ON G-RADICAL SUPPLEMENT SUBMODULES

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Abstract. In this work, we give some new properties of Rad-supplement and g-radical supplement submodules. Let $V$ be a g-radical supplement of $U$ in $M$ and $U$ or $V$ be essential submodule of $M$. Then $\text{Rad}_g V = V \cap \text{Rad}_g M$. Let $V$ be a g-radical supplement of $U$ in $M$, $U$ or $V$ be essential submodule of $M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$. In this work, some relations between Rad-supplement, g-radical supplement, $\beta^*$ and $\beta^*_g$ relations are also studied. Let $\beta^*_g M$ in $M$. If $V$ is a g-radical supplement of $X$ in $M$ and $V \subseteq M$, then $V$ is also a g-radical supplement of $Y$ in $M$. Let $M$ be an $R$-module. It is proved that $M$ is semilocal (g-semilocal) if every submodule of $M$ $\beta^*$ equivalent to a Rad-supplement (g-radical supplement) submodule in $M$.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let $R$ be a ring and $M$ be an $R$-module. We will denote a submodule $N$ of $M$ by $N \leq M$. Let $M$ be an $R$-module and $N \leq M$. If $L = M$ for every submodule $L$ of $M$ such that $M = N + L$, then $N$ is called a small submodule of $M$ and denoted by $N \ll M$. Let $M$ be an $R$-module and $N \leq M$. If there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K = 0$, then $N$ is called a direct summand of $M$ and it is denoted by $M = N \oplus K$. A submodule $N$ of an $R$-module $M$ is called an essential submodule of $M$, denoted by $N \triangleleft M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap K = 0$ implies that $K = 0$. Let $M$ be an $R$-module and $K$ be a submodule of $M$. $K$ is called a generalized small (briefly, g-small) submodule of $M$ if for every $T \leq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [13], it is called an e-small submodule of $M$ and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let $U$ and $V$ be submodules of $M$. If $M = U + V$ and $V$ is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then $V$ is called a supplement of $U$ in $M$. $M$ is called a supplemented module if every
submodule of $M$ has a supplement in $M$. Let $M$ be an $R$-module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \subseteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then $V$ is called a $g$-supplement of $U$ in $M$. $M$ is said to be $g$-supplemented if every submodule of $M$ has a $g$-supplement in $M$. The intersection of all maximal submodules of an $R$-module $M$ is called the radical of $M$ and denoted by $\text{Rad}(M)$. If $M$ have no maximal submodules, then we denote $\text{Rad}(M) = M$. $M$ is said to be semilocal if $M/\text{Rad}(M)$ is semisimple, i.e. every submodule of $M/\text{Rad}(M)$ is a direct summand of $M/\text{Rad}(M)$. Let $M$ be an $R$-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq \text{Rad}(V)$, then $V$ is called a generalized (radical) supplement (briefly $\text{Rad}$-supplement) of $U$ in $M$. $M$ is said to be generalized (radical) supplemented (briefly $\text{Rad}$-supplemented) if every submodule of $M$ has a $\text{Rad}$-supplement in $M$. More informations about supplemented modules are in [2, 8, 12]. More informations about $g$-small submodules and $g$-supplemented modules are in [3, 7, 10]. The definition of generalized supplemented modules and some properties of them are in [11]. The definition of $g$-semilocal modules and some properties of them are in [5].

Definition 1. Let $M$ be an $R$-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq \text{Rad}(V)$, then $V$ is called a $g$-radical supplement of $U$ in $M$. If every submodule of $M$ has a $g$-radical supplement in $M$, then $M$ is called a $g$-radical supplemented module. (See [4, 6].)

Lemma 1. Let $M$ be an $R$-module. The following assertions hold.

1. For every $m \in \text{Rad}(M)$, $\text{Rad}(m) \ll_g M$.
2. If $N \leq M$, then $\text{Rad}(N) \leq \text{Rad}(M)$.
3. $\text{Rad}(M) = \sum_{L \ll_g M} L$.

Proof. See [4, Lemma 2 and Lemma 3].
Clearly we can see that every g-supplemented module is g-radical supplemented. But the converse is not true in general. Every Rad-supplemented module is g-radical supplemented.

**Lemma 2.** Let $V$ be a g-radical supplement of $U$ in $M$ and $U \trianglelefteq M$. Then $\text{Rad}_g V = V \cap \text{Rad}_g M$.

**Proof.** By Lemma 1, $\text{Rad}_g V \leq V \cap \text{Rad}_g M$. Let $T$ be an essential maximal submodule of $V$. Then $U \cap V \leq \text{Rad}_g V \leq T$ holds. By \( \frac{M}{U + T} = \frac{V}{U + T} = \frac{V}{V \cap (U + T)} \), $U + T \leq M$, $U + T$ is an essential maximal submodule of $M$ and $\text{Rad}_g M \leq U + T$. Hence $V \cap \text{Rad}_g M \leq V \cap (U + T) = U \cap V + T = T$. Thus $\text{Rad}_g V = V \cap \text{Rad}_g M$, as desired. □

**Theorem 1.** Let $V$ be a g-radical supplement of $U$ in $M$, $U \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.

**Proof.**

$\implies$ Clear.

$\iff$ Since $Rx \ll_g M$, by Lemma 1, $Rx \leq \text{Rad}_g M$ and $x \in \text{Rad}_g M$. Then $x \in V \cap \text{Rad}_g M$. By Lemma 2, $\text{Rad}_g V = V \cap \text{Rad}_g M$. Hence $x \in \text{Rad}_g V$ and by Lemma 1, $Rx \ll_g V$. We can also prove this part as follows:

Let $T$ be an essential maximal submodule of $V$. Here $U \cap V \leq \text{Rad}_g V \leq T$. Assume that $Rx \nless T$. Then $Rx + T = V$ and $M = U + V = U + Rx + T$. Since $Rx \ll_g M$ and $U + T \leq M$, $U + T = M$. Then $V = V \cap M = V \cap (U + T) = U \cap V + T = T$, a contradiction. Hence $Rx \leq T$ for every essential maximal submodule $T$ of $V$ and $Rx \leq \text{Rad}_g V$. Thus $x \in \text{Rad}_g V$ and by Lemma 1, $Rx \ll_g V$.

□

**Corollary 1.** Let $V$ be a Rad-supplement of $U$ in $M$, $U \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.

**Proof.** Clear from Theorem 1. □

**Corollary 2.** Let $V$ be a Rad-supplement of $U$ in $M$, $U \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.

**Proof.** Clear from Theorem 1. □

**Theorem 2.** Let $V$ be a g-radical supplement of $U$ in $M$, $V \subseteq M$ and $x \in V$. The following assertions hold.

1. $\text{Rad}_g V = V \cap \text{Rad}_g M$.
2. $Rx \ll_g V$ if and only if $Rx \ll_g M$.

**Proof.**
(1) By Lemma 1, $\text{Rad}_g V \leq V \cap \text{Rad}_g M$. Let $T$ be an essential maximal submodule of $V$. Then $U \cap V \leq \text{Rad}_g V \leq T$ holds. Since $T \leq V$ and $V \leq M$, then $T \leq M$ and $U + T \leq M$. Then by $M = \frac{U + T + V}{U + T} \cong \frac{V}{V \cap (U + T)} = \frac{V}{U \cap V + T} = T$. $U + T$ is an essential maximal submodule of $M$ and $\text{Rad}_g M \leq U + T$. Hence $V \cap \text{Rad}_g M \leq V \cap (U + T) = U \cap V + T = T$. Thus $\text{Rad}_g V = V \cap \text{Rad}_g M$, as desired.

(2) $\implies$ Clear.

$\Leftarrow$ Since $Rx \leq \text{Rad}_g M$, by Lemma 1, $Rx \leq \text{Rad}_g M$ and $x \in \text{Rad}_g M$. Then $x \in V \cap \text{Rad}_g M$. By Theorem 2 (1), $\text{Rad}_g V = V \cap \text{Rad}_g M$. Hence $x \in \text{Rad}_g V$ and by Lemma 1, $Rx \leq \text{Rad}_g V$. We can also prove this part as follows:

Let $T$ be an essential maximal submodule of $V$. Here $U \cap V \leq \text{Rad}_g V \leq \leq T$. Assume that $Rx \subseteq T$. Then $Rx + T = V$ and $M = U + V = U + Rx + T$. Since $T \leq V$ and $V \leq M$, then $T \leq M$ and $U + T \leq M$. Since $Rx \leq \text{Rad}_g M$, $U + T = M$. Then $V = V \cap M = V \cap (U + T) = U \cap V + T = T$, a contradiction. Hence $Rx \leq T$ for every essential maximal submodule $T$ of $V$ and $Rx \leq \text{Rad}_g V$. Thus $x \in \text{Rad}_g V$ and by Lemma 1, $Rx \leq \text{Rad}_g V$.

\[\square\]

**Corollary 3.** Let $V$ be a Rad-supplement of $U$ in $M$ and $V \leq M$. Then $\text{Rad}_g V = V \cap \text{Rad}_g M$.

**Proof.** Clear from Theorem 2 (1).

\[\square\]

**Corollary 4.** Let $V$ be a Rad-supplement of $U$ in $M$, $V \leq M$ and $x \in V$. Then $Rx \leq \text{Rad}_g V$ if and only if $Rx \leq \text{Rad}_g M$.

**Proof.** Clear from Theorem 2 (2).

\[\square\]

**Example 1.** Consider the $\mathbb{Z}$-module $\mathbb{Q}$. For $\mathbb{Z} \leq \mathbb{Q}$, $\text{Rad}_g \mathbb{Z} = \text{Rad} \mathbb{Z} = 0$. Since $\text{Rad}_g \mathbb{Q} = \text{Rad} \mathbb{Q} = \mathbb{Q}$, $\mathbb{Z} \cap \text{Rad}_g \mathbb{Q} = \mathbb{Z} \cap \mathbb{Q} = \mathbb{Z}$. Hence $\text{Rad}_g \mathbb{Z} \neq \mathbb{Z} \cap \text{Rad}_g \mathbb{Q}$.

**Proposition 1.** Let $X\beta_g Y$ in $M$. If $V$ is a g-radical supplement of $X$ in $M$ and $V \leq M$, then $V$ is also a g-radical supplement of $Y$ in $M$.

**Proof.** By hypothesis, $M = X + V$ and $X \cap V \leq \text{Rad}_g V$. Since $X\beta_g Y$ and $V \leq M$, $Y + V = M$. Let $T$ be any essential maximal submodule of $V$. Since $T \leq V$ and $V \leq M$, then $T \leq M$. Assume that $Y \cap V \leq T$. Then $Y \cap V + T = V$. Here $M = Y + V = Y + Y \cap V + T = Y + T$ and since $X\beta_g Y$ and $T \leq M$, $X + T = M$. Then $V = V \cap M = V \cap (X + T) = V \cap X + T$ and since $X \cap V \leq \text{Rad}_g V \leq T$, $V = V \cap X + T$. This is a contradiction. Hence $Y \cap V \leq T$ for every essential maximal submodule of $V$ and $Y \cap V \leq \text{Rad}_g V$. Thus $V$ is a g-radical supplement of $Y$ in $M$.

\[\square\]

**Lemma 3.** Let $X\beta_g Y$ in $M$. If $X$ and $Y$ have Rad-supplements in $M$, then they have the same Rad-supplements in $M$. 


Proof. Let \( C \) be a Rad-supplement of \( X \) in \( M \). Then \( M = X + C \) and \( X \cap C \leq \text{Rad} \). Since \( X \beta^*Y, Y + C = M \). Let \( T \) be any maximal submodule of \( C \). Assume that \( Y \cap C \nmid T \). Then \( Y \cap C + T = C \). Here \( M = Y + C = Y + Y \cap C + T = Y + T \) and since \( X \beta^*Y, X + T = M \). Then \( C = C \cap M = C \cap (X + T) = X \cap C + T \) and since \( X \cap C \leq \text{Rad} \), \( C = X \cap C + T = T \). This is a contradiction. Hence \( Y \cap C \leq T \) for every maximal submodule of \( C \) and \( Y \cap C \leq \text{Rad} \). Thus \( C \) is a Rad-supplement of \( Y \) in \( M \). Similarly, the interchanging the roles of \( X \) and \( Y \), we can prove that every Rad-supplement of \( Y \) in \( M \) is also a Rad-supplement of \( X \) in \( M \). □

Corollary 5. Let \( X \) lies above \( Y \) in \( M \). If \( X \) and \( Y \) have Rad-supplements in \( M \), then they have the same Rad-supplements in \( M \).

Proof. Clear from Lemma 3. □

Lemma 4. Let \( X \beta^*Y \) in \( M \). If \( X \) has a g-radical supplement \( V \) in \( M \), then \( V \) is also a g-radical supplement of \( Y \) in \( M \).

Proof. By hypothesis, \( M = X + V \) and \( X \cap V \leq \text{Rad} \). Since \( X \beta^*Y, Y + V = M \). Let \( T \) be any essential maximal submodule of \( V \). Assume that \( Y \cap V \nmid T \). Then \( Y \cap V + T = V \). Here \( M = Y + V = Y + Y \cap V + T = Y + T \) and since \( X \beta^*Y, X + T = M \). Then \( V = V \cap M \cap (X + T) = X \cap V + T \) and since \( X \cap V \leq \text{Rad} \), \( V = V \cap X + T = T \). This is a contradiction. Hence \( Y \cap V \leq T \) for every essential maximal submodule of \( V \) and \( Y \cap V \leq \text{Rad} \). Thus \( V \) is a g-radical supplement of \( Y \) in \( M \). □

Corollary 6. Let \( X \) lies above \( Y \) in \( M \). If \( X \) and \( Y \) have g-radical supplements in \( M \), then they have the same g-radical supplements in \( M \).

Proof. Clear from Lemma 4. □

Lemma 5. Let \( X \beta^*Y \) and \( Y \) be a Rad-supplement of \( U \) in \( M \). Then \( U \cap X \leq \text{Rad} \).

Proof. Since \( Y \) is a Rad-supplement of \( U \) in \( M \), \( M = U + Y \) and \( U \cap Y \leq \text{Rad} \). Since \( M = U + Y \) and \( X \beta^*Y, M = U + X \). Let \( T \) be any maximal submodule of \( M \). Here \( U \cap Y \leq \text{Rad} \). Assume that \( U \cap X \nmid T \). Then \( U \cap X + T = M \) and since \( M = U + X \), by [2, Lemma 1.24], \( X + U \cap T = M \). Since \( X \beta^*Y, Y \cap T = M \) and since \( U + T = M \), by [2, Lemma 1.24] again, \( U \cap Y + T = M \). Then by \( U \cap Y \leq T \), \( M = U \cap Y + T = T \). This is a contradiction. Hence \( U \cap X \leq T \) for every maximal submodule \( T \) of \( M \) and \( U \cap X \leq \text{Rad} \). □

Corollary 7. Let \( X \) lies above \( Y \) and \( Y \) be a Rad-supplement of \( U \) in \( M \). Then \( U \cap X \leq \text{Rad} \).

Proof. Clear from Lemma 5. □

Lemma 6. Let \( M \) be an \( R \)-module. If every submodule of \( M \) is \( \beta^* \) equivalent to a Rad-supplement submodule in \( M \), then \( M \) is semilocal.
Theorem 3. Let $X \beta^* Y$ and $Y$ be a g-radical supplement of $U$ in $M$. Then $U \cap X \leq \text{Rad}_g M$.

Proof. Since $Y$ is a g-radical supplement of $U$ in $M$, $M = U + Y$ and $U \cap Y \leq \text{Rad}_g Y \leq \text{Rad}_g M$. Since $M = U + Y$ and $X \beta^* Y$, $M = U + X$. Let $T$ be any essential maximal submodule of $M$. Here $U \cap Y \leq \text{Rad}_g M \leq T$. Assume that $U \cap X \notin T$. Then $U \cap X + T = M$ and since $M = U + X$, by [2, Lemma 1.24], $X + U \cap T = M$. Since $X \beta^* Y$, $Y + U \cap T = M$ and since $U + T = M$, by [2, Lemma 1.24] again, $U \cap Y + T = M$. Then by $U \cap Y \leq T$, $M = U \cap Y + T = T$. This is a contradiction. Hence $U \cap X \leq T$ for every essential maximal submodule $T$ of $M$ and $U \cap X \leq \text{Rad}_g M$.

Corollary 9. Let $X$ lies above $Y$ and $Y$ be a g-radical supplement of $U$ in $M$. Then $U \cap X \leq \text{Rad}_g M$.

Proof. Clear from Theorem 3.

Theorem 4. Let $M$ be an $R$-module. If every submodule of $M$ lies above a g-radical supplement submodule in $M$, then $M$ is g-semilocal.

Proof. Let $X / \text{Rad}_g M \leq M / \text{Rad}_g M$. Since $X \leq M$, by hypothesis, there exists a g-radical supplement submodule $Y$ in $M$ such that $X \beta^* Y$. Let $Y$ be a g-radical supplement of $U$ in $M$. By Theorem 3, $U \cap X \leq \text{Rad}_g M$. Since $X \beta^* Y$ and $Y + U = M$, $X + U = M$. Then $\frac{M}{\text{Rad}_g M} = \frac{X + U}{\text{Rad}_g M} = \frac{X}{\text{Rad}_g M} + \frac{U + \text{Rad}_g M}{\text{Rad}_g M}$ and $\frac{X}{\text{Rad}_g M} \cap \frac{U + \text{Rad}_g M}{\text{Rad}_g M} = \frac{X \cap (U + \text{Rad}_g M)}{\text{Rad}_g M} = \frac{U \cap X + \text{Rad}_g M}{\text{Rad}_g M} = \frac{\text{Rad}_g M}{\text{Rad}_g M} = 0$. Hence $\frac{M}{\text{Rad}_g M} = \frac{X}{\text{Rad}_g M} \oplus \frac{U + \text{Rad}_g M}{\text{Rad}_g M}$ and $M / \text{Rad}_g M$ is semisimple. Thus $M$ is g-semilocal.

Corollary 10. Let $M$ be an $R$-module. If every submodule of $M$ lies above a g-radical supplement submodule in $M$, then $M$ is g-semilocal.

Proof. Clear from Theorem 4.
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