Considerations regarding the pressures distribution on leads of spur gears

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Abstract. The standard methodology to verify gears’ strength to rolling contact fatigue (RCF) is to compare the value for maximum Hertz pressure with a permissible value. However some experimental evidences can not be explained by this approach. Therefore the TCA was focused on both the surface distribution of contact pressures and depth distribution of von Mises stresses. Semi-analytic methods SAM and its variant for linear contacts with finite length SAM-C have been involved to obtain the pressures distribution, contact area and stresses state inside the stressed volume. For theoretical case of coincident line contact there are no edge effect but any modification of the relative position changes it to a non-coincident contact type where the end effects manifest.

The spur gears with pinion teeth having a longitudinal crowned profile have indicated a good capacity to avoid the end edge effect when the operating conditions included a misalignment. For operating conditions without misalignment, the lead modification by end relief causes a quasi-Hertzian pressures distribution; however, the existence even of a small misalignment changes it to a skewed distribution. The semi-analytical procedure appears as an accurate and very fast tool to establish the suitable values for parameters that define the flank modification of gears.

1. Introduction

Under real running conditions the meshing is altered by various deviations of the machine parts involved but also by elastic deformations [1-3], which determine transmission errors and perturbations of pressures distributions along the flank, especially at its end sides. The pressures concentrations accelerate the rolling contact fatigue (RCF) and wear phenomena resulting shorter lives for the machine parts subjected to rolling contact. The RCF may have various causes: the misalignment, figure 1a, or is originating from shallow layer cracks at a depth of (2.5…5) µm [4-7]. Nowadays the maximum value of the normal stress $\sigma_{zx}$ is admitted as critical stress for RCF of gears [8]. This value $\sigma_H$ is attained on the contact surface $\sigma_H = \sigma_{zz - max} = \sigma_{zz}(0,0,0)$ and diminishes sharply with depth, figure 1a and therefore can’t sustain the subsurface originated RCF. Furthermore, the maximum Hertz pressure $\sigma_H$ does not evaluate the effect of some operating factors like the misalignment, figure 2. Alternative fatigue criteria consider the maximum shear stress $\tau_{45}$, or maximum von Mises stress $\sigma_{vM}$ figure 1b, as critical stress responsible for the initiation of RCF [6, 9-13]. The equation (1) of von Mises equivalent stress reflects the contribution of each component of the stress tensor.

$$\sigma_{vM}(x, y, z) = \frac{1}{\sqrt{2}} \left[ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2} \quad (1)$$

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The stress level for each elementary volume of the entire significant loaded volume is important for studies focused on reliability of rolling contacts and evaluation of the corresponding fatigue lives [10-13]. Ioannides and Harris [10] proposed a stress-based model for RCF with its roots in rolling bearings fatigue theory [6, 10]. For each infinitesimal volumetric element of significantly stressed volume the new approach takes into account the probability of survival $S$ after a number $N$ of cycles:

$$\ln S \sim N^e \iiint_V \frac{(\sigma_{eq} - \sigma_u)^c}{z^h} \, dV$$

(2)

where $\sigma_{eq}$ is the equivalent stress value in the infinitesimal volumetric element, $\sigma_u$ is the fatigue limit stress, $c$ is the stress exponent, $z'$ is the stress-weighted average depth, $h$ is the depth exponent, $e$ is the Weibull slope and $V$ is the volume significantly stressed. The material constants $\sigma_u$, $e$, $c$ and $h$ need to be determined experimentally. Popinceanu et. al [9] accomplished a comparative analysis of different components of stress tensor that have been hypothesized as the critical stress in RCF. The analysis concluded that von Mises equivalent stress, derived from the energetic hypothesis, assures the best correlation with various theoretical and experimental findings. The maximum value of von Mises stress is attained under the contact surface, but von Mises stress has also a significant value on the contact surface figure1b. The maximum operational value of von Mises stresses may be under the flank surface or on flank surface, depending on the operating conditions as: steel cleanliness, surfaces roughness, lubrication quality, contamination level [13]. Nowadays the dynamic rating loads of rolling bearings are calculated based on von Mises stress hypothesis as critical stress in RCF [6, 11]. Zhu D. et. al [12], used von Mises stress to predict the pitting life in mixed lubricated line contacts. To evaluate the fatigue life of rolling contacts Morales-Espejel et. al [13] used von Mises equivalent stress in a general model for the surface-subsurface competing RCF mechanism. Therefore, in this paper the TCA has been directed to obtain: the surface distribution of contact pressures, contact area as well as the depth distribution of von Mises stresses.

For spur and helical gears some techniques as flank crowning figure 2a or flank relieving, figure 2b, are commonly used to modify the initial line contact to a localized bearing contact [8, 14, 15, 23]. The normal load acts along the contact line and a shared contact area develops. The shape and size of
Figure 2. The pressures concentration (a) and the effect of flank crowning (b).

the shared contact area, as well as the pressures distribution on this area, are unknowns and consequently the entire elastic state, as determined by the concentrated contact loading, is unknown. The concentrated contacts in gears meshing are no longer of Hertz type and no straight analytical solutions is available [14, 15]. The finite element method (FEM) proved to be time consuming in applications involving concentrated contacts problems. Semi-analytical methods (SAMs) have been developed to find the pressures distribution, contact area and depth stresses state developed in non-Hertzian concentrated contacts [3, 16-23].

2. Semi-analytical modelling of non-Hertz concentrated contacts

2.1 The semi-analytical model

The model used in the present paper was developed under hypotheses of normal and dry contact loading. For conditions of mixed elastohydrodynamic lubrication (EHL) D. Zhu et al [19] noticed that the stress concentration due to pressure peaks are similar with those from dry contact solutions presented Hu Y. et. al [20]. It was demonstrated [21] that solutions for pressures distribution obtained in EHL conditions gradually approach those of dry contact.

2.1.1. The algebraic equations of surface deformations. The semi-analytical method uses the initial, no load, separations between surfaces of the conjugate teeth that sustain the meshing process. A virtual rectangular contact area \( A_v \), is built on the common tangent plane, around the initial contact line and a Cartesian system \((x, y, z)\) is introduced, the \(x-O-y\) plane being the common tangent plane, figure [3, 22].

A uniformly spaced rectangular array is built on the virtual rectangular contact area with the grid sides parallel to the \(x\) and \(y\)-axes. The nodes of the grid are denoted by \((i, j)\), where indices \(i\) and \(j\) refer to the \(N_x\) grid columns and \(N_y\) grid rows, respectively. The real pressure distribution is approximated by a virtual pressure distribution, characterized by different, unknown, constant values \(p_{ij}\) inside of each \((i,j)\) patch.

The surface deformation is modelled by the following six linear algebraic equations:

a) geometric equation of the elastic contact:
\[ g_{ij} = h_{ij} + w_{ij} - \delta_0 \] (3)

where: \( g_{ij} \) is the gap between the normal loaded surfaces, \( h_{ij} \) is the separation between unloaded surfaces, \( w_{ij} \) is the elastic deformation of the two surfaces, measured along the normal load, and \( \delta_0 \) is the rigid displacement of the contacting bodies;

b) equation of the normal surface displacement:

\[ w_{ij} = \sum_{k=0}^{N_x-1} \sum_{j=0}^{N_y-1} (K_{i-k,j-l} \cdot p_{kl}) \]

where: the influence coefficients function \( K_{ij} \) represents the value of the surface deformation created in the point \((i, j)\) by the unit pressure acting in the elementary rectangle \((k, l)\), and \( p_{k,l} \) is the unknown value of the pressure acting in the same elementary rectangle, [22].

c) load balance equation:

\[ \Delta x \Delta y \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} p_{ij} = F \]

d) constraint equation of non-penetration:

\[ g_{ij} = 0, \quad \text{yields} \quad p_{ij} > 0, \quad (i, j) \exists A_r \]

ej) constraint equation of non-adhesion:

\[ g_{ij} > 0, \quad \text{yields} \quad p_{ij} = 0, \quad (i, j) \not\exists A_r \]

f) elastic-perfect plastic behaviour of the material:

\[ p_{ij} > p_Y \implies p_{ij} = p_Y \]

where \( p_Y \) is the value of the pressure able to initiate the plastic yielding.

2.1.2. Method for solving the system of surface deformations. A single-loop conjugate gradient method was used to solve the mentioned algebraic system of equations [16, 17].
2.1.3. Components of the stress tensor. For the elastic domain the superposition principle allows to use the method of influence coefficients:

\[ \sigma_{ij}(x,y,z) = \sum_{k=0}^{N_x-1} \sum_{l=0}^{N_y-1} (C_{ijkl} \cdot p_{kl}) \]

where the influence coefficients function \( C_{ijkl}(x,y,z) \) describes the stress component \( \sigma_{ij}(x,y,z) \) due to a unit pressure acting in the patch \((k, l)\), [23]. To increase the efficiency of the numerical algorithm, a dedicated discrete fast Fourier transform routine for 3D contact problems was used to operate the convolution products [17, 23].

2.2. Pressures distribution in linear contacts with finite length.
Hertz’s relations for the linear concentrated contacts have been derived from elastic half-space theory for the particular contact domain represented by an infinite long rectangle so that in any transverse section there is a plane state of deformations. Any real meshing contact between two standard spur gears has a finite length and the plane states of deformation exist in the midsections but both end width faces are free boundaries. However Hertz’ s approach provides plane states of stress for both width faces of gear, that evidently is not realistic.

2.2.1. Hetenyi’s recursive theory. To eliminate the non-realistic presence of two shear stresses and one normal stress in the width faces. Hetenyi [24, 25] proposed:
- a correction based on the application of mirror pressures, figure 4, able to eliminate the two shear stress, and
- an iterative computation to obtain the necessary correction for the normal stress.
Theoretically the number of iterations goes to infinite but practically the finite number of iterations depends on the accuracy needed.

![Figure 4](image-url) The objective pressures distribution and Hetenyi’s virtual pressures distributions.

De Mul, Kalker and Frederiksson [29] applied Hetenyi’s algorithm with good results; however due to its iterative character it proved to be time consuming.

2.2.2. Guilbault’s correction factor. To speed up the algorithm for pressures distribution in 2011 Guilbault [35] proposed the multiplication of pressures of virtual distributions by a factor \( \psi_G \) able to correct, in the same iterations, the normal stress besides the two shear stresses, figure 5.

\[ \psi_G = 1.29 - \frac{1}{1 - \nu} (0.08 - 0.5\nu) \]

2.3. Model validation
For the same input data (contact geometry, normal loading, materials characteristics) the computerized program has been validated by comparisons of its output data with those obtained by neutral sources.
Figure 5. The objective pressures distribution and Guilbault’s virtual pressures distributions.

The first way for model validation was to use Hertzian contacts and to compare code’s output with data obtained analytically by authors [23, 25] or from open literature [26-28]. In each case the output provided by the developed code were the same with theoretical values at least three significant figures.

The second way to validate the developed model was to apply it to non-Hertzian concentrated contacts and to compare code’s output with the data obtained using the finite element method (FEM) [23, 28, 29]. Very good fitness was obtained when the mesh used in FEM was sufficiently fine, less than 5 \( \mu m \).

Regarding the algorithm developed for linear contacts with finite length, the FEM results presented in 2015 de Najjari M. and Guilbault R. [36] have been considered. For both cases: the coincident linear contact and non-coincident linear contact the data provided by the corrected semi-analytical method (SAM-C) presented a very good fit with data resulted from FEM, figure 6.

Figure 6. (a)- Pressures distributions along cylindrical roller generatrix obtained with SAM-C and by FEM [36] for coincident contact arrangement; (b)-detail.

Figure 7. (a)- Pressures distributions along cylindrical roller generatrix obtained with SAM-C and those provided by FEM [36] for non-coincident contact arrangement; (b)-detail.
3. Matrix of separations

Spur involute gears are in line contact at every instant and consequently the virtual rectangular contact area is built having the contact line as median line. The half-width of the virtual rectangular area has a value $b_v$ related to the value $b_H$ obtained considering the Hertz hypothesis. The target is to determine the value $h_{ij}$ of the initial separation in each mesh point $(i, j)$ of the virtual rectangle. Two fixed coordinate systems, $O_{1v1w1}$ and $O_{2v2w2}$, are attached to pinion and gear figure 4a, [22, 30].

The position of the shared contact point C is given by the pressure angle $\alpha_{c2}$ corresponding to point C2 on the tooth profile of the driven gear. On pinion tooth the contact point C1 has the pressure angle $\alpha_{c1}$:

$$\alpha_{c1} = \tan^{-1}\left[\frac{1}{r_{b1}} (r_{b1} + r_{b2}) \cdot \tan(\alpha_w) - r_{b2} \cdot \tan(\alpha_{c2})\right]$$  \hspace{1cm} (4)

For the contact point C2 the position coordinates are:

$$v_{C2} = r_{C2} \cdot \sin(\varphi_{C2}) = \frac{r_{b2}}{\cos(\alpha_{c2})} \sin(\varphi_{C2})$$

$$w_{C2} = r_{C2} \cdot \cos(\varphi_{C2}) = \frac{r_{b2}}{\cos(\alpha_{c2})} \cos(\varphi_{C2})$$  \hspace{1cm} (5)

Figure 8. Geometric elements-(a) and coordinate systems-(b), [22].
A point \( M_2 \) located at radius \( r_{M_2} \) on the same involute will have the position angle \( \varphi_{M_2} \) and the Cartesian data as follows:

\[
\varphi_{M_2} = \varphi_{C_2} - \left[ \text{inv}(\alpha_{M_2}) - \text{inv}(\alpha_{C_2}) \right]
\]

\[
v_{M_2} = r_{M_2} \cdot \sin(\varphi_{M_2}) = \frac{r_{b_2}}{\cos(\alpha_{M_2})} \sin(\varphi_{M_2})
\]

\[
w_{M_2} = r_{M_2} \cdot \cos(\varphi_{M_2}) = \frac{r_{b_2}}{\cos(\alpha_{M_2})} \cos(\varphi_{M_2})
\]

(6)

Two coordinates transformations are worked out: the first coordinates transformation is a translation from the fixed system \( O_{X'y'z'} \) to the mobile coordinates system \( v'c'w' \), figure 4b.

\[
v'_{M_2} = v_{M_2} - v_{C_2} = \frac{r_{b_2}}{\cos(\alpha_{M_2})} \sin(\varphi_{M_2}) - \frac{r_{b_2}}{\cos(\alpha_{C_2})} \sin(\varphi_{C_2})
\]

\[
w'_{M_2} = w_{M_2} - w_{C_2} = \frac{r_{b_2}}{\cos(\alpha_{M_2})} \cos(\varphi_{M_2}) - \frac{r_{b_2}}{\cos(\alpha_{C_2})} \cos(\varphi_{C_2})
\]

(7)

The second coordinate transformation is a rotation with the angle \( \theta \) needed to obtain the \( C_z \) axis along the common normal of the teeth surfaces; \( Cy \) axis is in the tangent plane of the contact point, figure 4b.

\[
\theta = - \left( \frac{\pi}{2} - \alpha_w \right)
\]

\[
y_{M_2} = v'_{M_2} \cdot \cos(\theta) + w'_{M_2} \cdot \sin(\theta)
\]

\[
z_{M_2} = -v'_{M_2} \cdot \sin(\theta) + w'_{M_2} \cdot \cos(\theta)
\]

(9)

In the mesh grid of virtual rectangular area the point \((i,j)\) is located at the distance \( y_j \) from the contact line. A mobile point \( M_2 \), with the position angle \( \varphi_{M_2} \) figure 4b, is considered on the cross-section profile of gear 2. The target is to find the pressure angle \( \alpha_{M_2} \) for which \( y_{M_2} = y_j \). For the gear 2 and point \( M_2(x_i, y_j) \), the pressure angle \( \alpha_{M_2} \) is found as solution of the equation:

\[
f(\alpha_{M_2}) = v'_{M} \cdot \cos(\theta) + w'_{M} \cdot \sin(\theta) - y_j = 0
\]

\[
f(\alpha_{M_2}) = \frac{r_{b_2}}{\cos(\alpha_{M_2})} \sin(\theta + \varphi_{M_2}) - \frac{r_{b_2}}{\cos(\alpha_{C_2})} \sin(\theta + \varphi_{C_2}) - y_j = 0
\]

(10)

Newton-Raphson iterative method provides the solution \( \alpha_{M_2} \):

\[
(\alpha_{M_2})_{k+1} = (\alpha_{M_2})_k - \frac{f(\alpha_{M_2})_k}{f'(\alpha_{M_2})_k}
\]

For the point \((i,j)\) situated on virtual contact area the equations (9) provide the value of initial separation \( h_{2_{ij}} \) between the tangent plane and flank surface of gear 2:

\[
h_{2_{ij}} = z_{M_2[i,j]}
\]

(11)

A similar procedure is used to obtain the initial separation \( h_{1_{ij}} \) between the same point \((i,j)\) situated on the tangent plane and the flank surface of gear 1:

\[
h_{1_{ij}} = z_{M_1[i,j]}
\]

(12)

The normal separation between the surfaces of the meshing teeth is:

\[
h_{ij} = h_{1_{ij}} + h_{2_{ij}}
\]

(13)
The misalignment as well as flank crowning or end flank relief are further included in the computing process for the matrix of initial separations. The developed software SAM and SAM-C use the separations matrix as the initial data in an iterative process to obtain the pressures distribution, contact area and contact stress state in the shallow layers of loaded teeth [22, 30].

4. Load distribution model
The numerical analyses have been performed for a standard spur gears with the following data: 
\[ \alpha_0 = 20^\circ, \ h_{20} = 1, \ c_0 = 0.25, \ \rho_0 = 0.38 \] [31].
\[ z_1 = 23, \ z_2 = 51, \ m_n = 4 \text{ mm} , \ x_1 = 0 , \ x_2 = 0, \ B = 32 \text{ mm}. \]
The nominal power parameters were: \[ T_2 = 15 \text{ kW}, \ \omega_2 = 53.77 \text{ rad/s}. \]
The load distribution varies continuously because the meshing stiffness of the pair teeth changes while the contact point runs along the line of action. In this study the non-uniform model of load distribution proposed by Pedrero José et al. [32, 33] has been used. The critical contact point might be the inner limit of the contact interval or the inner point of single pair tooth contact. Numerical results obtained with Hertz equations indicated the inner point B of single pair tooth contact as the point where the Hertz pressure \( \sigma_h \) attains the highest value, figure 5 and Table1. In figure 5 \( \epsilon \) is the transverse contact ratio, and \( \xi \) represents the profile parameter [32]. A point \( X \) situated on involute at radius \( r_X \) has the profile parameter \( \xi_X \) defined as:

\[ \xi_X = \frac{Z}{2\pi} \left[ \frac{r_X^2}{r_b^2} - 1 \right]^{1/2} \]

where \( Z \) is the number of teeth and \( r_b \) is base radius. For pinion involute the first point \( A \) engaged in meshing process has the profile parameter denoted \( \xi_{1\text{inn}} \).

In the following all simulations were performed using the data corresponding to the meshing process in point B. In conditions of Hertz assumptions outcomes: the Hertzian stress \( \sigma_h = 584.2 \text{ MPa} \) and half-width of Hertzian contact area \( b_H = 0.103 \text{ mm} \). Different values for the misalignment angle \( \psi \) between gears axes, values for crowned parameter \( C_\beta \) or \( c \) parameter of end relief, have been considered.

![Figure 9. Load sharing ratio.](image-url)
5. TCA of standard spur gears
The Hertzian model of line contact assumes an uniformly distribution of the normal load and a constant value for the maximum pressure $\sigma_H$ along the entire length of the contact line, figure 6. The standards for gears consider the pressure $\sigma_H$ as critical stress for pitting failure of the involute spur and helical gears and used to derive the needed design equations [8].

5.1 Standard spur gears - the edge effect
The phenomenon of pressures concentration at the end sides of the line contact with finite length is called edge effect [6, 14, 15]. For the case of a flank with sharp edges the high values of pressures developed in the end areas of the flank might induce plastic deformations able to change locally the contact geometry and to attenuate the edge effect [34]. On the other part real gears have a finite value for the flank width so that at the end part of the flank some corrections are needed.

5.2 Standard gears operating in ideal conditions - numerical analysis
The semi-analytic method SAM-C including the correction by virtual pressures and supra-correction factor $\psi_G$, has been used to obtain the pressures distributions, shape and size of the contact area. Comparisons with results provided by uncorrected SAM are presented. Standard spur gears, having a straight line lead (SL-lead), obtained with good manufacturing practice and operating in ideal conditions, were first considered.

![3D Pressures distribution](image)

**Figure 10.** 3D pressures distribution for the case of coincident flanks ($\psi = 0$).

### Table 1. Values of Hertz pressure on the involute profile.

| Meshing point | Unit | A   | B   | C   | D   | E   |
|---------------|------|-----|-----|-----|-----|-----|
| $r_{X1}$      | mm   | 43.556 | 45.232 | 46 | 46.508 | 50 |
| $r_{X2}$      | mm   | 106 | 102.849 | 102 | 101.52 | 99.18 |
| Load sharing ratio |     | 0.33 | 1 | 1 | 1 | 0.33 |
| Normal load $F_n$ | N   | 960 | 2910 | 2910 | 2910 | 960 |
| Hertz pressure $\sigma_H$ | MPa | 481 | **584** | 556 | 543 | 296 |
5.2.1. Pressures distribution for coincident flanks. The 3D pressures distribution is presented in figure 10 and 2D distribution is depicted in figure 11. In Figure 12 are summarized similar results obtained using the SAM program without finite length corrections. To avoid the cumbersome calculations needed to evaluate the edge effect in standard spur gears, the present algorithm considered, for both sides of the flank, a chamfer with the radius $R_{ch}=0.2$ mm. The SAM program led to a pressures distribution with a peak at each end, figure 12. The magnitude of the pressure peak becomes higher for finer mesh. The correction introduced in SAM-C program led to smooth pressures distributions with no peak, figure 10 and figure 11.

5.2.2. Distribution of von Mises stresses. Some of plasticity criteria consider the value of von Mises stress as responsible for starting the material yielding inside the stressed volume [23], and for the initiation of the rolling contact fatigue [7, 12, 19]. The von Mises stresses distribution has been evaluated in two planes: the longitudinal plane xoz ($y=0$) figure 13 and the transverse plane yOz ($x=0$) figure 14. The longitudinal distribution exposes stress concentrations under the contact surface at the end edge.
zones. The figure 13 comprises also a detailed presentation of this zone where the maximum value of von Mises stress is \( \sigma_{vM} = 0.88 \sigma_H \).

The basic equation (2) points out the non-linear dependence between the probability of survival \( S \) and stress level of each infinitesimal elementary volume. The edge effects manifest locally and determine sharp diminishes for probabilities of survivals at a relative small percent from the total number of elementary volumes, but which have a harmful contribution to gear reliability.

The transverse distribution (plane yOz, \( x=0 \)) figure 14, reveals a symmetrical distribution versus Oz axis (there is no surface traction) with the maximum value \( \sigma_{vM} = 0.539 \sigma_H \) that is the value of maximum von Mises stress developed in a Hertzian line contact having a maximum Hertz pressure \( p_{yy0} \) [6, 22]:

\[
\sigma_{vM} = 0.539 \sigma_H = 0.539 \frac{p_{yy0}}{0.97} = 0.556 p_{yy0}
\]

**Figure 13.** Straight line lead, plane \( y=0 \): distributions of von Mises stresses, \( (\psi = 0) \).

**Figure 14.** Straight line lead, plane \( x=0 \): distributions of von Mises stresses \( (\psi = 0) \).
5.3 Standard gears operating in conditions of misalignment
The edge effect becomes much damaging when even a small misalignment exists.

5.3.1. Pressures distribution. Figure 15 and figure 16 present the pressures distribution obtained with SAM-C by addition a misalignment of \( \psi = 0.5 \text{ min.} \) to the operating data. As effect, the maximum value of pressure increased from \( 1.0064 \sigma_H \) to \( 1.32 \sigma_H \) but again the pressures distribution is smooth with no peak. The calculation with SAM without correction gave the results presented in figure 16. The results presented in figure 15 and figure 16 point out the strong and harmful influence of misalignment on pressure distribution.

![3D Pressures distribution](image)

![Longitudinal pressures distributions](image)

**Figure 15.** (a)-3D and (b)-2D pressures distributions for misalignment conditions \( \psi = 0.5 \text{ min.} \)

5.3.2. Von Mises stresses distribution. The maximum value of von Mises stress increased from \( 0.877 \sigma_H \) figure 13, to \( 1.3 \sigma_H \) figure 17.
Figure 16. Straight line lead, plane y=0: distributions of pressures ($\psi = 0.5 \text{ min.}$).

Figure 17. Straight line lead, plane y=0: distributions of von Mises stresses, ($\psi = 0.5 \text{ min.}$).

Figure 18. Straight line lead, contact area: (a) - operating without misalignment; (b) - operating with misalignment $\psi = 0.5 \text{ min.}$.
5.3.3. Evolution of the contact area. For operating conditions without misalignment, the contact area has a quasi rectangular shape with $2b$ width and which is broader at the ends ("dog bone" shape [14]) figure 18a. The shape changes to a trapezoidal or even a triangle when the gear is operating with a misalignment between axes, figure 18b.

5.4. Design solutions to attenuate the edge effect. Modified spur gears are commonly used to avoid the edge effect. For spur and helical gears some techniques as flank crowning figure 19 or flank end relief figure 20, are used to modify the initial line contact to a localized bearing contact [8, 14].

![Figure 19. Flank with longitudinal crowned profile.](image1)

![Figure 20. Flank with end relief profile.](image2)

6. Pinion modification by longitudinal crowning of flank profile
The parameter of crowned flank is represented by the end relief $C_{\beta}$ [8].

6.1 Modified spur gears with longitudinal crowned flank profile, case $\psi = 0$
For $C_{\beta} > 0$ and $\psi = 0$ the contact between the mating teeth is transformed into a point contact type, located evenly along the flank. The contact area is an elongated ellipse with the greater half-axis along the lead.

6.1.1. Pressure distribution. The pressure distribution is a quasi-ellipsoid and no edge effect develops if the value $C_{\beta}$ was well correlated with the normal load, figure 21. The maximum pressure $p_{\text{max}} = 1.28\sigma_H$ is located in middle of the ellipse representing the contact area. Comparing with the elliptic-cylindrical distribution developed by gears with straight line lead, the ellipsoidal distributions of contact pressures presented in figure 21 reveals a larger contact zone having pressures greater than $\sigma_H$ value.
An increase of the value for $C_{\beta}$ determines a reduction of the contact area and consequently increases for the surface pressures and von Mises stresses.
Figure 21. Pinion with crowned flank: pressures distributions \((C_\beta = 5 \, \mu m, \, \psi = 0.)\).

Figure 22. Pinion with crowned flank, plane \(y=0\): von Mises stresses distribution \((C_\beta = 5 \, \mu m, \, \psi = 0.)\).

6.1.2. Depth distribution of von Mises stresses. The maximum value of von Mises stress is \(\sigma_{vM-max} = 0.712\sigma_H\) figure 22, being lower than the maximum value \(0.88\sigma_H\) developed for the same operating conditions by gears having straight line lead, figure 8.

Figure 23 presents the pressures and von Mises stresses distributions for the case of meshing between a pinion having crowned profile with \(C_\beta = 20 \, \mu m\), and the conjugate gear with straight line lead profile. Comparing with the case presented in figure 21 the maximum value of pressure increased from \(p_{max} = 1.28\sigma_H\) to \(p_{max} = 1.65\sigma_H\).

A similar increase registered the maximum values of von Mises stress: from \(\sigma_{vM-max} = 0.712\sigma_H\) figure 22, to \(\sigma_{vM-max} = 0.924\sigma_H\) figure 23.
The crowned flank eliminates the edge effect but increases the percent of elementary volumes with the von Mises values over the ultimate stress $\sigma_u$, that might led to a lower probability of survival for the entire gear. On the other hand a higher crowning is needed when the gears operating conditions include a certain value for the misalignment.

6.2. Modified spur gears with longitudinal crowned flank profile, case $\psi > 0$.

6.2.1 Pressures distributions. When the operating conditions included a misalignment $\psi = 0.5$ min, the usage of a small value for the lead parameter $C_\beta = 5 \mu m$, led to a skewed pressures distribution figure 24 a smaller contact area and an emerging edge effect.

Figure 23. Pinion with crowned flank, plane $y = 0$: distributions of pressures and von Mises stresses, ($\psi = 0$, $C_\beta = 20 \mu m$).

Figure 24. Pinion with crowned flank: distributions of pressures ($\psi = 0.5$ min, $C_\beta = 5 \mu m$).
A good correlation of the value for crowning parameter $C_\beta$ with operating values of misalignment and radial load assures improved distributions for both the contact pressures and von Mises depth stress. For a large misalignment as $\psi = 3$ min., a crowned flank with the parameter $C_\beta = 20 \text{ \mu m}$ changes the location of pressures distribution figure 25, preserving the shapes and values from the case $C_\beta = 20 \text{ \mu m}$ and $\psi = 0$.

![Graph](image)

**Figure 25.** Gears with crowned pinion lead, plane $y=0$: distributions of pressures and von Mises stresses ($\psi = 3.0 \text{ min.},$ and $C_\beta = 20 \text{ \mu m}$).

6.2.2 Contact areas. For some cases presented in subsection 6.2.1, the corresponding contact areas are comparatively presented in figure 26. The figure 26 points out that a good value for the crowning parameter $C_\beta$ creates the contact area as a very long ellipse with one end of the greater axis located at gear’s width end, figures 26a, 26c, 26d. A too small value of crowning parameter $C_\beta$ produces a partial ellipse, figure 26b that indicates a pressures concentration due to the edge effect. A too high value of crowning parameter $C_\beta$ produces a contact area as an ellipse with its greater axis smaller than flank’s width, figure 26c but with an increased value for the maximum pressure.

7. Flank modification by end relief

The end relief is defined by parameters $b_{I(II)}$ and $c$, figure 20, representing the length of the modified flank and end relief of profile, respectively [8]. In this numerical analysis, the parameter $b_{I(II)}$ was maintained constant, $b_{I(II)} = 0.1B$ and the parameter $c$ was variable.

7.1. Modified spur gears by end relief of flank profile, case $\psi = 0$

7.1.1. Pressure distribution. The pressures distribution is close to an elliptic cylinder and weak edge develops if the value of parameter $c$ was correlated with the normal load, figures 27 and figures 28.
Figure 26. Gear with crowned pinion flank - the dependence of shape and position of the contact area on values of misalignment and $C_\beta$ parameter: (a) $\psi = 0$, $C_\beta = 5 \mu$m; (b) $\psi = 0.5$ min, $C_\beta = 5 \mu$m; (c) $\psi = 0$, $C_\beta = 20 \mu$m; (d) $\psi = 3$ min, $C_\beta = 20 \mu$m.

Over the most part of the contact area the pressures distribution is very close to the corresponding Hertzian distribution. For operating condition without misalignment between gears axes the longitudinal modification of pinion’s lead by end relief provides smaller values for the maximum pressure in comparison with the values provided by the longitudinal crowning of the tooth profile.

Figure 27. Pinion with end relief lead: distributions of pressures ($\psi = 0$ min., and $c = 5 \mu$m).
Figure 28. Pinion with end relief lead, plane y=0: distribution of von Mises stresses ($\psi = 0$ mm, and $c = 5$ μm).

7.2. Modified spur gears with end relief of flank profile, case $\psi > 0$

7.2.1 Distribution of pressures. When the operating conditions included a misalignment $\psi = 0.5$ mm, the pressures show a skewed distribution with a slight edge effect figure 29. With a value $c = 5$ μm for the end relief of pinion’s teeth and no misalignment the maximum pressure was $p_{\text{max}} = 1.14\sigma_H$, figure 27; when to operating conditions a misalignment $\psi = 0.5$ mm. was added the maximum value of pressures increased to $p_{\text{max}} = 1.45\sigma_H$, figure 29.

Figure 29. Pinion with end relief lead; distribution of pressures, ($\psi = 0.5$ mm, and $c = 5$ μm).
7.2.2. Distribution of von Mises stresses. The skewed distribution of pressures generates a lop-sided depth distribution of von Mises stresses, with the maximum value at an abscissa close to that of the maximum pressure, and below the contact surface, figure 30. For higher values of misalignment, the contact area is diminished drastically together with a severe increase of the maximum pressure. Figure 31 points out this damaging evolution for a pinion with an end relief value of \( c = 20 \) \( \mu \)m meshing with a standard gear under an axes misalignment of \( \psi = 3 \) min.

Figure 30. Pinion with end relief lead: distribution of von Mises stresses (\( \psi = 0.5 \) min., and \( c = 5 \) \( \mu \)m).

Figure 31. Pinion with end relief lead: distributions of pressures (\( \psi = 3 \) min., and \( c = 20 \) \( \mu \)m).

8. Conclusions
To derive the necessary design equations of spur and helical gears the standards assumed the maximum value of Hertzian contact pressure \( \sigma_H \) as the critical stress for rolling contact fatigue (RCF). This assumption might explain the surface origin of failures but is not valid for subsurface originated RCF. The assumption of maximum von Mises stress as critical stress can explain both the sub-surface and surface origins of RCF failures.

The paper’s purpose was to perform comparative analyses regarding the distribution of contact pressures, shape and size of contact area and distributions of von Mises stresses for both standard spur gears and modified spur gears.

The concentrated contacts in gears meshing are no longer of Hertz type and no straight analytical solutions is available to provide the elastic stress state. Semi-analytic methods SAM and its variant for linear contacts with finite length SAM-C have been developed to obtain the pressures distribution, contact area and stresses state inside the stressed volume.
For theoretical case of coincident line contact there are no edge effect but any modification of the relative position changes it to a non-coincident contact type where the end effects manifest.

The elements for the matrix of separations between the surfaces of the conjugate teeth have been obtained analytically considering the separation between teeth involutes, modification of pinion’s flank and value of misalignment.

The influence coefficients derived from Boussinesq integral equation have been used to obtain the pressures system of algebraic equations. The iterative method of conjugate gradients (CGM) with fast Fourier technique (FFT) was used to obtain the pressures distributions and contact areas. Further, in any point of the half-space the stress tensor components have been obtained by convolution using the influence coefficients based on Love’s equations for the half-space subjected on its boundary to a uniform pressure over a rectangular area.

The analyses of gears with straight line lead for both the pinion and driven gear, pointed out the harmful edge effect and necessity for suitable flank modifications. The teeth modifications were applied to pinion only.

Spur gears with pinion teeth having a crowned lead make an ellipsoidal distribution of pressures on an elliptical contact area. These gears have indicated a good capacity to avoid the end edge effect when the operating conditions included the misalignment between gears axes.

For operating conditions without misalignment, the spur gears with modified flank by end relief of longitudinal profile generate a pressures distribution close to the Hertz distribution for line contact. The presence of misalignment creates a skewed pressures distribution with severe pressures concentrations at one end of the tooth flank.

The teeth contact analysis is useful when the appropriate values for parameters that define the modified spur gears are required.

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