REMARKS ON SOME INTEGRAL FORMULAS FOR $G_2$-STRUCTURES

FRANCISCO MARTÍN CABRERA

Abstract. For seven-dimensional Riemannian manifolds equipped with a $G_2$-structure, we show in a full detailed way that all integral formulas and divergence equations, given by diverse authors, are agree with the ones displayed here in terms of the intrinsic torsion of the $G_2$-structure. Likewise the components of such an intrinsic torsion is expressed by means of exterior algebra.

Keywords and phrases: $G$-structure, intrinsic torsion, minimal connection, $G_2$-structure, vector cross product, $G_2$-map

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1. INTRODUCTION

By using different tools, diverse authors, Bryant in [4], Bor and Hernández Lamoned in [2], Friedrich and Ivanov in [8] and Niedziałomski in [15], have deduced integral formulas and divergence equations for Riemannian manifolds equipped with a $G_2$-structure. They express the scalar curvature of the manifold in terms related with components of the intrinsic torsion of the $G_2$-structure. Such terms are different in each formula. Thus it is natural to ask for the way in which all mentioned formulas are agree and the correlation between them.

Our purpose here is to show in a detailed way that all integral formulas, given by the above mentioned authors, are agree with

$$\int_M s \text{vol}_M = \int_M 9\|\xi_{(1)}\|^2 - \frac{3}{2}\|\xi_{(2)}\|^2 - \frac{3}{2}\|\xi_{(3)}\|^2 + \frac{15}{2}\|\xi_{(4)}\|^2 \text{vol}_M,$$

where $s$ is the scalar curvature, $M$ is a closed manifold equipped with a $G_2$-structure and $\xi$ denotes the intrinsic torsion. The terms $\xi_{(i)}$ denote $G_2$-components of $\xi$. Likewise, we also
prove that all divergence equations, given by such authors, are agree with
\[
 s = 9\|\xi_{(1)}\|^2 - \frac{3}{2}\|\xi_{(2)}\|^2 - \frac{3}{2}\|\xi_{(3)}\|^2 + \frac{15}{2}\|\xi_{(4)}\|^2 - 6 \text{ div } \xi_i e_i. 
\]

Correlations between the integral formulas and between the divergence equations have already been pointed out by Niedziałomski in [15]. Here we give a description of them. This is done following notations for \(G_2\)-structures used by Fernández and Gray in [6] and by the present author in [11–13]. In our view, the present text illustrates the advantages of such notations.

Using the map \(\xi \rightarrow -\xi \varphi = \nabla \varphi\), where \(\varphi\) is the fundamental three-form of the \(G_2\)-structure, it is found that all information about the intrinsic torsion is contained in the covariant derivative \(\nabla \varphi\). From this it is deduced that all information about the intrinsic torsion is contained in the exterior derivatives \(d \varphi\) and \(d \star \varphi\) (\(\star\) denotes Hodge star operator). This is described in [14] (see also Table I in [11]). Thus a useful alternative way to find such information is by studying \(d \varphi\) and \(d \star \varphi\). Because all the necessary tools are displayed in first sections, in last section we take the chance to express the components of the intrinsic torsion \(\xi\) in terms of \(d \varphi\) and of \(d \star \varphi\).

2. Preliminaries

Let \(\{e_0,\ldots,e_6\}\) denote the standard positive orthonormal basis of \(\mathbb{R}^7\). The exceptional Lie group \(G_2\) is the subgroup of \(SO(7)\) consisting of those elements which fix the three-form \(\varphi\) defined by
\[
\varphi = \sum_{i \in \mathbb{Z}_7} e_i^* \wedge e_{i+1}^* \wedge e_{i+3}^*,
\] (2.1)
where we have denoted by \(e_i^*\) the dual one-forms of \(e_i\). A Cayley basis is a \(\{u_0^*,\ldots,u_6^*\}\) for covectors such that, in terms of this basis, \(\varphi\) is written as in (2.1). The name Cayley basis is used because, in this context, it is well known the identification of \(\mathbb{R}^7\) with the imaginary octonions or Cayley numbers \(\text{Im} \mathbb{O}\). The dual basis, denoted by \(\{u_0,\ldots,u_6\}\) of a Cayley basis, is also called Cayley basis for vectors. Thus the standard basis \(\{e_0,\ldots,e_6\}\) is a Cayley basis.

Here we fix \(\text{Vol} = e_0^* \wedge \cdots \wedge e_6^*\) as volume form and, denoting the corresponding Hodge star operator by \(\star\), the four-form
\[
\star \varphi = -\sum_{i \in \mathbb{Z}_7} e_{i+2}^* \wedge e_{i+4}^* \wedge e_{i+5}^* \wedge e_{i+6}^*
\]
will play a relevant rôle along the present text.

The euclidean metric \(\langle \cdot , \cdot \rangle\) on \(\mathbb{R}^7\) induces the following inner product on skew symmetric \(p\)-forms:
\[
\langle \alpha, \beta \rangle = \frac{1}{p!} \sum_{i_1,\ldots,i_p \in \mathbb{Z}_7} \alpha(e_{i_1},\ldots,e_{i_p}) \beta(e_{i_1},\ldots,e_{i_p}),
\]
for all \(\alpha,\beta \in \Lambda^p \mathbb{R}_7^*\). The corresponding norm of \(\langle \cdot, \cdot \rangle\) will be denoted by \(| \cdot |\). The inner product \(\langle \cdot, \cdot \rangle\) is widely used in the literature and satisfies the identity
\[
\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \text{Vol}.
\]
It follows $(\varphi, \varphi) = (\star \varphi, \star \varphi) = 7$ and $\text{Vol} = \frac{1}{7} \varphi \wedge \star \varphi$. Note also that $(\star \alpha, \star \beta) = (\alpha, \beta)$.

For general tensors, it is used the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle \Psi, \Phi \rangle = \sum_{i_1, \ldots, i_r, j_1, \ldots, j_r \in \mathbb{Z}_7} \Psi_{i_1 \ldots i_r}^{j_1 \ldots j_r} \Phi_{i_1 \ldots i_r}^{j_1 \ldots j_r},$$

for all $(r, s)$-tensors $\Psi$ and $\Phi$ on $\mathbb{R}^7$, where $\Psi_{i_1 \ldots i_r}^{j_1 \ldots j_r}$ and $\Phi_{i_1 \ldots i_r}^{j_1 \ldots j_r}$ are computed by using an orthonormal basis for vectors and its dual one. The corresponding norm of $\langle \cdot, \cdot \rangle$ will be denoted by $\| \cdot \|$.

**Remark 2.1.** Let us recall the well known converse fact that if one has a three-form $\varphi$ on $\mathbb{R}^7$ and a basis $\{u_0, \ldots, u_6\}$ for vectors in $\mathbb{R}^7$ such that $\varphi$ is expressed as in (2.1), i.e.

$$\varphi = \sum_{i \in \mathbb{Z}_7} u_i^* \wedge u_{i+1}^* \wedge u_{i+3}^*,$$

then the group $G_2 = \{a \in GL(7) | a^* \varphi = \varphi\}$ preserves the metric such that $\{u_0, \ldots, u_6\}$ is orthonormal and the volume form defined by $\text{Vol} = u_0^* \wedge \ldots \wedge u_6^*$. It is also denoted by $G_2$ because it is isomorphic to the one above mentioned.

Using $\varphi$, a two-fold vector cross product $\times$ is defined by

$$(u, v) \rightarrow u \times v = \sum_{i \in \mathbb{Z}_7} \varphi(u, v, e_i) e_i \in \mathbb{R}^7$$

([3], [9], [6]). This product is skew-symmetric and satisfies

$$\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0, \quad \|u \times v\|^2 = \|u\|\|v\| - \langle u, v \rangle^2,$$

for all $u, v \in \mathbb{R}^7$, which are the conditions of the definition of such a product. From these initial conditions, the following identities are derived

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle,$$
$$u \times (u \times v) = \langle u, v \rangle u - \|u\|^2 v,$$
$$u \times (v \times w) = -v \times (u \times w) + \langle v, w \rangle u + \langle w, u \rangle v - 2\langle u, v \rangle w \quad (2.2)$$

Note that for a Cayley basis, as $\{e_0, \ldots, e_6\}$, one has

$$e_i \times e_{i+1} = e_{i+3}, \quad e_{i+3} \times e_i = e_{i+1}, \quad e_{i+1} \times e_{i+3} = e_i, \quad e_i \times e_i = 0, \quad i \in \mathbb{Z}_7.$$

Moreover, $G_2$ can be also described as the subgroup of $SO(7)$ consisting of those elements which fix the product $\times$, i.e.

$$G_2 = \{g \in SO(7) | g(u \times v) = gu \times gv, \forall u, v \in \mathbb{R}^7 \}.$$

As a consequence, the Lie algebra $\mathfrak{g}_2$ is described by

$$\mathfrak{g}_2 = \{a \in \mathfrak{so}(7) | a(u \times v) = a(u) \times v + u \times a(v), \forall u, v \in \mathbb{R}^7 \}.$$

Even the following description is also valid

$$\mathfrak{g}_2 = \{a \in \mathfrak{gl}(7) | a(u \times v) = a(u) \times v + u \times a(v), \forall u, v \in \mathbb{R}^7 \}.$$

By considering the identities

$$a(e_i) = a(e_{i+1} \times e_{i+3}) = a(e_{i+1}) \times e_{i+3} + e_{i+1} \times a(e_{i+3}), \quad i \in \mathbb{Z}_7,$$
and, denoting \( a_{ij} = \langle e_i, a(e_j) \rangle \) and so \( a = \sum_{i,j \in \mathbb{Z}_7} a_{ij} e_i \otimes e_j^* \), one deduces

\[
\mathfrak{g}_2 = \{ a \in \mathfrak{gl}(7) \mid a_{i+1,j+3} + a_{i+4,j+5} + a_{i+2,j+6} = 0, \quad a_{ij} = -a_{ji}, \forall i, j \in \mathbb{Z}_7 \}.
\]

Under the action of \( G_2 \) given by \( (ga)(v) = ga(g^{-1}v) \), for all \( v \in \mathbb{R}^7 \), the space \( \mathfrak{gl}(7) \cong \mathbb{R}^7 \otimes \mathbb{R}^{7*} \) (\( \cong \mathbb{R}^{7*} \otimes \mathbb{R}^7 \)) by using \( (\cdot, \cdot) \) is decomposed into the following irreducible \( G_2 \)-modules

\[
\mathfrak{gl}(7) = \mathbb{R} \text{Id}_{\mathbb{R}^7} \oplus S^2 \mathbb{R}^{7*} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp,
\]

where

\[
\begin{align*}
\mathbb{R} \text{Id}_{\mathbb{R}^7} &= \{ a \in \mathfrak{gl}(7) \mid a = \lambda \text{Id}_{\mathbb{R}^7}, \quad \lambda \in \mathbb{R} \}, \\
S^2 \mathbb{R}^{7*} &= \{ a \in \mathfrak{gl}(7) \mid a_{ij} = a_{ji}, \quad \sum_{i \in \mathbb{Z}_7} a_{ii} = 0, \quad \forall i, j \in \mathbb{Z}_7 \}, \\
\mathfrak{g}_2^\perp &= \{ a \in \mathfrak{gl}(7) \mid a_{ij} = -a_{ji}, \quad a_{i+1,j+3} + a_{i+4,j+5} + a_{i+2,j+6} = 0, \quad \forall i, j \in \mathbb{Z}_7 \}.
\end{align*}
\]

In the literature about \( G_2 \)-structures, it is frequently denoted \( X_1 = \mathbb{R} \text{Id}_{\mathbb{R}^7}, X_3 = S^2 \mathbb{R}^{7*}, X_2 = \mathfrak{g}_2 \) and \( X_4 = \mathfrak{g}_2^\perp \).

For \( a \in \mathfrak{g}_2^\perp \), one can consider the vector \( v = \sum_{i \in \mathbb{Z}_7} v_i e_i \), where \( v_i = a_{i+1,i+3} + a_{i+4,i+5} = a_{i+2,i+6} \). It is found that \( a_{ij} = v_i \phi(e_i, e_j) \), where \( \phi \) denotes the interior product. Thus one has a map \( \mathfrak{g}_2^\perp \longrightarrow \mathbb{R}^7, \ a \rightarrow v \). Reciprocally, for all \( v \in \mathbb{R}^7 \), one has the endomorphism \( A_v \) such that

\[
(A_v)_{i+1,i+3} = (A_v)_{i+4,i+5} = (A_v)_{i+2,i+6} = v_i, \quad (A_v)_{ij} = -(A_v)_{ji},
\]

for all \( i, j \in \mathbb{Z}_7 \). The map \( v \rightarrow A_v \) is the converse of the previous one \( a \rightarrow v \). Moreover, from the identity \( A_v(u) = u \times v \), one easily deduces that \( v \rightarrow A_v \) is a \( G_2 \)-map. Then \( \mathfrak{g}_2^\perp \cong \mathbb{R}^7 \) as \( G_2 \)-modules.

For a general \( a \in \mathfrak{gl}(7) \), we will determine its four \( G_2 \)-components \( a_{(1)}, a_{(2)}, a_{(3)} \) and \( a_{(4)} \) corresponding to the \( G_2 \)-modules \( X_1, X_2, X_3 \) and \( X_4 \), respectively. Also the norms of \( a_{(i)} \) will be computed.

To obtain \( a_{(4)} \), one considers the map \( p : \mathfrak{gl}(7) \longrightarrow \mathbb{R}^7 \) (already considered in \([4,15]\) defined by

\[
p(a) = \sum_{i \in \mathbb{Z}_7} a_{i+1,i+3} - a_{i+3,i+1} + a_{i+4,i+5} - a_{i+5,i+4} + a_{i+2,i+6} - a_{i+6,i+2}).
\]

We will denote \( p(a)_i = a_{i+1,i+3} - a_{i+3,i+1} + a_{i+4,i+5} - a_{i+5,i+4} + a_{i+2,i+6} - a_{i+6,i+2} \). It follows that, for all \( v \in \mathbb{R}^7 \), \( p(A_v) = 6v \). Therefore,

\[
a_{(4)} = \frac{1}{6} A_p(a).
\]

In a more developed way, \( a_{(4)} \) is given by

\[
6a_{(4)} = \sum_{i \in \mathbb{Z}_7} p(a)_i (e_{i+1} \otimes e_{i+3} - e_{i+3} \otimes e_{i+1} + e_{i+4} \otimes e_{i+5} - e_{i+5} \otimes e_{i+4} + e_{i+2} \otimes e_{i+6} - e_{i+6} \otimes e_{i+2}).
\]

Using \((2.2)\), one has

\[
36\|a_{(4)}\|^2 = \sum_{i \in \mathbb{Z}_7} \langle e_i \times p(a), e_i \times p(a) \rangle = -\sum_{i \in \mathbb{Z}_7} \langle e_i, p(a) \times (p(a) \times e_i) \rangle = -\left( \|p(a)\|^2 - 7\|p(a)\|^2 \right) = 6\|p(a)\|^2.
\]

Thus it is obtained

\[
\|a_{(4)}\|^2 = \frac{1}{6}\|p(a)\|^2.
\]
It is obvious that
\[
a_{(1)} = \frac{1}{7} \text{tr}(a) \text{Id}_{\mathbb{R}^7}, \quad \|a_{(1)}\|^2 = \frac{1}{7} \|\text{tr}(a)\|^2,
\]
where \(\text{tr}(a) = \sum_{i \in \mathbb{Z}_7} a_{ii}\). Taking this into account, one has
\[
a_{(3)} = \sum_{i \in \mathbb{Z}_7} (a_{ii} - \frac{1}{7} \text{tr}(a)) e_i \otimes e_i^* + \frac{1}{2} \sum_{i,j \in \mathbb{Z}_7, i \neq j} (a_{ij} + a_{ji})(e_i \otimes e_j^* + e_j \otimes e_i^*).
\]
Therefore,
\[
\|a_{(3)}\|^2 = \frac{1}{2} \sum_{i \in \mathbb{Z}_7} ((a_{i+1,i+3} + a_{i+3,i+1})^2 + (a_{i+4,i+5} + a_{i+5,i+4})^2 + (a_{i+2,i+6} + a_{i+6,i+2})^2)
\]
\[+ \sum_{i \in \mathbb{Z}_7} (a_{ii} - \frac{1}{7} \text{tr}(a))^2. \tag{2.7}
\]
One can check that \(\|a_{(1)} + a_{(3)}\|^2 = \|a_{(1)}\|^2 + \|a_{(3)}\|^2\) as it is expected.

For \(a_{(2)}\), one has
\[
6a_{(2)} = \sum_{i \in \mathbb{Z}_7} (3(a_{i+1,i+3} - a_{i+3,i+1})^2 + (a_{i+4,i+5} - a_{i+5,i+4})^2 + (a_{i+2,i+6} - a_{i+6,i+2})^2)
\]
\[+ \sum_{i \in \mathbb{Z}_7} (3(a_{i+4,i+5} - a_{i+5,i+4})^2 + (a_{i+2,i+6} - a_{i+6,i+2})^2)
\]
\[+ \sum_{i \in \mathbb{Z}_7} (3(a_{i+2,i+6} - a_{i+6,i+2})^2 + (a_{i+4,i+5} - a_{i+5,i+4})^2) \tag{2.8}
\]
Therefore,
\[
\|a_{(2)}\|^2 = \frac{1}{2} \sum_{i \in \mathbb{Z}_7} ((a_{i+1,i+3} - a_{i+3,i+1})^2 + (a_{i+4,i+5} - a_{i+5,i+4})^2 + (a_{i+2,i+6} - a_{i+6,i+2})^2)
\]
\[- \frac{1}{6} \|p(a)\|^2. \tag{2.9}
\]
Note also that \(\|a_{(2)} + a_{(4)}\|^2 = \|a_{(2)}\|^2 + \|a_{(4)}\|^2\) as it is expected.

3. SOME INVARIANTS OF AN ENDO MORPHISM IN \(\mathfrak{gl}(7)\) UNDER THE ACTION OF \(G_2\)

For our purposes, it is necessary to consider several invariants (used in [15]) of an endomorphism \(a \in \mathfrak{gl}(7)\) under the action of \(G_2\). There are some of them which are also invariant under the wider action of \(SO(7)\).

A first one is the norm of \(a\). In fact, \(\|a\|^2 = \|ga\|^2\), for all \(g \in SO(7)\). Other invariants are the coefficients of the characteristic polynomial. They are denoted by \(\sigma_i(a)\), where
\[
\det(a - \lambda \text{Id}) = \sum_{i=1}^{7} (-1)^i \sigma_{7-i}(a) \lambda^i.
\]
Thus, \(\sigma_0(a) = 1\), \(\sigma_1(a) = \text{tr}(a)\), \(\sigma_2(a) = \frac{1}{2} \sum_{i,j \in \mathbb{Z}_7, i \neq j} (a_{ii}a_{jj} - a_{ij}a_{ji})\).

In relation with the \(G_2\)-components of \(a\), one has the following result.

**Lemma 3.1.** For all \(a \in \mathfrak{gl}(7)\), it is satisfied
\[
(\sigma_1(a))^2 = 7\|a_{(1)}\|^2,
\]
\[
2\sigma_2(a) = 6\|a_{(1)}\|^2 + \|a_{(2)}\|^2 - \|a_{(3)}\|^2 + \|a_{(4)}\|^2.
\]
Proof. The expression for \((\sigma_1(a))^2\) follows directly from (2.5). To see the expression for \(\sigma_2(a)\), taking
\[
\|a(2) + a(4)\|^2 = \frac{1}{2} \sum_{i,j \in \mathbb{Z}_7, i \neq j} (a_{i,j}^2 + a_{j,i}^2) - \frac{1}{2} \sum_{i,j \in \mathbb{Z}_7, i \neq j} a_{i,j}a_{j,i},
\]
\[
\|a(1) + a(3)\|^2 = \frac{1}{2} \sum_{i,j \in \mathbb{Z}_7, i \neq j} (a_{i,j}^2 + a_{j,i}^2) + \frac{1}{2} \sum_{i,j \in \mathbb{Z}_7, i \neq j} a_{i,j}a_{j,i} + \sum_{i \in \mathbb{Z}_7} a_i^2
\]
into account, it is obtained
\[
\|a(2)\|^2 + \|a(4)\|^2 - \|a(1)\|^2 - \|a(3)\|^2 = - \sum_{i,j \in \mathbb{Z}_7, i \neq j} a_{i,j}a_{j,i} - \sum_{i \in \mathbb{Z}_7} a_i^2.
\]
Since \(\sigma_1(a)^2 = \sum_{i \in \mathbb{Z}_7} a_i^2 + \sum_{i,j \in \mathbb{Z}_7, i \neq j} a_{i,j}a_{j,i},\) one deduces
\[
\|a(2)\|^2 + \|a(4)\|^2 - \|a(1)\|^2 - \|a(3)\|^2 = 2\sigma_2(a) - \sigma_1(a)^2.
\]
Finally, using \(\sigma_1(a)^2 = 7\|a(1)\|^2,\) the identity for \(\sigma_2(a)\) is obtained. \(\square\)

Now, let us restrict our attention to the action of \(G_2\). There are the following invariants associated with an endomorphism \(a:\)
\[
i_0(a) = \sum_{i,j \in \mathbb{Z}_7} \langle a(e_i) \times a(e_j), e_i \times e_j \rangle,
\]
\[
i_1(a) = \sum_{i,j \in \mathbb{Z}_7} \langle a(e_i) \times e_i, a(e_j) \times e_j \rangle,
\]
\[
i_2(a) = \sum_{i,j \in \mathbb{Z}_7} \langle a(e_i) \times e_j, a(e_j) \times e_i \rangle.
\]

Lemma 3.2. For all \(a \in \mathfrak{gl}(7),\) it is satisfied
\[
i_0(a) = 6\|a(1)\|^2 + 3\|a(2)\|^2 - \|a(3)\|^2 - 3\|a(4)\|^2,
\]
\[
i_1(a) = 6\|a(4)\|^2,
\]
\[
i_2(a) = -6\|a(1)\|^2 + 3\|a(2)\|^2 + \|a(3)\|^2 - 3\|a(4)\|^2,
\]
\[
6\|a\|^2 = \sum_{i,j \in \mathbb{Z}_7} \|e_i \times a(e_j)\|^2.
\]

Proof. For \(i_1(a),\) one has the identity
\[
i_1(a) = \sum_{i \in \mathbb{Z}_7} \langle a(e_i) \times e_i, a(e_i) \times e_i \rangle + 2 \sum_{i \in \mathbb{Z}_7} \langle a(e_{i+1}) \times e_{i+1}, a(e_{i+3}) \times e_{i+3} \rangle + 2 \sum_{i \in \mathbb{Z}_7} \langle a(e_{i+4}) \times e_{i+4}, a(e_{i+5}) \times e_{i+5} \rangle + 2 \sum_{i \in \mathbb{Z}_7} \langle a(e_{i+2}) \times e_{i+2}, a(e_{i+6}) \times e_{i+6} \rangle.
\]
Now we will compute all these summands. By one hand, one has
\[
\sum_{i \in \mathbb{Z}_7} \langle a(e_i) \times e_i, a(e_i) \times e_i \rangle = - \sum_{i \in \mathbb{Z}_7} \langle a(e_i), e_i \times (e_i \times a(e_i)) \rangle.
\]
Therefore, by using (2.2), it is obtained
\[
\sum_{i \in \mathbb{Z}_7} \langle a(e_i) \times e_i, a(e_i) \times e_i \rangle = \|a\|^2 - \sum_{i \in \mathbb{Z}_7} a_i^2.
\]
By the other hand, one has\[
\sum_{i \in \mathbb{Z}_7} \langle a(e_{i+1}) \times e_{i+1}, a(e_{i+3}) \times e_{i+3} \rangle = - \sum_{i \in \mathbb{Z}_7} a_{i+1} + 3a_{i+3} + a_{i+1} \times a_{i+3} - \sum_{i \in \mathbb{Z}_7} a_{i+3} + 3a_{i+1} + 2a_{i+6} - \sum_{i \in \mathbb{Z}_7} a_{i+5} + 4a_{i+2} + 6a_{i+6} + \sum_{i \in \mathbb{Z}_7} a_{i+1} + 3a_{i+4} + 5a_{i+5} + \sum_{i \in \mathbb{Z}_7} a_{i+5} + 4a_{i+6} + 2a_{i+2}.
\]
\[
\sum_{i \in \mathbb{Z}_7} (a(e_{i+4}) \times e_{i+4}, a(e_{i+5}) \times e_{i+5}) = -\sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+6} - \sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+6} + \sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+6} + \sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+6} - \sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+6}.
\]

\[
\sum_{i \in \mathbb{Z}_7} (a(e_{i+2}) \times e_{i+2}, a(e_{i+6}) \times e_{i+6}) = -\sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6} - \sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6} + \sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6} - \sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6}.
\]

From these identities, it follows
\[
i_1(a) = \sum_{i \in \mathbb{Z}_7} (a(e_{i+1}) \times e_{i+1}) + 2 \sum_{i \in \mathbb{Z}_7} (a(e_{i+1}) \times e_{i+3}, a(e_{i+3}) \times e_{i+1}) + 2 \sum_{i \in \mathbb{Z}_7} (a(e_{i+4}) \times e_{i+5}, a(e_{i+5}) \times e_{i+4}) + 2 \sum_{i \in \mathbb{Z}_7} (a(e_{i+2}) \times e_{i+6}, a(e_{i+6}) \times e_{i+2}).
\]

For \(i_2(a)\), one has the identity
\[
i_2(a) = \sum_{i \in \mathbb{Z}_7} (a(e_{i+1}) \times e_{i+1}, a(e_{i+3}) \times e_{i+1}) + 2 \sum_{i \in \mathbb{Z}_7} (a(e_{i+1}) \times e_{i+3}, a(e_{i+3}) \times e_{i+1}) + 2 \sum_{i \in \mathbb{Z}_7} (a(e_{i+4}) \times e_{i+5}, a(e_{i+5}) \times e_{i+4}) + 2 \sum_{i \in \mathbb{Z}_7} (a(e_{i+2}) \times e_{i+6}, a(e_{i+6}) \times e_{i+2}).
\]

The first sum was already computed above. The following three sums are given by
\[
\sum_{i \in \mathbb{Z}_7} (a(e_{i+1}) \times e_{i+1}, a(e_{i+3}) \times e_{i+1}) = -\sum_{i \in \mathbb{Z}_7} a_{i+1i+1}a_{i+3i+3} - \sum_{i \in \mathbb{Z}_7} a_{i+1i+1}a_{i+3i+3} + \sum_{i \in \mathbb{Z}_7} a_{i+1i+1}a_{i+3i+3} + \sum_{i \in \mathbb{Z}_7} a_{i+1i+1}a_{i+3i+3}.
\]

\[
\sum_{i \in \mathbb{Z}_7} (a(e_{i+4}) \times e_{i+5}, a(e_{i+5}) \times e_{i+4}) = -\sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+5} - \sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+5} + \sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+5} + \sum_{i \in \mathbb{Z}_7} a_{i+4i+5}a_{i+4i+5}.
\]

\[
\sum_{i \in \mathbb{Z}_7} (a(e_{i+2}) \times e_{i+6}, a(e_{i+6}) \times e_{i+2}) = -\sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6} - \sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6} + \sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6} + \sum_{i \in \mathbb{Z}_7} a_{i+2i+6}a_{i+2i+6}.
\]

Therefore,
\[
i_2(a) = \sum_{i \in \mathbb{Z}_7} \left( (a_{i+1i+3} + a_{i+3i+1}) + a_{i+4i+5} + a_{i+5i+4} + a_{i+2i+6} + a_{i+6i+2} \right)
\]

\[-2 \sum_{i \in \mathbb{Z}_7} (a_{i+1i+3} + a_{i+3i+1}) + a_{i+4i+5} + a_{i+5i+4} \]

\[-2 \sum_{i \in \mathbb{Z}_7} (a_{i+1i+3} + a_{i+3i+1}) + a_{i+6i+2} \]

\[-2 \sum_{i \in \mathbb{Z}_7} (a_{i+4i+5} + a_{i+5i+4}) + a_{i+2i+6} + a_{i+6i+2}.\]
From this, it follows
\[
\begin{align*}
i_2(a) &= \|a\|^2 - 7\|a_{(1)}\|^2 + 2\|a_{(2)} + a_{(4)}\|^2 - 6\|a_{(4)}\|^2 \\
&= -6\|a_{(1)}\|^2 + \|a_{(3)}\|^2 + 3\|a_{(2)}\|^2 - 3\|a_{(4)}\|^2.
\end{align*}
\]

For \(i_0(a)\), using (2.3), one has
\[
\begin{align*}
i_0(a) &= -\sum_{i,j \in Z} \langle a(e_i), a(e_j) \times (e_j \times e_i) \rangle \\
&= \sum_{i,j \in Z} \langle a(e_i) \times e_j, a(e_j) \times e_i \rangle - \|a\|^2 - \sum_{i,j \in Z} a_{i_j}a_{i_j} + 2\sum_{i,j \in Z} a_{i_j}a_{i_j}.
\end{align*}
\]
This identity can be written in the following way
\[
\begin{align*}
i_0(a) &= i_2(a) - \|a\|^2 + 2\sigma_2(a) - \sum_{i \in Z} a_{i_i}^2 + 2\sum_{i \in Z} a_{i_i}^2 + \sum_{i,j \in Z, i \neq j} a_{i_i}a_{i_j} \\
&= i_2(a) - \|a\|^2 + 2\sigma_2(a) + \sigma_1(a)^2
\end{align*}
\]
(note that this is one identity given in [15] and displayed below). Finally, using the expressions for \(i_2(a), \sigma_2(a)\) and \(\sigma_1(a)^2\), it follows
\[
\begin{align*}
i_0(a) &= -6\|a_{(1)}\|^2 + 3\|a_{(2)}\|^2 + \|a_{(3)}\|^2 - 3\|a_{(4)}\|^2 - \|a\|^2 \\
&\hspace{1cm} + 6\|a_{(1)}\|^2 + \|a_{(2)}\|^2 - \|a_{(3)}\|^2 + \|a_{(4)}\|^2 - 7\|a_{(1)}\|^2 \\
&= 6\|a_{(1)}\|^2 + 3\|a_{(2)}\|^2 - \|a_{(3)}\|^2 - 3\|a_{(4)}\|^2.
\end{align*}
\]
The fourth identity in Lemma follows by using (2.2) in
\[
\sum_{i,j \in Z} \|e_i \times a(e_j)\|^2 = -\sum_{i,j \in Z} \langle a(e_j), e_i \times (e_i \times a(e_j)) \rangle.
\]
\[\square\]

From Lemma 3.1 and Lemma 3.2, it is direct to check the identities proved by Niedziałomski.

**Lemma 3.3** ([15]). For all \(a \in \mathfrak{gl}(7)\), the following relations hold:
\[
\begin{align*}
i_1(a) &= -i_0(a) + \|a\|^2 + 4\sigma_2(a) - \sigma_1(a)^2, \\
i_2(a) &= i_0(a) + \|a\|^2 - 2\sigma_2(a) - \sigma_1(a)^2.
\end{align*}
\]
In particular,
\[
i_1(a) - i_2(a) = -2i_0(a) + 6\sigma_2(a).
\]

Finally, it is interesting to note that, reciprocally, the norms of the \(G_2\)-components of an endomorphism \(a\) can be expressed in terms of its invariants. In fact, the identities in next lemma easily follow from Lemma 3.1, Lemma 3.2 and Lemma 3.3.

**Lemma 3.4.** For all \(a \in \mathfrak{gl}(7)\), it is satisfied
\[
\begin{align*}
\|a_{(1)}\|^2 &= \sigma_1(a)^2, \\
6\|a_{(2)}\|^2 &= i_0(a) + i_1(a) + i_2(a) = -6\sigma_2(a) + 3i_0(a) + 2i_1(a), \\
6\|a_{(3)}\|^2 &= 12\sigma_1(a)^2 - i_0(a) + i_2(a) = 12\sigma_1(a)^2 - 6\sigma_2(a) + i_0(a) + i_1(a), \\
6\|a_{(4)}\|^2 &= i_1(a).
\end{align*}
\]
4. $G_2$-structures

We recall briefly some facts about $G_2$-structures and display some results we will need later.

A $G_2$-structure on a Riemannian seven-manifold $(M, (\cdot, \cdot))$ is by definition a $G_2$-reduction $\mathcal{P}(M)$ of the orthonormal frame $SO(7)$-bundle $SO(M)$. This is equivalent to the existence of a global three-form $\varphi$ which may be locally written as in (2.1). For all $m \in M$, the tangent space $T M$ is then associated to the representation $\mathbb{R}^7$ of $G_2$. Hence, in each point of a Riemannian manifold with a $G_2$-structure, there are local orthonormal coframe fields $\{e_0, \ldots, e_6\}$ such that $\varphi$ is written as in (2.1), we call them Cayley coframes. The dual frame, denoted also by $\{e_0, \ldots, e_6\}$, of a Cayley coframe is called a Cayley frame.

When $M$ is equipped with a $G_2$-structure, one can also consider the two-fold vector cross product $\times : TM \times TM \to TM$ given by $(X \times Y, Z) = \varphi(X, Y, Z)$ [6].

If $M$ is equipped with a $G_2$-structure, then there always exists a $G_2$-connection $\tilde{\nabla}$ defined on $M$. Doing the difference $\tilde{\xi}_X = \tilde{\nabla}_X - \nabla_X$, where $\nabla_X$ is the Levi-Civita connection of $\langle \cdot, \cdot \rangle$, a tensor $\tilde{\xi}_X \in \mathfrak{so}(M)$ is obtained. Decomposing $\tilde{\xi}_X = (\tilde{\xi}_X)_{g_2^1} + (\tilde{\xi}_X)_{g_2^2}$, $(\tilde{\xi}_X)_{g_2^1} \in \mathfrak{g}_2$ and $(\tilde{\xi}_X)_{g_2^2} \in \mathfrak{g}_2^+$, a new $G_2$-connection $\nabla^{G_2}$, defined by $\nabla^{G_2}_X = \tilde{\nabla}_X - (\tilde{\xi}_X)_{g_2^2}$, can be considered.

Because the difference between two $G_2$-connections must be in $\mathfrak{g}_2$, $\nabla^{G_2}$ is the unique $G_2$-connection on $M$ such that its torsion $\xi_X = (\tilde{\xi}_X)_{g_2^2} = \nabla^{G_2}_X - \nabla_X$ is in $\mathfrak{g}_2^+$. $\nabla^{G_2}$ is called the minimal connection and $\xi$ is referred to as the intrinsic torsion of the $G_2$-structure on $M$ [5].

By one hand, as it is pointed out in [15], since $\xi_X \in \mathfrak{g}_2^+$, one has

$$\xi_X Y = A_{T(X)}(Y) = Y \times T(X),$$

where $T \in TM \otimes T^*M$ (see the description of $g_2^+$ given in Section 2).

On the other hand, the covariant derivative $\nabla \varphi$ satisfies $\nabla \varphi = -\xi \varphi$ and $\xi \rightarrow -\xi \varphi$ is a $G_2$-map. Thus the space of possible intrinsic torsions can be identified with the space $\mathfrak{X} \subseteq T^*M \otimes \Lambda^3 T^*M$ of covariant derivatives of $\varphi$. Consider now the $G_2$-map $r : \mathfrak{X} \to T^*M \times T^*M$ defined by [11, 12]

$$r(\alpha)(X, Y) = \frac{1}{4}\langle X, \omega_\alpha, Y, \cdot \rangle \varphi = \frac{1}{24}\langle X, \omega_\alpha, Y, \cdot \rangle \varphi.$$

Then an explicit expression [13] for $\xi$ is given by

$$\xi_X Y = -\frac{1}{3}\sum_{i \in \mathbb{Z}_7} r(\nabla \varphi)(X, e_i)e_i \times Y.$$

Thus, comparing with the identity $\xi_X Y = Y \times T(X)$, it is found [15]

$$T(X) = \frac{1}{3}\sum_{i \in \mathbb{Z}_7} r(\nabla \varphi)(X, e_i)e_i.$$

Note that $T_{ij} = \langle e_i, T(e_j) \rangle = \frac{1}{3}r(\nabla \varphi)(e_j, e_i)$.

The space $\mathfrak{X} = T^*M \otimes \mathfrak{g}_2^+ \subseteq T^*M \otimes (TM \otimes T^*M)$ of possible intrinsic torsions, under the action of $G_2$, is decomposed into four irreducible $G_2$-modules $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 \oplus \mathfrak{X}_3 \oplus \mathfrak{X}_4$. This gives place to a natural classification of types of $G_2$-structure. This was shown by Fernández...
and Gray, by using $\nabla \varphi$, in [6]. Using the composition of maps $\xi \rightarrow -\xi \varphi = \nabla \varphi \rightarrow r(-\xi \varphi) \rightarrow T$, it is derived $\mathfrak{X} \cong \mathfrak{gl}(TM)$, $\mathfrak{X}_1 \cong \mathbb{R} \text{Id}_{TM}$, $\mathfrak{X}_3 \cong S_0^2T^*M$, $\mathfrak{X}_2 \cong \mathfrak{g}_2$, and $\mathfrak{X}_4 \cong \mathfrak{g}_2^\perp$.

5. The different integral formulas

By using different tools, some authors have derived integral formulas for $G_2$-structures. Our purpose here is to display the correlations between them. The existence of theses correlations was indicated by Kiedziałomski in [15]. Firstly, it is shown the integral formula more recently proved.

**Theorem 5.1** ([15]). On a closed (i.e., compact without boundary) manifold $M$ equipped with a $G_2$-structure, the following integral formula holds

$$\frac{1}{6} \int_M s \text{vol}_M = \int_M -\frac{3}{2} i_0(T) + 6\sigma_2(T) \text{vol}_M,$$

where $s$ denotes the scalar curvature.

This is shown by deducing the divergence formula given by

$$\text{div} \sum_{i \in \mathbb{Z}_7} \xi_i e_i = -\frac{1}{6} s - \frac{3}{2} i_0(T) + 6\sigma_2(T) \tag{5.1}$$

(see Lemma 2.2, Lemma 4.1 and Lemma 4.2 in [15]). For the vector field $\sum_{i \in \mathbb{Z}_7} \xi_i e_i$, one has

$$\sum_{i \in \mathbb{Z}_7} \xi_i e_i = -p(T) = -\frac{1}{6}(p^* \varphi)^2, \tag{5.2}$$

where $p^* \varphi = * (\ast d \varphi \wedge \varphi) = -\ast (\ast d \varphi \wedge \ast \varphi)$ [11,12] and we have used the musical isomorphism $\langle \alpha^2, x \rangle = \alpha(x)$, for $\alpha \in T^*_mM$, $x \in T_mM$ and $m \in M$. In fact, considering a Cayley frame $\{e_0, \ldots, e_6\}$ (this frame will be also used as Cayley frame in the sequel),

$$\sum_{i \in \mathbb{Z}_7} \xi_i e_i = \sum_{i \in \mathbb{Z}_7} \xi_i \times T(e_i)$$

$$= \sum_{i \in \mathbb{Z}_7} (T_{i+1}e_i \times e_{i+1} + T_{i+2}e_i \times e_{i+2} + T_{i+3}e_i \times e_{i+3} + T_{i+4}e_i \times e_{i+4}$$

$$+ T_{i+5}e_i \times e_{i+5} + T_{i+6}e_i \times e_{i+6})$$

$$= \sum_{i \in \mathbb{Z}_7} (T_{i+1}e_{i+1} + T_{i+2}e_{i+2} - T_{i+3}e_{i+1} + T_{i+4}e_{i+2} - T_{i+5}e_{i+3} - T_{i+6}e_{i+2})$$

$$= -\sum_{i \in \mathbb{Z}_7} (T_{i+1}e_{i+1} + T_{i+2}e_{i+2} - T_{i+3}e_{i+1} + T_{i+4}e_{i+2} - T_{i+5}e_{i+3} - T_{i+6}e_{i+2}) e_i$$

$$= -p(T).$$

Finally, note that $p(T) = -\frac{1}{6} p(r(\nabla \varphi)) = \frac{1}{6}(p^* \varphi)^2$ (see [13]).

Now, taking Lemma 3.1 and 3.2 into account, it is obtained the following divergence equation and integral formula in terms of the $G_2$-components of $T$

$$\frac{1}{6} s = 9\|T_{(1)}\|^2 - \frac{3}{2} \|T_{(2)}\|^2 - \frac{3}{2} \|T_{(3)}\|^2 + \frac{15}{2} \|T_{(4)}\|^2 + \text{div} (p(T)). \tag{5.3}$$

As consequence, one has the following result.

**Theorem 5.2.** On a closed manifold $M$ equipped with a $G_2$-structure, the following integral formula holds

$$\frac{1}{6} \int_M s \text{vol}_M = \int_M 9\|T_{(1)}\|^2 - \frac{3}{2} \|T_{(2)}\|^2 - \frac{3}{2} \|T_{(3)}\|^2 + \frac{15}{2} \|T_{(4)}\|^2 \text{vol}_M. \tag{5.4}$$
By the fourth identity given in Lemma 3.2, it is obtained \( \| \xi \|^2 = 6 \| T \|^2 \). Taking this and
\[ p(T) = - \sum_{i \in \mathbb{Z}_7} \xi_i e_i \]
into account in follows next divergence equation
\[ s = 9 \| \xi(1) \|^2 - \frac{3}{2} \| \xi(2) \|^2 - \frac{3}{2} \| \xi(3) \|^2 + \frac{15}{2} \| \xi(4) \|^2 - 6 \div \sum_{i \in \mathbb{Z}_7} \xi_i e_i. \]
Hence one has next consequence.

**Corollary 5.3.** On a closed manifold \( M \) equipped with a \( G_2 \)-structure with intrinsic torsion \( \xi \), the following integral formula holds
\[ \int_M s \, \text{vol}_M = \int_M 9 \| \nabla \varphi \|^2 - \frac{3}{2} \| \nabla \varphi \|^2 - \frac{3}{2} \| \nabla \varphi \|^2 + \frac{15}{2} \| \nabla \varphi \|^2 \, \text{vol}_M. \]  

Now we will compare this identity with the integral formula given by Bor and Hernández Lamoneda.

**Theorem 5.4** ([2]). On a closed manifold \( M \) equipped with a \( G_2 \)-structure, the following integral formula holds
\[ \int_M s \, \text{vol}_M = \int_M 9 \| (\nabla \varphi)_{(1)} \|^2 - \frac{3}{2} \| (\nabla \varphi)_{(2)} \|^2 - \frac{3}{2} \| (\nabla \varphi)_{(3)} \|^2 + \frac{15}{2} \| (\nabla \varphi)_{(4)} \|^2 \, \text{vol}_M. \]  

Note the coincidence relative to the coefficient numbers. However, \( \| (\nabla \varphi)_{(i)} \|^2 \) is not equal to \( \| \xi(i) \|^2 \). What is happening? The answer is based in the conventions followed in [2] for the exterior product, the exterior derivative, etc. Such conventions are those ones fixed in [10]. Here we follow those ones fixed, for instance, in [0,16]. Because of this, one has \( \varphi = 3! \tilde{\varphi} \), where \( \tilde{\varphi} \) is the three-form considered in [2]. Below we will prove that \( \| (\nabla \varphi) \|^2 = 36 \| \xi \|^2 \). Therefore, \( \| (\nabla \varphi) \|^2 = \frac{1}{36} \| \nabla \varphi \|^2 = \| \xi \|^2 \) and formulas (5.5) and (5.6) are agree. Thus, Bor and Hernández Lamoneda are really right when they say intrinsic torsion in the introduction of [2].

Taking \( 3 T_{ji} = r(\nabla \varphi)(e_i, e_j) \) and Lemma 2.1 of [11] into account, \( \nabla \varphi \) can be expressed as
\[ \nabla \varphi = 3 \sum_{i,j \in \mathbb{Z}_7} T_{ji} e_i \otimes e_j \ast \ast \varphi. \]
This implies \( \| (\nabla \varphi) \|^2 = 9 \cdot 4 \cdot 6 \| T \|^2 = 36 \| \xi \|^2 \) as it was claimed.

In [4], Bryant has also given an integral formula for \( G_2 \)-structures. This is expressed in terms of \( \tau_0, \tau_1, \tau_2 \) and \( \tau_3 \) which are defined by
\[ d \varphi = \tau_0 \ast \varphi + 3 \tau_1 \wedge \varphi + \ast \tau_3, \]
\[ d \ast \varphi = 4 \tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi, \]
where \( \ast \) is the Hodge star operator induced by the volume form \( \text{vol}_M \). The function \( \tau_0 \) determines \( \xi(1) \), the one-form \( \tau_1 \) determines \( \xi(4) \), the two-form \( \tau_2 \in \Lambda^2_{(14)} T^* M \cong \mathfrak{g}_2 \) determines \( \xi(2) \) and the three-form \( \tau_3 \in \Lambda^3_{(27)} T^* M \cong S^3_0(T^* M) \) determines \( \xi(3) \). The volume form \( \text{vol}_M \), fixed by Bryant, is such that \( \text{vol}_M = - \text{vol}_M \). Hence \( \ast = - \ast \). But this is not a problem, because the corresponding norms are the same. Hence we will write
\[ d \varphi = \tau_0 \ast \varphi + 3 \tau_1 \wedge \varphi + \ast \tau_3, \]
\[ d \ast \varphi = 4 \tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi, \]
and the integral formula given by Bryant is displayed in next result.
Theorem 5.5 ([2]). On a closed manifold \( M \) equipped with a \( G_2 \)-structure, the following integral formula holds
\[
\int_M s \text{vol}_M = \int_M \frac{21}{8} \tau_0^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_2|^2 - \frac{1}{2}|\tau_3|^2 \text{vol}_M. \tag{5.7}
\]

It is denoted \(| \cdot |\), because it is used \((\cdot, \cdot)\). Likewise, the corresponding divergence formula is given by
\[
s = 12 d^* \tau_1 + \frac{21}{8} \tau_0^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_2|^2 - \frac{1}{2}|\tau_3|^2. \tag{5.8}
\]
Note that, despite of different volume forms, the one-form \( \tau_1 \) will be the same and, for the coderivatives, one has \( d^* = d^\flat \). We recall that \( d^* \tau_1 = -\text{div} \tau_1^2 \).

Lemma 5.6. The following identities hold
\[
\tau_0^2 = \frac{12^2}{T^2} \sigma_1(T)^2 = \frac{12^2}{T}\|T(1)\|^2, \quad |\tau_1|^2 = \frac{1}{4}\|p(T)\|^2 = \frac{3}{2}\|T(4)\|^2, \quad |\tau_3|^2 = 18\|T(3)\|^2, \quad |\tau_2|^2 = 18\|T(2)\|^2.
\]

Proof. In [11], the exterior derivative \( d\varphi \) is expressed as
\[
d\varphi = -3 \sum_{i \in \mathbb{Z}_7} (T_{i+2} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+2}^{*} \wedge e_{i+4}^{*} \wedge e_{i+5}^{*} \wedge e_{i+6}^{*}
+ 3 \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+1}^{*} \wedge e_{i+2}^{*} \wedge e_{i+4}^{*}
+ 3 \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+2}^{*} \wedge e_{i+3}^{*} \wedge e_{i+5}^{*}
+ 3 \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+3}^{*} \wedge e_{i+4}^{*} \wedge e_{i+6}^{*}
+ 3 \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+4}^{*} \wedge e_{i+5}^{*} \wedge e_{i+6}^{*} \tag{5.9}
\]
Denoting alternation by \(\text{alt} \), from \( (d\varphi)_{(1)} = \text{alt} (\nabla \varphi)_{(1)} = \tau_0 \ast \varphi = \frac{12}{7} \sigma_1(T) \ast \varphi \) and \( (d\varphi)_{(4)} = \text{alt} (\nabla \varphi)_{(4)} = 3 \tau_1 \wedge \varphi = -\frac{9}{7} p(T)^{\flat} \wedge \varphi \), it is obtained \( \tau_0 = \frac{12}{7} \sigma_1(T) \), \( \tau_1 = -\frac{1}{7} p(T)^{\flat} \), where it is used the musical isomorphism \( x^\flat (y) = \langle x, y \rangle \), for \( x, y \in T_m M, \ m \in M \). Therefore,
\[
\tau_3 = (d\varphi)_{(3)} = d\varphi - (d\varphi)_{(1)} - (d\varphi)_{(4)}
= -3 \sum_{i \in \mathbb{Z}_7} (T_{i+2} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+2}^{*} \wedge e_{i+4}^{*} \wedge e_{i+5}^{*} \wedge e_{i+6}^{*}
+ \frac{3}{2} \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+1}^{*} \wedge e_{i+2}^{*} \wedge e_{i+4}^{*}
+ \frac{3}{2} \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+2}^{*} \wedge e_{i+3}^{*} \wedge e_{i+5}^{*}
+ \frac{3}{2} \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+3}^{*} \wedge e_{i+4}^{*} \wedge e_{i+6}^{*}
+ \frac{3}{2} \sum_{i \in \mathbb{Z}_7} (T_{i+1} + T_{i+4} + T_{i+5} + T_{i+6}) e_i^{*} \wedge e_{i+4}^{*} \wedge e_{i+5}^{*} \wedge e_{i+6}^{*} \wedge e_{i+1}^{*}.
\]
By one hand, one has \( \tau_0^2 = \frac{12^2}{T^2} \sigma_1(T)^2 = \frac{12^2}{T}\|T(1)\|^2 \), and \( |\tau_1|^2 = \frac{1}{4}\|p(T)\|^2 = \frac{3}{2}\|T(4)\|^2 \).
On the other hand,
\[ |\tau_3|^2 = 9 \sum_{i \in \mathbb{Z}_7} \left( (T_{i+1} + T_{i+3})^2 + (T_{i+4} + T_{i+5})^2 + (T_{i+6})^2 \right) + 9 \sum_{i \in \mathbb{Z}_7} \left( (T_{i+1} - \frac{1}{7} \sigma_1(T))^2 + (T_{i+4} - \frac{1}{7} \sigma_1(T))^2 \right) + 9 \sum_{i \in \mathbb{Z}_7} \left( (T_{i+5} - \frac{1}{7} \sigma_1(T))^2 + (T_{i+6} + \frac{1}{7} \sigma_1(T))^2 \right) + 9.4 \sum_{i \in \mathbb{Z}_7} (T_{i+1} - \frac{7}{7} \sigma_1(T)) (T_{i+3} - \frac{4}{7} \sigma_1(T))
\]
\[ + 9.4 \sum_{i \in \mathbb{Z}_7} (T_{i+4} - \frac{1}{7} \sigma_1(T)) (T_{i+5} - \frac{1}{7} \sigma_1(T)) + 9.4 \sum_{i \in \mathbb{Z}_7} (T_{i+4} + \frac{1}{7} \sigma_1(T)) (T_{i+5} + \frac{1}{7} \sigma_1(T)). \]

Now, taking (2.7) into account, one has
\[ |\tau_3|^2 = 18 \cdot \frac{7}{2} \sum_{i \in \mathbb{Z}_7} \left( (T_{i+1} + T_{i+3})^2 + (T_{i+4} + T_{i+5})^2 + (T_{i+6})^2 \right) + 9.4 \sum_{i \in \mathbb{Z}_7} (T_{i+1} - \frac{1}{7} \sigma_1(T))^2 + 9.2 \sum_{i,j \in \mathbb{Z}_7, i \neq j} (T_{i} - \frac{1}{7} \sigma_1(T)) (T_{j} - \frac{1}{7} \sigma_1(T)) \]
\[ = 18 \|T(3)\|^2 + 18 \left( \sum_{i \in \mathbb{Z}_7} (T_{i} - \frac{1}{7} \sigma_1(T))^2 \right) = 18 \|T(3)\|^2. \]
Hence \(|\tau_3|^2 = |\ast \tau_3|^2 = 18 \|T(3)\|^2\).

In [11], it is shown an expression for \(d \ast \varphi = - \ast d \ast \varphi\). From such an expression one has
\[ (d \ast \varphi)_{(4)} = 4 \tau_1 \wedge \varphi = -2p(T)^3 \wedge \varphi, \]
the remaining \(G_2\)-component of \(d \ast \varphi\) is given by
\[ (d \ast \varphi)_{(2)} = \tau_2 \wedge \varphi = d \ast \varphi - (d \ast \varphi)_{(4)}. \]

Therefore,
\[ (d \ast \varphi)_{(4)} = -2 \sum_{i \in \mathbb{Z}_7} p(T)_i \ast (e_{i+1}^* \wedge e_{i+3}^* + e_{i+4}^* \wedge e_{i+5}^* + e_{i+6}^* \wedge e_{i+6}^*), \]
and
\[ (d \ast \varphi)_{(2)} = \sum_{i \in \mathbb{Z}_7} (3(T_{i+1} - T_{i+3}) - p(T)_i) \ast (e_{i+1}^* \wedge e_{i+3}^*) + \sum_{i \in \mathbb{Z}_7} (3(T_{i+4} - T_{i+5}) - p(T)_i) \ast (e_{i+4}^* \wedge e_{i+5}^*) + \sum_{i \in \mathbb{Z}_7} (3(T_{i+2} - T_{i+6}) - p(T)_i) \ast (e_{i+2}^* \wedge e_{i+6}^*). \]

Hence it is obtained
\[ \tau_2 = \frac{1}{2} \sum_{i \in \mathbb{Z}_7} (6(T_{i+1} - T_{i+3}) - 2p(T)_i) e_{i+1}^* \wedge e_{i+3}^* + \frac{1}{2} \sum_{i \in \mathbb{Z}_7} (6(T_{i+4} - T_{i+5}) - 2p(T)_i) e_{i+4}^* \wedge e_{i+5}^* + \frac{1}{2} \sum_{i \in \mathbb{Z}_7} (6(T_{i+2} - T_{i+6}) - 2p(T)_i) e_{i+2}^* \wedge e_{i+6}^*. \]
In summary, \( \tau_2 = 6T^b_{(2)} \), where \( T^b_{(2)}(x,y) = \langle x, T_{(2)}(y) \rangle \). From this, it is computed

\[
|\tau_2|^2 = \sum_{i \in \mathbb{Z}_7} (3(T_{i+1} + T_{i+3} - T_{i+3} - T_{i+1} - T_{i+5} + T_{i+5} - T_{i+6} + T_{i+6}) - p(T_i))^2 \\
+ \sum_{i \in \mathbb{Z}_7} (3(T_{i+2} + T_{i+4} - T_{i+4} - T_{i+2}) - p(T_i))^2.
\]

This implies \( |\tau_2|^2 = 36|T^b_{(2)}|^2 = 18\|T_{(2)}\|^2 \).

Replacing the values for \( \tau_i \) given in last Lemma, one found that Bryant’s formula \((5.7)\) is agree with the one given in \((5.4)\). Likewise, writing the divergence equation \((5.8)\) in terms of the components \( T_{(i)} \), it is checked that they are agree with \((5.3)\).

In \cite{7}, Friedrich and Ivanov considered types of \( G \)-structures which admit \( G \)-connection with totally skew-symmetric torsion. They showed that for \( G = G_2 \) such assumption is satisfied if and only if the \( G_2 \)-structure is of type \( X_1 \oplus X_3 \oplus X_4 \). In fact, they proved that for such a type of \( G_2 \)-structure, there is only one \( G_2 \)-connection with totally skew-symmetric torsion. Such a torsion \( \mathcal{T} \) is given (following the conventions fixed here) by

\[
\mathcal{T} = -\frac{1}{6}(d\varphi, \star \varphi) \varphi + \star d\varphi + 2 \star (p(T)^b \wedge \varphi).
\]

In order to compute \( |\mathcal{T}|^2 \), we consider the identity

\[
\star \mathcal{T} = -\frac{1}{6}(d\varphi, \star \varphi) \star \varphi + d\varphi + 2p(T)^b \wedge \varphi.
\]

From this, one has

\[
\star \mathcal{T} = -2\sigma_1(T) \star \varphi + (d\varphi)_{(1)} + (d\varphi)_{(3)} + (d\varphi)_{(4)} + 2p(T)^b \wedge \varphi.
\]

Therefore, it is finally obtained

\[
\star \mathcal{T} = -\frac{2}{7}(d\varphi, \star \varphi) \varphi + \star \tau_3 + \frac{1}{2}p(T)^b \wedge \varphi
\]

and

\[
|\mathcal{T}|^2 = |\star \mathcal{T}|^2 = \frac{4}{7}\sigma_1(T)^2 + |\tau_3|^2 + \|p(T)\|^2.
\]

This is equivalent to

\[
|\mathcal{T}|^2 = 4\|T_{(1)}\|^2 + 18\|T_{(3)}\|^2 + 6\|T_{(4)}\|^2.
\]

Friedrich and Ivanov have also deduced a divergence equation \cite{8} given by

\[
s = \frac{1}{18}(d\varphi, \star \varphi)^2 + 4\|p(T)\|^2 - \frac{1}{2}|\mathcal{T}|^2 + 6 \text{div}(p(T)).
\]

Note that \( 6|\mathcal{T}|^2 = \|\mathcal{T}\|^2 \). Now, taking \( (d\varphi, \star \varphi)^2 = \frac{122}{7}\sigma_1(T)^2 = \frac{122}{7}\|T_{(1)}\|^2 \), \( \|p(T)\|^2 = 6\|T_{(4)}\|^2 \) and equation \((5.14)\) into account, it can be seen that the divergence equations \((5.3)\) and \((5.15)\) are agree.
6. THE INTRINSIC TORSION IN TERMS OF THE EXTERIOR DERIVATIVES $d\varphi$ AND $d \star \varphi$

All information about the intrinsic torsion of a $G_2$ is contained in the covariant derivative $\nabla \varphi$. A useful alternative way to find such information is by means the exterior algebra. This is possible for $G_2$-structures by studying $d\varphi$ and $d \star \varphi$ as it is described in [14] (see Table I in [11]). Because we have already displayed all the necessary tools, now we will express the intrinsic torsion $\xi$ in terms of $d\varphi$ and of $d \star \varphi$.

For $T_{(1)}$ and $\xi_{(1)}$, from the expression (5.9) for $d\varphi$, one has

$$(d\varphi, \star \varphi) = 3 \sum_{i \in \mathbb{Z}_3} (T_{i+2, i+2} + T_{i+4, i+4} + T_{i+5, i+5} + T_{i+6, i+6}) = 12\sigma_1(T).$$

Hence, taking (2.5) into account, it is obtained

$$T_{(1)} = \frac{1}{84}(d\varphi, \star \varphi) \text{Id}_{TM}, \quad \xi_{(1)} X Y = \frac{1}{84}(d\varphi, \star \varphi) Y \times X,$$

For $\xi_{(4)}$, one considers the one-form $pd\varphi = \star(\star d\varphi \wedge \varphi) = -\star(\star d \star \varphi \wedge \star \varphi)$ and we know that $p(T) = \frac{1}{6}(pd\varphi)^2$ by (5.2). Hence, taking (2.4) into account, it is obtained

$$T_{(4)} = \frac{1}{36} A_{(pd\varphi) \varphi}, \quad \xi_{(4)} X Y = \frac{1}{36} Y \times A_{(pd\varphi) \varphi}(X) = \frac{1}{36} Y \times (X \times (pd\varphi)^2).$$

For $T_{(3)}$ and $\xi_{(3)}$, for all $i \in \mathbb{Z}_7$, one has $\xi_{(3)} X Y = Y \times T_{(3)}(X)$, where

$$T_{(3)} = \frac{1}{6} \sum_{i \in \mathbb{Z}_7} (d\varphi, e_i^* \wedge (e_{i+1} \star \varphi) - \frac{1}{4} \star \varphi) e_i \otimes e_i^*$$

$$+ \frac{1}{12} \sum_{i, j \in \mathbb{Z}_7, i \neq j} \left( d\varphi, e_i^* \wedge (e_{i+3} \star \varphi) + e_j^* \wedge (e_{i+3} \star \varphi) \right) (e_i \otimes e_j^* + e_j \otimes e_i^*).$$

All of this can be checked by using the identity (5.9) and taking (2.6) into account. Note that, for all $i \in \mathbb{Z}_7$,

$$(pd\varphi)_{i}^2 = 6p(T)_i = -(d\varphi, e_{i+1}^* \wedge (e_{i+3, j} \star \varphi) - e_{i+3}^* \wedge (e_{i+1} \star \varphi))$$

$$= -(d\varphi, e_{i+1}^* \wedge (e_{i+5} \star \varphi) - e_{i+5}^* \wedge (e_{i+4} \star \varphi))$$

$$= -(d\varphi, e_{i+2}^* \wedge (e_{i+6} \star \varphi) - e_{i+6}^* \wedge (e_{i+2} \star \varphi)).$$

Finally, for $T_{(2)}$ and $\xi_{(2)}$, one has $\xi_{(2)} X Y = Y \times T_{(2)}(X)$, where

$$T_{(2)} = \frac{1}{12} \sum_{i, j \in \mathbb{Z}_7, i \neq j} \left( d \star \varphi + \frac{1}{3} pd\varphi \wedge \star \varphi, \star (e_i^* \wedge e_j^*) \right) (e_i \otimes e_j^* - e_j \otimes e_i^*).$$

This can be checked by using the identities (5.10), (5.11) and (5.12), and taking (2.8) into account.

Conversely, if we use the map $k : TM \otimes T^* M \to \Lambda^4 T^* M$, defined by

$$a = \sum_{i, j \in \mathbb{Z}_7} a_{i, j} e_i \otimes e_j^* \longrightarrow k(a) = \sum_{i, j \in \mathbb{Z}_7} a_{i, j} e_i^* \wedge (e_{j} \star \varphi),$$

we will have $(d\varphi)_{(1)} = 3k(T_{(1)}), (d\varphi)_{(3)} = 3k(T_{(3)}),$ and $(d\varphi)_{(4)} = 3k(T_{(4)}).$ Note that $k(T_{(2)}) = 0.$
Now we consider the map $m : T^*M \otimes TM \rightarrow \Lambda^5 T^*M$, defined by

$$a = \sum_{i,j \in \mathbb{Z}_7} a_{i,j} e_i \otimes e_j^* \rightarrow m(a) = \sum_{i,j \in \mathbb{Z}_7} a_{i,j} \ast (e_i^* \wedge e_j^*).$$

It is obtained $(d\ast \varphi)_{(2)} = 3m(T(2))$ and $(d\ast \varphi)_{(4)} = 3m(T(4))$. Note that $m(T(1)) = m(T(3)) = 0$.

The map $k$ is closely related with map (2.17) in [4] given by $i : TM \otimes T^*M \rightarrow \Lambda^3 T^*M$,

$$a = \sum_{i,j \in \mathbb{Z}_7} a_{i,j} e_i \otimes e_j^* \rightarrow i(a) = \sum_{i,j \in \mathbb{Z}_7} a_{i,j} e_i^* \wedge (e_j \varphi)$$

In fact, it can be checked that

$$k(a) + \ast i(a^t) = \sigma_1(a) \ast \varphi,$$

or, equivalently,

$$\ast k(a) + i(a^t) = \sigma_1(a) \varphi,$$

for all $a \in TM \otimes T^*M$, where $a^t = \sum_{i,j \in \mathbb{Z}_7} a_{j,i} e_i \otimes e_j^*$. This lead us to claim that for two vectors $x, y \in T_m M$, $m \in M$, it is satisfied the identities

$$\ast (x^* \wedge (y \ast \varphi)) + y^* \wedge (x \ast \varphi) = \langle x, y \rangle \varphi,$$

and

$$x^* \wedge (y \ast \varphi) + \ast (y^* \wedge (x \ast \varphi)) = \langle x, y \rangle \ast \varphi,$$

where, for $x = \sum_{i \in \mathbb{Z}_7} x_i e_i$, one defines $x^* = \sum_{i \in \mathbb{Z}_7} x_i e_i^*$.

It is also interesting to have a look to map (2.18) in [4]. It is given by $j : \Lambda^3 T^*M \rightarrow TM \otimes T^*M$ with

$$j(\gamma) = \sum_{i,j \in \mathbb{Z}_7} (e_i \wedge e_j \varphi) \wedge \gamma e_i \otimes e_j^*. $$

The image of $j$ is the set of symmetric endomorphisms, i.e. $\text{Im} j = \mathfrak{X}_1 \oplus \mathfrak{X}_3$. For all $a \in \mathfrak{X}_1 \oplus \mathfrak{X}_3 \subset TM \otimes T^*M$, it is satisfied

$$j(i(a)) = -4a - 2\sigma_1(a) \text{Id}.$$

In particular, one has $j(i(\text{Id})) = -18 \text{Id}, i(\text{Id}) = 3 \varphi$ and $j(\varphi) = -6 \text{Id}$. On the other hand, for all $\gamma \in \mathfrak{X}_1 \oplus \mathfrak{X}_3 \subset \Lambda^3 T^*M$, it is satisfied

$$i(j(\gamma)) = 4 \gamma - 2(\gamma, \varphi) \varphi.$$

This is deduced by denoting $\gamma = i(a)$, checking $(\gamma, \varphi) = 3 \sigma_1(a)$ and using the previous identity for $j(i(a))$. By similar arguments, for all $a \in \mathfrak{X}_1 \oplus \mathfrak{X}_3 \subset TM \otimes T^*M$ and $\gamma \in \mathfrak{X}_1 \oplus \mathfrak{X}_3 \subset \Lambda^3 T^*M$, it can be deduced the identities

$$j(\ast k(a)) = -4a - 4\sigma_1(a) \text{Id}, \quad k(j(\gamma)) = 4 \ast \gamma + \frac{7}{3}(\gamma, \varphi) \ast \varphi.$$

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(Francisco Martín Cabrera) **DEPARTAMENTO DE MATEMÁTICAS, ESTADÍSTICA E INVESTIGACIÓN OPERATIVA, UNIVERSIDAD DE LA LAGUNA, 38200 LA LAGUNA, TENERIFE, SPAIN**

*Email address: fjmartincabrera@gmail.com*