On the lower semicontinuity and subdifferentiability of the value function for conic linear programming problems

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Abstract

Lemma 1 from the paper [N.E. Gretsky, J.M. Ostroy, W.R. Zame, Subdifferentiability and the duality gap, Positivity 6: 261–274, 2002] asserts that the value function $v$ of an infinite dimensional linear programming problem in standard form is lower semicontinuous whenever $v$ is proper and the involved spaces are normed vector spaces. In this note one shows that this statement is false even in finite-dimensional spaces, one provides an example of linear programming problem in Hilbert spaces whose (proper) value function is not lower semicontinuous (hence it is not subdifferentiable) at any point in its domain, one shows that the restriction of the value function to its domain in Kretschmer’s gap example is not bounded on any neighborhood of any point of the domain, and discuss other assertions done in the same paper.

1 Introduction

The following conical linear programming problem

$$(P) \text{ minimize } c^*(x) \text{ s.t. } x \in P, \ Ax - b \in Q,$$

and its dual

$$(D) \text{ maximize } y^*(b) \text{ s.t. } y^* \in Q^+, \ c^* - A^* y^* \in P^+,$$

are studied in [2], where $X, Y$ are Hausdorff locally convex spaces, $X^*$ and $Y^*$ are their topological dual spaces, $A : X \rightarrow Y$ is a continuous linear operator, $A^* : Y^* \rightarrow X^*$ is the adjoint of $A$, $P \subset X$ and $Q \subset Y$ are convex cones, $P^+ \subset X^*$ and $Q^+ \subset Y^*$ are the positive dual cones of $P$ and $Q$, $b \in Y$ and $c \in X^*$.

The main results from [2] are: Theorem 1 which states that the value function $v$ associated to $(P)$ is subdifferentiable at $b$ if and only if $(D)$ has optimal solutions and there is no duality gap, and its use for proving the Duffin–Karlovitz no-gap theorem; Lemma 1 which states that $v$ is lower semicontinuous whenever it is proper and the involved spaces are normed vector spaces; the modification of Kretschmer’s gap example in order to get a convex function which is subdifferentiable at a point but is not continuous there; Proposition 2 which provides sufficient conditions, adequate for the assignment model, to ensure that $v$ is Lipschitz on $Q$.

Unfortunately, [2, Lem. 1] is not true even in finite dimensional spaces, which makes its use to be not adequate in the proof of the Duffin–Karlovitz no-gap theorem, while the proof of [2, Prop. 2] needs, in our opinion, serious clarifications; moreover, there are also other

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inaccuracies in the paper. Having in view the remark that “This paper should be on the reading list of any advanced mathematical economics course which has a focus on extremal methods in infinite-dimensional spaces” from the review MR1932651 (2003i:90111) of [2] in Mathematical Reviews, we consider that there is a strong motivation for an attentive reading of this paper.

The paper is organized as follows. Having in view that the value function associated to problem (P) is positively homogeneous and subadditive, in Section 2 we underline some specific properties of such functions, pointing out the differences between the (lower-, upper-, local Lipschitz) continuity of such functions and their restrictions to the domain. In Section 3 we essentially discuss the proof of the Duffin–Karlovitz no-gap theorem, while in Section 4 we provide two counter-examples to [2] Lem. 1, the first in finite-dimensional spaces and the second in Hilbert spaces. Section 5 is dedicated to Kretschmer’s gap example, while in Section 6 we comment the proof of [2] Prop. 2.

Below, we introduce the basic notations and some preliminary results used in the paper.

Throughout this note, the considered spaces are real Hausdorff locally convex spaces (H.l.c.s. for short) if not mentioned explicitly otherwise. Having $X$ an H.l.c.s., $X^*$ is its topological dual endowed with its weak-star topology $w^* := \sigma(X^*, X)$. The value $x^*(x)$ of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^* \rangle$. It is well known that $(X^*, w^*)^*$ can be identified with $X$, what we do in the sequel. Having $(\emptyset \neq) K \subseteq X$ a convex cone (that is, $x + x' \in K$ and $tx \in K$ for all $x, x' \in K$ and $t \in \mathbb{R}_+ := [0, \infty[$), we set $x \leq_K x'$ (equivalently $x' \geq_K x$) for $x, x' \in X$ with $x' - x \in K$; clearly $\leq_K$ is a preorder on $X$, that is, $\leq_K$ is reflexive and transitive. For $E \subseteq X$, one denotes by $\text{span} \ E$, $\text{aff} \ E$, $\text{icr} \ E$, $\text{cor} \ E$, $\text{int} \ E$ and $\text{cl} \ E$ the linear hull, the affine hull, the intrinsic core, the core, the interior and the closure of $E$, respectively.

For $\emptyset \neq A \subseteq X$ (and similarly for $\emptyset \neq B \subseteq X^*$) we set $A^+ := \{x^* \in X^* \mid \forall x \in A : \langle x, x^* \rangle \geq 0\}$ for the positive dual cone of $A$; it is well known that $A^+ (\subseteq X^*)$ is a $w^*$-closed convex cone and $(K^+)^\ast = \text{cl} K$ whenever $K$ is a convex cone.

Having a function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, its domain is the set $\text{dom} f := \{x \in X \mid f(x) < \infty\}$; $f$ is proper if $\text{dom} f \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in X$; $f$ is convex if its epigraph $\text{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is convex; $f$ is positively homogeneous if $f(tx) = tf(x)$ for all $t \in \mathbb{R} := \mathbb{R} \setminus \{0\}$ and $x \in X$; $f$ is subadditive if $f(x + x') \leq f(x) + f(x')$ for all $x, x' \in \text{dom} f$; $f$ is sublinear if $f$ is positively homogeneous, subadditive and $f(0) = 0$; $f$ is lower semicontinuous (l.s.c. for short) at $x \in X$ if $\lim \inf_{x' \to x} f(x') \geq f(x)$, where $\overline{\mathbb{R}}$ is endowed with its usual topology (for example, the topology induced by the metric $d$ defined by $d((t, t'), (s, s')) := |\arctan t - \arctan t'|$ with $\arctan(\pm \infty) := \pm \pi/2$); $f$ is l.s.c. if $f$ is l.s.c. at any $x \in X$; the l.s.c. envelope of $f$ is the function $\overline{f} : X \to \overline{\mathbb{R}}$ such that $\text{epi} \overline{f} = \text{cl}(\text{epi} f)$, and so $\overline{f}$ is convex if $f$ is so; $f$ is upper semicontinuous (u.s.c. for short) at $x \in X$ (resp. on $X$) if $-f$ is l.s.c. at $x \in X$ (resp. on $X$); the subdifferential of $f$ at $x \in X$ with $f(x) \in \overline{\mathbb{R}}$ is the set

$$\partial f(x) := \{x^* \in X^* \mid \forall x' \in X : \langle x' - x, x^* \rangle \leq f(x') - f(x)\},$$

and $\partial f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$; $f$ is subdifferentiable at $x \in X$ if $\partial f(x) \neq \emptyset$. The conjugate of $f$ is the function $f^*: X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup \{\langle x, x^* \rangle - f(x) \mid x \in X\} = \sup \{\langle x, x^* \rangle - f(x) \mid x \in \text{dom} f\} \quad (x^* \in X^*),$$

where $\sup \emptyset := -\infty$; clearly, $f^*$ is a $w^*$-l.s.c. convex function. Having $g : X^* \to \overline{\mathbb{R}}$, its conjugate $g^* : X \to \overline{\mathbb{R}}$ is defined similarly. Notice that $f^* = (\overline{f})^*$; moreover, for $x \in X$ and
\(x^* \in X^*\) one has
\[
x^* \in \partial f(x) \iff [f(x) \in \mathbb{R} \land f(x) + f^*(x^*) = \langle x, x^* \rangle] \Rightarrow \overline{f}(x) = f(x) \in \mathbb{R} \Rightarrow \partial \overline{f}(x) = \partial f(x).
\] (1)

The indicator function of \(E \subset X\) is \(\iota_E : X \to \overline{\mathbb{R}}\) defined by \(\iota_E(x) := 0\) for \(x \in E\) and \(\iota_E(x) := \infty\) for \(x \in X \setminus E\); notice that \(\iota_E\) is l.s.c. iff \(E\) is closed, and \(\iota_E\) is convex iff \(E\) is convex.

2 Some properties of sublinear functions

Because the value function of a linear programming problem is positively homogeneous and subadditive (hence sublinear when it vanishes at 0), it is useful to point out some specific properties of such functions.

Let \(g : X \to \overline{\mathbb{R}}\) be positively homogeneous and subadditive. First observe that
\[
\forall x, x' \in \text{dom } g, \forall \lambda \in \mathbb{P} : x + x' \in \text{dom } g \land \lambda x \in \text{dom } g;
\]
consequently, for \(x, x' \in \text{dom } g\) and \(\lambda \in [0, 1]\) one gets
\[
g(\lambda x + (1 - \lambda)x') \leq g(\lambda x) + g((1 - \lambda)x') = \lambda g(x) + (1 - \lambda)g(x'),
\]
and so \(g\) is convex.

Because \(g\) is positively homogeneous, one has \(g(0) = g(\lambda 0) = \lambda g(0)\) for \(\lambda \in \mathbb{P}\), and so \(g(0) \in \{-\infty, 0, \infty\}\). Moreover, if \(g(x_0) = -\infty\) for some \(x_0 \in X\), then \(g(x + \lambda x_0) \leq g(x) + \lambda g(x_0) = -\infty\), whence \(g(x + \lambda x_0) = -\infty\), for all \(\lambda \in \mathbb{P}\) and \(x \in \text{dom } g\); consequently, \(g(x) = -\infty\) for all \(x \in \text{dom } g\) if \(g(0) = -\infty\).

Assume that \(g(0) = \infty\); then \(\overline{g} : X \to \overline{\mathbb{R}}\) defined by \(\overline{g}(x) := g(x)\) for \(x \neq 0\) and \(\overline{g}(0) := 0\) is sublinear. Indeed, take \(x', x'' \in \text{dom } \overline{g}\). If \(x', x'' \in \text{dom } g\), then \(x' + x'' \in \text{dom } g\) (hence \(x' + x'' \neq 0\)), and so \(\overline{g}(x' + x'') = g(x' + x'') \leq g(x') + g(x'') = \overline{g}(x') + \overline{g}(x'')\); if \(x' = 0\) (and similarly for \(x'' = 0\)), then \(\overline{g}(x' + x'') = \overline{g}(x') = \overline{g}(x') + \overline{g}(x'')\). Because \(\overline{g}\) is clearly positively homogeneous, \(\overline{g}\) is sublinear.

In the rest of this section, \(g : X \to \overline{\mathbb{R}}\) is a sublinear function; hence \(g(0) = 0\). Using [7, Th. 2.4.14] one obtains that
\[
[\partial g(0) \neq \emptyset \iff g \text{ is l.s.c. at } 0], \quad g^* = \iota_{\partial g(0)}, \quad \text{dom } g^* = \partial g(0),
\] (2)
\[
\partial g(0) \neq \emptyset \Rightarrow [\forall x \in X : \overline{g}(x) = \sup \{\langle x, x^* \rangle \mid x^* \in \partial g(0)\} = g^{**}(x)],
\] (3)
\[
\forall x \in X : \partial g(x) = \{x^* \in \partial g(0) \mid \langle x, x^* \rangle = g(x)\}.
\] (4)

Also note that \([g \text{ is u.s.c. at } 0] \iff [\text{dom } g = X \text{ and } g \text{ is continuous on } X]\); moreover, \([g \text{ is l.s.c. at } 0] \iff [g|_{\text{dom } g} \text{ is l.s.c. at } 0]\), where \(\text{dom } g\) is endowed with its trace (induced) topology.

Assume now that \(X\) is a normed vector space; then one also has:
\[
g \text{ is l.s.c. at } 0 \iff [\exists L > 0, \forall x \in X : g(x) \geq -L \|x\|];
\] (5)
\[
g \text{ is u.s.c. at } 0 \iff [\exists L > 0, \forall x \in X : g(x) \leq L \|x\|]
\] (6)
\[
\iff \text{dom } g = X \text{ and } g \text{ is } (L-)\text{lipschitz on } X.
\] (7)

Indeed, the implications \(\Leftarrow\) from (5)–(7) are obvious because \(g(0) = 0\). Assume that \(g\) is l.s.c. at 0. Then there exists \(r > 0\) such that \(g(x) \geq -1\) for \(x \in X\) with \(\|x\| \leq r\). Taking
\( x \in X \setminus \{0\} \) and \( x' := \frac{r}{\|x\|} x \), one has \( \|x'\| \leq r \), whence \(-1 \leq g(x') = \frac{r}{\|x\|} g(x) \), and so \( g(x) \geq -r^{-1} \|x\| \); hence the implication \( \Rightarrow \) holds in (\([5]\)). The proof of the implication \( \Rightarrow \) from (\([6]\)) is similar. Assume now that \( g(x) \leq L \|x\| \) for \( x \in X \); hence \( \text{dom } g = X \). Taking \( x \in X \), one has \( 0 = g(0) = g(x + (-x)) \leq g(x) + g(-x) \), whence \( g(x) \in \mathbb{R} \). Take now \( x, x' \in X \); we (may) assume that \( g(x) \geq g(x') \). Then

\[
g(x) = g((x - x') + x') \leq g(x - x') + g(x') \leq L \|x - x'\| + g(x'),
\]

whence \( |g(x) - g(x')| = g(x) - g(x') \leq L \|x - x'\| \). Therefore, \( g \) is \( L \)-Lipschitz.

In what concern the continuity and the upper semicontinuity of \( g|_{\text{dom } g} \), in a similar way as for (\([6]\)), one obtains:

\[
g|_{\text{dom } g} \text{ is u.s.c. at } 0 \iff \exists L > 0, \forall x \in \text{dom } g : g(x) \leq L \|x\|,
g|_{\text{dom } g} \text{ is continuous at } 0 \iff [\exists L > 0, \forall x \in \text{dom } g : |g(x)| \leq L \|x\|].
\]

The next example shows the big differences among the continuity properties of the functions \( g \) and \( g|_{\text{dom } g} \).

**Example 1** Let \( X \) be an infinite-dimensional normed vector space (n.v.s. for short) and let \( \varphi : X \to \mathbb{R} \) be a linear, not continuous functional. Consider the proper sublinear functions \( g_1, g_2, g_3 : X \to \mathbb{R} \) defined by

\[
g_1 := \max\{0, \varphi\}, \quad g_2(x) := \begin{cases} 0 & \text{if } x \in [\varphi \leq 0], \\ \varphi(x) & \text{if } x \in [\varphi > 0], \\ \infty & \text{if } x \in [\varphi > 0], \end{cases} \quad g_3(x) := \begin{cases} \varphi(x) & \text{if } x \in [\varphi \leq 0], \\ \infty & \text{if } x \in [\varphi > 0], \end{cases}
\]

where \([\varphi \leq 0] := \{x \in X \mid \varphi(x) \leq 0\}\) and similarly for \([\varphi > 0]\) (and the like). TFAH:

(i) \( \text{dom } g_1 = X, \partial g_1(x) = \{0\} \) for \( x \in [\varphi \leq 0], \partial g_1(x) = \emptyset \) for \( x \in [\varphi > 0], (g_1)^* = \iota_{\{0\}}, \overline{g_1} = 0, \) and so \( g_1 \) is l.s.c. at \( x \) iff \( x \in [\varphi \leq 0] \); moreover, \( g_1 \) is not u.s.c. at each \( x \in X \).

(ii) \( \text{dom } g_2 = [\varphi \leq 0], \partial g_2(x) = \{0\} \) for \( x \in [\varphi \leq 0], \partial g_2(x) = \emptyset \) for \( x \in [\varphi > 0], (g_2)^* = \iota_{\{0\}}, \overline{g_2} = 0, \) and so \( g_2 \) is l.s.c. at \( x \) iff \( x \in [\varphi \leq 0] \); moreover, \( g_2 \) is u.s.c. at \( x \) iff \( x \in [\varphi > 0] \) and \( g_2|_{\text{dom } g_2} \) is Lipschitz.

(iii) \( \text{dom } g_3 = [\varphi \leq 0], \partial g_3(x) = \emptyset \) for \( x \in X, (g_3)^* = \infty, \overline{g_3} = -\infty, \) and so \( g_3 \) is not l.s.c. at each \( x \in X \); moreover, \( g_3 \) is u.s.c. at \( x \) iff \( x \in [\varphi > 0] \), \( g_3|_{\text{dom } g_3} \) is u.s.c. at \( x \) iff \( x \in [\varphi = 0] \), and \( g_3|_{\text{dom } g_3} \) is not l.s.c. at each \( x \in \text{dom } g_3 \).

Proof. (i) It is clear that \( \text{dom } g_1 = X \) and \( 0 \in \partial g_1(0) \). Consider \( x^* \in \partial g_1(0) \); then, obviously, \( \langle x, x^* \rangle \leq g_1(x) = 0 \) for \( x \in [\varphi = 0] \), and so \( \langle x, x^* \rangle = 0 \) for \( x \in [\varphi = 0] \). Using (\([5]\)) Lem. 3.9], there exists \( \lambda \in \mathbb{R} \) such that \( x^* = \lambda \varphi \), and so \( \lambda = 0 \) because \( \varphi \) is not continuous. Hence \( \partial g_1(0) = \{0\} \), whence, \( \partial g_1(x) = \{0\} \iff g_1(x) = 0 \iff \varphi(x) \leq 0 \), and \( \partial g_1(x) = \emptyset \) when \( \varphi(x) > 0 \). Using (\([2]\)) one gets \( (g_1)^* = \iota_{\{0\}} \), while using (\([2]\)) one gets \( \overline{g_1} = 0 \), and so \( g_1 \) is l.s.c. at \( x \) and if only if \( x \in [\varphi \leq 0] \). Assume that \( g_1 \) is u.s.c. at \( x \in X \); then \( g_1 \) is Lipschitz on \( \text{dom } g_1 = X \) by (\([2]\)), and so \( g_1 \) is l.s.c. on \( X \); this contradiction proves that \( g_1 \) is not u.s.c. at each \( x \in X \).

(ii) It is obvious that \( \text{dom } g_2 = [\varphi \leq 0] \) and \( 0 \in \partial g_2(0) \). Similar to the proof of (i) one gets that \( \partial g_2(x) = \{0\} \) if \( x \in [\varphi \leq 0] \) and \( \partial g_2(x) = \emptyset \) otherwise, that \( \overline{g_2} = 0 \), and so \( g_2 \) is l.s.c. at \( x \) if and only if \( x \in [\varphi \leq 0] \). Because \( g_2(x) = \infty \geq g_2(x') \) for all \( x \in [\varphi > 0] \) and \( x' \in X \), \( g_2 \) is obviously u.s.c. at \( x \in [\varphi > 0] \). Assuming that \( g_2 \) is u.s.c. at some \( x \in \text{dom } g_2 \), one obtains that \( x \in \text{int } [\varphi \leq 0] \), and so one gets the contradiction that \( \varphi \) is continuous; hence \( g_2 \) is u.s.c. at \( x \) iff \( x \in [\varphi > 0] \). Because \( g_2|_{\text{dom } g_2} = 0 \), \( g_2|_{\text{dom } g_2} \) is Lipschitz.
(iii) Clearly, \( \text{dom}g_3 = [\varphi \leq 0] \). One has that \( x^* \in \partial g_3(0) \) \iff \( \langle x, x^* \rangle \leq \varphi(x) \) for all \( x \in [\varphi \leq 0] \); using similar arguments to those in the proof of (i), one gets \( \partial g_3(0) = \emptyset \), and so \( \partial g_3(x) = \emptyset \) for every \( x \in X \). Because \( \partial g_3(0) = \emptyset \), one has that \( (g_3)^* = \iota_0 = \infty \), and that \( g_3 \) is not l.s.c. at 0, whence \( \overline{g_3}(0) = -\infty \), and so \( \overline{g_3}(x) = -\infty \) for all \( x \in \text{cl(dom}g_3) = X \) because \( [\varphi \leq 0] \) is dense in \( X \) as \( \varphi \) is not continuous. Hence \( \overline{g_3} = -\infty \), and so \( g_3 \) is not l.s.c. at any \( x \in X \). As in the proof of (ii), \( g_3 \) is u.s.c. at \( x \) \iff \( x \in [\varphi > 0] \). Because \( g_3|_{\text{dom}g_3}(x) \geq g_3|_{\text{dom}g_3}(x') \) for all \( x \in [\varphi = 0] \) and \( x' \in \text{dom}g_3 \), \( g_3|_{\text{dom}g_3} \) is u.s.c. at each \( x \in [\varphi = 0] \).

Because \( \varphi \) is not continuous, \( \sup\{\varphi(x) \mid x \in S_X\} = \infty \), where \( S_X := \{x \in X \mid \|x\| = 1\} \), and so there exists a sequence \((x_n)_{n \geq 1} \subset S_X\) such that \( \varphi(x_n) \geq n \) for \( n \in \mathbb{N}^* \). Consider \( x \in [\varphi < 0] \) and take \( x_n' := x - [\varphi(x)/\varphi(x_n)]x_n \) for \( n \geq 1 \). Then \( \varphi(x_n') = 0 \) (whence \( x_n' \in \text{dom}g_3 \)) for \( n \in \mathbb{N}^* \), \( x_n' \to x \), and \( \limsup_{x_n' \to x'} \varphi(x_n') \geq \limsup_{n \to \infty} \varphi(x_n') = 0 \). \( \varphi(x) = g_3(x) \). Hence \( g_3|_{\text{dom}g_3} \) is not u.s.c. at \( x \). Therefore, \( g_3|_{\text{dom}g_3} \) is not u.s.c. at each \( x \in [\varphi < 0] \), proving so that \( g_3|_{\text{dom}g_3} \) is u.s.c. at \( x \in \text{dom}g_3 \) iff \( x \in [\varphi = 0] \). Consider now \( x \in [\varphi \leq 0] \) and take \( x_n'' := x - [1/\varphi(x_n)]x_n \) for \( n \geq 1 \). Then \( \varphi(x_n'') = \varphi(x) - 1 \leq 0 \) (whence \( x_n'' \in \text{dom}g_3 \)) for \( n \in \mathbb{N}^* \), \( x_n'' \to x \), and \( \liminf_{x_n'' \to x} \varphi(x_n'') \leq \liminf_{n \to \infty} \varphi(x_n'') = \varphi(x) - 1 < \varphi(x) = g_3(x) \). Hence \( g_3|_{\text{dom}g_3} \) is not l.s.c. at each \( x \in \text{dom}g_3 \).

The next result seems to be quite relevant in the context of [2].

**Proposition 2** Let \( X \) be a n.v.s. and \( g : X \to \overline{\mathbb{R}} \) be a proper sublinear function. Assume that \( x \in \text{dom}g \) and \( \delta, L > 0 \) are such that \( |g(x') - g(x'')| \leq L \|x' - x''\| \) for all \( x', x'' \in B(x, \delta) \cap \text{dom}g \), where \( B(X, \delta) := \{x' \in X \mid \|x' - x\| < \delta\} \). Then

\[
\forall \gamma \in \mathbb{P}, \forall x', x'' \in B(\gamma x, \gamma \delta) \cap \text{dom}g : |g(x') - g(x'')| \leq L \|x' - x''\|;
\]

in particular, \( g|_{\text{dom}g} \) is \( L \)-Lipschitz if \( x = 0 \) (or, more generally, \( \|x\| < \delta \)).

Proof. Take \( x', x'' \in B(\gamma x, \gamma \delta) \cap \text{dom}g \). Then \( \gamma^{-1} x', \gamma^{-1} x'' \in B(x, \delta) \cap \text{dom}g \), and so

\[
\gamma^{-1}\left|\frac{g(x') - g(x'')}{\gamma^{-1}}\right| = \left|\frac{g(x') - g(x'')}{\gamma^{-1}}\right| \leq L \left\|\gamma^{-1} x' - \gamma^{-1} x''\right\| = L \gamma^{-1} \|x' - x''\|,
\]

whence \( |g(x') - g(x'')| \leq L \|x' - x''\|. \)

Notice that \( g|_{\text{dom}g} \) is locally Lipschitz on \( \text{icr(dom}g) \) whenever \( g \) is a proper sublinear function and \( \text{dim}X < \infty \) (or, more generally, \( \text{dim}X_g < \infty \), where \( X_g := \text{span}(\text{dom}g) = \text{dom}g - \text{dom}g \)); this is because for \( \bar{g} := g|_{X_g} \in \Lambda(X_g) \), one has \( \text{dom} \bar{g} = \text{dom}g \) and \( \text{icr(dom}g) = \text{cor(dom} \bar{g}) = \text{int}(\text{dom} \bar{g}) \) and so the proper convex function \( \bar{g} \) is continuous (and so locally Lipschitz) on \( \text{int}(\text{dom} \bar{g}) \).

## 3 An alternative proof for Theorem 1 from [2]

If not mentioned explicitly otherwise, the problems (P) and (D), as well as the corresponding data, are as in the Introduction. Moreover, the preorder definitions by \( P, Q, P^+ \) and \( Q^+ \) are simply denoted by \( \leq \).

The value function associated to problem (P) is

\[
v : Y \to \overline{\mathbb{R}}, \quad v(y) := \inf\{\langle x, c^* \rangle \mid Ax \geq y, \ x \geq 0\} = \inf\{\langle x, c^* \rangle \mid x \in P, \ y \in Ax - Q\},
\]

\[\text{V}_n = V \text{ dom } \chi_{\text{int}(\text{dom } g)} = \text{ dom } V \text{ dom } \chi_{\text{int}(\text{dom } g)} \]
where \( \inf \emptyset := \infty \). It is clear that \( \text{dom } v = A(P) - Q \), \( v \) is positively homogeneous and convex, \( v(0) \leq 0 \), and \( v(y_1) \leq v(y_2) \) whenever \( y_1 \leq y_2 \); hence \( v(0) \in \{0, -\infty\} \), and so \( v(y) = -\infty \) for \( y \in \text{dom } v \) if \( v(0) = -\infty \). Moreover, by the definition of \( v \), we get

\[
v^*(y^*) = \sup \{ \langle y, y^* \rangle + \sup \{ \langle x, -c^* \rangle \mid x \in P, Ax - y = q \in Q \} \mid y \in Y \}
\]

\[
= \sup \{ \langle Ax - y, q \rangle + \langle x, -c^* \rangle \mid x \in P, q \in Q \}
\]

\[
= \sup \{ \langle x, A^*y^* - c^* \rangle + \langle q, -y^* \rangle \mid x \in P, q \in Q \}
\]

\[
= \langle \iota_{P^+}(c^* - A^*y^*), \iota_{Q^+}(y^*) \rangle
\]

for every \( y^* \in Y^* \); hence

\[
\text{dom } v^* = \{ y^* \in Q^+ \mid c^* - A^*y^* \in P^+ \} = Q^+ \cap (A^*)^{-1}(c^* - P^+), \quad v^* = \iota_{\text{dom } v^*}, \quad (8)
\]

and so \( \text{dom } v^* \) is the feasible set of problem (D). It follows that

\[
v^{**}(y) = \sup \{ \langle v, y^* \rangle - v^*(y^*) \mid y^* \in Y^* \} = \sup \{ \langle y, y^* \rangle \mid y^* \in Q^+, c^* - A^*y^* \in P^+ \}
\]

for all \( y \in Y \). Hence \( \text{val}(D) = v^{**}(b) \leq v(b) = \text{val}(P) \).

**Proposition 3** For the problems (P) and (D) above, one has: \( \partial v(b) \neq \emptyset \iff \{ v(P) = \text{val}(D) \} \in \mathbb{R} \) and (D) has optimal solutions]; moreover, if \( \partial v(b) \neq \emptyset \), then \( \partial v(b) \) is the set \( \text{Sol}(D) \) of the optimal solutions of the dual problem (D).

**Proof.** Assume that \( \partial v(b) \neq \emptyset \) and take \( y^* \in \partial v(b) \). Then \( v(b) \in \mathbb{R} \) and \( v(b) + v^*(y^*) = \langle b, y^* \rangle \) by (1). Consequently, \( \text{val}(P) = v(b) = \langle b, y^* \rangle - v^*(y^*) \leq \text{val}(D) \), and so \( \text{val}(P) = \text{val}(D) \); hence \( y^* \in \text{Sol}(D) \). Conversely, assume that \( \{ v(b) = \text{val}(P) = \text{val}(D) \} \in \mathbb{R} \) and (D) has an optimal solution \( y^* \). Then \( (\mathbb{R} \ni v(b) = \langle b, y^* \rangle - v^*(y^*) \), and so \( y^* \in \partial v(b) \); hence \( \text{Sol}(D) \subseteq \partial v(b) \). Therefore, \( \text{Sol}(D) = \partial v(b) \) whenever \( \partial v(b) \neq \emptyset \). \( \square \)

Proposition 3 is essentially [6 Prop. 2.5]; its first part is established in [2 Th. 1]. In this context it is worth recalling [2 Rem. 1, p. 274]:

Q1 – “REMARK 1. The Lipschitz property of the value function \( v \) in the assignment model ensures that the set of dual solutions coincides with the subdifferential of the value function. This, of course, need not be true in more general economic models.” (Our emphasis.)

The second part of Proposition 3 shows that, for the general conic linear programming problem (P), the set of solutions of the dual problem (D) coincides with the subdifferential of the value function whenever the latter is nonempty, not only for “the value function \( v \) in the assignment model”.

In [2 p. 266], one says “we will consider only LP problems for which the value function is proper”; hence, in [2], \( v \) is a proper sublinear function with \( \text{dom } v = A(P) - Q \). Of course, in this case (that is, \( v \) is proper), if \( \dim Y < \infty \) then \( v \) is subdifferentiable on \( \text{icr}(\text{dom } v) \) (\( \neq \emptyset \)), and so \( \emptyset \neq \partial v(y) \subseteq \partial v(0) \) by (4) for \( y \in \text{icr}(\text{dom } v) \), whence \( c^* \in P^+ + A^*(Q^+) \).

The following statement seems to be an important result from [2], even if it is not mentioned explicitly throughout this article.

Q2 – “LEMMA 1. If the value function \( v \) for a linear programming problem in standard form on ordered normed linear spaces is proper, then \( v \) is a lower semicontinuous extended real-valued (convex and homogeneous) function.” (Our emphasis.)
We shall see in the next section that [2, Lem. 1] is false even in finite dimensional spaces.

Immediately after the proof of [2, Lem. 1] one finds the following text:

Q3 – “As promised in the Introduction, we can quickly derive the Duffin–Karlovitz [3] and the Charnes–Cooper–Kortanek [1, 2] no-gap theorems from Theorem 1. Let \( X \) and \( Y \) be ordered normed linear spaces and consider a linear programming problem in standard form with data \( A, b, c^\ast \). The Duffin–Karlovitz theorem asserts that if the positive cone \( Y^+ \) has non-empty interior, if there is a feasible solution \( \tilde{x} \) for the primal problem such that \( \tilde{x} \geq 0 \) and \( A\tilde{x} - b \) is in the interior of \( Y^+_+ \), and if the value of the primal is finite, then the dual problem has a solution and there is no gap.\(^2\) Since \( \tilde{x} \) is feasible for \( b \), \( b \) is in the domain of the value function \( v \) and, hence, \( v(b) < +\infty \). By hypothesis there is an open ball \( U \) around \( A\tilde{x} - b \) within \( F_+ \). Hence, \( \tilde{x} \) is feasible for \( b + u \) for all \( u \in U \); viz. \( b \) is an interior point of the domain of \( v \). The hypothesis that the value of the primal is finite is just the statement that \( v(b) > -\infty \). Consequently, \( v \) is subdifferentiable at \( b \) and we conclude by Theorem 1 that there exists a dual solution and there is no gap, as asserted.” (Our emphasis.)

We have to understand that the statement of the Duffin–Karlovitz theorem is given by the emphasized text from Q3. Let us analyze the given proof. We agree with the facts that \( b + U \subset \text{dom } v \) [and so \( b \in \text{int(dom } v) \)] and \( v(b) \in \mathbb{R} \). From this, without providing a motivation, one concludes that \( \partial v(b) \neq \emptyset \). Even if not mentioned, it is true that \( v(y) \in \mathbb{R} \) for every \( y \in \text{dom } v \), and so \( v \) is proper. Using Lemma 1\(^3\) it follows that \( v \) is lower semicontinuous. Why is \( \partial v(b) \) nonempty? Without being mentioned explicitly, probably, one uses the following statement from [2, p. 266]:

Q4 – “Suppose that \( f : Y \to \mathbb{R} \cup \{+\infty\} \) is a proper convex function defined on the normed linear space \( Y \) and that \( b \in \text{dom } f \). Each of the following conditions implies the next and the last is equivalent to the subdifferentiability of \( f \) at \( b \).

1. \( f \) is lower semicontinuous and \( b \) is an interior point of \( \text{dom } f \);
2. \( f \) is locally Lipschitz at the point \( b \), i.e. there exists \( \delta > 0 \) such that \( f \) is Lipschitz on \( \text{dom } f \cap B(b; \delta) \);
3. \( f \) has bounded steepness at the point \( b \), i.e. the quotients \( (f(b) - f(y))/\|y - b\| \) are bounded above.”

So, using [2, Lem. 1] and the above implications 1. \( \Rightarrow \) 2. \( \Rightarrow \) 3., one gets \( \partial v(b) \neq \emptyset \). Is the implication 1. \( \Rightarrow \) 2. true?

On page 267 of [2] one mentions:

Q5 – “It is well-known (see Phelps [9]\(^4\)) that an extended real-valued proper lower semicontinuous convex function is locally Lipschitz and locally bounded on the interior of its domain.”

We did not succeed to find this assertion in (our reference) [4], but we found the following two related results:

---

\(1\)This is our reference [4].

\(2\)We did not find an assertion equivalent to the mentioned “Duffin–Karlovitz theorem” in [4]; in fact, in [4] p. 123, one says: “This theory makes very little use of topology so it is more like the theory of finite linear programming than like the theories given in [2] and [5]. The desirability of omitting topological considerations is emphasized by the paper of Charnes, Cooper and Kortanek. (However, in another paper [4] a topological approach to this and similar problems will be treated.)”.

\(3\)The string “Lemma 1” appears only once in [2], more precisely in the text from Q2.

\(4\)This is our reference [4].
Q6 – “Proposition 1.6. If the convex function $f$ is continuous at $x_0 \in D$, then it is locally Lipschitzian at $x_0$, that is, there exist $M > 0$ and $\delta > 0$ such that $B(x_0; \delta) \subset D$ and $|f(x) - f(y)| \leq M \|x - y\|$ whenever $x, y \in B(x_0; \delta)$.

Proposition 3.3. Suppose that $f$ is a proper lower semicontinuous convex function on a Banach space $E$ and that $D = \text{int dom}(f)$ is nonempty; then $f$ is continuous on $D$.”

In fact, this version of “the Duffin–Karlovitz theorem” is contained in [3, Th. 3] because $P := X_+$ and $Q := Y_+$ are tacitly assumed to be closed in [2]. Note that “the Duffin–Karlovitz theorem” is true even for $P$ and $Q$ not necessarily closed; for this one could apply [7, Th. 2.7.1] under its condition (iii) for $\Phi : X \times Y \to \mathbb{R}$ defined by $\Phi(x, y) := \langle x, c^* \rangle + \iota_P(x) + \iota_Q(Ax - y - b)$.

4 Two examples

In the sequel, the topological duals of Hilbert spaces (including $\mathbb{R}^m$ with $m \in \mathbb{N}^*$) are identified with themselves using Riesz’ theorem.

The next example shows that [2, Lem. 1] is not true even in finite dimensional spaces.

Example 4 Consider $X := \mathbb{R}^2$, $Y := \mathbb{R}^2 \times \mathbb{R}$, $A : X \to Y$ with $A(x_1, x_2) := (x_1, x_2, 0)$, $c^* := (0, 1) \in \mathbb{R}^2$,

$$P := \mathbb{R} \times \mathbb{R}_+, \quad Q := \{(y_1, y_2, y_3) \in Y \mid y_1, y_3 \in \mathbb{R}_+, \, (y_2)^2 \leq 2y_1y_3\},$$

the conic linear programming problem

(P) minimize $\langle x, c^* \rangle$ s.t. $x \in P$, $Ax - b \in Q$,

and $v : Y \to \mathbb{R}$ defined by $v(y) := \inf \{\langle x, c^* \rangle \mid x \in P, \, Ax - b \in Q\}$.

Then $v(y) = y_2$ if $y := (y_1, y_2, y_3) \in \mathbb{R} \times \mathbb{R}_+ \times \{0\}$, $v(y) = 0$ for $y \in \mathbb{R} \times \mathbb{R} \times (-P)$, and $v(y) = \infty$ elsewhere. Consequently, $v$ is not lower semicontinuous at any $y \in \mathbb{R} \times \mathbb{R} \times \{0\}$.

Proof. Observe that $\langle x, c^* \rangle = x_2 \geq 0$ for $x \in P$, and so $v(y) \geq 0$ for every $y \in \text{dom } v = A(P) - Q$. Take $y \in Y$ and $x \in P$; if $Ax - y = (x_1 - y_1, x_2 - y_2, -y_3) \in Q$, then $x_2 \geq 0$, $x_1 \geq y_1$, $y_3 \leq 0$ and $(x_2 - y_2)^2 \leq -2y_3(x_1 - y_1)$.

Hence $y \notin \text{dom } v$ for $y_3 > 0$, and so $v(y) = \infty$. Take $y_3 = 0$; then necessarily $x_2 = y_2$. Hence $y \notin \text{dom } v$ for $y_2 < 0$, and so $v(y) = \infty$. If $y_2 \geq 0$ then $x_2 = y_2$; taking $x_1 = y_1$, $x = (x_1, x_2)$ is feasible for (P), and so one obtains $v(y) = y_2$ in this case.

Take now $y_3 < 0$; then $x = (x_1, 0)$ with $x_1 = y_1 - \frac{1}{2}(y_2)^2/y_3$ is feasible for (P), and so $v(y) = 0$ in this case.

Consequently $v(y) = y_2$ if $y \in \mathbb{R} \times \mathbb{R}_+ \times \{0\}$, $v(y) = 0$ for $y \in \mathbb{R} \times \mathbb{R} \times (-P)$, and $v(y) = \infty$ elsewhere; hence,

$$\text{dom } v = (\mathbb{R} \times \mathbb{R}_+ \times \{0\}) \cup (\mathbb{R} \times \mathbb{R} \times (-P)).$$

Clearly, $v$ is convex (in fact sublinear), but $v$ is not l.s.c. at any $y \in \mathbb{R}^3$ with $y_2 > 0$ and $y_3 = 0$; indeed, in this case, $\text{dom } v \ni \zeta_n := (y_1, y_2, -1/n) \to y := (y_1, y_2, 0)$ and $v(\zeta_n) = 0 \to 0 < y_2 = v(y)$. □

The next example is an adaptation of [8, Examp. 2.3] to the present context. It shows that the value function $v$ can be proper and not lower semicontinuous at each $y \in \text{dom } v$. 

8
\textbf{Example 5} Let $X$ be a separable infinite-dimensional real Hilbert space with the orthonormal basis $(e_n)_{n \geq 1}$ and

$$P := \left\{ \sum_{n \geq 1} \lambda_n z_n \mid (\lambda_n) \in (\ell_2)^+ \right\} \subset X, \quad c^* := \sum_{n \geq 1} \eta_n e_{2n},$$

with $z_n := \eta_n e_{2n-1} - \mu_n e_{2n}$, where $\eta_n, \mu_n \in [0,1]$ are such that $\eta_n^2 + \mu_n^2 = 1$ for every $n \geq 1$ and $(\eta_n)_{n \geq 1} \in \ell_2$. Consider $\text{Pr}_L : X \to \ell_2$ the orthogonal projection on $L := \text{span} \{ e_{2n-1} \mid n \geq 1 \} = \{ \sum_{n \geq 1} \lambda_n e_{2n-1} \mid (\lambda_n) \in \ell_2 \}$, $A : X \to Y := X$ defined by $Ax := \text{Pr}_L x$, the conic linear programming problems

(P) \quad minimize $\langle x, c^* \rangle$ \quad s.t. $x \in P$, $Ax = b$, and $v : Y \to \mathbb{R}$ defined by $v(y) := \inf\{ \langle x, c^* \rangle \mid x \in P, Ax = y \}$ for $y \in Y$.

Then

$$\text{dom } v = \left\{ y := \sum_{n \geq 1} \gamma_n e_{2n-1} \mid (\gamma_n)^{-1} (\eta_n)_{n \geq 1} \in (\ell_2)^+ \right\} \subset L, \quad (9)$$

$$v(y) = -\sum_{n \geq 1} \mu_n \gamma_n \leq 0 = v(0), \quad \forall y := \sum_{n \geq 1} \gamma_n e_{2n-1} \in \text{dom } v. \quad (10)$$

More precisely, (P) has a unique feasible solution (hence a unique optimal solution) for every $b \in \text{dom } v$, and so $v$ is a proper sublinear function. Moreover, the dual problem (D) has no feasible solutions for every $b \in Y$, proving so that $v$ is not l.s.c. at any $b \in \text{dom } v$; in particular, $\partial v(b) = \emptyset$ for every $b \in Y$.

Proof. Note that $\langle z_n, z_m \rangle = \delta_{nm}$ for $n, m \geq 1$ ($\delta_{nm}$ being the Kronecker’s symbols). Clearly, $P$ is a closed convex cone. Consider $Q := \{0\} \subset Y$; then $Q^+ = Y$. Because $\text{Pr}_L = (\text{Pr}_L)^*$, one obtains that

$$\text{dom } v = A(P) - Q = \text{Pr}_L(P), \quad \text{dom } v^* = Q^+ \cap (A^*)^{-1}(c^* - P^+) = \text{Pr}_L^{-1}(c^* - P^+).$$

Consider now $y \in \text{dom } v (\subset L)$; hence $y = \sum_{n \geq 1} \gamma_n e_{2n-1}$ with $(\gamma_n) \in \ell_2$, and there exists $x \in P$ such that $y = Ax$; hence $x = \sum_{n \geq 1} \lambda_n z_n = \sum_{n \geq 1} \lambda_n (\eta_n e_{2n-1} - \mu_n e_{2n})$ for some $(\lambda_n) \in (\ell_2)^+$. Therefore, $\gamma_n = \lambda_n \eta_n \geq 0$, whence $\lambda_n = \gamma_n / \eta_n$, for $n \geq 1$; this shows that the set $\{ x \in P \mid y = Ax \}$ is a singleton $\{ x_y \}$ for (every) $y \in \text{dom } v$ and so

$$v(y) = \langle c^*, x_y \rangle = \sum_{n \geq 1} \eta_n (-\lambda_n \mu_n) = \sum_{n \geq 1} \eta_n (-\mu_n \cdot \gamma_n / \eta_n) = -\sum_{n \geq 1} \mu_n \gamma_n.$$

Consequently, (9) and (10) hold.

Assume that $x \in \text{dom } v^*$; then there exists $(\lambda_n) \in \ell_2$ such that $u := \text{Pr}_L x = \sum_{n \geq 1} \lambda_n e_{2n-1} \in c^* - P^+$, that is, $c^* - u \in P^+$; hence

$$0 \leq \langle c^* - u, z_k \rangle = -\lambda_k \eta_k - \eta_k \mu_k = -\eta_k (\lambda_k + \mu_k) \quad \forall k \geq 1.$$

It follows that $\lambda_k + \mu_k \leq 0$ for all $k \geq 1$, contradicting the fact that $\lambda_n \to 0$ and $\mu_n \to 1$. Hence $\text{dom } v^* = \emptyset$, and so (D) has no feasible solutions for every $y \in Y$. Consequently, $v^{**} = -\infty$, proving so that $\overline{v}(y) = -\infty$ for every $y \in \text{dom } v$, and so $v$ is not l.s.c. at any $y \in \text{dom } v$. 

5 On Kretschmer’s gap example in linear programming

In [3] pp. 230, 231, Kretschmer considers $Y := L^2 := L^2[0,1]$ (with respect to the Lebesgue measure $\mu$ on $[0,1]$) endowed with the usual inner product and ordered by $Q := L^2_+$, as well as $X := L^2 \times \mathbb{R}$ endowed with the inner product defined by $\langle (x,r), (x',r') \rangle := \langle x, x' \rangle + rr'$ and ordered by $P := L^2_+ \times \mathbb{R}_+$; obviously, $P^+ = P$ and $Q^+ = Q$. Moreover, one takes $A : X \to Y$ with $A(x,r) := y + re_0$, with $y(t) := \int_t^1 x(s)ds$ for $t \in [0,1]$ and $e_0 \in L^2$ with $e_0(t) := 1$ for $t \in [0,1]$. Furthermore, $A$ is a continuous linear operator and $A^* : Y \to X$ is given by $A^* y = (x,r) \in X$ with $x(t) := \int_0^t y(s)ds$ for $t \in [0,1]$ and $r = \int_0^1 y(s)ds$.

Let $c^* := c^*_\alpha : X \to \mathbb{R}$ be defined by $c^*(x,r) := \int_0^1 tx(t)dt + \alpha r = \langle (x,r), (e_1, \alpha) \rangle$, where $\alpha \in \mathbb{R}_+$ and $e_1(t) := t$ for $t \in [0,1]$; clearly, $c^* \in P^+$.

Consider the problem $(P) := (P_\alpha)$ and its dual $(D) := (D_\alpha)$ defined by

(P) minimize $\langle (x,r), c^* \rangle$ s.t. $(x,r) \geq 0, A(x,r) - b \geq 0$,

(D) maximize $\langle z,b \rangle$ s.t. $z \geq 0, A^* z \leq c^*$,

as well as the value function

$v := v_\alpha : Y \to \mathbb{R}, \quad v_\alpha(y) := \inf \{ \langle (x,r), c^*_\alpha \rangle \mid (x,r) \in F(y) \}$

where

$F(y) := \{ (x,r) \in X \mid (x,r) \geq 0, A(x,r) \geq y \}$

is the feasible set of the problem $(P)$; notice that $F(y)$ is the same for all $\alpha \in \mathbb{P}$. Clearly, $v(y) \geq 0 = v(0)$ for $y \in Y$ because $c^* \in P^+$; hence $0 \in \partial v(0)$. In fact, by [3], (2) and (3), one has

$\partial v(0) = Q^+ \cap (A^*)^{-1}(c^* - P^+) = Q \cap (A^*)^{-1}(c^* - P)$ and $v^{**} = \tau$.

Let us denote by $\mathcal{A}$ the class of measurable subsets of $[0,1]$. Without loss of generality we assume that $y(t) \in \mathbb{R}$ for all $y \in L^2$ and $t \in [0,1]$. For $y \in L^2$ and $\gamma \in \mathbb{R}$ we set $[y \geq \gamma] := \{ t \in [0,1] \mid y(t) \geq \gamma \}$ and $y_* := \max\{y,0\}$; clearly, $[y \geq \gamma] \in \mathcal{A}$ and $y_* \in L^2_+$. Moreover, the characteristic function of $E \subset [0,1]$ is the function $\chi_E : [0,1] \to \mathbb{R}$ defined by $\chi_E(t) := 1$ for $t \in E$ and $\chi_E(t) := 0$ for $t \in [0,1] \setminus E$.

Lemma 6 The following assertions hold:

(i) One has

$\text{dom } v = \{ y \in Y \mid \text{ess sup } y < \infty \} = \{ y \in L^2 \mid y_+ \in L^\infty \};$

in particular $L^\infty \subset \text{dom } v$, and so $\text{cl(dom } v) = Y$.

(ii) Let $A \in \mathcal{A}$ be such that $\beta := \mu(A) > 0$. Then there exists a sequence $(A_n)_{n \geq 1} \subset \mathcal{A}$ such that $A = \bigcup_{n \geq 1} A_n$, $A_n \cap A_m = \emptyset$ for $n \neq m$ and $\mu(A_n) = 2^{-n}\beta$ for $n \geq 1$.

(iii) Let $A$ and $(A_n)_{n \geq 1}$ be as in (ii) and consider

$\bar{y}_n := \sum_{k=1}^n 2^{k/4} \chi_{A_k} \geq 0 \quad (n \geq 1), \quad \bar{y} := \sup_{n \geq 1} \bar{y}_n \geq 0.$

Then $\text{ess sup } \bar{y}_n = 2^{n/4} \to \infty$, $\bar{y} \in L^2$, $\|\bar{y}_n\| < \|\bar{y}\| = \left[ \beta(\sqrt{2} + 1) \right]^{1/2}$ and $\|\bar{y}_n - \bar{y}\| \to 0$; consequently $\bar{y}_n \in L^2_+ \subset L^2_+$ for $n \geq 1$ and $\bar{y} \in L^2_+ \setminus L^\infty.$
Proof. (i) Let \( y \in \text{dom} v \); then there exists \((x, r) \in P\) such that \( y \leq A(x, r)\), and so \( y(t) \leq \int_0^1 x(s)ds + r \leq \int_0^1 x(s)ds + r \leq \|x\| + r \) for \( t \in [0, 1]\), whence \( \text{ess sup} y < \infty \). Conversely, if \( y \in L^2\) is such that \( r := \text{ess sup} y < \infty\), then \( y \leq \chi_{0} r e_0 \leq \chi_{0} r e_0 = A(0, r_+)\), where \( r_+ := \max(0, r)\), and so \( v(y) \leq \langle (0, r_+), e^* \rangle = or_+; \) hence \( y \in \text{dom} v\).

(ii) Because \( \mu \) has not atoms, there exists \( A_1 \subset A \) such that \( A_1 \in \mathcal{A} \) and \( \mu(A_1) = 2^{-1} \beta \) \((\in [0, \mu(A)]\)\). Hence \( A'_1 := A \setminus A_1 \in \mathcal{A} \) and \( \mu(A'_1) = \mu(A) - \mu(A_1) > 2^{-2} \beta\), and so there exists \( A_2 \subset A'_1 \) such that \( A_2 \in \mathcal{A} \) and \( \mu(A_2) = 2^{-2} \beta \) \((\in [0, \mu(A'_1)]\)\); clearly, \( A_1 \cap A_2 = \emptyset\).

Continuing in the same way we get the sequence \((\mu(B_n), t_n)\) for all \( t_n \in A_n \) and \( s \in [0, 1]\), and so \( \text{ess sup} \gamma_n = \text{ess sup} \tilde{y}_n = 2^{n/4} \) for \( n \geq 1\). Hence \( \gamma_n \in L^\infty \). \( \square \)

**Proposition 7** Assume that \( \alpha > 0 \). Then for every \( y \in \text{dom} v \) and every \( \rho > 0 \), \( v \) is not bounded on \( B(y, \rho) \cap \text{dom} v \); in particular, \( v|_{\text{dom} v} \) is not continuous at each \( y \in \text{dom} v \).

Proof. Consider \( y \in \text{dom} v \) and \( \rho > 0\), as well as \( \alpha > 0\) and \( \beta > 0\) and \( \gamma := -k_0\), \( A := E_{k_0} \setminus B = E_{k_0} \cap \eta_0, \eta_1 \subset ]\eta_0, \eta_1[ \) and \( \beta := \mu(A)\). Clearly, \( E_{k_0} \subset A \cup B\), and so \( \beta > 0\); set also \( \delta := \langle \beta(\sqrt{2} + 1) \rangle^{1/2} > 0\).

Consider now the sets \( A_n \) and the functions \( \gamma_n \) for \( n \geq 1\) provided by assertions (ii) and (iii) of Lemma \( \square \) as well as \( \gamma_n := \text{sup}_{n \geq 1} \gamma_n\); hence

\[
L^\infty \ni \gamma_n \to \|\gamma\| = \|\gamma_n\| \to \gamma_n \in L^2_+ \setminus L^\infty \ \text{and} \ \forall n \geq 1: \|\gamma_n\| < \|\gamma\| = \delta.
\]

Consider also \( 0 < \varepsilon < \rho/\delta \) and set \( y_n := y + \varepsilon \gamma_n \) for \( n \in \mathbb{N}^*\); clearly \( y_n \in L^2 \) and \( \text{ess sup} y_n \leq \text{ess sup} y + \varepsilon \text{ess sup} \gamma_n \), whence \( y_n \in \text{dom} v \) and \( \|y_n - y\| = \|\varepsilon \gamma_n\| < \varepsilon \delta < \rho \) for \( n \geq 1\). Moreover, \( y_n \not\in \text{dom} v \). Therefore,

\[
(y_n)_{n \geq 1} \subset B(y, \rho) \cap \text{dom} v \ \text{and} \ B(y, \rho) \cap (Y \setminus \text{dom} v) \neq \emptyset. \tag{11}
\]
Let $n \geq 1$ be fixed and consider $(x, r) \in F(y_n)$; hence
\[ x \geq 0, \quad r \geq 0, \quad \text{and} \quad \int_t^1 x(s)ds + r \geq y(t) + \varepsilon y_n(t) \quad \text{a.e.} \ t \in [0, 1]. \]
Because $A_n \subset A = E_{\gamma_0} \cap [\eta_0, \eta_1 \subset [y \geq \gamma])$, one has
\[ \int_t^1 x(s)ds + r \geq y(t) + \varepsilon y_n(t) \geq \gamma + 2^{n/4} \varepsilon \quad \text{for a.e.} \ t \in A_n, \]
and so, for a.e. $t \in A_n$, one has
\[ \langle (x, r), c^* \rangle = \int_0^1 sx(s)ds + \alpha r \geq \int_0^1 sx(s)ds + \alpha r \geq t \int_t^1 x(s)ds + \alpha r \geq \eta_0(\gamma + 2^{n/4} \varepsilon - r) + \alpha r = \eta_0(\gamma + 2^{n/4} \varepsilon) + r(\alpha - \eta_0) \geq \eta_0(\gamma + 2^{n/4} \varepsilon); \]
hence $\langle (x, r), c^* \rangle \geq \eta_0(\gamma + 2^{n/4} \varepsilon)$. Because $(x, r) \in F(y_n)$ is arbitrary, it follows that $v(y_n) \geq \eta_0(\gamma + 2^{n/4} \varepsilon)$. Therefore, $v(y_n) \geq \eta_0(\gamma + 2^{n/4} \rho/\delta)$ for every $n \geq 1$, and so $v(y_n) \to \infty$. Taking into account (14), it follows that $v$ is not bounded on $B(y, \rho) \cap \text{dom} \ v$; moreover, $y \notin \text{int}(\text{dom} \ v)$ because $B(y, \rho) \cap (Y \setminus \text{dom} \ v) \neq \emptyset$ for every $\rho > 0$, proving that $\text{int}(\text{dom} \ v) = \emptyset$.

Observe that the case $\alpha = 0$ is very special. Indeed, as seen in the proof of Lemma (5.1), for $y \in \text{dom} \ v$, $(0, r_+) \in F(y)$, where $r := \text{ess sup} \ y$, and so $0 \leq v(y) \leq ((0, r_+), (e_1, 0)) = 0$. Hence $v(y) = 0$ and the value $v(y)$ is attained. Therefore, $v = \iota_{\text{dom} \ v}$. On the other hand, for $y \in Y$, $z$ is feasible for the dual problem $(\mathcal{D}_y)$ if and only if $z \geq 0$, $\int_0^t z(s)ds \leq e_1(t)$ a.e. $t \in [0, 1]$ and $\int_0^1 z(s)ds \leq \alpha = 0$, and so $z = 0$ is the only feasible (hence optimal) solution of $(\mathcal{D}_y)$. Hence $v^*(y) = 0 = \overline{v}(y)$ for every $y \in Y$. Because $v = \iota_{\text{dom} \ v}$, one has $\overline{v} = \iota_{\text{cl}(\text{dom} \ v)}$, confirming so that $\text{cl}(\text{dom} \ v) = Y$.

Taking $\alpha := 2$ and $b := e_0$, one obtains [3, Examp. 5.1]; this is also considered in [2, Examp. 1], as well as the one in which $b := b_0 := \chi_{[0, 1/2]}$. The next two results are slight extensions of those related to the “modification” of [3, Examp. 5.1] used in [2, p. 270], the proofs using similar arguments to those in [2].

**Proposition 8** Consider $\alpha \in \mathbb{P}$ and $b := \chi_{[\gamma, 1)}$ with $I := [0, \delta]$, $J := [\gamma, 1]$, where $0 \leq \delta < \gamma < 1$. Then $\text{val}(P) = \alpha$, $\text{val}(\mathcal{D}) = \min \{1, \alpha\}$, and $(P)$, $(\mathcal{D})$ have optimal solutions; moreover, $\text{val}(P) = \text{val}(\mathcal{D}) \iff \alpha \in [0, 1] \iff \partial v(\chi_{[\gamma, 1]}) \neq \emptyset$.

**Proof.** Clearly, if $(x, r) \in P$ is feasible then $\int_t^1 x(s)ds + r \geq 1$ a.e. $t \in [\gamma, 1]$; because $\lim_{\gamma, 1 \to \infty} \int_t^1 x(s)ds = 0$, one gets $r \geq 1$. Because $(0, 1)$ is feasible for $(P)$, one has that $0$ is optimal solution for $(P)$ and $\text{val}(P) = \alpha$.

Observe that for $z \geq 0$ with $\int_0^t z(s)ds \leq t$ for $t \in [0, 1]$ one has $\int_0^t z(s)ds \leq 1$, and so, when $z$ is feasible for $(\mathcal{D})$ one has $\int_0^1 \chi_{[\gamma, 1]} z = \int_0^1 \gamma \leq \int_0^1 z \leq \min \{1, \alpha\}$. Hence $0 \leq \text{val}(\mathcal{D}) \leq \min \{1, \alpha\} =: \mu$. Take $\eta \in [\gamma, 1]$ and $z := \mu(1 - \eta)^{-1} \chi_{[\eta, 1]}$ ($\geq 0$); then $\int_0^t z(s)ds = 0$ for $t \in [0, \eta]$ and $\int_0^1 z(s)ds = \mu(1 - \eta)^{-1} \int_0^1 ds = \mu \frac{1 - \eta}{\eta} \leq \mu t \leq t$ for $t \in [\eta, 1]$ and so $z$ is feasible for $(\mathcal{D})$. Moreover, $\int_0^1 z(t)dt = \mu$, and so $z$ is an optimal solution for $(\mathcal{D})$, whence $\text{val}(\mathcal{D}) = \min \{1, \alpha\}$. Consequently, both problems have optimal solutions, and $\partial v(\chi_{[\gamma, 1]}) \neq \emptyset$ if and only if $\alpha \in [0, 1]$.

Taking $\alpha := 2$ and $\delta := \gamma = 0$ one (re)obtains (as already mentioned) the example from [3, Examp. 5.1], as well as the one from [2, p. 270] and the conclusions from there, that is, both problems have optimal solutions, but there is a (positive) duality gap.
Consequently, the previous example shows not only that \( v|_{\text{dom } v} \) is not locally Lipschitz, but also that \( v|_{\text{dom } v} \) is not l.s.c. on its domain; therefore, [2] Examp. 1 provides a counterexample to [2] Lem. 1.

**Proposition 9** Take \( \alpha \in \mathbb{P} \) and \( b := \chi_{[0,\delta]} \) with \( \delta \in ]0,1[ \). Then \( \text{val}(P) = \text{val}(D) = \min\{\delta, \alpha\} \) and (D) has optimal solutions; consequently, \( \partial v(\chi_{[0,\delta]}) \neq \emptyset \). Furthermore, (P) has optimal solutions iff \( \alpha \leq \delta \).

Proof. First observe that for \((x,r) \in P\), the following assertions are equivalent: \((x,r)\) is feasible for (P); \((x \cdot \chi_{[\delta,1]}, r)\) is feasible for (P); \( \int_0^1 x(s)ds + r \geq 1; \ r \geq (1 - \int_0^1 x(s)ds)\). Set \( F_1 := \{x \in L^2_\alpha \mid \int_0^1 x(s)ds \geq 1\}, \ F_2 := \{x \in L^2_\alpha \mid \int_0^1 x(s)ds \leq 1\}. \)

Notice that \( F_1 \cap F_2 \neq \emptyset \) and \( 0 \in F_2\); moreover, \((x,0)\) is feasible when \( x \in F_1 \) and \((x,1 - \int_0^1 x(s)ds)\) is feasible when \( x \in F_2\). It follows that \( \text{val}(P) = \min\{v_1, v_2\} \), where \( v_1 := \inf_{x \in F_1} \int_0^1 tx(t)dt \) and \( v_2 := \inf_{x \in F_2} \left( \int_0^1 tx(t)dt + \alpha - \alpha \int_0^1 x(t)dt \right) = \inf_{x \in F_2} \left( \alpha + \int_0^1 (t - \alpha)x(t)dt \right) = \alpha - \sup_{x \in F_2} \int_0^1 (\alpha - t)x(t)dt \leq \alpha. \)

For \( x \in L^2_\alpha \) and \( t \in [\delta,1] \) one has \( (\alpha - t)x(t) \leq (\alpha - \delta)x(t) \), and so \( \int_0^1 (\alpha - t)x(t)dt \leq (\alpha - \delta) \int_0^1 x(t)dt \), with equality iff \( x \cdot \chi_{[\delta,1]} = 0 \). Assume that \( x \in F_2\); for \( \alpha > \delta \) one has \( \int_0^1 (\alpha - t)x(t)dt \leq \alpha - \delta \), while for \( \alpha \leq \delta \) one has \( \int_0^1 (\alpha - t)x(t)dt \leq 0 \). Therefore, \( v_2 \geq \delta \) if \( \alpha \geq \delta \) and \( v_2 = \alpha \) if \( \alpha \leq \delta \), \( v_2 \) being attained for \( x = 0 \) in the latter case.

In what concerns \( v_1 \), one has

\[
v_1 = \inf_{x \in F_1} \left( \int_0^1 tx(t)dt + \int_0^1 x(s)ds \right) = \inf_{x \in F_1} \int_0^1 tx(t)dt \geq \delta \inf_{x \in F_1} \int_0^1 x(t)dt \geq \delta.
\]

For \( \varepsilon \in ]0,1 - \delta[ \) and \( x := \varepsilon^{-1}\chi_{[\delta,\delta + \varepsilon]} \), one has \( x \in F_1 \) and \( \int_0^1 tx(t)dt = \varepsilon^{-1} \int_0^{\delta + \varepsilon} tdt = \delta + \varepsilon/2 \), and so \( v_1 = \delta \). Consequently, \( v_1 = \min\{\alpha, \delta\} \); moreover, if \( \alpha > \delta \) then (P) has not optimal solutions, and \( x = 0 \) is solution of (P) if \( \alpha \leq \delta \).

If \( z \) is feasible for (D), then \( \int_0^\delta z(t)dt \leq \delta \) and \( \int_0^\delta z(t)dt \leq \int_0^1 z(t)dt \leq \alpha \), and so \( \text{val}(D) \leq \min\{\delta, \alpha\} \). Clearly, \( z := \chi_{[0,\min\{\alpha, \delta\}]} \) is an optimal solution of (D), and so \( \text{val}(D) = \min\{\delta, \alpha\} \).

Therefore, \( \text{val}(P) = \text{val}(D) = \min\{\delta, \alpha\} \) and (D) has optimal solutions; consequently, \( \partial v(\chi_{[0,\delta]}) \neq \emptyset \). Furthermore, (P) has optimal solutions iff \( \alpha \leq \delta \).\[\square\]

**Corollary 10** Let \( \alpha \in ]1, \infty[ \) and \( \delta \in ]0,1[ \), and consider the problems

\( (P_y) \) minimize \( \int_0^1 tx(t)dt + \alpha r \) s.t. \( x \geq 0, \ r \geq 0, \ \int_t^1 x(s)ds + r \geq y(t) \) a.e. \( t \in [0,1] \),

\( (D_y) \) maximize \( \int_0^1 y(t)z(t)dt \) s.t. \( z \geq 0, \ \int_0^1 z(s)ds \leq t \) a.e. \( t \in [0,1] \), \( \int_0^1 z(s)ds \leq \alpha. \)

Then \( \partial v(\chi_{[0,\delta]}) \neq \emptyset \) and \( v|_{\text{dom } v} \) is not continuous at \( \chi_{[0,\delta]} \).

Proof. By Proposition 9 one has that \( v(\chi_{[0,\delta]} = \delta \) and \( \partial v(\chi_{[0,\delta]}) \neq \emptyset \), while from Proposition 8 one has that \( v(\chi_{[0,\delta]}) = \alpha \) for every \( \gamma \in ]0,1[ \). Because \( \|\chi_{[0,\delta]} - \chi_{[0,\delta]}\|_2 = \|\chi_{[\gamma,1]}\|_2 = (1 - \gamma)^{1/2} \to 0 \) for \( \gamma \to 1 \), confirming that \( v|_{\text{dom } v} \) is not continuous at \( \chi_{[0,\delta]} \).\[\square\]
In the paragraph before [2, Examp. 1, p. 269], one says:

Q7 – “We give an example of a convex function which is subdifferentiable but not locally Lipschitz by exhibiting a linear programming problem for which the value function has this property. The example takes place in the Banach lattice $L^2[0,1]$ (a space for which the positive cone has empty interior) and for which $\text{dom } v \supseteq L^2[0,1]_+$. (Our emphasis.)

This text is completed by the following ones from [2, p. 270]:

Q8 – “On the other hand, we will establish that $v$ is not locally Lipschitz at $b_0$; in fact, $v$ is not even continuous there (or anywhere).” (Our emphasis.)

Q9 – “Similar perturbations show that $v$ is not continuous anywhere on $L^2[0,1]_+$. (Of course, $v$ is lower semicontinuous.” (Our emphasis.)

As seen in Lemma 6, one has $L^\infty_+ \subset L^\infty \subset \text{dom } v \not\supset L^2[0,1]_+$, which shows that the inclusion $\text{dom } v \supseteq L^2[0,1]_+$, mentioned in Q7, is not true.

Having in view the texts from Q7, Q8 and Q9, one may wonder what is meant in [2] by continuity and lower semicontinuity of $v$ at some point in $Y$, as well as by local Lipschitzness and subdifferentiability.

In what concerns the local Lipschitzness, it is quite clear that this is meant in the sense from condition 2. in Q4; related to “subdifferentiability”, this is not at any point $b$ with $v(b) \in \mathbb{R}$ as suggested by Q7, but just at a certain point $b_0$ as in Q8. As seen in Section 2 the are important differences among the continuity properties of $g$ and $g|_{\text{dom } g}$ at points from $\text{dom } g$. In fact, inspecting the proof of [2, Lem. 1] and the discussion of the modified version of [2, Examp. 1], in [2] one has in view the continuity and the lower semicontinuity of $v|_{\text{dom } v}$ at points in $\text{dom } v$.

Having in view Proposition 7 we agree with the remark “$v$ is not even continuous there (or anywhere)” from Q8. In what concerns Q9, on one hand, we would like to see those “similar perturbations” which “show that $v$ is not continuous anywhere on” $L^2[0,1]_+ \cap \text{dom } v$; on the other hand, as already mentioned, we do not agree with the remark “Of course, $v$ is lower semicontinuous”, which is surely based on [2, Lem. 1].

6 Some comments on Proposition 2 from [2]

In Section 6 of [2] one establishes two results on the Lipschitzness of the value function in infinite-dimensional linear programming; the second one, Proposition 2, is applied to the assignment model in [2, Sect. 7]. Our aim is to discuss the proof of [2, Prop. 2]; for easy reference, we quote its statement and proof, as well as its preamble:

Q10 – “Another structural condition is useful for application to the assignment model. We will use the condition in the context of a maximization problem and will state it as such.

**PROPOSITION 2.** Let $X$ and $Y$ be Banach lattices and let $A, b$, and $c$ be the data for an LP maximization problem. Assume that

- $A$ is a positive operator which maps the positive cone $X_+$ onto $Y_+$;
- the order interval $[0, x_0]$ is mapped onto the order interval $[0, Ax_0]$ for every $x_0 \geq 0$;
- $A$ is bounded below on the positive cone $X_+$, i.e. there exists a constant $M > 0$ such that $\|Ax\| \geq M \|x\|$ for all $x \geq 0$.

\[5\text{Recall that } v|_{\text{dom } v} \text{ is not l.s.c. at every } y \in \mathbb{R} \times P \times \{0\} \text{ in Example 4, and } v|_{\text{dom } v} \text{ is not l.s.c. at every } y \in \text{dom } v \text{ in Example 5.}\n\]

\[6\text{Of course, it is } c^* \text{ instead of } c.\]
Then the value function is Lipschitz on the positive cone $X_+^*$.

Proof. Start with $b_1 \geq 0$ and $b_2 \geq 0$. First, consider the case that $b_2 \leq b_1$. Given $\varepsilon > 0$, there is an almost optimal $x_1$ for $b_1$, viz. there is $x_1 \geq 0$ with $Ax_1 = b_1$, and $c^*(x_1) + \varepsilon > v(b_1)$.

Since $0 \leq b_2 \leq b_1 = Ax_1$ and since the positive operator $A$ maps $[0, x_1]$ onto $[0, Ax_1]$, there is $x_2$ such that $0 \leq x_2 \leq x_1$ with $Ax_2 = b_2$. Clearly, $x_2$ is feasible for $b_2$; hence, $v(b_2) \geq c^*(x_2)$.

We compute
\[
v(b_1) - v(b_2) \leq c^*(x_1) + \varepsilon - c^*(x_2) \leq \|c^*\| \|x_1 - x_2\| + \varepsilon \leq \|c^*\| \|Ax_1 - Ax_2\| + \varepsilon \leq \|c^*\| \|b_1 - b_2\| + \varepsilon
\]
Since this true for arbitrary $\varepsilon > 0$, we have that $v(b_1) - v(b_2) \leq \frac{1}{M} \|c^*\| \|b_1 - b_2\|.$

Switching the roles of $b_1$ and $b_2$ gives us $|v(b_1) - v(b_2)| \leq \frac{1}{M} \|c^*\| \|b_1 - b_2\|$ as desired.

For the general case in which we do not assume any order dominance between $x_1$ and $x_2$, define $x_3 = x_1 \wedge x_2$. Then $b_3 = b_1 - (b_1 - b_2)^+; i.e., b_1 - b_3 = (b_1 - b_2)^+$. Consequently,
\[
\|b_1 - b_3\| \leq \|(b_1 - b_2)^+\| \leq \|b_1 - b_2\|.
\]
Since $0 \leq b_3 \leq b_2$ and $v$ is an increasing function, we have that $v(b_1) - v(b_3) \leq c \leq \|c^*\| \frac{1}{M} \|b_1 - b_2\|.$ The same $x_3$ works for $v(b_2) - v(b_1)$ and we have shown that $v$ is Lipschitz on $Y^k$.

(Our emphasis.)

Remarks:

1) Even if not clearly stated, the considered problem is: maximize $c^*(x)$ s.t. $Ax \leq b$ and $x \geq 0$; compare with problem (P) on page 273 to which Proposition 2 is applied. This is also confirmed by the argument “Since $0 \leq b_3 \leq b_2$ and $v$ is an increasing function, we have that ...” from the end of the proof.

Set $F(b) := \{x \in X \mid x \geq 0, \ Ax \leq b\}$ (the feasible set corresponding to $b \in Y_+$).

2) (One had to) Observe first that $F(b)$ is bounded, and so $v(b) \in \mathbb{R}_+$, for every $b \in Y_+

3) By 2) and the definition of $v(b_1)$, for each $\varepsilon > 0$ there exists $x_1 \in F(b_1)$ such that $c^*(x_1) + \varepsilon > v(b_1)$; hence $x_1 \geq 0$ and $Ax_1 \leq b_1$.

So, why $Ax_1 = b_1$? Without having $Ax_1 = b_1$ one cannot find (using the hypotheses) $x_2 \in [0, x_1]$ such that $Ax_2 = b_2$ because $b_2$ could be outside $[0, Ax_1]$. How is the argument continued?

4) Assume that for each $\varepsilon > 0$ one finds $x_1 \in F(b_1)$ such that $Ax_1 = b_1$ and $c^*(x_1) + \varepsilon > v(b_1)$. “Switching the roles of $b_1$ and $b_2$”, will $b_2$ have the same property, that is, for each $\varepsilon > 0$ one finds $x_2 \in F(b_2)$ such that $Ax_2 = b_2$ and $c^*(x_2) + \varepsilon > v(b_2)$? If so, we agree with the estimate $|v(b_1) - v(b_2)| \leq \frac{1}{M} \|c^*\| \|b_1 - b_2\|.$

5) a) The particular case was the one in which $(0 \leq) b_2 \leq b_1$, that is, the case in which $b_1$ and $b_2$ are comparable.

b) Hence, the general case must be “the one in which we do not assume any order dominance between $b_1$ and $b_2$.

c) Under 5b), which are $x_1$ and $x_2$ here? and which is $b_3$? is it $Ax_3$?

d) We agree with $x_3 = x_1 \wedge x_2 \Rightarrow x_3 = x_1 - (x_1 - x_2)^+$. Assume that $b_k = Ax_k$ for $k \in \{1, 2, 3\}$ (which could be envisaged because one had already $b_k = Ax_k$ for $k \in \{1, 2\}$). Because $b_3 = b_1 - (b_1 - b_2)^+ = b_1 \wedge b_2$, one must have $A(x_1 \wedge x_2) = (Ax_1) \wedge (Ax_2)$ for $x_1, x_2 \in X_+$ (or, equivalently, for $x_1, x_2 \in X$). Do the imposed conditions on the data of [2] Prop. 2] ensure that $A$ is a homomorphism of Banach lattices?

\footnote{In fact, it is $Y_+$ instead of $X_+$.}

\footnote{Of course, it must be $Y_+$ instead of $Y$.}
6) Probably, $c$ from the inequality $v(b_1) - v(b_3) \leq c$ is $\|c^*\| \frac{1}{M} \|b_1 - b_3\|$, gotten because $0 \leq b_3 \leq b_1$.

Having in view the above remarks, we consider that the proof of [2] Prop. 2 needs several clarifications.

So, in our opinion, the authors of [2] did not succeed to accomplish their goal that emerges from the following text taken from the beginning of Section 2 of [2]:

Q11 – “The present study was motivated by the problem of showing that there was no gap in the infinite-dimensional linear programming problem that arose in our studies of the continuum assignment problem in [5]. The no-gap argument given there was incomplete; the current paper rectifies that omission.” (Our emphasis.)

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