Integrability of dispersionless Hirota type equations and the symplectic Monge-Ampère property

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Abstract

We prove that integrability of a dispersionless Hirota type equation implies the symplectic Monge-Ampère property in any dimension $\geq 4$. In 4D this yields a complete classification of integrable dispersionless PDEs of Hirota type through a list of heavenly type equations arising in self-dual gravity. As a by-product of our approach we derive an involutive system of relations characterising symplectic Monge-Ampère equations in any dimension.

Moreover, we demonstrate that in 4D the requirement of integrability is equivalent to self-duality of the conformal structure defined by the characteristic variety of the equation on every solution, which is in turn equivalent to the existence of a dispersionless Lax pair. We also give a criterion of linearisability of a Hirota type equation via flatness of the corresponding conformal structure, and study symmetry properties of integrable equations.

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1 Introduction and the main results

1.1 Dispersionless Hirota type equations

Let $u(x^0, \ldots, x^n)$ be a function of $n + 1$ independent variables. A dispersionless Hirota type equation is a scalar second-order PDE for $u$ of the form

$$F(U) = 0$$

(1)

where $U = Hess(u) = \{u_{\alpha\beta}\}$ is the Hessian matrix of $u$ ($u_{\alpha\beta} = \partial_{x^\alpha} \partial_{x^\beta} u$, $0 \leq \alpha \leq \beta \leq n$). Equations of type (1) appear in a wide range of applications including the following:

- **Integrable systems.** In this context, Hirota type equations arise as differential relations for $\tau$-functions of various 3D hierarchies of the dispersionless Kadomtsev-Petviashvili/Toda type, see e.g. [10, 62, 68, 69].

- **General relativity.** Symplectic Monge-Ampère equations, which constitute a subclass of equations (1), are known to arise as heavenly type equations governing self-dual Ricci-flat metrics in 4D [49, 30, 28].

- **Differential geometry.** In geometric context equations (1), also known as Hessian equations, appear as relations involving symmetric functions of the eigenvalues of $U$. Their analytical and global aspects were thoroughly investigated in [64, 67, 11, 45], see also references therein.
• **Submanifolds in Grassmanians.** Equation (1) can be viewed as the defining equation of a hypersurface $X$ in the Lagrangian Grassmannian $\Lambda$, locally parametrised by $(n+1) \times (n+1)$ symmetric matrices $U$. This point of view has been developed in [24, 57] leading to remarkable connections with integrable $GL(2, \mathbb{R})$ geometry. Integrability aspects of dispersionless systems related to Grassmann geometries were recently studied in [15, 16].

In what follows we assume that equation (1) is non-degenerate in the sense that the corresponding characteristic variety,

$$\sum_{\alpha \leq \beta} \frac{\partial F}{\partial u_{\alpha \beta}} p_{\alpha \beta} = 0,$$

defines a non-degenerate quadric of rank $n + 1$. This gives rise to the conformal structure $[g] = g_{\alpha \beta} dx^\alpha dx^\beta$ where $(g_{\alpha \beta})$ is the inverse to the matrix $(1 + \delta_{\alpha \beta} \frac{\partial F}{\partial u_{\alpha \beta}})$ of the above quadratic form. It will be demonstrated that integrability of non-degenerate Hirota type equations can be interpreted in terms of the conformal geometry of $[g]$ (see [1] for geometry of a special class of degenerate parabolic Monge-Ampère equations).

1.2 Equivalence group

Although we will be primarily interested in the 4D case corresponding to $n = 3$, the following properties hold in any dimension. The class of equations (1) is invariant under the action of the symplectic group $Sp(2n + 2, k)$, where $k = \mathbb{R}$ or $\mathbb{C}$ depending on the context. An element of this group is a block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $(n + 1) \times (n + 1)$ matrices $A, B, C, D$ satisfying the defining relations $A^t C = C^t A$, $B^t D = D^t B$, $A^t D - C^t B = id$, with the action on the Lagrangian Grassmannian $\Lambda$ defined as

$$U \mapsto \tilde{U} = (AU + B)(CU + D)^{-1}. \quad (2)$$

Transformations of this type preserve the integrability, and constitute a natural equivalence group of the problem. The corresponding infinitesimal generators are as follows:

$$X_{\alpha \beta} = \frac{\partial}{\partial u_{\alpha \beta}}, \quad L_{\alpha \beta} = \sum_{\gamma} u_{\beta \gamma} \frac{\partial}{\partial u_{\alpha \gamma}} + u_{\alpha \beta} \frac{\partial}{\partial u_{\alpha \alpha}},$$

$$P_{\alpha \beta} = 2 \sum_{\gamma} u_{\alpha \gamma} u_{\beta \gamma} \frac{\partial}{\partial u_{\gamma \gamma}} + \sum_{\gamma \neq \beta} u_{\alpha \gamma} u_{\beta \delta} \frac{\partial}{\partial u_{\gamma \delta}};$$

note that $X_{\alpha \beta} = X_{\beta \alpha}$ and $P_{\alpha \beta} = P_{\beta \alpha}$, while $L_{\alpha \beta} \neq L_{\beta \alpha}$. Thus, we have $(n + 1)(n + 2)/2$ operators $X_{\alpha \beta}$, $(n + 1)^2$ operators $L_{\alpha \beta}$ and $(n + 1)(n + 2)/2$ operators $P_{\alpha \beta}$. Altogether, they form the Lie algebra $\mathfrak{sp}(2n + 2)$ of dimension $(n + 1)(n + 3)$. Let us represent equation (1) in evolutionary form,

$$u_{00} = f(u_0, \ldots, u_n, u_{11}, u_{12}, \ldots, u_{nn}). \quad (3)$$

The action of the equivalence group $Sp(2n + 2)$ on hypersurfaces in $\Lambda$ induces a (local) action of the same group (equivalently, its Lie algebra) on the space $J^1(\mathbb{R}^{2(n+1)})$ of 1-jets of the function.
of variables \( u_{0i}, u_{ij} \) (\( 1 \leq i \leq j \leq n \)). This space has dimension \( n(n + 3) + 1 \) with coordinates \( u_{0i}, u_{ij}, f, f_{0i}, f_{0j} \).

It is easy to see that the induced action has a unique Zariski open orbit (its complement consists of 1-jets of degenerate systems). This property allows one to assume that all sporadic factors depending on first-order derivatives of \( f \) that arise in the process of Gaussian elimination in the proofs of our main results in Section 3 are nonzero. This considerably simplifies the arguments by eliminating unessential branching. Furthermore, in the verification of various polynomial identities involving higher-order partial derivatives of \( f \) one can, without any loss of generality, give the first-order derivatives of \( f \) any numerical values corresponding to a non-degenerate 1-jet: this often renders otherwise impossible computations manageable.

1.3 Integrability by the method of hydrodynamic reductions

Integrability of Hirota type equations (1) can be approached based on the method of hydrodynamic reductions [26, 27, 21, 22, 23, 24]. In the most general set-up (for definiteness, we restrict to the 4D case), it applies to quasilinear systems of the form

\[ A_0(v)v_{x0} + A_1(v)v_{x1} + A_2(v)v_{x2} + A_3(v)v_{x3} = 0, \]  

(4)

where \( v = (v^1, ..., v^m) \) is an \( m \)-component column vector of the dependent variables \( x^\alpha \) and \( A_\alpha \) are \( l \times m \) matrices where the number \( l \) of the equations is allowed to exceed the number \( m \) of the unknowns. Note that equation (1) can be cast into quasilinear form (4) by choosing \( u_{\alpha\beta} \) as the new dependent variables \( v \) and writing out the compatibility conditions among them, see Section 3. The method of hydrodynamic reductions consists of seeking multi-phase solutions in the form

\[ v = v(R^1, ..., R^N) \]  

(5)

where the phases \( R^I(x) \), whose number \( N \) is allowed to be arbitrary, are required to satisfy a triple of consistent \((1 + 1)\)-dimensional systems,

\[ R^I_{x2} = \mu^I(R)R^I_{x1}, \quad R^I_{x3} = \nu^I(R)R^I_{x1}, \quad R^I_{x0} = \lambda^I(R)R^I_{x1}, \]  

(6)

known as systems of hydrodynamic type. The corresponding characteristic speeds must satisfy the commutativity conditions [65, 66],

\[ \frac{\partial J \mu^I}{\mu^J - \mu^I} = \frac{\partial J \nu^I}{\nu^J - \nu^I} = \frac{\partial J \lambda^I}{\lambda^J - \lambda^I}, \]  

(7)

here \( I \neq J, \partial J = \partial_{R^J}, \ I, J = 1, ..., N \). Equations (6) are said to define an \( N \)-component hydrodynamic reduction of system (4). System (4) is said to be integrable if, for every \( N \), it possesses infinitely many \( N \)-component hydrodynamic reductions parametrised by \( 2N \) arbitrary functions of one variable [21, 23]. This requirement imposes strong constraints (integrability conditions) on the matrix elements of \( A_\alpha(v) \).

The method of hydrodynamic reductions has been successfully applied to the class of 3D Hirota type equations, leading to extensive classification results and remarkable geometric relations [24]. In the present paper we directly apply the method to the class of 4D Hirota type equations. The 4D situation turns out to be far more restrictive, in particular, we demonstrate that the requirement of integrability implies the symplectic Monge-Ampère property.
1.4 Symplectic Monge-Ampère equations

A symplectic Monge-Ampère equation is obtained by equating to zero a linear combination of minors (of all possible orders) of the Hessian matrix $U = Hess(u)$. These equations constitute a proper subclass of Hirota type equations (1). Geometrically, the corresponding hypersurface $X \subset \Lambda$ is a hyperplane section of the Plücker embedding of the Lagrangian Grassmannian $\Lambda$. Among the most well-studied examples one should primarily mention the equations

$$\det U = \text{tr} U \quad \text{and} \quad \det U = 1,$$

governing special Lagrangian submanifolds and affine hyperspheres, respectively [31, 8] (both non-integrable for $n \geq 2$).

In 2D, any symplectic Monge-Ampère equation is linearisable [37]. In 3D, integrability of a symplectic Monge-Ampère equation is equivalent to its linearisability [24]. In 4D, non-degenerate integrable symplectic Monge-Ampère equations were classified in [13]:

**Theorem 1** Over the field of complex numbers, any 4D integrable non-degenerate symplectic Monge-Ampère equation is $Sp(8)$-equivalent to one of the 6 normal forms:

1. $u_{00} - u_{11} - u_{22} - u_{33} = 0$ (linear wave equation);
2. $u_{02} + u_{13} + u_{00}u_{11} - u_{01}^2 = 0$ (second heavenly equation [49]);
3. $u_{02} - u_{01}u_{33} + u_{03}u_{13} = 0$ (modified heavenly equation [13]);
4. $u_{02}u_{13} - u_{03}u_{12} - 1 = 0$ (first heavenly equation [49]);
5. $u_{00} + u_{11} + u_{02}u_{13} - u_{03}u_{12} = 0$ (Husain equation [30]);
6. $\alpha u_{01}u_{23} + \beta u_{02}u_{13} + \gamma u_{03}u_{12} = 0$ (general heavenly equation [53]), $\alpha + \beta + \gamma = 0$.

Equations 2-6 are known to be non-linearisable, and contact non-equivalent. All of them originate from self-dual Ricci-flat geometry, and have been thoroughly investigated in the literature. Thus, bi-Hamiltonian formulation of heavenly type equations was established in [46, 47, 55, 56]. Twistor-theoretic aspects of the associated hierarchies were discussed in [60, 18, 19, 3, 5]. The integrability by the method of hydrodynamic reductions was demonstrated in [22, 23]. Symmetries and recursion operators were constructed in [61, 59, 54, 35, 36, 50, 43]. A $\bar{\partial}$-approach and a novel version of the inverse scattering transform were developed in [4, 41, 42].

It was conjectured in [13] that in 4D, the requirement of integrability of equation (1) implies the symplectic Monge-Ampère property. The proof of this conjecture, which is the main result of our paper, is given in Section 3. Together with Theorem 1, this completes the classification of integrable Hirota type equations in 4D.

The Monge-Ampère property comes as the result of a rather challenging calculation: starting with evolutionary form (3), we derive the integrability conditions, which constitute complicated differential relations that are linear in the third-order, and quadratic in the second-order partial derivatives of $f$. In 3D, these relations can be uniquely solved for all third-order partial derivatives of $f$ resulting in an involutive system of integrability conditions [24]. The first remarkable
phenomenon of the 4D case is the appearance, along with third-order relations, of a whole set of additional second-order relations that are quadratic in the second-order partial derivatives of \( f \). The second remarkable phenomenon is that the ideal generated by these quadratic relations possesses a linear radical responsible for the Monge-Ampère property.

To establish the Monge-Ampère property we need the corresponding set of differential constraints for \( f \). These have only been known in low dimensions \([52, 12, 29]\). In Section 2 we derive these constraints in any dimension by using formal theory of differential equations and representation theory.

**Theorem 2** Equation (3) is of symplectic Monge-Ampère type if and only if \( d^2 f \) is a linear combination of the second fundamental forms of the Plücker embedding of the Lagrangian Grassmannian \( \Lambda \) restricted to the hypersurface defined by (3). This property is characterised by \( N(n) = \frac{1}{24} n(n + 1)(n + 2)(n + 7) \) relations \((18)-(24)\) from Section 2 which are second-order quasilinear PDEs for \( f \).

In Section 2 we provide a characterisation of symplectic Monge-Ampère equations represented in implicit (non-evolutionary) form \((29)\) by an alternative set of linear differential constraints, see Theorem 12.

### 1.5 Integrability, self-duality and Lax pairs

In 4D, the key invariant of a conformal structure \([g]\) is its Weyl tensor \( W \). A conformal structure is said to be self-dual if, with a proper choice of orientation,

\[
W = *W,
\]

where \(*\) is the Hodge star operator. Integrability of the conditions of self-duality by the twistor construction is due to Penrose \([48]\) who observed that self-duality of \([g]\) is equivalent to the existence of a 3-parameter family of totally null surfaces (\(\alpha\)-surfaces). In Section 3 we prove that integrability of a 4D equation \((1)\) is equivalent to the requirement that the conformal structure \([g]\) defined by the characteristic variety of the equation must be self-dual on every solution. Thus, for the second heavenly equation we have

\[
g = dx^0 dx^2 + dx^1 dx^3 - u_{11} (dx^2)^2 + 2u_{01} dx^2 dx^3 - u_{00} (dx^3)^2,
\]

and a direct calculation shows that the conformal structure \([g]\) is indeed self-dual on every solution. Summarising, solutions to integrable systems carry integrable conformal geometry.

It is known that all equations from Theorem 1 possess dispersionless Lax pairs, that is, there exist vector fields \( X, Y \) depending on \( u_{\alpha\beta} \) and an auxiliary parameter \( \lambda \) such that the commutativity condition \([X, Y] = 0\) holds identically modulo the equation (and its differential consequences). For the second heavenly equation we have

\[
X = \partial_3 + u_{11} \partial_1 - u_{01} \partial_0 + \lambda \partial_0, \quad Y = \partial_2 - u_{01} \partial_1 + u_{11} \partial_0 - \lambda \partial_1,
\]

here \( \partial_\alpha = \frac{\partial}{\partial x_\alpha} \). Integral surfaces of the involutive distribution \([X, Y]\) provide null surfaces of the corresponding conformal structure \([g]\), thus establishing its self-duality.
More generally, for Hirota type equations (1), dispersionless integrability (i.e. integrability by the method of hydrodynamic reductions) is equivalent to the existence of a Lax pair in commuting vector fields. Due to the characteristic property of Lax pairs established in [9], this implies self-duality of the corresponding conformal structure \([g]\) on every solution. In fact, one can say more: the absence of \(\partial_\lambda\) in the vector fields \(X, Y\) defining the Lax pair (which is the case for all integrable Monge-Ampère equations from Theorem 1) implies that the corresponding conformal structure \([g]\) is hyper-Hermitian [20].

1.6 Hirota type equations: summary of results

In 4D we obtain a complete classification of integrable non-degenerate Hirota type equations:

**Theorem 3** For non-degenerate equations (1) in 4D, the following conditions are equivalent:

(a) Equation (1) is integrable by the method of hydrodynamic reductions.
(b) Conformal structure \([g]\) defined by the characteristic variety of equation (1) is self-dual on every solution (upon complexification, or in real signatures \((4,0), (2,2)\)).
(c) Equation (1) possesses a dispersionless Lax pair.
(d) Equation (1) is \(\text{Sp}(8)\)-equivalent (over \(\mathbb{C}\)) to one of the 6 canonical forms of integrable symplectic Monge-Ampère equations classified in [13].

The proof of Theorem 3 is given in Section 3.2. As a corollary, we obtain the following characterisation of linearisable equations (Section 3.3):

**Theorem 4** A non-degenerate Hirota type equation in 4D is linearisable by a transformation from the equivalence group \(\text{Sp}(8)\) if and only if the associated conformal structure \([g]\) is flat on every solution (this statement is true in both real and complex situations).

In addition, we investigate symmetry aspects of integrability in 4D. As noted in [13], every four-dimensional integrable symplectic Monge-Ampère equation is invariant under a subgroup of the equivalence group \(\text{Sp}(8)\), of dimension at least 12. As another corollary of Theorem 3 we obtain the following result (Section 3.4):

**Theorem 5** Let \(X^9 \subset \Lambda^{10}\) be a hypersurface in the Lagrangian Grassmannian corresponding to an integrable Hirota type equation in 4D. Then it is almost homogeneous: there exists a subgroup of the equivalence group that acts on \(X^9\) with a Zariski open orbit.

The fact that every integrable Hirota type equation in 4D possesses nontrivial symmetries from the equivalence group \(\text{Sp}(8)\) (as well as many more from the general contact group) is in sharp contrast with the situation in 3D [24] where a generic integrable Hirota type equation was shown to possess no continuous symmetries from the equivalence group \(\text{Sp}(6)\). We also note that for the two-component first-order systems in 4D studied in [16] the integrability implies a certain amount of symmetry from the corresponding equivalence group, yet in general this does not make the equation \(X\) almost homogeneous.
Finally, we obtain the following generalisation of one of the statements from Theorem 3. According to it, any integrable Hirota type equation in 4D is necessarily of Monge-Ampère type (in 3D this is not true, see [24]). This property persists in higher dimensions (Section 3.5):

**Theorem 6** In all dimensions higher than 3, the integrability of a non-degenerate Hirota type equation implies the symplectic Monge-Ampère property.

In particular, in higher dimensions \( n + 1 \geq 4 \), the local integrability constraints on a smooth hypersurface \( X \subset \Lambda \) imply its global algebraicity.

# 2 Characterisation of symplectic Monge-Ampère equations

In this section we consider Hirota type equations represented in evolutionary form (3). The proof of Theorem 3 will require differential constraints for the right-hand side \( f \) that characterise symplectic Monge-Ampère equations. In dimensions \( \leq 4 \) these were obtained in [6, 52, 12, 29] based on linear degeneracy (complete exceptionality) of Monge-Ampère equations. Here we complete the case of general dimension which was left open.

We adopt a differential-geometric point of view that identifies equation (3) with a hypersurface \( X \) in the Lagrangian Grassmannian \( \Lambda \) [24, 13]. The Monge-Ampère property is equivalent to the requirement that osculating spaces to \( X \) span a hyperplane in the projective space of the Plücker embedding of \( \Lambda \). The last condition can be represented as a simple relation among the second fundamental forms of \( X \), leading to the required differential constraints. The presentation below follows [14], see also [29].

Once the constraints characterising the symplectic Monge-Ampère property are derived we show their completeness by demonstrating involutivity of the corresponding system of PDEs. Theorem 2 then follows from Section 2.3. We also derive, in Section 2.4, a linear system characterising the symplectic Monge-Ampère property in implicit form.

## 2.1 Monge-Ampère equations in 2D

Let us begin with the two-dimensional situation which however contains all essential ingredients of the general case. Setting \( r = u_{00}, \ s = u_{01}, \ t = u_{11} \) we represent equation (3) as

\[
    r = f(s, t). \tag{8}
\]

Equation (8) specifies a surface \( X^2 \) in the Lagrangian Grassmannian \( \Lambda^3 \) which is identified with \( 2 \times 2 \) symmetric matrices,

\[
    U = \begin{pmatrix} r & s \\ s & t \end{pmatrix}.
\]

**Proposition 7** Equation (8) is of Monge-Ampère type if and only if the symmetric differential \( d^2 f \) is proportional to the quadratic form \( dr dt - ds^2 \) restricted to \( X^2 \):

\[
    d^2 f \in \text{span}(df dt - ds^2). \tag{9}
\]
**Proof:** The Plücker embedding $\Lambda^3 \hookrightarrow \mathbb{P}^4$ is a quadric with position vector $(t, s, r, tr - s^2)$. The induced embedding of $X^2$ has position vector

$$R = (t, s, f, tf - s^2).$$

To prove that equation (8) is of Monge-Ampère type we need to show that components of $R$ satisfy a linear relation with constant coefficients or, equivalently, that the Plücker image of $X^2$ belongs to a hyperplane in $\mathbb{P}^4$. This means that the union of all osculating spaces to $X^2$ must be 3-dimensional. Since the tangent space of $X^2$, which is spanned by the vectors

$$R_s = (0, 1, f_s, tf_s - 2s), \quad R_t = (1, 0, f_t, tf_t + f),$$

is already 2-dimensional, we have to show that the union of the second and third-order osculating spaces (spanned by the second and third-order partial derivatives of the position vector $R$ with respect to $s, t$), is only 1-dimensional. As higher-order derivatives of $R$ have zeros in the first two positions, we obtain the following rank condition:

$$\begin{pmatrix}
    f_{ss} & tf_{ss} - 2 \\
    f_{st} & tf_{st} + f_s \\
    f_{tt} & tf_{tt} + 2f_t \\
    f_{sss} & tf_{sss} \\
    f_{sst} & tf_{sst} + f_{ss} \\
    f_{ttt} & tf_{ttt} + 3f_{tt}
\end{pmatrix} = \begin{pmatrix}
    f_{ss} & -2 \\
    f_{st} & f_s \\
    f_{tt} & 2f_t \\
    f_{sss} & 0 \\
    f_{sst} & f_{ss} \\
    f_{ttt} & 3f_{tt}
\end{pmatrix} = 1.$$

In other words, the second column of the last matrix should be proportional to the first one. Denoting by $p$ the proportionality coefficient, this can be represented in compact form as

$$d^2 f = 2p \cdot (df dt - ds^2), \quad \text{(10)}$$

$$d^3 f = 3p \cdot dt d^2 f. \quad \text{(11)}$$

Calculating the (symmetric) differential of (10) and comparing the result with (11) we obtain the following equation for $p$:

$$dp = p^2 dt. \quad \text{(12)}$$

Equations (10) and (12) constitute a closed involutive differential system for $f$ which characterises symplectic Monge-Ampère equations in 2D. It remains to point out that (12) can be obtained from the consistency conditions of equations (10), without invoking (11). In other words, equations (10) imply both (11) and (12). This finishes the proof of Proposition 7. \(\square\)

**Remark:** Proposition 7 has a clear projective-geometric interpretation. The second fundamental forms of the surface $X^2 \subset \Lambda^3 \subset \mathbb{P}^4$ are spanned by $d^2 f$ and $df dt - ds^2$. Here the last form is the restriction to $X^2$ of the second fundamental form of the Grassmannian $\Lambda^3$ itself, namely, $dr dt - ds^2$. The claim is thus that the only ‘essential’ second fundamental form of $X^2$ is the one coming from the second fundamental form of $\Lambda^3$. This property is clearly a necessary condition for $X^2$ to be a hyperplane section, and the above proof shows that it is also sufficient.
Remark: Elimination of $p$ from (10) leads to the following system of PDEs for $f$:

\[ f_t f_{ss} + f_{tt} = 0, \quad f_s f_{ss} + 2f_{st} = 0. \] (13)

The general solution of this system is given by the formula

\[ f = \frac{s^2 - \beta s - \gamma t - \delta}{t + \alpha}, \]

which indeed specifies a Monge-Ampère equation, $rt - s^2 + \alpha r + \beta s + \gamma t + \delta = 0$.

2.2 Monge-Ampère equations in higher dimensions: the defining relations

A 3D equation \((3)_{n=2}\) specifies a hypersurface $X^5$ in the Lagrangian Grassmannian $\Lambda^6$. We will work in the affine chart of $\Lambda^6$ identified with the space of $3 \times 3$ symmetric matrices

\[ U = \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ u_{01} & u_{11} & u_{12} \\ u_{02} & u_{12} & u_{22} \end{pmatrix}. \]

The minors of $U$ define the Plücker embedding $\Lambda^6 \hookrightarrow \mathbb{P}^{13}$. The second fundamental forms of this embedding are given by $2 \times 2$ minors of the matrix $dU$. Since the second osculating space of $\Lambda^6 \subset \mathbb{P}^{13}$ is only 12-dimensional, there is also a third fundamental form, namely $\det dU$.

For general $n$ equation \((3)\) defines a hypersurface $X$ in the Lagrangian Grassmannian $\Lambda$ of dimension $d(n + 1)$, where $d(n) = \frac{n(n + 1)}{2}$ is the number of entries of a symmetric $n \times n$ matrix. The Lagrangian Grassmannian is embedded via the Plücker map into projective space of dimension $p(n + 1) - 1$, where $p(n) = \frac{2(2n+1)!}{n(n+2)!}$ is the number of independent minors of a symmetric $n \times n$ matrix:

\[ X^{d(n+1)-1} \subset \Lambda \hookrightarrow \mathbb{P}^{p(n+1)-1}. \] (14)

Proposition 8 Equation \((3)\) is of Monge-Ampère type if and only if $d^2 f$ belongs to the span of the second fundamental forms of the Plücker embedding of $\Lambda$ restricted to the hypersurface $X$.

We will prove this fully in the 3D case, the general case can be proved similarly for successive dimensions $n$. The condition of the proposition is clearly necessary, for general $n$ the sufficiency will follow from the theorem in the next section.

Proof: For $n = 2$ we have to require that the induced Plücker embedding of $X^5$ belongs to a hyperplane in $\mathbb{P}^{13}$, i.e. the union of all osculating spaces to $X^5$ is 12-dimensional. Calculations analogous to that from Section 2.1 yield the following expansions of the differentials of $f$:

\[
d^2 f = 2a_0(df du_{12} - du_{01} du_{02}) + 2a_1(df du_{22} - (du_{02})^2) + 2a_2(df du_{11} - (du_{01})^2) + 2b_0(du_{11} du_{22} - (du_{12})^2) + 2b_1(du_{01} du_{12} - du_{02} du_{11}) + 2b_2(du_{02} du_{12} - du_{01} du_{22}), \] (15)

\[ d^3 f = 3\omega d^2 f + 6s \det dU|_{u_{00}=f}, \] (16)
where \( \omega = a_0 du_{12} + a_1 du_{22} + a_2 du_{11} \). Compatibility conditions for the relations (15)-(16) implies:

\[
\begin{align*}
da_0 &= a_0 \omega - 2 s du_{12}, & da_1 &= a_1 \omega + s du_{11}, & da_2 &= a_2 \omega + s du_{22}, \\
db_0 &= b_0 \omega + 2sf, & db_1 &= b_1 \omega + 2 s du_{02}, & db_2 &= b_2 \omega + 2 s du_{01}, & ds &= s \omega.
\end{align*}
\] (17)

One can verify that \( d \omega = 0 \). Equations (15) and (17) constitute an involutive differential system for \( f \) which characterises Monge-Ampère equations. It remains to point out that equations (17) can be obtained from the consistency of equations (15) alone, without invoking (16). In other words, equations (15) imply both (16) and (17). This finishes the proof of Proposition 8. \( \square \)

Proposition 8 leads to a system of PDEs for \( f \) (for \( n = 2 \) these can be obtained by eliminating coefficients \( a_i, b_i \) from equations (15)). First of all, for every index \( i = 1, \ldots, n \) one has the analogues of (13),

\[
f_{u_{ij}} f_{u_{ij} u_{ii}} + f_{u_{ii} u_{ii}} = 0, \quad f_{u_{ij}} f_{u_{ij} u_{ii}} + 2 f_{u_{ij} u_{ii}} = 0.
\] (18)

Secondly, for every pair of indices \( i \neq j \in \{1, \ldots, n\} \) one has the relations

\[
f_{u_{ij}} f_{u_{ij} u_{ii}} + 2 f_{u_{ij} u_{jj}} + 2 f_{u_{ij} u_{jj}} + 2 f_{u_{ij} u_{ij}} = 0, \quad f_{u_{ij}} f_{u_{ij} u_{ii}} + 2 f_{u_{ij} u_{ii}} + 2 f_{u_{ij} u_{ij}} = 0,
\] (19)

\[
f_{u_{ij}} f_{u_{j} u_{ij}} + f_{u_{ii} u_{ij}} + 2 f_{u_{ij} u_{ij} + 2 f_{u_{ij} u_{ij}} + 2 f_{u_{ij} u_{ij}} + 2 f_{u_{ij} u_{ij}} = 0.
\] (20)

\[
f_{u_{ij}} f_{u_{ij} u_{j}} + f_{u_{ij} u_{ij}} + 2 f_{u_{ij} u_{ij}} + 2 f_{u_{ij} u_{ij}} + 2 f_{u_{ij} u_{ij}} + 2 f_{u_{ij} u_{ij}} = 0.
\] (21)

Based on a different approach, for \( n = 2 \) these relations were derived in [52], see also [29]. Furthermore, for every triple of distinct indices \( i \neq j \neq k \in \{1, \ldots, n\} \) one has the relations

\[
f_{u_{ij}} f_{u_{ik} u_{ij}} + f_{u_{ij} f_{u_{ik} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ij}} = 0,
\] (22)

\[
f_{u_{ij}} f_{u_{j} u_{ik}} + f_{u_{ij} f_{u_{ik} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} = 0.
\] (23)

For \( n = 3 \) the above relations (18)-(23) were obtained in [12] (25 relations altogether). Finally, for every four distinct indices \( i \neq j \neq k \neq l \in \{1, \ldots, n\} \) one has the relations

\[
f_{u_{ijkl}} f_{u_{ij} u_{kl}} + f_{u_{ij} f_{u_{ik} u_{kl}} + f_{u_{ij} u_{ik}} + f_{u_{ik} f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} + f_{u_{ij} u_{ik}} = 0.
\] (24)

**Remark:** Relations (18)-(24) can be obtained from relations (13) via a traveling wave reduction. Let \( u(x^0, \ldots, x^n) \) solve a Monge-Ampère equation represented in evolutionary form (3). Consider the traveling wave ansatz \( u(x^0, \ldots, x^n) = u(x^0, \xi) \) where

\[
\xi = \alpha_1 x^1 + \cdots + \alpha_n x^n, \quad \alpha_i = const.
\]

Using \( u_{0i} = \alpha_i u_{0 \xi} \), \( u_{ij} = \alpha_i \alpha_j u_{\xi \xi} \) we can reduce (3) to a 2D Monge-Ampère equation of type (8). Imposing relations (13) we obtain expressions that are quartic in \( \alpha \)'s. Equating similar terms we recover all relations (18)-(24).
2.3 Monge-Ampère equations in evolutionary form: proof of Theorem 2

In this section we apply the Spencer machinery [58] to show that the system of relations (18)-(24) defines the class of Monge-Ampère equations. Note that every differential system can be described by either its defining relations (PDEs) or jets of its solutions. In the latter case the system is involutive, but we do not have control over the defining relations. In the former case we have control but do not know compatibility a priori. To demonstrate that these two descriptions coincide we first prove that Monge-Ampère equations have defining relations of the second order only, then by dimensional reasons we conclude that these must coincide with relations (18)-(24).

**Theorem 9** Hirota type equation (3) is of Monge-Ampère type if and only if the right hand side $f$ satisfies relations (18)-(24).

**Proof.** Let $\mathcal{E} \subset J^\infty(\mathbb{R}^d)$ denote the system of PDEs for $f$ characterising the Monge-Ampère property. Here $d = d(n + 1) - 1$, $\mathbb{R}^d$ is the space of independent arguments of $f$ and $\mathcal{E}_0 = J^0 = \mathbb{R}^{d+1}$ is an open chart in $\Lambda$ (here $d(n + 1)$ and $p(n + 1)$ are the same as in Section 2.2). Locally, we make the identification $X = \text{graph}(f) \subset J^0$. Referring to [32, 33] for the basics of jet-theory and the formal theory of PDEs, we identify the system $\mathcal{E}$ with a co-filtered subset in jets, meaning that the sequence $\mathcal{E}_k \subset J^k(X)$ forms a tower of bundles $\pi_{k,l} : \mathcal{E}_k \to \mathcal{E}_l$ for $k > l$. Clearly $\mathcal{E}_1 = J^1$. We want to prove that $\mathcal{E}_2$ is generated by a system of relations $\psi_j = 0$ on 2-jets given by (18)-(24). It is however easier to study the properties of $\mathcal{E}$ by looking at its solutions.

Thus we identify $\mathcal{E}_k$ as the space of $k$-jets of the set of all Monge-Ampère equations written in the evolutionary form (3). By definition the symbol spaces of $\mathcal{E}$ are the subspaces $g_k = \text{Ker}(d\pi_{k,k-1} : T\mathcal{E}_k \to T\mathcal{E}_{k-1}) \subset S^k\tau^*$ where $\tau = T_{o}X$ is the model tangent space. For $k = 0, 1$ we have: $g_0 = \mathbb{R}$, $g_1 = \tau^*$. For $k \geq 2$ the symbols $g_k$ can be interpreted in terms of the Hessian matrix $U$. Indeed, any Monge-Ampère equation is a relation of the form

$$M_0 + M_1 + \cdots + M_n + M_{n+1} = 0,$$

where $M_i$ is a linear combination of $i \times i$ minors of $U$. Linearising this at the identity matrix $I$, i.e. setting $U = I + \epsilon A$ and truncating the higher-order terms in $\epsilon$, the symbol $g_k$ for $k \geq 2$ can be interpreted as the space generated by linearly independent minors of $A$ of size $k$.

It was noted in [44] that the number of independent $k \times k$ minors of a symmetric $n \times n$ matrix is $b(k,n) = \frac{1}{k!} \binom{n}{k} \binom{n+1}{k}$. The discussion above implies that $\dim g_k = b(k,n+1)$ for all $k \in [0, n+1]$ with the exception of $k = 1$, in which case $\dim g_1 = b(1,n+1) - 1 = d$ due to the relation $u_{00} = f$. For $k > n + 1$ the symbol vanishes, $g_k = 0$, signifying that the system $\mathcal{E}$ is of finite type. Its solution space $S$, which can be identified with the dual projective space to $\mathbb{P}^{n(n+1)−1}$ from (14), is therefore finite-dimensional, and

$$\dim S = \sum_{k=0}^{\infty} \dim g_k = \sum_{k=0}^{n+1} b(k,n+1) - 1 = p(n+1) - 1.$$  

We claim that $\mathcal{E}_2$ is precisely the locus of relations (18)-(24). These relations are independent and vanish on every Monge-Ampère equation. Thus $\{\psi_j\}$ contain the derived equations. On the other hand, their count is as follows: $n + n$ relations (18), $n(n − 1)$ relations (19), $n(n − 1)$ relations (20).
relations (20), \( \binom{n}{2} \) relations (21), \( \binom{n}{3} \) relations (22), \( 3 \binom{n}{3} = \frac{n(n-1)(n-2)}{2} \) relations (23), and \( \binom{n}{4} \) relations (24). These numbers sum up to \( N(n) = \binom{d(n+1)}{2} - b(2, n + 1) \) which is the codimension of \( g_2 \subset S^2 T^* X \). This count along with the quasi-linearity of relations \( \psi_j \) implies the claim.

To finish the proof we observe that the higher symbols \( g_{k+2} \) for \( k > 0 \) coincide with the prolongations \( \hat{g}_2^{(k)} := S^{k+2} T^* \cap S^k r^* \otimes g_2 \), this is the statement of Lemma 11 below. Thus the number of relations specifying the prolongation \( \hat{E}_2^{(k)} \) (the locus of the prolonged equations \( D_\sigma \psi_j = 0 \) for all multi-indices \( \sigma \) of length \( |\sigma| \leq k \)) is no less than that for \( \hat{E}_{k+2} \). But it cannot be bigger because otherwise the solution space of the prolonged system \( \hat{E}_2 \) will be less than that of \( E \) (which contains all Monge-Ampère equations). Thus \( \hat{E}_2^{(k)} = \hat{E}_{k+2} \), hence the system \( \hat{E}_2 \) given by relations (18)-(24) is formally integrable, and hence locally solvable for any admissible Cauchy data due to its finite type.

To justify the above proof it remains to compute prolongations of the symbols used above. For this we exploit the subalgebra \( A_n = s l_{n+1} \) in the Lie algebra \( C_{n+1} = g \) of the equivalence group \( G = Sp(2n + 2, \mathbb{C}) \): in the \( |1| \)-grading \( g = g_{-1} \oplus g_0 \oplus g_1 \) corresponding to the parabolic subalgebra \( p = p_{n+1} \) (numeration: the last node on the Dynkin diagram of \( C_{n+1} \) crossed) we have \( g_0 = gl_{n+1} = s l_{n+1} \otimes \mathbb{R} \) and this naturally acts on the tangent space to the Lagrangian Grassmannian \( \Lambda = G/P \) (with \( p = \text{Lie}(P) \)). Thus the tangent and symbol spaces are all \( A_n \)-modules. Below \( \Gamma_\mu \) indicates the irreducible \( A_n \)-representation with the highest weight \( \mu \) that we decompose by the fundamental weights \( \lambda_i \).

Denote by \( \hat{E} \) the equation-manifold describing Monge-Ampère equations in implicit form (25). Interpreted in jet-formalism (similar to the above proof) as a tower of bundles \( \hat{E}_k \), we obtain their symbol spaces \( \hat{\gamma}_k = \text{Ker}(\pi_{k,k-1} : \hat{E}_k \to \hat{E}_{k-1}) \); note that we do not need to pass to tangent spaces as the equation \( \hat{E} \) (as well as its solution spaces) is linear. Since the equation \( \hat{E} \) is specified by its solutions, it is involutive. However, a-priori, it can have PDE-generators of different orders. We will show that this is not the case. The Lie algebra \( A_n \) acts naturally on \( \hat{E}_k \) and hence the symbols \( \hat{\gamma}_k \) are \( A_n \)-modules.

**Proposition 10** The defining equations of \( \hat{E} \) have second order. In other words, the PDEs of higher order \( k > 2 \) specifying \( \hat{E}_k \) are prolongations of the second-order PDEs.

**Proof.** The statement is equivalent to the equality of symbolic prolongations: \( \hat{\gamma}_k = \hat{\gamma}_2^{(k-2)} \) for \( k > 2 \). Here, similarly to the preceding proof we identify \( \hat{\gamma}_k \subset S^k T^* \) where \( T = T_0 \Lambda \) is the model tangent space. As is standard in the formal theory of differential equations [32, 33], the above equality of prolongations is equivalent to the successive identities \( \hat{\gamma}_{k+1} = \hat{\gamma}_k^{(k)} \), \( k \geq 2 \), and this is equivalent to the vanishing of the cohomology in the second non-trivial term of the Spencer \( \delta \)-complex

\[
0 \to \hat{\gamma}_{k+1} \xrightarrow{\delta} T^* \otimes \hat{\gamma}_k \xrightarrow{\delta} \Lambda^2 T^* \otimes \hat{\gamma}_{k-1} \to \ldots
\]

Namely \( H_1^{(k)}(\hat{\gamma}) \) is the cohomology at the term \( T^* \otimes \hat{\gamma}_k \). It is well-known [58, 40] that the Spencer cohomology complex dualises over \( \mathbb{R} \) to the Koszul homology complex

\[
0 \leftarrow \hat{\gamma}_{k+1}^* \xleftarrow{\partial} T \otimes \hat{\gamma}_k^* \xleftarrow{\partial} \Lambda^2 T \otimes \hat{\gamma}_{k-1}^* \leftarrow \ldots
\]
Our claim is equivalent to the vanishing of the corresponding homology: \( H_{1,k}(\hat{g}^*) = 0 \) for \( k \geq 2 \).

From the preceding proof and [44] it follows that the symbols, considered as \( A_n \)-modules, are

\[
\hat{g}_k = S^k S^2 V^*_n \cap S^2 \Lambda^k V^*_n = \Gamma_{2k-n-1},
\]

where \( V_n = \Gamma_{\lambda_1} = \mathbb{R}^n \) is the standard representation and \( V^*_n = \Gamma_{\lambda_n} \) is its dual. For \( k = 0 \) we get \( \hat{g}_0 = \Gamma_0 = \mathbb{R} \). Dualising the symbol we get \( \hat{g}_k = \Gamma_{2k} \) (this is the main advantage: the computations will be visibly \( n \)-independent); for \( k = 0 \) again \( \hat{g}_0 = \mathbb{R} \). Also \( T = T_{2\lambda_1} \).

At this point we start working over \( \mathbb{C} \): since the complexification does not change the rank of the cohomology, this simplification does not restrict the generality. The Littlewood-Richardson rule yields the following tensor decompositions for the second nonzero term of the Koszul complex (it applies for \( 0 < k \leq n \) and requires a modification otherwise):

\[
\Gamma_{2\lambda_1} \otimes \Gamma_{2\lambda_k} = \Gamma_{2\lambda_{k+1}} + \Gamma_{\lambda_1+\lambda_k} + \lambda_{k+1} + \Gamma_{2\lambda_1+2\lambda_k}.
\]  \( (27) \)

Similarly for the third nonzero term, using the plethysm \( \Lambda^2 T = \Gamma_{2\lambda_1+\lambda_2} \), we have for \( k \geq 2 \) (the agreement is that \( \lambda_0 = \lambda_{n+1} = 0 \); entries in parentheses below are equal for \( k = 2 \) and add without multiplicity; for \( k = n + 1 \) the second and fifth terms disappear):

\[
\Gamma_{2\lambda_1+\lambda_2} \otimes \Gamma_{2\lambda_{k-1}} = \Gamma_{\lambda_2+2\lambda_k} + \Gamma_{\lambda_1+\lambda_k} + \lambda_{k+1} + (\Gamma_{\lambda_1+\lambda_k+\lambda_{k-1}+\lambda_k} + \Gamma_{2\lambda_1+2\lambda_k}) + \Gamma_{2\lambda_1+\lambda_{k-1}+\lambda_{k+1}} + (\Gamma_{2\lambda_1+\lambda_k+\lambda_{k-1}+\lambda_k}) + \Gamma_{3\lambda_1+\lambda_{k-1}+\lambda_k}.
\]  \( (28) \)

Similarly, using the plethysm \( \Lambda^3 T = \Gamma_{3\lambda_2} + \Gamma_{2\lambda_1+\lambda_3} \), we decompose the next term in \( (26) \):

\[
\Lambda^3 T \otimes \Gamma_{2\lambda_{k-2}} = \Gamma_{\lambda_2+2\lambda_k} + \Gamma_{\lambda_2+\lambda_{k-1}+\lambda_k} + \Gamma_{\lambda_1+\lambda_{k-1}+\lambda_k} + 2\Gamma_{\lambda_1+\lambda_2+\lambda_{k-1}+\lambda_k} + \Gamma_{\lambda_1+\lambda_2+\lambda_{k-2}+\lambda_{k-1}} + \Gamma_{\lambda_1+\lambda_{k-1}+\lambda_{k-2}+\lambda_{k-1}} + \Gamma_{\lambda_1+\lambda_{k-2}+\lambda_{k-1}+\lambda_k} + \Gamma_{\lambda_2+\lambda_{k-1}+\lambda_{k-2}+\lambda_{k-1}} + \Gamma_{\lambda_2+\lambda_{k-2}+\lambda_{k-1}+\lambda_k} + \Gamma_{\lambda_3+\lambda_{k-1}+\lambda_{k-2}+\lambda_{k-1}} + \Gamma_{\lambda_{k-1}+\lambda_{k-2}+\lambda_{k-1}} + \Gamma_{\lambda_{k-2}+\lambda_{k-1}+\lambda_k}.
\]

Again, in the special cases \( k \in \{2, 3, n+1, n+2\} \) this decomposition requires a modification. By Shur’s lemma a homomorphism \( \Gamma_{\mu} \to \Gamma_{\nu} \) is either zero or an isomorphism in the case \( \mu = \nu \).

It can be checked that Young symmetrisers are nontrivial on the common terms of \( (27) \) and \( (28) \), i.e. the last two terms of \( (27) \) are isomorphic images under the Koszul differential \( \partial \) of the same type modules from the decomposition \( (28) \). The first term \( \Gamma_{2\lambda_{k+1}} \) in the right-hand-side of \( (27) \) does not come however as the image of the second differential \( \partial \) and it does not go to zero under the first differential \( \partial \) in \( (26) \), i.e. it is mapped isomorphically to \( g^*_{k+1} \). Special cases \( k = 2, k = n \) have to be considered separately, but they lead to the same conclusion.

Thus the first Koszul homology vanishes, \( H_{1,k}(\hat{g}^*) = 0 \), and by dualisation the first Spencer cohomology does the same: \( H^{1,k}(\hat{g}) = 0, k > 1 \). This implies the claim of the proposition. In the same vein, \( H_{2,k}(\hat{g}^*) = 0 \) and hence \( H^{2,k}(\hat{g}^*) = 0 \) for \( k \geq 2 \). \( \square \)

As a corollary of this proposition we deduce the last building block for the main theorem of this section, which will therefore finish the proof of Theorem 2.

**Lemma 11** For \( k > 2 \) the following holds: \( g_k = g_2^{(k-2)} \).
Proof. Let us first note that the symbols of the two considered equations agree except at degree one: \( \hat{g}_k = g_k \) for \( k \geq 2 \) (also \( \hat{g}_0 = R = g_0 \)). However they form symbolic complexes over different vector spaces: \( T \) for \( \hat{g} \) and \( \tau \) for \( g \). The relation between these spaces is given by the exact sequence

\[
0 \rightarrow \tau \rightarrow T \rightarrow R \rightarrow 0
\]

that is induced by the embedding \( X \hookrightarrow \Lambda \), identifying the normal bundle with \( R \). Dualisation of this 3-sequence gives a relation between \( \hat{g}_1 = T^* \) and \( g_1 = \tau^* \); it is used in the diagram below.

We unite the Spencer \( \delta \)-complexes for \( \hat{g} \) and \( g \) into the following commutative diagram with exact vertical sequences:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_k & \tau^* \otimes g_{k-1} & \Lambda^2 \tau^* \otimes g_{k-2} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
0 & \hat{g}_{k+1} & T^* \otimes \hat{g}_k & \Lambda^2 T^* \otimes \hat{g}_{k-1} & \Lambda^3 T^* \otimes \hat{g}_{k-2} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & g_{k+1} & \tau^* \otimes g_k & \Lambda^2 \tau^* \otimes g_{k-1} & \Lambda^3 \tau^* \otimes g_{k-2} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

This diagram should be modified in the column with central term \( \Lambda^k T^* \otimes \hat{g}_1 \), as the kernel of the projection map gets an additional factor, namely it becomes \( \Lambda^k - 1 \tau^* \otimes g_1 \otimes \Lambda^k T^* \). By the standard diagram chase we know that \( H^{1,k}(g) = H^{1,k-1}(g) \) provided \( H^{1,k}(\hat{g}) = H^{2,k-1}(\hat{g}) = 0 \). This and the result of Proposition 10 imply the following relations:

\[
H^{1,1}(g) \oplus \Lambda^2 T^* \supset H^{1,2}(g) = H^{1,3}(g) = \ldots
\]

Since \( H^{1,n+2}(g) = 0 \) by dimensional reasons, the above sequence stabilises at zero and hence \( H^{1,k}(g) = 0 \) for \( k \geq 2 \). Vanishing of this Spencer cohomology is equivalent to the equality \( g_{k+1} = g_{k}^{(1)} \), and the result follows. \( \square \)

2.4 Monge-Ampère equations in implicit form

In implicit form, symplectic Monge-Ampère equations can be expressed as linear relations among minors of the Hessian matrix \( U \), namely \( F = 0 \) with

\[
F = \sum_{\sigma} a_{\sigma} \det U_{\sigma},
\]

(29)

where \( \sigma \) encodes minors of a symmetric \( (n+1) \times (n+1) \) matrix of any size \( |\sigma| \in [0, n+1] \). Note that there are relations among minors starting from \( n = 3 \), so a basis of minors should be chosen. Equations of this form can be uniquely characterised by their defining relations.
Theorem 12 The following linear system $\hat{\mathcal{E}}$ of PDEs of the second order,

\[
\begin{align*}
F_{u_{ii}u_{ii}} &= 0, & F_{u_{i}u_{ij}} &= 0, & 2F_{u_{ij}u_{ij}} &= 0, \\
F_{u_{ij}u_{kk}} + F_{u_{ik}u_{jk}} &= 0, & F_{u_{ij}u_{kl}} + F_{u_{ik}u_{jl}} + F_{u_{ij}u_{jk}} &= 0,
\end{align*}
\]

is involutive. Its solution space is generated by Monge-Ampère equations (29) (all indices $i, j, k, l$ run from 0 to $n$ and are assumed pairwise distinct).

Proof. An elementary approach to obtain this system is to calculate second-order partial derivatives of the function $F$ defined by (29) and to eliminate the parameters $a_{\sigma}$. This works well for small $n \leq 4$ and gives the required relations (in low dimensions one can verify directly that the defining relations for $F$ of order $> 2$ are prolongations of the relations of order 2).

The necessity of these equations follows from the fact that every traveling wave reduction of a Monge-Ampère equation is a Monge-Ampère equation in lower dimensions, and the reduction to $\mathbb{R}^4(x_1, x_2, x_3, x_4)$ yields the claim.

As for the sufficiency (recall that $u$ is a function of $n+1$ arguments), let us first note that the number of equations in the system is $\hat{N}(n+1)$ where $\hat{N}(n) = n + n(n - 1) + (\frac{n}{3}) + n(\frac{n+1}{2}) + (\frac{n}{4}) = \frac{(n+3)^2}{4}$, so the 2-symbol of $\hat{\mathcal{E}}$ has dimension \( \frac{d(n+1)^2}{2} - \hat{N}(n+1) \). Since this is equal to the dimension of the $A_n$-module $\Gamma_{2\lambda_2}$, namely dim $\Gamma_{2\lambda_2}$ = $\text{dim} S^2 \Lambda^2 T - \text{dim} \Lambda^4 T = \frac{n+1}{2} \frac{(n+2)}{3}$ for dim $T = n + 1$, it follows that the symbol space coincides with $\hat{\mathcal{G}}_2$ as discussed in the previous section (the fact that $\hat{\mathcal{G}}_2$ is a subspace of the 2-symbol of $\hat{\mathcal{E}}$ follows from the necessity of the above relations).

This in turn implies that $\hat{\mathcal{E}}$ coincides with the equation (also denoted $\hat{\mathcal{E}}$) from the previous section (note that they are formally different: the equation from the previous section is given by explicit linear relations and their prolongations). Indeed, this equation is involutive by Proposition 10, in particular the first Spencer cohomology vanishes, $\hat{H}^{1,k}(\hat{\mathcal{E}}) = H^{1,k}(\hat{\mathcal{G}}) = 0$ for $k > 1$, meaning that all equations of order $k + 2$ in the system describing $\hat{\mathcal{E}}$ are obtained by $k$-differentiations of the equation given in the formulation of the theorem. This finishes the proof.

Remark: Complex (26) yields that $H^{1,1}(\hat{\mathcal{G}}) = \Gamma_{4\lambda_1}$ is an irreducible module. In particular, all equations for $F$ in Theorem 12 follow from any one of them via $A_n$-representation, cf. the Remark at the end of Section 2.2.

Remark: There is yet another form of characterising symplectic Monge-Ampère property by the system whose solutions have the same locus $F = 0$ in $J^2(\mathbb{R}^{n+1})$ as (29). Namely, if we allow arbitrary reparametrisations $F \mapsto \Psi(F)$ for a diffeomorphism $\Psi \in \text{Diff}(\mathbb{R})$, the equation takes the form $F = \text{const}$. Making this substitution in the system of Theorem 12 in 2D gives the following nonlinear system (where we again use the notation $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$):

\[
\begin{align*}
F_{rr} &= \frac{F^2_r}{2F_rF_t + F^2_s}(2F_{rt} + F_{ss}), & F_{rs} &= \frac{F_r F_s}{2F_rF_t + F^2_s}(2F_{rt} + F_{ss}), \\
F_{tt} &= \frac{F^2_t}{2F_rF_t + F^2_s}(2F_{rt} + F_{ss}), & F_{st} &= \frac{F_s F_t}{2F_rF_t + F^2_s}(2F_{rt} + F_{ss}).
\end{align*}
\]
In higher-dimensional cases the situation is similar, but equations become somewhat more cumbersome. The resulting system is of infinite type, with the general solution depending on an arbitrary function of one argument. In 2D the general solution of the above system is

\[ F = \Psi(a_0 + a_1 r + a_2 s + a_3 t + a_4 (rt - s^2)) \]

where the constants \( a_i \) and the function \( \Psi \) are arbitrary.

Since the pseudogroup \( G = \text{Diff}_{\text{loc}}(\mathbb{R}) \) acts on the above system of PDEs, it is possible to compute the quotient. One way to do this is to derive scalar differential invariants and rational syzygies between them as explained in [34]. Another possibility is to fix a cross-section of the orbit foliation by choosing a proper normalisation. Making the solution either polynomial of order \( n + 1 \) (this defines \( F \) up to multiplication by a nonzero constant) or expressing \( F = 0 \) in the evolutionary form \( u_{00} = f \) we obtain determining equations \( \mathcal{E} \) or \( \mathcal{E} \), respectively.

3 Integrability and geometry of Hirota type equations

After a few remarks on the action of the equivalence group we prove the main result about Hirota type equations in 4D. All calculations are based on computer algebra systems Mathematica and Maple (these only utilise symbolic polynomial algebra over \( \mathbb{Q} \), so the results are rigorous). The programmes are available from the arXiv version of this paper. At the end of this section we use the reduction argument to conclude that integrability in dimensions higher than four also implies the Monge-Ampère property.

3.1 Action of the equivalence group

The group \( G = \text{Sp}(2n + 2, \mathbb{C}) \) acts naturally on \( \mathbb{C}^{2n+2} = T^*(\mathbb{C}^{n+1}) \), with the stabiliser of a Lagrangian plane \( \mathbb{C}^{n+1} \subset \mathbb{C}^{2n+2} \) being a parabolic subgroup \( P \) corresponding to the Dynkin diagram

Here the cross indicates the parabolic subgroup. This leads to the homogeneous representation of the (complex) Lagrangian Grassmannian \( \Lambda = G/P \). The stabiliser of a point \( o \in \Lambda \) is \( P \), and it acts on jets of hypersurfaces \( X \) through this point. In particular, \( G \) acts on the space \( J^k_1(\Lambda) \) of \( k \)-jets of codimension 1 submanifolds \( X \subset \Lambda \) (an affine chart of this is the standard jet-space \( J^k(\mathbb{C}^d) \)) and \( P \) acts on the space \( J^k_1(\Lambda)_o \) of \( k \)-jets of codimension 1 submanifolds through \( o \).

This action on \( J^k_1(\Lambda) \) has a unique open orbit corresponding to \( 1 \)-jets of non-degenerate hypersurfaces \( X \). Indeed, this is equivalent to the uniqueness of non-degenerate linear second-order equations up to complex symplectic transformations (over \( \mathbb{R} \) there are \( \lfloor \frac{n+1}{2} \rfloor + 1 \) open orbits). This justifies our computational trick described in the introduction, namely that we can evaluate any function on \( J^k_1(\Lambda) \) by restricting it to the fibre over a non-degenerate \( 1 \)-jet.

We claim that transformations from the equivalence group \( G \) correspond to special contact transformations from a different jet-space \( J^2(\mathbb{R}^{n+1}) \) that naturally act on equations of type (1), cf. [24, 13] (at this point we are not concerned with the classification and switch to the real case). Let us clarify this in the affine chart \( U \subset S^2\mathbb{R}^{n+1} \subset \Lambda \), where the generators of the Lie algebra \( \mathfrak{g} = \text{Lie}(G) = \text{sp}(2n + 2) \) are \( X_{\alpha\beta}, L_{\alpha\beta}, P_{\alpha\beta} \) as in Section 1.2. Consider the space of 2-jets of
functions $u = u(x^0, x^1, \ldots, x^n)$, namely $J^2(\mathbb{R}^{n+1}) \simeq \mathbb{R}^{n+1+d(n+2)}$ with coordinates $(x^i, u, u_j, u_{ij})$. The algebra $g$ acts naturally on $T^* (\mathbb{R}^{n+1}) \simeq \mathbb{R}^{2n+2} (x^i, u_j)$ by linear symplectic transformations, so it is contained in the algebra of contact vector fields in $J^2$. Indeed, on restriction to fibres of the bundle $J^2(\mathbb{R}^{n+1}) \rightarrow J^1(\mathbb{R}^{n+1})$, prolongations of the point vector fields $\zeta_{\alpha\beta} = \frac{1}{1+\delta_{\alpha\beta}} x^\alpha x^\beta \partial_a$, $\eta_{\alpha\beta} = -x^\alpha \partial_\alpha$ and the contact vector fields $\zeta_{\alpha\beta} = -u_\alpha \partial_\beta - u_\beta \partial_\alpha - u_{\alpha\beta} \partial_a$ coincide with the vector fields $X_{\alpha\beta}$, $L_{\alpha\beta}$ and $P_{\alpha\beta}$, respectively. Thus, $\xi_{\alpha\beta}(2) F(U) = X_{\alpha\beta} F(U)$, $\eta_{\alpha\beta}(2) F(U) = L_{\alpha\beta} F(U)$ and $\zeta_{\alpha\beta}(2) F(U) = P_{\alpha\beta} F(U)$ for all functions $F = F(U)$ of type (1).

### 3.2 Integrability: proof of Theorem 3

In this and the next two sections we restrict to the four-dimensional case ($n = 3$).

**Equivalence** $(a) \iff (d)$. Here we apply the method of hydrodynamic reductions to a general Hirota type equation written in evolutionary form (3). Our strategy is to derive a set of constraints for the right-hand side $f$ that are necessary and sufficient for integrability. As outlined in [24], in 3D this leads to an involutive system of third-order differential constraints for $f$. The crucial difference occurring in the 4D case is the appearance, along with third-order differential constraints, of additional second-order integrability conditions that imply the Monge-Ampère property. The rest follows from the classification of integrable symplectic Monge-Ampère equations in 4D [13]. Thus, the requirement of integrability in 4D is far more rigid than that in 3D.

The proof of the implication $(a) \implies (d)$ is as follows. Representing Hirota type equation in evolutionary form (3) \(n=3\) and introducing the notations

\[
\begin{align*}
  u_{01} &= d, \quad u_{02} = r, \quad u_{03} = n, \quad u_{11} = a, \quad u_{12} = b, \quad u_{13} = c, \quad u_{22} = p, \quad u_{23} = q, \quad u_{33} = m, \\
  u_{00} &= f(d, r, n, a, b, c, p, q, m),
\end{align*}
\]

we transform (3) \(n=3\) into quasilinear form (4) \(n=3\) by adding the compatibility conditions \((u_{\alpha\beta})_\gamma = (u_{\alpha\gamma})_\beta\), i.e.

\[
d_\alpha = f_x, \quad d_\lambda = a_\lambda, \quad d_\nu = b_\nu, \quad d_\alpha = c_\alpha, \quad \text{etc.}
\]

Ansatz (5) requires that the new dependent variables $d, r, n, a, b, c, p, q, m$ are sought as functions of the phases $R_I$, $I = 1, \ldots, N$, which themselves satisfy a triple of hydrodynamic type systems (6). This implies the relations

\[
\begin{align*}
  \partial_I b &= \mu^I \partial_I a, \quad \partial_I c = \nu^I \partial_I a, \quad \partial_I d = \lambda^I \partial_I a, \\
  \partial_I r &= \lambda^I \mu^I \partial_I a, \quad \partial_I n = \lambda^I \nu^I \partial_I a, \quad \partial_I q = \mu^I \nu^I \partial_I a, \quad \partial_I p = (\mu^I)^2 \partial_I a, \quad \partial_I m = (\nu^I)^2 \partial_I a, \\
  \partial_I = \partial_{R^I}, \quad \text{no summation assumed.}
\end{align*}
\]

Furthermore, the characteristic speeds $\mu^I, \nu^I, \lambda^I$ must satisfy the dispersion relation,

\[
(\lambda^I)^2 = f_a + f_b \mu^I + f_c \nu^I + f_d \lambda^I + f_r \lambda^I \mu^I + f_n \lambda^I \nu^I + f_q \lambda^I \nu^I + f_p (\mu^I)^2 + f_m (\nu^I)^2.
\]

Differentiating the dispersion relation by $R^I$, $J \neq I$, we obtain

\[
\frac{\partial_J \mu^I}{\mu^I - \mu^J} = \frac{\partial_J \nu^I}{\nu^I - \nu^J} = \frac{\partial_J \lambda^I}{\lambda^I - \lambda^J} = B_{IJ} \partial_J a,
\]

18
(no summation) where $B_{IJ}$ are rational expressions in $\mu^I, \mu^J, \nu^I, \nu^J, \lambda^I, \lambda^J$ whose coefficients depend on partial derivatives of $f$ up to the second order (we do not present them here explicitly). Calculating consistency conditions for relations (30) we obtain the symmetry condition $B_{IJ} = B_{JI}$ (which is satisfied identically), as well as the following equations for $a$:

$$\partial_I \partial_J a = 2B_{IJ} \partial_I a \partial_J a.$$  \hspace{1cm} (33)

Finally, the consistency conditions for relations (32) and (33) take the form

$$\partial_K B_{IJ} = (B_{IK} B_{JK} - B_{IK} B_{IJ} - B_{IJ} B_{JK}) \partial_K a,$$  \hspace{1cm} (34)

$I \neq J \neq K$ (without any loss of generality one can set $I = 1, J = 2, K = 3$). Calculating the left-hand side of (34) via (30), (32), (33), and utilising the dispersion relation (31) to eliminate all higher powers of $\lambda^I$, one can reduce (34) to a polynomial expression in $\mu^I, \mu^J, \mu^K, \nu^I, \nu^J, \nu^K$, which also depends linearly on $\lambda^I, \lambda^J, \lambda^K$; note that the common factor $\partial_K a$ will cancel. Equating to zero the coefficients of this polynomial we obtain a system $S$ of third-order PDEs for $f$ – the required integrability conditions. This system will be linear in the third-order partial derivatives of $f$, and quadratic in the second-order derivatives.

In the 3D case system $S$ can be uniquely solved for all of the 35 third-order partial derivatives of $f$, resulting in the 35 integrability conditions that are in involution: there will be no second-order relations left [24]. The first remarkable phenomenon of the 4D case is that, after solving system $S$ for all of the 165 third-order partial derivatives of $f$, there will still be numerous homogeneous quadratic relations in the second-order derivatives of $f$ remaining (over 2000 quadratic relations). The second remarkable phenomenon is that the radical of the ideal generated by these quadratic relations contains all of the 25 linear (in 2-jets of $f$) relations characterising Monge-Ampère systems in 4D, see Section 2.3. This establishes the Monge-Ampère property, and thus finishes the proof of the implication $(a) \implies (d)$ due to the existing classification of integrable symplectic Monge-Ampère equations in 4D [13]. Let us note that the computation of the radical of the quadratic ideal simplifies dramatically if one gives some generic constant numerical values, see Section 1.2 for the discussion. We have chosen $f_5 = f_n = 1$, all other $f_i = 0$.

The converse implication, $(d) \implies (a)$, is a straightforward calculation based on normal forms of symplectic Monge-Ampère equations from Theorem 1, see e.g. [22, 23].

**Equivalence** $(b) \iff (d)$. The proof of the implication $(b) \implies (d)$ is based on a direct computation of the Weyl tensor of the conformal structure $[g]$. First we demonstrate that either of the half-flatness conditions, $W_- = 0$ or $W_+ = 0$, implies that the 4D equation under study must be of symplectic Monge-Ampère type. Here $W_- = \frac{1}{2}(W - \ast W)$, $W_+ = \frac{1}{2}(W + \ast W)$. As above we use the 1-jet of $f$ defined as $f_b = f_n = 1$, all other $f_i = 0$. Let us substitute this 1-jet into one of the half-flatness conditions, say $W_- = 0$, reduce it modulo (3), and equate to zero coefficients at the fourth-order derivatives $u_{ijkl}$. This will give a linear system in the 2-jet of $f$ (30 linear relations altogether). A direct verification shows that this linear system implies all of the 25 conditions characterising Monge-Ampère equations in 4D (in which we substitute the same 1-jet of $f$). For $W_+ = 0$ considerations are essentially the same. Thus, we have established the Monge-Ampère property.
Furthermore, due to the half-flatness of \( g \), equation (1) possesses a dispersionless Lax pair [9]. Thus, any travelling wave reduction of this equation to 3D is a symplectic Monge-Ampère equation possessing a Lax pair; hence, the reduction must be linearisable. The rest follows from the classification of integrable symplectic Monge-Ampère equations in 4D possessing linearisable travelling wave reductions [13]. Let us note that the half-flatness conditions, \( W_+ = 0 \) and \( W_- = 0 \), are not equivalent: in fact, only one of them leads to integrable Monge-Ampère equations, while the other one is much more overdetermined, and implies linearisability. There is however no invariant way to distinguish between them (due to the lack of a canonically defined orientation), so we just state that conformal half-flatness implies the Monge-Ampère integrability.

The converse implication, \((d) \implies (b)\), is a straightforward computation based on normal forms of symplectic Monge-Ampère equations from Theorem 1: it was explicitly noted in Section 8 of [25].

Equivalence \((b) \iff (c)\).
This is a particular case of the general result of [9] relating self-duality of the conformal structure \([g]\) to the existence of a dispersionless Lax pair.

Let us give a few more details on the implication \((c) \implies (b)\). The main technical result of [9] is that any nontrivial Lax pair must be characteristic, i.e. null with respect to the conformal structure. For every solution \(u\) of (1) the congruence of null two-planes defined by the Lax pair uniquely lifts into the correspondence space \(\hat{M}_u \to M_u\), where \(M_u = \text{graph}(u) \subset \mathbb{R}^5(x^1, x^2, x^3, x^4, u)\) and \(\hat{M}_u \simeq M \times \mathbb{P}(\lambda)\) is the bundle of null \(\alpha\)-planes (self-dual 2-planes). The corresponding 2-distribution in \(\hat{M}_u\) is Frobenius-integrable for every solution \(u\), and thus by projection we obtain a 3-parameter family of \(\alpha\)-surfaces, i.e. totally null surfaces of the conformal structure \([g]\) on \(M_u\). According to Penrose [48] this is equivalent to self-duality.

The converse implication is easily seen by transitivity, \((b) \Rightarrow (d) \Rightarrow (c)\). Indeed, since \((b) \implies (d)\) is already established, the claim follows from the fact that all equations from Theorem 1 are known to possess dispersionless Lax pairs, see e.g. [13].

This finishes the proof of Theorem 3.

3.3 Linearisability: proof of Theorem 4

It is clear that if a second-order PDE is linearisable (more precisely, transformable to a constant-coefficient linear form) by a contact transformation, then the corresponding conformal structure is flat on every solution. Conversely, suppose that the Weyl tensor \(W\) vanishes on every solution. Then also \(W_- = 0\), so the conformal structure is self-dual on every solution. Therefore, by Theorem 3 the equation must be integrable, and of symplectic Monge-Ampère type. Moreover, up to a transformation from the equivalence group \(\text{Sp}(8)\) it reduces to one of the six normal forms from Theorem 1. A straightforward computation shows that \(W \neq 0\) on a generic solution for the last five equations from the list. Thus, the equation must be of the first type, and hence linearisable.

Corollary: Hirota type equation (1) is linearisable by a transformation from the equivalence group \(\text{Sp}(8)\) if and only if it is linearisable by a contact transformation.
3.4 Symmetry: proof of Theorem 5

For each of the integrable symplectic Monge-Ampère equations from Theorem 1, their symmetry algebras \( \mathfrak{s} \) inside \( \mathfrak{sp}(8) \) were computed in [13]. The full (infinite-dimensional) contact symmetry algebras \( \mathfrak{sym} \) of the same equations were computed in [36]. However, in neither of these references the Lie algebra structure of \( \mathfrak{s} = \mathfrak{sym} \cap \mathfrak{sp}(8) \) was investigated. Here we fill this gap by a straightforward application of the LieAlgebras package of Maple and the standard Lie theory.

The following Table summarises the results. Note that we have changed the linear hyperbolic equation from the list of Theorem 1 to the ultra-hyperbolic form \( u_{00} + u_{11} - u_{22} - u_{33} = 0 \) with the conformal structure of signature \((2,2)\): for this signature the null planes are real. Though the classification in Theorem 1 is over \( \mathbb{C} \), we provide finer Lie algebra structures over \( \mathbb{R} \), writing \( \mathfrak{sl}_2 = \mathfrak{sl}(2,\mathbb{R}) \) and so on. The results over \( \mathbb{C} \) are obtained by complexification using \( \mathfrak{so}(2,2)^\mathbb{C} = \mathfrak{so}(4,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C})^\mathbb{C} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \), etc.

| Equation | dim(\( \mathfrak{s} \)) | Levi decomposition of \( \mathfrak{s} \) |
|----------|----------------------|-------------------------------------|
| Linear ultrahyperbolic | 16 | \( \mathfrak{s} = \mathfrak{so}(2,2) \times S^2 \mathbb{R}^4 \approx \mathfrak{so}(2,2) \times S^2 \mathbb{R}^4 \) |
| Second heavenly | 14 | \( \mathfrak{s} = \mathfrak{sl}_2 \times \mathfrak{ta}_0 \); \( \mathfrak{ta}_0 = \mathbb{R}^2 + V_1 + V_2 + V_3 \) as \( \mathfrak{sl}_2 \)-module \( \langle \mathbb{R}^2 \rangle \) is a trivial module; \( V_i \) are 3D irreps; As Lie algebra: \( \mathbb{R}^2 = \mathfrak{so}_2 \langle \mathfrak{s}, \mathfrak{t} : [\mathfrak{s}, \mathfrak{t}] = \mathfrak{t} \rangle \), \( \text{ad}_s | V_i = 0 \), \( \text{ad}_t | V_i = V_i+1 \), \( [V_i, V_j] = V_{i+j} \) |
| Modified heavenly | 13 | \( \mathfrak{s} = \mathfrak{sl}_2 \oplus (\mathfrak{sl}_2 \times \mathfrak{ta}_0) \), where \( \mathfrak{ta}_0 = \mathbb{R} + V_1 + V_2 \) as \( \mathfrak{sl}_2 \)-module \( \langle \mathbb{R} \rangle \) is a trivial module; \( V_i \) are 3D irreps; As Lie algebra: \( \mathbb{R} = \langle \mathfrak{s} \rangle \), \( \text{ad}_s | V_i = 0 \), \( [V_i, V_j] = V_{i+j} \) |
| First heavenly | 13 | \( \mathfrak{s} = (\mathfrak{sl}_2 \oplus \mathfrak{sl}_2) \times \mathfrak{ta}_0 \), where \( \mathfrak{ta}_0 = \mathbb{R} + V_1 + V_2 \) as \( \mathfrak{sl}_2 \)-module \( \langle \mathbb{R} \rangle \) is trivial; \( V_i/V_2 \) are 3D irreps of the first/second copy of \( \mathfrak{sl}_2 \); As Lie algebra: \( \mathbb{R} = \langle \mathfrak{s} \rangle \), \( \text{ad}_s | V_i = (-1)^i \), \( [V_i, V_j] = 0 \) |
| Husain equation | 12 | \( \mathfrak{s} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2 \times \mathfrak{V} \) where \( \mathfrak{V} \) is a 3D irrep |
| General heavenly | 12 | \( \mathfrak{s} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \) |

It is apparent from the Table that the symmetry algebra of every equation contains \( \mathfrak{sl}_2 \), and that the minimal dimension of the symmetry algebra is 12. It was shown in [13] that each equation-manifold \( X^9 \subset \Lambda^{10} \) contains a subvariety \( X^4 \) along which \( X^9 \) is singular. Thus, \( X^4 \) is intrinsic to the problem and so is invariant under the symmetry group \( \mathcal{S} \). Therefore the action of \( \mathcal{S} \) on \( X^9 \) is not transitive.

**Proof of Theorem 5.** Since the symmetry generators are known explicitly, it is straightforward to verify that the action of the symmetry algebra \( \mathfrak{s} \) has an open orbit. As the action is algebraic and \( X \) is irreducible, it is a Zariski open orbit. Moreover, singular orbits form an algebraic stratified submanifold of \( X^9 \) of positive codimension. Since such submanifolds do not separate domains in \( X^9 \), there is precisely one Zariski open orbit of the symmetry group \( \mathcal{S} \). \( \square \)

Two remarks about this proof are in order. First, the set of singular orbits is strictly bigger than the singular variety \( X^4 \subset X^9 \). Second, in the complex case the unique Zariski open orbit is connected in the usual topology. However, when doing classification over \( \mathbb{R} \) (in this case the classification from Theorem 1 will contain more normal forms), the set of regular points can be
topologically disconnected. For instance, for the modified heavenly equation the rank of vector fields from $s$ drops to $8$ on the hypersurface $\{u_{03} = 0\} \subset X^9$, and this hypersurface separates $X^9$ into two open pieces (in the set-theoretic topology).

### 3.5 Integrability in higher dimensions: proof of Theorem 6

The Monge-Ampère property of higher-dimensional integrable Hirota type equations is a direct consequence of the analogous result in 4D. First of all, a generic 4D traveling wave reduction of a multi-dimensional integrable non-degenerate Hirota type equation will also be non-degenerate and integrable, and hence of the symplectic Monge-Ampère type by Theorem 3. In particular, all generic traveling wave reductions to 2D will be of the symplectic Monge-Ampère type. Thus, as noted in the Remark at the end of Section 2.2 (and also in the Remark at the end of Section 2.4), the equation will satisfy all relations (18)-(24), and therefore will itself be of the symplectic Monge-Ampère type by Theorem 2.

**Remark:** Although the classification of higher-dimensions integrable equations is still open, the lack of non-trivial examples makes it tempting to conjecture that all multi-dimensional (5D and higher) non-degenerate integrable Hirota type equations must be linearisable. We emphasize that the well-known integrable 6D version of the second heavenly equation [60, 51],

$$u_{15} + u_{26} + u_{13}u_{24} - u_{14}u_{23} = 0,$$

does not constitute a counterexample to this conjecture: the corresponding symmetric bi-vector $\partial_1\partial_5 + \partial_2\partial_6 + u_{13}\partial_2\partial_4 + u_{24}\partial_1\partial_3 - u_{14}\partial_2\partial_3 - u_{23}\partial_1\partial_4$ has rank 4, and therefore the characteristic variety of the equation (and hence the equation itself) is degenerate.

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