Stabilities of Shape Identification Inverse Problems in a Bayesian Framework

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Abstract
A general shape identification inverse problem is studied in a Bayesian framework. This problem requires the determination of the unknown shape of a domain in the Euclidean space from finite-dimensional observation data with some Gaussian random noise. Then, the stability of posterior is studied for observation data. For each point of the space, the conditional probability that the point is included in the unknown domain given the observation data is considered. The stability is also studied for this probability distribution.

As a model problem for our inverse problem, a heat inverse problem is considered. This problem requires the determination of the unknown shape of cavities in a heat conductor from temperature data of some portion of the surface of the heat conductor. To apply the above stability results to this model problem, one needs the measurability and some boundedness of the forward operator. These properties are shown.

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1. Introduction

In this paper, we study a general shape identification inverse problem. This problem requires the determination of unknown shape of a domain $D$
in \( \mathbb{R}^d \) from finite-dimensional observation data \( y \in \mathbb{R}^m \), that is,

\[
y = F(D) + \eta.
\]

Denote by \( \mathcal{D} \) the set of all possible domains. Then, \( F \) is the “forward” operator taking one instance of input \( D \in \mathcal{D} \) into a set of observations (an \( m \)-dimensional vector), and \( \eta \) is an \( m \)-dimensional Gaussian random noise variable. For this problem, we use the Bayesian formulation given by [10] and [22]. Although the set of possible domains is not a function space of the same type as in their theory, we can consider our problem as based on this previous work as mentioned in subsection 2.1. This gives a probability distribution on \( \mathcal{D} \) corresponding to the statistical properties of the observational noise.

We can consider many inverse problems as model problems of our general shape identification inverse problem. In this study, we consider the following model problem. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) and let \( C \) be the union of several (one or more) disjoint connected domains in \( \Omega \) such that \( \overline{C} \subset \Omega \). Then, \( D := \Omega \setminus \overline{C} \) is a heat conductor with cavities \( C \), and the shape of \( \partial C \) is unknown. We consider the following initial and mixed boundary value problem for the heat equation:

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \Delta u(t, x) \quad \text{if} \ (t, x) \in [0, T] \times D, \\
\frac{\partial u}{\partial \nu}(t, x) &= \psi(t, x) \quad \text{if} \ (t, x) \in [0, T] \times \partial \Omega, \\
u(t, x) &= 0 \quad \text{if} \ (t, x) \in [0, T] \times \partial C, \\
u(0, x) &= 0 \quad \text{if} \ x \in D.
\end{align*}
\]

Here, \( T \) is a positive real number, \( \Delta \) is the standard Laplacian, \( \nu \) is the unit normal to \( \partial \Omega \) directed into the exterior of \( \Omega \), and \( \psi \) is a given function on \([0, T] \times \partial \Omega\). Then, \( F(D) \) is a finite sampling of values of \( u \) on a subset \( A \) of \( \partial \Omega = \partial D \setminus \partial C \). We can consider \( F = G_2 \circ G_1 \), where \( G_1(D) \) are the values of \( u \) on \([0, T] \times A\) and \( G_2 \) is a finitization operator. Then, the inverse problem for \( G_1(D) \) is the problem of finding \( D \) from the observation data \( G_1(D) \). Many researchers have studied such inverse problems in cases where the boundary condition is a Dirichlet condition, Neumann condition, Robin condition, or mixed condition, e.g., [3], [4], [5], [7], [9], [11], [12], [14], [13], [16], and [19].

In general shape identification inverse problems and in our model problem, the regularity (i.e., smoothness) of the unknown shape is important. We intend to proceed with discussions under as general an assumption as
possible. We consider our model problem under the assumption that the unknown shape is Lipschitz continuous. [16] showed the uniqueness for \( G_1(D) \) of our model problem for the case where the unknown shape is Lipschitz continuous. [3] and [9] showed such inverse problems have only the logarithmic stability for the observational data in the case of a Dirichlet problem, and a Robin problem, for the case where the unknown shape is slightly smoother. (The logarithmic stability means that the Hausdorff distance between domains can be estimated by some negative power of the logarithm of the \( L^2 \) distance between data boundary value functions.) This logarithmic stability gives the largest possible error of the estimates.

However, in the Bayesian formulation, we can obtain the (locally) Lipschitz stability of the probability distribution of unknown domains for our general shape identification inverse problem and our model problem, even if the observation data are a finite set and the inverse problem is underdetermined. In addition to the probability distribution on \( D \), we consider for each point \( x \), the conditional probability that the point \( x \) is included in the unknown domain given the observation data \( y \), and we call this probability the “domain ratio”. We also study the stability of the domain ratio in the Bayesian framework.

The Bayesian formulation for our inverse problem is given by

\[
P(\text{domain}|\text{data}) \propto P(\text{data}|\text{domain})P(\text{domain}),
\]

and the posterior, \( P(\text{domain}|\text{data}) \), is the probability distribution on \( D \). In our framework, some properties of \( F(D) \) are important, where \( F(D) \) is included in the likelihood, \( P(\text{data}|\text{domain}) \). To consider our inverse problem in the Bayesian formulation, we need the measurability of the function \( F \). Moreover, to obtain the stabilities of the posterior and domain ratio considered in this study, we need the boundedness of the image of \( D \) under \( F \). For our model problem, we show such properties using some results of [24] that studied probabilistic representations of solutions to parabolic equations, that is, Feynman-Kac type formulas. Many reconstruction methods have been considered for our model problem and similar problems, e.g., [5], [7], [11], [12], [14], [15], and [19]. The (locally) Lipschitz stability of the posterior density ensures that the estimation result obtained by such a method with sufficient data is generally reliable, if the observational noise is Gaussian and \( F \) has the above measurability and boundedness.

Although it seems that there are no research results for our model problem in the Bayesian formulation, many researchers have studied inverse problems
arising from partial differential equations in the Bayesian formulation, e.g., [1], [2], [3], [13], [18], [20], [25], [26], [27], and [28]. In particular, for Bayesian geometric inverse problems, [13] has proposed an excellent Bayesian level set method with the preconditioned Crank-Nicolson MCMC algorithm. To apply this method to our model problem, we have to establish almost sure continuity of the level set map. It is an open problem. Whether applying the Bayesian level set method or the shape identification approach of this paper to our model problem with an MCMC algorithm, we have to solve (1) using some numerical algorithm to calculate the acceptance probability. Then, if the proposal is accepted, the shape of \( \partial C \) is changed and we have to solve (1) on the new domain. It has a high computational complexity (cf. [15]). Furthermore, if we use our shape identification approach, how to select a proposal is also an open problem.

The remainder of this paper is organized as follows. In section 2, we state our general shape identification inverse problem, model problem, and their Bayesian formulations. In section 3, we show the (locally) Lipschitz stabilities of the posterior density and domain ratio with respect to the observation data of our problems. Section 4 is an appendix. In this section, some definitions and a remark are described.

The contents of this paper are outlined in Appendix B of [24].

2. Shape identification inverse problems in a Bayesian framework and model problem

2.1. Shape identification inverse problems in a Bayesian framework

Let \( d \) be a positive integer and denote by \( \mathcal{D}^d \) the set of bounded domains (connected open subsets) in \( \mathbb{R}^d \). We consider a finite or infinite subfamily \( \mathcal{D} \) of \( \mathcal{D}^d \). Let \( m \) be a positive integer and \( F \) be a map from \( \mathcal{D} \) to \( \mathbb{R}^m \). We call \( F \) a *forward operator*. We assume that, for a given domain \( D \in \mathcal{D} \), we can observe data \( y \in \mathbb{R}^m \) with noise \( \eta \in \mathbb{R}^m \),

\[
y = F(D) + \eta,
\]

where \( \eta \) is an \( m \)-dimensional Gaussian random variable. Many researches have considered the case that \( \eta \) follows an \( m \)-dimensional Gaussian distribution with some covariance matrix or considered more general situations (e.g. [10]). For simplicity, we assume that the elements \( \{ \eta_i : i = 1, 2, \ldots, m \} \) of \( \eta = (\eta_i) \) are mutually independent and \( \eta_i \sim \mathcal{N}(0, \sigma^2) \), that is, each \( \eta_i \) follows the Gaussian distribution with mean 0 and variance \( \sigma^2 \).
We are concerned with the following inverse problem.

**Problem 2.1.** Infer \( D \in \mathcal{D} \) from given data \( y \) of (2).

We do not necessarily assume the injectivity of \( F \). Therefore, Problem 2.1 may be underdetermined in general. In this study, we consider Problem 2.1 in a Bayesian framework based on [22] and [10]. In fact, \( \mu_0, \Psi, \mu \) and \( \mu^y \) defined below play the following roles:

- \( \mu_0 \) is the prior distribution of \( D \);
- \( \Psi \) is the likelihood;
- \( \mu \) is the joint distribution of \((D, y)\);
- \( \mu^y \) is the posterior distribution of \( D \) given \( y \),

where \( \mu^y \) is obtained as a disintegration of \( \mu \).

Let \( \mathcal{B} \) be a \( \sigma \)-field on \( \mathcal{D} \). We always require that \( F \) is a measurable map on \((\mathcal{D}, \mathcal{B})\). Let \( \mu_0 \) be a probability measure on \((\mathcal{D}, \mathcal{B})\) and denote by \( \| \cdot \| \) the standard Euclidean norm. Define

\[
\Psi(D; y) := \frac{1}{(2\pi\sigma^2)^{m/2}} \exp \left( -\frac{1}{2\sigma^2} \| y - F(D) \|^2 \right) \quad (D \in \mathcal{D}, y \in \mathbb{R}^m)
\]

(3)

and

\[
Z_\Psi(y) := \int_{\mathcal{D}} \Psi(D; y) \mu_0(dD) \quad (y \in \mathbb{R}^m).
\]

Denote by \( dy \) the Lebesgue measure on \( \mathbb{R}^m \) and define a measure \( Q_0 \) on \( \mathbb{R}^m \) by

\[
Q_0(dy) := Z_\Psi(y) dy.
\]

As

\[
\int_{\mathbb{R}^m} Q_0(dy) = \int_{\mathbb{R}^m} \left\{ \int_{\mathcal{D}} \Psi(D; y) \mu_0(dD) \right\} dy = \int_{\mathcal{D}} \left\{ \int_{\mathbb{R}^m} \Psi(D; y) dy \right\} \mu_0(dD) = 1
\]

from Fubini’s theorem, \( Q_0 \) is a probability measure. Define a measure \( \mu \) on \( \mathcal{D} \times \mathbb{R}^m \) by

\[
\mu(dD, dy) := \frac{1}{Z_\Psi(y)} \Psi(D; y) \mu_0(dD) Q_0(dy) = \Psi(D; y) \mu_0(dD) dy.
\]
Then, \( \mu \) is a probability measure, the joint distribution of \((D, y)\), and \(D\) and \(y\) are not independent with respect to \(\mu\). For \( y \in \mathbb{R}^m \), define a measure \( \mu^y \) on \(D\) by
\[
\mu^y(dD) := \frac{1}{Z_\Psi(y)} \Psi(D; y) \mu_0(dD). \tag{4}
\]
Then, \( \mu^y \) is absolutely continuous with respect to \( \mu_0 \), and \( \mu^y \) satisfies the following from Fubini’s theorem:

2.a) \( \mu^y \) is a probability measure on \(D\) for each \( y \in \mathbb{R}^m \);
2.b) for every nonnegative measurable function \( f \) on \(D \times \mathbb{R}^m\), the function \( y \mapsto \int_D f(D, y) \mu^y(dD) \)
is a measurable function on \(\mathbb{R}^m\);
2.c) for every nonnegative measurable function \( f \) on \(D \times \mathbb{R}^m\), the equation
\[
\int_{D \times \mathbb{R}^m} f(D, y) \mu(dD, dy) = \int_{\mathbb{R}^m} \left\{ \int_D f(D, y) \mu^y(dD) \right\} Q_0(dy)
\]
holds.

Therefore, \( \mu^y \) is a disintegration of \( \mu \) (see [6] p.292 for the definition of disintegration) and \( \mu^y \) is a probability measure of the conditional random variable \( D \mid y \) from the viewpoint of Kolmogorov’s approach to conditioning (see [17] V, Sect. 1, cf. [6] p.293).

**Lemma 2.1.** Let \( \lambda^y \) be a measure on \(D\) indexed by \( y \in \mathbb{R}^m \). If \( \lambda^y \) is a disintegration of \( \mu \), that is, if \( \lambda^y \) satisfies \((2.a)\) \((2.b)\), and \((2.c)\) for every nonnegative measurable function \( f \) on \(D \times \mathbb{R}^m\), the measurable set
\[
N_f := \left\{ y \in \mathbb{R}^m : \int_D f(D, y) \mu^y(dD) \neq \int_D f(D, y) \lambda^y(dD) \right\}
\]
satisfies \( Q_0(N_f) = 0 \).

**Proof.** Put
\[
N^+_f := \left\{ y \in \mathbb{R}^m : \int_D f(D, y) \mu^y(dD) > \int_D f(D, y) \lambda^y(dD) \right\},
\]
\[
N^-_f := \left\{ y \in \mathbb{R}^m : \int_D f(D, y) \mu^y(dD) < \int_D f(D, y) \lambda^y(dD) \right\}.
\]
The indicator functions $\chi_{N^+_f}(y)$ and $\chi_{N^-_f}(y)$ are measurable. If $Q_0(N^+_f) > 0$, we have
\[
\int_{\mathbb{R}^m} \left\{ \int_{D} \chi_{N^+_f}(y)f(D, y)\nu(y)\mu(D) \right\} Q_0(dy) > \int_{\mathbb{R}^m} \left\{ \int_{D} \chi_{N^-_f}(y)f(D, y)\nu(y)\lambda(D) \right\} Q_0(dy).
\]
As $\nu(y)$ and $\lambda(y)$ are disintegrations of $\mu$, we have
\[
\int_{\mathbb{R}^m} \left\{ \int_{D} \chi_{N^+_f}(y)f(D, y)\nu(y)dD \right\} Q_0(dy) = \int_{\mathbb{R}^m \times \mathbb{R}^m} \chi_{N^+_f}(y)f(D, y)d\lambda(D, dy).
\]
Thus, we have $Q_0(N^+_f) = 0.$ Similarly, we also have $Q_0(N^-_f) = 0.$

From the general theory as described in [4], we cannot directly obtain the strong uniqueness of the disintegration of $\mu$, that is, the property that $\cup_f N_f$ is measurable and $Q_0(\cup_f N_f)$ is zero. However, $\nu(y)$ is a probability measure of the conditional random variable $D|y$ and we can consider (4) as a Bayesian formula as stated above. Note that $\Psi(D; y)$ and $Z\Psi(y)$ have the same factor $1/(2\pi\sigma^2)^{m/2}$. Here and in the following, we use the same representation of $\mu^y$ as used in [22] and [10], that is, we define
\[
\Phi(D; y) := \exp \left( -\frac{1}{2\sigma^2} \|y - F(D)\|^2 \right) \quad (5)
\]
and
\[
Z(y) := \int_{D} \Phi(D; y)\mu_0(dD). \quad (6)
\]
Then, we have
\[
\nu^y(dD) = \frac{1}{Z(y)}\Phi(D; y)\mu_0(dD). \quad (7)
\]
We call $\Phi$ the potential.

We also consider, for each $x \in \mathbb{R}^d$, the conditional probability that the point $x$ is included in the unknown domain given the observation data $y$. Define
\[
\rho(x|y) := \int_{D} \chi_{D}(x)\nu^y(dD) \quad (x \in \mathbb{R}^d),
\]
where
\[
\chi_{D}(x) := \begin{cases} 
1 & (x \in D) \\
0 & (x \in D^c),
\end{cases}
\]
where $D^c$ is the complement set of $D$. We call $\rho(x|y)$ the domain ratio of $x$ under $y$.

In this study, the conditional probability $\mu^y$ and the domain ratio $\rho(\cdot|y)$ play major roles, that is, Problem 2.1 is reformulated as follows.

**Problem 2.2.** Obtain $\mu^y$ on $D$ and $\rho(\cdot|y)$ on $\mathbb{R}^d$ from given data $y$ of (2).

### 2.2. Hausdorff distance

A Lipschitz domain is a domain such that its boundary is Lipschitz continuous (more precisely, see Definition 4.1). Our model problem in this study is defined on bounded Lipschitz domains (see subsection 2.3). In the following, we denote by $D_{\text{Lip}}$ the family of bounded Lipschitz domains in $\mathbb{R}^d$. Then, $D_{\text{Lip}} \subset D^\#$. When we consider our model problem, the considered set $D$ is a subfamily of $D_{\text{Lip}}$, and we assume that $D$ is equipped with the Hausdorff distance. For $\epsilon > 0$ and $A \subset \mathbb{R}^d$, put $O_\epsilon(A) := \{x \in \mathbb{R}^d : d(x, A) < \epsilon\}$, where $d(x, A)$ is the Euclidean distance between $x$ and $A$. Then, the Hausdorff distance between domains $D_1$ and $D_2$ of $D$, $d_H(D_1, D_2)$, is defined by

$$d_H(D_1, D_2) := \inf \{\epsilon > 0 : D_1 \subset O_\epsilon(D_2), \ D_2 \subset O_\epsilon(D_1)\}.$$ 

We assume that $\mathcal{B}$ is the topological $\sigma$-field ($\sigma$-field generated by the open sets of $D$) with respect to $d_H$, and require that $F$ is measurable on $(D, \mathcal{B})$. Note that $D$ is in general not a complete metric space if $D$ is an infinite set. To apply the results of [22] and [10] to our problem directly, $D$ must be complete. This completeness ensures that we can use the theory of disintegration (cf. [6]), and then the strong uniqueness of the disintegration holds. When $D$ is not a complete metric space, we cannot directly obtain the strong uniqueness from the general theory as described in [6]. However, we can consider our problem based on [22] and [10] as mentioned in subsection 2.1.

**Remark 2.1.** We do not consider the completion of $D$. The reason is that $F$ can not be extended naturally to a function on the completion in general.

### 2.3. Model problem

In this study, we consider the following example of Problem 2.2, and we call it a model problem. For a subset $U$ of $\mathbb{R}^d$, denote by $\overline{U}$ the closure of $U$. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ and let $C$ be the union of several (one or more) disjoint simply-connected Lipschitz domains (that are not tangential to each other) such that $\overline{C} \subset \Omega$. Set $D := \Omega \setminus \overline{C}$. Let $D$
denote a heat conductor, and let $C$ denote the union of one or more cavities inside $D$. In the following, we fix $\Omega$ and attempt to identify the unknown cavity/cavities from all the possible cavities. Define $D = \mathcal{D}_{hc} \subset \mathcal{D}_{Lip}$ as the family of such heat conductors with cavities. We consider the initial boundary value problem for the heat equation, (1), on $D \in \mathcal{D}_{hc}$. In (1), $T$ is a positive real number, $\Delta$ is the standard Laplacian (the coefficient $1/2$ is suitable for a Feynman-Kac type formula), $\nu$ is the unit normal to $\partial \Omega$ directed into the exterior of $\Omega$, and $\psi$ is a given $L^2$ function on $[0, T] \times \partial \Omega$.

The existence and uniqueness of weak solutions to the initial boundary value problem (1) are well established (see Definition 4.2 for the definition of a weak solution). We denote the weak solution by $u^D$. Let $A$ be an open subset of $\partial \Omega$ (an accessible portion of $\partial \Omega$). Then, we define a map $G_1$ by

$$G_1 : \mathcal{D}_{hc} \longrightarrow L^2([0, T] \times A), \quad D \mapsto u^D \big|_A.$$ 

Let $G_2$ be a suitable finitization (discretization) map, that is, a bounded linear operator

$$G_2 : L^2([0, T] \times A) \longrightarrow \mathbb{R}^m \quad (m \in \mathbb{N}).$$

Then, we define a forward operator $F$ by $F := G_2 \circ G_1$. For example, $G_2$ is constructed as follows. (Other construction methods of $G_2$ have been considered. For example, see [13].) Discretize $[0, T]$ into $m_T$ intervals,

$$[0, T] = \bigcup_{i=0}^{m_T-1} [t_i, t_{i+1}],$$

where $t_0 = 0$, $t_{m_T} = T$, and $t_{i+1} - t_i = T/m_T$ ($i = 0, 1, \ldots, m_T - 1$). Discretize $A$ into $m_A$ subsets,

$$A = \bigcup_{j=1}^{m_A} A_j,$$

where the Lebesgue measures of $A_1, \ldots, A_{m_A}$ are equal and the Lebesgue measure of $A_i \cap A_j$ is zero if $i \neq j$. Let $m = m_T m_A$, and define $\{v_{i,j} : i = 0, 1, \ldots, m_T - 1; j = 1, \ldots, m_A\}$ by

$$v_{ij} := \int_{t_i}^{t_{i+1}} dt \int_{A_j} v(t, x) dx \quad (8)$$

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for $v \in L^2([0,T] \times A)$. This gives

$$G_2 : L^2([0,T] \times A) \rightarrow \mathbb{R}^m, \quad v \mapsto \{v_{ij}\}.$$ 

Denote by $|A|$ the Lebesgue measure of $A$. Then, we have

$$\|G_2(v)\|^2 = \sum_{i=0}^{m_T-1} \sum_{j=1}^{m_A} G_2(v)_{ij}^2 \leq m|A|\|v\|^2_{L^2([0,T] \times A)}. \quad (9)$$

We define the forward operator by

$$F := G_2 \circ G_1 : \mathcal{D} \mapsto u^D |_{A} \mapsto \left\{ (u^D |_{A})_{ij} \right\} \quad (10)$$

and consider Problem 2.2. To consider this problem, we assume that $\mathcal{D} = \mathcal{D}_{hc}$ is equipped with the Hausdorff distance and $\mathcal{B}$ is the topological $\sigma$-field. As stated in subsection 2.2, $F$ must be a measurable map on $(\mathcal{D},\mathcal{B}) = (\mathcal{D}_{hc},\mathcal{B})$.

For $\alpha \in (0,1]$, we say that a bounded domain is of class $C^{2,\alpha}$ if its boundary is of class $C^{2,\alpha}$ (more precisely, the function $\varphi$ of Definition 4.1 is of class $C^{2,\alpha}$, that is, $\varphi$ is of class $C^2$ and all the second-order derivatives are $\alpha$-Hölder continuous). We consider the following assumption.

**Assumption 2.1.** There exists $\alpha \in (0,1]$ such that $\Omega$ is of class $C^{2,\alpha}$ and $\psi$ is Lipschitz continuous.

In this study, we use a stochastic representation (Feynman-Kac type formula) of $u^D$, because this formula gives a direct relation among $u^D$, $\psi$, and cavities $C$ (see (11) and (13)). Under Assumption 2.1, [24] gave the following formula of $u^D$,

$$u^D(t,x) = E_x \left[ \int_0^{t \wedge \tau(D)} \psi(t-r,X(r))L(dr) \right] \quad ((t,x) \in [0,T] \times \overline{D}), \quad (11)$$

where $(X(t), L(t))$ is a pair of stochastic processes. The symbols on the right-hand side of (11) are defined as follows.

The stochastic process $X$ with $L$ is an $\overline{\Omega}$-valued diffusion process with a normal reflection on $\partial \Omega$ starting at $x \in A$, i.e., $X = \{X(t)\}_{0 \leq t \leq T}$ is an $\overline{\Omega}$-valued continuous stochastic process with $X(0) = x$, and $L = \{L(t)\}_{0 \leq t \leq T}$.
is a continuous, increasing stochastic process (local time) such that the pair \((X, L)\) satisfies
\[
\begin{align*}
    dX(t) &= dB(t) + N(X(t))dL(t) \quad \text{for } 0 \leq t \leq T, \\
    L(t) &= \int_0^t 1_{\partial \Omega}(X(r))L(dr) \quad \text{for } 0 \leq t \leq T,
\end{align*}
\]
(12)

where \(B\) is a standard Brownian motion on \(\mathbb{R}^d\) and \(N(X(t))\) is the unit inward normal vector at \(X(t) \in \partial \Omega\). The existence and uniqueness of such \((X, L)\), a strong solution of (12), have been proved by [21] and [23]. Here, \(\tau(D)\) is defined by
\[
\tau(D) := \inf \left\{ t : 0 \leq t \leq T, \; X(t) \in \overline{\mathcal{C}} = \Omega \setminus D \right\} \quad (\inf \emptyset := \infty), \tag{13}
\]

\(E_x\) means the expectation considering \(X(0) = x\), and \(a \wedge b := \min\{a, b\}\).

Then, Theorem 3.1 of [24] implies the following theorem.

**Theorem 2.1.** If Assumption 2.1 holds, then for \(D \in \mathcal{D}_{hc}\), the weak solution \(u^D\) of (7) is represented by (11) and \(u^D\) is continuous on \([0, T] \times \overline{\mathcal{D}}\).

Theorem 4.1 of [24] ensures the continuity of \(G_1\) with respect to the Hausdorff distance if Assumption 2.1 is satisfied. From this fact and (9), we obtain the following theorem.

**Theorem 2.2.** Under Assumption 2.1, the forward operator \(F = G_2 \circ G_1\) is continuous with respect to the Hausdorff distance on \(\mathcal{D}_{hc}\).

From this theorem, \(F\) is a measurable map on \((\mathcal{D}, \mathcal{B}) = (\mathcal{D}_{hc}, \mathcal{B})\) under Assumption 2.1.

3. Stabilities of the posterior density and domain ratio with respect to observational data

3.1. Stabilities for the general shape identification inverse problem

Let \(\mathcal{D}\) be a subfamily of \(\mathcal{D}^\sharp\). For given data \(y\) and \(y' \in \mathbb{R}^m\), define
\[
\sigma(y, y') := \sup \left\{ \|y - F(D)\| \vee \|y' - F(D)\| : D \in \mathcal{D} \right\},
\]
where \(a \vee b := \max\{a, b\}\).

**Assumption 3.1.** There exists \(C_F > 0\) such that \(\|F(D)\| < C_F\) for every \(D \in \mathcal{D}\).
Note that, under this assumption, \( \sigma(y, y') < \infty \) and
\[
\min (Z(y), Z(y')) \geq \int_D \exp \left( -\frac{1}{2\sigma^2} \sigma(y, y') \right) \mu_0(dD) = \exp \left( -\frac{1}{2\sigma^2} \sigma(y, y') \right) > 0.
\]
The Hellinger distance between \( \mu^y \) and \( \mu^{y'} \) is defined by
\[
d_{\text{Hell}}(\mu^y, \mu^{y'}) := \sqrt{\frac{1}{2} \int_D \left( \frac{d\mu^y}{d\mu_0}(D) - \frac{d\mu^{y'}}{d\mu_0}(D) \right)^2 \mu_0(dD)}
= \sqrt{\frac{1}{2} \int_D \left( \frac{1}{\sqrt{Z(y)}} \Phi(D; y) - \frac{1}{\sqrt{Z(y')}} \Phi(D; y') \right)^2 \mu_0(dD)}.
\]
We have the following (locally) Lipschitz stabilities of the posterior density and domain ratio with respect to the data for our general shape identification inverse problem. Theorem 3.1 is similar to Corollary 4.4 of [22], and Theorem 3.2 is a type of stability as stated in Remark 4.6 of [10].

**Theorem 3.1.** If the forward operator \( F \) satisfies Assumption 3.1, then for given data \( y \) and \( y' \in \mathbb{R}^m \),
\[
d_{\text{Hell}}(\mu^y, \mu^{y'}) \leq \exp \left( \frac{3\sigma(y, y')^2}{4\sigma^2} \right) \frac{\sigma(y, y') \| y - y' \|}{\sigma}.
\]
Furthermore, for every \( r > 0 \), there exists \( C(r) > 0 \) such that
\[
d_{\text{Hell}}(\mu^y, \mu^{y'}) \leq C(r) \| y - y' \|
\]
for every \( y, y' \) with \( \| y \| < r, \| y' \| < r \).

**Proof.** We can prove this theorem in the same manner as the proof of Theorem 4.2 of [22]. Here, we only prove (14). For every \( D \in D \)
\[
\left\{ \frac{1}{\sqrt{Z(y)}} \exp \left( -\frac{1}{4\sigma^2} \| y - F(D) \|^2 \right) - \frac{1}{\sqrt{Z(y')}} \exp \left( -\frac{1}{4\sigma^2} \| y' - F(D) \|^2 \right) \right\}^2
\]
\[
\leq \frac{2}{Z(y)} \left\{ \exp \left( -\frac{1}{4\sigma^2} \| y - F(D) \|^2 \right) - \exp \left( -\frac{1}{4\sigma^2} \| y' - F(D) \|^2 \right) \right\}^2
+ 2 \exp \left( -\frac{1}{2\sigma^2} \| y' - F(D) \|^2 \right) \left( \frac{1}{\sqrt{Z(y)}} - \frac{1}{\sqrt{Z(y')}} \right)^2.
\]
We estimate the right-hand side. First,

$$\left\{ \exp \left( -\frac{1}{4\sigma^2} \|y - F(D)\|^2 \right) - \exp \left( -\frac{1}{4\sigma^2} \|y' - F(D)\|^2 \right) \right\}^2 \leq \left\{ \frac{1}{4\sigma^2} \|y - F(D)\|^2 - \frac{1}{4\sigma^2} \|y' - F(D)\|^2 \right\}^2 \tag{15}$$

$$= \frac{1}{4^2\sigma^4} \{(y - y') \cdot (y + y' - 2F(D))\}^2 \leq \frac{1}{4^2\sigma^4} \{(\|y - y'\| \cdot \|y + y' - 2F(D)\|)\}^2 \leq \frac{1}{4^2} \left\| \frac{\sigma(y, y')}{\sigma} \right\|^2 \cdot \sigma(y, y')^2$$

Next,

$$\left( \frac{1}{\sqrt{Z(y)}} - \frac{1}{\sqrt{Z(y')}} \right)^2 = \frac{(Z(y') - Z(y))^2}{Z(y)Z(y') \left( \sqrt{Z(y)} + \sqrt{Z(y')} \right)^2},$$

and similar to (15),

$$|Z(y) - Z(y')| \leq \left| \exp \left( -\frac{1}{2\sigma^2} \|y - F(D)\|^2 \right) - \exp \left( -\frac{1}{2\sigma^2} \|y' - F(D)\|^2 \right) \right|$$

$$\leq \frac{1}{2\sigma^2} \|y - y'\| \cdot (\|y - F(D)\| + \|y' - F(D)\|) \leq \frac{\|y - y'\| \cdot \sigma(y, y')}{\sigma} \cdot \frac{\sigma(y, y')}{\sigma}.$$

Moreover,

$$\min\{Z(y), Z(y')\} \geq \exp \left( -\frac{\sigma(y, y')^2}{2\sigma^2} \right) \tag{17}$$

and

$$Z(y)Z(y') \left( \sqrt{Z(y)} + \sqrt{Z(y')} \right)^2 \geq Z(y)Z(y') (Z(y) + Z(y')) \geq 2 \exp \left( -\frac{3\sigma(y, y')^2}{2\sigma^2} \right).$$

Therefore,

$$\left\{ \frac{1}{\sqrt{Z(y)}} \exp \left( -\frac{1}{4\sigma^2} \|y - F(D)\|^2 \right) - \frac{1}{\sqrt{Z(y')}} \exp \left( -\frac{1}{4\sigma^2} \|y' - F(D)\|^2 \right) \right\}^2$$
Furthermore, for every \(r > 0\).

Thus, we have (14). □

**Theorem 3.2.** If the forward operator \(F\) satisfies Assumption 3.1, then for given data \(y, y' \in \mathbb{R}^m\), and for every \(x \in \mathbb{R}^d\) we have

\[
|\rho(x|y) - \rho(x|y')| \leq 2 \exp\left(\frac{3\sigma(y, y')^2}{2\sigma^2}\right) \frac{\sigma(y, y')}{\sigma^2} \frac{\|y - y'\|^2}{\sigma^2}.
\]

Furthermore, for every \(r > 0\), there exists \(C(r) > 0\) such that

\[
|\rho(x|y) - \rho(x|y')| \leq C(r)\|y - y'\|
\]

for every \(y, y'\) with \(\|y\| < r, \|y'\| < r\).

**Proof.** We only prove (18). It holds that

\[
|\rho(x|y) - \rho(x|y')| = \left| \int_{\mathcal{D}} \chi_D(x) \frac{d\mu_y}{d\mu_0}(D) \mu_0(dD) - \int_{\mathcal{D}} \chi_D(x) \frac{d\mu_y'}{d\mu_0}(D) \mu_0(dD) \right|
\]

\[
\leq \int_{\mathcal{D}} \chi_D(x) \left| \frac{d\mu_y}{d\mu_0}(D) - \frac{d\mu_y'}{d\mu_0}(D) \right| \mu_0(dD)
\]

\[
\leq \int_{\mathcal{D}} \left| \frac{d\mu_y}{d\mu_0}(D) - \frac{d\mu_y'}{d\mu_0}(D) \right| \mu_0(dD) =: \left\| \mu_y - \mu_y' \right\|_{L^1(\mathcal{D})}.
\]

We can estimate this using \(\|\mu_y - \mu_y'\|_{L^1(\mathcal{D})} \leq 2\sqrt{2} \ d_{\text{Hell}}(\mu_y, \mu_y')\) and Theorem 3.1. Here, we estimate \(\|\mu_y - \mu_y'\|_{L^1(\mathcal{D})}\) directly:

\[
\left\| \mu_y - \mu_y' \right\|_{L^1(\mathcal{D})} = \int_{\mathcal{D}} \left| \frac{d\mu_y}{d\mu_0}(D) - \frac{d\mu_y'}{d\mu_0}(D) \right| \mu_0(dD)
\]

\[
= \int_{\mathcal{D}} \left| \frac{\Phi(D; y)}{Z(y)} - \frac{\Phi(D; y')}{Z(y')} \right| \mu_0(dD)
\]

\[
\leq \frac{1}{Z(y)Z(y')} \int_{\mathcal{D}} \{\Phi(D; y')|Z(y) - Z(y')| + |Z(y')|\Phi(D; y) - \Phi(D; y')\} \mu_0(dD)
\]

\[
= \frac{1}{Z(y)} \left\{ |Z(y) - Z(y')| + \int_{\mathcal{D}} |\Phi(D; y) - \Phi(D; y')| \mu_0(dD) \right\}
\]

\[
\leq \frac{2}{Z(y)} \int_{\mathcal{D}} |\Phi(D; y) - \Phi(D; y')| \mu_0(dD).
\]
Then, in the same manner as (16) with (17), we have
\[
\left\| \mu^y - \mu^{y'} \right\|_{L^1(D)} \leq 2 \exp \left( \frac{\sigma(y, y')^2}{2\sigma^2} \right) \frac{\sigma(y, y')}{\sigma} \| y - y' \|.
\] (19)
Thus, we have (18). \(\square\)

As the data \(y\) and \(y'\) are normally distributed with mean \(F(D)\) and the components of \(y\) and \(y'\) have variance \(\sigma^2\), we have
\[
E \left[ \| y - F(D) \|^2 \right] = m\sigma^2.
\]
Then, we probably have \(\| y - y' \| \gtrsim \sigma\). In that case, Theorems 3.1 and 3.2 are meaningless. Therefore, we consider these theorems as follows. Let \(N\) be a sufficiently large integer and let \(y^{(1)}, y^{(2)}, \ldots, y^{(N)}\) be data. Set
\[
y := \frac{1}{N} \sum_{i=1}^{N} y^{(i)}
\]
and \(y' := F(D)\) (the unknown true value). Then, \(\| y - y' \| \ll \sigma\) and \(\sigma(y, y') \ll \sigma\). Thus, we can consider Theorems 3.1 and 3.2 as giving meaningful results about the stabilities of \(\mu^y\) and \(\rho(x|y)\).

3.2. Case of the model problem

Here, we will apply the results of subsection 3.1 to the model problem of subsection 2.3.

**Theorem 3.3.** If Assumption 2.1 holds, then for the forward operator \(F\) defined by (10), there exists \(C_F > 0\) such that \(\| F(D) \| < C_F \) for every \(D \in D_{hc}\). Therefore, \(F\) satisfies Assumption 3.1 on \(D_{hc}\).

**Proof.** Decompose \(\psi\) into \(\psi(t, x) = \psi_+(t, x) + \psi_-(t, x)\), where
\[
\psi_+(t, x) := \begin{cases} 
\psi(t, x) & (\psi(t, x) \geq 0) \\
0 & (\psi(t, x) < 0)
\end{cases}, \quad \psi_-(t, x) := \begin{cases} 
\psi(t, x) & (\psi(t, x) < 0) \\
0 & (\psi(t, x) \geq 0)
\end{cases}.
\]
Then, from (11),
\[
E_x \left[ \int_0^{t \wedge \tau(D)} \psi_-(t - r, X(r)) L(dr) \right] \leq u^D(t, x) \leq E_x \left[ \int_0^{t \wedge \tau(D)} \psi_+(t - r, X(r)) L(dr) \right].
\]
Therefore, for every $D \in \mathcal{D}_{hc}$,

$$E_x \left[ \int_0^t \psi_-(t - r, X(r)) L(\,dr) \right] \leq u^D(t, x) \leq E_x \left[ \int_0^t \psi_+(t - r, X(r)) L(\,dr) \right].$$

Thus, using (9), we obtain the theorem.

Owing to this theorem, we can apply the results of subsection 3.1 to the model problem of subsection 2.3 on $\mathcal{D} = \mathcal{D}_{hc}$, that is, Theorems 3.1 and 3.2 hold for the problem.

4. Appendix. Definitions and Remark

Definition 4.1 (Lipschitz domain). Let $D$ be a bounded domain of $\mathbb{R}^d$. We call $D$ a bounded Lipschitz domain if, for every $x \in \partial D$, there exist a neighborhood $U(x)$ of $x$ and a function $\varphi : \mathbb{R}^{d - 1} \to \mathbb{R}$ such that:

(i) there exists $C_\varphi > 0$ such that

$$|\varphi (\xi) - \varphi (\xi')| \leq C_\varphi \|\xi - \xi'\|$$

for every $\xi, \xi' \in \mathbb{R}^{d - 1}$;

(ii) there exists an open neighborhood $O$ of the origin of $\mathbb{R}^d$ such that $D \cap U(x)$ can be transformed into

$$O \cap \{(\xi, \xi_d) \in \mathbb{R}^d = \mathbb{R}^{d - 1} \times \mathbb{R} : \xi_d > \varphi (\xi)\}$$

by a rigid motion $T_x$, that is, by a rotation plus a translation, and $T_x(x)$ is the origin.

Definition 4.2 (Weak solution to problem (1)). Put

$$\partial_t := \frac{\partial}{\partial t}, \quad \partial_{x_j} := \frac{\partial}{\partial x_j} \ (j = 1, \ldots, d).$$

Define

$$H^1(D) := \{ f \in L^2(D) : \partial_{x_j} f \in L^2(D) \ (j = 1, \ldots, d) \},$$

$$C([0, T]; L^2(D)) := \{ f : [0, T] \to L^2(D) : f \text{ is continuous} \},$$

$$V^{0,1}((0, T) \times D) := C([0, T]; L^2(D)) \cap L^2((0, T); H^1(D)),$$

$$H^{1,1}((0, T) \times D) := \{ f \in L^2((0, T) \times D) : \partial_t f, \partial_{x_j} f \in L^2((0, T) \times D) \ (j = 1, \ldots, d) \}.$$
and
\[ \nabla_x f := (\partial_{x_1} f, \ldots, \partial_{x_d} f). \]
The function spaces \( H^1(D) \) and \( H^{1,1}((0,T) \times D) \) are equipped with the usual Sobolev norms, and the function space \( V^{0,1}((0,T) \times D) \) is equipped with the norm \( \| \cdot \|_D \) defined by
\[ \|u\|_D := \max_{0 \leq t \leq T} \|u(t,\cdot)\|_{L^2(D)} + \|\nabla_x u\|_{L^2((0,T) \times D)}. \]

Denote by \( \gamma \) the boundary trace operator from \( H^1(D) \) into \( L^2(\partial D) \). A weak solution \( u^D \) to problem (1) is a function such that it belongs to \( V^{0,1}((0,T) \times D) \) and satisfies the weak form of (1), that is:

(i) \( u^D|_{\partial C} = 0 \), that is, \( \gamma u^D(t, \cdot) = 0 \) on \( \partial C \) for almost every \( t \in (0,T) \);
(ii) for every \( \eta \in H^{1,1}(D) \) with \( \eta|_{\partial C} = 0 \) and \( \eta(T, \cdot) = 0 \), it holds that
\[ \int_D u^D \partial_t \eta \, dx - \int_D \nabla_x u^D \cdot \nabla_x \eta \, dx + \int_{\partial \Omega} \psi \gamma \eta \, dt S(dx) = 0, \]
where \( S(dx) \) is the Lebesgue measure on \( \partial \Omega \).

Remark 4.1 (Stochastic representation formula for a general parabolic problem). A rather more general (backward) parabolic boundary value problem than (1) was considered in [24]. In particular, the Dirichlet boundary and the Neumann boundary are allowed to meet, and the Dirichlet boundary is allowed to vary with time. For the weak solution to the problem, the stochastic representation and continuity property, which are generalizations of Theorems 2.1 and 2.2, were shown through the coupled martingale formulation for the basic diffusion process. If the weak solution \( u^D \) to (1) has a smooth extension belonging to \( C^{1,2}([0,T] \times \mathbb{R}^d) \), we can prove Theorem 2.1 using Itô’s formula. However, in a general case, we have to apply the formula to the weak solution via a suitable smooth approximation procedure for the solution as in [24].

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Abstract
The strong uniqueness of disintegration holds if the \( \sigma \)-algebra on the considered space is countably generated. This strong uniqueness yields the strong uniqueness of the posterior as a disintegrating measure in the Bayesian inverse problem discussed in [4].

keywords: Disintegration, Uniqueness, Bayesian inverse problem, Posterior

2010 MSC: 28A50, 35R30, 60A10, 62F15

1 Introduction

In this addendum to [4], we consider the weak and strong uniqueness of disintegration and an application of strong uniqueness to the Bayesian inverse problem discussed in [4]. Experts will already know or be able to show the results of this addendum, but the author believes that the results are worth stating for a more general audience.

As stated in [2] on p.294, we can consider a conditional probability as a disintegrating measure. The weak uniqueness of disintegrations yields the well-definedness of conditional probability measures from the viewpoint of Kolmogorov’s approach to conditioning (see [5] V, Sect. 1, cf. [2] p.293). In [4], the author considered shape identification inverse problems in a Bayesian framework. The posterior is considered as a disintegration of the joint distribution. In [4], this posterior is constructed specifically (therefore, the existence of the posterior is ensured) and only weak uniqueness is proved. Thus, strong uniqueness is the remaining problem.

According to Theorem 1 (existence theorem) in [2], a \( \sigma \)-finite Radon measure on a metric space has a disintegration and the strong uniqueness of that disintegration holds. It is unclear whether or not the joint distribution in [4] is a Radon measure. However, it can be shown that the strong uniqueness of disintegration holds generally if the \( \sigma \)-algebra on a considered space is countably generated (see Theorem 2.1). It can be also shown that the \( \sigma \)-algebra on the considered (product) space in [4] is countably generated (see Lemma 3.1 and the proof of Theorem 3.1). Therefore, the posterior in [4] satisfies strong uniqueness as a disintegrating measure (see Theorem 3.1).
2 Uniqueness theorem of disintegration

Let \((\mathcal{X}, \mathcal{A}_X, \mu_X)\) and \((\mathcal{Y}, \mathcal{A}_Y, \mu_Y)\) be measure spaces, where \(\mathcal{A}_X\) and \(\mathcal{A}_Y\) are \(\sigma\)-algebras, and \(\mu_X\) and \(\mu_Y\) are \(\sigma\)-finite measures. Let \(T : (\mathcal{X}, \mathcal{A}_X) \to (\mathcal{Y}, \mathcal{A}_Y)\) be a measurable map. Assume that \(\{y\} \in \mathcal{A}_Y\) for every \(y \in \mathcal{Y}\). Let \(\mathcal{M}\) denote the set of nonnegative measurable functions on \(\mathcal{X}\).

Then, a family of measures on \(\mathcal{A}_X\), \(\{\mu_y\} (y \in \mathcal{Y})\), is called a \((T, \mu_Y)\)-disintegration of \(\mu_X\) if the following properties hold (cf. Definition 1 in [2]):

(i) \(\mu_y\) is a \(\sigma\)-finite measure on \(\mathcal{A}_X\) and \(\mu_y(\mathcal{X} \setminus T^{-1}(y)) = 0\) for \(\mu_Y\)-almost every \(y\);

and, for each \(f \in \mathcal{M}\),

(ii) \(y \mapsto \int_{\mathcal{X}} f(x) \mu_y(dx)\) is a measurable function;

(iii) \(\int_{\mathcal{X}} f(x) \mu_X(dx) = \int_{\mathcal{Y}} \left\{ \int_{\mathcal{X}} f(x) \mu_y(dx) \right\} \mu_Y(dy)\).

Let \(\mu_1^y\) and \(\mu_2^y\) be \((T, \mu_Y)\)-disintegrations of \(\mu_X\). For each \(f \in \mathcal{M}\), we set

\[ N_f := \left\{ y \in \mathcal{Y} : \int_{\mathcal{X}} f(x) \mu_1^y(dx) \neq \int_{\mathcal{X}} f(x) \mu_2^y(dx) \right\}. \]

According to [ii], \(N_f\) is a measurable set. We define the weak uniqueness and strong uniqueness of the \((T, \mu_Y)\)-disintegration of \(\mu_X\) as follows:

- Weak uniqueness implies that
  \[ \mu_Y(N_f) = 0 \]
  holds for every \(f \in \mathcal{M}\) and every pair \((\mu_1^y, \mu_2^y)\) of \((T, \mu_Y)\)-disintegrations of \(\mu_X\).

- Strong uniqueness implies that \(\bigcup_{f \in \mathcal{M}} N_f\) is a measurable set and
  \[ \mu_Y\left( \bigcup_{f \in \mathcal{M}} N_f \right) = 0 \]
  holds for every pair \((\mu_1^y, \mu_2^y)\) of \((T, \mu_Y)\)-disintegrations of \(\mu_X\).

**Theorem 2.1** The weak uniqueness of the \((T, \mu_Y)\)-disintegration of \(\mu_X\) always holds. If \(\mathcal{A}_X\) is countably generated (i.e., there exists a countable subset \(\mathcal{C}\) of \(\mathcal{A}_X\) such that \(\sigma(\mathcal{C}) = \mathcal{A}_X\)), then the strong uniqueness of the \((T, \mu_Y)\)-disintegration of \(\mu_X\) also holds.

**Proof.** First, we prove weak uniqueness. Let \(\mu_1^y\) and \(\mu_2^y\) be \((T, \mu_Y)\)-disintegrations of \(\mu_X\). For each \(f \in \mathcal{M}\), we set

\[ N_f^+ := \left\{ y \in \mathcal{Y} : \int_{\mathcal{X}} f(x) \mu_1^y(dx) > \int_{\mathcal{X}} f(x) \mu_2^y(dx) \right\}, \]

\[ N_f^- := \left\{ y \in \mathcal{Y} : \int_{\mathcal{X}} f(x) \mu_1^y(dx) < \int_{\mathcal{X}} f(x) \mu_2^y(dx) \right\}. \]

2
From [ii], \( N^+_f \) and \( N^-_f \) are measurable sets. Let \( \chi_A \) denote the indicator function of a set \( A \). From [i]

\[
N^+_f = \left\{ y \in \mathcal{Y} : \int_X \chi_{T^{-1}(y)}(x)f(x)\mu_1(dx) > \int_X \chi_{T^{-1}(y)}(x)f(x)\mu_2(dx) \right\}.
\]

Note that for \( j = 1, 2 \) and almost every \( y \in N^+_f \),

\[
\int_X \chi_{T^{-1}(y)}(x)f(x)\mu_j(dx) = \int_X \chi_{T^{-1}(N^+_f)}(x)f(x)\mu_j(dx).
\]

Therefore, if \( \mu_{\mathcal{Y}}(N^+_f) \neq 0 \), then

\[
\int_{\mathcal{Y}} \left\{ \int_X \chi_{T^{-1}(N^+_f)}(x)f(x)\mu_1(dx) \right\} \mu_{\mathcal{Y}}(dy) > \int_{\mathcal{Y}} \left\{ \int_X \chi_{T^{-1}(N^+_f)}(x)f(x)\mu_2(dx) \right\} \mu_{\mathcal{Y}}(dy).
\]

In contrast, from [iii]

\[
\int_{\mathcal{Y}} \left\{ \int_X \chi_{T^{-1}(N^+_f)}(x)f(x)\mu_2(dx) \right\} \mu_{\mathcal{Y}}(dy) = \int_{\mathcal{Y}} \left\{ \int_X \chi_{T^{-1}(N^+_f)}(x)f(x)\mu_1(dx) \right\} \mu_{\mathcal{Y}}(dy).
\]

Therefore, \( \mu_{\mathcal{Y}}(N^+_f) = 0 \). Similarly, \( \mu_{\mathcal{Y}}(N^-_f) = 0 \). Finally, we have \( \mu_{\mathcal{Y}}(N_f) = 0 \), which proves weak uniqueness.

Next, we prove strong uniqueness. It is equivalent to that

\[
\mu_{\mathcal{Y}}(\{y \in \mathcal{Y} : \mu_1(y) \neq \mu_2(y)\}) = 0 \tag{2.1}
\]

holds if \( \mu_1 \) and \( \mu_2 \) are (\( T, \mu_{\mathcal{Y}} \))-disintegrations of \( \mu_X \). Let \( \tilde{A}_X \) be the algebra generated by \( \mathcal{C} \). Then, \( A_X = \sigma(\tilde{A}_X) \) and \( \tilde{A}_X \) is countable because the family of all finite subsets of a countable set is also countable. We set

\[
N := \left\{ y \in \mathcal{Y} : \exists A \in \tilde{A}_X, \mu_1(y(A)) \neq \mu_2(y(A)) \right\}.
\]

Then, based on weak uniqueness, we have \( \mu_{\mathcal{Y}}(N) = 0 \). For every \( y \notin N \) and every \( A \in \tilde{A}_X \),

\[
\mu_1(y(A)) = \mu_2(y(A)).
\]

Because \( \mu_1 \) and \( \mu_2 \) are \( \sigma \)-finite and \( A_X = \sigma(\tilde{A}_X) \), we have

\[
\mu_1(y(A)) = \mu_2(y(A))
\]

for every \( y \notin N \) and every \( A \in A_X \) according to Dynkin’s extension theorem (Lemma 1.1 in [3], Theorem 8.5 in [1]) or the extension theorem called Hahn-Kolmogorov’s theorem (Theorem 1.7.8 in [7]), Carathéodory’s theorem (Theorem 6.2 in [1]), E. Hopf’s theorem (5.5 in [6]), or other names (cf. 5.16 in [6]). Therefore, we have (2.1), which proves strong uniqueness.
Remark 2.1 In [2], the strong uniqueness of the disintegration of a \( \sigma \)-finite Radon measure on a metric space \( \mathcal{X} \) is proved by reducing it to the case where \( \mathcal{X} \) is a compact metric space. Because the Borel field (i.e., the \( \sigma \)-algebra generated by the open sets) on a compact metric space is countably generated, this result also follows from Theorem 2.1.

3 Application of Theorem 2.1 to the framework in [4]

From Subsection 2.2 and onward in [4], the disintegration is considered in the following context. Let \( d \) be a positive integer and \( \mathcal{D}^\# \) be the family of all bounded domains (i.e., bounded connected open subsets) in \( \mathbb{R}^d \). Assume that \( \mathcal{D}^\# \) is equipped with the Hausdorff distance \( d_H \). Let \( \mathcal{D} \) be a metric subspace of \( \mathcal{D}^\# \), \( \mathcal{B}_D \) be the Borel field of \( \mathcal{D} \), and \( \mathcal{B}_{\mathbb{R}^m} \) be the Borel field of \( \mathbb{R}^m \), where \( m \) is a positive integer. Assume that \( \mathcal{X} = \mathcal{D} \times \mathbb{R}^m \) (the product metric space), \( \mathcal{A}_\mathcal{X} = \mathcal{B}_D \times \mathcal{B}_{\mathbb{R}^m} \) (the product \( \sigma \)-algebra), \( \mathcal{Y} = \mathbb{R}^m \), \( \mathcal{A}_\mathcal{Y} = \mathcal{B}_{\mathbb{R}^m} \), \( T : \mathcal{X} \rightarrow \mathcal{Y} \) is defined by \( T(D, y) = y \) (\( (D, y) \in \mathcal{D} \times \mathbb{R}^m \)), and \( \mu_\mathcal{X} \) and \( \mu_\mathcal{Y} \) are probability measures (\( \mu_\mathcal{X} \) is a joint distribution). Then, a \( (T, \mu_\mathcal{Y}) \)-disintegration \( \mu_\mathcal{Y} \) of \( \mu_\mathcal{X} \) is the posterior, that is, the probability measure of the conditional random variable \( D|y \), and we have

\[
\int_{\mathcal{X}} f(D, y) \mu_\mathcal{Y}(dD) = \int_{T^{-1}(y)} f(D, y) \mu_\mathcal{Y}(dD)
\]

\[
= \int_{\mathcal{D} \times \{y\}} f(D, y) \mu_\mathcal{Y}(dD) = \int_{\mathcal{D}} f(D, y) \mu_\mathcal{Y}(dD)
\]

for \( \mu_\mathcal{Y} \)-almost every \( y \) and every \( f \in \mathcal{M} \).

In [4], the author only proved the weak uniqueness of the disintegration in this context. However, the following theorem holds.

Theorem 3.1 In the context described above, the strong uniqueness of the \( (T, \mu_\mathcal{Y}) \)-disintegration of \( \mu_\mathcal{X} \) holds.

This theorem is derived from the following Lemma 3.1 and Theorem 2.1.

Lemma 3.1 Every subspace \( \mathcal{D} \) of \( \mathcal{D}^\# \) is a separable metric space.

Proof. If \( \mathcal{D}^\# \) is separable, then every subspace \( \mathcal{D} \) is also separable. We will now prove the separability of \( \mathcal{D}^\# \). For \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \) and \( \epsilon > 0 \), we set

\[
Q(a, \epsilon) := \lfloor a_1 \epsilon, (a_1 + 1)\epsilon \rfloor \times \cdots \times \lfloor a_d \epsilon, (a_d + 1)\epsilon \rfloor.
\]

For \( D \in \mathcal{D}^\# \), we set

\[
Q(D, \epsilon) := \left\{ Q(a, \epsilon) : a \in \mathbb{Z}^d, Q(a, \epsilon) \cap D \neq \emptyset \right\}.
\]

Then, \( Q(D, \epsilon) \) is a finite set for every \( \epsilon > 0 \). We set

\[
S(D, \epsilon) := \text{Int} \left( \bigcup_{Q(a, \epsilon) \in Q(D, \epsilon)} Q(a, \epsilon) \right),
\]
where “Int” indicates the interior. Then, we have $d_H(D, S(D, \epsilon)) < \sqrt{d\epsilon}$. Note that

$$Q := \left\{ Q \left( a, \frac{1}{n} \right) : a \in \mathbb{Z}^d, \; n \in \mathbb{N} \right\}$$

is a countable set. Therefore, $Q$ can be rewritten as $Q = \{Q_m : m \in \mathbb{N}\}$. We define

$$Q^* := \left\{ \text{Int} \left( \bigcup_{m \in I} Q_m \right) : I \subset \mathbb{N}, \; |I| < \infty, \; Q_m \in Q, \text{Int} \left( \bigcup_{m \in I} Q_m \right) \text{ is connected} \right\},$$

where $|I|$ is the number of the elements of $I$. Then, $Q^* \subset D^\sharp$ is also a countable set and it is dense in $D^\sharp$. Therefore, $D^\sharp$ is separable. □

Proof of Theorem 3.1. According to Lemma 3.1, the family of open subsets of $D$ is countably generated. Therefore, $B_D$ is countably generated, and $A_X$ is also countably generated. Therefore, the result follows from Theorem 2.1. □

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