Nonequilibrium Josephson effect in short-arm diffusive SNS interferometers.

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We study non-equilibrium Josephson effect and phase-dependent conductance in three-terminal diffusive interferometers with short arms. We consider strong proximity effect and investigate an interplay of dissipative and Josephson currents co-existing within the same proximity region. In junctions with transparent interfaces, the suppression of the Josephson current appears at rather large voltage, \( eV \sim \Delta \), and the current vanishes at \( eV \geq \Delta \). Josephson current inversion becomes possible in junctions with resistive interfaces, where it occurs within a finite interval of the applied voltage. Due to the presence of considerably large and phase-dependent injection current, the critical current measured in a current biased junction does not coincide with the maximum Josephson current, and remains finite when the true Josephson current is suppressed. The voltage dependence of the conductance shows two pronounced peaks, at the bulk gap energy, and at the proximity gap energy; the phase oscillation of the conductance exhibits qualitatively different form at small voltage \( eV < \Delta \), and at large voltage \( eV > \Delta \).

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I. INTRODUCTION

Multiterminal superconductor - normal metal - superconductor (SNS) junctions are interesting devices where an interplay between the dissipative normal electron current and non-dissipative Josephson current can be studied. The simplest device of this type consists of two superconducting reservoirs and one normal reservoir connected by a small normally conducting T-shape bridge, see Fig. 1(a). A mesoscopic size of the bridge is essential to keep the coherence of the current transport over the whole device. During the last decade, a large amount of interesting experiments have been done with such kind of devices (for the review see Ref. 1 and references therein, further references can be found in Ref. 2).

Non-equilibrium state in multiterminal SNS junctions exhibits two closely related major phenomena: the interferometer effect, which concerns the dependence of normal conductance of the device on the phase difference between the superconducting reservoirs,1 and the Josephson transistor effect, which concerns the dependence of the Josephson current on the current injected from the normal reservoir.1,2 The interferometer effect gives rise to a number of so called Josephson-like effects.2,3

The interferometer effect has received the most of attentions; it has been extensively studied experimentally1,4,5 and theoretically,6,7 and presently this effect is rather well understood. The proximity of the superconducting reservoirs leads to a modification of the density of states and transport properties of the normal bridge (proximity effect), which therefore become sensitive to the phase difference at the reservoirs, and exhibit oscillating behavior as the function of the phase difference.

The Josephson transistor effect has been also observed in experiment.10,11,12 The non-equilibrium Josephson effect has been first predicted13 and then theoretically studied in the ballistic junctions.14,15 Non-equilibrium population of the Andreev states16 induced by the current injection leads to a full-scale variation of the Josephson current with the applied voltage and to the inversion of the current direction. Similar transistor effect has been also investigated in diffusive junctions.17,18,19,20

The most of experiments with multiterminal junctions have been done using diffusive metallic bridges, whose length was large compared to the superconducting coherence length \( 0 \). In such long junctions, the proximity effect is suppressed, which results in small amplitude of the conductance oscillation (typically few percents of the full conductance value), and small magnitude of the Josephson current. In theoretical analysis of the interferometer effect in diffusive junctions, the presence of the Josephson current is usually ignored. Similarly, the theoretical studies of the nonequilibrium Josephson effect17,18,19 are restricted to the regime of “weak” prox-
inity effect when the induced gap in the normal bridge is much smaller than those in the superconducting reservoirs. Such a regime is relevant for long diffusive junctions and for junctions with high-resistance NS interfaces. In addition, the non-equilibrium Josephson current was calculated for specific four-terminal circuit geometries where the dissipative normal current and the Josephson current were spatially separated.

Meanwhile, it is conceptually interesting to investigate the problem of interplay of dissipative and Josephson currents flowing in the same diffusive lead under a strong proximity effect. In this paper, we address this problem by studying the non-equilibrium Josephson effect in 3-terminal SNS interferometers with short arms, whose lengths are smaller than the coherence length \( \xi_0 \).

Co-existence of dissipative and non-dissipative currents in the same proximity region makes it difficult to identify the Josephson current component. In equilibrium, the total current flowing through a proximity lead is the Josephson current, which is entirely determined by the supercurrent spectral density and the population numbers of the relevant states. The presence of the normal injection current and related gradients of the distribution functions violates the local conservation of the supercurrent component which varies along the lead, and hence its direct connection to the Josephson current is lost. Nevertheless, as we will show in the paper, a simple picture of the non-equilibrium Josephson effect as the result of non-equilibrium population of the states with the same current spectral density as in equilibrium can be justified for some particular cases (see also Ref. \( \text{[12]} \)). However, a general situation seems to be more complex.

In the short-arm SNS interferometers, the proximity effect is strong when the interfaces are transparent, but it also may be strong when the interfaces are highly resistive. The measure of the strength of the proximity effect is the magnitude of the induced energy gap, which is comparable with the superconducting energy gap in the leads, in which we neglect the inelastic collision term,\(^{\text{[20]}}\)

\[
\left[ \sigma_z E, \hat{G} \right] = i \hbar D \partial \hat{J}, \quad \hat{J} = \hat{G} \partial \hat{G}, \quad \hat{G}^2 = \hat{I},
\]

\[
\hat{G} = \left( \begin{array}{cc} \hat{g}_R & \hat{g}_K \\ 0 & \hat{g}_A \end{array} \right), \quad \hat{g}_K = \hat{g}_R \hat{J} - \hat{J} \hat{g}_A.
\]

Here \( \hat{g}_{R,A} \) are the retarded and advanced Green functions, \( \hat{J} = f_+ + \sigma_z f_- \) is the matrix distribution function, and \( \partial \) denotes spatial derivative. At the junction node, the matrix current \( \hat{J} \) obeys the Kirchhoff’s rule,\(^{\text{[20]}}\)

\[
\hat{J}_{x=-0} = \hat{J}_{x=+0} + \hat{J}_{y=+0}.
\]

The Keldysh component \( \hat{J}^K \) of the matrix current \( \hat{J} \) determines the electric currents in the leads,\(^{\text{[24]}}\)

\[
\hat{J} = \frac{\sigma}{4e} \int_0^\infty dE \text{Tr} \sigma_z \hat{J}^K = \frac{\sigma}{e} \int_0^\infty dE I_-(E),
\]

where \( \sigma \) is the normal conductivity. The current spectral density \( I_-(E) \) in Eq. \( \text{(4)} \) has three components,\(^{\text{[24]}}\)

\[
I_- \equiv (1/4) \text{Tr} \sigma_z \hat{J}^K = D_+ \partial f_- + I_s f_+ - I_{an} \partial f_+.
\]

The interplay of the injected and Josephson currents in the strong proximity regime is important for the interpretation of experiments with current biased junctions, Fig. \( \text{(1a)} \). The magnitude of the non-equilibrium critical Josephson current measured with this setup is different, as we will show, from the one derived from the current-phase relation measured by an rf-SQUID, Fig. \( \text{(1b)} \). This phenomenon has been earlier noticed in the absence of true Josephson current in the weak proximity regime.\(^{\text{[25]}}\)

The structure of the paper is the following. After introducing a basic formalism in Section II, we discuss the spectral functions in Section III, and the distribution functions in Section IV. Sections V and VI are devoted to a discussion of the non-equilibrium Josephson effect; the interferometer effect is considered in Section VII.

\section{II. Basic Equations}

The junction we are going to investigate is sketched in Fig. \( \text{(1a)} \). It consists of two superconducting reservoirs and a normal reservoir connected by mesoscopic T-shape diffusive metallic bridge. Such a geometry can be realized in experiment, e.g., by using nanowire technology.\(^{\text{[24]}}\) The superconducting reservoirs are assumed to have equal potentials, and the superconducting phase difference between the reservoirs is \( \phi \). The distance between the superconductors, \( 2d \) \((-d < x < d)\), and the length of the injection lead, \( L \) \((0 < y < L)\), are assumed to be small compared to the superconducting coherence length \( \xi_0 = \sqrt{\hbar D/\Delta} \) \((D \text{ is the diffusion coefficient)}\), however, the relation between these lengths can be arbitrary. For simplicity, we assume the cross sections and normal conductivities of all wires to be equal, and the current from the voltage-biased normal reservoir to be injected in the middle of the SNS junction.

Neglecting spatial variations of all quantities across the leads, we use one-dimensional static equations\(^{\text{[24]}}\) for the \( 4 \times 4 \) matrix Keldysh-Green function \( \hat{G} \) in the normal leads, in which we neglect the inelastic collision term,\(^{\text{[20]}}\)

\[
[\sigma_z E, \hat{G}] = i \hbar D \partial \hat{J}, \quad \hat{J} = \hat{G} \partial \hat{G}, \quad \hat{G}^2 = \hat{I},
\]

\[
\hat{G} = \left( \begin{array}{cc} \hat{g}_R & \hat{g}_K \\ 0 & \hat{g}_A \end{array} \right), \quad \hat{g}_K = \hat{g}_R \hat{J} - \hat{J} \hat{g}_A.
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Here \( \hat{g}_{R,A} \) are the retarded and advanced Green functions, \( \hat{J} = f_+ + \sigma_z f_- \) is the matrix distribution function, and \( \partial \) denotes spatial derivative. At the junction node, the matrix current \( \hat{J} \) obeys the Kirchhoff’s rule,\(^{\text{[20]}}\)

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\]

where \( \sigma \) is the normal conductivity. The current spectral density \( I_-(E) \) in Eq. \( \text{(4)} \) has three components,\(^{\text{[24]}}\)

\[
I_- \equiv (1/4) \text{Tr} \sigma_z \hat{J}^K = D_+ \partial f_- + I_s f_+ - I_{an} \partial f_+.
\]
The first term in Eq. (3) describes a dissipative current which provides usual Drude conductivity in the normal state. The second term gives conventional Josephson current in equilibrium, while the third term, the anomalous current, appears in non-equilibrium superconducting junctions. Another diagonal component of $\hat{J}^K$, 

$$I_+ \equiv \frac{1}{4} \text{Tr} \hat{J}^K = D_+ \partial f_+ + I_0 f_- + I_{an} \partial f_-, \quad (6)$$

has the meaning of the net quasiparticle current (the sum of the electron and hole probability currents). Explicit equations for the spectral characteristics of the junction, $D_\pm$, $I_\pm$, and $I_{an}$, are conveniently written in terms of the following parametrization of the matrix $\hat{g}$,

$$\hat{g} = \hat{u} + \hat{v} = \sigma_z u + \exp(i\sigma_z \psi)i\sigma_y v, \quad u^2 - v^2 = 1, \quad (7)$$

The function $u$ is related to the quasiparticle density of states (DOS), normalized by its value in the normal state, $N(E, x) = Re u^R(E, x)$, while the function $v$ is related to the spectral density of the condensate. The complex phase $\psi$ appears in the presence of a supercurrent. In these notations, the diffusion coefficients read

$$D_\pm = (1/4) \text{Tr} \left( 1 - \hat{u}^R \hat{u}^A \mp \hat{v}^R \hat{v}^A \right) \quad (8)$$

$$= (1/2) \left[ 1 + |u|^2 \mp |v|^2 \cosh (2 \text{Im} \psi^R) \right]. \quad (9)$$

In the normal state, $D_\pm$ turn to unity.

The spectral densities of the supercurrent and anomalous current are given by equations,

$$I_s = (1/4) \text{Tr} \sigma_z \left( \hat{v}^R \partial \hat{v}^R - \hat{\sigma}^A \partial \hat{v}^A \right) = - \text{Im} (v^2 \partial \psi)^R, \quad (9a)$$

$$I_{an} = (1/4) \text{Tr} \sigma_z \hat{v}^R \partial \hat{v}^A = - |v|^2 \sinh (2 \text{Im} \psi^R)/2, \quad (9b)$$

In Eqs. (9a) and (9b), the relations $(u, v)^A = -(u, v)^R_s$ and $\psi^A = \psi^R_s$ are used, which follow from the general relationship $\hat{g}^A = -\sigma_x \hat{g}^R \sigma_x$. Calculation of the electric current in Eq. (4) involves the two steps: first one has to solve the Usadel equations for the Green functions $\hat{g}^R, A$, and then to solve the kinetic equations to find the distribution functions.

III. SPECTRAL FUNCTIONS

The Green function components of Eq. (11) represent the Usadel equations for the spectral functions $\hat{g}^R, A$,

$$2Ev = i\hbar D \left[ \partial (u \partial v - v \partial u) - uv (\partial \psi)^2 \right], \quad (10)$$

$$v^2 \partial \psi = I, \quad (11)$$

where the spatial constant $I(E)$ is related to the supercurrent spectral density in Eq. (9), $I_s = -\text{Im} I^R$. In terms of the spectral angle $\theta$ related to the spectral functions as $u = \cosh \theta$, $v = \sinh \theta$, Eq. (11) takes the form,

$$\partial^2 \theta = (2E/i\hbar D) \sinh \theta + I^2 \cosh \theta \sinh^3 \theta. \quad (12)$$

The two terms in r.h.s. of Eq. (12) are related to two different depairing mechanisms, which provide spatial decrease of $\theta$ towards the middle of the junction. The first term is associated with the dephasing between the electron and hole wave functions at finite energy $E$. The second term describes depairing caused by the time-reversal symmetry breaking due to supercurrent flow.

The solution of Eqs. (11) and (12) reads

$$x_i = \int_{\theta_0}^{\theta(E, x_i)} \frac{d\theta}{\sqrt{R(E, \theta)}}, \quad x_i = x, y, \quad (13)$$

$$\psi(E, x_i) = \psi_0 + I \int_{\theta_0}^{x_i} \frac{dz}{\sqrt{v^2(E, z)}}, \quad (14)$$

$$R(E, \theta) = C + (4E/i\hbar D) \cosh \theta - (I/\sinh \theta)^2, \quad (15)$$

where $C(E)$ is the integration constant, and $\psi_0$ are the spectral functions at the junction node. The boundary conditions for Eqs. (11) and (12) are imposed by the conservation law for the matrix current in Eq. (3),

$$\partial x_i \theta|_{x=-0} = \partial x_i \theta|_{x=+0} + \partial y \theta|_{y=0}, \quad (16)$$

and similar equation for the function $\psi$. At the normal electrode, the current $I$ turns to zero, which means that $I \equiv 0$ along the injection lead, and therefore $\psi(y) \equiv$ const. Hence, the derivative of $\psi$ is continuous in the horizontal lead, whereas $\theta(x)$ has a kink at $x = 0$. In what follows, we consider symmetric junction, in which $\theta(x)$ is even, and $\psi(x)$ is odd function. In this case, the phase $\psi$, together with the anomalous current $I_{an}$, turns to zero at the junction node, and the kink in $\theta(x)$ is symmetric, $\theta'_0 \equiv \partial_x \theta|_{x=+0} = -\partial_x \theta|_{x=-0}$.

The boundary conditions at the NS interfaces depend on the interface resistance. Below we analyze the two different situations related to perfect and high-resistive interfaces, respectively.

A. Transparent interfaces

If the interface electric resistance $R_{NS}$ is much smaller than the normal resistance $R_N$ of the horizontal lead, one can assume the spectral functions to be continuous at $x = \pm d$, namely, $\psi(\pm d) = \pm \psi_0/2$, and $\theta(\pm d) = \theta_\pm \equiv \arctanh (\Delta/E)$; at the normal electrode $\theta = 0$. In this limit, the second term in the r.h.s. of Eq. (15) can be neglected due to large gradients of the spectral functions along the leads, which results in linear change of $\theta$ along the injection lead, $\theta(E, y) = \theta_0 (1 - y/L)$, and the boundary condition in Eq. (16) takes the form

$$\theta_0 = 2\theta'_0 L. \quad (17)$$

The analytical expressions for the spectral functions in the horizontal lead within this approximation have been found in Ref. 21. In the right lead, they are given by

$$u(E, x) = \tilde{u}_0 \cosh (\alpha + \Lambda x/d), \quad (18)$$

$$\psi(E, x) = \text{arctan} \left( \tilde{u}_0^{-1} \tanh (\alpha + \Lambda x/d) \right) - p. \quad (19)$$
Eqs. (18) and (19) can be parameterized as of the signs of \( u \), \( \alpha \), and \( p \). The spatial constants in Eqs. (18) and (19) can be parameterized as

\[
I = \frac{\tilde{v}_0}{d} \Lambda, \quad \Lambda = \text{Arcosh} \left( \frac{u_S}{u_0} \right) - \text{Arcosh} \left( \frac{u_0}{u_S} \right),
\]

(20)

\[
u_0 = \text{cosh} \theta_0 = \tilde{u}_0 \text{cosh} \alpha = u_S(E, \Delta),
\]

(21)

\[
\tilde{u}_0 = u_S(E, \Delta), \quad \tilde{\alpha} = E \text{ cos } p, \quad \tilde{\Delta} = \Delta \cos(\phi/2 + p),
\]

(22)

where \( u_S(E, \Delta) = E/\sqrt{E^2 - \Delta^2} \), and expressed via a single parameter \( p \), which is to be evaluated from the equation following from Eq. (17),

\[
\tilde{u}_0 \Lambda \sin p = a \theta_0.
\]

(23)

The magnitude of the parameter \( p \) is controlled by the parameter \( a = d/2L \). When \( a \) decreases, i.e., the resistance of the injection lead increases, \( p \) turns to zero, according to Eq. (23), and the spectral functions approach their values in a closed short SNS junctions. In the limit \( a = 0 \), DOS has the proximity gap \( |\Delta_0| \), where \( \Delta_0 = \Delta \cos(\phi/2) \), and reveals a BCS-like singularity at the gap edge,

\[
N(E, 0) = E \Theta(E - |\Delta_0|)/\sqrt{E^2 - \Delta^2_0},
\]

(24)

where \( \Theta(x) \) is the Heaviside step function. The supercurrent spectral density \( I_s(E) \) spreads over the region \( |\Delta_0| \leq E \leq \Delta_0 \), and has the singularity at the proximity gap edge as well,

\[
I_s(E) = \frac{\pi}{2d} \frac{\Delta_0 \Theta(E - |\Delta_0|)}{\sqrt{E^2 - \Delta^2_0}} \Theta(E - \Delta).
\]

(25)

In a general case, \( a \neq 0 \), the proximity of the normal reservoir leads to finite DOS at all energies, as shown in Fig. 2(a), though it is noticeably suppressed at \( E < |\Delta_0| \) for \( L \gg d \). The supercurrent spectral density at finite \( d/L \) extends over the whole subgap region [see Fig. 2(b)], while at \( E > \Delta \), both \( I_s \) and \( I_{an} \) turn to zero. Thus, in short diffusive junctions with transparent interfaces, the supercurrent is carried exclusively by the bound Andreev states confined to the potential well formed by the junction. However, this result is only correct to zero approximation with respect to the small parameter \( d/\xi_0 \).

### B. Opaque interfaces

The effect of the interface becomes important when the interface resistance \( R_{NS} \) exceeds the resistance of the normal conductor \( R_N = 2d/\sigma, r = R_{NS}/R_N \gg 1 \). In particular, the magnitude of the Josephson current is determined by the \( R_{NS} \) rather than by \( R_N \) in the limit \( r \gg 1 \). At the same time, as we will see below, the suppression of the proximity effect is governed by much smaller parameter \( r(d/\xi_0)^2 \ll r \), and the proximity effect can be strong even when \( r \gg 1 \).

A high-resistive interface can be modelled by an effective tunnel barrier characterized by its resistance \( R_{NS} \) in the normal state, which results in the following boundary conditions for the Green functions

\[
\sigma R_{NS} \partial \psi_N = u_N v_S \cos(\phi/2 - \psi_N) - u_N v_N, \quad (26a)
\]

\[
\sigma R_{NS} I = v_N v_S \sin(\phi/2 - \psi_N), \quad (26b)
\]

and similar for \( x = -d \) (the indices \( N \) and \( S \) refer to the normal and superconducting sides of the interface).

In the limit \( r \gg 1 \), the spatial variation of the spectral phase is strongly non-homogeneous: the phase drops at the barriers and is small in the normal region, \( \psi_N \ll 1 \), along with the spectral current \( I \). The spatial variation of the spectral function \( u \) is small and can be approximated by weakly varying parabolic function,

\[
u(E, x) \approx u_0 \left[ 1 + (\beta/2) (x/d)^2 \right].
\]

(27)

In Eq. (27), we neglected the effect of the injection lead, assuming its resistance to be larger than \( R_{NS} \), \( 1/a \gg r \). The coefficient \( \beta \ll 1 \) is to be found from Eq. (17), in which the electron-hole dephasing effect has to be taken into account because it now becomes comparable with the small current-induced depairing,

\[
\beta = -2i \frac{E}{\Delta} \left( \frac{d}{\xi_0} \right)^2 \frac{u_0^2}{v_0^2} + \left( \frac{p}{v_0} \right)^2.
\]

(28)

In Eq. (28), we may neglect spatial variations of the integrand which results in a linear spatial dependence of the phase, \( \psi(E, x) \approx 1x/v_0^2 \sim r^{-1} \).

By making use of Eqs. (27) and (28), the boundary conditions in Eqs. (23) give the equation for the spectral functions \( u_N \) and \( v_N \),

\[
\frac{u_S}{u_N} = \frac{i E}{u_N} + \frac{\gamma}{v_N} \cos \frac{\phi}{2} - r \psi_N^2, \quad \gamma = 2r (d/\xi_0)^2, \quad (29)
\]

and the expression for the spectral current \( I \),

\[
I = \frac{v_N v_S}{\sigma R_{NS}} \sin \frac{\phi}{2}.
\]

(30)
in which we omitted the small phase $\psi_N$ from the trigonometric functions. Equation (29) describes three mechanisms of depairing. The first term in the r.h.s. represents the electron-hole dephasing within the normal metal. The parameter $\gamma$ determines the magnitude of the energy gap $E_g \sim \Delta/(1 + \gamma)$ in the spectrum of the horizontal lead (see below). The second term describes suppression of the condensate function $\psi_N$ due to rapid change of the spectral phase across the tunnel barrier. This effect is similar to the mechanism which produces the Andreev bound states in the vicinity of the tunnel junction.

This is the effect of enhanced electron-hole dephasing, whose solution can be well approximated by a simple relation $E_g = |\Delta_\phi|/(1 + \gamma)$.

Thus, the regime of strong proximity effect with the proximity gap being of the order of $\Delta$ [Fig. 3(a)], persists in short junction despite of high-resistive interfaces, $R_{NS} \gg R_N$, as long as the condition $\gamma \lesssim 1$ holds.

In this case, the supercurrent spectral density,

$$I_s(E, \phi) = -\Im I = -\frac{\sin(\phi/2)}{2\sigma R_{NS}} \Im \left[ \frac{\Delta \psi_N(E, \phi)}{\sqrt{E^2 - \Delta^2}} \right];$$

extends over all quasiparticle states above the proximity gap, including the continuum states above the bulk energy gap $\Delta$, where $I_s(E)$ is negative [Fig. 3(b)]. In the limit $\gamma \ll 1$ (recall that we nevertheless assume here the interface resistance to be large, $r \gg 1$), the dephasing effect becomes negligibly small and the energy gap approaches $|\Delta_\phi|$, similar to the perfect SNS junction discussed above. However, the shape of the supercurrent spectral density essentially differs from Eq. (25),

$$I_s(E) = \frac{\sin \phi}{2\sigma R_{NS}} \frac{\Delta^2 \Theta(E - |\Delta_\phi|) \Theta(\Delta - E)}{\sqrt{E^2 - \Delta^2 \sqrt{\Delta^2 - E^2}}}.$$

It is interesting to note that the expression for the equilibrium Josephson current obtained from Eq. (33),

$$j_s = \frac{\Delta \sin \phi}{2e R_{NS} \gamma} K(|\sin(\phi/2)|), \quad \gamma \ll 1 \quad (T = 0)$$

($K$ is the elliptic integral), can be reproduced with the arguments of the scattering theory, similar to the case of perfect diffusive SNS junctions by applying the transmissivity distribution for a normal symmetric double-barrier structure. Such a possibility is explained by the absence of electron-hole dephasing in this limit.

The proximity gap is strongly suppressed, $E_g \ll \Delta$, only in the limit of very large barrier strength, $\gamma \gg 1$. This is the effect of enhanced electron-hole dephasing, similar to the case of a long diffusive SNS junction, where the proximity gap is also reduced due to the dephasing effect and closes at $\phi = \pi$. This situation is qualitatively different from the case of the tunnel junction with a single barrier, giving the Josephson current in junctions with two strong barriers,

$$j_s = \frac{\Delta \sin \phi}{2e R_{NS} \gamma} \ln \left[ \frac{4\gamma}{|\cos(\phi/2)|} \right], \quad \gamma \gg 1 \quad (T = 0),$$

to be much smaller than the result of the tunnel model.

**IV. KINETIC EQUATIONS**

In the absence of inelastic collisions, the kinetic equations in each lead have the form of conservation laws for the spectral currents $I_{\pm}(E)$,

$$D_{\pm} \partial f_{\pm} + I_s f_{\pm} \pm I_{an} \partial f_{\pm} = I_{\pm}(E) = \text{const.}$$

At the junction node, the conservation law for the matrix currents in Eq. (30) imposes the boundary condition

$$I_{\pm}^L = I_{\pm}^R + I_{\pm}^V,$$

where the indices $L$, $R$, and $V$ refer to the left, right, and injection leads, respectively. At the transparent interfaces, the distribution functions are determined by the local-equilibrium population in the reservoirs,

$$f_{\pm}(\pm d) = \tanh \frac{E}{2T}, \quad (E > \Delta), \quad f_{-}(\pm d) = 0, \quad (38a)$$

$$f_{\pm}(L) = n_{\pm} \equiv \frac{1}{2} \left[ n_{\pm} \left\{ \tanh \frac{E + eV}{2T} \pm \tanh \frac{E - eV}{2T} \right\} \right]^{\text{max}} \quad (38b)$$

At $E < \Delta$, the quasiparticle population in the leads is disconnected from the superconducting reservoirs due to
complete Andreev reflection, and the quasiparticle density function $f_+$ is determined by the condition of the absence of the net probability current, $I_n = 0$. Due to the conservation law in Eq. (37), the subgap probability current $I_s$ turns to zero within the entire device.

In the energy region $E > \Delta$, where the currents $I_s$ and $I_{an}$ turn to zero, and the diffusion coefficient $D_+$ turns to unity, the kinetic equations have a simple solution,

$$ I^R_L = \frac{n_0 - n_+}{R_+ + 2R'_+}, \quad I^R_R = -\frac{n_0 - n_-}{R_- + 2R'_-}, \quad (39) $$

Here the quantities $R_\pm = d$, $R^L_\pm = L$, $R_- = d(1/D^R_0)$, $R^L_- = L/(1/D^L_0)$ play the role of effective resistances of the leads for the spectral currents $I_{s,an}$, and the angle brackets denote spatial averaging along the leads. The currents in the left lead are equal by magnitude but flow in opposite directions, $I^L_L = -I^R_L$, and therefore the currents in the injection lead are twice the currents in the left lead. Combining this result with the relation $f_+(0) - n_0 = -R_+ I^L_+$, following from the kinetic equations, we find that in the limit of long injection lead the boundary condition at the junction node becomes independent of applied voltage,

$$ f_+(0) = n_0, \quad L \gg d, \quad (40) $$

which implies that the quasiparticles in horizontal leads are in equilibrium with the superconducting reservoirs.

Within the subgap energy region, $E < \Delta$, the situation is more complex due to appearance of the currents $I_s$ and $I_{an}$; the only simplification is due to the zero quasiparticle current, $I_n = 0$. By this reason, $f^L_+ = \text{const} = n_+ = f^L_- = \text{const} = n_-$ within the entire injection lead, including the junction node. Thus, the boundary conditions for the distribution functions in the horizontal leads read,

$$ f_+(0) = n_+, \quad f_-(0) = n_- = -R^V_L f^V_+, \quad f_-(\pm d) = 0. \quad (41) $$

Taking advantage of the symmetry of the quantities $D_\pm(x) = D_\pm(-x)$, $I_{an}(x) = -I_{an}(-x)$, we separate the even and odd parts of the distribution functions, $f_{s,a}(x) = [f(x) \pm f(-x)]/2$, in Eqs. (40), which then become split in the two independent pairs of kinetic equations. One pair that couples $f^L_+$ and $f^L_-$,

$$ D_+ \partial f^R_+ + I_s f^R_- - I_{an} \partial f^R_- = 0, \quad (42a) $$

$$ D_- \partial f^R_- + I_s f^R_+ + I_{an} \partial f^R_+ = (I^R_L + I^R_-)/2, \quad (42b) $$

has a constant solution, $f^R_+ = n_+$, $f^R_- = 0$, consistent with the boundary conditions, which yields the relation $I_s n_+ = (I^R_L + I^R_-)/2$. As we will see later, Eq. (42), the non-equilibrium Josephson current $j_s$ has the form

$$ j_s = (j^R + j^L)/2, $$

and taking into account Eq. (40), we arrive at the following result,

$$ j_s = \frac{\sigma N}{e} \int_0^\Delta dE I_s(E)n_+(E). \quad (43) $$

The second pair of kinetic equations couples the functions $f^V_+$ and $f^V_-,$

$$ D_+ \partial f^V_+ + I_s f^V_- - I_{an} \partial f^V_- = 0, \quad (44a) $$

$$ D_- \partial f^V_- + I_s f^V_+ + I_{an} \partial f^V_+ = -I^V_/2. \quad (44b) $$

Since the source term and also the boundary conditions to these equations, Eq. (44), depend on $I^V_/$, these functions determine the dissipative current. The solution to Eq. (44) in general case must be found numerically.

At zero temperature, it is possible to further extend the analysis. By making use of a step-wise shape of the distribution functions $n_{\pm} = \Theta(\pm(E - eV))$, we find that a trivial solution, $f^V_+ = 0$, $f^V_- = 0$, $I^V_/$ = 0, satisfies Eq. (44) and all the boundary conditions at $E > eV$. Thus the dissipative current vanishes in this energy interval. On the other hand, at $E < eV$, where $n_+ = 0$ and $n_- = 1$, equations (44) have a non-trivial solution, which implies that the dissipative current exists at these energies, while the Josephson current is zero, according to Eq. (43). Thus at zero temperature the dissipative and non-dissipative currents flow within the separate energy regions, which do not overlap: The injection current spreads over the energy region $0 < E < eV$,

$$ j^V_0 = \frac{\sigma N}{e} \int_0^{eV} dE I^V_/E), \quad (45) $$

while the supercurrent occupies the region $eV < E < \Delta$,

$$ j_s = \Theta(\Delta - eV)\frac{\sigma N}{e} \int_{eV}^\Delta dE I_s(E). \quad (46) $$

The analysis for the subgap region also applies to the case of resistive interfaces, $r \gg 1$. However, at the energies $E > \Delta$, the supercurrent $I_s$ and anomalous current $I_{an}$ are nonzero and give additional contribution to the Josephson current in Eq. (46).

V. Nonequilibrium Josephson Current

In equilibrium, the Josephson current is given by the second term in Eq. (5), as it was mentioned in Sec. II. Under non-equilibrium conditions, this connection becomes ambiguous and needs reconsideration. The reason is that the appearance of the dissipative currents and related gradients of the distribution functions in Eq. (5) will lead to spatial variation of the supercurrent term along the horizontal lead. To find an appropriate equation for the observable non-equilibrium Josephson current, we refer to a generic definition of the dc Josephson effect, as a current flow through a junction without any dissipation. In our case, the rate of the energy transfer from a voltage source to the junction is given by equation,

$$ W = \int_{-d}^d dx j(x) \frac{dV}{dx} = j^L V^L + j^R V^R, \quad (47) $$

where $V^L,R$ are the voltage drops at the left/right leads. The stationary Josephson effect assumes zero voltage drop between the superconducting electrodes, $V^L + V^R = 0$, thus the non-dissipative current component must satisfy the equation $j^L_s = j^L_s = j_s$. Combining this equation
with the Kirchhoff’s rule, we arrive at the following definition of the Josephson current through the currents in the left, right, and injection leads,

\[ j^{L,R} = j_s \pm \frac{j^V}{2}. \] (48)

Thus, in order to evaluate observable Josephson current in a general case, it is necessary to calculate the injection current and the current in one of the horizontal leads, and then apply Eq. (48). In particular case of symmetric junction, this procedure leads to Eq. (49).

Persistent current in a SQUID is the most fundamental manifestation of the Josephson effect. The Josephson current in Eq. (48) coincides with the circulating current, and it can be directly measured by measuring the induced flux with an external magnetometer.

Let us first consider junctions with transparent interfaces, where the non-equilibrium Josephson current is given by Eq. (49). Since the spectral density \( I_s \) is positive in this case, as it is found from numerical solution of Eq. (28), and the population of the subgap states is depleted with increasing voltage, the injection will suppress the Josephson current and block it completely at \( eV > \Delta \) [Fig. 4(a)]; however, the current direction cannot be reversed. The Josephson current weakly depends on the applied voltage and is close to the equilibrium value as long as the voltage is smaller than the proximity gap value, \( eV < |\Delta_0| \). We note that this equilibrium value differs from that in closed SNS junctions, it is reduced due to the proximity of the normal reservoir and therefore depends on the length of the injection lead, as shown in Fig. 4(b). At larger voltage, the Josephson current-voltage dependence \( j_s(V) \) becomes more steep, especially for the small phase differences.

For a long injection lead, Eq. (49) takes the form,

\[ j_s(\phi, V) = \frac{\pi \Delta_0}{2eR_N} \int_{|\Delta_0|}^\Delta dE \frac{n_s(E)}{\sqrt{E^2 - \Delta_0^2}}. \] (49)

\[ j_s(\phi, V) = \frac{\pi \Delta_0}{2eR_N} \int_{|\Delta_0|}^\Delta dE \frac{n_s(E)}{\sqrt{E^2 - \Delta_0^2}}. \] (49)

At zero temperature, the integration in Eq. (49) can be explicitly performed:

\[ j_s(\phi, V) = \frac{\pi \Delta_0}{2eR_N} \ln \frac{1 + \sin(\phi/2)}{2eR_N f(V) + eV/\cos^2(\phi/2)} \] (50)

\[ f(V) = \max[eV/\Delta, \cos(\phi/2)], \quad eV > |\Delta_0|. \] (51)

This current-voltage dependence is shown in Fig. 5(a) by a dashed line. In this case, the Josephson current at \( eV < |\Delta_0| \) is constant and equal to the equilibrium value.

To estimate the efficiency of the Josephson transistor let us consider the most steep part of the current-voltage characteristic, \( j_s(\phi, V) \), at small phase and at large voltage. For example, at \( \phi = 0.3\pi \), when the equilibrium Josephson current approaches about of 0.7 of its critical value \( j_c \), the switching effect occurs within a small voltage interval \( \delta V \sim 0.1\Delta/e \). The current gain in this case, \( \delta j_s/\delta V \sim 0.7 j_c/G\delta V \sim 7(L/d) \), exceeds unity even for comparable lengths of the leads, and it can be further enhanced by making the injection lead longer. The upper bound for the gain is imposed by the condition of small quasiparticle dwelling time \( L^2/D \), compared to the quasiparticle relaxation time \( \tau \), \( L^2/D \ll \tau \).

The non-equilibrium Josephson current-phase dependence for the junctions with transparent interfaces and high-resistive injection lead is shown in Fig. 5(a) for different applied voltages. The kinks on the graphs correspond to the phase values, at which the applied voltage equals the proximity gap, \( \phi_0(V) = 2 \arccos(eV/\Delta) \). At smaller phases, \( 0 < \phi < \phi_0(V) \), the current-phase dependence has an equilibrium form, while at larger phases it is considerably distorted. Correspondingly, the critical current \( j_c(V) \) remains independent of applied voltage until \( \phi_0(V) \) exceeds the value \( \phi_m = 1.97 \), at which the equilibrium supercurrent approaches its maximum value \( j_{c0} = j_c(0) = 0.66\pi\Delta/2eR \). At larger voltage, \( eV > \Delta \cos(\phi_m/2) = 0.55\Delta \), the critical current decreases and turns to zero at \( eV \geq \Delta \), as shown in Fig. 5(b),

\[ \frac{j_c(V)}{j_c(0)} = \frac{j_s[\phi_0(V), 0]}{j_s(0)} = 1.51 \frac{j_c}{j_c(0)} \frac{eV}{\Delta} \arccosh \frac{\Delta}{eV}. \] (52)
then becomes independent of applied voltage at this small dissipative and anomalous components can be omitted from the current spectral density, the small, and the function 

L/d

Fig. 6(a). Detailed analysis of the crossover region can be made for the junctions with high-resistive injection lead, L/d ≫ r ≫ 1, and at zero temperature. In this case, in the horizontal leads, the distribution function f− is small, and the function f+ is approximately constant and approaches the equilibrium value n0(E) = 1 in the superconducting reservoir [see Eq. (40)]. By these reasons, the small dissipative and anomalous components can be omitted from the current spectral density, L− ≈ I−, which then becomes independent of applied voltage at E > Δ. This results in the following modification of Eq. (46),

\[ j_s = \frac{\alpha}{e} \int_{\min(eV, \Delta)}^{\infty} dE I_s(E), \]  

where I∗(E) is to be found from Eqs. (32) and (31).

As follows from Eq. (53), the critical voltage, at which the current turns to zero, depends on the phase, and therefore the crossover extends over a certain, in fact rather small, voltage interval, as shown in Fig. 6(b). When the voltage approaches the critical region, a new current node in the current-phase dependence splits from the node at \( \phi = \pi \), then it moves towards smaller \( \phi \); the process ends when the extra node approaches \( \phi = 0 \). Such a fine structure of the Josephson current inversion has been observed experimentally in long SNS junctions. At very large interface resistance, \( \gamma \gg 1 \), this fine structure becomes irresolvable because of in this limit the phase dependence in \( I_s(E, \phi) \) for relevant energies is given by a prefactor \( \sin \phi \), and therefore the compensation effect appears simultaneously at all phases.

### VI. CRITICAL CURRENT IN CURRENT BIASED JUNCTION

In experiment, the current bias setup is often employed for investigation of the dc Josephson current, Fig. 1(c). In equilibrium, the maximum value of the current flowing through the junction without creating a voltage drop coincides with the maximum current in the current-phase dependence. This is not the case for a non-equilibrium junction with current injection: the “critical” current is contributed by both the non-equilibrium Josephson current and the injection current.

Suppose the voltage is applied between the injection electrode and left superconducting electrode, Fig. 1(c), then the external transport current \( j_T \) is equal to the current \( j^R \) in the right lead. In this case, the problem of the critical current evaluation is reduced to the analysis of the phase dependence of \( j^R = j_s - j^V/2 \) at given voltage. The requirement of zero potential difference between the superconducting electrodes is automatically fulfilled in our calculation (time-independent phase difference). For simplicity, we consider the junction with perfect interfaces, where the currents are given by Eqs. (10), (10).

The numerical results of such analysis are shown in Fig. 7(a,c). They are obtained by solving numerically Eq. (23) for the spectral functions and Eq. (36) for the distribution functions, which determine the magnitude of the injection current. At \( eV < \Delta \), when the supercurrent is allowed to flow through the junction, the current-phase relations are similar to that depicted in Fig. 5(a). At these voltages, the Josephson current coexists with the normal current flowing out of the injection lead. At larger voltages, \( eV > \Delta \), the supercurrent is blocked, however, the transport current still flows through the junction without voltage drop across it, within a certain range of the current magnitudes determined by the amplitude of the dependence \( j^R(\phi) \). The existence of such Josephson-like regime without real Josephson current has been first pointed out for a 4-terminal SNS junction with opaque interfaces.

To understand this phenomenon, it is important to remember that the injection current in NS interferometers is not uniquely determined by the bias voltage, but also depends on the superconducting phase. In principle, a similar regime with zero voltage drop across the junction may appear even for normal reservoirs, at the transport current \( j_T = j^V(\phi)/2 \). This value is unique for the given injection voltage, and therefore the corresponding dependence \( j_T(V) \) is represented by a straight line. In the superconducting junctions, such line broadens to a stripe, \( j_T = j^V(\phi)/2 \), due to the presence of the free parameter \( \phi \). The phase adjusts the injection current for given injection voltage and transport current to provide zero voltage drop across the junction. The width of the stripe is determined by the amplitude of the injection current oscillation with the phase. At \( eV < \Delta \), this effect is hidden by the presence of the true supercurrent [large shaded regions in Fig. 7(b,d)], however, it is fully revealed.
at $eV > \Delta$, where the supercurrent is suppressed [shaded stripes in Fig. 7(b,d)]. In fact, at large voltage, the width of the shaded stripes is determined by the amplitude of phase oscillations of the excess injection current. The qualitative difference between the phase dependence of the excess current ($eV > \Delta$) and the Josephson current ($eV < \Delta$), is clearly seen in Fig. 7(a,c). It is interesting that the “critical current” has different sign for positive and negative voltages [the shaded stripes in Fig. 7(b,d) are differently oriented with respect to the straight line]. This is consistent with the fact that the excess current changes sign along with the applied voltage.

VII. INTERFEROMETER EFFECT

In this Section, we investigate the conductance of the injection lead as a function of the bias voltage and superconducting phase focusing on its properties due to the strong proximity effect. At small temperatures, $T \ll eV$, the overall voltage dependence of the differential conductance is given by Eq. (54), $G(V, \phi) = dJ^V/dV = \sigma I^V$, where the injection current $I^V$ is to be calculated by numerical solution of Eq. (24) for the spectral functions and the kinetic equations (50). As shown in Fig. 8 the conductance has two peaks, at voltages $eV = \Delta$ and $eV \approx \Delta\phi$. The first peak is associated with enhanced transmissivity of the junction due to the DOS peak at the bulk gap edge. This peak has a logarithmic singularity at $\phi = 0$, and it becomes smeared and decreases while $\phi$ departs from zero. The second peak manifests a rapid change in the spectral functions in the vicinity of the spectrum edge in a short SNS junction (see Fig. 7), and it can be interpreted as a resonance transmission due to enhanced DOS at the proximity gap edge $|\Delta\phi|$. As soon as $L$ increases, this resonance becomes more sharp because of the singularities in the spectral functions at $E = |\Delta\phi|$ become more pronounced, whereas the peak at the bulk gap edge, $eV = \Delta$, decreases and vanishes at $d/L \to 0$, as shown in Fig. 8(b). Furthermore, the conductance exhibits the reentrance effect: $G(V)$ approaches the value $G_N = \sigma/(d/2 + L)$ in the normal state both at small and large voltages, as it was predicted for NS point contacts.

We notice that the differential conductance $G(V)$ deviates from $G_N$ in the short-arm interferometers at the characteristic energy of the order of $\Delta$, in contrast to the long-arm SNS junctions ($d \gg \xi_0$), where the conductance peak appears at the Thouless energy $E_{\text{Th}} = h D/(2d)^2 \ll \Delta^2$.

At zero phase difference, $\phi = 0$, the function $G(V)$ can be found analytically. In this case, $I_s = I_m = 0$, and the function $f_-$ obeys a simple equation $D_-\partial f_- = I_-\partial$ in each lead, with the diffusion coefficients

$$D_-^R,L = \cosh^2\left[\frac{\text{Re} \theta_\Sigma(E) + a|x|/d}{1 + a}\right],$$

$$D^-_S = \cosh^2\left[\frac{\text{Re} \theta_\Sigma(E) - y/L}{1 + a}\right].$$

From Eqs. (50) and (51), we obtain

$$G(V) = G_N \eta(eV) \quad (\phi = 0),$$

$$\eta(E) = z \text{Arctanh} z^{-1}, \quad z = (E/\Delta)\text{sign}(E-\Delta).$$

The oscillations of the conductance peak at $eV = \Delta$ with the phase can be found from the following arguments. At this energy, the diffusion coefficient $D_-$ turns to infinity in the horizontal lead which therefore becomes non-resistive with respect to the normal current, $R_-(\Delta) = 0$. Thus, the differential conductance at $eV = \Delta$ is completely determined by the resistance of the injec-
Fig. 9: Differential conductance vs phase at normal value current turn to zero, and only the resistances spectral densities of the superconducting and anomalous junction with the length horizontal lead coincides with the resistance of a short NS

tance \( R \) is involved in the calculation. Since the condensate \( \Delta \) is substituted for \( \Delta \) in the function

\[
G(\phi) \equiv G(\Delta/e, \phi) = G_N \frac{1 + a}{\cos(\phi/2)} \ln \cot \frac{\phi}{4}.
\]

According to Eq. (58), the peak height approaches

\[
G_N(1 + a) \text{ at } \phi = \pi.
\]

At this point, like at \( \phi = 0 \), the spectral densities of the superconducting and anomalous current turn to zero, and only the resistances \( R_- \) and \( R^C \) are involved in the calculation. Since the condensate function becomes completely suppressed in the middle of the junction, \( \theta_0 = 0 \), the injection lead behaves as a normal wire, and therefore the resistance \( R^C \) approaches its normal value \( L \). Correspondingly, the resistance of each horizontal lead coincides with the resistance of a short NS junction with the length \( d \), \( R_\perp = d \tan(\text{Re } (\theta_S)) / \text{Re } (\theta_S) \), and therefore \( G(V) \) at \( \phi = \pi \) is given by

\[
G(V) = G_N \frac{1 + a}{1 + a/\eta(eV)} \text{ (} \phi = \pi \text{).}
\]

In the limit of long injection lead, \( d/L \to 0 \), its resistance \( R_\perp = L \tan(\text{Re } \theta_0) / \text{Re } \theta_0 \), \( \theta_0 = \arctan(\Delta_0/E) \), completely determines the injection current, and the voltage dependence of the differential conductance shown in Fig. 9(b) is approximately described by Eq. (55), with \( |\Delta_0| \) substituted for \( \Delta \) in the function \( \eta(eV) \).

The differential conductance exhibits full-scale 2π-periodic oscillations with the phase difference \( \phi \) (Andreev interferometer effect). The form of the oscillations is qualitatively different for the subgap bias region, \( eV < \Delta \), and for \( eV > \Delta \), as shown in Fig. 9(b). In the latter case, the phase dependence of \( G \) has a comparatively simple form, with maxima at \( \phi = 2\pi n \) and minima at \( \phi = (2n + 1)\pi \). At \( eV < \Delta \), the differential conductance approaches minima both at even and odd multiples of \( \pi \), which reflects the interplay between the position and amplitude of the resonance at \( eV = |\Delta_0| \).

**VIII. SUMMARY**

We have developed a theory of the non-equilibrium Josephson effect in a three-terminal diffusive interferometer with short SNS junction having the length \( 2d \) much smaller than the superconducting coherence length \( \xi_0 \). We focused on the case of strong proximity effect, when the proximity energy gap in the normal region is of the order of \( \Delta \). For the junction with transmissive NS interfaces, the density of states \( N(E) \) and the supercurrent spectral density \( I_s(E) \) extend over the whole subgap region, \( 0 < E < \Delta \), due to the proximity to a normal reservoir, and exhibit a considerable enhancement at the energy equal to the proximity gap \( |\Delta_0| = |\cos(\phi/2)| \) in the spectrum of a closed SNS junction. The supercurrent spectral density is positive at all relevant energies. We demonstrated a possibility of the strong proximity effect in a junction with opaque interfaces whose resistance \( R_{NS} \) is much larger than the normal resistance \( R_N \) of the junction arms. In such case, the suppression of the proximity gap, \( E_g(\phi) = |\Delta_0|/(1 + \gamma) \), is controlled by the parameter \( \gamma = (2R_{NS}/R_N)(d/\xi_0)^2 \), which could be small in short junctions, \( d \ll \xi_0 \), even at large interface resistance, \( R_{NS} \gg R_N \). In contrast to the case of transmissive interfaces, the supercurrent spectral density \( I_s(E) \) is negative above the bulk gap value.

In three-terminal interferometers, the supercurrent generally coexists with the dissipative current flowing out from the injection electrode. In such situation, we defined the non-equilibrium Josephson current \( j_s \) as the non-dissipative component of the current flowing between the superconducting electrodes. In symmetric junctions, within the subgap energy region, this component coincides with its intuitive representation through the supercurrent spectral density \( I_s(E) \) and the quasiparticle population imposed by non-equilibrium injection, because the Andreev reflection blocks quasiparticle exchange with equilibrium superconducting reservoirs. In junctions with transmissive interfaces the Josephson current becomes completely blocked at \( eV = \Delta \) at zero temperature, while in junctions with high-resistive interfaces, the Josephson current undergoes inversion at \( eV \lesssim \Delta \), which spreads over a finite voltage interval. At the energies above the bulk energy gap, \( E > \Delta \), the population in the junction arms is basically determined by the equilibrium population in the superconducting reservoirs. By this reason, the Josephson current becomes voltage independent at \( eV > \Delta \).

We notice that spectroscopy of the supercurrent spectral density at the subgap energies is possible at zero temperature, similar to the tunnel spectroscopy of \( N(E) \), because the derivative of the Josephson current over applied voltage, \( \partial j_s/dV \), is proportional to \( I_s(eV) \).

The critical current \( j_c \) of the three-terminal junction, defined as the maximum value of the transport current \( j_T \) flowing through the junction without creating a voltage drop, does not coincide with the maximum in \( j_s(\phi) \). This is due to the presence of phase dependent injection current \( j^V(\phi) \) which contributes to \( j_c \), along with the Josephson current, and adjusts its magnitude providing zero voltage drop across the junction. At large voltage, where the Josephson current is suppressed, the domain
of the Josephson-like regime is determined by the amplitude of phase oscillations of the excess current in the injection electrode.

The behavior of the injection current is highly sensitive to the quasiparticle spectrum of the junction and can be used to detect the position of the phase-dependent proximity gap. In particular, the differential conductance of the junction with perfect interfaces exhibits sharp peaks at the bulk gap value, \( eV = \Delta \), and at the proximity gap, \( eV = |\Delta_\phi| \); the latter becomes more pronounced as the resistance of the injection lead increases. Furthermore, the differential resistance exhibits full-scale oscillations with the phase difference; at \( eV < \Delta \), the shape of the oscillations becomes rather complex, due to the interplay between the position and amplitude of the proximity gap resonance.

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1. C. J. Lambert and R. Raimondi, J. Phys. Condens. Matter. 10, 901 (1998).
2. P. Samuelsson, J. Lantz, V. S. Shumeiko and G. Wendin, Phys. Rev. B 62, 1309 (2000).
3. F. W. J. Hekking and Yu. V. Nazarov, Phys. Rev. Lett. 71, 1625 (1993); H. Nakano and H. Takayanagi, Phys. Rev. B 47, 7986 (1993).
4. B. J. van Wees, K. M. H. Lenssen, and C. J. P. M. Hamans, Phys. Rev. B 44, 470 (1991).
5. G. Wendin, and V. S. Shumeiko, Phys. Rev. B 53, R6006 (1996).
6. A. F. Volkov, and V. V. Pavlovskii, JETP Lett. 64, 670 (1996).
7. A. F. Volkov, and H. Takayanagi, Phys. Rev. Lett. 76, 4026 (1996); Phys. Rev. B 56, 11184 (1997); R. Shaikhaozdarov, A. F. Volkov, H. Takayanagi, V. T. Petrashe, and P. Delsing, ibid. 62, R14649 (2000).
8. V. T. Petrashev, V. N. Antonov, P. Delsing and T. Clae-son, Phys. Rev. Lett. 70, 347 (1993); P. G. N. de Vevar, T. A. Fulton, W. H. Malinson, and R. E. Miller, ibid. 73, 1416 (1994); H. Pothier, S. Gueron, D. Esteve, and M. H. Devoret, ibid. 73, 2488 (1994).
9. A. F. Volkov, and A. V. Zaitsev, Phys. Rev. B 53, 9267 (1996); A. V. Zaitsev, A. F. Volkov, S. W. D. Bailey, and C. J. Lambert, ibid. 60, 3559 (1999).
10. A. F. Morpurgo, T. M. Klapwijk, and B. J. van Wees, Appl. Phys. Lett. 72, 966 (1998); J. J. A. Baselmans, A. F. Morpurgo, B. J. van Wees, and T. M. Klapwijk, Nature 397, 43 (1999).
11. J. J. A. Baselmans, T. T. Heikilä, B. J. van Wees, and T. M. Klapwijk, Phys. Rev. Lett. 89, 207002 (2002).
12. J. Huang, F. Pierre, T. T. Heikilä, F. K. Wilhelm, and N. O. Birge, Phys. Rev. B 66, 020507(R) (2002).
13. P. Samuelsson, V. S. Shumeiko, and G. Wendin, Phys. Rev. B 56, R5763 (1997).
14. L. F. Chang and P. F. Bagwell, Phys. Rev. B 55, 12678 (1997).
15. P. Samuelsson, A. Ingerman, V. Shumeiko, and G. Wendin, Physica C 352, 1-4, 82 (2001).
16. A. F. Andreev, Sov. Phys. JETP 19, 1228 (1964).
17. A. F. Volkov, Phys. Rev. Lett. 74, 4730 (1995).
18. F. K. Wilhelm, G. Schön, and A. D. Zaikin, Phys. Rev. Lett. 81, 1682 (1998).
19. S. K. Yip, Phys. Rev. B 58, 5803 (1998).
20. T. T. Heikilä, J. Särkkä, and F. K. Wilhelm, Phys. Rev. B 66, 184513 (2002).
21. A. F. Volkov, and T. M. Klapwijk, Phys. Lett. A 168, 217 (1992); A. F. Volkov, A. V. Zaitsev, and T. M. Klapwijk, Physica C 210, 21 (1993).
22. R. Seviour, and A. F. Volkov, Phys. Rev. B 61, 9273 (2000).
23. A. Furusaki, and M. Tsukada, Phys. Rev. B 43, 10164 (1991).
24. A. Yu. Kazumov, I. I. Khodos, P. M. Aiayan, and C. Collix, Europhys. Lett. 34, 429 (1996); A. Bezryadin, C. N. Lau, and M. Tinkham, Nature (London) 404, 971 (2000).
25. A. I. Larkin, and Yu. N. Ovchinnikov, Sov. Phys. JETP 41, 960 (1975); 46, 155 (1977).
26. Yu. V. Nazarov, Phys. Rev. Lett. 73, 1420 (1994); Superlattices Microstruct. 25, 1221 (1999).
27. K. D. Usadel, Phys. Rev. Lett. 25, 507 (1970).
28. Similar solution for a diffusive point contact between superconductors was found in Ref. [22] by Matsubara technique.
29. To next order, one should take into account the electron-hole dephasing at finite energy [second term in the r.h.s. of Eq. (15)], which results in a small contribution to the supercurrent from the continuum states, \( E > \Delta \), where the spectral density \( I_s(E) \) is negative.
30. M. Yu. Kupriyanov, and V. F. Lukichev, Sov. Phys. JETP 67, 89 (1988).
31. E. V. Bezuglyi, E. N. Bratus', and V. P. Galaiko, Low Temp. Phys. 25, 167 (1999).
32. C. W. J. Beenakker, Phys. Rev. Lett. 67, 3836 (1991); ibid. 68, 1442 (1992).
33. M. J. M. de Jong, Phys. Rev. B 54, 8144 (1996).
34. V. Ambegaokar, and A. Baratoff, Phys. Rev. Lett. 10, 486 (1963).
35. Similar formula has been applied for calculation of the Josephson current in Ref. [12].
36. I. O. Kulik, A. N. Omelyanchouk, Sov. J. Low Temp. Phys. 5, 296 (1978).
37. S. N. Artemenko, A. F. Volkov, and A. V. Zaitsev, Solid State Commun. 30, 771 (1979).
38. Yu. V. Nazarov, and T. H. Stoof, Phys. Rev. Lett. 76, 823 (1996); A. F. Volkov, N. Allsopp, and C. J. Lambert, J. Phys. Condens. Matter 8, 45 (1996); A. A. Golubov, F. K. Wilhelm, and A. D. Zaikin, Phys. Rev. B 55, 1123 (1997).