Effect of minimal length on Landau diamagnetism and de Haas–van Alphen effect

Md. Abhishek1,2,3,a, Bhabani Prasad Mandal1,b

1 Department of Physics, Institute of Science, Banaras Hindu University, Varanasi 221005, India
2 Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India
3 Homi Bhabha National Institute, Training School Complex, Anushaktinagar, Mumbai 400094, India

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Abstract We study Landau diamagnetism in the framework of generalised uncertainty principle (GUP). We calculate the correction to magnetisation and susceptibility by constructing the grand partition function of diamagnetic material in this framework. We explicitly show that Curie’s law gets a temperature-independent correction which vanishes when quantum gravity effects are neglected. We further consider the low-temperature limit to find how GUP affects the de Haas–van Alphen effect.

1 Introduction

All the basic forces except gravity are well described in the quantum framework whereas the theory of gravity was understood based on Einstein’s General theory of relativity which is formulated in classical physics. There were certain areas where we need to merge quantum mechanics and general relativity to develop the quantum theory of gravitation. The existence of a minimum possible length, the Planck length $l_p \approx 10^{-35} m$ [1–7], was predicted by all approaches of quantum gravity theory, doubly special relativity (DSR), perturbative string theory, black hole physics, etc. The usual Heisenberg’s uncertainty principle (HUP) needs essential modifications to incorporate the existence of minimum length scale. The new uncertainty principle, known as the generalised uncertainty principle (GUP) [1, 8–11], is as follows:

$$\Delta x \Delta p \geq \frac{1}{2}(1 + \alpha(\Delta p)^2 + \alpha(p)^2),$$  

(1)

where $\Delta x$ and $\Delta p$ are the uncertainties in position and momentum, respectively. The positive parameter $\alpha = \frac{a_0^2}{\hbar} = \frac{a_0}{M_p l_p}$ does not depend on $\Delta x$ and $\Delta p$ but it may depend on the expectation values of $x$ and $p$. $a_0$ is known as the GUP parameter which is a positive dimensionless parameter and the Planck mass $M_p \approx 10^{19} Gev$. According to Eq. (1), a nonzero minimal uncertainty in length is $\Delta x_{\text{min}} \approx l_p \sqrt{a_0}$. The parameter $a_0$ is considered to be 1 in most of the GUP calculations. There are several ways to construct the modified Heisenberg algebra due to GUP, and the most commonly used modified Heisenberg algebra due to GUP approach, via the Jacobi identity [11, 12], is given by,

$$[x_i, p_j] = i(\delta_{ij} + \alpha \delta_{ij} p^2 + 2\alpha p_i p_j), \quad [x_i, x_j] = 0, \quad [p_i, p_j] = 0.$$  

(2)

To be consistent with Eq. (1), we can define the position and the momentum operators in the following way [7, 12, 13]

$$x_i \equiv x_{0i}, \quad p_i \equiv p_0(1 + \alpha \hat{p}_0^2),$$  

(3)

where $\hat{p}_0^2 = \sum_{j=1}^3 p_{0j} p_{0j}$. The operators $x_{0i}$ and $p_0(= -id/dx_{0i})$, at low energy, obey the commutation relation,

$$[x_{0i}, p_{0j}] = i\delta_{ij}.$$  

(4)

In the presence of potential $V(\vec{r})$, the Hamiltonian of any quantum mechanical system should be modified due to GUP in the following form [13–36],

$$H = \frac{\hat{p}_0^2}{2m} + V(\vec{r}) + \frac{\alpha}{m} \hat{p}_0^4 + O(\alpha^2).$$  

(5)

During the past several years, GUP and its rich consequences have been studied extensively in almost all branches of physics, including Black hole physics [11, 15, 18, 19, 30], Cosmology [17], Relativistic quantum mechanics [21, 31, 36], Non-Hermitian

a e-mail: mdabhishek@hri.res.in
b e-mail: bhabani.mandal@gmail.com (corresponding author)
theories [23, 25], Squeezed and Coherent states [26–29], and Statistical mechanics [37–40]. In quantum optics, experiments with a noticeable quantum gravity effect were proposed [41, 42]. Higher-order GUP and its effect have been considered in [22, 39, 40]. In the region of minimum length, exact solutions to several relativistic and non-relativistic problems along with a review of the subject have been obtained [14]. The effect of minimum length on electron magnetism using the ideas of quantum statistical mechanics was considered in [37] to remove the degeneracy of Landau levels. More recently in [40], the thermodynamics of ideal gas, Unruh–Davies–Dewitt–Fulling effect, blackbody radiation has been investigated using the statistical mechanical formulation with GUP. Various GUP-modified thermodynamical quantities have been compared using three dissimilar forms of GUP in the formulation of statistical mechanics [38, 43]. New higher-order GUP formulation in statistical mechanics is considered to calculate the canonical partition function for an ideal gas in one dimension, and hence, GUP-modified thermodynamical quantities were calculated in [39]. In this article, we revisit the effect of minimum length on diamagnetism by constructing the grand partition function and studying Curie’s law and the de Haas–van Alphen effect.

In this section, we study the diamagnetic property of material under the GUP effect. For this, we first calculate the grand-canonical partition function of the diamagnetic material without GUP effect [44, 45]. Now, using the modified commutation relation due to quantum gravity, the single-particle energy levels of electrons are given by [13, 32],

$$\epsilon_j = \frac{p_z^2}{2m} + 2\mu B \left( j + \frac{1}{2} \right) + \alpha \left[ \frac{p_z^4}{m} + 16m\mu^2B^2\left( j + \frac{1}{2} \right)^2 \right] + \mathcal{O}(\alpha^2).$$

(6)

These quantised energy levels with quantum number $j$ are degenerate. Now, the density of the one particle states [46],

$$g_j = \frac{1}{4\pi^2} \int_{p_j}^{p_{j+1}} dp_x dp_y dxdy \left[ \frac{1}{(1 + \alpha p_z^2)^3} \right]$$

$$= \frac{V^{2/3}}{\pi} \mu B[1 - 6\alpha m\mu B(2j + 1)] + \mathcal{O}(\alpha^2),$$

(7)

where $p_j = \sqrt{4m\mu B}$. As opposed to the low-energy case, we can see that $g_j$ is dependent on the quantum number $j$. At finite temperature $T$, the grand-canonical partition function of the electron gas becomes,

$$\ln Z_G = \int_{-\infty}^{+\infty} \frac{V^{1/3} dp_z}{2\pi(1 + \alpha p_z^2)^{3/2}} \sum_{j=0}^{\infty} g_j \ln \left[ 1 + z \exp(-\beta \epsilon_j) \right]$$

(8)

At a high temperature or equivalently at low $\beta (= \frac{1}{T})$, we can take the fugacity, $z << 1$, and the system is effectively Boltzmannian, thus,

$$\ln Z_G = \int_{-\infty}^{+\infty} \frac{z V^{1/3} dp_z}{2\pi(1 + \alpha p_z^2)^{3/2}} \sum_{j=0}^{\infty} g_j \exp(-\beta \epsilon_j)$$

$$= \frac{V z}{\lambda^3} \cosh x \left[ 1 - \frac{m\alpha}{\beta} (5 + 6x \coth x + 4x^2 (\coth^2 x + \cosech^2 x)) \right].$$

(9)

To make the above expression compact, we have introduced the thermal wavelength $\lambda \equiv \sqrt{\frac{2\pi}{mT}}$ and a new variable $x \equiv \frac{\mu B}{T}$. In case of $\alpha \to 0$, the above equation is reduced to the grand-canonical partition function of the diamagnetic material without GUP effect [44, 45]. Now, the average number of electrons in the system is,

$$\bar{N} = \left( z \frac{\partial}{\partial z} \ln Z_G \right)_{T, V, B}.$$  

(10)
Fig. 1 Specific heat at constant volume, \( C_V/\bar{N} \) vs. \( x = \frac{\mu B}{T} \) plot of a diamagnetic system with electron mass, \( m = 0.5 \times 10^{-3} \text{GeV} \). In (a), we set the GUP-dependent parameter, \( \alpha m_B m_B = 10^{-3} \), and show the full \( C_V/\bar{N} \) vs. \( x = \frac{\mu B}{T} \) curve along with \( \alpha = 0 \) curve. (b) shows the zoomed view nature of the two curves with \( \alpha m_B m_B = 0 \) and \( \alpha m_B m_B = 10^{-8} \) at extremely high temperature.

At high-temperature limit, fugacity \( z \to 0 \) and if the field intensity \( B \) and the temperature \( T \) are such that \( \mu B \ll T \) then, \( \bar{N} \simeq \frac{V z}{\lambda^3} [1 - 19 \frac{\alpha}{\beta} m] \).

(11)

To study thermodynamic properties of the system at high temperature, let us compute the internal energy of the system in terms of variable \( x \),

\[
U = -\left( \frac{\partial}{\partial \beta} \ln Z_G \right)_{V,z} = -\mu B \left( \frac{\partial}{\partial x} \ln Z_G \right)_{V,z} = V z \left( \frac{m}{2\pi} \right)^{3/2} (\mu B)^{3/2} \left( \frac{\text{cosech} x}{2 x^{3/2}} \right) \left[ (x + 2x^2 \coth x) - \alpha m_B \left\{ 15 + 8x^2 \coth^2 x + 8x^3 \coth^3 x + 8x^2 \text{cosech}^2 x + 8x \coth x (2 + 5x^2 \text{cosech}^2 x) \right\} \right].
\]

(12)

Now, the specific heat at constant volume,

\[
C_V = \left( \frac{\partial U}{\partial T} \right)_V = -\frac{\mu B}{T^2} \left( \frac{\partial U}{\partial x} \right)_V = \frac{\bar{N}}{4} \left[ (3 + 4x \coth x + 4x^2 \coth^2 x + 4x^2 \text{cosech}^2 x) \right. \\
- \frac{4\alpha m_B}{x} \left[ 15 + 10x \coth x - 8x^3 \coth^3 x + 2x^2 \text{cosech}^2 x + 16x^4 \text{cosech}^4 x + \coth^2 x (-4x^2 + 64x^4 \text{cosech}^2 x) \right].
\]

(13)

The GUP-corrected specific heat \( (C_V) \) differs from without GUP value at very high temperatures in a constant magnetic field. But without the quantum gravity effect, \( C_V \) acquires a constant value of \( \frac{15}{4} \bar{N} \) as follows,

\[
C_V = \frac{\bar{N}}{4} \left[ 3 + 4x \coth x + 4x^2 (\coth^2 x + \text{cosech}^2 x) \right] \Rightarrow \frac{15}{4} \bar{N} \text{ for } \mu B \ll T.
\]

(14)
Let us now study the magnetic behavior of the system in the presence of quantum gravity. Magnetic moment $M$ of the gas is given by,

$$ M = \frac{1}{\beta} \left( \frac{\partial}{\partial B} \ln Z_G \right)_{T,V,z}. $$

(15)

For high-temperature limit, the magnetic moment per unit volume, i.e., the magnetisation is,

$$ \mathcal{M} = \frac{M}{V} = -\frac{1}{3} \frac{z \mu_B}{V} \left[ 1 + \frac{m \alpha}{\beta} \right]. $$

(16)

Now at high temperature, the magnetic susceptibility per unit volume as a function of temperature $T$ and specific volume $V = V/\bar{N}$ becomes,

$$ \chi = \left( \frac{\partial \mathcal{M}}{\partial B} \right)_{T,V,z} $$

$$ = -\frac{\mu_B^2}{3VT} \left[ 1 + 20 \alpha m T \right]. $$

(17)

The effect of GUP on this phenomenon is very tiny to detect experimentally in the laboratory. To demonstrate this small GUP corrections graphically, we have taken the GUP parameter $\alpha = 1 GeV^{-2}$ in Fig. 1. The $1/T^2$ dependence in Fig. 1 confirms that Curie’s law still holds with the effect of minimum length. Also, the Curie constant ($\mu_B^2/3V$) reflects the quantisation of the orbits. The negative sign in Eq.(17) implies the property of diamagnetism ($\chi < 0$) is not affected by quantum gravity. In the case of $\alpha \to 0$, the above equation is reduced to the low-energy susceptibility per unit volume of the electron gas (Fig. 2).

3 Low-temperature de Haas–van Alphen effect

At a very low temperature ($T \to 0$), the susceptibility of an ideal Fermi gas discontinuously changes with the varying magnetic field $B$, this phenomenon is known as the de Haas–van Alphen effect [44]. In this section, we study the effect of GUP on this quantum mechanical effect.

We consider $E_0$ to be the ground-state energy of an ideal Fermi gas at absolute zero, and $E_0$ is a function of the field strength $B$. To simplify our calculation, we ignore the motion of the electrons along the direction of magnetic field $B$, i.e. we take $p_z = 0$. In other words, we consider a two-dimensional Fermi gas whose single-particle energy levels are,

$$ \epsilon_j = \mu B(2j + 1)[1 + 4\alpha m \mu B(2j + 1)], $$

(18)

which is $g_j$-fold degenerate, with,

$$ g_j = g[1 - 6\alpha m \mu B(2j + 1)], $$

(19)

where $g = \frac{\sqrt{2\pi}}{2\pi} e B$ is constant low-energy density of states. The ground-state energy $E_0$ is the sum of $\epsilon_j$ over the lowest $N$ single-particle states. Since $j$-th level degeneracy $g_j$ depends on the field strength $B$, the maximum number of particles that can have the
energy $\epsilon_j$ depends on $B$. If the field $B$ is such that $g_0 \geq N$, then all particles can occupy the lowest energy level ($j = 0$) and $E_0$ becomes,

$$E_0(B) = N\epsilon_0 = N\mu B[1 + 4\alpha m\mu B],$$

$$\frac{E_0(B)}{N} = \mu B_0 y + 4\alpha m\mu^2 B_0^2 y^2; \quad \text{for } g_0 \geq N,$$

where we have defined two new variables $B_0 := \frac{NB}{V}$ and $y := \frac{B}{B_0}$. The variable $B_0$ depends on the system, the volume ($V$), and total number of electrons ($N$). For a particular value of field $B$, suppose $\sum_{i=0}^{j} g_i < N < \sum_{i=0}^{j+1} g_i$, then the $(j + 1)$ number of lowest levels is completely filled with $g_i$ number of particles in every $i$th level. Also, the $(j + 1)$st level is only partially filled, and the higher levels are empty. In this case, the ground-state energy in terms of $B_0$ and $y$ becomes,

$$E_0(B) = \sum_{i=0}^{j} g_i \epsilon_i + \left[ N - \sum_{i=0}^{j} g_i \right] \epsilon_{j+1}$$

$$= g\mu B \left[ \sum_{i=0}^{j} (2i + 1 - 2\alpha m\mu B \sum_{i=0}^{j} (2i + 1)^2 \right] + \mu B \left[ N - \left\{ \sum_{i=0}^{j} g - 6\alpha m\mu B \sum_{i=0}^{j} (2i + 1) \right\} \right]$$

$$\times \left[ (2j + 3) + 4\alpha m\mu B (2j + 3)^2 \right]$$

$$= g\mu B \left[ (j + 1)^2 - \frac{2}{3} \alpha m\mu B (j + 1)(2j + 1)(2j + 3) \right] + \mu B \left[ N - g \left\{ (j + 1) - 6\alpha m\mu B (j + 1)^2 \right\} \right]$$

$$\times \left[ (2j + 3) + 4\alpha m\mu B (2j + 3)^2 \right],$$

$$\frac{E_0(B)}{N} = \mu B_0 y (2j + 3 - y(j + 1)(j + 2)) + \frac{2}{3} \alpha m B_0^2 \mu^2 y^2 \left[ 6(2j + 3)^2 - 5y(j + 1)(j + 2)(2j + 3) \right];$$

for $\sum_{i=0}^{j} g_i < N < \sum_{i=0}^{j+1} g_i$. \hspace{1cm} (21)

As calculated in the previous section, the magnetisation of the system is given by,

$$\mathcal{M} = -\frac{1}{V} \frac{\partial E_0}{\partial B}.$$

Now, there are two situations when the total number of electrons is greater than the degeneracy of the lowest level with $j = 0$, or greater than the degeneracy of any $j$-th level of the system as discussed above. So, the magnetisation,

$$\mathcal{M} = -\frac{\mu}{V} \left[ 1 + 8\alpha m B_0 y \right]; \quad \text{for } g_0 \geq N$$

$$\mathcal{M} = \frac{\mu}{V} \left[ (2j + 1)(j + 2)y - (j + 3) \right] - 2\alpha m B_0 \mu y \left[ 4(2j + 3)^2 - 5y(j + 1)(j + 2)(2j + 3) \right];$$

for $\sum_{i=0}^{j} g_i < N < \sum_{i=0}^{j+1} g_i$. \hspace{1cm} (22)

Figure 3 shows that the GUP-corrected magnetisation still changes discontinuously with the magnetic field. Also interestingly, the curves approach the low-energy results in higher $j$ values.

Now, the susceptibility per unit volume is,

$$\chi = \frac{\partial \mathcal{M}}{\partial B} = -\frac{1}{V} \frac{\partial^2 E_0}{\partial B^2}. \hspace{1cm} (25)$$

Depending upon the electron numbers, we have,

$$\chi = -\frac{8\alpha}{V} m^2 \mu^2; \quad \text{for } g_0 \geq N$$

$$\chi = \frac{2\mu}{VB_0} \left( j + 1 \right)(j + 2) - \frac{2\alpha}{V} m^2 \mu^2 \left[ 4(2j + 3)^2 - 10y(j + 1)(j + 2)(2j + 3) \right]; \quad \text{for } \sum_{i=0}^{j} g_i < N < \sum_{i=0}^{j+1} g_i. \hspace{1cm} (26)$$

As opposed to the magnetisation case, in Fig. 4, the susceptibility approaches to the low-energy values in higher $j$ values.

From the above results, it is straightforward to see that the magnetisation ($\mathcal{M}$) and the magnetic susceptibility per unit volume ($\chi$) are reduced to the result without minimal length effect as $\alpha \to 0$. As we have discussed in the previous section, the effect of GUP on this quantum phenomenon is practically undetectable, for the visualisation of results graphically, we have taken the value of the GUP parameter to be very large in Figs. 3 and 4.
Fig. 3 Reduced magnetisation, $M/(\mu V)$ vs. magnetic field, $y = B/B_0$ plot in de Haas–Van Alphen effect, with the GUP-dependent parameter, $amB_0\mu = 10^{-2}$. The blue curve indicates without GUP correction, and the yellow one indicates the result with GUP correction.

Fig. 4 Reduced susceptibility, $\chi/(2\mu V B_0)$ vs. magnetic field, $y = B/B_0$ plot in de Haas–Van Alphen effect, with the GUP-dependent parameter, $amB_0\mu = 10^{-2}$. The blue curve indicates without GUP correction, and the yellow one indicates the result with GUP correction.

4 Conclusions

The effect of gravity becomes very interesting at an extremely small length scale, which is very high energy. Thus to address fundamental areas of physics, we must resort to quantum gravity theories. In this article, we have studied how the theory of Landau diamagnetism gets modified when we consider the effect of quantum gravity theories. Our analysis shows that the overall behavior of the magnetic susceptibility for diamagnetic material is the same. It gets only a temperature-independent constant shift in the first order (first order in $\alpha$) correction due to GUP. In the absence of quantum gravity effects, it will reduce to $1/T$ dependence (usual Curie’s law). We also investigate the thermodynamical properties of the system at high temperature by calculating the specific heat ($C_V$). GUP effect on $C_V$ is very prominent at extremely high temperature [Fig. 1]. At a very low temperature, the susceptibility of an ideal Fermi gas changes discontinuously with the magnetic field, commonly known as the de Haas–van Alphen effect. We observed a modified de Haas–van Alphen effect in the presence of GUP as explained in Fig. 4. Although the effect of GUP on our system is very small to detect it experimentally in the laboratory, we have taken the value of the GUP parameter to be very large to show the corrections graphically. A similar analysis using non-commutative algebra (NC) was done in [47], which may help to further investigate the relationship between NC algebra and GUP for this simple system of an ideal Fermi gas.

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