On the homotopy groups of $E(n)$–local spectra with unusual invariant ideals

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Let $E(n)$ and $T(m)$ for nonnegative integers $n$ and $m$ denote the Johnson–Wilson and the Ravenel spectra, respectively. Given a spectrum whose $E(n)_*$–homology is $E(n)_*(T(m))/(v_1, \ldots, v_{n-1})$, then each homotopy group of it estimates the order of each homotopy group of $L_n T(m)$. We here study the $E(n)$–based Adams $E_2$–term of it and present that the determination of the $E_2$–term is unexpectedly complex for odd prime case. At the prime two, we determine the $E_\infty$–term for $\pi_*(L_2 T(1))/(v_1)$, whose computation is easier than that of $\pi_*(L_2 T(1))$ as we expect.

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1 Introduction

In [4], Ravenel has constructed the homotopy associative commutative ring spectrum $T(m)$ as a summand of $p$–component of the Thom spectrum associated with the map $\Omega SU(p^m) \to BU$. It is extensively used in [4, Section 7] to compute the homotopy groups of spheres in terms of “the method of infinite descent”. The Adams–Novikov $E_2$–term converging to the stable homotopy groups $\pi_*(T(m))$ is described by use of the Hopf algebroid $(BP_*, \Gamma(m+1))$ (cf [4, Definition 7.1.1]). In particular, the 0–th line is

$$\text{Ext}^0_{\Gamma(m+1)}(BP_*, BP_*) = \mathbb{Z}_p[v_1, \ldots, v_m] \subset BP_* = \mathbb{Z}_p[v_1, \ldots],$$

and the more the value of $m$, the more primitives we obtain. Since $v_k$ for $1 \leq k \leq m$ is a permanent cycle of the spectral sequence, we obtain spectra $T(m)/(v_k)$ and $T(m)/(v_k, v_l)$ for $1 \leq k, l \leq m$ (see Lemma 3.7.) Here $T(m)/J$ for an ideal $J$ of $BP_*$ denotes a spectrum such that $BP_*(T(m)/J) = BP_*/J$.

Let $BP\langle n \rangle$ denote the Johnson–Wilson ring spectrum with $BP\langle n \rangle_*= \mathbb{Z}_p[v_1, \ldots, v_n]$ and put $E(n) = v_n^{-1}BP\langle n \rangle$ as usual. Then we have the $E(n)$–based Adams spectral sequence $E_1^{s,t}(X) \Rightarrow \pi_*(L_n X)$ for a spectrum $X$, whose $E_2$–term is $E_2^s(X) = \text{Ext}^{s}\text{Ext}_E^{s+n}(E(n)_*, E(n)_*) (E(n)_*, E(n)_*(X))$. Here $L_n$ denotes the Bousfield localization functor with respect.
Theorem 1.1

For $E_2$–term for a spectrum $X$ with $E(n)_*(X) = E(n)_*/J[t_1, \ldots, t_m]$ for an ideal $J$ of $E(n)_*$, we introduce the generalized Johnson–Wilson spectrum $E_*(n) = v_n^{-1}BP(n + m)$. Then

$$\Sigma(n, m) = E_m(n)_* \otimes_{BP_*} BP_*[t_{m+1}, t_{m+2}, \ldots] \otimes_{BP_*} E_m(n)_*$$

is a Hopf algebroid over $E_m(n)_*$, and the $E(n)$–based Adams $E_2$–term $E_2^*(X)$ is isomorphic to $\text{Ext}_{\Sigma(n,m+1)}(E_m(n)_*, E_m(n)_*/J)$, which we denote $\text{Ext}^*(E_m(n)_*/J)$, by a similar change-of-rings theorem of Hovey and Sadofsky [1].

Consider $J_n$ be the sequence $v_1, v_2, \ldots, v_{n-1}$. Then $T(m)/(J_n)$ exists if $n \leq 2$ as commented above. Besides, if $L_nT(m)/J$ exists, then the $E(n)$–based Adams $E_2$–term for $\pi_*(L_nT(m)/J)$ is isomorphic to an Ext group $\text{Ext}^*(E(n)_*/J)$. Consider the long exact sequence of Ext groups associated to the short exact sequence

$$0 \rightarrow E_m(n)_*/(J_n) \rightarrow p^{-1}E_m(n)_*/(J_n) \rightarrow E_m(n)_*/(p^\infty, J_n) \rightarrow 0.$$

Since $\text{Ext}^*(p^{-1}E_m(n)_*/(J_n)) = \mathbb{Q}$, Corollary 4.5 implies our first theorem:

**Theorem 1.1** The Ext group $\text{Ext}^0(E_m(n)_*/(J_n))$ is isomorphic to $\mathbb{Z}(p)$, and the group $E_2^0(E_m(n)_*/(J_n))$ is isomorphic to the direct sum of the cyclic module over the ring $\mathbb{Z}(p)[v_1^{\pm 1}, v_{n+1}, \ldots, v_m]$ generated by

$$\frac{v_1^{e_1} \cdots v_n^{e_n}}{p^{1+\nu(e)}}$$

of order $p^{1+\nu(e)}$ with $\nu(e) = \min\{\nu(e_1), \ldots, \nu(e_n)\}$, where the integer $\nu(\ell)$ denotes the maximal power of $p$ that divides $\ell$.

For the case where $n > m$, we have an example which has a similar result to the above theorem (cf Proposition 4.7):

**Proposition 1.2** The $E(2)$–based Adams $E_2$–term $E_2^0(T(1)/(v_1))$ is isomorphic to $\mathbb{Z}(p)$ and $E_2^0(T(1)/(v_1))$ is the direct sum of the cyclic module over $\mathbb{Z}(p)$ generated by $v_2^{mp} v_3^{p^j} / p^{1+\min(i,j)}$ of order $p^{1+\min(i,j)}$.

In these cases, we did not determine $E_2^s$ for $s > 1$ since there is an obstruction, which comes from the generators known as $b_{ij}$ (see (3–2)). This is what we did not expect. For $p = 2$, we have the relation $b_{ij} = h_{ij}^2$, which makes possible to compute for $s > 1$. Since the $E(2)$–based Adams differentials are read off from Mahowald and Shimomura [2], we obtain the $E_\infty$–term.
Theorem 1.3  Let $p = 2$. The $E(2)$–based Adams $E_\infty$–term for $\pi_*(L_2T(1)/(v_1))$ is isomorphic to $\mathbb{Z}(2)$ if $s = 0$ and is isomorphic to the tensor product of $\Lambda(\rho_2)$ and the direct sum of

1. $v_2A[h_20]$, $v_3B[h_30]/(h_3^3)$ and $v_3Bh_30h_31$ whose elements are of order two,
2. $M^0$ and $M^1$.

Here the modules are given in Section 5.

In Section 2, we consider the Hopf algebroid $(E_m(n)_s, \Sigma(n, m + 1))$ and show a variation of the change-of-rings theorem given in Hovey and Sadofsky [1]. In Section 3, we exhibit the formulas for the structure maps (the right unit $\eta_R$ and the diagonal maps $\Delta$). We then observe the existence of spectra of the form $T(m)/J$. Section 4 is devoted to prove Theorem 1.1 and Proposition 1.2. In Section 5, we determine the $E_\infty$–term for $\pi_*(L_2T(1)/(2^\infty, v_1))$. The homotopy groups $\pi_*(L_2T(1))$ is determined easily if $p$ is odd, and stays undetermined if $p = 2$. The result of this section is the first step to understand $\pi_*(L_2T(1))$ at the prime two.

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2  A generalized Johnson–Wilson theory

Let $BP$ and $BP\langle n \rangle$ denote the Brown–Peterson and the Johnson–Wilson spectra characterized by $\pi_*(BP) = BP_* = \mathbb{Z}(p)[v_1, \ldots, v_n, \ldots]$ and $\pi_*(BP\langle n \rangle) = BP\langle n \rangle_* = \mathbb{Z}(p)[v_1, \ldots, v_n] \subset BP_*$ with $|v_n| = |t_n| = 2(p^n - 1)$. Then the $BP_*$–homology of $BP$ is $BP_*(BP) = BP_*[t_1, \ldots, t_n, \ldots]$. We put

$$E_m(n) = v_n^{-1}BP\langle n + m \rangle$$

for nonnegative integers $n$ and $m$. Then

$$E_m(n)_s = E(n)_s[v_{n+1}, \ldots, v_{n+m}] \subset v_n^{-1}BP_*.$$

We notice that $E_0(n)$ is the localized Johnson–Wilson spectrum $E(n)$.

Let $\Gamma(m + 1)$ (cf Ravenel [4, 7.1.1]) be the $BP_*(BP)$–comodule defined by

$$\Gamma(m + 1) = BP_*(BP)/(t_1, \ldots, t_m) = BP_*[t_{m+1}, t_{m+2}, \ldots].$$
Then the pair \((BP_*, \Gamma(m + 1))\) has the structure of the Hopf algebroid inherited from \((BP_*, BP_*BP))\). Put
\[
\Sigma_m(n, i) = E_m(n)_* \otimes_{BP_*} \Gamma(i) \otimes_{BP_*} E_m(n)_*.
\]
In particular, we write
\[
\Sigma(n, m + 1) = \Sigma(n, m + 1) = E_m(n)_* \otimes_{BP_*} \Gamma(m + 1) \otimes_{BP_*} E_m(n)_*.
\]

The pair \((E_m(n)_*, \Sigma_m(n, i))\) is a Hopf algebroid with the structure maps inherited from those of the Hopf algebroid \((BP_*, \Gamma(i))\) for all \(i > 0\). Consider the map between Hopf algebroids \((E_m(n)_*, \Sigma_m(n, 1)) \rightarrow (E_m(n)_*, \Sigma(n, m + 1))\) induced from the projection from \(BP_*BP\) to \(\Gamma(m + 1)\). The map is normal and that
\[
(2\text{-}1)
E_m(n)_*(T(m)) = E_m(n)_* \Box_{\Sigma(n, m + 1)} \Sigma_m(n, 1)
\]
if \(m > 0\). Here, \(T(m)\) denotes the Ravenel spectrum [4, 6.5.1], which is an associative commutative ring spectrum characterized by \(BP_*(T(m)) = BP_*[t_1, \ldots, t_m]\).

Since \(\Sigma_m(n, 1)\) is \(E_m(n)_*(E_m(n))\), the change-of-rings theorem [4, A1.3.12] shows the following:

**Lemma 2.2** There is an isomorphism
\[
\text{Ext}_{E_m(n)_*}(E_m(n)_*, E_m(n)_*(T(m))) = \text{Ext}_{\Sigma(n, m + 1)}(E_m(n)_*, E_m(n)_*).
\]

**Remark 2.3** In general, equation \((2\text{-}1)\) does not hold if we work on \(E(n)_*E(n)\)–comodules. For example, if we set \((n, i) = (2, 3)\), then
\[
\Sigma_{0}(2, 3) = E(2)_*[t_3, t_4, \ldots]/(\eta_R(v_2) : k > 2).
\]

In the right hand side we have the relation \(v_2 t_1^{k^2} \equiv v_2^k t_1 \mod (p)\) since \(\eta_R(v_2) = 0\). On the other hand, we do not have any relation on \(t_1\) in \(E(2)_*T(2) = E(2)_*[t_1, t_2]\).

Since \(E_m(n)_*\) is a free \(E(n)_*\)–module over the bases \(v^E = v_{n+1}^{e_1} \cdots v_{n+m}^{e_m}\) for \(E = (e_1, \ldots, e_m)\) with \(e_k \geq 0\), there is a homotopy equivalence \(E_m(n) = \bigvee E \Sigma[E] E(n)\). This shows that the \(E(n)\)–based and the \(E_m(n)\)–based Adams spectral sequences agrees from the \(E_2\)–term (cf Hovey and Sadofsky [1]).

### 3 Existence of some spectra

An ideal \(I = (a_0, a_1, \ldots, a_{n-1})\) of \(BP_*\) is called invariant if \(\eta_R(a_i) \equiv a_i \mod (a_0, a_1, \ldots, a_{i-1})\) for each \(0 \leq i < n\) as a \(BP_*BP\)–comodule. It is well known that if
there is a spectrum $X$ such that $BP_*(X) = BP_*/I$, then $I$ is invariant. Consider now the Ravenel spectrum $T(m)$. Then the $E_2$-term of the Adams–Novikov spectral sequence for $\pi_*(W \wedge T(m))$ for a spectrum $W$ is isomorphic to an Ext group over the Hopf algebroid $(BP_*, \Gamma(m + 1))$. We call an ideal $J = (w_0, w_1, \ldots, w_{n-1})$ of $BP_*$ unusual if it is not invariant and $\eta_R(w_i) \equiv w_i \mod (w_0, w_1, \ldots, w_{i-1})$ for each $0 \leq i < n$ as a $\Gamma(m + 1)$–comodule. In the same manner as above, if there is a spectrum $X$ such that $BP_*(X) = BP_*/[t_1, \ldots, t_m]$ for $m > 0$, then $J$ is invariant or unusual. In this section, we study the existence of a spectrum $X$ with $BP_*$–homology (resp. $E(n)_*$–homology)

$$BP_*(X) = BP_*/[t_1, \ldots, t_m]$$

(resp. $E(n)_*(X) = E(n)_*/[t_1, \ldots, t_m]$)

for an unusual ideal $J$. We write $T(m)/J$ (resp. $L_q T(m)/J$) for such $X$.

The next lemma is verified by Hazewinkel’s and Quillen’s formulas (see Miller, Ravenel and Wilson [3, (1.1)–(1.3)]):

**Lemma 3.1** Assume that $n \leq m$. Let $J_n$ denote the ideal $(v_1, \ldots, v_{n-1})$ of $BP_*$. Then the structure maps in $(BP_*, \Gamma(m + 1))$ act as

\begin{align*}
\eta_R(v_k) & \equiv v_k & \text{for } n \leq k \leq m, \\
\eta_R(v_{m+k}) & \equiv v_{m+k} + pt_{m+k} & \text{for } 0 < k \leq n, \\
\Delta(t_{m+k}) & \equiv t_{m+k} \otimes 1 + 1 \otimes t_{m+k} & \text{for } 0 \leq k \leq n, \\
\Delta(t_{m+n+1}) & \equiv t_{m+n+1} \otimes 1 + 1 \otimes t_{m+n+1} + v_n b_{m+1,n-1}
\end{align*}

mod $J_n$, where

\begin{align*}
(3–2) \quad b_{i,j} = \left(\eta^i_{\varphi^j} \otimes 1 + 1 \otimes \eta^i_{\varphi^j} - (t_i \otimes 1 + 1 \otimes t_i)^{\varphi^j}\right)/p.
\end{align*}

By this lemma, we read off the behavior of the structure maps $\eta_R$ and $\Delta$ mod $J_n$ of the Hopf algebroid $(E(m)_*, \Sigma(n, m + 1))$. For $n > m$, we only consider the case where $n = 2$ and $m = 1$.

**Lemma 3.3** The structure maps in $(BP_*, \Gamma(2))$ acts as

\begin{align*}
\eta_R(v_i) & \equiv v_i + pt_i & \text{for } i = 2 \text{ and } 3, \\
\eta_R(v_4) & \equiv v_4 + v_2 t_2 + pt_4 + v_2 c_{21} - \eta_R(v_2) t_2, \\
\eta_R(v_5) & \equiv v_5 + v_3 t_3 + v_2 t_2 + pt_5 + v_2 c_{31} + v_3 c_{22} - \eta_R(v_2) t_2 - \eta_R(v_2) t_3, \\
\Delta(t_i) & \equiv t_i \otimes 1 + 1 \otimes t_i & \text{for } i = 2 \text{ and } 3, \\
\Delta(t_4) & \equiv t_4 \otimes 1 + 1 \otimes t_4 + t_2 \otimes t_2 + v_2 b_{21}, \\
\Delta(t_5) & \equiv t_5 \otimes 1 + 1 \otimes t_5 + t_3 \otimes t_3 + t_2 \otimes t_3 + v_2 b_{31} + v_3 b_{22}
\end{align*}
We consider the Adams–Novikov spectral sequence

\[ E_2^{*,*}(X) = \text{Ext}_B^{*,*}(BP_*, BP_*(X)) \implies \pi_*(X). \]

By the change-of-rings theorem [4, A1.3.12], we have an isomorphism

\[ E_2^*(T(m)/I_n) = \text{Ext}_{\Gamma(n+1)}^0(BP_*/I_n). \]

Hereafter we use the abbreviation:

\[ \text{Ext}_T(A, -) = \text{Ext}_I(-) \quad \text{for a Hopf algebroid} \ (A, \Gamma). \]

**Lemma 3.6** For \( 0 \leq k \leq m \),

\[ v_{n+k} \in E_2^0(T(m)/I_n) = \text{Ext}_{\Gamma(n+1)}^0(BP_*/I_n), \]

where \( I_n = (p) + J_n \).

**Lemma 3.7** Let \( M \) be a \( T(m) \)-module spectrum. If \( \alpha \) and \( \beta \) \( \in E_2(T(m)) \) are permanent cycles in the spectral sequence (3–4), then there exist spectra of the form \( M/(\alpha^a) \) and \( M/(\alpha^a, \beta^b) \) for positive integers \( a \) and \( b \). In particular, we have \( T(m)/(v_k^a) \) and \( T(m)/(v_k^a, v_k^b) \) for \( i, j, k < m + 2 \).

**Proof** Since \( M \) is a \( T(m) \)-module spectrum, the elements \( \alpha \) and \( \beta \) yield the self maps on \( M \), which we also denote by \( \alpha \) and \( \beta \). Now \( M/(\alpha^a) \) is a cofiber of the self map \( \alpha^a \), and the \( M/(\alpha^a, \beta^b) \) is obtained by use of Verdier’s axiom on the equation \( \alpha^a \beta^b = \beta^b \alpha^a \) in \( [M, M]_* \).

Since the reduced comodule \( \Gamma(m+1) \) is \( (2p^{m+1} - 3) \)-connected, we have the vanishing line \( E_2^t(T(m)) = 0 \) for \( t < 2s(p^{m+1} - 1) \) by (3–5). It follows that \( v_k \in E_2^*(T(m)) \) in Lemma 3.6 is permanent if \( k < m + 2 \).

The existence of a spectrum with \( BP_*/\text{homology} \ BP_*/I_n \) is problematic and we still have little information for such a spectrum, which we usually call the \((n-1)\)st Smith–Toda spectrum and is denoted by \( V(n-1) \) (eg Smith [6], Toda [7] and Ravenel [4]). For \( n \leq 3 \), it is shown that \( V(n) \) exists if and only if \( p > 2n \). On the other hand, \( L_n V(n-1) \) exists if \( n^2 + n < 2p \) [5]. The smash products \( T(m) \) and these Smith–Toda spectra show the following:

**Proposition 3.8** If \( p > 2n \), \( T(m)/I_n \) exists, and if \( n^2 + n < 2p \), \( L_n T(m)/I_n \) exists.

*Geometry & Topology Monographs* 10 (2007)
4 \ Ext_{\Sigma(n,m+1)}^{s}(E_m(n)/J_n) for small s

In this section, let J_n denote the sequence v_1, \ldots, v_{n-1} of elements of E_m(n). Applying Ext to the short exact sequence

\[ 0 \to E_m(n)/(p,J_n) \xrightarrow{1/p} E_m(n)/(p^\infty,J_n) \xrightarrow{p} E_m(n)/(p^\infty,J_n) \to 0, \]

we have the long exact sequence of Ext groups with connecting homomorphism

\[ (4-1) \quad \delta : \ Ext(E_m(n)/(p^\infty,J_n)) \to \ Ext(E_m(n)/(p,J_n)). \]

By [4, Theorem 6.5.6], we know the structure of Ext(E_m(n)/(p,J_n)), which means that Ext(E_m(n)/(J_n)) is a computable object.

To compute Ext(E_m(n)/(p^\infty,J_n)), we redefine the class h_{m+k,0} (0 < k \leq n) by

\[ (4-2) \quad h_{i,0} = \left[ \frac{\log(1 + p\nu_i^{-1}t_i)}{p} \right] = \left[ \sum_{n>0} (-1)^{n-1} \frac{(p\nu_i^{-1}t_i)^n}{pn} \right]. \]

**Lemma 4.3** For 0 < k \leq n, the connecting homomorphism \( \delta \) in (4-1) acts for all \( \ell \) as \( \delta(h_{m+k,0}/p^\ell) = 0. \)

**Proof** It suffices to show that \( ph_{m+k,0} = d(\log(v_{m+k})) \). By Lemma 3.1, we have \( \eta_R(v_{m+k}) = v_{m+k} + pt_{m+k} \) for 0 < k \leq n, so the equation

\[ \log(1 + p\nu_{m+k}^{-1}t_{m+k}) = \log(\eta_R(v_{m+k})) - \log(v_{m+k}) = d(\log(v_{m+k})) \]

holds.

The element \( v_{m+k}^k \) is well-defined in \( \Sigma(n,m+1)/(p^k) \), although the representative \( x = \log(1 + p\nu_{m+k}^{-1}t_{m+k})/p \) of \( h_{m+k,0} \) has negative exponents of \( v_{m+k} \) in the coefficient.

An easy computation with Lemma 3.1 shows the following:

**Lemma 4.4** Put \( \nu(e_k) = \min\{\nu(e_1), \ldots, \nu(e_n)\} \). Then we have

\[ \delta\left(\frac{v_{m+1}^{e_1} \cdots v_{m+n}^{e_n}}{p^{1+\nu(e_k)}}\right) = v_{m+1}^{e_1} \cdots v_{m+n}^{e_n}h_{m+k,0} + \cdots \]

in \( \text{Ext}^1(E_m(n)/J_n) \) up to unit. For \( \nu \), see Theorem 1.1.
Corollary 4.5  \( \text{Ext}^0(E_m(n)_s/(p\infty, J_n)) \) is the direct sum of

1. the cyclic \( \mathbb{Z}_p[v_n^{\pm 1}, v_{n+1}, \ldots, v_m] \)-module generated by

\[
\frac{v_n^{e_1}, \ldots, v_m^{e_{m+n}}}{p^{1+\nu(e_k)}}
\]

of order \( p^{1+\nu(e_k)} \) with \( \nu(e_k) = \min \{ \nu(e_1), \ldots, \nu(e_n) \} \) and

2. \( \mathbb{Q}/\mathbb{Z}_p[v_n^{\pm 1}, v_{n+1}, \ldots, v_m] \).

Example 4.6  For \( m = n = 2 \), we have

\[
\delta \left( \frac{v_3^{sp} v_4^{sp}}{p^{1+\min(i,j)}} \right) = \begin{cases} 
    v_3^{sp} v_4^{sp} h_{40} & \text{for } i > j \\
    v_3^{sp} v_4^{sp} h_{30} & \text{for } i < j \\
    v_3^{sp} v_4^{sp} (h_{30} + ah_{40}) & \text{for } i = j
\end{cases}
\]

in \( \text{Ext}^1(E_2(2)_s/(p, v_1)) \) up to unit (where \( a \in (\mathbb{Z}/(p))^\times \)), and \( \text{Ext}^0(E_2(2)_s/(p\infty, v_1)) \) is the direct sum of

1. the cyclic module over \( \mathbb{Z}_p[v_2^{\pm 1}] \) generated by \( v_3^{sp} v_4^{sp}/P^{1+\min(i,j)} \) of order \( p^{1+\min(i,j)} \) and

2. \( \mathbb{Q}/\mathbb{Z}_p[v_2^{\pm 1}] \).

In the computations for \( \delta(h_{31}) \) and \( \delta(h_{41}) \), the elements \( b_{ij} \) (cf Lemma 3.1) occur, which are hard to express in terms of generators appearing in [4, Theorem 6.5.6]. We observe that the specific property \( b_{ij} = h_{2,ij}^2 \) at \( p = 2 \) makes the computations easy.

We consider the spectrum \( L_n T(m)/(J_n) \) for \( (n, m) = (2, 1) \), which is the simplest case satisfying \( n > m \), and compute \( \text{Ext}^0_{\Sigma(2,2)}(E_1(2)_s/(v_1)) \) for \( s < 2 \) for an odd prime. We consider the case for \( p = 2 \) in the next Section 5. Since \( p \) is odd, the condition of [4, Theorem 6.5.6] is always satisfied and \( \text{Ext}^0_{\Sigma(2,2)}(E_1(2)_s/(p, v_1)) \) is obtained as

\[
K(2)_s[v_3] \otimes \Lambda(h_{ij} : 2 \leq i, j \leq 3) \in \mathbb{Z}/2).
\]

Starting from this, \( \text{Ext}^0_{\Sigma(2,2)}(E_1(2)_s/(p\infty, v_1)) \) is determined by computing the connecting homomorphism (4–1) for \( (m, n) = (1, 2) \) as follows Corollary 4.5:

Proposition 4.7  For \( sp^i \in \mathbb{Z} \) and \( tp^j \geq 0 \), we have

\[
\delta(v_3^{sp} v_4^{sp}/P^{1+\min(i,j)}) = \begin{cases}
    sv_3^{sp-1} v_4^{sp} h_{20} & \text{if } i < j, \\
    tv_3^{sp} v_4^{sp-1} h_{30} & \text{if } i > j, \\
    v_3^{sp-1} v_4^{sp-1} (sv_3 h_{20} + tv_3 h_{30}) & \text{if } i = j,
\end{cases}
\]

and \( \text{Ext}^0_{\Sigma(2,2)}(E_1(2)_s/(p\infty, v_1)) \) is the direct sum of...
We begin with recalling the result of Mahowald and Shimomura [2]:

(1) the cyclic \( \mathbb{Z}(p) \)-module generated by \( v_2^{p^j} v_3^{p^j} / p^{1+\min(i,j)} \) of order \( p^{1+\min(i,j)} \) and

(2) \( \mathbb{Q}/\mathbb{Z}(p) \).

5 The homotopy groups \( \pi_*(L_2 T(1)/(v_1)) \) at the prime two

We begin with recalling the result of Mahowald and Shimomura [2]:

\[
\text{Ext}(E_1(2)_*/(2, v_1)) = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2)
\]

where \( \rho_2 \) is the generator of degree 0 represented by the cocycle \( v_2^{-5}t_4 + v_2^{-10}t_4^2 \). We see that (4–2) for \( p = 2 \) is also a cocycle with leading term \( v_i^{-2}t_i^2 \), and replace the representative cocycles by

\[
h_{i,0} = [t_i] \quad \text{and} \quad h_{i,1} = \left[ \sum_{n>0} (-1)^{n-1} \frac{(2v_i^{-1}t_i)^n}{2n} \right].
\]

Setting \( B = \mathbb{Z}/2[v_2^\pm 2, v_3^2] \), we rewrite the right hand side of (5–1) as

\[
B \otimes \Lambda(v_3) \otimes \Lambda(h_{21}, h_{30}, h_{31}) \otimes \Lambda(v_2) \otimes \mathbb{Z}/2[h_{20}] \otimes \Lambda(\rho_2).
\]

Since

\[
h_{21}h_{31} = v_2^{-1}v_3^2h_{20}h_{21} + v_2^2h_{30}^2 + v_2h_{20}h_{31}
\]

by [2, p 243 (1)], we replace \( h_{21}h_{31} \) (resp. \( h_{21}h_{30}h_{31} \)) with \( h_{30}^2 \) (resp. \( h_{30}^3 \)).

**Lemma 5.2** As the \( \mathbb{Z}/2 \)-module, \( \text{Ext}(E_1(2)_*/(2, v_1)) \) is isomorphic to

\[
A \otimes \Lambda(v_2) \otimes \mathbb{Z}/2[h_{20}] \otimes \Lambda(\rho_2)
\]

where

\[
A = B \otimes \Lambda(v_3) \otimes (\mathbb{Z}/2[h_{30}]/(h_{30}^4) \oplus \mathbb{Z}/2[h_{21}, h_{31}] \otimes \Lambda(h_{30}))
\]

and

\[
B = \mathbb{Z}/2[a_2^{\pm 1}, a_3] \quad \text{with} \quad a_i = v_i^2.
\]

**Lemma 5.3** The connecting homomorphism (4–1) for \( (m, n) = (1, 2) \) acts as

\[
\delta(v_i^s/2) = v_i^{s-1}h_{i,0} \quad \text{and} \quad \delta(a_i^{2s}/2^{n+2}) = a_i^{2s}h_{i,1} \quad (i = 2, 3)
\]

for odd \( s \) and \( n \geq 0 \).

**Proof** It follows from

\[
(v_i^s) \equiv 2v_i^{-1}t_i \quad \text{mod} \ (4),
\]

\[
d(a_i^{2s}) \equiv 2^{n+2}v_i^{2s+1}(v_i^{-1}t_i + v_i^{-2}t_i^2) \quad \text{mod} \ (2^{n+3}).
\]

\[\square\]
Hirofumi Nakai and Katsumi Shimomura

\[ \text{Ext}(E_1(2) \times (2, v_1)) \] is decomposed into the following four summands tensoring with \( \Lambda(\rho_2) \):

\[
\begin{align*}
&v_2A \oplus \Lambda(v_2) \otimes A \otimes \mathbb{Z}/p(h_{20})h_{20} \\
v_3B \oplus \Lambda(v_3) \otimes B\{h_{30}, h_{30}^2, h_{30}^3\} \oplus v_3h_{30}h_{31}B \\
&\quad \oplus B\{h_{21}, h_{31}\} \oplus v_3h_{21}h_{30}B \\
&Bh_{30}\{h_{21}, h_{31}\} \oplus v_3B\{h_{21}, h_{31}\}
\end{align*}
\]

With respect to each summand, we construct a long exact sequence in Lemma 5.4, Lemma 5.5 and Lemma 5.6. We often use the replacement

\[
h_{31} = [v_3^{-1}t_3 + v_3^{-2}t_3^2] = v_3^{-1}h_{30} + \cdots .
\]

If we define \( P_i (i \geq 0) \) and \( Q_j (j > 0) \) by

\[
\begin{align*}
P_i &= \mathbb{Z}_{(2)}\{a_2^{2i}s, a_3^{2j}r : 0 \leq j \leq i, 0 \neq s \in \mathbb{Z}, t \geq 0\}, \\
Q_j &= \mathbb{Z}_{(2)}\{a_2^{2i}s, a_3^{2j}r : 0 < i, s \in \mathbb{Z}, t > 0\},
\end{align*}
\]

then we decompose \( B \) into

\[
B = \left( \bigoplus_{i \geq 0} P_i \right) \oplus \left( \bigoplus_{j > 0} Q_j \right).
\]

Define \( M^0 \) and \( M^1 \) by

\[
\begin{align*}
M^0 &= \left( \bigoplus_{i \geq 0} P_i\left\{\frac{1}{2^{j+2}}\right\} \right) \oplus \left( \bigoplus_{j > 0} Q_j\left\{\frac{1}{2^{j+2}}\right\} \right) \oplus \mathbb{Q}/\mathbb{Z}_{(2)}, \\
M^1 &= \left( \bigoplus_{i \geq 0} P_i\left\{\frac{h_{21}}{2^{j+2}}\right\} \right) \oplus \left( \bigoplus_{j > 0} Q_j\left\{\frac{h_{31}}{2^{j+2}}\right\} \right).
\end{align*}
\]

Then we have the following results:

**Lemma 5.4** We have two long exact sequences

\[
\begin{array}{ccc}
B & \longrightarrow & M^0 \\
\downarrow & & \downarrow 2 \quad \delta \\
B\{h_{21}, h_{31}\} & \longrightarrow & M^0
\end{array}
\]

\[
\begin{array}{ccc}
& v_3h_{21}h_{30}B & \\
\delta & & \delta \\
\downarrow & & \downarrow 2 \quad \delta \\
& v_3B\{h_{21}, h_{31}\} & \\
\downarrow & & \downarrow 2 \quad \delta \\
& (v_3/2)B\{h_{21}, h_{31}\} & \\
\downarrow & & \downarrow 2 \quad \delta \\
& (v_3/2)B\{h_{21}, h_{31}\} & \\
& Bh_{30}\{h_{21}, h_{31}\}.
\end{array}
\]

*Geometry & Topology Monographs 10 (2007)*
We also see that we have the first sequence. The second sequence is obvious.

We have a long exact sequence

\[ \begin{align*}
\delta(a_2^{2i}a_3^{2j}/2^{2+\min(i,j)}) &= \begin{cases} 
    a_2^{2s}a_3^{2j}h_{21} & (i < j) \\
    a_2^{2s}a_3^{2i}h_{31} & (i > j) \\
    a_2^{2s}a_3^{2i}(h_{21} + h_{31}) & (i = j)
\end{cases}
\end{align*} \]

We also see that \( \delta(a_2^{2i}a_3^{2j}h_{31}/2^{i+2}) \) for \( i < j \), \( \delta(a_2^{2i}a_3^{2j}h_{21}/2^{j+2}) \) for \( i > j \), and \( \delta(a_2^{2i}a_3^{2j}h_{21}/2^{i+2}) \) for \( i = j \), and we have the first sequence. The second sequence is obvious. \( \square \)

**Lemma 5.5** We have a long exact sequence

\[ \begin{align*}
&v_3B \xrightarrow{\delta} (v_3/2)B \\
&h_{30}B \otimes \Lambda(v_3) \xrightarrow{\delta} (v_3h_{30}/2)B
\end{align*} \]

\[ \begin{align*}
&h_{30}^{2}B \otimes \Lambda(v_3) \xrightarrow{\delta} (v_3h_{30}^{2}/2)B \\
&\oplus (v_3h_{30}h_{31})B \xrightarrow{\delta} (v_3h_{30}h_{31}/2)B
\end{align*} \]

\[ \begin{align*}
&h_{30}^{3}B \otimes \Lambda(v_3).
\end{align*} \]

**Proof** It follows from

\[ \begin{align*}
\delta(a_2^{2s}a_3^{2j}v_3h_{30}^{k}/2) &= a_2^{2s}a_3^{2j}h_{30}^{k+1} \quad \text{for } 0 \leq k \leq 2,
\delta(a_2^{2s}a_3^{2j}v_3h_{30}h_{31}/2) &= a_2^{2s}a_3^{2j}h_{30}^{2}h_{31} = a_2^{2s}a_3^{2j}v_3h_{30}^{2} + \cdots \quad \square
\end{align*} \]

**Lemma 5.6** We have a long exact sequence

\[ \begin{align*}
v_2A &\xrightarrow{\delta} (v_2/2)A \\
h_{20}A \otimes \Lambda(v_2) &\xrightarrow{\delta} (v_2h_{20}/2)A
\end{align*} \]

\[ \begin{align*}
h_{20}^{2}A \otimes \Lambda(v_2) &\xrightarrow{\delta} (v_2h_{20}^{2}/2)A
\end{align*} \]

**Proof** Notice that each exponent of \( v_2 \) in \( (v_2h_{20}^{k}/2)A \) is odd. Since we have \( d(x) = 0 \) for \( x \in A \) in the cobar complex, we have

\[ d(v_2^{2s+1}v_3^{j}x) = d(v_2^{2s+1}v_3^{j}) \otimes x. \]
We see that
\[ d(v_2^{2n+1}v_3) = \begin{cases} 2v_2^{2n}v_3 t_2 + \cdots & \text{for } t = 2n, \\ 2v_2^{2n}v_3^2(v_3 t_2 + v_2 t_3) + \cdots & \text{for } t = 2n + 1. \end{cases} \]

In both cases we obtain
\[ \delta \left( \frac{v_2^{2n+1}v_3' x}{2} \right) = v_2^{2n}v_3' h_{20} x \]

replacing \( v_3 h_{20} \) by \( v_3 h_{20} = [v_3 t_2 + v_2 t_3] \) only for the case \( t = 2n + 1 \)

By the above three lemmas, we obtain the chart of differentials

Thus we conclude the following:

**Lemma 5.7** \( \text{Ext}_{\Sigma(2,2)}(E_1(2)_*, E_1(2)_*/(2^\infty, v_1)) \) is the tensor product of \( \Lambda(\rho_2) \) and the direct sum of
(1) \( v_2 A[h_{20}], v_3 B[h_{30}]/(h_{30}^3) \) and \( v_3 B h_{30} h_{31} \) whose elements are of order two,

(2) \( M^0 \) and \( M^1 \).

Let \( E_\infty^s(X) \) for a spectrum \( X \) denote the \( E_\infty \)–term of the \( E(2) \)–based Adams spectral sequence converging to the homotopy groups \( \pi_*(L_2 X) \).

**Theorem 5.8** The \( E_\infty \)–term \( E_\infty^s(L_2 T(1)/(2^\infty, v_1)) \) is the tensor product of \( \Lambda(v_2) \) and the direct sum of

(1) \( \widetilde{v}_2 A[h_{20}], v_3 B[h_{30}]/(h_{30}^3) \) and \( v_3 B h_{30} h_{31} \) whose elements are of order two,

(2) \( M^0 \) and \( M^1 \),

where \( \widetilde{v}_2 A[h_{20}] \) denotes the module

\[
\left( \mathbb{Z}/2[v_2^2, v_3^2] \otimes \Lambda(v_3) \otimes \left( \mathbb{Z}/2[h_{30}]/(h_{30}^3) \oplus \mathbb{Z}/2\{h_{21}, h_{31}\} \otimes \Lambda(h_{30}) \right) \right)[h_{20}]/(h_{30}^3).
\]

**Proof** In [2], the differentials of \( E(2) \)–based Adams spectral sequence for \( L_2 T(1)/I_2 \) (written as \( D \) in [2]) are determined as

\[
d_3(v_3) = 0 \quad \text{and} \quad d_3(v_3^k) = v_2^2 v_3^{k-2} h_{30}^3 \quad \text{for} \quad 2 \leq k \leq 3,
\]

and \( d_3(v_3^k x) = d_3(v_3^k) x \) for \( x = h_{20}, h_{21}, h_{30} \) and \( h_{31} \). Note that for each element \( w a_3^{2r+1} \in v_2 A[h_{20}] \), we see that

\[
d_3(w a_3^{2r+1}/2) = w a_3^{2r} h_{30}^3/2 \in v_2 A[h_{20}].
\]

This shows the structure of \( \pi_*(L_2 T(1)/(2^\infty, v_1)) \), since it has a horizontal vanishing line. \( \square \)

**Proof of Theorem 1.3** Consider the cofiber sequence

\[
T(1)/(v_1) \longrightarrow T(1)/(v_1) \wedge S Q \longrightarrow T(1)/(2^\infty, v_1).
\]

Then the homotopy groups of \( T(m)/(v_1) \wedge S Q \) and \( T(1)/(2^\infty, v_1) \) are determined in [4, Corollary 6.5.6] and Theorem 5.8, respectively. \( \square \)
Hirofumi Nakai and Katsumi Shimomura

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Geometry & Topology Monographs 10 (2007)