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Fluctuation-dissipation relation, Maxwell-Boltzmann statistics, equipartition theorem, and stochastic calculus

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Abstract
We derive the fluctuation-dissipation relation and explore its connection with the equipartition theorem and Maxwell-Boltzmann statistics through the use of different stochastic analytical techniques. Our first approach is the theory of backward stochastic differential equations, which arises naturally in this context, and facilitates the understanding of the interplay between these classical results of statistical mechanics. Moreover, it allows to generalize the classical form of the fluctuation-dissipation relation. The second approach consists in deriving forward stochastic differential equations for the energy of an electric system according to both Itô and Stratonovich stochastic calculus rules. While the Itô equation possesses a unique solution, which is the physically relevant one, the Stratonovich equation admits this solution along with infinitely many more, none of which has a physical nature. Despite of this fact, some, but not all of them, obey the fluctuation-dissipation relation and the equipartition of energy.

1. Introduction

The classical papers by Einstein [1] and Langevin [2] devised two significant tools that have been extensively employed in the study of stochastic processes relevant in the field of statistical physics along the twentieth century. Despite the huge influence of these papers, it is nonetheless quite remarkable that their methods were actually anticipated by Bachelier in the field of mathematical finance [3]. These tools are, of course, partial differential equation methods [1] and stochastic analytical methods [2, 4]. Their connection has been well established and can be found, for instance, in [5]. In a sense, partial differential equation methods have been more popular in the study of some aspects of statistical physics [6]. Nevertheless, stochastic methods yield additional insights into statistical mechanical systems that could be, at times, even more enlightening [7]. The aim of this work is to approach the classical fluctuation-dissipation relation with stochastic methods that yield additional information to the classical ones [2, 7].

In section 2 we approach the fluctuation-dissipation relation through the Langevin model for the random dispersal of a Brownian particle. According to this model the position $X_t$ of such a particle obeys Newton second law [8]

$$m \frac{d^2X_t}{dt^2} = -\gamma \frac{dX_t}{dt} + \sigma \xi_t,$$

$$X_{t \mid t=0} = X_0,$$

$$\frac{dX_t}{dt} \bigg|_{t=0} = V_0,$$

where $\xi_t$ is Gaussian white noise and $m, \gamma, \sigma > 0$ denote respectively the mass of the particle, the viscosity of the medium in which the particle is immersed, and the strength of the thermal fluctuations. Given that we are discussing the one dimensional problem, the stochastic process $X_t$ encodes the (signed) distance between the
Brownian particle and the origin, and it is therefore a real valued process. The random variables $X_0$ and $V_0$ denote the initial position and velocity of the particle. The traditional approach focuses on analyzing the long time behavior of the forward stochastic differential equation for the velocity of this particle

$$V_t = \frac{dX_t}{dt};$$

and the fluctuation-dissipation relation arises as a consequence of Maxwell-Boltzmann statistics in this long time limit. Therefore it is natural to study this problem posed backwards in time, with these statistics imposed on the final condition. This is the program carried out in section 2. However, as natural as it may sound to consider the final value problem for the velocity, the naïf approach to this violates physical causality and the theory of backward stochastic differential equations has to be invoked, as explained in appendix A.

The fluctuation-dissipation relation does not only manifest itself in the random dispersal of a particle subjected to a heat bath and embedded in a viscous medium. Another paradigmatic example of this relation arises in electric circuits, in which the stochasticity enters through the fluctuations in the electric current known as Johnson noise [7, 9, 10]. In this system, the fluctuation-dissipation relation appears as a consequence of the equipartition of energy in the long time limit. While building a theory à la Langevin for this phenomenon requires considering a stochastic differential equation for the electric current [7], it is natural to directly consider the equation for the energy instead. This is the approach developed in section 3, where we explore the technical difficulties that such a method implies. In particular, the equation for the current presents additive noise, and the equation for the energy shows multiplicative noise, and therefore it must be interpreted. While the Itô interpretation yields no problems, the Stratonovich interpretation is affected by an infinite multiplicity of solutions, of which only one has physical nature. We analyze how some of these spurious solutions still obey the fluctuation-dissipation relation, while others do not. The link of the results in this section with one of the most classical versions of the Itô versus Stratonovich dilemma is shown in the following section 4.

Finally, in section 5, we draw our main conclusions. On one hand, we discuss the possible interest of the present stochastic methods to address these physical problems. On the other hand, we comment on how our results could fit in the physics literature. All in all, we hope we can offer new viewpoints on such a classical and important result as the fluctuation-dissipation relation.

2. A backward stochastic differential equations approach to the fluctuation-dissipation relation

Before embarking into technical details we mention that the theory concerning backward stochastic differential equations can be consulted, for instance, in [11–13]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space completed with the $P-$null sets in which a Wiener Process $\{W_t\}_{t \geq 0}$ is defined; moreover assume $\mathcal{F}_T \supseteq \sigma(\{W_s; 0 \leq s \leq t\})$. This is a classical setting in which stochastic differential equations can be posed [14, 15]. Throughout this section we consider the random dispersal of a Brownian particle that will be assumed to be of unit mass, that is $m = 1$, without loss of generality. We consider the backward stochastic differential equation

$$dV_t = -\gamma V_t dt + \sigma_t dW_t,$$

where $0 \leq t \leq T$ and $F$ is $\mathcal{F}_{T-}$ measurable, that is, the fluctuations that take place in $[0, T]$ affect $F_t$, what is just a consequence of physical causality. Note that this is just a final value problem for the velocity of the particle, defined by equation (2), whose position is described by equation (1), but with the amplitude of the fluctuations promoted from constant to stochastic process, what is necessary to solve the problem. In other words, our plan is to solve the Langevin equation backwards in time for both the velocity $V_t$ and the diffusion coefficient $\sigma_t$. At this point, we mention to the reader the discussion in appendix A, which informally describes the nontrivial character of the theory of backward stochastic differential equations. Solving a single backward stochastic differential equation for two stochastic processes is key for keeping the adaptability of the solutions [13], what in turn guarantees physical causality (i.e. guarantees that future fluctuations do not affect the present). This is the main reason that underlies the consideration of the amplitude of the fluctuations as a stochastic process rather than a constant: it is a necessary fact to solve the backward stochastic differential equation. As strange as it may sound for newcomers to the field, and contrary to what happens to forward stochastic differential equations, a single backward stochastic differential equation has to be solved for two stochastic processes in order to keep the adaptability of the solutions, what in turn implies physical causality. This computation will additionally show that assuming the constancy of $\sigma_t$, as in [7] is not strictly needed, at least from the mathematical viewpoint, although this result can be recovered as a particular case in the present setting. Let us also note that many generalizations of the classical Langevin equation have been studied along the years, since classical developments [16, 17] to more recent extensions [18, 19], but however none of them, to the best of our knowledge, includes the
backward stochastic differential equation approach considered herein. On the other hand, the use of a stochastic amplitude of fluctuations seems not to be frequently used in this field, while stochastic volatility models are popular in finance [20], a field in which backward stochastic differential equations have been commonly employed too [12]. From the physical viewpoint, one of the factors that may yield a stochastic amplitude of fluctuations is a fluctuating temperature.

Furthermore assume $F \in L^2(\Omega)$. Following the theory developed in [13] we know equation (3) possesses a unique solution; moreover the stochastic process $V_t$, that is, the velocity of the Brownian particle, admits the following explicit representation:

$$V_t = e^{\gamma(T-t)} \mathbb{E}[F|\mathcal{F}_t].$$

Clearly, the fluctuation-dissipation relation can only be established through the explicit computation of the diffusion $\sigma_t$. We complete this calculation using Malliavin calculus [21] and proceed in two steps; first we make the change of variables

$$U_t = e^{\gamma(T-t)} V_t,$$

and therefore

$$dU_t = \gamma e^{\gamma(T-t)} V_t + e^{\gamma(T-t)} dV_t = \gamma e^{\gamma(T-t)} V_t - \gamma e^{\gamma(T-t)} V_t + e^{\gamma(T-t)} \sigma_t dW_t,$$

thus

$$dU_t = e^{\gamma(T-t)} \sigma_t dW_t,$$

or equivalently

$$U_t = U_0 + \int_0^t e^{\gamma(T-t)} \sigma_t dW_t$$

by the zero mean property of the Itô integral. Our second step is to use the generalization of the Clark-Ocone formula for $L^2(\Omega)$ random variables [21] to find

$$e^{\gamma(T-t)} \sigma_t = \mathbb{E}[D_tF|\mathcal{F}_t],$$

where $D_tF$ is the Malliavin derivative of $F$, so we may conclude

$$\sigma_t = e^{\gamma(T-t)} \mathbb{E}[D_tF|\mathcal{F}_t].$$

Summarizing, we have found the unique solution pair to the backward stochastic differential equation

$$V_t = e^{\gamma(T-t)} \mathbb{E}[F|\mathcal{F}_t],$$

$$\sigma_t = e^{\gamma(T-t)} \mathbb{E}[D_tF|\mathcal{F}_t].$$

To recover the classical results we need to assume that the velocity is Maxwell-Boltzmann distributed at the terminal time, that is

$$F \sim \mathcal{N}(0, k_B\tau),$$

where $k_B$ is the Boltzmann constant and $\tau > 0$ is the absolute temperature. Furthermore assume the representation of this random variable by means of an Itô integral

$$F = \sqrt{k_B\tau} \int_0^T \psi_t dW_t \left( \int_0^T \psi_s^2 ds \right)^{1/2},$$

where $\psi_t \in L^2(0, T)$ is an arbitrary deterministic function, to find

$$V_t = \sqrt{k_B\tau} e^{\gamma(T-t)} \int_0^t \psi_t dW_t \left( \int_0^T \psi_s^2 ds \right)^{1/2},$$

(5a)
Note that expressions (4a)–(4b) and (5a)–(5b) show that drag and fluctuation are interrelated, and in this sense constitute a generalization of the classical fluctuation-dissipation relation. In particular, they can be considered as an out of equilibrium extension of this relation; one that can be established even in those cases in which an equilibrium is never reached, see appendix B. On the other hand these expressions reveal that the classical fluctuation-dissipation relation is not recovered by simply assuming an approach to equilibrium driven by linear viscous damping and Maxwell-Boltzmann statistics once there, and more assumptions are necessary. To recover it let us moreover assume the Ornstein-Uhlenbeck form

\[ \psi_t \propto e^{\gamma(t-T)} \]

to obtain

\[ V_t = \int_0^t e^{\gamma(t-T)} \, dW_t \]
\[ \sigma_t = \frac{e^{\gamma(t-T)}}{1 - e^{-2\gamma T}} \]

Then we finally conclude

\[ V_t = \int_0^t e^{\gamma(t-T)} \, dW_t \]
\[ \sigma_t = \frac{1}{1 - e^{-2\gamma T}} \]

where the convergence is, obviously, uniform in \( t \). So we have recovered the classical results after assuming that, not only the approach to equilibrium, but also the dynamics once there, are of Ornstein-Uhlenbeck form; also, that the equilibrium is governed by Maxwell-Boltzmann statistics and happens in the distant future.

Another example of the use of backward stochastic differential equations to detect the connections between drag and stochasticity, or in other words fluctuation–dissipation relations, is illustrated in appendix B. There we consider the case of a Brownian motion with drift (including the zero drift case), that cannot be treated via the traditional approach of [7] due to the lack of stationary distribution for such a process. As shown in this appendix, the present approach is still able to deal with these cases and unveil the presence of those relations.

3. Energy, power, and forward stochastic differential equations

Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a filtered probability space completed with the \( \mathbb{P} \)-null sets in which a Wiener process \( \{B_t\}_{t \geq 0} \) is defined; moreover assume \( \mathcal{F}_t \supseteq \sigma(\{B_s\mid 0 \leq s \leq t\}) \). Now let us consider the circuit equation

\[ L \, dB_t = -R I_t \, dt + V \, dB_t, \]
\[ I_{t=0} = I_0 \]

for a rigid wire loop, where \( L, R, V > 0 \) are, respectively, the self-inductance, the resistance, and the amplitude of the thermal fluctuations. The fact that \( V \) does not vanish reflects the presence of Johnson noise in the circuit. The initial condition \( I_0 \in L^1(\Omega) \) is a \( \mathcal{F}_0 \)-measurable random variable. Under these conditions it is clear that this equation has a unique strong solution that is global in time [14, 15]. The approach developed in [7] consists in solving this forward stochastic differential equation in terms of the Ornstein-Uhlenbeck process, and then imposing that the energy of the circuit, which is given by

\[ E_t = \frac{1}{2} L I_t^2, \]

obeys the equipartition theorem in the long time limit. Since, in this case, the fluctuation-dissipation relation could be seen as a consequence of the interplay of Langevin dynamics and equipartition of energy, it seems natural to work directly with the energy instead of with the current. We can use stochastic calculus to find the forward stochastic differential equation that \( E_t \) obeys, but at this moment we have to confront an old dilemma.
On one hand, we can select Itô calculus \([22, 23]\) to find
\[
dE_t = \frac{V^2}{2L} dt - \frac{2}{L} E_t dt + \sqrt{\frac{V^2}{L}} E_t dB_t;
\]
on the other hand, if the selection is Stratonovich calculus \([24]\) we get
\[
dE_t = -2 \frac{R}{L} E_t dt + \sqrt{\frac{V^2}{L}} E_t dB_t;
\]
where we have considered the two interpretations of noise that have been historically more important by far. Actually, this double approach was studied before in the context of the random dispersal of the Brownian particle. In \([25]\) it is argued that the Stratonovich approach is superior to the Itô approach. In \([26]\), however, the equality of both approaches from the methodological viewpoint is discussed, but the preference for the Stratonovich one in those physical systems in which the fluctuations are external, such as in the cases studied in the present work, is concluded (we use the term external fluctuations in the sense of van Kampen, as discussed in appendix A). The results of this article were supported in \([27]\), where the preference towards the Stratonovich interpretation in continuous physical systems, again such as the ones studied herein, is mentioned. On the contrary, in \([28]\), the simpler character of the Itô interpretation in precisely this problem is defended. We shall further elaborate those arguments from now on. But before moving on to that goal, let us mention that there is a formula to equation (8) we would be tempted to say that it is equivalent to equation (8). However, this formula requires the continuously differentiability of the diffusion term, a fact that does not hold in the present case.

Indeed, such a formal application would change the number solutions as shown in \([28, 30]\) and the remainder of this section.

Let us start with equation (8), which admits the explicit solution
\[
E_t = \frac{1}{2} L \left[ e^{-(R/L)t} \sqrt{\frac{2E_0}{L}} + \frac{V}{L} \int_0^t e^{(R/L)(t-s)} dB_s \right]^2,
\]
a fact that is a direct consequence of the Itô stochastic calculus rules, indeed
\[
dE_t = LL dI_t + \frac{1}{2} L (dI_t)^2,
\]
where
\[
I_t = e^{-(R/L)t} \sqrt{\frac{2E_0}{L}} + \frac{V}{L} \int_0^t e^{(R/L)(t-s)} dB_s
\]
is the solution to equation (7). Moreover the solution is unique by the Watanabe-Yamada theorem \([31]\) as was claimed in \([28]\). Also, one can see that the expected energy obeys the ordinary differential equation
\[
\frac{d\mathbb{E}(E_t)}{dt} = \frac{V^2}{2L} - 2 \frac{R}{L} \mathbb{E}(E_t),
\]
where we have used the zero mean property of the Itô integral and the linearity of both drift and expectation; this is an ordinary differential equation that can be solved to yield
\[
\mathbb{E}(E_t) \rightarrow \frac{V^2}{4R} \exp\left(-2 \frac{R}{L} t\right)
\]
of course, the same result can be derived from (10) by means of the Itô isometry \([15]\), see \([28]\). Now we apply the equipartition theorem
\[
\frac{V^2}{4R} = \lim_{t \to \infty} \mathbb{E}(E_t) = \frac{1}{2} k_B T,
\]
so we conclude
\[
V = \sqrt{2 k_B T R},
\]
in perfect agreement with the result that is derived using the current rather than the energy \([7]\).

On the other hand, the Watanabe-Yamada theorem is not applicable to Stratonovich stochastic differential equations \([30]\). This theorem implies that diffusion terms with \(1/2 - \text{Hölder} \) continuity, or more regularity, do not jeopardize the uniqueness of solution to Itô stochastic differential equations; but counterexamples to this fact for Stratonovich stochastic differential equations are known \([30]\). As a consequence of this, equation (9) possesses an infinite number of solutions apart from (10) as proven in \([28]\). Now consider the family of solutions...
\[
E_t = \frac{1}{2} \left[ e^{-(R/L)t} \sqrt{\frac{2E_0}{L}} + V \int_0^t e^{(R/L)(t-\tau)} \, dB_\tau \right]^2 I_{t < T_0(\omega)} \\
+ \frac{V^2}{2L} \sum_{n=1}^N \left[ \int_0^t \mu_n + T_n(\omega) e^{(R/L)(t-\tau)} \, dB_\tau \right]^2 I_{\mu_n + T_n(\omega) < t < T_{n+1}(\omega)} \\
+ \frac{V^2}{2L} \int_0^t \mu_n + T_n(\omega) e^{(R/L)(t-\tau)} \, dB_\tau \right]^2 I_{t > \mu_n + T_n(\omega)} (11)
\]

for any set \( \{\mu_n\}_{n=1}^{N+1}, N = 1, 2, \ldots \), of almost surely non-negative, \( L^0(\Omega) \), and \( F_{T_n(\omega)} \)-measurable random variables, where

\[
T_0 := \inf \{ t > T_{n-1} + \mu_{n-1} + \lambda_{n-1} : E_t = 0 \}, \quad n = 1, 2, \ldots, N + 1,
\]

with \( T_0 := 0 =: \mu_0 \) and where \( \{\lambda_n\}_{n=0}^N \) is an arbitrary set of \( a.s. \) positive (including \( \lambda_0 \)), \( E_0 = 0 \) is allowed, \( L^0(\Omega) \), and \( F_{T_n(\omega)} \)-measurable random variables. These solutions are built by locking down the solution (10) at zero for random times following the procedure outlined in [28]; however all of these concrete solutions are new and were not previously reported. In particular, the random times \( \mu_n \)'s denote the duration of the lock downs at zero, while the random times \( \lambda_n \)'s are needed in order to separate consecutive lock downs; without them, given the definition of the passage times \( T_n \)'s, the solution would never leave \( \{0\} \) once there.

**Remark 3.1.** Note that the solutions in (11) depend on the passage times \( T_n \)'s, which depend in turn on the solution. This might create the appearance of a circular definition of them. However, the solutions are sequentially defined, and therefore free from this problem. Another way or realizing this is via the alternative definition of the passage times:

\[
T_1 := \inf \{ t > \lambda_0 : \sqrt{2L} E_0 + V \int_0^t e^{(R/L)\tau} \, dB_\tau = 0 \}, \\
T_n := \inf \{ t > T_{n-1} + \mu_{n-1} + \lambda_{n-1} : \int_0^t \mu_n + T_n(\omega) e^{(R/L)\tau} \, dB_\tau = 0 \}, \quad n = 2, 3, \ldots, N + 1.
\]

Written in this manner, it is clear that the only source of stochasticity that affects the passage times is the Brownian motion \( B_\tau \), and so happens to the solutions. In consequence, the solutions \( E_t \) are well defined. Note that exactly the same remark applies to the solutions \( E_t \) that will be introduced later in this section. In this second case we will continue to use the same notation as before, as it is comparatively easier to handle with.

Now we select a subclass of this family of solutions and analyze its behavior for long times.

**Theorem 3.2.** Let \( E_t \) be as in (11) and assume \( \{\lambda_n\}_{n=0}^N \) and \( \{\mu_n\}_{n=1}^{N+1} \) are finite \( a.s. \). Then

\[
\mathbb{E}(E_t) \sim \frac{V^2}{4R} \quad t \to \infty.
\]

**Proof.** The proof proceeds in four steps of which the first three are: samplewise convergence, boundedness in \( L^1(\Omega) \), and uniform integrability of a suitably chosen process. The fourth and final step allows us to conclude by means of the Vitali convergence theorem.

First note that the stochastic process

\[
E_t = \frac{V^2}{2L} \left[ \int_0^t \mu_n + T_n(\omega) e^{(R/L)(t-\tau)} \, dB_\tau \right]^2 I_{t > \mu_n + T_n(\omega)}, \\
= \frac{1}{2} \left[ e^{-(R/L)t} \sqrt{\frac{2E_0}{L}} + V \int_0^t e^{(R/L)(t-\tau)} \, dB_\tau \right]^2 I_{t < T_0(\omega)} \\
+ \frac{V^2}{2L} \sum_{n=1}^N \left[ \int_0^t \mu_n + T_n(\omega) e^{(R/L)(t-\tau)} \, dB_\tau \right]^2 I_{\mu_n + T_n(\omega) < t < T_{n+1}(\omega)} \\
\xrightarrow{t \to \infty} 0 \quad a.s.
\]

since all the \( T_i \)'s, \( i = 1, 2, \ldots \), are finite \( a.s. \) [28]. This completes the first step, that is, the samplewise convergence of the target process.
To get the boundedness in $L^2(\Omega)$ we estimate

$$E_t = \frac{V^2}{2L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)}$$

$$\leq 2E_0 e^{-2(R/L)t} + \frac{V^2}{L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2$$

$$+ \sum_{n=1}^{N} \frac{V^2}{2L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)}$$

and therefore

$$E \left\{ E_t - \frac{V^2}{2L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} \right\}$$

$$\leq 2E_0 e^{-2(R/L)t} + \frac{V^2}{2L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2$$

$$+ \sum_{n=1}^{N} \frac{V^2}{4R} \left( 1 - e^{-2R/L} \right)$$

$$\leq 2E_0 \left( 1 + \frac{N}{2} \right)$$

$$< \infty,$$

where the last bound is of course uniform in $t$, and where we have used the Itô isometry along with the string of equalities and final inequality

$$E \left\{ \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} \right\}$$

$$= E \left\{ \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \mid \mathcal{F}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} \right\} \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)}$$

$$= E \left\{ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{2R/L}(t-1) \, ds \mid \mathcal{F}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} \right\} \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)}$$

$$= \frac{L}{2R} E \left\{ (1 - \exp \left[ \frac{2R}{L} (\mu_n + \tau_{T_{x_0}}(\omega) - t) \right]) \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} \right\}$$

$$\leq \frac{L}{2R},$$

which follow from the tower property of conditional expectation and the Itô isometry.

For the uniform integrability consider $A \in \mathcal{F}$ with $P \{ A \} = \delta$. We have the estimate

$$E \left\{ E_t - \frac{V^2}{2L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \mathbf{1}_{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} \mid A \right\}$$

$$\leq 2E_0 \delta^2 e^{-2(R/L)t} + \frac{V^2}{L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \delta$$

$$+ \sum_{n=1}^{N} \frac{V^2}{2L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \delta$$

$$\leq 2E_0 \delta^2 \frac{V^2}{2R} \delta^{1/2}$$

$$+ \sum_{n=1}^{N} \frac{V^2}{2L} \left[ \int_0^{\tau_{x_0} + \tau_{T_{x_0}}(\omega)} e^{(R/L)(t-1)} \, dB_t \right]^2 \delta$$

$$\leq 2E_0 \delta^2 \frac{V^2}{2R} \delta^{1/2} + \sum_{n=1}^{N} \frac{\sqrt{3} V^2}{4R} \delta^{1/2}$$

$$= \left[ 2E_0 \delta^2 + \frac{\sqrt{3} V^2}{2R} \left( 1 + \frac{N}{2} \right) \right] \delta^{1/2},$$
which follows from the Hölder inequality
\[ E(\sigma_0 \lambda_t) \leq E(\sigma_0)^{1/2} E(\lambda_t)^{1/2} \]
\[ = E(\sigma_0)^{1/2} E(\lambda_t)^{1/2} \]
\[ = E(\sigma_0)^{1/2} \mathbb{P}(\mathcal{A})^{1/2}, \]
and analogous Hölder inequalities for the other terms, the tower property, and the inequalities
\[ E\left\{ \left( \int_0^t e^{(R/L)(t-\varepsilon)} dB_t \right)^4 \right\} = \frac{3L^2}{4R^2} (1 - e^{-2R/L})^2 \]
\[ \leq \frac{3L^2}{4R^2} \]
and
\[ E\left\{ \left( \int_{t_n+T_0}\omega} e^{(R/L)(t-\varepsilon)} dB_t \right)^4 \right\} \leq \frac{3L^2}{4R^2}. \]
since respectively
\[ \int_0^t e^{(R/L)(t-\varepsilon)} dB_t \sim \mathcal{N}\left( 0, \frac{L}{2R} \right) \]
and
\[ \int_{t_n+T_0}\omega} e^{(R/L)(t-\varepsilon)} dB_t \sim \mathcal{N}\left( 0, \frac{L}{2R} \right) \]
Then consequently
\[ E\left\{ E_t - \frac{V^2}{2L} \int_{t_n+T_0}\omega} e^{(R/L)(t-\varepsilon)} dB_t \right\} \leq \varepsilon \]
for all \( \varepsilon > 0 \), after choosing
\[ \delta = \frac{\varepsilon^2}{2E(\sigma_0)^{1/2} + \frac{3V^2}{4R} (1 + \frac{N}{2})} \]
Next, by the Vitali convergence theorem \([32]\)
\[ \lim_{t \to \infty} E\left\{ E_t - \frac{V^2}{2L} \int_{t_n+T_0}\omega} e^{(R/L)(t-\varepsilon)} dB_t \right\} = 0. \]
Finally, this implies
\[ \lim_{t \to \infty} E(\sigma_t) = \frac{V^2}{2L} \int_{t_n+T_0}\omega} e^{(R/L)(t-\varepsilon)} dB_t \int_{1 \geq \mu_n + T_0(\omega)} \left( \int_{t_n+T_0}\omega} e^{(R/L)(t-\varepsilon)} dB_t \right) \leq \varepsilon \]
by the tower property, the Itô isometry, and the dominated convergence theorem. \(\square\)
Remark 3.3. Note that the proof of the theorem actually implies the long time asymptotic behavior
\[
E_t \sim \frac{V^2}{2L} \left[ \int_{\mu'_{n+1}+T_n+T_{n+1}} e^{(R/L)(t-\tau)} d\mu(t) \right]^2 I_{\tau < \mu'_{n+1}+T_n+T_{n+1}}(\omega),
\]
understood in the sense of $L^1(\Omega)$.

Remark 3.4. Note that the theorem states the convergence
\[
\mathbb{E}(E_t) \xrightarrow{t/\tau} \frac{V^2}{4R},
\]
but its proof does not imply the stronger convergence
\[
\mathbb{E}\left( E_t - \frac{V^2}{4R} \right) \xrightarrow{t/\tau} 0.
\]
If it were so, that would imply that $E_t$ becomes deterministic asymptotically in time, in contradiction to the previous remark, which shows that it never stops fluctuating, not even asymptotically.

Note that all of these solutions are not physical, as they imply a zero energy, and therefore zero current, during some intervals of time, despite the presence of non-vanishing thermal fluctuations. But, in spite of their unphysical character, their long time behavior is still compatible with the equipartition theorem provided we assume $V = \sqrt{2k_B}\tau R$, which is nothing but the fluctuation-dissipation relation.

Let us now move to a second family of solutions to equation (9), in particular to the one given by the explicit formula
\[
E_t = \frac{1}{2L} \left[ \frac{V^2}{2L} + \frac{V}{L} \int_{\mu'_{n+1}}^{\mu'_{n+1}+T_n+T_{n+1}} e^{(R/L)(t-\tau)} d\mu(t) \right]^2 I_{\tau < \mu'_{n+1}+T_n+T_{n+1}}(\omega)
\]
for any set $\{\mu'_{n+1}, n \in \mathbb{N}\} \cup \{\mathbb{N}\}$, of almost surely non-negative, $L^1(\Omega)$, and $F_{\mu'_{n+1}}(\omega)$ measurable random variables, where
\[
\tau_n = \inf\{ t > \tau_n : \tau_{n-1} + \mu'_{n-1} + \lambda'_{n-1}; E_t = 0 \}, \quad n = 1, 2, \cdots, N + 1,
\]
with $\tau_0 := 0 := \mu'_{-1}$, and where $\{\lambda'_{n+1} n \in \mathbb{N}\}$ is an arbitrary set of a.s. positive (including $\lambda'_{0}$, so the initial condition $E_0 = 0$ is allowed), $L^1(\Omega)$, and $F_{\mu'_{n+1}}(\omega)$ measurable random variables. Note that some of these solutions are also new and were not reported in [28] (precisely, all the cases with $E_0 = 0$), but they have been built using the same philosophy. As before, the random times $\mu'_{n+1}$ denote the duration of the lock downs at the origin, while the random times $\lambda'_{n}$ serve as a separation between consecutive lock downs.

To carry out the long time analysis assume moreover that $N < \infty$ and $\{\lambda'_{n+1} n \in \mathbb{N}\}$ and $\{\mu'_{n+1} n \in \mathbb{N}\}$ are finite almost surely. Because every stopping time $\tau_{n+1}$, with $i = 1, 2, \cdots, N$, is finite a.s. [28], we deduce the long time behavior
\[
E_t \xrightarrow{t/\tau} 0 \quad \text{almost surely}
\]
for whatever absolute temperature $\tau$; consequently these solutions lead to zero energy, and consequently zero current, in the long time with probability one, despite of the presence of non-vanishing thermal fluctuations. Thus we conclude that these solutions are unphysical, and also that both the equipartition theorem and fluctuation-dissipation relation are meaningless for them.

To finish this section we would like to emphasize that these results are not a consequence of strictly considering the energy of the circuit, but could also be reached from the dynamics of other physical quantities. The power dissipated by the circuit is given by
\[
D_t = R I_t^2
\]
and, as we did before for the energy, we can use stochastic calculus to find the forward stochastic differential equation it obeys. If we select $b_0$ calculus we find
\[
dD_t = \frac{RV^2}{L^2} dt - 2 \frac{R}{L} D_t dt + 2 \frac{RV^2}{L^2} D_t dB_t;
while if the selection is Stratonovich calculus we get
\[ dD_t = -2\frac{R}{L} D_t \, dt + 2\sqrt{\frac{R V^2}{L^2}} \, d\omega_t. \]

On the other hand, the fluctuation-dissipation relation implies
\[ \lim_{t \to \infty} \mathbb{E}(D_t) = \lim_{t \to \infty} \frac{2R}{L} \mathbb{E}(E_t) = \frac{k_B \tau R}{L}, \]
a result that is achievable from the unique solution of the Itô equation, just as we did before in the case of the energy. Also note that arguing as in theorem 3.2 we can find infinitely many solutions to the Stratonovich forward stochastic differential equation that fulfill the convergence
\[ \mathbb{E}(D_t) \to \frac{RV^2}{L^2}, \]
and therefore the equipartition theorem whenever the fluctuation-dissipation relation \( V = \sqrt{2k_B \tau R} \) holds. But we can find as well infinitely many solutions that fulfill
\[ D_t \to 0 \quad \text{almost surely} \]
for any triplet of positive parameters \( \{L, V, R\} \), and for any absolute temperature \( \tau \), and therefore neither the equipartition theorem nor the fluctuation-dissipation relation make sense for them.

4. Ideal circuit and fluctuating thermal amplitude

The aim of this section is to show how the situation in the previous one is exactly the same as the most basic formulation of the Itô versus Stratonovich dilemma, that is, selecting the precise meaning of a formal basic stochastic integral. For this we consider a formal Gaussian white noise process \( \xi_t \) and an associated formal circuit equation
\[ \frac{L}{V} dI_t = VB_t \xi_t, \]
\[ I_{t|t=0} = I_0, \]
which models a fluctuating amplitude of the thermal fluctuations in an ideal circuit with no resistance. That means we are considering again Johnson noise but, instead of the constant temperature assumed before, we let the temperature fluctuate as a Brownian motion (which is one of the simplest possibilities from a mathematical viewpoint). If we consider the white noise process to be the formal derivative of the Brownian motion, i.e.
\[ \xi_t = \frac{\partial B_t}{\partial t}, \]
we can rewrite equation (12) as
\[ \frac{L}{V} dI_t = VB_t \partial B_t, \]
\[ I_{t|t=0} = I_0, \]
where \( \partial \) denotes a formal stochastic integration scheme. This equation can be formally solved to find
\[ I_t = I_0 + \frac{L}{V} \int_0^t B_s \partial B_s, \]
where the last integral is still to be defined. If we defined this integral in the sense of Riemann-Stieltjes we would find
\[ \int_0^t B_s \partial B_s = \mathbb{P} - \lim_{|\Pi_n| \to 0} \sum_{i=1}^n B_{t_i}^* (B_{t_i} - B_{t_{i-1}}), \]
where \( \Pi = \{0:=t_0, t_1, t_2, \ldots, t_n:=t\} \) is a partition of the interval \([0, t]\), \( |\Pi_n| = \max_{1 \leq i \leq n} (t_i - t_{i-1}) \), and \( t_i^* \in [t_{i-1}, t_i] \) is given by the convex linear combination \( t_i^* = \alpha t_i + (1 - \alpha) t_{i-1} \) with \( \alpha \in [0, 1] \) arbitrary. Such a constraint in the value of \( \alpha \) comes from our definition of stochastic integral as a limit (in probability) of Riemann sums; if a different definition were used such a constraint may be modified [29]. Of course, if the integral existed in the Riemann-Stieltjes sense the result would be independent of \( \alpha \). However, it is well known that the result is \( \alpha \) dependent and reads [33]
Therefore the solution to equation (12) is

\[ I_t = I_0 + \frac{L}{V} \left[ \frac{B^2}{2} + \left( \alpha - \frac{1}{2} \right) t \right], \quad \alpha \in [0, 1]; \]

consequently it is a multivalued stochastic process. However, causality in physics imposes the solution to be a single-valued stochastic process, and thus, a unique process has to be selected from this one-parameter family of solutions. Of course, this selection has to be done solely on physical grounds. Since the nature of the fluctuations is completely random, the induced current should be isotropic on average, and therefore we should have \( \mathbb{E}(I_t) = 0 \) for all \( t \geq 0 \). This imposes \( \alpha = 0 \) or, in other words, the Itô interpretation of noise.

This is a very basic formulation of the Itô versus Stratonovich dilemma. According to van Kampen, (12) is not actually an equation but a pre-equation, and it only becomes an actual equation when a noise interpretation is given [26]. However, the only fundamental difficulty associated to (12), once we have assumed its integral form (13)-(14), is its infinite multiplicity of solutions. Adding an interpretation of noise is equivalent to choosing one solution out of this set; something that can only be done using physical arguments. The same reasoning is applicable to the majority of the Itô versus Stratonovich dilemmas one finds in the physical literature. The situation is, in essence, identical to that of equation (9); it admits infinitely many solutions and only one has to be chosen, with this selection strictly based on physical grounds. In other words, and using the terminology of van Kampen, adding an interpretation of noise does not necessarily transform a pre-equation into an equation. This could be so in many situations, but there are known counterexamples both in the Stratonovich case (such as in section 3, [30], and [28]) and in the Itô one (see for instance [34]). Finally, it is convenient to emphasize that there is a correct unique solution to problems (9) and (12) only if they are intended to describe a physical phenomenon and not simply considered as abstract mathematical models. Equivalently, there is nothing mathematically wrong about their respective infinite solution sets; all these solutions are equally admissible. Moreover, if one changed the physical meaning of these models, the right physical solutions would in general differ from those highlighted herein.

5. Conclusions

In this work, we have examined the classical fluctuation-dissipation relation in the light of stochastic analytical methods that have not been applied before, to the best of our knowledge, to this problem. Although the methodology in this field has expanded notably [35], our double approach still lies out of the usual set of methods. We have started considering the random dispersal of the Brownian particle subjected to a thermal bath. Instead of studying the Langevin equation posed forward in time—a classical theoretical approach to this problem—we have studied this equation backwards in time. This is a natural approach to the problem (but see appendix A), since as the fluctuation-dissipation relation arises as a consequence of Maxwell-Boltzmann statistics in the long-time limit, we can impose this statistics to the final condition. Moreover, since a single backward stochastic differential equation has to be solved for two stochastic processes in order to keep the adaptability of the solution, this enables us to solve this equation simultaneously for the velocity of the Brownian particle and the strength of the thermal fluctuations, so any relation among them can be studied in detail. Our results indicate that the classical fluctuation-dissipation relation is only recovered after assuming Ornstein-Uhlenbeck-type dynamics at equilibrium and the occurrence of the final time in the distant future. Under these assumptions we found in section 2 that

\[
V_t = \sqrt{2\gamma \kappa_0 \tau} e^{-\gamma t} \int_0^t e^{\gamma \tau} dW_t, \\
\sigma_t = \sqrt{2\gamma \kappa_0 \tau},
\]

which is what one would classically expect. Note however that we have also obtained the additional pairs of results (4a)-(4b), (5a)-(5b), and (6a)-(6b), which generalize this one when the second or both of these two assumptions are removed. To the best of our knowledge these pairs of results are new, and in our opinion illustrate the role of these assumptions in a more transparent way than other approaches. We regard them as some of the advantages of the use of backward rather than forward stochastic differential equations in this problem. Note also that all of these pairs of relations show the interdependence of drag and dissipation, what illustrates that the classical form of the fluctuation-dissipation relation is not necessarily the most fundamental one. The results in appendix B point in the same direction. In this appendix, the relation between the drift and the amplitude of the fluctuations is unveiled for the drifted Brownian motion by means of backward stochastic differential equations. As noted there, this system was not accessible to the classical methodology since it lacks a
stationary distribution. However, this other methodology reveals such a connection in what is, therefore, an obvious generalization of the traditional result.

Subsequently we have studied the fluctuation-dissipation relation in the context of Johnson noise in electric circuits. In this case, instead of relying on the more traditional approach of studying the Langevin equation for the electric current, we have derived forward stochastic differential equations for the energy of the circuit. If such an equation is derived using Itô calculus, this leads to a unique solution that perfectly reproduces the classical results. If alternatively Stratonovich calculus is used, we end up with infinitely many solutions, of which only one is physical, precisely the solution derived using Itô calculus. Despite of the spurious character of these solutions, some them (in fact, infinitely many) still obey equipartition of energy and the fluctuation-dissipation relation in the long time limit. While for others (in fact, infinitely many too), both of these classical results are meaningless. The same conclusions arise if instead of the energy we focus on the power dissipated by the circuit. One obvious corollary of this result is that the physical validity of a solution is not necessarily established by the fluctuation-dissipation relation and the equipartition of energy.

Our present results are important in the light of the established consensus regarding the interpretation of noise. In [36] one finds this consensus summarized as ‘computers are Itô and circuits are Stratonovich’ in the field of electric engineering. However, we have found that circuits can be Itô too; in fact, a given circuit can be both Itô and Stratonovich, depending on the parameters, as theoretically and experimentally shown in [37]. A similar conclusion to that of [36] is reached in [38], where the authors claim that ‘the Stratonovich results, however, accurately describe what actually happens in nature’ in contrast to the Itô ones. But however, sometimes the unique solution to an Itô equation could be of physical nature, while the Stratonovich equation possesses an infinite number of unphysical solutions. While in [39] we can read ‘in most areas of physical science (...) Stratonovich calculus is preferred, mainly due to the consistency with the results emerging from the Fokker-Planck equation and the fluctuation-dissipation theorem’; nevertheless we have found solutions to a Stratonovich stochastic differential equation which are totally inconsistent with the fluctuation-dissipation relation, unlike the Itô solution. Furthermore, we have also learned in this section that unphysical solutions may still obey the fluctuation-dissipation relation. Also, a central role in this discussion has been played by the Wong-Zakai theorem. This theorem has been invoked many times to justify the Stratonovich interpretation; for instance in [40] we can read ‘if the stochastic differential equation has been obtained as the white-noise limit of a real noise equation, choose the Stratonovich interpretation’. According to this, since the white noise appearing in equations (8) and (9) is a mathematical idealization of a real noise, equation (9) should be chosen. Nevertheless, this rule of thumb forgets those cases in which such a theorem cannot be applied, like the present one. Indeed, if one substitutes the white noise in equation (9) by a correlated noise, one still finds the infinite multiplicity of solutions; this can be immediately checked by considering the case $E_0 = 0$ and repeating the same arguments of section 3. Moreover, the same happens even for deterministic forcings, as illustrated in appendix C. And this inapplicability of the Wong-Zakai theorem is present in a model that, far from being an artificial mathematical counterexample, has been considered as a benchmark in the physical theory that concerns the Itô versus Stratonovich dilemma [26]. In summary, we agree with the conclusions in [39] in that the Itô versus Stratonovich dilemma should be resolved on a case by case basis. We agree as well with conclusions in [40] such as ‘the ultimate test is the confrontation of the analytical results with the experimental facts’ and ‘there are no universally valid theoretical a priori reasons why one or the other interpretation of a stochastic differential equation should be preferred’; notwithstanding, one should also test the mathematical consistency of the model under consideration prior to its experimental validation and there are indeed theoretical reasons that impede the use of some models (or interpretations). Overall, on different problems from the one studied herein, one should in principle consider the two classical interpretations of noise along with other possible meanings associated to a stochastic differential equation model [41, 42]. Our main conclusion from this section is that general guidelines can be useful at times for the noise interpretation dilemma, but one should be aware of possible counterexamples too. Moreover, the connection between these results and the most basic version of the Itô versus Stratonovich dilemma was highlighted in section 4, which shows that both essentially face the same question. Finally, although we have herein focused on multiplicity of solutions, let us note that a change of interpretation may also lead to nonexistence of solution; for instance in [43] it is shown that the global solution of an Itô stochastic differential equation may blow up in finite time with a positive probability when the interpretation is shifted to that of Stratonovich.

The problem of the interpretation of noise has some reflection also in the second section. A backward stochastic differential equation cannot be interpreted in the sense of Itô as this would lead to the lack of adaptability, and therefore to the non-existence, of the solution. The interpretation in the sense of Pardoux and Peng [1, 3] guarantees the adaptability of the solution, what is a physical requisite in the problems at hand. However, at least mathematically speaking, such an equation is well posed, even if its solution is not adapted, if interpreted according to some anticipating stochastic integral. The problem of the interpretation of noise in the
anticipating setting is summarized in [44–48]. We leave as an open question the application of such techniques in systems in which the fluctuation-dissipation relation is of interest.

In summary:

- The results in section 2 and appendix B show that the classical fluctuation-dissipation relation can be generalized by means of backward stochastic differential equations. Even cases that lack a stationary distribution can be approached with this methodology.

- Spurious solutions to physical models under the form of a stochastic differential equation that obey the fluctuation-dissipation relation may exist. This has been proven by the direct construction of such solutions to the equation for the energy of an electric circuit interpreted according to the Stratonovich convention in section 3. Since the Itô interpretation is free from that problem, this highlights it is advantageous in this particular physically relevant case.

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**Data availability statement**

No new data were created or analysed in this study.

**Appendix A**

This appendix is devoted to clarify some terminology and methodological issues that arise throughout this work. Before coming into the details, we make a general recommendations to references [14, 15, 21, 33] for the theory underlying the stochastic analytical developments that are presented in this article.

**A.1. Internal versus external fluctuations**

The physical literature is not completely consistent with the meaning of the adjectives *internal* and *external* when applied to fluctuations. Even for a given author, the definitions of both types of fluctuations are not precise in the mathematical sense, and this could be a source of confusion. However, for particular systems, one can deduce which of these two types a particular problem belongs to. For instance, van Kampen in the abstract of [26] says ‘systems with internal noise, however, can only be properly described by a master equation (…)’; such a description never appears neither in [26] nor in [25] when treating Langevin models such as the ones we consider along this work. Moreover, at the beginning of chapter XVI in [49] he mentions that ‘stochastic differential equations are the appropriate tool for describing systems with external noise’ and ‘one example of a stochastic differential equation is the Langevin equation’. But perhaps the most illuminating discussion is that found at the end of chapter XVII in [49]: ‘In the classical Brownian motion the fluctuations are the internal noise of the total system, but that is not how Langevin proceeded. He considered the Brownian particle by itself as the system to be studied and the surrounding fluid as the external bath. After splitting off that part of the interaction with the bath that causes damping, the remaining part is treated as external noise described by a Langevin force.’

Notably, more recent articles use a different classification. In [50] the notion of internal fluctuations would include those cases in which the fluctuation-dissipation relation holds, since it relates fluctuations with viscous damping. In this work we use term *external fluctuations* in the sense of van Kampen, since our work connects with previous ones such as [26] and [49], which use this terminology. However, we would be considering internal fluctuations according to [50]. Anyway, our results connect in a transparent way with previous results in stochastic analysis, such as the Watanabe-Yamada theorem (see section 3), and within this context there should be no ambiguities.

**A.2. Final value problems for stochastic differential equations**

In the classical theory of ordinary differential equations one finds many types of problems: initial, final, intermediate, and boundary value problems for a given ordinary differential equation or system of them. For instance, posing a final value problem for Newton equation of motion presents no essential mathematical difference from the corresponding initial value problem. However, things change dramatically when one moves to the realm of stochastic differential equations. For them, only initial value problems can be posed, at least if we use the classical stochastic integration theory of Itô or a perturbation of it, such as the Stratonovich theory or
variants of it. What follows is an informal discussion of this topic just aimed to clarify the motivation of the developments in the current work; but it is complemented with references in which the interested reader can find the precise results. If one looks at the classical definition of solution to a stochastic differential equation, for instance definition 10.3.1 in [14] (see also the discussions in chapters 3 and 5 of [15]), one finds that the notion solution of a stochastic differential equation coincides identically with the notion of solution to its initial value problem. This is not by chance, and it is related to the fact that a stochastic differential equation is actually a stochastic integral equation and that we require adaptability of the solution in order to make sense of the stochastic integral. This requirement, the fact that the solution should be adapted to a suitable filtration (see definition 4.3.4 in [14] and definition 3.1.6 in [15]), is much more than a mathematical technicality. It is a necessary condition for the solution of a stochastic differential equation to be a Markov process and for the existence of a Fokker-Planck equation [6], which has been extensively employed in the physical literature dealing with stochastic processes, for its probability density, see chapter 10 in [14] and chapters 7 and 8 in [15]. Given its crucial role in physics, this concept has also been discussed in chapter 4.2.4 in [31]; note that adaptability implies the nonanticipating character of a stochastic integrand, see chapter 4.3 in [14] and chapter 3.1 in [15]. Among other things, we want adapted integrands in order to have solutions to stochastic differential equations that are not influenced by future fluctuations; of course, this is a necessary condition in order to enforce causality in the vast majority of physical systems [31]. If we naïvely pose a final value problem for a stochastic differential equation, and use a random variable for the final value, we would end up with a solution that depends on future fluctuations through its dependence on the final value (for which the future fluctuations of the solution are past fluctuations), thus violating causality; of course, such a pitfall never arises in initial value problems. The theory of backward stochastic differential equations is introduced in order to have a mathematical framework in which we can pose final value problems without violating causality [13]. This is the reason why it is the right framework to be employed in section 2. Since, as we have already pointed out, the naïf approach to this question fails, the theory of backward stochastic differential equations is far from naïf. Nevertheless, it has been successfully employed in financial applications [12]. As a final note, let us mention that it is possible to pose stochastic differential equations with anticipating stochastic integrals, and such models are relevant when future events affect the present, see for instance [21,44–48].

Appendix B

This appendix shows another example that is complementary to that of section 2. We consider the backward stochastic differential equation for the Brownian motion with constant drift $\eta$:

$$
\begin{align*}
    dV_t &= \eta dt + \sigma_t dW_t, \\
    V_T &= F,
\end{align*}
$$

where as before $0 \leq t \leq T$ and $F \in L^2(\Omega)$ is $\mathcal{F}_T$-measurable. Again following [13] we know that this equation possesses a unique solution and that $V_t$ is given by the explicit formula

$$
V_t = \eta(t - T) + \mathbb{E}[F|\mathcal{F}_T].
$$

To compute the diffusion $\sigma_t$, consider

$$
\begin{align*}
    V_t &= V_0 + \eta t + \int_0^t \sigma_t dW_s \\
    \implies V_t &= \mathbb{E}[V_t] + \int_0^t \sigma_t dW_s \\
    \implies V_T &= \mathbb{E}[V_T] + \int_0^T \sigma_t dW_s \\
    \implies F &= \mathbb{E}[F] + \int_0^T \sigma_t dW_s,
\end{align*}
$$

by the zero mean of Itô integrals. Now use again the generalization of the Clark-Ocone formula for $L^2(\Omega)$ random variables [21] to conclude

$$
\sigma_t = \mathbb{E}[D_t F|\mathcal{F}_t],
$$

where $D_t F$ is the Malliavin derivative of $F$.

Now the unique solution pair

$$
\begin{align*}
    V_t &= \eta(t - T) + \mathbb{E}[F|\mathcal{F}_t], \\
    \sigma_t &= \mathbb{E}[D_t F|\mathcal{F}_t],
\end{align*}
$$

can be simplified if we add further assumptions, in particular the Maxwell-Boltzmann distribution for the terminal velocity.
and moreover that this random variable is given by the Itô integral
\[ F = \sqrt{k_B \tau} \int_0^T \psi_t dW_t \left( \int_0^T \psi_s^2 ds \right)^{1/2}, \]
where \( \psi_t \in L^2(0, T) \) is as before an arbitrary deterministic function. In this case
\[ V_t = \eta(t - T) + \sqrt{k_B \tau} \int_0^T \psi_t dW_t \left( \int_0^T \psi_s^2 ds \right)^{1/2}, \]
\[ \sigma_t = \sqrt{k_B \tau} \psi_t \left( \int_0^T \psi_s^2 ds \right)^{1/2}. \]

A further simplification is possible if, following section 2, we assume \( \psi_t = \psi \) is a constant. In such a case
\[ V_t = \eta(t - T) + \sqrt{k_B \tau} W_t, \]
\[ \sigma_t = \sqrt{k_B \tau}, \]
which is a more classical result. Note that there is no convergence, except to the trivial solution in the case \( \eta = 0 \), in the limit \( T \to \infty \). This of course agrees with the fact that Brownian motion possesses no stationary distribution.

In every step of this derivation we have shown that drag and fluctuation are interrelated, and they even became equal in the last step. In this sense, all these expressions constitute different manifestations of the fluctuation-dissipation relation. We note that the formalism of backward stochastic differential equations have made possible to find these relations, which are not accessible to the classical derivation that uses forward stochastic differential equations \([7]\), as it relies on the existence of a stationary distribution that, as is well known, does not exist for Brownian motion.

### Appendix C

In this appendix we consider the deterministic counterpart of equation (7):
\[ \frac{dL}{dt} = -RL_t + V \sin(\omega t), \]
\[ L_{b=0} = I_0, \]
for some frequency \( \omega \neq 0 \). After assuming \( I_0 = 0 \), we can solve it to get the current
\[ I_t = V \left( L\omega e^{-RL/L} - L\omega \cos(\omega t) + R \sin(\omega t) \right) \]
\[ R^2 + L^2 \omega^2 \]
and the energy
\[ E_t = \frac{1}{2} LL_t^2 \]
\[ = \frac{L V^2}{2} \left[ \frac{L\omega e^{-RL/L} - L\omega \cos(\omega t) + R \sin(\omega t)}{R^2 + L^2 \omega^2} \right]^2. \]

However, if we carry out the direct derivation of the differential equation for the energy, we find
\[ \frac{dE_t}{dt} = -2 R L E_t + \sqrt{2 \frac{V^2}{L} E_t \sin(\omega t)}, \]
which is to be subjected to \( E_t |_{b=0} = 0 \). This equation admits the computed solution along with \( E_t = 0 \). Moreover, we can combine both following a procedure akin to that in section 3 to construct an uncountable number of solutions.

This example illustrates how the infinite multiplicity of solutions already affects the equation for the energy when the noise is replaced by a trigonometric function. Since this phenomenon takes place even at the deterministic level, it should not be expected that an argument close to the Wong-Zakai one can be employed for this model, as mentioned in section 5. Nevertheless, it is remarkable that this multiplicity of solutions is cured.
when the noise is introduced, but only if Itô calculus is used, not when the Stratonovich interpretation is employed. All of this serves as an example of how the theory of stochastic differential equations presents fundamental differences from the theory of ordinary differential equations, and in this particular case the difference is the Watanabe–Yamada theorem.

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