ON THE L.C.M. OF RANDOM TERMS OF BINARY RECURRENCE SEQUENCES

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Abstract. For every positive integer \( n \) and every \( \delta \in [0, 1] \), let \( B(n, \delta) \) denote the probabilistic model in which a random set \( A \subseteq \{1, \ldots, n\} \) is constructed by choosing independently every element of \( \{1, \ldots, n\} \) with probability \( \delta \). Moreover, let \( (u_k)_{k \geq 0} \) be an integer sequence satisfying \( u_k = a_1 u_{k-1} + a_2 u_{k-2} \), for every integer \( k \geq 2 \), where \( u_0 = 0, u_1 \neq 0, \) and \( a_1, a_2 \) are fixed nonzero integers; and let \( \alpha \) and \( \beta \), with \( |\alpha| \geq |\beta| \), be the two roots of the polynomial \( X^2 - a_1 X - a_2 \). Also, assume that \( \alpha/\beta \) is not a root of unity.

We prove that, as \( \delta n / \log n \to +\infty \), for every \( A \) in \( B(n, \delta) \) we have

\[
\log \text{lcm}(u_a : a \in A) \sim \delta \frac{\log(1 - \delta)}{1 - \delta} \frac{\log(1/\delta)}{\pi^2} \cdot n^2,
\]

with probability \( 1 - o(1) \), where \( \text{lcm} \) denotes the lowest common multiple, \( \log(1 - \delta) \) is the dilogarithm, and the factor involving \( \delta \) is meant to be equal to \( 1 \) when \( \delta = 1 \).

This extends previous results of Akiyama, Tropak, Matiyasevich, Guy, Kiss and Mátýás, who studied the deterministic case \( \delta = 1 \), and is motivated by an asymptotic formula for \( \text{lcm}(A) \) due to Cilleruelo, Rué, Šarka, and Zumalacárregui.

1. Introduction

It is well known that the Prime Number Theorem is equivalent to the asymptotic formula

\[
\log \text{lcm}(1, 2, \ldots, n) \sim n,
\]
as \( n \to +\infty \), where \( \text{lcm} \) denotes the lowest common multiple.

For every positive integer \( n \) and every \( \delta \in [0, 1] \), let \( B(n, \delta) \) denote the probabilistic model in which a random set \( A \subseteq \{1, \ldots, n\} \) is constructed by choosing independently every element of \( \{1, \ldots, n\} \) with probability \( \delta \). Motivated by (1), Cilleruelo, Rué, Šarka, and Zumalacárregui [8] proved the following result (see also [5] for a more precise version, and [6, 7] for others results of similar flavor).

**Theorem 1.1.** Let \( A \) be a random set in \( B(n, \delta) \). Then, as \( \delta n \to +\infty \), we have

\[
\log \text{lcm}(A) \sim \frac{\delta \log(1/\delta)}{1 - \delta} \cdot n,
\]

with probability \( 1 - o(1) \), where the factor involving \( \delta \) is meant to be equal to \( 1 \) for \( \delta = 1 \).

Let \( (u_k)_{k \geq 0} \) be an integer sequence satisfying \( u_k = a_1 u_{k-1} + a_2 u_{k-2} \), for every integer \( k \geq 2 \), where \( u_0 = 0, u_1 \neq 0, \) and \( a_1, a_2 \) are two fixed nonzero integers. Moreover, let \( \alpha \) and \( \beta \), with \( |\alpha| \geq |\beta| \), be the two roots of the polynomial \( X^2 - a_1 X - a_2 \). We assume that \( \alpha/\beta \) is not a root of unity, which is a necessary and sufficient condition to have \( u_k \neq 0 \) for all integers \( k \geq 1 \).

Akiyama [1] and, independently, Tropak [14] proved the following analog of (1) for the sequence \( (u_k)_{k \geq 1} \).

**Theorem 1.2.** We have

\[
\log \text{lcm}(u_1, u_2, \ldots, u_n) \sim \frac{3\log(\sqrt{\frac{a_1^2}{a_2^2}})}{\pi^2} \cdot n^2,
\]

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as $n \to +\infty$.

Special cases of Theorem 1.2 were previously proved by Matiyasevich, Guy [11], Kiss and Mátyás [10]. Furthermore, Akiyama [2, 3] generalized Theorem 1.2 to sequences having some special divisibility properties, while Akiyama and Luca [4] studied $\text{lcm}(u_{f(1)}, \ldots, u_{f(n)})$ when $f$ is a polynomial, $f = \varphi$ (the Euler’s totient function), $f = \sigma$ (the sum of divisors function), or $f$ is a binary recurrence sequence.

Motivated by Theorem 1.1, we give the following generalization of Theorem 1.2.

**Theorem 1.3.** Let $A$ be a random set in $B(n, \delta)$. Then, as $\delta n / \log n \to +\infty$, we have

$$\text{lcm}(u_a : a \in A) \sim \frac{\delta \text{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\alpha/\sqrt{a_1^2, a_2}|}{\pi^2} \cdot n^2,$$

with probability $1 - o(1)$, where $\text{Li}_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ is the dilogarithm and the factor involving $\delta$ is meant to be equal to 1 when $\delta = 1$.

When $\delta = 1/2$ all the subsets $A \subseteq \{1, \ldots, n\}$ are chosen by $B(n, \delta)$ with the same probability. Hence, Theorem 1.3 together with the identity $\text{Li}_2\left(\frac{1}{2}\right) = (\pi^2 - 6(\log 2)^2)/12$ (see, e.g., [15]) give the following result.

**Corollary 1.1.** As $n \to +\infty$, we have

$$\text{lcm}(u_a : a \in A) \sim \frac{1}{4} \left(1 - \frac{6(\log 2)^2}{\pi^2}\right) \cdot \log \left|\frac{\alpha}{\sqrt{a_1^2, a_2}}\right| \cdot n^2,$$

uniformly for all sets $A \subseteq \{1, \ldots, n\}$, but at most $o(2^n)$ exceptions.

2. **Notation**

We employ the Landau–Bachmann “Big Oh” and “little oh” notations $O$ and $o$, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. For real random variables $X$ and $Y$, we say that “$X \sim Y$ with probability $1 - o(1)$” if $P(|X - Y| \geq \varepsilon|Y|) = o_\varepsilon(1)$ for every $\varepsilon > 0$. We write $\text{lcm}(S)$ for the lowest common multiple of the elements of $S \subseteq \mathbb{Z}$, with the convention $\text{lcm}(\emptyset) := 1$. We also let $[a, b]$ and $(a, b)$ denote the lowest common multiple and the greatest common divisor, respectively, of two integers $a$ and $b$. Throughout, the letters $p$ is reserved for prime numbers, and $\nu_p$ denotes the $p$-adic valuation. As usual, we write $\Lambda(n)$, $\varphi(n)$, $\tau(n)$, and $\mu(n)$, for the von Mangoldt function, the Euler’s totient function, the number of divisors, and the Möbius function of a positive integer $n$, respectively.

3. **Preliminaries on Lehmer sequences**

Let $\zeta$ and $\eta$ be complex numbers such that $c_1 := (\zeta + \eta)^2$ and $c_2 := \zeta \eta$ are nonzero coprime integers and $\zeta/\eta$ is not a root of unity. Also, assume $|\zeta| \geq |\eta|$. The Lehmer sequence $(\tilde{u}_k)_{k \geq 0}$ associated to $\zeta$ and $\eta$ is defined by

$$\tilde{u}_k := \begin{cases} 
(\zeta^k - \eta^k)/(\zeta - \eta) & \text{if } k \text{ is odd}, \\
(\zeta^k - \eta^k)/(\zeta^2 - \eta^2) & \text{if } k \text{ is even}, 
\end{cases}$$

for every integer $k \geq 0$. It is known that $(\tilde{u}_k)_{k \geq 1}$ is an integer sequence. For every positive integer $m$ coprime with $c_2$, let $\varrho(m)$ be the rank of appearance of $m$ in the Lehmer sequence $(\tilde{u}_k)_{k \geq 0}$, that is, the smallest positive integer $k$ such that $m \mid \tilde{u}_k$. It is known that $\varrho(m)$ exists. Moreover, for every prime number $p$ not dividing $c_2$, put $\kappa(p) := \nu_p(\tilde{u}_{\varrho(p)})$.

We need the following properties of the rank of appearance.

**Lemma 3.1.** We have:

(i) $m \mid \tilde{u}_k$ if and only if $(m, c_2) = 1$ and $\varrho(m) \mid k$, for all integers $m, k \geq 1$.

(ii) $\varrho(p^k) = p^{\max(k - \kappa(p))} \varrho(p)$, for all primes $p$ not dividing $2c_2$ and all integers $k \geq 1$. 
(iii) $\varrho(2^k) = 2^{\max(k - v_2(\tilde{u}_h(\varphi)))} \varrho(4)$, for all integers $k \geq 2$.

Proof. (i) We have $(\tilde{u}_k, c_2) = 1$ for all integers $k \geq 1$ [12, Lemma 1]. Also, $(\tilde{u}_k, \tilde{u}_h) = \tilde{u}_{(k,h)}$ for all integers $k, h \geq 1$ [12, Lemma 3]. Hence, on the one hand, if $m \mid \tilde{u}_k$ then $(m, c_2) = 1$ and $m \mid (\tilde{u}_k, \tilde{u}\varphi(m)) = \tilde{u}_{(k,\varphi(m))}$, which in turn implies that $\varrho(m) \mid k$, by the minimality of $\varrho(m)$. On the other hand, if $(c_2, m) = 1$ and $\varrho(m) \mid k$ then $m \mid \tilde{u}_\varphi(m) = \tilde{u}_{(k,\varphi(m))} = (\tilde{u}_k, \tilde{u}\varphi(m))$, so that $m \mid \tilde{u}_k$.

(ii) If $p \mid \tilde{u}_m$, for some positive integer $m$, then $p \mid \tilde{u}_\varphi(m)$ [12, Lemma 5]. Hence, it follows by induction on $h$ that $\nu_p(\tilde{u}_p^h\varphi(p)) = \kappa(p) + h$, for every integer $h \geq 0$. At this point, the claim follows easily from (i).

(iii) If $4 \mid \tilde{u}_m$, for some positive integer $m$, then $2 \mid \tilde{u}_\varphi(m)$ [12, Lemma 5]. The proof proceeds similarly to the previous point. $\square$

Hereafter, in light of Lemma 3.1(i), in subscripts of sums and products the argument of $\varrho$ is always tacitly assumed to be coprime with $c_2$.

Let us define the cyclotomic numbers $(\phi_k)_{k \geq 1}$ associated to $\zeta$ and $\eta$ by

$$
\phi_k := \prod_{1 \leq h \leq k \atop (h,k) = 1} \left( \zeta - e^{\frac{2\pi i k}{\eta}} \right),
$$

for every integer $k \geq 0$. It can be proved that $\phi_k \in \mathbb{Z}$ for every integer $k \geq 3$. Moreover, from (4) it follows easily that

$$
\zeta^k - \eta^k = \prod_{d \mid k} \phi_d,
$$

which in turn, applying Möbius inversion formula and taking into account (3), gives

$$
\phi_k = \prod_{d \mid k} \left( \zeta^d - \eta^d \right)^{\mu(k/d)} = \prod_{d \mid k} \tilde{u}_d^{\mu(k/d)},
$$

for all integers $k \geq 3$. We need the following result about $\phi_k$.

Lemma 3.2. For every integer $k \geq 13$, we have

$$
|\phi_k| = \lambda_k \cdot \prod_{\varphi(p) = k, p \mid k} \varrho(p),
$$

where $\lambda_k$ is equal to 1 or to the greatest prime factor of $k/(k,3)$.

Proof. Let $p$ be a prime number not dividing $c_2$. By the definition of $\varrho(p)$, we have that $p \nmid \tilde{u}_h$ for each positive integer $h < \varrho(p)$. Hence, by (5), we obtain that $\nu_p(\tilde{u}_p^h\varphi(p)) = \kappa(p) + h$. In particular, $p \mid \phi_p$. Let $k \geq 3$ be an integer and suppose that $p$ is a prime factor of $\phi_k$. On the one hand, if $\varrho(p) = k$ then, by the previous consideration, $\nu_p(\phi_p) = \kappa(p)$. On the other hand, if $\varrho(p) \neq k$ then $p \mid (\phi_p, \phi_k)$. Finally, for $k \geq 13$ and for every integer $h \geq 3$ with $h \neq k$, we have that $(\phi_h, \phi_k)$ divides the greatest prime factor of $k/(k,3)$ [12, Lemma 7]. $\square$

We conclude this section with a formula for a sum involving the von Mangoldt function.

Lemma 3.3. We have

$$
\sum_{\varrho(m) = r} \Lambda(m) = \varphi(r) \log |\zeta| + O_{\zeta, \eta}(\tau(r) \log(r + 1))
$$

and, in particular,

$$
\sum_{\varrho(m) = r} \Lambda(m) \ll_{\zeta, \eta} \varphi(r),
$$

for every positive integer $r$. 

Proof. Clearly, we can assume \( r \geq 13 \). Write \( m = p^k \), where \( p \) is a prime number not dividing \( c_2 \) and \( k \) is a positive integer. First, suppose that \( p > 2 \). By Lemma 3.1(ii), we have that \( \varrho(m) = p^{\max(k-\kappa(p), 0)} \varrho(p) \). Hence, \( \varrho(m) = r \) if and only if \( k \leq \kappa(p) \) and \( \varrho(p) = r \), or \( k > \kappa(p) \) and \( p^{k-\kappa(p)} \varrho(p) = r \). In the first case, the contribution to the sum in (6) is exactly \( \kappa(p) \log p \).

In the second case, \( p \mid r \) and, since \( k \) is determined by \( p \) and \( r \), the contribution to the sum in (6) is \( \log p \). Using Lemma 3.1(iii), the case \( p = 2 \) can be handled similarly. Therefore,

\[
\sum_{\varrho(m) = r} \Lambda(m) = \sum_{\varrho(p) = r} \kappa(p) \log p + O \left( \sum_{p \mid r} \log p \right) = \log |\phi_r| + O(\log r),
\]

where we used Lemma 3.2. Furthermore, from (5) and the identity \( \sum_{d \mid r} \mu(r/d) d = \phi(r) \), it follows that

\[
\log |\phi_r| = \phi(r) \log |\zeta| + O \left( \sum_{d \mid r} \log \left| 1 - \left( \frac{\eta}{\zeta} \right)^d \right| \right).
\]

If \( |\eta/\zeta| < 1 \) then \( \log \left| 1 - \left( \frac{\eta}{\zeta} \right)^d \right| = O_{\zeta, \eta}(1) \). If \( |\eta/\zeta| = 1 \) then, since \( \eta/\zeta \) is an algebraic number that is not a root of unity, it follows from classic bounds on linear forms in logarithms (see, e.g., [9, Lemma 3]) that \( \log \left| 1 - \left( \frac{\eta}{\zeta} \right)^d \right| = O_{\zeta, \eta}(\log(d + 1)) \). Consequently,

\[
\log |\phi_r| = \phi(r) \log |\zeta| + O_{\zeta, \eta}(\tau(r) \log(r + 1)).
\]

Putting together (8) and (9), we get (6). Finally, the upper bound (7) follows since \( \tau(k) \leq k^\varepsilon \) and \( \varphi(k) \geq k^{1-\varepsilon} \), for all \( \varepsilon > 0 \) and every integer \( k \gg \varepsilon \) [13, Ch. I.5, Corollary 1.1 and Eq. 12].

4. Further preliminaries

We need two estimates involving the Euler's totient function. Define

\[
\Phi(x) := \sum_{n \leq x} \varphi(n),
\]

for every \( x \geq 1 \).

**Lemma 4.1.** We have

\[
\Phi(x) = \frac{3}{\pi^2} x^2 + O(x \log x) \quad \text{and} \quad \sum_{n \leq x} \frac{\varphi(n)}{n} \ll x,
\]

for every \( x \geq 2 \).

**Proof.** The first formula is well known [13, Ch. I.3, Thm. 4] and implies

\[
\sum_{n \leq x} \frac{\varphi(n)}{n} \leq \sum_{n \leq x/2} 1 + \sum_{x/2 < n \leq x} \frac{\varphi(n)}{x/2} \ll x,
\]

as desired. \( \square \)

The following lemma is an easy inequality that will be useful later.

**Lemma 4.2.** It holds \( 1 - (1 - x)^k \leq kx \), for all \( x \in [0, 1] \) and all integers \( k \geq 0 \).

**Proof.** The claim is \((1 + (-x))^k \geq 1 + k(-x)\), which follows from Bernoulli's inequality. \( \square \)
5. Proof of Theorem 1.3

Henceforth, all the implied constants may depend by \(a_1\), \(a_2\), and \(u_1\). It is well known that the generalized Binet’s formula

\[
(10) \quad u_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} u_1,
\]

holds for every integer \(k \geq 0\). We put \(\zeta := \alpha/\sqrt{b}\) and \(\eta := \beta/\sqrt{b}\), where \(b := (a_1^2, a_2^2)\). Note that indeed \(c_1 = a_1^2/b\) and \(c_2 = -a_2^2/b\) are nonzero relatively prime integers, \(\zeta/\eta = \alpha/\beta\) is not a root of unity, and \(|\zeta| \geq |\eta|\). Moreover, from (3) and (10), it follows easily that

\[
u_k = \begin{cases} 
  b^{(k-1)/2} u_1 \tilde{u}_k & \text{if } k \text{ is odd}, \\
  a_1 b^{k/2-1} u_1 \tilde{u}_k & \text{if } k \text{ is even},
\end{cases}
\]

for every integer \(k \geq 0\). Therefore, for every \(A \subseteq \{1, \ldots, n\}\), we have

\[
\log \lcm(u_a : a \in A) = \log \lcm(\tilde{u}_a : a \in A) + O(n).
\]

Note that \(O(n)\) is a “little oh” of the right-hand side of (2), as \(\delta n/\log n \to +\infty\). Hence, it is enough to prove Theorem 1.3 with \(\log \lcm(\tilde{u}_a : a \in A)\) in place of \(\log \lcm(u_a : a \in A)\), and this will be indeed our strategy.

Hereafter, let \(A\) be a random set in \(B(n, \delta)\), and put \(L := \lcm(\tilde{u}_a : a \in A)\) and \(X := \log L\). For every positive integer \(m\) coprime with \(c_2\), let us define

\[
I_A(m) := \begin{cases} 
  1 & \text{if } \varphi(m) \mid a \text{ for some } a \in A, \\
  0 & \text{otherwise}.
\end{cases}
\]

The following lemma gives an expression for \(X\) in terms of \(I_A\) and the von Mangoldt function.

**Lemma 5.1.** We have

\[
X = \sum_{\varphi(m) \leq n} \Lambda(m) I_A(m).
\]

**Proof.** For every prime power \(p^k\) with \(p \nmid c_2\), we know from Lemma 3.1(i) that \(p^k \mid L\) if and only if \(\varphi(p^k) \mid a\) for some \(a \in A\) and, in particular, \(\varphi(p^k) \leq n\). Hence,

\[
X = \sum_{p^k \mid L} \log p = \sum_{\varphi(p^k) \leq n} (\log p) I_A(p^k) = \sum_{\varphi(m) \leq n} \Lambda(m) I_A(m),
\]

as claimed.

The next lemma provides two expected values involving \(I_A\) and needed in later arguments.

**Lemma 5.2.** We have

\[
\mathbb{E}(I_A(m)) = 1 - (1 - \delta)^{\lfloor n/\varphi(m) \rfloor}
\]

and

\[
\mathbb{E}(I_A(m)I_A(\ell)) = 1 - (1 - \delta)^{\lfloor n/\varphi(m) \rfloor} - (1 - \delta)^{\lfloor n/\varphi(\ell) \rfloor}
\]

\[
\quad + (1 - \delta)^{\lfloor n/\varphi(m) \rfloor + \lfloor n/\varphi(\ell) \rfloor - \lfloor n/(\varphi(m)\varphi(\ell)) \rfloor},
\]

for all positive integers \(m\) and \(\ell\) with \((m\ell, c_2) = 1\).

**Proof.** By the definition of \(I_A\), we have

\[
\mathbb{E}(I_A(m)) = \mathbb{P}(\exists a \in A : \varphi(m) \mid a) = 1 - \mathbb{P} \left( \bigwedge_{t \leq n/\varphi(m)} (\varphi(m)t \notin A) \right) = 1 - (1 - \delta)^{\lfloor n/\varphi(m) \rfloor},
\]

which is the first claim. On the one hand, by linearity of expectation and by (11), we obtain

\[
\mathbb{E}(I_A(m)I_A(\ell)) = \mathbb{E}(I_A(m) + I_A(\ell) - 1 + (1 - I_A(m))(1 - I_A(\ell)))
\]

\[
= \mathbb{E}(I_A(m)) + \mathbb{E}(I_A(\ell)) - 1 + \mathbb{E}((1 - I_A(m))(1 - I_A(\ell)))
\]

as claimed. \(\square\)
\[ \begin{align*}
&= 1 - (1 - \delta)^{\lceil n/g(m) \rceil} - (1 - \delta)^{\lceil n/g(\ell) \rceil} + \mathbb{E}((1 - I_A(m))(1 - I_A(\ell))).
\end{align*} \]

On the other hand, by the definition of \( I_A \),
\[ \mathbb{E}((1 - I_A(m))(1 - I_A(\ell))) = \mathbb{P}(\forall a \in A : g(m) \nmid a \text{ and } g(\ell) \nmid a) \]
\[ = \mathbb{P} \left( \bigwedge_{\substack{k \leq n \\lvert \, g(m) \mid k \text{ or } g(\ell) \mid k}} (k \notin A) \right) = (1 - \delta)^{\lceil n/g(m) \rceil + \lceil n/g(\ell) \rceil - \lfloor n/[g(m), g(\ell)] \rfloor}, \]
and the second claim follows too. \( \square \)

Now we give an asymptotic formula for the expected value of \( X \).

**Lemma 5.3.** We have
\[ \mathbb{E}(X) = \frac{\delta \text{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\zeta|}{\pi^2} \cdot n^2 + O(\delta n (\log n)^3), \]
for all integers \( n \geq 2 \). In particular,
\[ \mathbb{E}(X) \sim \frac{\delta \text{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\zeta|}{\pi^2} \cdot n^2, \]
as \( n \to +\infty \), uniformly for \( \delta \in (0, 1] \).

**Proof.** From Lemma 5.1 and Lemma 5.2, it follows that
\[ \mathbb{E}(X) = \sum_{\phi(m) \leq n} \Lambda(m) \mathbb{E}(I_A(m)) \]
\[ = \sum_{\phi(m) \leq n} \Lambda(m) \left( 1 - (1 - \delta)^{\lceil n/g(m) \rceil} \right) \]
\[ = \sum_{r \leq n} (1 - (1 - \delta)^{\lceil n/r \rceil}) \sum_{\phi(m) = r} \Lambda(m). \]
Consequently, thanks to Lemma 3.3 and Lemma 4.2, we obtain
\[ \mathbb{E}(X) = \sum_{r \leq n} \left( 1 - (1 - \delta)^{\lceil n/r \rceil} \right) \varphi(r) \log |\zeta| + O \left( \delta n \sum_{r \leq n} \frac{\tau(r) \log (r + 1)}{r} \right) \]
\[ = \sum_{r \leq n} (1 - (1 - \delta)^{\lceil n/r \rceil}) \varphi(r) \log |\zeta| + O(\delta n (\log n)^3), \]
where we used the fact that
\[ \sum_{r \leq n} \frac{\tau(r)}{r} \leq \left( \sum_{s \leq n} \frac{1}{s} \right)^2 \ll (\log n)^2. \]
Note that \( \lfloor n/r \rfloor = j \) if and only if \( r \in (n/(j + 1), n/j] \). Hence,
\[ \sum_{r \leq n} (1 - (1 - \delta)^{\lfloor n/r \rfloor}) \varphi(r) = \sum_{j \leq n} (1 - (1 - \delta)^{j}) \sum_{n/(j+1) \leq r \leq n/j} \varphi(r) \]
\[ = \sum_{j \leq n} (1 - (1 - \delta)^{j}) \left( \Phi \left( \frac{n}{j} \right) - \Phi \left( \frac{n}{j + 1} \right) \right) \]
\[ = \delta \sum_{j \leq n} (1 - \delta)^{j-1} \Phi \left( \frac{n}{j} \right) \]
\[ = \delta \sum_{j \leq n} \frac{(1 - \delta)^{j-1}}{j^2} \cdot \frac{3}{\pi^2} \cdot n^2 + O \left( \delta \sum_{j \leq n} \frac{n}{j} \log \left( \frac{n}{j} \right) \right) \]
where we used Lemma 4.1. Finally, putting together (12) and (13), we get the desired claim. □

The next lemma is an upper bound for the variance of $X$.

**Lemma 5.4.** We have
\[ \mathbb{V}(X) \ll \delta n^3 \log n, \]
for all integers $n \geq 2$.

**Proof.** On the one hand, by Lemma 5.1, we have
\[
\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sum_{g(m), g(\ell) \leq n} \Lambda(m)\Lambda(\ell)(\mathbb{E}(I_A(m)I_A(\ell)) - \mathbb{E}(I_A(m))\mathbb{E}(I_A(\ell))).
\]

On the other hand, from Lemma 5.2 and Lemma 4.2, it follows that
\[
\mathbb{E}(I_A(m)I_A(\ell)) - \mathbb{E}(I_A(m))\mathbb{E}(I_A(\ell)) = (1 - \delta)^{\lfloor n/g(m)\rfloor + \lfloor n/g(\ell)\rfloor - \lfloor n/[g(m), g(\ell)]\rfloor}(1 - (1 - \delta)^{\lfloor n/[g(m), g(\ell)]\rfloor}) \leq \frac{\delta n}{[g(m), g(\ell)]}.
\]

Therefore,
\[
\mathbb{V}(X) \leq \delta n \sum_{g(m), g(\ell) \leq n} \frac{\Lambda(m)\Lambda(\ell)}{[g(m), g(\ell)]} = \delta n \sum_{r,s \leq n} \frac{1}{[r,s]} \sum_{g(m) = r, g(\ell) = s} \Lambda(r) \sum_{g(\ell) = s} \Lambda(\ell) \ll \delta n \sum_{r,s \leq n} \frac{\varphi(r)\varphi(s)}{[r,s]} = \delta n \sum_{r,s \leq n} (r,s) \frac{\varphi(r)\varphi(s)}{rs},
\]
where we used Lemma 3.3 and the identity $[r,s] = rs/(r,s)$. At this point, writing $r = dr'$ and $s = ds'$, where $d := (r,s)$, we obtain
\[
\sum_{r,s \leq n} (r,s) \frac{\varphi(r)\varphi(s)}{rs} = \sum_{d \leq n} d \sum_{r',s' \leq n/d} \varphi(dr')\varphi(ds') \leq \sum_{d \leq n} d \left( \sum_{t \leq n/d} \varphi(t) \right)^2 \ll \sum_{d \leq n} d \left( \frac{n}{d} \right)^2 \ll n^2 \log n,
\]
where we used Lemma 4.1 and the inequality $\varphi(dm) \leq d\varphi(m)$, holding for every integer $m \geq 1$. Finally, putting together (14) and (15), we get the desired claim. □

**Proof of Theorem 1.3.** By Chebyshev’s inequality, Lemma 5.3, and Lemma 5.4, we have
\[
P\left( |X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X) \right) \leq \frac{\mathbb{V}(X)}{(\varepsilon \mathbb{E}(X))^2} \ll \frac{\log n}{\varepsilon^2 \delta n} = o(1),
\]
as $\delta n/\log n \to +\infty$. Hence, again by Lemma 5.3, we have
\[
X \sim \frac{\delta \text{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3\log \zeta}{\pi^2} \cdot n^2,
\]
with probability $1 - o(1)$, as desired. □
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