Calculating the output distribution of stack filters that are erosion-dilation cascades, in particular \textit{LULU}-filters

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Abstract

Two procedures to compute the output distribution \(\phi_S\) of certain stack filters \(S\) (so called erosion-dilation cascades) are given. One rests on the disjunctive normal form of \(S\) and also yields the rank selection probabilities. The other is based on inclusion-exclusion and e.g. yields \(\phi_S\) for some important \textit{LULU}\textsuperscript{-}operators \(S\). Properties of \(\phi_S\) can be used to characterize smoothing properties.

1 Introduction

The \textit{LULU} operators are well known in the nonlinear multiresolution analysis of sequences. The notation for the basic operators \(L_n\) and \(U_n\), where \(n \in \mathbb{N}\) is a parameter related to the window size, has given rise to the name \textit{LULU} for the theory of these operators and their compositions. Since the time they were introduced nearly thirty year ago, while also being used in practical problems, they slowly led to the development of a new framework for characterizing, evaluating, comparing and designing nonlinear smoothers. This framework is based on concepts like idempotency, co-idempotency, trend preservation, total variation preservation, consistent decomposition.

As opposed to the deterministic nature of the above properties, the focus of this paper is on properties of the \textit{LULU} operators in the setting of random sequences. More precisely, this setting can be described as follows: Suppose that \(X\) is a bi-infinite sequence of random variables \(X_i\) \((i \in \mathbb{Z})\) which are independent and with a common (cumulative) distribution function \(F_X(t)\) from \(L^1([0,1],[0,1])\). Let \(S\) be a smoother. Then we consider the following two questions:

1. Find a map \(\phi_S : [0,1] \rightarrow [0,1]\) such that the common output distribution \(F_{SX}(t)\) of \((SX)_i\) \((i \in \mathbb{Z})\) equals \(F_{SX} = \phi_S \circ F_X\). The function \(\phi_S\) is also called \textit{distribution transfer}.

2. Characterize the smoothing effect an operator \(S\) has on a random sequence \(X\) in terms of the properties of the common distribution of \((SX)_i\) \((i \in \mathbb{Z})\).
With regard to the first question we present a new technique which one may call "expansion calculus" which uses a shorthand notation for the probability of composite events and a set of rules for manipulation. Using this technique we provide new elegant proofs of the earlier results in [1] for the distribution transfer of the operators $L_n U_n$ and (dually) $U_n L_n$. The power of this approach is further demonstrated by deriving the distribution transfer maps for the alternating sequential filters $C_n = L_n U_n L_{n-1} U_{n-1} \ldots L_1 U_1$ and (dually) $F_n = U_n L_n U_{n-1} L_{n-1} \ldots U_1 L_1$.

With regard to the second question, we may note that it is reasonable to expect that a smoother should reduce the standard deviation of a random sequence. Indeed, for simple distributions (e.g. uniform) and filters with small window size (three point average, $M_1, L_1 U_1, U_1 L_1$) when the computations can be carried out a significant reduction of the standard deviation is observed (for the uniform distribution the mentioned filters reduce the standard deviation respectively by factors of 3, 5/3, 1.293, 1.293). However, in general, obtaining such results is to a large extent practically impossible due to the technical complexity particularly when nonlinear filters are concerned. In this paper we propose a new concept which characterizes the probability of the occurrence of outliers rather then considering the standard deviation. We call it robustness. Upper robustness characterizes the probability of positive outliers while the lower robustness characterizes the probability of negative outliers. In general, the higher the order of robustness of a smoother the lower the probability of occurrence of outliers in the output sequence. In terms of this concept it is easy to characterize a smoother given its distribution transfer function.

The paper is structured as follows. In the next section we give the definitions of the LULU operators with some fundamental properties. The concept of robustness is defined and studied in Section 3. In Section 4 we demonstrate the application of inclusion-exclusion principle for deriving the distribution transfer function of erosion-dilation cascades, a kind of operator frequently used in Mathematical Morphology. This method is considerably refined in Section 5 where it is applied to LULU-operators which are in fact particular cases of such cascades. Formulas for the major LULU-operators are obtained explicitly or recursively. Using these results the robustness of these operators is also analyzed. Section 6 proposes to substitute inclusion-exclusion by some principle of exclusion which is better suited for erosion-dilation cascades that don't allow the refinements of inclusion-exclusion possible for LULU-operators.

### 2 The basics of the LULU theory

Given a bi-infinite sequence $x = (x_i)_{i \in \mathbb{Z}}$ and $n \in \mathbb{N}$ the basic LULU operators $L_n$ and $U_n$ are defined as follows

\[
(L_n x)_i := (x_{i-n} \land x_{i-n+1} \land \cdots \land x_i) \lor (x_{i-n+1} \land \cdots \land x_{i+1}) \lor \cdots \lor (x_i \land \cdots \land x_{i+n}) \quad (1)
\]

\[
(U_n x)_i := (x_{i-n} \lor x_{i-n+1} \lor \cdots \lor x_i) \land (x_{i-n+1} \lor \cdots \lor x_{i+1}) \land \cdots \land (x_i \lor \cdots \lor x_{i+n}) \quad (2)
\]

where $\alpha \land \beta := \min(\alpha, \beta)$, and $\alpha \lor \beta := \max(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{R}$. Central to the theory is the concept of separator, which we define below. For every $a \in \mathbb{Z}$ the operator $E_a : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}$ given by

\[(E_a x)_i = x_{i+a}, \quad i \in \mathbb{Z},\]

is called a shift operator.

**Definition 1** An operator $S : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}$ is called a separator if
\( S \circ E_a = E_a \circ S, \ a \in \mathbb{Z} \); (horizontal shift invariance)

\( P(f + c) = P(f) + c, \ f, c \in \mathbb{R}^Z; \) \( c \)-constant function (vertical shift invariance);

\( P(\alpha f) = \alpha P(f), \ \alpha \in \mathbb{R}, \ \alpha \geq 0, \ f \in \mathbb{R}^Z; \) (scale invariance)

\( P \circ P = P; \) (Idempotence)

\( (id - P) \circ (id - P) = id - P. \) (Co-idempotence)

The first two axioms in Definition 1 and partially the third one were first introduced as required properties of nonlinear smoothers by Mallows, [4]. Rohwer further made the concept of a smoother more precise by using the properties (i)–(iii) as a definition of this concept. The axiom (iv) is an essential requirement for what is called a \textit{morphological filter}, [11], [12], [13]. In fact, a \textit{morphological filter} is exactly an increasing operator which satisfies (iv). The co-idempotence axiom (v) in Definition 1 was introduced by Rohwer in [8], where it is also shown that it is an essential requirement for operators extracting signal from a sequence. More precisely, axioms (iv) and (v) provide for consistent separation of noise from signal in the following sense: Having extracted a signal \( Sx \) from a sequence \( x \), the additive residual \( (I - S)x \), the noise, should contain no signal left, that is \( S \circ (I - S) = 0 \). Similarly, the signal \( Sx \) should contain no noise, that is \( (I - S) \circ S = 0 \). It was shown in [8] that \( L_n, U_n \) and their compositions \( L_nU_n, U_nL_n \) are separators.

The smoothing effect of \( L_n \) on an input sequence is the removal of picks, while the smoothing effect of \( U_n \) is the removal of pits. The composite effect of the two \textit{LU-operators} \( L_nU_n \) and \( U_nL_n \) is that the output sequence contains neither picks nor pits which will fit in the window of the operators. These are the so called \( n \)-monotone sequences, [8]. Let us recall that a sequence \( x \) is \( n \)-\textit{monotone} if any subsequence of \( n+1 \) consecutive elements is monotone. For various technical reasons the analysis is typically restricted to the set \( \mathcal{M}_1 \) of absolutely summable sequences. Let \( \mathcal{M}_n \) denote the set of all sequences \( x \in \mathcal{M}_1 \) which are \( n \)-monotone. Then

\[ \mathcal{M}_n = \text{Range}(L_nU_n) = \text{Range}(U_nL_n) \]

is the set of signals.

The power of the \textit{LU}-operators as separators is further demonstrated by their trend preservation properties. Let us recall, see [8], that an operator is called \textit{neighbor trend preserving} if \( (Sx)_i \leq (Sx)_{i+1} \) whenever \( x_i \leq x_{i+1}, \ i \in \mathbb{N} \). An operator \( S \) is \textit{fully trend preserving} if both \( S \) and \( I - S \) are neighbor trend preserving. The operators \( L_n, U_n \) and all their compositions are fully trend preserving. With the total variation of a sequence,

\[ TV(x) = \sum_{i \in \mathbb{N}} |x_i - x_{i+1}|, \ x \in \mathcal{M}_1 \]

a generally accepted measure for the amount of contrast present, since it is a semi-norm on \( \mathcal{M}_1 \), any separation may only increase the total variation. More precisely, for any operator \( S : \mathcal{M}_1 \to \mathcal{M}_1 \) we have

\[ TV(x) \leq TV(Sx) + TV((I - S)x). \] (3)

All operators \( S \) that are fully trend preserving have variation preservation, in that

\[ TV(x) = TV(Sx) + TV((I - S)x). \] (4)
We mention these properties because they provide but few of the motivation for studying the robustness of operators, when the popular medians are optimal in that respect. We intend to show that some \( LULU \)-composition are nearly as good as the medians, but have superiority most important aspects.

An operator \( S \) satisfying property (4) is called total variation preserving, [6]. As mentioned already, the \( LU \)-operators are total variation preserving.

### 3 Distribution transfer and degree of robustness of a smoother

Suppose that \( X \) is a bi-infinite sequence of random variables \( X_i \ (i \in \mathbb{Z}) \) which are independent and with a common (cumulative) distribution function \( F_X \). Let \( S \) be a smoother. As stated in the introduction we seek a function \( \phi_S : [0, 1] \to [0, 1] \), called a distribution transfer function such that

\[
F_{SX} = \phi_S \circ F_X \tag{5}
\]

is the common distribution of \( (SX)_i \ (i \in \mathbb{Z}) \). We should note that for an arbitrary smoother the existence of such a distribution transfer function is not obvious. However, for the smoothers typically considered in nonlinear signal processing (i.e. stack filters of which the LULU operators are particular cases) such a function does not only exist but it is a polynomial. For example, it is shown in [5] that the distribution transfer function of the ranked order operators

\[
(R_{nk}x)_i = \text{the } k\text{th smallest value of } \{x_{i-n}, \ldots, x_{i+n}\}
\]

is given by

\[
\phi_{R_{nk}}(p) = \sum_{j=k}^{2n+1} \binom{2n+1}{j} p^j (1 - p)^{2n+1-j}. \tag{6}
\]

The popular median smoothers \( M_n, n \in \mathbb{N} \), are particular cases of the ranked order operators, namely \( M_n = R_{n,n+1} \). Hence we have

\[
\phi_{M_n}(p) = \sum_{j=n+1}^{2n+1} \binom{2n+1}{j} p^j (1 - p)^{2n+1-j}. \tag{7}
\]

Note that in terms of (5) the common distribution function of \( (M_nX)_i, i \in \mathbb{Z}, \) is

\[
F_{M_nX}(t) = \sum_{j=n+1}^{2n+1} \binom{2n+1}{j} F_X^j(t)(1 - F_X(t))^{2n+1-j}, \quad t \in \mathbb{R}.
\]

Using that

\[
\frac{d}{dz} \phi_{M_n}(p) = (2n+1) \binom{2n}{n} p^n (1 - p)^n \tag{8}
\]

its density is

\[
f_{M_nX}(t) = \frac{d}{dz} \phi_{M_n}(F_X(t)) f_X(t) = (2n+1) \binom{2n}{n} F_X^n(t)(1 - F_X(t))^n f_X(t)
\]
where \( f_X(t) = \frac{d}{dt}F_X(t), \ t \in \mathbb{R} \), is the common density of \( X_i, \ i \in \mathbb{Z} \). The distribution of the output sequence of the basic smoothers \( L_n \) and \( U_n \) is derived in [8]. Equivalently these results can be formulated in terms of distribution transfer. More precisely we have

\[
\phi_{L_n}(p) = 1 - (n + 1)(1 - p)^{n+1} + n(1 - p)^{n+2} \tag{9}
\]

\[
\phi_{U_n}(p) = (n + 1)p^{n+1} - np^{n+2} \tag{10}
\]

A primary aim of the processing of signals through nonlinear smoothers is the removal of impulsive noise. Therefore, the power of such a smoother can be characterized by how well it eliminates outliers in a random sequence. The concepts of robustness of a smoother introduced below are aimed at such characterization.

**Definition 2** A smoother \( S : \mathbb{R}^\mathbb{Z} \rightarrow \mathbb{R}^\mathbb{Z} \) is called lower robust of order \( r \) if there exists a constant \( \alpha > 0 \) such that for every bi-infinite sequence \( X \) of identically distributed random variables \( X_i \ (i \in \mathbb{Z}) \) there exists \( t_0 \in \mathbb{R} \) such that \( P(X_i < t) < \varepsilon \) implies \( P((SX)_i < t) < \alpha \varepsilon^k \) for all \( t < t_0 \) and \( \varepsilon > 0 \).

Similarly, a smoother \( S : \mathbb{R}^\mathbb{Z} \rightarrow \mathbb{R}^\mathbb{Z} \) is called upper robust of order \( r \) if there exists a constant \( \alpha > 0 \) such that for every bi-infinity sequence \( X \) of identically distributed random variables \( X_i \ (i \in \mathbb{Z}) \) there exists \( t_0 \in \mathbb{R} \) such that \( P(X_i > t) < \varepsilon \) implies \( P((SX)_i > t) < \alpha \varepsilon^k \) for all \( t > t_0 \) and \( \varepsilon > 0 \).

A smoother which is both lower robust of order \( r \) and upper robust of order \( r \) is called robust of order \( k \).

The reasoning behind these concepts is simple: If a distribution density is heavy tailed, there is a probability \( \varepsilon \) that the size of a random variable is excessively large (larger than \( t \)) in absolute value. Using a non-linear smoother we would aim to restrict this to an acceptable probability \( \alpha \varepsilon^k \) that such an excessive value can appear in \( SX \), by choosing a smoother with the order of robustness \( k \).

Clearly there is a general problem of smoothing: a trade-off to be made between making a smoother more robust, and the (inevitable) damage to the underlying signal preservation. (A smoother clearly cannot create information, but only selectively discard it.) This is fundamental. There are two main reasons for using one-sided robustness: Firstly, the unreasonable pulses often are only in one direction, as in the case of ”glint” in signals reflected from objects with pieces of perfect reflectors, and there clearly are no reflections of negative intensity possible. Secondly, we may choose smoothers that are not symmetric, as are the \( LU \)-operators, for reasons that are of primary importance. In this case the robustness is determined from the sign of the impulse.

The robustness of a smoother can be characterized through its distribution transfer function as stated in the theorem below.

**Theorem 3** Let the smoother \( S \) have a distribution transfer function \( \phi_S \). Then

a) \( S \) is lower robust of order \( r \) if and only if \( \phi_S(p) = O(p^r) \) as \( p \rightarrow 0 \).
b) $S$ is upper robust of order $r$ if and only if $\phi_S(1 - p) - 1 = O(p^r)$ as $p \to 0$.

**Proof.** Points a) and b) are proved using similar arguments. Hence we prove only a). Let $\phi_S(p) = O(p^r)$ as $p \to 0$. This means that there exists $\alpha > 0$ and $\delta > 0$ such that $\phi_S(p) < \alpha p^k$ for all $p \in [0, \delta)$. Let $X$ be a sequence of identically distributed random variables with common distribution function $F_X$. Since $\lim_{t \to -\infty} F_X(t) = 0$, there exists $t_0$ such that $F_X(t_0) < \delta$. Let $t < t_0$ and $\varepsilon > 0$ be such that $P(X_i < t) < \varepsilon$. The monotonicity of $F_X$ implies that $F_X(t) \in [0, \delta)$.

Then
\[ P((SX)_i < t) = F_{SX}(t) = \phi_S(F_X(t)) < \alpha(F_X(t))^k < \alpha \varepsilon^k, \]
which proves that $S$ is lower robust of order $k$. It is easy to see that the argument can be reversed so that the stated condition is also necessary.

In the common case when the distribution transfer function is a polynomial, conditions a) and b) can be formulated in a much simpler way as given in the next corollary.

**Corollary 4** Let the distribution transfer function of a smoother $S$ be a polynomial $\phi_S$. Then

a) $S$ is lower robust of order $r$ if and only if $p = 0$ is a root of order $r$ of $\phi_S$.

b) $S$ is upper robust of order $r$ if and only if $p = 1$ is a root of order $r$ of $\phi_S - 1$.

Using the distribution transfer functions given in (9) and (10) it follows from Corollary 4 that $U_n$ is lower robust of order $n + 1$ and that $L_n$ is upper robust of order $n + 1$.

The robustness of the median filter $M_n$ can be obtained from (7). Obviously $p = 0$ is a root of order $n + 1$. Furthermore, $\phi_{M_n}(1) = 1$. Then using also that $p = 1$ is a root of order $n$ of $\frac{d}{dx} \phi_{M_n}$, see [5], we obtain that $p = 1$ is a root of order $n + 1$ of $\phi_{M_n} - 1$. Therefore, $M_n$ is robust of order $n + 1$.

Clearly with symmetric smoothers, in that $S(-x) = -S(x)$, the concepts of lower and upper robustness are not needed, as is the case for example with $M_n$. However, we have to recall in this regard that the operators $L_n$, $U_n$ and their compositions, which are the primary subject of our investigation, are not symmetric. A useful feature of the lower and upper robustness is that it can be induced through the point-wise defined partial order between the operators. Let us recall that given the maps $A, B : \mathbb{R}^Z \to \mathbb{R}^Z$, the relation $A \leq B$ means that $Ax \leq Bx$ for all $x \in \mathbb{R}^Z$.

**Theorem 5** Let $A, B : \mathbb{R}^Z \to \mathbb{R}^Z$ be smoothers. If $A \leq B$ then $\phi_B \leq \phi_A$.

**Proof.** Let $X$ be a sequence of independent random variables $X_i (i \in \mathbb{Z})$ uniformly distributed on $[0, 1]$. Let $p \in [0, 1]$. If $t$ is such that $p = F_X(t)$ then

\[ \phi_B(p) = \phi_B(F_X(t)) = F_{BX}(t) = P((BX)_i \leq t) \leq P((AX)_i \leq t) = F_{AX}(t) = \phi_A(F_X(t)) = \phi_A(p). \]
As a direct consequence of Theorem 5 and Theorem 3 we obtain the following theorem.

**Theorem 6** Let \( A, B : \mathbb{R}^Z \rightarrow \mathbb{R}^Z \) be smoothers such that \( A \leq B \). Then

a) If \( A \) is lower robust to the order \( k \), then so is \( B \).

b) If \( B \) is upper robust to the order \( k \), then so is \( A \).

Using Theorem 6 one can derive statements about the lower robustness and the upper robustness of the \( LU \)-operators:

\[
U_nL_n \leq M_n \leq L_nU_n \tag{11}
\]

Therefore, \( U_nL_n \) inherits the upper-robustness of \( M_n \), while \( L_nU_n \) inherits the lower-robustness of \( M_n \). More precisely

- \( U_nL_n \) is upper robust of order \( n + 1 \);
- \( L_nU_n \) is lower robust of order \( n + 1 \)

One may expect that, since \( L_n \) is upper robust of order \( n + 1 \) and \( U_n \) is lower robust of order \( n + 1 \), their compositions should be both lower and upper robust of order \( n + 1 \). However, as we will see later, this is not the case. The problem is the following. The definition of robustness requires that the random variables in the sequence \( X \) are identically distributed but they are not necessarily independent. However, the distribution transfer functions \( \phi_{L_n} \) and \( \phi_{U_n} \) are derived under the assumption of such independence. Noting that entries in the sequences \( L_nX \) are not independent, it becomes clear that the common distribution of \( U_nL_nX \) cannot be obtained by applying \( \phi_{U_n} \) to \( F_{L_nX} \). More generally, since the distribution transfer functions are derived for sequences of independent identically distributed random variables the equality \( \phi_{AB} = \phi_A \circ \phi_B \) does not hold for arbitrary operators \( A \) and \( B \). Therefore the order of robustness of \( B \) is not necessarily preserved by the composition \( AB \).

Observe that another concept of robustness is introduced in [10]. Other than Definition 2 it only applies to stack filters. The concept is similar in that it also based on certain probabilities (in this case “selection probabilities”).

4 The output distribution of arbitrary erosion-dilation cascades

Here we present a method for obtaining output distributions of so called erosion-dilation cascades (defined below). It essentially uses the inclusion-exclusion principle for the probability of simultaneous events. For convenience we recall this principle below. For \( n = 2 \) the easy proof is given later on along the way.

**Lemma 7** For any random variables \( Z_1, Z_2 \cdots Z_n \) it holds that
$$P(Z_1, \ldots, Z_n \leq t) = 1 - \sum_{i=1}^{n} P(Z_i > t) + \sum_{1 \leq i < j \leq n} P(Z_i, Z_j > t) - \sum_{1 \leq i < j < k \leq n} P(Z_i, Z_j, Z_k > t) + \cdots + (-1)^n P(Z_1, \ldots, Z_n > t)$$

Let us recall that in the general setting of Mathematical Morphology [12] the basic operators $L_n$ and $U_n$ are morphological opening and closing respectively. As such they are compositions of an erosion and a dilation. More precisely, for a sequence $x = (x_i)_{i \in \mathbb{Z}}$ we have

$$(L_n x)_i := (\lor^n (\land^n x))_i$$
$$(U_n x)_i := (\land^n (\lor^n x))_i$$

where $$(\land^n x)_i := x_{i-n} \land x_{i-n+1} \land \cdots \land x_i$$
is an erosion with structural element $W = \{-n, -n-1, \ldots, 1, 0\}$ and

$$(\lor^n x)_i := x_i \lor x_{i+1} \lor \cdots \lor x_{i+n}$$
is a dilation with structural element $W' = \{0, 1, \ldots, n\}$. Generalizing the $LU$-operators $L_n U_n$ and $U_n L_n$, call a $LULU$-operator any composition of the basic smoothers $L_n$ and $U_n$, such as $L_3 U_4 L_2 U_1 U_5$. In particular, each $LULU$-operator is a composition of dilations and erosions, that is, an cascade of dilations and erosions (CDE). More generally, each alternating sequential filter (ASF), by which definition is a composition of morphological openings and closings with structural elements of increasing size, is a CDE with the extra property of featuring the same number of erosions and dilations. See also [3].

We will demonstrate our method on two examples of cascades - the first in one dimension, the second in two dimensions. This method is considerably refined in the next section.

**Example 1.** Consider $S := \lor^1 \land^2 \lor^3$. It is a a cascade of an erosion $\land^2$ and dilations $\lor^1, \lor^3$. It is not an ASF. To compute the distribution transfer of $S$, let $X$ be a bi-infinite sequence of independent identically distributed random variables $X_i$. Put

$$Y_i = \left(\lor^1 X\right)_i, \quad Z_i := \left(\land^2 Y\right)_i, \quad A_i := \left(\lor^3 Z\right)_i. \quad (13)$$

Thus $Y, Z, A$ are again bi-infinite sequences of identically distributed (though dependent) random variables. Let $t \in \mathbb{R}$ and $p = F_X(t)$. Then

$$\phi_S(p) = F_{S X}(t) = F_A(t) = P(A_0 \leq t) = P(Z_0 \lor Z_1 \leq t) = P(Z_0 \leq t \text{ and } Z_1 \leq t)$$

In order to reduce the $Z_i$'s to the $Y_i$'s we switch all $\leq t$ to $> t$ by using exclusion-inclusion (the case $n = 2$ in Lemma 7):

$$P(Z_0, Z_1 \leq t) = P(Z_0 \leq t) - P(Z_0 \leq t, Z_1 > t)$$
\begin{align*}
= P(Z_0 \leq t) - (P(Z_1 > t) - P(Z_1, Z_0 > t)) \\
= 1 - P(Z_0 > t) - P(Z_1 > t) + P(Z_1, Z_0 > t)
\end{align*}

Since our \(Z_i\)'s are identically distributed we have \(P(Z_0 > t) = P(Z_1 > t)\) and hence

\[
\phi_S(p) = 1 - 2P(Z_0 > t) + P(Z_1, Z_0 > t)
\]
\[
= 1 - 2P(Y_{-2} \land Y_{-1} \land Y_0 > t) + P(Y_{-1} \land Y_0 \land Y_1, Y_{-2} \land Y_{-1} \land Y_0 > t)
\]
\[
= 1 - 2P(Y_{-2}, Y_{-1}, Y_0 > t) + P(Y_{-2}, Y_{-1}, Y_0, Y_1 > t)
\]
\[
= 1 - 2P(Y_0, Y_1, Y_2 > t) + P(Y_0, Y_1, Y_2, Y_3 > t)
\]

By the dual of Lemma 7 and because e.g. \(P(Y_0, Y_1 \leq t) = P(Y_1, Y_2 \leq t) = P(Y_2, Y_3 \leq t)\) we get

\[
\phi_S(p) = 1 - 2 \left(1 - 3P(Y_0 \leq t) + 2P(Y_0, Y_1 \leq t) + P(Y_0, Y_2 \leq t) - P(Y_0, Y_1, Y_2 \leq t)\right)
\]
\[
+ \left(1 - 4P(Y_0 \leq t) + 3P(Y_0, Y_1 \leq t) + 2P(Y_0, Y_2 \leq t) + P(Y_0, Y_3 \leq t) - 2P(Y_0, Y_1, Y_2 \leq t)\right)
\]
\[
- 2P(Y_0, Y_1, Y_3 \leq t) + P(Y_0, Y_1, Y_2, Y_3 \leq t)\right)
\]
\[
= 2P(Y_0 \leq t) - P(Y_0, Y_1 \leq t) + P(Y_0, Y_3 \leq t) - 2P(Y_0, Y_1, Y_3 \leq t) + P(Y_0, Y_1, Y_2, Y_3 \leq t)
\]
\[
= 2P(X_0, X_1, X_2, X_3 \leq t) - P(X_0, X_1, X_2, X_3, X_4 \leq t) + P(X_0, X_1, \cdots, X_6 \leq t)
\]
\[
- 2P(X_0, X_1, \cdots, X_6 \leq t) + P(X_0, X_1, \cdots, X_6 \leq t)
\]
\[
= 2p^4 - p^5 + p^7 - 2p^7 + p^7
\]
\[
= 2p^4 - p^5
\]

**Example 2.** Let \(S\) be an opening on \(\mathbb{R}^{Z \times Z}\) with defining structural element a \(2 \times 2\) square. Let now \(X\) be an infinite 2-dimensional array of independent identically distributed random variables \(X_{(i,j)}\) where \((i,j)\) ranges over \(Z \times Z\). In order to derive the output distribution of \(S\) we put
\[ Y_{(i,j)} := X_{(i,j)} \land X_{(i-1,j)} \land X_{(i,j+1)} \land X_{(i-1,j+1)} \]

\[ Z_{(i,j)} := Y_{(i,j)} \lor Y_{(i+1,j)} \lor Y_{(i,j-1)} \lor Y_{(i+1,j-1)} \]

Let \( t \in \mathbb{R} \) and \( p = F_X(t) \). The output distribution of \( S \) is

\[ \phi_S(p) = P(Z_{(0,0)} \leq t) \]

\[ = P(Y_{(0,0)}, Y_{(1,0)}, Y_{(0,-1)}, Y_{(1,-1)} \leq t) \]

Following [1], which introduced that handy notation in the 1-dimensional case, we abbreviate the latter as

\[ ((0,0), (1,0), (0,-1), (1,-1))_Y \]

If say \( ((0,0), (1,0), (0,-1), (1,-1))_Y \) means

\[ P(Y_{(1,0)} \leq t, Y_{(0,0)}, Y_{(0,-1)} > t) \]

then it follows from Lemma 7 and from translation invariance (e.g. \( ((0,0), (1,0)) = ((0,-1), (1,-1)) \)) that

\[ \phi_L(p) = ((0,0), (1,0), (0,-1), (1,-1))_Y \]

\[ = 1 - 4(0,0)_Y + 2((0,0), (1,0))_Y + 2((0,0), (0,-1))_Y + ((1,0), (0,-1))_Y \]

\[ + ((0,0), (1,-1))_Y - ((0,0), (0,-1), (1,-1))_Y - ((0,0), (1,0), (1,-1))_Y \]

\[ - ((0,0), (0,-1), (1,0))_Y - ((1,0), (0,-1), (1,-1))_Y + ((0,0), (1,0), (0,-1), (1,-1))_Y \]

According to the definition of \( Y_{(i,j)} \) we e.g. have

\[ ((0,0), (0,-1))_Y = ( (0,0), (-1,0), (0,1), (-1,1), (0,-1), (-1,-1), (0,0), (-1,0) )_X \]

\[ = ( (0,0), (-1,0), (0,1), (-1,1), (0,-1), (-1,-1))_X \]

Putting \( q = 1 - p = P(X_{(0,0)} > t) \) the latter contributes a term \( q^6 \) to

\[ \phi_L(p) = 1 - 4q^4 + 2q^6 + 2q^6 + q^7 + q^7 - q^8 - q^8 - q^8 - q^8 + q^9 \]

\[ = 1 - 4q^4 + 4q^6 + 2q^7 - 4q^8 + q^9 \]
5 Formulas for the distribution transfer of the major \textit{LULU}-operators

As it was already done in the preceding section it is often convenient to use the notation \( q = 1 - p \). For example the output distribution of \( M_n, L_n \) and \( U_n \) given in (7), (9) and (10) respectively can be written in the following shorter form:

\[
\phi_{M_n}(p) = \sum_{j=n+1}^{2n+1} \binom{2n+1}{j} p^j q^{2n+1-j},
\]

\[
\phi_{L_n}(p) = 1 - (n + 1)q^{n+1} + nq^{n+2},
\]

\[
\phi_{U_n}(p) = (n + 1)p^{n+1} - np^{n+2}.
\]

Theorem 11 below deals with the output distribution of \( L_n U_n \) and \( U_n L_n \). They were first derived in [2], but the statement of the theorem was also independently proved by Butler [1]. In 5.1 we present a proof using Butler’s ”expansion calculus”. In 5.2 this method is applied to more complicated situations.

5.1 The output distribution of the \textit{LU}-operators

First, observe that instead of winding up with full blown inclusion-exclusion when switching \textit{all} inequalities \( > t \) to \( \leq t \) (dual of Lemma 7), one can be economic and only switch \textit{some} inequalities:

\[
(\overline{0}, \overline{1}, \cdots, \overline{n})_X = (\overline{0}, \cdots, \overline{n-1})_X - (\overline{0}, \cdots, \overline{n-1}, n)_X
\]

\[
= (\overline{0}, \cdots, \overline{n-2})_X - (\overline{0}, \cdots, \overline{n-2}, n - 1)_X - (\overline{0}, \cdots, \overline{n-1}, n)_X
\]

\[
\vdots
\]

\[
= (\overline{0})_X - (\overline{0}, 1)_X - (\overline{0}, \overline{1}, 2)_X - \cdots - (\overline{0}, \cdots, \overline{n-1}, n)_X
\]

\[
= 1 - (0)_X - \sum_{i=0}^{n-1} (\overline{0}, \cdots, \overline{i}, i + 1)_X
\]

Lemma 8 [1, Corollary 4]: Let \( X \) be a bi-infinite sequence of random variables. Then

\[
(\overline{0}, \overline{1}, \cdots, \overline{n})_X = 1 - [n + 1](0)_X + \sum_{i=0}^{n-1} [n - i](0, \overline{1}, \cdots, \overline{i}, i + 1)_X
\]

Note that for \( i = 0 \) we get the summand \( n(0,1)_X \).
Proof: From
\[(k+1)X = (k+1,k)X + (k+1,\overline{k})X\]
\[= (k+1,k)X + (k+1,\overline{k},k-1)X + (k+1,\overline{k},k-1)X = \cdots\]
\[= (k+1,k)X + (k+1,\overline{k},k-1)X + \cdots + (k+1,\overline{k},\cdots,\overline{1})X + (k+1,\overline{k},\cdots,\overline{0})X\]

follows, by translation invariance, that
\[(0,\cdots,\overline{k},k+1)X = (k+1,\overline{k}+1)X - \sum_{i=0}^{k-1} (k-i-1,\overline{k}-i,\cdots,\overline{k},k+1)X\]
\[= (0)_X - (0,1)_X - \sum_{i=0}^{k-1} (0,\overline{1},\cdots,\overline{i+1},i+2)_X\]

Using (17) one derives for (say) \(n = 4\) that
\[(0,\overline{1},\overline{2},\overline{3},\overline{4})_X = 1 - (0)_X - \sum_{k=0}^{3} (0,\cdots,\overline{k},k+1)_X\]
\[= 1 - (0)_X - \sum_{k=0}^{3} \left[ (0)_X - (0,1)_X - \sum_{i=0}^{k-1} (0,\overline{1},\cdots,\overline{i+1},i+2)_X \right]\]
\[= 1 - 5(0)_X + 4(0,1)_X\]
\[+ (0,\overline{1},2)_X\]
\[+ (0,\overline{1},2)_X + (0,\overline{1},\overline{2},3)_X\]
\[+ (0,\overline{1},2)_X + (0,\overline{1},\overline{2},3)_X + (0,\overline{1},\overline{2},\overline{3},4)_X\]
\[= 1 - 5(0)_X + \sum_{i=0}^{3} [4-i](0,\overline{1},\cdots,\overline{i},i+1)_X\]

Unsurprisingly, for \(dependently\) distributed random variables \(B_i\) certain combinations of \(B_i\)'s being \(\leq t\) and simultaneously other \(B_j\)'s being \(> t\), are impossible, i.e. have probability 0. More specifically:

**Lemma 9** [1, Theorem 10]: Let \(A\) be a bi-infinite identically distributed sequence of random variables and let \(B = \bigvee r A\). Then

\[(0,\overline{1},\cdots,\overline{n-1},n)_B = \begin{cases} 0 & , \quad n \leq r + 1 \\ (0,\cdots,r,r+1,\overline{n-1},n,\cdots,n+r)_A & , \quad r + 1 < n < 2r + 4 \end{cases}\]
For instance, for \( n = 5, r = 1 \) we have \( r + 1 < n < 2r + 4 \), and so

\[
(0, 1, 2, 3, 4, 5) = (0, 1, 2, 3, 5, 6)
\]

Let us give an ad hoc argument which conveys the spirit of the proof. In view of \( B_i = A_i \lor A_{i+1} \) one e.g. has that \( B_5 \leq t \iff A_5, A_6 \leq t \). Using inclusion-exclusion we get

\[
(0, 5, 2, 3, 4, 3)_B = (0, 5, 2, 3)_B - (0, 5, 2, 1)_B - (0, 5, 2, 3)_B + (0, 5, 2, 1, 3)_B
\]

\[
= P(A_0, A_1, A_5, A_6 \leq t, B_2, B_4 > t) - P(A_0, A_1, A_2, A_5, A_6 \leq t, B_2, B_4 > t) - P(A_0, A_1, A_3, A_4, A_5, A_6 \leq t, B_2, B_4 > t) + P(A_0, A_1, \cdots A_6 \leq t, B_2, B_4 > t)
\]

Since \( A_4, A_5 \leq t \) is incompatible with \( B_4 = A_4 \lor A_5 > t \), the last two terms are 0. Furthermore, given that \( A_5 \leq t \), the statement \( B_4 > t \) amounts to \( A_4 > t \). Ditto, given that \( A_2 \leq t \), the statement \( B_2 > t \) amounts to \( A_3 > t \). Hence

\[
(0, 5, 2, 3, 4, 3)_B
\]

\[
= P(A_0, A_1, A_5, A_6 \leq t, A_4 > t, B_2 > t) - P(A_0, A_1, A_2, A_5, A_6 \leq t, A_4 > t, A_3 > t)
\]

\[
= (P(A_0, A_1, A_5, A_6 \leq t, A_4 > t) - P(A_0, A_1, A_5, A_6 \leq t, A_4 > t, A_2 \leq t, A_3 \leq t))
\]

\[
- P(A_0, A_1, A_5, A_6 \leq t, A_4 > t, A_2 \leq t, A_3 \geq t)
\]

\[
= P(A_0, A_1, A_5, A_6 \leq t, A_4 > t) - P(A_0, A_1, A_5, A_6 \leq t, A_4 > t, A_2 \leq t)
\]

\[
= (0, 1, 2, 3, 4, 5, 6)_A
\]

Dualizing Lemma 9 yields:

**Lemma 10** [1, Corollary 11]: Let \( B = \bigwedge^r A \). Then

\[
(\overline{0}, 1 \cdots, n - 1, \overline{n})_B = \begin{cases} 
0 & , n \leq r + 1 \\
(\overline{0}, \cdots, \overline{r}, r + 1, n - 1, \overline{n}, \cdots, n + r)_A & , r + 1 < n \leq 2r + 4
\end{cases}
\]

**Theorem 11** The distribution transfer functions of \( L_n U_n \) and \( U_n L_n \) are:

\[
\phi_{L_n U_n}(p) = p^{n+1} + np^{n+1}q + p^{2n+2}q + \frac{1}{2}(n-1)(n+2)p^{2n+2}q^2,
\]

\[
\phi_{U_n L_n}(p) = 1 - \phi_{L_n U_n}(q) = 1 - q^{n+1} - npq^{n+1} - pq^{2n+2} - \frac{1}{2}(n-1)(n+2)p^2q^{2n+2}.
\]
Proof: Since \( L_n U_n = (\bigvee^n \bigwedge^n)(\bigwedge^n \bigvee^n) = \bigvee^n \bigwedge^{2n} \bigvee^n \) we put
\[
A = \bigvee^n X, \quad B = \bigwedge^{2n} A, \quad C = \bigvee^n B
\]
and calculate
\[
\phi_{L_n U_n}(p) = P(C_0 \leq t) = (0)_C = (0, \cdots, n)_B
\]
\[
= 1 - [n + 1](\overline{0})_B + \sum_{i=0}^{n-1} [n - i](\overline{0}, 1, \cdots, i, i+1)_B \quad \text{(dual of Lemma 8)}
\]
\[
= 1 - [n + 1](\overline{0}, \cdots, 2n)_A + n(\overline{0}, \cdots, 2n+1)_A + \sum_{i=1}^{n-1} 0 \quad \text{(Lemma 10, } r = 2n)
\]
\[
= 1 - [n + 1] \left[ 1 - [2n + 1](0)_A + \sum_{i=0}^{2n-1} [2n - i](0, \overline{1}, \cdots, \overline{i}, i+1)_A \right] + n \left[ 1 - [2n + 2](0)_A + \sum_{i=0}^{2n} [2n + 1 - i](0, \overline{1}, \cdots, \overline{i}, i+1)_A \right] \quad \text{(Lemma 8)}
\]
\[
= [n + 1](0)_A + \sum_{i=0}^{2n} [i - n](0, \overline{1}, \cdots, \overline{i}, i+1)_A
\]
\[
= [n + 1](0)_A - n(0, 1)_A + \sum_{i=1}^{n} [i - n](0, \overline{1}, \cdots, \overline{i}, i+1)_A
\]
\[
+(0, \overline{1}, \cdots, \overline{n+1}, n+2)_A + \sum_{i=n+2}^{2n} [i - n](0, \overline{1}, \cdots, \overline{i}, i+1)_A
\]
\[
= [n + 1](0, \cdots, n)_X - n(0, \cdots, n+1)_X + 0 + (0, \cdots, n, \overline{n+1}, n+2, \cdots, 2n+2)_X
\]
\[
+ \sum_{i=n+2}^{2n} [i - n](0, \cdots, n, \overline{n+1}, \overline{i}, i+1, \cdots, i+1+n)_X \quad \text{(Lemma 9)}
\]
\[
= (n + 1)p^{n+1} - np^{n+2} + p^{2n+2}q + \sum_{i=n+2}^{2n} (i - n)p^{2n+2}q^2
\]
\[
= p^{n+1} + np^{n+1} p + p^{2n+2}q + (2 + 3 + \cdots + n)p^{2n+2}q^2
\]
\[
= p^{n+1} + np^{n+1} q + p^{2n+2}q + \frac{1}{2}(n - 1)(n + 2)p^{2n+2}q^2
\]
\[
\square
\]
From (18) it is clear that \( p^{n+1} \) is the highest power of \( p \) dividing \( \phi_{L_n U_n} \). An easy calculation confirms that, as a polynomial in \( p \), the right hand side of (19) is \((2n + 3)p^2 + (\cdots)p^3 + \cdots \).
From Corollary 4 hence follows that \( L_n U_n \) is lower robust of order \( n + 1 \), but upper robust only of order 2.
5.2 The output distributions of the LULU-operators $C_n$ and $F_n$

We consider next the specific, mutually dual LULU-operators

$$C_n = L_n U_n L_{n-1} U_{n-1} \cdots L_1 U_1,$$

$$F_n = U_n L_n U_{n-1} L_{n-1} \cdots U_1 L_1.$$  

In view of

$$C_{n-1} = \bigvee_{n-1}^{2n-2} \bigwedge_{n-1}^{2n-2} C_{n-2}$$

$$C_n = \bigvee_{n-1}^{2n} \bigwedge_{n-1}^{2n} C_{n-1}$$

$$= \bigvee_{n-1}^{2n} \bigwedge_{n-1}^{2n} \bigvee_{n-1}^{2n-1} \bigwedge_{n-1}^{2n-1} C_{n-2}$$

we define the following doubly infinite sequences of identically distributed random variables. Starting with a sequence $X$ of i.i.d. random variables, put

$$A := \bigvee_{n-1}^{2n-2} C_{n-2} X$$

$$B := \bigwedge_{n-1}^{2n-2} A$$

$$C' := \bigvee_{n-1}^{2n-2} B$$

$$C := \bigvee_{n-1}^{2n-2} B$$

$$D := \bigwedge_{n-1}^{2n} C$$

$$E := \bigvee_{n-1}^{2n} D$$

**Theorem 12** [1, Theorem 14] With $A, B$ as defined above the output distribution $\phi_{C_n}$ of $C_n$ can be computed recursively as follows:

$$\phi_{C_n} = \phi_{C_{n-1}} + n(G_{2n} - G_{2n-1}),$$

where

$$G_{2n} := (0, \cdots, 2n - 1, 2n, 2n + 1, \cdots, 4n)_B$$

$$G_{2n-1} := (0, \cdots, 2n-2, 2n - 1, 2n, \cdots, 4n - 2)_A$$
Proof: First, one calculates
\[
\phi_{C_{n-1}} = (0)_{C'} = (0, \ldots, n-1)_B
\]
\[
= 1 - n(\overline{0})_B + \sum_{i=0}^{n-2} [n - 1 - i](\overline{0}, 1, \ldots, i, i + 1)_B \quad \text{(dual of Lemma 8)}
\]
\[
= 1 - n(\overline{0}, \ldots, 2n-2)_A + [n - 1](\overline{0}, \ldots, 2n-1)_A + \sum_{i=1}^{n-2} 0 \quad \text{(Lemma 10, } r = 2n - 2)
\]
\[
= 1 - n(\overline{0}, \ldots, 2n-2)_A + [n - 1](\overline{0}, \ldots, 2n-1)_A
\]  \hspace{1cm} (20)

The expansion of \(\phi_{C_n}\) is driven a bit further:
\[
\phi_{C_n} = (0)_E = (0, \ldots, n)_D
\]
\[
= 1 - [n + 1](0)_D + \sum_{i=0}^{n-1} [n - i](\overline{0}, 1, \ldots, i, i + 1)_D \quad \text{(dual of Lemma 8)}
\]
\[
= 1 - [n + 1](0, \ldots, 2n)_C + n(\overline{0}, \ldots, 2n+1)_C + \sum_{i=1}^{n-1} 0 \quad \text{(Lemma 10, } r = 2n)
\]
\[
= 1 - [n + 1] \left[ 1 - [2n + 1](0)_C + \sum_{i=0}^{2n-1} [2n - i](0, \overline{1}, \ldots, \overline{7}, i + 1)_C \right]
\]
\[
+ n \left[ 1 - [2n + 2](0)_C + \sum_{i=0}^{2n} [2n + 1 - i](0, \overline{1}, \ldots, \overline{7}, i + 1)_C \right] \quad \text{(Lemma 8)}
\]
\[
= [n + 1](0)_C - \sum_{i=0}^{2n} [n - i](0, \overline{1}, \ldots, \overline{7}, i + 1)_C \quad \text{(easy arithmetic)}
\]
\[
= [n + 1](0, \ldots, 2n - 1)_B - n(0, \ldots, 2n)_B - \sum_{i=1}^{2n-1} 0
\]
\[
+ n(0, \ldots, 2n - 1, \overline{2n}, 2n + 1, \ldots, 4n)_B \quad \text{(Lemma 9, } r = 2n - 1)
\]
\[
= [n + 1](0, \ldots, 2n - 1)_B - n(0, \ldots, 2n)_B + nG_{2n}
\]  \hspace{1cm} (21)

This yields
\[
\phi_{C_n} - nG_{2n} = [n + 1](0, \ldots, 2n - 1)_B - n(0, \ldots, 2n)_B \quad \text{(by (21))}
\]
\[
= [n + 1] \left[ 1 - 2n(\overline{0})_B + \sum_{i=0}^{2n-2} [2n - 1 - i](\overline{0}, 1, \ldots, i, i + 1)_B \right]
\]
\[
- n \left[ 1 - [2n + 1](\overline{0})_B + \sum_{i=0}^{2n-1} [2n - i](\overline{0}, 1, \ldots, i, i + 1)_B \right] \quad \text{(dual of Lemma 8)}
\]
\[= 1 - n(\overline{0})_B + \sum_{i=0}^{2n-1} [n - 1 - i](\overline{0}, 1, \ldots, i, \overline{i + 1})_B \quad \text{(easy arithmetic)} \]

\[= 1 - n(\overline{0}, \ldots, 2n - 2)_A + [n - 1](\overline{0}, \ldots, 2n - 1)_A + \sum_{i=1}^{2n-2} 0 \]

\[-n(\overline{0}, \ldots, 2n - 2, 2n - 1, \overline{2n}, \ldots, 4n - 2)_A \quad \text{(Lemma 10, } r = 2n - 2) \]

\[= \phi_{C_{n-1}} - nG_{2n-1} \quad \text{(by (20))} \]

which, upon adding \(nG_{2n}\) on both sides, gives the claimed formula for \(\phi_{C_n}\). \(\square\)

As an example, let us compute the output distribution of \(C_2\). From \(A = \bigvee^1 C_0 X = \bigvee^1 X\) follows \(B = \bigwedge^2 A = \bigwedge^2 \bigvee X\). Using expansion calculus the reader may verify that

\[G_{2n} = G_4 = (0, 1, 2, 3, 4, 5, 6, 7, 8)_B = p^4 q^2 [p + p^2 q]^2 \]

Similarly one gets

\[G_{2n-1} = G_3 = p^2 q^2 (1 - p^2)^2. \]

Therefore

\[\phi_{C_2} = \phi_{C_1} + 2(G_4 - G_3) \]

\[= 3p^3 + 3p^4 - 9p^5 + 4p^6 + 4p^7 - 10p^8 + 4p^9 + 8p^{10} - 8p^{11} + 2p^{12} \]

As to robustness, from the above representation of \(\phi_{C_2}\) and by using Corollary 4 we obtain that \(C_2\) is lower robust of order 3 like \(U_2\). Similar to \(L_2 U_2\) discussed in 5.1, the upper robustness of \(C_2\) is not inherited from \(L_2\). Indeed, we have

\[\phi_{C_2} - 1 = q^2 (2p^{10} - 4p^9 - 2p^8 + 4p^7 + 4p^4 - p^3 - 3p^2 - 2p - 1) \]

which implies that \(C_2\) is upper robust only of order 2. However, upper robustness is not constantly 2; these results were obtained from Theorem 12:

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| lower robustness | 2 | 3 | 4 | 4 | 5 | 6 |
| upper robustness  | 2 | 2 | 3 | 3 | 4 | 4 |

We mention that some handy closed formula for \(G_{2n}\) and \(G_{2n-1}\) is conjectured in [1, section 4.5.6].
6 Using some principle of exclusion as opposed to inclusion-exclusion

As witnessed by 5.2, one can sometimes exploit symmetry to tame the inherent exponential complexity of inclusion-exclusion. However, without the possibility to clump together many identical terms, the number of summands in Lemma 7 is $2^n$, which is infeasible already for $n = 20$ or so.

In [14] on the other hand, some multi-purpose principle of exclusion (POE) is employed. When POE is aimed at calculating the output distribution of a stack filter $S$, a prerequisite is that the stack filter $S$ be given as a disjunction of conjunctions $K_i$, i.e. in disjunctive normal form (DNF). The POE then starts off with the set Mod$_1$ of all 0,1-strings that satisfy $K_1$, then from Mod$_1$ one excludes all 0,1-strings that violate $K_2$. This yields Mod$_2 \subseteq$ Mod$_1$, from which all 0,1-strings are excluded that violate $K_3$, and so on. The feasibility of the POE hinges on the compact representation of the sets Mod$_i$.

The details being given in [14], here we address the question of how one gets the DNF in the first place. Specifically we consider the frequent case that our stack filter $S$ is a CDE (section 4) whose structural elements are provided. Let us go in media res by reworking $S = \bigvee^1 \bigwedge^2 \bigvee^2$ of Example 1:

\[(S X)_0 = A_0 = Z_0 \lor Z_1 \quad \text{(DNF)} \]
\[= (Y_{-2} \land Y_{-1} \land Y_0) \lor (Y_{-1} \land Y_0 \land Y_1) \quad \text{(blowup)} \]
\[= Y_0 \land Y_{-1} \land (Y_{-2} \lor Y_1) \quad \text{(get CNF)} \]
\[= (X_0 \lor X_1 \lor X_2 \lor X_3) \land (X_{-1} \lor X_0 \lor X_1 \lor X_2) \quad \text{(blowup)} \]
\[\quad \land ((X_{-2} \lor X_{-1} \lor X_0 \lor X_1) \lor (X_1 \lor X_2 \lor X_3 \lor X_4)) \]
\[= (X_0 \lor X_1 \lor X_2 \lor X_3) \land (X_{-1} \lor X_0 \lor X_1 \lor X_2) \quad \text{(condense)} \]
\[\quad \land (X_{-2} \lor X_{-1} \lor X_0 \lor X_1 \lor X_2 \lor X_3 \lor X_4) \]
\[= (X_0 \lor X_1 \lor X_2 \lor X_3) \land (X_{-1} \lor X_0 \lor X_1 \lor X_2) \quad \text{(condense further)} \]
\[\quad \land (X_{-2} \lor X_{-1} \lor X_0 \lor X_1 \lor X_2 \lor X_3 \lor X_4) \]
\[= X_0 \lor X_1 \lor X_2 \lor (X_3 \land X_{-1}) \quad \text{(get DNF)} \]

The last line is the sought DNF of $S$. It is obtained by starting with the DNF $Z_0 \lor Z_1$. This gets “blown up” to a DNF in terms of $Y_i$’s (using definition (13) of $Z_0$ and $Z_1$). This DNF needs to be switched to CNF (= conjunctive normal form). This in turn is blown up to a CNF in terms of $X_i$’s. Usually the result can and must be condensed in obvious ways (“condense further” meant that only the inclusion-minimal index sets carry over). Like this one takes turns switching DNF’s

\[\*$\text{More precisely, the positive Boolean function that underlies the stack filters must be given in DNF.}\]
\[\text{† How one switches between DNF and CNF of a positive Boolean function (i.e. one without negated variables like here) is a well researched topic which we don’t touch upon here.} \]
with CNF’s, and blowing up expressions. This is done as often as there are structural elements. As a bonus the so-called rank selection probabilities \( rsp[i] \) are calculated. This is defined as the probability that the filter selects the \( i \)-th smallest pixel in the \( w \)-element sliding window. For instance here \( w = 5 \) and \( rsp[1] = rsp[2] = rsp[3] = 0, rsp[4] = 0.4, rsp[5] = 0.6 \).

There exists a Mathematica Demonstration Project whereby the user provides the structural elements of any desired CDE \( S \) (also 2-dimensional). From this the program first calculates the DNF, which then serves to calculate the output polynomial \( \phi_S(p) \). Alternatively the DNF of any stack filter (CDE or not) can be fed directly by the user. Albeit Wild’s algorithm is multi-purpose, it managed to calculate \( \phi_{C_5}(p) \) up to \( n = 5 \) (and the result agreed with Butler’s). Written out as CDE we have \( C_5 = \bigvee^5 \bigwedge^{10} \bigvee^9 \cdots \bigwedge^2 \bigvee \). The corresponding structural elements \( \{0,1,2,3,4,5\}, \{0,-1,\ldots,-10\}, \{0,1,\ldots,9\} \) and so forth triggered the calculation of a DNF comprising a plentiful 12018 conjunctions (time: 168224 sec). From this \( \phi_{C_5}(p) \) was calculated in 45069 sec. Here it is:

\[
12x^5 + 7x^6 - 23x^7 + 19x^8 - 130x^9 + 194x^{10} - 59x^{11} - 142x^{12} \\
+ 460x^{13} - 787x^{14} + 715x^{15} - 7x^{16} - 1030x^{17} \\
+ 1959x^{18} - 2216x^{19} + 208x^{20} + 3711x^{21} - 6748x^{22} \\
+ 8412x^{23} - 7587x^{24} + 2023x^{25} + 4680x^{26} \\
- 7903x^{27} + 8839x^{28} - 13540x^{29} + 30009x^{30} \\
- 51715x^{31} + 50159x^{32} - 7686x^{33} - 51417x^{34} \\
+ 78198x^{35} - 50589x^{36} + 6900x^{37} - 7680x^{38} \\
+ 56330x^{39} - 86905x^{40} + 43710x^{41} + 49540x^{42} \\
- 114680x^{43} + 103390x^{44} - 40555x^{45} - 15370x^{46} \\
+ 33955x^{47} - 25460x^{48} + 11790x^{49} - 3645x^{50} \\
+ 740x^{51} - 90x^{52} + 5x^{53}
\]

7 Conclusion

The main result of this paper is the calculation of the output distribution of the \( LULU \)-operators \( C_n \) and \( F_n \). The basic technique for obtaining the output distribution in closed or recursive form may also apply to other highly symmetric erosion-dilation cascades; otherwise the algorithm of [14] does the job. Further we introduced the concept of robustness which is useful in characterizing the properties of a smoother in terms of its output distribution.
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