Periodic body motions along a horizontal rough surface by moving two internal masses

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Abstract. The analysis of the translational motion of a body carrying two moving internal masses along a horizontal rough surface is provided. The masses perform harmonic oscillations in the vertical plane, the frequencies of which are equal, and the amplitudes and phases can be arbitrary. It is assumed that the values of the problem parameters are chosen in such a way that the body moves without separation from the surface. Friction between a body and a surface is described by Coulomb's law. The classification of possible modes of body motion with periodically varying velocity is carried out. In the three-dimensional space of the parameters of the problem, regions in which the periodic motions are of a qualitatively different nature were constructed.

1. Introduction
The study of the dynamics of mechanical systems with movable internal masses, which are separated from the environment by the body frame, is of interest for the design and creation of capsule robots that are resistant to the environment. The problem of the motion of a rigid body on a horizontal surface by means of internal moving masses has been considered in many works in various formulations. A rigorous theoretical study of the dynamics and optimal motion control of such mechanical systems was initiated in [1] and [2]. In [3], a study of stationary periodic modes of motion was started. In [4], the optimal control of the body motion is constructed with the rectilinear relative motion of the two internal masses. In [5, 6] a complete qualitative study of the dynamics of an oscillatory system consisting of a rigid body and a particle moving along a circle inside the body is given. The latter system is a special case of the system considered in this paper, with a fixed value of one of the parameters.

2. Statement of the problem
We consider a body moving in a vertical plane along a horizontal rough surface. The motion of the body occurs due to the relative motion of two internal masses along mutually perpendicular guides. Let us introduce an absolute coordinate system, the abscissa axis of which is directed horizontally, along the surface towards the direction of the body motion and a coordinate system rigidly connected to the body, the axes of which are directed along the guides of the internal masses (see Fig. 1).

The law of relative motion of masses is given by the following equations

\[
\begin{align*}
\xi &= a \cos(\omega t + \phi_0) \\
\eta &= b \sin \omega t
\end{align*}
\]
Let us denote by \( x \) and \( y \) the coordinates of the body center of mass in the absolute coordinate system. Then by the center of mass motion theorem, we have

\[
\begin{align*}
(m_0 + m_1 + m_2)\ddot{x} + m_1\ddot{\xi} &= F_{fr}, \\
(m_0 + m_1 + m_2)\ddot{y} + m_2\ddot{\eta} &= N - (m_0 + m_1 + m_2)g.
\end{align*}
\]  
(2)

Here \( m_0 \) is the mass of the body, \( m_1 \) and \( m_2 \) are masses moving along the horizontal and vertical guides, respectively. \( N \) is a support reaction force. \( F_{fr} \) is a Coulomb friction force

\[
F_{fr} = \begin{cases} 
-fN\text{sign}\dot{x}, & \dot{x} \neq 0 \\
m_1\ddot{\xi}, & \dot{x} = 0 \text{ and } |m_1\ddot{\xi}| \leq fN \\
fN\text{sign}\xi, & \dot{x} = 0 \text{ and } |m_1\ddot{\xi}| > fN
\end{cases}
\]  
(3)

We consider the motion of the body without separation from the surface. In this case \( \ddot{y} \equiv 0 \) and from the second equation in (2) we have the following relation

\[
N = (m_0 + m_1 + m_2)g + m_2\ddot{\eta} \geq 0.
\]  
(4)

Taking into account (1) and denoting \( m_0 + m_1 + m_2 = m \) the latter relation can be rewritten as

\[
N = mg - m_2b\omega^2\sin\omega t \geq 0.
\]  
(5)

In what follows, we suppose that the parameters of the system satisfy the relation (5)

Let us pass to new dimensionless variables and new time according to the formulas

\[
x = \frac{am_1}{m}x', \quad y = \frac{bm_2}{m}y', \quad \xi = a\xi', \quad \eta = b\eta', \quad t = \frac{t'}{\omega}.
\]  
(6)

For the convenience of further calculations, we omit the primes and adhere to the previous notation.

Then the motion equation takes the form

\[
\ddot{x} + \ddot{\xi} = \Phi_c.
\]  
(7)
Here
\[ \Phi_c = \begin{cases} 
-k(\mu + \eta) \text{sign} \dot{x}, & \dot{x} \neq 0 \\
\dot{\xi}, & \dot{x} = 0 \text{ and } |\dot{\xi}| \leq k(\mu + \eta) \\
k(\mu + \eta) \text{sign} \dot{\xi}, & \dot{x} = 0 \text{ and } |\dot{\xi}| > k(\mu + \eta)
\end{cases}, \tag{8} \]

where
\[ \begin{cases} 
\xi = \cos(t + \phi_0) \\
\eta = \sin t
\end{cases}. \tag{9} \]

Here we have introduced the following notation
\[ k = f \frac{m_2 b}{m_1 a}, \quad \mu = \frac{mg}{bm_2 \omega^2}. \tag{10} \]

Thus, the problem under consideration has three parameters \( k, \mu \) and \( \phi_0 \). In terms of the introduced parameters, the condition (5) can be written as
\[ \mu \geq 1. \]

In this paper we determine the domains of parameter values in which body motions with a periodically varying velocity are possible.

3. Zones of acceleration and deceleration

Consider the motion of the body according to (7). Since time here is an argument of 2\( \pi \)-periodic functions, we can restrict ourselves to analyzing the acceleration of the body in one period. Let the value of inertia force for a period of time do not exceed the the maximum value of the friction force
\[ |\cos(t + \phi_0)| < k(\mu - \sin t), \quad 0 \leq t \leq 2\pi. \tag{11} \]

Then, if the initial velocity of the body is zero, the friction force is balanced by the force of inertia and is equal to it in absolute value. The acceleration of the body is zero, the body continues to be at rest and is not able to start moving. If the initial velocity of the body is nonzero, then the friction force acting on it exceeds the value of the inertial force, the body moves with deceleration to a complete stop and falls into the conditions of the first assumption. Thus, for the existence of motions with a periodically varying velocity, the existence of such a moment in time (let us call it \( t^* \)) is necessary at which the inertial force reached the maximum value of the friction force
\[ \cos(t^* + \phi_0) = \pm k(\mu - \sin t^*). \tag{12} \]

The last equation can be rewritten down as
\[ \begin{align*}
\cos t^* \cos \phi_0 - (\sin \phi_0 - k) \sin t^* - k\mu &= 0 \\
\cos t^* \cos \phi_0 - (\sin \phi_0 + k) \sin t^* + k\mu &= 0 
\end{align*} \tag{13} \]

Let us denote \( X = \cos t^* \), \( Y = \sin t^* \). Then two equations (13) can be represented in the equivalent form of two systems of algebraic equations
\[ \begin{cases} 
\cos \phi_0 X - (\sin \phi_0 - k) Y - k\mu = 0 \\
X^2 + Y^2 = 1
\end{cases}, \tag{14} \]
\[ \begin{cases} 
\cos \phi_0 X - (\sin \phi_0 + k) Y + k\mu = 0 \\
X^2 + Y^2 = 1
\end{cases}. \tag{15} \]

It is possible to give a visual geometric interpretation to the solutions of these systems - they represent the points of intersection of lines and a circle (see Fig 2).
Figure 2. The geometrical interpretation of the parameters of the problem

Let us denote the straight line described by the first equation from system (14) by \( \alpha_1 \), and the straight line corresponding to the first equation from system (15) by \( \alpha_2 \), respectively. If the parameter \( k \) is not equal to zero, then these lines intersect and the ordinate of the point of their intersection is equal to \( \mu \). The angle between the line passing through the origin and the intersection point and the abscissa axis is equal to \( \pi/2 - \phi_0 \). The points of intersection of lines \( \alpha_1 \) and \( \alpha_2 \) with the abscissa axis are determined by the values

\[
X_1 = -X_2 = \frac{k\mu}{\cos \phi_0}.
\]

Let us consider this geometric interpretation in the context of body motion. The current time is plotted on the unit circle shown in the Fig. 2, starting from the initial moment of time \( t_0 \) with coordinates \([1, 0]\) and then counterclockwise. Straight lines \( \alpha_1 \) and \( \alpha_2 \) cut off some time intervals when they intersect the unit circle. Consider the interval \((t_1, t_2)\). Here the following inequality takes place

\[
\cos(t + \phi_0) - k(\mu - \sin t) > 0.
\]

Suppose that the body is moving to the right, that is, the velocity vector is directed to the right, \( \dot{x} > 0 \). Then the equation of motion takes the form

\[
\ddot{x} = \cos(t + \phi_0) - k(\mu - \sin t).
\]

Thus, the acceleration in this interval is positive (when moving to the right, the body is accelerated), and negative or equal to zero throughout the rest of the circle.

Reasoning in a similar way, we find that when moving to the left on the interval \((t_3, t_4)\)

\[
\ddot{x} = \cos(t + \phi_0) + k(\mu - \sin t),
\]

the body accelerates, and slows down on the rest of the circle.

Thus, we have determined that if straight lines \( \alpha_1 \) and \( \alpha_2 \) intersect with a circle when moving to the right on the interval \((t_1, t_2)\), the body is accelerated (hereinafter, such intervals are called the acceleration zones) when moving to the right, the interval \((t_3, t_4)\) is the acceleration zone when moving to the left, but on the intervals \((t_2, t_3)\) and \((t_4, t_1)\) the acceleration is always opposite to the direction of motion or vanishes, since it is here that relation (11) is satisfied and this means that if the body stops at a given interval, it cannot start moving until one of the acceleration zones begins. Hereinafter, these intervals are called deceleration zones, and it is worth noting that they play an important role in the dynamics of body motion.
4. Possible cases of mutual arrangement of lines and a circle.
Let the initial phase $\phi_0$ belong to the interval $(0, \pi/2)$. In this case, point of intersection $P$ lies to the right of the ordinate axis and has a positive coordinate along the abscissa axis. Then, there are three different cases of mutual arrangement of straight lines $\alpha_1$, $\alpha_2$ and the unit circle.

![Figure 3. The first case. One deceleration zone.](image1)

![Figure 4. The second case. One deceleration and one acceleration zones.](image2)

![Figure 5. The third case. Two deceleration and two acceleration zones.](image3)

(i) Straight lines $\alpha_1$, $\alpha_2$ do not intersect the circle (see Fig 3). In this case, there is one deceleration zone on the entire circle. The boundary case occurs when the line $\alpha_2$ touches the circle. It can be shown that for this case the parameters of the problem satisfy the relation

$$\mu > \frac{\sqrt{1 + 2 \sin \phi_0 k} + k^2}{k}.$$  

(ii) Line $\alpha_2$ intersects the circle, line $\alpha_1$ does not intersect. In this case, there is one deceleration zone and one acceleration zone when the body moving in the negative direction (see Fig. 4). The boundary case is reached when the line $\alpha_1$ touches the circle. Here the parameters of the problem satisfy the relation

$$\frac{\sqrt{1 - 2 \sin \phi_0 k} + k^2}{k} < \mu < \frac{\sqrt{1 + 2 \sin \phi_0 k} + k^2}{k}.$$  

(iii) Both lines intersect the circle. There are two deceleration zones (see Fig. 5). For this case should be fulfilled inequality

$$\mu < \frac{\sqrt{1 - 2 \sin \phi_0 k} + k^2}{k}.$$  

Fig. 6 shows the in 3D space domains corresponding to the three cases described. For convenience, at $\phi_0$ equal to 0, $\pi/4$, and $\pi/2$, the cross-sections are given in which Roman numerals correspond to the case number for the domain shown.

Further analysis of the parameter domains in the intervals $\pi/2 < \phi_0 \leq \pi$ and $\pi < \phi_0 \leq 2\pi$ showed that the inequalities obtained in the interval $0 \leq \phi_0 \leq \pi/2$ and determining the regions of three possible cases of mutual arrangement of straight lines and the unit circle can be expanded to the remaining values of the initial phase as follows.

The domain which is determined by the relation

$$\mu > \frac{\sqrt{1 + 2 |\sin \phi_0| k} + k^2}{k}$$
Figure 6. Domains in three-dimensional space corresponding to the three found cases. $0 \leq \phi_0 \leq \pi/2$

corresponds to the first case, where none of the lines intersect the circle. The next domain where

$$\frac{\sqrt{1 - 2|\sin\phi_0|k + k^2}}{k} \leq \mu < \frac{\sqrt{1 + 2|\cos\phi_0|k + k^2}}{k}$$

corresponds to the second case where one of the lines intersect the circle and one does not. And the domain for the third case, where both lines intersect the circle, is given by the relation

$$\mu < \frac{\sqrt{1 - 2|\sin\phi_0|k + k^2}}{k}.$$  

These domains for the $\phi_0$ in interval $(0, 2\pi)$ are shown on the Fig.7.

5. Possible periodic types of motions in the three identified cases.
Consider the types of motions that are possible in each of the cases found.

The first case is of no interest, because even with a certain initial velocity, the body moves with deceleration and eventually stop. Since in this range of parameters none of the straight lines $\alpha_1, \alpha_2$ intersects the unit circle, there is no time moment $t^*$ when the body begins to move from a state of rest.

In the second case, a periodic regime of body motion is possible. Let’s demonstrate how to build it. Without loss of generality, consider the case $0 \leq \phi_0 \leq \pi/2$. Suppose that the initial moment of time $t_0$ does not belong to the acceleration zone $(t_3, t_4)$, the body is at rest. When the time reaches the moment $t_3$, the body begins to move to the left with the negative acceleration, when the moment $t_4$ is reached, the body acquires the maximum possible value of velocity, and the acceleration becomes zero. On the interval $(t_4, t_3 + 2\pi)$, the body moves with deceleration. Let us show that under the assumptions made, the body definitely stops at this interval. As already shown, the acceleration in the interval $(t_4, t_3 + 2\pi)$ when moving to the
Figure 7. Domains in three-dimensional space corresponding to the three found cases.

\[ 0 \leq \phi_0 \leq 2\pi \]

left is positive, or equal to zero, therefore, the function of the velocity on time, which can be obtained by integrating equation (17)

\[ v(t) = \int_{t_3}^{t} [\cos(\tau + \phi_0) + k(\mu - \sin \tau)] d\tau \]

is a monotonically increasing function. Since at the moment of time \( t_4 \) this function is negative, and at the moment \( t_3 + 2\pi \) the function \( v(t_3 + 2\pi) = 2\pi k \mu \) is positive, then on the interval \( (t_4, t_3 + 2\pi) \), there should be such a moment of time at which this function is equal to zero and the body stops. As this interval is a deceleration zone, then, as mentioned above, the body cannot start moving before the moment \( t_3 + 2\pi \). Thus it enters the acceleration zone \( (t_3 + 2\pi, t_4 + 2\pi) \) at rest and we get a periodic mode of motion.

It is easy to show that if the initial moment of time \( t_0 \) lies within the acceleration zone, and the body begins to move from a state of rest, then in the deceleration zone there is necessarily a stop, which leads to the conditions of the first assumption and a periodic mode of motion after one period of time.

Now let us assume that the motion begins in such a way that the body enters the acceleration zone \( (t_3, t_4) \) with a negative velocity \( v(t_3) \) that is less than the value \(-2\pi k \mu\). Then, after one period of time, the body enters the acceleration zone with velocity \( v(t_3) + 2\pi k \mu \) and with each period this value increases until it becomes positive, which means that in the previous period of time the body in the deceleration zone stopped and entered a periodic mode of motion.

Thus, in this case, only one variant of motion is possible, which, within a finite time interval, reaches the periodic mode. This case is interesting because when the value of the parameter \( \phi_0 \) is equal to \( \pi n \), \( n \in \mathbb{Z} \) the corresponding region of parameters \( \mu \) and \( k \) degenerates into a curve (see Fig. 6), and thus the case becomes unrealizable (see, for example, [6]).

Finally, consider the last third case. It is the most difficult to research. Numerical analysis showed that there are the following types of motion.

(i) Motion with complete stops in both deceleration zones (motion to the right in the first acceleration zone \( (t_1, t_2) \), deceleration and complete stop in the first deceleration zone \( (t_2, t_3) \), rollback to the left in the second acceleration zone \( (t_3, t_4) \), deceleration and complete stop in the second deceleration zone \( (t_4, t_1 + 2\pi) \))
(ii) Motion with a full stop in one deceleration zone. Since the deceleration zones are qualitatively different from each other (for example, due to the fact that the point of intersection of straight lines $\alpha_1$ and $\alpha_2$ is always in the upper half-plane, the interval $(t_2, t_3)$ is less than the interval $(t_4, t_1)$), then a type of motion may arise in which the body totally stops only in one of the deceleration zones, for example, in the zone $(t_4, t_1 + 2\pi)$. Thus the body accelerates to the right in the first acceleration zone, brakes in the first deceleration zone, but does not stop and enters the second acceleration zone, still moving to the right, continues slow down, but as soon as motion to the right stops, the rollback to the left begins with acceleration. In the second deceleration zone, braking occurs until the body comes to a complete stop. Here, it is also possible that the body slips through the second acceleration zone and for the first time stop in the second deceleration zone, in this case there is no rollback to the left.

In any case, the first two types of motion are periodic in nature and reach a periodic regime in a finite time.

(iii) Motion without full stops. In this case, the body does not stop in the deceleration zones, but overshoots them and changes the direction of motion in the acceleration zones. In this case, the motion is not periodic, but, possibly, asymptotically approaches it over time.

These types of motions are in good agreement with ones obtained in [6], where their detailed analysis was carried out for the value of the parameter $\phi_0 = 0$.

6. Conclusions

As a result of this paper, the domain of parameters was determined in which body motions with periodically varying velocity are possible. This domain was divided into two qualitatively different subdomains. In the first of them, the existence of a single type of motion was proved - a periodic mode with one stop in the deceleration zone. For the values of the parameters from the second subdomain, three different modes of motion were numerically found, two of which are also motions with periodically varying velocity.

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