Deep Structured Teams in Arbitrary-Size Linear Networks: Decentralized Estimation, Optimal Control and Separation Principle

Jalal Arabneydi and Amir G. Aghdam

Abstract—In this article, we introduce decentralized Kalman filters for linear quadratic deep structured teams. The agents in deep structured teams are coupled in dynamics, costs and measurements through a set of linear regressions of the states and actions (also called deep states and deep actions). The information structure is decentralized, where every agent observes a noisy measurement of its local state and the global deep state. Since the number of agents is often very large in deep structured teams, any naive approach to finding an optimal Kalman filter suffers from the curse of dimensionality. Moreover, due to the decentralized nature of information structure, the resultant optimization problem is non-convex, in general, where non-linear strategies can outperform linear ones. However, we prove that the optimal strategy is linear in the local state estimate as well as the deep state estimate and can be efficiently computed by two scale-free Riccati equations and Kalman filters. We propose a bi-level orthogonal approach across both space and time levels based on a gauge transformation technique to achieve the above result. We also establish a separation principle between optimal control and optimal estimation. Furthermore, we show that as the number of agents goes to infinity, the Kalman gain associated with the deep state estimate converges to zero at a rate inversely proportional to the number of agents. This leads to a fully decentralized approximate strategy where every agent predicts the deep state by its conditional and unconditional expected value, also known as the certainty equivalence approximation and (weighted) mean-field approximation, respectively.

I. INTRODUCTION

In recent years, there has been a growing interest in large-scale systems such as social networks, epidemics, smart grids, economics and robotics, to name only a few. In such systems, a large number of decision-makers interact with each other in order to minimize a common cost function, where the global decision process (i.e., system-level decision) is a manifestation of many different local decision processes (i.e., agent-level decisions). To be able to coordinate the global process based on the local processes, the standard approach is to consider centralized information. On the other hand, centralized information is not desirable in large-scale systems due to physical and economic limitations. Hence in practice, the information set of each local decision-maker is often different, leading to a discrepancy in perspective.

Since it is conceptually challenging to establish coordination among agents with different viewpoints, decentralized systems do not typically admit a globally optimal solution.

In general, information structures can be categorized into three classes: classical (centralized), partially nested (semi-centralized) and non-classical (decentralized). When every agent knows the history of the actions and observations of all agents, the information structure is called classical. On the other hand, when every agent knows the history of the actions and observations of all agents whose actions affect its observations, the information structure is called partially nested; any other information structure is called non-classical [1], [2]. It is well-known that the optimal strategy is an affine function of the observations in linear quadratic Gaussian (LQG) systems under classical and partially-nested information structures. However, this is not the case for non-classical information structures, in general. For example, it is shown in [3] that solving a simple two-agent LQG model with decentralized information ends up with a non-convex optimization problem where non-linear strategies outperform linear ones. Even if the attention is restricted to linear strategies, the best linear solution is not necessarily a convex optimization problem solution, except for a few special cases such as quadratic invariance [4]. In addition, the best linear strategy might not even have a finite-dimensional representation [5]. For more counterexamples on the complexity of decentralized linear problems, see [6], [7].

In this paper, we study a newly emergent class of large-scale decentralized control systems called deep structured teams [8]–[15], which may be viewed as a generalization of the notion of weighted mean-field teams introduced in [16]. Since the closest model to such systems is feed-forward deep neural networks, we refer to them as deep structured teams/games. In particular, agents in deep structured models interact with each other through a set of linear regressions of the states and actions of all agents, which is similar in spirit to the interaction of neurons in feed-forward deep neural networks. In addition, we show in [11] that the secret ingredient of finding a low-dimensional solution in such models is related to invariance/equivariance symmetry, which is the backbone of deep learning. Furthermore, we demonstrate in [9] that today’s most common feed-forward deep neural networks (i.e., those with rectified linear unit activation function) may be viewed as a special case of deep structured teams, where layers are time steps and neurons are simple integrator agents whose goal is to collaborate in order to minimize a common loss (cost) function. For more applications of deep structured models, the reader is
referred to reinforcement learning [8], [13], [14], nonzerosum game [12], [15], minmax optimization [17], leader-followers [18], [19], epidemics [20], smart grids [21], mean-field teams [22]–[25], and networked estimation [26], [27].

Herein, we focus on LQG deep-structured systems under a non-classical information structure that is neither partially nested nor quadratically invariant. In particular, we introduce a bi-level orthogonal approach to finding a low-dimensional solution using the gauge transformation technique, initially proposed in [16]. More precisely, we prove that the optimal strategy is an affine function of the observations, where controllers’ and observers’ gains are computed by two scale-free local and global Riccati equations and Kalman filters, respectively. The derivation of the proposed decentralized Kalman filters is different from the standard one in [28], [29]. For example, our proof technique holds only for i.i.d. random variables and naturally for those non-i.i.d. variables that are conditionally i.i.d. relative to some latent variables. For instance, exchangeable random variables (that are not necessarily i.i.d.) behave as conditionally i.i.d. variables in the infinite-population model according to De Finetti’s theorem [30]. To the best of our knowledge, this is the first result establishing a tractable optimal strategy under noisy measurements for a class of large-scale control systems with non-classical information structure. However, at this stage, it is unclear whether or not such a low-dimensional representation exists for general non-i.i.d. random variables.

The rest of the paper is organized as follows. In Section II we formulate an LQG deep structured team problem with noisy observations. In Section III we derive our Kalman filters in five steps. In Section IV we present the main result of the paper followed by two extensions to infinite-population approximation and least square estimation. In Section V we summarize and conclude the paper.

II. PROBLEM FORMULATION

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ refer to the sets of real and natural numbers, respectively. Given any $n \in \mathbb{N}$, $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and $x_{1:n} = \{x_1, \ldots, x_n\}$. For vectors $x, y, z$, $\text{vec}(x, y, z) = [x^\top, y^\top, z^\top]^\top$. $\text{Cov}(x) = \mathbb{E}[(x - \mathbb{E}[x])(x - \mathbb{E}[x])^\top]$ is the covariance matrix. $\mathcal{N}(\mu, \Sigma)$ is a multi-dimensional Gaussian probability distribution with mean $\mu$ and covariance matrix $\Sigma$. In addition, $0$, $1$, $\mathbb{I}$ represent matrix zero where all arrays are zero, matrix one where all arrays are one, and identity matrix, respectively. $\otimes$ is Kronecker product. Given vector $x = \text{vec}(x_1, \ldots, x_n) \in \mathbb{R}^n$, $x^{-1} \in \mathbb{R}^{n^{-1}}$ denotes $x$ without the $i$-th component.

Consider a system consisting of $n \in \mathbb{N}_n$ agents. Let $x_t \in \mathbb{R}^{d_x}$, $u_t \in \mathbb{R}^{d_u}$ and $w_t \in \mathbb{R}^{d_w}$, $d_x, d_u, d_w \in \mathbb{N}$, denote the state, action and noise of agent $i \in \mathbb{N}_n$ at time $t \in \mathbb{N}$, respectively. For each agent $i \in \mathbb{N}_n$ define

$$
\bar{x}_t := \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_t^i, \quad \bar{u}_t := \frac{1}{n} \sum_{i=1}^{n} \alpha_i u_t^i, \quad \bar{w}_t := \frac{1}{n} \sum_{i=1}^{n} \alpha_i w_t^i.
$$

Without loss of generality, we assume that the influence factors are normalized as follows: $\frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 = 1$. Let the dynamics of agent $i \in \mathbb{N}$ at time $t \in \mathbb{N}$ be described by

$$
x_{t+1}^i = A_t x_t^i + B_t u_t^i + E_t w_t^i + \alpha_i (\bar{A}_t \bar{x}_t + \bar{B}_t \bar{u}_t + \bar{E}_t \bar{w}_t),
$$

where matrices $A_t$, $B_t$, $E_t$, $\bar{A}_t$, $\bar{B}_t$ and $\bar{E}_t$ have appropriate dimensions. Denote by $y_t^i \in \mathbb{R}^{d_y}$ and $v_t^i \in \mathbb{R}^{d_v}$ the state observation and measurement noise of agent $i$ at time $t$, respectively, such that

$$
y_t^i = C_t x_t^i + S_t v_t^i + \alpha_i (\bar{C}_t \bar{x}_t + \bar{S}_t \bar{v}_t),
$$

where $C_t$, $S_t$, $\bar{C}_t$ and $\bar{S}_t$ have appropriate dimensions and

$$
y_t := \frac{1}{n} \sum_{i=1}^{n} \alpha_i y_t^i, \quad \bar{v}_t := \frac{1}{n} \sum_{i=1}^{n} \alpha_i v_t^i.
$$

Following the terminology of deep structured models [8], [11], aggregate variables (linear regressions) defined in (1) and (3) are called deep variables due to the fact that their evolutions across time horizon are similar to feed-forward deep neural networks. Hence, for ease of reference, we refer to $\bar{x}_t$ as deep state at time $t$ in the sequel.

We consider a non-classical (decentralized) information structure called imperfect deep state sharing (IDSS) such that the action of agent $i$ is given by

$$
u_t^i = g_t^i(y_{t-1}^i; \bar{y}_{t-1}^i),$$

(IDSS)

where $g_t^i : \mathbb{R}^{2d_y} \to \mathbb{R}^{d_u}$. When $n \geq 3$, a salient property of IDSS structure is that it provides natural encryption of data in terms of noisy deep state (linear regression) so that no agent knows the local (private) state of other agents. Let $x_t := \{x_t^1, \ldots, x_t^n\}$, $u_t := \{u_t^1, \ldots, u_t^n\}$, $w_t := \{w_t^1, \ldots, w_t^n\}$, $y_t := \{y_t^1, \ldots, y_t^n\}$, and $v_t := \{v_t^1, \ldots, v_t^n\}$. The random variables $\{x_t, w_t; t \geq 1\}$ are defined on a common probability space and are mutually independent across agents and control horizon. Furthermore, $x_t \sim \mathcal{N}(\mu_t^x, \Sigma_t^x)$, $w_t \sim \mathcal{N}(0, \Sigma_t^w)$, $v_t \sim \mathcal{N}(0, \Sigma_t^v)$, $t \in \mathbb{N}_T$. Let $g := \{(g_t^i)_{i=1}^n\}_{t=1}^T$ denote the strategy of the system. We now define the team cost as follows.

$$J_n(g) := \mathbb{E}\left[\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} c_t^i\right],$$

where $c_t^i : \mathbb{R}^{2d_y} \to \mathbb{R}$.
where the per-step cost function of agent $i \in \mathbb{N}_n$ is

$$c_i^t := (x_i^t)^T Q_i x_i^t + (u_i^t)^T R_i u_i^t + x_i^t \tilde{Q}_i \bar{x}_i^t + u_i^t \tilde{R}_i \bar{u}_i^t. \quad (4)$$

It is straightforward to consider cross terms of the form $\alpha_i x_i^t Q_i \bar{x}_i^t$ and $\alpha_i u_i^t \tilde{R}_i \bar{u}_i^t$ in (4); see also [9, Remark 1] for other straightforward extensions.

We are interested in the following optimization problem.

**Problem 1.** Given any $n \in \mathbb{N}$, find the optimal strategy $g^*$ such that for any strategy $g$: $J_n(g^*) \leq J_n(g)$.

### III. DERIVATION OF KALMAN FILTERS

Prior to delving into details, we present 3 standard lemmas.

**Lemma 1.** Let $x$ and $y$ be Gaussian random variables with zero mean. The best non-linear strategy is equal to the best linear one as far as the minimum mean-square error is concerned, i.e., $\mathbb{E}[x | y] = \mathbb{E}[xg^T]$ if $\mathbb{E}[yy^T]^{-1}$ is positive definite. In addition, $\mathbb{E}[(x-\mathbb{E}[x | y])(x-\mathbb{E}[x | y])^T] = \mathbb{E}[xx^T] - \mathbb{E}[xy^T]\mathbb{E}[yy^T]\mathbb{E}[yx^T]$.

Furthermore, $x$ and $y$ are independent if and only if $\mathbb{E}[xy^T] = 0$, i.e. $x \perp y$.

**Lemma 2.** Let $x$, $y_1$, and $y_2$ be Gaussian random variables. Let also $y_1$ and $y_2$ be independent and have zero mean. Then,

$$\mathbb{E}[x | y_1, y_2] = \mathbb{E}[x | y_1] + \mathbb{E}[x | y_2].$$

**Proof.** The proof follows from the fact that $\mathbb{E}[y_1y_2^T] = 0$.

At any time $t \in \mathbb{N}_T$, denote by $\mathcal{H}_t^y := \{y_{1:t}, u_{1:t}\}$ and $\mathcal{H}_t := \{y_{1:t}, u_{1:t-1}\}$ the history sets with and without joint action $u_t$, respectively.

**Lemma 3.** The conditional expectation of the joint state at time $t \in \mathbb{N}_T$ given $\mathcal{H}_t^y$ does not depend on the strategy $g$, i.e. $\mathbb{E}[x_t | \mathcal{H}_t^y] = \mathbb{E}[x_t | \mathcal{H}_t]$.

**Proof.** The proof follows from the fact that the following equality holds irrespective of strategy $g$, i.e. $\mathbb{P}(x_t | \mathcal{H}_t^y) = \mathbb{P}(x_t | \mathcal{H}_t)$, where by Bayes’ rule,

$$\begin{align*}
\mathbb{P}(x_t | \mathcal{H}_t^y) &= \frac{\mathbb{P}(x_t, u_t | \mathcal{H}_t^y)}{\int_{x_t} \mathbb{P}(x_t, u_t | \mathcal{H}_t) d(x_t)} \\
&= \frac{\mathbb{P}(u_t | \mathcal{H}_t) \mathbb{P}(x_t | \mathcal{H}_t)}{\int_{x_t} \mathbb{P}(u_t | \mathcal{H}_t) \mathbb{P}(x_t | \mathcal{H}_t) d(x_t)} \\
&= \frac{\mathbb{P}(u_t = g_t(\mathcal{H}_t)) \mathbb{P}(x_t | \mathcal{H}_t) d(x_t)}{\mathbb{P}(u_t = g_t(\mathcal{H}_t)) \int_{x_t} \mathbb{P}(x_t, u_t | \mathcal{H}_t) d(x_t)} = \mathbb{P}(x_t | \mathcal{H}_t).
\end{align*}$$

We now take 5 steps to derive a low-dimensional solution.

**A. Step 1: Gaussian transformation**

Define the following auxiliary variables using the gauge transformation presented in [11] such that for every $i \in \mathbb{N}_n$:

$$\Delta x_i^t := x_i^t - \alpha_i \bar{x}_i^t, \ \Delta u_i^t := u_i^t - \alpha_i \bar{u}_i^t, \ \Delta w_i^t := w_i^t - \alpha_i \bar{w}_i^t, \ \Delta y_i^t := y_i^t - \alpha_i \bar{y}_i^t, \ \Delta v_i^t := v_i^t - \alpha_i \bar{v}_i^t. \quad (5)$$

**Lemma 4 (Linear dependence between auxiliary variables).** The gauge transformation (5) introduces the following relations: $\sum_{i=1}^n \alpha_i \Delta x_i^t = 0$, $\sum_{i=1}^n \alpha_i \Delta u_i^t = 0$, $\sum_{i=1}^n \alpha_i \Delta w_i^t = 0$, $\sum_{i=1}^n \alpha_i \Delta y_i^t = 0$ and $\sum_{i=1}^n \alpha_i \Delta v_i^t = 0$.

**Proof.** The proof directly follows from (1), (3) and (5).

From Lemma 4 and equations (1) and (2), one arrives at:

$$\Delta x_{i+1} = A_i \Delta x_i + B_i \Delta u_i + E_i \Delta w_i,$$

and for every $i \in \mathbb{N}_n$:

$$\Delta y_i = C_i \Delta x_i + \text{S} \Delta u_i.$$

**Lemma 5 (Orthogonal relation in cost function).** Form the gauge transformation (5), the per-step cost function (3) can be expressed in terms of auxiliary variables and aggregate (deep) variables as follows:

$$\frac{1}{n} \sum_{i=1}^n ((\Delta x_i^t)^T Q_i \Delta x_i^t + (\Delta u_i^t)^T R_i \Delta u_i^t) = \tilde{x}_i^t Q_i \tilde{x}_i + \tilde{u}_i^t R_i \tilde{u}_i$$

*Proof. From (1) and (5), $x_i = \Delta x_i^t + \alpha_i \bar{x}_i^t$ and $u_i = \Delta u_i^t + \alpha_i \bar{u}_i^t$, $i \in \mathbb{N}_n$, can be replaced by the auxiliary and deep variables in the cost function. The proof follows from the fact that $\sum_{i=1}^n \alpha_i \Delta x_i^t Q_i \bar{x}_i^t = 0$ and $\sum_{i=1}^n \alpha_i \Delta u_i^t R_i \bar{u}_i^t = 0$.

**Lemma 6 (Relations between primitive random variables).** The following holds for every finite $n \in \mathbb{N}$, $t \in \mathbb{N}_t$ and $i \neq j \in \mathbb{N}_n$:

1. $\Delta x_i^t \perp \Delta x_j^t$ and $\mathbb{E}[(\Delta x_i^t)(\Delta x_j^t)^T] = -\frac{\alpha_i \alpha_j}{n} \Sigma^x \neq 0$.
2. $\Delta w_i^t \perp \Delta w_j^t$ and $\mathbb{E}[(\Delta w_i^t)(\Delta w_j^t)^T] = -\frac{\alpha_i \alpha_j}{n} \Sigma^w \neq 0$.
3. $\Delta v_i^t \perp \Delta v_j^t$ and $\mathbb{E}[(\Delta v_i^t)(\Delta v_j^t)^T] = -\frac{\alpha_i \alpha_j}{n} \Sigma^v \neq 0$.
4. $x_i^t \perp \tilde{x}_i^t$ and $\mathbb{E}[(x_i^t)(\tilde{x}_i^t)^T] = \frac{\alpha_i}{n} \Sigma^x \neq 0$.
5. $w_i^t \perp \tilde{w}_i^t$ and $\mathbb{E}[(w_i^t)(\tilde{w}_i^t)^T] = \frac{\alpha_i}{n} \Sigma^w \neq 0$.
6. $v_i^t \perp \tilde{v}_i^t$ and $\mathbb{E}[(v_i^t)(\tilde{v}_i^t)^T] = \frac{\alpha_i}{n} \Sigma^v \neq 0$.
7. $\mathbb{E}[(\Delta x_i^t)(\tilde{x}_i^t)^T] = (1 - \frac{\alpha_i}{n}) \Sigma^x \neq 0$.
8. $\mathbb{E}[(\Delta w_i^t)(\tilde{w}_i^t)^T] = (1 - \frac{\alpha_i}{n}) \Sigma^w \neq 0$.
9. $\mathbb{E}[(\Delta v_i^t)(\tilde{v}_i^t)^T] = (1 - \frac{\alpha_i}{n}) \Sigma^v \neq 0$.
10. $\Delta x_i^t \perp \tilde{x}_i^t$ and $\mathbb{E}[(\Delta x_i^t)(\tilde{x}_i^t)^T] = 0$.
11. $\Delta w_i^t \perp \tilde{w}_i^t$ and $\mathbb{E}[(\Delta w_i^t)(\tilde{w}_i^t)^T] = 0$.
12. $\Delta v_i^t \perp \tilde{v}_i^t$ and $\mathbb{E}[(\Delta v_i^t)(\tilde{v}_i^t)^T] = 0$.

The non-orthogonal relations (1)–(6) simplify to the orthogonal ones, as $n \to \infty$.

**Proof.** The proof follows from (1), (3), (5) and the fact that driving noises and measurement noises are i.i.d. random vectors with zero mean.

In the perfect sharing and deep state sharing, the certainty equivalence principle simplifies the analysis and results in two standard decoupled Riccati equations [8], [11]. In the imperfect observation case, however, the certainty equivalence principle does not hold. To see this, notice that
although the dynamics and cost of the auxiliary subsystems are decoupled, their uncertainties are coupled (correlated) as shown in Lemma 6. This makes the analysis more difficult.

B. Step 2: Innovation processes

Define the following global variables:

\[ \begin{align*}
  z_{t+1} &:= E[\hat{x}_{t+1} | H_t], \\
  z_{t+1|t} &:= E[\hat{x}_{t+1} | H_t^t], \\
  \xi_{t+1} &:= \tilde{x}_t - z_{t+1|t}, \\
  \xi_{t+1|t} &:= \tilde{x}_t - z_{t+1|t}, \\
  \Sigma_{t+1} &:= E[\xi_{t+1}^T H_t], \\
  \Sigma_{t+1|t} &:= E[\xi_{t+1|t}^T H_t^t]
\end{align*} \]

From Lemma 3 and equations (1), (2) and (7), the dynamics of the deep-state estimate \( z_{t+1|t} \) is given by

\[ z_{t+1|t} = E[\tilde{x}_{t+1} | H_t^t] = E[(A_t + \tilde{A}_t)\tilde{x}_t + (B_t + \tilde{B}_t)\tilde{u}_t + (E_t + \tilde{E}_t)\tilde{w}_t | H_t^t] = (A_t + \tilde{A}_t)z_{t|t} + (B_t + \tilde{B}_t)\tilde{u}_t. \]  

(8)

We now define local variables for each step:

\[ \Delta \tilde{x}_{t+1|t} := E[\Delta x_{t+1} | H_t^t], \quad \Delta \xi_{t+1|t} := \Delta \tilde{x}_{t+1|t} - \Delta \tilde{x}_{t+1|t}, \quad \Sigma_{t+1|t} := E[\xi_{t+1|t}^T | H_t^t], \quad \Sigma_{t+1|t}^{1|t} := E[\xi_{t+1|t}^T (\xi_{t+1|t})^T | H_t^t]. \]

From Lemma 3 and (1), (2), (5) and (9), it results that

\[ \Delta \tilde{x}_{t+1|t} = E[\tilde{x}_{t+1} | H_t^t] = E[(A_t + \tilde{A}_t)\tilde{x}_t + (B_t + \tilde{B}_t)\tilde{u}_t + (E_t + \tilde{E}_t)\tilde{w}_t | H_t^t] = A_t \Delta \tilde{x}_t + B_t \Delta \tilde{u}_t. \]

We decompose the above innovation process into the gauge transformation such that

\[ \begin{align*}
  \Delta \tilde{p}_{t+1} &:= \Delta \tilde{x}_{t+1|t} - \alpha_t \Delta \tilde{y}_{t+1} - E[\Delta \tilde{y}_{t+1} | H_t^t], \\
  \Delta \tilde{p}_{t+1}^{1|t} &:= \Delta \tilde{y}_{t+1} - \Delta \tilde{y}_{t+1}^{1|t}.
\end{align*} \]

(10)

In the sequel, we refer to \( \Delta \tilde{p}_{t|t} \) as the auxiliary innovation process of agent \( i \) and to \( \tilde{p}_{t|t} \) as the global innovation process.

Remark 1. Note that the following non-orthogonal relations hold for each agent \( i \in N \) and \( t + 1 \neq j \in N_a; \)

1) \( \Delta \tilde{p}_{t+1} \perp \Delta \tilde{p}_{t+1}^{1|t} \),
2) \( \tilde{p}_{t+1} \perp \Delta \tilde{p}_{t+1}^{1|t} \),
3) \( \tilde{p}_{t+1} \perp \Delta \tilde{p}_{t+1} \).

The above non-orthogonal relations in (1)–(3) simplify to the orthogonal ones, as \( n \to \infty \).

Lemma 7 (Orthogonality in the transformed space). Given \( H_t^t \), the following holds for every \( n \in N \) and \( i \in N_a; \)

1) \( \Delta \tilde{p}_{t+1} \perp \Delta \tilde{p}_{t+1} \),
2) \( \tilde{p}_{t+1} \perp \Delta \tilde{p}_{t+1}^{1|t} \),
3) \( \Delta \tilde{p}_{t+1} \perp \Delta \tilde{p}_{t+1} \).

Proof: We first prove that \( \{\Delta \tilde{x}_{t+1}, \Delta \tilde{y}_{t+1} | \ \forall i \in N_a\} \) and \( \{\tilde{x}_{t+1}, \tilde{y}_{t+1}\} \) are two independent sets. In particular, given the history set \( H_t \), we show that the randomness of \( \Delta \tilde{x}_{t+1} \) and \( \Delta \tilde{y}_{t+1} \) is completely characterized by the set \( \{\Delta \tilde{x}_{t+1}, \Delta \tilde{u}_{t+1} \} \) as follows:

\[ \begin{align*}
  \Delta \tilde{x}_{t+1} &= \left( \prod_{k=1}^{t} \frac{1}{A_k + \tilde{A}_k} \right) \Delta x_{t+1} + \sum_{k=1}^{t} \Delta A(t - k) B_k \Delta u_k \\
  \Delta \tilde{y}_{t+1} &= C_{t+1} \Delta x_{t+1} + S_{t+1} \Delta u_{t+1},
\end{align*} \]

where \( A(0) := I \) and \( \Delta A(t - k) := \prod_{k=1}^{t-1} (A_k + \tilde{A}_k) \).

Similarly, the randomness of \( \tilde{x}_{t+1} \) and \( \tilde{y}_{t+1} \) is completely characterized by the set \( \{\tilde{x}_{t+1}, \tilde{u}_{t+1}\} \) such that

\[ \begin{align*}
  \tilde{x}_{t+1} &= \left( \prod_{k=1}^{t} (A_k + \tilde{A}_k) \right) \tilde{x}_{t+1} + \sum_{k=1}^{t} \tilde{A}(t - k) (B_k + \tilde{B}_k) \tilde{u}_k \\
  \tilde{y}_{t+1} &= (C_{t+1} + \tilde{C}_{t+1}) \tilde{x}_{t+1} + (S_{t+1} + \tilde{S}_{t+1}) \tilde{u}_{t+1},
\end{align*} \]

where \( \tilde{A}(0) := I \) and \( \tilde{A}(t - k) := \prod_{k=1}^{t-1} (A_k + \tilde{A}_k) \).

C. Step 3: Covariance matrices

From (1), (2) and (3), the estimation error \( \xi_{t+1} \) and covariance matrix \( \Sigma_{t+1} \) evolve at time \( t \in \mathbb{N}_T \) as follows:

\[ \begin{align*}
  \xi_{t+1} &= (A_t + \tilde{A}_t) \xi_t + (E_t + \tilde{E}_t) \tilde{w}_t, \\
  \Sigma_{t+1} &= (A_t + \tilde{A}_t) \Sigma_t (A_t + \tilde{A}_t)^T + \text{Cov}((E_t + \tilde{E}_t) \tilde{w}_t).
\end{align*} \]

Similarly, the dynamics of local estimation error \( e_{t+1} \) and covariance matrix \( \Sigma_{t+1}^{1|t} \) for every \( i \in N_a \) at time \( t \in \mathbb{N}_T \) are

\[ \begin{align*}
  e_{t+1} &= A_t e_t + E_t \Delta w_t, \\
  \Sigma_{t+1|t} &= A_t \Sigma_t A_t^T + \text{Cov}(E_t \Delta w_t).
\end{align*} \]

Given any \( i \in N_a \), one can define index-invariant covariance matrices as follows:

\[ \Sigma_{t+1} := (1 - \frac{\alpha_i^2}{n})^{-1} \Sigma_{t+1}^{i|t}, \quad \Sigma_{t} := (1 - \frac{\alpha_i^2}{n})^{-1} \Sigma_{t}^{i|t} \]

(11)

where

\[ \Sigma_{t+1}^{i|t} = A_t \Sigma_{t} A_t^T + E_t \Sigma_{t} E_t^T. \]

Denote by \( \Delta \tilde{P}_{t+1} \) and \( \alpha \) the joint vector consisting of \( \Delta \tilde{P}_{t+1} \) and \( \alpha_j \), \( \forall j \neq i \in N_a \), respectively.

Lemma 8. For any \( i \neq j \in N_a \), the following equalities hold at any time \( t \in \mathbb{N}_T; \)

1) \( \text{E}(e_{t+1}^T(e_{t+1}^iT)) = (1 - \frac{\alpha_i^2}{n}) \Sigma_{t+1|t}. \)
Lemma 10 (Global update). The update of the estimate of the deep state given $H_{t+1}^{L}$ can be computed as follows:

$$E[x_{t+1} | H_{t+1}^{L}] = E[x_{t+1} | \tilde{p}_{t+1}] = \Sigma_{t+1 | t}(C_{t+1} + \tilde{C}_{t+1})^\top \times \left( (C_{t+1} + \tilde{C}_{t+1}) \Sigma_{t+1 | t}(C_{t+1} + \tilde{C}_{t+1})^\top + (S_{t+1} + \tilde{S}_{t+1}) \Sigma_{t+1 | t} \tilde{S}_{t+1} \tilde{C}_{t+1}^{-1} \right)^{-1} \times (\tilde{y}_{t+1} - (C_{t+1} + \tilde{C}_{t+1}) z_{t+1 | t}).$$

Proof. From Lemmas 1 and 7 and equation (12), one has

$$E[x_{t+1} | H_{t+1}^{L}] = E[x_{t+1} | \tilde{p}_{t+1}] = E[x_{t+1} \tilde{p}_{t+1}^\top | Cov(\tilde{p}_{t+1})^{-1} \tilde{p}_{t+1}].$$

The one step update of the covariance matrix of the global innovation process is calculated as

$$Cov(\tilde{p}_{t+1})^{-1} \tilde{p}_{t+1}. $$

From Lemmas 1 and 7, we obtain

$$0 = \frac{1}{n} \sum_{i=1}^{n} (\alpha_i)^2 = 1. $$

Thus, for every $i \in \mathbb{N}$, it follows that

$$(I_{n} \times n) - \frac{1}{n} (\alpha_i)^2 (\alpha_i)^\top = \left( (I_{n} \times n) - (\alpha_i)^2 (\alpha_i)^\top \right)^{-1}.$$
From equations (12) and (13) and update rules in Lemmas 10 and 11 one gets two low-dimensional (scale-free) Kalman filters as follows. For every $i \in \mathbb{N}_n$,

$$
\begin{align*}
\Delta \hat{x}_{i+1|t+1} &= \Delta \hat{x}_{i+1|t} + L_{t+1}(\Delta y_{i+1} - C_{t+1} \Delta \hat{x}_{i+1|t}), \\
\Delta \hat{z}_{i+1|t+1} &= A_t \Delta \hat{x}_{i+1|t} + B_t \Delta u_t, \\
\Sigma_{i+1|t+1} &= (I - L_{t+1} C_{t+1} \Sigma_{i+1|t}), \\
\Sigma_{t+1|t+1} &= A_t \Sigma_{i|t} A_t^T + E_t \Sigma_{i|t} E_t^T, \\
L_{t+1} &= \Sigma_{i+1|t+1}(C_{t+1} \Sigma_{i+1|t+1} + S_{t+1} \Sigma_{i+1|t+1} S_{t+1}^T)^{-1},
\end{align*}
$$

(15)

with the initial conditions $\Delta \hat{x}_{i|0} = (1 - \frac{1}{N} \sum_{j=1}^{n} \alpha_j) \mu^x$ and $\Sigma_{i|0} = \Sigma^x$. Furthermore,

$$
\begin{align*}
\hat{z}_{t+1|t} &= \hat{z}_{t+1|t} + \hat{L}_{t+1}(\tilde{y}_{t+1} - (C_{t+1} + \hat{C}_{t+1}) \hat{z}_{t+1|t}), \\
\Sigma_{t+1|t} &= (A_t + A_t \hat{A}_t) \Sigma_{t|t} (A_t + A_t \hat{A}_t)^T + \frac{1}{n} (E_t + E_t) \Sigma_{i|t} (E_t + E_t)^T, \\
\tilde{L}_{t+1} &= \Sigma_{t+1|t}(C_{t+1} + \hat{C}_{t+1})^T((C_{t+1} + \hat{C}_{t+1}) \Sigma_{t+1|t} \\
&\quad \times (C_{t+1} + \hat{C}_{t+1})^T + \frac{1}{n} (S_{t+1} + \hat{S}_{t+1}) \Sigma_{i+1|t}^{-1}(S_{t+1} + \hat{S}_{t+1})^{-1},
\end{align*}
$$

(16)

with the conditions $\hat{z}_{i|0} = (1 - \frac{1}{N} \sum_{j=1}^{n} \alpha_j) \mu^z$ and $\Sigma_{i|0} = \frac{1}{N} \Sigma^z$.

IV. MAIN RESULTS

We impose the following standard assumption.

**Assumption 1.** Let matrices $Q_t$ and $Q_t + \tilde{Q}_t$ be symmetric and positive semi-definite and matrices $R_t$ and $R_t + \tilde{R}_t$ be symmetric and positive definite for every time $t \in \mathbb{N}$.

For any $t \in \mathbb{N}_T - 1$, define the following Riccati equations:

$$
\begin{align*}
P_t &= Q_t + A_t^T P_{t+1} A_t - A_t^T P_{t+1} B_t (B_t^T P_{t+1} B_t + R_t)^{-1} \times B_t^T P_{t+1} A_t, \\
\tilde{P}_t &= Q_t + A_t (A_t + \tilde{A}_t)^T \tilde{P}_{t+1} (A_t + \tilde{A}_t)^T - (A_t + \tilde{A}_t)^T \tilde{P}_{t+1} (B_t + \tilde{B}_t)^T \tilde{P}_{t+1} (B_t + \tilde{B}_t) + (B_t + \tilde{B}_t)^T \tilde{P}_{t+1} (B_t + \tilde{B}_t)^T, \\
&\quad \times (B_t + \tilde{B}_t)^T \tilde{P}_{t+1} (A_t + \tilde{A}_t),
\end{align*}
$$

(17)

where $P_T = Q_T$ and $\tilde{P}_T = Q_t + \tilde{Q}_T$.

A. Solution of Problem 7

**Theorem 1** (Decentralized estimation, optimal control, and separation principle). Let Assumption 1 hold. The optimal strategy of agent $i \in \mathbb{N}_n$ at time $t \in \mathbb{N}$ under the decentralized information structure IDSS is given by

$$
u_{i|t}^x = \theta_i^x \hat{z}_{i|t} + \alpha_i (\tilde{\theta}_i - \theta_i^z) \hat{z}_{i|t},
$$

(18)

where local and global estimates are computed by two scale-free Kalman filters in (15) and (16) as follows:

$$
\begin{align*}
\hat{x}_{i+1|t+1} &= \mathbb{E}[x_{i+1|t+1}\|H_t] = A_t \hat{x}_{i|t} + B_t u_t + \alpha_i (\hat{A}_t \hat{z}_{i|t} + \hat{B}_t \tilde{u}_t), \\
\tilde{x}_{i+1|t+1} &= \mathbb{E}[x_{i+1|t+1}\|H_t] = \hat{x}_{i+1|t} + \Lambda_{i+1|t}(y_{i+1} - C_{i+1} \hat{x}_{i+1|t}) \\
&\quad + \alpha_i (\tilde{L}_{i+1|t}(\tilde{y}_{i+1} - (C_{i+1} + \hat{C}_{i+1}) \hat{z}_{i+1|t})), \\
\hat{z}_{i+1|t+1} &= \mathbb{E}[\hat{z}_{i+1|t} + \tilde{z}_{i+1|t}\|H_t] = \hat{z}_{i+1|t} \\
&\quad + \tilde{L}_{i+1|t}(\tilde{y}_{i+1} - (C_{i+1} + \hat{C}_{i+1}) \hat{z}_{i+1|t}),
\end{align*}
$$

where $\theta_i^x = -(B_t^T P_{t+1} B_t + R_t)^{-1} B_t^T P_{t+1} A_t, \tilde{\theta}_i = -(B_t + \tilde{B}_t)^T \tilde{P}_{t+1} (B_t + \tilde{B}_t) + R_t + \tilde{R}_t)^{-1}\times (B_t + \tilde{B}_t)^T \tilde{P}_{t+1} (A_t + \tilde{A}_t).

In addition, the local and global gains $\theta_i^z$ and $\tilde{\theta}_i$ are obtained by two scale-free Riccati equations in [17] as

**Proof.** Proof follows by using the completion-of-square method on the transformed cost function in Lemma 5 and replacing the state values with their conditional expectations, defined in (7) and (9). On the other hand, we showed that the conditional expectations could be computed recursively by Kalman filters in (15) and (16), irrespective of the control strategy (also known as the separation principle [1], [31]–[33]). Note that the separation principle is weaker than the certainty equivalence principle [34]. The remaining problem is an optimal LQ deep structured team with (perfect) deep state sharing whose solution is obtained from the Riccati equations in (17); see [8], [9], [11] for more details.

From Theorem 1 the optimal solution of agent $i \in \mathbb{N}_n$ can be implemented in a distributed manner. In particular, prior to the operation of the system, agent $i$ solves two scale-free Riccati equations and Kalman filters to obtain $\{L_{1:T}, L_{1:T}, \theta_{1:T}, \tilde{\theta}_{1:T}\}$. At any time $t \in \mathbb{N}_T$, agent $i$ estimates its local state $x_i^T$ by $\hat{x}_{i|t}$ and global state $\hat{x}_i$ by $z_{i|t}$ based on local and global noisy observations $y_i^T$ and $\tilde{y}_i$. Then, given its private (influence factor) $\alpha_i$, agent $i$ calculates its optimal strategy according to (13).

**Remark 2.** The only information shared among agents at each time instant $t$ is noisy deep state $\tilde{y}_i \in \mathbb{R}^{d_y}$ whose size (dimension) is independent of the number of agents $n$.

Establishing Theorem 1 for the special case of infinite population (i.e. $n = \infty$) is straightforward because auxiliary and global (deep) variables are mutually orthogonal according to Lemma 6 and Remark 1. This makes it significantly easier to develop a low-dimensional Kalman filter for $n = \infty$. The same argument holds for the finite-population case where deep state is observed perfectly (i.e. $x_i \in \mathcal{H}_t$); see mean-field teams in [16], [22] that considers such a special case for homogeneous influence factors, where $\alpha_i = 1$, $\forall i \in \mathbb{N}_n$.

**Theorem 2.** As $n \to \infty$, covariance matrices of the global Kalman filter (16) converge to zero at rate $1/n$. As a result, the followings hold at any $t \in \mathbb{N}_T$ for the model with $n = \infty$.

- **Blind optimal global estimator:** The global state $\hat{x}_i$ is almost surely equal to its conditional and unconditional expectation, i.e. $\hat{x}_i = z_{i|t} = \mathbb{E}[\hat{x}_i]$, which may be viewed as the certainty equivalence approximation [34] and (weighted) mean-field approximation [35], respectively.

- **Optimal fully decentralized strategy:** Agent $i \in \mathbb{N}_n$ requires access only to its local observation $y_i^T$ to calculate (15), leading to a fully decentralized strategy, where the global observation is perfectly predicted as follows:

$$
\tilde{y}_{t+1} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i y_{i+1} = (C_{t+1} + \hat{C}_{t+1}) \lim_{n \to \infty} \tilde{x}_{t+1} = (C_{t+1} + \hat{C}_{t+1}) \tilde{x}_{t+1|t}.
$$


Proof. The proof follows by noting that the covariance matrix of the summation of any \( n \) uniformly bounded independent random variables goes to zero with the rate \( 1/n \), and that \( z_{t+1|t+1} = z_{t+1|t} \) for \( n = \infty \), according to (16). □

C. Extension to least-square estimation

The results of Theorems 1 and 2 naturally hold for the best linear strategy minimizing the least-square estimation error without imposing any Gaussian assumption [29]. In such a case, the proof uses Hilbert space analysis with inner product \( \langle x, y \rangle = E[xy^T] \), where \( x \) and \( y \) are random variables.

V. Conclusions

In this paper, a new class of large-scale decentralized multi-agent systems, called deep structured teams, with noisy measurements was studied. A novel transformation-based approach was proposed to introduce a bi-level orthogonal relationship between the agents across both state space and time horizon. The optimal solution was shown to be linear in the local and global estimates and computed by two standard Riccati equations and Kalman filters. In addition, a fully decentralized sub-optimal strategy was developed, whose performance converges to that of the optimal one at a rate inversely proportional to the number of agents. The main results of this paper naturally extend to a model with multiple sub-populations and multiple features in a fashion similar to the one proposed in [9, Section IV] and [11].

REFERENCES

[1] H. Witsenhausen, “Separation of estimation and control for discrete time systems,” Proc. of IEEE, vol. 59, no. 11, pp. 1557–1566, Nov. 1971.
[2] Y. C. Ho and K. h. Chu, “Team decision theory and information structures in optimal control problems—part I,” IEEE Transactions on Automatic Control, vol. 17, no. 1, pp. 15–22, 1972.
[3] H. Witsenhausen, “A counterexample in stochastic optimum control,” SIAM Journal on Control and Optimization, vol. 6, pp. 131–147, Dec. 1968.
[4] M. Rotkowitz and S. Lall, “A characterization of convex problems in decentralized control,” IEEE Transactions on Automatic Control, vol. 51, no. 2, pp. 274–286, 2006.
[5] P. Whittle and J. Rudge, “The optimal linear solution of a symmetric team control problem,” Journal of Applied Probability, pp. 377–381, 1974.
[6] G. M. Lipsa and N. C. Martins, “Optimal memoryless control in Gaussian noise: A simple counterexample,” Automatica, vol. 47, no. 3, pp. 552–558, 2011.
[7] S. Yuksel and S. Tatikonda, “A counterexample in distributed optimal sensing and control,” IEEE Transactions on Automatic Control, vol. 54, no. 4, pp. 841–844, April 2009.
[8] J. Arabneydi, M. Roudneshin, and A. G. Aghdam, “Reinforcement learning in deep structured teams: Initial results with finite and infinite valued features,” in IEEE Conference on Control Technology and Applications, 2020.
[9] J. Arabneydi and A. G. Aghdam, “Receding horizon control in deep structured teams: A provably tractable large-scale approach with application to swarm robotics,” in Proceedings of the 66th IEEE Conference on Decision and Control, 2021.
[10] ——, “Deep teams: Decentralized decision making with finite and infinite number of agents,” IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2020.2966035, 2020.
[11] ——, “Deep structured teams with linear quadratic model: Partial equivariance and gauge transformation,” [Online]. Available at https://arxiv.org/abs/1912.03951, 2019.
[12] J. Arabneydi, A. G. Aghdam, and R. P. Malhamé, “Explicit sequential equilibria in LQ deep structured games and weighted mean-field games: A unified non-standard Riccati equation,” [Online]. Available at https://arxiv.org/abs/1912.03951, 2020.
[13] V. Fathi, J. Arabneydi, and A. G. Aghdam, “Reinforcement learning in linear quadratic deep structured teams: Global convergence of policy gradient methods,” in Proceedings of the 59th IEEE Conference on Decision and Control, 2020.
[14] M. Roudneshin, J. Arabneydi, and A. G. Aghdam, “Reinforcement learning in nonzero-sum Linear Quadratic structured games: Global convergence of policy optimization,” in Proceedings of the 59th IEEE Conference on Decision and Control, 2020.
[15] J. Arabneydi and A. G. Aghdam, “Deep structured teams and games with markov-chain model: Finite and infinite number of players,” Submitted, 2021.
[16] J. Arabneydi, “New concepts in team theory: Mean field teams and reinforcement learning.” Ph.D. dissertation, Department of Electrical and Computer Engineering, McGill University, Montreal, Canada, 2016.
[17] M. Baharloo, J. Arabneydi, and A. G. Aghdam, “Minmax mean-field team approach for a leader-follower network: A saddle-point strategy,” IEEE Control Systems Letters, vol. 4, no. 1, pp. 121–126, 2019.
[18] J. Arabneydi, M. Baharloo, and A. G. Aghdam, “Optimal distributed control for leader-follower networks: A scalable design,” in Proceedings of the 31st IEEE Canadian Conference on Electrical and Computer Engineering, 2018, pp. 1–4.
[19] M. Baharloo, J. Arabneydi, and A. G. Aghdam, “Near-optimal control strategy in leader-follower networks: A case study for linear quadratic mean-field teams,” in Proceedings of the 57th IEEE Conference on Decision and Control, 2018, pp. 3288–3293.
[20] J. Arabneydi and A. G. Aghdam, “A mean-field team approach to minimize the spread of infection in a network,” in Proceedings of American Control Conference, 2019, pp. 2747–2752.
[21] ——, “Optimal dynamic pricing for binary demands in smart grids: A fair and privacy-preserving strategy,” in Proceedings of American Control Conference, 2018, pp. 5368–5373.
[22] J. Arabneydi and A. Mahajan, “Team-optimal solution of finite number of mean-field coupled LQ subsystems,” in Proceedings of the 54th IEEE Conference on Decision and Control, 2015, pp. 5308 – 5313.
[23] J. Arabneydi and A. G. Aghdam, “A certainty equivalence result in team-optimal control of mean-field coupled Markov chains,” in Proceedings of the 56th IEEE Conference on Decision and Control, 2017, pp. 3125–3130.
[24] J. Arabneydi and A. Mahajan, “Team optimal control of coupled subsystems with mean-field sharing,” in Proceedings of the 53rd IEEE Conference on Decision and Control, 2014, pp. 1669–1674.
[25] ——, “Linear quadratic mean field teams: Optimal and approximately optimal decentralized solutions,” Available at https://arxiv.org/abs/1609.00056, 2016.
[26] J. Arabneydi and A. G. Aghdam, “Data collection versus data estimation: A fundamental trade-off in dynamic networks,” IEEE Transactions on Network Science and Engineering, vol. 7, no. 3, pp. 2000–2015, 2020.
[27] ——, “Near-optimal design for fault-tolerant systems with homogeneous components under incomplete information,” in Proceedings of the 61st IEEE International Midwest Symposium on Circuits and Systems, 2018, pp. 809–812.
[28] R. E. Kalman, “A new approach to linear filtering and prediction problems,” Transactions of the ASME—Journal of Basic Engineering, pp. 35–45, 1960.
[29] D. Simon, Optimal state estimation: Kalman, H infinity, and nonlinear approaches. John Wiley & Sons, 2006.
[30] P. Diaconis and D. Freedman, “de Finetti’s theorem for Markov chains,” The Annals of Probability, JSTOR, pp. 115–130, 1980.
[31] M. Aoki, Optimization of stochastic systems: topics in discrete-time systems. Academic Press, 1967.
[32] K. J. Åström, Introduction to stochastic control theory. Courier Corporation, 2012.
[33] Y. Bar-Shalom and E. Tse, “Dual effect, certainty equivalence, and separation in stochastic control,” IEEE Transactions on Automatic Control, vol. 19, no. 5, pp. 494–500, 1974.
[34] H. Van de Water and J. Willems, “The certainty equivalence property in stochastic control theory,” IEEE Transactions on Automatic Control, vol. 26, no. 5, pp. 1080–1087, 1981.
[35] G. Parisi, Statistical field theory. Addison-Wesley, 1988.