Flatness-based control of a single qubit gate

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Abstract

This work considers the open-loop control problem of steering a two level quantum system from an initial to a final condition. The model of this system evolves on the state space $\mathcal{X} = SU(2)$, having two inputs that correspond to the complex amplitude of a resonant laser field. A symmetry preserving flat output is constructed using a fully geometric construction and quaternion computations. Simulation results of this flatness-based open-loop control are provided.

Index Terms

Quantum control, nonlinear systems, geometric control, flatness, qubit gate.

I. INTRODUCTION

Take a single qubit, i.e. a two level quantum system. Denote by $\omega_0$ its transition frequency. Assume that it is controlled via a resonant laser field $v \in \mathbb{R}$:

$$v = u \exp(-i\omega_0 t) + u^* \exp(i\omega_0 t)$$

where $u = u_1 + u_2 \in \mathbb{C}$, $(u_1, u_2) \in \mathbb{R}^2$, is its complex amplitude. In general, the frequency $\omega_0$ is large and the time variation of $u$ is slow: $|\dot{u}| \ll \omega_0 |u|$. In the interaction frame, after the rotating wave approximation and up to some scaling (see e.g., [1]), the Hamiltonian reads $u_1 \sigma_1 + u_2 \sigma_2$ where $\sigma_1$ and $\sigma_2$ are the first two Pauli matrices (see appendix). The gate generation problem then reads: take a transition time $T > 0$ such that $\omega_0 T \ll 1$ and a goal matrix $\bar{U} \in SU(2)$; find
a smooth laser impulsion \([0,T] \ni t \mapsto u(t) \in \mathbb{C}\) with \(u(0) = u(T) = 0\) such that the solution \([0,T] \ni t \mapsto U(t) \in SU(2)\) of the initial value problem
\[
\frac{d}{dt} U(t) = (u_1(t)\sigma_1 + u_2(t)\sigma_2) \ U(t), \quad U(0) = I_2
\] (2)
reaches \(\bar{U}\) at time \(T\), i.e., \(U(T) = \bar{U}\). This motion planning problem admits a well known elementary solution\(^1\). It relies on the fact that \(\bar{U} = \exp(-i\gamma\sigma_1) \ \exp(-i\beta\sigma_2) \ \exp(-i\alpha\sigma_1)\), for all \(\bar{U} \in SU(2)\), for convenient \((\alpha,\beta,\gamma) \in \mathbb{R}^3\) (see, e.g., [2]). An obvious steering control \(u(t)\) is decomposed into three elementary and successive pulses: for the first (resp. third) pulse, \(u_2 = 0\) and \(u_1\) is such that its integral over the pulse interval equals \(\alpha\) (resp. \(\gamma\)); for the second pulse, \(u_1 = 0\) and the integral of \(u_2\) is \(\beta\).

Here, we propose another solution where \(u_1\) and \(u_2\) vary simultaneously, i.e., the steering control \(u(t)\) is contained in a single pulse. Our solution does not rely on optimal control techniques (see for instance [3] and the references therein) and is explicit. It does not rely on numerical resolution scheme. It provides controls that can be chosen to be \(C^0\) or \(C^\infty\) function of \(t\). As far as we know, such explicit solution is new and could be of some interest for reducing the transition time \(T\) while still respecting the rotating wave approximation. Our approach is based on the fact that the system dynamics is differentially flat [4]. The flat output constructed in this paper has a clear geometrical interpretation.

In section [II], theorem [I] shows, using a quaternion description of (2), that this system is flat. We propose a coordinate free definition of the flat-output that lives in the homogenous space \(SU(2)/\exp(i\mathbb{R}\sigma_1)\). This geometric construction preserves invariance with respect to right translations. In the sense of [5], the flat output is compatible with right translations. The proposed construction can be seen as the analogue of the geometric construction based on the Frenet formula for the car system, where the steering angle is directly related to the curvature of the path followed by the flat-output curve [6]. In section [III] we show how to use such geometric flatness parameterization to solve analytically the motion planning problem corresponding to an arbitrary quantum gate. Simulations illustrate theorem [2] and the interest of such explicit open-loop steering control. In section [IV] some conclusions are briefly stated. Some material has been deferred to the appendix.\(^1\) In part A one finds the basics properties of Pauli matrices.

\(^1\)The so-called ZYZ quantum logic gate.
and their associated quaternions as well as the correspondence between $SU(2)$ and quaternions of length one. In part B one finds a proof of the fact that the motion planning algorithm has no singularities.

II. A SYMMETRY PRESERVING FLAT OUTPUT

The dynamics (2) reads in quaternion notation (see appendix A)

$$\frac{d}{dt}q = (u_1 e_1 + u_2 e_2)q$$

where $q \in \mathbb{H}_1$ is a quaternion of length one and where $(u_1, u_2) \in \mathbb{R}^2$ is the control relative to the modulation of a coherent laser field ($u_1 + u_2$ is the complex field amplitude). This system is a driftless system on the Lie Group $\mathbb{H}_1$. It is controllable (see, e.g., [7]). Moreover, this control system is invariant with respect to right translations in the sense of [5], [8]:

- the group $G = \mathbb{H}_1$ acts on the state space $\mathcal{X} = \mathbb{H}_1$ via right multiplication $\phi_g : q \mapsto qg$ where $q \in \mathbb{H}_1$.
- the dynamics is $G$-invariant: if $t \mapsto (q(t), u_1(t), u_2(t))$ is a solution of (3) then $t \mapsto (q(t)g, u_1(t), u_2(t))$ is also a solution of (3) for any $g \in G$.

The controllability structure of this system is in fact of a very special kind. Around any point $\bar{q} \in \mathbb{H}_1$, (3) can be seen in local coordinates as a driftless controllable system with 3 states and 2 controls. Thus, as known since [9] (see also [10]), such system is differentially flat and the flat output function can be chosen to depend only on the state. More precisely, the flat output for the controllable system $\frac{d}{dt}x = u_1 f_1(x) + u_2 f_2(x)$ with $\dim(x) = 3$ is obtained by the rectifying coordinates of any vector field $f(x) = \alpha_1(x)f_1(x) + \alpha_2(x)f_2(x)$ which is a linear combination of the two control vector fields $f_1$ and $f_2$ ($\alpha_1$, $\alpha_2$ are any scalar functions of $x$).

We propose here a coordinate free and symmetry preserving construction of the flat output via the previous procedure. Thus we are looking for a flat output map $h : \mathbb{H}_1 \mapsto \mathcal{Y}$, where $\mathcal{Y}$ is the output space, a compact manifold of dimension 2, and $G$-compatible in the sense of [5]. This means that the output map $h$ must satisfy the following constraint: there exists an action of $G = \mathbb{H}_1$ on the flat output space $\mathcal{Y}$ described by the transformation group $\rho_g : y \mapsto \rho_g(y)$ such

\[\text{Take e.g., the exponential map: } (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \exp(x_1 e_1 + x_2 e_2 + x_3 e_3)\bar{q} \text{ that maps a neighborhood of } 0 \in \mathbb{R}^3 \text{ to a neighborhood of } \bar{q} \text{ in } \mathbb{H}_1.\]
that \( \rho_g(h(q)) = h(qg) \) for any \( q \in \mathbb{H}_1 \). The following construction will be based on the control vector field associated to \( u_1 \), and hence to \( e_1 \).

Denote by \( K = \{ \exp(\phi e_1) \}_{\phi \in [0,2\pi]} \) the one dimensional subgroup of \( \mathbb{H}_1 \) generated by \( e_1 \). We can consider the action of \( K \) on \( \mathbb{H}_1 \) via left multiplication: to any \( k \in K \), we have the diffeomorphism \( \mathbb{H}_1 \ni q \mapsto kq \in \mathbb{H}_1 \). Two elements of \( \mathbb{H}_1 \), \( q \) and \( p \), belong to the same orbit if and only if there exists \( k \in K \) such that \( kq = p \). Denote by \( \mathcal{Y} \) the set of the orbits. This set is a compact manifold of dimension 2 and the output function \( h \) is the map that associates to any \( q \), the orbit \( h(q) \) to which \( q \) belongs. This map is a smooth submersion, and \( \mathcal{Y} \) is called an homogenous space (see, e.g., [11]). If \( q \) and \( p \) belong to the same orbit, \( qg \) and \( pg \) also belong to the same orbit for any \( g \in \mathbb{H}_1 \). Therefore, this output map is \( G \)-compatible in the sense of [5].

Assume that \( y(t) \) is a curve on \( \mathcal{Y} = \mathbb{H}_1 / K \), at least of class \( C^2 \). Since the map \( h : \mathbb{H}_1 \to \mathcal{Y} \) is a submersion, in adequate local coordinates one has \( h(x_1,x_2,x_3) = (x_1,x_2) \). Assume, without loss of generality, that the open neighborhood of definition of \( h \) is rectangular and contains \((0,0,0)\). Define locally the smooth map \( g : U \subset \mathcal{Y} \to V \subset \mathbb{H}_1 \), where \( U, V \) are open sets and \( g(x_1,x_2) = (x_1,x_2,0) \). Note that \( g \) is smooth, and \( Y(t) = g(y(t)) \) is such that \( h(Y(t)) = y(t) \). Then, locally, there exist smooth maps \( g^{(1)} \) and \( g^{(2)} \) such that \( \dot{Y}(t) = g^{(1)}(y(t),\dot{y}(t)) \) and \( \ddot{y}(t) = g^{(2)}(y(t),\dot{y}(t),\ddot{y}(t)) \).

Let us show now that the map \( h \) defines a flat output. This means that the inverse of system (3) with output \( y = h(q) \) has no dynamics\(^3\). Thus we have to consider the following implicit system

\[
\frac{d}{dt} q = (u_1 e_1 + u_2 e_2) q, \quad y = h(q)
\]

where \( t \mapsto y(t) \) is a known function of time and where the quaternion \( q(t) \in \mathbb{H}_1 \) and the control \((u_1(t), u_2(t))\) are the unknown quantities.

The problem is how to manipulate \( h \), since only a geometric construction for \( h \) is available. Knowing the function \( t \mapsto y(t) \) means that we have at our disposal a smooth function \( t \mapsto Y(t) \in \mathbb{H}_1 \) such that \( y(t) = h(Y(t)) \). Hence, to have \( y(t) = h(q(t)) \) means that \( q \) and \( Y \) belongs to the same orbit for each time \( t \). Therefore, there exists \( k(t) = \exp(\phi(t)e_1) \) in \( K \) such that \( q = kY \).

\(^3\)This is equivalent to say that the state \( q \) and the input \( u = (u_1, u_2) \) can be written respectively as \( q = \mathcal{A}(y, \dot{y}, \ddot{y}, ..., y^{(\alpha)}) \) and \( u = \mathcal{B}(y, \dot{y}, \ddot{y}, ..., y^{(\beta)}) \) for convenient smooth maps \( \mathcal{A} \) and \( \mathcal{B} \).
Since $k(t) = q(t)Y^*(t)$, then $k(t)$ is smooth. Thus, we have
\[ \frac{d}{dt} q = \left( \frac{d}{dt} k \right) Y + k \frac{d}{dt} Y. \]

But $\frac{d}{dt} k = \omega e_1 k$ where $\omega = \frac{d}{dt} \phi$. Using (3), we get the following equation $k \frac{d}{dt} Y = ((u_1 - \omega) e_1 + u_2 e_2) k Y$, that is
\[ k \left( \frac{d}{dt} Y \right) Y^* k^* = (u_1 - \omega) e_1 + u_2 e_2. \]

This quaternion equation gives in fact $k$ as a function of $\left( \frac{d}{dt} Y \right) Y^*$. Left and right multiplication by $e_3$ yields
\[ e_3 k \left( \frac{d}{dt} Y \right) Y^* k^* e_3 = (u_1 - \omega) e_1 + u_2 e_2 \]
since $e_3 e_i e_3 = e_i$ for $i = 1, 2$. Hence, we have the following relation (without the controls and $\omega$):
\[ e_3 k \left( \frac{d}{dt} Y \right) Y^* k^* = k \left( \frac{d}{dt} Y \right) Y^* k^*. \]

Assume that
\[ \left( \frac{d}{dt} Y \right) Y^* = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \]
where the $\omega_i$’s are known smooth real functions of time. Thus, we get
\[ k \left( \frac{d}{dt} Y \right) Y^* k^* = \omega_1 e_1 + k^2 (\omega_2 e_2 + \omega_3 e_3) \]
since $e_1 k^* = k^* e_1$, $k k^* = 1$ and $e_i k^* = k e_i$ for $i = 2, 3$. Therefore, (4) reads:
\[ k^4 (\omega_2 e_2 + \omega_3 e_3) = (\omega_2 e_2 - e_3 e_3) \]
since $e_3 k^2 = (k^*)^2 e_3$ and $k^{-1} = k^*$. Right multiplication by $e_2$ yields the following algebraic equation defining $k$
\[ k^4 (\omega_2 + \omega_3 e_1) = (\omega_2 - \omega_3 e_1). \]

Since $k = \cos \phi + \sin \phi e_1$, we have the following equation for the angle $\phi$
\[ (\cos 4\phi + \sin 4\phi e_1)(\omega_2 + \omega_3 e_1) = (\omega_2 - \omega_3 e_1) \]
which is equivalent to $\exp(4\phi i) = \frac{z^2}{|z|^2}$ where $z = \omega_2 - \omega_3 i$ is a known complex number. Thus, there are four distinct possibilities for $k$:
\[ k = \pm \exp \left( \frac{\theta}{2} e_1 \right), \quad k = \pm e_1 \exp \left( \frac{\theta}{2} e_1 \right) \]
where $\theta$ is the argument of $\omega_2 - \omega_3 t$. The controls $u_1$ and $u_2$ associated to one of these four trajectories are obtained by

$$e_3 \kappa \frac{d}{dt} YY^* k^* e_3 = (u_1 - \omega)e_1 + u_2 e_2$$

where $2\omega = \frac{d}{dt}\theta$ is given via simple algebraic formulae based on $\omega_2, \omega_3, \frac{d}{dt}\omega_2$ and $\frac{d}{dt}\omega_3$:

$$\omega = \frac{\omega_2 \frac{d}{dt}\omega_2 - \omega_2 \frac{d}{dt}\omega_3}{2(\omega_2^2 + \omega_3^2)}.$$

For the two branches $k = \pm \exp(\frac{\theta}{2} e_1)$ we get

$$\begin{align*}
  u_1 &= \omega_1 + \frac{\omega_3 \frac{d}{dt}\omega_2 - \omega_2 \frac{d}{dt}\omega_3}{2(\omega_2^2 + \omega_3^2)} \\
  u_2 &= \sqrt{\omega_2^2 + \omega_3^2}
\end{align*}$$

and for the two other ones $k = \pm e_1 \exp(\frac{\theta}{2} e_1)$ we get

$$\begin{align*}
  u_1 &= \omega_1 + \frac{\omega_3 \frac{d}{dt}\omega_2 - \omega_2 \frac{d}{dt}\omega_3}{2(\omega_2^2 + \omega_3^2)} \\
  u_2 &= -\sqrt{\omega_2^2 + \omega_3^2}
\end{align*}$$

where just the sign of $u_2$ is changed. All the previous computations are valid when $\omega_2 - \omega_3 t \neq 0$, i.e., when $\frac{d}{dt}y \neq 0$: $(\omega_2^2 + \omega_3^2)$ does not depends on $Y(t)$ such that $h(Y(t)) = y(t)$; it depends only on $y(t)$ and vanishes if, and only if, $\frac{d}{dt}y(t) = 0$. To summarize, we have proved the following result:

**Theorem 1:** Take $T > 0$ and an arbitrary $C^2$ curve $[0, T] \ni t \mapsto y(t)$ on $\mathcal{Y}$ such that $\frac{d}{dt}y(t) \neq 0$ for any $t \in [0, T]$. For any smooth curve $t \mapsto Y(t) \in \mathbb{H}_1$ such that $h(Y(t)) = y(t)$, set $z = \omega_2(t) - \omega_3(t) \neq 0$ for all $t \in [0, T]$ where $(\frac{d}{dt}Y) Y^* = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$. Then there exists a smooth function $[0, T] \ni t \mapsto \theta(t) \in \mathbb{R}$ such that $\exp(\theta(t)) = \frac{z}{|z|}$ and any smooth solution $t \mapsto (q(t), u_1(t), u_2(t))$ of (3) satisfying $h(q(t)) = y(t)$ for all $t \in [0, T]$ is one of the four following trajectories indexed by $n \in \{0, 1, 2, 3\}$:

$$\begin{align*}
  q(t) &= (e_1)^n \exp\left(\frac{\theta(t)}{2} e_1\right) Y(t) \\
  u_1 &= \omega_1 + \frac{\omega_3 \frac{d}{dt}\omega_2 - \omega_2 \frac{d}{dt}\omega_3}{2(\omega_2^2 + \omega_3^2)} \\
  u_2 &= (-1)^n \sqrt{\omega_2^2 + \omega_3^2}
\end{align*}$$

(7)
Recall that some \( Y(t) \) such that \( y(t) = h(Y(t)) \) is locally given by \( Y(t) = g(y(t)) \), and furthermore \( \dot{Y}(t) = g^{(1)}(y, \dot{y}) \) and \( \dot{Y}(t) = g^{(2)}(y, \dot{y}, \ddot{y}) \), where \( g, g^{(1)} \) and \( g^{(2)} \) are smooth maps. In particular, the last theorem proves that \( y = h(q) \) is a flat output.

The flat output \( y = h(q) \) is obtained with \( e_1 \) playing a specific role. In fact one can see that any map \( h_\eta : \mathbb{H}_1 \mapsto \mathbb{H}_1/K_\eta \ (\eta \in S^1) \) corresponding to the subgroup \( K_\eta = \exp(\mathbb{R}(\cos \eta e_1 + \sin \eta e_2)) \) also defines a flat output. It just corresponds to a rotation by the angle \( \eta \) of \( (q_1, q_2) \) and \( (u_1, u_2) \).

If we set
\[
e_1 = \cos \eta \tilde{e}_1 + \sin \eta \tilde{e}_2, \quad e_2 = -\sin \eta \tilde{e}_1 + \cos \eta \tilde{e}_2
\]
the imaginary quaternions \((e_1, e_2, e_3)\) and \((\tilde{e}_1, \tilde{e}_2, e_3)\) satisfy exactly the same commutation relations. Thus, if \( t \mapsto q(t) \) is a solution of \( (3) \) with the control \((u_1(t), u_2(t))\) then
\[
t \mapsto q_0(t) + (\cos \eta q_1(t) - \sin \eta q_2(t))e_1 + (\sin \eta q_1(t) + \cos \eta q_2(t))e_2 + q_3(t)e_3
\]
is also a solution of \( (3) \) with the control
\[
\tilde{u}_1 = \cos \eta u_1(t) - \sin \eta u_2(t), \quad \tilde{u}_2 = \sin \eta u_1(t) + \cos \eta u_2(t).
\]
This symmetry and the fact that, as stated in theorem \([1] \) \( h = h_0 \) is a flat output, implies directly that \( h_\eta \) is also a flat-output. The family \((h_\eta)_{\eta \in S^1}\) is made of flat outputs all compatible versus right translations.

### III. Motion planning

In this section, we will use \((7)\) with \( n = 0 \) to propose an explicit solution for the motion planning problem stated in the introduction: for any \( T > 0 \) and any final state \( \bar{q} \in \mathbb{H}_1 \), find a smooth control \( [0, T] \ni t \mapsto u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2 \) with \( u(0) = u(T) = 0 \), such that the solution \( [0, T] \ni t \mapsto q(t) \in \mathbb{H}_1 \) of \((3)\) starting from \( q(0) \) reaches \( \bar{q} \) at time \( T \): i.e., \( q(T) = \bar{q} \).

As the system is driftless, every time re-parameterization of a solution is also a solution. In fact, consider the equation
\[
\frac{d}{ds} \bar{q}(s) = (\tilde{u}_1(s)e_1 + \tilde{u}_2(s)e_2)\bar{q}(s)
\]
Let \( \zeta : [0, T] \to [0, 1] \) be an increasing diffeomorphism. Then \( \bar{q}(s) \) is a solution of the previous equation defined on \([0, 1]\), with input \((\tilde{u}_1(s), \tilde{u}_2(s))\) if and only if \( q(t) = \bar{q}(\zeta(t)) \) is a solution of \((3)\) defined on \([0, T]\) with input \((u_1(t), u_2(t)) = \frac{d \zeta}{dt}(\tilde{u}_1(\zeta(t)), \tilde{u}_2(\zeta(t)))\). One concludes that, without loss of generality, one may always state the motion planning problem with the (virtual)
time \( s \) belonging to the interval \([0, 1]\) and after that, one may “control the clock” by choosing a convenient bijection \( s = \zeta(t) \). Thus, it is enough to solve the motion planning problem in the \( s \) scale where we can disregard the fact that the control has to vanish at the beginning and at the end: it is enough to take for example \( \zeta(t) = 3 \left( \frac{t}{T} \right)^2 - 2 \left( \frac{t}{T} \right)^3 \) to get \( u \) equal to zero at \( t = 0 \) and at \( t = T \), since \( \frac{d}{dt} \zeta(0) = \frac{d}{dt} \zeta(T) = 0 \).

In the sequel we propose a solution in the \( s \)-scale. For clarity’s sake, we will remove the \( \tilde{} \) when \( u \) and \( q \) are considered as function of \( s \). The derivation in \( s \) will be denoted by ‘: \( du/ds = u' \), \( dq/ds = q' \), …

Thus, we have to find a smooth control \([0, 1] \ni s \mapsto u(s)\) such that the solution of
\[
q'(s) = (u_1(s)e_1 + u_2(s)e_2)q(s), \quad q(0) = 1
\]
satisfies \( q(1) = \bar{q} \), where \( \bar{q} \) is any goal state in \( \mathbb{H}_1 \).

We can always assume that
\[
\bar{q} = \bar{q}_0 + \sqrt{\bar{q}_1^2 + \bar{q}_2^2} (\sin \eta e_1 + \cos \eta e_2) + \bar{q}_3
\]
for some angle \( \bar{\eta} \in [0, 2\pi] \). Thus, as explained at the end of last section, up to a rotation of angle \( \bar{\eta} \) of the control, we can assume that \( \bar{q}_1 = 0 \). More precisely, if \( \bar{q}_1 \neq 0 \), set \( \bar{\eta} \) to be the argument of the complex \( \bar{q}_2 + \bar{q}_1 i \). If \( s \mapsto (u_1(s), u_2(s)) \) steers \( q \) from \( q(0) = 1 \) to \( q(1) = \bar{q}_0 + \sqrt{\bar{q}_1^2 + \bar{q}_2^2} e_2 + \bar{q}_3 e_3 \), then the control
\[
s \mapsto (\cos \bar{\eta} u_1(s) + \sin \bar{\eta} u_2(s), -\sin \bar{\eta} u_1(s) + \cos \bar{\eta} u_2(s))
\]
steers \( q \) from \( q(0) = 1 \) to \( q(1) = \bar{q} \).

Thus up-to a rotation of angle \( \bar{\eta} \) of the control, we can assume that \( \bar{q}_1 = 0 \) and \( \bar{q}_2 \geq 0 \). Thus we can define two angles \( \tilde{\alpha} \in [0, \pi] \) and \( \tilde{\beta} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) such that
\[
\bar{q} = \cos \tilde{\alpha} + \sin \tilde{\alpha} (\cos \tilde{\beta} e_2 + \sin \tilde{\beta} e_3).
\]
If the control \( s \mapsto u(s) \) steers the system from \( q(0) = 1 \) to \( q(1) = \cos \tilde{\alpha} + \sin \tilde{\alpha} (\cos \tilde{\beta} e_2 + \sin \tilde{\beta} e_3) \), the same control steers the system from
\[
q(0) = \cos \tilde{\lambda} + \sin \tilde{\lambda} (\cos \tilde{\beta} e_2 + \sin \tilde{\beta} e_3)
\]
to
\[
q(1) = \cos (\tilde{\lambda} + \tilde{\alpha}) + \sin (\tilde{\lambda} + \tilde{\alpha}) (\cos \tilde{\beta} e_2 + \sin \tilde{\beta} e_3).
\]
This is a direct consequence of right translation invariance of (3) and right multiplication by $\cos \bar{\lambda} + \sin \bar{\lambda} (\cos \bar{\beta} e_2 + \sin \bar{\beta} e_3)$.

Take now the formulae (7) in the $s$-scale with

$$Y(s) = \cos(\alpha(s)) + \sin(\alpha(s)) (\cos(\beta(s)) e_2 + \sin(\beta(s)) e_3)$$

where $\alpha(s)$ and $\beta(s)$ are smooth functions such that

$$\alpha(0) = \bar{\lambda}, \quad \alpha(1) = \bar{\lambda} + \bar{\alpha}, \quad \beta(0) = \beta(1) = \bar{\beta}.$$ (9)

Set, as in theorem 1

$$Y'Y^* = \omega_1(s) e_1 + \omega_2(s) e_2 + \omega_3(s) e_3.$$ (8)

Simple computations shows that

$$z = \omega_2 - \omega_3 i = \exp(-i\beta)(\alpha' - i\beta' \cos \alpha \sin \alpha).$$

Now we shall construct (8) such that $q(s) = \exp(\phi(s) e_1) Y(s), s \in [0, 1]$ is a trajectory of the system. We will assume that $q(0) = Y(0)$ and $q(1) = Y(1)$. So we must have $\phi(0) = \phi(1) = 0$. Furthermore, if we can ensure that $s \mapsto z(s)$ never vanishes, and $\theta(0) = \theta(1) = 0$, then the trajectory of (7) with $n = 0$ will provide a steering control $u$.

Let us now show in detail how to design the functions $\alpha(s)$ and $\beta(s)$ satisfying these constraints. First of all we have the initial and final constraints (9). By taking

$$\bar{\lambda} = \left\{ \begin{array}{ll}
-\frac{\bar{\alpha}}{2}, & \text{for } \bar{\alpha} \in \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right]; \\
\frac{\pi}{4} - \frac{\bar{\alpha}}{2}, & \text{otherwise;}
\end{array} \right.$$ we always have $\cos \alpha \sin \alpha$ far from 0 when $s = 0$ and $s = 1$. Thus we can impose the following initial and final constraints for $\beta'$:

$$\beta'(0) = -\frac{\alpha \sin \bar{\beta}}{\sin \bar{\lambda} \cos \bar{\lambda}} \quad \beta'(1) = -\frac{\alpha \sin \bar{\beta}}{\sin(\bar{\lambda} + \bar{\alpha}) \cos(\bar{\lambda} + \bar{\alpha})}$$ and for $\alpha'$

$$\alpha'(0) = \alpha'(1) = \cos \bar{\beta} \bar{\alpha}.$$ (10)

Then $\alpha(s)$ and $\beta(s)$ are the polynomials of degree $\leq 3$ satisfying these initial and final constraints. Since $\bar{\alpha} > 0$ and $|\bar{\beta}| \leq \frac{\pi}{2}$, $s \mapsto \alpha(s)$ can be a strictly increasing function on $[0, 1]$ and $\alpha' > 0$ for $s \in [0, 1]$ (see appendix B). Thus the complex number

$$z = \exp(-i\beta)(\alpha' - i\beta' \cos \alpha \sin \alpha)$$

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never vanishes for \( s \in [0, 1] \). For \( s = 0 \) and \( s = 1 \), we have

\[
\alpha' - i\beta' \cos \alpha \sin \alpha = \exp(i\bar{\beta})\bar{\alpha}.
\]

Thus \( z(0) = z(1) = \bar{\alpha} > 0 \). To summarize the closed path \([0, 1] \ni s \mapsto z(s) \in \mathbb{C} \) never passes through 0 nor turns around 0. We satisfy the assumption of theorem [I] in the \( s \)-scale. Moreover we can set \( z(s) = r(s) \exp(i\theta(s)) \) with \( r(s) > 0 \) and \( \theta(s) \) smooth functions on \([0, 1]\) with \( \theta(0) = \theta(1) = 0 \).

We avoid with such design of \( \alpha(s) \) and \( \beta(s) \) the monodromy problem associated to the resolution of \( (\exp(\varphi))^4 = x^2 / |z|^2 \). Finally we have proved the following result.

**Theorem 2:** Take \( \bar{q} = \bar{q}_0 + \bar{q}_1 e_1 + \bar{q}_2 e_2 + \bar{q}_3 e_3 \in \mathbb{H}_1 \) with \( \bar{q} \neq 1 \). Chose \( \bar{\eta} \in [0, 2\pi] \) such that \( q_1e_1 + q_2e_2 = \sqrt{\bar{q}_1^2 + \bar{q}_2^2}(\sin \bar{\eta}e_1 + \cos \bar{\eta}e_2) \). Define \( \bar{\alpha} \in ]0, \pi] \) and \( \bar{\beta} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) such that

\[
\bar{q}_0 + \sqrt{\bar{q}_1^2 + \bar{q}_2^2} e_2 + \bar{q}_3 e_3 = \cos \bar{\alpha} + \sin \bar{\alpha}(\cos \bar{\beta}e_2 + \sin \bar{\beta}e_3).
\]

Set \( \bar{\lambda} = -\frac{\bar{\alpha}}{2} \) if \( \bar{\alpha} \in \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right] \) and \( \bar{\lambda} = \frac{\pi}{2} - \frac{\bar{\alpha}}{2} \) otherwise. Define \( \alpha(s) \) and \( \beta(s) \) as being the unique polynomial functions of degree \( \leq 3 \) such that \( ' \) stands for \( d/ds \)

\[
\begin{align*}
\alpha(0) &= \bar{\lambda}, & \alpha(1) &= \bar{\lambda} + \bar{\alpha}, & \alpha'(0) &= \alpha'(1) = \bar{\alpha} \cos \bar{\beta} \\
\beta(0) &= \beta(1) = \bar{\beta} \\
\beta'(0) &= -\frac{\bar{\alpha} \sin \bar{\beta}}{\sin \bar{\lambda} \cos \bar{\lambda}}, & \beta'(1) &= -\frac{\bar{\alpha} \sin \bar{\beta}}{\sin(\bar{\lambda} + \bar{\alpha}) \cos(\bar{\lambda} + \bar{\alpha})}
\end{align*}
\]

Define \( \omega_1(s), \omega_2(s) \) and \( \omega_3(s) \) by

\[
\omega_1 = (1 - 2\cos^2(\alpha))\beta'
\]

\[
\omega_2 - i\omega_3 = \exp(-i\beta)(\alpha' - i\beta' \sin \alpha \cos \alpha).
\]

Then \( \omega_2 \) and \( \omega_3 \) never vanish simultaneously and the control

\[
\begin{pmatrix}
u_1(t) \\ u_2(t) \end{pmatrix} = \frac{d}{dt} \zeta(t) \begin{pmatrix} \cos \eta & \sin \eta \\ -\sin \eta & \cos \eta \end{pmatrix} \begin{pmatrix} \omega_1 + \frac{\omega_3}{2(\omega_2^2 + \omega_3^2)} \\ \sqrt{\omega_2^2 + \omega_3^2} \end{pmatrix}_{s = \zeta(t)}
\]

steers system (3) from \( q(0) = \bar{q} \) to \( q(T) = \bar{q} \) with \( t \mapsto \zeta(t) \in [0, 1] \) being a \( C^k \) increasing bijection between \([0, T] \) and \([0, 1] \) \( k \geq 1 \). When in addition \( \frac{d^n}{dt^n}|_{s = 0} = 0 \) for \( s = 0 \) and \( s = 1 \), and \( n = 1, \ldots, k \), the control \( t \mapsto u(t) \) is \( C^{k-1} \) with \( \frac{d^{n-1}}{dt^{n-1}}|_{s = 0} = 0 \) for \( s = 0 \) and \( s = 1 \).

Figure [I] illustrates the steering control described by theorem [II] with \( T = 2, \bar{q}_0 = e_3 \), and \( \zeta(t) = 3(t/T)^2 - 2(t/T)^3 \). We see that the control is a smooth function with maxima around
Fig. 1. The steering control and trajectory derived from theorem 2 with $T = 2$, $q = e_3$ and $\zeta(t) = 3(t/T)^2 - 2(t/T)^3$. The control magnitude is very close to an ZYZ control design with two separated $\frac{\pi}{2}$ pulses. The simulation code (matlab m-file and scilab sci-file) can be downloaded from http://cas.ensmp.fr/~rouchon/publications/PR2007/CodeMatlabScilabQubit.zip.

$\pi/2$, a value close to the ZYZ design based on two successive pulses: $(u_1, u_2) = (0, \frac{\pi}{2})$ for $t \in [0,1]$ and $(u_1, u_2) = (\frac{\pi}{2}, 0)$ for $t \in [1,2]$. Thus our flatness based design yields, with the same transition time and control magnitude, smooth control actions.

IV. CONCLUDING REMARKS

The results of this paper holds if the laser matches exactly the resonant frequency. If we have a frequency offset of $\Delta_r$ from resonance, then this offset leads to the following drift (see, e.g., [1]):

$$\frac{d}{dt}q = (u_1 e_1 + u_2 e_2 + \Delta_r e_3)q.$$ 

It is still interesting to notice that $h(q)$ is also a flat output. In this case, the key relation (4) becomes

$$e_3 k \left( \frac{d}{dt} Y \right) Y^* k^* e_3 = k \left( \frac{d}{dt} Y \right) Y^* k^* + 2\Delta_r e_3.$$
and $k = \exp(\phi e_1)$ is a root of the following polynomial

$$k^4(\omega_2 e_2 + \omega_3 e_3) + 2k^2 \Delta_r e_3 - (\omega_2 e_2 - \omega_3 e_3) = 0.$$  

Then one could try to apply similar techniques for solving the motion planning problem for this system, although the time-scale $s = \zeta(t)$ cannot be considered in this case.

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**APPENDIX**

*A – Pauli Matrices and Quaternions*

The Hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the three Pauli matrices. They satisfy $\sigma_k^2 = 1$, $\sigma_k \sigma_j = -\sigma_j \sigma_k$ for $k \neq j$, and

$$\sigma_1 \sigma_2 = i\sigma_3, \quad \sigma_2 \sigma_3 = i\sigma_1, \quad \sigma_3 \sigma_1 = i\sigma_2.$$
Any matrix $U$ in $SU(2)$ reads

$$U = q_0 - q_1 \sigma_1 - q_2 \sigma_2 - q_3 \sigma_3$$

with $(q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ such that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. By setting

$$e_1 = -i \sigma_1, \quad e_2 = -i \sigma_2, \quad e_3 = -i \sigma_3$$

one can identify $SU(2)$ with the set of quaternions

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$$

of length one. This set is denoted by $\mathbb{H}_1$ and corresponds to quaternions $q \in \mathbb{H}$ such that $qq^* = 1$ where $q^* = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3$ is the conjugate quaternion of $q$. Thus the dynamics (2) becomes (3) with $q$ corresponding to $U$. Notice that $\mathbb{H}_1$ is a compact Lie group of dimension 3.

We recall here some useful relations for $k = 1, 2, 3$, $j \neq k$ and $\phi \in \mathbb{R}$:

$$e_k^2 = -1, \quad e_k e_j = -e_j e_k, \quad \exp(\phi e_k) = \cos \phi + e_k \sin \phi$$

$$\exp(\phi e_k) e_j = e_j \exp(-\phi e_k)$$

$$e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2$$

**B – Proof that $z = \omega_2 - i \omega_3$ never vanishes for $s \in ]0, 1[$**

Since $\omega_2 - i \omega_3 = \exp(i\beta)(\alpha' - i\beta' \sin \alpha \cos \alpha)$, it suffices to show that $\alpha' > 0$ for $s \in ]0, 1[$. For this, let $\delta = \tilde{\alpha} - \alpha'(0) = \tilde{\alpha}(1 - \cos \tilde{\beta}) \geq 0$. A simple exercise shows that the polynomial $\alpha(s) = as^3 + bs^2 + cs + d$ meeting the restrictions $\alpha'(0) = \alpha'(1)$ and $\alpha(1) - \alpha(0) = \tilde{\alpha}$ is such that $a = -2\delta$, $b = 3\delta$, $c = \alpha'(0)$ and $d = \alpha(0)$. In particular $\alpha'(s) = -6\delta s(s - 1) + \alpha'(0)$. If $\cos \beta \neq 1$, then $-6\delta s(s - 1) > 0$, for $s \in ]0, 1[$. As $\alpha'(0) \geq 0$, then $\alpha' > 0$ for $s \in ]0, 1[$. If $\cos \tilde{\beta} = 1$, then $\delta = 0$ and $\alpha'(0) = \tilde{\alpha} \cos \tilde{\beta} > 0$. So $\alpha' > 0$ for $s \in [0, 1]$. 

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