Quantum Gravity and Inflation

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ABSTRACT

We review some perturbative results obtained in quantum gravity in an accelerating cosmological background. We then describe a class of non-local, purely gravitational models which have the correct structure to reproduce the leading infrared logarithms of quantum gravitational back-reaction during the inflationary regime. These models end inflation in a distinctive phase of oscillations with slight and short violations of the weak energy condition and should, when coupled to matter, lead to rapid reheating. By elaborating this class of models we exhibit one that has the same behaviour during inflation, goes quiescent until the onset of matter domination, and induces a small, positive cosmological constant of about the right size thereafter. We also briefly comment on the primordial density perturbations that this class of models predict.

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1 Introduction

- **FRW Cosmology and Inflation:**

  On scales larger than about $100\ Mpc$ the universe is well described by the FRW geometry:

  \[ ds^2 = -dt^2 + a^2(t) \, dx \cdot dx \ . \]  

  The time variation of the scale factor $a(t)$ gives the instantaneous values of the *Hubble parameter* $H(t)$ and the *deceleration parameter* $q(t)$:

  \[ H(t) \equiv \frac{\dot{a}(t)}{a(t)} = \frac{d}{dt} \ln a(t) \ , \]  

  \[ q(t) \equiv -\frac{\dot{a}(t) \ddot{a}(t)}{a^2(t)} = -1 - \frac{\dot{H}(t)}{H^2(t)} \equiv -1 + \epsilon(t) \ . \]  

  Their current values are:

  \[ H_{\text{now}} \simeq (73.8 \pm 2.4) \text{km/sec Mpc} \simeq 2.4 \times 10^{-18} \text{Hz} \]  

  and $\epsilon_{\text{now}} \simeq 0.33 \pm 0.13$ \cite{2,3}.

  There is overwhelming evidence that the history of the universe included a period of accelerated expansion known as inflation and defined by $H > 0$ with $\epsilon < 1$ \cite{4,5}. This expansion occurred very early -- $t \sim 10^{-33} \text{sec}$ -- and the latest data \cite{3,6}, plus the assumption of single scalar inflation imply:

  \[ H_I \lesssim 1.7 \times 10^{38} \text{Hz} \text{ with } \epsilon_I \lesssim 0.011 \]  

- **The Horizon Problem:**

  The strongest evidence in favor of primordial inflation is the fact that we can detect epochs of cosmological history during which the observable universe was in thermal equilibrium. Without an early phase of primordial acceleration there is no way such distant regions can even have exchanged a single photon, much less interacted strongly enough to have equilibrated. To understand why, let us use the fact that photons travel on paths with zero invariant interval to calculate the size of our horizon:

  \[ ds^2 = -dt^2 + a^2(t) \, dr^2 = 0 \implies dr = \frac{dt}{a(t)} \ . \]  

  Now consider some past time $t_{\text{past}}$, and compare the coordinate distance $R_{\text{past}}$ we can observe at $t_{\text{now}}$ with the coordinate radius of light which propagated from the beginning of the universe at $t_{\text{initial}}$ to $t_{\text{past}}$:

  \[ R_{\text{past}} = \int \frac{dt}{a(t)} \ , \text{ for } t_{\text{past}} < t < t_{\text{now}} \]
\[ R_{\text{future}} = \int \frac{dt}{a(t)} , \quad \text{for } t_{\text{initial}} < t < t_{\text{past}} . \] (6)

For the universe at \( t_{\text{past}} \) to have reached thermal equilibrium by causal processes requires:

\[ \left( \frac{R_{\text{past}}}{R_{\text{future}}} \right)^2 \leq 1 . \] (7)

Suppose the universe expanded with constant \( \epsilon \equiv -\dot{H}H^{-2} \):

\[
\text{constant } \epsilon \implies H = \frac{1}{\epsilon t} \implies a \sim t^{\frac{1}{\epsilon}} \]
(8)

\[
\implies \int \frac{dt}{a(t)} = \frac{1}{(\epsilon - 1)Ha} .
\] (9)

If the universe was decelerating throughout its existence the upper limit of (9) dominates over the lower one:

\[
\epsilon > 1 \implies Ha \sim t^{1 - \frac{1}{\epsilon}} \text{ falls} \]
(10)

\[
\implies R_{\text{past}} \sim \left. \frac{1}{(\epsilon - 1)Ha} \right|_{\text{now}} , \quad R_{\text{future}} \sim \left. \frac{1}{(\epsilon - 1)Ha} \right|_{\text{past}} \]
(11)

\[
\implies R_{\text{future}} \ll R_{\text{past}} .
\] (12)

For instance, at recombination – when the universe is observed to be in thermal equilibrium to one part in \( 10^5 \) – and at nucleosynthesis, respectively:

\[
\left( \frac{R_{\text{past}}}{R_{\text{future}}} \right)^2 \sim 2000 \text{ and } 10^9 . \]
(13)

Hence the observed equilibrium during these epochs could not have come about by causal processes; it would have had to be an accidental feature of the way the universe began. No one knows how the universe began, but assuming it began in a very high degree of thermal equilibrium seems problematic. This sort of unsatisfactory conclusion can be avoided if we assume the universe went through a phase of acceleration before \( t_{\text{past}} \). In that case the integral (9) is dominated by its lower limit and we can make the past light-cone arbitrarily large by assuming \( t_{\text{initial}} \) is close to zero:

\[
\epsilon < 1 \implies Ha \sim t^{1 - \frac{1}{\epsilon}} \text{ grows} \]
(14)

\[
\implies R_{\text{future}} \sim \left. \frac{1}{(\epsilon - 1)Ha} \right|_{\text{initial}} \]
(15)

\[
\implies \text{for } t_{\text{initial}} \to 0 : R_{\text{future}} \gg R_{\text{past}} .
\] (16)
• **Single-Scalar Inflation:**

Although the evidence for a phase of primordial inflation is very strong [8], there is no compelling mechanism for making it happen [9]. The simplest model consists of gravity plus a minimally coupled scalar field (called the *inflaton*) whose Lagrangian is [10]:

\[
\mathcal{L} = \sqrt{-g} \left( -\frac{1}{2} g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) + \frac{R}{16\pi G} \right). \tag{17}
\]

Note that this model is general enough to support *any* expansion history \(a(t)\), provided only that \(\dot{H}(t) \leq 0\) throughout. To see this, note that the nontrivial Einstein equations are:

\[
3H^2 = 8\pi G \left[ \frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 + V(\varphi) \right], \tag{18}
\]

\[
-2\dot{H} + 3H^2 = 8\pi G \left[ \frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 - V(\varphi) \right]. \tag{19}
\]

One would usually take the scalar potential \(V(\varphi)\) as given and use these equations to determine the expansion history, but let us adopt the opposite perspective. That is, we will assume \(a(t)\) is known and we then use the equations to reconstruct the potential \(V(\varphi)\) which supports that geometry. By adding (19) to (18) we get the inflaton as a function of time:

\[
-2\dot{H} = 8\pi G \left( \frac{d\varphi}{dt} \right)^2 \implies \varphi(t) = \varphi_I + \int^t dt' \left( -\frac{\dot{H}(t')}{{4\pi G}} \right)^{\frac{1}{2}}. \tag{20}
\]

By inverting this relation we get the time as a function of the inflaton: \(t = t(\varphi)\). Now subtract (19) from (18) to find the potential which gives the desired expansion history:

\[
6H^2 = 16\pi G V(\varphi) \implies V(\varphi) = \frac{3}{8\pi G} H^2[t(\varphi)]. \tag{21}
\]

This construction seems to have first appeared in [11], and independently in [12] and [13].

• **Scalar Inflation Problems:**

As we have seen, the potential energy of a minimally coupled scalar field
can cause inflation, but this mechanism involves assumptions which seem unlikely and are sometimes contradictory:

- That the universe began with the scalar field approximately spatially homogeneous over more than a Hubble volume $V(\varphi) > H^{-3}$ [14].
- That the scalar field potential must be flat enough make inflation last a long time [9, 10].
- That the minimum of the scalar field potential has just the right value $V_{\text{min}} \approx 0$ to leave the post-inflationary universe with only the small amount of vacuum energy we detect today [15, 16].
- That the scalar field couples enough to ordinary matter so that its kinetic energy can create a hot, dense universe at the end of inflation, but not so much that loop corrections from ordinary matter compromise the flatness of the inflaton potential [17].

- **Gravity-Driven Inflation:**
  A more natural mechanism for inflation can be found within gravitation – which, after all, plays the dominant role in shaping cosmological evolution – by supposing that the bare cosmological constant $\Lambda$ is not unnaturally small but rather large and positive. Here “large” means a $\Lambda$ induced by some matter scale which might be as high as $10^{18}$ GeV. Then, the value of the dimensionless coupling constant would be $G\Lambda \sim 10^{-4}$, rather than the putative value of $10^{-122}$ [15, 16].

  Because $\Lambda$ is constant in *space*, no special initial condition is needed to start inflation. We also dispense with the need to employ a new, otherwise undetected scalar field. However, $\Lambda$ is constant in *time* as well, and classical physics can offer no natural mechanism for stopping inflation once it has begun [18]. Quantum physics can: accelerated expansion continually rips virtual infrared gravitons out of the vacuum [19] and these gravitons attract one another, thereby slowing inflation [20]. This is a very weak effect for $G\Lambda \ll 1$, but a cumulative one, so inflation would last a long time for no other reason than that gravity is a weak interaction [20].

- **Graviton Physical Modes:**
  In terms of the full metric field $g_{ij}(x)$, the fluctuating graviton field $h_{ij}^{TT}(x)$ is defined as:

  $$
g_{ij}(t, x) = a^2(t) \left[ \delta_{ij} + \sqrt{32\pi G} h_{ij}^{TT}(t, x) \right].$$

  (22)
The free field expansion of the graviton field is:

\[ h_{ij}^{TT}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \left\{ u(t, k) e^{ik \cdot \mathbf{x}} \epsilon_{ij}(\mathbf{k}, \lambda) \alpha(\mathbf{k}, \lambda) + (c.c.) \right\} , \quad (23) \]

where \((c.c.)\) denotes complex conjugation, \(\epsilon_{ij}(\mathbf{k}, \lambda)\) are the same transverse and traceless polarization tensors as in flat space, \(\alpha(\mathbf{k}, \lambda)\) is the annihilation operator, and \(u(t, k)\) are the mode functions which obey:

\[ \ddot{u}(t, k) + 3H(t) \dot{u}(t, k) + \frac{k^2}{a^2(t)} u(t, k) = 0 . \quad (24) \]

The mechanism we have sketched is that inflation rips gravitons out of the vacuum, and then the self-gravitation of these particles slows inflation. Let us first estimate the energy \(E(t, k)\) which is present in a single polarization of a single wave vector \(\mathbf{k}\) at time \(t\). Because the precise definition of energy is subtle for gravitons we base this estimate on a massless, minimally coupled scalar field \(\varphi(x)\), whose mode equation is the same as \(24\). The scalar field Lagrangian density is:

\[ \mathcal{L}(x) = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = \frac{1}{2} a^3(t) \dot{\varphi}^2 - \frac{1}{2} \nabla \varphi \cdot \nabla \varphi . \quad (25) \]

The Langrangian diagonalizes in momentum space:

\[ L(t) = \int d^3x \, \mathcal{L}(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2} a^3(t) \left| \dot{\varphi}(t, \mathbf{k}) \right|^2 - \frac{1}{2} a(t) k^2 \left| \varphi(t, \mathbf{k}) \right|^2 \right\} \]

so that any mode with wavenumber \(k\) evolves independently as a harmonic oscillator \(q(t)\) with a time-dependent mass \(m(t) \sim a^3(t)\) and angular frequency \(\omega(t) \equiv k a^{-1}(t)\):

\[ q(t) = u(t, k) A + u^*(t, k) A^\dagger , \quad [A, A^\dagger] = 1 , \quad (27) \]

\[ E(t, k) = \frac{1}{2} a^3(t) \dot{q}^2(t) + \frac{1}{2} a(t) k^2 q^2(t) . \quad (28) \]

For the special case of de Sitter the mode functions are given by:

\[ u(t, k) = \frac{H}{\sqrt{2k^3}} \left[ 1 - \frac{ik}{H a(t)} \right] \exp \left( \frac{ik}{H a(t)} \right) . \quad (29) \]
Although our conclusions are quite generic, it will simplify the subsequent analysis if we make this assumption of de Sitter.

At any instant $t$ the minimum energy is $E_{\text{min}}(t,k) = \frac{1}{2}ka^{-1}(t)$. However because both the mass and angular frequency are time-dependent, the state with minimum energy at one instant is not the minimum energy state at later times; there is particle production. Bunch-Davies vacuum $|\Omega\rangle$ is the minimum energy state in the distant past, and the expectation value of the energy operator (28) in this state is:

$$
\langle \Omega | E(t,k) | \Omega \rangle = \frac{a^3(t)}{2} |\dot{u}(t,k)|^2 + \frac{k^2a(t)}{2} |u(t,k)|^2
$$

By setting this equal to $\left(\frac{1}{2} + N\right)\hbar\omega$, one can read off the instantaneous occupation number $N(t,k)$:

$$
N(t,k) = \left[ \frac{H a(t)}{2k} \right]^2.
$$

We can consider $N(t,k)$ to be the number of gravitons with one polarization and wave vector $k$ that have been created by time $t$.

At this point a short digression is useful on the significance of the co-moving wave number $k$ in an expanding universe. Because $k = \frac{2\pi}{\lambda}$ is the inverse of a coordinate length, the physical wave number is $ka^{-1}(t)$. This falls exponentially during inflation. Horizon crossing is when the physical wave number equals the Hubble parameter:

$$
\text{Horizon Crossing} \implies k_{\text{phys}} = k a^{-1}(t) = H.
$$

It is natural to separate modes into “infrared” and “ultraviolet” depending upon whether or not they have experienced horizon crossing:

$$
\text{Infrared} \implies H < k < H a(t),
$$

$$
\text{Ultraviolet} \implies k > H a(t).
$$

From (32) we see that there is negligible production of ultraviolet gravitons, whereas the number of infrared gravitons in even a single wave vector grows exponentially. This is a crucial observation because it means that the physics
of this effect is controlled by the known, low energy theory of gravity, without regard to its still unknown ultraviolet completion.

The energy density induced by both polarizations of these infrared gravitons equals:

$$\rho_{IR} = \frac{2}{a^3(t)} \int^{H_a} d^3k \frac{d^3k}{(2\pi)^3} N(t, k) \frac{k}{a(t)} = \frac{H^4}{8\pi^2}.$$  \hspace{1cm} (36)

This is much less than the energy density of the cosmological constant:

$$\rho_{\Lambda} = \frac{3H^2}{8\pi G} \implies \frac{\rho_{IR}}{\rho_{\Lambda}} = \frac{G H^2}{3\pi} \lesssim 10^{-11}. \hspace{1cm} (37)$$

One may wonder if the gravitational self-interaction of \( \rho_{IR} \) can even screen itself, much less \( \rho_{\Lambda} \). To see that it can, note that even a small energy density can induce significant screening if it interacts over a sufficiently large volume. A simple way to see this is to consider the total energy density \( \rho_{\text{tot}} \) produced by a static energy density \( \rho_{\text{bare}} \) distributed throughout a sphere of radius \( R \).

For simplicity, we follow ADM [21] in using the Newtonian formula assuming it is the total mass \( \frac{4}{3} \pi \rho_{\text{tot}} c^{-2} R^3 \) that gravitates:

$$\rho_{\text{tot}} \approx \rho_{\text{bare}} - \frac{4\pi G \rho_{\text{tot}}^2 R^2}{5c^4} \implies \rho_{\text{tot}} \approx \frac{5c^4}{8\pi G R^2} \left[ \sqrt{1 + \frac{16\pi G \rho_{\text{bare}} R^2}{5c^4}} - 1 \right]. \hspace{1cm} (38)$$

As \( R \) goes to infinity the screening becomes total — i.e., \( \rho_{\text{tot}} \) goes to zero — independent of how small \( \rho_{\text{bare}} \) is.

Equation (38) means the gravitational self-interaction of infrared gravitons can screen \( \rho_{IR} \), but what about the vastly larger energy density \( \rho_{\Lambda} \) of the cosmological constant? The key observation for realizing that even \( \rho_{\Lambda} \) can be screened is that the gravitational self-interaction hasn’t had time to reach a static limit. Indeed, most of the universe is not even now in causal contact, and never will be if the current phase of accelerated expansion persists. The lower bound of \( \rho_{\text{tot}} = 0 \) implicit in the static result (38) arises because it is the instantaneous value of \( \rho_{\text{tot}} \) which gravitates, so making it smaller by screening also cuts off the effect. But that cutoff disappears when one takes account of the causal nature of the interaction. The source for the gravitational field at time \( t \) is not the instantaneous energy density of infrared gravitons but rather its value far back in the past light-cone. That is not reduced by the instantaneous energy density becoming small; indeed,
the effect of screening is to make the past light-cone open outwards, which exposes more of the early times when the energy density of infrared gravitons was high.

This discussion does not prove the viability of gravity-driven inflation. Because inflationary particle production is itself a 1-loop effect, the gravitational response to it cannot occur at less than 2-loop order. Two-loop computations in quantum gravity are not simple around flat space background, and they are considerably tougher around de Sitter. Then there is the delicate gauge issue of how to invariantly quantify screening [22]. Good physicists on both sides of the question have debated whether or not there is a significant screening effect from the mechanism we have described [23], or from any of the related relaxation mechanisms which have been proposed [24, 25, 26]. There is even disagreement about the basic formalism of perturbative quantum gravity on de Sitter background [27, 28, 29]. The aim of this introduction has been merely to establish the plausibility of the mechanism. Having hopefully done that, we will henceforth explore a simple class of effective field equations that might describe it.

2 Model Building

• Perturbative Results:
Let use first review some perturbative results on de Sitter:

\[ \text{de Sitter Inflation} \implies a(t) = e^{Ht}. \]  

(39)

The gravitational Lagrangian is:

\[ L_{\text{gr}} = \frac{1}{16\pi G}(R - 2\Lambda)\sqrt{-g}. \]  

(40)

It turns out that quantum corrections cannot grow faster than powers of \( \ln(a) = Ht [30] \). We are interested in the regime of \( \ln(a) \gg 1 \), in which case one needs only the leading logarithm contributions at any loop order \( L \) which contain the most factors of \( \ln(a) \). Explicit computations [20, 31], and general counting rules [30], give the following behaviour for the leading logarithm contributions to the energy density induced by quantum gravitational effects:

\[ \rho_1 \sim +\Lambda^2, \]  

(41)

\[ \rho_2 \sim -G\Lambda^3\ln[a(t)], \]  

(42)

\[ \rho_L \sim -\Lambda^2\left(G\Lambda\ln[a(t)]\right)^{L-1}. \]  

(43)
Because stress-energy is separately conserved at each loop order, the quantum gravitationally induced pressure must be that of negative vacuum energy, up to small subleading logarithm corrections:

\[ \dot{\rho}_L = -3H(\rho_L + p_L) \implies p_L(t) \sim -\rho_L(t) \quad . \]  

(44)

Hence the general form of the pressure is:

\[ p(t) \sim \Lambda^2 f[G\Lambda \ln(a)] \quad . \]  

(45)

Perturbation theory is valid only if the effective dimensionless coupling constant \( G\Lambda \ln(a) \) of the theory is small. Thus, perturbation theory breaks down after a large number of e-foldings – \( N \equiv H t = \ln(a) \sim (GA)^{-1} \). However, if we had the effective field equations, at least for a general FRW geometry, it would be possible to evolve arbitrarily far in the future. So we shall try to guess these equations based on some general principles, and on what we know from perturbation theory.

* **Guessing the Effective Field Equations:**

The classical gravitational equations of motion coming from (40) are:

\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} \quad . \]  

(46)

The equations of motion in the presence of the quantum induced stress-energy tensor \( T_{\mu\nu}[g] \) are:

\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}[g] \quad . \]  

(47)

The full quantum induced stress-energy encodes all information about quantum gravity. For example, variations of it about flat space – with \( \Lambda = 0 \) – give all scattering amplitudes to all orders in perturbation theory. There is absolutely no chance we can guess this, nor is there any need to do so. We require only the most cosmologically significant part of the full effective quantum gravitational equations; that is, a functional of the FRW scale factor \( a(t) \).

A few basic principles can be used to guide us [32]:

(i) **Correspondence:** The form of \( T_{\mu\nu}[g] \) must of course reproduce the known results from perturbation theory about de Sitter space.

(ii) **Non-locality:** It is easy to show that a purely local \( T_{\mu\nu}[g](x) \) can only lead to a constant change in the cosmological constant. Note first that such a local \( T_{\mu\nu}[g](x) \) must be composed of the Riemann tensor and its derivatives.
Now consider the de Sitter geometry for an arbitrary Hubble parameter \( H' \), not necessarily equal to the one associated with \( \Lambda = 3H^2 \). The Riemann tensor for this geometry reduces to a constant times sums of products of the metric, and any covariant derivative of it therefore vanishes:

\[
R_{\rho\sigma\mu\nu} = H'^2 [g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}] \implies D_\alpha R_{\rho\sigma\mu\nu} = 0 .
\] (48)

Hence any local stress-energy must reduce, for this geometry, to \( H'^4 g_{\mu\nu} \), and the effective field equation would become:

\[
G_{\mu\nu} = -3H^2 g_{\mu\nu} + \#8\pi GH'^4 g_{\mu\nu} = -3H'^2 \left( \frac{H^2}{H'^2} - \frac{8}{3\pi G H'^2} \right) g_{\mu\nu} .
\] (49)

This amounts to merely a renormalization of \( \Lambda' = \frac{9}{16\pi G} \left[ \sqrt{1 + \frac{32}{9\pi G \Lambda}} - 1 \right] .
\] (50)

If one began in this geometry – which our actual renormalization condition would require – then there would never be any deviation form it. We conclude that screening requires a non-local \( T_{\mu\nu}[g] \).

(iii) Causality: The quantum induced stress-energy must be both conserved and causal, in the sense that \( T_{\mu\nu}[g](x) \) depends only upon metrics on or within the past light-cone of the point \( x^\mu \). We would normally ensure conservation by defining the stress-energy from the variation of an invariant effective action:

\[
T_{\mu\nu}[g](x) = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma[g]}{\delta g^{\mu\nu}(x)} \implies D^\nu T_{\mu\nu} = 0 .
\] (51)

However, this procedure conflicts with causality for the sort of non-local contributions of greatest interest to us.

To understand the problem, consider the action of a point particle \( q(t) \). Suppose the action contains a non-local term of the form \( q(s) \times q(s - \Delta t) \). One might think that its non-locality is safely confined to the past of \( q(s) \), but any variation must also affect the term \( q(s - \Delta t) \). This gives rise to an equation which depends on the future as well as the past:

\[
\Gamma[q] = \int ds \ q(s) \ q(s - \Delta t) \implies
\]

\[
\frac{\delta \Gamma}{\delta q(t)} = \int ds \left[ \delta(s - t) \ q(s - \Delta t) + q(s) \ \delta(s - \Delta t - t) \right] = q(t - \Delta t) + q(t + \Delta t) .
\] (54)
This same problem must afflict any variation such as (51) which is based on a non-local effective action that contains only a single field.

The proper way to derive non-local effective field equations which are both causal and conserved is by varying the Schwinger-Keldysh effective action [33]. This avoids the single field conundrum by employing two fields $g^\pm_{\mu\nu}$; with the $+$ sign corresponding to the background metric during forward evolution and the $-$ sign to backwards evolution. The stress-energy tensor of the Schwinger-Keldysh formalism is the variation with respect to either field, after which the two fields are set equal:

$$T_{\mu\nu}[g](x) = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma[g^+, g^-]}{\delta g^+_{\mu\nu}(x)} \bigg|_{g^\pm = g}. \quad (55)$$

One can show that the $+$ and $-$ contributions from fields at any point $x''^\mu$ exactly cancel unless $x''^\mu$ is on or within the past light-cone of $x'^\mu$.

The Schwinger-Keldysh effective action is what one should use to derive the correct effective field equations. However, deriving anything is tough in quantum gravity. The point of this exercise was to try guessing the most cosmologically significant part of the effective field equations. Because it is those equations we seek, not the effective action, we shall adopt the shortcut of simply making an appropriately non-local and causal ansatz for them, and then enforce conservation directly.

**Perfect Fluid Ansatz:**

The ansatz must apply to all FRW cosmologies. The “perfect fluid” form of $T_{\mu\nu}$ can represent any cosmology and in addition provides enough free parameters to enforce conservation and correspondence with perturbative results:

$$T_{\mu\nu}[g] = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \quad (56)$$

Our stress-energy is defined by specifying three things:

(i) the energy density $\rho$ as a functional of the metric tensor $\rho[g](x)$,

(ii) the pressure $p$ as a functional of the metric tensor $p[g](x)$, and

(iii) the 4-velocity field $u_\mu$ as a functional of the metric tensor $u_\mu[g](x)$, chosen to be timelike and normalized:

$$g^{\mu\nu} u_\mu u_\nu = -1 \quad \implies \quad u^\mu u_{\mu;\nu} = 0. \quad (57)$$

Because of the normalization (57), only three of the components of $u_\mu$ are algebraically independent. Hence our ansatz consists of five independent
functionals in total. Stress-energy conservation:

\[ D^\mu T_{\mu\nu} = 0 \]  

(58)

provides four equations and allows us to determine any four of these functionals in terms of the fifth. It turns out to be most convenient to specify the induced pressure functional \( p[g] \) and then use conservation to obtain the form of the induced energy density \( \rho[g] \) and the 4-velocity \( u_\mu[g] \), up to their initial value data.

• Building \( p[g] \):
We want the pressure \( p[g](x) \) to be a causal, non-local functional of the metric which reduces to the form (45) in the de Sitter limit. A very simple ansatz along these lines is:

\[ p[g](x) = \Lambda^2 f \left( -G \Lambda X[g](x) \right) , \]  

(59)

where \( -X[g](x) \) is a dimensionless, non-local functional of the metric that grows like \( \ln(a) \) when the metric is de Sitter. A natural way of incorporating causal non-locality is through the inverse of some differential operator. The simplest choice for this operator is the covariant scalar d’Alembertian:

\[ \Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( g^{\mu\nu} \sqrt{-g} \partial_\nu \right) . \]  

(60)

To make \( X[g](x) \) dimensionless, we need to act the inverse of \( \Box \) on a curvature scalar, the simplest choice for which is the Ricci scalar \( R \). We are therefore led to consider \( X[g] = \Box^{-1}R \), with the inverse defined using retarded boundary conditions.

To see that this simple ansatz has the right properties, we specialize \( \Box \) and \( R \) to a general FRW geometry:

\[ \Box = - \left( \partial_t^2 + 3H \partial_t \right) , \quad R(t) = 12H^2(t) + 6\dot{H}(t) . \]  

(61)

Hence the specialization of \( X[g](x) \) to FRW is:

\[ X = \frac{1}{\Box} R = - \int_0^t dt' a^{-3} \int_0^{t'} dt'' a^{3} \left[ 12H^2 + 6\dot{H} \right] . \]  

(62)

For de Sitter spacetime \( a(t) = e^{Ht} \) with constant \( H \) – we get the correct correspondence limit:

\[ \frac{1}{\Box} R = -4\ln(a) + \frac{4}{3} \left[ 1 - e^{-3Ht} \right] . \]  

(63)
More generally, expression (62) implies that \(-X[g](x)\) will grow during the inflationary regime of large Ricci curvature, and then freeze in to a constant during the radiation dominated era of \(R(t) = 0\). As long as the function \(f(x)\) in (59) grows monotonically and without bound, this ansatz for the pressure is bound to produce enough screening to end inflation in roughly the right way.

**Numerical Results:**

There is no hope of deriving an analytic solution for \(a(t)\) when the pressure is as complicated as (59) with (62), but this is a simple problem to solve numerically. Figures 1-2 give the evolution of the non-local source \(X(t)\), figures 3-4 present the Ricci scalar \(R(t)\), and figures 5-6 show the Hubble parameter \(H(t)\). These results were generated for the choice \(f(x) = \exp(x) - 1\) – the “exponential model” – although any function \(f(x)\) which grows monotonically and without bound gives the same qualitative behaviour, including even \(f(x) = x\). To avoid a long preliminary evolution with negligible effect, we set the unrealistically high value of \(G\Lambda = 1/200\). Again, the behaviour is qualitatively the same for any choice of \(G\Lambda\).

The following basic features emerge from our numerical work [32]:

- During the era of inflation, the source \(-X(t)\) grows while the curvature
scalar $R(t)$ and the Hubble parameter $H(t)$ decrease.

- Inflationary evolution dominates roughly until we reach the critical point $X_{cr}$ defined by:

$$1 - 8\pi G \Lambda f [-G \Lambda X_{cr}] \equiv 0 .$$

(64)

- The epoch of inflation ($q < 0$) ends slightly before $X(t)$ reaches $X_{cr}$. This is most directly seen from the deceleration parameter because initially $q(t = 0) = -1$, while at criticality $q(t = t_{cr}) = +\frac{1}{2}$.

- The source $X(t)$ oscillates with constant period and decreasing amplitude.

- Oscillations in $R(t)$ become significant as we approach the end of inflation; they are centered around $R = 0$, their frequency is given by:

$$\omega = G\Lambda H_0 \sqrt{72\pi f'_{cr}} ,$$

(65)

and their amplitude decreases like the inverse of the number of oscillations.

- While there is net expansion during the era of oscillations, the Hubble parameter $H(t)$ attains small negative values for short time intervals. Of course negative $H(t)$ corresponds to a compressing universe, which should lead to rapid reheating when matter couplings are included.
- **Analytic Results:**
Although one cannot obtain analytic results for the full evolution of $a(t)$, it is possible to give an approximate treatment for the period of oscillations. We use the evolution equation:

$$2\dot{H} + 3H^2 = \Lambda \left\{ 1 - 8\pi G\Lambda f[-G\Lambda X] \right\} , \quad X \equiv \frac{1}{\Box} R . \quad (66)$$

Recall that we assume only that the function $f(x)$ grows monotonically and without bound. Hence there must exist a critical point $X_{cr}$ such that:

$$1 - 8\pi G\Lambda f[-G\Lambda X_{cr}] = 0 \implies X_{cr} = -\frac{1}{G\Lambda} f^{-1}\left(\frac{1}{8\pi G\Lambda}\right) . \quad (67)$$

Inflationary evolution dominates roughly until we reach the critical point. Close to the critical point the induced pressure $p$ is nearly constant and, thus, it makes sense to expand $f$ around its critical point:

$$f \simeq f_{cr} - G\Lambda \Delta X(t) f'_{cr} , \quad \Delta X(t) \equiv X(t) - X_{cr} . \quad (68)$$

Now consider the linearized evolution equation:

$$2\dot{H} + 3H^2 \simeq 8\pi (GA)^2 \Lambda (X - X_{cr}) f'[-G\Lambda X_{cr}] . \quad (69)$$
Figure 4: The evolution of the curvature scalar $R(t)$ during the oscillatory regime for the exponential model.

Figure 5: The evolution of the Hubble parameter $H(t)$ over the full range for the exponential model.
Using (61) we can express the co-moving time derivative of the Hubble parameter as:

$$\dot{H} = \frac{1}{6} R - 2H^2 . \quad (70)$$

Because the amplitudes of both $R(t)$ and $H(t)$ fall like $t^{-1}$ during the era of oscillations, the second term in (70) is irrelevant. Consequently, the evolution equation (69) becomes:

$$-R + 3H^2 \simeq -24\pi (G\Lambda)^2 \Lambda (X - X_{cr}) f'_{cr} , \quad (71)$$

where we have defined:

$$f'_{cr} \equiv f'[-G\Lambda X_{cr}] \equiv -\frac{1}{G\Lambda} \frac{d}{dX} f[-G\Lambda X]|_{X=X_{cr}} . \quad (72)$$

Action of the d’Alembertian operator (61) on (71) gives:

$$\ddot{R} + 2H \dot{R} + (\omega^2 - \dot{H}) R + [3H^2 R - 36H^4] \simeq 0 , \quad (73)$$

where we define:

$$\omega^2 \equiv 24\pi (G\Lambda)^2 \Lambda f'_{cr} , \quad (74)$$
We can again neglect the various “small” terms in (73) to infer:

\[
\ddot{R} + 2H \dot{R} + \omega^2 R \simeq 0 \quad \Rightarrow \quad R(t) \simeq \frac{\sin(\omega t)}{a(t)}.
\]  

(75)

This reveals the presence of oscillations. Note also that the frequency \(74\) agrees with numerical results.

**Generic Results:**

It is also possible to derive approximate analytic results for the period of inflation. If \(N\) is the number of e-foldings before criticality, the various geometrical parameters are \([34, 35]\):

\[
a(t) = a_{cr} e^{-N},
\]

(76)

\[
H(t) \simeq \frac{1}{3} \omega \sqrt{4N + \frac{4}{3}},
\]

(77)

\[
\epsilon(t) \simeq \frac{2}{4N + \frac{4}{3}}.
\]

(78)

During the oscillatory era it is best to describe these same parameters using the time \(\Delta t \equiv t - t_{cr}\) since criticality. The following approximate relations hold \([34, 35]\):

\[
a(t) = a_{cr} C_2 \left[ C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi) \right],
\]

(79)

\[
H(t) = \frac{\omega \left[ 1 - \sqrt{2} \sin(\omega \Delta t + \phi) \right]}{C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi)},
\]

(80)

\[
\epsilon(t) = 1 + \frac{\sqrt{2} \cos(\omega \Delta t + \phi) \left[ C_1 + \omega \Delta t + \sqrt{2} \cos(\omega \Delta t + \phi) \right]}{\left[ 1 - \sqrt{2} \sin(\omega \Delta t + \phi) \right]^2}.
\]

(81)

The constants \(\phi, C_1\) and \(C_2\) in relations \((79)\)-(81) are chosen to match the two epochs at criticality \((N = 0\) and \(\Delta t = 0\)):

\[
\phi = \arcsin \left( \frac{\sqrt{2} - \sqrt{2970}}{56} \right) \approx -\frac{\pi}{2},
\]

(82)

\[
C_1 = \frac{\sqrt{27}}{2} - \frac{\sqrt{27}}{2} \sin \phi - \sqrt{2} \cos \phi \approx 3,
\]

(83)

\[
C_2 = \frac{1}{C_1 + \sqrt{2} \cos \phi} \approx \frac{1}{6}.
\]

(84)
• Primordial Density Perturbations:
  
  (i) Scalar perturbations. Initially ultraviolet modes in scalar driven inflation oscillate and redshift, and then become approximately constant around the time of horizon crossing [5]. The behaviour of scalar perturbations in this model differs in two significant ways. First, initially ultraviolet modes merely redshift, they do not oscillate. Like scalar driven inflation, the modes of this model become approximately constant around the time of horizon crossing. However, all super-horizon modes in this model begin oscillating with the same frequency $\omega$ at the end of inflation [36]. Because there are so many of these super-horizon modes after a long period of inflation, the fact that all of them start to oscillate at the end of inflation should lead to very rapid reheating, without the need to invoke anything other than the usual gravitational couplings to matter. After the universe reaches radiation domination one can show that the oscillations stop [20], which is consistent with an approximately scale invariant power spectrum. What we cannot do is to evaluate the normalization. That is fixed by canonical quantization in scalar driven inflation, but we only have the effective field equations for this model. Recall that the combination of causality and non-locality means our effective field equations cannot derive from a conventional action principle.

  (ii) Tensor perturbations. The analysis of tensor perturbations in this class of models is much simpler than that of scalar perturbations [35]. The reason is that our perfect fluid stress-energy has no effect on the tensor perturbations $h_{ij}^{TT}$. Therefore the resulting power spectrum $\Delta_h^2$ has the usual form:

  $\Delta_h^2(k) \simeq \frac{16G\mathcal{H}^2(t_k)}{\pi}$,  \hspace{1cm} (85)

  but with the expansion history peculiar to our model. There is nothing unconventional about our expansion history [66,78] during the epoch of inflation, so our prediction for the $B$ mode of polarization in the cosmic microwave background is not distinct from that of scalar driven inflation. The period for which our model differs is the phase of oscillations [79,81], during which the usual Hubble “friction” term actually changes sign. Because the end of inflation comes about 50 e-foldings after the horizon crossing of the observable part of the cosmic microwave background, the corresponding enhancement in the stochastic background of gravitational radiation will be at the uncomfortably high frequency of $f \sim 10^9 Hz$ [35]. No current gravity wave detector has sensitivity at this frequency but one has been proposed [37].
3 Post-Inflationary Evolution

We assume that energy flows from the gravitational to the matter sector, leading to a radiation dominated universe at \( t = t_r \). Because our model is sourced by the Ricci scalar, which vanishes during radiation domination, the quantum induced stress-energy simply cancels the bare cosmological constant. There is no deviation from conventional cosmology until the onset of matter domination at \( t = t_m \). By that time the scales are so much below those of primordial inflation that only very small changes occur in \( X(t) \), and we can use first order perturbation theory to compute the total pressure:

\[
p_{\text{tot}} \equiv - \frac{\Lambda}{8\pi G} + p[g](x) \tag{86}
\]

\[
= - \frac{\Lambda}{8\pi G} \left\{ 1 - 8\pi G \Lambda f \left[ - \Lambda (X_{cr} + \Delta X) \right] \right\} \tag{87}
\]

\[
\simeq - \frac{\Lambda}{G} \times (\Lambda)^2 f'_{cr} \Delta X . \tag{88}
\]

The simple source (59) grows according to the formula:

\[
\Delta X(t) \equiv X(t) - X_{cr} = - \frac{4}{3} \ln \left[ 1 + \frac{3}{2} H_m(t-t_m) \right] + O(1) . \tag{89}
\]

These facts give rise to two fatal problems for the model:

- **The Sign problem:** Because \( f \) is monotonically increasing and unbounded:

\[
p_{\text{tot}} > 0 \quad \text{when} \quad X(t) < X_{cr} \ll 0 . \tag{90}
\]

The observation of late time acceleration [15, 16] implies the opposite.

- **The Magnitude problem:** The magnitude of the total pressure produced is unacceptably large:

\[
\frac{p_{\text{tot}}}{p_{\text{now}}} \simeq \left( \frac{G \Lambda H_I}{H_{\text{now}}} \right)^2 f'_{cr} \Delta X \simeq 10^{86} \times f'_{cr} \times \Delta X , \tag{91}
\]

where we have used:

\[
p_{\text{now}} \simeq - \frac{3}{8\pi G} H_{\text{now}}^2 , \quad H_I \sim 10^{13} GeV , \quad H_{\text{now}} \sim 10^{-33} eV . \tag{92}
\]
Improved Ansatz:
Both problems can be addressed by changing the source [62]. What we need
to do is add an extra curvature $S$ inside the inverse d’Alembertian, divided
by $\Lambda$ to keep things dimensionless [38]:

$$p[g](x) = \Lambda^2 f[-G\Lambda X](x) ,$$

(93)

$$-G\Lambda X = -G\Lambda \frac{1}{\Lambda} R \quad \Rightarrow \quad -G \frac{1}{\Lambda} (R \times S) = - \frac{G\Lambda}{\Lambda} \left( R \times \frac{S}{\Lambda} \right)$$

(94)

In this way the magnitude falls with cosmological evolution so that $\Delta X(t)$
 experiences only an acceptably small change at the onset of matter domination.
To keep inflation ending successfully it is necessary to evaluate this curvature $S$
 far back in the past of the Ricci scalar $R$. We obtained acceptable results
with a factor of ten.

That suffices for the magnitude problem. To solve the sign problem we
note that the curvature scalar is positive during both inflation $-R = +12H^2$
 and matter domination $-R = +3H^2$. A simple choice for $S$ that changes
its sign is $R_{00}$ which equals $-3H^2$ during inflation and $+\frac{1}{2}H^2$ during matter
domination [38]. Note that we can invariantly select the 00 component of
$R_{\mu\nu}$ using the timelike 4-velocity field $u^\mu$, which is just $\delta_0^\mu$ for FRW. Hence
the specialization of the improved ansatz to FRW is:

$$p[g](x) = \Lambda^2 f[-G\Lambda Y](x) ,$$

(95)

$$Y[g](t) = -\frac{1}{\Lambda} \frac{1}{\Lambda} \left[ R(t) \times R_{00}(\frac{1}{10}t) \right] \equiv X_{cr} + \Delta Y .$$

(96)

Late Time Acceleration:
Finally, we compute the total pressure in the improved ansatz [38]:

$$p_{\text{tot}} \simeq -G\Lambda^3 f_{cr}' \Delta Y \simeq - 200 G\Lambda^2 f_{cr}' H_m^2 .$$

(97)

For the exponential model:

$$f(x) = e^x - 1 \quad \Rightarrow \quad f_{cr}' = \frac{1}{8\pi G\Lambda} ,$$

(98)

the pressure ratio is:

$$t \gg t_m \quad \Rightarrow \quad \frac{p_{\text{tot}}}{p_{\text{now}}} \simeq \frac{200}{3} 8\pi (G\Lambda)^2 \times f_{cr}' \times \left( \frac{H_m}{H_{\text{now}}} \right)^2$$

(99)

$$\simeq \frac{200}{3} 8\pi (G\Lambda)^2 \times f_{cr}' \times 10^{10}$$

(100)

$$\simeq \frac{2}{3} \times 10^{12} \times G\Lambda .$$

(101)
It is evident that for physically reasonable values of $G\Lambda = M^4 M_{Pl}^{-4}$ we can achieve the desired equality of $p_{tot}$ with $p_{now}$ whose ratio is given by (101).

4 Conclusions

There is very strong evidence that the universe underwent a very early phase of accelerated expansion known as primordial inflation. One can devise a scalar inflaton (17) to support this geometry but this entails positing a new and otherwise undetected degree of freedom, as well as making some unrealistic and sometimes contradictory assumptions about the inflaton’s potential and its initial condition. On the other hand, there is no question that inflation results in the production of a vast sea of infrared gravitons, nor is there any question that these gravitons attract one another to some extent. Explicit results from perturbation theory indicate that this attraction grows stronger with time, until perturbation theory eventually breaks down.

Great controversy surrounds this final claim but, if it can be established, the phenomenological payoff is enormous. For then it becomes possible to dispense with the scalar inflaton and to make a virtue out of what is usually regarded as a terrible problem: namely, the fact that the observed cosmological constant is more than 120 orders of magnitude below its natural scale. We propose that the bare cosmological constant is not unnaturally small but instead only a few orders of magnitude below the Planck scale. What is being measured today is not this bare cosmological constant but rather the expansion rate, and we propose that the effect of the bare cosmological constant on the current expansion rate is subject to almost perfect screening by the self-gravitation of gravitons produced during a very long period of $\Lambda$-driven inflation.

We believe it is possible to use perturbation theory to establish the reality of quantum gravitational screening. We also feel one can resum the series of leading infrared logarithms to derive what happens at late times. However, neither thing will be easy, nor will they be quickly attained. In the meantime, we have devised a class of non-local effective field equations which might describe the eventual result of such a derivation. At this stage, one is free to dismiss our motivation from quantum gravitational inflation and simply regard these effective field equations in the same light as another classical model of inflation. They are at least no worse than scalar inflaton models, and they do have some remarkable and quite generic features. Chief
of these are that inflation ends in a phase of oscillations which violate the weak energy condition, and for which there is participation from every super-horizon mode, not just the zero mode. The former feature may have left an observable signature in the stochastic background of gravitational radiation [35]. And the last feature should lead to almost instantaneous reheating using only the universal gravitational coupling to matter [36].

Although the simplest of our models breaks down after the onset of matter domination, it can be easily fixed. Indeed, this can be done in such a way as to explain the current phase of cosmic acceleration. It will be interesting to see if any of these models can be derived from fundamental theory.

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