The Parametrized Relativistic Particle and the Snyder space-time

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Abstract

Using the parametrized relativistic particle we obtain the noncommutative Snyder space-time. In addition, we study the consistency conditions between the boundary conditions and the canonical gauges that give origin to noncommutative theories. Using these results we construct a first order action, in the reduced phase-space, for the Snyder particle with momenta fixed on the boundary.

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1 Introduction

Recently it has been studied the possibility of consider noncommutative space-times, i.e. spaces that satisfy relations of the following type,

\[[\hat{X}^\mu, \hat{X}^\nu] = i\hbar \theta^{\mu\nu},\]

\[\text{(1)}\]
with $\theta^{\mu\nu}$ an antisymmetric tensor. For example, it is possible to show in string theory that for some backgrounds, the low energy limit of the theory implies field theories in noncommutative spaces [1]. In this case $\theta^{\mu\nu}$ is a constant tensor that implies violations of the Lorentz symmetry [2].

However, long time ago H. Snyder [3] proposed a noncommutative space-time, where $\theta^{\mu\nu}$ depends on the space-time. This space is quite interesting since is discrete and compatible with the Lorentz symmetry. Also, it is interesting to mention that a space-time of the Snyder type it is obtained from the quantum gravity in $2+1$–dimensions [4]. This gives indications that it is possible to regard that Quantum Gravity in $3 + 1$–dimensions implies noncommutative spaces of the Snyder type. Another interesting property of the Snyder space is that exist a mapping between this space and the $\kappa$–Minkowski space-time [5], and this space-time is one of the arenas of the Doubly Special Relativity (DSR).

In the literature exist some realizations of the Snyder space, in one of these realizations is used a theory with two times [6]. Another proposal [7] is based essentially on the ordinary free relativistic particle, see section 3. Other realization is based on the non-relativistic particle with interactions [8]. In this paper we propose a different realization of the Snyder space-time, that is based in some way on a combination of the proposals [8] and [7]. Our starting point is to use a generalization of the Klein-Gordon equation proposed originally by Fock [9], rediscovered by Stueckelberg, Nambu [10], and Feynman [11]. In particular we use the action of this particle in the massless case [12] and fixing a gauge we obtain a representation of the Snyder space-time. The interesting points of this proposal are: It is a more direct form to obtain the Snyder space-time, it is also covariant from the beginning and allow us to construct an explicit action for the particle in the momentum space. To understand more clearly our proposal we start from the analysis of the non-relativistic particle in Section 2, where we study the consistency conditions that impose the boundary conditions to the action. In Section 3 we introduce our realization and the action for the particle in the momentum space-time.
2 Parametrized non-relativistic particle

The parametrized non-relativistic particle is the generally covariant system obtained by including the non relativistic time \( t \) among the dynamical variables. The action for paths obeying the boundary conditions

\[ x^i(\tau_1) = x^i_1, \quad t(\tau_1) = t_1, \quad x^i(\tau_2) = x^i_2, \quad t(\tau_2) = t_2, \quad (2) \]

is given by

\[ S_{nr} = \int_{\tau_1}^{\tau_2} d\tau \left( p_i \dot{x}^i + p_t \dot{t} - \lambda \left( p_t + H(p, x) \right) \right), \quad (3) \]

where

\[ \chi_1 = p_t + H(p, x), \quad (4) \]

is a first class constraint, where \( H(p, x) \) is the Hamiltonian of the system. To recover the dynamics of the relativistic particle we usually impose a canonical gauge condition of the form

\[ \chi_2 = t - f(\tau) \approx 0, \quad (5) \]

where \( f(\tau) \) is fixed in such way that the boundary conditions \( (2) \) are satisfied, see [13]. Fixing the gauge \( (5) \), the phase space action \( (3) \) is reduced to

\[ S_{nr} = \int_{\tau_1}^{\tau_2} d\tau \left( p_i \dot{x}^i - H(p, x) \dot{f} \right), \quad (6) \]

and from the elimination of the parametrization we finally obtain

\[ S_{nr} = \int_{f_1}^{f_2} df \left( p_i \frac{dx^i}{df} - H(p, x) \right). \quad (7) \]

This is the action of the non-relativistic particle with time \( f \) and boundary conditions

\[ x^i(f_1) = x^i_1, \quad x^i(f_2) = x^i_2. \quad (8) \]

Now we want to construct noncommutative theories, by fixing gauge conditions. Several authors (see for example [14]) have been considered gauge conditions of the following form,

\[ \chi_2 = t + \theta^i p_i - f(\tau) \approx 0. \quad (9) \]

The gauge condition \( (9) \) is correct in the Dirac sense, since the set of constraints \( (\chi_1, \chi_2) \) forms a good set of second class constraints and the Dirac
brackets can be built. Now, following Dirac’s method for second class constraints [15], we define the Dirac brackets. Given two phase space functions $A$ and $B$ these brackets are given by

$$\{A, B\}^* = \{A, B\} - \{A, \chi^a\}C^{ab}\{\chi^b, B\}, \quad (10)$$

with $C^{ab}$ the inverse matrix of $C_{ab} = \{\chi^a, \chi^b\}$. The gauge condition (9) implies the following Dirac brackets for the canonical variables,

$$\{t, x^i\}^* = \frac{\theta^i}{1 - \theta^k \frac{\partial H}{\partial x^k}}, \quad \{t, p_i\}^* = 0, \quad \{p_i, p_j\}^* = 0, \quad (11)$$

$$\{x^i, x^j\}^* = \frac{\partial H}{\partial p_i} \frac{\theta^j}{1 - \theta^k \frac{\partial H}{\partial x^k}} \frac{\partial H}{\partial p_j} - \frac{\theta^i}{1 - \theta^k \frac{\partial H}{\partial x^k}} \frac{\partial H}{\partial x^j}, \quad \{x^i, p_j\}^* = \delta^i_j + \frac{\theta^i \frac{\partial H}{\partial x^j}}{1 - \theta^k \frac{\partial H}{\partial x^k}}. \quad (12)$$

However, to impose this gauge condition in the action (3), it does not have sense. The reason of that is the fact that we will obtain an action with no well defined boundary conditions. For example, we will get the simultaneous fixing of the coordinates $x^i$ and the momenta $p_i$. Then, we will need an action where from the beginning we fix variables on the boundary that are consistent with the gauge condition. In this case, we see that a possibility is to fix on the boundary the variables $(t, p_i)$ since this set is a complete set of commuting variables, according to the Dirac brackets (11), then the respective action is given by,

$$S_{nr1} = \int_{\tau_1}^{\tau_2} dt \left( -x^i \dot{p}_i + p_t \dot{t} - \lambda (p_t + H(p, x)) \right). \quad (12)$$

Now fixing the gauge (9) and with the elimination of the parametrization we get finally,

$$S_{nr1} = \int_{f_1}^{f_2} df \left( \left( \theta^i H(p, x) - x^i \frac{dp_i}{df} \right) - H(p, x) \right), \quad (13)$$

with boundary conditions,

$$p_i(f_1) = p_{i1}, \quad p_i(f_2) = p_{i2}. \quad (14)$$

The action (13) with the boundary conditions (14) is a well defined physical problem that we can quantize using for example the path integral. So we learnt that to fix gauge conditions of the type (9), we must be careful that the gauge condition be consistent with the boundary conditions, and in consequence with the base selected to quantize the theory.
3 Snyder space-time

To obtain a realization of the Snyder space-time in $d+1$-dimensions is useful to start from a space-time with one extra dimension. For example in the Ref. [7] it was shown that the Snyder space-time appears by imposing a particular gauge condition to the action

$$S = \int d\tau \left( \dot{X}_M P^M - \frac{\lambda_1}{2} (P^M P_M - \kappa^2) - \lambda_2 (P_4 - M) \right),$$  \hspace{1cm} (15)

where $X^M = (X^\mu, X^4)$, and $P^M = (P^\mu, P^4)$, with $\mu = 0,\ldots,3$. Furthermore, the background metric is flat with signature $\text{sig}(\eta) = (-,+,\ldots,+)$. It is interesting to observe that the previous action is directly equivalent to the relativistic particle, we can see this from the elimination in the the action (15) of the auxiliary momenta $P_4$ and the Lagrange’s multiplier $\lambda_2$ associated to the second constraint, we obtain

$$S = \int \left( \dot{X}^\mu P_\mu - \frac{\lambda_1}{2} (P^\mu P_\mu - \tilde{m}^2) + M \dot{X}_4 \right), \quad \tilde{m} = \sqrt{\kappa^2 - M^2}$$  \hspace{1cm} (16)

Thus, we obtain the action of a free relativistic particle with mass $\tilde{m}$, plus a total derivative of the extra dimension that does not interact with the additional variables. We also see from (16), that from the definition of the momentum for the $X_4$ variable we get immediately the second constraint of (15) and we recover this action.

Now our propose is to obtain another realization of the Snyder space-time where the extra dimension plays different role in such way we can extract some physical content of this new dimension. To that end we start from the action

$$S = \int d\tau \left( \frac{\dot{X}_\mu \dot{X}^\mu}{2\dot{\eta}} \right).$$  \hspace{1cm} (17)

Here the extra dimension is given by the parameter $\eta$. We will assume that this parameter is an invariant scalar under Lorentz transformations of the $3+1$-dimensional space-time [12].

The equations of motion of this system are

$$\frac{d}{d\tau} \left( \frac{\dot{X}_\mu}{\dot{\eta}} \right) = 0,$$  \hspace{1cm} (18)

$$\frac{d}{d\tau} \left( \frac{\dot{X}_\mu \dot{X}^\mu}{2\dot{\eta}^2} \right) = 0.$$  \hspace{1cm} (19)
From these equations we get,

\[ \frac{\dot{X}_\mu}{\dot{\eta}} = c_\mu, \quad c_\mu = \text{constant}, \quad (20) \]

\[ \frac{\dot{X}_\mu \dot{X}^\mu}{2\eta^2} = \frac{\mu_c^\mu}{2}. \quad (21) \]

In this case the canonical momenta are given by

\[ P_\eta = \frac{-\dot{X}_\mu \dot{X}^\mu}{2\eta^2}, \quad (22) \]

\[ P_\mu = \frac{\dot{X}_\mu}{\dot{\eta}}. \quad (23) \]

From the definition of these momenta we obtain the first class constraint

\[ \phi = P_\eta + \frac{1}{2} P_\mu P^\mu = 0. \quad (24) \]

This means that the physical states of the theory in the base \(|x^\mu, \eta\rangle\) satisfy the following quantum evolution equation

\[ \left( \hat{P}_\eta + \frac{1}{2} \hat{P}_\mu \hat{P}^\mu \right) \psi = 0, \quad \text{with} \quad \hat{P}_\eta = -i\hbar \partial_\eta, \quad \hat{P}_\mu = -i\hbar \partial_\mu. \quad (25) \]

This expression is a generalization of the Klein-Gordon equation, that was originally proposed by V. Fock [9], and also used by Stueckelberg and Nambu [10]. An interesting property of this system is that integrating his propagator over the \(\eta\) parameter we recover the usual propagator of the Klein-Gordon equation [11]. Furthermore, the extra dimension \(\eta\) gives an alternative mechanism to give mass to the particle [16]. Solving this equation for \(\eta\) we can obtain the Klein-Gordon equation, but with the characteristic that the mass \(m^2\) could be positive or negative so we also have tachyons. Is interesting to notice that for a given time dependent potential \(V(X^0)\) we can obtain solutions to the equation (25) where the mass is time dependent, and these models have been used as toy models to analyze spacelike singularities in string theory [17].

Another form to see these facts is to consider a gauge condition to the system. The natural one is of the form,

\[ \chi = \eta - \tau \approx 0. \quad (26) \]
In this case the equations of motion are the following

\begin{align}
\dddot{X}_\mu &= 0, \\
\dot{X}_\mu \dot{X}^\mu &= c^2.
\end{align}

For \( c^2 = 0 \), we have the equations of motion of a massless relativistic. For \( c^2 < 0 \), we get a relativistic particle and for \( c^2 > 0 \), we obtain a tachyon.

The canonical form of the action (17), consistent with the gauge condition (26) is given by

\[ S = \int d\tau \left( P_\eta \dot{\eta} + P_\mu \dot{X}^\mu - \lambda \left( P_\eta + \frac{1}{2} P_\mu P^\mu \right) \right). \tag{29} \]

To obtain a noncommutative theory taking as starting point the action (29) we follow the procedure established in [14], where using a selected gauge condition is possible to obtain a noncommutative theory from the Dirac brackets. However, we need to take into account that the boundary conditions must be consistent with the new gauge condition. This fact can imply a modification of the kinetic term of the action, as we saw in the case of the non-relativistic particle.

To get the Snyder space-time we propose the following gauge condition

\[ \chi_2 = f(\tau) + \eta - \alpha X^\mu P_\mu \approx 0, \quad \alpha = \text{constant}. \tag{30} \]

We must notice that this gauge condition is not unique, but the other possible choices are related by a canonical transformation. Furthermore, the gauge condition (30) is a good canonical gauge condition in the Dirac sense since, if we define \( \chi_1 = \phi \), we obtain

\[ \{\chi_1, \chi_2\} = - [1 + 2\alpha P_\eta]. \tag{31} \]

Then, the constraints \((\chi_1, \chi_2)\) form a good set of canonical second class constraints. In particular, for the Dirac brackets of the reduced phase space variables \(X_\mu\) and \(P_\nu\) we obtain

\begin{align}
\{X_\mu, X_\nu\}^* &= \frac{1}{l^2} L_{\mu\nu}, \\
L_{\mu\nu} &= X_\mu P_\nu - P_\mu X_\nu, \tag{32} \\
\{X_\mu, P_\nu\}^* &= \eta_{\mu\nu} + \frac{1}{l^2} P_\mu P_\nu, \tag{33} \\
\{P_\mu, P_\nu\}^* &= 0. \tag{34}
\end{align}
where
\[ l^2 = \frac{1}{\alpha} (1 + 2\alpha P_\eta) = \frac{1}{\alpha} (1 - \alpha P_\mu P^\mu). \] (35)

Using these brackets we can compute the Dirac brackets for \( \eta \) and \( P_\eta \) using the constraint (24) and the gauge condition (30). In particular we obtain that \( P_\eta \) commute with all the momenta \( P_\mu \). Now, to quantize this theory we must promote the Dirac brackets to commutators and we get for the coordinates of the phase space a realization of the Snyder space-time [3]. We must notice that in the usual formulation of Snyder we have that \( l \) is a constant and is in some sense a minimal distance to which the space-time is discrete. In our case, \( l \) depends of the additional momentum \( P_\eta \) or using in strong way the constraints, in the momenta of the particle, see (35), and it is not in the center of the algebra. However, \( l \) commutes with,

\[ \{ l^2, L_{\mu\nu} \} = \{ l^2, P_\mu P_\nu \} = 0. \] (36)

Then, we do not have ordering problems in the commutators (32)-(34). Now, to construct the action of this particle we need to fix the appropriated boundary conditions that are consistent with the Dirac brackets (32)-(34) and in consequence with the constraint (24) and the gauge condition (30). From this brackets we see that the momenta \((P_\eta, P_\mu)\) form a complete set of commuting variables of the extended phase space, then we can fix these variables on the boundary. The corresponding action will be in this case,

\[ S_{sp} = \int_{\tau_1}^{\tau_2} d\tau \left(-\eta \dot{P}_\eta - X^\mu \dot{P}_\mu - \lambda \left( P_\eta + \frac{1}{2} P_\mu P^\mu \right) \right). \] (37)

Now, imposing in strong way the second class constraints we obtain the action in the reduced phase space

\[ S_{rsp} = \int_{\tau_1}^{\tau_2} d\tau \left(X^\mu \left( \alpha P_\mu P^\beta - \delta^\beta_\mu \right) \dot{P}_\beta - f(\tau) P_\mu \dot{P}^\mu \right). \] (38)

with boundary conditions,

\[ P_\mu(\tau_1) = P_{\mu 1}, \quad P_\mu(\tau_2) = P_{\mu 2}. \] (39)

This is a first order action for the free relativistic particle in the Snyder space-time. From the variation of \( X^\mu \) we get that the particle must satisfy,

\[ \left( \alpha P_\mu P^\beta - \delta^\beta_\mu \right) \dot{P}_\beta = 0. \] (40)
Using the fact that \((\alpha P_{\mu}P^{\beta} - \delta_{\mu}^{\beta})\) is an invertible matrix we get as result the equation of motion of the free relativistic particle \(\dot{P}_{\beta} = 0\). It is easy to see that this equation is consistent with the equation (20) and the definition of the momentum (23).

In conclusion, in this paper we analyze the consistency condition between the boundary condition and the gauge conditions used to obtain noncommutative theories. This analysis impose strong conditions in the kinetic term of the Hamiltonian action and can avoid the elimination of all the momenta to construct a second order action in the reduced space. Furthermore this procedure is consistent with the election of a complete base of commuting observables at the level of quantum mechanics. As a second interesting point of our article we construct using the massless parametrized relativistic particle an action for a particle in the Snyder space-time. We check the consistency conditions between the boundary conditions and the gauge condition. In this way we obtain a first order action for the Snyder particle in the reduced phase-space and in consequence this action is consistent with the second order action of the parametrized relativistic particle. So the main result of our paper is that the more useful way to analyze the Snyder particle is to use the action of the massless parametrized relativistic particle.

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