Combinatorics

A solution to one of Knuth’s permutation problems

Une solution d’un problème de permutation de Knuth

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1. Introduction

In a recent talk [4], D. Knuth posed the following problem. Consider the $n$-dimensional box $B = [0, W_1] \times \cdots \times [0, W_n]$, where $W_1 < W_2 < \cdots < W_n$. If $\pi$ is a permutation in $S_n$, the symmetric group on $n$ letters, define the region

$$C_\pi = \{x \in B \mid x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(n)}\}.$$

In other words, we dissect $B$ by cutting it along the planes $x_i = x_j$, for $1 \leq i < j \leq n$. Each $C_\pi$ is a piece of this dissection. Let us view the volume of $C_\pi$ as a polynomial in the $W_i$. How many distinct volumes are there amongst the $C_\pi$, and which $C_\pi$ have the same volume?

See Fig. 1 for the case $n = 3$, in which $C_{132}$ and $C_{123}$ have the same volume and all others have distinct volumes. The left-hand image shows the original problem; the other two images show $B$ being dissected further along the planes $x_i = W_j$, so that the volumes may be more easily computed.

Definition 1.1. Let $\mathcal{P}$ denote the set of all partitions. Let $\pi$ be a permutation with matrix $[a_{ij}]$ acting on the right. Define $\psi : S_n \to \mathcal{P}$ to be the map which sends $\pi$ to the partition whose Young diagram is

$$\{(i', j') : a_{ij} = 0 \text{ for all } i \leq i' \text{ and } j \leq j'\}.$$
Fig. 1. Knuth’s problem in dimension 3.

Fig. 2. Three of the constructions described in this article, applied to the permutation $\pi = 42531$. From left to right: $\lambda_{\max}(\pi) = (4, 2, 2, 2, 1)$, $\psi(\pi) = (3, 1, 1, 1)$, and the diagram of $\pi$.

In other words, we cross out all matrix entries which lie weakly below and/or to the right of every one in the permutation matrix for $\pi$ (see Fig. 2, center image). The entries which are not crossed out form the Young diagram of $\psi(\pi)$. Note that our permutation matrices always act on the right.

**Theorem 1.2.** If $\pi$ and $\sigma$ are permutations, then $\text{Vol}(C_\pi) = \text{Vol}(C_\sigma)$ if and only if $\psi(\pi) = \psi(\sigma)$.

We defer the proof of this theorem to the end of the paper. However, there is an immediate corollary, if we appeal to a few results in the literature:

**Corollary 1.3.** The number of distinct elements of the set \{Vol($C_\pi$): $\pi \in S_n$\} is $C_n = \frac{1}{n+1} \binom{2n}{n}$, the $n$th Catalan number.

**Proof.** Observe that $\psi(\pi)$ is closely related to a well-known construction, namely that of the diagram of the permutation $\pi$. To construct the diagram of $\pi$, one crosses out all entries directly below and directly to the right of each of the ones in the matrix for $\pi$. The result need not be a Young diagram (see Fig. 2, right image). As observed by Reifegerste [5], this procedure yields a Young diagram (and hence coincides with our $\psi(\pi)$) precisely when $\pi$ is 132-avoiding. In other words, our $\psi$ map yields precisely the rank-zero piece of Fulton’s essential set [3,1]; the entire essential set has rank zero precisely when $\pi$ is 132-avoiding. Alternatively, one can see directly that boundary of the Young diagram for $\psi(\pi)$ is always a Dyck path [2]. Both 132-avoiding permutations and Dyck paths are enumerated by the Catalan numbers. $\square$

We do not know of a good reason why this problem, or our solution, should have anything to do with combinatorial representation theory; the map $\psi$ as defined above arises naturally in our solution.

We note that Knuth’s original setting of the problem [4] is slightly different. Namely, fix weights $W_1 < \cdots < W_n$, and let $X_1, \ldots, X_n$ be uniform random variables on $[0,1]$. We rank the quantities $x_i = W_i X_i$ from smallest to largest. If $\pi$ is a permutation on $n$ letters, define the event $E_\pi$: $x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(n)}$. Knuth observed that when $n \geq 3$, certain of these events $E_\pi$ occur with the same probability regardless of the choice of $W_i$. Theorem 1.2 now classifies the events $E(\pi)$ which occur with the same probability.

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2. A refinement of the dissection

We will proceed in the manner suggested in Fig. 1: we subdivide the box $B$ further, along the hyperplanes $x_i = W_j$. Once this is done, all pieces have very simple shapes, and are easily understood.

**Definition 2.1.** Let $W_0 = 0$, and define

$$a_i = W_i - W_{i-1} > 0 \quad \text{for } 1 \leq i \leq n,$$

$$B = \{1, 2, \ldots, n\}^n,$$

$$I = \{1\} \times \{1, 2\} \times \{1, 2, 3\} \times \cdots \times \{1, 2, 3, \ldots, n\} \subseteq B$$
and for \( \rho = (\rho_1, \ldots, \rho_n) \in B \), we define the open box

\[
B_\rho = (W_{\rho_1-1}, W_{\rho_1}) \times (W_{\rho_2-1}, W_{\rho_2}) \times \cdots \times (W_{\rho_n-1}, W_{\rho_n}).
\]

Note that the dimensions of \( B_\rho \) are \( a_{\rho_1} \times a_{\rho_2} \times \cdots \times a_{\rho_n} \). Observe that those boxes \( B_\rho \) for which \( \rho \in I \) lie within \( B \), and indeed partition \( B \) up to a set of volume zero (namely, the boundaries of the boxes). Also, note that if \( \rho \in B \) and \( \rho_i = \rho_j \) for some \( i \neq j \), then \( B_\rho \) is symmetric about the hyperplane \( \{ x_i = x_j \} \), whereas if \( \rho_i < \rho_j \), then the hyperplane \( \{ x_i = x_j \} \) does not intersect \( B_\rho \) at all.

The motivation for all of these definitions is to simplify the computation of the volumes of the \( C_\pi \). We begin with the following immediate observation:

**Lemma 2.2.** For any \( \rho \in B \) and any \( \pi \in S_n \), \( \rho(\pi(1)) \geq \rho(\pi(2)) \geq \cdots \geq \rho(\pi(n)) \) if and only if all points \( x \in B_\rho \) satisfy \( x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(n)} \).

The symmetric group \( S_n \) acts on \( B \) by permuting coordinates. Each box \( B_\rho \) has a stabilizer \( G_\rho \subseteq S_n \) under this action. In fact, \( G_\rho \) is isomorphic to a product of symmetric groups

\[
G_\rho \cong S_{n_1} \times \cdots \times S_{n_k}
\]

where \( n_j \) is the number of occurrences of the number \( j \) in \( \rho \). Observe that \( G_\rho \) also acts faithfully on \( B_\rho \) by permuting coordinates, and so partitions \( B_\rho \) into \(|G_\rho| \) equal-volume fundamental domains. We thus have the following volume computation:

**Lemma 3.1.** \( C_\pi \) meets \( B_\rho \) if and only if \( \lambda(i) = \pi(i) \)

**Proof.** By Lemma 2.2, \( C_\pi \) meets \( B_\rho \) if and only if \( i \in I \) and \( \rho(\pi(1)) \geq \cdots \geq \rho(\pi(n)) \). Now, \( \rho \in I \iff \rho_i \leq i \iff \lambda_i \leq \pi(i) \); similarly, \( \rho(\pi(1)) \geq \cdots \geq \rho(\pi(n)) \) is equivalent to \( \lambda_1 \geq \cdots \geq \lambda_n \). \( \square \)

Recall that the set of integer partitions forms a distributive lattice, Young’s lattice, under the partial order of inclusion of Young diagrams. See, for example, [6, Section 7.2] for an introduction to Young’s lattice.

**Definition 3.2.** Let \( \lambda_{\max}(\pi) = \bigcup \{ \mu \in \mathcal{P} : \mu \text{ is a partition with } n \text{ parts and } \mu_i \leq \pi(i) \} \), where \( \bigcup \) denotes union of Young diagrams (the least upper bound in Young’s lattice).

**Lemma 3.3.** \( C_\pi \) meets \( B_\rho \) if and only if \( \lambda(i) = \pi(i) \) and \( \lambda \subseteq \lambda_{\max}(\pi) \)

**Proof.** It is easy to check that if \( \lambda \) and \( \mu \) are partitions which meet the condition of Lemma 3.1, then so is \( \lambda \cup \mu \) (their union as Young diagrams). Moreover, if \( \nu \subseteq \lambda \), then \( \nu \) meets the conditions of Lemma 3.1. As such, the condition of Lemma 3.1 is equivalent to \( \lambda \subseteq \lambda_{\max}(\pi) \). \( \square \)

**Proof of Theorem 1.2.** If \( \lambda \) is a partition, write \( \rho(\lambda) = (\lambda_{\pi^{-1}(1)}, \ldots, \lambda_{\pi^{-1}(n)}) \). Taking \( \rho = \rho(\lambda) \) and applying Lemmas 2.3 and 3.3, we see that

\[
\text{Vol}(C_\pi) = \sum_{\lambda \subseteq \lambda_{\max}(\pi)} \frac{1}{|G_\rho(\lambda)|} \prod_i a_{\lambda_i} = \sum_{\lambda \subseteq \lambda_{\max}(\pi)} \frac{1}{|G_\rho(\lambda)|} \prod_i a_{\lambda_i}.
\]

The latter equality holds because \( G_\rho \) is isomorphic to \( G_{\sigma,\rho} \) for any permutation \( \sigma \in S_n \). As such, \( \text{Vol}(C_\pi) = \text{Vol}(C_\pi') \) if and only if \( \lambda_{\max}(\pi) = \lambda_{\max}(\pi') \).

Next, we need a concrete description of \( \lambda_{\max}(\pi) \). Let \( \lambda \) be a partition such that \( \lambda_i \leq \pi(i) \). In particular,
\(\lambda_1 \leq \pi(1),\)
\(\lambda_2 \leq \min\{\lambda_1, \pi(2)\} \leq \min\{\pi(1), \pi(2)\},\)
\[\vdots\]
\(\lambda_n \leq \min\{\lambda_{n-1}, \pi(n)\} \leq \min\{\pi(1), \ldots, \pi(n)\}.\)

Now, \(\lambda\) is maximal in Young’s lattice if we replace all of the above inequalities with equalities. Therefore, \(\lambda^\text{max}_i = \min\{\pi(1), \ldots, \pi(i)\}.\)

Recalling Definition 1.1, we now compare \(\lambda^\text{max}(\pi)\) to \(\psi(\pi)\). Observe that the permutation matrix for \(\pi\) has ones in positions \((i, \pi(i))\) and zeros elsewhere, so the \(i\)th part of \(\psi(\pi)\) is \(\min\{\pi(1), \pi(2), \ldots, \pi(i)\} - 1\). In other words, one obtains \(\psi(\pi)\) by deleting the first column of the Young diagram of \(\lambda^\text{max}\); this column is necessarily of height \(n\), so one can also reconstruct \(\lambda^\text{max}(\pi)\) given \(\psi(\pi)\) (see Fig. 2, left and center images). We conclude that if \(\pi, \pi'\) are permutations in \(S_n\), then

\[
\text{Vol}(C_\pi) = \text{Vol}(C_{\pi'}) \iff \lambda^\text{max}(\pi) = \lambda^\text{max}(\pi') \iff \psi(\pi) = \psi(\pi'). \quad \square
\]

References

[1] Kimmo Eriksson, Svante Linusson, Combinatorics of Fulton’s essential set, Duke Mathematical Journal 85 (1) (1996) 61–76.
[2] Markus Fulmek, Enumeration of permutations containing a prescribed number of occurrences of a pattern of length 3, Advances in Applied Mathematics 30 (4) (2003) 607–632, arXiv:math/0112092v3.
[3] W. Fulton, Flags, Schubert polynomials, degeneracy loci, and determinantal formulas, Duke Math. J 65 (3) (1992) 381–420.
[4] Donald E. Knuth, Permutation problems, Talk (1056-05-168) at the Joint Meetings of the AMS-CMS, San Francisco, 2010.
[5] Astrid Reifegerste, On the diagram of 132-avoiding permutations, European Journal of Combinatorics 24 (6) (2003) 759–776, arXiv:math/0208006v3.
[6] R.P. Stanley, Enumerative Combinatorics, Univ. Press, Cambridge, 2001.