Decentralized Online Regularized Learning
Over Random Time-Varying Graphs

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Abstract

We study the decentralized online regularized linear regression algorithm over random time-varying graphs. At each time step, every node runs an online estimation algorithm consisting of an innovation term processing its own new measurement, a consensus term taking a weighted sum of estimations of its own and its neighbors with additive and multiplicative communication noises and a regularization term preventing over-fitting. It is not required that the regression matrices and graphs satisfy special statistical assumptions such as mutual independence, spatio-temporal independence or stationarity. We develop the nonnegative supermartingale inequality of the estimation error, and prove that the estimations of all nodes converge to the unknown true parameter vector almost surely if the algorithm gains, graphs and regression matrices jointly satisfy the sample path spatio-temporal persistence of excitation condition. Especially, this condition holds by choosing appropriate algorithm gains if the graphs are uniformly conditionally jointly connected and conditionally balanced, and the regression models of all nodes are uniformly conditionally spatio-temporally jointly observable, under which the algorithm converges in mean square and almost surely. In addition, we prove that the regret upper bound is $O(T^{1-\tau} \ln T)$, where $\tau \in (0.5, 1)$ is a constant depending on the algorithm gains.

Index Terms

Decentralized online linear regression, regularization, random time-varying graph, persistence of excitation, regret analysis.

This work was supported by the National Natural Science Foundation of China under Grant 62261136550. (Corresponding author: Tao Li.)

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I. INTRODUCTION

Empirical risk minimization is an important criterion to judge the predictive ability of models in statistical learning. Problems in machine learning are usually ill-posed, and the solutions obtained by using the empirical risk minimization principle are unstable, easily overfitted and usually have large norms. To solve this problem, Poggio, Girosi et al. [1]-[3] introduced regularization methods in inverse problems. Regularization is an effective tool for dealing with the complexity of a model, and its role is to choose a model with less empirical risk and complexity at the same time. In addition, this method can transform the original ill-posed problem into a well-posed one, reduce the norm of estimations of the unknown true parameter vector, and keep the sum of the squared errors small [4]. Regularization methods are widely used in various research areas, such as classification in machine learning, online state estimation for power systems and image reconstruction [5]-[7]. There are two classic regularization methods: Ivanov regularization [8], which restricts the hypothesis space, and Tikhonov regularization, which restricts certain parameters in the loss function, i.e., adds regularization terms which are also called penalty terms into the algorithm. In learning theory, ridge regression adopts Tikhonov regularization [9].

An offline algorithm usually needs to acquire a finite amount of data generated by an unknown, stationary probability distribution in advance, which heavily relies on the memory of the system especially when the dataset is far too large, while an online algorithm processes infinite data streams that are continuously generated at rapid rates, where the data is discarded after it has been processed, and the unknown data generation process is possibly non-stationary. So far, the centralized online regularized algorithms have been widely studied in [10]-[14]. In a centralized algorithm, there is an information fusion center collecting the measurements of all nodes and giving the global estimation. In reality, many learning tasks process very large datasets, and thus decentralized parallel processing of data by communicating and computing units in the network is necessary, see e.g. [15]-[16] and references therein. Besides, if the data contains sensitive private information (e.g. medical and social network data, etc.), they may come from different units in the network, and transmitting all these data subsets to the fusion center may lead to potential privacy risks [17]-[18]. Therefore, decentralized learning is needed, which can improve the efficiency of communication and protect the privacies of users.

At present, the non-regularized decentralized linear regression problems have been widely studied in [19]-[29]. Xie and Guo [24]-[25] considered the time-varying linear regression with
measurement noises, where the cooperative information condition on the conditional expectations of the regression matrices was proposed over a deterministic, undirected and strongly connected graph. Chen et al. [27] proposed a saturated innovation update algorithm for the decentralized estimation under sensor attacks, where the interagent communication is noiseless. They proved that if the communication graph is undirected and fixed, the nodes are locally observable, and the number of attacked nodes is less than half of the total, then all nodes’ estimations converge to the unknown true parameter with a polynomial rate. Wang et al. [29] investigated a consensus plus innovation based decentralized linear regression algorithm over random networks with random regression matrices. Some scholars have also considered both measurement and communication noises among nodes, e.g. [20]-[21] and [23].

Based on the advantages of the decentralized information structure, the online algorithm and the regularization method, we propose a decentralized online regularized algorithm for the linear regression problem over random time-varying graphs. The algorithm of each node contains an innovation term, a consensus term and a regularization term. In each iteration, the innovation term is used to update the node’s own estimation, the consensus term is the weighted sum of estimations of its own and its neighbors with additive and multiplicative communication noises, and the regularization term is helpful for constraining the norm of the estimation in the algorithm and preventing the unknown true parameter vector from being overfitted. Although regularization is an effective method to deal with linear regression problems, it brings essential difficulties to the convergence analysis of the algorithm. Compared with the non-regularized decentralized linear regression algorithm, the estimation error equation of this algorithm contains a non-martingale difference term with the regularization parameter, which cannot be directly analyzed by using martingale convergence theorem as [30]. We no longer require that the sequences of regression matrices and graphs satisfy special statistical properties, such as mutual independence, spatio-temporal independence and stationarity. Compared with the case with i.i.d. data, dependent observations and data contain less information and therefore lead to more unstable learning errors as well as the performance degradation [31]. Besides, we consider both additive and multiplicative communication noises in the process of the information exchange among nodes. All these challenges make it difficult to analyze the convergence and performance of the algorithm, and the methods in the existing literature are no longer applicable. For example, the methods in [19]-[22] and [26] are applicable for the case that the graphs, regression matrices and noises are i.i.d. and mutually independent and it is required that the expectations of the regression matrices
be known in [20]-[21]. Liu et al. [17] studied the decentralized regularized gossip gradient descent algorithm for linear regression models, where the method is applicable for the case that only two nodes exchange information at each instant. In addition, they require that the graphs be strongly connected and the observation vectors and the noises be i.i.d. and bounded. Wang et al. [29] studied the non-regularized decentralized online algorithm, where the communication noises were not considered, and the method therein is only applicable for the case that the inhomogeneous part of the estimation error equation is a martingale difference sequence.

To overcome the difficulties mentioned above, we develop the nonnegative supermartingale inequality of the estimation error, and further establish the sample path spatio-temporal persistence of excitation condition by combining information of the regression matrices, graphs and algorithm gains, under which sufficient conditions for the convergence of the algorithm are obtained. We prove that if the algorithm gains and the sequences of regression matrices and Laplacian matrices of the graphs jointly satisfy the sample path spatio-temporal persistence of excitation condition, then the estimations of all nodes converge to the unknown true parameter vector almost surely. Furthermore, we give an intuitive sufficient condition of the sample path spatio-temporal persistence of excitation condition, i.e., the graphs are conditionally balanced and uniformly conditionally jointly connected, and the regression models of all nodes are uniformly conditionally spatio-temporally jointly observable, which drives the algorithm to converge in mean square and almost surely by the proper choice of the algorithm gains.

Historically, Guo [32] was the first to propose the stochastic persistence of excitation condition for analyzing the centralized Kalman filtering algorithm, which was then refined in [33]. Whereafter, the cooperative information condition on the conditional expectations of the regression matrices over the deterministic connected graph for the decentralized adaptive filtering algorithms was proposed in [24]-[25]. The stochastic spatio-temporal persistence of excitation condition on the conditional expectations of the regression matrices and graphs was proposed for the non-regularized decentralized online estimation algorithm over the random time-varying graphs in [29]. In this paper, this excitation condition is further weakened to the sample path spatio-temporal persistence of excitation condition, which only requires that the infinite series composed of the minimum eigenvalues of the information matrices diverge for almost all sample paths. Meanwhile, we extend the work of [30] from distributed averaging to decentralized online learning and generalize the network model of Xie and Guo [24]-[25] from deterministic, time-invariant and undirected graphs to random and time-varying digraphs. Xie and Guo [24]-[25]
considered tracking time-varying parameters by using constant algorithm gains and obtained the $L_p$-boundedness of the tracking errors. In [24]-[25], the homogeneous part of the estimation error equation is $L_p$-exponentially stable. Different from [24]-[25], we consider estimating time-invariant parameters. To ensure the strong consistency of the algorithm, we use decaying algorithm gains, which result in that the homogeneous part of the estimation error equation is not $L_p$-exponentially stable. To ensure not only boundedness but also strong consistency of the algorithm, the decaying rates of the algorithm gains and regularization parameter need to be designed precisely.

We use the regret to evaluate the performance of the decentralized optimization algorithm, which has been investigated in [28] and [34]-[35]. Yuan et al. [28] studied the non-regularized decentralized online linear regression problem over the fixed graph. Bedi et al. [34] considered the multi-agent stochastic optimization problems with constraints in heterogeneous networks. Dixit et al. [35] studied the decentralized online dynamic optimization problems over deterministic and time-varying graphs. Compared with [28] and [34]-[35], which assumed the graphs to be deterministic, strongly connected and balanced, we consider the decentralized online regularized linear regression algorithm over the random time-varying graphs, and our techniques are helpful to further study the optimization problems over random time-varying graphs. We prove that the upper bound of the regret is $O(T^{1-\tau}\ln T)$, where $\tau \in (0.5, 1)$ is a constant depending on the decaying algorithm gains.

The rest of this paper is organized as follows. Section II proposes the algorithm. Section III gives the convergence analysis. Section IV gives some numerical examples. Section V concludes the full paper and gives some future topics.

Notation and symbols: $\mathbb{R}^n$: the $n$ dimensional real vector space; $\mathbb{R}^{m \times n}$: the $m \times n$ dimensional real matrix space; $\otimes$: the Kronecker product; $\text{diag}(A_1, \cdots, A_n)$: the block diagonal matrix with entries being $A_1, \cdots, A_n$; $\|A\|$: the 2-norm of matrix $A$; $\lambda_{\text{min}}(A)$: the minimum eigenvalue of real symmetric matrix $A$; $\lambda_2(A)$: the second smallest eigenvalue of real symmetric matrix $A$; $A \succeq B$: the matrix $A - B$ is positive semi-definite; $A \succeq B$: the matrix $A - B$ is nonnegative; $I_n$: the $n$ dimensional identity matrix; $O_{n \times m}$: the $n \times m$ dimensional zero matrix; $1_N$: the $N$ dimensional vector with all ones; $\lfloor x \rfloor$: the largest integer less than or equal to $x$; $\lceil x \rceil$: the smallest integer greater than or equal to $x$; For a sequence of $n \times n$ dimensional matrices $\{Z(k), k \geq 0\}$,
denote
\[
\Phi_Z(j, i) = \begin{cases} 
Z(j) \cdots Z(i), & j \geq i \\
I_n, & j < i.
\end{cases}, \quad \prod_{k=i}^{j} Z(k) = \Phi_Z(j, i).
\]

II. Decentralized Regularized Linear Regression

Suppose that \(x_0 \in \mathbb{R}^n\) is the unknown true parameter vector. We consider a network modeled by a sequence of random digraphs with \(N\) nodes \(\{G(k) = \{V, E_G(k), A_G(k)\}, k \geq 0\}\), where \(V\) is the node set, \(E_G(k)\) is the edge set at instant \(k\), and \(A_G(k)\) is the weighted adjacency matrix. The regression model of node \(i\) at instant \(k\) is given by
\[
y_i(k) = H_i(k)x_0 + v_i(k), \quad k \geq 0, \quad i \in V,
\]
where \(H_i(k) \in \mathbb{R}^{n_i \times n}\) is the random regression matrix of node \(i\) at instant \(k\), \(v_i(k) \in \mathbb{R}^{n_i}\) is the additive measurement noise, and \(y_i(k) \in \mathbb{R}^{n_i}\) is the observation data.

**Remark 1.** The regression model (1) with random regression matrices over random graphs have been widely studied in [17], [19]-[23], [26], [29] and [36]-[37]. There are various uncertainties in real-word networks, where intermittent sensing failures and packet dropouts may occur at random times [21]. (i) Node/link failures can be modeled by a sequence of random communication graphs [36]. (ii) Node sensing failures or measurement losses can be modeled by a sequence of random observation/regression matrices, for example, \(H_i(k) = \frac{1}{p} \mu_i(k)C_i\), where \(\{\mu_i(k), k \geq 0\}\) is a sequence of zero-one i.i.d. Bernoulli variables accounting for sensor failures, \(p > 0\) is the sensing probability, and \(C_i\) models the normal operation of the sensor [20]-[21]. Besides, in decentralized parameter identification [29], all the nodes over graphs cooperatively identify the parameters of an auto-regressive (AR) model from noisy data, and each node’s measurement equation is given by
\[
y_i(k) = \sum_{j=1}^{d} c_j y_i(k-j) + v_i(k), \quad k \geq 0, \quad i \in V,
\]
where \(\{c_j \in \mathbb{R}, 1 \leq j \leq d\}\) are the model parameters to be identified, and \(v_i(k) \in \mathbb{R}\) is the additive zero-mean white noise. Here, the unknown parameter \(x_0 = [c_1, \cdots, c_d]^T \in \mathbb{R}^d\) and the random regression matrix \(H_i(k) = [y_i(k-1), \cdots, y_i(k-d)] \in \mathbb{R}^{1 \times d}\).

**Remark 2.** In this paper, the network structure is modeled by a sequence of random digraphs \(G(k, \omega) = \{V, E_G(k, \omega), A_G(k, \omega)\}, k \geq 0\), where \(\omega\) is the sample path, \(A_G(k, \omega) = [w_{ij}(k, \omega)]_{N \times N}\) is...
an \(N\)-dimensional random weighted adjacency matrix with zero diagonal elements, and \(E_{G(k, \omega)}\) is the edge set, where each edge represents a communication link. Denote the neighborhood of node \(i\) at instant \(k\) by \(N_i(k, \omega) = \{ j \in V \mid (j, i) \in E_{G(k, \omega)} \}\). There is a one-to-one correspondence between \(G(k, \omega)\) and \(A_{G(k, \omega)}\). Here, the sample path \(\omega\) is omitted. The in-degree of node \(i\) at instant \(k\) \(\text{deg}_{\text{in}}^i(k) = \sum_{j \in V} \omega_{ij}(k)\) and the out-degree of node \(i\) at instant \(k\) \(\text{deg}_{\text{out}}^i(k) = \sum_{j \in V} \omega_{ji}(k)\). If \(\text{deg}_{\text{in}}^i(k) = \text{deg}_{\text{out}}^i(k), \forall i \in V\), then \(G(k)\) is balanced. We call \(L_{G(k)} = D_{G(k)} - A_{G(k)}\) the Laplacian matrix of \(G(k)\), where \(D_{G(k)} = \text{diag}(\text{deg}_{\text{in}}^1(k), \ldots, \text{deg}_{\text{in}}^N(k))\).

We now provide the following real-world example of decentralized linear regression with random regression matrices over random graphs.

**Example 1.** In the decentralized multi-area state estimation in power systems [29], the power grid is partitioned into multiple geographically non-overlapping areas, where the communication topology is modeled by random graphs. The state of the grid \(x_0\) to be estimated consists of the amplitude of the voltages and the phase angles of all the buses. The measurement \(y_i(k)\) in each region consists of the active power flow and the reactive power flow. After a DC power flow approximation [38], the grid state to be estimated degenerates into a vector of phase angles of all buses. The relationship between the measurement in the \(i\)-th region and the grid state is represented by

\[y_i(k) = s_i(k)H_i(k)x_0 + v_i(k), \quad k \geq 0,\]

where \(\{v_i(k), k \geq 0\}\) is the sensing noise sequence, \(\{s_i(k) \in \mathbb{R}, k \geq 0\}\) is a sequence of i.i.d. Bernoulli variables, which represents the intermittent sensor failures, and \(\{H_i(k), k \geq 0\}\) is the sequence of observation matrices without sensing failures.

Denote \(y(k) = [y^T_1(k), \ldots, y^T_N(k)]^T, H(k) = [H^T_1(k), \ldots, H^T_N(k)]^T, H_k = \text{diag}(H_1(k), \ldots, H_N(k)), v(k) = [v^T_1(k), \ldots, v^T_N(k)]^T\). Rewrite (1) by the compact form

\[y(k) = H(k)x_0 + v(k), \quad k \geq 0.\]  

(3)

The estimation of \(1_N \otimes x_0\) can be obtained by minimizing the loss function \(\Psi(\cdot)\) as

\[\hat{x}_0 = \arg \min_{x \in \mathbb{R}^N} \Psi(x) \triangleq \frac{1}{2} \left( \mathbb{E} \left[ \|y(k) - H(k)x\|^2 \right] + \mathbb{E} \left[ \left( (L_{G(k)} \otimes I_n) x, x \right) \right] + \lambda \|x\|^2 \right), \quad \lambda > 0,\]

if the sequences of the graphs, regression matrices and noises are identically distributed, respectively, the mean graph is undirected, and the sequences of regression matrices and noises...
are mutually independent. The loss function $\Psi(\cdot)$ consists of three parts: the mean squared error of the estimation $\mathbb{E}[\|y(k) - \mathcal{H}(k)x\|^2]$, the consensus error $\mathbb{E}[\langle (\mathcal{L}_{G(k)} \otimes I_n)x, x\rangle]$ and the regularization term $\lambda \|x\|^2$. To solve the above optimization problem, we consider the stochastic gradient descent (SGD) algorithm:

$$x(k + 1) = x(k) + a(k)\mathcal{H}^T(k) (y(k) - \mathcal{H}(k)x(k)) - b(k) (\mathcal{L}_{G(k)} \otimes I_n) x(k) - \lambda(k)x(k), \quad (4)$$

where $\lambda(k) \triangleq \lambda c(k)$. Let $x(k) = [x_1^T(k), \ldots, x_n^T(k)]^T$, then $x_i(k) \in \mathbb{R}^n$ is the estimation of the node $i$ at instant $k$. From (4), it follows that

$$x_i(k + 1) = x_i(k) + a(k)\mathcal{H}_i^T(k) (y_i(k) - \mathcal{H}_i(k)x_i(k)) + b(k) \sum_{j \in \mathcal{N}_i(k)} w_{ij}(k)(x_j(k) - x_i(k))$$

$$- \lambda(k)x_i(k), \quad i \in \mathcal{V},$$

where $a(k)\mathcal{H}_i^T(k)(y_i(k) - \mathcal{H}_i(k)x_i(k))$ is the innovation term updating the estimation $x_i(k)$ with the innovation gain $a(k)$, $b(k) \sum_{j \in \mathcal{N}_i(k)} w_{ij}(k)(x_j(k) - x_i(k))$ is the consensus term taking a weighted sum of estimations of its own and its neighbors with the consensus gain $b(k)$, and $\lambda(k)x_i(k)$ is the regularization term constraining the estimation $x_i(k)$ with the regularization gain $\lambda(k)$, which avoids overfitting the estimation of the unknown true parameter vector $x_0$. In the practical algorithm, there are communication noises among nodes. Specifically, node $j$ acquires the estimations of its neighbors by

$$\mu_{ji}(k) = x_j(k) + f_{ji}(x_j(k) - x_i(k))\xi_{ji}(k), \quad j \in \mathcal{N}_i(k), \quad (5)$$

where $\xi_{ji}(k)$ is the communication noise, and $f_{ji}(x_j(k) - x_i(k))$ is the noise intensity function.

Therefore, the decentralized online regularization algorithm is given by

$$x_i(k + 1) = x_i(k) + a(k)\mathcal{H}_i^T(k) (y_i(k) - \mathcal{H}_i(k)x_i(k)) + b(k) \sum_{j \in \mathcal{N}_i(k)} w_{ij}(k)(\mu_{ji}(k) - x_i(k))$$

$$- \lambda(k)x_i(k), \quad i \in \mathcal{V}. \quad (6)$$

Here, by assuming that (i) the sequences of the graphs, regression matrices and noises are identically distributed, respectively; (ii) the mean graph is undirected; (iii) the sequences of regression matrices and noises are mutually independent, we obtain the above algorithm. In fact, even if the mean graphs are directed, and the graphs and regression matrices do not satisfy special statistical properties such as independence and stationarity, the algorithm (6) will still be proved to converge.
Remark 3. The communication model (5) with relative-state-dependent communication noises $f_{ji}(x_j(k) - x_i(k)) \xi_{ji}(k)$ are reasonable for many realistic applications. (i) In some multiple robots or UAV systems, due to the non-reliability of the communications, the transmission states are more prone to noises whose density functions depend on the distances between the transmitter and receiver, i.e., the relative distances between agents \cite{39}. (ii) In consensus problems with quantized measurements of relative states \cite{40}, the measurement of $x_j(k) - x_i(k)$ is given by $z_{ji}(k) = \xi_{ij}(k)(x_j(k) - x_i(k))$, where $\Delta_{ji}(k)$ is the quantization uncertainty. (iii) In distributed averaging with Gaussian fading channels \cite{41}, the measurement of $x_j(k) - x_i(k)$ is given by $z_{ji}(k) = \gamma_{ij}(x_j(k) - x_i(k)) + \Delta_{ij}(k)(x_j(k) - x_i(k))$ with $\Delta_{ij}(k) = \xi(k) - \gamma_{ij}$ being independent zero-mean Gaussian noises, which is a special case of (5).

Denote $\mathcal{F}(k) = \sigma(\mathcal{A}_\xi(s), H_i(s), v_i(s), \xi_{ji}(s), j, i \in V, 0 \leq s \leq k)$ with $\mathcal{F}(-1) = \{\Omega, \emptyset\}$, and $\xi(k) = [\xi_{11}(k), \ldots, \xi_{NN}(k), \xi_{1N}(k), \ldots, \xi_{N1}(k)]^T$. For the algorithm (6), we need the following assumptions.

\begin{enumerate}
\item [(A1)] For the noise intensity function $f_{ji}(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$, there exist constants $\sigma$ and $b$, such that $|f_{ji}(x)| \leq \sigma \|x\| + b, \forall x \in \mathbb{R}^n$.
\end{enumerate}

Remark 4. Assumption (A1) indicates that the communication noises in (5) cover both additive and multiplicative noises, where $\sigma$ and $b$ are multiplicative and additive noise intensity coefficients, respectively.

\begin{enumerate}
\item [(A2)] The noises $\{v(k), \mathcal{F}(k), k \geq 0\}$ and $\{\xi(k), \mathcal{F}(k), k \geq 0\}$ are both martingale difference sequences and independent of $\{\mathcal{H}(k), \mathcal{A}_\xi(k), k \geq 0\}$. There exists a constant $\beta_v$, such that $\sup_{k \geq 0} \mathbb{E}[\|v(k)\|^2 + \|\xi(k)\|^2 | \mathcal{F}(k - 1)] \leq \beta_v$ a.s.
\end{enumerate}

Remark 5. Different from \cite{19}-\cite{22}, the measurement noises are only assumed to be a martingale difference sequence and independent of the graphs and regression matrices in Assumption (A2). In this paper, neither mutual independence nor spatio-temporal independence is assumed on the regression matrices and graphs. This is applicable to complex scenarios where regression matrices and graphs are spatio-temporal dependent.
TABLE I: Summary of assumptions, excitation conditions and main results in relevant works.

|                                                                 | Regression matrices                  | Graphs                                      | Excitation conditions                             | Results                                |
|-----------------------------------------------------------------|--------------------------------------|---------------------------------------------|--------------------------------------------------|----------------------------------------|
| R.T.V.                                                          | random and time-varying              | deterministic, undirected and time-invariant| connected graph and cooperative information      | \(L_p\)-stable tracking errors         |
| [24], [25]                                                     |                                      |                                             |                                                  |                                        |
| R.T.I.P.                                                        | spatio-temporally independent        | deterministic and time-invariant            | stability of the information matrix              | mean and m.s. convergence              |
| [42]                                                           |                                      |                                             |                                                  |                                        |
| R.T.I.P.                                                        | i.i.d.                               | balanced digraphs and homogeneous ergodic Markov chain | global observability and a spanning tree contained in the union of the graphs | a.s. and m.s. convergence              |
| [36]                                                           |                                      |                                             |                                                  |                                        |
| R.T.I.P.                                                        | i.i.d.                               | undirected and i.i.d.                       | global observability and connected mean graph    | a.s. convergence                       |
| [21]                                                           |                                      |                                             |                                                  |                                        |
| R.T.I.P.                                                        | temporally strictly stationary and temporally correlated | deterministic and time-invariant            | spatially joint observability                    | m.s. convergence                       |
| [29]                                                           |                                      |                                             |                                                  |                                        |
| R.T.I.P.                                                        | time-invariant and deterministic     | undirected and i.i.d.                       | global observability and connected mean graph    | a.s. convergence                       |
| [27]                                                           |                                      |                                             |                                                  |                                        |
| R.T.I.P.                                                        | random and time-varying              | random and time-varying digraphs            | stochastic spatio-temporal persistence of excitation | a.s. and m.s. convergence              |
| This paper                                                      | random and time-varying              | random and time-varying digraphs            | sample path spatio-temporal persistence of excitation | a.s. and m.s. convergence              |

The problems of decentralized online regression over graphs have been investigated in most of the literature, including regression with time-varying unknown parameters (R.T.V.P.) (e.g. [24]-[25]) and regression with time-invariant unknown parameters (R.T.I.P.) (e.g. [19]-[22], [26], [29], [36] and [42]-[43]), where different assumptions of models and excitation conditions have been required. Here, we sum up in Table I the assumptions, excitation conditions that they required and their main results.

III. MAIN RESULTS

The convergence analysis of the algorithm (6) are presented in this section. First, Lemma 1 gives a nonnegative supermartingale type inequality of the squared estimation error. Based on which, Theorem 1 proves the almost sure convergence of the algorithm. Then, Theorem 2 gives intuitive convergence conditions for the case with balanced conditional digraphs by Lemma 2. Finally, Theorem 3 establishes an upper bound for the regret of the algorithm and Theorem 4 gives a non-asymptotic rate for the algorithm. The proofs of theorems are in Appendix A, and those of lemmas in this section are in Appendix B.

Denote
\[ f_i(k) = \text{diag}(f_{1i}(x_1(k) - x_i(k)), \cdots, f_{Ni}(x_N(k) - x_i(k)));
\]
\[ Y(k) = \text{diag}(f_1(k), \cdots, f_N(k));
\]
\[ M(k) = Y(k) \otimes I_n; W(k) = \text{diag}(\alpha_1^T(k) \otimes I_n, \cdots, \alpha_N^T(k) \otimes I_n), \]
where \(\alpha_i^T(k)\) is the
$i$-th row of $A_{G(k)}$. A compact form equivalent to (6) is given by

$$
x(k + 1) = [(1 - \lambda(k))I_{NN} - b(k)L_{G(k)} \otimes I_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k))]x(k) + a(k)\mathcal{H}^T(k)y(k) + b(k)W(k)M(k)\xi(k).
$$

(7)

Denote $\delta(k) = x(k) - 1_N \otimes x_0$ as the global estimation error. Noting that $(L_{G(k)} \otimes I_n)(1_N \otimes x_0) = (L_{G(k)}1_N) \otimes x_0 = 0$ and $\mathcal{H}(k)(1_N \otimes x_0) = H(k)x_0$, subtracting $1_N \otimes x_0$ from both sides of (7) gives

$$
\delta(k + 1) = ((1 - \lambda(k))I_{NN} - b(k)L_{G(k)} \otimes I_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k))x(k) + a(k)\mathcal{H}^T(k)y(k) + b(k)W(k)M(k)\xi(k) - 1_N \otimes x_0
$$

$$
= \Phi_P(k, 0)\delta(0) + \sum_{i=0}^k a(i)\Phi_P(k, i + 1)\mathcal{H}^T(i)v(i) + \sum_{i=0}^k b(i)\Phi_P(k, i + 1)W(i)M(i)\xi(i)
$$

$$
- \sum_{i=0}^k \lambda(i)\Phi_P(k, i + 1)(1_N \otimes x_0),
$$

(8)

where $P(k) = (1 - \lambda(k))I_{NN} - b(k)L_{G(k)} \otimes I_n - a(k)\mathcal{H}^T(k)\mathcal{H}(k)$.

Denote $\hat{L}_{G(k)} = \frac{L_{G(k)} + L_{G(k)}^T}{2}$. For any given positive integers $h$ and $k$, denote

$$
\Lambda_k^h = \min \left\{ \sum_{i=kh}^{(k+1)h-1} b(i)\mathbb{E}\left[\hat{L}_{G(i)} | \mathcal{F}(kh - 1) \right] \otimes I_n + a(i)\mathbb{E}\left[\mathcal{H}^T(i)\mathcal{H}(i) | \mathcal{F}(kh - 1) \right] \right\}.
$$

Denote $V(k) = ||\delta(k)||^2$. A nonnegative supermartingale type inequality of the squared estimation error $V(k)$ is obtained in the following lemma, which plays a key role in the convergence and performance analysis of the algorithm.

**Lemma 1.** For the algorithm (6), if Assumptions (A1)-(A2) hold, the algorithm gains $a(k)$, $b(k)$ and $\lambda(k)$ monotonically decrease to zero, and there exists a positive integer $h$ and a positive constant $\rho_0$, such that $\sup_{k \geq 0}(||L_{G(k)}|| + (\mathbb{E}[||\mathcal{H}^T(k)\mathcal{H}(k)||^{2\max(h,2)} | \mathcal{F}(k - 1)]))^\frac{1}{2\max(h,2)} \leq \rho_0$ a.s., then there exists a positive integer $k_0$, such that

$$
\mathbb{E}[V((k + 1)h)|\mathcal{F}(kh - 1)] \leq (1 + \Omega(k))V(kh) - 2 \left( \Lambda_k^h + \sum_{i=kh}^{(k+1)h-1} \lambda(i) \right) V(kh)
$$

$$
+ \Gamma(k) \text{ a.s., } k \geq k_0,
$$

where $\{\Omega(k), k \geq 0\}$ and $\{\Gamma(k), k \geq 0\}$ are nonnegative deterministic real sequences satisfying $\Omega(k) + \Gamma(k) = O(a^2(kh) + b^2(kh) + \lambda(kh))$. 

November 4, 2024
**Remark 6.** The entries of $A_{G(k)}$ represent the weights of all edges of the graph. It is reasonable to assume that the weights are uniformly bounded with respect to the sample paths.

It is realistic to assume that the regression matrices are conditionally bounded. For example, consider the AR model (2) with i.i.d. Gaussian white noises. For this case, $\mathbb{E}[\|H_i^T(k)H_i(k)\|^{\mathcal{F}(k-1)}]$ is bounded but $H_i(k)$ is not uniformly bounded with respect to the sample paths.

The main results are as follows. Firstly, we analyze the convergence of the algorithm.

**Theorem 1.** For the algorithm (6), if Assumptions (A1)-(A2) hold, the algorithm gains $a(k)$, $b(k)$ and $\lambda(k)$ decrease monotonically, and there exists a positive integer $h$ and a positive constant $\rho_0$, such that (i) $\sum_{k=0}^{\infty} \lambda_k = \infty$ a.s. with $\inf_{k \geq 0} (\lambda_k + \lambda(k)) \geq 0$ a.s.; (ii) $\sum_{k=0}^{\infty} (a^2(k) + b^2(k) + \lambda(k)) < \infty$; (iii) $\sup_{k \geq 0} (\|L_{G(k)}\| + (\mathbb{E}[\|\mathcal{H}^T(k)\mathcal{H}(k)\|^{2\max(h,2)}|\mathcal{F}(k-1)])^{\frac{1}{2\max(h,2)}}) \leq \rho_0$ a.s., then $\lim_{k \to \infty} x_i(k) = x_0$, $i \in \mathcal{V}$ a.s.

The following proof sketch provides the main steps in the proof of Theorem 1 from which we have solved the limitations of prior works.

**Proof sketch.** The non-commutative, non-independent and non-stationary random matrices pose intrinsic difficulties for analyzing the convergence of the online algorithm (6), for which our proof mainly consists of the following three steps. (I) We start by introducing the tools of conditional mathematical expectations in probability theory, and then derive the upper bound of the state transition matrix, i.e., $\|\mathbb{E}[\Phi_p^T((k+1)h-1, kh)\Phi_p((k+1)h-1, kh)|\mathcal{F}(kh-1)]\| \leq 1 - 2(\lambda_k^h + \sum_{i=kh}^{(k+1)h-1} \lambda(i)) + O(a^2(kh) + b^2(kh) + \lambda^2(kh))$ a.s. through the binomial expansion techniques, which helps us to transform the analysis of random matrix products into that of the minimum eigenvalue of the information matrix $\Lambda_k^h$ and we no longer need to separate the product of the random matrices by assuming independence as (20)-(21). (II) Different from the existing techniques in (20)-(21) and (29), which directly analyzed the mean squared error $\mathbb{E}[V(k)]$, we next turn to consider the squared estimation error $V(kh)$. By applying the inequality techniques in probability theory, we reveal the relation between $\mathbb{E}[V((k+1)h)|\mathcal{F}(kh-1)]$ and $V(kh)$ and derive the nonnegative supermartingale type inequality of the squared estimation error in Lemma 1 (III) Then, by the probabilistic structure of the estimation error obtained in Lemma 1 and the convergence theorem for nonnegative supermartingales, we conclude that $V(kh)$ and the infinite series $\sum_{k=0}^{\infty} (\lambda_k^h + \sum_{i=kh}^{(k+1)h-1} \lambda(i))V(kh)$ almost surely converge by choosing appropriate algorithm gains and the regularization parameter. (IV) Finally, we prove that $V(kh) \to 0$ a.s. as
by the condition (i) in Theorem \ref{thm:main} where we no longer require the minimum eigenvalue of the information matrix $\Lambda_k^h$ has a positive and deterministic lower bound as in \cite{29}. As a consequence, our results not only remove the reliance on the special statistical properties of regression matrices and graphs, but also weaken the conditions in our previous works.

**Remark 7.** The choices of the algorithm gains $a(k), b(k)$ and the regularization parameter $\lambda(k)$ is crucial for the nodes to estimate the unknown true parameter vector $x_0$. The conditions (i) and (iii) in Theorem \ref{thm:main} imply $\sum_{k=0}^{\infty} (a(k) + b(k)) = \infty$, which means that the algorithm gains can not be too small. Besides, to reduce the effects of perturbing the innovations and consensus by the communication noises, the monotonically decreasing algorithm gains and the condition (ii) in Theorem \ref{thm:main} ensure that all nodes avoid making excessive changes to the current estimate when acquiring new noisy data. The conditions (i) and (ii) give what need to be satisfied for the algorithm gains.

It is worth noting that the information matrix $\Lambda_k^h$ in the condition (i) is coupled with the gains, graphs and regression matrices. For some special cases, Theorem \ref{thm:main} later provides acceptable ranges on the values of these gains.

**Remark 8.** Most existing literature on decentralized online regression suppose that the mean graphs are balanced and strongly connected (e.g. \cite{20}-\cite{21} and \cite{28}). Here, the condition (i) in Theorem \ref{thm:main} may still hold even if the mean graphs are unbalanced and not strongly connected. For example, consider a fixed digraph $G = \{V = \{1, 2\}, A_G = [w_{ij}]_{2 \times 2}\}$ with $w_{12} = 1$ and $w_{21} = 0$. Obviously, $G$ is unbalanced and not strongly connected. Suppose $H_1 = 0, H_2 = 1$. Choose $a(k) = b(k) = \frac{1}{k+1}$. We have $\lambda_{\min}(b(k)\hat{L}_G + a(k)H^T H) = \frac{1}{k+1} \lambda_{\min}(\hat{L}_G + H^T H) = \frac{1}{2k+2}$. Then, the condition (i) holds with $h = 1$ and $\sum_{k=0}^{\infty} \Lambda_k^h = \sum_{k=0}^{\infty} \frac{1}{2k+2} = \infty$.

For the special case without regularization, we directly obtain the following corollary by Theorem \ref{thm:main}.

**Corollary 1.** For the algorithm (6) with $\lambda(k) \equiv 0$, if Assumptions (A1)-(A2) hold, and there exists a positive integer $h$ and a positive constant $\rho_0$, such that (i) $\sum_{k=0}^{\infty} \Lambda_k^h = \infty$ a.s. with $\inf_{k \geq 0} \Lambda_k^h \geq 0$ a.s.; (ii) $\sum_{k=0}^{\infty} (a^2(k) + b^2(k)) < \infty$; (iii) $\sup_{k \geq 0} \left( \|L_G(k)\| + (\mathbb{E}[\|H^T(k)H(k)\|^{2\max\{h, 2\}}]|\mathcal{F}(k-1)|)^{\frac{1}{2\max\{h, 2\}}} \right) \leq \rho_0$ a.s., then $\lim_{k \to \infty} x_i(k) = x_0$, $i \in V$ a.s.
Remark 9. The conditions for the regularized algorithm in Theorem 1 are less conservative than those for the non-regularized algorithm in Corollary 1. For example, consider the digraph sequence \( \{G(k) = \{V = \{1, 2\}, A_{G(k)} = [w_{ij}(k)]_{2 \times 2}, k \geq 0\} \) with \( w_{12}(k) = \begin{cases} 1, & k = 2m; \\ \frac{1}{k}, & k = 2m + 1 \end{cases} \)
and \( w_{21}(k) = 0 \), and the regression matrices \( H_1(k) = 0, H_2(k) = \begin{cases} 1, & k = 2m; \\ \frac{1}{\sqrt{ak}}, & k = 2m + 1 \end{cases} \), \( m \geq 0 \). Choose \( a(k) = b(k) = \frac{1}{k+1} \). Then, we have \( \sum_{k=0}^{\infty} A_k^1 = \infty \). Obviously, the condition (i) in Corollary 1 does not hold since \( \inf_{k\geq0} A_k^1 \leq \inf_{k\geq0} A_{2k+1}^1 = \inf_{k\geq0}(-\frac{\sqrt{4k^2-6}}{10(2k+1)^2}) < 0 \). Choose \( \lambda(k) = \frac{1}{10(k+1)^2} \). Then, the condition (i) in Theorem 1 holds with \( \inf_{k\geq0}(A_k^1 + \lambda(k)) = 0 \).

Subsequently, we list the conditions on the algorithm gains that may be needed later.

(C1) The algorithm gains \( \{a(k), k \geq 0\}, \{b(k), k \geq 0\} \) and \( \{\lambda(k), k \geq 0\} \) are all nonnegative sequences monotonically decreasing, and satisfy \( \sum_{k=0}^{\infty} \min\{a(k), b(k)\} = \infty, \sum_{k=0}^{\infty}(a^2(k) + b^2(k) + \lambda(k)) < \infty, a(k) = O(a(k+1)), b(k) = O(b(k+1)) \) and \( \lambda(k) = o(\min\{a(k), b(k)\}) \).

(C2) The algorithm gains \( \{a(k), k \geq 0\}, \{b(k), k \geq 0\} \) and \( \{\lambda(k), k \geq 0\} \) are all nonnegative sequences monotonically decreasing to zero, and satisfy \( \sum_{k=0}^{\infty} \min\{a(k), b(k)\} = \infty, a(k) = O(a(k+1)), b(k) = O(b(k+1)) \) and \( a^2(k) + b^2(k) + \lambda(k) = o(\min\{a(k), b(k)\}) \).

Remark 10. As mentioned previously, the regularization gain \( \lambda(k) \) has the function of constraining the estimation \( x_i(k+1) \) in the algorithm (6). To ensure that the algorithm can cooperatively estimate the unknown true parameter vector \( x_0 \) by means of the innovation and consensus terms, Conditions (C1)-(C2) require the regularization gain \( \lambda(k) \) to decay faster than the innovation gain \( a(k) \) and the consensus gain \( b(k) \).

Next, we consider the sequence of balanced conditional digraphs

\[ \Gamma_1 = \{\{G(k), k \geq 0\} \mid \mathbb{E}[A_{G(k)}|F(k-1)] \preceq \Omega_{N \times N} \text{ a.s.}, G(k|k-1) \text{ is balanced a.s.}, k \geq 0\} \].

Here, \( \mathbb{E}[A_{G(k)}|F(m)], m < k, \) is called the conditional generalized weighted adjacency matrix of \( A_{G(k)} \) w.r.t. \( F(m) \), and its associated random graph is called the conditional digraph of \( G(k) \) w.r.t. \( F(m) \), denoted by \( G(k|m) \), i.e., \( G(k|m) = \{V, \mathbb{E}[A_{G(k)}|F(m)]\} \).
For any given positive integers \( h \) and \( k \), we denote
\[
\tilde{\Lambda}^h_k = \lambda_{\min}\left[\sum_{i=kh}^{(k+1)h-1} \mathbb{E}\left[\hat{L}_{G(i)} \otimes I_n + \mathcal{H}^T(i)\mathcal{H}(i)\big| \mathcal{F}(kh - 1)\right]\right].
\]

Then \( \tilde{\Lambda}^h_k \) contains information of both the Laplacian matrices of the graphs and regression matrices. As shown in Remark 7, the information matrix \( \Lambda^h_k \) in the condition (i) in Theorem 1 is coupled with the gains, graphs and regression matrices, it is generally difficult to decouple the algorithm gains from \( \Lambda^h_k \) for general graphs and regression matrices since \( \hat{L}_{G(k)} \) is indefinite. For the case with the conditionally balanced digraphs \( G(k) \in \Gamma_1 \), we have \( \Lambda^h_k \geq \min\{a(i), b(i), kh \leq i < (k + 1)h\} \tilde{\Lambda}^h_k \) a.s. The following lemma gives a lower bound of \( \tilde{\Lambda}^h_k \), where one can see how the conditionally balanced graphs and regression matrices of all nodes affect the lower bound, respectively.

**Lemma 2.** For the algorithm (6), suppose that \( \{G(k), k \geq 0\} \in \Gamma_1 \), and there exists a positive constant \( \rho_0 \), such that
\[
\sup_{k \geq 0} \mathbb{E}\left[\left\|\mathcal{H}^T(k)\mathcal{H}(k)\right\|\big| \mathcal{F}(k - 1)\right] \leq \rho_0 \text{ a.s.}
\]
Then, for any given positive integer \( h \),
\[
\tilde{\Lambda}^h_k \geq \lambda_2 \left[\sum_{i=kh}^{(k+1)h-1} \mathbb{E}\left[\hat{L}_{G(i)}\big| \mathcal{F}(kh - 1)\right]\right] \geq \frac{\lambda_2 \left(\mathcal{L}_G\right)}{2Nh\rho_0 + N\lambda_2 \left(\mathcal{L}_G\right)} \times \lambda_{\min}\left[\sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} \mathbb{E}\left[H_i^T(j)H_i(j)\big| \mathcal{F}(kh - 1)\right]\right] \text{ a.s.}
\]

**Remark 11.** If the network structure degenerates into a deterministic, undirected and connected graph \( G \), and \( \mathcal{H}^T_i(k)\mathcal{H}_i(k) \leq \rho_0 I_n, i \in \mathcal{V} \text{ a.s.} \), then the inequality in Lemma 2 degenerates to
\[
\tilde{\Lambda}^h_k \geq \frac{\lambda_2 \left(\mathcal{L}_G\right)}{2Nh\rho_0 + N\lambda_2 \left(\mathcal{L}_G\right)} \times \lambda_{\min}\left[\sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} \mathbb{E}\left[H_i^T(j)H_i(j)\big| \mathcal{F}(kh - 1)\right]\right] \text{ a.s.}
\]
which is given in [24]-[25].

Then, we give intuitive convergence conditions for the case with balanced conditional digraphs. We first introduce the following definitions.
Definition 1. For the random undirected graph sequence \( \{G(k), k \geq 0\} \), if there exists a positive integer \( h \) and a positive constant \( \theta \), such that
\[
\inf_{k \geq 0} \lambda_2 \left( \sum_{i=kh}^{(k+1)h-1} \mathbb{E} \left[ \mathcal{L}_{G(i)} | \mathcal{F}(kh - 1) \right] \right) \geq \theta \text{ a.s.,}
\]
then \( \{G(k), k \geq 0\} \) is said to be uniformly conditionally jointly connected.

Definition 2. For the sequence of regression matrices \( \{H_i(k), i = 1, \cdots, N, k \geq 0\} \), if there exists a positive integer \( h \) and a positive constant \( \theta \), such that
\[
\inf_{k \geq 0} \lambda_{ \min} \left( \sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ H_i^T(j)H_i(j) | \mathcal{F}(kh - 1) \right] \right) \geq \theta \text{ a.s.,}
\]
then \( \{H_i(k), i = 1, \cdots, N, k \geq 0\} \) is said to be uniformly conditionally spatio-temporally jointly observable.

Denote the symmetrized graph of \( G(k) \) by \( \tilde{G}(k) = \{V, E_{G(k)} \cup E_{\tilde{G}(k)}, \mathcal{A}_{G(k)} + \mathcal{A}_{\tilde{G}(k)}^T \} \), where \( \tilde{G}(k) \) is the reversed digraph of \( G(k) \) \(^{[30]}\).

Theorem 2. For the algorithm (6), suppose that \( \{G(k), k \geq 0\} \in \Gamma_1 \), Assumptions (A1)-(A2) hold, and there exists a positive integer \( h \) and a positive constant \( \rho_0 \), such that (i) \( \{\tilde{G}(k), k \geq 0\} \) is uniformly conditionally jointly connected; (ii) \( \{H_i(k), i \in \mathcal{V}, k \geq 0\} \) is uniformly conditionally spatio-temporally jointly observable; (iii) \( \sup_{k \geq 0} (\|L_{G(k)}\| + (\mathbb{E}[\|H^T(k)H(k)\|^2_{\text{max}}(h, 2)} | \mathcal{F}(k - 1)\]^{\frac{1}{2}}_{\text{max}}(h, 2)) \leq \rho_0 \) a.s.

(I). If Condition (C1) holds, then \( \lim_{k \to \infty} x_i(k) = x_0, \ i \in \mathcal{V} \) a.s.

(II). If Condition (C2) holds, then \( \lim_{k \to \infty} \mathbb{E}[\|x_i(k) - x_0\|^2] = 0, \ i \in \mathcal{V} \).

The combination of the conditions (i)-(ii) in Theorem 2 with Condition (C1) or Condition (C2) gives an intuitive sufficient condition for the condition (i) in Theorem 1 to hold.

The condition (i) in Theorem 1 is called the sample path spatio-temporal persistence of excitation condition, which is an indispensable part of the convergence and performance analysis of the algorithm. Specifically, spatio-temporal persistence of excitation means that the infinite series of the minimum eigenvalues of the information matrices composed of the graphs and regression matrices in fixed-length time intervals diverges for almost all sample paths, i.e., \( \sum_{k=0}^{\infty} \Lambda_k = \infty \) a.s., which is to avoid the failure of estimating the unknown true parameter vector due to lack of
effective measurement information or sufficient information exchange among nodes. To illustrate this, let us consider the extreme case that the regression matrix of each node at each instant is always zero, then $\Lambda^h_k = 0$ for any given positive integer $h$. In this case, no matter how to design the algorithm gains, one can not obtain any information about the unknown true parameter vector since there is no measurement information and no information interflow. Here, \textit{spatio-temporality} focuses on the state of the information matrices consisting of the graphs and regression matrices of all nodes over a fixed-length time period rather than the state at each instant, where the temporality is captured by $h$. From Theorem 2, we know that neither the \textit{locally temporally joint observability} of each node, i.e., \[ \inf_{k \geq 0} \lambda_{\min}(\mathbb{E}[\sum_{j=kh}^{(k+1)h-1} H_i^T(j)H_i(j) | \mathcal{F}(kh-1)]) \] being uniformly bounded away from zero for each node $i$, nor the \textit{instantaneously globally spatially joint observability} of all the regression models, i.e., \[ \inf_{k \geq 0} \lambda_{\min}(\mathbb{E}[\sum_{i=1}^{N} H_i^T(j)H_i(j) | \mathcal{F}(kh-1)]) \] being uniformly bounded away from zero for each instant $j$, is necessary.

At present, most results on decentralized online linear regression algorithms (e.g. [20]-[21] and [36]-[37]) all require that the regression matrices and graphs satisfy some special statistical properties, such as i.i.d., spatio-temporal independence or stationarity, etc. However, these special statistical assumptions are difficult to be satisfied if the regression matrices are generated by the auto-regressive models. In order to solve this problem, in the past several decades, many scholars have proposed the persistence of excitation condition based on the conditional expectations of the regression matrices. The stochastic persistence of excitation condition was first proposed in the analysis of the centralized Kalman filter algorithm in [32] and then refined in [33]. For the decentralized adaptive filtering algorithms in [24]-[25], the cooperative information condition on the conditional expectations of the regression matrices was proposed for the case with deterministic connected graphs. For the decentralized online estimation algorithms over random time-varying graphs in [29], the stochastic spatio-temporal persistence of excitation condition was proposed. The stochastic spatio-temporal persistence of excitation condition in [29] means that the minimum eigenvalue of the matrix consisting of the spatial-temporal observation matrices and Laplacian matrices in each time period $[kh, (k+1)h - 1]$ has a positive lower bound $c(k)$ independent of the sample paths, i.e., $\Lambda^h_k \geq c(k)$ a.s. with $\sum_{k=0}^{\infty} c(k) = \infty$. Here, in Corollary 1, we show that this is actually not needed. Therefore, the \textit{sample path spatio-temporal persistence of excitation} condition in this paper is more general than the stochastic spatio-temporal persistence of excitation condition in [29].

We give an example for which the stochastic spatio-temporal persistence of excitation condition
does not hold but the *sample path spatio-temporal persistence of excitation* condition hold.

**Example 2.** Consider a fixed digraph $G = \{V = \{1, 2\}, A_G = [w_{ij}]_{2 \times 2}\}$ with $w_{12} = 1$ and $w_{21} = 0$. Let $H_1 = 0$ and $H_2 = \sqrt{x}$, where the random variable $x$ is uniformly distributed in $(0.25, 1.25)$. Choose $a(k) = b(k) = \frac{1}{k+1}$. For any given positive integer $h$, if Assumption (A2) holds, then the *sample path spatio-temporal persistence of excitation* condition holds and there does not exist a positive real sequence $\{c(k), k \geq 0\}$ satisfying $\Lambda^h_k \geq c(k)$ a.s., i.e., the stochastic spatio-temporal persistence of excitation condition in [29] does not hold.

Therefore, the *sample path spatio-temporal persistence of excitation* condition weakens the stochastic spatio-temporal persistence of excitation condition in [29]. To our best knowledge, we have obtained the most general persistence of excitation condition ever.

To appraise the performance of the algorithm (6), the regret upper bound of the decentralized online regularized algorithm will be taken into account. The loss function of node $j$ at instant $t$ is defined by

$$l_{j,t}(x) \triangleq \frac{1}{2} \|H_j(t)x - y_j(t)\|^2, \ x \in \mathbb{R}^n.$$  

The performance of the algorithm (6) is appraised, at node $i$, through its regret defined by

$$\text{Regret}_{\text{LMS}}(i, T) \triangleq \sum_{t=0}^{T} \sum_{j=1}^{N} \mathbb{E}[l_{j,t}(x_{i}(t)) - l_{j,t}(x^*_{\text{LMS}})],$$

where

$$x^*_{\text{LMS}} \triangleq \arg \min_{x \in \mathbb{R}^n} \sum_{t=0}^{T} \sum_{j=1}^{N} \frac{1}{2} \mathbb{E}[\|H_j(t)x - y_j(t)\|^2]$$

is the linear optimal estimated parameter.

The following theorems give an upper bound of $\text{Regret}_{\text{LMS}}(i, T)$, $i \in \mathcal{V}$, and a non-asymptotic rate for the algorithm.

**Theorem 3.** For the algorithm (6), suppose that $\{\hat{G}(k), k \geq 0\} \in \Gamma_1$ and Assumptions (A1)-(A2) hold. If $a(k) = b(k) = \frac{c_1}{(k+1)^\tau}$ and $\lambda(k) = \frac{c_2}{(k+1)^\tau}$, $c_1 > 0$, $c_2 > 0$, $0.5 < \tau < 1$, and there exists a positive integer $h$ and a positive constant $\rho_0$, such that (i) $\{\hat{G}(k), k \geq 0\}$ is *uniformly conditionally jointly connected*; (ii) $\{H_i(k), i \in \mathcal{V}, k \geq 0\}$ is *uniformly conditionally spatio-temporally jointly observable*; (iii) $\sup_{k \geq 0}(\|L_{\hat{G}(k)}\| + (\mathbb{E}[\|H^T(k)H(k)\|^{2}\max\{h,2\} | \mathcal{F}(k-1)\})^{\frac{1}{2\max\{h,2\}}}) \leq \rho_0$ a.s., then the regret of node $i$ satisfies

$$\text{Regret}_{\text{LMS}}(i, T) = O\left(T^{1-\tau} \ln T\right), \ i \in \mathcal{V}. \quad (9)$$
Remark 12. Yuan et al. [28] studied the non-regularized decentralized online linear regression algorithm over the fixed graph. In this paper, we consider the decentralized online regularized linear regression algorithm over the random time-varying graphs. Theorem 4 shows that the upper bound of the regret is $O(T^{1-\tau} \ln T)$, where $\tau \in (0.5, 1)$ is a constant depending on the decaying algorithm gains, which improves the regret upper bound $O(\sqrt{T})$ with the fixed gain in [28].

Theorem 4. For the algorithm (6), suppose that $\{G(k), k \geq 0\} \in \Gamma_1$ and Assumptions (A1)-(A2) hold. If $a(k) = b(k) = \frac{1}{(k+1)^\tau}$ and $\lambda(k) = \frac{1}{(k+1)^{2\tau}}$, $0.5 < \tau < 1$, and there exists a positive integer $h$ and a positive constant $\rho_0$, such that (i) $\{\hat{G}(k), k \geq 0\}$ is uniformly conditionally jointly connected; (ii) $\{H_i(k), i \in \mathcal{V}, k \geq 0\}$ is uniformly conditionally spatio-temporally jointly observable; (iii) $\sup_{k \geq 0}(\|L_{\hat{G}}(k)\| + (\mathbb{E}[\|H^T(k)H(k)\|^2_{\max(h, \lambda)} | \mathcal{F}(k - 1)])^{\frac{1}{\max(h, \lambda)}} \leq \rho_0$ a.s., then there exist constants $c_i$, $i = 1, \ldots, 7$, such that $\mathbb{E}[V(k)]$ is bounded by

$$
2^h \left( \frac{25c_4 \ln(k + 1)}{c_5 \left( \frac{k}{2n} \right)^\tau} + c_6 e^{-c_5 \left( \frac{k}{2n} \right)^{1-\tau} - (k+2)^{1-\tau}} \right) + \frac{c_7}{k^{2\tau}},
$$
when $k$ is larger than

$$
\left( \frac{12}{c_5(1-\tau)} \ln \left( \frac{4}{c_5(1-\tau)} \right) \right)^{\frac{1}{1-\tau}} + \frac{4c_2^2}{c_3^2} + c_3^2 + 2.
$$

The proof of Theorem 4 is similar to Theorem 3 in which we further analyze the nonnegative supermartingale type inequality of $V(k)$ given by Lemma 11 and obtain the upper bound of $\mathbb{E}[V(k)]$ by Lemma 2.9 in [44]. The detailed proof is put in Appendix A.

IV. NUMERICAL EXAMPLE

We consider the graphs composed of 10 nodes, and the states are $x_i(k)$, $i = 1, \ldots, 10$, $k \geq 0$. Each node estimates the unknown true parameter vector $x_0 = [5, 4, 3]^T$.

The local observation is given by $y_i(k) = H_i(k)x_0 + v_i(k)$, $i = 1, \ldots, 10$. Here, the regression matrices are taken as

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \tilde{h}_{s+2,t,k} \\
\tilde{h}_{s,t,k} & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & \tilde{h}_{5,t,k} & 0 \\
0 & 0 & 0 \\
0 & 0 & \tilde{h}_{6,t,k}
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \tilde{h}_{6,t,k} & 0
\end{pmatrix},
$$

where $\tilde{h}_{s,t,k} = (-1)^s h_{s,t}(k) + (-1)^t 0.5$, $\tilde{h}_{s+2,t,k} = (-1)^s h_{s+2,t}(k) + (-1)^t 0.5$, $\tilde{h}_{5,t,k} = (-1)^t h_{5,t}(k)$ $+ (-1)^t 0.5$, $\tilde{h}_{6,t,k} = (-1)^{t+1} h_{6,t}(k)$ $+ (-1)^{t+1} 0.5$, and $h_{i,j}(k), i = 1, 2, 3, 4, 5, 6, j = 1, 2,$ are
TABLE II: Settings of the algorithm gains

| Setting  | $a(k)$          | $b(k)$          | $\lambda(k)$ |
|----------|-----------------|-----------------|--------------|
| Setting I| $(k + 1)^{-0.6}$ | $(k + 1)^{-0.6}$ | $(k + 1)^{-2}$ |
| Setting II| $(k + 1)^{-0.6}$ | $(k + 1)^{-0.6}$ | $0.1(k + 1)^{-3}$ |
| Setting III| $(k + 1)^{-0.8}$ | $(k + 1)^{-0.8}$ | $(k + 1)^{-2}$ |
| Setting IV| $(k + 1)^{-0.8}$ | $(k + 1)^{-0.8}$ | 0            |

independent random variables with uniform distribution on $(0, 1)$. The signal at the receiver of the communication channel is described by (5), where $f_{x_i}(x_j(k) - x_i(k)) = 0.1||x_j(k) - x_i(k)|| + 0.1$, $i, j = 1, \cdots, 10$. The updating rules of the estimation states follow algorithm (6), where the random weights $\{w_{ij}(k), i, j = 1, \cdots, 10, k \geq 0\}$ are selected by the following rules. When $k = 2m, m \geq 0$, the random weights are uniformly distributed in the interval $[0, 1]$; when $k \neq 2m, m \geq 0$, the random weights are uniformly distributed in $[-0.5, 0.5]$. So the random weights may be negative at some time instants. Here, $\{w_{ij}(k), i, j = 1, \cdots, 10, k \geq 0\}$ are spatio-temporally independent. Thus, when $k = 2m, m \geq 0$, the average graph is balanced and connected, and when $k \neq 2m, m \geq 0$, the average graph is empty. Suppose that the measurement noises $\{v_i(k), i = 1, \cdots, 10, k \geq 0\}$ and the communication noises $\{\xi_{ij}(k), i, j = 1, \cdots, 10, k \geq 0\}$ are normally distributed r.v.s and independent of the random graphs. Different settings of the algorithm gains are shown in Table II where we take $a(k) = b(k)$ for simplicity. To verify the sample path spatio-temporal persistence of excitation condition, Fig. 1 presents the curves of the quantity $R(k) \triangleq (\sum_{i=0}^{k} A_i^2)^{-1} = (\sum_{i=0}^{k} \lambda_{\min}\left(\sum_{j=2i}^{2i+1} (b(j)E[\hat{c}_j(g_{i,j})]| \mathcal{F}(2i-1)\right) \otimes I_3 + a(j)E[\mathcal{H}^T(j)\mathcal{H}(j)| \mathcal{F}(2i-1)]))^{-1}$ over time $k$ with different settings in Table II.

Here, all the conditions of Theorem 1 hold with $h = 2$ and $\rho_0 = 5$. The trajectories of the estimation errors are shown in Fig. 2. It can be seen that as time goes on, the estimations of each node’s algorithms with different algorithm gains in Table II converge to the true value of $x_0$ almost surely. We can see from Fig. 1 and Fig. 2 that larger $\sum_{i=0}^{k} A_i^2$ leads to faster convergence of the algorithm. It shows in Fig. 2 that (i) for the same gains $a(k)$ and $b(k)$, it is possible that the algorithm with larger regularization parameter $\lambda(k)$ converges faster than that with smaller one; (ii) for the same regularization parameter, larger gains accelerate the convergence of the algorithm.

Meanwhile, the trajectories of the mean value of the norms of agents’ states are plotted in Fig. 3 from which we can see that compared with the non-regularized algorithms, the regularization
Fig. 1: The sample paths of $R(k)$ with different settings.

is effective for reducing the magnitudes of estimations of the unknown true vector, which helps to reduce the model complexity and to avoid overfitting.

Finally, all the conditions of Theorem 2 hold with setting III in Table II. Fig. 4 shows that the estimation state of each node converges to $x_0$ in mean square.

V. CONCLUSIONS

We study the decentralized online regularized linear regression algorithm over random time-varying graphs. For the generalized regression models and graphs, we obtain the sufficient conditions for the almost sure and mean square convergences of the algorithm. We prove that if the algorithm gains, the sequences of regression matrices and Laplacian matrices of the graphs jointly satisfy the sample path spatio-temporal persistence of excitation condition, then the estimations of all nodes converge to the unknown true parameter vector almost surely. Especially, if the graphs are conditionally balanced and uniformly conditionally jointly connected, and the regression models of all nodes are uniformly conditionally spatio-temporally jointly observable, then the sample path spatio-temporal persistence of excitation condition holds and the estimations of all nodes converge to the unknown true parameter vector almost surely and in mean square by properly choosing the algorithm gains. To appraise the performance of the algorithm, we prove that the regret upper bound is $O(T^{1-\tau} \ln T)$, where $\tau \in (0.5, 1)$ is a constant depending on the algorithm gains.
Fig. 2: The sample paths of errors with different settings.
Fig. 3: Trajectories of the mean value of norms of agents’ states.

Fig. 4: Mean squared errors for Setting III.

There are a number of interesting open problems. For examples, (i) how to design the appropriate regularization parameter to achieve the optimal convergence rate for decentralized online regularized algorithm; (ii) It is interesting to develop tools for the case with more general state-dependent communication noises, i.e. the multiplicative term does not go to zero in communication model (5); (iii) it is interesting to study the case with time-varying unknown parameters by combining our method with those in Xie and Guo [24]-[25]; (iv) the case with random time delays as in Wang et al. [29].
APPENDIX A

Proofs of Theorems 1-4 and Verification of Example 2

Proof of Theorem 1. It follows from the condition (iii) and Lemma 1 that there exists a positive integer \( k_0 \), such that

\[
\mathbb{E}[V((k + 1)h) | \mathcal{F}(kh - 1)] \leq (1 + \Omega(k))V(kh) - 2 \left( \Lambda_k^h + \sum_{i=kh}^{(k+1)h-1} \lambda(i) \right) V(kh) + \Gamma(k) \text{ a.s., } k \geq k_0,
\]

where \( \Omega(k) + \Gamma(k) = O(a^2(kh) + b^2(kh) + \lambda(kh)) \) is nonnegative. It follows from the condition (i) and \( V(k) = \|\delta(k)\|^2 \) that \( \Lambda_k^h + \sum_{i=kh}^{(k+1)h-1} \lambda(i) \) and \( V(kh) \) are both nonnegative and adapted to \( \mathcal{F}(kh - 1) \) for \( k \geq k_0 \). By the condition (ii), we have

\[
\sum_{k=0}^{\infty} \Omega(k) + \sum_{k=0}^{\infty} \Gamma(k) < \infty,
\]

which together with above and Theorem 1 in [45] leads to

\[
\sum_{k=0}^{\infty} \left( \Lambda_k^h + \sum_{i=kh}^{(k+1)h-1} \lambda(i) \right) V(kh) < \infty \text{ a.s.,}
\]

and \( V(kh) \) converges almost surely. This along with the condition (i) gives \( \lim \inf_{k \to \infty} V(kh) = 0 \text{ a.s., which further leads to} \)

\[
\lim_{k \to \infty} \delta(kh) = 0 \text{ a.s.} \quad \text{(A.1)}
\]

Denote \( m_k = \lfloor \frac{k}{h} \rfloor \). For any \( \varepsilon > 0 \), by Markov inequality and the conditions (ii)-(iii), we have

\[
\sum_{k=0}^{\infty} \mathbb{P}\{\|b(k)\mathcal{L}_{\mathcal{G}(k)} \otimes I_n + a(k)\mathcal{H}^T(k)\mathcal{H}(k)\| \geq \varepsilon\} < \infty \text{ a.s.,}
\]

which together with Borel-Cantelli Lemma gives

\[
\mathbb{P}\{\|b(k)\mathcal{L}_{\mathcal{G}(k)} \otimes I_n + a(k)\mathcal{H}^T(k)\mathcal{H}(k)\| \geq \varepsilon \text{ i.o.}\} = 0,
\]

i.e., \( \|b(k)\mathcal{L}_{\mathcal{G}(k)} \otimes I_n + a(k)\mathcal{H}^T(k)\mathcal{H}(k)\| \) converges to zero almost surely, which gives

\[
\sup_{k \geq 0} \|b(k)\mathcal{L}_{\mathcal{G}(k)} \otimes I_n + a(k)\mathcal{H}^T(k)\mathcal{H}(k)\| < \infty \text{ a.s.}
\]

Noting that \( 0 \leq k - m_k h < h \), we have

\[
\Phi_0 \triangleq \sup_{k \geq 0} \sup_{m_k h \leq i \leq k} \|\Phi_P(k - 1, i)\| < \infty \text{ a.s.} \quad \text{(A.2)}
\]
By definitions of $W(k)$ and $M(k)$, we know that $\|W(k)\| \leq \sqrt{N}\|A_{G(k)}\|$ and $\|M(k)\| \leq \sqrt{4\sigma^2V(k) + 2b^2} \leq 2\sigma\delta(k) + \sqrt{2b}$, it follows from the estimation error equation (8) and (A.2) that

$$\|\delta(k)\| \leq f(k) + \sum_{i=mkh}^{k-1} g(i)\|\delta(i)\|,$$

where

$$
\begin{align*}
    f(k) &= \Phi_0(\|\delta(mkh)\| + \sum_{i=mkh}^{k-1} (a(i)\|v(i)\|\|\mathcal{H}^T(i)\| + b\sqrt{2N}b(i)\|\xi(i)\|\|A_{g(i)}\| + \sqrt{N}\|x_0\|\lambda(i))), \\
g(i) &= 2\sigma\sqrt{N}\Phi_0 b(i)\|\xi(i)\|\|A_{g(i)}\|.
\end{align*}
$$

By Gronwall inequality, we get

$$\|\delta(k)\| \leq f(k) + \sum_{i=mkh}^{k-1} f(i)g(i) \prod_{j=i+1}^{k-1} (1 + g(j)). \quad (A.3)$$

For any $\varepsilon > 0$, it follows from Assumption (A2), the conditions (ii)-(iii) and Markov inequality that

$$\sum_{k=0}^{\infty} \mathbb{P}\{b(k)\|\xi(k)\|\|A_{G(k)}\| \geq \varepsilon\} < \infty \text{ a.s.},$$

which shows that $b(k)\|\xi(k)\|\|A_{G(k)}\|$ converges to zero almost surely by Borel-Cantelli Lemma. Following the same way, we know that

$$\lim_{k \to \infty} a(k)\|v(k)\|\|\mathcal{H}^T(k)\| = 0 \text{ a.s.},$$

which together with (A.1)-(A.2) gives $\lim_{k \to \infty} f(k) = 0$ a.s. By (A.2), we also have $\lim_{k \to \infty} g(k) = 0$ a.s. It follows from the above and (A.3) that

$$\lim_{k \to \infty} \delta(k) = 0 \text{ a.s.}$$

**Proof of Theorem 2.** It follows from the conditions (i)-(ii) and Definitions 1-2 that there exist positive constants $\theta_1$ and $\theta_2$, such that $\forall k \geq 0,$

$$\lambda_2 \left( \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ \hat{\mathcal{L}}_{g(j)} | \mathcal{F}(kh - 1) \right] \right) \geq \theta_1 \text{ a.s.}$$

and

$$\lambda_{\min} \left( \sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ H^T_i(j) H_i(j) | \mathcal{F}(kh - 1) \right] \right) \geq \theta_2 \text{ a.s.}$$
Denote
\[ T_k = \theta_1 \theta_2 \min\{a((k+1)h), b((k+1)h)\} + \frac{(k+1)h-1}{2Nh\rho_0 + N\theta_1} + \sum_{i=kh}^{(k+1)h-1} \lambda(i). \]

By Condition (C1), we have
\[ \sum_{k=0}^{\infty} T_k = \infty. \]

Hence, by Condition (C1) and (I) of Lemma A.3 we have proved (I) of Theorem 2.

Next, we will prove (II) of Theorem 2. It follows from Condition (C2) that
\[ \sum_{k=0}^{\infty} T_k = \infty. \]

Noting that \( a(k) = O(a(k+1)) \) and \( b(k) = O(b(k+1)) \), we know that there exist positive constants \( C_1 \) and \( C_2 \), such that
\[ a(k) \leq C_1 a(k+1), \quad b(k) \leq C_2 b(k+1). \]

Denote \( C = \max\{C_1, C_2\} \), we have
\[ \min\{a(k), b(k)\} \leq C \min\{a(k+1), b(k+1)\}, \]
which leads to
\[ \frac{\min\{a(kh), b(kh)\}}{\min\{a((k+1)h), b((k+1)h)\}} = \prod_{i=0}^{h-1} \left[ \frac{\min\{a(kh+i), b(kh+i)\}}{\min\{a(kh+i), b(kh+i+1)\}} \right] \leq C^h, \]
thus, we obtain
\[ \min\{a(kh), b(kh)\} \leq C^h \min\{a((k+1)h), b((k+1)h)\}, \]
which together with Condition (C2) gives
\[ a^2(kh) + b^2(kh) + \lambda(kh) = o(T_k). \]

It follows from (II) of Lemma A.3 that
\[ \lim_{k \to \infty} \mathbb{E}[V(k)] = 0. \]

**Verification of Example 2** By Assumption (2), we have \( \mathbb{E}[\mathcal{H}^T \mathcal{H} | \mathcal{F}(kh-1)] = \mathbb{E}[\mathcal{H}^T \mathcal{H} | \sigma(\mathcal{H})] = \mathcal{H}^T \mathcal{H} \) a.s. For any given positive integer \( h \), by the definition of Laplacian matrix, we get
\[ \Lambda^h_k = \sum_{i=kh}^{(k+1)h-1} \frac{x+1 - \sqrt{x^2 - 2x + 2}}{2(i+1)} \quad \text{a.s., } \forall k \geq 0. \tag{A.4} \]
Noting that \( x \) is uniformly distributed in \((0.25, 1.25)\), we obtain \( x + 1 - \sqrt{x^2 - 2x + 2} > 0 \) a.s., which together with (A.4) gives

\[
\sum_{k=0}^{\infty} \Lambda_k^h = \left( x + 1 - \sqrt{x^2 - 2x + 2} \right) \sum_{k=0}^{\infty} \sum_{i=kh}^{(k+1)h-1} \frac{1}{2(i+1)} = \infty \text{ a.s.}
\]

Suppose that there exists a positive real sequence \( \{c(k), k \geq 0\} \) satisfying \( \Lambda_k^h \geq c(k) \) a.s. For any given integer \( k_0 > 0 \), denote \( \mu = \frac{(k_0h+1)c(k_0)}{h} \). It follows from \( \Lambda_k^h \geq c(k) \) a.s. that

\[
0 < \mu \leq \frac{x + 1 - \sqrt{x^2 - 2x + 2}}{2} < 1 \text{ a.s.,}
\]

which leads to \( x \geq \mu + \frac{1}{4(1 - \mu)} \) a.s. Hence, by (A.5), we have

\[
P \{ \Lambda_{k_0}^h \geq c(k_0) \} \leq P \left\{ \mu + \frac{1}{4(1 - \mu)} \leq x \leq \frac{5}{4} \right\} = \frac{5}{4} - \mu - \frac{1}{4(1 - \mu)} < 1,
\]

which is contradictory to \( \Lambda_k^h \geq c(k) \) a.s.

**Proof of Theorem 3.** It follows from \( \{G(k), k \geq 0\} \in \Gamma_1 \) that

\[
\sum_{i=kh}^{(k+1)h-1} \mathbb{E} \left[ b(i) \hat{L}_{G(i)} \otimes I_n + a(i) \mathcal{H}^T(i) \mathcal{H}(i) \right| \mathcal{F}(kh - 1) \]
\[
\geq \min\{a((k + 1)h), b((k + 1)h)\} \sum_{i=kh}^{(k+1)h-1} \mathbb{E} \left[ \hat{L}_{G(i)} \otimes I_n + \mathcal{H}^T(i) \mathcal{H}(i) \right| \mathcal{F}(kh - 1) \] a.s.\]

It follows from the conditions (i)-(iii), Definitions 1-2 and Lemma 2 that there exist positive constants \( \theta_1 \) and \( \theta_2 \), such that

\[
\Lambda_k^h \geq (2Nh\rho_0 + N\theta_1)^{-1}\theta_1\theta_2 \min\{a((k + 1)h), b((k + 1)h)\} \text{ a.s.}
\]

Denote

\[
L(k) = (2Nh\rho_0 + N\theta_1)^{-1}\theta_1\theta_2 \min\{a((k + 1)h), b((k + 1)h)\} + \sum_{i=kh}^{(k+1)h-1} \lambda(i),
\]

we have

\[
\Lambda_k^h + \sum_{i=kh}^{(k+1)h-1} \lambda(i) \geq L(k) \text{ a.s.,}
\]

which together with Lemma 1 shows that there exists a positive integer \( k_0 \), such that

\[
\mathbb{E}[V((k + 1)h)|\mathcal{F}(kh - 1)] \leq (1 + \Omega(k)) V(kh) - 2L(k)V(kh) + \Gamma(k) \text{ a.s., } k \geq k_0, \quad (A.6)
\]
where \( \Omega(k) + \Gamma(k) = \mathcal{O}(a^2(kh) + b^2(kh) + \lambda(kh)) \), which together with the choice of the algorithm gains shows that there exists a positive constant \( u \) such that

\[
\Gamma(k) \leq u(kh + 1)^{-2\tau}, \quad k \geq 0.
\]

Noting that \( \Omega(k) = o(L(k)) \) and \( L(k) = o(1) \), without loss of generality, we suppose that \( 0 < \Omega(k) \leq L(k) < 1, \quad k \geq k_0 \). Denote

\[
v = (4Nh\rho_0 + 2N\theta_1)^{-1}c_1\theta_1\theta_2,
\]

it follows that

\[
L(k) \geq v(kh + h + 1)^{-\tau}, \quad k \geq k_0.
\]

By taking mathematical expectation on both sides of (A.6), we get

\[
\mathbb{E}[V((k + 1)h)] \leq \prod_{i=k_0+1}^{k} \left(1 - \frac{v}{(ih + h + 1)^\tau}\right) \sup_{k \geq 0} \mathbb{E}[V(k)] + \sum_{i=k_0}^{k} \frac{u}{(ih + 1)^{2\tau}} \prod_{j=i+1}^{k} \left(1 - \frac{v}{(jh + h + 1)^\tau}\right), \quad k > k_0. \tag{A.7}
\]

Noting that

\[
\sum_{j=i+1}^{k} \frac{v}{(jh + h + 1)^\tau} \geq \int_{i+1}^{k} \frac{v}{(xh + h + 1)^\tau} dx = \frac{v}{h(1 - \tau)} ((kh + h + 1)^{1-\tau} - (ih + 2h + 1)^{1-\tau}), \quad i \geq 0,
\]

we have

\[
\prod_{j=i+1}^{k} \left(1 - \frac{v}{(jh + h + 1)^\tau}\right) \leq \exp \left(-\frac{v}{h(1 - \tau)} ((kh + h + 1)^{1-\tau} - (ih + 2h + 1)^{1-\tau}) \right), \quad i \leq k. \tag{A.8}
\]

By Theorem 2 we have \( \lim_{k \to \infty} \mathbb{E}[V(k)] = 0 \), which shows that

\[
\sup_{k \geq 0} \mathbb{E}[V(k)] < \infty.
\]

It follows from (A.8) that

\[
\exp \left(-\frac{v}{h(1 - \tau)} (kh + h + 1)^{1-\tau}\right) = o \left((kh + h)^{-\tau}\ln(kh + h)\right),
\]
thus, we have\[ \prod_{i=k_0+1}^{k} \left(1 - \frac{v}{(ih + h + 1)^\tau} \right) \sup_{k \geq 0} \mathbb{E}[V(k)] = o \left( (kh + h)^{-\tau} \ln(kh + h) \right). \tag{A.9} \]

Denote\[ \epsilon(k) = \left\lceil \frac{2}{v} (kh + h + 1)^\tau \ln(kh + h + 1) \right\rceil, \]
then we have \( \epsilon(k) = o(k) \). Noting that \( \epsilon(k) \to \infty \) as \( k \to \infty \), without loss of generality, we suppose that \( k_0 < \epsilon(k) \leq 2\epsilon(k) \leq k \). On one hand, for the case with \( k_0 \leq i \leq k - 1 - \epsilon(k) \), we know that\[ ih + 2h + 1 \leq kh + h + 1 - \epsilon_1(k), \tag{A.10} \]
where \( \epsilon_1(k) = \left\lceil \frac{2}{v} (kh + h + 1)^\tau \ln(kh + h + 1) \right\rceil \), which directly gives\[ (kh+h+1)^{1-\tau} - (kh+h+1-\epsilon_1(k))^{1-\tau} \geq (kh+h+1)^{-\tau} \epsilon_1(k)(1-\tau) \geq v^{-1}2h(1-\tau)\ln(kh+h+1). \]
This together with (A.10) gives\[ (h(1 - \tau))^{-1}v \left( (kh + h + 1)^{1-\tau} - (ih + 2h + 1)^{1-\tau} \right) \geq 2\ln(kh + h + 1). \]

Then, it follows from (A.8) that\[ \sum_{i=k_0}^{k-1-\epsilon(k)} \frac{u}{(ih + 1)^{2\tau}} \prod_{j=i+1}^{k} \left(1 - \frac{v}{(jh + h + 1)^\tau} \right) = O \left( k^{-1} \right). \tag{A.11} \]

On the other hand, for the case with \( k - \epsilon(k) \leq i \leq k \), we have \( k \leq 2k - 2\epsilon(k) \leq 2i \), which shows that\[ \frac{u}{(ih + 1)^{2\tau}} \leq \frac{4^\tau u}{(kh + 2)^{2\tau}}, \quad k - \epsilon(k) \leq i \leq k. \]

Then it follows from\[ \prod_{j=i+1}^{k} \left(1 - \frac{v}{(jh + h + 1)^\tau} \right) \leq 1 \]
that\[ \sum_{i=k-\epsilon(k)}^{k} \frac{u}{(ih + 1)^{2\tau}} \prod_{j=i+1}^{k} \left(1 - \frac{v}{(jh + h + 1)^\tau} \right) = O \left( (kh + h)^{-\tau} \ln(kh + h) \right). \tag{A.12} \]
Thus, by (A.11)-(A.12), we obtain\[ \sum_{i=k_0}^{k} \frac{u}{(ih + 1)^{2\tau}} \prod_{j=i+1}^{k} \left(1 - \frac{v}{(jh + h + 1)^\tau} \right) = O \left( (kh + h)^{-\tau} \ln(kh + h) \right). \tag{A.13} \]
Combining (A.7), (A.9) and (A.13) directly shows that

$$
E[V(kh)] = O((kh + h)^{-\tau} \ln(kh + h)).
$$

By Lemma [A.1] there exists a positive integer $k_1$ such that

$$
E[V(k)] \leq 2^h E[V(m_k h)] + h^2 \gamma(m_k h), \quad k \geq k_1,
$$

where $\gamma(k) = O((k + 1)^{-2\tau})$ and $m_k = \lfloor \frac{k}{h} \rfloor$. Thus, we get

$$
T \sum_{t=0}^{T} E[V(t)] = O\left(\sum_{t=0}^{T} (t + 1)^{-\tau} \ln(t + 1)\right).
$$

Noting that

$$
T \sum_{t=0}^{T} (t + 1)^{-\tau} \ln(t + 1) = O\left(\int_{0}^{T} (x + 1)^{-\tau} \ln(x + 1) dx\right),
$$

we get (9).

**Proof of Theorem 4.** It follows from $\{G(k), k \geq 0\} \in \Gamma_1$ that $E[\hat{L}_{G(k)}|\mathcal{F}(k - 1)]$ is positive semi-definite. Noting that

$$
E\left[\hat{L}_{G(k)}|\mathcal{F}(mh - 1)\right] = E\left[E\left[\hat{L}_{G(k)}|\mathcal{F}(k - 1)\right]|\mathcal{F}(mh - 1)\right], \quad k \geq mh,
$$

we know that $E[\hat{L}_{G(k)}|\mathcal{F}(mh - 1)]$ is also positive semi-definite, which shows that

$$
\sum_{i=kh}^{(k+1)h-1} E\left[b(i)\hat{L}_{G(i)} \otimes I_n + a(i)\mathcal{H}^T(i)\mathcal{H}(i)|\mathcal{F}(kh - 1)\right]
\geq \min\{a((k + 1)h), b((k + 1)h)\} \sum_{i=kh}^{(k+1)h-1} E\left[\hat{L}_{G(i)} \otimes I_n + \mathcal{H}^T(i)\mathcal{H}(i)|\mathcal{F}(kh - 1)\right] \text{ a.s.}
$$

It follows from the conditions (i)-(ii) and Definitions [1,2] that there exist positive constants $\theta_1$ and $\theta_2$, such that

$$
\inf_{k \geq 0} \lambda_2 \left(\sum_{j=kh}^{(k+1)h-1} E\left[\hat{L}_{G(j)}|\mathcal{F}(kh - 1)\right]\right) \geq \theta_1 \text{ a.s.}
$$

and

$$
\inf_{k \geq 0} \lambda_{\text{min}} \left(\sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} E\left[H_i^T(j)H_i(j)|\mathcal{F}(kh - 1)\right]\right) \geq \theta_2 \text{ a.s.},
$$

respectively, which together with the condition (iii) and Lemma [2] gives

$$
\Lambda_k^h \geq (2Nh_0 + N\theta_1)^{-1} \theta_1 \theta_2 \min\{a((k + 1)h), b((k + 1)h)\} \text{ a.s.}
$$
Denote
\[ L(k) = (2Nh\rho_0 + N\theta_1)^{-1}\theta_1\theta_2 \min\{a((k+1)h), b((k+1)h)\} + \sum_{i=k}^{(k+1)h-1} \lambda(i), \]
we have
\[ \Lambda_k^h + \sum_{i=k}^{(k+1)h-1} \lambda(i) \geq L(k) \text{ a.s.}, \quad (A.14) \]

By the proof of Lemma 1, we have
\[
\Omega(k) = (h+1)(9^h - 1 - 4h)(\rho_0 a(kh) + \lambda(kh))^2 + 2^{h+3}hN^2\beta_v\sigma^2\rho_0^2a^2(kh) \\
+ 2^{h+5}hN^2\beta_v\sigma^2\rho_0^2a^2(kh) + \sum_{i=k}^{(k+1)h-1} \lambda(i) \\
\leq (9^h - 4h - 1)(h+1)(\rho_0 + 1)^2 + 2^{h+3}hN^2\beta_v\sigma^2\rho_0^2 + 2^{h+5}hN^2\beta_v\sigma^2\rho_0^2 + h \\
\triangleq \frac{c_1}{(kh + 1)^{2\tau}}, \quad \forall \ k \geq 0,
\]
Noting that
\[ L(k) \geq \frac{\theta_1\theta_2}{(2Nh\rho_0 + N\theta_1)((k+1)h+1)^\tau} \triangleq \frac{c_2}{((k+1)h+1)^\tau}, \quad \forall \ k \geq 0, \]
and
\[ L(k) \leq \frac{\theta_1\theta_2}{(2Nh\rho_0 + N\theta_1)(kh+1)^\tau} + \frac{h}{(kh+1)^\tau} \triangleq \frac{c_3}{(kh+1)^\tau}, \quad \forall \ k \geq 0, \]

Denote
\[ k_0 = \left\lceil \frac{4c_2^2}{c_3^2} \right\rceil + 1. \]

We can verify that \( 0 < \Omega(k) \leq L(k) < 1, \ k \geq k_0. \) By \( (A.14) \) and the proof of Lemma 1, we obtain
\[ \mathbb{E}[V((k+1)h)|\mathcal{F}(kh-1)] \leq (1 + \Omega(k))V(kh) - 2L(k)V(kh) + \Gamma(k) \text{ a.s., } k \geq k_0, \quad (A.15) \]
where
\[
\Gamma(k) = 2h\rho_0\beta_v a^2(kh) + (2^{h+3}h\sigma^2 + 4b^2)hN^2\beta_v\rho_0^2b^2(kh) + 8ha(kh)b(kh)(\beta_v \rho_0 \\
+ N^2\beta_v\rho_0^2(2^{h+2}h\sigma^2 + 2b^2)) + 2hN\|x_0\|^2\lambda(kh) + 2Nh^2\|x_0\|^2\lambda^2(kh) \\
\leq 2h\rho_0\beta_v a^2(kh) + (2^{h+3}h\sigma^2 + 4b^2)hN^2\beta_v\rho_0^2a^2(kh) + 8ha^2(kh)(\beta_v \rho_0 \\
+ N^2\beta_v\rho_0^2(2^{h+2}h\sigma^2 + 2b^2)) + 2hN\|x_0\|^2\lambda^2(kh) + 2Nh^2\|x_0\|^2\lambda^2a^2(kh) \\
\triangleq \frac{c_4}{(kh + 1)^{2\tau}}, \quad \forall \ k \geq 0.
\]
Denote 
\[ c_5 = (2N\max\{\rho_0, 1\} + N\theta_1)h^\tau)^{-1}\min\{\theta_1, 1\}\min\{\theta_2, 1\}, \]
it follows that 
\[ L(k) \geq c_5(kh + h + 1)^{-\tau}, \quad k \geq k_0. \]

By taking mathematical expectation on both sides of (A.15), we get
\[ \mathbb{E}[V((k + 1)h)] \leq \left(1 - \frac{c_5}{(kh + h + 1)^\tau}\right)\mathbb{E}[V(kh)] + \frac{c_4}{(kh + 1)^2}\tau, \quad \forall \ k > k_0. \]  
(A.16)
By Theorem 2 we have \( \lim_{k \to \infty} \mathbb{E}[V(k)] = 0 \), which shows that there exists a constant \( c_6 > 0 \), such that \( \sup_{k \geq 0} \mathbb{E}[V(k)] \leq c_6 \), which together with Lemma 2.9 in [44] gives
\[ \mathbb{E}[V(kh)] \leq \frac{25c_4\ln(k + 1)}{c_5(k + 1)^\tau} + c_6e^{-c_5\frac{(k+1)^{1-\tau} - (k_0+2)^{1-\tau}}{1-\tau}}, \]
where
\[ k \geq \left[ \frac{12}{c_5(1-\tau)}\ln\left(\frac{4}{c_5(1-\tau)}\right) \right]^{1/\tau} + 1 + k_0. \]

Denote \( m_k = \lfloor \frac{k}{h} \rfloor, \quad \forall \ k \geq 0 \). Noting that \( m_k h \leq k < (m_k + 1)h \), then for
\[ k \geq \left[ \frac{12}{c_5(1-\tau)}\ln\left(\frac{4}{c_5(1-\tau)}\right) \right]^{1/\tau} + \frac{4c_1^2}{c_2^2} + c_3^2 \] + 2,
it follows from (A.16) and Lemma [A.1] that
\[
\begin{align*}
\mathbb{E}[V(k)] & \leq 2^h\mathbb{E}[V(m_k h)] + h^{2^h}\gamma(m_k h) \\
& \leq 2^h\left(\frac{25c_4\ln(m_k + 1)}{c_5(m_k + 1)^\tau} + c_6e^{-c_5\frac{(m_k+1)^{1-\tau} - (k_0+2)^{1-\tau}}{1-\tau}}\right) + h^{2^h}\gamma(m_k h) \\
& = 2^h\left(\frac{25c_4\ln(m_k + 1)}{c_5(m_k + 1)^\tau} + c_6e^{-c_5\frac{(m_k+1)^{1-\tau} - (k_0+2)^{1-\tau}}{1-\tau}}\right) + h^{2^h}(4b^2a^2(m_k h)N^2\rho_0^2\beta_v \]
\[ + 2a^2(m_k h)\rho_0\beta_v + (N\|x_0\|^2 + a^2(m_k h)\rho_0^2)\lambda(m_k h) + N\|x_0\|^2\lambda^2(m_k h)) \]
\[ \leq 2^h\left(\frac{25c_4\ln(m_k + 1)}{c_5(m_k + 1)^\tau} + c_6e^{-c_5\frac{(m_k+1)^{1-\tau} - (k_0+2)^{1-\tau}}{1-\tau}}\right) + h^{2^h}(4b^2a^2(m_kh)N^2\rho_0^2\beta_v \]
\[ + 2a^2(m_k h)\rho_0\beta_v + (N\|x_0\|^2 + \rho_0^2)\lambda^2\left(\frac{k}{2}\right) + N\|x_0\|^2a^2\left(\frac{k}{2}\right)) \]
\[ \Delta 2^h\left(\frac{25c_4\ln(k + 1)}{c_5\left(\frac{k}{2h}\right)^\tau} + c_6e^{-c_5\frac{(\left(\frac{k}{2h}\right)^{1-\tau} - (k_0+2)^{1-\tau}}{1-\tau}}\right) + \frac{c_7}{(k + 2)^2}\tau \]
\[ \leq 2^h\left(\frac{25c_4\ln(k + 1)}{c_5\left(\frac{k}{2h}\right)^\tau} + c_6e^{-c_5\frac{(\left(\frac{k}{2h}\right)^{1-\tau} - (k_0+2)^{1-\tau}}{1-\tau}}\right) + \frac{c_7}{k^{2\tau}.} \]
\]
APPENDIX B

PROOFS OF LEMMAS [12]

Proof of Lemma [1]. By the estimation error equation (8), we have

\[ \mathbb{E}[V((k + 1)h)|\mathcal{F}(kh - 1)] = \mathbb{E}\left[ \sum_{i=1}^{4} A_i^T(k)A_i(k) + 2 \sum_{1 \leq i < j \leq 4} A_i^T(k)A_j(k)|\mathcal{F}(kh - 1) \right], \quad (B.1) \]

where \( A_1(k) = \Phi_P((k + 1)h - 1, kh)\delta(kh), \ A_2(k) = \sum_{i=kh}^{(k+1)h-1} a(i)\Phi_P((k + 1)h - 1, i + 1)\mathcal{H}^T(i)v(i), \ A_3(k) = \sum_{i=kh}^{(k+1)h-1} b(i)\Phi_P((k + 1)h - 1, i + 1)W(i)M(i)\xi(i), \) and \( A_4(k) = \sum_{i=kh}^{(k+1)h-1} \lambda(i)\Phi_P((k + 1)h - 1, i + 1)\mathcal{H}(i)\).

We now consider the RHS of (B.1) term by term. It follows from Assumption (A2) that \( \Phi_P^T((k+1)h-1, kh)\Phi_P((k+1)h-1, i+1)\mathcal{H}^T(i) \) and \( v(i) \) are independent, \( kh \leq i \leq (k+1)h-1 \), which further shows that \( \Phi_P^T((k+1)h-1, kh)\Phi_P((k+1)h-1, i+1)\mathcal{H}^T(i) \) and \( v(i) \) are conditionally independent w.r.t. \( \mathcal{F}(kh - 1) \) by Lemma A.1 in [30]. Noting that \( \delta(kh) \in \mathcal{F}(kh - 1) \), we have

\[ \mathbb{E} \left[ A_1^T(k)A_2(k)|\mathcal{F}(kh - 1) \right] = 0. \quad (B.2) \]

Similarly, we also have

\[ \mathbb{E} \left[ A_1^T(k)A_3(k) + A_2^T(k)A_4(k) + A_3^T(k)A_4(k)|\mathcal{F}(kh - 1) \right] = 0. \quad (B.3) \]

By Lemma A.2 there exist positive integers \( k_2 \) and \( k_3 \), such that

\[ \mathbb{E} \left[ A_1^T(k)A_4(k)|\mathcal{F}(kh - 1) \right] \leq \frac{1}{2} \left( \sum_{i=kh}^{(k+1)h-1} \lambda(i) \right) (1 + p(k))V(kh) + Nh\|x_0\|^2\lambda(kh) \text{ a.s., } k \geq \max\{k_2, k_3\}, \quad (B.4) \]

where \( p(k) = (9h - 1 - 4h)(\rho_0 \max\{a(kh), b(kh)\} + \lambda(kh))^2 \). For \( kh \leq i \neq j \leq (k+1)h - 1 \), by Assumption (A2) and Lemma A.1 in [30], we know that

\[ \mathbb{E} \left[ v^T(i)\mathcal{H}(i)\Phi_P^T((k+1)h-1, i+1) \times \Phi_P((k+1)h-1, j+1)W(j)M(j)\xi(j)|\mathcal{F}(kh - 1) \right] = 0. \quad (B.5) \]

Therefore, by (B.5), Assumptions (A1)-(A2), Lemmas A.1-A.2 and Lemma A.1 in [30], there exists a positive integer \( k_1 \), such that

\[ \mathbb{E} \left[ A_2^T(k)A_3(k)|\mathcal{F}(kh - 1) \right] = 0. \]
\[ \begin{aligned}
&\leq 4 \sum_{i=kh}^{(k+1)h-1} a(i)b(i)E[\|H(i)\|^{2}\|v(i)\|^{2} + \|W(i)\|^{2}\|M(i)\|^{2}\|\xi(i)\|^{2}|\mathcal{F}(kh - 1)] \\
&\leq 4h(\beta v \rho_{0} + N^{2}\beta v \rho_{0}^{2}(2^{h+2}\sigma^{2}V(kh) + 2^{h+2}h\sigma^{2} + 2b^{2}))a(kh)b(kh) \ \text{a.s.,} \ k \geq \max\{k_{1}, k_{3}\}(\text{B.6})
\end{aligned} \]

Similar to (B.5), we get
\[ \begin{aligned}
\mathbb{E}\left[ v^{T}(i)H(i)\Phi_{P}^{T}\left((k+1)h - 1, i + 1\right)\Phi_{P}\left((k+1)h - 1, j + 1\right)H^{T}(j)v(j)|\mathcal{F}(kh - 1)\right] &= 0, \\
kh \leq j \neq i \leq (k+1)h - 1,
\end{aligned} \]

which together with Assumption (A2), Lemma A.2 and Lemma A.1 in [30] gives
\[ \begin{aligned}
\mathbb{E}\left[ A_{2}^{T}(k)A_{2}(k)|\mathcal{F}(kh - 1)\right] &\leq 2h\beta v \rho_{0}a^{2}(kh) \ \text{a.s.} \quad (\text{B.7})
\end{aligned} \]

for \( k \geq k_{3} \). Following the same way as (B.5), which along with Assumption (A2), Lemmas A.1-A.2 and Lemma A.1 in [30] leads to
\[ \begin{aligned}
\mathbb{E}\left[ A_{3}^{T}(k)A_{3}(k)|\mathcal{F}(kh - 1)\right] &\leq 2hN^{2}\beta v \rho_{0}^{2}(2^{h+2}\sigma^{2}V(kh) + 2^{h+2}h\sigma^{2} + 2b^{2})b^{2}(kh) \ \text{a.s.}, \quad (\text{B.8})
\end{aligned} \]

for \( k \geq \max\{k_{1}, k_{3}\} \). By Lemma A.2 we know that
\[ \begin{aligned}
\mathbb{E}\left[ A_{4}^{T}(k)A_{4}(k)|\mathcal{F}(kh - 1)\right] &\leq 2Nh^{2}\|x_{0}\|^{2}\lambda^{2}(kh) \ \text{a.s.}, \quad (\text{B.9})
\end{aligned} \]

for \( k \geq k_{3} \). By Lemma A.2 we have
\[ \begin{aligned}
\mathbb{E}\left[ A_{1}^{T}(k)A_{1}(k)|\mathcal{F}(kh - 1)\right] &\leq \left(1 - 2\Lambda_{k}^{h} - 2 \sum_{i=kh}^{(k+1)h-1} \lambda(i) + p(k)\right)V(kh) \ \text{a.s.,} \quad (\text{B.10})
\end{aligned} \]

for \( k \geq k_{2} \). Noting that \( \lambda(k) \) converges to zero, we know that there exists a positive integer \( k_{4} \), such that \( \lambda(k) \leq 1, \ k \geq k_{4} \). Substituting (B.2)-(B.4) and (B.6)-(B.10) into (B.1) leads to
\[ \begin{aligned}
\mathbb{E}\left[ V((k + 1)h)|\mathcal{F}(kh - 1)\right] &\leq (1 + \Omega(k))V(kh) - \left(2\Lambda_{k}^{h} + 2 \sum_{i=kh}^{(k+1)h-1} \lambda(i)\right)V(kh) + \Gamma(k) \ \text{a.s.,} \ k \geq k_{0},
\end{aligned} \]
where
\[
\begin{aligned}
k_0 &= \max\{k_1, k_2, k_3, k_4\}, \\
\Omega(k) &= p(k) + 2^{h+3}hN^2\beta_v\sigma^2\rho^2(b \lambda(k)) + 2^{h+5}hN^2\beta_v\sigma^2 \\
&\quad \times \rho^2a(kh)b(kh) + hp(k) + \sum_{i=kh}^{(k+1)h-1} \lambda(i), \\
\Gamma(k) &= 2hp_0\beta_va^2(kh) + (2^{h+3}h\sigma^2 + 4b^2)hN^2\beta_v\rho^2b^2(kh) \\
&+ 8ha(kh)b(kh)(\beta_v p_0 + N^2\beta_v p_0^2(2^{h+2}h\sigma^2 + 2b^2)) \\
&+ 2hN\|x_0\|^2\lambda(k) + 2Nh^2\|x_0\|^2\lambda^2(kh).
\end{aligned}
\]

Noting that \( p(k) = O(a^2(kh) + b^2(kh) + \lambda^2(kh)) \), we get \( \Omega(k) + \Gamma(k) = O(a^2(kh) + b^2(kh) + \lambda(kh)) \).

**Proof of Lemma** It follows from \( \mathcal{L}_{\hat{G}(k)} = D_{\hat{G}(k)} - A_{\hat{G}(k)} \) that \( 1_N \) is the eigenvector corresponding to the zero eigenvalue of \( \mathcal{L}_{\hat{G}(k)} \). \( \mathbb{E}[\hat{L}_{\hat{G}(k)}|\mathcal{F}(k-1)] \) has a unique zero eigenvalue with eigenvector \( 1_N \) since \( \hat{G}(k) \) is conditionally balanced. Therefore, \( \mathbb{E}[\sum_{i=kh}^{(k+1)h-1} \hat{L}_{\hat{G}(i)} \otimes I_n|\mathcal{F}(kh-1)] \) has only \( n \) zero eigenvalues, whose corresponding eigenvectors consist of \( s_1(k) = \frac{1}{\sqrt{N}} 1_N \otimes e_1, \cdots, s_n(k) = \frac{1}{\sqrt{N}} 1_N \otimes e_n \), where \( e_i \) is the standard orthonormal basis in \( \mathbb{R}^n \). Suppose that the remaining eigenvalues of \( \mathbb{E}[\sum_{i=kh}^{(k+1)h-1} \hat{L}_{\hat{G}(i)} \otimes I_n|\mathcal{F}(kh-1)] \) are \( \gamma_{n+1}(k), \cdots, \gamma_{Nn}(k) \), and the corresponding unit orthogonal eigenvectors are \( s_{n+1}(k), \cdots, s_{Nn}(k) \). Then, given a positive integer \( k \), for any unit vector \( \eta \in \mathbb{R}^{Nn} \), there exist \( r_i(k) \in \mathbb{R}, 1 \leq i \leq Nn \), such that
\[
\eta = \eta_1(k) + \eta_2(k),
\]
where
\[
\begin{aligned}
\eta_1(k) &= \sum_{i=1}^{n} \frac{1}{\sqrt{N}} r_i(k)s_i(k), \\
\eta_2(k) &= \sum_{i=n+1}^{N} r_i(k)s_i(k), \\
\sum_{i=1}^{Nn} r_i^2(k) &= 1.
\end{aligned}
\]
Denote $H_k = \text{diag}(H_{k,1}, \ldots, H_{k,N})$, where

$$
H_{k,i} = \sum_{j=kh}^{(k+1)h-1} \mathbb{E}[H_i^T(j) H_i(j) | \mathcal{F}(kh-1)],
$$

$$
\mathcal{L}_k = \sum_{i=kh}^{(k+1)h-1} \mathbb{E}[\tilde{G}_i(i) | \mathcal{F}(kh-1)],
$$

$$
u_{st}(k) = \eta_s^T(k) H_k \eta_t(k), \quad s, t = 1, 2.
$$

It follows that

$$
\eta^T \left( \sum_{i=kh}^{(k+1)h-1} \mathbb{E} \left[ \tilde{G}_i(i) \otimes I_n + H^T(i) H(i) | \mathcal{F}(kh-1) \right] \right) \eta
$$

$$
= \sum_{i=1}^{2} (u_{ii}(k) + \tilde{w}_{ii}(k)) + 2u_{12}(k) + 2\tilde{w}_{12}(k). \tag{B.11}
$$

Denote

$$
\zeta(k) = \frac{2h\rho_0}{2h\rho_0 + \lambda_1(\mathcal{L}_k)}.
$$

Noting that $H_k$ is positive semi-definite, by Cauchy inequality, we have

$$
2 \left| \eta_1^T(k) H_k \eta_2(k) \right| = 2 \left| \eta_1^T(k) H_k^2 \frac{1}{2} \eta_2(k) \right| \leq \zeta(k) \eta_1^T(k) H_k \eta_1(k) + \frac{1}{\zeta(k)} \eta_2^T(k) H_k \eta_2(k). \tag{B.12}
$$

Substituting (B.12) into (B.11) leads to

$$
\eta^T \left( \sum_{i=kh}^{(k+1)h-1} \mathbb{E} \left[ \tilde{G}_i(i) \otimes I_n + H^T(i) H(i) | \mathcal{F}(kh-1) \right] \right) \eta
$$

$$
\geq (1 - \zeta(k)) u_{11}(k) + \left( 1 - \frac{1}{\zeta(k)} \right) u_{22}(k) + \tilde{w}_{11}(k) + \tilde{w}_{22}(k) + 2\tilde{w}_{12}(k). \tag{B.13}
$$

We now consider the RHS of (B.13) item by item. Denote $A_k = [r_1(k), \ldots, r_n(k)]^T$ and $B_k = [s_1(k), \ldots, s_n(k)]$, we have

$$
u_{11}(k) = \eta_1^T(k) H_k \eta_1(k) = A_k^T B_k H_k B_k A_k. \tag{B.14}
$$

Noting that $H_k = \text{diag}(H_{k,1}, \ldots, H_{k,N})$ and $s_i(k) = \frac{1}{\sqrt{N}} I_N \otimes e_i$, $1 \leq i \leq n$, it follows that

$$
B_k^T H_k B_k = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ H_i^T(j) H_i(j) | \mathcal{F}(kh-1) \right].
$$

By (B.14), we get

$$
u_{11}(k) \geq \frac{1}{N} \lambda_{\min} \left( \sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ H_i^T(j) H_i(j) | \mathcal{F}(kh-1) \right] \left( \sum_{i=1}^{n} r_i^2(k) \right) \right). \tag{B.15}
$$
It can be verified that
\[ u_{22}(k) \leq \| H_k \| \| \eta_2(k) \|^2 \leq h \rho_0 \sum_{i=n+1}^{Nn} r_i^2(k) \text{ a.s.} \] (B.16)

Noting that \( 1 - \frac{1}{\zeta(k)} < 0 \), by (B.16), we have
\[ \left( 1 - \frac{1}{\zeta(k)} \right) u_{22}(k) \geq h \rho_0 \left( 1 - \frac{1}{\zeta(k)} \right) \sum_{i=n+1}^{n} r_i^2(k). \] (B.17)

It follows from
\[ \left( \mathbb{E} \left[ \hat{L}_{G(i)} | \mathcal{F} \left( m_i h - 1 \right) \right] \otimes I_n \right) \left( 1_N \otimes e_j \right) = \left( \mathbb{E} \left[ \hat{L}_{G(i)} | \mathcal{F} \left( m_i h - 1 \right) \right] 1_N \right) \otimes e_j = 0 \]
that
\[ \tilde{w}_{11}(k) = \tilde{w}_{12}(k) = 0. \] (B.18)

Since \( G(k \mid k - 1) \) is balanced, then
\[ \lambda_{n+1} (\mathcal{L}_k \otimes I_n) = \lambda_2 (\mathcal{L}_k) > 0, \]
where \( \lambda_{n+1}(\cdot) \) denotes the \((n + 1)\)-th smallest eigenvalue. Noting that \( \{ s_i(k), n + 1 \leq i \leq Nn \} \) is an orthonormal system, we know that
\[ \tilde{w}_{22}(k) \geq \lambda_2(\mathcal{L}_k) \left( 1 - \sum_{i=1}^{n} r_i^2(k) \right) \text{ a.s.} \] (B.19)

Denote
\[
F_k(x) = \left( \frac{1 - \zeta(k)}{N} \right) \lambda_{\min} \left( \sum_{i=1}^{N} \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ H_i^T(j) H_i(j) \mid \mathcal{F}(kh - 1) \right] \right) - \lambda_2(\mathcal{L}_k) - h \rho_0 + \frac{h \rho_0}{\zeta(k)} x \\
+ \lambda_2(\mathcal{L}_k) + h \rho_0 - \frac{h \rho_0}{\zeta(k)}, \quad x \in \mathbb{R}.
\]

Then, by (B.13) and (B.15)-(B.19), we have
\[ \eta^T \left( \sum_{i=kh}^{(k+1)h-1} \mathbb{E} \left[ \hat{L}_{G(i)} \otimes I_n + \mathcal{H}_i^T H_i \mid \mathcal{F}(kh - 1) \right] \right) \eta \geq F_k \left( \sum_{i=1}^{n} r_i^2(k) \right) \text{ a.s.,} \] (B.20)

which along with \( \zeta(k) = \frac{2h \rho_0}{2h \rho_0 + \lambda_2(\mathcal{L}_k)} \) gives
\[ \frac{dF_k(x)}{dx} \leq 0 \text{ a.s.} \]

Hence, the function \( F_k(x) \) is monotonically decreasing. It follows from (B.20) and
\[ 0 \leq \sum_{i=1}^{n} r_i^2(k) \leq 1 \]
that

\[ \eta^T \left( \sum_{i=kh}^{(k+1)h-1} \mathbb{E} \left[ \mathcal{L}_{G(i)} \otimes I_n + \mathcal{H}^T(i) \mathcal{H}(i) | \mathcal{F}(kh-1) \right] \right) \eta \geq F_k(1) \text{ a.s.}, \]

which completes the proof.

**APPENDIX C**

**KEY LEMMAS**

**Lemma A.1.** For the algorithm \( (\mathcal{G}) \), if Assumptions (A1)-(A2) hold, the algorithm gains \( a(k) \), \( b(k) \) and \( \lambda(k) \) monotonically decrease to zero, and there exists a positive constant \( \rho_0 \), such that \( \sup_{k \geq 0} (\| \mathcal{L}_{G(k)} \| + (\mathbb{E}[\| \mathcal{H}^T(k) \mathcal{H}(k) \|^2 | \mathcal{F}(k-1)]^{1/2} \) \( \leq \rho_0 \) a.s., then there exist nonnegative deterministic sequences \( \{\alpha(k), k \geq 0\} \) and \( \{\gamma(k), k \geq 0\} \) satisfying \( \alpha(k) = o(1) \) and \( \gamma(k) = \mathcal{O}(a^2(k) + b^2(k) + \lambda(k)) \), such that \( \mathbb{E}[V(k) | \mathcal{F}(m_kh-1)] \leq \prod_{i=m_kh}^{k-1} (1 + (1 + \alpha(j))V(m_kh) + \sum_{i=m_kh}^{k-1} \gamma(i) \prod_{j=i+1}^{k-1} (1 + \alpha(j)) \text{ a.s.}, h \geq 1, \text{ where } m_k = \lfloor \frac{k}{h} \rfloor, \text{ especially, there exists a positive integer } k_1, \text{ such that } \mathbb{E}[V(k) | \mathcal{F}(m_kh-1)] \leq 2^k V(m_kh) + h^2 \gamma(m_kh) \text{ a.s.}, k \geq k_1. \)

**Proof.** By the estimation error equation \( (\mathcal{S}) \), we have

\[
V(k + 1) = (1 - \lambda(k))^2 V(k) - (1 - \lambda(k)) \delta^T(k) (D(k) + D^T(k)) \delta(k) + \delta^T(k) D^T(k) D(k) \delta(k) + S^T(k) S(k) + 2 S^T(k) ((1 - \lambda(k)) I_N - D(k)) \delta(k) + \lambda^2(k) \| 1_N \otimes x_0 \|^2 - 2 \lambda(k) \delta^T(k) ((1 - \lambda(k)) I_N - D(k)) (1_N \otimes x_0) - 2 \lambda(k) S^T(k) (1_N \otimes x_0) \leq (1 - \lambda(k))^2 V(k) + (1 - \lambda(k)) \| \delta^T(k) (D(k) + D^T(k)) \delta(k) \| + \| D(k) \|^2 \| \delta(k) \|^2 + \| S(k) \|^2 - 2 \lambda(k) S^T(k) (1_N \otimes x_0) + 2 S^T(k) ((1 - \lambda(k)) I_N - D(k)) \delta(k) + N \lambda^2(k) \| x_0 \|^2 + 2 \lambda(k) \| \delta^T(k) ((1 - \lambda(k)) I_N - D(k)) (1_N \otimes x_0) \| , \tag{C.1}
\]

where \( S(k) = a(k) \mathcal{H}^T(k) \psi(k) + b(k) W(k) M(k) \xi(k), D(k) = b(k) \mathcal{L}_{G(k)} \otimes I_n + a(k) \mathcal{H}^T(k) \mathcal{H}(k). \)

We now consider the conditional mathematical expectation of each term on the RHS of \( (C.1) \) w.r.t. \( \mathcal{F}(m_kh-1) \). Noting that \( k - 1 \geq m_kh - 1 \), by Assumption (A2), we get

\[
\mathbb{E} \left[ S^T(k) ((1 - \lambda(k)) I_N - D(k)) \delta(k) | \mathcal{F}(m_kh-1) \right] = \mathbb{E} \left[ a(k) \psi^T(k) \mathcal{H}(k) ((1 - \lambda(k)) I_N - D(k)) \delta(k) | \mathcal{F}(m_kh-1) \right] + \mathbb{E} \left[ b(k) \xi^T(k) M^T(k) W^T(k) ((1 - \lambda(k)) I_N - D(k)) \delta(k) | \mathcal{F}(m_kh-1) \right] = \mathbb{E} \left[ a(k) \psi^T(k) \mathcal{H}(k) ((1 - \lambda(k)) I_N - D(k)) \delta(k) | \mathcal{F}(k-1) \right] | \mathcal{F}(m_kh-1) \]

\[
+ \mathbb{E} \left[ b(k) \xi^T(k) M^T(k) W^T(k) ((1 - \lambda(k)) I_N - D(k)) \delta(k) | \mathcal{F}(k-1) \right] | \mathcal{F}(m_kh-1) \]
\[= a(k) \mathbb{E} \left[ v^T(k) | \mathcal{F}(k-1) \right] \mathbb{E} \left[ \mathcal{H}(k)((1 - \lambda(k)) I_{N_k} - D(k)) | \mathcal{F}(k-1) \right] \delta(k) | \mathcal{F}(m_k h - 1) \]
\[+ b(k) \mathbb{E} \left[ \xi^T(k) | \mathcal{F}(k-1) \right] M^T(k) \]
\[\times \mathbb{E} \left[ W^T(k)((1 - \lambda(k)) I_{N_k} - D(k)) | \mathcal{F}(k-1) \right] \delta(k) | \mathcal{F}(m_k h - 1) \]
\[= 0, \qquad (C.2)\]

where the penultimate equality is due to \( \delta(k) \in \mathcal{F}(k-1) \), \( M(k) \in \mathcal{F}(k-1) \) and Lemma A.1 in \([30]\). Following the same way as above, we have
\[ \mathbb{E} \left[ S^T(k)(1_N \otimes x_o) | \mathcal{F}(m_k h - 1) \right] \]
\[= \mathbb{E} \left[ a(k) v^T(k) \mathcal{H}(k) + b(k) \xi^T(k) M^T(k) W^T(k) | \mathcal{F}(m_k h - 1) \right] (1_N \otimes x_o) \]
\[= a(k) \mathbb{E} \left[ v^T(k) | \mathcal{F}(k-1) \right] \mathbb{E} \left[ \mathcal{H}(k) | \mathcal{F}(k-1) \right] | \mathcal{F}(m_k h - 1) \]
\[+ b(k) \mathbb{E} \left[ \xi^T(k) | \mathcal{F}(k-1) \right] M^T(k) \mathbb{E} \left[ W^T(k) | \mathcal{F}(k-1) \right] | \mathcal{F}(m_k h - 1) \]
\[= 0. \qquad (C.3)\]

Denote \( q(k) = \max\{a(k), b(k)\} \). Noting that \( V(k) \in \mathcal{F}(k-1) \), we obtain
\[ \mathbb{E} \left[ \| D(k) \|^2 | \mathcal{F}(m_k h - 1) \right] \]
\[\leq \mathbb{E} \left[ (b(k) \| L_{\mathcal{G}(k)} \| + a(k) \| H^T(k) \mathcal{H}(k) \|)^2 | \mathcal{F}(m_k h - 1) \right] \]
\[\leq q^2(k) \mathbb{E} \left[ ( \| L_{\mathcal{G}(k)} \| + \| H^T(k) \mathcal{H}(k) \|)^2 V(k) | \mathcal{F}(k-1) \right] | \mathcal{F}(m_k h - 1) \]
\[= q^2(k) \mathbb{E} \left[ ( \| L_{\mathcal{G}(k)} \| + \| H^T(k) \mathcal{H}(k) \|)^2 | \mathcal{F}(k-1) \right] V(k) | \mathcal{F}(m_k h - 1) \]
\[\leq 2q^2(k) \rho_0^2 \mathbb{E}[V(k) | \mathcal{F}(m_k h - 1)] \text{ a.s.} \qquad (C.4)\]

It follows from the definition of \( M(k) \) that \( \mathbb{E}[\| M(k) \|^2 | \mathcal{F}(m_k h - 1)] \leq 4\sigma^2 \mathbb{E}[V(k) | \mathcal{F}(m_k h - 1)] + 2b^2 \), which together with Assumption (A2) and Lemma A.1 in \([30]\) gives
\[ \mathbb{E} \left[ \| S(k) \|^2 | \mathcal{F}(m_k h - 1) \right] \]
\[\leq 2q^2(k) \mathbb{E} \left[ \| H^T(k) \|^2 | \mathcal{F}(k-1) \right] \]
\[\leq 2q^2(k) \mathbb{E} \left[ \| W(k) \|^2 | \mathcal{F}(k-1) \right] \]
\[\leq 2q^2(k) \mathbb{E} \left[ \| W(k) \|^2 | \mathcal{F}(k-1) \right] \mathbb{E} \| L_{\mathcal{G}(k)} \| | \mathcal{F}(k-1) \]
\[\leq 2q^2(k) \mathbb{E} \left[ \| W(k) \|^2 | \mathcal{F}(k-1) \right] \mathbb{E} \| \xi(k) \|^2 | \mathcal{F}(k-1) \]
\[\leq 2q^2(k) \rho_0 \beta_v + 2q^2(k) N^2 \rho_0^2 \beta_v (4\sigma^2 \mathbb{E}[V(k) | \mathcal{F}(m_k h - 1)] + 2b^2) \text{ a.s.} \qquad (C.5)\]

Noting that
\[ \mathbb{E} \left[ \| \delta^T(k) (D(k) + D^T(k)) \delta(k) \| | \mathcal{F}(m_k h - 1) \right] \]
\[\leq 2 \mathbb{E} \left[ V(k) (b(k) \| L_{\mathcal{G}(k)} \| + a(k) \| H^T(k) \mathcal{H}(k) \|) | \mathcal{F}(m_k h - 1) \right] \]
\[= 2 \mathbb{E} \left[ V(k) (b(k) \| L_{\mathcal{G}(k)} \| + a(k) \| H^T(k) \mathcal{H}(k) \|) | \mathcal{F}(k-1) \right] | \mathcal{F}(m_k h - 1) \]
For the algorithm (6), if the algorithm gains Lemma A.2.

\[
\text{by mean value inequality, we have}
\]

\[
\begin{align*}
\mathbb{E} \left[ \delta^T (k) \left( (1 - \lambda(k)) I_{Nn} - D^T (k) \right) (1_N \otimes x_0) \right] & \leq \mathbb{E} \left[ \left\| \delta (k) \right\| \left\| (1 - \lambda(k)) I_{Nn} - D^T (k) \right\| 1_N \otimes x_0 \right] \mathcal{F}(m_k h - 1) \\
& \leq \mathbb{E} \left[ \left\| \delta (k) \right\| \left\| (1 - \lambda(k)) I_{Nn} - D^T (k) \right\| \right] 1_N \otimes x_0 \mathcal{F}(m_k h - 1) \\
& \leq \mathbb{E} \left[ \left\| \delta (k) \right\| \left\| (1 - \lambda(k)) + b(k) \right\| \mathcal{L}_G(k) \right] + a(k) \left\| \mathcal{H}^T (k) \mathcal{H}(k) \right\| \left\| \mathcal{F}(m_k h - 1) \right\| \\
& \leq \left( \mathbb{E}[V(k)|\mathcal{F}(m_k h - 1)] + N \|x_0\|^2 \right) \\
& + \frac{1}{2} \left( N \|x_0\|^2 \mathbb{E}[V(k)|\mathcal{F}(m_k h - 1)] + q^2(k) \rho_0^2 \right) \text{ a.s.,}
\end{align*}
\]

which together with (C.1)-(C.7) leads to

\[
\mathbb{E}[V(k+1)|\mathcal{F}(m_k h - 1)] \leq (1 + \alpha(k)) \mathbb{E}[V(k)|\mathcal{F}(m_k h - 1)] + \gamma(k) \text{ a.s.,}
\]

where \( \alpha(k) = \lambda^2 (k) + \lambda(k) (1 + N \|x_0\|^2) + 2q^2(k) \rho_0^2 + 8q^2(k) N^2 \rho_0^2 \beta_0 \sigma^2 + 2\rho_0 |1 - \lambda(k)| q(k) \)
and \( \gamma(k) = 4b^2 q^2(k) N^2 \rho_0^2 \beta_0 + 2q^2(k) \rho_0 \beta_0 + (N \|x_0\|^2 + q^2(k) \rho_0^2) \lambda(k) + N \|x_0\|^2 \lambda^2 (k) \).

Thus, by (C.8), we get

\[
\mathbb{E}[V(k)|\mathcal{F}(m_k h - 1)] \leq \prod_{i=m_k h}^{k-1} (1 + \alpha(i)) V(m_k h) + \sum_{i=m_k h}^{k-1} \gamma(i) \prod_{j=i+1}^{k-1} (1 + \alpha(j)) \text{ a.s.}
\]

Since \( a(k), b(k) \) and \( \lambda(k) \) converge to 0, then \( \alpha(k) \to 0 \) and \( \gamma(k) \to 0 \) as \( k \to \infty \), from which we know that there exists a positive integer \( k_1 \) such that \( \alpha(i) \leq 1, i \geq k_1 \). Noting that \( 0 \leq k - m_k h < h \), the lemma is proved by (C.9). \( \square \)

**Lemma A.2.** For the algorithm (6), if the algorithm gains \( a(k), b(k) \) and \( \lambda(k) \) monotonically decrease satisfying \( \sum_{k=0}^{\infty} (a^2 (k) + b^2 (k) + \lambda^2 (k)) < \infty \), and there exists a positive integer \( h \) and a positive constant \( \rho_0 \), such that \( \sup_{k \geq 0} (\| \mathcal{L}_G(k) \| + (\mathbb{E}[\| \mathcal{H}^T (k) \mathcal{H}(k) \| \mathcal{F}(k - 1)])^{\frac{1}{2}})^{\max(1, 2)} \leq \rho_0 \) a.s., then there exits a positive integer \( k_2 \), such that \( \| \mathbb{E}[\Phi_p((k + 1) h - 1, k h) \Phi_p((k + 1) h - 1, k h) | \mathcal{F}(k h - 1)] \| \leq 1 - 2\lambda_i - 2 \sum_{i=k h}^{(k+1) h - 1} \lambda(i) + p(k) \) a.s., \( k \geq k_2 \), where \( p(k) = (9 h - 1 - 4 h)(\rho_0 \max \{a(k h), b(k h)\} + \lambda(k h))^2 \), especially, there exists a positive integer \( k_3 \), such that \( \| \mathbb{E}[\Phi_p((k + 1) h - 1, i + 1) \Phi_p((k + 1) h - 1, i + 1) | \mathcal{F}(k h - 1)] \| \leq 2 \) a.s., \( k \geq k_3, \forall i \in [k h, (k + 1) h - 1] \).
Proof. By the definitions of $D(k)$ and $P(k)$, it follows that

$$
\| \mathbb{E}[\Phi_p^T((k+1)h-1, kh)\Phi_p((k+1)h-1, kh)|\mathcal{F}(kh-1)] \|
$$

$$
= \| \mathbb{E}[(1 - \lambda(kh))I_{nn} - D^T(kh)) \times \cdots \times ((1 - \lambda((k+1)h-1))I_{nn} - D^T((k+1)h-1))
\times ((1 - \lambda((k+1)h-1))I_{nn} - D((k+1)h-1)) \times \cdots
\times ((1 - \lambda(kh))I_{nn} - D(kh))] |\mathcal{F}(kh-1)| \|
$$

$$
= \left\| I_{nn} - \sum_{i=kh}^{(k+1)h-1} \mathbb{E} [D^T(i) + D(i) + 2\lambda(i)I_{nn}|\mathcal{F}(kh-1)] + \mathbb{E} \left[ \sum_{i=2}^{2h} M_i(k)|\mathcal{F}(kh-1) \right] \right\|
$$

$$
\leq \left\| I_{nn} - \sum_{i=kh}^{(k+1)h-1} \mathbb{E} [D^T(i) + D(i) + 2\lambda(i)I_{nn}|\mathcal{F}(kh-1)] \right\| + \left\| \mathbb{E} \left[ \sum_{i=2}^{2h} M_i(k)|\mathcal{F}(kh-1) \right] \right\|
$$

(C.10)

where $M_i(k), i = 2, \cdots, 2h$ represent the $i$-th order terms in the binomial expansion of $\Phi_p^T((k+1)h-1, kh)\Phi_p((k+1)h-1, kh)$. By the definition of spectral radius, we have

$$
\max_{1 \leq i \leq Nn} |\lambda| \left( \sum_{j=kh}^{(k+1)h-1} \mathbb{E} [D^T(j) + D(j) + 2\lambda(j)I_{nn}|\mathcal{F}(kh-1)] \right)
$$

$$
\leq \left\| \sum_{j=kh}^{(k+1)h-1} \mathbb{E} [D^T(j) + D(j) + 2\lambda(j)I_{nn}|\mathcal{F}(kh-1)] \right\|
$$

$$
\leq 2 \sum_{j=kh}^{(k+1)h-1} \mathbb{E} [b(j) |\mathcal{G}(j)|] + a(j) |\mathcal{H}^T(j)\mathcal{H}(j)| + \lambda(j)|\mathcal{F}(kh-1)|
$$

$$
\leq 2 \sum_{j=kh}^{(k+1)h-1} \max\{a(j), b(j)\} \rho_0 + 2 \sum_{j=kh}^{(k+1)h-1} \lambda(j)
$$

$$
\leq 2 \sum_{j=kh}^{(k+1)h-1} (\rho_0 a(j) + \rho_0 b(j) + \lambda(j)) \text{ a.s.}
$$

(C.11)

Noting that algorithm gains decrease to zero, we know that the RHS of the last inequality of (C.11) converges to zero as $k \to \infty$, which is independent of the sample paths. Hence, there exists a positive integer $l_1$, such that

$$
\lambda_i \left( \sum_{j=kh}^{(k+1)h-1} \mathbb{E} [D^T(j) + D(j) + 2\lambda(j)|\mathcal{F}(kh-1)] \right) \leq 1, \ i = 1, \cdots, Nn, \ k \geq l_1, \text{ a.s.}
$$

from which we get

$$
\left\| I_{nn} - \sum_{j=kh}^{(k+1)h-1} \mathbb{E} [D^T(j) + D(j) + 2\lambda(j)I_{nn}|\mathcal{F}(kh-1)] \right\|
$$
By Cr-inequality, we have
\[
= \rho \left( I_{N_n} - \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ D^T(j) + D(j) + 2\lambda(j)I_{N_n} | \mathcal{F}(kh-1) \right] \right)
\]
\[
= \max_{1 \leq i \leq N_n} \left| \lambda_i \left( I_{N_n} - \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ D^T(j) + D(j) + 2\lambda(j)I_{N_n} | \mathcal{F}(kh-1) \right] \right) \right|
\]
\[
= 1 - \lambda_{\min} \left( \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ D^T(j) + D(j) + 2\lambda(j)I_{N_n} | \mathcal{F}(kh-1) \right] \right)
\]
\[
= 1 - 2\lambda_{\min} \left( \sum_{j=kh}^{(k+1)h-1} \mathbb{E} \left[ b(j)\bar{L}_{G(j)} \otimes I_n + a(j)H^T(j)H(j) + \lambda(j)I_{N_n} | \mathcal{F}(kh-1) \right] \right)
\]
\[
= 1 - 2\lambda_k^h - 2 \sum_{j=kh}^{(k+1)h-1} \lambda(j), \quad k \geq l_1. \quad (C.12)
\]

which together with conditional Hölder inequality and conditional Lyapunov inequality leads to
\[
\mathbb{E} \left[ \left| \prod_{j=1}^{r} \left( D(n_j) + \lambda(n_j)I_{N_n} \right) \right|^2 | \mathcal{F}(kh-1) \right]
\]
\[
\leq \left[ \mathbb{E} \left[ \left| b(j)L_{G(j)} \otimes I_{N_n} + a(j)H^T(j)H(j) \right|^2 | \mathcal{F}(kh-1) \right] \right] \frac{1}{2} \cdot \left[ \mathbb{E} \left[ \left| D(n_r) + \lambda(n_r)I_{N_n} \right|^2 | \mathcal{F}(kh-1) \right] \right] \frac{1}{2}
\]
\[
\leq 2\left( \rho_0q(r) + \lambda(r) \right) \left[ \mathbb{E} \left[ \left| \prod_{j=1}^{r-1} \left( D(n_j) + \lambda(n_j)I_{N_n} \right) \right|^2 | \mathcal{F}(kh-1) \right] \right] \frac{1}{2}
\]
\[
\leq 2 \rho_0q(n_r) + \lambda(n_r) \leq 2 \rho_0q(n_r) + \lambda(n_j) \quad \text{a.s.,} \quad 1 \leq r \leq h, \quad kh \leq j \leq (k+1)h-1. \quad (C.13)
\]

where \(kh \leq n_1 \leq \cdots \leq n_r \leq (k+1)h-1\). On one hand, we have
\[
\mathbb{E}[|M_i(k)| | \mathcal{F}(kh-1)]
\]
\[
= \mathbb{E} \left[ \left| \sum_{s+t=i}^{s} \prod_{w=1}^{t} \left( D^T(n_t) + \lambda(n_t)I_{N_n} \right) \right| \prod_{w=1}^{t} \left( D(v_{t+1-w}) + \lambda(v_{t+1-w})I_{N_n} \right) | \mathcal{F}(kh-1) \right]
\]
\[
\leq \sum_{s+t=i} \mathbb{E} \left[ \left| \prod_{t=1}^{s} \left( D^T(n_t) + \lambda(n_t)I_{N_n} \right) \right| \left| \prod_{w=1}^{t} \left( D(v_{t+1-w}) + \lambda(v_{t+1-w})I_{N_n} \right) \right| | \mathcal{F}(kh-1) \right]
\]
where \( kh \leq n_1 \leq \cdots \leq n_s \leq (k + 1)h - 1 \), \( kh \leq v_1 \leq \cdots \leq v_t \leq (k + 1)h - 1 \). Noting that there exists a positive integer \( l_2 \), such that \( \rho_0(q) + \lambda(k) \leq 1, k \geq l_2 \). By (C.14), we get

\[
\mathbb{E}[\|M_i(k)\| \mathcal{F}(kh - 1)] \leq C_{2h}^i(\rho_0(qkh) + \lambda(kh))^2 \text{ a.s., } 2 \leq i \leq h, \quad k \geq l_2. \tag{C.15}
\]

On the other hand, for the case with \( h < i \leq 2h \), each term of \( M_i(k) \) is multiplied by at most \( i \) (\( 2 \leq i \leq h \)) different elements, where the matrix and its transpose are regarded as the same element. Noting that \( \|A^T A\| = \|A\|^2 \) for any matrix \( A \). By (C.13), we get

\[
\mathbb{E}[\|M_i(k)\| \mathcal{F}(kh - 1)] \leq C_{2h}^i(\rho_0(qkh) + \lambda(kh))^2 \text{ a.s., } h < i \leq 2h, \quad k \geq l_2. \tag{C.16}
\]

By (C.15) and (C.16), we have

\[
\left\| \mathbb{E} \left[ \sum_{i=2}^{2h} M_i(k) \mathcal{F}(kh - 1) \right] \right\| \leq \sum_{i=2}^{2h} C_{2h}^i(\rho_0(qkh) + \lambda(kh))^2 = (9^h - 1 - 4h)(\rho_0(qkh) + \lambda(kh))^2 \text{ a.s., } k \geq l_2. \tag{C.17}
\]

Denote the RHS of (C.17) by \( p(k) \), we know that \( \sum_{k=0}^{\infty} p(k) < \infty \). Denote \( k_2 = \max\{l_1, l_2\} \).

By (C.10), (C.12) and (C.17), we get

\[
\left\| \mathbb{E} \left[ \Phi_P^T((k + 1)h - 1, kh)\Phi_P((k + 1)h - 1, kh) \mathcal{F}(kh - 1) \right] \right\| \leq 1 - 2\Lambda_h^k - 2 \sum_{i=kh}^{(k+1)h-1} \lambda(i) + p(k) \text{ a.s., } k \geq k_2.
\]

Furthermore, denote \( G_j(k, i) \) the \( j \)-th order terms in the binomial expansion of \( \Phi_P^T((k + 1)h - 1, i + 1)\Phi_P((k + 1)h - 1, i + 1) \). Similar to (C.16), we get \( \mathbb{E}[\|G_j(k, i)\| \mathcal{F}(i)] \leq C_{2((k+1)h-i-1)}^j(\rho_0(q(kh) + \lambda(kh))^2 \text{ a.s., } k \geq k_2, \quad kh \leq i \leq (k + 1)h - 1, \quad 2 \leq j \leq 2((k+1)h-i-1), \) which along with (C.11) gives

\[
\left\| \mathbb{E} \left[ \Phi_P^T((k + 1)h - 1, i + 1)\Phi_P((k + 1)h - 1, i + 1) | \mathcal{F}(i) \right] \right\| = \left\| I_{nn} - \sum_{j=i+1}^{(k+1)h-1} \mathbb{E} \left[ D^T(j) + D(j) + 2\lambda(j)I_{nn} | \mathcal{F}(i) \right] + \mathbb{E} \left[ \sum_{j=2}^{2((k+1)h-i-1)} G_j(k, i) | \mathcal{F}(i) \right] \right\| \leq \left\| I_{nn} - \sum_{j=i+1}^{(k+1)h-1} \mathbb{E} \left[ D^T(j) + D(j) + 2\lambda(j)I_{nn} | \mathcal{F}(i) \right] + \mathbb{E} \left[ \sum_{j=2}^{2((k+1)h-i-1)} G_j(k, i) | \mathcal{F}(i) \right] \right\|.
\]
\[ \leq 1 + \sum_{j=k+1}^{k+1} \mathbb{E} \left[ D^T(j) + D(j) + 2\lambda(j)I_{Nn} \right| \mathcal{F}(i) \right] + \sum_{j=2}^{2(k+1)-1} \mathbb{E} \left[ \|G_j(k, i)\|_2 \right| \mathcal{F}(i) \] 

\[ \leq 1 + \sum_{j=k+1}^{k+1} \left[ \mathbb{E} \left[ D^T(j) + D(j) + 2\lambda(j)I_{Nn} \right| \mathcal{F}(i) \right] + \sum_{j=2}^{2(k+1)-1} \mathbb{E} \left[ \|G_j(k, i)\|_2 \right| \mathcal{F}(i) \] 

\[ \leq 1 + \sum_{j=k+1}^{k+1} \left[ (\rho_0a(j) + \rho_0b(j) + \lambda(j)) + \sum_{j=2}^{2(k+1)-1} C_{2j}^j (\rho_0q(kh) + \lambda(kh))^2 \right] 

\[ \leq 1 + \sum_{j=k+1}^{k+1} \left[ (\rho_0a(j) + \rho_0b(j) + \lambda(j)) + \sum_{j=2}^{2(k+1)-1} C_{2j}^j (\rho_0q(kh) + \lambda(kh))^2 \right] 

\[ = 1 + \sum_{j=k+1}^{k+1} \left[ (\rho_0a(j) + \rho_0b(j) + \lambda(j)) + \rho(k) \text{ a.s., } k \geq k_2, \right] 

which further shows that there exists a positive integer \( k_3 \), such that \( \mathbb{E}[\Phi^k((k+1)h - 1, i + 1)\Phi_p((k+1)h - 1, i + 1)|\mathcal{F}(i)] \leq 2 \text{ a.s., } k \geq k_3. \]

**Lemma A.3.** For the algorithm (6), suppose that \( \{G(k), k \geq 0 \} \in \Gamma_1 \), Assumptions (A1)-(A2) hold, and there exist positive constants \( \rho_0, \theta_1 \) and \( \theta_2 \) and a positive integer \( h \), such that (i) \( \inf_{k \geq 0} \frac{\lambda(1)}{\rho_0} \left( \sum_{i=k+1}^{k+1} \mathbb{E}[\hat{L}_{G(i)}(i)|\mathcal{F}(k-1)] \right) \geq \theta_1 \text{ a.s.; (ii) } \inf_{k \geq 0} \lambda(\min(\sum_{i=k}^{k+1} \mathbb{E}[\hat{L}_{G(i)}(i)|\mathcal{F}(k-1)]) \geq \theta_2 \text{ a.s.; (iii) } \sup_{k \geq 0} (\|L_{G(i)}(i)\| + \mathbb{E}[\|H^T(k)H(k)\|^{2\max(h,2)}|\mathcal{F}(k-1)])^{1/2} \leq \frac{1}{\rho_0} \text{ a.s.} 

\[ G(k) = \frac{\theta_1\theta_2}{2Nh\rho_0 + N\theta_1} \min\{a((k+1)h), b((k+1)h)\} + \sum_{i=k+1}^{k+1} \lambda(i) \] 

(I) If \( \sum_{k=0}^{\infty} G(k) = \infty \) and \( \sum_{k=0}^{\infty} \{a^2(k) + b^2(k) + \lambda(k)\} < \infty \), then \( \lim_{k \to \infty} x_i(k) = x_0, \ i \in \mathcal{V} \) a.s.

(II) If \( \sum_{k=0}^{\infty} G(k) = \infty \) and \( a^2(kh) + b^2(kh) + \lambda(kh) = o(G(k)) \), then \( \lim_{k \to \infty} \mathbb{E}[\|x_i(k) - x_0\|^2] = 0, \ i \in \mathcal{V} \).

**Proof.** Noting that \( \{G(k), k \geq 0 \} \in \Gamma_1 \), it follows that \( \mathbb{E}[\hat{L}_{G(i)}(i)|\mathcal{F}(k-1)] \) is positive semi-definite. Noting that \( \mathbb{E}[\hat{L}_{G(i)}(i)|\mathcal{F}(mh-1)] = \mathbb{E}[\hat{L}_{G(i)}(i)|\mathcal{F}(k-1)] \mathcal{F}(mh-1), k \geq mh \), then \( \mathbb{E}[\hat{L}_{G(i)}(i)|\mathcal{F}(mh-1)] \) is also positive semi-definite, which shows

\[ \sum_{i=k}^{k+1} \mathbb{E} \left[ b(i)\hat{L}_{G(i)}(i) \otimes I_n + a(i)\mathcal{H}(i)\right| \mathcal{F}(kh - 1) \right] \]

\[ \geq \min\{a((k+1)h), b((k+1)h)\} \sum_{i=k+1}^{k+1} \mathbb{E} \left[ \hat{L}_{G(i)}(i) \otimes I_n + \mathcal{H}(i)\right| \mathcal{F}(kh - 1) \right] .(C.18) \]
By Lemma 2, condition (ii) and (C.18), we get
\[ \Lambda^h_k \geq \frac{\theta_1 \theta_2}{2N\theta_0 + N\theta_1} \min\{a((k+1)h), b((k+1)h)\} \text{ a.s.,} \]
which leads to
\[ \Lambda^h_k + \sum_{i=kh}^{(k+1)h-1} \lambda(i) \geq G(k). \] (C.19)

We first prove (I) of Lemma A.3. Here, the algorithm gains guarantee that the sample path spatio-temporal persistence of excitation condition holds, then the algorithm converges almost surely by Theorem 1.

Next, we will prove (II) of Lemma A.3. By (C.19) and Lemma 1, there exists a positive integer \( k_0 \), such that
\[ \mathbb{E}[V((k+1)h)|\mathcal{F}(kh-1)] \leq (1+\Omega(k))V(kh) - 2G(k)V(kh) + \Gamma(k) \text{ a.s., } k \geq k_0, \] (C.20)
where \( \Omega(k) + \Gamma(k) = O(a^2(kh) + b^2(kh) + \lambda(kh)) \). Noting that \( G(k) = o(1) \) and \( \Omega(k) = o(G(k)) \), without loss of generality, we suppose that \( 0 < \Omega(k) \leq G(k) < 1, k \geq k_0 \). Taking mathematical expectation on both sides of (C.20), we get
\[ \mathbb{E}[V((k+1)h)] \leq \left(1 - G(k)\right)\mathbb{E}[V(kh)] + \Gamma(k), \]
\( k \geq k_0, \) On one hand, we know that \( \Gamma(k) = o(G(k)) \) and \( \sum_{k=0}^{\infty} G(k) = \infty \), which together with Lemma A.3 in [30] gives
\[ \lim_{k \to \infty} \mathbb{E}[V(kh)] = 0. \] (C.21)
On the other hand, by Lemma A.1, we know that there exists a positive integer \( k_1 \), such that
\[ \mathbb{E}[V(k)] \leq 2^h \mathbb{E}[V(m_kh)] + h2^h \gamma(m_kh), \]
where \( \gamma(k) = o(1) \) and \( m_k = \lfloor \frac{k}{h} \rfloor \). Noting that \( 0 \leq k - m_kh < h \), by (C.21), we get
\[ \lim_{k \to \infty} \mathbb{E}[V(k)] = 0. \]

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