Chaotic string-capture by black hole

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Abstract

We consider a macroscopic charge-current carrying (cosmic) string in the background of a Schwarzschild black hole. The string is taken to be circular and is allowed to oscillate and to propagate in the direction perpendicular to its plane (that is parallel to the equatorial plane of the black hole). Numerical investigations indicate that the system is non-integrable, but the interaction with the gravitational field of the black hole anyway gives rise to various qualitatively simple processes like "adiabatic capture" and "string transmutation".
1 Introduction

Since Witten’s discovery of charge-current carrying topological defects (cosmic strings) in a $U(1) \times U(\tilde{1})$ gauge theory [1], a large amount of work has been devoted to the study of their astrophysical and mathematical aspects. Concerning the mathematical aspects of the charge-current carrying strings it has been of special interest to consider the questions of integrability and separability of the equations of motion in certain electromagnetic and gravitational backgrounds. This seems to be a natural extension of the work devoted to the study of charged point particles in such backgrounds.

It is of course well-known that the equations of motion for the Nambu-Goto string are generally extremely complicated if the string world-sheet is embedded in a curved 4-dimensional spacetime, and in fact the complete solution is only known in a very few special cases like conical spacetime [2], gravitational shock-wave background [3] and a few others. When charges and currents are introduced on the string the situation is further complicated and the analysis of the equations of motion for an arbitrary string configuration in an arbitrary electromagnetic and gravitational background becomes extremely problematic. A simple way to proceed is to consider only certain families of string configurations in certain families of backgrounds. Following this approach Carter, Frolov and their collaborators considered infinitely long stationary open strings in black hole backgrounds [4,5]. The main results can be found in Ref. 5:

Starting from a charge-current carrying string described by the action [6]:

$$S = \int L(\omega) \sqrt{-\det G_{\alpha\beta}} d\tau d\sigma,$$

(1.1)

where:

$$G_{\alpha\beta} = g_{\mu\nu} x^\mu_{,\alpha} x^\nu_{,\beta}$$

(1.2)

is the induced metric on the world-sheet, and $L$ is the Lagrangian taken to be a function of the world-sheet projection of the gauge covariant derivative of a world-sheet scalar field $\Phi$ [6]:

$$\omega = G^{\alpha\beta}(\Phi,_{\alpha} + A_\mu x^\mu_{,\alpha})(\Phi,_{\beta} + A_\mu x^\mu_{,\beta}),$$

(1.3)

they made a suitable Ansatz for the infinitely long stationary open charge-current carrying string, and were then able to write the equations of motion.
in a simple "point particle" Hamiltonian form. This was a considerable simplification of the problem and they were finally able to show that the system could be separated for so-called non-dispersive strings in a Kerr-de Sitter background.

A similar analysis was carried out by the present author [7] for a family of circular strings. Although the formal calculations were quite similar to those of Ref. 5, it turned out that the questions of separability and integrability were less transparent. In this paper we continue the analysis of the charge-current carrying circular string in a black hole background. To be more precise we consider the model [8] originally introduced by Witten [1], which in the notation of Refs. 5 and 6 is obtained by the choice:

$$L(\omega) = 1 + \omega/2$$  \hspace{1cm} (1.4)

and for simplicity we take a background consisting of the simplest kind of black hole, namely the Schwarzschild one:

$$ds^2 = -(1 - 2m/r)dt^2 + \frac{1}{1 - 2m/r}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (1.5)

We have considered other string models also, for instance the Kaluza-Klein model of Nielsen [9]. The results in these cases turn out to be quantitatively similar to the results we will describe below for the model (1.4), also for more general black hole backgrounds like the Reissner-Nordström or the Kerr-Newman black holes [10].

The idea is then to make an Ansatz that describes charge-current carrying circular strings and to look at the equations of motion obtained from the action (1.1) with $L(\omega)$ given by (1.4) and with the background potentials given by (from (1.5)):

$$A_{\mu} = 0, \quad g_{\mu\nu} = diag\left(-(1 - 2m/r), (1 - 2m/r)^{-1}, r^2, r^2 \sin^2 \theta\right).$$  \hspace{1cm} (1.6)

The circular string in this background will essentially be described by the same parameters as a charged point particle and will essentially have the same physical degrees of freedom. Our analysis will therefore be a natural generalization of the analysis of the geodesics of point particles in black hole backgrounds (see for instance Ref. 11). In that spirit the background (1.6) is considered to be fixed, i.e. we do not include possible backreactions from
the string in our analysis. Furthermore we do not discuss the question of a possible critical (maximal) current [8] on the string. We take the more mathematical point of view and consider both charges and currents as completely arbitrary quantities.

The paper is organized as follows: In section 2 we consider the equations of motion for the circular string. We review [7] how a ”point particle” Hamiltonian for the physical degrees of freedom can be obtained, which simplifies the further analysis considerably. In section 3 we analyse the effective potential of the Hamiltonian of section 2 with respect to local and global extrema. In section 4 we continue with some numerical investigations. We consider Poincaré-plots (surfaces of section) of the string-dynamics and we look at different string-trajectories representing different types of capture and scattering. Finally section 5 contains our conclusions.

We use sign conventions of Misner, Thorne, Wheeler [11] and units in which beside $G = 1, c = 1$ the string tension $(2\pi\alpha')^{-1} = 1$. 
2 The circular string

We now consider the circular string in the model described by Eqs. (1.1)-(1.6). The Ansatz is:

\[ t = t(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = \sigma, \quad (2.1) \]
\[ \Phi = f(\tau) + n\sigma, \quad (2.2) \]

where \( f \) is an arbitrary function of \( \tau \) and \( n \) is a constant that is related to the electromagnetic current on the string [7]. The Ansatz (2.1) for the spacetime coordinates describes a plane circular string which is allowed to oscillate and to propagate in the direction perpendicular to its plane. For \( \theta = \pi/2 \) the string is winding around the black hole in the equatorial plane and for general \( \theta \) it is winding around the ”Z-axis” keeping its plane parallel to the equatorial plane. The Ansatz (2.2) for the scalar field \( \Phi \) gives rise to a uniformly charged and current carrying string [7]. In Ref. 7 it was shown that after a suitable redefinition of the string time \( \tau \) the equations of motion for the 4 spacetime coordinates can be obtained as Hamilton equations for the Hamiltonian:

\[ H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu + \frac{1}{2} \left( r^2 \sin^2 \theta + n^2 + \Omega^2 \right) + \frac{(\Omega^2 - n^2)^2}{8r^2 \sin^2 \theta}, \quad (2.3) \]

with the constraints:

\[ H = 0, \quad P_\phi = -n\Omega, \quad (2.4) \]

where \( \Omega \) is the constant of motion corresponding to the cyclic coordinate \( \Phi \), and can be interpreted as the constant charge density of the string [7].

It was already noted in Ref. 7 that because of the \( \theta \)-dependence of the potential in Eq. (2.3) the Hamiltonian is not in the Hamilton-Jacobi separable form [12] (the kinetic energy term is of course separable; it is just the usual kinetic energy for a massless point particle in the Schwarzschild background). In principle this could be just a defect of the coordinate system. Actually the Hamiltonian (2.3) is not even in the separable form in flat spacetime. In that case the system is however certainly integrable and the Hamiltonian is in the separable form after transformation to cylinder coordinates. The same is unfortunately not true in the black hole background considered here. The point is that when the string is outside the equatorial plane (\( \theta = \pi/2 \)) it experiences central gravitational forces directed towards
the center of the black hole \((r = 0)\) as well as non-central forces from the string tension and the electromagnetic self-interaction directed towards the center of the circular string \((Z = r \cos \theta \neq 0)\). Thus neither spherical nor cylindrical coordinates for the 3 dimensional space are ”good” coordinates for the dynamical system considered here. In section 4 we will give numerical evidence for the non-integrability of the equations of motion indicating that the Hamiltonian \((2.3)\) can not be separated in any other coordinate system neither.

Returning to the form of the Hamiltonian \((2.3)\) we can eliminate the cyclic coordinates \(t\) and \(\phi\) by:

\[
P_\phi = -n\Omega, \quad P_t = -\mathcal{E}_o,
\]

where \(\mathcal{E}_o\) is the energy of the string. Then we get the 2 dimensional Hamiltonian:

\[
H = \frac{1}{2} (1 - 2m/r) P_r^2 + \frac{1}{2r^2} P_{\theta}^2 + \frac{1}{2} (r \sin \theta + \frac{N^2}{2r \sin \theta})^2 - \frac{\mathcal{E}_o^2}{2(1 - 2m/r)},
\]

where we also introduced the notation:

\[
N^2 = n^2 + \Omega^2.
\]

Following the point particle case \([11]\) we can write the equation \(H = 0\) as:

\[
\alpha_o \mathcal{E}_o^2 + \gamma_o - (\Delta^2 P_r^2 + \Delta^2 P_{\theta}^2) = 0,
\]

where:

\[
\alpha_o = r^4,
\]

\[
\gamma_o = -\Delta (r^2 \sin \theta + \frac{N^2}{2 \sin \theta})^2,
\]

and:

\[
\Delta = r^2 - 2mr.
\]

We can then define an effective potential \(U(r, \theta)\) \([11]\):

\[
\alpha_o U^2(r, \theta) + \gamma_o = 0.
\]
If $U(r, \theta) = \mathcal{E}_o$ we have a turning point ($P_r = P_\theta = 0$). The string, which is now described by the point particle coordinates $r$ and $\theta$, is then restricted to the area where $U(r, \theta) \leq \mathcal{E}_o$. The explicit expression for $U(r, \theta)$ is:

$$U(r, \theta) = \sqrt{\Delta} (\sin \theta + \frac{N^2}{2r^2 \sin \theta}). \quad (2.13)$$

In the next section we will analyse this potential in some detail. It is then convenient also to express it in terms of cylinder coordinates ($R = r \sin \theta$, $Z = r \cos \theta$):

$$\tilde{U}(R, Z) = (R + \frac{N^2}{2R})\sqrt{1 - \frac{2m}{\sqrt{R^2 + Z^2}}} \quad (2.14)$$

In these coordinates $R$ is the radius of the string loop and $|Z|$ measures the distance from the equatorial plane of the black hole to the plane of the string. Note that $\tilde{U}(2m, 0) = 0$, i.e. the global minimum of the potential outside the horizon of the black hole is at the horizon in the equatorial plane.
3 The effective potential

In this section we analyse the potential (2.14) in more detail. A typical picture of the potential is shown in Fig.1. We see that the potential is essentially an elongated valley between 2 mountain chains. The mountain chain to the right (larger $R$) represents the string tension trying to collapse the circular string, while the mountain chain to the left (smaller $R$) represents the electromagnetic self-interactions trying to expand the circular string. The most interesting observation is however that near the equatorial plane the gravitational attraction between the black hole and the string may overcome the other forces involved. This gives rise to a mountain pass from the valley through the horizon into the black hole. The only chance for the charged and/or current carrying circular string to collapse is therefore to go through this mountain pass and to fall into the black hole.

Let us now consider the critical points of the potential:

$$\frac{\partial \tilde{U}}{\partial R} = (1-\frac{N^2}{2R^2})(1-\frac{2m}{\sqrt{R^2 + Z^2}})^{1/2} + \frac{mR}{(R^2 + Z^2)^{3/2}}(R + \frac{N^2}{2R})(1-\frac{2m}{\sqrt{R^2 + Z^2}})^{-1/2} = 0,$$  \hspace{1cm} (3.1)

$$\frac{\partial \tilde{U}}{\partial Z} = \frac{mZ}{(R^2 + Z^2)^{3/2}}(R + \frac{N^2}{2R})(1-\frac{2m}{\sqrt{R^2 + Z^2}})^{-1/2} = 0.$$  \hspace{1cm} (3.2)

Since we are looking for solutions outside the horizon ($R^2 + Z^2 = r^2 \geq 4m^2$) we find that necessarily $Z = 0$. It follows that $R = r$ is determined by:

$$r^3 - mr^2 - \frac{N^2}{2} r + \frac{3}{2} mN^2 = 0.$$  \hspace{1cm} (3.3)

The second derivatives are given by:

$$\frac{\partial^2 \tilde{U}}{\partial R \partial Z} |_{\text{crit.}} = 0,$$  \hspace{1cm} (3.4)

$$\frac{\partial^2 \tilde{U}}{\partial Z^2} |_{\text{crit.}} = \frac{m(r + \frac{N^2}{r})}{r^2\sqrt{r^2 - 2mr}} |_{\text{crit.}} > 0,$$  \hspace{1cm} (3.5)

$$\frac{\partial^2 \tilde{U}}{\partial R^2} |_{\text{crit.}} = \frac{1}{r^2(r^2 - 2mr)^{3/2}} \left( (N^2 - m^2) r^2 - 6mN^2 r + \frac{15}{2} m^2 N^2 \right) |_{\text{crit.}}.$$  \hspace{1cm} (3.6)
So the equatorial plane is a stable plane and the critical points obtained as solutions to Eq. (3.3) are stable (unstable) if (3.6) is positive (negative). In the case illustrated in Fig.1. there is obviously both a stable and an unstable critical point outside the horizon. This is actually the most interesting case since we can then have a stationary circular string winding around the black hole in the equatorial plane [10]. By carefully analysing the solutions of the cubic equation (3.3) and the corresponding signs of expression (3.6) we find that for:

\[
N^2 > (13\sqrt{13} + 47)m^2 \equiv N_{\text{crit.}}^2 \quad (3.7)
\]

there is a stable and an unstable critical point outside the horizon (Fig.1.). If \(N^2 < N_{\text{crit.}}^2\) there are no critical points outside the horizon and therefore a string in the equatorial plane will always collapse and fall into the black hole. Finally for \(N^2 = N_{\text{crit.}}^2\) we get the (unstable) critical point closest to the horizon:

\[
\frac{\partial^2 \tilde{U}}{\partial R^2} \bigg|_{\text{crit.}} = \frac{\partial \tilde{U}}{\partial R} \bigg|_{\text{crit.}} = 0, \quad (3.8)
\]

with solution:

\[
N^2 = N_{\text{crit.}}^2, \quad r_{\text{crit.}} = \frac{1}{2}(5 + \sqrt{13})m, \quad (3.9)
\]

i.e. whatever the charges and currents are we can never have stationary strings with \(r \leq r_{\text{crit.}}\) winding around the black hole.

In the rest of this paper we will only consider the case (3.7). So we have a global minimum outside the horizon at \((R, Z) \equiv (r_+, 0)\) and a mountain pass (a saddle point) from the valley towards the black hole at \((R, Z) \equiv (r_-, 0)\):

\[
\frac{\partial \tilde{U}}{\partial R} \bigg|_{R=r_+, Z=0} = \frac{\partial \tilde{U}}{\partial R} \bigg|_{R=r_-, Z=0} = 0,
\]

\[
\frac{\partial^2 \tilde{U}}{\partial R^2} \bigg|_{R=r_+, Z=0} > 0, \quad \frac{\partial^2 \tilde{U}}{\partial R^2} \bigg|_{R=r_-, Z=0} < 0,
\]

\[
\frac{\partial^2 \tilde{U}}{\partial Z^2} \bigg|_{R=r_+, Z=0} > 0, \quad \frac{\partial^2 \tilde{U}}{\partial R \partial Z} \bigg|_{R=r_+, Z=0} = 0.
\]

From equation (3.3) we find:

\[
r_+ = \frac{m}{3} \left( 1 + \sqrt{4 + 6 \frac{N^2}{m^2} \cos\left(\frac{\psi}{3} + \frac{4\pi}{3}\right)} \right), \quad (3.10)
\]

\[
r_- = \frac{m}{3} \left( 1 + \sqrt{4 + 6 \frac{N^2}{m^2} \cos\frac{\psi}{3}} \right), \quad (3.11)
\]
where:
\[
\cos \psi = \frac{\sqrt{8m(m^2 - 18N^2)}}{(2m^2 + 3N^2)^{3/2}} \in [-1, 0].
\]

(3.12)

We close this section by considering strings infinitely far away from the black hole. The mountain chain due to the tension of the string becomes infinitely high:
\[
\tilde{U}(R, Z) \to \infty, \quad \text{for} \quad R \to \infty,
\]
so that the only way for a string with finite energy to escape from the black hole is to go in the \(Z\)-direction:
\[
\tilde{U}(R, \pm \infty) = R + \frac{N^2}{2R}.
\]

(3.14)

A stationary circular configuration at \(Z = \pm \infty\) will be in the minimum of this potential, which is situated at \(R = N\sqrt{2}\). The corresponding potential energy is:
\[
\tilde{U}\left(\frac{N}{\sqrt{2}}\right) = \sqrt{2}N.
\]

(3.15)

It is interesting to compare the potential energy (3.15) with the corresponding energies at the points \((r_\pm, 0)\) given in Eqs. (3.10)-(3.11). Obviously:
\[
\tilde{U}\left(\frac{N}{\sqrt{2}}, \pm \infty\right) > \tilde{U}(r_+, 0),
\]

(3.16)

\[
\tilde{U}(r_-, 0) > \tilde{U}(r_+, 0),
\]

(3.17)

and numerically we find:
\[
\tilde{U}\left(\frac{N}{\sqrt{2}}, \pm \infty\right) \geq \tilde{U}(r_-, 0) \iff |N| \leq (13.22...)m.
\]

(3.18)

We will return to these conditions in the following section.

4 Numerical solutions

We now return to the equations of motion corresponding to the 2 dimensional Hamiltonian (2.6). In explicit form they read:
\[
\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} - \sin \theta \cos \theta (1 - \frac{N^4}{4r^4 \sin^4 \theta}),
\]

(4.1)
\begin{align*}
\dot{r} &= (r - 3m)\dot{\theta}^2 - (r - m)\sin^2 \theta - \frac{mN^2}{r^2} + \frac{(r - 3m)N^4}{4r^4 \sin^2 \theta}. 
\tag{4.2}
\end{align*}

From the constraint $H = 0$ we have the first integral:
\begin{align*}
\mathcal{E}_o^2 &= \dot{r}^2 + (r^2 - 2mr)\dot{\theta}^2 + (r^2 - 2mr)(\sin \theta + \frac{N^2}{2r^2 \sin \theta})^2. 
\tag{4.3}
\end{align*}

Unfortunately we have not been able to find an other first integral, necessary to separate Eqs. (4.1)-(4.2). We will therefore turn to numerical methods and in that process we will actually provide some evidence that no second first integral exists. To integrate Eqs. (4.1)-(4.2) numerically it is convenient to introduce the dimensionless quantities $(x, y, \bar{N})$:
\begin{align*}
r &\equiv xm, \quad \theta \equiv y, \quad N \equiv \bar{N}m, 
\tag{4.4}
\end{align*}
and to write the equations of motion in first order form:
\begin{align*}
\dot{x} &= p, 
\tag{4.5}
\dot{y} &= q, 
\tag{4.6}
\dot{p} &= (x - 3)q^2 - (x - 1)\sin^2 y - \frac{\bar{N}^2}{x^2} + \frac{(x - 3)\bar{N}^4}{4x^4 \sin^2 y}, 
\tag{4.7}
\dot{q} &= -\frac{2pq}{x} - \sin y \cos y \left(1 - \frac{\bar{N}^4}{4x^4 \sin^4 y}\right). 
\tag{4.8}
\end{align*}

We have integrated this system of equations using the fourth order Runge-Kutta method for a variety of initial data, of which we now present a few.

First let us consider the situation where the string is initially at rest (non-propagating and non-oscillating) at $Z = -\infty$, i.e. it is described by Eq. (3.15). Because of the gravitational interaction the string will start moving towards the equatorial plane of the black hole. Since the string is coming from $Z = -\infty$ it apriori has enough energy to escape to $Z = +\infty$ after having passed the black hole. However, this will generally not happen. The point is that part of the translational energy of the c.o.m. of the string is transformed into oscillational energy, and the string will therefore be "trapped" in the vicinity of the equatorial plane. This is a typical "adiabatic invariance" phenomenon [12] for motion in a 2 dimensional elongated potential. This is shown in Fig.2a. There are now 2 possibilities depending on the charges and...
currents on the string. If $|N| \leq (13.22...)m$ the string has enough energy to pass the mountain pass towards the center of the black hole (see Eq. (3.18)). After "jumping" around the equatorial plane for some time the string will eventually hit the mountain pass and collapse into the black hole. If on the other hand $|N| > (13.22...)m$ the string will be "adiabatically trapped" around the equatorial plane forever (Fig.2a), but will never actually fall into the center of the black hole. In the latter case it is convenient to consider the surface of section defined by:

$$\theta = \pi/2, \quad \dot{\theta} \geq 0,$$

and to plot $(\dot{R}, R)$ (so-called Poincaré plots) for various values of $N$. A typical plot is shown in Fig.2b. for $\bar{N} = 14$. The result is a completely irregular collection of points, that strongly indicates that no second integral of motion exists for the system (4.5)-(4.8), i.e. it is non-integrable (compare with similar plots for say the Hénon-Heiles system [13]).

The "adiabatic capture" processes described above are obviously most relevant for strings initially at rest. If the string is instead in an initial state of oscillation and/or propagation the typical picture is that the interaction with the black hole gives rise to a change in the distribution of translational and oscillational energy, i.e. if the string is in one oscillating state at $Z = -\infty$ it will move towards the equatorial plane, pass the black hole and continue towards $Z = +\infty$ in another oscillating state. Such processes we denote as "string transmutation" and 2 examples are shown in Fig.3. Finally it is interesting to remark that similar processes were found in a completely different context by de Vega and Sánchez [14]. They considered the quantum scattering of microscopic fundamental strings on a black hole whereas we consider classical scattering of macroscopic charge-current carrying circular cosmic strings on a black hole. Note that in describing these processes we have neglected such phenomena as electromagnetic and gravitational radiation from the oscillating string. As explained in the Introduction treating such effects is out of the scope of this paper.

5 Conclusion

In this paper we continued our investigations of charge-current carrying circular strings in black hole backgrounds [7,10]. Following the approach of
Ref. 7 we obtained a 2 dimensional "point particle" Hamiltonian (2.6) determining the dynamics of the string. From this Hamiltonian we defined an effective potential which was analysed with respect to local and global extrema in section 3. An important result of this analysis was the discovery of a critical charge-current (3.7) implying the existence of stable stationary strings winding around the black hole in the equatorial plane. Numerical investigations in section 4 indicated that, in contrary to the somewhat similar point particle case, the equations of motion (4.5)-(4.8) are non-integrable and we gave an example where the string is in fact jumping chaotically around the equatorial plane of the black hole (Fig.2.). Finally we presented different kinds of string trajectories representing "adiabatic capture" and "string transmutation" processes.

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Figure captions

Fig. 1. The effective potential $\tilde{U}(R, Z)$. In the equatorial plane ($Z = 0$) we find the global minimum and the mountain pass connecting the elongated valley and the center of the black hole.

Fig. 2. "Adiabatic capture". $(R, Z)$-trajectory (a) for a string with $\tilde{N} = 14$ that is initially in a stationary state at $Z = -\infty$, and the corresponding Poincaré plot (b) indicating non-integrability of the system (4.5)-(4.8).

Fig. 3. "String transmutation". $(R, Z)$-trajectories for strings that are initially in oscillating but non-propagating states at $(Z = -\infty)$. The gravitational interaction with the black hole changes the state of oscillation.