Oscillations of simple networks

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Abstract

To describe the flow of a miscible quantity on a network, we introduce the graph wave equation where the standard continuous Laplacian is replaced by the graph Laplacian. This is a natural description of an array of inductances and capacities, of fluid flow in a network of ducts and of a system of masses and springs. The structure of the graph influences strongly the dynamics which is naturally described using the basis of the eigenvectors. In particular, we show that if two outer nodes are connected to a common third node with the same coupling, then this coupling is an eigenvalue of the Laplacian. Assuming the graph is forced and damped at specific nodes, we derive the amplitude equations. These are analyzed for two simple non trivial networks: a tree and a graph with a cycle. Forcing the network at a resonant frequency reveals that damping can be ineffective if applied to the wrong node, leading to a disastrous resonance and destruction of the network. These results could be useful for complex physical networks and engineering networks like power grids.
1 Introduction

The flow of a scalar quantity in a network is an important problem for fundamental science and engineering applications. The latter include gas, water or power distribution networks. Other examples are a simplified version of road traffic and the flow of nerve impulse in the brain. The static aspect of the problem has long been studied within the framework of operations research, see for example [1]. However in many cases the dynamic character is crucial. Take for example the prediction of traffic jams in a given road network or the prediction of a flood in a river basin.

The basic equations describing the flow of a miscible quantity are the well-known conservation laws of mathematical physics. These laws are building blocks for studying physical systems. They are universal and can be found in mechanics, in electromagnetism .. Each conservation law has the form of a flux relation

$$u_t + \partial_x q = 0,$$

where the first term is the time derivative of a density $u$ and the second one is the space derivative of the flux $q$ along the relevant coordinate. The most important conserved quantities in mechanics are the mass, momentum and energy. In electromagnetism the current and voltage obey conservation laws.

To describe the flow of a given quantity on a network it is natural to try and generalize these conservation laws. For that we introduce the graph representing the network and the generalized gradient $\nabla$ or its transpose, the incidence matrix. To write this it is important to orient the branches of the graph in a fixed way. This can be arbitrary. An important class of flows are the ones such that there is no dissipation along the branches but only at the vertices. A typical example is a small power grid for which the power line dissipation can be neglected and where the only power input and outputs occur at given nodes. For some models it is possible to reduce the dynamics to what has been called a graph wave equation by Friedman and Tillich [2]. Here the usual Laplacian is replaced by its discrete analog the graph laplacian $\nabla^T \nabla$.

Here we show how the conservation laws lead to the graph wave equation. We illustrate this in three different physical contexts: a network of inductances and capacities, its equivalent mechanical analog represented by masses and springs and an array of fluid ducts. For the latter, note the study by Maas [3] who
considered graphs obtained by linking elementary graphs. He established in particular inequalities for the first non zero eigenvalue of the Laplacian of these graphs. Note also that the static problem of gas or fluid transport is usually solved using optimisation techniques [4]. The graph laplacian was also considered recently by Burioni et al for the thermodynamics of the Hubbard model to describe static configurations of a Josephson junction array [5]. Another problem considered by that group is to use the graph as a controlled obstacle to obtain a desired reflection of discrete nonlinear Schrödinger solitons [6]. Here we adopt a different point of view. We consider that the network is fixed and is submitted to forcing and damping on specific nodes. This formulation is particularly interesting because the graph Laplacian i.e. the spatial part of the equation is a symmetric matrix so that its eigenvalues are real and its eigenvectors are orthogonal. It is then natural to describe the dynamics of the network by projecting it on the basis of the eigenvectors. Here we examine these normal modes for two simple graphs, a tree and a graph with a cycle. We write down amplitude equations for the normal modes of the system. Finally we force the graph on a given node and damp it on another. We choose the forcing frequency so that the system resonates. We illustrate two specific effects: first that damping can be ineffective if applied to the wrong node. The second is that we can have a multiple eigenvalue corresponding to different eigenvectors. Then depending how the system is excited we can get a different response. We consider a periodic forcing for simplicity. The result for other types of solicitations can be derived from this study using a superposition argument because the system is linear.

The article is organized as follows, section 2 presents the derivation of the graph wave equation in different physical contexts. In section 3 we compute the eigenmodes for two specific simple graphs, a tree and a graph with a cycle. Section 4 introduces the equations for the amplitudes of the normal modes when the graph is forced. These equations are analyzed in section 5. There we force the simple graphs of section 3 at resonance and analyze their response. We conclude in section 6.
Figure 1: Schematic drawing of the complex graph oscillator. The arrows on the branches are oriented arbitrarily.

2 The model: graph wave equation

We now introduce the basic notions from graph theory following the presentation of [7]. A graph $\mathcal{G}(V, E)$ is the association of a vertex set $V$ and an edge set $E$ where an edge is an unordered pair of distinct vertices. We assume the vertices and edges to be numbered $V = \{1, 2, \ldots, n\}$ and $E = \{1, 2, \ldots, m\}$. The latter are oriented with an arbitrary but fixed orientation. We consider for simplicity only simple graphs which do not have multiple edges. Fig. 1 shows such a graph with $n = 4$ vertices and $m = 3$ edges. The basic tool for expressing a flux is the so-called incidence matrix $C(n, m)$ defined as

\begin{align*}
C_{xe} &= -1 \quad \text{edge } e \text{ starts from vertex } x, \quad \text{(2)} \\
C_{xe} &= 1 \quad \text{edge } e \text{ finishes at vertex } x, \quad \text{(3)} \\
C_{xe} &= 0 \quad \text{otherwise} \quad \text{(4)}
\end{align*}

For the example shown in Fig. 1 we have

\[
\begin{pmatrix}
-1 & 0 & 0 \\
1 & -1 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The transpose $C^T = \nabla$ is a discrete differential operator (gradient of graph). To see this consider a function $f : V \rightarrow \mathbb{R}$. The vector $(\nabla f)(e)$ is the difference of the values of $f$ at the end points of vertex $e$ (with orientation). In the example
above, we have
\[ \nabla f = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} f_2 - f_1 \\ f_3 - f_2 \\ f_4 - f_2 \end{pmatrix}, \tag{5} \]
which is the discrete gradient of \( f \) associated to the graph.

We now consider the specific inductance-capacity electrical network shown in the left panel of Fig. 2. The equations of motion in terms of the (node) voltages and (branch) currents are the conservation of current and voltage
\[ Cv_t - \nabla^T i = s, \tag{6} \]
\[ Li_t - \nabla v = 0, \tag{7} \]
where
\[ C = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix} \]
are respectively the diagonal matrix of capacities and the diagonal matrix of inductances and \( s \) represents the currents applied to each vertex. (similar to boundary conditions in the continuum case). Note that the equations (6) also describe in the linear limit, shallow water waves in a network of canals and fluid flow in a pipe network. In the first situation \( v \) corresponds to a surface elevation and \( q \) to a flow. For the second example \( v \) is the pressure and \( q \) the flow. From the two equations (6) one obtains the generalized wave equation. For this take the derivative of the 1st equation and substitute the second to obtain
\[ Cv_{tt} - \nabla^T L^{-1} \nabla v = s_t, \tag{8} \]
where
\[ \nabla^T L^{-1} \nabla = \begin{pmatrix} L_1^{-1} & -L_1^{-1} & 0 & 0 \\ -L_1^{-1} & L_1^{-1} + L_2^{-1} + L_3^{-1} & -L_2^{-1} & -L_3^{-1} \\ 0 & -L_2^{-1} & L_2^{-1} & 0 \\ 0 & -L_3^{-1} & 0 & L_3^{-1} \end{pmatrix} \tag{9} \]
A similar equation arises when describing the other physical systems shown in the right panel of Fig. 2, the collection of four masses \( m_i, \ i = 1 - 4 \) connected
Figure 2: Two different physical networks corresponding to the graph shown in Fig. 1. The left panel shows a network of inductances and capacities and the right panel shows the mechanical analog in terms of masses and springs by springs of stiﬀnesses $k_i$, $i = 1 − 3$. Here the variables are the displacements $y_i$, $i = 1 − 4$ of each mass. The equations of motion are

$$
\begin{align*}
    m_1 \ddot{y}_1 &= k_1 (y_2 - y_1) \\
    m_2 \ddot{y}_2 &= k_1 (y_1 - y_2) + k_2 (y_3 - y_2) + k_3 (y_4 - y_2) \\
    m_3 \ddot{y}_3 &= k_2 (y_2 - y_3) \\
    m_4 \ddot{y}_4 &= k_3 (y_2 - y_4).
\end{align*}
$$

Notice the correspondence capacities / masses and inverse inductances / stiﬀnesses. The matrix on the right hand side is symmetric. This symmetry has important consequences as shown below.

In the following we will consider the graph shown in Fig. 3 which includes an additional branch between nodes 3 and 4. We assume that the masses are equal, that $k_1 = k_3 = k$, $k_2 = \alpha^2 k$ and $k_4 = \beta^2 k$. 

Figure 3: The specific network that will be studied throughout the article.

We normalize times by the natural frequency
\[
\omega = \sqrt{\frac{k}{m}}, \quad t' = \omega t.
\] (10)

Omitting the primes, the equations can be written in matrix form as
\[
\begin{pmatrix}
\ddot{y}_1 \\
\ddot{y}_2 \\
\ddot{y}_3 \\
\ddot{y}_4
\end{pmatrix} =
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
1 & -2 - \alpha & \alpha & 1 \\
0 & \alpha & -\alpha - \beta & \beta \\
0 & 1 & \beta & -1 - \beta
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix},
\] (11)

which we will write formally as
\[
Y_{tt} = GY,
\] (12)

where \(G\) is the graph Laplacian \([7]\), the equivalent of (9). The matrix \(G\) is symmetric. If we had assumed different masses on the nodes, we would have lost this property. For electrical networks this corresponds to the same capacity. In the rest of the article we keep the masses the same.

Notice that \(G\) is a singular matrix since the sum of its lines (resp. columns) gives a 0 line (resp. column). Therefore \(G\) will have a zero eigenvalue which corresponds to the Goldstone mode. The solutions of the linear system \(GY = S\) are given up to a constant. The linear evolution problem (12) gives rise to periodic solutions \(Y(t) = Z \exp i\omega t\) where the \(Z\) verify the spectral problem
\[
GZ = -\omega^2 Z.
\] (13)

Since the matrix \(G\) is symmetric, its eigenvalues are real and the eigenvectors are orthogonal. They provide a basis of \(R^n\) which is adapted to describe the
evolution of $Y$ on the graph. Specifically we arrange the eigenvalues $\lambda_i = -\omega_i^2$ as
\[
|\lambda_1| = 0 \leq |\lambda_2| \leq \cdots \leq |\lambda_n|.
\] (14)
We label the associated eigenvectors $v_1, v_2, \ldots, v_n$. These verify
\[
Gv_i = -\omega_i^2 v_i.
\] (15)
and are orthogonal with respect to the standard scalar product. We then normalize them so $<v_i, v_j> = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker symbol.

It is natural to write the equation of motion (12) in terms of the amplitudes of the normalized eigenvectors $v_i$, $Y = \sum_i \alpha_i v_i$. We obtain the standard result
\[
\ddot{\alpha}_i + \omega_i^2 \alpha_i = 0,
\] (16)
so that the normal modes do not exchange energy. The problem is then Hamiltonian with
\[
H = \frac{1}{2} \sum_i \dot{\alpha}_i^2 + \omega_i^2 \alpha_i^2.
\] (17)
When a perturbation acts on the system the equations need to be modified. This is the object of section 4.

In this article, to make things precise, we have chosen to study the network shown in Fig. 3. The dynamics on this simple yet not trivial example already shows some specific effects like the influence of damping and the multiplicity of eigenvectors for a given eigenvalue. When $\beta = 0$ the graph is called a tree (a connected graph with no cycle). We will consider this situation first. We will see that for a graph with a cycle ($\beta \neq 0$) the dynamics will be quite different than for the tree.

## 3 Computation of the eigenmodes

### 3.1 The case of a tree

Throughout this section we assume that the branch 4 is absent so that $\beta = 0$. Let us first analyze the degenerate case $\alpha = 1$. Then the eigenvalues are
\[
\lambda_4 = -4 \quad \lambda_3 = -1 \quad \lambda_2 = -1 \quad \lambda_1 = 0.
\] (18)
As expected we have the Goldstone mode $\lambda_1 = 0$ with its flat eigenvector $(1,1,1,1)$. The eigenvector associated to $\lambda_3 = \lambda_2$ are $(1,0,0,-1)$ and $(0,0,1,-1)$. The former is the antisymmetric mode which will be preserved when perturbing the graph around $y_3$. The eigenvector for $\lambda_4 = -4$ is $(1,-3,1,1)$. The eigenvalue -1 is double. The case $\alpha = 1$ is a special situation unlikely to occur for real systems for which one branch will always have a different stiffness from the other. For this reason we will assume $\alpha \neq 1$.

For a general $\alpha$ the eigenvalues and eigenvectors can be computed analytically. As previously we find the Goldstone mode $\lambda_1 = -\omega_1^2 = 0$ with the usual eigenvector. The other eigenvalues and eigenvectors are

$$\lambda_4 = -\omega_4^2 = -\frac{1}{2} \left( \sqrt{4\alpha^2 - 4\alpha + 9 + 2\alpha + 3} \right), \quad (19)$$

$$v_4^T = \left( 1, -\frac{1}{2} \left( \sqrt{4\alpha^2 - 4\alpha + 9 + 2\alpha + 1} \right), \frac{1}{2} \left( \sqrt{4\alpha^2 - 4\alpha + 9 + 2\alpha - 3} \right), 1 \right), \quad (20)$$

and

$$\lambda_3 = -\omega_3^2 = \frac{1}{2} \left( \sqrt{4\alpha^2 - 4\alpha + 9 - 2\alpha - 3} \right), \quad (21)$$

$$v_3^T = \left( 1, \frac{1}{2} \left( \sqrt{4\alpha^2 - 4\alpha + 9 - 2\alpha + 1} \right), -\frac{1}{2} \left( \sqrt{4\alpha^2 - 4\alpha + 9 - 2\alpha + 3} \right), 1 \right), \quad (22)$$

and

$$\lambda_2 = -\omega_2^2 = -1, \quad (23)$$

$$v_2^T = (1,0,0,-1). \quad (24)$$

The frequencies $\omega_i$ are plotted as a function of the parameter $\alpha$ in Fig. 4. Note how the eigenvalue $\omega_2$ is independent of $\alpha$, $\omega_2 = 1$. We will explain this lower in the section.
Figure 4: Plot of the eigenvalues $\omega_i$, $i = 1 - 4$ as a function of $\alpha$ for the graph shown in Fig. 3 with $\beta = 0$ i.e. a tree.

Notice how the mode $v_2$ is unchanged, it is shown schematically in Fig. 5.

Figure 5: Schematic representation of the constant mode $v_2$, corresponding to $\omega_2 = 1$

Fig. 6 shows the plot of the components of the non trivial eigenvectors as a function of $\alpha$. Note how the nodes 1, 2 and 4 are in phase for $v_3$ for $\alpha < 1$. For $\alpha > 1$ node 2 becomes out of phase with nodes 1 and 4. Interestingly for $\alpha = 1$ the second component of $v_3 = 0$, this will have important consequences for the dynamics as we will see below. The eigenvector $v_4$ (right panel of Fig. 6) is simpler, nodes 1, 3 and 4 are always in phase. The modes $v_2$ and $v_3$ evolve continuously from $\alpha = 0.1$ to 0.9. Only the amplitudes change. When $\alpha = 5$. 

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the mode $v_2$ has changed, the nodes 2 and 3 are in phase while the nodes 1 and 4 are out of phase.

Figure 6: Plot of the components of the non trivial eigenvectors $v_3$ (left panel) and $v_4$ (right panel) as a function of $\alpha$. 

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Figure 7: Schematic representation of the non trivial eigenvectors $v_3, v_4$. The top panel is for $\alpha = 0.1$. It shows the modes $v_3$ corresponding to $\omega_3 = 0.36$ (left) and $v_4$ corresponding to $\omega_4 = 1.75$ (right). The middle panel is for $\alpha = 0.9$ and shows the modes $v_3$ corresponding to $\omega_3 = 0.964$ (left) and $v_4$ corresponding to $\omega_4 = 1.9671$ (right). The bottom panel corresponds to $\alpha = 5$. It shows the modes $v_3$ corresponding to $\omega_3 = 1.335$ (left) and $v_4$ (right) corresponding to $\omega_4 = 3.35$. 
The Figs. 5 and 7 show a schematic representation of the non trivial eigenvectors for \( \alpha = 0.1, 0.9 \) and 5.

\[\text{3.2 A general result for the eigenvalue 1}\]

The tree studied in the previous section has a Laplacian whose eigenvalue is equal to 1 corresponding to an eigenvector independent of \( \alpha \). This is a general property for any graph such that two outer nodes (termed leaves in graph theory) are connected to a common node with the same coupling. Let us assume this coupling is 1 for now.

We consider the situation shown in Fig. 8 where the outer nodes 1 and 2 are connected to a common node 3 which has \( p \) connections to the rest \( G' \) of the graph.

Figure 8: A graph such that two outer nodes are connected to a common node with the same coupling.

The characteristic polynomial associated to the Laplacian matrix \( G \) is

\[
\det(G - \lambda I) = \begin{vmatrix}
1 - \lambda & 0 & -1 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 1 - \lambda & -1 & 0 & \ldots & \ldots & \ldots & 0 \\
-1 & -1 & p + 2 - \lambda & -1 & \ldots & -1 & 0 & \ldots & 0 \\
0 & 0 & -1 & | & | & | & | & | \\
| & | & | & | & | & | & | \\
| & | & -1 & | & | & | & | \\
| & | & | & 0 & | & | & | \\
| & | & | & | & 0 & | & | \\
0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0
\end{vmatrix}.
\]

(25)
Adding all the lines to the first line, we obtain $-\lambda$ for all coefficients of the line. Factoring it, we get

$$\det(G - \lambda I) = -\lambda$$

We then add the first line to the third line to obtain

$$\det(G - \lambda I) = -\lambda$$

We can then expand using the first column and the second column successively so that the factor $(1 - \lambda)$ appears in $\det(G - \lambda I)$.

Remark that if the couplings between nodes 1 and 3 and nodes 1 and 2 are equal to $\alpha$ instead of 1, we would get $\alpha$ as an eigenvalue.

Using a similar argument it can be shown that if $k$ leaves are connected to a common node with the same coupling $\alpha$ then $\alpha$ is an eigenvalue of multiplicity $k - 1$.

The eigenvector associated to the eigenvalue 1 (or $\alpha$) is such that $x_3 = 0$. This can be seen by inspection of the first line of $(G - \lambda I)$, see formula (25).

The third line of $(G - \lambda I)$ shows that

$$\sum_{i \in \Gamma(3)} x_i = 0,$$

where $\Gamma(3)$ is the set of the nodes adjacent to node 3.
3.3 The graph with a cycle

We now add branch 4 to the graph and form a cycle. We assume for simplicity $\alpha = \beta$. For this more complex graph, the eigenvalues and eigenvectors must be computed numerically. We have used Matlab. As for the tree case we have chosen $\alpha = 0.1, 0.9$ and 5. The plot of the eigenvalues as a function of $\alpha$ is shown in Fig. 9.

![Plot of the eigenvalues](image)

Figure 9: Plot of the eigenvalues $\omega_i$, $i = 1 - 4$ as a function of $\alpha$ for the graph shown in Fig. 3 with $\alpha = \beta$ i.e. a graph with a cycle.

As expected from the theory [8], the eigenvalues of the graph with a cycle $\mu$ and the eigenvalues of the tree $\lambda$ are interlaced such that

$$0 = \lambda_1 = \mu_1 \leq \lambda_2 \leq \mu_2 \leq \lambda_3 \leq \mu_3 \leq \lambda_4 \leq \mu_4.$$  \hfill (26)

Note that $\lambda_4 = \mu_4$ for $\alpha < 1$. The fact that $\lambda_2$ and $\lambda_3$ are close confines $\mu_2$. This has important consequences for the dynamics.

The dependency of the eigenvectors on the coupling parameter $\alpha$ is shown in Fig. 10. In the next section, we study it in detail and compare it to the one for the tree.
Figure 10: Plot of the components of the non trivial eigenvectors $v_2, v_3, v_4$ from top to bottom as a function of $\alpha$. 
3.4 Comparison of the eigenmodes between the tree and the graph

Let us compare the modes for different values of the coupling parameter $\alpha$. Consider the small value $\alpha = 0.1$. For the mode 2, the tree and the graph with a cycle show similar eigenmodes 2 and 3. For the 4th eigenmode the cycle breaks the symmetry between the nodes 1 and 4 but this change remains small as shown in table 4.

![Diagram](image)

Figure 11: Strong coupling $\alpha = 0.9$. Comparison of the eigenmodes 2, 3 and 4 from left to right for the tree (top panels) and the graph with a cycle (bottom panels).

The modes 2 3 and 4 for $\alpha = 0.9$ are shown in Fig. 11 for the tree (top panels) and for the graph with a cycle (bottom panels). For this value of $\alpha$ the branches 2 and 4 of the graph with a cycle are almost equivalent so that we expect a large difference between the modes for the graph with a cycle and the modes for the tree. This is indeed the case. The modes 3 are close as shown by the frequencies. On the contrary the mode 2 for the graph with a cycle only involves nodes 3 and 4, its frequency is almost double the one for the mode 2 for the tree. The mode 4 for the graph with a cycle couples nodes 3 and 4 together in opposition to node 1 and its frequency is almost equal to the one for the tree.

Finally, we discuss the high value of the coupling $\alpha = 5$. Fig. 12 shows the modes 2, 3 and 4 for the tree (top panels) and for the graph with a cycle (bottom panels). The mode 2 has a frequency of 1.3 for the tree and 3.90 for
the graph with cycle. There the link between nodes 3 and 4 is inactive. For the
mode 3 activity switches from the node 3 to the node 4 as one adds the link
between 3 and 4. Finally note how the mode 4 is modified for the graph with a
cycle. The nodes 3 and 4 are now strongly coupled and oscillate in phase. The
frequency 1.10 remains close to 1.

Figure 12: \( \alpha = 5 \), comparison of the eigenmodes 2, 3 and 4 from left to right
for the tree (top panels) and the graph with a cycle (bottom panels).

4 Forcing the network: amplitude equations

For a general (nonlinear) evolution problem of the form

\[ Y_{tt} = GY + N(Y), \]  

(27)

it is natural to expand \( Y \) using the eigenvectors as

\[ Y(t) = a_1(t)v_1 + a_2(t)v_2 + \cdots + a_n(t)v_n. \]  

(28)

Inserting (28) into (27) and projecting on each mode \( v_i \) we get the system of
coupled equations

\[ a_{1tt} + \omega_1^2 a_1 = < N(Y)v_1 >, \]  

(29)

\[ a_{2tt} + \omega_2^2 a_2 = < N(Y)v_2 >, \]  

(30)

\[ \cdots \]  

(31)

\[ a_{ntt} + \omega_n^2 a_n = < N(Y)v_n >. \]  

(32)
We expect this decomposition to be more adapted to describe the dynamics of \( Y \) on the graph. In particular it should explain some of the unexpected couplings that are observed between the modes.

Let us now assume that the network is forced at some node \( n_f \) and damped at some node \( n_d \). The motion can be represented as

\[
Y_{tt} = GY + M_d (-dY_t) + M_f f 1, \quad (33)
\]

where the \((n,n)\) matrices \( M_d, M_f \) are everywhere 0 except for \( M_d(n_d,n_d) = 1, M_f(n_f,n_f) = 1 \). The vector \( 1 = (1,1,\ldots,1)^T \). Assuming the linear combination for \( Y \) \((28)\) and projecting we get

\[
\ddot{a}_j = -\omega_j^2 a_j - d \sum_{k=1}^{n} < M_d v_k | v_j > \dot{a}_k + f < M_f 1 | v_j > . \quad (34)
\]

In terms of components we obtain

\[
\ddot{a}_j = -\omega_j^2 a_j - d v_{j,n_d} \sum_{k=1}^{n} v_{k,n_d} \dot{a}_k + f v_{j,n_f}, \quad j = 1, n \quad (35)
\]

where \( v_{j,n_d}, v_{j,n_f} \) are respectively the \( n_d, n_f \) components of the normal vector \( v_j \). From these equations one can see that exciting one node will cause disturbances to propagate all through the network in a precise way. The forcing will act on mode \( j \) only if the component \( v_{j,n_f} \neq 0 \). The damping of mode \( j \) will be effective only if \( v_{j,n_d} \neq 0 \). In the next section we will show examples of the dynamics of the network when it is excited periodically at a given node and damped at another. We will examine this for both a tree \( \beta = 0 \) and a graph with a closed cycle.

5 Numerical results: forcing the network

As an example of the dynamics of the network, we consider that the graph is forced periodically at a given node \( n_f \) and damped at a node \( n_d \), this happening for a time duration \( 100 < t < 300 \). This could model an electrical power grid where the power stations input energy. The nodes where damping occurs correspond to cities where the energy is absorbed.

As seen above the coordinates of the eigenvectors will determine whether a given node will contribute to the mode amplitude or not. So a first interesting
effect is that damping can be completely ineffective if it is applied to the wrong
node. To illustrate this effect we choose a tree with $\alpha = 0.1$, $\beta = 0$ and the
same graph but with a cycle so that $\alpha = 0.1$, $\beta = 0.1$. The first set of figures
correspond to exciting the tree on node 4 and damping it on node 1 and on
node 2. For the damping on node 1 the corresponding amplitude equations are

\[
\ddot{a}_1 = d(-0.25\dot{a}_1 - 0.15\dot{a}_3 - 0.2\dot{a}_4 - 0.35\dot{a}_2) + 0.5f, 
\]

\[
\ddot{a}_2 + a_2 = d(-0.35\dot{a}_1 - 0.21\dot{a}_3 - 0.29\dot{a}_4 - 0.5\dot{a}_2) - 0.7f, 
\]

\[
\ddot{a}_3 + 0.13a_3 = d(-0.15\dot{a}_1 - 0.1\dot{a}_3 - 0.12\dot{a}_4 - 0.21\dot{a}_2) + 0.3f, 
\]

\[
\ddot{a}_4 + 3.1a_4 = d(-0.2\dot{a}_1 - 0.12\dot{a}_3 - 0.16\dot{a}_4 - 0.28\dot{a}_2) + 0.4f. 
\]

The 2nd equation is resonant if $f = \sin t$ ($\omega = 1$) but it is damped. This
gives rise to an increase of the amplitude of mode 2 with a saturation as shown
in the left panel of Fig. 13. The same happens if the node 4 is damped. The
final value of $a_2$ can be obtained by a Green’s function approach on the 4th
differential equation. This equation can be written

\[
\ddot{a} + \omega_0^2a = f(t) - d\dot{a}. 
\]

The Green’s function $G(t - t_0)$ solves

\[
\ddot{G} + \omega_0^2G + d\dot{G} = \delta(t - t_0), 
\]

where $\delta(t - t_0)$ is the Dirac function at $t_0$. From [9] we have

\[
G(t - t_0) = e^{-\frac{d}{2}(t-t_0)}\frac{\sin \sqrt{\omega_0^2 - \frac{d^2}{4}}(t-t_0)}{\sqrt{\omega_0^2 - \frac{d^2}{4}}}. 
\]

The solution is the given by

\[
a(t) = \int_0^t G(t - t_0)f(t_0)dt_0. 
\]

On the contrary when damping is applied to nodes 3 or 4 the equation for $a_2$
becomes

\[
\ddot{a}_2 + a_2 = -0.7f, 
\]

which is not damped at all. The amplitude of the mode 2 then grows linearly
as expected. This is shown in the right panel of Fig. 13. The final amplitude in
Figure 13: Example of the influence of damping applied to different nodes, the graph is a tree with $\alpha = 0.1$, $\beta = 0$. Plot of $a_2(t)$ (blue online), $a_3(t)$ (red online) and $a_4(t)$ (green online) when the node 4 is excited with a frequency $\omega = 1$. The left (resp. right) panel corresponds to the node 1 (resp. 2) being damped.

Fig. 13 can be easily found by seeing that the solution of the linear resonance equation

\begin{equation}
\ddot{a} + a = f \sin t,
\end{equation}

is

\begin{equation}
a(t) = -\frac{f}{2}t \cos t.
\end{equation}

In the example above one gets

$$max(a_2(300)) = -\frac{0.7}{2}(300 - 100) = 70,$$

which is in excellent agreement with the value on the right panel of Fig. 13.

When the graph has a cycle so that $\beta = 0.1$ we get a slightly different picture because the linear resonance observed previously is not exact. There is a small damping due to the non zero second coordinate of the eigenvector $v_3$. We excite node 4 at a frequency $\omega = 1$ and damp the nodes 1,2,3 and 4 respectively. We find two groups of behaviors shown in Fig. 14 the left panel corresponds to damping on nodes 2 and 3. One can see the typical beat at frequency $(\omega_3 - \omega)/2$ which yields a half-period $T/2 = 157$. The mode 2 which is the initial condition is strongly reduced. When the damping is applied to the nodes 1 or 4 we have a damped resonance. This is shown in the right panel of
Fig. 14. Note that the amplitude of mode 3 is practically constant during and after the forcing/damping region.

Figure 14: Similar plot as in Fig. 13 except that the graph now has a cycle $\alpha = 0.1, \beta = 0.1$. Plot of $a_3(t)$ (blue online) and $a_2(t)$ (red online) when the node 4 is excited with a frequency $\omega = 1$. The left (resp. right) panel corresponds to the node 2 (resp. 1) being damped.

Finally note that if we excite nodes 2 or 3 with $\omega = 1$ we do not get any significant response. This is because $v_2$ has zero components on nodes 2 and 3.

Another effect occurs because close frequencies correspond to two different eigenmodes. So depending on the way the network is excited or damped we get the two eigenmodes or just one. As an example consider the tree graph with $\alpha = 0.9, \beta = 0$. As shown in Fig. 14 we have two close eigenvalues

$$
\omega_2 = 1, \quad v_2 = \begin{pmatrix} 0.71 \\ 0 \\ 0 \\ -0.71 \end{pmatrix}, \quad \omega_3 = 0.964, \quad v_3 = \begin{pmatrix} 0.4 \\ 0.028 \\ -0.82 \\ 0.4 \end{pmatrix}.
$$
When node 1 is damped, both modes $v_2$ and $v_3$ can be excited because they have non zero components on that node. This is shown in the left panel of Fig. 15. On the contrary when node 2 is damped (middle panel of Fig. 15) no damping occurs for the amplitude $a_2$ causing an unbounded linear growth. The mode $v_3$ is excited but in a much smaller way because it is weakly damped, since $v_3$ has a small non zero component on node 2. When node 3 is damped, the mode 3 is strongly damped so that there is practically only mode 2. Therefore one sees that applying damping to node 3 will result in mode 2 only being excited while applying damping on node 1 will result in the presence of both modes 2 and 3.

6 Conclusion

To describe the flow of a miscible quantity on a network, we introduced the graph wave equation where the standard continuous Laplacian is replaced by the graph Laplacian. We showed that a natural example is an electrical network of inductances on the branches and capacities on the nodes. This system can also describe shallow water waves on a network of canals or fluid flow in a
network of pipes. There is also a mechanical analog in terms of masses and springs.

Since the graph Laplacian is a symmetric matrix, its eigenvectors are orthogonal and provide a natural basis to describe the flow in terms of amplitudes on each mode. We derived such amplitude equations when the network is forced and damped on a given node. The eigenvalues and components of the eigenvectors are important elements of these amplitude equations. We analyzed them for two simple non trivial networks: a tree and a graph with a cycle. For a tree such that at least two outer modes are connected to a common third node with the same coupling $\alpha$, we get the general result that one eigenvalue is equal to $\alpha$.

The numerical analysis of the amplitude equations shows in particular that damping can be ineffective if applied to the wrong node. This could cause disastrous resonance and destruction of the network. Another effect is that we have multiple eigenvalues corresponding to different eigenvectors. Therefore exciting the system on different nodes can cause one or the other eigenvector to appear.

These results could be useful for complex physical networks like arrays of Josephson junctions. They could also have important applications in engineering situations like for a power grid.

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