APPROXIMATION PROPERTIES FOR NON-COMMUTATIVE $L_p$-SPACES ASSOCIATED WITH DISCRETE GROUPS

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Abstract. Let $1 < p < \infty$. It is shown that if $G$ is a discrete group with the approximation property introduced by Haagerup and Kraus, then the non-commutative $L_p(VN(G))$ space has the operator space approximation property. If, in addition, the group von Neumann algebra $VN(G)$ has the QWEP, i.e. is a quotient of a $C^*$-algebra with Lance’s weak expectation property, then $L_p(VN(G))$ actually has the completely contractive approximation property and the approximation maps can be chosen to be finite-rank completely contractive multipliers on $L_p(VN(G))$. Finally, we show that if $G$ is a countable discrete group having the approximation property and $VN(G)$ has the QWEP, then $L_p(VN(G))$ has a very nice local structure, i.e. it is a $\mathcal{COL}_p$ space and has a completely bounded Schauder basis.

1. Introduction

Approximation properties for Banach spaces were first studied by Grothendieck [17]. The corresponding non-commutative analogue of Grothendieck’s program has been developed and successfully applied to operator algebras, and more recently to operator spaces. These non-commutative approximation properties have played a crucial role in the study of von Neumann algebras and $C^*$-algebras (particularly, for group von Neumann algebras and group $C^*$-algebras). For example, it is well-known (from [20], 8 and 9) that a discrete group $G$ is amenable if and only if the reduced group $C^*$-algebra $C^*_{red}(G)$ is nuclear (respectively, the group von Neumann algebra $VN(G)$ is injective). Some weaker conditions (i.e. weak amenability and approximation property) for locally compact groups have been studied by Haagerup and Kraus (see [19] and [20]). It was shown in [20] that a discrete group $G$ has the approximation property (or simply, AP) if and only if $C^*_{red}(G)$ has the operator space approximation property (or simply, OAP) of Effros and Ruan [4] (respectively, $VN(G)$ has the weak* OAP of Kraus [28]). The approximation property for a group $G$ is also closely related to its Fourier algebra $A(G)$, which can be identified with the operator predual of $VN(G)$. Since a dual operator space $V^*$ has the weak* OAP if and only if $V$ has the OAP (see [7]), we can conclude that a discrete group
G has the AP if and only if its Fourier algebra $A(G)$ has the OAP (see §2 for a direct proof).

Non-commutative $L_p$-spaces (for $1 \leq p < \infty$) are also very important in the study of non-commutative harmonic analysis and non-commutative probability theory. Given a von Neumann algebra $R$, we may isometrically identify $L_1(R)$ with the predual $R_\ast$ of $R$. However, it is very important to note that since we usually use the trace duality between $R$ and $R_\ast$, the correct operator space matrix norm on $L_1(R)$ should be given by the opposite operator space matrix norm on $R_\ast$, i.e. we should completely isometrically identify $L_1(R)$ with $(R_\ast)^{\text{op}} = (R^{\text{op}})_\ast$ (see detailed explanations in §3). Then we may use Pisier’s complex interpolation method to obtain a canonical operator space matrix norm on the non-commutative $L_p$ space. The purpose of this paper is to study the approximation properties for non-commutative $L_p(VN(G))$ spaces associated with discrete groups $G$. Our main results can be stated as follows.

**Theorem 1.1.** Let $1 < p < \infty$. If $G$ is a discrete group with the AP, then $L_p(VN(G))$ has the OAP.

Using a similar technique, we can also prove (in Proposition 3.5) that if $G$ is a weakly amenable discrete group and $1 < p < \infty$, then $L_p(VN(G))$ has the completely bounded approximation property.

It is known by Grothendieck [17] (see [31]) that if a Banach space dual $V$ is separable and has the Grothendieck’s approximation property, then $V$ actually has the contractive approximation property. The separability can be removed if $V$ is a reflexive space (see [11]). At this moment, we can not obtain such a general result for operator space duals. However, we may obtain the following result with an additional assumption that $VN(G)$ has the QWEP introduced by Kirchberg [25], i.e. $VN(G)$ is a quotient of a $C^\ast$-algebra with Lance’s weak expectation property. A $C^\ast$-algebra $A$ is said to have the weak expectation property if for the universal representation $\pi : A \rightarrow B(H)$, there is a completely positive and contractive map $P : B(H) \rightarrow A^{\ast\ast}$ such that $P \circ \pi = id_A$ (see Lance [29]).

**Theorem 1.2.** Let $1 < p < \infty$. If $G$ is a discrete group with the AP and $VN(G)$ has the QWEP, then $L_p(VN(G))$ has the completely contractive approximation property.

Combining Theorem 1.2 and the recent results in [24], we can show in Theorem 1.2 that if $G$ is a countable discrete group satisfying the conditions given in Theorem 1.2, then $L_p(VN(G))$ has a very nice local structure, i.e. it is a $\text{COL}_p$ space and has a completely bounded Schauder basis (see definitions in §5). This is a quite surprising result, and it is only true for $1 < p < \infty$. Indeed, for $p = 1$, it is known from [11] and [24] that $L_1(VN(G)) = VN(G)^{\text{op}}$ (equivalently, $VN(G)_\ast$) is a $\text{COL}_1$ space if and only if $VN(G)$ is an injective von Neumann algebra. For $p = \infty$, you can provide the text in a more readable format.
we need to consider the reduced group $C^*$-algebra $C^*_{\text{red}}(G)$, and it is known from [11] that $C^*_{\text{red}}(G)$ is a $\mathcal{O}C\mathcal{L}_\infty$ space if and only if $C^*_{\text{red}}(G)$ is a nuclear $C^*$-algebra. Therefore, in the case of $p = 1$ or $\infty$, this can only happen when a discrete group $G$ is amenable.

The paper is organized as follows. In §2, we recall some necessary notions and results on operator spaces and completely bounded multipliers of Fourier algebras, and we clarify some simultaneous convergence properties for completely bounded multipliers of $C^*_{\text{red}}(G)$ and $A(G)$ (see Propositions 2.2 and 2.3). In §3, we recall the complex interpolation for operator spaces introduced by Pisier [33] and recall non-commutative $L_p(R)$ spaces arising from von Neumann algebras $R$. Since we are mainly interested in the case of group von Neumann algebras $VN(G)$ for discrete groups $G$, it suffices to consider von Neumann algebras with normal faithful tracial states. The readers are referred to [27], [40], [22] and [16] for non-commutative $L_p$-spaces arising, by complex interpolation, from general von Neumann algebras. We prove Theorem 1.1 in §3, and prove Theorem 1.2 in §4. Motivated by an argument given in [20, Theorem 2.1], we are able to show in Theorem 4.4 that the completely contractive approximation maps in Theorem 1.2 can actually be chosen to be completely contractive multipliers on $L_p(VN(G))$. This result will be very useful in non-commutative harmonic analysis. We end the paper by §5, in which we study the local structure and completely bounded Schauder basis for $L_p(VN(G))$ spaces. We include some interesting examples of countable residually finite discrete groups $G$ with the AP, for which the $L_p(VN(G))$ spaces are $\mathcal{O}CL_p$ spaces and have completely bounded Schauder bases.

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2. Preliminaries

We assume that the readers are familiar with the basic notions in operator algebras and operator spaces. The readers are referred to Takesaki’s book [39] and Stratila’s book [38] for details on operator algebras, and are referred to Paulsen’s book [32], the recent book of Effros and Ruan [12], and Pisier’s book [33] for the details on operator spaces and completely bounded maps. We recall that an operator space $V$ is said to have the completely bounded approximation property (or simply, CBAP) if there exists a net of finite-rank maps $T_\alpha : V \to V$ such that $\|T_\alpha\|_{cb} \leq \lambda$ for some constant $\lambda$ and $T_\alpha \to id_V$ in the point-norm topology on $V$, i.e. we have $\|T_\alpha(x) - x\| \to 0$ for all $x \in V$. We let

$$\Lambda(V) = \inf\{\lambda\}$$

(2.1)

denote the CBAP constant of $V$. An operator space $V$ is said to have the completely contractive approximation property (or simply, CCAP) if $\Lambda(V) = 1$. In [1],
Effros and the second author studied the operator space analogue of Grothendieck’s approximation property for Banach spaces. We recall that an operator space $V$ is said to have the operator space approximation property (or simply, OAP) if there exists a net of finite-rank maps $T_{\alpha} : V \to V$ such that $T_{\alpha} \to \text{id}_V$ in the stable point-norm topology on $V$, i.e. we have $\|T_{\alpha} \otimes \text{id}_{K_\infty}(x) - x\| \to 0$ for all $x \in V \otimes K_\infty$, where $\otimes$ denotes the operator space injective tensor product. It is easy to see that CBAP implies OAP.

Let us assume that $G$ is a discrete group. The left regular representation $\lambda : G \to B(\ell_2(G))$ is defined by

$$\lambda(s)\xi(t) = \xi(s^{-1}t)$$

for all $\xi \in \ell_2(G)$ and $s, t \in G$. If we let $\lambda(\mathbb{C}[G]) = \text{span}\{\lambda(s), s \in G\}$, then the reduced group C*-algebra $C^*_r(G)$ and the group von Neumann algebra $VN(G)$ are the norm closure and the weak* closure of $\lambda(\mathbb{C}[G])$ in $B(\ell_2(G))$, respectively. The Fourier algebra

$$A(G) = \{f : f(t) = \langle \lambda(t)\xi | \eta \rangle \text{ for some } \xi, \eta \in \ell_2(G)\}$$

is the space of all coefficient functions of the left regular representation $\lambda$. Given $f \in A(G)$, its norm is given by

$$\|f\| = \inf\{\|\xi\| \|\eta\| : f(t) = \langle \lambda(t)\xi | \eta \rangle\}.$$ 

It was shown by Eymard [15] that $A(G)$ with this norm and the pointwise multiplication $m$ is a (contractive) commutative Banach algebra. Moreover, there is a natural operator space matrix norm on $A(G)$ obtained by identifying $A(G)$ with the operator predual $VN(G)_*$ of $VN(G)$. With this operator space matrix norm, the multiplication $m : A(G) \times A(G) \to A(G)$ is completely contractive in the sense that

$$\|[m(f_{ij}, h_{kl})]\| \leq \|[f_{ij}]\| \|[h_{kl}]\|$$

for all $[f_{ij}] \in M_m(A(G))$ and $[h_{kl}] \in M_n(A(G))$, and thus $A(G)$ is a completely contractive Banach algebra (see [13] and [22]).

A function $\varphi$ on $G$ is called a multiplier of $A(G)$ if $m_\varphi(f) = \varphi f$ maps $A(G)$ into $A(G)$. It is easy to show from the closed graph theorem that if $\varphi$ is a multiplier of $A(G)$, then the map $m_\varphi : A(G) \to A(G)$ is automatically bounded. If the map $m_\varphi$ is completely bounded on $A(G)$, we call $\varphi$ a completely bounded multiplier of $A(G)$. We let $M_0A(G)$ denote the space of all completely bounded multipliers of $A(G)$, which is equipped with the cb-norm on $A(G)$. It is clear that if $\varphi \in A(G)$, then the multiplication map $m_\varphi : A(G) \to A(G)$ is completely bounded with

$$\|m_\varphi\|_{cb} \leq \|\varphi\|.$$ 

Therefore, we have a norm decreasing inclusion $A(G) \hookrightarrow M_0A(G)$. If $G$ is an amenable group then $A(G) \hookrightarrow M_0A(G)$ is an isometric inclusion.
Amenability is one of the most important subjects in harmonic analysis. We recall that a group \( G \) is amenable if and only if \( A(G) \) has a contractive approximate identity. If we let \( A_c(G) \) denote the space of all elements in \( A(G) \) with compact supports, then \( A_c(G) \) is norm dense in \( A(G) \) and thus the amenability of \( G \) is equivalent to the existence of a net of \( \varphi_\alpha \in A_c(G) \) such that \( \| \varphi_\alpha \| \leq 1 \) and \( m_{\varphi_\alpha} \to id_{A(G)} \) in the point-norm topology. Haagerup introduced a weaker amenability condition for \( G \) in \([19]\). Let us recall that a group \( G \) is said to be weakly amenable if there exists a net of \( \varphi_\alpha \in A_c(G) \) such that \( \| m_{\varphi_\alpha} \|_{cb} \leq \lambda \) for some constant \( \lambda \) and \( m_{\varphi_\alpha} \to id_{A(G)} \) in the point-norm topology. It is clear from the definition that amenability implies weak amenability, but the converse is not true (see \([21], [4] \) and \([20, §3]\)).

It is also known that \( M_0 A(G) \) equipped with the cb-norm is a dual space. It has a predual \( Q(G) \), which is the closure of \( \ell_1(G) \) under the norm given by

\[
\|f\|_Q = \sup \{ | \int_G f(t) \varphi(t) \, dt | : \varphi \in M_0 A(G), \| \varphi \|_{cb} \leq 1 \}
\]

(see \([21], [3] \) and \([20, §1]\)). A group \( G \) is said to have the approximation property (or simply, \( AP \)) if there exists a net of \( \{ \varphi_\alpha \} \) in \( A_c(G) \) such that \( m_{\varphi_\alpha} \to id_{A(G)} \) in the \( \sigma(M_0 A(G), Q(G)) \) topology. It was shown by Haagerup and Kraus \([20]\) that weak amenability implies \( AP \), but the converse is not true (see \( G = \mathbb{Z}^2 \times SL(2, \mathbb{Z}) \)).

Given \( \varphi \in M_0 A(G) \), the adjoint map \( M_\varphi = m_\varphi^* : V N(G) \to V N(G) \) is a weak* continuous completely bounded map on \( V N(G) \), which satisfies

\[
M_\varphi(\lambda(t)) = \varphi(t) \lambda(t).
\]

Since \( M_\varphi \) maps \( C^*_{red}(G) \) into \( C^*_{red}(G) \), it also induces a completely bounded map \( \overline{\varphi} : C^*_{red}(G) \to C^*_{red}(G) \). All these maps have the same cb-norm, i.e. we have

\[
\| m_\varphi \|_{cb} = \| M_\varphi \|_{cb} = \| \overline{\varphi} \|_{cb}.
\]

Let us write \( M_\infty = B(\ell_2), K_\infty = K(\ell_2) \) and \( T_\infty = K(\ell_2)^* = B(\ell_2)_* \), and let \( \bar{\otimes} \) and \( \hat{\otimes} \) denote the normal spatial and operator space projective tensor products, respectively. Then for any \( \varphi \in M_0 A(G) \), we may obtain completely bounded maps \( m_\varphi \otimes id_\infty, \overline{\varphi} \otimes id_\infty \) and \( M_\varphi \otimes id_\infty \) on \( A(G) \hat{\otimes} K_\infty, C^*_{red}(G) \hat{\otimes} K_\infty \) and \( V N(G) \bar{\otimes} M_\infty \), respectively. Given \( a \in V N(G) \bar{\otimes} M_\infty \) and \( f \in A(G) \hat{\otimes} T_\infty \), we may define a bounded linear functional \( \omega_{a,f} \) on \( M_0 A(G) \) given by

\[
\omega_{a,f}(\varphi) = \langle M_\varphi \otimes id_\infty(a), f \rangle.
\]

We may similarly define bounded linear functionals on \( M_0 A(G) \) by considering

\[
\hat{\omega}_{a,f}(\varphi) = \langle \overline{M}_\varphi \otimes id_\infty(a), f \rangle
\]

for \( a \in C^*_{red}(G) \hat{\otimes} K_\infty \) and \( f \in (C^*_{red}(G) \hat{\otimes} K_\infty)^* \), and

\[
\tilde{\omega}_{a,f}(\varphi) = \langle a, m_\varphi \otimes id_\infty(f) \rangle.
\]
for \( f \in A(G) \hat{\otimes} K_\infty \) and \( a \in (A(G) \hat{\otimes} K_\infty)^* \). Haagerup and Kraus proved in [20, Propositions 1.4 and 1.5] that the bounded linear functionals defined in (2.3) and (2.4) are all contained in \( Q(G) \), and on the other hand, all linear functionals in \( Q(G) \) have such forms. We note that this is also true for (2.3). We summarize these in the following proposition.

**Proposition 2.1.** (Haagerup and Kraus [20]) Let \( G \) be a discrete group. Then the bounded linear functionals defined in (2.3), (2.4) and (2.5) are all contained in \( Q(G) \), and we actually have

\[
Q(G) = \{ \omega_{a,f} : a \in VN(G) \overline{\otimes} M_\infty, f \in A(G) \hat{\otimes} T_\infty \} = \{ \tilde{\omega}_{a,f} : a \in C^*_{red}(G) \hat{\otimes} K_\infty, f \in (C^*_{red}(G) \hat{\otimes} K_\infty)^* \} = \{ \tilde{\omega}_{a,f} : f \in A(G) \hat{\otimes} K_\infty, a \in (A(G) \hat{\otimes} K_\infty)^* \}.
\]

**Proof.** We only need to study (2.6) and the last equality. Given \( f \in A(G) \hat{\otimes} K_\infty \), \( a \in (A(G) \hat{\otimes} K_\infty)^* = VN(G) \overline{\otimes} T_\infty \) and \( \varepsilon > 0 \), it is known from the operator space theory (see [1] and [12]) that we may write

\[
(2.6) \quad a = \alpha \hat{\otimes} \beta = [\sum_{j,k} \alpha_{ij} \hat{\otimes} j_{jk} \beta_{kl}]
\]

for some Hilbert-Schmidt matrices \( \alpha = [\alpha_{ij}], \beta = [\beta_{kl}] \in HS_\infty \) with \( \|\alpha\|_2 = \|\beta\|_2 = 1 \) and \( \tilde{\alpha} = [\hat{\otimes} j_{jk}] \in VN(G) \overline{\otimes} M_\infty \) with \( \|\tilde{\alpha}\| < \|a\| + \varepsilon \). Then for any \( \varphi \in M_0 A(G) \),

\[
\tilde{\omega}_{a,f}(\varphi) = \langle a, m_{\varphi} \otimes id_\infty(f) \rangle = \langle \alpha \hat{\otimes} \beta, m_{\varphi} \otimes id_\infty(f) \rangle = \| M_{\varphi} \otimes id_\infty(\tilde{\alpha}, \alpha^{tr} f \beta^{tr}) = \omega_{\tilde{a},f} \in Q(G) \).
\]

Since \( \alpha^{tr} f \beta^{tr} \in A(G) \hat{\otimes} T_\infty \), we can conclude from the first equality that \( \tilde{\omega}_{a,f} = \omega_{\tilde{a},f} \) is an element in \( Q(G) \).

On the other hand, given any \( \omega \in Q(G) \), it is known from [20, Proposition 1.5] that we can write \( \omega = \omega_{a,f} \) for some \( a \in VN(G) \overline{\otimes} M_\infty \) and \( f \in A(G) \hat{\otimes} T_\infty \).

Using a similar calculation as that given in (2.6), we may write \( f = \alpha f \beta \) for some \( \alpha, \beta \in HS_\infty \) and \( \tilde{f} \in A(G) \hat{\otimes} K_\infty \), and thus

\[
\omega = \omega_{a,f} = \tilde{\omega}_{\tilde{a},f}
\]

with \( \tilde{a} = \alpha^{tr} a \beta^{tr} \in VN(G) \overline{\otimes} T_\infty = (A(G) \hat{\otimes} K_\infty)^* \).

Proposition 2.1 shows that a net of completely bounded multipliers \( \{ \varphi_{\alpha} \} \) converges to \( \varphi \) in the \( \sigma(M_0 A(G), Q(G)) \) topology if and only if the corresponding net of completely bounded maps \( \{ m_{\varphi_{\alpha}} \} \) converges to \( m_{\varphi} \) (respectively, \( \{ \tilde{M}_{\varphi_{\alpha}} \} \) converges to \( \tilde{M}_{\varphi} \)) in the stable point-weak topology. If \( G \) is a discrete group, then every element \( \varphi \in A_c(G) \) has a finite support, and thus \( m_{\varphi} \) is a finite-rank map on \( A(G) \) (respectively, \( \tilde{M}_{\varphi} \) is a finite-rank map on \( C^*_{red}(G) \)). It follows that \( G \) has the AP if and only if there exists a net of \( \{ \varphi_{\alpha} \} \) in \( A_c(G) \) such that the finite-rank maps \( m_{\varphi_{\alpha}} \) converge to \( id_{A(G)} \) (respectively, \( \tilde{M}_{\varphi_{\alpha}} \) converge to \( id_{C^*_{red}(G)} \)) in
the stable point-weak topology. By a standard convexity argument we may obtain a net of \( \{ \tilde{\varphi}_c \} \) in \( A_c(G) \) such that \( m_{\tilde{\varphi}_c} \to id_{A(G)} \) in the stable point-norm topology. This gives a direct proof that \( G \) has the AP if and only if \( A(G) \) has the OAP. Similarly, we can also find a (possibly different) net of \( \{ \tilde{\psi}_\beta \} \) in \( A_c(G) \) such that \( \overline{M}_{\tilde{\psi}_\beta} \to id_{C^*_{red}(G)} \) in the stable point-norm topology. The following proposition shows that we can actually choose the same net of \( \{ \varphi_c \} \) in \( A_c(G) \) such that \( m_{\varphi_c} \to id_{A(G)} \) and \( \overline{M}_{\varphi_c} \to id_{C^*_{red}(G)} \) simultaneously.

**Proposition 2.2.** Let \( G \) be a discrete group with the AP. Then there exists a net of \( \{ \varphi_c \} \) in \( A_c(G) \) such that \( m_{\varphi_c} \to id_{A(G)} \) in the stable point-norm topology on \( A(G) \) and \( \overline{M}_{\varphi_c} \to id_{C^*_{red}(G)} \) in the stable point-norm topology on \( C^*_{red}(G) \).

**Proof.** It suffices to show that given any \( f \in A(G) \otimes K_\infty \), \( a \in C^*_r(G) \otimes K_\infty \) and \( \varepsilon > 0 \), we can find an element \( \varphi \in A_c(G) \) such that

\[
\| m_{\varphi} \otimes id_\infty(f) - f \| < \varepsilon \quad \text{and} \quad \| \overline{M}_{\varphi} \otimes id_\infty(a) - a \| < \varepsilon.
\]

If \( F = \{ f_1, \cdots, f_n \} \) is any (non-empty) finite subset of \( (C^*_r(G) \otimes K_\infty)^* \), then we have \( \tilde{\omega}_{a, f_i} \in Q(G) \), and it follows from Proposition 2.1 that there exist \( \tilde{\alpha}_i \in VN(G) \otimes T_\infty \) and \( \tilde{f}_i \in A^c(G) \otimes K_\infty \) such that

\[
\tilde{\omega}_{a, f_i} = \tilde{\omega}_{\tilde{\alpha}_i, \tilde{f}_i}.
\]

Since \( G \) has the AP, there exists \( \varphi_F \in A_c(G) \) such that

\[
\| m_{\varphi_F} \otimes id_\infty(f_i) - f_i \| < \varepsilon
\]

and

\[
\| m_{\varphi_F} \otimes id_\infty(\tilde{f}_i) - \tilde{f}_i \| < \frac{1}{n(K + 1)} \quad \text{for} \quad 1 \leq i \leq n,
\]

where we let \( K = \max\{ \| \tilde{\alpha}_i \| \} \). It follows that

\[
| \langle \overline{M}_{\varphi_F} \otimes id_\infty(a) - a, f_i \rangle | = | \tilde{\omega}_{a, f_i}(\varphi_F - 1) | = | \tilde{\omega}_{\tilde{\alpha}_i, \tilde{f}_i}(\varphi_F - 1) | = | \langle \tilde{\alpha}_i, m_{\varphi_F} \otimes id_\infty(\tilde{f}_i) - \tilde{f}_i \rangle | < \frac{1}{n}.
\]

Then we get a net of elements \( \{ \varphi_F \} \) in \( A_c(G) \), which is indexed by finite subsets \( F \) of \( (C^*_r(G) \otimes K_\infty)^* \), such that

\[
\| m_{\varphi_F} \otimes id_\infty(f) - f \| < \varepsilon
\]

for all \( \varphi_F \) and \( \overline{M}_{\varphi_F} \to id_{C^*_{red}(G)} \) in the stable point-weak topology. By a standard convexity argument, we may find an element \( \varphi \in A_c(G) \) such that

\[
\| m_{\varphi} \otimes id_\infty(f) - f \| < \varepsilon \quad \text{and} \quad \| \overline{M}_{\varphi} \otimes id_\infty(a) - a \| < \varepsilon.
\]
Haagerup [19] proved that if $G$ is weakly amenable, then
\begin{equation}
\Lambda(G) = \Lambda(A(G)) = \Lambda(C^*_{\text{red}}(G)) < \infty.
\end{equation}

Using a similar argument to that given in Proposition 2.2, we can obtain the following proposition for weakly amenable groups.

**Proposition 2.3.** Let $G$ be a weakly amenable discrete group. Then there exists a net of $\{\varphi_\alpha\}$ in $A_c(G)$ such that
\begin{align*}
\|m_{\varphi_\alpha}\|_{cb} &= \|M_{\varphi_\alpha}\|_{cb} \leq \Lambda(G) \\
\text{and } m_{\varphi_\alpha} &\to \text{id}_{A(G)} \text{ and } M_{\varphi_\alpha} \to \text{id}_{C^*_{\text{red}}(G)} \text{ in the point-norm topologies on } A(G) \text{ and } C^*_{\text{red}}(G), \text{ respectively.}
\end{align*}

**Proof.** Let us outline the proof. First, let us fix arbitrary $f \in A(G)$ and $a \in C^*_{\text{red}}(G)$. If $F = \{f_1, \ldots, f_n\}$ is a finite subset of $C^*_{\text{red}}(G)^*$, we have $\tilde{\omega}_{a, fi} \in \hat{Q}(G)$ and thus there exists $\tilde{a}_i \in VN(G) \hat{\otimes} T_\infty$ and $\tilde{f}_i \in A(G) \hat{\otimes} K_\infty$ such that
\[ \tilde{\omega}_{a, fi} = \tilde{\omega}_{\tilde{a}_i, \tilde{f}_i}. \]

Since $G$ is weakly amenable, for any $\varepsilon > 0$ there exists $\varphi_F \in A_c(G)$ such that
\begin{align*}
\|m_{\varphi_F}(f) - f\| < \varepsilon
\end{align*}
and
\begin{align*}
\|m_{\varphi_F} \otimes \text{id}_\infty(\tilde{f}_i) - \tilde{f}_i\| &< \frac{1}{m(K + 1)} \quad \text{(for } 1 \leq i \leq n),
\end{align*}
where we let $K = \max\{\|\tilde{a}_i\|\}$. Then we can show as the proof given in Proposition 2.2 that $M_{\varphi_F} \to \text{id}_{C^*_{\text{red}}(G)}$ in the point-weak topology, and thus there exists $\varphi$ in the convex hull of $\{\varphi_F\}$ for which
\begin{align*}
\|m_{\varphi}\|_{cb} &= \|M_{\varphi}\|_{cb} \leq \Lambda(G) \\
\text{and } \|m_{\varphi}(f) - f\| &< \varepsilon \text{ and } \|M_{\varphi}(a) - a\| < \varepsilon.
\end{align*}
A similar argument can be applied to arbitrary finite collection of elements in $A(G)$ and $C^*_{\text{red}}(G)$, and this completes the proof.

3. Approximation Properties for $L_p(VN(G))$

Let us first briefly recall some basic notions from the complex interpolation theory (see [1]). A pair of Banach spaces $(V, W)$ is said to be a compatible couple if there is a topological vector space $X$ and continuous inclusions $V \hookrightarrow X$ and $W \hookrightarrow X$, which allow us to identify $V \cap W$ and $V + W$ in $X$. Then there is a canonical Banach space norm on $V \cap W$ given by
\[ \|x\|_{V \cap W} = \max\{\|x\|_V, \|x\|_W\}, \]
and a canonical Banach space norm on $V + W$ given by

$$\|x\|_{V+W} = \inf\{\|v\|_V + \|w\|_W : x = v + w\}.$$ 

Let $S$ denote the strip $\{z \in \mathbb{C} : 0 \leq \text{Re}z \leq 1\}$ in the complex plane $\mathbb{C}$ and let $S^0$ denote the interior of $S$. We let $\mathcal{F}$ denote the collection of all continuous and bounded functions $f : S \to V + W$ such that

1. $f$ is analytic on $S^0$,
2. $f(it) \in V$ and $f(1 + it) \in W$ for all $t \in \mathbb{R}$,
3. $f(it) \to 0$ and $f(1 + it) \to 0$ as $t \to \infty$.

It is clear that $\mathcal{F}$ is a vector space. Actually, $\mathcal{F}$ is a Banach space with the norm given by

$$\|f\|_F = \max\{\sup\{\|f(it)\|_V : t \in \mathbb{R}\}, \sup\{\|f(1 + it)\|_W : t \in \mathbb{R}\}\}.$$ 

For $0 \leq \theta \leq 1$, the space

$$(V, W)_\theta = \{a \in V + W : a = f(\theta) \text{ for some } f \in \mathcal{F}\}$$

is called the complex interpolation of $(V, W)$. This is a Banach space with the norm given by

$$\|a\|_\theta = \inf\{\|f\|_F : a = f(\theta), f \in \mathcal{F}\}.$$ 

If $V$ and $W$ are operator spaces, Pisier \[33\] showed that there is a canonical operator space matrix norm on $(V, W)_\theta$ given by

$$(3.1) \quad M_n((V, W)_\theta) = (M_n(V), M_n(W))_\theta$$

for all $n \in \mathbb{N}$.

Let $R$ be a von Neumann algebra with a normal faithful tracial state $\tau$. For $1 \leq p < \infty$, the non-commutative $L_p(R)$ space is defined to be the closure of $R$ under the $p$-norm

$$\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}.$$ 

We usually write $L_\infty(R) = R$. The trace $\tau$ induces a canonical contractive embedding $j_1 : R \to R_*$ given by

$$(3.2) \quad \langle j_1(x), y \rangle = \tau(xy).$$

With this embedding, $(R, R_*)$ is a compatible couple of Banach spaces, and we actually have the isometry

$$(3.3) \quad L_p(R) = (R, R_*)_{\frac{1}{p}}$$

for $1 \leq p < \infty$ (see Kosaki \[27\]).
However, we have to be very careful about their operator space matrix norms. In the operator space theory it is customary to use the following parallel duality pairing

\[(\beta_{ij}, \alpha_{ij}) = \sum_{i,j=1}^{n} \beta_{ij} \alpha_{ij} = \text{Tr}(\beta \alpha^{op})\]  

between \(T_n = M_n^\ast\) and \(M_n\). The reason to use this parallel duality pairing instead of the trace duality pairing

\[\langle [\beta_{ij}], [\alpha_{ij}] \rangle = \sum_{i,j=1}^{n} \beta_{ij} \alpha_{ji} = \text{Tr}(\beta \alpha)\]  

is that we will be able to get the complete isometry

\[M_n(V)^\ast = T_n \hat{\otimes} V^\ast\]  

for every operator space \(V\). Therefore, if we wish to use the trace duality pairing \(3.2\), we should define

\[L_1(R) = (R^{op})_\ast = (R_\ast)^{op},\]  

for which we can obtain the complete isometry

\[L_1(M_n \hat{\otimes} R) = T_n \hat{\otimes} L_1(R).\]  

The opposite operator space \((R_\ast)^{op}\) can be isometrically identified with \(R_\ast\), but is equipped with the opposite operator space matrix norm

\[\|f^{op}_{ij}\| = \|f_{ji}\|\]  

for all \([f^{op}_{ij}] \in M_n((R_\ast)^{op})\). Then we may use \(3.1\) to obtain a canonical operator space matrix norm on \(L_p(R)\) given by

\[M_n(L_p(R)) = (M_n(R), M_n(L_1(R)))_{\frac{1}{p}}.\]  

The following proposition shows that we can replace \(R\) in \(3.3\) and \(3.6\) by a weak\(^\ast\) dense \(C^\ast\)-subalgebra of \(R\).

**Proposition 3.1.** Let \(R\) be a von Neumann algebra with a normal faithful tracial state \(\tau\) and let \(B\) be a weak\(^\ast\) dense \(C^\ast\)-subalgebra of \(R\). For \(1 < p < \infty\), we have the complete isometry

\[L_p(R) = (B, L_1(R))_{\frac{1}{p}}.\]  

**Proof.** Let us first consider the isometry case. Given any \(1 < p < \infty\), there exists a positive number \(q\) such that \(1 < q < p\). It follows from the reiteration theorem (see [1] §4.6) that we have the isometry

\[L_p(R) = ((R, L_1(R))_0, (R, L_1(R))_{\frac{1}{q}})_{\frac{1}{p}} = (R, L_q(R))_{\frac{1}{q}}.\]  

Since \(L_q(R)\) is a reflexive space, we also have the isometry

\[(B, L_q(R))_{\frac{1}{p}} = (B^\ast, L_q(R^\ast))_{\frac{1}{p}} = (L_{q'}(R), B^\ast)_{1 - \frac{p}{q}}.\]
(see [1] §4.5), where \( q' = \frac{q}{q-1} \) is the conjugate exponent to \( q \). Since \( B \subseteq R \), there is a canonical contractive inclusion

\[
(B, L_q(R))_{\frac{q}{q}} \to (R, L_q(R))_{\frac{q}{q}} = L_p(R),
\]

from which we obtain a canonical contraction (by taking the adjoint)

\[
L_{q'}(R) = L_p(R)^{\ast} \to (L_{q'}(R), B^*)_{1 - \frac{q}{q'}}.
\]

If we let \( F_0 \) denote the space of elements having the form

\[
f(z) = \exp(\lambda \varphi) \sum_{n=1}^{N} \exp(\lambda_n z) x_n
\]

with \( \lambda > 0, N \in \mathbb{N}, \lambda_n \in \mathbb{R} \) and \( x_n \in L_{q'}(R) \cap B^* \subseteq L_1(R) \), then it is known from [1] that \( F_0 \) is dense in \( F \) (with respect to the compatible couple \( (L_{q'}(R), B^*) \)) and thus the space of elements \( f(1 - \frac{q}{q'}) \) with \( f \in F_0 \) is norm dense in \( (L_{q'}(R), B^*)_{1 - \frac{q}{q'}} \).

If \( f \in F_0 \) then for every \( t \in \mathbb{R} \), we have

\[
\|f(1 + it)\|_{B^*} = \sup\{|\tau(f(1 + it)y)| : y \in B, \|y\| \leq 1\}
= \sup\{|\tau(f(1 + it)y)| : y \in R, \|y\| \leq 1\} = \|f(it)\|_{L_1(R)}
\]

by the Kaplansky’s density theorem. This shows that we actually have the isometry

\[
L_{q'}(R) = (L_{q'}(R), B^*)_{1 - \frac{q}{q'}},
\]

and thus the isometry

\[
L_p(R) = (B, L_q(R))_{\frac{q}{q}}.
\]

It was shown in Wolff [42, Theorem 2] that if we are given Banach spaces \( V_i \) (\( i = 1, 2, 3, 4 \)) such that \( V_1 \cap V_4 \) is norm dense in \( V_2 \) and \( V_3 \), and \( (V_2, V_4)_\theta = V_3 \), \( (V_1, V_3)_\phi = V_2 \) for some \( 0 < \theta, \phi < 1 \), then we have \( (V_1, V_4)_\xi = V_2 \) for \( \xi = \frac{\phi - \theta}{1 - \phi + \phi \theta} \) and \( \phi = \frac{q}{p} \), then we obtain \( \xi = \frac{1}{p} \) and the isometry

\[
L_p(R) = (B, L_1(R))_{\frac{q}{p}}
\]

by Wolff’s result.

For each \( n \in \mathbb{N} \), we have the contractive linear isomorphisms

\[
M_n((B, L_1(R))_{\frac{q}{p}}) = (M_n(B), M_n(L_1(R)))_{\frac{q}{p}} \to (M_n(R), M_n(L_1(R)))_{\frac{q}{p}} = M_n(L_p(R)),
\]

and

\[
M_n((B, L_1(R))_{\frac{q}{p}}) = (M_n(B), M_n(L_1(R)))_{\frac{q}{p}} \to (M_n(R), M_n(L_1(R)))_{\frac{q}{p}} = M_n(L_{q'}(R)).
\]

Then by duality, we must have the isometry

\[
M_n(L_p(R)) = (M_n(B), M_n(L_1(R)))_{\frac{q}{p}}.
\]

\[\square\]
If $G$ is a discrete group, then we have a canonical normal faithful tracial state $\tau$ on $VN(G)$ given by

$$\tau(x) = \langle x\delta_e | \delta_e \rangle,$$

where $\delta_e$ is the characteristic function at the unit element $e \in G$. In this case, $(C^{*}_{red}(G), L_1(VN(G)))$ is a compatible couple with the canonical embedding $j_1 : C^{*}_{red}(G) \to L_1(VN(G))$ given by

$$j_1(\lambda(s)) = \delta_{s^{-1}} = \check{\delta}_s$$

for all $s \in G$. Given a function $\varphi : G \to \mathbb{C}$, we let $\check{\varphi}$ be the function on $G$ defined by $\check{\varphi}(t) = \varphi(t^{-1})$. For $1 < p < \infty$, we have the complete isometry

$$L_p(VN(G)) = (C^{*}_{red}(G), L_1(VN(G)))_{\frac{1}{p}}$$

by Proposition 3.1, and have a canonical (contractive) inclusion

$$j_p : C^{*}_{red}(G) \to L_p(VN(G)),$$

for which the range space $j_p(C^{*}_{red}(G))$ is norm dense in $L_p(VN(G))$. If we let $p'$ be the conjugate exponent to $p$, then the duality between $L_p(VN(G))$ and $L_{p'}(VN(G))$ is given by

$$\langle j_p(\lambda(s)), j_{p'}(\lambda(t)) \rangle = \tau(\lambda(s)\lambda(t))$$

for all $s, t \in G$.

In this case, the space $L_1(VN(G))$ can be explicitly expressed as follows. First let us recall that there is a normal $*$-anti-automorphism $\kappa$ on the group von Neumann algebra $VN(G)$ defined by

$$\kappa\left(\sum_i a_i \lambda(t_i)\right) = \sum_i a_i \lambda(t_i^{-1})$$

(see [18] and [14]). Since $\kappa$ is a normal $*$-anti-automorphism on $VN(G)$, we must have

$$\|\kappa_n([x_{ij}])\| = \|[x_{ji}]\|$$

for all $[x_{ij}] \in M_n(VN(G))$. Its pre-adjoint map $\kappa_*$ induces an isometry on $A(G)$ which satisfies

$$\|(\kappa_*)_n([f_{ij}])\| = \|[f_{ji}]\|$$

for all $[f_{ij}] \in M_n(A(G))$. Then we may completely isometrically identify $L_1(VN(G))$ with $\kappa_*(A(G)) = \{\kappa_*(f) = \check{f} : f \in A(G)\}$ and thus obtain

$$\|\check{f}_{ij}\| = \|[f_{ji}]\|$$

for $[\check{f}_{ij}] \in M_n(L_1(VN(G)))$. 
If \( \varphi \) is a completely bounded multiplier on \( A(G) \), we can prove that \( \tilde{\varphi} \) is a completely bounded multiplier on \( L_1(VN(G)) \) such that \( \|m_{\tilde{\varphi}}\|_{cb} = \|m_\varphi\|_{cb} \). To see this, let us assume that \( \varphi \in M_0A(G) \). Then for any \( \tilde{f} \in L_1(VN(G)) \), we have

\[
m_{\tilde{\varphi}}(\tilde{f}) = \tilde{\varphi} \tilde{f} = (\varphi \tilde{f}) = \kappa_*(\varphi f) \in L_1(VN(G)).
\]

This shows that

\[
(3.10) \quad m_{\tilde{\varphi}} = \kappa_* \circ m_\varphi \circ \kappa_*
\]

is a well-defined multiplier map on \( L_1(VN(G)) \). Since

\[
\|m_{\tilde{\varphi}}(\tilde{f}_{ij})\| = \|\kappa_*(\varphi f_{ij})\| = \|\varphi f_{ij}\| = \|m_\varphi(f_{ij})\|
\]

for every \( [\tilde{f}_{ij}] \in M_n(L_1(VN(G))) \), we can conclude that

\[
(3.11) \quad \|m_{\tilde{\varphi}}\|_{cb} = \|m_\varphi\|_{cb}.
\]

In general if \( T \) is a completely bounded map (respectively, a complete isometry) on \( A(G) \), then \( T \) induces a completely bounded map (respectively, a complete isometry)

\[
\tilde{T} = \kappa_* \circ T \circ \kappa_*
\]

on \( L_1(VN(G)) \), which satisfies

\[
\|\tilde{T}\|_{cb} = \|T\|_{cb}.
\]

As a consequence we may replace the approximation maps \( m_\varphi \) on \( A(G) \) in Proposition \( \Xi \) and Proposition \( \Xi^* \) by the corresponding maps \( m_{\tilde{\varphi}} \) on \( L_1(VN(G)) \) and obtain the following modified result.

**Proposition 3.2.** If \( G \) is a discrete group with the AP, then there exists a net of \( \{\varphi_\alpha\} \) in \( A_c(G) \) such that \( m_{\varphi_\alpha} \to id_{L_1(VN(G))} \) in the stable point-norm topology on \( L_1(VN(G)) \) and \( \overline{M_{\varphi_\alpha}} \to id_{C^*_r(G)} \) in the stable point-norm topology on \( C^*_r(G) \).

If \( G \) is a weakly amenable discrete group, then there exists a net of \( \{\varphi_\alpha\} \) in \( A_c(G) \) such that

\[
\|m_{\varphi_\alpha}\|_{cb} = \|\overline{M_{\varphi_\alpha}}\|_{cb} \leq \Lambda(G)
\]

and \( m_{\varphi_\alpha} \to id_{L_1(VN(G))} \) and \( \overline{M_{\varphi_\alpha}} \to id_{C^*_r(G)} \) in the point-norm topologies on \( L_1(VN(G)) \) and \( C^*_r(G) \), respectively.

**Proposition 3.3.** Given \( \varphi \in M_0A(G) \), \( (\overline{M_{\varphi}}, m_{\tilde{\varphi}}) \) is a compatible pair of completely bounded maps on the compatible couple \((C^*_r(G), L_1(VN(G)))\).

**Proof.** Assume that we are given \( \varphi \in M_0A(G) \). For any \( s, t \in G \), we have

\[
\langle j_1(\overline{M_{\varphi}}(\lambda(s))), \lambda(t) \rangle = \langle j_1(\varphi(s)\lambda(s)), \lambda(t) \rangle = \varphi(s)\tau(\lambda(s)\lambda(t))
\]

and

\[
\langle m_{\tilde{\varphi}}(j_1(\lambda(s))), \lambda(t) \rangle = \varphi(t^{-1})\tau(\lambda(s)\lambda(t)) = \varphi(s)\tau(\lambda(s)\lambda(t)).
\]
This shows that
\[ j_1(\overline{M}_\varphi(\lambda(s))) = m_\varphi(j_1(\lambda(s))) \]
for all \( s \in G \). Since \( \lambda(C[G]) \) is norm dense in \( C^*_\text{red}(G) \), we obtain
\[ j_1 \circ \overline{M}_\varphi = m_\varphi \circ j_1, \]
and thus \((\overline{M}_\varphi, m_\varphi)\) is compatible pair on \((C^*_\text{red}(G), L_1(VN(G)))\).
\[ \square \]

**Remark 3.4.** Let \( \otimes^h \) denote the Haagerup tensor product for operator spaces (see [12]). If \((V_i, W_i)\) with \((i = 1, 2)\) are compatible couples of operator spaces, Pisier proved in [13] Theorem 2.3 that \((V_1 \otimes^h V_2, W_1 \otimes^h W_2)\) is again a compatible couple of operator spaces and we have the complete isometry
\[ (3.13) \quad (V_1 \otimes^h V_2, W_1 \otimes^h W_2)_\theta = (V_1, W_1)_\theta \otimes^h (V_2, W_2)_\theta. \]
It is also known that for arbitrary operator space \( V \), we have the complete isometry
\[ (3.14) \quad V \otimes K_\infty = C \otimes^h V \otimes^h R, \]
where \( C \) and \( R \) are column and row operator Hilbert spaces over \( \ell_2 \). It follows from (3.13) and (3.14) that we have the complete isometry
\[ (V, W)_\theta \otimes K_\infty = (V \otimes K_\infty, W \otimes K_\infty)_\theta \]
for arbitrary compatible couple of operator spaces \((V, W)\). In particular for \( 1 < p < \infty \), we have the complete isometry
\[ L_p(VN(G)) \otimes K_\infty = (C^*_\text{red}(G) \otimes K_\infty, L_1(VN(G)) \otimes K_\infty)_{\frac{1}{p}}. \]

**Proof of Theorem 1.1.** We need to show that for every \( a \in L_p(VN(G)) \otimes K_\infty \) and \( \varepsilon > 0 \), there exists a finite-rank map \( T \) on \( L_p(VN(G)) \) such that
\[ \|T \otimes \text{id}_K(a) - a\| < \varepsilon. \]
Since \( L_p(VN(G)) \otimes K_\infty = (C^*_\text{red}(G) \otimes K_\infty, L_1(VN(G)) \otimes K_\infty)_{\frac{1}{p}} \), there exists a continuous and bounded map
\[ f : S \to C^*_\text{red}(G) \otimes K_\infty + L_1(VN(G)) \otimes K_\infty \]
in \( \mathcal{F} \) such that \( a = f(\frac{1}{t}) \). Since \( f(it) \in C^*_\text{red}(G) \otimes K_\infty \) and \( f(it) \to 0 \) as \( t \to \infty \), the set \( \{f(it)\}_{t \in \mathbb{R}} \) is contained in a compact subset of \( C^*_\text{red}(G) \otimes K_\infty \). Then there exists an element \((x_n) \in C^*_\text{red}(G) \otimes K_\infty \otimes_\varepsilon C_0\) such that \( \{f(it)\}_{t \in \mathbb{R}} \) is contained in the norm closure of the convex hull of \( \{x_n\} \) in \( C^*_\text{red}(G) \otimes K_\infty \). Similarly, there exists an element \((y_n) \in L_1(VN(G)) \otimes K_\infty \otimes_\varepsilon C_0\) such that the set \( \{f(1 + it)\}_{t \in \mathbb{R}} \) is contained in the norm closure of the convex hull of \( \{y_n\} \) in \( L_1(VN(G)) \otimes K_\infty \). Since \( G \) has the AP and we may identify \( K_\infty \otimes C_0 \) with a closed subspace of \( K_\infty \otimes K_\infty \cong K_\infty \), it follows from Proposition 12 that there exists \( \varphi \in A_c(G) \) such that
\[ \|\overline{M}_\varphi \otimes \text{id}_K(f(it)) - f(it)\| < \varepsilon. \]
and
\[ \| m \varphi \otimes \text{id}_\infty (f(1 + it)) - f(1 + it) \| < \varepsilon \]
for all \( t \in \mathbb{R} \). We wish to thank Quanhua Xu, who pointed out this trick to the second author.

Since \((M \varphi, m \hat{\varphi})\) is a compatible pair of finite-rank maps on \((C^*_\text{red}(G), L_1(VN(G)))\), this induces a well-defined finite-rank map \( T \) on \( C^*_\text{red}(G) + L_1(VN(G)) \) such that \((T \otimes \text{id}_\infty) \circ f \in F\) and \( T \otimes \text{id}_\infty (a) = T \otimes \text{id}_\infty (f(1 + it)) \). Then we have
\[ \| T \otimes \text{id}_\infty (a) - a \|_{L_p(VN(G)) \otimes K_\infty} \leq \| (T \otimes \text{id}_\infty) \circ f - f \|_F \]
\[ = \max \{ \sup \{ \| M \varphi \otimes \text{id}_\infty (f(it)) - f(it) \|_{C^*_\text{red}(G) \otimes K_\infty} \}, \sup \{ \| m \hat{\varphi} \otimes \text{id}_\infty (f(1 + it)) - f(1 + it) \|_{L_1(VN(G)) \otimes K_\infty} \} \} \leq \varepsilon. \]
This completes the proof.

Using a similar argument, we can easily prove the following proposition.

**Proposition 3.5.** Let \( G \) be a weakly amenable discrete group and \( 1 < p < \infty \). Then \( L_p(VN(G)) \) has the CBAP with
\[ \Lambda(L_p(VN(G))) \leq \Lambda(G). \]

**Proof.** If \( G \) is weakly amenable, then there exists a net of \( \{ \varphi_\alpha \} \) in \( A_c(G) \) such that
\[ \| m \varphi_\alpha \|_{cb} = \| M \varphi_\alpha \|_{cb} \leq \Lambda(G) \]
and \( m \hat{\varphi}_\alpha \to \text{id}_{L_1(VN(G))} \) and \( M \varphi_\alpha \to \text{id}_{C^*_\text{red}(G)} \) in the point-norm topology. For each \( \alpha \), it is known from the complex interpolation theory (see Pisier [33, Proposition 2.1]) that \((M \varphi_\alpha, m \hat{\varphi}_\alpha)\) induces a completely bounded finite-rank map \( T_\alpha \) on \( L_p(VN(G)) \) such that
\[ \| T_\alpha \|_{cb} \leq \| M \varphi_\alpha \|_{cb}^{1-\frac{1}{p}} \| m \hat{\varphi}_\alpha \|_{cb}^{\frac{1}{p}} \leq \Lambda(G). \]
The fact that \( T_\alpha \to \text{id}_{L_p(VN(G))} \) in the point-norm topology follows from a similar argument that given in the proof of Theorem 1.1.

4. Completely Contractive Approximation Property for \( L_p(VN(G)) \)

Given an operator space \( V \), we let \( F(V, V) \) denote the space of all (completely) bounded finite-rank maps on \( V \). Then we may identify the algebraic tensor product \( V^* \otimes V \) with \( F(V, V) \) via the map
\[ u \in V^* \otimes V \to F_u = \sum f_i \otimes x_i. \]
The \( \lambda \)-ball of \( F(V, V) \) is refered to as the set of all elements in \( T \in F(V, V) \) with \( \| T \|_{cb} \leq \lambda \). We first need the following lemma.
Lemma 4.1. Let $V$ be an operator space and $1 \leq \lambda < \infty$. Then $\Lambda(V) \leq \lambda$ if and only if for every $\varepsilon > 0$, $id_V$ is contained in the stable point-norm closure of the $(1 + \varepsilon)\lambda$-ball of $F(V, V)$.

Proof. $\Leftarrow$: Given any finite subset $S = \{v_1, \cdots, v_n\}$ of $V$ and $\varepsilon > 0$, we let $I = \{\alpha\}$ denote the index set consisting of all such $\alpha = \{S, \varepsilon\}$. Then there is a canonical partial order on $I$ given by $\alpha \preceq \alpha'$ if and only if $\varepsilon' \leq \varepsilon$ and $S \subseteq S'$.

Given any $\alpha = \{S, \varepsilon\} \in I$, we let

$$M = \max\{\|v_i\| : v_i \in S\} + 1.$$ 

Since $id_V$ is contained in the stable point-norm closure of the $(1 + \frac{1}{M})\lambda$-ball of $F(V, V)$, there exists a finite-rank map $T : V \to V$ such that $\|T\|_{cb} \leq (1 + \frac{1}{M})\lambda$ and $\|T(v_i) - v_i\| < \varepsilon$ for all $v_i \in S$. Then $T_\alpha = \frac{1}{1 + \frac{M}{\varepsilon}}T$ is a finite-rank map on $V$

such that $\|T_\alpha\|_{cb} \leq \lambda$ and

$$\|T_\alpha(v_i) - v_i\| \leq \frac{\varepsilon}{1 + \frac{M}{\varepsilon}}\|T(v_i)\| + \|T(v_i) - v_i\| < (\lambda + 1)\varepsilon.$$ 

This shows that $T_\alpha \to id_V$ in the point-norm topology, and thus $\Lambda(V) \leq \lambda$.

$\Rightarrow$: This is obvious since if $\Lambda(V) \leq \lambda$, then there exists a net of finite-rank maps $T_\alpha$ on $V$ such that $\|T_\alpha\|_{cb} \leq \lambda$ and $T_\alpha \to id_V$ in the point-norm topology. This implies that $T_\alpha \to id_V$ in the stable point-norm topology, and thus $id_V$ is contained in the stable point-norm closure of the $\lambda$-ball of $F(V, V)$. \hfill $\square$

Given operator spaces $V$ and $W$, we have the (complete) isometric inclusions

$$CB(V, W) \hookrightarrow CB(V, W^{**}) \cong (V \hat{\otimes} W^*)^*.$$ 

Then every $u \in V \hat{\otimes} W^*$ determines a bounded linear functional $F_u$ on $CB(V, W)$ with

$$\|F_u\| \leq \|u\|_{V \hat{\otimes} W^*}.$$ 

To see this, suppose that we are given an element $u \in V \hat{\otimes} W^*$. We define $F_u : CB(V, W) \to \mathbb{C}$ to be given by

$$F_u(T) = \alpha\langle g_{kl}(T(v_{ij}))\rangle \beta$$

if we can write $u = \alpha(v \otimes g)\beta$ for some $\alpha \in M_{1, \infty^2}$, $v = [v_{ij}] \in V \hat{\otimes} K_\infty$, $g = [g_{kl}] \in W^* \hat{\otimes} K_\infty$ and $\beta \in M_{\infty^2, 1}$. It is clear that this is a well-defined linear functional on $CB(V, W)$ and

$$|F_u(T)| \leq \|\alpha\| \|v\| \|g\| \|\beta\| \|T\|_{cb}$$

for all $T \in CB(V, W)$. This shows that

$$\|F_u\| \leq \inf\{\|\alpha\| \|v\| \|g\| \|\beta\| : u = \alpha(v \otimes g)\beta = \|u\|_{V \hat{\otimes} W^*}\}.$$
Moreover, it was shown in [10] that these maps \( F_u \) with \( u \in V \hat{\otimes} W^* \) are exactly the continuous linear functionals with respect to the stable point-norm topology on \( CB(V, W) \).

**Theorem 4.2.** Let \( R \) be a von Neumann algebra with a normal faithful tracial state and with the QWEP, and let \( 1 < p < \infty \). If \( V \) is a complemented operator subspace of \( L_p(R) \) and \( V \) has the OAP, then \( V \) has the CBAP such that

\[
\Lambda(V) \leq \|P\|_{cb},
\]

where \( P \) is a completely bounded projection from \( L_p(R) \) onto \( V \).

**Proof.** From Lemma 4.1, it suffices to show that for every \( \varepsilon > 0 \) the identity map \( id_V \) is contained in the stable point-norm closure of the \((1 + \varepsilon)\|P\|_{cb}\)-ball of \( F(V, V) \). Suppose not, i.e. suppose that there exists an \( \varepsilon > 0 \) such that \( id_V \) is not contained in the stable point-norm closure of the \((1 + \varepsilon)\|P\|_{cb}\)-ball of \( F(V, V) \). Then there exists a stable point-norm continuous linear functional \( F \) on \( CB(V, V) \) such that \( F(id_V) = 1 \) and \(|F(T)| \leq 1\) for all \( T \in F(V, V) \) with \( ||T||_{cb} \leq (1 + \varepsilon)\|P\|_{cb} \). The latter condition is equivalent to saying that we have

\[
|F(T)| \leq \frac{||T||_{cb}}{(1 + \varepsilon)\|P\|_{cb}}
\]

for all \( T \in F(V, V) \). Since \( V^* \hat{\otimes} V \) can be identified with the cb-norm closure of \( V^* \otimes V \) in \( CB(V, V) \), \( F \) can be identified with a bounded linear functional on \( V^* \hat{\otimes} V \), which is still denoted by \( F \), with norm

\[
||F||_{V^* \otimes V} \leq \frac{1}{(1 + \varepsilon)\|P\|_{cb}}.
\]

We let \( I(V^*, V^*) \) denote the space of all *completely integral maps* from \( V^* \) into \( V^* \), and let \( \mathcal{N}(V^*, V^*) \) denote the space of all *completely nuclear maps* from \( V^* \) into \( V^* \) (see details in [10] and [12]). For \( 1 < p < \infty \), \( L_p(R) \) is a reflexive space. Then the closed subspace \( V \) of \( L_p(R) \) is also reflexive, and thus we have the isometry

\[
I(V^*, V^*) = (V^* \hat{\otimes} V)^*.
\]

Since \( V \) has the OAP, we have the isometry

\[
\mathcal{N}(V^*, V^*) = V^{**} \hat{\otimes} V^* = V \hat{\otimes} V^*.
\]

Together with the QWEP condition on \( R \), Junge [23] proved that we actually have the isometry

\[
I(V^*, V^*) = \mathcal{N}(V^*, V^*) = V \hat{\otimes} V^*.
\]

Then we may choose an element \( u \in V \hat{\otimes} V^* \) such that \( F = F_u \) and

\[
||u||_{V \hat{\otimes} V^*} = ||F|| \leq \frac{1}{(1 + \varepsilon)\|P\|_{cb}}.
\]
Since $V$ has the OAP, there exists a net of finite-rank maps $T_\alpha \in \mathcal{F}(V, V)$ such that $T_\alpha \to id_V$ in the stable point-norm topology. Then we obtain a contradiction since

$$1 = F(id_V) = \lim_{\alpha} |F(T_\alpha)| = \lim_{\alpha} |F_u(T_\alpha)|$$

$$= |F_u(id_V)| \leq \|u\|_{V \hat{\otimes} V^*} \cdot \|id_V\|_{cb} \leq \frac{1}{(1 + \varepsilon)\|P\|_{cb}} < 1.$$  

This shows that $\Lambda(V) \leq \|P\|_{cb}$.

Since CBAP implies OAP, we may obtain the following proposition from Theorem 4.2 by considering $V = L^p(R)$ and $P = id_{L^p(R)}$.

**Proposition 4.3.** Let $R$ be a von Neumann algebra with a normal faithful tracial state and with the QWEP. For $1 < p < \infty$, the following are equivalent:

1. $L^p(R)$ has the CCAP,
2. $L^p(R)$ has the CBAP,
3. $L^p(R)$ has the OAP.

**Proof of Theorem 1.2.** Let $G$ be a discrete group with the AP and let $VN(G)$ have the QWEP. Then for $1 < p < \infty$, $L_p(NV(G))$ has the OAP by Theorem 1.1, and thus has the CCAP by Proposition 4.3. □

Let $G$ be a discrete group. Then there is a unital normal $\ast$-isomorphic injection $\pi: VN(G) \to VN(G \times G)$, which is defined by

$$\pi(\lambda(s)) = \lambda(s) \otimes \lambda(s),$$

and is known as the *co-multiplication* of the Hopf von Neumann algebra $VN(G)$. The map $\pi$ is invariant with respect to the tracial states on $VN(G)$ and $VN(G \times G)$, i.e. we have

$$(\tau \otimes \tau) \circ \pi(x) = \tau(x)$$

for all $x \in VN(G)$. We may use $\pi$ to identify $VN(G)$ with the von Neumann subalgebra $\pi(VN(G))$ of $VN(G \times G)$. Since $\tau \otimes \tau$, restricted to $\pi(VN(G))$, is again a normal faithful tracial state on $\pi(VN(G))$, there is a unique $\tau \otimes \tau$-invariant normal conditional expectation $\mathcal{E}$ from $VN(G \times G)$ onto $\pi(VN(G))$ (see [38]) which satisfies

$$\mathcal{E}(\sum_{s,t} \alpha_{s,t} \lambda(s) \otimes \lambda(t)) = \sum_s \alpha_{s,s} \lambda(s) \otimes \lambda(s)$$

for all $\sum_{s,t} \alpha_{s,t} \lambda(s) \otimes \lambda(t) \in VN(G \times G)$. If we let

$$\rho = \pi^{-1} \circ \mathcal{E} : VN(G \times G) \to VN(G),$$

then $\rho$ is a normal complete contraction such that $\rho \circ \pi = id_{VN(G)}$, and

$$\tau \circ \rho(y) = \tau \otimes \tau(\mathcal{E}(y)) = \tau \otimes \tau(y)$$
for all \( y \in VN(G \times G) \). We note that
\begin{equation}
(4.4) \quad \kappa \circ \rho \circ (\kappa \otimes \kappa) = \rho
\end{equation}
on \( VN(G \times G) \) since for every \( x = \sum_{s,t} \alpha_{s,t} \lambda(s) \otimes \lambda(t) \in VN(G \times G) \) we have
\[
\kappa \circ \rho \circ (\kappa \otimes \kappa)(x) = \kappa \circ \rho \Big( \sum_{s,t} \alpha_{s,t} \lambda(s^{-1}) \otimes \lambda(t^{-1}) \Big) \\
= \kappa \Big( \sum_s \alpha_{s,s} \lambda(s^{-1}) \Big) \\
= \sum_s \alpha_{s,s} \lambda(s) = \rho(x).
\]
The pre-adjoint map \( \rho_1 : A(G) \to A(G \times G) \) is a completely isometric inclusion, and induces a complete isometry
\[
\tilde{\rho}_1 = (\kappa_1 \otimes \kappa_1) \circ \rho_1 \circ \kappa_1 : L_1(VN(G)) \to L_1(VN(G \times G)).
\]
It follows from (4.4) that we may identify \( \tilde{\rho}_1 \) with \( \rho_1 \) and obtain
\[
(\tilde{\rho}_1(\lambda(s)), x) = (\hat{j}_1(\lambda(s)), \rho(x)) = \tau(\lambda(s) \pi^{-1} \circ E(x)) \\
= \tau \otimes \tau(\pi(\lambda(s)))x = (\langle j_1 \otimes j_1 \rangle \circ \pi(\lambda(s)), x)
\]
for all \( s \in G \) and \( x \in VN(G \times G) \). This shows that if we let
\[
\pi_{\infty} = \pi_1|_{C^*_r(G)} : C^*_r(G) \to C^*_r(G \times G)
\]
and
\[
\pi_1 = \tilde{\rho}_1 : L_1(VN(G)) \to L_1(VN(G \times G)),
\]
then \((\pi_{\infty}, \pi_1)\) satisfies
\[
\pi_1 \circ j_1 = (j_1 \otimes j_1) \circ \pi_{\infty},
\]
and thus is a compatible pair of complete contractions from the compatible couple \((C^*_r(G), L_1(VN(G)))\) into the compatible couple \((C^*_r(G \times G), L_1(VN(G \times G)))\). By complex interpolation, we obtain, for each \( 1 < p < \infty \), a complete contraction
\[
\pi_p : L_p(VN(G)) \to L_p(VN(G \times G)),
\]
which satisfies
\begin{equation}
(4.5) \quad \pi_p(j_p(\lambda(s))) = j_p(\lambda(s) \otimes j_p(\lambda(s)).
\end{equation}
On the other hand, we let
\[
\rho_{\infty} = \rho_1|_{C^*_r(G \times G)} : C^*_r(G \times G) \to C^*_r(G)
\]
and
\[
\rho_1 = \hat{\pi}_1 : L_1(VN(G \times G)) \to L_1(VN(G)),
\]
where \( \hat{\pi}_1 = \kappa_1 \circ \pi_1 \circ (\kappa_1 \otimes \kappa_1) \). Then we may identify \( \hat{\pi}_1 \) with \( \pi_1 \) and show that \((\rho_{\infty}, \rho_1)\) is a compatible pair of complete contractions from the compatible couple.
By complex interpolation, we obtain, for each $1 < p < \infty$, a complete contraction

$$\rho_p : L_p(VN(G \times G)) \to L_p(VN(G)),$$

which satisfies that $\rho_p \circ \pi_p = id_{L_p(VN(G))}$. This shows that $\pi_p$ is actually a completely isometric embedding from $L_p(VN(G))$ into $L_p(VN(G \times G))$, and the range space $\pi_p(L_p(VN(G))$ is completely contractively complemented in $L_p(VN(G \times G))$.

Let $T$ be a completely bounded map on $L_p(VN(G))$. If $VN(G)$ has the QWEP, Junge [23] proved that $T$ determines a completely bounded map $id_{L_p(VN(G))} \otimes T$ on $L_p(VN(G \times G))$, which satisfies

$$\|id_{L_p(VN(G))} \otimes T\|_{cb} \leq \|T\|_{cb}.$$

From this, we may construct another completely bounded map

$$\tilde{T} = \pi_p^* \circ (id_{L_p(VN(G))} \otimes T) \circ \pi_p$$

on $L_p(VN(G))$. We note that this construction is motivated by an argument given in [20, Theorem 2.1]. Using Theorem 1.2 and a similar technique used in [20, Theorem 2.1], we may obtain the following result.

**Theorem 4.4.** Let $1 < p < \infty$. If $G$ is a discrete group with the AP and $VN(G)$ has the QWEP, then there exists a net of completely contractive finite-rank maps $T_\alpha : L_p(VN(G)) \to L_p(VN(G))$ such that $T_\alpha \to id_{L_p(VN(G))}$ in the point-norm topology, and for each $\alpha$,

$$T_\alpha(j_p(\lambda(s))) = \varphi_\alpha(s)j_p(\lambda(s))$$

for some $\varphi_\alpha$ in $A_c(G)$.

**Proof.** With the hypothesis, we know from Theorem 1.2 that $L_p(VN(G))$ has the CCAP. Let $\{T_\alpha\}$ be a net of finite-rank complete contractions on $L_p(VN(G))$ such that $T_\alpha \to id_{L_p(VN(G))}$ in the point-norm topology. Since $j_p(\lambda(\mathbb{C}[G]))$ is norm dense in $L_p(VN(G))$, we may assume, without loss of generality, that the range of $T_\alpha$ is contained in $j_p(\lambda(\mathbb{C}[G]))$ (consider a small perturbation if necessary). Using (4.6), we get a net of completely bounded maps

$$\tilde{T}_\alpha = \pi_p^* \circ (id_{L_p(VN(G))} \otimes T_\alpha) \circ \pi_p$$

on $L_p(VN(G))$ such that $\|\tilde{T}_\alpha\|_{cb} \leq \|T_\alpha\|_{cb} \leq 1$.

For each $\alpha$, let $\varphi_\alpha$ be the function on $G$ defined by

$$\varphi_\alpha(s) = \langle T_\alpha(j_p(\lambda(s)), j_p(\lambda(s^{-1})) \rangle.$$

Since the range of $T_\alpha$ is contained in $j_p(\lambda(\mathbb{C}[G]))$, we can write

$$T_\alpha = \sum_i f_i^\alpha \otimes j_p(\lambda(s_i^\alpha))$$
for some $f^*_i \in \overline{L_p(VN(G))^*}$ and $s^*_i \in G$. Then it is easy to see that $\varphi_\alpha$ has a finite support and thus is an element in $A_c(G)$. Moreover, we have

$$\hat{T}_\alpha(j_p(\lambda(s))) = \varphi_\alpha(s)j_p(\lambda(s))$$

because

$$\langle \hat{T}_\alpha(j_p(\lambda(s))), j_{p'}(\lambda(t)) \rangle = \langle (id_{\overline{L_p(VN(G))}} \otimes T_\alpha) \circ \sigma_p(j_p(\lambda(s))), \pi_{p'}(j_{p'}(\lambda(t))) \rangle$$

$$= \langle j_p(\lambda(s)) \otimes T_\alpha(j_p(\lambda(s))), j_{p'}(\lambda(t)) \otimes j_{p'}(\lambda(t)) \rangle$$

$$= \langle T_\alpha(j_p(\lambda(s))), j_{p'}(\lambda(t)) \rangle \tau(\lambda(s)\lambda(t))$$

$$= \langle T_\alpha(j_p(\lambda(s))), j_{p'}(\lambda(s^{-1})) \rangle \tau(\lambda(s)\lambda(t))$$

$$= \langle \varphi_\alpha(s)j_p(\lambda(s)), j_{p'}(\lambda(t)) \rangle$$

for all $s, t \in G$. This shows that for each $\alpha$, $\hat{T}_\alpha = m_{\varphi_\alpha}$ is a finite-rank multiplier map on $L_p(VN(G))$. Since $T_\alpha \to id_{\overline{L_p(VN(G))}}$ in the point-norm topology, we have

$$\varphi_\alpha(s) = \langle T_\alpha(j_p(\lambda(s))), j_{p'}(\lambda(s^{-1})) \rangle \to \langle j_p(\lambda(s)), j_{p'}(\lambda(s^{-1})) \rangle = \tau(\lambda(s)\lambda(s^{-1})) = 1$$

for all $s \in G$. This implies that $\hat{T}_\alpha \to id_{\overline{L_p(VN(G))}}$ in the point-norm topology. □

5. Local Structure and Completely Bounded Schauder Basis

In a recent paper [24], we studied the complemented $\mathcal{OL}_p$ spaces and completely bounded Schauder basis for operator spaces. We recall that an operator space $V$ is said to be a $\mathcal{COL}_p$ space if there exists a constant $\lambda \geq 1$ such that for every finite-dimensional subspace $E \subseteq V$ there exists a finite-dimensional $C^*$-algebra $A$ and a diagram of completely bounded maps

$$
\begin{array}{ccc}
V & \xrightarrow{r} & L_p(A) \\
\downarrow & & \downarrow id \\
L_p(A) & \xrightarrow{s} & L_p(A)
\end{array}
$$

with $\|r\|_{cb}\|s\|_{cb} \leq \lambda$ and $E \subseteq r(L_p(A))$. This definition is slightly stronger than the notion of $\mathcal{OL}_p$ spaces introduced in [13].

A sequence of elements $\{x_n\}$ in a separable operator space $V$ is a Schauder basis for $V$ if every element $x \in V$ can be uniquely written as $x = \sum_{n=1}^{\infty} \alpha_n x_n$, where the sum converges in the norm topology. It is well-known from Banach space theory that if $\{x_n\}$ is a basis for $V$, then the natural projections

$$P_k : x \in V \to \sum_{n=1}^{k} \alpha_n x_n \in \text{span}\{x_1, \ldots, x_k\}$$

are uniformly bounded, i.e. $\sup \{\|P_k\|\} < \infty$. We say that $\{x_n\}$ is a completely bounded Schauder basis for $V$ if $\sup \{\|P_k\|_{cb}\} < \infty$. It was shown in [24] that if $R$ is an injective von Neumann algebra with a separable predual, then for $1 \leq p < \infty$, $L_p(R)$ is a $\mathcal{COL}_p$ space and has a completely bounded Schauder basis. In general, if $R$ is a von Neumann algebra with a separable predual and has the QWEP, it
was shown in [24] for $1 < p < \infty$, $L_p(R)$ has the CBAP if and only if $L_p(R)$ is a $\mathcal{COL}_p$ space (respectively, $L_p(R)$ has a completely bounded Schauder basis). As a consequence of this result and Theorem 1.2 (or Proposition 1.3), we can conclude the following result for discrete groups.

**Theorem 5.1.** Let $G$ be a countable discrete group and $1 < p < \infty$. If $G$ has the AP and $VN(G)$ has the QWEP, then $L_p(VN(G))$ is a $\mathcal{COL}_p$ space and has a completely bounded Schauder basis.

It is known that for $2 \leq n < \infty$, the free group of $n$ generators $F_n$ is weakly amenable with $\Lambda(F_n) = 1$ (see [5] and [18]) and $VN(F_n)$ has the QWEP (see [25]). Then $L_p(VN(F_n))$ is a $\mathcal{COL}_p$ space and has a completely bounded Schauder basis. In general, we wish to consider residually finite discrete groups with the AP. We recall that a discrete group $G$ is residually finite if for any finitely many distinct elements $s_1, \ldots, s_n \in G$, there is a homomorphism $\theta$ from $G$ onto a finite group $H$ such that the elements $\theta(s_1), \ldots, \theta(s_n)$ are distinct in $H$. It was shown by Wassermann [41] that every residually finite discrete group $G$ has the property (F) (see definition in [41] and [25]), and thus its group von Neumann algebra $VN(G)$ has the QWEP (see Kirchberg [25, §7]). From this, we can conclude the following proposition.

**Proposition 5.2.** Let $G$ be a countable residually finite discrete group with the AP, then for $1 < p < \infty$, $L_p(VN(G))$ has the CCAP, and thus is a $\mathcal{COL}_p$ space and has a completely bounded Schauder basis.

In the following let us discuss some examples of countable residually finite discrete groups for which the non-commutative $L_p(VN(G))$ spaces are $\mathcal{COL}_p$ spaces and have completely bounded Schauder bases.

**Example 5.3.** Since the integer valued special linear group $SL(2, \mathbb{Z})$ is a closed discrete subgroup of $SL(2, \mathbb{R})$, it is known from [3] that $SL(2, \mathbb{Z})$ is weakly amenable with $\Lambda(SL(2, \mathbb{Z})) = 1$. This group is also residually finite since for any finitely many distinct elements $s_1, \ldots, s_n \in SL(2, \mathbb{Z})$, there is a sufficiently large prime number $p$ and a natural homomorphism $\theta_p$ from $SL(2, \mathbb{Z})$ onto the finite group $SL(2, \mathbb{Z}_p)$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, such that the elements $\theta_p(s_1), \ldots, \theta_p(s_n)$ are distinct in $SL(2, \mathbb{Z}_p)$.

**Example 5.4.** Let us consider the discrete group $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$, the semi-direct product of $\mathbb{Z}^2$ by $SL(2, \mathbb{Z})$. It is known from [20] that this group has the AP, but is not weakly amenable. On the other hand, it is easy to see that this group is residually finite since we may consider the natural homomorphisms $\theta_p$ from $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ onto the finite groups $\mathbb{Z}_p^2 \rtimes SL(2, \mathbb{Z}_p)$ for prime numbers $p$. 


Example 5.5. It is known (by Borel and Harish-Chandra [2]) that every arithmetic subgroup \( \Gamma \) of \( \text{Sp}(1,n) \) (with \( n \in \mathbb{N} \)) is a lattice of \( \text{Sp}(1,n) \), i.e. a closed discrete subgroup of \( \text{Sp}(1,n) \) for which \( \text{Sp}(1,n)/\Gamma \) has a bounded \( \text{Sp}(1,n) \)-invariant measure. In this case, \( \Gamma \) is weakly amenable with
\[
\Lambda(\Gamma) = \Lambda(\text{Sp}(1,n)) = 2n - 1
\]
(see [13] and [4]). Let \( \mathbb{H}_{\text{int}} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k \) denote the quaternionic integers ring, and let \( \Gamma_n \) denote the subgroup of \( \text{Sp}(1,n) \) consisting of all \((n+1) \times (n+1)\) matrices with values in \( \mathbb{H}_{\text{int}} \), which leave the sesquilinear form
\[
Q(x, y) = \bar{y}_0 x_0 - \sum_{m=1}^{n} \bar{y}_m x_m \quad (x_m, y_m \in \mathbb{H}_{\text{int}})
\]
invariant. Then \( \Gamma_n \) is a lattice of \( \text{Sp}(1,n) \) such that \( \Lambda(\Gamma_n) = \Lambda(\text{Sp}(1,n)) = 2n - 1 \) (see [4] §6). Considering the canonical representations of \( \Gamma_n \) into the finite ring \( M_{n+1}(\mathbb{Z}_p + \mathbb{Z}_p i + \mathbb{Z}_p j + \mathbb{Z}_p k) \) for prime numbers \( p \), we see that \( \Gamma_n \) is residually finite, and thus \( VN(\Gamma_n) \) has the QWEP.

Example 5.6. Let \( \Gamma_n \) be the lattice of \( \text{Sp}(1,n) \) discussed in (3), and let \( G = \bigoplus_n \Gamma_n \) be the discrete subgroup of the infinite product group \( \prod_n \Gamma_n \), which consists of all sequences \( g = (g_n) \) with \( g_n \in \Gamma_n \) such that all but finitely many \( g_n \) are equal to the unit element \( 1_n \) of \( \Gamma_n \). Then for each \( n \in \mathbb{N} \), we may identify the normal subgroup
\[
G_n = \{(g_i) : g_i = 1, \text{ for } i > n\}
\]
of \( G \) with the finite product group \( \Gamma_1 \times \cdots \times \Gamma_n \), and identify
\[
VN(G_n) = VN(\Gamma_1) \otimes \cdots \otimes VN(\Gamma_n)
\]
with a von Neumann subalgebra of \( VN(G) \). Since \( \{G_n\} \) is an increasing sequence of normal subgroups in \( G \) and \( G = \bigcup_n G_n \), we get an increasing sequence of finite von Neumann algebras
\[
VN(G_1) \hookrightarrow \cdots \hookrightarrow VN(G_n) \hookrightarrow \cdots \hookrightarrow VN(G),
\]
which preserve the tracial states and have a weak* dense union \( \bigcup_n VN(G_n) \) in \( VN(G) \). Then there is an increasing sequence of (completely contractive) normal conditional expectations \( \mathcal{E}_n \) from \( VN(G) \) onto \( VN(G_n) \) such that \( \mathcal{E}_n \to id_{VN(G)} \) in point-weak* topology (and thus in stable point-weak* topology).

It was shown by Cowling and Haagerup [4] that the groups \( G_n \) are weakly amenable with
\[
\Lambda(G_n) = \Lambda(VN(G_n)) = \Lambda(VN(\Gamma_1)) \cdots \Lambda(VN(\Gamma_n)) = 1 \cdot 3 \cdots (2n - 1).
\]
Then we can conclude that
\[
\Lambda(G) = \sup\{\Lambda(G_n)\} = \infty.
\]
This shows that the group $G$ is not weakly amenable. But $G$ has the AP. This follows from the fact that any point $u \in VN(G) \otimes M_\infty$ can approximated by $\{(E_n \otimes id_\infty)(u)\}$ in the weak* topology on $VN(G) \otimes M_\infty$, and for each $n$, $VN(G_n)$ has the weak* CBAP (and thus weak* OAP). Moreover, for each $n \in \mathbb{N}$, $G_n \cong \Gamma_1 \times \cdots \times \Gamma_n$ is residually finite and thus $G$ is residually finite. This provides another example of a discrete residually finite group which has the AP, but is not weakly amenable.

References

[1] J. Bergh and J. Lofstrom, Interpolation Spaces. An Introduction, Springer, New York 1976.
[2] A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups, Ann. of Math. 75 (1962), 485–535.
[3] A. Connes, Classification of injective factors, Ann. of Math. 104(1976), 73–115.
[4] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math. 96(1989), 507–549.
[5] J. De Cannière and U. Haagerup, Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups, Amer. J. Math. 197(1984), 455–500.
[6] J. Diestel and J. Uhl, Vector measures, Amer. Math. Soc. Math. Surveys 15, Providence, 1977.
[7] E. Effros, J. Kraus and Z-J. Ruan, On two quantized tensor products, in Operator Algebras, Mathematical Physics, and Low Dimensional Topology, (Istanbul 1991), pp 125-145, Research Notes in Math. 5, A K Peters, Wellesley, MA 1993.
[8] E. Effros and C. Lance, Tensor product of operator algebras, Adv. Math. 25(1977), 1–34.
[9] E. Effros and Z-J. Ruan, On approximation properties for operator spaces, Internat. J. Math. 1 (1990), 163–187.
[10] E. Effros and Z-J. Ruan, Mapping spaces and liftings for operator spaces, Proc. London Math. Soc. 69 (1994), 171–197.
[11] E. Effros and Z-J. Ruan, $\mathcal{O}L_p$ spaces, Contemporary Math. Amer. Math. Soc. 228 (1998), 51–77.
[12] E. Effros and Z-J. Ruan, Operator Spaces, London Math. Soc. Monographs, New Series 23, Oxford University Press, New York 2000.
[13] E. Effros and Z-J. Ruan, Operator space tensor products and Hopf convolution algebras, J. Op. Theory, to appear.
[14] M. Enock and J.-M. Schwartz, Kac Algebras and Duality of Locally Compact Groups, Springer-Verlag, 1992.
[15] P. Eymard, L’algebre de Fourier d’un groupe localement compact, Bull. Soc. Math. France 92(1964), 181–236.
[16] F. Fidaleo, Canonical operator space structures on non-commutative $L^p$ spaces, J. Funct. Anal. 169(1999), 226–250.
[17] A. Grothendieck, Produits Tensoriels Topologiques et Espaces Nucléaires, Mem. Amer. Math. Soc. No. 16, 1955.
[18] U. Haagerup, The reduced $C^*$-algebra of the free group on two generators, in 18th Scandinavian Congress of Mathematicians, Proceedings 1980, pp 321–335, Progress in Mathematics 11, Birkhauser Verlag, 1981.
[19] U. Haagerup, Group $C^*$-algebras without the completely bounded approximation property, preprint.
[20] U. Haagerup and J. Kraus, Approximation properties for group $C^*$-algebras and group von Neumann algebras, Trans. Amer. Math. Soc. 344(1994), 667–699.
[21] C.S.Herz, *Une généralisation de la notion de transformée de Fourier-Stieltjes*, Ann. Inst. Fourier (Grenoble), 24(1974), 145–157.
[22] H. Izumi, *Construction of non-commutative $L^p$ spaces with a complex parameter arising from modular actions*, International J. Math. 8(1997), 163–187.
[23] M. Junge, *Applications of Fubini’s theorem for noncommutative $L_p$ spaces*, preprint.
[24] M. Junge, N. Nielsen, Z-J. Ruan and Q. Xu, *CO$L_p$ spaces - The local structure of non-commutative $L_p$ spaces*, preprint.
[25] E. Kirchberg, *On non-semisplit extensions, tensor products, and exactness of group $C^*$-algebras*, Invent. Math. 112 (1993), 449–489.
[26] E. Kirchberg, *Discrete groups with Kazhdan’s property T and factorization property are residually finite*, Math. Ann. 299(1994), 551–563.
[27] H. Kosaki, *Application of the complex interpolation method to a von Neumann algebra: non-commutative $L^p$ spaces*, J. Funct. Anal. 56 (1984), 29–78.
[28] J. Kraus, *The slice map problem and approximation properties*, J. Funct. Anal. 102(1991), 116–155.
[29] C. Lance, *On nuclear $C^*$-algebras*, J. Funct. Anal. 12 (1973), 157–176.
[30] H. Leptin, *Sur l’algèbre de Fourier d’un groupe localement compact*, C. R. Acad. Sci. Paris Sér. A 266(1979), 39–59.
[31] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Berlin and New York, 1977.
[32] V. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Research Notes in Mathematics Series, No. 146, Longman Scientific and Technical, Essex, 1986.
[33] G. Pisier, *The Operator Hilbert Space $OH$, Complex Interpolation and Tensor Norms*, Mem. Amer. Math. Soc. 122 (1996), No. 585.
[34] G. Pisier, *Non-commutative vector valued $L_p$-spaces and completely $p$-summing maps*, Astérisque No. 247(1998).
[35] G. Pisier, *An introduction to the Theory of Operator Spaces*, preprint.
[36] Z-J. Ruan, *Subspaces of $C^*$-algebras*, J. Funct. Anal. 76 (1988), 217–230.
[37] Z-J. Ruan, *The operator amenability of $A(G)$*, Amer. J. Math. 117(1995), 1449–1474.
[38] S. Stratila, *Modular Theory in Operator Algebras*, Abacus Press, Tunbridge Wells, 1981.
[39] M. Takesaki, *Theory of Operator Algebras, I*, Springer-Verlag, New York 1979.
[40] A. Terp, *Interpolation spaces between a von Neumann algebra and its predual*, J. Operator Theory 8(1982), 327–360.
[41] S. Wassermann, *Exact $C^*$-algebras and Related Topics*, Lecture Note Series 19, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul 1994.
[42] T. Wolff, *A note on interpolation spaces*, in Harmonic Analysis, (Minneapolis, Minn. 1981), pp 199–204, Lecture Notes in Math. 908 Springer, Berlin and New York, 1982.