Fourier spectral approximation for the convective Cahn-Hilliard equation in 2D case

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Abstract
In this paper, we consider the Fourier spectral method for numerically solving the 2D convective Cahn-Hilliard equation. The semi-discrete and fully discrete schemes are established. Moreover, the existence, uniqueness and the optimal error bound are also considered.

Keywords: Fourier spectral method, convective Cahn-Hilliard equation, error estimate.

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1. Introduction
Suppose that $\Omega = [0, L_1] \times [0, L_2]$, $L_1, L_2 > 0$. We consider the following problem for the 2D convective Cahn-Hilliard equation. We seek a real-valued function $u(x, y, t)$ defined on $\Omega \times [0, T]$

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u = \Delta \varphi(u) + \nabla \cdot \psi(u), \quad (x, y) \in \Omega, \quad t \in (0, T],$$

where $\varphi(u) = \gamma_2 u^3 + \gamma_1 u^2 - u$, $\psi(u) = u^2$. $\gamma > 0$, $\gamma_2 > 0$ and $\gamma_1$ are constants. On the basis of physical considerations, as usual Eq. (1) is supplemented with the following boundary value conditions

$$u(x, y, t) = \Delta u(x, y, t) = 0, \quad (x, y) \in \partial \Omega,$$

and the initial value condition

$$u(x, y, 0) = u_0(x), \quad (x, y) \in \Omega.$$
Eq. (1) is a typical fourth order parabolic equation, which arises naturally as a continuous model for the formation of facets and corners in crystal growth, see [1,2]. Here \( u(x,t) \) denotes the slope of the interface. The convective term \( \nabla \cdot \psi(u) \) (see [1]), stems from the effect of kinetic that provides an independent flux of the order parameter, similar to the effect of an external field in spinodal decomposition of a driven system.

During the past years, many authors have paid much attention to the convective Cahn-Hilliard equation. It was K. H. Kwek [3] who first studied the convective Cahn-Hilliard equation for the case with convection, namely, \( \psi(u) = u \). By some a priori estimates, he proved the existence of a classical solution, and gave the error estimates by the discontinuous Galerkin method. Zarksm et al [4] investigate bifurcations of stations periodic solutions of a convective Cahn-Hilliard equation, they described phase separation in driven systems, and studied the stability of the main family of these solutions. Eden and Kalantarov [5,6] considered the convective Cahn-Hilliard equation with periodic boundary conditions in one space dimension and three space dimension. They established some result on the existence of a compact attractor. Recently, Gao and Liu [7] studied the instability of the traveling waves of the 1D convective Cahn-Hilliard equation. Zhao and Liu [8,9] considered the optimal control problem for the convective Cahn-Hilliard equation in 1D and 2D case. For more recent results on the convective Cahn-Hilliard equation, we refer the reader to [10,11,12] and the references therein.

It is known to all, spectral methods are essentially discretization methods for the approximate solution of partial differential equations. They have the natural advantage in keeping the physical properties of primitive problems [13,14,15]. On the other hand, until to now, there’s no numerical results on the convective Cahn-Hilliard equation by spectral methods. So, in this paper, a Fourier spectral method for numerically solving problem (1)-(3) is developed.

**Remark 1.** For the classical Cahn-Hilliard equation (see [14,15,16,17]), there are two important features: conservation of mass and the existence of Lyapunov functional. These two properties play important roles both in Cahn-Hilliard equation’s mathematical theoretical analysis and its numerical analysis. They are used to estimate the absolute pointwise maximum value of the solution. However, for problem (1)-(3), the two important properties might not be existent. This means that we should find another useful approach to estimate the absolute pointwise maximum value of the solution.
We now consider the Fourier spectral method for the problem (1)-(3), the existence of a solution locally in time is proved by the standard Picard iteration, global classical existence results can be found in [12]. Adjusted to our needs, the results is given in the following form:

**Theorem 2.** Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), Then problem (7)-(8) admits a unique solution \( u \) such that

\[
    u \in L^2([0, T]; H^4(\Omega)) \cap L^\infty([0, T]; H^2(\Omega)), \quad \forall T > 0,
\]

This paper is organized as follows. In the next section, we consider a semi-discrete Fourier spectral approximation, prove its existence and uniqueness of the numerical solution and derive the error bound. In section 3, we consider the full-discrete approximation for problem (1)-(3). Furthermore, we prove convergence to the solution of the associated continuous problem. In the last section, some numerical experiments which confirm our results are performed.

Throughout this paper, we denote \( L^2, L^p, L^\infty, H^k \) norm in \( \Omega \) simply by \( \| \cdot \| \), \( \| \cdot \|_p \), \( \| \cdot \|_\infty \) and \( \| \cdot \|_{H^k} \).

2. Semi-discrete approximation

In this section, we consider the semi-discrete approximation for problem (1)-(3). First of all, we recall some basic results on the Fourier spectral method which will be used throughout this paper. Let \( L_1 = L_2 = 2\pi \), \( 2N_1, 2N_2 \) be any positive integers. In the continuation of this work, let \( N_1 = N_2 = N \), \( h = \frac{2\pi}{N} \), \( x_i = ih, y_j = jh, i, j \in \Lambda \), where \( \Lambda = 1, 2, \cdots, 2N \). For any integer \( 2N > 0 \), we introduce the finite dimensional subspace of \( H^2(\Omega) \cap H^1_0(\Omega) \):

\[
    S_N = \text{span} \{ \sin k_1 x \sin k_2 y, \quad k_1, k_2 \in \Lambda \},
\]

Let \( P_N : L^2(\Omega) \to S_N \) be an orthogonal projecting operator which satisfies:

\[
    (u - P_N u, v) = 0, \quad \forall v \in S_N. \tag{4}
\]

For operator \( P_N \) and functions in \( S_N \), we have the following results (see [13, 15, 17]):

(B1) \( P_N \) commutes with derivation on \( H^2 \), i. e.

\[
    P_N \Delta u = \Delta P_N u, \quad \forall u \in H^2(\Omega) \cap H^1_0(\Omega).
\]
(B2) For any real $0 \leq \mu \leq \sigma$, there is a constant $c$, such that

$$\|u - P_N u\|_\mu \leq c N^{\mu - \sigma} \|\nabla^\sigma u\|, \ \forall u \in H^\sigma(\Omega).$$

We define the Fourier spectral approximation: For each $N \geq 1$, find

$$u_N(t) = \sum_{j=1}^{N} a_j(t) \sin k_1 x \sin k_2 y \in S_N$$

such that $\forall v_N \in S_N$,

$$\left( \frac{\partial u_N}{\partial t}, v_N \right) + \gamma (\Delta u_N, \Delta v_N) = \left( \varphi(u_N), \Delta v_N \right) + \left( \nabla \cdot \psi(u_N), v_N \right), \ (5)$$

for all $t \in [0,T]$ with $u_N(0) = P_N u_0$.

Now, we are going to establish the existence, uniqueness et. al. of the Fourier spectral approximation solution $u_N(t)$ for $t \geq 0$.

**Lemma 3.** Suppose that $u_0 \in L^2(\Omega)$. Then, (5) has a unique solution $u_N(t)$ satisfying the following inequalities:

$$\|u_N(t)\|^2 \leq e^{c_1 t} \|u_0\|^2, \ \forall t \in (0,T), \ (6)$$

and

$$\int_0^t \|\Delta u_N(\tau)\|^2 d\tau \leq \left( \frac{c_1 t e^{c_1 t} + 1}{\gamma} \right) \|u_0\|^2, \ t \in (0,T), \ (7)$$

where $c_1$ is a positive constant depends only on $\gamma, \gamma_1, \gamma_2$ and the domain.

**Proof.** Set $v_N = \sin j_1 x \sin j_2 y$ in (5) for each $j$ ($1 \leq j \leq N$) to obtain

$$\frac{d}{dt} a_j(t) = f_j(a_1(t), a_2(t), \cdots, a_j(t)), \ j = 1, 2, \cdots, N, \ (8)$$

where all $f_j : \mathbb{R}^N \rightarrow \mathbb{R}$ ($1 \leq j \leq N$) are smooth and locally Lipschitz continuous. Noticing that $u_N(0) = P_N u_0$, then

$$a_j(0) = (u_0, \phi_j), \ j = 1, 2, \cdots, N. \ (9)$$

Using the theory of initial-value problems of the ordinary differential equations, there is a time $T_N > 0$ such that the initial-value problem (8)-(9) has a unique smooth solution $(a_1(t), a_2(t), \cdots, a_N(t))$ for $t \in [0, T_N]$. 

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Setting \( v_N = u_N \) in (3), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_N\|^2 + \gamma \|\Delta u_N\|^2 = (\varphi(u_N), \Delta u_N) + (\nabla \cdot \psi(u_N), u_N).
\]
Note that
\[
\varphi'(u_N) = 3\gamma_2 u_N^2 + 2\gamma_1 u_N - u_N \geq -c_0 = -\frac{\gamma_1^2}{3\gamma_2} - 1.
\]
Thus
\[
(\varphi(u_N), \Delta u_N) = -(\varphi'(u_N) \nabla u_N, \nabla u_N) \leq c_0 \|\nabla u_N\|^2
\]
\[
= -c_0(u_N, \Delta u_N) \leq \frac{\gamma}{2} \|\Delta u_N\|^2 + \frac{c_0^2}{2\gamma} \|u_N\|^2.
\]
On the other hand, a simple calculation shows that
\[
(\nabla \cdot \psi(u_N), u_N) = \int_\Omega \nabla \cdot (u_N^2) u_N dx = 0,
\]
Summing up, we get
\[
\frac{d}{dt} \|u_N\|^2 + \gamma \|\Delta u_N\|^2 \leq \frac{c_0^2}{\gamma} \|u_N\|^2.
\]
Using Gronwall’s inequality, we obtain
\[
\|u_N\|^2 \leq e^{\frac{c_0^2}{\gamma} t} \|u_N(0)\|^2 \leq e^{\frac{c_0^2}{\gamma} t} \|u_0\|^2, \ t \in (0, T).
\]
Setting \( c_1 = \frac{c_0^2}{\gamma} \), we get the conclusion (6). Integrating (10) from 0 to \( t \), we get
\[
\int_0^t \|\Delta u_N(\tau)\|^2 d\tau \leq \frac{1}{\gamma} \left( c_1 \int_0^t \|u_N(\tau)\|^2 d\tau + \|u_N(0)\|^2 \right)
\]
\[
\leq \left( \frac{c_1 t}{\gamma} e^{c_1 t} + \frac{1}{\gamma} \right) \|u_0\|^2.
\]
Hence, Lemma 3 is proved. \( \square \)

**Lemma 4.** Suppose that \( u_0 \in H_0^1(\Omega) \). Then, (3) has a unique solution \( u_N(t) \) satisfying
\[
\|\nabla u_N(t)\|^2 \leq e^{c_2 t} \|\nabla u_0\|^2 + \tilde{c}_2, \ t \in (0, T),
\]
(11)
and
\[
\int_0^t \| \nabla \Delta u_N(\tau) \|^2 d\tau \leq \frac{\tilde{c}_3 t}{\gamma} + \frac{1}{\gamma} \| \nabla u_0 \|^2, \quad t \in (0, T),
\]
(12)
where \( c_2 \) is a positive constant depends only on \( \gamma, \gamma_1, \gamma_2 \) and the domain.

**Proof.** Setting \( v_N = \Delta u_N \) in (5), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_N \|^2 + \gamma \| \nabla \Delta u_N \|^2 = - \int_\Omega \Delta \varphi(u_N) \Delta u_N dx - \int_\Omega \nabla \cdot \psi(u_N) \Delta u_N dx.
\]
Note that
\[
\Delta \varphi(u_N) = (3\gamma_2 u_N^2 + 2\gamma_1 u_N - 1) \Delta u_N + (6\gamma_2 u_N + 2\gamma_1) |\nabla u_N|^2.
\]
Hence
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_N \|^2 + \gamma \| \nabla \Delta u_N \|^2 + \gamma \| \nabla u_N \|^2 \Delta u_N
\]
\[
= - \int_\Omega (2\gamma_2 u_N^2 + 2\gamma_1 u_N - 1) |\Delta u_N|^2 dx + \int_\Omega u_N^2 \nabla \Delta u_N dx
\]
\[
- \int_\Omega 2\gamma_1 |\nabla u_N|^2 \Delta u_N dx - 6 \int_\Omega \gamma_2 u_N |\nabla u_N|^2 \Delta u_N dx
\]
\[
\leq \gamma_2 \int_\Omega u_N^2 |\Delta u_N|^2 dx + c \int_\Omega |\nabla u_N|^4 dx + c \int_\Omega |\Delta u_N|^2 dx
\]
\[
+ \frac{\gamma}{8} \int_\Omega |\nabla \Delta u_N|^2 dx + c \int_\Omega u_N^4 dx.
\]
On the other hand, by Nirenberg’s inequality, we have
\[
\| \nabla u_N \|^4 \leq \left( c'_1 \| \nabla \Delta u_N \|^{\frac{1}{2}} \| u_N \|^{\frac{1}{2}} + c'_2 \| u_N \| \right)^4 \leq \frac{\gamma}{8} \| \nabla \Delta u_N \|^2 + c_3,
\]
\[
\| u_N \|^4 \leq \left( c'_1 \| \nabla \Delta u_N \|^{\frac{1}{2}} \| u_N \|^{\frac{1}{2}} + c'_2 \| u_N \| \right)^4 \leq \frac{\gamma}{8} \| \nabla \Delta u_N \|^2 + c_4,
\]
and
\[
\| \Delta u_N \|^2 \leq \left( c'_1 \| \nabla \Delta u_N \|^{\frac{1}{2}} \| u_N \|^{\frac{1}{2}} + c'_2 \| u_N \| \right)^2 \leq \frac{\gamma}{8} \| \nabla \Delta u_N \|^2 + c_5.
\]
Summing up, we immediately obtain
\[
\frac{d}{dt} \| \nabla u_N \|^2 + \gamma \| \nabla \Delta u_N \|^2 \leq 2(c_3 + c_4 + c_5). \tag{13}
\]
Using Nirenberg’s inequality again, we get
\[
\| \nabla u_N \| \leq c'_1 \| \nabla \Delta u_N \|^{\frac{3}{4}} \| u_N \|^{\frac{1}{4}} + c'_2 \| u_N \|. \tag{14}
\]
Adding (13) and (14) together gives
\[
\frac{d}{dt} \| \nabla u_N \|^2 + c'_3 \| \nabla u_N \|^2 \leq c'_4. \tag{15}
\]
Therefore, Gronwall’s inequality shows that
\[
\| \nabla u_N \|^2 \leq e^{c'_3 t} \| \nabla u_N(0) \|^2 + \frac{c'_4}{c'_3}.
\]
Setting \( c_2 = c'_3, \tilde{c}_2 = \frac{c'_4}{c'_3} \), we get the conclusion (11). Integrating (13) from 0 to \( t \), we deduce that
\[
\int_0^t \| \nabla \Delta u_N(\tau) \|^2 d\tau \leq \frac{2}{\gamma} (c_3 + c_4 + c_5)t + \frac{1}{\gamma} \| \nabla u_N(0) \|^2.
\]
Setting \( \tilde{c}_3 = 2(c_3 + c_4 + c_5) \), we obtain (12). Lemma 4 is proved. \( \square \)

**Remark 5.** Based on Lemmas 3-4, we obtain the \( H^1 \)-norm estimate of the numerical solution \( u_N(t) \) for problem (5). Noticing that we consider the problem in 2D case, by Sobolev’s embedding theorem, we have \( H^1(\Omega) \to L^p(\Omega) \) for all \( p < \infty \). Hence, \( \| u_N(t) \|_p \leq c \| u_N(t) \|_{H^1} \leq c_6, \forall p \in ]1, \infty[, \) where \( c_6 \) is a positive constant depends only on \( \gamma, \gamma_1, \gamma_2 \) and the domain.

**Lemma 6.** Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \). Then, (7) has a unique solution \( u_N(t) \) satisfying
\[
\| \Delta u_N(t) \|^2 \leq e^{c_7 t} \| \Delta u_0 \|^2 + c_8, \quad t \in (0, T), \tag{16}
\]
and
\[
\int_0^t \| \Delta^2 u_N(\tau) \|^2 d\tau \leq \tilde{c}_7 t + \tilde{c}_8 \| \Delta u_0 \|^2, \quad t \in (0, T), \tag{17}
\]
where \( c_7, c_8 \) are positive constants depend only on \( \gamma, \gamma_1, \gamma_2 \) and the domain.
Proof. Setting $v_N = \Delta u_N$ in (5), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\Delta u_N\|^2 + \gamma \|\Delta^2 u_N\|^2 = \int_{\Omega} \Delta^2 u_N \Delta \varphi(u_N) dx + \int_{\Omega} \Delta^2 u_N \nabla \cdot \varphi(u_N) dx.
\]
By Hölder’s inequality, we derive that
\[
\frac{1}{2} \frac{d}{dt} \|\Delta u_N\|^2 + \gamma \|\Delta^2 u_N\|^2 \leq \frac{1}{\gamma} \|\Delta \varphi(u_N)\|^2 + \frac{1}{\gamma} \|\nabla \psi(u_N)\|^2 + \frac{\gamma}{2} \|\Delta^2 u_N\|^2.
\]
Noticing that
\[
\|\Delta \varphi(u_N)\|^2 \leq 2(\int_{\Omega} |\varphi'(u_N)|^2 |\Delta u_N|^2 dx + \int_{\Omega} |\varphi''(u_N)|^2 |\nabla u_N|^4 dx)
\]
\[
\leq 2[(\int_{\Omega} |\varphi'(u_N)|^3 dx)^{\frac{2}{3}}(\int_{\Omega} |\Delta u_N|^6 dx)^{\frac{1}{3}} + (\int_{\Omega} |\varphi''(u_N)|^6 dx)^{\frac{1}{2}}(\int_{\Omega} |\nabla u_N|^6 dx)^{\frac{1}{2}}]
\]
\[
\leq c_9[(\int_{\Omega} |\Delta u_N|^6 dx)^{\frac{3}{2}} + (\int_{\Omega} |\nabla u_N|^6 dx)^{\frac{3}{2}}],
\]
where $c_9$ is a positive constant depends only on $\gamma$, $\gamma_1$, $\gamma_2$ and the domain.
On the other hand, we have
\[
\|\nabla \psi(u_N)\|^2 = \int_{\Omega} u_N^2 |\nabla u_N|^2 dx \leq \frac{1}{2} \int_{\Omega} u_N^4 dx + \frac{1}{2} \int_{\Omega} |\nabla u_N|^4 dx \leq \frac{c_4^4}{4} + \frac{1}{2} \|\nabla u_N\|^4.
\]
Using Nirenberg’s inequality, we have
\[
\|\nabla u_N\|^4_4 \leq \left( c_1' \|\Delta^2 u_N\|^\frac{4}{7} \|\nabla u_N\|^\frac{4}{3} + c_2' \|\nabla u_N\| \right)^4 \leq \varepsilon \|\Delta^2 u_N\|^2 + c_\varepsilon,
\]
\[
\|\nabla u_N\|^4_6 \leq \left( c_1' \|\Delta^2 u_N\|^\frac{4}{7} \|\nabla u_N\|^\frac{4}{3} + c_2' \|\nabla u_N\| \right)^4 \leq \varepsilon \|\Delta^2 u_N\|^2 + c_\varepsilon,
\]
and
\[
\|\Delta u_N\|^2 \leq \left( c_1' \|\Delta^2 u_N\|^\frac{2}{7} \|\nabla u_N\|^\frac{2}{3} + c_2' \|\nabla u_N\| \right) \leq \varepsilon \|\Delta^2 u_N\|^2 + c_\varepsilon.
\]
Summing up, we derive that
\[
\frac{d}{dt} \|\Delta u_N\|^2 + \left[ \gamma - \frac{4c_9}{\gamma} \left( \frac{1}{\gamma} + \frac{1}{\gamma} \right) \varepsilon \right] \|\Delta^2 u_N\|^2 \leq \frac{4c_9c_\varepsilon}{\gamma} + c_\varepsilon + \frac{c_4^4}{\gamma},
\]
(18)
where $\varepsilon$ is small enough, it satisfies $\gamma - \left(\frac{4c_9}{\gamma} + \frac{1}{\gamma}\right)\varepsilon > 0$. By the Calderon-Zygmund type estimate, we get

$$\frac{d}{dt} \|\Delta u_N\|^2 + \tilde{c}_4 (\|\Delta u_N\|^2 + \|\nabla \Delta u_N\|^2) \leq \tilde{c}_5.$$ 

Therefore, Gronwall’s inequality shows that

$$\|\Delta u_N\|^2 \leq e^{\tilde{c}_4 t} \|\Delta u_N(0)\|^2 + \tilde{c}_5.$$ 

Setting $c_7 = \tilde{c}_4, c_8 = \tilde{c}_5$, we obtain (16). Integrating (18) form 0 to $t$, we obtain

$$\int_0^t \|\Delta^2 u_N(\tau)\|^2 d\tau \leq \frac{4c_9 c_\varepsilon + c_\varepsilon + c_6^4}{\gamma^2 - (4c_9 + 1)\varepsilon} t + \frac{\gamma}{\gamma^2 - (4c_9 + 1)\varepsilon} \|\Delta u_0\|^2,$$

where $\tilde{c}_7 = \frac{4c_9 c_\varepsilon + c_\varepsilon + c_6^4}{\gamma^2 - (4c_9 + 1)\varepsilon}$ and $\tilde{c}_8 = \frac{\gamma}{\gamma^2 - (4c_9 + 1)\varepsilon}$. Hence, we get (17). Lemma 6 is proved.

**Remark 7.** Based on Lemmas 3-6, we obtain the $H^2$-norm estimate of the numerical solution $u_N(t)$ for problem (5). Noticing that we consider the problem in 2D case, by Sobolev’s embedding theorem, we have $H^2(\Omega) \rightarrow L^\infty(\Omega)$, that is

$$\|u_N(t)\|_\infty \leq c\|u_N(t)\|_{H^2} \leq c_{10},$$

where $c_{10}$ is a positive constant depends only on $\gamma, \gamma_1, \gamma_2$ and the domain.

**Theorem 8.** Suppose that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. Then for any $T > 0$, problem (3) admits a unique solution $u_N(x,t)$, such that

$$u_N(x,t) \in L^\infty(0,T; H^2_{\text{per}}(\Omega)) \cap L^2(0,T; H^4_{\text{per}}(\Omega)).$$

**Proof.** We are going to apply the Leray-Schauder fixed point theorem to complete the proof.

Define the linear space

$$X = \left\{ u_N \in L^\infty(0,T; H^2_{\text{per}}(\Omega)) \cap L^2(0,T; H^4_{\text{per}}(\Omega)); u|_{\partial\Omega} = 0, u(x,y,0) = u_0 \right\}.$$ 

Clearly, $X$ is a Banach space. Define the associated operator $T$,

$$T : X \rightarrow X, \quad u_N \rightarrow w_N,$$
where \( w \) is determined by the following linear problem:

\[
\left( \frac{\partial w_N}{\partial t}, v_N \right) + \gamma (\Delta w_N, \Delta v_N) = (\varphi(u_N), \Delta v_N) + (\nabla \cdot \psi(u_N), v_N), \quad \forall v_N \in S_N,
\]

\[
\frac{\partial w_N}{\partial n}|_{\partial \Omega} = \frac{\partial \Delta w_N}{\partial n}|_{\partial \Omega} = 0, \quad w(x, y, 0) = u_0.
\]

From the discussions in Lemmas 3-6 and by the contraction mapping principle, \( T \) has a unique fixed point \( u \), which is the desired solution of problem \( \mathcal{P} \).

Because the proof of the uniqueness of the solution is easy, we omit it here.

Then, we complete the proof.

\[\square\]

Now, we estimate the error \( \|u(t) - u_N(t)\| \). Denote \( \eta_N = u(t) - P_N u(t) \) and \( e_N = P_N u(t) - u_N(t) \). From (11) and (5), we get:

\[
(e_N, v_N) + \gamma (\Delta e_N, \Delta v_N) = (\varphi(u) - \varphi(u_N), \Delta v_N) + (\nabla \cdot (\psi(u) - \psi(u_N)), v_N), \quad \forall v_N \in S_N.
\]

Set \( v_N = e_N \) in (19), we derive that

\[
\frac{1}{2} \frac{d}{dt} \|e_N\|^2 + \gamma \|\Delta e_N\|^2 = (\varphi(u) - \varphi(u_N), \Delta e_N) - (\psi(u) - \psi(u_N), \nabla \cdot e_N).
\]

By Theorem 2, we have \( \sup_{x \in \Omega} \|u(x, t)\| \leq c_{11} \), where \( c_{11} \) is a positive constant depends only on \( \gamma, \gamma_1, \gamma_2 \) and the domain. Then

\[
(\varphi(u) - \varphi(u_N), \Delta e_N) = \gamma_2 ((u - u_N)(u^2 + uu_N + u_N^2) - (u - u_N, \Delta e_N))
\]

\[+ \gamma_1 (u + u_N)(u - u_N) - (u - u_N, \Delta e_N) \leq \gamma_2 \sup_{x \in \Omega} |u^2 + uu_N + u_N^2| \cdot \|e_N + \eta_N\| \|\Delta e_N\|
\]

\[+ \gamma_1 \sup_{x \in \Omega} |u + u_N| \cdot \|e_N + \eta_N\| \|\Delta e_N\| + \|e_N + \eta_N\| \|\Delta e_N\|
\]

\[\leq 2\gamma_2 (c_{10}^2 + c_{10} c_{11} + c_{11}^2) (\|e_N\| \|\Delta e_N\| + \|\eta_N\| \|\Delta e_N\|)
\]

\[+ 2\gamma_1 (c_{10} + c_{11}) (\|e_N\| \|\Delta e_N\| + \|\eta_N\| \|\Delta e_N\|)
\]

\[+ 2 (\|e_N\| \|\Delta e_N\| + \|\eta_N\| \|\Delta e_N\|)
\]

\[= [2\gamma_2 (c_{10}^2 + c_{10} c_{11} + c_{11}^2) + 2\gamma_1 (c_{10} + c_{11}) + 2] (\|e_N\| \|\Delta e_N\| + \|\eta_N\| \|\Delta e_N\|)
\]

\[\leq \frac{\gamma}{4} \|\Delta e_N\|^2 + c_{12} (\|e_N\|^2 + \|\eta_N\|^2),
\]
where \( c_{12} = \frac{2}{\gamma} [\gamma_2 (c_{10}^2 + c_{10} c_{11} + c_{11}^2) + \gamma_1 (c_{10} + c_{11}) + 1]^2 \). On the other hand, we have

\[
-(\psi(u) - \psi(u_N), \nabla e_N) = -((u - u_N)(u + u_N), \nabla e_N) \
\leq \sup_{x \in \Omega} |u + u_N| \cdot \|e_N + \eta_N\| \|\nabla e_N\|
\]

\[
\leq 2(c_{10} + c_{11})(\|e_N\| \|\nabla e_N\| + \|\eta_N\| \|\nabla e_N\|)
\]

\[
\leq -\frac{\gamma}{4} (e_N, \Delta e_N) + c_{13} \|e_N\|^2 + \|\eta_N\|^2
\]

\[
\leq \frac{\gamma}{4} \|\Delta e_N\|^2 + (c_{13} + \frac{\gamma}{16}) \|e_N\|^2 + c_{13} \|\eta_N\|^2,
\]

where \( c_{13} = \frac{4}{\gamma} (c_{10} + c_{11})^2 \). From Theorem 2 and (B2), we have

\[
\|\eta_N\| \leq c N^{-2} \|\Delta u\| \leq c_{14} N^{-2}.
\]

Summing up, we immediately obtain

\[
\frac{d}{dt} \|e_N\|^2 + \gamma \|\Delta e_N\|^2 \leq (2c_{12} + 2c_{13} + \frac{\gamma}{8}) \|e_N\|^2 + 2(c_{12} + c_{13}) c_{14} N^{-4}.
\]

Therefore, by Gronwall’s inequality, we deduce that

\[
\|e_N\| \leq c(\|e_N(0) + N^{-2}\|).
\] 

(20)

Thus, we obtain the following theorem:

**Theorem 9.** Suppose that \( u_0 \in H^2_{per}(\Omega) \), \( u(x,t) \) is the solution of problem (1)-(3) and \( u_N(x,t) \) is the solution of semi-discrete approximation (B). Then, there exists a constant \( c \), independent of \( N \), such that

\[
\|u(x,t) - u_N(x,t)\| \leq c(N^{-2} + \|u_0 - u_N(0)\|).
\]

### 3. Fully discrete scheme

In this section, we set up a full-discretization scheme for problem (1)-(3) and consider the fully discrete scheme which implies the pointwise bounded of the solution.

Let \( \Delta t = T/M \), for a positive integer \( M \), \( \partial_t u^k = \frac{u^k - u^{k-1}}{\Delta t} \). Note that \( \varphi(s) = \gamma_2 s^2 + \gamma_1 s - 1 \) and \( \psi(s) = s^2 \). The full-discretization spectral method
for problem (1)-(3) is read as: find \( u^j_N \in S_N \) \( (j = 0, 1, 2, \cdots, k) \) such that for any \( v_N \in S_N \), there hold
\[
\begin{align*}
\left( \frac{u^k_N - u^{k-1}_N}{\Delta t}, v_N \right) + \gamma(\Delta u^k_N, \Delta v_N) + (\varphi'(u^{k-1}_N)\nabla u^k_N, \nabla v_N) \\
- \frac{2}{3} (u^{k-1}_N \nabla \cdot u^k_N, v_N) + \frac{2}{3} (u^{k-1}_N u^k_N, \nabla \cdot v_N) = 0.
\end{align*}
\] (21)
for all \( T > 0 \) and \( t \in [0, T] \) with \( u_N(0) = P_N u_0 \).

The solution \( u^k_N \) has the following property:

**Lemma 10.** Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( u^k_N \) is a solution of problem (21), then there exists positive constants \( c_{15}, c_{16}, c_{17}, c_{18} \) depend only on \( \gamma, \gamma_1, \gamma_2 \) and \( u_0 \), such that
\[
\begin{align*}
\|u^k_N\| \leq c_{15}, & \quad \|\nabla u^k_N\| \leq c_{16}, \quad \|\Delta u^k_N\| \leq c_{17}, \quad \|u^k_N\|_\infty \leq c_{18}.
\end{align*}
\]

**Proof.** Let \( v_N = u^k_N \) in (21), we derive that
\[
\begin{align*}
\frac{1}{2} \bar{\partial}_t \|u^k_N\|^2 + \frac{\tau}{2} \|\bar{\partial}_t u^k_N\|^2 + \gamma \|\Delta u^k_N\|^2 + (\varphi'(u^{k-1}_N)\nabla u^k_N, \nabla u^k_N) \\
- \frac{2}{3} (u^{k-1}_N \nabla \cdot u^k_N, u^k_N) + \frac{2}{3} (u^{k-1}_N u^k_N, \nabla \cdot u^k_N) = 0.
\end{align*}
\] (22)
Note that
\[
\varphi'(u^{k-1}_N) = 3\gamma_2(u^{k-1}_N)^2 + 2\gamma_1 u^{k-1}_N - u^{k-1}_N \geq -2c_0 = -\frac{\gamma_1^2}{3\gamma_2} - 1.
\]
Thus
\[
(\varphi'(u^{k-1}_N)\nabla u^k_N, \nabla u^k_N) \geq -c_0 \|\nabla u^k_N\|^2.
\]
On the other hand, we have
\[
- \frac{2}{3} (u^{k-1}_N \nabla \cdot u^k_N, u^k_N) + \frac{2}{3} (u^{k-1}_N u^k_N, \nabla \cdot u^k_N) = 0.
\]
Therefore
\[
\bar{\partial}_t \|u^k_N\|^2 + \tau \|\bar{\partial}_t u^k_N\|^2 + 2\gamma \|\Delta u^k_N\|^2 \leq c_0 \|\nabla u^k_N\|^2.
\] (22)
Note that
\[
c_0 \|\nabla u^k_N\|^2 \leq \frac{c_0^2}{4\gamma} \|u^k_N\|^2 + \gamma \|\Delta u^k_N\|^2.
\] (23)
Summing up, we get
\[ \frac{\|u_N^k\|^2 - \|u_N^{k-1}\|^2}{\Delta t} + \gamma \|\Delta u_N^k\|^2 \leq \frac{c_0^2}{4\gamma} \|u_N^k\|^2, \]
that is
\[ \|u_N^k\|^2 \leq \frac{4\gamma}{4\gamma - c_0^2 \Delta t} \|u_N^{k-1}\|^2 \leq \left( \frac{4\gamma}{4\gamma - c_0^2 \Delta t} \right)^k \|u_N^0\|^2 = c_{15}. \]

Let \( \varphi = \Delta u_N^k \) in (21), we derive that
\[ \frac{1}{2} \partial_t \|\nabla u_N^k\|^2 + \frac{\tau}{2} \|\partial_t \nabla u_N^k\|^2 + \gamma \|\nabla \Delta u_N^k\|^2 \]
\[ = (\varphi'(u_N^{k-1}) \nabla u_N^k, \nabla \Delta u_N^k) - \frac{4}{3} \left( u_N^{k-1} \nabla \cdot u_N^k, \Delta u_N^k \right) - \frac{2}{3} \left( \nabla u_N^{k-1} u_N^k, \Delta u_N^k \right). \]

By Young’s inequality, Sobolev’s interpolation inequality and (25), we get
\[ (\varphi'(u_N^{k-1}) \nabla u_N^k, \nabla \Delta u_N^k) \]
\[ \leq c(\|u_N^{k-1}\|_{L^4}^2 + 1) \|\nabla u_N^k\|_{L^6} \|\nabla \Delta u_N^k\| \]
\[ \leq c(\|\nabla u_N^{k-1}\| \|u_N^{k-1}\| + 1)(\|\nabla \Delta u_N^k\| \|u_N^k\|^{\frac{2}{3}} \|u_N^k\|^{\frac{1}{3}} + 1) \|\nabla \Delta u_N^k\| \]
\[ \leq \frac{\gamma}{4} \|\nabla \Delta u_N^k\|^2 + c(\|\nabla u_N^{k-1}\|^2 + 1). \]

Using Young’s inequality and Sobolev’s interpolation inequality again, we get
\[ -\frac{4}{3} \left( u_N^{k-1} \nabla \cdot u_N^k, \Delta u_N^k \right) \]
\[ \leq c(\|u_N^{k-1}\|_{L^4} \|\nabla u_N^k\|_{L^4} \|\Delta u_N^k\|_{L^4}) \]
\[ \leq c(\|u_N^{k-1}\| \|\nabla \Delta u_N^k\|^{\frac{2}{3}} \|u_N^k\|^{\frac{1}{3}} \|\nabla \Delta u_N^k\| \|u_N^k\|^{\frac{2}{3}} \|u_N^k\|^{\frac{1}{3}}) \]
\[ \leq \frac{\gamma}{8} \|\nabla \Delta u_N^k\|^2 + c, \]
and
\[ -\frac{2}{3} \left( \nabla u_N^{k-1} u_N^k, \Delta u_N^k \right) \leq \frac{2}{3} \|\nabla u_N^{k-1}\| \|u_N^k\| \|\Delta u_N^k\| \]
\[ \leq \frac{2}{3} \|\nabla u_N^{k-1}\| \|\nabla \Delta u_N^k\| \|u_N^k\| \]
\[ \leq \frac{\gamma}{8} \|\nabla \Delta u_N^k\|^2 + c \|u_N^k\|^2 \|\nabla u_N^{k-1}\|^2 \]
\[ \leq \frac{\gamma}{8} \|\nabla \Delta u_N^k\|^2 + c \|u_N^k\|^2. \]
Hence, (20) can be rewritten as
\[
\frac{\| \nabla u_N^k \|^2 - \| \nabla u_N^{k-1} \|^2}{\Delta t} + \gamma \| \nabla \Delta u_N^k \|^2 \leq c(\| \nabla u_N^{k-1} \|^2 + 1). \tag{27}
\]

Using discrete Gronwall’s inequality, we deduce that
\[
\| \nabla u_N^k \|^2 \leq (\| \nabla u_0 \|^2 + ct_n) e^{\gamma t_n} \leq c_{16}. \tag{28}
\]

By Sobolev’s embedding theorem, we have
\[
\| u_N^k \|_{L^p} \leq c_{19}, \quad 1 < p < \infty. \tag{29}
\]

Let \( \varphi = \Delta^2 u_N^k \) in (21), we have
\[
\frac{1}{2} \bar{\theta}_t \| \Delta u_N^k \|^2 + \frac{\tau}{2} \| \bar{\theta}_t \Delta u_N \|^2 + \gamma \| \Delta^2 u_N^k \|^2 = (\nabla \cdot [\varphi'(u_N^{k-1}) \nabla \cdot u_N^k], \Delta^2 u_N^k) + \frac{2}{3} (u_N^{k-1} \nabla \cdot u_N^k, \Delta^2 u_N^k)
+ \frac{2}{3} (\nabla \cdot (u_N^{k-1} u_N^k), \Delta^2 u_N^k).
\]

Based on the above results and Sobolev’s interpolation inequality, we deduce that
\[
(\nabla \cdot [\varphi'(u_N^{k-1}) \nabla \cdot u_N^k], \Delta^2 u_N^k)
= (\varphi'(u_N^{k-1}) \Delta u_N^k, \Delta^2 u_N^k) + (\varphi''(u_N^{k-1}) |\nabla u_N^k|^2, \Delta^2 u_N^k)
\leq |\varphi'(u_N^{k-1})|_{L^4} |\Delta u_N^k|_{L^4} \| \Delta^2 u_N^k \| + |\varphi''(u_N^{k-1})|_{L^6} \| \nabla u_N^k \|_{L^6} \| \Delta^2 u_N^k \|
\leq c \| \Delta u_N^k \|_{L^4} \| \Delta^2 u_N^k \| + c \| \nabla u_N^k \|_{L^6} \| \Delta^2 u_N^k \|
\leq c \| \Delta^2 u_N^k \|^{\frac{2}{3}} \| \nabla u_N^k \|^{\frac{2}{3}} + c \| \Delta^2 u_N^k \|^{\frac{1}{2}} \| \nabla u_N^k \|^{\frac{1}{2}}
\leq \frac{\gamma}{4} \| \Delta^2 u_N^k \|^2 + c(c_{15}, c_{16}, c_{19}).
\]

We also have
\[
\frac{2}{3} (u_N^{k-1} \nabla \cdot u_N^k, \Delta^2 u_N^k) \leq \frac{2}{3} \| u_N^{k-1} \|_{L^4} \| \nabla u_N^k \|_{L^4} \| \Delta^2 u_N^k \|
\leq c \| \nabla u_N^k \|_{L^4} \| \Delta^2 u_N^k \| \leq c \| \Delta^2 u_N^k \|^{\frac{3}{2}} \| \nabla u_N^k \|^{\frac{1}{2}}
\leq \frac{\gamma}{8} \| \Delta^2 u_N^k \|^2 + c(c_{15}, c_{16}, c_{19}),
\]
and
\[
\frac{2}{3}(\nabla \cdot (u^{k-1}_N u_N^k), \Delta^2 u_N^k)
\]
\[
= \frac{2}{3}[(u_N^k \nabla \cdot u_N^{k-1}, \Delta^2 u_N^k) + (u_N^{k-1} \nabla \cdot u_N^k, \Delta^2 u_N^k)]
\]
\[
\leq \frac{2}{3} \|u_N^{k-1}\|_{\infty} \|\nabla u_N^{k-1}\| \|\Delta^2 u_N^k\| + \frac{2}{3} \|u_N^{k-1}\|_{L^4} \|\nabla u_N^k\|_{L^4} \|\Delta^2 u_N^k\|
\]
\[
\leq c \|\nabla u_N^{k-1}\| \|\Delta^2 u_N^k\|^\frac{3}{2} \|u_N^k\|^\frac{1}{2} + c \|\Delta^2 u_N^k\|^\frac{3}{2} \|\nabla u_N^k\|^\frac{1}{2}
\]
\[
\leq \frac{\gamma}{8} \|\Delta^2 u_N^k\|^2 + c(c_{15}, c_{16}, c_{19}).
\]
Summing up, we derive that
\[
\frac{\|\Delta u_N^k\|^2 - \|\Delta u_N^{k-1}\|^2}{\Delta t} + \gamma \|\Delta^2 u_N^k\|^2 \leq c.
\] (30)
Therefore
\[
\|\Delta u_N^k\|^2 \leq c \Delta t + \|\Delta u_N^{k-1}\|^2 \leq cT + \|\Delta u_0\|^2 = c_{17}.
\] (31)
By Sobolev’s embedding theorem, we have
\[
\|u_N^k\|_{\infty} \leq c_{18}.
\] (32)
Then, the proof is complete.

In the following, we analyze the error estimates between the numerical solution $u_N^k$ and the exact solution $u(t_k)$.

We introduce a linear problem as follows: $\forall v \in S_N$,
\[
\begin{cases}
\left(\frac{w_N^k - w_N^{k-1}}{\Delta t} + \gamma \Delta^2 w_N^k + \gamma w_N^k - \Delta \varphi(u^k) - \nabla \cdot \psi(u^k), v_N\right) = (\gamma u_N^k, v_N), \\
w_N^0 = P_N u_0.
\end{cases}
\] (33)
First of all, we study the error estimates between $u(t_k)$ and $w_N^k$. Set $u^k = u(t_k)$, $\eta^k = u^k - P_N u^k$ and $\theta^k = P_N u^k - w_N^k$. Then, we have
\[
u^k - w_N^k = u^k - P_N u^k + P_N u^k - w_N^k = \eta^k + \theta^k.
\]
Lemma 11. Suppose that $u^k = u(t_k)$ is the solution of problem (1)-(3) and $w_N^k$ is the solution of problem (33). Suppose further that $u_{tt} \in L^2(0, T; L^2(\Omega))$. Then, we have

$$\|u^k - w_N^k\|^2 \leq \|u^0 - w_N^0\|^2 + c(\Delta t)^2.$$  

Proof. Note that $u^k - w_N^k = \eta^k + \theta^k$. $\theta^k$ satisfies

$$\left(\frac{\theta^k - \theta^{k-1}}{\Delta t} + \gamma \Delta^2 \theta^k + \gamma \theta^k - \left(\frac{u^k - u^{k-1}}{\Delta t} - u_t\right), v_N\right) = 0 \tag{34}$$

Set $v_N = \theta^k$, we get

$$\frac{1}{2} \frac{\|\theta^k\|^2 - \|\theta^{k-1}\|^2}{\Delta t} + \gamma \|\Delta \theta^k\|^2 + \gamma \|\theta^k\|^2 = \left(\frac{u_N^k - u_N^{k-1}}{\Delta t} - u_t^k, \theta^k\right).$$

Note that

$$\left(\frac{u_N^k - u_N^{k-1}}{\Delta t} - u_t^k, \theta^k\right) \leq \|\theta^k\| \left\|\frac{u_N^k - u_N^{k-1}}{\Delta t} - u_t^k\right\|$$

$$\leq \gamma \|\theta^k\|^2 + \frac{1}{(\Delta t)^2} \left\|\int_{t_{k-1}}^{t_k} (s - t_{k-1})u_{tt}(\xi) ds\right\|^2$$

$$\leq \gamma \|\theta^k\|^2 + \frac{1}{(\Delta t)^2} \int_{t_{k-1}}^{t_k} (s - t_{k-1})^2 ds \int_{t_{k-1}}^{t_k} \|u_{tt}(\xi)\|^2 ds$$

$$\leq \gamma \|\theta^k\|^2 + \Delta t \int_{t_{k-1}}^{t_k} \|u_{tt}(\xi)\|^2 ds,$$

where $t_{k-1} < \xi < t_k$. Summing up, we derive that

$$\frac{\|\theta^k\|^2 - \|\theta^{k-1}\|^2}{\Delta t} \leq \Delta t \int_{t_{k-1}}^{t_k} \|u_{tt}(\xi)\|^2 ds,$$

that is

$$\|\theta^k\|^2 \leq \|\theta^{k-1}\|^2 + (\Delta t)^2 \int_{t_{k-1}}^{t_k} \|u_{tt}(\xi)\|^2 ds.$$  

Therefore

$$\|\theta^k\|^2 \leq \|\theta^0\|^2 + (\Delta t)^2 \sum_{i=1}^{k} \int_{t_{k-1}}^{t_k} \|u_{tt}(\xi^i)\|^2 ds,$$  

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where \( i = 1, 2, \cdots, k \) and \( t_{i-1} < \xi^i < t_i \). Set \( \| u_{tt}(\xi) \| = \max\{\| u_{tt}(\xi^i) \| \} \), we have
\[
\| \theta^k \|^2 \leq \| \theta^0 \|^2 + (\Delta t)^2 \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \| u_{tt}(\xi) \|^2 \, ds
\]
\[
= \| \theta^0 \|^2 + (\Delta t)^2 \int_0^T \| u_{tt}(\xi) \|^2 \, ds
\]
\[
\leq \| \theta^0 \|^2 + c(\Delta t)^2.
\]
Hence, the proof is complete.

Secondly, we study the error estimate between \( w^k_N \) and \( u^k_N \). Set \( w^k_N - u^k_N = e^k_N \). We have the following lemma:

**Lemma 12.** Suppose that \( w^k_N \) is the solution of problem (33) and \( u^k_N \) is the solution of the full-discrete scheme (21). Suppose further that \( u_t \in L^2(0,T; L^2(\Omega)) \). Then, we have
\[
\| w^k_N - u^k_N \|^2 \leq c((\Delta t)^2 + N^{-4}).
\]

**Proof.** Combining (21) and (33) together gives
\[
\begin{align*}
\left( e^k - e^{k-1} \right) \Delta t + \gamma \Delta^2 e^k - \gamma \theta^k, v_N \right) + (\nabla \varphi(u^k) - \varphi'(u_N^{k-1}) \nabla u_N^k, \nabla v_N) \\
= (\nabla \cdot (u^k)^2, v_N) - \frac{2}{3}(u_N^{k-1} \nabla \cdot u_N^k, v_N) + \frac{2}{3}(u_N^{k-1} u_N^k, \nabla \cdot v_N).
\end{align*}
\]

(35)

Set \( v_N = e^k \) in (35), we get
\[
\frac{1}{2} \| e^k \|^2 - \| e^{k-1} \|^2 + \gamma \| \Delta e^k \|^2 = \gamma(\theta^k, e^k) - (\nabla \varphi(u^k) - \varphi'(u_N^{k-1}) \nabla u_N^k, \nabla e^k)
\]

\[
+ (\nabla \cdot (u^k)^2, e^k) - \frac{2}{3}(u_N^{k-1} \nabla \cdot u_N^k, e^k) + \frac{2}{3}(u_N^{k-1} u_N^k, \nabla \cdot e^k).
\]
Note that

\[-(\nabla \varphi(u^k) - \varphi'(u_N^{-1})\nabla u_N^k, \nabla e^k) = -(\varphi'(u^k)\nabla u^k - \varphi'(u_N^{-1})\nabla u_N^k, \nabla e^k)\]

\[= - (\varphi'(u^k)\nabla u^k - \varphi'(u_N^k)\nabla u_N^k, \nabla e^k) + \varphi'(u_N^{-1})\nabla u_N^k, \nabla e^k)\]

\[= - (\varphi'(u^k)(\nabla u^k - \nabla u_N^k), \nabla e^k) - ([\varphi'(u^k) - \varphi'(u_N^{-1})]\nabla u_N^k, \nabla e^k)\]

\[= - ([\varphi'(u_N^{-1}) - \varphi'(u_N^{-1})]\nabla u_N^k, \nabla e^k)\]

\[\Delta I_1 + I_2 + I_3.\]

\[I_1 = (\varphi'(u^k)\Delta e^k + \varphi''(u^k)(\nabla u^k\nabla e^k, u^k - u_N^k)\]

\[\leq ||\varphi'(u^k)||_\infty ||\Delta e^k|| + ||\varphi''(u^k)||_\infty ||\nabla u^k||_{L^4} ||\nabla e^k||_{L^4} ||u^k - u_N^k||\]

\[\leq c||u^k - u_N^k||c(||e^k|| + ||\Delta e^k||)\]

\[\leq c||e^k|| + ||\eta^k|| + ||\theta^k|| (||e^k|| + ||\Delta e^k||)\]

\[\leq \epsilon ||\Delta e^k||^2 + c(||e^k|| + ||\theta^k|| + ||e^k||).\]

\[I_2 = - (\varphi''(\phi^k u^k + (1 - \phi^k)u_N^{-1})\nabla u_N^k (u^k - u_N^k), \nabla e^k)\]

\[\leq ||\varphi''(\lambda_2 u^k + (1 - \lambda_2)u_N^{-1})||_\infty ||\nabla u_N^k||_{L^4} ||u^k - u_N^{-1}|| ||\nabla e^k||_{L^4}\]

\[\leq c(||\varphi''(\lambda_2 u^k + (1 - \lambda_2)u_N^{-1})||_\infty ||\nabla u_N^k||_{L^4} ||u^k - u_N^{-1}|| (||e^k|| + ||\Delta e^k||)\]

\[\leq c(||e^k|| + ||\Delta e^k||) ||u^k - u_N^{-1}||\]

\[\leq \epsilon ||\Delta e^k||^2 + c \Delta t \int_{t_{k-1}}^{t_k} ||u_t||^2 ds,\]

\[I_3 = - (\varphi''(\lambda_2 u_N^{-1} + (1 - \lambda_2)u_N^{-1})\nabla u_N^k (u^k - u_N^{-1}), \nabla e^k)\]

\[\leq ||\varphi''(\lambda_2 u_N^{-1} + (1 - \lambda_2)u_N^{-1})||_\infty ||\nabla u_N^k||_{L^4} ||u^k - u_N^{-1}|| ||\nabla e^k||_{L^4}\]

\[\leq c(||e^k|| + ||\Delta e^k||) ||u^k - u_N^{-1}||\]

\[\leq \epsilon ||\Delta e^k||^2 + c(||\eta^k||^2 + ||\theta^k||^2 + ||e^k||^2),\]

where \(\lambda_1, \lambda_2 \in (0, 1)\). Hence

\[(\nabla \varphi(u^k) - \varphi'(u_N^{-1})\nabla u_N^k, \nabla e^k)\]

\[\leq 3\epsilon ||\Delta e^k||^2 + c(||\eta^k||^2 + ||\theta^k||^2 + ||e^k||^2 + ||\eta^k||^2 \]

\[+ ||\theta^k||^2 + ||e^k||^2 + \Delta t \int_{t_{k-1}}^{t_k} ||u_t||^2 ds).\]
We also have
\[
\begin{align*}
(\nabla \cdot (u^k)^2, e^k) - \frac{2}{3}(u_N^{k-1} \nabla \cdot u_N^k, e^k) + \frac{2}{3}(u_N^k \nabla \cdot e^k)
&= \frac{2}{3}(u^k \nabla \cdot u^k - u_N^{k-1} \nabla \cdot u_N^k, e^k) + \frac{4}{3}(u^k \nabla \cdot u^k, e^k) - \frac{2}{3}(u_N^{k-1} u_N^k, \nabla \cdot e^k) \\
&= \frac{2}{3}(u^k \nabla u^k - u_N^{k-1} \nabla u_N^k, e^k) - \frac{2}{3}([u^k]^2 - u_N^{k-1} u_N^k, \nabla \cdot e^k) \\
&= \frac{2}{3}[(u^k - u_N^k, e^k \nabla u^k) + (u_N^k - u_N^{k-1}, e^k \nabla u_N^k) - (u^k - u_N^{k-1}, u_N^k \nabla e^k) - (u_N^k - u_N^{k-1}, u_N^k \nabla e^k)] \\
&\leq \varepsilon([e^k]^2 + \|\Delta e^k\|^2) + c([\|\eta^k\|^2 + \|\theta^k\|^2 + \|e^k\|^2 + \|\theta^k\|^2 + \|\eta^k\|^2] \\
&\quad + \|\theta^k\|^2 + \|e^k\|^2 + \Delta t \int_{t_{k-1}}^{t_k} \|u_t\|^2 ds),
\end{align*}
\]
and
\[
\gamma(\theta^k, e^k) \leq \varepsilon\|e^k\|^2 + c\|\theta^k\|^2.
\]
Summing up, we immediately obtain
\[
\begin{align*}
\frac{\|e^k\|^2 - \|e^{k-1}\|^2}{\Delta t} + 2(\gamma - 5\varepsilon)\|\Delta e^k\|^2
&\leq c([\|\eta^k\|^2 + \|\theta^k\|^2 + \|e^k\|^2 + \|\theta^k\|^2 + \|\eta^k\|^2] \\
&\quad + \|\theta^k\|^2 + \|e^k\|^2 + \Delta t \int_{t_{k-1}}^{t_k} \|u_t\|^2 ds),
\end{align*}
\]
where \(\varepsilon\) is small enough, which satisfies \(\gamma - 5\varepsilon > 0\). Thus
\[
(1 - c\Delta t)\|e^k\|^2 + 2(\gamma - 5\varepsilon)\Delta t\|\Delta e^k\|^2
\leq (1 + c\Delta t)\|e^{k-1}\|^2 \\
\quad + c\Delta t([\|\eta^k\|^2 + \|\theta^k\|^2 + \|\eta^k\|^2] \|\theta^k\|^2 + \|\eta^k\|^2 + \Delta t \int_{t_{k-1}}^{t_k} \|u_t\|^2 ds),
\]
that is
\[
\|e^k\|^2 \leq \frac{1 + c\Delta t}{1 - c\Delta t}\|e^{k-1}\|^2 \\
\quad + \frac{c\Delta t}{1 - c\Delta t}([\|\eta^k\|^2 + \|\theta^k\|^2 + \|\eta^k\|^2 + \|\theta^k\|^2 + \|\eta^k\|^2 + \Delta t \int_{t_{k-1}}^{t_k} \|u_t\|^2 ds).
Applying the discrete Gronwall's inequality with sufficient small $\Delta t$ such that $1 - c\Delta t > 0$, we get
$$\|e^n\| \leq c(\Delta t + N^{-2}).$$
Hence, we complete the proof.

Furthermore, we have the following theorem:

**Theorem 13.** Suppose that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $u(x, t)$ is the solution of problem (1)-(3) satisfying
$$u_t \in L^2(0, T; L^2(\Omega)), \quad u_{tt} \in L^2(0, T; L^2(\Omega)).$$
Suppose further that $u^k_N \in S_N$ ($k = 0, 1, 2, \cdots$) is the solution for problem (21) and the initial value $u^0_N$ satisfies
$$\|u^0_N - P_N u_0\| \leq c N^{-2} \|\Delta u\|.$$
Then, there exists a positive constant $c$ depends on $\gamma, \gamma_1, \gamma_2, T$ and $u_0$, independent of $N$ such that
$$\|u(x, t_k) - u^k_N\| \leq c(\Delta t + N^{-2}), \quad j = 0, 1, 2, \cdots, N.$$

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