Abstract:

Partition functions of some two-dimensional statistical models can be represented by means of Grassmann integrals over loops living on two-dimensional torus. It is shown that those Grassmann integrals are topological invariants, which depend only on the winding numbers of the loops. The fact makes possible to evaluate the partition functions of the models and the statistical mean values of certain topological characteristics (indices) of the configurations, which behave as the (topological) order parameters.
1. INTRODUCTION

The integral topological invariants play an important role in modern mathematical physics. Among the most illustrative examples we may mention the Pontryagin index, familiar from the gauge fields physics [8], or the Euler characteristic of the manifold, entering the analysis of the string perturbation theory [5]. Some physicists tend to assign to topological methods and ideas a privileged role in the structure of physical theories [10]. We do not wish to discuss the importance of topological methods in general, nevertheless, in this contribution, we present some statistical models where topological considerations enter in a nontrivial way. We shall study, in what follows, the partition functions of gases of loops living on two-dimensional torus. In the cases studied, we can express those partition functions by means of certain Grassmann integrals over the loops. The starting point of our treatment will be the Berezin’s work [2], in which the free fermionic representation of the partition function of the Ising model in the plane was studied. Berezin has proved the lemma, that certain Grassmann integral over the loops has the same value for all nonselfintersecting loops living in the plane. He remarked also, that the lemma does not hold, when the plane is replaced by the torus. On the other hand, quite recently it has arisen the problem in the string statistics whether the nontrivial winding modes should be summed over [3,7,9], when putting the strings in the finite box with periodic boundary conditions. The present author has constructed the toy string model [6], in which the winding modes played a crucial role. He has studied the gas of loops (or classical strings) living on the two dimensional toroidal lattice and found the phase transition with the phases differing by the parities of the winding numbers of the dominant configurations. In this contribution, we reproduce the results of [6] expressing the main characteristics of the model by means of the Grassmann integrals introduced by Berezin. The key ingredient of the method constitute in the generalization of the Berezin’s lemma to the case of the loops living on the torus. We prove, in fact, that the value of the appropriately constructed Grassmann integral over the loops is the function of the winding numbers of the loops only, hence, it is the topological invariant. In particular, for all loops in the plane the value of the integral is the same and we recover the result of Berezin. The topological invariance of the constructed Grassmann integral will enable us to evaluate the partition function and the “topological” order parameter” of the model. This order parameter measures the dominance of the configurations with different parities as the function of the temperature. The quantity turns out to jump at the temperature of the phase transition.

The paper is organized as follows: In Sec.2 we introduce the Grassmann integrals over loops on the two-dimensional toroidal lattice and show their invariance with respect to small deformations. Then we pick up a representative from each class of loops with given winding numbers and evaluate the value of the integral. In Sec.3 we introduce the toy two-dimensional string model [6] and evaluate its partition function. We also introduce and evaluate the
topological order parameter, mentioned above.
We shall end up with short conclusions.

2. GRASSMANN INTEGRALS OVER CLOSED LOOPS.

Consider a square toroidal two-dimensional lattice with $N_1$ horizontal and $N_2$ vertical sites. The distance between the neighbouring sites is set to 1 and the sites $(i_1, i_2)$ are parametrized by integers $i_1 \in (1, \ldots, N_1)$ and $i_2 \in (1, \ldots, N_2)$. We assign to each site four Grassmann variables $x_{i_1, i_2}, x^*_{i_1, i_2}, y_{i_1, i_2}, y^*_{i_1, i_2}$ where each variable correspond to one half-link attached to the site in the way described in Fig.1. We assign also to each pair of half-links attached to the site the (ordered) product of the corresponding Grassmann variables according Fig.2.

Draw now a closed nonselfintersecting loop $C$ on the lattice. We assign it the Grassmann integral $I(C)$ as follows

$$I(C) = \int \prod_{\text{links of } C} d\mu_{\text{link}} \prod_{\text{sites of } C} A_r(\text{site}), \quad r \in (3, \ldots, 8) \quad (2.1)$$

where the index $r$ denotes the way in which the loop runs over the site (see Fig.2) and the measure $d\mu_{\text{link}}$ is given by

$$d\mu_{\text{link}} \equiv dx_{i_1+1, i_2}^* dx_{i_1, i_2}, \quad (2.2a)$$

for the horizontal links and

$$d\mu_{\text{link}} \equiv dy_{i_1, i_2+1}^* dy_{i_1, i_2}, \quad (2.2b)$$

for the vertical links.

We now show that the integral (2.1) is invariant for all loops with the fixed horizontal and vertical winding numbers respectively. Consider first the case of topologically trivial loops $C_{\text{triv}}$, e.g. those with both winding numbers equal to zero. Each such loop can be regarded as the closed loop in the "covering" plane of the torus (see Fig.3). The original torus is obtained by the identification of the points $(i_1 + k_1 N_1, i_2 + k_2 N_2)$ where $k_1, k_2$ are integers. Since the loop is nonselfintersecting, the value of $I(C)$ is obviously unchanged in this picture where to each point of the covering plane we associate its own quadruple of the Grassmann variables. The closed loop in the plane is the boundary of some domain. Now it is obvious that this loop can be deformed by the succesion of small deformations to the boundary of a single plaquette. Indeed, we can take away from the domain a plaquette after plaquette until remains a single one. We do it in such a way that the removed plaquette participates on the
nonempty connected piece of the boundary of the domain (see Fig.4). It is a simple exercise to
demonstrate that such a plaquette always exists and that respecting this rule of removing the
plaquettes the boundary of each intermediate domain will consist of a single loop. To prove
the invariance of the integral \( I(C_{\text{triv}}) \) it is enough to show that \( I(C_{\text{triv}}) \) does not change,
removing a single plaquette from the domain. This can be done by the simple inspection of
all possibilities (see Fig.5 and Lemma 1 of the Appendix for the details). The value of \( I(C) \)
is therefore the same for all loops with both winding numbers equal to zero and is given by
\( I(C_{\text{plaquette}}) \) i.e.

\[
I(C_{\text{plaquette}}) = \int dy_{i_1,i_2+1}dy_{i_1,i_2}dx_{i_1,i_2+1,i_2}+1 dy_{i_1+1,i_2+1}dx_{i_1,i_2+1,i_2+1} dy_{i_1,i_2,1}dx_{i_1,i_2,1,i_2,1} = 1 = I(C_{\text{triv}}) \tag{2.3}
\]

We remind here the Berezin rules

\[
\int x_\alpha dx_\alpha = 1, \quad \int dx_\alpha = 0, \tag{2.4}
\]

\[
\{x_\alpha, x_\beta\}_+ = \{x_\alpha, dx_\beta\}_+ = \{dx_\alpha, dx_\beta\}_+ = 0, \tag{2.5}
\]

where the (multi)index \( \alpha \) denotes the variable (e.g. \( y_{i_1,i_2}^* \)).

The loop \( C_{k,l} \) with (at least one nonzero) winding numbers \([k,l]\) also can be regarded as
the contour in the covering plane, starting at \((0,0)\) and ending at \((kN_1,lN_2)\). We can compute
easily \( I(C_{k,l}) \) as follows. First we perform explicitly the integration over the star-variables in
(2.1) which gives

\[
I(C) = \int \prod_{\text{sites of } C} B_r(site), \tag{2.6}
\]

where \( B_r \) are associated with the pairs of half-links entering a site according the Fig.2 and are
given by

\[
B_3 = dx_{i_1-1,i_2}x_{i_1,i_2}, \quad B_4 = dy_{i_1,i_2-1}y_{i_1,i_2}, \quad B_5 = dx_{i_1-1,i_2}y_{i_1,i_2},
\]

\[
B_6 = dy_{i_1,i_2-1}x_{i_1,i_2}, \quad B_7 = dx_{i_1-1,i_2}dy_{i_1,i_2-1}, \quad B_8 = x_{i_1,i_2}y_{i_1,i_2} \tag{2.7}
\]

Then we realize that each nonselfintersecting contour \( C_{k,l} \) can be completed to a closed
nonselfintersecting loop \( C_{k,l}^c \) in the covering plane in such a way, that the part, which completes
the original contour, is the union of three straight lines (see Fig.8 and, for the proof, Lemma 5 of the Appendix). For concreteness, we shall work with the case \( l \neq 0 \), in which two of three completing lines are horizontal and they join the original contour at the vertex of the type \( A_6 \) (see Fig.8). Now denote \( C_{k,l}^+ \) the piece of \( C_{k,l}^c \) which completes \( C_{k,l} \) in the covering plane. We can write (see Fig.8)

\[
I(C_{k,l}^c) = \int \left[ \prod_{\text{sites of } C_{k,l}^c} B_r(\text{site}) \right] dx_{-1,0} dx_{0,0} dx_{kN_1-lN_2} dy_{kN_1,lN_2-1} \\
= \int \left[ \prod_{\text{sites of } C_{k,l}^+} B_r \right] \left[ \prod_{\text{sites of } C_{k,l}^c} B_r \right] dx_{-1,0} dx_{kN_1-lN_2} dy_{kN_1,lN_2-1} x_{0,0} \\
= I(C_{k,l}) \times \int \left[ \prod_{\text{sites of } C_{k,l}^+} B_r(\text{site}) \right] dx_{-1,0} dx_{kN_1-lN_2} = 1
\]

Hence

\[
I^{-1}(C_{k,l}) = \int \left[ \prod_{\text{sites of } C_{k,l}^+} B_r(\text{site}) \right] dx_{-1,0} dx_{kN_1-lN_2} = -1. \tag{2.9}
\]

All remaining cases (\( l = 0 \) or joining the original contour at the vertex of different type) can be treated in full analogy with this one and we get for all \([k, l]\), except \([0, 0]\),

\[
I(C_{k,l}) = -1. \tag{2.10}
\]

We conclude with the formula valid for all \([k, l]\) i.e.

\[
I(C_{k,l}) = (-1)^{k+l+kl}, \tag{2.11}
\]

(2.11) holds due to the fact that nonselfintersecting loops on the toroidal lattice with both winding numbers even are necessarily topologically trivial. The proof of this statement we present in the Lemma 2 of the Appendix.
Consider the following Grassmann integral, defined in the Grassmann algebra associated with the toroidal lattice in the way described in the preceding section

\[ S_{N_1, N_2}(\rho'_r) = \int \left( \prod_{\text{sites } r=1}^8 \rho'_r A_r(\text{site}) \right) \times \exp \sum_{i_1, i_2} \left[ x_{i_1, i_2} x_{i_1, i_2+1}^* + y_{i_1, i_2} y_{i_1, i_2+1}^* \right] \prod_{\text{links}} d\mu_{\text{link}} \]  

(3.1)

where we set

\[ A_1 = A_7 A_8, \quad A_2 = 1 \]  

(3.2)

The numbers \( \rho'_r \) are taken from the interval \( <0, 1> \). Expanding the product in (3.1) we get the sum of the terms of the type

\[ \prod_{i_1, i_2} A_r(i_1, i_2) \rho'_{r(i_1, i_2)} \]  

(3.3)

Each such term can be graphically represented as the lattice with the vertices \( A_r \) marked according Fig.2. Clearly not all such terms will contribute to \( S_{N_1, N_2}(\rho'_r) \). Indeed, if there is a link with just one marked half, the Grassmann variable corresponding to the other half of the link is absent in the integrand and, due to Berezin rules (2.4), the term possessing such a link gives zero contribution. Now it is easy to see, that terms, the graphical representation of which do not have the halfoccupied links, do contribute to \( S_{N_1, N_2}(\rho'_r) \), because if there is a link with no marked half, the corresponding pair of the Grassmann variables sitting at the exponent in (3.1) makes the contribution nonzero. As the example of the graphical representation of a contributing term, we can take Fig.4 in which the vertex \( A_1 \) is drawn as the union of the vertices \( A_7 \) and \( A_8 \). Such a rule of drawing \( A_1 \) enables us to consider the contributing terms as the weighted configurations of nonselfintersecting and mutually nonintersecting loops living on two dimensional toroidal lattice. Moreover, we can write for \( S_{N_1, N_2}(\rho'_r) \)

\[ S_{N_1, N_2}(\rho'_r) = \sum_{\text{configurations of loops}} \left( \prod_{r=1}^8 \rho'_{r} \right) \prod_{j} I(C_j) \]  

(3.4)

where \( a_r \) are numbers of \( r \)-vertices in the configuration, \( I(C) \) is the integral (2.1) and pieces of loops meeting each other at one vertex should avoid each other in the way depicted in Fig.4. If \( I(C) \) were always +1, \( S_{N_1, N_2}(\rho'_r) \) would give us precisely the partition function of the eight-vertex model with the Boltzmann weights \( \rho'_r \) (generally in an external field) [1]. As we see
from (2.11), the noncontractible loops spoil the interpretation of $S_{N_1,N_2}(\rho'_r)$ as the statistical partition function of some model, since there are configurations entering the sum with negative weights. We can save the day, however, as follows. Take odd the both dimensions $N_1$ and $N_2$ of the lattice and set

$$
\rho'_1 = \lambda_1 \lambda_2 \rho_1, \quad \rho'_2 = \rho_2, \quad \rho'_3 = \lambda_1 \rho_3, \quad \rho'_4 = \lambda_2 \rho_4,
$$

$$
\rho'_5 = \lambda_2 \rho_5, \quad \rho'_6 = \lambda_1 \rho_6, \quad \rho'_7 = \lambda_1 \lambda_2 \rho_7, \quad \rho'_8 = \rho_8.
$$

Then

$$
\left( \prod_{r=1}^{8} \rho'_r^{a_r} \right) \prod_j I(C_j) = \left( \prod_{r=1}^{8} \rho_r^{a_r} \right) \prod_j \left[ I(C_j) \lambda_1^{h_j} \lambda_2^{v_j} \right],
$$

where $h_j$ and $v_j$ are the numbers of the horizontal and the vertical links of the $j$-th loop respectively. If $N_1$ and $N_2$ are odd then

$$
(-1)^{h_j} = (-1)^{k_j},
$$

$$
(-1)^{v_j} = (-1)^{l_j},
$$

where $[k_j, l_j]$ are the winding numbers of the loops, respectively. Then we realize (for the proof see Lemma 4 of the Appendix), that for the configuration of nonselfintersecting and mutually nonintersecting loops $C_j$ on the lattice, it holds

$$
\prod_j I(C_j) = (-1)^{\sum k_j + \sum l_j + (\sum k_j)(\sum l_j)}
$$

Define

$$
Z_{N_1,N_2}(\lambda_1, \lambda_2) \equiv \frac{1}{2} \left[ S_{N_1,N_2}(-\lambda_1, -\lambda_2) + S_{N_1,N_2}(\lambda_1, \lambda_2) + S_{N_1,N_2}(\lambda_1, -\lambda_2) - S_{N_1,N_2}(\lambda_1, \lambda_2) \right]
$$

Combining (3.6),(3.7) and (3.8), it follows

$$
Z_{N_1,N_2}(\lambda_1 = 1, \lambda_2 = 1) = \sum_{\text{configurations of loops}} \left( \prod_{r=1}^{8} \rho_r^{a_r} \right),
$$

7
or, in other words, the combination (3.9) of \( S_{N_1,N_2}(\pm \lambda_1, \pm \lambda_2) \) gives the partition function of the eight-vertex model. Without loss of generality, setting \( \rho_2 = 1 \) we may rewrite (3.1) as follows

\[
S_{N_1,N_2}(\rho_r') = \int \exp \left[ \sum_{\text{sites}} \left( \sum_{r=3}^{8} \rho'_r A_r(\text{site}) + (\rho'_1 + \rho'_3\rho'_4 - \rho'_5\rho'_6 - \rho'_7\rho'_8)A_7(\text{site})A_8(\text{site}) \right) \right] 
\times \exp \left[ \sum_{\text{sites}} (x_{i_1,i_2}x_{i_1+1,i_2}^* + y_{i_1,i_2}y_{i_1+1,i_2}^*) \right] \prod_{\text{links}} d\mu_{\text{link}}
\]  

(3.11)

Note, that if

\[
\rho'_1 = \rho'_5\rho'_6 + \rho'_7\rho'_8 - \rho'_3\rho'_4,
\]  

(3.12)

the expression in the exponent is the quadratic form in the Grassmann algebra with the cyclic matrix and, therefore, \( S_{N_1,N_2}(\rho'_r) \) can be computed easily in this case. The constraint (3.12) is well-known and gives the so-called free fermionic sector of the eight-vertex model [1]. If (3.12) does not hold, we have the quartic term in the exponent, hence, we may call the formulas (3.11),(3.5) and (3.9) interacting fermionic representation of the eight-vertex model.

In what follows we shall study the particular case* in which

\[
\rho_3 = \rho_4 = 0, \quad \rho_2 = \rho_5 = \rho_6 = \rho_7 = \rho_8 = 1, \quad \rho_1 = 2,
\]  

(3.13)

\[
\lambda_1 = \lambda_2 = e^{-\beta}.
\]  

(3.14)

(3.12) is obviously satisfied, therefore we can easily compute \( S_{N_1,N_2}(\rho'_r) \). Before doing that, however, let us look more closely on the set of configurations when conditions (3.13) apply. The vertices \( A_3 \) and \( A_4 \) are absent, thus the loops locally have a "zig-zag" shape i.e. at every vertex of the lattice the loop has to change its direction. Moreover, the vertex \( A_1 \) enters with the weight \( \rho_1 = 2 \). Without endangering our results, in this particular case, we may abandon our (conventional) way of drawing the vertex \( A_1 \), as described in Fig.4. The only aim of this convention constituted in representing the contributing configurations to \( S_{N_1,N_2}(\rho'_r) \) as the configurations of nonintersecting loops. If the vertex \( A_1 \) itself enters with the weight \( \rho_1 = 2 \), we can view it as corresponding to two ways of connecting the half-links of the \( A_1 \) vertex (see Fig.6). The factor \( e^{-\beta} \) causes that the weight \( w_c \), which the loop enters the sum with, is given by

\* This "six-vertex" model differs from the usual one, in which \( \rho_7 = \rho_8 = 0 \)
\[ w_c = e^{-L_c\beta}, \quad (3.15) \]

where \( L_c \) is the length of the loop. Indeed, each link is accompanied by one of the factors \( \lambda_1 \) or \( \lambda_2 \), as it follows from (3.5). Summarizing, we may look at the particular case (3.13-14) as at the lattice regularization of the model of free classical strings (loops) with energies proportional to their lengths and with the configurations \( 6_a \) and \( 6_b \) considered as different (corresponding to splitting and joining of strings).

The free fermionic representation (3.9) and (3.11) of the partition function of such string model and the value (2.11) of the Grassmann integral topological invariant make possible to evaluate the statistical mean values of some topological characteristics of the strings such as the functions of the winding numbers are. In our case, we actually have a constraint in the space of loops, since the lattice regularization respects the zig-zag rule and the dimensions \( N_1 \) and \( N_2 \) of the lattice are both odd. In fact, the zig-zag rule means that

\[ h = v, \quad (3.16) \]

where \( h(v) \) is the horizontal (vertical) length of the string, respectively. Combining this fact with (3.7), we see

\[ (-1)^k = (-1)^l, \quad (3.17) \]

where \( k(l) \) is the horizontal (vertical) winding number of the string, respectively. Therefore, the parities of both winding numbers of the string are the same.

In what follows, we shall calculate the quantity

\[ < (-1)^K >_{\beta,V \to \infty}, \quad (3.18) \]

where \( V \equiv N_1N_2 \) is the volume of the system and \( K \) is the total horizontal winding number of the configuration of strings.** It is given by

\[ < (-1)^K > = \frac{\sum_{configurations} (-1)^{k_j} e^{-\beta \sum L_j}}{\sum_{configurations} e^{-\beta \sum L_j}} \quad (3.19) \]

where \( k_j(L_j) \) is the horizontal winding number (total length) of the \( j \)-th string of the configuration. In the case of our "zig-zag" string model (3.8) and (3.17) imply

** The strings are nonoriented, nevertheless \((-1)^K\) can be defined unambiguously assigning to each string whatever orientation.
\[
\prod_j I(C_j) = (-1)^{\sum k_j},
\]

(3.20)

hence, following (3.4), we may write

\[
< (-1)^K > = \frac{S_{N_1,N_2}(\lambda_1 = e^{-\beta}, \lambda_2 = e^{-\beta})}{Z_{N_1,N_2}(\lambda_1 = e^{-\beta}, \lambda_2 = e^{-\beta})},
\]

(3.21)

where \(Z_{N_1,N_2}(\lambda_1, \lambda_2)\) is the partition function of the model and it is given by (3.9). In our "zig-zag" case

\[
S_{N_1,N_2}(-\lambda_1, \lambda_2) = S_{N_1,N_2}(\lambda_1, -\lambda_2),
\]

(3.22a)

\[
S_{N_1,N_2}(-\lambda_1, -\lambda_2) = S_{N_1,N_2}(\lambda_1, \lambda_2),
\]

(3.22b)

as it follows from (3.4),(3.6) and (3.16). Using (3.9) then we have

\[
Z_{N_1,N_2}(\lambda_1, \lambda_2) = S_{N_1,N_2}(-\lambda_1, \lambda_2)
\]

(3.23)

and

\[
< (-1)^K > = \frac{S_{N_1,N_2}(\lambda_1 = e^{-\beta}, \lambda_2 = e^{-\beta})}{S_{N_1,N_2}(\lambda_1 = -e^{-\beta}, \lambda_2 = e^{-\beta})}.
\]

(3.24)

Actual computation of \(S_{N_1,N_2}\) is easy, since the model fulfills the free fermionic constraint (3.11). Using the discrete Fourier transformation we can easily diagonalize the matrix of the quadratic form sitting at the exponent of (3.10) and obtain

\[
S_{N_1,N_2}(-e^{-\beta}, e^{-\beta}) = (1 + 2e^{-2\beta}) \prod_{(p,q)\neq(0,0)} (4e^{-4\beta} + 1 + 4e^{-2\beta} \cos \frac{2\pi}{N_1} p \cos \frac{2\pi}{N_2} q)^{1/2}
\]

(3.25a)

and

\[
S_{N_1,N_2}(e^{-\beta}, e^{-\beta}) = (1 - 2e^{-2\beta}) \prod_{(p,q)\neq(0,0)} (4e^{-4\beta} + 1 - 4e^{-2\beta} \cos \frac{2\pi}{N_1} p \cos \frac{2\pi}{N_2} q)^{1/2},
\]

(3.25b)

where the square roots in (3.25) should be taken positive. Note that at certain critical inverse temperature
\[
\beta_c = \ln \sqrt{2}
\]

\( S_{N_1, N_2}(e^{-\beta}, e^{-\beta}) \) changes its sign! Moreover, the free energy per site of the model

\[
F(\beta) = 1 - \frac{1}{2\pi \beta} \int_0^\pi dx \text{Arch} \left[ \frac{\cosh(2\beta - \ln 2)}{\cos x} \right]
\]

is nonanalytic at \( \beta_c \) and we have the second-order phase transition in the system.

The computation of \( < (-1)^K > \) is not difficult. We use the formulas \([4,6]\)

\[
2^{n-1} \prod_{r=0}^{r=n-1} \cos \left( \theta + \frac{2\pi}{n} r \right) = \cos n\theta,
\]

\[
2^{n-1} \prod_{r=0}^{r=n-1} \{ \cosh \phi - \cos \left( \theta + \frac{2\pi}{n} r \right) \} = \cosh n\phi - \cos n\theta
\]

and write

\[
S_{N_1, N_2}(e^{-\beta}, e^{-\beta}) = e^{-\beta N_1} N_2^{2N_1 + N_2} \cosh N_1(\beta - \ln \sqrt{2})
\]

\[
\prod_{\frac{\pi}{N_2} q \in (0, \frac{\pi}{2})} \cosh^2 \left[ \frac{N_1}{2} \text{Arch} \left( \frac{\cosh (2\beta - \ln 2)}{\cos \frac{2\pi}{N_2} q} \right) \right] \prod_{\frac{\pi}{N_2} q \in (\frac{\pi}{2}, \pi)} \sinh^2 \left[ \frac{N_1}{2} \text{Arch} \left( \frac{\cosh (2\beta - \ln 2)}{\cos \frac{2\pi}{N_2} q} \right) \right]
\]

\[
S_{N_1, N_2}(e^{-\beta}, e^{-\beta}) = e^{-\beta N_1} N_2^{2N_1 + N_2} \sinh N_1(\beta - \ln \sqrt{2})
\]

\[
\prod_{\frac{\pi}{N_2} q \in (0, \frac{\pi}{2})} \sinh^2 \left[ \frac{N_1}{2} \text{Arch} \left( \frac{\cosh (2\beta - \ln 2)}{\cos \frac{2\pi}{N_2} q} \right) \right] \prod_{\frac{\pi}{N_2} q \in (\frac{\pi}{2}, \pi)} \cosh^2 \left[ \frac{N_1}{2} \text{Arch} \left( \frac{\cosh (2\beta - \ln 2)}{\cos \frac{2\pi}{N_2} q} \right) \right]
\]

Dividing (3.29a) by (3.29b) we get from (3.24)

\[
< (-1)^K >_{\beta, N_1 \to \infty} = \text{sign}(\beta - \ln \sqrt{2}),
\]

hence

\[
< (-1)^K >_{\beta, V \to \infty} = \text{sign}(\beta - \ln \sqrt{2}).
\]
We observe that \( <(-1)^K > \) can be interpreted as the topological order parameter which says that at low temperatures the "even" configurations dominate while at high temperatures the "odd" (and necessarily topologically nontrivial) configurations are dominant. Note that from the technical point of view, we could obtain the result particularly due to the invariant character of the integral (2.1) and due to formula (2.11).

4. CONCLUSIONS.

In this contribution, we have constructed and evaluated the Grassmann integral topological invariants and found the applications of these results in the field of statistical physics. In particular, we were able to introduce and to evaluate the topological order parameter in the toy string model constructed by the author previously in [6]. We believe that our results can be generalized in two directions. First one, more mathematical, would constitute in further study of the Grassmann integrals, either with more complicated integrands or over the loops living on more complicated manifolds. The second direction would constitute in studying the existence and the behaviour of the topological order parameters in various physical systems. We intend to pursue these problems with the hope of obtaining new interesting results.
APPENDIX.

**Lemma 1:** Small deformations of loops in the plane, induced by removing a single plaquette from the domain bounded by the loop, do not change the value of the integral (2.1), provided the removed plaquette participates on the nonempty connected piece of the boundary of the domain.

**Proof:** In Fig.5a-e we list all possibilities of removing the plaquette in the way described in the formulation of the lemma. The plaquette to be removed is depicted. The term ”rotations” in Fig.5 means three other possible orientations of the drawing rotated by the multiple of $\frac{\pi}{2}$ with respect to the depicted one. The new deformed loop follows the dashed line. Both original and deformed loops have (large) common part alluded by the half-links. In Fig.5b,c,d there are two half-links attached to each ”connecting” vertex indicating two possible ways of the continuation of the loop from the vertex. The proof of the lemma is performed by the simple inspection of all possibilities. For concreteness, we present the proof for the case of Fig.5b in the original (i.e. nonrotated) orientation and with both continuing half-links pointing to the horizontal directions. We use formulae (2.6),(2.7) and write

$$I(C_{\text{original}}) =$$

$$= \int \ldots (dx_{i_1-1,i_2}dy_{i_1,i_2-1})(x_{i_1,i_2-1}y_{i_1,i_2-1})(dy_{i_1+1,i_2-1}x_{i_1+1,i_2}) \ldots$$

$$= \int \ldots dx_{i_1-1,i_2}x_{i_1+1,i_2} \ldots = \int \ldots (dx_{i_1-1,i_2}x_{i_1,i_2})(dx_{i_1,i_2}x_{i_1+1,i_2}) \ldots = I(C_{\text{deformed}})$$

(A.1)

where the dots stand instead of terms common to both loops. Analogously we can verify all remaining cases, thus proving the lemma.

**Lemma 2:** Let $C$ be the nonselfintersecting noncontractible loop on the toroidal lattice of the type $[am, an], a \in Z$. Then $|a| = 1$. In particular, the nonselfintersecting loops on the toroidal lattice with both winding numbers even are necessarily topologically trivial.

**Proof:** Consider the nonselfintersecting noncontractible loop $C$ on the toroidal lattice, with the winding numbers $[k, l]$. To each point $P_0 \equiv (p_1N_1, p_2N_2); p_1, p_2 \in Z$ we associate the (infinitely long) ”covering” contour $C_{\text{cov}}(P)$, which connects the point $P_b \equiv (p_1N_1 + bkN_1, p_2N_2 + blN_2)$ with the point $P_{b+1} \equiv (p_1N_1 + (b+1)kN_1, p_2N_2 + (b+1)lN_2)$ for all integer $b$ and, for a given $b$, $C_{\text{cov}}(P)$ is given by the canonical mapping of the loop $C$ to the covering plane. The contour $C_{\text{cov}}(P)$ divides the covering plane in two pieces $L_+(C_{\text{cov}}(P))$
and \( L(C_{\text{cov}}(P)) \) defined as follows (see Fig. 7)

\[
\lim_{t \to \pm \infty} (-tlN_2, tkN_1) \in L_{\pm}(C_{\text{cov}}(P)), \quad t \in \mathbb{Z}. \quad (A.2)
\]

The piece of the contour \( C_{\text{cov}}(P) \) between the points \( P_b \) and \( P_{b+1} \) is finite, hence the contour \( C_{\text{cov}}(P) \) is necessarily contained in the strip, which is finitely thick in the transverse direction given by the "normal" vector \((-lN_2, kN_1)\). The definition \((A.2)\), therefore, is self-consistent.

Let \( P \equiv (p_1N_1, p_2N_2), Q \equiv (q_1N_1, q_2N_2) \) be two points in the covering plane. We show that

\[
Q \in L_{\pm}(C_{\text{cov}}(P)) \iff P \in L_{\mp}(C_{\text{cov}}(Q)). \quad (A.3)
\]

Indeed, if \( Q \in L_{\pm}(C_{\text{cov}}(P)) \) then \( C_{\text{cov}}(Q) \subset L_{\pm}(C_{\text{cov}}(P)) \), since the loop \( C \) is nonselfintersecting. Then, obviously, it follows \( C_{\text{cov}}(P) \subset L_{\mp}(C_{\text{cov}}(Q)) \), hence \( P \in L_{\mp}(C_{\text{cov}}(Q)) \). In complete analogy the inverse implication is valid.

Define

\[
P >_C Q \iff Q \in L_-(C_{\text{cov}}(P)), \quad (A.4a)
\]

\[
P =_C Q \iff Q \in C_{\text{cov}}(P), \quad (A.4b)
\]

\[
P <_C Q \iff Q \in L_+(C_{\text{cov}}(P)). \quad (A.4c)
\]

Obviously \( P >_C Q \) and \( Q >_C R \) implies \( P >_C R \), since \( R \in L_-(C_{\text{cov}}(Q)) \) and \( L_-(C_{\text{cov}}(Q)) \subset L_-(C_{\text{cov}}(P)) \).

Now consider the nonselfintersecting noncontractible loop \( C \) with the winding numbers \([k = am, l = an], a \in \mathbb{Z}\), the points \( P_0 = (0,0), P_1 = (|a|mN_1, |a|nN_2), Q_1 = (mN_1, nN_2), Q_2 = (2mN_1, 2nN_2), \ldots, Q_{|a|−1} = ((|a| - 1)mN_1, (|a| - 1)nN_2)\) and the covering contour \( C_{\text{cov}}(P_0) \). Suppose \( P_0 >_C Q_1 \). Due to invariance with respect to the shifts of the covering plane induced by the vectors \((rN_1, sN_2); r, s \in \mathbb{Z}\), we have \( Q_1 >_C Q_2 >_C \ldots >_C Q_{a−1} >_C P_1 \). Hence \( Q_1 >_C P_1 \). On the other hand, \( C_{\text{cov}}(P_0) \equiv C_{\text{cov}}(P_1) \), which, together with \((A.3)\), implies \( P_0 <_C Q_1 \) and we ended up with the contradiction. Analogously, the assumption \( P_0 <_C Q_1 \) leads to the contradiction \( P_0 >_C Q_1 \). It remains the possibility \( P_0 =_C Q_1 \). In this case \( Q_1 \in C_{\text{cov}}(P_0) \), hence, since \( C \) is nonselfintersecting, \( Q_1 \equiv P_1 \) and the loop has the winding numbers \([am, an]\) where \(|a| = 1\).
Lemma 3: Let $C$ be a noncontractible loop of the type $[k, l]$. Let $P \in C$, $P \equiv (0, 0)$ be the point in the covering plane of the torus; $t, p, q, x, y$ are integers. Then (for the notation see the proof of Lemma 2)

$$
\lim_{t \to +\infty} (tpN_1 + x, tqN_2 + y) \in L_\pm(C_{cov}(P)) \iff \lim_{t \to -\infty} (tpN_1, tqN_2) \in L_\mp(C_{cov}(P)), \quad (A.5)
$$

unless $[p, q] = [ck, cl]; c \in Z$. In other words, following the line $(tpN_1 + x, tqN_2 + y)$, where $t$ varies, we have to connect two pieces $L_+ (C_{cov}(P))$ and $L_- (C_{cov}(P))$ of the covering plane, unless $[p, q] = [ck, cl]$.

Proof: The contour $C_{cov}(P)$ runs through the points $(bkN_1, blN_2); b \in Z$. Between the points $(bkN_1, blN_2)$ and $((b + 1)kN_1, (b + 1)lN_2)$ the length of the contour is finite, since the original loop $C$, living on the torus, has finite length. That means that the strip $S$ exists in the covering plane, containing $C_{cov}(P)$ and dividing the covering plane $L$ in three pieces i.e. $S, S_+$ and $S_-$, such that $S_\pm \subset L_\pm(C_{cov}(P))$. Since the boundary lines between $S_+$ and $S_-$ and between $S_+$ and $S_-$ have the tangent vector $[kN_1, lN_2]$, the line with the tangent vector $[pN_1, qN_2]$ has to connect $S_+$ with $S_-$ (and, therefore, $L_+ (C_{cov}(P))$ with $L_- (C_{cov}(P))$) unless $[p, q] = [ck, cl]$.

Lemma 4: For the configuration of nonselfintersecting and mutually nonintersecting loops $C_j$ on the toroidal lattice with the winding numbers $[k_j, l_j]$, it holds

$$
\prod_j I(C_j) = (-1)^{\sum k_j + \sum l_j + (\sum k_i)(\sum l_j)}. \quad (A.6)
$$

Proof: If at most one loop of the configuration is noncontractible, the proposition obviously holds. Now let $C_1$ and $C_2$ be two noncontractible mutually nonintersecting loops at the toroidal lattice, of the types $[k_1, l_1]$ and $[k_2, l_2]$ respectively. Take two points $P_1 \in C_1$ and $P_2 \in C_2$ and lift them to the fundamental domain of the covering plane. Consider then two contours $C_{1,cov}(P_1)$ and $C_{2,cov}(P_2)$. Without loss of generality, we set $P_1 = (0, 0), P_2 = (x, y)$. The contour $C_{2,cov}(P_2)$ runs through the points $(bk_2N_1 + x, bl_2N_2 + y); b \in Z$. Following Lemma 3, unless $[k_2, l_2] = [ck_1, cl_1], c \in Z$, it has to connect $L_+ (C_{1,cov}(P_1))$ with $L_- (C_{1,cov}(P_1))$ and, consequently, to intersect $C_{1,cov}(P_1)$. Since the loops are nonintersecting, this means $[k_2, l_2] = [ck_1, cl_1]$. From Lemma 2 it follows, however, that $|c| = 1$. In conclusion, the loops $C_1$ and $C_2$ have the same winding numbers (up to the sign). Analogously it can be shown that $n$ mutually nonintersecting noncontractible loops on the toroidal lattice must have the same winding numbers. The formula (A.6) then trivially follows.
Lemma 5: Each noncontractible nonselfintersecting loop $C$ on the toroidal lattice can be completed to the closed nonselfintersecting loop in the covering plane, in such a way that the part, which completes the original contour, is the union of three straight lines (see Fig. 8).

Proof. Consider the loop $C$ of the type $[k, l]; l \neq 0$. Following Lemma 3, each (infinite) horizontal line has to intersect $C_{\text{cov}}(P)$. Pick up two horizontal lines, $H_0$ and $H_1$, with the mutual vertical distance equal to $N_2$. Travelling from the left these lines intersect $C_{\text{cov}}(P)$ for the first time at the points $B_0$ and $B_1$, respectively. Now there exist two points $A_0$ and $A_1$, belonging to the first and the second line, respectively, such that they both lie to the left from the points $B_0$ and $B_1$, respectively, their horizontal coordinate is the same and the line $A_0 - A_1$ does not intersect $C_{\text{cov}}(P)$. The loop $C^c$ connecting the points $A_0 - A_1 - B_1 - B_0 - A_0$ is the closed nonselfintersecting loop in the covering plane, with the required properties. If $l = 0$, we take $V_0$ and $V_1$ to be the vertical lines with the mutual horizontal distance equal $N_1$ and construct the loop $C^c$ in the full analogy with the previous case.
FIGURE CAPTIONS.

Fig.1: The association of the Grassmann variables to the halflinks of the lattice.

Fig.2: The association of the (ordered) products of the Grassmann variables to each even subset of the halflinks attached to a single site of the lattice.

Fig.3: The covering plane of the torus. The rectangle $(0, 0) - (N_1, 0) - (N_1, N_2) - (0, N_2)$ is the fundamental domain. Other rectangles are its copies.

Fig.4: The illustration of the allowed way of the deformations of loops by removing the plaquettes. The plaquette marked by ”n” participates on the nonempty disconnected piece of the boundary of the domain, hence, it must not to be removed. The plaquette ”y”, instead, can be removed yielding the deformed loop 4b. Note the way of drawing the vertex of the type $A_1$.

Fig.5: The ways of deforming the loops by removing a single plaquette. The plaquette to be removed is depicted. The term ”rotations” means three other possible orientations of the drawing, rotated by the multiple of $\frac{\pi}{2}$ with respect to the depicted one. The new deformed loop follows the dashed line. Both original and deformed loops have (in general large) common part, alluded by the halflinks. In Fig.b,c,d there are two halflinks attached to each ”connecting” vertex, indicating two possible ways of the continuation of the loop from the vertex.

Fig.6: Two ways of connecting the halflinks of the $A_1$ vertex, corresponding to splitting and joining of strings.

Fig.7: The illustration of the construction of the contour $C_{cov}(P)$. The contour divides the covering plane in two pieces, marked by $L_+$ and $L_-$. The ”normal” line to $C_{cov}(P)$ is also depicted. Both winding numbers are chosen to be positive.

Fig.8: Completing of the contour in the covering plane by the union of three straight lines.
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