Abstract. Let $C$ be a model category with an initial object $\emptyset$ and functorial factorizations. Let $S : C \to C$ be the suspension functor. An object $X$ of $C$ is said to be charged if it comes equipped with a map $S\emptyset \to X$. If $Y$ is any object of $C$, then $SY$ has a preferred charge, given by applying suspension to the map $\emptyset \to Y$. This motivates the question of whether a given charged object is a suspension up to a weak equivalence in a way that preserves charge structures. We study this question in the context of spaces over a given space, where we give a complete obstruction in a certain metastable range. As an application we show how this can be used to study when an embedding into a smooth manifold of the form $N \times I$ compresses to an embedding into $N$.

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1. Introduction

If $Y$ is a topological space then its unreduced suspension

$$SY = C_-Y \cup_Y C_+Y$$

has two preferred basepoints

$$-, + \in SY$$

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given by the vertices of each cone $C_{\pm}Y$. We say that a space $X$ is charged if it comes equipped with two base points, which we represent as a map $S^0 \to X$, in which $S^0 = \{-1, +1\}$. The map $S^0 \to X$ is called the structure map of $X$. A space equipped with a choice of charge is called a charged space.

Let $T$ be the category of spaces and let $T(S^0 \to *)$ be the category whose objects are charged spaces, where a morphism is a map of underlying spaces that is compatible with the structure maps. A morphism of charged spaces is said to be a weak equivalence if it is a weak homotopy equivalence when considered as a map of spaces. Unreduced suspension can be regarded as a functor $S:T \to T(S^0 \to *)$.

The (unreduced) desuspension problem for $X$ asks whether there is a space $Y$ and a weak equivalence of charged spaces $SY \simeq X$. This is not generally the same as the corresponding desuspension problem for based spaces: for example, the zero sphere $S^0$ is the unreduced suspension of the empty space, but it is not weak equivalent to the reduced suspension of any based space.

We are really interested in a fiberwise version of the desuspension problem. In order to formulate it, fix a map of spaces $f:A \to B$ and let

$$T(A \overset{f}{\to} B)$$

be the category of spaces which factorize $f$. This is the category whose objects are triples $(Y, r, s)$ in which $Y$ is a space, and $r:Y \to B$, $s:A \to Y$ are maps such that $f = r \circ s$. To avoid clutter, it is standard to omit the structure maps $r, s$ from the notation, letting $Y$ refer to $(Y, r, s)$. When $f$ is understood, we often denote the category by $T(A \to B)$. Throughout this paper, we will always assume that $B$ is a path connected space.

A morphism $(Y, r, s) \to (Y', r', s')$ is a map of spaces $u:Y \to Y'$ such that $r = r' \circ u$ and $s' = u \circ s$. A morphism $Y \to Y'$ is said to be a weak equivalence if it is a weak homotopy equivalence of underlying spaces. More generally, $Y$ and $Y'$ are weakly equivalent, written $Y \simeq Y'$, if there is a finite zig-zag of weak equivalences connecting $Y$ to $Y'$. By slight abuse of language, we say in this case there is a weak equivalence $Y \simeq Y'$.

In the special case of the projection map $B \times S^0 \to B$, an object $X \in T(B \times S^0 \to B)$ is called a fiberwise charged space over $B$. Consider the unreduced fiberwise suspension functor

$$(1) \quad S_B:T(\emptyset \to B) \to T(B \times S^0 \to B)$$
which maps an object $Y$ to the object

$$S_B Y = (B \times \{-1\}) \cup Y \times [-1, 1] \cup (B \times \{1\}) ,$$

where the right side is the double mapping cylinder of the structure map $Y \to B$ with itself. The map $B \times S^0 \to S_B Y$ is given by the two summands appearing in the mapping cylinder. We are now in the position of being able state the main problem of interest.

\textit{Fiberwise Desuspension Problem:} Let $X \in T(B \times S^0 \to B)$ be an object. Find an object $Y \in T(\emptyset \to B)$ and a weak equivalence of fiberwise charged spaces

$$X \simeq S_B Y .$$

\textbf{Remark 1.1.} This problem is about the degree to which the \textit{un}reduced fiberwise suspension functor $S_B : T(\emptyset \to B) \to T(B \times S^0 \to B)$ is surjective up to weak equivalence. At the risk of belaboring the point, we wish to emphasize that this problem is not the same as the based version. To see this, we set

$$R(B) = T(B \xrightarrow{id} B) .$$

This is sometimes called the category of \textit{retractive spaces} over $B$. The based version of the desuspension problem considers the extent to which the \textit{reduced} fiberwise suspension functor

$$\Sigma_B : R(B) \to R(B)$$

is surjective on objects up to weak equivalence (for the definition of $\Sigma_B$ see §2). A solution $Y$ to the unbased version of the fiberwise suspension problem need not admit a section $B \to Y$, whereas of course a solution to the based version comes equipped with a section.

Even in the case when $X \in T(B \times S^0 \to B)$ is in the image of the forgetful functor $R(B) \to T(B \times S^0 \to B)$ (where the structure map is induced by the projection $B \times S^0 \to B$), it could well be the case that $X$ can be written in the form $S_B Y$ up to weak equivalence, but $X$ might not be of the form $\Sigma_B Z$ up to weak equivalence.

\textit{Example 1.2.} Let $X$ be the Klein bottle. This fibers over $S^1$, with fiber $S^1$, and it is the unreduced fiberwise suspension of the degree two map $\times 2 : S^1 \to S^1$. Moreover $X \in T(S^1 \times S^0 \to S^1)$ lifts to an object of $R(S^1)$.

However, $X$ is not weakly equivalent to the reduced fiberwise suspension of an object $Y \in R(S^1)$. If it were, then $Y \to S^1$ would be weakly equivalent to a fibration over $S^1$ equipped with section and fiber $S^0$. Such a fibration is always fiber homotopically trivial and this would imply that $X$ is weak homotopy equivalent to a torus. This yields a
contradiction, so $X$ is not a reduced fiberwise suspension up to weak equivalence.

In [K2] the first author proved a Freudenthal suspension theorem for the functor (1). In the current paper we will extend this result to a certain metastable range where there will be an obstruction. To formulate the main result, we say that $X \in T(B \times S^0 \to B)$ is $r$-connected if the structure map $X \to B$ is $(r + 1)$-connected. We say that $X$ has dimension $\leq k$ if $X$ can be obtained up to weak equivalence from $B \times S^0$ by attaching cells of dimension $\leq k$. In this case we write $\dim X \leq k$.

The retractive space category $R(B)$ is a pointed simplicial model category ($\S 2$). Hence for objects $U, V \in R(B)$ we can talk about the abelian group of stable retractive fiberwise homotopy classes

$$\{U, V\}_{R(B)}.$$ 

The trivial element is represented by the class of the composite $U \to B \to V$.

To any object $X \in T(B \times S^0 \to B)$, we can associate a pair of objects

$$i_- X, i_+ X \in R(B)$$

in which $i_- X$ is just $X$ with section $B \to X$ given by $B \times -1 \to B \times S^0 \to B$, and $i_+ X$ is similarly defined by restricting to $B \times +1 \to B \times S^0$. The category $R(B)$ has internal (fiberwise) smash products, and we therefore consider the object $i_+ X \wedge_B i_- X \in R(B)$. We will also need to consider the object of $R(B)$ given by the amalgamated union

$$X^+ = X \cup_{B \times S^0} B.$$ 

In $\S 3$ we construct a fiberwise reduced diagonal map,

$$\tilde{\Delta}: X^+ \to i_+ X \wedge_B i_- X,$$

which is a morphism of $R(B)$. Note that one should “derive” these constructions to ensure homotopy invariance. To obtain homotopy invariance for (2), it suffices to assume at the outset that $B \times S^0 \to X$ is a cofibration and $X \to B$ is a fibration. In the following discussion, this will be assumed.

It is not difficult to check that the homotopy class of $\tilde{\Delta}$ is trivial whenever $X$ is weakly equivalent to a fiberwise suspension $S_B Y$. Our first main result provides a partial converse, which one can view as an unbased fiberwise version of a result of Berstein-Hilton and Ganea [BH], [Ga].
Theorem A. Let $X \in T(B \times S^0 \to B)$ be an object. Assume $X$ is $r$-connected, $\dim X \leq 3r$ and
$$[\tilde{\Delta}] \in \{X^+, i_+ X \wedge_B i_- X\}_{R(B)}$$
is trivial. Then $X \simeq S_B Y$.

Remark 1.3. According to Lemma 2.2 below, there is a preferred weak equivalence $\Sigma B i_- X \simeq \Sigma B i_+ X$. Consequently, there is an isomorphism of abelian groups
$$\{X^+, i_+ X \wedge_B i_- X\}_{R(B)} \cong \{X^+, i_+ X \wedge_B i_+ X\}_{R(B)}.$$

The relative case. Suppose that $A \to B$ is a map and $X \in T(S_B A \to B)$ is an object. The relative case of the fiberwise desuspension problem asks the extent to which $X$ lies in the image of the functor
$$S_B : T(A \to B) \to T(S_B A \to B)$$
up to weak equivalence (we recover the absolute case when $A = \emptyset$). In this more general context, we redefine
$$X^+ := X \cup_{S_B A} B.$$

With respect to this notational convention, the diagonal obstruction in the relative case can be regarded as lying in $[\tilde{\Delta}] \in \{X^+, i_+ X \wedge_B i_- X\}_{R(B)}$. In the current context, we say that $X$ is $r$-connected if $X \to B$ is $(r + 1)$-connected, and we write $\dim X \leq k$ if $X$ is obtained up to weak equivalence from $S_B A$ by attaching cells of dimension at most $k$. In the following, we will need to assume that the structure map $S_B A \to X$ is a cofibration and the structure map $X \to B$ is a fibration.

Addendum B. With respect to these conventions, Theorem A holds in the relative case.

Embedding up to homotopy type. Let $N$ be a compact smooth manifold of dimension $n$. Suppose that $f : K \to N$ is a map, in which $K$ is a finite CW complex of dimension $\leq k$. An $h$-embedding of $f$ consists of a pair $(U, h)$ in which $U \subset \text{int} N$ is a compact codimension zero submanifold and $h : K \to U$ is a homotopy equivalence such that the composite
$$K \xrightarrow{h} U \subset N$$
is homotopic to $f$. Heuristically, we think of $U$ as a “compact regular neighborhood” of $K$ in $N$, in the sense that $U \subset N$ is a codimension zero compact manifold model for $K$ up to homotopy.

There is also a notion of concordance of $h$-embeddings of $f$. Let $(U_0, h_0)$ and $(U_1, h_1)$ be $h$-embeddings of $f : K \to N$. A concordance
between them consists of a compact submanifold \( W \subset N \times [0, 1] \) and a weak equivalence \( H: K \times [0, 1] \to W \) such that

- There is a decomposition
  \[ \partial W = U_0 \cup \partial_1 W \cup U_1; \]
  where \( U_0, \partial_1 W \) and \( U_1 \) are compact codimension zero submanifolds of \( \partial W \) such that \( \partial_1 W \) is an \( h \)-cobordism between \( \partial U_0 \) and \( \partial U_1 \).
- \( W \) meets \( N \times \{i\} \) transversely with intersection \( U_i \) for \( i = 0, 1 \);
- \( H \) extends \( h_0 \amalg h_1 \);
- the composite \( K \times [0, 1] \xrightarrow{H} W \to N \times [0, 1] \) is homotopic to \( f \times \text{id} \).

The operation of concatenation shows that concordance defines an equivalence relation on \( h \)-embeddings of \( f \).

The above notion of \( h \)-embedding often appears in the literature as “embedding up to homotopy,” or as “embedded thickening.” The topic was first systematically studied by Wall [W1], Stallings (in the PL case) [St] and Mazur [M]. The work of Wall and Stallings showed that \( h \)-embeddings exist whenever \( f \) is \((2k-n+1)\)-connected and \( k \leq n-3 \).

Habegger [H] extended this result to one more dimension (i.e., \( f \) is \((2k-n)\)-connected) by exhibiting a necessary and sufficient obstruction lying in a quotient of a singular cohomology group (cf. Remark 6.3 below) . The case when \( N \) is a sphere was studied by Connolly and Williams [CW] where it is shown that the operation which sends an \( h \)-embedding \((U, h)\) of \( K \) in \( S^n \) to its complement data \( S^n \setminus U \) induces, in a wide range, a bijection between the set of concordance classes of \( h \)-embeddings of \( K \) in \( S^n \) and the set of homotopy types of the Spanier-Whitehead \((n-1)\)-duals of \( K \). More recently, the second author’s Ph. D. thesis has extended the Connolly-Williams program to arbitrary compact manifolds \( N \) [J].

When \( K \) itself is a closed manifold of dimension \( \leq n-3 \) and \( n \geq 6 \), the foundational results of surgery theory imply that \( f \) is homotopic to a smooth embedding provided that an \( h \)-embedding of \( f \) exists and the stable normal bundle of \( f \), i.e., \( f^*\nu_N - \nu_K \), destabilizes in a suitable way to a vector bundle of rank \( n-k \) (cf. [W2, chap. 11]). In the metastable range \( 3(k+1) \leq 2n \), there is no obstruction to finding the bundle destabilization, so \( f \) is homotopic to an embedding if and only if an \( h \)-embedding of \( f \) exists.

Given an \( h \)-embedding of \( f \), one can associate an \( h \)-embedding of the composite

\[ f_1: K \xrightarrow{f} N \times 0 \subset N \times D^1 \]
as follows: we let $J = [-1/2, 1/2] \subset D^1$. Then we have $U \times J \subset \text{int}(N \times [-1, 1])$ and the map

$$h_1: K \times 0 \to U \times 0 \subset U \times J$$

is a homotopy equivalence. Hence,

$$(U \times J, h_1)$$

is an $h$-embedding of $f_1$. This is called the decomposition of $(U, h)$. Conversely, we say $(U, h)$ is the compression of $(U \times J, h_1)$.

Let $C$ be the closure of the complement of $U$ in $N$ and let $W$ denote the closure of the complement of $U \times J$ in $N \times D^1$. Then $W \in T(N \times S^0 \to N)$ is an object and there is a weak equivalence

$$W \simeq S_B C$$

(see e.g., [K2]). Consequently, a necessary condition for an $h$-embedding of $f_1: K \to N \times D^1$ to decompress (up to concordance) is that the complement data must be weak equivalent to an unreduced fiberwise suspension of some object $C \in T(\emptyset \to N)$.

In what follows we set

$$(K \times K)^+ := (K \times K) \amalg (N \times N) \in R(N \times N)$$

If $\Delta: N \to N \times N$ is the diagonal map and $Y \in R(N)$ is an object, we set

$$\Delta_* (Y) := Y \cup_{\Delta} (N \times N) \in R(N \times N)$$

Lastly, $\Sigma_N^{\tau - \epsilon} Y$ denotes the fiberwise suspension of $Y$ with respect to the virtual vector bundle $\tau - \epsilon$, where $\tau$ is the tangent bundle of $N$ and $\epsilon$ is the trivial bundle of rank one (cf. §2).

**Theorem C (Compression).** Let $(U, h)$ be an $h$-embedding of $f_1: K \to N \times D^1$. Then there is an obstruction

$$\theta(U, h) \in \{(K \times K)^+, \Delta_* (\Sigma_N^{\tau - \epsilon} i_N S_N K)\}_{R(N \times N)}$$

which vanishes whenever $(U, h)$ compresses into $N$. Conversely, if we assume in addition that

$$r \geq \max(3k - 2n + 3, \frac{2k - n + 3}{2}),$$

$k \leq n - 3$ and $n \geq 6$, then the vanishing of $\theta(U, h)$ is sufficient to compressing $(U, h)$ up to concordance.

**Remark 1.4.** In the special case $N = D^n$, it is not hard to deduce Theorem C from the main result of [K5].
**Outline.** §2 is mostly about language. In §3 which introduce the concept of a charged co-$H$ space. This is subsequently used to give a proof of Theorem A. In §4 we recall the notion of fiberwise duality introduced in [K4]. §5 contains the proof of Theorem C.

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### 2. Preliminaries

Henceforth, we let $T$ denote the category of compactly generated weak Hausdorff spaces which we consider as enriched over simplicial sets. Products are retopologized with respect to the compactly generated topology. We assume that the reader is familiar with the basic ideas of model categories. We give $T$ the simplicial model structure in which a weak equivalence is a weak homotopy equivalence, a fibration is a Serre fibration and a cofibration satisfies the left lifting property with respect to the acyclic fibrations.

As in the introduction, if $f: A \to B$ is a map, we let $T(A \xrightarrow{f} B)$ be the category of factorizations. When $f$ is understood, we abbreviate the notation to $T(A \to B)$. Define fibrations, cofibrations and weak equivalences in $T(A \to B)$ by applying the forgetful functor $T(A \to B) \to T$. Since

$$T(A \to B) \cong f\backslash(T/B),$$

it follows from [Q, II.2.8, prop. 6] that $T(A \to B)$ is a simplicial model category with respect to these choices. For objects $U, V \in T(A \to B)$ we therefore have the homotopy set

$$[U, V]_{T(A \to B)},$$

which is given by the homotopy classes of morphisms $U^c \to V^f$ in which $U^c$ is a cofibrant approximation of $U$ and $V^f$ is a fibrant approximation of $V$.

**Finiteness.** Let $Y \in T(A \to B)$ be an object. Given a commutative diagram

$$
\begin{array}{ccc}
S^{j-1} & \xrightarrow{\alpha} & Y \\
\downarrow & & \downarrow \\
D^j & \xrightarrow{} & B
\end{array}
$$

in which the structure map appears on the right, we can form the object

$$Y \cup_{\alpha} D^j \in T(A \to B).$$
This is called *attaching a cell* to $Y$ along $\alpha$.

An object $X \in T(A \to B)$ is said to be *finite* if it is, up to isomorphism, obtained from $A$ by the iterated attachment of a finite number of cells. More generally, $X$ is *homotopy finite* if it is homotopy equivalent to a finite object. Lastly, $X$ is *finitely dominated* if it is a retract of a homotopy finite object.

**Adjunction rules.** For $f: A \to B$ as above, consider the faithful embedding

$$i: R(A) \to T(A \to B)$$

given by the identity. This has a right adjoint $j: T(A \to B) \to R(A)$ given by $Y \mapsto A \times_B Y$ in which the latter denotes the fiber product.

Let

$$f_z: R(B) \to T(A \to B)$$

be defined by $X \mapsto X_z$, in which $X_z$ denotes $X$ with structure map $A \to X$ given by the composite $A \to B \to X$. Then $f_z$ has a left adjoint $f^+: T(A \to B) \to R(B)$ given by $X \mapsto X^+$, where

$$X^+ = X \cup_A B.$$

There is a third adjunction to consider. Let

$$f^!: R(B) \to R(A)$$

given by mapping $Y \in R(B)$ to the fiber product $Y' := Y \times_B A$. This is right adjoint to the functor $f_1: R(A) \to R(B)$ given by $X \mapsto X_1$, where $X_1 = X \cup_A B$. Then $f_1 = f^+ \circ i$ and $f^! = j \circ f_z$. The above may be summarized in the diagram

\[
\begin{array}{ccc}
R(A) & \xrightarrow{j} & T(A \to B) & \xleftarrow{i} & R(B) \\
& \searrow f^+ & \swarrow f_z & \searrow f^! & \\
& & & & \\
& f_1 & & & \\
\end{array}
\]

in which the right pointing arrows are the right adjoints to the corresponding left pointing ones.

**Corollary 2.1.** (1). For a cofibrant object $X \in T(A \to B)$ and a fibrant object $Y \in R(B)$ there is a natural isomorphism of pointed sets

$$[X, Y_z]_{T(A \to B)} \cong [X^+, Y]_{R(B)}.$$
(2). For a cofibrant object \( Z \in R(A) \), there is an isomorphism of pointed sets

\[ [Z, Y]_{R(A)} \cong [Z!, Y]_{R(B)}. \]

**Fiberwise smash products.** If \( X, Y \in R(B) \) are objects, then the *internal smash product*

\[ X \smash{\!}_B Y \in R(B) \]

is given by

\[ \text{colim}(B \leftarrow X \cup_B Y \rightarrow X \times_B Y) \]

If \( X \in R(B) \) and \( Z \in R(B') \), then the *external smash product*

\[ X \smash{\!} Z \in R(B \times B') \]

is given by

\[ \text{colim}(B \times B' \leftarrow B \times Z \cup_{B \times B'} X \times B' \rightarrow X \times Z) \]

Note that when \( B = B' \), we have

\[ X \smash{\!}_B Y \cong \Delta^!(X \smash{\!} Y), \]

where \( \Delta : B \to B \times B \) is the diagonal map. Also note that for \( K \in R(B), L \in R(B') \), we have

\[ (K \times L)^+ \cong K^+ \smash{\!} L^+ \]

as objects of \( R(B \times B') \), where \( (K \times L)^+ = (K \times L) \amalg (B \times B') \), \( K^+ = K \amalg B \), and \( L^+ = L \amalg B' \).

**Fiberwise suspension.** For any map \( A \to B \) one can think of unreduced fiberwise suspension as a functor

\[ S_B : T(A \to B) \to T(S_B A \to B). \]

Similarly, the *reduced* fiberwise suspension functor \( \Sigma_B : R(B) \to R(B) \) is defined as

\[ \Sigma_B Y := \text{colim}(B \leftarrow S_B B \to S_B Y), \]

where on the right we are considering \( Y \) as an object of \( T(\emptyset \to B) \) by means of the forgetful functor. As in the introduction, let \( i_-, i_+ : T(B \times S^0 \to B) \to R(B) \) be the functors restricting the structure map using the inclusions \( B \times \{\pm 1\} \subset B \times S^0 \).

**Lemma 2.2.** Let \( X \in T(B \times S^0 \to B) \) be an object. Then there is a weak equivalence

\[ \Sigma_B i_- X \simeq \Sigma_B i_+ X. \]
Proof. The object $S_B X \in T(B \times S^1 \to B)$ can be considered as an object of $R(B)$ by choosing the north pole of $S^1$. With respect to this choice, the evident quotient maps $S_B X \to \Sigma_B i_- X$ and $S_B X \to \Sigma_B i_+ X$ are weak equivalences of $R(B)$.

**Bundle suspensions.** For a vector bundle $\xi$ with base space $B$, let $S^\xi_B \in R(B)$ be the fiberwise one-point compactification of $\xi$. For an object $Y \in R(B)$, set

$$\Sigma^\xi_B Y = S^\xi_B \wedge_B Y \in R(B).$$

If $\xi = \epsilon$ is a trivial bundle of rank one, then we recover the reduced fiberwise suspension.

**Remark 2.3.** Let $c: B \to \ast$ be the constant map to a point. Then

$$c_! S^\xi_B := B^\xi \in R(\ast)$$

is the Thom space of $\xi$.

If $\xi$ and $\eta$ are a pair of vector bundles over $B$, and $X, Y \in R(B)$ are (cofibrant) objects, then we define

$$[X, \Sigma_B^{\xi-\eta} Y]_{R(B)} := [\Sigma_B^\xi X, \Sigma_B^\eta Y]_{R(B)},$$

and similarly,

$$\{X, \Sigma_B^{\xi-\eta} Y\}_{R(B)} := \{\Sigma_B^\eta X, \Sigma_B^\xi Y\}_{R(B)}.$$

**Remark 2.4.** Alternatively, there is a fiberwise spectrum $S^\xi_B - \eta$ over $B$ whose fiber at $b \in B$ is the function spectrum $F(S^\eta_b, S^\xi_b)$. One may then define a fiberwise spectrum $\Sigma_B^{\xi-\eta} Y$ to be $S^\xi_B - \eta \wedge_B Y$. This is a model for the fiberwise suspension of $Y$ with respect to the virtual bundle $\xi - \eta$.

**The Adjoint to $S_B$.** The functor $S_B: T(\emptyset \to B) \to T(B \times S^0 \to B)$ has a right adjoint

$$O_B: T(B \times S^0 \to B) \to T(\emptyset \to B).$$

For $Y \in T(B \times S^0 \to B)$, one defines $O_B Y$ to be the space of paths $\gamma: D^1 \to Y$ which project to a constant path $D^1 \to Y \to B$ and which satisfy the condition $\gamma(\pm 1) \in B \times \{\pm 1\}$. The structure map $O_B Y \to B$ is given by mapping such a path in $Y$ to its associated constant path in $B$. 
Equivariant versus fiberwise spaces. Let $G$ be a topological group whose underlying space is cofibrant. Let $R^G(*)$ be the category of based left $G$-spaces. Then $R^G(*)$ is a simplicial model category in which a fibration and weak equivalence are defined by the forgetful functor $R^G(*) \to T$ and cofibrations are defined by the left lifting property with respect to the acyclic fibrations.

Then there is a Quillen equivalence

$$R^G(*) \xrightarrow{f} R(BG) \xleftarrow{g} R^G(*)$$

in which $f(X) = B(*; G; X)$ is the two-sided bar construction and $g(Y) = \text{map}_{R(BG)}(EG, Y)$ is the space of maps $EG \to Y$ which cover the identity map of $BG$ (here $f$ is the left adjoint to $g$). For details, see [Sh, cor. 8.7]. In particular the homotopy categories of $R^G(*)$ and $R(BG)$ are equivalent.

Remark 2.5. As briefly mentioned in the introduction, when making various functorial constructions it is often crucial to “derive” them by applying fibrant and/or cofibrant replacements when needed to ensure that the result is homotopy invariant (this occurs especially often in §4). In order to avoid notational clutter, we assume that this has been done wherever necessary, but we will not usually indicate it in the notation. We hope this does not lead to any confusion. The reader is forewarned.

3. Charged co-H structures

Let $X \in T(B \times S^0 \to B)$ be an object. We can then form the space

$$i_+X \vee_B i_-X,$$

which is the pushout of the diagram $X \xleftarrow{s_-} B \xrightarrow{\partial} X$, where $s_\pm$ are the restrictions of the structure map $B \times S^0 \to X$ to each summand. We wish to consider $i_+X \vee_B i_-X$ as an object of $T(B \times S^0 \to B)$. This can be achieved by considering the commutative diagram of spaces over $B$

$$
\begin{array}{ccc}
B & \xrightarrow{s_-} & \emptyset & \xrightarrow{g} & B \\
\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{i_-X} & B & \xrightarrow{i_+X} & X.
\end{array}
$$

The pushout of the top line is $B \times S^0$ whereas the pushout of the bottom line is $i_+X \vee_B i_-X$. So we have a charge structure $B \times S^0 \to i_+X \vee_B i_-X$. 
Let $i_+X \times_B i_-X$ be the fiber product of $i_+X$ with $i_-X$. There is an evident inclusion

$$i_+X \vee_B i_-X \xrightarrow{\subseteq} i_+X \times_B i_-X.$$  

If we give $i_+X \times_B i_-X$ the induced charge structure, the inclusion becomes a morphism of $T(B \times S^0 \rightarrow B)$. There is also a diagonal morphism

$$\Delta : X \rightarrow i_+X \times_B i_-X.$$  

**Definition 3.1.** A charged co-$H$ structure on $X$ is a morphism

$$p : X \rightarrow i_+X \vee_B i_-X$$

of $T(B \times S^0 \rightarrow B)$ such that the composition

$$X \xrightarrow{\delta} i_+X \vee_B i_-X \xrightarrow{\subseteq} i_+X \times_B i_-X$$

coincides with $\Delta$ up to homotopy, i.e., in $[X, i_+X \times_B i_-X]_{T(B \times S^0 \rightarrow B)}$.

Recall that $i_+X \wedge_B i_-X$ is the pushout of the diagram

$$i_+X \vee_B i_-X \xrightarrow{\subseteq} i_+X \times_B i_-X \xrightarrow{\delta} i_+X \wedge_B i_-X$$

Then the charge structure $B \times S^0 \rightarrow i_+X \wedge_B i_-X$ factors through $B$. In other words, we can legitimately consider $i_+X \wedge_B i_-X$ to be an object of $R(B)$ without any loss of information.

Hence, if $X \in T(B \times S^0 \rightarrow B)$ is an object, then the composition

$$X \xrightarrow{\Delta} i_+X \times_B i_-X \rightarrow i_+X \wedge_B i_-X$$

factors through $X^+ := X \cup_{B \times S^0} B$, and the resulting map

$$\tilde{\Delta} : X^+ \rightarrow i_+X \wedge_B i_-X$$

is a morphism of $R(B)$.

**Lemma 3.2.** Assume $X$ is fibrant and cofibrant. If $X$ can be equipped with a charged co-$H$ structure, then

$$[\tilde{\Delta}] \in \{X^+, i_+X \wedge_B i_-X\}_{R(B)}$$

is trivial. Conversely, if $X$ is $r$-connected, $\dim X \leq 3r$ and $\tilde{\Delta}$ is trivial, then $X$ can be equipped with a charged co-$H$ structure.

**Proof.** (Sketch). The diagram

$$i_+X \vee_B i_-X \xrightarrow{\subseteq} i_+X \times_B i_-X \rightarrow i_+X \wedge_B i_-X$$

is a cofibration sequence of $T(B \times S^0)$. It follows that if $X$ has a charged co-$H$ structure, then the composite

$$X \rightarrow i_+X \times_B i_-X \rightarrow i_+X \wedge_B i_-X$$
is null-homotopic when considered as a morphism of $T(B \times S^0 \to B)$. Now apply the functor $\oplus : T(B \times S^0 \to B) \to R(B)$ to obtain the first part of the lemma.

For the second part, use the Blakers-Massey theorem to show that the diagram (4) forms a fiber sequence up through dimension $\leq 3r + 1$. In particular, the sequence

$$[X, i_+ X \vee_B i_- X]_{T(B \times S^0 \to B)} \to [X, i_+ X \times_B i_- X]_{T(B \times S^0 \to B)} \to [X, i_+ X \wedge_B i_- X]_{T(B \times S^0 \to B)}$$

is exact, where the third term is a pointed set. By the adjunction property Corollary 2.1, there is an isomorphism of pointed sets

$$[X, i_+ X \wedge_B i_- X]_{T(B \times S^0 \to B)} \simeq [X^+, i_+ X \wedge_B i_- X]_{R(B)}$$

Furthermore, another application of Blakers-Massey theorem shows that stabilization induces an isomorphism

$$[X^+, i_+ X \wedge_B i_- X]_{R(B)} \to \{X, i_+ X \wedge_B i_- X\}_{R(B)} .$$

□

For an object $X \in T(B \times S^0 \to B)$ consider the canonical morphism

$$c_X : S_B O_B X \to X .$$

**Proposition 3.3.** Assume $X$ is fibrant and cofibrant. Then $X$ can be given a charged co-$H$ structure if and only if $c_X$ admits a section up to homotopy inside $T(B \times S^0 \to B)$.

**Proof.** It is enough show that there is an $\infty$-cartesian square of $T(B \times S^0 \to B)$ of the form

$$
\begin{array}{ccc}
S_B O_B X & \longrightarrow & i_+ X \vee_B i_- X \\
\downarrow^{c_X} & & \downarrow \cap \\
X \longrightarrow & i_+ X \times_B i_- X
\end{array}
$$

To do this we convert $\Delta$ into a fibration. Let $W_B X$ be the space of fiberwise paths in $X$. This is the space whose points are paths $\lambda : [-1, 1] \to X$ such that the projection to $B$ is constant. The inclusion $X \to W_B X$ given by the constant paths is a weak equivalence of $T(B \times S^0 \to B)$. Furthermore, the map $X \to i_+ X \times_B i_- X$ factors as

$$X \xrightarrow{q} W_B X \xrightarrow{q} i_+ X \times_B i_- X ,$$

where $q$ is the fibration given by $\lambda \mapsto (\lambda(0), \lambda(1))$. It is therefore enough to identify the pullback of the diagram

(5) $$W_B X \to i_+ X \times_B i_- X \leftarrow i_+ X \vee_B i_- X$$
with $S_BOBX$ up to weak equivalence. The pullback of (5) may explicitly computed as an amalgmated union

$$U \cup V$$

in which $U$ is the space of paths $\lambda: [-1, 1] \to X$ such that $\lambda(-1) \in B \times \{-1\}$ and the projection $[-1, 1] \to X \to B$ is constant. Similarly $V$ is the space of paths $\lambda: [-1, 1] \to X$ such that $\lambda(1) \in B \times \{+1\}$ and the projection $[-1, 1] \to X \to B$ is constant. The intersection $U \cap V$ is the space of paths $\lambda: [-1, 1] \to X$ such that $\lambda(-1) \in B \times \{-1\}$, $\lambda(1) \in B \times \{+1\}$ and the projection $[-1, 1] \to X \to B$ is constant. Clearly, $U \cap V = OBX$. Furthermore, each of the projections $U \to B$, $V \to B$ is a weak equivalence. In other words, $W_BX$ coincides up to weak equivalence with the homotopy pushout of the diagram

$$B \leftarrow OBX \to B.$$  

that is, with $S_BOBX$.

**Theorem 3.4.** Let $X \in T(B \times S^0 \to B)$ be an object and assume $B$ is connected. Assume $X$ is charged co-$H$, is $r$-connected, $\dim X \leq 3r$ and $r \geq 1$. Then there is an object $Y \in T(\emptyset \to B)$ and a weak equivalence $SBY \simeq X$.

**Proof.** In what follows we may assume $X$ is both fibrant and cofibrant. We adapt the proof of [K1, thm. 2.1]. Since $X$ is charged co-$H$, we can choose a section up to homotopy $s: X \to S_BOBX$. Applying $OB$ gives a map $OBS: OBX \to OBSOBX$. We also have a canonical map $u_{OBX}: OBX \to OBSOBX$. Let $Z$ be the homotopy pullback of the diagram

$$OBX \xrightarrow{u_{OBX}} OBSOBX \xleftarrow{OBS} OBX.$$  

Then $Z$ is an object of $T(\emptyset \to B)$, and we have an $\infty$-cartesian square

$$
\begin{array}{c}
Z \\
\downarrow^j \\
OBX \\
\downarrow^i \\
OBSOBX
\end{array}
\xrightarrow{OBS} \begin{array}{c}
OBSOBX \\
\downarrow^j \\
OBSOBX
\end{array}
$$

Each map of this square is $(2r - 1)$-connected, by a straightforward Blakers-Massey argument which which omit. Again by the Blakers-Massey theorem, we find that this square is $(4r - 1)$-cocartesian. Consider the adjoint $\hat{j}: S_BZ \to X$.

**Claim:** Assume $r \geq 1$. Then the map $\hat{j}: S_BZ \to X$ is $3r$-connected.
The bottom composite appearing in (8) is the identity map. The left-hand square of (8) is (4r)-cocartesian because it is the suspension of the (4r − 1)-cocartesian square (7).

If we can show that the right-hand square of (8) is (3r+1)-cocartesian, then it will follow that the outer square of (8) is also (3r+1)-cocartesian. Since the bottom composite is the identity map, and each vertical map of the square is 2-connected it will follow from [K2, lem. 5.6] that the top composite, which is \( \tilde{j} \), is (3r)-connected, yielding the claim.

The right-hand square of (8) is (3r + 1)-cocartesian by a direct application of the dual Higher Blakers-Massey theorem [Go, th. 2.6] to the 3-cube

\[
\begin{array}{ccc}
O_B X & \rightarrow & B \\
\uparrow & & \uparrow \\
B & \rightarrow & X \\
\uparrow & & \uparrow \\
O_B S_B O_B X & \rightarrow & B \\
\downarrow & & \downarrow \\
B & \rightarrow & S_B O_B X \\
\end{array}
\]

in which the map from the top face to the bottom one is induced by \( s \). We leave the details of this to the reader. This establishes the claim.

To complete the proof of Theorem 3.4, we only need to apply [K2, th. 4.2] to the (3r)-cocartesian square

\[
\begin{array}{ccc}
Z & \rightarrow & B \\
\downarrow & & \downarrow \\
B & \rightarrow & X \\
\end{array}
\]

This gives a space \( Y \) and a (3r − 2)-connected map \( Y \rightarrow Z \) such that the composite

\[ S_B Y \rightarrow S_B Z \rightarrow X \]

is a weak equivalence. \( \square \)
Proof of Theorem A and Addendum B. We only prove Theorem A, as the extension to the relative case is basically the same. Let \( X \in T(B \times S^0 \to B) \). If \( X \simeq S_BY \) then \( X \) admits a charged co-\( H \)-structure so \([\tilde{\Delta}]\) is trivial by the first part of Lemma 3.2.

Conversely, assume \([\tilde{\Delta}] = 0\), with \( X \) \( r \)-connected and \( \dim X \leq 3r \). By the second part of Lemma 3.2, \( X \) admits a charged co-\( H \) structure and by Theorem 3.4, \( X \simeq S_BY \). □

4. Duality

\( N \)-duality. Suppose \( N \) is a connected compact manifold with boundary \( \partial N \) (or more generally, a finite Poincaré pair). Then \( N \in T(\partial N \to N) \), and \( N^+ \in R(N) \) is just the double \( N \cup_{\partial N} N \) (this is also weakly equivalent to \( i_-(S_N \partial N) \)).

Suppose we are given finitely dominated objects \( U, U^* \in R(N) \) (which we can assume to be fibrant and cofibrant), and an element
\[
d \in \{ \Sigma^j_N N^+, U \wedge_N U^* \}_{R(N)}.\]

**Definition 4.1** (cf. [K4]). The element \( d \) is said to be an \( N \)-duality if the operation \( f \mapsto (f \wedge_N \text{id}) \circ d \) induces an isomorphism
\[
\{ U, E \}_{R(N)} \cong \{ \Sigma^j_N N^+, E \wedge_N U^* \}_{R(N)}
\]
for all objects \( E \in R(N) \) which are fibrant and cofibrant. The integer \( j \) is called the **indexing parameter** of \( d \).

**Remark 4.2.** The object \( U \wedge_N U^* \) need not be fibrant. Hence, it is too much to hope for \( d \) to be represented by a fiberwise stable morphism \( \Sigma^j_N N^+ \to U \wedge_N U^* \). In general, \( d \) may be represented by a fiberwise stable morphism
\[
\Sigma^j_N N^+ \to (U \wedge_N U^*)^f,
\]
where the target is a fibrant approximation of \( U \wedge_N U^* \). In this case, we call any representative (10) an \( N \)-duality map.

If \( N \) is connected, the above formulation can be re-expressed in terms of the Quillen equivalence (3): choose a universal principal bundle
\[
p: \tilde{N} \to N
\]
with structure group \( G \) (in particular, this identifies \( N \) with \( BG \), so \( G \) models the loop space \( \Omega N \)). Then, with respect to the Quillen equivalence (3), \( U, U^* \) correspond to objects \( \tilde{U}, \tilde{U}^* \in R^G(\ast) \) and \( d \) corresponds to an equivariant stable homotopy class
\[
\tilde{d} \in \{ \Sigma^j \tilde{N}/\partial \tilde{N}, \tilde{U} \wedge \tilde{U}^* \}_{R^G(\ast)},
\]
where $\partial \tilde{N}$ is the pullback of the universal bundle along $\partial N \to N$. The $N$-duality condition (9) can then be re-expressed as follows: for all objects $\tilde{E} \in R^G(*)$ the operation $f \mapsto (f \wedge \text{id}) \circ \tilde{d}$ yields an isomorphism

$$\{ \tilde{U}, \tilde{E} \}_{R^G(*)} \cong \{ \Sigma^j \tilde{N}/\partial \tilde{N}, \tilde{E} \wedge \tilde{U}^* \}_{R^G(*)}. \tag{12}$$

There is a technical advantage in the equivariant setting: all objects are fibrant. So $\tilde{d}$ admits a representative equivariant stable map

$$\Sigma^j \tilde{N}/\partial \tilde{N} \to \tilde{U} \wedge \tilde{U}^*.$$

When there is no confusion we will abuse notation and denote the representative by $\tilde{d}$.

In some of the proofs appearing below, it will be more convenient to rephrase (12) in terms of stable function spaces. For cofibrant objects $A, B \in R^G(*)$, we let $F^\text{st}(A, B)^G$ be the function space of stable, based, equivariant maps; a point in this space is represented by a based equivariant map $\Sigma^k A \to \Sigma^k B$ for some $k \geq 0$. Then the set of path components of this space is identified with $\{ A, B \}_{R^G(*)}$, and it’s not hard to see that (12) is equivalent to the statement that the operation $f \mapsto (f \wedge \text{id}) \circ \tilde{d}$ yields a weak homotopy equivalence of function spaces

$$F^\text{st}(\tilde{U}, \tilde{E})^G \simeq F^\text{st}(\Sigma^j \tilde{N}/\partial \tilde{N}, \tilde{E} \wedge \tilde{U}^*)^G \tag{13}$$

for any object $\tilde{E} \in R^G(*)$.

**Equivariant/fiberwise duality.** There is another kind of duality which the first author has called *equivariant duality* [K6]. Suppose we are given cofibrant objects $\tilde{X}, \tilde{Y} \in R^G(*)$ and a (stable) map of based spaces

$$\delta: S^d \to \tilde{X} \wedge_G \tilde{Y},$$

where $\tilde{X} \wedge_G \tilde{Y}$ is the orbit space of $G$ acting diagonally on the smash product. We say that $\delta$ is an *equivariant duality map* if for all objects $\tilde{E} \in R^G(*)$, the operation $f \mapsto (f \wedge \text{id}) \circ \delta$ induces an isomorphism of abelian groups

$$\{ \tilde{X}, \tilde{E} \}_{R^G(*)} \cong \{ S^d, \tilde{E} \wedge_G \tilde{Y} \}_{R(*)} \tag{14}.$$

In terms of the Quillen equivalence (3), we can reformulate (14) in the fiberwise setting. and $X, Y \in R(N)$ correspond to $\tilde{X}, \tilde{Y} \in R^G(*)$, then $\tilde{X} \wedge_G \tilde{Y}$ is identified with $(X \wedge_N Y)/N$, i.e., the (homotopy) cofiber of the structure map $N \to X \wedge_N Y$. Hence, with respect to the identifications $\delta$ can be rewritten as

$$\delta': S^d \to (X \wedge_N Y)/N,$$
and the isomorphism (14) becomes

\[
\{X, E\}_{R(N)} \cong \{S^{d}, (E \wedge_N Y)/N\}_{R(\ast)}
\]

for all objects \(E \in R(N)\). In this case, we say that \(\delta'\) is a fiberwise duality map.

**Relating the two kinds of duality.** We now explain how \(N\)-duality is related to equivariant duality. For the following description, we assume the reader is familiar with the basics of the theory of fiberwise spectra. Let \(\tau\) denote the tangent bundle of \(N\) and recall that \(S^\tau_N \in R(N)\) is the fiberwise one-point compactification of \(\tau\). If we apply functions into the sphere spectrum \(S^0\) fiberwise, we obtain a fiberwise spectrum \(S^{-\tau}_N\) and we can consider the fiberwise smash product \(S^{-\tau}_N \wedge_N N^+\) which we can conveniently denote as \(\Sigma^{-\tau}_N N^+\). If we collapse the “zero section” \(N \subset \Sigma^{-\tau}_N N^+\) to a point, we obtain \(N^{-\tau}/(\partial N)^{-\tau}\), where \(N^{-\tau}\) is the Thom spectrum of the stable normal bundle of \(N\) and \((\partial N)^{-\tau}\) is the Thom spectrum of the pullback of \(-\tau\) to \(\partial N\). Hence we have a degree one stable map

\[
\alpha: S^0 \to (\Sigma^{-\tau}_N N^+)/N
\]

which represents the fundamental class of \(N\) (since \(H_0((\Sigma^{-\tau}_N N^+)/N)\) is isomorphic to \(H_d(N, \partial N)\), where \(d = \dim N\), and coefficients are twisted by the orientation bundle).

Suppose now that \(d: \Sigma^j_N N^+ \to (U \wedge_N U^*)^f\) is any stable morphism of \(R(N)\). Consider the composite

\[
\delta: S^0 \xrightarrow{\alpha} (\Sigma^{-\tau}_N N^+)/N \xrightarrow{(\Sigma^{-\tau}_N \delta)/N} (U \wedge_N \Sigma^{-\tau}_N U^*)^f/N \simeq (U \wedge_N \Sigma^{-\tau}_N U^*)/N
\]

**Lemma 4.3.** The map \(d\) is an \(N\)-duality if and only if \(\delta\) is a fiberwise duality.

**Proof.** Consider the diagram of abelian groups

\[
\begin{array}{ccc}
\{X, E\}_{R(N)} & \xrightarrow{a} & \{\Sigma^j_N N^+, E \wedge_N Y\}_{R(N)} \\
& \downarrow{b} & \\
\{\Sigma^j_N \Sigma^{-\tau}_N N^+, E \wedge_N \Sigma^{-\tau}_N Y\}_{R(N)} & \xrightarrow{c} & \{S^j, (E \wedge_N \Sigma^{-\tau}_N Y)/N\}_{R(\ast)},
\end{array}
\]

where the homomorphism \(a\) is induced by \(f \mapsto (f \wedge_N \text{id}) \circ d\), the homomorphism \(b\) is given by fiberwise smashing with \(\Sigma^{-\tau}_N\) and the homomorphism \(c\) is induced by collapsing out \(N\) and pre-composing with \(\alpha\). It is easy to see that the map \(b\) is an isomorphism. The composite \(c \circ b \circ a\) is the homomorphism induced by \(\delta\). Consequently, the lemma will follow.
if we can show that $c \circ b$ is an isomorphism. By the Quillen equivalence (3), the homomorphism $c \circ b$ corresponds to the isomorphism

$$\{ \Sigma^j N/\partial \tilde{N}, \tilde{E} \wedge \tilde{Y} \}_R \cong \{ S^j, \tilde{E} \wedge_G \Sigma^{-\tau} Y \}_R$$

induced by the equivariant duality map

$$S^j \to \tilde{N}/\partial \tilde{N} \wedge_G S^{-\tau}$$

that induces Poincaré duality for $N$ (see [K6], [K3]). Here, $S^{-\tau}$ is the spectrum with $G$-action which corresponds to $S_N^{-\tau}$ under the Quillen equivalence (3).

□

We list some basic properties of $N$-duality maps.

(a) If $d: \Sigma^j N^+ \to (U \wedge_N U^*)^f$ is an $N$-duality map, then so is the map

$$d^f: \Sigma^j N^+ \to (U \wedge_N U^*)^f \xrightarrow{\text{twist}} (U^* \wedge_N U)^f.$$ 

Hence, there is no ambiguity in saying that $U$ and $U^*$ are $N$-dual to each other.

(b) Suppose that $d': \Sigma^k N^+ \to (V \wedge_N V^*)^f$ is another $N$-duality map. Then $d$ and $d'$ induce an umkehr correspondence

$$(16) \quad \{ U, V \}_R \cong \{ \Sigma^j N^*, \Sigma^k U^* \}_R.$$ 

(c) Given a finitely dominated object $U \in R(N)$, there is an integer $j \gg 0$, a finitely dominated object $U^* \in R(N)$ and an $N$-duality map $d: \Sigma^j N^+ \to U \wedge_N U^*$.

(d) If $U^*$ is $N$-dual to $U$ and $U'$ is another $N$-dual to $U$ (with respect to the same indexing parameter $j$), then $U^*$ and $U'$ are stably weak equivalent in $R(N)$. This means that there is a weak equivalence $\Sigma^k U^* \simeq \Sigma^k U'$ for $k$ suitably large. In particular, the stable $N$-dual is unique (in fact, up to contractible choice).

(e) Stably, fiberwise duals preserve homotopy cofiber sequences. This means that if $U \to V \to W$ is a homotopy cofiber sequence of $R(N)$, then there is a homotopy cofiber sequence of corresponding $N$-duals $W^* \to V^* \to U^*$ (where the indexing parameter $j$ is large and is the same for all three objects).

(f) If $U, U^* \in R(N \times I)$ are $(N \times I)$-dual with indexing parameter $j = 0$ and $p: \tilde{N} \times I \to \tilde{N}$ is the projection, then $p_* U$ and $p_* U^*$ are $N$-dual with indexing parameter $j = 1$ (the indexing parameter changes because $p_*(N \times I)^+ \cong \Sigma_N N^+$).
The proofs of most these properties have appeared elsewhere (cf. [K4], [K3], [K6]). We now explain the umkehr correspondence (16), since crucial use is made of it in this paper. The idea is to apply the duality isomorphisms simultaneously to get

\[\{U, V\} \xrightarrow{\delta_1} \{\Sigma^j N^+, V \wedge_N U^*\} \xleftarrow{\delta_2} \{\Sigma^j V^*, \Sigma^k U^*\}\],

where the first isomorphism is given by \((\delta_1: f \mapsto f \wedge \text{id}_{U^*}) \circ d\) and the second one by \((g \mapsto (\text{id}_V \wedge \text{id}_N g) \circ d')\).

**Remark 4.4.** Although similar in spirit, the above notions of duality should not be confused with the fiberwise duality theory of Becker and Gottlieb [BG]— the latter uses a different concept of finiteness: an object \(U \in \mathcal{R}(N)\) is Becker-Gottlieb finitely dominated if the homotopy fiber of the map \(U \to N\) is a finitely dominated space. Our notion of finiteness is more general: for example, if \(N\) is closed and of positive dimension, then the wedge \(N \vee S^j \in \mathcal{R}(N)\) is finite in our sense but not even finitely dominated in the Becker-Gottlieb sense.

The main source of examples arises from embedding theory. Suppose we have a codimension zero compact manifold decomposition of \((N, \partial N)\) as

\[(U, \partial_0 U) \cup (V_0, \partial_0 V),\]

in which \(\partial U = \partial_0 U \cup_{\partial_0 V} \partial_1 U\), \(\partial V = \partial_0 V \cup_{\partial_0 V} \partial_1 V\) and

\[(U, \partial U) \cap (V, \partial V) = (\partial_1 U, \partial_0 V) = (\partial_1 V, \partial_0 V).\]

In particular, when \(\partial_0 U = \emptyset\) we can think of this as giving a codimension zero embedding of \(U\) in \(N\). For \(i = 0, 1\), define \(U \vee \partial_i U \in \mathcal{R}(N)\) to be the pushout of \(N \leftarrow \partial_i U \to U\). Then there is an \(N\)-duality map

\[(17) \quad N^+ = N \vee \partial N \rightarrow U \vee \partial_0 U \wedge_N U \vee \partial_1 U\]

which arises as follows: there is an evident fiberwise diagonal map

\[(U, \partial U) \rightarrow (U \times_N U, \partial_0 U \times_N U \cup U \times_N \partial_1 U)\]

inducing a fiberwise map \(U \vee \partial U \rightarrow U \vee \partial_0 U \wedge_N U \vee \partial_1 U\). If we precompose this with the fiberwise “collapse” map

\[N \vee \partial N \rightarrow N \vee (V \cup \partial_0 U) \cong U \vee \partial U\]

we obtain a map of the form (17). The proof that this map is an \(N\)-duality essentially follows from Lemma 4.3 in conjunction with [K6],[K3],[K4]; we omit the details.

Now suppose that \(f: K \to N\) is a fixed map and \((U, h)\) is an \(h\)-embedding of \(f\), where \(U \subset N\) and \(h: K \xrightarrow{\sim} U\). Let \(C\) be the closure
of the complement of $U$. Then we obtain a diagram
\begin{equation}
\begin{array}{ccc}
\partial U & \rightarrow & C \\
\downarrow & & \downarrow \\
K \xrightarrow{h} U & \rightarrow & N
\end{array}
\end{equation}
in which the displayed square is a pushout. In particular, we obtain a manifold decomposition

$$(N, \partial N) = (U, \emptyset) \cup (C, \partial N)$$

Using $f$, we can identify $K^+ = K \amalg N$ with $U^+$. Furthermore, “excision” gives a weak equivalence $U/\partial U \simeq i_+(S_N C)$. Using these identifications together with the $N$-duality map (17), we infer

**Lemma 4.5.** $K^+ := K \amalg N$ is $N$-dual to $i_+(S_N C)$ with indexing parameter $j = 0$.

If we reverse the roles of $K$ and $C$, we immediately obtain

**Lemma 4.6.** $C^+ = C \cup \partial N$ is $N$-dual to $i_+(S_N K)$ with indexing parameter $j = 0$.

Suppose next that we are given an $h$-embedding $(U, h)$ of $f_1: K \rightarrow N \times I$. Let $W$ be the closure of the complement of $U$. Then we obtain a diagram similar to (18):

\begin{equation}
\begin{array}{ccc}
\partial U & \rightarrow & W \\
\downarrow & & \downarrow \\
K \xrightarrow{h} U & \rightarrow & N \times I
\end{array}
\end{equation}

It now follows from Lemma 4.6 that $W \cup_{\partial (N \times I)} N \times I \in R(N \times I)$ is $(N \times I)$-dual to $i_+ S_{N \times I} K$ with indexing parameter $j = 0$. Consequently, by property (f) above, we conclude

**Lemma 4.7.** The object

$W^+ := W \cup_{\partial (N \times I)} N = p_*(W \cup_{\partial (N \times I)} N \times I)$

is $N$-dual to $p_* i_+ S_{N \times I} K \simeq i_+ S_N K$ with indexing parameter 1.

**Lemma 4.8.** Let $d \in \{ N^+, U \wedge_N U^* \}_{R(N)}$ and $d' \in \{ N^+, V \wedge_N V^* \}_{R(N)}$ be $N$-dualities. Then

$$d \wedge d' \in \{ N^+ \wedge N^+, (U \wedge_N U^*) \wedge (V \wedge_N V^*) \}_{R(N \times N)}$$

$$\simeq \{ (N \times N)^+, (U \wedge V) \wedge_{N \times N} V^* \wedge U^* \}_{R(N \times N)}$$

is an $(N \times N)$-duality.
Proof. Using the Quillen equivalence (3), we saw in (11) that representatives of both \( d \) and \( d' \) correspond to \( G \)-equivariant maps

\[
d: \tilde{N}/\partial\tilde{N} \to \tilde{U} \wedge \tilde{U}^*, \quad d': \tilde{N}/\partial\tilde{N} \to \tilde{V} \wedge \tilde{V}^*,
\]

where \( G \) is a suitable topological group model for the loop space \( \Omega N \). It suffices to show that the \( (G \times G) \)-equivariant duality map

\[
(20) \quad \tilde{N}/\partial\tilde{N} \wedge \tilde{N}/\partial\tilde{N} \xrightarrow{\tilde{d} \wedge \tilde{d}'} \tilde{U} \wedge \tilde{U}^* \wedge \tilde{V} \wedge \tilde{V}^* \cong \tilde{U} \wedge \tilde{V} \wedge \tilde{V}^* \wedge \tilde{U}^*
\]

satisfies the equivariant duality condition (13) with respect to the group \( G \times G \). Here we are identifying the domain of (20) with \( \tilde{N} \times \tilde{N}/\partial(\tilde{N} \times \tilde{N}) \).

Let \( \tilde{E} \in R^{G \times G}(\ast) \) be a cofibrant object. Then

\[
F^{\ast}(\tilde{U} \wedge \tilde{V}, \tilde{E})^{G \times G} \cong F^{\ast}(\tilde{U}, F^{\ast}(\tilde{V}, \tilde{E})^{1 \times G})^{G \times 1},
\]

\[
\cong F^{\ast}(\tilde{U}, F^{\ast}(\tilde{N}/\partial\tilde{N}, \tilde{E} \wedge \tilde{V}^*^{1 \times G})^{G \times 1}),
\]

\[
\cong F^{\ast}(\tilde{N}/\partial\tilde{N}, F^{\ast}(\tilde{U}, \tilde{E} \wedge \tilde{V}^*)^{G \times 1})^{1 \times G},
\]

\[
\cong F^{\ast}(\tilde{N}/\partial\tilde{N}, F^{\ast}(\tilde{N}/\partial\tilde{N}, \tilde{E} \wedge \tilde{V}^* \wedge \tilde{U}^*)^{G \times 1})^{1 \times G},
\]

\[
\cong F^{\ast}(\tilde{N}/\partial\tilde{N} \wedge \tilde{N}/\partial\tilde{N}, \tilde{E} \wedge \tilde{V}^* \wedge \tilde{U}^*)^{G \times G},
\]

where the three isomorphisms listed above are given by the evident adjunctions, and the second and fourth lines are deduced by the duality condition (13). It is not difficult to check that the composite of these identifications is induced by the operation arising from the map (20).

\[ \square \]

Let \( \tau_N \) denote the tangent bundle of \( N \). Let \( \Delta: N \to N \times N \) be the diagonal map. Recall that \( \Delta_*: R(N) \to R(N \times N) \) is the functor given by \( Y \mapsto Y \cup \Delta \) (\( N \times N \)).

**Lemma 4.9.** Suppose that \( d \in \{ N^+, U \wedge N U^* \}_{R(N)} \) is an \( N \)-duality. Then there is preferred \( (N \times N) \)-duality

\[
\hat{d} \in \{(N \times N)^+, \Delta_* \Sigma_N U \wedge_{N \times N} \Delta_* U^* \}_{R(N \times N)}.
\]

**Proof.** Fix a universal principal bundle over \( N \) with structure group \( G \). By Lemma 4.3 and the Quillen equivalence (3), it is enough to find a \( (G \times G) \)-equivariant duality map

\[
S^0 \to (\Sigma^{-\tau} U \wedge_G (G \times G)_+) \wedge_{G \times G} (U^* \wedge_G (G \times G)_+)
\]

(in the above display, \(-\tau\) appears instead of \( \tau \) since Lemma 4.3 involves twisted suspension by \( -\tau_{N \times N} \)). Let \( E \in R^{G \times G}(\ast) \) be an object. In what follows, \( E \) will also be considered as an object of \( R^G(\ast) \) by means
of restriction along the diagonal $G \to G \times G$. Then we have a chain of isomorphisms
\[
\{(\Sigma^{-\tau}U) \wedge_G (G \times G)_+, E\}_{R \times G_{(s)}} \cong \{\Sigma^{-\tau}U, E\}_{R^{(s)}},
\cong \{S^0, E \wedge_G U^*\}_{R^{(s)}},
\cong \{S^0, E \wedge_{G \times G} (U^* \wedge_G (G \times G)_+)\}_{R^{(s)}},
\]
in which the first and last of these is given by extension of scalars and the middle one arises from the equivariant duality between $\Sigma^{-\tau}U$ and $U^*$. Specializing to $E = (\Sigma^{-\tau}U) \wedge_G (G \times G)_+$, we obtain an isomorphism of abelian groups
\[
\{(\Sigma^{-\tau}U) \wedge_G (G \times G)_+, \Sigma^{-\tau}U \wedge_G (G \times G)_+\}_{R \times G_{(s)}} \cong \{S^0, ((\Sigma^{-\tau}U) \wedge_G (G \times G)_+) \wedge_{G \times G} (U^* \wedge_G (G \times G)_+)\}_{R^{(s)}}.
\]
Since the left side contains a preferred element given by the identity map of $(\Sigma^{-\tau}U) \wedge_G (G \times G)_+$, it follows that the abelian group on the right possesses a preferred element as well. Represent this preferred element as a stable map
\[
S^0 \to ((\Sigma^{-\tau}U) \wedge_G (G \times G)_+) \wedge_{G \times G} (U^* \wedge_G (G \times G)_+).
\]
Then it is straightforward to check that the latter satisfies the equivariant duality condition. □

**Corollary 4.10.** The object $\Delta_*(W^+) := W \cup_{\partial(N \times I)} (N \times N)$ is $(N \times N)$-dual to
\[
\Delta_*(\Sigma^{-\tau}\iota_+ S_N K)
\]
with indexing parameter $j = 1$.

**Proof.** This follows from Lemma 4.7 and Lemma 4.9. □

**5. Proof of Theorem C**

**Proof of Theorem C.** Suppose that $(U, h)$ is an $h$-embedding of $f_1 : K \to N \times D^1$, where $N$ is a compact manifold with boundary $\partial N$. Let $W \subset T(N \times S^0 \to N)$ denote the complement. If $(U, h)$ compresses into $N$ then there is an object $C \subset T(\partial N \to N)$ and a weak equivalence $W \simeq S_N C$ of $T(S_N \partial N \to N)$.

The main result of the second author’s Ph. D. thesis says that the converse is in fact true provided that
\[
2r \geq 2k - n + 3
\]
k \leq n - 3 and $n \geq 6$, where $f : K \to N$ is $r$-connected and $K$ has the homotopy type of a CW complex of dimension $\leq k$ (cf. [J]). Hence, to
compress \((U, h)\) into \(N\) we only need to find a fiberwise desuspension of \(W\).

Using Poincaré duality, a cohomology computation shows that \(W\) is an \((n - k - 1)\)-connected object having dimension \(\dim W \leq n - r\) (cf. [K2, p. 615]). Applying Theorem A, we see that \(W\) fiberwise desuspends if and only if the diagonal obstruction

\[ [\tilde{\Delta}] \in \{W^+, i_+W \wedge_N i_-W\}_{R(N)} \]

vanishes. Here we are assuming in addition that \(n - r \leq 3(n - k - 1)\), i.e., \(r \geq 3k - 2n + 3\).

Using the adjunction property, there is an isomorphism of abelian groups

\[(21) \quad \{W^+, i_+W \wedge_N i_-W\}_{R(N)} \cong \{\Delta_*(W^+), i_+W \hat{\wedge} i_-W\}_{R(N \times N)}\]

The idea now is to apply the umkehr correspondence (16) to \([\tilde{\Delta}]\).
By Corollary 4.10, \(\Delta_*(W^+)(N \times N)\)-dual to \(\Delta_*(\Sigma_N^i i_+S_NK)\) with indexing parameter \(j = 1\). By Lemma 4.5, \(K^+\) is \(N\)-dual to \(i_+S_NW\) with indexing parameter \(j = 1\). As in the proof of Lemma 2.2, \(i_+S_NW\) can be identified with \(\Sigma_N i_+W\). Consequently, \(K^+\) is \(N\)-dual to \(i_+W\) with indexing parameter \(j = 0\). By Lemma 2.2, \(\Sigma_N i_-W \cong \Sigma_N i_+W\), so \(K^+\) is also \(N\)-dual to \(i_-W\). By Lemma 4.8, we see that \(i_-W \hat{\wedge} i_+W\) is \((N \times N)\)-dual to \(K^+ \hat{\wedge} K^+ \cong (K \times K)^+\) with indexing parameter \(j = 0\). Applying the umkehr correspondence and the isomorphism (21) we obtain an isomorphism of abelian groups

\[(22) \quad \{W^+, i_+W \wedge_N i_-W\}_{R(N)} \cong \{(K \times K)^+, \Delta_*(\Sigma_N^i i_+S_NK)\}_{R(N \times N)}.\]

We can therefore take

\[ \theta(U, h) \in \{(K \times K)^+, \Delta_*(\Sigma_N^i i_+S_NK)\}_{R(N \times N)} \]

to be the unique element that corresponds to \([\tilde{\Delta}]\) with respect to the isomorphism (22).

6. IDENTIFICATION OF THE CRITICAL GROUP

Suppose that \(f : K \to N\) is \((2k - n)\)-connected. We will identify the obstruction group

\[ \{(K \times K)^+, \Delta_*(\Sigma_N^i i_+S_NK)\}_{R(N \times N)} \]

in terms of singular cohomology.

The following result follows from classical obstruction theory. We omit the proof.
Lemma 6.1. Let \( X, Y \in R(B) \) be objects. Suppose \( Y \) is an \( j \)-connected object and \( \dim X \leq j + 1 \). Let \( r_Y : Y \to B \) denote the structure map. Then

\[
\{X, Y\}_{R(B)} \cong H^{j+1}(X, B; \pi_{j+1}(r_Y)),
\]

where the coefficients are twisted with respect to the \( \pi_1(B) \)-module \( \pi_{j+1}(r_Y) \).

Recall that \( p : \tilde{N} \to N \) is a choice of universal principal bundle with structure group \( G \). If we set \( \Gamma = \pi_0(G) \), then \( \Gamma \) is identified with the fundamental group of \( N \). Let \( w : \Gamma \to \mathbb{Z}/2 \) be the homomorphism given by the first Stiefel-Whitney class of the tangent bundle \( \tau \) of \( N \) (this uses the identification \( \hom(\Gamma, \mathbb{Z}/2) \cong H^1(N; \mathbb{Z}/2) \)). Define a \( (\Gamma \times \Gamma) \)-module structure on \( \pi_{2k-n+1}(f) \otimes \mathbb{Z}[\Gamma] \) as follows: if \( (g, h) \in G \times G \), \( x \in \pi_{2k-n+1}(f) \) and \( y \in \mathbb{Z}[\Gamma] \), then

\[
(g, h) \cdot (x, y) := w(g_1)g_x x \otimes g_2 y g_1^{-1}
\]

(compare \([H]\)).

Corollary 6.2. Assume \( f : K \to N \) is \((2k - n)\)-connected. Then there is an isomorphism of abelian groups

\[
\{(K \times K)^+, \Delta_* (\sum^N_{N} i^* S_N K)\}_{R(N \times N)} \cong H^{2k}(K \times K; \pi_{2k-n+1}(f) \otimes \mathbb{Z}[\Gamma]),
\]

where coefficients are twisted by the \( (\pi_1(K) \times \pi_1(K)) \)-module structure on \( \pi_{2k-n+1}(f) \otimes \mathbb{Z}[\Gamma] \) that is induced by the pullback along the homomorphism \( f_* \times f_* : \pi_1(K) \times \pi_1(K) \to \Gamma \times \Gamma \).

Proof. Set \( Y = \sum^N_{N} \pi_{2k-n+1}(f) \otimes \mathbb{Z}[\Gamma] \). Then

\[
\Delta_* Y = Y \cup_{\Delta} (N \times N) \simeq \hocolim(Y \leftarrow N \xrightarrow{\Delta} N \times N).
\]

As homotopy pullbacks over the same base commute with (homotopy) colimits, the pullback to \( \Delta_* Y \) of the universal bundle \( p \times p : \tilde{N} \times \tilde{N} \to N \times N \) may be identified up to homotopy with the based \( (G \times G) \)-space

\[
\text{colim}(\tilde{Y} \times G^{\text{ad}} \leftarrow * \times G^{\text{ad}} \to *) \simeq \tilde{Y} \wedge G^+_x
\]

where

- \( \tilde{Y} := Y \times_N \tilde{N} \) is the fiber product of \( Y \) and \( \tilde{N} \) over \( N \). This comes equipped with the structure of a based \( G \)-space. We make it into a based \( (G \times G) \)-space by letting the left factor of \( G \times G \) act with the given \( G \)-action and declaring that the right factor act trivially.
- The space \( G^{\text{ad}} \) is a copy of \( G \) with \( (G \times G) \)-action \( (g, h) \cdot x := gxh^{-1} \).
- We give \( \tilde{Y} \wedge G^+_x \) the diagonal \( (G \times G) \)-action.
Note that the underlying space of \( \bar{Y} \wedge \Gamma^{ad} \) is identified with the homotopy fiber of the structure map \( r_{\Delta,Y} : \Delta_* Y \to N \times N \).

In particular, since \( \bar{Y} \) is \((2k - 2)\)-connected, we see that \( \bar{Y} \wedge \Gamma^{ad} \) is \((2k - 2)\)-connected, i.e., \( r_{\Delta,Y} \) is a \((2k - 1)\)-connected map. By Lemma 6.1, it follows that

\[
\{(K \times K)^+, \Delta_* Y\}_{R(N \times N)} \cong H^{2k}(K \times K; \pi_{2k}(r_{\Delta,Y}))
\]

It suffices to show that \( \pi_{2k}(r_{\Delta,Y}) \) is isomorphic to \( \pi_{2k-n+1}(f) \otimes \mathbb{Z}[\Gamma] \) as \((\Gamma \times \Gamma)\)-modules. But this is straightforward to check using the identification (23).

**Remark 6.3.** If \( f : K \to N \) is \((2k - n)\)-connected, \( 2k - n \geq 2 \) and \( k \leq n - 3 \), then Habegger exhibits a necessary and sufficient obstruction to finding an \( h \)-embedding of \( f \) in \( N \) [H]. The obstruction lies in the group

\[
H^{2k}(K \times K; \pi_{2k-n+1}(f) \otimes \mathbb{Z}[\Gamma])_{\mathbb{Z}_2}
\]

which is the coinvariants of a certain involution of the obstruction group appearing in Corollary 6.2. In comparing his result with our Theorem C, note that Habegger’s assumptions \( 2k - n \geq 2, k \leq n - 3 \) are more restrictive than our \( k \leq n - 3, n \geq 6 \).

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