On the Exponential Diophantine Equation

\[
\left( F^{(k)}_{m+1} \right)^x - \left( F^{(k)}_{m-1} \right)^x = F^{(k)}_n
\]

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Abstract

In this paper, we explicitly find all solutions of the title Diophantine equation, using lower bounds for linear forms in logarithms and properties of continued fractions. Further, we use a version of the Baker-Davenport reduction method in Diophantine approximation, due to Dujella and Pethő. This paper extends the previous work of [19].

Keywords: Generalized Fibonacci numbers, linear forms in logarithms, continued fraction, reduction method.

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1 Introduction

Let \( \{F_n\}_{n \geq 0} \) be the Fibonacci sequence given by

\[ F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0 \]

with initials \( F_0 = 0 \) and \( F_1 = 1 \).

The Fibonacci numbers are celebrated for possessing wonderful and amazing properties [11]. Hundreds of Fibonacci properties have been developed over the centuries by numerous mathematicians and number enthusiasts. Among such fabulous properties, in 1876, E. Lucas [15] showed that for all \( n \geq 1 \),

\[
\begin{align*}
F^2_n + F^2_{n+1} &= F_{2n+1}, \\
F^2_{n+1} - F^2_{n-1} &= F_{2n}.
\end{align*}
\]

(1.1) (1.2)

Diophantine equations related to sums of powers of two terms of a given linear recurrence sequence, were studied by several authors. For instance, motivated by the naive identity [11], which tells us that the sum of the square of two consecutive Fibonacci numbers is still a Fibonacci number, Marques and Togbé [10] showed that, if \( x \geq 1 \) is an integer such that \( F^x_n + F^x_{n+1} \) is a Fibonacci number for all sufficiently large \( n \), then \( x \in \{1, 2\} \). Later, Luca and Oyono [13] solved this problem completely by showing that the Diophantine equation \( F^s_m + F^s_{m+1} = F_n \) has no solutions \((m, n, s)\) with \( m \geq 2 \) and \( s \geq 3 \). Subsequently, Chaves and Marques [4] proved the Diophantine equation

\[
\left( F^{(k)}_m \right)^s + \left( F^{(k)}_{m+1} \right)^s = F^{(k)}_n
\]

in \( k \)-generalized Fibonacci numbers, showing that it has no positive integer solution \((n, m, k)\) with \( k \geq 3 \) and \( n \geq 1 \). Then Ruiz and Luca [21] solved this Diophantine equation completely by showing that it has no solutions.
Indeed, observe that recursion (1.4) implies the three-term recursion
\[ F^x_{n+1} - F^x_{n-1} = F^y_m, \]
has only non-negative integer solutions \((m, n, x) = (2n, n, 2), (1, 1, x), (2, 1, x), (0, n, 0)\). Consequently, Gómez et al. [8] considered an extra exponent \(y\) in (1.3) and investigated the equation
\[ F^x_{n+1} - F^x_{n-1} = F^y_n \]
in positive integers \((n, m, x, y)\). Patel and Teh [20] found all the solutions of the exponential Diophantine equation \(F^x_{n+1} + F^x_{n} - F^x_{n-1} = F^m_n\) in non-negative integers \((m, n, x)\), which tells us that the sum of the power of three consecutive Fibonacci numbers is still a Fibonacci number.

Let \(k \geq 2\) be an integer. One of numerous generalizations of the Fibonacci sequence, which is called the \(k\)-generalized Fibonacci sequence \(\{F^{(k)}_n\}_{n \geq -(k-2)}\) is given by the recurrence
\[ F^{(k)}_n = F^{(k)}_{n-1} + F^{(k)}_{n-2} + \cdots + F^{(k)}_{n-k} = \sum_{i=1}^{k} F^{(k)}_{n-i} \quad \text{for all } n \geq 2, \]  
with initial conditions \(F^{(k)}_{-(k-2)} = F^{(k)}_{-(k-3)} = \cdots = F^{(k)}_0 = 0\) and \(F^{(k)}_1 = 1\). Here, \(F^{(k)}_n\) denotes the \(n\)th \(k\)-generalized Fibonacci number.

Note that for \(k = 2\), we have \(F^{(2)}_n = F_n\), the \(n\)th Fibonacci number. For \(k = 3\), we have \(F^{(3)}_n = T_n\), the \(n\)th Tribonacci numbers. They are followed by the Tetranacci numbers for \(k = 4\), and so on.

The first observation is that the first \(k + 1\) non-zero terms in \(F^{(k)}_n\) are powers of 2, namely
\[ F^{(k)}_1 = 1, F^{(k)}_2 = 1, F^{(k)}_3 = 2, F^{(k)}_4 = 4, \ldots, F^{(k)}_{k+1} = 2^{k-1}, \]
while the next term in the above sequence is \(F^{(k)}_{k+2} = 2^k - 1\). Thus, we have that
\[ F^{(k)}_n = 2^{n-2} \quad \text{holds for all } 2 \leq n \leq k + 1. \]

Indeed, observe that recursion (1.4) implies the three-term recursion
\[ F^{(k)}_n = 2F^{(k)}_{n-1} - F^{(k)}_{n-k-1} \quad \text{for all } n \geq 3, \]
which shows that the \(k\)-Fibonacci sequence grows at a rate less than \(2^{n-2}\). In fact, the inequality \(F^{(k)}_n < 2^{n-2}\) holds for all \(n \geq k + 2\) (see [1], Lemma 2).

In this paper, we study an analogue of (1.3) when the Fibonacci sequence is replaced by \(k\)-generalized Fibonacci sequence. More precisely, our main result is the following.

**Theorem 1.1.** The Diophantine equation
\[ \left( F^{(k)}_{m+1} \right)^x - \left( F^{(k)}_{m-1} \right)^x = F^{(k)}_n \]  
has no positive integer solutions \(m, n, k\) and \(x \geq 2\) with \(3 \leq k \leq \min\{m, \log x\}\).

Before getting into details, we give a brief description of our method. We first use Matveev’s result [17] on linear forms in logarithms to obtain an upper bound for \(x\) in terms of \(m\). When \(m\) is small, we use Dujella and Pethő’s result [6] to decrease the range of possible values that allow us to treat our problem computationally. When \(m\) is large, we use a lower bound for linear forms in two logarithms [12] to get an absolute upper bound for \(x\). In the final step, we use continued fractions to lower the bounds and then complete the calculations.
2 Auxiliary results

We recall some of the facts and properties of the $k$-generalized Fibonacci sequence which will be used after. Note that the characteristic polynomial of the $k$-generalized Fibonacci sequence is

$$
\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.
$$

$\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle. It is real and positive, so it satisfies $\alpha(k) > 1$. The other roots are strictly inside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single root, which is located between $2(1 - 2^{-k})$ and $2$ (see [15]). To simplify notation, we will omit the dependence on $k$ of $\alpha$.

Dresden [5] gave a simplified Binet-like formula for $F_n^{(k)}$:

$$
F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1},
$$

where $\alpha = \alpha_1, \ldots, \alpha_k$ are the roots of $\Psi_k(x)$. He also showed that the contribution of the roots which are inside the unit circle to the right-hand side of (2.6) is very small. More precisely, he proved that

$$
\left| F_n^{(k)} - \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \right| < \frac{1}{2} \text{ for all } n \geq 1.
$$

The following inequality is proved by Bravo and Luca [11 Lemma 1].

**Lemma 2.1.** The inequality

$$
\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}
$$

holds for all $n \geq 1$.

The following result is derived in [4 Eq. 3.14].

**Lemma 2.2.** Let $y_m := \frac{|E_m^{(k)}|s}{g^m a}$ for $m \geq 1395$. Then

$$
|E_m^{(k)} - g^s\alpha^{(m-1)s}| < 2y_m g^s \alpha^{(m-1)s}.
$$

The following result is proved by Sánchez and Luca [22 Lemma 7].

**Lemma 2.3.** If $r \geq 1, T > (4r^2)^r$, and $T > a/(\log a)^r$. Then

$$
a < 2^r T (\log T)^r.
$$

The following lemma is useful in the subsequent result.

**Lemma 2.4.** For all $m \geq 3$ and $k \geq 3$, we have

$$
F_m^{(k)} / F_{m+1}^{(k)} \leq 3/7.
$$

**Proof.** Indeed, we have $7F_m^{(k)} \leq 3F_{m+1}^{(k)} = 3 \left( F_m^{(k)} + F_{m-1}^{(k)} + \cdots + F_{m-(k-1)}^{(k)} \right)$. Further simplification, we obtain

$$
4F_{m-1}^{(k)} = 4 \left( F_{m-2}^{(k)} + F_{m-3}^{(k)} + \cdots + F_{m-(k+1)}^{(k)} \right) \leq 3F_m^{(k)} + 3 \left( F_{m-2}^{(k)} + F_{m-3}^{(k)} + \cdots + F_{m-(k-1)}^{(k)} \right)
$$

and then $F_{m-1}^{(k)} + 3F_{m-k}^{(k)} + 3F_{m-(k+1)}^{(k)} \leq 3F_m^{(k)}$. Furthermore,

$$
0 \leq 2F_{m-1}^{(k)} + 3 \left( F_{m-2}^{(k)} + F_{m-3}^{(k)} + \cdots + F_{m-(k+2)}^{(k)} \right)
$$

for $m \geq 3$, which follows the result. \[\square\]
In order to prove our main result, we use a few times a Baker-type lower bound for a non-zero linear forms in logarithms of algebraic numbers. We state a result of Matveev [17] about the general lower bound for linear forms in logarithms, but first, recall some basic notations from algebraic number theory.

Let \( \eta \) be an algebraic number of degree \( d \) with minimal primitive polynomial

\[
f(X) := a_0 X^d + a_1 X^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],
\]

where the \( a_i \)'s are relatively prime integers, \( a_0 > 0 \), and the \( \eta^{(i)} \)'s are conjugates of \( \eta \). Then

\[
h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \left( \max\{|\eta^{(i)}|, 1\} \right) \right)
\]

is called the logarithmic height of \( \eta \).

With the established notations, Matveev (see [17] or [2, Theorem 9.4]), proved the ensuing result.

**Theorem 2.5.** Assume that \( \gamma_1, \ldots, \gamma_t \) are positive real algebraic numbers in a real algebraic number field \( K \) of degree \( D \), \( b_1, \ldots, b_t \) are rational integers, and

\[
\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,
\]

is not zero. Then

\[
|\Lambda| \geq \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B)A_1 \cdots A_t\right),
\]

where

\[
B \geq \max\{|b_1|, \ldots, |b_t|\},
\]

and

\[
A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \text{ for all } i = 1, \ldots, t.
\]

When \( t = 2 \) and \( \gamma_1, \gamma_2 \) are positive and multiplicatively independent, we use the following result of Laurent, Mignotte and Nesterenko [12].

Let \( B_1, B_2 \) be real numbers larger than 1 such that

\[
\log B_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D} \right\} \text{ for } i = 1, 2.
\]

Let \( b_1, b_2 \) be integers not both zero. Put

\[
\Gamma := b_1 \log \gamma_1 + b_2 \log \gamma_2
\]

and

\[
b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.
\]

**Theorem 2.6** ([12], Corollary 2, pp. 288). With the above notations and assumptions, we have

\[
\log |\Gamma| > -24.34 D^4 \left( \max \left\{ \log b' + 0.14, \frac{21}{D} \cdot \frac{1}{2} \right\} \right)^2 \log B_1 \log B_2.
\]
Note that \( \Gamma \neq 0 \) since \( \gamma_1 \) and \( \gamma_2 \) are multiplicatively independent and \( b_1 \) and \( b_2 \) are integers not both zero.

The following criterion of Legendre, a well-known result from the theory of Diophantine approximation, is used to reduce the upper bounds on variables which are too large.

**Lemma 2.7.** Let \( \tau \) be an irrational number, \( \frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots \) be all the convergents of the continued fraction of \( \tau \), and \( M \) be a positive integer. Let \( N \) be a non-negative integer such that \( q_N > M \). Then putting \( a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\} \), the inequality

\[
\left| \tau - \frac{r}{s} \right| > \frac{1}{(a(M) + 2)s^2},
\]

holds for all pairs \((r, s)\) of positive integers with \(0 < s < M\).

Another result which will play an important role in our proof is due to Dujella and Pethő [6, Lemma 5 (a)].

**Lemma 2.8.** Let \( M \) be a positive integer, let \( \frac{p}{q} \) be a convergent of the continued fraction of the irrational \( \gamma \) such that \( q > 6M \), and let \( A, B, \mu \) be some real numbers with \( A > 0 \) and \( B > 1 \). Let \( \epsilon := ||\mu q|| - M||\gamma q|| \), where \( || \cdot || \) denotes the distance from the nearest integer. If \( \epsilon > 0 \), then there exists no solution to the inequality

\[
0 < |u\gamma - v + \mu| < AB^{-u},
\]

in positive integers \( u \) and \( v \) with

\[
u \leq M \quad \text{and} \quad u \geq \frac{\log(Aq/\epsilon)}{\log B}.
\]

### 3 Proof of Theorem 1.1

By Lemma 2.1, we obtain

\[
\alpha^{n-1} \geq F_n^{(k)} = (F_{m+1}^{(k)})^x - (F_{m-1}^{(k)})^x \geq \alpha^{(m-1)x} - \alpha^{(m-3)x} = \alpha^{(m-2)x} (\alpha^x - \alpha^{-x}) > \alpha^{(m-2)x+1}
\]

and

\[
\alpha^{n-2} \leq F_n^{(k)} = (F_{m+1}^{(k)})^x - (F_{m-1}^{(k)})^x \leq \alpha^{mx} - \alpha^{(m-2)x} < \alpha^{mx},
\]

where we used that \( \alpha^x - \alpha^{-x} > \alpha \) for \( x \geq 2 \). Thus,

\[
(m-2)x + 2 < n < mx + 2.
\]

From (2.6), we can write

\[
F_n^{(k)} = g\alpha^{n-1} + E_n^{(k)}, \quad \text{where} \quad |E_n^{(k)}| < 1/2.
\]

Hence, (1.5) can be written as

\[
g\alpha^{n-1} - (F_{m+1}^{(k)})^x = -(F_{m-1}^{(k)})^x - E_n^{(k)}.
\]
Dividing (3.10) by \((F_{m+1})^x\) and taking absolute values and by Lemma 2.4 we obtain
\[
\left| g\alpha^{n-1}(F_{m+1})^{-x} - 1 \right| < 2 \left( \frac{F_{m-1}}{F_{m+1}} \right)^x < \frac{2}{2.3x} \tag{3.11}
\]

In order to use Theorem 2.5 we take \(t := 3\),
\[
\gamma_1 := F_{m+1}, \gamma_2 := \alpha, \gamma_3 := g,
\]
and
\[
b_1 := -x, b_2 := n - 1, b_3 := 1.
\]
Hence,
\[
A_1 := g\alpha^{n-1}(F_{m+1})^{-x} - 1.
\]
To see \(A_1 \neq 0\), suppose \(g\alpha^{n-1}(F_{m+1})^{-x} - 1 = 0\), we would get a relation \((F_{m+1})^x = g\alpha^{n-1}\), which is contradict the identity (3.9).

Note that \(K := \mathbb{Q}(\alpha)\) contains \(\gamma_1, \gamma_2, \gamma_3\) and has \(D = [K : \mathbb{Q}] = k\). By the properties of the dominant root of \(\Psi_k(x)\) and Lemma 2.1 we have \(h(\gamma_1) = \log(F_{m+1}) \leq m \log \alpha < 0.7m\), and \(h(\gamma_2) = (\log \alpha)/k < 0.7/k\). Bravo and Luca [11] showed that \(h(\gamma_3) < 3 \log k\). Thus we can take \(A_1 := 0.7km, A_2 := 0.7\) and \(A_3 := 3k \log k\). From (3.8), we can take \(B := n - 1\).

From Matveev’s theorem, we have a lower bound for \(|A_1|\), which together with the upper bound given by (3.11) gives
\[
\exp \left( -1.4 \cdot 30^{6} \cdot 3^{4.5} \cdot k^{2} \cdot (1 + \log k) \cdot (1 + \log(n - 1)) \cdot (0.7mk) \cdot 0.7 \cdot (3k \log k) \right) < \frac{2}{2.3x}.
\]
Hence, noting that \(1 + \log(n - 1) < 2 \log(mx + 1)\), is true for all \(n \geq 4\) and performing the calculations, we get
\[
x < 1.04 \cdot 10^{12} mk^4 (\log k)^2 \log(mx + 1),
\]
or equivalently
\[
\frac{mx + 1}{\log(mx + 1)} < 1.1 \cdot 10^{12} m^2 k^4 (\log k)^2.
\]
Now using Lemma 2.3 for \(a := mx + 1\) and \(T := 1.1 \cdot 10^{12} m^2 k^4 (\log k)^2\), we have that
\[
x < 7.1 \cdot 10^{13} m^5 (\log m)^3, \tag{3.12}
\]
where we used the fact that \(28 + 6 \log m + 2 \log \log m < 32 \log m\) for \(m \geq 3\) and the presumption \(k \leq m\).

### 3.1 The case for small \(m \in [3, 1457]\)
In this case, we have
\[
x < 7.1 \cdot 10^{13} \cdot (1457)^5 \cdot \log(1457)^3 \leq 1.81 \cdot 10^{32}
\]
for \(m \in [3, 1457]\). Thus, we obtain \(n \leq mx + 1 < 2.63 \cdot 10^{35}\) and \(k \leq \log x \leq 74\). Also note that \(n \leq mx + 1\), gives us \(x \geq (n - 1)/1457\).

We set
\[
\Gamma_1 := (n - 1) \log \alpha - \log \left( \frac{1}{g} \right) - x \log F_{m+1}^{(k)}.
\]
Thus, $\Lambda_1 = e^{\Gamma_1} - 1$. Recall that, from (3.10), we have $\Lambda_1 < 0$, which implies $\Gamma_1 < 0$. Now, since $|\Lambda_1| < 2 / (2.3)^2 \leq 0.379$ for $x \geq 2$, it follows that $e^{|\Gamma_1|} < 1.5$. Hence, we get

$$0 < |\Gamma_1| < e^{|\Gamma_1|} - 1 \leq e^{|\Gamma_1|} |e^{\Gamma_1} - 1| < \frac{3}{2.3^x}.$$ 

Dividing the last inequality by $\log F_{m+1}^{(k)}$, and using that $x > (n - 1)/1457$, we get

$$0 < n \left( \frac{\log \alpha}{\log F_{m+1}^{(k)}} \right) - x - \left( \frac{\log(\alpha/g)}{\log F_{m+1}^{(k)}} \right) < 3.01 \cdot (2.3)^{\frac{n}{1457}}. \quad (3.13)$$ 

In order to use the reduction method, take

$$\gamma_{m,k} := \frac{\log \alpha}{\log F_{m+1}^{(k)}}, \quad \mu_{m,k} := -\frac{\log(\alpha/g)}{\log F_{m+1}^{(k)}}, \quad A := 3.01, \quad B := (2.3)^{\frac{1}{1457}}.$$ 

The fact that $\alpha$ is a unit in $O_K$, the ring of integers of $K$, ensures that $\gamma_{m,k}$ is an irrational number. Let $q_{(t,m,k)}$ be the denominator of the $t$th convergent of the continued fraction of $\gamma_{m,k}$.

Taking $M := 2.64 \cdot 10^{35}$, we use Mathematica to get

$$\min_{3 \leq k \leq 74 \atop 3 \leq m \leq 1457} q_{(700,m,k)} > 10^{319} > 6M \quad \text{and} \quad \max_{3 \leq k \leq 74 \atop 3 \leq m \leq 1457} q_{(700,m,k)} < 2.1 \cdot 10^{425}.$$ 

The maximal value of $M \|\gamma_{m,k} \cdot q_{(700,m,k)}\| < 10^{-284}$, whereas the minimal value of $\|\mu_{m,k} \cdot q_{(700,m,k)}\| > 5.29 \cdot 10^{-214}$. Also, for

$$\varepsilon_{700,m,k} := \|\mu_{m,k} \cdot q_{(700,m,k)}\| - 2.64 \cdot 10^{35} \|\gamma_{m,k} \cdot q_{(700,m,k)}\|,$$

we obtain that

$$\varepsilon_{700,m,k} > 5.29 \cdot 10^{-214},$$

which means that $\varepsilon_{700,m,k}$ is always positive (this is not true for the denominator of 600th convergent). Hence by Lemma 2.8, there are no integer solutions for (3.13) when

$$\left\lfloor \frac{\log(3.01 \cdot 2.1 \cdot 10^{425}/5.29 \cdot 10^{-214})}{\log((2.3)^{\frac{1}{1457}})} \right\rfloor \leq n \leq 2.63 \cdot 10^{35}.$$ 

It follows that $850196 \leq n \leq 2.64 \cdot 10^{35}$ and therefore we have $n \leq 850195$. Consequently $x \leq 850192$, since $x \leq (n - 3)/(m - 2)$. Also, using that $k \leq \log x$, we get $k \leq 13$.

A computer search with Mathematica revealed that there are no solutions to (1.5) in the following range:

$$3 \leq m \leq 1457, \quad 3 \leq k \leq 13, \quad 21 \leq x \leq 850192 \quad \text{and} \quad (m - 2)x + 2 \leq n \leq mx + 1.$$ 

This completes the analysis of the case when $m$ is small.
3.2 The case of large $m$

Set $y_m := \frac{|E_m(k)|x}{g^m}$. From (3.12) and $m \geq 1458$, we have

$$y_m < \frac{|E_m(k)|\cdot 7.1 \cdot 10^{13}n^5 \log m^3}{g^{m-1}} < \frac{1}{\alpha^{(m-1)/2}},$$  \hspace{1cm} (3.14)

where $7.1 \cdot 10^{13}n^5 \log m^3 < (7/4)^{m-1/2} < \frac{\alpha^{m-1}}{2}$ holds for $m \geq 1458$. In particular, $y_m < \alpha^{-728} < 10^{-30}$. Similarly,

$$y_{m-1} = \frac{|E_{m-1}(k)|x}{g^{m-2}} < \frac{1}{\alpha^{(m-1)/2}} \quad \text{and} \quad y_{m+1} = \frac{|E_{m+1}(k)|x}{g^{m}} < \frac{1}{\alpha^{(m-1)/2}}.$$  

Lemma 2.2 is true if we replace $m$ by $m-1$ and $m+1$. Thus

$$|(F_{m-1}^{(k)})^x - g^x \alpha^{(m-2)x}| < 2y_{m-1}g^x \alpha^{(m-2)x} \quad \text{and} \quad |(F_{m+1}^{(k)})^x - g^x \alpha^{mx}| < 2y_{m+1}g^x \alpha^{mx}.$$  \hspace{1cm} (3.15)

Now, we need to make a few algebraic manipulations in order to apply Theorem 2.6. Rewrite (1.5) as follows,

$$g\alpha^{n-1} + E_n(k) = \left((F_{m+1}^{(k)})^x - g^x \alpha^{mx}\right) - \left((F_{m-1}^{(k)})^x - g^x \alpha^{(m-2)x}\right) + g^x \alpha^{mx} - g^x \alpha^{(m-2)x},$$

which gives

$$|g\alpha^{n-1} - g^x \alpha^{mx}(1 - \alpha^{-2x})| \leq |(F_{m+1}^{(k)})^x - g^x \alpha^{mx}| + |(F_{m-1}^{(k)})^x - g^x \alpha^{(m-2)x}| + |E_n(k)|$$

$$< 2y_{m+1}g^x \alpha^{mx} + 2y_{m-1}g^x \alpha^{(m-2)x} + \frac{1}{2}.$$  

Dividing $g^x \alpha^{mx}$ on both sides, we have

$$|g^{1-x}\alpha^{n-(mx+1)} - (1 - \alpha^{-2x})| < 2y_{m+1} + 0.1y_{m-1} + \frac{1}{2g^x \alpha^{mx}},$$  \hspace{1cm} (3.16)

where we used the fact $2\alpha^{-2x} < 2 \left(\frac{7}{4}\right)^{-6} < 0.1$. Since

$$2g^x \alpha^{mx-\frac{m-2}{2}} > 2 \left(\frac{1}{2}\right)^x \left(\frac{7}{4}\right)^{\frac{mx-m-2}{2}} > 10^{531} > 10^3,$$

we have $(2g^x \alpha^{mx})^{-1} < 0.001/\alpha^{(m-2)/2}$. Using the last inequalities with (3.16), we obtain

$$|g^{1-x}\alpha^{n-(mx+1)} - (1 - \alpha^{-x})| < \frac{2}{\alpha^{x^2}} + \frac{0.1}{\alpha^{m-2}} + \frac{0.001}{\alpha^{m-2}}$$

$$< \frac{2.11}{\alpha^{m-2}}.$$  \hspace{1cm} (3.17)

Hence, we conclude that

$$|g^{1-x}\alpha^{n-(mx+1)} - 1| < \frac{2.11}{\alpha^{m-2}} + \frac{1}{\alpha^{xx}}$$

$$< \frac{3.11}{\alpha^4},$$  \hspace{1cm} (3.18)
where \( l := \min\{x, \frac{m-2}{2}\} \).

We apply Theorem 2.6 with \( t := 2, \lambda_1 := g, \lambda_2 := \alpha \) and \( c_1 := 1 - x, c_2 := n - (mx + 1) \). The fact that \( \lambda_1 = g \) and \( \lambda_2 = \alpha \) are multiplicatively independent follows because \( \alpha \) is a unit and \( g \) is not. To see that this is not so, we perform a norm calculation of the element \( g \) in \( \mathbb{L} := \mathbb{Q}(\alpha) \). The norm of \( g \) has been determined for all \( k \geq 2 \) in (7) and the formula is

\[
|N_{\mathbb{L}/\mathbb{Q}}(g_k(\alpha))| = \frac{(k - 1)^2}{2k+1k^2 - (k + 1)^{k+1}}.
\]

One can check that \( |N_{\mathbb{L}/\mathbb{Q}}(g_k(\alpha))| < 1 \) for all \( k \geq 2 \) and therefore \( g \) is not a unit for any \( k \). So we can take \( \mathbb{K} := \mathbb{Q}(\alpha) \) which has degree \( D = k, h(g) < 3 \log k, h(\alpha) = (\log \alpha)/k \). Moreover, we can take

\[
\log B_1 = 4 \log k > \max \left\{ h(g), \frac{|\log(g)|}{1}, \frac{1}{k} \right\} \quad \text{and} \quad \log B_2 = \max \left\{ h(\alpha), \frac{|\log(\alpha)|}{1}, \frac{1}{k} \right\} = \frac{1}{k}.
\]

Observe that \( n \geq (m - 2)x + 3 \), which implies \( n - (mx + 1) \geq -2(x - 1) \) and \( n \leq mx + 1 \), yields \( n - (mx + 1) \leq 2(x - 1) \). Hence \( |n - (mx + 1)| < 2(x - 1) < 2x \). Thus

\[
b' = \frac{|(1 - x)|}{k(\frac{1}{4})} + \frac{|n - (mx + 1)|}{4k \log k} < x + \frac{2x}{4k \log k} < 1.2x.
\]

By Theorem 2.6 we get

\[
|\Lambda_2| > \exp \left( -24.34 \cdot k^4 \left( \max \left\{ \log(1.2x) + 0.14, \frac{21}{k}, \frac{1}{2} \right\} \right)^2 \cdot 4 \log k \left( \frac{1}{k} \right) \right).
\]

Thus

\[
|\Lambda_2| > \exp \left( -97.4 \cdot k^3 \log k \left( \max \left\{ \log(1.4x), \frac{21}{k}, \frac{1}{2} \right\} \right)^2 \right),
\]

where we used the fact that \( \log(1.2x) + 0.14 = \log(1.2 \exp(0.14)x) < \log(1.4x) \). The last inequality together with (3.18) we get

\[
\frac{3.11}{\alpha^l} > \exp \left( -97.4 \cdot k^3 \log k \left( \max \left\{ \log(1.4x), \frac{21}{k}, \frac{1}{2} \right\} \right)^2 \right)
\]

yielding

\[
l \log \alpha - \log 3.11 < 97.4 \cdot k^3 \log k \left( \max \left\{ \log(1.4x), \frac{21}{k}, \frac{1}{2} \right\} \right)^2.
\]

(3.19)

Since \( \log(1.4x) > 1.02 \), the maximum in the right-hand side cannot be \( \frac{1}{2} \). Let the maximum in the right-hand side of (3.19) be \( 21/k \), then we get

\[
l < 174.1 \cdot k^3 \log k \left( \frac{21}{k} \right)^2
\]

yielding

\[
l < 7.7 \cdot 10^4k \log k.
\]

If \( l = x \), then

\[
x < 7.7 \cdot 10^4k \log k.
\]

(3.20)
If \( l = \frac{m-2}{2} \), then
\[
\frac{m - 2}{2} < 7.7 \cdot 10^4 k \log k
\]
thus,
\[
m < 1.55 \cdot 10^5 k \log k, \quad (3.21)
\]
where we used that \( m - 2 > \frac{m}{1.002} \) for \( m \geq 1458 \).

Now, assume that the maximum in the right-hand side of (3.19) is \( \log(1.4x) \) then,
\[
l < 174.1 \cdot k^3 \log k (\log(1.4x))^2
\]
yielding
\[
l < 3.5 \cdot 10^2 k^3 \log k (\log x)^2.
\]
For the above inequality, we used that \( \log(1.4x) < 1.4 \log x \) holds for \( x \geq 2 \). If \( l = x \), then
\[
x < 3.5 \cdot 10^2 k^3 \log k (\log x)^2
\]
thus,
\[
\frac{x}{(\log x)^2} < 3.5 \cdot 10^2 k^3 \log k.
\]
Now we apply Lemma 2.3 with \( r := 2 \) and \( T := 3.5 \cdot 10^2 k^3 \log k \) to get
\[
x < 1.4 \cdot 10^3 k^3 \log k (3 \log k + \log \log k + \log(3.5 \cdot 10^2))^2
\]
thus,
\[
x < 1.01 \cdot 10^5 k^3 (\log k)^3, \quad (3.22)
\]
where we used that \( (3 \log k + \log \log k + \log(3.5 \cdot 10^2))^2 < 72 (\log k)^2 \).

If \( l = \frac{m-2}{2} \), then we use (3.12) to get
\[
\frac{m - 2}{2} < 3.5 \cdot 10^2 k^3 \log k (\log x)^2
\]
\[
< 3.5 \cdot 10^2 k^3 \log k (\log(7.1 \cdot 10^{13} m^5 (\log m)^3))^2
\]
\[
< 3.5 \cdot 10^2 k^3 \log k (\log(7.1 \cdot 10^{13}) + 5 \log m + 3 \log \log m)^2
\]
thus,
\[
\frac{m}{(\log m)^2} < 7.72 \cdot 10^4 k^3 \log k
\]
where we used that \( (\log(7.1 \cdot 10^{13}) + 5 \log m + 3 \log \log m)^2 < 1.1 \cdot 10^2 (\log m)^2 \) and \( m - 2 > m/1.002 \) for \( m \geq 1458 \).

Using Lemma 2.3 again we get an upper bound for \( m \) in terms of \( k \)
\[
m < 5.6 \cdot 10^7 k^3 (\log k)^3, \quad (3.23)
\]
where we used that \( (\log(7.7 \cdot 10^{14}) + 3 \log k + \log \log k)^2 < 1.8 \cdot 10^2 (\log k)^2 \) and \( m - 2 > m/1.002 \) for \( m \geq 1458 \).

Comparing inequalities (3.20) with (3.22) and (3.21) with (3.23), respectively, we conclude that (3.22) and (3.23) always hold.

Now by (3.23) combined with (3.12), we get
\[
x < 1.85 \cdot 10^{56} k^{15} (\log k)^{18} \quad (3.24)
\]
where we used that
\[(\log(5.6 \cdot 10^7) + 3 \log k + 3 \log \log k)^3 < 4.7 \cdot 10^3(\log k)^3 \text{ and } m - 2 > m/1.002 \text{ for } m \geq 1458 .\]
Finally, comparing (3.22) with (3.24), we conclude that (3.24) always hold and gives us an upper bound for \(x\) in terms of \(k\). Since \(k \leq \log x\), we have
\[x < 1.85 \cdot 10^{56}(\log x)^{15}(\log \log x)^{18}\]
which is true only for \(x < 2.27 \cdot 10^{105} \), so \(k < \log(2.27 \cdot 10^{105}) < 242\).

The previous bounds are too large, so we need to reduce them by using a criterion due to Legendre. First, we go to (3.18) and using that \(x \geq 20 \) and \(m \geq 1458\), we get the following upper bound
\[|\Lambda_2| < \frac{1}{\alpha^{2x}} + \frac{2.11}{\alpha^{m/2}} < \frac{1}{\alpha^{40}} + \frac{2.11}{\alpha^{150}} < 1.9 \cdot 10^{-10} .\] (3.25)

Let
\[\Gamma_2 := (x - 1) \log(g^{-1}) - (mx + 1 - n) \log \alpha.\]
Equation (3.25) can be rewritten as \(|e^{\Gamma_2 - 1}| < 1.9 \cdot 10^{-10} \). Since \(|e^{\Gamma_2 - 1}| < 1.9 \cdot 10^{-10} \), we get that \(|e^{\Gamma_2}| < 1.9 \cdot 10^{-10} + 1 \). Thus
\[|\Gamma_2| \leq e^{\Gamma_2}|e^{\Gamma_2 - 1}| < (1.9 \cdot 10^{-10} + 1)|\Lambda_2| < (1.9 \cdot 10^{-10} + 1) \left(\frac{1}{\alpha^{2x}} + \frac{2.11}{\alpha^{(m-2)/2}}\right).\]

By replacing \(\Gamma_2\) in the above inequality by it’s formula and dividing through by \((x - 1) \log \alpha\), we obtain
\[\left|\frac{\log(g^{-1})}{\log \alpha} - \frac{mx + 1 - n}{x - 1}\right| < \frac{1.9 \cdot 10^{-10} + 1}{(x - 1) \log \alpha} \left(\frac{1}{\alpha^{2x}} + \frac{2.11}{\alpha^{(m-2)/2}}\right).\] (3.26)

Since \(m \geq 1458\), we have \(\alpha^{m/2} > (\frac{7}{4})^{728} > 2.32 \cdot 10^{71}x\). Assume that \(x > 150\). Then \(\alpha^{2x} > 2.32 \cdot 10^{71}x\). Now (3.26) becomes
\[\left|\frac{\log(g^{-1})}{\log \alpha} - \frac{mx + 1 - n}{x - 1}\right| < \frac{1}{4.14 \cdot 10^{109}(x - 1)^2}.\] (3.27)

By the Lemma (2.7), we infer that \((mx + 1 - n)/(x - 1)\) is a convergent of the continued fraction of \(\beta_k = (\log(g^{-1}))/\log \alpha = [a_0^{(k)}, a_1^{(k)}, \ldots]\) and \(p_t^{(k)}/q_t^{(k)}\) its \(t\)th convergent. Thus, \((mx + 1 - n)/(x - 1) = p_t^{(k)}/q_t^{(k)}\) for some \(t_k\). Therefore \(q_t^{(k)}|x - 1\), and so \(x - 1 \geq q_t^{(k)}\). On the other hand, with the help of Mathematica, we get that
\[\min q_{230}^{(k)} > 3.88 \cdot 10^{109} > 2.27 \cdot 10^{105} > x - 1,\]
therefore \(1 \leq t_k \leq 230\), for all \(3 \leq k \leq 242\). Using Mathematica, we observe that \(a_{t_k} + 1 \leq \max\{a_t^{(k)}\} < 4.09 \cdot 10^{70}\), for \(k \in \{3, \ldots, 242\}\) and \(t \in \{1, \ldots, 231\}\). From the properties of continued fractions, we have
\[\left|\beta_k - \frac{mx + 1 - n}{x - 1}\right| = \left|\beta_k - \frac{p_t^{(k)}}{q_t^{(k)}}\right| > \frac{1}{4.09 \cdot 10^{70}(x - 1)^2}\]
which contradicts (3.27). So, \(x \leq 150\) and \(k \in \{3, 4, 5\}\).
Now divide \((1 + \alpha^{-2x})\) in \((3.17)\), we get
\[
\left| \alpha^{n-(mx+1)g^{1-x}(1+\alpha^{-2x})^{-1}} - 1 \right| < \frac{2.11}{\alpha^{(m-2)/2}}.
\] (3.28)

Next, we put \(t := mx + 1 - n\). Using \((3.25)\), we obtain
\[
g^{1-x} \alpha^{-t} - 1 < 1.9 \cdot 10^{-10},
\]
yielding
\[
t > \frac{(x - 1) \log g^{-1}}{\log \alpha} - \frac{\log(1 + 1.9 \cdot 10^{-10})}{\log \alpha} > 0.68x - 0.69
\]
and
\[
g^{1-x} \alpha^{-t} - 1 > -1.9 \cdot 10^{-10},
\]
yielding
\[
t < \frac{(x - 1) \log g^{-1}}{\log \alpha} - \frac{\log(1 - 1.9 \cdot 10^{-10})}{\log \alpha} < 1.27x - 1.26.
\]

Therefore, \(t \in ([0.68x - 0.69], [1.27x - 1.26])\). After performing a calculation for the range \(20 \leq x \leq 150, 3 \leq k \leq 5\) and \(0.68x - 0.69 < t < 1.27x - 1.26\), we get
\[
\min \left\{ \left| \frac{\alpha^{-t}g^{1-x}}{(1 + \alpha^{-x})} - 1 \right| \right\} > 0.0003,
\]
which implies
\[
\frac{2.11}{\alpha^{(m-2)/2}} > 0.0003 \Rightarrow m \leq 33
\]
and that contradicts the fact \(m \geq 1458\). Hence, the result is proved.

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