Red–shifts near black holes

Adam D Helfer

Department of Mathematics, Mathematical Sciences Building, University of Missouri, Columbia, Missouri 65211, U.S.A.

Abstract. A simple ordinary differential equation is derived governing the red–shifts of wave–fronts propagating through a non–stationary spherically symmetric space–time. Approach to an event horizon corresponds to approach to a fixed point; in general, the phase portrait of the equation illuminates the qualitative features of the geometry. In particular, the asymptotics of the red–shift as a horizon is approached, a critical ingredient of Hawking’s prediction of radiation from black holes, are easily brought out. This asymptotic behavior has elements in common with the universal behavior near phase transitions in statistical physics. The validity of the Unruh vacuum for the Hawking process can be understood in terms of this universality. The concept of surface gravity is extended to to non–stationary spherically symmetric black holes. Finally, it is shown that in the non–stationary case, Hawking’s predicted flux of radiation from a black hole would be modified.

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1. Introduction

By definition, a black hole cannot be directly observed by someone in its exterior. For this reason, it is the geometry of the neighborhood of the horizon, as measured by later and later observers, which is of interest in the external physics of the holes. A basic element of this geometry is ray–tracing: looking in from far away towards where the hole will form (and neglecting opacity), what would one see?

An estimate of just this, for rays sufficiently close to the horizon in the Schwarzschild (or more generally Kerr–Newman\(^1\)) geometries, is a cornerstone of Hawking’s (1974, 1975) prediction of thermal radiation from black holes. In the Schwarzschild case, Hawking argued that the asymptotic ray–tracing relation between surfaces of constant phase labeled by retarded time \(u\) in the future to those of constant phase labeled by advanced time \(v\) in the past would be

\[
v(u) \simeq C \exp \left( -\frac{u}{4m} \right),
\]

as \(u \to +\infty\), that is, as the horizon is approached. Here \(m\) is the mass in geometric units. It is \(v(u)\) which is the key function in the Hawking analysis; for example, the Hawking temperature comes out of the calculation as \(T_H = - (1/2\pi) \ddot{v} / \dot{v}\).

The present paper is concerned with the justification of the relation (1), and the derivation of similar relations for more general situations. It will be well to first explain what is known.

Hawking’s argument for the relation (1) was brief and perhaps elliptic; the taking into account of the effects of tracing the rays back through the collapsing matter was not really spelled out. The argument relied on specific properties of the Schwarzschild (or Kerr–Newman) solutions. A fuller treatment of the spherically symmetric case, assuming a static exterior, was given by Birrell and Davies (1982). Their argument was by computation in a particular coordinate system, and its relation to invariant geometry is not manifest.

These arguments leave open two sorts of issues. First, one would like an invariant explanation of where the relation (1) comes from; and second, one would like to be able to treat a broader class of objects. For example, one would like to know what happens for a black hole accompanied by an accretion disk, or a spherically symmetric hole by a (non–static) nebula. What happens if the collapsing matter is not sharply bounded (so the event horizon may not be in a static exterior)?\(^2\) If we are concerned (as we are in the case of the Hawking process) with a star that has contracted to within atomic or even nuclear dimensions of its Schwarzschild radius, are we justified

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\(^1\) In what follows, it will be understood that the Reissner–Nordstrom and Kerr solutions are special cases of Kerr–Newman.

\(^2\) There is a loose sense in which many astrophysical black holes are expected to be Kerr–Newman (“black hole uniqueness”). However, this is because for this class of holes one assumes that the hole approaches a stationary state and is isolated (that is, has only the electromagnetic stress–energy in its exterior).
in treating it as a classical bounded object? And what about bodies which are not quite black holes? What happens if a body collapses to \( r = 2m(1+\epsilon) \) and stays there? To investigate conceptual questions about the Hawking process, it is desirable to have an invariant way of understanding and treating these issues.

The aims of the present paper are: (a) to bring out very clearly where the formula (1) comes from; (b) to show that it has a flavor of “universal” behavior in the sense of statistical physics; (c) to show that a similar result will hold without any assumption of asymptotic stationarity or vacuum exterior for the hole; and (d) to present a general framework for the study of similar questions.

In the course of this analysis, we clarify some general features of the structure of black holes. We shall find a way of extending the concept of surface gravity to non-stationary spherically symmetric black holes. This new measure of surface gravity will be a global one, determined by red-shifts of light rays passing close to the horizon. While global, it is given by a well-defined integral formula in terms of the local geometry. This enables us to ask which details of the geometry the surface gravity is sensitive to: which contribute to the integral?

We shall see that, as the hole is approached, all contributions from the regime prior to the formation of the event horizon are suppressed, and so the surface gravity is independent of the details of the formation of the hole, as one would expect.

We shall also see that the surface gravity is essentially insensitive to the geometry of the horizon! This is in apparent conflict with the usual view that the surface gravity is determined by the gradient of the norm of a Killing vector at the horizon. There is no real conflict, however. The usual definition of the surface gravity is really a global one, since Killing’s equation serves to evolve the Killing field throughout the space-time, and the Killing field must be normalized at infinity. Thus Killing’s equation provides a sort of rigidity which connects data in one part of the space-time to data in another. This rigidity makes it impossible to use the Killing field to localize contributions to the surface gravity. Our formula, on the other hand, expresses it as an integral, and one can say which elements contribute more substantially than others. What the surface gravity turns out to depend on is how the neighborhood of the horizon is linked to future null infinity, not on what happens at the horizon itself.

Of course, in the stationary case, the understanding of the surface gravity developed here, and the usual one, are simply alternative ways of regarding the same thing. In the non-stationary case, we have a new candidate definition for the surface gravity. One would not expect this to be constant. For example, if a black hole were enclosed in a shell of matter (or radius far larger than the gravitational radius of the hole), one would expect all physics within the shell to be red-shifted due to the shell’s potential. Thus the surface gravity of the hole should appear smaller from outside the shell than if the shell were not there. If the shell were moved slowly (“adiabatically”), one would expect to a time-dependent surface gravity, with time-dependence given by the time-dependent red-shift due to the varying potential of the shell.

Our formula for the surface gravity is well-defined in such circumstances, and
beyond the adiabatic approximation. It remains independent of the details of the formation of the hole, and (in the sense of the second paragraph previous) of the geometry of the horizon.

The techniques we use are fairly elementary ones from differential geometry and ordinary differential equations. We shall show that the function \( v(u) \) is governed by a certain system of ordinary differential equations. (These equations are equivalent to some of the familiar optical equations.) Approach to an event horizon corresponds to approach to a certain kind of fixed point in the phase diagram; the asymptotics of the equations near the fixed point are what lead to the constancy of the surface gravity.

Most of this paper is concerned with the spherically symmetric case. Section 2 describes the universal character of the asymptotic behavior of the red–shift. Section 3 derives the main equations governing the evolution of the red–shift, and Section 4 discusses their consequences, in particular, the behavior of red–shifts near an event horizon. Section 5 contains comments on the non–spherically symmetric case. Section 6 sketches the implications for Hawking’s proposal in the case of non–stationary black holes.

Conventions. The conventions are those of Penrose and Rindler (1984–6). The metric has signature \(+−−−\), and the curvature satisfies \( \nabla_a \nabla_b v^d - \nabla_b \nabla_a v^d = R_{abc}{}^d v^c \). We use natural units.

2. The Universality

I will begin with a simple physical picture which brings out the “universal” character of the red–shift, without actually establishing its functional form. (Precisely what is meant by this will appear shortly.) This argument contains some restrictive assumptions, but its simplicity makes it worthwhile.

Consider a spherically symmetric collapsing object in general relativity. We shall assume the object has a sharp boundary, or limb, and that exterior to this is vacuum. Then the exterior will be a portion of the Schwarzschild solution. Inside the collapsing object is another metric, the details of which will be unimportant. We will only need to assume that the interior metric joins to the exterior metric suitably smoothly. (A metric of class \( C^1 \) would be more than enough.) We shall assume a black hole forms, so that as \( t \to +\infty \) the coordinate \( r(t) \) of the limb will satisfy \( r(t) \to 2m \).

Now trace backwards in time a radially–directed wave–packet. The wave packet in the distant future (near future null infinity, \( I^+ \)), has some characteristic wave–length with respect to the frame defined by the spherical symmetry at \( I^+ \). We are going to follow it backwards in time, through the collapsing object, and out to the distant past (\( I^- \)). We do this in two stages, which have different physical significances.
Stage I. This covers tracing the packet back from $I^+$ to the limb of the collapsing object, which is (say) at coordinate $r(t_{\text{packet out}})$. If $r(t_{\text{packet out}}) \approx 2m$, then the wave packet is deep in the potential well and has been very red–shifted. The precise measurement of the red–shift depends on the choice of the observer’s frame at $r(t_{\text{packet out}})$. However, relative to, say, observers freely falling into the hole, the red–shift is large, and becomes larger as later and later packets are used, since for these $r(t_{\text{packet out}}) \to 2m$. As later and later packets are used, this portion of the trajectory corresponds to a divergent red–shift.

Stage II. This covers tracing the packet back through the collapsing object, out the other side, and off to past null infinity $I^-$. We would expect this propagation to,
to some extent, undo the red–shift of the journey from \( I^+ \) to the limb. However, it cannot undo all of the red–shift, in the following sense. Throughout the second stage of the trip, there is no propagation which can give rise, even in the limit of later and later times, to an infinite red–shift. On this portion of the trip, the red–shift as the packet is propagated through the collapsing body remains finite, since this is only propagation over a finite non–singular region. Also the propagation from the collapsing body to \( I^- \) introduces no divergent red–shift, since on this segment of the trip, the radius of the body stays bounded away from \( 2m \) so no infinitely deep potential well is involved.

The total red–shift is the product of the red–shifts for Stages I and II, and thus has the form of a part which diverges (as later and later packets are used) times a part which remains regular. The divergent part depends only on the propagation through the vacuum region exterior to the body, and so is independent of the details of the collapsing body, that is, universal for the class of spherical sharply–bounded bodies of given mass.

We may extract the invariant information in this universality class as follows. Write the red–shift factor as \( \dot{v}(u) = w_1 w_{II} \) for the product of the two stages; then, since \( w_{II} \) will tend to a constant, we see that the limiting behavior of \( (\log \dot{v}) = \dot{v}/\dot{v} \) is independent of the trip in Stage II. It is also independent of the particular choice of Lorentz frame for measuring red–shifts near \( r = 2m \), since changes of frame will only contribute smooth multiplicative factors with finite limits. At the present level of analysis, the limiting form of \( (\log \dot{v}) \) could in principle be anything of the form \( f(u/m)/m \), where \( u \) is the retarded time and \( f \) is a dimensionless function. Explicit computations show that \( -(\log \dot{v}) \to \kappa = 1/4m \), the surface gravity of the black hole.

While we have emphasized that there is an element of physics here in common with the universal behavior familiar in statistical mechanics, it would be wrong to think that the parallel extends, in any obvious way, very much further. The picture drawn here of the formation of a black hole differs from that of a conventional thermodynamic system fundamentally. First, we are concerned here with an essentially non–stationary process, since the time–dependence of the limb’s trajectory is crucial. Second, the space–time is not a homogeneous extensive system. For example, one would not have a correlation length in the usual sense, although at a formal level one can treat the red–shift factor \( \dot{v} \) as such.

3. The Case of Spherical Symmetry

In this section we derive the differential equations governing the evolution of the red–shift, in the case of spherically symmetric space–times.

We shall assume \( (M, g_{ab}) \) is a spherically symmetric space–time, which is asymptotically flat at past and future null infinity in a sense strong enough to guarantee the existence of \( I = I^+ \cup I^- \) with their usual differential–geometric structures. Choose Bondi systems \((u, \theta, \varphi)\) on \( I^+ \) and \((v, \theta, \varphi)\) on \( I^- \) respecting the spherical symmetry. Then \( u \) will be called the retarded time coordinate, and \( v \) the
advanced time coordinate. Fix too a null vector field $n^a$ tangent to the generators of $\mathcal{I}^+$ and a null covector field $l_a$ orthogonal to the Bondi cuts at $\mathcal{I}^+$, with $l_a n^a = 1$. The normalization $n^a \nabla_a u = 1$ is assumed.

Caution. In most papers it is conventional to use $n^a$ for a tangent vector to $\mathcal{I}^+$ and $l^a$ for a tangent vector to $\mathcal{I}^-$. We shall not do this. The arguments here are global ones involving parallel transport between $\mathcal{I}^-$ and $\mathcal{I}^+$, and it is most natural to take $n^a$ and $l^a$ defined at $\mathcal{I}^+$ and then extend them to $\mathcal{I}^-$ by transporting along certain paths. This gives the opposite-to-usual senses for the vectors on $\mathcal{I}^-$.  

Note. A rigorous treatment of the differential geometry at $\mathcal{I}$ would involve conformally rescaling the metric and also rescaling the components of tangent vectors so that their limits took finite values on $\mathcal{I}$. This seemed overly formal for the relatively simple geometric issues to be considered here. Thus all calculations in this paper “at $\mathcal{I}$” should be understood as done close to $\mathcal{I}$, in the asymptotically flat regime. The sophisticated reader will have no difficulty recasting the results and arguments here in a more formal mold.

3.1. Parallelism at Infinity

One might hope that very far away from an isolated system one can use parallel transport to provide an essentially unambiguous way of identifying vectors at one point with vectors at another. In general, the situation is not quite so simple (because, while the gravitational field is locally weak, the regime far away from the source contains points very distant from each other). It turns out however that this is a reasonable hypothesis in the spherically symmetric case. (It is gravitational radiation which interferes with the use of a parallel transport to identify vectors at different points of $\mathcal{I}$; in a spherically symmetric space–time, there is no gravitational radiation.) In other words, we assume that over $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$ we have a natural way of identifying vectors at different points, which we call a parallelism.

One important consequence of this is that the red–shift of a radial null geodesic is well–defined. Suppose that we send in a wave packet with wave–vector $l^a_{\text{in}}$ in the distant past (at $\mathcal{I}^-$), and it emerges as $l^a_{\text{out}}$ at $\mathcal{I}^+$. In order to compare $l^a_{\text{in}}$ and $l^a_{\text{out}}$ we must have some way of identifying the tangent spaces at the point of emission on $\mathcal{I}^-$ and the point of arrival on $\mathcal{I}^+$. It is this which the parallelism provides. Using the parallelism, we can regard $l^a_{\text{in}}$ and $l^a_{\text{out}}$ as elements of the same space. Then the symmetry of the situation implies that that must be multiples of each other, and the red–shift is simply the constant of proportionality.

It should be noted that the red–shift factor can be given directly. Take a $u = \text{const}$ surface near $\mathcal{I}^+$ and trace it backwards until it emerges as $v = v(u) = \text{const}$ near $\mathcal{I}^-$. Then the function $v = v(u)$ is the mapping of surfaces of constant phase, and $\dot{v}(u)$ is precisely the red–shift factor.\(^3\)

\(^3\) While it may seem that $\dot{v}(u)$ gives a definition of the red–shift independent of the existence of a parallelism, this is not really so. For in order to interpret $\dot{v}(u)$ as a red–shift, one needs to know how to relate the clocks near $\mathcal{I}^+$ (relative to which the coordinate $u$ is normalized) to those near $\mathcal{I}^-$ (relative
3.2. Holonomy

In this subsection, we derive one of the two main equations for the red shift. This equation arises from a simple holonomy argument, and at first seems to be all that is necessary for understanding the red–shift. We shall see, however, in the next subsection, that there is a subtlety, and this equation must be supplemented by another in order to understand the black–hole state.

The red–shift arises by comparing the result of propagating light rays through space–time from $I^-$ to $I^+$ with the parallelism at infinity. Mathematically, this corresponds to a holonomy, that is, parallel propagation around a certain closed path $\Gamma_u$. In fact, it will be convenient to begin by defining the oppositely oriented path $-\Gamma_u$, as follows.

The path at infinity

\begin{figure}
\centering
\includegraphics{holonomy_path}
\caption{The path $\Gamma_u$ for defining the holonomy is the geodesic $\gamma_u$ ‘closed by a path at infinity’.
}
\end{figure}

\begin{figure}
\centering
\includegraphics{holonomy_path}
\caption{The path $\Gamma_u$ for defining the holonomy is the geodesic $\gamma_u$ ‘closed by a path at infinity’.
}
\end{figure}

to which $v$ is normalized.)
(a) Fix some angular coordinates \((\theta_0, \phi_0)\), and for each \(u\) follow the radial null geodesic \(\gamma_u(s)\) inwards from \((u, \theta_0, \phi_0)\) through the space–time and out to \((v, \theta_1, \phi_1)\) on \(\mathcal{I}^-\), where \(v = v(u)\), with \(v(u)\) the mapping of surfaces of constant phase.

(b) Close the path “at infinity.” What is really meant by this is that we use the parallelism at infinity to identify the tangent space at \((v, \theta_1, \phi_1)\) with that at \((u, \theta_0, \phi_0)\). We abuse terminology slightly by imagining that this parallelism arises from transport along some mathematically fictitious path at infinity.

Transport \(l^a\) and \(n^a\) parallel along \(\gamma_u(s)\).

Let \(H^a{}_b(u)\) be the holonomy around the path \(\Gamma_u\), or rather the part of the holonomy acting on vectors orthogonal to the surfaces of spherical symmetry, that is, those spanned by \(l^a\) and \(n^a\). Since parallel transport must preserve null vectors, the vectors \(l^a\) and \(n^a\) must return to multiples of themselves, or of each other. The latter possibility would require a non–trivial topology and will be ignored. Thus we must have

\[
H^a{}_b = hl^a n_b + h^{-1} n^a l_b
\]

(2)

for a scalar function \(h(u)\) which characterizes the holonomy.

We now derive a differential equation for \(h\). The relative increment in the holonomy as we move from \(\Gamma_u\) to \(\Gamma_{u+\delta u}\) is \(\delta u\) times

\[
H^{-1} \dot{H} = \dot{H} H^{-1} = (\dot{h}/h) (l^a n_b - n^a l_b) .
\]

(3)

On the other hand, this quantity may be computed by integrating the curvature tensor along \(\gamma_u\). If we parallel transport \(l^a\) and \(n^a\) along \(\gamma_u\), then \(l^a\) will be tangent to \(\gamma_u\) and \(n^a \delta u\) will be a connecting vector to \(\gamma_{u+\delta u}\), so we shall have

\[
H^{-1} \dot{H} = \int R^a pb^a l^p n^b ds
\]

(4)

where \(s\) is an affine parameter normalized to \(l^a \nabla_a s = 1\) (and we understand that the integration implicitly relies on parallel transport to identify tensor fields along \(\gamma_u\)). Comparing (3) and (4), we have

\[
\dot{h}/h = \int_{\gamma_u} R^a pb^a l^p n^a n_b ds .
\]

(5)

We next recast this as a differential equation for \(v(u)\).

By definition, the result of transporting the vector \(n^a = \partial_u\) at \(\mathcal{I}^+\) backwards along \(\gamma_u\) to \(\mathcal{I}^-\) is a vector \(\dot{v}(u) \partial_v\) at \(\mathcal{I}^-\). If this transport is then closed “at infinity,” we come back to \(\dot{v}(u) \partial_u\), and thus

\[
\dot{v}(u) n^a = H^{-1} n^a = h n^a,
\]

(6)

or

\[
\dot{v}(u) = h .
\]

(7)
Note that this means $\dot{v}(u) > 0$ for all $u$. Thus we have

$$\left(\log \dot{v}(u)\right)' = \int_{\gamma_u} R_D \, ds,$$

(8)

where $R_D = R_{abcd}l^a n^b l^c n^d$ is the sectional curvature. This is the first differential equation we have been seeking.

### 3.3. Interpretation and Geometry: Evolution of $ds$

The equation (8) evidently governs the evolution of the red–shift, and so at first one would think that this is all that is necessary for understanding the approach to the black–hole state. However, as mentioned above, there is a subtlety. To understand this, note the following:

(a) The sectional curvature $R_D$ is independent of the choices of scale of $l^a$ and $n^a$ as long as the normalization $l_a n^a = 1$ is preserved. Thus $R_D$ is well–defined by the local geometry of the space–time, and independent of issues of normalization at infinity.

(b) On the other hand, the measure $ds$ is normalized by the requirement $l^a \nabla_a s \big|_{I^+} = 1$. In other words, as one–forms on the geodesic $\gamma_u$, we have $ds = n_a$ at $I^+$. Notice this means that $ds$ is not normalized relative to the Bondi frame at $I^-$, rather it has been red–shifted.

(c) The measure $ds$ is dynamic; it evolves as $u$ increases.

The key feature of the geometry turns out to be (c). We shall see that as the geodesic $\gamma_u$ approaches the event horizon, the measure $ds$ tends exponentially to zero except on the portion of $\gamma_u$ connecting $I^+$ to the horizon. It is this, coupled with equation (8), which gives rise to the universality of the collapse, for it suppresses all contributions of the geometry from times prior to the formation of the hole.

We may work out the dynamics of $ds$ easily enough. Let us put

$$n^b \nabla_b l^a = c_l l^a$$

$$n^b \nabla_b n^a = c_n n^a.$$  

(9)

(Here $c_l$, $c_n$ would be written $2\Re \gamma$, $2\Re \gamma'$ in the Newman–Penrose formalism. However, we shall reserve $\gamma$ for the geodesic.) Then short computations give

$$l^a \nabla_a c_l = R_D = -l^a \nabla_a c_n,$$

(10)

and so

$$c_l(\gamma_u(s)) = -c_n(\gamma_u(s)) = -\int_{\gamma_u(s)}^{I^+} R_D(\gamma_u(s')) \, ds'. $$

(11)

The evolution of $ds$ is given by

$$\mathcal{L}_n ds = n^b \nabla_b ds = c_n ds.$$  

(12)
Equation (12) shows that as long as $c_n$ is negative and bounded away from zero in a neighborhood of the event horizon, the affine measure $d\mathbf{s}$ along the null geodesics will tend exponentially quickly to zero as the event horizon is approached. Likewise, the measure $d\mathbf{s}$ on the portions of $\gamma_u$ tending to the past of the event horizon will vanish exponentially quickly, because they are related to that on the event horizon by finite parallel-transport factors.\footnote{4}

Note that the argument above shows that, as the horizon is approached, we have $c_n \to \int_{\gamma_u} R_D \, ds$.

4. Analysis and Applications

In this section, we look at the consequences of the formulas derived above, and in particular study what happens when a black hole forms. We first summarize the main definitions and formulas.

The function $v(u)$ is the mapping of surfaces of constant phase, so $\dot{v}(u)$ is the red-shift of a wave packet propagating through the space-time along the radial null geodesic $\gamma_u$ terminating at $\mathcal{I}^+$ at retarded time $u$. The evolution of the red-shift is governed by equation (8)

$$\left(\log \dot{v}(u)\right) = \ddot{v} / \dot{v} = \int_{\gamma_u} R_D \, ds . \quad \tag{8}$$

Here $R_D = R_{abcd} l^a n^b \ell^c n^d$ is the sectional curvature, the null vectors $l^a$ (tangent to the geodesic) and $n^a$ normalized by $l_a n^a = 1$. The definition of $R_D$ does not depend on further specializations of $l^a$ and $n^a$, but other quantities in general do, and we assume that $l^a$ and $n^a$ are transported parallel along the geodesic and are normalized at $\mathcal{I}^+$. The measure $d\mathbf{s}$ is the restriction of the one-form $n_a$ to the geodesic, and so it is also transported parallel along the geodesic and normalized at $\mathcal{I}^+$. The evolution of $d\mathbf{s}$ is given by equation (12)

$$\mathcal{L}_n ds = n^b \nabla_b ds = c_n ds , \quad \tag{12}$$

where

$$c_n(\gamma_u(s)) = \int_{\gamma_u(s')}^{\mathcal{I}^+} R_D(\gamma_u(s')) \, ds' . \quad \tag{11(b)}$$

\footnote{4 There is implicit here an assumption that the geometry becomes asymptotically flat at a reasonable rate in a neighborhood of $\mathcal{I}^-$, so that the exponential decrease of $d\mathbf{s}$ is not compensated by a growth of curvature in a neighborhood of $\mathcal{I}^-$. One would certainly expect this assumption to hold for a space-time corresponding to an isolated collapsing object.}
4.1. Incoming and Outgoing Contributions

Our equation (8) for the evolution of the red–shift involves an integral over the whole of the geodesic $\gamma_u$. Near an event horizon, the contributions from different portions of this are very unequal. Essentially, as $u$ increases, it is the portions of the geodesic connecting the outside of the region where the black hole will form to $I^+$ which are important, the others being suppressed in the limit. (This portion may be thought of as corresponding to Stage I of the discussion in Section 2, whereas the contribution of the earlier part of the geodesic corresponds to Stage II.)

It is therefore convenient to re–write equation (8) by dividing the integral into two parts. We choose a point $\gamma_u(s(u))$ on the geodesic $\gamma_u$, and divide the geodesic into a ray $\gamma_{u+}$ outwards from $\gamma_u(s(u))$ to $I^+$ and a ray $\gamma_{u-}$ from $I^-$ to $\gamma_u(s(u))$. (We shall discuss the choice of $\gamma_u(s(u))$ shortly.) It will also be convenient to normalize the measure on $\gamma_{u-}$ relative to $I^-$; this measure is $d\sigma_- = (\dot{v})^{-1}ds$. We write $d\sigma_+ = ds$ for the measure on $\gamma_{u+}$. Then we let

$$a_{\pm}(u) = \int_{\gamma_{u\pm}} R_D d\sigma_{\pm}. \quad (13)$$

It should be emphasized that as long as there is no extreme injection of matter from $I^-$, or ejection of matter to $I^+$, one would expect the quantities $a_{\pm}(u)$ to remain finite and bounded and (as long as $\gamma_u(s(u))$ does not go off to infinity) generically non–zero. Then equation (8) becomes

$$(\log \dot{v}(u))' = a_-(u)\dot{v} + a_+(u) \quad (14)$$

or equivalently

$$\ddot{v} = a_-(u)(\dot{v})^2 + a_+(u)\dot{v}. \quad (15)$$

4.2. Phase Portrait

Equation (15) can be regarded as an evolution equation for $\dot{v}$. If we temporarily ignore the $u$–dependence of $a_{\pm}$, then the equation has two fixed points, one at $\dot{v} = 0$ and the other at $\dot{v} = -a_+/a_-$. The fixed point $\dot{v} = 0$ is attractive for $a_+ < 0$ and repulsive for $a_+ > 0$; the other fixed point always has the opposite character.

In general, there are no restrictions on the signs of $a_{\pm}$. The stationary case is exceptional in that it turns out that there $a_- = a_+ = 0$, if $\gamma_u(s(u))$ is chosen to be the spatial origin.\(^5\) This is clearly a non–representative situation as far as the behavior governed by equation (15); this points up the importance of studying the generic, non–stationary case.

\(^5\) Here is an outline of the proof. If there is a timelike Killing vector $\xi^a$, then the world–line of the spatial origin must be a geodesic with tangent $\xi^a$, so $\xi^b \nabla_b \xi^a$ vanishes at the origin. However, a short computation by resolving $\xi^a$ into components along $l^a$ and $n^a$ and using Killing’s equation shows that $\xi^b \nabla_b \xi^a$ comes out to be $c_n = a_+$ times a non–zero spacelike vector. Thus $a_+ = 0$; by symmetry, then $a_- = 0$. 

According to the sign of $a_+$ and the position of $\dot{v}$ relative to $-a_+/a_-$, then, various sorts of qualitative behavior are possible. (Remember that $\dot{v} > 0$ always.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.pdf}
\caption{Some possible instantaneous phase portraits for $\dot{v}$. The fixed points at zero and $-a_+/a_-$ are indicated, with the arrows near them indicating their attractive or repulsive character. Cases (a) and (d) correspond to collapse, with $\dot{v}$ being driven to zero. Case (b) is stable, with $\dot{v}$ tending to track $-a_+/a_-$. In case (c), one would have $\dot{v}$ driven towards $+\infty$.}
\end{figure}

Approach to an event horizon means $\dot{v} \to 0$, and in the simplest cases one would expect this to mean that $a_+(u)$ becomes negative and that $\dot{v}a_+$ dominates $(\dot{v})^2a_-$. Precisely, if there are constants $a$ and $k$ such that $a_+(u) < a < 0$ and (if $a_-(u) > 0$) $|\dot{v}a_+/a_+| < k < 1$, then we shall have $\ddot{v}/\dot{v} < a(1 - k)$, and hence $\dot{v} \to 0$ at least as fast as $\sim \exp a(1 - k)u$. One can think of $\dot{v}$ as collapsing to zero in this case. (See figure 3(a),(d).)

On the other hand, if $a_+ > 0$ and $a_- < 0$, then $\dot{v} = 0$ is a repulsive fixed point and $\dot{v} = -a_+/a_-$ is an attractive one. This is a stable situation. No collapse will occur, and $\dot{v}$ will tend to track the value $-a_+/a_-$. (Figure 3(b).)

A situation with $a_+, a_- > 0$ would drive $\dot{v}$ larger; if this were to persist it could drive $\dot{v} \to +\infty$ in finite retarded time. Indeed, by comparing equation (15) with

$$\ddot{v} = (\dot{v})^2a_-(u) , \tag{16}$$

which is easily integrated, one sees that $\dot{v} \to +\infty$ at or before the time $u$ such that

$$\int_{u_0}^u a_-(u')du' = 1/\dot{v}(u_0) . \tag{17}$$

It is hard to see that such a situation could be compatible with physical requirements like global hyperbolicity and reasonable asymptotic behavior at $\mathcal{I}$. It would mean
that by the retarded time \( u \), distant observers have already received all of the data incoming from \( \mathcal{I}^- \). (Figure 3(c).)

One can also have more complicated situations, in which the signs of \( a_\pm \) change. Some qualitative understanding of these can be pieced together from the observations just given, but in general one needs to understand the functions \( a_\pm(u) \) quantitatively. For example, we saw in the previous paragraph that \( \dot{v} \) could be driven to \(+\infty\) in finite time. This would mean that whatever \( a_\pm(u) \) would do beyond that would be “too late” to affect the behavior of \( \dot{u} \).

4.3. Approach to An Event Horizon

We have already indicated that in the simplest cases one would expect approach to an event horizon to correspond to a situation where \( a_+(u) < a < 0 \) (and \(|\dot{v}a_+/a_+| < k < 1 \) if \( a_- > 0 \)). Here we discuss this situation in more detail.

(We first mention that one can verify that these inequalities hold in a simple example. This is constructed as follows. One starts with a Schwarzschild solution of mass \( m_- \) and static, say perfect fluid, interior. Then one introduces an incoming pulse of radiation of energy \( \Delta m \) at a fixed advanced time (say \( v = 0 \)), so that one has in effect a time–reversed Vaidya solution in the exterior of the fluid. If one takes \( \gamma_u(s(u)) \) to be the spatial origin, then one finds

\[
\begin{align*}
  a_-(u) &= 0 \\
  a_+(u) &= -\frac{m_+}{r_0^2} + \frac{r_0 - 2m_+}{r_0^2} \cdot \frac{m_-}{r_0^2},
\end{align*}
\]

where \( m_+ = m_- + \Delta m \) is the total mass and \( r_0 \) is the value of the radius at which \( \gamma_- \) crosses \( v = 0 \). We have \( a_+ < 0 \). If a black hole forms, then \( r_0 \to 2m_+ \) and \( a_+ \to -1/4m_+ \), (minus) the surface gravity.)

Notice that as long as \( \gamma_u(s(u)) \) is chosen to continuously approach a finite point in space–time as \( u \to +\infty \), the integral \( a_-(u) = \int_{\gamma_-} R_D ds_- \) will remain bounded. Since this integral is multiplied by \( \dot{v} \) in equation (14), it will not contribute to the limiting behavior of \( \ddot{v}/\dot{v} \). This is true not just for \( \gamma_u(s(u)) \) being, for example, the spatial origin, but even if \( \gamma_u(s(u)) \) tends to any finite point on the event horizon. This shows that the limiting behavior of \( \ddot{v}/\dot{v} \) is independent of all details of formation of the black hole, and even of the geometry of any finite point on the horizon.

While this behavior is easy enough to understand analytically, it is not captured very well by any of the standard infinite regimes common in relativity. What is relevant here is compressed to a point in the usual conformal diagram: it is the “gap” between the event horizon and \( \mathcal{I}^+ \).

---

6 Again, this is true with mild asymptotic assumptions at \( \mathcal{I}^- \). For example, assuming that \( R_D \) has the usual peeling behavior uniformly in \( v \) for a neighborhood of \( \mathcal{I}^- \) near the limiting value of \( v \) would be sufficient.
Redshifts near black holes

Figure 4. The regime relevant to the asymptotic behavior of the fractional acceleration of the redshift $\ddot{v}/\dot{v}$, indicated by arrows. It is the gap between the event horizon and future null infinity. This regime is compressed to a single point in a conformal diagram, such as this one.

Let us recall that the evolution of $ds$ is given by

$$\mathcal{L}_n ds = n^b \nabla_b ds = c_n ds,$$

where

$$c_n(\gamma_u(s)) = \int_{\gamma_u(s)}^{T^+} R_D(\gamma_u(s')) ds'.$$

In the situation we are considering, this approaches a constant negative value if $\gamma_u(s)$ approaches a finite point on the event horizon. This means that $ds$ vanishes exponentially quickly as the event horizon is approached along the vector field $n^a$.\(^7\)

\(^7\) Since $n^a$ is itself being exponentially compressed by the same factor, the measure $ds$ would vanish linearly in local coordinates.
This is really the feature which makes the limiting behavior of

\[ \frac{\dot{v}}{\dot{\nu}} = \int_{\gamma_u} R_D \, ds \]  

(19)

independent of all details of the formation of the hole or of the geometry of finite portions of the horizon.

Let us recall that this subsection has been concerned with the formation of an event horizon only in the simple case \( a_+(u) < a < 0 \), and this probably corresponds to what happens in a “normal” collapse of a spherically symmetric star. However, it is quite possible that other, more complicated, situations occur, for example in critical collapse.

4.4. Surface Gravity

Usually, the surface gravity of a black hole is only considered to be defined if the hole is stationary. Indeed, the surface gravity is defined in terms of the Killing vector \( \xi^a \) associated with the stationarity. We shall show here that in the case of a stationary black hole, the surface gravity is simply (minus) \( \ddot{v}/\dot{v} \), and so this quantity is a candidate for the notion of “surface gravity” in general.

Let us write

\[ \xi^a = \lambda l^a + \nu n^a , \]  

(20)

for scalar functions \( \lambda \) and \( \nu \). (These are not the Newman–Penrose quantities often denoted by those symbols.) We have \( \lambda, \nu > 0 \) in the stationary region, and \( \lambda, \nu \to 1 \) at \( \mathcal{I}^+ \) (normalization). The components of Killing’s equation in the \( l^a, n^a \) plane are

\[ l^a \nabla_a \nu = 0 \]

\[ c_l \lambda + n^a \nabla_a \lambda = 0 \]

\[ l^a \nabla_a \lambda + c_n \nu + n^a \nabla_a \nu = 0 \]  

(21)

The first implies \( \nu = 1 \) everywhere. (The remaining system is of course overdetermined, giving a consistency requirement for a Killing vector to exist.) Now, we may use these equations in any of the standard formulas for the surface gravity. For example, one has

\[ - (1/2) \nabla_a \xi^b \xi_b = \kappa \xi_a \]  

(22)

as the horizon is approached. Contracting this with \( n^a \), say, and substituting (21) gives

\[ \kappa = c_l = -c_n \quad \text{at the horizon.} \]  

(23)

We saw in the previous subsection that this limiting behavior of \( c_n \) was

\[ \frac{\ddot{v}}{\dot{v}} = \int_{\gamma_u} R_D \, ds , \]  

(19)
and thus we identify (minus) this with the surface gravity, whether the exterior is asymptotically stationary or not.

Two remarks are in order. First, in the non-stationary case, one would not usually expect $-\ddot{v}/\dot{v}$ to tend to a constant value, and this is the reason for the terminology “limiting behavior” rather than “limiting value.” Second, it is arguable whether this quantity ought to be called the surface gravity, since it depends on the global structure and not just on the local geometry of the horizon. (However, this is true even when the surface gravity is defined by means of a Killing field, for then the definition depends on the normalization of the field at infinity.) In any event, the identification made here of this quantity with $-\ddot{v}/\dot{v}$ shows that it is interpretable as a relative acceleration even in the non-stationary case; whatever one calls it, is it the key quantity in the geometric-optics approximation to scattering by the black hole.

5. Comments on the Non-Spherically Symmetric Case

Perhaps surprisingly, much of the analysis above can be extended to the non-spherically symmetric case without great difficulty. The results are formulas for the gravitational acceleration of wave-vectors by the black hole; these are no longer quantifiable by a scalar red-shift, however. We shall only sketch these ideas here.

Suppose that $(M, g_{ab})$ is a general space-time, which is well-enough behaved that $I^-$ and $I^+$ exist as null hypersurfaces, and $\gamma$ is a null geodesic from $I^-$ to $I^+$. (Actually, all considerations here are local to the geodesic, so it is enough to have small portions of $I^\pm$ near the endpoints of the geodesic; one does not need much of the global structure of $I$.)

Let $l^a$ be a parallel propagated tangent vector to $\gamma_u$. This will be interpreted as the nominal wave vector of a wave packet in the geometric-optics approximation, so we want to understand how $l^a$ is affected by its passage from $I^-$ to $I^+$. In order to do this, we must have some way of comparing vectors on $I^-$ with those on $I^+$. In general, there is no simple way to do this, for in the presence of gravitational radiation one does not have covariantly constant vector fields at $I$, and this leaves aside the issue of relating vectors at $I^-$ to those at $I^+$.

It is possible to overcome these difficulties by various technical means, but at the present level of discussion it turns out that we do not have to address this issue. We suppose that a Lorentz transformation $L_u$ is chosen identifying the tangent space at the past end-point of $\gamma_u$ with that at the future end-point. In general, there will be different choices of $L_u$ possible, and presumably the correct choice in any situation is dictated by the physics.

We may then define a closed path $\Gamma_u$ as before, one portion of which is the geodesic $\gamma_u$, and the other portion of which is a mathematical fiction, parallel transport along the second portion being effected by $L_u^{-1}$. Let us write $P(u)$ for the parallel transport along $\gamma_u$; then the holonomy around the closed path beginning and ending at the end-point at $I^-$ is $Q = L^{-1}P$. A wave vector $l^a_{in}$ at $I^-$ is taken to $l^a_{out} = P^{a}_{\phantom{a}b}l^b_{in}$ at
In order to get a measure of the red shift, this must be compared with $L^a u^b_{\text{in}}$, the result of transporting the vector from $I^-$ to $I^+$ by the fictional “path at infinity.” In the sense of this fictional transport, and referring all vectors to $I^+$, we may identify a “red–shift operator” $Q = PL^{-1}$. Then we have

$$
\dot{Q} = \dot{PL}^{-1} - PL^{-1}\dot{LL}^{-1} = \dot{PP}^{-1}Q - Q\dot{LL}^{-1}
$$

(24)

In this formula, we have

$$
\dot{P}^a_c (P^{-1})^c_b = \int_{\gamma_u} R_{pqab} p^a n^b ds
$$

(25)

the integration of the tensor field being done by identifying the tensor field with a tensor in the tangent space to the end–point of $\gamma_u$ on $I^+$ by parallel transport along $\gamma_u$.

If $p$ is a point at which the event horizon is smooth, and near $p$ one varies the geodesic $\gamma_u$ to approach (from the past) the generator of the event horizon through $p$, then the normal field $n^a$ to the geodesic becomes compressed as $p$ is approached. This means that $n_a$, which restricts to $ds$ on the geodesic, tends to zero. Thus contributions to $\dot{PP}^{-1}$ from the geometry in the neighborhood of the event horizon are suppressed, and the late–time behavior of $\dot{PP}^{-1}$ depends only on the portion of the space–time connecting the event horizon to $I^+$, in the same sense as for radial symmetry.

6. The Hawking Process

The foregoing results have implications for the usual view of the Hawking process. These will be outlined here, although there are a number of technicalities and caveats which we shall only mention, their full treatment requiring more extensive discussion than can be given in this space. We consider only the spherically symmetric case here.

The first point to be made is that to leading order, Hawking’s model of the emission of radiation by an object collapsing to form a black hole is isomorphic to a moving–mirror model in two–dimensional Minkowski space. This comes about as follows.

The physical question at hand is: given a massless quantum field quiescent in the far past and propagating through the space–time of a gravitationally collapsing body, what does the field look like at late retarded times far away from the body? This is a scattering problem, the object being to work out the field operators near $I^+$ in terms of those near $I^-$.

We shall restrict our attention to the s–wave sector, which can be shown to be by far the main contribution. This leaves us with s–waves propagating through the collapsing object. For these, for those field modes of moderate wavelength near $I^+$, the geometric–optics approximation is a good one, since these arise from very blue–shifted modes near the surface of the collapsing object and on the rest of the space–time. If,
as usual, we rescale the fields $\phi$ to have finite (operator–valued) limits $\phi_0$ near $I$ by putting $\phi = \phi_0/r$, then we have

$$
\phi_0^+(u) = -\phi_0^- (v(u)),
$$

(26)

where $v(u)$ is the mapping of surfaces of constant phase, and the minus sign comes from reflection through the spatial origin.

The relation (26) is formally identical to that for scattering of a massless field on one side of a mirror with trajectory $v = v(u)$ (in standard null coordinates $u = t - x$, $v = t + x$) in two–dimensional Minkowski space. To check that the quantum field theories are in fact the same, one has to verify that canonical commutation relations and field representations are the same; this is routine, using the fact that the original space–time was asymptotically flat near $I^-$.

We can immediately write down the expectation of the stress–energy near $I^+$ from knowledge of the corresponding result in the moving–mirror theory (cf. Birrell and Davies 1982):

$$
\langle \hat{T}^{\text{ren}}_{ab} \rangle = (12\pi r^2)^{-1} \left[ \frac{3}{4} \left( \frac{\ddot{v}}{v} \right)^2 - \frac{1}{2} \frac{\nu^{(3)}}{\dot{v}} \right] l_a l_b
$$

(27)

for the $s$–wave contribution to the stress–energy near $I^+$. Of course, this equation is only valid insofar as: (a) we may neglect all contributions other than the $s$–wave one; and (b) the geometric–optics approximation (here expressed by equation (26)) is legitimate. The usual view is that these do give a good treatment of the leading physics. See Page (1976a, 1976b, 1977) for the validity of the $s$–wave approximation. See Hawking (1975, pp. 210–211) for an argument that the inclusion of backscattering only alters the result (27) by a geometric factor independent of the details of the formation of the hole; for another approach to this issue, see Fredenhagen and Haag (1990).

If there is an event horizon with surface gravity $\kappa = -\dot{v}/\ddot{v}$, then at late retarded times (and for $r \gg m$, the mass of the object), we will have

$$
\langle \hat{T}^{\text{ren}}_{ab} \rangle = (12\pi r^2)^{-1} \left[ \frac{1}{4} \kappa^2 + \frac{1}{2} \dot{\kappa} \right] l_a l_b.
$$

(28)

In particular, if $\kappa \to 0$, or even if the time over which $\kappa$ changes significantly is much smaller that $\kappa^{-1}$, we have

$$
\langle \hat{T}^{\text{ren}}_{ab} \rangle \simeq (48\pi r^2)^{-1} \kappa^2 l_a l_b
$$

(29)

which is the standard result for asymptotically stationary black holes. But the formula (28) is valid even for non–stationary black holes, insofar as the general framework of Hawking’s analysis is valid.

Several comments about this are in order:

(a) Equation (28) can be viewed as giving us corrections to the adiabatic approximation for the variation of the emission rate with the surface gravity. One
would expect that approximation to break down when the time scale over which $\kappa$ changes to become comparable to $\kappa^{-1}$, and that is just what appears in (28).

(b) In particular, there would be no significant modification of Hawking’s prediction in the case of an isolated Schwarzschild black hole, since in this case $|\dot{\kappa}| \ll \kappa^2$ until the hole is of Planck dimensions, when any semiclassical treatment is dubious.

(c) It is possible for $\dot{\kappa}$ to be negative; indeed, one would expect $\dot{\kappa} < 0$ for an accreting black hole.

(d) It is possible that in some cases $(1/4)\kappa^2 + (1/2)\dot{\kappa} < 0$. In this case $\langle \hat{T}^{\text{ren}}_{ab} \rangle$ would correspond to a flux of null negative–energy particles. Such behavior is familiar, if not wholly understood, in the moving–mirror case.

(e) Whatever the sign of $\dot{\kappa}$, equation (28) shows that Hawking’s prediction will be substantially modified if the surface gravity changes very rapidly.

(f) We can understand the validity of the Unruh (1976) vacuum for computing the Hawking effect, in terms of the universality discussed above. The point is that the Hawking effect arises from the singular behavior of $\dot{v}$, and changing the early part of the space–time in any way which only prepends a non–singular transmission of null rays will not affect this. Unruh’s choice of vacuum is simply such a choice which is mathematically natural.

(g) Finally, one of the most important lessons of the moving–mirror models is that it is not possible to understand their energy budget without taking into account the energy of the mirror and its driving engine as quantum operators, nor without taking into account the entanglement of the field state with that of the mirror and driving engine (Parentani 1996, Helfer 2001). The entanglement occurs because the field energies depend on the mirror and its driving engine, which are themselves quantum systems: the field energies are functionals of the acceleration of the mirror, which is a certain quantum operator. This quantum character cannot be ignored for understanding the system’s energetics, because observation of the field energy requires driving the mirror/engine system into a superposition of energy eigenstates. If there is a parallel with black holes, then the treatment of the collapsing object and its space–time geometry as classical is questionable.

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