Multiple Half-Quantum Vortices in Rotating Superfluid $^3$He

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Half-quantum vortices and ordinary vortices in a rotating thin film superfluid $^3$He under a strong magnetic field are considered. It is shown that $2n + 1$ half-quantum vortices interpolate between $n$ singular vortices and $(n+1)$ singular vortices as the angular velocity is changed. The phase diagram of the vortex configurations in the angular velocity-magnetic field space is obtained for a paramagnon parameter $\delta = 0.05$.

Superfluid $^3$He exhibits extremely exotic and interesting properties due to its complex order parameter $[1-3]$, which attracted much attention not only of condensed matter physicists but also of particle theorists and gravitational physicists. One of the manifestations of such exotic properties is a vortex having a half amount of vortex quantum called a half vortex quantum, abbreviated as HQV hereafter, whose existence was predicted first by Volovik and Mineev in 1976 $[4, 5]$. A HQV is also expected to be present in BEC of alkali atoms $[6-8]$ and spin-triplet superconductors $[9-11]$, among other physical systems. In spite of extensive theoretical $[12-14]$ and experimental $[15, 16]$ research on HQV in superfluid $^3$He since then, its existence is yet to be experimentally confirmed.

Recently, we investigated a rotating superfluid $^3$He in a slab geometry under a strong magnetic field $[17]$, in which we have shown that a HQV is energetically stable compared to a singular vortex (SV) in the $A_2$-phase side (i.e., lower temperature side) in the vicinity of the $A_1$-$A_2$ phase boundary. In this part of the phase diagram, a HQV will nucleate first as the angular velocity of the rotation is increased from zero. Let us summarize our results obtained in $[17]$ to establish notations and convention. Consider a rotating thin film of superfluid $^3$He in a cylindrical slab geometry under a strong magnetic field $H$. In the presence of a magnetic field, the superfluid has different populations between the spin up-up $(\uparrow\uparrow)$ condensate and the spin down-down $(\downarrow\downarrow)$ condensate, where the spin direction is measured with respect to the magnetic field. This phase is called the $A_2$ phase. The angular velocity $\Omega$ is parallel to the $z$-axis and the film is perpendicular to the $z$-axis. The magnetic field is taken parallel to the rotation axis. The thickness and the radius of the film are denoted by $d$ and $R$, respectively, where $d$ must be less than the dipole coherence length so that the $d$-vector stays in the $xy$-plane throughout the condensate. We use the Ginzburg-Landau free energy $[1]$, in which the correction of the fourth order coefficients $\beta_i$ of the bulk free energy by the paramagnon parameter $\delta$ is taken into account, to find the most stable vortex configuration for given parameters $\delta$, $\Omega$ and $H$.

Instead of expanding the order parameter in terms of the standard Cartesian base $\{e_i\} = \{e_x, e_y, e_z\}$, we expand it in terms of $\{e_\pm\} = \{e_\pm, e_0\}$ base defined by

$$e_\pm = \pm \frac{1}{\sqrt{2}} (e_x \pm i e_y), \quad e_0 = e_z.$$

The boundary condition $i = \pm \hat{z}$ forces $A_{\nu\pm}$ have non-vanishing values in the bulk. Here the first subscript of $A$ is the spin index while the second one is the orbital index. A strong magnetic field along the $z$-axis further forces the order parameter to have only four nonvanishing components $A_{\pm\pm}$ in the bulk. Let $l = 1 - T/T_c$, $T_c$ being the critical temperature, and $\alpha = \alpha'$, where $\alpha$ is the coefficient of the second order term of the bulk free energy and define the scaled magnetic field $h$ by $h = \eta H/\alpha'$, where $\eta$ is a constant coupling strength between $H$ and the condensate. It turns out to be convenient to further scale $h$ as $\hat{h} = h/t$. The bulk order parameter is found by minimizing the uniform Ginzburg-Landau free energy. We assume the vortex is embedded in a $\hat{l} = \pm \hat{z}$ texture, for concreteness, and the order parameter has only nonvanishing components $A_{\pm\pm}$ at $r \gg 1$, where the length is scaled by the coherence length with vanishing external magnetic field. We parameterize the components as $A_{\mu\nu} = \mu_\nu(r)e^{i n_{\mu\nu}\phi}$ assuming the cylindrical symmetry, where $\phi$ is the azimuthal angle in the $xy$-plane and $n_{\mu\nu} \in \mathbb{Z}$. It turns out that $n_{\mu\nu}$ satisfy the quantization condition $n_{-\nu} = n_{\mu+} + 2$ due to the coupling between $A_{\mu+}$ and $A_{\mu-}$ through the gradient free energy $[17]$.

When the HQV order parameter is expanded in $\{e_\nu\}$, it is found that the order parameter is a superposition of a $(\uparrow\downarrow)$ condensate with no winding number and a $(\downarrow\uparrow)$ condensate with a unit winding number or the other way around. Such a HQV has a free energy

$$F_{\text{HQV}}^{(\pm)} = 2\pi \int_0^R rdr (F - F_0) = 4\pi \left( A_{\pm\pm}^{(0)} \right)^2 \left( \ln R + C_{\pm} \right),$$

where $F_0$ is the bulk free energy without a vortex. Here $A_{\pm\pm}^{(0)}$ stands for the amplitude of the bulk order parameter with orbital state $l_z = \pm 1$ of the up-up $(\uparrow\uparrow)$ or the down-down $(\downarrow\downarrow)$ spin condensate, while $C_{\pm}$ is the vortex core energy of the $(\pm)$ condensate. The parameters $C_{\pm}$ are obtained as functions of the paramagnon parameter $\delta$ (i.e., the pressure) and the external magnetic field.
valid and energies of the vortices in the (+)-condensate is satisfied for all pairs $i$. We fix the pressure so that making use of the system with more than one vortex is evaluated by terns of the stable configurations of these vortices. We increased from zero.

The parameter $C_+$ is scaled by $\hbar/2mR^2$. Figure 1 depicts the parameters $C_{\pm}$ and $C_S$ for the paramagnon parameter $\delta = 0.05$, corresponding to low pressure, as functions of $\hat{h}$. Observe that when $\hat{h} > 0$, there is a range in the diagram where $C_S > C_-$, which implies that a single HQV, having a phase factor $e^{i\phi}$ in the ($-$) condensate, nucleates first as $\Omega$ is gradually increased from zero.

In this Letter, we consider the case in which the angular velocity $\Omega$ is further increased to investigate how many HQVs and SVs exist in the superfluid and the patterns of the stable configurations of these vortices. We again assume that the $\hat{t} = +\hat{z}$ at $r \gg 1$. The free energy of the system with more than one vortex is evaluated by making use of $C_{\pm}$ and $C_S$ numerically obtained in Fig. 1. We fix the pressure so that $\delta = 0.05$ for numerical calculations throughout this Letter. Let $r_i$ be the position of the $i$th vortex center. When the condition $|r_i - r_j| \gg 1$ is satisfied for all pairs $i \neq j$, the London approximation is valid and energies of the vortices in the (+)-condensate and that of the vortices in the ($-$)-condensate may be evaluated independently since the coupling between two condensates appears only through the gradient free energy. The hydrodynamic energy associated with the flow around vortices has been evaluated previously for superfluid $^4$He [18].

Suppose the number of vortices $n$ satisfies $n \leq 5$. Then the vortices distribute uniformly on a circle with the radius $r \approx \sqrt{(n-1)/2\Omega}$ centered at the origin of the cylinder, where $\Omega$ is scaled by $\hbar/2mR^2$ as before. Then the free energy of $n$ SVs in the $A_2$ phase of superfluid $^3$He in the rotating frame takes the form

$$F_n^{(S)}(\Omega, u) = 4\pi \left[ \left( A_{++}^{(0)} \right)^2 + \left( A_{--}^{(0)} \right)^2 \right] F_n(\Omega, u), \quad (3)$$

where $u = r/R$ and

$$F_n(\Omega, u) = n[\ln R + C_S + \ln(1 - u^{2n})] - (n - 1)\ln u - \ln(n - \Omega(1 - u^2)]. \quad (4)$$

Here the core energy of the vortices has been taken into account in the definition of $F_n$. It has been shown that the function $F_n(\Omega, u)$ has a minimum at $u$ in the physical region $(0, 1)$ when $\Omega$ is greater than some critical value $\Omega_0(n)$ [18]. Let

$$f_n(\Omega) = \min_{u \in (0, 1)} F_n(\Omega, u) \approx n[\ln R + C_S - \Omega] + \frac{1}{2}n(n - 1)[1 + \ln(2\Omega) - \ln(n - 1)] \quad (5)$$

be the minimum value, where the approximate value $u \approx \sqrt{(n-1)/2\Omega}$ has been used. This approximation is verified numerically to be quite accurate in the given parameter range when $n \geq 2$. Then the energy of the stable configuration of $n$ singular vortices is given by

$$F_n^{(S)}(\Omega) = 4\pi \left[ \left( A_{++}^{(0)} \right)^2 + \left( A_{--}^{(0)} \right)^2 \right] f_n(\Omega) \quad (6)$$

for $n \leq 5$.

Next, let us consider the case in which $n \geq 6$. It was shown for superfluid $^4$He that a stable configuration for $6 \leq n \leq 8$ is $n-1$ vortices distributing uniformly in a circle centered at the origin plus a single vortex at the origin [18]. Vortex configuration with less symmetry is expected as the angular velocity is further increased beyond $n = 8$. These patterns are verified both experimentally [19] and by numerical simulation [20]. When $6 \leq n \leq 8$, $F_n(\Omega, u)$ in Eq. (3) takes the form [18]

$$F_n(\Omega, u) = n[\ln R + C_S] + (n - 1)[\ln(1 - u^{2n})] - n\ln u - \ln(n - 1) - \Omega(1 - u^2)] - \Omega. \quad (7)$$

With the same approximation employed to obtain Eq. (5), the free energy minimizing configuration is given numerically [17]. The inequality $C_- < C_+$ is always satisfied since the coherence lengths $\xi_{\pm}$ of the condensates ($\pm$) satisfy $\xi_+ < \xi_-$, form which we find a HQV, carrying a vortex of unit winding number in the ($-$) condensate has less energy compared to that with a vortex in the ($+$) condensate. Let $L^{(-)} = 4\pi(2m/\hbar)(A_{++}^{(0)})^2R^2$ be the angular momentum of the system. By considering the free energy $F^{(-)} - \Omega L^{(-)}$ in the rotating frame, we find that a HQV nucleates at $\Omega = \xi_+^{(-)} = \ln R + C_-$, where $\Omega$ is scaled by $\hbar/2mR^2$.

In a SV, both ($+$) and ($-$) condensates carry a vortex with a unit winding number and these components are superposed so that the vortex cores overlap exactly. A SV has the free energy

$$F_{SV} = 4\pi \left[ \left( A_{++}^{(0)} \right)^2 + \left( A_{--}^{(0)} \right)^2 \right] (\ln R + C_S). \quad (2)$$

The parameter $C_S$ is the singular vortex core energy and is a function of $\delta$ and $\hat{h}$. A singular vortex nucleates at a critical angular velocity $\Omega_S = \ln R + C_S$.

FIG. 1: (Color online) $\hat{h}$-dependence of parameters $C_+$ (red), $C_-$ (blue) and $C_S$ (black). See Eqs. (1) and (2) for definitions of these parameters. The paramagnon parameter $\delta$ is set to 0.05.
by $u \approx \sqrt{n/2\Omega}$, which gives the minimum energy

$$f_n(\Omega) = \min_{u \in (0,1)} F_n(\Omega, u)$$

$$= n(\ln R + C_S - \Omega) - (n - 1) \ln(n - 1)$$

$$+ \frac{1}{2}(n - 1)[1 + \ln(2\Omega) - \ln n].$$

Similarly the free energies of $n$ HQVs with vortices in (+)-condensate and (−)-condensate are evaluated as

$$F_n^{(\pm)}(\Omega) = 4\pi \left( A_{\pm}^{(0)} \right)^2 (f_n(\Omega) + n\Delta C_{\pm}),$$

where $\Delta C_{\pm} = C_{\pm} - C_S$. Equation (9) applies to cases in which $2 \leq n \leq 8$.

It is expected that HQVs appear in the vicinity of the parameter domain where the free energy difference between the $n$ singular vortices and $n+1$ singular vortices is small. We expect there are $n$ HQVs with vortices in the (+)-condensate and $n+1$ HQVs with vortices in the (−)-condensate in the process of the transition from $n$ singular vortices to $n+1$ singular vortices. We denote this configuration of HQVs as $\text{HQV}(n+1, n)$, while a configuration with $n$ singular vortices is denoted as $\text{SV}(n)$. In a sense, $\text{HQV}(n+1, n)$ is roughly regarded as “$\text{SV}(n+1/2)$”. Note that the number of HQVs with vortices in (−)-condensate is larger than that of HQVs with vortices in (+)-condensate due to the inequality $C_{-} < C_{+}$ (see Fig. 1); it is energetically favorable to have an extra vortex in the (−)-condensate rather than in the (+) component.

The conditions under which $\text{HQV}(n+1, n)$ is stabilized are

$$F_{n+1}^{(-)}(\Omega) + F_n^{(+)}(\Omega) < F_n^{(S)}(\Omega)$$

and

$$F_{n+1}^{(-)}(\Omega) + F_n^{(+)}(\Omega) < F_{n+1}^{(S)}(\Omega)$$

simultaneously. More explicitly, these conditions are written as

$$\left( A_{-}^{(0)} \right)^2 (f_{n+1}(\Omega) - f_n(\Omega))$$

$$+ \left( A_{+}^{(0)} \right)^2 n\Delta C_+ + \left( A_{-}^{(0)} \right)^2 (n + 1)\Delta C_- < 0$$

and

$$- \left( A_{+}^{(0)} \right)^2 (f_{n+1}(\Omega) - f_n(\Omega))$$

$$+ \left( A_{+}^{(0)} \right)^2 n\Delta C_+ + \left( A_{-}^{(0)} \right)^2 (n + 1)\Delta C_- < 0,$$

from which the necessary condition for the existence of a stable $\text{HQV}(n+1, n)$ configuration is found to be

$$\Delta F_n \equiv \left( A_{+}^{(0)} \right)^2 n\Delta C_+ + \left( A_{-}^{(0)} \right)^2 (n+1)\Delta C_- < 0.$$
FIG. 3: (Color online) Phase diagram of various types of vortices for $\Omega > 0$ for the paramagnon parameter $\delta = 0.05$. SV$(n)$ is the domain where $n$ singular vortices are the most stable configuration while the wedge shaped domain between blue solid curves, denoted HQV$(\nu + 1, \nu)$, is a region where a configuration of $\nu + 1$ HQVs of ($-$)-vortex and $\nu$ HQV of ($+$)-vortex are most stable. A dashed red line is a boundary between two types of SVs while a solid blue line is a boundary between SV and HQV. The point $P_i$ denotes the point of the same symbol in Fig. 2.

diagram for $0 \leq n \leq 5$ has been plotted in the $\hat{h}$-$\Omega$ plane. We expect these textures may be experimentally observable by using NMR for example. Strong magnetic field may be disturbing for NMR measurement. However turning off or reducing the magnetic field may not be a serious problem; transition to different phases of texture, after the magnetic field is turned off or made small, involves topology change of the order parameter and hence unstable texture may persist for considerable length of time [21]. Dynamics of the texture after the magnetic field is reduced is beyond the scope of the present Letter and will be studied in a separate publication. Analysis of the oscillation modes of the vortices on a circle is also an interesting problem, which might have some observable consequence [22].

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FIG. 4: (a) Configuration of HQV$(4, 3)$ for $\hat{h} = 0.5$ and $\Omega = 12.9$. The inner circle of radius $u_+ \approx 0.28$ supports three ($+$)-HQVs, denoted by $\oplus$, while the outer circle of radius $u_- \approx 0.34$ supports four ($-$)-HQVs, denoted by $\ominus$. The relative orientation of ($+$) HQVs and ($-$) HQVs is arbitrary. The wall of the cylinder at $u = 1$ is not shown. (b) Configuration of HQV$(6, 5)$ for $\hat{h} = 0.5$ and $\Omega = 16.3$. There is a ($-$)-HQV at the center and five ($-$)-HQVs on a circle with radius $u_- \approx 0.43$ and five ($+$)-HQVs on a circle with radius $u_+ \approx 0.35$.

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