Collinear Asymptotic Dynamics for Massive Particles. Multi-Channel Regge Behaviour.*

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Abstract

The high-energy behaviour in a multi-channel system is investigated in the framework of collinear asymptotic dynamics for massive particles. We consider the most general trilinear coupling of $N$ different scalar fields. We find Regge behaviour and obtain closed expressions for the Regge trajectories and couplings. The results are corroborated in the multi-channel Bethe-Salpeter approach. The scattering processes dominated by single Regge trajectories are explicitly given.

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1 Introduction

The framework of collinear asymptotic dynamics for massive particles is a natural extension of methods that use large-time scale Hamiltonians to treat massless particles. The existence of massless fields in the theory may generally demand a special treatment of infrared and collinear divergencies. In theories and processes with massive particles collinear configurations play a role in the high-momentum regime only. In this context it has been shown that our approach is suitable for the description of Regge behaviour in the $\varphi^3$-theory. Since, to the best of our knowledge, multi-channel Regge behaviour has not been studied in the literature, we investigate the high-energy behaviour for the most general trilinear coupling of $N$ hermitian scalar fields, $\varphi_i$, $i = 1, \ldots, N$,

$$H_1(x) = -\frac{1}{3!} \sum_{i,j,k=1}^N c_{ijk} :\varphi_i(x)\varphi_j(x)\varphi_k(x):$$ (1.1)

The totally symmetric symbol $c_{ijk}$ denotes the coupling constants.

Starting from a time-ordered exponential approximation of the time-evolution operator, we derive a matrix formulation for the multi-channel amplitudes at high energies. At large time scales the dynamics is dominated by collinear processes. In this regime we obtain multi-channel Regge behaviour.

We find a generally non-orthogonal base of states in the crossed channel leading to a separation of the various Regge trajectories. This fact implies a set of linear combinations of the amplitudes, the high-energy behaviour of which is expressed by one Regge trajectory only.

The feasibility of this approach we find encouraging for further studies of more realistic interactions. In this situation it is of particular interest to confront our approach with a Bethe-Salpeter formulation of the problem in the crossed channel. We generalize methods successful for the single-channel system to the multi-channel case.

The results of the Bethe-Salpeter treatment confirm the findings of our $s$-channel treatment of the collinear asymptotic dynamics.

2 Collinear Asymptotic Dynamics

We consider the scattering matrix element $T_{fi}((\vec{p}_1, r), (\vec{p}_2, a) \rightarrow (\vec{p}_3, s), (\vec{p}_4, b))$, where $r, a, s$ and $b$ denote the particle type. Let $p^{(j)} = (E^{(j)}(\vec{p}), \vec{p})$, $E^{(j)}(\vec{p}) = (\vec{p}^2 + m_j^2)^{\frac{1}{2}}$, stand for the four momentum and energy of a particle of type $j$, respectively. The scattering-matrix element can be written as

$$T_{fi} = T + T^{cr},$$ (2.1)

$$T = -i \int d^4x d^4y e^{-i(p_2^{(a)} y - p_4^{(b)} x)} \Theta(x^0 - y^0) \langle \vec{p}_3, s| j_b H(x) j_a H(y) | \vec{p}_1, r \rangle$$ (2.2)

$$= -i \int d^4x d^4y e^{-i(p_2^{(a)} y - p_4^{(b)} x)} \Theta(x^0 - y^0)$$

$$\times \langle \vec{p}_3, s| U^\dagger(x^0, -\infty) j_b(x) U(x^0, y^0) j_a(y) U(y^0, -\infty) | \vec{p}_1, r \rangle,$$ (2.3)

and

$$T^{cr} = T(a \leftrightarrow b, p_2^{(a)} \leftrightarrow -p_4^{(b)}).$$
In (2.3) the current for a particle of type \(i\) is given by

\[
  j_i(x) = \frac{1}{2!} \sum_{j,k} c_{ijk} \varphi_j(x) \varphi_k(x): 
\]

(2.4)

The matrix element \(T\) will be calculated in the laboratory frame of the particle with momentum \(p_2^{(a)}; p_3^{(r)}\) and \(p_3^{(s)}\) are large momenta with small transfer \(t = (p_3^{(r)} - p_3^{(s)})^2\).

As in our previous paper \([\text{I}]\) we use as an asymptotic approximation the Hamiltonian

\[
  \hat{H}_I = H_I^{(+)} + H_I^{(-)}
\]

with

\[
  H_I^{(+)} = -\frac{1}{2} \int d^3x \sum_{i,j,k} c_{ijk} \varphi_i^{(-)} \varphi_j^{(-)} \varphi_k^{(+)} , \quad H_I^{(-)} = (H_I^{(+)})^\dagger .
\]

(2.5)

Here \(\varphi_i^{(+)}\) and \(\varphi_i^{(-)}\) are the annihilation and creation parts of \(\varphi_i\), respectively. As in the case of one particle type, we can replace \(j_i(x)\) by

\[
  j_i^{(S)} = \sum_{j,k} c_{ijk} \varphi_j^{(-)} \varphi_k^{(+)} .
\]

(2.6)

Following the arguments given in \([\text{I}]\), we replace \(U(x^0,y^0) = 1\) in (2.3) and insert a complete set of states between \(j_\alpha(x)\) and \(j_\alpha(y)\). In order to make the calculation closely connected to the one in our previous paper, we use the following decomposition of the unit operator:

\[
  1 = |0\rangle \langle 0| + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n=1}^{N} \int \prod_{l=1}^{n} \frac{d^3k_l}{2(2\pi)^3 E^{(i_l)}(\vec{k}_l)} \times |\vec{k}_1, i_1\rangle, \ldots, |\vec{k}_n, i_n\rangle |\vec{k}_1, i_1\rangle, \ldots, |\vec{k}_n, i_n\rangle |\langle \vec{k}_1, i_1|, \ldots, |\langle \vec{k}_1, i_1|, \ldots, |\langle \vec{k}_n, i_n| |\langle \vec{k}_n, i_n| .
\]

(2.7)

The validity of this decomposition can be readily verified. Using this expression, we get

\[
  T = (2\pi)^3 \delta(p_1^{(r)} + p_2^{(a)} - p_3^{(s)} - p_4^{(b)}) \mathcal{M}_{s,b}^{\tau,a} ,
\]

(2.8)

\[
  \mathcal{M}_{s,b}^{\tau,a} = (2\pi)^3 \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n=1}^{N} \int \prod_{l=1}^{n} \frac{d^3k_l}{2(2\pi)^3 E^{(i_l)}(\vec{k}_l)} \times \delta(\vec{p}_1 + \vec{p}_2 - \sum_{l=1}^{n} \vec{k}_l) \times \frac{1}{E^{(r)}(\vec{p}_1) + E^{(a)}(\vec{p}_2) - \sum_{l=1}^{n} E^{(i_l)}(\vec{k}_l) + \epsilon} \times G_a^{(n)}((\vec{p}_1, r); (\vec{k}_1, i_1), \ldots, (\vec{k}_n, i_n))G_b^{(n)*}((\vec{p}_3, s); (\vec{k}_1, i_1), \ldots, (\vec{k}_n, i_n))
\]

with

\[
  G_a^{(n)}((\vec{p}, i); (\vec{k}_1, i_1), \ldots, (\vec{k}_n, i_n)) = ((\vec{k}_1, i_1), \ldots, (\vec{k}_n, i_n)) j_\alpha^{(S)}(0) U(0, -\infty) |\vec{p}, i\rangle
\]

(2.9)

\[
  = \sum_{\nu=1}^{n} \sum_{d} c_{ai_{\nu,d}}((\vec{k}_1, i_1), \ldots, (\vec{k}_\nu, i_{\nu}), \ldots, (\vec{k}_n, i_n)) |\varphi_d^{(+)}(0) U(0, -\infty) |\vec{p}, i\rangle ,
\]

(2.10)

where the symbol \((\vec{k}_{\nu}, i_{\nu})\) indicates that the corresponding particle has to be omitted from the state.
As in our previous paper, the time-evolution operator \( U(0, -\infty) \) is dominated by the operator
\[
W_1(0, -\infty) = \mathcal{T} \exp \left\{ -i \int_{-\infty}^{0} H_1^{(+)}(t) \, dt \right\} . \quad (2.11)
\]
In the time-ordered expansion of \( W_1 \) only the term of order \( n - 1 \) contributes to \( G_a^n \), i.e.
\[
\langle (\vec{k}_1, t_1), \ldots, (\vec{k}_\nu, t_\nu), \ldots, (\vec{k}_n, t_n) | \varphi_d^{(+)}(0) W_1(0, -\infty) | \vec{p}, i \rangle \quad (2.12)
\]
\[
= (-i)^{n-1} \int_{-\infty}^{0} dt_1 \cdots dt_{n-1} \Theta(t_1, \ldots, t_{n-1})
\times \langle 0 | \varphi_d^{(+)}(0) a_i(\vec{k}) \cdots a_{\nu}(\vec{k}_\nu) \cdots a_i(\vec{k}_n) H_1^{(+)}(t_1) \cdots H_1^{(+)}(t_{n-1}) | \vec{p}, i \rangle .
\]
In this expression, \( \Theta(t_1, \ldots, t_{n-1}) = \Theta(t_1 - t_2) \cdots \Theta(t_{n-2} - t_{n-1}) \), and \( a_i(\vec{k}) \) is the annihilation operator of a particle of type \( i \). With the definition
\[
D_i(t, \vec{k}) = -i[a_i(\vec{k}), H_1^{(+)}(t)] = i \int_{x^a = t} d^3x \, e^{i(k_i x)} x^i \langle x \rangle ,
\]
\( \text{cf. (2.6)} \), the dominant contribution to (2.12) can be written as
\[
\int_{-\infty}^{0} dt_1 \cdots dt_{n-1} \Theta(t_1, \ldots, t_{n-1})
\times \sum_{\pi_\nu} \langle 0 | \varphi_d^{(+)}(0) D_{\pi(1)}(t_1, \vec{k}_{\pi(1)}) D_{\pi(2)}(t_2, \vec{k}_{\pi(2)}) \cdots D_{\pi(n)}(t_{n-1}, \vec{k}_{\pi(n)}) | \vec{p}, i \rangle ,
\]
where the sum runs over all permutations of the integers 1, 2, \ldots, \( \nu - 1, \nu + 1, \ldots, n \). Since
\[
D_i(t, \vec{k})|\vec{p}, j\rangle = i \sum_{l} e^{i(E_l(\vec{p} - \vec{k}) + E^i_l(\vec{k}) - E^i_l(\vec{p}))t} \frac{2E_l(\vec{p} - \vec{k})}{|\vec{p} - \vec{k}|} |\vec{p} - \vec{k}, l\rangle ,
\]
the result for \( G_a^n \) is
\[
G_a^n((\vec{p}, r); (\vec{k}_1, i_1), \ldots, (\vec{k}_n, t_n)) \quad (2.15)
\]
\[
= \sum_{\nu=1}^{n} \sum_{\pi_\nu} u_{\nu_1, \nu_2, \ldots, \nu_n} c_{\pi(\nu)} u_{\nu} \cdots c_{\pi(\nu-1) u_{n-1}} u_{n-1} \cdots
\times \frac{c_{u_{\nu+1} i_\nu} u_{\nu+1}}{(\vec{p}^2 - \vec{q}_{\nu+1, \nu}^2)^2 + m^2_{\nu+1}} \frac{c_{u_{\nu+1} i_{\nu-1} u_{\nu-1}} u_{\nu-1}}{(\vec{p}^2 - \vec{q}_{\nu-1, \nu}^2)^2 + m^2_{\nu-1}} \cdots \frac{c_{u_{i} i_{1} u_{1}} u_{1}}{(\vec{p}^2 - \vec{q}_{i, 1}^2)^2 + m^2_{i}}
\]
with
\[
\vec{q}_{i, \nu} = \sum_{j = i, j \neq \nu}^{n} \vec{k}_{\pi(j)},
\]
and
\[
p^\parallel \gg k^\parallel_{\pi(n)} \gg k^\parallel_{\pi(n-1)} \gg \ldots \gg k^\parallel_{\pi(\nu+1)} \gg k^\parallel_{\pi(\nu-1)} \gg \ldots \gg k^\parallel_{\pi(1)} \gg k^\parallel_{\nu} \gg o(m) . \quad (2.16)
\]
Equations (2.15) and (2.16) are a consequence of collinear asymptotic dynamics [1].
In what follows, the decomposition of momenta in longitudinal and transverse parts is to be done with respect to \( \vec{p}_1 \). In this way we obtain

\[
\mathcal{M} = (2\pi)^3 \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1, \ldots, i_n=1}^{N} \int \prod_{l=1}^{n} \frac{dk_{\|l}^0 d^2k_{\perp l}^0}{2(2\pi)^3 E^{(i_l)}(k_{\| l}^0) E^{(r)}(p_1^\|)} \frac{\delta(\vec{p}_1 - \sum_{l=1}^{n} \vec{k}_l^0)}{\sum_{l=1}^{n} E^{(i_l)}(k_{\| l}^0)} \prod_{\nu=1}^{n} \frac{1}{\prod_{\nu=1}^{n} \Theta(k_{\| \nu}^0, \ldots, k_{\perp \nu}^0)} \sum_{u_1, \ldots, u_{\nu-1}} \sum_{u_{\nu+1}, \ldots, u_n} \frac{c_{\nu\pi(u)} u_n c_{\nu\pi(u-1)u_{n-1}}}{q_{\|\nu,u}^0 + m_{u_n}^2 q_{\perp\nu-\nu}^0 + m_{u_{n-1}}^2} \ldots \\
\times \frac{c_{\nu u_2 i_1} u_1}{(q_{\|\nu-\nu}^0 + m_{u_1}^2)(q_{\|\nu-\nu}^0 + m_{u_1}^2)} c_{\nu i_1 a} c_{\nu i_1 b}.
\]

Note that this expression factorizes into longitudinal and transverse parts. Because of the strong ordering (2.16) only the same permutations of momenta in both factors of (2.17) give a contribution to \( \mathcal{M} \). Therefore we get

\[
\mathcal{M} = (2\pi)^3 \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1, \ldots, i_n=1}^{N} \int \prod_{l=1}^{n} \frac{dk_{\|l}^0 d^2k_{\perp l}^0}{2(2\pi)^3 E^{(i_l)}(k_{\| l}^0) E^{(r)}(p_1^\|)} \frac{\delta(\vec{p}_1 - \sum_{l=1}^{n} \vec{k}_l^0)}{\sum_{l=1}^{n} E^{(i_l)}(k_{\| l}^0)} \prod_{\nu=1}^{n} \frac{1}{\prod_{\nu=1}^{n} \Theta(k_{\| \nu}^0, \ldots, k_{\perp \nu}^0)} \sum_{u_1, \ldots, u_{\nu-1}} \sum_{u_{\nu+1}, \ldots, u_n} \frac{c_{\nu\pi(u)} u_n c_{\nu\pi(u-1)u_{n-1}}}{q_{\|\nu,u}^0 + m_{u_n}^2 q_{\perp\nu-\nu}^0 + m_{u_{n-1}}^2} \ldots \\
\times \frac{c_{\nu u_2 i_1} u_1}{(q_{\|\nu-\nu}^0 + m_{u_1}^2)(q_{\|\nu-\nu}^0 + m_{u_1}^2)} c_{\nu i_1 a} c_{\nu i_1 b}.
\]

In this form, the integration over transverse momenta can be factorized:

\[
\mathcal{M} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1, \ldots, i_n=1}^{N} \int \prod_{l=1}^{n} \frac{dk_{\|l}^0 d^2k_{\perp l}^0}{2(2\pi)^3 E^{(i_l)}(k_{\| l}^0) E^{(r)}(p_1^\|)} \frac{\delta(\vec{p}_1 - \sum_{l=1}^{n} \vec{k}_l^0)}{\sum_{l=1}^{n} E^{(i_l)}(k_{\| l}^0)} \prod_{\nu=1}^{n} \frac{1}{\prod_{\nu=1}^{n} \Theta(k_{\| \nu}^0, \ldots, k_{\perp \nu}^0)} \sum_{u_1, \ldots, u_{\nu-1}} \sum_{u_{\nu+1}, \ldots, u_n} \frac{c_{\nu\pi(u)} u_n c_{\nu\pi(u-1)u_{n-1}}}{q_{\|\nu,u}^0 + m_{u_n}^2 q_{\perp\nu-\nu}^0 + m_{u_{n-1}}^2} \ldots \\
\times F_{u_1}^{u_2}(\vec{p}_3^\|) \cdots c_{\nu u_2 i_1} u_1 c_{\nu u_2 i_1} u_1 F_{u_1}^{u_2}(\vec{p}_3^\|) c_{\nu i_1 a} c_{\nu i_1 b},
\]

where

\[
F_{u_1}^{u_2}(\vec{p}_3^\|) = \frac{1}{2(2\pi)^3} \int \frac{d^2q_{\perp}}{(q_{\perp}^2 + m_{u_1}^2)(q_{\perp}^2 + m_{u_2}^2)}.
\]

The longitudinal integral can be carried out in leading logarithmic approximation, resulting in \( (s = 2m_a p_1^\|) \)

\[
\int \prod_{l=1}^{n} \frac{dk_{\|l}^0 d^2k_{\perp l}^0}{E^{(i_l)}(k_{\| l}^0) E^{(r)}(p_1^\|) + m_a - \sum_{l=1}^{n} E^{(i_l)}(k_{\| l}^0)} = \frac{2 (\ln s)^{n-1}}{s (n-1)!}.
\]
for all permutations. Thus all permutations for all \( \nu \) give the same contribution to \( \mathcal{M} \), so that, if we define

\[
C_{u_1,u_2}^{w_1,w_2} = \sum_{i=1}^{N} c_{u_1 i u_2} c_{w_1 i w_2} ,
\]

(2.20)

the matrix element can be written as

\[
\mathcal{M} = \mathcal{M}_{s,b}^{r,a} = \frac{1}{s} \sum_{n=1}^{\infty} \frac{(\ln s)^{n-1}}{(n-1)!} \sum_{u_1,\ldots,u_n=1}^{w_1,\ldots,w_n=1} C_{u_1,\ldots,u_n}^{w_1,\ldots,w_n} \mathcal{C}_{u_1,u_2}^{w_1,w_2} \cdots \mathcal{C}_{u_2,u_1}^{w_2,w_1} \mathcal{C}_{u_1,u_2}^{w_1,w_2} .
\]

(2.21)

This formula has a simple interpretation in terms of processes in the crossed channel. To see this, we introduce an index \( I \),

\[
(r,s) \rightarrow I(r,s) = 1, \ldots, N^2
\]

characterizing the states in the crossed channel, and define matrices

\[
\mathcal{M}_{I(r,s)I(a,b)} = \mathcal{M}_{s,b}^{r,a} ,
\]

(2.22)

\[
\mathcal{C}_{I(r,s)I(a,b)} = \mathcal{C}_{s,b}^{r,a} ,
\]

(2.23)

\[
F_{IJ} = \delta_{IJ} F_I , \quad F_{I(u,w)} = F^u_w .
\]

(2.24)

With this, we get

\[
\mathcal{M} = \frac{1}{s} \exp(FC \ln s) = \frac{1}{s} \exp(FC \ln s) \mathcal{C} = \frac{1}{s} \sqrt{F}^{-1} \left( \tilde{\mathcal{C}} \exp(\tilde{\mathcal{C}} \ln s) \right) \sqrt{F}^{-1} ,
\]

(2.25)

\[
\tilde{\mathcal{C}} = \sqrt{F} \mathcal{C} \sqrt{F} .
\]

**Example**

Consider the interaction

\[
\mathcal{H}_1 = -\frac{g_1}{2!} : \varphi^2 \sigma : - \frac{g_2}{3!} : \sigma^3 : ,
\]

(2.26)

which is a special case of (1.1) for \( N = 2 \), \( \varphi_1 = \varphi , \varphi_2 = \sigma \), and \( c_{112} = c_{121} = c_{211} = g_1 \), \( c_{222} = g_2 \), \( c_{111} = c_{122} = c_{212} = c_{221} = 0 \). Let \( I(1,1) = 1 \), \( I(2,2) = 2 \), \( I(1,2) = 3 \), and \( I(2,1) = 4 \), then the matrix \( \mathcal{C} \) is block diagonal,

\[
\mathcal{C} = \left( \begin{array}{cc} C' & 0 \\ 0 & \mathcal{C}'' \end{array} \right) , \quad C' = \left( \begin{array}{cc} g_{11}^2 & g_{12}^2 \\ g_{21}^2 & g_{22}^2 \end{array} \right) , \quad \mathcal{C}'' = \left( \begin{array}{cc} g_{11} g_2 & g_{11}^2 \\ g_{12} g_2 & g_{22} \end{array} \right) ,
\]

and \( \mathcal{F} \) is diagonal with \( F_1 = F_3^1 \), \( F_2 = F_2^2 \), \( F_3 = F_4 = F_1^2 = F_2^1 \), where \( F^u_w \) is given by (2.18). The symmetric matrix \( \tilde{\mathcal{C}} \) is

\[
\tilde{\mathcal{C}} = \left( \begin{array}{cc} \tilde{C}' & 0 \\ 0 & \tilde{C}'' \end{array} \right) , \quad \tilde{C}' = \left( \begin{array}{cc} F_1 g_{11}^2 & \sqrt{F_1 F_2 g_{11}^2} \\ \sqrt{F_1 F_2 g_{11}^2} & F_2 g_{22}^2 \end{array} \right) , \quad \tilde{C}'' = F_3 \mathcal{C}'' .
\]

Therefore, also \( \mathcal{M} \) is block diagonal. Let us only consider the upper left part, which is given by

\[
\mathcal{M}' = \frac{1}{s} \sqrt{F}^{-1} C' \exp(\tilde{\mathcal{C}} \ln s) \sqrt{F}^{-1} .
\]
By diagonalizing \( \hat{C}' \), it is easy to see that, e. g.,

\[
\mathcal{M}_{\varphi,\varphi \to \varphi,\varphi} = M'_{11}
= \frac{g_2^2}{\lambda^{[1]} - \lambda^{[2]}} \left( (\lambda^{[1]} + F_2(y_1^2 - y_2^2)) s^\lambda^{[1]} - 1 - (\lambda^{[2]} + F_2(y_1^2 - y_2^2)) s^\lambda^{[2]} - 1 \right),
\]

(2.27)

with the eigenvalues

\[
\lambda^{[1,2]} = \frac{1}{2} \left( F_1 g_1^2 + F_2 g_2^2 \pm \sqrt{(F_1 g_1^2 - F_2 g_2^2)^2 + 4 F_1 F_2 g_1^2} \right)
\]

(2.28)

of \( \hat{C}' \). Thus the high-energy behaviour of \( \mathcal{M}_{\varphi,\varphi \to \varphi,\varphi} \) is governed by two Regge terms.

We will come back to this example in sec. 4.

3 Multi-Channel Bethe-Salpeter Equation

Equation (2.25) shows that the amplitude \( T \) possesses an exponential \( \ln s \) dependence, which is a typical feature of Regge behaviour. In order to see that this result really reflects the existence of Regge trajectories in the crossed channel, we will now formulate the multi-channel Bethe-Salpeter equation and study its solution. Although the one-channel Bethe-Salpeter equation was considered by Lee and Sawyer and many others long ago, to the best of our knowledge there is no treatment of the multi-channel case.

Let us start with the off-shell amplitude

\[
T((q_1, i_1), (q_3, i_3) \to (q_2, i_2), (q_4, i_4)) = (2\pi)^4 \delta(q_1 + q_3 - q_2 - q_4) t_{i_3, i_4}^{i_1, i_2}(P_{\text{tot}}, q_{13}, q_{24})
\]

(3.1)

with \( P_{\text{tot}} = q_1 + q_3 = q_2 + q_4 \) and \( q_{ij} = (q_i - q_j)/2 \).

The Bethe-Salpeter equation for \( t_{i_3, i_4}^{i_1, i_2}(P_{\text{tot}}, q_{13}, q_{24}) \) with the interaction (1.1) in the ladder approximation reads

\[
t_{i_3, i_4}^{i_1, i_2}(P_{\text{tot}}, q_{13}, q_{24}) = \sum_i \frac{c_{i_1 i_2} c_{i_3 i_4}}{(q_{13} - q_{24})^2 - m_i^2 + i\varepsilon} + \frac{1}{(2\pi)^4} \sum_{i, j, k} \int d^4 q \frac{c_{i_1 j} c_{i_3 k}}{(q_{13} - q)^2 - m_i^2 + i\varepsilon} \times \frac{1}{(P_{\text{tot}}/2 + q)^2 - m_j^2 + i\varepsilon} \frac{1}{(P_{\text{tot}}/2 - q)^2 - m_k^2 + i\varepsilon} t_{k, i_4}^{j, i_2}(P_{\text{tot}}, q, q_{24}).
\]

(3.2)

In the c. m. frame \( (P_{\text{cm}}^{\text{tot}} = (W, 0)) \) the partial-wave decomposition of \( t_{i_3, i_4}^{i_1, i_2} \) is given by

\[
t_{i_3, i_4}^{i_1, i_2}(P_{\text{cm}}^{\text{tot}}, q_{13}, q_{24}) = \frac{1}{4\pi |q_{13}| |q_{24}|} \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(|\cos \vartheta|) t_{\ell}^{i_1, i_2 i_3, i_4}(W, q_{13}^0, |q_{13}|, q_{24}^0, |q_{24}|)
\]

(3.3)

(\( \vartheta \) is the angle between \( q_{13} \) and \( q_{24} \), so that

\[
t_{\ell}^{i_1, i_2 i_3, i_4}(W, q_{13}^0, |q_{13}|, q_{24}^0, |q_{24}|) = 2\pi |q_{13}| |q_{24}| \int d\cos \vartheta P_{\ell}(|\cos \vartheta|) t_{i_3, i_4}^{i_1, i_2}(P_{\text{cm}}^{\text{tot}}, q_{13}, q_{24})
\]

(3.4)
and we end up with a separable form of the Bethe-Salpeter equation,

\[ (2.23) \quad (2.20) \]

\[ Q = \frac{c_{i12}c_{i34}c_{i14}Q_{\ell}(\beta(\rho_{13}, |\rho_{13}|, \rho_{24}, |\rho_{24}|))}{(P_{tot}^m/2 + q^2 - m_j^2 + i\varepsilon)[(P_{tot}^m/2 - q)^2 - m_k^2 + i\varepsilon]} , \]

where \( Q_{\ell} \) is a Legendre function of the second kind, and its argument is given by

\[ \beta(\rho_{13}, |\rho_{13}|, \rho_{24}, |\rho_{24}|) = \frac{-(q_{13}^2 - q_{24}^2) + |q_{13}|^2 + |q_{24}|^2 + m_j^2 - i\varepsilon}{2|\rho_{13}||\rho_{24}|} . \]

As we are only interested in the Regge trajectories in the leading logarithmic approximation and near \( \ell = -1 \), we adopt the method used in [4] for the one-channel case, which amounts to replacing \( Q_{\ell} \) by its leading \( \ell \)-plane singularity,

\[ Q_{\ell}(\beta) \to \frac{1}{\ell + 1} . \]

Inserting this into (3.5) and using the matrix notation for products of \( c_{ijk} \) introduced in (2.23) and (2.20),

\[ C_{I(i,j)I(k,l)} = \sum_{m=1}^{N} c_{imk}c_{jml} , \]

and

\[ t_{I(i,j)I(k,l)} = t_{i,j;k,l} , \]

we end up with a separable form of the Bethe-Salpeter equation,

\[ (\ell + 1)t_{IJJ}(W, \rho_{13}, |\rho_{13}|, \rho_{24}, |\rho_{24}|) \]

\[ = -2\pi C_{I,IJ} - \frac{i}{(2\pi)^3} \sum_{K=1}^{N^2} \int dq^0 d|q| C_{IK} f_{KJ}(W, q^0, |q|) t_{IJK}(W, q^0, |q|, \rho_{24}, |\rho_{24}|) , \]

where

\[ f_{I(j,k)}(W, q^0, |q|) = \frac{1}{(P_{tot}^m/2 + q^2 - m_j^2 + i\varepsilon)(P_{tot}^m/2 - q)^2 - m_k^2 + i\varepsilon} . \]

In matrix form, the solution of this equation is

\[ t_{I,J} = -2\pi C_{I,J} \frac{1}{\ell + 1 - \hat{F}} = -2\pi \sqrt{\hat{F}}^{-1} \frac{1}{\ell + 1 - \hat{C}} \sqrt{\hat{F}}^{-1} , \]

\[ \hat{C} = \sqrt{\hat{F}} \sqrt{\hat{F}} . \]

Here the matrix \( \hat{F} \) is given by

\[ \hat{F}_{I,J}(W) = \frac{-i\delta_{IJ}}{(2\pi)^3} \int dq^0 d|q| f_{I,J}(W, q^0, |q|) . \]
Since \( \hat{C} \) is symmetric, we can diagonalize it: Let \( v[i] \) denote the normalized eigenvectors,
\[
\hat{C}v[i] = \lambda[i]v[i] , \quad i = 1, \ldots, N^2 ,
\]
then
\[
t_{\ell} = -2\pi \sqrt{\hat{F}}^{-1} \left( \sum_{i} \frac{v[i] \lambda[i] v[i]^{\dagger}}{\ell + 1 - \lambda[i]} \right) \sqrt{\hat{F}}^{-1} .
\]
(3.10)

Thus, the Regge trajectories are given by
\[
\alpha[i] = \ell_{\text{pole}} = \lambda[i] - 1 .
\]
(3.12)

With (3.11) we are ready to use the Mandelstam-Sommerfeld-Watson representation \[4, 6, 7\] to obtain the high-energy amplitude in the crossed channel \((t = (q_1 + q_3)^2 = W^2 \text{ fixed}, s = (q_1 - q_2)^2 \to \infty)\), with the result
\[
t^\text{poles} \overset{s \to \infty, t \text{ fixed}}{=} \frac{1}{s} \sqrt{\hat{F}}^{-1} \left( \sum_{i} v[i] \lambda[i] s^{\lambda[i]} v[i]^{\dagger} \right) \sqrt{\hat{F}}^{-1} = \frac{1}{s} \sqrt{\hat{F}}^{-1} \left( \hat{C} \exp(\hat{C} \ln s) \right) \sqrt{\hat{F}}^{-1} .
\]
(3.13)

It can be shown by elementary integration that \( \hat{F} \) in (3.13) and \( F \) in (2.25) coincide. This shows that the approaches of sec. \ref{sec:2} and the present section are equivalent.

### 4 Base States

The states \( v[i] \) are eigenstates of the hermitian matrix \( \hat{C} = \hat{C} \), thus they are orthogonal. This is, however, no longer true for the states \( \sqrt{\hat{F}}^{-1} v[i] \). As the dual base for these states we introduce the states
\[
u[i] = \sqrt{\hat{F}} v[i] \quad \text{with} \quad \nu[i]^{\dagger} \hat{F}^{-1} \nu[j] = \delta_{ij} .
\]
(4.1)

The only matrix elements of \( M = t_{\text{poles}} \) in this base that are non-zero are
\[
u[i]^{\dagger} M \nu[i] , \quad i = 1, \ldots, N^2 .
\]

The high-energy behaviour of each of these matrix elements is governed by only one trajectory,
\[
u[i]^{\dagger} M \nu[j] = \delta_{ij} \lambda[i] s^{\lambda[i]-1} ,
\]
(4.2)
although the states \( \nu[i] \) are in general not orthogonal. Equation (4.2) leads to the following sum rules for the scattering amplitudes:
\[
\sum_{i,j=1}^{N^2} \nu[i]^{\dagger} \nu[j] \sqrt{\hat{F}_I \hat{F}_J} M_{I,J} = \delta_{ij} \lambda[i] s^{\lambda[i]-1} .
\]
(4.3)

Note that some of these relations simplify whenever \( \lambda[i] = 0 \). In this case,
\[
M \nu[i] = 0 \quad \text{for} \quad \lambda[i] = 0
\]
(4.4)
(or equivalently \( C \nu[i] = 0 \)), i.e. states corresponding to fixed Regge singularities are transparent in our approximation.
As an illustration, we again consider the example of sec. 2, eq. (2.26):

The four eigenvalues are given by (2.28) and

$$\lambda^{[3,4]} = F_2 g_1 (g_2 \pm g_1) ,$$

the corresponding eigenvectors are

$$v^{[1]} = \frac{1}{\sqrt{d}} \begin{pmatrix} \sqrt{a} \\ 0 \end{pmatrix}, \quad v^{[2]} = \frac{1}{\sqrt{d}} \begin{pmatrix} \sqrt{b} \\ 0 \end{pmatrix}, \quad v^{[3]} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v^{[4]} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

with $d = \sqrt{(F_1 g_1^2 - F_2 g_2^2)^2 + 4F_1 F_2 g_1^4}$, $a = (d + F_1 g_1^2 - F_2 g_2^2)/2$, and $b = (d - F_1 g_1^2 + F_2 g_2^2)/2$. Out of the ten relations described by (4.3), four are trivial due to the fact that the corresponding eigenvectors are diagonal. The remaining relations read (using the symmetry of $M$)

$$a F_1 M_{\varphi,\varphi \rightarrow \varphi,\varphi} + 2F_1 F_2 g_1^2 M_{\varphi,\varphi \rightarrow \varphi,\varphi} + b F_2 M_{\sigma,\sigma \rightarrow \sigma,\sigma} = (\lambda^{[1]} - \lambda^{[2]}) M_{\lambda^{[1]} s} M_{\lambda^{[1]} - 1} ,$$

$$b F_1 M_{\varphi,\varphi \rightarrow \varphi,\varphi} - 2F_1 F_2 g_1^2 M_{\varphi,\varphi \rightarrow \varphi,\varphi} + a F_2 M_{\sigma,\sigma \rightarrow \sigma,\sigma} = (\lambda^{[1]} - \lambda^{[2]}) M_{\lambda^{[2]} s} M_{\lambda^{[2]} - 1} ,$$

$$M_{\varphi,\varphi \rightarrow \sigma,\sigma} + 2M_{\varphi,\varphi \rightarrow \varphi,\varphi} + M_{\sigma,\sigma \rightarrow \varphi,\varphi} = 2g_1 (g_1 + g_2) s M_{\lambda^{[3]} - 1} ,$$

$$M_{\varphi,\varphi \rightarrow \sigma,\sigma} - 2M_{\varphi,\varphi \rightarrow \varphi,\varphi} + M_{\sigma,\sigma \rightarrow \varphi,\varphi} = 2g_1 (g_2 - g_1) s M_{\lambda^{[4]} - 1} ,$$

$$F_1 g_1^2 M_{\varphi,\varphi \rightarrow \varphi,\varphi} + (F_2 g_2^2 - F_1 g_1^2) M_{\varphi,\varphi \rightarrow \varphi,\varphi} - F_2 g_2^2 M_{\sigma,\sigma \rightarrow \sigma,\sigma} = 0 ,$$

$$M_{\varphi,\varphi \rightarrow \sigma,\sigma} - M_{\sigma,\sigma \rightarrow \varphi,\varphi} = 0 ,$$

where, e. g., $M_{\varphi,\varphi \rightarrow \varphi,\varphi}$ is short for $M((\vec{p}_1, \varphi), (\vec{p}_2, \sigma) \rightarrow (\vec{p}_3, \varphi), (\vec{p}_4, \sigma))$.

Of course, these relations are only valid in the Regge limit $s \rightarrow \infty$, $t$ fixed, and apply only to Regge behaviour, so that a vanishing combination of amplitudes means that this combination does not exhibit Regge behaviour.

In the example, eigenvalues $\lambda^{[i]}$ can vanish only for $g_2 = \pm g_1$. If, say, $g_2 = g_1$ (i. e. $\lambda^{[1]} = g_1^2 (F_1 + F_2)$, $\lambda^{[2]} = 0$, $\lambda^{[3]} = 2g_1^2 F_3$, $\lambda^{[4]} = 0$), the relations (4.6) – (4.11) simplify to

$$M_{\varphi,\varphi \rightarrow \varphi,\varphi} = M_{\sigma,\sigma \rightarrow \sigma,\sigma} = M_{\varphi,\varphi \rightarrow \varphi,\varphi} = g_1^2 s M_{\lambda^{[1]} - 1} ,$$

$$M_{\varphi,\varphi \rightarrow \sigma,\sigma} = M_{\sigma,\sigma \rightarrow \varphi,\varphi} = M_{\varphi,\varphi \rightarrow \varphi,\varphi} = g_1^2 s M_{\lambda^{[3]} - 1} .$$

5 Concluding Remarks

Our results obtained in the s-channel within the collinear asymptotic approach agree with the t-channel Bethe-Salpeter treatment. This puts on firm ground our interpretation of Regge behaviour as a consequence of collinear three-particle dynamics in the s-channel. It opens the way for the application of the collinear asymptotic Hamiltonian in a time-ordered exponential approximation to realistic theories. Especially, we have in mind effective QCD-inspired hadron interactions at large distances, and on the other hand jet production in the deep inelastic region.
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