Effective Computational Methods for Solving the Jeffery-Hamel Flow Problem

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Abstract

In this paper, the effective computational method (ECM) based on the standard monomial polynomial has been implemented to solve the nonlinear Jeffery-Hamel flow problem. Moreover, novel effective computational methods have been developed and suggested in this study by suitable base functions, namely Chebyshev, Bernstein, Legendre, and Hermite polynomials. The utilization of the base functions converts the nonlinear problem to a nonlinear algebraic system of equations, which is then resolved using the Mathematica®12 program. The development of effective computational methods (D-ECM) has been applied to solve the nonlinear Jeffery-Hamel flow problem, then a comparison between the methods has been shown. Furthermore, the maximum error remainder (MERₙ) has been calculated to exhibit the reliability of the suggested methods. The results persuasively prove that ECM and D-ECM are accurate, effective, and reliable in getting approximate solutions to the problem.

Keywords: Approximate solution, Bernstein polynomials, Chebyshev polynomials, Hermite polynomials, Legendre polynomials.

Introduction:

In several fields of engineering and applied sciences, nonlinear ordinary differential equations (NODE) play a significant role in simulating many real-life issues. Many phenomena, including engineering, fluid mechanics, physics, chemical matters, biology, and electrostatics, have been mathematically formulated using these types of equations. The exact solution for nonlinear problems is difficult or sometimes cannot be obtainable. Therefore authors want to develop efficient either numerical or approximate methods to solve these types of problems.¹⁻⁴

Several analytical and approximate methods have been proposed by researchers to solve nonlinear differential equations, such as the Adomian decomposition method (ADM) and Direct Homotopy Analysis Method (DHAM) ⁵, the Bernoulli collocation method ⁶, the Hemite polynomial method ⁷, the Taylor collocation method ⁸, and the Gegenbauer wavelet method ⁹. In particular, Singh ¹⁰ has used the Jacobi collocation method to solve the fractional advection-dispersion equation. Ganji et al. ¹¹ have used the fifth-kind Chebyshev polynomials to solve differential equations with multiple variable orders and non-local and non-singular kernels. Also, Singh et al. ¹² used Boubaker polynomials to solve a class of fractional optimal control problems. Yuttanan et al. ¹³ solved the non-linear distributed fractional differential equations using the Legendre wavelets method and some other approximation methods, see ¹⁴⁻¹⁶.

One of the most important applications in fluid mechanics and biomechanical engineering is the flow between two nonparallel plates ¹⁷. Jeffery ¹⁸ and Hamel ¹⁹ introduced incompressible viscous fluid movement in convergent and divergent channels, and this is known as Jeffery-Hamel flow.

Many researchers have attempted to develop analytical approximations methods to solve the Jeffery-Hamel flow: such as optimal iterative perturbation technique ²⁰, Bernstein collocation method (BCM) ²¹, modified Adomian decomposition method (MADM) ²²⁻²³, Homotopy analysis method (HAM) ²⁴, Homotopy perturbation method (HPM) ²⁵, Bernoulli collocation method ²⁶, Hermite wavelet method ²⁷, differential transform method (DTM) ²⁸. More recently, AL-Jawary et al. ²⁹, has implemented three semi-analytical iterative
This paper is organized as follows: The mathematical description of the Jeffery-Hamel flow problem is presented in section two. Section three explains the basic concepts of the proposed methods. Solving the Jeffery-Hamel flow problem by the proposed methods will be given in section four. In section five, the numerical results will be displayed and explained. Finally, in section six, a conclusion will be presented.

The Mathematical Formulation of Jeffrey Hamel’s Flow Problem

The Jeffery-Hamel flow problem represented by the NODE is the steady flow of a viscous, conductive, incompressible fluid in two dimensions at the intersection of two plane rigid and non-parallel walls that get together at an angle $2\alpha$. It is assumed that the flow is perfectly radial and symmetric. Therefore, the velocity field is only along the radial direction and depends on $r$ and $\theta$, so it can be given by $V(u(r, \theta), 0)$, as illustrated in (Fig. 1).

![Jeffrey-Hamel flow's geometry](image)

The continuity equations and the Navier-Stokes equations can be expressed in polar coordinates as follows:

1. $\rho \frac{\partial}{\partial r}(ru(r, \theta)) = 0,$
2. $u(r, \theta) \frac{\partial u(r, \theta)}{\partial r} - \frac{1}{\rho} \frac{\partial p}{\partial r} + v \left[ \frac{\partial^2 u(r, \theta)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u(r, \theta)}{\partial \theta^2} \right] - \frac{\sigma B_0^2}{\rho r^2} u(r, \theta)$
3. $- \frac{1}{\rho r^2} \frac{\partial p}{\partial \theta} + 2v \frac{\partial u(r, \theta)}{\partial \theta} = 0,$

where $u(r, \theta)$ is the radial velocity, $B_0$ is denoted by the electromagnetic induction and $\sigma$ is a fluid’s conductivity, $P$ is the pressure of the fluid, $\rho$ is the fluid density constant, and $v$ is the kinematic viscosity parameter.

Eq.1 can be written as:

4. $\frac{g(\theta)}{g_{\text{max}}} = r u(r, \theta),$ 

By using dimensionless parameters $\frac{\theta}{\alpha},$ so

5. $w(x) = \frac{g(\theta)}{g_{\text{max}}},$ where, $x = \frac{\theta}{\alpha}.$
By eliminating $P$ term from Eq.2 and Eq.3, and using the formulas given in Eq.4 and Eq.5, a nonlinear third-order ODE is obtained:

$$w''(x) + 2\alpha \text{Re} \ w(x) w'(x) + (4-\text{Ha}) \alpha^2 \ w'(x) = 0, \quad 6$$

with the boundary conditions as follows:

$$w(0) = 1, \quad w'(0) = 0, \quad w(1) = 0, \quad 7$$

where, $\text{Re} = \frac{u_{\text{max}}}{v}$, and $\text{Ha}^2 = \frac{\sigma \beta L}{\rho v}$, are the Reynolds number and the Hartmann number’s square, respectively.

The Basic Concepts of the Proposed Methods

A description of the suggested methods will be presented in this section. Also, orthogonal polynomials and the operational matrices will be offered, which are used in the development of the ECM algorithm to get the approximate solution to the problem.

The Basic Concepts of ECM

Consider $m^{\text{th}}$-order non-linear ODE as follows

$$f(x,y,y',y'',...,y^{(m)}) = h(x), \quad \alpha \leq x \leq \beta, \quad 8$$

with either the I.C:

$$y^{(i)}(\alpha) = \omega_i, \quad 0 \leq i \leq m-1, \quad 9$$

or the following B.C:

$$y^{(i)}(\beta) = \delta_i, \quad 0 \leq i \leq \frac{m}{2} - 1, \quad 10$$

where $h(x)$ is a function that is known and $\omega_i, \mu_i, \delta_i$, are constants. The essential assumption is that Eq.8 has a unique solution with the initial or boundary conditions given in Eq.9 or Eq.10. Moreover, a function $y(x) \in L^2[0,1]$ can be expressed by a linear combination of $m^{\text{th}}$-order function series based on the classical standard monomial polynomials as:

$$y(x) = \sum_{i=0}^{m} c_i \varphi_i(x), \quad 11$$

where $c_i$ are the coefficients whose values will be found by giving the following definitions

$$X = [\varphi_0, \varphi_1, \varphi_2, ..., \varphi_m], \quad C = [c_0, c_1, c_2, ..., c_m]^T$$

where $\varphi_m$ represents the base functions from the classical polynomials $31$. By using the dot product, the $m^{\text{th}}$ order approximation of the series solution provided in Eq.11 is as follows:

$$y(x) = \sum_{i=0}^{m} c_i \varphi_i(x) = X \ C, \quad 12$$

Assume that the derivative of vector $X$ will be defined as below

$$D[X] = X \ B,$$

where $B_{(m+1) \times (m+1)}$ is the operational auxiliary matrix with the given entries in classical monomials:

$$B = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix}^{(m+1) \times (m+1)}$$

Also, the higher derivatives can be written as,

$$D^m[X] = X \ B^m$$

Therefore, Eq.13 can be used to write the derivatives in the following format:

$$y^{(m)}(x) = X \ B^m \ C \quad m \geq 1, \quad 14$$

Now, substituting the Eqs.12, and 14 in Eqs.8-10, the matrix equation with the restrictions $31$, can be obtained:

$$f(x, X \ C, X B \ C, X B^2 \ C, ..., X B^m \ C) = h(x), \quad m = 1, 2, ... \quad 15$$

and

$$X(0) B^i C = \omega_i, \quad 0 \leq i \leq m-1, \quad 16$$

Consider the Hilbert space $H = L^2[0,1]$, which has the inner product as follows:

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x) dx, \quad 17$$

Assume a set of functions that are linearly independent in $H$

$$\psi = \{\psi_0, \psi_1, ..., \psi_m\}, \quad 18$$

where $\psi_m$ be the base function of a standard monomial polynomials $x^i, \forall \ i = 0, 1, 2, ..., m$ or any other type of polynomial $31,32$. Then, by applying the inner product given in Eq.17 with the elements of $\psi$ defined in Eq.18, the following matrix equation $33$ will be shown:

$$G = E, \quad 19$$

The $i^{\text{th}}$ row of $G$ and $E$, respectively, is made up of:

$$\langle \psi_i, f(x, X C, X B C, X B^2 C, ..., X B^m C) \rangle, \quad \langle \psi_i, h(x) \rangle, \quad 0 \leq i \leq m. \quad 20$$

In addition, by applying the initial or boundary conditions in Eqs.15, and 16, some entries of Eq.19 are modified from the left-hand side $G$ and the corresponding right-hand side $E$ $35$. Thus, a system of $(m+1)$ nonlinear algebraic equations for unknown $C$ will be obtained. By solving the resulting system numerically or sometimes analytically, unique values can be obtained for unknown elements $c_0, c_1, c_2, ..., c_m$, this will be substituted in Eq.12 to obtain an approximate solution to Eq.8.
First Kind Chebyshev Polynomials
The first kind of Chebyshev polynomials \( T_i(x) \) of degree \( i \) is defined by:
\[
T_i(x) = \sum_{j=0}^{i} (-1)^{i-j} 2^{j} \binom{i+j-1}{i-j} (x^2 - 1)^j / (i-j)! (2j)! x^j.
\]
The unknown function \( y(x) \) can be represented as:
\[
y(x) = \sum_{i=0}^{\infty} c_i T_i(x),
\]
where,
\[
c_i = \langle y, T_i \rangle = (2i+1) \int_{-1}^{1} y(x) P_i(x) dx; \quad i \geq 0.
\]
In general, only the first \((m+1)\) terms of the Chebyshev polynomials have been expressed \(^{36}\), so
\[
y(x) = \sum_{i=0}^{m} c_i T_i(x) = C^T \Phi(x), \tag{22}
\]
where \( c_i = \langle y, T_i \rangle = (2i+1) \int_{-1}^{1} y(x) P_i(x) dx; \quad i \geq 0 \).

Bernstein Polynomials
The degree \( n \) Bernstein polynomials in \([0, 1]\) are defined by \(^{44}\):
\[
B_{j,n}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad 0 \leq j \leq n \tag{27}
\]
There is \((n+1)\) degree of the Bernstein Polynomials. Also, these polynomials have two most significant properties \(^{30}\):
1) Property of unity partition, \( \sum_{j=0}^{n} B_{j,n}(x) = 1, \quad 0 \leq x \leq 1 \)
2) Positivity property, \( B_{j,n}(x) \geq 0, \text{ for } 0 \leq j \leq n \text{ and } B_{j,n}(x) = 0 \text{ if } j < 0 \text{ or } n < j \).

In general, the \( y(x) \) can be approximated by the linear combination of Bernstein polynomial shown in the following formula below:
\[
y(x) = \sum_{j=0}^{n} c_j B_{j,n}(x) = C^T \Phi(x), \tag{28}
\]
where \( C^T = [c_0 c_1 c_2 \ldots c_m] \) and \( \Phi(x) = [T_0(x), T_1(x), \ldots, T_m(x)]^T \).
Moreover, the derivatives of \( \Phi(x) \) can be considered as:
\[
D[\Phi(x)] = D_T \Phi(x), \quad D^2[\Phi(x)] = D_T^2 \Phi(x), \ldots, D^m[\Phi(x)] = D_T^m \Phi(x), \tag{23}
\]
where \( D_T (m+1) \times (m+1) \), is the operational matrix of the provided derivative, which is defined as follows:
\[
 D_T = (di,j) = \begin{cases} \frac{2i}{\rho_j}, & \text{for } j = i - k, \\ 0, & \text{otherwise}, \end{cases}
\]
where, \( k = 1, 3, 5, \ldots, m - 1 \) if \( m \) is even, or \( k = 1, 3, 5, \ldots, m \) if \( m \) is odd, \( \rho_0 = 2 \), and \( \rho_k = 1 \) for all \( k \geq 1 \).
For example, if \( m \) is even then the \( D_T \) is expressed as follows:
\[
 D_T = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 \\ 5 & 0 & 10 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ m - 1 & 0 & 2(m - 1) & 0 & \cdots & 2(m - 1) & 0 & 0 \\ 0 & 2m & 0 & 2m & 0 & \cdots & 0 & 2m \end{pmatrix}
\]
In addition, if \( m \) is odd then the matrix \( D_T \) is defined as follows:
\[
 D_T = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2(m - 1) & 0 & 2(m - 1) & \cdots & 2(m - 1) & 0 & 0 \\ m & 0 & 2m & 0 & \cdots & 0 & 2m & 0 \end{pmatrix}
\]
Hence, the derivatives can be written by using Eq.23 in the following form:
\[
\frac{dy}{dx} = C^T D_T \Phi(x), \quad \frac{d^2 y}{dx^2} = C^T D_T^2 \Phi(x), \ldots, \frac{d^m y}{dx^m} = C^T D_T^m \Phi(x). \tag{26}
\]
The Hermite polynomials, \( H_m(x) \), on \((-\infty, \infty)\) of \(m^{th}\) order are defined as \(^{36}\):

\[
H_m(x) = \sum_{j=0}^{K} \frac{(-1)^j}{j! (m - 2j)} (2x)^{m-2j}
\]

where \(K = \frac{m-1}{2}\) if \(m\) is odd and \(K = \frac{m}{2}\) if \(m\) is even. Also, the Hermite polynomials \(H_m(x)\) can be written as follows:

\[
H_m(x) = \sum_{j=0}^{K} \frac{(-1)^j}{j!} m(m-1) \ldots (m-2j+1) (2x)^{m-2j}
\]
The function \( y(x) \) is defined by a truncated Hermite polynomials \( H_m(x) \), as:

\[
y(x) = \sum_{j=0}^{K} c_j H_j(x) = \mathcal{O}(x) \ C,
\]

where, \( \mathcal{O}(x) = [H_0(x), H_1(x), ..., H_K(x)] \) and, \( C = [c_0, c_1, c_2, ..., c_K]^T \). On the other hand, Hermite polynomials \( H_m(x) \) and the powers \( x^m \) are related to the following relation \(^{45}\),

\[
x^{2m} = \frac{(2m)!}{2^{2m}} \sum_{m=0}^{S} \frac{H_{2m}(x)}{(s-m)! (2m)!}, \quad 0 \leq x \leq 1
\]

and

\[
D_M = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & 0 & \cdots & 0 \\
0 & \frac{3}{4} & 0 & \frac{1}{8} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{K!}{2^{K-1} K!} & 0 & \frac{K!}{2^{K-2} K!} & \cdots & 0
\end{pmatrix}
\]

and for even \( K \), the matrix \( D_M \) is defined as follows \(^{45}\):

\[
D_M = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & 0 & \cdots & 0 \\
0 & \frac{3}{4} & 0 & \frac{1}{8} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{K!}{2^{K-1} K!} & 0 & \frac{K!}{2^{K-2} K!} & \cdots & 0
\end{pmatrix}
\]

From above, the expression of \( \mathcal{O}(x) \) will be written as follows:

\[
\mathcal{O}(x) = X(x)(D_M)^{-1} \mathcal{T}
\]

and,

\[
(\mathcal{O}(x))^{(n)} = X^{(n)}(x)(D_M)^{-1} \mathcal{T} \quad n = 1, 2, ..., \]

Furthermore, the below relation can be applied to obtain the \( X^{(n)}(x) \) by using terms of the \( X(x) \) \(^{36}\):

\[
X^{(1)}(x) = X(x) \ G, \quad X^{(2)}(x) = X(x) \ G^2,
\]

\[
X^{(n)}(x) = X(x) \ G^n
\]

where \( G = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & K \\
0 & 0 & 0 & \cdots & 0 & (K+1)(K+1)
\end{pmatrix}
\]

Similarly, the derivatives \( y^{(n)}(x) \) can be expressed as:

\[
\frac{d^n y}{d x^n} = (\mathcal{O}(x))^{(n)} \ C = X(x) \ G^n ((D_M)^{-1})^T \mathcal{T}, \quad \text{where } n = 1, 2, ..., \]

Solving the Jeffery-Hamel Flow Problem by the ECM and D-ECM

The proposed methods from section three will be implemented in this section to provide approximate solutions to the Jeffery-Hamel flow problem.

The D-ECM depends on the base functions of different polynomials such as Chebyshev, Bernstein, Legendre, and Hermite polynomials that are given in the Eqs.21, 27, 31, 35, respectively, and applying the operational matrices corresponding to these polynomials represented on Eqs.24, 25, 29, 33, 39, 40, respectively. To increase the accuracy and efficiency of ECM, these polynomials are used
in two steps of the suggested approach procedure. Firstly, to describe the unknown function \(y(x)\) and its derivatives; secondly, to process of calculating the inner product to solve the left and right sides of the matrix equation, which are given in Eq.19.

By substituting the initial or boundary conditions in Eqs.15, and 16, some entries of Eq.19 are modified. Thereafter, \((m + 1)\) nonlinear algebraic equations for unknown \(C\) can be obtained by solving this system numerically by Mathematica\textsuperscript{\textregistered}12, where unique values are given for unknown elements \(c_0, c_1, c_2, \ldots c_m\), to achieve the approximate solution to the problem.

The ECM and D-ECM procedures can be used to solve Eq.6 with boundary conditions Eq.7, by using Eqs.12, 14, replacing unknown function \(w(x)\) with its derivatives as matrices, for ECM:

\[
X B^2 C + 2a Re (X C)(X B C) \\
+ (4 - Ha) \alpha^2 (X B C) = 0,
\]

\[
(X C)(0) = 1, (X B C)(0) = 0, (X C)(1) = 0
\]

Then, the process has been used as presented in Eqs.19, 20, so:

\[
\langle x^i, X B^2 C + 2a Re (X C)(X B C) + (4 - Ha) \alpha^2 (X B C) \rangle = \langle x^i, 0 \rangle,
\]

\[\forall i = 0,1,2,\ldots, m.\]

Applying Eqs.22, 26 for D-ECM based on the first kind of Chebyshev polynomials, it follows:

\[
C^T D_T^3 \phi(x) \\
+ 2a Re (C^T \phi(x))(C^T D_T \phi(x)) \\
+ (4 - Ha) \alpha^2 (C^T D_T \phi(x)) = 0,
\]

\[C^T \phi(0) = 1, C^T D_T \phi(0) = 0, C^T \phi(1) = 0\]

Using the procedures as given in the Eqs.19, and 20, hence:

\[
\langle T_i(x), C^T D_T^3 \phi(x) \rangle \\
+ 2a Re (C^T \phi(x))(C^T D_T \phi(x)) \\
+ (4 - Ha) \alpha^2 (C^T D_T \phi(x)) \rangle = \langle T_i(x), 0 \rangle,
\]

\[\forall 0 \leq i \leq m\]

By setting the Eqs.28, and 30 for D-ECM based on the Bernstein polynomials, the following is obtained:

\[
C^T D_B^3 \phi(x) + 2a Re (C^T \phi(x))(C^T D_B \phi(x)) \\
+ (4 - Ha) \alpha^2 (C^T D_B \phi(x)) = 0,
\]

\[C^T \phi(0) = 1, C^T D_B \phi(0) = 0, C^T \phi(1) = 0\]

By implementing the processes as presented in Eqs.19, 20, Eq.47 will be shown

\[
\langle B_{j,n}(x), C^T D_B^3 \phi(x) \rangle \\
+ 2a Re (C^T \phi(x))(C^T D_B \phi(x)) \\
+ (4 - Ha) \alpha^2 (C^T D_B \phi(x)) \rangle = \langle B_{j,n}(x), 0 \rangle,
\]

\[\forall j = 0,1,2,\ldots, n.\]

Substituting the Eqs.32, and 34 for D-ECM based on the Legendre polynomials, it follows that:

\[
C^T D_P^3 \phi(x) + 2a Re (C^T \phi(x))(C^T D_P \phi(x)) \\
+ (4 - Ha) \alpha^2 (C^T D_P \phi(x)) = 0,
\]

\[C^T \phi(0) = 1, C^T D_P \phi(0) = 0, C^T \phi(1) = 0\]

Moreover, using the techniques given in the Eqs.19, and 20, the following equation will be obtained:

\[
\langle P_i(x), C^T D_P^3 \phi(x) \rangle \\
+ 2a Re (C^T \phi(x))(C^T D_P \phi(x)) \\
+ (4 - Ha) \alpha^2 (C^T D_P \phi(x)) \rangle = \langle P_i(x), 0 \rangle,
\]

\[\forall 0 \leq i \leq m\]

Furthermore, applying Eqs.36, 41 for D-ECM based on the Hermite polynomials, it follows:

\[
X(x)G^3 ((D_M)^{-1})^T C \\
+ 2a Re (\phi(x) \phi(x))(X(x) G ((D_M)^{-1})^T C) \\
+ (4 - Ha) \alpha^2 (X(x) G ((D_M)^{-1})^T C) = 0
\]

\[\langle D_M \phi(0) = 1, X(0) G ((D_M)^{-1})^T C = 0, \phi(1) C = 0\]

Then, using the procedures as given in Eqs.19, 20, so:

\[
\langle H_i(x), X(x)G^3 ((D_M)^{-1})^T C \rangle \\
+ 2a Re (\phi(x) \phi(x))(X(x) G ((D_M)^{-1})^T C) \\
+ (4 - Ha) \alpha^2 (X(x) G ((D_M)^{-1})^T C) \rangle = \langle H_i(x), 0 \rangle, \forall i = 0,1,\ldots, K.
\]

Then, the values of \(C = [c_0, c_1, c_2, \ldots c_m]^T\) are calculated by solving the algebraic system obtained by the inner product for the left and right sides, from Eqs.43, 45, 47, 49, and 51, respectively. Subsequently, applying the boundary conditions on the Eqs.42, 44, 46, 48, and 50 leads to obtaining the approximate solution.

The approximate polynomials for the Jeffery-Hamel flow problem when the parameter values are as follows: \(a = 5^\circ, Re = 10, Ha = 0\) as in \(30\), with \(n=12\), will be:

- By using ECM based on the standard monomial polynomial,

\[
w(x) \approx 1.12597 x^2 + 8.4681 * 10^{-7} x^3
\]

\[
+ 0.166615 x^4
\]

\[
+ 0.0000643873 x^5
\]

\[
- 0.0470176 x^6
\]

\[
+ 0.000792892 x^7
\]

\[
+ 0.00575024 x^8
\]

\[
+ 0.00218839 x^9
\]

\[
- 0.0035073 x^{10}
\]

\[
+ 0.00126349 x^{11}
\]

\[
- 0.000175929 x^{12}
\]

- By using D-ECM based on the first kind of the Chebyshev polynomials,
\[ w(x) \approx 1.12597 x^2 + 8.79819 \times 10^{-8} x^3 + 0.166622 x^4 + 0.000024296 x^5 - 0.046882 x^6 + 0.000494395 x^7 + 0.00618549 x^8 + 0.00177057 x^9 - 0.00325326 x^{10} + 0.00117476 x^{11} - 0.000162363 x^{12}. \]

- By using D-ECM based on the Bernstein polynomials,
  \[ w(x) \approx 1.12597 x^2 + 5.81122 \times 10^{-7} x^3 + 0.166618 x^4 + 0.000498819 x^5 - 0.046966 x^6 + 0.000676843 x^7 + 0.00592368 x^8 + 0.00201225 x^9 - 0.00339754 x^{10} + 0.00122292 x^{11} - 0.000169476 x^{12}. \]

- By using D-ECM based on the Legendre polynomials,
  \[ w(x) \approx 1.12597 x^2 + 8.93943 \times 10^{-8} x^3 + 0.166622 x^4 + 0.0000245082 x^5 - 0.0468829 x^6 + 0.000496724 x^7 + 0.00618166 x^8 + 0.0017746 x^9 - 0.0032559 x^{10} + 0.00117574 x^{11} - 0.000162521 x^{12}. \]

- By using D-ECM based on the Hermite polynomials,
  \[ w(x) \approx 1.12597 x^2 + 6.8438 \times 10^{-8} x^3 + 0.166623 x^4 + 0.0000209462 x^5 - 0.046871 x^6 + 0.000454542 x^7 + 0.00625298 x^8 + 0.00169763 x^9 - 0.00320442 x^{10} + 0.00115627 x^{11} - 0.000159336 x^{12}. \]

The Numerical Results and Discussion:
In this section, an example is presented when the value of \( n = 3, \alpha = 5^\circ, Re = 10, \) and \( Ha = 0, \) to illustrate the approach of the proposed methods to solve the Jeffery-Hamel flow problem.

To explain the technique of ECM, by using the Eqs.12, and 14, it follows:
  \[ w(x) = X C \]
  \[ = \varphi_0 c_0 + \varphi_1 c_1 + \varphi_2 c_2 \]
  \[ + \varphi_3 c_3, \]
where, \( \varphi_0 = 1, \varphi_1 = x, \varphi_2 = x^2, \varphi_3 = x^3, \) and the derivatives of \( w(x) \) as matrices, expressed as:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix} [c_0 \ c_1 \ c_2 \ c_3]^T,
\]

and,

\[
\begin{bmatrix}
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} [c_0 \ c_1 \ c_2 \ c_3]^T.
\]

Now, substituting the \( w'(x), w''(x) \) in Eqs.6, 7, and applying the inner product to solve the left and right sides of the matrix equation given in Eq.43, with boundary conditions Eq.42, four nonlinear algebraic equations for unknown \( c_0, c_1, c_2, c_3, \) can be obtained as:

\[
\begin{align*}
\frac{100}{3} c_1^2 + \frac{100}{3} c_0 c_1 + 25 \deg c_1^2 + 50\degree^2 c_2^2 \\
+ 50\degree^2 c_0 c_2 + 60\degree c_1 c_2 + \frac{5}{27} \pi c_2^2 \\
+ 2 c_3 + 60\degree^2 c_3 + 60\degree c_0 c_3 \\
+ \frac{10}{27} \pi c_1 c_3 + \frac{25}{63} \pi c_2 c_3 \\
+ \frac{1}{24} \pi c_3^2 = 0, \\
c_0 = 1, \\
c_1 = 0,
\end{align*}
\]
By solving this system numerically by Mathematica®12, unique values of \( c_0, c_1, c_2, c_3 \) are given as follows:
\[
c_0 = 0, \quad c_1 = 0, \quad c_2 = -1.14626, \quad c_3 = 0.146262.
\]
Hence, the values of \( c_0, c_1, c_2, c_3 \), will be substituted in Eq.52 to obtain an approximate solution to the Eq.6, as:
\[
w(x) \approx 1 - 1.14626 x^2 + 0.146262 x^3.
\]

Using the Eqs.22, and 26, the following is a description of D-ECM based on the first kind of the Chebyshev polynomials technique:
\[
w(x) = c^T \varphi(x)
\]
\[
= c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x) + c_3 T_3(x),
\]
\[53\]
where, \( T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = -1 + 2x^2, \quad T_3(x) = -3x + 4x^3 \), and the derivatives \( w'(x), w''(x) \) as matrices, can be given as:
\[
w'(x) = C^T D_T \varphi(x) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
3 & 0 & 6 & 0 \\
\end{bmatrix} [T_0(x), T_1(x), T_2(x), T_3(x)]^T,
\]
Also,
\[
w''(x) = C^T D_T^3 \varphi(x) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 \\
\end{bmatrix} [T_0(x), T_1(x), T_2(x), T_3(x)]^T.
\]

Therefore, substituting the \( w'(x), w''(x) \) in Eqs.6, 7, and employing the inner product of the matrix equation given in Eq.45, with boundary conditions Eq.44, four nonlinear algebraic equations for unknown \( c_0, c_1, c_2, c_3 \), are achieved as:
\[
- \frac{100}{3} c_0^2 + \frac{100}{3} c_0 c_1 + \frac{220}{3} c_1 c_2 - 200 c_2^2 
+ \frac{200}{27} \pi c_2^2 - 8 c_3 + 180^2 c_3 
+ 180^2 c_0 c_3 - 400^2 c_1 c_3 
+ \frac{400}{27} \pi c_1 c_3 - 1100^2 c_2 c_3 
+ \frac{63}{20} \pi c_2 c_3 + 1200^2 c_3^2 
- \frac{20}{3} \pi c_3^2 = 0,
\]
\[
c_0 = 1, \quad c_1 = 0, \quad c_2 = -3 c_3 = 0,
\]
\[
c_0 + c_1 + c_2 + c_3 = 0.
\]

Solving this system numerically by Mathematica®12, the following unique values of \( c_0, c_1, c_2, c_3 \) will be obtained:
\[
c_0 = 0.419839, \quad c_1 = 0.120242, \quad c_2 = -0.580161, \quad c_3 = 0.400086.
\]

Hence, the values of \( c_0, c_1, c_2, c_3 \), will be substituted in Eq.53 to obtain an approximate solution to the Eq.6, as:
\[
w(x) \approx 1 - 1.16032 x^2 + 0.160322 x^3.
\]

By implementing the Eqs.28, 30, for D-ECM based on the Bernstein polynomials, the following is obtained:
\[
w(x) = c^T \varphi(x)
\]
\[
= c_0 B_{0,3} + c_1 B_{1,3} + c_2 B_{2,3} + c_3 B_{3,3},
\]
\[54\]
where, \( B_{0,3} = 1 - 3x + 3x^2 - x^3, \quad B_{1,3} = 3x - 6x^2 + 3x^3, \quad B_{2,3} = 3x^2 - 3x^3, \quad B_{3,3} = x^3 \), as matrices, the derivatives \( w'(x), w''(x) \) may be written as:
\[
w'(x) = C^T D_B \varphi(x) = \begin{bmatrix}
-3 & -1 & 0 & 0 \\
3 & -1 & -2 & 0 \\
0 & 2 & 1 & -3 \\
0 & 0 & 1 & 3 \\
\end{bmatrix} [B_{0,3}, B_{1,3}, B_{2,3}, B_{3,3}]^T,
\]
and,
\[
w''(x) = C^T D_B^3 \varphi(x) = \begin{bmatrix}
-6 & -6 & -6 & -6 \\
18 & 18 & 18 & 18 \\
-18 & -18 & -18 & -18 \\
6 & 6 & 6 & 6 \\
\end{bmatrix} [B_{0,3}, B_{1,3}, B_{2,3}, B_{3,3}]^T.
\]

Thus, if the \( w'(x), w''(x) \) substituting into Eqs.6, 7, and using the inner product of the matrix equation from Eq.47 with the boundary conditions from Eq.46, four nonlinear algebraic equations for unknown \( c_0, c_1, c_2, c_3 \), are attained as follows:
The following unique values of \( c_0, c_1, c_2, c_3 \) will be found by numerically solving this system with Mathematica:\(^{12}\):

\[
c_0 = 1, \quad c_1 = 1, \quad c_2 = 0.60497, \quad c_3 = 0.
\]

To achieve an approximate solution to Eq.6, the values of \( c_0, c_1, c_2, c_3 \) will be substituted in Eq.54, as follows:

\[
w(x) \approx 1 - 1.18509 x^2 + 0.185091 x^3
\]

By applying the Eqs.32, and 34, for D-ECM based on the Legendre polynomials, the following is achieved:

\[
w(x) = C^T \emptyset(x)
\]

\[
= c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)
\]

\[
+ c_3 P_3(x),
\]

where, \( P_0(x) = 1, \) \( P_1(x) = x, \) \( P_2(x) = -\frac{1}{2} + \frac{3x^2}{2}, \)

\[
P_3(x) = -\frac{3x}{2} + \frac{5x^3}{2},
\]

and the derivatives of \( w(x) \), can be written as matrices:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
1 & 0 & 5 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
15 & 0 & 0 & 0
\end{bmatrix}
\]

In addition, four nonlinear algebraic equations with unknowns \( c_0, c_1, c_2, c_3 \), are obtained by substituting \( w'(x) \) and \( w''(x) \) in Eqs. 6, 7, and applying the inner product of the matrix equation from Eq.49 with the boundary conditions from Eq.48:

\[
\begin{align*}
-\frac{3c_0}{2} - 15^2 c_0 - \frac{25}{7} c_0^2 + \frac{9c_1}{2} - 15^2 c_1 \\
- 75^2 c_1 - 45^2 c_2 + 2 \\
- \frac{25}{14} c_0 c_2 - \frac{25}{14} c_1 c_2 + \frac{75}{14} c_2 c_3 \\
+ 75^2 c_3 - \frac{25}{14} c_0 c_3 + \frac{75}{14} c_2 c_3 + \frac{25}{2} c_3^2 = 0, \\
c_0 = 1, \\
-3c_0 + 3c_1 = 0, \\
c_3 = 0.
\end{align*}
\]

Then, using Mathematica\(^ {12}\), to solve this system numerically, the following unique values of \( c_0, c_1, c_2, c_3 \), will be obtained:

\[
w'(x) = \begin{bmatrix} H_0(x) H_1(x) H_2(x) H_3(x) \end{bmatrix}
\begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} c_0 \ c_1 \ c_2 \ c_3 \end{bmatrix},
\]

Also,

\[
\begin{align*}
c_0 &= 0.648645, \quad c_1 = 0.0324384, \quad c_2 \\
&= -0.702709, \quad c_3 = 0.0216256.
\end{align*}
\]

As a result, the values \( c_0, c_1, c_2, c_3 \), will be substituted in Eq.55 to give an approximate solution to Eq.6, as follows:

\[
w(x) \approx 1 - 1.05406 x^2 + 0.0540639 x^3
\]

Moreover, by using the Eqs. 36, 41, a description of the D-ECM based on the Hermite polynomials procedure follows:

\[
w(x) = \emptyset(x) C
\]

\[
= H_0(x) c_0 + H_1(x) c_1 + H_2(x) c_2 + H_3(x) c_3,
\]

where, \( H_0(x) = 1, \) \( H_1(x) = 2x, \) \( H_2(x) = -2 + 4x^2, \) \( H_3(x) = -12x + 8x^3, \)

\[
\text{and the derivatives } w'(x), w''(x) \text{ as matrices can be obtained as:}
\]

\[
\begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} c_0 \ c_1 \ c_2 \ c_3 \end{bmatrix},
\]

\[
\begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} c_0 \ c_1 \ c_2 \ c_3 \end{bmatrix},
\]
Substituting $w'(x), w''(x)$ into Eqs. 6, 7, and using the inner product of the matrix equation from Eq. 51 with the boundary conditions from Eq. 50, yields four nonlinear algebraic equations with unknowns $c_0, c_1, c_2, c_3$:

$$
\begin{align*}
-\frac{400}{3}x^2 c_1 &- \frac{400}{3}c_0 c_1 + \frac{1760}{3}c_1 c_2 \\
+ \frac{1600}{3}c_2^2 &- 32c_3 + 1120x^2 c_3 \\
+ 1120c_3 &+ \frac{3200}{3}c_1 c_3 \\
- \frac{9920}{7}c_2 c_3 &- 3200c_3^2 = 0,
\end{align*}
$$

$c_0 - 2c_2 = 1,
2c_1 - 12c_3 = 0,$
$c_0 + 2c_1 + 2c_2 - 4c_3 = 0.$

Then, using Mathematica $12$, solve this system numerically to acquire the following unique values of $c_0, c_1, c_2, c_3$:

$c_0 = 0.419839, c_1 = 0.120242, c_2 = -0.290081, c_3 = 0.0200403.$

As a consequence, the values $c_0, c_1, c_2, c_3$, will be swapped in Eq.56 to get the following approximate solution to Eq.6:

$$w(x) \approx 1.16032x^2 + 0.160322x^3.$$

Furthermore, the maximal error remainder $MER_n$ has been introduced in this section because there is no exact solution available to the problem, as well as to verify the accuracy and reliability of the approximate solution obtained by ECM and D-ECM. The $MER_n$ is calculated by:

$$MER_n = \max_{0 \leq x \leq 1} |w^{(n)}(x) + 2\alpha Re w(x) w'(x) + (4-Ha) \alpha^2 w^2(x)|$$

Fig. 2 presents the logarithmic plots for the $MER_n$ values, obtained by the ECM based on the standard monomial polynomial, as well as, by the D-ECM based on the Chebyshev, Bernstein, Legendre, and Hermite polynomials, for parameters $Re = 10, Ha = 0$ and $\alpha = 5^\circ$, as is evident from the figure, good agreements have been obtained for all proposed methods.

Moreover, in Table 1 the values of $MER_n$ for the approximate solution is given by using ECM and D-ECM with $n = 12$ and parameters $Re = 10, Ha = 0$ and versus the value of $\alpha$, which appears the efficiency of these methods. In addition, it can be noted that D-ECM based on the Hermite polynomials method produces better accuracy with the lowest errors compared to the other methods.

| $\alpha$ | ECM $n=12$ | D-ECM $n=12$ | D-ECM $n=12$ | D-ECM $n=12$ | D-ECM $n=12$ |
|---------|------------|--------------|--------------|--------------|--------------|
|         | Standard   | Chebyshev    | Bernstein    | Legendre     | Hermite      |
| $3^\circ$ | $1.78573 \times 10^{-6}$ | $1.55044 \times 10^{-7}$ | $1.07042 \times 10^{-6}$ | $1.57835 \times 10^{-7}$ | $1.16736 \times 10^{-7}$ |
| $-3^\circ$ | $3.09536 \times 10^{-6}$ | $2.15838 \times 10^{-7}$ | $1.55861 \times 10^{-6}$ | $2.20333 \times 10^{-7}$ | $1.60927 \times 10^{-7}$ |
| $-5^\circ$ | $0.0000152937$ | $1.01151 \times 10^{-6}$ | $7.35397 \times 10^{-6}$ | $1.03335 \times 10^{-6}$ | $8.15967 \times 10^{-7}$ |
Furthermore, in Table 2 the comparisons of \( MER_{12} \) values are presented when \( Re = 10, Ha = 0, \alpha = 5^\circ \), for the solutions by proposed methods and by the Chebyshev and the Bernstein operational matrices methods according to previous studies \(^{30}\). Better accuracy can be realized by using the suggested methods.

Also, Figs.4-7 illustrate the velocity profiles for the Jeffery–Hamel problem in the cases \( \alpha = 5^\circ, \alpha = -5^\circ \) with fixed \( Re = 50 \) and increasing values of \( Ha \), as chosen in \(^{17}\). The velocity is noted to be increased by increasing \( Ha \) values in all the figures. The curvature of the curves also increases with increasing \( Ha \) values.

### Table 2. The comparison between the \( MER_{12} \) when \( Re = 10, Ha = 0, \alpha = 5^\circ \) by proposed methods and by Chebyshev and Bernstein \(^{30}\).

| ECM Standard | D-ECM Chebyshev | D-ECM Bernstein | D-ECM Legendre | D-ECM Hermite | Chebyshev \(^{30}\) | Bernstein \(^{30}\) |
|--------------|-----------------|-----------------|----------------|----------------|-----------------|-----------------|
| 5.08086      | 5.27892         | 3.48673         | 5.36366        | 4.10628        | 3.3003          | 9.68873         |

The approximate solution is accurate and efficient even within a few orders of polynomials. In addition, the \( MER_n \) has been calculated for the proposed methods and compared with the Chebyshev and the Bernstein operational matrices methods that are available in the literature, the results obtained showed that the proposed methods have produced better accuracy with less errors. Moreover, it can be concluded that the results of the \( MER_n \) by the proposed methods D-ECM decreased significantly compared to ECM, which gives higher accuracy and efficiency. Furthermore, it was found that the results of D-ECM based on the Hermite polynomials are better than the other methods.

The present methods can also be extended to partial differential equations and fractional differential equations, which certainly require extensive further analysis.

### Conclusion:

The effective computational method and novel computational methods with suitable base functions, namely Chebyshev, Bernstein, Legendre, and Hermite polynomials, have been presented in this paper for solving the Jeffery-Hamel problem. The nonlinear problems are reduced to the solution of a nonlinear algebraic system of equations, which is processed using Mathematica\(^\text{®}12\).
Authors’ declaration:
- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for republication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors’ Contributions:
OMS contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript. MA.AJ interpreted, drafted, revised, proofread, and verified the analytic approximate methods of the manuscript. The authors discussed the results and contributed to the final manuscript.

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طريق حسابي فعالة لحل مشكلة تدفق جيفري-هامل

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الخلاصة:
في هذا البحث، تم تنفيذ الطريقة الحسابية الفعالة (ECM) المتسطدة إلى متعددة الحدود القياسية الأحادية لحل مشكلة تدفق جيفري-هامل غير الخطية. علاوة على ذلك، تم تطوير واتخاذ الطريقة الحسابية الفعالة الجديدة في هذه الدراسة من خلال واتتخاذ أساسية مناسبة وهي متعدد الحدود تشبيهيف، هيرمت، ليجندر، هيرمت، يوتي استخدام الدوال الأساسية إلى تحويل المسألة غير الخطية إلى نظام جيري غير خطئي من المعادلات، والذي يتم حل بعد ذلك باستخدام برنامج ماهماتيكا. تم تطبيق طرق حسابية فعالة (D-ECM) تدفق جيفري-هامل غير الخطية، ثم عرض مقارنة بين الطريقة. علاوة على ذلك، تم حساب الحد الأقصي للخطأ المتبقي (MER)، لإظهار دقة وفعالية وموثوقية الحصول على حل تقريب للمشكلة.

الكلمات المفتاحية: الحل التقريبي، متعدد حدود بيرنشتاين، متعدد حدود تشبيهيف، متعدد حدود هيرمت، متعددات حدود ليجندر.