Mixed-integer quadratic programming is in NP

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Abstract Mixed-integer quadratic programming is the problem of optimizing a quadratic function over points in a polyhedral set where some of the components are restricted to be integral. In this paper, we prove that the decision version of mixed-integer quadratic programming is in NP, thereby showing that it is NP-complete.

This is established by showing that if the decision version of mixed-integer quadratic programming is feasible, then there exists a solution of polynomial size. This result generalizes and unifies classical results that quadratic programming is in NP (Vavasis in Inf Process Lett 36(2):73–77 [17]) and integer linear programming is in NP (Borosh and Treybig in Proc Am Math Soc 55:299–304 [1], von zur Gathen and Sieveking in Proc Am Math Soc 72:155–158 [18], Kannan and Monma in Lecture Notes in Economics and Mathematical Systems, vol. 157, pp. 161–172. Springer [9], Papadimitriou in J Assoc Comput Mach 28:765–768 [15]).

Keywords Quadratic programming · Integer programming · Complexity

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1 Introduction

Mixed-integer quadratic programming (MIQP) is the problem of optimizing a quadratic function over points in a polyhedral set that have some components integer, and others continuous. More formally, a MIQP problem is an optimization problem of the form:

\[
\begin{align*}
\min & \quad x^\top H x + c^\top x \\
\text{s.t.} & \quad x \in \mathcal{P} \\
& \quad x \in \mathbb{Z}^p \times \mathbb{R}^q,
\end{align*}
\]

where \( H \in \mathbb{Q}^{n \times n} \) and is symmetric, \( c \in \mathbb{Q}^n \), \( \mathcal{P} \) is a polyhedron \( \mathcal{P} = \{ x : Ax \leq b \} \), \( A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m \), and \( n = p + q \). The decision version of this problem is: Does there exist a feasible solution to \( \mathcal{F}(H, c, d, \mathcal{P}) \) where \( \mathcal{F}(H, c, d, \mathcal{P}) \) is the set of \( x \) satisfying

\[
\begin{align*}
& x^\top H x + c^\top x + d \leq 0 \\
& x \in \mathcal{P} \\
& x \in \mathbb{Z}^p \times \mathbb{R}^q.
\end{align*}
\]

The special case of MIQP when all variables are required to be integral \( (q = 0) \) is called integer quadratic programming (IQP). It is well known that IQP is NP-hard. We show that the decision versions of IQP and MIQP lie in NP and are therefore NP-complete. This result generalizes and unifies classical results that quadratic programming is in NP [17] and integer linear programming is in NP [1,9,15,18]. It is unknown if any type of higher degree polynomial programming problems lie in NP.

Recently, Del Pia and Weismantel [4] showed that IQP can be solved in polynomial time when \( n = 2 \). It is a major open question whether IQP can be solved in polynomial-time for fixed dimension.

1.1 Statement of result and discussion

Given a set \( S \subseteq \mathbb{R}^n \), conv\( (S) \) denotes the convex hull of \( S \), and cone\( (S) \) denotes the conic hull of \( S \). Given a rational scalar/vector/matrix, its complexity is the bit-size of its smallest binary encoding. The complexity of a linear inequality \( ax \leq \beta \) or equation \( ax = \beta \) is the sum of the complexity of \( a \) and of \( \beta \). The facet complexity of a rational polyhedron \( \mathcal{P} \subseteq \mathbb{R}^n \) is the smallest number \( \varphi \) such that: \( \varphi \geq n \) and there exists a system \( Ax \leq b \) of rational linear inequalities defining \( \mathcal{P} \), where each inequality in \( Ax \leq b \) has size at most \( \varphi \). The vertex complexity of \( \mathcal{P} \) is the smallest number \( \nu \) such that: \( \nu \geq n \) and there exist rational vectors \( v^1, \ldots, v^\ell, r^1, \ldots, r^m \) with \( \mathcal{P} = \text{conv}\{v^1, \ldots, v^\ell\} + \text{cone}\{r^1, \ldots, r^m\} \), where each of \( v^1, \ldots, v^\ell, r^1, \ldots, r^m \) has size at most \( \nu \). It is well-known that facet complexity \( \varphi \) and vertex complexity \( \nu \) of a rational polyhedron \( \mathcal{P} \) satisfy \( \nu \leq 4n^2\varphi \) and \( \varphi \leq 4n^2\nu \). (See [16] for more details.)

In this work, the complexity of a rational polyhedron \( \mathcal{P} \) is the maximum between its
Mixed-integer quadratic programming is in NP facet complexity and its vertex complexity. The complexity of a finite set of objects is the sum over the complexity of the constituent objects. We will prove the following result.

**Theorem 1** Let $n$, $p$ be positive integers. Let $P$ be a rational polyhedron, let $H \in \mathbb{Q}^{n \times n}$, $c \in \mathbb{Q}^n$, $d \in \mathbb{Q}$, and let $\phi$ be the complexity of $\{H, c, d, P\}$. If $\mathcal{F}(H, c, d, P)$ is non-empty, then there exists $x^0 \in \mathcal{F}(H, c, d, P)$ such that the complexity of $x^0$ is bounded from above by $f(\phi)$ where $f$ is a polynomial function.

Theorem 1 directly implies the following.

**Corollary 1** The decision versions of IQP and MIQP are NP-complete.

**Proof** Given a graph $G = (V, E)$ and an integer $k$, determining whether there is a cut of cardinality at least $k$ in $G$ is NP-complete [5,10]. It is well known that such problem can be written as the decision IQP problem

$$\sum_{v_i,v_j \in E} (x_i + x_j - 2x_ix_j) \geq k$$

$$x_i \in \{0, 1\}^n \quad \forall v_i \in V.$$ 

Hence any problem in NP can be polynomially transformed to a decision IQP. Theorem 1 proves that there is a polynomial-length certificate for yes-instances of decision MIQP, showing that decision IQP and MIQP are in NP. \hfill \Box

We end this section by contrasting the result of Theorem 1 with several well-known negative results when one considers a more general version of decision IQP by varying the number of quadratic inequalities.

1. ‘Many’ general quadratic inequalities: By using a simple reduction from the problem of determining the feasibility of a quartic equation in 58 non-negative integer variables, we obtain that determining the feasibility of a system with $2 \left(\binom{58}{2} + 58 + 1\right)$ quadratic inequalities and 58 linear inequalities in $\left(\binom{58}{2} + 58\right)$ continuous variables and 58 integer variables is undecidable (see Theorem 3.2 and Theorem 3.3(i) in [12], also see [8]). Therefore already with 3424 quadratic inequalities, 58 linear inequalities, 58 integer variables, 1711 continuous variables, it is not possible to bound the size of smallest feasible solution.

2. Two general quadratic inequalities: In the presence of two quadratic inequalities (in fact one quadratic equation) in two variables, there exist examples (the so called Pellian and anti-Pellian equations) where the minimal binary encoding length of any feasible integral solution is exponential in the complexity of the instance [13].

3. ‘Many’ convex quadratic inequalities: Consider the following system of inequalities [11]:

$$x_1 \geq 2$$

$$x_i \geq x_{i-1}^2 \quad \forall i \in \{2, \ldots, n\}$$

$$x \in \mathbb{Z}^n.$$
It is clear that for this system, the minimal binary encoding length of any feasible integral solution is exponential in the complexity of the instance.

The above examples serve to highlight the fact that the result of Theorem 1 (a feasible system of inequalities with exactly one quadratic inequality always has at least one integer feasible solution of small size) is tight with respect to the number of quadratic inequalities.

The rest of the paper is organized as follows. Section 2 collects all notation and results that are needed to prove Theorem 1. Section 3 presents a proof of Theorem 1. Finally, in Sect. 4 we present some algorithmic consequences of Theorem 1.

1.2 High-level idea of the proof

To obtain an intuition about the proof it is useful to focus on the pure-integer setting where \( q = 0 \) in (2). In this case, it suffices to prove there is a feasible solution for (2) of polynomially (with respect to input complexity) small length. If there is a (low-complexity) direction \( \tilde{r} \) in the recession cone of \( \mathcal{P} \) where the higher-order term \( \tilde{r}^\top H \tilde{r} \) is negative, then we can start with a low-complexity solution \( \tilde{x} \) for the continuous relaxation of \( \mathcal{P} \) and show that \( \tilde{x} + \lambda \tilde{r} \) is an integral solution of \( \mathcal{P} \) for a not-too-large scaling \( \lambda \), and we are done.

If all directions in the recession cone of \( \mathcal{P} \) have non-negative higher-order term value, then the situation is more complicated. If all these directions had strictly positive higher-order term value, then we could use this fact to argue that all solutions for (2) must have a polynomially bounded length, since they have to satisfy \( x^\top H x + c^\top x + d \leq 0 \) (see Lemma 4). The problem occurs in directions \( r \) with zero higher-order term value \( r^\top H r = 0 \); in, and around, these directions we cannot provide an effective upper bound on the solutions length. We then need to carefully “remove” from \( \mathcal{P} \) these directions making sure we do not eliminate feasible solutions of small length (using the decompositions from Proposition 1, Lemma 2, and Proposition 2). Based on their lower-order value, we either ignore these direction or use them to find a feasible solution, as in the previous paragraph.

2 Preliminaries

2.1 Notation

Throughout this paper, we use \( e^i \) to represent the \( i \)th vector of the standard basis of \( \mathbb{R}^n \), \( \text{sign}(u) \) to represent the sign of a real number \( u \). Given a set \( S \subseteq \mathbb{R}^n \), \( \dim(S) \) denotes the affine dimension of \( S \), \( \text{rec}(S) \) denotes the recession cone of \( S \), and \( \text{int.cone}(S) = \{ \sum_{r \in S} \lambda_j r^j : \lambda_j \in \mathbb{Z}_+ \forall j \} \). Moreover, we denote by \( S|_p \) the projection of \( S \) to the first \( p \) coordinates. Given a vector \( x \in \mathbb{R}^n \), we write \( x|_p \) instead of \( \{x\}|_p \).

Given an object \( \mathcal{O} \) and another object \( f(\mathcal{O}) \) that is a function of it, we say that \( f(\mathcal{O}) \) has \( \mathcal{O} \) if the complexity of \( f(\mathcal{O}) \) is at most a polynomial function of the complexity of \( \mathcal{O} \) (or more precisely, there is a polynomial \( p \) such that for every input object \( \mathcal{O} \), the complexity of \( f(\mathcal{O}) \) is at most \( p(\text{complexity}(\mathcal{O})) \)). Furthermore, given a set \( \mathcal{F} \)
as in (2), we say that a set has $F$-small complexity if it has $\{H, c, d, P\}$. We note here that, using the definition of complexity of a rational polyhedron, it is straightforward to verify that if two rational polyhedra are $\{H, c, d, P\}$ then their intersection and well as Minkowski sum are also $\{H, c, d, P\}$.

### 2.2 Quadratic programming is in NP

Quadratic programming (QP) is the special case of MIQP when all variables are continuous ($p = 0$). Vavasis [17] proved that the decision version of QP is in NP. The following theorem is slightly stronger than the one presented by Vavasis, but follows directly from his proof.

**Theorem 2** The feasibility problem over the continuous relaxation of (2) is in NP. Moreover, suppose that the continuous relaxation of (1) has a global optimal solution. Then there exists a system of rational linear equations of $\{H, c, d, P\}$-small complexity whose unique solution is one of the global optimal solutions.

### 2.3 Mixed-integer linear programming is in NP

We will need the following generalization of a classical result that can be used to prove that the decision version of mixed integer linear programming (MIP) is in NP. We say that a pointed polyhedral cone $C \subseteq \mathbb{R}^p$ is a simple cone if the number of extreme rays is equal to the dimension of the cone.

**Proposition 1** Let $P = \{x : Ax \leq b\} \subseteq \mathbb{R}^{p+q}$ be a rational pointed polyhedron. Then there is a polytope $Q$ and a finite family $\{R_K\}_{K \in \mathcal{K}}$ of subsets of extreme rays of $P$ with the following properties:

1. $P = Q + \bigcup_{K \in \mathcal{K}} \text{cone}(R_K) = \bigcup_{K \in \mathcal{K}} (Q + \text{cone}(R_K))$;
2. The polytope $Q$ and each vector in $R_K$, for each $K \in \mathcal{K}$, have $P$-small complexity;
3. For each $K \in \mathcal{K}$, all vectors in $R_K$ are integral;
4. Each cone $\text{cone}(R_K)$ is simple.

**Proof** Assume $P$ is non-empty, otherwise there is nothing to prove. By standard polyhedral theory, there is a set of vectors $\{v^1, \ldots, v^\ell\}$ (the vertices of $P$) and a set of integral vectors $\{r^1, \ldots, r^m\}$ (a scaling of the the extreme rays of $P$) such that $P = \text{conv}\{v^1, \ldots, v^\ell\} + \text{cone}\{r^1, \ldots, r^m\}$ and the $v^i$'s and $r^j$'s have $P$-small complexity.

Define $Q = \text{conv}\{v^1, \ldots, v^\ell\}$ and for a subset $K \subseteq \{1, \ldots, m\}$, let $R_K = \{r^j : j \in K\}$ be the set of extreme rays indexed by $K$. Let $\mathcal{K}$ be the set of $K$’s such that the cone $\text{cone}(R_K)$ is simple. Properties 2, 3, and 4 of the proposition follow directly by construction.

It follows from Caratheodory’s Theorem that each vector in $\text{cone}\{r^1, \ldots, r^m\}$ belongs to $\text{cone}(R_K)$ for some $K \in \mathcal{K}$, and thus $P = Q + \bigcup_{K \in \mathcal{K}} \text{cone}(R_K)$. Using distributivity of union over Minkowski sums then gives Property 1, thus concluding the proof.  

\[\square\]
Now we refine this decomposition to obtain the main result of this section.

**Proposition 2** Let \( P = \{ x : Ax \leq b \} \subseteq \mathbb{R}^{p+q} \) be a rational pointed polyhedron. Then there is a finite family \( \{ P_i \}_i \) of polytopes, and a finite family \( \{ R_K \}_{K \in \mathcal{K}} \) of subsets of extreme rays of \( P \) with the following properties:

1. \( P \cap (\mathbb{Z}^p \times \mathbb{R}^q) = \bigcup_{i, K \in \mathcal{K}} (P_i + \text{int.cone}(R_K)); \)
2. Each polytope \( P_i \) and each vector in \( R_K \) has \( P \)-small complexity;
3. For each \( K \in \mathcal{K} \), all vectors in \( R_K \) are integral;
4. Each cone \( \text{cone}(R_K) \) is simple.

**Proof** Consider the decomposition of \( P \) given by Proposition 1, and define the set \( B = \bigcup_{K \in \mathcal{K}} B^K \) where

\[
B^K = \left\{ x \in \mathbb{Z}^p \times \mathbb{R}^q : x = v + \sum_{j \in K} \mu_j r^j, \ v \in Q, \ \mu_j \in [0, 1] \ \forall j \right\}.
\]

Notice that \( B \) is a union of the polytopes given by the fibers \( \{ \tilde{y} \} \times \{ z \in \mathbb{R}^q : (\tilde{y}, z) \in B^K \} \) ranging over all \( K \in \mathcal{K} \) and \( \tilde{y} \in B^K \cap \mathbb{Z}^p \subseteq \mathbb{Z}^p \). Using the fact that \( Q \) and the \( r^j \)'s have \( P \)-small complexity and \( |K| \leq p + q \), we get that all points in \( B^K \cap \mathbb{Z}^p \) have \( P \)-small complexity. Hence each of these fibers also has \( P \)-small complexity, since it is the intersection of two \( P \)-small complexity polyhedra \( \{ x \in \mathbb{R}^p \times \mathbb{R}^q : x = v + \sum_{j \in K} \mu_j r^j, \ v \in Q, \ \mu_j \in [0, 1] \ \forall j \} \) and \( \{ x \in \mathbb{R}^p \times \mathbb{R}^q : x \cap \mathbb{Z}^p = \tilde{y} \} \). Let \( \{ P_i \}_i \) be this collection of fibers.

By construction, Properties 2, 3 and 4 of the proposition are satisfied, so it suffices to show Property 1. By using distributivity of union over Minkowski sums, notice \( \bigcup_{i, K \in \mathcal{K}} (P_i + \text{int.cone}(R_K)) = B + \bigcup_{K \in \mathcal{K}} \text{int.cone}(R_K) \), so we prove that \( B + \bigcup_{K \in \mathcal{K}} \text{int.cone}(R_K) \) equals \( P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \).

To show \( P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \subseteq B + \bigcup_{K \in \mathcal{K}} \text{int.cone}(R_K) \), take a point \( x \in P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \). We can write it as \( x = v + r \) for \( v \in Q \) and \( r \in \text{cone}(R_K) \) for some \( K \in \mathcal{K} \).

Then consider multipliers \( \mu_j \in \mathbb{R}^+ \) for \( j \in K \) such that \( r = \sum_{j \in K} \mu_j r^j \). Breaking up the multipliers into their fractional and integer parts, we get that

\[
x = v + \sum_{j \in K} (\mu_j - [\mu_j]) r^j + \sum_{j \in K} [\mu_j] r^j.
\]

Clearly the last term belongs to \( \text{int.cone}(R_K) \). Moreover, this term is integer (since the \( r^j \)'s are integer) and \( x \in \mathbb{Z}^p \times \mathbb{R}^q \), thus the remaining part \( v + \sum_{j \in K} (\mu_j - [\mu_j]) r^j = x - \sum_{j \in K} [\mu_j] r^j \) belongs to \( \mathbb{Z}^p \times \mathbb{R}^q \) and hence to \( B \). Thus \( x \in B + \text{int.cone}(R_K) \), concluding this part of the proof.

We now show the reverse direction \( P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \supseteq B + \bigcup_{K \in \mathcal{K}} \text{int.cone}(R_K) \). It is easy to see that \( P \supseteq B + \bigcup_{K \in \mathcal{K}} \text{int.cone}(R_K) \), since \( P = Q + \bigcup_{K \in \mathcal{K}} \text{cone}(R_K) \supseteq B \) and \( \text{int.cone}(R_K) \subseteq \text{cone}(R_K) \). Moreover, \( B \subseteq \mathbb{Z}^p \times \mathbb{R}^q \) and \( \text{int.cone}(R_K) \subseteq \mathbb{Z}^{p+q} \) (again since the \( r^j \)'s are integral), and hence \( B + \bigcup_{K \in \mathcal{K}} \text{int.cone}(R_K) \subseteq \mathbb{Z}^p \times \mathbb{R}^q \). This concludes the proof. \( \square \)
One way of interpreting this decomposition is the following: Notice that each set \( \text{int.cone}(R_K) \), for \( K \in \mathcal{K} \), is linearly isomorphic to \( \mathbb{Z}^{|K|}_+ \); this proposition then asserts that we can decompose any mixed-integer linear set into (overlapping) sets that are affinely isomorphic to some \( \mathbb{Z}^n_+ \) (we have one such set for each point in \( \bigcup_i P_i \) and each \( K \in \mathcal{K} \)).

Also, notice that this proposition proves that the decision version of MIP is in NP. Note that for the pure integer case this is well-known (see [16]).

**Proposition 3** The decision version of MIP is in NP.

Indeed, consider the decision version of a MIP: Given a rational polyhedron \( P \subseteq \mathbb{R}^{p+q} \), does there exist a point in \( P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \)? If the answer to this decision problem is yes, then one can present a vertex of one of the polytopes \( P_i \) as a certificate.

### 3 Proof of main result

We begin this section with some technical lemmas.

**Lemma 1** (Normalizing hyperplane) Let \( C = \text{cone}\{r^1, \ldots, r^s\} \subseteq \mathbb{R}^n \) be a pointed rational cone. Then there exists a hyperplane \( H = \{ x : f^T x = 1 \} \) such that:

1. The complexity of \( f \) is polynomially bounded by the maximum complexity of \( r^i \), for \( i = 1, \ldots, s \);  
2. If \( x \in C \) and \( \|x\| = 1 \), then \( f^T x \geq \frac{1}{R} \) where \( R = \max_{i \in \{1, \ldots, s\}} \|r^i\| \); 
3. \( C \cap H \) is bounded.

**Proof** Let \( \{r^{s+1}, \ldots, r^t\} \) be a minimal subset of \( \{e^1, \ldots, e^n\} \) with the property that the cone \( C' = \text{cone}\{r^1, \ldots, r^s, r^{s+1}, \ldots, r^t\} \) is full-dimensional. Clearly \( |\{r^{s+1}, \ldots, r^t\}| = n - d \), where \( d \) is the dimension of \( C \), and \( C' \) is pointed. Let \( f \) be an extreme point of the following polyhedron: \( \{w : w^T r^i \geq 1 \ \forall i \in \{1, \ldots, t\} \} \) (an extreme point exists since the rank of the matrix defining the polyhedron is \( n \)). Then \( f \) satisfies the first and third criteria. Suppose \( \hat{x} \in C \) and \( \|\hat{x}\| = 1 \). There exists \( 0 < \mu \leq 1 \) such that \( \mu \hat{x} \) belongs to the polytope defined as the convex hull of \( \{ \frac{r^i}{\|r^i\|} : i = 1, \ldots, s \} \) (since the norm of any vector in this polytope is at most 1). Thus there exist nonnegative \( \lambda_i, \ i = 1, \ldots, s, \lambda_i = 1 \) such that \( f^T \hat{x} \geq f^T (\mu \hat{x}) = \sum_{i=1}^s \lambda_i \frac{1}{\|r^i\|} f^T r^i \geq \frac{1}{R} \). \( \square \)

Sometimes we will apply Lemma 1 to a pointed cone \( C \), without giving explicitly the set of rays \( \{r^1, \ldots, r^s\} \). By definition of complexity, there exist vectors \( r^1, \ldots, r^s \), each of \( C \)-small complexity, such that \( C = \text{cone}\{r^1, \ldots, r^s\} \). Hence, in this case, Lemma 1 implies that there exists a normalizing hyperplane \( H = \{ x : f^T x = 1 \} \) such that:

1. \( f \) has \( C \)-small complexity;
2. For every nonzero \( x \in C \), there exists \( \mu > 0 \) such that \( \mu x \in H \).

The following lemma outlines a crucial decomposition strategy for searching integer feasible points.
Lemma 2 Let $C$ be a simple, rational, pointed, polyhedral cone such that $x^\top H x \geq 0$ for every $x \in C$. Let $\mathcal{H} = \{x : f^\top x = 1\}$ be the normalizing hyperplane from Lemma 1. Then there exists a finite family of simple, rational, pointed, polyhedral cones $C_i$, $i \in I$ such that:

(a) $\bigcup_{i \in I} C_i = C$, and the dimension of each $C_i$ is equal to the dimension of $C$,
(b) For every $i \in I$, if a face $C'$ of $C_i$ satisfies $\min\{x^\top H x : x \in C' \cap \mathcal{H}\} = 0$, then there exists an extreme ray $v$ of $C'$ with $v^\top H v = 0$,
(c) For every $i \in I$, $C_i$ has $\{H,C\}$-small complexity.

Proof The proof is by induction on the dimension $d$ of the cone. If the cone has dimension one, then the claim is trivially true.

Since $C \cap \mathcal{H}$ is a polytope, by Theorem 2 there exists an optimal solution $\tilde{x}$ of the problem $\min\{x^\top H x : x \in C \cap \mathcal{H}\}$ that has $\{H, C, \mathcal{H}\}$-small complexity. As $\mathcal{H}$ has $C$-small complexity, $\tilde{x}$ has $\{H, C\}$-small complexity. If the minimum value is strictly positive then the trivial family of cones $\{C\}$ satisfies the desired properties. So we now assume that the minimum value is zero.

Let $\{F^j\}_{j \in J}$ be the faces of $C$ that do not contain $\tilde{x}$. By induction, for every $j \in J$, there exist finitely many simple cones (of dimension $d - 1$, and with $d - 1$ extreme rays) $\{C^j_i\}_{i \in I(j)}$ that satisfy (a) and (b) with respect to the $d - 1$ dimensional cone $F^j$. We show that the family of cones $\tilde{C}^j_i := \text{cone}(C^j_i \cup \{\tilde{x}\})$, for $j \in J$ and $i \in I(j)$, satisfies the desired properties. First note that for each $j$, $\tilde{x}$ is affinely independent from all the vectors in $F_j$, thus each $\tilde{C}^j_i$ is a simple, rational, pointed, polyhedral cone.

To prove Property (b), by the induction hypothesis this holds for all the faces of the $C^j_i$'s. But then this property also holds for all faces of the cones $\{\tilde{C}^j_i\}_{j,i \in I(j)}$, since each of them is either a face of some $C^j_i$, or it contains $\tilde{x}$ as an extreme ray.

Condition (a) is also intuitively clear: since $F^j = \bigcup_{i \in I(j)} C^j_i$ for each $j$, the collection $\{\tilde{C}^j_i\}_{i,j}$ is performing a triangulation of the cone $C$. More formally, it is clear that $\bigcup_{j,i \in I(j)} \tilde{C}^j_i \subseteq C$, so we prove the other containment. So consider $y \in C$. If $y$ is a scaling of $\tilde{x}$, it belongs to all the $\tilde{C}^j_i$'s and we are done. Otherwise let $\lambda y$ (for $\lambda > 0$) be such that $\lambda y \in \mathcal{H}$, and since $\tilde{x} \in \mathcal{H}$ our assumption implies $\lambda y \neq \tilde{x}$. Since $\bigcup_{j,i \in I(j)} \tilde{C}^j_i$ is a cone, it suffice to show $\lambda y$ belongs to it. For that, consider the unique line $L$ passing through the points $\tilde{x}$ and $\lambda y$. Since both $\tilde{x}$ and $\lambda y$ belong to the polytope $C \cap \mathcal{H}$, this line intersects a face of $C \cap \mathcal{H}$ (and hence of $C$) that does not contain $\tilde{x}$; let $C'$ be this face of $C$ and $z = L \cap C'$ be the intersection point. Since the $C^j_i$'s cover the facets of $C$ that do not contain $\tilde{x}$, let $C^j_i$ be a cone containing $z$. We then have that $\lambda y \in \text{conv}(\{z\} \cup \{\tilde{x}\}) \subseteq \text{cone}(C^j_i \cup \{\tilde{x}\}) = \tilde{C}^j_i$. This concludes this part of the proof.

The above proof of (a) and (b) can be seen as a constructive algorithm that recursively constructs the simple cones $\tilde{C}^j_i$'s. In order to show that (c) holds, we just need to prove that all the $d$ extreme rays of the cones constructed in such way have $\{H, C\}$-small complexity. Note that such extreme rays are either extreme rays of $C$, which have $C$-small complexity, or optimal solutions of a problem $\min\{x^\top H x : x \in C' \cap \mathcal{H}\}$, for a face $C'$ of $C$, which have $\{H, C\}$-small complexity. This concludes the proof. $\square$
Now we are ready to present a proof of our main result.

**Proof (Proof of Theorem 1)** Consider a feasible set $\mathcal{F} = \mathcal{F}(H, c, d, P)$ and let $Q(x) := x^T H x + c^T x + d$ denote the quadratic form, so $\mathcal{F} = P \cap \{x \in \mathbb{Z}^p \times \mathbb{R}^q : Q(x) \leq 0\}$.

Without loss of generality, we assume that the polyhedron $\mathcal{P}$ is pointed: If not, consider the partition of the feasible region into $2^n$ pieces as $x \in \mathcal{F}$

$$
\begin{align*}
  x_i &\geq 0 \quad i \in S \subseteq \{1, \ldots, n\} \\
  x_i &\leq 0 \quad i \in \{1, \ldots, n\} \setminus S,
\end{align*}
$$

for every $S \subseteq \{1, \ldots, n\}$; note that the complexity of the additional constraints is $O(n)$ and therefore each part in this partition has $\mathcal{F}$-small complexity, so we can restrict to a non-empty part.

If $\mathcal{P}$ is bounded, then for every $x \in \mathcal{F}$, the integral vector $x|_p$ has $\mathcal{F}$-small complexity. Therefore there exists a vector $\tilde{y} \in \mathbb{Z}^p$ of $\mathcal{F}$-small complexity such that the set \{ $x : x \in \mathcal{F}, x|_p = \tilde{y}$ \} is nonempty. In this case the result then follows from Theorem 2. Thus from now on we assume that $\mathcal{P}$ is unbounded.

Notice that an external description of $\text{rec}(\mathcal{P})$ can be obtained from an external description of $\mathcal{P}$ by replacing all the right-hand sides with a zero, hence $\text{rec}(\mathcal{P})$ has $\mathcal{F}$-small complexity. Using our assumption that $\text{rec}(\mathcal{P})$ is pointed, let $H := \{x : f^T x = 1\}$ be the normalizing hyperplane from Lemma 1 for $\text{rec}(\mathcal{P})$. Examine the optimization problem:

$$
\min \ r^T H r \\
\text{s.t.} \ r \in \text{rec}(\mathcal{P}) \cap \mathcal{H}.
$$

Since $\text{rec}(\mathcal{P}) \cap \mathcal{H}$ is compact, there exists a global optimal value. We break up into two cases depending on the sign of the optimal value.

**Case 1:** The optimum for (3) is strictly negative. We construct a feasible solution of $\mathcal{F}$ as follows. Since $\text{rec}(\mathcal{P})$ and $H$ have $\mathcal{F}$-small complexity, Theorem 2 asserts that there is an optimal solution $r^*$ for (3) which has $\mathcal{F}$-small complexity. Then let $\tilde{r}$ be an integer vector with $\mathcal{F}$-small complexity obtained by scaling $r^*$. Also, by Proposition 3, let $\tilde{x}$ be a point in the mixed-integer linear set $\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$ with $\mathcal{F}$-small complexity.

For every $\lambda \in \mathbb{Z}_+$, the point $\tilde{x} + \lambda \tilde{r}$ belongs to $\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$. Expanding the quadratic form we obtain

$$
Q(\tilde{x} + \lambda \tilde{r}) = \lambda^2 \tilde{r}^T H \tilde{r} + \lambda \left(2 \tilde{x}^T H \tilde{r} + c^T \tilde{r}\right) + \tilde{x}^T H \tilde{x} + c^T \tilde{x} + d.
$$

We define $v_1 := \tilde{r}^T H \tilde{r}$, $v_2 := 2\tilde{x}^T H \tilde{r} + c^T \tilde{r}$, and $v_3 := \tilde{x}^T H \tilde{x} + c^T \tilde{x} + d$, so that $Q(\tilde{x} + \lambda \tilde{r}) = \lambda^2 v_1 + \lambda v_2 + v_3$. Since $v_1 < 0$, this is a strictly concave polynomial in $\lambda$, and so setting $\lambda$ larger than its larger root gives $Q(\tilde{x} + \lambda \tilde{r}) < 0$. Explicitly, let
Let \( P = \bigcup_{K \in \mathcal{K}} (\tilde{Q} + \text{cone}(R_K)) \) be the decomposition from Proposition 1. By the feasibility of the instance \( \mathcal{F} \), there is \( \tilde{K} \in \mathcal{K} \) such that \( (\tilde{Q} + \text{cone}(R_{\tilde{K}})) \cap (\mathbb{Z}^p \times \mathbb{R}^q) \cap \{ x : Q(x) \leq 0 \} \) is non-empty. Notice \( \tilde{Q} \) and \( R_{\tilde{K}} \) have \( \mathcal{F} \)-small complexity. Because we are in Case 2 and since cone\( (R_{\tilde{K}}) \) is simple and pointed (since we assumed \( P \) pointed), we can use Lemma 2 to refine cone\( (R_{\tilde{K}}) \) into the family of cones with extreme rays \( \{ R_{\tilde{K},j} \} \); again let \( \tilde{j} \) be such that \( (\tilde{Q} + \text{cone}(R_{\tilde{K},\tilde{j}})) \cap (\mathbb{Z}^p \times \mathbb{R}^q) \cap \{ x : Q(x) \leq 0 \} \) is non-empty. Notice cone\( (R_{\tilde{K},\tilde{j}}) \) has \( \mathcal{F} \)-small complexity.

Finally, we employ Proposition 2 to the polyhedron \( \tilde{Q} + \text{cone}(R_{\tilde{K},j}) \) to obtain a family of polytopes \( \{ P_i \} \) and subsets of extreme rays \( \{ R_K \} \) such that

\[
(\tilde{Q} + \text{cone}(R_{\tilde{K},j})) \cap (\mathbb{Z}^p \times \mathbb{R}^q) = \bigcup_{i, K \in \mathcal{K}} (P_i + \text{int.cone}(R_K)).
\]

Again there are indices \( \tilde{i} \) and \( \tilde{K} \) such that \( (P_{\tilde{i}} + \text{int.cone}(R_{\tilde{K}})) \cap \{ x : Q(x) \leq 0 \} \) is non-empty. Notice \( P_{\tilde{i}} \) and \( R_{\tilde{K}} \) have \( \mathcal{F} \)-small complexity, because \( \tilde{Q} \) and cone\( (R_{\tilde{K},\tilde{j}}) \) have \( \mathcal{F} \)-small complexity and hence the input \( \tilde{Q} + \text{cone}(R_{\tilde{K},j}) \) has \( \mathcal{F} \)-small complexity, and because of the guarantees of Proposition 2.

Notice these reductions guarantee that \( P_{\tilde{i}} + \text{int.cone}(R_{\tilde{K}}) \leq P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \). Therefore, to conclude the proof it suffices to show that \( (P_{\tilde{i}} + \text{int.cone}(R_{\tilde{K}})) \cap \{ x : Q(x) \leq 0 \} \) contains a solution with \( \mathcal{F} \)-small complexity.

In addition to maintaining low complexity, these reductions are motivated by the following property we obtain: since \( R_{\tilde{K},j} \) are precisely the extreme rays of \( \tilde{Q} + \text{cone}(R_{\tilde{K},j}) \) (the input for reduction step 3), the output \( R_K \) is a subset of (possibly scalings of) \( R_{\tilde{K},j} \); the construction of the latter in step 2 then guarantees that cone\( (R_K) \) satisfies property (b) of Lemma 2 (we use \( C_i = \text{cone}(R_{\tilde{K}}) \)).

Now we construct the desired solution. To simplify the notation, let \( P := P_{\tilde{i}} \), let \( F = \text{cone}(R_{\tilde{K}}) \), and enumerate \( R_{\tilde{K}} = \{ r^j \} \). In addition, for an index \( i \), we exclude ray \( r^j \) to define the face \( F_i := \text{cone}(r^j)_{j \neq i} \), and similarly for a set of indices \( J \), let \( F_J := \text{cone}(r^j)_{j \in J} \). Finally, we define the int.cone version of these cones, namely \( F^I := \text{int.cone}(r^j)_{j \neq i} \) and \( F^J := \text{int.cone}(r^j)_{j \notin J} \). So we want to exhibit a \( \mathcal{F} \)-small complexity solution in the non-empty set

\[
Q(x) \leq 0 \quad x \in P + F^I.
\]

\[\blacksquare\]
We first analyze the behavior of $Q$ over a single direction $r^j$. Any point in $P + F^I$ can be written as $x^j + \mu r^j$ for $x^j \in P + F^I_j$ and $\mu \in \mathbb{Z}_+$. We partition the rays $r^j$ into those that attain zero or strictly positive higher-order value $(r^j)^\top H r^j$, so define $J := \{ j : (r^j)^\top H r^j = 0 \}$. Then $Q$ has linear behavior along the directions $r^j$ with $j \in J$: for all $j \in J$, $x^j \in P + F^I_j$, and $\mu \in \mathbb{Z}_+$

$$Q(x^j + \mu r^j) = \mu \cdot \left(2(x^j)^\top H r^j + c^\top r^j\right) + (x^j)^\top H x^j + c^\top x^j + d. \quad (5)$$

Hence, if there is $j \in J$ and a point $x^j \in P + F^I_j$ such that the first term $2(x^j)^\top H r^j + c^\top r^j$ is negative, then we can find a large scaling $\mu$ such that the point $x^j + \mu r^j$ satisfies (4); in fact, we can construct such a point in a way that it has $F$-small complexity. Clearly if $x^j$ has $F$-small complexity and the linear optimization problem

$$\min \{ 2(x^j)^\top H r^j + c^\top r^j : x^j \in P + F^I_j \} \quad \text{has negative objective value.}$$

**Lemma 3** Consider $j \in J$ and the linear optimization problem $\min \{2(x^j)^\top H r^j + c^\top r^j : x^j \in P + F^I_j\}$. If this problem is unbounded, or the optimum is negative, then

there is a point $\tilde{x}^j \in P + F^I_j$ which has $F$-small complexity and negative objective value.

**Proof** To simplify the notation, let $\text{obj}(x) = 2x^\top H r^j + c^\top r^j$ denote the objective function. Let $\tilde{p} \in \text{argmin}\{\text{obj}(p) : p \in P\}$. (Note that $P$ is bounded and non-empty, so $\tilde{p}$ is well-defined.) Since $P$ has $F$-small complexity, it follows that $\tilde{p}$ has $F$-small complexity. Clearly if $\text{obj}(\tilde{p}) < 0$, then set $\tilde{x}^j$ to $\tilde{p}$ as the desired point in $P + F^I_j$, concluding the proof. Otherwise, by linearity of $\text{obj}$ and the definition of $F^I_j$ there exists some $i \neq j$ such that $2(r^i)^\top H r^j < 0$. Then let $\tilde{n}_i$ be the smallest non-negative integer satisfying

$$\text{obj}\left(\tilde{p} + r^i \tilde{n}_i\right) \leq -1$$

Clearly $\tilde{n}_i$ has $F$-small complexity and therefore $\tilde{x}^j = \tilde{p} + \tilde{n}_i r^i$ is the desired point in $P + F^I_j$, concluding the proof. \qed

Then suppose there is $j \in J$ such that $\min \{2(x^j)^\top H r^j + c^\top r^j : x^j \in P + F^I_j\}$ is unbounded or negative. From Lemma 3, let $\tilde{x}^j \in P + F^I_j$ have $F$-small complexity such that $\tilde{v} := 2(\tilde{x}^j)^\top H r^j + c^\top r^j < 0$; notice that $\tilde{v}$ has $F$-small complexity. Given (5), we set $\mu = \frac{(\tilde{x}^j)^\top H \tilde{x}^j + c^\top \tilde{x}^j + d}{|\tilde{v}|}$ to get that $\tilde{x}^j + \mu r^j$ is feasible for the problem (4) and has $F$-small complexity; this concludes the proof in this case.

Finally, consider the case where for all $j \in J$ we have $\min \{2(x^j)^\top H r^j + c^\top r^j : x^j \in P + F^I_j\}$ non-negative. In this case, the feasibility of (4) implies that the following is feasible:

$$Q(x) \leq 0$$

$$x \in P + F^I_j. \quad (6)$$
The advantage is that now all extreme rays $r$ in $F_J$ have strictly positive higher-order value $r^\top Hr > 0$. Moreover, since the cone $F$ satisfies property (b) of Lemma 2, this implies that for all non-zero $r$ in the face $F_J$ we get strictly positive higher-order value $r^\top Hr > 0$. The next lemma uses this fact to upper bound the length of the solutions in this set.

**Lemma 4** There is a rational number $v^*$ of $\mathcal{F}$-small complexity such that for all $x$ satisfying $Q(x) \leq 0$ and $x \in P + F_J$, we have $\|x\| \leq v^*$.

**Proof** Let $\mathcal{H} = \{x : f^\top x = 1\}$ be the normalizing hyperplane given by Lemma 1 for the cone $F$. Now consider any point of the form $\bar{x} + \bar{r} \in P + F_J$ (with $\bar{x} \in P$ and $\bar{r} \in F_J$) such that $Q(\bar{x} + \bar{r}) \leq 0$; also consider the vector in direction $\bar{r}$ that belongs to $\mathcal{H}$, namely let $\bar{r} = \lambda \bar{r}$ for $\bar{r} \in F_J \cap \mathcal{H}$ and $\lambda > 0$. We upper bound the norm of $\bar{x} + \bar{r}$, starting by bounding $\lambda$.

As mentioned above, $r^\top Hr > 0$ for all $r \in F_J$. Let $v^*_1 = \min\{r^\top Hr : r \in F_J \cap \mathcal{H}\}$ (notice that $F_J \cap \mathcal{H}$ is compact). Since $F_J$, $\mathcal{H}$ and $H$ have $\mathcal{F}$-small complexity, it follows from Theorem 2 that $v^*_1$ also has $\mathcal{F}$-small complexity.

Evaluating $Q$ over $\bar{x} + \bar{r}$ we have

$$Q(\bar{x} + \bar{r}) = \lambda^2 \bar{r}^\top H \bar{r} + \lambda \left(2\bar{r}^\top H \bar{x} + c^\top \bar{r}\right) + \left(\bar{x}^\top H \bar{x} + c^\top x + d\right).$$

Let $v^*_2 := \min\{2r^\top Hx + c^\top r : x \in P, r \in F_J \cap \mathcal{H}\}$ and $v^*_3 := \min\{x^\top Hx + c^\top x + d : x \in P\}$. Note that, by Theorem 2, also $v^*_2$ and $v^*_3$ have $\mathcal{F}$-small complexity. From (7), we obtain $Q(\bar{x} + \bar{r}) \geq \lambda^2 v^*_2 + \lambda v^*_2 + v^*_3$. Since $v^*_3 > 0$, the polynomial $\lambda^2 v^*_2 + \lambda v^*_2 + v^*_3$ is strictly convex (as a function of $\lambda$), and since $Q(\bar{x} + \bar{r}) \leq 0$, we have that $\lambda$ cannot be larger than the largest of its roots; explicitly, $\lambda \leq \sqrt{-v^*_2 \pm \sqrt{(v^*_2)^2 - 4v^*_2 v^*_3}} / 2v^*_3$. Moreover, this bound is independent of our choice of point $\bar{x} + \bar{r}$ and is an $\mathcal{F}$-small complexity value.

We can finally bound the norm of $\bar{x} + \bar{r}$. By triangle inequality, $\|\bar{x} + \bar{r}\| \leq \|\bar{x}\| + \lambda \|\bar{r}\|$. Let $v^*_4$ be the ceiling of the infinity norm of $P$, $v^*_4$ has $\mathcal{F}$-small complexity, because it can be obtained as the ceiling of the infinity norm of $P$; $v^*_4$ has $\mathcal{F}$-small complexity, because it can be obtained as the ceiling of the infinity norm of $P$; $v^*_4$ has $\mathcal{F}$-small complexity, because it can be obtained as the ceiling of $\max_i \{\max\{(e^i)^\top x : x \in P\}, \min\{(e^i)^\top x : x \in P\}\}$. Therefore we can bound $\|\bar{x}\| \leq \sqrt{n} \cdot v^*_4$. Also, since $\bar{r} \in \mathcal{H}$, and by the definition of $\mathcal{H}$ (see Lemma 1), we have $f^\top \bar{r} = 1$ and $f^\top \bar{r} / \|\bar{r}\| \geq 1 / \max_j \|r_j\|$, which imply $\|\bar{r}\| \leq \max_j \|r_j\|$. Together, these bounds give an upper bound for $\|\bar{x} + \bar{r}\|$ by an $\mathcal{F}$-small complexity value which is independent of $\bar{x} + \bar{r}$; this concludes the proof. $\square$

Now we construct a $\mathcal{F}$-small complexity solution in (2). Let $\bar{x}$ be a solution for (6) and recall that $\bar{x} \in \mathbb{Z}^p \times \mathbb{R}^q$. By the bound of Lemma 4, and using integrality, we have that the first $p$ components of $\bar{x}$ have $\mathcal{F}$-small complexity. Then we fix these values and consider the optimization over the other components $\min\{Q(x) : x \in P + F_J, x_i = \bar{x}_i \forall i \leq p\}$. First, we claim that this optimization problem has an optimal solution. Since $\bar{x}$ is a feasible solution for this problem attaining value at most 0, the level set $Q(x) \leq 0$ is non-empty. Since this level set is contained in (6), Lemma 4 implies that this level set is bounded; this gives that the problem has an optimal solution.
Furthermore, Theorem 2 then guarantees that it has an optimal solution $\tilde{x}$ that has $\mathcal{F}$-small complexity. Moreover, note that $\tilde{x}$ is a feasible mixed-integer point, since the first $p$ components are integral and $P + FJ$ is contained in $\mathcal{P}$. Since $Q(\tilde{x}) \leq Q(x) \leq 0$, $\tilde{x}$ is the desired feasible solution for (6). This concludes the proof. \qed

4 Algorithmic consequences

In this section we highlight some algorithmic consequences of Theorem 1. Let $\mathcal{P} = \{x : Ax \leq b\} \subseteq \mathbb{R}^{p+q}$, and let $\mathcal{P}_I$ be its mixed-integer hull, i.e. $\mathcal{P}_I = \text{conv}(\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q))$. We will consider the following continuous quadratic problem:

$$\begin{align*}
\min & \quad x^\top Hx + c^\top x \\
\text{subject to} & \quad x \in \mathcal{P}_I.
\end{align*}$$

(8)

**Proposition 4** The MIQP (1) is bounded if and only if (8) is bounded.

**Proof** Clearly if the MIQP is unbounded, then so is (8), thus we now assume that (8) is unbounded and we show that (1) is unbounded as well. By Vavasis [17], there exists a rational ray $\{\bar{x} + \lambda \bar{r} : \lambda \geq 0\}$ contained in $\mathcal{P}_I$ such that $\lim_{\lambda \to \infty} (\bar{x} + \lambda \bar{r})^\top H (\bar{x} + \lambda \bar{r}) + c^\top (\bar{x} + \lambda \bar{r}) = -\infty$. By expanding the latter, we obtain:

$$\lim_{\lambda \to \infty} \lambda^2 (\bar{r}^\top H \bar{r}) + \lambda (2 \bar{r}^\top H \bar{x} + c^\top \bar{r}) + (\bar{x}^\top H \bar{x} + c^\top \bar{x}) = -\infty$$

(9)

Since the vector $\bar{x}$ is in $\mathcal{P}_I$, it can be written as a convex combination of rational points in $\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$. Let $\bar{x}$ be one of those points with $2 \bar{r}^\top H \bar{x} + c^\top \bar{r} \leq 2 \bar{r}^\top H \bar{x} + c^\top \bar{x}$. Consider now the ray $\{\bar{x} + \lambda \bar{r} : \lambda \geq 0\}$, and the objective value of the points on it:

$$(\bar{x} + \lambda \bar{r})^\top H (\bar{x} + \lambda \bar{r}) + c^\top (\bar{x} + \lambda \bar{r}) = \lambda^2 (\bar{r}^\top H \bar{r}) + \lambda (2 \bar{r}^\top H \bar{x} + c^\top \bar{r}) + (\bar{x}^\top H \bar{x} + c^\top \bar{x})$$

(10)

As the limit in (9) is $-\infty$, we have that $\bar{r}^\top H \bar{r} < 0$, or $\bar{r}^\top H \bar{x} = 0$ and $2 \bar{r}^\top H \bar{x} + c^\top \bar{r} < 0$. In both cases, the quadratic polynomial in the right hand side of (10) has negative leading term, which implies that $\lim_{\lambda \to \infty} (\bar{x} + \lambda \bar{r})^\top H (\bar{x} + \lambda \bar{r}) + c^\top (\bar{x} + \lambda \bar{r}) = -\infty$. The statement follows, because the rationality of $\bar{x}$ and of $\bar{r}$ implies that for every $\tilde{\lambda} > 0$, there exist infinitely many points in $\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$ in the ray $\{\bar{x} + \lambda \bar{r} : \lambda \geq \tilde{\lambda}\}$. \qed

**Theorem 3** The decision problem “is an IQP problem unbounded?” is NP-hard.

**Proof** Consider the homogeneous IQP:

$$\begin{align*}
\text{minimize} & \quad x^\top Hx \\
\text{subject to} & \quad x \geq 0 \\
& \quad x \in \mathbb{Z}^n,
\end{align*}$$

(11)
where $H$ is an integer square matrix of order $n$. By Proposition 4, problem (11) is unbounded if and only if the following QP is:

\[
\begin{align*}
\text{minimize} & \quad x^\top H x \\
\text{subject to} & \quad x \geq 0.
\end{align*}
\] (12)

Murty and Kabadi [14] showed that deciding if (12) is unbounded is NP-complete. \(\square\)

Clearly, Theorem 3 implies that the same decision problem is NP-hard also for MIQPs.

Even though by Theorem 3 it is hard to decide if a MIQP is bounded, in the next theorem we show that, when we already know if the problem is bounded or not, we can bound the size of the corresponding certificates. Compared to Theorem 1 the main feature of the following is that it bounds the complexity of an optimal solution to the optimization problem (1). Its proof follows from our proof of Theorem 1.

**Theorem 4** There is a polynomial function $\bar{f}$ such that the following holds. Consider an instance of the MIQP (1) and let $\bar{\phi}$ be the complexity of this instance. Then setting $\psi = \bar{f}(\phi)$ we have:

(i) If the instance is bounded, then there is an optimal solution of the problem that has complexity at most $\psi$.

(ii) If the instance is unbounded, then there is a ray $\{\bar{x} + \lambda \bar{r} : \lambda \geq 0\}$, with $\bar{x} \in \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$, along which the objective goes to minus infinity, with complexity at most $\psi$.

**Proof (Sketch of proof)** By Proposition 3, if $\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$ is nonempty, it contains a point $\bar{x}$ of polynomial complexity. Let $-d$ be a number of polynomial complexity that upper bounds the objective value of $\bar{x}$, and define $Q(x) := x^\top H x^\top + c^\top x + d$. Let $\psi = f(\phi)$, where $\phi$ is the complexity of $\{H, c, d, \mathcal{P}\}$, and $f$ is the function in the statement of Theorem 1. Using the definition of $d$, we get that $\psi$ is a polynomial function of the complexity $\bar{\phi}$ of the optimization problem (1), as desired. We show that $\psi$ satisfies our statement.

In Case 1 of the proof of Theorem 1, the given ray $\{\bar{x} + \lambda \bar{r} : \lambda \geq 0\}$ has complexity at most $\psi$ and implies that MIQP is unbounded.

In Case 2, there exists an element $P + F^I$ of our decomposition, and an index $i \in J$ such that $\min\{2(x^i)^\top H r^i + c^\top r^i : x^i \in P + F^I_i\}$ in Lemma 3 is unbounded or the optimum is negative. In this case, the given ray $\{\bar{x}^j + \mu r^j : \mu \geq 0\}$ has complexity at most $\psi$ and implies that MIQP is unbounded.

Otherwise, for every element $P + F^I$ of our decomposition, and for all $j \in J$, we have $\min\{2(x^j)^\top H r^j + c^\top r^j : x^j \in P + F^I_j\} \geq 0$. It follows that an optimum point on the set $P + F^I$ lies in the subset $P + F^I_j$, and Lemma 4 implies that there is a rational number $v^*$ of complexity at most $\psi$ such that for all $x$ satisfying $Q(x) \leq 0$, $x \in P + F^I_j$ we have $\|x\| \leq v^*$. The result now follows by the last paragraph of the proof of Theorem 1. \(\square\)

The next theorem, that we will need later, is a direct consequence of Vavasis’ proof in [17].
Mixed-integer quadratic programming is in NP

Theorem 5 There is a finite algorithm for QP. If the problem is bounded, the algorithm returns a rational optimal solution of polynomial size. If it is unbounded, it returns a ray of polynomial size \( \{ \bar{x} + \lambda \bar{r} : \lambda \geq 0 \} \) in \( \mathcal{P} \), along which the objective goes to minus infinity. Moreover, if the dimension is fixed, such algorithm runs in polynomial time.

Proof (Sketch of proof) If the optimum is finite, Vavasis’ algorithm first divides the polyhedron \( \mathcal{P} \) into polyhedra \( \mathcal{P}^j, j = 1, \ldots, 2^n \) by intersecting \( \mathcal{P} \) with each orthant. It then considers each face \( F \) of each \( \mathcal{P}^j \) such that the restriction of the objective function on the affine hull of \( F \) is positive definite. Then it finds, via a closed form, the optimum of the objective function over the affine hull of \( F \). An optimal solution of QP is then the best feasible point among the ones considered. (See §2 in [17] for more details.) Vavasis’ algorithm also detects if the problem is unbounded or not, and, if it is unbounded, it finds a ray \( \{ \bar{x} + \lambda \bar{r} : \lambda \geq 0 \} \) contained in \( \mathcal{P} \) along which the objective goes to minus infinity. This is done essentially as above, after having intersected \( \mathcal{P} \) with a parametric box centered in the origin of increasing diameter. (See §3 in [17].)

If we assume that the dimension is \( p + q \) of a MIQP problem (1) is fixed, then we obtain the following results.

Theorem 6 Assume that the dimension \( p + q \) is fixed. Then we can decide in polynomial time if MIQP (1) is bounded or not. If it is unbounded, we can find in polynomial time a ray \( \{ \tilde{x} + \lambda \bar{r} : \lambda \geq 0 \} \), with \( \tilde{x} \in \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q) \), along which the objective goes to minus infinity.

Proof It is well-known that in fixed dimension, by discretization of the continuous variables, we can obtain in polynomial time a description of \( \mathcal{P}_I \) (see [6] for the result in the integer case, and [2,7] for the discretization).

By Theorem 5, in fixed dimension, there exists a polynomial algorithm to solve (8). If the problem is bounded, it returns an optimal solution \( \bar{x} \). If the problem is unbounded, it returns a ray \( \{ \bar{x} + \lambda \bar{r} : \lambda \geq 0 \} \) contained in \( \mathcal{P}_I \) along which the objective goes to minus infinity.

If (8) is unbounded, then following the proof of Proposition 4, we can find in polynomial time a ray \( \{ \bar{x} + \lambda \bar{r} : \lambda \geq 0 \} \), with \( \bar{x} \in \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^q) \), along which the objective goes to minus infinity.

Theorems 4 and 6 allow us to apply any algorithm for polytopes to the unbounded case. In particular we obtain the following corollaries:

Corollary 2 There exists a pseudo-polynomial time algorithm for MIQP in fixed dimension.

Proof We first apply Theorem 6 to decide if MIQP is bounded. If it is unbounded, we obtain a ray along which the objective goes to minus infinity, in which case we are done. Otherwise, by Theorem 4, we find a number \( \psi \) such that an optimal solution of the problem has complexity bounded from above by \( \psi \). The value \( \psi \) is polynomial in the complexity of the input \( \{ H, c, \mathcal{P} \} \) of the MIQP. Since the dimension is fixed, the number \( 2^\psi \) is polynomial in the numeric value of the input \( \{ H, c, \mathcal{P} \} \). We can now
partition our MIQP (1) into a polynomial number of continuous problems. In order to do so, for every integral \( x^I \) in the box \([−2^\psi, 2^\psi]^p\), consider the continuous quadratic problem obtained from (1) by adding the constraints \( x_j = x^I_j \), for \( j = 1, \ldots, p \). By Theorem 5, we can now solve in polynomial time each of these continuous quadratic problems.

**Corollary 3** There exists a weak fully polynomial time approximation scheme (weak FPTAS) algorithm for MIQP in fixed dimension.

**Proof** De Loera, Hemmecke, Köppe, and Weismantel [3] gave a weak FPTAS for optimizing polynomials over the mixed-integer points of polytopes in fixed dimension. The statement for more general polyhedra then directly follows from Theorem 4 and Theorem 6.

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