Graph state basis for Pauli Channels

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Abstract

We introduce graph state basis diagonalization to calculate the coherent information of a quantum code passing through a Pauli channel. The scheme is 5000 times faster than the best known one for some concatenated repetition codes, providing us a practical constructive way of approaching the quantum capacity of a Pauli channel. The calculation of the coherent information of non-additive quantum code can also be greatly simplified in graph state basis.

The quantum coding theorem for noisy channels [1] [2] [3] [4] states that the quantum capacity $Q(N)$ of a channel $N$ is given by regularized coherent information:

$$Q(N) = \lim_{n \to \infty} \frac{1}{n} \max_{\rho_n} I(\rho_n, N^{\otimes n}),$$

(1)

the r.h.s term has long been known an upper bound for $Q(N)$, which is the content of the converse coding theorem [2] [3]. The direct coding theorem, stating that $Q_n$ is the von Neumann entropy, of approaching the quantum capacity of a Pauli channel. The calculation of the coherent information of non-additive quantum code can also be greatly simplified in graph state basis.

The coherent information $I(\rho, N)$ of a state $\rho$ with respect to the noise $N$ is defined by

$$I(\rho, N) = S(N(\rho)) - S(\mathcal{I}_A \otimes N(\Psi_{AP})),$$

(2)

where $S(\rho) = -\text{tr} \rho \log_2 \rho$ is the von Neumann entropy, $\Psi_{AP}$ is a purification of $\rho$, and $\mathcal{I}_A$ is the identity operation on the ancilla system $A$. The last term, $S(\mathcal{I}_A \otimes N(\Psi_{AP}))$, is the entropy exchange $S_c(\rho, N)$ of $\rho$ with respect to $N$.

For a code state $\rho^C$ of $n$ qubits, denote $\sigma^{AC} = N^{\otimes n}(\rho^C), \sigma^{AC} = \mathcal{I}_A \otimes N^{\otimes n}(\Psi^{AC} \Psi^{AC})$, where $\Psi^{AC}$ is the purification of $\rho^C$, the coherent information of the state $\rho^C$ with respect to the noise $N^{\otimes n}$ per qubit will be $I^{C^N} = \frac{1}{n}(S(\sigma^{AC}) - S(\sigma^{AC}))$. Thus $I^{C^N}$ is the lower bound of $Q(N)$ according to quantum noisy coding theorem. It is known that the one-shot capacity $Q_1(N) = \max_\rho I(\rho, N)$ is exactly the maximum rate achievable with a non-degenerate code for Pauli channel $N$. That $Q(N) > Q_1(N)$ is then established by the construction of a massively non-degenerate code, this was accomplished in the work of [5] [6] for depolarizing channel and [7] [8] for some Pauli channels. It is not known which quantum code achieves the quantum capacity for a channel that is neither degradable nor anti-degradable. Pauli channel with proper channel parameters is an example of such channels. So we need to check all possible codings to seek the maximal coherent information, this is an awful work in the viewpoint of just working out the quantum capacity. However, the history of classical communication tells us that coding is the really important thing even when the capacity is known. The aim of this paper is to provide a scheme to work out the coherent information for a quantum code with respect to a Pauli channel.

**Graph state basis.**— A graph $G = (V; \Gamma)$ is composed of a set $V$ of vertices and a set of edges specified by the adjacency matrix $\Gamma$, which is an $n \times n$ symmetric matrix with vanishing diagonal entries and $\Gamma_{ab} = 1$ if vertices $a, b$ are connected and $\Gamma_{ab} = 0$ otherwise. The neighborhood of a vertex $a$ is denoted by $N_a = \{ v \in V | \Gamma_{av} = 1 \}$, i.e. the set of all the vertices that are connected to $a$.

Graph states [9] [10] are useful multipartite entangled states that are essential resources for the one-way computing [11] and can be experimentally demonstrated [12]. To associate the graph state to the underlying graph, we assign each vertex with a qubit, each edge represents the interaction between the corresponding two qubits. More physically, the interaction may be Ising interaction of spin qubits. Let us denote the Pauli matrices at the qubit $a$ by $X_a, Y_a, Z_a$ and identity by $I_a$. The graph state related to graph $G$ is defined as

$$|G\rangle = \prod_{\Gamma_{ab} = 1} U_{ab} |+\rangle^V = \frac{1}{\sqrt{2^n}} \sum_{\mu} (-1)^{\mu} \Psi_{\Gamma\mu} |\mu\rangle$$

(3)

where $|\mu\rangle$ is the joint eigenstate of Pauli operators $Z_a$ ($a \in V$) with eigenvalues $(-1)^{\mu_a}$, $|+\rangle^V$ is the joint +1 eigenstate of Pauli operators $X_a$ ($a \in V$) and $U_{ab}$ ($U_{ab} = \text{diag}(1, 1, 1, -1)$ in the $Z$ basis) is the controlled phase gate between qubits $a$ and $b$. Graph state can also be viewed as the result of successively performing 2-qubit Control-Z operations $U_{ab}$ to the initially unconnected $n$ qubit state $|+\rangle^V$. It can be shown that graph state is the joint +1 eigenstate of the $n$ vertices stabilizers

$$K_a = X_a \prod_{b \in N_a} Z_b := X_a Z_{N_a}, \quad a \in V.$$

(4)

Meanwhile, the graph state basis are $|G_{k_1, k_2, \ldots, k_n}\rangle = \prod_{a \in V} Z_{k_a} |G\rangle$, with $k_a = 0, 1$. Thus

$$K_a |G_{k_1, k_2, \ldots, k_n}\rangle = (-1)^{k_a} |G_{k_1, k_2, \ldots, k_n}\rangle.$$

(5)

**Output state in graph state basis.**— Consider the input code state $\rho^C$ which is diagonal in graph state basis, that is,

$$\rho^C = \sum_{k=0}^{2^n-1} \pi_k |G_k\rangle \langle G_k|,$$

(6)
with $\pi_k$ the probability of state the graph state $|G_k\rangle$, $0 \leq \pi_k \leq 1$ and $\sum_k \pi_k = 1$, we have denoted $|G_{k_1,k_2,\ldots,k_n}\rangle = |G_k\rangle$ with the conventions of $k = \sum_i k_i 2^{n-1}$ and

$$k \oplus j = \sum_i (k_i \oplus j_i)2^{n-i}. \quad (7)$$

The purification state can be $|\Psi^{AC}\rangle = \sum_k \sqrt{\pi_k} |G_k\rangle_A |G_k\rangle$. In Krauss representation, Pauli channel map $N$ acting on qubit state $\rho$ can be written as $N(\rho) = f_0 \rho + p_1 X \rho X + p_2 Y \rho Y + p_3 Z \rho Z$, where $p_{x(y,z)} \in [0,1]$ are the probabilities, $f = 1 - p_0 - p_1 - p_2$ is the fidelity of the channel, and $X, Y, Z$ are the Pauli operators. For depolarizing channel, $p_x = p_y = p_z = f = 1 - 3p$. For $n$ use of channels with $n$ qubits input state $\rho_n$, we have the output state $N^{\otimes n}(\rho_n) = \sum_{a} \eta(a) E_{a} \rho_{a} E_{a}^{\dagger}$, with $\eta(a) = f^{n-i-j-1} p_1 p_2 p_3^{1-i-j}$ for $E_a = X^{i} Y^{j} Z^{l}$. Then the joint output state of $\rho^{C}$ and the ancilla is $\sigma^{AC} = \sum_{a \in V} \eta(a) G_{A}(G_{A})_{A} \otimes N^{\otimes n}(\rho_{a}) = \sum_{a \in V} \eta(a) G_{A}(G_{A})_{A} \otimes (E_{a} |G_{a}\rangle \langle G_{a}| E_{a}^{\dagger} |G_{a}\rangle \langle G_{a}|)$. In graph state basis, we have

$$\sigma^{AC}_{a,im,jl} = \frac{1}{\sqrt{\pi_{l} \pi_{j}}} \sum_{a} \eta(a) G_{a} E_{a}^{\dagger} Z^{(i)} |G\rangle \langle G| \langle G| Z^{(j)} E_{a} Z^{(l)} |G\rangle,$$  

(8)

where we have denoted $Z^{(k)} = \prod_{v \in V} Z_{v}^{k_{v}}$. According to the orthogonality of graph state basis, $\langle G | Z^{(m)} E_{a} Z^{(i)} |G\rangle = 0$ except $Z^{(m)} E_{a} Z^{(i)} = K_{a}$ up to a factor of $\pm 1$, $i$ (the factor will be omit hereafter because it has no effect in the elements of $\sigma^{AC}$), for some $K_{a} \in K$ (the group with generators of all $K_{a}, a \in V$, an Abelian group, the vertices stabilizer group). Thus we have $E_{a} = Z^{(m)} K_{a} Z^{(i)}$, so $Z^{(j)} E_{a}^{\dagger} Z^{(l)} = Z^{(j)} Z^{(l)} E_{a}^{\dagger} Z^{(i)} K_{a} Z^{(m)}$. It can be written as $Z^{(j)} Z^{(l)} E_{a}^{\dagger} Z^{(i)} = \pm K_{a}$. For a non-zero $\langle G | Z^{(j)} E_{a}^{\dagger} Z^{(l)} |G\rangle$, we should get

$$i \oplus j \oplus m \oplus l = 0. \quad (9)$$

Let $k = i \oplus m$, then $l = i \oplus j \oplus m = j \oplus k$, and $m = i \oplus k$ so the possible non-zero elements are in the form of

$$\sigma^{AC}_{i(jk),j(jk)} = \frac{1}{\sqrt{\pi_{l}}} \sum_{a \in E_{a} \in Z^{(k)}} (-1)^{P_{a}} \eta(a). \quad (10)$$

where $P_{a} = 0$ with $E_{a} = Z^{(i+k)} K_{a} Z^{(i)}$ such that $Z^{(i+j)}$ communicates with $K_{a}$, and $P_{a} = 1$ with $E_{a} = Z^{(i+k)} K_{a} Z^{(i)}$ such that $Z^{(i+j)}$ anti-communicates with $K_{a}$. Notice that the joint output state can be block diagonalized according to $k$. So that what left is to diagonalizing each block with give $k$. The routine of calculating the non-zero elements is like this: (1) to list all the elements of $K$, (2) to determine $P$ according to the commutator of $Z^{(i+j)}$ and $K_{a}$ (3) to multiply $Z^{(k)}$ to obtain the coset $E = Z^{(k)} K$ of the Pauli group with respect to its subgroup $K$ and to determine $\eta$.

Notice that for $i' \oplus j' = i \oplus j$, we have $\sigma^{AC}_{i'(jk),j'(jk)|i(jk),j(jk)|} = \sigma^{AC}_{i(jk),j(jk)|i'(jk),j'(jk)|}$. This property is very useful in further diagonalizing the submatrix for stabilizer code of equal probability.

**Stabilizer code.**—In a graphical quantum error-correction code, each codeword can be written as $|G_k\rangle = Z^{(k)} |G\rangle = \prod_{v \in V} Z_{v}^{k_{v}} |G\rangle$. To encode is to properly choose some of the $|G_k\rangle$ in order to form the code. A code is thus completely characterized by the set of $k$ for a underneath given graph. For stabilizer code encoding $q$ qubits into $n$ qubits, all the $2^{q}$ chosen $Z^{(k)}$ forms a group, each $Z^{(k)}$ is self inverse. Without loss of generality, we use the binary vector $(k_{1}, k_{2}, \ldots, k_{n})$ to characterize the codeword of the stabilizer code. Then quantum stabilizer encoding is an encoding of classical binary serial $(k_{1}, k_{2}, \ldots, k_{n})$ into binary serial $(k_{1}, k_{2}, \ldots, k_{n})$. Denote $k = \sum_{i=1}^{n} k_{i} 2^{i-1}$, we have $i' \oplus j' = i \oplus j$ if $i' \oplus j' = i \oplus j$ when $i, j, i', j'$ correspond to codewords. A matrix $M$ with $M^{2} = M$ can be diagonalized with Hadamard matrix $H_{q}$. Its eigenvalues will be $\frac{1}{\sqrt{2}} (H_{q} M, H_{q})$. Each block of $\sigma^{AC}$ can be written in the form of $M$ for a code with a priori equal probability $\pi_{a} = 2^{-q}$. Hence, the eigenvalues of each block of $k$ of $\sigma^{AC}$ will be

$$\lambda_{q} = \frac{1}{\sqrt{2}} (H_{q} M, H_{q}). \quad (11)$$

The channel output state is simply $\sigma^{C} = Tr_{A} \sigma^{AC}$, the matrix element of $\sigma^{C}$ in graph state basis is

$$\sigma^{C}_{km} = \sum_{i} \sigma^{AC}_{i(i \oplus k),i(i \oplus k)} \delta_{km} \quad (12)$$

In graph state basis, the output state of a Pauli channel with diagonal input is still diagonal, as mentioned in [13]. Thus far, we have obtained all the eigenvalues for calculating the coherent information of the input of a priori uniform distributed stabilizer code with respect to Pauli channel.

**Concatenated repetition codes in depolarizing channel.**—Depolarizing channel is a special case of Pauli channel, we will mainly deal with the depolarizing channel, the results can easily be extended to generic Pauli channel. One way to show $Q(N) > Q_{l}(N)$ is to find $Q(N) > 0$ for very noisy channel $N$ where $Q_{l}(N) = 0$. Some codes that were shown to allow correction in the range of $Q_{l}(N) = 0$ consist of an $n_1$ qubit bit flip code concatenated with an $n_2$ qubit phase flip code [7] [8]. These have been called “$n_1$ in $n_2$” codes [8], since each of the $n_2$ blocks of the phase flip code consists of an $n_1$ qubit bit flip code. One of the examples is the famous Shor [9, 1, 3] code which is the “3 in 3” code. The codewords of “$n_1$ in $n_2$” code can be
where \( \frac{1}{\sqrt{2n_2}} (|0^{\otimes n_1}\rangle + |1^{\otimes n_1}\rangle) \) are the \( n_1 \)-partite GHZ states. The “\( n_1 \) in \( n_2 \)” codeword is the repetition of GHZ state. 

Comparing with the definition of graph state, we find that the underneath graph for the code can be the “forest” graph. The graph contains \( n_2 \) independent and identical subgraphs, each subgraph of \( n_1 \) vertex has a tree structure with the root vertex connecting with all the other vertices and no other links. The vertices will be numbered in the following order: the \( j_i \) th leaf of the \( j_2 \)-th tree is numbered as the \( (j_2(n_1-1)+j_i) \)-th vertex. 

The vertices stabilizer group \( K^{c} \) can be divided into its subgroup \( K^{c0} \) with generators \( K_{1}, \ldots, K_{n_2-1} \) and the coset \( K = K^{c} K_{n_2} \), so that all the elements of \( K^{c0} \) commute with \( Z_{n_1} \) and all the elements of \( K^{c} \) anti-commute with \( Z_{n_1} \).

Denote the “forest” vertices stabilizer group as \( K_{j_2} \), its subgroup \( K_{j_2'} \) for \( j_2 \)-th tree. Let’s split the “forest” vertices stabilizer group \( K \) into two parts according to the commutators of the elements and \( Z_{n_1} Z_{n_2} \cdots Z_{n_m} \). The commutator of the element of \( K \) and \( Z_{n_1} Z_{n_2} \cdots Z_{n_m} \) for “forest” graph can be reduced to the product the commutators of the corresponding piece of \( K_{j_2} \). When the element of \( K \) commutes with \( Z_{n_1} Z_{n_2} \cdots Z_{n_m} \), the number of the trees with anti-commutator of its corresponding section of the element or \( Z_{n_1} \) should be even. When the element of \( K \) anti-commutates with \( Z_{n_1} Z_{n_2} \cdots Z_{n_m} \), the number of that should be odd. For a given coset \( E = \mathbb{Z}^{(k)}K \) of the “forest”, the “forest” coset head \( \mathbb{Z}^{(k)} \) is composed of \( n_2 \) sections, each section is the coset head of the tree graph case. Since the trees are identical, the types of the sections can be denoted with \( I, Z_{n_1}, Z_{1} Z_{n_1}, Z_{1} Z_{2} Z_{n_1}, \ldots, Z_{1} \cdots Z_{1} \cdots Z_{1} Z_{n_1} \). Here type \( Z_{1} \cdots Z_{1} \cdots Z_{m} \) \((1 \leq m \leq n_1 - 1)\) represents all the cosets which \( Z \) operator numbers are \( m \). When the eigenvalues of the cosets \( Z_{1} \cdots Z_{1} \cdots Z_{m} \) and \( Z_{1} \cdots Z_{1} \cdots Z_{m} \cdots Z_{n_1} \) are equal, we can further simplify the types of coset head sections as \( I, Z_{n_1}, Z_{1}, Z_{1} \cdots Z_{1} \cdots Z_{1} \cdots Z_{n_1-1}, \) with degeneracies 1, 1, 2, 2, 2, 2, respectively.
with coset head $I$ is $l_0$, the number of trees with coset head type $Z_{n_1}$ is $l_1$, the number of trees with coset head type $Z_1Z_2 \cdots Z_{m}$ is $l_{m+1}$. The total number of the trees is $n_2 = \sum_{m=1}^{\infty} l_{m+1}$. For a particular element of $K$, consider the type $Z_1Z_2 \cdots Z_{m}$ trees, suppose there be $s_{m+1}$ of the trees with their corresponding piece of $K_{j_{m+1}}$ anti-commuting with their own $Z_{j_{m+1}}$, this element of $K$ should contribute to the eigenvalues of the joint output state of the concatenated code with

$$t_{0} s_{0} l_{1} = l_{1} \sum_{m=1}^{\infty} \frac{1}{l_{m+1}} \prod_{m=1}^{n_1-1} l_{m+1}^2 l_{m+1} s_{m+1} n_{1-m} = \frac{1}{t_{0}} \sum_{m=1}^{n_1-1} l_{m+1} s_{m+1} n_{1-m}$$ (19)

Summing upon all elements of $K$, we arrive at

$$\eta' + \eta'' = \left(t_0 + t_{n_1+1}\right)^n (t_1 + t_{n_1+1})^l$$

$$\times \prod_{m=1}^{n_1-1} l_{m+1} s_{m+1} n_{1-m}$$

$$\eta' - \eta'' = \left(t_0 - t_{n_1+1}\right)^n (t_1 - t_{n_1+1})^l$$

$$\times \prod_{m=1}^{n_1-1} l_{m+1} s_{m+1} n_{1-m}$$

Denote $a_{0, \pm} = t_0 \pm t_{n_1+1}, a_{1, \pm} = t_1 \pm t_{n_1+1}, a_{m+1} = t_{m+1} \pm t_{n_1+1}$ for $2 \leq m \leq n_1-1$, notice that $a_{m+1} = \pm a_{n_1-m+1}$ for $2 \leq m \leq n_1-1$, we may only use $a_{m+1}$ with $1 \leq m \leq \lceil \frac{n_1-1}{2} \rceil$ to specify the eigenvalues besides $a_{0, \pm}, a_{1, \pm}$. Consider the factors $(l_{m+1} + t_{n_1+1})^{l_{m+1}}$ and $(l_{m+1} - t_{n_1+1})^{l_{m+1}}$ in Eq. (20), their degeneracies are $(2C_{n-1}^{m})^{l_{m+1}}$ and $(2C_{n-1}^{m-1})^{l_{m+1}}$, respectively. For $1 \leq m \leq \lceil \frac{n_1-1}{2} \rceil$, let $m_{\mu} = l_{m+1} - t_{n_1+1}$, the degeneracy of $a_{m+1}$ should be $\sum_{m_{\mu}} (2C_{n-1}^{m})^{l_{m+1}} = (2C_{n-1}^{m+1})^{l_{m+1}}$. For even $n_1$ and $m = \frac{n_1}{2}$, let $m_{\mu} = l_{m+1}$ the degeneracy of $a_{m+1}$ is $2l_{m+1}$, will be $(2C_{n_1-1}^{m})^{l_{m+1}} = (C_{n_1}^{m+1})^{l_{m+1}}$. Thus the number of coset that gives the same $\eta' + \eta''$ is

$$d(\mu) = \frac{2^{n_1} n_1!}{\mu! (C_{n_1}^{\mu+1})^{l_{\mu+1}}}$$

where vector $\mu = (\mu_1, \mu_2, \ldots, \mu_{\lceil n_1-1/2 \rceil})$, $h = \sum_{m=1}^{\lceil n_1-1/2 \rceil} \mu_{m+1}$, with $\sum_{m=1}^{\lceil n_1-1/2 \rceil} \mu_{m+1} = n_2$ and $\mu_0 = 0 = 0$. Let $n_3 = \lceil (n_1 - 2)/2 \rceil + 1$, the eigenvalues of the joint output state $\sigma_{AC}$ of "7 in $n_2$" code can be written as

$$\eta_{\pm}(\mu) = \frac{1}{2} \sum_{m=0}^{n_3} a_{m+1}^{\mu_0} a_{m+1}^{\mu_1} \prod_{m=2}^{n_3} a_{m+1}^{\mu_m}$$

with degeneracy $d(\mu)$. According to Eq. (12), the eigenvalues of the output state $\sigma_{C}$ are

$$\eta'_{\pm}(\mu) = \frac{1}{2} \sum_{m=0}^{n_3} a_{m+1}^{\mu_0} a_{m+1}^{\mu_1} \prod_{m=2}^{n_3} a_{m+1}^{\mu_m}$$

also with degeneracy $d(\mu)$. The coherent information of "7 in $n_2$" code per channel use should be

$$I^{CN} = \frac{1}{n_1 n_2} \sum_{\mu} d(\mu) [-\eta'_{\pm}(\mu) \log_2 \eta'_{\pm}(\mu)]$$

$$+ \eta_{\pm}(\mu) \log_2 \eta_{\pm}(\mu) + \eta_{-}(\mu) \log_2 \eta_{-}(\mu)$$

### Table 1

| $n_1$ | $n_2$ | $3p_{\text{max}}$ |
|-------|-------|------------------|
| 9     | 334   | 0.19080163947    |
| 10    | 113   | 0.19005842449    |
| 11    | 828   | 0.19047623466    |
| 12    | 309   | 0.18981106971    |
| 14    | 812   | 0.18953710664    |

### Examples.
Consider "7 in $n_2$" code, we have $d(\mu) = \frac{1}{\mu_0} (2^{n_2} - 1) 2^{n_2} - 1$ for even $n_2$, $d(\mu) = \frac{1}{\mu_0} (2^{n_2} - 1) 2^{n_2} - 1$ for odd $n_2$, where $p_{\text{max}}$ is the critical value of channel noise such that $I^{CN}(p_{\text{max}}) = 0$, and for $p < p_{\text{max}}$ we have positive coherent information. The "13 in $n_2$" code has an optimal $n_2 > 1020$. Too large $n_2$ makes the storage of $C_{n_2/2}^{n_2/2}$ overflow.

The density matrix of the joint output can also be block diagonalizable for a non-additive code input which is diagonal in graph state basis. For example, the eigenvalues of the output and joint output of (5, 6, 2) code with respect to depolarizing channel can be obtained analytically when the input is diagonal and with equal probability in graph state basis. The density matrix of the joint output can be block diagonalized as $2^6$ blocks, each block is a $6 \times 6$ matrix and can be diagonalized eventually.

In summary, we have block diagonalized the output density matrix of a code and the joint output density of the code and the ancilla system with respect to Pauli channel when the input quantum code is diagonal in graph state basis. For a $(n, L, d)$ code which is diagonal in graph state basis and encoding $L$ states into $n$ qubits with distance $d$, the joint output density matrix is reduced to $2^n$ blocks, each is a $L \times L$ matrix. For a stabilizer code $[n, l, d]$ input which is diagonal in graph
state basis and with an equal prior probability for all $2^l$ encoded states, each block of joint output density matrix of a depolarizing channel can be further diagonalized with the Hadamard matrix. The eigenvalues are obtained in closed form. "$n_1$ in $n_2$" concatenated repetition codes are used to illustrate the details.

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