EFFECTS OF DISKS ON GRAVITATIONAL LENSING BY SPIRAL GALAXIES

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ABSTRACT

Gravitational lensing of a quasar by a spiral galaxy should often be accompanied by damped Ly$\alpha$ absorption and dust extinction due to the intervening gaseous disk. In nearly edge-on configurations, the surface mass density of the gas and stars in the disk could by itself split the quasar image and contribute significantly to the overall lensing cross section. We calculate the lensing probability of a disk-plus-halo mass model for spiral galaxies, including the cosmic evolution of the lens parameters. A considerable fraction of the lens systems produce two images with subarcsecond separation, straddling a nearly edge-on disk. Because of that, extinction by dust together with observational selection effects (involving a minimum separation and a maximum flux ratio for the lensed images) suppress the detection efficiency of spiral lenses in optical wave bands by at least an order of magnitude. The missing lenses could be recovered in radio surveys. In modifying the statistics of damped Ly$\alpha$ absorbers, the effect of extinction dominates over the magnification bias due to lensing.

Subject headings: cosmology: theory — galaxies: spiral — gravitational lensing — quasars: absorption lines

1. INTRODUCTION

Gravitational lensing by a spiral galaxy occurs when the line of sight to a background quasar passes within a few kiloparsecs from the center of the galactic disk. Since galactic disks are rich in neutral hydrogen (H$\text{I}$), the quasar spectrum is likely to show a damped Ly$\alpha$ absorption trough at the lens redshift. Therefore, the efficiency of blind searches for gravitational lensing with subarcsecond splitting can be enhanced by more than an order of magnitude by selecting a subset of all bright quasars that show a low-redshift ($z \lesssim 1$) damped Ly$\alpha$ absorption with a high H$\text{I}$ column density, $\gtrsim 10^{21}$ cm$^{-2}$ (Bartelmann & Loeb 1996). Moreover, multiply imaged quasars could be identified spectroscopically through their multiple-trough absorption spectra. The composite spectrum of a lensed quasar is a superposition of the spectra received from the different images that intersect the absorbing disk at different locations, probe different H$\text{I}$ column densities, and hence acquire different widths in their damped Ly$\alpha$ troughs (Loeb 1997).

The magnification bias due to lensing changes the statistics of damped Ly$\alpha$ absorbers (DLAs) in quasar spectra by bringing into view quasars that would otherwise fall below the detection threshold (Bartelmann & Loeb 1996; Smette, Claeskens, & Surdej 1997). For optical observations, this effect is counteracted by dust extinction in the lensing galaxy (Malhotra, Rhoads, & Turner 1997; Perna, Loeb, & Bartelmann 1997). The combination of lensing and dust extinction results in a net distortion of the H$\text{I}$ column density distribution of DLAs.

Since galactic disks are thin, their own surface mass density could exceed the critical value necessary for image splitting when they happen to be projected nearly edge-on. In such orientations, a pair of quasar images will straddle the symmetry axis of the disk rather than its center. When averaged over all possible disk orientations, the total lensing cross section of the disk-plus-halo mass distribution should still be roughly the same as that of a halo with the same spherically averaged mass profile (Wang & Turner 1997; Keeton & Kochanek 1998). However, when lensing by the disk is included in the calculation, the effects of dust extinction and H$\text{I}$ absorption are substantially different because of the new predominant image configurations straddling edge-on disks.

Previous theoretical studies have either ignored the possibility of lensing by the disk itself (Bartelmann & Loeb 1996; Smette et al. 1997) or else ignored the effects of dust extinction or H$\text{I}$ absorption on the lensing statistics (Maller, Flores, & Primack 1997; Wang & Turner 1997; Keeton & Kochanek 1998). In addition, these discussions did not examine the implications of evolution in the spiral lens population for its overall lensing properties and statistics. In this paper, we include all of the above ingredients. Our model evolves the number density of galaxies based on the Press-Schechter (1974) theory for dark matter halos, combined with simple evolutionary scaling laws for the disk properties inside these halos (Mo, Mao, & White 1997).

In § 2 we describe our model for the mass and H$\text{I}$ distributions in galactic disks, as well as our model for their dust content. We also summarize the adopted scaling laws for the evolution of the lens population with redshift. In § 3 we compute the lensing cross sections, imaging probabilities, and lensing statistics of our model spiral galaxies. The impact of dust on the lensing statistics and the effect of lensing on the statistics of DLAs are examined in § 4. Finally, § 5 summarizes the main conclusions from this work.

2. A LENS MODEL FOR SPIRAL GALAXIES

2.1. Mass Model and Lensing Properties

We use a mass model for spiral lenses that was recently suggested by Keeton & Kochanek (1998). The model consists of oblate, ellipsoidal, isothermal building blocks for
individual mass components. They have the axisymmetric density distribution

$$\rho(R, z) = \frac{v_c^2}{4\pi G} \frac{e}{s} \frac{1}{R^2 + s^2 + x_3^2/q_3^2} ,$$  

where $R$ is the distance from the symmetry axis of the ellipsoid, $x_3$ is the distance from its midplane, $q_3$ is the axis ratio, $s$ is the core radius which softens the mass distribution, $v_c$ is the asymptotic circular velocity, and $e$ is the eccentricity of the mass distribution,

$$e = (1 - q_3^2)^{1/2} .$$

Hence, each mass component is described by one parameter quantifying its dynamical properties, namely, the circular velocity $v_c$, and two parameters characterizing its shape, namely, the core radius $s$ and the oblateness $q_3$. The projection of the three-dimensional density $q$ onto the plane of sight, with $x = (x_1, x_2)$, and $b_p = 2\pi \int (v_c)^2 \frac{D_d D_h}{D_s} \frac{e}{\sin^{-1} e} = b \frac{e}{\sin^{-1} e} ,$$

where $b$ is the Einstein radius of a singular isothermal sphere with a circular velocity $v_c$. Finally, $q$ is the projected axis ratio

$$q = (q_3^2 \sin^2 i + \cos^2 i) \frac{1}{2} ,$$

where $i$ is the inclination angle of the disk relative to the line of sight, with $i = 0$ for a face-on orientation.

As shown by Keeton & Kochanek (1998), the lensing potential of the surface-mass density (3) is

$$\psi(s, q_3) = x_1 a_1 + x_2 a_2 - b_p s$$

$$\times \ln [(\varrho + s)^2 + (1 - q^2)x_3^2]^{1/2} ,$$

with

$$\varrho = \left[ q^2(x_1^2 + s^2) + x_3^2 \right]^{1/2} ,$$

$$a_1 = \left( \frac{b_p}{(1 - q^2)^{1/2}} \tan^{-1} \left[ \frac{(1 - q^2)^{1/2}x_1}{\varrho + s} \right] \right) ,$$

$$a_2 = \left( \frac{b_p}{(1 - q^2)^{1/2}} \tan^{-1} \left[ \frac{(1 - q^2)^{1/2}x_2}{q^2s} \right] \right) .$$

We now combine three of the isothermal oblate ellipsoids to a mass model consisting of a maximal disk and a surrounding halo. The total lensing potential then reads

$$\psi = \psi(s_d, q_{3d}) - \psi(r_d, q_{3d}) + \psi(s_h, 1) ,$$

where $s_{d,h}$ are the core radii of disk and halo, respectively, $q_{3d}$ is the disk axis ratio, and $r_d$ is the disk truncation radius, or disk radius for simplicity. The first term is the potential of a disk with an asymptotically flat rotation curve, axis ratio $q_{3d}$, and core radius $s_d$. The second term truncates that disk at radius $r_d$. The third term adds the surrounding spherical halo necessary to maintain a flat rotation curve beyond the disk truncation radius. This is the maximal truncated Mestel disk model introduced by Keeton & Kochanek (1998). All lensing properties of the combined model can now be calculated in terms of the potential $\psi$.

To completely define the model, we need to specify five parameters, namely, the circular velocity $v_c$, the two core radii $s_{d,h}$, the disk radius $r_d$, and the disk oblateness $q_{3d}$. For reference, we define $v_{c,h} = 220$ km s$^{-1}$, $r_{d,h} = 8$ h$^{-1}$ kpc, and $q_{3d} = 0.03$. Requiring a flat rotation curve, we must then use $s_{d,h} \approx 0.72r_{d,h}$ (Keeton & Kochanek 1998). Finally, we choose $s_{d,h} = 0.2h^{-1}$ kpc. The rotation curve of this model is plotted in Figure 1. As the figure shows, the rotation curve is flat to better than 5% beyond a radius of $\sim 2h^{-1}$ kpc.

To illustrate the lensing properties of the combined disk-plus-halo model described by equation (9), we show in Figure 2 the caustics and critical curve configuration for a disk inclination angle of $i = 80^\circ$. The lens has two critical curves and caustics. The inner caustic, which is the image of the outer critical curve, has the familiar asteroid shape with four cusps connected by folds.

### 2.2. Neutral Hydrogen Disk

Next, we address the effects of the gaseous component of the disk. To evaluate the level of dust extinction and H I absorption, we assume that the H I density, $n_{H I}$, follows a double-exponential profile

$$n_d(R, x_3) = n_{H I,0} \exp \left( -\frac{|x_3|}{H} \right) \exp \left( -\frac{R}{R_0} \right) .$$

The three parameters that define this profile are the central neutral hydrogen density $n_{H I,0}$, the disk scale height $H$, and the disk scale length $R_0$. In order to keep the

For $s_d \rightarrow 0$, the disk becomes a Mestel (1963) disk.
number of free parameters small in our model, we assume $H = q_3 r_d$ and $R_0 = r_d$, so that the $\text{H}$ scale height is given by the short axis of the oblate ellipsoid used to describe the lensing disk. We also identify the $\text{H}$ scale radius with the truncation radius for the lensing disk. The radial shape of the resulting radial $\text{H}$ profile is similar to that observed in local disk galaxies (Broeils & van Woerden 1994). We choose $n_{\text{H}1,0}$ so that the face-on $\text{H}$ column density, $N_0 = 2Hn_{\text{H}1,0}$, corresponds to the characteristic observed value (Broeils & van Woerden 1994), $N_0 \approx 11.25 \times 10^{20}$ cm$^{-2}$. Hereafter, we use the notation $N = 10^{20} N_{20}$ cm$^{-2}$. Note that with these parameter choices, the solar neighborhood values for the $\text{H}$ column density ($N_{\text{H}1} = 7.4 \times 10^{20}$ cm$^{-2}$; Kulkarni & Heiles 1987) and $\text{H}$ scale height (a few hundred pc; Knapp 1987) of the Milky Way disk are reproduced reasonably well. Changing the disk scale height up or down by a factor of 2 does not significantly alter the results.

We ignore the consumption of $\text{H}$ gas by star formation. A complete model for the evolution of $\text{H}$ in spiral galaxies should include various feedback effects of stars on the disks in which they form, such as heating, ionization, or expulsion of gas by supernova-driven winds. For the purposes of this study, we prefer to avoid the highly uncertain nature of these effects and focus on a simple model.

2.3. Extinction by Dust

We assume for simplicity that the distribution of dust follows equation (10) with a constant ratio of dust to $\text{H}$ gas. This implies dust scale heights that are somewhat larger than typically observed (e.g., Kylafis & Bahcall 1987), but there are counterexamples (Xilouris et al. 1997) with larger dust scale height.

We adopt the scattering and absorption cross sections due to silicates and graphites derived by Draine & Lee (1984). The total extinction cross section is the sum of the contributions from scattering and absorption, and the relative proportion of these components is chosen so as to fit best the observed Galactic extinction law. Figure 3 shows the sum of the extinction cross sections from graphites and silicates as a function of wavelength, assuming a dust-to-gas ratio of 1:100 by mass (Whittet 1992).

The dust optical depth is given by

$$\tau_{\text{ext}}(\lambda) = N\sigma_{\text{ext}}(\lambda),$$

so that the extinction in magnitudes is

$$\Delta m(\lambda) = [2.5 \log_{10} \{\exp \{-N\sigma_{\text{ext}}(\lambda)\}\}] \approx 1.09N\sigma_{\text{ext}}(\lambda).$$

In quantifying the influence of dust, we restrict our attention to the observer’s Johnson $B$ band. In the rest frame of a lens at a redshift $z_l$, this band is centered on the wavelength $\lambda_l = 0.435 \mu\text{m} (1+z_l)^{-1}$. For $z_l = 0.3$, Figure 3 yields $\sigma_{\text{ext}} N \approx 0.084N_{20}$. Hence, for $N = 10^{21}$ cm$^{-2}$, $\Delta m \approx 0.9$ magnitudes in the $B$ band. Note that $N$ can easily exceed this value for edge-on disks, which are favored both geometrically and due to the magnification bias. The value of $N$ can be as large as $N_0 R/H = N_0/q_3 \approx 3.4 \times 10^{22}$ cm$^{-2}$, yielding $\Delta m \approx 31$ for a lens at $z_l = 0.3$. This implies that the influence of dust on the imaging properties of spiral lenses can be severe.

One would expect the dust-to-gas ratio of spiral disks to decline with increasing redshift, in accordance with their metalliclicity history. Indeed, Pei, Fall, & Bechtold (1991) infer that the dust-to-gas ratio in DLAs at redshifts $2 \leq z \leq 3$ is only a tenth of the Milky Way value, although with considerable scatter (Fall & Pei 1993). If we parameterize the dust content to have a power-law dependence on redshift, $(1+z)^{-\delta}$, then a reduction by a factor of $\sim 10$ from the present time to $z \approx 2.3$ implies $\delta \approx 2$. Equation (12) is then...
changed to
\[ \Delta m(\lambda, z) \approx 1.09(1 + z)^{-2}N_{20} \sigma_{tot}(\frac{\lambda}{1 + z}). \] (13)

This dependence will be referred to as the dust evolution model in the discussion that follows. Strictly speaking, we should adapt our model to reproduce the observed optical depth of dust rather than the dust-to-gas ratio, but since our model well reproduces the observed H I column density distribution (see Fig. 10 and Tytler 1987; Lanzetta et al. 1991), we can safely calibrate the dust content in the way described.

2.4. Scalings of Galaxy Properties with Luminosity

We assume that the lens population admits the Schechter luminosity function at present with a number density per unit luminosity,
\[ \frac{dn(l)}{dl} = n_1 l^\alpha \exp (-l) dl, \] (14)

where \( l \equiv L/L_\odot \) is the scaled galaxy luminosity. For spiral galaxies, \( n_1 = 1.5 \times 10^{-2} h^4 \text{Mpc}^{-3} \) and \( v = -0.81 \) (Marzke et al. 1994). The luminosity is related to the circular velocity through the Tully-Fisher (1977) relation,
\[ \frac{v_c}{v_*} = l^{1/4}, \] (15)

where \( \alpha \) varies between \( \sim 2 \) in the B band and \( \sim 4 \) in the H band (see review by Strauss & Willick 1995, and references therein).

Equations (5) and (15) imply that the Einstein radius scales as \( b \propto l^{1/3} \). The computational effort required later on is substantially reduced if the scale radii of the disk-plus-halo model change with l in the same way as b does. For convenience, we therefore adopt the infrared value of \( \alpha = 4 \) and assume that the scale radii behave like \( r_d \propto l^{1/2} \). It is then sufficient to calculate all cross sections for one reference luminosity only and later scale the result to the desired \( l \) by changing the Einstein radius. If \( \sigma_{tot} \) is the cross section for some arbitrary image properties for the reference luminosity \( L_\odot \), then the average cross section for the entire population of spiral galaxies is
\[ \langle \sigma \rangle = n_\odot \int_0^\infty dl \sigma_{tot}(l^{1/3}) \exp (-l) \]
\[ = n_\odot \Gamma(1 + v + 4/3)\sigma_{tot}. \] (16)

2.5. Evolution of the Spiral Galaxy Population

The simplest assumption about the evolution of the spiral population is that they maintain a constant comoving number density, \( n(z) = n(0)(1 + z)^3 \), and constant scale radii. This is the no-evolution model. Alternatively, Mo et al. (1997) recently suggested a model for the evolution of spirals, based on four assumptions: (1) the disk mass is a fixed fraction \( m_d \) of the halo mass; (2) the disk angular momentum is a fixed fraction \( j_d \) of the halo’s angular momentum; (3) the radial disk profile is exponential, and the disk is centrifugally supported; and (4) the disk is dynamically stable. These assumptions yield a set of simple and straightforward scaling relations, which we reproduce from Mo et al. (1997; see also Fall & Efstathiou 1980).

We assume that the halo is a singular isothermal sphere with a radial density profile,
\[ \rho = \frac{v_c^2}{4\pi G \Delta}. \] (17)

Taking the virial radius \( r_{200} \) as the size of the halo, the halo mass is \( M = \frac{v_c^2}{2} r_{200}/G \). Here \( r_{200} \) is the radius within which the average halo density is 200 times the critical density of the universe,
\[ \frac{3M}{4\pi r_{200}^3} = 200 \times \frac{3H^2(z)}{8\pi G}, \] (18)

and hence
\[ r_{200}(z) = \frac{v_c}{10H(z)} \approx 7.8 h^{-1} \text{Mpc} \times \left( \frac{v_c}{220 \text{ km s}^{-1}} \right)^3 \left( \frac{H(z)}{H_0} \right)^{-1}. \] (19)

where \( H_0 = 100 \text{ km s}^{-1} \text{Mpc}^{-1} \) is the Hubble constant, and \( H(z) \) is the Hubble parameter at redshift \( z \). Similarly,
\[ M(z) = \frac{v_c^2}{10GH(z)} \approx 2.5 \times 10^{12} h^{-1} M_\odot \times \left( \frac{v_c}{220 \text{ km s}^{-1}} \right)^3 \left( \frac{H(z)}{H_0} \right)^{-1}. \] (20)

The disk radius \( r_d \) is related to the spin parameter of the halo, \( \lambda \equiv J/E^{1/2} GM^{-5/2} \), where \( J \) and \( E \) are the total angular momentum and energy of the halo, respectively, by
\[ r_d = \frac{\lambda v_c}{10 \sqrt{2H(z)} m_d} \approx 7.8 h^{-1} \text{Mpc} \times \left( \frac{\lambda}{0.05} \right) \]
\[ \times \left( \frac{v_c}{220 \text{ km s}^{-1}} \right)^3 \left( \frac{H(z)}{H_0} \right)^{-1}, \] (21)

and the central surface mass density of the disk is\(^2\)
\[ \Sigma_0 = \frac{10 m_d v_c H(z)}{G \lambda^{3/2} j_d} \left( \frac{m_d}{0.05} \right)^2 \]
\[ \approx 6.8 \times 10^{-2} \text{ g cm}^{-2} \times \left( \frac{m_d}{0.05} \right)^2 \left( \frac{\lambda}{0.05} \right)^{-2} \]
\[ \times \left( \frac{v_c}{220 \text{ km s}^{-1}} \right)^3 \left( \frac{H(z)}{H_0} \right). \] (22)

\(^2\) Since \( r_d \propto l^{1/2} \) implies constant surface brightness, the chosen scaling reflects Freeman’s law (Freeman 1970; Holmberg 1975; Peterson, Strom, & Strom 1979; van der Kruit 1987; Lauberts & Valenlijn 1989). Thus, we effectively ignore the scatter in Freeman’s law and the influence of low surface brightness (LSB) galaxies on the lensing cross section (e.g., Bothun, Impey, & McGaugh 1997 and references therein). This omission is justified here because (1) the average circular velocities of LSB galaxies is somewhat lower than that of high–surface brightness (HSB) galaxies, and the lensing cross section is a sensitive function of the circular velocity, and (2) LSB galaxies are less compact than LSB galaxies and thus less efficient lenses. For the purposes of strong lensing, it therefore seems safe to neglect any contribution from LSB galaxies.

\(^3\) Different realizations of galaxy formation in different tidal environments could result in disks of different sizes for the same halo properties, due to variations in the spin parameter acquired by the baryons. As discussed before, we ignore this scatter in the disk properties here.
In summary, scale radii scales as $n(z) H(z)^{-1}$, and the surface mass density scales as $n(z) H(z)$. Since the characteristic values of $\lambda$, $j$, and $m_h$ are expected to depend very weakly on redshift, we use these simple proportionality relations with $n(z)$ and $H(z)$ in scaling the properties of the local spiral population to higher redshifts. Since $H(z)/H_0 > 1$ for $z > 0$, scale radii decrease with increasing $z$, while the surface mass density increases with increasing $z$.

The evolution in the number density of spiral galaxies can be expressed in terms of the Press-Schechter distribution function for the mass $M$, given by equation (20). We evolve the number density of galaxies given by equations (14), (15), and (20) with the factor $n_{ps}(M, z)/n_{ps}(M, z = 0)$, where $n_{ps}(M, z)$ is the Press-Schechter number density of halos with mass $M$ at redshift $z$. Because of the inherent uncertainty in modeling galaxy evolution, we present numerical results for three models, assuming (1) no evolution, (2) evolutionary model for galaxies based solely on the Press-Schechter mass function and the local luminosity function, (3) evolutionary model for galaxies out to redshift $z$ beyond the galactic mass scale (e.g., Frenk, &Navarro, 1996), and with the factor $n_{ps}(M, z)/n_{ps}(M, z = 0)$, where $n_{ps}(M, z)$ is the Press-Schechter number density of halos with mass $M$ at redshift $z$. Since the number density and scale radii.

It is well known that the present-day Press-Schechter mass function extends out to halo masses that are well beyond the galactic mass scale (e.g., Navarro, Frenk, & White 1995) in all viable models of structure formation. This is because of the fact that the nonlinear mass scale at present (which defines the exponential break in the Press-Schechter mass function) corresponds to a much larger circular velocity than $L_\star$ in the Schechter function does, based on the Tully-Fisher relation (see eqs. [14] and [15]). This implies that nongravitational processes (such as inefficient cooling or expulsion of gas by supernovae) prevented disk formation inside supergalactic halos at the present time. An evolutionary model for galaxies based solely on the Press-Schechter approach is therefore incomplete. The simplest interpretation of the discrepancy between the Press-Schechter mass function and the local luminosity function of galaxies is that massive galaxies with $L \gg L_\star$ have not changed their dynamical properties since the redshift of galaxy formation ($z \sim 2$–4) when the nonlinear mass scale was comparable to their mass. Indeed, recent observations imply no significant evolution in the population of massive galaxies out to redshifts $z \sim 1$ (Ellis 1997 and references therein; but see Kaufmann, Charlot, & White 1996). However, the same observations reveal many more dwarf galaxies at high redshift than are found locally. Since the lensing probability is dominated by $L_\star$ galaxies at $z \lesssim 0.7$, the no-evolution model might be more appropriate for calculations of lensing by disk galaxies. However, to bracket the other extreme of complete evolution we also show results for the Press-Schechter prediction. Since we use the Press-Schechter mass function only to correct the overall normalization of the number density of spiral galaxies relative to its present-day value, our approach should be less affected by the incompleteness of the Press-Schechter treatment.

In all numerical calculations, we use the cosmological parameters $\Omega_0 = 0.3$, $\Omega_\Lambda = 0$, and $h = 0.7$. For the Press-Schechter mass function, we use the standard CDM power spectrum (Bardeen et al. 1986) with the normalization $\sigma_8 h^{-1/2} = 1$. A cosmological model with these parameters reproduces the local abundance of rich galaxy clusters (White, Elstathiou, & Frenk 1993; Eke, Cole, & Frenk 1996; Viana & Liddle 1996), has the shape parameter $\Omega_0 h = 0.21$ preferred by analyses of galaxy clustering (Peacock & Dodds 1994), but has a somewhat higher normalization than that derived from the COBE data (e.g., Ratra et al. 1997). It also agrees with the observed abundance of giant luminous arcs in galaxy clusters (Bartelmann et al. 1998).

### 3. LENSING CROSS SECTIONS

Next, we proceed to calculate the magnification cross section of the disk-plus-halo lenses. This calculation must be done numerically. First, we cover the source plane with a grid of source positions. Far from the lens center, the resolution of this grid can be low, while close to the caustic curves where the highest magnifications arise, the resolution should be high. We therefore use an adaptive grid in the source plane whose resolution increases toward the caustic curves. Then, for each source position, all image positions need to be found. For this purpose, we use the algorithm described by Schneider, Ehlers, & Falco (1992). Briefly, it is based on covering the image plane with a uniform grid, which is then mapped back to the source plane. All grid cells on the image plane whose mapping on the source plane contains the source position are taken to contain images of the source. In regions of strong lensing, the parity of the cell might be flipped when it is mapped into the source plane; i.e., corners of the cell might be interchanged. One should therefore start with triangular rather than rectangular cells in the image plane, because the image of a triangle remains a convex figure whose interior is well defined. This can be easily achieved by splitting each rectangular cell along one of its diagonals. Here again, we use an adaptive-grid approach in order to achieve high resolution at a reasonable computational cost. First, the image positions are searched on a coarse grid, constraining the regions on the lens plane where the images are located. Then each of these regions is covered with a fine grid on which the final image positions are localized. This way, all the images corresponding to every segment on the source-plane grid are identified.

The magnification of each image can then be computed from the total lensing potential (eq. [9]),

$$
\mu_{GL} = \text{det}^{-1} \left( \delta_{jk} - \frac{\delta^2 \psi}{\partial x_i \partial x_j} \right),
$$

evaluated at each of the image positions $x_i$. When dust extinction is included, the effective magnification of an image at position $x_i$ is

$$
\mu(x_i) = \mu_{GL}(x_i) \exp \left[ -N(x_i) \sigma_{ext} \right],
$$

where $N(x_i)$ is the $H_\alpha$ column density at the position of the image. The net magnification is the sum of the moduli of the effective magnifications for all of its $N$ images,

$$
\mu = \sum_{i=1}^{N} |\mu(x_i)|.
$$

This approach yields maps of the source magnification in the source plane. The magnification cross sections can be extracted from these maps as the area in the source plane within which sources are magnified by a factor $\mu$. The magnification cross section depends on the orientation of the disk relative to the line of sight. Denoting the cross section for a magnification $\mu$ and a disk inclination angle
flux ratio between the images, the lensing statistics of spiral lenses is well described by the simple spherical halo model. We will demonstrate below what happens when further constraints are imposed on the image properties.

In the limit of small magnifications \( \mu \rightarrow 1 \), the cross section for the disk-plus-halo model becomes numerically incomplete; this follows from the fact that the cross section increases rapidly as \( \mu \rightarrow 1 \), while the simulation is spatially bounded. This limit is irrelevant for our purposes, because we always impose further imaging constraints. In particular, when we require the image separation and flux ratio to be bounded to reasonable limits, the cross section is confined to the multiple-imaging region, which is entirely contained within our simulated section of the source plane.

### 3.1. Imaging Probabilities

Given the inclination-averaged cross sections, the probability for a (point) source at redshift \( z_s \) to be imaged with magnification \( \geq \mu \) is obtained by the line-of-sight integral over the density of lenses times their lensing cross section, \( \sigma(\mu) \). For the model without number-density evolution, this integral yields

\[
P_{\text{GL}}(\mu) = n_0 b_*^4 \varepsilon_s \left( 1 + \frac{v}{c} + \frac{4}{z} \right) \int_0^{z_s} dz (1 + z)^3 \int_0^1 \frac{c dt}{dz} \sigma(\mu),
\]

where \( |c dt/dz| \) is the proper-distance interval corresponding to the redshift interval \( dz \). The factor \( n_0 b_*^4 \varepsilon_s (1 + z)^3 \) comes from integrating over the luminosity distribution of the spiral galaxies (see eq. [16]). When evolution of the galaxy population is taken into account as described in § 2.5, the factor \( (1 + z)^3 \) above is changed to \( n_{00}(M_* z)/n_{00}(M_* 0) (1 + z)^3 \), where \( M_*(z) \) is the mass of a galaxy with circular velocity \( v_\text{c} \) at the corresponding redshift.

Now let \( dN_{\text{QSO}}/dS|\leq S|dS \) be the intrinsic number density of quasars at redshift \( z_s \), with a flux between \( S \) and \( S + dS \). Accounting for magnification bias, the number of lensed quasars is

\[
N_{\text{QSO}}(S) = \int_0^\infty dS' P_{\text{GL}}(S'/S) \frac{dN_{\text{QSO}}}{dS'} (S'),
\]

and hence the probability for a quasar at redshift \( z_s \) to be detected with flux \( > S \) is

\[
P_{\text{GL}}(S) = \frac{1}{N_{\text{QSO}}(S)} \int_0^\infty dS' P_{\text{GL}}(S'/S) \frac{dN_{\text{QSO}}}{dS'} (S').
\]

We approximate the observed quasar number counts in the \( B \) band by a broken power law,

\[
\frac{dN_{\text{QSO}}}{dS} = \left( \frac{S}{S_0} \right)^\alpha \text{ for } S \leq S_0,
\]

\[
\left( \frac{S}{S_0} \right)^\beta \text{ for } S > S_0,
\]

where \( S_0 \) corresponds to \( B_{\text{QSO}} \approx 19 \). For quasar redshifts \( z_s \approx 2, a = -1.64 \) and \( b = -3.52 \) (e.g., Pei 1995).

Figure 5 shows \( P_{\text{GL}}(\mu) \), and Figure 6 shows \( P_{\text{GL}}(S) \) for the no-evolution galaxy model. These figures include curves that illustrate the influence of nonevolving and evolving dust. All curves in Figure 5 were calculated under the additional constraints that the two brightest images be separated by \( \geq 0.3 \) and that their flux ratio be \( \leq 20 \). The solid and long-dashed curves in Figure 5 are for the disk-plus-halo...
model and for the singular isothermal sphere, respectively. Evidently, the disk-plus-halo model is less efficient at producing multiple images with the specified properties than the singular isothermal sphere, despite the fact that the total magnification cross sections of the two models are close to each other (Fig. 4). The reason for this difference is that the images produced by the spiral lens model are typically closer to each other than those of the singular isothermal sphere and are occasionally below the 0.3'' threshold. This occurs because a significant fraction of the lensing cross section is contributed by the disk in nearly edge-on orientations. The image separation on either side of the disk is then smaller than the Einstein diameter of the corresponding singular isothermal sphere.

The imposed constraints on the minimum image separation and maximum flux ratio approximately reflect characteristic thresholds for detecting a multiply imaged system through space-based observations with finite resolution and dynamical range (see Kochanek 1993). Figure 5 demonstrates that these additional constraints reduce the detection efficiency of spiral lenses considerably compared to the singular isothermal sphere model. The figure also quantifies the severe effect that dust extinction has on the detection efficiency. Besides suppressing the detection probability by 1 or 2 orders of magnitude for evolving and nonevolving dust, respectively, the dust extinction allows for values of $\mu = (S_{\text{observed}}/S_{\text{intrinsic}})$ smaller than unity, which are otherwise unphysical.

![Figure 6](image1)

**Fig. 6.** Probability $P_{\text{GL}}(B_{\text{QSO}})$ for a quasar to be multiply imaged with image separation $>0.3$ and flux ratio $<20$, as a function of $B_{\text{QSO}}$. The mass model includes (a) no evolution, (b) evolution of disk scale lengths, and (c) evolution of scale lengths and number density. The four curves per panel are for the disk model without dust (solid curve), with nonevolving dust (dotted curve), and with evolving dust (short-dashed curve), and the singular isothermal sphere model (long-dashed curve).

![Figure 7](image2)

**Fig. 7.** Cumulative distribution of multiple images with separation $\theta > \theta_{\text{min}}$, as a function of $\theta_{\text{min}}$, for a constant maximum flux ratio $r_{\text{max}} = 20$ and quasar magnitude $B_{\text{QSO}} = 18$. The disk model includes (a) no evolution, (b) evolution of scale lengths, and (c) evolution of scale lengths and disk number density. The three curves per panel are without dust extinction (solid line) and with extinction by nonevolving (dotted) and evolving (short-dashed) dust. The long-dashed curve describes the singular isothermal sphere model with no dust.
Figure 6 then shows the probability $P_{GL}(S)$ for quasars at redshift $z_s = 2$ to be lensed with image separation $\geq 0.3$ and flux ratio $\leq 20$ as a function of the quasar $B$ magnitude. Again, the dotted and short-dashed curves are calculated including dust extinction, and the solid and long-dashed curves are without dust for the disk-plus-halo model and for the singular isothermal sphere model, respectively. The three panels in the figure are for different levels of cosmological evolution as detailed in the figure caption. The figure shows that bright quasars, with $B_{QSO} \leq 19$, are much more likely to be multiply imaged than faint quasars, because of the magnification bias. At $B_{QSO} = 18$, the multiple-imaging probability is about a factor of 5 higher than for $B_{QSO} \sim 19$. As already shown in Figure 5, dust extinction severely reduces the imaging probability, and it also leads to a weaker rise in the lens detection probability with increasing quasar brightness.

3.2. Image Statistics

In order to appreciate the significance of the selection effects in identifying spiral lenses, it is instructive to examine the distribution of image separations and flux ratios. Figure 7 shows the cumulative distribution of multiple images as a function of the minimum image separation $\theta_{\min}$. The distribution is arbitrarily normalized to unity at $\theta_{\min} = 0.1$. As before, the three panels in the figure are for disks (a) with no evolution, (b) with evolution of scale radii only, and (c) with evolution of scale radii and galaxy number density. The three curves per panel are for models without dust (solid line), with nonevolving dust (dotted line), and with evolving.
dust (short-dashed line). The long-dashed curve in each panel shows the singular isothermal sphere model without dust for comparison.

Figure 7 implies that the median image separation is larger for the singular isothermal sphere than for the disk-plus-halo model. Images produced by the spiral lens model are on average closer to each other by $\sim 0.2$ than expected from the singular isothermal sphere model. This is because multiple images in the dust-plus-halo model are predominantly produced by the disk rather than by the halo, with a preference for nearly edge-on disk orientations.

Dust increases the average image separation because images with small separations occur close to the disk, where their flux is heavily suppressed by extinction. Disk evolution reduces the average image separation slightly, because the disks get more compact with increasing redshift. Number-density evolution according to the Press-Schechter model increases the abundance of field spirals at redshifts $z \sim 1-2$ (since some of these galaxies have subsequently merged and disappeared by now) and reduces the image separation further because the lenses are, on average, farther away.

Figure 8 shows the cumulative distribution of images as a function of the maximum flux ratio $r_{max}$, arbitrarily normalized to unity at $r_{max} = 50$. The notation is the same as in Figure 7. Generally, the curves for the disk-plus-halo model without dust are the flattest, showing that the images are usually of comparable brightness. When dust is included, the average flux ratio increases because of the strong extinction gradient around the disk.

4. EFFECTS OF THE GASEOUS DISK

4.1. Impact of Dust on Lensing Statistics

To further examine the influence of dust on the detectability of lensed quasars, we show in Figure 9 the ratio between the lensing probability $P_{GL}(B_{QSO})$ with and without dust. The three panels are again for a galaxy model (a) with no evolution, (b) with length-scale evolution only, and (c) with length-scale plus number-density evolution. The two curves in each panel are for nonevolving and evolving dust (dotted and solid lines, respectively).

Figure 9 shows that the majority of all quasars that are lensed by spirals are undetectable in the $B$ band. The deficit caused by dust increases for brighter quasars as a result of the magnification bias. At $B_{QSO} \approx 18$, only about 10% of all lensed quasars are observable, quite independent of the degree of evolution that is included in the calculation. If dust does not evolve with redshift, the deficit of lensed quasars is more severe. For faint quasars, $B_{QSO} \gtrsim 20$, the reduction in the lens detection probability is least severe in the no-evolution model. This is mainly because when disk evolution is included, the surface density of the disk increases with $z$, leading to more extensive dust extinction than the no-evolution model predicts.

Figure 9 implies that the fraction of quasars lensed by spiral galaxies in radio surveys should be higher by about an order of magnitude relative to that found in optical surveys.

4.2. Statistics of Damped Ly$\alpha$ Absorption

Based on the neutral hydrogen column density at the position of each image, we can investigate the influence of lensing and dust on the statistics of damped Ly$\alpha$ absorption by spirals. Let $P_{GL}(S, N)$ be the probability of observing a quasar with a flux $\geq S$ that is imaged by the population of spirals and shows an H$\alpha$ column density $> N$ in its spectrum. When there is more than one image, we take $N$ to be the H$\alpha$ column density in the brightest image, i.e., that which dominates the absorption trough. As shown by Bartelmann & Loeb (1996), the observed column density distribution of neutral hydrogen is then given by

$$f(N) = \frac{c}{H_0} \frac{1}{\Delta X} \left| \frac{\partial P_{GL}(S, N)}{\partial N} \right|, \tag{31}$$

where $\Delta X$ is the absorption distance scanned by the DLAs in the survey (Bahcall & Peebles 1969). Examples for the distribution of $Nf(N)$ are plotted in Figure 10.

![Figure 10](image-url)

**Fig. 10**—H$\alpha$ column-density distribution, $Nf(N)$, for lensed quasars with $B = 18$ at $z_2 = 2$ as a function of $N_{20} \equiv (N/10^{20}\text{cm}^{-2})$. The disk model includes (a) no evolution, (b) evolution of scale lengths, and evolution of scale lengths and (c) number density. The three curves were calculated without dust, with nonevolving dust, and with evolving dust (solid, dotted, and short-dashed curves, respectively). For reference, the long-dashed curve shows the result without lensing or dust extinction.
The inferred cosmological density parameter in neutral hydrogen, $\Omega_{\text{HI}}$, traced by DLAs is given by

$$\Omega_{\text{HI}} = \frac{H_0}{c} \frac{\bar{m}}{\rho_{\text{cr},0}} \int_{N_0}^{\infty} dN f(N),$$

(32)

where $\rho_{\text{cr},0}$ is the present-day critical density of the universe, $N_0 \approx 10^{20}$ cm$^{-2}$ is the minimum column density for damped Ly$\alpha$ absorption, and $\bar{m}$ is the mean molecular mass. Because of the magnification bias, $f(N)$ depends on the quasar magnitude $B_{\text{QSO}}$, and so does the inferred $\Omega_{\text{HI}}$. Figure 11 shows the ratio between the inferred and true values of $\Omega_{\text{HI}}$ for spiral galaxies. The different panels and line types are the same as those in previous figures. Similarly to $P_{\text{G},1}$, $f(N)$ and $\Omega_{\text{HI}}$ depend on the redshift range of the absorbers (see eq. [27]). We use the full redshift range, $0 \leq z \leq z_s = 2$, for Figure 10 and the upper panels of Figure 11, and the high-redshift range $1 \leq z \leq z_s$ for the lower panels of Figure 11.

Models without dust cause $\Omega_{\text{HI}}$ to be overestimated by factors of up to a few in samples of bright quasars. The magnification bias brings into view quasars that were otherwise too faint to be detected. The lensed quasar images occur close to the lens center, where the H I column density is higher than average. However, when dust is included, the net effect is reversed. Now those quasars whose images are close to the disk are obscured, and the number of quasar spectra with high H I column density is lower. For evolving dust and $B_{\text{QSO}} \approx 18$, only about 20%-30% of the neutral hydrogen is detected.

At face value, the evolution model for spiral galaxies sketched in § 2.5 predicts decreasing $\Omega_{\text{HI}}$ with increasing redshift. In contrast, for fixed source redshift, the observed $\Omega_{\text{HI}}$ predicted by our model rises with absorber redshift, reaches a peak at intermediate redshifts, and falls toward the source redshift (see Fig. 3 in Bartelmann & Loeb 1996). This is a consequence of gravitational lensing, because the lensing efficiency is optimal when the lenses are located roughly halfway between observer and source. The observed change of $\Omega_{\text{HI}}$ with redshift (e.g., Storrie-Lombardi, McMahon, & Irwin 1996) is thus qualitatively reproduced by our model, although the intrinsic $\Omega_{\text{HI}}$ decreases with absorber redshift.

5. CONCLUSIONS

We investigated the lensing effects of spiral galaxies, modeled as maximal truncated Mestel disks embedded in spherical halos. This model makes the disk as massive as possible. Our results therefore represent the opposite extreme to the simple isothermal sphere models. Intermediate models with less massive disks would shift these results toward those obtained from the spherical model.

Although a disk-plus-halo configuration yields an inclination-averaged lensing cross section similar to that of the spherically symmetric mass distribution (Fig. 4), its
typical image separations are considerably smaller (Fig. 7). A substantial contribution to this cross section comes from lensing by nearly edge-on configurations of the disk, in which a pair of images straddles the disk. Because the images are often close to the disk, they suffer strong extinction by dust. The extinction lowers the detection efficiency of spiral lenses in the optical band by an order of magnitude. Moreover, disk lenses are often characterized by small image separation (Fig. 7) and large differential extinction (Fig. 8). As a result, the selection effects imposed by the finite angular resolution and dynamic range of observations set limits on the minimum image separation and their maximum flux ratio and substantially lower the probability for observing spiral lenses (see Fig. 9).

We find that dust dominates over the magnification bias because of lensing in modifying the statistics of damped Lyz absorption by spirals (Fig. 11). If most of the damped absorption systems are spirals, the inferred value of the cosmological density parameter in H1 at z ≲ 1 could be underestimated by a factor of a few.

The spiral-lens system B0218 + 357 shows compelling evidence for strong extinction. O’Dea et al. (1992) find that the quasar spectrum is red, Wiklind & Combes (1995) and Menten & Reid (1996) find strong molecular lines, and Grundahl & Hjorth (1995) find that image A is much fainter than image B in the optical in contrast to the radio observations, arguing for a substantial extinction of image A. Jaunsen & Hjorth (1997) have recently argued for the existence of disk lensing and associated extinction also in the lens system B1600 + 434, which has recently been modeled by Maller et al. (1997) and Koopmans, de Bruyn, & Jackson (1998). More generally, Malhotra et al. (1997) argue for a systematic reddening of lensed quasars relative to the rest of the quasar population, but they may have overestimated the effect (cf. Falco, Kochanek, & Muñoz 1998).

The optical depth to gravitational lensing of quasars was shown to be a very effective tool for setting constraints on the cosmological constant (Kochanek 1996 and references therein). In singular isothermal models, ~80%–90% of the lensing probability is contributed by early-type galaxies (e.g., Fukugita & Turner 1991). However, if a significant fraction of the present-day early-type galaxies were in the form of spiral building blocks at z ≃ 1 (as argued by Kaufmann et al. 1996), then the depletion of spiral lenses due to dust extinction and selection effects could weaken the current lensing constraints on the cosmological constant. This issue was addressed by Mao & Kochanek (1994) and Rix et al. (1994). They found that estimates of the cosmological constant from the statistics of strong lensing can be significantly changed only when the early-type galaxies seen today evolved dramatically below a redshift z ≃ 1.

In this context, and in view of ongoing lensing surveys, it is instructive to examine which fraction of the total lensing optical depth is contributed by spirals rather than by early-type galaxies. To calculate this, we assume that the total galaxy population today is composed of 25% early-type galaxies and 75% spirals. We model spirals as described above and early-type galaxies as singular isothermal spheres with a characteristic velocity dispersion of $\sigma_\text{e} = v_\text{e}/2^{1/2} = 220 \text{ km s}^{-1}$, following the Schechter luminosity function (14) and the Faber-Jackson (1976) relation. For simplicity, we use the same Schechter-function parameters as for the spirals. We consider two scenarios, one in which the number density of early-type galaxies stays constant with redshift, and another in which early-type galaxies are assembled by merging of spirals. In the latter scenario, we assume that the number density of early-type galaxies changes with redshift as a power law, $n_\text{e}(z) = n_\text{e}(0)(1 + z)^{1.6}$, with $\epsilon \sim -1.6$, chosen such that $\sim 3/4$ of the early-type galaxies are in the form of spiral building blocks at $z \sim 1$ (Kaufmann et al. 1996). We further assume that mergers conserve mass. In both scenarios, we can then compute the lensing optical depths for both spirals and early-type galaxies $\phi_{\text{GL,E}}$ from equation (27) and the corresponding imaging probability $P_{\text{GL,E}}$ from equation (29) (under the constraints that the images be separated by $\gtrsim 0.3$ and have a flux ratio $\lesssim 20$).

Figure 12 shows the fraction of the total lensing probability contributed by spirals, namely $\phi_{\text{GL,E}}/\phi_{\text{GL,total}} = P_{\text{GL,E}}/(P_{\text{GL,E}} + P_{\text{GL,L}})$. Figure 12a shows results for early-type galaxies with constant comoving number density, while Figure 12b examines the scenario in which early-type galaxies merge out of spirals. The three curves per panel are for spirals without dust, with nonevolving dust, and with evolving dust (solid, dotted, and dashed lines, respectively). The no-dust results are valid for radio-selected lenses. The solid line in panel (a) reproduces the familiar result that dust-free spirals contribute at most ~20% of the multiple-imaging probability, even for the small image-separation cutoff of $\theta_{\text{min}} = 0.3''$; this fraction is lower for higher $\theta_{\text{min}}$. In the presence of dust, the fraction contributed by spirals is considerably smaller. In the merger scenario, the spiral lens fraction rises to ~50% without dust, ~9% with evolving dust, for quasars with $B \sim 18$. Thus, up to half of the lenses found in radio surveys (which are sensitive to small image separations) might be imaged by spirals.

Figure 9 implies that radio surveys should be about a factor of 5–10 more efficient at detecting spiral lenses than optical surveys. The recent CLASS and JVAS radio surveys indeed provide preliminary hints of a more substantial population of spiral lenses than found in optical surveys (cf. Table 1 in Browne et al. 1997; Jackson, Nair, & Browne 1997; but see Fassnacht & Cohen 1998). To date, the combined CLASS and JVAS samples encompass more than $10^4$ flat-spectrum radio sources, 11 of which have been identified as lens systems. A substantial fraction of those have been classified as being lensed by spirals or S0 galaxies. Because of their small image separation and simple geometry, lenses containing edge-on disks might offer a unique opportunity for estimating the masses of galactic disks at high redshifts and also for constraining the Hubble constant based on the time delay between the flux variations in their images. Indeed, the time delay in B0218 has been measured to 12 ± 3 days, translating to a Hubble constant of $H_0 \sim 60 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (Corbett et al. 1996), and B1608 + 656 offers another promising lens system for this purpose (Myers et al. 1995; Fassnacht et al. 1996).

The above results were obtained under the assumption that all spirals possess the same dust-to-gas ratio throughout their disks at any given redshift. If the scatter in the dust content of different galaxies is large or if the dust distribution is patchy, then the quantitative impact of dust on the lensing statistics would be altered. Also, we have assumed

1 See Myers and Williams & Schechter in Proceedings of the Workshop on Golden Lenses at Jodrell Bank (1997) at http://multivac.jb.man.ac.uk/~3000/ceres/workshop1/proceedings.html.
that the scale heights of dust and gas are the same and that they are constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius. On physical grounds, one would expect the scale heights of dust and gas to increase with radius, because their sound speed (which is set by the balance between cooling and heating) should not vary much with radius. In addition, if spiral disks at set by the balance between cooling and heating) should not increase with radius, because their sound speed (which is constant across the disk radius.
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