A Modification Fractional Homotopy Analysis Method for Solving Partial Differential Equations Arising in Mathematical Physics

Hassan Kamil Jassim, Mayada Gassab Mohammed, Hossein Ali Eaued
Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq
hassankamil@utq.edu.iq

Abstract: In this paper, we apply a new technique, namely fractional Sumudu homotopy analysis method (FSHAM) on fractional partial differential equations to obtain the analytical approximate solutions. The fractional derivative is described in the Caputo sense. This method is the combination of the Sumudu transform (ST) and homotopy analysis method (HAM). The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

Keyword: Fractional differential equations; Sumudu transform; homotopy analysis method; Caputo fractional derivative.

1. Introduction
The idea of fractional calculus coincides with that of classical calculus. Leibniz and Hopital first raised this issue in 1695 and in 1730 Euler’s attention was drawn to it, followed by Lagrange in 1772 and Laplace in 1812. The first concept of arbitrary derivation was introduced by Lacroix and later by Fourier, Abel, Liouville, Grunewald, Letnikov and Riemann. Various fractional derivatives were thus introduced. Grunewald and Krug introduced the work of Riemann and Liouville and introduced another integral and derivative called Riemann Liouville. Caputo introduced a new derivative by rewriting the Riemann Liouville formula [1].
Fractional calculus have been attractive to many researchers because they play an important role in describing many phenomena arising in physics, chemistry, biology, aerodynamics, control theory, finance, and social sciences. Especially, boundary value problems of fractional differential equations are often regarded as valuable mathematical models in the study of various physical, biological, and chemical processes, such as heat conduction, chemical engineering, thermo-elasticity, computational fluid dynamics, and bacterial self-regularization, and represent very interesting results [2-5].

Many numerical and analytical techniques have been suggested for the solutions of linear and nonlinear partial differential equations of fractional order such as homotopy analysis technique [6], variational iteration method [7,8], homotopy perturbation method [9-11], Laplace homotopy perturbation method [12], Laplace decomposition method [13], Sumudu variational iteration method [14], variation iteration transform method [15], reduce differential transform method [16], series expansion method [17], and another methods [18,19]. This paper considers the efficiency of fractional Sumudu homotopy analysis method (F.SHAM) to solve time-fractional partial differential equations. The F.SHAM is a graceful coupling of two powerful techniques namely homotopy analysis method and Sumudu transform methods and gives more refined convergent series solution.

2. Preliminaries

Some fractional calculus definitions and notation needed [2,4,14] in the course of this work are discussed in this section.

**Definition 2.1.** A real function \( u(t), t > 0 \), is said to be in the space \( C_{\theta}, \theta \in R \) if there exists a real number \( q, (q > \theta) \), such that \( u(t) = t^q u_1(t) \), where \( \varphi (\mu) \in C [0, \infty) \), and it is said to be in the space \( C_{\theta} \) if \( u^{(m)} \in C_{\theta}, m \in N \).

**Definition 2.2.** The Riemann Liouville fractional integral operator of order \( \alpha \geq 0 \), of a function \( u(t) \in C_{\theta}, \theta \geq -1 \) is defined as

\[
I^\alpha u(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau) d\tau, & \alpha > 0, t > 0, \\
I^0_0 u(t) = u(t), & \alpha = 0,
\end{cases}
\]
where \( \Gamma(\cdot) \) is the well-known Gamma function.

Properties of the operator \( I^\alpha \), which we will use here, are as follows

For \( u \in C_\vartheta, \vartheta \geq -1, \alpha, \sigma \geq 0, \)

1. \( I^\alpha I^\sigma u(t) = I^{\alpha+\sigma} u(t). \)
2. \( I^\alpha I^\sigma u(t) = I^\sigma I^\alpha u(t) \)
3. \( I^\alpha_t^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m}. \)

**Definition 2.3.** The fractional derivative of \( u(x,t) \) in the Caputo sense is defined as

\[
D^\alpha u(t) = I^{m-\alpha} D^m u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(\tau)d\tau, \tag{2}
\]

for \( m-1 < \alpha \leq m, m \in N, t > 0, \varphi \in C_{-1}. \)

The following are the basic properties of the operator \( D^\alpha: \)

1. \( D^\alpha I^t u(x,t) = u(x,t). \)
2. \( I^\alpha D^\alpha u(x,t) = u(x,t) - \sum_{k=0}^{m-1} u^{(k)}(x,0) \frac{t^k}{k!}. \)

**Definition 2.4.** The Mittag–Leffler function \( E_\alpha \) with \( \alpha > 0 \) is defined as

\[
E_\alpha (z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\alpha + 1)} \tag{3}
\]

**Definition 2.5.** The Sumudu transform is defined over the set of function

\[ A = \{ u(t) \mid \exists M, \omega_1, \omega_2 > 0, |u(t)| < Me^{\omega |t|}, if \ t \in (-1)^j \times [0, \infty) \} \]

by the following formula

\[
S[u(t)] = \int_0^\infty e^{-\omega t} u(\omega t) dt, \ \ \omega \in (-\omega_1, \omega_2). \tag{4}
\]

**Definition 2.6.** The Sumudu transform of the Caputo fractional derivative is defined as
\[ S[D_t^{m\alpha} u(x,t)] = \omega^{-m\alpha} S[u(x,t)] \]
\[ - \sum_{k=0}^{m-1} \omega^{-(m\alpha+k)} u^{(k)}(x,0), \quad m - 1 < m\alpha < m. \] (5)

3. Fractional Sumudu Homotopy Analysis Method (FSHAM)

Let us consider a general fractional nonlinear partial differential equation of the form:
\[ D_t^\alpha u(x,t) + R u(x,t) + N u(x,t) = g(x,t), \quad 0 < \alpha \leq 1, \quad x \in R, \quad t > 0 \] (6)
with the initial condition
\[ u(x,0) = f(x) \] (7)
where \( D_t^\alpha u(x,t) \) is the Caputo fractional derivative of the function \( u(x,t) \) defined as:
\[ D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \]
\[ = \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \frac{\partial^m u(x,s)}{\partial t^m} \, ds, & m - 1 < \alpha < m \\
\frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N 
\end{cases} \]
and \( R \) is the linear differential operator, \( N \) represents the general nonlinear differential operator, and \( g(x,t) \) is the source term.

Now taking the Sumudu transform of both sides of equation (6) we have
\[ S[D_t^\alpha u(x,t)] + S[R u(x,t)] + S[N u(x,t)] = S[g(x,t)]. \] (8)
Using the differentiation properties of the Sumudu transform and above initial condition, we have
\[ \frac{S[u(x,t)]}{w^{\alpha}} - \frac{u(x,0)}{w^{\alpha}} + S[R u(x,t)] + S[N u(x,t)] = S[g(x,t)], \] (9)
or
\[ S[u(x,t)] - u(x,0) + w^{\alpha} \{ S[R u(x,t)] + S[N u(x,t)] - S[g(x,t)] \} = 0. \] (10)
We define the nonlinear operator
\[ N[\phi(x,t,q)] = S[\phi(x,t,q)] - u(x,0) + w^{\alpha}[S[R\phi(x,t,q)] + S[N\phi(x,t,q)] - S[g(x,t)]] \] (11)
where \( q \in [0,1] \) and \( \phi(x,t,q) \) is a real function of \( x,t,q \).

The so-called zero-order deformation equation of the (11) has the form
\[ (1-q)S[\phi(x,t,q)] - u_{0}(x,t) = qhH(x,t)N[\phi(x,t,q)] \] (12)
where \( S \) is the sumudu transform, \( q \in [0,1] \) is the embedding parameter, \( H(x,t) \) denotes a nonzero auxiliary function, \( h \neq 0 \) is an auxiliary parameter.

\( u_{0}(x,t) \) is an initial guess of \( u(x,t) \) and \( \phi(x,t,q) \) is an unknown function.

Obviously, when the parameter \( q = 0 \) and \( q = 1 \), it holds
\[ \phi(x,t,0) = u_{0}(x,t), \phi(x,t,1) = u(x,t) \] (13)
respectively. Thus as \( q \) increases from 0 to 1,
the solution \( \phi(x,t,q) \) varies from the initial guess \( u_{0}(x,t) \) to the solution \( u(x,t) \).

Expanding \( \phi(x,t,q) \) in Taylor's series with respect to \( q \),
We have
\[ \phi(x,t,q) = u_{0}(x,t) + \sum_{m=1}^{\infty} u(x,t)q^{m} \] (14)
Where
\[ u_{m}(x,t) = \frac{1}{m!} \frac{\partial^{m} \phi(x,t,q)}{\partial q^{m}} \bigg|_{q=0} \] (15)

If the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \), and the auxiliary function are properly chosen,
The series (14) converges at \( q = 1 \), then we have
\[ u(x,t) = u_{0}(x,t) + \sum_{m=1}^{\infty} u_{m}(x,t) \] (16)
Which must be one of the solution of the original nonlinear equations.
According to the definition (16), the governing equation can be deduced from the zero-order deformation (12)

Define the vectors

\[ u_m^\tau(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\} \] (17)

Differentiating the zero-order deformation equation (12) \(m\) times with respect to \(q\) and then dividing by \(m!\) and finally setting \(q=0\) we get the following \(m^{th}\)-order deformation equation:

\[ S[u_m(x,t) - x_m u_{m-1}(x,t)] = h H(x,t) R_m(u_{m-1}(x,t)) \] (18)

Applying the inverse sumudu transform, we have

\[ u_m(x,t) = x_m u_{m-1}(x,t) + S^{-1}[h H(x,t) R_m(u_{m-1}(x,t))] \] (19)

Where

\[ R_m(u_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x,t,q)]}{\partial q^{m-1}} \right|_{q=0} \] (20)

And

\[ X_m = \begin{cases} 0 & , x \leq 1 \\ 1 & , x > 1 \end{cases} \] (21)

In this way, it is easily to obtain \(u_m(x,t)\) for \(m \geq 1\), at \(m^{th}\)-order,

We have

\[ u(x,t) = \sum_{m=0}^{M} u_m(x,t) \] (22)

We get an accurate approximation of the original Eq.(6) When \(M \to \infty\)

4. Applications

Example 4.1. First, we consider the time-fractional Burger's equation

\[ D_t^\alpha u(x,t) + uu_x = u_{xx} \] (23)

with condition \(u(x,0) = x\) (24)
Applying the Sumudu transform on both sides in Eq.(23) and after using the differentiation property of Sumudu transform for fractional derivative we get:

\[
\frac{S[u(x, t)]}{w^\alpha} - \frac{u(x, 0)}{w^\alpha} + S[u u_x - u_{xx}] = 0
\] (25)

On simplifying and using the Eq.(24) we have

\[
S[u(x, t)] - x + w^\alpha [S[u u_x - u_{xx}]] = 0
\] (26)

we now define a nonlinear operator is

\[
N[\phi(x, t, q)] = S[\phi(x, t, q)] - x + w^\alpha \left[ S\left[ \phi(x, t, q) \frac{\partial \phi(x, t, q)}{\partial x} - \frac{\partial^2 \phi(x, t, q)}{\partial x^2} \right] \right]
\] (27)

and thus

\[
R_m(u_{m-1}, x, t) = S[u_{m-1}(x, t)] - (1 - x_m) x + w^\alpha \left[ S\left( \sum_{i=0}^{m-1} u_i u_{(m-1-i)x} \right) - S\left( \frac{\partial^2 u_{m-1}}{\partial x^2} \right) \right]
\] (28)

\[
S[u_m(x, t) - x_m u_{m-1}(x, t)] = hH(x, t)R_m(u_{m-1}, x, t)
\] (29)

Applying the inverse Sumudu transform we have

\[
u_m(x, t) = x_m u_{m-1}(x, t) + h S^{-1} H(x, t)R_m(u_{m-1}, x, t)
\] (30)

Solving above the Eq.(30) for m=1,2,… and choosing H(x,t)=1

Let us take the initial condition

\[
u_0(x,t) = x
\] (31)

the other components are given by

\[
u_1(x, t) = x_1 u_0(x, t) + h S^{-1} [R_1(u_0^-)]
\]

\[
= (0)(x) + h S^{-1} \left[ S(u_o) - (1 - x_1)x + w^\alpha \left[ S(u_o u_{ox}) - S\left( \frac{\partial^2 u_o}{\partial x^2} \right) \right] \right]
\]
\[ = hS^{-1}[x - (1 - 0)x + w^\alpha(x - 0)] \]

\[ = hS^{-1}[w^\alpha(x)] \]

\[ = hx \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

\[ u_2(x,t) = x_2 u_1(x,t) + hS^{-1}\left\{ R_2(u_1^\rightarrow) \right\} \]

\[ = \frac{hxt^\alpha}{\Gamma(\alpha + 1)} + hS^{-1}\left[ S(u_1) - (1 - x_2)x + w^\alpha \left\{ S(u_{\alpha_1}u_{\alpha_2} + u_{\alpha_3}u_{\alpha_4}) - S \left( \frac{\partial^2 u_1}{\partial x^2} \right) \right\} \right] \]

\[ = \frac{hxt^\alpha}{\Gamma(\alpha + 1)} + hS^{-1}\left[ \frac{h\Gamma(\alpha + 1)w^\alpha}{\Gamma(\alpha + 1)} + w^\alpha \left\{ S \left( \frac{h\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \right) + \left( \frac{hxt^\alpha}{\Gamma(\alpha + 1)} \right) \right\} \right] \]

\[ = \frac{hxt^\alpha}{\Gamma(\alpha + 1)} + hS^{-1}\left[ hxw^\alpha + w^\alpha \left\{ 2 \frac{xh\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \right\} \right] \]

\[ = \frac{hxt^\alpha}{\Gamma(\alpha + 1)} + hS^{-1}\left[ hxw^\alpha + \frac{2xh\Gamma(\alpha + 1)w^2\alpha}{\Gamma(\alpha + 1)} \right] \]

\[ = \frac{hxt^\alpha}{\Gamma(\alpha + 1)} + \frac{h^2xt^\alpha}{\Gamma(\alpha + 1)} + \frac{2h^2xt^2\alpha}{\Gamma(2\alpha + 1)} \] \hspace{1cm} (33)

and so on

\[ u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \]

\[ = x + \frac{hxt^\alpha}{\Gamma(\alpha + 1)} + \frac{hxt^\alpha}{\Gamma(\alpha + 1)} + \frac{h^2xt^\alpha}{\Gamma(\alpha + 1)} + \frac{2h^2xt^2\alpha}{\Gamma(2\alpha + 1)} + \cdots \] \hspace{1cm} (34)

Put \( h = -1 \)

\[ u(x,t) = x - \frac{xt^\alpha}{\Gamma(\alpha + 1)} + \frac{2xt^2\alpha}{\Gamma(2\alpha + 1)} + \frac{6xt^3\alpha}{\Gamma(3\alpha + 1)} + \cdots \] \hspace{1cm} (35)
Put \( \alpha = 1 \)

\[
u(x,t) = x \left(1 - t + t^2 - t^3 + \cdots \right)
= \frac{x}{1 + t}
\]  

(36)

The Figure 1 show the graphs of the approximate and the exact solutions among different values of \( t \) and \( \alpha \) when \( x \) is fixed for nonlinear Burger equation in the Caputo fractional operator. In Figures 2,3,4 we plotted the graphs of the approximate solutions among different values of \( x \) and \( t \) when \( \alpha = 0.7, 0.9, 1 \). In Figure 5 we plotted the graphs of the exact solution among different values of \( x \) and \( t \).
Example 4.2. Consider the following nonlinear time-fractional Klein-Gordon equation:

\[ D_\alpha^\alpha u(x,t) = u_{xx} - u^2 \]  

\[ u(x,0) = 1 + \sin(x) \quad 0 < \alpha \leq 1, \; t \geq 0 \]  

Applying the Sumudu transform on both sides in Eq.(37) and after using the differentiation property of Sumudu transform for fractional derivative we get:

\[ \frac{S[u(x,t)]}{w^\alpha} - \frac{u(x,0)}{w^\alpha} - S[u_{xx} - u^2] = 0 \]  

On simplifying and using the Eq. (38) we have

\[ S[u(x, t)] - (1 + \sin(x)) - w^\alpha[S[u_{xx} - u^2]] = 0 \]  

We now define a nonlinear operator as:

\[ N[\phi(x,t,q)] = S[\phi(x,t,q)] - (1 + \sin(x)) - w^\alpha \left[ S \left\{ \frac{\partial^2 \phi(x,t,q)}{\partial x^2} - \phi^2(x,t,q) \right\} \right] \]  

And thus
\[ R_m\left(u_{m-1}, x, t\right) = S[u_{m-1}] - (1 - x_m)(1 + \sin(x)) - w^a\left[S\left(\frac{\partial^2 u_{m-1}}{\partial x^2}\right) - S\left(\sum_{i=0}^{m-1} u_i u_{(m-1)}\right)\right] \] (42)

The \( m^{th} \) order deformation Eq. is

\[ S[u_m(x,t) - x_m u_{m-1}(x,t)] = hH(x,t)R_m\left(u_{m-1}, x, t\right) \] (43)

Applying the inverse Sumudu transform we have

\[ u_m(x,t) = x_m u_{m-1}(x,t) + hS^{-1}H(x,t)\left[R_m\left(u_{m-1}, x, t\right)\right] \] (44)

Solving above the Eq.(44) for \( m=1,2,\ldots \) and choosing \( H(x,t) = 1 \)

Let us take the initial condition

\[ u_0(x,0) = 1 + \sin(x) \] (45)

The other components are given by

\[ u_1(x,t) = x_1 u_0(x,t) + hS^{-1}\left[R_1\left(u_0^{-}\right)\right] \]

\[ = (0)(1 + \sin(x)) + hS^{-1}\left\{S(u_0) - (1 - x_1)(1 + \sin(x)) - w^a\left[S\left(\frac{\partial^2 u_0}{\partial x^2}\right) - S\left(u_0 u_0\right)\right]\right\} \]

\[ = h\left\{\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^\alpha}{\Gamma(\alpha+1)} 3\sin(x) + \frac{t^\alpha}{\Gamma(\alpha+1)} \sin^2(x)\right\} \]

\[ = \frac{ht^\alpha}{\Gamma(\alpha+1)} \left\{1 + 3\sin(x) + \sin^2(x)\right\} \] (46)

\[ u_2(x,t) = x_2 u_1(x,t) + hS^{-1}\left[R_2\left(u_1^{-}\right)\right] \]

\[ = \frac{ht^\alpha}{\Gamma(\alpha+1)} \left\{1 + 3\sin(x) + \sin^2(x)\right\} + hS^{-1}\left\{S(u_1) - (1 - x_2)(1 + \sin(x)) - w^a\left[S\left(\frac{\partial^2 u_1}{\partial x^2}\right) - S(u_0 u_1 + u_1 u_0)\right]\right\} \]

\[ = u_1 + hS^{-1}\left\{\frac{h\Gamma(\alpha+1) w^a}{\Gamma(\alpha+1)} \left(1 + 3\sin(x) + \sin^2(x)\right) - (1 - 1)(1 + \sin(x))\right\} \]
\[-w^\alpha \left[ \frac{h\Gamma(\alpha+1)w^\alpha}{\Gamma(\alpha+1)} \left( -3\sin x - 2\sin^2 x + 2\cos^2 x \right) - 2 \left( \frac{h\Gamma(\alpha+1)w^\alpha}{\Gamma(\alpha+1)} \left( 1 + 4\sin x + 4\sin^2 x + \sin^3 x \right) \right) \right] \]

\[= u_1 + hS^{-1} \left[ hw^\alpha \left( 1 + 3\sin(x) + \sin^2(x) \right) - h w^{2\alpha} \left( -3\sin(x) - 2\sin^2(x) + 2\cos^2(x) \right) \right] + 2 \left( hw^{2\alpha} \left( 1 + 4\sin(x) + 4\sin^2(x) + \sin^3(x) \right) \right) \]

\[= u_1 + h^2 \frac{t^\alpha}{\Gamma(\alpha+1)} \left[ 1 + 3\sin(x) + \sin^2(x) \right] - h^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \left[ -3\sin(x) - 2\sin^2(x) + 2\cos^2(x) \right] + 2 \left( h^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \left( 1 + 4\sin(x) + 4\sin^2(x) + \sin^3(x) \right) \right) \]

\[= u_1 + \frac{h^2 t^{\alpha\alpha}}{\Gamma(\alpha+1)} + \frac{h^2 t^{\alpha\alpha}}{\Gamma(\alpha+1)} \sin^2(x) + \frac{h^2 t^{\alpha\alpha}}{\Gamma(\alpha+1)} \sin^2(x) + \frac{h^2 t^{2\alpha\alpha}}{\Gamma(2\alpha+1)} \sin^3(x) \]

\[+ \frac{h^2 t^{2\alpha\alpha}}{\Gamma(2\alpha+1)} 2\sin^2(x) - 2 \frac{h^2 t^{2\alpha\alpha}}{\Gamma(2\alpha+1)} \sin^2(x) + 2 \frac{h^2 t^{2\alpha\alpha}}{\Gamma(2\alpha+1)} \sin^3(x) \]

\[= \left( h + h^2 \right) \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} \left( 1 + 3\sin x + \sin^2 x \right) + \frac{h^2 t^{2\alpha\alpha}}{\Gamma(\alpha+1)} \left( 11\sin x + 12\sin^2 x + 2\sin^3 x \right) \right] \]

and so on.

\[u(x,t) = u_0 + u_1 + u_2 + \ldots \]

\[= 1 + \sin(x) + \frac{h^\alpha}{\Gamma(\alpha+1)} \left( 1 + 3\sin(x) + \sin^2(x) \right) + \left( h + h^2 \right) \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} \left( 1 + 3\sin(x) + \sin^2(x) \right) \right] + \frac{h^2 t^{2\alpha\alpha}}{\Gamma(2\alpha+1)} \left( 11\sin(x) + 12\sin^2(x) + 2\sin^3(x) \right) \]

(47)

Put \( h = -1 \)
\[
\begin{align*}
    u(x,t) = & 1 + \sin(x) - \frac{t^\alpha}{\Gamma(\alpha+1)} \left\{ 1 + 3\sin(x) + \sin^2(x) \right\} + (1 + (-1)) \left\{ \frac{t^\alpha}{\Gamma(\alpha+1)} \left[ 1 + 3\sin(x) + \sin^2(x) \right] \right\} \\
    & + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \left\{ 11\sin(x) + 12\sin^2(x) + 2\sin^3(x) \right\}
\end{align*}
\]

(49)

Put \( \alpha = 1 \)

\[ u(x,t) = 1 + \sin(x) - t(1 + 3\sin(x) + \sin^2(x)) + \cdots \]  

(50)

The Figure 6 show the graphs of the approximate and the exact solutions among different values of \( t \) and \( \alpha \) when \( x \) is fixed for nonlinear Burger equation in the Caputo fractional operator. In Figures 7,8,9 we plotted the graphs of the approximate solutions among different values of \( x \) and \( t \) when \( \alpha = 0.7, 0.9, 1 \). In Figure 10 we plotted the graphs of the exact solution among different values of \( x \) and \( t \).

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Figure 6
Figure 7

Figure 8
Example 4.3. Consider the following system of nonlinear time-fractional PDEs:

\[ D_t^{\alpha} u - v_x + v + u = 0 \]
\[ D_t^{\beta} v - u_x + v + u = 0 \]  \hspace{1cm} (51)

Subject to initial conditions

\[ u(x, 0) = \sinh(x) \]
\[ v(x, 0) = \cosh(x) \]  \hspace{1cm} (52)
Applying the Sumudu transform on both sides in Eq.(51) and after using the differentiation property of Sumudu transform for fractional derivative we get:

\[
\frac{S[u(x,t)]}{w^\alpha} - \frac{u(x,0)}{w^\alpha} + S[-v_x + v + u] = 0
\]

\[
(53)
\]

\[
\frac{S[v(x,t)]}{w^\beta} - \frac{v(x,0)}{w^\beta} + S[-u_x + v + u] = 0
\]

On simplifying and using the Eq. (52) we have

\[
S[u(x,t)] - \sinh(x) + w^\alpha \left[ S(-v_x + v + u) \right] = 0
\]

\[
(54)
\]

\[
S[v(x,t)] - \cosh(x) + w^\beta \left[ S(-u_x + v + u) \right] = 0
\]

We now define a nonlinear operator as:

\[
N_1 \left[ \phi_1 (x,t,q) \right] = S[\phi_1 (x,t,q)] - \sinh(x) + w^\alpha \left[ S\left( -\frac{\partial \phi_2}{\partial x} + \phi_2 + \phi_1 \right) \right]
\]

\[
(55)
\]

\[
N_2 \left[ \phi_2 (x,t,q) \right] = S[\phi_2 (x,t,q)] - \cosh(x) + w^\beta \left[ S\left( -\frac{\partial \phi_1}{\partial x} + \phi_2 + \phi_1 \right) \right]
\]

and thus

\[
R_{1m} \left( u_{m-1}, v_{m-1} \right) = S[u_{m-1}] - (1-x_m)\sinh(x) + w^\alpha \left[ S\left( -\frac{\partial v_{m-1}}{\partial x} \right) + S(v_{m-1}) + S(u_{m-1}) \right]
\]

\[
R_{2m} \left( u_{m-1}, v_{m-1} \right) = S[v_{m-1}] - (1-x_m)\cosh(x) + w^\beta \left[ S\left( -\frac{\partial u_{m-1}}{\partial x} \right) + S(v_{m-1}) + S(u_{m-1}) \right]
\]

\[
(56)
\]

The \( m^{th} \) order deformation Eq. is

\[
S[u_m (x,t) - x_m u_{m-1} (x,t)] = hH(x,t) R_{1m} \left( u_{m-1}, v_{m-1} \right)
\]

\[
S[v_m (x,t) - x_m v_{m-1} (x,t)] = hH(x,t) R_{2m} \left( u_{m-1}, v_{m-1} \right)
\]

\[
(57)
\]
Applying the inverse Sumudu transform we have:

\[
u_m(x, t) = x_m u_{m-1}(x, t) + h S^{-1} H(x, t) [R_m(u_{m-1}, v_{m-1})]
\]

(58)

\[
u_m(x, t) = x_m u_{m-1}(x, t) + h S^{-1} H(x, t) [R_m(u_{m-1}, v_{m-1})]
\]

Solving above the Eq.(58) for \(m=1,2,\ldots\) and choosing \(H(x, t) = 1\)

Let us take the initial condition

\[
u_0(x, 0) = \sinh(x)
\]

(59)

\[
u_0(x, 0) = \cosh(x)
\]

The other components are given by

\[
u_1(x, t) = x_1 u_0(x, t) + h S^{-1} [R_{11}(u_0, v_0)]
\]

\[
u_1(x, t) = x_1 v_0(x, t) + h S^{-1} [R_{21}(u_0, v_0)]
\]

\[
= h S^{-1} \left\{ S(u_0) - (1 - x_m) \sinh(x) + \alpha S\left( - \frac{\partial v_0}{\partial x} \right) + S(v_0) + S(u_0) \right\}
\]

\[
= h S^{-1} \left\{ S(v_0) - (1 - x_m) \cosh(x) + \beta S\left( - \frac{\partial u_0}{\partial x} \right) + S(v_0) + S(u_0) \right\}
\]

\[
= 0 + h S^{-1} \left\{ \sinh(x) - (1 - 0) \sinh(x) + \alpha \left[ - \sinh(x) + \cosh(x) + \sinh(x) \right] \right\}
\]

\[
= 0 + h S^{-1} \left\{ \cosh(x) - (1 - 0) \cosh(x) + \beta \left[ - \cosh(x) + \cosh(x) + \sinh(x) \right] \right\}
\]

\[
= h S^{-1} \left\{ \alpha \cosh(x) \right\}
\]

\[
= h S^{-1} \left\{ \beta \sinh(x) \right\}
\]

(60)
\[ u_2 (x,t) = x_2 u_1 (x,t) + hS^{-1} [R_{12} (u_1^+, v_1^+)] \]

\[ v_2 (x,t) = x_2 v_1 (x,t) + hS^{-1} [R_{22} (u_1^-, v_1^-)] \]

\[
= (1) \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + hS^{-1} \left\{ S(u_1) - (1 - x_2) \sinh(x) + w^\alpha \left[ S\left( -\frac{\partial v_1}{\partial x} \right) + S(v_1) + S(u_1) \right] \right\}
\]

\[
= (1) \frac{h^t\beta}{\Gamma(\beta + 1)} \sinh(x) + hS^{-1} \left\{ S(v_1) - (1 - x_2) \cosh(x) + w^\beta \left[ S\left( -\frac{\partial u_1}{\partial x} \right) + S(v_1) + S(u_1) \right] \right\}
\]

\[
= \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + hS^{-1} \left\{ h^w\alpha \cosh(x) - h^w\alpha+\beta \cosh(x) + h^w\alpha+\beta \sinh(x) + h^w2\alpha \cosh(x) \right\}
\]

\[
= \frac{h^t\beta}{\Gamma(\beta + 1)} \sinh(x) + hS^{-1} \left\{ h^w\beta \sinh(x) - h^w\alpha+\beta \sinh(x) + h^w2\beta \sinh(x) + h^w\alpha+\beta \cosh(x) \right\}
\]

\[
= \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + \frac{h^z\alpha}{\Gamma(\alpha + 1)} \cosh(x) - \frac{h^z\alpha+\beta}{\Gamma(\alpha+\beta+1)} \cosh(x) + \frac{h^z2\alpha+\beta}{\Gamma(2\alpha+1)} \cosh(x) + \frac{h^z2\alpha}{\Gamma(2\alpha+1)} \cosh(x)
\]

\[
= \frac{h^t\beta}{\Gamma(\beta + 1)} \sinh(x) + \frac{h^z\beta}{\Gamma(\beta + 1)} \sinh(x) - \frac{h^z\alpha+\beta}{\Gamma(\alpha+\beta+1)} \sinh(x) + \frac{h^z2\beta}{\Gamma(2\beta+1)} \sinh(x) + \frac{h^z2\alpha+\beta}{\Gamma(\alpha+\beta+1)} \cosh(x)
\]

\[
= \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x)
\]

and so on.

Therefore, the solution is

\[ u(x,t) = u_0 + u_1 + u_2 + \ldots \]

\[ v(x,t) = v_0 + v_1 + v_2 + \ldots \]

\[ u(x,t) = \sinh(x) + \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) + \frac{h^t\alpha}{\Gamma(\alpha + 1)} \cosh(x) \]
\[ -\frac{h^{2}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \cosh(x) + \frac{h^{2}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \sinh(x) + \frac{h^{2}t^{2\alpha}}{\Gamma(2\alpha+1)} \cosh(x) \]  
\[ v(x,t) = \cosh(x) + \frac{ht^{\beta}}{\Gamma(\beta+1)} \sinh(x) + \frac{ht^{\beta}}{\Gamma(\beta+1)} \sinh(x) + \frac{h^{2}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \cosh(x) \]  
\[ -\frac{h^{2}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \sinh(x) + \frac{h^{2}t^{2\beta}}{\Gamma(2\beta+1)} \sinh(x) + \frac{h^{2}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \cosh(x) \]  

Put \( h = -1 \)

\[ u(x,t) = \sinh(x) - \frac{t^{\alpha}}{\Gamma(\alpha+1)} \cosh(x) \]  
\[ -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \cosh(x) + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \sinh(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \cosh(x) \]  
\[ v(x,t) = \cosh(x) - \frac{t^{\beta}}{\Gamma(\beta+1)} \sinh(x) \]  
\[ -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \sinh(x) + \frac{t^{2\beta}}{\Gamma(2\beta+1)} \sinh(x) + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \cosh(x) \]  

\[ \text{Setting } \alpha = \beta = 1, \text{ we reproduce the solution of the problem as follows:} \]

\[ u(x,t) = \sinh(x) \left[ 1 + \frac{t^{2}}{2!} + \cdots \right] - \cosh(x) \left[ t + \frac{t^{3}}{3!} + \cdots \right] \]
\[ v(x,t) = \cosh(x) \left[ 1 + \frac{t^{2}}{2!} + \cdots \right] - \sinh(x) \left[ t + \frac{t^{3}}{3!} + \cdots \right] \]  

\[ \text{(64)} \]

This solution is equivalent to the exact solution in closed form:
\[ u(x,t) = \sinh(x-t) \cdot \]  
\[ v(x,t) = \cosh(x-t) \cdot \]  

\[ \text{(65)} \]

5. Conclusion

The coupling of homotopy analysis method (HAM) and the Sumudu transform method in the sense of Caputo fractional derivative, proved very effective to solve fractional partial differential equations arising in mathematical physics. The proposed algorithm provides the solution in a series form that converges rapidly to the exact
solution if it exists. From the obtained results, it is clear that the FSHAM yields very accurate solutions using only a few iterates. As a result, the conclusion that comes through this work is that FSHAM can be applied to other fractional partial differential equations of higher order, due to the efficiency and flexibility in the application as can be seen in the proposed examples.

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