Virial Coefficients from Unified Statistical Thermodynamics of Quantum Gases
Trapped under Generic Power Law Potential in \( d \) Dimension and Equivalence of Quantum Gases

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Abstract From the unified statistical thermodynamics of quantum gases, the virial coefficients of ideal Bose and Fermi gases, trapped under generic power law potential are derived systematically. From the general result of virial coefficients, one can produce the known results in \( d = 3 \) and \( d = 2 \). But more importantly we found that, the virial coefficients of Bose and Fermi gases become identical (except the second virial coefficient, where the sign is different) when the gases are trapped under harmonic potential in \( d = 1 \). This result suggests the equivalence between Bose and Fermi gases established in \( d = 1 \) (J. Stat. Phys. DOI 10.1007/s10955-015-1344-4). Also, it is found that the virial coefficients of two-dimensional free Bose (Fermi) gas are equal to the virial coefficients of one-dimensional harmonically trapped Bose (Fermi) gas.

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1 Introduction
After the demonstration by Einstein\(^{11-2}\) that there is a possibility of condensation of free bosons, bulk behavior of ideal free Bose gas are studied by many authors.\(^{3-11}\) At the same time, theoretical investigations are also done for the free Fermi gas\(^{12-16}\) and remarkable property of degeneracy pressure of the free fermions are noticed.\(^4\) Dimensional dependence of the thermodynamic quantities of both types of ideal quantum gases are investigated in detail\(^3,10-11\) and interesting dimensional dependence of thermodynamic quantities was reported. For instance, BEC is noticed in free Bose gas only when \( d > 2 \), whether the specific heat is found discontinuous if \( d \geq 4 \). But, it was May,\(^6\) who first noticed an equivalence of specific heat for two-dimensional ideal free bosons and fermions. Later, Lee\(^{17}\) generalized this equivalence between two-dimensional quantum gases for other thermodynamic quantities as well. It is reported in the paper that, one can obtain this equivalence, if the fugacities of Bose and Fermi gases are related by Euler transformation.\(^{17-18}\) He also found a way to present the thermodynamic quantities of both of Fermi and Bose gases in a unified approach. Vieferes et al.\(^{19}\) showed that all the virial coefficients are equal in two-dimensional free quantum gases (except the second virial coefficients for which the signs are opposite).

The subject of quantum gases drew more attention after it was possible to experimentally detect BEC\(^{20-22}\) and Fermi degeneracy\(^{23}\) in trapped quantum gases. Since then, a lot of studies are done on Bose and Fermi gas trapped under generic power\(^{24-31}\) law potential (\( U = \sum_{i=1}^{d} c_i |x_i/a_i|^m \)) in arbitrary dimension. Some drastic changes are noted in the characteristics of both Bose and Fermi gases due to trapping potential.\(^{24-31}\) For instance, Bagnoto and Kleppner demonstrated the possibility of BEC of an ideal Bose gas confined by one- and two-dimensional power-law traps\(^{32-33}\) (\( d < 3 \)) with appropriate trapping potential, which is not a property of free Bose gas.\(^{25,30}\) In addition, the equivalence of two-dimensional free quantum gases do not remain valid for any trapping potential.\(^{34}\) Very recently Mehedi\(^{34}\) was able to present the thermodynamics of both types of quantum gases trapped under generic power law potential in a unified way. Remarkably an equivalence between them in \( d = 1 \) was found for harmonically trapped systems.\(^{34}\) Turning our attention towards virial coefficients, it would be intriguing to check if the virial coefficients become equal, for harmonically trapped quantum gases in one dimension.

Virial coefficients appear to relate the pressure of a many-particle system in powers of the number density in a given \( d \)-dimensional volume, providing a systematic treatment in the corrections of the laws of ideal gases. The \( d \)-dimensional virial coefficients of quantum gases trapped under generic power-law potentials are not yet reported. In this paper, we have first calculated the virial coeffi-
ponents of both types of ideal quantum gases trapped under
generic power law potential \(U = \sum_{i=1}^{d} c_i |x_i/a_i|^{n_i}\) in a
unified approach in arbitrary dimension. But the main
motivation of this investigation is to check out whether the
virial coefficients of Bose and Fermi gases become same for
when they are harmonically trapped in \(d = 1\). Now, from
the more general result one should be able to reproduce
the known solutions such as text book results of virial
coefficients of bosons and fermions in \(d = 3\) and the outcome
of Vieferes et al.’s calculation relating virial coefficients of
two-dimensional free Bose and Fermi gases. But more im-
portantly, we would be able to find out, whether the virial
coefficients are also the same for one-dimensional quantum
gases trapped in a harmonic potential, suggesting the
equivalence found by Mehedi.[34]

2 Virial Coefficients of Ideal Quantum Gases
Trapped under Generic Power Law Potential

For a quantum gas, the average number of particles
occupying the \(i\)-th single particle energy eigenstate and
the grand potential are given by
\[
\bar{n}_i = \frac{1}{z^{-1} e^{\beta \epsilon_i} + 1},
\]
(1)
\[
q = \frac{1}{d} \sum_{i=1}^{c_i} \ln(1 + az e^{-\beta \nu}_i),
\]
(2)
where, \(a = -1(1)\) stands for a Bose systems (Fermi sys-
tems), \(z\) is the fugacity and \(\beta = 1/KT\) is the Boltz-
mann constant. Let us consider an ideal quantum system
trapped in a generic power law potential in \(d\)-dimensional
space with a single particle Hamiltonian of the form,
\[
\epsilon(p, x_i) = bp^l + \sum_{i=1}^{d} c_i \left| \frac{x_i}{a_i} \right|^{n_i},
\]
(3)
where, \(b, l, a_i, c_i\), and \(n_i\) are all positive constants, \(p\)
is the momentum and \(x_i\) is the \(i\)-th component of the coor-
dinates of a particle. Here, \(c_i, a_i, n_i\) determine the depth
and confinement power of the potential, \(l\) is the kinematical
parameter, and \(x_i < a_i\). As \(|x_i/a_i| < 1\), the potential
term goes to zero as all \(n_i \rightarrow \infty\). Using \(l = 2, b = 1/2m\)
one can get the energy spectrum of the hamiltonian used
Refs. [3-4, 25]. If one uses \(l = 1\) and \(b = c\) one finds the
hamiltonian of massless systems such as photons.[4]
Now replacing the sum by integration, the grand potential for
the quantum gases become,[34]
\[
q = \text{sign}(\sigma) \frac{V_d' \Gamma(\sigma)}{\lambda' \Gamma(\lambda+1)(\sigma)},
\]
(4)
where, \(\lambda'\) and \(V'\) are the effective thermal wavelength
and effective volume,[1] and \(\Gamma(\lambda)(\sigma)\) is the polylog function.
Here,
\[
V_d' = V_d \prod_{i=1}^{d} \left( \frac{K^T}{c_i} \right)^{1/n_i} \Gamma \left( \frac{1}{n_i} + 1 \right),
\]
(5)

\[
\lambda' = \frac{\hbar b^{l/2} \beta^l}{\pi^{1/2}} \left( \Gamma(d/2 + 1) \right)^{1/d}.
\]
(6)
And a useful representation of polylog is[17]
\[
\text{Li}_q(m) = \frac{1}{\Gamma(q)} \int_{0}^{m} \left( \frac{\ln(m)}{\eta} \right)^{q-1} \frac{d\eta}{1-\eta},
\]
(7)
for Re\((m) < 1\). And quantity \(\lambda'\) and \(\sigma\) are,
\[
\lambda = \frac{d}{l} + \sum_{i=1}^{d} \frac{1}{n_i},
\]
(8)
\[
\sigma = \left\{ \begin{array}{ll}
-z, & \text{Fermi system,} \\
\frac{l}{d}, & \text{Bose system.}
\end{array} \right.
\]
(9)
where \(z\) is the fugacity. Now the density can be calculated
from the grand potential,
\[
N = z \left( \frac{\partial Q}{\partial \beta} \right)_V \Rightarrow \rho = \frac{N}{V_d} = \text{sign}(\sigma) \frac{1}{\lambda'^d} \text{Li}_\lambda(\sigma).
\]
(10)
And the pressure,
\[
P = \frac{1}{\beta} \frac{\partial Q}{\partial V_d} \Rightarrow \beta P = \frac{1}{\lambda'^d} \text{Li}_{\lambda+1}(\sigma).
\]
(11)
Now writing the pressure and density equation as series[4]
\[
\beta P = \frac{1}{\lambda'^d} \left( \sum_{j=1}^{\infty} b_j z^j \right),
\]
(12)
\[
\rho = \frac{1}{\lambda'^d} \left( \sum_{j=1}^{\infty} j b_j z^j \right),
\]
(13)
where the coefficients \(b_j\) are defined as,
\[
b_j = \begin{cases} 
(-1)^{j+1} i^{-j-1}, & \text{Fermi system,} \cr 
\frac{i}{j-(-1)^{j+1}}, & \text{Bose system.}
\end{cases}
\]
(14)
Next we need to express the fugacity in terms of density
so that one can write the pressure as a series of density.
Writing the density equation explicitly,
\[
\lambda'^d \rho = b_1 z + 2b_2 z^2 + 3b_3 z^3 + 4b_4 z^4 + \cdots
\]
(15)
In the high temperature limit we can approximate fugacity,
\[
z = \sum_{j=1}^{\infty} a_j (\rho \lambda'^d)^j
\]
\[
= a_1 (\lambda'^d) + a_2 (\lambda'^d)^2 + a_3 (\lambda'^d)^3 + \cdots
\]
(16)
Putting \(z\) into the expression \(\lambda'^d\),
\[
\lambda'^d \rho = b_1 (a_1 (\lambda'^d) + a_2 (\lambda'^d)^2 + a_3 (\lambda'^d)^3 + \cdots)
+ 2b_2 (a_1 (\lambda'^d)^2 + a_2 (\lambda'^d)^2)^2 + a_3 (\lambda'^d)^3)^2 + a_4 (\lambda'^d)^3)^3 + \cdots
\]
\[
= \lambda'^d \rho (b_1 a_1 + (\lambda'^d)^2 (b_1 a_2 + 2b_2 a_1^2)
+ (\lambda'^d)^3 (b_1 a_3 + 4b_2 a_2 a_1 + 3b_3 a_1^3) + (\lambda'^d)^4
\times (b_1 a_4 + 2b_2 a_3 a_1 + 9b_3 a_2 a_2 + 4b_4 a_1^4)
+ (\lambda'^d)^5 (b_1 a_5 + 4b_2 a_4 a_1 a_1 + 6b_3 a_3 a_1 + 9b_4 a_1 a_2 + 4b_4 a_1^2)
+ 9b_5 a_1 a_3 a_1^2 + 16b_5 a_4 a_2 + 5b_6 a_3 a_1^3)\cdots
\]
(17)
Now comparing Eqs. (15) and (17),
\[
b_1 = 1, \quad b_4 a_2 + 2b_2 a_1^2 = 0,
\]
above two equations we can calculate the virial coefficients of ideal quantum gases trapped under generic potential the virial coefficients, Eqs. (22)–(28) will change upon boson or fermion as well as the type of trapping potential. This also means that, these results are quite general for ideal quantum gases. 

Turning our attention towards free massive Fermi gas in three-dimensional space, the virial coefficients are, 

\[
A_1 = 1, \quad A_2 = -2b_2, \quad A_3 = 4b_2^2 - 2b_3, \quad A_4 = -20b_2^4 + 18b_2b_3 - 3b_4, \quad A_5 = 112b_2^4 - 144b_2b_3 + 18b_3^2 + 32b_2b_4 - 4b_5, \quad A_6 = -672b_2^4 + 1120b_2^3b_3 - 315b_2b_4^2 - 280b_5^2b_4 + 60b_2b_4 + 50b_2b_5 - 5b_6, \quad A_7 = 4224b_2^6 - 8640b_2^5b_3 + 3888b_2^4b_3^2 - 216b_2^4b_5 + 320b_2b_5^2 - 216b_2^3b_5 + 320b_2^2b_5 - 5b_6b_7, 
\]

where, the \(b_j\) are defined in Eq. (14). Point to note that, these results are quite general for ideal quantum gas trapped under generic power law potential. Depending upon boson or fermion as well as the type of trapping potential the virial coefficients, Eqs. (22)–(28) will change accordingly. Also, the result of Eqs. (22)–(28) coincides well with Ref. [25].

### 3 Results

The results of virial coefficients presented in Eqs. (22)–(28) are for ideal quantum gases trapped under generic power law potential in any dimension. In this section we first present the final results for some specific situation.

\[
\beta P = \frac{1}{\lambda^d} \sum_{j=1}^{\infty} b_j \beta^j = \frac{1}{\lambda^d} (b_1 \beta + b_2 \beta^2 + b_3 \beta^3 + \cdots)
\]

The virial expansion is defined as, 

\[
\beta P = \sum_{k=1}^{\infty} A_k \lambda^d (k-1),
\]

where, \(A_k\) are the virial coefficients. Thus comparing the above two equations we can calculate the virial coefficients,

\[
A_1 = 1,
\]

\[
A_2 = -2b_2,
\]

\[
A_3 = 4b_2^2 - 2b_3,
\]

\[
A_4 = -20b_2^4 + 18b_2b_3 - 3b_4,
\]

\[
A_5 = 112b_2^4 - 144b_2b_3 + 18b_3^2 + 32b_2b_4 - 4b_5,
\]

\[
A_6 = -672b_2^4 + 1120b_2^3b_3 - 315b_2b_4^2 - 280b_5^2b_4 + 60b_2b_4 + 50b_2b_5 - 5b_6,
\]

\[
A_7 = 4224b_2^6 - 8640b_2^5b_3 + 3888b_2^4b_3^2 - 216b_2^4b_5 + 320b_2b_5^2 - 216b_2^3b_5 + 320b_2^2b_5 - 5b_6b_7,
\]

We have later presented two tables containing the results of virial coefficients of bosons and fermions, trapped under different power law potentials.

#### 3.1 Free Quantum Gases in \(d = 3\)

At first, we present the virial coefficients for the free massive bosons \((l = 2)\) in three-dimensional space. So, choosing \(d = 3\) and all \(n_i \rightarrow \infty\) which leaves us \(\chi = \frac{3}{2}\). In this case Eqs. (22)–(28) dictates, the virial coefficients of bosons are,

\[
A_1 = 1, \quad A_2 = -0.177, \quad A_3 = -0.033, \quad A_4 = -0.00011, \quad A_5 = -0.00000354,
\]

\[
\vdots
\]

Turning our attention towards free massive Fermi gas in three-dimensional space, the virial coefficients are,

\[
A_1 = 1, \quad A_2 = 0.177, \quad A_3 = 0.033, \quad A_4 = 0.00011, \quad A_5 = 0.00000354,
\]

\[
\vdots
\]

The results of Eqs. (29) and (30) concur with Ref. [4]. It is interesting to point out that, in the case of Fermi gases in \(d = 3\) the virial coefficients alters sign while all the virial coefficients of Bose gases are negative. Point to note, any negative (positive) virial coefficient \(A_n\) indicates that, the \(n\)-particle wave function is symmetric (asymmetric). As the bosonic \(n\)-particle wave function is always symmetric in three-dimensional space, we see the virial coefficients of bosons are negative. And as we know in three dimensions, the \(n\) particle fermion wavefunctions (even \(n\)) are anti-symmetric. \(A_n\) are positive for even \(n\), in \(d = 3\). But any \(n\)-particle Fermi wavefunction is symmetric in \(d = 3\) while \(n\) is odd. So, we have found negative virial coefficient \(A_n\), for odd \(n\). Thus we find out the reason due to
which we get the alternating sign in virial coefficients of Fermi gas in $d = 3$.

### 3.2 Free Quantum Gases in $d = 2$

Turning our focus on the virial coefficients of free massive quantum gases in lower dimensions we choose $d = 2$ and all $n_i \to \infty$, and find out that $\chi = 2$. Equations (22)–(28) dictate the virial coefficients for bosons are,

\[
A_2 = -0.25, \quad A_3 = 0.0278, \quad A_4 = 0, \\
A_5 = -0.000278, \quad A_6 = 0, \quad A_7 = 0.00000472,
\]

while the virial coefficients of fermions are,

\[
A_1 = 1, \quad A_2 = 0.25, \quad A_3 = 0.0278, \quad A_4 = 0, \\
A_5 = -0.000278, \quad A_6 = 0, A_7 = 0.00000472.
\]

(31)

The results match precisely with the viebers et al.\cite{19}

Please note that, all the virial coefficients for Bose gases are equal to Fermi gases (except for the second virial coefficient, for which the signs are opposite). This important result turns out to indicate the equivalence of free Bose and Fermi gases in two dimension.\cite{17} And the opposite signs of second virial coefficient indicate that, the only difference in pressures between free Fermi and Bose gases is the Fermi degeneracy pressure.\cite{19} Also, note that all the even virial coefficients are zero after $A_2$. Both of these features are explained later, more carefully. But it is reported in Ref.\cite{34} the equivalence found in Two-dimensional quantum gases is lost when they are trapped with potential. So, let us now turn our attention towards the virial coefficients of the trapped system.

### 3.3 Effect of Trapping Potential on Virial Coefficients

To see the effect of different trapping potential in the various dimension we introduce two tables presenting virial coefficients of Bose (Table 1) and Fermi gases (Table 2) below. In this case we have considered symmetric potential, i.e. $v_1 = v_2 = \cdots = v_d = n$. It is clear from the tables that the trapping potential greatly affects the virial coefficients.

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $n = 2$ | $d = 1$ | $-2.50 \times 10^{-1}$ | $2.78 \times 10^{-2}$ | $0$ | $-2.78 \times 10^{-3}$ | $0$ | $4.72 \times 10^{-6}$ |
| $n = 3$ | $d = 1$ | $-3.08 \times 10^{-1}$ | $6.955 \times 10^{-2}$ | $-1.15 \times 10^{-2}$ | $9.99 \times 10^{-3}$ | $1.10 \times 10^{-4}$ | $-6.17 \times 10^{-5}$ |
| $n = 4$ | $d = 1$ | $-3.30 \times 10^{-1}$ | $9.04 \times 10^{-2}$ | $-2.06 \times 10^{-2}$ | $3.61 \times 10^{-3}$ | $-3.46 \times 10^{-4}$ | $-5.60 \times 10^{-5}$ |
| $n = \infty$ | $d = 1$ | $-3.54 \times 10^{-1}$ | $1.15 \times 10^{-1}$ | $-3.41 \times 10^{-2}$ | $9.01 \times 10^{-3}$ | $-2.0 \times 10^{-3}$ | $3.12 \times 10^{-4}$ |

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $n = 2$ | $d = 2$ | $-2.50 \times 10^{-1}$ | $-9.07 \times 10^{-3}$ | $-2.71 \times 10^{-3}$ | $-1.08 \times 10^{-3}$ | $-5.03 \times 10^{-4}$ | $-2.59 \times 10^{-4}$ |
| $n = 3$ | $d = 2$ | $-3.08 \times 10^{-1}$ | $8.12 \times 10^{-2}$ | $-1.62 \times 10^{-2}$ | $2.23 \times 10^{-3}$ | $-6.19 \times 10^{-5}$ | $-7.59 \times 10^{-5}$ |
| $n = 4$ | $d = 2$ | $-3.30 \times 10^{-1}$ | $9.04 \times 10^{-2}$ | $-2.06 \times 10^{-2}$ | $3.61 \times 10^{-3}$ | $-3.46 \times 10^{-4}$ | $-5.60 \times 10^{-5}$ |
| $n = \infty$ | $d = 2$ | $-3.54 \times 10^{-1}$ | $1.15 \times 10^{-1}$ | $-3.41 \times 10^{-2}$ | $9.01 \times 10^{-3}$ | $-2.0 \times 10^{-3}$ | $3.12 \times 10^{-4}$ |

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $n = 2$ | $d = 3$ | $-2.50 \times 10^{-1}$ | $-3.30 \times 10^{-3}$ | $-1.11 \times 10^{-4}$ | $-3.54 \times 10^{-6}$ | $-8.39 \times 10^{-8}$ | $-3.66 \times 10^{-10}$ |
| $n = 3$ | $d = 3$ | $-3.08 \times 10^{-1}$ | $8.12 \times 10^{-2}$ | $-1.62 \times 10^{-2}$ | $2.23 \times 10^{-3}$ | $-6.19 \times 10^{-5}$ | $-7.59 \times 10^{-5}$ |
| $n = 4$ | $d = 3$ | $-3.30 \times 10^{-1}$ | $9.04 \times 10^{-2}$ | $-2.06 \times 10^{-2}$ | $3.61 \times 10^{-3}$ | $-3.46 \times 10^{-4}$ | $-5.60 \times 10^{-5}$ |
| $n = \infty$ | $d = 3$ | $-3.54 \times 10^{-1}$ | $1.15 \times 10^{-1}$ | $-3.41 \times 10^{-2}$ | $9.01 \times 10^{-3}$ | $-2.0 \times 10^{-3}$ | $3.12 \times 10^{-4}$ |
More importantly, it is seen in the tables that the virial coefficients of one-dimensional harmonically trapped Bose gases are equal to the one-dimensional harmonically trapped Fermi gases, except the second virial coefficients for which the signs are opposite. This outstanding property, earlier seen in two-dimensional free quantum gases, suggests high-temperature behavior of two-dimensional free quantum gases (Bose and Fermi) is similar to one-dimensional harmonically trapped quantum gases (Bose and Fermi).

(i) Equivalence of Harmonically Trapped Quantum Gases in \( d = 1 \)

It is very important to point out in both the cases where this equivalence of Bose and Fermi gases are noticed takes the value \( \chi = 1 \). Now, focusing over the harmonically trapped quantum gases in one dimension, we find the density of such quantum gases

\[
\rho = \text{sgn}(\sigma) \frac{1}{\lambda} \text{Li}_1(\sigma) = \frac{1}{\lambda} \log(1 \pm z),
\]

where, the upper sign is for fermions and the other is for boson. From this we can explicitly write the fugacity, which is

\[
z = \mp 1 \pm e^{\pm \rho \lambda'}.
\]

The pressure equation leads us to

\[
\beta P = \mp \frac{1}{\lambda} \text{Li}_2(1 \pm z).
\]

Now using the techniques of partial derivative

\[
\left( \frac{\partial P}{\partial \rho} \right)_T = \left( \frac{\partial P}{\partial \mu} \right)_T \left( \frac{\partial \mu}{\partial \rho} \right)_T,
\]

we get from the above two equations,

\[
\left( \frac{\partial P}{\partial \rho} \right)_T = \mp \frac{\lambda'}{\beta} \frac{1}{e^{\rho \lambda'} - 1}.
\]

Now the equation of state is thus given by,

\[
\beta P = \mp \int_0^\rho \frac{\gamma \lambda'}{e^{\gamma \rho \lambda'} - 1} d\gamma.
\]

Notably this functional form can be expressed in terms of Bernoulli numbers,

\[
\frac{p}{e^p - 1} = \sum_{n=0}^\infty B_n \frac{p^n}{n!},
\]

where, \( 0 < |p| < 2\pi \). \( B_n \) are known as Bernoulli numbers and be defined as,

\[
B_n = \begin{cases} 0, & n = 0, \\ -\frac{1}{2}, & n = 1, \\ (-1)^{n/2-1} \frac{2(2n)!}{(2\pi)^n} \zeta(2n), & \text{even } n, \\ 0, & \text{odd } n. \end{cases}
\]

Again rewriting Eq. (21), choosing \( d = 1 \) and \( n = 2 \),

\[
\beta P = \sum_{k=1}^\infty A_k \rho^k \lambda^{(k-1)}.
\]

Using the above equations we conclude virial coefficients in this case,

\[
A_1 = (\mp)^{|l-1|} B_{l-1} \frac{1}{l!}.
\]

Incidentally this is the same result obtained in two-dimensional free quantum gas as well.\(^{[19]}\) Now as odd Bernoulli numbers \( B_n \) are zero except \( (n = 1) \) we find out even virial coefficients \( A_n \) will be zero for both Bose and Fermi gases, except \( (n = 2) \). As it is seen from above beside the second virial coefficient, all the virial coefficients do become the same when there is an equivalence, let us focus on this phenomena in a more detail. The pressure for one-dimensional harmonically trapped quantum gases from Eq. (38),

\[
\beta P = \rho \pm \frac{1}{4} \rho^2 \lambda' + \frac{1}{36} \rho^3 \lambda' \lambda^2 - \frac{1}{3600} \rho^5 \lambda^4 + \frac{1}{211680} \rho^6 \lambda^6.
\]

So, the difference in pressure of Bose and Fermi gas in this case is,

\[
P_F - P_B = \frac{1}{2} \rho^2 \lambda' = \frac{1}{2 \beta} \frac{N}{V^{1/2}} \lambda' .
\]

Now, with harmonic trapping potential \( V'' \propto T^{1/2} \) and \( \lambda' \propto T^{-1/2} \), the right hand side of the above equation is a temperature independent quantity. As it turns out it is nothing but the ground state pressure also known as degeneracy pressure in Fermi gases.\(^{[34]}\) Similar situation is also observed in Ref. \([19]\) in the case of two-dimensional free quantum gases. The reason of such phenomena is due to Landen relation of dilog functions,\(^{[17]}\)

\[
\text{Li}_2(x_1) = -\text{Li}_2(x_2) - \frac{1}{2} \text{Li}_1(x_2)^2,
\]

where, \( x_1 \) and \( x_2 \) are related as \( x_2 = x_1/(1-x_1) \), known as Euler transformation.\(^{[18]}\) As fugacities of Bose and Fermi gas can be connected as an Euler transformation Eqs. (10) and (11) dictate the relation described by Eq. (43)

4 Conclusion

The virial coefficients of both types of ideal quantum gases trapped under generic power law potentials are calculated from unified statistical thermodynamics. The general results of this paper coincide with the known results\(^{[4,19]}\) in \( d = 2 \) and \( d = 3 \) with appropriate choice
of \( n_i \). The equivalence\(^{17}\) of two-dimensional ideal free Bose and Fermi gases revealed a remarkable property\(^{19}\) that their virial coefficient are same (except the second virial coefficient, where the sign is different). We further showed that, the recently established equivalence (in \( d = 1 \)) between the harmonically trapped ideal Bose and Fermi gases also maintains this property. Hence it can be concluded that, in both of the cases, where equivalence relation can be established between Bose and Fermi gas, bosons and fermions do carry identical virial coefficients. Also from the table, one can see that the virial coefficients of two-dimensional free quantum gases are identical to the virial coefficients of one-dimensional harmonically trapped quantum gases. This interesting result suggests that the high-temperature behavior of bosons and fermions in these two types of systems should be indistinguishable. Lastly, since the calculation in this paper is done in thermodynamic limit, the virial coefficients are yet to be done beyond the thermodynamic limit. We are currently doing this using Yukalov’s semiclassical approximation.\(^{37}\) It will be equally intriguing to examine the behavior of virial coefficients for relativistic quantum gases by taking into account the effect of antiparticles.

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References

[1] S.N. Bose, Z. Physik 26 (1924) 178.
[2] A. Einstein, Berl. Ber 22 (1924) 261.
[3] R.M. Ziff, G.E Uhlenbeck, and M. Kac, Phys. Rep. 32 (1977) 169.
[4] R.K. Pathria, Statistical Mechanics, Elsevier, Amsterdam (2004).
[5] H.E. Haber and H.A. Weldon, Phys. Rev. Lett. 46 (1981) 1497.
[6] R.H. May, Phys. Rev. A 1515 (1964) 135.
[7] J.E. Robinson, Phys Rev. E 83 (1951) 678.
[8] E.H. Lieb and W. Liniger, Phys. Rev. 130 (1964) 1605.
[9] E.H. Lieb, Phys. Rev. 130 (1963) 1606.
[10] D.S. Fisher and P.C. Hohenberg, Phys. Rev. B 37 (1988) 4936.
[11] R. Beckmann, F. Karch, and D.E. Miller, Phys. Rev. Lett. 43 (1979) 1277.
[12] C.M. Bender, S. Boettcher, and L. Lipatov, Phys. Rev. D 46 (1992) 5557.
[13] C.M. Bender and S. Boettcher, Phys. Rev. D 51 (1994) 1875.
[14] C.M. Bender and S. Boettcher, and L.R. Mead, J. Math. Phys. 35 (1994) 368.
[15] C.M. Bender and K.A. Milton, Phys. Rev. D 50 (1994) 6547.
[16] C. Grosche and F. Steiner, J. Math. Phys. 36 (1995) 2354.
[17] M.H. Lee, Phys. Rev. E 55 (1997) 1518.
[18] M.H. Lee, Acta Phys. Polonica 40 (2009) 1279.
[19] S. Viefer, F. Ravndal, and T. Haugset, American Journal of Physics 63 (1995) 369, doi:10.1119/1.17922.
[20] C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Phys. Rev. Lett. 75 (1995) 1687.
[21] M.H. Anderson, J.R. Esher, M.R. Mathews, C.E. Wie- man, and E.A. Cornell, Science 269 (1995) 195.
[22] K.B. Davis, M.O. Mewes, M.R. Andrew, N.J. Van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. 1687 (1995) 75.
[23] B. DeMarco and D.S. Jin, Science 285 (1999) 1703.
[24] S. Biswas, J. Mitra, and S. Bhattacharyya, J. Stat. Mech. P 03013 (2015) 15.
[25] Luca Salasnich, J. Math. Phys. 41 (2000) 8016.
[26] Z. Yan, Phys. Rev. A 59 (1999) 4057.
[27] Z. Yan, Phys. Rev. A 61 (2000) 063607.
[28] Z. Yan, Ming-Zhe Li, L. Chen, C. Chen, and J. Chen, J. Phys. A: Math. Gen. 32 (1999) 4069.
[29] Z. Yan, Eur. J. Phys. 21 (2000) 625.
[30] M.M. Faruk and G.M. Bhuiyan, Acta Physica Polonica B 46 (2015) 12.
[31] M.M. Faruk, Acta Phys. Polonica B 46 (2015) 12.
[32] V. Bagnato and D. Kleppner, Phys. Rev. A 44 (1991) 7439.
[33] W.S. Dai and M. Xie, Phys. Rev. A 67 (2003) 027601.
[34] M.M. Faruk, J. Stat. Phys. DOI 10.1007/s10955-015-1344-4.
[35] Kazumoto Iguchi, Mod. Phys. Lett. B 11 (1997) 765.
[36] M. Abramowitz and I.A. Stegun, eds. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th Printing, Dover, New York (1972) pp. 804-806.
[37] V.I. Yukalov, Phys. Rev. A 72 (2005) 033608.