Consistent analytic approach to the efficiency of collisional Penrose process

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Abstract

We propose a consistent analytic approach to the efficiency of collisional Penrose process in the vicinity of a maximally rotating Kerr black hole. We focus on a collision with arbitrarily high center-of-mass energy, which occurs if either of the colliding particles has its angular momentum fine-tuned to the critical value to enter the horizon. We show that if the fine-tuned particle is ingoing on the collision, the upper limit of the efficiency is \((2 + \sqrt{3})(2 - \sqrt{2}) \simeq 2.186\), while if the fine-tuned particle is bounced back before the collision, the upper limit is \((2 + \sqrt{3})^2 \simeq 13.93\).

Despite earlier claims, the former can be attained for inverse Compton scattering if the fine-tuned particle is massive and starts at rest at infinity, while the latter can be attained for various particle reactions, such as inverse Compton scattering and pair annihilation, if the fine-tuned particle is either massless or highly relativistic at infinity. We discuss the difference between the present and earlier analyses.

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I. INTRODUCTION

If two particles collide in the ergosphere of a rotating black hole, a product particle can escape to infinity with energy greater than the total energy of the incident particles. This is called the collisional Penrose process, and was pioneered by Piran et al. [1, 2]. The study of this phenomenon has recently been revived since Bañados, Silk and West (BSW) [3] revealed that maximally rotating black holes can accelerate particles to arbitrarily high energy if their angular momenta are fine-tuned. More precisely, if two particles start at rest at infinity and if the angular momentum of either of the two is fine-tuned to the threshold value to enter the horizon, the center-of-mass energy of the two colliding particles can be arbitrarily high.

It is a fundamental question whether particle acceleration by a rotating black hole has anything to do with energy extraction from the rotating black hole. Although it was claimed [4] that there is no energy extraction for the BSW process of two equal masses, where both the fine-tuned particle and the generic particle are ingoing on the collision, it was later shown that the energy extraction efficiency can reach $\simeq 1.3$ for the BSW process if the masses of the product particles are not equal to the incident particles [5, 6]. In Ref. [6], the authors including two of the present ones also claimed that $\simeq 1.4$ is a common upper limit for any particle reactions. More recently, Schnittman [7] has shown that the upper limit can reach $\simeq 14$, which is analytically given by $(2 + \sqrt{3})^2$ [8, 9], if the collision occurs immediately after the ingoing fine-tuned particle is bounced back outwardly near the horizon by a potential barrier. In Ref. [9], the present authors discussed that the process of such high efficiency would require heavy particle production. Berti et al. [10] pointed out that the efficiency can be arbitrarily high if we allow a particle with the subcritical value of angular momentum to start with an outgoing initial velocity in the vicinity of a black hole, although the physical motivation of such a particle is controversial [11].

In the current article, we propose a consistent analytic approach and show that the assumption made in the previous analytic analyses in Refs. [6, 9] is too restricted. Under a physically reasonable assumption, we show that the upper limit for the BSW process is given by $(2 + \sqrt{3})(2 - \sqrt{2}) \simeq 2.186$, which can be attained for inverse Compton scattering if the incident fine-tuned particle is massive and starts at rest at infinity. We also show that the upper limit for Schnittman’s process is given by $(2 + \sqrt{3})^2 \simeq 13.93$, which can be attained for various particle reactions if the incident fine-tuned particle is massless or starts...
with a highly relativistic velocity at infinity.

II. ANALYTIC APPROACH

We consider the reaction of incident particles 1 and 2 colliding near the horizon to two product particles 3 and 4 in the equatorial plane of a Kerr black hole of mass $M$ and spin $a$. The energy and angular momentum conservation yields

$$E_1 + E_2 = E_3 + E_4, \quad L_1 + L_2 = L_3 + L_4.$$  \hfill (2.1)

The conservation of the total radial momentum immediately before and after the reaction is given by

$$p^r_1 + p^r_2 = p^r_3 + p^r_4$$ \hfill (2.2)

at the collision point $r = r_c$. The radial momentum $p^r$ of the particle is given by

$$p^r = \sigma \sqrt{-2V(r)},$$ \hfill (2.3)

where $\sigma = \pm 1$ and

$$V(r) = -\frac{Mm^2}{r} + \frac{L^2 - a^2(E^2 - m^2)}{2r^2} - \frac{M(L - aE)^2}{r^3} - \frac{E^2 - m^2}{2}.$$ \hfill (2.4)

Hereafter, we concentrate on maximally rotating black holes, i.e., $a = M$ for simplicity. For this case, $L = 2ME$ gives the critical value of angular momentum for an ingoing particle to enter the horizon from outside.

There are 3 parameters $(E, L, m)$ for each of the particles except for $\sigma$. We have 3 equations and so $3 \times 4 - 3 = 9$ degrees of freedom are remaining except for $\sigma_i \ (i = 1, 2, 3, 4)$. If we specify the species of the incident particles, we can fix $m_1$ and $m_2$. Thus, $9 - 2 = 7$ degrees of freedom are remaining. If we specify the remaining parameters $E_1$, $L_1$, $E_2$ and $L_2$ for the incident particles, we have $7 - 4 = 3$ degrees of freedom remaining. Moreover, we can fix $m_3$ and $m_4$ for a particle reaction we know. Thus, only one degree of freedom is remaining. We can take $\delta$ for an escaping particle as this, where $\delta$ parameterizes the ratio of $L$ and $E$, i.e.,

$$L = (2 + \delta)ME.$$ \hfill (2.5)

We should note that $\delta$ essentially determines the orbit of the particle and, hence, whether it can escape to infinity or not. If $0 < \delta < \delta_{\text{max}}(r_c)$ and $E \geq m$, the particle will escape to
infinity whether it is initially outgoing or ingoing, while if $\delta_{\min}(r_c) < \delta < 0$, it must start with an outgoing initial velocity, where $\delta_{\max}(r)$ and $\delta_{\min}(r)$ are determined by the turning point condition $V(r) = 0$. This can be seen in Fig. 1 of Ref. [9]. Clearly, we cannot control $\delta$, which corresponds to the direction of the initial velocities of the product particles. We take the energy and angular momentum of the escaping particle as functions of those of the incident particles, the collision point $r_c$ and $\delta$. That is, identifying particle 3 with that of the escape to infinity, we have

$$E_3 = E_3(E_1, L_1, m_1, \sigma_{1c}; E_2, L_2, m_2, \sigma_{2c}; m_3, \sigma_{3c}; m_4, \sigma_{4c}; \delta_3; r_c),$$  \hspace{1cm} (2.6)

$$L_3 = L_3(E_1, L_1, m_1, \sigma_{1c}; E_2, L_2, m_2, \sigma_{2c}; m_3, \sigma_{3c}; m_4, \sigma_{4c}; \delta_3; r_c),$$  \hspace{1cm} (2.7)

where $\sigma_{ic} (i = 1, 2, 3, 4)$ are the values of $\sigma_i$ immediately before and after the collision. The energy extraction efficiency $\eta$ is defined as $\eta := E_3/(E_1 + E_2)$.

To investigate the collisional Penrose process with arbitrarily high center-of-mass energy, we assume that particles 1 and 2 are critical and subcritical, respectively; i.e., $\tilde{L}_1 = 2E_1$ and $\tilde{L}_2 < 2E_2$, where $\tilde{L} := L/M$ and for which the center-of-mass energy behaves as $E_{cm} \propto (r - r_c)^{-1/2}$ for $0 < r - r_c \ll M$ [12, 13]. We assume $\sigma_{2c} = -1$ and $\sigma_{4c} = -1$ on collision to specify the process. To go further, we express $p^t$ and $p^r$ in terms of $\epsilon$, where $r = M/(1 - \epsilon)$. We note that for fixed $E$ and $L$, $p^t$ is given by

$$p^t = \frac{1}{\epsilon^2}[2(2E - \tilde{L}) + 2(-4E + 3\tilde{L})\epsilon + (7E - 6\tilde{L})\epsilon^2 + 2(-E + \tilde{L})\epsilon^3],$$  \hspace{1cm} (2.8)

while $(p^r)^2$ is given by

$$(p^r)^2 = [(2E - \tilde{L}) - 2(E - \tilde{L})\epsilon]^2 - [m^2 - (E - \tilde{L})(3E - \tilde{L})]\epsilon^2 - 2(E - \tilde{L})\epsilon^3. \hspace{1cm} (2.9)$$

Denoting $r_c = M/(1 - \epsilon_c)$, we can find that Eqs. (2.1) and (2.2) imply that $\sigma_{3c} < 0$ or $2E_3 - \tilde{L}_3 = O(\epsilon_c)$. In the former case, we can expect particle 3 to eventually escape to infinity only if $2E_3 - \tilde{L}_3 < 0$, while the forward-in-time condition $p^t > 0$ implies $2E_3 - \tilde{L}_3 > (4E_3 - 3\tilde{L}_3)\epsilon_c + O(\epsilon_c^2)$. Therefore, whether $\sigma_{3c} = 1$ or $-1$, we conclude $2E_3 - \tilde{L}_3 = O(\epsilon_c)$, so that particle 3 must be near-critical.

Since $E_3$ and $\tilde{L}_3$ are functions of the radius of the collision point $r_c$, we can assume that $E_3$ and $\tilde{L}_3$ are expandable in terms of $\epsilon_c$, i.e.,

$$E_3 = E_3(0) + E_3(1)\epsilon_c + E_3(2)\epsilon_c^2 + \ldots,$$  \hspace{1cm} (2.10)

$$\tilde{L}_3 = \tilde{L}_3(0) + \tilde{L}_3(1)\epsilon_c + \tilde{L}_3(2)\epsilon_c^2 + \ldots.$$  \hspace{1cm} (2.11)
We expand $E_i$ and $\tilde{L}_i$ in terms of $\epsilon_c$ for the product particles $i = 3, 4$, but not for incident particles $i = 1, 2$. This looks asymmetric but is suitable for the present physical setting. Equivalently, instead of $\tilde{L}_3$, it is more convenient to expand $\delta_3$ as follows:

$$\delta_3 = \delta_{3(1)} \epsilon_c + \delta_{3(2)} \epsilon_c^2 + \ldots \tag{2.12}$$

If particle 3 is ingoing immediately after the collision with $E_3 \geq m_3$ and to escape to infinity, it must be bounced back by a potential barrier inside the collision point, for which

$$0 < \delta_{3(1)} \leq \delta_{(1),\text{max}}, \quad \delta_{(1),\text{max}} := \frac{2E_{3(0)} - \sqrt{E_{3(0)}^2 + m_3^2}}{E_{3(0)}}. \tag{2.13}$$

If particle 3 is outgoing immediately after the collision with $E_3 \geq m_3$ and to escape to infinity, it must not encounter a potential barrier outside the collision point and this is guaranteed for a near-critical particle.

Since we have already seen the terms of $O(1)$ in Eq. (2.2), we proceed to the terms of $O(\epsilon_c)$ in the same equation. Together with Eq. (2.1), we obtain

$$A - E_{3(0)}(2 - \delta_{3(1)}) = \sigma_3c \sqrt{E_{3(0)}^2(3 - \delta_{3(1)})(1 - \delta_{3(1)}) - m_3^2}, \tag{2.14}$$

where $A := 2E_1 + \sigma_1c \sqrt{3E_1^2 - m_1^2} > 0$. Squaring the both sides of Eq. (2.14), we find

$$2 - \delta_{3(1)} = \frac{A^2 + E_{3(0)}^2 + m_3^2}{2AE_{3(0)}}. \tag{2.15}$$

Substituting the above into the left-hand side of Eq. (2.14), we find

$$A - \frac{E_{3(0)}^2 + m_3^2}{A} = 2\sigma_3c \sqrt{E_{3(0)}^2(3 - \delta_{3(1)})(1 - \delta_{3(1)}) - m_3^2}. \tag{2.16}$$

For $\sigma_3c = 1$, we immediately find

$$E_{3(0)} \leq \lambda_0, \quad \lambda_0 := \sqrt{A^2 - m_3^2}. \tag{2.17}$$

For $\sigma_3c = -1$, where $\delta_{3(1)} \geq 0$ must be satisfied for particle 3 to escape to infinity, Eq. (2.15) yields

$$E_{3(0)}^2 - 4AE_{3(0)} + A^2 + m_3^2 \leq 0. \tag{2.18}$$

Thus, we find

$$\lambda_- \leq E_{3(0)} \leq \lambda_+, \quad \lambda_\pm := 2A \pm \sqrt{3A^2 - m_3^2}, \tag{2.19}$$
where \( E_{3(0)} = \lambda_+ \) is realized only for \( \delta_{3(1)} = 0 \).

Therefore, for given \( E_1, E_2 \) and \( m_i \ (i = 1, 2, 3, 4) \), we find for \( \sigma_{3c} = 1 \)
\[
\eta_{\text{max}} = \frac{\sqrt{(2E_1 + \sigma_{1c}\sqrt{3E_1^2 - m_1^2})^2 - m_3^2}}{E_1 + E_2},
\]
(2.20)
while for \( \sigma_{3c} = -1 \)
\[
\eta_{\text{max}} = \frac{2(2E_1 + \sigma_{1c}\sqrt{3E_1^2 - m_1^2}) + \sqrt{3(2E_1 + \sigma_{1c}\sqrt{3E_1^2 - m_1^2})^2 - m_3^2}}{E_1 + E_2},
\]
(2.21)
where \( \sigma_{1c} = -1 \) and 1 correspond to the BSW and Schnittman processes, respectively. We can see that the upper limit for \( \sigma_{3c} = -1 \) is always greater than that for \( \sigma_{3c} = 1 \).

The terms of \( O(\epsilon^2) \) in Eq. (2.2) yield
\[
\sigma_{1c} \frac{E_1^2}{\sqrt{3E_1^2 - m_1^2}} + \frac{(3E_2 - \bar{L}_2)(E_2 - \bar{L}_2) - m_2^2}{2(2E_2 - \bar{L}_2)} = -\sigma_{3c} \frac{E_{3(0)}^2}{\sqrt{E_{3(0)}^2(3 - \delta_{3(1))}(1 - \delta_{3(1))} - m_3^2}}
\]
\[
+ \frac{2E_2 - \bar{L}_2}{2} - E_{3(0)}(2\delta_{3(1))} - \delta_{3(2))} - E_3(1)(2 - \delta_{3(1))} - \frac{(E_1 + E_2 - E_{3(0)})^2 + m_3^2}{2(2E_2 - \bar{L}_2)},
\]
(2.22)
If we fix \( E_{3(0)} \) and \( \delta_{3(1))} \), we find the relation between \( E_{3(1)} \) and \( \delta_{3(2))} \) from the above equation. Since both \( E_{3(1)} \) and \( \delta_{3(2))} \) appear only linearly, we can always solve the above equation for \( E_{3(1)} \) in terms of \( \delta_{3(2))} \). We do not obtain any additional condition to the lower-order terms.

Here we still take \( m_i \ (i = 1, 2, 3, 4) \) as fixed parameters but \( E_1 \) and \( E_2 \) as free ones in the ranges \( E_1 \geq m_1 \) and \( E_2 \geq m_2 \), respectively. For the BSW process with \( m_1 > 0 \) and \( m_2 > 0 \), the maximum efficiency is attained for \( E_1 = m_1 \) and \( E_2 = m_2 \) as we can see from Eqs. (2.20) and (2.21) with \( \sigma_{1c} = -1 \). The upper limit of the efficiency for \( \sigma_{3c} = 1 \) is given by
\[
\eta_{\text{max}} = \frac{\sqrt{(2 - \sqrt{2})^2m_1^2 - m_3^2}}{m_1 + m_2},
\]
(2.23)
which is less than unity, while for \( \sigma_{3c} = -1 \) it is given by
\[
\eta_{\text{max}} = \frac{2(2 - \sqrt{2})m_1 + \sqrt{3(2 - \sqrt{2})^2m_1^2 - m_3^2}}{m_1 + m_2}.
\]
(2.24)
The above expression applies also for \( m_2 = 0 \). For perfectly elastic collision of equal masses, we find \( \eta_{\text{max}} = (7 - 4\sqrt{2})/2 < 1 \) even for \( \sigma_{3c} = -1 \). But if we change the masses of
the particles, the upper limit can be greater than unity. For pair annihilation, $\eta_{\text{max}} = (2 - \sqrt{2})(2 + \sqrt{3})/2 \simeq 1.093$ for $\sigma_{3c} = -1$, which agrees very well with the numerical result given in Ref. [5]. For inverse Compton with $m_2 = m_3 = 0$, $\eta_{\text{max}} = (2 - \sqrt{2})(2 + \sqrt{3}) \simeq 2.186$, which is realized for $E_1 = m_1 \gg E_2$. This is the maximum upper limit of the efficiency for the BSW process. For the BSW process with $m_1 = 0$, the upper limit of the efficiency for $\sigma_{3c} = 1$ is given by $\eta_{\text{max}} = 2 + \sqrt{3} \simeq 3.732$, while for $\sigma_{3c} = -1$ the upper limit is given by $\eta_{\text{max}} = 1$. These upper limits are realized for $E_1 \gg \text{max}(E_2, m_3)$. In particular, for inverse Compton with $m_1 = m_3 = 0$, we can see $\eta_{\text{max}} = 1$.

For Schnittman’s process ($\sigma_{1c} = 1$), the situation is very different and much simpler. The upper limit can be attained for $E_1 \gg \text{max}(m_1, m_3)$ and $E_1 \gg E_2$ as we can see from Eqs. (2.20) and (2.21) with $\sigma_{1c} = 1$. In this case, we find $\eta_{\text{max}} = 2 + \sqrt{3} \simeq 3.732$ and $(2 + \sqrt{3})^2 \simeq 13.93$ for $\sigma_{3c} = 1$ and $-1$, respectively. The latter is a universal upper limit irrespective of the details of the particle reaction or the masses of the particles.

III. DISCUSSION AND SUMMARY

Here, we compare the current analysis with earlier ones [6, 9]. For example, if we fix the parameters such as $\sigma_{1c} = 1$, $\sigma_{3c} = -1$ and $\delta_{3(1)} = 0$, i.e., those for Schnittman’s process, and solve Eq. (2.22) for $m_4^2$, we find

$$m_4^2 = 2(2E_2 - \tilde{L}_2)F + E_2^2 + m_2^2 - (E_1 + E_2 - E_{3(0)})^2,$$

where

$$F = -\frac{E_1^2}{\sqrt{3}E_1^2 - m_1^2} - \frac{E_{3(0)}^2 + E_{3(0)}\delta_{3(2)}(2E_{3(0)} - \sqrt{3E_{3(0)}^2 - m_3^2})}{\sqrt{3E_{3(0)}^2 - m_3^2}} + \frac{E_{3(1)}(3E_{3(0)} - 2\sqrt{3E_{3(0)}^2 - m_3^2})}{\sqrt{3E_{3(0)}^2 - m_3^2}}.$$ (3.2)

If we assume $E_{3(1)} \geq 0$ and $\delta_{3(2)} \geq 0$, we can conclude that $F \leq 0$ and, hence, $E_2$ is bounded from below. This places a strong limit on the upper limit. However, if we allow $E_{3(1)}$ to be negative, the sign of $F$ is indefinite and there is no constraint on $E_2$ and $m_2$ except for the initial assumption $E_2 \geq m_2$. In Refs. [6, 9], the authors assumed that $E_3$ does not depend on $\epsilon_c$, which is equivalent to $E_{3(1)} = 0$ in the current analysis. This assumption is
apparently too restricted. In Ref. [6], it gives a rather smaller upper limit, while in Ref. [9], the possibility of heavy particle production has to be discussed. The current analysis implies that $E_{3(1)}$ is negative for the upper limit case and suggests that the upper limit is realized only in the near-horizon limit of Schnittman’s process.

Bejger et al. [5] numerically demonstrated that for the BSW process of pair annihilation the upper limit can reach 1.295, which is greater than our near-horizon limit 1.093. Note that their value is not realized in a simple near-horizon limit. We believe that their limit value 1.295 will be obtained if the angular momentum of particle 2 is also fine-tuned so that $2E_2 - \tilde{L}_2 = O(\epsilon_c)$, i.e., particle 2 is near-critical. It would be interesting to pursue this direction further in the generalization of the present analytic approach.

In summary, we propose a consistent analytic approach to the efficiency of the collisional Penrose process with the expansion in powers of the small parameter $\epsilon_c$, which parameterizes the closeness of the collision point to the horizon. According to this systematic approach, we find that the upper limits of the collisional Penrose process restricted in the equatorial plane near the horizon are given by $(2 + \sqrt{3})(2 - \sqrt{2}) \simeq 2.186$ and $(2 + \sqrt{3})^2 \simeq 13.93$ for the BSW and Schnittman processes, respectively. The former is realized for inverse Compton scattering, while the latter can be universally attained for various reaction of particles. In spite of the earlier claims [6, 9], these upper limits can be realized for standard particle reactions such as inverse Compton scattering and pair annihilation.

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