KAM Theory for secondary tori

L. Biasco & L. Chierchia
Dipartimento di Matematica e Fisica
Università degli Studi Roma Tre
Largo San L. Murialdo 1 - 00146 Roma, Italy
biasco@mat.uniroma3.it, luigi@mat.uniroma3.it

July 21, 2018

Contents

1 Functional setting and Main Theorem 3

2 Geometry of resonances 5

3 Normal Forms 9
  3.1 Normal form in $f_0$ (non–resonant regime) 12
  3.2 Normal form in $f_1$ (simple resonances) 12
    3.2.1 The effective potential 14
    3.2.2 Rescalings 15

4 The nearly–integrable structure at simple resonances 16
  4.1 A class of Morse non-degenerate functions 16
  4.2 The Structure Theorem 17

5 Proof of Part I of the Structure Theorem 19
  5.1 Critical points and critical energies of the “unperturbed potential” $F^0$ 19
  5.2 The slow angle 20
  5.3 The auxiliary Hamiltonian 20
  5.4 A special group of symplectic transformations 21
  5.5 An intermediate transformation 21
  5.6 The integrating transformation 23
  5.7 Properties of the actions as functions of the energy 25
  5.8 The final canonical transformation 26
  5.9 Conclusion of the proof of part one of the Structure Theorem 28

6 Proof of Part II of the Structure Theorem 29

7 Proof of the Main Theorem 38
  7.1 Application of the Structure Theorem 39
  7.2 Application of the KAM theorem 41
A Properties of the class of non–degenerate potentials 43
B Proof of the Normal Form Lemma 3.1 45
C On action–angle variables for 1D mechanical systems with parameters 49
   C.1 The “unperturbed case” .......................... 50
   C.2 The action as a function of the angle at constant energy .............. 50
   C.3 The domains of definition of action angle variables .................. 52
   C.4 Definition of action variables ................................ 53
   C.5 Properties of the actions as functions of the energy and viceversa .... 54
D Miscellanea 55

Abstract

(i) In [3] (Rend. Lincei Mat. Appl. 26 (2015), 1–10; see also arXiv:1503.08145 [math.DS]) the following result has been announced:

**Theorem** Consider a real-analytic nearly–integrable mechanical system with potential $f$, namely, a Hamiltonian system with real-analytic Hamiltonian

$$H(y, x) = \frac{1}{2} \sum_{i=1}^{n} y_i^2 + \varepsilon f(x),$$

$(y, x) \in \mathbb{R}^n \times \mathbb{T}^n$ being standard action–angle variables. For “general non–degenerate” potentials $f$’s there exists $\varepsilon_0, a > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, then the Liouville measure of the complementary of $H$–invariant tori is smaller than $\varepsilon |\log \varepsilon|^a$.

In this paper we provide a proof of such result.

(ii) The class of “general non–degenerate” potentials $\mathcal{P}_s$ (defined in §1) is, for any given $s > 0$, an open and dense subset of real–analytic functions on $\mathbb{T}^n$ having holomorphic extensions on $\{x \in \mathbb{C}^n | |Im x_i| < s\}$, the topology being that induced by the weighted Fourier norm $|f|_s := \sup_{k \in \mathbb{Z}^n} |f_k|e^{|k|s}$. The class $\mathcal{P}_s$ is also of “full measure” in natural ways (compare Proposition 1.1 proven in Appendix A).

(iii) The above Theorem is based on an extension of KAM Theory to a suitable $\varepsilon$–dependent neighborhood of simple resonances $\{y \in B : y \cdot k = 0\}$, with $|k| \leq K \sim |\log \varepsilon|^b$, for a suitable $b > 0$ and any given ball $B \subset \mathbb{R}^n$. The main issue is giving a quantitative analytic description of the integrability structure of the averaged Hamiltonian at simple resonances suitable for application of KAM methods. Such analytic properties are summarized in the “Structure Theorem” of §4, whose proof occupies the main part of this paper, namely, Sect’s 5, 6 and Appendix C, where action–angle variables for generic parameter–depending systems are discussed.

(iv) In view of the Structure Theorem one can then apply simultaneously (for $|k| \leq K$) explicit classical KAM measure estimates (as given, e.g., in [5]) and conclude the proof of the main Theorem.
1 Functional setting and Main Theorem

In this paper we consider real–analytic functions, which are “non–degenerate” in a suitable sense\(^1\).
Let \( s > 0 \) and consider the real–analytic functions on \( \mathbb{T}^n \) having zero average and finite “sup–Fourier norm”
\[
|f|_s := \sup_{k \in \mathbb{Z}^n} |f_k|e^{|k|s} < \infty ,
\]
where \( f_k \) denotes Fourier coefficients and, as usual, \(|k|\), for integer vectors, denotes the 1-norm \( \sum |k_j| \).
Denote by \( \mathcal{A}_s^n \) the Banach space of such functions.
Let \( \mathbb{Z}_n^j \) denote the set of integer vectors \( k \neq 0 \) in \( \mathbb{Z}^n \) such that the first non–null component is positive:
\[
\mathbb{Z}_n^j := \{ k \in \mathbb{Z}^n : k \neq 0 \text{ and } k_j > 0 \text{ where } j = \min \{ i : k_i \neq 0 \} \},
\]
and denote by \( \mathbb{Z}_n^0 \) the generators of one–dimensional maximal lattices, namely, the set of vectors \( k \in \mathbb{Z}_n^0 \) such that the greater common divisor of their components is 1, namely
\[
\mathbb{Z}_n^0 := \{ k \in \mathbb{Z}_n^* : \gcd(k_1, \ldots, k_n) = 1 \} ;
\]
then, the list of one–dimensional maximal lattices is given by the sets \( \mathbb{Z}_n^0 k \) with \( k \in \mathbb{Z}_n^0 \).

We can, now, decompose the Fourier expansion of any function \( f \in \mathcal{A}_s^n \) as sum of real–analytic functions of one–variable, which are the projection of \( f \) onto the one–dimensional maximal lattice \( \mathbb{Z}_n^0 k \) (with \( k \in \mathbb{Z}_n^0 \)), as follows:
\[
f(x) = \sum_{k \in \mathbb{Z}_n^0} F^k(k \cdot x) , \quad \text{where } F^k(t) := \sum_{j \in \mathbb{Z}^n \setminus \{0\}} f_{jk} e^{ijt}
\]
\( f_{jk} \) being the Fourier coefficient of \( f \) with Fourier index \( jk \in \mathbb{Z}^n \). Notice that, since \( f \in \mathcal{A}_s^n \), the functions \( F^k \) belong to \( \mathcal{A}_{|k|s}^1 \).

**Definition 1.1 (The class \( \mathcal{P}_s \) of non–degenerate potentials)** Let \( s > 0, 0 < \delta \leq 1 \) and let
\[
K_s(\delta) := c \max \left\{ 1, \frac{1}{s}, \frac{1}{s} \log \frac{1}{s \delta} \right\},
\]
where \( c > 1 \) is a suitably large constant to be chosen below (see \((253)\)), depending only on \( n \).
Let \( \mathcal{P}_s(\delta) \) be the set of functions in \( \mathcal{A}_s^n \) such that, for \( k \in \mathbb{Z}_n^0 \), the following holds
\[(P1) \ |f_k| \geq \delta |k| e^{-|k|s} \text{ if } |k| > K_s(\delta) ; \]
\[(P2) \ \min_{\xi \in \mathbb{R}} (|\partial_\xi F^k(t)| + |\partial_\xi^2 F^k(t)|) > 0 \text{ if } |k| \leq K_s(\delta) ; \]
\[(P3) \ F^k(t_1) \neq F^k(t_2) \text{ for every } 0 \leq t_1 < t_2 < 2\pi \text{ such that } \partial_\xi F^k(t_1) = \partial_\xi F^k(t_2) = 0 \text{ and } |k| \leq K_s(\delta). \]
Finally, \( \mathcal{P}_s := \bigcup_{\delta > 0} \mathcal{P}_s(\delta) \).

\(^1\)It would be easier to consider larger function spaces of smooth functions. However, the natural (both from the theoretical and applicative point of view) and most challenging setting is, we believe, that of real–analytic potentials.
An example of function $f \in \mathcal{P}_s(\delta)$, as it is immediate to verify, is given by
\[
f(x) = 2\delta \sum_{k \in \mathbb{Z}^n_k} e^{-|k|s} \cos(k \cdot x), \quad \text{i.e.,} \quad f_k = \begin{cases} \delta e^{-|k|s} & \text{if } k \in \mathbb{Z}^n_s \\ 0 & \text{otherwise} \end{cases}.
\] (6)

The class $\mathcal{P}_s$ is “general” in several ways: from a probabilistic, topological and measure theoretical points of view.

To describe the probabilistic point of view, let us denote by $\ell^n_\infty$ the Banach space of complex sequences $z = \{z_k\}_{k \in \mathbb{Z}^n_k}$ with finite sup-norm $|z|_\infty := \sup_{k \in \mathbb{Z}^n_k} |z_k|$. The map
\[
 j : f \in \mathcal{A}_s^n \rightarrow \{f_k e^{[k|s]} \}_{k \in \mathbb{Z}^n_k} \in \ell^n_\infty
\] (7)
is an isomorphism of Banach spaces\(^2\), which allows to identify functions in $\mathcal{A}_s^n$ with points in $\ell^n_\infty$ and the Borellians of $\mathcal{A}_s^n$ with those of $\ell^n_\infty$. Now, consider the probability measure given by the standard normalized Lebesgue–product measure on the unit closed ball of $\ell^n_\infty$, namely, the unique probability measure $\mu$ on the Borellians of $\{z \in \ell^n_\infty : |z|_\infty \leq 1\}$ such that, given Lebesgue measurable sets $A_k$ in the unit complex disk $A_k \subseteq D := \{w \in \mathbb{C} : |w| \leq 1\}$ with $A_k \neq D$ only for finitely many $k$, one has
\[
\mu\left(\prod_{k \in \mathbb{Z}^n_k} A_k\right) = \prod_{\{k \in \mathbb{Z}^n_k : A_k \neq D\}} \text{meas}(A_k)
\]
where “meas” denotes the normalized Lebesgue measure on the unit complex disk $D$. Denote by $\mathbb{B}$ the closed ball of radius one in $\mathcal{B}_s^n$ and by $\mathcal{B}$ the Borellians in $\mathbb{B}$. Then, the isometry $j$ in (7) naturally induces a measure $\mu_s$ on the Borellians $\mathcal{B}$.

The properties of $\mathcal{P}_s$ are collected in the following proposition, whose simple proof is given in Appendix A.

**Proposition 1.1** Let $s > 0$. The set $\mathcal{P}_s \subseteq \mathcal{A}_s^n$ contains an open dense set, is prevalent, $\mathcal{P}_s \cap \mathbb{B} \in \mathcal{B}$ and $\mu_s(\mathcal{P}_s \cap \mathbb{B}) = 1$.

Let $\| \cdot \|$ be the standard Euclidean norm on $\mathbb{R}^n$. Then, one has the following

**Theorem 1.1** Let $s > 0$ and let $\Omega$ be a bounded region in $\mathbb{R}^n$ with $n \geq 2$. Let $f \in \mathcal{P}_s$ and consider the Hamiltonian
\[
H := \frac{1}{2} \|y\|^2 + \varepsilon f(x),
\] (8)

There exist $\varepsilon_0 > 0$ and $\kappa > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, the measure of the set of $H$–trajectories in $\Omega \times \mathbb{T}^n$, which do not lie on an invariant Lagrangian (Diophantine) torus, is bounded by $\varepsilon |\log \varepsilon|^\kappa$.

We remark that the constants $\varepsilon_0$ and $\kappa$ depend only on $n$, $s$, and $F^k$ with $k \in \mathbb{Z}^n_s$, $|k| \leq K_s(\delta)$.

**Remark 1.1** In proving Theorem 1.1 we will assume that
\[
\Omega = B_1(0) := \{\|y\| < 1\} \quad \text{and} \quad |f|_s = 1.
\] (9)

This is not restrictive since we can always consider a large enough ball $B_R(0) \supseteq \Omega$ and rescale the action and the time in order to obtain (9) (suitably renaming $\varepsilon$ and $f$).

\(^2\)Recall that since the functions in $\mathcal{A}_s^n$ are real-analytic one has the reality condition $f_k = \bar{f}_{-k}$. 


2 Geometry of resonances

In this section we construct a covering of $\mathbb{R}^n$ (thought of as frequency space) by three regions: a non–resonant region $\Omega_0$, a neighborhood of simple resonances $\Omega_1$ and a region $\Omega_2$ of “small” measure containing all other resonances. The sets $\Omega_1$ and $\Omega_2$ will be described in terms of linear maps $L_k$, $k \in \mathbb{Z}^n$, that depend on a given resonance $\{y \cdot k = 0\}$: such maps $L_k$ will later be associated to generating functions $S(J, x) = x \cdot L_k J$, whose corresponding symplectic maps have the role of “straighten out” the geometry.

Fix $k \in \mathbb{Z}^n \backslash \{0\}$ with $\gcd(k_1, \ldots, k_n) = 1$. Then, there exists a matrix $A_k \in \text{Mat}_{n \times n}(\mathbb{Z})$ such that

$$A_k = \left(\hat{A}_k \right)_k,$$
$$\hat{A}_k \in \text{Mat}_{(n-1) \times n}(\mathbb{Z}), \quad \det A_k = 1, \quad \|\hat{A}_k\|_\infty \leq |k|_\infty,$$  \hspace{1cm} (10)

and

$$\|A_k^{-1}\|_\infty \leq c |k|^{n-1};$$  \hspace{1cm} (11)

the existence of such a matrix is guaranteed by an elementary result of linear algebra based on Bezout’s Lemma (see Lemma D.8 in appendix D).

We then define a linear map $L_k : \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$L_k : J = (\hat{J}, J_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mapsto L_k J := J_n k + p_k^\perp \hat{A}_k^T \hat{J},$$  \hspace{1cm} (12)

where $p_k^\perp$ is the orthogonal projection on the subspace perpendicular to $k$. Observe that $L_k$ can be also written as composition of two linear maps:

$$L_k = A_k^T U_k$$  \hspace{1cm} (14)

where $U_k$ acts as the identity on the first $(n-1)$ components and:

$$U_k : (\hat{J}, J_n) \mapsto (\hat{J}, J_n - \frac{1}{\kappa}(\hat{A}_k k) \cdot J), \quad \text{i.e.} \quad U_k = \begin{pmatrix} I_{n-1} & 0 \\ -\kappa^{-1} \hat{A}_k k & 1 \end{pmatrix},$$ \hspace{1cm} (15)

where $I_m$ denotes the $(m \times m)$–identity matrix and where

$$\kappa := \|k\|^2 = k_1^2 + \cdots + k_n^2 \in \mathbb{N}.$$ \hspace{1cm} (16)

Note that, by (10), we get

$$\|U_k\|, \|U_k^{-1}\| \leq c.$$ \hspace{1cm} (17)

---

3Here, $k$ is a row vector. Normally we do not distinguish between row and column vectors since it will be clear from context. The notation $|M|_\infty$, with $M$ matrix or vector, denotes the maximum norm $\max_{ij} |M_{ij}|$ or, respectively, $\max_i |M_i|$.

4Recalling that for any $n \times n$ matrix $M$, one always has $|\det M| \leq n^{n/2}|M|_\infty^n$, (11) follows by the D’Alembert expansion of determinants (with $c = (n-1)(n-1)/2$).

5Without further notice, we shall always identify linear maps with the associated matrices.

6Explicitly, for $y \in \mathbb{R}^n$,

$$p_k^\perp y := y - \frac{1}{\kappa}(y \cdot k)k.$$ \hspace{1cm} (13)

7Here and in the following, we shall denote by “$c$” suitable constants (which, in general, differ from formula to formula) depending only on $n$. 

5
Elementary properties of $L_k$ are the following\footnote{Eq. (18) follows from (10), (14) and (15). Identities (19), (20), (21) and the first bound in (22) follow directly from definition (12); the bound on $\|L_k^{-1}\|$ follows by observing that $\|L_k^{-1}\| = \|U_k^{-1}A_k^{-T}\| \leq \|U_k^{-1}\|\|A_k^{-T}\|$ and that $\|U_k^{-1}\| \leq c$ and that $\|A_k^{-T}\| \leq c\|k\|^{-n-1}$ (as it follows by bounding the norm of a matrix by a constant times the maximum of its entries and using the co-factor representation for the inverse of $A_k$, and taking into account (10)).}:

$$\det L_k = 1,$$

$$L_k \cdot k = \kappa J_n,$$  \hspace{1cm} (18)

$$\|L_k J\|^2 = \kappa J_n^2 + \|p_k^+ \hat{A}_k \hat{J}\|^2,$$  \hspace{1cm} (19)

$$\hat{\Omega} \subseteq \mathbb{R}^{n-1}, \ a > 0 \implies L_k(\hat{\Omega} \times (-\frac{a}{\kappa}, \frac{a}{\kappa})) = \{|y \cdot k| < a\} \cap \{y : p_k^+ y = p_k^\perp \hat{A}_k^T \hat{\Omega}\},$$  \hspace{1cm} (20)

$$\|L_k\| \leq c \|k\|, \quad \text{and} \quad \|L_k^{-1}\| \leq c \|k\|^{-n-1}.$$  \hspace{1cm} (21)

Further more interesting properties of $L_k$ are given in the following simple

**Lemma 2.1** (i) The map $p_k^+ \hat{A}_k : \mathbb{R}^{n-1} \to k^\perp$ is a linear isomorphism.

(ii) For any $a > 0$, $L_k(\mathbb{R}^{n-1} \times (-\frac{a}{\kappa}, \frac{a}{\kappa})) = \{|y \cdot k| < a\}.$

**Proof** (i): Let $\hat{A}_k = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{pmatrix}$, with $a^i \in \mathbb{Z}^n$. Then $p_k^+ \hat{A}_k \mathbb{R}^{n-1} = \text{span}\{ p_k^+ a^1, ..., p_k^+ a^{n-1} \}$. But the vectors $p_k^+ a^i$ are linearly independent (if $0 = \sum \lambda_i p_k^+ a^i = p_k^+ (\sum \lambda_i a^i)$, then there exists $c$ such that $\sum \lambda_i a^i = ck$, which implies that $\lambda_1 = \cdots = \lambda_{n-1} = c = 0$ since $A_k = (\hat{A}_k)$ has determinant one and hence the vectors $a^1, ..., a^{n-1}, k$ are linearly independent). The claim then follows from the rank–nullity theorem of linear algebra.

(ii): Let $W := L_k(\mathbb{R}^{n-1} \times (-a/\kappa, a/\kappa))$. From (19) it follows that $W \subseteq \{y | y \cdot k| < a\}$. Now, let $y \in \mathbb{R}^n$ be such that $|y \cdot k| < a$ and define $J_n := (y \cdot k)/\kappa$. Then, $|J_n| < a/\kappa$ and, furthermore, $y - J_n k \in k^\perp$. Thus, by part (i) of this Lemma, there exists $\hat{J} \in \mathbb{R}^{n-1}$ such that $y - J_n k = p_k^+ \hat{A}_k \hat{J}$, hence, $y = L_k(\hat{J}, J_n)$, proving that $\{y | y \cdot k| < a\} \subseteq W$ and thus, $\{y | y \cdot k| < a\} = W$. \hfill $\blacksquare$

Set\footnote{Recall (3).}$Z^n_{\kappa, K} := \{k \in \mathbb{Z}^n, \ |k| \leq K\}.$  \hspace{1cm} (23)

To quantify neighborhoods of resonances, as standard, we introduce two $\varepsilon$–dependent Fourier cut-offs $K, K$ and an $\varepsilon$–dependent “width” $\alpha$ of simple resonance, by setting:

$$K := \log^2 \frac{1}{\varepsilon}, \quad K := K^2, \quad \alpha := \sqrt{\varepsilon} K^{n+1},$$  \hspace{1cm} (24)

where

$$\nu > n + 1$$  \hspace{1cm} (25)
is a suitable constant to be fixed later. Let us assume that $\varepsilon$ is so small that (recall (22))

$$\|L_k^{-1}\| \leq K^n.$$ (26)

Set

$$\hat{D} := B_{K^n}(0)$$ (27)

For $k \in \mathbb{Z}_{n,K}^*$, define

$$\hat{Z}_k := \left\{ \hat{J} \in \hat{D} : \min_{l \in \mathbb{Z}_{n,K}^*, \ l \neq \hat{Z}_k} \left| \left( p_k \hat{A}_k \hat{J} \right) \cdot l \right| \geq 3\alpha K \frac{\|l\|}{\|k\|} \right\}, \quad Z_k^2 := \hat{Z}_k \times (-\frac{\alpha}{2K}, \frac{\alpha}{2K}) \subseteq \mathbb{R}^n, \quad (28)$$

$$Z_k := \hat{Z}_k \times (-\frac{\alpha}{K}) \subseteq \mathbb{R}^n, \quad \text{and} \quad Z'_k := (\hat{D} \setminus \hat{Z}_k) \times (-\frac{\alpha}{K}) \subseteq \mathbb{R}^n, \quad (29)$$

$$\Omega^0 := \{ \|y\| < 1 : \min_{k \in \mathbb{Z}_{n,K}^*} |y \cdot k| \geq \alpha/2 \}, \quad \Omega^1 := \bigcup_{k \in \mathbb{Z}_{n,K}^*} L_k Z_k^2, \quad \Omega^2 := \bigcup_{k \in \mathbb{Z}_{n,K}^*} L_k Z'_k.$$ (30)

**Remark 2.1** The set $\Omega^0$ is a non–resonant set. The set $\Omega^1$, by (21) and Lemma 2.1, is seen to be a suitable neighborhood of simple resonances $\{y \cdot k = 0\}$ with $k \in \mathbb{Z}_{n,K}^*$; finally, $\Omega^2$ is a neighborhood of order–two or higher resonances. Next Lemma clarifies and quantifies these observations.

**Proposition 2.1**

(i) $\Omega^0 \cup \Omega^1 \cup \Omega^2 \supseteq B_1(0)$.

(ii) The set $\Omega^0$ is $(\alpha/2, K)$ completely non–resonant, i.e.,

$$y \in \Omega^0 \quad \implies \quad |y \cdot k| \geq \alpha/2, \quad \forall 0 < |k| \leq K. \quad (31)$$

(iii) For each $k \in \mathbb{Z}_{n,K}^*$, the set $L_k Z_k$ is $(2\alpha K/\|k\|, K)$ non–resonant modulo $\mathbb{Z}k$, i.e.,

$$y \in L_k Z_k \quad \implies \quad |y \cdot l| \geq 2\alpha K/\|k\|, \quad \forall \ l \in \mathbb{Z}^n, \ l \notin \mathbb{Z}k, \ |l| \leq K. \quad (32)$$

(iv) There exists a constant $c > 0$ depending only on $n$ such that:

$$\text{meas}(\Omega^2) \leq c \alpha^2 K^{n^2-n-1} K^{n+2}. \quad (33)$$

**Remark 2.2** Since $\alpha$ has been chosen as $\sqrt{c} K^c$ (see (24)), from (33) it follows that the region of second or higher order resonances $\Omega^2$ has Lebesgue measure smaller than $(c \varepsilon |\log \varepsilon|^c)$ and therefore no further analysis on $\Omega^2$ is needed.

**Remark 2.3** In the definition of $\hat{Z}_k$ in (28) one could also use a smaller set $\hat{D}_k \subset \hat{D}$, such that Proposition 2.1 still holds true. We have chosen a unique $\hat{D}$ for every $k$, just for simplicity.

---

\[\text{Note:}\]

\[\text{We follow the terminology introduced in [14]: given } \alpha, K > 0 \text{ and a sublattice } \Lambda \subseteq \mathbb{Z}^n, \text{ one says that } D \subseteq \mathbb{R}^n \text{ (or } D \subseteq \mathbb{C}^n) \text{ is } \langle(\alpha, K) \text{–resonant modulo } \Lambda \rangle \text{ if } |y \cdot \hat{k}| \geq \alpha \text{ for all } y \in D \text{ and } \hat{k} \in \mathbb{Z}^n \setminus \Lambda \text{ with } |\hat{k}| \leq K; \text{ if } \Lambda = \{0\} \text{ is the trivial lattice, then } D \text{ is said to be } \langle(\alpha, K) \text{–completely non–resonant}.}\]
Lemma 2.1

Proof of Proposition 2.1.

(i): If \( y \notin \Omega^0 \) with \(|y| < 1 \), there exists a \( k \in \mathbb{Z}_{n,K}^+ \) such that \(|y \cdot k| < \alpha/2\); but then, in view of Lemma 2.1, \( y \) belongs to

\[
\{ y' \in \mathbb{R}^n : |y' \cdot k| < \alpha/2 \} \cap B_1(0) = L_k(\mathbb{R}^{n-1} \times (-\alpha/2, \alpha/2)) \cap B_1(0)
\]

\[
\subseteq L_k\left( \mathbb{R}^{n-1} \times (-\alpha/2, \alpha/2) \right) \cap L_k^{-1}B_1(0)
\]

\[
\subseteq L_k\left( \mathbb{R}^{n-1} \times (-\alpha/2, \alpha/2) \right) \cap B_{K^+}(0)
\]

\[
\subseteq L_k(\mathbb{Z}^1_k \cup \mathbb{Z}^1_k) \subset \Omega_1 \cup \Omega^2.
\]

(ii): Let \( y \in \Omega^0 \) and let \( 0 < |k| \leq K \). Then, there exist \( j \in \mathbb{Z} \setminus \{0\} \) and \( k' \in \mathbb{Z}_{n,K}^+ \) with \( k = jk' \), so that

\[
|y \cdot k| = |j||y \cdot k'| \geq |y \cdot k'| \geq \alpha/2,
\]

proving (31).

(iii): Let \( y = L_k J = L_k(J, J_n) \) for some \( k \in \mathbb{Z}_{n,K}^+, \bar{J} \in \mathbb{Z}_K \) and \(|J_n| < \alpha/\kappa\). Let, also, \( l \in \mathbb{Z}^n, l \notin \mathbb{Z}_K \) with \(|l| \leq K\). As above, there exists \( j \in \mathbb{Z} \setminus \{0\} \) and \( l' \in \mathbb{Z}_{n,K}^+ \) such that \( l = j l' \). Then,

\[
|y \cdot l| = |L_k J \cdot l| \geq |J_n k \cdot l + (p_k^T \tilde{A}_k^T \bar{J}) \cdot l' - (p_k^T \tilde{A}_k^T \bar{J}) \cdot l'| - |k | \cdot |J_n|
\]

\[
\geq |j||l'\left( p_k^T \tilde{A}_k^T \bar{J} \right) - |k | \cdot |J_n|
\]

\[
\geq 3\alpha K \|l\| - \alpha K \|l\| \geq 2\alpha K \|l\| \geq 2\alpha K \|l\|,
\]

proving (32).

(iv): Then (denoting Lebesgue measure by “meas”), from the definition of \( \Omega^2 \) in (30) it follows:

\[
\text{meas} (\Omega^2) = \text{meas} \left( \bigcup_{k \in \mathbb{Z}_{n,K}^+} L_k Z_k' \right) \leq \sum_{k \in \mathbb{Z}_{n,K}^+} \text{meas} (L_k Z_k')
\]

\[
= \sum_{k \in \mathbb{Z}_{n,K}^+} | \det L_k | \text{meas} (Z_k') \tag{34}
\]

Remark 2.4 The geometry of resonances here is different from the geometry of resonances (in the convex case) as discussed, e.g., in [14]. In fact, in [14] more resonances are disregarded in the non-resonant set, namely, the resonances with \(|k| \leq (1/\varepsilon)^a, a > 0\). Furthermore, the neighborhood of simple resonances in [14] has width \( \varepsilon^b, 0 < b < 1/2 \), and, as a consequence, the set of double resonances has measure greater than \( \varepsilon^{2b} \), which is a set not negligible for our purposes. On the other hand, in Nekhoroshev’s theorem one can average out the perturbation up to an exponentially high order \( e^{-\text{const}(1/\varepsilon)^n} \), while we will get only \( e^{\log |\varepsilon|} \).

\[\text{(11)}\] Indeed, \( j = \pm \text{gcd} \{ k_1, \ldots, k_n \} \).
Moreover\(^\text{12}\)

\[
\text{meas} (Z'_k) \leq c \alpha \kappa^{-1} \sum_{l \in Z_k^\times, l \notin Z_k} \text{meas} \left\{ \| \hat{J} \| \leq K^n : \| (p_k^l \hat{A}_k^T \hat{J}) \cdot l \| < 3\alpha \kappa \| l \| \| k \| \right\}. \tag{35}
\]

Now, denoting by

\[
v_{k,l} := \| k \|^2 l - (l \cdot k) k = \| k \|^2 p_k^l l,
\]

we see that

\[
\| (p_k^l \hat{A}_k^T \hat{J}) \cdot l \| < 3\alpha \kappa \| l \| \| k \| \quad \Longleftrightarrow \quad |\hat{J} \cdot \hat{A}_k v_{k,l}| < 3\alpha \| l \| \| k \| K,
\]

so that (35) reads

\[
\text{meas} (Z'_k) \leq c \alpha \kappa^{-1} \sum_{l \in Z_k^\times, l \notin Z_k} \text{meas} \left\{ \| \hat{J} \| \leq K^n : |\hat{J} \cdot \hat{A}_k v_{k,l}| < 3\alpha \| l \| \| k \| K \right\}. \tag{38}
\]

Now, observe that \(v_{k,l} \in \mathbb{Z}^n \setminus \{0\}\) (since \(l \notin \mathbb{Z}k\)) and that \(v_{k,l} \in k^\perp\). But then \(\hat{A}_k v_{k,l} \neq 0\) (indeed, \(\hat{A}_k v_{k,l} = 0\) implies that \(\hat{A}_k v_{k,l} = (\hat{A}_{k^\perp} v_{k,l}) = 0\), contradicting the invertibility of \(\hat{A}_k\), hence (since \(\hat{A}_k v_{k,l} \in \mathbb{Z}^n \setminus \{0\}\)), \(\| \hat{A}_k v_{k,l} \| \geq 1\). Thus, from (38) there follows\(^\text{13}\)

\[
\text{meas} (Z'_k \cap B_{R_k}) \leq c \alpha \sum_{l \in Z_k^\times, l \notin Z_k} \alpha \kappa \| l \| K^{n(n-2)} \leq c \alpha^2 K^{n(n-2)}K^{n+2} \| k \|^{-1}, \tag{39}
\]

which, together with (34), yields (33).

\section{Normal Forms}

In this section we describe a normal form lemma, which allows to average out non–resonant Fourier modes of the perturbation on suitable non–resonant regions, and then apply it on \(\Omega^0\) and \(\Omega^1\).

We remark that such normal form lemma is not standard as, for technical reasons which will be clarified later, we shall need estimates in a complex domain very close to the initial one.

\textbf{Notation}

Given a set \(D \subseteq \mathbb{R}^m, r > 0\) we denote by \(D_r \subseteq \mathbb{C}^m\) the complex open neighborhood of \(D\) formed by points \(z \in \mathbb{C}^m\) such that \(\|z - y\| < r\), for some \(y \in D\).

Given \(s > 0\), we denote by \(\mathbb{T}^n_s\) the open complex neighborhood of \(\mathbb{T}^n\) given by

\[
\mathbb{T}^n_s := \{ x \in \mathbb{C}^n : \max_{1 \leq j \leq n} |\text{Im} x_j| < s \}/2\pi \mathbb{Z}^n
\]

\(^{12}\) Denoting Lebesgue measure on \(\mathbb{R}^{n-1}\) again by \(\text{meas}\).

\(^{13}\) In general, fixed a positive integer \(m\), there exists a constant \(c > 0\) such that for every \(w \in \mathbb{R}^m \setminus \{0\}\) and \(b > 0\) one has: \(\text{meas} \{ y \in \mathbb{R}^m : \| y \| \leq r \} \) and \(|y \cdot w| < b\) \(\leq c r^{m-1} b/\|w\|\).
Given a real–analytic function \( f : D_r \times T^n_s \rightarrow \mathbb{C} \), \( f(y, x) = \sum_{k \in \mathbb{Z}^n} f_k(y)e^{ik \cdot x} \), we consider the weighted sup–norm
\[
|f|_{r,s} := \sup_{k \in \mathbb{Z}^n} \left( \sup_{y \in D_r} |f_k(y)|e^{k|s|} \right);
\]
if the (real) domain need to be specified, we let:
\[
|f|_{D_r,r,s} := |f|_{r,s}.
\]
Given \( f(y, x) = \sum_{k \in \mathbb{Z}^n} f_k(y)e^{ik \cdot x} \) and a sublattice \( \Lambda \) of \( \mathbb{Z}^n \), we denote by \( p_\Lambda \) the projection on the Fourier coefficients in \( \Lambda \), namely
\[
p_\Lambda f := \sum_{k \in \Lambda} f_k(y)e^{ik \cdot x},
\]
and by \( p_\Lambda^\perp \) its “orthogonal” operator (projection on the Fourier modes in \( \mathbb{Z}^n \setminus \Lambda \)):
\[
p_\Lambda^\perp f := \sum_{k \notin \Lambda} f_k(y)e^{ik \cdot x}.
\]
Finally, given \( N > 0 \), we introduce the following “truncation” and “high–mode” operators
\[
T_N f := \sum_{|k| \leq N} f_k(y)e^{ik \cdot x}, \quad T_N^\perp f := \sum_{|k| > N} f_k(y)e^{ik \cdot x}.
\]
For later use, we point the following elementary decay property of analytic function with vanishing low modes:
\[
T_N f = 0 \ , \ 0 < \sigma < s \quad \implies \quad |f|_{r,s–\sigma} \leq e^{-N\sigma}|f|_{r,s}.
\]
We are, now, ready to state the Normal Form Lemma we need. In order not to introduce too many symbols we shall denote by \( H = h + f \) (but without \( \varepsilon \)) the Hamiltonian and by \( \alpha \) and \( K \) (which have been already fixed in (24)) the non–resonance parameters, however the lemma applies to arbitrary \( H \), \( \alpha \) and \( K \).

**Lemma 3.1 (Normal Form Lemma)** Let \( r, s, \alpha \) be positive numbers, \( K \geq 2 \), \( D \subset \mathbb{R}^n \), and let \( \Lambda \) be a sublattice of \( \mathbb{Z}^n \). Let \( H(y, x) = h(y) + f(y, x) \) be real–analytic on \( D_r \times T^n_s \) with \( |f|_{r,s} < \infty \). Assume that \( D_r \) is \((\alpha, K)\) non–resonant modulo \( \Lambda \), namely
\[
|h'(y) \cdot k| \geq \alpha, \quad \forall y \in D_r, k \notin \Lambda, \ |k| \leq K
\]
and that
\[
\vartheta_s := \frac{2^9 n K^3}{\alpha \sigma} \ |f|_{r,s} < 1.
\]
Then, there exists a real–analytic symplectic change of variables
\[
\Psi : D_{r_s} \times T^n_{s_s} \rightarrow D_r \times T^n_s \quad \text{with} \quad r_s := r/2, \ s_s := s(1 - 1/K)
\]
such that
\[
H \circ \Psi = h + f^\circ + f_s, \quad f^\circ := p_\Lambda f + T_K^\perp p_\Lambda^\perp f
\]
with
\[
|f_s|_{r_s,s_s} \leq 2\vartheta_s |f|_{r,s}, \quad |T_K p_\Lambda^\perp f|_{r_s,s_s} \leq (\vartheta_s/2)^K |f|_{r,s}.
\]

\[\]
Proof: See Appendix B.

Remark 3.1 (i) Having information on non–resonant Fourier modes up to order $K$, the best one can do is to average out the non–resonant Fourier modes up to order $K$, namely, to “kill” the term $T_K p_A f$ of the Fourier expansion of the perturbation. This explains the “flat” term $f^\flat = p_A f + T_K p_A f$ surviving in (47) and which cannot be removed in general. Now, think of the remainder term $f_*$ as

$$f_* = p_A f_* + (T_K p_A f_* + T_K p_A f_*) ;$$

then, $p_A f_*$ is a $(\partial_* |f|_{r,s})$–perturbation of the part in normal form (i.e., with Fourier modes in $\Lambda$), while $T_K p_A f_*$ is, by (43), a term exponentially small with $K$ (see also below) and $T_K p_A f_*$ is a very small remainder bounded by $(\partial_* / 2)^K |f|_{r,s}$.

(ii) The “novelty” of this lemma is that the bounds in (48) hold on the large angle domain $\mathbb{T}^n_s$, with $s = s(1 - 1/K)$. In particular, it will be important in our analysis (precisely in order to obtain (70) below) the first estimate in (48). The drawback of the gain in angle–analyticity is that the power of $K$ in the smallness condition (45) is not optimal: for example in [14] the power of $K$ is one (and $s_* = s/6$).

(iii) To compare with more standard formulations, such as the Normal Form Lemma in § 2 of [14], write (47) as

$$H \circ \Psi = h + g + f_* \quad \text{with} \quad p_A h = g, \quad p_A f_* = 0 .$$

Then, $g = p_A f + p_A f_*$, $f_* = T_K p_A f + p_A f_*$ and $T_K p_A f$ is the following bounds hold

$$|g - p_A f|_{r,s} \leq 2\partial_* |f|_{r,s} \quad \Rightarrow |f_*|_{r,s} \leq 2\partial_* |f|_{r,s} ,$$

(50)

provided

$$\partial_* \leq e^{-s/2} , \quad K \geq 2$$

(51)

(which will be henceforth assumed). To check (50), notice that by (48) and (43) (used with $N = K$, $s$ replaced by $s_*$ and $\sigma = \frac{1}{2} - \frac{r}{K}$ so that $s_* - \sigma = s/2$ and $e^{-K\sigma} = e^{-K s/2} \cdot e^s$), one gets

$$|f_*|_{r,s} \leq |T_K p_A f_*|_{r,s} + |T_K p_A (f_* + f)|_{r,s} \leq |T_K p_A f_*|_{r,s} + |T_K p_A (f)_{r,s} - \sigma + |T_K p_A f|_{r,s} - \sigma|$$

$$\leq |T_K p_A f|_{r,s} - \sigma + |T_K p_A f + e^{-K s/2} (e^{-s} \partial_* + 1) f|_{r,s}$$

$$\leq 2e^{-K s/2} |f|_{r,s} .$$

(51)

(iv) In our applications $\alpha$ and $K$ (or $K$) are as in (24), $r \sim \alpha / K$ and $f$ is replaced by $\varepsilon f$. Thus,

$$\partial_* \sim |\log \varepsilon|^{-4(n-1)} \quad \Rightarrow \quad \partial_*^K \ll \varepsilon |\log \varepsilon| ,$$

(52)

which is smaller than any power of $\varepsilon$ (but not exponentially small with $1/\varepsilon$).

(v) If a set $D \subset \mathbb{R}^n$ is $(\alpha, K)$ non–resonant $(mod \Lambda)$ for $h = \|y\|^2/2$, then the complex domain $D_r$ is $(\alpha - r K, K)$ non–resonant $(mod \Lambda)$, provided$^{15} r K < \alpha$.

We now apply the Normal Form Lemma to the Hamiltonian $H$ in (8) in the non–resonant and simple resonant regions.

---

$^{15}$Indeed, if $y \in D_r$ there exists $y_0 \in D$ such that $\|y - y_0\| < r$ and $|y_0 \cdot k| \geq \alpha$ for all $k \in \mathbb{Z}^n \setminus \Lambda$, $|k| \leq K$. Thus, $|y \cdot k| = |y_0 \cdot k - (y_0 - y) \cdot k| \geq |y_0 \cdot k| - r K \geq \alpha - r K$. 

11
3.1 Normal form in Ω\(^0\) (non–resonant regime)

Recalling the definition of \(\alpha\) given in (24), we set
\[
    r(0) := \frac{\alpha}{4K} = \frac{1}{4} \sqrt{\varepsilon} K^\nu .
\] (53)

We can apply Lemma 3.1 to \(H\) in (8) with\(^{16}\):
\[
f \sim \varepsilon f , \quad D \sim \Omega^0 , \quad r \sim r(0) , \quad \Lambda \sim \{0\} , \quad \alpha \sim \alpha/4 ,
\]
and
\[
    \vartheta_s \sim \vartheta_s := 2^{11n} \frac{K^3 \varepsilon |f|_s}{\alpha r(0)^{s/2}} \geq \frac{213n}{s K^{2\nu - 2}} .
\] (54)

By (24), \(\vartheta_s < 1\), provided \(\varepsilon\) is small enough depending on \(s\) and \(n\) (recall that \(\nu > n + 1\)). Thus, there exists a symplectic change of variables
\[
    \Psi_{(0)} : \Omega^0_{r(0)/2} \times T^n_s \to \Omega^0_{r(0)/2} \times T^n_s , \quad s_s := s(1 - 1/K)
\] (55)
(recall (46)) such that \(H\) is transformed in
\[
    H_{(0)} := H \circ \Psi_{(0)} = \|I\|^2/2 + \varepsilon g^{(0)}(I) + \varepsilon f^{(0)}_{\ast\ast}(I, \varphi), \quad \langle f^{(0)}_{\ast\ast} \rangle = 0 ,
\] (56)
where \(\langle \cdot \rangle = p_{(0)}\), denotes the average with respect to the angles \(\varphi\); by (50) and (9), one has:
\[
    \sup_{\omega r(0)} |g^{(0)}| - \langle f \rangle \leq 2 \vartheta_{(0)} , \quad |f^{(0)}_{\ast\ast}|_{r(0)/2, s/2} \leq 2 e^{-K s/2} \quad \varepsilon \leq 2 e^{2\varepsilon |\log \varepsilon|}
\] (57)
provided \(\varepsilon\) is small enough (depending on \(s\) and \(n\)) so that (51) is satisfied.

3.2 Normal form in Ω\(^1\) (simple resonances)

In order to construct normal forms near simple resonances, recall that \(\Omega^1\) is the union of sets\(^{17}\) \(L_kZ_k\), with \(k \in \mathbb{Z}_{\nu,K}^n\), which are \((2\alpha K/\|k\|, K)\) non–resonant modulus the one–dimensional lattice \(\mathbb{Z}k\); compare Proposition 2.1, (iii). Therefore, fixed \(k \in \mathbb{Z}_{\nu,K}^n\), we let
\[
    r_k := \frac{\alpha}{\|k\|} = \frac{\sqrt{\varepsilon} K^\nu + 1}{\|k\|} ,
\] (58)
and apply the normal form Lemma 3.1 with\(^{18}\)
\[
f \sim \varepsilon f , \quad D \sim D^k := L_kZ_k , \quad r \sim r_k , \quad \alpha \sim \alpha K/\|k\| , \quad K \sim K , \quad \Lambda \sim \mathbb{Z}k
\] (59)
and
\[
    \vartheta_s \sim \vartheta_k := 2^9 n \frac{K^2 \|k\|^2 \varepsilon |f|_s}{\alpha^2 s} \geq \frac{29 n K^2}{s K^{2\nu - 2}} ,
\] (60)

\(^{16}\)The set \(\Omega^0\) is defined in (30). By Remark 3.1–(v) and (31), the domain \(\Omega^0_{r(0)}\) is \((\alpha/4, K)\) completely non–resonant.

\(^{17}\)Recall the definitions given in (12), (29), (30).

\(^{18}\)The symbol “\(a \sim b\)” reads “with \(a\) replaced by \(b\)”.

12
Notice that by Remark 3.1-(v) and (32), the domain $D_{r_k}^k$ is $(2\alpha K/\|k\| - r_k K, K) = (\alpha K/\|k\|, K)$ non-resonant modulus $\mathbb{Z}k$. Again, by (24), $\vartheta_k < 1$, provided $\varepsilon$ is small enough (depending on $s$ and $n$). Thus, there exists a symplectic change of variables$^{19}$

$$
\Psi_k : D_{r_k/2}^k \times \mathbb{T}_s^n \to D_{r_k}^k \times \mathbb{T}_s^n, \quad s_* := s(1 - 1/K) \tag{61}
$$
such that $H$ in (8) is transformed in

$$
H \circ \Psi_k = \|I\|^2/2 + \varepsilon g^k(I, \varphi) + \varepsilon f_{s_*}^k(I, \varphi), \tag{62}
$$
where

$$
g^k = p_{k\varepsilon} g^k, \quad p_{k\varepsilon} f_{s_*}^k = 0, \tag{63}
$$
with the following estimates holding (recall (50) and (9)):

$$
|g^k - p_{k\varepsilon} f_{|r_k/2, s_*}| \leq 2\vartheta, \quad |f_{s_*}^k|_{r_k/2, s/2} \leq 2e^{-\kappa s/2} = 2e^{2|\varphi|} \leq 2|\varepsilon|^2 |\log |\varepsilon|^3, \tag{64}
$$
where

$$
\vartheta := \frac{2^9 n}{sK^{\nu-1}}, \quad (\nu > n + 1). \tag{65}
$$
Note that $g^k$ and $p_{k\varepsilon} f$ depend, effectively, only on one angle $t \in \mathbb{T}^1$: more precisely, setting

$$
F_j^k := f_{jk}, \quad G_j^k(I) := g_{jk}^k(I), \quad \text{and} \quad F^k(t) := \sum_{j \in \mathbb{Z}} F_j^k e^{ijt}, \quad G^k(I, t) := \sum_{j \in \mathbb{Z}} G_j^k(I)e^{ijt}, \tag{66}
$$
we have (recall (4))

$$
p_{k\varepsilon} f(\varphi) = F^k(k, \varphi), \quad g^k(I, \varphi) = G^k(I, k \cdot \varphi). \tag{67}
$$
We also remark that since $f \in A^n_\kappa$, the functions $F^h$ belong to $A_{|h|s}$ for every $h \in \mathbb{Z}^n_s$, with

$$
|F^h|_{|h|s} \leq |f|_s \tag{9}, \tag{68}
$$
 Analogously, by (64)

$$
|G^k - F^k|_{r_k/2, |k|s} \leq 2\vartheta. \tag{69}
$$
For later use (compare (72) and (75) below), we point out that

$$
\frac{1}{|f_k|} |G^k_1 - f_k|_{r_k/2} \leq \frac{2\vartheta e^{-|k|s}}{|f_k|} \tag{69} \leq \frac{2e^{s|k|/K} \vartheta e^{-|k|s}}{|f_k|} \leq \frac{2\vartheta e^{(1-|k|)s}}{|f_k|}. \tag{P1}
$$

$$
\leq \frac{2\vartheta e^{s|k|/K} \frac{n+1}{\delta}}{\delta} \leq \frac{2\vartheta e^s K^{\frac{n+1}{\delta}}}{\delta} \tag{65} \leq \frac{2^{10} n e^s}{\delta s K^{\frac{2s-1}{2s-n-1}}}, \tag{70}
$$
which is small if $\varepsilon$ is small (recall that $n \geq 2$ and $\nu > n + 1$).

$^{19}$Recall (46) and Remark 3.1, (iii).
3.2.1 The effective potential

We now show that for $|k|$ large, the “effective potential” $G^k$ (defined in (66)) behaves, essentially, as a cosine; compare, in particular, Eq. (72) below.

Recalling the definition of $T_h$ given in (42), we have that

$$T_1 F^k(\psi_n') = f_k e^{i\psi_n'} + f_k e^{-i\psi_n'} = 2|f_k| \cos(\psi_n' + \psi_n^{(k)})$$

for suitable constants $\psi_n^{(k)}$.

**Remark 3.2** We can assume, up to translation, that $\psi_n^{(k)} = \pi$. So, from now on, we assume that

$$T_1 F^k(\psi_n') = f_k e^{i\psi_n'} + f_k e^{-i\psi_n'} = -2|f_k| \cos(\psi_n').$$ (71)

**Lemma 3.2**

$$G^k(I, \psi_n') = 2|f_k| \left( - \cos(\psi_n') + R^k(I, \psi_n') \right),$$

$$|R^k(\sqrt{2e/\kappa A_k} T^k J', \psi_n')|_{D^k, r_k/2, |k| s/3} \leq \frac{2|k| n+3 e^{-|k| s/4}}{\delta} + \frac{2^{10} ne^s}{\delta s K^{\nu - n + 1}}.$$ (72)

**Proof** Indeed,

$$\frac{1}{|f_k|} |F^k(\psi_n') + 2|f_k| \cos(\psi_n')||_{|k| s/3} \leq \frac{1}{|f_k|} |T_1 F^k||_{|k| s/3} = \frac{1}{|f_k|} \sup_{|j| \geq 2} |f_{kj}| e^{|k| s j/3}$$

$$\leq 1 \sup_{|j| \geq 2} e^{-2|k| s j/3} = \frac{1}{|f_k|} e^{-4|k| s j/3}$$ (P1) \leq \frac{|k| n+3 e^{-|k| s/4}}{\delta}. \quad (73)

Also,

$$\frac{1}{|f_k|} \left| T_1 (G^k(I, \psi_n') - F^k(\psi_n')) \right|_{D^k, r_k/2, |k| s/3} \leq \frac{2\delta}{|f_k|} \sup_{|j| \geq 2} e^{-2|k| s j/3}$$

$$= \frac{2\delta}{|f_k|} e^{-2|k| s j/3}$$ (P1) \leq \frac{2\delta}{|f_k|} e^{-4|k| s j/3} \leq \frac{2\delta}{|f_k|} \frac{|k| n+3 e^{-|k| s/4}}{\delta}, \quad (74)$$

provided $K \geq 24$. Then (72) follows by (70), (73) and (74). □

Moreover by (70), (74), we have also that

$$|f_k|^{-1} |G^k(I, \psi_n') - F^k(\psi_n')|_{D^k, r_k/2, |k| s/3} \leq \frac{4\delta e^s K^{n+3}}{\delta} = \frac{2^{11} ne^s}{\delta s K^{\nu - n + 1}}, \quad (75)$$

which is small if $\varepsilon$ is small (recall that $n \geq 2$ and $\nu > n + 1$).
3.2.2 Rescalings

Recalling the definition of $K_s(\delta)$ in (5), we set

$$\delta_k := \begin{cases} 1 & \text{if } |k| \leq K_s(\delta) \\ 2|f_k| & \text{if } |k| > K_s(\delta) \end{cases}.$$  (76)

Note that by (9)

$$\delta_k \leq 1.$$  (77)

Define the conformally symplectic transformation

$$\Phi^{(0)} : (I', \varphi') \mapsto (I, \varphi) = (\varsigma_k I', \varphi'),$$  (78)

where

$$\varsigma_k := \sqrt{2 \delta_k \varepsilon} = \sqrt{2 \delta_k \varepsilon} \|k\|^2.$$  (79)

Then, the flow of $H \circ \Psi_k$ (recall (62) and (67)) is equivalent to the flow of the Hamiltonian

$$\frac{1}{\varsigma_k} H \circ \Psi_k \circ \Phi^{(0)}(I', \varphi') = \sqrt{\frac{\delta_k \varepsilon}{2 \kappa}} \|I'\|^2 + \sqrt{\frac{\kappa \varepsilon}{2 \delta_k}} (G_k(\varsigma_k I', k \cdot \varphi') + f_k^*(\varsigma_k I', \varphi')).$$  (80)

Dividing such Hamiltonian by $\sqrt{\delta_k \varepsilon / 2}$ (which corresponds to a time rescaling), we are lead to study the Hamiltonian

$$H_k(I', \varphi') := \frac{1}{\kappa} \|I'\|^2 + \frac{1}{\delta_k} (G_k(\varsigma_k I', k \cdot \varphi') + f_k^*(\varsigma_k I', \varphi')),$$  (81)

which is defined on the domain

$$D_k' \times \mathbb{T}_n^a,$$  (82)

with

$$r_k' := \frac{r_k}{2\varsigma_k} = \frac{K^{v+1}}{\sqrt{8\delta_k}} \quad \text{and} \quad D_k' := \frac{1}{\varsigma_k} D_k^{(59)} = \frac{1}{\varsigma_k} L_k Z_k^{(29), (24), (79)} = L_k \left( \frac{1}{\varsigma_k} \tilde{Z}_k \times \left( - \frac{K^{v+1}}{\sqrt{2\delta_k} \|k\|}, \frac{K^{v+1}}{\sqrt{2\delta_k} \|k\|} \right) \right).$$  (83)

Note that, by (64)

$$|\delta_k^{-1} f_k^*(\varsigma_k I', \varphi')|_{D_k', r_k', s/2} \leq \frac{2}{\delta} |k| \frac{2^{2+3} e^{2|k|s} - K s / 4}{e^{2|k|s} - 1} \leq \frac{2}{\delta} e^{K s / 4} = \frac{1}{\delta} e^{\frac{1}{2} |\log \varepsilon|^3}$$  (84)

for $\varepsilon$ small enough.

Recalling (28) we set

$$D_k^{(4)} := \frac{1}{\varsigma_k} L_k Z_k^{(4)}.$$  (85)

Note that

$$\Phi^{(0)}(D_k^{(4)} \times \mathbb{T}^a) = L_k Z_k^{(4)} \times \mathbb{T}^a.$$  (86)

Then by (30)

$$\Phi^{(0)} \left( \bigcup_{k \in \mathbb{Z}_+^n \setminus K} D_k^{(4)} \times \mathbb{T}^a \right) = \bigcup_{k \in \mathbb{Z}_+^n \setminus K} \Phi^{(0)}(D_k^{(4)} \times \mathbb{T}^a) = \Omega^1 \times \mathbb{T}^a.$$  (87)

---

20See Lemma D.2, (ii) in Appendix D).
21See, again, Lemma D.2.
4 The nearly–integrable structure at simple resonances

Given a bounded holomorphic function \( f : D \times T^n \to \mathbb{C} \), with \( D \subseteq \mathbb{R}^n \) we set

\[
\|f\|_{D,r,s} = \|f\|_{r,s} := \sup_{D \times T^n} |f|.
\]

The following relation between the two norms \( |\cdot| \) and \( \|\cdot\| \) holds: for \( \sigma > 0 \), we have\(^{22}\)

\[
|f|_{r,s} \leq \|f\|_{r,s} \leq \coth^n \left( \frac{\sigma}{2} \right)|f|_{r,s+\sigma} \leq (1 + 2/\sigma)^n |f|_{r,s+\sigma}.
\]

4.1 A class of Morse non-degenerate functions

Let \( s_0 > 0 \) and let us consider a bounded holomorphic function \( F_0 : T^{s_0} \to \mathbb{C} \), with \( \|F_0\|_{s_0} < \infty \).\(^{23}\)

**Definition 4.1** \( \beta, M > 0 \). We say that \( F_0 \) as in (88) is \((\beta, M)\)-Morse-non-degenerate if \( \|F_0\|_{s_0} \leq M \) and

\[
\min_x (|(F_0)'(x)| + |(F_0)''(x)|) \geq \beta, \quad \min_{1 \leq i < j \leq 2N} |F_0(x_i) - F_0(x_j)| \geq \beta,
\]

where \( x_i^0, 1 \leq i \leq 2N \) are the critical points of \( F_0 \) in \((-\pi, \pi]\).

We note that, by (89), the function \( F_0 \) has only non-degenerate critical points: let us say \( N \) minima: \( x_{2j-1}^0 \), and \( N \) maxima: \( x_{2j}^0 \), in \((-\pi, \pi]\), for some integer \( N \geq 1 \) and \( 1 \leq j \leq N \). It is immediate to realize that \( N \) is uniformly bounded by a constant depending only on \( s_0, M \) and the minimum appearing in (89).

The corresponding critical energies are

\[
E_i^0 := F_0(x_i^0), \quad 1 \leq i \leq 2N.
\]

By (90), \( E_i^0 \) are all different.

**Definition 4.2** We say that \( F_0 \) as in (88) is \( \gamma \)-cosine-like\(^{23}\) if

\[
\|F_0(x) + \cos x\|_{s_0} \leq \gamma, \quad \text{for some} \quad 0 < \gamma \leq \frac{1}{4} \min \{1, s_0^2 \}.
\]

**Lemma 4.1** If \( F_0 \) is \( \gamma \)-cosine-like, then it is also \((\beta, M)\)-Morse-non-degenerate with

\[
\beta = 1/4, \quad M = \gamma + \cosh s_0 \leq \frac{1}{4} + \cosh s_0.
\]

Moreover \( F_0 \) has only two non-degenerate critical points (a maximum and a minimum).\(^{23}\)

\(^{22}\) Since \( \sum_{k \in \mathbb{Z}} e^{-|k|\sigma} = \left( \sum_{k \in \mathbb{Z}} e^{-|k|} \right)^n = \left( 1 + 2 \sum_{j \geq 1} e^{-j\sigma} \right)^n = \left( \frac{e^{\sigma} + 1}{e^{\sigma} - 1} \right)^n = \coth^n (\sigma/2) \).

\(^{23}\) Actually we should say minus-cosine-like.
Proof We have, by Cauchy estimates,

\[ |(F^0)'(x)| + |(F^0)''(x)| \geq |\sin x| + |\cos x| - \frac{\gamma}{s_0} - 2\frac{\gamma}{s_0^2} \geq 1 - \frac{\gamma}{s_0} - 2\frac{\gamma}{s_0^2} \geq \frac{1}{4}. \]

We can choose \( M \) as above since \( \| \cos x \|_{s_0} = \cos s_0 \). Regarding the last sentence we note that for \( x \in (-\pi, \pi) \) we have only two critical points, a minimum in \((-\pi/6, \pi/6)\) and a maximum in \((-\pi, -5\pi/6) \cup (5\pi/6, \pi]\). Indeed we have that, setting \( g(x) := F^0(x) + \cos x \), \((F^0)'(x) = \sin x + g'(x)\), so that

\[ (F^0)'(x) = \sin x + g'(x) \geq \sin x - \frac{\gamma}{s_0} \geq \sin x - 1/4. \]

This implies that \((F^0)'(\pi/6) \geq 1/4, (F^0)'(-\pi/6) \leq -1/4\). Then, by continuity, there exists a critical point of \( F^0 \) in \((-\pi/6, \pi/6)\). Moreover such point is a minimum and there are no other critical points in \((-\pi/6, \pi/6)\) since there \( F^0 \) is strictly convex:

\[ (F^0)''(x) = \cos x + g''(x) \geq \frac{\sqrt{3}}{2} - 2\frac{\gamma}{s_0^2} \geq \frac{\sqrt{3}}{2} - 1/2 > 0. \]

Similarly in \((-\pi, -5\pi/6) \cup (5\pi/6, \pi]\) there is only one critical point, which is a maximum. Finally, by (92), \((F^0)'(x) \geq 1/4 \) for \( x \in [\pi/6, 5\pi/6] \) and, analogously, \((F^0)'(x) \leq -1/4 \) for \( x \in [-5\pi/6, -\pi/6]\); so that there are no other critical points.

### 4.2 The Structure Theorem

We start introducing a parameter

\[ \theta \geq 0, \]  

that will be chosen in Section 7 as a function of \( \epsilon \). We also say that a function \( \theta \to E(\theta) \subseteq \mathbb{R}^m \) is decreasing w.r.t. \( \theta \) if \( \theta < \theta' \) implies \( E(\theta) \supseteq E(\theta') \).

In light of (80) we are now going to study the behavior of the “effective Hamiltonian” close to a simple resonance identified by a fixed \( k \in \mathbb{Z}^n \), namely we are considering Hamiltonian of the form

\[ \mathcal{H}(I', \phi') := \frac{1}{\kappa}||I'||^2 + \Phi(I', k \cdot \phi'). \]  

### Assumptions on the “effective potential” \( \Phi \)

Consider the parameters

\[ s_0, r_0 > 0, \quad r' := cn|k|_{\infty}r_0, \]  

c > 1 being the constant defined in (17), which depends only on \( n \). We will make the following assumptions:

(A1) There exists

\[ F^0 \in \mathcal{A}_{s_0}^1 \]

such that

\[ \| \Phi - F^0 \|_{\mathcal{D}, r', s_0} \leq \eta_*, \]  

with\(^{24}\)

\[ \mathcal{D} = L_k(\dot{D} \times (-R_0, R_0)) \cup \mathcal{D} \subseteq \mathbb{R}^{n-1}, \quad R_0 \geq 4 + \cosh s_0; \]

\(^{24}\)Recall the definition of \( L_k \) given in (12).
\((A2)\) \(F^0\) is \((\beta, M)\)-Morse-non-degenerate with \(\max\{2\sqrt{M}, 4\} \leq R_0.\) (99)

In alternative to \((A2)\) we will assume, when it holds, the following stronger (recall Lemma 4.1 and (98)) hypothesis:

\((A3)\) \(F^0\) is \(\gamma\)-cosine-like with \(\gamma \leq \varsigma(s_0) := \varsigma_0 \min\{1, s_0^4\},\) (100)

where \(0 < \varsigma_0 \leq 1/4\) is a suitably small positive constant to be chosen below (see Lemma C.5).

Recalling (98) we set

\[ D_\sharp := L_k(\hat{D} \times (-R_0/2, R_0/2)), \quad \hat{D} \subset \mathbb{R}^{n-1}. \] (101)

**Theorem 4.1 (Integrable structure at simple resonances)**

**Part I.** Assume that \(G\) in (94) satisfies (A1) and (A2) or (A3). Then there exist a suitably large constant \(c > 1,\) which, when (A3) holds, depends only on \(n, s_0, r_0,\) otherwise it depends also on \(F^0,\)

such that if \(\eta_* \leq 1/c\) (102)

the following holds. For every \(0 \leq i \leq 2N\) there exist

i) disjoint open subsets \(C^i(\theta) \subseteq D \times \mathbb{T}^n\) decreasing w.r.t. \(\theta,\) with

\[ \text{meas}\left( (D_\sharp \times \mathbb{T}^n) \setminus \bigcup_{0 \leq i \leq 2N} C^i(\theta) \right) \leq c\theta|\ln \theta|; \] (103)

ii) open subsets \(B^i(\theta) \subseteq \mathbb{R}^n,\) decreasing w.r.t. \(\theta\) with\(^{28}\)

\[ \text{diam}(B^i(0)) \leq 2c(R_0 + \text{diam}(\hat{D})), \quad \forall 0 \leq i \leq 2N; \] (104)

iii) a symplectomorphism

\[ \Psi^i : (p, q) \in B^i(0) \times \mathbb{T}^n \rightarrow C^i(0) \ni (I', \varphi'), \quad \text{with} \quad \Psi^i\left(B^i(\theta) \times \mathbb{T}^n\right) = C^i(\theta), \quad \forall \theta \geq 0, \] (105)

with holomorphic extension

\[ \Psi^i : \left( B^i(\theta) \right)_{\rho_*} \times \mathbb{T}^n_{\sigma_*} \rightarrow D_{\rho_*} \times \mathbb{T}^n_{\sigma_*}, \quad \text{with} \quad \rho_* := \frac{\theta}{c|k|^{n-1}}, \quad \sigma_* := \frac{1}{c|k|^{n-1}|\log \theta|}; \] (106)

such that

\[ \mathcal{H} \circ \Psi^i(p, q) =: h^{(i)}(p). \] (107)

Moreover

\[ \|\partial_{pp} h^{(i)}\|_{B^i(\theta), \rho_*} \leq c/\theta, \quad \text{for} \quad 0 \leq i \leq 2N. \] (108)

\(^{25}\) In any case it is independent of \(\hat{D}.\)

\(^{26}\) \(2N\) being the number of critical points of \(F^0.\) Note that, when (A3) holds, \(N = 1\) by Lemma 4.1.

\(^{27}\) The set \(\bigcup C^i(0)\) contains \(D_\sharp \times \mathbb{T}^n\) up to the connected components of the critical energy level containing critical points.

\(^{28}\) Where \(c\) is the constant defined in (17).
Part II. Assuming \( 0 < \eta_* \leq 1/c\|k\|^{2n} \), \( 0 < \mu \leq 1/c\|k\|^{2n} \), we have that for every there exist open subsets \( \tilde{B}^i(\mu) \subseteq B^i(0) \), decreasing w.r.t. \( \mu \), such that
\[
\text{meas}(B^i(0) \setminus \tilde{B}^i(\mu)) \leq c\|k\|^{4n} \mu^{1/c}
\]and
\[
|\det (\partial_{pp} h^{(i)}(\mu))| > \mu, \quad \forall \ 0 \leq i \leq 2N, \ |k| \leq K, \ \forall \ p \in \tilde{B}^i(\mu).
\]
The following two sections are devoted to the proof of Theorem 4.1 part I and part II, respectively.

5 Proof of Part I of the Structure Theorem

In this section we will prove Theorem 4.1 part I.

5.1 Critical points and critical energies of the “unperturbed potential” \( F^0 \)
We order the critical points of \( F^0 \) (recall (96)) in the following way (where\(^{29} x^0_0 := x^0_{2N} - 2\pi \))
\[
x^0_0 < x^0_1 < x^0_2 < \ldots < x^0_{2N-1} < x^0_{2N}, \quad x^0_{2j-1} \text{ minimum}, \quad x^0_{2j} \text{ maximum}, \quad 1 \leq j \leq N.
\]Fix \( 1 \leq j \leq N \) and consider a minimum point \( x^0_{2j-1} \), thanks to (89) the function \( F^0 \) is strictly increasing, resp. strictly decreasing, in the interval \( [x^0_{2j-1}, x^0_{2j}] \), resp. \( [x^0_{2j-2}, x^0_{2j-1}] \), then we can invert \( F^0 \) on the above intervals obtaining two functions
\[
X^0_{2j} : [E^0_{2j-1}, E^0_{2j}] \rightarrow [x^0_{2j-1}, x^0_{2j}] \quad \text{and} \quad X^0_{2j-1} : [E^0_{2j-1}, E^0_{2j-2}] \rightarrow [x^0_{2j-2}, x^0_{2j-1}]
\]such that
\[
F^0(X_i^0(E)) = E, \quad X_i^0(F^0(\psi_n)) = \psi_n, \quad \forall 1 \leq i \leq 2N.
\]
Note that \( X^0_i \) is increasing, resp. decreasing, if \( i \) is even, resp. odd.
Set
\[
E_i^{(1),0} := E_i, \quad E_i^{(2j-1),0} := \min\{E_{2j-2}, E_{2j}^0\} \quad \text{for} \ 1 \leq j \leq N, \quad E_i^{(2j),0} := \min\{E_{2j-2}, E_{2j-1}^0\} \quad \text{for} \ 1 \leq j < N, \quad E_{2N+1}^{(0),0} = E_{2N}^{(0),0} = +\infty,
\]
where
\[
j_- := \max\{i < j \ s.t. \ E_{2i}^0 > E_{2j}^0\}, \quad j_+ := \min\{i > j \ s.t. \ E_{2i}^0 > E_{2j}^0\}.
\]
\(^{29}\)Similarly we will set \( E_0^0 := E_{2N}^0 \) below.
5.2 The slow angle

Let us, now, perform the linear symplectic change of variables \( \Phi^{(1)} : (J', \varphi') \mapsto (I', \varphi') \) generated by 
\[
S(J', \varphi') := A_k \varphi' \cdot J',
\]
\[
\Phi^{(1)} : (J', \varphi') \mapsto (I', \varphi') = (A_k^T J', A_k^{-1} \varphi') = (k J_n' + \hat{A}_k^T \hat{J}', A_k^{-1} \varphi').
\]  
(117)

Note that \( \Phi^{(1)} \) does not mix actions with angles, its projection on the angles is a diffeomorphism of \( T^n \) onto \( T^n \), and, most relevantly,
\[
\psi_n' = k \cdot \varphi'
\]  
(118)
is the canonical angle associated to the one-dimensional “secular system” near the simple resonance \( \{y \cdot k = 0\} \) (i.e., the one-dimensional system governed by the Hamiltonian obtained disregarding the small term \( f_{k^*} \) in (80)).

In the \((J', \varphi')\)-variables, we have:
\[
\tilde{H}(J', \varphi') := \mathcal{H} \circ \Phi^{(1)}(J', \varphi') = \frac{1}{k} \|A_k^T J'\|^2 + \mathcal{G}(A_k^T J', \psi_n').
\]  
(119)

As for the \((J', \varphi')\)-domain, we see that the real \( J' \)-domain is given by
\[
\tilde{D} := A_k^{-T} D = A_k^{-T} L_k (\hat{D} \times (-R_0, R_0)) \overset{(14)}{=} U_k (\hat{D} \times (-R_0, R_0)),
\]  
(120)

with \( U_k \) in (15). We also set
\[
\tilde{D}_\delta := U_k (\hat{D} \times (-R_0/2, R_0/2)).
\]  
(121)

Then, if we choose
\[
\tilde{r} := \frac{r'}{n|k|_\infty}, \quad \tilde{s} := \frac{s_0}{c|k|_\infty},
\]  
(122)

(for a suitable \( c \) depending only on \( n \)) we see that \( \Phi^{(1)} \) has holomorphic extension on the complex domain (recall Lemma D.4)
\[
\Phi^{(1)} : \tilde{D}_\delta \times \mathbb{T}^n \rightarrow \mathcal{D}_{\tilde{r}, \delta} \times \mathbb{T}^n
\]  
(123)

indeed: by (10), \( \|A_k\| = \|A_k\| \leq n|k|_\infty \), so that \( \|A_k\| \tilde{r} \leq r' \), while, for every \( 1 \leq i \leq n \),
\[
\sum_{1 \leq j \leq n} |(A_k^{-1})_{ij}| \leq n|A_k^{-1}|_\infty \overset{(11)}{=} c|k|_\infty^{-1},
\]  
(11)

Note that by (97)
\[
\|\mathcal{G}(A_k^T J', \psi_n') - F^0(\psi_n')\|_{\tilde{D}_\delta, \tilde{s}, 0} \leq \eta^*.
\]  
(124)

5.3 The auxiliary Hamiltonian

A crucial role will be played by the auxiliary Hamiltonian
\[
H^* := (J_n'')^2 + F^*(J'', \psi_n''), \quad \text{where} \quad F^*(J'', \psi_n'') := \mathcal{G}(L_k J'', \psi_n'').
\]  
(125)
This Hamiltonian represents a one dimensional mechanical system depending on the parameter \( \hat{J}' \). The relation between \( \tilde{H} \) (defined in (119)) and \( H^* \) is the following: recalling (12), (14) and the change \( J' = U_k J'' \), \( U_k \) defined in (15), it results

\[
H^*(J'', \psi'') = \tilde{H}(U_k J'', \psi'') - \frac{1}{\kappa} \| p_k^T \hat{A}_k \hat{J}'' \|^2. \tag{126}
\]

Recalling (120), the potential \( F^* \) in (125) is defined for

\[
(J'', \psi'') \in D_{\tau_0} \times T_{s_0}, \quad \text{where} \quad D := \hat{D} \times (-R_0, R_0), \tag{127}
\]

(95), (122) and (17). Note that, by (124),

\[
\| F^* - F^0 \|_{D_{\tau_0}, s_0} \leq \eta_* . \tag{128}
\]

### 5.4 A special group of symplectic transformations

In the following, symplectic transformations will have a special form, namely, they will belong to a special group \( G \), formed by symplectic transformations \( \Phi \) satisfying

\[
\hat{I} = \hat{J}, \quad I_n = I_n(J, \psi_n), \quad \hat{\varphi} = \hat{\psi} + \hat{\varphi}'(J, \psi_n), \quad \varphi_n = \varphi_n(J, \psi_n), \tag{129}
\]

where, in general, \( \varphi, \psi \) may belong either to \( \mathbb{T}^n \) or to \( \mathbb{R}^n \). For a transformation \( \Phi \) as in (129) we let \( \tilde{\Phi} \) denote the map

\[
\tilde{\Phi}(J, \psi_n) := (\hat{J}, I_n(J, \psi_n), \varphi_n(J, \psi_n)). \tag{130}
\]

Some general properties of \( G \) are discussed in Appendix D.

### 5.5 An intermediate transformation

To simplify geometry, we now introduce a symplectic transformation that removes the dependence upon \( J_n \) from the potential. Since (102) holds\(^{30} \), by Lemma D.10 in Appendix D, one can find a symplectomorphism \( \Phi_{(2\text{bis})} \in G \) satisfying

\[
\Phi_{(2\text{bis})} : (J, \psi) \rightarrow (J'', \psi''), \quad \hat{J}' = \hat{J}, \quad J''_n = J_n + a_s(\hat{J}, \psi_n), \quad \hat{\psi}' = \hat{\psi} + b_s(\hat{J}, \psi_n), \quad \psi''_n = \psi_n, \tag{131}
\]

with (taking \( c \) large enough)

\[
\Phi_{(2\text{bis})} : D_{\tau_0/2} \times T_{s_0/2} \rightarrow D_{\tau_0} \times T_{s_0}. \tag{132}
\]

and

\[
\|a_s\|_{\hat{D}_{\kappa, \tau_0, s_0}} \leq 4 \eta_*/r_{\tau_0}, \quad \|b_s\|_{\hat{D}_{\kappa, \tau_0/2, s_0}} \leq (16 \pi + 8) \eta_*/r_0^2, \tag{133}
\]

and such that

\[
H_{\text{pend}}(J, \psi_n) := H^* \circ \Phi_{(2\text{bis})} = (1 + b(J, \psi_n))(J_n - J_n^*)(\hat{J})^2 + F(\hat{J}, \psi_n), \quad \tag{134}
\]

\[
F(\hat{J}, \psi_n) = F^0(\psi_n) + G(\hat{J}, \psi_n). \]

\(^{30}\) Note that this implies (349).
Furthermore (see (356) below)

\[\|J_n^\ast\|_{\mathcal{D}, r_0} \leq 2 \eta_\ast/r_0 \leq \eta r_0, \quad \|G\|_{\mathcal{D}, r_0, s_0} \leq (1 + 4/r_0^2) \eta_\ast \leq \eta,\]

\[\|(1 + |J_n - J_n^\ast(\hat{J})|b(J, \psi_n))\|_{\mathcal{D}, r_0/2, s_0} \leq \left(4 + \frac{34}{r_0^2}\right) \eta_\ast \leq \eta,\]

\[\|J_n - J_n^\ast(\hat{J})\|_{\mathcal{D}, r_0/2, s_0} \leq \frac{48}{r_0^2} \eta_\ast \leq \eta,\]

where

\[\eta := \left(4 + \frac{48}{r_0^2}\right) \eta_\ast\]

**Notations** For brevity we introduce the following notations.

\[p := (n, r_0, s_0, \beta, M).\]

We say that

\[a \prec b \quad \text{if} \quad \exists C = C(p) > 0 \quad \text{s.t.} \quad a \leq C b.\]

We also say that, given \(F^0\) satisfying (A1) and (A2),

\[a \prec_{F^0} b \quad \text{if} \quad \exists C = C(F^0) > 0 \quad \text{s.t.} \quad a \leq C b.\]

**Remark 5.1** Note that \(a \prec b\) implies \(a \prec_{F^0} b\). Note also that if (A3) holds, then, by Lemma 4.1, \(p\) reduces to \((n, r_0, s_0)\)

Let us assume that

\[\eta \leq \eta_0 = \eta_0(p),\]

for a suitable small \(\eta_0\). By (89), for \(\eta_0\) small enough, we can continue the critical points \(x^0_j\) (defined in (113)), resp. critical energies \(E^0_j\), of \(F^0\) obtaining critical points \(x_j(\hat{J})\), resp. critical energies \(E_j(\hat{J})\), of \(F(\hat{J}, \cdot)\), solving the implicit function equation

\[\partial_{\psi_n} F(\hat{J}, x_j(\hat{J})) = 0\]

and then evaluating

\[F(\hat{J}, x_j(\hat{J})) =: E_j(\hat{J}),\]

respectively. Note that \(x_j(\hat{J})\), and \(E_j(\hat{J})\) are analytic functions of \(\hat{J} \in \hat{D}_{r_0}\). By (89)

\[\sup_{J \in \hat{D}_{r_0}} |x_j(\hat{J}) - x^0_j|, \quad \sup_{J \in \hat{D}_{r_0}} |E_j(\hat{J}) - E^0_j| \leq \eta.\]

\[\text{To find } x_j := x^0_j + \chi_j \text{ we have to solve for every } \hat{J} \text{ the equation } \partial_{\psi_n} F(\hat{J}, x^0_j + \chi_j) = 0. \text{ Since }\]

\[\partial_{\psi_n} F(\hat{J}, x^0_j + \chi_j) = \partial^2_{\psi_n, \psi_n} F^0(x^0_j) \chi_j + O(\chi_j^2) + O(\eta_0)\]

the equation reduces, by (89), to find \(\chi_j\) solving the fixed point \(\chi_j = O(\chi_j^2) + O(\eta_0)\). Moreover since \(F\) is an analytic function of \(\hat{J} \in \hat{D}_{r_0}\), the same holds for \(\chi_j(\hat{J})\).
Therefore we note that \( x_j(\hat{J}) \) and \( E_j(\hat{J}) \) maintain the same order of \( x^0_j \) and \( E^0_j \). In particular, recalling 115

\[
E^{(i)}_-(\hat{J}) := E_i(\hat{J}) , \quad E^{(2j-1)}_+(\hat{J}) := \min\{E_{2j-2}(\hat{J}), E_{2j}(\hat{J})\} \quad \text{for} \quad 1 \leq j \leq N , \\
E^{(2j)}_+(\hat{J}) := \min\{E_{2j-1}(\hat{J}), E_{2j+1}(\hat{J})\} \quad \text{for} \quad 1 \leq j < N , \quad E^{(2N)}_+(\hat{J}) = E^0_+(\hat{J}) = +\infty ,
\]

where \( j_\pm \) were defined in (116).

By (89), (90) for \( \eta_0 \) small enough we get

\[
\inf_{J \in \mathcal{D}_{\eta_0}} \min_{\psi_n \in \mathbb{R}} \left( \left| \partial_{\psi_n} F(\hat{J}, \psi_n) \right| + \left| \partial_{\psi_n} \psi_n F(\hat{J}, \psi_n) \right| \right) \geq \frac{\beta}{2} , \quad \inf_{J \in \mathcal{D}_{\eta_0} \setminus \hat{J}} \min \left| E_j(\hat{J}) - E_i(\hat{J}) \right| \geq \frac{\beta}{2} . \quad (145)
\]

Reasoning as above, by (145), for \( \hat{J} \in \hat{D} \) (namely \( \hat{J} \) real) and \( \eta_0 \) small enough, we can “continue” also the functions \( X^0_{2j}, X^0_{2j-1}, \) obtaining

\[
X_{2j}(\cdot, \hat{J}) := \left[ E_{2j-1}(\hat{J}), E_{2j}(\hat{J}) \right] \rightarrow [x_{2j-1}(\hat{J}), x_{2j}(\hat{J})] , \\
X_{2j-1}(\cdot, \hat{J}) := \left[ E_{2j-1}(\hat{J}), E_{2j-2}(\hat{J}) \right] \rightarrow [x_{2j-2}(\hat{J}), x_{2j-1}(\hat{J})] ,
\]

solving the implicit function equations

\[
F(\hat{J}, X_i(E, \hat{J})) = E , \quad X_i(F(\hat{J}, \psi_n), \hat{J}) = \psi_n , \quad \forall 1 \leq i \leq 2N . \quad (147)
\]

Note that \( X_i \) is increasing, resp. decreasing (as a function of \( E \)), if \( i \) is even, resp. odd. Note also that

\[
\partial_E X_i(E, \hat{J}) = 1/\partial_{\psi_n} F(\hat{J}, X_i(E, \hat{J}))
\]

and

\[
X_{2j-1}(E_{2j-2}(\hat{J}), \hat{J}) = x_{2j-2}(\hat{J}) , \quad X_{2j-1}(E_{2j-1}(\hat{J}), \hat{J}) = x_{2j-1}(\hat{J}) , \\
X_{2j}(E_{2j}(\hat{J}), \hat{J}) = x_{2j}(\hat{J}) . \quad (149)
\]

### 5.6 The integrating transformation

**Proposition 5.1** Let \( H_{\text{pend}} \) be as in (134), (135), (102). There exist a suitably large constant \( C > 1 \), which, when (A3) holds, depends only on \( n, s_0, r_0 \), otherwise depend also on \( F^0 \) (introduced in (96)) such that, if

\[
\eta \leq 1/C , \quad 0 \leq \theta \leq 1/C , \quad (150)
\]

the following holds. There exist:

i) disjoint open connected sets

\[
\mathcal{C}^i(\theta) = \mathcal{C}^i(\theta) \times \mathbb{T}^{n-1} , \quad 0 \leq i \leq 2N , \quad (151)
\]

decreasing w.r.t. \( \theta \) and satisfying (recall (127))

\[
\hat{D} \times (-R_0/2, R_0/2) \times \mathbb{T}^n \subset \bigcup_{0 \leq i \leq 2N} \mathcal{C}^i(0) \subset \hat{D} \times (-R_0, R_0) \times \mathbb{T}^n , \quad (152)
\]
is proved in \(C\) one easily recognizes that

\[
\text{diam} \left( \mathcal{P}^i(0) \right) \leq 2 \left( R_0 + \text{diam}(\hat{D}) \right), \quad \forall 0 \leq i \leq 2N; \tag{154}
\]

\(\text{ii) open connected sets } \mathcal{P}^i(\theta) \text{ decreasing w.r.t. } \theta \text{ with}
\]

\[
\text{diam} \left( \mathcal{P}^i(0) \right) \leq 2 \left( R_0 + \text{diam}(\hat{D}) \right), \quad \forall 0 \leq i \leq 2N; \tag{154}
\]

\(\text{ii) symplectomorphisms}
\]

\[
\mathfrak{F}^i : \mathcal{P}^i(0) \times \mathbb{T}^n \supseteq (P, Q) \to (J, \psi) \in \mathcal{C}^i(0) \tag{155}
\]

in \(\mathcal{G}\) such that\(^{32}\)

\[
\mathfrak{F}^i(\mathcal{P}^i(\theta) \times \mathbb{T}^n) = \mathcal{C}^i(\theta), \quad \hat{\mathfrak{F}}^i(\mathcal{P}^i(\theta) \times \mathbb{T}^n) = \hat{\mathcal{C}}^i(\theta) \tag{156}
\]

with \(\hat{\mathfrak{F}}^i\) injective.

Furthermore, \(\mathfrak{F}^i\) have the following form

\[
\begin{align*}
\hat{J} &= \hat{P}, & J_n &= v^i(P, Q_n), & \hat{\psi} &= \hat{Q} + z^i(P, Q_n), & \psi_n &= u^i(P, Q_n), & \text{for } 1 \leq i \leq 2N - 1, \\
\hat{J} &= \hat{P}, & J_n &= v^i(P, Q_n), & \hat{\psi} &= \hat{Q} + z^i(P, Q_n), & \psi_n &= Q_n + u^i(P, Q_n), & \text{for } i = 0, 2N,
\end{align*}
\]

with \(v^i, z^i, u^i\), 2\(\pi\)-periodic in \(Q_n\), \(|z^i| \leq C\theta\) and, for \(i = 0, 2N\), \(\sup |\partial_{Q_n} u^i| < 1\); for \(\theta > 0\), \(\mathfrak{F}^i\) have holomorphic extension

\[
\mathfrak{F}^i : (\mathcal{P}^i(\theta) \times \mathbb{T}^n) \ni T_\alpha \to D_{r_0} \times \mathbb{T}_{s_0}, \tag{159}
\]

with

\[
\rho = \theta/C, \quad \sigma = 1/C|\log \theta|. \tag{160}
\]

Finally, \(\mathfrak{F}^i\) “integrates” \(H_{\text{pend}}\), namely\(^{33}\):

\[
H_{\text{pend}} \circ \mathfrak{F}^i(P, Q) = H_{\text{pend}} \circ \hat{\mathfrak{F}}^i(P, Q) =: \mathcal{E}^i(P). \tag{161}
\]

Proposition 5.2 (i) Actually, as standard in the theory of integrable systems, one first introduces the action \(P_n^{(i)}\) through line integrals \(\oint pqd\) as function of energy \(E\) and then defines the integrated Hamiltonian \(\mathcal{E}^i\) inverting such function; in particular, one has

\[
P_n^{(i)} \left( \mathcal{E}^i(P_n, \hat{P}), \hat{P} \right) = P_n; \tag{162}
\]

for more details, see Appendix C.

(ii) Recalling the definition of \(E_{\pm}^{(i)}\) in (144) and setting

\[
\begin{align*}
a_{-}^{(2j-1)} &= 0, & a_{+}^{(2j-1)} &= P_n^{(2j-1)}(E_{-}^{(2j-1)}(\hat{P}) - 2\theta, \hat{P}), & 1 \leq j \leq N, \\
a_{-}^{(2j)} &= P_n^{(2j)}(E_{-}^{(2j)}(\hat{P}) + 2\theta, \hat{P}), & a_{+}^{(2j)} &= P_n^{(2j)}(E_{+}^{(2j)}(\hat{P}) - 2\theta, \hat{P}), & 1 \leq j < N, \\
a_{-}^{(0)} &= P_n^{(0)}(R_0^2 - M - 2\theta, \hat{P}), & a_{+}^{(0)} &= P_n^{(0)}(E_{-}^{(2N)}(\hat{P}) + 2\theta, \hat{P}), \\
a_{-}^{(2N)} &= P_n^{(2N)}(E_{-}^{(2N)}(\hat{P}) + 2\theta, \hat{P}), & a_{+}^{(2N)} &= P_n^{(2N)}(R_0^2 - M - 2\theta, \hat{P}),
\end{align*}
\]

one easily recognizes that

\[
\mathcal{P}^i(\theta) = \left\{ P = (\hat{P}, P_n) \mid \hat{P} \in \hat{D}, \quad a_{-}^{(i)}(\hat{P}, \theta) < P_n < a_{+}^{(i)}(\hat{P}, \theta) \right\} \subseteq \hat{D} \times \mathbb{R} \subseteq \mathbb{R}^n. \tag{164}
\]

\(^{32}\)Recall the notation introduced in (130).

\(^{33}\)The function \(\mathcal{E}^i(P)\) is actually the inverse of the action function \(E \to P_n^{(i)}(E, \hat{P})\); see (162) below and Appendix C.
5.7 Properties of the actions as functions of the energy

Next proposition, which is proved in [4], contains the fundamental properties of $P_n^{(i)}$ (defined in (162)), which will be heavily exploited in the following.

Let $P_n^{(i),0}$ be the function in (162) when $\eta = 0$, namely when $b, J_n, G$ in (134) vanish (see (304) below).

**Proposition 5.2** There exist suitably small, resp. large, constant $r > 0$, resp. $C > 1$, which, when (A3) holds, depend only on $n, s_0, r_0$, otherwise depend also on $F^0$ (introduced in (96)) such that, if $\eta \leq 1/C$ then the following holds. There exist real-analytic functions $\phi_{\pm}^{(i)}(\zeta, \hat{J})$, $\chi_{\pm}^{(i)}(\zeta, \hat{J})$, defined for $|\zeta| < r$, $\hat{J} \in \mathcal{D}_{r_0}$, with

$$
\sup_{|\zeta| < r, \hat{J} \in \mathcal{D}_{r_0}} \left( |\phi_{\pm}^{(i)}| + |\chi_{\pm}^{(i)}| \right) < C, \quad \sup_{|\zeta| < r, \hat{J} \in \mathcal{D}_{r_0}} \left( |\partial_j \phi_{\pm}^{(i)}| + |\partial_j \chi_{\pm}^{(i)}| \right) < C\eta,
$$

such that\(^{34}\)

$$
P_n^{(i)}\left( E_{\pm}^{(i)}(\hat{J}) \mp \zeta, \hat{P} \right) = \phi_{\pm}^{(i)}(\zeta, \hat{J}) + \zeta \log \zeta \chi_{\pm}^{(i)}(\zeta, \hat{J}), \quad \text{for } 0 < \zeta < r, \ \hat{J} \in \mathcal{D}.
$$

Moreover

$$
\chi_{\pm}^{(2j-1)}(0) = 0,
$$

and\(^{35}\)

$$
|\chi_{\pm}^{(i)}(0, \hat{J})| \geq \frac{1}{4\pi \sqrt{\|\partial_{\psi_0} \psi \|_{D,r_0,s_0}}} \geq 1/C > 0.
$$

Notice that (167) implies that $P_n^{(2j-1)}(E, \hat{P})$ has holomorphic extension on $\{|E - E_{\pm}^{(2j-1)}(\hat{J})| < r\} \times \mathcal{D}_{r_0}$, as well as $P_n^{(2j-1),0}(E)$ has holomorphic extension on $\{|E - E_{\pm}^{(2j-1),0}| < r\}$. Furthermore

$$
\hat{D} \times (E_{\pm}^{(i),0} + r/4, E_{\pm}^{(i),0} - r/4) \subset \{(\hat{P}, E) \text{ s.t. } \hat{P} \in \hat{D}, \ E_{\pm}^{(i)}(\hat{P}) < E < E_{\pm}^{(i)}(\hat{P})\}
$$

and

$$
\sup_{\hat{D}_{r_0} \times \{E_{\pm}^{(i),0} + (-1)^j r/2, E_{\pm}^{(i),0} - r/2\}} |P_n^{(i)}(\hat{P}, E) - P_n^{(i),0}(E)| < C\eta.
$$

Finally by Lemma C.4 we get

$$
\|\partial_{PP} E^{(i)}(\psi, \theta, \rho) \|_{\psi(\theta), \rho} \leq C/\rho, \quad \text{for } 0 \leq i \leq 2N.
$$

\(^{34}\)For $0 \leq i \leq 2N$ except $i = 0, 2N$ and the $+$ sign, since $E_{\pm}^{(0)}(\hat{J}) = E_{\pm}^{(2N)}(\hat{J}) = +\infty$ (recall (144)).

\(^{35}\)Recall the definition of $F$ in (134).

\(^{36}\)Recall (115).
5.8 The final canonical transformation

Let us define
\[ C_i^i(\theta) := \Phi^{(2\text{bis})}(C_i(\theta)) \], \[ C_i^i(\theta) := \Phi^{(2\text{bis})}(\hat{C}_i(\theta)) \].
(172)

We have that
\[ C_i^i(\theta) = \hat{C}_i(\theta) \times \mathbb{T}^{n-1} \],
(173)
since
\[ C_i^i(\theta) = \Phi^{(2\text{bis})}(C_i(\theta)) = \Phi^{(2\text{bis})}(\hat{C}_i(\theta)) \times \mathbb{T}^{n-1} = \hat{C}_i(\theta) \times \mathbb{T}^{n-1} . \]

Let us also define
\[ \tilde{\mathcal{F}}_i := \Phi^{(2\text{bis})} \circ \mathcal{F}_i . \]
(174)

By (157), (158) we have that \( \tilde{\mathcal{F}}_i \) has the form
\[ j = \hat{p} , \quad J_n = v_i(P, Q_n) , \quad \psi = \hat{Q} + z_i(P, Q_n) , \quad \psi_n = u_i(P, Q_n) , \quad \text{for } 1 \leq i \leq 2N - 1 , \]
\[ j = \hat{p} , \quad J_n = v_i(P, Q_n) , \quad \psi = \hat{Q} + z_i(P, Q_n) , \quad \psi_n = Q_n + u_i(P, Q_n) , \quad \text{for } i = 0, 2N . \]
(175)
(176)

Let us define the linear symplectic transformation of the form in (129) \( \Phi_{\text{lin}} : \mathbb{R}^{2n} \ni (J'', \psi'') \rightarrow (J', \psi') \in \mathbb{R}^{2n} \) generated by the generating function \( J' \cdot \psi'' + (J_n' + \frac{1}{\kappa}(\hat{A}k) \cdot J')\psi_n'' \) namely (recalling (15))
\[ J' = U_k J'' , \quad \text{with} \quad \dot{j}' = \dot{j}'' , \quad J_n' = J_n'' + \frac{1}{\kappa}(\hat{A}k) \cdot J' , \quad \psi'' = \psi' + \frac{\psi_n''}{\kappa} \hat{A}k , \quad \psi_n' = \psi_n'' . \]
(177)

Remark 5.3 Note that such map is only \( 2\pi \kappa \)-periodic in \( \psi_n'' \).

Note also that its inverse is
\[ \Phi_{\text{lin}}^{-1} \quad \text{with} \quad \dot{j}'' = \dot{j}' , \quad J_n'' = J_n' + \frac{1}{\kappa}(\hat{A}k) \cdot J' , \quad \psi'' = \psi' - \frac{\psi_n'}{\kappa} \hat{A}k , \quad \psi_n'' = \psi_n' \]
(178)
and that the operatorial norms of \( \Phi_{\text{lin}}, \Phi_{\text{lin}}^{-1} \) are bounded by some constant \( c(n) > 0 \)
\[ \| \Phi_{\text{lin}} \| , \| \Phi_{\text{lin}}^{-1} \| \leq c(n) . \]
(179)

Recalling the notation in (130) we introduce the volume preserving map \( \hat{\Phi}_{\text{lin}} : \mathbb{R}^n \times \mathbb{T}^1 . \) Recalling (126) we have that \( H^* \) defined in (125) satisfies
\[ H^*(J'', \psi_n'') = (\hat{H} \circ \hat{\Phi}_{\text{lin}})(J'', \psi_n'') - \frac{1}{\kappa} \| p^T \hat{A}^T \| ^2 , \quad \text{(namely (126))} \]
(180)
where \( \hat{H} \) was defined in (119). Moreover, recalling (134), we have
\[ (\hat{H} \circ \hat{\Phi}_{\text{lin}} \circ \hat{\Phi}^{(2\text{bis})})(J, \psi_n) = H_{\text{pend}}(J, \psi_n) + \frac{1}{\kappa} \| p^T \hat{A}^T \| ^2 \]
(181)
and, recalling (161)

\[
(H \circ \Phi_{\text{lin}} \circ \hat{\Phi}^{(2\text{bis})} \circ \hat{\Phi}^i)(P, Q_n) = E^{(i)}(P) + \frac{1}{\kappa} \| p^\perp A^T \hat{P} \|^2
\]  

(182)

Recall also the definition of $C'_i(\theta) = \hat{C}'_i(\theta) \times \mathbb{T}^{n-1}$ given in (172), (173) and of $\mathcal{P}^i(\theta)$ given in Proposition 5.1.

Set

\[
C^i(\theta) := \tilde{C}'(\theta) \times \mathbb{T}^{n-1}, \quad \tilde{C}'(\theta) := \Phi_{\text{lin}}(C'_i(\theta)) \times \mathbb{T}^{n-1}
\]

(183)

and

\[
C^i(\theta) := \Phi^{(1)}(C^i(\theta)).
\]

(184)

By (153) we get

\[
\operatorname{meas} \left( \left( U_k(\hat{D} \times (-R_0/2, R_0/2)) \times \mathbb{T}^n \right) \setminus \bigcup_{0 \leq i \leq 2N} C^i(\theta) \right) \lesssim \theta |\log \theta|.
\]

(185)

Recalling (164), one sees that one can define the sets $B^i(\theta)$ appearing in 2) of Theorem 4.1 as

\[
B^i(\theta) := \begin{cases} \mathcal{P}^i(\theta) & \text{if } 1 \leq i \leq 2N - 1 \\ U_k \mathcal{P}^i(\theta) & \text{if } i = 0, 2N \end{cases}
\]

(186)

Note that (104) follows by (154) and (17).

1) The oscillatory case: $1 \leq i \leq 2N - 1$

Fix $1 \leq i \leq 2N - 1$. Let us define the symplectomorphism of the form (129)

\[
\Phi_i : B^i(\theta) \times \mathbb{T}^n \rightarrow C^i(\theta), \quad \text{defined as}
\]

\[
\hat{J}' = \hat{p}, \quad \hat{J}'_{n} = v_1^i(p, q_n) - \frac{1}{\kappa} (\hat{A} k) \cdot \hat{p},
\]

\[
\hat{v}' = \hat{q} + z_1^i(p, q_n) + \frac{u_1^i(p, q_n)}{\kappa} \hat{A} k, \quad \hat{v}'_{n} = u_1^i(p, q_n),
\]

(187)

with $v_1^i, z_1^i, u_1^i$ defined in (175), (157). The fact that it is symplectic can be seen directly by (187) but also noting that locally

\[
\Phi_i = \Phi_{\text{lin}} \circ \Phi^{2\text{bis}} \circ \hat{\Phi}^i = \Phi_{\text{lin}} \circ \hat{\Phi}^i
\]

(188)

with $\Phi_{\text{lin}}$ defined in (177), $\Phi^{2\text{bis}}$ defined in (131) and $\hat{\Phi}^i$ defined in (155) (recall also (157)). By (183), (156) and (186), $\Phi_i$ is surjective on $C^i(\theta)$, namely

\[
\Phi_i(B^i(\theta) \times \mathbb{T}^n) = C^i(\theta).
\]

(189)

The injectivity is obvious. Note that, by Lemma D.5, (159), (160), (179), $\Phi_i$ has a holomorphic extension on

\[
\Phi_i : (B^i(\theta))_{\rho_0} \times \mathbb{T}^n_{\sigma_0} \rightarrow D_{\rho_0} \times \mathbb{T}^n_{\sigma_0}, \quad \rho_0 := c \frac{\theta}{C}, \quad \sigma_0 := c \frac{1}{C |\log \theta|}
\]

(190)

where $c$ is a (small) constant depending only on $n$. Applying again Lemma D.5, we also prove that

\[
\Phi_i : (B^i(\theta))_{\rho_*} \times \mathbb{T}^n_{\sigma_*} \rightarrow D_c \times \mathbb{T}^n_{\hat{s}},
\]

(191)

with $\rho_*, \sigma_*$, resp. $\hat{r}, \hat{s}$, defined in (106), resp. (122), taking $c$ large enough.

\[\text{Note that } U_k(\hat{D} \times (-R_0, R_0)) = U_k \hat{D} = \hat{D}^k \text{ defined in (120).}\]
2) The libration case: $i = 0, 2N$

Fix $i = 0, 2N$. Let us define the symplectomorphism in $G$ (recall (183) and (186))

$$
\Phi^i : \mathcal{B}^i(\theta) \times \mathbb{T}^n \to \mathcal{C}^i(\theta), \quad \text{defined as}
$$

$$
j' = \hat{p}, \quad J'_n = v^i_{*, k}(U_k^{-1} p, q_n) - \frac{1}{\kappa}(\hat{A}k) \cdot \hat{p},
$$

$$
\psi' = \hat{q} + z^i_{*, k}(U_k^{-1} p, q_n) + \frac{U_k^{-1}p, q_n}{\kappa} \hat{A}k, \quad \psi'_n = q_n + u^i(U_k^{-1} p, q_n),
$$

(192)

where, recall (15), $U_k^{-1}p = \left(\hat{p}, p_n + \frac{1}{\kappa}(\hat{A}k) \cdot \hat{p}\right)$,

with $v^i_{*}, z^i_{*}, u^i$ defined in (175), (177). The fact that it is symplectic can be seen directly by (192) but also noting that \textit{locally}

$$
\Phi^i = \Phi^{lin} \circ \Phi^{2bis} \circ \tilde{\mathfrak{g}} \circ \tilde{\Phi}^{lin} = \Phi^{lin} \circ \tilde{\mathfrak{g}} \circ \tilde{\Phi}^{lin},
$$

(193)

with $\Phi^{lin}$ defined in (177), $\Phi^{2bis}$ defined in (131) and $\tilde{\mathfrak{g}}$ defined in (155) (recall also (157)).

Note that $\Phi^i$ is injective, as it directly follows by the fact that so are $\Phi^{2bis}$ and $\tilde{\mathfrak{g}}$, and also surjective, namely (189) holds; indeed

$$
\Phi^i(\mathcal{B}^i(\theta) \times \mathbb{T}^n) \overset{(186)}{=} \Phi^i(\mathcal{B}^i(\theta) \times \mathbb{T}^1) \times \mathbb{T}^{n-1} \overset{(183)}{=} \Phi^{lin}(\tilde{\mathfrak{g}}(\Phi^i(\mathcal{B}^i(\theta) \times \mathbb{T}^1))) \times \mathbb{T}^{n-1}
$$

$$
= \Phi^{lin}(\tilde{\mathfrak{g}}(\Phi^i(\mathcal{B}^i(\theta) \times \mathbb{T}^1))) \times \mathbb{T}^{n-1} \overset{(156)}{=} \Phi^i(\mathcal{C}^i(\theta)) \times \mathbb{T}^{n-1}
$$

$$
\overset{(187)}{=} \Phi^{lin}(\mathcal{C}^i(\theta)) \times \mathbb{T}^{n-1} \overset{(183)}{=} \mathcal{C}^i(\theta) \times \mathbb{T}^{n-1} = \mathcal{C}^i(\theta)
$$

Reasoning as in the case $1 \leq i < 2N$, we get that $\Phi^i$ has a holomorphic extension as in (190). (191) holds as well.

### 5.9 Conclusion of the proof of part one of the Structure Theorem

We set

$$
\Psi^i := \Phi^{(1)} \circ \Phi^i,
$$

(194)

where $\Phi^{(1)}$ was defined in (117); therefore (106) holds by (191) and (123).

Then (103) follows by (185) recalling (14), (101) and since $\Phi^{(1)}$ preserve volume being symplectic. (105) follows by (189).

Recalling (119) and (182) we have that $h^{(i)} := \mathcal{H} \circ \Phi^{(1)} \circ \Phi^i$ can be written as

$$
h^{(i)}(p) = \begin{cases} 
    E^{(i)}(p) + \hat{h}_k(\hat{p}) & \text{if } 1 \leq i < 2N, \\
    (E^{(i)} + \hat{h}_k)(U_k^{-1} p) & \text{if } i = 0, 2N.
\end{cases}
$$

(195)

where

$$
\hat{h}_k(\hat{p}) := \kappa^{-1} \| p^\perp A_k^T \hat{p} \|^2.
$$

(196)

(107) follows. Finally (108) follows by (171), (195), (196), (10), (17).

This concludes the proof of part one of the Structure Theorem.
6 Proof of Part II of the Structure Theorem

A crucial fact is that the integrable Hamiltonian $h^{(i)}$ defined in (195) twists, namely (112) (together with the measure estimate (111)) holds. This will be a direct consequence of the following

**Proposition 6.1** Under the assumptions of Theorem 4.1 (in particular (109)), for any $\mu$ satisfying (110), there exists a subset $\tilde{P}^i(\mu) \subseteq P^i(0)$, satisfying

\[
\text{meas}(P^i(0) \setminus \tilde{P}^i(\mu)) \leq c \|k\|^4 n^{1/c},
\]

such that

\[
\left| \det \left[ \partial_{PP} \left( E^{(i)}(P) + \hat{h}_k(\hat{P}) \right) \right] \right| > \mu, \quad \forall \ 0 \leq i \leq 2N, \quad \forall \ P \in \tilde{P}^i(\mu).
\]

**Proof of (112).** One sees that, in analogy to (186), one can define the sets $\tilde{B}^i(\mu)$ appearing in 3) of Theorem 4.1 as

\[
\tilde{B}^i(\mu) := \begin{cases} 
\tilde{P}^i(\mu) & \text{if } 1 \leq i \leq 2N - 1, \\
U_k \tilde{P}^i(\mu) & \text{if } i = 0, 2N.
\end{cases}
\]

Recalling (186) and (17), by (197), we get (111). Finally (112) follows by (198), Lemma D.1 and noting that $\det U_k = 1$ by (15).

**Proof of Proposition 6.1.**

First we note that

\[
\det (\partial_{PP} \hat{h}_k) = 2^{n-1} \kappa^{-n}.
\]

Indeed by (20)

\[
\hat{h}_k(\hat{P}) + (P_n)^2 = \kappa^{-1} \|L_k P\|^2,
\]

(with $L_k$ defined in (12)) and, by Lemma D.1 and (18), we get

\[
da_k := \det \left( \partial_{PP} \left( \hat{h}_k(\hat{P}) + (P_n)^2 \right) \right) = 2^n \kappa^{-n}
\]

and, therefore, (200) follows.

The “twist” determinant as a function of the energy

Let us fix $0 \leq i \leq 2N$. Consider the analytic function

\[
d_k^i(E, \hat{P}) := \det \left[ \partial_{PP} \hat{h}_k(\hat{P}) + \partial_{PP} E^{(i)}(\hat{P}, P_n^{(i)}(\hat{P}, E)) \right],
\]

where, $P_n^{(i)}(E, \hat{P})$ is, by definition, the inverse map of $E^{(i)}(\hat{P}, P_n)$ (recall (162)). For brevity we will often omit to write the indexes $k$ and/or $^{(i)}$.

---

38 Decreasing w.r.t. $\mu$.

39 $c$ defined in Theorem 4.1.

40 This map can be obviously explicitly constructed, see subsection C.4 below.
The case $i$ odd; close to a maximum of the potential

Instead of use the variable $E$ we use, in the case of odd $i = 2j - 1$,

$$\zeta := E_{+}^{(2j-1)}(\hat{P}) - E,$$

(203)

where $E_{+}^{(2j-1)}(\hat{P})$ was defined in (144). In the variable $\zeta$, (note that $\partial E$ is equal to $\partial \zeta$, up to sign), recalling (166),

$$P_n(E, \hat{P}) = P_n(E_{+}^{(2j-1)}(\hat{P}) - \zeta, \hat{P}) = 1 \oplus \zeta \log \zeta,$$

(204)

where by $g = g_1 \oplus g_2$ we mean that there exist two functions $\varphi_1(\zeta, \hat{P}), \varphi_2(\zeta, \hat{P})$, analytic in a complex neighborhood of zero times $\hat{D}$ (depending only on $p$ defined in (137)), such that $g(\zeta, \hat{P}) = g_1(\zeta)\varphi_1(\zeta, \hat{P}) + g_2(\zeta)\varphi_2(\zeta, \hat{P})$. We get (for $i, j = 1, \ldots, n - 1$)

$$\partial_{E} P_n = 1 \oplus \log \zeta, \quad \partial_{P} P_n = \eta(1 \oplus \zeta \log \zeta), \quad \partial_{EE} P_n = \log \zeta \oplus \zeta^{-1}, \quad \partial_{EP} P_n = \eta(1 \oplus \log \zeta), \quad \partial_{P^2} P_n = \eta(1 \oplus \zeta \log \zeta).$$

(205)

Recalling (162), by the chain rule we get\(^{41}\)

$$\partial_{E} P_n = \frac{1}{\partial_{E} P_n}, \quad \partial_{P} P_n = -\frac{\partial_{P} P_n}{\partial_{EE} P_n}, \quad \partial_{P_n P_n} E = -\frac{\partial_{EE} P_n}{(\partial_{EE} P_n)^3},$$

$$\partial_{P_n P_n} E = \frac{\partial_{EE} P_n \partial_{P} P_n}{(\partial_{EE} P_n)^3} - \frac{\partial_{EE} P_n}{(\partial_{EE} P_n)^2} \in \mathbb{R}^{n-1},$$

$$\partial_{P_n P_n} E = -\frac{\partial_{P} P_n}{\partial_{EE} P_n} + 2 \frac{\partial_{P} P_n \partial(\partial_{EE} P_n)}{(\partial_{EE} P_n)^2} - \frac{\partial_{EE} P_n \partial_{P} P_n \partial_{P} P_n}{(\partial_{EE} P_n)^3} \in \text{Mat}_{(n-1) \times (n-1)},$$

(206)

where $E$ and $P_n$ are evaluated in $(P_n(E, \hat{P}), \hat{P})$ and $(E, \hat{P})$, respectively\(^{42}\).

By (205) and (206) we get (for $i, j = 1, \ldots, n - 1$)

$$\zeta(\partial_{E} P_n)^3 \partial_{P_n P_n} E = -\zeta \partial_{EE} P_n = 1 \oplus \zeta \log \zeta,$$

$$\zeta(\partial_{E} P_n)^3 \partial_{P_n P_n} E = \zeta \partial_{EE} P_n \partial_{P_n} P_n - \zeta \partial_{EE} P_n \partial_{P_n} P_n \partial_{E} E P_n = \eta(1 \oplus \zeta \log \zeta \oplus \zeta \log^2 \zeta),$$

$$\zeta(\partial_{E} P_n)^3 \partial_{P_n P_n} E = -\zeta(\partial_{EE} P_n)^2 \partial_{P_n} P_n + 2\zeta \partial_{EE} P_n \partial_{P_n} P_n \partial_{P_n} E P_n - \zeta \partial_{EE} E P_n \partial_{P_n} P_n \partial_{P_n} P_n$$

$$= \eta(1 \oplus \zeta \log \zeta \oplus \zeta \log^2 \zeta \oplus \zeta \log^3 \zeta).$$

(207)

The “rescaled” determinant $f$

Recalling that, by (203) we have

$$E = E_{+}^{(2j-1)}(\hat{P}) - \zeta,$$

we set

$$f(\zeta, \hat{P}) = f^{(i)}(\zeta, \hat{P}) := \zeta^n \left(\partial_{E} P_n(E_{+}^{(2j-1)}(\hat{P}) - \zeta, \hat{P}) \right)^3 \partial_{E} E_{+}^{(2j-1)}(\hat{P}) - \zeta, \hat{P}).$$

(208)

---

\(^{41}\)By $\partial_{P}$ we mean the row vector $(\partial_{P_1}, \ldots, \partial_{P_{n-1}})$, then $\partial_{P}^T$ is a column vector.

\(^{42}\)Or, which is equivalent, in $P$ and $(E(P), \hat{P})$, respectively.
We will omit the dependence on $k, i$. Note that
\[
\begin{align*}
d_k^i &= \det \left( \partial_{PP} h_k + \partial_{PE} E_k^{(i)} \right) = \det \left( \begin{array}{cc} \partial_{PP} h + \partial_{PE} E & \partial_T \left( \partial_{PP} E \right) \\ \partial_P \left( \partial_{PP} E \right) & \partial_P \left( \partial_{PP} E \right) \end{array} \right) \\
&= \partial_{PP} E \det \left( \partial_{PP} h + \partial_{PE} E \right) + \det \left( \begin{array}{cc} \partial_{PP} h + \partial_{PE} E & \partial_T \left( \partial_{PP} E \right) \\ \partial_P \left( \partial_{PP} E \right) & 0 \end{array} \right)
\end{align*}
\]
devloping the determinat w.r.t. the last column. By (207) we get
\[
\begin{align*}
\zeta^n \left( \partial_{EPn} \right)^3 \partial_{PP} E \det \left( \partial_{PP} h + \partial_{PE} E \right)
&= \zeta^{n-1} \sum_{\ell=0}^{3n-3} \varphi_\ell^{(1)} \left( \zeta, \hat{P} \right) \log^\ell \zeta + \eta \sum_{\ell=0}^{3n-5} \varphi_\ell^{(2)} \left( \zeta, \hat{P} \right) \log^\ell \zeta + R^{(1)},
\end{align*}
\]
where
\[
\inf_{\hat{P} \in \hat{D}} \left| \varphi_{3n-3} \right| \geq c(p) d_k > 0,
\]
with $d = d_k$ defined in (201) and
\[
R^{(1)} = O(\zeta^n) = \zeta^n \sum_{\ell=0}^{3n-2} \varphi_\ell^{(3)} \left( \zeta, \hat{P} \right) \log^\ell \zeta,
\]
for suitable $\varphi_\ell^{(1)}, \varphi_\ell^{(2)}, \varphi_\ell^{(3)}$, analytic and uniformly bounded\(^{43}\) in a uniform neighborhood of zero (and $\hat{P} \in \hat{D}$).

**Remark 6.1** We stress that, as a consequence of (165), all the estimates of this section are uniform in $\hat{P} \in \hat{D}$.

Moreover
\[
\begin{align*}
\zeta^n \left( \partial_{EPn} \right)^3 \partial_{PP} E \det \left( \partial_{PP} h + \partial_{PE} E \right)
&= \eta^2 \left( 1 + \zeta \log \zeta + \zeta^2 \log^2 \zeta \right)^2 \left( 1 + \zeta \log \zeta + \zeta^2 \log^2 \zeta + \zeta \log^3 \zeta \right)^{n-2} \\
&= \eta^2 \sum_{\ell=0}^{3n-4} \varphi_\ell^{(4)} \left( \zeta, \hat{P} \right) \log^\ell \zeta + R^{(2)},
\end{align*}
\]
where
\[
R^{(2)} = O(\zeta^n) = \zeta^n \sum_{\ell=0}^{3n-2} \varphi_\ell^{(5)} \left( \zeta, \hat{P} \right) \log^\ell \zeta,
\]
for suitable $\varphi_\ell^{(4)}, \varphi_\ell^{(5)}$, analytic and uniformly bounded in a uniform neighborhood of zero (and $\hat{P} \in \hat{D}$). Therefore
\[
f = f_k^{(i)} = \zeta^{n-1} \sum_{\ell=0}^{3n-3} \varphi_\ell \left( \zeta, \hat{P} \right) \log^\ell \zeta + \eta \sum_{\ell=0}^{3n-4} \chi_\ell \left( \zeta, \hat{P} \right) \log^\ell \zeta + R
\]  
\(^{43}\)Here and in the following of this section by “uniformly bounded” we mean “bounded by a constant depending only on $p$ in

31
where, by (209),
\[
\inf_{\hat{P} \in \hat{D}} |\varphi_{3n-3}(0, \hat{P})| \geq c(p) \delta_k > 0,
\]
and
\[
R = O(\zeta^n) = \zeta^n \sum_{\ell=0}^{3n-2} \psi_{\ell}(\zeta, \hat{P}) \log^\ell \zeta,
\]
for suitable $\varphi_{\ell}, \chi_{\ell}, \psi_{\ell}$ analytic and uniformly bounded in a uniform neighborhood of zero (and $\hat{P} \in \hat{D}$).

**A class of useful linear operators**

We now consider the linear operator
\[
L := \zeta \partial_\zeta
\]
and, recursively, $L^\ell := L \circ L^{\ell-1}$. We have
\[
L \left( \zeta^h \sum_{\ell=0}^{\ell_0} g_{\ell}(\zeta, \hat{P}) \log^\ell \zeta \right) = \zeta^h \sum_{\ell=0}^{\ell_0} \tilde{g}_{\ell}(\zeta, \hat{P}) \log^\ell \zeta, \quad \text{with} \quad \tilde{g}_{\ell_0}(0, \hat{P}) = h g_{\ell_0}(0, \hat{P}),
\]
\[
L^{\ell_0} \left( \sum_{\ell=0}^{\ell_0} g_{\ell}(\zeta, \hat{P}) \log^\ell \zeta \right) = \ell_0! g_{\ell_0}(\zeta, \hat{P}) + \zeta \sum_{\ell=0}^{\ell_0} \tilde{g}_{\ell}(\zeta, \hat{P}) \log^\ell \zeta,
\]
\[
L^{\ell_1} \left( \sum_{\ell=0}^{\ell_0} g_{\ell}(\zeta, \hat{P}) \log^\ell \zeta \right) = \zeta \sum_{\ell=0}^{\ell_0} \tilde{g}_{\ell}(\zeta, \hat{P}) \log^\ell \zeta, \quad \text{when} \quad \ell_1 > \ell_0,
\]
for (different) suitable $\tilde{g}_{\ell}$. Moreover
\[
\partial_\zeta \left( \zeta^h \sum_{\ell=0}^{\ell_0} g_{\ell}(\zeta, \hat{P}) \log^\ell \zeta \right) =
\]
\[
h g_{\ell_0}(\zeta, \hat{P}) \zeta^{h-1} \log^{\ell_0} + \zeta^{h-1} \sum_{\ell=0}^{\ell_0-1} \left( h g_{\ell}(\zeta, \hat{P}) + (\ell + 1) g_{\ell+1}(\zeta, \hat{P}) \right) \log^\ell \zeta + \zeta^h \sum_{\ell=0}^{\ell_0} \tilde{g}_{\ell}(\zeta, \hat{P}) \log^\ell \zeta.
\]
In particular by (213), (215), (216) we get
\[
(\partial_\zeta \circ L^{\ell_1}) \left( \sum_{\ell=0}^{\ell_0} g_{\ell}(\zeta, \hat{P}) \zeta^h \log^\ell \zeta \right) = \sum_{\ell=0}^{\ell_0} \tilde{g}_{\ell}(\zeta, \hat{P}) \zeta^h \log^\ell \zeta, \quad \text{when} \quad \ell_1 > \ell_0,
\]
for suitable $\tilde{g}_{\ell}$.

Introduce the linear differential operator (w.r.t. $\zeta$) of order $3n^2 - 3n$:
\[
\mathcal{L} := L^{3n-3} (\partial_\zeta \circ L^{3n-3})^{n-1}.
\]
Non-degeneracy of the derivatives of $f$

Let us decompose $f$ in \((210)\) as

$$f = \zeta^{n-1} \phi_{3n-3}(0, \hat{P}) \log^{3n-3} \zeta + \tilde{f} + \tilde{R},$$

with

$$\tilde{f} := \zeta^{n-1} \sum_{\ell=0}^{3n-4} \phi_\ell(\zeta, \hat{P}) \log^\ell \zeta + \eta \sum_{\ell=0}^{3n-4} \chi_\ell(\zeta, \hat{P}) \log^\ell \zeta,$$

and\(^44\)

$$\tilde{R} := \zeta^n \left( \frac{\phi_{3n-3}(\zeta, \hat{P}) - \phi_{3n-3}(0, \hat{P})}{\zeta} \right) \log^{3n-3} \zeta + R$$

We want to evaluate $L f$. We have

$$L \left( \zeta^{n-1} \phi_{3n-3}(0, \hat{P}) \log^{3n-3} \zeta \right) = (3n-3)!((n-1)!)^{3n-2} \phi_{3n-3}(0, \hat{P}) + \zeta \sum_{\ell=0}^{3n-2} \tilde{\phi}_\ell(\zeta, \hat{P}) \log^\ell \zeta,$$

for suitable $\tilde{\phi}_\ell$. By \((215)\) and \((217)\) we get

$$L \tilde{f} = \zeta \sum_{\ell=0}^{3n-4} \tilde{\chi}_\ell(\zeta, \hat{P}) \log^\ell \zeta,$$

for suitable $\tilde{\chi}_\ell$. Finally, by \((213)\) and \((216)\), we have that

$$L \tilde{R} = \zeta \sum_{\ell=0}^{3n-2} \tilde{\psi}_\ell(\zeta, \hat{P}) \log^\ell \zeta,$$

(with $R$ defined in \((212)\)) for suitable $\tilde{\psi}_\ell$. Recollecting we get

$$\left(L f\right)(\zeta, \hat{P}) = c_k(\hat{P}) + \zeta \sum_{\ell=0}^{3n-2} \tilde{f}_\ell(\zeta, \hat{P}) \log^\ell \zeta,$$

for suitable $\tilde{f}_\ell(\zeta, \hat{P})$ analytic and uniformly bounded in a uniform neighborhood of zero (and $\hat{P} \in \hat{D}$) and where

$$c_k(\hat{P}) := (3n-3)!((n-1)!)^{3n-2} \phi_{3n-3}(0, \hat{P}),$$

satisfies, by \((211)\),

$$\inf_{\hat{P} \in \hat{D}} |c_k(\hat{P})| \geq c(p) d_k > 0.$$ \(\text{(221)}\)

By \((219)\)

$$\inf_{0 < \zeta \leq \zeta_0} \inf_{\hat{P} \in \hat{D}} |\left(\mathcal{L}f\right)(\zeta, \hat{P})| \geq c(p) d_k / 2 > 0,$$

for a suitable $\zeta_0 = \zeta_0(p) > 0$. \(\text{(222)}\)

\(^{44}\)Note that the function $\frac{\phi_{3n-3}(\zeta, \hat{P}) - \phi_{3n-3}(0, \hat{P})}{\zeta}$ is analytic.
By (218) we have that
\[(\mathcal{L}f)(\zeta, \hat{P}) = \sum_{d=0}^{m_n} a_d(\zeta) \partial^d f(\zeta, \hat{P}), \quad \text{where } m_n := 3n^2 - 3n, \tag{223}\]
for suitable polynomials \(a_d(\zeta)\). Then by (222), taking in case \(\zeta_0\) smaller, we get
\[\inf_{0 < \zeta \leq \zeta_0} \sup_{\hat{P} \in \hat{D}} \max_{1 \leq d \leq m_n} |\partial^d f(\zeta, \hat{P})|/d! \geq c(p) d_k > 0 \tag{224}\]
(with a smaller \(c(p)\)).

**The measure of sublevels of \(f\)**

We need the following result, which is proved in the appendix

**Lemma 6.1** Let \(f \in C^{m+1}([a, b])\) and assume that for some \(m \geq 1\)
\[\min_{x \in [a, b]} \max_{1 \leq d \leq m} |\partial^d f(x)|/d! =: \xi_m > 0. \tag{225}\]
Then for \(0 < \mu < 1\)
\[\text{meas}\{x \in [a, b] : |f(x)| \leq \mu\} \leq \frac{m(M + 1)(b - a + 2\mu^{1/m+1})}{\xi_m} \mu^{1/(m+1)}, \]
with \(M := \max_{x \in [a, b]} 2 \leq d \leq m+1 |\partial^d f(x)|/d!\)

We apply Lemma 6.1 with
\[f \rightsquigarrow f, \quad m \rightsquigarrow m_n, \quad a \rightsquigarrow \zeta_1, \quad b \rightsquigarrow \zeta_0, \quad \xi_m \rightsquigarrow c(p) d_k \quad \text{(in (224))}, \quad x \rightsquigarrow \zeta, \quad \text{with } \zeta_1 \text{ to be chosen later. By (210) we have that}^{45}\]
\[\sup_{\hat{P} \in \hat{D}} \sup_{|\zeta| \leq |\zeta_0|/2} |f(\zeta, \hat{P})| \ll |\log^{2n-2} \zeta_1| \leq 1/\zeta_1 \]
for \(\zeta_1\) small enough. Then, by Cauchy estimates, we get that \(M\) in Lemma 6.1 satisfies
\[M \ll 1/\zeta_1^{m_n+2}. \]
By Lemma 6.1 we get, for \(0 < \mu < 1\),
\[\text{meas}\{\zeta \in [\zeta_1, \zeta_0] : |f(\zeta, \hat{P})| \leq \mu\} \ll \mu^{1/(m_n+1)} /d_k \zeta_1^{m_n+2}, \]
for every\(^{46} \hat{P} \in \hat{D}\). We can optimize the choice of \(\zeta_1\) taking
\[\zeta_1 = \mu^{1/(m_n+1)} /\zeta_1^{m_n+2}, \quad \text{namely } \zeta_1 := \mu^{1/(m_n+1)} /\zeta_1^{m_n+2}, \]
\(^{45}\)Denoting, as usual, the \(\zeta_1/2\)-complex-neighborhood of the real interval \([\zeta_1, \zeta_0]\) by \([\zeta_1, \zeta_0]_{|\zeta_1|/2}\).
\(^{46}\)Note that the hidden constant in \(\ll\) is independent of \(\hat{P} \in \hat{D}\).
so that

$$\text{meas}(\{\zeta \in (0, \zeta_0] \ : \ |f_k(\zeta, \hat{P})| \leq \mu\}) \leq \frac{\mu^{\frac{n}{m(n+1)(m+n+3)}}}{d_k},$$

for every $\hat{P} \in \hat{D}$. Since by the first equality in (205)

$$\zeta^n (\partial_E P_n(\hat{E}^{(2j-1)}_+ (\hat{P}) - \zeta, \hat{P}))^{3n} \leq 1$$

for $\zeta_0$ small, we get, recalling (208),

$$\text{meas}(\{\zeta \in (0, \zeta_0] \ : \ |d_k(\hat{E}_{++}^{(2j-1)}(\hat{P}) - \zeta_0, \hat{P})| \leq \mu\}) \leq \frac{\mu^{\frac{n}{m(n+1)(m+n+3)}}}{d_k}, \quad (226)$$

for every $\hat{P} \in \hat{D}$, where $d_k$ was defined in (202). Recalling (203), we get

$$\text{meas}(\{E \in [\hat{E}^{(2j-1)}_+ (\hat{P}) - \zeta_0, \hat{E}^{(2j-1)}_+ (\hat{P})] \ : \ |d_k(E, \hat{P})| \leq \mu\}) \leq \frac{\mu^{\frac{n}{m(n+1)(m+n+3)}}}{d_k}, \quad (227)$$

for every $\hat{P} \in \hat{D}$.

Conclusions of the proof in the case $i$ odd, close to maxima of the potential

Recalling (202) and (161), we have

$$\det \left[ \partial_PP \left( \hat{h}_k(\hat{P}) + E_k^{(i)}(P) \right) \right] = d_k(E_k^{(i)}(P), \hat{P}). \quad (228)$$

Recalling that, by (163), we have $E^{(2j-1)}_+(\hat{P}) = E^{(2j-1)}_+(\hat{P}, a^{(2j-1)}_+(\hat{P}))$, we set

$$r_0(\hat{P}) := a^{(2j-1)}_+(\hat{P}) - P_n \left( E^{(2j-1)}_+(\hat{P}) - \zeta_0, \hat{P} \right) = P_n \left( E^{(2j-1)}_+(\hat{P}) - \zeta_0, \hat{P} \right) - P_n \left( E^{(2j-1)}_+(\hat{P}) - \zeta_0, \hat{P} \right). \quad (229)$$

Note that by (229) and (333) (which relies on Proposition 5.2) there exists $r_0 > 0$, which, when (A3) holds, depends only on $n, s_0, r_0$, otherwise depend also on $F^0$, such that

$$r_0(\hat{P}) \geq r_0, \quad \hat{P} \in \hat{D}. \quad (230)$$

For every $\hat{P} \in \hat{D}$, we now consider the change of variable $P_n = P_n(E, \hat{P})$. By (227) and (228), and noting that (recall (204))

$$0 < \partial_E P_n(E, \hat{P}) \leq |\log(E^{(2j-1)}_+(\hat{P}) - E)|, \quad (231)$$

35
we get\(^47\)
\[
\text{meas}\left(\{ P_n \in [a_n^{(2j-1)}(\hat{P}) - r_0, a_n^{(2j-1)}(\hat{P})] \mid \det \left[ \partial_{PP}(\hat{h}_k(\hat{P}) + E^{(2j-1)}(P)) \right] \leq \mu \} \right) \ll \mu^a d_k^{-2},
\]
\[
a_n := \frac{1}{27n^6} < \frac{1}{m_n(m_n + 1)(m_n + 3)} \tag{232}
\]
(recall (223)) uniformly in \(\hat{P} \in \hat{D}\). Then by Fubini theorem we get
\[
\text{meas}\left(\{ P \mid P_n \in [a_n^{(2j)}(\hat{P}) - r_0, a_n^{(2j)}(\hat{P})] \mid \hat{P} \in \hat{D}, \mid \det[\partial_{PP}(\hat{h}_k + E^{(2j-1)})] \leq \mu \} \right) \ll \mu^a d_k^{-2}. \tag{233}
\]
This conclude the proof close to the maxima of the odd case \(i = 2j - 1\).

**The case \(i\) even close to maxima**

The proof is analogous to the odd case, leading to
\[
\text{meas}\left(\{ P \mid P_n \in [a_n^{(2j)}(\hat{P}) - r_0, a_n^{(2j)}(\hat{P})] \mid \hat{P} \in \hat{D}, \mid \det[\partial_{PP}(\hat{h}_k + E^{(2j)})] \leq \mu \} \right) \ll \mu^a d_k^{-2}, \tag{234}
\]
\[
\text{meas}\left(\{ P \mid P_n \in (a_n^{(2j)}(\hat{P}), a_n^{(2j)}(\hat{P}) + r_0] \mid \hat{P} \in \hat{D}, \mid \det[\partial_{PP}(\hat{h}_k + E^{(2j)})] \leq \mu \} \right) \ll \mu^a d_k^{-2}. \tag{235}
\]

**Far away from maxima**

We now study the point far away from maxima, where we note that the second derivatives of the functions \(E^{(i)}\) are uniformly bounded (see Lemma C.3). We have
\[
\det \left[ \partial_{PP}(\hat{h}_k(\hat{P}) + E^{(i)}(P)) \right] = \det \left[ \partial_{PP}\hat{h}_k(\hat{P}) \right] \cdot \partial_{PP}E^{(i)}(P) + O(\vartheta) \tag{209} = 2^{n-1}n^{-n} \partial_{PP}E^{(i)}(P) + O(\vartheta),
\]
valid in the sets\(^48\)
\[
\begin{align*}
\left\{ P = (\hat{P}, P_n) \mid P_n \in (0, a_n^{(2j-1)}(\hat{P}) - r_0/2], \hat{P} \in \hat{D} \right\}, \tag{237}
\left\{ P = (\hat{P}, P_n) \mid P_n \in [a_n^{(2j)}(\hat{P}) + r_0/2, a_n^{(2j)}(\hat{P}) - r_0/2], \hat{P} \in \hat{D} \right\}, \tag{238}
\end{align*}
\]
when \(i\) is odd, respectively even.

We have to distinguish the cases when (A2) or (A3) hold.

\(^47\) Let \(E(\hat{P}) \subseteq [E_n^{(2j-1)}(\hat{P}) - \zeta_0, E_n^{(2j-1)}(\hat{P})]\) and \(P(\hat{P}) \subseteq [a_n^{(2j-1)}(\hat{P}) - r_0, a_n^{(2j-1)}(\hat{P})]\) be the sets whose measures are estimated in (227) and (232); so that \(P(\hat{P}) = P_n(\hat{E}(\hat{P}), \hat{P})\). Then, denoting by \(Z(\hat{P}) \subseteq (0, \zeta_0)\) the set whose measure \(\mu_0 : = \text{meas}(Z(\hat{P}))\) is estimated in (226), we have
\[
\text{meas}(P(\hat{P})) = \int_{P(\hat{P})} dP_n = \int_{Z(\hat{P})} \partial_{PP}P_n(E, \hat{P})dE \tag{231} \leq \int_{Z(\hat{P})} |\log(E_n^{(2j-1)}(\hat{P}) - E)|dE
\]
\[
\leq \int_{Z(\hat{P})} |\log E|dK = \int_{Z(\hat{P}) \cap (0, \mu_0]} |\log E|dK + \int_{Z(\hat{P}) \cap (\mu_0, \zeta_0]} |\log E|dK \leq \mu_0 \log \mu_0 + |\log \zeta_0| \mu_0 \leq \mu_0 \log \mu_0.
\]
Therefore by (226)
\[
\text{meas}(P(\hat{P})) \ll \left( \frac{\mu_0 m_{n+1}}{m_{n+1}^2 + \mu_0 \log \mu_0} / d_k \right).
\]

\(^48\) Recall that \(a_n^{(2j-1)}(\hat{P}) = 0\) as defined in (163).
The case when (A2) holds

We note that, by Proposition 5.2, we can extend the Hamiltonian $E^{(2j-1)}(P)$ in a complex 4$r_0$-neighborhood\(^{49}\) of zero so that $E^{(2j-1)}(P)$ has holomorphic extension on the complex domain

$$\hat{D}_{r_0} \times \left[ -2r_0, a_{+}^{(2j-1),0} - \frac{r_0}{4} \right]_{r_0/8},$$

(239)

Then (236) holds in the domain (239).

Let us define the intervals

$$\Gamma^{(2j-1)} := \left[ -\frac{r_0}{4}, a_{+}^{(2j-1),0} - \frac{r_0}{4} \right], \quad \Gamma^{(2j)} := \left[ a_{+}^{(2j),0} + 3r_0/4, a_{+}^{(2j)}(\hat{P}) - 3r_0/4 \right]$$

(240)

Since we are far away from hyperbolic equilibria, by Lemma C.3 we get that

$$\sup_{\hat{D}_{r_0} \times \Gamma^{(2j-1)}_{r_0/8}} \| \partial_{P\mu} E^{(i)}(P) \| \leq 1, \quad 0 \leq i \leq 2N.$$  

(241)

Note that (236) holds in $\hat{D} \times \Gamma^{(i)}$ for every $i$ and also that, for $\eta$ small enough,

$$\left( \hat{D} \times \Gamma^{(2j-1)} \right) \cup \left\{ P \mid P_n \in \left[ a_{+}^{(2j-1)}(\hat{P}) - r_0, a_{+}^{(2j-1)}(\hat{P}) \right], \hat{P} \in \hat{D} \right\} \supseteq \Psi^{(2j-1)}(0)$$

(242)

$$\left( \hat{D} \times \Gamma^{(2j)} \right) \cup \left\{ P \mid P_n \in \left( a_{+}^{(2j)}(\hat{P}), a_{+}^{(2j)}(\hat{P}) + r_0 \right] \cup \left[ a_{+}^{(2j)}(\hat{P}) - r_0, a_{+}^{(2j)}(\hat{P}) \right), \hat{P} \in \hat{D} \right\} = \Psi^{(2j)}(0)$$

(defined in (164)).

Let us consider now the finitely many non constant analytic functions\(^{50}\)

$$P_n \to \partial_{P_n} E^{(i),0}(P_n), \quad 0 \leq i \leq 2N.$$  

By analyticity we have that there exist $m = m(F^0) \geq 1$ and $\xi = \xi(F^0) > 0$, depending on the function $F^0$ introduced in (96), such that

$$\min_{P_n \in \Gamma^{(i)}} \max_{1 \leq d \leq m} | \partial_{P_n} E^{(i),0}(P_n) |/d! \geq 2\xi > 0, \quad \forall 0 \leq i \leq 2N.$$  

By Lemma C.3

$$\sup_{\hat{D}_{r_0} \times \Gamma^{(i)}_{r_0/8}} \| \partial_{P\mu} E^{(i)}(P) - \partial_{P_n} E^{(i),0}(P_n) \| \leq C\eta, \quad \forall 0 \leq i \leq 2N,$$

(243)

where $C > 1$ was defined in Proposition 5.2. By Cauchy estimates we get, for $\eta$ small enough depending on $F^0$,

$$\inf_{P \in D \times \Gamma^{(i)}} \max_{1 \leq d \leq m} | \partial_{P_n} E^{(i)}(P) |/d! \geq \xi > 0, \quad \forall 0 \leq i \leq 2N.$$  

(244)

Recalling (236) we have

$$\det \left[ \partial_{PP}(\hat{h}_{k}(\hat{P}) + E^{(i)}(P)) \right] = 2^{n-1} \kappa^{-n} \partial_{P_n} E^{(i)}(P) + O(\partial) = 2^{n-1} \kappa^{-n} \left( \partial_{P_n} E^{(i)}(P) + O(\kappa^n\eta) \right).$$

\(^{49}\)Reducing, in case, $r_0 = r_0(\eta) > 0$.

\(^{50}\)E^{(i),0} being the inverse of $T^{(i),0}_n$.  

37
By (109) we get that the term\(^51\) \(O(\kappa^n \eta)\) above is negligible, together with its derivatives of any order; then by (244) we obtain
\[
\inf_{P \in \hat{D} \times \mathbb{I}^{(i)}} \max_{1 \leq d \leq m} \left| \frac{d}{d!} \det \left[ \partial_{PP} \left( \hat{h}_k(\hat{P}) + E^{(i)}(P) \right) \right] \right| \geq 2^{n-2} \kappa^{-n} \xi > 0, \quad \forall 0 \leq i \leq 2N. \tag{245}
\]
We can apply Lemma 6.1 uniformly in \(\hat{P} \in \hat{D}\) with
\[
f(\cdot) \sim \det \left[ \partial_{PP} \left( \hat{h}_k(\hat{P}) + E^{(i)}(\hat{P}, \cdot) \right) \right], \quad x \sim P_n, \quad [a, b] \sim \mathbb{I}^{(i)}, \quad \xi_m \sim 2^{n-2} \kappa^{-n} \xi
\]
and \(M < 1\) (by (241) and Cauchy estimates); then we get
\[
\text{meas} \left( \left\{ P_n \in \mathbb{I}^{(i)} \mid \left| \det \left[ \partial_{PP} \left( \hat{h}_k(\hat{P}) + E^{(i)}(P) \right) \right] \right| \leq \mu \right\} \right) \leq C\kappa^n \mu^{\frac{1}{m(m+1)}},
\]
uniformly in \(\hat{P} \in \hat{D}\). Then by Fubini theorem we have
\[
\text{meas} \left( \left\{ P \in \hat{D} \times \mathbb{I}^{(i)} \mid \left| \det \left[ \partial_{PP} \left( \hat{h}_k(\hat{P}) + E^{(i)}(P) \right) \right] \right| \leq \mu \right\} \right) \leq C\kappa^n \mu^{\frac{1}{m(m+1)}}, \tag{246}
\]
Recalling the definition of \(a_n\) in (232) we choose
\[
c \geq \max\{6^3 n^6, m(m+1)\}
\]
Recalling (242) by (233),(234), (235) and (246) we get (197) when (A2) holds.

**The case when (A3) holds**

We are in the cosine-like case (recall (100)); by Lemma C.5 below (see (341)), (236) and (109) we have that
\[
\kappa^{-n} \leq \left| \det \left[ \partial_{PP} \left( \hat{h}_k(\hat{P}) + E^{(i)}(P) \right) \right] \right|
\]
for all the values of \(P\) in (237), (238) respectively. Then, by (110)
\[
\left| \det \left[ \partial_{PP} \left( \hat{h}_k(\hat{P}) + E^{(i)}(P) \right) \right] \right| > \mu \tag{247}
\]
again for all the values of \(P\) in (237), (238) respectively. Then (232), (233), (235), (247) prove Proposition 6.1 in the case when (A3) holds. This concludes the proof of Proposition 6.1.

## 7 Proof of the Main Theorem

In this final section we will show the existence of a high density (almost exponential) of Kolmogorov’s tori of different topologies in all neighbourhoods of simple resonances.

---

\(^51\)More precisely \(|O(\kappa^n \eta)| \leq C\kappa^n \eta.\)
7.1 Application of the Structure Theorem

Fix \( k \in \mathbb{Z}_{\geq 0}^n \). In this subsection we will apply the Structure Theorem 4.1 to the “effective part” of the Hamiltonian \( H_k \) in (80). Namely we apply Theorem 4.1 with

\[
\mathcal{H}(I', \varphi') \sim \frac{1}{r_c} \|I'\|^2 + \frac{1}{\delta_k} G^k(\varsigma_k I', k \cdot \varphi'), \quad \mathcal{G}(I', x) \sim \frac{1}{\delta_k} G^k(\varsigma_k I', x), \quad F^0 \sim \frac{1}{\delta_k} F^k,
\]

\[
\mathcal{D} \sim D^k, \quad \dot{D} \sim \frac{1}{\delta_k} \dot{Z}_k, \quad R_0 \sim R_{0,k} := \frac{K^{\nu+1}}{\sqrt{28K}|k|}, \quad r_0 \sim 4, \quad s_0 \sim s/4. \tag{248}
\]

where \( \delta_k, G^k \) and \( F^k, \varsigma_k, D^k, \dot{Z}_k \), were defined in (76), (66), (79), (81), (29). We note that with such positions we have that \( r'_k \) defined in (81) satisfies

\[
r'_k = K^{\nu+1}/\sqrt{8\delta_k} \geq r' = cn|k|_{\infty}r_0, \tag{249}
\]

where \( r' \) was defined in (95). Indeed (249) follows by (77) taking \( \varepsilon \) small enough (recall (24)). We also note that \( s_* \) defined in (61) satisfies

\[
s_* \geq 3s_0, \tag{250}
\]

taking \( \varepsilon \) small enough (recall again (24)). Note that, by (87)

\[
\|g\|_{s_0} \leq \coth^n \left( s'/s_0 - 2 \right) |g|_{s'}, \quad \forall g \in A_{s'}^n. \tag{251}
\]

**Remark 7.1** Here we take \( \varepsilon \leq \varepsilon_0 \) small enough. Obviously \( \varepsilon_0 \) is small uniformly on \( k \).

We now verify that the hypotheses of Theorem 4.1 hold.

Let us start with (A1). We see that (97) follow by (52) (69), resp. (75), if \( |k| \leq K_s(\delta) \), resp. \( K_s(\delta) < |k| \leq K \), taking

\[
\eta_* := \coth(s/24) \frac{2^{10} n e^s}{\delta s K^{\nu+1}}. \tag{252}
\]

Moreover the inequality in (98) holds (recall the definition of \( R_0 \) in (248)) taking \( \varepsilon \) small enough (recall (24)).

We now check that (A3) holds when \( K_s(\delta) < |k| \leq K \), taking the constant \( c \) in (5) large enough. Indeed we have that by (72) and (251) (with \( n = 1 \), also recall (76)) \( \delta_k^{-1} F^k \) is \( \gamma \)-cosine-like taking

\[
\gamma := \coth(s/24) \left( \frac{2|k|^{n+3} e^{-|k|s/4}}{\delta} + \frac{2^{10} n e^s}{\delta s K^{2\nu+1}} \right). \tag{253}
\]

Then (100) holds taking \( c \) in (5) large enough depending on \( n \) in order to adjust the first addendum and, then, \( \varepsilon \) small enough in order to adjust the second addendum.

We finally check that (A2) holds when \( |k| \leq K_s(\delta) \). We first note that in this case \( \delta_k^{-1} F^k = F^k \) since \( \delta_k = 1 \) (recall (76)). By (68) and (87) we get

\[
\|F^k\|_{s_0} \leq \coth(3s/8) =: M. \tag{52}
\]

\[\text{\textsuperscript{52}}\text{Recall also (81) and (251) with } n = 1.\]
Moreover every $F^k$ is $(\beta_k, M)$-Morse-non-degenerate for some $\beta_k > 0$ by (P2) and (P3) of Definition 1.1. Then we can take
\[
\beta := \min_{|k| \leq K, \epsilon} \beta_k > 0,
\]
since we are taking the minimum over a finite set. We conclude that every $F^k$ is $(\beta, M)$-Morse-non-degenerate.

**Remark 7.2** A crucial fact is that we can choose the constant $c$ appearing in Theorem 4.1, uniformly in $k \in Z_{s,K}^n$. Indeed for all the cases with $K_{s}(\delta) < |k| \leq K$, $c$ only depends on $n, s_0, r_0$, namely, in view of (248), on $n$ and $s$. In the finite number of cases $|k| \leq K_{s}(\delta)$ the constant $c$ actually depends on $k$ (since it depends on the particular for of $F^k$); however we can the minimum over all this finite number of constants obtaining a constant $c = c(n, s) > 0$ for which Theorem 4.1 holds uniformly in $k \in Z_{s,K}^n$.

Finally we have that (109) and, a fortiori (102), are simultaneously, namely for every $k \in Z_{s,K}^n$, satisfied taking $\epsilon$ small enough (recall (24)). Then, as a corollary of Theorem 4.1, we get the following result (where we denote by $N_k$ the number of maxima/minima of $F^k$).

**Proposition 7.1** Let $\epsilon$ be small enough. Let $\theta > 0$ and $53$
\[
0 < \mu \leq 1/c K^{2n},
\]
where $c = c(n, s) > 0$ was defined in Remark 7.2. For every $k \in Z_{s,K}^n$ and $0 \leq i \leq 2N_k$ there exist

i) disjoint subsets $C^\epsilon_k(\theta, \mu) \subseteq D^\epsilon_k \times T^n$ decreasing w.r.t. $\theta$ and $\mu$, with $54$
\[
\text{meas}\left( D^\epsilon_k \times T^n \setminus \bigcup_{0 \leq i \leq 2N_k} C^\epsilon_k(\theta, \mu) \right) \leq c(\theta |\ln \theta| + K^{4\nu} \mu^{1/\epsilon}) ;
\]
ii) $B^\epsilon_k(\theta, \mu) \subseteq \mathbb{R}^n$, decreasing w.r.t. $\theta$ and $\mu$, with $55$
\[
diam(B^\epsilon_k(\theta, \mu)) \leq 2c(R_0, k + \epsilon_k^{-1} \text{diam}(Z_k)) ;
\]
iii) holomorphic symplectomorphisms $56$
\[
\Psi^i_k : (B^i_k(\theta, \mu))_{\rho^i} \times T^n, \rightarrow D^\epsilon_k \times T^n, \quad \text{with} \quad \rho^i := \frac{\theta}{cK^{n-1}}, \quad \sigma^i := \frac{1}{cK^{n-1}|\log \theta|},
\]
\[
\Psi^i_k \left( B^i_k(\theta, \mu) \times T^n \right) = \mathcal{C}_k(\theta, \mu),
\]
\[
\text{such that} \quad H_k \circ \Psi^i(p, q) = h_k^{(i)}(p) + f_k^{(i)}(p, q).
\]

---

53 The constant $c$ is defined in Remark 7.2.
54 $D^\epsilon_k$ was defined in (83).
55 With $R_0, k$ defined in (248) and $c$ in (17) respectively.
56 $v^i_k$ defined in (81).
57 $H_k$ defined in (80).
with
\[ \| f_k^{(i)} \|_{\mathcal{B}_k(\theta, \mu, \rho', \sigma')} \leq e^{\frac{\delta}{\log \epsilon^3}} \] (260)

Furthermore
\[ \left| \det \left( \partial_{pp} h_k^{(i)}(p) \right) \right| \geq \mu, \quad \forall \ 0 \leq i \leq 2N, \ |k| \leq K, \ \forall \ p \in \mathcal{B}_k(\theta, \mu). \] (261)

Finally
\[ \| \partial_{pp} h^{(i)} \|_{\mathcal{B}_k(\theta, \mu, \rho')} \leq c/\theta, \quad \text{for} \ 0 \leq i \leq 2N. \] (262)

7.2 Application of the KAM theorem

Let us start by stating a quantitative version, suitable for our purposes, of the classical KAM Theorem; for references, discussions and extensions we refer to [1] and references therein; see also [6] for a nice divulgative account of KAM theory.

**Theorem 7.1** Fix \( n \geq 2 \) and \( \tau > n - 1 \). Let \( D \) be any non–empty, bounded subset of \( \mathbb{R}^n \). Let
\[ H(p, q) := h(p) + f(p, q) \]
real–analytic on \( D_{r_0} \times T^n \), for some \( r_0 > 0 \) and \( 0 < s \leq 1 \), and having finite norms:
\[ M := \| \partial_{pp} h \|_{D, r_0}, \quad \| f \|_{D, r_0, s}. \] (263)

Assume that the frequency map \( p \in D \rightarrow \omega = \partial_p h \) is a local diffeomorphism, namely, assume:
\[ d := \inf_D | \det \partial_{pp} h | > 0. \] (264)

Define
\[ m := \frac{d}{M^s} \leq 1. \] (265)

Then there exists a positive constant \( c < 1 \), depending only on \( n \) and \( \tau \), such that, if
\[ \epsilon := \frac{\| f \|_{D, r_0, s}}{M r_0^2} \leq c m^s s^{4s+4}, \] (266)
then the following holds. Define
\[ \alpha := \frac{c}{m s^{3s+3}} (M r_0) \sqrt{\epsilon}, \quad \hat{\tau} := m^2 r_0, \quad r_{\epsilon} := \frac{1}{c m} \sqrt{\epsilon} r_0. \] (267)

Then, there exists a positive measure set \( T_\alpha \subseteq (D_{r_{\epsilon}} \cap \mathbb{R}^n) \times T^n \) formed by “primary” Kolmogorov’s tori; more precisely, for any point \((p, q) \in T_\alpha\), \( \phi_{p, q}^\tau(p, q) \) covers densely an \( H \)–invariant, analytic, Lagrangian torus, with \( H \)–flow analytically conjugated to a linear flow with \((\alpha, \tau)\)–Diophantine frequencies \( \omega = h_p(p_0) \), for a suitable \( p_0 \in D \); each of such tori is a graph over \( T^n \) \( r_{\epsilon} \)–close to the unperturbed trivial.
Finally, the Lebesgue outer measure of \((D \times \mathbb{T}^n) \setminus T_\alpha\) is bounded by:

\[
\text{meas} \left( (D \times \mathbb{T}^n) \setminus T_\alpha \right) \leq C \sqrt{\epsilon}
\]

with

\[
C := \left( \max \left\{ \varepsilon^n r_0, \text{diam} D \right\} \right)^n \cdot \frac{1}{c m^{n+5} s^{3+3}}.
\]

**Remark 7.3** The statement of the above quantitative KAM theorem is as Theorem 1 in [5] with the following minor simplification. In Theorem 1 of [5] appear the quantity \(\lambda := L M\), where \(M\) is defined in (263) and \(L\) denotes a suitable uniform Lipschitz constant of the local complex inverse of the “frequency map” \(p \rightarrow \omega = \partial_p h(p)\) (compare formula (9) of [5]); since one can show that \(1 \leq \lambda \leq 2 \cdot n! \varepsilon^{-1}\) (see formula (14) of [5]), we substitute everywhere \(\lambda\) with 1 or \(2 \cdot n! \varepsilon^{-1}\) in Theorem 7.1, obtaining a slightly weaker formulation of Theorem 1 of [5].

**KAM tori in \(\Omega^0\)**

We now apply Theorem 7.1 to the Hamiltonian \(H_{(0)}\) in (56). It is immediate to see, thanks to (57), that KAM tori cover all \(\Omega^0\) (defined in (30)) up to a set of measure \(\varepsilon^{5 \log \varepsilon}\).

**KAM tori in \(\Omega^1\)**

We now want to apply Theorem 7.1 to the Hamiltonians \(h_k^{(i)}(p) + f_k^{(i)}(p, q)\) defined in (259), for all \(k \in \mathbb{Z}^n_{0,K}, 0 \leq i \leq 2N_k\).

The objects appearing in Theorem 7.1 have to be replaced by the following:

\[
h \leadsto h_k^{(i)}, \quad f \leadsto f_k^{(i)}, \quad D \leadsto B_k^i(\theta, \mu), \quad r_0 \leadsto \rho' = \frac{\theta}{cK^{n-1}}, \quad s \leadsto \sigma' := \frac{1}{cK^{n-1}},
\]

(recall (257)).

By (260), it follows immediately that

\[
\|f\|_{b, r_0, s} \leq \varepsilon^{5 \log \varepsilon}. \tag{271}
\]

By (262) we get

\[
M \leq c / \theta, \tag{272}
\]

where, here and in the following,

\[
c = c(n, s) \geq 1,
\]

are suitably large (different) constants depending only on \(n\) and \(s\). By (261) we have that \(d\) defined in (264) satisfies

\[
d \geq \mu.
\]

So we get that \(m\) in (265) satisfies

\[
m = \frac{d}{M^n} \geq \mu \theta^n / c. \tag{273}
\]

42
Now, we choose the parameters $\mu$ and $\theta$ as follows\footnote{Where $c$ is the constant defined in Proposition 7.1.}
\[
\mu = \theta := \varepsilon |\log \varepsilon|^2.
\] (274)

With such choices the condition of the KAM Theorem 7.1 are met: in particular (265) follows by (273) and also (266), which is implied by the stronger condition
\[
\|f\|_{D,r_0,s} \leq c \mu^{8n+1} K^{2-2n} s^{4r+1},
\]
which holds by (271) (recall also (24)), taking $\varepsilon$ small enough.

Noting that, by (256), (248) and (79),
\[
\text{diam}(D) \leq c \frac{K^{\nu+1}}{\varepsilon_k} + \zeta^{-1} \text{diam}(\tilde{Z}_k) \leq c \frac{K^{\nu+1} \sqrt{\varepsilon}}{\zeta_k} + \zeta^{-1} \text{diam}(\tilde{D}) \leq c \frac{K^{\nu+1} \sqrt{\varepsilon}}{\zeta_k} + \zeta^{-1} K^n,
\] (275)
the maximum in (269) can be estimated by
\[
\frac{c}{\zeta_k} \varepsilon \|f\|_{D,r_0,s} T^{n(3n+1)-1/2} d^n s^{4r+3} \leq \frac{c}{\zeta_k} \varepsilon |\log \varepsilon|^3 \leq \frac{1}{\zeta_k} \frac{1}{\varepsilon} \varepsilon |\log \varepsilon|^3,
\]
for $\varepsilon$ small enough. Then by (234) (and (284)) we get that the measure of the non-torus set in $(D^k \times \mathbb{T}^n)$ is estimated by\footnote{Note that $\zeta_k \leq 1$, see (79).}
\[
\frac{1}{\zeta_k} \varepsilon^2 |\log \varepsilon|,
\]
for $\varepsilon$ small enough. Therefore the measure of the non-torus set in $\bigcup_{k \in \mathbb{Z}_n^*} D^k \times \mathbb{T}^n$ is estimated by
\[
\frac{1}{\zeta_k} \varepsilon |\log \varepsilon|,
\]
for $\varepsilon$ small enough (recall (24)). By (85), (78) and (79) we get that the measure of the non-torus set in $\Omega^1 \times \mathbb{T}^n$ is bounded by $\varepsilon |\log \varepsilon|$.

Acknowledgment. We are indebted to V. Kaloshin, G. Loddi, A. Neishtadt and A. Sorrentino.

A Properties of the class of non–degenerate potentials

Proof of Proposition 1.1.

- $\mathcal{P}_s \cap \mathbb{B}_s^n \in \mathcal{B}$ and $\mu_s(\mathcal{P}_s \cap \mathbb{B}_s^n) = 1$

We shall prove that, for every $\delta > 0$, the measure of the sets of potentials $f$ that do not satisfy, respectively, (P1), (P2), (P3), (P4) is, respectively, $O(\delta^2), 0, 0, 0$, the result will follow letting $\delta \to 0$.

First, by the identification (7), the measure of the set of potentials $f$ that do not satisfy (P1) with a given $\delta$ is bounded by $\delta^2 \sum_{k \in \mathbb{Z}_n^*} |k|^{-n-3}$.\footnote{Note that $\zeta_k \leq 1$, see (79).}
Next, recall that properties (P2), (P3) and (P4) concern only a finite number of $k$, i.e., $k \in \mathbb{Z}_+^n$, $|k| \leq K_\ast(\delta)$.

To show that the set of potentials that do not satisfy (P2) has $\mu_\ast$-measure zero it is enough to check that, for every $k \in \mathbb{Z}_+^n$, $|k| \leq K_\ast(\delta)$, the set $\mathcal{E}^{(k)}$ of $f$’s for which $F^k$ has a degenerate critical point has zero $\mu_\ast$-measure.

Fix $k \in \mathbb{Z}_+^n$, $|k| \leq K_\ast(\delta)$ and denote points in $E^{(k)}$ by $(\zeta, \varphi)$, where $\zeta = f_k$ and $\varphi = \{f_h\}_{h \neq k}$. Write

$$F^k(\xi) = \zeta e^{i\xi} + \bar{\zeta} e^{-i\xi} + G(\xi), \quad \text{where} \quad \zeta := f_k \quad \text{and} \quad G(\xi) := \sum_{|j| \geq 2} f_{jk} e^{ij\xi}. \quad (276)$$

Now, one checks immediately that $\partial_\xi F^k(\xi_0) = 0$ is equivalent to $\zeta = \zeta(\xi_0, \varphi) = \frac{1}{2} e^{-i\xi_0}(iG'(\xi_0) + G''(\xi_0))$, which, as $\xi_0$ varies in $\mathbb{T}$, describes a smooth closed “critical” curve in $\mathbb{C}$, as a side remark, notice that $\zeta$ depends on $\varphi$ only through the Fourier coefficients $f_{jk}$ with $|j| \geq 2$. Thus the section $E^{(k)}_\varphi = \{\zeta \in D : (\zeta, \varphi) \in E^{(k)}\}$ is (a piece of) a smooth curve in $D = \{z \in \mathbb{C} : |z| \leq 1\}$, hence $\mu_\ast(E^{(k)}_\varphi) = 0$ for every $\varphi$ and by Fubini’s theorem $\mu_\ast(E^{(k)}) = 0$, as claimed.

An analogous result $^{63}$ holds true for (P3).

Regarding (P4) we have that the three real equations

$$\partial_\xi F^k(\xi_0) = 0 = \partial_\xi^2 F^k(\xi_0), \quad F^k(\xi_1) - F^k(\xi_2) = 0, \quad \text{for} \quad \xi_1, \xi_2 \in \mathbb{T},$$

can be rewritten (recall (276)) the complex equation

$$\zeta = \zeta(\xi_1, \xi_2, \varphi) = \frac{i}{2(e^{i\xi_1} - e^{i\xi_2})} \left( G'(\xi_1) - G'(\xi_2) + iG(\xi_1) - iG(\xi_2) \right) \quad (277)$$

and the real one

$$\frac{1}{2} (e^{i(\xi_1 - \xi_2)} - e^{-i(\xi_1 - \xi_2)}) \left( G'(\xi_1) - G'(\xi_2) + iG(\xi_1) - iG(\xi_2) \right) - (e^{i\xi_1} - e^{-i\xi_2}) \left( e^{-i\xi_2} G'(\xi_1) - e^{-i\xi_1} G'(\xi_2) \right) = g(\xi_1, \xi_2, \varphi) = (1 - \cos(\xi_1 - \xi_2)) \left( G'(\xi_1) + G'(\xi_2) \right) - \sin(\xi_1 - \xi_2) \left( G(\xi_1) - G(\xi_2) \right) = 0 \quad (278)$$

We claim that, for every fixed $\varphi$, the analytic function $(\xi_1, \xi_2) \mapsto g(\xi_1, \xi_2, \varphi)$ is not identically zero and, therefore, the set $R_\varphi$ of its zeros has zero measure. Assume by contradiction that $g$ is identically zero. Then $g(\xi_2 + \varepsilon, \xi_2, \varphi) \equiv 0$ for every $\xi_2$ and $\varepsilon$, in particular, evaluating the order fourth term of the Taylor expansion in $\varepsilon$ around $\varepsilon = 0$, we get $\frac{1}{12} \left( G''''(\xi_2) + G''(\xi_2) \right) = 0, \forall \xi_2$. The general (real) solution of the above equation is $G(\xi_2) = ce^{i\xi_2} + ce^{-i\xi_2} + c_0$, with $c \in \mathbb{C}$, $c_0 \in \mathbb{R}$, which contradicts the expression of $G$ in (276). Therefore, for every fixed $\varphi$, the image of the zero measure set $R_\varphi$ through the Lipschitz function $(\xi_1, \xi_2) \mapsto (\zeta(\xi_1, \xi_2, \varphi)$ (defined in (277)) has zero measure in $D$. Then we conclude as in the case (P2) above.

\[ \bullet \ \mathcal{P}_s \text{ contains an open subset } \mathcal{P}_s' \text{ which is dense in the unit ball of } \mathcal{A}_n. \]

Let us define $\mathcal{P}_s'$ as $\mathcal{P}_s$ but with the difference that (P1) is replaced by the stronger condition $^{64}$

---

$^{62}$Recall the definition of $F^k$ in (4).

$^{63}$In this case the critical curve is given by $\{\zeta = (-b(\xi) \pm \sqrt{b^2(\xi) - c(\xi)} + iG'(\xi))e^{-i\xi} / 2, \xi \in \mathbb{R}, b^2(\xi) \geq c(\xi)\}$, where $b(\xi) := (G'''(\xi) - G''(\xi)) / 2$ and $c(\xi) := -G''(\xi)G'''(\xi) + 5G'(\xi) + (G''(\xi) + G'''(\xi))^2 / 3$.

$^{64}$Note that $\mu_\ast(\mathcal{P}_n) = 0$. 

44
Let us first prove that \( P_s' \) is open. Let \( f \in P_s' \). We have to show that there exists \( \rho > 0 \) such that if \( |g|_s < \rho \), then \( f + g \in P_s' \). Fix \( \delta > 0 \) such that \( (P1') \) holds and choose \( \rho < \delta \) small enough such that \( |K_s(\delta)| > K_s(\delta') - 1 \), where \( \delta' := \delta - \rho \) and \( \lfloor \cdot \rfloor \) denotes integer part. Then, it is immediate to verify that \( |\theta| > K_s(\delta') \implies |\theta| > K_s(\delta') \), namely \( f + g \) satisfies \( (P1') \) (with \( \delta' \) instead of \( \delta \)). Since \( (P2) \), \( (P3) \) and \( (P4) \) are “open” conditions and regard only a finite number of \( k \) it is simple to see that they are satisfied also by \( f + g \) for \( \rho \) small enough. Then \( f + g \in P_s' \) for \( \rho \) small enough.

Let us now show that \( P_s' \) is dense in the unit ball of \( A^n_s \). Take \( f \) in the unit ball of \( A^n_s \) and \( 0 < \theta < 1 \). We have to find \( \tilde{f} \in P_s' \) with \( |\tilde{f} - f|_s \leq \theta \). Let \( \delta := \theta/4 \) and denote by \( f_k \) and \( \tilde{f}_k \) (to be defined) be the Fourier coefficients of, respectively, \( f \) and \( \tilde{f} \). We, then, let \( \tilde{f}_k = f_k \) unless one of the following two cases occurs:

- \( k \in \mathbb{Z}^n_s, |k| > K_s(\delta) \) and \( |f_k|e^{|k|s} < \delta \), in which case, \( \tilde{f}_k = \delta e^{-|k|s} \),

- \( k \in \mathbb{Z}^n_s, |k| \leq K_s(\delta) \) and \( F^k \) (defined as in (4)) does not satisfy either \( (P2) \), \( (P3) \) or \( (P4) \), in which case, \( \tilde{f}_k \) is chosen at a distance less than \( \theta e^{-|k|s} \) from \( f_k \) but outside the critical curves defined above.

At this point, it is easy to check that \( \tilde{f} \in P_s' \) and is \( \theta \)-close to \( f \).

- \( P_s \) is prevalent.

Consider the following compact subset of \( \ell^n_{\mathbb{C}} \): let \( K := \{ z = \{ z_k \} \in \mathbb{Z}^n_\mathbb{C} : z_k \in D_{1/|k|} \} \), where \( D_{1/|k|} := \{ w \in \mathbb{C} : |w| \leq 1/|k| \} \), and let \( \nu \) be the unique probability measure supported on \( K \) such that, given Lebesgue measurable sets \( A_k \subseteq D_{1/|k|} \), with \( A_k \neq D_{1/|k|} \) only for finitely many \( k \), one has

\[
\nu \left( \prod_{k \in \mathbb{Z}^n_\mathbb{C}} A_k \right) := \frac{1}{\prod_{k \in \mathbb{Z}^n_\mathbb{C} \setminus A_k \neq D_{1/|k|}} |k|^2 \pi \text{meas}(A_k) .
\]

The isometry \( j_s \) in (7) naturally induces a probability measure \( \nu_s \) on \( A^n_s \) with support in the compact set \( K_s := j_s^{-1} K \). Now, for \( \delta > 0 \), let \( P_{s,\delta} \) be the set of \( \delta \)'s in the unit ball of \( A^n_s \) satisfying \( (P1) \)–\( (P4) \), so that \( P_s = \bigcup_{\delta > 0} P_{s,\delta} \). Reasoning as in the proof of \( \mu_s(P_s) \equiv 1 \), one can show that \( \nu_s(P_{s,\delta}) \geq 1 - \text{const} \delta^2 \). It is also easy to check that, for every \( g \in A^n_s \), the translated set \( P_{s,\delta} + g \) satisfies \( \nu_s(P_{s,\delta} + g) \geq \nu_s(P_{s,\delta}) \). Thus, one gets \( \nu_s(P_{s,\delta} + g) = \nu_s(P_s) = 1 \), \( \forall g \in A^n_s \), which means that \( P_s \) is prevalent. (recall footnote ??)

### B Proof of the Normal Form Lemma 3.1

Given a function \( \phi \) we denote by \( X^t_\phi \) the hamiltonian flow at time \( t \) generated by \( \phi \) and by “ad” the linear operator \( u \mapsto \text{ad}_\phi u := \{ u, \phi \} \) and \( \text{ad}^\ell \) its iterates:

\[
\text{ad}_\phi^0 u := u, \quad \text{ad}_\phi^\ell u := \text{ad}^{\ell-1}_\phi u, \quad \ell \geq 1,
\]

45
as standard, \( \{ \cdot, \cdot \} \) denotes Poisson bracket\(^{65} \).

Recall the identity (“Lie series expansion”)
\[
\left. u \circ X^1_\phi \right|_{t=0} = \sum_{\ell \geq 0} \frac{1}{\ell!} \text{ad}_\phi^\ell u = \sum_{\ell=0}^\infty \frac{\partial^\ell_t \left( u \circ X^1_t \phi \right)}{\ell!} \bigg|_{t=0},
\]
valid for analytic functions and small \( \phi \).

By standard Cauchy estimates, we get (compare, e.g., Lemma B4 of [14])
\[\text{Lemma B.1}\]

For \( 0 < r - \rho < r_0 \), \( 0 < s - \sigma < s_0 \), \( \rho, \sigma > 0 \)
\[\{ f, g \}|_{r-\rho, s-\sigma} \leq \frac{n}{\rho \sigma} \left( \frac{1}{(r_0 - r + \rho) \sigma} + \frac{1}{(s_0 - s + \sigma) \rho} \right) |f|_{r_0, s_0} |g|_{r, s}. \]

Summing the Lie series in (279) and using Lemma B5 of [14], we get, also,
\[\text{Lemma B.2}\]

Let \( 0 < \rho < r \leq r_0 - \rho \) and \( 0 < \sigma < s \leq s_0 - \sigma \).
\[\hat{\vartheta} := \frac{4n|\phi|_{r_0, s_0}}{\rho \sigma} \leq 1. \]

Then,
\[|u \circ X^1_\phi - u|_{r-\rho, s-\sigma} \leq \sum_{\ell \geq 1} \frac{1}{\ell!} |\text{ad}_\phi^\ell u|_{r-\rho, s-\sigma} \leq \hat{\vartheta}|u|_{r, s}. \]

Given \( K \geq 2 \) and a lattice \( \Lambda \), recall the definition of \( f^b \) in (47) and define
\[f^K := f - f^b = T_K \mathbb{P}_\Lambda^* f,\]
so that we have the decomposition (valid for any \( f \)):
\[f = f^b + f^K, \quad f^b := \mathbb{P}_\Lambda f + T_K \mathbb{P}_\Lambda^* f, \quad f^K := T_K \mathbb{P}_\Lambda^* f. \]

\[\text{Lemma B.3}\]

Consider a real-analytic Hamiltonian
\[H = H(y, x) = h(y) + f(y, x) \quad \text{analytic on } D_r \times T^n_s. \]

Suppose that \( D_r \) is \((\alpha, K)\) non-resonant modulo \( \Lambda \) for \( h \) (with \( K \geq 2 \)). Assume that
\[\hat{\vartheta} := \frac{2^{\frac{5}{2}} nK^3}{\alpha rs} |f^K|_{r, s} \leq 1. \]

Then there exists a real-analytic symplectic change of coordinates
\[\Psi : D_{r_+} \times T^n_{s_+} \to D_r \times T^n_s, \quad r_+ := r(1 - 1/2K), \quad s_+ := s(1 - 1/K^2),\]
such that
\[H \circ \Psi = h(y) + f_+(y, x), \quad f_+ := f^b + f_* \]
with
\[|f_*|_{r_+, s_+} \leq 2\hat{\vartheta}|f|_{r, s}. \]

\(^{65}\)Explicitly, \( \{ u, v \} = \sum_{i=1}^n (u_{x_i}v_{y_i} - u_{y_i}v_{x_i}) \).
Notice that, by (284) and (288) (and the fact that $|f - f^K|_{r,s} \leq |f|_{r,s}$), one has
\[ f^+_K = f^*_K, \quad |f^+_r|_{r,s} = |f^* + f - f^K|_{r,s} \leq |f^*|_{r,s} + |f|_{r,s} \leq (1 + 2\tilde{\vartheta})|f|_{r,s}. \tag{289} \]
Notice also that
\[ f^*_r - f^* \overset{(287)}{=} f^*_r \implies |f^*_r - f^*|_{r,s} \leq |f^*|_{r,s} \leq 2\tilde{\vartheta}|f|_{r,s}. \tag{290} \]

**Proof** (of Lemma B.3) Let us define
\[ \phi = \phi(y, x) := \sum_{|m| \leq K, m \notin \Lambda} \frac{f_m(y)}{h(y) \cdot m} e^{im \cdot x}, \quad \Psi := X_\phi^1, \]
and note that $\phi$ solves the homological equation
\[ \{h, \phi\} + f^K = 0. \tag{291} \]
Since $D_r$ is $(\alpha, K)$ non–resonant modulo $\Lambda$
\[ |\phi|_{r,s} \leq |f^K|_{r,s}/\alpha. \tag{292} \]
Then, one has
\[ H \circ \Psi = h + f^* + f_* \]
with
\[ f_* = (h \circ \Psi - h - \{h, \phi\}) + (f \circ \Psi - f). \]
In order to estimate $f_*$ we now use Lemma B.2 with parameters
\[ r_0 \sim r, \quad s_0 \sim s, \quad r \sim r(1 - 1/4K), \quad s \sim s(1 - 1/2K^2), \quad \rho \sim r/4K, \quad \sigma \sim s/2K^2. \]
With these choices it is $\check{\vartheta} = \tilde{\vartheta}$, and, by (286) $\check{\vartheta} \leq 1$. Thus, (281) holds and Lemma B.2 applies. By (282) we get (288) noting that
\[ h \circ \psi - h - \{h, \phi\} = \sum_{\ell \geq 2} \frac{1}{\ell!} \text{ad}_{\phi}^\ell h - \sum_{\ell \geq 1} \frac{1}{(\ell + 1)!} \text{ad}_{\phi}^\ell \{h, \phi\} \overset{(291)}{=} -\sum_{\ell \geq 1} \frac{1}{(\ell + 1)!} \text{ad}_{\phi}^\ell f^K, \]
which implies (again by (282)) that
\[ |h \circ \psi - h - \{h, \phi\}|_{r,s} \leq \check{\vartheta}|f^K|_{r,s} \leq \tilde{\vartheta}|f|_{r,s}. \]
Finally, applying again Lemma B.2 with $u = f$, by (282), we get $|f \circ \Psi - f|_{r,s} \leq \check{\vartheta}|f|_{r,s}$, concluding the proof of Lemma B.3.

**Proof of the Normal Form Lemma 3.1** Denote by
\[ \bar{K} := \lceil K \rceil := \min\{n \in \mathbb{Z} : n \geq K\}, \tag{293} \]
47
the ceiling function of $K$. The idea is to construct $\Psi$ by applying $\bar{K}$ times Lemma B.3. To do this, fix $1 \leq j < K$ and make the following inductive assumption:

Let

$$
\begin{align*}
    f_0 & := f, \quad H_0 := h + f_0 = H, \quad \rho := \frac{r}{4K}, \quad \sigma := \frac{s}{2KK}, \\
    r_i & := r - 2i\rho, \quad s_i := s - 2i\sigma, \quad |\cdot|_i := |\cdot|_{r_i,s_i},
\end{align*}
$$

and assume that there exist, for $1 \leq i \leq j$, real–analytic symplectic transformations

$$
\Psi_{i-1} : D_{r_i} \times T^n_{s_i} \to D_{r_{i-1}} \times T^n_{s_{i-1}},
$$

such that

$$
H_i := H_{i-1} \circ \Psi_{i-1} =: h + f_i
$$

satisfies, for $1 \leq i \leq j$, the estimates

$$
\vartheta_i \leq (4\delta |f|_{r,s})^{i+1}, \quad |f_i^2 - f_{i-1}^2|_i \leq 2\vartheta_{i-1} |f_{i-1}|_{i-1},
$$

where

$$
\vartheta_i := \delta |f^K|_i \quad \text{with} \quad \delta := \frac{2^{\frac{5}{2}}nK^3}{\nu r s}.
$$

Notice that, recalling (45), it is

$$
\vartheta_* = 2^4 \delta |f|_{r,s} \implies 4\delta |f|_{r,s} = \frac{\vartheta_*}{4} < \frac{1}{4} < 1.
$$

Let us first show that the inductive hypothesis is true for $j = 1$. Indeed, by (298), $\delta |f^K|_0 \leq \delta |f|_0 < 1/16 < 1$, therefore, by the definition of $\delta$ and $\vartheta$ in, respectively, (297) and (286), we see that we can apply Lemma B.3 with $f = f_0$, being $\vartheta = \vartheta_0 = \delta |f^K|_0$. Thus, we obtain the existence of $\Psi_0$ so that $H_1 := H_0 \circ \Psi_0 = h + f_1$ and, by (289) and (288),

$$
\vartheta_1 = \delta |f^K|_1 \leq (2\vartheta_0 |f|_0) = 2 \delta^2 |f^K|_0 |f|_0 \leq 2(\delta |f|_0)^2 \leq (4\delta |f|_0)^2,
$$

showing that the first inequality in (296) holds for $i = 1$, the second inequality follows from (290).

Now, let us assume that the inductive hypothesis holds true for $1 \leq i < j < K$ and let us prove that it holds also for $i = j + 1$. First, let us check that

$$
|f_i| \leq 2 |f|_{r,s}, \quad \forall 1 \leq i \leq j.
$$

Indeed, by the estimate in (289), one has that $|f_i| \leq (1 + 2\vartheta_i)|f_{i-1}|_{i-1}$, for all $1 \leq i \leq j$, which, iterated, yields

$$
\begin{align*}
|f_i| & \leq |f_0| \prod_{\ell=1}^i (1 + 2\vartheta_\ell) = |f|_{r,s} \exp \left( \sum_{\ell=1}^i \log(1 + 2\vartheta_\ell) \right) \\
& \leq |f|_{r,s} \exp \left( 2 \sum_{\ell=1}^i 2^{-2\ell} \right) \leq |f|_{r,s} \exp \left( 2 \sum_{\ell=1}^\infty 2^{-2\ell} \right) \leq 2 |f|_{r,s}.
\end{align*}
$$

48
Now, by (297), (296) with \( i = j \) (inductive assumption) and (298), we have that \( \vartheta_j < 1 \). Thus, we can apply Lemma B.3 to \( f_j \) (with \( \tilde{\vartheta} = \vartheta_j \)) and get a symplectic transformation \( \Psi_j \) such that

\[
\vartheta_{j+1} := \delta|f_{j+1}|_{j+1} \leq \delta(2\vartheta_j|f_j|_j) \leq (4\delta|f_\ast|_\ast) \vartheta_j \leq (4\delta|f_\ast|_\ast)^{j+2},
\]

which is the first inequality in (296) with \( i = j+1 \), the second inequality comes from (290). This completes the proof of the induction.

Now, we can conclude the proof of Lemma 3.1: recall (293) and define

\[
\Psi := \Psi_0 \circ \cdots \circ \Psi_{K-1}.
\]

Notice that, by (294), \( r_\ast = r/2 = r_\ast \) and \( s_\ast = s(1-1/K) = s_\ast \) and notice that, by the induction, it is

\[
H \circ \Psi = H_{K-1} \circ \Psi_{K-1} \overset{(295)}{=} h + f_\ast =: h + f^0 + f_\ast.
\]

But, then, since \( T_K P_N f^0 = (f^0)^K = 0 \) (for any \( f \)), using that \( K \geq 2 \), we have

\[
|T_K f_\ast |_{r_\ast s_\ast} = |f^0|_{K} \overset{(297)}{=} |f^0|_K \leq \delta^{-1} \vartheta_K \leq \delta^{-1} (4\delta|f_\ast|_\ast)^K |f_\ast|_0 \leq (2^3 \delta|f_\ast|_0)^K |f_\ast|_0 \overset{(298)}{=} (2^{-1} \vartheta_\ast)^K |f_\ast|_0 \leq (2^{-1} \vartheta_\ast)^K |f_\ast|_0,
\]

proving the second estimates in (48).

Finally, (using again that \( K \geq 2 \) and that \( \vartheta_\ast < 1 \))

\[
|f_\ast|_{r_\ast s_\ast} \leq \quad \overset{(301)}{\leq} \quad \overset{(300)}{=} \quad \overset{(298)}{=} \quad \overset{(296)}{=} \quad \overset{(284)}{=} \quad \overset{(287)}{=} \quad \overset{(296)}{=} \quad \overset{(298)}{=} \quad \overset{(296)}{=} \quad \overset{(298)}{=}
\]

which proves also the first estimate in (48).

\[\square\]

\section{C On action–angle variables for 1D mechanical systems with parameters}

We will use the notations of sections 4 and 5, in particular subsections 5.1 and 5.5.
C.1 The “unperturbed case”

Consider the “unperturbed case” when \( \eta = \eta_* = 0 \) (recall (136)). Namely consider the one dimensional Hamiltonian

\[
H_{\text{pend}}^0(J_n, \psi_n) = J_n^2 + F_0^0(\psi_n), \quad \text{with } F_0^0 \text{ satisfying (99)}.
\] (302)

In the particular important case in which \( F_0^0 \) is minus cosine we can explicitly evaluate \( F_0(\psi_n) = -\cos \psi_n \) \( \Rightarrow M = \cosh s_0, \ N = 1, \ x_0^0 = 0, \ x_0^2 = \pi, \ E_1^0 = -1, \ E_2^0 = 1, \ \beta = 1 \). (303)

For \( E \in (E_-^{(i),0}, E_+^{(i),0}) \), let us define the functions \( P_n^{(i),0}(E) \) as

\[
P_n^{(2j-1),0}(E) := \frac{1}{\pi} \int_{X_{2j-1}^0(E)}^{X_{2j}^0(E)} \sqrt{E - F_0^0(x)} \, dx,
\]

\[
P_n^{(2j),0}(E) := \frac{1}{\pi} \int_{X_{2j-1}^0(E)}^{X_{2j+1}^0(E)} \sqrt{E - F_0^0(x)} \, dx,
\]

\[
P_n^{(2N),0}(E) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{E - F_0^0(x)} \, dx,
\]

\[
P_n^{(0),0}(E) := -\frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{E - F_0^0(x)} \, dx.
\] (304)

In the following we will use the notations \( p \) and \( < \) introduced in (137) and (138).

**Lemma C.1** For real \( E \), we have that

\[
\min_{1 \leq i \leq 2N - 1} E \in (E_-^{(i),0}, E_+^{(i),0}) \inf_{E \in (E_-^{(i),0}, E_+^{(i),0})} \partial_E P_n^{(i),0}(E) =: C_{F_0} > 0.
\] (305)

In particular\(^\text{66}\)

\[
\text{if } F_0 \text{ satisfies (A3) then } C_{F_0} \geq 1/2.
\] (306)

**Proof** See [4]. \([\blacksquare]\)

C.2 The action as a function of the angle at constant energy

Let us consider now the Hamiltonian \( H_{\text{pend}} \) defined in (134).

For \( \eta \) small enough we can solve, w.r.t. \( J_n \), the implicit function equation

\[
J_n^*(\hat{J}) + \frac{z}{\sqrt{1 + b(J, \psi_n)}} - J_n = 0,
\] (307)

finding

\[
J_n = \mathcal{J}_n(z, \psi_n, \hat{J}),
\]

\(^{66}\) In the special case in which \( F_0(\psi_n) = -\cos \psi_n \) (note that \( N = 1 \)), the minimum is \( 1/\sqrt{2} \).
where \( \mathcal{J}_n \) is the analytic function \( \mathcal{J}_n : (-R_0, R_0)_{r_0/4} \times \mathbb{T}_{s_0} \times \hat{D}_{r_0} \)

\[
\mathcal{J}_n(z, \psi_n, \hat{J}) = J^*_n(\hat{J}) + z + \tilde{J}_n(z, \psi_n, \hat{J})
\]

whit \( \tilde{J}_n \) solving the fixed point equation

\[
\tilde{J}_n = \Phi(\tilde{J}_n; z, \psi_n, \hat{J}) := \frac{1}{\sqrt{1 + b(\hat{J}, J^*_n(\hat{J}) + z + \tilde{J}_n, \psi_n)}} - 1.
\]

We are going to solve (309) in the closed ball

\[
\|\tilde{J}_n\|_B \leq \eta,
\]

of the Banach space \( B \) of analytic functions \( \phi : (-R_0, R_0)_{r_0/4} \times \mathbb{T}_{s_0} \times \hat{D}_{r_0} \rightarrow \mathbb{C} \) endowed with the sup-norm

\[
\|\phi\|_B := \sup_{z \in (-R_0, R_0)_{r_0/4}} \|\phi(z, \cdot, \cdot)\|_{\mathcal{D}, s_0}. \]

We first note that by (308), (310), (135) we get

\[
\|\mathcal{J}_n(z, \cdot, \cdot)\|_B \leq r_0 \eta + R_0 + r_0/4 + \eta \leq R_0 + 3 r_0/8,
\]

assuming

\[
\eta \leq \min\{1, r_0\}/32. \tag{312}
\]

For \(|t| \leq 1/4\), we have that \( \frac{|t|}{4 \sqrt{-1 \cdot \hat{J}}^3} \leq 1 \), then we get by (135), (312), (311) and Cauchy estimates

\[
\|\Phi(\tilde{J}_n)\|_B \leq \eta, \quad \|D_{\tilde{J}_n} \Phi(\tilde{J}_n)\|_{\mathcal{L}(B, B)} \leq \frac{\partial_z b(\hat{J}, J^*_n + z + \tilde{J}_n, \psi_n)}{2(1 + b(\hat{J}, J^*_n + z + \tilde{J}_n, \psi_n))^{3/2}} \leq \frac{8 \eta}{r_0} \leq \frac{1}{4} \tag{313}
\]

in the closed ball of \( \tilde{J}_n \) satisfying (310).

Obviously\(^{67}\)

\[
J_n = \mathcal{J}_n\left( \pm \sqrt{E - F(\hat{J}, \psi_n, \hat{J})} \right) \quad \text{solves} \quad \text{(w.r.t. } J_n) \quad H_{\text{pend}}(\hat{J}, J_n, \psi_n) = E,
\]

according to \( \pm (J_n - J^*_n(\hat{J})) \geq 0 \), for every \((\text{real}) E\) such that\(^{68}\)

\[
E + M < R_0^2. \tag{315}
\]

By (309) we get

\[
\partial_z \tilde{J}_n = -\left( 1 + \frac{\partial_z b}{2(1 + b)^{3/2}} \right)^{-1} \frac{\partial_z b}{2(1 + b)^{3/2}}
\]

so that, recalling (313),

\[
\|\partial_z \tilde{J}_n\|_B \leq \frac{4 \eta}{3 r_0} \leq \frac{1}{3}.
\]

Then \( \mathcal{J}_n \) is an increasing function of \((\text{real}) z\), indeed by (308) we obtain \( \partial_z \mathcal{J}_n = 1 + \partial_z \tilde{J}_n \).

\textbf{Remark C.1} In the following we will often omit the explicit dependence on \( \hat{J} \), for brevity.

\(^{67}\)For real values of \( \hat{J}, \psi_n, E. \)

\(^{68}\)So that \( \mathcal{J}_n\left( \pm \sqrt{E - F(\hat{J}, \psi_n, \hat{J})} \right) \) is well defined. Recall (99).
C.3 The domains of definition of action angle variables

Outside the zero measure set formed by the connected components in the set of critical energies \( \{ H_{\text{pend}} = E_i \}, 1 \leq i \leq 2N \), containing the critical points \( x_i \), the phase space \( \mathbb{R}^n \times \mathbb{T}^n \) is composed by \( 2N + 1 \) open connected components \( C^i \), \( 0 \leq i \leq 2N \), defined as

\[
C^i := \tilde{C}^i \times \mathbb{T}^{n-1},
\]

where

\[
\tilde{C}^i \subseteq \hat{D} \times \mathbb{R} \times \mathbb{T}^1 \subseteq \mathbb{R}^n \times \mathbb{T}^1
\]

are defined as follows.\(^{69}\)

For \( i = 2j - 1 \) odd, \( 1 \leq j \leq N \), \( \tilde{C}^i \) is defined as

\[
\tilde{C}^i_{2j-1} := \left\{ J_n \left( -\sqrt{E_+^{(2j-1)}(\hat{J}) - F(\psi_n), \psi_n} \right) < J_n < J_n \left( \sqrt{E_+^{(2j-1)}(\hat{J}) - F(\psi_n), \psi_n} \right), \right. \\
X_{2j-1}(E_+^{(2j-1)}(\hat{J})) < \psi_n < X_{2j}(E_+^{(2j-1)}(\hat{J})), \quad \hat{J} \in \hat{D} \right. \\
\left. \setminus \left\{ J_n = J_n^c, \psi_n = x_{2j-1} \right\} \right\}
\]

For \( i = 2j \) even, \( 1 \leq j \leq N - 1 \), \( \tilde{C}^i \) is still a normal set with respect to the variable \( J_n \):

\[
\tilde{C}^i_{2j} := \left\{ J_n \left( -\sqrt{E_+^{(2j)}(\hat{J}) - F(\psi_n), \psi_n} \right) < J_n < J_n \left( \sqrt{E_+^{(2j)}(\hat{J}) - F(\psi_n), \psi_n} \right), \right. \\
X_{2j-1}(E_+^{(2j)}(\hat{J})) < \psi_n < X_{2j}(E_+^{(2j)}(\hat{J})), \quad \hat{J} \in \hat{D} \right. \\
\left. \setminus \left\{ J_n = J_n^c, \psi_n = x_{2j} \right\} \right\}
\]

where \( j_-, j_+ \) were defined in (116).

Finally

\[
\tilde{C}^i_{2N} := \left\{ J_n > J_n \left( \sqrt{E_+^{(2N)}(\hat{J}) - F(\psi_n), \psi_n} \right), \quad \psi_n \in \mathbb{T}, \quad \hat{J} \in \hat{D} \right\}
\]

\[
\tilde{C}^i_0 := \left\{ J_n < J_n \left( -\sqrt{E_+^{(2N)}(\hat{J}) - F(\psi_n), \psi_n} \right), \quad \psi_n \in \mathbb{T}, \quad \hat{J} \in \hat{D} \right\}
\]

Note that actually in \( \tilde{C}_i \) with \( 1 \leq i < 2N \), \( \psi_n \) is not an angle!

Let us introduce the (small) parameter

\[
\theta \geq 0.
\]

\(^{69}\)Omitting to write, for brevity, the explicit dependence of \( J_n, F, X_i \) on \( \hat{J} \)
Recalling (315), we define the following subsets of $\tilde{C}^i$ (defined in (316),(317),(318), (319))

$$\tilde{C}^{2j-1}(\theta) := \tilde{C}^{2j-1} \cap \{E^{(2j-1)}_+(\tilde{J}) < H_{\text{pend}} < E^{(2j-1)}_+(\tilde{J}) - 2\theta, \tilde{J} \in \tilde{D} \}, \quad \text{for } 1 \leq j \leq N,$$

$$\tilde{C}^{2j}(\theta) := \tilde{C}^{2j} \cap \{E^{(2j)}_+(\tilde{J}) + 2\theta < H_{\text{pend}} < E^{(2j)}_+(\tilde{J}) - 2\theta, \tilde{J} \in \tilde{D} \}, \quad \text{for } 1 \leq j < N,$$

$$\tilde{C}^i(\theta) := \tilde{C}^i \cap \{E^{(2N)}_+(\tilde{J}) + 2\theta < H_{\text{pend}} < R_0^2 - M - 2\theta, \tilde{J} \in \tilde{D} \}, \quad \text{for } i = 0, 2N,$$

where $H_{\text{pend}}$ was defined in (134). Note that $\tilde{C}^i(0) = \tilde{C}^i$ for $1 \leq i < 2N$.

Finally we define\textsuperscript{70}

$$\tilde{C}^i(\theta) = \tilde{C}^i(\theta) \times T^{n-1}, \quad \tilde{C}^i(\theta) \ni (J, \psi_n), \quad T^{n-1} \ni \hat{\psi}. \quad (322)$$

The sets $\tilde{C}^i(\theta)$ and, therefore, $\tilde{C}^i(\theta)$ have different homotopy: for every fixed $\tilde{J}$ the set

$$\tilde{C}^i(\theta) := \{(J_n, \psi_n) \mid (J, \psi_n) \in \tilde{C}^i(\theta) \} \subseteq \mathbb{R}^1 \times T^1$$

is contractible for $1 \leq i \leq 2N - 1$ and is not contractible for $i = 0, 2N$. Note that, recalling (99),

$$\hat{D} \times (-R_0/2, R_0/2) \times T^n \subseteq \bigcup_{0 \leq i \leq 2N} \tilde{C}^i(0) \subset \hat{D} \times (-R_0, R_0) \times T^n. \quad (323)$$

\textbf{C.4 Definition of action variables}

On the above connected components $C_i$, $0 \leq i \leq 2N$, we want to define action angle variables integrating $H_{\text{pend}}$. We first define the action variables as a function of the energy $E$ and of the dummy variable $\tilde{J}$. More precisely, for $0 \leq i \leq 2N$, we are going to define the functions

$$P_n^{(i)} : (E, \tilde{J}) \in \mathcal{E}^i \rightarrow \mathbb{R}, \quad \text{where } \mathcal{E}^i := \mathcal{E}^i(0)$$

and

$$\mathcal{E}^{2j-1}(\theta) := \{(E, \tilde{J}) \mid \text{s.t. } E^{(2j-1)}_+(\tilde{J}) < E < E^{(2j-1)}_+(\tilde{J}) - 2\theta, \tilde{J} \in \tilde{D} \}, \quad 1 \leq j \leq N,$$

$$\mathcal{E}^{2j}(\theta) := \{(E, \tilde{J}) \mid \text{s.t. } E^{(2j)}_+(\tilde{J}) + 2\theta < E < E^{(2j)}_+(\tilde{J}) - 2\theta, \tilde{J} \in \tilde{D} \}, \quad 1 \leq j < N,$$

$$\mathcal{E}^{2N}(\theta) = \mathcal{E}^0(\theta) := \{(E, \tilde{J}) \mid \text{s.t. } E^{(2N)}_+(\tilde{J}) + 2\theta < E < R_0^2 - M - 2\theta, \tilde{J} \in \tilde{D} \}, \quad (324)$$

where the positive parameter $\theta$ was introduced in (320) (and recall (315)). We also introduce the complex $\theta$-neighborhoods\textsuperscript{71}

$$\mathcal{E}^0_\theta(\tilde{J}) := (\mathcal{E}^i(\tilde{J}))_\theta \subseteq \mathbb{C}^n. \quad (325)$$

The functions $P_n^{(i)}$ are defined as follows\textsuperscript{72}.

For $i = 2j - 1$ odd, $1 \leq j \leq N$, and $E^{(2j-1)}_+(\tilde{J}) < E < E^{(2j-1)}_+(\tilde{J})$, we set

$$P_n^{(2j-1)}(E) = P_n^{(2j-1)}(E, \tilde{J}) \quad (326)$$

$$:= \frac{1}{2\pi} \int_{X_{2j-1}(E)} \left[ \mathcal{J}_n \left( \sqrt{E - F(x)}, x \right) - \mathcal{J}_n \left( -\sqrt{E - F(x)}, x \right) \right] dx$$

$$= \frac{1}{\pi} \int_{X_{2j-1}(E)} \sqrt{E - F(x)} \left( 1 + b_\psi \left( \sqrt{E - F(x)}, x \right) \right) dx,$$

\textsuperscript{70}With a little abuse of notation we invert the order of $\psi_n$ and $\hat{\psi}$.

\textsuperscript{71}Recall the notation on page 9.

\textsuperscript{72}Sometimes omitting, for brevity, to write the explicit dependence on $\tilde{J}$. 

53
where the last equality holds recalling (307) and defining \( b_z = b_z(\hat{J}, z, x) \) as follows:

\[
2 + 2b_z(\hat{J}, z, x) := \frac{1}{\sqrt{1 + b(\hat{J}, \mathcal{J}_n(z, x), x)}} + \frac{1}{\sqrt{1 + b(\hat{J}, \mathcal{J}_n(-z, x), x)}}
\]  

(327)

(b defined in (134)). Note that

\[
b_z \text{ is even w.r.t. } z \quad \text{and} \quad \sup_{z \in (-R_0, R_0)} \| b_z(\hat{J}, z, x) \|_{D, r_0, s_0} < \eta.
\]  

(328)

For \( i = 2j \) even, \( 1 \leq j \leq N - 1 \) and \( E^{(2j)}_{\hat{J}}(\hat{J}) < E < E^{(2j)}_{\hat{J}}(\hat{J}) \), we set (recall (116))

\[
P^{(2j)}(E) = P^{(2j)}(E, \hat{J}) := \frac{1}{2\pi} \int_{X_{2j+1}(E)} \mathcal{J}_n \left( \sqrt{E - F(x)}, x \right) dx
\]  

(329)

\[
P^{(2j)}(E) = P^{(2j)}(E, \hat{J}) = \frac{1}{2\pi} \int_{X_{2j+1}(E)} \sqrt{E - F(x)} \left( 1 + b_z(\sqrt{E - F(x)}, x) \right) dx,
\]  

where \( j-, j+ \) were defined in (116).

Finally for \( E > E^{(2N)}_{\hat{J}}(\hat{J}) \) we set

\[
P^{(2N)}(E) = P^{(2N)}(E, \hat{J}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{J}_n \left( \sqrt{E - F(x)}, x \right) dx,
\]  

(330)

\[
P^{(0)}(E) = P^{(0)}(E, \hat{J}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{J}_n \left( -\sqrt{E - F(x)}, x \right) dx.
\]  

(331)

\[C.5 \quad \text{Properties of the actions as functions of the energy and viceversa}\]

\[\text{Lemma C.2 Assume that} \quad \frac{\eta}{c_{F_0}} \leq \eta_0(p), \]

with \( c_{F_0} \) defined in (305) and \( \eta_0 = \eta_0(p) \) small enough. Then for every \( 1 \leq i \leq 2N - 1 \)

\[
\inf \partial_E P^{(i)}(E, \hat{J}) \geq c_{F_0}/2 > 0,
\]  

(333)

\[\text{while}^73\]

\[
\frac{1}{8\sqrt{E}} \leq \partial_E P^{(2N)}(E, \hat{J}), -\partial_E P^{(0)}(E, \hat{J}) \leq \frac{2}{\sqrt{E}}, \quad \forall 2M \leq E \leq R_0^2 - M, \quad \hat{J} \in \hat{D},
\]  

(334)

\[
\partial_E P^{(2N)}(E, \hat{J}), -\partial_E P^{(0)}(E, \hat{J}) \geq \frac{1}{8\sqrt{2M}}, \quad \forall E_{2N} \leq E \leq 2M, \quad \hat{J} \in \hat{D}
\]  

(335)

\[M \text{ defined in (302)}.\]

\[^73\text{Recall (315)}\]
Proof It essentially follows from Proposition 5.2; see [4] for details. □

By (333) we have that, for every fixed \( \hat{J} \in \hat{D} \), the function \( E \mapsto P_n^{i}(E, \hat{J}) \) is strictly monotone and, therefore, invertible with inverse \( E^{(i)}(\hat{J}, P_n) \) such that
\[
E^{(i)}(\hat{J}, P_n^{i}(E, \hat{J})) = E \quad \text{and} \quad P_n^{i}(E^{(i)}(\hat{J}, P_n), \hat{J})) = P_n.
\] (336)
As a corollary of Proposition 5.2 and of the chain rule applied to (336), giving
\[
\partial P_n E^{(i)}(P) = \frac{1}{\partial E P_n}, \quad \partial P E = -\frac{\partial E P_n}{\partial E P_n},
\]
by Lemma C.2 we get the following. For every \( 1 \leq i \leq 2N - 1 \)
\[
0 < \partial P_n E^{(i)} \leq \frac{2}{c_{F_{0}}},
\] (337)
(\( c_{F_{0}} \) defined in (305)). Moreover
\[
\frac{\sqrt{E}}{2} \leq \partial P_n E^{(2N)} - \partial P_n E^{(0)} \leq 8 \sqrt{E}, \quad \forall 2M \leq E \leq R_0^2 - M, \quad \hat{P} \in \hat{D},
\]
\[
\partial P_n E^{(2N)} - \partial P_n E^{(0)} \leq 8 \sqrt{2M}, \quad \forall E^{(0)} - E^{2N} < E \leq 2M, \quad \hat{P} \in \hat{D}
\] (338)
(\( M \) defined in (302)).

The proofs of the following lemmata essentially follows from Proposition 5.2; see [4] for details.

Lemma C.3 Let \( C > 1 \) as in Proposition 5.2.
\[
\sup_{D_{\gamma_{0}} \times \Gamma^{(i)}_{\gamma_{0}/8}} \| \partial P_P E^{(i)}(P) \| \leq C, \quad \sup_{D_{\gamma_{0}} \times \Gamma^{(i)}_{\gamma_{0}/8}} | \partial P_n E^{(i)}(P) - \partial P_n E^{(i)}(P_n) | \leq C \eta, \quad \forall 0 \leq i \leq 2N,
\] (340)
where the intervals \( \Gamma^{(i)} \) where defined in (240) and \( x_{0} > 0 \) in (230).

Lemma C.4 (171) holds.

Lemma C.5 Assume that \( F \) is cosine-like according to Definition 4.2, with \( \epsilon \) (namely \( \epsilon_{*} \) defined in (100)) small enough. Then
\[
\inf_{E_1 < E < E_2, \hat{P} \in \hat{D}} \left| \partial P_n P_n E^{(1)} \right| \geq c_{4}, \quad \inf_{E_1 < E < R_0^2 - 1, \hat{P} \in \hat{D}} \left| \partial P_n P_n E^{(2)} \right| \geq c_{4},
\] (341)
for a suitable (absolute constant) \( c_{4} > 0 \).

D Miscellanea

Composition of maps

Lemma D.1 Let \( y = Ly' \), where \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is linear. Then for every function \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) we have
\[
\det \left( \partial y^{y'} (h \circ L(y')) \right) = (\det L)^2 \det (\partial y h)|_{y = Ly'}. \]
Lemma D.1 follows immediately observing that $\partial_{y'} (h(Ly')) = L^T (\partial_{yy} h) |_{y = Ly'} L$.
Given a Hamiltonian $H(J, \psi)$ we denote by $\Phi_H^t$ its flow at time $t$.

**Lemma D.2** (i) **[Time rescaling]** Let $c > 0$. Then $\Phi_H^t = \Phi_{c^{-1} H}^t$.
(ii) **[Action rescaling]** Consider the conformally symplectic change of variables

$$ (J, \psi) = \Phi(J, \tilde{\psi}) := (c \tilde{J}, \tilde{\psi}) $$

and set $\tilde{H} := H \circ \Phi$ and $\tilde{\tilde{H}} := c^{-1} H \circ \Phi$. Then $\Phi \circ \Phi_H^{t/c} = \Phi_{\tilde{H}}^t \circ \Phi$ and $\Phi \circ \Phi_H^{t/c} = \Phi_{\tilde{\tilde{H}}}^t \circ \Phi$.

The proof is a straightforward check.

**Lemma D.3** Consider $s_1 > 0, s > s_2 > 0$ and an holomorphic map $\Phi : T_s T_{s_2} \rightarrow T_{s_1} T_{s_2}$ and an holomorphic function $f$ with $|f|_s < \infty$. Then

$$ |f \circ \Phi|_{s_1} \leq \coth^n \left( \frac{s - s_2}{2} \right) |f|_s \leq \left( 1 + \frac{2}{s - s_2} \right)^n |f|_s . $$

**Proof** By (87) we get $|f \circ \Phi|_{s_1} \leq \|f \circ \Phi\|_{s_1} \leq \|f\|_{s_2} \leq \coth^n \left( \frac{s - s_2}{2} \right) |f|_s$. 

**Lemma D.4** Given a matrix $M \in \text{Mat}_{n \times n}(\mathbb{Z})$ with $\det M = \pm 1$, consider the symplectic linear map $\Phi : \mathbb{R}^n \times \mathbb{T}^n \rightarrow \mathbb{R}^n \times \mathbb{T}^n$ defined as

$$ (J', \psi') = \Phi(J, \psi) := (M^T J, M^{-1} \psi) . $$

Let $D \subseteq \mathbb{R}^n$ and $r, s > 0$. Set $D' := (M^T)^{-1} D$. Then

$$ \Phi(D'_r \times T^n_{s_0}) \subseteq D_r \times T^n_s, \quad \text{where} \quad r' := r/\|M^T\|, \quad s' := s/\sup_{1 \leq i \leq n, 1 \leq j \leq n} |(M^{-1})_{ij}| . $$

Moreover, given a function $f : D_{r_0} \times T^n_{s_0}$ with $r_0 \geq r, s_0 > s$, we have

$$ |f \circ \Phi|_{D', r', s'} \leq \coth^n \left( \frac{s_0 - s}{2} \right) |f|_{s_0} \leq \left( 1 + \frac{2}{s_0 - s} \right)^n |f|_{s_0} . $$

The first part is obvious; the second part follows from Lemma D.4.

**Restrictions of maps**

**Lemma D.5** Let $\Phi : D' \times \mathbb{T}^n \rightarrow D \times \mathbb{T}^n$ be a real analytic map with holomorphic extension

$$ \Phi : D'_r \times \mathbb{T}^n_{s'} \rightarrow D_r \times \mathbb{T}^n_s $$

for some $r, r' s, s' > 0$. There exists a suitably small constant $c$ depending only on $n$ such that

$$ \Phi(D'_{car} \times \mathbb{T}^n_{cas'}) \subseteq D_{ar} \times \mathbb{T}^n_{as'}, \quad \forall 0 < a \leq 1 . $$

**Proof** By Cauchy estimates, applied to the various components of $\Phi$. 

56
A group of parameter–dependent symplectic transformations

Let us consider the group $G$ introduced in (129).

**Lemma D.6** Given a symplectic transformation of the form (129), we have that, for every fixed $\hat{J}$, the restriction

$$(J_n, \psi_n) \mapsto (I_n(J, \psi_n), \varphi_n(J, \psi_n))$$

is also symplectic.

**Proof** Note that by the conservation of the symplectic form $dI \wedge d\varphi = dJ \wedge d\psi$ follows that

$$\partial_{J_n}I_n\varphi_n - \partial_{J_n}\varphi_n\partial_{\psi_n}I_n = 1.$$ 

Recall the definition given in (130) and note that

$$(\Phi_1 \circ \Phi_2)^- = \Phi_1 \circ \Phi_2.$$  \hspace{1cm} (344)

Furthermore, obviously, one has

$$\hat{\phi}(E) \times T^{n-1} = \phi(E \times T^{n-1}) , \quad \forall \phi \in G , \quad \forall E \subseteq \mathbb{R}^n \times T^1 .$$  \hspace{1cm} (345)

By Lemma D.6 we have the following

**Lemma D.7** If $\Phi \in G$, then $\hat{\Phi}$ is volume-preserving.

An elementary result in linear algebra

**Lemma D.8** Given $k \in \mathbb{Z}^n$, $k \neq 0$ there exists a matrix $A = (A_{ij})_{1 \leq i, j \leq n}$ with integer entries such that $A_{nj} = k_j \ \forall 1 \leq j \leq n$, $\det A = d := \gcd(k_1, \ldots, k_n)$, and $|A|_\infty = |k|_\infty$.

**Proof** The argument is by induction over $n$. For $n = 1$ the lemma is obviously true. For $n = 2$, it follows at once from Bezout’s Lemma. Given two integers $a$ and $b$ not both zero, there exist two integers $x$ and $y$ such that $ax + by = d := \gcd(a, b)$, and such that $\max\{|x|, |y|\} \leq \max\{|a|/d, |b|/d\}$.

Indeed, if $x$ and $y$ are as in Bezout’s Lemma with $a = k_1$ and $b = k_2$ one can take $A = \begin{pmatrix} y & -x \\ k_1 & k_2 \end{pmatrix}$.

Now, assume, by induction for $n \geq 3$ that the claim holds true for $(n-1)$ and let us prove it for $n$. Let $\bar{k} = (k_1, \ldots, k_{n-1})$ and $\bar{d} = \gcd(k_1, \ldots, k_{n-1})$ and notice that $\gcd(\bar{d}, k_n) = d$. By the inductive assumption, there exists a matrix $\bar{A} = \begin{pmatrix} \bar{A} \\ \bar{k} \end{pmatrix} \in \text{Mat}_{(n-1) \times (n-1)}(\mathbb{Z})$ with $\bar{A} \in \text{Mat}_{(n-2) \times (n-1)}(\mathbb{Z})$, such

---

The first statement in this formulation of Bezout’s Lemma is well known and it can be found in any textbook on elementary number theory; the estimates on $x$ and $y$ are easily deduced from the well known fact that given a solution $x_0$ and $y_0$ of the equation $ax + by = d$, all other solutions have the form $x = x_0 + k(b/d)$ and $y = y_0 - k(a/d)$ with $k \in \mathbb{Z}$ and by choosing $k$ so as to minimize $|x|$.
that $\det \tilde{A} = \tilde{d}$ and $|\tilde{A}|_\infty = |\tilde{k}|_\infty$. Now, let $x$ and $y$ be as in Bezout’s Lemma with $a = \tilde{d}$, and $b = k_n$.

We claim that $A$ can be defined as follows:

$$A = \begin{pmatrix}
\tilde{k} & \tilde{x} \\
\tilde{d} & 0 \\
\vdots & \vdots \\
0 & k_n
\end{pmatrix}, \quad \tilde{k} = (\frac{1}{-1})^n y \frac{\tilde{k}}{d}, \quad \tilde{x} := (\frac{1}{-1})^{n+1} x.$$  

(346)

First, observe that since $\tilde{d}$ divides $k_j$ for $j \leq (n-1)$, $\tilde{k} \in \mathbb{Z}_{n-1}$. Then, expanding the determinant of $A$ from last column, we get

$$\det A = (\frac{1}{-1})^{n+1} \tilde{x} \det \tilde{A} + k_n (\frac{1}{-1})^{n-2} \frac{\tilde{y}}{d} \det \tilde{A} = x \tilde{d} + k_n y = \tilde{d}.$$  

Finally, by Bezout’s Lemma, we have that $\max\{|x|, |y|\} \leq \max\{|\tilde{d}/d, |k_n|/d\}$, so that

$$|\tilde{k}|_\infty = |y| \frac{|\tilde{k}|_\infty}{d} \leq |\tilde{k}|_\infty \leq |k|_\infty, \quad |\tilde{x}| = |x| \leq \frac{|k_n|}{d} \leq |k|_\infty,$$

which, together with $|\tilde{A}|_\infty = |\tilde{k}|_\infty$, shows that $|A|_\infty = |k|_\infty$.

Measure of sub–levels of smooth non–degenerate functions

Here we prove Lemma 6.1. We start by recalling an elementary result, whose proof can be found in [2]:

**Lemma D.9** Let $g(x)$ a monic polynomial of degree $d$. Then

$${\text{meas}}\left\{ x \in \mathbb{R} : |g(x)| \leq \gamma \right\} \leq 2d\gamma^{1/d}.$$  

We now prove Lemma 6.1. Let us divide the interval $[a, b]$ in disjoint intervals of length $2r := 2\mu^{1/m+1}$. Let $I$ one of such intervals and let $x_0$ is middle point. By (225) let $1 \leq d \leq m$ such that

$$|\partial^d f(x_0)|/d! \geq \xi_m.$$  

(347)

By the Taylor remainder formula we get, for $x \in I$, $|f(x) - P^d_{x_0}(x)| \leq M r^{d+1} = M \mu^{\frac{d+1}{m+1}}$,

where $P^d_{x_0}(x) := \sum_{0 \leq j \leq d} \frac{\partial^j f(x_0)}{j!} (x - x_0)^j$ is the Taylor polynomial of degree $d$. Then we have that

$$\{x \in I : |f(x)| \leq \mu\} \subseteq \{x \in I : |P^d_{x_0}(x)| \leq (M + 1) \mu^{\frac{d+1}{m+1}}\}.$$  

(348)
We now apply Lemma D.9 to the monic polynomial \( g(x) := d!F_{x_0}(x)/\partial_x^d f(x_0) \) with
\[
\gamma := (M+1)\mu^{d+1}/\xi_m \geq d!(M+1)\mu^{d+1}/|\partial_x^d f(x_0)|.
\]
By (348) and Lemma D.9 we get (recall \( 1 \leq d \leq m \))
\[
\text{meas}\{x \in I : |f(x)| \leq \mu\} \leq 2d^{1/d} \leq \frac{2m(M+1)\mu^{1/m}}{\xi_m}.
\]
Since the number of disjoint intervals is smaller that \( \frac{b-a}{\eta^{d/m+1}} + 1 \), this concludes the proof of Lemma 6.1.

Canonical form of generalized pendula

The following lemma describes how to make independent of the action \( J_n \) a pendulum depending on parameters.

**Lemma D.10** Let
\[
H^*(y, x) := y_n^2 + F^0(x_n) + G^*(y, x_n),
\]
with \( \|G^*\|_{D,r_0,s_0} \leq \eta_* \). Assume that
\[
\eta_* \leq \frac{r_0}{4}.
\] (349)
Then the fixed point equation
\[
y(y, x_n) = \frac{1}{2}\partial_{x_n} G^*(y, y(y, x_n), x_n)
\] (350)
has a unique solution \( y = y(y, x_n) \) with
\[
\|y\|_{D,r_0,s_0} \leq 2\eta_*/r_0 \leq r_0/8.
\] (351)
Set
\[
J_n^*(\hat{y}) := \langle y(\hat{y}, x_n) \rangle, \quad a_*(\hat{y}, x_n) := y(\hat{y}, x_n) - \langle y(\hat{y}, x_n) \rangle,
\]
where \( \langle \cdot \rangle \) denotes the average w.r.t. \( x_n \). Let \( \phi = \phi(\hat{y}, x_n) \) the unique function satisfying \( a_* = \partial_{x_n} \phi \) with \( \langle \phi \rangle = 0 \). Consider the canonical transformation \( \Psi \)
\[
y_n = Y_n + a_*(\hat{Y}, X_n) = Y_n - J_n^*(\hat{Y}) + y(\hat{Y}, X_n), \quad x_n = X_n, \quad \hat{y} = \hat{Y}, \quad \hat{x} = \hat{X} + b_*(\hat{Y}, X_n),
\] (352)
obtained by the generating function \( Y_n x_n + \hat{Y} \hat{x} + \phi(\hat{Y}, x_n) \), with \( b_* = -\partial_{x_n} \phi \)
and
\[
\|J_n^*\|_{D,r_0} \leq 2\eta_*/r_0 \leq r_0/8, \quad \|a_*\|_{D,r_0,s_0} \leq 4\eta_*/r_0, \quad \|b_*\|_{D,r_0/2,s_0} \leq (16\pi + 8)\eta_*/r_0^2.
\] (353)
Note that
\[
\Psi : D_{r_0/2} \times \mathbb{T}^{n-1} \times \mathbb{T}_{s_0} \rightarrow D_{r_0} \times \mathbb{T}^{n-1} \times (\mathbb{T}_{s_0} + (16\pi + 8)\eta_*/r_0^2) \times \mathbb{T}_{s_0}.
\] (354)
Then (352) casts $H^*$ into

$$
\left( Y_n - J_n^* (\hat{Y}) + y(\hat{Y}, X_n) \right)^2 + F^0(X_n) + G^* (\hat{Y}, Y_n - J_n^* (\hat{Y}) + y(\hat{Y}, X_n), X_n)
$$

$$
= (1 + b(Y, X_n)) (Y_n - J_n^*(\hat{Y}))^2 + F(\hat{Y}, X_n),
$$

with

$$
F = F^0 + G, \quad G := G^* (\hat{Y}, y(\hat{Y}, X_n), X_n) + (y(\hat{Y}, X_n))^2
$$

and 75 omitting, for brevity, the dependence on $\hat{Y}, X_n$.

$$
b = \frac{G^*(y + Y_n - J_n^*) - G^*(y) - \partial_{Y_n} G^*(y)(Y_n - J_n^*)}{(Y_n - J_n^*)^2} = \int_0^1 (1 - t) \partial_{Y_n} G^*(y + t(Y_n - J_n^*)) dt. \quad (355)
$$

Finally

$$
\|G\|_{D, r_0, s_0} \leq \left(1 + 4/r_0^2\right) \eta_s, \quad \|(1 + |Y_n - J_n^*|)b(Y, X_n)\|_{D, r_0/2, s_0} \leq \left(4 + \frac{34}{r_0^2}\right) \eta_s,
$$

$$
\|Y_n - J_n^* |\partial_{Y_n} b(Y, X_n)\|_{D, r_0/2, s_0} \leq \frac{48}{r_0^2} \eta_s. \quad (356)
$$

**Proof** (350) is solved by the standard Fixed Point Theorem for $y$ in the ball in (351), (353) and the first estimate in (356) follow by (349), (351) and Cauchy estimates. By (355), (349), (351) and Cauchy estimates we get

$$
\|b\|_{D, r_0/2, s_0} \leq \frac{16}{r_0} \eta_s \quad (357)
$$

and, therefore, dividing the cases $|Y_n - J_n^*| \leq 1$ and $|Y_n - J_n^*| > 1$, we get the second estimate in (356). Finally, by (355), we have

$$
\partial_{Y_n} b = \frac{\partial_{Y_n} G^*(y + Y_n - J_n^*) - \partial_{Y_n} G^*(y)}{(Y_n - J_n^*)^2} = \frac{2b}{Y_n - J_n^*}.
$$

Then

$$
\|\partial_{Y_n} b\|_{Y_n - J_n^*} \leq \frac{\|\partial_{Y_n} G^*(y + Y_n - J_n^*) - \partial_{Y_n} G^*(y)\|_{Y_n - J_n^*}}{|Y_n - J_n^*|} + 2|b| \leq \frac{48}{r_0^2} \eta_s. \quad \blacksquare
$$

**References**

[1] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. Mathematical aspects of classical and celestial mechanics, volume 3 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, third edition, 2006. [Dynamical systems. III]. Translated from the Russian original by E. Khukhro.

[2] L. Biasco, and F. Coglitore. Periodic orbits accumulating onto elliptic tori for the (N + 1)-body problem. Celestial Mech. Dynam. Astronom. 101 (2008), no. 4, 349-373.

[3] L. Biasco, and L. Chierchia. On the measure of Lagrangian invariant tori in nearly-integrable mechanical systems. Rend. Lincei Mat. Appl. 26 (2015), 1–10

75 Using (350).
[4] L. Biasco, and L. Chierchia. Action-angle for a one dimensional mechanical systems with parameters. Preprint 2017.

[5] L. Biasco, and L. Chierchia. Explicit estimates on the measure of primary KAM tori. arXiv:1612.01903 [math.DS] (Dec 2016)

[6] H. S. Dumas, The KAM Story, World Scientific, 2014

[7] B. R. Hunt, V. Y. Kaloshin, Prevalence, chapter 2, Handbook in dynamical systems, edited by H. Broer, F. Takens, B. Hasselblatt, Vol. 3, 2010, pg. 43-87

[8] A. N. Kolmogorov, On conservation of conditionally periodic motions for a small change in Hamilton’s function. (Russian) Dokl. Akad. Nauk SSSR (N.S.) 98, (1954). 527-530.

[9] V. F. Lazutkin, Concerning a theorem of Moser on invariant curves (Russian), Vopr. Dinamich. Teor. Rasprostr. Seism. Vols. 14 (1974), 109-120.

[10] A.G. Medvedev, A.I. Neishtadt, D.V. Treschev, Lagrangian tori near resonances of near–integrable Hamiltonian systems, Nonlinearity, 28:7 (2015), 2105–2130

[11] A. I. Neishtadt, Estimates in the Kolmogorov theorem on conservation of conditionally periodic motions, J. Appl. Math. Mech. 45 (1981), no. 6, 766-772

[12] J. Pöschel, Integrability of Hamiltonian systems on Cantor sets, Comm. Pure Appl. Math., v. 35 (1982), no. 1, 653-695

[13] J. Pöschel, A lecture on the classical KAM theorem. Smooth ergodic theory and its applications (Seattle, WA, 1999), 707-732, Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001

[14] J. Pöschel, Nekhoroshev estimates for quasi–convex hamiltonian systems. Math. Z. 213, pag. 187 (1993).