Secrecy Capacity under Limited Discussion Rate for Minimally Connected Hypergraphical Sources

Qiaqiao Zhou and Chung Chan

Abstract—We investigate the secret key generation in the multiterminal source model, where the users discuss under limited rate. For the minimally connected hypergraphical sources, we give an explicit formula of the maximum achievable secret key rate, called the secrecy capacity, under any given total discussion rate. Besides, we also partially characterize the region of achievable secret key rate and discussion rate tuple. When specializes to the hypertree sources, our results give rise to a complete characterization of the region.

I. INTRODUCTION

We study the secret key generation problem among multiple users [1], in which each user observes a component of some correlated discrete sources. The users discuss over a noiseless public channel, possibly interactively in several rounds, to agree on a common secret key that is independent of their discussion. While the maximum achievable secret key rate with unfettered discussion was characterized in [1], it remains open when the discussion has limited rate.

The secret key generation under limited discussion rate was first studied by Csiszár and Narayan for discrete sources in the two-user case with a helper [2]. For the one-way discussion, they characterized the optimal trade-off between the secret key rate and discussion rate, which was subsequently extended to gaussian sources in [3, 4]. The minimum rate of interactive discussion required to generate a secret key of maximum rate was examined by [5, 6] in the two-user case and [7] in the multiterminal case. In [8], the optimal trade-off between the secret key rate and discussion rate tuple was characterized for a variant of the multiterminal source model. In [9], a hypergraphical source model [10] was considered, and each user observes one realization of the source. They determined the minimum amount of discussion needed to generate a secret key of a given size when the discussion is restricted to be linear function of the source. In [11], the optimal trade-off between the secret key rate and total discussion rate was characterized for the pairwise independent network (PIN) model proposed in [12, 13]. Chan et al. [11] also gave an outer bound on the region of achievable secret key rate and discussion rate tuple. The bound was shown to be tight for PIN model on a tree, but remains unknown whether it is tight for other sources. Besides, although the expression is single-letter, it may take exponential time to compute the bound since you have to evaluate all the subsets.

In this paper, we show that the bound is also tight for the minimally connected hypergraphical (MCH) sources [10], generalizing the result of [11]. We obtain a partial characterization of the region of achievable secret key rate and discussion rate tuple. Our result is stronger than the existing result for its being more explicit and can be computed more efficiently. In particular, for the hypertree sources, the region can be completely characterized. More importantly, for the MCH sources, we obtain an explicit formula of the maximum achievable secret key rate, called the secrecy capacity, under any given total discussion rate. The main property in deriving the results is the alternative characterization of two special classes of hypergraph, which is established via the notion of path and partition connectivity of hypergraph.

II. PROBLEM FORMULATION

Let $V := \{|V|\} := \{1, 2, \ldots, |V|\}$ be a finite set of $|V| \geq 2$ users. The users have access to a correlated discrete memoryless multiple source $Z_V := (Z_i \mid i \in V)$ taking values from a finite set $Z_V := \prod_{i \in V} Z_i$. N.b., we use the sans serif font for random variable and the normal font for its corresponding alphabet set. Each user $i \in V$ observes an $n$ i.i.d. samples $Z_i^n := (Z_{it} \mid t \in [n])$ of the source $Z_i$. Then, each user $i \in V$ generates a random variable $U_i$ independent of other sources, i.e., $H(U_i|Z_V) = \sum_{i \in V} H(U_i)$. Following these observations, the users are allowed to discuss interactively in ascending order of user indices over a noiseless channel for $\ell$ number of rounds. More specifically, at round $t \in [\ell]$, each user $i \in V$ reveals a message that is a function of its accumulated observations, $F_{it} := f_{it} (U_i, Z_i^n, F_{i[t-1]}^{-1}, F_V^{-1})$, where $F_{i[t]} := (F_{jt} \mid j < i)$ denotes all the previous messages in the same round, and $F_V^{-1} := (F_{ir} \mid i \in V, \tau < t)$ denotes all the messages in the previous rounds. We will write $F_i := (F_{it} \mid t \in [\ell])$ and $F := (F_i \mid i \in V)$ to denote, respectively, the collection of messages from user $i \in V$ and all users. After the public discussion, each user $i \in V$ then try to extract a common secret key $K$ from its accumulated observations. The secret key is required to satisfy

$$\lim_{n \to \infty} \Pr(\exists i \in V, K \neq \theta_i (U_i, Z_i^n, F)) = 0,$$

(2.1)

$$\lim_{n \to \infty} \frac{1}{n} \log |K| - H(K|F) = 0,$$

(2.2)

for some function $\theta_i$ for $i \in V$.  

Q. Zhou (email: zq115@ie.cuhk.edu.hk) is with the Institute of Network Coding and the Department of Information Engineering, The Chinese University of Hong Kong.

C. Chan (email: chung.chan@cityu.edu.hk) is with the Department of Computer Science, City University of Hong Kong.
We say a secret key rate \( r_K \) and discussion rate tuple \( r_V := (r_i | i \in V) \) is achievable if there exists a sequence \((U,V,F,K)\) satisfying

\[
\begin{align*}
\frac{1}{n} \log |K|, \text{ and } r_i \leq \liminf_{n \to \infty} \frac{1}{n} \log |F_i|,
\end{align*}
\]

in addition to (2.1) and (2.2). Then, the achievable rate region \( \mathcal{R} \) is defined as

\[
\mathcal{R} := \{(r_K, r_V) | (r_K, r_V) \text{ is achievable}\}. \tag{2.4}
\]

The secrecy capacity under a total public discussion rate \( R \geq 0 \) is defined as

\[
C_S(R) := \max \{ r_K | (r_K, r_V) \in \mathcal{R}, r(V) \leq R \}, \tag{2.5}
\]

where \( r(B) := \sum_{i \in B} r_i, \forall B \subseteq V \) for notational convenience. The unconstrained secrecy capacity \( C_S \) characterized in [1] is defined as

\[
C_S := \lim_{R \to \infty} C_S(R) \tag{2.6}
\]

The communication complexity \( R_S \) [7] is the minimum total discussion rate required to achieve the unconstrained secrecy capacity, namely,

\[
R_S := \min \{ R \geq 0 | C_S(R) = C_S \} \tag{2.7}
\]

Characterizing \( C_S(R) \) or even \( R_S \) for the general multiterminal source model appears intractable [7], let alone \( \mathcal{R} \). To simplify the problem, the following hypergraphical source model, which generalizes the PIN model [12, 13], has been considered in [7, 9–11, 14].

**Definition 2.1 ([10])** \( Z_V \) is a hypergraphical source if there is a hypergraph \( \mathcal{H} = (V,E,\xi) \) with an edge function \( \xi : E \to 2^V \setminus \{\emptyset\} \) and some independent edge random variables \( X_e \) for \( e \in E \) with \( H(X_e) > 0 \), such that

\[
Z_i := (X_e | e \in E, i \in \xi(e)) \quad \forall i \in V. \tag{2.8}
\]

The weight function \( w : E \to \mathbb{R} \) of the hypergraph is defined as

\[
w(e) := H(X_e), \quad \forall e \in E. \tag{2.9}
\]

A hypergraph \( \mathcal{H} \) is minimally connected if \( \mathcal{H} \) becomes disconnected after removing any edge. We will further simplify the problem by restricting ourselves to the following special hypergraphical source model.

**Definition 2.2** \( Z_V \) is a minimally connected hypergraphical (MCH) source if it is a hypergraphical source and the corresponding hypergraph is minimally connected. \( \square \)

Our goal is to characterize or bound \( C_S(R) \) and \( \mathcal{R} \) for the above MCH source model.
Similarly, we use $\mathcal{H} \setminus C$ to denote a subhypergraph obtained from $\mathcal{H}$ by removing the vertex in $C$ from $V$ and $E$, and then discarding the empty edges. More precisely,

**Definition 3.3** $\mathcal{H} \setminus C$ is a hypergraph where the set of vertices is $V(\mathcal{H} \setminus C) = V(\mathcal{H}) \setminus C$, the set of edges is $E(\mathcal{H} \setminus C) = \{e \setminus C \mid e \in E(\mathcal{H}), \xi(e) \cap C \neq \emptyset\}$, and the edge function is $\xi_{\mathcal{H} \setminus C}(e) = \xi(e) \setminus C$, for $e \in E(\mathcal{H})$. \hfill \Box

Let $\Pi(V)$ be the set of partitions of $V$ into non-empty disjoint subsets. For $P \in \Pi(V)$, $\mathcal{H}[P]$ is a hypergraph obtained from $\mathcal{H}$ by agglomerating the vertices and edges with reference to (w.r.t.) $P$ in the following.

**Definition 3.4** $\mathcal{H}[P]$ is a hypergraph where the set of vertices is $V(\mathcal{H}[P]) = P$, the set of edges is $E(\mathcal{H}[P]) = \{\{ \cup_{C \in P, C \subset \xi(e) \neq \emptyset} e \} \mid e \in E(\mathcal{H})\}$, and the edge function is $\xi_{\mathcal{H}[P]}(e) = \{C \mid C \in P, \xi(e) \cap C \neq \emptyset\}$, for $e \in E(\mathcal{H})$. \hfill \Box

A simple example that illustrates the above operations is as follows:

**Example 3.2** Consider the hypergraph $\mathcal{H}$ in Fig. 1. Let $C = \{1, 2, 3\}$ and $P = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$. $\mathcal{H}_C$ is a hypergraph with $V(\mathcal{H}_C) = \{1, 2, 3\}$ and $E(\mathcal{H}_C) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. $\mathcal{H} \setminus C$ is a hypergraph with $V(\mathcal{H} \setminus C) = \{4, 5, 6\}$ and $E(\mathcal{H} \setminus C) = \{\{4\}, \{5\}, \{6\}\}$. $\mathcal{H}[P]$ is a hypergraph with $V(\mathcal{H}[P]) = \{\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 6\}\}$. These hypergraphs are visualised in Fig. 2. \hfill \Box

The last graph-theoretic notion we shall introduce is the partition connectivity of a hypergraph. Let $\Pi'(V)$ denote the set of all partitions of $V$ into at least two non-empty disjoint subsets, i.e.,

$$\Pi'(V) = \{P \in \Pi(V) \mid |P| > 1\} = \Pi(V) \setminus \{\{V\}\}.$$  

With $|V(\mathcal{H})| \geq 2$, (which will be assume hereafter), the *partition connectivity* of a hypergraph $\mathcal{H}$ is defined as

$$I(\mathcal{H}) := \min_{P \in \Pi'(V)} \frac{1}{|P| - 1} |E_P(\mathcal{H})|,$$  

where

$$E_P(\mathcal{H}) := \sum_{C \in P} \{|e \mid \xi_{\mathcal{H}}(e) \cap C \neq \emptyset\} - |E(\mathcal{H})|.$$  

which corresponds to the number of edges that cross the partition $P$. The partition connectivity we defined here is the multivariate mutual information [15] that specialises to the hypergraph without considering the weight of edges. It was pointed out in [15] that the set of optimal solutions to (3.3a) forms a lattice w.r.t. the partial order “$\leq$” on partitions defined below. We say a partition $P$ is finer than another partition $P'$, denoted as $P \preceq P'$, iff

$$\forall C \in P, \exists C' \in P' : C \subseteq C'.$$  

In other words, $P$ can be obtained from $P'$ by further partitioning some parts of $P'$. Hence, there is a unique finest optimal partition, denoted by $P^*(\mathcal{H})$ and referred to as the *fundamental partition*. Note that both $I(\mathcal{H})$ and $P^*(\mathcal{H})$ can be computed in strongly polynomial time [15]. The fundamental partition has various properties and operational meanings. In particular, we will rely on the following property, which has an elegant interpretation in data clustering.

**Proposition 3.1** ([15, Lemma 5.1 and Theorem 5.2]) The fundamental partition $P^*(\mathcal{H})$ of a hypergraph $\mathcal{H}$ satisfies

$$P^*(\mathcal{H}) \setminus \{\{i\} \mid i \in V\} = \text{maximal}\{C \subseteq V \mid \xi(e) \cap C \neq \emptyset\},$$  

$$I(\mathcal{H}_C) > I(\mathcal{H})$$  

where maximal $\mathcal{F}$ denotes the collection of inclusion-wise maximal sets in a set family $\mathcal{F}$, i.e., maximal $\mathcal{F} := \{B \in \mathcal{F} \mid \exists B' \supseteq B, B' \in \mathcal{F}\}$. \hfill \Box

**IV. MAIN RESULTS**

Our main result is an explicit formula of $C_S(\mathcal{R})$ and a partial characterization of $\mathcal{R}$ for the MCH sources.

**Theorem 4.1** For a MCH source $Z_V$ (see Definition 2.2), we have $(r_K, r_V) \in \mathcal{R}$ only if

$$r(C) \geq \deg_{\mathcal{H}}(C) - 1 \times r_K, \quad \forall C \in P^*(\mathcal{H}),$$  

where $\deg_{\mathcal{H}}(C)$ (3.1b) is the degree of vertices $C$ in $\mathcal{H}$ and $P^*(\mathcal{H})$ is the fundamental partition. Moreover, for all $C \in P^*(\mathcal{H})$, the above are simultaneously achievable for any secret key rate below the unconstrained secrecy capacity

$$C_S = \min_{e \in E(\mathcal{H})} w(e).$$
In particular, this gives the optimal trade-off

\[ C_S(R) = \min \left\{ \frac{R}{|E(H)| - 1}, C_S \right\} \quad (4.3) \]

and so \( R_S = (|E(H)| - 1)C_S \).

**Proof** See Appendix.

Our result is stronger than existing result:

1) Eq. (4.1) gives the exact minimum total discussion rate of users in each \( C \in P^+(H) \). However, for the discussion rate tuple obtained in [11], it remains unknown whether it is tight.

2) We obtain an explicit formula of \( C_S(R) \), which is not covered by any existing results.

**Example 4.1** Let \( V = \{1, 2, 3, 4, 5, 6\} \) and

\[
\begin{align*}
Z_1 &:= (X_a, X_c), \\
Z_2 &:= (X_a, X_b), \\
Z_3 &:= (X_b, X_c), \\
Z_4 &:= X_a, \\
Z_5 &:= X_b, \\
Z_6 &:= X_c,
\end{align*}
\]

where \( X_i \)'s are independent with \( H(X_a) = 1, H(X_b) = 1.5, \) and \( H(X_c) = 2 \). It is a MCH source where the corresponding hypergraph is \( H \) in Fig. 1 with weight \( w(\{1, 2, 4\}) = H(X_a) = 1, w(\{2, 3, 5\}) = H(X_b) = 1.5, \) and \( w(\{1, 3, 6\}) = H(X_c) = 2 \). It can be show that \( P^+(H) = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6\}\} \). By (4.1) and (4.2), we have \( (r_K, (r_1, r_2, r_3, r_4)) = (0, 0, 0, 0) \in \mathcal{R} \). Finally, we have \( C_S(R) = \min \left\{ \frac{R}{2}, 1 \right\} \) and so \( R_S = 2 \).

Applying the above result to a hypergraphical source \( Z_V \), where the corresponding hypergraph is a hypertree, gives a complete characterization of \( \mathcal{R} \).

**Corollary 4.1** For a hypergraphical source \( Z_V \) w.r.t. a hypertree \( H \) with weight \( w \), we have

\[
\begin{align*}
\mathcal{R} &= \{(r_K, r_V) \mid r_K \in [0, C_S], r_i \geq (\deg_H(i) - 1) r_K, i \in V\}, \\
C_S &= \min_{e \in E(H)} w(e),
\end{align*}
\]

and \( \deg_H(i) \) (3.1a) is the degree of node \( i \) in hypertree \( H \).

**Proof** See Appendix.

In [11], the region \( \mathcal{R} \) has been completely characterized for PIN model on a tree. Since hypertree contains tree as a special case, our result generalizes [11].

**Example 4.2** Let \( V = \{1, 2, 3, 4, 5\} \) and

\[
\begin{align*}
Z_1 &:= (X_a, X_c), \\
Z_2 &:= X_a, \\
Z_3 &:= (X_a, X_b), \\
Z_4 &:= X_b, \\
Z_5 &:= X_a, \\
Z_6 &:= X_c,
\end{align*}
\]

where \( X_i \)'s are independent with \( H(X_a) = 1.5, H(X_b) = H(X_c) = 1 \). This is a hypergraphical source w.r.t. the hypertree in Fig. 4 with weight \( w(\{1, 2, 3\}) = H(X_a) = 1.5, w(\{3, 4\}) = H(X_b) = 1, \) and \( w(\{4, 5\}) = H(X_c) = 1 \).

By (4.4), we have

\[
\begin{align*}
&\mathcal{R} = \{(r_K, (r_1, r_2, r_3)) \mid r_K \in [0, 1], r_1 \geq r_K, r_2 \geq 0, r_3 \geq r_K, \}
&\quad \text{ and } r_4 \geq 0, r_5 \geq 0.
\end{align*}
\]

This simple result is not directly covered by any existing results.

To prove Theorem 4.1, we will rely on the following property of the minimally connected hypergraph (MCH). It characterizes the relation of degree, number of connected component, and number of edges in MCH.

**Lemma 4.1** \( H \) is a MCH iff \( \mathcal{C}(H) \) (see Definition 3.4) is a hypertree without singleton edges, that is, loopless. Furthermore, we have

\[
\mathcal{C}(H \setminus C) = \deg_H(C), \quad \forall C \in P^+(H),
\]

and

\[
\sum_{C \in P^+(H)} (\deg_H(C) - 1) = |E(H)| - 1,
\]

where \( \mathcal{C}(H \setminus C) \) is the number of connected components of \( H \) after removing vertices \( C, \) and \( \deg_H(C) \) (3.1b) is the degree of vertices \( C \) in \( H \).

**Proof** See Appendix.

**Example 4.3** Consider the MCH \( H \) in Fig. 1. Recall that \( P^+(H) = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6\}\} \). Then, by Definition 3.4, \( H[P^+(H)] \) is a hypergraph with vertices \( V(H[P^+(H)]) = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6\}\} \) and edges \( E(H[P^+(H)]) = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}\} \). See Fig. 3, \( H[P^+(H)] \) is indeed a loopless hypertree. It can be verified that

\[
\mathcal{C}(H \setminus C) = \deg_H(C) = \begin{cases} 3, & C = \{1, 2, 3\} \\
1, & C = \{i\}, i \in \{4, 5, 6\}\end{cases}
\]

Therefore, we have \( \sum_{C \in P^+(H)} (\deg_H(C) - 1) = 2 \), which equals \( |E(H)| - 1 \).
To prove Corollary 4.1 and Lemma 4.1, we will use the following alternative characterization of the hypertree through the partition connectivity and fundamental partition.

**Lemma 4.2** A hypergraph $\mathcal{H}$ is a hypertree (see Definition 3.1) iff

\[
I(\mathcal{H}) = 1, \quad \text{and} \quad P^*(\mathcal{H}) = \{\{v\} \mid v \in V\}
\]

i.e., singleton partition is the fundamental partition. \hfill \Box

**PROOF** See Appendix.

**Example 4.4** Consider the hypertree in Fig. 4. It can be shown that $I(\mathcal{H}) = 1$ and $P^*(\mathcal{H}) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$, which is exactly (4.7).

\[\square\]

V. CONCLUSION

In this paper, we give an explicit formula of the secrecy capacity under any given total public discussion rate for sources which correspond to minimally connected hypergraph. A key property in the derivation of the result is the alternative characterization of two special classes of hypergraph, which is established through the notion of path and partition connectivity of hypergraph. While we also partially characterize the region of achievable secret key rate and discussion rate tuple, a complete characterization still remains unknown, and will be a interesting future work.

**REFERENCES**

[1] I. Csiszár and P. Narayan, “Secrecy capacities for multiple terminals,” IEEE Trans. Inf. Theory, vol. 50, no. 12, pp. 3047–3061, Dec. 2004.

[2] I. Csiszár and P. Narayan, “Common randomness and secret key generation with a helper,” IEEE Trans. Inf. Theory, vol. 46, no. 2, pp. 344–366, Mar. 2000.

[3] S. Watanabe and Y. Oohama, “Secret key agreement from correlated gaussian sources by rate limited public communication,” IEICE Trans. Fundamentals, vol. E93-A, no. 11, pp. 1976–1983, Nov. 2010.

[4] ——, “Secret key agreement from vector gaussian sources by rate limited public communication,” IEEE Trans. Inf. Forensics Security, vol. 6, no. 3, pp. 541–550, Sep. 2011.

[5] J. Liu, P. Cuff, and S. Verdú, “Secret key generation with limited interaction,” IEEE Trans. Inf. Theory, vol. 63, no. 11, pp. 7358–7381, Nov. 2017.

[6] H. Tyagi, “Common information and secret key capacity,” IEEE Trans. Inf. Theory, vol. 59, no. 9, pp. 5627–5640, Sep. 2013.

[7] M. Mukherjee, N. Kashyap, and Y. Sankarasubramaniam, “On the public communication needed to achieve SK capacity in the multiterminal source model,” IEEE Trans. Inf. Theory, vol. 62, no. 7, pp. 3811–3830, Jul. 2016.

[8] J. Liu, P. Cuff, and S. Verdú, “Secret key generation with one communicator and a strong converse via hypercontractivity,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Hong Kong, Jun. 2015, pp. 710–714.

[9] T. A. Courtade and T. R. Halsford, “Coded cooperative data exchange for a secret key,” IEEE Trans. Inf. Theory, vol. 62, no. 7, pp. 3785–3795, Jul. 2016.

[10] C. Chan and L. Zheng, “Mutual dependence for secret key agreement,” in Proc. 44th Annua. Conf. Inf. Sci. Syst. (CISS), Princeton, NJ, USA, Mar. 2010, pp. 1–6.

[11] C. Chan, M. Mukherjee, N. Kashyap, and Q. Zhou, “Secret key agreement under discussion rate constraints,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Aachen, Germany, Jun. 2017, pp. 1519–1523.

[12] S. Nitinawarat, C. Ye, A. Barg, P. Narayan, and A. Reznik, “Secret key generation for a pairwise independent network model,” IEEE Trans. Inf. Theory, vol. 56, no. 12, pp. 6482–6489, Dec. 2010.

[13] S. Nitinawarat and P. Narayan, “Perfect omniscience, perfect secrecy, and Steiner tree packing,” IEEE Trans. Inf. Theory, vol. 56, no. 12, pp. 6490–6500, Dec. 2010.

[14] M. Mukherjee, C. Chan, N. Kashyap, and Q. Zhou, “Bounds on the communication rate needed to achieve SK capacity in the hypergraphical source model,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Barcelona, Spain, Jul. 2016, pp. 2504–2508.

[15] C. Chan, A. Al-Bashabsheh, J. Ebrahimi, T. Kaced, and T. Liu, “Multivariate mutual information inspired by secret-key agreement,” Proc. IEEE, vol. 103, no. 10, pp. 1883–1913, Oct. 2015.

**APPENDIX A**

**PROOFS**

**Proof (Theorem 4.1)** To prove Theorem 4.1, we will make use of the following outer bound on the region $\mathcal{R}$ in [11].

**Proposition 1.1 (11, Theorem 4.1)** For any $(r_K, r_V) \in \mathcal{R}$, we have

\[
r(V \backslash B) \geq (|P| - 1)[r_K - I_P(Z_B)], \quad \text{where} \quad I_P(Z_B) := \frac{1}{|P| - 1} \left[ \sum_{C \in P} H(Z_C) - H(Z_B) \right] \tag{1.1a}
\]

for any $B \subseteq V$ with size $|B| > 1$ and $P \in \Pi'(B)$. \hfill \Box

Now, for any $C \in P^*(\mathcal{H})$, let $P$ be the set of equivalent classes of hypergraph $\mathcal{H} \setminus C$. It is clear that $P \in \Pi(V \setminus C)$ and $|P| = \mathbb{C}(\mathcal{H} \setminus C)$. Since $\mathbb{C}(\mathcal{H} \setminus C)$ is a positive integer, we have the following two cases and will show that (4.1) holds for both.

Case 1: $\mathbb{C}(\mathcal{H} \setminus C) = |P| = 1$, i.e., $\mathcal{H}$ remains connected after removing vertices $C$. Then, for the R.H.S. of (4.1),

\[
(\deg_{\mathcal{H}}(C) - 1)r_K = \mathbb{C}(\mathcal{H} \setminus C) - 1)r_K = 0
\]

where the first equality follows from (4.5). Therefore, (4.1) holds trivially.

Case 2: $\mathbb{C}(\mathcal{H} \setminus C) = |P| > 1$, i.e., $\mathcal{H}$ will be disconnected after removing vertices $C$. It follows that $P \in \Pi'(V \setminus C)$ and $I_P(Z_{V \setminus C}) = 0$.

Applying the lower bound (1.1) with $B = V \setminus C$ and the partition $P$ of $B$, we have

\[
r(C) \geq (|P| - 1)[r_K - I_P(V \setminus C)]
\]

\[
= \mathbb{C}(\mathcal{H} \setminus C) - 1)r_K
\]

\[
= (\deg_{\mathcal{H}}(C) - 1)r_K,
\]

where the last equality follows from (4.5) in Lemma 4.1. This completes the proof of (4.1).

Next, we consider to prove (4.2). Let $e^*$ be the optimal solution to the R.H.S. of (4.2) and $P$ be the set of equivalent classes of $\mathcal{H}$ after removing edge $e^*$. It follows that $P \in \Pi'(V)$ due to the fact that $\mathcal{H}$ is minimally connected. Then,

\[
C_{S} \leq I_P(Z_V) = w(e^*)
\]
where (a) follows from [15, (4.1) and (4.3)]; (b) is because $e^*$ is the only edge that crosses $P$. Therefore, we have proved the converse “$\leq$” of (4.2). The achievability “$\geq$” of (4.2) can be proved by utilizing the secret key agreement scheme in [10]. More precisely, reducing the weight of every edge $e \in E(\mathcal{H})$ to $w(e^*)$. Since $\mathcal{H}$ is connected, $\exists i \in \xi_{\mathcal{H}}(e^*)$ such that $i \in \xi_{\mathcal{H}}(e)$ for some $e \in E(\mathcal{H}) \setminus \{e^*\}$. User $i$ discusses in public $X_e \otimes X_{e^*}$ which is independent of $X_{e^*}$. Then, all users $j \in \xi_{\mathcal{H}}(e) \setminus \{i\}$ can recover $X_e$ securely. Now we have achieved a common secret key $X_e$ among users in $\xi_{\mathcal{H}}(e^*) \cup \xi_{\mathcal{H}}(e)$. Doing this repeatedly for the remaining edges, we can agree on a common secret key $X_e$ among all users $V$ since $\mathcal{H}$ is connected. This gives the achievability “$\geq$” of (4.2).

Indeed, the above scheme achieves a common secret key $X_e$ of $H(X_{e^*})$ bits by using $(|E(\mathcal{H})| - 1)H(X_{e^*})$ bits of public discussion, which also gives the achievability “$\geq$” of (4.3). For the converse “$\leq$” of (4.3), we have $C_S(R) \leq C_S$ trivially. Then, summing the (4.1) over $C \in P^*(\mathcal{H})$, we have

$$r(V) \geq \sum_{C \in P^*(\mathcal{H})} \deg_{\mathcal{H}}(C) - 1 |r_K|,$$

where the equality follows from (4.6) in Lemma 4.1. Hence, we have $C_S(R) = \frac{R}{|E(\mathcal{H})| - 1}$, which completes the proof of (4.3), and (4.1) is simultaneously achievable for all $C \in P^*(\mathcal{H})$. In particular, when $C_S(R) = C_S$, we have $R_S = (|E(\mathcal{H})| - 1)C_S$ as desired.

**Proof (Corollary 4.1)** For $H$ being a hypertree, we have $P^*(\mathcal{H}) = \{\{i\} | i \in V\}$ by (4.7b) in Lemma 4.2. Then, by (4.1), we have

$$r_i \geq \deg_{\mathcal{H}}(i) - 1 |r_K|, \quad \forall i \in V,$$

which are simultaneously achievable for any key rate $r_K \leq C_S = \min_{e \in E(\mathcal{H})} w(e)$ as suggested by Theorem 4.1. This completes the proof of Corollary 4.1.

**Proof (Lemma 4.1)”if” case: Suppose $\mathcal{H}[P^*(\mathcal{H})]$ is a hypertree without singleton edges. Note that $I(\mathcal{H}C) > I(\mathcal{H})$, $\forall C \in P^*(\mathcal{H}) : |C| > 1$ by Proposition 3.1. This implies $I(\mathcal{H}) = I(\mathcal{H}[P^*(\mathcal{H})])$ by (4.7b) and [15, Corollary 5.3]. By (4.7a), we have $I(\mathcal{H}C) > 1$, therefore, $\mathcal{H}C$ is connected. Since $\mathcal{H}[P^*(\mathcal{H})]$ is a hypertree and therefore connected for its vertices $P^*(\mathcal{H}) \setminus C$, i.e., $C' \sim_{\mathcal{H}} P^*(\mathcal{H})$, $\forall C', C'' \in P^*(\mathcal{H}) \setminus C$. Altogether, we have $\mathcal{H}$ is also connected. Since $\mathcal{H}C$ is loopless, every edge $e \in E(\mathcal{H}[P^*(\mathcal{H})])$ is incident on at least two distinct vertices in $\mathcal{H}[P^*(\mathcal{H})]$, say $C'$ and $C''$. Then, $\exists v' \in C'$, $\exists v'' \in C''$ such that $v', v'' \in \xi_{\mathcal{H}}(e)$ by Definition 3.4 of $H(P^*(\mathcal{H}))$. Hence, $\{v', e, v''\}$ is a path in $\mathcal{H}$. It is unique because $\{C', C''\}$ is a unique path in the hypertree $\mathcal{H}[P^*(\mathcal{H})]$. Removing edge $e$ from $\mathcal{H}$ disconnects $v'$ and $v''$. Therefore, $\mathcal{H}$ is minimally connected.

“only if” case: If $\mathcal{H}$ is minimally connected, then $\mathcal{H}[P^*(\mathcal{H})]$ is connected by Definition 3.4. Let $P$ be the set of equivalent classes (or connected components) of $\mathcal{H}$ after removing any edge $e$. Since $\mathcal{H}$ is minimally connected, we have $P \in \Pi(V)$. Then, $I(H) \leq \frac{1}{|P| - 1} \sum_{P \in \Pi(V)} I(H) = E_P(H) \leq E_P(\mathcal{H}) = 1,$

where the equality follows from the fact that only edge $e$ crosses $P$. Suppose to the contrary that there is a cycle in $\mathcal{H}[P^*(\mathcal{H})]$, then there is also a cycle in $\mathcal{H}$, say $\{v_1, v_2, \ldots, v_{11}, v_1\}$, that crosses $P^*(\mathcal{H})$, i.e., $\{v_1, \ldots, v_{11}\} \notin C, \forall C \in P^*(\mathcal{H})$. We have $I(H) \geq I(H) \geq (a) \geq I(H)$

where (a) is as argued in (1.7); (b) as is argued above. This violates the Proposition 3.1. Therefore, $\mathcal{H}[P^*(\mathcal{H})]$ is a hypertree. It remains to prove $\mathcal{H}[P^*(\mathcal{H})]$ is loopless. Suppose to the contrary that $\mathcal{H}[P^*(\mathcal{H})]$ has a singleton edge $e' \in E(\mathcal{H}[P^*(\mathcal{H})])$ incident on $C \in P^*(\mathcal{H})$. Then, $\exists e \in E(\mathcal{H})$ such that $\xi_{\mathcal{H}}(e) \subseteq \xi_{\mathcal{H}[P^*(\mathcal{H})]} = C$. Removing $e$ from $\mathcal{H}$ does not disconnect $\mathcal{H}_e$, otherwise, $I(H) \leq 1 = I(\mathcal{H}[P^*(\mathcal{H})]) = I(\mathcal{H})$ contradicts Proposition 3.1. Doing so disconnects the edges $P^*(\mathcal{H}) \setminus C$ in $\mathcal{H}[P^*(\mathcal{H})]$ either, since $\mathcal{H}[P^*(\mathcal{H})]$ is a hypertree. Therefore, $\mathcal{H}$ remains connected after removing $e$, contradicting the fact that $\mathcal{H}$ is minimally connected. Hence, $\mathcal{H}[P^*(\mathcal{H})]$ is loopless.

Next, we proceed to prove (4.5). For any $C \in P^*(\mathcal{H})$, consider two distinct $C', C'' \in P^*(\mathcal{H}) \setminus C$. If there is a path in $\mathcal{H} \setminus C$ between any $v' \in C'$ and any $v'' \in C''$, then there is a path in $\mathcal{H}[P^*(\mathcal{H})] \setminus C$ between $C'$ and $C''$ by Definition 3.4 of $\mathcal{H}[P^*(\mathcal{H})]$. The contrapositive statement implies

$$\mathcal{E}(\mathcal{H} \setminus C) \geq \mathcal{E}(\mathcal{H}[P^*(\mathcal{H})] \setminus C).$$

Note that $\mathcal{H}'_C$ is connected for all $C' \in P^*(\mathcal{H})$. If there is a path in $\mathcal{H}[P^*(\mathcal{H})] \setminus C$ between $C'$ and $C''$, then there is a path in $\mathcal{H} \setminus C$ between all $v' \in C'$ and all $v'' \in C''$. The contrapositive statement implies

$$\mathcal{E}(\mathcal{H} \setminus C) \leq \mathcal{E}(\mathcal{H}[P^*(\mathcal{H})] \setminus C).$$

Hence, we have

$$\mathcal{E}(\mathcal{H} \setminus C) = \mathcal{E}(\mathcal{H}[P^*(\mathcal{H})] \setminus C).$$

By Definition 3.4 and (3.1b), we have

$$\deg_{\mathcal{H}[P^*(\mathcal{H})]}(C) = \deg_{\mathcal{H}}(C), \forall C \in P^*(\mathcal{H}).$$

Combining (1.2), (1.3) and (1.4) gives (4.5) as desired.

It remains to prove (4.6). Since $\mathcal{H}[P^*(\mathcal{H})]$ is a hypertree, by Lemma 4.2, we have

$$I(\mathcal{H}[P^*(\mathcal{H})]) = 1,$$

$$P^*(\mathcal{H}[P^*(\mathcal{H})]) = \{\{C\} | C \in P^*(\mathcal{H})\}.$$
Then, by (3.3),
\[
1 = \frac{1}{|P^*(\mathcal{H})| - 1} \left[ \sum_{C \in P^*(\mathcal{H})} \deg_{\mathcal{H}[P^*(\mathcal{H})]}(C) - |E(\mathcal{H}[P^*(\mathcal{H})])| \right]
\]
Note that by Definition 3.4, we have
\[
|E(\mathcal{H}[P^*(\mathcal{H})])| = |E(\mathcal{H})|.
\]
Substituting (1.4) and (1.6) into the above equation gives the desired result (4.6). This completes the proof of Lemma 4.1.

PROOF (LEMMA 4.2) “if” part: Consider a hypergraph \( \mathcal{H} \) that satisfies (4.7). \( I(\mathcal{H}) = 1 \) implies that \( \mathcal{H} \) is connected. It remains to argue the path between any two distinct vertices in \( \mathcal{H} \) is unique, i.e., no cycles. Suppose to the contrary that there exists a cycle in \( \mathcal{H} \), say \((v_1, e_1, v_2, \ldots, e_{\ell-1}, v_\ell = v_1)\) with \( \ell \geq 3 \). Define a graph \( \mathcal{G} = (V, E, \xi) \) with
\[
V(\mathcal{G}) = \{v_1, \ldots, v_\ell\},
E(\mathcal{G}) = \{e_1, \ldots, e_{\ell-1}\},
\xi(e_i) = \begin{cases} \{v_i, v_{i+1}\}, & i \in \{1, \ldots, \ell-2\}, \\ \{v_i, v_1\}, & i = \ell - 1. \end{cases}
\]
i.e., \( \mathcal{G} \) is obtained from \( \mathcal{H}_{\{v_1, \ldots, v_\ell\}} \) by further shrinking the hyperedges into edges and removing the hyperedges not appear in the sequence. Then,
\[
I(\mathcal{H}_{\{v_1, \ldots, v_\ell\}}) \stackrel{(a)}{\geq} I(\mathcal{G}) \stackrel{(b)}{>} 1 \stackrel{(c)}{=} I(\mathcal{H})
\]
where (a) is because processing the edges can only decrease the partition connectivity [15]; (b) is because \( I(\mathcal{G}) = \frac{\ell - 1}{\ell - 2} > 1 \), which is achieved by \( P^*(\mathcal{G}) = \{\{v_i\} \mid i \in \{1, \ldots, \ell - 1\}\} \); (c) is because \( I(\mathcal{H}) = 1 \) by the assumption. Therefore, by Proposition 3.1, the fundamental partition is not the singleton partition, which contradicts our assumption (4.7b).

“only if” part: Suppose \( \mathcal{H} \) is a hypertree. For any \( \mathcal{P} \in \mathcal{P}'(V) \), there are exactly \( |\mathcal{P}| - 1 \) edges crossing \( \mathcal{P} \) since \( \mathcal{H} \) is a hypertree, i.e., \( E_{\mathcal{P}}(\mathcal{H}) = |\mathcal{P}| - 1 \). Therefore, (4.7a) holds. For any \( C \subseteq V \) with \( |C| > 1 \), \( \mathcal{H}_C \) is a hypergraph with at most one path between any two of its vertices since \( \mathcal{H} \) is a hypertree, i.e., \( \mathcal{H}_C \) is either a hypertree or is disconnected. For \( \mathcal{H}_C \) being a hypertree, \( I(\mathcal{H}_C) = 1 \) as argued above. For \( \mathcal{H}_C \) being disconnected, \( I(\mathcal{H}_C) = 0 \). Altogether, \( I(\mathcal{H}_C) \leq 1 \), \( \forall C \subseteq V \). By Proposition 3.1, we have (4.7b) as desired.