From quantized spins to rotating black holes

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Abstract: It has been shown recently that classical-spin limit of minimally coupled spinning particles can reproduce the dynamics of Kerr black holes. This raises the intriguing question of how particles with finite (quantized) spins can capture the classical potential of black holes, especially since their Wilson coefficients in the framework of one-particle effective theory (EFT) were shown to be distinct. In this paper we first derive the 1 PM potential for the general EFTs to all orders in spin operators. When applied to minimal coupling, we demonstrate that in the chiral spinor basis it factorizes into a spin-independent universal piece and a spin-dependent combinatorial factor. By simply replacing the combinatoric factor by its infinite-spin limit counterpart, universality allows us to extract black hole observables via finite-spin particles. We also discuss the possible constraint universality imposes on the gravitational Compton amplitude, relevant for the 2 PM potential, as well as extending such property to non-minimal interactions.
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1 Introduction

While there has been a long history of progress in the extraction of classical quantities in gravitational systems from observables of quantum field theory, such as the metric from vacuum expectation values [1] or form factors [2, 3] and two-body potentials from scattering amplitudes [4–6], recently there has been a surge of interest in applying modern on-shell techniques to these problems. For example, the simplification of loop-level gravitational scattering amplitudes either through the double copy [7] or BCJ relations [8], has been utilized for the computation of classical potentials [9, 10]. Furthermore, the massive spinor-helicity formalism introduced by one of the authors [11], has enabled a more streamlined approach to the computation of spin-effects in the classical potential [12, 13] and scattering angle [14].

An important aspect in the success of applying amplitudes for black hole physics is that for long range forces, a black hole is well described as a point particle. In essence, this is a reflection of the no hair theorem: for the asymptotic observer, a black hole is completely characterized by its mass, spin and charge, much like that of an elementary particle. Indeed, this feature is well appreciated in the context of Schwarzschild black holes and minimally (gravitationally) coupled massive scalar particle. For spinning black holes, the natural counterpart would be spinning particles. However, this extension immediately raises two questions: 1. Which spins should one choose, and what principle governs their coupling to the gravitational degrees of freedom. 2. As there are no known examples of elementary particles with spin beyond two, what is the preferred description for the higher-spin extensions?[1]

Earlier work in deriving spin-effects of the classical potential from minimally coupled spinning particles appears to give conflicting results. While the spin-orbit coupling computed from spin-$\frac{1}{2}$, 1 particles [15, 16] matches with that of effective field theory computations [17], they differ starting from terms quadratic in spin operators. The difference can in principle be attributed to the fact that the expectation values of classical spin operators commute, while

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[1] Here we are considering isolated massive higher-spin particles in flat space. By isolated, we are referring to the property that there are no nearby states at a given mass level. String theory and Kaluza Klein reduction introduces an infinite tower of states at the scale of the first massive state.
their quantum counterparts do not. Indeed recent work in matching the angular impulse of spin-1 particles with that of Kerr black hole [18], demonstrates that the two matches up to commutator terms.

On another front, the new massive spinor-helicity formalism introduced in [11], allows one to identify a set of couplings for arbitrary spins that are universal. For the three point amplitude of massive spin-$s$ states coupled to a positive helicity graviton the amplitude is given as:

$$\frac{\kappa}{2} \frac{(12)^{2s}}{m^{2s-1}}$$

where the definition of $x$ is of kinematic origin:

$$x^{\lambda} = \frac{\lambda_q \tilde{\lambda}_{\dot{q}} \alpha \bar{\alpha} \gamma_1}{m},$$

where $\lambda_q, \tilde{\lambda}_q$ are the spinors of the massless leg. For $s \leq \frac{3}{2}$ the amplitude matches to that of minimal coupling to gravity, and thus can be viewed as the higher-spin extension of minimal coupling, although this is in general different from the usual notion of minimal coupling in QFT where one simply covariantize the derivatives in the free kinetic term, as is shown in appendix A. Remarkably, the particle description of Kerr black hole can be identified as the classical-spin limit of the above minimal coupling, i.e. $s \to \infty, \hbar \to 0$ with $s \hbar$ fixed. Indeed this was confirmed in the matching of the three-point amplitude induced from minimal coupling with the worldline formalism with black hole Wilson coefficients [13] or the coupling to the Kerr black hole stress-tensor [14], as well as through its effect on the impulse [19].

The comparison of the three-point amplitudes for minimal coupling and worldline formalism, which is termed one-particle effective field theory (EFT), allows us to keep track of the difference between the classical-spin limit and finite (quantum) spins. In particular, it was shown that the Wilson coefficients for Kerr black hole differs from finite-spin minimal coupling by $O(1/s)$ terms [13]. This easily explains the mismatch in the computations of classical potential in earlier work [15, 16]. However, in the on-shell chiral basis, we’ve shown in [13] that the correct 1 PM potential can be reproduced from finite spins with suitable prescription. In this paper, we derive the origin of this prescription. We show that

When cast in the chiral basis, the 1 PM potential for minimal coupling factorizes into a spin-dependent combinatoric factor and a spin-independent kinematic term.

In other words, the classical-spin limit can be obtained from any finite spin computation by simply replacing the combinatoric factors by their infinite spin asymptotic form. This turns out to be the prescription presented in [13]. We verify this by first computing the 1 PM potential to all order in spins for general EFTs. Taking the Wilson coefficients to be Kerr black hole, we demonstrate that the result matches with that derived from our prescription.

We then consider whether such universality phenomenon persists at 2 PM. This can potentially serve as a guiding principle for finding the correct gravitational Compton amplitude
for higher spins, which is non-unique. We find, that both the BCFW non-local amplitude, and the local extension constructed in [13], can be made consistent with universality in the holomorphic classical limit of [12], while giving us distinct predictions for terms with degree five and beyond in spin operators. Note that although universality does not provide us with a useful criteria to seek out the correct Compton amplitude, it does allow us to conjecture that such property must hold to all PM order.

Finally we go back and consider whether universality can be made to work for general EFTs. We find that indeed it can be made the case by specially engineered three-point coupling in the chiral basis, which both preserves universality and reproduce the correct Wilson coefficient in the classical-spin limit.

This paper is organized as follows. In section 2, we review the matching of one-particle EFT to on-shell three-point amplitudes. Next, we compute the classical potential between two bodies at 1 PM leading PN to all orders in spin in section 3. The justification for the prescription given in [13] is outlined in section 4 for 1 PM order and in section 5 for 2 PM order. We also present an extension in section 6 such that non-black hole bodies can be incorporated into our formalism. We conclude our paper with section 7.
2 Three point amplitude of general EFT

We aim to derive the classical potential for general EFTs utilizing the on-shell approach, where the potential is identified with the four-point amplitude in the limit where the square of the momentum transfer is zero. This requires us to convert from operators in the EFT language to the on-shell matrix elements. In the following we will compute the amplitude obtained by taking the operators from the EFT and acting them on the Hilbert space of a spin-s particle.

2.1 One-particle effective action to on-shell amplitudes

We begin by considering the effective action of a point particle coupled to a gravitational background. Such a formulation was introduced by Goldberger and Rothstein [20] with inclusion of spin by Porto [21]. A more rigorous treatment of redundant spin variables have been given by works of Levi and Steinhoff [22], and the approach has become one of the main techniques for computing the post-Newtonian effects of gravity [21–26]; consult the review [27] for a more complete list of references. This is an effective action where the gravitational field is decomposed into modes with different scaling properties and modes shorter than the scale \( r_s \) of the compact object has been removed, thus allowing us to approximate the black hole as an isolated compact object. One then starts with the following worldline action:

\[
S = \int d\sigma \left\{ -m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} + L_{SI} [u^\mu, S_{\mu\nu}, g_{\mu\nu}(y^\mu)] \right\}
\]  

(2.1)

where \( u^\mu \equiv \frac{dy^\mu}{d\sigma} \), \( S_{\mu\nu} \) correspond to the spin-operator, and \( \Omega_{\mu\nu} \) is the angular velocity. In this section we choose the covariant Spin Supplementary Condition (SSC), \( P^\mu S_{\mu\nu} = 0 \), where \( P^\mu = (P_1 - P_2)_{\mu} \) for the spin-operator\(^2\). This is solved by identifying \( S_{\mu\nu} = -\frac{1}{m} \epsilon^{\mu\nu\rho\sigma} P_{\rho} S_{\sigma} \) with the spin vector operator.

The first two terms of the EFT Lagrangian eq. (2.1) are called minimal coupling and are universal, irrespective of the details of the point-like particle, while the terms in \( L_{SI} \) correspond to spin-interaction terms that are beyond minimal coupling, and depend on the inner structure of the particle. The angular velocity \( \Omega^{\mu\nu} \) is defined as \( \Omega^{\mu\nu} := e^\mu_{\lambda} D^\lambda_{\sigma} \), where \( e^\mu_{\lambda}(\sigma) \) is the tetrad attached to the worldline of the particle. Generalising the quadrupole moment operator introduced in [23], the non-minimal spin-interaction terms can be parameterized as [27]:

\[
L_{SI} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} C_{ES}^{2n} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^\mu_1 S^\mu_2 \cdots S^\mu_{2n-1} S^\mu_{2n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)!} C_{BS}^{2n+1} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^\mu_1 S^\mu_2 \cdots S^\mu_{2n} S^\mu_{2n+1}.
\]  

(2.2)

\(^2\)The difference \( P_1 - P_2 \) was used to define average momentum to comply with conventions of amplitude literature; all momenta are considered to be incoming. It is shown in appendix B that switching to reasonably chosen NW SSC does not change three-point amplitudes.
where $E$ and $B$ are the electric and magnetic components of the Weyl tensor defined as:

$$E_{\mu\nu} := R_{\mu\alpha\nu\beta} u^\alpha u^\beta$$
$$B_{\mu\nu} := \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} R_{\delta\nu}^{\alpha\beta} u^\gamma u^\delta,$$  

(2.3)

and the covariant derivatives act on the Riemann tensors. Here the Riemann tensors contain linear perturbations around flat space, and the information with regards to non-trivial backgrounds is encoded in the Wilson coefficients $C_{S^n}$, which is set to 1 for Kerr black-holes.

Since each Riemann tensor is linear in the perturbed metric, these operators represent the coupling of the worldline particle to a graviton, which translates to a three-point amplitude involving two identical massive spin-$s$ state and the emission of a graviton:

$$\langle q \langle q | S \rangle \rangle = M^{-2}_s.$$

(2.4)

To construct the amplitude we first identify $S^\mu$ as the Pauli-Lubanski pseudo-vector $S^\mu = -\frac{1}{2m} u^\mu P^\rho J_{\rho\mu}$. The explicit form of the Lorentz generator $J_{\rho\mu}$ then depends on the representation of the external state under $SL(2, \mathbb{C})$. For example, the generator for spin-$s$ in $(s, 0)$ representation we write:

$$(J_{\mu\nu})_{\alpha_1,\alpha_2,\ldots,\alpha_{2s}} \beta_1\beta_2,\ldots,\beta_{2s} = \sum_i (J_{\mu\nu})_{\alpha_i^\dagger} \beta_i^\dagger \bar{i}_i = 2s (J_{\mu\nu})_{\alpha_1^\dagger} \beta_1^\dagger \bar{i}_1, \quad (J_{\mu\nu})_{\alpha^\dagger} \beta = \frac{i}{2} (\sigma_{\mu\nu})_{\alpha^\dagger} \beta,$$

(2.5)

where $\bar{i}_i = \delta_{\alpha_1^\dagger} \cdots \delta_{\alpha_{i-1}^\dagger} \delta_{\alpha_{i+1}^\dagger} \cdots \delta_{\alpha_{2s}^\dagger}$, with a similar form for the conjugate representation. The sign $\dagger$ means the RHS can be used instead of LHS of $= \dagger$ as $SL(2, \mathbb{C})$ indices are symmetrised, but the proper definition for $J_{\mu\nu}$ is the expression between $= \dagger$ and $\dagger$. Using this, we find that

$$m (S_{\mu})_{\alpha^\dagger} \beta = \frac{1}{4} [\sigma_\mu (P \cdot \tilde{\sigma}) - (P \cdot \sigma) \tilde{\sigma}]_{\alpha^\dagger} \beta,$$

(2.6)

$$m (S_{\mu})_{\alpha} \beta^\dagger = -\frac{1}{4} [\tilde{\sigma}_\mu (P \cdot \sigma) - (P \cdot \tilde{\sigma}) \sigma]_{\alpha} \beta^\dagger.$$

(2.7)

When contracted with the momentum of the graviton, $q$, one finds:

$$\langle q \cdot S \rangle_{\alpha^\dagger} \beta = \frac{x}{2} \lambda_{\alpha^\dagger} \lambda_\beta \equiv \frac{x}{2} |q\rangle \langle q|$$

(2.8)

$$\langle q \cdot S \rangle_{\alpha} \beta^\dagger = -\frac{\tilde{\lambda}_{\alpha} \tilde{\lambda}_\beta}{2x} \equiv -\frac{|q||q|}{2x}.$$

Finally, we introduce the external polarization tensors $\epsilon_s = \frac{1}{m^2} (\prod_{i=1}^s \lambda_{\alpha_i} \tilde{\lambda}_{\alpha_i} \lambda_{\beta_i} \tilde{\lambda}_{\beta_i})_{sym}$, where $sym$ indicates the full symmetrization of the $SU(2)$ indices reflecting the transverse traceless

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3The vacuum Einstein equation reduces to $R_{\mu\nu} = 0$, therefore the Riemann tensor is equal to the Weyl tensor $R_{\mu\nu\lambda\sigma} = C_{\mu\nu\lambda\sigma}$ in this backgroundt.
nature of $\epsilon_s$. Note that for fixed $s$, the polarization tensor is in $(\frac{s}{2}, \frac{s}{2})$ representation containing $s$ chiral and $s$ anti-chiral $SL(2, \mathbb{C})$ indices, and thus can only transform non-trivially under at most $2s$ spin vector operators $S^\mu$. Thus $M_s^{2n}$ will receive contributions from the terms in the one-particle effective action with $S^n$ where $n \leq 2s$.

Putting everything together, we find the following three-point amplitude [13]

$$M_s^{2n} = \epsilon^* (2) \left[ \sum_{n=0}^{\infty} \frac{\kappa m x^{2n}}{2} \frac{C_{S^n}}{n!} \left( - \frac{q \cdot S}{m} \right)^n \right] \epsilon (1)$$

for integer spin $s$, where $\eta = +1$ for positive helicity graviton and $\eta = -1$ for negative helicity graviton. The first two Wilson coefficients are fixed as unity from the universal terms in eq.(2.1), while $C_{S^n}$ are simply the electric and magnetic Wilson coefficients $C_{ES^n}$ and $C_{BS^{2n}+1}$ of eq.(2.2). Note that since the polarization tensors are written as

$$\epsilon_{\mu_1 \mu_2 \cdots \mu_s} \rightarrow \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_s \dot{\alpha}_1 \dot{\alpha}_2 \cdots \dot{\alpha}_s} = \frac{1}{m^s} \lambda^{\alpha_1} \lambda^{\alpha_2} \cdots \lambda^{\alpha_s} \tilde{\lambda}^{\dot{\alpha}_1} \tilde{\lambda}^{\dot{\alpha}_2} \cdots \tilde{\lambda}^{\dot{\alpha}_s}$$

(2.10)

where it’s symmetrized with respect to the $2s$ SU(2) little group indices $\{I_i\}$, it is also often referred to as the mixed basis, as it involves equal number of chiral and anti-chiral spinors for each leg. We will instead refer to it as the polarization tensor basis. The coupling constant $\kappa$ is defined as $\sqrt{32 \pi G}$ where $G$ is the gravitational constant. The on-shell form of the spin operators is defined by (appropriate tensor sum) of eq.(2.8) as in eq.(2.5). For example, explicitly expanding the positive helicity three-point amplitude gives the following expression.

$$M_s^2 = \sum_{a+b \leq s} \frac{\kappa m x^2}{2} C_{S^{a+b}} n_{a,b}^{s} (21)^s-a \left( - \frac{x \langle 2q \rangle \langle q1 \rangle}{2m} \right)^a (21)^{s-b} \left( \frac{[2q][q1]}{2mx} \right)^b$$

(2.11)

Let us summarize; when projecting the EFT to on-shell amplitudes, we necessarily project onto amplitudes with fixed spin for the asymptotic states. This introduces $s$-dependence for the amplitude, and thus to extract classical observables, one must analytically take $s \gg 1$. There are two sources of $s$-dependence, 1. the action of the spin operators on the spin-$s$ state 2. the maximal degree of spin operators that are present in the amplitude. To illustrate this fact, let us demonstrate the equivalence of the Kerr black hole and the $s \gg 1$ limit of minimal coupled spinning particle.

### 2.2 Minimal coupling in $s \gg 1$ limit as BHs

As shown directly in [14, 19], the classical spin-limit of minimal coupling reproduces various classical observable of Kerr black holes, such as stress-tensor form factor and impulse. In the context of one-particle effective action, it was shown in [13] that the Wilson coefficients of minimal coupling at finite-spin deviate from that of Kerr black hole ($C_{S^n} = 1$) by $\sim \frac{1}{s}$.
Thus in the limit $s \to \infty$ we recover Kerr black hole. Here we give a brief review of the map between the Wilson coefficient in the EFT basis and the coupling constants, $g_i s$, defined kinematically in the (anti)chiral spinor basis as follows. Begin with the general form of three-point amplitudes introduced in \[11\]:

\[
M^{+2}_s = \frac{\kappa m x^2}{2m^{2s}} \left[ g_0 \langle 21 \rangle^{2s} + g_1 \langle 21 \rangle^{2s-1} \frac{x \langle 2q \rangle \langle q1 \rangle}{m} + \cdots + g_{2s} \frac{(x \langle 2q \rangle \langle q1 \rangle)^{2s}}{m^{2s}} \right],
\]

\[
M^{-2}_s = \frac{\kappa m x^{-2}}{2m^{2s}} \left[ g_0 \langle 21 \rangle^{2s} + g_1 \langle 21 \rangle^{2s-1} \frac{[2q][q1]}{xm} + \cdots + g_{2s} \frac{([2q][q1])^{2s}}{x^{2s}m^{2s}} \right],
\]

(2.12)

Here, we’ve expressed the coupling to the positive helicity graviton in the chiral spinor basis and the negative helicity graviton in the anti-chiral basis. For these choices, the minimal coupling simply corresponds to setting all couplings except $g_0$ to zero:

\[
M^{+2}_{s,\text{min}} = \frac{\kappa m x^2}{2} \frac{(21)^{2s}}{m^{2s}}, \quad M^{-2}_{s,\text{min}} = \frac{\kappa m x^{-2}}{2} \frac{[21]^{2s}}{m^{2s}}.
\]

(2.13)

The minimal nature of the coupling can be seen in the high energy limit where all momenta are approximately massless, the expression matches to the minimal derivative three-point amplitude. Caution: the minimal coupling of eq.(2.13) for $s > 2$ is different from the usual usage of minimal coupling in the QFT literature where the derivatives of the kinetic term are simply covariantized, as shown in appendix A.

In [13] it was shown that at large $s$, the minimal couplings are matched to one-particle EFT with Wilson coefficients $C^{S^n}_{S^n} = 1 + \mathcal{O}(1/s)$. To show this, we first work out the map between $C^{S^n}_{S^n}$ and $g_i$, by converting the EFT amplitude in eq.(2.11), into the chiral basis in eq.(2.12). This requires us to convert the square brackets to chiral spinors using the following two identities:

\[
[21] = \langle 21 \rangle + \frac{x \langle 2q \rangle \langle q1 \rangle}{m}, \quad [2q][q1] = -\frac{x \langle 2q \rangle \langle q1 \rangle}{m}.
\]

(2.14)

We then arrive at:

\[
M^{+2}_s = \sum_{a+b \leq 2s} x^2 C^{S^n}_{S^n} n_{a,b} \lambda_2^{2s} \left[ \langle a \rangle \langle b \rangle - \frac{x \langle 2q \rangle \langle q1 \rangle}{2m} \right] \lambda_1^{2s}.
\]

(2.15)

Comparing with eq.(2.12) gives the following relation between $C^{S^n}_{S^n}$ and $g_i$

\[
g_i = \sum_{n=0}^{i} F^{s}_{i,n} C^{S^n}_{S^n}
\]

(2.16)

\[
F^{s}_{i,n} = \frac{1}{(-2)^n (i-n)!} \sum_{m=0}^{n} \frac{1}{(s-m)!(s+m-i)!m!(n-m)!}
\]

(2.17)
Now minimal coupling corresponds to setting \( g_i = 0 \) for \( i \neq 0 \). Since

\[
g_0 = C_{S^0}, \quad g_1 = s(C_{S^0} - C_{S^1}), \quad g_2 = \frac{s^2(C_{S^2} - 2C_{S^1} + C_{S^0})}{2} + \frac{s(2C_{S^1} - 2C_{S^0} - C_{S^2})}{4}
\]  

(2.18)

Normalizing \( g_0 = 1 \) and the vanishing of \( g_1 \) then sets \( C_{S^0} = C_{S^1} = 1 \), while \( g_2 = 0 \) sets

\[
C_{S^2} = \frac{2s}{2s - 1}.
\]  

(2.19)

which indeed tends to unity as one approaches the \( s \gg 1 \) limit. Similarly the vanishing of \( g_3 \) sets \( C_{S^3} = \frac{2(s+1)}{2s-1} \). Thus in summary we see that the Wilson coefficients for minimally coupled particles deviate from that of Kerr black holes:

\[
C_{S^n}^{Min,s} = C_{S^n}^{Kerr,s} + O\left(\frac{1}{s}\right).
\]  

(2.20)

It is not hard to work out the precise coefficients for \( \frac{1}{s} \) corrections. A brief outline is given in appendix C. Up to \( O(s^{-2}) \) order it can be shown analytically that

\[
C_{S^n}^{Min,s} = 1 + \frac{n(n-1)}{4s} + \frac{(n^2 - 5n + 10)n(n-1)}{32s^2} + O(s^{-3}).
\]  

(2.21)
3 Classical potential for general EFT at 1 PM leading PN

Here we derive the classical potential for general one-particle EFT at 1 PM leading PN order, but including all orders in spin operators. The relevant information is contained in the exchange of single virtual graviton between two sources, where the cubic coupling is governed by the operators in the one body EFT linear in the Riemann tensor. Since the classical potential correspond to long range effects, it is determined by the analytic structure of the amplitude around $q^2 = 0$, where $q$ is the transfer momenta. In other words, it is given by the graviton exchange in the $t$-channel as shown in fig.1:

$$V(p, q) = \frac{M_4(s, t)}{4E_aE_b} \bigg|_{t \to 0} = \frac{\text{Res}_t}{4E_aE_bq^2}$$

where all $q^0$ terms are dropped and $\text{Res}_t$ is the $t$-channel residue.\(^4\)

In general the residue of an amplitude is given by the products of three-point amplitudes, which we have computed for general EFTs. However, in the center of mass and the non-relativistic limit, $q$ is space-like and hence $q^2 \to 0$ implies $q \to 0$ for real kinematics, i.e. the soft momentum limit instead of the on-shell limit. To this end one instead considers complex kinematics, which allows us to approach the $q^2 \to 0$ limit while maintaining non-zero $q$. This leads us to the holomorphic classical limit (HCL) introduced by Guevara [12], where the on-shell momenta for external particles are chosen as follows:

$$P_1 = (E_a, \vec{p} + q/2) = |\hat{\eta}\rangle\langle\hat{\lambda}| + |\bar{\lambda}\rangle\langle\hat{\eta}|$$
$$P_2 = (E_a, \vec{p} - q/2) = \beta'|\hat{\eta}\rangle\langle\bar{\lambda}| + \frac{1}{\beta'}|\bar{\lambda}\rangle\langle\hat{\eta}| + |\bar{\lambda}\rangle\langle\bar{\lambda}|$$
$$P_3 = (E_b, -\vec{p} - q/2) = |\eta\rangle\langle\lambda| + |\lambda\rangle\langle\eta|$$
$$P_4 = (E_b, -\vec{p} + q/2) = \beta|\eta\rangle\langle\lambda| + \frac{1}{\beta}|\lambda\rangle\langle\eta| + |\lambda\rangle\langle\lambda|.$$ \(^{(3.2)}\)

\(^4\)We ignore non-analytic singularity present in the amplitude as it corresponds to higher PM/PN effects.
Note that each momentum is complex. In this frame, the transverse momentum is then

\[ q^\mu = (P_1 - P_2)^\mu = (0, \vec{q}) = -|\lambda| \langle \lambda | + \mathcal{O}(\beta - 1) = |\lambda| \langle \lambda | + \mathcal{O}(\beta - 1) \tag{3.3} \]

where \( q^2 \to 0 \) limit corresponds to \( \beta \to 1 \) yet \( q^\mu \neq 0 \). The spinors are constrained by the conditions \( \langle \lambda \eta \rangle = [\lambda \eta] = m_a \) and \( \langle \lambda \eta \rangle = [\lambda \eta] = m_b \). The spatial momenta are defined as \( \vec{p} = \vec{p}_a = -\vec{p}_b \).

Since in the HCL limit \( q \) is null, the four-particle kinematics reduces to a product of two copies of three-particle kinematics,

\[
\langle 21 \rangle = -\left( \frac{[2\lambda][\bar{\lambda}1]}{x_1 m_a} \right), \quad x_1 \langle 2\lambda \rangle \langle \lambda 1 \rangle = -\left( \frac{[2\lambda][\bar{\lambda}1]}{x_1 m_a} \right)
\]

\[
\langle 43 \rangle = -\left( \frac{[4\lambda][\bar{\lambda}3]}{x_3 m_b} \right), \quad x_3 \langle 4\lambda \rangle \langle \lambda 3 \rangle = -\left( \frac{[4\lambda][\bar{\lambda}3]}{x_3 m_b} \right)
\]

The \( x \)-factors \( x_1 = \frac{[\bar{\lambda}P_1|\zeta]}{m_a(-\bar{\lambda}\zeta)} \) and \( x_3 = \frac{[\bar{\lambda}P_3|\zeta]}{m_b(\lambda\zeta)} \) are defined as usual. The product of \( x \)-factors can be expressed as

\[
\frac{x_1}{x_3} = \frac{u}{m_a m_b} = \rho + \sqrt{\rho^2 - 1}, \quad \frac{x_3}{x_1} = \frac{v}{m_a m_b} = \rho - \sqrt{\rho^2 - 1}. \tag{3.5}
\]

where the variables \( u \) and \( v \) are defined as follows:

\[
u = [\lambda|P_1|\eta], \quad v = [\eta|P_3|\lambda], \quad \rho = \frac{P_1 \cdot P_3}{m_a m_b} = \frac{u + v}{2 m_a m_b}. \tag{3.7}
\]

The static limit corresponds to the limit \( \rho \to 1 \).\(^6\) Expanding the expression around \( \rho = 1 \) is needed for sorting out leading PN contributions.

Note that in the HCL limit, local operators may become proportional to each other. For example, in the HCL limit we have:

\[
\epsilon_{\mu \nu \lambda \sigma} P_1^\mu P_3^\nu g^\lambda S_\sigma^\alpha \xrightarrow{\text{HCL}} - i m_a^2 m_b \sqrt{\rho^2 - 1} \left( \frac{g \cdot S_a}{m_a} \right). \tag{3.8}
\]

Thus the two operators, namely \( c(P_1, P_3, q, S_a) \) and \( q \cdot S_a \), can only be disentangled by keeping track of which order in \( \sqrt{\rho^2 - 1} \) do they appear. These two local operators and their cousins

5These variables are subject to following constraints by on-shell conditions of external particles.

\[
[\eta|P_1|\eta]|\lambda|P_1|\lambda| = uv - m_a^2 m_b^2
\]

\[
[\lambda|P_1|\lambda| = \frac{(\beta - 1)^2}{\beta} m_b^2 + (1 - \beta)v + \frac{\beta - 1}{\beta} u. \tag{3.6}
\]

6The same variable was denoted as \( r \) in [12, 13]. To avoid confusion with non-relativistic variable conjugate to \( q \), the letter \( \rho \) was chosen.
for $S_b$ forms the basis for leading PN potential. After identifying the local operators we simply take the non-relativistic limit by identifying:

$$\frac{i}{m_a m_b} \epsilon_{\mu' \nu' \lambda' \sigma'} P_{\mu' 3} P_{\nu' 3} q^\lambda S^\sigma \xrightarrow{\text{non-rel}} \left( \frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \times \vec{S} \cdot (-i\vec{q})$$  \hspace{1cm} (3.9)

We now compute the leading order PN classical potential for general EFT.

### 3.1 Computing the classical potential

The HCL $\text{Res}_l$ is obtained by simply gluing three-point amplitudes computed from the one-particle effective theory eq. (2.9) and summing over all possible intermediate polarisation channels.

$$\text{Res}_l = A_{3aA_{3b}^+}^+ + A_{3aA_{3b}^-}^-$$

$$\begin{align*}
&= (-1)^{s_a + s_b} \sum_{i=0}^{2 s_a} \sum_{j=0}^{2 s_b} \alpha^2 m_a^2 m_b^2 \frac{C_{s_a} C_{s_b}}{i l j!} \left[ \frac{x_3^2}{x_1^3} \epsilon(2) \left( \frac{q \cdot S_a}{m_a} \right)^i \epsilon(1) \right] \left[ \epsilon(4) \left( \frac{q \cdot S_b}{m_b} \right)^j \epsilon(3) \right] \\
&\quad + \frac{x_3^2}{x_1^3} \epsilon(2) \left( -\frac{q \cdot S_a}{m_a} \right)^i \epsilon(1) \left[ \epsilon(4) \left( -\frac{q \cdot S_b}{m_b} \right)^j \epsilon(3) \right]
\end{align*}$$

$$\begin{align*}
&= (-1)^{s_a + s_b} \sum_{i=0}^{2 s_a} \sum_{j=0}^{2 s_b} B_{i,j} \left[ \epsilon(2) \left( \frac{q \cdot S_a}{m_a} \right)^i \epsilon(1) \right] \left[ \epsilon(4) \left( -\frac{q \cdot S_b}{m_b} \right)^j \epsilon(3) \right] \\
&= \frac{(-1)^i x_2^2 x_4^2}{x_3^2} + \frac{(-1)^j x_3^2 x_4^2}{x_3^2} \alpha^2 m_a^2 m_b^2 C_{s_a} C_{s_b},
\end{align*}$$

(3.10)

The coupling constant $\alpha$ is defined as $\alpha = \kappa/2 = \sqrt{8\pi G}$. The spins $s_a$ and $s_b$ are assumed to be integers. The sign factor $(-1)^{s_a + s_b}$ appears due to the choice of mostly minus metric signature; $|\epsilon(P)|^2 = (-1)^s$. This sign factor is irrelevant when computing the classical potential. The $B_{i,j}$ coefficient is worked out to be

$$B_{i,j} = \frac{(-1)^i x_2^2 x_4^2}{i l j!} \alpha^2 m_a^2 m_b^2 C_{s_a} C_{s_b},$$

(3.11)

which we find in the limit $\rho \to 1$ as

$$B_{i,j}|_{\rho \to 0} = \frac{|(-1)^i + (-1)^j| - 2\sqrt{\rho^2 - 1} |(-1)^i - (-1)^j|}{i l j!} \alpha^2 m_a^2 m_b^2 C_{s_a} C_{s_b} + O(\rho^2 - 1).$$

(3.12)

When $i + j$ is even, the first term is the leading PN contribution. When $i + j$ is odd, $\sqrt{\rho^2 - 1}$ term is the leading PN contribution. The two cases are treated separately.

- $i + j$ even: $(-1)^i = (-1)^j$ can be used to simplify the expression.

$$\frac{B_{i,j}}{t} \left( \frac{q \cdot S_a}{m_a} \right)^i \left( -\frac{q \cdot S_b}{m_b} \right)^j = (-1)^{\frac{i+j}{2}} \frac{2\alpha^2 m_a^2 m_b^2}{i l j! q^2} C_{s_a} C_{s_b} \left( \frac{-i\vec{q} \cdot \vec{S}_a}{m_a} \right)^i \left( \frac{-i\vec{q} \cdot \vec{S}_b}{m_b} \right)^j$$

(3.13)
In position space, the expression becomes

\[
-(1)^{\frac{i+j}{2}} \frac{4\alpha^2 m_a^2 m_b^2}{2\pi i j!} C_{S_a} C_{S_b} \left( \frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right)^i \left( \frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right)^j \frac{1}{r}.
\] (3.14)

With non-relativistic flux normalisation \( \frac{1}{4E_a E_b} \simeq \frac{1}{4m_a m_b} \),

\[
-(1)^{\frac{i+j}{2}} \frac{4\alpha^2 m_a^2 m_b^2}{i j! q^2} C_{S_a} C_{S_b} \left( \frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right)^i \left( \frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right)^j \frac{G_{m_a m_b}}{r}.
\] (3.15)

- \( i + j \) odd: Due to the following vector identity [13], any of \( S_a \) or \( S_b \) can be converted to spin-orbit coupling term.

\[
\left[ \vec{v} \times \vec{S}_a \cdot \vec{q} \right] \left[ \vec{S}_b \cdot \vec{q} \right] = \left[ \vec{S}_a \cdot \vec{q} \right] \left[ \vec{v} \times \vec{S}_b \cdot \vec{q} \right] + q^2 \vec{v} \cdot \vec{S}_a \times \vec{S}_b
\] (3.16)

The \( q^2 \) dependent part will combine with \( q^2 \) of the denominator to yield \( q^0 \) order expression, which does not contribute to long-distance effects. Therefore, there is a freedom for choosing which of \( S_a \) or \( S_b \) acquires spin-orbit coupling. The convention we choose is to attach spin-orbit coupling to odd powered spin. For convenience, let us treat the cases separately.

For odd \( i \) and even \( j \), attach spin-orbit factor to \( S_a \). The expression \( \frac{B_{i,j}}{r} \left( \frac{q S_a}{m_a} \right)^i \left( -\frac{q S_b}{m_b} \right)^j \) is then evaluated as follows.

\[
-(1)^{\frac{i+j}{2}} \frac{4\alpha^2 m_a^2 m_b^2}{i j! q^2} C_{S_a} C_{S_b} \left[ \left( \frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \times \frac{\vec{S}_a}{m_a} \cdot (-i\vec{q}) \right] \left( -i\vec{q} \cdot \frac{\vec{S}_a}{m_a} \right)^{i-1} \left( \frac{\vec{S}_b}{m_b} \cdot \vec{q} \right)^j
\] (3.17)

Going to position space and including non-relativistic flux normalisation factors, the contribution is evaluated as follows.

\[
-\frac{2(1)^{\frac{i+j}{2}}}{i j!} C_{S_a} C_{S_b} \left[ \left( \frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \times \frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right] \left( \frac{\vec{S}_a}{m_a} \cdot \vec{\nabla} \right)^{i-1} \left( \frac{\vec{S}_b}{m_b} \cdot \vec{\nabla} \right)^j \frac{G_{m_a m_b}}{r}
\] (3.18)

For even \( i \) and odd \( j \), attach spin-orbit factor to \( S_b \). The expression \( \frac{B_{i,j}}{r} \left( \frac{q S_a}{m_a} \right)^i \left( -\frac{q S_b}{m_b} \right)^j \) is then evaluated as follows.

\[
-(1)^{\frac{i+j}{2}} \frac{4\alpha^2 m_a^2 m_b^2}{i j! q^2} C_{S_a} C_{S_b} \left[ \left( \frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \times \frac{\vec{S}_b}{m_b} \cdot (-i\vec{q}) \right] \left( -i\vec{q} \cdot \frac{\vec{S}_a}{m_a} \right)^i \left( \frac{\vec{S}_b}{m_b} \cdot \vec{q} \right)^{j-1}
\] (3.19)
Going to position space and including non-relativistic flux normalisation factors, the contribution becomes the following.

\[
2(-1)^{ij+kl} C_{S_i} C_{S_j} \left[ \left( \frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \times \vec{S}_b \cdot \vec{\nabla} \right] \left( \frac{\vec{S}_a \cdot \vec{\nabla}}{m_a} \right)^i \left( \frac{\vec{S}_b \cdot \vec{\nabla}}{m_b} \right)^j \frac{G m_a m_b}{r} \right)
\]

(3.20)

3.2 Hilbert space matching

The polarisation tensor for particle 1 represents a state vector living in the Hilbert space \( H_1 \) of particle 1, which is defined with respect to momentum \( P_1 \) of particle 1. Thus, the amplitude can be thought as a map \( A_4 : H_1 \otimes H_3 \to H_2 \otimes H_4 \). This is problematic since the Hamiltonian \( H \) is expected to be an endomorphism of the Hilbert space \( H_a \otimes H_b \), where \( H_a(H_b) \) is the Hilbert space defined with respect to momentum \( P_a(P_b) \); \( H : H_a \otimes H_b \to H_a \otimes H_b \). This gap can be filled by forcing the polarisation tensors of each particles to live in the representative Hilbert space \( H_a \otimes H_b \). Since the original polarisation tensors are foreign to the Hilbert space \( H_a \otimes H_b \), this forcing procedure produces effects that must be taken into account when computing the classical potential.

Define \( P_1^\mu = P_1^\mu + \bar{P}_2^\mu \), \( q_1^\mu = P_1^\mu - P_2^\mu = (0, \vec{q}) \). In terms of average momentum and momentum transfer, the polarisation tensors can be expressed as follows.

\[
\epsilon(P_2) = \epsilon(P_a) - \frac{1}{2} q_1^\mu \frac{\partial}{\partial P_a^\mu} \epsilon(P_a) + \cdots \\
\epsilon(P_1) = \epsilon(P_a) + \frac{1}{2} q_1^\mu \frac{\partial}{\partial P_a^\mu} \epsilon(P_a) + \cdots
\]

(3.21)

(3.22)

Textbook definition constructs polarisation tensors explicitly in one reference frame and then uses boosts to extend to arbitrary momentum. Based on this definition, the polarisation tensor \( \epsilon(P) \) can schematically be written as follows where \( P_0 \) is the reference momentum.

\[
\epsilon(P) = G(P; P_0) \epsilon(P_0)
\]

(3.23)

Thus, the derivative on polarisation tensor can be represented as

\[
\frac{\partial}{\partial P_\mu} \epsilon(P) = \lim_{\delta p \to 0} \frac{G(P + \delta p; P_0) G^{-1}(P; P_0) - I \epsilon(P)}{\delta p}
\]

(3.24)

In the non-relativistic limit with \( P_0 = (m, \vec{0}) \), the following relations can be derived which holds up to linear order in momentum.

\[
G(P; P_0) = e^{-i\vec{\lambda}(\vec{p}) \cdot \vec{K}} \approx e^{i \frac{1}{m} \vec{p} \cdot \vec{K}}
\]

(3.25)

\[
\vec{K} = J^{00} = S^{00}
\]

(3.26)

\[
\frac{\partial}{\partial P_\mu} \epsilon(P) \approx \lim_{\delta p \to 0} \frac{e^{i \frac{1}{m} (\vec{p} + \delta \vec{p}) \cdot \vec{K}} e^{-i \frac{1}{m} \vec{p} \cdot \vec{K}} - I \epsilon(P)}{\delta p} \approx \frac{i}{m} \vec{K} \epsilon(P)
\]

(3.27)
\( \vec{\lambda}(\vec{p}) \) is the rapidity vector defined by the relation \( \vec{\lambda} = -\sinh^{-1}(\frac{p_p}{m}) \). Using Newton-Wigner SSC (NW SSC)\(^7\), \( S^{\mu\nu}(P_\nu + m\delta_0^\nu) = 0 \), the following relation can be derived for \( S^{i0} \):

\[
S^{i0}(P_0 + m) = -S^{ij}P_j = \frac{P_0}{m} \epsilon^{ijk} p^j s^k
\]

(3.28)

\( \vec{K} = S^{i0} = \frac{P_0}{(P_0 + m)m} \vec{p} \times \vec{s} \simeq \frac{1}{2m} \vec{p} \times \vec{s} \)

(3.29)

Therefore, the derivative can be represented as follows in the non-relativistic limit.

\[
q^\mu \frac{\partial}{\partial P^\mu} \epsilon(P_a) = \vec{q} \cdot \frac{\partial}{\partial \vec{p}_a} \epsilon(P_a) \simeq -\frac{i\vec{q}}{2} \cdot \left( \frac{\vec{p}_a}{m_a} \times \frac{\vec{s}_a}{m_a} \right) \epsilon(P_a)
\]

(3.30)

Summing up, the polarisation tensors can be represented as

\[
\epsilon(P_2) = \epsilon(P_a) - \left( \frac{\vec{p}_a}{m_a} \times \frac{\vec{s}_a}{m_a} \right) \frac{i\vec{q}}{4} \epsilon(P_a) + \cdots
\]

(3.31)

\[
\epsilon(P_1) = \epsilon(P_a) + \left( \frac{\vec{p}_a}{m_a} \times \frac{\vec{s}_a}{m_a} \right) \frac{i\vec{q}}{4} \epsilon(P_a) + \cdots
\]

(3.32)

\[
\epsilon^\ast(P_2) \epsilon(P_1) = \epsilon^\ast(P_a) \left[ 1 - \left( \frac{\vec{p}_a}{m_a} \times \frac{\vec{s}_a}{m_a} \right) \frac{i\vec{q}}{2} + \cdots \right] \epsilon(P_a)
\]

(3.33)

For particle \( b \), there is an additional sign factor due to definition of \( \vec{q} \), which is consistent with the dictionary provided in [15].

\[
\epsilon^\ast(P_4) \epsilon(P_3) = \epsilon^\ast(P_b) \left[ 1 + \left( \frac{\vec{p}_b}{m_b} \times \frac{\vec{s}_b}{m_b} \right) \frac{i\vec{q}}{2} + \cdots \right] \epsilon(P_b)
\]

(3.34)

In sum, the overall effect is to multiply all the results obtained in the previous sections by the factor

\[
1 - \frac{1}{2} \left[ \frac{\vec{p}_a}{m_a} \times \frac{\vec{s}_a}{m_a} - \frac{\vec{p}_b}{m_b} \times \frac{\vec{s}_b}{m_b} \right] \cdot (i\vec{q}) = 1 - \frac{1}{2} \left[ \frac{\vec{p}_a}{m_a} \times \frac{\vec{s}_a}{m_a} - \frac{\vec{p}_b}{m_b} \times \frac{\vec{s}_b}{m_b} \right] \cdot \vec{\nabla}
\]

(3.35)

and truncating to leading PN order. This effect can be compared to \( V_{\text{kin}} \) of eq.(3.32) in [29], which is an augmentation of \( V_{\text{el}} + V_{\text{mag}} \) in eq.(3.31) by the factor \( \left[ 1 - \frac{1}{2} (\vec{v}_1 \times \vec{a}_1 - \vec{v}_2 \times \vec{a}_2) \right] \) followed by truncation to leading PN order.

\(^7\)In classical mechanics, Poisson bracket relations of \( P^\mu \) and \( S^{\mu\nu} \) depend on the choice of SSC. NW SSC is the choice that corresponds to canonical Poisson brackets [28].
3.3 The general/Kerr 1 PM leading PN potential

Combining everything, the general 1 PM leading PN all order in spin classical potential is given as

\[
V_{cl} = - \sum_{m,n=0}^\infty \frac{(-1)^n C_{S_{a}^{m}} C_{S_{b}^{m}}}{(2n-m)!m!} \left( \frac{S_{a}}{m_a} \cdot \vec{\nabla} \right)^{2n-m} \left( \frac{S_{b}}{m_b} \cdot \vec{\nabla} \right)^{m} \frac{G_{ma} m_b}{r} \\
- \sum_{m,n=0}^\infty \frac{2(-1)^{m+n} C_{S_{a}^{m+1}} C_{S_{b}^{n+1}}}{(2m+1)!(2n)!} \left[ \left( \frac{\mathbf{p}_a}{m_a} - \frac{\mathbf{p}_b}{m_b} \right) \times \frac{\mathbf{S}_{a}}{m_a} \cdot \vec{\nabla} \right] \left( \frac{\mathbf{S}_{a}}{m_a} \cdot \vec{\nabla} \right)^{2m} \left( \frac{\mathbf{S}_{b}}{m_b} \cdot \vec{\nabla} \right)^{2n} \frac{G_{ma} m_b}{r} \\
- \sum_{m,n=0}^\infty \frac{2(-1)^{m+n} C_{S_{a}^{m}} C_{S_{b}^{n+1}}}{(2m)!(2n+1)!} \left[ \left( \frac{\mathbf{p}_a}{m_a} - \frac{\mathbf{p}_b}{m_b} \right) \times \frac{\mathbf{S}_{b}}{m_b} \cdot \vec{\nabla} \right] \left( \frac{\mathbf{S}_{b}}{m_b} \cdot \vec{\nabla} \right)^{2m} \left( \frac{\mathbf{S}_{a}}{m_a} \cdot \vec{\nabla} \right)^{2n-m} \frac{G_{ma} m_b}{r} \\
+ \sum_{m,n=0}^\infty \frac{(-1)^{n} C_{S_{a}^{m}} C_{S_{b}^{m}}}{2(2n-m)!m!} \left( \left[ \left( \frac{\mathbf{p}_a}{m_a} \times \frac{\mathbf{S}_{a}}{m_a} - \frac{\mathbf{p}_b}{m_b} \times \frac{\mathbf{S}_{b}}{m_b} \right) \cdot \vec{\nabla} \right] \left( \frac{\mathbf{S}_{a}}{m_a} \cdot \vec{\nabla} \right)^{2n-m} \left( \frac{\mathbf{S}_{b}}{m_b} \cdot \vec{\nabla} \right)^{m} \frac{G_{ma} m_b}{r} \right)
\]

\[ (\vec{S} \cdot \vec{\nabla})^2 \frac{1}{r} \equiv -(\vec{S} \times \vec{\nabla})^2 \frac{1}{r} \]

\[ (\vec{S} \cdot \vec{\nabla})^2 \frac{1}{r} \equiv -(\vec{S} \times \vec{\nabla})^2 \frac{1}{r} \]

The above expression reproduces \( \cosh(a_0 \times \vec{\nabla}) \frac{m_1 m_2}{r} \) of eq.(3.31) in [29], where the following notation had been adopted.

\[ (\vec{S} \cdot \vec{\nabla})^2 \frac{1}{r} \equiv -(\vec{S} \times \vec{\nabla})^2 \frac{1}{r} \]

The difference of both sides does not contribute to long-distance dynamics as it is some multiple of Dirac delta. Using the notation eq.\((3.38)\) on eq.\((3.37)\) yields

\[ -\sum_{n=0}^\infty \frac{1}{(2n)!} \left[ \left( \frac{\mathbf{S}_{a}}{m_a} + \frac{\mathbf{S}_{b}}{m_b} \right) \times \vec{\nabla} \right] \frac{2n}{r} \frac{G_{ma} m_b}{r} = -\cosh \left[ \left( \frac{\mathbf{S}_{a}}{m_a} + \frac{\mathbf{S}_{b}}{m_b} \right) \times \vec{\nabla} \right] \frac{G_{ma} m_b}{r}. \]

For the second and third line, once again we can use eq.\((3.38)\) to simplify it to the form:

\[ -\frac{2^{C_{S_{a}^{m}} C_{S_{b}^{n}}}}{ilj!} \left[ \left( \frac{\mathbf{p}_a}{m_a} - \frac{\mathbf{p}_b}{m_b} \right) \cdot \vec{\nabla} \right] \left( \frac{S_{a}}{m_a} \times \vec{\nabla} \right)^i \left( \frac{S_{b}}{m_b} \times \vec{\nabla} \right)^j \frac{G_{ma} m_b}{r}. \]

Setting all Wilson coefficients to unity gives:

\[ -\sum_{n=0}^\infty \frac{2}{(2n+1)!} \left( \frac{\mathbf{p}_a}{m_a} - \frac{\mathbf{p}_b}{m_b} \right) \cdot \left[ \left( \frac{\mathbf{S}_{a}}{m_a} + \frac{\mathbf{S}_{b}}{m_b} \right) \times \vec{\nabla} \right] \frac{2n+1}{r} \frac{G_{ma} m_b}{r} \]

\[ (3.38) \]
which can be formally written as

$$-2 \left( \frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \cdot \sinh \left[ \left( \frac{\vec{S}_a}{m_a} + \frac{\vec{S}_b}{m_b} \right) \times \vec{\nabla} \right] \frac{G m_a m_b}{r}.$$  

(3.42)

This expression matches $-2(\vec{v}_1 - \vec{v}_2) \cdot \sinh(\hat{a} \times \vec{\nabla}) \frac{m_1 m_2}{R}$ of eq.(3.31) in [29]. Applying similar identities to the last line we find the complete 1 PM leading PN potential for rotating black holes:

$$V_{BBN}^{cl} = \left( -\cosh \left[ \left( \frac{\vec{S}_a}{m_a} + \frac{\vec{S}_b}{m_b} \right) \times \vec{\nabla} \right] -2 \left( \frac{\vec{p}_a}{m_a} - \frac{\vec{p}_b}{m_b} \right) \cdot \sinh \left[ \left( \frac{\vec{S}_a}{m_a} + \frac{\vec{S}_b}{m_b} \right) \times \vec{\nabla} \right] \right) \frac{G m_a m_b}{r}$$

$$+ \frac{1}{2} \left( \left[ \frac{\vec{p}_a}{m_a} \times \frac{\vec{S}_a}{m_a} - \frac{\vec{p}_b}{m_b} \times \frac{\vec{S}_b}{m_b} \right] \cdot \vec{\nabla} \right) \cosh \left[ \left( \frac{\vec{S}_a}{m_a} + \frac{\vec{S}_b}{m_b} \right) \times \vec{\nabla} \right] \frac{G m_a m_b}{r}.$$  

(3.43)

The above result can be matched to eq.(3.31) and eq.(3.32) in [29]. The first few terms of eq.(3.43) are;

$$V_{cl} = -\frac{G m_a m_b}{r} + \frac{G}{r^2} \hat{n} \cdot \left[ \frac{4m_a + 3m_b}{2m_a} \vec{p}_a \times \vec{S}_a - (a \leftrightarrow b) \right]$$

$$- \frac{G}{r^3} (\delta_{i,j} - 3 \hat{n}_i \hat{n}_j) \left[ \frac{m_b}{2m_a} C_{S_a} S_i S_j^i + (a \leftrightarrow b) + S_i S_j^i \right]$$  

(3.44)

where $\hat{n} = \vec{\xi} / r$.

We end this section with an interesting observation; the classical potential eq.(3.36) without the contributions coming from Hilbert space matching eq.(3.35) matches to intermediate results in EFT computations where higher time derivatives has not yet been eliminated. In EFT computations the classical potential is obtained by fixing spin variable gauges and doing Feynman diagram computations. The result obtained at this point will in general contain higher time derivatives such as $\dot{\vec{S}}$ and $\dot{\vec{v}}$, and neglecting these higher time derivatives will give an expression equivalent to eq.(3.36) without the last line; for example, compare the results given in this section with eq.(48) of [21] and eq.(71) of [25] for spin-orbit interactions, terms proportional to $C_{1(BS)}$ in eq.(3.10) of [26] for $(S_a)^3$ interactions, and the sum of first two terms proportional to $C_{1(ES^2)}$ in eq.(3.10) of [26] for $(S_a)^2 S_b$ interactions. The last procedure of EFT computations is eliminating higher time derivatives through redefinition of variables. Since the last line of eq.(3.36) is generated through Hilbert space matching procedure, this procedure generates the terms corresponding to terms generated from redefining variables to eliminate higher time derivatives.
4 From quantized spins to Kerr Black hole

As mentioned in the introduction, Holstein and Ross [15] computed the classical potential up to quartic order in spin, arising from the graviton exchange of minimally coupled massive spin-$\frac{1}{2}$, 1 particles. While the spin-orbit terms matches with that computed from EFT computations, deviations occur starting from quadratic order and beyond. Furthermore, it was observed that the spin-orbit terms are identical for the two choices of spins. From the discussion in the previous section, it is clear that finite spin results must deviate from Black holes due to the difference in their Wilson coefficients. Universality in the spin-orbit term can be viewed as simply a consequence of $C_{S^0} = C_{S^1} = 1$ for any spin, and the universal contributions from Hilbert space matching.

Naively one would conclude that it is impossible to extract the spin-operator dependent part of the potential from scattering amplitudes, which are usually formulated with finite spin asymptotic states. However, as we will now show, the classical-spin limit can be derived from finite spin applying a new form of universality:

\begin{equation}
A_{s_a,s_b}^{s_i,s_j} = \frac{(2s_a)!}{(2s_a - i)! (2s_b - j)!},
\end{equation}

where $s_a$ and $s_b$ is the spin of the two particles. Importantly as long as $A_{i,j}$ is non-vanishing, which requires $i \leq 2s_a$ and $j \leq 2s_b$, it is independent of spin.

Since the classical spin-limit of minimal coupling yields the dynamics of Kerr black hole, we can extract the Kerr black hole potential simply by computing the universal piece from finite spins, then dressing the result with the exponentiated factor. This is indeed the approach taken in [13], which reproduced the classical potential up to quartic order in spin. Here we present a detailed derivation.

4.1 The classical limit $s \gg 1$ in polarisation tensor basis

In previous section, the potential was extracted from the gluing of three-point interactions in the polarization basis. This has the advantage of being able to separate the spin-dependent piece that is inherent in the definition of polarization tensors, which are used for converting the operators of the EFT to on-shell matrix elements, from that of the spin-effects of the operators in the EFT. On the other hand for minimal coupling the computation is the simplest in the (anti)chiral basis. Let us begin by constructing an explicit map between the two basis directly in terms of on-shell variables.
The leading $q^2 \to 0$ residue for general EFT \(^8\) given by eq.(3.10) can be recast into on-shell matrix elements using eq.(2.8)

\[
Res_t = (-1)^{s_a + s_b} \sum_{i=0}^{2s_a} \sum_{j=0}^{2s_b} B_{i,j}^{\text{min}} \sum_{k=0}^{\infty} \tilde{n}_{i,k}^{s_a} (21)^{s_a-k} \left( -\frac{x_1 \langle 2\tilde{\lambda} \rangle \langle \lambda 1 \rangle}{ma} \right)_k [21]^{s_a-i+k} \left( \frac{2\tilde{\lambda} \langle \lambda 1 \rangle}{x_1 ma} \right)^{i-k} \\
\times \sum_{l=0}^{\infty} \tilde{n}_{j,l}^{s_b} (43)^{s_b-l} \left( -\frac{x_3 \langle 4\lambda \rangle \langle \lambda 3 \rangle}{mb} \right)_l [43]^{s_b-j+l} \left( \frac{4\lambda \langle \lambda 3 \rangle}{x_3 mb} \right)^{j-l},
\]

where $B_{i,j}^{\text{min}}$ are defined through eq.(3.10) with the Wilson coefficients set to minimal coupling, i.e. $C_{S0} = C_{S1} = 1$ and $C_{S2}$ given in eq.(2.19). The coefficients $\tilde{n}_{i,k}^{s_a}$ are defined as

\[
\tilde{n}_{i,k}^{s_a} = \frac{\gamma}{2} \left( \begin{array}{c} \gamma \\ k \\ i - k \end{array} \right).
\]

To convert into the anti-chiral basis, we use eq.(3.4) to convert all angle brackets to square brackets. For brevity, we denote $[21]$ and $[43]$ as $I$, $[\lambda 1]$ as $x$, and $[\lambda 3]$ as $y$. We find;

\[
Res_t = \sum_{i=0}^{2s_a} \sum_{j=0}^{2s_b} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} B_{i,j}^{\text{min}} \tilde{n}_{i,k}^{s_a} \tilde{n}_{j,l}^{s_b} (I + x)^{s_a-k} x^i (I + y)^{s_b-l} y^l.
\]

Now let us take $s_a, s_b \to \infty$, and denote the minimal coupling $B_{i,j}^{\text{min}}$ coefficients in this limit as $B_{i,j}^{\text{min,\infty}}$, which simply correspond to setting all Wilson coefficients to 1. We introduce formal parameters $\tilde{x} = 2s_a x$ and $\tilde{y} = 2s_b y$. Holding $\tilde{x}$ and $\tilde{y}$ finite while sending $s_a, s_b \to \infty$ yields the following expression.

\[
\lim_{s_a, s_b \to \infty} Res_t = \sum_{i,j=0}^{\infty} B_{i,j}^{\text{min,\infty}} \frac{\tilde{x}^i \tilde{y}^j e^{\tilde{x}/2 + \tilde{y}/2}}{2^{i+j}}
\]

This expression will be matched to an equivalent expression computed in the spinor basis. Importantly, after the identification of the polynomial in $\tilde{x}, \tilde{y}$ with the operators in the classical potential, the coefficients of these operators are determined by $B_{i,j}^{\text{min,\infty}}$. Thus the question now becomes, can we derive $B_{i,j}^{\text{min,\infty}}$ from amplitudes of finite spin particles? As we show in the next section, the answer is yes.

4.2 The classical limit $s \gg 1$ in chiral-basis and universality

Now let us compute the same residue through the (anti)chiral basis:

\[
Res_t = A_{3a}^+ A_{3b}^- + A_{3a}^- A_{3b}^+ = \alpha^2 m_a^2 m_b^2 \left( \frac{x_1^2}{x_3^2} \right)^2 \left( \frac{(-1)^{2s_b} [21]^{2s_a} [43]^{2s_b}}{x_3^2 \left( -1 \right)^{2s_a} [21]^{2s_a} [43]^{2s_b}} \right).
\]

\(^8\)The spins $s_a$ and $s_b$ are assumed to be integers. The sign factor $(-1)^{s_a+s_b}$ appears due to the choice of mostly minus metric signature.
The sign factors \((-1)^{2s_a}\) and \((-1)^{2s_b}\) are remnants of taking momenta \(P_2\) and \(P_4\) as outgoing. Again using eq.(3.4), this can be converted into

\[
\text{Res}_t = \sum_{i=0}^{2s_a} \sum_{j=0}^{2s_b} A_{i,j}^{s_a,s_b} \left( [2]^{2s_a} \left( \frac{[\lambda][\bar{\lambda}]}{x_1 m_a} \right)^i [1]^{2s_a} \right) \left( [4]^{2s_b} \left( \frac{[\lambda][\bar{\lambda}]}{x_3 m_b} \right)^j [3]^{2s_a} \right)
\]

\[= \sum_{i=0}^{2s_a} \sum_{j=0}^{2s_b} A_{i,j}^{s_a,s_b} x^i y^j \tag{4.7} \]

Note that only \(A_{i,0}\) and \(A_{0,j}\) are non-vanishing. Importantly, the only dependence of \(A_{i,j}^{s_a,s_b}\) on spins of external particles \(s_a\) and \(s_b\) is through combinatoric factors. In other words, defining:

\[A_{i,j}^{s_a,s_b} = A_{i,j}^{N_{i,j}^{s_a,s_b}}, \quad N_{i,j}^{s_a,s_b} = (-1)^{2s_a+2s_b} \frac{(2s_a)!}{(2s_a-i)!} \frac{(2s_b)!}{(2s_b-j)!} \]

the factors \(A_{i,j}^{s_a,s_b}\) are independent of \(s_a\) and \(s_b\). Note that for generic values of non-zero \(g_i\) the above property will not hold! It is instructive to understand the origin of such universality. The combinatoric factors \(N_{i,j}^{s_a,s_b}\) takes its root from the coefficients of a binomial expansion. Indeed, it originated from:

\[\langle 21 \rangle^s \to (\bar{\lambda}+x)^s, \quad \langle 43 \rangle^s \to (\bar{\lambda}+y)^s \tag{4.9} \]

Since the only spin-operator dependence stems from \(x, y\), and the only spin dependence stems from the external spinor wave functions, \([1]^{2s_a}, [2]^{2s_a}, [3]^{2s_a}, [4]^{2s_a}\), the coefficient for a given term with fixed degree of spin-operators must differ only by combinatoric factors when computed from particles of distinct \(s\). This viewpoint on combinatoric factors will echo throughout the rest of the paper, especially in section 6.

Returning to eq.(4.7), taking the \(s_a, s_b \to \infty\) limit becomes simple in terms of \(A_{i,j}\). As before we take the \(s_a, s_b \to \infty\) limit with the variables \(\bar{x}, \bar{y}\) held fixed, yielding

\[
\lim_{s_a,s_b \to \infty} \text{Res}_t = \sum_{i,j=0}^{\infty} A_{i,j}^{s_a,s_b} \bar{x}^i \bar{y}^j \tag{4.10} \]

The above expression is the spinor basis equivalent of eq.(4.5). Again, we stress that \(A_{i,j}\) defined through eq.(4.7) and eq.(4.8) is independent of spin. More precisely, for any given spin \(\{s_a, s_b\}\), for \(i \leq 2s_a\) and \(j \leq 2s_b\) we have non-vanishing \(A_{i,j}\) given by an overall spin-dependent combinatoric factor \(N_{i,j}^{s_a,s_b}\) multiplied to \(A_{i,j}\).

Now let’s match eq.(4.5) to eq.(4.10):

\[
\sum_{i,j=0}^{\infty} B_{i,j}^{\min,\infty} \frac{B_{i,j}^{\max,\infty}}{2i+j} \bar{x}^i \bar{y}^j e^{\bar{x}^2/2+\bar{y}^2/2} = \sum_{i,j=0}^{\infty} A_{i,j}^{s_a,s_b} \bar{x}^i \bar{y}^j \tag{4.11} \]

\(\text{The } A_{i,j} \text{ coefficient here is the same as the } \tilde{A}_{i,j} \text{ in } [13]\)
The equation above can be solved for $B_{i,j}^{\text{min},\infty}$ as a function of $A_{i,j}$:

$$B_{i,j}^{\text{min},\infty} = 2^{i+j} \sum_{k,l} \frac{A_{i-k,j-l}}{(-2)^{k+l}k!l!}$$

(4.12)

As one can see the classical potential, given by $B_{i,j}^{\text{min},\infty}$, can be computed from $A_{i,j}$ which are spin independent. In other words due to the universal behaviour of minimal coupling, in that the $A_{i,j}$ are independent of spins, *allows us to obtain the classical-spin limit from finite-spin amplitudes*! Indeed the RHS of eq.(4.12) is what was used to compute the potential in [13].
5 Universality at 2 PM

At 2 PM the classical potential can be extracted from the one-loop amplitude with graviton exchange. As pointed out long ago by Holstein and Donoghue [30], the presence of both massless and massive particles in loops can generate non-trivial classical pieces. At one loop, these take the form of an non-analytic piece proportional to:

\[ \frac{1}{\sqrt{q^2}} \]  

In four-dimension, any 1-loop amplitude can be reduced to a scalar integral basis including bubble, triangle and box integrals. At four-points, only \( t \)-channel triangles can generate such terms, hence the scalar triangle coefficient yields the complete classical contribution at one-loop.

In this section, we will only be interested in the \( G^2h^0|q|^{-1} \) effects which can be cleanly captured by the \( t \)-channel triangle in the HCL limit. To compute the integral coefficient, we apply the unitarity cut approach [31, 32] especially by Forde [33], where the contributions of each integral is separated by their distinct set of propagators. By putting these propagators on-shell unitarity dictates that the result must be given by the product of tree-amplitudes. In our case, the triangle cut, we have the product of two minimal coupling three-point amplitude and a gravitational Compton amplitude, as illustrated in fig.2. The triangle coefficient can then be captured by removing the contributions from the box integrals.

Compared to 1 PM, the new feature at 2 PM is the gravitational Compton amplitude,
involving two massive spinning particles and two massless gravitons:

For Kerr black holes, we will be interested in the Compton amplitude which correspond to the four-point extension of the three-point minimal coupling. More precisely, the residue of the massive pole must yield the product of three-point minimal coupling discussed previously. However as shown in [11] and [13] for \( s > 2 \), due to polynomial ambiguities, factorization constraints do not uniquely determine the gravitational Compton amplitude. More precisely, by matching to the factorization pole in all three channels, for \( s \leq 2 \) we can find a solution:

\[
M_4^{(s \leq 2)}(1^s, -2^s, k_3^{-2}, k_4^{+2}) = -\frac{\langle k_3|P_1|k_4 \rangle^4}{\langle k_3|P_1|k_3 \rangle \langle k_4|P_1|k_4 \rangle \langle k_4|k_3|k_4 \rangle} \left( -\langle 1k_3 \rangle \langle 2k_4 \rangle + \langle 2k_3 \rangle \langle 1k_4 \rangle \right)^{2s}
\]

where \( |k_3 \rangle, |k_4 \rangle \) and \( |k_3 \rangle, |k_3 \rangle \) are massless spinors for the massless propagators. Importantly, the result does not contain any \( \frac{1}{m} \) factors. This has two important implications: 1. One can take \( m \to 0 \) limit smoothly, indicating that the spinning particle has a point-like description. 2. Since pure polynomial terms must have \( \frac{1}{m^n} \) factors simply on dimensional grounds, they can be considered as finite size effects and do not mix with eq.(5.2). For \( s > 2 \), the situation is drastically different. The amplitude takes the form (see [13]):

\[
M_4^{(s > 2)}(1^s, -2^s, k_3^{-2}, k_4^{+2}) = -\frac{\langle k_3|P_1|k_4 \rangle^4}{\langle k_3|P_1|k_3 \rangle \langle k_4|P_1|k_4 \rangle \langle k_4|k_3|k_4 \rangle} \left( \frac{\langle 12 \rangle - \langle 12 \rangle}{m} + \ldots \right)^{2s} + \ldots,
\]

where we’ve only listed the leading term in propagators and \( \frac{1}{m} \) expansion. We see that unlike \( s \leq 2 \), here the leading piece already contains non-trivial \( \frac{1}{m} \) dependence, thus making the separation of finite size effects from that of what is dictated by factorization operationally meaningless.

However, motivated by the universality found in our 1 PM discussion, it is natural to conjecture that the correct Compton amplitude must be the one that preserves universality. In the following we will begin by considering spin-1 and 2, where the Compton amplitude is known. We will demonstrate that the HCL limit of the triangle integral, along with its coefficient, indeed satisfy universality, i.e. it is given by a product of spin-dependent combinatoric factor and a spin-independent piece, reproducing the pattern in eq.(4.8). This provides supporting evidence that universality should be expected beyond 1 PM. Next we compare two possible higher-spin extension for Compton amplitude, the non-local BCFW recursion, and the local completion. We will demonstrate that in both case, universality can be achieved.
5.1 2 PM Classical-spin limit and universality for $s \leq 2$

It is known that the spin $s$ amplitude can be used to probe $S^{n \leq 2s}$ effects in the classical potential. Given the discussions in the previous sections, the classical-spin limit can be derived from finite spin applying a new form of universality. Thus in this section, we examine the validity of universality for 2 PM. For $s \leq 2$ particles, we insert the Compton amplitude eq.(5.2) into the computation of the triangle integral in the HCL limit.

Taking spin 1 and spin 2 as an example, the triangle integral in the HCL limit yields the following coefficients for the anti-chiral basis (defined in eq.(4.7)).

1. $S^0_aS^0_b$

\[
\frac{A_{0,0}^{s=1}}{m_a^2m_b^2} = -\frac{24\pi^2G^2(m_a + m_b)}{|q|} + O(\epsilon^2)
\]

\[
\frac{A_{0,0}^{s=2}}{m_a^2m_b^2} = -\frac{24\pi^2G^2(m_a + m_b)}{|q|} + O(\epsilon^2)
\]

2. $S^1_aS^0_b$

\[
\frac{A_{1,0}^{s=1}}{m_a^2m_b^2} = \frac{4\pi^2G^2(4m_a + 3m_b)}{|q|\epsilon} + \frac{24\pi^2G^2(m_a + m_b)}{|q|} + O(\epsilon)
\]

\[
\frac{A_{1,0}^{s=2}}{m_a^2m_b^2} = \frac{8\pi^2G^2(4m_a + 3m_b)}{|q|\epsilon} + \frac{48\pi^2G^2(m_a + m_b)}{|q|} + O(\epsilon)
\]

3. $S^1_aS^1_b$

\[
\frac{A_{1,1}^{s=1}}{m_a^2m_b^2} = \frac{4\pi^2G^2(m_a + m_b)}{|q|\epsilon^2} + \frac{4\pi^2G^2(m_b - m_a)}{|q|\epsilon} + \frac{14\pi^2G^2(m_a + m_b)}{|q|} + O(\epsilon)
\]

\[
\frac{A_{1,1}^{s=2}}{m_a^2m_b^2} = \frac{16\pi^2G^2(m_a + m_b)}{|q|\epsilon^2} + \frac{16\pi^2G^2(m_b - m_a)}{|q|\epsilon} + \frac{56\pi^2G^2(m_a + m_b)}{|q|} + O(\epsilon)
\]

4. $S^2_aS^0_b$

\[
\frac{A_{2,0}^{s=1}}{m_a^2m_b^2} = -\frac{\pi^2G^2(m_a + m_b)}{|q|^2\epsilon^2} - \frac{2\pi^2G^2(4m_a + 3m_b)}{|q|^2\epsilon} - \frac{\pi^2G^2(34m_a + 27m_b)}{|q|^2} + O(\epsilon)
\]

\[
\frac{A_{2,0}^{s=2}}{m_a^2m_b^2} = -\frac{6\pi^2G^2(m_a + m_b)}{|q|^2\epsilon^2} - \frac{12\pi^2G^2(4m_a + 3m_b)}{|q|^2\epsilon} - \frac{3\pi^2G^2(34m_a + 27m_b)}{|q|^2} + O(\epsilon)
\]

\[\text{Here, } \epsilon = \sqrt{\rho^2 - 1}\]
5.2 2 PM Classical-spin limit and universality for $s > 2$

Given that universality is found to hold for minimal coupling with $s \leq 2$, we naturally ask whether there exists some higher spin extension of minimally coupled gravitational Compton amplitude, such that universality is respected. A straightforward BCFW construction of the Compton amplitude yields a non-local expression for $s > 2$ (see also [34]). However since we are interested in only the classical part of the triangle integral, a priori it is not clear whether this would yield a problematic potential. Here we will keep an open mind and simply analyze whether the result respects universality. On the other hand, a manifestly local, albeit non-unique, higher-spin extension was given in eq.(5.24) of [13]. We also analyze what universality has to say, and whether or not it can be utilized to constrain the ambiguity.

\begin{align}
\mathcal{A}_{s=1}^{a=1} &= \frac{\pi^2 G^2 (m_a + m_b)}{m_a^2 m_b^2} - \frac{\pi^2 G^2 (m_a + 6m_b)}{2|\vec{q}|\epsilon} - \frac{\pi^2 G^2 (4m_a + 11m_b)}{2|\vec{q}|} + O(\epsilon) \\
\mathcal{A}_{s=2}^{a=1} &= \frac{12\pi^2 G^2 (m_a + m_b)}{|\vec{q}|\epsilon^2} - \frac{6\pi^2 G^2 (m_a + 6m_b)}{|\vec{q}|} + O(\epsilon) \\
\mathcal{A}_{s=1}^{a=2} &= \frac{\pi^2 G^2 (m_a + m_b)}{8|\vec{q}|\epsilon^2} + \frac{\pi^2 G^2 (m_b - m_a)}{4|\vec{q}|\epsilon} + \frac{13\pi^2 G^2 (m_a + m_b)}{32|\vec{q}|} + O(\epsilon) \\
\mathcal{A}_{s=2}^{a=2} &= \frac{9\pi^2 G^2 (m_a + m_b)}{2|\vec{q}|\epsilon^2} + \frac{9\pi^2 G^2 (m_b - m_a)}{|\vec{q}|\epsilon} + \frac{117\pi^2 G^2 (m_a + m_b)}{8|\vec{q}|} + O(\epsilon)
\end{align}

Importantly, the $\mathcal{A}$ coefficients satisfy the following identity:

\begin{equation}
\mathcal{A}_{i\leq 2, j\leq 2}^{a=2} = \frac{4!^2}{(4 - i)! (4 - j)!} \frac{(2 - i)! (2 - j)!}{2!^2} \mathcal{A}_{i\leq 2, j\leq 2}^{a=1}.
\end{equation}

The factors in front of $\mathcal{A}_{i\leq 2, j\leq 2}^{a=1}$ are the combinatoric factors defined in eq.(4.8), of spin 2 divided by that of spin 1. Thus we again find that at 2 PM, the classical contribution extracted from minimally coupled spin 1 and 2 respects universality as defined in eq.(4.8). Although we’ve only displayed up to $O(\epsilon)^0$ for brevity in eq.(5.4a) to eq.(5.4f), the universality behaviour is satisfied to all orders in $\epsilon$. Similarly, we’ve also checked that universality also holds for spin-1/2 and 3/2.
5.2.1 The BCFW Compton amplitude

First, we start from the BCFW representation of the Compton amplitude. The BCFW amplitude here is constructed from the \( \langle k_3, k_4 \rangle \) shift:

\[
M^\text{BCFW}_4(1^s, 2^s, k_3^{-2}, k_4^+2) = \frac{M_3(1^s, -\hat{P}_{14}^s, \hat{k}_3^{-2})M_3(2^s, -\hat{P}_{14}^s, \hat{k}_3^{-2})}{\langle k_4|P_1|k_4 \rangle} + \frac{M_3(1^s, -\hat{P}_{13}^s, \hat{k}_3^{-2})M_3(2^s, -\hat{P}_{13}^s, \hat{k}_4^+2)}{\langle k_3|P_1|k_3 \rangle} \tag{5.6}
\]

We present the details in appendix D. Taking \( P_2 \to -P_2 \), it indeed recovers (5.2). Note that while for \( s > 2 \) the expression is non-local, it factorizes correctly on all three channels. While the \( s, u \)-channel factorization matching is simply by construction, it is non-trivial that the result satisfies \( t \)-channel residue as well. Since it is just an extrapolation of eq.(5.2) to \( s > 2 \), not surprisingly the HCL limit of the triangle integral satisfy universality for all spins. Take \( A_{4,0} \) as an example:

\[
\frac{A_{4,0}^s}{m^2 a^s b^s} = \frac{225\pi^2G^2(m_a + m_b)}{8|\vec{q}| \epsilon^2} - \frac{75\pi^2G^2(4m_a + 3m_b)}{2|\vec{q}| \epsilon} - \frac{225\pi^2G^2(23m_a + 16m_b)}{16|\vec{q}|} + O(\epsilon)
\]

\[
\frac{A_{4,0}^{s=2}}{m^2 a^2 b^2} = \frac{15\pi^2G^2(m_a + m_b)}{8|\vec{q}| \epsilon^2} - \frac{5\pi^2G^2(4m_a + 3m_b)}{2|\vec{q}| \epsilon} - \frac{15\pi^2G^2(23m_a + 16m_b)}{16|\vec{q}|} + O(\epsilon)
\]

We find that \( A_{4,0}^{s=3} = 15A_{4,0}^{s=2} = \frac{6^2}{(6-4)!(6-0)!} \frac{(4-4)!(4-0)!}{4!^2} A_{4,0} \), which indeed satisfy the universality relation.

An interesting self-consistency test is the following; assuming universality also holds at 2 PM, then the computations done for \( s \leq 2 \) would be sufficient to capture the correct potential involving operators \( S^{n \leq 4} \) by RHS of eq.(4.12). Here, we take the BCFW Compton amplitude in the polarization tensor basis and take the classical-spin limit of the triangle integral. In the basis of eq.(4.2) we find:

- \( S_0^0 \)

\[
B^\text{BCFW}_{0,0} = \frac{24\pi^2G^2 m_a^2 m_b^2 (m_a + m_b)}{|\vec{q}|} + O(\epsilon)^2 \tag{5.8}
\]

- \( S_1^0 \)

\[
B^\text{BCFW}_{1,0} = -\frac{4\pi^2G^2 m_a m_b^2 (4m_a + 3m_b)}{|\vec{q}| \epsilon} - \frac{12\pi^2G^2 m_a^2 m_b^2 (4m_a + 3m_b)}{|\vec{q}|} \epsilon + O(\epsilon)^3 \tag{5.9}
\]
\( S_a S_b \)

\[
B_{2,0}^{\text{BCFW}} = \frac{2\pi^2 G^2 m_b^2 (m_a + m_b)}{|q|^2} + \frac{\pi^2 G^2 m_b^2 \left[ \left( 22 + \frac{12}{2s-1} \right) m_a + \left( 15 + \frac{12}{2s-1} \right) m_b \right]}{|q|} + O(\epsilon)^2
\]

\[
= \frac{2\pi^2 G^2 m_b^2 (m_a + m_b)}{|q|^2} + \frac{\pi^2 G^2 m_b^2 (22m_a + 15m_b)}{|q|} + O \left( \frac{1}{s} \right) + O(\epsilon)^2
\]

(5.10)

\( S_a S_b \)

\[
B_{3,0}^{\text{BCFW}}
\]

\[
= -\frac{\pi^2 G^2 m_b^2 \left( 1 + \frac{2}{2s-1} \right) (4m_a + 3m_b)}{|\vec{q}|m_a} - \frac{\pi^2 G^2 m_b^2 \left[ \left( 11 + \frac{24}{2s-1} \right) m_a + \left( \frac{13}{2} + \frac{18}{2s-1} \right) m_b \right]}{|\vec{q}|m_a} \epsilon + O(\epsilon)^3
\]

\[
= -\frac{\pi^2 G^2 m_a^2 (4m_a + 3m_b)}{|\vec{q}|m_a} \epsilon - \frac{\pi^2 G^2 m_b^2 (22m_a + 13m_b)}{2|\vec{q}|m_a} \epsilon + O \left( \frac{1}{s} \right) + O(\epsilon)^3
\]

(5.11)

\( S_a S_b \)

\[
B_{4,0}^{\text{BCFW}}
\]

\[
= \frac{\pi^2 G^2 \left( \frac{1}{2} + \frac{1}{2s-1} \right) m_a^2 (m_a + m_b)}{|\vec{q}|m_a^2 \epsilon^2} + \frac{\pi^2 G^2 m_b^2 \left[ \frac{152s^3 - 28s^2 - 390s + 87}{8(1-2s)^2(2s-3)} m_a + \frac{3(4s^3 - 11s + 1)}{(1-2s)^2(2s-3)} m_b \right]}{|\vec{q}|m_a^2} + O(\epsilon)^2
\]

\[
= \frac{\pi^2 G^2 m_a^2 (m_a + m_b)}{4|\vec{q}|m_a^2 \epsilon^2} + \frac{\pi^2 G^2 m_b^2 (19m_a + 12m_b)}{8m_a^2 |\vec{q}|} + O \left( \frac{1}{s} \right) + O(\epsilon)^2
\]

(5.12)

We find that it indeed reproduces the RHS of eq.(4.12).

Finally, we discuss where the universality of the BCFW representation comes from. In section 4.2, we established that universality stems from the external wave functions \(|1\rangle^{2s}, |2\rangle^{2s}, |3\rangle^{2s}, |4\rangle^{2s}\). The coefficient for a fixed degree of spin operators differ only by combinatoric factors when computed from different values of \(s\). We explicitly show that the BCFW representation of the Compton amplitude can be written in the form:

\[
(\Pi + \chi)^{2s}
\]

(5.13)

Looking at the terms in the parenthesis of eq.(5.6) with \(P_2 \rightarrow -P_2\), it can be rewritten by the following identity:

\[
\frac{-\langle 1k_3 \rangle |2k_4 \rangle + \langle 2k_3 \rangle |1k_4 \rangle}{\langle k_3 | P_1 | k_4 \rangle}
\]

\[
= \frac{1}{2} \left( -\frac{|12 \rangle}{m} + \frac{\langle 1k_4 \rangle |k_4 2 \rangle + \langle 2k_4 \rangle |k_4 1 \rangle}{2m^2} - \frac{\langle k_3 | P_1 | k_3 \rangle |k_4 2 \rangle - \langle k_4 | P_1 | k_4 \rangle (|k_3 1 \rangle |k_4 2 \rangle)}{2m^2 (k_4 | P_1 | k_4 ^2)} \right)
\]

(5.14)
Parameterizing under each term in the HCL, one finds

\[
\frac{1}{2} \left( -\frac{12}{m} + \frac{12}{m} + \frac{[1k_4]⟨k_4|2⟩ + [2k_4]⟨k_4|1⟩}{2m^2} - \frac{[1k_3]⟨k_3|3⟩ + [2k_3]⟨k_3|1⟩}{2m^2} \right) \rightarrow -1 - \frac{(-1 + y)^2 |\hat{\lambda}|}{4y} m_a
\]

where \( y \) is part of the parameterization of \( |k_4⟩, |k_3⟩ \) and \( |k_4⟩, |k_3⟩ \), such that the cut conditions are satisfied. For details see [13]. In other words, after inserting the cut solutions, the BCFW representation indeed takes the form:

\[
(1 + \alpha x)^{2s_a}
\]

where \( \alpha \) is some function of \( y, m, \varepsilon \). Again, as \( x \) is the only source for spin-vector operator on leg \( a \), following the same argument as before leads to the conclusion that universality is manifestly satisfied.

### 5.2.2 The \( s > 2 \) local Compton amplitude

Now, we turn to the local representation of the Compton amplitude given in eq.(5.24) of [13]. We first ask if universality is respected when comparing with the known Compton amplitudes \( (s \leq 2) \) for coefficients \( A_{0,0} \) to \( A_{4,4} \), and if not can the situation be rectified by inclusion of suitable polynomial terms.

We analyze the coefficients \( A_{i,j} \) extracted from the gravitational Compton amplitude for \( s = 6 \) in [13]. For \( i, j \leq 4 \), we expect to match with that computed from \( s \leq 2 \) via universality. In the following, we give a few examples.

1. \( i, j < 4 \) \((S_a^{i<4} S_b^{j<4})\):

\[
\frac{A_{\text{local},s=6}^{2,0}}{m_a^2 m_b^2} = -\frac{66\pi^2 G^2 (m_a + m_b)}{|\vec{q}| \varepsilon^2} - \frac{132\pi^2 G^2 (4m_a + 3m_b)}{|\vec{q}| \varepsilon} - \frac{33\pi^2 G^2 (34m_a + 27m_b)}{|\vec{q}|} + O(\varepsilon)
\]

Comparing with eq.(5.4d), one finds

\[
A_{\text{local},s=6}^{2,0} = \frac{12!}{(12 - 2)!} \cdot \frac{(4 - 2)!}{4!} A_{s=2}^{2,0} = \frac{12!}{(12 - 2)!} \cdot \frac{(2 - 2)!}{2!} A_{s=1}^{2,0}
\]

}\]

\[
(5.18)
\]
where universality between the local amplitude and eq. (5.2) is satisfied. Similarly, comparing between different higher spins, we also find universality. For example for $s = 4, 6$

$$
\frac{A^{\text{local, } s=4}_{2,0}}{m_a^2 m_b^2} = -\frac{28\pi^2 G^2 (m_a + m_b)}{|q| \epsilon^2} - \frac{56\pi^2 G^2 (4m_a + 3m_b)}{2|q| \epsilon} - \frac{14\pi^2 G^2 (34m_a + 27m_b)}{6|q|} + O(\epsilon)
$$

$$
= \frac{8}{(8 - 2)!} \frac{(12 - 2)!}{12!} \frac{A^{\text{local, } s=6}_{2,0}}{m_a^2 m_b^2}
$$

(5.19)

2. $i = 4, j < 4$ ($S^i_a = 4, S^j_b < 4$) effects:

Operators with $S^4$ are the highest degree for which spin-2 particles can probe. Here, unlike the BCFW higher-spin extension, we indeed find discrepancy with that from universality. Take for example $A_{4,0}$:

$$
\frac{A^{\text{local, } s=6}_{4,0}}{m_a^2 m_b^2} = -\frac{7425\pi^2 G^2 (m_a + m_b)}{8|q| \epsilon^2} - \frac{2475\pi^2 G^2 (4m_a + 3m_b)}{2|q| \epsilon} - \frac{1485\pi^2 G^2 (460m_a + 317m_b)}{64|q|} + O(\epsilon)
$$

$$
= \left\{ 495 - \frac{297\epsilon^2 m_b}{8 (m_a + m_b)} + O(\epsilon^3) \right\} \frac{A^{s=2}_{4,0}}{m_a^2 m_b^2}
$$

(5.20)

We see that the $s = 6$ and $s = 2$ results no longer differ by an overall combinatoric factor. In other words, universality is lost. However, as discussed in [13], given the polynomial ambiguity of Compton amplitudes beyond $s > 2$, we can restore universality simply by adding local contact terms. Indeed by adding

$$
M^{s=6}_{\text{contact}} = -\frac{3}{2} (8\pi G) m^2 \left( \frac{12}{4} \right) \left( \frac{\langle k_3 \rangle [k_4 1] + \langle k_3 1 \rangle [k_4 2]}{2m^2} \right)^4 \left( \frac{\langle 12 \rangle - [12]}{2m} \right)^8
$$

(5.21)

to the spin-6 Compton amplitude, we find:

$$
\frac{A^{\text{local+contact, } s=6}_{4,0}}{m_a^2 m_b^2} = -\frac{7425\pi^2 G^2 (m_a + m_b)}{8|q| \epsilon^2} - \frac{2475\pi^2 G^2 (4m_a + 3m_b)}{2|q| \epsilon} - \frac{1485\pi^2 G^2 (23m_a + 16m_b)}{16|q|} + O(\epsilon)
$$

$$
= \frac{12! (4 - 4)!}{(12 - 4)! 4!} \times \frac{A^{s=2}_{4,0}}{m_a^2 m_b^2}
$$

(5.22)

We now see that a correct contact term makes $A_{i=4, j<4} = A^{\text{local+contact}}_{i=4, j<4}$. For generic spins, the suitable contact term is:

$$
M^{s}_{\text{contact}} = -\frac{3}{2} (8\pi G) m^2 \left( \frac{2s}{4} \right) \left( \frac{\langle k_3 \rangle [k_4 1] + \langle k_3 1 \rangle [k_4 2]}{2m^2} \right)^4 \left( \frac{\langle 12 \rangle - [12]}{2m} \right)^{2s-4}
$$

(5.23)
Note that this contact term is constructed in a way that it only modifies the behaviour of $S_a^{i>4}S_b^{j>4}$ and we do not need to worry about breakdown of universality for $S_a^iS_b^j<4$.

3. $i > 4, j > 4$ ($S_a^{i>4}S_b^{j>4}$):

Having patched up our local expression such that universality is respected for $A_{i,j}$ with $i, j \leq 4$, we now turn to its fate for $i, j > 4$. Again, focusing on spin-6, we find

$$A_{5,0}^{\text{local+contact}, s=6} = \frac{m_a^2m_b^2}{2|\vec{q}|\epsilon^2} + \frac{17325\pi^2G^2(4m_a + 3m_b)}{8|\vec{q}|\epsilon} + \frac{693\pi^2G^2(107m_a + 72m_b)}{4|\vec{q}|} + O(\epsilon)$$

$$A_{5,0}^{\text{local+contact}, s=3} = \frac{m_a^2m_b^2}{8|\vec{q}|\epsilon^2} + \frac{525\pi^2G^2(4m_a + 3m_b)}{32|\vec{q}|\epsilon} + \frac{21\pi^2G^2(107m_a + 72m_b)}{16|\vec{q}|} + O(\epsilon)$$

$$= \frac{6!}{(6-5)!} \frac{(12-5)!}{12!} \frac{A_{5,0}^{\text{local+contact}, s=6}}{m_a^2m_b^2}$$

(5.24)

where universality is still preserved. Here, we conclude that even with the modification from the contact terms eq.(5.23) universality is still preserved by all spins.

Thus in conclusion, both the non-local BCFW construction and the local Compton amplitude derived previously, augmented by suitable contact terms, serve as viable extensions of higher-spin Compton amplitude in the context of universality. The two extensions give the same $A_{i,j}$ for $i, j \leq 4$, but differ beyond. In other words, universality and minimal coupling is insufficient to determine the 2 PM potential, even in the HCL limit. However, when combined with other arguments, the BCFW Compton amplitude appears to be favored, which we will discuss in the conclusion.

Before ending this section, we briefly discuss the universality of both the local Compton amplitude and the contact term. For the contact term, it is clear that universality comes from the binomial expansion of

$$\left(\frac{12}{2m} - \left[\frac{12}{2m}\right]\right)^{2s} \rightarrow -1 + \frac{\hat{\lambda}\hat{\lambda}}{2m_a}$$

(5.25)

However, for the local Compton amplitude eq.(5.24) in [13], it appears to be a remarkable fact that it retains universality.

### 6 Extending universality beyond black holes

From the discussion so far, we’ve seen that the classical potential of minimal coupling exhibit universality both at 1 and 2 PM. The origin of this universality reflects the fact that the
minimal coupling three-point amplitude for general spins can be viewed as a one parameter family with a special feature: the 3-pt amplitude of spin-$s$ is simply $2s$ power of that of spin-$\frac{1}{2}$.

An interesting question is if the same can be replicated in general context. More precisely, given a set of Wilson coefficients $C$, can we find a set of couplings in the chiral basis $g_i(s)$ such that universality is preserved and in the classical-spin limit, it reproduces the correct classical potential computed directly from the Wilson coefficients in eq.(3.36). Remarkably the answer is yes, which we now show.\footnote{The reader might wonder given that we have already computed the 1 PM potential in eq.(3.36), why are we interested in searching for universality to simplify computations? The aim is of course to set up potential tools for higher PM results.}

6.1 Spin-dependence factorisation beyond minimal coupling

Given a fixed set of Wilson coefficients, consider the following couplings in the chiral basis:

$$g_i(s) \equiv \left( \frac{2s}{i} \right) \hat{g}_i, \quad \hat{g}_i = \frac{1}{2i} \sum_{m=0}^{i} \binom{i}{m} (-1)^m C_{S_m}$$

(6.1)

Note that $g_0 = \hat{g}_0$; minimal coupling already satisfies this condition. Note that this differs from the couplings obtained by converting the one-body EFT from the polarization basis to the chiral basis, i.e. eq.(2.16) and eq.(2.17), which we denote as $g_{1,\text{bd}}^i$ here to make distinctions.

However, if we extrapolate to $s \to \infty$ when $\hat{g}_i \neq 0$, we find

$$\lim_{s \to \infty} \frac{g_{1,\text{bd}}^i}{g_i(s)} = 1.$$ 

(6.2)

In other words, computations using the couplings defined in eq.(6.1), will match that of using the Wilson coefficients directly in the classical spin limit.

Now let’s consider the $t$-channel residue arising from such three-point coupling:

$$\text{Res}_t = \sum_{i,j} (-1)^{2s_a+2s_b} \alpha^2 m_a^2 m_b^2 A_{i,j} \hat{x}_i^i \hat{y}_j^j$$

$$A_{i,j} = \frac{x_2^x}{x_3^y} \sum_{k=0}^i (-1)^k \binom{i}{k} \hat{g}_k^a + \frac{x_2^y}{x_3^x} \sum_{l=0}^j (-1)^l \binom{j}{l} \hat{g}_l^b.$$ 

(6.3)

Factoring the $N_{i,j}^{s_a,s_b}$ factor from $(-1)^{2s_a+2s_b} A_{i,j}$ gives the following spin-independent expression:

$$A_{i,j} = \sum_{k=0}^i (-1)^k \frac{x_2^x}{x_3^y} \hat{g}_k^a \hat{g}_j^b + \sum_{l=0}^j (-1)^l \frac{x_2^y}{x_3^x} \hat{g}_i^a \hat{g}_l^b.$$ 

(6.4)

Thus we find that the 1PM potential derived from the couplings in eq.(6.1) satisfies universality just like that of minimal coupling. Combined with eq.(6.2), this tells us that for general Wilson coefficients, one can use the couplings in eq.(6.1) to compute the spin-independent universal factors, and then reproduce the correct potential from eq.(4.12). Indeed one can straightforwardly verify that this reproduces eq.(3.36).
6.2 Manifest universality for general EFT

Having seen that universality for general Wilson coefficients can be satisfied by defining couplings \( g_i \) as in eq.(6.1), it is natural to ask whether the resulting three-point amplitude can be written in a form that manifest universality. In other words, can we write the three-point amplitude for general \( s \), as \( 2s \) powers of some fundamental spin-\( \frac{1}{2} \) amplitude? The answer once again, is yes.

Adopt following definitions.

\[
\bar{u}(P_2) u(P_1) = \frac{[21] - \langle 21 \rangle}{2m}
\]

\[
S_{1/2}^\mu = \frac{1}{2} \bar{u}(P_2) \gamma^\mu \gamma_5 u(P_1) = -\frac{1}{4m} \left( [2|\sigma^\mu|1] + \langle 2|\sigma^\mu|1 \rangle \right)
\]

The definition for spin vector has been adopted from Holstein and Ross [15] with a sign choice that matches to our conventions. An extra factor of \( \frac{1}{2m} \) has been inserted as a normalisation condition \( \bar{u}_I(P_2) u_J(P_1) = \delta_{IJ} \). One then finds the following candidate expression for the one-particle EFT amplitude eq.(2.9)

\[
M_{2s}^{2\eta} = \frac{\kappa m x^{2\eta}}{2} \oint \frac{dz}{2\pi i z} \left( \sum_{n=0}^{\infty} C_S^n z^n \right) \left( \bar{u}(P_2) u(P_1) - \eta \frac{q \cdot S_{1/2}}{m z} \right)^{2s}
\]

\[
= \frac{\kappa m x^{2\eta}}{2} \oint \frac{dz}{2\pi i z} \left( \sum_{n=0}^{\infty} C_S^n z^n \right) \left( \frac{[21] - \langle 21 \rangle}{2m} + \eta \frac{[2|q|1] + \langle 2|q|1 \rangle}{4m^2} \right)^{2s}.
\]

Here the contour encircles the origin, and the contour integral merely serves the auxiliary function of extracting the right combinatoric factors. For positive helicity \( \eta = +1 \), this expression becomes

\[
M_{s}^{+2} = (-1)^{2s} \frac{\kappa m x^{2s}}{2} \oint \frac{dz}{2\pi i z} \left( \sum_{n=0}^{\infty} C_S^n z^n \right) \left( \frac{\langle 21 \rangle}{m} + \frac{z - 1}{z} \frac{x \langle 23 \rangle \langle 31 \rangle}{2m^2} \right)^{2s}.
\]

Using the binomial expansion leaves the following residue integral to be worked out.

\[
\oint \frac{dz}{2\pi i z} z^n \left( \frac{z - 1}{z} \right)^i = (-1)^i \sum_{j=0}^{i} \oint \frac{dz}{2\pi i z} \left( \frac{i}{j} \right) (-z)^j z^{n-i} = (-1)^n \left( \frac{i}{n} \right).
\]

One then finds:

\[
M_{s}^{+2} = (-1)^{2s} \frac{\kappa m x^{2s}}{2} \sum_{i=0}^{2s} \left( \frac{2s}{i} \right) \left[ \frac{1}{2^i} \sum_{n=0}^{\infty} (-1)^n \left( \frac{i}{n} \right) C_S^n \right] \left( \frac{\langle 21 \rangle}{m} \right)^{2s-i} \left( \frac{x \langle 23 \rangle \langle 31 \rangle}{m^2} \right)^i
\]

\[
= (-1)^{2s} \frac{\kappa m x^{2s}}{2} \sum_{i=0}^{2s} \left( \frac{2s}{i} \right) \hat{g}_i \left( \frac{\langle 21 \rangle}{m} \right)^{2s-i} \left( \frac{x \langle 23 \rangle \langle 31 \rangle}{m^2} \right)^i
\]

where hatted coupling \( \hat{g}_i \) is precisely the coupling defined in eq.(6.1)!
Another advantage of the representation eq.(6.6) is that evaluation of cuts become simple by the following identity.

\[
\sum_{\mathbf{P}^{2s}} \mathcal{M}_1(1, \mathbf{P}, q_1, z_1)^{2s} \mathcal{M}_2(\mathbf{P}, 2, q_2, z_2)^{2s} = \left[ \sum_{\mathbf{P}} \mathcal{M}_1(1, \mathbf{P}, q_1, z_1) \mathcal{M}_2(\mathbf{P}, 2, q_2, z_2) \right]^{2s} \tag{6.10}
\]

In the above identity, \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are arbitrary expressions bilinear in the massive spinor-helicity variables schematically written as bold variables. The sum \(\sum_{\mathbf{P}^{2s}}\) on the LHS of the identity denotes the sum over all intermediate states of a particle with spin-\(s\), while the sum \(\sum_{\mathbf{P}}\) on the RHS of the identity denotes the sum over 2 states of a particle with spin-\(\frac{1}{2}\). This identity can be proved by writing the sum as the sum over overcomplete basis of spin coherent states. The identity also implies that combinatoric structures of tree amplitudes propagate to higher loop orders, suggesting that spin-dependence factorisation may not only be a property of 1 PM and 2 PM computations but also a property of arbitrary higher PM order computations.

Finally, the expression eq.(6.6) allows us to straightforwardly take the infinite spin-limit and connect to one-particle EFT three-point amplitude. This can be done by noting that classical spin \(S^{\mu}_s\) is \(2s\) times the spin of spin-\(\frac{1}{2}\) particle; \(\frac{1}{2s} S^{\mu}_s = S^{\mu}_{1/2}\). Formally writing \(\bar{u}u = 1\) and suppressing the subscript \(s\) of \(S^{\mu}_s\),

\[
\lim_{s \to \infty} M^{2s}_s = \frac{k m x^{2n}}{2} \int \frac{dz}{2\pi i z} \left( \sum_{n=0}^{\infty} C^n S^n z^n \right) \left( 1 - \frac{1}{2s} \eta \cdot \frac{q \cdot S}{m z} \right)^{2s} \tag{6.11}
\]

which is the one-particle EFT amplitude eq.(2.9).

### 7 Conclusions

In this paper, we study the approach of utilizing minimally coupled spinning particles to extract spin-effects of the classical potential for Kerr black holes. Earlier attempts failed beyond leading order in spin operators [15, 16], due to the fact that quantum spin operators do not commute as their classical counterpart. In the language of modern on-shell approaches, the obstruction is manifested in that their Wilson coefficients \(C^n_{S^n}\) differ starting at \(n > 1\). On the other hand, recent works have shown that the classical-spin limit of minimal coupling do in fact reproduces the dynamics of Kerr black hole [13, 14, 19].
Here we study the effects of general Wilson coefficients carefully and derive the 1 PM potential to all orders in spin in the general setup. Utilized to minimal coupling, we find that for finite spins, the potential factorizes into a spin-dependent combinatoric factor and a spin-independent function. We term such factorization as universality, reflecting the fact that the spin-independent function is universal for all spins. This allows us to compute this factor using finite spins, and then simply replace the combinatoric factors with its classical-spin limit. This yields the formula first proposed in [13], which was argued intuitively. As a proof of its validity, we reproduce the 1 PM black hole potential to all orders in spin operators.

We analyze the situation at 2 PM for spin-1 and 2 particles, which requires a new ingredient, the gravitational Compton amplitude. We find that the resulting potential for both cases satisfies universality. Furthermore, universality is maintained when extending beyond spin-2 using the non-local BCFW construction. While such a behavior was found to be also present for the local representation in [13], thus casting doubt on its utility to determine the correct higher-spin Compton amplitude, it does provide supporting evidence that universality persists to all orders in PM for minimally coupled particles.

However, when combined with other evidences seems to favor the BCFW Compton amplitude for the purpose of obtaining the classical potential, even though as an amplitude it contains spurious singularity. Firstly, in [11] the three-point amplitude already induces the complex shift that allows one to convert the full Schwarzschild metric to that of Kerr found by Newman and Janis [35], indicating that only the knowledge of three-point is necessary for the extraction of classical effects to all order. Furthermore, non-black hole effects appears to be associated with $O(\frac{1}{m^n})$ local contact terms for the amplitude. Take for example the Love number, which describes the deformation of stellar objects under external gravitational force. On the worldline this correspond to an operator of the form $E^2$, which translate to a contact contribution for the Compton amplitude for scalars

$$\langle 12 \rangle_4^4 \over M_{pl}^2 m^2$$

(7.1)

where we give its contribution to the negative helicity graviton on legs 1,2. For Schwarzschild black holes the Love number is zero [36, 37]. Thus in this case, one can simply state that the “correct” Compton amplitude for scalars, which corresponds to Schwarzschild black holes, are the ones without $O(\frac{1}{m^n})$ contributions. Indeed if we add an $R^3$ term to the action, the Compton amplitude for a scalar would be modified by $\frac{1}{m^n}$ terms, where $m$ is the scale for which $R^3$ is suppressed, and without missing a beat it was shown in [38] that the inclusion of $R^3$ will generate non-zero Love number. Thus if the criteria for the correct Compton amplitude is that it is completely determined by three point amplitudes, having consistent factorization in all three channels satisfy universality and without $O(\frac{1}{m^n})$ contributions, BCFW is the unique solution. For further argument from double copy, see [34].

Finally, the source of such universality lies in the fact that minimal coupling spin-s three-point amplitude is a $2s$-power of the fundamental spin-$\frac{1}{2}$ particle, which also explains...
its presence in the 1 PM and 2 PM potential when the BCFW representation was used. This leads us to the question whether such phenomenon can be extended to general Wilson coefficients. In other words, whether one can find a family of finite spin couplings such that in the classical-spin limit, it degenerates to the amplitude of the given Wilson coefficients \textit{and} its 1 PM potential exhibit universality. We find the answer to be affirmative.

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A Lagrangian description of minimal coupling

For $s \leq 2$, the minimal coupling eq.(2.13) coincides with the minimal coupling for Dirac fermion, $F^{\mu \nu}F_{\mu \nu}$ for vectors, and Rarita-Schwinger for spin-$\frac{3}{2}$ and KK gravity for spin-2. These reflect the fact that in the high-energy limit where the kinematics becomes massless, the amplitude becomes the minimal coupling for self-interacting fields. Beyond spin-2 no such construction is known in flat-space. In conventional QFT literature minimal coupling is simply the covariantization of the kinetic terms, which is simply the d’alembertian operator plus terms that enforces the state to be transverse traceless. The goal of this appendix is to explicitly demonstrate that our higher spin minimal coupling do not match with such case.

A.1 Kinetic term

The kinetic term for arbitrary integer spin field can be written as follows.

$$S = \int \sqrt{-g} \frac{(-1)^s}{2} \left( D_{\mu} \phi^{\nu_{1} \cdots \nu_{s}} D_{\mu} \phi_{\nu_{1} \cdots \nu_{s}} - m^{2} \phi^{\nu_{1} \cdots \nu_{s}} \phi_{\nu_{1} \cdots \nu_{s}} \right)$$  \hspace{1cm} (A.1)

The sign factor $(-1)^s$ is there to make sure that the kinetic term for physical degrees of freedom have the right sign. Although unphysical polarisations are present in this Lagrangian, effects of such unphysical degrees of freedom can be remedied by using physical polarisation tensors for external legs and Feynman propagators with projection tensors projecting onto physical degrees of freedom for internal legs. That unphysical degrees of freedom will be irrelevant for computing on-shell three-particle amplitude will be the subject of the following section.
The sources for graviton coupling takes the following form for this symmetric tensor field $\phi_{\nu_1 \cdots \nu_s}$\(^{12}\).

$$T_{\mu\nu} = (-1)^s \left[ (\partial_{\mu} \phi^{\sigma_1 \cdots \sigma_s})(\partial_{\nu} \phi_{\sigma_1 \cdots \sigma_s}) + s(\partial^\lambda \phi_{\mu \sigma_2 \cdots \sigma_s})(\partial_{\nu} \phi_{\sigma_2 \cdots \sigma_s}) - sm^2 \phi_{\mu \sigma_2 \cdots \sigma_s} \phi_{\nu \sigma_2 \cdots \sigma_s} \right] - \eta_{\mu\nu} \mathcal{L}$$

$$G^{\mu\nu\lambda} = \frac{(-1)^s s}{2} \left( \phi^{\nu\sigma_2 \cdots \sigma_s} \partial^\mu \phi_{\sigma_2 \cdots \sigma_s} + \phi^{\mu\sigma_2 \cdots \sigma_s} \partial^\nu \phi_{\sigma_2 \cdots \sigma_s} - \phi^{\lambda\sigma_2 \cdots \sigma_s} \partial^\mu \phi_{\sigma_2 \cdots \sigma_s} + (\mu \leftrightarrow \nu) \right)$$  \hspace{1cm} (A.2)

$T_{\mu\nu}$ comes from coupling to $g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \cdots$, while $G^{\mu\nu\lambda}$ comes from coupling to $\Gamma^\lambda_{\mu\nu}$. These two sources will contribute to the three-point amplitude as following terms.

$$-\frac{\kappa}{2} h_{\mu\nu} T_{\mu\nu} \rightarrow \frac{\kappa}{2} \frac{x^2}{m^{2s-2}} \left[ 21 \right]^s \left[ 21 \right]^s \left[ 21 \right]^s \left[ 23 \right]^s [31]$$  \hspace{1cm} (A.3)

In the chiral (undotted) basis, the two terms combine into the following expression.

$$M_3 = \frac{\kappa x^2}{2m^{2s-2}} \sum_{i=0}^{2s} g_{i}^{\text{kin}} \left[ 21 \right]^{2s-i} \left( \frac{x \left[ 23 \right] \left[ 31 \right]}{m} \right)^i$$  \hspace{1cm} (A.4)

$$g_{i}^{\text{kin}} = \binom{s}{i} - s \binom{s-1}{i-1} = -(i-1) \binom{s}{i}$$  \hspace{1cm} (A.5)

In [13] it was noted that $g_1 = g_{1}^{\text{kin}} = 0$ cannot be altered.

### A.2 Killing unphysical degrees of freedom and the condition $g_1 = 0$

Terms in the Lagrangian that kill unphysical degrees of freedom on flat space will in general contain terms where derivatives on the fields are contracted to one of the indices of the fields that derivatives act on.

$$(\partial_{\mu} \phi_{\nu_1 \cdots \nu_{s-1}})(\partial_{\lambda} \phi^{\lambda \nu_1 \cdots \nu_{s-1}}).$$  \hspace{1cm} (A.6)

Doing integration by parts, this term can be cast as follows.

$$(\partial_{\mu} \phi^{\lambda \nu_1 \cdots \nu_{s-1}})(\partial_{\lambda} \phi_{\mu \nu_1 \cdots \nu_{s-1}}).$$  \hspace{1cm} (A.7)

In the above expression, derivatives are contracted to the field that it does not act on. These expressions can be promoted to curved space as follows.

$$(\partial_{\mu} \phi^{\lambda \nu_1 \cdots \nu_{s-1}})(\partial_{\lambda} \phi_{\mu \nu_1 \cdots \nu_{s-1}}) \rightarrow g^{\alpha \beta} g^{\mu \nu} g^{\lambda_1 \sigma_1} \cdots g^{\lambda_{s-1} \sigma_{s-1}} (D_{\alpha} \phi_{\beta \lambda_1 \cdots \lambda_{s-1}})(D_{\mu} \phi_{\nu \sigma_1 \cdots \sigma_{s-1}})$$  \hspace{1cm} (A.8)

$$(\partial_{\mu} \phi_{\nu_1 \cdots \nu_{s-1}})(\partial_{\lambda} \phi^{\mu \nu_1 \cdots \nu_{s-1}}) \rightarrow g^{\alpha \beta} g^{\mu \nu} g^{\lambda_1 \sigma_1} \cdots g^{\lambda_{s-1} \sigma_{s-1}} (D_{\alpha} \phi_{\mu \lambda_1 \cdots \lambda_{s-1}})(D_{\nu} \phi_{\beta \sigma_1 \cdots \sigma_{s-1}})$$  \hspace{1cm} (A.9)

\(^{12}\text{Lower indices are used as defining indices for the symmetric tensor field } \phi_{\nu_1 \cdots \nu_s}.\)
Up to surface terms, eq. (A.8) and eq. (A.9) differ by a term linear in the curvature tensor $R_{\mu\nu} = [D_{\mu}, D_{\nu}]$. Therefore, any expression of the form eq. (A.9) can be converted to the form eq. (A.8) by introducing extra Riemann tensor couplings.

$$\sqrt{-g}(g^{\alpha\beta} g^{\mu\nu} - g^{\mu\nu} g^{\alpha\beta})g^\lambda_1\sigma_1 \ldots g^\lambda_{s-1}\sigma_{s-1} (D_\alpha \phi_\beta \lambda_1 \ldots \lambda_{s-1}) (D_\mu \phi_\nu \sigma_1 \ldots \sigma_{s-1})$$

$$\propto \partial[\ldots] + \sqrt{-g}g^{\alpha\beta} g^{\mu\nu} g^\lambda_1\sigma_1 \ldots g^\lambda_{s-1}\sigma_{s-1} (\phi_\beta \lambda_1 \ldots \lambda_{s-1}) ([D_\alpha, D_\mu] \phi_\nu \sigma_1 \ldots \sigma_{s-1}) \quad \text{(A.10)}$$

Linear coupling to $h$ obtained from the substitution $g^{\mu\nu} \rightarrow \eta^{\mu\nu} - \kappa h^{\mu\nu}$ and $D_\mu \rightarrow \partial_\mu + \Gamma_\mu$ on eq. (A.8) will not contribute to on-shell three-point amplitude, due to transverse nature of on-shell physical DOF; $p_\mu \epsilon(p)^\mu = 0$. Also, terms linear in the curvature tensor cannot affect $g_1$ and only can affect $g_{i \geq 2}$. This shows $g_1 = 0$ is a constraint that cannot be changed for coupling to gravitons.

To remove the ambiguity coming from eq. (A.10), expression of the form eq. (A.8) and its generalisation to multiple derivatives will be considered as the canonical expression for terms introduced to kill unphysical degrees of freedom.

### A.3 Effects of Electric coupling

The first nontrival electric coupling for spin 2 particle will be generated by the following interaction term.

$$R_{\mu\nu} \phi^\mu \phi^\nu \rightarrow \kappa \left( \left[ e^+ \cdot v(2) \right] k_3 \cdot v(1) \right) - \left[ e^+ \cdot v(1) \right] \left[ k_4 \cdot v(2) \right] \right)^2$$

$$= \frac{\kappa}{2} \frac{\langle 21 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2}{m^4} = \frac{\kappa x^2}{2m^2} \left( \frac{x \langle 23 \rangle \langle 31 \rangle}{m} \right)^2 \quad \text{(A.11)}$$

As a generalisation, consider the following coupling of the curvature tensor to higher-spin fields.

$$\left( \partial_{\mu_1} \ldots \partial_{\mu_{\nu(j-2)}} \partial_{\nu_1} \ldots \partial_{\nu_{j-2}} R_{\mu_{j-1} \nu_{j-1} \mu_{j} \nu_j} \right) \phi^{\mu_1 \ldots \mu_{\nu(j-2)} \nu_1 \ldots \nu_{j-2}} \phi^{\nu_1 \ldots \nu_j \sigma_1 \ldots \sigma_{s-j}} \quad \text{(A.12)}$$

Although in principle covariant derivatives $D$ must be used, considering them as partial derivatives $\partial$ suffices for analysing 3pt amplitudes. Recycling the computation eq. (A.11) produces the following contribution to the 3pt amplitude.

$$\frac{2j-4}{(\partial \ldots \partial R) \phi} \phi \rightarrow - \frac{\kappa}{2} \frac{\langle 21 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2}{m^4} \left( \frac{\langle 23 \rangle \langle 31 \rangle \langle 23 \rangle \langle 31 \rangle}{2m^2} \right) \cdot \frac{(-1)^s \langle 21 \rangle^{s-j} \langle 21 \rangle^{s-j}}{m^{2s-2j}}$$

$$= \frac{\kappa}{2j-1} \frac{x^{s+j} \langle 21 \rangle^{s+2-j} \langle x \langle 23 \rangle \langle 31 \rangle \rangle^{2j-2} \langle 21 \rangle^{s-j}}{m^{2s+2-j}} \quad \text{(A.13)}$$

This piece is related to the Wilson coefficient $C_{ES}\langle 2j \rangle$ appearing in one-particle effective action for point particles. The contribution of this coupling to the 3pt amplitude in the chiral
(undotted) basis is given below.

\[
\frac{(\partial \cdots \partial R) \phi}{m^{2n-2}} \rightarrow \frac{\kappa x^2}{2m^{2s-2}} \sum_{i=2n}^{s} g^E_{i} \langle 21 \rangle^{2s-i} \left( \frac{x\langle 23 \rangle \langle 31 \rangle}{m} \right)^i
\]  
(A.14)

\[
g^E_{i} = \frac{(-1)^s(-1)^n}{2^{n-1}} \left( \frac{s-n-1}{i-2n} \right), \quad n \leq s-1
\]  
(A.15)

As \( n \) is restricted to the range \( n \leq s-1 \) for eq.(A.12), the coupling that affects \( C^{E_{2s}} \) needs to be introduced independently. The following coupling will do the job.

\[
(\partial_{\mu_1} \cdots \partial_{\mu_{s-1}} \partial_{\nu_1} \cdots \partial_{\nu_{s-1}} R_{\mu_1 \nu_1 \cdots \mu_s \nu_s}) \phi^{\mu_1 \cdots \mu_s} (\partial^{\nu_1 \cdots \nu_s} \phi) (\partial^{\nu_1 \cdots \nu_s} \phi)
\]  
(A.16)

The contribution to the 3pt amplitude coming from this coupling is given below.

\[
\frac{2s-2}{(\partial \cdots \partial R)(\partial \phi)(\partial \phi)} \rightarrow \frac{\kappa x^2}{m^{2s-2}} \left( \frac{1}{2s} \right) \left( \frac{x\langle 23 \rangle \langle 31 \rangle}{m} \right)^{2s}
\]  
(A.17)

The two couplings generate all electric couplings.

### A.4 Effects of magnetic coupling

The first nontrivial magnetic coupling for spin 2 particle will be generated by the following interaction term.

\[
(\partial_{\mu} R_{\nu \lambda \delta}) \phi^{\nu \sigma} (\partial^{\delta \phi}) \rightarrow \frac{\kappa x^2}{4} \langle 21 \rangle \left( \frac{x\langle 23 \rangle \langle 31 \rangle}{m} \right)^3
\]  
(A.18)

As a generalisation, consider the following coupling to the curvature tensor.

\[
(\partial_{\mu_1} \cdots \partial_{\mu_{j-2}} \partial_{\nu_1} \cdots \partial_{\nu_{s-j}} R_{\mu_1 \nu_1 \cdots \mu_{j-1} \nu_{j-1} \mu_{j+1} \nu_{j+1} \cdots \mu_s \nu_s}) \phi^{\mu_1 \cdots \mu_j \nu_1 \cdots \nu_{j-1}} (\partial^{\nu_{j+1} \cdots \nu_s} \phi) (\partial^{\nu_{j+1} \cdots \nu_s} \phi)
\]  
(A.19)

The computation eq.(A.18) can be recycled to compute the 3pt amplitude contribution for this coupling.

\[
\frac{(\partial \cdots \partial R) \phi}{m^{2s-2}} \rightarrow -\frac{\kappa x^2}{4} \left( \frac{1}{2s} \right)^3 \left( \frac{[23][31][23][31]}{2m^2} \right)^{j-2} (-1)^s \frac{(21)^{s-j}[21]^{s-j}}{m^{2s-2j}}
\]  
(A.20)

This piece is related to the Wilson coefficient \( C_{BS_{2j-1}} \). Summing up, this coupling’s contribution to the 3pt amplitude in the chiral (undotted) basis becomes the following.

\[
\frac{(\partial \cdots \partial R) \phi}{m^{2n}} \rightarrow \frac{\kappa x^2}{2m^{2s-2}} \sum_{i=2n+1}^{2s} g^B_{i} \langle 21 \rangle^{2s-i} \left( \frac{x\langle 23 \rangle \langle 31 \rangle}{m} \right)^i
\]  
(A.21)

\[
g^B_{i} = \frac{(-1)^s(-1)^n}{2^n} \left( \frac{s-n-1}{i-2n-1} \right), \quad n \leq s-1
\]  
(A.22)
All magnetic couplings up to $C_{BS^{2s-1}}$ are covered by this coupling.

A.5 Fixing the coefficients

Having found all $g_i^{\#}$s, the problem of finding the Lagrangian that corresponds to the minimal coupling reduces to finding a general expression for $c_\#$ such that $\sum c_\# g_i^{\#} = \delta_{i,0}$. This is a linear algebra problem which is easily solved by Gauss elimination, which results in the following iterative solution for $c_\#$.

\[ c_{ES^{2n}} = -\frac{g_{2n}^{kin}}{2n} + \sum_{m=1}^{n-1} \left( g_{2n}^{ES^{2m}} c_{ES^{2m}} + g_{2n}^{BS^{2m+1}} c_{BS^{2m+1}} \right) \]
\[ c_{BS^{2n+1}} = -\frac{g_{2n+1}^{kin}}{2n+1} + \sum_{m=1}^{n-1} \left( g_{2n+1}^{ES^{2m}} c_{ES^{2m}} + g_{2n+1}^{BS^{2m+1}} c_{BS^{2m+1}} \right) + g_{2n+1}^{ES^{2n}} c_{ES^{2n}} \]

First few solutions are:

\[ \bar{c}_{ES^2} = \frac{s(s-1)}{2}, \quad \bar{c}_{BS^3} = \frac{s(s-1)(s-2)}{3}, \quad \bar{c}_{ES^4} = -\frac{(s+1)s(s-1)(s-2)}{12}, \]
\[ \bar{c}_{BS^5} = -\frac{(s+1)\cdots(s-3)}{30}, \quad \bar{c}_{ES^6} = -\frac{(s+2)\cdots(s-3)}{180}, \quad \bar{c}_{BS^7} = -\frac{(s+2)\cdots(s-4)}{630}, \]
\[ \bar{c}_{ES^8} = -\frac{(s+3)\cdots(s-4)}{5040}, \quad \bar{c}_{BS^9} = -\frac{(s+3)\cdots(s-5)}{22680}, \cdots \]

where the definition $\bar{c}_\# = (-1)^n c_\#$ has been adopted for simplicity. Non-vanishing results demonstrate that minimal coupling of eq.(2.13) cannot be equivalent to substituting partial derivatives to gauge-covariant derivatives. Empirically these coefficients seem to obey the following pattern.

\[ \bar{c}_{ES^{2n}} = (-2)^{n-1} \frac{(s+n-1)\cdots(s-n)}{(2n)!} = (-2)^{n-1} \binom{s+n-1}{2n} \]
\[ \bar{c}_{BS^{2n+1}} = -(-2)^n \frac{(s+n-1)\cdots(s-n-1)}{(2n+1)!} = -(-2)^n \binom{s+n-1}{2n+1} \]

This pattern assumes the condition $n \leq s - 1$. Inserting these coefficients into the following sum

\[ g_i^{\min} = g_i^{\kin} + \sum_{n=0}^{s-1} \left( g_i^{ES^{2n}} c_{ES^{2n}} + g_i^{BS^{2n+1}} c_{BS^{2n+1}} \right) \]
\[ = -(i-1) \binom{s}{i} + \sum_{n=1}^{s-1} \left[ \binom{s-n-1}{i-2n} \binom{s+n-1}{2n} - \binom{s-n-1}{i-2n} \binom{s+n-1}{2n+1} \right] \]
\[ = \delta_{i,0} \]

which holds for $0 \leq i \leq 2s$ shows that there is no need to introduce the coupling in eq.(A.17) for producing minimal coupling.
B Switching spin supplementary conditions

Switching SSC is equivalent to shifting the “centre” of the body \[22\]. Since covariant SSC is well-defined, let us take it as the starting point for other SSCs. Consider the following shift from covariant SSC to a new SSC:

\[
S_{\text{cov}}^{\mu\nu} \rightarrow S^{\mu\nu} = S_{\text{cov}}^{\mu\nu} - X^\mu P^\nu + P^\mu X^\nu \tag{B.1}
\]

The following choice of \(X^\mu\) shifts from covariant SSC to NW SSC

\[
S^{\mu\nu}(P^\nu + m e^\nu) = 0,
\]

where \(e^\mu\) is a unit time-like vector and \(P^2 = m^2\).

\[
X^\mu = \frac{S_{\text{cov}}^{\mu\nu} e^\nu}{m + P \cdot e} \tag{B.2}
\]

The change in spin length \(\frac{1}{2}S^{\mu\nu}S_{\text{cov}}^{\mu\nu}\) under this switching of SSC is \(\frac{1}{2}S^{\mu\nu}S_{\text{cov}}^{\mu\nu} = \frac{1}{2}S_{\text{cov}}^{\mu\nu}S_{\text{cov}}^{\mu\nu} + m^2 X^2\). It may not be obvious that spatial displacement of the centre \(X^2 < 0\) reduces spin length, but this is because the increase in mass dipole moment \(S^{0i}\) is far greater than the increase in spin \(S^{ij}\).

B.1 Invariance of three-point amplitude

The changes in the three-particle amplitude induced from switching SSC by eq.(B.1) is proportional to the following expression.

\[
\delta(\Omega_{\mu\nu}S^{\mu\nu}) \propto (q^\mu \xi^\nu - \xi^\mu q^\nu)X^\mu P^\nu \tag{B.3}
\]

Due to three-particle kinematics \(q \cdot P = 0\), only the first term needs to be considered. To avoid breaking Lorentz invariance of the three-point amplitude, the unit time-like vector of eq.(B.2) can only be chosen to be a linear combination\(^\text{13}\) of \(P^\mu\) and \(q^\mu\), but due to covariant SSC the vector \(X^\mu\) is proportional to \(S_{\text{cov}}^{\mu\nu}q^\nu\). In sum,

\[
\delta(\Omega_{\mu\nu}S^{\mu\nu}) \propto q_\mu X^\mu \propto S_{\text{cov}}^{\mu\nu}q^\mu q^\nu = 0. \tag{B.4}
\]

Choice of \(X^\mu \propto q^\mu\) will also make the contribution vanish from the on-shell condition \(q^2 = 0\). Therefore, changing the SSC does not change the three-point amplitude deduced from one-particle EFT.

\(^{13}\)To keep this term at most linear in spin, the remaining four-vector \(S_{\text{cov}}^{\mu\nu}q^\nu\) that can be constructed from given variables was not considered.
C Wilson coefficients for minimal coupling

To compute $1/s$ corrections to Wilson coefficients for minimal coupling, the inverse matrix of the matrix $F_{i,n}$ in eq.(2.17) is needed. $F_{i,n}$ can be expanded as an asymptotic series in $1/s$:

$$F_{i,n}^s = \frac{(-1)^n}{2^n} \frac{1}{(i-n)!n!} \sum_{m=0}^{\infty} \binom{n}{m} \frac{(s!)^2}{(s-m)!(s+m-i)!}$$

$$= \frac{(-1)^n}{2^n} \frac{s^i}{(i-n)!n!} \sum_{m=0}^{\infty} \binom{n}{m} \frac{e^{-\frac{c^2-(2m+1)+2m^2}{4s} + O(s^{-2})}}{\frac{1}{2} i^2 - 2(n+1)i + n(n+1)} + O(s^{-2})$$

Note that in the $s \to \infty$ limit, $F_{i,n}^s$ scales as $O(s^i)$. This motivates us to introduce $\tilde{g}_i$ and $\tilde{F}_{i,n}^s$ as finite $s \to \infty$ quantity:

$$\tilde{g}_i = \frac{i!}{s^i} g_i, \quad \tilde{F}_{i,n}^s = \frac{i!}{s^i} F_{i,n}^s$$

This particular scaling allows a simple expression for $\tilde{F}_{i,n}^s$ in the asymptotic limit $s \to \infty$.

$$\tilde{F}_{i,n} \equiv \lim_{s \to \infty} \tilde{F}_{i,n}^s = \binom{i}{n} (-1)^n$$

As a matrix, $\tilde{F}_{i,n}$ is a lower triangular infinite matrix which squares to the identity, i.e. $\sum_{n=0}^{\infty} \tilde{F}_{i,n} \tilde{F}_{n,j} = \delta_{i,j}$. Therefore, in the asymptotic limit $s \to \infty$

$$\tilde{g}_i = \sum_{n=0}^{\infty} \tilde{F}_{i,n} C_{S^n} \quad \rightarrow \quad C_{S^n} = \sum_{i=0}^{\infty} \tilde{F}_{n,i} \tilde{g}_i$$

Inserting $\tilde{g}_0 = 1$ and $\tilde{g}_{i>0} = 0$ into the above equation indeed yields $C_{S^n} = 1$, which is the leading result in $1/s$. The subleading $1/s$ terms of $\tilde{F}_{i,n}^s$ is;

$$\tilde{F}_{i,n}^s = \tilde{F}_{i,n} - \frac{2i^2 - 2(n+1)i + n(n+1)}{4s} \tilde{F}_{i,n} + O(s^{-2})$$

The inverse matrix up to the same asymptotic order can be computed using the formal matrix identity $(\mathbb{1} - h)^{-1} = \sum_{i=0}^{\infty} h^i$.

$$\left(\tilde{F}_{i,n}^s\right)^{-1} = \tilde{F}_{i,n} + \frac{1}{s} \sum_{j,k} \frac{2j^2 - 2(k+1)j + k(k+1)}{4} \tilde{F}_{n,j} \tilde{F}_{j,k} \tilde{F}_{k,i} + O(s^{-2})$$

$$(C.6)$$
Therefore, the Wilson coefficients for minimal coupling up to this order is

\[ C_{S_n}^{\text{Min},s} = \left( \tilde{F}_s \right)_{n,0}^{-1} = 1 + \frac{n}{s} \sum_{j,k} (-1)^{j+k} \frac{2j^2 - 2(k + 1)j + k(k + 1)}{4(n - j)!(j - k)!k!} + \mathcal{O}(s^{-2}) \]

\[ = 1 + \frac{n(n - 1)}{4s} + \mathcal{O}(s^{-2}). \]  

(C.7)

It is also possible to work out \( \mathcal{O}(s^{-2}) \) order corrections. For brevity, we only report the result for \( C_{S_n}^{\text{Min},s} \):

\[ C_{S_n}^{\text{Min},s} = 1 + \frac{n(n - 1)}{4s} + \frac{(n^2 - 5n + 10)n(n - 1)}{32s^2} + \mathcal{O}(s^{-3}). \]  

(C.8)

D BCFW construction of the Compton amplitude

Here we give a brief derivation of the the BCFW construction of the Compton amplitude. Here, all momenta are incoming. To make contact with eq.(5.2), one just flip \( P_2 \rightarrow -P_2 \). We first perform the shift

\[ |\hat{4}\rangle = |4\rangle - z|3\rangle \]

\[ |\hat{3}\rangle = |3\rangle + z|4\rangle \]  

(D.1)

This shift is chosen such that the known Compton amplitude for \( s \leq 2 \) does not develop boundary terms as \( z \rightarrow \infty \). Then we simply have:

\[ M_4^{\text{BCFW}}(1^s, 2^s, \hat{k}_{3}^{-2}, \hat{k}_{4}^{+2}) = \frac{\hat{M}_3(1^s, -\hat{P}_{13}^{s}, \hat{k}_{3}^{-2})\hat{M}_3(2^s, \hat{P}_{13}^{s}, \hat{k}_{3}^{-2})}{\langle k_4|P_1|k_1 \rangle} + \frac{\hat{M}_3(1^s, -\hat{P}_{13}^{s}, \hat{k}_{3}^{-2})\hat{M}_3(2^s, \hat{P}_{13}^{s}, \hat{k}_{4}^{+2})}{\langle k_3|P_1|k_3 \rangle} \]  

(D.2)

We will compute the two channels separately, and combine them in the end.

• \( \hat{P}_{13} \) channel

\[ \hat{M}_3(1^s, -\hat{P}_{13}^{s}, \hat{k}_{3}^{-2})\hat{M}_3(2^s, \hat{P}_{13}^{s}, \hat{k}_{4}^{+2}) = \left( \frac{1}{\hat{x}_3} \right)^2 \frac{(-1)^{2s}(1\hat{P}_{13}^{s})^{2s}}{m^{2s-2}} \varepsilon_{IJ}^{2s} \left( \frac{\hat{x}_4}{\hat{x}_3} \right)^2 \frac{(\hat{P}_{13}^{s})^{2s}}{m^{2s-2}} \]  

\[ = \left( \frac{\hat{x}_4}{\hat{x}_3} \right)^2 \frac{(2\hat{P}_{13}^{s})^{2s}}{m^{4s-4}} \]  

(D.3)

Solving the condition \( \hat{P}_{13}^{s} = m^2 \) yields the following identities:
\[
 z = -\frac{\langle 3|P_1|3 \rangle}{\langle 3|P_1|4 \rangle} \\
 \hat{P}_{13} = P_1 + \frac{\langle 3|34|3|P_1 \rangle}{\langle 3|P_1|4 \rangle} \\
 \langle 2|\hat{P}_{13}|1 \rangle = -m^2 \langle 3|P_1|4 \rangle \langle 2|34|3|P_1 \rangle + \langle 23|14 \rangle \\
 (D.4)
\]

So that the \( P_{13} \) channel contribution is:

\[
\left( \frac{x_4}{x_3} \right)^2 \frac{\langle 2|\hat{P}_{13}|1 \rangle^{2s}}{m^{4s-1}\langle 3|P_1|3 \rangle} = \frac{\langle 3|P_1|4 \rangle^4}{\langle 3|k_3|3 \rangle^2 \langle 3|P_1|3 \rangle} \left( -\frac{\langle 13|24 \rangle + \langle 23|14 \rangle}{\langle 3|P_1|4 \rangle} \right)^{2s} \\
(D.5)
\]

- \( \hat{P}_{14} \) channel

\[
\hat{M}_3(1^s, -\hat{P}_{14}, \hat{k}_{4}^2)\hat{M}_3(2^s, \hat{P}_{14}, \hat{k}_{3}^2) = \left( \frac{x_4'^2}{x_3'^2} \frac{1}{m^{2s-2}} \right) \varepsilon_{iJ} \left( \frac{1}{x_3'} \frac{\langle 1|\hat{P}_{14}|2 \rangle^{2s}}{m^{2s-2}} \right) \\
= \left( \frac{x_4'}{x_3'} \right)^2 \frac{\langle 1|\hat{P}_{14}|2 \rangle^{2s}}{m^{4s-4}} \\
(D.6)
\]

Solving the condition \( \hat{P}_{13}^2 = m^2 \) yields the following identities:

\[
 z' = \frac{\langle 4|P_1|4 \rangle}{\langle 3|P_1|4 \rangle} \\
 \hat{P}_{14} = P_1 + \frac{\langle 3|P_1|4 \rangle}{\langle 3|P_1|4 \rangle} \\
 \langle 1|\hat{P}_{14}|2 \rangle = m^2 \frac{\langle 13|42 \rangle + \langle 23|41 \rangle}{\langle 3|P_1|4 \rangle} \\
 (D.7)
\]

\[
\hat{P}_{14} = P_1 + \frac{\langle 3|P_1|4 \rangle}{\langle 3|P_1|4 \rangle} \\
 x_4' = \frac{\langle 3|P_1|3 \rangle}{\langle 3|P_1|4 \rangle} \\
 \frac{1}{x_3'} = \frac{\langle 3|P_1|4 \rangle}{\langle 3|P_1|4 \rangle} \\
(D.7)
\]

So that the \( P_{14} \) channel contribution is:

\[
\left( \frac{x_4'}{x_3'} \right)^2 \frac{\langle 1|\hat{P}_{14}|2 \rangle^{2s}}{m^{4s-4}\langle 3|P_1|3 \rangle} = \frac{\langle 3|P_1|4 \rangle^4}{\langle 3|k_4|3 \rangle^2 \langle 4|P_1|4 \rangle} \left( -\frac{\langle 13|42 \rangle + \langle 23|41 \rangle}{\langle 3|P_1|4 \rangle} \right)^{2s} \\
(D.8)
\]
Finally, putting everything together, the $\langle k_4, k_3 \rangle$ shift Compton amplitude is:

$$
\left( \frac{\langle 3 \rangle P_1 | 4 \rangle^4}{\langle 3 \rangle k_4 | 3 \rangle^2 \langle 4 \rangle P_1 | 4 \rangle} + \frac{\langle 3 \rangle P_1 | 4 \rangle^4}{\langle 3 \rangle k_4 | 3 \rangle^2 \langle 3 \rangle P_1 | 3 \rangle} \right) \left( \frac{\langle 13 \rangle|22| + \langle 23 \rangle|41\rangle}{\langle 3 \rangle P_1 | 4 \rangle} \right)^{2s} = - \frac{\langle 3 \rangle P_1 | 4 \rangle^4}{\langle 3 \rangle k_4 | 3 \rangle \langle 4 \rangle P_1 | 4 \rangle \langle 3 \rangle P_1 | 3 \rangle} \left( \frac{\langle 13 \rangle|42| + \langle 23 \rangle|41\rangle}{\langle 3 \rangle P_1 | 4 \rangle} \right)^{2s}
$$

which is consistent with eq.(5.2) so that it captures the correct residues on all three channels.

As a consistency check, let us see if the amplitude eq.(5.2) satisfies the factorisation property on the $t$-channel. The first way the $t$-channel could be approached is to send $|3 \rangle \rightarrow y|4\rangle$, $y$ carrying the necessary little group weights. Factorisation property then implies the following.

$$
\text{Res} \ M_4 = \left( \frac{\langle \langle -\hat{P} \rangle \rangle^6}{\langle -\hat{P} \rangle 4 \rangle^2 \langle 34 \rangle^2} \right) \left( \frac{\langle \hat{P} \rangle P_1 | \zeta \rangle^2 \langle 21 \rangle^{2s}}{m^2 \langle \hat{P} \rangle \zeta \rangle^2 m^{2s-2}} \right)
$$

The on-shell momentum $\hat{P} = k_3 + k_4$ can be written as $\hat{P} = (|4\rangle + y|3\rangle)|4\rangle$, since $|3\rangle = y|4\rangle$ in this limit. Gauge-fixing by $|\zeta \rangle = |3\rangle$ results in the following expression.

$$
\text{Res} \ M_4 = \frac{\langle 3 \rangle P_1 | 3 \rangle^2}{y^4} \left( \frac{\langle 21 \rangle}{m} \right)^{2s}
$$

In the limit $|3\rangle \rightarrow y|4\rangle$ it can be shown that

$$
\frac{\langle 14 \rangle|32| + \langle 13 \rangle|42|}{\langle 3 \rangle P_1 | 4 \rangle} \rightarrow \frac{\langle 1 \rangle k_3 P_2 - P_1 k_3 | 2 \rangle}{m \langle 3 \rangle P_1 | 3 \rangle} = \frac{\langle 1 \rangle k_3 P_2 - P_2 k_3 | 2 \rangle}{m \langle 3 \rangle P_1 | 3 \rangle} = - \frac{\langle 12 \rangle}{m}
$$

so that eq.(5.2) reduces to

$$
t M_4 \rightarrow \frac{\langle 3 \rangle P_1 | 3 \rangle^2}{y^4} \left( \frac{\langle 21 \rangle}{m} \right)^{2s}
$$

consistent with the expected residue eq.(D.11). That the same factorisation behaviour holds for the other limit $|3\rangle \rightarrow y^{-1}|4\rangle$ can be shown similarly. In this limit,

$$
\frac{\langle 14 \rangle|32| + \langle 13 \rangle|42|}{\langle 3 \rangle P_1 | 4 \rangle} \rightarrow \frac{\langle 1 \rangle k_4 P_2 - P_1 k_4 | 2 \rangle}{m \langle 4 \rangle P_1 | 4 \rangle} = \frac{\langle 1 \rangle k_4 P_2 - P_2 k_4 | 2 \rangle}{m \langle 4 \rangle P_1 | 4 \rangle} = - \frac{\langle 12 \rangle}{m}
$$

$$
t M_4 \rightarrow \frac{\langle 4 \rangle P_1 | 4 \rangle^2}{y^4} \left( \frac{\langle 21 \rangle}{m} \right)^{2s}
$$

and factorisation property requires the following relations with $\hat{P} = (|4\rangle + y^{-1}|3\rangle)|4\rangle$.

$$
\text{Res} \ M_4 = \left( \frac{\langle \langle -\hat{P} \rangle \rangle^6}{\langle -\hat{P} \rangle 3 \rangle^2 \langle 43 \rangle^2} \right) \left( \frac{\langle \hat{P} \rangle P_1 | \zeta \rangle^2 \langle 21 \rangle^{2s}}{m^2 \langle \hat{P} \rangle \zeta \rangle^2 m^{2s-2}} \right)
$$

The gauge choice $|\zeta \rangle = |4\rangle$ gives the desired relation.

$$
\text{Res} \ M_4 = \frac{\langle 4 \rangle P_1 | 4 \rangle^2}{y^4} \left( \frac{\langle 21 \rangle}{m} \right)^{2s}
$$
References

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