ORBITAL STABILITY OF SOLITARY WAVES FOR GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS IN THE ENDPOINT CASE

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Abstract. We consider the following generalized derivative nonlinear Schrödinger equation

\[ i \partial_t u + \partial_x^2 u + i |u|^{2\sigma} \partial_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R} \]

where \( \sigma \in (0, 1) \). The equation has a two-parameter family of solitary waves

\[ u_{\omega,c}(t, x) = \Phi_{\omega,c}(x)e^{i\omega t + \frac{c}{2} x - \frac{c^2}{4} \int_{-\infty}^{x} \Phi_{\omega,c}(y)^2 dy}, \]

with \( (\omega, c) \) satisfying \( \omega > c^2/4 \), or \( \omega = c^2/4 \) and \( c > 0 \). The stability theory in the frequency region \( \omega > c^2/4 \) was studied previously. In this paper, we prove the stability of the solitary wave solutions in the endpoint case \( \omega = c^2/4 \) and \( c > 0 \).

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1. Introduction

The derivative nonlinear Schrödinger (DNLS) equation

\[ i \partial_t v + \partial_x^2 v + i \partial_x (|v|^2 v) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R} \]

appears in the long wave-length approximation of Alfvén waves propagating in plasma [1, 2, 3]. Applying the gauge transformation

\[ u(t, x) = v(t, x)e^{i \int_{-\infty}^{x} |v(t, y)|^2 dy}, \]

the equation (1.1) has the Hamiltonian form

\[ i \partial_t u + \partial_x^2 u + i |u|^2 \partial_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \]

The Cauchy problem for (1.2) has been studied by many researchers. The locally well-posedness theory in the energy space \( H^1(\mathbb{R}) \) was studied in [16, 17, 18, 28]. Local well-posedness in low-regularity spaces \( H^s(\mathbb{R}) \), \( s \geq \frac{1}{2} \) was investigated by Takaoka [26] using the Fourier restricted method. Biagioni and Linares [4] proved that when \( s < \frac{1}{2} \), the solution map from \( H^s(\mathbb{R}) \) to \( C([-T, T] : H^s(\mathbb{R})) \), \( T > 0 \) for (1.2) is not locally uniformly continuous.

The problem of global well-posedness has attracted the attention of a number of authors. Hayashi and Ozawa [18, 24] proved the global existence in \( H^1(\mathbb{R}) \) with \( \|u_0\|_{L^2}^2 < 2\pi \). Wu
29, 30 showed it holds for initial data \( u_0 \) having the mass \( \| u_0 \|_{L^2}^2 \) less than threshold \( 4\pi \). For the initial data with low regularity, Colliander, Keel, Staffilani, Takaoka, and Tao 6 proved that the \( H^s \)-solution is global if \( \| u_0 \|_{L^2}^2 < 2\pi \) when \( s > 1/2 \) by the I-method (see also 5, 29). Miao, Wu and Xu 23 showed that \( H^{1/2} \)-solution is global if \( \| u_0 \|_{L^2}^2 < 2\pi \). Guo and Wu 13 improved this result to obtain that \( H^{1/2} \)-solution is global if \( \| u_0 \|_{L^2}^2 < 4\pi \).

Despite the amount of studies devoted to (1.2), existence of blowing up solutions remains an open problem.

It is known that (1.2) has a two-parameter family of the solitary waves \( u_{\omega,c}(t, x) = e^{i\omega t}\phi_{\omega,c}(x - ct) \), where \( (\omega, c) \) satisfies \( \omega > c^2/4 \) or \( \omega = c^2/4 \) and \( c > 0 \). Boling Guo and Yaping Wu 12 proved that the solitary waves \( u_{\omega,c} \) are orbitally stable when \( \omega > c^2/4 \) and \( c > 0 \) by the abstract theory of Grillakis, Shatah, and Strauss 10, 11 and the spectral analysis of the linearized operators. Colin and Ohta 7 proved that the solitary waves \( u_{\omega,c} \) are orbitally stable when \( \omega > c^2/4 \) by characterizing the solitary waves from the viewpoint of a variational structure. The case of \( \omega = c^2/4 \) and \( c > 0 \) is treated by Kwon and Wu 19.

Recently, the stability of the multi-solitons is studied by Miao, Tang, and Xu 22 and Le Coz and Wu 20.

Liu, Simpson, and Sulem 21 introduced an extension of (1.2) with general power non-linearity, which is the so-called generalized derivative nonlinear Schrödinger equation:

\[
\begin{cases}
  i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \\
  u(0, x) = u_0(x)
\end{cases}
\]

(1.3)

where \( \sigma > 0 \). The equation of (1.3) is invariant under the scaling transformation

\[ u_\gamma(t, x) = \gamma^{1/2}\gamma u(\gamma^2 t, \gamma x), \quad \gamma > 0, \]

which implies that its critical Sobolev exponent is \( s_c = \frac{1}{2} - \frac{1}{2\sigma} \). Hayashi and Ozawa 15 proved local well-posedness in \( H^1(\mathbb{R}) \) when \( \sigma \geq 1 \) and showed that the following quantities are conserved:

\[
M(u) = \| u \|_{L^2}^2 = M(u_0),
\]

(1.4)

\[
E(u) = \| \partial_x u \|_{L^2}^2 - \frac{1}{\sigma + 1}Im \int_\mathbb{R} |u|^{2\sigma} u\overline{\partial_x u}dx = E(u_0),
\]

(1.5)

\[
P(u) = Re \int_\mathbb{R} i\partial_x u\overline{\partial_x u}dx = Im \int_\mathbb{R} u\overline{\partial_x u}dx = P(u_0).
\]

(1.6)

Moreover, they proved global well-posedness for small initial data. They also constructed global solutions for any initial data in the \( L^2 \)-subcritical case \( 1/2 \leq \sigma < 1 \). Recently, Fukaya, Hayashi, Inui 9 and Miao, Tang, Xu 22 investigate the global well-posedness for (1.3) in the case \( \sigma > 1 \) by variational argument.

Similar to the equation (1.2), by 21, (1.3) has a two-parameter family of solitary waves

\[ u_{\omega,c}(t, x) = e^{i\omega t}\phi_{\omega,c}(x - ct), \]
with \((\omega, c)\) satisfying \(\omega > c^2/4\), or \(\omega = c^2/4\) and \(c > 0\),

\[
\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)e^{\frac{ix}{\sqrt{2}} \int_0^x \Phi_{\omega,c}(y)^{2\sigma} dy}
\]  

(1.7)

and

\[
\Phi_{\omega,c}(x) = \begin{cases} 
\left( \frac{(\sigma+1)(4\omega-c^2)}{2\sqrt{c}\cosh(\sqrt{4\omega-c^2}x-c)} \right)^{\frac{1}{2\sigma}}, & \text{if } \omega > \frac{c^2}{4}, \\
\left( \frac{2(\sigma+1)c}{c^2 + 1} \right)^{\frac{1}{2\sigma}}, & \text{if } \omega = \frac{c^2}{4} \text{ and } c > 0.
\end{cases}
\]

(1.8)

Moreover, \(\Phi_{\omega,c}\) is the positive even solution of

\[
-\partial_x^2 \Phi + (\omega - \frac{c^2}{4})\Phi + \frac{c}{2}\Phi^{2\sigma} \Phi - \frac{2\sigma + 1}{(2\sigma + 2)^2} |\Phi|^{4\sigma} \Phi, \quad x \in \mathbb{R},
\]

(1.9)

and the complex-valued function \(\phi_{\omega,c}\) satisfies

\[
-\partial_x^2 \phi + \omega \phi + ic\partial_x \phi - i|\phi|^{2\sigma} \partial_x \phi = 0, \quad x \in \mathbb{R}.
\]

Liu, Simpson and Sulem [21] proved that when \(1 < \sigma < 2\), for some \(z_0 = z_0(\sigma) \in (0, 1)\), if \(-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}\), the solitary waves are orbitally stable and if \(2z_0\sqrt{\omega} < c < 2\sqrt{\omega}\), they are orbitally unstable. They also showed that the solitary waves for all \(\omega > c^2/4\) are orbitally unstable when \(\sigma \geq 2\) and orbitally stable when \(0 < \sigma < 1\). In [8], it is proved that the solitary waves are orbitally unstable if \(c = 2z_0\sqrt{\omega}\) when \(3/2 < \sigma < 2\). Tang and Xu [27] investigated stability of the two sum of solitary waves for (1.3). We also refer to [14, 25] for some lower regularity results.

In this present work, we consider the stability of solitary wave solutions of (1.3) in the endpoint case \(\omega = c^2/4\) when \(\sigma \in (0, 1)\). For simplicity, we denote \(\phi_c = \phi_{c^2/4,c}\), which solves

\[
-\partial_x^2 \phi + \frac{c^2}{4} \phi + ic\partial_x \phi - i|\phi|^{2\sigma} \partial_x \phi = 0, \quad x \in \mathbb{R}.
\]

(1.10)

More precisely, we prove that when \(\omega = c^2/4\), the solitary waves (1.7) are orbitally stable in the sense of the following.

**Theorem 1.1.** For any \(\varepsilon > 0\), there exists some \(\delta = \delta(\varepsilon)\) such that if

\[
\|u_0 - \phi_c\|_{H^1} \leq \delta,
\]

(1.11)

then there exist \(\theta(t) \in [0, 2\pi), y(t) \in \mathbb{R}\) such that the solution \(u(t)\) of the equation (1.3) satisfies that, for any \(t \in \mathbb{R}\),

\[
\|u(t) - e^{i\theta(t)}\phi(\cdot - y(t))\|_{H^1} \leq \varepsilon.
\]

We use a variational method to prove Theorem 1.1, but it is not standard and the difficulty is from the “zero mass” property of (1.10) in this endpoint case.

The rest of the paper is organized as follows. In section 2, we give a variational characterization of solitary wave solutions. Then by a variational argument, we prove Theorem 1.1 in section 3.
In this section, we give a variational characterization of the solitary wave solution $\phi_c$ of (1.10) defined by (1.7). Note that it is not standard because of the “zero mass” of the equation (1.10). Our approach is inspired by Kwon, Wu [19]. In addition to (1.4), (1.5) and (1.6), we define several other variational functionals as follows:

\[ S_c(u)(t) = E(u) + cP(u) + \frac{c^2}{4} M(u) \]

\[ = \|\partial_x u\|_{L^2}^2 + cIm \int \overline{u} \partial_x u dx + \frac{c^2}{4} \|u\|_{L^2}^2 - \frac{1}{\sigma + 1} Im \int |u|^{2\sigma} \overline{u} \partial_x u dx, \]

\[ K_c(u)(t) = \|\partial_x u\|_{L^2}^2 + cIm \int \overline{u} \partial_x u dx + \frac{c^2}{4} \|u\|_{L^2}^2 - Im \int |u|^{2\sigma} \overline{u} \partial_x u dx. \]

We also denote for convenience that

\[ L_c(u)(t) = \|\partial_x u\|_{L^2}^2 + cIm \int \overline{u} \partial_x u dx + \frac{c^2}{4} \|u\|_{L^2}^2 \]

and

\[ N(u)(t) = Im \int |u|^{2\sigma} \overline{u} \partial_x u dx, \]

which imply that

\[ S_c(u) = L_c(u) - \frac{1}{\sigma + 1} N(u), \quad K_c(u) = L_c(u) - N(u). \]

Using a standard argument as in Berestycki and Lion [4], we can obtain the uniqueness result as follows.

Lemma 2.1. If $\Psi \in H^1(\mathbb{R}) \setminus \{0\}$ is a solution of

\[ -\partial_x^2 \Psi + \frac{c}{2} |\Psi|^{2\sigma} \Psi - \frac{2\sigma + 1}{(2\sigma + 2)^2} |\Psi|^{4\sigma} \Psi = 0, \quad x \in \mathbb{R}, \]

then there exists some $(\theta, x_0)$ such that

\[ \Psi(x) = e^{i\theta} \Phi_c(x - x_0) \]

with

\[ \Phi_c(x) = \left( \frac{2(\sigma + 1)c}{\sigma^2 (cx)^2 + 1} \right)^{\frac{1}{2\sigma}}. \]

The difficulty is that we have no $L^2$-control from $L_c(u)$. A counterpart result is as follows.

Lemma 2.2. If $\psi \in H^1(\mathbb{R}) \setminus \{0\}$ is a solution of (1.10) then there exists some $(\theta, x_0)$ such that

\[ \psi(x) = e^{i\theta} \phi_c(x - x_0). \]
We consider the following minimization problem:
\[ d(c) = \inf\{ S_c(u) : u \in H^1(\mathbb{R}) \setminus \{0\}, K_c(u) = 0 \}. \]  
(2.6)

Letting
\[ \tilde{L}_c(f) := L_c(e^{\xi ix} f), \quad \tilde{N}_c(f) := N(e^{\xi ix} f), \]
then
\[ \tilde{L}_c(f) = \|\partial_x f\|_{L^2}^2, \quad \tilde{N}_c(f) = -\frac{c}{2}\|f\|_{L^{2\sigma+2}}^{2\sigma+2} + \text{Im} \int_{\mathbb{R}} |f|^{2\sigma} \bar{f} \partial_x f \, dx, \]
and
\[ \tilde{S}_c(f) := S_c(e^{\xi ix} f) = \tilde{L}_c(f) - \frac{1}{\sigma+1} \tilde{N}_c(f), \]
\[ \tilde{K}_c(f) := K_c(e^{\xi ix} f) = \tilde{L}_c(f) - \tilde{N}_c(f). \]

Hence, equivalently, we have
\[ d(c) = \inf\{ \tilde{S}_c(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, \tilde{K}_c(v) = 0 \}. \]  
(2.7)

We need the following result to give the characterization of \( d(c) \).

**Lemma 2.3.** Assume that \( f \in H^1(\mathbb{R}) \setminus \{0\} \) satisfies
\[ \|\partial_x f\|_{L^2(\mathbb{R})}^2 \leq \frac{\sigma+1}{\sigma} d(c), \]
then \( \tilde{K}_c(f) \geq 0 \).

**Proof.** We argue by contradiction to assume that there exists some \( f \in H^1(\mathbb{R}) \setminus \{0\} \) such that \( \tilde{K}_c(f) < 0 \). Then, there exists some \( \gamma \in (0,1) \) such that \( \tilde{K}_c(f_{\gamma}) = 0 \), with \( f_{\gamma}(x) = \gamma^{\frac{1}{2}} f(\gamma x) \). Indeed, since
\[ \tilde{K}_c(f_{\gamma}) = \gamma^{\frac{2}{\sigma}+1} \left[ \|\partial_x f\|_{L^2}^2 + \frac{c}{2}\|f\|_{L^{2\sigma+2}}^{2\sigma+2} - \text{Im} \int_{\mathbb{R}} |f|^{2\sigma} \bar{f} \partial_x f \, dx \right], \]
\( \tilde{K}_c(f) < 0 \) implies that we may choose
\[ \gamma = \frac{\|\partial_x f\|_{L^2}^2 + \frac{c}{2}\|f\|_{L^{2\sigma+2}}^{2\sigma+2}}{\text{Im} \int_{\mathbb{R}} |f|^{2\sigma} \bar{f} \partial_x f \, dx} < 1 \]
to get \( \tilde{K}_c(f_{\gamma}) = 0 \). Therefore, by definition of \( d(c) \) in (2.7), \( \tilde{S}_c(f_{\gamma}) \geq d(c) \), which gives then
\[ \|\partial_x f_{\gamma}\|_{L^2}^2 = \frac{\sigma+1}{\sigma} \left( \tilde{S}_c(f_{\gamma}) - \frac{1}{\sigma+1} \tilde{K}_c(f_{\gamma}) \right) \geq \frac{\sigma+1}{\sigma} d(c), \]
or \( \gamma^{\frac{2}{\sigma}+1}\|\partial_x f\|_{L^2}^2 \geq \frac{\sigma+1}{\sigma} d(c) \). Since \( \gamma < 1 \), this contradicts the assumption \( \|\partial_x f\|_{L^2(\mathbb{R})} \leq \frac{\sigma+1}{\sigma} d(c) \) and we conclude the proof.

Now we give the characterization of \( d(c) \).

**Lemma 2.4.** It holds that
\[ S_c(\phi_c) = d(c) \]
with \( \phi_c \) defined by (1.1) with \( w = c^2/4 \).
Proof. We first claim that $d(c) > 0$.

Indeed, considering
\[
\tilde{S}_c(v) - \frac{1}{\sigma + 1} \tilde{K}_c(v) = (1 - \frac{1}{\sigma + 1}) \tilde{L}_c(v) = (1 - \frac{1}{\sigma + 1}) \| \partial_x v \|^2_{L^2} \geq 0.
\]
Now, if $d(c) = 0$, then we obtain some minimizing sequence $\{v_n\} \subset H^1(\mathbb{R}) \setminus \{0\}$, such that
\[
\tilde{S}_c(v_n) \to 0, \quad \text{and} \quad \tilde{K}_c(v_n) = 0.
\]
Thus, we have $\| \partial_x v_n \|^2_{L^2} \to 0$ and
\[
\tilde{N}_c(v_n) \to 0.
\]
that is
\[
\int_{\mathbb{R}} \left( \frac{c}{2} |v_n|^{2\sigma + 2} - Im |v_n|^{2\sigma} \overline{v_n} \partial_x v_n \right) dx \to 0. \tag{2.8}
\]
Note that
\[
\left| Im \int_{\mathbb{R}} |v_n|^{2\sigma} \overline{v_n} \partial_x v_n dx \right| \leq \| \partial_x v_n \|_{L^2} \| v_n \|^{2\sigma + 1}_{L^{2\sigma + 2}} \tag{2.9}
\]
\[
\leq \| \partial_x v_n \|^{\theta}_{L^2} \cdot \| v_n \|^{2\sigma + 2 - \theta}_{L^{2\sigma + 2}} \leq \frac{c}{4} \| v_n \|^{2\sigma + 2}_{L^{2\sigma + 2}} + A \| \partial_x v_n \|^{2\sigma + 2}_{L^2}
\]
with some $\theta \in (1, 2\sigma + 2)$ and some constants $A > 0$. From (2.8), (2.9) and $\| \partial_x v_n \|^2_{L^2} \to 0$, we obtain that
\[
\| v_n \|^{2\sigma + 2}_{L^{2\sigma + 2}} \to 0
\]
which gives by interpolation that
\[
\| v_n \|_{L^\infty} \to 0, \quad \text{as} \quad n \to \infty. \tag{2.10}
\]
Using (2.10), we obtain that
\[
0 = K_c(u_n) = \| \partial_x v_n \|^2_{L^2} + \frac{c}{2} \int_{\mathbb{R}} |v_n|^{2\sigma + 2} dx - Im \int_{\mathbb{R}} |v_n|^{2\sigma} \overline{v_n} \partial_x v_n dx \tag{2.11}
\]
\[
\geq \| \partial_x v_n \|^2_{L^2} + \frac{c}{2} \int_{\mathbb{R}} |v_n|^{2\sigma + 2} dx - \frac{1}{2} \| \partial_x v_n \|^2_{L^2} - \frac{1}{2} \int_{\mathbb{R}} |v_n|^{4\sigma + 2} dx
\]
\[
= \frac{1}{2} \| \partial_x v_n \|^2_{L^2} + \int_{\mathbb{R}} |v_n|^{2\sigma + 2} (\frac{c}{2} - \frac{1}{2} |v_n|^{2\sigma}) dx
\]
\[
\geq \frac{1}{2} \| \partial_x v_n \|^2_{L^2} + \frac{c}{4} \int_{\mathbb{R}} |v_n|^{2\sigma + 2} \geq 0,
\]
which gives $v_n = 0$. This is a contradiction and gives the claim $d(c) > 0$.

Next, let $\{v_n\} \subset H^1(\mathbb{R}) \setminus \{0\}$ be the minimizing sequence such that as $n \to \infty$,
\[
\tilde{S}_c(v_n) = \| \partial_x v_n \|^2_{L^2} + \frac{1}{\sigma + 1} \left( \frac{c}{2} \int_{\mathbb{R}} |v_n|^{2\sigma + 2} dx - Im \int_{\mathbb{R}} |v_n|^{2\sigma} \overline{v_n} \partial_x v_n dx \right) \to d(c),
\]
and
\[
\tilde{K}_c(v_n) = \| \partial_x v_n \|^2_{L^2} + \frac{c}{2} \int_{\mathbb{R}} |v_n|^{2\sigma + 2} dx - Im \int_{\mathbb{R}} |v_n|^{2\sigma} \overline{v_n} \partial_x v_n dx = 0.
\]
There holds then \( \| \partial_x v_n \|_{L^2}^2 \to \frac{\sigma}{\sigma - 1} d(c) \). Moreover, by a similar argument of (2.11), there exists some absolute constant \( C > 0 \) such that
\[
\| v_n \|_{L^{2\sigma + 2}} \leq C.
\]

Now we apply the profile decomposition to the uniformly bounded sequence \( \{v_n\} \) in \( \dot{H}^1(\mathbb{R}) \cap L^{2\sigma + 2}(\mathbb{R}) \) to obtain that there exist some sequences \( \{V^j\}_{j=1}^\infty \) and \( \{x_n^j\}_{n,j=1}^\infty \) such that, up to some subsequence, for each \( L \geq 1 \),
\[
v_n = \sum_{j=1}^L V^j(-x_n^j) + R_n^L \tag{2.12}
\]
with
\[
\forall k \neq j, \quad |x_n^j - x_n^k| \to \infty, \quad \text{as} \quad n \to \infty
\]
and
\[
\lim_{L \to \infty} \left( \lim_{n \to \infty} \| R_n^L \|_{L^q(\mathbb{R})} \right) = 0, \quad \forall q > 2\sigma + 2. \tag{2.13}
\]

Moreover,
\[
\| \partial_x v_n \|_{L^2}^2 = \sum_{j=1}^L \| \partial_x V^j \|_{L^2}^2 + \| \partial_x R_n^L \|_{L^2}^2 + o_n(1), \tag{2.14}
\]
\[
\tilde{S}_c(v_n) = \sum_{j=1}^L \tilde{S}_c(V^j) + \tilde{S}_c(R_n^L) + o_n(1), \tag{2.15}
\]
and
\[
\tilde{K}_c(v_n) = \sum_{j=1}^L \tilde{K}_c(V^j) + \tilde{K}_c(R_n^L) + o_n(1). \tag{2.16}
\]

Thus by Lemma 2.3, \( \tilde{K}_c(V^j) = K_c(e^{\tilde{S}_c V^j}) \geq 0 \) or \( V^j = 0 \). Since also \( \tilde{K}_c(R_n^L) \geq 0 \), from (2.16) and \( \tilde{K}(v_n) = 0 \), we get that for any \( j = 1, 2, \cdots, L \) there must hold that \( \tilde{K}_c(V^j) = 0 \), which means that \( \tilde{S}_c(V^j) \geq d(c) \) or \( V^j = 0 \). Now from (2.14), there exists only one \( j \), say \( j = 1 \), such that \( \tilde{S}_c(V^1) = d(c) \) and \( V^j = 0 \) for \( j = 2, \cdots, L \). Hence, \( V^1 \) is the function such that
\[
\tilde{S}_c(V^1) = d(c), \quad \tilde{K}_c(V^1) = 0.
\]

Then there exists some Lagrange constant \( \rho \) such that
\[
\tilde{S}_c'(V^1) = \rho \tilde{K}_c'(V^1),
\]
which implies
\[
\langle \tilde{S}_c'(V^1), V^1 \rangle = \rho \langle \tilde{K}_c'(V^1), V^1 \rangle.
\]
That is to say
\[
(\rho(\sigma + 1) - 1) \tilde{K}_c(V^1) = \rho \sigma \| \partial_x V^1 \|_{L^2}^2,
\]
which implies that \( \rho = 0 \). Therefore, we have that
\[
\tilde{S}_c'(V^1) = 0, \quad \text{or} \quad S_c'(e^{\tilde{S}_c V^1}) = 0
\]
and then \( e^{\tilde{S}_c V^1}(x) \) solves the equation (1.10). Hence by Lemma 2.2,
\[
e^{\tilde{S}_c V^1}(x) = e^{i\theta} \phi_c(x - x_0).
\]
It follows that
\[
d(c) = \tilde{S}_c(V^1) = S_c(e^{i\theta} \phi_c(\cdot - x_0)) = S_c(\phi_c).
\]
Finally in this section, we prove the following lemma, which is useful to show our main result.

**Lemma 2.5.** There hold that
\[
P(\phi_c) + \frac{c}{2} M(\phi_c) > 0, \quad \partial_c P(\phi_c) + \frac{c}{2} \partial_c M(\phi_c) > 0.
\]

(2.17)

**Proof.** First of all, by the definition of (1.7) and (1.8) and straight calculation, it follows that
\[
P(\phi_c) + \frac{c}{2} M(\phi_c) = \frac{1}{2\sigma + 2} \Phi_c^{2\sigma + 2} > 0.
\]

(2.18)

Next, letting \( \phi := \phi_1 \), it follows that
\[
\int |\phi_c|^2 dx = c^{\frac{1}{\sigma}} \|\phi\|^{\frac{2}{\sigma}}_{L^2}, \quad P(\phi_c) = c^{\frac{1}{\sigma}} P(\phi).
\]

Hence,
\[
\partial_c P(\phi_c) + \frac{c}{2} \partial_c M(\phi_c) = c^{\frac{1}{\sigma} - 1} \left( \frac{1}{\sigma} P(\phi) + \frac{1}{2} \frac{1}{\sigma} - 1 \right) M(\phi).
\]

Note that
\[
P(\phi_c) = -\frac{1}{2} c \|\Phi_c\|^{2}_{L^2} + \frac{1}{2\sigma + 2} \|\Phi_c\|^{2\sigma + 2}_{2\sigma + 2}
\]
and recall
\[
\Phi(x) := \Phi_1(x) = \left( \frac{2(\sigma + 1)}{(\sigma x)^2 + 1} \right)^{\frac{1}{\sigma}}.
\]

We have then
\[
\frac{1}{\sigma} P(\phi) + \frac{1}{2} \frac{1}{\sigma} - 1 M(\phi)
= -\frac{1}{2\sigma} \int_{\mathbb{R}} \Phi^2 dx + \frac{1}{2\sigma(\sigma + 1)} \int_{\mathbb{R}} \Phi^{2\sigma + 2} dx + \left( \frac{1}{2\sigma} - \frac{1}{2} \right) \int_{\mathbb{R}} \Phi^2 dx
= \frac{1}{2\sigma(\sigma + 1)} \int_{\mathbb{R}} \Phi^{2\sigma + 2} dx - \frac{1}{2} \int_{\mathbb{R}} \Phi^2 dx
= \frac{(2\sigma + 2)^{\frac{1}{\sigma}}}{2} \int_{\mathbb{R}} \left( \frac{2}{\sigma(\sigma x^2 + 1)^{\frac{\sigma + 1}{\sigma}}} - \frac{1}{(\sigma x^2 + 1)^{\frac{1}{\sigma}}} \right) dx
= \frac{(2\sigma + 2)^{\frac{1}{\sigma}}}{2\sigma} \int_{\mathbb{R}} \left( \frac{2}{\sigma(x^2 + 1)^{\frac{\sigma + 1}{\sigma}}} - \frac{1}{(x^2 + 1)^{\frac{1}{\sigma}}} \right) dx.
\]

\footnote{Due to a private discussion with Cui Ning.}
Finally, we are sufficed to show the integration
\[ I = I(\sigma) = \int_R \left( \frac{2}{\sigma(x^2 + 1)^{\frac{\sigma + 1}{\sigma}}} - \frac{1}{(x^2 + 1)^{\frac{1}{\sigma}}} \right) dx > 0. \] (2.19)

In fact, by straight calculation,
\[
\frac{d}{dx} \left[ x(x^2 + 1)^{-\frac{1}{\sigma}} \right] = (x^2 + 1)^{-\frac{1}{\sigma}} - \frac{1}{\sigma} \cdot 2x^2 \cdot (x^2 + 1)^{-\frac{1}{\sigma} - 1} = (1 - \frac{2}{\sigma})(x^2 + 1)^{-\frac{1}{\sigma}} + \frac{2}{\sigma}(x^2 + 1)^{-\frac{1}{\sigma} - 1},
\]
which implies that if \( \sigma < 2 \),
\[
\int_R \left[ (1 - \frac{2}{\sigma})(x^2 + 1)^{-\frac{1}{\sigma}} + \frac{2}{\sigma}(x^2 + 1)^{-\frac{1}{\sigma} - 1} \right] dx = 0
\]
or
\[
\int (x^2 + 1)^{-\frac{1}{\sigma} - 1}dx = (1 - \frac{\sigma}{2}) \int (x^2 + 1)^{-\frac{1}{\sigma}}dx.
\]
Since \( \sigma < 1 \), we obtain that
\[
I(\sigma) = \int_R \left( \frac{2}{\sigma(x^2 + 1)^{\frac{\sigma + 1}{\sigma}}} - \frac{1}{(x^2 + 1)^{\frac{1}{\sigma}}} \right) dx = 2(\frac{1}{\sigma} - 1) \int (x^2 + 1)^{-\frac{1}{\sigma}}dx > 0.
\]

As a result, we conclude Lemma 2.5 by (2.18) and (2.19). \( \Box \)

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. The proof is based on the variational characterization of solitary wave solutions in section 2. Using the notations defined in the above section, we set
\[
\mathcal{A}^+ = \{ u \in H^1(\mathbb{R}) \setminus \{0\} : S_c(u) < S_c(\phi_c), L_c(u) < \frac{\sigma + 1}{\sigma} d(c) \},
\]
\[
\mathcal{A}^- = \{ u \in H^1(\mathbb{R}) \setminus \{0\} : S_c(u) < S_c(\phi_c), L_c(u) > \frac{\sigma + 1}{\sigma} d(c) \}.
\]

Lemma 3.1. The sets \( \mathcal{A}^+ \) and \( \mathcal{A}^- \) are invariant under the flow of (1.3), i.e., if \( u_0 \in \mathcal{A}^+ \) (resp. \( \mathcal{A}^- \)), then the solution \( u(t) \) of (1.3) with \( u(0) = u_0 \) belongs to \( \mathcal{A}^+ \) (resp. \( \mathcal{A}^- \)) as long as \( u(t) \) exists.

Proof. Let \( u_0 \in \mathcal{A}^+ \) and \( I = (-T_*, T^*) \) be the maximal existence interval of the solution \( u(t) \) of (1.3) with \( u(0) = u_0 \). By \( u_0 \neq 0 \) and the conservation laws (1.5), (1.4) and (1.6), we have that \( u(t) \neq 0 \) for \( t \in I \). By definition of \( S_c(u) \), \( S_c \) is also conserved, which means that \( S_c(u(t)) = S_c(u_0) < d(c) \) for \( t \in I \). By continuity of the function \( t \mapsto L_c(u(t)) \), we assume that there exists some \( t_0 \) such that
\[
L_c(u(t_0)) = \tilde{L}_c(e^{-\frac{2}{\sigma} i x} u(t_0)) = \frac{\sigma + 1}{\sigma} d(c).
\]
Thus, from
\[ d(c) > S_c(u(t_0)) = \frac{\sigma}{\sigma + 1} L_c(u(t_0)) + \frac{1}{\sigma + 1} K_c(u(t_0)) = d(c) + \frac{1}{\sigma + 1} K_c(u(t_0)), \]
we get that \( \tilde{K}_c(e^{-\frac{\sigma}{\sigma + 1} i x} u(t_0)) = K_c(u(t_0)) < 0 \). On the other hand, by Lemma 2.3, it holds that
\[ \tilde{L}_c(e^{-\frac{\sigma}{\sigma + 1} i x} u(t_0)) = L_c(u(t_0)) > \frac{\sigma}{\sigma + 1} d(c) \]
and we get a contradiction. Hence \( \mathcal{A}^+ \) is invariant under the flow of (1.3).

In the same way, we see that \( \mathcal{A}^- \) is invariant under the flow of (1.3).

\[ \square \]

Lemma 3.2. Let \( w \in H^1(\mathbb{R}) \). For any \( \varepsilon > 0 \), there exists some \( \delta > 0 \) such that if
\[ |S_c(w) - S_c(\phi_c)| + |K_c(w)| < \delta, \tag{3.1} \]
then
\[ \inf_{(\theta, y) \in \mathbb{R}^2} \| w - e^{i\theta} \phi_c(\cdot - y) \|_{H^1} < \varepsilon. \]

Proof. By contradiction, we assume that there exist \( \varepsilon_0 > 0 \) and some sequences \( \{w_n\} \subset H^1(\mathbb{R}) \) such that
\[ S_c(w_n) \to d(c), \quad K_c(w_n) \to 0, \quad as \ n \to \infty, \]
but
\[ \inf_{(\theta, y) \in \mathbb{R}^2} \| w_n - e^{i\theta} \phi_c(\cdot - y) \|_{H^1} > \varepsilon_0. \tag{3.2} \]

On the other hand, we follow the the proof of Lemma 2.3, using the profile decomposition, to find that
\[ e^{-\frac{\sigma}{\sigma + 1} i x} w_n - V^1(\cdot - x_n^1) \to 0, \quad in \quad \dot{H}^1(\mathbb{R}) \]
i.e.
\[ w_n(\cdot + x_n^1) - e^{\frac{\sigma}{\sigma + 1} i x} V^1 \to 0, \quad in \quad \dot{H}^1(\mathbb{R}) \]
and \( e^{\frac{\sigma}{\sigma + 1} i x} V^1(x) = e^{i\theta} \phi_c(x - x_0) \) solves the equation (1.10). Hence, we obtain that for large \( n \) it holds that
\[ \inf_{(\theta, y) \in \mathbb{R}^2} \| w_n - e^{i\theta} \phi_c(\cdot - y) \|_{H^1} \leq \| w_n - e^{i\theta_0} \phi_c(\cdot - x_0 - x_n) \|_{H^1} < \varepsilon_0, \]
which is a contradiction with (3.2).

\[ \square \]

Lemma 3.3. For any \( \varepsilon > 0 \), there exists some \( \delta > 0 \) such that if \( \| u_0 - \phi_c \|_{H^1} < \delta \), then the solution \( u(t) \) of (1.3) with \( u(0) = u_0 \) satisfies
\[ |S_c(u(t)) - S_c(\phi_c)| + |K_c(u(t))| < \varepsilon, \quad for \quad t \in I \]
where \( I \) is the maximal lifespan.
Proof. For sufficiently small $\delta > 0$, which will be determined later, it follows that $\| u_0 - \phi_c \|_{H^1} < \delta$ implies

$$S_{c+\lambda}(u_0) = S_{c+\lambda}(\phi_c) + O(\delta) \quad (3.3)$$

where $\lambda$ will be chosen later such that $|\lambda|$ is small. Applying Taylor expansion to the function $S_{c+\lambda}(\phi_{c+\lambda})$ of $\lambda$, we obtain that

$$S_{c+\lambda}(\phi_{c+\lambda}) = E(\phi_{c+\lambda}) + (c + \lambda) P(\phi_{c+\lambda}) + \frac{(c + \lambda)^2}{4} M(\phi_{c+\lambda})$$

$$= S_c(\phi_{c+\lambda}) + \lambda(P(\phi_{c+\lambda}) + \frac{c}{2} M(\phi_{c+\lambda})) + \frac{\lambda^2}{4} M(\phi_{c+\lambda})$$

$$= S_c(\phi_{c+\lambda}) - S_c(\phi_c) + \lambda \left( P(\phi_{c+\lambda}) - P(\phi_c) + \frac{c}{2} M(\phi_{c+\lambda}) - \frac{c}{2} M(\phi_c) \right)$$

$$+ \frac{\lambda^2}{4} (M(\phi_{c+\lambda}) - M(\phi_c)) + S_c(\phi_c) + \lambda \left( P(\phi_c) + \frac{c}{2} M(\phi_c) \right) + \frac{\lambda^2}{4} M(\phi_c)$$

$$= \frac{\lambda^2}{2} \langle S''_c(\phi_c) \partial_c \phi_c, \partial_c \phi_c \rangle + \lambda^2 \left( \langle P'(\phi_c), \partial_c \phi_c \rangle + \frac{c}{2} \langle M'(\phi_c), \partial_c \phi_c \rangle \right)$$

$$+ S_{c+\lambda}(\phi_c) + o(\lambda^2)$$

$$= -\frac{\lambda^2}{2} \langle S''_c(\phi_c) \partial_c \phi_c, \partial_c \phi_c \rangle + S_{c+\lambda}(\phi_c) + o(\lambda^2),$$

where we use the formular

$$\langle S''_c(\phi_c) \partial_c \phi_c, \partial_c \phi_c \rangle = -\langle P'(\phi_c), \partial_c \phi_c \rangle - \frac{c}{2} \langle M'(\phi_c), \partial_c \phi_c \rangle = -\partial_c P(\phi_c) - \frac{c}{2} \partial_c M(\phi_c)$$

which is negative by Lemma 2.5. Thus, combined with (3.3), we obtain that

$$S_{c+\lambda}(u_0) = S_{c+\lambda}(\phi_{c+\lambda}) + \frac{\lambda^2}{2} \langle S''_c(\phi_c) \partial_c \phi_c, \partial_c \phi_c \rangle + o(\lambda^2) + O(\delta).$$

For any $\lambda$ satisfying $|\lambda| \in (0, \lambda_0)$ with some $\lambda_0 > 0$ small enough, we may choose $\delta > 0$ small such that

$$S_{c+\lambda}(u_0) < S_{c+\lambda}(\phi_{c+\lambda}). \quad (3.4)$$

Now we deal with $L_c$. Note that

$$L_{c+\lambda}(u_0) = L_{c+\lambda}(\phi_c) + O(\delta).$$
By Taylor expansion, we estimate

\[ L_{c+\lambda}(u_0) = L_{c+\lambda}(\phi_c) + O(\delta) \]

\[ = L_{c+\lambda}(\phi_c) - L_c(\phi_c) + L_c(\phi_c) + O(\delta) \]

\[ = \lambda \left( P(\phi_c) + \frac{c}{2} M(\phi_c) \right) + \frac{\sigma + 1}{\sigma} d(c) + o(\lambda) + O(\delta) \]

\[ = \frac{\sigma + 1}{\sigma} d(c + \lambda) - \frac{\sigma + 1}{\sigma} (d(c + \lambda) - d(c)) + \lambda \left( P(\phi_c) + \frac{c}{2} M(\phi_c) \right) + o(\lambda) + O(\delta) \]

which, by choosing \( \lambda_0 \) and \( \delta > 0 \) smaller, is small than \( \frac{\sigma + 1}{\sigma} d(c + \lambda) \), i.e.

\[ L_{c+\lambda}(u_0) < \frac{\sigma + 1}{\sigma} d(c + \lambda), \quad \forall \lambda \in (0, \lambda_0). \]

Similarly, it must hold that for small \( \lambda_0 \) and \( \delta > 0 \),

\[ L_{c-\lambda}(u_0) > \frac{\sigma + 1}{\sigma} d(c - \lambda), \quad \forall \lambda \in (0, \lambda_0). \]

In view of the invariant sets \( A^\pm \), these estimates, combined with (3.4), imply that for any \( t \in I = (-T^*, T^*) \),

\[ L_{c+\lambda}(u(t)) < \frac{\sigma + 1}{\sigma} d(c + \lambda), \quad L_{c-\lambda}(u(t)) > \frac{\sigma + 1}{\sigma} d(c - \lambda). \]  \( (3.5) \)

Therefore, we can choose sufficiently small \( \lambda_0 > 0 \) and \( \delta > 0 \) such that if \( \|u_0 - \phi_c\|_{H^1} < \delta \),

\[ |S_c(u(t)) - S_c(\phi_c)| + |L_c(u(t)) - L_c(\phi_c)| < \frac{\epsilon}{A}, \quad \text{for} \ t \in I \]

with some constant \( A > 0 \) large enough. Since \( K_c = (\sigma + 1)S_c - \sigma L_c \), we finally obtain that

\[ |S_c(u(t)) - S_c(\phi_c)| + |K_c(u(t))| < \epsilon, \quad \text{for} \ t \in I. \]

Now we prove Theorem 1.1.

**The Proof Theorem 1.1.** By contradiction, we assume that there exists some \( \epsilon_0 > 0 \) such that for any small \( \delta > 0 \) there exists some sequence \( \{t_n\} \) satisfying \( \|u_0 - \phi_c\|_{H^1} < \delta \), but

\[ \inf_{(\theta, y) \in \mathbb{R}^2} \|u(t_n) - e^{i\theta} \phi_c(\cdot - y)\|^2_{H^1} \geq \epsilon_0. \]  \( (3.6) \)

Taking \( \delta \) small enough, we get from Lemma 3.3 that

\[ |S_c(u(t_n)) - S_c(\phi_c)| + |K_c(u(t_n))| < \tilde{\delta}, \]

where \( \tilde{\delta} > 0 \) can be chosen sufficiently small such that, using Lemma 3.2, it holds that

\[ \inf_{(\theta, y) \in \mathbb{R}^2} \|u(t_n) - e^{i\theta} \phi_c(\cdot - y)\|^2_{H^1} < \frac{\epsilon_0}{2}. \]
Moreover, by mass conservation, we have
\[ \|u(t_n)\|_{L^2}^2 - \|\phi_c\|_{L^2}^2 = O(\delta). \]

Choosing \(\delta\) smaller, we obtain then
\[ \inf_{(\theta, y) \in \mathbb{R}^2} \|u(t_n) - e^{i\theta} \phi_c(\cdot - y)\|_{L^2}^2 < \frac{\varepsilon_0}{2}. \]

Finally, there holds that
\[ \inf_{(\theta, y) \in \mathbb{R}^2} \|u(t_n) - e^{i\theta} \phi_c(\cdot - y)\|_{H^1}^2 < \varepsilon_0, \]
which contradicts (3.6). \( \square \)

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