Electrodynamic behaviour of a two-junction quantum interferometer enclosing a superconducting nanocylinder

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Abstract. The electrodynamic response of a two-junction symmetric quantum interferometer, containing two equal 0-junctions and enclosing a superconducting nanocylinder, is studied by means of a first order perturbation analysis of the dynamical equations obtained for the composite system in the absence of noise. It is shown that, when an appropriate number of fluxons is trapped in the inner superconducting cylinder, the system behaves like a π-SQUID.

1. Introduction
Two-junctions quantum interferometer models have been adopted as model systems of Superconducting QUantum Interference Devices (SQUIDs) [1-2] for a long time and are now being considered, for their peculiar properties, as possible candidates in realizing elementary memory cells in quantum computing [3-4]. The interest in these systems, however, rests also upon their intrinsic analytical properties. In particular, it has been shown [5-6] that a perturbation analysis, aimed to solving the coupled nonlinear ordinary differential equations governing the dynamics of the two superconducting phase differences in the device, \( \phi_1 \) and \( \phi_2 \), gives rise to a single-junction equivalent model, for small values of the parameter \( \beta = \frac{L I_{J_0}}{\Phi_0} \), where \( L \) is the inductance of a single branch of the device, \( I_{J_0} \) is the maximum Josephson current of the two equal junctions, and \( \Phi_0 \) is the elementary flux quantum. This model was already known in the literature in the rather simple case of \( \beta = 0 \). In the latter case, indeed, a symmetric quantum interferometer containing identical junctions can be described by a differential equation analogous to the one governing the dynamics of a single junction. In this equivalence, however, we need to specify that the superconducting phase difference \( \phi \) across the equivalent junction is given by the average of the two superconducting phases and that the effective current-phase relation is of the conventional type \( I = I_J \sin \phi \), with a maximum Josephson junction \( I_J \) modulated by the externally applied normalized magnetic flux \( \psi_{ex} \) as follows: \( I_J = I_{J_0} \cos(\pi \psi_{ex}) \).

In case the parameter \( \beta \) is finite a different analysis of the dynamical system is required. For small values of the parameter \( \beta \), indeed, one can first realize that, by a mere change in variables, the

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system can be analyzed in terms of the average superconducting phase $\phi$ and of the fluxon number $\psi$, which represents the total magnetic flux linked to the superconducting loop normalized to the elementary flux quantum $\Phi_0$. This change of variables is necessary also to discriminate between slow and fast dynamics in the system. Indeed, for small values of the parameter $\beta$, the flux number $\psi$ possesses a much more rapid dynamical behaviour than the average superconducting phase $\phi$. In this way, the dynamical equation for $\psi$, in which a coupling term in $\cos \phi$ appears, can be solved as if the coupling term would be a fixed parameter. When the solution of this equation is found by a perturbation analysis to first order in $\beta$ and the result is substituted in the dynamical equation for the slow variable $\phi$, one obtains [6, 7] the following nonlinear ordinary differential equation

$$\frac{d\phi}{dt} + \cos(\pi \psi_{\text{ex}}) \sin \phi + \pi \beta \sin^2(\pi \psi_{\text{ex}}) \sin 2\phi = \frac{i_B}{2},$$

where $\tau = \frac{2\pi R I_{J_0}}{\Phi_0}$, $R$ being the resistive parameter of the junctions, and $i_B$ is the bias current injected in the device, normalized to $I_{J_0}$. As one can see from equation (1), the single junction equivalent model described above is recovered by setting $\beta = 0$. Carrying out the perturbation analysis to first order in $\beta$, however, we notice the appearance of second harmonic term in the effective current-phase relation for the equivalent junction. This term is not due to intrinsic junction properties, since the current-phase relation (CPR) for each junction is of conventional type. On the contrary, the second harmonic sine term appears because of the inductively mediated electrodynamic coupling between the junctions.

In what follows we shall carry out a similar perturbation analysis for a two junction quantum interferometer containing two identical conventional $0$-junctions and a superconducting nanocylinder entrapping magnetic flux as shown in figure 1. We shall notice that, already at a stage where the dynamical equations is written for the superconducting phases $\phi_1$ and $\phi_2$, one can argue that the composite system could behave like a $\pi$-SQUID, which is realized by substituting one of the two $0$-junctions with a $\pi$-junction in a conventional superconducting quantum interference device. This behaviour is obtained by virtue of flux quantization into the superconducting nanocylinder. We finally exhibit the reduced model for this system and explore the possibility of realizing an experiment which could confirm the prediction made in the present work.

![Figure 1. Schematic representation of the composite system.](image-url)

2. The dynamical model
In the present section we first find the dynamical equations for the superconducting phase differences $\phi_1$ and $\phi_2$ across the two junctions in the interferometer enclosing a superconducting nanocylinder
with entrapped magnetic flux. Starting from these equations and following the analysis described in references [5] and [6], the reduced model for the composite system is derived.

The Resistively Shunted Junction (RSJ) model [1-2] is used to describe the dynamics of the two identical overdamped junctions, which symmetrically interrupt the interferometer superconducting loop. In this way, the dynamical equations in the absence of noise are written as follows:

\[ \frac{d\phi_1}{d\tau} + \sin \phi_1 = \frac{i_g + i_s}{2}, \quad (2a) \]

\[ \frac{d\phi_2}{d\tau} + \sin \phi_2 = \frac{i_g - i_s}{2}, \quad (2b) \]

where \( i_s = \frac{i_g - i_g}{2} \) is the circulating current. In order to express the above equations in terms of the variables \( \phi \) and \( \psi \), we first proceed to the following transformation of variables: \( \phi = \frac{\phi_1 + \phi_2}{2} \), \( \zeta = \frac{\phi_1 - \phi_2}{2} \). Furthermore, by fluxoid quantization [1], we may express \( \zeta \) in terms of \( \psi \) as follows: \( \zeta = \pi(n - \psi) \), where \( n \) is an integer. Let us now express the currents in terms of all fluxes in the system. For this purpose, let us define the geometric fluxes \( \psi_{ex}' = \frac{\mu_0 HS}{\Phi_0} \) and \( \psi_{ex}' = \frac{\mu_0 HS_0}{\Phi_0} = \frac{r^2}{R^2} \psi_{ex} \), where \( H \) is the applied field and \( S \) and \( S_0 \) are, respectively, the total circular area of radius \( R \) of the region enclosed by the interferometer loop and the superconducting cylinder section of radius \( r \). On the other hand, the flux numbers \( \psi \) and \( \psi' = k \), \( k \) being an integer owing to flux quantization in the superconducting cylinder, represent the actual flux linked to \( S \) and \( S_0 \), respectively. By elementary electrodynamics and simple algebraic manipulations, we finally write the dynamical equations for the composite system in terms of \( \phi \) and \( \psi \) in the following way [8]:

\[ \frac{d\phi}{d\tau} + (-1)^n \cos(\pi \psi) \sin \phi = \frac{i_g}{2}, \quad (3a) \]

\[ \pi \frac{d\psi}{d\tau} + (-1)^n \cos \phi \sin(\pi \psi) + \frac{\psi}{2\beta} = \frac{1}{2\beta} \left[ 1 - \frac{r^2}{R^2} \right] \psi_{ex} + k \frac{\beta}{\gamma}. \quad (3b) \]

where \( \beta \) and \( \gamma \) are the characteristic parameters of the system. Indeed, denoting by \( 2L \) and \( L' \) the self inductance of the interferometer loop and of the inner superconducting cylinder, respectively, and by \( 2M \) the value of the mutual inductance between these two elements of the composite system, we may define: \( \beta = \frac{l}{\Phi_0} \), with \( l = \frac{LL' - M^2}{L'} \), and \( \gamma = \frac{m}{\Phi_0} \), with \( m = \frac{LL' - M^2}{M} \). In this way, \( \beta \) is analogous to the usual interferometer parameter \( \beta \), while the ratio \( \frac{\beta}{\gamma} = \frac{M}{L} \) takes account of the electrodynamic interaction between the inner superconducting cylinder and the interferometer loop.
Notice also that equation (3a) is formally identical to the corresponding equations for a superconducting quantum interferometer, while only equation (3b) differs, when $M \neq 0$, from the corresponding dynamical equation for the flux number of an ordinary superconducting quantum interferometer.

3. Some properties of the composite system

In the present section we investigate the electrodynamic properties of the composite system. We start by analyzing some characteristic features coming from equation (3b). We first notice that this equation clearly reduces to the usual expression for a superconducting quantum interferometer when we set $M = 0$. In this case, indeed, the parameter $\beta$ goes to $\beta$ and the ratio $\frac{\beta}{\gamma}$ vanishes. Interesting new behaviour of the composite system is thus expected only for $M \neq 0$. In order to simplify the analysis, let us assume $r \ll R$ in such a way that we can make use of the approximated mutual inductance expression

$$M = \frac{\mu_0 \pi r^2}{4R}. \quad (4)$$

By taking $L' = \frac{\mu_0 \pi r^2}{d}$, where $d$ is the cylinder height, we have that $\frac{\bar{\beta}}{\bar{r}} = \frac{M}{L'} \approx \frac{d}{4R}$. By choosing $d$ of the same order of magnitude of $r$, we notice that $\frac{r^2}{R^2} \frac{\bar{\beta}}{\gamma} = O\left(\frac{r^3}{R^3}\right)$ and equation (3b) can be written as follows

$$\pi \frac{d\psi}{d\tau} + (-1)^n \cos \phi \sin(\pi \psi) + \frac{\psi}{2\beta} = \frac{1}{2\beta} \left(\psi_{ex} + \frac{d}{4R} k\right). \quad (5)$$

The above equation, apart from a redefinition of the characteristic parameter $\beta$ ($\beta \rightarrow \bar{\beta}$), presents a shift in the normalized external magnetic flux $\psi_{ex}$, which, as specified in reference [8], can give rise to $\pi$-SQUID-like behaviour of the composite system if this shift is equal to a half integer number.

First of all, let us notice that, in order to obtain an integer number $k$ of fluxons inside the inner superconducting cylinder, we need to apply a magnetic field $H_N$ before cooling the cylinder itself below its critical temperature. Therefore, we have:

$$k = \Omega \left[\frac{\mu_0 H_N S_0}{\Phi_0}\right], \quad (6)$$

where $\Omega$ is a nonlinear operator, which gives the closest integer value of the quantity inside the square brackets. Therefore, if we chose to trap a given number $k$ of fluxons inside the superconducting cylinder, we need to have:

$$\left(k - \frac{1}{2}\right) \frac{\Phi_0}{S_0} < \mu_0 H_N < \left(k + \frac{1}{2}\right) \frac{\Phi_0}{S_0}. \quad (7)$$

In this way, in order to get $\pi$-SQUID-like behaviour of the composite system, we may set $\frac{d}{4R} k = \frac{2s+1}{2}$, with $s$ integer, so that equation (7) can be recast in the following form:
We should notice, however, that the range of applied magnetic field values, obtained for various integers $s$, need to be compatible with the trapping capabilities of the superconducting cylinder. Therefore, one needs to compute the saturation number $n_s$ corresponding to a given hole size, giving the maximum number of trapped fluxons in a cylindrical cavity. In type-II superconductors. For example, in the limit of $r >> \lambda$, with $\lambda$ the penetration depth of the superconducting material, it can be shown that [9]

$$n_s \simeq \frac{r^2}{3\sqrt{3}\xi\lambda}.$$  \hfill (9)

where $\xi$ is the coherence length of the superconducting cylinder. In this way, since we can only take values of $k$ less than $n_s$, we might introduce the following constraint to equation (8):

$$k = (2s + 1)\frac{2R}{d} \leq \frac{r^2}{3\sqrt{3}\xi\lambda}.$$  \hfill (10)

For a practical realization of the system, let us choose $s = 0$, $R = 1.0\, \text{mm}$, $d = 10.0\, \mu\text{m}$, giving $k = 200$. We need to find a hollow cylindrical type-II superconductor whose geometry and whose properties are such to satisfy equation (10). By choosing, for instance, Niobium at temperatures close to $4.2\, \text{K}$, with $\lambda \approx 40\, \text{nm}$ and $\xi \approx 9.0\, \text{nm}$, and by taking $r = 800\, \text{nm}$, we get $n_s \approx 342$, which makes the constraint in (10) compatible with the quantities chosen above. We can thus safely evaluate the field range in which trapping of 200 fluxons in the superconducting cylinder is possible:

$$0.411\, \text{T} < \mu_0 H_N < 0.412\, \text{T}.$$  \hfill (11)

4. The reduced model for the composite system

We now consider again equations (3a) and (3b), in order to get a single junction equivalent model of the composite system. Let us first set $f = \frac{d}{4R}k$, so that the dynamical equations for the variables $\phi$ and $\psi$ can be written as follows:

$$\frac{d\phi}{d\tau} + (-1)^n \cos(\pi \psi) \sin \phi = \frac{i_n}{2},$$  \hfill (12a)

$$\pi \frac{d\psi}{d\tau} + (-1)^n \cos \phi \sin(\pi \psi) + \frac{\psi}{2\tilde{\beta}} = \frac{1}{2\tilde{\beta}} [\psi_{ex} + f].$$  \hfill (12b)

Let us now assume that the parameter $\tilde{\beta}$ is small enough to allow a perturbation solution of equation (12b) for the variable $\psi$, so that [6,7]:

$$\psi = \psi_0 + \tilde{\beta} \psi_1 = \psi_{ex} + f - 2\tilde{\beta} (-1)^n \sin[\pi(\psi_{ex} + f)] \cos \phi.$$  \hfill (13)

By substituting (13) in (12), we get the following single-junction equivalent dynamical equation
the only difference with respect to the usual model is the $f$-shift of the applied magnetic flux.

\[
\frac{d\phi}{d\tau} + (-1)^n \cos[\pi(\psi_{ex} + f)]\sin \phi + \pi \tilde{\beta} \sin^2[\pi(\psi_{ex} + f)]\sin 2\phi = \frac{i_n}{2}, \quad (14)
\]

We might then exhibit the effective CPR of this device, for various values of $f$, as in figures 2 and 3. In particular, in figure 2 we show the effective superconducting current $i_f = (-1)^n \cos[\pi(\psi_{ex} + f)]\sin \phi + \pi \tilde{\beta} \sin^2[\pi(\psi_{ex} + f)]\sin 2\phi$ versus the average phase difference.
\( \phi \) for \( n \) even, \( \bar{\beta} = 0.05, \psi_{ex} = 0.1 \) and for various values of the parameter \( f \). The same is done in figure 3, for the same values of the parameters \( n, \bar{\beta} \) and \( \psi_{ex} \), and for still different values of \( f \).

5. Conclusions

We have studied the electrodynamic behaviour of a composite system consisting of a two-junction symmetric quantum interferometer enclosing a concentric superconducting cylinder of nanoscopic dimensions. It has been shown that the physical properties of this system can be described by a single-junction equivalent model with two characteristic parameters: \( \bar{\beta} \), which is the analogous of the parameter \( \beta \) for d. c. SQUIDs and \( f \), proportional to the number of trapped fluxons in the inner cylinder. It has been noted that, for specific numbers of trapped fluxons inside the inner superconducting cylinder, the composite system behaves like a \( \pi \)-SQUID. Furthermore, deviation of the CPR from the usual sinusoidal behaviour is detected mainly for two reasons. First of all, when the characteristic parameter \( \bar{\beta} \) is different from zero, a second harmonic term appears in the effective CPR of the single-junction equivalent model. This feature is also present in two-junction quantum interferometers. It has also been shown that, for \( \bar{\beta} \neq 0 \), by keeping the externally applied field fixed, the CPR may be different from its sinusoidal behaviour by making the parameter \( f \) vary.

The composite system has been studied in the absence of noise. Further work will be devoted to the analysis of the physical properties of the system in the presence of thermal fluctuations.

6. References

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