dimensional fractals generated by the same kind of IFS. For a set of points $D \subset \mathbb{R}^n$ with cardinality $k^2$, let $F^n(k, D)$ denote the fractal generated by an IFS of the form \( \{ f_d(x) = \frac{x + d}{k} \mid d \in D \} \). A Sierpinski tetrahedron is $F^3(2, D_S)$ and its projected image by a projection $\varphi$ is $F^2(2, \varphi(D_S))$. Positively-measured fractals of the form $F^2(k, D)$ are special cases of self-affine tiles which have been intensively studied from the 1990’s [1, 2, 6, 18, 19, 9, 11], and Kenyon proved in [6] that for a four point set $D = \{ O, P, Q, R \}$, the fractal $F^2(2, D)$ has positive Lebesgue measure if and only if $p \vec{OP} + q \vec{OQ} + r \vec{OR} = 0$ for odd integers $p, q, r$. Since projections of a Sierpinski tetrahedron have this form, his result characterizes the directions from which a Sierpinski tetrahedron is projected to sets with positive measures.

We reformulate Kenyon’s proof through the notion of projections of differenced radix expansion sets, and obtain similar results for other fractal imaginary cubes. We call a fractal imaginary cube expressible as $F^3(k, D)$ for a digit set $D$ of cardinality $k^2$ a (dilational) fractal imaginary cube of degree $k$. As we will explain in the next section, Sierpinski tetrahedron is the only fractal imaginary cubes of degree 2. There are two fractal imaginary cube of degree 3, which are called H-fractal and T-fractal ([14, 16], see Fig. 5(b, c)). We characterize the directions from which these two fractal imaginary cubes are projected to sets with positive measures.

We introduce imaginary cubes and fractal imaginary cubes in the next section and explain our results in Section 3. In section 4, we overview the properties of self-affine tiles we use and give the key idea of projection of differenced radix expansion sets. Then, we give proofs in Section 5. By a projection, we mean not only an orthogonal projection but a projection to any plane that is not parallel to the direction of the projection. This does not introduce ambiguity to our results because the positivity of Lebesgue measure of a projected image is independent of the choice of the plane.

## 2 Fractal Imaginary Cubes

Fractal objects with square projections along three orthogonal directions were investigated in [14], and the more general notion of an imaginary cube was introduced in [15].

**Definition 1.** An *imaginary cube* is an object that is projected to squares
Regular tetrahedron

H

T

Figure 2: Examples of Imaginary cubes.

Figure 3: Objects in Fig. 2 located in unit cubes.

along three orthogonal directions so that, for each direction, the four edges of
the projected square are parallel to the other two directions.

An imaginary cube is contained in a cube defined by the three square pro-
jections. When we need to specify this cube \( C \), we say that it is an imaginary
cube of \( C \). Note that a regular octahedron also has square projections along
three orthogonal directions, but it is not an imaginary cube.

A regular tetrahedron is an example of a polyhedral imaginary cube. There
are two more polyhedral imaginary cubes that are relevant to this paper: hexag-
onal bipyramid imaginary cube (H in short) and triangular antiprismoid imag-
inary cube (T in short), which are defined in Fig. 2 and cubes of which they
are imaginary cubes are given in Fig. 3. H and T have some remarkable prop-
erties. H is a double imaginary cube in that it is an imaginary cube of two
different cubes and thus it has six square projections. T has the property that
the three diagonals of T are orthogonal to each other, and H and T form a
three-dimensional tiling of the space \( \mathbb{R}^3 \).

The three imaginary cubes: regular tetrahedron, H and T, have the common
property that they are convex hulls of fractal imaginary cubes. According to
Hutchinson \[5\], for contractions \( f_i : \mathbb{R}^n \to \mathbb{R}^n \) \( (i = 1, \ldots, m) \), an IFS (iterated
function system) \( \{f_i \mid i = 1, 2, \ldots, m\} \) defines a self-similar fractal object as the
fixed point of the following contraction map on the metric space of non-empty
compact subsets of $\mathbb{R}^n$.

$$F(X) = \bigcup_{i=1}^{m} f_i(X).$$

In this article, we only consider the simple case $n = 2$ or $3$ and an IFS has the form $\{ f_d(x) = \frac{x + d}{k} \mid d \in D \}$ for some integer $k > 1$ and some $D \subset \mathbb{R}^3$ with $|D| = k^2$ called the digit set. That is, the components of an IFS are dilations with ratio $\frac{1}{k}$ and center $\frac{d}{k^2}$. In this case, $F(X)$ can be expressed as

$$F(X) = \frac{X + D}{k}$$

for $+$ the Minkowski sum. We denote the induced fractal object by $F(k, D)$, or by $F^n(k, D)$ when it is appropriate to make the dimension of the space clear.

Fractals of the form $F(k, D)$ have the following simple description. Let $A$ be any convex set that contains centers of $f_d$ for $d \in D$. We define a sequence $(A_i)_{0\leq i}$ of compact sets as $A_0 = A$ and $A_{i+1} = F(A_i)$. We have $f_d(A) \subseteq A$ by the fact that $A$ is a convex set that contains the fixed point of $f_d$. Therefore $A_1 = F(A_0) = \bigcup_{d \in D} f_d(A) \subseteq A$. Thus, $F$ is monotonic and we have a decreasing sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ of compact sets. Their intersection $\cap_{i \geq 0} A_i$ is a non-empty compact set that is a fixed point of $F$ (because any point $x'$ in $F(\cap_{i \geq 0} A_i)$ has the form $\frac{x' - d}{k}$ for $x \in \cap_{i \geq 0} A_i$, and thus $x' \in \cap_{i \geq 0} \frac{A_i + D}{k} = \cap_{i \geq 0} A_{i+1}$). Thus, we have the description $F(k, D) = \cap_{i \geq 0} A_i$. As the starting object $A$, it is natural to consider the polyhedron obtained as the convex hull of $\{ \frac{d}{k^2} \mid d \in D \}$, which is also the convex hull of the fractal $F(k, D)$.

Note that $F(k, D)$ admits $k$-adic expansion $(d_i)_{i \geq 0}$ for $d_i \in D$ as follows:

$$F(k, D) = \left\{ \sum_{i \geq 0} \frac{d_i}{k^i} \mid d_i \in D \right\}.$$

Since our IFS consists of similarity maps that do not perform rotation, an image of a three dimensional fractal $F^3(k, D)$ by a projection $\varphi$ is a two-dimensional fractal $F^2(k, \varphi(D))$. In order that $F^2(k, \varphi(D))$ has a positive Lebesgue measure, its Hausdorff dimension should be 2. On the other hand, the Hausdorff dimension of $F^2(k, \varphi(D))$ is no more than $\log_k |D|$. Since we are interested in fractals with minimum number of IFS-components, we only study the case $\log_k |D| = 2$, i.e., $|D| = k^2$. We call an imaginary cube of the form $F(k, D)$ for $|D| = k^2$ a dilational fractal imaginary cube of degree $k$, or simply a fractal imaginary cube of degree $k$.

We study a condition on the digit set $D$ so that $F^3(k, D)$ becomes a fractal imaginary cube. If $F^3(k, D)$ is an imaginary cube, then there is a cube $C$ of which $F^3(k, D)$ is an imaginary cube. Since $C$ is a convex set containing $F^3(k, D)$, we can apply the above construction for $A = C$. Therefore, for the sequence $C = C_0 \supset C_1 \supset C_2 \ldots$ defined as $C_{i+1} = F(C_i)$, we have $F^3(k, D) = \cap_{i \geq 0} C_i$. Since $F^3(k, D)$ and $C$ have the same three square projections, $C_i$ are all imaginary cubes of $C$. In particular, $C_1 = F(C)$ is an imaginary cube of $C$. In the following lemma, we show that it is also a sufficient condition.
Lemma 2. If a digit set $D$ and a cube $C$ have the property that $F(C)$ is an imaginary cube of $C$ for $F$ in $[1]$, then $F^3(k, D)$ is a fractal imaginary cube of $C$.

Proof. We first show by induction that $C = C_0 \supset C_1 \supset \ldots$ for $C_{i+1} = F(C_i)$ are all imaginary cubes. Suppose that $C_i$ is an imaginary cube of $C$ and $\varphi$ is a projection along an edge of $C$. Then $C_i$ and $C$ are projected to the same square by $\varphi$. Thus, $f_d(C_i)$ and $f_d(C)$ are projected to the same square by $\varphi$. Therefore, $C_{i+1} = \cup_{d \in D} f_d(C_i)$ and $C_1 = \cup_{d \in D} f_d(C)$ are projected to the same figure by $\varphi$, which is a square because $C_1$ is an imaginary cube of $C$. Thus, $C_{i+1}$ is an imaginary cube. Now, we show that $F^3(k, D) = \cap_{i \geq 0} C_i$ is also an imaginary cube. Let $B = \varphi(C)$ be the projected square. For every point $p \in B$ and $i \geq 0$, $\varphi^{-1}(p) \cap C_i$ is a non-empty compact set and $\varphi^{-1}(p) \cap C_{i+1}$. Therefore, $\varphi^{-1}(p) \cap (\cap_{i \geq 0} C_i) = \cap_{i \geq 0} (\varphi^{-1}(p) \cap C_i)$ is non-empty. Thus, $\varphi$ projects $F^3(k, D)$ onto $B$. 

If $D$ a digit set of cardinality $k^2$ and $F(C) = \cup_{d \in D} f_d(C)$ is an imaginary cube, then $F(C)$ is a union of $k^2$ cubes selected from the $k^3$ cubes obtained by cutting $C$ into $k \times k \times k$ small cubes so that the $k^2$ cubes do not overlap when viewed from the three face-directions of $C$. Such a selection exists corresponding to a Latin square of size $k$, which is a $k \times k$ matrix of $K = \{0, 1, 2, \ldots, k-1\}$ such that all the numbers appear on all the columns and rows. Thus, we have the following theorem, where a Latin square is expressed as a function $h : K \times K \to K$.

Theorem 3 ([14]). Let $D \subset \mathbb{R}^3$ be a set with cardinality $k^2$. $F^3(k, D)$ is an imaginary cube of the unit cube $[0, 1]^3$ if and only if $D = \{(i, j, h(i, j)) \mid 0 \leq i, j \leq k - 1\}$ for a Latin square $h$.

When $k = 2$, there are two digit sets of this form but they generate congruent arrangements of the four cubes $F(C)$ and therefore there is only one fractal imaginary cube of degree 2. We define the digit set

$$D_S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}. $$

Fig. 1(a) shows $F(C) = \cup_{d \in D_S} f_d(C)$ with the set $D_S$ marked as red circles. $D_S$ forms the set of vertices of the regular tetrahedron $S$ in Fig. 5(a) and $F(2, D_S)$ is a Sierpinski tetrahedron.

When $k = 3$, there are only two arrangements of such nine cubes modulo congruence, and therefore there are two fractal imaginary cubes of degree 3. This time, we consider the unit cube $[-1/2, 1/2]^3$ centered at the origin and consider Latin squares of $K = \{-1, 0, 1\}$ for the ease of developments in the following sections. We define the following two digit sets $D_H$ and $D_T$ in $\{-1, 0, 1\}^3$:

$$D_H = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \mp 1 \\ \mp 1 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \mp 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ (double signs of the same orders)}$$

$$D_T = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. $$

5
The corresponding sets $\cup_{d \in D_H} f_d(C)$ and $\cup_{d \in D_T} f_d(C)$ are depicted in Fig. 4(b, c). Here, the fixed points of $f_d$ for $d \in D_H$ and $f_d$ for $d \in D_T$ are indicated as red circles, and they are the sets $D_H^2$ and $D_T^2$, respectively. $D_H^2$ is the set of eight vertices and the center of the dodecahedron H in Fig. 4(b). $D_T^2$ is the set of six vertices and the three middle points of the edges of the larger regular triangle face of the octahedron T in Fig. 4(c). $F(3, D_H)$ and $F(3, D_T)$ are fractal imaginary cubes whose convex hulls are H and T, respectively. We denote $F(2, D_S)$, $F(3, D_H)$ and $F(3, D_T)$ by $S_\infty$, $H_\infty$ and $T_\infty$, respectively. Fig. 5 shows the first two levels of the approximations of these three fractals starting with their convex hulls. We call the shapes (i.e., equivalence class module congruence and similarity) of $F(3, D_H)$ and $F(3, D_T)$ H-fractal and T-fractal, respectively. Note that an H-fractal has six-fold symmetry and it is...
Figure 6: Projected images of (a) $S_\infty$ from $(1, -1, 1)$; (b) $H_\infty$ from $(1, -1, 1)$; (c) $T_\infty$ from $(1, -1, 1)$; (d) $S_\infty$ from $(1, 2, 0)$; (e) $H_\infty$ from $(1, 1, 0)$; (f) $T_\infty$ from $(1, 1, 0)$.

3 Main results

As imaginary cubes, the fractals $S_\infty$, $H_\infty$, and $T_\infty$ have orthogonal projections to squares along the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. In addition, since $H_\infty$ is a double imaginary cube, it is projected to squares along $(-1, 2, 2), (2, -1, 2)$ and $(2, 2, -1)$ which are obtained by rotating the above directions by 180 (or equivalently 60) degrees around the axis $(1, 1, 1)$.

In addition, these fractals are projected to figures with positive measures in many other directions. For example, $S_\infty$ is projected to Fig. 6(a) from the vertices of the cube, and $H_\infty$ and $T_\infty$ are projected to (b) and (c), respectively, along $(-1, 1, 1), (1, -1, 1)$ and $(1, 1, -1)$. Also, $H_\infty$ and $T_\infty$ are projected to (e) and (f), respectively, along $(0, 1, 1), (1, 0, 1)$ and $(1, 1, 0)$. $S_\infty$ is projected to (d) along $(2, 1, 0)$ and its permutations. In addition, $H_\infty$ is projected to (b) along $(1, 1, -5), (1, -5, 1)$ and $(-5, 1, 1)$ and (e) along $(4, 1, 1), (1, 4, 1)$ and $(1, 1, 4)$ which are obtained by rotating the above directions by 180 degree.

Our main theorem is the characterization of the directions along which Sierpinski tetrahedron, H-fractal, T-fractal are projected to figures with positive Lebesgue measures:

**Theorem 4.** (1) Let $a, b, c$ be coprime integers.

(a) $S_\infty$ is projected along the vector $(a, b, c)$ to a set with positive Lebesgue measure if and only if $a + b + c$ is an odd number.
(b) $T_{\infty}$ is projected along the vector $(a, b, c)$ to a figure with positive Lebesgue measure if and only if $3 \nmid a + b + c$.

(c) $H_{\infty}$ is projected along the vector $(a, b, c)$ to a figure with positive Lebesgue measure if and only if $3 \nmid a + b + c$ or $a \equiv b \equiv c \equiv -\frac{a+b+c}{3} \not\equiv 0 \mod 3$.

(2) For real numbers $a, b, c$ such that $a/b, b/c,$ or $c/a$ exist and are irrational, $S_{\infty}, T_{\infty},$ and $H_{\infty}$ are projected along $(a, b, c)$ to null-sets.

As Theorem 4(1c) says, the directions along which $H_{\infty}$ is projected to sets with positive measures are divided into two disjoint sets. The first one is the same as that of $T_{\infty}$ and the second one is obtained by rotating the first one by 180 degree around the axis $(1, 1, 1)$. We first show this fact.

Lemma 5. An integer vector $(a, b, c)$ such that $3 \nmid a + b + c$ is rotated by a 180-degree rotation around the axis $(1, 1, 1)$ to $\frac{1}{3}(\alpha, \beta, \gamma)$ for integers $\alpha, \beta, \gamma$ such that $\frac{\alpha + \beta + \gamma}{3} \not\equiv 0 \mod 3$, and vice versa.

Proof. Since $\frac{1}{3}(-1, 2, 2), \frac{1}{3}(2, -1, 2)$ and $\frac{1}{3}(2, 2, -1)$ are obtained by rotating $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ respectively, we set

$$(\alpha, \beta, \gamma) = a(-1, 2, 2) + b(2, -1, 2) + c(2, 2, -1)$$

for $a, b, c \in \mathbb{R}$ and show that $a, b, c$ are integers and $3 \nmid a + b + c$ if and only if $\alpha, \beta, \gamma$ are integers and $\alpha \equiv \beta \equiv \gamma \equiv -\frac{\alpha + \beta + \gamma}{3} \not\equiv 0 \mod 3$.

If $a + b + c \equiv 1 \mod 3$, we have $\alpha + \beta + \gamma = 3(a + b + c) \equiv 3 \mod 9$ and thus $\frac{\alpha + \beta + \gamma}{3} \equiv 1 \mod 3$. Since $\alpha = 2(a + b + c) - 3a$, we have $\alpha \equiv -1 \mod 3$. In the same way, $\alpha \equiv \beta \equiv \gamma \equiv -1 \mod 3$.

Similarly, if $a + b + c \equiv -1 \mod 3$, we have $\alpha + \beta + \gamma \equiv -3 \mod 9$ and therefore $\frac{\alpha + \beta + \gamma}{3} \equiv -1 \mod 3$ and $\alpha \equiv \beta \equiv \gamma \equiv 1 \mod 3$.

On the other hand, if $\alpha \equiv \beta \equiv \gamma \equiv -\frac{\alpha + \beta + \gamma}{3} \not\equiv 0 \mod 3$, then $a + b + c = \frac{\alpha + \beta + \gamma}{3} \not\equiv 0 \mod 3$. $a = \frac{1}{3}(2\alpha + \beta + \gamma - \alpha)$ is an integer, and similarly for $b$ and $c$.

If the direction of the projection is parallel to the plane $x + y + z = 0$, then the projected images of $S_{\infty}, H_{\infty},$ and $T_{\infty}$ are obviously null-sets and the direction $(a, b, c)$ of the projection satisfies $a + b + c = 0$. Therefore, Theorem 4 holds for this case.

If the direction is not parallel to $x + y + z = 0$, then one can restate Theorem 4 as a condition on $a, b$ and $c$ so that the projected image to the plane $x + y + z = 0$, that has the form $F^2(k, D')$ for some two-dimensional digit set $D'$ parameterized by $a, b$ and $c$, has positive measure. Note that for a non-degenerate affine transformation $\psi$ and a two-dimensional digit set $D'$ we have $\mathcal{F}(k, \psi(D')) = \psi(\mathcal{F}(k, D'))$, and thus $\mathcal{F}(k, D')$ has positive measure if and only if $\mathcal{F}(k, \psi(D'))$ does. Therefore, we can further restate it as a property of an affine-transformed image of the projected image. In addition, we can reduce the number of parameters to two and use two real parameters $u$ and $v$ to express the shape of the projected image because we have solved the case $a + b + c = 0$. 

8
With this idea, we define the following three series of digit sets in $\mathbb{R}^2$.

$$D^u,v_S = \{ \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} u \\ v \end{array} \right) \}$$

$$D^u,v_H = \{ \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} \pm 1 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} \pm 1 \\ \mp 1 \end{array} \right), \left( \begin{array}{c} \pm u \\ \mp v \end{array} \right) \}$$

$$D^u,v_T = \{ \left( \begin{array}{c} -1 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} -1 + u \\ v \end{array} \right), \left( \begin{array}{c} u \\ -1 + v \end{array} \right), \left( \begin{array}{c} u \\ v \end{array} \right) \}$$

For $D_S$, we consider projections to the plane $x + y + z = 1$. This plane contains three points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of $D_S$, which are fixed by projections. Therefore, we consider an (affine-transformed) two-dimensional coordinate on $x + y + z = 1$ that assigns $(0, 0), (1, 0), (0, 1)$ to $(0, 0, 1), (1, 0, 0), (0, 1, 0)$, respectively. $D^u,v_S$ is an affine-transformed projected image of $D_S$ by setting $(u, v)$ to be the address of the projected image of $(1, 1, 1)$.

$D^u,v_H$ is similarly defined from $D_H$ for the plane $x + y + z = 0$. The plane $x + y + z = 0$ contains seven points of $D_H$, which are fixed by projections, so we consider an (affine-transformed) two-dimensional coordinate on $x + y + z = 0$ that assigns $(0, 0), (1, 0), (0, 1)$ to $(0, 0, 0), (-1, 0, 1), (-1, 0, 1)$, respectively. $D^u,v_H$ is an affine-transformed projected image of $D_H$ by setting $(u, v)$ to be the address of the projected image of $(1, 1, 1)$.

For $D^u,v_T$, the plane $x + y + z = -1$ contains six points of $D_T$ which are fixed by projections, so we consider an (affine-transformed) two-dimensional coordinate on $x + y + z = -1$ that assigns $(0, 0), (1, 0), (0, 1)$ to $(0, 0, -1), (1, 0, 0), (0, 1, 0)$, respectively. $D^u,v_T$ is an affine-transformed projected image of $D_T$ by setting $(u, v)$ to be the address of the projected image of $(1, 1, 0)$.

**Theorem 6.**

(a) $\mathcal{F}(2, D^u,v_S)$ has a positive measure if and only if $u = \frac{p}{r}$ and $v = \frac{q}{r}$ for odd coprime integers $p, q, r$.

(b) $\mathcal{F}(3, D^u,v_T)$ has a positive measure if and only if $u = \frac{p}{r}$ and $v = \frac{q}{r}$ for coprime integers $p, q, r$ such that $p \equiv q \equiv r \mod 3$.

(c) $\mathcal{F}(3, D^u,v_H)$ has a positive measure if and only if $u = \frac{p}{r}$ and $v = \frac{q}{r}$ for coprime integers $p, q, r$ such that $p \equiv q \equiv r \mod 3$ or $p \equiv q \equiv -r \mod 3$.

Theorem 6(a) was proved by Kenyon in [9, Theorem 14]. Note that the following corollary holds since an arbitrary non-colinear four point set $D \subset \mathbb{R}^2$ is affine transformed to $D^u,v_S$.

**Corollary 7.** Let $D = \{O, P, Q, R\} \subset \mathbb{R}^2$ be a non-colinear four-points set. $\mathcal{F}(2, D)$ has a positive Lebesgue measure if and only if $pOP + qOQ + rOR = 0$ for odd integers $p, q, r$.

**Proposition 8.** Theorem 6 and Theorem 4 are equivalent.

*Proof.* If $\varphi$ is parallel to the line $x + y + z = 1$, that is, if $a + b + c = 0$, then Theorem 4 holds as we have noted. We show the equivalence for other cases.
Case $S_\infty$: One can calculate
\[ u = \frac{-a + b + c}{a + b + c}, \quad v = \frac{a - b + c}{a + b + c}. \]
In addition, for $p = -a + b + c$, $q = a - b + c$ and $r = a + b + c$, we have $a = \frac{r - p}{2}$, $b = \frac{r - q}{2}$ and $c = \frac{r + q}{2}$. Therefore, one can easily show that (1) $u = \frac{p}{r}$ or $v = \frac{q}{r}$ is irrational if and only if $\frac{p}{a}$ or $\frac{q}{a}$ exists and is irrational; (2) $a, b, c$ are coprime integers such that $a + b + c$ is odd if and only if $p, q, r$ are odd coprime integers.

Case $T_\infty$: One can calculate
\[ u = \frac{-2a + b + c}{a + b + c}, \quad v = \frac{a - 2b + c}{a + b + c}. \]
In addition, for $p = -2a + b + c$, $q = a - 2b + c$ and $r = a + b + c$, we have $a = \frac{r - 2p}{2}$, $b = \frac{r - q}{2}$ and $c = \frac{r + q}{2}$. From these facts, we can prove the above (1) and the following (2'): $a, b, c$ are coprime integers such that $3 \nmid a + b + c$ if and only if $p, q, r$ are coprime integers such that $p \equiv q \equiv r \mod 3$.

Case $H_\infty$: We have the same $u$ and $v$ as $T_\infty$. Thus, we have (1) and (2').

In addition, there is another direction along which $\mathcal{F}^3(3, D_H)$ is projected to $\mathcal{F}^2(3, D_H^{u,v})$. That is, $(-1, -1, -1)$ is mapped to $(u, v)$ and $(1, 1, 1)$ is mapped to $(-u, -v)$. In this case,
\[ u = \frac{2a - b - c}{a + b + c}, \quad v = \frac{-a + 2b - c}{a + b + c}. \]
Then, for $p = \frac{2a - b - c}{3}$, $q = \frac{-a + 2b + c}{3}$ and $r = \frac{a + b + c}{3}$, we have $a = r + p$, $b = r + q$, and $c = p + q + r$. In this case, one can show (2''): $a, b, c$ are coprime integers such that $a \equiv b \equiv c \equiv -s + b + c \mod 3$ if and only if $p, q, r$ are coprime integers such that $p \equiv q \equiv -r \mod 3$.

4 Differences expansion sets

As is mentioned in the introduction, positively-measured fractals of the form $\mathcal{F}^2(k, D)$ are special cases of self-affine tiles. A self-affine tile is a compact set $T$ in $\mathbb{R}^2$ of positive Lebesgue measure such that $A(T) = \cup_{d \in D} (T + d)$ for an expansive matrix $A$ and a digit set $D \subset \mathbb{R}^2$ of cardinality $|\det(A)|$, and $\mathcal{F}^2(k, D)$ is a special case that $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

Characterization of pairs $(A, D)$ that generate self-affine tiles was studied by Bandit [1], Kenyon [6], Lagarias and Wang [9], and others. In our study, we use the following characterization in [9]. Here, we state it only for self-affine tiles that have the form $\mathcal{F}^2(k, D)$.

**Theorem 9** (Theorem 1.1 of [9]). Let $k \geq 2$, $D \subset \mathbb{R}^2$ be a set of cardinality $k^2$ and $0 \in D$. The following four conditions are equivalent.

(i) $\mathcal{F}^2(k, D)$ has positive Lebesgue measure.
(ii) $\mathcal{F}^2(k, D)$ has nonempty interior.
(iii) $\mathcal{F}^2(k, D)$ is the closure of its interior, and its boundary has Lebesgue measure zero.
(iv) For each \( t \geq 1 \), all the \((k,D)\)-expansions of length \( t \) designate distinct points in \( \mathcal{E}^2(k,D,t) \), and \( \mathcal{E}^2(k,D) \) is a uniformly discrete set, i.e., there exists \( \delta > 0 \) such that \( |x - x'| > \delta \lor x = x' \) for all \( x, x' \in \mathcal{E}^2(k,D) \).

Here, \((k,D)\)-expansion, \( \mathcal{E}(k,D,t) \subset \mathbb{R}^n \) and \( \mathcal{E}(k,D) \subset \mathbb{R}^n \) are defined as follows. If \( x \in \mathbb{R}^n \) is expressed as \( x = \sum_{j=0}^{t-1} k^j d_j \) for \( d_j \in D \), then we say that the sequence \((d_j)_{0 \leq j < t}\) is a \((k,D)\)-expansion of \( x \). We define

\[
\mathcal{E}(k,D,0) = \{0\}, \\
\mathcal{E}(k,D,t) = k\mathcal{E}(k,D,t-1) + D \quad (t > 0)
\]

We have \( \mathcal{E}(k,D,0) \subseteq \mathcal{E}(k,D,1) \subseteq \ldots \) and

\[
\mathcal{E}(k,D) = \bigcup_{t=1}^{\infty} \mathcal{E}(k,D,t).
\]

We sometimes write \( \mathcal{E}^n(k,D) \) for \( \mathcal{E}(k,D) \) to make the dimension of the space clear. We have

\[
\mathcal{E}(k,D,t) = \{ \sum_{j=0}^{t-1} k^j d_j \mid d_j \in D \}
\]

and thus \( \mathcal{E}(k,D,t) \) is the set of points that have \((k,D)\)-expansions of length \( t \).

**Corollary 10** (Corollary 1.1 of [9]). Let \( D \subset \mathbb{Z}^2 \) be a set of cardinality \( k^2 \) and \( 0 \in D \). If \( D \) forms a complete residue system of \( \mathbb{Z}^2/(k\mathbb{Z} \times k\mathbb{Z}) \), then \( \mathcal{F}^2(k,D) \) has a positive Lebesgue measure.

**Proof.** We use Theorem 9 (iv) \( \rightarrow \) (i). Suppose that \( D \) forms a complete residue system of \( \mathbb{Z}^2/(k\mathbb{Z} \times k\mathbb{Z}) \). \( \mathcal{E}^2(k,D) \) is a uniformly discrete set because \( D \) is an integral digit set. If two \((k,D)\)-expansions \((d_i)_{i<t}\) and \((e_i)_{i<t}\) designate the same point, i.e., \( \sum_{i=0}^{t-1} k^i d_i = \sum_{i=0}^{t-1} k^i e_i \), then \( k(\sum_{i=1}^{t-1} k^{i-1}(d_i-e_i)) + (d_0-e_0) = 0 \) and therefore, \( d_0 - e_0 \in k\mathbb{Z} \times k\mathbb{Z} \) which implies \( d_0 = e_0 \). We can inductively show that \( d_i = e_i \) for \( 0 \leq i < t \). \( \square \)

With this Corollary, one can derive the ‘if’ part of Theorem 6.

**Lemma 11.** The ‘if’ parts of Theorem 6 (a), (b), (c) hold.

**Proof.** For (a), consider the digit set

\[
D' = \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} r \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ r \end{array} \right), \left( \begin{array}{c} p \\ q \end{array} \right) \right\}
\]

that generates \( r\mathcal{F}(2,D_0^{u,v}) \) for \( u = \frac{p}{q} \) and \( v = \frac{q}{p} \). If \( p, q, r \) are odd, then \( D' \) is a complete residue system of \( \mathbb{Z}^2/(2\mathbb{Z} \times 2\mathbb{Z}) \). Proofs of (b) and (c) are similar. \( \square \)

For an integral digit set \( D \subset \mathbb{Z}^n \), a **differenced digit set** \( \Delta(D) \) is the digit set

\[
\Delta(D) = D - D = \{ x - y \mid x, y \in \mathcal{E}(k,D) \}
\]

and **differenced radix expansion set** is the set \( \mathcal{E}(k, \Delta(D)) \). Note that \( 0 \in \Delta(D) \) even if \( 0 \notin D \). Differenced radix expansion set is an important tool for investigating tiling properties of self-affine tiles 6,9.
Theorem 9 says that we can prove that $F^2(k, D)$ is a null-set by showing the existence of different $(k, D)$-expansions of the same point, that is, the existence of different sequences $(d_i)_{0 \leq i < t}$ and $(e_i)_{0 \leq i < t}$ of $D$ such that

$$
\sum_{i=0}^{t-1} k^i d_i = \sum_{i=0}^{t-1} k^i e_i,
$$
or equivalently,

$$
\sum_{i=0}^{t-1} k^i (d_i - e_i) = 0.
$$

It means that $0$ has a $(k, \Delta(D))$-expansion of length $t$ other than $(0)_{0 \leq i < t}$. Thus, one can show that $F^2(k, D)$ is a null-set through the observation of differenced radix expansions of $0$, and the proof of Theorem 6(a) in [6] by Kenyon was based on this idea. Here, we reformulate his proof so that it uses projection of three-dimensional differenced radix expansion sets.

For a three-dimensional digit set $D \subset \mathbb{R}^3$ and a projection $\varphi$, we have

$$
\Delta(\varphi(D)) = \varphi(\Delta(D)).
$$

In addition, we have

$$
\mathcal{E}^2(k, \varphi(D)) = \varphi(\mathcal{E}^3(k, D)).
$$

Therefore, we have

$$
\mathcal{E}^2(k, \Delta(\varphi(D))) = \varphi(\mathcal{E}^3(k, \Delta(D))).
$$

Moreover, any differenced radix expansion of $(0, 0)$ by $\varphi(D)$ is obtained as a projection of a differenced radix expansion of $\varphi^{-1}((0, 0))$ by $D$. If $\varphi$ is a projection along $(a, b, c)$ for coprime integers $a, b, c$, then $\varphi^{-1}((0, 0)) = \{(ma, mb, mc) \mid m \in \mathbb{Z}\}$. Therefore, we have the following lemma. Note that $\mathcal{E}^2(k, \varphi(D))$ is always a uniformly discrete set for an integral digit set $D$ and a projection $\varphi$ along an integral vector.

**Lemma 12.** Let $a, b, c$ be integers and $\varphi$ be a projection along $(a, b, c)$. $F^3(k, D)$ is projected to a null-set by $\varphi$ if and only if $(ma, mb, mc) \in \mathcal{E}^3(k, \Delta(D))$ for some $m \neq 0$ or $0 \in \mathbb{R}^3$ has a $(k, \Delta(D))$-expansion other than $(0)_{0 \leq i < t}$.

With this lemma, for $D \in \{D_S, D_T, D_H\}$, one can study the directions along which $F^2(k, D)$ is projected to a null-set by analyzing the shape of the differenced radix expansion set $\mathcal{E}^3(k, \Delta(D))$. Here, since these expansion sets are rather complicated, we consider other digit sets $D_S', D_T', D_H'$ that are also projected to $D_S^u, D_T^u, D_H^u$, respectively, and analyze $\mathcal{E}^3(k, \Delta(D))$ for $D \in \{D_S', D_T', D_H'\}$.

**5 Proofs of the main theorem**

We study the ‘only if’ parts of Theorem 6.
Figure 7: (a) The digit set $D_{S'}$; (b) The differenced digit set $\Delta(D_{S'}) = D_{S'} + (-D_{S'})$.

Figure 8: $E(2, \Delta(D_{S'}))$ for $-2 \leq x, y \leq 6, -1 \leq z \leq 4$. Points in $2E(2, \Delta(D_{S'}))$ are circled and translations of $\Delta(D_{S'})$ by $(0, 0, 0)$ and $(0, 0, 4)$ are colored red.

5.1 Sierpinski Tetrahedron case:

We consider the three-dimensional digit set

$$D_{S'} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

which is projected on the $xy$-plane along $(a, b, c)$ to $D_{u,v}^{S'}$ for $u = -\frac{a}{c}$ and $v = -\frac{b}{c}$. The sets $D_{S'}$ and $\Delta(D_{S'})$ are depicted in Fig. 7. We define

$$B = \{(2^n x, 2^n y, 2^n z) \mid x, y, z \text{ are odd}\},$$

$$C = \mathbb{Z}^3 \setminus B.$$

Lemma 13. $E(2, \Delta(D_{S'})) = C$.

Proof. $E(2, \Delta(D_{S'})) \subseteq C$ is proved by induction on $E(2, \Delta(D_{S'}))$. For the proof of $E(2, \Delta(D_{S'})) \supseteq C$, we prove

$$(x, y, z) \in C \rightarrow (x, y, z) \in E(2, \Delta(D_{S'}))$$

by well-founded induction on $|x| + |y| + |z|$. It holds for the case $|x| + |y| + |z| = 0$ because $(0, 0, 0) \in C$ and $(0, 0, 0) \in E(2, \Delta(D_{S'}))$. For $n > 0$, suppose that it holds for the case $|x| + |y| + |z| < n$ and show that it holds for the case $|x| + |y| + |z| = n$. Suppose that $(x, y, z) \in C$ and $|x| + |y| + |z| = n > 0$. We define $(x', y', z') \in C$ such that $|x'| + |y'| + |z'| < n$ as follows.

If $x$, $y$, and $z$ are all even, then $(x', y', z') = (x/2, y/2, z/2)$. If $x$, $y$, and $z$ are all odd, then $(x, y, z) \notin C$. For the other cases, without loss of generality, we consider the following two cases.
Case $x$ and $y$ are even and $z$ is odd: If $(x, y, z) \in \{(0, 0, 1), (0, 0, -1)\}$, then $(x', y', z') = (0, 0, 0)$. Otherwise, at least one of $(x/2, y/2, (z - 1)/2)$ or $(x/2, y/2, (z + 1)/2)$ is in $C$ and let $(x', y', z')$ be such an element.

Case $x$ is even and $y$ and $z$ are odd: If $(x, y, z) \in \{(0, 1, -1), (0, -1, 1)\}$, then $(x', y', z') = (0, 0, 0)$. Otherwise, at least one of $(x/2, (y - 1)/2, (z + 1)/2)$ and $(x/2, (y + 1)/2, (z - 1)/2)$ is in $C$, and let $(x', y', z')$ be such an element.

By induction hypothesis, $(x', y', z') \in \mathcal{E}(2, \Delta(D_{S'}))$. Since $(x, y, z) - 2(x', y', z') \in \Delta(D_{S'}^v)$ for every case, $(x, y, z) \in \mathcal{E}(2, \Delta(D_{S'}^v))$.

Now, we prove the 'only if' part of Theorem 6(a):

**Lemma 14.**

1. If $u = \frac{2}{q}$ and $v = \frac{2}{r}$ for coprime integers $p, q, r$ such that not all of $p, q, r$ are odd, then $\mathcal{F}^2(2, D_{S,u,v}^v)$ is a null-set.
2. If $u$ or $v$ is irrational, then $\mathcal{F}^2(2, D_{S,u,v}^v)$ is a null-set.

**Proof.** (1) Suppose that $p, q, r$ are coprime and not all of $p, q, r$ are odd. Set $a = -p$, $b = -q$, $c = r$. If $c$ is even then $a$ or $b$ is odd. On the other hand, if $c$ is odd then $a$ or $b$ is even. Therefore, $(a, b, c) \in C$. Therefore, by Lemma 13 $(a, b, c) \in \mathcal{E}^3(2, \Delta(D_{S'}))$. Thus, by Lemma 12 $\mathcal{F}(2, D_{S'})$ is projected to a null-set along $(a, b, c)$. Along the vector $(a, b, c)$, $\mathcal{F}^3(2, D_{S'})$ is projected to $\mathcal{F}^2(2, D_{S,u,v}^v)$ for $u = -\frac{2}{q} = \frac{p}{r}$ and $v = -\frac{2}{r} = \frac{p}{q}$. Therefore, $\mathcal{F}^2(2, D_{S,u,v}^v)$ is a null-set.

(2) Suppose that $u$ or $v$ is irrational. Let $\varphi$ be the projection along $(u, v, -1)$ that maps $D_{S'}$ to $D_{S,u,v}^v$. We show that $\mathcal{E}^3(2, \varphi(D_{S'}))$ is not uniformly discrete. That is,

\[
\forall \delta > 0 \ (\exists x, x' \in \mathcal{E}^3(2, \varphi(D_{S'}))) \ (|x - x'| \leq \delta \land x \neq x')
\]

\[
\iff \forall \delta > 0 \ \exists x \in \mathcal{E}^3(2, \Delta(D_{S'})) \ (|x| \leq \delta \land x \neq 0)
\]

\[
\iff \forall \delta > 0 \ \exists x \in \varphi(\mathcal{E}^3(2, \Delta(D_{S'}))) \ (|x| \leq \delta \land x \neq 0)
\]

\[
\iff \forall \delta > 0 \ \exists y \in \mathcal{E}^3(2, \Delta(D_{S'})) \ (|\varphi(y)| \leq \delta \land y \neq 0)
\]

The last equivalence holds because, for $y \in \mathbb{Z}^3$, $\varphi(y) = 0$ iff $y = 0$. Let $[x]$ and $\{x\}$ be the integer and fractional parts of $x \in \mathbb{R}$, respectively. We have $|\varphi([2iu], [2iv], -2i)| = |\varphi((2iu - [2iu], 2iv - [2iv], -2i)| = \{(2iu, 2iv)\}$. If $u$ and $v$ are irrational, then $\{(2iu, 2iv)\}$ is a null-set for some $i$ because of the density of $\{2ix\} \ i \in \mathbb{Z}$ for an irrational $x$. It is also the case if one of $u$ or $v$ is irrational because $\{i \ i \{2ix\} - 2i\} \in \mathcal{E}^3(2, \Delta(D_{S'}))$. Obviously, one can choose such $i$ so that $\{(2iu, 2iv, -2i) \in \mathcal{E}^3(2, \Delta(D_{S'}))$.}

**5.2 T-fractal case**

We consider the differenced radix expansion set of the following three-dimensional digit set.

\[
D_{T'} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}
\]
Figure 9: (a) The digit set $D_{T'}$; (b) The differenced digit set $\Delta(D_{T'}) = D_{T'} + (-D_{T'})$.

The sets $D_{T'}$ and $\Delta(D_{T'})$ are depicted in Fig. 9. $D_{T'}$ is projected on the $xy$-plane along $(a, b, c)$ to $D_{T'}^u$ for $u = 1 - \frac{a}{c}$ and $v = 1 - \frac{b}{c}$. For $A \subseteq \mathbb{R}^3$ and $c \in \mathbb{Z}$, we denote by $A_c$ the slice $\{(a, b) \mid (a, b, c) \in A\}$. One can see that

$$\Delta(D_{T'})_1 = \{(i, j) \mid i, j < 3 \wedge i + j > 0\},$$

$$\Delta(D_{T'})_0 = \{(i, j) \mid |i|, |j|, |i+j| < 3\},$$

$$\Delta(D_{T'})_{-1} = \{(i, j) \mid i, j > -3 \wedge i + j < 0\}.$$

From the inductive definition of $\mathcal{E}(3, \Delta(D_{T'}))$, we have

$$\mathcal{E}(3, \Delta(D_{T'}))_{3n-1} = 3\mathcal{E}(3, \Delta(D_{T'}))_n + \Delta(D_{T'})_{-1},$$

$$\mathcal{E}(3, \Delta(D_{T'}))_{3n} = 3\mathcal{E}(3, \Delta(D_{T'}))_n + \Delta(D_{T'})_0,$$

$$\mathcal{E}(3, \Delta(D_{T'}))_{3n+1} = 3\mathcal{E}(3, \Delta(D_{T'}))_n + \Delta(D_{T'})_1.$$

Thus, $\mathcal{E}(3, \Delta(D_{T'}))_z$ can be described through balanced trinary expansion of $z \in \mathbb{Z}$, which is the sequence $(a_i)_{0 \leq i < t}$ for $a_i \in \{-1, 0, 1\}$ such that $z = \Sigma_{i=0}^{t-1} a_i 3^i$. $\mathcal{E}(3, \Delta(D_{T'}))_z$ for $z = -1, 0, 1, 2, 3, 6$ are shown in Fig. 10. As these figures suggest, $\mathcal{E}(3, \Delta(D_{T'}))$ has a bit complicated structure. We only characterize $\mathcal{E}(3, \Delta(D_{T'}))_{3m(3^n-1)}$ for $m, n \geq 0$. We define

$$A_m = \{(i, j) \mid i, j \leq 0 \wedge i + j > = -3^{m+1}\},$$

$$B = 3\mathbb{Z}^2.$$

**Lemma 15.**

1. $\mathcal{E}(3, \Delta(D_{T'}))_0 = \mathbb{Z}^2$.

2. $\mathcal{E}(3, \Delta(D_{T'}))_{3n} = \mathbb{Z}^2 \setminus 3^n B$ for $n \geq 0$.

3. $\mathcal{E}(3, \Delta(D_{T'}))_{3m(3^n-1)} = \mathbb{Z}^2 \setminus 3^m B \setminus (3^{m+n} B + A_m)$ for $m \geq 0, n > 0$.

**Proof.**

1. One can expand $(x, y) \in \mathbb{Z}^2$ as $(x, y) = a_i 3^i + \ldots + a_0$ for $a_i \in \{-1, 0, 1\}^2$. Since $\{-1, 0, 1\}^2 \subseteq \Delta(D_{T'})_0$, $(a_i)_{i \geq 0}$ forms a $(3, \Delta(D_{T'})_0)$-expansion of $(x, y)$.

2. Induction on $n$. Case $n = 0$ : We have

$$\mathcal{E}(3, \Delta(D_{T'}))_1 = 3\mathcal{E}(3, \Delta(D_{T'}))_0 + \Delta(D_{T'})_1 = B + \Delta(D_{T'})_1.$$

It is equal to $\mathbb{Z}^2 \setminus B$. See Fig. 10. $z = 1$. 15
Case $n > 0$: By induction hypothesis
\[
E(3, \Delta(D_T'))_{3^n} = 3E(3, \Delta(D_T'))_{3^{n-1}} + \Delta(D_T')_0
\]
\[
= 3(\mathbb{Z}^2 \setminus 3^{n-1}B) + \Delta(D_T')_0
\]
\[
= B \setminus 3^nB + \Delta(D_T')_0.
\]
It is equal to $\mathbb{Z}^2 \setminus 3^nB$. See Fig. 10, $z = 3$.

(3) Induction on $m$. Case $m = 0$: For $n > 0$,
\[
E(3, \Delta(D_T'))_{3^m(3^{n-1})} = 3E(3, \Delta(D_T'))_{3^m(3^{n-1})} + \Delta(D_T')_{-1}
\]
\[
= 3(\mathbb{Z}^2 \setminus 3^{m-1}B) + \Delta(D_T')_{-1}
\]
\[
= B \setminus 3^mB + \Delta(D_T')_{-1}.
\]
It is equal to $\mathbb{Z}^2 \setminus B \setminus (3^nB + A_0)$. See Fig. 10, $z = 2$.

Case $m > 0$: By induction hypothesis
\[
E(3, \Delta(D_T'))_{3^m(3^{n-1})} = 3E(3, \Delta(D_T'))_{3^m(3^{n-1})} + \Delta(D_T')_0
\]
\[
= 3(\mathbb{Z}^2 \setminus 3^{m-1}B \setminus (3^{m-1+n}B + A_{m-1})) + \Delta(D_T')_0
\]
\[
= B \setminus 3^mB \setminus (3^{m+n}B + 3A_{m-1}) + \Delta(D_T')_0
\]
It is equal to $\mathbb{Z}^2 \setminus 3^mB \setminus (3^{m+n}B + A_m)$. See Fig. 10, $z = 6$. 

We prove the ‘only if’ part of Theorem 6(b). In the proof of Lemma 14, $(a, b, c) \in E^3(k, \Delta(D_{S'}))$ for $a = -p$, $b = -q$ and $c = r$, and thus we could
apply Lemma 12 for $m = 1$. However, in the proof of Lemma 16 below, $(a, b, c) \notin \mathcal{E}^3(k, \Delta(D_{T^*}))$ in general and therefore the same argument does not apply. Instead, we show that $(ma, mb, mc) \in \mathcal{E}^3(k, \Delta(D_{T^*}))$ for some $m > 0$ to apply Lemma 12.

Lemma 16.

(1) If $u = \frac{p}{r}$ and $v = \frac{q}{r}$ for coprime integers $p, q, r$ such that $r > 0$ and $\neg(p \equiv q \equiv r \mod 3)$, then $\mathcal{F}^2(3, D^n_{T^*})$ is a null-set.

(2) If $u$ or $v$ is irrational, then $\mathcal{F}^2(3, D^n_{T^*})$ is a null-set.

Proof. (1): Let $c = r, a = r - p, b = r - q$. Suppose that $p \equiv q \equiv r \mod 3$ does not hold. Then, we have $\neg((3 \mid a) \wedge (3 \mid b))$. Let $m$ and $c'$ be $3^n c' = c$ for $3 \not| c'$. By Euler’s theorem, $3^{\phi(c')} \equiv 1 \mod c'$ for $\phi$ Euler's function. Thus, for $n = \phi(c')$ and some $s > 0$, we have $sc' = 3^n - 1$ and thus $sc = 3^n(3^n - 1)$. Let $t_0 = 1$ and $t_k = (3^{2k-1}n + 1)(3^{2n} + 1)$ for $k > 0$. We have $t_1sc = 3^{m}(3^n + 1)(3^{n} - 1) = 3^{m}(3^{2n} - 1)$, and one can inductively show $t_ksc = 3^{m}(3^{2^n} - 1)$ for $k \geq 0$.

Suppose that $(t_ksa, t_ksb, t_ksc) \notin \mathcal{E}(3, \Delta(D_{T^*}))$ for every $k \geq 0$. Then, by Lemma 15, $(t_ksa, t_ksb) \notin (3mB \cup (3m + 2^n)B + A_m)$. Since $\neg((3 \mid a) \wedge (3 \mid b))$ and $3 \not| st_k$, we have $(t_ksa, t_ksb) \notin B$ and thus $(t_ksa, t_ksb) \notin 3mB$. Therefore, $(t_ksa, t_ksb) \in 3m + 2^nB + A_m$.

Let $j$ satisfy $2jn \geq m + 1$. We have $(t_jsa, t_jsb) \in 3m + 2jnB + A_m$ and $(t_{j+1}sa, t_{j+1}sb) \in 3m + 2^{j+1}nB + A_m$, which is impossible because $\frac{t_{j+1} - t_j}{t_j} = 2^{j+1}n + 1 \geq 3^{m+1} = \text{diam}(A_m)$ for diam($A_m$) the diameter of $A_m$.

Thus, $(t_ksa, t_ksb, t_ksc) \in \mathcal{E}(3, \Delta(D_{T^*}))$ for some $k$. Therefore, $\mathcal{F}^3(3, D_{T^*})$ is projected to a null-set along $(a, b, c)$ by Lemma 12.

(2): As in Lemma 14 we assume that $u$ or $v$ is irrational and prove

$$\forall l > 0 \exists y \in \mathcal{E}(3, \Delta(D_{T^*}) \setminus \{0\}) \ | \varphi(y) \leq \frac{1}{3^l}.$$

Let $u = (u, v) \in \mathbb{R}^2$. For $a \in \mathbb{R}$, we write $[a]$ and $\{a\}$ for the integer and fractional part $a$, respectively, and for $a = (a_0, a_1) \in \mathbb{R}^2$, we write $[a]$ and $\{a\}$ for $([a_0], \{a_1\})$ and $\{(a_0), \{a_1\}\}$, respectively. For $l > 0$, we divide $[0, 1]^2$ into $3^{2l}$ disjoint regions $r(s, t) = [\frac{s}{3^l}, \frac{s + 1}{3^l}) \times [\frac{t}{3^l}, \frac{t + 1}{3^l}) \ (0 \leq s, t < 3^l)$. By the pigeonhole principle, there is a region $r(s, t)$ such that the set $P = \{i \mid (3^i u) \in r(s, t)\}$ is infinite. For the enumeration $0 \leq p_0 < p_1 < \ldots$ of $P$, we define $n = p_0$, $m = p_1$ and $n_i = p_{i+2} - m$ for $i = 0, 1, \ldots$.

Let $k_i = [3^{m+n_i}u] - [3^nu], (a_i, b_i) = k_i$, and $y_i = (a_i, b_i, 3^m(3^n - 1))$. For each $i$, $y_i$ satisfies $|\varphi(y_i)| \leq \frac{1}{3^l}$ because

$$|\varphi(y_i)| = |(a_i, b_i) - 3^m(3^n - 1)u| = |k_i - 3^{m+n_i}u + 3^mu| = |-3^{m+n_i}u + 3^mu| < \frac{1}{3^l}.$$

Therefore, we complete the proof by showing that $y_i \in \mathcal{E}(3, \Delta(D_{T^*}))$ for some $i$. By Lemma 15, it means $k_i \in \mathbb{Z}^2 \setminus 3mB \setminus (3^{m+n_i}B + A_m)$. Therefore, we show that for some $i$, $k_i \not\in 3mB$ and $k_i \not\in 3^{m+n_i}B + A_m$. 

17
Suppose that \( k_i \in 3^m B \) or \( k_i \in 3^{m+n_i} B + A_m \) for every \( i \).

\[
k_i = [3^{m+n_i} u] - [3^m u] \in 3^{m+n_i} B + A_m
\]

for infinite number of \( i \). Then,

\[
3^{m+n_i} u - 3^m u \in 3^{m+n_i} B + A_m + \left[-\frac{1}{3^2},\frac{1}{3^2}\right]^2.
\]

Therefore,

\[
u \in \frac{3^{ni}}{3^{ni} - 1} B + \frac{1}{3^{ni} - 1} A_m + \left[-\frac{1}{3^2},\frac{1}{3^2}\right]^2 \subseteq \frac{3^{ni}}{3^{ni} - 1} B + \frac{1}{3^{ni} - 1} [-1,4]^2.
\]

Since it holds for arbitrary large \( n_i \) and the right-hand side converges to \( B \), we have \( u \in B \), but it contradicts the fact that at least one component of \( u \) is irrational. Therefore, \( k_i \in 3^{m+n_i} B + A_m \) for a finite number of \( i \), so let \( I \) be their maximum.

We have \( k_i \in 3^m B \) for \( i > I \). Let \( (u_j)_{j \geq 0} \) and \( (v_j)_{j \geq 0} \) be the trinary expansions of the fractional parts of \( u \) and \( v \), respectively, and let \( \sigma, \tau \in \{0,1,2\} \) be the trinary expressions of \( s \) and \( t \), respectively. From the construction, \( (u_j)_{j \geq 0} \) and \( (v_j)_{j \geq 0} \) contain \( \sigma \) and \( \tau \), respectively, as subsequences starting at \( h, m \) and \( m + n_i \) \( (i = 0,1,\ldots) \).

Since \( k_i = [3^{m+n_i} u] - [3^m u] \in 3^n B \), the last \( m \) digits of the trinary expansions of \( 3^{m+n_i} u \) and \( 3^m u \) coincide. That is, \( (u_j)_{0 \leq j \leq m} \) and \( (u_j)_{n_i \leq j \leq m+n_i} \) coincide and thus \( (u_j)_{0 \leq j \leq m+t} \) and \( (u_j)_{n_i \leq j \leq m+n_i+t} \) coincide because \( (u_j)_{m \leq j \leq m+t} = (u_j)_{m+n_i \leq j \leq m+n_i+t} = \sigma \). Therefore, since \( (u_j)_{0 \leq j \leq m+t} \) contains \( \sigma \) at \( h \) and \( m \), \( (u_j)_{n_i \leq j \leq m+n_i+t} \) contains \( \sigma \) at \( h + n_i \) and \( m + n_i \). It is also the case for \( (v_j)_{j \geq 0} \) and \( \tau \). Since \( m + n_i - 1 \) is next to the last index at which \( \sigma \) and \( \tau \) are contained in \( (u_j)_{0 \leq j \leq m+n_i+t} \) and \( (v_j)_{0 \leq j \leq m+n_i+t} \) simultaneously, we have \( m + n_i - 1 = h + n_i \) and thus \( n_i - n_i - 1 = m - h \) and \( (u_j)_{h \leq j \leq m} = (u_j)_{n_i - 1 \leq j \leq n_i} \) and \( (v_j)_{h \leq j \leq m} = (v_j)_{n_i - 1 \leq j \leq n_i} \). Since it holds for \( i > I \), \( u \) and \( v \) are rational numbers. It contradicts the fact that \( u \) or \( v \) is irrational.

\[\square\]

5.3 H-fractal case

Let

\[D_H = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \right\},\]

which is projected on the \( xy \)-plane along \((a,b,c)\) to \( D_H^{a,b,c} \) for \( u = -\frac{b}{c} \) and \( v = -\frac{a}{c} \). The sets \( D_H \) and \( \Delta(D_H) \) are depicted in Fig. [11]

We define

\[A'_m = \{(i, j) \mid (i,j) \geq -3^m \land i + j \leq 2(3^m) \} \cup \{(i,j) \leq 3^m \land i + j \geq -2(3^m) \},\]

\[C'_m = \{(i,j) \mid \sigma(i,j), |i| + |j|, i + j | 3^m \},\]

\[B'_m = \{(i,j) \mid i \equiv j \not\equiv 0 \pmod{3} \} = 3Z^2 + \{(-1,-1),(1,1)\} \].

18
Figure 11: (a) The digit set $D_H$; (b) The differenced digit set $\Delta(D_H') = D_H + (-D_H)$.

We also define $J = \{(i, j) \mid |i|, |j|, |i + j| \leq 1\}$. We have

\[ \Delta(D_H'_{-2}) = \Delta(D_H')_2 = \{(0, 0)\}, \]
\[ \Delta(D_H'_{-1}) = \Delta(D_H')_1 = J, \]
\[ \Delta(D_H')_0 = G'_1. \]

From the inductive definition of $\mathcal{E}(3, \Delta(D_H'))$, we have

\[ \mathcal{E}(3, \Delta(D_H'))_{3n-1} = (3\mathcal{E}(3, \Delta(D_H'))_n + \Delta(D_H')_{n-1} + \Delta(D_H')_2), \]

\[ \mathcal{E}(3, \Delta(D_H'))_{3n} = 3\mathcal{E}(3, \Delta(D_H'))_n + \Delta(D_H')_0, \]

\[ \mathcal{E}(3, \Delta(D_H'))_{3n+1} = (3\mathcal{E}(3, \Delta(D_H'))_n + \Delta(D_H')_1) \cup (3\mathcal{E}(3, \Delta(D_H'))_{n+1} + \Delta(D_H')_{-2}). \]

**Lemma 17.**

1. $\mathcal{E}(3, \Delta(D_H'))_0 = \mathbb{Z}^2$.
2. $\mathcal{E}(3, \Delta(D_H'))_{3n} = \mathbb{Z}^2 \setminus 3^n B'$ for $n \geq 0$.
3. $\mathcal{E}(3, \Delta(D_H'))_{3n+3n-1} = (\mathbb{Z}^2 \setminus 3^n B' \setminus (3^{m+n} B' + A'_m)) \cup (3^{m+n} B' + G'_m)$ for $m \geq 0, n > 0$.

**Proof.**

1. The same as Lemma 15.
2. Induction on $n$. Case $n = 0$ (see Fig. 12 z=1): We have

\[ \mathcal{E}(3, \Delta(D_H'))_1 = (3\mathcal{E}(3, \Delta(D_H'))_0 + \Delta(D_H')_1) \cup (3\mathcal{E}(3, \Delta(D_H'))_1 + \Delta(D_H')_{-2}) \]

\[ = 3\mathbb{Z}^2 + J = \mathbb{Z}^2 \setminus B'. \]

Case $n > 0$ (see Fig. 12 z=3): By induction hypothesis

\[ \mathcal{E}(3, \Delta(D_H'))_{3n} = 3\mathcal{E}(3, \Delta(D_H'))_{3n-1} + \Delta(D_H')_0 \]

\[ = 3(\mathbb{Z}^2 \setminus 3^{n-1} B') + \Delta(D_H')_0 \]

\[ = 3\mathbb{Z}^2 \setminus 3^n B' + \Delta(D_H')_0 \]

\[ = \mathbb{Z}^2 \setminus 3^n B'. \]
Figure 12: \(E(3, \Delta(D_{H'})\) for \(-4 \leq x, y \leq 16, z \in \{-1, 0, 1, 2, 3, 6\}\). Digits of \(3E(3, \Delta(D_{H'}))\) are circled and translations of \(\Delta(D_{H'})\) by \((0, 0, 0)\), \((3, 12, 0)\), \((9, 6, 3)\) are colored red.

(3) We prove by induction on \(m\). Case \(m = 0\) (see Fig. 12, \(z=2\)):

For \(n = 1\),

\[
E(3, \Delta(D_{H'}))_2 = (3E(3, \Delta(D_{H'}))_1 + \Delta(D_{H'})_{-1}) \cup (3E(3, \Delta(D_{H'}))_0 + \Delta(D_{H'})_2) \\
= (3Z^2 \setminus B') + J \cup 3Z^2 \\
= (3Z^2 \setminus 3B' + J) \cup (3Z^2 \setminus 3B') \cup 3B' \\
= (3Z^2 \setminus 3B' + J) \cup 3B' \\
= Z^2 \setminus B' \setminus (3B' + A'_0) \cup (3B' + G'_0)
\]

For \(n > 1\),

\[
E(3, \Delta(D_{H'}))_n = (3E(3, \Delta(D_{H'}))_{n-1} + \Delta(D_{H'})_{-1}) \cup (3E(3, \Delta(D_{H'}))_{n-1} + \Delta(D_{H'})_2) \\
= (3Z^2 \setminus 3B' + J) \cup 3Z^2 \setminus B' \setminus (3Z^2 \setminus 3B' + A'_0) \cup (3^n B' + G'_0)) \\
= (3Z^2 \setminus 3^n B' + J) \cup 3Z^2 \setminus 3^n B' \setminus (3^n B' + 3A'_0) \cup 3^n B' \\
= (3Z^2 \setminus 3^n B' + J) \cup 3^n B' \\
= Z^2 \setminus B' \setminus (3^n B' + A'_0) \cup 3^n B'.
\]
Case $m > 0$ (see Fig. $12$, $z=6$):

$$\mathcal{E}(3, \Delta(D_{H^u}))_{3^m(3^m-1)} = 3\mathcal{E}(3, \Delta(D_{H^u}))_{3^{m-1}(3^m-1)} + \Delta(D_{H^u})_0$$

$$= 3((\mathbb{Z}^2 \setminus 3^{m-1}B' \setminus (3^{m+n-1}B' + A'_{m-1})) \cup (3^{m+n-1}B' + G'_{m-1})) + G'_{m-1}$$

$$= (3\mathbb{Z}^2 \setminus 3^mB' \setminus (3^{m+n}B' + 3A'_{m-1}) \cup (3^{m+n}B' + 3G'_{m-1})) + G'_{m-1}$$

$$= (3\mathbb{Z}^2 \setminus 3^mB' \setminus (3^{m+n}B' + 3A'_{m-1}) + G'_{m-1}) \cup (3^{m+n}B' + 3G'_{m-1} + G'_{m})$$

$$= (\mathbb{Z}^2 \setminus 3^mB' \setminus (3^{m+n}B' + A'_m)) \cup (3^{m+n}B' + G'_m).$$

Now, we prove the 'only if' part of Theorem 6(c):

**Lemma 18.**

1. If $u = \frac{p}{q}$ and $v = \frac{s}{t}$ for coprime integers $p, q, r$ such that $r > 0$ and $-p \equiv q \equiv r \mod 3$ and $-p \equiv q \equiv -r \mod 3$, then $\mathcal{F}(3, D_{H^u}^{n,v})$ is a null-set.

2. If $u$ or $v$ is irrational, then $\mathcal{F}(3, D_{H^u}^{n,v})$ is a null-set.

**Proof.** (1): Let $c = r, a = -p, b = -q$ and $3^mc' = c$ for $3 \nmid c'$. Suppose that neither $p \equiv q \equiv r \mod 3$ nor $p \equiv q \equiv -r \mod 3$ hold. Then, $m > 0$ and $-((3 \mid a) \land (3 \not| b))$ or $m = 0$ and $-((a \equiv b \not\equiv 0 \mod 3)$. We define $s, t, k (k \geq 0)$ as in the proof of Lemma 16. We have $t_{ksc} = 3^n(3^k - 1)$ for $k \geq 0$. We show that $(t_{ksc}, t_{ksc}, t_{ksc}) \in \mathcal{E}(3, \Delta(D_{H^u}))$ for some $k \geq 0$.

Suppose that $(t_{ksc}, t_{ksc}, t_{ksc}) \not\in \mathcal{E}(3, \Delta(D_{H^u}))$ for every $k \geq 0$. By Lemma 17(3), $(t_{ksc}, t_{ksc}, t_{ksc}) \in (3^mB' \cup (3^{m+n+2}B' + A'_m))$ and $(t_{ksc}, t_{ksc}, t_{ksc}) \not\in (3^{m+n+2}B' + G'_m)$. If $m = 0$, then $-(a \equiv b \not\equiv 0 \mod 3)$. That is, $(a, b) \not\equiv B'$. In addition, $t_{ksc} \not\equiv 0 \mod 3$. Therefore, $(t_{ksc}, t_{ksc}) \not\equiv B'$. If $m > 0$, then $-(3 \mid a) \land (3 \not| b)$. Therefore, $-(3 \mid t_{ksc}) \land (3 \not| t_{ksc})$ and thus $(t_{ksc}, t_{ksc}) \not\equiv B'$. In both cases, $(t_{ksc}, t_{ksc}) \not\equiv 3^mB'$. Therefore, $(t_{ksc}, t_{ksc}) \in 3^{m+n+2}B' + A'_m$.

The rest of the proof is the same as that of Lemma 16(1) with $A_{m}$ and $B$ substituted to $A'_{m}$ and $B'$, respectively.

(2): As in Lemma 16, we assume that $u$ or $v$ is irrational and prove

$$\forall l > 0 \ (\exists y \in \mathcal{E}(3, \Delta(D_{H^u})) \setminus \{0\}) \ |\varphi(y)| \leq \frac{2}{3^l}.$$  

The proof is essentially the same. Note that $k_{j} \in 3^mB'$ also means that the last $m$ digits of the trinary expansions of $3^{m+n_i}u$ and $3^m u$ coincide.

\[\square\]

### 6 Concluding remarks

We characterized directions from which Sierpinski tetrahedron, T-fractal, and H-fractal are projected to sets with positive measures.

A natural question is whether our result can be generalized to fractal imaginary cubes of degree $\geq 4$. According to [14], there are 36 non-congruent fractal imaginary cubes of degree 4 (including Sierpinski tetrahedron that already appeared at degree 2), and 3482 fractal imaginary cubes of degree 5.
Since digit sets of fractal imaginary cubes of degree $k$ exist corresponding to Latin squares of size $k$ and it is shown in [17] that the number of Latin squares of size $k$ is lower bounded by $(k!)^{2k}/k^k$ the number of fractal imaginary cubes increases rapidly. Among the 36 fractal imaginary cubes of degree 4, some have positively-measured symmetric projected images along the line $x = y = z$ ([14]), but some do not. We only have the following simple observation about fractal imaginary cubes in general.

**Proposition 19.** A degree-$k$ fractal imaginary cube $F(k, D)$ of the unit cube $[0, 1]^3$ is projected to a set with positive measure along $(nk, mk, 1)$ for integers $m, n, k$, and their permutations.

**Proof.** By Theorem 3 $D$ satisfies

$$\{(x, y) \mid (x, y, z) \in D\} = \{0, \ldots, k - 1\}^2.$$ 

Therefore, the projected image of $D$ to the $xy$-plane, which is $\{(x - nkz, y - mkz) \mid (x, y, z) \in D\}$, is a complete residue system of $\mathbb{Z}^2/k \mathbb{Z} \times k \mathbb{Z}$. The statement holds by Corollary 10.

Among fractal imaginary cubes, H-fractal has the distinct property that it is a fractal double imaginary cube, and this additional symmetricity ‘doubles’ the number of projected images with positive measure compared to the case of T-fractal. It is shown in [14] that all convex double imaginary cubes are variants of H, and one can easily observe from this fact that the convex hull of a fractal double imaginary cube is always the polyhedron H. It would be interesting to know whether H-fractal is the only fractal double imaginary cube.

As related works, Kenyon studied in [7] line-projections of the one-dimensional Sierpinski gasket and characterized directions from which it is projected to sets with positive measures. It is interesting to know whether one can obtain his result also through the method presented in this paper. Hochman showed that Hausdorff dimensions of irrational projections of the one-dimensional Sierpinski gasket are zero in [4], and Hausdorff dimension of projected images of fractals are surveyed in [3] and [13]. Hausdorff dimensions of the projected images of fractals studied in this paper are left open.

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22
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A  Pictures of a H-fractal model
B Pictures of a T-fractal model