A LOW-DEGREE STRICTLY CONSERVATIVE FINITE ELEMENT METHOD FOR INCOMPRESSIBLE FLOWS

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Abstract. In this paper, a new $P_2 - P_1$ finite element pair is proposed for incompressible fluid. For this pair, the discrete inf-sup condition and the discrete Korn’s inequality hold on general triangulations. It yields exactly divergence-free velocity approximations when applied to models of incompressible flows. The robust capacity of the pair for incompressible flows are verified theoretically and numerically.

1. Introduction

The property of conservation plays a key role in the modeling of many physical systems. For the Stokes problem, for example, if a stable finite element pair can inherit the mass conservation, the approximation of the velocity can be independent of the pressure and the method does not suffer from the locking effect with respect to large Reynolds’ numbers (c.f., e.g., [7]). The importance of conservative schemes is also significant in, e.g., the nonlinear mechanics [2, 3] and the magnetohydrodynamics [23, 25, 26]. In this paper, we focus on the conservative scheme for the Stokes-type problems. Actually, Stokes-type problems are immediately connected to, e.g., Osceen problem and others and can be applied widely for not only fluid problems but also elastic models such as the earth model with a fluid core [12], and their conservative schemes can be relevant to more other equations.

Most classical stable Stokes pairs relax the divergence-free constraint by enforcing the condition in the weak sense, and the conservation can be preserved strictly only for special examples. Though, during the past decade, the conservative schemes have been recognized more clearly as pressure robustness and widely studied and surveyed in, e.g., [17, 21, 28, 35]. This conservation is also connected to other key features like “viscosity-independent” [39], “gradient-robustness” [30], etc for numerical schemes. There have been various successful examples along different technical approaches. Efforts have been devoted to the construction of conforming conservative pairs, and extra structural assumptions are generally needed for the subdivision and finite element functions. Examples include conforming elements designed for special meshes, such as $P_k - P_{k-1}$ triangular elements for $k \geq 4$ on singular-vertex-free meshes [36] and for smaller $k$ constructed on composite grids [1, 34, 36, 42, 45], and the pairs given in [15, 21] which work for general triangulations but with extra smoothness requirement and more complicated shape function spaces. A natural way to relax the constraints is to use $H(\text{div})$-conforming but $(H^1)^2$-nonconforming finite element

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functions for the velocity; for example, in [32], an incomplete cubic polynomial space is used for the velocity and piecewise constant for the pressure, and the pair is both stable and conservative on general triangulations. Several sequel conservative pairs are constructed by adding some terms (usually being polynomials of high degree) to $H(\text{div})$ conforming elements (locally) to enforce weak continuity of tangential component of the velocity approximation, see, e.g., [19, 38, 40]. For conservative pairs in three-dimension, we refer to, e.g., [22, 44, 48] where composite grids are required, as well as Refs. [20] and [47] where high degree local polynomials are utilized. We refer to [9, 27, 46] for rectangular grids and [33] for cubic grids where full advantage of the geometric symmetry of the cells are taken. Beside these finite element methods mentioned above, an alternative is to construct specially discrete variational forms onto $H(\text{div})$ functions where extra stabilisations may play roles; works such as the discontinuous Galerkin method, the weak Galerkin method, and the virtual element method all fall into this category; there have been many valuable works of these types, but we do not seek to give a complete survey and thus will not discuss them in the present paper.

The method given here falls into nonconforming methods which use $H(\text{div})$-conforming and $(H^1)^2$-nonconforming finite element functions for the velocity. Precisely, we propose a new $P_2-P_1$ finite element pair; for the velocity field, we use piecewise quadratic $H(\text{div})$ functions whose tangential component is continuous in the average sense, and for the pressure, we use discontinuous piecewise linear functions. The pair is stable and immediately strictly conservative on general triangulations. Further, a discrete Korn’s inequality holds for the velocity. To the best of our knowledge, this is the lowest-degree conservative stable pair for the Stokes problem on general triangulations.

For the newly designed space for velocity, all the degrees of freedom are located on edges of the triangulation. It is thus impossible to construct a commutative interpolator with a non-constant pressure space. To prove the inf-sup condition, we adopt Stenberg’s macroelement technique [37]. On every macroelement, the surjection property of the divergence operator is confirmed by figuring out its kernel space. The structure of Stokes complex is thus utilized in a way different from the general way in the study of conservative pairs [15, 21] and in the study of biharmonic finite elements [14, 43, 49, 50].

The capability of the pair is verified both theoretically and numerically. When applied to the Stokes and the Darcy–Stokes–Brinkman problems, the approximation of the velocity is independent of the small parameters and thus locking-free; numerical experiments verify the validity of the theory. We note that, as the tangential component of the velocity function is continuous only in the average sense, the convergence rate can only be proved to be of $O(h)$ order. This is suboptimal as piecewise quadratic and linear polynomials are respectively used for the velocity and the pressure. However, since the pair is conservatively stable on general triangulations, it plays superior to some $O(h^2)$ schemes numerically in robustness with respect to triangulations and in avoiding locking for small parameters. The performance of the pair on the Navier–Stokes equation is also illustrated numerically.

This proposed pair is related to the pair in [32] and some sequel works. The main difference is in the fact that we use quadratic polynomials and Ref. [32], for example, uses piecewise incomplete
cubic polynomials for velocity; namely, the degree is reduced in our pair. Natural connections between the proposed pair and DG-type methods may be expected under the framework of [24]; along the lines of [13], these connections can be expected helpful for the construction of optimal solvers for the DG schemes. The possible generalization of the proposed pair to three-dimensional case will also be discussed.

The rest of the paper is organized as follows. At the remaining of this section, some notations are given. In Section 2, a new $P_2 - P_1$ element method is proposed, and significant properties of it are presented. In Sections 3, the convergence analysis of the element applied to the Stokes problem and the Darcy-Stokes-Brinkman problem is provided. In Section 4, numerical experiments are presented to reflect the efficiency of the strictly conservative method when compared with some classical elements. A meticulous proof of a significant lemma devoted to the verification of the inf-sup condition is put in Appendix A.

1.1. Notations. Throughout this paper, $Ω$ is a bounded and connected polygonal domain in $\mathbb{R}^2$. We use $\nabla$, $Δ$, $\text{div}$, $\text{rot}$, $\text{curl}$ to denote the gradient, Laplace, divergence, rotation, and curl operators, respectively. As usual, we use $L^p(Ω)$, $H^s(Ω)$, $H(\text{div},Ω)$, $H(\text{rot},Ω)$, $H^s_0(Ω)$, and $H_0(\text{div},Ω)$ for standard Sobolev spaces. Denote $L^2_0(Ω) := \{w ∈ L^2(Ω) : \int_Ω w \, dΩ = 0\}$. We use “∼” for vector valued quantities. Specifically, we denote $L^p(Ω) := (L^p(Ω))^2$, $H^s(Ω) := (H^s(Ω))^2$, $H(\text{div},Ω) := (H(\text{div},Ω))^2$, and $H(\text{rot},Ω) := (H(\text{rot},Ω))^2$. Denote, by $H^s(Ω)$ and $H^s(Ω)$, the dual spaces of $H^s(Ω)$ and $H^s(Ω)$, respectively. We utilize the subscript “· ·” to indicate the dependence on grids. Particularly, an operator with the subscript “· ·” implies the operation is done cell by cell. We denote $⟨· , ·⟩$ and $⟨· , ·⟩$ as the usual inner product and the dual product, respectively. Finally, $\leq$, $\geq$, and $\equiv$ respectively denote $\leq$, $\geq$, and $= \text{ up to some multiplicative generic constant}$ [41], which only depends on the domain and the shape-regularity of subdivisions.

Let $\{T_h\}$ be a family of triangular grids of domain $Ω$. The boundary $\partial Ω = \Gamma_D \cup \Gamma_N$. Let $N_h$ be the set of all vertices, $N_h^i = N_h^i \cup N_h^b$, with $N_h^i$ and $N_h^b$ comprising the interior vertices and the boundary vertices, respectively. Similarly, let $E_h = E_h^i \cup E_h^b$ be the set of all the edges, with $E_h^i$ and $E_h^b$ comprising the interior edges and the boundary edges, respectively. For an edge $e$, $n_e$ is a unit vector normal to $e$ and $t_e$ is a unit tangential vector of $e$ such that $n_e \times t_e > 0$. On the edge $e$, we use $[\cdot]_e$ for the jump across $e$. If $e ∈ \partial Ω$, then $[\cdot]_e$ is the evaluation on $e$. The subscript $\cdot_e$ can be dropped when there is no ambiguity.

Suppose that $T$ represents a triangle in $T_h$. Let $h_T$ and $ρ_T$ the circumscribed radius and the inscribed circles radius of $T$, respectively. Let $h := \max_{T ∈ T_h} h_T$ be the mesh size of $T_h$. Let $P_l(T)$ denote the space of polynomials on $T$ of the total degree no more than $l$. Similarly, we define the space $P_l(e)$ on an edge $e$. We assume that $\{T_h\}$ is a family of regular subdivisions, i.e.,

\begin{equation}
\max_{T ∈ T_h} \frac{h_T}{ρ_T} \leq γ_0,
\end{equation}

where $γ_0$ is a generic constant independent of $h$. 
2. A NEW NONCONFORMING FINITE ELEMENT METHOD

2.1. A new finite element pair. Let $T$ be a triangle with nodes $\{a_i, a_j, a_k\}$, and $e_i$ be an edge of $T$ opposite to the $i$-th vertex $a_i$; see Figure 1. Denote a unit vector normal to $e_i$ and a unit tangential vector of $e_i$ as $\mathbf{n}_{T,e_i}$ and $\mathbf{t}_{T,e_i}$, respectively.

The new vector $P_2$ element is defined by the triple $(T, P_T, D_T)$:

1. $T$ is a triangle;
2. $P_T := (P_2(T))^2$;
3. for any $v \in (H^1(T))^2$, the degrees of freedom on $T$, denoted by $D_T$, are

$$\begin{align*}
\left\{ \int_{e_i} v \cdot \mathbf{n}_{T,e_i} \, ds, \int_{e_i} v \cdot \mathbf{n}_{T,e_i}(\lambda_j - \lambda_k) \, ds, \int_{e_i} v \cdot \mathbf{n}_{T,e_i}(-\lambda_j \lambda_k + \frac{1}{6}) \, ds, \int_{e_i} v \cdot \mathbf{t}_{T,e_i} \, ds \right\}_{j=1:3},
\end{align*}$$

where $\{\lambda_i, \lambda_j, \lambda_k\}$ represent the barycentric coordinates on $T$. The above triple is $P_T$-unisolvent. Particularly, we use $\varphi_{n_{T,e_i},0}$, $\varphi_{n_{T,e_i},1}$, $\varphi_{n_{T,e_i},2}$, and $\varphi_{t_{T,e_i},0}$ to represent the nodal basis functions associated with DOFs on $e_i$, and then

$$\begin{align*}
\varphi_{n_{T,e_i},0} &= \lambda_j(3\lambda_j - 2)\frac{\mathbf{t}_k}{(\mathbf{n}_i, \mathbf{t}_k)} + \lambda_k(3\lambda_k - 2)\frac{\mathbf{t}_j}{(\mathbf{n}_i, \mathbf{t}_j)} + 6\lambda_j \lambda_k \mathbf{n}_i; \\
\varphi_{n_{T,e_i},1} &= 3\lambda_j(3\lambda_j - 2)\frac{\mathbf{t}_k}{(\mathbf{n}_i, \mathbf{t}_k)} - 3\lambda_k(3\lambda_k - 2)\frac{\mathbf{t}_j}{(\mathbf{n}_i, \mathbf{t}_j)}; \\
\varphi_{n_{T,e_i},2} &= 30\lambda_j(3\lambda_j - 2)\frac{\mathbf{t}_k}{(\mathbf{n}_i, \mathbf{t}_k)} + 30\lambda_k(3\lambda_k - 2)\frac{\mathbf{t}_j}{(\mathbf{n}_i, \mathbf{t}_j)}; \\
\varphi_{t_{T,e_i},0} &= 6\lambda_j \lambda_k \mathbf{t}_i.
\end{align*}$$

The corresponding finite element space is defined by

$$V_h := \{v_h \in L^2(\Omega) : v_h|_T \in (P_2(T))^2, \forall T \in \mathcal{T}_h; v_h \cdot \mathbf{n}_e \text{ and } \int_{e} v_h \cdot \mathbf{t}_e \, ds \text{ are continuous } \forall e \in \mathcal{E}_h^i\}.$$

Note that $V_h \subset H(\text{div}, \Omega)$ but $V_h \not\subset H^1(\Omega)$.

The nodal interpolation operator $\Pi_h : H^1(\Omega) \rightarrow V_h$ is defined such that for any $e \in \mathcal{E}_h$,

$$\begin{align*}
\int_{e} \Pi_h v \cdot \mathbf{n}_e \, w \, ds &= \int_{e} v \cdot \mathbf{n}_e \, w \, ds, \quad \forall w \in P_2(e), \\
\int_{e} \Pi_h v \cdot \mathbf{t}_e \, ds &= \int_{e} v \cdot \mathbf{t}_e \, ds.
\end{align*}$$
The operator $\Pi_h$ is locally defined, and the local space $V_h(T)$ restricted on $T$ is invariant under the Piola’s transformation, i.e., it maps $V_h(T)$ onto $V_h(\hat{T})$, where $\hat{T}$ represents a reference triangle. Moreover, $\Pi_h$ preserves quadratic functions locally. Therefore, a combination of Lemmas 1.6 and 1.7 in [7], standard scaling arguments, and the Bramble-Hilbert lemma leads to the following approximation property of $\Pi_h$.

**Proposition 2.1.** If $0 \leq k \leq 1 \leq s \leq 3$, then
\begin{equation}
|v - \Pi_h v|_{k, h} \leq h^{s-k} |v|_{s, \Omega}, \quad \forall v \in H^s(\Omega).
\end{equation}
Moreover, the following low order estimate is valid
\begin{equation}
||v - \Pi_h v||_{0, \Omega} \leq h^{1/2} ||v||_{0, \Omega} ||v||_{1, \Omega}.
\end{equation}
Assume $\Gamma_D$ to be a part of the boundary $\partial \Omega$. Define
\begin{equation}
V_{hD} := \{v_h \in L^2(\Omega) : v_h|_T \in (P_2(T))^2, \forall T \in T_h; \}
v_h \cdot n_e \text{ and } \int_e v_h \cdot t_e \, ds \text{ are continuous for any } e \in E^*_h \text{ and vanish for any } e \subset \Gamma_D\}.
\end{equation}
Specially, if $\Gamma_D = \partial \Omega$, $V_{hD}$ is written as $V_{h0}$. Define
\begin{equation}
Q_h := \{q \in L^2(\Omega) : q|_T \in P_1(T), \forall T \in T_h\}, \quad \text{and } Q_{hs} := Q_h \cap L^2(\Omega).
\end{equation}
Evidently, $\text{div} V_h \subset Q_h$. Therefore, $V_{hD} \times Q_h$ and $V_{h0} \times Q_{hs}$ each forms a conservative pair. The stability and discrete Korn’s inequality also hold. We firstly introduce an assumption on the triangulations.

**Assumption A.** Every triangle in $T_h$ has at least one vertex in the interior of $\Omega$.

The theorems below, which will be proved in the sequel subsections, hold on triangulations that satisfy **Assumption A.**

**Theorem 2.2 (Inf-sup conditions).** Let $\{T_h\}$ be a family of triangulations satisfying **Assumption A.** Then
\begin{equation}
\sup_{v_h \in V_{h0}} \frac{\int_\Omega \text{div} v_h q_h \, d\Omega}{||v_h||_{1, h}} \geq ||q||_{0, \Omega}, \quad \forall q \in Q_h, \quad \text{if } \Gamma_D \neq \partial \Omega,
\end{equation}
\begin{equation}
\sup_{v_h \in V_{h0}} \frac{\int_\Omega \text{div} v_h q_h \, d\Omega}{||v_h||_{1, h}} \geq ||q||_{0, \Omega}, \quad \forall q \in Q_{hs}.
\end{equation}

**Theorem 2.3 (Discrete Korn’s inequality).** Let $\{T_h\}$ be a family of triangulations satisfying **Assumption A.** Let $\epsilon(v) := \frac{1}{2}[\nabla v + (\nabla v)^T]$. Then
\begin{equation}
\sum_{T \in T_h} \int_T |\epsilon(v)|^2 \, dT \geq ||v||_{1, h}^2, \quad \forall v \in V_{hD}.
\end{equation}

2.2. **Proof of inf-sup conditions.** Note that the the commutativity $\text{div} \Pi_h w = P_{Q_h} \text{div} w$ does not hold for all $w \in H^1_0(\Omega)$, where $P_{Q_h}$ represents the $L^2$ projection onto $Q_h$. To prove the inf-sup conditions (2.4) and (2.5), we adopt the macroelement technique by Stenberg [37] We postpone the proof of Theorem 2.2 after some technical preparations.
2.2.1. Stenberg’s macroelement technique. A macroelement is a connected set of at least two cells in \( \mathcal{T}_h \). And a macroelement partition of \( \mathcal{T}_h \), denoted by \( \mathcal{M}_h \), is a set of macroelements such that each triangle in of \( \mathcal{T}_h \) is covered by at least one macroelement in \( \mathcal{M}_h \).

**Definition 2.4.** Two macroelements \( M_1 \) and \( M_2 \) are said to be equivalent if there exists a continuous one-to-one mapping \( G : M_1 \to M_2 \), such that

- (a) \( G(M_1) = M_2 \);
- (b) if \( M_1 = \bigcup_{i=1}^{m} T_i^{1} \), then \( T_i^{2} = G(T_i^{1}) \) with \( i = 1 : m \) are the cells of \( M_2 \).
- (c) \( G|_{T_i^{1}} = F_{T_i^{1}} \circ F_{T_i^{2}}^{-1} \), \( i = 1 : m \), where \( F_{T_i^{1}} \) and \( F_{T_i^{2}} \) are the mappings from a reference element \( \hat{T} \) onto \( T_i^{1} \) and \( T_i^{2} \), respectively.

A class of equivalent macroelements is a set of all the macroelements which are equivalent to each other. Given a macroelement \( M \), we denote

\[
\mathcal{V}_{h,0,M} := \mathcal{V}_{h,0}(M), \quad \mathcal{Q}_{h,M} := \mathcal{Q}_{h}(M), \quad \text{and} \quad \mathcal{Q}_{h,s,M} := \mathcal{Q}_{h,s}(M).
\]

And we denote

\[
N_M := \{ q_h \in \mathcal{Q}_{h,M} : \int_M \nabla \cdot \mathbf{v}_h \, dM = 0, \ \forall \mathbf{v}_h \in \mathcal{V}_{h,0,M} \}.
\]

Stenberg’s macroelement technique can be summarized as the following proposition.

**Proposition 2.5.** [37, Theorem 3.1] Suppose there exist a macroelement partitioning \( \mathcal{M}_h \) with a fixed set of equivalence classes \( \mathbb{E}_i \) of macroelements, \( i = 1, 2, \ldots, n \), a positive integer \( N \) (\( n \) and \( N \) are independent of \( h \)), and an operator \( \Pi : H^1_0(\Omega) \to \mathcal{V}_{h,0} \), such that

- \( (C_1) \) for each \( M \in \mathbb{E}_i \), \( i = 1, 2, \ldots, n \), the space \( N_M \) defined in (2.7) is one-dimensional, which consists of functions that are constant on \( M \);
- \( (C_2) \) each \( M \in \mathcal{M}_h \) belongs to one of the classes \( \mathbb{E}_i \), \( i = 1, 2, \ldots, n \);
- \( (C_3) \) each \( e \in \mathcal{E}_h \) is an interior edge of at least one and no more than \( N \) macroelements;
- \( (C_4) \) for any \( \mathbf{v} \in H^1_0(\Omega) \), it holds that

\[
\sum_{T \in \mathcal{T}_h} h_T^{-2} ||\mathbf{v} - \Pi \mathbf{v}||^2_{L^2} + \sum_{e \in \mathcal{E}_h} h_e^{-1} ||\mathbf{v} - \Pi \mathbf{v}||^2_{L^2} \leq ||\mathbf{v}||^2_{L^2} \quad \text{and} \quad ||\Pi \mathbf{v}||_{L^2} \leq ||\mathbf{v}||_{L^2}.
\]

Then the stability (2.5) is valid.

2.2.2. Technical lemmas. In general, the main difficulty to design a stable mixed element stems from \( (C_1) \). We use the specific type of macroelements as below.

**Definition 2.6.** A macroelement, denoted by \( M \), being a union of \( m \) cells which share exactly one common vertex in the interior of the macroelement, is called an **m-cell vertex-centred macroelement**, and **m-macroelement** for short.

The set of interior edges and cells of \( M \) are denoted by \( \{e_i\}_{i=1}^{m} \) and \( \{T_i\}_{i=1}^{m} \), respectively. Denote the lengths of interior edges by \( \{d_i\}_{i=1}^{m} \) and the areas of cells by \( \{S_i\}_{i=1}^{m} \). Figure 2 gives an illustration of a 6-macroelement.

Below are concrete definitions of some local defined spaces on \( M \) introduced in the previous context.

\[
\mathcal{V}_{h,0,M} := \{ \psi_h \in L^2(M) : \psi_h|_T \in (P_2(T))^2, \forall T \subset M, \psi_h \cdot \mathbf{n}_e \text{ and } \int_e \psi_h \cdot \mathbf{t}_e \, ds \text{ are continuous across interior edges and vanish on } \partial M \},
\]
Let $M$ be an $m$-macroelement. Then $N_M$ is a one-dimensional space consisting of constant functions on $M$.

**Proof.** First we are to prove $\text{Im}(\text{div}, V_{h0,M}) = Q_{h^v,M}$. As $\text{dim}V_{h0,M} = \text{dim}(\text{ker}(\text{div}, V_{h0,M})) + \text{dim}(\text{Im}(\text{div}, V_{h0,M}))$, we obtain $\text{dim}(\text{Im}(\text{div}, V_{h0,M})) \geq 3m - 1$, where we utilize $\text{dim}V_{h0,M} = 4m$ by definition and $\text{dim}(\text{ker}(\text{div}, V_{h0,M})) \leq m + 1$ by Lemma 2.7. From $\text{dim}(Q_{h^v,M}) = 3m - 1$ and $\text{Im}(\text{div}, V_{h0,M}) \subset Q_{h^v,M}$, we derive $\text{Im}(\text{div}, V_{h0,M}) = Q_{h^v,M}$. Then, for any $q_h \in N_M$, it holds that $\int_M p^*_h q_h \, dM = 0$, $\forall p^*_h \in Q_{h^v,M}$. Let $q_h = \overline{q}_h + q^*_h$, where $\overline{q}_h = \int_M q_h \, dM$ and $q^*_h \in Q_{h^v,M}$. Then $\int_M p^*_h q^*_h \, dM = 0$, $\forall p^*_h \in Q_{M^v,*}$, which yields $q^*_h = 0$ and hence $q_h = \overline{q}_h$. Therefore, $N_M$ is a one-dimensional space consisting of constant functions on $M$. \qed

**Lemma 2.9.** Let $\{T_h\}$ be a family of triangulations satisfying Assumption A. Each macroelement in $M_h$ has one interior vertex. Then conditions $C_2$, $C_3$, and $C_4$ in Theorem 2.5 are satisfied.

**Proof.** From the Assumption A and the regularity (1.1) of $T_h$, there exists a generic constant $n$, independent of $h$, such that condition $C_2$ holds. If $e \in \mathcal{E}_h^i$, then at least one endpoint of $e$ is an interior vertex. Hence $e$ is an interior edge of at least one macroelement of $M_h$. On the other hand, $e$ is interior to at most two macroelements, which occurs if both endpoints of $e$ are interior in $\Omega$. Therefore, condition $C_3$ also holds. By Proposition 2.1 and the well-known trace theorem (see, e.g., [5, Theorem 1.6.6]), condition $C_4$ can be obtained. \qed

2.2.3. **Proof of Theorem 2.2.** By Lemma 2.8, Lemma 2.9, and Theorem 2.5, it holds that $V_{h0} \times Q_h$ satisfies the inf-sup condition (2.5). The inf-sup stability of $V_{hD} \times Q_h$ is proved utilizing the technique introduced in [29] by the following four steps.
Step 1. Given $q_h \in Q_h$, let $\overline{q}_h = \frac{1}{|\Omega|} \int_{\Omega} q_h \, d\Omega$ and $q_h^* = q_h - \overline{q}_h$. Then $q_h^* \in L^2(\Omega)$ and
\begin{equation}
\|q_h^*\|_{L^2(\Omega)}^2 = \|\overline{q}_h\|_{L^2(\Omega)}^2 + \|q_h\|_{L^2(\Omega)}^2.
\end{equation}

Step 2. By (2.5), there exists some $v_h^* \in V_{h0}$, such that
\begin{equation}
(\text{div } v_h^*, q_h^*) \geq C_1 \|q_h^*\|_{L^2(\Omega)}^2 \quad \text{and} \quad |v_h^*|_{1,h} = \|q_h^*\|_{L^2(\Omega)},
\end{equation}
where $C_1 > 0$ is a generic constant independent of $h$.

Step 3. Let $\Gamma_N := \partial \Omega \setminus \Gamma_D$. Notice that $\overline{q}_h$ is constant in $\Omega$. Let $\overline{v}_h \in V_{hD}$ satisfy $\overline{v}_h \cdot \mathbf{n}_e = C_0 \overline{q}_h|_e$ for any $e \subset \Gamma_N$, and other degrees of freedom vanish. The value of $C_0$ is chosen such that $|\overline{v}_h|_{1,h} = \|\overline{q}_h\|_{L^2(\Omega)}$. Then it holds with $C_2 := C_0 \frac{|\Omega|}{|\Gamma|}$ that
\begin{equation}
(\text{div } \overline{v}_h, q_h) = \sum_{T \in \mathcal{T}_h} \int_T \text{div } \overline{v}_h \, \overline{q}_h \, dT = \sum_{e \subset \Gamma_N} \int_e \overline{v}_h \cdot \mathbf{n}_e \, ds = C_2 \|\overline{q}_h\|_{L^2(\Omega)}^2.
\end{equation}

Step 4. Let $v_h = v_h^* + \kappa \overline{v}_h$ with $\kappa = \frac{2C_1\kappa}{(C_2)^2 + 2}$. By the Schwarz inequality, the elementary inequality, and (2.11)–(2.13), we obtain
\begin{align*}
(\text{div } v_h, q_h) &= (\text{div } v_h^*, q_h^*) + \kappa(\text{div } \overline{v}_h, q_h) + \kappa(\text{div } \overline{v}_h, q_h^*) \geq C_1 \|q_h^*\|_{L^2(\Omega)}^2 + C_2 \kappa \||\overline{q}_h\|_{L^2(\Omega)}^2 + \kappa(\text{div } \overline{v}_h, q_h^*)
\end{align*}
\begin{align*}
& \geq C_1 \|q_h^*\|_{L^2(\Omega)}^2 + C_2 \kappa \|\overline{q}_h\|_{L^2(\Omega)}^2 - \sqrt{2} \kappa \left( \frac{C_2}{\sqrt{2}} \|\overline{v}_h\|_{L^2(\Omega)}^2 + \frac{\sqrt{2}}{2C_2} \|q_h^*\|_{L^2(\Omega)}^2 \right)
\end{align*}
\begin{align*}
& = (C_1 - \frac{C_2 \kappa}{C_2}) \|q_h^*\|_{L^2(\Omega)}^2 + \frac{C_2 \kappa}{2} \|\overline{q}_h\|_{L^2(\Omega)}^2 = \frac{C_1}{(C_2)^2 + 2} \|q_h\|_{L^2(\Omega)}^2.
\end{align*}
From $|v_h^*|_{1,h} = \|q_h^*\|_{L^2(\Omega)}$, $|\overline{v}_h|_{1,h} = \|\overline{q}_h\|_{L^2(\Omega)}$, and the Poincaré inequality, we have $|v_h|_{1,h} \leq \|q_h\|_{L^2(\Omega)}$. This completes the proof of (2.4) and Theorem 2.2.

2.3. Proof of discrete Korn’s inequality. To verify the discrete Korn’s inequality, we follow the lines of [29] and firstly introduce an auxiliary element scheme constructed by adding element bubble functions to the standard Bernardi-Raugel element [4]. Denote $P_T := (P_1(T))^2 \oplus \text{span}[\lambda_2 \lambda_3 \mathbf{n}_1, \lambda_3 \lambda_1 \mathbf{n}_2, \lambda_1 \lambda_2 \mathbf{n}_3] \oplus \text{span}[\lambda_1 \lambda_2 \lambda_3]^2$. Define
\begin{equation*}
C_h := \left\{ z_h \in H^1(\Omega) : z_{h,T} \in P_T, \forall T \in \mathcal{T}_h, \right. \quad \left. z_h(a) \text{ is continuous at any } a \in N_h^l, \text{ and } \int_e z_h \cdot \mathbf{n}_e \, ds \text{ is continuous across any } e \subset \mathcal{E}_h^l \right\},
\end{equation*}
and $C_h^N := \left\{ z_h \in C_h : z_h(a) = 0, \forall a \in \Gamma_N \right\}$.

2.3.1. Technical lemmas.

Lemma 2.10. The element pair $C_h^N \times Q_h$ satisfies the inf-sup condition
\begin{equation}
\sup_{z_h \in C_h^N} \frac{\int_\Omega \text{div } z_h \, q_h \, d\Omega}{|z_h|_{1,h}} \gtrsim \|q_h\|_{L^2(\Omega)}, \quad \forall q_h \in Q_h.
\end{equation}
Proof. Let $H^1_N(\Omega) := \{ \mathbf{v} \in H^1(\Omega) : \mathbf{v} = 0 \text{ on } \Gamma_N \}$. Define $\Pi_C : H^1_N(\Omega) \mapsto C^N_h$ by

\[
\Pi_C \mathbf{v}(a) = R_h \mathbf{v}(a), \quad \forall a \in N_h,
\]

\[
\int_e (\Pi_C \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_e ds = 0, \quad \forall e \in \mathcal{E}_h,
\]

\[
\int_T x \text{ div } (\Pi_C \mathbf{v} - \mathbf{v}) dT = 0, \quad \int_T y \text{ div } (\Pi_C \mathbf{v} - \mathbf{v}) dT = 0, \quad \forall T \in \mathcal{T}_h,
\]

where $R_h$ represents the local $L^2$-projection given in [18, (A.53)–(A.54)]. It can be verified directly that $(\text{div} \Pi_C \mathbf{v}, q_h) = (\text{div} \mathbf{v}, q_h)$ for any $q_h \in Q_h$, and $|\Pi_C \mathbf{v}|_{1,h} \leq |\mathbf{v}|_{1,h}$. Hence the stability (2.14) is valid [16, Propositions 4.1–4.2].

Lemma 2.11. For any $\mathbf{v} \in V_{dD}$ and $\mathbf{z} \in C^N_h$, it holds that

\[
\sum_{T \in \mathcal{T}_h} \int_T \nabla \mathbf{v} : \text{curl } \mathbf{z} dT = 0.
\]

Proof. Let subscripts “$1$” and “$2$” represent the components of the vector in the $x$ and $y$ directions, respectively. By integration by parts and direct calculation, we obtain

\[
\sum_{T \in \mathcal{T}_h} \int_T \nabla \mathbf{v} : \text{curl } \mathbf{z} dT = \sum_{T \in \mathcal{T}_h} \sum_{e \in \partial T} \int_e (v_1 \nabla z_1 \cdot \mathbf{t}_{T,e} + v_2 \nabla z_2 \cdot \mathbf{t}_{T,e}) ds
\]

\[
= \sum_{T \in \mathcal{T}_h} \sum_{e \in \partial T} \int_e (\mathbf{v} \cdot \mathbf{n}_{T,e})(g_{T,e} \cdot \mathbf{t}_{T,e}) + (\mathbf{v} \cdot \mathbf{t}_{T,e})(g_{T,e} \cdot \mathbf{n}_{T,e}) ds,
\]

where $g_{T,e} := \nabla \mathbf{z} \cdot \mathbf{t}_{T,e}$, $\mathbf{t}_{T,e}$ is the counter-clockwise unit tangent vector of $T$ on $e$ and $\mathbf{n}_{T,e}$ represents the unit outer normal vector. Notice that $\mathbf{z}_e \in (P_1)^2 + \text{span}\{\phi_{T,e} \cdot \mathbf{n}_{T,e}\}$, and $\phi_{T,e}$ is the quadratic bubble function associated with $e$ in $T$, we derive that $g_{T,e} \cdot \mathbf{t}_{T,e}$ is constant on each $e \subset \partial T$. We verify (2.15) with respect to the following three cases.

**Case 1.** For $e \in \mathcal{E}_h$ with $T_1 \cap T_2 = e$. Utilizing the continuity of $\mathbf{v} \cdot \mathbf{n}$, $\mathbf{n}_{T_1,e} = -\mathbf{n}_{T_2,e}$, $\mathbf{t}_{T_1,e} = -\mathbf{t}_{T_2,e}$, and $g_{T_1,e} = -g_{T_2,e}$ by $\mathbf{z} \in H^1(\Omega)$, we obtain

\[
\int_e (\mathbf{v} \cdot \mathbf{n}_{T_1,e})(g_{T_1,e} \cdot \mathbf{n}_{T_1,e}) + \int_e (\mathbf{v} \cdot \mathbf{n}_{T_2,e})(g_{T_2,e} \cdot \mathbf{n}_{T_2,e}) = \int_e \mathbf{v} \cdot \mathbf{n}_{e,T_1}(g_{T_1,e} \cdot \mathbf{n}_{T_1,e}) ds = 0.
\]

At the same time, utilizing the continuity of $\int_e \mathbf{v} \cdot \mathbf{t} ds$ across interior edges, and noticing that $g_{T_1,e} \cdot \mathbf{t}_{T_1,e} = g_{T_2,e} \cdot \mathbf{t}_{T_2,e} = c$, where $c$ represent a constant on $e$, we have

\[
\int_e (\mathbf{v} \cdot \mathbf{t}_{T_1,e})(g_{T_1,e} \cdot \mathbf{t}_{T_1,e}) + \int_e (\mathbf{v} \cdot \mathbf{t}_{T_2,e})(g_{T_2,e} \cdot \mathbf{t}_{T_2,e}) = (g_{T_1,e} \cdot \mathbf{t}_{T_1,e} + c) \int_e \mathbf{v} \cdot \mathbf{t}_{T_1,e} ds + \int_e \mathbf{v} \cdot \mathbf{t}_{T_2,e} ds = 0.
\]

**Case 2.** For $e \subset \Gamma_D$, $\mathbf{v} \cdot \mathbf{n}_e = \int_e \mathbf{v} \cdot \mathbf{t}_e ds = 0$, and $g_{T,e} \cdot \mathbf{t}_e$ is constant on $e \subset T$. Therefore,

\[
\int_e (\mathbf{v} \cdot \mathbf{n}_{T,e})(g_{T,e} \cdot \mathbf{n}_{T,e}) + (\mathbf{v} \cdot \mathbf{t}_{T,e})(g_{T,e} \cdot \mathbf{t}_{T,e}) ds = 0, \quad \forall e \subset \Gamma_D.
\]

**Case 3.** For $e \subset \Gamma_N$, we have $\mathbf{z}_e = 0$ by the definition of $B^N_h$. Hence $g_{T,e} = 0$ for $e \subset T$, and

\[
\int_e (\mathbf{v} \cdot \mathbf{n}_{T,e})(g_{T,e} \cdot \mathbf{n}_{T,e}) + (\mathbf{v} \cdot \mathbf{t}_{T,e})(g_{T,e} \cdot \mathbf{t}_{T,e}) ds = 0, \quad \forall e \subset \Gamma_N.
\]

Combining Cases 1–3, the right-hand-side of (2.16) is zero. The proof is completed.\]
2.3.2. **Proof of Theorem 2.3.** For any \( v \in V_h \), a direct calculation leads to \( \varepsilon(v)|_T = (\nabla v - \frac{1}{2} \text{rot } v \chi)|_T \), where \( \chi = (0^T - 1^T 0^T) \). From (2.14) and \( \text{rot } v \in Q_h \), there exists some \( z \in C_h^{N} \), such that

\[
\int_{\Omega} \nabla q \cdot \phi \, d\Omega = \sum_{T \in T_h} \int_T \text{rot } v q \, dT, \quad \forall q \in Q_h \quad \text{and} \quad |z|_{1,h} \leq ||\text{rot } v||_{0,\Omega} \lesssim |v|_{1,h}.
\]

Therefore, \( ||\nabla v - \nabla z||_{0,\Omega} \leq |v|_{1,h} + |z|_{1,h} \leq |v|_{1,h} \), and

\[
\sum_{T \in T_h} \int_T \varepsilon(v) : (\nabla v - \nabla z) 
\]

(2.17) \[
\sum_{T \in T_h} \int_T \varepsilon(v) : (\nabla v - \nabla z) \, dT = \sum_{T \in T_h} \int_T \nabla v \cdot (\nabla v - \nabla z) \, dT = \sum_{T \in T_h} \int_T |\nabla v|^2 \, dT = |v|_{1,h}^2.
\]

Finally, \( \sum_{T \in T_h} \int_T |\varepsilon(v)|^2 \, dT \geq \sum_{T \in T_h} \int_T |\varepsilon(v)|^2 \, dT \geq |v|_{1,h} \), and the proof is completed.

3. **Application to Stokes problems**

3.1. **Application to the Stokes equations.** Consider the stationary Stokes system:

\[
\begin{cases}
-\varepsilon^2 \Delta \mathbf{u} + \nabla p = f, & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} = g, & \text{in } \Omega, \\
\mathbf{u} = 0, & \text{on } \Gamma_D.
\end{cases}
\]

(3.1)

For simplicity of presentation, we only consider the case where \( \partial \Omega = \Gamma_D \) here. Extensions to other boundary conditions follows directly.

The discretization scheme of (3.1) reads: Find \( (\mathbf{u}_h, p_h) \in V_{h0} \times Q_h \), such that

\[
\begin{cases}
\varepsilon^2 (\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}_h) - (\nabla \cdot \mathbf{u}_h, p_h) = (f, \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h, \\
(\nabla u_h, q_h) = (g, q_h), & \forall q_h \in Q_h
\end{cases}
\]

(3.2)

Based on the discussions in Section 2, Brezzi’s conditions can be easily verified, and (3.2) is uniformly well-posed with respect to \( \varepsilon \) and \( h \).

**Theorem 3.1.** Let \( (\mathbf{u}, p) \) and \( (\mathbf{u}_h, p_h) \) be the solutions of (3.1) and (3.2), respectively. The following estimates hold with \( 0 < r \leq 2 \):

\[
|\mathbf{u} - \mathbf{u}_h|_{1,h} \leq h|\mathbf{u}|_{r+1,\Omega} + h|\mathbf{u}|_{2,\Omega},
\]

\[
||p - p_h||_{0,\Omega} \leq h^r|p|_{r,\Omega} \lesssim (h^r|\mathbf{u}|_{r+1,\Omega} + h|\mathbf{u}|_{2,\Omega}).
\]

**Proof.** Since the mixed element is inf-sup stable, divergence-free, and \( \mathbf{v}_h \cdot \mathbf{n} \) is continuous, the following estimates are standard [6, 7, 11]:

\[
|\mathbf{u} - \mathbf{u}_h|_{1,h} \leq \inf_{\mathbf{w}_h \in V_h} |\mathbf{u} - \mathbf{w}_h|_{1,h} + \frac{\sum_{\epsilon \in \mathcal{E}_h} \varepsilon^2 \int_e (\nabla \mathbf{u} \cdot \mathbf{n}_e) \cdot [\mathbf{v}_h] \, ds}{\sum_{\epsilon \in \mathcal{E}_h} \varepsilon^2 \int_e (\nabla \mathbf{u} \cdot \mathbf{n}_e) \cdot [\mathbf{v}_h] \, ds},
\]

\[
||p - p_h||_{0,\Omega} \leq \varepsilon^2 |\mathbf{u} - \mathbf{u}_h|_{1,h} + \inf_{q_h \in Q_h} ||p - q_h||_{0,\Omega} + \sup_{\mathbf{w}_h \in V_h} \frac{\sum_{\epsilon \in \mathcal{E}_h} \varepsilon^2 \int_e (\nabla \mathbf{u} \cdot \mathbf{n}_e) \cdot [\mathbf{v}_h] \, ds}{||\mathbf{v}_h||_{1,h}}.
\]
The term $\inf_{\psi_h \in V_h} |u - \psi_h|_{1,h}$ is bounded by the interpolation error. Since $\int_{\Omega} \psi_h \, ds$ is continuous across interior edges and vanish on $\partial \Omega$, a standard estimate similar to that of the Crouzeix and Raviart element [10, Lemma 3] leads to

$$\sum_{e \in E_h}^e \varepsilon^2 \int_{\partial e} (\nabla u \cdot n_e) \cdot [\psi_h] \, ds \lesssim \varepsilon^2 h^2 |u|_{2,\Omega} |\psi_h|_{1,h}.$$  

Hence we derive

$$|u - u_h|_{1,h} \lesssim h' |u|_{r+1,\Omega} + h |u|_{2,\Omega} \quad \text{with} \quad 0 < r \leq 2.$$  

The above estimates together with $\inf_{q_h \in Q_h} \|p - q_h\|_{0,\Omega} \lesssim h' |p|_{r,\Omega}$ lead to that

$$\|p - p_h\|_{0,\Omega} \lesssim h' |p|_{r,\Omega} + \varepsilon^2 (h' |u|_{r+1,\Omega} + h |u|_{2,\Omega}) \quad \text{with} \quad 0 < r \leq 2.$$  

The proof is completed. \qed

3.2. Application to the Darcy–Stokes–Brinkman equations. Consider the Darcy–Stokes–Brinkman equations:

$$\begin{aligned}
-\varepsilon^2 \Delta u + \nabla p &= f \quad \text{in} \quad \Omega, \\
\text{div } u &= g \quad \text{in} \quad \Omega, \\
\n \cdot n &= 0, \quad \varepsilon u \cdot t = 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}$$

where $\varepsilon \in (0, 1]$ is a parameter. When $\varepsilon$ is not too small and $g = 0$, it is a Stokes problem with an additional lower order term. When $\varepsilon = 0$, the first equation becomes the Darcy’s law for porous medium flow. Most classic mixed elements fail to converge uniformly with respect to $\varepsilon$ when applied to (3.4) [32].

The discretization scheme of (3.1) reads: Find $(u_h, p_h) \in V_h \times Q_{h^+}$, such that

$$\begin{aligned}
\varepsilon^2 (\nabla u_h, \nabla v_h) + (u_h, v) - (\text{div } u_h, p_h) &= (f, v) \quad \forall v \in V_h, \\
(\text{div } u_h, q_h) &= (g, q) \quad \forall q \in Q_h.
\end{aligned}$$

Since the finite element pair is stable and conservative, Brezzi’s conditions can be easily verified for (3.5), and it is uniformly well-posed with respect to $\varepsilon$ and $h$, provided $\int_{\Omega} g \, d\Omega = 0$. Robust convergence can be obtained both for smooth continuous solutions and for the case that the effect of the $\varepsilon$-dependent boundary layers is taken into account later.

**Theorem 3.2.** If $u \in H^{r+1}(\Omega) \cap H^1_0(\Omega)$ and $p \in H^r(\Omega) \cap L^2(\Omega)$ with $0 < r \leq 2$, then

$$\begin{aligned}
\|\text{div } u - \text{div } u_h\|_{0,\Omega} &\lesssim h' |u|_{r+1,\Omega}, \\
\|u - u_h\|_{0,\Omega} + \varepsilon |u - u_h|_{1,h} &\lesssim h' (\varepsilon + h) |u|_{r+1,\Omega} + \varepsilon h |u|_{2,\Omega}, \\
\|p - p_h\|_{0,\Omega} &\lesssim h' |p|_{r,\Omega} + h' (\varepsilon + h) |u|_{r+1,\Omega} + \varepsilon h |u|_{2,\Omega}.
\end{aligned}$$

**Proof.** Evidently $\text{div } u_h = P_{Q_{h^+}} (\text{div } u)$, where $P_{Q_{h^+}}$ represents the $L^2$-projection into $Q_{h^+}$. Therefore, the first inequality (3.6) follows from the estimation of the $L^2$-projection. For this conservative pair, the following estimates are standard (see, e.g., [7] and [32]),

$$\begin{aligned}
\|u - u_h\|_{0,\Omega} + \varepsilon |u - u_h|_{1,h} &\lesssim \\
\inf_{\psi_h \in V_h} (|u - \psi_h|_{0,\Omega} + \varepsilon |u - \psi_h|_{1,h}) + \sup_{\psi_h \in E_h(0)} \frac{\sum_{e \in E_h}^e (\nabla u \cdot n_e) \cdot [\psi_h] \, ds}{\varepsilon |\psi_h|_{1,h}}.
\end{aligned}$$
Theorem 3.3. In [32], we can obtain the following uniform convergence estimate.

\[
\| p - p_h \|_{0,\Omega} \leq \| u - u_h \|_{0,\Omega} + \varepsilon \| u - u_h \|_{1,h} + \sum_{e \in E_h} \varepsilon^2 \int_e (\nabla u \cdot n_e) \cdot \| v_h \| \, ds
\]
\[
+ \inf_{q_h \in Q_h} \| p - q_h \|_{0,\Omega} + \sup_{y_h \in V_h} \frac{\| v_h \|_{1,h}}{\varepsilon}. \]

Hence the estimates (3.7) and (3.8) are derived in a similar way as those in Theorem 3.1.

As is mentioned in [32], it may happen that \| u \|_{2,\Omega} and \| u \|_{3,\Omega} blow up when \varepsilon tends to 0. In this case, the convergence estimates given in Theorem 3.2 will deteriorate, especially when the solution of (3.4) has boundary layers. To derive a uniform convergence analysis of the discrete solutions, we assume that \( \Omega \) is a convex polygon. Let \{a_j := (x_j, y_j)\} denote the set of corner nodes of \( \Omega \). Define

\[
H^1_+(\Omega) := \{ g \in H^1(\Omega) \cap L^2_0(\Omega) : \int_\Omega \frac{|g(x,y)|}{(x-x_j)^2 + (y-y_j)^2} \, d\Omega < \infty, \ j = 1, 2, \ldots, l \},
\]

with associated norm

\[
\| g \|_{1,+}^2 := \| g \|_{1,\Omega}^2 + \sum_{j=1}^l \int_\Omega \frac{|g(x,y)|}{(x-x_j)^2 + (y-y_j)^2} \, d\Omega.
\]

Let \((u^0, p^0)\) solves (3.4) in the case of \( \varepsilon = 0 \). Then it is proved in [32] that

\[
\varepsilon^2 \| u \|_{2,\Omega} + \| u \|_{1,\Omega} + \| u - u^0 \|_{0,\Omega} + \| p - p^0 \|_{1,\Omega} + \varepsilon^2 \| u^0 \|_{1,\Omega} + \varepsilon^2 \| p^0 \|_{1,\Omega} \leq \varepsilon \frac{1}{2}(\| f \|_{\text{rot}} + \| g \|_{1,+}),
\]

where \( \| \cdot \|_{\text{rot}} := \| \cdot \|_{0,\Omega} + \| \text{rot}(\cdot) \|_{0,\Omega} \) is the norm defined in \( H(\text{rot}, \Omega) \). Following the technique in [32], we can obtain the following uniform convergence estimate.

**Theorem 3.3.** Let \((u, p)\) be the exact solution of (3.4) and \((u_h, p_h)\) be its approximation in \( V_{h0} \times Q_{h*} \). If \( f \in H(\text{rot}, \Omega) \) and \( g \in H^1_+(\Omega) \), then

\[
\| \text{div} u - \text{div} u_h \| \leq h \| g \|_{1,\Omega},
\]

\[
\| u - u_h \|_{0,\Omega} + \varepsilon \| u - u_h \|_{1,h} \leq h^2(\| f \|_{\text{rot}} + \| g \|_{1,+}),
\]

\[
\| p - p_h \|_{0,\Omega} \leq h^2(\| f \|_{\text{rot}} + \| g \|_{1,+}).
\]

**Proof.** The first estimate is direct since \( \text{div} u = g \). To obtain the second inequality, we first analyze the interpolation error. By (2.2), (2.3), and (3.11), we have

\[
\| u - \Pi_h u \|_{0,\Omega} \leq \| (I - \Pi_h)(u - u^0) \|_{0,\Omega} + \| u^0 - \Pi_h u^0 \|_{0,\Omega}
\]
\[
\leq h^2(\| u - u^0 \|_{0,\Omega})^2 + h^2(\| u - u^0 \|_{1,\Omega})^2 + h^2(\| u^0 \|_{1,\Omega}) \leq h^2(\| f \|_{\text{rot}} + \| g \|_{1,+}).
\]

At the same time,

\[
\varepsilon \| u - \Pi_h u \|_{1,h} \leq \varepsilon \| u^0 \|_{1,\Omega}^2 + \varepsilon \| u - u^0 \|_{1,h}^2 \leq \varepsilon h^2(\| u \|_{0,\Omega}^2 + \| u^0 \|_{0,\Omega}^2) \leq h^2(\| f \|_{\text{rot}} + \| g \|_{1,+}),
\]

where we utilize \( \varepsilon \| u^0 \|_{1,\Omega}^2 + \varepsilon \| u^0 \|_{0,\Omega}^2 \leq \| f \|_{\text{rot}} + \| g \|_{1,+} \).
By the continuity of $v_\epsilon \cdot n_e$ and $\int_{e} v_\epsilon \cdot t_e \, ds$, a standard estimate (see, e.g., [32, Lemma 5.1]) yields
$$\sum_{e \in \mathcal{E}_h} \epsilon^2 \int_e (\nabla u_\epsilon \cdot n_e) \cdot \llbracket v_\epsilon \rrbracket \, ds \lesssim \epsilon^2 h^2 \|u_\epsilon\|_{1/2, \Omega} \|u_\epsilon\|_{1, \Omega} \|v_\epsilon\|_{1/2, \Omega}. $$

Then we derive
$$\left| \sum_{e \in \mathcal{E}_h} \epsilon^2 \int_e (\nabla u_\epsilon \cdot n_e) \cdot \llbracket v_\epsilon \rrbracket \, ds \right| \lesssim \epsilon h^{1/2} [u_\epsilon]_{1/2, \Omega} [u_\epsilon]_{2, \Omega} \lesssim h^{1/2} (\|f\|_{\text{rot}} + \|g\|_{1, +}).$$

A combination of (3.9), (3.15), (3.16), and (3.17) leads to
$$\|u - u_h\|_{0, \Omega} + \epsilon \|u - u_h\|_{1/2, \Omega} \lesssim h^{1/2} (\|f\|_{\text{rot}} + \|g\|_{1, +}).$$

Again from (3.11) and notice that $\epsilon < 1$, we have
$$\|p - p_{\Pi_{Q_0}} p\|_{0, \Omega} \lesssim h |p|_{1, \Omega} \lesssim h \|p - p^0\|_{1, \Omega} + h |p^0|_{1, \Omega} \lesssim h (\|f\|_{\text{rot}} + \|g\|_{1, +}).$$

Hence, by (3.10), (3.18), and (3.19), the last estimate (3.14) is derived. \hfill \Box

4. Numerical Experiments

In this section, we carry out numerical experiments to validate the theory and illustrate the capacity of the newly proposed element pair. Examples are given as illustrations from different perspectives.

- Examples 1 and 2 test the method with the Stokes problem, especially its robustness with respect to the Reynolds’ number and to the triangulations;
- Examples 3 and 4 test the method with the Darcy–Stokes–Brinkman equation, especially the robustness with respect to the the small parameter, for smooth solutions as well as solutions with sharp layers;
- Examples 5 and 6 test the method with the incompressible Navier–Stokes equation, regarding evolutionary and steady states.

Three kinds of $P_2 - P_1$ pairs are involved in the experiments, namely,

- TH: the Taylor-Hood element pair with continuous $P_2$ functions for the velocity space and continuous $P_1$ functions for the pressure space;
- SV: the Scott-Vogelius element pair with continuous $P_2$ functions for the velocity space and discontinuous $P_1$ functions for the pressure space;
- NPP: the newly proposed $P_2 - P_1$ element pair.

All simulations are performed on uniformly refined grids. For the SV pair, an additional barycentric refinement is applied on each grid to guarantee the stability.

Example 1. This example was suggested in [28] to illustrate the non-pressure-robustness of classical elements. Let $\Omega = (0, 1)^2$. Consider the Stokes equations (3.1) with $\epsilon^2 = 1$, $g = 0$, and $f = (0, Ra(1 - y + 3y^2))^T$, where $Ra > 0$ represents a parameter. No-slip boundary conditions are imposed on $\partial \Omega$. The exact solution pair is $u = 0$ and $p = Ra(y^3 - \frac{y^2}{2} + y - \frac{7}{12})$. 
For the continuous problem, different values of $Ra$ result in different exact pressures and the same exact velocity vector. As is shown in Figure 3, for both the SV and NPP pairs, the numerical velocities are very close to zero for different values of $Ra$. However, for the TH pair, the discrete velocity is far from zero, even when $Ra = 1$. It demonstrates the advantage of pressure-robust pairs especially for problems with large pressures.

**Example 2.** This example was also introduced in [28]. Let $\Omega = (0,4) \times (0,2) \setminus [2,4] \times [0,1]$. Consider a flow with Coriolis forces with the following form

$$
\begin{cases}
-\varepsilon^2 \Delta u + \nabla p + 2 w \times u = f, & \text{in } \Omega \\
\text{div } u = 0, & \text{in } \Omega
\end{cases}
$$

where $w = (0,0,w)^T$ is a constant angular velocity vector. Changing the magnitude $w$ will change only the exact pressure, and not the true velocity solution. Dirichlet boundary conditions are imposed on $\partial \Omega$; see Figure 4 (Left). The computed domain and initial unstructured grid are depicted in Figure 4. Simulations were performed with $\varepsilon^2 = 0.01$, while $w = 100$ or $w = 1000$.
Figure 5. Example 2 ($w = 100$ and $\varepsilon^2 = 0.01$): Speed obtained by the TH pair (left column), the SV pair (middle column), and the NPP pair (right column); rows 1 – 3 are results on meshes 0 – 2.

Figure 6. Example 2 ($w = 1000$ and $\varepsilon^2 = 0.01$): Speed obtained by the TH pair (left column), the SV (middle column), and the NPP pair (right column); rows 1 – 3 are results on meshes 0 – 2.

of good quality and commonly used. As is shown in Figure 8, the simulation by the SV pair turns out to be not reliable on the grid, while the NPP pair plays fine.
Figure 7. Forward facing step domain and structured non-barycentric mesh.

Figure 8. Example 2 ($w = 100$ and $\varepsilon^2 = 0.01$): Simulation by the SV pair (middle) is not as good as the TH pair (left) nor the NPP pair (right) on non-barycentric mesh.

Example 3. Let $\Omega = (0, 1)^2$. We consider the Darcy–Stokes–Brinkman problem such that

$$ u = \text{curl} \left( \sin^2(\pi x) \sin^2(\pi y) \right) = \pi \begin{pmatrix} \sin^2(\pi x) \sin(2\pi y) \\ -\sin^2(\pi y) \sin(2\pi x) \end{pmatrix}; \quad p = \frac{2}{\pi} - \sin(\pi x). $$

The force $f$ is computed by $f = -\varepsilon^2 \Delta u + u + \nabla p$, and $g = \text{div} u = 0$. The solution is smooth and independent of $\varepsilon$. We start from an unstructured initial triangular grid, and it is successively refined to maintain the quality of grids.

In Figure 9, we draw convergence curves of the NPP pair with different values of $\varepsilon$, where curves represents actual error declines while triangles illustrates theoretic convergence rates correspondingly. As is shown, when $0 < \varepsilon < 1$, errors in $L^2$-norm are of $O(h^3)$ order and errors in $H^1$-norm are of $O(h)$ order. In the limiting case of $\varepsilon = 0$, the $L^2$-norm error reaches $O(h^3)$ order and $H^1$-norm error reaches $O(h^2)$ order, which is due to the fact that $V_{h0}$ is a conforming subspace of $H(\text{div}, \Omega)$.

Figure 9. Example 3: Velocity errors in the $L^2$-norm (left) and in the $H^1$-norm (right) by the NPP method.
In Figure 10, we present the errors in the norm \( \| \cdot \|_{e,h} \) by the TH pair and the NPP pair when \( \varepsilon = 2^{-8} \). Here \( \| \varepsilon \|_{e,h} := \varepsilon^2 \| \cdot \|_{1,h}^2 + \| \varepsilon \|_{0,\Omega} + \| \text{div} \, \varepsilon \|_{0,\Omega}^2 \) is the commonly used norm which combines the Stokes and Darcy problems. Although the convergence rate of the NPP pair is one order lower than that of the TH pair, the error of the former is smaller (several magnitudes) than that of the latter in the figure where millions of DOFs have been used on the finest grid. For the NPP pair, the associated energy error of velocity is close to \( 10^{-3} \), while for the TH pair, it does not reach an error of \( 10^{-3} \) even on the eighth level mesh. However, as remarked in the figure, the degrees of freedom of the TH pair (on the eighth level mesh) is over 500 times more than the NPP pair (on the third level mesh).

**Example 4.** Let \( \Omega = (0, 1)^2 \). Consider the Darcy–Stokes–Brinkman problem such that:

\[
\mathbf{u} = \varepsilon \text{curl} \left( e^{-\frac{\varepsilon}{x^2}} \right) = \begin{pmatrix} -xe^{-\frac{\varepsilon}{x^2}} \\ ye^{-\frac{\varepsilon}{x^2}} \end{pmatrix}; \quad p = -\varepsilon e^{-\frac{\varepsilon}{x^2}}.
\]

The boundary layers of both components of the exact velocity \( \mathbf{u} \) are shown in Figure 11.

**Figure 10.** Example 3: Errors of velocity in the energy norm by the NPP pair and the TH pair when \( \varepsilon = 2^{-8} \).

**Figure 11.** Example 4: \( x \)-component (left) and \( y \)-component (right) of the exact velocity with boundary layers when \( \varepsilon = 2^{-4} \).
Table 1. Example 4 (with boundary layers): Errors of velocity in the energy norm by the NPP element.

| $h$   | $2.599E-1$ | $1.300E-01$ | $6.498E-02$ | $3.249E-02$ | $1.625E-02$ | Rate |
|-------|------------|-------------|-------------|-------------|-------------|------|
| $2^{-4}$ | 2.998E-02 | 1.147E-02 | 4.958E-03 | 2.430E-03 | 1.228E-03 | 1.15 |
| $2^{-6}$ | 6.589E-02 | 3.000E-02 | 1.159E-02 | 4.228E-03 | 1.753E-03 | 1.33 |
| $2^{-8}$ | 1.061E-01 | 6.246E-02 | 3.238E-02 | 1.438E-02 | 5.504E-03 | 1.07 |
| $2^{-10}$ | 1.171E-01 | 7.906E-02 | 5.147E-02 | 3.061E-02 | 1.601E-02 | 0.71 |
| $2^{-12}$ | 1.234E-01 | 8.455E-02 | 5.688E-02 | 3.848E-02 | 2.529E-02 | 0.57 |

Table 2. Example 4 (with boundary layers): Errors of pressure in the $L^2$-norm by the NPP element.

| $h$   | $2.599E-1$ | $1.300E-01$ | $6.498E-02$ | $3.249E-02$ | $1.625E-02$ | Rate |
|-------|------------|-------------|-------------|-------------|-------------|------|
| $2^{-4}$ | 2.260E-03 | 8.080E-04 | 2.884E-04 | 1.211E-04 | 5.702E-05 | 1.34 |
| $2^{-6}$ | 2.779E-03 | 7.880E-04 | 2.696E-04 | 9.042E-05 | 2.938E-05 | 1.63 |
| $2^{-8}$ | 6.283E-03 | 2.044E-03 | 5.366E-04 | 1.273E-04 | 3.448E-05 | 1.90 |
| $2^{-10}$ | 6.730E-03 | 3.056E-03 | 1.339E-03 | 4.607E-04 | 1.235E-04 | 1.43 |
| $2^{-12}$ | 6.710E-03 | 3.044E-03 | 1.440E-03 | 7.024E-04 | 3.216E-04 | 1.09 |

Theorem 3.3. From Table 2, the discrete pressure exhibits $O(h^2)$ order of convergence, higher than the theoretical estimation $O(h^{1/2})$ order.

Example 5. Let $\Omega = (0,1)^2$. Consider the incompressible Navier–Stokes equations

\[
\begin{aligned}
\partial_t u - \varepsilon^2 \Delta u + (u \cdot \nabla)u + \nabla p &= f & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega,
\end{aligned}
\]

with prescribed solution

\[
\begin{pmatrix}
\sin(1-x) \sin(y+t) \\
- \cos(1-x) \cos(y+t)
\end{pmatrix}; \quad p = -\cos(1-x) \sin(y+t).
\]

In this example, the Crank–Nicolson scheme is used for time discretization, and the Newton linearization is adopted to handle the nonlinear term. To isolate the spatial error, let the time-step $dt = 10^{-3}$ and the final time be $10^{-2}$. Unstructured subdivisions illustrated in Example 2 are utilized.

As is depicted in Table 3 with $\varepsilon^2 = 10^{-6}$, solutions by the TH pair converge with $O(h^{3/2})$ order in the $L^2$ norm, and by the SV pair they converge with $O(h^2)$ order. It is analyzed in [31] that the TH pair loses order mainly because it is not pressure-robust, while the suboptimal result of the SV pair is due to additional error sources arising from the nonlinear term. The NPP pair exhibits a convergence rate of $O(h^2)$ order which is consistent with its theoretical analysis, and it gives even a more accurate approximation than the SV pair in this case.
Table 3. Example 5 ($\varepsilon^2 = 10^{-6}$): Errors of velocity in the $L^2$-norm.

| $h$   | $\|(u - u_h)(T)\|_{0,\Omega}$ Rate | $\|(u - u_h)(T)\|_{0,\Omega}$ Rate | $\|(u - u_h)(T)\|_{0,\Omega}$ Rate |
|-------|-------------------------------------|-------------------------------------|-------------------------------------|
| 2.599E-01 | 1.746E-04 – | 1.031E-04 | 7.128E-05 – |
| 1.300E-01 | 6.006E-05 1.54 | 1.362E-05 2.92 | 9.407E-06 2.92 |
| 6.498E-02 | 2.158E-05 1.48 | 1.989E-06 2.78 | 1.377E-06 2.77 |
| 3.249E-02 | 7.583E-06 1.51 | 3.561E-07 2.48 | 2.488E-07 2.47 |
| 1.625E-02 | 2.524E-06 1.59 | 7.790E-08 2.19 | 5.455E-08 2.19 |

Example 6. Let $\Omega = (0, 1)^2$ be a square domain. Consider the Navier–Stokes equations (4.1) with boundary conditions $u = (-1, 0)^T$ on the side $y = 1$ and $u = (0, 0)^T$ on the other three sides. Take the viscosity as $\varepsilon^2 = 10^{-3}$.

The backward-Euler time-stepping scheme and the Picard iteration are adopted for this example. Set the time step to be $dt = 0.1$. Consider a long time simulation with the final time equals 90 to derive a steady solution. Indeed, as the solution is steady, the choice of time scheme has little influence on the accuracy of the final solution. Referenced data in a benchmark work [8] are involved to make a reliable comparison, where the solutions are derived on a rather fine mesh, i.e., $1024 \times 1024$ rectangular subdivision of domain $\Omega$. We wish to see whether major features of the steady-state flow can be captured on a coarse mesh with $43 \times 43 \times 2$ cells.

Isolines of the streamfunction, vorticity, and pressure fields are displayed in Figures 12, 13, and 14, respectively. Compared with the TH pair, the shapes of contour maps derived by the NPP pair are closer to the reference solution. Specially, the colormap of pressure obtained by the NPP pair and the TH pair are quite different; note the difference between the sidebars of them. By the values given in [8, Table 1], the NPP pair method gives more accurate approximation of pressure then the TH pair does.

Figure 12. Example 6 (streamfunction): Isolines given by the NPP pair (middle) is closer to the reference solution (right) than the TH pair (left).

Moreover, the extremes of the streamfunction and the vorticity are depicted in Tables 4 and 5, respectively. Both of them indicate that the NPP pair gives closer results to the benchmark reference results.
Figure 13. Example 6 (vorticity): Isolines derived by the NPP pair (middle) is closer to the reference solution (right) than the TH pair (left).

Figure 14. Example 6 (pressure): The extreme values of the pressure by the TH pair (left) is notably different with these by the NPP pair (middle), and the latter is closer to the reference values [8].

Table 4. Example 6 (streamfunction): Values on the primary and the lower-left secondary vortices.

| Scheme  | Mesh  | Primary x   | y   | Secondary x | y   |
|---------|-------|-------------|-----|-------------|-----|
| TH      | $43 \times 43$ | $1.0862E-01$ | 0.4688 | $0.5703$ | $-1.3882E-03$ | 0.1328 | 0.1094 |
| NPP     | $43 \times 43$ | $1.1733E-01$ | 0.4688 | $0.5703$ | $-1.6221E-03$ | 0.1406 | 0.1094 |
| Ref.    | $1024 \times 1024$ | $1.1892E-01$ | 0.4688 | $0.5654$ | $-1.7292E-03$ | 0.1367 | 0.1123 |

Table 5. Example 6 (vorticity): Values on the primary and the lower-left secondary vortices.

| Scheme  | Mesh  | Primary x   | y   | Secondary x | y   |
|---------|-------|-------------|-----|-------------|-----|
| TH      | $43 \times 43$ | $1.8976E+00$ | 0.4688 | $0.5703$ | $-9.1294E-01$ | 0.1328 | 0.1094 |
| NPP     | $43 \times 43$ | $2.0615E+00$ | 0.4688 | $0.5703$ | $-9.8718E-01$ | 0.1406 | 0.1094 |
| Ref.    | $1024 \times 1024$ | $2.0674E+00$ | 0.4688 | $0.5654$ | $-1.1120E+00$ | 0.1367 | 0.1123 |

The velocity along the centerlines of the cavity is also an important quantity of concern in the literature. We can see, from Figure 15, that the results computed by the NPP pair are in better agreement with the reference results in Ref. [8] than the TH pair.
Appendix A. Dimension of the local space \( \ker(\text{div}, V_{h0}(M)) \)

This appendix is devoted to analyze the basis functions of \( \ker(\text{div}, V_{h0}(M)) \) defined on a macroelement \( M \). Lemma 2.7 is proved at the end of this section.

A.1. Local structure of divergence-free functions. Let \( T \) be a triangle with nodes \( \{a_i, a_j, a_k\} \) and edges \( \{e_i, e_j, e_k\} \). Denote \( n_{T,e_l} \) as a unit outward vector normal to \( e_l \) and \( t_{T,e_l} \) as a unit tangential vector of \( e_l \) such that \( n_{T,e_l} \times t_{T,e_l} > 0 \), where \( l \in \{i, j, k\} \). Denote the lengths of edges by \( \{l_i, l_j, l_k\} \), and the area of \( T \) by \( S \).

\[ W_{T,e_j,e_k} := \left\{ \psi \in (P_2(T))^2 : \text{div} \psi = 0, \int_{e_i} \psi \cdot t_{e_i} = 0, \int_{e_i} \psi \cdot n_{e_i} q = 0, \forall q \in P_2(e_i) \right\} \]

namely, \( W_{T,e_j,e_k} \) consists of quadratic polynomials that are divergence-free and all nodal parameters associated with \( e_j \) equal to zero. Denote

\[ W^n_{T,e_j,e_k} := \left\{ \psi \in W_{T,e_j,e_k} : \int_{e_j} \psi \cdot n_{e_j} = 0, \int_{e_k} \psi \cdot n_{e_k} = 0 \right\} \]

and

\[ W_{T,e_k} := W_{T,e_j,e_k} \cap W_{T,e_l,e_i} \]

Direct calculation leads to the following results.

**Lemma A.1.** \( \dim(W_{T,e_k}) = 1 \), \( \dim(W^n_{T,e_j,e_k}) = 4 \), and \( \dim(W_{T,e_j,e_k}) = 5 \).
A set of basis functions can be constructed explicitly for the local spaces. Recall that $\varphi_{n, e_i, 0}$, $\varphi_{n, e_i, 1}$, $\varphi_{n, e_i, 2}$, and $\phi_{e_i, 0}$ represent the basis functions on $e_i \in \{i, j, k\}$. Let

$$W_{T, e_j} = \frac{2S}{3l_j} \varphi_{n, e_j, 1} - \varphi_{T, e_j, 0}, \quad W_{T, e_k} = - \frac{2}{3l_k} \varphi_{n, e_k, 1} + \varphi_{T, e_k, 0}, \quad W_{T, e_0} = - \frac{1}{l_0} \varphi_{n, e_0, 1} + \frac{1}{10l_0} \varphi_{n, e_0, 2} - \frac{2}{3l_k} \varphi_{n, e_0, 1},$$

Then

$$W_{T, e_i} = \text{span}\{W_{T, e_j}, W_{T, e_k}, W_{T, e_0}, W_{T, e_i}\}, \quad W_{T, e_i} = \text{span}\{W_{T, e_i}\}.$$

**A.2. Divergence-free functions on sequentially connected cells.** Let $T_1$ and $T_2$ be two adjacent cells such that $\overline{T_1} \cap \overline{T_2} = e_2$. Denote

$$\mathcal{V}_{h}(T_1 \cup T_2) := \{\psi_h \in L^2(T_1 \cup T_2) : \psi_h|_{T_i} \in (P_2(T_i))^2, \ l = 1, 2, \ \psi_h \cdot n_e \text{ and } \int_e \psi_h \cdot t_e \ ds \text{ are continuous across } e_2\},$$

$$\mathcal{V}_{h,0}(T_1 \cup T_2) := \{\psi_h \in \mathcal{V}_{h}(T_1 \cup T_2) : \psi_h \cdot n_e \text{ and } \int_e \psi_h \cdot t_e \text{ vanish on } \partial(T_1 \cup T_2)\},$$

$$\mathcal{V}_{h,e_3}(T_1 \cup T_2) := \{\psi_h \in \mathcal{V}_{h}(T_1 \cup T_2) : \psi_h \cdot n_e \text{ and } \int_e \psi_h \cdot t_e \text{ vanish on } \partial(T_1 \cup T_2) \setminus e_3\}.$$

**Lemma A.2.** $\dim(\mathcal{V}_{h,0}(T_1 \cup T_2)) = 0$, $\dim(\mathcal{V}_{h,e_3}(T_1 \cup T_2)) = 2$.

![Figure 17. Two adjacent cells; Degrees of freedom vanish on dotted edges.](image-url)

**Figure 17.** Two adjacent cells; Degrees of freedom vanish on dotted edges.

**Proof.** Given $\psi_h \in \mathcal{V}_{h,0}(T_1 \cup T_2)$, $\psi_h|_{T_1} = \alpha \varphi_{T_1, e_2}$ and $\psi_h|_{T_2} = \beta \varphi_{T_2, e_2}$. It is easy to verify that the weak continuity conditions imposed on $e_2$ makes $\alpha = \beta = 0$, namely $\psi_h = 0$.

Similarly, we can prove $\dim(\mathcal{V}_{h,e_3}(T_1 \cup T_2)) = 2$. Specifically, two basis functions are

$$\psi_h^1 \text{ such that } \psi_h^1|_{T_1} = W_{T, e_2} \text{ and } \psi_h^1|_{T_2} = W_{T, e_2} + \frac{S_1 + S_2}{d_2} W_{T, e_3, e_2}$$

and

$$\psi_h^2 \text{ such that } \psi_h^2|_{T_1} = 0 \text{ and } \psi_h^2|_{T_2} = W_{T, e_3}.$$

□
To admit a nontrivial basis function, at least four cells are needed.

**Lemma A.3.** Let $\omega$ be a subdomain composed of three continuous cells, and its first and last cells are not connected; see Figure 18. If $\mathbf{v}_h \in \mathbf{V}_h(\omega)$, $\text{div}(\mathbf{v}_h) = 0$, and degrees of freedom of $\mathbf{v}_h$ vanish on $\partial \omega$, then $\mathbf{v}_h = 0$.

![Figure 18. No trivial divergence-free function on three continuous cells](image1)

**Lemma A.4.** Let $\omega$ be a subdomain composed of four continuous cells, and its first and last cells are not connected; see Figure 19. A local space on $\omega$ is defined as

$$
\mathbf{V}_{h0,\omega} := \{ \mathbf{v}_h \in L^2(\omega) : \mathbf{v}_h|_T \in (P_2(T))^2, \forall T \in \omega, \mathbf{v}_h \cdot \mathbf{n}_e \text{ and } \int_e \mathbf{v}_h \cdot \mathbf{t}_e \, ds \text{ are continuous across interior edges and vanish on edges lying on } \partial \omega \},
$$

then $\dim(\ker(\text{div}, \mathbf{V}_{h0,\omega})) = 1$.

![Figure 19. Patch $\omega$ composed of four continuous cells sharing an only vertex $a_1$.](image2)

**Proof.** We will complete the proof in four steps.

**Step 1.** Consider $T_1$ and $T_4$. We have

$$
\mathbf{v}_h|_{T_1} = r \mathbf{w}_{T_1,e_2}, \quad \text{and} \quad \mathbf{v}_h|_{T_4} = s \mathbf{w}_{T_4,e_4}.
$$

with constants $r$ and $s$.

**Step 2.** Consider two adjacent cells $T_1 \cup T_2$. From Lemma A.2, it holds that

$$
\mathbf{v}_h|_{T_2} = r \mathbf{w}_{T_2,e_2} + \left( \frac{S_1 + S_2}{d_2} r \right) \mathbf{w}_{T_2,e_1,e_2} + b_4 \mathbf{w}_{T_2,e_3}.
$$

with $b_4$ to be determined.
Step 3. Consider two adjacent cells $T_3 \cup T_4$. Similar to Step 2, we have
\begin{equation}
\varphi_h|_{T_3} = c_1 w_{T_3,e_3} - \left( \frac{S_4 + S_3}{d_4} s \right) w_{T_3,e_3,e_4} + s w_{T_3,e_4},
\end{equation}
with $c_1$ to be determined.

Step 4. Consider two adjacent cells $T_2 \cup T_2$. Utilizing the continuity of $\varphi_h \cdot n$ and $\int_{e_3} \varphi_h \cdot t$ on $e_3$, and the representation of $\varphi_h$ in (A.4) – (A.6), we derive
\begin{equation}
b_4 = c_1 = 0, \quad s = -\frac{(S_1 + S_2)d_4}{(S_4 + S_3)d_2} r.
\end{equation}
Therefore, $\varphi_h$ is determined once the constant $r$ is given and $\dim(\ker(\text{div}, \varphi_h)) = 1$.

Remark A.5. Figure 18 can degenerate to a patch which admits only trivial function; see Figure 20.

Remark A.6. Let $T_1$, $T_2$ and $T_3$ be three adjacent cells, such that $\overline{T_1} \cap \overline{T_2} = e_2$, $\overline{T_2} \cap \overline{T_3} = e_3$, and $\overline{T_3} \cap \overline{T_1} = e_1$. We may treat this triple of cells as a degenerate case of a four cell sequence $T_1 - T_2 - T_3 - T_1$, and the degenerate patch may inherit the divergence-free basis function on the patch $T_1 - T_2 - T_3 - T_1$.

Figure 20. Degenerate case: three cells form a patch

Figure 21. The first and the last cells of the left pattern overlap to form the right pattern
Take \( T_1 \) to be the overlapping cell. The function (corresponding to (A.8)), denoted by \( \psi_1 \), satisfies
\[
\psi_1|_{T_1} = \frac{-d_2}{S_1 + S_2} \psi_{T_1,e_2} + \frac{d_1}{S_3 + S_1} \psi_{T_1,e_1},
\]
and
\[
\psi_1|_{T_3} = -\psi_{T_3,e_3,e_1} + \frac{d_1}{S_3 + S_1} \psi_{T_3,e_1}.
\]
(A.9)

The key ingredient is that \( \psi_1|_{T_1} \) consists of \( \psi_{T_1,e_2} \) and \( \psi_{T_1,e_1} \); note that this is just the dominant ingredients of the function (corresponding to \( T_1 - T_2 - T_3 - T_1 \)) on the two end cells.

### A.3. Structure of the kernel space on an \( m \)-macroelement

Let \( M = \bigcup_{s=1}^{m} T_s \) be an \( m \)-macroelement, and \( \overline{T}_s \cap \overline{T}_{s+1} = e_{s+1} \). Particularly, \( T_{m+1} = T_1 \) and \( e_{m+1} = e_1 \). In the following context, the subscript \( s \) actually refers to \( s - m \) if it is calculated to be great than \( m \).

![A macroelement composed of six cells with one interior vertex](image)

**Figure 22.** A macroelement composed of six cells with one interior vertex

**Definition A.7.** Let a subdomain \( \omega_i \) be composed of continuous cells \( T_i \cup T_{i+1} \cup T_{i+2} \cup T_{i+3} \), and \( e_{i+1}, e_{i+2}, \) and \( e_{i+3} \) be its three interior edges. If

(i) for \( m \geq 4 \), \( \psi|_{\omega_i} \in \text{ker} \langle \text{div}, V_{h0,\omega_i} \rangle \) satisfies condition (A.8), and \( \psi \) vanishes on \( M \setminus \omega_i \),

(ii) for \( m = 3 \), \( \psi|_{\omega_i} \in \text{ker} \langle \text{div}, V_{h0,\omega_i} \rangle \) satisfies condition (A.9),

then \( \psi \) is called an atom function on \( M \).

Therefore, there exist \( m \) atom functions on an \( m \)-macroelement. Recall
\[
\text{ker} \langle \text{div}, V_{h0,M} \rangle = \{ \psi_h \in V_{h0,M} : \text{div} \psi_h = 0 \}
\]
and denote
\[
Z_{h0}^n(M) := \{ \psi_h \in \text{ker} \langle \text{div}, V_{h0,M} \rangle : \int_{e_i} \psi_h \cdot n_{e_i} = 0, \ 1 \leq i \leq m \},
\]
(A.11)

For an atom function \( \psi_j \) on the \( m \)-macroelement \( M \), \( \text{supp}(\psi_j) := T_i \cup T_{i+1} \cup T_{i+2} \cup T_{i+3} \) \( (m \geq 4) \) or \( \text{supp}(\psi_j) := T_i \cup T_{i+1} \cup T_{i+2} \) \( (m = 3) \). \( T_{i+s} \) is called the \( (s + 1) \)-th cell of \( \text{supp}(\psi_j) \).

**Lemma A.8.** It holds that \( Z_{h0}^n(M) = \text{span} \{ \psi_j \} \}_{1 \leq i \leq m} \).

**Proof.** Here we adopt a sweeping procedure (c.f., Ref. [50]) to conduct the proof. Let \( T \) be an arbitrary cell in \( M \), and \( T_R \cup T_L \) be three cells in \( M \) arranged in a clockwise direction. Let \( e_R := \overline{T}_R \cap \overline{T} \) and \( e_L := \overline{T} \cap \overline{T}_L \). Use \( S_L \) and \( S_R \) to represent the areas of \( T_L \) and \( T_R \), respectively. Let \( d_L \) and \( d_R \) represent the lengths of \( e_L \) and \( e_R \).
Given \( \psi_h \in Z_{h0}^{n} \), there exists \( \alpha_i \in \mathbb{R} \), \( 1 \leq i \leq 4 \), such that
\[
\psi_h|_T = \alpha_1 w_{T,el} + \alpha_2 w_{T,el,er} + \alpha_3 w_{T,el,er,el} + \alpha_4 w_{T,er}.
\]
Let \( \psi_{k_1} \), \( \psi_{k_2} \), and \( \psi_{k_3} \) be three atom functions satisfy that \( T \) is the first, second, and third cell of \( \text{supp}(\psi_{k_1}) \), \( \text{supp}(\psi_{k_2}) \), and \( \text{supp}(\psi_{k_3}) \), respectively. Then
\[
\psi_{k_1}|_T = \frac{-d_L}{S + S_L} w_{T,el} \quad \text{or} \quad \psi_{k_1}|_T = \frac{-d_L}{S + S_L} w_{T,el} + \frac{d_R}{S + S_R} w_{T,er} \quad \text{when} \quad m = 3,
\]
\[
\psi_{k_2}|_T = \frac{-d_R}{S + S_R} w_{T,el} - \frac{d_L}{S + S_L} w_{T,er} \quad \text{and} \quad \psi_{k_3}|_T = \frac{-d_R}{S + S_R} w_{T,el} - \frac{d_l}{S + S_L} w_{T,er,el}.
\]
Notice that \( w_{T,er,k_L} = 0 \) in the sense of DOF, i.e., the degrees of freedom associated with \( e_L \) of \( w_{T,er} \) all vanish. Set
\[
\tilde{z}_0 = \left( -\frac{S + S_L}{d_L} \psi_{k_1} - \alpha_2 \psi_{k_2} - \alpha_3 (\psi_{k_1} + \psi_{k_3}) \right) := r_1 \psi_{k_1} + r_2 \psi_{k_2} + r_3 \psi_{k_3},
\]
then \( (\psi_h - \tilde{z}_0)|_{e_L} = 0 \) in the sense of DOF.

(i) If \( m = 3 \), it holds \( \tilde{z}_h - \tilde{z}_0 \) vanish on \( M \) by Remark A.5. Hence \( \psi_h = \sum_{l=1}^{m} r_l \psi_{k_l} \).

(ii) If \( m > 3 \), consider the left cell \( T_L \) adjacent to \( T \). There exists \( \psi_{k_4} \) such that \( T_L \) is the first cell of \( \text{supp}(\psi_{k_4}) \). Therefore, \( (\psi_h - \tilde{z}_0)|_{T_L} \in \text{span}(\psi_{k_1}|_{T_L}) \). Hence there exists a constant \( r_4 \), such that
\[
(\psi_h - \tilde{z}_0)|_{T_L} = r_4 \psi_{k_4},
\]
and then \( (\psi_h - \sum_{l=1}^{4} r_l \psi_{k_l}) \) vanishes on \( T_L \). Therefore, the number of supporting cells is reduced to \( m - 1 \). Conduct a similar analysis to the next left cell, and the number of supporting cells can also be reduced by one. Repeat this process until the number of supporting cells is smaller than three, and they form a pattern as shown in Remark A.5 (left). Finally it can be derived that \( \tilde{z}_h = \sum_{l=1}^{m} r_l \psi_{k_l} \).

Proof of Lemma 2.7. It suffices for us to show \( \dim(\ker(\text{div}, \nu_{h,0,M})) \leq \dim(Z_{h0}^{n}(M)) + 1 \), and Lemma 2.7 follows by Lemma A.8.

Let \( n_{e_L} \) be the unit normal vector on an interior edge \( e_L \) with \( l = 1 : m \), whose direction is from \( a_0 \) to \( a_m \). Given \( \psi_h \in \ker(\text{div}, \nu_{h,0,M}) \), then the divergence theorem leads to \( \int_{\nu_{h} \cdot n_{e_L}} = \int_{\nu_{h} \cdot n_{e_i}} \) with \( 1 \leq i, j \leq m \). Assume there exists a function \( \psi_h \in \ker(\text{div}, \nu_{h,0,M}) \), such that \( \int_{\nu_{h} \cdot n_{e_i}} = 1 \) with \( i = 1 : m \). Then, \( \psi_h \in \ker(\text{div}, \nu_{h,0,M}) \) can be uniquely decomposed into \( \psi_h = \alpha \psi_{h} + \psi_{h}^{1} \) with \( \psi_{h}^{1} \in Z_{h0}^{n} \), where \( \alpha \) represents a constant. Namely, \( \ker(\text{div}, \nu_{h,0,M}) = Z_{h0}^{n}(M) + \text{span}(\psi_{h}) \). If such a function \( \psi_{h}^{1} \) does not exist, then \( \ker(\text{div}, \nu_{h,0,M}) = Z_{h0}^{n}(M) \). In any event, \( \dim(\ker(\text{div}, \nu_{h,0,M})) \leq \dim(Z_{h0}^{n}(M)) + 1 \). The proof is completed.

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