Strong Lefschetz Elements of the Coinvariant Rings of Finite Coxeter Groups

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Abstract For the coinvariant rings of finite Coxeter groups of types other than $H_4$, we show that a homogeneous element of degree one is a strong Lefschetz element if and only if it is not fixed by any reflections. We also give the necessary and sufficient condition for strong Lefschetz elements in the invariant subrings of the coinvariant rings of Weyl groups.

Keywords Coxeter group · Weyl group · Coinvariant ring · Strong Lefschetz property · Flag variety · Hard Lefschetz theorem

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1 Introduction

Let \( W \) be a finite Coxeter group generated by reflections acting on an \( n \)-dimensional \( \mathbb{R} \)-vector space \( V \). The polynomial ring \( \mathbb{R}[V] \) is then equipped with the action of \( W \). The coinvariant ring of \( W \) is defined by \( R = \mathbb{R}[V]/J \), where \( J \) is the graded ideal of \( \mathbb{R}[V] \) generated by the \( W \)-invariant polynomials without constant terms. The first purpose of this paper is to give the necessary and sufficient condition for the strong Lefschetz elements (Definition 3 below) of \( R \):

**Theorem 1** Let \( W \) be a finite Coxeter group which does not contain irreducible components of type \( H_4 \). Then a homogeneous element \( \ell \) of degree one is a strong Lefschetz element if and only if \( \ell \) is not fixed by any reflections of \( W \).

The second purpose is to give the necessary and sufficient condition for strong Lefschetz elements of the parabolic invariant \( R^{W_S} \), which is by definition the invariant subring of \( R \) under the action of a parabolic subgroup \( W_S \) of \( W \):

**Theorem 2** Let \( W \) be a Weyl group, and \( W_S \) a parabolic subgroup of \( W \). Then a homogeneous element \( \ell \) of degree one of the parabolic invariant \( R^{W_S} \) is a strong Lefschetz element if and only if \( \ell \) is not fixed by any reflections in \( W \setminus W_S \).

The notion of the strong Lefschetz property for commutative Artinian graded rings is an abstraction of a property of the cohomology ring \( H^*(X, \mathbb{R}) \) of a compact Kähler manifold \( X \). The Hard Lefschetz Theorem (Proposition 7, see [4], e.g.) tells us that the multiplication by the class of the Kähler form induces an isomorphism between \( H^i(X, \mathbb{R}) \) and \( H^{2 \dim X - i}(X, \mathbb{R}) \). In view of such a property of the cohomology ring, the strong Lefschetz property is defined as follows:

**Definition 3** A graded ring \( R = \bigoplus_{d=0}^{m} R_d \) having a symmetric Hilbert function is said to have the strong Lefschetz property, if there exists an element \( \ell \in R_1 \) such that the multiplication map \( \times^{m-2i} : R_i \to R_{m-i} (f \mapsto \ell^{m-2i} f) \) is bijective for every \( i = 0, 1, \ldots, |m/2| \). In this case, \( \ell \) is called a strong Lefschetz element.

It should be remarked that we can define the strong Lefschetz property for graded rings with non-symmetric Hilbert functions (see [6], e.g.). In this paper, we only consider the strong Lefschetz property for symmetric Hilbert functions.

Our interest is the condition for an element in \( R_1 \) to be a strong Lefschetz element. It is not a trivial problem even in the case of the cohomology ring \( H^*(X, \mathbb{R}) \) of a Kähler manifold \( X \). In fact, the Kähler cone of \( X \) is only part of the set of the strong Lefschetz elements in \( H^*(X, \mathbb{R}) \).

When \( W \) is a Weyl group (i.e. crystallographic Coxeter group), its coinvariant ring \( R \) is isomorphic to the cohomology ring of the flag variety \( G/B \), where \( G \) is the Lie group corresponding to \( W \), and \( B \) is a Borel subgroup of \( G \). Moreover the parabolic invariant \( R^{W_S} \) is isomorphic to the cohomology ring of the partial flag variety \( G/P \), where \( P \) is the parabolic subgroup of \( G \) whose Weyl group is \( W_S \). Thus \( R \) and \( R^{W_S} \) have the strong Lefschetz property for Weyl groups \( W \) from the geometric result. By an additional algebraic argument, Theorem 1 (resp. Theorem 2) determines the whole set of the strong Lefschetz elements of \( R \) (resp. \( R^{W_S} \)) for Weyl groups \( W \), and the set of the strong Lefschetz elements includes the Kähler cone as one of the connected components.