The implications of $\mathcal{N} = 1$ superconformal symmetry for four dimensional quantum field theories are studied. Superconformal covariant expressions for two and three point functions of quasi-primary superfields of arbitrary spin are found and connected with the operator product expansion. The general formulae are specialised to cases involving a scalar superfield $L$, which contains global symmetry currents, and the supercurrent, which contains the energy momentum tensor, and the consequences of superconformal Ward identities are analysed. The three point function of $L$ is shown to have unique completely antisymmetric or symmetric forms. In the latter case the superspace version of the axial anomaly equation is obtained. The three point function for the supercurrent is shown to have two linearly independent forms. A linear combination of the associated coefficients for the general expression is shown to be related to the scale of the supercurrent two point function through Ward identities. The coefficients are given for the two free field superconformal theories and are also connected with the parameters present in the supercurrent anomaly for supergravity backgrounds. Superconformal invariants, which are possible even in three point functions, are discussed.

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1. Introduction

The very first paper \cite{1} in the western literature on supersymmetry in four space-time dimensions in fact introduced the $\mathcal{N} = 1$ superconformal group. Nevertheless, given the inevitable breakdown of conformal invariance in perturbative treatments of non trivial quantum field theories, most subsequent discussions of supersymmetric theories were concerned with theories invariant under just the restricted supersymmetry group which is the minimal extension of the Poincaré group, whose elements are standard Lorentz and translation transformations, and for which there are no perturbative quantum anomalies. However in the last few years the work of Seiberg \cite{2} and others have shown that there should exist a host of non trivial superconformal field theories which in many cases can be identified with renormalisation group fixed points where the $\beta$-function vanishes. Just as two-dimensional conformal field theories have a very rich mathematical structure, with applications in string theories and statistical physics, it is now possible to hope for similar elegant exact results in the as yet relatively unexplored case of four dimension field theories. Virtually all the new results depend essentially on constraints imposed by supersymmetry and in consequence superconformal field theories are the most promising immediate candidates for potential extension of some of the two-dimensional results for conformal field theories to higher dimensions.

Although $\mathcal{N} = 2$ and 4 superconformal theories in four space-time dimensions have considerable interest, and also remarkably there may also be possible superconformal field theories in five and six dimensions, we here consider just the $\mathcal{N} = 1$ case in four dimensions using standard superspace formalism. Some relevant results were obtained long ago \cite{3,4} and recently there has been extensive work by Howe and West \cite{5}, on which we attempt to build (although much of their discussion was concerned with $\mathcal{N} > 1$). Furthermore Anselmi and co-workers \cite{6} have undertaken specific calculations in $\mathcal{N} = 1$ supersymmetric Yang Mills theory, exploring analogues in four-dimensional superconformal theories of the two-dimensional Virasoro central charge $c$. Here we extend previous results \cite{7,8} which give explicit forms for two and three point functions in conformal field theories for operators of arbitrary spin to the $\mathcal{N} = 1$ superconformal case. The resulting expressions determine the forms of the operator product expansions for quasi-primary operators and although essentially kinematic are, in our view, a necessary precursor to dynamical investigations. We also analyse the supercurrent \cite{9}, which contains the energy momentum tensor amongst its component fields, and its Ward identities which reflect superconformal invariance. One of the main concerns of this paper is to analyse in detail the two and three point functions of the supercurrent and their relation through Ward identities and
also their connection to anomalies present on curved supergravity backgrounds. We also
discuss possible superconformal invariants which can appear in the general expression for
four-point functions. A similar analysis was described in \[14\] but the present discussion is
perhaps more complete and differs in some details.

In the next section we establish notation and review $\mathcal{N} = 1$ superconformal transfor-
mations on superspace in four space-time dimensions. We discuss primarily infinitesimal
superconformal transformations which are super-diffeomorphisms restricted by a natural
condition playing a similar role to the conformal Killing equation for ordinary confor-
mal transformations. We identify the associated Lie algebra with that for the supergroup
$Sl(4|1)$ with suitable reality conditions. From the discussion of the action of superconfor-
mal transformations on superspace we construct variables which transform homogeneously
and may be used in a simple construction of two and three point functions. Unlike the non
supersymmetric case there is even a superconformal invariant for three points. In section
3 we describe how quasi-primary superfields may be defined in general by a simple trans-
formation rule, depending on the scale dimension and $U(1)$ $R$-symmetry charge, as well
as its particular spin representation. We further show how derivatives of quasi-primary
superfields in particular special cases are also quasi-primary. Such results demonstrate the
consistency of conservation conditions on the supercurrent with superconformal invariance
and are important in the subsequent analysis. A corollary of these results is that the
Bianchi identity for a $\mathcal{N} = 1$ supersymmetric gauge theory is consistent with superconfor-
mal invariance only if the scale dimension and $R$-charge are those of the free abelian theory.
We also describe, in terms of the results of section 2, general superconformal covariant con-
structions for two and three point functions of quasi-primary superfields. The result for
the three point function depends on a homogeneous function on superspace coordinates
which can be directly related to the leading coefficient of the term in the operator product
expansion associated with the operators appearing in the three point function. In section
4 we introduce the supercurrent by a variant of Noether’s construction which is used to
find the corresponding Ward identities. In section 5 we consider first the application of
the general formalism to the simple cases involving chiral scalar superfields. After showing
how conditions flowing from the conservation equations may be imposed on the general
form for the superfield three point function in section 6 we consider the three point func-
tion for superfields containing an internal symmetry current. We discuss the associated
Ward identities and also the anomalies which are supersymmetric extension of the usual
axial current anomaly. In section 7 we consider the supercurrent and obtain results for the
three point functions involving two supercurrents and a scalar superfield and also three
supercurrents. For the latter case we show that there are two possible linearly independent forms although one linear combination is shown to be related to the coefficient of the two point function through a Ward Identity. In section 8 general results are then restricted to the case of free fields which give two different trivial superconformal theories in four dimensions. In section 9 we show how superconformal invariance allows for the evaluation of integrals, generalising old results in the non supersymmetric case, and in section 10 we discuss possible superconformal invariants that may be present in higher point correlation functions which include generalisations of the usual invariant cross ratios as well as the Grassmann valued invariants present for just three points. Finally in a conclusion we relate the coefficients which are present in the general supercurrent three point function to the coefficients $c, a$ appearing in the supergravity extension of the energy momentum tensor trace for a curved space background. Both $c, a$ are possible generalisations of the Virasoro central charge to four dimensional superconformal theories. An appendix contains some details concerning the transformation properties of the supercurrent in free theories.

2. Superconformal Transformations

The conformal group is defined by the intersection of the group of diffeomorphisms with local rescalings of the metric. The superconformal group can be obtained in a variety of equivalent ways but here we consider it as a reduction of those super-diffeomorphisms which leave the chiral subspaces of superspace invariant. With standard superspace coordinates $z^A = (x^a, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \in \mathbb{R}^{4|4}$ the chiral restrictions are given by

$$z_+^A = (x_+^a, \theta^\alpha), \quad z_-^A = (x_-^a, \bar{\theta}^{\dot{\alpha}}), \quad x_\pm^a = x^a \pm i \theta^a \bar{\theta},$$

so that $D_\tilde{\alpha} z_+ = 0$, $D_\alpha z_- = 0$. For a general diffeomorphism preserving the chiral decomposition of superspace we may write

$$\delta x_+^a = v^a(z_+), \quad \delta \theta^\alpha = \lambda^\alpha(z_+), \quad \delta x_-^a = \bar{v}^a(z_-), \quad \delta \bar{\theta}^{\dot{\alpha}} = \bar{\lambda}^{\dot{\alpha}}(z_-).$$

We use the notation of Wess and Bagger [1], with minor emendations, thus $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ are regarded as row, column vectors and we let $\tilde{\theta}_\alpha = \epsilon_{\alpha\dot{\beta}} \theta^{\dot{\beta}}, \tilde{\bar{\theta}}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\beta} \bar{\theta}^\beta$ form associated column, row vectors, $\tilde{\theta}^2 = \theta \tilde{\theta}, \tilde{\bar{\theta}}^2 = \bar{\theta} \tilde{\bar{\theta}}$, as usual 4-vectors are identified with $2 \times 2$-matrices using the hermitian $\sigma$-matrices $\sigma_\alpha, \tilde{\sigma}_\alpha, \sigma_\alpha \tilde{\sigma}_\beta = -\eta_{\alpha\dot{\beta}} 1, x^a \rightarrow x_\alpha^\alpha = x^a (\sigma_a)_{\alpha\dot{\alpha}}$, $\tilde{x}^{\dot{\alpha}\alpha} = x^a (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} = \epsilon^\alpha_{\dot{\beta}} \epsilon^{\dot{\alpha}\beta} x_{\beta\beta}$, with inverse $x^a = -\frac{1}{2} \text{tr}(\sigma^a \tilde{x})$. Hence (2.1) gives $\tilde{x}_\pm = \tilde{x} \pm 2 i \theta \bar{\theta}$, $x_\pm = x \mp 2 i \theta \bar{\theta}$. The associated spinor derivatives satisfy $\{D_\alpha, \tilde{D}_{\tilde{\alpha}}\} = -2i(\sigma^a \partial_a)_{\alpha\dot{\alpha}}$. 

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1 We use the notation of Wess and Bagger [1], with minor emendations, thus $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ are regarded as row, column vectors and we let $\tilde{\theta}_\alpha = \epsilon_{\alpha\dot{\beta}} \theta^{\dot{\beta}}, \tilde{\bar{\theta}}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\beta} \bar{\theta}^\beta$ form associated column, row vectors, $\tilde{\theta}^2 = \theta \tilde{\theta}, \tilde{\bar{\theta}}^2 = \bar{\theta} \tilde{\bar{\theta}}$, as usual 4-vectors are identified with $2 \times 2$-matrices using the hermitian $\sigma$-matrices $\sigma_\alpha, \tilde{\sigma}_\alpha, \sigma_\alpha \tilde{\sigma}_\beta = -\eta_{\alpha\dot{\beta}} 1, x^a \rightarrow x_\alpha^\alpha = x^a (\sigma_a)_{\alpha\dot{\alpha}}$, $\tilde{x}^{\dot{\alpha}\alpha} = x^a (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} = \epsilon^\alpha_{\dot{\beta}} \epsilon^{\dot{\alpha}\beta} x_{\beta\beta}$, with inverse $x^a = -\frac{1}{2} \text{tr}(\sigma^a \tilde{x})$. Hence (2.1) gives $\tilde{x}_\pm = \tilde{x} \pm 2 i \theta \bar{\theta}$, $x_\pm = x \mp 2 i \theta \bar{\theta}$. The associated spinor derivatives satisfy $\{D_\alpha, \tilde{D}_{\tilde{\alpha}}\} = -2i(\sigma^a \partial_a)_{\alpha\dot{\alpha}}$. 

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3
The corresponding differential generators acting in chiral superfields are

\[ \mathcal{L}_+ = h^a \partial_a + \lambda^a D_a, \quad \mathcal{L}_- = \bar{h}^\alpha \partial_\alpha + \bar{\lambda}^\alpha \bar{D}^\alpha, \] (2.3)

where

\[ h^a(z) = v^a(z_+) - 2i\lambda(z_+)\sigma^a\bar{\theta}, \quad \bar{h}^\alpha(z) = \bar{v}^\alpha(z_-) + 2i\theta\sigma^a\bar{\lambda}(z_-). \] (2.4)

Alternatively we may define \( \mathcal{L}_\pm \) by the requirements [\( \bar{D}_\alpha, \mathcal{L}_+ \) = 0, [\( D_\alpha, \mathcal{L}_- \) = 0 which, assuming just the form (2.3), leads to

\[ \bar{D}_\alpha h^b = -2i(\lambda\sigma^b)\bar{\alpha}, \quad \bar{D}_\alpha \lambda^\beta = 0, \quad D_\alpha \bar{h}^b = 2i(\sigma^b\bar{\lambda})_\alpha, \quad D_\alpha \bar{\lambda}^\beta = 0, \] (2.5)

which are easily seen to be solved by (2.4). \( \lambda, \bar{\lambda} \) are thus determined in terms of \( h, \bar{h} \) with the remaining constraints on \( h, \bar{h} \) which follow from (2.3) may be written as

\[ \bar{D}(\bar{\alpha}\bar{\lambda}^\beta) = 0, \quad D(\alpha\bar{h}_\beta) = 0. \] (2.6)

We here define infinitesimal superconformal transformations as those diffeomorphisms of the form (2.2) where with (2.4) we have\[ h^a = \bar{h}^a, \] (2.7)

With this restriction it is easy to see from (2.3) that

\[ \partial_a h_b + \partial_b h_a = 2\rho \eta_{ab}, \quad \rho = \frac{1}{2}(D_\alpha \lambda^\alpha + \bar{D}_{\bar{\alpha}} \bar{\lambda}_{\bar{\alpha}}), \] (2.8)

which is just the standard conformal Killing equation. The solution of (2.8) is well known and for dimensions \( d > 2 \) gives the standard result for an infinitesimal conformal transformation in terms of a translation \( a^a \), rotation \( \omega^{ab} = -\omega^{ba} \), special conformal transformation \( b^a \) and scale parameter \( \lambda \). In the present case the solution becomes [13], with \( \lambda = \kappa + \bar{\kappa} \),

\[ v^a(y, \theta) = a^a + \omega^a b^b + (\kappa + \bar{\kappa}) y^a + b^a y^2 - 2y^a y \cdot b + 2i \theta \sigma^a \bar{\epsilon} - 2 \theta \sigma^a \bar{\eta} \eta, \]
\[ \lambda^a(y, \theta) = \epsilon^a - \theta^\beta \omega^\beta \alpha + \kappa \theta^a + (\theta b \bar{y})^\alpha - i(\bar{\eta} \bar{y})^\alpha + 2\bar{\eta}^\alpha \theta^2, \quad \omega^a = -\frac{1}{4}\omega^{ab}(\sigma_a \bar{\sigma}_b)_{\beta^\alpha} \]
\[ \bar{v}^\alpha(y, \bar{\theta}) = a^a + \omega^a b^b + (\kappa + \bar{\kappa}) y^a + b^a y^2 - 2y^a y \cdot b - 2i \bar{\epsilon} \sigma^a \bar{\theta} - 2 \bar{\eta} \bar{y} \sigma^a \theta, \]
\[ \bar{\lambda}^{\alpha}(y, \bar{\theta}) = \epsilon^{\alpha} + \omega^{\alpha} \beta \bar{\theta}^\beta + \bar{\kappa} \bar{\theta}^\alpha + (\bar{y} b \bar{\theta})^{\alpha} + i(\bar{\eta} \bar{y})^{\alpha} + 2\bar{\eta}^{\alpha} \bar{\theta}^2, \quad \bar{\omega}^{\alpha} = -\frac{1}{4}\omega^{ab}(\bar{\sigma}_a \sigma_b)^{\alpha}_{\beta}, \] (2.9)

\[ ^2 \text{Alternatively (2.6) with } h = \bar{h} \text{ can be regarded as the superconformal Killing equations, they were obtained previously by Conlong and West [12].} \]
Superconformal transformations are parameterised additionally by the supertranslations $\epsilon^\alpha$, $\tilde{\epsilon}^\dot{\alpha}$ and an extra Grassmann spinor $\eta_\alpha$, $\bar{\eta}_{\dot{\alpha}}$ (in our notational conventions $\tilde{\eta}^\alpha = \epsilon^{\alpha\beta} \eta_\beta$, $\tilde{\bar{\eta}}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\eta}_{\dot{\beta}}$), as well as, if $\kappa = \frac{1}{2}(\lambda + i\alpha)$, $\bar{\kappa} = \frac{1}{2}(\lambda - i\alpha)$ the angle $\alpha$ which corresponds to the $U(1)$ $R$-symmetry acting on $\theta$, $\bar{\theta}$.

The action of infinitesimal superconformal transformations on fields defined on superspace is then generated by

$$\mathcal{L} = h^a \partial_a + \lambda^\alpha D_\alpha + \tilde{\lambda}_{\dot{\alpha}} \tilde{D}^{\dot{\alpha}}. \tag{2.10}$$

From (2.4) it is easy to see that

$$[D_\alpha, \mathcal{L}] = (D_\alpha \lambda^\beta) D_\beta, \quad [\tilde{D}^{\dot{\alpha}}, \mathcal{L}] = (\tilde{D}^{\dot{\alpha}} \tilde{\lambda}_{\dot{\beta}}) \tilde{D}^{\dot{\beta}}. \tag{2.11}$$

Writing the superspace exterior derivative $d = d z^A \partial_A = e^A D_A$, which requires

$$e^a = dx^a + i \theta \sigma^a d \bar{\theta} - i d \theta \sigma^a \bar{\theta}, \tag{2.12}$$

then if $[\mathcal{L}, D_A] = -R^B_A D_B$ the associated variation of one-forms is given by $\delta e^A = e^B R^A_B$. The results in (2.11) imply that for superconformal transformations $R^\alpha_\beta = R^\dot{\alpha}_{\dot{\beta}} = 0$ and that in this case the variation of $e^a$ is homogeneous

$$\delta e^a = e^b \partial_b h^a, \tag{2.13}$$

since $R^b_a = \partial_a h^b$. Using (2.8) for $e^2 = e^a e_a$ therefore,

$$\delta e^2 = 2 \rho e^2. \tag{2.14}$$

The invariance of the square of the superspace interval, $e^2$ with $e^a$ given by (2.12), up to a local rescaling can be regarded as an alternative basic characterisation of superconformal transformations.

Besides transformations connected to the identity it is natural to extend the superconformal group by inversions $z \rightarrow z'$ where $[13],$

$$\tilde{x}'_+ = -x'^{-1}_+, \quad \tilde{\theta}' = -ix'^{-1}_+ \tilde{\theta}, \quad \theta' = \tilde{\theta} x'^{-1}_+, \quad \Rightarrow \quad \tilde{x}'_+ = x'^{-1}_+, \tag{2.15}$$

with $x'^{-1}_\pm = \tilde{x}'_\pm/x^2_\pm$. From (2.12) it is straightforward to find, as in (2.14), that $e^2$ is invariant up to a rescaling,

$$\tilde{e}' = x'^{-1}_+ e x'^{-1}_-, \quad e'^2 = \frac{e^2}{x'^2_+ x'^2_-}. \tag{2.16}$$
It is easy to verify from (2.17) that inversions are idempotent, \((z')' = z\).

The Lie algebra of the differential generators \(L\), given by (2.10), is easily calculated
\[
L' = [L_2, L_1],
\] (2.17)
so that, for instance,
\[
h'^a = [h_2, h_1]^a + 2i(\lambda_2 \sigma^a \lambda_1 - \lambda_1 \sigma^a \lambda_2).
\] (2.18)

It is straightforward to check that \(h' = \bar{h}', \lambda', \bar{\lambda}'\) satisfy (2.5). The superconformal algebra may be identified with that of supermatrices, in terms of the parameters in (2.9),
\[
M = \begin{pmatrix}
\omega - \frac{1}{3}(\kappa + 2\bar{\kappa})1 & -ib & 2\eta \\
-i\bar{a} & \bar{\omega} + \frac{1}{3}(2\kappa + \bar{\kappa})1 & 2\bar{\epsilon} \\
2\epsilon & 2\bar{\eta} & \frac{2}{3}(\kappa - \bar{\kappa})
\end{pmatrix},
\] (2.19)
since (2.17) corresponds exactly to
\[
M' = [M_1, M_2].
\] (2.20)

It is easy to see that \(\text{str}M = 0\) so that \(M\) belongs to Lie algebra \(sl(4|1)\) which is restricted to \(su(2,2|1)\) by the reality condition,
\[
M = -BM^\dagger B, \quad B = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\] (2.21)

The non-linear realisation of the superconformal group on superspace may be recovered by regarding \(z = (x, \theta, \bar{\theta})\) as coordinates on the coset \(SU(2,2|1)/G_0\), where \(G_0 \in SU(2,2|1)\) is the stability group of the point \(z = 0\) under superconformal transformations, generated by matrices \(M_0\) of the same form as in (2.19) with \(a = \epsilon = \bar{\epsilon} = 0\).\(^3\)

To describe the coset explicitly it is convenient to define
\[
\mathcal{V}(z) = \begin{pmatrix}
1 & 0 \\
-i\bar{x}_+ & 2\bar{\theta} \\
2\theta & 1
\end{pmatrix}, \quad \bar{\mathcal{V}}(z) = \mathcal{V}(z)^\dagger B = \begin{pmatrix}
i\bar{x}_- & 1 & -2\bar{\theta} \\
2\theta & 0 & -1
\end{pmatrix},
\] (2.22)
which are constrained by \(\bar{\mathcal{V}}(z)\mathcal{V}(z) = \begin{pmatrix}0 & 0 \\
0 & -1\end{pmatrix}\). For \(M\) given by (2.19) the associated \(\delta x_\pm, \delta \theta, \delta \bar{\theta}\) as given by (2.2) are then obtained by
\[
M\mathcal{V}(z) = \mathcal{V}(\delta z) + \mathcal{V}(z)\mathcal{H}(z), \quad \mathcal{V}(\delta z) = \mathcal{L}\mathcal{V}(z),
\]
\[
-\bar{\mathcal{V}}(z)M = \bar{\mathcal{V}}(\delta z) + \bar{\mathcal{H}}(z)\bar{\mathcal{V}}(z), \quad \bar{\mathcal{V}}(\delta z) = \mathcal{L}\bar{\mathcal{V}}(z),
\] (2.23)
\(^3\) For a general review of such constructions see [14].
if $\mathcal{H}, \bar{\mathcal{H}}$ are matrices of the form

$$
\mathcal{H} = \begin{pmatrix}
\hat{\omega} - \sigma & 1 \\
0 & 2(\sigma - \sigma)
\end{pmatrix}, \quad \bar{\mathcal{H}} = \begin{pmatrix}
-\hat{\omega} - \bar{\sigma} & 1 \\
\bar{\tau} & -2(\bar{\sigma} - \sigma)
\end{pmatrix},
$$

(2.24)

which are determined by requiring the structure of $\mathcal{V}, \bar{\mathcal{V}}$ in (2.23) to be preserved. The elements of $\mathcal{H}, \bar{\mathcal{H}}$ are given by

$$
D_\alpha \lambda^\beta = -\hat{\omega}^\alpha_\beta + \delta^\alpha_\beta (2\sigma - \sigma), \quad \bar{D}^\dot{\alpha} \bar{\lambda}_\beta = -\hat{\omega}^{\dot{\alpha}}_\beta + \delta^{\dot{\alpha}}_\beta (2\sigma - \sigma), \quad \hat{\omega}^\alpha_\alpha = \hat{\omega}^{\dot{\alpha}}_{\dot{\alpha}} = 0,
$$

$$
D_\alpha \sigma = -\frac{1}{3} D_\beta \hat{\omega}^\alpha_\beta = \tau_\alpha, \quad \bar{D}_{\dot{\alpha}} \bar{\sigma} = \frac{1}{3} \bar{D}_{\dot{\beta}} \hat{\omega}^{\dot{\alpha}}_{\dot{\beta}} = \bar{\tau}_{\dot{\alpha}},
$$

(2.25)

or explicitly, independent of $a, c, \bar{c}$,

$$
\hat{\omega}^\alpha_\beta (z_+) = \omega^\alpha_\beta + \frac{1}{2} (\bar{c} + b \bar{x}_+) \alpha^\beta + 4 \eta^\alpha_\beta + 2 \delta^\beta_\eta \theta \eta,
$$

$$
\sigma (z_+) = \frac{1}{3} (k + \bar{k}) + 2 \theta^\alpha_\beta - b x_+, \quad \bar{\tau}_{\dot{\alpha}} (z_+) = 2 (\bar{\eta} + i b^\dot{\alpha}),
$$

$$
\hat{\omega}^{\dot{\alpha}}_\beta (z_-) = \bar{\omega}^{\dot{\alpha}}_\beta + \frac{1}{2} (\bar{c} - b \bar{x}_-) \bar{\alpha}^\beta - 4 \bar{\theta}^{\dot{\alpha}} \bar{\eta}_\beta - 2 \delta^{\dot{\alpha}} \bar{\eta} \bar{\theta},
$$

$$
\bar{\sigma} (z_-) = \frac{1}{3} (2 k + \bar{k}) + 2 \bar{\theta}^\alpha - b \bar{x}_-, \quad \tau_{\alpha} (z_-) = 2 (\eta - i b \bar{\theta})_{\alpha}.
$$

(2.26)

It is easy to check that $M_0 \mathcal{V}(0) = \mathcal{V}(0) \mathcal{H}(0)$, $\mathcal{H}(0) \bar{\mathcal{V}}(0) = -\bar{\mathcal{V}}(0) M_0$. For general $z$ the important result that $\hat{\omega}, \sigma$ depend only on $z_+$ follows directly from (2.25) by using $\bar{D}_{\dot{\alpha}} \bar{D}^\dot{\alpha} \bar{\lambda}^\beta = \delta^\alpha_\beta \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\beta}} \bar{\lambda}^\beta$ which from (2.25) gives $\bar{D}_{\dot{\alpha}} \hat{\omega}^{\dot{\alpha}}_\beta = \bar{D}_{\dot{\alpha}} \sigma = 0$, and similarly for the dependence of $\hat{\omega}, \sigma$ on $z_-$. We may also note that $\text{str} \ \mathcal{H} (z) = -2 \sigma (z_-)$, $\text{str} \ \bar{\mathcal{H}} (z) = -2 \sigma (z_+)$. From (2.8) we find

$$
\check{\rho} = \sigma + \bar{\sigma},
$$

(2.27)

and using (2.5)

$$
\hat{\omega} = \frac{1}{4} \partial^{[a} h^{b]} \sigma_a \bar{\sigma}_b, \quad \check{\omega} = \frac{1}{4} \partial^{[a} h^{b]} \check{\sigma}_a \sigma_b.
$$

(2.28)

By using (2.17) and (2.20) in (2.23) we may find

$$
\mathcal{L}_2 \mathcal{H}_1 - \mathcal{L}_1 \mathcal{H}_2 + [\mathcal{H}_1, \mathcal{H}_2] = \mathcal{H}',
$$

$$
\mathcal{L}_2 \bar{\mathcal{H}}_1 - \mathcal{L}_1 \bar{\mathcal{H}}_2 - [\mathcal{H}_1, \mathcal{H}_2] = \bar{\mathcal{H}}'.
$$

(2.29)

It is straightforward to rewrite (2.23) for the non infinitesimal case. The element of the superconformal group formed by exponentiating $M$ corresponds to a finite superconformal transformation $z \rightarrow z'$ given by

$$
e^M \mathcal{V}(z) = \mathcal{V}(z') \mathcal{G}(z) \quad \bar{\mathcal{V}}(z) e^{-M} = \bar{\mathcal{G}}(z) \bar{\mathcal{V}}(z'),
$$

(2.30)

where $\mathcal{G}(z), \bar{\mathcal{G}}(z)$ are matrices of the same form as $\mathcal{H}, \bar{\mathcal{H}}$ in (2.24), with the group property that if $z \xrightarrow{e^M} z' \xrightarrow{e^{M_2}} z''$ and $e^{M_2} e^M = e^M$, so that $z \xrightarrow{e^M} z''$, then $\mathcal{G}_2(z') \mathcal{G}_1(z) = \mathcal{G}(z)$ while $\bar{\mathcal{G}}_2(z') \bar{\mathcal{G}}_1(z') = \bar{\mathcal{G}}(z)$. The action of finite conformal transformations on coordinates $x^a \in \mathbb{R}^4$ is globally well defined on a compactification of Minkowski space, $\mathbb{R}^4 \rightarrow S^3 \times S^1$, or some multiple covering\footnote{For a discussion of some global issues see [15].} and similar considerations apply in the superconformal case.
when the transformations act on $SU(2,2|1)/G_0$. However such issues are not relevant for the considerations of this paper.

The usefulness of the coset construction becomes manifest if we consider

$$\mathcal{V}(z_1)\mathcal{V}(z_2) = \begin{pmatrix} i\bar{x}_{12} & -2\bar{\theta}_{12} \\ 2\theta_{12} & -1 \end{pmatrix},$$

(2.31)

where

$$\bar{x}_{12} = \bar{x}_{1-} - \bar{x}_{2+} + 4i\bar{\theta}_1\theta_2, \quad \theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2.$$  
(2.32)

The expression (2.31) is a function of the supertranslation invariant interval given by

$$z_{12}^A = (x_{12}^a, \theta_{12}^a, \bar{\theta}_{12\bar{a}}), \quad y_{12} = x_1 - x_2 - i\theta_1\bar{\sigma}\theta_2 + i\theta_2\sigma\theta_1 = -y_{21},$$

(2.33)

since $x_{12} = y_{12} - i\theta_{12}\sigma\theta_{12}$. From (2.23) (2.31) transforms according to $\delta(\mathcal{V}(z_1)\mathcal{V}(z_2)) = -\mathcal{H}(z_1)\mathcal{V}(z_2) - \mathcal{V}(z_1)\mathcal{H}(z_2)$ and using the form for $\mathcal{H}$, $\bar{\mathcal{H}}$ given in (2.24) this gives

$$\delta\bar{x}_{12} = (\hat{\omega}(z_{1-}) + \bar{\sigma}(z_{1-})1)\bar{x}_{12} + \bar{x}_{12}( - \hat{\omega}(z_{2+}) + \sigma(z_{2+})1),$$

(2.34)

as well as

$$\delta\theta_{12} = \theta_{12}( - \hat{\omega}(z_{2+}) + \sigma(z_{2+})1) + 2(\bar{\sigma}(z_{1-}) - \sigma(z_{1+}))\theta_{12} - \frac{1}{\bar{\tau}}\bar{\tau}(z_{1+})\bar{x}_{12},$$

$$\delta\bar{\theta}_{12} = (\hat{\omega}(z_{1-}) + \bar{\sigma}(z_{1-})1)\bar{\theta}_{12} - 2(\bar{\sigma}(z_{2-}) - \sigma(z_{2+}))\bar{\theta}_{12} + \frac{1}{\tau}\bar{x}_{12}\tau(z_{2-}).$$

(2.35)

Combining this result for $\delta\theta_{12}$ with (2.34) gives a simpler expression for the variation in the form

$$\delta(\theta_{12}\bar{x}_{12}^{-1}) = (\theta_{12}\bar{x}_{12}^{-1})( - \hat{\omega}(z_{1-}) + \sigma(z_{1+})1 - 2\bar{\sigma}(z_{1-})1) - \frac{1}{\bar{\tau}}\bar{\tau}(z_{1+}),$$

(2.36)

and similarly from the result (2.33) for $\delta\bar{\theta}_{12}$ with $1 \leftrightarrow 2$

$$\delta(\bar{x}_{21}^{-1}\bar{\theta}_{12}) = ( - \hat{\omega}(z_{1+}) + \sigma(z_{1+})1 - 2\bar{\sigma}(z_{1-})1)(\bar{x}_{21}^{-1}\bar{\theta}_{12}) - \frac{1}{\tau}\tau(z_{1-}).$$

(2.37)

If we let

$$(x_{21})_{a\dot{a}} = -\epsilon_{a\beta}\epsilon_{\dot{a}\dot{\beta}}(\bar{x}_{12})^{\dot{\beta}\beta}, \quad x_{21} = x_{2+} - x_{1-} + 4i\bar{\theta}_2\theta_1,$$

(2.38)

then the inverse of $\bar{x}_{12}$ is given explicitly by

$$\bar{x}_{12}^{-1} = \frac{1}{x_{12}^2} x_{21},$$

(2.39)

since $\bar{x}_{12} x_{21} = x_{12}^2 1$. It is also useful to note that from (2.34)

$$\delta x_{21} = (\hat{\omega}(z_{2+}) + \sigma(z_{2+})1)x_{21} + x_{21}( - \hat{\omega}(z_{1-}) + \bar{\sigma}(z_{1-})1).$$

(2.40)
Under inversions as in (2.15) then with (2.32) and (2.38)
\[ \bar{x}_{12} \longrightarrow x_{12}^{-1} x_{12} x_{2}^{-1}. \] (2.41)

For two points in superspace \( z_1, z_2 \) then \( \bar{x}_{12} = x_{12}^{\alpha} \tilde{\sigma}_{\alpha} \), defined in (2.32) such that it depends just on \( z_{1-} \) and \( z_{2+} \) and satisfies \( \bar{x}_{12}^\dagger = -\bar{x}_{21} \), or equivalently \( x_{21} \), play a crucial role in the subsequent construction of superconformally covariant two and higher point amplitudes as a consequence of their homogeneous transformation properties in (2.34) or (2.40). Given the result (2.31) we may also define a scalar by
\[ \text{sdet} V(z_1) V(z_2) = -\left( x_{12} + 2i \theta_{12} \bar{\sigma}_{12} \right)^2 = -x_{21}^2. \] (2.42)

Under a finite transformation \( z \longrightarrow z' \) as in (2.30) we may find directly
\[ x'_{21} = \frac{x_{21}}{\Omega(z_{2-}) \Omega(z_{1+})}, \quad \bar{\Omega}(z_{-}) = \text{sdet} \mathcal{G}(z), \quad \Omega(z_{+}) = \text{sdet} \bar{\mathcal{G}}(z). \] (2.43)

In constructing conformally covariant two point functions both \( x_{12} \) and \( x_{21} \), which are related by \( z_1 \leftrightarrow z_2 \), are necessary. A symmetric scalar is given by, with \( y_{12} \) defined in (2.33),
\[ x_{12}^2 x_{21}^2 = (y_{12}^2 + \theta_{12}^2 \bar{\theta}_{12}^2)^2. \] (2.44)

For three points \( z_1, z_2, z_3 \) we may define 4-vectors \( X_{1}^\alpha, \bar{X}_{1}^\dot{\alpha} \) by
\[ \begin{align*}
X_1 &= \frac{x_{12} \bar{x}_{23} x_{31}}{x_{21}^2 x_{13}^2}, & \bar{X}_1 &= -\frac{x_{13} \bar{x}_{32} x_{21}}{x_{31}^2 x_{12}^2} = X_1^\dagger,
\end{align*} \] (2.45)
so that \( X_1 \) transforms homogeneously at \( z_1 \) according to
\[ \delta X_1 = (\dot{\omega}(z_{1+}) - \sigma(z_{1+})1) X_1 + X_1 (-\dot{\omega}(z_{1-}) - \bar{\sigma}(z_{1-})1), \] (2.46)

and similarly for \( \bar{X}_1 \). To achieve analogous results for the Grassmann variables we consider
\[ \begin{align*}
\tilde{\Theta}_1 &= i \left( \frac{1}{x_{21}^2} x_{12} \bar{\theta}_{12} - \frac{1}{x_{13}^2} x_{13} \bar{\theta}_{13} \right), & \tilde{\bar{\Theta}}_1 &= i \left( \frac{1}{x_{12}^2} \theta_{12} x_{21} - \frac{1}{x_{13}^2} \theta_{13} x_{31} \right),
\end{align*} \] (2.47)

since then the inhomogeneous terms involving \( \bar{r}, \tau \) in (2.36) and (2.37) cancel and \( \Theta_{1}^\alpha, \tilde{\Theta}_{1}^\dagger \) transform homogeneously as chiral, anti-chiral spinors at \( z_1 \),
\[ \begin{align*}
\delta \Theta_1 &= \Theta_1 (-\dot{\omega}(z_{1+}) + \sigma(z_{1+})1 - 2\bar{\sigma}(z_{1-})1), \\
\delta \tilde{\Theta}_1 &= (\dot{\omega}(z_{1-}) + \bar{\sigma}(z_{1-})1 - 2\sigma(z_{1+})1) \tilde{\Theta}_1.
\end{align*} \] (2.48)
Using formulae such as
\[
\tilde{x}_{23} = \tilde{x}_{21} + \tilde{x}_{13} + 4i \tilde{\theta}_{12} \theta_{13} ,
\]
we may find the relation
\[
X_1 - \tilde{X}_1 = 4i \tilde{\Theta}_1 \Theta_1 ,
\]
and for future reference we may note
\[
X_1^2 = \frac{x_{23}^2}{x_{21}^2} , \quad \tilde{X}_1^2 = \frac{x_{31}^2}{x_{32}^2}.
\]

With obvious cyclic permutation of indices in (2.45) and (2.47) we may similarly define \(X_2, \tilde{X}_2, \Theta_2, \tilde{\Theta}_2\) and \(X_3, \tilde{X}_3, \Theta_3, \tilde{\Theta}_3\) which transform homogeneously at \(z_2\) and \(z_3\) respectively. It is easy to see that
\[
\tilde{x}_{21}X_1\tilde{x}_{12} = \frac{1}{X_2} \tilde{X}_2 , \quad \tilde{x}_{21}\tilde{X}_1\tilde{x}_{12} = \frac{1}{X_2} \tilde{X}_2 ,
\]
and
\[
\frac{x_{12}^2}{x_{21}^2} \tilde{x}_{21} \tilde{\Theta}_1 = \frac{1}{X_2} \tilde{X}_2 \tilde{\Theta}_2 , \quad \frac{x_{31}^2}{x_{12}^2} \tilde{\Theta}_1 \tilde{x}_{12} = - \frac{1}{X_2} \tilde{\Theta}_2 \tilde{X}_2 ,
\]
with similar results giving \(X_3, \tilde{X}_3, \Theta_3, \tilde{\Theta}_3\). From (2.53) we have
\[
\Theta_1^2 = \left( \frac{x_{21}^2}{x_{12}^2} \right)^2 \frac{x_{32}^2}{x_{31}^2} \Theta_2^2 , \quad \tilde{\Theta}_1^2 = \left( \frac{x_{12}^2}{x_{21}^2} \right)^2 \frac{x_{23}^2}{x_{13}^2} \tilde{\Theta}_2^2 .
\]
A straightforward check on (2.52) and (2.53) are that the conformal transformation properties of both sides are consistent. From (2.52)
\[
\frac{X_1^2}{X_2^2} = \frac{X_2^2}{X_3^2} = \frac{X_3^2}{X_1^2} ,
\]
and from (2.46) this is a superconformal invariant. Under \(z_2 \leftrightarrow z_3, X_1 \leftrightarrow -\tilde{X}_1\), while \(X_2 \leftrightarrow -\tilde{X}_3\), so that
\[
I = \frac{1}{2} \left( \frac{X_1^2}{X_1^2} + \frac{\tilde{X}_1^2}{X_1^2} \right) - 1 = 4 \frac{\Theta_1^2 \tilde{\Theta}_1^2}{X_1^2} ,
\]
using (2.50), is a completely symmetric superconformal invariant.\(^{5}\) We may also construct an invariant which is completely antisymmetric by
\[
J = \frac{1}{2} \left( \frac{X_1^2}{X_1^2} - \frac{\tilde{X}_1^2}{X_1^2} \right) = -2i \Theta_1 \left( \frac{X_1}{X_1^2} + \frac{\tilde{X}_1}{X_1^2} \right) \tilde{\Theta}_1 .
\]
Such invariants, depending only on three points \(z_1, z_2, z_3\), do not exist with ordinary conformal symmetry, it is immediately evident from (2.56) and (2.57) that \(J^2 = 2I, I^2 = 0.\)

\(^5\) This invariant was found by Park [10].
3. Superfield Transformations

A quasi-primary superfield $O^i(z)$, with $i$ denoting vector or spinor indices, is here defined by requiring that it forms a representation under superconformal transformations induced from a finite dimension irreducible representation of $G_0$. Under an infinitesimal superconformal transformation, as described in the previous section,

$$\delta O^i(z) = -L O^i(z) + \frac{1}{2} \partial^a h^b(z) (s_{ab})^i_\nu(O^\nu(z) - 2q \sigma(z_+) O^i(z) - 2\bar{q} \bar{\sigma}(z_-) O^i(z), \quad (3.1)$$

where $s_{ab} = -s_{ba}$ are the generators of $O(3,1)$, or the associated spin group, for the representation $(j, \bar{j})$, $2j, 2\bar{j} = 0, 1, 2, \ldots$, defined by $O^i$, and $q, \bar{q}$ are parameters such that $q + \bar{q}$ is the scale dimension and $3(q - \bar{q})$ is the $U(1)$ $R$-symmetry charge of the field $O^i$. Thus the superfield representation may be labelled $(j, \bar{j}, q, \bar{q})$. Using (2.28) we may write

$$\frac{1}{2} \partial^a h^b(z) s_{ab} = \hat{\omega}^\alpha_\beta(z_+) s^\alpha_\beta + \hat{\omega}_\alpha^\beta(z_-) \bar{s}^\beta_\alpha, \quad (3.2)$$

where $s, \bar{s}$, $s^\alpha_\alpha = \bar{s}^\alpha_\alpha = 0$, act on undotted, dotted spinor indices and form spin $j, \bar{j}$ representations of the algebra,

$$[s^\alpha_\beta, s_\gamma^\delta] = \delta^\delta_\alpha s_\gamma^\beta - \delta^\beta_\gamma s_\alpha^\delta, \quad [s^\alpha_\beta, \bar{s}^\gamma_\delta] = \delta^\delta_\alpha \bar{s}^\gamma_\beta - \delta^\beta_\gamma \bar{s}^\alpha_\delta, \quad [s^\alpha_\beta, \bar{s}^\gamma_\delta] = 0, \quad (3.3)$$

which leads to $\left[ \frac{1}{2} \partial^a h^b(z) s_{ab}, \frac{1}{2} \partial^c h^d(z) s_{cd} \right] = [\hat{\omega}_1, \hat{\omega}_2]^{\alpha}_{\beta} s^\alpha_\beta + [\hat{\omega}_1, \hat{\omega}_2]^{\bar{\alpha}}_{\bar{\beta}} \bar{s}^\beta_\alpha$. Using this and (2.29) it is straightforward to check that the form (3.1) is consistent with the algebra in (2.17), $[\delta_1, \delta_2] O = -\delta' O$. If the field is chiral, depending only on $z_+$, then there must be no terms in (3.1) corresponding to $\hat{\omega}$, so that only $(j, 0)$ representations, when $O^i(z) \to O_{\alpha_1 \ldots \alpha_2 j}(z_+)$ totally symmetric in $\{\alpha_1 \ldots \alpha_2 j\}$, are possible, or to $\bar{\sigma}$ which requires $\bar{q} = 0$. For such chiral superfields, for which the representation may therefore be denoted $(j, q)_+$, and also anti-chiral superfields, labelled by $(\bar{j}, \bar{q})_-$, the scale dimension is therefore related to the $R$-charge \([16,17]\).

The field transformation defined by (3.1) in general defines an irreducible representation. However for particular $(j, \bar{j}, q, \bar{q})$ the superfield representation is reducible since then suitable derivatives also transform as quasi-primary fields [18]. These cases are physically significant in application to superconformal covariant conservation equations. As an illustration which is relevant subsequently we may consider a $\bar{j} = 0$ representation for which (3.1) becomes

$$\delta \phi_{\alpha_1 \ldots \alpha_2 j} = -L \phi_{\alpha_1 \ldots \alpha_2 j} + 2j \hat{\omega}_{(\alpha_1^{\beta}} \phi_{\alpha_2^{\ldots \alpha_2 j})_{\beta}} - (2q \sigma + 2\bar{q} \bar{\sigma}) \phi_{\alpha_1 \ldots \alpha_2 j}. \quad (3.4)$$
From (2.11) we have
\[ [D_\alpha, \mathcal{L}] = -\hat{\omega}_\alpha^\beta D_\beta + (2\bar{\sigma} - \sigma) D_\alpha, \quad (3.5) \]
and using
\[ D_\gamma \hat{\omega}_\alpha^\beta = -2\delta_\gamma^\beta D_\alpha \sigma + \delta_\alpha^\beta D_\gamma \sigma, \quad (3.6) \]
we may find
\[ D_\alpha [\delta \phi_{\alpha_1...\alpha_2j} = -\mathcal{L}(D_\alpha \phi_{\alpha_1...\alpha_2j}) + 2j \hat{\omega}^{(\alpha_1}_\beta D_{|\alpha|} \phi_{\alpha_2...\alpha_2j)\beta} + \hat{\omega}_\alpha^\beta D_\beta \phi_{\alpha_1...\alpha_2j}

- ((2q - 1)\sigma + 2(\bar{q} + 1)\bar{\sigma}) D_\alpha \phi_{\alpha_1...\alpha_2j}

- 4j D_{(\alpha_1} \phi_{\alpha_2...\alpha_2j)\alpha} + 2(j - q) D_\alpha \sigma \phi_{\alpha_1...\alpha_2j}. \quad (3.7) \]

From (3.7) we may easily see, by requiring cancellation of the \( D_\alpha \sigma \) terms, that
\[ \tilde{D}^\alpha \phi_{\alpha_1...\alpha_{2j-1} \alpha} \]
\[ \left\{ \begin{array}{l}
q = j + 1, \\
q = -j.
\end{array} \right. \quad (3.8) \]

The conditions in (3.8) for \( D \) derivatives to be quasi-primary in fact apply, without any modification of the argument since \( D\bar{\sigma} = D\tilde{\omega} = 0 \), to any \( \bar{j} \geq 0 \) representation. Hence in general the derivatives defined in (3.8) give
\[ (j, \bar{j}, j + 1, \bar{q}) \rightarrow (j - \frac{1}{2}, \bar{j}, j + \frac{1}{2}, \bar{q} + 1), \quad (3.9a) \]
\[ (j, \bar{j}, -j, \bar{q}) \rightarrow (j + \frac{1}{2}, \bar{j}, -j - \frac{1}{2}, \bar{q} + 1), \quad (3.9b) \]
and the kernels of these maps are invariant subspaces of the superfield representation spaces for these cases, if \( j = 0 \) in (3.9d) the kernel is just the space of \((j, \bar{q})_+\) anti-chiral superfields. Such results also obviously apply for \( \bar{D} \) derivatives if \( \bar{q} = \bar{j} + 1 \) or \( \bar{q} = -\bar{j} \). Similar arguments show that \( D^2 O^i \) is quasi-primary only for \((0, j, 1, \bar{q})\) representations, as expected since \( D^2 O^i \) is an anti-chiral \((j, \bar{q} + 2)_-\) superfield, and conversely for \( \bar{D}^2 O^i \) if \( j = 0, \bar{q} = 1 \).

The representation defined by (3.1) is unitary and has positive energy when \( q, \bar{q} \) are real with the following restrictions
\[
\begin{align*}
j, \bar{j} &\geq 0 \quad q \geq j + 1, \bar{q} \geq \bar{j} + 1; \\
j = q = 0, \bar{q} \geq \bar{j} + 1; \quad \bar{j} = \bar{q} = 0, \quad q \geq j + 1; \\
j = \bar{j} = q = \bar{q} = 0.
\end{align*}
\quad (3.10)\]

6 In [3] the invariant subspaces defined by the kernel of \( D^2 \) were considered, in these papers a representation of the superconformal generators acting on superfields based on the little group generated by matrices \( M \) in (2.19) with \( a = \epsilon = \bar{n} = 0\) is considered. This is related to the representation here by a Fourier transform with respect to \( \bar{\theta} \).
The results (3.8) are relevant when derivative constraints are imposed on superfields if they are to be superconformal covariant. For the vector supercurrent \( T_\alpha(z) \rightarrow T_\alpha\dot{\alpha}(z) \) then

\[
\tilde{D}^\alpha T_\alpha\dot{\alpha} = \tilde{D}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = 0,
\]

is superconformally covariant as a consequence of (3.9a) and its conjugate only for \( T \) belonging to the \((\frac{1}{2}, \frac{3}{2})^+ \) representation. Similarly for the scalar superfield \( L(z) \) containing a current amongst its components the supersymmetric conservation equations

\[
D^2 L = \bar{D}^2 L = 0,
\]

require \( L \) to belong to the \((0, 0, 1, 1) \) representation. We may also note that for chiral, anti-chiral spinor superfields \( W_\alpha, \bar{W}_{\dot{\alpha}} \) then

\[
\tilde{D}^\alpha W_\alpha = \bar{D}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}},
\]

is consistent with superconformal invariance only if \( W, \bar{W} \) belong to \((\frac{1}{2}, \frac{3}{2})^+ \), \((\frac{1}{2}, \frac{3}{2})^- \) representations respectively (from (3.9a) both sides of (3.13) are then \((0, 0, 2, 2) \) superfields). Eq. (3.13) is just the Bianchi identity for abelian supergauge theories so this result shows that it can be maintained at a superconformal point only if the scale dimensions of \( W, \bar{W} \) remain equal to their free field values. In consequence a non trivial superconformal gauge theory must violate the Bianchi identity which requires there to be both massless magnetically and electrically charged fields.\(^7\)

Using (3.11) and the conditions (2.6) and (2.7) we may define a scalar superfield

\[
L_h = \tilde{h}^{\dot{\alpha} \alpha} T_{\alpha\dot{\alpha}},
\]

which satisfies the conservation equations (3.12). Under a superconformal transformation we have

\[
\delta_1 L_{h_2} = \tilde{h}_2^{\dot{\alpha} \alpha} \delta_1 T_{\alpha\dot{\alpha}} = - (L_1 + 2\sigma_1 + 2\bar{\sigma}_1) L_{h_2} - L_{h'} ,
\]

where \( h' \) is as in (2.18).

\(^7\) Similar results hold in any dimension for purely bosonic gauge fields. If \( F_{ab} \) is the field strength, with scale dimension \( \eta \), and which is supposed to transform under conformal transformations according to \( \delta F_{ab} = -(h \cdot D + \eta \rho) F_{ab} + \tilde{\omega}^c \dot{F}_{cb} + \tilde{\omega}^c F_{ac} \) where \( \partial_a h_b = \eta_{ab} \rho - \tilde{\omega}_{ab}, \tilde{\omega}_{ab} = -\tilde{\omega}_{ba} \), then \( \partial_c \delta F_{ab} = -(h \cdot D + (\eta + 1) \rho) \partial_c F_{ab} + \tilde{\omega}_{c}^d \partial_d F_{ab} + \tilde{\omega}_{a}^d \partial_d F_{ab} + \tilde{\omega}_{b}^d \partial_d F_{ab} + 2(\eta - 2) b_{[c} F_{ab]} \). The inhomogeneous terms in the transformation of the Bianchi identity \( \partial_c F_{ab} = 0 \), involving the parameter \( b \) for special conformal transformations, vanish only if \( \eta = 2 \) which is the free field case.
An expression for the superfield \( L \) which trivially satisfies the conservation equations in (3.12) is given by
\[
L = \bar{D}^{\alpha} F_{\alpha}, \quad \bar{D}_{\dot{\alpha}} F_{\alpha} = 0, \tag{3.16}
\]
and for this to be superconformally covariant \( F_{\alpha} \) must be a \((\frac{1}{2}, \frac{3}{2})_+\) chiral superfield.

To construct an expression for the supercurrent \( T \) in which the conservation equations (3.11) are identically satisfied we consider a \((\frac{3}{2}, q)_+\) chiral superfield \( C^{\alpha\beta\gamma}(z_+) \) which transforms as
\[
\delta C^{\alpha\beta\gamma} = -\mathcal{L} C^{\alpha\beta\gamma} - 3 C^{'(\beta\gamma} \hat{\omega}_{\alpha^\prime)} - 2q \sigma C^{\alpha\beta\gamma}, \tag{3.17}
\]
and using (3.5) and
\[
[\partial_{\alpha\dot{\alpha}}, \mathcal{L}] = -\hat{\omega}_{\alpha}^\beta \partial_{\beta\dot{\alpha}} + \hat{\omega}_{\dot{\alpha}}^\beta \partial_{\alpha\beta} + (\sigma + \bar{\sigma}) \partial_{\alpha\dot{\alpha}} + i(D_{\alpha} \sigma D_{\dot{\alpha}} + D_{\dot{\alpha}} \bar{\sigma} D_{\alpha}), \tag{3.18}
\]
we may find
\[
D_{\beta} \partial_{\gamma\dot{\alpha}} \delta C^{\alpha\beta\gamma} = -\mathcal{L}(D_{\beta} \partial_{\gamma\dot{\alpha}} C^{\alpha\beta\gamma}) - D_{\beta} \partial_{\gamma\dot{\alpha}} C^{'(\beta\gamma} \hat{\omega}_{\alpha^\prime)} - D_{\dot{\beta}} \partial_{\gamma\dot{\alpha}} C^{\alpha\beta\gamma} \hat{\omega}_{\dot{\alpha}}^\beta - (2q \sigma + 3\bar{\sigma}) D_{\beta} \partial_{\gamma\dot{\alpha}} C^{\alpha\beta\gamma} + (3 - 2q)(D_{\beta} \sigma \partial_{\gamma\dot{\alpha}} C^{\alpha\beta\gamma} - D_{\dot{\beta}} C^{\alpha\beta\gamma} b_{\gamma\dot{\alpha}}). \tag{3.19}
\]
In consequence \( D_{\beta} \partial_{\gamma\dot{\alpha}} C^{\alpha\beta\gamma} \) is quasi-primary if \( q = \frac{3}{2} \) and it is easy to see that
\[
T_{\alpha\dot{\alpha}} = \epsilon_{\alpha\alpha'} D_{\beta} \partial_{\gamma\dot{\alpha}} C^{'\alpha'\beta\gamma}, \quad \bar{D}_{\dot{\alpha}} C^{\alpha\beta\gamma} = 0, \tag{3.20}
\]
then gives an expression for the supercurrent which satisfies (3.11) identically, and also has the correct superconformal representation \((\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2})\) for \( T \). Such a \((\frac{3}{2}, \frac{3}{2})_+\) superfield falls outside the restrictions given in (3.10) so \( C^{\alpha\beta\gamma} \) cannot exist in a unitary theory but the representation (3.20) is important in various cases subsequently.

For any quasi-primary superfield general constructions for two and three point functions, consistent with superconformal invariance, are possible using the results of section 2. For a superfield \( O^i \) we also consider its conjugate \( \bar{O}^i \) when \((j, j, q, \bar{q}) \rightarrow (\bar{j}, \bar{j}, \bar{q}, q)\). To define the two-point function for superfields at \( z_1, z_2 \) we introduce \( I^{i\bar{i}}(x_{12}, x_{\bar{1}2}) \) which transforms as a bilocal invariant tensor for the corresponding superfield representations at \( z_1, z_2 \) with \( q = \bar{q} = 0 \),
\[
(L_{z_1} + L_{z_2}) I^{i\bar{i}}(x_{12}, x_{\bar{1}2}) - \frac{1}{2} \partial^a h^b(z_1)(s_{ab})^i \bar{i} I^{i\bar{i}}(x_{12}, x_{\bar{1}2}) + I^{i\bar{i}}(x_{12}, x_{\bar{1}2})(\bar{s}_{ab})_{\bar{i} \bar{j}} \bar{i} \bar{j} \partial^a h^b(z_2) = 0, \tag{3.21}
\]
with \( \bar{s}_{ab} = -s_{ab} \) the generators of the conjugate representation. With \( g_{ii'}, \bar{g}_{ii'} \) group invariant tensors for these representations then we may also define
\[
\bar{I}_{i\bar{i}}(x_{21}, x_{\bar{2}1}) = \bar{g}_{i\bar{i}} g_{ii'} I^{i\bar{i}'}(x_{12}, x_{\bar{1}2}), \tag{3.22}
\]
14
and the normalisation of $I, \bar{I}$ is fixed by requiring

$$I^{ii}(x_{12}, x_{12})\bar{I}_{ii'}(x_{21}, x_{21}) = \delta^i_{i'}, \quad \bar{I}_{ii}(x_{21}, x_{21})I^{ii'}(x_{12}, x_{12}) = \delta^{i'}_{i}. \quad (3.23)$$

For the fundamental spinor representation then from (2.40) and (2.38) expressions satisfying (3.21) and also (3.22) may be obtained which depend just on $x_{12} = -x_{21},$

$$I_{a\bar{a}}(x_{12}) = i\frac{(x_{12})_{a\bar{a}}}{\sqrt{x_{21}^2}}, \quad \bar{I}^{\bar{a}a}(x_{21}) = \epsilon^{\bar{a}\bar{b}}\epsilon^{ab}I_{\beta\bar{\beta}}(x_{12}) = i\frac{(x_{21})^{\bar{a}a}}{\sqrt{x_{21}^2}}, \quad (3.24)$$

while for general representations explicit results are easily obtained by reduction of tensor products. Thus for 4-vectors, when $g_{ii'}, \bar{g}_{ii'} \rightarrow \eta_{ab},$

$$I_{ab}(x_{12}, x_{12}) = \bar{I}_{ba}(x_{21}, x_{21}) = \frac{\text{tr}(\sigma_a \bar{x}_{12}\sigma_b \bar{x}_{21})}{2\sqrt{x_{12}^2}x_{21}^2} = \frac{\text{tr}(\bar{\sigma}_a x_{12}\bar{\sigma}_b x_{21})}{2\sqrt{x_{12}^2}x_{21}^2}. \quad (3.25)$$

Here the denominator may be simplified by use of (2.44).

With the definitions (3.25) and (3.24) we may rewrite the transformations (2.52) and (2.53) as

$$I_{ab}(x_{21}, x_{21})X_1^b = \frac{1}{(X_2^2X_1^2x_{12}^2x_{21}^2)^{\frac{1}{2}}} X_2^f, \quad X_a^f \equiv I_{ab}(\bar{X}, X)X^b = -\left(\frac{X_2^2}{X_1^2}\right)^{\frac{1}{2}} \bar{X}_a, \quad (3.26)$$

and

$$I_{a\bar{a}}(x_{21})\bar{\Theta}_1^{\bar{a}} = \left(\frac{x_{21}^2}{X_2^2}\right)^{\frac{1}{2}} \frac{1}{x_{21}^2} \bar{\Theta}_2^{\bar{a}}, \quad \bar{\Theta}_a^f \equiv I_{a\bar{a}}(X)\bar{\Theta}^\bar{a} = i\frac{1}{(X^2)^{\frac{1}{2}}} (\bar{X}\bar{\Theta})_\alpha, \quad (3.27)$$

$$\Theta_1^{\alpha}I_{a\bar{a}}(x_{12}) = \left(\frac{x_{21}^2}{X_2^2}\right)^{\frac{1}{2}} \frac{1}{x_{21}^2} \Theta_2^f, \quad \Theta_\alpha^f \equiv \Theta^\alpha I_{a\bar{a}}(-X) = -i\frac{1}{(X^2)^{\frac{1}{2}}} (X\Theta)_\alpha. \quad (3.27)$$

From (2.45) we may also obtain

$$\bar{I}_{i_1i_2}(x_{12}, x_{12})I^{i_2i_3}(x_{23}, x_{23})\bar{I}_{i_3i_1}(x_{31}, x_{31}) = \bar{I}_{i_1i_1}(\bar{X}_1, X_1). \quad (3.28)$$

The significance of $I, \bar{I}$ becomes more apparent on considering the transformation of superfields under inversions, for which $z \rightarrow z'$ as in (2.13), when we require

$$O^i(z) \rightarrow O'^i(z) = \frac{1}{x'_{2q}x'_{-2q}} I^{ii}(-x'_-, -x'_{+}) \bar{O}_i(z'), \quad (3.29)$$

$$O_i(z) \rightarrow O'_i(z) = \frac{1}{x'_{2q}x'_{-2q}} \bar{I}_{ii}(-x'_{+}, -x'_-) O^i(z'). \quad (3.29)$$
Since all superconformal transformations can be generated by combining inversions with ordinary supersymmetry the transformations in (3.29) are sufficient to obtain any superfield superconformal transformation. As a consequence of (3.23) \( I^{\bar{i}i}(-x_-, -x_+) \bar{I}_{\bar{i}i'}(x_-, x_+) = \delta_{i'i} \) and using (2.15) \( x'_\pm = -x_\pm/x_\mp^2 \) it is easy to verify from (3.29) that two inversions leave the superfields invariant. In the purely bosonic case \( I_{ab}(x_{12}, x_{12}) \) reduces to the inversion tensor \( \eta_{ab} - 2x_{12a}x_{12b}/x_{12}^2 \) which played a crucial role in the discussion of conformal invariance in arbitrary dimensions [7,8].

A general superconformal covariant expression for the two point function of the superfield \( O \) and its conjugate \( \bar{O} \) is given in terms of \( I \) by

\[
\langle O^i(z_1)\bar{O}^{\bar{i}}(z_2) \rangle = C_O \frac{I^{\bar{i}i}(x_{12}, x_{12})}{x_{12}^2q_{12}^2}, \tag{3.30}
\]

with \( C_O \) an overall normalisation constant. In the denominator of (3.30), and in other expressions subsequently, the singular behaviour at \( x_{12}^2, x_{21}^2 = 0 \) has been modified in accord with the standard lore of quantum field theory, thus for a product of field operators \( \phi(x)\phi(0) \) the singular functions should depend on \( x^2 + i\epsilon x^0 \) while for time ordered products on \( x^2 + i\epsilon \). However in (3.30) and subsequently we leave such resolutions of light cone singularities implicit. The conditions (3.10) are necessary and sufficient for (3.30) to be expressible in terms of a sum over intermediate states of positive norm, as in any unitary theory.

For the three point function we may write a general form as

\[
\langle O_1^{i_1}(z_1)O_2^{i_2}(z_2)O_3^{i_3}(z_3) \rangle = \frac{I_1^{i_1\bar{i}_1}(x_{13}, x_{13})I_2^{i_2\bar{i}_2}(x_{23}, x_{23})}{x_{13}^2q_{13}x_{23}^2q_{23}x_{31}^2q_{31}x_{32}^2q_{32}} t_{i_1\bar{i}_1 i_2\bar{i}_2 i_3}(X_3, \Theta_3, \bar{\Theta}_3), \tag{3.31}
\]

where \( X_3, \Theta_3, \bar{\Theta}_3 \) are defined by appropriate modification of (2.45) and (2.47), and \( I_1, I_2 \) are the bilocal tensors introduced above for the representations defined by the quasi-primary superfields \( O_1, O_2 \). The expression (3.31) automatically has the correct transformation properties at \( z_1, z_2 \) and also at \( z_3 \) if \( t_{i_1\bar{i}_1 i_2\bar{i}_2 i_3} \) has the homogeneity properties

\[
t_{i_1\bar{i}_1 i_2\bar{i}_2 i_3}(\lambda\bar{\lambda}X, \lambda\Theta, \bar{\lambda}\bar{\Theta}) = \lambda^{2a}\bar{\lambda}^{2\bar{a}} t_{i_1\bar{i}_1 i_2\bar{i}_2 i_3}(X, \Theta, \bar{\Theta}), \tag{3.32}
\]

where

\[
a - 2\bar{a} = \bar{q}_1 + \bar{q}_2 - q_3, \quad \bar{a} - 2a = q_1 + q_2 - \bar{q}_3. \tag{3.33}
\]

Since \( \Theta, \bar{\Theta} \) are two-component Grassmann spinors we must have

\[
2(a - \bar{a}) = \frac{2}{3} \sum_i (\bar{q}_i - q_i) = 0, 1, 2, \tag{3.34}
\]

16
and if \( a = \bar{a} \) we may equivalently write, as a consequence of (2.50), \( t_{i_1 i_2}^{i_3}(X, \bar{X}) \). Otherwise \( t_{i_1 i_2}^{i_3}(X, \Theta, \bar{\Theta}) \) is required to transform according to the appropriate spin representations when \( X, \Theta, \bar{\Theta} \) transform infinitesimally as \( \delta X = \omega X - X\bar{\omega}, \delta \Theta = -\Theta \omega, \delta \bar{\Theta} = \bar{\omega} \Theta \).

Using (3.32) we may use the invariance properties of \( t_{i_1 i_2}^{i_3}(X, \Theta, \bar{\Theta}) \) to obtain the transformation formula

\[
I_1^{i_1 i_1}(x_{13}, x_{13}) I_2^{j_1 j_2}(x_{13}, x_{13}) I_3^{j_3 j_3}(x_{13}, x_{13}) t_{i_1 i_2}^{i_3}(X_{13}, \Theta_{3}, \bar{\Theta}_{3})
\]

\[
= \frac{x_{13}^{2(a-2a)} x_{31}^{2(\bar{a}-2a)} X_1^{2a} \overline{X}_1^{2a}}{X^{2(a+q_2)} X_1^{2(a+q_2)}} \bar{t}_{i_1 i_2}^{i_3}(X_{1}, \Theta_{1}, \bar{\Theta}_{1}),
\]

(3.35)

\[
\bar{t}_{i_1 i_2}^{i_3}(X^I, \Theta^I, \bar{\Theta}^I) = I_1^{i_1 i_1}(\bar{X}, \bar{X}) I_2^{j_1 j_2}(-X, X) I_3^{j_3 j_3}(-X, -X) t_{i_1 i_2}^{i_3}(X, \Theta, \bar{\Theta}),
\]

with \( X^I, \Theta^I, \bar{\Theta}^I \) defined in (3.26) and (3.27). This result, with the aid of (3.28) and (3.29), allows us to rewrite (3.31) in the equivalent form

\[
\langle O_1^{i_1}(z_1) O_2^{i_2}(z_2) O_3^{i_3}(z_3) \rangle = \frac{I_2^{i_1 i_1}(x_{21}, x_{21}) I_3^{i_3 i_3}(x_{31}, x_{31})}{x_{21}^{q_2} x_{12}^{q_2} x_{12}^{q_3} x_{13}^{q_3}} \bar{t}_{i_1 i_2 i_3}(X_{1}, \Theta_{1}, \bar{\Theta}_{1}),
\]

(3.36)

where

\[
\bar{t}_{i_1 i_2 i_3}(X, \Theta, \bar{\Theta}) = \frac{1}{X^{2(a+q_2)} X_1^{2(a+q_2)}} I_2^{i_1 i_2}(\bar{X}, X) \bar{t}_{i_1 i_2 i_3}(X^I, \Theta^I, \bar{\Theta}^I).
\]

(3.37)

It is clearly possible to obtain a third representation in which we have a function \( \bar{t}_{i_1 i_2 i_3}(X_2, \Theta_2, \bar{\Theta}_2) \). It is straightforward to verify that the result (3.36) satisfies the equivalent homogeneity properties to (3.32) and (3.33).

The relevance of these results becomes more apparent if we consider the short distance limit \( z_1 \to z_2 \). In this limit it is easy to see that

\[
X_{1}^{I} \sim -\frac{1}{(x_{21}^{q_2} x_{12}^{q_2})^{\frac{a}{2}}} x_{12}, \quad \Theta_{1}^{I} \sim -\frac{1}{(x_{21}^{q_2})^{\frac{a}{2}}} \bar{\Theta}_{12}, \quad \bar{\Theta}_{1}^{I} \sim -\frac{1}{(x_{12}^{q_2})^{\frac{a}{2}}} \bar{\Theta}_{12}.
\]

(3.38)

Using these limiting expressions in (3.36) with (3.37), and applying the homogeneity relation (3.32) once more, the leading behaviour has the form

\[
\langle O_1^{i_1}(z_1) O_2^{i_2}(z_2) O_3^{i_3}(z_3) \rangle \sim I_3^{i_3 i_3}(x_{31}, x_{31}) \bar{t}_{i_1 i_2 i_3}(x_{21}, \bar{\Theta}_{21}, \bar{\Theta}_{21}) \quad \text{for} \quad z_1 \sim z_2.
\]

(3.39)

Given the result (3.30) for the two point function we may therefore obtain for the contribution to the operator product expansion of \( O_1, O_2 \) involving \( O_3 \),

\[
O_1^{i_1}(z_1) O_2^{i_2}(z_2) \sim \frac{1}{C_{O_3}} \bar{t}_{i_1 i_2 i_3}(x_{21}, \bar{\Theta}_{21}, \bar{\Theta}_{21}) \bar{O}_3^{i_3}(z_2).
\]

(3.40)
This demonstrates how in the operator product expansion of two quasi-primary fields the most singular coefficient in the expansion involving a third operator, without any derivatives acting on it, determines completely the corresponding superconformally covariant three point function.

In the rest of this paper we apply the general results to various particular cases, mainly involving the supercurrent. It is important to recognise that the representations (3.16) and (3.20) may be used to provide alternative less singular forms. For the superfield \( L \), which contains a conserved current, then the general formula (3.30) gives simply

\[
\langle L(z_1) L(z_2) \rangle = C_L \frac{1}{x_{12}^2 x_{12}^2}. \tag{3.41}
\]

Making use of (3.16) an expression which satisfies (3.12) identically and reduces to (3.41) for \( z_1 \neq z_2 \) is then

\[
\langle L(z_1) L(z_2) \rangle = \frac{1}{4} C_L \frac{D_2^\alpha D_1^\alpha}{(x_{12}^2)^2}, \quad x_{12} = x_1 - x_2. \tag{3.42}
\]

In a similar fashion the general form for the two point function of the supercurrent is from (3.30)

\[
\langle T_a(z_1) T_b(z_2) \rangle = C_T \frac{I_{ab}(x_{12}^2, x_{12}^2)}{(x_{12}^2 x_{12}^2)^2} \quad \text{or} \quad \langle T_{\alpha \dot{\alpha}}(z_1) T_{\beta \dot{\beta}}(z_2) \rangle = 2 C_T \frac{(x_{12}^2)_{\alpha \dot{\alpha}} (x_{21}^2)_{\beta \dot{\beta}}}{(x_{12}^2 x_{12}^2)^2}, \tag{3.43}
\]

and with the aid of (3.20) this can be rewritten as

\[
\langle T_{\alpha \dot{\alpha}}(z_1) T_{\beta \dot{\beta}}(z_2) \rangle = -\frac{1}{16} C_T \partial_1 \partial_2 \delta_{\alpha \beta} \partial_{\dot{\alpha} \dot{\beta}} D_2 \partial_1 D_1 \left( E_{\gamma \dot{\gamma}} \frac{\theta_{12}^2}{(x_{12}^2)^2} \right), \tag{3.44}
\]

where \( E \) is the projector for symmetric three index spinors

\[
E_{\alpha \gamma \epsilon, \beta \delta \eta} = \delta_{(\alpha \beta} \delta_{\dot{\gamma} \dot{\delta}} \delta_{\epsilon \eta)}. \tag{3.45}
\]

The expressions (3.42) and (3.44) are no longer manifestly superconformally covariant but they allow the non integrable singularity at \( z_1 = z_2 \) to be easily regularised, consistent with the conservation equations, using the method of differential regularisation by replacing in (3.42) and (3.44)

\[
\mathcal{R} \left( \frac{1}{(x_{12}^2)^2} \right) = -\frac{1}{4} \partial^2 \left( \frac{1}{x_{12}^2} \ln (\mu^2 x_{12}^2) \right), \tag{3.46}
\]

with \( \mu \) an arbitrary mass scale.

\[\text{s} This result was essentially given in the first paper in ref. [6].\]
4. Supercurrent and Ward Identities

In order to derive the Ward identities which constrain correlation functions and operator product expansions involving the supercurrent we use a variant of the standard Noether construction of the supercurrent from symmetries of the action. For orientation we first consider a global continuous symmetry acting on the basic fields of the theory under which the action is invariant \( \delta_{\epsilon} S = 0 \), for \( \epsilon \) the parameter representing an infinitesimal group transformation. To define the associated conserved current in terms of a superfield we allow \( \epsilon \) to be extended to independent \( \epsilon(z_+), \bar{\epsilon}(z_-) \) which are defined locally on chiral, anti-chiral superspace (thus chiral fields transform according to \( \epsilon \) while anti-chiral fields according to \( \bar{\epsilon} \)). The action then transforms generally as

\[
\delta_{\epsilon, \bar{\epsilon}} S = i \int d^8z \left( \epsilon_i K_i - \bar{\epsilon}_i \bar{K}_i \right). \tag{4.1}
\]

For \( \epsilon_i = \bar{\epsilon}_i \), a constant, the variation must vanish and this requires that \( K_i - \bar{K}_i \) must be a total derivative so that in general we may therefore write \( K_i - \bar{K}_i = \bar{D}^\alpha U_{i\alpha} + \bar{D}_\alpha \bar{U}_i^\alpha \) for some \( U_{i\alpha}, \bar{U}_i^\alpha \). Furthermore in the definition (4.1) \( K_i \) is arbitrary up to \( K_i = K_i - D_\alpha \bar{U}_i^\alpha \) and similarly \( \bar{K}_i = \bar{K}_i + \bar{D}^\alpha U_{i\alpha} \). Using this freedom allows us to set \( K_i - \bar{K}_i \) to zero so that (4.1) then becomes

\[
\delta_{\epsilon, \bar{\epsilon}} S = i \int d^8z \left( \epsilon_i - \bar{\epsilon}_i \right) L_i. \tag{4.2}
\]

Since \( \delta S = 0 \) for arbitrary variations of the fields defines the equations of motion then, so long as these are satisfied, (4.2) must vanish for any \( \epsilon_i, \bar{\epsilon}_i \) which leads to \( L_i \) being required to obey (3.12). In a quantum field theory, with \( L_i \) an operator superfield, the Ward identities for correlation functions involving other superfields \( O \) may be formally derived from the functional integral using (4.2), assuming invariance of the measure or equivalently no anomalies, in the form

\[
- \int d^8z \left( \epsilon_i(z_+) - \bar{\epsilon}_i(z_-) \right) \langle L_i(z) \ldots O(z_r) \ldots \rangle + \sum_{O} \langle \ldots \delta_{\epsilon, \bar{\epsilon}} O(z_r) \ldots \rangle = 0, \tag{4.3}
\]

which leads to differential relations on taking functional derivatives with respect to \( \epsilon_i \) or \( \bar{\epsilon}_i \).

For the supercurrent we follow a similar analysis\(^9\) considering now the response of the action to local superspace diffeomorphisms preserving chiral superspace so that coordinates transform as in (2.2) with \( h, \bar{h} \) given by (2.4). For general \( h, \bar{h} \) satisfying (2.6) it is

\(^9\) For a very different application of Noether’s theorem to a derivation of the supercurrent see [21].
convenient to define

\[ \omega_{h,\alpha}^\beta - 3\sigma_h \delta_\alpha^\beta = D_\alpha \lambda^\beta + \frac{1}{2} \partial_{\alpha\dot{\alpha}} \bar{h}^{\dot{\alpha}\beta}, \quad \bar{\omega}_{h,\dot{\alpha}}^\dot{\beta} + 3\bar{\sigma}_{\dot{\alpha}} \delta_\dot{\alpha}^\dot{\beta} = \bar{D}_{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}} - \frac{1}{2} \partial_{\dot{\alpha}\ddot{\alpha}} \bar{h}^{\ddot{\alpha}\dot{\beta}}, \tag{4.4} \]

where \( \omega_{h,\alpha}^\alpha = \bar{\omega}_{h,\dot{\alpha}}^{\dot{\alpha}} = 0 \) and these satisfy the chirality conditions

\[ \bar{D}_{\dot{\alpha}} \omega_{h,\beta}^\beta = \bar{D}_{\dot{\alpha}} \sigma_h = 0, \quad D_\alpha \bar{\omega}_{h,\dot{\alpha}}^{\dot{\beta}} = D_\alpha \bar{\sigma}_h = 0. \tag{4.5} \]

Analogous to (1.1) we then assume that under transformations on the basic fields induced by such diffeomorphisms the action is assumed to transform as

\[ \delta_h S = -\frac{1}{2} i \int d^8 z \left(h^a J_a - \bar{h}^a \bar{J}_a\right) + \int d^6 z_+ \sigma_h \mathcal{J} + \int d^6 z_- \bar{\sigma}_h \bar{\mathcal{J}}, \tag{4.6} \]

where \( \mathcal{J}, \bar{\mathcal{J}} \) are chiral, anti-chiral superfields, satisfying \( \bar{D}_{\dot{\alpha}} \mathcal{J} = 0, D_\alpha \bar{\mathcal{J}} = 0. \) As a consequence also of (2.6) \( J, \bar{J} \) are arbitrary up to

\[ J_{\alpha\dot{\alpha}} \sim J_{\alpha\dot{\alpha}} + \bar{D}^\dot{\beta} K(\beta_\alpha)_{\dot{\alpha}}, \quad \bar{J}_{\alpha\dot{\alpha}} \sim \bar{J}_{\alpha\dot{\alpha}} + \bar{D}^{\dot{\beta}} \bar{K}(\dot{\alpha}_{\beta})_{\alpha}. \tag{4.7} \]

Using this freedom we show that improvement terms may be added to \( J, \bar{J} \) so that their difference can be transformed to zero.\(^{10}\)

To demonstrate this we make essential use of the invariance of \( S \) under the usual Poincaré and supersymmetry transformations on the fields. This requires that (4.6) should vanish for \( h^a = \bar{h}^a = h_L^a \), with \( h_L^a \) given by the restriction of (2.9) to the non superconformal case when \( \sigma_{h_L} = \bar{\sigma}_{h_L} = 0 \), so that

\[ \int d^8 z \ h_L^a \left(J_a - \bar{J}_a\right) = 0, \quad \bar{h}_L^{\dot{\alpha} \alpha} = \bar{a}^{\dot{\alpha} \alpha} + \omega_{\beta}^{\dot{\alpha} \beta} \bar{\chi}_+^{\dot{\alpha} \beta} - \bar{\chi}_-^{\dot{\alpha} \beta} \omega_\beta^{\alpha} - 4i \bar{\theta}^{\dot{\alpha}} \bar{\epsilon}^\alpha + 4i \bar{\epsilon}^{\dot{\alpha}} \theta^\alpha. \tag{4.8} \]

---

\(^{10}\) This is analogous to the usual introduction of improvement terms to obtain a symmetric traceless energy momentum tensor. In \( d \)-dimensions then for an action \( S \) depending on fields \( \phi \) then if these transform under translations, Lorentz and scale transformations according to \( \delta \phi = -h^a \partial_a \phi - \frac{1}{2} \omega_{ab} s_{ab} \phi - \eta \rho \phi \), with \( s_{ab} \) the spin generators and \( \eta \) the scale dimension, then for arbitrary local \( h^a(x), \omega^{ab}(x) = -\omega^{ba}(x), \rho(x) \) the action \( S \) may be supposed to transform as

\[ \delta S = -\int d^d x \left\{ (\partial_a h_b + \omega_{ab} - \rho \eta_{ab}) T_{ab}^{\text{imp}} + \partial_c \omega_{ab} X^{cab} + \partial_a \partial_b \rho S^{ab} \right\}, \]

where the further assumption that terms involving only single derivatives of \( \rho \) can be removed is also made. With this restriction if

\[ T_{ab}^{\text{imp}} = T_{ab} - \partial_d \left(X^{cab} - X_{acb} + X_{bac}\right) + \partial_b \partial_d \left(\eta_{ac} J^{db} + \eta_{bd} J^{ca} \right) + \eta_{bc} J^{da} + \eta_{bd} J^{ca} + 2 \eta_{ab} J^{cd} - 2 \eta^{cd} J^{ab} \]

for, if \( d > 2 \), \( J^a = \frac{\eta^{ab} \eta_{cd} S^{abcd}}{2(d-2)} \) then \( \delta S = -\int d^d x \left( \partial_a h_b + \omega_{ab} - \rho \eta_{ab} \right) T_{ab}^{\text{imp}} \), and since the variation with respect to \( h_a, \omega_{ab}, \rho \) must vanish independently, subject to the equations of motion, \( T_{ab}^{\text{imp}} \) is conserved, symmetric and traceless.

20
In consequence $J_a - \tilde{J}_a$ must be expressible as a total derivative which, in order to cancel the $a, \epsilon, \bar{\epsilon}$ terms in $h_L$, should be of the form
\begin{equation}
J_{\alpha\dot{\alpha}} - \tilde{J}_{\alpha\dot{\alpha}} = \tilde{D}^\beta Z_{(\beta\alpha)\dot{\alpha}} + \tilde{D}^\beta Z_{(\dot{\alpha}\beta)\alpha} + \tilde{D}^\beta \tilde{D}^\gamma X_{\beta\alpha\dot{\alpha}\beta} + \tilde{D}^\beta \tilde{D}^\gamma \tilde{X}_{\beta\alpha\dot{\alpha}\beta} \\
= \tilde{D}^\beta Z_{(\beta\alpha)\dot{\alpha}} + \tilde{D}^\beta Z_{(\dot{\alpha}\beta)\alpha} + D_\alpha \tilde{D}^\beta X_{(\dot{\alpha}\dot{\beta})} + D_\alpha \tilde{D}^\beta X_{(\beta\alpha)} + D_\alpha D_\dot{\alpha} X + D_\dot{\alpha} D_\alpha X,
\end{equation}
where the second line is obtained by decomposing $X_{\beta\alpha\dot{\alpha}\beta}$ and $\tilde{X}_{\beta\alpha\dot{\alpha}\beta}$ into irreducible components and wherever possible absorbing terms into a redefinition of $Z, \tilde{Z}$. With this particular form in (4.8) we find
\begin{equation}
-i \int d^8 z \, h_L^\alpha (J_a - J_a) = 4 \int d^8 z \, (\bar{\omega}^{\dot{\alpha}\beta} X_{(\beta\dot{\alpha})} - \omega^{\beta\alpha} X_{(\beta\alpha)}).
\end{equation}
In order to ensure such terms are absent we must further require
\begin{equation}
\tilde{X}_{(\beta\alpha)} = \tilde{D}^\gamma Y_{(\gamma\beta\alpha)} + D_{(\beta\gamma)} Y_{\alpha} + \tilde{D}^\gamma Y_{\beta\alpha\beta}, \quad Y_{\alpha\beta\dot{\beta}} = Y_{\beta\alpha\beta},
\end{equation}
and similarly for $X_{(\alpha\dot{\beta})}$. However we then find
\begin{equation}
\tilde{D}_\alpha \tilde{D}^\beta \tilde{X}_{(\beta\alpha)} = -\frac{3}{2} \tilde{D}_\alpha D_\beta \tilde{D}^\beta Y_{\beta\alpha} - \tilde{D}^\beta \tilde{D}^\gamma Y_{\beta\alpha\beta} + \tilde{D}^{\gamma} \left( \frac{4}{3} \tilde{D}_\alpha \tilde{D}^\gamma Y_{\beta\gamma\beta} + 2 \tilde{D}^\beta \tilde{D}_\alpha Y_{\beta\alpha\beta} \right).
\end{equation}
Thus these terms may be removed by a further redefinition of $Z, \tilde{Z}$ and also $\tilde{X}$ in (4.9). Hence, taking account of the freedom in (4.7), we may therefore in general write, for suitable $X, \tilde{X}$,
\begin{equation}
J_{\alpha\dot{\alpha}} - \tilde{J}_{\alpha\dot{\alpha}} \sim \frac{1}{2} [D_\alpha, \tilde{D}_\alpha] (X - \tilde{X}) - \frac{1}{2} \sigma_{\alpha\dot{\alpha}} (X + \tilde{X}).
\end{equation}
From the trace of the formulae in (4.4) we may find
\begin{equation}
\begin{aligned}
i \partial_{\alpha\dot{\alpha}} \left( \tilde{h}^{\dot{\alpha}\alpha} + \tilde{h}^{\dot{\alpha}\alpha} \right) &= -\frac{1}{6} [D_\alpha, \tilde{D}_\dot{\alpha}] (\tilde{h}^{\dot{\alpha}\alpha} - \tilde{h}^{\dot{\alpha}\alpha}) - 16i (\sigma_h + \tilde{\sigma}_h), \\
\frac{1}{2} [D_\alpha, \tilde{D}_\dot{\alpha}] (\tilde{h}^{\dot{\alpha}\alpha} + \tilde{h}^{\dot{\alpha}\alpha}) &= -3i \partial_{\alpha\dot{\alpha}} (\tilde{h}^{\dot{\alpha}\alpha} - \tilde{h}^{\dot{\alpha}\alpha}) - 48i (\sigma_h - \tilde{\sigma}_h),
\end{aligned}
\end{equation}
and then using this with (4.13) allows us to obtain finally
\begin{equation}
\delta_{h, \tilde{h}} S = \frac{1}{4} i \int d^8 z \, (\tilde{h}^{\dot{\alpha}\alpha} - \tilde{h}^{\dot{\alpha}\alpha}) T_{\alpha\dot{\alpha}} + 4 \int d^8 z \, (\sigma_h (2X - \tilde{X}) - \tilde{\sigma}_h (X - 2\tilde{X})) \\
+ \int d^6 z_+ \sigma_h J + \int d^6 z_- \tilde{\sigma}_h \tilde{J} \\
= -\frac{1}{2} i \int d^8 z \, (h^\alpha - \tilde{h}^\alpha) T_\alpha + \int d^6 z_+ \sigma_h T + \int d^6 z_- \tilde{\sigma}_h \tilde{T},
\end{equation}
where the supercurrent is now given by
\begin{equation}
T_{\alpha\dot{\alpha}} = \frac{1}{2} (J_{\alpha\dot{\alpha}} + \tilde{J}_{\alpha\dot{\alpha}}) - \frac{1}{12} [D_\alpha, \tilde{D}_\alpha] (X + \tilde{X}) + 3i \partial_{\alpha\dot{\alpha}} (X - \tilde{X}),
\end{equation}
\[21\]
and, using the chiral properties (4.5) of \( \sigma_h, \sigma_{\bar{h}} \),

\[
\mathcal{T} = \mathcal{J} - \bar{D}^2(2X - \bar{X}), \quad \bar{\mathcal{T}} = \bar{\mathcal{J}} + D^2(X - 2\bar{X}).
\]

(4.17)

The implicit definition (4.15) does not determine the supercurrent uniquely since if

\[
T_{\alpha\dot{\alpha}} \rightarrow T_{\alpha\dot{\alpha}} + D_{\dot{\alpha}} \bar{D}_\alpha \mathcal{S} - \bar{D}_\dot{\alpha} D_\alpha \mathcal{S}, \quad D_{\alpha} \mathcal{S} = 0, \quad \bar{D}_{\dot{\alpha}} \mathcal{S} = 0,
\]

(4.18)

then this can be compensated in (4.15) by taking

\[
\mathcal{T} \rightarrow \mathcal{T} + \frac{3}{2} \bar{D}^2 \mathcal{S}, \quad \bar{\mathcal{T}} \rightarrow \bar{\mathcal{T}} + \frac{3}{2} D^2 \mathcal{S}.
\]

(4.19)

To obtain the conservation equations it is convenient as usual to solve (2.6) in terms of unconstrained prepotentials \( L^\alpha, \bar{L}^{\dot{\alpha}} \) where

\[
\tilde{h}^{\dot{\alpha}\alpha} = 2 \tilde{D}^{\dot{\alpha}} L^\alpha, \quad \tilde{\bar{h}}^{\dot{\alpha}\alpha} = 2 \tilde{D}^{\dot{\alpha}} \bar{L}^\alpha \quad \Rightarrow \quad \sigma_h = \frac{1}{24} i \tilde{D}^2 D_\alpha L^\alpha, \quad \bar{\sigma}_{\bar{h}} = -\frac{1}{24} i D^2 \tilde{D}_{\dot{\alpha}} \bar{L}^{\dot{\alpha}}.
\]

(4.20)

Varying \( L^\alpha, \bar{L}^{\dot{\alpha}} \) in (4.15) now gives

\[
\tilde{D}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = \frac{1}{3} D_\alpha \mathcal{T}, \quad \tilde{D}_{\dot{\alpha}} T_{\alpha\dot{\alpha}} = \frac{1}{3} D_{\dot{\alpha}} \bar{\mathcal{T}}.
\]

(4.21)

For superconformal invariance the variation in (4.15) must vanish when (2.7) is satisfied, when \( \sigma_h = \sigma, \bar{\sigma}_{\bar{h}} = \bar{\sigma} \) as given by (2.25) and (2.28). Thus (4.15) should be independent of \( \sigma_h, \bar{\sigma}_{\bar{h}} \) or \( \mathcal{T} = \bar{\mathcal{T}} = 0 \), for a suitable choice of \( \mathcal{S}, \bar{\mathcal{S}} \) in (4.19), in this case and then (4.21) reduces to the superconformal covariant equation (3.11).

Although the above considerations are classical we can use them as previously in (4.3) to obtain the corresponding quantum field theory Ward identity, assuming superconformal invariance extends to the quantum theory, in the form

\[
\frac{1}{2} \int d^8 z \left( h^a(z) - \bar{h}^a(z) \right) \langle T_a(z) \ldots O(z_r) \ldots \rangle + \sum O \langle \ldots \delta_h \bar{h} O(z_r) \ldots \rangle = 0.
\]

(4.22)

For quasi-primary superfields the definition of \( \delta_h \bar{h} O \) need not be unique but it should reduce to (3.1) in the superconformal limit, \( h = \bar{h}, \) when (4.22) becomes to just the requirement of superconformal covariance of the correlation function \( \langle \ldots O(z_r) \ldots \rangle \).

For subsequent applications we apply these results to the trivial cases of free field theories. For chiral scalar fields \( \phi(z_+), \bar{\phi}(z_-) \) we take

\[
S = \int d^8 z \bar{\phi} \phi,
\]

(4.23)
and the fields are supposed to transform as

\[ \delta_h \phi = -\mathcal{L}_+ \phi - 2q \sigma_h \phi, \quad \bar{\delta}_h \bar{\phi} = -\mathcal{L}_- \bar{\phi} - 2\bar{q} \bar{\sigma}_h \bar{\phi}, \]

(4.24)

with \( \mathcal{L}_\pm \) given in (2.3). The variation of (4.23) can then be written as in (4.16) with

\[ J_{\alpha \dot{\alpha}} = \frac{1}{2} D_\alpha \bar{\phi} \bar{D}_{\dot{\alpha}} \bar{\phi} - i \partial_{\alpha \dot{\alpha}} \phi \bar{\phi}, \quad \bar{J}_{\dot{\alpha} \alpha} = \frac{1}{2} D_{\dot{\alpha}} \bar{\phi} \bar{D}_\alpha \bar{\phi} + i \phi \partial_{\alpha \dot{\alpha}} \bar{\phi}, \]

(4.25)

\[ \mathcal{J} = \frac{1}{2} q D^2 (\phi \bar{\phi}), \quad \bar{\mathcal{J}} = \frac{1}{2} \bar{q} \bar{D}^2 (\bar{\phi} \phi). \]

Clearly the difference is of the required form given by (4.13) with \( X = \bar{X} = \frac{1}{2} \phi \bar{\phi} \). Hence from (4.17) we have \( \mathcal{T} = \bar{\mathcal{T}} = 0 \) if \( q = \bar{q} = 1 \) and the supercurrent for this theory becomes

\[ T_{\alpha \dot{\alpha}} = \frac{1}{3} (D_\alpha \phi \bar{D}_{\dot{\alpha}} \bar{\phi} + 2i \phi \bar{\phi} \bar{D}_{\dot{\alpha} \alpha} \bar{\phi}). \]

(4.26)

The other example of a trivial superconformal theory is formed by the abelian gauge theory with action

\[ S = \frac{1}{4} \int d^6 z_+ W^2 + \frac{1}{4} \int d^6 z_- \bar{W}^2, \quad W_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha V, \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} D^2 \bar{D}_{\dot{\alpha}} V. \]

(4.27)

The action of superdiffeomorphisms on the scalar gauge superfield \( V \) is taken as

\[ \delta_{h, \bar{h}} V = -(\frac{1}{2} (h^a + \bar{h}^a) \partial_a + \lambda^a D_\alpha + \bar{\lambda}_\dot{\alpha} \bar{D}_{\dot{\alpha}}) V - \frac{1}{8} i (\bar{\lambda}^\dot{\alpha} - \bar{\lambda}^{\dot{\alpha} a}) [D_\alpha, \bar{D}_{\dot{\alpha}}] V. \]

(4.28)

This clearly reduces to the general form for a superconformal transformation (3.1) when \( h = \bar{h} \) and has the important property that it preserves gauge transformations,

\[ \delta V = -i \frac{1}{2} (\epsilon - \bar{\epsilon}) \quad \Rightarrow \quad \delta_{h, \bar{h}} \delta V = i \frac{1}{2} (\mathcal{L}_+ \epsilon - \mathcal{L}_- \bar{\epsilon}), \quad \bar{D}_\alpha \epsilon = 0, \quad D_\alpha \bar{\epsilon} = 0. \]

(4.29)

From this and using (4.4) the chiral fields \( W, \bar{W} \) transform as

\[ \delta_{h, \bar{h}} W_\alpha = -\mathcal{L}_+ W_\alpha + \omega_{\alpha \beta} W_\beta - 3\sigma_h W_\alpha - \frac{1}{8} i \epsilon_{\alpha \beta} D^2 (\bar{W}_\beta (\bar{h}^\dot{\beta} - \bar{h}^{\dot{\beta}})), \]

(4.30)

\[ \delta_{h, \bar{h}} \bar{W}_{\dot{\alpha}} = -\mathcal{L}_- \bar{W}_{\dot{\alpha}} - \bar{W}_{\dot{\beta}} \bar{\omega}_{\dot{\beta} \dot{\alpha}} - 3\bar{\sigma}_h \bar{W}_{\dot{\beta}} - \frac{1}{8} i \epsilon_{\dot{\beta} \dot{\alpha}} D^2 ((\bar{h}^{\dot{\beta}} - \bar{h}^\dot{\beta}) W_\beta), \]

which for \( h = \bar{h} \) automatically give that \( W, \bar{W} \) are \((\frac{1}{2}, \frac{3}{2})_+, (\frac{1}{2}, \frac{3}{2})_- \) superconformal superfields. From (4.30) we easily find

\[ \delta_{h, \bar{h}} W^2 = - (\mathcal{L}_+ + 6\sigma_h) W^2 + \frac{1}{4} i \bar{D}^2 ((\bar{h}^{\dot{\beta}} - \bar{h}^\dot{\beta}) W_\beta \bar{W}_\beta), \]

(4.31)

\[ \delta_{h, \bar{h}} \bar{W}^2 = - (\mathcal{L}_- + 6\bar{\sigma}_h) \bar{W}^2 + \frac{1}{4} i D^2 ((\bar{h}^{\dot{\beta}} - \bar{h}^\dot{\beta}) W_\beta \bar{W}_\beta), \]
and applying this in (4.27) gives the required form in (4.15), with \( T = \bar{T} = 0 \), where

\[
T_{\alpha\bar{\alpha}} = -2W_{\alpha}\bar{W}_{\bar{\alpha}}.
\]  

(4.32)

5. Chiral Superfields

The general results in section 3 simplify significantly if they are applied for cases involving chiral superfields so we consider these first. We denote a \((0, q)_+\) chiral scalar by \( \phi(z_+) \) and its anti-chiral \((0, \bar{q})_-\) partner, where \( q = \bar{q} \), by \( \bar{\phi}(z_-) \). From (3.30) the associated two point function is simply

\[
\langle \phi(z_+)\bar{\phi}(z_-) \rangle = C_{\phi} \frac{1}{x_{21}^{2q}}.
\]  

(5.1)

For two chiral scalar superfields and an anti-chiral superfield we may write the three point function from (3.31) as

\[
\langle \phi_1(z_+)\phi_2(z_+)\bar{\phi}_3(z_-) \rangle = C_{123} \frac{1}{x_{31}^{2q_1}x_{32}^{2q_2}}, \quad q_1 + q_2 = \bar{q}_3.
\]  

(5.2)

As in (3.40) this leads to the operator product

\[
\phi_1(z_+)\phi_2(z_+) \sim \frac{1}{C_{\phi}} C_{123} \phi_3(z_2+), \quad q_1 + q_2 = q_3,
\]  

(5.3)

without any singularities as \( z_1 \to z_2 \) so that the chiral scalar fields form a closed algebraic ring.

Results for two or three chiral fields are only possible in special cases. For the two point function we may write the conformally invariant form \([4]\),

\[
\langle \phi_1(z_+)\phi_2(z_+) \rangle = C_{12} \delta^4(x_{1+} - x_{2+}) \theta_{12}^2, \quad q_1 + q_2 = 3,
\]  

(5.4)

but this is a pure contact term and should be removable by suitable counterterms in the effective action. For three chiral scalar fields we may write from (3.31) (in (3.32) \( a = \bar{a} - 1 = q_3 - 2 \))

\[
\langle \phi_1(z_+)\phi_2(z_+)\phi_3(z_+) \rangle = C_{123} \frac{1}{x_{31}^{2q_1}x_{32}^{2q_2}} X_3^{2(q_3-2)}\Theta_3^2,
\]  

\[
= C_{123} \frac{1}{x_{12}^{2q_2}x_{13}^{2q_3}} X_1^{2(q_1-2)}\Theta_1^2, \quad q_1 + q_2 + q_3 = 3,
\]  

(5.5)
where consistency depends on the condition \( \sum_i q_i = 3 \) and in the second line we have transformed to the alternative form given by (3.36) using (3.37). The chirality properties are not manifest in (5.5) but in the second line the form of \( X^3 \) in (2.51) and of \( \Theta_1 \) in (2.47) demonstrate that this expression depends only on \( z_{3+} \) and also since \( f(X_1)\Theta_1^2 = f(\bar{X}_1)\bar{\Theta}_1^2 \) on \( z_{2+} \) whereas similar arguments from the first line of (5.3) demonstrate that it also depends only on \( z_{1+} \). We later give an equivalent expression in which the chirality properties are obvious but the conformal properties are less evident.\(^\text{11}\) Corresponding to (5.3) we have an operator product,

\[
\phi_1(z_{1+})\phi_2(z_{2+}) \sim -\frac{C_{123}}{C_{\phi \phi}} \frac{\theta_{12}^2}{((x_{1+} - x_{2+})^2)^2-q_3} \bar{\phi}_3(z_{2-}), \quad q_3 = 3 - q_1 - q_2 .
\]

For three point functions involving the supercurrent and chiral scalar fields we may write from (3.31) the unique expression

\[
\langle T_{\alpha\bar{\alpha}}(z_1) \phi(z_{2+}) \bar{\phi}(z_{3-}) \rangle = -iA \frac{(x_{13})_{\alpha\beta} (x_{31})_{\beta\bar{\alpha}}}{(x_{13}^2 x_{31}^2)^2} \frac{1}{x_{32}^2} \frac{\bar{X}^{\alpha\beta}}{(X^3)^2} (5.7a)
\]

\[
= -iA \frac{1}{x_{12}^2 x_{31}^2} \frac{x_{32}^2}{X_1^{2(q-1)}} . \quad (5.7b)
\]

where (5.7b) follows directly or from (3.36) with the general result (3.37) using \( a = \bar{a} = -\frac{3}{2} \) and taking \( t^{\dot{\alpha}\alpha}(X) = iA \bar{X}^{\dot{\alpha}\alpha} / (X^2)^2 \). This satisfies the hermeticity condition \( t^{\dot{\alpha}\alpha}(X)^\dagger = t^{\dot{\alpha}\alpha}(-X) \). From the definition of \( X_1 \) in (2.45), (5.7) clearly depends only on \( z_{2+}, z_{3-} \) as required from the form of the l.h.s. With the aid of (5.7a, b) we may easily find the leading contribution to the operator product expansion for the supercurrent and a chiral scalar field,

\[
T_{\alpha\bar{\alpha}}(z_1) \phi(z_{2+}) \sim iA \frac{1}{C_{\phi \phi}} \frac{(x_{21})_{\alpha\bar{\alpha}}}{(x_{12}^2)^2} \phi(z_{2+}) .
\]

To obtain the corresponding Ward identity we may take from (4.21) with (4.22)

\[
\delta_h \phi = \frac{1}{4} i \tilde{D} (L^\alpha D_\alpha \phi) = q_1 \frac{1}{12} i (\tilde{D}^2 D_\alpha L^\alpha) \phi ,
\]

and using this in (4.22) gives

\[
\tilde{D}_1^{\dot{\alpha}} \langle T_{\alpha\bar{\alpha}}(z_1) \phi(z_{2+}) \bar{\phi}(z_{3-}) \rangle
\]

\[
= \frac{2}{3} iq \frac{1}{D_1^{\alpha}} \delta^{\dot{\alpha}}(z_1 - z_2) \langle \phi(z_{2+}) \bar{\phi}(z_{3-}) \rangle + 2i \delta^{\dot{\alpha}}(z_1 - z_2) D_2^{\alpha} \langle \phi(z_{2+}) \bar{\phi}(z_{3-}) \rangle .
\]

\(^\text{11}\) A similar but not apparently identical form was given in [12], see also [22].
where the chiral delta function is

\[ \delta^6_+(z_1 - z_2) = \delta^4(x_{1+} - x_{2+}) \theta_{12}^2. \quad (5.11) \]

In the next section we show how \((5.7a)\) satisfies the condition that the r.h.s. of \((5.10)\) is zero for \(z_1 \neq z_2\). The delta functions appearing in the Ward identity \((5.10)\) arise from the singularities in \((5.7a, b)\) for \(z_1 \sim z_2\). The first term on the r.h.s. of \((5.10)\) is thus generated from the leading singular term in the operator product expansion \((5.8)\). The action of the derivative may be calculated by

\[ \tilde{D}_1 \alpha \frac{(x_{21})_{a\dot{a}}}{x_{12}^{2\lambda}} = 4i(2 - \lambda) \frac{1}{x_{12}^{2\lambda}} (\tilde{\theta}_{12})_a \xrightarrow{\lambda \to 2} 4\pi^2 \delta^4(x_{12}) (\tilde{\theta}_{12})_a = 2\pi^2 D_1 \alpha \delta^6_+(z_1 - z_2), \quad (5.12) \]

using the result that, as a distribution on \(\mathbb{R}^4\), \((x^2)^{-\lambda}\) has a pole as \(\lambda \to 2\) with a residue which is proportional to \(\delta^4(x)\). Using \((5.12)\) with \((5.8)\) in \((5.10)\) we must then require for consistency

\[ \frac{A}{C_\phi} = \frac{1}{3\pi^2} q. \quad (5.13) \]

Thus the Ward identity determines completely the overall coefficient of the three point function \((5.7a, b)\) involving chiral scalar fields and the supercurrent.

6. Ward Identities and Correlation Functions

If the general results of section 3 are applied to correlation functions involving the current superfield \(L\) or the supercurrent \(T_a\) then it is in general necessary to impose restrictions in order to satisfy the conservation equations \((3.12)\) and \((4.21)\) at non coincident points. Furthermore the Ward identities \((4.3)\) and \((4.22)\) lead to relations between a three point function containing \(L\) or \(T_a\) and the associated two point function without them, as exemplified in \((5.13)\).

To obtain simple results for the action of derivatives on three point functions we first exhibit how covariant spinor derivatives act on functions of \(X_3, \Theta_3, \bar{\Theta}_3\) by writing the conformally covariant formulae

\[ \tilde{D}_1 \dot{\alpha} f(X_3, \Theta_3, \bar{\Theta}_3) = -i \frac{1}{x_{31}^{2\lambda}} (\bar{x}_{13})^{\dot{\alpha}\alpha} D_3 \alpha f(X_3, \Theta_3, \bar{\Theta}_3), \]

\[ D_1 \alpha f(X_3, \Theta_3, \bar{\Theta}_3) = -i \frac{1}{x_{13}^{2\lambda}} (x_{13})_{\alpha\dot{\alpha}} \tilde{D}_3 \dot{\alpha} f(X_3, \Theta_3, \bar{\Theta}_3), \quad (6.1) \]
where, for \( X_3, \Theta_3, \bar{\Theta}_3 \rightarrow X, \Theta, \bar{\Theta} \) and \( D_3, \bar{D}_3 \rightarrow D, \bar{D} \),

\[
D_\alpha = \frac{\partial}{\partial \Theta^\alpha} - 2i(\sigma^a \bar{\Theta})_\alpha \frac{\partial}{\partial X^a}, \quad \bar{D}_\dot{\alpha} = -\frac{\partial}{\partial \bar{\Theta}^{\dot{\alpha}}}. \tag{6.2}
\]

With these definitions and from the relation (2.50) \( \bar{X} = X + 2i\Theta\sigma\bar{\Theta} \) it is easy to verify that \( D_\alpha \bar{X}^\alpha = 0 \) which is in accord with (6.1) since \( \bar{D}_{1\dot{\alpha}}X_3 = 0 \). From (6.1) we may then find

\[
\bar{D}_{1\dot{\alpha}} \left( \frac{1}{(x_{13})^2} (x_{31})_{\alpha \dot{\alpha}} F^\alpha (X_3, \Theta_3, \bar{\Theta}_3) \right) = -i \frac{1}{x_{31}^2 x_{13}^2} D_{3\alpha} F^\alpha (X_3, \Theta_3, \bar{\Theta}_3), \tag{6.3}
\]

and also

\[
\bar{D}_{1\dot{\alpha}} \left( \frac{1}{(x_{13})^2} f(X_3, \Theta_3, \bar{\Theta}_3) \right) = \frac{1}{(x_{31})^2} D_{3\alpha} f(X_3, \Theta_3, \bar{\Theta}_3), \tag{6.4}
\]

with a similar formula involving \( D_{1\dot{\alpha}} \).

With these results it is straightforward to check that (6.7a) satisfies the requirement from (5.10) that \( \bar{D}_{1\dot{\alpha}}^\dot{\alpha} (T_{\alpha \dot{\alpha}}(z_1)\phi(z_2+)\phi(z_3-)) = 0 \) at least for \( z_1 \neq z_2, z_3 \) since applying (5.3) in this case requires only that

\[
D_\alpha \frac{\bar{X}^{\dot{\alpha}}}{(X^2)^2} = 0, \quad \bar{D}_{\dot{\alpha}} \frac{\bar{X}^{\dot{\alpha}}}{(X^2)^2} = 0, \tag{6.5}
\]

which are easily verified.

We now apply the general result to the three point function for the internal symmetry current scalar superfield \( L_i \), where \( i \) is a group index. Applying (3.31), with \( q_L = q_L = 1 \), gives

\[
\langle L_i(z_1)L_j(z_2)L_k(z_3) \rangle = \frac{1}{x_{31}^2 x_{13}^2 x_{32}^2 x_{23}^2} t_{ijk}(X_3, \bar{X}_3), \tag{6.6}
\]

and \( t_{ijk}(X, \bar{X}) \) is homogeneous

\[
t_{ijk}(\rho X, \rho \bar{X}) = \rho^{-2} t_{ijk}(X, \bar{X}), \tag{6.7}
\]

and satisfies the symmetry relations

\[
t_{ijk}(X, \bar{X}) = t_{ijk}(-\bar{X}, -X) = t_{jki}(X^I, \bar{X}^I). \tag{6.8}
\]

Applying the conservation equation (3.12) leads from (6.4) to

\[
D^2 t_{ijk}(X, \bar{X}) = \bar{D}^2 t_{ijk}(X, \bar{X}) = 0. \tag{6.9}
\]
The solution of these conditions is straightforward\textsuperscript{12}
\begin{equation}
t_{ijk}(X, \bar{X}) = C_f \, i f_{ijk} \left( \frac{1}{X^2} - \frac{1}{\bar{X}^2} \right) + C_d \, d_{ijk} \left( \frac{1}{X^2} + \frac{1}{\bar{X}^2} \right), \tag{6.10}
\end{equation}
where $f_{ijk}$, $d_{ijk}$ are totally antisymmetric, symmetric group tensors.

We analyse first the contribution involving $f_{ijk}$ when (6.6) and (6.10) give
\begin{equation}
\langle L_i(z_1)L_j(z_2)L_k(z_3) \rangle_f = C_f \, i f_{ijk} \left( \frac{1}{x_{13}^2 \bar{x}_{32}^2 \bar{x}_{21}^2} - \frac{1}{x_{31}^2 \bar{x}_{23}^2 x_{12}^2} \right). \tag{6.11}
\end{equation}
To obtain Ward identities we assume that under infinitesimal group transformations as considered in (4.2), (4.3)
\begin{equation}
\delta \epsilon \, L_i = - f_{ijk} \frac{1}{2} (\epsilon_j + \bar{\epsilon}_j)L_k + i (\epsilon_j - \bar{\epsilon}_j)K_{ij}. \tag{6.12}
\end{equation}
and then (4.3), assuming $\langle K_{ij}L_k \rangle = 0$, gives
\begin{equation}
\frac{1}{d} \bar{D}_1^2 \langle L_i(z_1)L_j(z_2)L_k(z_3) \rangle + \frac{1}{d} f_{ijk} \delta_+^6(z_1 - z_2)\langle L_\ell(z_2)L_k(z_3) \rangle + \frac{1}{d} f_{ijk} \delta_+^6(z_1 - z_3)\langle L_j(z_2)L_\ell(z_3) \rangle = 0, \tag{6.13}
\end{equation}
with a similar equation involving $D_1^2$. From (3.41) we may take
\begin{equation}
\langle L_i(z_1)L_j(z_2) \rangle = C_L \, \delta_{ij} \frac{1}{x_{21}^2 x_{12}^2}, \tag{6.14}
\end{equation}
and using the counterpart to (5.12),
\begin{equation}
\bar{D}_1^2 \frac{1}{x_{12}^2} = -i 16 \pi^2 \delta_+^6(z_1 - z_2), \tag{6.15}
\end{equation}
it is easy to see that (6.11) is compatible with (6.13) and (6.14) if
\begin{equation}
8 \pi^2 C_f = C_L. \tag{6.16}
\end{equation}

The part of the three point function involving $d_{ijk}$ may be written as
\begin{equation}
\langle L_i(z_1)L_j(z_2)L_k(z_3) \rangle_d = C_d \, d_{ijk} \left( \frac{1}{x_{13}^2 x_{32}^2 x_{21}^2} + \frac{1}{x_{31}^2 x_{23}^2 x_{12}^2} \right) - 4 \pi^2 i C_d \, d_{ijk} \left( \delta^8(z_1 - z_2) \frac{1}{x_{23}^2 x_{32}^2} + \delta^8(z_2 - z_3) \frac{1}{x_{13}^2 x_{31}^2} + \delta^8(z_3 - z_1) \frac{1}{x_{21}^2 x_{12}^2} \right), \tag{6.17}
\end{equation}
\textsuperscript{12} This demonstrates that $\mathcal{N} = 1$ superconformal invariance leads to unique totally symmetric or antisymmetric expressions for the three point functions of conserved currents, as was conjectured earlier [23].
with
\[ \delta^8(z_1 - z_2) = \delta^4(x_1 - x_2) \theta_{12}^2 \bar{\theta}_{12}^2. \]  
(6.18)

Since \(-\frac{1}{4}D_1^2\delta^8(z_1 - z_2) = \delta^6_+(z_1 - z_2)\), the chiral \(\delta\)-function defined in (5.11), the second line of (6.17) removes terms involving a single delta function from \(\bar{D}^2_1\langle L_i(z_1) L_j(z_2) L_k(z_3)\rangle_d\), such as were present in (5.13). The potentiality of introducing such contact terms to impose the conservation equations is a reflection of the ambiguities in the precise definition of this three point function as a distribution arising from the singular behaviour at coincident points.

A more careful consideration reveals the supersymmetric counterpart of the well known axial anomalies when calculating the action of \(\bar{D}^2_1\) on (6.17). To demonstrate the necessary presence of such anomalies in the present formalism and to determine their form we make use of the representation (3.16) to impose the conservation equations (3.12) trivially at \(z_1, z_2\). Thus, suppressing group indices so that \(\delta_{i,j,k} \rightarrow 1\),

\[ \langle L(z_1) L(z_2) L(z_3) \rangle = \bar{D}_2^2 \bar{D}_1^2 \Gamma_{\alpha\beta}(z_1+, z_2+, z_3) + \Gamma_{\text{loc}}(z_1, z_2, z_3), \]  
(6.19)

where \(\Gamma_{\text{loc}}(z_1, z_2, z_3)\) is a purely local contact term which is necessary to ensure that the representation for the three point function for \(L\) in (6.19) is symmetric. Assuming superconformal invariance \(\Gamma_{\alpha\beta}(z_1+, z_2+, z_3) = -\Gamma_{\beta\alpha}(z_2+, z_1+, z_3)\) is then determined by requiring it to be a three point function of the general form in (3.31) with \(q_1 = q_2 = \frac{3}{2}, q_1 = q_2 = 0\) which gives in this case

\[ \Gamma_{\alpha\beta}(z_1+, z_2+, z_3) = C_d \frac{(x_{13})_{\alpha\hat{\alpha}} (x_{23})_{\beta\hat{\beta}} \varepsilon^{\hat{\alpha}\hat{\beta}} \bar{\Theta}_{13}^2}{2 X_{13}^2}, \quad \bar{\Theta}_{13}^2 = \bar{\Theta}_{32}^2 \]  
(6.20)

The overall coefficient is chosen so that, using (3.3), (6.19) gives the symmetric form

\[ \langle L(z_1) L(z_2) L(z_3) \rangle = C_d \left( \frac{1}{x_{13}^2 x_{23}^2 x_{21}^2} + \frac{1}{x_{31}^2 x_{23}^2 x_{12}^2} \right), \]  
(6.21)

at non coincident points. The singularities which appear in the expression (6.20) for \(\Gamma_{\alpha\beta}(z_1+, z_2+, z_3)\) at coincident points are integrable and hence possible ambiguities proportional to derivatives of \(\delta\)-functions, which arise for (6.21), are not present. By its construction the representation (5.19) ensures that anomalies arising from the first term on the r.h.s. are confined to the action of \(\bar{D}_3^2\) and \(D_3^2\). To obtain the anomalies explicitly it is convenient to rewrite (6.20) in the alternative form (3.30)

\[ \Gamma_{\alpha\beta}(z_1+, z_2+, z_3) = -C_d \frac{(x_{21})_{\beta\hat{\beta}}}{(x_{12}^2)^2} \frac{1}{x_{13}^2 x_{23}^2 x_{21}^2} \varepsilon^{\hat{\alpha}\hat{\beta}} X_{1\alpha\hat{\alpha}} \bar{\Theta}_{13}^2, \]  
(6.22)
so that, using the analogous result to (6.4) for $D_s^2$, reduces calculating the action of $D_s^2$ to $\check{D}_1^2 \check{\Theta}_1^2 = -4$ and hence

$$D_s^2 \Gamma_{\alpha\beta}(z_{1+}, z_{2+}, z_3) = -2C_d \frac{(x_{13})_{\alpha\beta} (x_{23})_{\beta\dot{\beta}}}{(x_{31}^2 x_{32}^2)^2} \varepsilon^{\dot{\alpha}\dot{\beta}}. \quad (6.23)$$

With the aid of

$$\check{D}_1^\alpha \frac{(x_{13})_{\alpha\dot{\alpha}}}{(x_{31}^2)^2} = 2\pi^2 \check{D}_1 \delta^\alpha_-(z_1 - z_3), \quad (6.24)$$

and similarly for $\check{D}_2^\beta$, (6.19) gives

$$D_s^2 (\check{D}_2^\beta \check{D}_1^\alpha \Gamma_{\alpha\beta}(z_{1+}, z_{2+}, z_3)) = 8\pi^4 C_d \check{D}_1 \delta^\alpha_-(z_1 - z_3) \check{D}_2 \delta^\alpha_-(z_2 - z_3). \quad (6.25)$$

A similar calculation for $\check{D}_s^2 \Gamma_{\alpha\beta}(z_{1+}, z_{2+}, z_3)$ naively gives zero but in this case it is necessary to be more careful in the treatment of singularities at coincident points. If we modify (6.19) to

$$\Gamma_{\alpha\beta}(z_{1+}, z_{2+}, z_3) = C_d \frac{(x_{13})_{\alpha\dot{\alpha}} (x_{23})_{\beta\dot{\beta}}}{(x_{31}^2)^2} \varepsilon^{\dot{\alpha}\dot{\beta}} \check{\Theta}_3^2 \frac{2X_3}{2}, \quad (6.26)$$

then

$$\check{D}_s^2 \Gamma_{\alpha\beta}(z_{1+}, z_{2+}, z_3) = 8C_d (\lambda_1 + \lambda_2) (\lambda_1 + \lambda_2 + 1) \theta_{13}^2 \theta_{23}^2 \frac{(x_{13})_{\alpha\dot{\alpha}} (x_{23})_{\beta\dot{\beta}}}{x_{12}^2 x_{13}^2 x_{23}^2} \frac{\varepsilon^{\dot{\alpha}\dot{\beta}}}{x_{12}^2 x_{13}^2 x_{23}^2}, \quad (6.27)$$

where here $x_{12} = x_{1+} - x_{2+}$. As $\lambda_1, \lambda_2 \to 0$ the factor on the r.h.s. of (6.27) depending on $x_{12}, x_{13}, x_{23}$ generates a pole in $\lambda_1 + \lambda_2$ with a residue $\propto \delta^4(x_{13}) \delta^4(x_{23})$ so that

$$\check{D}_s^2 \Gamma_{\alpha\beta}(z_{1+}, z_{2+}, z_3) = 8\pi^4 C_d \varepsilon_{\alpha\beta} \delta^6_+(z_1 - z_3) \delta^6_+(z_2 - z_3), \quad (6.28)$$

and hence, similar to (6.23),

$$\check{D}_s^2 (\check{D}_2^\beta \check{D}_1^\alpha \Gamma_{\alpha\beta}(z_{1+}, z_{2+}, z_3)) = 8\pi^4 C_d \check{D}_1 \delta^6_+(z_1 - z_3) \check{D}_2 \delta^6_+(z_2 - z_3). \quad (6.29)$$

To obtain a suitable expression for $\Gamma_{loc}(z_1, z_2, z_3)$ we first define

$$16f(z_1, z_2, z_3)$$

$$= \check{D}_{3\dot{a}} \delta^8(z_3 - z_1) D_3^2 \check{D}_3 \delta^8(z_3 - z_2) + \check{D}_3 \delta^8(z_3 - z_1) D_3^2 D_{3\dot{a}} \delta^8(z_3 - z_2) \quad (6.30)$$

$$= \check{D}_{3\dot{a}} \delta^8(z_3 - z_1) D_3^2 \check{D}_3 \delta^8(z_3 - z_2) + \check{D}_3 \delta^8(z_3 - z_1) D_3^2 D_{3\dot{a}} \delta^8(z_3 - z_2).$$
which has the properties

\[ D_3^2 f(z_1, z_2, z_3) = \tilde{D}_{1\dot{\alpha}} \delta^6 (z_1 - z_3) \tilde{D}_{2\dot{\delta}} \delta^6 (z_2 - z_3), \]

\[ \tilde{D}_3^2 f(z_1, z_2, z_3) = \tilde{D}^1_{\alpha} \delta^6 (z_1 - z_3) D_2\alpha \delta^6 (z_2 - z_3), \]

\[ D_2^2 f(z_1, z_2, z_3) = \tilde{D}_2^2 f(z_1, z_2, z_3) = 0, \quad f(z_1, z_2, z_3) = -f(z_3, z_2, z_1). \]

If we take

\[ \Gamma_{\text{loc}}(z_1, z_2, z_3) = -\frac{8}{3} \pi^4 C_d (f(z_1, z_2, z_3) + f(z_2, z_1, z_3)), \]

then (6.31) and (6.25), (6.29) give

\[ D_3^2 \langle L(z_1) L(z_2) L(z_3) \rangle = \frac{8}{3} \pi^4 C_d \tilde{D}_{1\dot{\alpha}} \delta^6 (z_1 - z_3) \tilde{D}_{2\dot{\delta}} \delta^6 (z_2 - z_3), \]

\[ \tilde{D}_3^2 \langle L(z_1) L(z_2) L(z_3) \rangle = \frac{8}{3} \pi^4 C_d \tilde{D}^1_{\alpha} \delta^6 (z_1 - z_3) D_2\alpha \delta^6 (z_2 - z_3), \]

and also the corresponding results required by symmetry of \( \langle L(z_1) L(z_2) L(z_3) \rangle \).

If an external real superfield \( V \) is coupled to \( L \) through an additional term in the action \( S_V = 2 \int d^8 z \, LV \) then the results in (6.33) can be summarised through the operator equations

\[ \tilde{D}^2 \langle L \rangle = -\frac{16}{3} \pi^4 C_d W^2, \quad D^2 \langle L \rangle = -\frac{16}{3} \pi^4 C_d \overline{W}^2, \]

where \( W, \overline{W} \) are as in (4.27). For the associated current \( J_\alpha \), defined by \( J_{\alpha \dot{\alpha}} = -\frac{1}{2} [D_{\alpha}, \tilde{D}_{\dot{\alpha}}] L \), then (6.33) gives

\[ \partial_\alpha \langle J^\alpha \rangle = \frac{1}{16} i [D^2, \tilde{D}^2] \langle L \rangle = -\frac{1}{3} \pi^4 C_d i (D^2 W^2 - \tilde{D}^2 \overline{W}^2), \]

which reduces to the standard form for the anomaly of the axial current in a \( U(1) \) gauge field background.

### 7. Supercurrent Correlation Functions

We here apply the general results of section 3 to a couple of particular non trivial cases involving the supercurrent. First we consider the three point function involving two supercurrents and a scalar superfield with \( q = \bar{q} \). Adapting the general form (3.31) to this case we have

\[ \langle T_{\alpha \dot{\alpha}}(z_1) T_{\beta \dot{\beta}}(z_2) O(z_3) \rangle = \frac{(x_{13})_{\alpha \gamma} (x_{31})_{\gamma \dot{\alpha}} (x_{23})_{\beta \delta} (x_{32})_{\delta \dot{\delta}}}{(x_{31}^2 x_{13}^2 x_{32}^2 x_{23}^2)^2} \ell^{\gamma \dot{\gamma} \delta \dot{\delta}} (X_3, \bar{X}_3). \]
It remains to determine the form of \( t_{ab}(X, \bar{X}) = \frac{1}{4}(\sigma_a)_{\alpha\bar{\alpha}}(\sigma_b)_{\beta\bar{\beta}}t^{\alpha\beta}(X, \bar{X}) \), using 4-vector notation for convenience, which is homogeneous of degree \( 2(q - 3) \). From the invariance of (7.1) under \( z_1 \leftrightarrow z_2 \), when \( X_3 \leftrightarrow -\bar{X}_3 \), and \( \alpha\dot{\alpha} \leftrightarrow \beta\dot{\beta} \) this satisfies the symmetry condition

\[
t_{ab}(X, \bar{X}) = t_{ba}(-\bar{X}, -X),
\]

and also the reality constraint

\[
t_{ab}(X, \bar{X})^* = t_{ab}(\bar{X}, X).
\]

A general form compatible with (7.2) and (7.3) is

\[
t_{ab}(X, \bar{X}) = \frac{\eta_{ab}}{(X \cdot \bar{X})^{3-q}} \left( A + B \frac{P^2}{X \cdot \bar{X}} \right) + X_{(a} \dot{X}_{b)} \frac{1}{(X \cdot \bar{X})^{4-q}} \left( C + D \frac{P^2}{X \cdot \bar{X}} \right)
+ E i\epsilon_{abcd} X^c \dot{X}^d \frac{1}{(X \cdot \bar{X})^{4-q}},
\]

where \( A, B, C, D, E \) are real coefficients and we have defined

\[
X_a - X_a = iP_a, \quad P_a P_b = \frac{1}{4} \eta_{ab} P^2, \quad P^2 = -8\Theta^2\Theta^2.
\]

Using the results in (6.3) and (6.2) the conservation equation following from applying \( \tilde{D}_1 \dot{\alpha} \) to (7.1) leads to

\[
\Theta \sigma^c \tilde{\sigma}^a \frac{\partial}{\partial X^c} t_{ab}(X, \bar{X}) = 0.
\]

The terms resulting from (7.6) which are \( O(P^0) \) give

\[
C = -\frac{3}{2} \frac{q}{3} - q \quad \text{A}, \quad E = -\frac{1}{2}(1 - q)C,
\]

while, using \( \Theta P^2 = 0 \) and \( \Theta P_a = \Theta^2 \tilde{\Theta} \tilde{\sigma}_a \), the \( O(P) \) terms determine \( B \) and \( D \),

\[
B = \frac{1}{8}(3 - q)(4 - q) \quad \text{A}, \quad D = \frac{1}{8}(3 - q)(4 - q) \quad \text{C}.
\]

The remaining conservation equations follow automatically as a consequence of the symmetry and reality conditions (7.2) and (7.3). Thus the three point function (7.1) is uniquely determined up to an overall constant although there are no Ward identities in this case which allow the constant to be determined.

It is also convenient to rewrite the result (7.1) in the form given by (3.36) so that

\[
\langle T_{\bar{\alpha}\dot{\alpha}}(z_1) T_{\beta\dot{\beta}}(z_2) O(z_3) \rangle = -\frac{(x_{21})_{\beta\gamma}}{(x_{12}^2 x_{21}^2)^2} \frac{1}{(x_{32}^2 x_{13}^2)^q} \tilde{t}_{\alpha\dot{\alpha}} \gamma(\bar{X}_1, \bar{X}_1).
\]

(7.9)
To obtain \( \tilde{t}_{ab}(X, \bar{X}) \) we first use (3.33) with the result (3.26) for \( X^I, \bar{X}^I \) and \( \det I = -1 \) to give

\[
\tilde{t}_{ab}(X^I, \bar{X}^I) = I_a^c(\bar{X}, X)I_b^d(\bar{X}, X) t_{cd}(X, \bar{X}) = t_{ab}(X, \bar{X}),
\]

(7.10)

and then applying (3.37)

\[
\tilde{t}_{ab}(X, \bar{X}) = \frac{1}{(X^2 \bar{X}^2)^{q/2}} I^c_b(\bar{X}, X) t_{ac}(X, \bar{X}).
\]

(7.11)

Using the explicit form for \( I_{ab} \),

\[
I_{ab}(\bar{X}, X) = \frac{1}{(X^2 \bar{X}^2)^{q/2}} (\eta_{ab} X \cdot \bar{X} - 2X(a \bar{X}_b) - i\epsilon_{abcd}X^c \bar{X}^d),
\]

(7.12)

and

\[
\frac{1}{(X^2 \bar{X}^2)^{\rho}} = \frac{1}{(X \cdot \bar{X})^2}(1 + \frac{2}{7} \frac{P^2}{X \cdot \bar{X}}),
\]

(7.13)

we find that if

\[
\tilde{t}_{ab}(X, \bar{X}) = \frac{\eta_{ab}}{(X \cdot \bar{X})^{q/2}} \left( \bar{A} + \bar{B} \frac{P^2}{X \cdot \bar{X}} \right) + X(a \bar{X}_b) \frac{1}{(X \cdot \bar{X})^{q+1}} \left( \bar{C} + \bar{D} \frac{P^2}{X \cdot \bar{X}} \right) + \bar{E} i\epsilon_{abcd}X^c \bar{X}^d \frac{1}{(X \cdot \bar{X})^{q+1}},
\]

(7.14)

then (7.11) implies

\[
\bar{A} = A, \quad \bar{C} = -C - 2A = -\frac{q}{q - \frac{3}{2}} A, \quad \bar{E} = E - A = \frac{1}{2}(q - 2)C,
\]

(7.15)

and

\[
\bar{B} = B + \frac{3}{4}(q - 1)A + \frac{1}{8}C - \frac{1}{2}E = \frac{1}{8}q(q + 1)A,
\]

\[
\bar{D} = -D - \frac{1}{4}(3q - 2)C - 2B - \frac{3}{2}(q - 1)A + \frac{1}{2}E = \frac{1}{8}q(q + 1)\bar{C}.
\]

(7.16)

The results (7.15) and (7.16) are similar in form to (7.7) and (7.8) with \( q \leftrightarrow 3 - q \) which is necessary for the analogous conservation equation to (7.6) to be satisfied by \( \tilde{t}_{ab}(X, \bar{X}) \). When \( q = 0 \), \( \tilde{t}_{ab} = A\eta_{ab} \), or \( \tilde{t}_{a\alpha}^{\beta\beta} = -2A\delta^\alpha_\beta\delta^\beta_\alpha \), and (7.14) reduces to the form in (3.43), as expected since \( O \) is then the identity operator. We may also verify that (7.14), with \( \bar{B}, \bar{D} \) determined by (7.16), obeys \( D^2 \tilde{t}_{ab}(X, \bar{X}) = -4\Phi^2 \partial X^2 \tilde{t}_{ab}(X, \bar{X}) = 0 \) which, for \( q = 1 \) and using the corresponding equation to (6.4), ensures that (7.8) satisfies the conservation equations in (3.12) which become necessary if we let \( O \to L \).

Using the general formula (3.40) the results for the three point function (7.1) are equivalent to determination of the coefficient of the contribution of the scalar superfield \( O \) to the operator product expansion of two supercurrents

\[
T_a(z_1)T_b(z_2) \sim \frac{1}{C_O} \tilde{t}_{ab}(x_{2\bar{1}}, x_{2\bar{1}}) O(z_2),
\]

(7.17)
where $\tilde{t}_{ab}$ is defined by \( (\ref{7.10}) \) and is given by the same solution of the constraints as $t_{ab}$ but with $E \to -E$.

Following a similar analysis we turn to the more intricate case of the three point function of the supercurrent by itself. The general result \( (3.31) \) now requires

$$
\langle T_{\alpha\dot{\alpha}}(z_1)T_{\beta\dot{\beta}}(z_2)T_{\gamma\dot{\gamma}}(z_3) \rangle = \frac{(x_{13})_{\alpha\dot{\alpha}}(x_{31})_{\beta\dot{\beta}}(x_{23})_{\gamma\dot{\gamma}}}{(x_{31}^2 x_{13}^2 x_{23}^2)^{1/2}} t_{\epsilon\dot{\epsilon},\eta\dot{\eta}}(X_3, \bar{X}_3),
$$

where it remains to determine $t_{abc}(X, \bar{X}) = -\frac{1}{8}(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}}(\bar{\sigma}_c)_{\gamma\dot{\gamma}} (X, \bar{X})$, which is homogeneous of degree $-3$. As well as $P_a$ given by \( (7.20) \) it is convenient to define also

$$
Q_a = \frac{1}{2}(\bar{X}_a + X_a),
$$

so that under inversion following \( (3.26) \) they transform as

$$
Q_a^I = I_a^b(\bar{X}, X)Q_b = -Q_a, \quad P_a^I = I_a^b(\bar{X}, X)P_b = P_a - 2\frac{Q \cdot P}{Q^2} Q_a
$$

Assuming the symmetry condition

$$
t_{abc}(X, \bar{X}) = t_{bac}(-\bar{X}, -X),
$$

we can write a general expression, depending on 9 coefficients, for it as

$$
t_{abc}(X, \bar{X}) = \frac{1}{(X \cdot X)^2} \epsilon_{abcd} Q^d \left( A + B \frac{P^2}{X \cdot X} \right) + \frac{1}{(X \cdot X)^2} \eta_{ab} \left( CP_c + D \frac{P \cdot Q}{X \cdot X} Q_c \right)
\begin{equation}
+ \frac{1}{(X \cdot X)^2} \left( EP_{(a} + F \frac{P \cdot Q}{X \cdot X} Q_{(a} \right) \eta_{b)c} + \frac{1}{(X \cdot X)^3} Q_{a} Q_{b} \left( GP_c + H \frac{P \cdot Q}{X \cdot X} Q_c \right)
\end{equation}
\begin{equation}
+ \frac{1}{(X \cdot X)^3} J(Q_a P_b + Q_b P_a) Q_c.
\end{equation}

From \( (3.35) \) we let

$$
t'_{abc}(X, \bar{X}) = \tilde{t}_{abc}(X^I, \bar{X}^I) = I_a^e(\bar{X}, X)I_b^f(\bar{X}, X)I_c^g(\bar{X}, X) t_{efg}(X, \bar{X}),
$$

where, using \( (7.20) \), $\tilde{t}_{abc}(X, \bar{X}) = t_{bac}(X, \bar{X})$ and $t'_{abc}(X, \bar{X})$ has the same form as $t_{abc}(X, \bar{X})$ in \( (7.22) \) but with

\[
(A', B', C', E', G', J') = (A, B, C, E, G, J),
\]

\[
D' = -D - 2C, \quad F' = -F - 2E, \quad H' = -H - 2G - 4J.
\]

\[\text{This case was also investigated in \cite{10} but with different conclusions.}\]
Since the three point function in (7.18) is totally symmetric we must now impose, in addition to (7.21), by virtue of (3.36) and (3.37),

\[ t_{bca}(X, \bar{X}) = I^{e} b(\bar{X}, X) t'_{ae}(X, \bar{X}). \]  

(7.25)

By explicit calculation

\[ I^{e} b(\bar{X}, X) t'_{ae}(X, \bar{X}) = \frac{1}{(X \cdot \bar{X})^2} \epsilon_{abcd} Q^{d} \left( A' + (B' + \frac{1}{4} A' - \frac{1}{4} C' + \frac{1}{8} E') \frac{P^2}{X \cdot \bar{X}} \right) \]

\[ + \frac{1}{(X \cdot \bar{X})^2} ((C' - A') \eta_{ab} P_c + \frac{1}{2} E' \eta_{ac} P_b + (A' + \frac{1}{2} E') \eta_{bc} P_a) \]

\[ + \frac{1}{(X \cdot \bar{X})^3} P \cdot Q ((A' + D') \eta_{ab} Q_c - (E' + \frac{1}{2} F') \eta_{ac} Q_b - (A' - \frac{1}{2} F') \eta_{bc} Q_a) \]

\[ + \frac{1}{(X \cdot \bar{X})^3} ((A' - 2C' - G') Q_a Q_b P_c + J' Q_a Q_c P_b - (A' + E' + J') Q_b Q_c P_a) \]

\[ - \frac{1}{(X \cdot \bar{X})^4} P \cdot Q Q_a Q_b Q_c (2D' + F' + H' + 2J'), \]  

(7.26)

and then, using (7.24), it is easy to read off the conditions necessary to satisfy (7.25)

\[ E = 2(C - A), \quad G + J = D + \frac{1}{2} F = A - 2C. \]  

(7.27)

For the conservation equations it is sufficient to impose just

\[ \Theta \sigma^e \delta^a \frac{\partial}{\partial X^e} t_{abc}(X, \bar{X}) = 0. \]  

(7.28)

Inserting (7.22) the equations split into those which are O(P^0),

\[ E = 2(C - A), \quad F = -2C - 5E, \quad G = -2A + \frac{1}{2} F, \quad J = 2A + D, \quad H = -2G - 6J, \]  

(7.29)

and also those which arise from terms which are O(P),

\[ 4C + D + 6E + \frac{3}{2} F + G + J = 0, \quad H = 4E + F - 2J, \quad 8B = 4A - 4E - F - J. \]  

(7.30)

In fact (7.29) and (7.30) together imply (7.27) and there remain two independent parameters which may be taken as A, C so that we may determine

\[ B = \frac{1}{2} A, \quad \frac{1}{2} D = E = 2(C - A), \quad F = 10A - 12C, \quad G = \frac{1}{2} H = -\frac{3}{2} J = 3(A - 2C). \]  

(7.31)
In consequence there are two linearly independent superconformal covariant forms for the three point function for the supercurrent (7.18). In contrast for the three point function of the energy momentum tensor in conformal field theories there are in general three linearly independent forms [7,8].

The result (7.22) with (7.31) is not very transparent but it can be recast more simply as

\[ t_{abc}(X, \bar{X}) = \tau_{abc}(X, \bar{X}) + \tau_{bac}(-X, -\bar{X}), \]

where

\[
\tau_{abc}(X, \bar{X}) = -\frac{iA}{(X^2)^2} \left( X_a\eta_{bc} + X_b\eta_{ac} - X_c\eta_{ab} + i\epsilon_{abcd}X^d \right) \\
+ \frac{1}{2}(2C - A) \frac{1}{(X^2)^3} \left( 2(X_aP_b + P_aX_b)X_c - 3X_aX_bP_c - 6\frac{P \cdot X}{X^2}X_aX_bX_c \right) \\
- P \cdot X \left( 3(X_a\eta_{bc} + X_b\eta_{ac}) - 2X_c\eta_{ab} \right) + \frac{1}{2}X^2 \left( P_a\eta_{bc} + P_b\eta_{ac} + P_c\eta_{ab} \right).
\]

As in previous cases we may relate the results for the three point function to the relevant coefficient in an associated operator product expansion

\[ T_{\alpha\dot{\alpha}}(z_1)T_{\beta\dot{\beta}}(z_2) \sim -\frac{1}{2CT}i\tilde{f}_{\alpha\dot{\alpha},\beta\dot{\beta}} \tilde{\gamma}\tilde{\gamma}(x_{2\bar{1}}, x_{2\bar{1}}) T_{\gamma\dot{\gamma}}(\bar{z}_2), \]

with, from (7.23),

\[ \tilde{f}_{\alpha\dot{\alpha},\beta\dot{\beta}} \tilde{\gamma}\tilde{\gamma}(X, \bar{X}) = (\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}}(\tilde{\sigma}^c)\tilde{\gamma}\tilde{\gamma}t_{bac}(X, \bar{X}). \]

Explicitly, with a similar decomposition to (7.32),

\[
\tau_{\alpha\dot{\alpha},\beta\dot{\beta}} \tilde{\gamma}\tilde{\gamma}(X, \bar{X}) = 2iA \frac{1}{(X^2)^2}X_{\beta\dot{\beta}} \delta_\alpha^\gamma \delta_\dot{\beta}^\dot{\gamma} \\
+ \frac{1}{2}(2C - A) \frac{1}{(X^2)^3} \left( 2(X_{\alpha\dot{\alpha}}\sigma^a + \sigma^aX_{\alpha\dot{\alpha}})\tilde{\chi}_{\beta\dot{\beta}} - 3X_{\alpha\dot{\alpha}}X_{\beta\dot{\beta}} \left( \tilde{P} \tilde{\gamma}\tilde{\gamma} + 2\frac{P \cdot X}{X^2} \tilde{X} \tilde{\gamma}\tilde{\gamma} \right) \\
+ 2(P \cdot X X_{\alpha\dot{\alpha}} - X^2P_{\alpha\dot{\alpha}})\delta_\beta^\gamma \delta_\dot{\beta}^\dot{\gamma} + 2(P \cdot X X_{\beta\dot{\beta}} - X^2P_{\beta\dot{\beta}})\delta_\alpha^\gamma \delta_\dot{\alpha}^\dot{\gamma} \\
+ (4P \cdot X X_{\alpha\dot{\beta}} + X^2P_{\alpha\dot{\beta}})\delta_\beta^\gamma \delta_\dot{\alpha}^\dot{\gamma} + (4P \cdot X X_{\beta\dot{\alpha}} + X^2P_{\beta\dot{\alpha}})\delta_\alpha^\gamma \delta_\dot{\beta}^\dot{\gamma} \right).
\]

The terms in (7.35) with coefficient $2C - A$ have an $O(X^{-4})$ singularity but closer analysis shows that $(X^2)^{-\lambda} \tilde{\tau}_{\alpha\dot{\alpha},\beta\dot{\beta}} \tilde{\gamma}\tilde{\gamma}(X, \bar{X})$ has no pole as $\lambda \to 0$ so this is integrable. The construction of $t_{abc}(X, \bar{X})$ guarantees that $\tilde{\tau}_{\alpha\dot{\alpha},\beta\dot{\beta}} \tilde{\gamma}\tilde{\gamma}(x_{2\bar{1}}, x_{2\bar{1}}) + \tilde{\tau}_{\beta\dot{\beta},\alpha\dot{\alpha}} \tilde{\gamma}\tilde{\gamma}(-x_{2\bar{1}}, -x_{2\bar{1}})$ satisfies the constraints obtaining from the conservation equations (4.21) for $z_1 \neq z_2$ (in (7.33) $P \to -2\theta_{x_1}\sigma\theta_{x_2}$ for this case). To take account of singularities at $z_1 = z_2$ we first note that

\[ \tilde{D}_1^\alpha \tilde{\tau}_{\alpha\dot{\alpha},\beta\dot{\beta}} \tilde{\gamma}\tilde{\gamma}(x_{2\bar{1}}, x_{2\bar{1}}) = 0, \quad \tilde{D}_2^\beta \tilde{\tau}_{\alpha\dot{\alpha},\beta\dot{\beta}} \tilde{\gamma}\tilde{\gamma}(x_{2\bar{1}}, x_{2\bar{1}}) = 0, \]

(7.36)
without any $\delta$-function contributions. However calculating the action of $\tilde{D}_1^\alpha$ requires a more careful treatment. Modifying the singularity in \((7.33)\) as in \((5.12)\) we find

$$
\tilde{D}_1^\alpha \left( \frac{1}{x_{12}^2} T_{\alpha \dot{\alpha}, \beta \dot{\beta}} (x_{21}, x_{22}) \right) = 8A \frac{1}{x_{12}^2} \frac{1}{\theta_{12}^2 \beta} \left( \delta_{\alpha}^\gamma \delta_{\beta}^\gamma \right) - 20i(2C - A) \frac{1}{x_{12}^2} \left( (x_{21} \theta_{12})_{\beta} \delta_{\alpha}^\gamma \delta_{\beta}^\gamma + (x_{21})_{\alpha, \beta} \delta_{\gamma}^\gamma (\theta_{12})^\gamma \right) + \frac{1}{x_{12}^2} \left( (x_{21} \theta_{12})_{\alpha} (x_{21})_{\beta, \gamma} (\bar{x}_{12})^\gamma \right) .
$$

Taking the limit $\lambda \to 0$ then reveals the local contributions with support when $z_1 = z_2$

$$
\tilde{D}_1^\alpha T_{\alpha \dot{\alpha}, \beta \dot{\beta}} (x_{21}, x_{22}) = 8\pi^2 i A \left( \theta_{12} \right)_{\beta} \delta^4 (x_{21}) \delta_{\alpha}^\gamma \delta_{\beta}^\gamma
- \frac{40}{3} \pi^2 (2C - A) \theta_{12}^2 \delta_{\alpha}^\gamma \delta_{\beta}^\gamma (x_{21}) \delta_{\gamma}^\gamma (\theta_{12})^\gamma
= 4\pi^2 i A \delta^6_+ (z_1 - z_2) \delta_{\alpha}^\gamma \delta_{\beta}^\gamma + \frac{20}{3} \pi^2 (2C - A) \delta_{\alpha}^\gamma \delta_{\beta}^\gamma (\theta_{12})^\gamma \tilde{D}_1^\delta \delta^8 (z_1 - z_2) .
$$

A similar result may also be derived for $\tilde{D}_2^\beta T_{\alpha \dot{\alpha}, \beta \dot{\beta}} (x_{21}, x_{22})$.

The association of the supercurrent with superconformal transformations allows the derivation of Ward identities which constrain one linear combination of the parameters in $t_{abc}$. To derive these from \((4.22)\) we need to define $\delta_{h, \bar{h}} T_a$. In the superconformal case, given by \((2.7)\), this must reduce to the particular case of \((3.1)\) appropriate for the supercurrent and therefore, based on particular examples, we postulate the form\(^{14}\)

$$
\delta_{h, \bar{h}} T_{\alpha \dot{\alpha}} = - \left( \frac{1}{2} (h^\alpha + \bar{h}^\alpha) \partial_\alpha + \lambda^\alpha D_\alpha + \bar{\lambda}_\dot{\alpha} \bar{D}^\dot{\alpha} + 3(\sigma_h + \bar{\sigma}_{\bar{h}}) \right) L_{\alpha \dot{\alpha}}
+ \omega_{h, \alpha} \beta \partial_{\beta \dot{\beta}} - T_{\alpha \beta} \bar{\omega}_{\alpha \dot{\alpha}} + (D(h - \bar{h}))_I \alpha \alpha O_I ,
$$

where $D$ represents the action of various derivatives and $O_I$ are a basis of superfield operators in the theory. These terms are model dependent, the results for free theories are given in an appendix. Nevertheless the fields which contribute may include the supercurrent itself so that these terms are relevant for Ward identities applied to the three point function of the supercurrent by itself. Using the prepotentials given in \((4.20)\) we have

$$
\omega_{h, \alpha} \beta - 3\delta_{\alpha}^\gamma \sigma_h = - \frac{1}{4} i \tilde{D}^2 D_\alpha L^\beta ,
\bar{\omega}_{\bar{h}, \alpha} \gamma + 3\delta_{\alpha}^\beta \bar{\sigma}_{\bar{h}} = - \frac{1}{4} i \tilde{D}^2 \bar{D}_\alpha \bar{L}^\beta ,
$$

we may obtain from \((4.22)\)

$$
\tilde{D}_1^\alpha (T_{\alpha \dot{\alpha}}(z_1) T_{\beta \dot{\beta}}(z_2) \ldots )
= (2i \tilde{D}_1^\alpha \delta^\alpha (z_1 - z_2) \delta_{\alpha}^\gamma \delta_{\beta}^\gamma + \tilde{D}_1^\alpha \chi_{\alpha \dot{\alpha}, \beta \dot{\beta}} \gamma \gamma (z_{12})) (T_{\gamma \gamma} \ldots ) + \ldots ,
\tilde{D}_1^\alpha (T_{\alpha \dot{\alpha}, \beta \dot{\beta}}(z_1) T_{\beta \dot{\beta}}(z_2) \ldots )
= (2i \tilde{D}_1^\alpha \delta^\alpha (z_1 - z_2) \delta_{\beta}^\gamma \delta_{\dot{\beta}}^\gamma + \tilde{D}_1^\alpha \chi_{\alpha \dot{\alpha}, \beta \dot{\beta}} \gamma \gamma (z_{12})) (T_{\gamma \gamma} \ldots ) + \ldots ,
$$

\(^{14}\) For an alternative approach see \(24\).
where additional terms representing contributions from other operators in the correlation function and also less singular terms, involving derivatives of \( T_{\gamma\gamma}(z_2) \), are not shown. The terms involving \( \chi_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(z_{12}) \), where \( z_{12} \) is defined in (2.33) and which is restricted to be a linear combination constructed from \( D_\delta D_3^\delta (z_1 - z_2) = -4 \delta^4(x_{12})\bar{\theta}_{12\delta}\bar{\theta}_{12\delta} \) and \( D_\delta \bar{D}_\delta \delta^8(z_1 - z_2) = 4 \delta^4(x_{12})\bar{\theta}_{12\delta}\bar{\theta}_{12\delta} \), arise from the model dependent \( h - \bar{h} \) contributions in (7.39). However such terms may also be viewed as a reflection of the arbitrariness of the operator product coefficient \( \tilde{t}_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(x_{21}, x_{2\bar{1}}) \) in (7.34) up to purely local \( \delta \)-function contributions and in consequence it is therefore possible to redefine it so as to remove the \( \chi_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(z_{12}) \) terms from (7.41).

To apply the results in (7.36) and (7.38) to the Ward identity we first introduce

\[
\begin{align*}
  h_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(z_{12}) &= h_{\beta\dot{\beta},\alpha\dot{\alpha}}\hat{\gamma}(z_{21}) \\
  &= [D_\alpha, \bar{D}_\dot{\alpha}]\delta^8(z_1 - z_2) \beta^\gamma\hat{\gamma}_\beta + [D_\beta, \bar{D}_\dot{\beta}]\delta^8(z_1 - z_2) \alpha^\gamma\hat{\gamma}_\dot{\alpha} \\
  &\quad - 2[D_\alpha, \bar{D}_\dot{\beta}]\delta^8(z_1 - z_2) \beta^\gamma\hat{\gamma}_\dot{\alpha} - 2[D_\beta, \bar{D}_\dot{\alpha}]\delta^8(z_1 - z_2) \alpha^\gamma\hat{\gamma}_\beta \\
  &\quad + 6i(\partial_{\beta\dot{\alpha}}\delta^8(z_1 - z_2) \beta^\gamma\hat{\gamma}_\dot{\alpha} - \partial_{\alpha\dot{\beta}}\delta^8(z_1 - z_2) \beta^\gamma\hat{\gamma}_\beta),
\end{align*}
\]

which satisfies

\[
\begin{align*}
  \bar{D}_1^\dot{\alpha} h_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(z_{12}) &= 12D_1^\beta\delta^6(x_1 - x_2) \beta^\gamma\hat{\gamma}_\beta - 8i\delta(\alpha^\gamma\partial_\beta)(\alpha^\gamma\partial_\dot{\beta})\bar{D}_1^\dot{\alpha}\delta^8(z_1 - z_2), \\
  \bar{D}_1^\alpha h_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(z_{12}) &= 12\bar{D}_1^\alpha\delta^6(x_1 - x_2) \beta^\gamma\hat{\gamma}_\dot{\alpha} + 8i\delta(\alpha^\gamma\partial_\beta)(\alpha^\gamma\partial_\dot{\beta})\bar{D}_1^\dot{\alpha}\delta^8(z_1 - z_2).
\end{align*}
\]

Redefining the operator product coefficient in (7.34) to be

\[
\tilde{t}_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(x_{21}, x_{2\bar{1}}) = t_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(x_{21}, x_{2\bar{1}}) + t_{\beta\dot{\beta},\alpha\dot{\alpha}}\hat{\gamma}\hat{\gamma}(\bar{x}_{2\bar{1}}, \bar{x}_{2\bar{1}})
\]

\[
- \frac{\pi^2}{6}i(2C - A) h_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(z_{12}),
\]

we then obtain

\[
\bar{D}_1^\dot{\alpha}\tilde{t}_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(x_{2\bar{1}}, x_{2\bar{1}}) = 2\pi^2i(2A - 5(2C - A))D_1^\beta\delta^6(z_1 - z_2) \delta^\gamma\hat{\gamma}_\beta.
\]

The additional term in (7.44) is equivalent to setting \( \chi_{\alpha\dot{\alpha},\beta\dot{\beta}}\hat{\gamma}\hat{\gamma}(z_{12}) = 0 \) in (7.41) and hence, using (7.45), we therefore find from the Ward identity the constraint

\[
\pi^2(10C - 7A) = 2C_T.
\]

Thus the general superconformal three point function for the supercurrent contains one new parameter beyond the coefficient \( C_T \) for the two point function.
8. Free Fields

The trivial realisations of superconformal field theories are given by free fields, i.e. for the chiral scalar superfield theory defined by the action (4.23) or the abelian gauge theory described by (4.27). As a consistency check we give the reduction of some of our general results to these cases. From (4.23) the two point function is easily found,

\[ \langle \phi(z_1^+) \bar{\phi}(z_2^-) \rangle = \frac{1}{4\pi^2 x_{21}^2} . \] (8.1)

For gauge superfield \( V \), and with covariant gauge fixing parameterised by \( \xi \), the normalisations in (4.27) require

\[ \langle V(z_1) V(z_2) \rangle = \frac{1}{16\pi^2} \left( \left( 1 - \frac{1}{\xi} \right) \ln y_{12}^2 - \left( 1 + \frac{1}{\xi} \right) \frac{1}{y_{12}^2} \theta_{12} \bar{\theta}_{12} \right) , \] (8.2)

where \( y_{12} \) is given by (2.33). With the definitions in (4.27) we may then show that

\[ \langle W^\alpha(z_1^+) W^\dot{\alpha}(z_2^-) \rangle = \frac{i}{\pi^2} \frac{(x_{12})^{\alpha\dot{\alpha}}}{x_{21}^2} , \] (8.3)

while \( \langle W^\alpha(z_1^+) W^\beta(z_2^+) \rangle = 0 \). Clearly (8.1) and (8.3) are in accord with the general superconformal expression (3.30).

With these results and the explicit forms (4.26) and (4.32) we may evaluate the coefficient of the two point function for the supercurrent, defined by (3.43), directly to be

\[ C_{T,\phi} = \frac{1}{6\pi^4} , \quad C_{T,V} = \frac{1}{2\pi^4} . \] (8.4)

The form required by (3.43) is trivial to obtain in the \( V \) case, using (8.3), but emerges in the \( \phi \) case after lengthy calculation. To obtain the coefficients in the supercurrent three point function it is sufficient to find the leading contributions to the operator product \( T_{\alpha\dot{\alpha}}(z_1) T_{\beta\dot{\beta}}(z_2) \). This is easy in the \( V \) case since it is evident from (4.32) and (8.3) that the first operator term in the short distance expansion is the supercurrent itself. This then gives

\[ A_V = 2C_V = -\frac{1}{\pi^2} C_{T,V} . \] (8.5)

In the operator product expansion for two supercurrents formed from free chiral fields, as in (4.26), the leading term is \( \phi \bar{\phi} \) and it is necessary to remove this and its derivatives after using Taylor expansions in the form

\[ \phi(z_{2^+}) = \phi(z_{1^+}) + (x_{21} \cdot \partial_1 + \theta_{21} \alpha D_1^\alpha) \phi(z_{1^+}) + \ldots , \]
\[ \bar{\phi}(z_{2^-}) = \bar{\phi}(z_{1^-}) + (x_{21} \cdot \partial_1 + \bar{\theta}_{21} \dot{\alpha} \bar{D}_1^\dot{\alpha}) \bar{\phi}(z_{1^-}) + \ldots . \] (8.6)
Noting that \( \frac{1}{2}[D_\alpha, \bar{D}_\dot{\alpha}]\phi\bar{\phi} = D_\alpha\phi \bar{D}_\dot{\alpha}\bar{\phi} - i \phi \bar{\phi} \bar{\partial}_{\alpha\dot{\alpha}}\bar{\phi} \), we then find

\[
A_S = \frac{2}{5} C_S = \frac{1}{9\pi^2} C_{T,\phi}.
\] (8.7)

Of course both (8.5) and (8.7) are in accord with the Ward identity (7.46).\(^\text{15}\)

As an example of a current superfield we consider \( L_i = \bar{\phi} t_i \phi \) for free chiral scalar superfields where \( t_i \) are hermitian matrices obeying the Lie algebra \([t_i, t_j] = if_{ijk}t_k\) and we take \( \text{tr}(t_i t_j) = T\delta_{ij} \). It is very easy to see that this gives the results (6.14) and (6.11) for the two and three point functions with

\[
C_L = \frac{T}{(4\pi^2)^2}, \quad C_f = \frac{T}{2(4\pi^2)^3}.
\] (8.8)

Manifestly these satisfy (6.16). If \( \frac{1}{2}\text{tr}(\{t_i, t_j\}t_k) = d_{ijk} \) then the symmetric form (6.17) is also given by

\[
C_d = \frac{1}{(4\pi^2)^3},
\] (8.9)

and this is appropriate in (6.34) or (6.35) to give the standard one loop anomaly result.

9. Superconformal Integrals

It was realised long ago \[25\] that integrals such as those appearing in field theoretic calculations may be significantly simplified in special cases as a consequence of the restrictions of conformal invariance. As general discussion was given by Symanzik \[26\] and we extend this here to particular superconformal examples.

If we define

\[
x_i^2 = (x_{i+} - 2i\theta_i\sigma\bar{\theta} - x)^2,
\] (9.1)

then the integrations over anti-chiral superspace we consider here are

\[
S_N = i \int \! d^4 x d^2 \bar{\theta} \prod_{i=1}^N \frac{1}{(x_i^2)^{q_i}}, \quad \sum_i q_i = 3.
\] (9.2)

\(^{15}\) The results for free fields may be used to relate the parameters \( A, C \) for the general superconformal supercurrent three point function to those specifying the conformal energy momentum tensor three point function. In terms of the parameters \( r, s, t \) in [7] we have \( 2r = 29A - 9C, 4s = -45A + 24C, t = 4A \).
Under a superconformal transformation $z \rightarrow z'$, $x_{i}^{2} = x_{i}^{2}/\Omega(z_{i})\bar{\Omega}(z)$, from (2.43), while the measure $d^{6}z' = d^{6}z/\Omega^{3}(z)$ so that the condition on $\sum_{i} q_{i}$ in (9.2) defines a superconformal covariant function. Using the standard result

$$\frac{1}{(x^2 + i\epsilon)^\alpha} = \frac{e^{-ix\epsilon}}{\Gamma(\alpha)} \int_{0}^{\infty} d\lambda \lambda^{\alpha-1} e^{i\lambda x^2},$$

(9.3)

then we find

$$S_{N} = \pi^{2} \prod_{i} \Gamma(q_{i}) \int_{0}^{\infty} \prod_{i} d\lambda_{i} \lambda_{i}^{q_{i}-1} \frac{1}{\Lambda^2} \int d^2 \theta e^{-\frac{1}{2} \sum_{i<j} \lambda_{i} \lambda_{j} x_{ij}^2}, \quad X_{i+j} = x_{i}^{+} - x_{j}^{+} - 2i\theta_{ij} \sigma \bar{\theta},$$

(9.4)

where $\Lambda = \sum_{i} \lambda_{i}$. Expanding the exponential allows the $\bar{\theta}$ integration to be performed giving

$$S_{N} = -\pi^{2} \prod_{i} \Gamma(q_{i}) \int_{0}^{\infty} \prod_{i} d\lambda_{i} \lambda_{i}^{q_{i}-1} \frac{1}{\Lambda^{2}} \sum_{j<k} \theta_{j} \sigma \cdot \partial_{j} \bar{\theta}_{k} \bar{\theta}_{k} e^{-\frac{1}{2} \sum_{i<j} \lambda_{i} \lambda_{j} x_{ij}^2},$$

(9.5)

for $x_{ij+} = x_{i}^{+} - x_{j}^{+}$. In (9.4) and (9.5) we have assumed that the integrals are initially defined in a region where $x_{ij+}^2 > 0$.

The crucial observation of Symanzik is that in an integral

$$\int_{0}^{\infty} \prod_{i} d\lambda_{i} \lambda_{i}^{\delta_{i}-1} \frac{1}{\Lambda^{p}} e^{-\frac{1}{2} \sum_{i<j} \lambda_{i} \lambda_{j} u_{ij}}, \quad u_{ij} = u_{ji}, \quad \sum_{i} \delta_{i} = 2p,$$

(9.6)

it is possible to transform $\Lambda$ to $\Lambda = \sum_{i} \kappa_{i} \lambda_{i}$ with arbitrary $\kappa_{i} > 0$, $\sum \kappa_{i} > 0$. This then allows the choice $\Lambda = \lambda_{i}$ for some $i$ and the integral, using contour techniques, written in terms of the conformal invariant cross ratios $u_{ij} u_{kl}/u_{ik} u_{jl}$. For the case $i = 1, 2, 3$, when there are no invariants, the integral is easily determined to be

$$\frac{\Gamma(p-\delta_{1})\Gamma(p-\delta_{2})\Gamma(p-\delta_{3})}{u_{12}^{p-\delta_{3}} u_{23}^{p-\delta_{1}} u_{31}^{p-\delta_{2}}},$$

(9.7)

There are various alternative ways of writing (9.5) in the desired form. One convenient representation is

$$S_{N} = \frac{4\pi^{2}}{\prod_{i} \Gamma(q_{i})} \int_{0}^{\infty} \prod_{i} d\lambda_{i} \lambda_{i}^{q_{i}-1} \frac{1}{\Lambda^{2}} e^{-\frac{1}{2} \sum_{i<j} \lambda_{i} \lambda_{j} x_{ij}^2}$$

$$\times \left( \sum_{jkl} \lambda_{j} \lambda_{k} \lambda_{l} \bar{x}_{jl+} \bar{x}_{lk+} \bar{\theta}_{k} - \frac{1}{2} \sum_{j} \lambda_{j} \lambda_{k} x_{jk+} \sum_{l} \lambda_{l} \theta_{l}^{2} \right),$$

(9.8)
which has the form required by (9.6) with \( p = 3 \). For \( N > 3 \) the integral involves non trivial functions of superconformal invariants but here we consider just \( N = 3 \) when the result by applying (9.7) is

\[
S_3 = -4\pi^2 \prod_i \frac{\Gamma(2 - q_i)}{\Gamma(q_i)} \frac{\theta_{12} \bar{\theta}_{13} x_{23+}^2 + \theta_{23} \bar{\theta}_{21} x_{31+}^2 + \theta_{31} \bar{\theta}_{32} x_{12+}^2}{(x_{12+}^2 - q_3)(x_{23+}^2 - q_1)(x_{31+}^2 - q_2)}. \quad (9.9)
\]

It is straightforward to reexpress this in the manifestly superconformal form expected from (5.5).

10. Superconformal Invariants

For analysis of higher point correlation functions it is necessary to understand what conformal invariants may occur. For an \( N \)-point function depending on \( z_r \in \mathbb{R}^{4_1^4} \), \( r = 1, \ldots, N \) we may use supertranslations to set \( z_1 = 0 \) and then superconformal transformations to set \( x_2 = \infty \) (in a suitable compactification) and also \( \theta_2, \bar{\theta}_2 = 0 \). Superconformal invariants are then given by those scalars formed from \( z_r, r = 3 \ldots N \) which are invariant under the residual symmetry group \( O(3,1) \times D \times U(1)_R \), where \( D \) denotes the group of scale transformations. For the \( x_r \) coordinates we may consider the \( \frac{1}{2}(N - 2)(N - 1) - 1 = \frac{1}{2}N(N - 3) \) scalars \( x_r x_s / x_3^2, s \geq r > 3, s > r = 3 \) and with the Grassmann coordinates we may define the \( (N - 2)^3 \) invariants \( \theta_s x_r \bar{\theta}_t / x_r^2 \). However for \( N > 6 \) the \( x_r \) are linearly dependent and we may restrict \( r = 3, 4, 5, 6 \), giving \( 4N - 15 \) c-number invariants (in this case the ordinary \( O(4,2) \) conformal group is acting transitively) and \( 4(N - 2)^2 \) invariants formed from \( \theta, \bar{\theta} \). When \( N = 3 \) there is clearly one Grassmann invariant which corresponds to \( J \) defined in (2.57).

To proceed we extend the definition of \( X \) for three points \( z_1, z_2, z_3 \) in (2.45) to a similar expression formed from \( z_r, z_s, z_t \)

\[
X_r(st) = \frac{x_{rs} \bar{x}_{sr} x_{tf} \bar{x}_{ft}}{x_{sr}^2 x_{tf}^2}, \quad \bar{X}_r(st) = -X_r(ts), \quad (10.1)
\]

and also, extending (2.47),

\[
\Theta_{r(st)} = i \left( \frac{1}{x_{sr}^2} \bar{\theta}_{rs} \bar{x}_{sr} - \frac{1}{x_{tf}^2} \bar{\theta}_{rt} \bar{x}_{ft} \right), \quad \bar{\Theta}_{r(st)} = i \left( \frac{1}{x_{rs}^2} \bar{x}_{rs} \bar{\theta}_{rs} - \frac{1}{x_{ft}^2} \bar{x}_{ft} \bar{\theta}_{rt} \right). \quad (10.2)
\]

These functions of \( z_r, z_s, z_t, r \neq s \neq t \), transform homogeneously at \( z_r \) according to (2.46) and (2.48). Trivially from (10.1) we have

\[
\Theta_{r(su)} = \Theta_{r(st)} + \Theta_{r(tu)} \quad \bar{\Theta}_{r(su)} = \bar{\Theta}_{r(st)} + \bar{\Theta}_{r(tu)}, \quad (10.3)
\]
and
\[ X_{r(su)} = X_{r(st)} + X_{r(tu)} - 2i\Theta_{r(st)}\sigma\bar{\Theta}_{r(tu)}. \]
(10.4)

As special case (10.4) reduces for \( u = s \) to \( X_{r(st)} + X_{r(ts)} = -2i\Theta_{r(st)}\sigma\bar{\Theta}_{r(st)} \), which is equivalent to (2.50).

In the limiting situation considered above, \( z_1 = (0,0,0), z_2 = (\infty,0,0) \), we have \( X_{1(2r)} = x_{r+}/x_{r+}^2, \Theta_{1(2r)} = i\tilde{\theta}_r \tilde{x}_{r-}/x_{r-}^2, \bar{\Theta}_{1(2r)} = -i\tilde{x}_{r+}\tilde{\bar{\theta}}_{r}/x_{r+}^2 \). Hence it is natural to construct a basis of superconformal invariants in terms of \( X_{1(2r)}, \Theta_{1(2r)}, \bar{\Theta}_{1(2r)} \) for \( r = 3, \ldots N \). Thus we may define a set of bosonic invariants formed in this fashion, in which the points \( z_1, z_2 \) play a privileged role, by

\[ u_r = \frac{X_{2(2r)}}{X_{1(23)}} = \text{det} \left( X_{1(23)}^{-1}X_{1(2r)} \right), \quad r > 3, \]
(10.5)

and

\[ v_{rs} = \frac{X_{1(2r)} \cdot X_{1(2s)}}{X_{1(2r)}} = \frac{1}{2} \text{tr} \left( X_{1(2r)}^{-1}X_{1(2s)} \right), \quad s > r \geq 3. \]
(10.6)

The definitions (10.5) and (10.6) are special cases of an extension to the superconformal case of the usual invariant cross ratios and a related invariant trace given by

\[ u_{rs, tu} = \frac{x_{rt}^2 x_{su}^2}{x_{ru}^2 x_{st}^2}, \quad v_{rs, tu} = \frac{1}{2} \text{tr} \left( \tilde{x}_{rt}^{-1} \tilde{x}_{su}^{-1} \tilde{x}_{tu} \right), \]
(10.7)

since it is easy to see, from the definitions (10.5) and (10.6) together with (10.1), that \( u_r = u_{12, 3r}, v_{rs} = v_{12, rs} \). From (10.7) it follows that

\[ u_{rs, tu} = u_{sr, ut} = \frac{1}{u_{sr, tu}}, \quad v_{rs, tu} = v_{sr, ut} = \frac{v_{sr, tu}}{u_{sr, tu}}, \]
(10.8)

which may also be derived from the definitions in (10.5) and (10.6) using the relation \( \tilde{x}_{sr} X_{r(st)} \tilde{x}_{fs} = X_{s(rt)}^{-1} \) which follows from (2.52).

Restricting to functions of four points the above discussion also suggests, in addition to those in (10.5) and (10.6), introducing an associated set of Grassmann invariants given by \( Q_{12, rs}, \bar{Q}_{12, rs}, r, s \geq 3 \) where in general we define

\[ \bar{Q}_{rs, tu} = 4i\Theta_{r(su)} \tilde{x}_{r(st)}^{-1} \tilde{\Theta}_{r(st)}, \quad Q_{rs, tu} = 4i\Theta_{r(st)} \tilde{x}_{r(ts)}^{-1} \tilde{\Theta}_{r(su)}. \]
(10.9)

Using the transformation relations in (2.53) as well as (2.52) we have

\[ Q_{sr, tu} = \bar{Q}_{rs, ut}. \]
(10.10)
From (10.4) we may show that
\[
\frac{X^2_{r(st)}}{X^2_{r(ts)}} = (1 + Q_{rs,tt})^{-1} = 1 + \tilde{Q}_{rs,tt},
\]
and, just in (2.57), we may define, using (10.10) and \( Q_{rs,tt} = \tilde{Q}_{rt,ss} \),
\[
J_{rst} = -\frac{1}{2} (Q_{rs,tt} - \tilde{Q}_{rs,tt}),
\]
as an totally antisymmetric invariant depending on \( z_r, z_s, z_t \).

Other invariants formed from four points \( z_r, z_s, z_t, z_u \) should be expressible in general in terms of the basis described above. Alternatively the invariants \( u_{rs,tu}, v_{rs,tu} \) and \( Q_{rs,tu}, \tilde{Q}_{rs,tu} \) obey various relations, besides those given in (10.8) and (10.10). Using (10.3) and (10.4) we may obtain
\[
u_{us,tr} = w_{rs,tu} (1 + \tilde{Q}_{ru,ts}), \quad w_{rs,tu} = 1 + u_{rs,tu} - 2v_{rs,tu},
\]
which allows the trace invariants given by (10.6) to be expressed in terms of the invariant cross ratios as given by (10.3), and also
\[
1 + \tilde{Q}_{rt,su} = (1 + \tilde{Q}_{rs,tu})(1 + Q_{rs,tt}).
\]
By combining (10.13) with \( w_{rs,tu} = w_{sr,ut} \), which follows from (10.8), we may obtain
\[
u_{tu,rs} = u_{rs,tu} \frac{1 + \tilde{Q}_{st,tu}}{1 + \tilde{Q}_{ur,ts}} = u_{rs,tu} \frac{1 + Q_{rs,tt}}{1 + Q_{rs,uu}},
\]
which also follows from (10.11) and (10.14). This results shows how \( u_{rs,tu} \) is related to its conjugate \( \bar{u}_{rs,tu} = u_{tu,rs} \).

The presence of the Grassmann invariants, such as given by (10.9), clearly complicates the analysis of superconformal \( N \) point functions for \( N \geq 4 \). For chiral fields the necessary additional conditions are more restrictive. As an illustration we consider a 4 point function for chiral scalar fields. If we express it in the form
\[
\langle \phi_1(z_1+)\phi_2(z_2+)\phi_3(z_3+)\phi_4(z_4+) \rangle = \frac{X_{1(23)}^2(q_1-2)}{x_1^{12}x_2^{2q_2}x_3^{2q_3}x_4^{2q_4}F_{12,34}(z_1, z_2, z_3+, z_4+)}F_{12,34}(z_1, z_2, z_3+, z_4+),
\]
then superconformal invariance, if \( \sum_i q_i = 3 \), requires
\[
F_{12,34}(z_1^{'}, z_2^{'}, z_3^{'+}, z_4^{'+}) = \Omega(z_1^{'})^{-2}\tilde{\Omega}(z_1^-)F_{12,34}(z_1, z_2, z_3^+, z_4^+).
\]
$F_{12,34}$ may be expanded as

$$F_{12,34}(z_1, z_2, z_3+, z_4+) = A\bar{\Theta}_{(23)}^2 + B\bar{\Theta}_{(24)}^2 + C\bar{\Theta}_{(23)}\bar{\Theta}_{(24)} + D\bar{\Theta}_{(23)}X_{1(23)}^{-1}X_{1(24)}\bar{\Theta}_{(24)},$$

so that, if the coefficients $A, B, C, D$ are functions of just the two invariants $u = u_{12,34}, w = w_{12,34}$, (10.16) and (10.18) give a generalisation of the result displayed in (5.3) for the 3 point function which has the required superconformal transformation properties and further depends manifestly only on $z_3+, z_4+$. Interchanging $z_3$ and $z_4$ so that in the corresponding expression to (10.16) we have $F_{12,43}(z_1, z_2, z_4+, z_3+)$ it is easy to see that

$$F_{12,43}(z_1, z_2, z_4+, z_3+) = u^{-q_1+2}F_{12,34}(z_1, z_2, z_3+, z_4+).$$

Writing now

$$F_{12,43}(z_1, z_2, z_4+, z_3+) = A'\bar{\Theta}_{(24)}^2 + B'\bar{\Theta}_{(23)}^2 + C'\bar{\Theta}_{(24)}\bar{\Theta}_{(23)} + D'\bar{\Theta}_{(24)}X_{1(24)}^{-1}X_{1(23)}\bar{\Theta}_{(23)},$$

then this leads to

$$A' = u^{-q_1+2}B, \quad B' = u^{-q_1+2}A, \quad C' = u^{-q_1+2}C, \quad D' = u^{-q_1+3}D.$$  

In a similar fashion if $z_1 \leftrightarrow z_2$ then we obtain $F_{21,34}(z_2, z_1, z_3+, z_4+)$ which may be expressed as

$$F_{21,34}(z_2, z_1, z_3+, z_4+) = A'\bar{\Theta}_{2(13)}^2 + B'\bar{\Theta}_{2(14)}^2 + C'\bar{\Theta}_{2(13)}\bar{\Theta}_{2(14)} + D'\bar{\Theta}_{2(14)}X_{2(13)}^{-1}X_{2(14)}\bar{\Theta}_{2(13)},$$

where the necessary relations for compatibility are

$$A' = Au^{q_1}, \quad B' = Bu^{q_1+1}, \quad C' = Du^{q_1+1}, \quad D' = Cu^{q_1+1}.$$  

Although in the expression given by (10.16) with (10.18) only $z_3+, z_4+$ appear further conditions are necessary to ensure that the whole result depends only on $z_1+, z_2+$. These may be obtained by considering $z_2 \leftrightarrow z_3$ applied to (10.16),

$$\langle \phi_1(z_{1+})\phi_2(z_{2+})\phi_3(z_{3+})\phi_4(z_{4+}) \rangle = \frac{X_{1(32)}^{2(q_1-2)}}{X_{12}^{2q_2}X_{13}^{2q_3}X_{14}^{2q_4}} E(z_1, z_2, z_3, z_4+),$$

45
where by using (10.11) we have

\[ E(z_1, z_2, z_3, z_{4+}) = (1 + \bar{Q}_{12,33})^{-2} F_{12,34}(z_1, z_2, z_{3+}, z_{4+}) = F_{13,24}(z_1, z_3, z_{2+}, z_{4+}) + G, \]

(10.25)

so that the dependence on \( z_{2-} \) is isolated in \( G \). To obtain an explicit form for \( G \) we first rewrite \( F_{12,34} \) as

\[
F_{12,34}(z_1, z_2, z_{3+}, z_{4+}) = A \bar{\Theta}_{1(32)}^2 + (B + \bar{Q}_{12,33} D) \bar{\Theta}_{1(34)}^2 + C \bar{\Theta}_{1(32)} \bar{\Theta}_{1(34)} \\
+ (1 + Q_{12,33}) D \bar{\Theta}_{1(32)} X_{1(32)}^{-1} X_{1(34)} \bar{\Theta}_{1(34)},
\]

(10.26)

where

\[
A = A + B + C + v D, \quad C = -(2B + C + D), \quad v = v_{12,34}.
\]

(10.27)

The invariants \( u, w \), on which \( A, B, C, D \) depend may also be re-expressed in terms of \( \dot{u} = u_{13,24}, \dot{w} = w_{13,24} \), which depend only on \( z_{2+} \), using (10.13) with (10.14) giving

\[
u = \dot{\bar{w}} \frac{1 + \bar{Q}_{12,43}}{1 + Q_{12,33}}, \quad w = \dot{u} \frac{1}{(1 + Q_{13,42})(1 + Q_{12,33})}.
\]

(10.28)

Hence the dependence on \( z_{2-} \) in (10.25) occurs only in terms involving \( \bar{Q}_{12,33}, \bar{Q}_{12,43}, \bar{Q}_{13,42} \). By combining such terms with (10.26) in (10.25) \( G \) may be reduced to the form

\[
G = (\alpha_1 \bar{Q}_{12,43} + \alpha_2 \bar{Q}_{21,43}) \bar{\Theta}_{1(32)}^2 + (\beta_1 \bar{Q}_{12,43} + \beta_2 \bar{Q}_{21,43}) \bar{\Theta}_{1(34)}^2 \\
+ \gamma \frac{1}{X_{1(34)}^2} \bar{\Theta}_{1(32)} \bar{\Theta}_{1(34)}^2,
\]

(10.29)

where \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \) are linear in \( A, B, C, D \). In consequence the dependence on \( z_{2-} \) may be eliminated by imposing the conditions that \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \) each vanish and hence in (10.24) \( E \to F_{13,24} \) given by

\[
F_{13,24}(z_1, z_3, z_{2+}, z_{4+}) = A \bar{\Theta}_{1(32)}^2 + B \bar{\Theta}_{1(34)}^2 + C \bar{\Theta}_{1(32)} \bar{\Theta}_{1(34)} + D \bar{\Theta}_{1(32)} X_{1(32)}^{-1} X_{1(34)} \bar{\Theta}_{1(34)},
\]

(10.30)

where in the arguments of \( A, B, C, D \) \( u, w \to \dot{\bar{w}}, \dot{u} \). In a similar fashion the necessary dependence only on \( z_{1+} \) may also be ensured. These conditions require relations between the functions \( A, B, C, D \) but it is clear from the results of the previous section that there is at least a single arbitrary function of \( u, v \) remaining in the general solution.
11. Conclusion

In the above we have endeavoured to generalise the kinematic analysis in [7,8] of conformal invariance and its implications in quantum field theory in general dimensions on flat space to the simplest case of \( \mathcal{N} = 1 \) supersymmetry in four dimensions. In the analysis of the two and three point functions of the energy momentum tensor in four dimensions the coefficients which appeared in the two and three point functions (for the latter there are in general three parameters which may be connected with the three trivial free conformal field theories in four dimensions) can be related to the coefficients which appear in the trace of the energy momentum tensor when a conformal field theory is extended to a curved space background. Here we describe the connections of the results obtained here with the similar parameters which may be defined when a superconformal theory is extended to a minimal \( \mathcal{N} = 1 \) supergravity background.

In this case the theory includes a superfield \( H^a(z) \), which contains the metric, such that the expectation of the energy momentum tensor may be defined by

\[
\langle T_a \rangle = \frac{\delta W}{\delta H^a} ,
\]  

where \( W \) is the connected vacuum functional for the curved background (in our conventions the functional integral gives \( e^{iW} \)). Assuming the theory is defined to preserve the usual supergravity superspace reparameterisation invariance we may extend the definitions in (4.15) to obtain

\[
\frac{1}{2i} \int d^8 z E^{-1} (h^a - \bar{h}^a) \frac{\delta W}{\delta H^a} = \int d^6 z_+ \hat{\varphi}^3 \hat{\sigma}_h \hat{T} + \int d^6 z_- \hat{\varphi}^3 \hat{\sigma}_{\bar{h}} \hat{\bar{T}} ,
\]

where \( d^8 z E^{-1} \) and \( d^6 z_+ \hat{\varphi}^3 \), \( d^6 z_- \hat{\varphi}^3 \) are the appropriate invariant integration measures on full superspace and its chiral, anti-chiral projections [13]. \( h^a, \bar{h}^a \) satisfy supercovariant generalisations of (2.6) while \( \mathcal{T}, \bar{\mathcal{T}} \) are covariantly chiral, anti-chiral scalars formed from \( H^a \) (\( \hat{T}, \bar{\hat{T}} \) are defined by transformation to a chiral representation when they depend only on \( z_+, z_- \) respectively). \( \mathcal{T}, \bar{\mathcal{T}} \) are formed from the supergravity curvatures, the chiral superfields \( W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}, R \), and their anti-chiral conjugates \( \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}, \bar{R} \), together with the real vector superfield \( G_a \), and supercovariant derivatives. The general form for \( \mathcal{T} \) can be written as [27]

\[
8\pi^2 \mathcal{T} = c W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} - a G + h(\bar{D}^2 - 4R)\bar{D}^2 R ,
\]

with \( G \) a topological density whose chiral superspace integral is related to the difference of the Euler and Pontryagin invariants,

\[
G = W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} - \frac{1}{4}(\bar{D}^2 - 4R)(G^a G_a + 2R) ,
\]
and where $D_\alpha, \bar{D}_{\dot{\alpha}}$ are supercovariant spinor derivatives. In (11.3) $h$ is arbitrary since it may be varied at will by adding a purely local term $\propto \int d^8zE^{-1}RR$ to $W$. The coefficients $c, a$ have a non trivial significance in any superconformal theory \cite{6} and for $n_S, n_V$ free superfields, as described by actions (4.23), (4.27), we have \cite{13}

$$c = \frac{1}{24}(3n_V + n_S), \quad a = \frac{1}{48}(9n_V + n_S). \quad (11.5)$$

There is a direct relation between $c, a$ and the parameters $A, C$ specifying the general superconformal supercurrent three point function as found in section 7. In principle the relation may be found from (11.2) first by obtaining

$$\tilde{D}^{\dot{\alpha}}\langle T_{\alpha\dot{\alpha}} \rangle = \frac{2}{3}D_\alpha T, \quad \bar{D}^\alpha\langle T_{\alpha\dot{\alpha}} \rangle = \frac{2}{3}\bar{D}_{\dot{\alpha}} \bar{T}, \quad (11.6)$$

and then taking two functional derivatives with respect to $H$ of both sides and restricting to flat space. This gives contributions to $\tilde{D}^{\dot{\alpha}}\langle T_{\alpha\dot{\alpha}}(z_1)T_{\beta\dot{\beta}}(z_2)T_{\gamma\dot{\gamma}}(z_3) \rangle$ and $\bar{D}^\alpha\langle T_{\alpha\dot{\alpha}}(z_1)T_{\beta\dot{\beta}}(z_2)T_{\gamma\dot{\gamma}}(z_3) \rangle$ which are proportional to various derivatives acting on $\delta^8(z_1 - z_2)\delta^8(z_1 - z_3)$. With careful regularisation the results from the r.h.s. of (11.6), depending on $c, a, h$ may be matched with results arising from explicit calculation using the general form (7.18). However such an analysis is not straightforward (an analogous investigation of the energy momentum tensor three point function assuming conformal invariance was undertaken in \cite{8}) although the necessary relations are easy to read off from the results for free fields (11.3) and (8.7), (8.5) with (8.4). This gives

$$A = \frac{8}{9\pi^6}(3c - 5a), \quad C = \frac{4}{9\pi^6}(6c - 7a). \quad (11.7)$$

As a consistency check we verify the relation between $c$ and the coefficient $C_T$ of the supercurrent two point function. For this it is sufficient to restrict to constant rescalings when we may take in (11.2) $\hat{\sigma}_h = \hat{\bar{\sigma}}_h = 1$. With $\mu$ an arbitrary renormalisation scale we may write

$$\mu \frac{\partial}{\partial \mu} W = \int d^6z_+ \hat{\varphi}^3 \hat{T} + \int d^6z_- \hat{\bar{\varphi}}^3 \hat{\bar{T}}. \quad (11.8)$$

In this the terms depending on $a$ are topological invariants while $h$ disappears since it is the coefficient of terms which are total derivatives. Furthermore the difference between the integrals of $W^\alpha\beta\gamma W_{\alpha\beta\gamma}$ and $\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ is also a topological invariant so that we may write from (11.8) and (11.1)

$$\mu \frac{\partial}{\partial \mu} \langle T_{\alpha\dot{\alpha}}(z_1)T_{\beta\dot{\beta}}(z_2) \rangle = 8c \frac{\delta^2}{\delta H^{\alpha\dot{\alpha}}(z_1)\delta H^{\dot{\beta}\beta}(z_2)} \int d^6z_+ \hat{\varphi}^3 W^\alpha\beta\gamma W_{\alpha\beta\gamma} \bigg|_{\text{flat space}}. \quad (11.9)$$
Using, to lowest order in expansion about flat space, $\delta W_{\alpha\beta\gamma} = -\frac{1}{16} \tilde{D}^2 D_\alpha \tilde{D}^\beta D_\beta \delta H_{\gamma\beta}$ this may be readily calculated giving
\[
\mu \frac{\partial}{\partial \mu} \langle T_\alpha(z_1) T_\beta(z_2) \rangle = 4c \partial_\alpha \partial_\gamma D_1 \partial_\delta \partial_\beta D_2 \delta_+^\epsilon (z_1 - z_2) \mathcal{E}^{\epsilon\gamma,\beta}_\delta, \tag{11.10}
\]
with $\mathcal{E}$ defined by (3.43). The result (11.10) may be compared with that obtained from the regularised version of (3.44) using, with the definition in (3.46),
\[
\mu \frac{\partial}{\partial \mu} R(\frac{\theta_2}{x_{12}^2}) = 2\pi \delta_+^6 (z_1 - z_2). \tag{11.11}
\]
It is then evident that we must have
\[
C_T = \frac{4}{\pi^4} c, \tag{11.12}
\]
which is compatible with the Ward identity result (7.46) and (11.7).

12. Note Added

A further consistency check may be found by reducing the results (7.18) and (7.22) to their $\theta, \tilde{\theta}$ independent forms. With $T^a(z) = R^a(x)$, the $R$-symmetry current, we have
\[
\langle R^a(x_1) R^b(x_2) R^c(x_3) \rangle = A \frac{I^a_{e}(x_{13}) I^b_{f}(x_{23})}{x_{13}^2 x_{23}^2} \frac{i \epsilon^{efcd}}{X_{3d}} \frac{X_{3d}}{X_{3d}^2}, \tag{12.1}
\]
where $I^a_{b}(x) = \delta^a_{b} - 2x^a x_b/x^2$ is the reduction of the inversion tensor given by (7.12). Using the standard form for the anomaly, such as obtained in [8] with symmetrisation and transforming to Minkowski space,
\[
\partial_3 \langle R^a(x_1) R^b(x_2) R^c(x_3) \rangle = -A \frac{1}{6} \pi^4 \partial_1 \epsilon \partial_2 \epsilon \epsilon^{abcd} \delta^4(x_{13}) \delta^4(x_{23}). \tag{12.2}
\]
From (11.6), for a general background, $iD_a \langle T^a \rangle = \frac{1}{2} (\tilde{D}^2 T - D^2 \tilde{T})$ which for flat space becomes [3] (adapting to the supergravity conventions of [13]), with $G_a(z) = \frac{4}{3} A_a(x)$,
\[
\partial_a \langle R^a \rangle = \frac{1}{54\pi^2} (3c - 5a) \epsilon^{abcd} F_{ab} F_{cd}, \quad F_{ab} = \partial_a A_b - \partial_b A_a. \tag{12.3}
\]
Since (11.1) now reduces to $\langle R^a \rangle = \delta W/\delta A_a$ the compatibility of (12.2) and (12.3) gives the first of eqs.(11.7).
Appendix A.

We here describe how the assumed transformation rule (7.39) for the supercurrent is realized for free fields. For the scalar case with the supercurrent given by (4.20) and the elementary chiral transforming as in (1.24) with \( q = \bar{q} = 1 \) we have

\[
\delta_{\lambda,\bar{\lambda}} T_{\alpha \bar{\alpha}} = - \left( \frac{1}{2}(h^a + \bar{h}^a) \partial_a + \lambda^\alpha D_\alpha + \bar{\lambda}_{\bar{\alpha}} \bar{D}_{\bar{\alpha}} + 3(\sigma_h + \bar{\sigma}_h) \right) T_{\alpha \bar{\alpha}} + \omega_{h\alpha}^\beta T_{\beta \alpha} - T_{\alpha \bar{\beta}} \bar{\omega}_h^\beta \bar{\alpha} \\
- X_{\alpha}^\beta T_{\beta \alpha} + T_{\alpha \bar{\beta}} X_{\alpha}^\beta + (h^a - \bar{h}^a) \frac{1}{4} D_\alpha \phi \bar{\partial}_a D_{\bar{\alpha}} \phi \\
- D_\alpha (h^a - \bar{h}^a) \frac{1}{3} \partial_\alpha \phi \bar{D}_{\bar{\alpha}} \phi + \bar{D}_\alpha (h^a - \bar{h}^a) \frac{1}{3} D_\alpha \phi \partial_\alpha \phi \\
+ i \partial_\alpha \bar{\phi} (h^a - \bar{h}^a) \frac{1}{3} \partial_\alpha (\phi \bar{\phi}) - i [D_\alpha, \bar{D}_\alpha] (\bar{h}^\beta - \bar{\bar{\beta}}^\beta) \frac{1}{12} i \phi \bar{D}_{\bar{\beta}} \phi \\
+ (i \partial_\alpha \bar{\lambda}^\beta + \delta_{\alpha}^\beta \bar{D}_\alpha \bar{\sigma}_h) \frac{2}{3} D_\beta \phi \bar{\phi} + (i \partial_\alpha \bar{\lambda}^\beta - \delta_{\alpha}^\beta \partial_\alpha \sigma_h) \frac{2}{3} \bar{\phi} \bar{D}_{\bar{\beta}} \phi \\
i \partial_\alpha \bar{\lambda}^\beta + \delta_{\alpha}^\beta \bar{\partial}_\alpha \bar{\sigma}_h = \frac{1}{10} i\epsilon_{\alpha \gamma} \bar{D}^2 D_\alpha (\bar{h}^\gamma - \bar{\bar{\beta}}^\gamma) \\
- \frac{1}{48} \delta_{\alpha}^\beta (4 \bar{D}_\alpha \partial_\gamma + i \epsilon_{\alpha \gamma} \bar{D}^2 D_\gamma) (\bar{h}^\gamma - \bar{\bar{\beta}}^\gamma), \\
i \partial_\alpha \bar{\lambda}^\beta - \delta_{\alpha}^\beta \partial_\alpha \sigma_h = \frac{1}{10} i\epsilon_{\alpha \gamma} \bar{D}^2 D_\gamma (\bar{h}^\gamma - \bar{\bar{\beta}}^\gamma) \\
- \frac{1}{48} \delta_{\alpha}^\beta (4 \bar{D}_\alpha \partial_\gamma + i \epsilon_{\alpha \gamma} \bar{D}^2 D_\gamma) (\bar{h}^\gamma - \bar{\bar{\beta}}^\gamma), \\
\partial_{\alpha \bar{\alpha}} (\sigma_h - \bar{\sigma}_h) = \frac{1}{48} (D_\alpha D_\alpha (D_\beta \bar{D}_\beta + 2 \bar{D}_\beta \bar{D}_\beta) + 3 D_\beta D_\alpha D_\beta D_\alpha) (\bar{h}^\beta - \bar{\bar{\beta}}^\beta). 
\]

For the case of free vector fields we may use (1.32) and (1.30) to easily obtain

\[
\delta_{\lambda,\bar{\lambda}} T_{\alpha \bar{\alpha}} = - \left( \frac{1}{2}(h^a + \bar{h}^a) \partial_a + \lambda^\alpha D_\alpha + \bar{\lambda}_{\bar{\alpha}} \bar{D}_{\bar{\alpha}} + 3(\sigma_h + \bar{\sigma}_h) \right) T_{\alpha \bar{\alpha}} + \omega_{h\alpha}^\beta T_{\beta \alpha} - T_{\alpha \bar{\beta}} \bar{\omega}_h^\beta \bar{\alpha} \\
- (h^a - \bar{h}^a) W_\alpha \delta_\beta \bar{W}_\alpha \\
+ \frac{1}{4} i \epsilon_{\alpha \beta} \bar{D}^2 (\bar{W}_\beta (\bar{h}^\beta - \bar{\bar{\beta}}^\beta)) \bar{W}_\alpha + \frac{1}{4} i \epsilon_{\alpha \beta} W_\alpha D^2 ((\bar{h}^\beta - \bar{\bar{\beta}}^\beta) W_\beta). 
\]

The extra terms in (A.2) and (A.3) may both be decomposed into quasi-primary operators and their derivatives. The crucial difference is that in the case of chiral superfields this includes the supercurrent itself. In this case they therefore contribute to the Ward identity for the supercurrent three point function, giving a non zero \( \chi_{\alpha \bar{\alpha}, \beta \bar{\beta}^\gamma} \) in (1.22) whereas such terms were absent in the vector case when the parameters \( A, C \) satisfied (3.5).

50
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