THE ORBIFOLD TOPOLOGICAL VERTEX

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ABSTRACT. We define Donaldson-Thomas invariants of Calabi-Yau orbifolds and we develop a topological vertex formalism for computing them. The basic combinatorial object is the orbifold vertex $V^G_{\lambda\mu\nu}$, a generating function for the number of 3D partitions asymptotic to 2D partitions $\lambda, \mu, \nu$ and colored by representations of a finite Abelian group $G$ acting on $\mathbb{C}^3$. In the case where $G \cong \mathbb{Z}_n$ acting on $\mathbb{C}^3$ with transverse $A_{n-1}$ quotient singularities, we give an explicit formula for $V^G_{\lambda\mu\nu}$ in terms of Schur functions. We discuss applications of our formalism to the Donaldson-Thomas Crepant Resolution Conjecture and to the orbifold Donaldson-Thomas/Gromov-Witten correspondence. We also explicitly compute the Donaldson-Thomas partition function for some simple orbifold geometries: the local football $\mathbb{P}^1_{a,b}$ and the local $B\mathbb{Z}_2$ gerbe.

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1. Introduction

The topological vertex is a powerful tool for computing the Gromov-Witten (GW) or Donaldson-Thomas (DT) partition function of any toric Calabi-Yau threefold (toric CY3). The vertex was originally discovered in physics using the duality between Chern-Simons theory and topological string theory \cite{1}. A mathematical treatment of the topological vertex in GW theory was given in \cite{18,17,21}, and the topological vertex for DT theory was developed in \cite{20}, where it was used to prove the DT/GW correspondence in the toric CY3 case.

In this paper we develop a topological vertex formalism which computes the DT partition function of an orbifold toric CY3.

The central object in our theory is the orbifold vertex $V_{G}^{\lambda \mu \nu}$. It is a generating function for the number of 3D partitions, colored by representations of $G$, and asymptotic to a triple of 2D partitions $(\lambda, \mu, \nu)$. Here $G$ is an Abelian group acting on $\mathbb{C}^3$ with trivial determinant and the action dictates a fixed coloring scheme for the boxes in the 3D partition (see §3.1). The usual topological vertex is the case where $G$ is the trivial group.

Associated to an orbifold toric CY3 $X$ is a trivalent graph whose vertices are the torus fixed points and whose edges are the torus invariant curves. There is additional data at the vertices describing the stabilizer group of the fixed point and there is additional data at the edges giving the degrees of the line bundles normal to the fixed curve. The general orbifold vertex formalism determines the DT partition function $DT(X)$ by a formula of the
form
\[
DT(\mathcal{X}) = \sum_{\text{edge assignments}} \prod_{e \in \text{Edges}} E(e) \prod_{v \in \text{Vertices}} \hat{V}_{G}^{\lambda \mu \nu}(v)
\]

where the sum is over all ways of assigning 2D partitions to the edges. The edge terms $E(e)$ are relatively simple and depend on the normal bundle of the corresponding curve as well as the partition assigned to the edge. The vertex terms $\hat{V}_{G}^{\lambda \mu \nu}(v)$ are given by the universal series $V_{G}^{\lambda \mu \nu}$ modified by certain signs with $G, \lambda, \mu, \nu$ obtained as the local group of the vertex $v$ and the partitions along the incident edges.

To make the above formula computationally effective, one needs a closed formula for the universal series $V_{G}^{\lambda \mu \nu}$. One of our main results is Theorem 12 which gives an explicit formula, in terms of Schur functions, for $V_{G}^{\lambda \mu \nu}$ in the case where $G$ is $\mathbb{Z}_n$ acting on $\mathbb{C}^3$ with weights $(1, -1, 0)$. This corresponds to the case where the orbifold structure of $\mathcal{X}$ occurs along smooth, disjoint curves which then necessarily have transverse $A_{n-1}$ singularities ($n$ can vary from curve to curve). We call this the transverse $A_{n-1}$ case and we make the above formula fully explicit in that instance (Theorem 10).

Besides providing a tool to compute DT partition functions of orbifolds, our orbifold vertex formalism gives insight into two central questions in the DT theory of orbifolds.

- How is the DT theory of an orbifold $\mathcal{X}$ related to the GW theory of $\mathcal{X}$?
- How is the DT theory of $\mathcal{X}$ related to the DT theory of $Y$, a Calabi-Yau resolution of $X$, the singular space underlying $\mathcal{X}$?

The four relevant theories can be arranged schematically in the diagram below:

\[
\begin{array}{ccc}
DT(X) & \xrightarrow{\text{DT/GW correspondence}} & GW(Y) \\
\downarrow \text{DT crepant resolution conjecture} & & \downarrow \text{GW crepant resolution conjecture} \\
DT(Y) & \xrightarrow{\text{ori-DT/GW correspondence}} & GW(X)
\end{array}
\]

In the transverse $A_{n-1}$ case, or more generally when $\mathcal{X}$ satisfies the Hard Lefschetz condition [9, Defn 1.1] c.f. [8, Lem 24], the (conjectural) equivalences of the four theories take on a particularly nice form. Namely, the (suitably renormalized) partition functions of the four theories are equal after a change of variables and analytic continuation. For the top equivalence,
this is the famous DT/GW correspondence of Maulik, Nekrasov, Okounkov, and Pandharipande [20], for the right equivalence, this is the Bryan-Graber version of the crepant resolution conjecture in GW theory [9].

In §4, we formulate the DT crepant resolution conjecture for $X$ satisfying the hard Lefschetz condition. In a forthcoming paper [16], we will use our orbifold vertex to prove the conjecture for the case where $X$ is toric with transverse $A_{n-1}$ orbifold structure. We will also formulate an orbifold version of the DT/GW correspondence. This correspondence can be proved for a large class of toric orbifolds with transverse $A_{n-1}$ structure by using the other three equivalences in the diagram: our proof of the DT correspondence, the (non-orbifold) DT/GW correspondence of [20], and a proof of the GW crepant resolution conjecture for a large class of toric orbifolds with transverse $A_{n-1}$ structure which has been obtained by Coates and Iritani[12].

Our paper is organized as follows. In §2 we define DT theory for orbifolds. In §3 we introduce the vertex formalism and give our main two results: Theorem 10, an explicit formula for the partition function of an orbifold toric CY3 with transverse $A_{n-1}$ orbifold structure and Theorem 12, a formula for the $\mathbb{Z}_n$ vertex in terms of Schur functions. In §4 we formulate the DT crepant resolution conjecture. We then use our vertex formalism to compute the partition function of the local football (Proposition 3) and the local $B\mathbb{Z}_2$-gerbe (§4.3). Each of these examples is used to illustrate the DT crepant resolution conjecture and the orbifold DT/GW correspondence. The derivation of the vertex formalism and the proof of Theorem 10 begins in §5. A key component of the proof is a K-theory decomposition of the structure sheaf of a torus invariant substack into edge and vertex terms (Propositions 4 and 5 and Lemma 15). The proof of Theorem 10 is finished in §6 where the signs in the vertex formula are derived. Finally, a proof of Theorem 12 is given in §7 using vertex operators. Necessary background on orbifold toric CY3s and orbifold Riemann-Roch is collected in two brief appendices.

2. ORBIFOLD CY3s AND DT THEORY

2.1. Orbifold CY3s. An orbifold CY3 is defined to be a smooth, quasi-projective, Deligne-Mumford stack $\mathcal{X}$ over $\mathbb{C}$ of dimension three having generically trivial stabilizers and trivial canonical bundle,

$$K_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}.$$

The definition implies that the local model for $\mathcal{X}$ at a point $p$ is $[\mathbb{C}^3/G_p]$ where $G_p \subset SL(3, \mathbb{C})$ is the (finite) group of automorphisms of $p$. 

2.2. **The $K$-theory of $\mathcal{X}$**. Our DT invariants will be indexed by compactly supported elements of $K$-theory, up to numerical equivalence. Let $K_c(\mathcal{X})$ be the Grothendieck group of compactly supported coherent sheaves on $\mathcal{X}$. We say that $F_1, F_2 \in K_c(\mathcal{X})$ are *numerically equivalent*,

$$F_1 \sim_{\text{num}} F_2,$$

if

$$\chi(E \otimes F_1) = \chi(E \otimes F_2)$$

for all sheaves $E$ on $\mathcal{X}$.

In this paper, $K$-theory will always mean compactly supported $K$-theory modulo numerical equivalence:

$$K(\mathcal{X}) = K_c(\mathcal{X})/\sim_{\text{num}}.$$

There is a natural filtration

$$F_0 K(\mathcal{X}) \subset F_1 K(\mathcal{X}) \subset F_2 K(\mathcal{X}) \subset K(\mathcal{X})$$

given by the dimension of the support. An element of $F_d K(\mathcal{X})$ can be represented by a formal sum of sheaves having support of dimension $d$ or less.

2.3. **The Hilbert scheme of substacks**. Given $\alpha \in K(\mathcal{X})$, we define

$$\text{Hilb}^\alpha(\mathcal{X})$$

to be the category of families of substacks $Z \subset \mathcal{X}$ having $[O_Z] = \alpha$. By a theorem of Olsson-Starr [26], $\text{Hilb}^\alpha(\mathcal{X})$ is represented by a scheme which we also denote by $\text{Hilb}^\alpha(\mathcal{X})$. Note that our indexing is slightly different than Olsson-Starr who index instead by the corresponding Hilbert function

$$E \mapsto \chi(E \otimes \alpha).$$

Note that the Hilbert scheme $\text{Hilb}^\alpha(\mathcal{X})$ is a scheme rather than just a stack, as its objects (substacks $Z \subset \mathcal{X}$) do not have automorphisms.

2.4. **Definition of DT invariants**. In [2], Kai Behrend defined an integer-valued constructible function

$$\nu_S : S \to \mathbb{Z}$$

associated to any scheme $S$ over $\mathbb{C}$.

**Definition 1.** The DT invariant of $\mathcal{X}$ in the class $\alpha \in K(\mathcal{X})$ is given by the topological Euler characteristic of $\text{Hilb}^\alpha(\mathcal{X})$, weighted by Behrend’s function $\nu : \text{Hilb}^\alpha(\mathcal{X}) \to \mathbb{Z}$. That is

$$DT_\alpha(\mathcal{X}) = e(\text{Hilb}^\alpha(\mathcal{X}), \nu)$$

$$= \sum_{k \in \mathbb{Z}} k e(\nu^{-1}(k))$$
where $e(-)$ is the topological Euler characteristic.

**Remark 2.** In the case where $X$ is compact and a scheme, and $\alpha \in F_1K(\mathcal{X})$, our definition coincides (via Behrend [2, Theorem 4.18]) with the definition given in [20] which uses a perfect obstruction theory. It should be possible to construct a perfect obstruction theory on $\text{Hilb}^\alpha(\mathcal{X})$ along the lines of [20, 28], but we don’t pursue that in this paper. One advantage of defining the invariants directly in terms of the weighted Euler characteristic is that $DT_\alpha(X)$ is well defined for non-compact geometries.

**Remark 3.** If $\alpha = [O_Z] \in F_1K(X)$ and $X = X$ is a scheme, we can recover the more familiar discrete invariants $n = \chi(O_Z)$ and $\beta = [Z] \in H_2(X)$ via the Chern character:

$$\text{ch}(O_Z) = [Z]^\vee + \chi(O_Z)[pt]^\vee.$$ 

2.5. DT partition functions. We define the DT partition function by

$$DT(\mathcal{X}) = \sum_{\alpha \in F_1K(\mathcal{X})} DT_\alpha(\mathcal{X})q^\alpha.$$ 

With an appropriate choice of a basis $e_1, \ldots, e_r$ for $F_1K(\mathcal{X})$, we can regard $DT(\mathcal{X})$ as a formal Laurent series in a set of variables $q_1, \ldots, q_r$ where

$$q^\alpha = q_1^{d_1} \cdots q_r^{d_r} \quad \text{for } \alpha = \sum_{i=1}^{r} d_i e_i.$$ 

We define the degree zero DT partition function by

$$DT_0(\mathcal{X}) = \sum_{\alpha \in F_0K(\mathcal{X})} DT_\alpha(\mathcal{X})q^\alpha,$$ 

and we define the reduced DT partition function by

$$DT'(\mathcal{X}) = \frac{DT(\mathcal{X})}{DT_0(\mathcal{X})}.$$ 

In the case where $\mathcal{X} = X$ is a scheme, Maulik, Nekrasov, Okounkov, and Pandharipande conjectured that the reduced DT partition function is equal to the reduced GW partition function after a change of variables [20, Conjecture 2].

3. THE ORBIFOLD VERTEX FORMALISM

In the case where $\mathcal{X} = X$ is a scheme and toric, the topological vertex formalism computes the DT partition function $DT(X)$ in terms of the topological vertex $V_{\lambda_\mu_\nu}$, a universal object which is a generating function for 3D partitions asymptotic to $(\lambda, \mu, \nu)$. We extend the vertex formalism to toric orbifolds, particularly in the case where $\mathcal{X}$ has transverse $A_{n-1}$ orbifold structure.
3.1. 3D partitions, 2D partitions, and the vertex.

**Definition 4.** Let \((\lambda, \mu, \nu)\) be a triple of ordinary partitions. A *3D partition* \(\pi\) *asymptotic to* \((\lambda, \mu, \nu)\) is a subset

\[ \pi \subset (\mathbb{Z}_{\geq 0})^3 \]

satisfying

1. if any of \((i + 1, j, k), (i, j + 1, k), \) and \((i, j, k + 1)\) is in \(\pi\), then \((i, j, k)\) is also in \(\pi\), and
2. (a) \((j, k)\) \(\in\) \(\lambda\) if and only if \((i, j, k)\) \(\in\) \(\pi\) for all \(i \gg 0\),
   (b) \((k, i)\) \(\in\) \(\mu\) if and only if \((i, j, k)\) \(\in\) \(\pi\) for all \(j \gg 0\),
   (c) \((i, j)\) \(\in\) \(\nu\) if and only if \((i, j, k)\) \(\in\) \(\pi\) for all \(k \gg 0\).

where we regard ordinary partitions as finite subsets of \((\mathbb{Z}_{\geq 0})^2\) via their diagram.

Intuitively, \(\pi\) is a pile of boxes in the positive octant of 3-space. Condition (1) means that the boxes are stacked stably with gravity pulling them in the \((-1, -1, -1)\) direction; condition (2) means that the pile of boxes is infinite along the coordinate axes with cross-sections asymptotically given by \(\lambda, \mu, \) and \(\nu\).

The subset \(\{(i, j, k) : (j, k) \in \lambda\} \subset \pi\) will be called the *leg* of \(\pi\) in the \(i\) direction, and the legs in the \(j\) and \(k\) directions are defined analogously. Let

\[ \xi_{\pi}(i, j, k) = 1 - \# \text{ of legs of } \pi \text{ containing } (i, j, k). \]

We define the renormalized volume of \(\pi\) by

\[ |\pi| = \sum_{(i, j, k) \in \pi} \xi_{\pi}(i, j, k). \]

Note that \(|\pi|\) can be negative.

**Definition 5.** The topological vertex \(V_{\lambda\mu\nu}\) is defined to be

\[ V_{\lambda\mu\nu} = \sum_{\pi} q^{|\pi|} \]

where the sum is taken over all 3D partitions \(\pi\) asymptotic to \((\lambda, \mu, \nu)\).

We regard \(V_{\lambda\mu\nu}\) as a formal Laurent series in \(q\). Note that \(V_{\lambda\mu\nu}\) is clearly cyclically symmetric in the indices, and reflection about the \(i = j\) plane yields

\[ V_{\lambda\mu\nu} = V_{\mu'\lambda'\nu'} \]

where \(^t\) denotes conjugate partition:

\[ \lambda' = \{(i, j) : (j, i) \in \lambda\}. \]
This definition of topological vertex differs from the vertex \( C(\lambda, \mu, \nu) \) of the physics literature by a normalization factor. Our \( V_{\lambda\mu\nu} \) is equal to \( P(\lambda, \mu, \nu) \) defined by Okounkov, Reshetikhin, and Vafa [25, eqn 3.16]. They derive an explicit formula for \( V_{\lambda\mu\nu} = P(\lambda, \mu, \nu) \) in terms of Schur functions [25, eqns 3.20 and 3.21].

The \( \mathbb{Z}_n \) orbifold vertex counts 3D partitions colored with \( n \) colors. We color the boxes of a 3D partition \( \pi \) according to the rule that a box \((i, j, k) \in \pi \) has color \( i - j \mod n \) (c.f. [4]).

**Definition 6.** The \( \mathbb{Z}_n \) vertex \( V_{\lambda\mu\nu}^n \) is defined by

\[
V_{\lambda\mu\nu}^n = \sum_{\pi} q_{|\pi|_0} \cdots q_{|\pi|_{n-1}}
\]

where the sum is taken over all 3D partitions \( \pi \) asymptotic to \((\lambda, \mu, \nu)\) and \(|\pi|_a\) is the (normalized) number of boxes of color \( a \) in \( \pi \). Namely

\[
|\pi|_a = \sum_{i,j,k \in \pi, i-j=0 \mod n} \xi_{\pi}(i, j, k)
\]

where \( \xi_{\pi} \) is defined in equation (2).

Note that the \( \mathbb{Z}_n \)-orbifold vertex \( V_{\lambda\mu\nu}^n \) has fewer symmetries than the usual vertex since the \( k \) axis is distinguished. However, reflection through the \( i = j \) plane yields

\[
V_{\lambda\mu\nu}^n(q_0, q_1, \ldots, q_{n-1}) = V_{\mu'\lambda'\nu'}^n(q_0, q_{n-1}, \ldots, q_1).
\]

In general, if \( F \) is a series in the variables \( q_k \) with \( k \in \mathbb{Z}_n \), we let \( \overline{F} \) denote the same series with the variable \( q_k \) replaced by \( q_{-k} \). So for example, the above symmetry can be written

\[
V_{\lambda\mu\nu}^n = \overline{V}_{\mu'\lambda'\nu'}^n.
\]

The \( G \) vertex is defined in general as follows. Given a finite Abelian group \( G \) acting on \( \mathbb{C}^3 \) via characters \( r_1, r_2, r_3 \) we define

\[
V_{\lambda\mu\nu}^G = \sum_{\pi} \prod_{r \in \hat{G}} q_{|\pi|_r}
\]

where the sum is over 3D partitions asymptotic to \((\lambda, \mu, \nu)\) and where \(|\pi|_r\) is the (normalized) number of boxes in \( \pi \) of color \( r \in \hat{G} \):

\[
|\pi|_r = \sum_{i,j,k \in \pi, r_1^i r_2^j r_3^k = r} \xi_{\pi}(i, j, k).
\]

One of our main results is an explicit formula for the \( \mathbb{Z}_n \)-orbifold vertex (see Theorem [12]).
3.2. Orbifolds with transverse $A_{n-1}$ singularities. Let $\mathcal{X}$ be a orbifold toric CY3 whose orbifold structure is supported on a disjoint union of smooth curves. Then the local group along each curve is $\mathbb{Z}_n$ (where $n$ can vary from curve to curve) and the coarse space $X$ has transverse $A_{n-1}$ singularities along the curves. By Lemma 40, $X$ is determined by its coarse space $X$.

The combinatorial data determining a toric variety $X$ is well understood and is most commonly expressed as the data of a fan (by the Lemma 40, we do not require the stacky fans of Borisov, Chen and Smith [5]). In the case of a orbifold toric CY3, it is convenient to use equivalent (essentially dual) combinatorial data, namely that of a $(p, q)$-web diagram. Web diagrams are discussed in more detail in §B.

Associated to $\mathcal{X}$ is a planar trivalent graph $\Gamma = \{\text{Edges}, \text{Vertices}\}$ where the vertices correspond to torus fixed points, the edges correspond to torus fixed curves, and the regions in the plane delineated by the graph correspond to torus fixed divisors. $\Gamma$ will necessarily have some non-compact edges; these correspond to non-compact torus fixed curves. We denote the set of compact edges by $\text{Edges}^{\text{cpt}}$.

It will be notationally convenient to choose an orientation on $\Gamma$:

**Definition 7.** Let $\Gamma$ be a trivalent planar graph. An orientation is a choice of direction for each edge and an ordering $(e_1(v), e_2(v), e_3(v))$ of the edges incident to each vertex $v$ which is compatible with the counterclockwise cyclic ordering.

Given an orientation on the graph $\Gamma$ associated to $\mathcal{X}$, let the two regions in the plane incident to an edge $e$ be denoted by $D(e)$ and $D'(e)$ with the convention that $D(e)$ lies to the right of $e$ (see Figure 1). We also use $D(e)$ and $D'(e)$ to denote the corresponding torus invariant divisors and we let $C(e) \subset \mathcal{X}$ denote the torus invariant curve corresponding to $e$. Let $p_0(e)$ and $p_\infty(e)$ denote the the torus fixed points corresponding to the initial and final vertices incident to $e$. Let $D_0(e)$ and $D_\infty(e)$ denote the torus invariant divisors meeting $C(e)$ transversely at $p_0(e)$ and $p_\infty(e)$ respectively. Let $D_1(v), D_2(v), D_3(v)$ be the regions (and the corresponding torus invariant divisors) opposite the edges $e_1(v), e_2(v), e_3(v)$.

Let

$$m = m(e) = \deg \mathcal{O}_{C(e)}(D(e))$$
$$m' = m'(e) = \deg \mathcal{O}_{C(e)}(D'(e)).$$

Define $n(e)$ such that $\mathbb{Z}_{n(e)}$ is the local group of $C(e) \subset \mathcal{X}$. If $n(e) \neq 1$, then $C(e)$ is a $B\mathbb{Z}_{n(e)}$ gerbe over $\mathbb{P}^1$ and

$$m, m' \in \frac{1}{n(e)} \mathbb{Z}.$$
with \[ m + m' = -2. \]

If \( n(e) = 1 \), then one of
\[ a = n(f), \quad a' = n(f'), \]
and/or one of
\[ b = n(g), \quad b' = n(g'), \]
is possibly greater than one and \( C(e) \) is a football: a \( \mathbb{P}^1 \) with root constructions of order \( \max(a, a') \) and \( \max(b, b') \) at 0 and \( \infty \).

We define
\[ \delta_0 = \begin{cases} 1 & \text{if } a > 1 \\ 0 & \text{if } a = 1. \end{cases} \]

We define \( \delta_0', \delta_\infty, \) and \( \delta'_\infty \) similarly according to the values of \( a', b, \) and \( b' \) respectively. Note that at least one of \( (\delta_0, \delta_0') \) is zero and likewise for \( (\delta_\infty, \delta'_\infty) \). Using the condition that \( O_C(D + D') = K_C = O_C(-p_0 - p_\infty) \), we can write
\[
O_C(D) = O_C(\tilde{m}p - \delta_0 p_0 - \delta_\infty p_\infty), \\
O_C(D') = O_C(\tilde{m}'p - \delta_0' p_0 - \delta'_\infty p_\infty),
\]
where
\[
m = \tilde{m} - \frac{\delta_0}{a} - \frac{\delta_\infty}{b}, \\
m' = \tilde{m}' - \frac{\delta_0'}{a'} - \frac{\delta'_\infty}{b'}
\]
since \( p_0, p_\infty \in C \) are orbifold points of order \( \max(a, a') \) and \( \max(b, b') \) respectively. Note that \( \tilde{m}, \tilde{m}' \in \mathbb{Z} \) and the Calabi-Yau condition implies
\[
\tilde{m} + \tilde{m}' = \delta_0 + \delta_0' + \delta_\infty + \delta'_\infty - 2.
\]
By convention, we define $\tilde{m} = m$ and $\tilde{m}' = m'$ if $n(e) > 1$ (but note that in this case, $\tilde{m}$ and $\tilde{m}'$ may not be integers).

3.3. **Generators for** $F_1 K(\mathcal{X})$. To write an explicit formula for $DT(\mathcal{X})$, we must choose generators for $F_1 K(\mathcal{X})$. Let $p \in \mathcal{X}$ be a generic point and let $p(e)$ be a generic point on the curve $C(e)$ (so $p(e) \cong B\mathbb{Z}_{n(e)}$). Let $\rho_k$, $k \in \{0, \ldots, n(e) - 1\}$ be the irreducible representations of $\mathbb{Z}_{n(e)}$ with the indexing chosen so that

$$\mathcal{O}_{p(e)}(-kD(e)) \cong \mathcal{O}_{p(e)} \otimes \rho_k.$$ 

We define the following classes in $F_1 K(\mathcal{X})$ and their associated variables.

| Class in $F_1 K(\mathcal{X})$ | Associated variable | Indexing set |
|-------------------------------|---------------------|--------------|
| $[\mathcal{O}_p]$            | $q$                 |              |
| $[\mathcal{O}_{p(e)} \otimes \rho_k]$ | $q_{e,k}$ | $e \in \text{Edges}, \ k \in \{0, \ldots, n(e) - 1\}$ |
| $[\mathcal{O}_{C(e)}(-1) \otimes \rho_k]$ | $v_{e,k}$ | $e \in \text{Edges}_{\text{cpt}}, \ k \in \{0, \ldots, n(e) - 1\}$ |

Pushforwards by the inclusions of $p$, $p(e)$, and $C(e)$ into $\mathcal{X}$ are implicit in the above. The class $[\mathcal{O}_{C(e)}(-1) \otimes \rho_k]$ is defined as follows. The curve $C(e)$ is a $B\mathbb{Z}_{n(e)}$ gerbe over $\mathbb{P}^1$. If $C(e) \cong \mathbb{P}^1 \times B\mathbb{Z}_{n(e)}$ is the trivial gerbe, then $\mathcal{O}_{C(e)}(-1)$ is pulled back from $\mathbb{P}^1$ and $\rho_k$ is pulled back from $B\mathbb{Z}_{n(e)}$.

More generally, let $\pi : \tilde{C}(e) \to C(e)$ be the degree $n$ cover obtained from the base change $\mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^n$. Then $\tilde{C}(e)$ is the trivial $B\mathbb{Z}_{n(e)}$ gerbe and we define $[\mathcal{O}_{C(e)}(-1) \otimes \rho_k]$ to be the class $\frac{1}{n} \pi_* [\mathcal{O}_{\tilde{C}(e)}(-1) \otimes \rho_k]$. In general, this class is not defined with $\mathbb{Z}$ coefficients.

The above classes generate $F_1 K(\mathcal{X})$ (over $\mathbb{Q}$) but there are relations. In particular, for each $e \in \text{Edges}$, there is the relation

$$(4) \quad [\mathcal{O}_p] = [\mathcal{O}_{p(e)} \otimes R_{\text{reg}}]$$

where $R_{\text{reg}} = \sum_k \rho_k$ denotes the regular representation of $\mathbb{Z}_{n(e)}$. This relation gives rise to the relation

$$q = \prod_{k=0}^{n(e)-1} q_{e,k}.$$ 

There may be additional relations among the classes supported on curves coming from the global geometry of $\mathcal{X}$. We leave relations among the corresponding variables implicit in all our formulas.
Remark 8. If $n(e) = 1$ for all edges $e$, then $\mathcal{X} = X$ is not an orbifold. In this case, the only variables are $q$ corresponding to $[\mathcal{O}_p]$ and $v_e$ corresponding to $\mathcal{O}_{C(e)}(-1)$. If $Z \subset X$ is a subscheme with $\chi(\mathcal{O}_Z) = n$ and

$$\beta = [Z] = \sum_i d_i [C(e_i)],$$

then

$$[\mathcal{O}_Z] = n[\mathcal{O}_p] + \sum_i d_i [\mathcal{O}_{C(e_i)}(-1)]$$

in K-Theory. Thus the associated DT invariant appears as the coefficient of $q^n v^\beta = q^n \prod_i v_i^{d_i}$ which is consistent with the notation of [20].

3.4. The vertex formula. Let

$$\lambda[k, n] = \{(i, j) \in \lambda : i - j = k \mod n\}$$

be the set of boxes in $\lambda$ of color $k \mod n$. Let

$$|\lambda|_k = |\lambda[k, n]|$$

denote the number of boxes of color $k$ in $\lambda$. Usually, $n$ is understood from the context, but if we need to make it explicit, we write $|\lambda|_{k, n}$.

Definition 9. An edge assignment on $\Gamma$ is a choice of a partition $\lambda(e)$ for each edge $e$ such that $\lambda(e) = \emptyset$ for every non-compact edge. An edge assignment is called multi-regular if each $\lambda = \lambda(e)$ satisfies $|\lambda|_k = \frac{1}{n} |\lambda|$ for all $k$.

Assume that $\Gamma$ has an orientation (Definition [7]). Given an edge assignment and a vertex $v$, we get a triple of partitions $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$ by setting $\lambda_i(v) = \lambda(e_i(v))$ if $e(v_i)$ has the orientation pointing outward from $v$ and $\lambda_i(v) = \lambda(e_i(v))'$ if $e_i(v)$ has the inward orientation. We also impose the convention that if any of the edges $e_i(v)$ have $n(e_i(v)) \neq 1$, then we fix the ordering so that this (necessarily unique) edge is given by $e_3(v)$. We will call such an edge the special edge and denote it also as simply $e(v)$.

The following quantities are used in the vertex formula. Let

$$C^\lambda_{\tilde{m}, \tilde{m}'} = \sum_{(i, j) \in \lambda} -\tilde{m}i - \tilde{m}' j + 1.$$

and let

$$C^\lambda_{\tilde{m}, \tilde{m}'}[k, n] = \sum_{(i, j) \in \lambda[k, n]} -\tilde{m}i - \tilde{m}' j + 1.$$

We define

$$A_\lambda(k, n) = \sum_{(i, j) \in \lambda} \left\lfloor \frac{i + k}{n} \right\rfloor.$$
Let $e = e(v)$ be the special edge associated to a vertex. We write

$$q_v = \begin{cases} (q_{e,0}, q_{e,1}, \ldots, q_{e,n(e)-1}) & \text{if } e \text{ is oriented outward from } v \text{ and} \\ (q_{e,0}, q_{e,n(e)-1}, \ldots, q_{e,1}) & \text{if } e \text{ is oriented inward toward } v. \end{cases}$$

We define

$$(5) \quad (-1)^{s(\lambda)} q_v$$

to be the same as $q_v$ but with the variable $q_{e,k}$ multiplied by the additional sign $(-1)^{s_k(\lambda)}$ where

$$s_k(\lambda) = |\lambda|_{k-1} + |\lambda|_{k+1}.$$ 

Note that this sign is trivial in the multi-regular case.

We also adopt a product convention for our variables. Namely, we set

$$v_e^{[\lambda]} := \prod_{k=0}^{n(e)-1} v_{e,k}^{[\lambda]},$$

$$q_e^{C^\lambda_{m,m'}} := \prod_{k=0}^{n(e)-1} q_{e,k}^{[k,n(e)]},$$

$$q_e^{A^{\lambda}} := \prod_{k=0}^{n(e)-1} q_{e,k}^{A^{\lambda}(k,n(e))}.$$ 

We will need an additional sign $(-1)^{\Sigma_{\lambda(e)}(e)}$ associated to each edge $e$. Let $\lambda = \lambda(e), n = n(e)$, and let

$$\Sigma_{\lambda(e)}(e) = \sum_{k=0}^{n-1} C_{m,m'}^{\lambda}[k,n] \left( |\lambda|_{k-1} - |\lambda|_{k+1} \right) + |\lambda|_k \left( 1 + (1 + \tilde{m} + \delta_0 + \delta_\infty)|\lambda|_{k-1} \right).$$

Note that in the multi-regular case this sign simplifies significantly:

$$(-1)^{\Sigma_{\lambda(e)}(e)} = (-1)^{(\tilde{m} + \delta_0 + \delta_\infty)|\lambda|}.$$ 

Finally, we need on more sign $(-1)^{\Sigma_{\pi(v)}}$ attached to each vertex partition. Here

$$\Sigma_{\pi(v)} = \sum_{k=0}^{n-1} |\lambda_3|_k \left( |\lambda_1|_k + |\lambda_2|_k + |\lambda_1|_{k-1} + |\lambda_2|_{k+1} \right)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the legs of $\pi(v)$ and the color of $(j, k) \in \lambda_1, (k, i) \in \lambda_2$, and $(i, j) \in \lambda_3$ is given by $i - j \mod n$. Note that in the multi-regular case, this sign is trivial. Indeed, then $|\lambda_3|_k$ is independent of $k$ and so the sum can be rearranged so that the other terms cancel mod 2 in pairs.

The following theorems provide an explicit formula for the DT partition function of a toric orbifold with transverse $A_{n-1}$ singularities.
**Theorem 10.** Let $\mathcal{X}$ be an orbifold toric CY3 with transverse $A_{n-1}$ singularities and let $\Gamma$ be the diagram of $\mathcal{X}$. Define $DT(\mathcal{X})$ to be

$$
\sum_{\text{edge assignments}} \prod_{e \in \text{Edges}} E_\lambda(e) \prod_{v \in \text{Vertices}} (-1)^{\Sigma_n(v)} V^{n_{e(v)}}_{\lambda_1(v) \lambda_2(v) \lambda_3(v)} \left( (-1)^{s(\lambda_3(v))} q_v \right)
$$

where

$$E_\lambda(e) = (-1)^{S_\lambda(e)} v_{e|\lambda}^{\lambda(e)} \prod_{\bar{m}'(e)} \left( q_1^{A_1} \delta_0 \left( q_1^{A_1'} \right) \delta_0' \left( q_2^{A_2} \right) \delta_\infty \left( q_3^{A_3} \right) \delta_\infty' \right)
$$

and where $(f, f', g, g')$ are the edges meeting $e$ arranged and oriented as in Figure 7. Then the DT partition function $DT(\mathcal{X})$ is obtained from $DT(\mathcal{X})$ by adding a minus sign to the variables $q_{e,0}$ (and hence also to $q$).

Note that for multi-regular edge assignments, the signs $(-1)^{\Sigma_{n(e)}}$ and $(-1)^{s(\lambda_3(v))}$ are both 1.

**Remark 11.** Switching the orientation of an edge $e$ has the effect of switching the variables $q_{e,k} \leftrightarrow q_{e,n(e)-k}$, for $k = 1, \ldots, n(e) - 1$. The edge term in the formula is written for the orientations in Figure 1, but is easily modified to an arbitrary orientation using this rule.

To make the above formula fully explicit, we give a closed formula for $E^{n}_{\lambda \mu \nu}(q_0, \ldots, q_{n-1})$. We first introduce a little more notation.

Consider the indices on the variables $q_0, \ldots, q_{n-1}$ to be in $\mathbb{Z}_n$ and define $q_t$ recursively by $q_0 = 1$ and

$$q_t = q_t \cdot q_{t-1}
$$

for positive and negative $t$, in other words

$$\{ \ldots, q_{-2}, q_{-1}, q_0, q_1, q_2, \ldots \} = \{ \ldots, q_0^{-1}, q_{-1}^{-1}, q_0^{-1}, 1, q_1, q_2, q_3, \ldots \}.
$$

Let

$$q = q_0 \cdots q_{n-1}
$$

and let

$$q_{\bullet} = \{ q_0, q_1, q_2, q_3, \ldots \} = \{ 1, q_1, q_1 q_2, q_1 q_2 q_3, \ldots \}.
$$

Given a partition $\nu = (\nu_0 \geq \nu_1 \geq \cdots)$, let

$$q_{\bullet - \nu} = \{ q_{-\nu_0}, q_{-\nu_1}, q_{-\nu_2}, q_{-\nu_3}, \ldots \}.
$$

**Theorem 12.** The $\mathbb{Z}_n$ vertex $V^{n}_{\lambda \mu \nu}(q_0, \ldots, q_{n-1})$ is given by the following formula:

$$V^{n}_{\lambda \mu \nu} = V^{n}_{000} \cdot q^{-A_\lambda} \cdot q^{-A_{\mu'}} \cdot H_\nu \cdot O_\nu \cdot \sum_{\eta} q_{-|\eta|}^{-|\eta|} \cdot s_\lambda(\eta) (q_{\bullet - \nu}) \cdot s_\mu(\eta) (q_{\bullet - \nu'}).$$
where \( s_{\alpha/\beta} \) is the skew Schur function associated to partitions \( \beta \subset \alpha \) (\( s_{\alpha/\beta} = 0 \) if \( \beta \not\subset \alpha \)), the overline denotes the exchange of variables \( q_k \leftrightarrow q_{-k} \), and

\[
H_{\nu'} = \prod_{(j,i) \in \nu'} \frac{1}{1 - \prod_{s=1}^{\infty} q_{h^s_{\nu'}(j,i)}}
\]

\( h^s_{\nu'}(j,i) \) is the number of boxes of color \( s \) in the \((j,i)\)-hook of \( \nu' \),

\[
O_{\nu} = \prod_{k=0}^{n-1} \mathcal{V}_{000}^n(q_k, q_{k+1}, \ldots, q_{n+k-1})^{-2|\nu|_{k}+|\nu|_{k+1}+|\nu|_{k-1}},
\]

\[
\mathcal{V}_{000}^n = M(1,q)^n \prod_{0<a\leq b<n} M(q_a \cdots q_0, q) M(q_a^{-1} \cdots q_b^{-1}, q),
\]

\[
M(v,q) = \prod_{m=1}^{\infty} \frac{1}{(1 - vq^m)^m}.
\]

Recall that by our product convention

\[
q^{-A_{\lambda}} = \prod_{k=0}^{n-1} q_{-A_{\lambda}(k,n)}.
\]

Note that in the multi-regular case, \( O_{\nu} = 1 \).

4. Applications of the Orbifold Vertex

4.1. The orbifold DT crepant resolution conjecture and the orbifold DT/GW correspondence. We give a brief description of the DT Crepant Resolution Conjecture which will be spelled out in detail in [16].

Let \( \mathcal{X} \) be an orbifold CY3 and let \( X \) be its coarse space. Let

\[
Y = \text{Hilb}^{[O_p]}(\mathcal{X})
\]

be the Hilbert scheme parameterizing substacks in the class \([O_p] \in F_0 K(\mathcal{X})\). \( Y \) is birational to \( X \) and admits a proper morphism \( \pi : Y \to X \). By a theorem of Bridgeland, King, and Reid [7], \( Y \) is a smooth CY3 and moreover, there is a Fourier-Mukai isomorphism \([7, 11]\)

\[
\Phi : K(\mathcal{X}) \to K(Y)
\]

defined by

\[
E \mapsto Rq_\ast p_\ast E
\]

where

\[
p : Z \to \mathcal{X}, \quad q : Z \to Y
\]

are the projections from the universal substack \( Z \subset \mathcal{X} \times Y \) onto each factor.

The Fourier-Mukai isomorphism does not respect the filtrations \( F_\bullet K(\mathcal{X}) \) and \( F_\bullet K(Y) \). However, if \( \mathcal{X} \) has transverse \( A_{n-1} \) orbifold structure, or
more generally satisfies the Hard Lefschetz condition \cite{9} Defn 1.1 c.f. \cite{8} Lem 24], then the image of $F_0K(X)$ under $\Phi$ is contained in $F_1K(Y)$. We call this image $F_{\text{exc}}K(Y)$; its elements can be represented by formal differences of sheaves supported on the exceptional fibers of $\pi : Y \to X$. We define the multi-regular part of $K$-theory, $F_{\text{mr}}K(X)$, to be the pre-image of $F_1K(Y)$ under $\Phi$. Its elements can be represented by formal differences of sheaves supported in dimension one where at the generic point of each curve in the support, the associated representation of the stabilizer of that point is a multiple of the regular representation. In summary, the following filtration is respected by the Fourier-Mukai isomorphism

$$F_0K(X) \subset F_{\text{mr}}K(X), \quad F_{\text{exc}}K(Y) \subset F_1K(Y).$$

We define the exception partition function of $Y$ and the multi-regular partition function of $X$ as follows

$$DT_{\text{exc}}(Y) = \sum_{\alpha \in F_{\text{exc}}K(Y)} DT_\alpha(Y)q^\alpha,$$

$$DT_{\text{mr}}(X) = \sum_{\alpha \in F_{\text{mr}}K(X)} DT_\alpha(Y)q^\alpha.$$

We then have our DT crepant resolution conjecture:

**Conjecture 1.** Let $X$ be an orbifold CY3 satisfying the Hard Lefschetz condition. Let $Y$ the the Calabi-Yau resolution of $X$ described above. Then using $\Phi$ to identify the variables we have an equality

$$\frac{DT_{\text{mr}}(X)}{DT_{\text{mr}}(X)} = \frac{DT(Y)}{DT_{\text{exc}}(Y)}.$$

The series $DT_0(X)$ and $DT_{\text{exc}}(Y)$ are not unrelated. The conjecture in \cite{4} Conjecture A.6] globalizes to

**Conjecture 2.** Using $\Phi$ to identify variables, we have the equality

$$DT_0(X) = \frac{DT_{\text{exc}}(Y)\widetilde{DT}_{\text{exc}}(Y)}{DT_0(Y)}$$

where $\widetilde{DT}_{\text{exc}}(Y)(q) = DT_{\text{exc}}(Y)(q^{-1})$.

Conjecture \cite{1} will be proven in the toric transverse $A_{n-1}$ case in \cite{16} using the orbifold vertex developed in this paper. Conjecture \cite{2} was proven in the transverse $A_{n-1}$ case in \cite{4}.

\footnote{The theorem in \cite{4} is for the local case $X = [\mathbb{C}^3/\mathbb{Z}_n]$. Conjecture \cite{2} is local in nature; extending from $X = [\mathbb{C}^3/\mathbb{Z}_n]$ to $X$ global is routine.}
We will see in the examples below that the series

\[ DT'_{mr}(X) = \frac{DT_{mr}(X)}{DT_0(X)} \]

which we call the reduced, multi-regular DT partition function of \( X \), is equal to the reduced GW partition function \( GW'(X) \) after a change of variables and analytic continuation. The general change of variables can be formulated in terms of Iritani’s stacky Mukai vector [15], but we will not formulate that explicitly here.

4.2. Example: the local football. Let

\[ X_{a,b} = \text{Tot}(O(-p_0) \oplus O(-p_\infty) \to \mathbb{P}^1_{a,b}) \]

be the total space of the bundle \( O(-p_0) \oplus O(-p_\infty) \) over the football \( \mathbb{P}^1_{a,b} \) which is by definition \( \mathbb{P}^1 \) with root constructions [10] of order \( a \) and \( b \) at the points \( p_0 \) and \( p_\infty \) respectively. \( X_{a,b} \) is a natural orbifold generalization of the resolved conifold which is the special case \( X_{1,1} \). We use our orbifold vertex formalism to derive a closed formula for the partition function \( DT(X_{a,b}) \).

Let \( O(D) = O(-p_0) \) and let \( O(D') = O(-p_\infty) \). Then the graph in Figure [1] is the whole graph of \( X_{a,b} \) and we have

\[ n(f) = a, \quad n(g') = b, \quad n(f') = n(g) = n(e) = 1, \quad \tilde{m} = \tilde{m}' = 0, \]

and so

\[ \tilde{m} + \delta_0 + \delta_\infty = 1. \]

We write our variables as follows:

\[ p_k = q_{f,k}, \quad k = 0, \ldots, a - 1 \]
\[ r_k = q_{g',k}, \quad k = 0, \ldots, b - 1 \]
\[ v = v_e \]

and of course

\[ q = p_0 \cdots p_{a-1} = r_0 \cdots r_{b-1}. \]

As in the usual conifold case, the variables \( v \) and \( q \) keep track of the degree and the holomorphic Euler characteristic of the curve respectively. Loosely speaking, the new variables \( p_k \) and \( r_l \) can be thought of as keeping track of embedded points on the stacky locus having representation \( k \in \hat{\mathbb{Z}}_a \) and \( l \in \hat{\mathbb{Z}}_b \) respectively.

Since the orbifold edges, namely \( f \) and \( g' \), are non-compact, the edge assignments are multi-regular and so only sign in the formula for \( DT(X_{a,b}) \) is the sign \((-1)^{(\tilde{m} + \delta_0 + \delta_\infty)|\lambda|}\). Thus

\[ DT(X_{a,b}) = \sum_{\lambda} E_\lambda \cdot V^a_{\lambda|0|}(p_0, \ldots, p_{a-1}) \cdot V^b_{\lambda'|0|}(r_0, \ldots, r_{b-1}) \]
where
\[ E_\lambda = (-1)^{|\lambda|} p^{\lambda} q^{\lambda} |p_0| p_A^{A_\lambda(0,a)} \cdots p_{a-1}^{A_\lambda(a-1,a)} r_0^{A_\lambda(0,b)} \cdots r_{b-1}^{A_\lambda(b-1,b)}. \]

Applying the formula in Theorem 12, we get
\[ V_{a,b}^\varnothing = V_{a,b}^\varnothing(p) \cdot p^{A_\lambda(0,a)} \cdots p_{a-1}^{A_\lambda(a-1,a)} \cdot s_\lambda(p) \]
\[ V_{a,b}^\varnothing(r) = V_{a,b}^\varnothing(r) \cdot r^{A_\lambda(0,b)} \cdots r_{b-1}^{A_\lambda(b-1,b)} \cdot s_\lambda(r), \]
where
\[ p = (p_0, \ldots, p_{a-1}), \quad p = (1, p_1, p_1 p_2, p_1 p_2 p_3, \ldots), \]
\[ r = (r_0, \ldots, r_{b-1}), \quad r = (1, r_1, r_1 r_2, r_1 r_2 r_3, \ldots). \]

The formula then reads
\[ DT(X_{a,b}) = V_{a,b}^\varnothing(p) V_{a,b}^\varnothing(r) \sum_{\lambda} s_\lambda(-vq \bar{p} \cdot s_\lambda(\bar{r}). \]

If we write \( Q = (1, q, q^2, q^3, \ldots) \), then we can rewrite the variables \( p \) and \( r \) as
\[ p = (1, p_1, p_1 p_2, p_1 p_2 p_3, \ldots), \quad \bar{p} = (Q, p_1 Q, p_1 p_2 Q, \ldots, p \cdots p_{a-1} Q) \]
\[ r = (1, r_1, r_1 r_2, r_1 r_2 r_3, \ldots), \quad \bar{r} = (Q, r_1 Q, r_1 r_2 Q, \ldots, r \cdots r_{b-1} Q) \]
and hence
\[ \bar{p} = (Q, p_{a-1} Q, p_{a-1} p_{a-2} Q, \ldots, p_1 \cdots p_{a-1} Q) \]
\[ \bar{r} = (Q, r_{b-1} Q, r_{b-1} r_{b-2} Q, \ldots, r_1 \cdots r_{b-1} Q) \]

Using the orthogonality of Schur functions [19 § I.4 (4.3')] and the fact that
\[ \prod_{i,j} (1 + x_i y_j) = M(w, q)^{-1} \]
if
\[ (x_1, x_2, x_3, \ldots) = -w q Q, \quad (y_1, y_2, y_3, \ldots) = Q, \]
we get
\[ DT(X_{a,b}) = V_{a,b}^\varnothing(p) V_{a,b}^\varnothing(r) \prod_{k=1}^{a} \prod_{l=1}^{b} M(v p_k \cdots p_{a-1} r_1 \cdots r_{b-1} q)^{-1}. \]

Using the formula for \( V_{a,b}^\varnothing \), we arrive at the following

**Proposition 3.** The DT partition function of the local football \( X_{a,b} \) is given by
\[ DT(X_{a,b}) = M(1, q)^{a+b} \prod_{w \in C_{a,b}^+} M(w, q) \prod_{u \in C_{a,b}^-} M(u, q)^{-1}. \]
where
\[ C_{a,b}^+ = \{ p_i \cdots p_j, p_i^{-1} \cdots p_j^{-1}, r_k \cdots r_l, r_k^{-1} \cdots r_l^{-1}, 0 < i < j < a, 0 < k \leq l < b \} \]
\[ C_{a,b}^- = \{ v p_k \cdots p_{a-1} r_l \cdots r_{b-1} : k = 1, \ldots, a, l = 1, \ldots, b \}. \]

Since the only stacky curves in \( X_{a,b} \) are non-compact, the reduced multi-regular DT partition function is equal to the usual reduced partition function:
\[ DT_{\text{mr}}'(X_{a,b}) = DT_{\text{mr}}(X_{a,b}) = \prod_{u \in C_{a,b}^-} M(u, -q)^{-1}. \]

The Calabi-Yau resolution \( Y \rightarrow X \) has a single \((-1, -1)\) curve given by the proper transform of the football to which are attached two chains of \((0, -2)\)-curves having \(a-1\) and \(b-1\) components each. Using the usual (non-orbifold) vertex formalism, one can verify that as predicted
\[ \frac{DT(Y)}{DT_{\text{exc}}(Y)} = \prod_{u \in C_{a,b}^-} M(u, -q)^{-1}, \]
where on \( Y \), the variables \( p_1, \ldots, p_{a-1} \) and \( r_1, \ldots, r_{b-1} \) correspond to the classes of the curves in each of the chains and \( v \) corresponds to the class of the \((-1, -1)\)-curve.

4.3. **Example: The local \( B\mathbb{Z}_2 \) gerbe.** Another example related to the conifold is the local \( B\mathbb{Z}_2 \) gerbe. In this case, \( X \) is the global quotient of the resolved conifold \( \text{Tot}(O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1) \) by \( \mathbb{Z}_2 \) acting fiberwise by \(-1\). The graph of \( X \) is again given by the one in Figure 1 but now with \( e \) being the only orbifold edge. The numerical invariants are
\[ n(e) = 2, \quad m = \tilde{m} = m' = \tilde{m}' = -1, \]
and the variables are
\[ q_0, q_1, v_0, v_1 \]

corresponding to the \( K \)-theory classes
\[ \mathcal{O}_p \otimes \rho_0, \quad \mathcal{O}_p \otimes \rho_1, \quad \mathcal{O}_C(-1) \otimes \rho_0, \quad \mathcal{O}_C(-1) \otimes \rho_1, \]

where \( p = p(e) \) is a point on the curve \( C = C(e) \).

The Calabi-Yau resolution \( Y \rightarrow X \) is given by local \( \mathbb{P}^1 \times \mathbb{P}^1 \), namely
\[ Y = \text{Tot} \left( O(-2, -2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \right). \]

Unlike the local football, there is not a nice closed formula for \( DT(X) \). However, our vertex formula does provide an explicit formula for the coefficients of the expansion of \( DT(X) \) as a series in \( v_0 \) and \( v_1 \). For applications to the DT/GW correspondence and the DT crepant resolution conjecture, we can restrict ourselves to curve classes whose generic point has a representation which is a multiple of the regular representation. This corresponds to
expanding $DT(\mathcal{X})$ about the variable $v = v_0v_1$, which in the vertex formula corresponds to summing over multi-regular edge assignments. Recall that this series is denoted $DT_{mr}(\mathcal{X})$. We compute with the vertex formula:

$$DT_{mr}(\mathcal{X}) = \sum_{\nu} E_{\nu} \left( V_{000}^2(q_0, q_1) \right)^2$$

where

$$E_{\nu} = v^{\nu_0} \cdot q_0^{\sum_{i,j \in \nu[0,2]} i \cdot \nu_i^0} \cdot q_1^{\sum_{i,j \in \nu[1,2]} i \cdot \nu_i^1}$$

and

$$V_{000}^2 = V_{000} \prod_{j,i \in \nu'} \frac{1}{1 - q_0^{h^0_{\nu', (j,i)}}} \frac{1}{q_1^{h^1_{\nu', (j,i)}}}$$

Noting that $DT_0(\mathcal{X}) = \left( V_{000}^2 \right)^2$, we get

$$\frac{DT_{mr}(\mathcal{X})}{DT_0(\mathcal{X})} = \sum_{d=0}^{\infty} v^d \sum_{\nu | \nu_0 = \nu_1 = d} q_0^{\sum_{i,j \in \nu[0,2]} i \cdot \nu_i^0} \frac{1}{q_0^{\sum_{i,j \in \nu[1,2]} i \cdot \nu_i^1}} \prod_{j,i \in \nu'} \left( 1 - q_0^{h^0_{\nu', (j,i)}} \frac{1}{q_1^{h^1_{\nu', (j,i)}}} \right)^2$$

We expand the above to order 3 in $v$. The linear term corresponds to the two partitions of size 2 and the quadratic term corresponds to the 5 partitions of size 4. The rational function in the $\nu$ sum is invariant under $\nu \leftrightarrow \nu'$ and is easily evaluated:

$$1 + v \frac{2q_0 q_1^2}{(1 - q_0 q_1)^2} + v^2 \left\{ \frac{2q_0 q_1^2}{(1 - q_0 q_1^2)^2} + \frac{2q_0^4 q_1^2}{(1 - q_0^2 q_1^2)^2} + \frac{2q_0^4 q_1^2}{(1 - q_0^2 q_1^2)^2} + \frac{2q_0^4 q_1^2}{(1 - q_0^2 q_1^2)^2} \right\} + O(v^3)$$

As predicted by Conjecture 1, the above series (after replacing $q_0$ with $-q_0$) matches with $DT(Y)/DT_{exc}(Y)$ under the change of variables

$$q = q_0 q_1, \quad v_s = q_1 v, \quad v_f = q_1.$$ 

Here $v_s$ and $v_f$ are the variables associated to the generating curve classes in $\mathbb{P}^1 \times \mathbb{P}^1$ (the section and fiber classes).

We note that it is noticeably more efficient to compute with the orbifold vertex than to compute on local $\mathbb{P}^1 \times \mathbb{P}^1$. 

The GW partition function of $\mathcal{X}$ is obtained from $DT_{mr}(\mathcal{X})/DT_0(\mathcal{X})$ by the change of variables

$$q_0q_1 = -e^{i\lambda}, \quad q_1 = -e^{ix}, \quad v = w,$$

where $\lambda$ is the genus parameter, $w$ is the degree parameter, and $x$ indexes the number of marked $\mathbb{BZ}_2$ points. So for example, if $GW_{1,g,n}(\mathcal{X})$ denotes the GW invariant of degree 1 maps whose domain curve is genus $g$ with $n$ marked $\mathbb{BZ}_2$ points, then

$$\sum_{n,g} GW_{1,g,n}(\mathcal{X}) \lambda^{2g-2} x^n = \frac{1}{2} \left( 2 \sin \frac{\lambda}{2} \right)^{-2} \sec^2 \frac{x}{2}.$$

5. Proof of Theorem

5.1. Overview. Our computation of the DT partition function of $\mathcal{X}$ uses a localization technique. The action of the torus $T$ on $\mathcal{X}$ induces a $T$ action on $\text{Hilb}^\alpha(\mathcal{X})$ with isolated fixed points. The fixed points are given by substacks of $\mathcal{X}$ defined by monomial ideals on each chart and these correspond to 3D partitions at each vertex. We use a theorem of Behrend and Fantechi [3, Theorem 3.4] which says that the weighted Euler characteristic of $\text{Hilb}^\alpha(\mathcal{X})$ is given by a signed count of fixed points:

$$DT_\alpha(\mathcal{X}) = e(\text{Hilb}^\alpha(\mathcal{X}), \nu_{\text{Hilb}^\alpha(\mathcal{X})}) = \sum_{p \in \text{Hilb}^\alpha(\mathcal{X})} (-1)^{\dim T_p \text{Hilb}^\alpha(\mathcal{X})}.$$

The above formula is also apparent from the point of view of virtual localization as used in [20], although we avoid non-compactness issues by the use of weighted Euler characteristics.

Thus the main two tasks are the following.

1. A combinatorial description of the $T$-fixed substacks and the computation of the $K$-theory class of a given $T$-fixed substack.
2. The computation of the parity of the tangent space to a fixed point in order to determine the sign.

Our approach to the above two tasks are quite different from [20] whose techniques do not readily generalize to the orbifold case. In fact our approach provides a substantial simplification in the non-orbifold case over the proof of [20]; in particular, we avoid the need for the combinatorial analysis in [20, §4.11].

To handle (1), we find a $K$-theory decomposition of $T$-invariant substacks into edge and vertex terms, and we use well chosen functions on $K$-theory to write the class in our basis. This is carried out in §5.2.
To handle (2), we exploit $T$-equivariant Serre duality and the Euler pairing in $K$-theory to determine the vertex and edge contributions to the signs. This is quite involved and is carried out in §6.

Our techniques yield a vertex formalism for an arbitrary orbifold toric CY3 $\mathcal{X}$ (not just the transverse $A_{n-1}$ case). Namely, we derive a formula of the form given by equation (1) where $E(e)$ is a signed monomial depending on $\lambda(e)$ and the local geometry of $C(e)$ and $\widehat{\mathcal{V}}^{\mu\nu}_{\lambda\mu\nu}$ is the generating function for 3D partitions asymptotic to $(\lambda, \mu, \nu)$, colored by representations of $G$ (as in equation (3)), except counted with the sign rule given in Theorem 21 (see Remark 24). Although the formula is completely combinatorial and can be made explicit, it is not as computational effective as the formula for the transverse $A_{n-1}$ case because we do not have an explicit formula for the general orbifold vertex $\widehat{\mathcal{V}}^{\mu\nu}_{\lambda\mu\nu}$ as we do in the transverse $A_{n-1}$ case. We also expect there to be an explicit formula for the vertex for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ but not in general.

5.2. The $K$-theory decomposition.

**Lemma 13.** Torus fixed points in

$$\bigsqcup_{\alpha \in F_1 K(\mathcal{X})} \text{Hilb}^\alpha(\mathcal{X})$$

are isolated and in bijective correspondence with sets $\{\lambda(e), \pi(v)\}$ where $\lambda(e)$ is an edge assignment (Definition 9) and $\{\pi(v) : v \in \text{Vertices}\}$ is a collection of 3D-partitions such that $\pi(v)$ is asymptotic to $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$.

**Proof.** Fix an orientation of $\Gamma$, the graph associated to $\mathcal{X}$. Recall that $D(e)$ and $D'(e)$ are the invariant divisors incident to $C(e)$ corresponding to the regions to the left and right of $e$ respectively. Recall that $(D_1(e), D_2(e), D_3(e))$ are the invariant divisors incident to $p(e)$ corresponding to the regions opposite of $(e_1(v), e_2(v), e_3(v))$ from $v$.

Let $Y \subset \mathcal{X}$ be a torus invariant substack of dimension at most one. We associate to $Y$ a collection $\{\lambda(e), \pi(v)\}$ as follows. Define $\lambda(e)$ to be the set $\langle i, j \rangle$ such that the composition

$$\mathcal{O}_{\mathcal{X}}(-iD(e) - jD'(e)) \to \mathcal{O}_X \to \mathcal{O}_Y$$

is non-zero at a general point of $C(e)$.

Similarly, we define $\pi(v)$ to be the set $\langle i, j, k \rangle$ such that the composition

$$\mathcal{O}_{\mathcal{X}}(-iD_1(v) - jD_2(v) - kD_3(v)) \to \mathcal{O}_X \to \mathcal{O}_Y$$

is non-zero at $p(v)$. The fact that $\pi(v)$ is asymptotic to $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$ follows easily from the construction and our conventions.

Conversely, given $\{\lambda(e), \pi(v)\}$, an edge assignment $\lambda(e)$ and a set $\{\pi(v)\}$ of 3D-partitions asymptotic to $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$, we construct a torus
invariant substack $Y \subset X$ as follows. Note that the edge assignment is uniquely determined by the $3D$-partitions $\{ \pi(v) \}$. For each $v$, consider the ideal sheaf $I_{\pi(v)} \subset O_X$ generated by the image of the maps

$$O_X(-iD_1(v) - jD_2(v) - kD_3(v)) \to O_X$$

for $(i, j, k)$ not contained in $\pi(v)$. This determines a torus invariant substack in the torus open invariant neighborhood of the point $p(v)$ for each $v$. By the compatibility of the edge partitions, these substacks agree on the overlaps and thus determine a global substack.

**Remark 14.** It will be convenient notation to identify an element $(i, j, k) \in \pi(v)$ with the corresponding divisor. Thus if we write $D \in \pi(v)$ we will mean

$$D = iD_1(v) + jD_2(v) + kD_3(v)$$

for the corresponding $(i, j, k) \in \pi(v)$. Similarly, $D \in \lambda(e)$ means

$$D = iD(e) + jD'(e)$$

for the corresponding $(i, j) \in \lambda(e)$. Our orientation conventions guarantee consistency between the divisors associated to elements of edge partitions and the divisors associated to elements of the legs of vertex partitions.

We write the $K$-theory class of the structure sheaf of a torus invariant substack as a sum over edge and vertex terms:

**Proposition 4.** Let $Y \subset X$ be a $T$-invariant substack of dimension no greater than one. Let $\{ \lambda(e), \pi(v) \}$ be the corresponding set of vertex and edge partitions. Then in $T$-equivariant compactly supported $K$-theory we have

$$O_Y = \sum_{e \in \text{Edges}} \sum_{D \in \lambda(e)} O_C(e)(-D) + \sum_{v \in \text{Vertices}} \sum_{D \in \pi(v)} \xi_{\pi(v)}(D)O_{p(v)}(-D).$$

**Proof.** For any $N \in \mathbb{N}$ let $N$ be the cubical $3D$-partition of size $N$, that is $N = \{(i, j, k) : 0 \leq i, j, k < N \}$. Let $Z_N$ be the $T$-invariant substack having empty edge partitions and vertex partitions all equal to $N$. Let $Y_N$ be the stack theoretic union of $Y$ and $Z_N$. Choose $N$ large enough so that for each $v$, $\pi(v)$ is contained in the the union of the legs of $\pi(v)$ with $N$.

We have embeddings $Y \subset Y_N$ and $Z_N \subset Y_N$ from which we get the following $K$-theory equalities:

$$I_Y - I_{Y_N} = \sum_{v \in \text{Vertices}} \sum_{D \in N \setminus \pi(v)} O_{p(v)}(-D)$$

$$I_{Z_N} - I_{Y_N} = \sum_{e \in \text{Edges}} \sum_{D \in \lambda(e)} O_C(e)(-D - ND_0(e) - ND_{\infty}(e)).$$
For any \( D \in \lambda(e) \), we have
\[
O_{C(e)}(-D) = O_{C(e)}(-D - ND_0(e) - ND_\infty(e)) \\
+ \sum_{k=0}^{N-1} O_{p_0(e)}(-D - kD_0(e)) + O_{p_\infty(e)}(-D - kD_\infty(e)).
\]

We note that if \( v \) is the initial vertex of \( e \), then
\[
\sum_{D \in \lambda(e)} \sum_{k=0}^{N-1} O_{p_0(e)}(-D - kD_0(e)) = \sum_{D \in N \cap \text{Leg}_e \pi(v)} O_{p(v)}(-D)
\]
where \( \text{Leg}_e \pi(v) \) is the leg of \( \pi(v) \) in the \( e \) direction. The similar statement holds for \( p_\infty(e) \).

Putting it all together we get:
\[
O_Y = O_Z_N - (I_Y - I_{Y_N}) + (I_{Z_N} - I_{Y_N})
\]
\[
= \sum_{v \in \text{Vertices}} \left( \sum_{D \in N} O_{p(v)}(-D) - \sum_{D \in N} O_{p(v)}(-D) \right) \\
+ \sum_{e \in \text{Edges}} \sum_{D \in \lambda(e)} O_{C(e)}(-D - ND_0(e) - ND_\infty(e)) \\
= \sum_{v \in \text{Vertices}} \sum_{D \in \pi(e) \cap N} O_{p(v)}(-D)
\]
\[
+ \sum_{e \in \text{Edges}} \sum_{D \in \lambda(e)} \left( O_{C(e)}(-D) - \sum_{k=0}^{N-1} (O_{p_0(e)}(-D - kD_0(e)) + O_{p_\infty(e)}(-D - kD_\infty(e))) \right)
\]
\[
= \sum_{v \in \text{Vertices}} \sum_{D \in \pi(v)} \xi_{\pi(v)}(D) O_{p(v)}(-D) + \sum_{e \in \text{Edges}} \sum_{D \in \lambda(e)} O_{C(e)}(-D)
\]
which proves the proposition. \( \square \)

In the case where \( X \) has transverse \( A_{n-1} \) orbifold structure, we can further refine our \( K \)-theory decomposition of \( O_Y \) into the basis described in \( \S 3.3 \). In the below lemmas, we write the decompositions of the vertex and the edge terms.

**Lemma 15.** The vertex terms decompose as follows
\[
\sum_{D \in \pi(v)} O_{p(v)}(-D) = \begin{cases} \\
\sum_{i,j,k \in \pi(v)} [O_{p(v)} \otimes \rho_{i-j}] & \text{if } e(v) \text{ is oriented outward,} \\
\sum_{i,j,k \in \pi(v)} [O_{p(v)} \otimes \rho_{j-i}] & \text{if } e(v) \text{ is oriented inward.}
\end{cases}
\]
Proof. This follows immediately from our conventions (§3.2) and our choice of the indexing of the representations $\rho_k$ of $\mathbb{Z}_{\nu(e))}$ (§3.3). □

Proposition 5. Let $e$ be a compact edge corresponding to a curve $C = C(e)$ and let $\lambda = \lambda(e)$ be an edge partition. Let $D = D(e)$, $D' = D'(e)$ and let $m = \deg(D)$, $m' = \deg(D')$. Assume that $e$ and its incident edges $f$, $f'$, $g$, $g'$ are oriented as in figure [1] Let $n = n(e)$, $a = n(f)$, $a' = n(f')$, $b = n(g)$, and $b' = n(g')$, then

$$\sum_{i,j \in \lambda} O_C(-iD - jD') = \sum_{k=0}^{n-1} |\lambda|_k \cdot [O_C(-1) \otimes \rho_k]$$

$$+ \sum_{k=0}^{n-1} C_{\bar{m}, \bar{m}'}[k,n] \cdot [O_{p(e)} \otimes \rho_k]$$

$$+ \delta_0 \sum_{k=0}^{a-1} A_\lambda(k,a) \cdot [O_{p(f)} \otimes \rho_k]$$

$$+ \delta_{a'} \sum_{k=0}^{a'-1} A_{\lambda'}(k,a') \cdot [O_{p(f')} \otimes \rho_k]$$

$$+ \delta_\infty \sum_{k=0}^{b-1} A_\lambda(k,b) \cdot [O_{p(g)} \otimes \rho_k]$$

$$+ \delta_{b'} \sum_{k=0}^{b'-1} A_{\lambda'}(k,b') \cdot [O_{p(g')} \otimes \rho_k].$$

Since $O_C(-iD - jD')$ is supported on $C$, it must be a combination of the classes $[O_C(-1) \otimes \rho_k] k = 0, \ldots, n-1$, $[O_{p(edge)} \otimes \rho_k] k = 0 \ldots, n(edge) - 1$ for $edge \in \{e, f, f', g, g'\}$, and $[O_{p}]$ since the remaining generators are always supported away from $C$. The classes $[O_{p(edge)} \otimes \rho_0]$ can be written in terms of the other classes using the relation (4). There are no further relations and hence the decomposition of $O_C(-iD - jD')$ into the above classes (without $[O_{p(edge)} \otimes \rho_0]$) has unique coefficients. We first compute the coefficients of that decomposition and then restore the classes with $\rho_0$ via the relation (4). Let $B$ be the set of such classes:

$$B = \left\{ [O_p], [O_C(-1) \otimes \rho_k]_{k=0, \ldots, n-1}, [O_{p(edge)} \otimes \rho_k]_{k=1, \ldots, n(edge), edge \in \{e, f, f', g, g'\}} \right\}.$$  

The coefficient of $[O_C(-1) \otimes \rho_k]$ in $\sum_{i,j \in \lambda} O_C(-iD - jD')$ is clearly $|\lambda|_k$ since each summand acts with weight $i - j \mod n$ at the generic point.

To determine the other coefficients, we construct functions on $K$-theory vanish on all the elements of $B$ except one. For example, the holomorphic
Euler characteristic $\chi$ which vanishes on all the classes in $B$ except $[\mathcal{O}_p]$ on which it is one.

We first suppose that $n > 1$. Then we have that $f = f' = g = g' = 1$ and the only point classes are $[\mathcal{O}_{p(e)} \otimes \rho_k]$ for $k = 1 \cdots n-1$ and $[\mathcal{O}_p]$. We define a function $\alpha_k$ on $K$-theory as follows. Let $C_l \subset IC$ be the component of the inertia stack corresponding to $l \in \mathbb{Z}/n$. Let $\omega = \exp \left( \frac{2\pi i}{n} \right)$ and let $\tau$ be the Toen operator (see Appendix A). We define

$$\alpha_k(E) = \sum_{l=0}^{n-1} \int_{C_l} (\omega^{-lk} - 1) \tau(E).$$

**Lemma 16.** The function $\alpha_k$ equals 0 on all classes of $B$ except for $[\mathcal{O}_{p(e)} \otimes \rho_k]$ on which it is 1.

**Proof.** Recall that by definition $[\mathcal{O}_C(-1) \otimes \rho_i] = \frac{1}{n} \pi_* [\mathcal{O}_{\tilde{C}}(-1) \otimes \rho_i]$ where $\pi : \tilde{C} \to C$ is an $n$-fold cover with $\tilde{C} \cong \mathbb{P}^1 \times B\mathbb{Z}/n$. By the functorial properties of the Toen operator (Theorem 35 in Appendix A), we have

$$\tau([\mathcal{O}_C(-1) \otimes \rho_i]) = \frac{1}{n} \pi_* \tau([\mathcal{O}_{\tilde{C}}(-1) \otimes \rho_i]).$$

However, since $\tau([\mathcal{O}_{\tilde{C}}(-1)])$ has vanishing $H^2$ terms on each component of $I\tilde{C}$, all the integrals in $\alpha_k([\mathcal{O}_C(-1) \otimes \rho_i])$ are zero. For the point classes, we compute (using example 36 from Appendix A)

$$\alpha_k([\mathcal{O}_{p(e)} \otimes \rho_j]) = \sum_{l=0}^{n-1} \int_{C_l} (\omega^{-lk} - 1) \tau([\mathcal{O}_{p(e)} \otimes \rho_j])$$

$$= \sum_{l=0}^{n-1} (\omega^{-lk} - 1) \int_{C_l} \omega^l \tau([\mathcal{O}_{p(e)} \otimes \rho_j])$$

$$= \frac{1}{n} \sum_{l=0}^{n-1} \omega^{l(k-j)} - \omega^l$$

$$= \delta_{k-j,0} - \delta_{j,0}.$$

Note that

$$\alpha_k(\mathcal{O}_p) = \alpha_k(\mathcal{O}_{p(e)} \otimes R_{reg}) = \sum_{j=0}^{n-1} \delta_{k-j,0} - \delta_{j,0} = 0$$

and the lemma is proved. □

By the above lemma, the coefficient of $\mathcal{O}_{p(e)} \otimes \rho_k$ in $\mathcal{O}_C(-iD - jD')$ is given by $\alpha_k(\mathcal{O}_C(-iD - jD'))$. Using example 37 in Appendix A, we can
compute as follows

\[ \alpha_k (\mathcal{O}_C(-iD - jD')) = \sum_{l=0}^{n-1} \int_{C_l} (\omega^{-lk} - 1) \tau(\mathcal{O}_C(-iD - jD')) \]

\[ = \sum_{l=0}^{n-1} (\omega^{-lk} - 1) \int_{C_l} \omega^{l(i-j)} \left(1 + [p(e)](1 - im - jm')\right) \]

\[ = (1 - im - jm') \frac{1}{n} \sum_{l=0}^{n-1} \omega^{l(i-j-k)} - \omega^{l(i-j)} \]

\[ = (1 - i\tilde{m} - j\tilde{m}')(\delta_{i-j,k} - \delta_{i-j,0}). \]

Therefore

\[ \text{Coeff}_{[\mathcal{O}_{p(e)} \otimes \rho_k]} \left( \sum_{i,j \in \lambda} \mathcal{O}_C(-iD - jD') \right) = C^\lambda_{\tilde{m},\tilde{m}'}[k, n] - C^\lambda_{\tilde{m},\tilde{m}'}[0, n]. \]

We also have

\[ \text{Coeff}_{[\mathcal{O}_p]} \left( \sum_{i,j \in \lambda} \mathcal{O}_C(-iD - jD') \right) = \chi \left( \sum_{i,j \in \lambda} \mathcal{O}_C(-iD - jD') \right) \]

\[ = \sum_{i,j \in \lambda} \sum_{l=0}^{n-1} \int_{C_l} \tau(\mathcal{O}_C(-iD - jD')) \]

\[ = \sum_{i,j \in \lambda} \frac{1}{n} \sum_{l=0}^{n-1} \omega^{l(i-j)} (1 - im - jm') \]

\[ = \sum_{i,j \in \lambda} (1 - im - jm') \delta_{i-j,0} \]

\[ = C^\lambda_{\tilde{m},\tilde{m}'}[0, n] \]

Using the relation

\[ [\mathcal{O}_p] = \sum_{k=0}^{n-1} [\mathcal{O}_{p(e)} \otimes \rho_k], \]

we find that

\[ \sum_{i,j \in \lambda} [\mathcal{O}_C(-iD - jD')] = \sum_{k=0}^{n-1} |\lambda_k| \cdot [\mathcal{O}_C(-1) \otimes \rho_k] \]

\[ + \sum_{k=0}^{n-1} C^\lambda_{\tilde{m},\tilde{m}'}[k, n] \cdot [\mathcal{O}_{p(e)} \otimes \rho_k] \]

and Proposition 5 is proved for the case of \( n > 1 \).
We now assume that \( n = 1 \). Recall the definitions of \( \delta_0, \delta'_0, \delta_\infty, \delta'_\infty, \tilde{m}, \) and \( \tilde{m}' \) from \( \S 3.4 \).

The holomorphic Euler characteristic of a general line bundle on the football \( C \) is given in example 38 in Appendix A:

\[
\chi(\mathcal{O}_C(dp + sp_0 + tp_\infty)) = d + 1 + \left\lfloor \frac{s}{\max(a, a')} \right\rfloor + \left\lfloor \frac{t}{\max(b, b')} \right\rfloor
\]

Thus

\[
\chi(\mathcal{O}_C(-iD - jD')) = \chi(\mathcal{O}_C((-i\tilde{m} - j\tilde{m}')(p + (i\delta_0 + j\delta_0')p_0 + (i\delta_\infty + j\delta_\infty')p_\infty)) - i\tilde{m} - j\tilde{m}' + 1 + \left\lfloor \frac{i\delta_0 + j\delta_0'}{\max(a, a')} \right\rfloor + \left\lfloor \frac{i\delta_\infty + j\delta_\infty'}{\max(b, b')} \right\rfloor
\]

where in the last equality we used the fact that either \( \delta_0 \) or \( \delta_0' \) is zero and that either \( \delta_\infty \) or \( \delta_\infty' \) is zero.

We conclude that

\[
\text{Coeff}_{[\mathcal{O}_p]} \left( \sum_{i,j \in \lambda} \mathcal{O}_C(-iD - jD') \right) = \lambda_{\tilde{m}, \tilde{m}'} + \delta_0 A_{\lambda}(0, a) + \delta_0' A_{\lambda'}(0, a') + \delta_\infty A_{\lambda}(0, b) + \delta_\infty' A_{\lambda'}(0, b')
\]

For \( k = 1, \ldots, \max(a, a') - 1 \) we define

\[
\mu_k(E) = \chi(E(kD_0)) - \chi(E).
\]

For \( k = 1, \ldots, \max(b, b') - 1 \) we define

\[
\nu_k(E) = \chi(E(kD_\infty)) - \chi(E).
\]

**Lemma 17.** The function \( \mu_k \) is zero on all the classes in \( \mathcal{B} \) except for \( \delta_0[\mathcal{O}_{p(f)} \otimes \rho_k] + \delta_0'[\mathcal{O}_{p(f')} \otimes \rho_k] \) on which it is 1. Likewise, the function \( \nu_k \) is zero on all the classes in \( \mathcal{B} \) except for \( \delta_\infty[\mathcal{O}_{p(g)} \otimes \rho_k] + \delta_\infty'[\mathcal{O}_{p(g')} \otimes \rho_k] \) on which it is 1.

**Proof.** Since \( \mathcal{O}_C(D_0) = \mathcal{O}_C(p_0) \), we have

\[
\mu_k(\mathcal{O}_C(-1)) = \chi(\mathcal{O}_C(-p + kp_0)) - \chi(\mathcal{O}_C(-p)) = \left\lfloor \frac{k}{\max(a, a')} \right\rfloor = 0.
\]

By our orientation conventions, the weight of the action of \( \mathcal{O}(kD_0) \) on \( \mathcal{O}_{p(f)} \) and \( \mathcal{O}_{p(f')} \) is \(-k\). Then for \( k, l \in \{1, \ldots, a - 1\} \)

\[
\mu_k(\mathcal{O}_{p(f)} \otimes \rho_l) = \chi(\mathcal{O}_{p(f)} \otimes \rho_{l-k}) - \chi(\mathcal{O}_{p(f)} \otimes \rho_k) = \delta_{l,k}
\]
and similarly for $k, l \in \{1, \ldots, a' - 1\}$ we have

$$
\mu_k(\mathcal{O}_{p(f')} \otimes \rho_l) = \delta_{l,k}.
$$

Finally, $\mu_k$ vanishes on $[\mathcal{O}_p], [\mathcal{O}_{p(g)} \otimes \rho_l], \text{ and } [\mathcal{O}_{p(g')} \otimes \rho_l]$ since these classes can be taken with support disjoint from $D_0$. This proves the assertions of the lemma for $\mu_k$; the proof for $\nu_k$ is similar. 

By the lemma, we can use $\mu_k$ and $\nu_k$ to determine the remaining coefficients of $\sum_{i,j} \mathcal{O}_C(-iD - jD')$ in the basis $\mathcal{B}$.

$$
\mu_k(\mathcal{O}_C(-iD - jD')) = \chi(\mathcal{O}_C((-i\tilde{m} - j\tilde{m'})p + (i\delta_0 + j\delta_0' + k)p_0 + (i\delta_\infty + j\delta_\infty')p_\infty)) \\
+ \chi(\mathcal{O}_C((-i\tilde{m} - j\tilde{m'})p + (i\delta_0 + j\delta_0' + k)p_0 + (i\delta_\infty + j\delta_\infty')p_\infty)) \\
= \left[\frac{i\delta_0 + j\delta_0' + k}{\max(a, a')}\right] - \left[\frac{i\delta_0 + j\delta_0'}{\max(a, a')}\right] \\
= \delta_0 \left(\left[\frac{i + k}{a}\right] - \left[\frac{i}{a}\right]\right) + \delta_0' \left(\left[\frac{j + k}{a'}\right] - \left[\frac{j}{a'}\right]\right)
$$

where in the last equality we use the fact that at least one of $\delta_0, \delta_0'$ is zero. Computing similarly, we get that

$$
\nu_k(\mathcal{O}_C(-iD - jD')) = \delta_\infty \left(\left[\frac{i + k}{b}\right] - \left[\frac{i}{b}\right]\right) + \delta_\infty' \left(\left[\frac{j + k}{b'}\right] - \left[\frac{j}{b'}\right]\right).
$$
Putting together the computations, we obtain

\[
\sum_{i,j \in \lambda} O_C(-iD - jD') = |\lambda| \cdot [O_C(-1)] \\
+ \left( C_{\hat{m},\hat{m}'}^{\lambda} + \delta_0 A_\lambda(0, a) + \delta'_0 A_{\lambda'}(0, a') \\
\quad + \delta_\infty A_\lambda(0, b) + \delta'_\infty A_{\lambda'}(0, b') \right) [O_p] \\
+ \sum_{k=1}^{a-1} (A_\lambda(k, a) - A_\lambda(0, a)) \cdot [O_{p(f)} \otimes \rho_k] \\
+ \sum_{k=1}^{a'-1} (A_{\lambda'}(k, a') - A_{\lambda'}(0, a')) \cdot [O_{p(f')} \otimes \rho_k] \\
+ \sum_{k=1}^{b-1} (A_\lambda(k, b) - A_\lambda(0, b)) \cdot [O_{p(g)} \otimes \rho_k] \\
+ \sum_{k=1}^{b'-1} (A_{\lambda'}(k, b') - A_{\lambda'}(0, b')) \cdot [O_{p(g')} \otimes \rho_k].
\]

Note that we can multiply the \(f\) (respectively \(f', g, g'\)) sum by \(\delta_0\) (respectively \(\delta'_0, \delta_\infty, \delta'_\infty\)) without changing the equality. Thus applying the relation (4), we get

\[
\sum_{i,j \in \lambda} O_C(-iD - jD') = |\lambda| \cdot [O_C(-1)] \\
+ C_{\hat{m},\hat{m}'}^{\lambda} \cdot [O_{p(e)} \otimes \rho_0] \\
+ \sum_{k=1}^{a-1} A_\lambda(k, a) \cdot [O_{p(f)} \otimes \rho_k] \\
+ \sum_{k=1}^{a'-1} A_{\lambda'}(k, a') \cdot [O_{p(f')} \otimes \rho_k] \\
+ \sum_{k=1}^{b-1} A_\lambda(k, b) \cdot [O_{p(g)} \otimes \rho_k] \\
+ \sum_{k=1}^{b'-1} A_{\lambda'}(k, b') \cdot [O_{p(g')} \otimes \rho_k].
\]
which proves Proposition 4 in the case where \( n = 1 \) and hence completes its proof.

6. THE SIGN FORMULA

Sign, sign, everywhere a sign
Blocking out the scenery, breaking my mind

—Five Man Electrical Band

6.1. Overview. By [3, Theorem 3.4] and Lemma 13, the invariant \( DT_0(\mathcal{X}) \) is given by a signed count of torus invariant ideal sheaves \( I \) where the sign is given by \( (-1)^{\text{Ext}^1_0(I,I)} \). This section is devoted to computing those signs and arranging them into vertex and edge terms. In §6.2 we derive a general sign formula, theorem 21, and in §6.3, we compute the sign formula in the case where \( \mathcal{X} \) has transverse \( A_{n-1} \) orbifold structure.

6.2. General Sign Formula. Let \( I \subseteq \mathcal{O}_X \) be the ideal sheaf of \( Y \). The Zariski tangent space to \( Y \) in \( \text{Hilb}(\mathcal{X}) \) is isomorphic to \( \text{Ext}^1_0(I,I) \). We want to compute its dimension modulo 2 in terms of the associated partitions \( \{\lambda(e)\} \) and \( \{\pi(v)\} \). Let \( T \) be the 3-dimensional torus acting on \( \mathcal{X} \).

For a \( T \)-representation \( V \), we use \( V^\vee \) to denote the dual representation. By equivariant Serre duality, we have
\[
\text{Ext}^i_F(G)^\vee = \text{Ext}^{3-i}_{G}(F \otimes \omega_X),
\]
and likewise for traceless \( \text{Ext} \). If \( w \in \text{Hom}(T, \mathbb{C}^*) \), we use the notation \( \mathbb{C}[w] \) to denote a 1-dimensional \( T \)-representation with weight \( w \).

Lemma 18. As a \( T \)-equivariant line bundle, \( \omega_X \cong \mathcal{O}_X \otimes \mathbb{C} \mathbb{C}[\mu] \) for some primitive weight \( \mu \).

Proof. The Calabi-Yau condition on \( \mathcal{X} \) implies that \( \omega_X \) must be an equivariant lift of \( \mathcal{O}_X \) and hence it is of the form \( \mathcal{O}_X \otimes \mathbb{C}[\mu] \). If \( \mu \) is not primitive, then the generic stabilizer of \( \mathcal{X} \) is non-trivial.

Definition 19. We define the shifted dual of a \( T \)-representation \( V \) by the formula
\[
V^* = V^\vee \otimes \mathbb{C}[-\mu].
\]

Note that the shifted dual induces a fixed-point free involution on characters of \( T \).

Proposition 6. The shifted dual satisfies the following properties.

1. For any \( T \)-equivariant sheaves \( F \) and \( G \),
\[
\text{Ext}^i(F,G)^* \cong \text{Ext}^{3-i}(G,F).
\]
(2) Let $V$ and $W$ be virtual $T$-representations such that

$$V - V^* = W - W^*.$$ 

Then the virtual dimensions of $V$ and $W$ are equal modulo 2.

**Proof.** The first statement is a restatement of equivariant Serre duality. The second statement follows by comparing the dimensions of the $\nu$ and $-\nu - \mu$ weight spaces of $V$ and $W$ as $\nu$ runs through half the characters of $T$. $\square$

**Definition 20.** Let $V$ be a virtual $T$-representation. We define $s(V) \in \mathbb{Z}/2\mathbb{Z}$ to be the dimension modulo 2 of $V$. We also define $\sigma(V - V^*) = s(V)$, where the input of $\sigma$ is required to be an anti-self shifted dual virtual representation. $\sigma$ is well-defined by Proposition 6.

Considered as $T$-representations, we have that

$$\text{Ext}^1_0(I, I) - \text{Ext}^2_0(I, I) = \chi(O_X, O_X) - \chi(I, I).$$

Using the exact sequence

$$0 \rightarrow I \rightarrow O_X \rightarrow O_Y \rightarrow 0,$$

we can write

$$\chi(O_X, O_X) - \chi(I, I) = \chi(O_X, O_Y) + \chi(O_Y, O_X) - \chi(O_Y, O_Y).$$

Since $\chi(O_X, O_Y)^* = -\chi(O_Y, O_X)$, we have

$$s(\text{Ext}^1_0(I, I)) = s(\chi(O_X, O_Y)) + \sigma(\chi(O_Y, O_Y)).$$

The first term is $\chi(O_Y)$ modulo 2, so we are left to compute the second term. For this we use the $K$-theory decomposition above.

Given any decomposition $O_Y = \sum_i K_i$ in $K_T(\mathcal{X})$, we have

$$\chi(O_Y, O_Y) = \sum_{i,j} \chi(K_i, K_j)$$

$$= \sum_i [(\text{Ext}^0(K_i, K_i) - \text{Ext}^1(K_i, K_i)) - (\text{Ext}^0(K_i, K_i) - \text{Ext}^1(K_i, K_i))^*]$$

$$+ \sum_{i<j} [\chi(K_i, K_j) - \chi(K_i, K_j)^*],$$

and therefore

$$\sigma(\chi(O_Y, O_Y)) = \sum_i s(\text{Hom}(K_i, K_i) - \text{Ext}^1(K_i, K_i)) + \sum_{i<j} s(\chi(K_i, K_j)).$$

We treat the first sum first, and call these the diagonal terms. It can be divided into edge terms and vertex terms.

**Proposition 7.** If $K$ and $L$ are supported on curves, then

$$\text{Ext}^1(K, L) \cong H^0(\text{Ext}^1(K, L)) \oplus H^1(\text{Hom}(K, L)).$$
Proof. The local-to-global spectral sequence degenerates at the $E_2$ term.

First we consider edge terms. Let $e$ be a compact edge and let $C = C(e)$, $D = D(e)$, and $D' = D'(e)$ so that $C = D \cap D'$. For $A \in \lambda(e)$ (recall Remark 14) we have

\[(6) \quad 0 \to \mathcal{O}_X(-A - D - D') \to \mathcal{O}_X(-A - D) \oplus \mathcal{O}_X(-A - D') \to \mathcal{O}_X(-A) \to \mathcal{O}_C(-A) \to 0.\]

If we apply the functor $\mathcal{H}om(\cdot, \mathcal{O}_C(-A))$ to this we obtain a complex which computes the local Ext sheaves.

1. $\mathcal{H}om(\mathcal{O}_C(-A), \mathcal{O}_C(-A)) = \mathcal{O}_C$
2. $\mathcal{E}xt^1(\mathcal{O}_C(-A), \mathcal{O}_C(-A)) = N_{C/X}$
3. $\mathcal{E}xt^2(\mathcal{O}_C(-A), \mathcal{O}_C(-A)) = \wedge^2 N_{C/X}$

Since $h^0(\mathcal{O}_C) = 1$ and $h^1(\mathcal{O}_C) = 0$ we deduce that each edge $e$ contributes $|\lambda(e)|(1 + h^0(N_{C/X}))$ to the diagonal terms.

We compute the vertex terms as follows. Let $v$ be a vertex and let $p = p(v)$ and $D_i = D_i(v)$. For $A \in \pi(v)$, we have the following exact sequence.

\[(7) \quad 0 \to \mathcal{O}_X(-A - \sum_i D_i) \to \bigoplus_{1 \leq i < j \leq 3} \mathcal{O}_X(-A - D_i - D_j) \to \bigoplus_{1 \leq i \leq 3} \mathcal{O}_X(-A - D_i) \to \mathcal{O}_X(-A) \to \mathcal{O}_p(-A) \to 0.\]

By a similar computation to the edge case, we see that every vertex $v$ contributes $|\pi(v)|(1 + h^0(N_{p/X}))$ to the diagonal terms. Note that $|\pi(v)|$ is not the cardinality of $\pi(v)$, but $\sum_{A \in \pi(v)} \xi_\pi(A)$.

Finally, we must compute the off-diagonal terms $s(\chi(K_i, K_j))$. These can be divided into edge terms, where $K_i$ and $K_j$ are supported on the same edge, and vertex terms, which come in three types:

1. $K_i$ and $K_j$ are supported at the same $p = p(v)$.
2. $K_j$ is supported at $p = p(v)$ and $K_i$ is supported along $C = C(e)$ where $e$ is incident to $v$.
3. $K_i$ is supported on $C = C(e)$ and $K_j$ is supported on $C' = C(e')$, where $e \neq e'$ have the vertex $v$ in common.
It is convenient to introduce an arbitrary total order on each partition \( \lambda(e) \) and \( \pi(v) \). For each \( A < B \) in \( \lambda(e) \), if we apply \( \mathcal{H}om(\cdot, \mathcal{O}_C(-B)) \) to \( (6) \), we obtain the complex which computes the local Ext sheaves:

\[
\mathcal{E}xt^i(\mathcal{O}_C(-A), \mathcal{O}_C(-B)) = \mathcal{O}_C(A - B) \otimes \wedge^i N_{C/X}.
\]

It follows that each edge \( C \in E \) contributes

\[
\sum_{A, B \in \lambda(e), A < B} \chi(\mathcal{O}_C(A - B) \otimes \lambda_{-1}(N_{C/X}))
\]

to the off-diagonal terms of \( \sigma(\chi(\mathcal{O}_Y, \mathcal{O}_Y)) \).

For each \( A < B \) in \( \pi(v) \), we can apply the same argument to \( (7) \) to obtain a contribution of

\[
\sum_{A, B \in \pi(v), A < B} \xi_{\pi(v)}(A) \xi_{\pi(v)}(B) \chi(\mathcal{O}_p(A - B) \otimes \lambda_{-1}(N_{p/X}))
\]

to the type 1 terms.

If \( v \) is incident to \( e \), \( A \in \lambda(e) \), and \( B \in \pi(v) \), then applying \( \mathcal{H}om(\cdot, \mathcal{O}_p(-B)) \) to \( (6) \) produces

\[
0 \to \mathcal{O}_p(A - B) \to \mathcal{O}_p(A - B) \otimes N_{C/X} \to \mathcal{O}_p(A - B) \otimes \wedge^2 N_{C/X} \to 0,
\]

which yields a type 2 vertex contribution at \( v \) of

\[
\sum_{i=1}^2 \sum_{A \in \lambda(e_i)} \sum_{B \in \pi(v)} \xi_{\pi(v)}(B) \chi(\mathcal{O}_p(A - B) \otimes \lambda_{-1}(N_{C(e_i)/X})).
\]

Finally, suppose \( C = C(e) = D \cap D', C' = C(f') = D \cap D'_0 \), and \( p = p(v) = C \cap C' \) (see figure 1). Let \( A \in \lambda(e) \), and \( B \in \lambda(f') \). If we apply \( \mathcal{H}om(\cdot, \mathcal{O}_{C'}(B)) \) to \( (6) \), we obtain the complex

\[
0 \to \mathcal{O}_{C'}(A - B) \to \mathcal{O}_{C'}(A - B + D) \oplus \mathcal{O}_{C'}(A - B + D') \to \mathcal{O}_{C'}(A - B + D + D') \to 0.
\]

Using the fact that \( \mathcal{O}_{C'} \to \mathcal{O}_{C'}(D') \) is injective, we compute the cohomology of the above complex to obtain

\[
(1) \quad \mathcal{H}om(\mathcal{O}_C(A), \mathcal{O}_{C'}(B)) = 0,
\]

\[
(2) \quad \mathcal{E}xt^1(\mathcal{O}_C(A), \mathcal{O}_{C'}(B)) = \mathcal{O}_p(A - B + D')
\]

\[
(3) \quad \mathcal{E}xt^2(\mathcal{O}_C(A), \mathcal{O}_{C'}(B)) = \mathcal{O}_p(A - B + D + D')
\]

Note that \( \mathcal{O}_p(D') = N_{p/C} \) and \( \mathcal{O}_p(D + D') = N_{p/C}^{-1} \) since by the Calabi-Yau condition, \( \mathcal{O}_p(D + D' + D_0) = \mathcal{O}_p \). Therefore

\[
s(\chi(\mathcal{O}_C(A), \mathcal{O}_{C'}(B))) = h^0(\mathcal{O}_p(A - B) \otimes N_{p/C}) + h^0(\mathcal{O}_p(A - B) \otimes N_{p/C}^\vee)
\]

\[
= h^0(\mathcal{O}_p(A - B) \otimes N_{p/C}) + h^0(\mathcal{O}_p(B - A) \otimes N_{p/C}).
\]
Now summing up over all contributions of this type we can write the type 3 off-diagonal vertex contribution of a vertex $v$ as
\[
\sum_{i \neq j} \sum_{A \in \lambda(e_i(v))} \sum_{B \in \lambda(e_j(v))} h^0(\mathcal{O}_p(v)(A - B) \otimes N_{p(v)/C(e_j(v)))}.
\]

Putting it all together yields the following sign formula.
\[
(8)
\]
\[
s(\text{Ext}_0^1(I, I)) = \chi(\mathcal{O}_Y) + \sum_{e \in \text{Edges}} |\lambda(e)| (1 + h^0(N_{C(e)/X})) + \sum_{v \in \text{Vertices}} |\pi(v)| (1 + h^0(N_{p(v)/X}))
+ \sum_{e \in \text{Edges}} \sum_{A,B \in \lambda(e)} \chi(\mathcal{O}_{C(e)}(A - B) \otimes \lambda_1(N_{C(e)/X}))
+ \sum_{v \in \text{Vertices}} \sum_{A,B \in \lambda(\pi(v))} \xi_{\pi(v)}(A)\xi_{\pi(v)}(B) h^0(\mathcal{O}_p(v)(A - B) \otimes \lambda_1(N_{p(v)/X}))
+ \sum_{v \in \text{Vertices}} \sum_{i=1}^3 \sum_{A \in \lambda(e_i(v))} \sum_{B \in \lambda(\pi(v))} \xi_{\pi(v)}(B) h^0(\mathcal{O}_p(v)(A - B) \otimes \lambda_1(N_{C(e_i)/X}))
+ \sum_{v \in \text{Vertices}} \sum_{i \neq j} \sum_{A \in \lambda(e_i(v))} \sum_{B \in \lambda(e_j(v))} h^0(\mathcal{O}_p(v)(A - B) \otimes N_{p(v)/C(e_j(v)))}.
\]

The above formula can be divided into three pieces. The first is an overall $\chi(\mathcal{O}_Y)$, the second is a sum over edges and the third is a sum over vertices. The contribution of an edge $e$ is
\[
|\lambda(e)| (1 + h^0(N_{C(e)/X})) + \sum_{A,B \in \lambda(e)} \chi(\mathcal{O}_{C(e)}(A - B) \otimes \lambda_1(N_{C(e)/X})).
\]

Recall that $<$ was an arbitrary total order. We can resymmetrize as follows.
Let $C = C(e)$, $D = D(e)$, and $D' = D'(e)$. We have that
\[
N_{C/X} = \mathcal{O}_C(D) + \mathcal{O}_C(D'),
\]
\[
\lambda_1 N_{C/X} = \mathcal{O}_C - \mathcal{O}_C(D) - \mathcal{O}_C(D') + K_C.
\]

By Serre duality
\[
\chi(\mathcal{O}_C(A - B)) = -\chi(\mathcal{O}_C(B - A) \otimes K_C),
\]
\[
\chi(\mathcal{O}_C(A - B + D)) = -\chi(\mathcal{O}_C(B - A + D')),
\]
\[
h^0(\mathcal{O}_C(D')) = h^1(\mathcal{O}_C(D)).
\]

This allows the second half of the edge contribution to be rewritten as a sum over all pairs $(A, B)$, where the diagonal terms are accounted for by the first
half. So the edge contribution is given by

$$\sum_{A, B \in \lambda(e)} \chi(\mathcal{O}_C(A - B) + \mathcal{O}_C(A - B + D)).$$

At each vertex $v$, we can do a similar cancellation with the terms

$$|\pi(v)|(1 + h^0(N_{p/X}) + \sum_{A, B \in \pi(v)} \sum_{A < B} \xi_{\pi(v)}(A)\xi_{\pi(v)}(B)h^0(\mathcal{O}_p(A - B) \otimes \lambda_{-1}(N_{p/X})).$$

using the fact that

$$\lambda_{-1}(N_{p/X}) = \sum_{i=1}^{3} (\mathcal{O}_p(-D_i) - \mathcal{O}_p(D_i)).$$

These terms become

$$|\pi(v)| + \sum_{A, B \in \pi(v)} \xi_{\pi(v)}(A)\xi_{\pi(v)}(B)h^0\left(\sum_{i=1}^{3} \mathcal{O}_p(A - B + D_i)\right).$$

The computations of this section are summarized by the following theorem.

**Theorem 21.** Let $I \subset \mathcal{O}_X$ be a torus fixed ideal corresponding to a substack $Y \subset \mathcal{X}$ and let $\{\lambda(e), \pi(v)\}$ be the corresponding sets of partitions. Then $s(\text{Ext}^1_{0}(I, I))$, the parity of the dimension of the Zariski tangent space of $I$ in $\text{Hilb}(\mathcal{X})$, is given by

$$s(\text{Ext}^1_{0}(I, I)) = \chi(\mathcal{O}_Y) + \sum_{e \in \text{Edges}} SE_{\lambda(e)}(v) + \sum_{v \in \text{Vertices}} SV_{\pi(v)}(v)$$

where

$$SE_{\lambda(e)} = \sum_{A, B \in \lambda} \chi\left(\mathcal{O}_{C(e)}(A - B) + \mathcal{O}_{C(e)}(A - B + D(e))\right)$$

and

$$SV_{\pi(v)} = |\pi| + \sum_{A, B \in \pi} \xi_{\pi}(A)\xi_{\pi}(B)h^0\left(\sum_{i=1}^{3} \mathcal{O}_{p(v)}(A - B + D_i(e))\right)$$

$$+ \sum_{i=1}^{3} \sum_{A \in \lambda(e_i(v))} \sum_{B \in \pi} \xi_{\pi}(B)h^0\left(\mathcal{O}_{p(v)}(A - B) \otimes \lambda_{-1}(N_{C(e_i(v))/\mathcal{X}})\right)$$

$$+ \sum_{i \neq j} \sum_{A \in \lambda(e_i(v))} \sum_{B \in \lambda(e_j(v))} h^0(\mathcal{O}_{p(v)}(A - B + D_j)).$$
Example 22. If $X$ is a scheme, then $SE_{\lambda}(e)$ simplifies to $m(e)\lambda$ and $SV_\pi(v)$ simplifies to 0 and we recover the signs of the classical topological vertex. The simplifications are straightforward:

$$SE_{\lambda}(e) = \sum_{A,B \in \lambda} \deg(A - B) + 1 + \deg(A - B + D(e)) + 1$$

$$= \sum_{A,B \in \lambda} \deg(D(e))$$

$$= m(e)|\lambda|^2 = m(e)|\lambda| \mod 2.$$ 

As for the vertex term, note that $\lambda - 1 N_{C(e_i(v))/X}$ restricted to $p(v)$ is zero, so

$$SV_\pi(v) = |\pi| + \sum_{A,B \in \pi} 3 \xi_\pi(A)\xi_\pi(B)$$

$$+ \sum_{i \neq j} \sum_{A \in \lambda(e_i(v))} \sum_{D \in \lambda(e_j(v))} 1$$

$$= |\pi| + 3|\pi|^2 + 2 \sum_{i < j} |\lambda(e_i(v))| \cdot |\lambda(e_j(v))|$$

$$= 0 \mod 2.$$

Example 23. If $X = [C^3/G]$ then there is a single vertex and each torus invariant ideal $I$ corresponds to a single (leg-less) partition $\pi$. Let $r_1, r_2, r_3 \in \hat{G}$ be the characters of $G$ given by $\mathcal{O}_p(D_i)$ and let $0 \in \hat{G}$ be the trivial character. Let $|\pi|_r$ be the number of boxes in $\pi$ colored by the character $r$. Then the sign associated to $I$ simplifies as follows.

$$s(\text{Ext}^1_0(I, I)) = \chi(\mathcal{O}_Y) + SV_\pi$$

$$= |\pi|_0 + |\pi| + \sum_{A,B \in \pi} 3 \sum_{i=1}^3 h^0(\mathcal{O}_p(A - B + D_i))$$

$$= |\pi|_0 + |\pi| + \sum_{r \in \hat{G}} |\pi|_r \left(|\pi|_{r+r_1} + |\pi|_{r+r_2} + |\pi|_{r+r_3}\right)$$

Remark 24. A general orbifold vertex formula can now be obtained. Using our identification of the torus fixed points (Lemma 13), our $K$-Theory decomposition of torus fixed ideas (Proposition 4), our general sign formula (Theorem 21), and the Behrend-Fantechi theorem, we get a combinatorial formula for $DT(X)$ of the form given by equation (1). The details of the formula, particularly the edge term, depend on the choice of generators for $F_1 K(X)$. 
6.3. **Sign formula in the transverse $A_{n-1}$ case.** In this section we simplify the sign formula from Theorem 21 in the case where $X$ has transverse $A_{n-1}$ orbifold structure.

We first simplify the edge term $SE_\lambda(e)$. First suppose that $n = n(e) > 1$ so that $C = C(e)$ is a $B\mathbb{Z}_n$ gerbe. Let $D = D(e)$. Then

$$SE_\lambda(e) = \sum_{k=0}^{n-1} \sum_{A,B \in \lambda} \chi(O_C(A - B)) + \chi(O_C(A - B + D)).$$

$$= \sum_{k=0}^{n-1} \left( \sum_{A,B \in \lambda[k,n]} (\deg(A) - \deg(B) + 1) \right)$$

$$+ \left( \sum_{A \in \lambda[k,n]} (\deg(A) + 1) - (\deg(B) + 1) + \deg(D) + 1 \right)$$

$$= \sum_{k=0}^{n-1} |\lambda_k|^2 + |\lambda_{k-1}C_{m,m'}^{\lambda}[k,n] - |\lambda|_kC_{m,m'}^{\lambda}[k-1,n] + (1 + m)|\lambda|_k|\lambda|_{k-1}$$

$$= \sum_{k=0}^{n-1} |\lambda_k|^2 + C_{m,m'}^{\lambda}[k,n] (|\lambda|_{k-1} - |\lambda|_{k+1}) + (1 + m)|\lambda|_k|\lambda|_{k-1}.$$

Now suppose that $n = 1$ so that $C$ is a football. Extracting the edge terms from equation (8), we get

$$SE_\lambda(e) = |\lambda| (1 + h^0(N_{C/X})) + \sum_{A,B \in \lambda} \chi(O_C(A - B) \otimes \lambda_{-1}N_{C/X}).$$

Since $C$ is a football, and $\lambda_{-1}N_{C/X}$ has rank and degree zero, it is trivial in $K$-theory and so the term in the sum is zero. Thus we compute (mod 2):

$$SE_\lambda(e) = |\lambda| (1 + \bar{h}^0(O_{C}(D) \oplus O_{C}(D')))$$

$$= |\lambda| (1 + \bar{h}^0(O_{C}(D)) + \bar{h}^1(O_{C}(-D' + K)))$$

$$= |\lambda| (1 + \chi(O_{C}(D)))$$

$$= |\lambda| (\tilde{m} + \delta_0 + \delta'_0).$$

The vertex term simplifies as follows. Writing $\lambda_i = \lambda(e_i(v))$ and using the facts that $\lambda_{-1}N_{p(v)/X} = 0$ and $\lambda_{-1}N_{C(e_i)/X} = 0$ if $i = 1$ or 2, the vertex
terms from equation (8) simplify to become
\[SV_{\pi} = |\pi| + \sum_{A \in \lambda_3} \sum_{B \in \pi} \xi_{\pi}(B) h^0 (O_p(A - B + D_1) + O_p(A - B + D_2))
+ \sum_{i \neq j} \sum_{A \in \lambda_i} \sum_{B \in \lambda_j} h^0 (O_p(A - B + D_j))
= \sum_{k=0}^{n-1} |\pi|_k (|\lambda_3|_{k-1} + |\lambda_3|_{k+1})
+ \sum_{k=0}^{n-1} |\lambda_3|_k (|\lambda_1|_k + |\lambda_2|_k + |\lambda_1|_{k-1} + |\lambda_2|_{k+1}).
\]

The above computations yield the following theorem.

**Theorem 25.** Let \(\mathcal{X}\) be a orbifold toric CY3 with transverse \(A_{n-1}\) orbifold structure. Then the sign formula in theorem 21 simplifies as follows
\[s(\text{Ext}^0(I, I)) = \chi(O_Y) + \sum_{e \in \text{Edges}} \text{SE}_{\lambda}(e) + \sum_{v \in \text{Vertices}} \text{SV}_{\pi(v)}(v)\]
where
\[\text{SE}_{\lambda}(e) = \sum_{k=0}^{n-1} C_{m,m'}^{\lambda} [k,n] (|\lambda|_{k-1} - |\lambda|_{k+1}) + |\lambda|_k (1 + (1 + m)|\lambda|_{k-1})\]
if \(n = n(e) > 1\),
\[\text{SE}_{\lambda}(e) = |\lambda| (\tilde{m} + \delta_0 + \delta_{\infty})\]
if \(n = 1\), and
\[SV_{\pi} = \sum_{k=0}^{n-1} |\pi|_k (|\lambda_3|_{k-1} + |\lambda_3|_{k+1})
+ \sum_{k=0}^{n-1} |\lambda_3|_k (|\lambda_1|_k + |\lambda_2|_k + |\lambda_1|_{k-1} + |\lambda_2|_{k+1}).\]

Theorem 10, our vertex formula for \(DT(\mathcal{X})\) in the transverse \(A_{n-1}\) case is now easily proved. By Lemma 13 and [3, Theorem 3.4], the partition function is given by a signed sum over edge assignments and compatible 3D partitions at the vertices. Using Proposition 4, Lemma 15, and Proposition 5, the variable associated to each term in the sum is assigned. Finally, the sign of each term is determined by Theorem 25: the \(\chi(O_Y)\) term is accounted for by adding a sign to the \(q\) variable and all the \(q_{e,0}\) variables. The \(\text{SE}_{\lambda}(e)\) term is accounted for by the \(\text{SE}_{\lambda}(e)\) term in the formula, the first term in \(SV_{\pi}(v)\) is accounted for by changing the signs on the vertex variables as
7. Proof of Theorem 12

The proof of Theorem 12 involves some intricate combinatorics, and thus we have broken it into several subsections.

7.1. Review of vertex operators.

Let \( \lambda \subset \mathbb{Z}_{\geq 0}^2 \) be a partition (considered as a Young diagram). The rows or parts of \( \lambda \) are the integers \( \lambda_j = \min \{ i \mid (i, j) \not\in \lambda \} \), for \( j \geq 0 \). Let \( \lambda \) and \( \mu \) be two partitions. We write \( \lambda \succ \mu \) if

\[
\lambda_0 \geq \mu_0 \geq \lambda_1 \geq \mu_1 \geq \cdots
\]

and note that \( \lambda \succ \mu \) if and only if as diagrams \( \mu \subset \lambda \) and \( \lambda \) and \( \mu \) are two adjacent diagonal slices in some 3D partition (see for example [25, §3]).

Fix \( n \), and let \( q_0, \ldots, q_{n-1} \) be indeterminates. Let \( R \) be the ring of formal Laurent series in \( q_0, \ldots, q_{n-1} \). Let \( \mathcal{P} \) be the set of all Young diagrams, and let \( R\mathcal{P} \) be the free \( R \) module generated by elements of \( \mathcal{P} \).

We define two types of operators on \( R\mathcal{P} \) in terms of their action upon an element of \( \mathcal{P} \).

**Definition 26.** Let \( x \) be a monomial in \( q_0, \ldots, q_{n-1} \). Then

\[
\Gamma_+(x)\lambda \overset{\text{def}}{=} \sum_{\mu \prec \lambda} x^{||\lambda|-|\mu||} \mu
\]

\[
\Gamma_-(x)\lambda \overset{\text{def}}{=} \sum_{\mu \succ \lambda} x^{||\mu|-|\lambda||} \mu
\]

\[
Q_i \lambda = q_i^{||\lambda||} \lambda \quad (0 \leq i \leq n-1)
\]

We will sometimes use the following shorthand notation:

\[
\Gamma_+ (x) = \Gamma_+(x) \quad \Gamma_- (x) = \Gamma_-(x) \quad Q = Q_0Q_1 \cdots Q_{n-1}
\]

**Lemma 27.** Let \( \{x_i \mid i \in \mathbb{Z}_{\geq 0}\} \) be monomials in \( q_0, \ldots, q_{n-1} \). Then

\[
\left\langle \mu \left| \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_- (x_i) \right| \lambda' \right\rangle = s_{\mu/\lambda'}(x_0, x_1, x_2, \ldots),
\]

\[
\left\langle \mu \left| \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_+ (x_i) \right| \lambda' \right\rangle = s_{\lambda'/\mu}(x_0, x_1, x_2, \ldots).
\]
Proof. By elementary properties of Schur functions, this reduces immediately to the case where $x_i = 0$ for $i > 1$ — which in turn follows from the semistandard Young tableau definition of the Schur function [27, Definition 7.10.1]. □

Corollary 28. Let $\{x_i\}, \{y_i\}$ be monomials in $q_0, \ldots, q_{n-1}$. Then

$$\langle \mu | \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_-(x_i) \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_+(y_i) | \lambda' \rangle = \sum_{\eta} s_{\mu/\eta}(\{x_i\}) s_{\lambda'/\eta}(\{y_i\}).$$

Proof. If $\eta$ is a partition, then let $\delta_{\eta}$ be the projection operator onto the space spanned by $|\eta\rangle$. Then

$$\langle \mu | \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_-(x_i) \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_+(y_i) | \lambda' \rangle = \sum_{\eta} \langle \mu | \left( \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_-(x_i) \right) \delta_{\eta} \left( \prod_{i \in \mathbb{Z}_{\geq 0}} \Gamma_+(y_i) \right) | \lambda' \rangle = \sum_{\eta} s_{\mu/\eta}(\{x_i\}) s_{\lambda'/\eta}(\{y_i\}).$$

□

It follows from Lemma [27] that these $\Gamma_{\pm}$ are the same vertex operators as used in [23, 24, 25]. We therefore have

Lemma 29. Let $\sigma, \tau = \pm 1$ and let $a, b$ be monomials in $q_0, \ldots, q_{n-1}$. Then

$$\Gamma_{\sigma}(a) \Gamma_{\tau}(b) = (1 - ab)^{-\tau+\sigma} \Gamma_{\tau}(b) \Gamma_{\sigma}(a).$$

Proof. The identity is derived by expressing $\Gamma_+$ and $\Gamma_-$ as the exponential of another operator, and then applying the Campbell-Baker-Hausdorff theorem. This is done for the case $\sigma = -\tau$ in [4, Lemma 31] and the other cases are essentially the same. □

Lemma 30. Let $z$ be a monomial in $q_0, \ldots, q_{n-1}$. Then

$$\Gamma_{\sigma}(z) Q_i = Q_i \Gamma_{\sigma}(z q_i^{\sigma}).$$
Proof. It is easy to check that, for any partitions $\lambda, \mu$,

$$
\langle \lambda | \Gamma_+ (z) Q_i | \mu \rangle = \langle \lambda | Q_i \Gamma_+ (z q_i^{+1}) | \mu \rangle = \begin{cases} 
  z^{\nu_0} q_i^{\mu_0}, & \lambda \subseteq \mu \\
  0, & \lambda \not\subseteq \mu,
\end{cases}
$$

$$
\langle \lambda | \Gamma_- (z) Q_i | \mu \rangle = \langle \lambda | Q_i \Gamma_- (z q_i^{-1}) | \mu \rangle = \begin{cases} 
  z^{\nu_0} q_i^{\mu_0}, & \mu \subseteq \lambda \\
  0, & \mu \not\subseteq \lambda.
\end{cases}
$$

\[\square\]

We must also establish some notation for the edge sequence of the partition $\nu$. Define the set $S(\nu)$ by

$$
S(\nu) = \{ \nu_j - j - 1 \mid j \geq 0 \}.
$$

We define the edge sequence of $\nu$: for $t \in \mathbb{Z}$,

$$
\nu(t) = \begin{cases} 
  +1 & \text{if } t \in S(\nu), \\
  -1 & \text{if } t \not\in S(\nu).
\end{cases}
$$

For example

$$
S(\emptyset) = \{-1, -2, -3, \ldots \}, \quad \emptyset(t) = \begin{cases} 
  +1 & t < 0, \\
  -1 & t \geq 0.
\end{cases}
$$

Note that the complement of $S(\nu)$ is given by

$$
S(\nu)^c = -S(\nu) - 1 = \{-\nu_j' + j \mid j \geq 0 \}.
$$

We use the following shorthand:

**Definition 31.** If $\alpha$ and $\beta$ are partitions, and $\sigma = \pm 1$, we write $\alpha \prec \succ \beta$ to mean

$$
\begin{cases} 
  \alpha \prec \beta & \text{if } \sigma = +1, \\
  \alpha \succ \beta & \text{if } \sigma = -1.
\end{cases}
$$

7.2. Writing $V_{\lambda\mu\nu}(q_0, \ldots, q_{n-1})$ as a vertex operator product.
Recall the following notation:

\[ A_\lambda(k, n) = \sum_{(i,j) \in \lambda} \left\lfloor \frac{i + k}{n} \right\rfloor, \]
\[ q^{-A_\lambda} = \prod_{k=0}^{n-1} q_k^{-A_\lambda(k, n)}, \]
\[ q_t = q^{-N} \prod_{k=-nN}^{-nN+1} q_k \quad \text{for large } N, \]
\[ q = q_0 \cdots q_{n-1}. \]

Recall also that an overline denotes the exchange of variables \( q_k \leftrightarrow q_{-k} \) with subscripts in \( \mathbb{Z}_n \).

We will apply the following conventions for products of possibly non-commuting operators. For operators \( \Phi_t \) depending on \( t \in S \subset \mathbb{Z} \) we let

\[ \overrightarrow{\prod}_{t \in S} \Phi_t \]

denote the product where \( t \) increases from left to right in the order the operators are written. We denote the retrograde expression as

\[ \overleftarrow{\prod}_{t \in S} \Phi_t. \]

**Proposition 8.** The orbifold vertex is given by the following vertex operator expression:

\[ V^n_{\lambda\mu\nu} = q^{-A_\lambda} q^{-A_{\mu'}} q_0^{-|\lambda|} \left\langle \mu \left| \overrightarrow{\prod}_t \Gamma_{\nu'(t)} \left( q_{-\nu'(t)} \right) \right| \lambda' \right\rangle. \]

**Proof.** We first make a slight refinement to the definition of \( V^n_{\lambda\mu\nu} \), as follows: Fix an integer \( N \), and set

\[ V^{n,N}_{\lambda\mu\nu} = \sum_{\pi} q_0^{\pi_0} \cdots q_{n-1}^{\pi_{n-1}} \]

where the sum is now taken over all 3D partitions \( \pi \) asymptotic to \((\lambda, \mu, \nu)\) such that any boxes \((i, j, k)\) not contained in the \( \lambda \)-leg or the \( \mu \)-leg satisfy \( i < nN \), \( j < nN \). It is clear that

\[ \lim_{N \to \infty} V^{n,N}_{\lambda\mu\nu} = V^n_{\lambda\mu\nu} \]

in the sense that the low order terms of \( V^{n,N}_{\lambda\mu\nu} \) and \( V^n_{\lambda\mu\nu} \) agree.
Following the strategy of [23, 4], we will calculate this generating function as a matrix coefficient in a product of vertex operators. The simplest case, \( \lambda = \mu = \nu = \emptyset \), is done in full detail in [4]. The case \( n = 1 \) but with \( \lambda, \mu, \nu \) arbitrary is handled in [23].

Consider, as a first approximation to \( V_{\lambda, \mu, \nu}^{n, N} \), the expression

\[
\langle \mu | \prod_{-nN+1 \leq t \leq nN-1} Q_t \Gamma_{\nu'(t)}(1) | \lambda' \rangle
\]

Observe that, for each \( t \),

\[
Q_t \Gamma_{\nu'(t)}(1) | \gamma \rangle = \sum_{\eta \prec \gamma} q_t^{\eta} | \eta \rangle.
\]

So, in other words, \( Q_t \Gamma_{\nu'(t)}(1) \) sends a partition \( | \gamma \rangle \) to a weighted formal sum of all partitions \( | \eta \rangle \) such that \( \gamma \) and \( \eta \) are the \( (t+1) \)st and \( t \)th slices, respectively, in a 3D partition. In this sum, each \( \eta \) is weighted by \( q_t^{\eta} \). Since \( Q_t \Gamma_{\nu'(t)}(1) \) is a linear operator, this property extends to linear combinations of such \( | \gamma \rangle \), so

\[
Q_{t+1} \Gamma_{\nu'(t+1)}(1) Q_t \Gamma_{\nu'(t)}(1) | \gamma \rangle = \sum_{\alpha \prec \gamma} Q_{t+1}^{\alpha} Q_t^{\beta} \langle \alpha | \gamma \rangle,
\]

and so forth. Since the indices on the \( Q_i \) operators are taken modulo \( n \), the \( \langle \mu \rangle \) coordinate of

\[
\prod_{-nN+1 \leq t \leq nN-1} Q_t \Gamma_{\nu'(t)}(1) | \lambda' \rangle
\]

counts sequences of \( \mathbb{Z}_n \)-weighted Young diagrams, interlacing according to \( \nu' \), beginning with \( \lambda' \) and ending with \( \mu \), as does \( V_{\lambda, \mu, \nu}^{n, N} \). However, there are two important differences between \( V_{\lambda, \mu, \nu}^{n, N} \) and (10).

First, the contribution of the box \((i, j, k)\) in the 3D partition \( \pi \) to \( V_{\lambda, \mu, \nu}^{n, N} \) is \( q_{i-j}^{\xi_{\pi}(i,j,k)} \), where recall that

\[
\xi_{\pi}(i, j, k) = 1 - \# \text{ of legs of } \pi \text{ containing } (i, j, k).
\]

By contrast, (10) assigns weight \( q_{i-j}^{\xi'_{\pi}(i,j,k)} \) where

\[
\xi'_{\pi}(i, j, k) = \begin{cases} 1 & (i, j, k) \in \pi \setminus \{ \nu \text{ leg} \}, \\ 0 & \text{otherwise}. \end{cases}
\]

The easiest way to see that we must use the edge sequence associated to \( \nu' \) and not \( \nu \) is to look at Figure 7 and note that the \( i \) and \( j \) axes not in the standard order so that we are looking at \( \nu \) “from the bottom” and hence getting \( \nu' \).
See Figure 2 for a comparison of $\xi$ and $\xi'$.

To account for this difference, we divide (10) by the weight of the $\lambda$ and $\mu$ legs, $(q_0 \cdots q_{n-1})^{N(|\lambda|+|\mu|)}$. This may be achieved with the $Q$ operators:

$$\langle \mu | Q^{-N} \prod_{-nN+1 \leq t \leq nN-1} Q_t \Gamma_{\nu'(t)}(1) Q^{-N} | \lambda' \rangle$$

Using the commutation relations of Lemma 30, we move the operators $Q_t$ to the left if $t \leq 0$, and to the right if $t > 0$, giving

(11) $$\langle \mu | \left( \prod_{-nN+1 \leq t \leq nN-1} \Gamma_{\nu'(t)}(q_t^{-\nu'(t)}) \right) Q^{-1}_0 | \lambda' \rangle.$$  

The second difference between (11) and $V_{\lambda \mu \nu}^{n,N}$ is that each partition in (10) has a contribution from the boxes which lie inside the $\lambda$ or $\mu$-leg, outside the region $i, j \leq n$, and inside the region $|i - j| \leq n$; these are the regions in Figure 3 at the left and right sides of the first picture, whose projections to the $xy$ plane are triangular, and whose cross-sections, when viewed from the left, are $\lambda$ and $\mu'$.

The weights contributed by these regions are $q^{A_{\lambda'}}$ and $q^{A_{\mu'}}$, as explained in Lemma 32 below. In the non-orbifold case, [23] refers to these constants as framing factors. The terms from the corresponding partitions in $V_{\lambda \mu \nu}^{n,N}$ do not have this contribution.

At this point we have nearly proven the proposition. We have

$$V_{\lambda \mu \nu}^{n,N} = q^{-A_{\lambda}} q^{-A_{\mu'}} q_0^{-|\lambda|} \langle \mu | \prod_{-nN+1 \leq t \leq nN-1} \Gamma_{\nu'(t)}(q_t^{-\nu'(t)}) | \lambda' \rangle + \text{error}$$
Figure 3. A 3D partition which fits within an $N \times N \times \infty$ box, compared with the corresponding sequence of $2N + 1$ Young diagrams.

where the expressions in both sides assign the same weight to a 3D partition. All that remains is to understand the “error” term: $V^{\alpha N}_{\lambda \mu \nu}$ and (11), written as formal sums over 3D partitions, are not supported on the same index set. In particular, (11) includes contributions from 3D partitions which have boxes outside of $[0, N] \times [0, N] \times [0, \infty]$ but inside the region $|x - y| < N$. However, the smallest such 3D partition grows without bound as $N$ grows large, so the error term disappears in the large-$N$ limit. □

Lemma 32. Let $L, M \subseteq (\mathbb{Z}_{\geq 0})^3$ be the regions

$L = \{(i, j, k) \mid (j, k) \in \lambda, i > nN - 1, i - j \leq nN - 1\},$

$M = \{(i, j, k) \mid (i, k) \in \mu', j > nN - 1, i - j \geq -nN + 1\}.$

Then

$$\prod_{(i,j,k) \in L} q_{i-j} = q^{A_{\lambda}}, \quad \prod_{(i,j,k) \in M} q_{i-j} = q^{A_{\mu'}}.$$

Proof. Let $L_t$ denote the diagonal slice

$L_t = \{(i, j, k) \in L \mid i - j = t\}.$

Observe that when $t > nN - 1$, $L_t$ is the empty set. Moreover, $L_{nN-1}$ is the largest of the $L_t$; it consists of boxes $(nN - 1 + j, j, k)$ where $(j, k) \in \lambda$ and $j \geq 1$ (see Figure 4). Each of these boxes contributes weight $q_{-1}$ to
\( \prod_L q_{i-j}, \) since the subscripts of \( q \) are taken mod \( n \). Similarly, for \( c > 0 \),

\[
L_{nN-c} = \{ (nN - c + j, j, k) \mid (j, k) \in \lambda, \ j \geq c \}
\]

where each box in \( L_{nN-c} \) has color \(-c\). It follows that

\[
\prod_{(i,j,k) \in L} q_{i-j} = \prod_{c=1}^{\infty} \prod_{(i,j,k) \in L_{nN-c}} q_{-c} = \prod_{m=0}^{\infty} \prod_{c=1}^{n} \prod_{(j,k) \in \lambda} q_{-c} = \prod_{\tilde{c}=1}^{n} \prod_{(j,k) \in \lambda} q_{-\tilde{c}} = \prod_{\tilde{c}=1}^{n} \prod_{(j,k) \in \lambda} q_{\left\lfloor \frac{j-\tilde{c}}{n} \right\rfloor + 1} = \prod_{c=0}^{n-1} \prod_{(j,k) \in \lambda} q_{c \left\lfloor \frac{j+c}{n} \right\rfloor + 1}.
\]

The second line uses the fact that the subscripts of the \( q_i \) are taken modulo \( n \). In the last line we changed variables by \( \tilde{c} \mapsto n - c \). The end result is precisely equal to \( q^{A_{\lambda}} \) as defined in Section 3.4.

Similarly, let

\[
M_t = \{ (i, j, k) \in M \mid i - j = t \}.
\]

When \( t < -nN + 1 \), \( M_t \) is empty; otherwise, for \( c > 0 \),

\[
M_{-nN+c} = \{ (i, nN - c + i, k) \mid (i, k) \in \mu', i \geq c \}.
\]
Figure 4. Computation of the framing factor associated to $\lambda$

\[ \prod_{(i,j,k) \in M} q_{i-j} = \prod_{c=1}^{\infty} \prod_{(i,j,k) \in M_{-nN+c}} q_c \]

\[ = \prod_{m=0}^{\infty} \prod_{\bar{c}=1}^{n} \prod_{i \geq nm + \bar{c}} q_{\bar{c}} \]

\[ = \prod_{\bar{c}=1}^{n} \prod_{(i,k) \in \mu'} q_{\bar{c}}^{\left\lfloor \frac{i - \bar{c}}{n} \right\rfloor + 1} \]

\[ = \prod_{c=0}^{n-1} \prod_{(i,k) \in \mu'} q_{-c}^{\left\lfloor \frac{i + c}{n} \right\rfloor} \]

\[ = q^{A_{\mu'}}. \]

7.3. $n$-quotient, $n$-core, and the retrograde.

7.3.1. Edge sequences and charge. Let $\nu: \mathbb{Z} \to \{\pm 1\}$ be a function satisfying $\nu(t) = -1$ for $t \gg 0$, and $\nu(t) = 1$ for $t \ll 0$. We say that $\nu(t)$ is an edge sequence, and to such a sequence we associate its slope diagram which consists of the graph of a continuous, piecewise linear function having slopes $\pm 1$, such that the slope of the function at $t$ is given by $\nu(t)$ and such the changes in slope occur at half-integers.

The slope diagram associated to a sequence $\nu$ determines a Young diagram and hence a partition. The Young diagram is given by rotating the slope diagram 135 degrees counterclockwise and translating so that the positive $x$ and $y$ axes eventually coincide with the rotated slope diagram. Note
Figure 5. A three-core \( \nu \), viewed as a partition and as a triple of integers \((3, -2, -1)\) summing to zero. Note that \( \nu_0, \nu_1, \nu_2 \) are empty partitions.

That this association is consistent with the edge sequence \( \nu(t) \) associated with a partition \( \nu \) as defined in equation (9). However, there are many edge sequences having the same associated partition. If \( \nu(t) \) is an edge sequence having associated partition \( \nu \), then there exists a unique integer \( c(\nu) \in \mathbb{Z} \) such that

\[
\nu = R^{c(\nu)} \nu
\]

where \( R \) is the right-shift operator, which acts on an edge sequence \( \eta \) by

\[
R\eta(t) = \eta(t - 1).
\]

We call \( c(\nu) \) the charge of \( \nu \). The edge sequence \( \nu(t) \) associated to a partition by equation (9) always has charge zero; we adopt the convention an edge sequence without an underline always has charge zero. The uniqueness of \( c(\nu) \) implies that the map

\[
(\nu(t) \mapsto (\nu, c(\nu(t)))
\]

is a bijection, so we will use these notations interchangeably.
7.3.2. Ribbons, the \( n \)-quotient, and \( n \)-core.

There is an operation known as adding a ribbon to an edge sequence \( \nu \). Fix \( t_1 < t_2 \) with \( \nu(t_1) = 1, \nu(t_2) = -1 \) (there are infinitely many such pairs \( (t_1, t_2) \)). Then construct a new edge sequence \( \rho \) such that

\[
\rho(t) = \begin{cases} 
-1 & t = t_1 \\
+1 & t = t_2 \\
\nu(t) & \text{otherwise} 
\end{cases}
\]

If \( \nu \) and \( \rho \) are the Young diagrams associated to \( \nu \) and \( \rho \), then the set-theoretic difference \( \rho - \nu \) is a connected strip of boxes which contains no \( 2 \times 2 \) region, commonly called a ribbon, border strip or rim hook in the combinatorics literature; we shall use the term to refer to either the strip of boxes or to the endpoints \( (t_1, t_2) \), according to whether we are speaking of Young diagrams or edge sequences. We say that the ribbon is of length \( t_2 - t_1 \) and to lie at position \( t_1 \). It is easy to check that adding a ribbon does not affect the charge of an edge sequence.

Observe that any charge-zero edge sequence can be constructed from \( \emptyset \) by adding ribbons of length 1. This corresponds to adding boxes to a Young diagram in such a way that the result remains a Young diagram.

If \( \nu \) is an edge sequence, we define its associated \( n \)-tuple \( (\nu_0, \ldots, \nu_{n-1}) \) of edge sequences by

\[
\nu_i(t) = \nu(nt + i).
\]

Letting \( (\nu_i, c_i) = \nu_j \) under the bijection \( (12) \), we then define the \( n \)-quotient and the \( n \)-core of \( \nu \) to be \( (\nu_0, \ldots, \nu_{n-1}) \) and \( (c_0, \ldots, c_{n-1}) \) respectively.

The process of passing from an edge sequence to its \( n \)-core and \( n \)-quotient is reversible: there is a unique way to construct an edge sequence \( \nu \) with a prescribed \( n \)-core and \( n \)-quotient. As such, we identify \( \nu \) with its \( n \)-quotient together with its \( n \)-core:

\[
\nu \leftrightarrow ((\nu_0, \ldots, \nu_{n-1}), (c_0, \ldots, c_{n-1})).
\]

If the edge sequence \( \nu \) is charge zero (i.e. it came from a partition), then \( \sum_i c_i = 0 \). Customarily, one only considers \( n \)-cores arising from partitions, and so unless otherwise stated, we will assume that all \( n \)-cores satisfy \( \sum_i c_i = 0 \). One special case is worthy of note. The partition whose \( n \)-quotient is \( c = (c_0, \ldots, c_{n-1}) \) and whose \( n \)-quotient is \( (\emptyset, \ldots, \emptyset) \) is often identified with \( c \), and is customarily also called an \( n \)-core.

Note that adding an \( n \)-hook to \( \nu \) at position \( t \equiv t_0 \) (mod \( n \)) corresponds to adding a 1-hook (i.e. a single box) to \( \nu_{t_0} \), without altering the \( n \)-core, or any of the other \( \nu_i \). It follows that the \( n \)-core of \( \nu \) is the (unique) partition obtained by iteratively removing \( n \)-hooks from \( \nu \) until it is impossible to do so.
Let $R_k$ be the operator which acts on an edge sequence $\nu$ by right-shifting the $k$th component of the associated $n$-tuple of $\nu$:

$$R_k(\nu_0, \nu_1, \ldots, \nu_{n-1}) = (\nu_0, \nu_1, \ldots, R\nu_k, \ldots, \nu_{n-1}).$$

Note that $R_k$ increases the charge of $\nu$ by one. It follows that the operator $R_k R_{k+1}^{-1}$ leaves the charge of $\nu$ unaffected, so it restricts to an operator on partitions and hence defines an operator on $RP$. The effect of $R_k R_{k+1}^{-1}$ is to leave the $n$-quotient of $\nu$ unaffected, while incrementing $c_k$ and decrementing $c_{k+1}$. Moreover, the operators $R_k R_{k+1}^{-1}$ and their inverses, acting on $\emptyset$, are sufficient to generate any $n$-core. Indeed, if $\nu$ is an $n$-core $(c_0, \ldots, c_{n-1})$, then the associated edge sequence is given by

$$\nu = \prod_{i=0}^{n-1} R_i^c \emptyset.$$

**Remark 33.** We can prove statements about partitions inductively, in the following manner. To prove the statement $P(\nu)$:

1. Prove $P(\emptyset)$.
2. Prove that $P(\nu) \Leftrightarrow P(R_k R_{k+1}^{-1} \nu)$ for each $k$.
3. Prove that $P(\nu) \Rightarrow P(\rho)$, where $\rho$ is any partition obtained from $\nu$ by adding a ribbon.

Proving (1) and (2) establishes $P$ for all $n$-core partitions, and then (3) extends the proof to all partitions.

### 7.3.3. Comparison of the operator with its retrograde.

**Proposition 9.** The operator expression appearing in Proposition 8 can be written in terms of its retrograde and a scalar operator, namely

$$\prod_t \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right) = V^n_{\emptyset\emptyset} \cdot O_\nu \cdot \text{Mon}_L^{-1} \cdot \prod_t \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right)$$
Figure 6. Applying $R_0R_1^{-1}$ to a 4-core generates a new 4-core, increasing the weight by $q_1^{-1}q$.

where

$$O_\nu = \prod_{k=0}^{n-1} V_{\emptyset \emptyset \emptyset}^n(q_k, q_{k+1}, \ldots, q_{n+k-1})^{-2|\nu|_k + |\nu|_{k+1} + |\nu|_{k-1}},$$

$$V_{\emptyset \emptyset \emptyset}^n = M(1, q)^n \prod_{0 < a \leq b < n} M(q_a \cdots q_b, q) M(q_a^{-1} \cdots q_b^{-1}, q),$$

$$M(v, q) = \prod_{m=1}^{\infty} \frac{1}{(1 - vq^m)^m},$$

$$\text{Mon}_{\nu'} = (-1)^{|\nu|} \prod_{(j, i) \in \nu'} \prod_{s=0}^{n-1} q_s h^s_{\nu'}(j, i), \text{ and}$$

$$h^s_{\nu'}(j, i) = \text{the number of boxes of color } s \text{ in the } (j, i)-\text{hook of } \nu'.$$

Proof. Replacing the product with its retrograde has the effect of reversing the order of every pair of operators $\Gamma_{\nu'(t)}(q_t^{-\nu'(t)}), \Gamma_{\nu'(t')} (q_{t'}^{-\nu'(t')})$ for $t' > t$. By Lemma 29, this introduces a scalar factor:

$$\prod_{t} \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right) = \prod_{t < t'} (1 - q_t^{-\nu'(t)} q_{t'}^{-\nu'(t')}) \frac{1}{2(\nu'(t') - \nu'(t))} \prod_{t} \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right)$$
so that to prove the Lemma, we must prove
\[ \prod_{t < t'} \left( 1 - q_t^{-\nu(t)} q_{t'}^{-\nu(t')} \right)^{\frac{1}{2}(\nu(t') - \nu(t))} = \text{Mon}_{\nu'}^{-1} \cdot V_{\emptyset \emptyset} \cdot O_{\nu}. \]

We begin by simplifying the right hand side. Let
\[ \text{Hook}_{\nu'} = \left\{ (t, t') \in \mathbb{Z}^2 \mid t < t', \nu(t) = -1, \nu'(t') = 1 \right\}. \]

Observe that \( \text{Hook}_{\nu'} \) is a finite set, and indeed is in bijection with the set of hooks of \( \nu' \), as the ordered pairs \( (t, t') \) represent the ends of the legs of a hook. In turn, each hook of \( \nu' \) corresponds uniquely to some \( (j, i) \in \nu' \) given by the corner of the hook. The product over the hooks then becomes:
\[ \prod_{(t, t') \in \text{Hook}_{\nu'}} \left( 1 - q_t^{-\nu(t)} q_{t'}^{-\nu(t')} \right)^{\frac{1}{2}(\nu(t') - \nu(t))} \]
\[ = \prod_{(t, t') \in \text{Hook}_{\nu'}} (1 - q_t^{+1} q_{t'}^{-1}) \]
\[ = \prod_{(t, t') \in \text{Hook}_{\nu'}} (-q_t^{+1} q_{t'}^{-1}) \prod_{(t, t') \in \text{Hook}_{\nu'}} (1 - q_t^{-1} q_{t'}^{-1}) \]
\[ = (-1)^{|\nu'|} \prod_{(t, t') \in \text{Hook}_{\nu'}} q_t^{+1} \cdots q_{t'}^{+1} \prod_{(t, t') \in \text{Hook}_{\nu'}} (1 - q_t^{-1} q_{t'}^{-1}) \]
\[ = \text{Mon}_{\nu'}^{-1} \prod_{(t, t') \in \text{Hook}_{\nu'}} (1 - q_t^{-1} q_{t'}^{+1}) \]

where the last equality follows from the fact that \( (t + 1, \ldots, t') \) are exactly the set of colors of the boxes in the \( (j, i) \)-hook of \( \nu' \) corresponding to \( (t, t') \).

Using the above, we can then write
\[ \prod_{t < t'} \left( 1 - q_t^{-\nu(t)} q_{t'}^{-\nu(t')} \right)^{\frac{1}{2}(\nu(t') - \nu(t))} = C(\nu') \cdot \text{Mon}_{\nu'}^{-1} \]

where
\[ C(\nu') = \prod_{t < t'} \left( 1 - q_t^{-1} q_{t'}^{+1} \right)^{\frac{1}{2}(\nu(t') - \nu(t))}. \]

We need to prove that \( C(\nu') = V_{\emptyset \emptyset} \cdot O_{\nu} \) and we will do so using the induction strategy described in Remark \(33\).

We first study the base case for this strategy, \( \nu' = \emptyset \).

\[ C(\emptyset) = \left( \prod_{t < 0} q_t^{-1} q_{t'}^{+1} \right)^{-1}. \]
The partition $\rho'$ is obtained from $\nu'$ by adding a length $n$ border strip at time $T = i - j$.

Letting $t = nt_0 + c$ and $t' = nt'_0 + d$ where $c, d \in \{0, \ldots, n - 1\}$ we see that

$$q_{t}^{-1}q_{t'}^{+1} = \begin{cases} 
q_{t'}^{t_0-t_0} \cdot q_{c+1} \cdots q_{d} & \text{if } d > c, \\
q_{t'}^{t_0-t_0} \cdot q_{d+1}^{-1} \cdots q_{c}^{-1} & \text{if } d = c, \\
q_{t'}^{t_0-t_0} \cdot q_{d}^{-1} \cdots q_{c}^{-1} & \text{if } d < c.
\end{cases}$$

Then writing $m = t'_0 - t_0$ we get

$$C(\emptyset) = \prod_{m=1}^{\infty} (1 - q^m)^{-mn} \prod_{0 < a \leq b < n} (1 - q_a \cdots q_b \cdot q^m)^{-m} \cdot \left(1 - q_a^{-1} \cdots q_b^{-1} \cdot q^m\right)^{-m}$$

$$= V_{0000}^n \cdot O_{\emptyset}$$

which proves the base case of the induction.

Observe that adding an $n$-hook to $\nu'$ leaves the quantity

$$-2|\nu'|_k + |\nu'|_{k-1} + |\nu'|_{k+1}$$

invariant, for each $k$: an $n$-border strip contains one box of each of the $n$ colors. As such, $O_{\nu'}$ depends only upon the $n$-core of $\nu'$. We will show that $C(\nu')$ also depends only upon the $n$-core of $\nu'$, which lets us reduce to the case where $\nu'$ itself is an $n$-core partition.

To do this, let $\rho'$ be a partition obtained by adding an $n$-border strip to $\nu'$ at position $T$ (see Figure 7). In particular, this means that

$$\rho'(T) = -1, \quad \nu'(T) = +1,$$

$$\rho'(T + n) = +1, \quad \nu'(T + n) = -1.$$

We will show that $C(\rho')/C(\nu') = 1$. First, it is helpful to rewrite $C(\nu')$ as follows:
\[ C(\nu') = \prod_{k \geq 0} \prod_{t \in \mathbb{Z}} \left( 1 - q_{t+k}^{-1} q_t^{-1} \right)^{\frac{1}{2}(t(t+1) - \nu(t))}. \]

Let

\[ K(t, k) = \frac{1}{2} \left[ (\rho(t+k) - \nu(t+k)) - (\rho(t) - \nu(t)) \right] \]

so that

\[ \frac{C'(\rho)}{C'(\nu)} = \prod_{k \geq 0} \prod_{t \in \mathbb{Z}} \left( 1 - q_{t+k}^{-1} q_t^{-1} \right)^{K(t, k)}. \]

Observe that \( K(t, k) = 0 \) unless \( \rho'(t) \neq \nu'(t) \) or \( \rho'(t+k) \neq \nu'(t+k) \). Moreover, the edge sequences of \( \rho' \) and \( \nu' \) differ only at \( T \) and at \( T+n \), since \( \rho' \) is the result of adding an \( n \)-border strip at position \( T \) to \( \nu' \). Therefore,

\[
\frac{C(\rho')}{C(\nu')} = \prod_{k \geq 0} \left( 1 - q_{T+k}^{-1} q_T^{-1} \right)^{K(T, k)} \prod_{k \geq 0} \left( 1 - q_{T+n+k}^{-1} q_{T+n}^{-1} \right)^{K(T+n, k)} \\
\cdot \prod_{k \geq 0} \left( 1 - q_{T-k}^{-1} q_{T-k}^{-1} \right)^{K(T-k, k)} \prod_{k \geq 0} \left( 1 - q_{T+n-k}^{-1} q_{T+n-k}^{-1} \right)^{K(T+n-k, k)} \\
= \prod_{k \geq 0} \left( 1 - q_{T+k}^{-1} q_T^{-1} \right)^{K(T, k)+K(T+n, k)} \\
\cdot \prod_{k \geq 0} \left( 1 - q_{T-k}^{-1} q_{T-k}^{-1} \right)^{K(T-k, k)+K(T+n-k, k)}. 
\]

We next examine the quantity \( K(T, k) + K(T + n, k) \). Consider first the case \( k \neq n \). In this case we have \( \rho(T+k) = \nu(T+k) \), \( \rho(T+k-n) = \nu(T+k-n) \), so

\[
2(K(T, k) + K(T + n, k)) = \rho'(T) - \nu'(T) + \rho'(T + n) - \nu'(T + n) \\
= 0 
\]

because \( \rho'(T) = -\rho'(T + n) \), \( \nu'(T) = -\nu'(T + n) \). As such, all terms other than possibly those where \( k = n \) cancel from the product, so

\[
\frac{C(\rho')}{C(\nu')} = (1 - q_{T+n}^{-1} q_T^{-1})^{2K(T,n)+K(T+n,n)} (1 - q_{T-n}^{-1} q_{T-n}^{-1})^{K(T-n,n)+K(T,n)} \\
= (1 - q)^{2K(T,n)+K(T+n,n)+K(T-n,n)}. 
\]
All of the terms in the exponent can now be computed explicitly, since they involve only known quantities:

\[ K(T + n, n) = \frac{1}{2}(-\rho'(T + n) + \nu'(T + n)) = -1, \]
\[ K(T, n) = \frac{1}{2}((\rho'(T + n) - \nu'(T + n)) - (\rho'(T) - \nu'(T))) = 2, \]
\[ K(T - n, n) = \frac{1}{2}(-\rho'(T + n) + \nu'(T + n)) = 1. \]

Thus we have \( C(\rho')/C(\nu') = 1 \). This means that adding an \( n \)-border strip to \( \nu' \) does not affect \( C(\nu') \), and as such \( C(\rho') \) depends only upon the \( n \)-core of \( \nu' \).

Thus to finish the proof of proposition 9 using the induction argument outlined in Remark 33, it remains only to prove the following lemma.

**Lemma 34.** Let \( \nu' \) be an \( n \)-core and let \( \rho' = R_k R_{k+1}^{-1} \nu' \), then

\[ \frac{C(\rho')}{C(\nu')} = \frac{C_{\rho}}{C_{\nu}}. \]

We prove the lemma by direct computation. To streamline the notation we will drop the primes from \( \nu' \) and \( \rho' \).

We define \( T_k \) to be the operator which cyclically permutes the variables by \( k \):

\[ (T_k F)(q_0, \ldots, q_{n-1}) = F(q_k, \ldots, q_{k+n-1}). \]

Note that it follows immediately from equation (13) that

\[ C(R^k \nu) = T_k C(\nu). \]

We begin with a computation:

\[
\frac{V_n}{T_1 V_0} = \prod_{0 < a \leq b < n} \frac{M(q_a \cdots q_b, q) M(q_{a+1}^{-1} \cdots q_{b+1}^{-1}, q)}{M(q_{a+1} \cdots q_{b+1}, q) M(q_{a+1}^{-1} \cdots q_{b+1}^{-1}, q)} = \frac{\prod_{m=1}^{n-1} \prod_{a=1}^{n-1} (1 - q_a \cdots q_{b+1} q^m) (1 - q_{a+1} \cdots q_{b+1}^{-1} q^{-m})}{\prod_{b=1}^{n-1} \prod_{1}^{n-1} (1 - q_1 \cdots q_b q^m) (1 - q_1^{-1} \cdots q_b^{-1} q^{-m})} = \frac{\prod_{m=1}^{n-1} \prod_{c=1}^{n-1} (1 - q_c \cdots q_{c+1} q^m) (1 - q_c q^{-m})}{\prod_{m=1}^{n-1} \prod_{c=1}^{n-1} (1 - q_{c+1} \cdots q_{c} q^{-m})},
\]
In the above, the equality from the second to the third line is because all the terms cancel except for those in the numerator with \((a, b) = (a, n - 1)\) and those in the denominator with \((a, b) = (1, b)\). The equality from the fourth to the last line uses the reindexing \(m \mapsto m - 1\) on the first terms in the numerator and denominator.

We now wish to compare \(C(\nu)\) to \(C(R_0 \nu)\). Since \(\nu = (c_0, \ldots, c_{n-1})\) is an \(n\)-core, we have that

\[
\nu(cn) = \begin{cases} 
+1 & c < c_0, \\
-1 & c \geq c_0.
\end{cases}
\]

Thus \(R_0 \nu = (c_0 + 1, c_1, \ldots, c_{n-1})\) differs from \(\nu\) (as an edge sequence) only at \(t = c_0 n\) where we have

\[
(R_0 \nu)(c_0 n) = 1 \\
\nu(cn) = -1.
\]

Thus

\[
\frac{C(\nu)}{C(R_0 \nu)} = \prod_{t < c_0 n} (1 - q_t^{-1} q_{cn}) \frac{1}{2}(-1-\nu(t)) - \frac{1}{2}(1-\nu(t)) \\
\cdot \prod_{c_0 n < t} (1 - q_t^{-1} q_{cn}) \frac{1}{2}(\nu(t)+1) - \frac{1}{2}(\nu(t)-1) \\
= \prod_{t < c_0 n} (1 - q_t^{-1} q_{cn})^{-1} \cdot \prod_{c_0 n < t} (1 - q_t^{-1} q_{cn}) + 1.
\]

In the above expression, we can rewrite the product over \(t > c_0 n\) as a product over \(m = 1, 2, \ldots\) and \(a = 1, \ldots, n\) by setting \(t = (c_0 + m-1)n + a\) so that

\[
q_{c_0 n}^{-1} q_t = q_{c_0 n+1} \cdots q_{(c_0 + m-1)n + a} = q_1 \cdots q_a \cdot q_1^{m-1}.
\]

Similarly, we can rewrite the product over \(t < c_0 n\) as a product over \(m = 1, 2, \ldots\) and \(a = n - 1, n - 2, \ldots, 0\) by \(t = n(c_0 - m) + a\) so that

\[
q_t^{-1} q_{cn} = q_{n(c_0 - m)+a+1} \cdots q_{cn} = q_{a+1} \cdots q_n \cdot q_a^{m-1}.
\]

Thus

\[
\frac{C(\nu)}{C(R_0 \nu)} = \prod_{m=1}^{\infty} \frac{\prod_{a=1}^{n-1} (1 - q_1 \cdots q_a q^{m-1})}{\prod_{a=0}^{n-1} (1 - q_a \cdots q_n q^{m-1})} \\
= \prod_{m=1}^{\infty} \prod_{a=1}^{n-1} \frac{(1 - q_1 \cdots q_a q^{m-1})}{(1 - q_a \cdots q_n q^{m-1})} \\
= \frac{V^n_{000}}{T^1 V^n_{000}}.
\]
and so we have shown
\[ C(R_0 \nu) = \frac{T_1 V^n_{000}}{V^n_{000}} \cdot C(\nu) \]
for any edge sequence \( \nu \) with empty \( n \)-quotient.

The operator \( R_k \) can be obtained from \( R_0 \) by conjugating with \( R^k \):
\[ R_k = R^k R_0 R^{-k} \]
and thus
\[ R_k R_{k+1}^{-1} = R^k R_0 R^1 R_{k+1}^{-1}. \]

We now compute using equations (14) and (15):
\[
\begin{align*}
C(R_k R_{k+1}^{-1} \nu) &= C(R^k R_0 R^1 R_{k+1}^{-1} \nu) \\
&= T_k C(R_0(R_0^{-1} R_{k+1}^{-1} \nu)) \\
&= T_k \left( \frac{T_1 V^n_{000}}{V^n_{000}} \cdot C(R_0^{-1} R_{k+1}^{-1} \nu) \right) \\
&= \frac{T_{k+1} V^n_{000}}{T_k V^n_{000}} \cdot T_{k+1} \left( \frac{V^n_{000}}{T_1 V^n_{000}} \cdot C(R_{k+1}^{-1} \nu) \right) \\
&= \frac{T_{k+1} V^n_{000}}{T_k V^n_{000}} \cdot \frac{T_{k+1} V^n_{000}}{T_{k+2} V^n_{000}} \cdot T_{k+1} T_{k+1} C(\nu).
\end{align*}
\]
and so
\[
\frac{C(R_k R_{k+1}^{-1} \nu)}{C(\nu)} = \frac{(T_{k+1} V^n_{000})^2}{T_k V^n_{000} \cdot T_{k+2} V^n_{000}}.
\]

On the other hand, we have
\[
\frac{O_{R_k R_{k+1}^{-1} \nu}}{O_\nu} = \prod_{i=0}^{n-1} (T_1 V^n_{000})^{\epsilon_i}
\]
where
\[
\epsilon_i = -2 \left( |R_k R_{k+1}^{-1} \nu|_i - |\nu|_i \right) + \left( |R_k R_{k+1}^{-1} \nu|_{i-1} - |\nu|_{i-1} \right) + \left( |R_k R_{k+1}^{-1} \nu|_{i+1} - |\nu|_{i+1} \right).
\]

The operation \( R_k R_{k+1}^{-1} \) adds one box of each color except for \( k + 1 \) to an \( n \)-core \( \nu \) (see figure 6). Therefore
\[
\epsilon_i = -2(1 - \delta_{k+1,l}) + (1 - \delta_{k+1,l-1}) + (1 - \delta_{k+1,l+1})
\]
\[ = +2\delta_{k+1,l} - \delta_{k+1,l-1} - \delta_{k+1,l+1}
\]
and so
\[
\frac{O_{R_k R_{k+1}^{-1} \nu}}{O_\nu} = \frac{(T_{k+1} V^n_{000})^2}{T_k V^n_{000} \cdot T_{k+2} V^n_{000}} = \frac{C(R_k R_{k+1}^{-1} \nu)}{C(\nu)}.
\]
This completes the proof of Lemma \ref{lem:34} and hence of Proposition \ref{prop:9}.

**Proposition 10.**

\[
\left\langle \mu \left| \prod_t \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right) \right| \lambda' \right\rangle = H_{\nu'} \cdot \text{Mon}_{\nu'} \cdot \text{Schur}_{\lambda \mu \nu}
\]

where

\[
H_{\nu'} = \prod_{(j,i) \in \nu'} \frac{1}{1 - \prod_{s=0}^{n-1} q_s h_{\nu'}^s(j,i)},
\]

\[
\text{Mon}_{\nu'} = (-1)^{|\nu|} \prod_{(j,i) \in \nu'} \prod_{s=0}^{n-1} q_s h_{\nu'}^s(j,i), \quad \text{and}
\]

\[
\text{Schur}_{\lambda \mu \nu} = \sum_{\eta} s_{\mu/\eta} \left( q_t^{\nu'(t)=-1} \right) s_{\lambda'/\eta} \left( q_t^{-1}^{\nu'(t)=+1} \right).
\]

**Proof.** We commute the operators so that all the \( \Gamma^+ \)s are on the right and all the \( \Gamma^- \)s are on the left. Using the commutation relations we obtain

\[
\prod_t \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right) = \prod_{t' > t} \frac{1}{1 - q_{t'} q_t^{-1}} \prod_{\nu'(t)=-1} \Gamma_-(q_t) \prod_{\nu'(t)=+1} \Gamma_+(q_t^{-1}).
\]

Observe that \( \nu'(t) = -1, \nu'(t') = 1 \) for \( t < t' \) if and only if there is a hook of \( \nu' \) with endpoints at \( t, t' \). Moreover, we can rewrite the monomials appearing in the scalar factor above as follows:

\[
q_t^{-1} q_t^{1} = q_{t+1}^{-1} \cdot q_{t+2}^{-1} \cdots q_{t'}^{-1} = \prod_{s=0}^{n-1} q_s^{h_{\nu'}^s(j,i)}
\]

where the hook corresponding to \( (t', t) \) has corner \( (j, i) \in \nu' \) and \( h_{\nu'}^s(j, i) \) is the number of boxes of color \( s \) in the hook. Clearing the denominators of inverses, we find that the scalar factor is exactly equal to

\[
H_{\nu'} \cdot \text{Mon}_{\nu'}.
\]

The equality

\[
\left\langle \mu \left| \prod_{\nu'(t)=+1} \Gamma_+(q_t^{-1}) \prod_{\nu'(t)=-1} \Gamma_-(q_t) \right| \lambda' \right\rangle = \text{Schur}_{\lambda \mu \nu}
\]

follows immediately from Corollary \ref{cor:28} and the lemma is proved.
We can now put it all together and complete the proof of Theorem 12. Combining Propositions 8 and 9 we get

\[ V^n_{\lambda \mu \nu} = V^n_{\emptyset \emptyset \emptyset} \cdot q^{-A_\lambda} \cdot O_\nu \cdot q_0^{-|\lambda|} \cdot \left( \prod_{\ell} \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right) \right)^\lambda \cdot \langle \left( \prod_{\ell} \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right) \right)^\lambda \rangle. \]

Applying Proposition 10 and using homogeneity of Schur functions, we get

\[ \left( \prod_{\ell} \Gamma_{\nu'(t)} \left( q_t^{-\nu'(t)} \right) \right)^\lambda = q_0^{-|\eta|} s_{\mu/\eta}(\left| q_t^{\nu'(t)} \right| = -1) s_{\lambda'/\eta}(\left| q_0^{-1} \cdot q_t^{-1} \right|_{\nu'(t) = +1}). \]

Finally, using

\[ S(\nu') = -S(\nu')^c - 1, \quad q_0^{-1} \cdot q_t^{-1} = q_{-1-t}, \]

we observe the following equalities of sets:

\[ \{ q_t : \nu'(t) = -1 \} = \{ q_t : t \in S(\nu')^c \} = q_{\bullet - \nu'} \]

\[ \{ q_0^{-1} \cdot q_t^{-1} : \nu'(t) = +1 \} = \{ q_{-1-t} : t \in S(\nu') \} = \{ q_{-1-t} : t \in -S(\nu')^c - 1 \} = \{ q_T : T \in S(\nu')^c \} = q_{\bullet - \nu'} \]

which, when substituted into equation (16), completes the proof of Theorem 12. □

**Appendix A. Grothendieck-Riemann-Roch for Orbifolds and the Toen Operator.**

We briefly review Grothendieck-Riemann-Rock for Deligne-Mumford stacks and we work out some examples needed in the paper. The basic reference is [29]; see also [30, Appendix A].

Let \( \mathcal{X} \) be a smooth Deligne-Mumford stack. Let \( I\mathcal{X} \) be the inertia stack of \( \mathcal{X} \). The objects of \( I\mathcal{X} \) are pairs \((x, g)\) where \( x \) is an object of \( \mathcal{X} \) and \( g \) is an automorphism of \( x \). There is a local immersion

\[ \pi : I\mathcal{X} \to \mathcal{X} \]

which forgets \( g \).
Let $E$ be a vector bundle on $I\mathcal{X}$. There is a canonical automorphism\footnote{induced by the canonical 2-morphism $\pi \Rightarrow \pi$ given by $(x, g) \mapsto g$.} of $E$ and consequently there is a decomposition

$$E = \bigoplus \omega E^\omega$$

where the sum is over roots of unity $\omega \in \mathbb{C}$ and the canonical automorphism acts by multiplication by $\omega$ on $E^\omega$.

We define an endomorphism $\rho$ of $K(I\mathcal{X}) \otimes \mathbb{C}$ by

$$\rho(E) = \sum \omega [E^\omega].$$

Let $N$ be the normal bundle to the local immersion $\pi : I\mathcal{X} \rightarrow \mathcal{X}$ and let

$$\lambda_{-1}(N^\vee) = \sum_i (-1)^i \Lambda^i N^\vee \in K(I\mathcal{X}).$$

We define the Toen operator

$$\tau_{\mathcal{X}} : K(\mathcal{X}) \rightarrow A(I\mathcal{X})$$

by

$$\tau_{\mathcal{X}}(E) = \frac{\text{ch}(\rho(\pi^* E))}{\text{ch}(\rho(\lambda_{-1} N^\vee))} \cdot \text{td}(T_{I\mathcal{X}})$$

where $\text{td}(T_{I\mathcal{X}})$ is the Todd class of $I\mathcal{X}$.

Toen’s Grothendieck-Riemann-Roch theorem for stacks asserts \cite{29} 4.10, 4.11 that $\tau$ is functorial with respect to proper pushforwards.

**Theorem 35** (Toen). Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of smooth Deligne-Mumford stacks, then for all $E \in K(\mathcal{X})$,

$$f_*(\tau_{\mathcal{X}}(E)) = \tau_{\mathcal{Y}}(f_* E).$$

In particular, for $f : \mathcal{X} \rightarrow \text{pt}$, we get

$$\chi(E) = \int_{I\mathcal{X}} \tau_{\mathcal{X}}(E).$$

**Example 36.** Let $\mathcal{X} = B\mathbb{Z}_n$, then $I\mathcal{X} = \cup_{i=0}^{n-1} \mathcal{X}^i$ where $\mathcal{X}^i \cong B\mathbb{Z}_n$. If $L_k \rightarrow \mathcal{X}$ is the line bundle determined by the 1-dimensional representation of $\mathbb{Z}_n$ having character $\omega^k$ where $\omega = \exp\left(\frac{2\pi i}{n}\right)$, then the canonical automorphism acts by multiplication with $\omega^{kl}$ on $L_k$ restricted to $\mathcal{X}^l$. Thus

$$\tau_{\mathcal{X}}(L_k)|_{\mathcal{X}^l} = \omega^{kl}.$$
and such that the restriction of $L_{k,m}$ to a point $B\mathbb{Z}_n \in \mathcal{C}$ is the bundle $L_k$ from example 36. Then $IC = \bigcup_{l=0}^{n-1} C_l$ and
\[ \tau(L_{k,m})|_{C_l} = \omega^{kl}(1 + m[pt]). \]

**Example 38.** Let $\mathbb{P}_{a,b}^1$ be the football, i.e. the stack given by root constructions [10] of orders $a$ and $b$ at the points $[0] \in \mathbb{P}^1$ and $[\infty] \in \mathbb{P}^1$ respectively. Let $[pt] \in \mathbb{P}_{a,b}^1$ be a non-stacky point. The following lemma gives a formula for the Euler characteristic of a line bundle on the football.

**Lemma 39.**
\[ \chi \left( \mathcal{O}_{\mathbb{P}_{a,b}^1}(d[pt] + s[0] + t[\infty]) \right) = d + 1 + \left\lfloor \frac{s}{a} \right\rfloor + \left\lfloor \frac{t}{b} \right\rfloor. \]

The inertia stack breaks into components as follows:
\[ I_{\mathbb{P}_{a,b}^1} = \mathbb{P}_{a,b}^1 \bigcup_{k=0}^{a-1} P_k \bigcup_{l=0}^{b-1} Q_l \]
where $P_k \cong B\mathbb{Z}_a$ and $Q_l \cong B\mathbb{Z}_b$. Let $\omega_a = \exp \left( \frac{2\pi i}{a} \right)$ and $\omega_b = \exp \left( \frac{2\pi i}{b} \right)$, then $N_{P_k/\mathbb{P}_{a,b}^1}$ is the line bundle on $B\mathbb{Z}_a$ with character $\omega_a^k$ and $N_{Q_l/\mathbb{P}_{a,b}^1}$ is the line bundle on $B\mathbb{Z}_b$ with character $\omega_b^l$. Therefore
\[ \tau \left( \mathcal{O}_{\mathbb{P}_{a,b}^1}(d[pt] + s[0] + t[\infty]) \right) \bigg|_{\mathbb{P}_{a,b}^1} = (1 + d[pt] + s[0] + t[\infty])(1 + \frac{1}{2}\lfloor [0] + [\infty] \rfloor), \]
\[ \tau \left( \mathcal{O}_{\mathbb{P}_{a,b}^1}(d[pt] + s[0] + t[\infty]) \right) \bigg|_{P_k} = \frac{ch\left( \rho(\mathcal{O}(s[0])|_{P_k}) \right)}{ch\left( \rho\left( 1 - N_{P_k/\mathbb{P}_{a,b}^1} \right) \right)} = \frac{\omega_a^{ks}}{1 - \omega_a^{-k}}, \]
and similarly
\[ \tau \left( \mathcal{O}_{\mathbb{P}_{a,b}^1}(d[pt] + s[0] + t[\infty]) \right) \bigg|_{Q_l} = \frac{\omega_b^{lt}}{1 - \omega_b^{-l}}. \]

Now integrating $\tau \left( \mathcal{O}(d[pt] + s[0] + t[\infty]) \right)$ over $I_{\mathbb{P}_{a,b}^1}$, we get
\[ \chi \left( \mathcal{O}_{\mathbb{P}_{a,b}^1}(d[pt] + s[0] + t[\infty]) \right) = d + \frac{s}{a} + \frac{t}{b} + \frac{1}{2a} + \frac{1}{2b} + \frac{1}{a} \sum_{k=1}^{a-1} \frac{\omega_a^{ks}}{1 - \omega_a^{-k}} + \frac{1}{b} \sum_{l=1}^{b-1} \frac{\omega_b^{lt}}{1 - \omega_b^{-l}}. \]
The lemma then follows from the identity\[4\]
\[
\frac{1}{a} \sum_{k=1}^{a-1} \omega_{a}^{ks} \quad \sum_{k=1}^{a-1} \omega_{a}^{-k} = \left| \frac{s}{a} \right| - \frac{s}{a} + \frac{a - 1}{2a}
\]
and its counterpart for the sum over \(l\).

**APPENDIX B. ORBIFOLD TORIC CY3S AND WEB DIAGRAMS**

A orbifold toric CY3 is a smooth toric Deligne-Mumford stack \(\mathcal{X}\) with generically trivial stabilizers and having trivial canonical bundle.

**Lemma 40.** A orbifold toric CY3 \(\mathcal{X}\) is uniquely determined by its coarse moduli space \(X\).

**Proof:** This follows from the classification result of Fantechi, Mann, and Nironi [13]. They show that if \(\mathcal{X}\) is a smooth Deligne-Mumford toric stack, then the structure morphism to the coarse space factors canonically via toric morphisms

\[
\mathcal{X} \to \mathcal{X}^{rig} \to \mathcal{X}^{can} \to X
\]

where \(\mathcal{X} \to \mathcal{X}^{rig}\) is an Abelian gerbe over \(\mathcal{X}^{rig}\), \(\mathcal{X}^{rig} \to \mathcal{X}^{can}\) is a fibered product of roots of toric divisors, and \(\mathcal{X}^{can} \to X\) is the minimal orbifold having \(X\) as its coarse moduli space. They prove that \(\mathcal{X}^{can}\) is unique and canonically associated to \(X\). Since we assume \(\mathcal{X}\) is an orbifold, we have \(\mathcal{X} = \mathcal{X}^{rig}\). Since we assume \(K_{\mathcal{X}}\) is trivial, the stacky locus in \(\mathcal{X}\) has codimension at least two and hence \(\mathcal{X} = \mathcal{X}^{can}\). □

The combinatorial data determining a toric variety is well understood and is most commonly expressed as the data of a fan (by the above lemma, we do not require the stacky fans of Borisov, Chen and Smith [5]). In the case of an orbifold toric CY3, it is convenient to use equivalent (essentially dual) combinatorial data, namely that of a web diagram.

**Definition 41.** A web diagram consists of the data

- A graph \(\Gamma\) which is trivalent and embedded in the plane. The graph is finite and necessarily has some non-compact edges.
- A marking \(\{x_{v,e}\}\), which consists of a non-zero vector \(x_{v,e} \in \mathbb{Z}^{2}\) for each pair \((v, e)\) where \(e\) is an edge incident to a vertex \(v\).

The data satisfies the following.

\[\text{You can have some fun and try to prove this elementary identity for yourself. If you get stuck, a complete proof can be found at:}\]

\[\text{www.math.ubc.ca/~jbryan/papers/identity.pdf.}\]
For each compact edge \( e \) with bounding vertices \( v \) and \( v' \),
\[
x_{v,e} + x_{v',e} = 0.
\]

For each vertex \( v \) with incident edges \((e_1, e_2, e_3)\),
\[
x_{v,e_1} + x_{v,e_2} + x_{v,e_3} = 0.
\]

Two markings \( \{x_{v,e}\} \) and \( \{x'_{v,e}\} \) are equivalent if there exists \( g \in SL_2(\mathbb{Z}) \) such that \( gx_{e,v} = x'_{v,e} \) for all \((v,e)\).

**Lemma 42.** Every orbifold toric CY3 \( \X \) determines a web diagram \( \Gamma_{\X} \), unique up to equivalence.

**Proof:** By lemma 40, \( \X \) is determined by its coarse space \( X \), a toric variety with Gorenstein finite quotient singularities and trivial canonical bundle. Such an \( X \) determines a simplicial fan \( \Sigma \subset N \otimes \mathbb{Q} \) with \( N \cong \mathbb{Z}^3 \). Since the canonical divisor is trivial, there exists a linear function \( l : N \to \mathbb{Z} \) such that \( l(v_i) = 1 \) for all the generators \( v_i \) of the one dimensional cones of \( \Sigma \). Thus \( \Sigma \) intersects the plane \( \{l = 1\} \) in a triangulation \( \hat{\Gamma} \) having integral vertices. Let \( \Gamma_{\X} = \Gamma \) be the graph dual to \( \hat{\Gamma} \) in the plane \( \{l = 1\} \). We define a marking of \( \Gamma \) as follows. Under duality, a vertex in \( \Gamma \) with incident edge \( e \) corresponds to a triangle \( \hat{\nu} \) in \( \hat{\Gamma} \) and a bounding edge \( \hat{e} \). Fixing an orientation on the plane, the edge \( \hat{e} \) inherits an orientation from the triangle \( \hat{\nu} \). The oriented edge defines an integral vector \( x_{v,e} \) in \( \{l = 0\} \). The set \( \{x_{v,e}\} \) satisfies the conditions of a marking by construction. \( \square \)

**Remark 43.** When we picture the web diagram \( \Gamma \) in relation to the triangulation \( \hat{\Gamma} \), we will use an element of \( SL_2(\mathbb{Z}) \) to rotate the vectors \( x_{v,e} \) counterclockwise by ninety degrees so that the edges of \( \Gamma \) are perpendicular to the edges of \( \hat{\Gamma} \). In Figure 8, we show the web diagrams and the dual fan triangulation for (1) local \( \mathbb{P}^1 \times \mathbb{P}^1 \), namely the total space of the canonical bundle over \( \mathbb{P}^1 \times \mathbb{P}^1 \) and, (2) local \( \mathbb{P}^1 \times B\mathbb{Z}_2 \), namely the orbifold quotient of the resolved conifold \( O(-1) \oplus O(-1) \to \mathbb{P}^1 \) by \( \mathbb{Z}_2 \) acting fiberwise. Note that the coarse space of local \( \mathbb{P}^1 \times B\mathbb{Z}_2 \) has a transverse \( A_1 \) singularity and its unique crepant resolution is given by local \( \mathbb{P}^1 \times \mathbb{P}^1 \).

**Remark 44.** The term web-diagram comes from physics (e.g. [1]). It is essentially the same as the data determining a tropical plane curve [22]. The tropical curve associated to \( \Gamma_{\X} \) may be interpreted as the tropicalization of the curve mirror to \( \X \) [14, § 4].

**Remark 45.** The vertices of \( \Gamma_{\X} \) correspond to torus fixed points in \( \X \), the edges correspond to torus invariant curves, and the regions in the plane delineated by the graph correspond to torus invariant divisors.
Let $v$ be the vertex of $\Gamma_X$, let $(e_1, e_2, e_3)$ be the three edges incident to $v$, and let $x_{v,e_i} = (a_i, b_i)$ be the markings. Then $X$ has an open neighborhood about the torus fixed point corresponding to $v$ given by $[\mathbb{C}^3/G]$ where $G$ is the subgroup of the torus $T = (\mathbb{C}^*)^3$ given by

$$t_1t_2t_3 = 1, \quad t_i^{a_j} = t_j^{a_i}, \quad t_i^{b_j} = t_j^{b_i}. $$

The action of $G$ on $\mathbb{C}^3$ is given by

$$(z_1, z_2, z_3) \mapsto (t_1z_1, t_2z_2, t_3z_3)$$

where the $z_i$ coordinate axis is the $T$ invariant curve corresponding to the edge $e_i$.

**Proof:** The local model is easily read off from the fan (e.g. [6, Eqn. 3]). The lemma is obtained by simply translating the fan data into the web diagram. \qed

For

$$x_i = (a_i, b_i)$$
we define
\[ x_i \wedge x_j = a_i b_j - a_j b_i. \]
We order the edges \((e_1, e_2, e_3)\) cyclically in the counterclockwise direction. Then it follows from the lemma that the order of \(G\) is given by:
\[ |G| = x_1 \wedge x_2 = x_2 \wedge x_3 = x_3 \wedge x_1. \]
Moreover, the order of \(H_i\), the stabilizer group of a generic point on the \(T\) invariant curve corresponding to \(e_i\) is given by
\[ |H_i| = \text{div}(x_i) \]
where \(\text{div}(x_i) = \gcd(a_i, b_i)\) is the divisibility of \(x_i\).

**B.2. Reading off the local data at a curve from the web diagram.** Let \(e\) be a compact edge in the web diagram and let \(C \subset \mathcal{X}\) be the corresponding torus invariant curve. By the Fantechi-Mann-Nironi classification, \(C\) is given by an Abelian gerbe over a football. There is a neighborhood of \(C\) in \(\mathcal{X}\) isomorphic to the total space of the normal bundle of \(C\) in \(\mathcal{X}\). The normal bundle is the sum of two line bundles, so to specify the neighborhood of \(C\) we must determine the two normal bundles. In the case where \(C\) is a scheme, a line bundle is determined by its degree. In general, the line bundles are determined by a slight generalization of the numerical degree, and we explain below how to extract this data from the web-diagram.

**Definition 47.** Let \(\mathbb{P}^1_{k_0, k_{\infty}}\) be the stack obtained from \(\mathbb{P}^1\) by root constructions \([10]\) of order \(k_0\) and \(k_{\infty}\) at the points \([0], [\infty] \in \mathbb{P}^1\) (the so-called “football”). Let
\[ \pi : C \to \mathbb{P}^1_{k_0, k_{\infty}} \]
be a \(\mathbb{Z}_h\) gerbe over the football \(\mathbb{P}^1_{k_0, k_{\infty}}\) and let \(L \to C\) be a line bundle. We define the **type** of \(L\) to be the triple of integers \((a_0, a_{\infty}, m)\) such that
\[ 0 \leq a_0 < k_0, \quad 0 \leq a_{\infty} < k_{\infty}, \]
and
\[ L^\otimes h \cong \pi^* \mathcal{O}_{\mathbb{P}^1_{k_0, k_{\infty}}}(a_0[0] + a_{\infty}[\infty] + m[p]) \]
where \([p] \in \mathbb{P}^1_{k_0, k_{\infty}}\) is a generic point. \(L\) is determined up to isomorphism by its type the **degree** of \(L\) to be
\[ \deg(L) = \frac{1}{h} \left( \frac{a_0}{k_0} + \frac{a_{\infty}}{k_{\infty}} + m \right). \]

The web diagram of \(\mathcal{X}\) near the edge \(e\) is given by the following diagram:
Since the divisibility of \( x_0 \) is \( h \), we may use the action of \( SL_2(\mathbb{Z}) \) to set \( x_1 = (h, 0) \) and thus \( x_\infty = (-h, 0) \). Since the order of the local groups at \( 0 \) and \( \infty \) is \( k_0 \) and \( k_\infty \) respectively, we know that \( x_3 \) and \( x_2 \) have the form

\[
x_3 = (\tilde{a}_0, -k_0) \quad x_2 = (-\tilde{a}_\infty, -k_\infty)
\]

for some integers \( \tilde{a}_0 \) and \( \tilde{a}_\infty \). We define \( a_0, a_\infty, \) and \( m \) such that

\[
a_0 = \tilde{a}_0 \mod k_0 \quad 0 \leq a_0 < k_0,
\]

\[
a_\infty = \tilde{a}_\infty \mod k_\infty \quad 0 \leq a_\infty < k_\infty,
\]

and

\[
m = \frac{\tilde{a}_0 - a_0}{k_0} + \frac{\tilde{a}_\infty - a_\infty}{k_\infty}.
\]

**Lemma 48.** The type of \( \mathcal{O}_C(D) \) is given by \( (a_0, a_\infty, m) \) and the numerical degree of \( \mathcal{O}_C(D) \) is given by

\[
\frac{1}{hk_0k_\infty} x_2^\infty \wedge x_3^0.
\]

**Proof:** The generators of the one dimensional cones in the fan of \( X \) corresponding to the divisors \( D', D, D_0, \) and \( D_\infty \) can be taken to be \( (1, 0, 0) \), \( (1, h, 0) \), \( (1, -\tilde{a}_0, k_0) \) and \( (1, -\tilde{a}_\infty, -k_\infty) \) respectively (c.f. proof of Lemma 42). Linear functions on the fan give rise to relations among the divisors [6, Theorem 4.10]. The linear functions corresponding to the second and third entries of the above vectors give rise to relations which we restrict to \( C \):

\[
\mathcal{O}_C(hD - \tilde{a}_0D_0 - \tilde{a}_\inftyD_\infty) \cong \mathcal{O}_C,
\]

\[
\mathcal{O}_C(k_0D_0 - k_\inftyD_\infty) \cong \mathcal{O}_C.
\]

Both relations pullback from \( \mathbb{P}^{1}_{k_0, k_\infty} \) where the second can be written

\[
\mathcal{O}_{\mathbb{P}^{1}_{k_0, k_\infty}}(k_0[0]) \cong \mathcal{O}_{\mathbb{P}^{1}_{k_0, k_\infty}}(k_\infty[\infty]) \cong \mathcal{O}_{\mathbb{P}^{1}_{k_0, k_\infty}}([p]).
\]

Then the first assertion of the lemma, which is equivalent to

\[
\mathcal{O}_C(hD) = \pi^*\mathcal{O}_{\mathbb{P}^{1}_{k_0, k_\infty}}(a_0[0] + a_\infty[\infty] + m[p]),
\]

follows from the definitions and the above relations.
Computing the degree from the above relation, we get
\[
\text{deg}(\mathcal{O}_C(D)) = \frac{1}{h} \left( \frac{a_0}{k_0} + \frac{a_{\infty}}{k_{\infty}} + m \right)
\]
\[
= \frac{1}{h} \left( \frac{\tilde{a}_0}{k_0} + \frac{\tilde{a}_{\infty}}{k_{\infty}} \right)
\]
\[
= \frac{1}{hk_0k_{\infty}}(k_{\infty}\tilde{a}_0 + k_0\tilde{a}_{\infty})
\]
\[
= \frac{1}{hk_0k_{\infty}}(-\tilde{a}_{\infty}, -k_{\infty}) \wedge (\tilde{a}_0, -k_0)
\]
\[
= \frac{1}{hk_0k_{\infty}}x_2 \wedge x_3.
\]

The Calabi-Yau condition implies
\[
\mathcal{O}_C(D + D') \cong \pi^*\mathcal{O}_{\mathbb{P}^1_{k_0,k_{\infty}}}(-[0] - [\infty]).
\]

In terms of the corresponding types \((a_0, a_{\infty}, m)\) and \((a'_0, a'_{\infty}, m')\), the condition is given by
\[
a_0 + a'_0 = -1 \mod k_0,
\]
\[
a_{\infty} + a'_{\infty} = -1 \mod k_{\infty},
\]

and
\[
\frac{a_0}{k_0} + \frac{a_{\infty}}{k_{\infty}} + m + \frac{a'_0}{k_0} + \frac{a'_{\infty}}{k_{\infty}} + m' = -\frac{1}{k_0} - \frac{1}{k_{\infty}}.
\]

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