The running quark mass in the SF scheme
and its two-loop anomalous dimension

\textbf{ALPHA}

Collaboration

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Abstract

The non-perturbatively defined running quark mass introduced by the ALPHA collaboration is based on the PCAC relation between correlation functions derived from the Schrödinger functional (SF). In order to complete its definition it remains to specify a number of parameters, including the ratio of time to spatial extent, $T/L$, and the angle $\theta$ which appears in the spatial boundary conditions for the quark fields. We investigate the running mass in perturbation theory and propose a choice of parameters which attains two desired properties: firstly the two-loop anomalous dimension $d_{\text{SF}}^1$ is reasonably small. This is needed in order to ease matching with the non-perturbative computations and to achieve a precise determination of the renormalization group invariant quark mass. Secondly, to one-loop order of perturbation theory, cut-off effects in the step-scaling function are small in $O(a)$ improved lattice QCD.
1 Introduction

This paper is part of a project by the ALPHA collaboration to determine the $A$-parameter and the renormalization group invariant quark masses with controlled errors, using hadronic observables as experimental input \cite{1,2}. For numerical simulations of the lattice regularized theory, the basic difficulty consists in the large difference of length scales, ranging from the long distances typical for hadronic physics to short distances where perturbation theory can be applied with confidence. The proposed solution \cite{1} combines an intermediate finite volume renormalization scheme with a finite size scaling technique \cite{3}, which allows to step up the energy ladder recursively.

The very definition of the renormalized running coupling and quark mass is of great importance for the method to be practical. The running parameters should be relatively easy to compute by numerical simulation and they should not be affected by large cutoff effects. It is then possible to perform reliable continuum extrapolations and trace the non-perturbative evolution of the running parameters directly in the continuum limit, covering a wide range of energy scales. At high energies the evolution may be compared with perturbation theory, and, once perturbative evolution has set in, one may use perturbation theory to evolve to infinite energy and determine the renormalization group invariant parameters. For this last step to be feasible, the matching to perturbation theory should not require extremely high energies. This can be regarded as a further requirement to be met by a sensible definition of the running parameters.

In this paper we carry out a perturbative investigation of a two-parameter family of running quark masses in the SF scheme \cite{1}, which depends on the ratio $\rho = T/L$ between time and spatial extent of the space-time manifold, as well as on the parameter $\theta$ appearing in the spatial boundary conditions on the quark fields. In particular we determine the two-loop quark mass anomalous dimension $\delta_{1}^{S\Phi}$ which is needed for a precise determination of the renormalization group invariant quark mass \cite{4}. It turns out that the parameters must be chosen with care for $\delta_{1}^{S\Phi}$ to be reasonably small. Taking the size of the one-loop cutoff effects in the step-scaling function as a further criterion suggests a specific choice of both parameters. This completes the definition of the running quark mass in the SF scheme which is expected to meet all of the above mentioned requirements.

Many technical details of our perturbative calculation have appeared in ref. \cite{5} and will not be repeated here. Parts of our results as well as of the corresponding non-perturbative study \cite{4} have already been published in ref. \cite{2}. This paper is organised as follows: In sect. 2 we recall some well-known facts
about the renormalization group, in particular the relation between running parameters and the renormalization group invariants. We then review the SF scheme and present our perturbative results for the renormalized quark mass in the continuum limit (sect. 3). Cutoff effects are discussed in sect. 4 and we end with a short summary. Finally, an appendix has been included to indicate the changes to appendix A of ref. [5] and appendix B of ref. [6] for our more general choice of parameters.

2 Renormalization group

In order to put the present work into its context we review some aspects of the renormalization group for QCD with $N$ colors and $N_f$ quark flavors and a diagonal quark mass matrix.

2.1 Callan Symanzik equation and running parameters

In the following it is assumed that the theory has been regularized, e.g. through the introduction of a space-time lattice, and that all renormalization conditions are independent of the quark masses. Any physical quantity $P$ is a renormalization group invariant, i.e. as a function of the normalization mass $\mu$, the renormalized coupling $g_R$ and the renormalized quark masses $m_{R,s}$, $s = 1, \ldots, N_f$, it satisfies the Callan-Symanzik equation,

$$\left\{ \frac{\partial}{\partial \mu} + \beta(g_R) \frac{\partial}{\partial g_R} + \tau(g_R) \sum_{s=1}^{N_f} m_{R,s} \frac{\partial}{\partial m_{R,s}} \right\} P = 0. \quad (2.1)$$

The renormalization group functions $\beta$ and $\tau$ can be derived from the relation between the bare and renormalized parameters, using the independence of the former upon $\mu$. The precise formulae depend on the details of the regularization and will not be needed in the following.

For small couplings $\beta$ and $\tau$ admit asymptotic expansions of the form

$$\beta(g) \xrightarrow{g \to 0} -g^3 \sum_{k=0}^{\infty} b_k g^{2k}, \quad (2.2)$$

$$\tau(g) \xrightarrow{g \to 0} -g^2 \sum_{k=0}^{\infty} d_k g^{2k}, \quad (2.3)$$

with coefficients which are renormalization scheme dependent in general. In the minimal (MS) or modified minimal ($\overline{\text{MS}}$) scheme of dimensional regularization $b_k$ and $d_k$ are known for $k \leq 3$ [7–9], the first few being given by (with
\[ C_F = (N^2 - 1)/2N \],

\[ b_0 = \left\{ \frac{11}{2} N - \frac{2}{3} N_f \right\}(4\pi)^{-2}, \quad (2.4) \]
\[ d_0 = 6 C_F (4\pi)^{-2}, \quad (2.5) \]
\[ b_1 = \left\{ \frac{3}{2} N^2 - \left( \frac{15}{2} N - N^{-1} \right) N_f \right\}(4\pi)^{-4}, \quad (2.6) \]
\[ d_1 = C_F \left\{ \frac{203}{6} N - \frac{3}{2} N^{-1} - \frac{10}{3} N_f \right\}(4\pi)^{-4}. \quad (2.7) \]

While the minimal schemes of dimensional regularization are only defined in a perturbative framework, we emphasize that \( \beta \) and \( \tau \) are in general non-perturbatively defined functions. If these are given, running parameters at the momentum scale \( q \) are obtained by integrating the equations

\[ q \frac{\partial \bar{g}}{\partial q} = \beta(\bar{g}), \quad q \frac{\partial \bar{m}_s}{\partial q} = \tau(\bar{g}) \bar{m}_s, \quad (2.8) \]

with the boundary conditions

\[ \bar{g}(\mu) = \bar{g}_R, \quad \bar{m}_s(\mu) = \bar{m}_{R,s}, \quad s = 1, \ldots, N_f. \quad (2.9) \]

The running parameters are related to the renormalization group invariant (RGI) quark masses \( M_s \) and the \( \Lambda \) parameter, through \( (s = 1, \ldots, N_f) \),

\[ \Lambda = q \left( b_0 \bar{g}^2 \right)^{-b_1/2b_0^2} \exp \left\{ -\frac{1}{2b_0 \bar{g}^2} \right\} \]
\[ \times \exp \left\{ - \int_0^{\bar{g}} dx \left[ \frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\}, \quad (2.10) \]
\[ M_s = \bar{m}_s \left( 2b_0 \bar{g}^2 \right)^{-d_0/2b_0} \exp \left\{ - \int_0^{\bar{g}} dx \left[ \frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x} \right] \right\}. \quad (2.11) \]

One easily checks that the above expressions for \( \Lambda \) and \( M_s \) are indeed solutions of the Callan-Symanzik equation (2.1).

### 2.2 Finite renormalizations

Any two mass independent renormalization schemes can be related by a finite parameter renormalization of the form

\[ \mu' = c \mu, \quad c > 0, \quad (2.12) \]
\[ g'_R = g_R \sqrt{\lambda(g_R)}, \quad (2.13) \]
\[ m'_{R,s} = m_{R,s} \lambda(g_R), \quad s = 1, \ldots, N_f, \quad (2.14) \]
where we assumed the quark mass matrix to be diagonal in both schemes. The invariance of a physical observable $P$ under such a change of variables implies the existence of a function $P'$ such that

$$P'(\mu', g'_R(g_R), \{m'_{R,s}(g_R, m_{R,s})\}) = P(\mu, g_R, \{m_{R,s}\}), \quad (2.15)$$

and $P'$ satisfies the Callan-Symanzik equation in the primed scheme with renormalization group functions $\beta'$ and $\tau'$, given by

$$\beta'(g'_R) = \left\{ \beta(g_R) \frac{\partial g'_R}{\partial g_R} \right\}_{g_R=g_R(g'_R)}, \quad (2.16)$$

$$\tau'(g'_R) = \left\{ \tau(g_R) + \beta(g_R) \frac{\partial}{\partial g_R} \ln \mathcal{X}_m(g_R) \right\}_{g_R=g_R(g'_R)}. \quad (2.17)$$

In perturbation theory, the finite renormalization constants $\mathcal{X}_g$ and $\mathcal{X}_m$ are expanded according to

$$\mathcal{X}(g_R) \overset{g_R \to 0}{\sim} 1 + \sum_{k=1}^{\infty} \mathcal{X}^{(k)} \frac{g_R^{2k}}{2k}. \quad (2.18)$$

Inserting into eqs. (2.16),(2.17) one finds that $b_0, b_1$ and $d_0$ are the same in both schemes (these are the “universal” coefficients), while all other coefficients are scheme dependent. In particular, the two-loop anomalous quark mass dimensions are related by

$$d'_1 = d_1 + 2b_0 \mathcal{X}_m^{(1)} - d_0 \mathcal{X}_g^{(1)}. \quad (2.19)$$

Renormalization group invariant parameters can also be formed in the primed scheme and one finds

$$\Lambda' = \Lambda \exp \left\{ \mathcal{X}_g^{(1)} / 2b_0 \right\}, \quad (2.20)$$

$$M'_s = M_s, \quad s = 1, \ldots, N_f. \quad (2.21)$$

Therefore, given these parameters in some renormalization scheme, they are \textit{exactly} known in any other scheme by just computing the one-loop relation between the renormalized coupling constants. Moreover, any physical observable can be considered a function of the renormalization group invariant parameters, i.e. there exists a function $\hat{P}$ such that

$$\hat{P}(\Lambda, \{M_s\}) = P(\mu, g_R, \{m_{R,s}\}). \quad (2.22)$$

It thus appears natural to regard the non-perturbatively defined renormalization group invariants $\{M_s\}_{s=1,\ldots,N_f}$ and $\Lambda$ (in some renormalization scheme) as the fundamental parameters of QCD.
2.3 Determination of the RGI parameters

In sect. 3 we will review the SF scheme, which is a non-perturbatively defined mass-independent renormalization scheme. The non-perturbative evolution of the running parameters in this scheme can be traced using numerical simulations and the functions $\bar{g}$ and $\bar{m}_s$ are then known up to some high energy scale $q$. One may look for the onset of perturbative evolution, and, once arrived in the perturbative regime, link the running parameters to the renormalization group invariants, using eqs. (2.10), (2.11) and the perturbative expansion of $\beta$ and $\tau$. In order to achieve the desired precision this requires the knowledge of the two-loop anomalous dimension $d_1^{SF}$ and the coefficient $b_2^{SF}$ of the $\beta$-function in the SF scheme [4].

The easiest method to obtain $d_1^{SF}$ consists in calculating the one-loop relation between the renormalized parameters in the SF scheme and some other scheme in which $d_1$ is already known [cf. eq. (2.19)]. For the purpose of our perturbative study the \(\overline{\text{MS}}\) scheme is an appropriate reference scheme and we may thus use the result (2.7).

3 The SF scheme

We consider QCD on a finite (Euclidean) space-time manifold of size $T \times L^3$ with Schrödinger functional boundary conditions for the fields [10–12]. All dimensionful quantities are defined in units of $L$. In particular, for a given correlation function, the ratio $\rho = T/L$ is assumed to be fixed so that there is (apart from quark masses) only a single scale, $L$, in the theory, which plays the rôles of the inverse normalization mass, $L = 1/\mu$ [1].

In addition to dependence on the geometrical ratio $\rho$, correlation functions will also be functions of the parameter $\theta$ appearing in the definition of the spatial boundary conditions on the quark fields (see e.g. eq.(4.8) in ref. [12]). One is completely free to employ different choices of the parameters $\rho$ and $\theta$ for the definitions of different physical quantities. This is a very convenient aspect of the SF framework which we will exploit; it can however lead to some slight abuse of notation which we hope is not confusing in our presentation.

3.1 Renormalized coupling

We start with the definition of the renormalized coupling constant in the SF scheme. This was introduced in ref. [14] for the pure gauge theory. The case of $N = 3$ colors, to which we restrict attention in this paper, was considered in detail in ref. [13], and we take over the background gauge field and choice of the parameter $\rho = 1$ specified there.
These choices were also made for the extension of the definition of the SF coupling to full QCD discussed in ref. [14]. There the coupling was renormalized at the physical value of the quark mass, thus leading to a quark mass dependent $\beta$ function. Here we will deviate from this definition and adopt a mass independent renormalization scheme as advocated in ref. [12]. Renormalizing the coupling at vanishing quark mass leads to the quark mass independent relation between the renormalized couplings

$$\bar{g}^2_{\text{SF}}(L) = \bar{g}^2_{\text{MS}}(q)X_g(\bar{g}_{\text{MS}}(q)),$$  (3.1)

The SF coupling has been computed to one-loop order of perturbation theory in refs. [13,14],

$$X^{(1)}_g = 2b_0 \ln(qL) - \frac{1}{4\pi}(c_{1,0} + c_{1,1}N_f),$$  (3.2)

where the coefficient $c_{1,1}$ depends on the parameter $\theta$. As a result of the detailed study in ref. [14] the particular choice $\theta = \pi/5$ was recommended for the definition of the SF coupling in QCD, in which case one obtains

$$c_{1,0} = 1.25563(4), \quad c_{1,1} = 0.039863(2).$$  (3.3)

In this paper we will also adopt this choice, but it will be clear where numerical results will change if in future simulations another choice of $\theta$ is employed for $g_{\text{SF}}$, and in fact it turns out that our general conclusions are not dependent on this.

### 3.2 The rôle of the PCAC relation

In QCD with $N_f \geq 2$ quark flavors the PCAC relation provides an attractive starting point for the non-perturbative definition of a renormalized quark mass. Denoting the isospin non-singlet axial current and density by $A^a_\mu$ and $P^a$, respectively, the PCAC relation (for mass degenerate quarks),

$$\partial_\mu A^a_\mu = 2mP^a,$$  (3.4)

is a local relation between composite fields which is expected to hold when inserted in Euclidean correlation functions up to contact terms.

The normalization of the axial current is conventionally fixed by requiring current algebra relations to assume their canonical form [15,16]. The non-linearity of these relations furthermore implies that the anomalous dimension of the axial current vanishes and so does the total anomalous dimension of the right hand side of eq. (3.4). Therefore, a renormalized quark mass can be
defined through the PCAC relation by providing an independent renormalization condition for the axial density. Below the running quark mass in the SF scheme will be defined along these lines by using correlation functions derived from the QCD Schrödinger functional.

3.3 The renormalized axial density

To define the renormalized axial density in the SF scheme we regularize the theory and choose the framework of $O(a)$ improved lattice QCD as discussed in ref. [12]. This choice is motivated by the corresponding non-perturbative studies [4], for which a perturbative investigation of the cutoff effects provides complementary information (cf. sect. 4). We emphasize, however, that the results for the anomalous dimension are independent of this choice, and e.g. dimensional regularization would have been a practical alternative. In the following we use notations and conventions as in ref. [12] without further notice.

The renormalized $O(a)$ improved axial density has the form

$$ (P_R)^a = Z_P (1 + b_p a m_q) P^a, \quad P^a = \bar{\psi} \gamma_5 \frac{1}{2} \tau^a \psi, \quad (3.5) $$

where $\tau^a$ are the Pauli matrices acting in flavor space. If chosen appropriately the improvement coefficient $b_p$ cancels cutoff effects in on-shell correlation functions which are proportional to the subtracted bare quark mass $m_q = m_0 - m_c$.

To define $Z_P$ we recall the definition of the bare correlation functions $f_P$ and $f_1$,

$$ f_P(x) = -a^6 \sum_{y,z} \frac{1}{3} \langle P^a(x) \bar{\zeta}(y) \gamma_5 \frac{1}{2} \tau^a \zeta(z) \rangle, \quad (3.6) $$

$$ f_1 = -a^{12} L^6 \sum_{u,v,y,z} \frac{1}{3} \langle \bar{\zeta}'(u) \gamma_5 \frac{1}{2} \tau^a \zeta'(v) \bar{\zeta}(y) \gamma_5 \frac{1}{2} \tau^a \zeta(z) \rangle. \quad (3.7) $$

The boundary source fields $\zeta, \bar{\zeta}$ and $\zeta', \bar{\zeta}'$ are renormalized multiplicatively with a common renormalization constant $Z_{\zeta} [17,12]$. One may therefore define the renormalization constant of the axial density through the ratio [12]

$$ Z_P(g_0, L/a) = c \frac{\sqrt{f_1}}{f_P(T/2)}, \quad (3.8) $$

at vanishing quark mass $m_q = 0$ and for vanishing boundary gauge fields. Here, the constant $c$ is chosen such that $Z_P = 1$ holds exactly at tree-level of
perturbation theory. Using the notation of refs. [6,5] with the modifications as indicated in the appendix we obtain

\[ c = \frac{u_0}{\sqrt{t_0}} = \sqrt{N} + O(a^2). \]  

(3.9)

In general, \( c \) is a computable constant for a given lattice size and values of \( \rho \) and \( \theta \).

The implicit dependence of \( Z_P \) upon \( \rho \) and \( \theta \) will be discussed later. Here we emphasize that \( Z_P \) is quark mass independent. Therefore, not only the \( \beta \) function but also the anomalous dimension

\[ \tau_{SF}(g_R) = L \left. \frac{\partial \ln Z_P(g_0, L/a)}{\partial L} \right|_{g_0 = g_0(g_R)} \]  

(3.10)

is quark mass independent. Here \( g_R \) stands for \( \bar{g}_{SF}(L) \), and its (mass independent) relation to \( g_0 \) is currently known to one-loop order [14] and to two-loop order in quenched QCD [18].

In bare perturbation theory the renormalization constant \( Z_P \) has an expansion

\[ Z_P(g_0, L/a) = 1 + \sum_{k=1}^{\infty} Z_P^{(k)}(L/a) g_0^{2k}, \]  

(3.11)

where in the limit \( a/L \to 0 \) the coefficients \( Z_P^{(k)} \) are polynomials in \( \ln(L/a) \) of degree \( k \) up to corrections of \( O(a/L) \). In particular the coefficient of the logarithmic divergence in \( Z_P^{(1)} \) is given by the one-loop anomalous quark mass dimension \( d_0 \), and thus we parametrize \( Z_P^{(1)} \) as

\[ Z_P^{(1)} = C_F z_p(\theta, \rho) - d_0 \ln(L/a) + O(a/L). \]  

(3.12)

Here we have made explicit the dependence of the cutoff independent term \( z_p \) on \( \theta \) and \( \rho \), which is inherited by any renormalized quantity involving the pseudoscalar density. The Feynman diagrams contributing to the correlation functions \( f_P, f_1 \) (and hence to \( Z_P \)) at one loop order have been discussed in detail in refs. [6,5]. Here we use these results to extract \( Z_P^{(1)} \) for various different choices of the parameters \( \rho = T/L \) and \( \theta \). Some typical results for \( z_p \) are collected in table 1. In section 4 we will also consider the \( \theta, \rho \) dependence of the remaining cutoff terms in (3.12).

### 3.4 Renormalized quark masses

With the \( O(a) \) improved bare axial current

\[ (A_1)^a_\mu = A_\mu^a + c_A a_\frac{1}{2}(\partial_\mu + \partial_\mu)P^a, \quad A_\mu^a = \bar{\psi} \gamma_\mu \gamma_5 \frac{1}{2} T^a \psi, \]  

(3.13)
the renormalized current takes the form,

$$(A_R)_\mu^a = Z_A(1 + b_A am_q)(A_1)_\mu^a.$$  \hfill (3.14)

To define the renormalized quark mass we first introduce a bare current quark mass through the PCAC relation between the unrenormalized fields,

$$m = \frac{1}{2}\left(\partial_0^* + \partial_0\right)f_A(x_0)\left|_{x_0=T/2}\right.$$  \hfill (3.15)

Here, the correlation function of the axial current is defined as $f_P$ in eq. (3.6), but with the axial density replaced by the zero component of the (improved) axial current (3.13).

The renormalized mass in the SF scheme is defined through the PCAC relation involving the renormalized $O(a)$ improved fields. This leads to the relation

$$\bar{m}_{SF}(L) = m\left(\frac{1 + b_A am_q}{1 + b_P am_q}\right)Z_A = mZ_A/Z_P + O(a).$$  \hfill (3.16)

In order to compute the two-loop anomalous dimension in this scheme we first relate $\bar{m}_{SF}(L)$ to the running mass in the $\overline{MS}$ scheme. To this end we start by combining the result for the axial current renormalization constant $[19,16]$

$$Z_A^{(1)} = -0.087344(1) \times C_F,$$  \hfill (3.17)

with the one-loop value of the ratio $m_q/m$, which we obtained in the course of calculations done in ref. [5]. We then arrive at

$$\bar{m}_{SF}(L) = m_q\left\{1 + g_0^2[d_0 \ln(L/a) - (z_p + 0.019458(1))C_F] + O(g_0^4)\right\}. \hfill (3.18)$$
The corresponding relation for the renormalized $\overline{\text{MS}}$ mass has first been obtained in ref. [19] and since then verified by many others, including one of the present authors. Then using the result

$$m_{\overline{\text{MS}}}(q) = m_q \{ 1 + g_0^2 [-d_0 \ln(aq) + 0.122282(1) \times C_F] + O(g_0^4) \}, \quad (3.19)$$

we obtain the one-loop coefficient

$$\lambda_m^{(1)} = d_0 \ln(qL) - (z_p + 0.141740(2)) C_F. \quad (3.20)$$

To finally obtain the two-loop anomalous dimension $d_{1i}^{\text{SF}}$ we may now use the known result for $d_1$ in the $\overline{\text{MS}}$ scheme [cf. eq. (2.7)] and combine it with the one-loop coefficients (3.2) and (3.20) according to eq. (2.19). Proceeding in this way, we have avoided to expand eq. (3.8) to order $g_0^4$ which would have required a two-loop computation. A few numerical values are given in table 1.

3.5 Zero-momentum gluon exchange

The results for $Z_P^{(1)}$ and $d_{1i}^{\text{SF}}$ with $\rho = 2$ show a strong dependence on the parameter $\theta$. In the corresponding non-perturbative study a similar behavior is only seen at very small couplings [3]. While this is not a problem in principle, it makes it more difficult to connect the perturbative regime to low energy physics along the lines of ref. [1].

A closer look into the one-loop computation reveals that this strong $\theta$-dependence is almost entirely due to the exchange of gluons with zero spatial momentum [20]. These contributions are gauge invariant by themselves and it is not too difficult to compute them analytically. Setting

$$w = \sqrt{3} \theta \rho, \quad (3.21)$$

we find

$$z_p(\theta, \rho)_{\text{mom}=0} = -\frac{\rho^3}{192w^4 \cos^2 w} \left\{ 72 + 24w^2 - 5w^4 - 96 \cosh w 
+ (3w^4 + 24) \cosh 2w - 12w \sinh 2w \right\}, \quad (3.22)$$

with the special case

$$z_p(0, \rho)_{\text{mom}=0} = -\frac{1}{32} \rho^3. \quad (3.23)$$

\footnote{Ref. [19], although analytically correct, contains a small error in the quoted numerical results, which is caused by setting $F_{0001} = 1.41$ rather than $F_{0001} = 1.310962...$}
One notices the overall factor $\rho^3$ which enhances the $\theta$-dependence of $z_p$ for large $\rho$ and results in the strong $\theta$-dependence of $d_{1}^{\beta}$ for $\rho = 2$. If a subtracted constant is defined through
\[
\Delta z_p = z_p - z_p|_{\text{mom}=0},
\] (3.24)
the remaining $\theta$-dependence is indeed very weak. We thus recommend to define the renormalized axial density with $\rho = 1$. Having fixed this parameter we furthermore choose $\theta = 0.5$ which leads to a conveniently small value of $d_{1}^{\beta}/d_0$ (cf. table 1). This completes the definition of the running parameters in the SF scheme.

4 One-loop cutoff effects in the step scaling function

An infinitesimal variation of $Z_P$ with the scale $L$ defines the anomalous dimension $\tau_{SF}(g_R)$ [cf. eq. (3.10)]. In the context of numerical simulations it is more convenient to consider finite variations of the scale, e.g. a change from $L$ to $sL$, with a scale factor $s$. This leads to the definition of the step scaling function
\[
\Sigma_P(s, g_R^2, a/L) = \frac{Z_P(g_0, sL/a)}{Z_P(g_0, L/a)}|_{g_0=g_0(g_R)},
\] (4.1)
with continuum limit
\[
\lim_{a \to 0} \Sigma_P(s, g_R^2, a/L) = \sigma_P(s, g_R^2),
\] (4.2)
to be taken at fixed $g_R = \bar{g}_{SF}(L)$.

We set the scale factor to $s = 2$ in the following. To one-loop order of perturbation theory we have
\[
\Sigma_P(2, g_R^2, a/L) = 1 + k(L/a)g_R^2 + O(g_R^4),
\] (4.3)
with
\[
k(L/a) = Z_P^{(1)}(2L/a) - Z_P^{(1)}(L/a).
\] (4.4)

Using the notation of refs. [3, 5] for the renormalized correlation functions $f_P$ and $f_1$, the one-loop coefficient on a finite lattice takes the form
\[
Z_P^{(1)}(L/a) = \frac{t_1}{2t_0} - \frac{u_1}{u_0} + c_i^{(1)} \left[ \frac{t_2}{2t_0} - \frac{u_2}{u_0} \right] + a m_i^{(1)} \left[ \frac{t_3}{2t_0} - \frac{u_3}{u_0} \right].
\] (4.5)
In order to see how fast the continuum limit,
\[ k(\infty) = -d_0 \ln(2), \tag{4.6} \]
is approached we define
\[ \delta_k(L/a) = k(L/a)/k(\infty) - 1. \tag{4.7} \]
This quantity contains all lattice artifacts at \( O(g_k^2) \). In the framework of \( O(a) \) improved lattice QCD these are expected to decrease asymptotically with a rate proportional to \( a^2/L^2 \). In the present case, \( O(a) \) improvement is achieved by setting \( c_{sw}^{(0)} = 1 \). In particular, the boundary counterterms proportional to \( c_t \) and \( \tilde{c}_t \) are not needed at this order of perturbation theory, owing, in the case of \( \tilde{c}_t \), to the identity
\[ \frac{t_2}{2t_0} - \frac{u_2}{u_0} = 0. \tag{4.8} \]

Some results for \( \delta_k \) are tabulated in table 2. The numerical values have been obtained by inserting the exact expressions for the coefficients \( u_0, u_1, u_3 \) and \( t_0, t_1, t_3 \) for the given lattice size, and the coefficient \( a m_c^{(1)} = -0.2025565(1) \times C_F \) into eq. (4.5).

Lattice artifacts appear to be reasonably small for all parameter choices considered. However, we observe that cutoff effects for \( \rho = 2 \) are generally larger than for \( \rho = 1 \). Furthermore the asymptotic \( O(a^2) \) decay seems to set in earlier for \( \rho = 2 \). It turns out that both effects are largely due to the zero spatial momentum gluon exchange contributions. Subtracting these contributions from \( \delta_k \) we define
\[ \Delta \delta_k \overset{\text{def}}{=} [k(L/a) - k(L/a)|_{\text{mom}=0}]/k(\infty) - 1. \tag{4.9} \]
and list the numerical values in table 2. The subtraction terms are computed numerically for the given lattice size. In the special case \( \theta = 0 \) we also obtained a compact analytical formula,
\[ k(L/a)|_{\text{mom}=0, \theta=0} = \left( \frac{3\rho a^2}{16L^2} - \frac{63a^3}{64L^3} \right) C_F. \tag{4.10} \]
For \( \rho = 2 \) one clearly sees that the cutoff effects are dominated by the zero-momentum contributions. In the case \( \theta = 0 \) it is the explicit factor \( \rho \) in eq. (4.10) which enhances the cutoff effects and also explains the early onset of the \( O(a^2) \) behavior for \( \rho = 2 \). We conclude by noting that cutoff effects with the parameters \( \rho = 1 \) and \( \theta = 0.5 \) are indeed quite small, a fact that partially motivated this choice.
Table 2: The one-loop cutoff effects $\delta_k$ in the step scaling function, with and without zero momentum gluon contributions, for the same choice of parameters as in sect. 3.

5 Summary

We have carried out a perturbative investigation of a two-parameter family of running quark masses in the SF scheme. Its definition is based on the PCAC relation between correlation functions derived from the Schrödinger functional, together with an independent renormalization condition for the axial density. At one-loop order of perturbation theory and for asymmetric space-time volumes with $T = 2L$, many correlation functions show a strong dependence on the parameter $\theta$. Its origin could be traced back to the contribution of gluon exchange with vanishing spatial momentum. Non-perturbatively this behavior is only matched at very short distances $[4]$, making it more difficult to apply the strategy of non-perturbative renormalization as outlined in refs. $[1,2]$. However, setting $T = L$ completely eliminates this problem and also leads to a reasonably small two-loop anomalous dimension $d^{\text{SF}}$. We believe that this situation is generic and thus generally recommend the choice of symmetric space-time volumes for the study of scale dependent renormalization constants in the SF scheme.

In the present case we made the additional choice of $\theta = 0.5$ thus com-
pleting the definition of the running quark mass. One-loop cutoff effects in
the step-scaling function are found to be reasonably small if \( O(a) \) improved
lattice QCD is used as a regularization. Finally, we mention that a corre-
sponding non-perturbative study of the running quark mass is in progress \[3\],
and preliminary results have been reported in ref. \[2\].

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Appendix A

The expansions of the functions $t_i, u_i, v_i, w_i, y_i \ i \neq 1$ up to corrections of order $(a/L)^2$ for general $\rho = T/L$ are as given in appendix B of ref. [3] and in appendix A of ref. [5] except that $s_i, c_i$ are now defined by

\begin{align}
    s_i &= \sinh(\sqrt{3} \theta \rho), \\
    c_i &= \cosh(\sqrt{3} \theta \rho),
\end{align}

and the expressions for $i = 3$ now read

\begin{align}
    t_3 &= -\frac{2N s_i}{\sqrt{3} \theta c_i^3} \frac{L}{a} + \frac{2N}{c_i^2} \\
    &\quad - \left\{ \frac{19N \theta s_i}{3\sqrt{3} c_i^3} - \frac{4\rho N \theta^2 (1 - 2 s_i^2)}{3 c_i^4} \right\} \frac{a}{L}, \tag{A.3}
\end{align}

\begin{align}
    u_3 &= -\frac{N s_i}{\sqrt{3} \theta c_i^2} \frac{L}{a} - \left\{ \frac{N \theta s_i}{6\sqrt{3} c_i^2} - \frac{2\rho N \theta^2 (1 - s_i^2)}{3 c_i^3} \right\} \frac{a}{L}, \tag{A.4}
\end{align}

\begin{align}
    v_3 &= -\frac{N s_i (c_i - 2)}{\sqrt{3} \theta c_i^3} \frac{L}{a} - \left\{ \frac{N \theta s_i (c_i - 2)}{6\sqrt{3} c_i^2} \\
    &\quad + \frac{2\rho N \theta^2}{3 c_i^4} (c_i^3 - 4 c_i^2 - 2 c_i + 6) \right\} \frac{a}{L}, \tag{A.5}
\end{align}

\begin{align}
    w_3 &= \frac{2N c_i}{co} \frac{L}{a} + \left\{ \frac{3N \theta^2}{c_i} + \frac{4\rho N \theta^3 s_i}{\sqrt{3} c_i^2} \right\} \frac{a}{L}, \tag{A.6}
\end{align}

\begin{align}
    y_3 &= -\frac{N s_i (c_i + 2)}{3\sqrt{3} \theta c_i^3} \frac{L}{a} - \left\{ \frac{N \theta s_i (c_i + 2)}{18\sqrt{3} c_i^2} \\
    &\quad + \frac{2\rho N \theta^2}{9 c_i^4} (c_i^3 + 4 c_i^2 - 2 c_i - 6) \right\} \frac{a}{L}. \tag{A.7}
\end{align}
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