Selection properties of the split interval and the Continuum Hypothesis

Taras Banakh\textsuperscript{1,2} \vspace{1em}

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Abstract
We prove that every usco multimap $\Phi : X \to Y$ from a metrizable separable space $X$ to a GO-space $Y$ has an $F_\sigma$-measurable selection. On the other hand, for the split interval $\mathbb{I}$ and the projection $P : \mathbb{I}^2 \to \mathbb{I}^2$ of its square onto the unit square $\mathbb{I}^2$, the usco multimap $P^{-1} : \mathbb{I}^2 \to \mathbb{I}^2$ has a Borel ($F_\sigma$-measurable) selection if and only if the Continuum Hypothesis holds. This CH-example shows that know results on Borel selections of usco maps into fragmentable compact spaces cannot be extended to a wider class of compact spaces.

Keywords Continuum hypothesis · Split interval · Measurable selection · Borel selection · usco multimap · Fragmentable compact · Rosenthal compact · GO-space

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1 Introduction

By a multimap $\Phi : X \to Y$ between topological spaces $X, Y$ we understand any subset $\Phi \subseteq X \times Y$, which can be thought as a function assigning to every point $x \in X$ the subset $\Phi(x) := \{y \in Y : \langle x, y \rangle \in \Phi\}$ of $Y$. For a subset $A \subseteq X$ we put $\Phi[A] = \bigcup_{x \in A} \Phi(x)$. Each function $f : X \to Y$ can be thought as a single-valued multimap $\{\langle x, f(x) \rangle : x \in X\} \subseteq X \times Y$.

For a multimap $\Phi : X \to Y$, its inverse multimap $\Phi^{-1} : Y \to X$ is defined by $\Phi^{-1} := \{\langle y, x \rangle : \langle x, y \rangle \in \Phi\}$.

A multimap $\Phi : X \to Y$ is called

- lower semicontinuous if for any open set $U \subseteq Y$ the set $\Phi^{-1}[U]$ is open in $X$;
- upper semicontinuous if for any closed set $F \subseteq Y$ the set $\Phi^{-1}[F]$ is closed in $X$;

\textsuperscript{1} Jan Kochanowski University in Kielce, Kielce, Poland
\textsuperscript{2} Ivan Franko National University in Lviv, Lviv, Ukraine
– **Borel-measurable** if for any Borel set \( B \subseteq Y \) the set \( \Phi^{-1}[B] \) is Borel in \( X \);
– **compact-valued** if for every \( x \in X \) the subspace \( \Phi(x) \) of \( Y \) is compact and non-empty;
– **usco** if \( \Phi \) is upper semicontinuous and compact-valued.

It is well-known that for any surjective continuous function \( f : X \to Y \) between compact Hausdorff spaces, the inverse multimap \( f^{-1} : Y \mapsto X \) is usco.

Let \( \Phi : X \to Y \) be a multimap between topological spaces. A function \( f : X \to Y \) is called a **selection** of \( \Phi \) if \( f(x) \in \Phi(x) \) for every \( x \in X \). The Axiom of Choice ensures that every multimap \( \Phi : X \to Y \) with non-empty values has a selection. The problem is to find selections possessing some additional properties like the continuity or measurability.

One of classical results in this direction is the following theorem of Kuratowski and Ryll-Nardzewski [12] (see also [16, §5.2] or [14, 6.12]).

**Theorem 1** Let \( X, Y \) be Polish spaces. Any Borel-measurable multimap \( \Phi : X \to Y \) with non-empty values has a Borel-measurable selection.

We recall that a function \( f : X \to Y \) between topological spaces is called **Borel-measurable** (resp. **\( F_\sigma \)-measurable**) if for every open set \( U \subseteq Y \) the preimage \( f^{-1}[U] \) is Borel (or type \( F_\sigma \)) in \( X \).

\( F_\sigma \)-Measurable selections of usco multimaps with values in non-metrizable compact spaces were studied by many mathematicians [4–9]. Positive results are known for two classes of compact spaces: fragmentable and linearly ordered.

Let us recall [3, 5.0.1] (see also [15, §6]) that a topological space \( K \) is **fragmentable** if \( K \) has a metric \( \rho \) such that for every \( \varepsilon > 0 \) each non-empty subset \( A \subseteq K \) contains a non-empty relatively open set \( U \subseteq A \) of \( \rho \)-diameter \( < \varepsilon \). By [3, 5.1.12], each fragmentable compact Hausdorff space contains a metrizable dense \( G_\delta \)-subspace.

The following selection theorem can be deduced from Theorem 1’ and Lemma 6 in [9].

**Theorem 2** (Hansell, Jayne, Talagrand) Any usco map \( \Phi : X \to K \) from a perfectly paracompact space \( X \) to a fragmentable compact space \( Y \) has an \( F_\sigma \)-measurable selection.

A similar selection theorem holds for usco maps into countably cellular GO-spaces. A Hausdorff topological space \( X \) is called a **generalized ordered space** (briefly, a **GO-space**) if \( X \) admits a linear order \( \leq \) such that the topology of \( X \) is generated by a base consisting of open order-convex subsets of \( X \). A subset \( C \) of a linearly ordered space \( X \) is called order-convex if for any points \( x \leq y \) in \( C \) the order interval \( [x, y] := \{ z \in X : x \leq z \leq y \} \) is contained in \( X \). We say that the topology of \( X \) is generated by the linear order \( \leq \) if the topology of \( X \) is generated by the subbase \( \{ (\leftarrow, a), (a, \to) : a \in X \} \) consisting the the order-convex sets \( (\leftarrow, a) := \{ x \in X : x < a \} \) and \( (a, \to) = \{ x \in X : a < x \} \).

A topological space \( X \) is **countably cellular** if every disjoint family of open sets in \( X \) is at most countable. It is easy to see that each separable topological space is countably cellular. A topological space is called **\( F_\sigma \)-perfect** if every open set in \( X \) is countably cellular.
of type $F_\sigma$ in $X$ (i.e., can be represented as the countable union of closed sets). For example, every metrizable space is $F_\sigma$-perfect.

The following selection theorem will be proved in Sect. 2.

**Theorem 3** Let $Y$ be a GO-space and $X$ be an $(F_\sigma$-perfect) topological space. If $X$ or $Y$ is countably cellular, then any usco map $\Phi : X \rightharpoonup Y$ has a Borel ($F_\sigma$-measurable) selection.

Theorems 2, 3 suggest the following problem.

**Problem 1** Is it true that any usco map $\Phi : M \rightharpoonup K$ from a compact metrizable space $M$ to a compact Hausdorff space $K$ has a Borel ($F_\sigma$-measurable) selection?

In this paper we prove that this problem has negative answer under the negation of the Continuum Hypothesis (i.e., under $\omega_1 < c$). A suitable counterexample will be constructed using the split square $\mathbb{I}^2$, which is the square of the split interval $\mathbb{I}$.

The *split interval* is the linearly ordered space $\mathbb{I} = [0, 1] \times \{0, 1\}$ whose topology is generated by the lexicographic order (defined by $\langle x, i \rangle \leq \langle y, j \rangle$ iff either $x < y$ or else $x = y$ and $i \leq j$). The split interval plays a fundamental role in the theory of separable Rosenthal compacta [17]. Let us recall that a topological space is called *Rosenthal compact* if it is homeomorphic to a compact subspace of the space $B_1(P)$ of functions of the first Baire class on a Polish space $P$. It is well-known (and easy to see) that the split interval is Rosenthal compact and so is its square. By Theorem 4 of Todorčević [17], each non-metrizable Rosenthal compact space of countable spread contains a topological copy of the split interval. A topological space has *countable spread* if it contains no uncountable discrete subspaces.

By Theorem 3, any usco map $\Phi : X \rightharpoonup \mathbb{I}$ from an $F_\sigma$-perfect topological space $X$ has an $F_\sigma$-measurable selection. In contrast, the split square $\mathbb{I}^2$ has dramatically different selections properties. Let $p : \mathbb{I} \rightarrow \mathbb{I}$, $p : \langle x, i \rangle \mapsto x$, be the natural projection of the split interval onto the unit interval $\mathbb{I} = [0, 1]$, and $P : \mathbb{I}^2 \rightarrow \mathbb{I}^2$, $P : \langle x, y \rangle \mapsto \langle p(x), p(y) \rangle$, be the projection of the split square $\mathbb{I}^2$ onto the unit square $\mathbb{I}^2$.

**Theorem 4** The following conditions are equivalent:

1. the usco multimap $P^{-1} : \mathbb{I}^2 \rightharpoonup \mathbb{I}^2$ has a Borel-measurable selection;
2. the usco multimap $P^{-1} : \mathbb{I}^2 \rightharpoonup \mathbb{I}^2$ has an $F_\sigma$-measurable selection;
3. $\omega_1 = c$.

The implication $(2) \Rightarrow (1)$ of Theorem 4 is trivial and the implications $(1) \Rightarrow (2)$ are proved in Lemmas 2 and 8, respectively.

Combining Theorem 4 with Theorem 4 in [17], we obtain the following consistent characterization of metrizable compacta.

**Corollary 1** Under $\omega_1 < c$ a Rosenthal compact space $K$ is metrizable if and only if $K$ has countable spread and each usco multimap $\Phi : \mathbb{I}^2 \rightharpoonup K^2$ has a Borel-measurable selection.
The “only if” part follows from Theorem 2. To prove the “if” part, assume that a Rosenthal compact $K$ is not metrizable but has countable spread. By Theorem 4 of [17], the space $K$ contains a topological copy of the split interval $]\mathbb{I}\mathbb{I}$. We lose no generality assuming that $]\mathbb{I}\mathbb{I} \subseteq K$. By Theorem 4, under $\omega_1 < c$, the usco multimap $P^{-1} : \mathbb{I}^2 \to \mathbb{I}^2 \subseteq K^2$ does not have Borel-measurable selections. \hfill \Box

Now we pose some open problems suggested by Theorem 4.

**Problem 2** Assume CH. Is it true that each usco map $\Phi : X \to \mathbb{I}^2$ from a metrizable (separable) space $X$ has a Borel-measurable selection?

Observe that the map $p : \mathbb{I} \to \mathbb{I}$ is 2-to-1 and its square $P : \mathbb{I}^2 \to \mathbb{I}^2$ is 4-to-1. A function $f : X \to Y$ is called $n$-to-1 for some $n \in \mathbb{N}$ if $|f^{-1}(y)| \leq n$ for any $y \in Y$. By Theorem 3 of Todorcevic [17], every Rosenthal compact space of countable spread admits a 2-to-1 map onto a metrizable compact space. Let us observe that the splitted square $\mathbb{I}^2$ contains a discrete subspace of cardinality continuum and hence has uncountable spread.

**Problem 3** Let $n \in \{2, 3\}$. Is there an $n$-to-1 map $f : K \to M$ from a (Rosenthal) compact space $K$ to a metrizable compact space $M$ such that the inverse multimap $f^{-1} : M \to \mathbb{I}^2$ has no Borel selections?

**Remark 1** hris Heunen asked on Mathoverflow [10] whether for any continuous surjective function $\pi : X \to Y$ between compact Hausdorff spaces and any continuous map $\gamma : \mathbb{I} \to Y$ there exists a Borel-measurable function $g : \mathbb{I} \to X$ such that $\gamma = \pi \circ g$. Theorem 4 provides a consistent counterexample to this problem of Heunen. Indeed, consider the projection $P : \mathbb{I}^2 \to \mathbb{I}^2$ and take any continuous surjective map $\gamma : \mathbb{I} \to \mathbb{I}^2$ (whose existence was proved by Giuseppe Peano in 1890). By Theorem 2, the inverse multimap $\gamma^{-1} : \mathbb{I}^2 \to \mathbb{I}$ has a Borel-measurable selection $s : \mathbb{I}^2 \to \mathbb{I}$. Assuming that there exists a Borel-measurable function $g : \mathbb{I} \to \mathbb{I}^2$ with $\gamma = P \circ g$, we conclude that $g \circ s : \mathbb{I}^2 \to \mathbb{I}^2$ is a Borel-measurable selection of the multimap $P^{-1}$, with contradicts Theorem 4 under CH.

### 2 Proof of Theorem 3

Theorem 3 follows from Lemmas 2 and 3, proved in this section.

First we prove one lemma, showing that our definition of a GO-space agrees with the original definition of Lutzer [13]. Probably this lemma is known but we could not find the precise reference in the literature.

**Lemma 1** The linear order $\leq$ of any GO-space $X$ is a closed subset of the square $X \times X$.

**Proof** Given two elements $x, y \in X$ with $x \not\leq y$, use the Hausdorff property of $X$ and find two disjoint order-convex neighborhoods $O_x, O_y \subseteq X$ of the points $x, y$, respectively. We claim that the product $O_x \times O_y$ is disjoint with the linear order $\leq$. Assuming that this is not true, find elements $x' \in O_x$ and $y' \in O_y$ such that $x' \leq y'$.
Taking into account that the sets \( O_x, O_y \) are disjoint and order-convex, we conclude that \( x' < y \) and \( x < y' \). It follows from \( x < y \) that \( y < x \). Then \( x' < y < x < y' \) and hence \( [y, x] \subseteq O_x \cap O_y = \emptyset \), which is a desired contradiction. This contradiction shows that the neighborhood \( O_x \times O_y \) of the pair \((x, y)\) is disjoint with \( \leq \) and hence \( \leq \) is a closed subset of \( X \times X \). □

**Lemma 2** Any usco multimap \( \Phi : X \rightarrow Y \) from an \((F_\sigma\text{-perfect})\) topological space \( X \) to a countably cellular GO-space \( Y \) has a Borel \((F_\sigma\text{-measurable})\) selection.

**Proof** Being a GO-space, \( Y \) has a base of the topology consisting of open order-convex subsets with respect to some linear order \( \leq \) on \( Y \). By Lemma 1, the linear order \( \leq \) is a closed subset of \( Y \times Y \). Then for every \( a \in Y \) the order-convex set \( (\leftarrow, a] = \{y \in Y : y \leq a\} \) is closed in \( Y \), which implies that each non-empty compact subset of \( Y \) has the smallest element.

Then for any usco multimap \( \Phi : X \rightarrow Y \) we can define a selection \( f : X \rightarrow Y \) of \( \Phi \) assigning to each point \( x \in X \) the smallest element \( f(x) \) of the non-empty compact set \( \Phi(x) \subseteq Y \). We claim that this selection is \( F_\sigma\)-measurable.

A subset \( U \subseteq Y \) is called upper if for any \( u \in U \) the order-convex set \( (\leftarrow, u] = \{y \in Y : y \leq u\} \) is contained in \( U \).

**Claim 1** For any upper open set \( U \subseteq Y \) the preimage \( f^{-1}[U] \) is open in \( X \).

**Proof** For any \( x \in f^{-1}[U] \) we get \( \Phi(x) \subseteq \{f(x), \rightarrow\} \subseteq U \). The upper semicontinuity of \( \Phi \) yields a neighborhood \( O_x \subseteq X \) such that \( \Phi(O_x) \subseteq U \). Consequently, \( f(O_x) \subseteq \Phi[O_x] \subseteq U \), witnessing that the set \( f^{-1}[U] \) is open in \( X \).

A subset \( L \subseteq Y \) is lower if for every \( a \in L \) the order-convex set \( (\leftarrow, a] = \{y \in Y : y \leq a\} \) is contained in \( L \).

**Claim 2** For any closed lower set \( L \subseteq Y \) the preimage \( f^{-1}[L] \) is closed in \( X \).

**Proof** Observe the the complement \( X \setminus L \) is an open upper set in \( Y \). By Claim 1, the preimage \( f^{-1}[X \setminus L] \) is open in \( X \) and hence its complement \( X \setminus f^{-1}[X \setminus L] = f^{-1}[L] \) is closed in \( X \).

**Claim 3** For any lower set \( L \subseteq Y \) the preimage \( f^{-1}(L) \) is of type \( F_\sigma \) in \( X \).

**Proof** If \( L \) has a largest element \( \lambda \), then \( L = (\leftarrow, \lambda] \) and \( f^{-1}[L] = f^{-1}[(\leftarrow, \lambda]] \) is closed by Claim 2. So, we assume that \( L \) does not have a largest element. Then the countable cellularity of \( Y \) implies that \( L \) has a countable cofinal subset \( C \subseteq L \) (which means that for every \( x \in L \) there exists \( y \in C \) with \( x < y \)). By Lemma 2, for every \( c \in C \) the preimage \( f^{-1}[(\leftarrow, c]] \) is closed in \( X \). Since \( L = \bigcup_{c \in C} (\leftarrow, c] \), the preimage \( f^{-1}[L] = \bigcup_{c \in C} f^{-1}[(\leftarrow, c]] \) is of type \( F_\sigma \) in \( X \).

**Claim 4** For any open order-convex subset \( U \subseteq Y \) the preimage \( f^{-1}[U] \) is a Borel subset of \( X \) (of type \( F_\sigma \) if the space \( X \) is \( F_\sigma\text{-perfect} \)).

**Proof** The order-convexity of \( U \) implies that \( U = \overleftarrow{U} \cap \overrightarrow{U} \) where \( \overleftarrow{U} = \bigcup_{u \in U}(\leftarrow, u] \) and \( \overrightarrow{U} = \bigcup_{u \in U}[u, \rightarrow) \). Taking into account that \( Y \) has a base of order-convex sets,
one can show that the upper set \( \overrightarrow{U} \) is open in \( X \). By Claim 1, the preimage \( f^{-1}[\overrightarrow{U}] \) is open in \( X \) (of type \( F_\sigma \) if the space \( X \) is \( F_\sigma \)-perfect). By Claim 3, the preimage \( f^{-1}[\overrightarrow{U}] \) is of type \( F_\sigma \) in \( X \). Then \( f^{-1}(U) = f^{-1}[\overrightarrow{U}] \cap f^{-1}[\overleftarrow{U}] \) is Borel (of type \( F_\sigma \) if \( X \) is \( F_\sigma \)-perfect).

\[ \square \]

Claim 5 For every open set \( U \subseteq Y \) the preimage \( f^{-1}(B) \) is Borel subset of \( X \) (of type \( F_\sigma \) if \( X \) is \( F_\sigma \)-perfect).

Proof By the definition of the topology of \( Y \), each point \( x \in U \) has an open order-convex neighborhood \( O_x \subseteq U \). By the Kuratowski-Zorn Lemma, each open order-convex subset of \( U \) is contained in a maximal open order convex subset of \( U \). Let \( C \subseteq B \) be the family of maximal open order-convex subsets of \( U \). Observe that \( U = \bigcup C \) and any distinct sets \( C, D \in C \) are disjoint: otherwise the union \( C \cup D \) would be an open order convex subset of \( U \) and by the maximality of \( C \) and \( D \), \( C = C \cup D = D \). Since the space \( Y \) is countably cellular, the family \( C \) is at most countable. By Claim 4, for every \( C \in \mathcal{C} \) the preimage \( f^{-1}(C) \) is Borel (of type \( F_\sigma \)-set if \( X \) is \( F_\sigma \)-perfect) and so is the countable union \( f^{-1}(U) = \bigcup_{C \in \mathcal{C}} f^{-1} \).

Claim 5 completes the proof of Lemma 2.

\[ \square \]

Lemma 3 Every usco multimap \( \Phi : X \rightarrow Y \) from a countably cellular (\( F_\sigma \)-perfect) topological space \( X \) into a GO-space \( Y \) has a Borel (\( F_\sigma \)-measurable) selection.

Proof The Kuratowski-Zorn Lemma implies that the usco map \( \Phi \) contains a minimal usco map \( \Psi : X \rightarrow Y \). We claim that the image \( \Psi[X] \subseteq Y \) is a countably cellular subspace of \( Y \). Assuming the opposite, we can find an uncountable disjoint family \( (U_\alpha)_{\alpha \in \omega_1} \) of non-empty open subsets in \( \Psi[X] \). For every \( \alpha \in \omega_1 \), find \( x_\alpha \in X \) such that \( \Phi(x_\alpha) \cap U_\alpha \neq \emptyset \). By Lemma 3.1.2 [3], the minimality of the usco map \( \Psi \) implies that \( \Psi(V_\alpha) \subseteq U_\alpha \) for some non-empty open set \( V_\alpha \subseteq X \). Taking into account that the family \( (U_\alpha)_{\alpha \in \omega_1} \) is disjoint, we conclude that the family \( (V_\alpha)_{\alpha \in \omega_1} \) is disjoint, witnessing that the space \( X \) is not countably cellular. But this contradicts our assumption. This contradiction shows that the GO-subspace \( \Psi[X] \) of \( Y \) is countably cellular. By Lemma 2, the usco map \( \Psi : X \rightarrow \Psi[X] \) has a Borel (\( F_\sigma \)-measurable) selection, which is also a selection of the usco map \( \Phi \).

\[ \square \]

Finally, let us prove one selection property of the split interval, which will be used in the proof of Lemma 8.

Lemma 4 Any selection \( s : \mathbb{I} \rightarrow \mathbb{I} \) of the multimap \( p^{-1} : \mathbb{I} \rightarrow \mathbb{I} \) is \( F_\sigma \)-measurable.

Proof Given any open subset \( U \subseteq \mathbb{I} \), we need to show that \( s^{-1}[U] \) is of type \( F_\sigma \) in \( \mathbb{I} \). For every \( x \in s^{-1}[U] \), find an open order-convex set \( I_x \subseteq U \) containing \( s(x) \). It is well-known (see e.g. [2, 3.10.C(a)]) that the split interval \( \mathbb{I} \) is hereditarily Lindelöf. Consequently, there exists a countable set \( C \subseteq s^{-1}[U] \) such that \( \bigcup_{x \in s^{-1}[U]} I_x = \bigcup_{x \in C} I_x \) and hence \( s^{-1}[U] = \bigcup_{x \in C} s^{-1}[I_x] \). For every \( x \in C \) the order-convexity of the interval \( I_x \subseteq \mathbb{I} \) implies that its preimage \( s^{-1}[I_x] \) is a convex subset of \( \mathbb{I} \), containing \( x \). Since convex subsets of \( \mathbb{I} \) are of type \( F_\sigma \), the countable union \( s^{-1}[U] = \bigcup_{x \in C} I_x \) is an \( F_\sigma \)-set in \( \mathbb{I} \).

\[ \square \]
3 Selection properties of the split square $\mathbb{I}^2$ under the negation of CH

In this section we study the selection properties of the split square $\mathbb{I}^2$ under the negation of the Continuum Hypothesis.

By $\langle x, y \rangle$ we denote the ordered pair of points $x, y$. In this way we distinguish ordered pairs from the order intervals $(x, y) := \{z : x < z < y\}$ in linearly ordered spaces.

The split interval $\mathbb{I} = \mathbb{I} \times \{0, 1\}$ carries the lexicographic order defined by $\langle x, i \rangle \leq \langle y, j \rangle$ iff either $x < y$ or $(x = y$ and $i \leq j)$. It is well-known that the topology generated by the lexicographic order on $\mathbb{I}$ is compact and Hausdorff, see \[2, 3.10.C(b)\].

By $p : \mathbb{I} \to \mathbb{I}, p : \langle x, i \rangle \mapsto x$, we denote the coordinate projection and by $P : \mathbb{I}^2 \to \mathbb{I}^2$, $P : \langle x, y \rangle \mapsto \langle p(x), p(y) \rangle$ the square of the map $p$.

The following lemma proves the implication $(1) \Rightarrow (3)$ of Theorem 4.

**Lemma 5** If $\omega_1 < c$, then the multimap $P^{-1} : \mathbb{I}^2 \to \mathbb{I}^2$ has no Borel selections.

**Proof** To derive a contradiction, assume that the multimap $P^{-1}$ has a Borel-measurable selection $s : \mathbb{I}^2 \to \mathbb{I}^2$.

For a real number $x \in \mathbb{I}$ by $x_0$ and $x_1$ we denote the points $\langle x, 0 \rangle$ and $\langle x, 1 \rangle$ of the split interval $\mathbb{I}$. Then $\mathbb{I} = \mathbb{I}_0 \cup \mathbb{I}_1$ where $\mathbb{I}_i = \{x_i : x \in \mathbb{I}\}$ for $i \in \{0, 1\}$.

For any numbers $i, j \in \{0, 1\}$ consider the set

$$Z_{ij} = \{z \in \mathbb{I}^2 : s(z) \in \mathbb{I}_i \times \mathbb{I}_j\}$$

and observe that $\mathbb{I}^2 = \bigcup_{i, j=0}^1 Z_{ij}$.

For a point $a \in \mathbb{I}$, let $[0_0, a)$ and $(a, 1_1]$ be the order intervals in $\mathbb{I}$ with respect to the lexicographic order. Observe that for any $x \in \mathbb{I}$ we have

$$p([0_0, x_0)) = [0, x), \quad p([0_0, x_1)) = [0, x],$$

$$p([x_0, 1_1]) = [x, 1], \quad p((x_1, 1_1]) = (x, 1].$$

For every $a \in (0, 2) \subseteq \mathbb{R}$ consider the lines

$$L_a = \{(x, y) \in \mathbb{R}^2 : x + y = a\} \quad \text{and} \quad \Gamma^a = \{(x, y) \in \mathbb{R}^2 : y - x = a\}$$

on the plane.

**Claim 6** For every $a \in \mathbb{R}$ the intersection $L_a \cap Z_{00}$ is at most countable.

**Proof** If for some $a \in \mathbb{R}$ the intersection $L_a \cap Z_{00}$ is uncountable, then we can choose a non-Borel subset $B \subseteq L_a \cap Z_{00}$ of cardinality $|B| = \omega_1$. For every point $\langle x, y \rangle \in B \subseteq Z_{00}$, the definition of the set $Z_{00}$ ensures that $s(\langle x, y \rangle) = \langle x_0, y_0 \rangle$ and hence the set $U_{\langle x, y \rangle} = [0_0, x_1) \times [0_0, y_1) = [0_0, x_0] \times [0_0, y_0]$ is an open neighborhood of $s(\langle x, y \rangle)$ in $\mathbb{I}^2$. Observe that $\langle x, y \rangle \in s^{-1}(U_{\langle x, y \rangle}) \subseteq p(U_{\langle x, y \rangle}) = [0, x] \times [0, y]$ and hence $L_a \cap s^{-1}(U_{\langle x, y \rangle}) = \{(x, y)\}$. Then for the open set $U = \bigcup_{\langle x, y \rangle \in B} U_{\langle x, y \rangle}$ the preimage $s^{-1}(U)$ is not Borel in $\mathbb{I}^2$ because the intersection $s^{-1}(U) \cap L_a = B$ is not Borel. But this contradicts the Borel measurability of $s$. \[ Springer\]
By analogy we can prove the following claims.

**Claim 7** For every \( a \in \mathbb{R} \) the intersection \( L_a \cap Z_{11} \) is at most countable.

**Claim 8** For every \( b \in \mathbb{R} \) the intersection \( \Gamma^b \cap (Z_{01} \cup Z_{10}) \) is at most countable.

Now fix any subset set \( \Omega \subseteq [\frac{1}{2}, \frac{3}{2}] \) of cardinality \( |\Omega| = \omega_1 \). By Claims 6, 7, for every \( a \in \Omega \) the intersection \( L_a \cap (Z_{00} \cup Z_{11}) \) is at most countable. Consequently the union

\[
U = \bigcup_{a \in \Omega} L_a \cap (Z_{00} \cup Z_{11})
\]

has cardinality \( |U| \leq \omega_1 \). Since \( |U| \leq \omega_1 < \mathfrak{c} \), there exists a real number \( b \in [\frac{1}{2}, \frac{3}{2}] \) such that the line \( \Gamma^b \) does not intersect the set \( U \). Since \( \{b\} \cup \Omega \subseteq [\frac{1}{2}, \frac{3}{2}] \) for every \( a \in \Omega \) the intersection \( \Gamma^b \cap L_a \cap \mathbb{I}^2 \) is not empty. Then the set \( X = \bigcup_{a \in \Omega} L_a \cap \Gamma^b \subseteq \mathbb{I}^2 \) is uncountable and \( X \subseteq \Gamma^b \setminus U \subseteq \Gamma^b \cap (Z_{01} \cup Z_{10}) \). But this contradicts Claim 8.

### 4 Selection properties of the split square \( \mathbb{I}^2 \) under the Continuum Hypothesis

In this section we shall prove that under the continuum hypothesis the usco multimap \( P^{-1} : \mathbb{I}^2 \to \mathbb{I}^2 \) has an \( F_\sigma \)-measurable selection.

First we introduce some terminology related to monotone functions.

A subset \( f \subseteq \mathbb{I}^2 \) is called

- a **function** if for any \( \langle x, y \rangle, \langle x', y' \rangle \in f \) the equality \( x = x' \) implies \( y = y' \);
- **strictly increasing** if for any \( \langle x, y \rangle, \langle x', y' \rangle \in f \) the strict inequality \( x < x' \) implies \( y < y' \);
- **strictly decreasing** if for any \( \langle x, y \rangle, \langle x', y' \rangle \in f \) the inequality \( x < x' \) implies \( y > y' \);
- **strictly monotone** if \( f \) is strictly increasing or strictly decreasing.

**Lemma 6** Each strictly increasing function \( f \subseteq \mathbb{I}^2 \) is a subset of a Borel strictly increasing function \( \bar{f} \subseteq \mathbb{I}^2 \).

**Proof** It follows that the strictly increasing function \( f \) is a strictly increasing bijective function between the sets \( \text{pr}_1[f] = \{ x \in \mathbb{I} : \exists y \in \mathbb{I} \ : \langle x, y \rangle \in f \} \) and \( \text{pr}_2[f] = \{ y \in \mathbb{I} : \exists x \in \mathbb{I} \ : \langle x, y \rangle \in f \} \). It is well-known that monotone functions of one real variable have at most countably many discontinuity points. Consequently, the sets of discontinuity points of the strictly monotone functions \( f \) and \( f^{-1} \) are at most countable. This allows us to find a countable set \( D_f \subseteq \mathbb{I} \) such that the set \( f \setminus D_f \) coincides with the graph of some increasing homeomorphism between subsets of \( \mathbb{I} \). Replacing \( D_f \) by a larger countable set, we can assume that \( D_f = f \cap (\text{pr}_1[D_f] \times \text{pr}_2[D_f]) \), where \( \text{pr}_1, \text{pr}_2 : \mathbb{I}^2 \to \mathbb{I} \) are coordinate projections. By the Lavrentiev Theorem [11, 3.9], the homeomorphism \( f \setminus D_f \) extends to a (strictly increasing) homeomorphism \( h \subseteq \mathbb{I}^2 \) between \( G_\delta \)-subsets of \( \mathbb{I}^2 \) such that \( f \setminus D_f \) is dense in \( h \). It is easy to check that the Borel subset \( \bar{f} = (h \setminus (\text{pr}_1[D_f] \times \text{pr}_2[D_f])) \cup D_f \) is a strictly increasing function extending \( f \). \( \square \)
By analogy we can prove

**Lemma 7** Each strictly decreasing function \( f \subset \mathbb{I}^2 \) is a subset of a Borel strictly decreasing function \( \tilde{f} \subset \mathbb{I}^2 \).

Now we are ready to prove the main result of this section.

**Lemma 8** Under \( \omega_1 = c \) the multifunction \( \mathcal{P}^{-1} : \mathbb{I}^2 \to \mathbb{I}^2 \) has an \( F_\sigma \)-measurable selection.

**Proof** Let \( \mathcal{M} \) be the set of infinite strictly monotone Borel functions \( f \subset \mathbb{I}^2 \). Since \( \omega_1 = c \), the set \( \mathcal{M} \) can be written as \( \mathcal{M} = \{ f_\alpha \}_{\alpha < \omega_1} \). It is clear \( \bigcup_{\alpha < \omega_1} f_\alpha = \mathbb{I}^2 \).

So, for any point \( z \in \mathbb{I}^2 \) we can find the smallest ordinal \( \alpha_z < \omega_1 \) such that \( z \in f_\alpha \). Consider the sets

\[
L := \{ z \in \mathbb{I}^2 : f_\alpha \text{ is strictly increasing} \} \quad \text{and} \quad \Gamma := \{ z \in \mathbb{I}^2 : f_\alpha \text{ is strictly decreasing} \} = \mathbb{I}^2 \setminus L.
\]

Define a selection \( s : \mathbb{I}^2 \to \mathbb{I}^2 \) of the multimap \( \mathcal{P}^{-1} : \mathbb{I}^2 \to \mathbb{I}^2 \) letting

\[
s(\langle x, y \rangle) = \begin{cases} 
\langle x_1, y_1 \rangle & \text{if } \langle x, y \rangle \in L, \\
\langle x_1, y_0 \rangle & \text{if } \langle x, y \rangle \in \Gamma,
\end{cases}
\]

for \( \langle x, y \rangle \in \mathbb{I}^2 \).

We claim that the function \( s : \mathbb{I}^2 \to \mathbb{I}^2 \) is \( F_\sigma \)-measurable. Given any open set \( U \subset \mathbb{I}^2 \), we should prove that its preimage \( s^{-1}[U] \) of type \( F_\sigma \) in \( \mathbb{I}^2 \). Consider the open subset \( V := U \cap (0, 1/2)^2 \subset \mathbb{I}^2 \) of \( U \). Using Lemma 4, it can be shown that the set \( s^{-1}[U \setminus V] \subset \mathbb{I}^2 \setminus (0, 1/2)^2 \) is of type \( F_\sigma \) in \( \mathbb{I}^2 \). Therefore, it remains to show that the preimage \( s^{-1}[V] \) is of type \( F_\sigma \) in \( \mathbb{I}^2 \).

Let \( \mathbb{Q} := \{ \frac{n}{m} : n, m \in \mathbb{N}, \ n < m \} \) be the set of rational numbers in the interval \((0, 1)\).

Consider the subsets \( L_V := L \cap s^{-1}(V) \) and \( \Gamma_V := \Gamma \cap s^{-1}(V) \). For every \( \langle x, y \rangle \in L_V \) we have \( s(\langle x, y \rangle) = \langle x_1, y_1 \rangle \in V \) and by the definition of the topology of the split interval, we can find rational numbers \( a(x, y), b(x, y) \in \mathbb{Q} \) such that \( x < a(x, y), y < b(x, y) \) and \( s(\langle x, y \rangle) = \langle x_1, y_1 \rangle \in \langle x_1, a(x, y) \rangle \times \langle y_1, b(x, y) \rangle \subset V \). Then

\[
[x, a(x, y)] \times [y, b(x, y)] = s^{-1}[[x_1, a(x, y)_0) \times [y_1, b(x, y)_0) \subset s^{-1}[V].
\]

On the other hand, for every \( \langle x, y \rangle \in \Gamma_V \) there are rational numbers \( a(x, y), b(x, y) \in \mathbb{Q} \) such that \( x < a(x, y), b(x, y) < y \) and \( s(\langle x, y \rangle) = \langle x_1, y_0 \rangle = \langle x_1, a(x, y) \rangle \times (b(x, y)_1, y_0) \subset V \). In this case

\[
[x, a(x, y)] \times (b(x, y), y) = s^{-1}[[x_1, a(x, y)_0) \times (b(x, y)_1, y_0) \subset s^{-1}[V].
\]

It follows that

\[
s^{-1}[V] = \left( \bigcup_{\langle x, y \rangle \in L_V} [x, a(x, y)] \times [y, b(x, y)] \right)
\]
\[
\bigcup_{(x,y) \in \Gamma_V} \left[ x, a(x, y) \right] \times \left[ b(x, y), y \right].
\]

This equality and the following claim imply that the set \( s^{-1}[V] \) is of type \( F_\sigma \) in \( \mathbb{I}^2 \).

**Claim 9** There are countable subsets \( L' \subseteq L_V \) and \( \Gamma' \subseteq \Gamma_V \) such that

\[
\bigcup_{(x,y) \in L_V} \left[ x, a(x, y) \right] \times \left[ y, b(x, y) \right] = \bigcup_{(x,y) \in L'} \left[ x, a(x, y) \right] \times \left[ y, b(x, y) \right]
\]

and

\[
\bigcup_{(x,y) \in \Gamma_V} \left[ x, a(x, y) \right] \times \left( b(x, y), y \right] = \bigcup_{(x,y) \in \Gamma'} \left[ x, a(x, y) \right] \times \left( b(x, y), y \right].
\]

We shall show how to find the countable set \( L' \subseteq L_V \). The countable set \( \Gamma' \subseteq \Gamma_V \) can be found by analogy.

For rational numbers \( r, q \in \mathbb{Q} \), consider the set

\[
L_{r,q} = \{ (x, y) \in L_V : a(x, y) = r, b(x, y) = q \}
\]

and observe that \( L_V = \bigcup_{r,q \in \mathbb{Q}} L_{r,q} \).

**Claim 10** For any rational numbers \( r, q \in \mathbb{Q} \) there exists a countable subset \( L'_{r,q} \subseteq L_{r,q} \) such that

\[
\bigcup_{(x,y) \in L'_{r,q}} [x,r] \times [y,q] = \bigcup_{(x,y) \in L_{r,q}} [x,r] \times [y,q].
\]

**Proof** For every rational numbers \( r' \leq r \) and \( q' \leq q \), consider the numbers

\[
y(r') := \inf\{ y : (x, y) \in L_{r,q}, x < r' \} \quad \text{and} \quad x(q') := \inf\{ x : (x, y) \in L_{r,q}, y < q' \}.
\]

Choose countable subsets \( L^0_{r',q}, L^0_{r,q} \subseteq L_{r,q} \) such that

\[
y(r') = \inf\{ y : (x, y) \in L^0_{r',q}, x < r' \} \quad \text{and} \quad x(q') = \inf\{ x : (x, y) \in L^0_{r,q}, y < q' \}
\]

and moreover,

\[
y(r') = \min\{ y : (x, y) \in L^0_{r',q}, x < r' \} \quad \text{if} \quad y(r') = \min\{ y : (x, y) \in L_{r,q}, x < r' \}
\]

and

\[
x(q') = \min\{ x : (x, y) \in L^0_{r,q}, y < q' \} \quad \text{if} \quad x(q') = \min\{ x : (x, y) \in L_{r,q}, y < q' \}.
\]
Consider the countable subset
\[
L''_{r, q} := \bigcup \{ L'_{r, q} \cup L_{r, 1}^{0, q'} : r', q' \in \mathbb{Q}, \ r' < r, \ q' < q \}
\]
of \( L_{r, q} \).

**Claim 11**
\[
\bigcup \{ (x, r) \times [y, q) \} \setminus \{(x, y)\} \subseteq \bigcup \{ (x, r) \times [y, q) \}.
\]

**Proof** Fix any pairs \( \langle x, y \rangle \in L_{r, q} \) and \( \langle x', y' \rangle \in \big( [x, r) \times [y, q) \big) \setminus \{(x, y)\} \). Three cases are possible:
1. \( x < x' < r \) and \( y < y' < q \);
2. \( x = x' \) and \( y < y' < q \);
3. \( x < x' < r \) and \( y = y' \).

In the first case there exist rational numbers \( r', q' \) such that \( x < r' < x' < r \) and \( y < q' < y' < q \). The definition of \( \bar{x}(q') \) ensures that \( \bar{x}(q') \leq x < x' \). By the choice of the family \( L_{r, q}^{0, q} \), there exists \( \langle x'', y'' \rangle \in L_{r, q}^{0, q} \subseteq L''_{r, q} \) such that \( x'' < x' < r \) and \( y'' < q' < y' < q \). Then \( \langle x', y' \rangle \in [x'', r) \times [y'', q) \).

Next, assume that \( x = x' \) and \( y < y' < q \). In this case we can choose a rational number \( q' \) such that \( y < q' < y' \). It follows that \( \bar{x}(q') \leq x = x' \). If \( \bar{x}(q') < x' \), then by the definition of the family \( L_{r, q}^{0, q} \), there exists \( \langle x'', y'' \rangle \in L_{r, q}^{0, q} \subseteq L_{r, q} \) such that \( x'' < x' < r \) and \( y'' < q' < y' < q \). Then \( \langle x', y' \rangle \in [x'', r) \times [y'', q) \).

So, we assume that \( \bar{x}(q') = x' = x \) and hence \( \bar{x}(q') = x = \min\{x' : \langle x'', y'' \rangle \in L_{r, q} : y'' < q' \} \). In this case \( x' = \bar{x}(q') = x'' \) for some \( \langle x'', y'' \rangle \in L_{r, q}^{0, q} \subseteq L_{r, q}^{0, q} \) with \( y'' < q' < y' < q \). Then \( \langle x', y' \rangle \in [x'', r) \times [y'', q) \).

By analogy, in the third case \( x < x' < r \) and \( y = y' \) we can find a pair \( \langle x'', y'' \rangle \in L_{r, q}^{0, q} \) such that \( \langle x', y' \rangle \in [x'', r) \times [y'', q) \).

Claim 11 implies that the set
\[
D_{r, q} = \left( \bigcup \{ (x, r) \times [y, q) \} \right) \setminus \left( \bigcup \{ (x, r) \times [y, q) \} \right)
\]
is contained in \( L_{r, q} \).

**Claim 12** The set \( D_{r, q} \) is a strictly decreasing function.

**Proof** First we show that \( D_{r, q} \) is a function. Assuming that \( D_{r, q} \) is not a function, we can find two pairs \( \langle x, y \rangle, \langle x, y' \rangle \in D_{r, q} \) with \( y < y' \). Applying Claim 11, we conclude that
\[
\langle x, y' \rangle \in \big( [x, r) \times [y, q) \big) \setminus \{(x, y)\} \subseteq \bigcup \{ [x'', r) \times [y'', q) \}
\]
and hence \( \langle x, y' \rangle \notin D_{r, q} \), which contradicts the choice of the pair \( \langle x, y' \rangle \). This contradiction shows that \( D_{r, q} \) is a function.
Assuming that \( D_{r,q} \) is not strictly decreasing, we can find pairs \( \langle x, y \rangle, \langle x', y' \rangle \in D_{r,q} \) such that \( x < x' \) and \( y \leq y' \). Applying Claim 11, we conclude that

\[
\langle x', y' \rangle \in \big( [x, r) \times [y, q) \big) \setminus \{ \langle x, y \rangle \} \subseteq \bigcup_{\langle x'', y'' \rangle \in L''_{r,q}} [x'', r) \times [y'', q)
\]

and hence \( \langle x', y' \rangle \notin D_{r,q} \), which contradicts the choice of the pair \( \langle x, y \rangle \). This contradiction shows that \( D_{r,q} \) is strictly decreasing. \( \square \)

**Claim 13** The set \( D_{r,q} \) is at most countable.

**Proof** To derive a contradiction, assume that \( D_{r,q} \) is uncountable. By Lemma 7, the strictly decreasing function \( D_{r,q} \) is contained in some Borel strictly decreasing function, which is equal to \( f_\alpha \) for some ordinal \( \alpha < \omega_1 \). Since the intersection of a strictly increasing function and a strictly decreasing function contains at most one point, the set

\[
D'_{r,q} = \bigcup \{ D_{r,q} \cap f_\beta : \beta \leq \alpha, \ f_\beta \text{ is strictly increasing} \}
\]

is at most countable. We claim that \( D_{r,q} = D'_{r,q} \). To derive a contradiction, assume that \( D_{r,q} \setminus D'_{r,q} \) contains some pair \( z = \langle x, y \rangle \). It follows from \( z \in D_{r,q} \subseteq f_\alpha \) that \( \alpha z \leq \alpha \). Since \( z \notin D'_{r,q} \), the strictly monotone function \( f_\alpha \) is not strictly increasing and hence \( f_\alpha_z \) is strictly decreasing. Then the definition of the set \( L \) guarantees that \( z \notin L \), which contradicts the inclusion \( z \in D_{r,q} \subseteq L_{r,q} \subseteq L \). \( \square \)

Now consider the countable subset \( L'_{r,q} := L''_{r,q} \cup D_{r,q} \) of \( L_{r,q} \) and observe that

\[
\bigcup_{\langle x,y \rangle \in L_{r,q}} ( [x, r) \times [y, q) ) \subseteq \bigcup_{\langle x,y \rangle \in L'_{r,q}} ( [x, r) \times [y, q) )
\]

This completes the proof of Claim 10.

**Claim 14** There exists a countable subset \( L' \subseteq L_V \) such that

\[
\bigcup_{\langle x,y \rangle \in L'} ( [x, a(x, y)) \times [y, b(x, y)) = \bigcup_{\langle x,y \rangle \in L_V} ( [x, a(x, y)) \times [y, b(x, y))
\]

**Proof** By Claim 10, for any rational numbers \( r, q \in \mathbb{Q} \) there exists a countable subset \( L'_{r,q} \subseteq L_{r,q} \) such that

\[
\bigcup_{\langle x,y \rangle \in L'_{r,q}} ( [x, a(x, y)) \times [y, b(a, y)) ) = \bigcup_{\langle x,y \rangle \in L'_{r,q}} ( [x, r) \times [y, q) ) =
\]

\[
= \bigcup_{\langle x,y \rangle \in L_{r,q}} ( [x, r) \times [y, q) ) = \bigcup_{\langle x,y \rangle \in L_{r,q}} ( [x, a(x, y)) \times [y, b(x, y)) )
\]

Since \( L_V = \bigcup_{r,q \in \mathbb{Q}} L_{r,q} \), the countable set \( L' := \bigcup_{r,q \in \mathbb{Q}} L'_{r,q} \) has the required property. \( \square \)
By analogy with Claim 14 we can prove

**Claim 15**  *There exists a countable subset* \( \Gamma' \subseteq \Gamma_V \) *such that*

\[
\bigcup_{(x,y) \in \Gamma'} \left( [x, a(x, y)) \times (b(x, y), y] \right) = \bigcup_{(x,y) \in \Gamma_V} \left( [x, a(x, y)) \times (b(x, y), y] \right).
\]

Claims 14 and 15 complete the proof of Claim 9 and the proof of Lemma 8.

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