The Dieck-Temperley-Lieb algebras in Brauer algebras

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Abstract

In this paper, we will study the Dieck-Temlerley-Lieb algebras of type $B_n$ and $C_n$. We compute their ranks and describe a basis for them by using some results from corresponding Brauer algebras and Temperley-Lieb algebras.

1 Introduction

The Temperley-Lieb algebras play an important role in representation theory and knot theory. The classical $TL$ algebra first came out in [23], and in [16], Fan extended it to other types as a quotient of Hecke algebras and described a basis for each type. Dieck have defined an diagrammatic $TL$ algebra of type $B$ and compute its rank in [13] and [14]. Now there are some works in $TL$ category related to $TL$ algebra, which can be found in [1] and [19].

The spherical Coxeter groups are very classical and important topics in Lie theory. The irreducible spherical Coxeter groups can be classified as simply-laced types and non-simply laced types. Tits has described how to obtain the non-simply laced types from the simply laced types in [24] (also see [4]), which has been applied in many related fields. From studying the invariant theory for orthogonal groups, Brauer discovered Brauer algebras of type $A(3)$; Cohen, Frenk and Wales extended it to the definition of simply laced type in [10], including type $D_n$. In [6] and [7], Cohen, Liu and Yu applied the similar method to obtain the Brauer algebras of type $B_n$ and $C_n$. In [6] and [7], the Brauer algebras of type $C_n$ and $B_n$ are described as the subalgebras of Brauer algebras of type $A_{2n-1}$ and $D_{n+1}$ spanned by the submonoids invariant under the classical Dynkin diagram automorphisms, repectively.

In this paper, we will apply the same method on $TL$ algebras of type $A_{2n-1}$

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and $D_{n+1}$ to obtain the $TL$ algebras of type $C_n$ and $B_n$. And our $TL$ algebras of type $C_n$ coincide with the $TL$ algebras of type $B_n$ given by Dieck. Therefore we call our $TL$ algebras Dieck-Temperley-Lieb algebras (DTL in short).

In this paper, we first recall some results about Brauer algebras of simply-laced types and obtain some results about the height functions under the diagram automorphisms. We apply these to the $TL$ algebras of simply-laced types which can be considered as subalgebras of corresponding Brauer algebras. We prove the isomorphism between $DTL(C_n)$ and a subalgebra of $TL(A_{2n-1})$. We describe a rewriting forms for $DTL(B_n)$ and compute the rank of $DTL(B_n)$ being $C_n + C_{n+1} - 1$, where $C_n$ and $C_{n+1}$ are Catalan numbers.

The paper is sketched as the following. In Section 2, we recall the definition of Brauer algebras of simply-laced types and the classical Brauer algebras. In Section 3, we prove that the classical diagram automorphisms to obtain non-simply laced Brauer algebras from simply-laced Brauer algebras are height-invariant automorphisms. In Section 4, we present the definition of Dieck-Temperley-Lieb algebras, and show some algebra homomorphisms between the DTL algebras and some corresponding Brauer algebras. We recall some notions and results about Brauer algebras of simply-laced type, such as admissible root sets, Brauer monoid actions and rewriting forms in Section 5. In Section 6, we use a combinatorial method to prove the isomorphism of $DTL(C_n)$ and $STL(A_{2n-1})$. In Section 7, we present a rewriting forms for $DTL(B_n)$. In Section 8, we recall the diagram representations for Brauer algebras of type $D_n$, and show an diagram representation for $DTL(B_n)$ to prove that the rewriting forms in Section 7 are linearly independent and $DTL(B_n)$ has rank $C_n + C_{n+1} - 1$.

## 2 The Brauer algebras of simply-laced types

First we recall the definition of simply-laced Brauer algebra from [10]. In order to avoid confusion with the generators in Section 4, the symbols of [10] have been capitalized.

**Definition 2.1.** Let $Q$ be a graph. The Brauer monoid $BrM(Q)$ is the monoid generated by the symbols $R_i$ and $E_i$, for each node $i$ of $Q$ and $\delta$, $\delta^{-1}$ subject to the following relation, where $\sim$ denotes adjacency between nodes of $Q$.

\[
\delta \delta^{-1} = 1 \quad \text{(2.1)}
\]
\[
R_i^2 = 1 \quad \text{(2.2)}
\]
The Brauer algebra \( \text{Br}(Q) \) is the the free \( \mathbb{Z} \)-algebra for Brauer monoid \( \text{BrM}(Q) \).

The Brauer algebras \( \text{Br}(Q) \) has been well studied in [10], where the basis and ranks of finite types are given. Usually we call \( R_i \)s Coxeter generators, and \( E_i \)s Temperley-Lieb generators.

**Remark 2.2.** From [10], we know that the following relations in \( \text{Br}(A_m) \) hold for \( i \sim j \sim k \).

\[
R_i R_j R_i = E_i E_j \quad (2.11) \\
R_j E_i E_j = R_i E_j \quad (2.12) \\
E_i R_j E_i = E_i \quad (2.13) \\
E_j E_i R_j = E_j R_i \quad (2.14) \\
E_i E_j E_i = E_i \quad (2.15) \\
E_j E_i R_k E_j = E_j R_i E_k E_j \quad (2.16) \\
E_j R_i R_k E_j = E_j E_i E_k E_j \quad (2.17)
\]

**Remark 2.3.** In [3], Brauer shows a diagram description for a basis of Brauer monoid of type \( A_m \), which is just a diagram monoid with \( 2m + 2 \) dots and \( m + 1 \) strands, and for each dot, there is a unique strand connecting it with another dot. Here we suppose the \( 2m + 2 \) dots have coordinates \((i, 0)\) and \((i, 1)\) in \( \mathbb{R}^2 \) with \( 1 \leq i \leq m + 1 \). The multiplication of two diagrams is given by concatenation, where any closed loops formed are replaced by a factor of \( \delta \), and we give one example in Figure 1. The generators of \( \text{BrM}(A_m) \) of the form \( R_i \) and \( E_i \) correspond to the diagrams indicated in Figure 2. Each Brauer diagram can be written as a product of elements from \( \{R_i, E_i\}_{i=1}^m \). Henceforth, we identify \( \text{BrM}(A_m) \) with its diagrammatic version. It makes clear that \( \text{Br}(A_m) \) is a free algebra over \( \mathbb{Z}[\delta^{\pm 1}] \) of rank \( (m + 1)!! \), the product of the first \( m + 1 \) odd integers. The monomials of \( \text{BrM}(A_m) \) that correspond to diagrams will be referred to as diagrams.
3 Height and automorphisms

Remark 3.1. We keep notation as in [8, Section 2] and first introduce some basic conceptions. Let $Q$ be the diagram of a connected finite simply laced Coxeter group (type $A_n, D_n, E_6, E_7, E_8$), and $\text{BrM}(Q)$ is the associated Brauer monoid as Definition 4.1. An element $a \in \text{BrM}(Q)$ is said to be of height $t$ if the minimal number of $R_i$ occurring in an expression of $a$ is $t$, denoted by $\text{ht}(a)$.

Proposition 3.2. Let $Q$ be the diagram of a connected finite simply laced...
Coxeter group and \( \text{BrM}(Q) \) is the associated Brauer monoid. Let \( \sigma \) be an automorphism on \( \text{BrM}(Q) \), which is induced by permutation on Coxeter generators \( R_i \) and Temperley-Lieb generators \( E_i \), respectively. Then for each monomial \( a \in \text{BrM}(Q) \), it follows that

\[
ht(a) = ht(\sigma(a)).
\]

**Proof.** Let \( a \in \text{BrM}(Q) \) and \( t = ht(a) \). So \( a \) has a reduced word which exactly has \( t \) Coxeter generators \( R_{i_1}, \ldots, R_{i_t} \). Therefore it follows that \( \sigma(a) \) can be reduced to a word with exactly \( t \) Coxeter generators \( \sigma(R_{i_1}), \ldots, \sigma(R_{i_t}) \). Therefore \( ht(a) \geq ht(\sigma(a)) \). Because \( \sigma \) is an automorphism, we can obtain that \( ht(a) \leq ht(\sigma(a)) \). Hence \( ht(a) = ht(\sigma(a)) \).

\( \square \)

**Remark 3.3.** In classical finite Weyl groups, we can define automorphisms on Dynkin diagrams of simply-laced Weyl groups to obtain the Weyl groups of non-simply laced types listed in Figure 3, and we have already applied these automorphisms on simply-laced Brauer algebras to define and study Brauer algebras of non-simply laced types, which can be found in [6] for type \( C_n \) from \( A_{2n-1} \), [7] for type \( B_n \) from \( D_{n+1} \), [21] for type \( F_4 \) from \( E_6 \), [22] for type \( G_2 \) from \( D_4 \). These conclusions are contained in [20] for completing the project of obtaining Brauer algebras of non-simply laced type from simply-laced types.

Let \( M \) and \( Q \) be the non-simply laced type and simply-laced type, respectively, from the above remark, and we list them in the table below and the diagram automorphisms.

| \( M \) | \( Q \) |
|---|---|
| \( C_n \) | \( A_{2n-1} \) |
| \( B_n \) | \( D_{n+1} \) |
| \( F_4 \) | \( E_6 \) |
| \( G_2 \) | \( D_4 \) |

In the [20], to obtain the Brauer algebra of type \( M(\text{Br}(M)) \) from Brauer algebra of type \( Q (\text{Br}(Q)) \), we always define an automorphism \( \sigma \) on \( \text{Br}(Q) \) which just extends the classical automorphism on Weyl groups on to the Temperley-Lieb generators \( E_i \), and implies that \( \sigma(E_i) = E_j \) if \( \sigma(R_i) = R_j \). By Proposition 3.2, the following holds.

**Corollary 3.4.** The automorphism \( \sigma \) on \( \text{Br}(Q) \) in Remark 3.3 is a height invariant automorphism.
4 The Dieck-Temperley-Lieb algebras

Let $M$ be a connected double laced or simply laced Dynkin diagram of finite type, namely type $A_n$, $B_n$, $C_n$, $D_n$, $E_n (n = 6, 7, 8)$, $F_4$. We list their Dynkin diagrams in Table I.

| type   | diagram |
|--------|---------|
| $A_n$  | $n$ $n-1$ $n-2$ $\ldots$ $2$ $1$ |
| $D_n$  | $n$ $n-1$ $4$ $3$ $1$ $2$ |
| $E_n$, 6 $\leq$ n $\leq$ 8 | $n$ $n-1$ $5$ $4$ $3$ $1$ $2$ |
| $B_n$  | $n-1$ $n-2$ $2$ $1$ $0$ |
| $C_n$  | $n-1$ $n-2$ $2$ $1$ $0$ |
| $F_4$  | $1$ $2$ $3$ $4$ |

Table 1: Coxeter diagrams of spherical types
Definition 4.1. Let $R$ be a commutative ring with invertible element $\delta$. For $n \in \mathbb{N}$, the reduced Temperley-Lieb algebra of type $M$ over $R$ with loop parameter $\delta$, denoted by $\text{DTL}(M, R, \delta)$, is the $R$-algebra generated by $\{e_i\}_{i \in M}$ subject to the following relations. For each $i \in M$,

$$e_i^2 = \delta^{\kappa_i} e_i; \quad (4.1)$$

for $i, j \in M$ not adjacent to each other, namely $\circ_i \circ_j$,

$$e_i e_j = e_j e_i; \quad (4.2)$$

for $i, j \in M$ and $\circ_i \circ_j$,

$$e_i e_j e_i = e_i; \quad (4.3)$$

for $i, j \in M$ and $\overrightarrow{i} \overrightarrow{j}$,

$$e_j e_i e_j = \delta e_j. \quad (4.4)$$

The parameter $\kappa_i \in \mathbb{N}$ is given below,

- for type $A_n$, $D_n$, $E_n$, $\kappa_i = 1$ for $1 \leq i \leq n$,
- for type $C_n$, $\kappa_0 = 1$, $\kappa_i = 2$ for $1 \leq i \leq n - 1$;
- for type $B_n$, $\kappa_0 = 2$, $\kappa_i = 1$ for $1 \leq i \leq n - 1$;
- for type $F_4$, $\kappa_1 = \kappa_2 = 2$, $\kappa_3 = \kappa_4 = 1$;

If $R = \mathbb{Z}[\delta^\pm 1]$ we write $\text{DTL}(M)$ instead of $\text{DTL}(M, R, \delta)$ and speak of the Dieck-Temperley-Lieb algebra of type $M$. The submonoid of the multiplicative monoid of $\text{DTL}(M)$ generated by $\delta, \delta^{-1}$ and $\{e_i\}_{i \in M}$ is denoted by $\text{DTLM}(M)$. It is the monoid of monomials in $\text{DTL}(M)$ and will be called the Dieck-Temperley-Lieb monoid of type $M$.

Remark 4.2. When $Q$ is of simply laced type, it can be seen that $\text{DTL}(Q)$ is the classical Temperley-Lieb algebra of type $Q$, then we denote it by $\text{TL}(Q)$ to replace $\text{DTL}(Q)$. Similarly, we replace $\text{DTLM}(Q)$ by $\text{TLM}(Q)$. When $Q$ is of type $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$, the algebra $\text{TL}(Q)$ is a subalgebra of $\text{Br}(Q)$, and the monomials of height 0 form a basis of $\text{TL}(Q)$\footnote{\cite{[8]}}.

Similar to the case for Brauer algebras, there is a natural anti-involution on $\text{DTL}(M, R, \delta)$ linearly induced by

$$x_1 x_2 \ldots x_n \mapsto x_n \ldots x_2 x_1$$

with each $x_i$ being the generator of $\text{DTL}(M, R, \delta)$.

Proposition 4.3. The identity map on $\{\delta, e_i \mid i \in M\}$ extends to a unique anti-involution on the algebra $\text{DTL}(M, R, \delta)$.
Let’s recall the definition of Brauer algebras of double-laced types in the following from [22].

**Definition 4.4.** Let $R$ be a commutative ring with invertible element $\delta$ and $M$ be a Dynkin diagram of Weyl type. For $n \in \mathbb{N}$, the Brauer algebra of type $M$ over $R$ with loop parameter $\delta$, denoted by $\text{Br}(M, R, \delta)$, is the $R$-algebra generated by $\{r_i, e_i\}_{i \in M}$ subject to the following relations. For each $i \in M$,

\begin{align*}
    r_i^2 &= 1, \quad (4.5) \\
    r_i e_i &= e_i r_i = e_i, \quad (4.6) \\
    e_i^2 &= \delta^{\kappa_i} e_i; \quad (4.7)
\end{align*}

for $i, j \in M$ not adjacent to each other, namely $\underset{i}{\circ} \underset{j}{\circ}$,

\begin{align*}
    r_i r_j &= r_j r_i, \quad (4.8) \\
    e_i r_j &= r_j e_i, \quad (4.9) \\
    e_i e_j &= e_j e_i; \quad (4.10)
\end{align*}

for $i, j \in M$ and $\underset{i}{\circ} \underset{j}{\circ}$,

\begin{align*}
    r_i r_j r_i &= r_j r_i r_j, \quad (4.11) \\
    r_j r_i e_j &= e_i e_j, \quad (4.12) \\
    r_i e_j r_i &= r_j e_i r_j; \quad (4.13)
\end{align*}

for $i, j \in M$ and $\underset{i}{\circ} \underset{j}{\circ}$,

\begin{align*}
    r_j r_i r_j r_i &= r_i r_j r_i r_j, \quad (4.14) \\
    r_j r_i e_j &= r_i e_j, \quad (4.15) \\
    r_j e_j r_i &= e_i e_j e_i, \quad (4.16) \\
    (r_j r_i r_j) e_i &= e_i (r_j r_i r_j), \quad (4.17) \\
    e_j r_i e_j &= \delta e_j, \quad (4.18) \\
    e_i e_j e_i &= \delta e_j, \quad (4.19) \\
    e_j r_i r_j &= e_j r_i, \quad (4.20) \\
    e_j e_i r_j &= e_j e_i; \quad (4.21)
\end{align*}

The parameter $\kappa_i \in \mathbb{N}$ is the same as Definition 4.1.

If $R = \mathbb{Z}[\delta^{\pm 1}]$ we write $\text{Br}(M)$ instead of $\text{Br}(M, R, \delta)$ and speak of the Brauer algebra of type $M$. The submonoid of the multiplicative monoid of $\text{Br}(M)$ generated by $\delta$, $\delta^{-1}$ and $\{r_i, e_i\}_{i \in M}$ is denoted by $\text{BrM}(M)$. It is the monoid of monomials in $\text{Br}(M)$ and will be called the Brauer monoid of type $M$. 

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By easy verification and [22 Table 1], we can obtain the conclusion below.

**Theorem 4.5.** There is an algebra homomorphism $\phi : \text{DTL}(M) \to \text{Br}(M)$ by mapping the generators $e_i$ of $\text{DTL}(M)$ to the generators $e_i$ of $\text{Br}(M)$. Furthermore we have the commutative diagram below.

![Commutative Diagram]

**Remark 4.6.** Now we consider the case $M = C_n$, and $Q = A_{2n-1}$. By [13], up to some parameter, the algebra $\text{DTL}(C_n)$ is isomorphic to the Temperley-Lieb algebras of type $B_n$ defined by the author, where the rank and the diagram representation are given. From [13], the algebra $\text{DTL}(C_n)$ has rank $\binom{2n}{n}$, and the diagram generators of $\text{DTL}(C_n)$ are given in Figure 4 with the multiplication laws as the classical Brauer algebras. Then by the diagram version of [6 Theorem 1.1], we find that the four morphisms in the commutative diagrams of Theorem 4.5 are injective.

![Figure 4: The diagram generators of DTL(C_n)]

As in [6, 7, 21, 22], The following can be verified.

**Corollary 4.7.** The diagram automorphisms in Figure 3 induce automorphisms on the corresponding simply-laced Temperley-Lieb algebras.
As in [6, 7, 21, 22], we define STL(Q) is the subalgebra of TL(Q) generated by the \(\sigma\)-invariant submomid of TLM(Q), where Q can be \(A_{2n-1}\), \(D_{n+1}\) or \(E_6\).

5 Some conclusions of Brauer algebras of simply-laced type

Let Q be a spherical Coxeter diagram of simply laced type, i.e., its connected components are of type A, D, E as listed in Table 1. This section is to summarize some results in [11].

When Q is \(A_n\), \(D_n\), \(E_6\), \(E_7\), or \(E_8\), we denote it as \(Q \in ADE\). Let \((W,T)\) be the Coxeter system of type Q with \(T = \{R_1, \ldots, R_n\}\) associated to the diagram of Q in Table 1. Let \(\Phi\) be the root system of type Q, let \(\Phi^+\) be its positive root system, and let \(\alpha_i\) be the simple root associated to the node \(i\) of Q. We are interested in sets \(B\) of mutually commuting reflections, which has a bijective correspondence with sets of mutually orthogonal roots of \(\Phi^+\), since each reflection in \(W\) is uniquely determined by a positive root and vice versa.

Remark 5.1. The action of \(w \in W\) on \(B\) is given by conjugation in case \(B\) is described by reflections and given by \(w\{\beta_1, \ldots, \beta_p\} = \Phi^+ \cap \{\pm w\beta_1, \ldots, \pm w\beta_p\}\), in case \(B\) is described by positive roots. For example, \(R_4R_1R_2R_1\{\alpha_1 + \alpha_2, \alpha_4\}\) = \(\{\alpha_1 + \alpha_2, \alpha_4\}\), where \(Q = A_4\).

For \(\alpha, \beta \in \Phi\), we write \(\alpha \sim \beta\) to denote \(|(\alpha, \beta)| = 1\). Thus, for \(i\) and \(j\) nodes of Q, we have \(\alpha_i \sim \alpha_j\) if and only if \(i \sim j\).

Definition 5.2. Let \(B\) be a \(W\)-orbit of sets of mutually orthogonal positive roots. We say that \(B\) is an admissible orbit if for each \(B \in \mathfrak{B}\), and \(i, j \in Q\) with \(i \neq j\) and \(\gamma, \gamma - \alpha_i + \alpha_j \in B\) we have \(r_iB = r_jB\), and each element in \(\mathfrak{B}\) is called an admissible root set.

This is the definition from [11], and there is another equivalent definition in [10]. We also state it here.

Definition 5.3. Let \(B \subset \Phi^+\) be a mutually orthogonal root set. If for all \(\gamma_1, \gamma_2, \gamma_3 \in B\) and \(\gamma, \gamma - \alpha_i + \alpha_j \in B\) we have \(2\gamma + \gamma_1 + \gamma_2 + \gamma_3 \in B\), then \(B\) is called an admissible root set.

By these two definitions, it follows that the intersection of two admissible root sets are admissible. It can be checked by definition that the intersection of two admissible sets are still admissible. Hence for a given set \(X\) of mutually orthogonal positive roots, the unique smallest admissible set containing \(X\) is called the admissible closure of \(X\), and denoted as \(X^{cl}\) (or \(X\)). Up to the
action of the corresponding Weyl groups, all admissible root sets of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ have appeared in [10], [12] and [8], and are listed in Table 2. In the table, the set $Y(t)^*$ consists of all $\alpha^*$ for $\alpha \in Y(t)$, where $\alpha^*$ is the unique positive root orthogonal to $\alpha$ and all other positive roots orthogonal to $\alpha$ for type $D_n$ with $n > 4$. For type $D_n$, if we consider the root systems are realized in $\mathbb{R}^n$, with $\alpha_1 = \varepsilon_2 - \varepsilon_1$, $\alpha_2 = \varepsilon_2 + \varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$, for $3 \leq i \leq n$, then $\Phi^+ = \{\varepsilon_j \pm \varepsilon_i|1 \leq j \leq n\}$, then $(\varepsilon_j \pm \varepsilon_i)^* = \varepsilon_j \mp \varepsilon_i$. For $D_4$, the $t$ can be 0, 1, 2, 3, which means the number of nodes in the coclique. When $t = 2$, although in the Dynkin diagram $\{\alpha_1, \alpha_2\}$ and $\{\alpha_1, \alpha_4\}$ are symmetric, they are in the different orbits under the Weyl group’s actions. Then the admissible root sets for $D_4$ can be written as the $W(D_4)$’s orbits of $\emptyset$, $\{\alpha_3\}$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_4\}$, and $\{\alpha_1, \alpha_2, \alpha_4, \alpha_1 + \alpha_2 + \alpha_4 + 2\alpha_3\}$.

**Example 5.4.** If $Q = D_4$, the root set $\{\alpha_1, \alpha_2, \alpha_4\}$ is mutually orthogonal but not admissible, and its admissible closure is $\{\alpha_1, \alpha_2, \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4\}$.

**Definition 5.5.** Let $\mathcal{A}$ denote the collection of all admissible subsets of $\Phi$ consisting of mutually orthogonal positive roots. Members of $\mathcal{A}$ are called admissible sets.

Now we consider the actions of $R_i$ on an admissible $W$-orbit $\mathfrak{B}$. When $R_i B \neq B$, We say that $R_i$ lowers $B$ if there is a root $\beta \in B$ of minimal height among those moved by $R_i$ that satisfies $\beta - \alpha_i \in \Phi^+$ or $R_i B < B$. We say that $R_i$ raises $B$ if there is a root $\beta \in B$ of minimal height among those moved by $R_i$ that satisfies $\beta + \alpha_i \in \Phi^+$ or $R_i B > B$. By this we can set an partial order on $\mathfrak{B} = WB$. The poset $(\mathfrak{B}, <)$ with this minimal ordering is called the monoidal poset (with respect to $W$) on $\mathfrak{B}$ (so $\mathfrak{B}$ should be admissible for the poset to be monoidal). If $\mathfrak{B}$ just consists of sets of a single root, the order is determined by the canonical height function on roots. There is an important conclusion in [11], stated below. This theorem plays a crucial role in obtaining a basis for Brauer algebra of simply laced type in [10].

### Table 2: Admissible root sets of simply laced type

| $Q$ | representatives of orbits under $W(Q)$ |
|-----|----------------------------------------|
| $A_n$ | $\{\alpha_{2i-1}\}_{i=1}^t, 0 \leq t \leq \lfloor (n+1)/2 \rfloor$. |
| $D_n$ | $Y(t) = \{\alpha_{n+2-2i}, \alpha_{n-2}, \ldots, \alpha_{n+2-2t}\}$, $0 \leq t \leq \lfloor n/2 \rfloor$. |
|     | $Y(t) \cup Y(t)^*, 0 \leq t \leq \lfloor n/2 \rfloor$. |
| $E_6$ | $\emptyset$, $\{\alpha_6\}$, $\{\alpha_6, \alpha_4\}$, $\{\alpha_6, \alpha_2, \alpha_3\}^{cl}$. |
| $E_7$ | $\emptyset$, $\{\alpha_7\}$, $\{\alpha_7, \alpha_5\}$, $\{\alpha_5, \alpha_5, \alpha_2\}$, $\{\alpha_7, \alpha_2, \alpha_3\}^{cl}$, $\{\alpha_7, \alpha_5, \alpha_2, \alpha_3\}^{cl}$. |
| $E_8$ | $\emptyset$, $\{\alpha_8\}$, $\{\alpha_8, \alpha_6\}$, $\{\alpha_8, \alpha_2, \alpha_3\}^{cl}$, $\{\alpha_8, \alpha_5, \alpha_2, \alpha_3\}^{cl}$. |
Theorem 5.6. There is a unique maximal element in $\mathcal{B}$.

For any $\beta \in \Phi^+$ and $i \in \{1, \ldots, n\}$, there exists a $w \in W$ such that $\beta = w\alpha_i$. Then $R_\beta := wR_iw^{-1}$ and $E_\beta := wE_iw^{-1}$ are well defined (this is well known from Coxeter group theory for $R_\beta$; see [10, Lemma 4.2] for $E_\beta$). If $\beta, \gamma \in \Phi^+$ are mutually orthogonal, then $E_\beta$ and $E_\gamma$ commute (see [10, Lemma 4.3]). Hence, for $B \in \mathcal{A}$, we define the product

$$E_B = \prod_{\beta \in B} E_\beta,$$

which is a quasi-idempotent, and the normalized version

$$\hat{E}_B = \delta^{-|B|}E_B,$$

which is an idempotent element of the Brauer monoid. For a mutually orthogonal root subset $X \subset \Phi^+$, we have

$$E_{X^{cl}} = \delta^{|X^{cl}\setminus X|}E_X.$$

Let $C_X = \{i \in Q \mid \alpha_i \perp X\}$ and let $W(C_X)$ be the subgroup generated by the generators of nodes in $C_X$. The subgroup $W(C_X)$ is called the centralizer of $X$. The normalizer of $X$, denoted by $N_X$, can be defined as

$$N_X = \{w \in W \mid E_Xw = wE_X\}.$$  

We let $D_X$ denote a set of right coset representatives for $N_X$ in $W$.

In [10, Definition 3.2], an action of the Brauer monoid $BrM(Q)$ on the collection $\mathcal{A}$ of admissible root sets in $\Phi^+$ was indicated below, where $Q \in$ ADE.

Definition 5.7. There is an action of the Brauer monoid $BrM(Q)$ on the collection $\mathcal{A}$. The generators $R_i \ (i = 1, \ldots, n)$ act by the natural action of Coxeter group elements on its positive root sets as in Remark 5.1, and the element $\delta$ acts as the identity, and the action of $E_i \ (i = 1, \ldots, n)$ is defined by

$$E_iB := \begin{cases} 
B & \text{if } \alpha_i \in B, \\
(B \cup \\{\alpha_i\})^{cl} & \text{if } \alpha_i \perp B, \\
R_\beta R_iB & \text{if } \beta \in B \setminus \alpha_i^+. 
\end{cases}$$

We will refer to this action as the admissible set action. This monoid action plays an important role in getting a basis of $BrM(Q)$ in [10]. For the basis, we state one conclusion from [10, Proposition 4.9] below.

Proposition 5.8. Each element of the Brauer monoid $BrM(Q)$ can be written in the form

$$\delta^kuvE_Xzv,$$

where $X$ is the highest element from one $W$-orbit in $\mathcal{A}$, $u, v^{-1} \in D_X$, $z \in W(C_X)$, and $k \in \mathbb{Z}$. 

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Remark 5.9. There is a more general version for simply laced types in \[8\]. We keep notations as in \[8, Section 2\] and first introduce some basic conceptions. Let \(Q\) be the diagram of a connected finite simply laced Coxeter group (type \(A_n, D_n, E_6, E_7, E_8\)). Then \(\text{BrM}(Q)\) is the associated Brauer monoid as in Definition 4.1. Recall an element \(a \in \text{BrM}(Q)\) is said to be of height \(t\) if the minimal number of \(R_i\) occurring in an expression of \(a\) is \(t\), denoted by \(\text{ht}(a)\). By \(B_Y\) we denote the admissible closure of \(\{\alpha_i | i \in Y\}\), where \(Y\) is a coclique of \(Q\). The set \(B_Y\) is a minimal element in the \(W(Q)\)-orbit of \(B_Y\) which is endowed with a poset structure induced by the partial ordering \(<\) defined on \(W(Q)\)-orbits in \(A\). If \(d\) is the Hasse diagram distance for \(W(Q)B_Y\) from \(B_Y\) to the unique maximal element, then for \(B \in W(Q)B_Y\) the height of \(B\), already used in Definition notation \(\text{ht}(B)\), is \(d - l\), where \(l\) is the distance in the Hasse diagram from \(B\) to the maximal element. The Figure 5 is a Hasse diagram of admissible sets of type \(A_4\) with 2 mutually orthogonal positive roots. As indicated in Theorem 5.6 the set \(\{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}\) is the maximal root set in its \(W(A_4)\)-orbit.

**Figure 5:** A Hasse diagram of type \(A_4\).
Theorem 5.10. ([8, Theorem 2.7]) Each monomial $a$ in $\text{BrM}(Q)$ can be uniquely written as $\delta^i a_B E_Y h a_{B'}^\text{op}$, for some $i \in \mathbb{Z}$ and $h \in W(Q_Y)$, where $W(Q_Y)$ is the group of invertible elements in $\hat{E}_Y W(Q) \hat{E}_Y$, $B = a \emptyset$, $B' = \emptyset a$, $a_B \in \text{BrM}(Q)$, $a_{B'}^\text{op} \in \text{BrM}(Q)$ and

(i) $a \emptyset = a_B \emptyset = a_B B_Y$, $\emptyset a = \emptyset a_{B'}^\text{op} = B_Y a_{B'}^\text{op}$,
(ii) $\text{ht}(B) = \text{ht}(a_B)$, $\text{ht}(B') = \text{ht}(a_{B'}^\text{op})$.

6 The isomorphism of $\text{DTL}(C_n)$ and $\text{STL}(A_{2n-1})$

In this section, we focus on type $C_n$. First recall the Dynkin diagram of type $C_n$.

$$C_n = \begin{array}{cccccccc}
\circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \\
n-1 & n-2 & 2 & 1 & 0 
\end{array}.$$  

From [6], the automorphism $\sigma$ on $\text{BrM}(A_{2n-1})$ has a diagram explanation, which means the symmetry to the middle axis. Therefore, a $\sigma$-invariant monomial is a diagram which is symmetric to the middle axis. The same explanation also can be applied to the $\text{TLM}(A_{2n-1})$, a submonoid of $\text{BrM}(A_{2n-1})$. Combining the diagram representation of $\text{TL}(A_{2n-1})$ and the symmetry of $\sigma$, therefore the algebra $\text{STL}(A_{2n-1})$ has a basis consisting of the diagram monomials of $\text{BrM}(A_{2n-1})$ which have no intersections and are symmetric to the middle axis. We give one example in the Figure 6. By the diagram

![Diagram](image)

Figure 6: $\phi(e_1e_0)$ in $\text{STL}(A_3)$

images of generators of $\text{TL}(C_n)$, we can obtain the following lemma.

Lemma 6.1. The algebra $\phi(\text{TL}(C_n))$ is a subalgebra of $\text{STL}(A_{2n-1})$. 
Let \( m \geq 1 \). The root system of the Coxeter group \( W(A_m) \) of type \( A_m \) is denoted by \( \Phi \). It is realized as \( \Phi := \{ \epsilon_i - \epsilon_j \mid 1 \leq i, j \leq m+1, i \neq j \} \) in the Euclidean space \( \mathbb{R}^{m+1} \), where \( \epsilon_i \) is the \( i \)th standard basis vector. Put \( \alpha_i := \epsilon_i - \epsilon_{i+1} \). Then \( \{ \alpha_i \}_{i=1}^{m} \) is called the set of simple roots of \( \Phi \). Denote by \( \Phi^+ \) the set of positive roots in \( \Phi \) with respect to these simple roots; that is, \( \Phi^+ := \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq m+1 \} \).

An admissible set \( B \) corresponds to a Brauer diagram top in the following way: for each \( \beta \in B \), where \( \beta = \epsilon_i - \epsilon_j \) for some \( i, j \in \{1, \ldots, m+1\} \) with \( i < j \), draw a horizontal strand in the corresponding Brauer diagram top from the dot \( i \) to the dot \( j \) in the top. All horizontal strands on the top are obtained this way, so there are precisely \(|B|\) horizontal strands.

We will refer to this action as the admissible set action. Alternatively, this action can be described as follows for a monomial \( a \): complete the top corresponding to \( B \) into a Brauer diagram \( b \), without increasing the number of horizontal strands in the top. Now \( aB \) is the top of the Brauer diagram \( ab \).

Now we will prove the following theorem.

**Theorem 6.3.** The algebra \( DTL(C_n) \) is isomorphic to \( STL(A_{2n-1}) \) under \( \phi \).

**Proof.** By Remark 4.6 and Lemma 6.1 it remains to prove that the algebra morphism \( \phi : DTL(C_n) \to STL(A_{2n-1}) \) is surjective to accomplish the proof. Consider the symmetry of \( \sigma \), we modify those \( Y \) in the Table 2 as the following. For \( i \in \{0, \ldots, n\} \) we let \( Y_i \in \mathcal{A} \) be the following set of nodes of size \( i \):

\[
Y_i = \begin{cases} 
\{n, n \pm 2, n \pm 4, \ldots, n \pm (i - 1)/2\} & \text{if } i \equiv 1 \pmod{2} \\
\{n \pm 1, n \pm 3, \ldots, n \pm i/2\} & \text{if } i \equiv 0 \pmod{2}
\end{cases}
\]

The corresponding set of positive roots \( \{\alpha_y \mid y \in Y\} \) is denoted by \( B_i \).

Let \( B_{Y_i} \) be the root sets corresponding to \( Y_i \). It can be seen that each \( Y_i \) is of height 0 in the sense of Remark 5.9. If we replace those \( Y \)s in Table 2 by those \( Y_i \)s, which are \( \sigma \)-invariant and of height 0, by the Theorem 5.10 the proof is reduced to the following problem.

If we have an admissible set \( B \) which are \( \sigma \)-invariant, of size \( i \) and of height 0, there exists an element \( a \in \phi(TLM(C_n)) \), such that \( aY_i = B \).

We prove this fact by induction. First, it can be easily verified when \( n = 1 \), or 2.

Considering the diagram version for \( B \), if there is no horizontal strands having ends \( i \) and \( 2n \)(since \( B \) is symmetric to the middle axis), then it is totally reduced to the case \( n - 1 \). Since \( B \) has height 0, namely the diagram version of \( B \) has no intersection, so there exist two possible cases for \( B \) displayed in the Figure 7.
Now we can consider case 1 in the Figure 7. Since $B$ has height zero, all the strands having ends in $\{1, 2, \ldots, j\}$ must be horizontal, and the other ends must be also in $\{1, 2, \ldots, j\}$. Then we can see that $j$ must be even. We divide the set $B$ into two disjoint subsets $B_1$ and $B_2$, where $B_1$ are those roots having ends in $\{1, 2, \ldots, j\}$ or $\{2n + 1 - j, 2n + 2 - j, \ldots, 2n\}$, and $B = B_1 \bigsqcup B_2$. It can be seen that the subalgebra $A$ of $\text{DTL}(C_n)$ generated by $\{e_i\}_{i=1}^{n-1}$ is isomorphic to $\text{TL}(A_{n-1})$. Similarly, we do the same division on $Y_i$, such that $Y_i = Y_{i, 1} \bigsqcup Y_{i, 2}$, where $Y_{i, 1}$ is the most left $\frac{j}{2}$ dots and the most right $\frac{j}{2}$ dots. Similarly, $B_{Y_i} = B_{Y_{i, 1}} \bigsqcup B_{Y_{i, 2}}$. By the property of $A(\text{TL}(A_{n-1}))$ (Proposition 4) and the symmetry to the middle axis, we can find some Temperly-Lieb monomial $a_1$ in $A$ such that $\phi(a_1)B_{Y_{i, 1}} = B_1$, $\phi(a_1)B_{Y_{i, 2}}$ and $B_2$ are lowered to the case of $\text{DTL}(C_{n-j+1})$ which is generated by $\{e_i\}_{i=0}^{n-j}$. By induction, then we can find one monomial $a_2$ in this $\text{DTL}(C_{n-j+1})$, such that $\phi(a_2a_1)B_{Y_{i, 2}} = B_2$, $\phi(a_2)B_1 = B_1$. Therefore $\phi(a_2a_1)B_{Y_{i}} = B$.

Now we consider the case 2 in the Figure 7. The set $B$ has no intersection, then $|B| = n$. Let $\alpha = \epsilon_{2n} - \epsilon_1$ be the root represented by the strands from 1 to $2n$ and $\alpha' = \epsilon_{2n-2} - \epsilon_3$ be the root represented by the strands from 3 to $2n - 2$ (Figure 8). Let $B'$ be a height 0 and symmetric to the middle axis as displayed in the top of Figure 8. By induction on the $\text{DTL}(C_{n-2})$ generated by $\{e_i\}_{i=0}^{n-3}$, we can see that there exist a monomial $b_1$ in this $\text{DTL}(C_{n-2})$ such that $\phi(b_1)B_{Y_{n-2}} = B' \setminus \{\alpha_1, \alpha_{2m-1}\}$. Therefore $\phi(b_1)B_{Y_{n}} = B'$. When the element $\phi(e_{n-2})$ acts on $B'$, we have

$$\phi(e_{n-2})B' = (B' \setminus \{\alpha_1, \alpha_{2n-1}, \alpha'\}) \cap \{\alpha_2, \alpha_{2n-2}, \alpha\}.$$ 

Now we can reduce $(\phi(e_{n-2})B') \setminus \{\alpha\}$ and $B \setminus \{\alpha\}$ to $\text{DTL}(C_{n-1})$ generated by $\{e_i\}_{i=0}^{n-2}$, then we can find a monomial $b_2$ in this $\text{DTL}(C_{n-1})$ such that

$$\phi(b_2)((\phi(e_{n-2})B') \setminus \{\alpha\}) = B \setminus \{\alpha\}$$

and

$$\phi(b_2e_{n-2}b_1)B_{Y_{n}} = B.$$
7 The rewriting forms for $\text{DTL}(B_n)$

First recall the Dynkin diagram of type $B_n$.

$$B_n = \begin{array}{cccccccc}
& & & & & & & \\
& n-1 & n-2 & \cdots & 2 & 1 & 0 & \\
\end{array}$$

In [7, Remark 6.4], we have forecasted that the algebra $\text{DTL}(B_n)$ has rank $C_n + C_{n+1} - 1$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$, the Catalan number. Here we will give a precise proof for this claim. To prove this, we define

\[
\begin{align*}
\hat{e}_0 &= e_0 \\
\hat{e}_1 &= e_1 e_0 e_1 \\
\hat{e}_{i+1} &= e_{i+1} e_i e_{i+1}, \quad 1 \leq i \leq n - 2.
\end{align*}
\]

Lemma 7.1. In the algebra $\text{DTL}(B_n)$, we have the following equalities.

\[
\begin{align*}
\hat{e}_i \hat{e}_{i+1} &= \delta \hat{e}_i e_{i+1}, \quad 0 \leq i \leq n - 2 \quad (7.1) \\
\hat{e}_i \hat{e}_{i+1} &= \delta \hat{e}_i e_{i+1}, \quad 1 \leq i \leq n - 2 \quad (7.2) \\
\hat{e}_i e_{i+1} \hat{e}_i &= \delta \hat{e}_i, \quad 0 \leq i \leq n - 1 \quad (7.3) \\
\hat{e}_{i+1} e_i \hat{e}_{i+1} &= \delta \hat{e}_{i+1}, \quad 1 \leq i \leq n - 2 \quad (7.4) \\
\hat{e}_i^2 &= \delta^2 \hat{e}_i, \quad 0 \leq i \leq n - 1 \quad (7.5) \\
\hat{e}_i \hat{e}_j &= \hat{e}_j \hat{e}_i, \quad |i - j| > 1 \quad (7.6) \\
\hat{e}_i e_j &= e_j \hat{e}_i, \quad |i - j| > 1 \quad (7.7) \\
\hat{e}_i \hat{e}_j &= \delta \hat{e}_i, \quad |i - j| = 1 \quad (7.8) \\
\hat{e}_i \hat{e}_j &= \delta e_i \hat{e}_j, \quad |i - j| > 1 \quad (7.9)
\end{align*}
\]
Proof. For (7.1), it can be verified easily when \( i = 0 \), and we have
\[
\hat{e}_i \hat{e}_{i+1} = e_i \hat{e}_{i-1}(e_i e_{i+1} e_i) \hat{e}_{i-1} e_i e_{i+1}
\]
then by induction
\[
\delta e_i \hat{e}_{i-1} e_i e_{i+1} = \delta \hat{e}_i e_{i+1}.
\]

For (7.2), we have
\[
e_i \hat{e}_{i+1} = (e_i e_{i+1} e_i) \hat{e}_{i-1} e_i e_{i+1}
\]
then by (4.3)
\[
\hat{e}_i e_{i+1} e_i = \hat{e}_i e_{i+1} e_i = \delta \hat{e}_i.
\]

For (7.3), by (4.4), it is true when \( i = 0 \), and we have
\[
\hat{e}_i e_{i+1} \hat{e}_i = e_i \hat{e}_{i-1}(e_i e_{i+1} e_i) \hat{e}_{i-1} e_i
\]
then by induction
\[
\delta e_i \hat{e}_{i-1} e_i = \delta \hat{e}_i.
\]

For (7.4), we have
\[
\hat{e}_{i+1} e_i \hat{e}_{i+1} = e_{i+1} \hat{e}_i (e_{i+1} e_i e_{i+1}) \hat{e}_i e_{i+1}
\]
then by (4.3)
\[
\hat{e}_{i+1} e_i e_{i+1} = \hat{e}_{i+1} e_i e_{i+1} = \delta \hat{e}_{i+1}.
\]

For (7.5), we have
\[
\hat{e}_i^2 = e_i \hat{e}_{i-1}(e_i e_i) \hat{e}_{i-1} e_i
\]
then by (4.1)
\[
\delta e_i \hat{e}_{i-1} e_i^2 = \delta \hat{e}_i.
\]

For (7.6), we first consider \( \hat{e}_i \hat{e}_{i+2} \). Then we have
\[
\hat{e}_i \hat{e}_{i+2} = (\hat{e}_i e_{i+2}) e_{i+1} \hat{e}_i e_{i+1} e_{i+2}
\]
then by (4.2)
\[
\hat{e}_{i+2} \hat{e}_i e_{i+1} e_{i+2} = \delta e_{i+2} \hat{e}_i e_{i+1} e_{i+2}
\]
and we can obtain the following by Proposition 4.3
\[
\hat{e}_{i+2} \hat{e}_i = \delta e_{i+2} \hat{e}_i = \delta \hat{e}_i e_{i+2} = \hat{e}_i \hat{e}_{i+2}.
\]
Then for the general case of (7.6), it can be verified by induction and Proposition 4.3.

The formula (7.7) follows from the above proof of (7.6).

The formula (7.8) follows from (7.1), (7.2), (7.3), and (7.4).

For (7.9), if $j - i > 1$, we have

$$e_i \hat{e}_j = e_j \ldots e_{i+2} e_i \hat{e}_{i+1} e_{i+2} \ldots e_j$$

(7.2)

$$\delta^{-1} e_j \ldots e_{i+2} \hat{e}_i e_{i+2} \ldots e_j$$

(7.7) - 4.3

$$\delta^{-1} \hat{e}_i \hat{e}_j;$$

if $i - j > 1$, we have

$$\hat{e}_i \hat{e}_j = e_i \ldots e_{j+2} (\hat{e}_{j+1} \hat{e}_j) e_{j+2} \ldots e_i$$

(7.3) - 4.3

$$\delta e_i \ldots e_{j+2} e_{j+1} \hat{e}_j e_{j+2} \ldots e_i$$

(7.1)

$$\delta e_i \ldots e_{j+2} e_{j+1} e_{j+2} \ldots e_i \hat{e}_j$$

(4.3)

$$\delta e_i \hat{e}_j.$$

Let $\psi$ be the algebra morphism from $\text{DTL}(B_n)$ to $\text{TL}(D_{n+1})$.

**Lemma 7.2.** The subalgebra $Y_1$ of $\text{DTL}(B_n)$ generated by $\{e_i\}_{i=1}^{n-1}$ is isomorphic to $\text{TL}(A_{n-1})$. Then $Y_1$ has rank $C_n$ over the ground ring.

**Proof.** By easy verification, there is a surjective algebra morphism $\psi'$ from $\text{TL}(A_{n-1})$ to $Y_1$. By $\psi$, we see that $\psi(e_i) = E_{i+2}$, for $1 \leq i \leq n - 1$, which are Temperley-Lieb generators of $\text{TL}(D_{n+1})$. By [3] Proposition 4, the subalgebra of $\text{TL}(D_{n+1})$ generated by $\{E_{i+2}\}_{i=1}^{n-1}$ is isomorphic to $\text{TL}(A_{n-1})$. Now we have

$$\text{TL}(A_{n-1}) \xrightarrow{\psi'} Y_1 \xrightarrow{\psi} \text{TL}(D_{n+1}),$$

then $\text{TL}(A_{n-1}) \cong Y_1$.

**Lemma 7.3.** The subalgebra $Y_2$ of $\text{DTL}(B_n)$ generated by $\{\hat{e}_i\}_{i=0}^{n-1}$ can be spanned by $C_{n+1}$ monomials.

**Proof.** We write down the relations about the generators $\{\hat{e}_i\}_{i=0}^{n-1}$ from Lemma 7.1 below,

$$\hat{e}_i e_{i+1} \hat{e}_i = \delta \hat{e}_i, \quad |i - j| = 1,$$

$$\hat{e}_i \hat{e}_j = \hat{e}_j \hat{e}_i, \quad |i - j| > 1,$$

$$\hat{e}_i \hat{e}_j \hat{e}_i = \delta^2 \hat{e}_i, \quad |i - j| = 1.$$
If we define $\delta = 1$, we can define a surjective algebra morphism from $\text{TL}(A_n)$ to algebra $\mathcal{Y}_2$. Then by the monomial reduction of $\text{TL}(A_n)$, the algebra $\mathcal{Y}_2$ can have the same spanning elements as the reduced monomials of $\text{TL}(A_n)$ with generators $E_1, \ldots, E_n$ replaced by $e_0, \ldots, e_{n-1}$, respectively. Therefore $\mathcal{Y}_2$ can be spanned by $C_{n+1}$ monomials, which is the rank of $\text{TL}(A_n)$. \qed

**Proposition 7.4.** It follows that

$$\text{DTL}(B_n) = \mathcal{Y}_1 + \mathcal{Y}_2.$$  

Then $\text{DTL}(B_n)$ can be spanned by $C_n + C_{n+1} - 1$ elements.

**Proof.** Because the generators of $\text{DTL}(B_n)$ are in $\mathcal{Y}_1 + \mathcal{Y}_2$, we just need to prove that $\mathcal{Y}_1 + \mathcal{Y}_2$ are closed under multiplication. Since $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are algebras and we have the natural involution in Proposition 4.3, it remains to prove that $y_1y_2 \in \mathcal{Y}_1 + \mathcal{Y}_2$ for $y_1 \in \mathcal{Y}_1$ and $y_2 \in \mathcal{Y}_2$. We claim that $y_1y_2 \in \mathcal{Y}_2$ for any $y_1 \in \mathcal{Y}_1$ and $y_2 \in \mathcal{Y}_2$. By induction, it can be reduced to to $e_i \hat{e}_j \in \mathcal{Y}_2$. Therefore this holds for (7.1), (7.2), (7.9) and Proposition 4.3. \qed

### 8 The Rank of $\text{DTL}(B_n)$

In this section, we try to prove that the rank of $\text{DTL}(B_n)$ is exactly $C_n + C_{n+1} - 1$. To prove this, we mainly use the diagram representation of Brauer algebra of type $D_{n+1}$ from [12]. Now we recall the diagram representations here. Divide $2n+2$ points into two sets $\{1, 2, \ldots, n+1\}$ and $\{\hat{1}, \hat{2}, \ldots, \hat{n+1}\}$ of points in the (real) plane with each set on a horizontal line and point $i$ above $\hat{i}$. An $n+1$-connector is a partition on $2n+2$ points into $n+1$ disjoint pairs. It is indicated in the plane by a (piecewise linear) curve, called strand from one point of the pair to the other. A decorated $n+1$-connector is an $n+1$-connector in which an even number of pairs are labeled 1, and all other pairs are labeled by 0. A pair labeled 1 will be called decorated. The decoration of a pair is represented by a black dot on the corresponding strand.

**Remark 8.1.** Denote $T_{n+1}$ the set of all decorated $n+1$-connectors. Denote $T^0_{n+1}$ the subset of $T_{n+1}$ of decorated $n+1$-connectors without decorations and denote $T^-_{n+1}$ the subset of $T_{n+1}$ of decorated $n+1$-connectors with at least one horizontal strand.

Let $H$ be the commutative monoid with presentation

$$H = \langle \delta^{\pm 1}, \xi, \theta \mid \xi^2 = \delta^2, \xi\theta = \delta\theta, \theta^2 = \delta^2\theta \rangle = \langle \delta^{\pm 1} \rangle \{1, \xi, \theta\}.$$  

A *Brauer diagram* of type $D_{n+1}$ is the scalar multiple of a decorated $n$-connector by an element of $H$ belonging to $\langle \delta^{\pm 1} \rangle (T_{n+1} \cup \xi T^-_{n+1} \cup \theta(T^0_{n+1} \cap \ldots)$.
The Brauer diagram algebra of type $D_{n+1}$, denoted $\text{BrD}(D_{n+1})$, is the $\mathbb{Z}[\delta^{\pm 1}]$-linear span of all Brauer diagrams of type $D_{n+1}$ with multiplication laws defined in [12, Definition 4.4]. The corresponding monoid is denoted $\text{BrMD}(D_{n+1})$.

The scalar $\xi\delta^{-1}$ appears in various products of $n+1$-connectors described in [12, Definition 4.4] and two consecutive black dots on a strand are removed. Also, the scalar $\theta\delta^{-1}$ appears in various products of $n+1$-connectors in which a dotted circle appears, as described in [12, Figure 16]. The multiplication is an intricate variation of the multiplication in classical Brauer diagrams, where the points of the bottom of one connector are joined to the points of the top of the other connector, so as to obtain a new connector. In this process, closed strands appear which are turned into scalars by translating them into elements of $H$ as indicated in Figure 9.

$$
\begin{align*}
\bigcirc &= \delta, & \bigcirc \bullet \bigcirc &= \theta, & \begin{array}{c}
\includegraphics{circle_cross}\end{array} &= \xi
\end{align*}
$$

Figure 9: The closed loops corresponding to the generators of $H$

$$
\begin{align*}
\psi(R_i) &= \begin{array}{c}
\includegraphics{psi Ri}\end{array}, & \psi(E_i) &= \begin{array}{c}
\includegraphics{psi Ei}\end{array}, & \psi(R_i) &= \begin{array}{c}
\includegraphics{psi Ri}\end{array}, & \psi(E_i) &= \begin{array}{c}
\includegraphics{psi Ei}\end{array}, & 2 \leq i \leq n+1
\end{align*}
$$

Figure 10: The images of the generators of $\text{Br}(D_{n+1})$ under $\psi$

Let $c_1$ and $c_2$ decorated $n+1$-connectors, and $\kappa_1, \kappa_2 \in \{1, \xi, \theta\}$. Now we describe the product $\kappa_1 c_1 \kappa_2 c_2$ in [12, Definition 4.4] being the form of $\kappa c$ where $c$ is a decorated $n+1$-connector and $\kappa \in H$.

(i) As the classical case, draw the diagram $c_1$ and $c_2$, and stack them.
(ii) Determine the pairing of $c$: for a point at the top of $c_1$ or the bottom of $c_2$, follow the strand until it ends in a point at the top of $c_1$ or the bottom of $c_2$. This results in a new pairing for $c$.

(iii) Set $\kappa = \kappa_1 \kappa_2$. For each straightening step in a concatenation of pairs as carried out in the previous step, check if the pattern shrunk to a straight horizontal line segment occurs as the left hand-sides of the first 20 relations in Figure 11. If so, multiply $\kappa$ by $\xi \delta^{-1}$; otherwise, $\kappa$ is not changed. (Compare with the left-hand picture of the last two relations in Figure 11; this pattern as well as each triple of straight line segments forming a shape appearing in the first 20 relations in Figure 11 but whose decoration pattern does not appear in the first 20 relations in Figure 11 does not change $\kappa$.)

(iv) At this stage, only closed loops remains. Closed loops come from strands which have no endpoints in $c$. First simplify loops by removing crossing as in (iii), i.e. by use of the first 20 relations in Figure 11 again, the configurations not appearing in the figure do not give $\xi \delta^{-1}$) and shrink them using the rules on the bottom lines of the first 20 relations in Figure 11 (at this stage, factor $\xi \delta^{-1}$ may emerge). Next, replace each closed loops without decoration by $\delta$ (that is, remove the loop and multiply $\kappa$ by $\delta$) and each pair of disjoint closed decoration loops by $\theta$. As the number of decorated pairs is even, what might remain is a simple decorated loop in the presence of a decorated pair; if so, undecorate the pair, remove the decorated loop by multiply $\kappa$ by $\theta \delta^{-1}$. (Compare with the right-hand side of the last two relations in Figure 11.)

(v) If $\theta$ is a factor of $\kappa$, remove all decorations from $c$.

In [12], the algebra BrD(D_{n+1}) is proved to be isomorphic to Br(D_{n+1}) by means of the isomorphism $\psi : Br(D_{n+1}) \rightarrow BrD(D_{n+1})$ defined on generators as in Figure 10. It is free over $\mathbb{Z}[\delta^{\pm 1}]$ with basis $T_{n+1} \cup \xi T_{n+1} \cup \theta(T_{n+1}^0 \cap T_{n+1}^\infty)$.

**Theorem 8.2.** The algebra DTL(B_n) has rank $C_n + C_{n+1} - 1$.

**Proof.** Suppose that the canonical Temperley-Lieb basis for $\mathcal{Y}_1$ in lemma 7.2 is $\mathcal{K}_1$ and the spanning set for $\mathcal{Y}_2$ in lemma 7.3 is $\mathcal{K}_2$. We see that $\psi \phi(e_i) = \psi(E_{i+2})$, which is drawn in Figure 8 for $i = 1, \ldots, n-1$. Then we see those $\{\psi(E_{i+2})\}_{i=1}^{n-1}$ generate the canonical Temperley-Lieb algebra of type $A_{n-1}$, which has rank $C_n$, and $\mathcal{K}_1 \subset T_{n+1}$, which represents $C_n$ different diagrams. For $\mathcal{K}_2 \setminus \{1\}$, we have $\mathcal{K}_2 \setminus \{1\} \subset \theta(T_{n+1}^0 \cap T_{n+1}^\infty)$, up to some powers of $\delta$. Because

\[
\begin{align*}
\psi \phi(\hat{e}_i) &= \theta \delta \psi(E_{i+2}), \quad \text{for} \quad i = 1, \ldots, n-1, \\
\psi \phi(\hat{e}_0) &= \theta \delta \psi(E_1),
\end{align*}
\]
Figure 11: 22 reduction relations for Brauer diagram algebra of type $D_{n+1}$

which are generators of $TL(A_n)$, without considering $\theta$ and $\delta$. So $K_2 \setminus \{1\} \subset \theta(T_{n+1}^0 \cap T_{n+1}^\circ)$, and represents $C_{n+1} - 1$ different diagrams in $\theta(T_{n+1}^0 \cap T_{n+1}^\circ)$. Therefore $DTL(B_n)$ has rank $C_n + C_{n+1} - 1$.

Using the cellular structure of $Br(B_n)$ ([7]) and $Br(C_n)$ ([6]), we can obtain
the following Theorem.

**Theorem 8.3.** The algebras $DL(B_n)$ and $DL(C_n)$ are cellular algebras in the sense of [18].

**Remark 8.4.** By [16], it is known that the Temperley-Lieb algebras of type $F_n$, $TL(F_n)$ for $n \geq 4$ is of finite rank. We can verify that the $DL(F_n)$ is a quotient algebra of $TL(F_n)$, so $DL(F_n)$ is also of finite rank. But it is not easy to give the precise rank here, we will leave it for some further research.

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