GENERIC TROPICAL INITIAL IDEALS OF COHEN-MACaulAY ALGEBRAS

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Abstract. We study the generic tropical initial ideals of a positively graded Cohen-Macaulay algebra $R$ over an algebraically closed field $k$. Building on work of Römer and Schmitz, we give a formula for each initial ideal, and we express the associated quasivaluations in terms of certain $I$-adic filtrations. As a corollary, we show that in the case that $R$ is a domain, every initial ideal coming from the codimension 1 skeleton of the tropical variety is prime, so “generic presentations of Cohen-Macaulay domains are well-poised in codimension 1.”

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1. Introduction

In this paper we study initial ideals of generic tropical variety of a homogeneous ideal in a polynomial ring. In particular, we show that when the ring is Cohen-Macaulay and the initial ideal comes from any codimension 1 cone in the tropical variety then the corresponding initial ideal is prime. This work can be considered as a follow-up to [KMM].

We begin by explaining some background material. Let $k$ be an algebraically closed field and let $k[x] := k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ indeterminates. Take $f(x) = \sum_\alpha c_\alpha x^\alpha \in k[x]$. Recall that for $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, the initial form $in_w(f)$ is the polynomial $\sum_\beta c_\beta x^\beta$ where the sum is over all $\beta$ such that the inner product $\langle w, \beta \rangle$ is minimum. For an ideal $J \subset k[x]$, $in_w(J)$ is the ideal generated by $in_w(f)$, $\forall f \in J$. We also recall that the the tropical variety $\text{Trop}(J) \subset \mathbb{R}^n$ is defined as:

$$\text{Trop}(J) = \{ w \in \mathbb{R}^n \mid \text{in}_w(J) \text{ contains no monomials} \}.$$ 

It is well-known that $\text{Trop}(J)$ is the support of a polyhedral fan (see [MS15, Theorem 3.3.5]).

Fix a homogeneous ideal $J \subset k[x]$ and consider the ideal $I = g \circ J$ obtained by making a coordinate transformation $g \in \text{GL}_n(k)$. In [RS12] it is shown that for $g$ in a nonempty Zariski open, $\text{Trop}(I)$ is independent of the choice of $g$. Hence, $\text{Trop}(I)$ for $g$ generic is known
as the \textit{generic tropical variety} of $J$. In this paper, when $R = k[x]/J$ is Cohen-Macaulay, we describe the initial ideals $in_w(I)$ for $w$ in the the generic tropical variety $\text{Trop}(I)$.

First, let us consider the case of a principal ideal. Take a homogeneous polynomial 
\[\sum_{\deg(\alpha) = r} c_{\alpha} x^\alpha = f \in k[x] \] with $c_{\alpha} \neq 0$, $\forall \alpha$, that is, every monomial of degree $r$ appears in $f$ with nonzero coefficient. For an index set $A \subseteq [n] := \{1, \ldots, n\}$ we let $C_A \subseteq \mathbb{Q}^n$ be the polyhedral cone of tuples $w = (w_1, \ldots, w_n)$ with the property that $w_i = \min\{w_j \mid 1 \leq j \leq n\}$ for all $i \in A^c := [n] \setminus A$, and $\tilde{C}_A \subset C_A$ its relative interior. It is straightforward to show that, for any $w \in \tilde{C}_A$, the initial form $in_w(f)$ is the sum of those monomial terms $c_{\beta} x^\beta$ such that support of $\beta$ is a subset of $A^c$. Thus, $in_w(f)$ does not contain any indeterminate $x_i$, $i \in A$. It follows that the quotient algebra $k[x]/\langle in_w(f) \rangle$ is a polynomial ring in $|A|$ variables over the algebra $k[x_i \mid i \in A^c]/\langle in_w(f) \rangle$. If $|A^c| > 2$ and the coefficients $c_{\alpha}$ are chosen sufficiently generically, then both $f$ and $in_w(f)$ are irreducible, so both $k[x]/\langle f \rangle$ and its initial degeneration $k[x]/\langle in_w(f) \rangle$ are domains. We generalize these observations to any ideal $J \subset k[x]$ such that $k[x]/J$ is Cohen-Macaulay.

Now fix a homogeneous ideal $J \subset k[x]$ and consider the ideal $I = g \circ J$ for generic $g \in \text{GL}_n(k)$. In \cite{RS12} it is shown that if $k[x]/I$ has Krull dimension $d$, the tropical variety $\text{Trop}(I) \subseteq \mathbb{Q}^n$ is the support of the polyhedral fan $W^a_n$ composed of the cones $C_A$ with $|A^c| = n - d + 1$. Moreover, $\text{Trop}(I)$ detects whether $k[x]/I$ is Cohen-Macaulay, and in this case the fan structure on $W^a_n$ induced from the Gröbner fan of $I$ coincides with the fan structure defined by the cones $C_A$ (see \cite[Corollary 4.7]{RS10}).

Each point $w \in \text{Trop}(I)$ can be used to construct a discrete homogeneous quasivaluation $\nu_w : R \setminus \{0\} \to \mathbb{Q}$ and an associated graded algebra $gr_w(R)$ (see Section 2 as well as \cite[Section 2.4]{KM19}). Here, $gr_w(R) \cong k[x_1, \ldots, x_n]/in_w(I)$, where $in_w(I)$ is the initial ideal of $I$ with respect to $w$. In the following let $I = g \circ J$ for $g \in \text{GL}_n(k)$ chosen from an appropriate Zariski open subset, and let $y_i = \pi(x_i) \in k[x]/I \cong R$. Our first theorem sharpens the picture provided in \cite{RS10} by giving an explicit description of $gr_w(R)$. Each $y_i$ gives a $(y_i)$-adic filtration $\langle y_i \rangle \supset \langle y_i^2 \rangle \supset \cdots$ which in turn gives a quasivaluation ord, on $R$ (see Section 2). We compute $\nu_w$ in terms of the functions ord, for $1 \leq i \leq n$. We write $\min(w)$ as a shorthand for $\min\{w_i \mid 1 \leq i \leq n\}$.

\begin{theorem}
With notation as above, let $R$ be Cohen-Macaulay. Let $A \subset [n]$ with $|A^c| \leq n - d + 1$ and $w \in \tilde{C}_A$, then:
\begin{enumerate}
\item $gr_w(R) \cong (R/\langle y_i \mid i \in A \rangle)[t_i \mid i \in A]$,
\item $\nu_w = (\min(w) \circ \deg) \oplus \bigoplus_{i \in A^c} ((w_i - \min(w)) \circ \text{ord}_i)$.
\end{enumerate}
\end{theorem}

Here $\deg$ denotes the quasivaluation on $R$ given by homogeneous degree, and the operations $\oplus$ and $\circ$ are described in Section 2.

As a corollary of Theorem 1.1 we obtain a description of each initial ideal $in_w(I)$. For a subset $A \subseteq [n]$ let $I_A \subset k[x_j \mid j \in A^c]$ be the kernel of the induced presentation $\pi_A : k[x_j \mid j \in A^c] \to R/\langle \{y_i \mid i \in A\} \rangle$.

\begin{corollary}
Let $w \in \tilde{C}_A$ be as above, then $in_w(I) = I_A k[x_1, \ldots, x_n]$.
\end{corollary}

In particular, Corollary 1.2 recovers and refines \cite[Proposition 4.6]{RS10}.

Let $\tilde{y} = \{y_1, \ldots, y_n\} \subset R$. The image of the generating set $\tilde{y}$ generates each associated graded algebra $gr_w(R)$. Following \cite[Section 2]{KM19}, $\tilde{y}$ is said to be a \textit{Khovanskii basis} for $(R, \nu_w)$. In fact all $\mathbb{Q}$-quasivaluations with Khovanskii basis $\tilde{y}$ are of the form $\nu_w$ for some $w \in \mathbb{Q}^n$. Such a quasivaluation is a valuation precisely when the initial ideal $in_w(I)$ is prime. A relatively open cone $C \subset \text{Trop}(I)$ with $in_w(I) = in_{w'}(I)$ for all $w, w' \in C$ is said
to be a prime cone if \( i_w(I) \) is a prime ideal for all \( w \in \hat{C} \) (see [KM19]). The next result gives the state of affairs for prime cones in a generic tropicalization \( \text{Trop}(I) \).

**Theorem 1.3.** Let \( R, J, \text{ and } I \) be as above. The ideal \( J \) is radical if and only if \( i_w(I) \) is radical for all \( w \in W_n^d \). The ideal \( J \) is prime if and only if \( i_w(I) \) is prime for all \( w \in W_{d-1}^n \subset W_d^n \).

Let 1 denote the all 1’s vector. From [KM19, Theorem 1] and Theorem 1.3 it follows that if \( R \) is a domain then any linearly independent collection \( w = \{w_1, \ldots, w_{d-2}, 1\} \subset C_A \) with \( |A| = d - 2 \) defines an integral, rank \( d - 1 \) valuation \( v_w : R \setminus \{0\} \to \mathbb{Z}^{d-1} \) with a finite Khovanskii basis. The associated graded algebra \( gr_w(R) \) can be computed with Theorem 1.1. Valuations of this type define flat degenerations of \( \text{Proj}(R) \) to complexity-1 \( T \)-varieties, and are studied in [KMM].

In general the initial ideals coming from points in the tropical variety \( \text{Trop}(I) \) are not all prime. When this happens \( I \) is said to be well-poised. This property was defined in [IM19], where it was shown that the so-called semi-canonical embeddings of rational, complexity-1 \( T \)-varieties are always well-poised. Other examples are the Plücker embeddings of Grassmannian varieties of 2-planes and any monomial-free linear ideal. Theorem 1.3 shows that the generic tropicalization of a Cohen-Macaulay domain almost has this property.

**Corollary 1.4.** Let \( R \) be Cohen-Macaulay, with \( I \) as above. Then \( \text{Trop}(I) \) is well-poised in codimension 1. That is, if \( w \) belongs to a codimension 1 cone in \( \text{Trop}(I) \), for the fan structure coming from the Gröbner fan, then \( i_w(I) \) is prime.

**Remark 1.5.** For a positively graded domain \( R \) (not necessarily Cohen-Macaulay) [KMM] uses the Bertini irreducibility theorem to construct a valuation with corank 1 whose associated graded algebra is finitely generated. The above corollary shows that, when \( R \) is Cohen-Macaulay, in fact one can construct this valuation using any codimension 1 cone in the generic tropical variety. This makes a direct connection between constructions in [KM19] and [KMM].

With above assumptions, it turns out that the additional property of being well-poised is a strong requirement.

**Corollary 1.6.** Let \( R \) be Cohen-Macaulay, with \( I \) as above. Then \( I \) is well-poised if and only if it is a linear ideal. In particular, in this case \( R \) must be isomorphic to a polynomial ring over \( k \).

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2. **Filtrations and quasivaluations**

Let \( \Gamma \) be an ordered group, and let \( \bar{\Gamma} = \Gamma \cup \{\infty\} \). Recall that a *quasivaluation* \( v : R \to \bar{\Gamma} \) over \( k \) is a function satisfying the following axioms for all \( f, g \in R \):

- \( v(fg) \geq v(f) + v(g) \),
- \( v(f + g) \geq \min\{v(f), v(g)\} \),
- \( v(Cf) = v(f) \), \( \forall C \in k \setminus \{0\} \).

We say \( v \) is a *valuation* if \( v(fg) = v(f) + v(g) \). Let \( S(R, v) \subset \bar{\Gamma} \) be the set of values of \( v \); this is a semigroup under the group operation in \( \bar{\Gamma} \) if \( v \) is a valuation, and is referred
to as the value semigroup of \( v \) in this case. The kernel of a quasivaluation is the vector space of elements which are sent to \( \infty \). We say a quasivaluation \( v \) is homogeneous with respect to the grading on \( R \) if the value \( v(f) \) of an element \( f \in R \) is always achieved on one of its homogeneous components. In the sequel we deal with groups \( \Gamma \subseteq \mathbb{Q} \), and we assume that the subgroup \( \langle S(R, v) \rangle \subseteq \Gamma \) is discrete. Moreover, we take quasivaluations to be homogeneous (when this makes sense), and we assume the kernel is \( \{0\} \), i.e. \( v(f) = \infty \) if and only if \( f = 0 \).

For each quasivaluation \( v \) we obtain an \( \Gamma \)-algebra filtration of \( R \) given by the spaces

\[
F_r(v) = \{ f \mid v(f) \geq r \}.
\]

In particular, for \( r \leq s \) we have \( F_r(v) \supseteq F_s(v) \) and \( F_r(v)F_s(v) \subseteq F_{r+s}(v) \). The filtration \( F(v) \) is homogeneous in the sense that \( F_r(v) = \bigoplus_{n \geq 0} F_{r}F_n \cap R_n \). In particular, the set of homogeneous elements of degree \( d \) with value greater than or equal to \( r \) is always a finite dimensional vector space. Under this operation, the kernel of \( v \) is the vector space \( \bigcap_{r \in \Gamma} F_r(v) \). Conversely, if we start with a homogeneous algebra filtration \( F \) of \( R \) with \( \bigcap_{r \in \Gamma} F_r = \{0\} \), we obtain a quasivaluation \( v_F : R \rightarrow \Gamma \) with kernel \( \{0\} \), where \( v_F(f) = \max \{ r \mid v(f) \geq r \} \). For any algebra filtration \( F \) the associated-graded algebra is defined as follows:

\[
gr_F(R) = \bigoplus_{r \in \Gamma} F_r/F_{r+1}.
\]

**Example 2.1.** Any homogeneous ideal \( I \subseteq R \) has a corresponding \( I \)-adic filtration by the powers \( I^r \subseteq R \). We let \( v_I : R \rightarrow \mathbb{Z} \) denote the corresponding quasivaluation and \( gr_I(R) \) be the associated graded algebra.

**Proposition 2.2.** Let \( v : R \rightarrow \Gamma \) be a quasivaluation with no kernel. If \( gr_v(R) \) is reduced, then \( R \) is reduced, and if \( gr_v(R) \) is a domain, then \( R \) is a domain.

**Proof.** We show that if \( R \) has a nilpotent or a zero divisor, then \( gr_v(R) \) does as well. Suppose \( 0 \neq f \in R \), and \( v(f) = r \in \Gamma \). Let \( \tilde{f} \) be the image of \( f \in F_r(v) \) under the projection \( F_r(v) \rightarrow F_r(v)/F_{r+1}(v) \). By definition, \( \tilde{f}^n \) is computed by taking the image of \( f^n \in F_{nr}(v) \) under the projection \( F_{nr}(v) \rightarrow F_{nr}(v)/F_{nr+1}(v) \). If \( f \neq 0 \) and \( f^n = 0 \), then \( \tilde{f} \neq 0 \) and \( \tilde{f}^n = 0 \), since \( f^n \in F_{nr}(v) \). Similarly, if \( f, g \neq 0 \) with \( v(f) = r, v(g) = s \) and \( fg = 0 \), then \( \tilde{f}, \tilde{g} \neq 0 \) and \( \tilde{f}g = 0 \) since \( fg \in F_{r+s+1}(v) \).

Now we define the operation \( \oplus \) on the set of quasivaluations on \( R \), this will be used in the proof of Theorem 1.1.

**Definition 2.3.** For quasivaluations \( v_1, v_2 \) on \( R \), define \( v_1 \oplus v_2 \) to be the quasivaluation defined by the filtration composed of the following spaces:

\[
F_r(v_1 \oplus v_2) = \bigoplus_{r_1 + r_2 = r} F_{r_1}(v_1) \cap F_{r_2}(v_2).
\]

Suppose \( \Gamma \) is a \( \mathbb{Q} \)-vector space. For \( w \in \mathbb{Q}_{\geq 0} \) we let \( w \odot v \) denote the quasivaluation obtained by scaling the values of \( v \) by \( w \), that is, \( (w \odot v)(f) = wv(f) \), \( \forall f \in R \). It is straightforward to check that \( F_r(w \odot v) = F_{\frac{r}{w}}(v) \) and that \( n \odot v = \bigoplus_{i=1}^n v \) for \( n \in \mathbb{Z}_{\geq 0} \).

From now on we will speak of \( \oplus \) on filtrations and quasivaluations interchangeably.

We say that a vector space basis \( \mathbb{B} \subset R \) is an adapted basis for a filtration \( F \) if \( F_r \cap \mathbb{B} \) is a basis for \( F_r \), for all \( r \).

The operation \( \oplus \) is not associative in general. However, \( \oplus \) is associative on filtrations \( F^1, F^2, \ldots, F^t \) if the \( F^i \) have a common adapted basis \( \mathbb{B} \).
Proposition 2.4. Let $F^1, \ldots, F^\ell$ be homogeneous filtrations corresponding to discrete quasi-valuations on $R$. The subspaces $\{F^i_r \mid 1 \leq i \leq \ell, r \in \Gamma\}$ generate a distributive lattice of subspaces of $R$ if and only if the $F^i$ share a common adapted basis. In this case, $\oplus$ defines an associative operation on the set of quasi-valuations defined by taking multiples and $\oplus$-sums of $\nu_{F^1, \ldots, F^\ell}$. Finally, if $F, G$ are filtrations sharing a common basis $B$ then $\nu_{F \oplus G}(b) = \nu_F(b) + \nu_G(b)$ for any $b \in B$.

Proof. If the $F^i$ share a common adapted basis $B$, then each space $F^i$ corresponds to a subset $B^i = B \cap F^i$ and the operations of intersection and sum correspond to intersection and union of these subsets, respectively. It follows that the $F^i$ generate a distributive lattice in the subspaces of $R$. Conversely, if the $F^i$ generate such a lattice, then the same holds for their intersection with any graded component $R_n$. A distributive lattice of subspaces of a finite dimensional vector space always has an adapted basis (see [DR JS18, Section 3]). Let $B_n$ be this basis for $n \in \mathbb{Z}_{\geq 0}$, then $B = \coprod_{n \geq 0} B_n$ is adapted to each of the $F^i$.

Now, for any $r \in \mathbb{Q}$ we have:

$$\sum_{s+s_3=r} \left( \sum_{s_1+s_2=s} F^i_{s_1} \cap F^j_{s_2} \right) \cap F^k_{s_3} = \sum_{s_1+s'=r} F^i_{s_1} \cap \left( \sum_{s_2+s_3=s'} F^j_{s_2} \cap F^k_{s_3} \right),$$

this implies that $\left((\nu_{F^i} \oplus \nu_{F^j}) \oplus \nu_{F^k}\right)^{-1}(r) = (\nu_{F^i} \oplus (\nu_{F^j} \oplus \nu_{F^k}))^{-1}(r)$. Observe that any collection of filtrations obtained by $\oplus$ and scaling from the $F^i$ also share the basis $B$, so the first part of this proof applies. Finally, we let $b \in B \subset R$ and we suppose that $F, G$ are filtrations adapted to $B$. If $b \in F_r, b \notin F_{<r}$ and $b \in G_s, b \notin G_{<s}$ then $b \in F_r \cap G_s \subset \sum_{t_1+t_2=r+s} F_{t_1} \cap G_{t_2}$. If $t_1 + t_2 < r + s$ then without loss of generality we may assume that $t_1 < r$. It follows that $b \notin F_{t_1} \cap G_{t_2}$. The set $B$ is an adapted basis to each space $F_{t_1} \cap G_{t_2}$ and their sum, it follows that $b$ is in $\sum_{t_1+t_2<r+s} F_{t_1} \cap G_{t_2}$ if and only if $b$ is in one of the $F_{t_1} \cap G_{t_2}$, a contradiction. \hfill $\Box$

Now we assume that $\Gamma = \mathbb{Z}$ or $\mathbb{Q}$ with the usual total ordering. An element $u \in \Gamma^n$ determines a valuation $\bar{v}_u : k[x] \to \Gamma$ defined by sending a monomial $x^a$ to the inner product $(u, a)$, and a polynomial $\sum c_a x^a$ to $\min\{\bar{v}_u(x^a) \mid c_a \neq 0\}$. A homogeneous presentation $\pi : k[x] \to R$ then determines an associated weight quasi-valuation on $R$ by the pushforward operation: $v_u = \pi_* \bar{v}_u$. In particular for $f \in R$ we have $v_u(f) = \max\{\bar{v}_u(p(x)) \mid \pi(p(x)) = f\}$ (see [KM19, Definition 3.1]).

The associated graded algebra $gr_u(R)$ of $v_u$ is presented by the initial ideal $in_u(I) \subset k[x]$ ([KM19, Lemma 3.4]), where $I = \mathrm{ker}(\pi)$. In particular, $gr_u(R)$ is presented by the images of the generators $\pi(x_i) = b_i \in R$, $1 \leq i \leq n$. We say that the set $B = \{b_1, \ldots, b_n\} \subset R_1 \subset R$ is a Khovanskii basis of $v_u$ ([KM19, Definition 1]). By [KM19, Proposition 3.7], any quasi-valuation with Khovanskii basis $B$ is of the form $v_u$ for some $u \in \Gamma^n$.

Now we let $\Gamma = \mathbb{Q}$. The behavior of the weight quasi-valuations $v_u$ is governed by the Gröbner fan $\Sigma(I)$. To a total monomial ordering $<$ we associate a closed cone $\tau_\prec \in \Sigma(I)$. This is the set of $u$ such that $in_\prec(in_u(I) = in_\prec(I))$. The initial ideal $in_\prec(I)$ is a monomial ideal; we let $\mathbb{B}_\prec \subset R$ be the set of images of monomials not contained in $in_\prec(I)$. It is well-known that $\mathbb{B}_\prec$ forms a vector space basis of $R$. For the basics of Gröbner bases, and a proof of the following see [Stu96] and [KM19, Proposition 3.3].

Proposition 2.5. Let $I \subset k[x_1, \ldots, x_n]$ be a homogeneous ideal with Gröbner fan $\Sigma(I)$, then:

1. For a monomial order $<$ and $u \in \tau_\prec$, the set $\mathbb{B}_\prec \subset R$ is an adapted basis of $v_u$.
2. If $u, w \in \tau_\prec$ then $v_u \oplus v_w = v_{u+w}$.
Proposition 3.7] implies that

The fact that a dense, open subset of $M$ rings are characterized by the fact that every $x$ has the appropriate size ($U$-Cohen-Macaulay so the image of any $\{ \}$ be the intersection of all of the inverse images of the $X$-projection $ji$ are polynomial generators of the coordinate ring $k$. Let $Z = X \times_k M_{k \times n}(k)$ and $X^* \subset Z$ be the 0-locus of the forms $F_j = \sum_{i=1}^{n} x_i y_{ji}$, where the $y_{ji}$ are polynomial generators of the coordinate ring $k[M_{k \times n}(k)]$. Let $\mathcal{L}$ be the restriction of $\mathcal{O}(1)$ on $P(V^*)$ to $X$, then the $x_i \in V$ define sections $s_i \in H^0(X, \mathcal{L})$. Consider the projection $p : X^* \to X$, and let $U \subset X$ be an affine open subset with coordinate ring $A = k[U]$. Without loss of generality we may arrange that $\mathcal{L} |_U = s_1 \mathcal{O}_U$ with $s_i = f_i s_1$, so that $p^{-1}(U) \subset X^*$ is the 0-locus of the equations $y_{j1} = \sum_{i>1} f_i y_{ji}$ in $U \times_k M_{k \times n}, (k)$. It follows that $p^{-1}(U) \cong U \times_k M_{k \times n-1}(k)$. As a consequence, we see that if $X$ is reduced or irreducible, $X^*$ is as well.

Now we have a surjective map of schemes $q : X^* \to M_{k \times n}(k)$, with $M_{k \times n}(k)$ regular. Furthermore, the fiber $X^*_m$ over a closed point $m \in M_{k \times n}(k)$ is isomorphic to the 0-locus of the equations $f_j = \sum_{i=1}^{n} x_i m_{ji} \in k[x]/I$. So we may argue with the generic principle to understand these subschemes (see [FOV99 Chapter 3]). Irreducibility of $X^*_m$ for $m$ in a dense, open subset of $M_{k \times n}(k)$ follows from the last part of the argument in [FOV99 3.4.10], provided we choose $|A| < d - 1$ so that $X^*_m$ has the correct dimension.

3. Proof of main theorems

The proof of Theorem [11] involves a Bertini-type construction and a theorem of Rees, both of which we introduce now. We also use the fact that positively graded Cohen-Macaulay rings are characterized by the fact that every homogeneous system of parameters (hsop) is a regular sequence.

Proposition 3.1. Let $k$ be an algebraically closed field. Let $J \subset k[x]$ be a homogeneous ideal such that the positively graded algebra $R = k[x]/J$ is a $d$-dimensional Cohen-Macaulay algebra. Then there is a dense, open subset $U \subset GL_n(k)$ such that for any $g \in U$, and any $A \subset [n]$ with $|A| \leq d - 1$, the ideal $I_A = (I, x_i \mid i \in A)$ has height $d - |A|$, where $I = g \circ J$. Any such set $\{ x_i \mid i \in A \}$ defines a regular sequence in $R \cong k[x]/I$. Moreover, if $J$ is radical then $I_A$ is radical, and if $J$ is irreducible then $I_A$ is irreducible for all $|A| < d - 1$.

Proof. Let $X = \text{Proj}(R) \subset P(V^*)$, where $V$ is the space of linear forms in $k[x]$. For $A \subset [n]$ let $V_A$ be the span of $\{ x_i \mid i \in A \}$. By the Kleinman-Bertini theorem there is a dense, open subset $W_A \subset GL_n(V)$ such that $g \circ X \cap P(F_A)$ has dimension $d + |A| - n = d - |A|$ for all $g \in W_A$. If we let $W = \bigcap_{|A| \leq d - 1} W_A$ and $I = g \circ J$ for $g \in W$, then the image of any set $\{ x_i \mid i \in A \}$ in $k[x]/I$ is part of a system of parameters for $R \cong k[x]/I$. The algebra $R$ is Cohen-Macaulay so the image of any $\{ x_i \mid i \in A \}$ in $k[x]/I$ is a regular sequence.

To prove reduced and irreducible, we argue as in the proof of the Bertini theorem. [FOV99 3.4.8]. We will show that the 0-locus of a choice of $k$ generic forms, thought of as the rows of an element of $M_{k \times n}(k)$, has the desired property (reduced, irreducible, respectively) when $k$ has the appropriate size ($\leq d - 1, < d - 1$, respectively). If $|A| = k$ it follows that there is a dense, open subset $U_A \subset V^A$ which has the required property. To prove the theorem, we let $U \subset GL_n(k)$ be the intersection of all of the inverse images of the $U_A$ under the projections $V^n \to V^A$ with $GL_n(k)$.

Let $Z = X \times_k M_{k \times n}(k)$ and $X^* \subset Z$ be the 0-locus of the forms $F_j = \sum_{i=1}^{n} x_i y_{ji}$, where the $y_{ji}$ are polynomial generators of the coordinate ring $k[M_{k \times n}(k)]$. Let $\mathcal{L}$ be the restriction of $\mathcal{O}(1)$ on $P(V^*)$ to $X$, then the $x_i \in V$ define sections $s_i \in H^0(X, \mathcal{L})$. Consider the projection $p : X^* \to X$, and let $U \subset X$ be an affine open subset with coordinate ring $A = k[U]$. Without loss of generality we may arrange that $\mathcal{L} |_U = s_1 \mathcal{O}_U$ with $s_i = f_i s_1$, so that $p^{-1}(U) \subset X^*$ is the 0-locus of the equations $y_{j1} = \sum_{i>1} f_i y_{ji}$ in $U \times_k M_{k \times n-1}(k)$. It follows that $p^{-1}(U) \cong U \times_k M_{k \times n-1}(k)$. As a consequence, we see that if $X$ is reduced or irreducible, $X^*$ is as well.

Now we have a surjective map of schemes $q : X^* \to M_{k \times n}(k)$, with $M_{k \times n}(k)$ regular. Furthermore, the fiber $X^*_m$ over a closed point $m \in M_{k \times n}(k)$ is isomorphic to the 0-locus of the equations $f_j = \sum_{i=1}^{n} x_i m_{ji} \in k[x]/I$. So we may argue with the generic principle to understand these subschemes (see [FOV99 Chapter 3]). Irreducibility of $X^*_m$ for $m$ in a dense, open subset of $M_{k \times n}(k)$ follows from the last part of the argument in [FOV99 3.4.10], provided we choose $|A| < d - 1$ so that $X^*_m$ has the correct dimension.
Similarly, if $k$ is characteristic 0, the reduced property is an immediate consequence of \[FOV99, 3.3.15\]. If not, we argue as in \[FOV99, 3.4.13\]. First, we may adapt the proof of \[FOV99, 3.4.12\]. Let $U$ be as above, then for $m$ in some dense, open subset of $M_{k,n}(k)$, and sections $n_i = (d\ell_i, f_i) \in H^0(U, \Omega_U \oplus O_U)$, the sections $h_j = \sum n_i m_{ij}$ specialize to linearly independent sections over every fiber. If $x \in U$ is in the 0-locus of the corresponding $\ell_j$, it follows that the differentials $d\ell_j$ are all linearly independent in the fiber $\Omega_U(x)$. Moreover, by the same Kleiman-Bertini argument we may choose the dense/open subset of $M_{k,n}(k)$ so that the intersection of $\mathcal{X}_m$ with the singular components of $X$ strictly lowers the dimension. As a consequence, if $X$ is Serre’s condition $R_t$, then so is $\mathcal{X}_m$ for $m$ chosen in this subset. Since $X$ is Cohen-Macaulay, $\mathcal{X}$ and $\mathcal{X}_m$ are $S_t$ for all $t$, so if $X$ is reduced, so is $\mathcal{X}_m$.

We have proved that generic $\mathcal{X}_m$ are geometrically irreducible and reduced, so the affine scheme $\text{Spec}(k[x]/I_A)$ is generically reduced, and irreducible. But this scheme is also Cohen-Macaulay, so it is unmixed, and generically reduced implies reduced everywhere. In particular, if $I$ is prime, so is $I_A$ for all $|A| < d - 1$. \hfill \Box

Next we use the following result, due originally to Rees (see \[Ree57, Theorem 2.1\]).

**Proposition 3.2.** Let $\vec{y} = \{y_1, \ldots, y_k\}$ be a regular sequence in an algebra $R$, and consider the $J$-adic filtration of $R$, where $J = (y_1, \ldots, y_k)$, with associated graded algebra $gr_J(R)$. Then we have $gr_J(R) \cong (R/J)[\ell_1, \ldots, \ell_k]$. The isomorphism is defined by sending each $\ell_i \in (R/J)[\ell_1, \ldots, \ell_k]$ to the image of $y_i$ in $gr_J(R)$. Moreover, $R/J$, in the righthand side, is identified with $R/J$ as the zeroth degree part of $gr_J(R)$ in the lefthand side.

As before let $R$ be a positively graded algebra. From now on we let $\vec{y} = \{y_1, \ldots, y_n\} \subset R$ be chosen as in Proposition 3.1, and we let $I \subset k[x]$ be the corresponding homogeneous ideal. We select a subset $A \subset [n]$ with $|A| \leq d - 1$, and we let $J_A = (y_i \mid i \in A)$. We let $\text{ord}_A : R \to \mathbb{Z}$ be the quasivaluation obtained from the $J_A$-adic filtration of $R$, and $gr_A(R)$ be the corresponding associated-graded algebra. Finally, we let $\epsilon_A \in \mathbb{Q}^n$ be the $(0,1)$-vector with a 1 for each $i \in A$ and a 0 for $j \in A^c$.

**Proposition 3.3.** We have $\epsilon_A = (\text{ord}_A(y_1), \ldots, \text{ord}_A(y_n)) \in \mathcal{W}_d^n$. In particular, $\epsilon_A \in \text{Trop}(I)$, $\nu_{\epsilon_A} = \text{ord}_A$ and $\text{in}_{\epsilon_A}(I) = I_A k[x]$.

**Proof.** Observe that $y_i \in J_A \setminus J_A^2$ for $i \in A$ and $y_j \in R \setminus J_A$ for $j \in A^c$. This and the definition of $\mathcal{W}_d^n$ prove the first claim. Proposition 3.2 implies that the set $\vec{y}$ is a Khovanskii basis of $\text{ord}_A$, so by \[KM19, Proposition 3.7\], $\text{ord}_A = \nu_{\epsilon_A}$ since both quasivaluations take the same values on $\vec{y}$. Proposition 3.2 also shows that $\text{in}_{\epsilon_A}(I) = I_A k[x]$. \hfill \Box

Now we fix $A \subset [n]$ with $|A| = d$, and consider the weight vectors $\epsilon_i$ for $i \in A$. We show that these all live in a common cone of the Gröbner fan.

**Lemma 3.4.** Let $A$ and $i \in A$ be as above, then $\text{in}_{\epsilon_A \setminus \{i\}}(\text{in}_{\epsilon_i}(I)) = \text{in}_{\epsilon_A}(I)$.

**Proof.** By Proposition 3.3, $\text{in}_{\epsilon_i}(I) = I_i k[x_1, \ldots, x_n]$, where $I_i \subset k[x_1, \ldots, x_i, \ldots, x_n]$ presents $R/\langle y_i \rangle$. The algebra $R/\langle y_i \rangle$ is also Cohen-Macaulay, so the images of $\{y_j \mid j \in A \setminus \{i\}\}$ also form a regular sequence. It follows that $gr_{\epsilon_A \setminus \{i\}}(gr_{\{i\}}(R)) \cong (R/J_A)[\ell_i \mid i \in A]$, and so $\text{in}_{\epsilon_A \setminus \{i\}}(\text{in}_{\epsilon_i}(I)) = \text{in}_{\epsilon_A}(I)$. \hfill \Box

Now we prove Theorem 1.1 and Corollary 1.2.

**Proof of Theorem 1.1** and **Corollary 1.2** We observe that $(\text{deg}(y_1), \ldots, \text{deg}(y_n)) = (1, \ldots, 1)$, and that $\text{in}_{w(1,\ldots,1)}(J) = J$ for any ideal we encounter, as everything is homogeneous. In
particular, $deg = v_{(1,\ldots,1)}$. For $w \in \mathcal{C}_A$ we then get $(\min(w) \odot deg) \oplus (\bigoplus_{i \in A} (w_i - \min(w))) \odot \text{ord}_{i} = v_{\min(w)(1,\ldots,1)} \oplus (\bigoplus_{i \in A} v_{(w_i, \min(w))}) = v_w$, where the latter equality is a consequence of Proposition \ref{prop:order}. By Lemma \ref{lem:positive} the $e_i$ for $i \in A$ all lie in the same cone of $\Sigma(I)$. Moreover, for any $w_i > 0$ with $w = \sum_{i \in A} w_i e_i$, we have $in_w(I) = in_{\mathcal{C}_A}(I)$, as $w$ and $e_A$ are in the relative interior of this cone. It follows that $gr_w(R) \cong gr_{\mathcal{C}_A}(R) \cong R/J_A[t_i \mid i \in A]$. Moreover, $\oplus$-ing with some multiple of degree does not change this calculation. \hfill \Box

Now we use Propositions \ref{prop:order} and \ref{prop:positive} to prove Theorem \ref{thm:main} and Corollary \ref{cor:prime}.

Proof of Theorem \ref{thm:main} and Corollary \ref{cor:prime}. Let $w \in \mathcal{C}_A \subset \mathcal{W}_d^n$, then $gr_w(R) \cong k[x]/in_w(I) \cong k[x]/in_{\mathcal{C}_A}(I) \cong R/J_A[t_i \mid i \in A]$. If $J$ is radical, then $I$ and $I_A$ are radical, so $R/J_A$ is reduced and $gr_w(R)$ is reduced. The same reasoning holds if $w \in \mathcal{W}_d^{n-1}$ and $I$ is prime. Now if $in_w(I)$ is prime (resp. radical), Proposition \ref{prop:positive} implies that $J$ is prime (resp. radical). \hfill \Box

To prove Corollary \ref{cor:prime}, we show a slightly stronger result.

Proposition 3.5. Let $R$, $J$, and $I$ be as above, then the following are equivalent:

1. $\mathcal{C}_A$ is a prime cone for some $A \subseteq [n]$ with $|A| = d - 1$.
2. $I$ is well-poised.
3. $I$ is a linear ideal.

Proof. Clearly (3) $\implies$ (2) $\implies$ (1), so we show that (1) $\implies$ (3). Suppose that $C_A$ were a prime cone for some $|A| = d - 1$, then by Theorem \ref{thm:main} $gr_w(R) \cong R/J_A[t_i \mid i \in A]$ is a domain. It must be the case that $R/J_A$ is a positively graded domain of dimension 1 which is generated by its degree 1 component, so $R/J_A \cong k[t]$. As a consequence, the monomials in the generators $\{y_i \mid i \in A\} \cup \{t\}$ form a homogeneous $k$-vector space basis of $R$. Let $I_1 \subseteq I$ be the linear part of $I$. The Hilbert functions of $R/I$ and $R/I_1$ agree, so $I_1 = I$. \hfill \Box

References

[DRJS18] Sandra Di Rocco, Kelly Jabbusch, and Gregory G. Smith. Toric vector bundles and parliaments of polytopes. Trans. Amer. Math. Soc., 370(11):7715–7741, 2018.
[FOV99] H. Flenner, L. O’Carroll, and W. Vogel. Joins and intersections. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.
[IM19] Nathan Ilten and Christopher Manon. Rational complexity-one $T$-varieties are well-poised. Int. Math. Res. Not. IMRN, (13):4198–4232, 2019.
[KM19] Kiumars Kaveh and Christopher Manon. Khovanskii bases, higher rank valuations, and tropical geometry. SIAM J. Appl. Algebra Geom., 3(2):292–336, 2019.
[KMM] Kiumars Kaveh, Christopher Manon, and Takuya Murata. On degenerations of projective varieties to complexity-one $T$-varieties. arXiv:1708.02698 [math.AG].
[MS15] Diane Maclagan and Bernd Sturmfels. Introduction to tropical geometry, volume 161 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015.
[Ree57] D. Rees. The grade of an ideal or module. Proc. Cambridge Philos. Soc., 53:28–42, 1957.
[RS10] Tim Römer and Kirsten Schmitz. Algebraic properties of generic tropical varieties. Algebra Number Theory, 4(4):465–491, 2010.
[RS12] Tim Römer and Kirsten Schmitz. Generic tropical varieties. J. Pure Appl. Algebra, 216(1):140–148, 2012.
[Stu96] Bernd Sturmfels. Gröbner bases and convex polytopes, volume 8 of University Lecture Series. American Mathematical Society, Providence, RI, 1996.
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