Definition of Magnetic Monopole Numbers for $SU(N)$ Lattice Gauge-Higgs Models

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Abstract

A geometric definition for a magnetic charge of Abelian monopoles in $SU(N)$ lattice gauge theories with Higgs fields is presented. The corresponding local monopole number defined for almost all field configurations does not require gauge fixing and is stable against small perturbations. Its topological content is that of a 3-cochain. A detailed prescription for calculating the local monopole number is worked out.

Our method generalizes a magnetic charge definition previously invented by Phillips and Stone for $SU(2)$.

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1 Introduction

The investigation of the complicated vacuum structure of quantized Yang-Mills theories requires the use of non-perturbative methods. The lattice regularization is the most convenient in this respect. Among other applications it allows to identify those excitations (monopoles, instantons, etc.) which are expected to dominate the Euclidean path integral at large scales and to explain important features like chiral symmetry breaking and quark confinement in QCD.

Monopoles in which we take interest in this paper form the basis for the dual superconductor scenario of confinement [1, 2]. According to this scenario the condensation of (anti-) monopoles causes the color-electric flux to be squeezed into string-like flux tubes (dual Meissner effect). The consequence is a linearly rising potential between static quarks.

Whether extended non-Abelian ’t Hooft-Polyakov-like monopoles (see [3]) or singular Abelian Dirac-like monopoles play the major role in the confinement phenomenon has not been settled yet. The majority of investigations concentrates on the latter kind of excitations. These can be identified by Abelian projection after an appropriate gauge has been fixed [4]. Up till now the maximally Abelian gauge was considered to be the most favourable gauge [4, 5, 6] (for a recent review see e.g. [7]). For this gauge the resulting $U(1)$ degrees of freedom as well as the corresponding Abelian monopoles were shown to reproduce almost completely the string tension of the full theory (so-called Abelian and monopole dominance, respectively) [8, 9].

Other gauges – defined for instance by diagonalizing gauge observables as parallel transporters along closed paths (e.g. the Polyakov loop) which transform like Higgs fields in the adjoint representation – have been studied, too [10]. Based on appropriately defined order parameters according to a construction of Fröhlich and Marchetti [11] the condensation of Abelian monopoles in the confinement phase was demonstrated to occur widely independent of the gauge chosen [12].

But a general gauge independent proof for the validity of the dual superconductor picture is still missing.

Therefore, it would be worthwhile to have a gauge-independent and – at the same time – geometric lattice definition of the monopole charge. This, in general, requires the existence of a Higgs field in the adjoint representation. Magnetic fluxes and the magnetic charge, respectively, can easily be obtained by projecting the non-Abelian plaquette variables onto the local colour direction of the Higgs field. However, the naive discretization does not provide any geometric meaning of the resulting charge. Nevertheless, corresponding investigations of magnetic fluxes have provided useful insight into the phase structure of the 4d $SU(2)$ Georgi-Glashow model [13].

A first geometrical definition for the monopole number has been given in [14, 15]. The lattice gauge field together with the Higgs field determines local maps from the boundary of cubes into the coset space $SU(2)/U(1)$. The winding numbers of these maps provide the local integer-valued monopole charge, for each cube, or alternatively speaking, a closed magnetic current on the dual lattice. In our present work we are going to generalize the definition to a $SU(N)$ gauge-Higgs theory. The Higgs fields employed are considered to be $\mathbb{CP}^{N-1}$-valued fields. We shall present all the details necessary for developing a computer code for the calculation of the winding numbers from the local lattice field data.
The resulting magnetic charge definition can be employed for explorations of the phase structure of $SU(N)$ gauge-Higgs models. But, it might be worthwhile also for the application to pure $SU(N)$ gauge theories, when an auxiliary Higgs field is determined as proposed for the recently invented Laplacian gauge fixing [17].

Recently, in the context of investigations of the electro-weak phase transition for the $SU(2)$ theory another version of an integer-valued, gauge independent lattice magnetic charge definition was invented [18]. It remains an interesting task to work out the inter-relation between the different definitions, if there is any.

Section 2 will provide the general description and the necessary notations. In Section 3 we present the computation of the monopole number. Some detailed proofs are shifted into the Appendix. In Section 4 we shall draw the conclusions.

## 2 Description of the model and notation

In this work we want to consider a model consisting of an $SU(N)$ lattice gauge field $U = \{U_{ij}\}$ coupled to a Higgs field $\Phi = \{\phi_i\}$ transforming with respect to the fundamental representation. $i, j, \ldots$ denote the lattice sites and pairs $\langle ij \rangle$ the corresponding links of a 4-dimensional simplicial lattice. Without loss of generality for our construction of the local monopole number we work with a Higgs field of fixed length, $\|\phi_i\| = 1$. In the following we denote by $U_{ijk\ldots} = U_{ij}U_{jk}\ldots$ parallel transporters longer than one link.

As the phase of the vector $\phi_i \in \mathbb{C}^N$ will be irrelevant for the monopole charge, it is sufficient to identify the corresponding equivalence class $[\phi_i]$. The latter can be represented by the $N \times N$ projection matrix $\Phi_i = \phi_i \otimes \phi_i^*$ projecting on the subspace spanned by the vector $\phi_i$. In this sense the Higgs field takes values in a $\mathbb{C}P^{N-1}$ manifold.

The action of a complex invertible matrix $V$ on elements of $\mathbb{C}P^{N-1}$ will be denoted by ‘$*$’, e.g. $V * \Phi$ means the projector $V\phi \otimes (V\phi)^*/\|V\phi\|^2$ or the class $[V\phi]$. For the lattice we assume an ordering of the lattice sites and denote the different kinds of simplices (sites, links, triangles, tetrahedra, 4-tetrahedra) by ordered tuples of vertices such as $\langle ij\ldots k \rangle$.

If we consider a site $i$ of a given simplex $\langle ij\ldots k \rangle$ as the coordinate origin, a cube $c_{\langle ij\ldots k \rangle}$ parametrized by coordinates $0 \leq s_j, \ldots, s_k \leq 1$ can be defined.

## 3 Definition of the local monopole number

Our aim in this section is to define for a given tetrahedron $\Delta = \langle ijkl \rangle$ a map $X_\Delta$ from the boundary of the 3-dimensional cube $c_{\langle ijkl \rangle}$ to $\mathbb{C}P^{N-1}$. This map is constructed from the link and Higgs variables at the particular tetrahedron $\Delta$.

Since $\Pi_2(\mathbb{C}P^{N-1}) \cong \mathbb{Z}$, the homotopy class of such a map is represented by an integer $n_\Delta$. This integer will be referred to as the local monopole number at the particular simplex. It will be shown to have the following properties:

(a) **Gauge invariance:** $n_\Delta$ is independent of a particular choice of gauge.

(b) **Local conservation:**
The assignment $\Delta \mapsto n_\Delta$ represents a cochain in the sense that

$$(\delta n)_T = \sum_{\Delta \in \partial T} (-)\Delta n_\Delta = 0,$$

the sign depending on the relative orientation of $\Delta$ in the boundary of a 4-tetrahedron $T$. If $\mu$ is the link dual to $\Delta$ we may equivalently say that $M_\mu = n_\Delta$ is a conserved (divergence free) magnetic current on the dual lattice.

(c) **Geometric significance:**

The collection of the $X_\Delta$’s define a section of a $SU(N)/U(N-1)$-bundle over the 2-skeleton of the lattice. This section may be regarded as an interpolation of the Higgs field which is as horizontal as possible with respect to the lattice gauge field.

The maps $X_\Delta$ are worked out stepwise for higher simplices. In order to state their relevant properties, we need $GL(N)$-valued parallel transport functions $V_{(ij...kt)}(s_j, \ldots, s_k)$ as introduced in [16]. These functions essentially measure to what extent the lattice gauge field fails to be flat. They are an important ingredient in the construction of $SU(N)$ bundles associated to lattice gauge fields. In our case, the relevant formulae read

$$
V_{(ij)} = U_{ij},
$$

$$
V_{(ijk)} = s_j U_{ijk} + (1 - s_j) U_{ik},
$$

$$
V_{(ijkl)} = s_j s_k U_{ijkl} + (1 - s_j) s_k U_{ikl} + s_j (1 - s_k) U_{ijl} + (1 - s_j)(1 - s_k) U_{il},
$$

where $0 \leq s_j, s_k \leq 1$. Let us assume for simplicity that the simplex $\Delta$ in question is $\langle 01 \ldots k \rangle$. Then $X_\Delta$ is assumed to meet the following requirements:

$$
X_{\langle 0\ldots k \rangle}(\ldots, s_r = 1, \ldots) = V_{\langle 0\ldots r \rangle}(s_1, \ldots) \ast X_{\langle r\ldots k \rangle}(s_r, \ldots), \tag{1}
$$

$$
X_{\langle 0\ldots k \rangle}(\ldots, s_r = 0, \ldots) = X_{\langle 0\ldots r-1 \rangle}(s_1, \ldots, s_r, \ldots) \tag{2},
$$

and likewise for all other simplices, with the obvious replacements. It follows from the results in [16] that such a collection of maps defines a section over the 2-skeleton in a bundle with principal fiber $\mathbb{C}P^{N-1}$. It is also seen that the local winding number of this section at a tetrahedron $\Delta = \langle ijk \ell \rangle$ is just the winding number of $X_{(ijkl)} : \partial C_{(ijkl)} \rightarrow \mathbb{C}P^{N-1}$. Let us now explicitly construct this map.

Requirements (1) and (2) suggest to define the $X_\Delta$’s inductively for simplices of increasing dimension. This is done in four steps:

**Step 0:** As the $X_\Delta$’s essentially interpolate the Higgs field, we set $X_{(i)} = \Phi_i$ for the lattice sites.

**Step 1:** On the far end $s_j = 1$ of the interval $c_{(ij)}$, we must have, by requirement (1)

$$
X_{(ij)}(s_j = 1) = V_{(ij)} \ast X_{(j)}.
$$

On the base $s_j = 0$, $X_{(ij)}$ must be $X_{(i)}$, by requirement (2). We thus interpolate geodesically in $\mathbb{C}P^{N-1}$ for intermediate values of $s_j$.

**Step 2:** On the far sides $s_j = 1$ or $s_k = 1$ of the square $c_{(ijk)}$, we must have, by requirement (1)

$$(A) \quad X_{(ijk)}(s_j = 1, s_k) = V_{(ij)} \ast X_{(jk)}(s_k),$$

$$(B) \quad X_{(ijk)}(s_j, s_k = 1) = V_{(ijk)}(s_j) \ast X_{(k)}.$$
Again, on the base $s_j = s_k = 0, X_{(ijk)}$ must be $X_{(i)}$, by requirement (2). For intermediate values we interpolate geodesically from the the far sides to the base.

**Step 3:** In the last step we proceed as before. Again, on the far faces $s_j = 1, s_k = 1$ or $s_l = 1$ of the cube $c_{(ijkl)}$, we must have, by requirement (3)

\[
\begin{align*}
(I) & \quad X_{(ijkl)}(s_j = 1, s_k, s_l) = V_{(ij)} \ast X_{(jkl)}(s_k, s_l), \\
(II) & \quad X_{(ijkl)}(s_j, s_k = 1, s_l) = V_{(ijk)}(s_j) \ast X_{(kl)}(s_l), \\
(III) & \quad X_{(ijkl)}(s_j, s_k, s_l = 1) = V_{(ijkl)}(s_j, s_k) \ast X_{(l)}.
\end{align*}
\]

As above, on the base $s_j = s_k = s_l = 0, X_{(ijkl)} = X_{(i)}$, by requirement (2). Unlike in the previous cases, we cannot interpolate from the the far sides to the base since there might be an obstruction. Nevertheless, we might do so on the boundary of our cube. This completes our construction. Requirements (1) and (3) are more or less obviously satisfied.

In the next chapter we shall see how the winding number of $X_{(ijkl)}$ may be calculated.

### 4 Calculation of the monopole number

We have to determine the class of $X_\Delta$. Take a simplex $\Delta = \langle 0123 \rangle$. The boundary of the unit cube $c_\Delta$ with base $\langle 0 \rangle$ is the union of six faces $C_k^0, C_k^1 (k = 1, 2, 3)$, where

\[
C_k^0 = \{(s_1, s_2, s_3) : \quad s_k = 0\}, \quad C_k^1 = \{(s_1, s_2, s_3) : \quad s_k = 1\}.
\]

The monopole number may be computed as an intersection number [19],

\[
n_\Delta = X_\Delta(\partial c_\Delta) : E^\perp,
\]

$E^\perp$ denoting the $(2N - 4)$-dimensional space of all projectors perpendicular to some rank one projector $E$. The intersection number does not depend on the particular projector $E$ chosen. To facilitate the calculations, we take it to be $\Phi_0$. The strategy is to compute the intersection number separately for the six different faces of the surface of our cube. As geodesic interpolation is needed for the definition of $X_\Delta$, let us collect some facts about the geometry of complex projective spaces (see also [20], [21], [22]). Two points $P, Q$ in the complex projective space may be joined by a unique shortest geodesic if and only if $PQ \neq 0$. Furthermore, for such points it holds:

**Lemma 4.1** The unique shortest geodesic between two points $P, Q$ intersects $E^\perp$ if and only if $\text{Re} \, \text{tr}[PQE] \leq 0$ and $\text{Im} \, \text{tr}[PQE] = 0$. Moreover, such an arc can intersect $E^\perp$ only once.

We shall also need the following fact about triangles.

**Lemma 4.2** Let $P, Q, R$ be three points such that there is a unique shortest geodesic between any two of them. Let $\Delta$ be the surface spanned by all geodesics between $P$ and the geodesic arc $QR$. This surface intersects $E^\perp$ if and only if the equation $\text{Im} \, \varphi(t) = 0$ has a solution $0 \leq t \leq 1$ satisfying $\text{Re} \, \varphi(t) \leq 0$. Here

\[
\varphi(t) = t^2 \text{tr}[QR]^{1/2} \text{tr}[ERP] + (1 - t)^2 \text{tr}[QR]^{1/2} \text{tr}[EQP] + t(1 - t) \text{tr}[E(QR + RQ)P].
\]
The sign of such an intersection is given by

$$\text{sign \ Im} \left[ \frac{\partial \varphi(t)}{\partial t} \right].$$

Lemma 4.1 and 4.2 are proven in the appendix. Let us first turn our attention to the faces $C^0_k$. With our particular choice of $E$ the following lemma holds.

**Lemma 4.3** Except for a measure zero set of configurations $(U, \Phi)$, the surfaces $X_{(0123)}(C^0_k)$ do not intersect with $\Phi^\perp_0$.

The proof of this lemma is given in the appendix. It follows that we only have to consider the faces $C^1_1, C^1_2, C^1_3$, schematically shown in Fig. 1 as quadrilaterals (I), (II), (III) respectively. On these faces, the map $X_{(0123)}$ has the form given in Eqs. (I), (II), (III). The monopole number will be the sum of the intersection numbers obtained in each case.

Figure 1: The image of the surfaces $C^1_k$ as treated in cases (I), (II), (III).

**Case (I):** The image of $C^1_1$ is described by Eq. (I). As illustrated in Fig. 2, it is naturally divided into two triangles $A, B$, according to the structure of Eqs. (A) and (B). Let us first consider triangle $A$. The corners of this triangle are the points $P = U_{01} \ast \Phi_1, Q = U_{012} \ast \Phi_2, R = U_{0123} \ast \Phi_3$. The surface enclosed by these three points is swept out by the set of all geodesics between $P$ and the geodesic arc $QR$, its orientation being fixed by the order $P, Q, R$ of the vertices, see Fig. 4. This is the situation described in Lemma 4.2, except for degenerate cases and we use this result to calculate the intersection number of $A$ with $\Phi^\perp_0$.

Inserting the above expressions for $P, Q, R$ into Eq. (4), we have to consider

$$\varphi_{IA}(t) = t^2 \langle \phi_0 | U_{0123} \phi_3 \rangle \langle U_{123} \phi_3 | \phi_1 \rangle \langle U_{01} \phi_1 | \phi_0 \rangle \langle \phi_2 | U_{23} \phi_3 \rangle.$$  (4)
\[ \begin{align*} 
&+ (1-t)^2 \langle \phi_0 | U_{012} \phi_2 \rangle \langle U_{12} \phi_2 | \phi_1 \rangle \langle U_{01} \phi_1 | \phi_0 \rangle \langle \phi_2 | U_{23} \phi_3 \rangle \\
&+ t(1-t) \langle \phi_0 | U_{012} \phi_2 \rangle \langle \phi_2 | U_{23} \phi_3 \rangle \langle U_{123} \phi_3 | \phi_1 \rangle \langle U_{01} \phi_1 | \phi_0 \rangle \\
&+ \langle \phi_0 | U_{0123} \phi_3 \rangle \langle U_{23} \phi_3 | \phi_2 \rangle \langle U_{12} \phi_2 | \phi_1 \rangle \langle U_{01} \phi_1 | \phi_0 \rangle), 
\end{align*} \]

where the standard notation for the scalar product in \( \mathbb{C}^N \) has been used.

Figure 2: The image of \( C_1 \) according to Eq. (I).

In order to find the intersections, we must look for solutions \( 0 \leq t \leq 1 \) of \( \text{Im} \varphi_{IA}(t) = 0 \) for which \( \text{Re} \varphi_{IA}(t) \leq 0 \). According to Lemma 4.2, such solutions will contribute with the sign

\[ \text{sign} \ \text{Im} \left[ \frac{\partial \varphi_{IA}(t)}{\partial t} \right] \]

to the intersection number.

Triangle \( B \) is swept out by geodesics from the dotted line Fig. 2 to the point \( P = U_{01} * \Phi_1 \). This arc is parametrized by \( V_{(01)} V_{(123)}(s_2) * \Phi_3 \). Hence by Lemma 4.1, we have to find points \( 0 \leq s_2 \leq 1 \) such that \( \text{Im} \varphi_{IB}(s_2) = 0 \) and \( \text{Re} \varphi_{IB}(s_2) \leq 0 \), where \( \varphi_{IB} \) is the function

\[ \text{tr} \left[ X_{(0)} (V_{(01)} * X_{(1)}) (V_{(123)}(s_2) * X_{(3)}) \right]. \]

Explicitly, in terms of the parallel transporters and Higgs fields, this may be written as

\[ \varphi_{IB}(s_2) = s_2^2 \langle \phi_0 | U_{01} \phi_1 \rangle \langle \phi_1 | U_{123} \phi_3 \rangle \langle U_{0123} \phi_3 | \phi_0 \rangle \]

\[ + (1-s_2)^2 \langle \phi_0 | U_{01} \phi_1 \rangle \langle \phi_1 | U_{13} \phi_3 \rangle \langle U_{013} \phi_3 | \phi_0 \rangle \]

\[ + s_2(1-s_2) \langle \phi_0 | U_{01} \phi_1 \rangle \langle \phi_1 | U_{123} \phi_3 \rangle \langle U_{013} \phi_3 | \phi_0 \rangle \]

\[ + \langle \phi_1 | U_{13} \phi_3 \rangle \langle U_{0123} \phi_3 | \phi_0 \rangle), \]

which is explicitly seen to be gauge invariant (we have suppressed a positive overall factor which obviously does not affect our considerations). Values of \( s_2 \) with the above properties
may thus be found by elementary means. In the appendix, case α) we show that they make the following contribution to the intersection number, i.e. have the following sign for the relative orientation:

\[
\text{sign } \text{Im} \left[ \frac{\partial \varphi_{II}(s_2)}{\partial s_2} \right].
\]

**Case (II):** The image of \( C_2^1 \) is a square. It is obtained by ‘moving down’ the geodesic arc \( PQ \) to the arc \( XW \), see Fig. 4. From the parametrization of this surface, Eq. (II) one concludes that it will intersect the manifold \( \Phi_0^\perp \) if and only if

\[
(V_{(012)}(s_1) \ast X_{(23)}(s_3))X_{(0)} = 0,
\]

\[
\iff (V_{(210)}(s_1) \ast X_{(0)})X_{(23)}(s_3) = 0,
\]

for some \( 0 \leq s_1, s_3 \leq 1 \). From Eq. (6) we see that this will happen precisely if the geodesic arc parametrised by \( X_{(2)}(s_2) \) intersects the manifold \( (V_{(210)}(s_1) \ast \Phi_0^\perp \rangle \) for some value \( 0 \leq s_1 \leq 1 \). The arc has endpoints \( X_{(2)} \) and \( V_{(23)} \ast X_{(3)} \). We may apply Lemma 4.1 to conclude that for such an intersection we must have \( \text{Im} \varphi_{II}(s_1) = 0 \) and \( \text{Re} \varphi_{II}(s_1) \leq 0 \), where \( \varphi_{II} \) is the function

\[
\text{tr} \left[ X_{(2)}(V_{(23)} \ast X_{(3)}) (V_{(210)}(s_1) \ast X_{(0)}) \right].
\]

Expressing everything in terms of parallel transporters and Higgs fields, we find

\[
\varphi_{II}(s_1) = s_1^2 \langle \phi_2 | U_{23} \phi_3 \rangle \langle U_{0123} \phi_3 | \phi_0 \rangle \langle \phi_0 | U_{012} \phi_2 \rangle + (1 - s_1)^2 \langle \phi_2 | U_{23} \phi_3 \rangle \langle U_{023} \phi_3 | \phi_0 \rangle \langle \phi_0 | U_{02} \phi_2 \rangle + s_1(1 - s_1) \langle \phi_2 | U_{23} \phi_3 \rangle \langle U_{0123} \phi_3 | \phi_0 \rangle \langle \phi_0 | U_{02} \phi_2 \rangle + \langle U_{023} \phi_3 | \phi_0 \rangle \langle \phi_0 | U_{012} \phi_2 \rangle,
\]

and this is directly seen to be gauge invariant. Again as in case (I), we have omitted an irrelevant positive overall factor. It is shown in the appendix, case β) that such an intersection contributes with the sign

\[
\text{sign } \text{Im} \left[ \frac{\partial \varphi_{II}(s_1)}{\partial s_1} \right].
\]

**Case (III):** From Eq. (III) we can see that the image of \( C_3^1 \) is a square, parametrized by \( V_{(0123)}(s_1, s_2) \ast X_{(3)} \), where \( 0 \leq s_1, s_2 \leq 1 \). This square will intersect the manifold \( \Phi_0^\perp \) precisely for values \( s_1, s_2 \) such that

\[
X_{(0)}(V_{(0123)}(s_1, s_2) \ast X_{(3)}) = 0.
\]

We have to find such values and determine their contribution to the intersection number. Writing everything in terms of parallel transporters and Higgs fields, the above condition reads \( \varphi_{III}(s_1, s_2) = 0 \), where

\[
\varphi_{III}(s_1, s_2) = \langle \phi_0 | U_{0123} \phi_3 \rangle s_1 s_2 + \langle \phi_0 | U_{013} \phi_3 \rangle s_1 (1 - s_2)
\]

\[
+ \langle \phi_0 | U_{023} \phi_3 \rangle (1 - s_1) s_2 + \langle \phi_0 | U_{03} \phi_3 \rangle (1 - s_1)(1 - s_2),
\]

\[
(8)
\]
and this expression is manifestly gauge invariant. Condition (8) constitutes a system of two real equations in two real variables, which can easily be solved. In the appendix, case $\gamma$) it is demonstrated that any point $(s_1, s_2)$ satisfying Eq. (8) contributes the value

$$\text{sign } \text{Im} \left[ \frac{\partial \phi_{III}(s_1, s_2)}{\partial s_1} \frac{\partial \phi_{III}(s_1, s_2)}{\partial s_2} \right]$$

to the intersection number.

5 Conclusions and outlook

In this work we have given a geometric definition of local monopole numbers for an $SU(N)$ lattice gauge-Higgs system. The definition yields a closed magnetic current which is stable under perturbations of the configuration in question. It is explicitly worked out how to calculate the monopole number for each tetrahedron of a simplicial 4-dimensional lattice. On the practical side, one has to solve four quadratic equations in one variable for each such tetrahedron. The coefficients of these equations are given by traces of parallel transporters and Higgs variables. The prescription is explicitly gauge invariant and can be directly implemented in a computer code.

While the construction gives an integer local monopole number for almost all field configurations, the question is on what kind of monopole the construction actually triggers. In the $SU(2)$ case for the Georgi-Glashow model it has been shown [15] that the construction is able to detect a discretized 't Hooft-Polyakov monopole, provided the configuration is smooth enough. This question remains to be discussed for $N > 2$. On the other hand, for rough lattice fields as they are generated in a Monte-Carlo simulation, one has to expect nonvanishing monopole charges which do not correspond to physical excitations but are lattice artefacts (dislocations). It is then necessary to find out whether these configurations become sufficiently suppressed in the continuum limit.

Appendix

Points in $\mathbb{CP}^{N-1}$ are identified with equivalence classes $[a] = Ca$ (lower case letters) or projectors $A$ (capital letters) projecting on the subspace spanned by $a$. Two points $[p], [q]$ are joined by the unique shortest geodesic $[(1 - t)\langle p|q\rangle p + t\langle p|q\rangle^2 q]$, provided $\langle p|q\rangle \neq 0$ (see [20]).

Proof of Lemma 4.1:
From the form of the geodesics it is clear that the geodesic arc $\overline{PQ}$ will intersect $E^\perp$ iff $(1 - t)\langle e|p\rangle\langle p|q\rangle + t\langle p|q\rangle^2 \langle e|q\rangle = 0$ for some $0 \leq t \leq 1$. This is easily seen to be equivalent to

$$\text{Re}[\langle e|p\rangle\langle p|q\rangle\langle q|e\rangle] \leq 0, \quad \text{Im}[\langle e|p\rangle\langle p|q\rangle\langle q|e\rangle] = 0,$$

proving the lemma.
Proof of Lemma 4.3:

We parametrize the geodesic arc \( \overline{QR} \) by \([p'(t)]\) as above. Now the triangle \( \triangle \) intersects \( E^\perp \) iff some arc \( P'(t)P \) does. According to lemma 4.1 this can happen if and only if \( \text{Im} \text{tr}[PP'(t)E] = 0 \) and \( \text{Re} \text{tr}[PP'(t)E] \leq 0 \). Inserting the expression for \( P'(t) \), we see that these are just the conditions given in the lemma. The statement for the sign of the intersection is obtained in case \( a.A \) below.

Proof of Lemma 4.2:

The image under the map \( X_{(0123)} \) of the faces \( C_k^0 \) is obtained by coning the boundary of Fig. 4 to the point \( E := \Phi_0 \). Let \( A \) be any point in this boundary. In order that the geodesic arc \( AE \) intersects \( E^\perp \), it is necessary that \( \text{Re} \text{tr} AE \leq 0 \), by Lemma 4.1. But this is easily seen to imply \( AE = 0 \), i.e. \( A \in E^\perp \). So the image of the faces \( C_k^0 \) intersects \( E^\perp \) if and only if the boundary of the triangle in Fig. 2 does. On dimensional grounds this can happen only for a measure zero set of configurations.

We now prove the various claims about intersection numbers made above. Let us briefly recall how the relative orientation of two oriented, embedded transversally intersecting manifolds is defined. Suppose \( X, Y \subset Z \) are oriented embedded manifolds, intersecting transversally at a point \( p \). The union of two oriented frames in \( T_pX \) and \( T_pY \) then gives a frame in \( T_pZ \) whose orientation may be compared to the orientation of \( Z \). The sign \( X : Y \) is defined to be \( \pm 1 \) according to whether these orientations coincide resp. do not coincide. Going over to the case at hand, we identify \( C_N^N \) with the real vector space \( \mathbb{R}^{2N} \) and Euclidean inner product \( \langle \cdot, \cdot \rangle = \text{Re} \langle \cdot, \cdot \rangle \). The tangent space \( T_A \mathbb{C}P^{N-1} \) is canonically identified with \( \{ x \in C_N^N \cong \mathbb{R}^{2N} | Ax = 0 \} \). Each such tangent space may be equipped with the (real) \((2N-2)\)-form \( * (a \wedge ia) \), defining thus a canonical orientation of the complex projective space. Here, ‘\(*\)’ denotes the operation of taking the Hodge dual of an alternating form with help of the above Euclidean inner product. In the same fashion, the tangent space \( T_A E^\perp \), \( EA = 0 \) is identified with \( \{ x \in C_N^N \cong \mathbb{R}^{2N} | Ax = Ex = 0 \} \) and the (real) \((2N-4)\)-form \( * (e \wedge ie \wedge a \wedge ia) \) defines an orientation. Suppose we are given a surface in projective space parametrized by \([\sigma(s_1, s_2)]\). With the above identifications, the tangent space at a point \([\sigma_0] \) of this surface is given by the real linear span of \( x_1, x_2 \), where

\[
x_i = \frac{\partial \sigma}{\partial s_i} - \left\langle \sigma_0, \frac{\partial \sigma}{\partial s_i} \right\rangle \sigma_0,
\]

and the 2-form \( x_1 \wedge x_2 \) defines an orientation. Now let \([\sigma_0] \) be also in \( E^\perp \), i.e. an intersection point. Then by definition, \( \langle \sigma_0 | x_i \rangle = \langle \sigma_0 | e \rangle = 0 \) and one calculates

\[
x_1 \wedge x_2 \wedge *(e \wedge ie \wedge \sigma_0 \wedge i\sigma_0) = \text{Im} \left[ \left\langle e, \frac{\partial \sigma}{\partial s_1} \right\rangle \left\langle \frac{\partial \sigma}{\partial s_2}, e \right\rangle \right] \ast (\sigma_0 \wedge i\sigma_0).
\]

Hence the sign of this intersection point is given by

\[
\text{sign Im} \left[ \left\langle e, \frac{\partial \sigma}{\partial s_1} \right\rangle \left\langle \frac{\partial \sigma}{\partial s_2}, e \right\rangle \right], \tag{9}
\]

We are now ready to prove the claims made above about the intersection numbers. The maps given in Eqs. \((I), (II), (III)\) define a surface in \( \mathbb{C}P^{N-1} \) which is schematically drawn.
Case \( \alpha.A \): By definition the surface \( IA \) is parametrized by \([\sigma(s, t)]\), where
\[
\sigma(t, s) = (1 - s)\langle \phi(t)|U_{01}\phi_1\rangle \phi(t) + s|\langle \phi(t)|U_{01}\phi_1\rangle|^2 U_{01}\phi_1, \\
\phi(t) = (1 - t)\langle \phi_2|U_{23}\phi_3\rangle U_{012}\phi_2 + t|\langle \phi_2|U_{23}\phi_3\rangle|^2 U_{0123}\phi_3.
\]
The orientation of this surface is given by the order of its vertices \( P, Q, R \), defined above. It is easily seen that this means that this orientation coincides with \( \eta \wedge \xi \), where
\[
\eta = \frac{\partial \sigma}{\partial t} - \left(\sigma_0 \left| \frac{\partial \sigma}{\partial t} \right| \sigma_0, \quad \xi = \frac{\partial \sigma}{\partial s} - \left(\sigma_0 \left| \frac{\partial \sigma}{\partial s} \right| \sigma_0.
\]
Now using Eq. (9) and the definition of \( \varphi_{IA} \) in Eq. (4), we see that the sign of an intersection is given by
\[
\text{sign Im} \left[ \frac{\partial \varphi_{IA}(s_2)}{\partial s_2} \right].
\]
Case \( \alpha.B \): The surface \( IB \) is a triangle, parametrized by \([\sigma(s_2, s_3)]\), where
\[
\sigma(s_2, s_3) = (1 - s_3)\langle V_{123}(s_2)|\phi_3\rangle U_{01}V_{(123)}(s_2)\phi_3 + s_3|\langle V_{123}(s_2)|\phi_3\rangle|^2 U_{01}\phi_1.
\]
Now setting \( e = \phi_0 \), using Eq. (9) and the definition of \( \varphi_{IB} \) in Eq. (5), it is straightforward to calculate that an intersection has sign
\[
\text{sign Im} \left[ \frac{\partial \varphi_{IB}(s_2)}{\partial s_2} \right].
\]
Case \( \beta \): Surface \( II \) is a square, parametrized by \([\sigma(s_1, s_3)]\), where \( x_1 \wedge x_3 \) is its orientation and
\[
\sigma(s_1, s_3) = (1 - s_3)\langle \phi_2|U_{23}\phi_3\rangle V_{(012)}(s_1)\phi_2 + s_3|\langle \phi_2|U_{23}\phi_3\rangle|^2 V_{(012)}(s_1)U_{23}\phi_3.
\]
Now setting \( e = \phi_0 \), using Eq. (9) and the definition of \( \varphi_{II} \) in Eq. (7), we get the result after a short computation.
Case \( \gamma \): Surface \( III \) is a square, parametrized by \([\sigma(s_1, s_2)]\), where \( x_1 \wedge x_2 \) is its orientation and
\[
\sigma(s_1, s_2) = V_{(0123)}(s_1, s_2)\phi_3.
\]
Now setting \( e = \phi_0 \), it follows immediately from the definition of \( \varphi_{III} \) in Eq. (8) and from Eq. (9) that the sign of the intersection is given by
\[
\text{sign Im} \left[ \frac{\partial \varphi_{III}(s_1, s_2)}{\partial s_1} \frac{\partial \varphi_{III}(s_1, s_2)}{\partial s_2} \right].
\]
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