ON THE GENERIC EXISTENCE OF PERIODIC ORBITS IN HAMILTONIAN DYNAMICS

VIKTOR L. GINZBURG AND BAŞAK Z. GÜREL

Abstract. We prove several generic existence results for infinitely many periodic orbits of Hamiltonian diffeomorphisms or Reeb flows. For instance, we show that a Hamiltonian diffeomorphism of a complex projective space or Grassmannian generically has infinitely many periodic orbits. We also consider symplectomorphisms of the two-torus with irrational flux. We show that such a symplectomorphism necessarily has infinitely many periodic orbits whenever it has one and all periodic points are non-degenerate.

CONTENTS

1. Introduction and main results 1
   1.1. Introduction 1
   1.2. Periodic orbits of Hamiltonian diffeomorphisms 2
   1.3. Periodic orbits of symplectomorphisms of the two-torus 5
   1.4. Periodic orbits of Reeb flows 6
   1.5. Acknowledgments 8
2. Proofs and remarks 8
   2.1. Hamiltonian diffeomorphisms: Proofs of Theorems 1.2 and 1.4 8
   2.2. Symplectomorphisms of $\mathbb{T}^2$: Proof of Theorem 1.7 9
   2.3. Reeb flows: Proofs of Theorems 1.11 and 1.13 10
   2.4. Discussion 11
References 12

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. This paper focuses on the problem of $C^\infty$-generic existence of infinitely many periodic orbits for Hamiltonian (or symplectic) diffeomorphisms and Reeb flows. The topology of the underlying symplectic or contact manifold plays an essential role in this question and we prove several generic existence results which apply, among other manifolds, to complex projective spaces and Grassmannians. To put these results into perspective, recall that for a broad class of closed symplectic manifolds, including all symplectically aspherical ones, every Hamiltonian diffeomorphism has infinitely many periodic orbits; see [FH, Hi2, Gi, GG2]. However, in contrast with the symplectically aspherical manifolds, complex projective
spaces and Grassmannians admit Hamiltonian diffeomorphisms with only finitely many periodic orbits, and hence the existence of infinitely many periodic orbits for these manifolds can be expected to only hold $C^\infty$-generically or under suitable additional requirements on the diffeomorphisms.

On the contact side, our results apply to many classes of “fillable” contact forms. These include, among other examples, various sets of contact forms on the unit cotangent bundles and spheres, equipped with standard contact structures. In particular, we recover the $C^\infty$-generic existence of infinitely many closed characteristics on convex hypersurfaces in $\mathbb{R}^{2n}$, [Ek], and the $C^\infty$-generic existence of infinitely many closed geodesics, [Hi1, Ra1, Ra2].

In both the Hamiltonian and contact cases, the proof is based on the fact, established in [GG2, GK], that indices and/or actions of periodic orbits must satisfy certain relations when the diffeomorphisms or the flow in question has only finitely many periodic orbits. Roughly speaking, the argument is that these relations are fragile and can be destroyed by a $C^\infty$-small perturbation, and hence such a perturbation must create infinitely many periodic orbits. (In the contact case, the resonance relations that we utilize generalize those from [Ek, EH, Vi1]. This reasoning is akin to the argument used in [Ek, Ra2].)

It is worth pointing out that a similar generic existence result (Proposition 1.6) holds when $H_{\text{bad}}(M; \mathbb{Z}) \neq 0$ and is an easy consequence of the Birkhoff–Moser fixed point theorem, [Mo], and Floer’s theory; see Section 2.4. However, in this case, there are no examples of Hamiltonian diffeomorphisms with finitely many periodic orbits.

Finally, we also consider symplectomorphisms $\varphi: \mathbb{T}^2 \to \mathbb{T}^2$ with irrational flux or, equivalently, Hamiltonian perturbations of an irrational shift of $\mathbb{T}^2$. We show that $\varphi$ necessarily has infinitely many periodic orbits whenever it has one and all its periodic points are non-degenerate. The proof of this theorem relies on the theory of Floer–Novikov homology developed in [LO, On].

Remark 1.1 (Smoothness). Throughout the paper, for the sake of simplicity, all maps and vector fields are assumed to be $C^\infty$-smooth and the spaces of maps or vector fields are equipped with the $C^\infty$-topology, unless specified otherwise. However, in all our results $C^\infty$ can be replaced by $C^k$ with $k \geq 2$. The only exception is Proposition 1.6 where one has to require that $k \geq 4$.

1.2. Periodic orbits of Hamiltonian diffeomorphisms. Consider a closed symplectic manifold $(M^{2n}, \omega)$, which throughout this paper is assumed to be weakly monotone; see, e.g., [HS] or [MS] for the definition. (This condition can be eliminated by utilizing the machinery of virtual cycles.) Recall that $M$ is said to be monotone (negative monotone) if $\langle \omega \rangle |_{\pi_2(M)} = \lambda c_1(M) |_{\pi_2(M)}$ for some non-negative (respectively, negative) constant $\lambda$ and $M$ is called rational if $\langle [\omega], \pi_2(M) \rangle = \lambda_0 \mathbb{Z}$, i.e., the integrals of $\omega$ over spheres in $M$ form a discrete subgroup of $\mathbb{R}$. (When $\langle [\omega], \pi_2(M) \rangle = 0$, we set $\lambda_0 = \infty$.) The constants $\lambda$ and $\lambda_0 \geq 0$ are referred to as the monotonicity and rationality constants. The positive generator $N$ of the discrete subgroup $\langle c_1(M), \pi_2(M) \rangle \subset \mathbb{R}$ is called the minimal Chern number of $M$. When this subgroup is zero, we set $N = \infty$. The manifold $M$ is called symplectically aspherical if $c_1(M) |_{\pi_2(M)} = 0 = [\omega] |_{\pi_2(M)}$. A symplectically aspherical manifold is monotone and a monotone or negative monotone manifold is rational.

A one-periodic (in time) Hamiltonian $H: \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ determines a time-dependent vector field $X_H$ on $M$ via Hamilton’s equation $i_{X_H} \omega = -dH$. Let
\( \varphi = \varphi_H \) be the Hamiltonian diffeomorphism of \( M \), given as the time-one map of \( X_H \). Recall that there is a one-to-one correspondence between \( k \)-periodic points of \( \varphi \) and \( k \)-periodic orbits of \( H \). In this paper, we restrict our attention exclusively to periodic points of \( \varphi \) such that the corresponding periodic orbits of \( H \) are contractible. A fixed point \( x \) of \( \varphi \) is said to be non-degenerate if \( D\varphi_x : T_x M \to T_x M \) has no eigenvalues equal to one. Recall also that \( \varphi \) is called non-degenerate if all its fixed points are non-degenerate. When all periodic points of \( \varphi \) are non-degenerate we will refer to \( \varphi \) as a strongly non-degenerate Hamiltonian diffeomorphism. Clearly, \( \varphi \) is strongly non-degenerate if and only if all iterations \( \varphi^k, k \geq 1 \), are non-degenerate.

In what follows, we will denote the group of Hamiltonian diffeomorphisms of \( M \), equipped with the \( C^\infty \)-topology, by \( \text{Ham}(M, \omega) \).

Our first result concerns the generic existence of infinitely many periodic orbits on symplectic manifolds with large \( N \).

**Theorem 1.2.** Assume that \( n+1 \leq N < \infty \). Then strongly non-degenerate Hamiltonian diffeomorphisms with infinitely many periodic orbits form a \( C^\infty \)-residual set in \( \text{Ham}(M, \omega) \).

**Example 1.3.** The only known monotone manifold to which this theorem applies is \( \mathbb{CP}^n \). This manifold admits Hamiltonian diffeomorphisms with finitely many fixed points. The simplest of such diffeomorphisms is an irrational rotation of \( S^2 \). Similar diffeomorphisms, arising from Hamiltonian torus actions, exist in higher dimensions; see, e.g., [GK]. These examples show that the genericity assumption in Theorem 1.2 is essential. Note also that a Hamiltonian diffeomorphism with finitely many periodic orbits need not be associated with a Hamiltonian torus action. For instance, there exists a Hamiltonian perturbation \( \varphi \) of an irrational rotation of \( S^2 \) with exactly three ergodic invariant measures: the Lebesgue measure and the two measures corresponding to the fixed points of \( \varphi \); [AK, FK]. There exist also a multitude of negative monotone manifolds satisfying the hypotheses of the theorem. However, to the best of the authors’ knowledge there are neither examples of negative monotone manifolds admitting Hamiltonian diffeomorphisms with finitely many periodic orbits nor any results asserting that such manifolds do not exist.

The theorem also holds when \( N = \infty \), i.e., \( c_1(TM) \mid_{\pi_2(M)} = 0 \), as can be easily seen by arguing as in [SZ]. However, in this case we expect the Conley conjecture to hold, i.e., every Hamiltonian diffeomorphism to have infinitely many periodic orbits. For instance, when, in addition, \( M \) is rational, this is proved in [GG2]; see also [Gi]. Furthermore, when \( M = S^2 \), a much stronger generic existence result is established by the methods of two-dimensional dynamics in [We]. Namely, a \( C^\infty \)-generic Hamiltonian diffeomorphism of \( S^2 \) has positive topological entropy and, as a consequence, infinitely many hyperbolic periodic points. (The proof relies on Pixton’s theorem asserting, roughly speaking, the generic existence of hyperbolic periodic points of a particular type; see [Pi].)

Theorem 1.2, proved in Section 2.1, is an easy consequence of the mean index resonance relations established in [GK]. A different type of relations, involving both the mean indices and actions and proved in [GG2], leads to our next, more technical, result. Denote by \( \Lambda \) the Novikov ring of \( M \), equipped with the valuation \( L_\omega(A) := -\langle \omega, A \rangle, A \in \pi_2(M) \), and by * the pair-of-pants product in the quantum homology \( HQ_*(M) \). (We refer the reader to, e.g., [MS] for a detailed discussion of these notions; throughout this paper we adhere to the conventions from [GG2].)
Section 2.) Recall also that the Hofer norm $\|\varphi\|$ of $\varphi \in \text{Ham}(M, \omega)$ is defined as

$$\|\varphi\| = \inf_H \int_0^1 (\max_M H_t - \min_M H_t) dt,$$

where the infimum is taken over all $H$ such that $\varphi_H = \varphi$; see, e.g., [Po].

**Theorem 1.4.** Assume that $M$ is monotone or negative monotone with monotonicity constant $\lambda$ satisfying $|\lambda| < \infty$.

(i) Then strongly non-degenerate Hamiltonian diffeomorphisms with infinitely many periodic orbits form a $C^\infty$-residual set in the Hofer ball

$$\mathcal{B} = \{\varphi \in \text{Ham}(M, \omega) \mid \|\varphi\| < \lambda_0\},$$

where $\lambda_0$ is the rationality constant of $M$.

(ii) Assume in addition that there exists $u \in H^*_{<2n}(M)$ and $w \in H^*_{<2n}(M)$ and $\alpha \in \Lambda$ such that

$$[M] = (\alpha u) \ast w$$

and one of the following requirements is satisfied

- (a) $I_\omega(\alpha) = \lambda_0$;
- (b) $2n - \deg u < 2N.$

Then strongly non-degenerate Hamiltonian diffeomorphisms with infinitely many periodic orbits form a $C^\infty$-residual set in $\text{Ham}(M, \omega)$.

This theorem is proved in Section 2.1.

**Example 1.5.** Let us list some of the manifolds satisfying the hypotheses of Theorem 1.4(ii); see, e.g., [GG2] and references therein for details.

- The complex projective spaces $\mathbb{C}P^n$ and complex Grassmannians satisfy (1.1) and both (a) and (b).
- Assume that $M$ satisfies (1.1) and (a) or (b) and $P$ is symplectically aspherical. Then $M \times P$ satisfies (1.1) and (a) or, respectively, (b).
- The product $M \times W$ of two rational manifolds satisfies (1.1) and (a) whenever $M$ does and $\lambda_0(W) = m\lambda_0(M)$, where $m$ is a positive integer or $\infty$. For instance, (a) holds for the products $\mathbb{C}P^n \times \mathbb{C}P^{m_1} \times \ldots \times \mathbb{C}P^{m_r}$ with $m_1 + 1, \ldots, m_r + 1$ divisible by $n + 1$ and equally normalized symplectic structures.
- The monotone product $\mathbb{C}P^n \times W$, where $W$ is monotone and $\gcd(n + 1, N(W)) \geq 2$, satisfies (1.1) and (b). For instance, this is the case for the monotone product $\mathbb{C}P^{m_1} \times \ldots \times \mathbb{C}P^{m_r}$ if $\gcd(n_1 + 1, \ldots, n_r + 1) \geq 2$.

While the proofs of Theorems 1.2 and 1.4 are global and rely on relations between the mean indices and/or actions of the periodic orbits, a local argument based on the Birkhoff–Moser fixed point theorem (see [Mo]) combined with some input from Floer’s theory, yields the following.

**Proposition 1.6.** Assume that $M$ is weakly monotone and $H_{\omega_{\text{std}}}(M; \mathbb{Z}) \neq 0$. Then strongly non-degenerate Hamiltonian diffeomorphisms with infinitely many periodic orbits form a $C^\infty$-residual set in $\text{Ham}(M, \omega)$.

This proposition, proved in Section 2.4, also covers the second case of Example 1.5. Note that here the requirement that $M$ be weakly monotone is purely technical and can be eliminated completely with the use of virtual cycles. It is also worth pointing out that in the setting of the proposition, we have no examples of
Hamiltonian diffeomorphisms with finitely many periodic points. It is possible that the Conley conjecture holds for symplectic manifolds $M$ with $\text{Hom}(M; \mathbb{Z}) \neq 0$, i.e., every Hamiltonian diffeomorphism of $M$ has infinitely many periodic points.

Drawing on Theorems 1.2 and 1.4 and Proposition 1.6, we conjecture that the existence of infinitely many periodic points is a $C^\infty$-generic property, at least when $M$ is monotone or negative monotone. Furthermore, it is perhaps illuminating to look at these results in the context of the closing lemma asserting, in particular, that the existence of a dense set of periodic orbits is $C^1$-generic for both Hamiltonian diffeomorphisms and flows; see [PR]. Thus, once the $C^\infty$-topology is replaced by the $C^1$-topology a much stronger result than the generic existence of infinitely many periodic orbits holds – the dense existence. However, this is no longer true for the $C^k$-topology with $k > \dim M$ as the results of M. Herman show (see [He1, He2]) and the above conjecture on the $C^\infty$-generic existence of infinitely many periodic orbits can be viewed as a viable form of a $C^\infty$-closing lemma.

Furthermore, the aforementioned results of Herman (or rather the argument from [He1, He2]) also suggest that a $C^\infty$-small Hamiltonian perturbation $\varphi$ of the shift $R_\theta$ of the standard symplectic torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ never has periodic orbits when $\theta$ satisfies a certain Diophantine condition. (In other words, here $R_\theta(x) = x + \theta$, where $x \in \mathbb{T}^{2n}$, and $\varphi R_\theta^{-1}$ is a $C^\infty$-small Hamiltonian diffeomorphism.) The situation however changes dramatically, as we will see in the next section, when $\varphi$ is required to have at least one periodic point, at least in the non-degenerate case and in dimension two.

1.3. Periodic orbits of symplectomorphisms of the two-torus. Let us consider a symplectomorphism $\varphi: \mathbb{T}^2 \to \mathbb{T}^2$ such that $\varphi R_\theta^{-1}$ is a Hamiltonian diffeomorphism for some $\theta \in \mathbb{T}^2$. It is easy to see that this requirement is equivalent to that the flux of $\varphi$ is $\theta$; see [Ba].

**Theorem 1.7.** Assume that $\varphi$ has at least one fixed point, all fixed points of $\varphi$ are non-degenerate, and that at least one of the components $\theta_1$ or $\theta_2$ of $\theta$ is irrational. Then $\varphi$ has infinitely many periodic orbits. Moreover, for any sufficiently large prime $k$, there is a simple $k$-periodic orbit.

This theorem, proved in Section 2.2, is a symplectomorphism (two-dimensional) version of the non-degenerate Conley conjecture established in [SZ]; see also [CZ]. As an immediate consequence of Theorem 1.7, we obtain the following result.

**Corollary 1.8.** Assume that $\varphi$ is strongly non-degenerate and has at least one periodic point, and that at least one of the components $\theta_1$ or $\theta_2$ of $\theta$ is irrational. Then $\varphi$ has infinitely many periodic orbits.

**Remark 1.9.** The corollary still holds (without any non-degeneracy or fixed point requirement) when the flux $\theta$ is rational. Indeed, in this case a suitable iteration of $\varphi$ is Hamiltonian and one can apply the Conley conjecture, which in this case is proved in [FH]

One can view Corollary 1.8 as a generic existence result similar to Theorem 1.2 and 1.4, with the class of Hamiltonian diffeomorphisms replaced by symplectomorphisms with flux $\theta$ and at least one homologically non-trivial periodic point. However, this similarity is probably superficial and we conjecture that any symplectomorphism of the standard $\mathbb{T}^{2n}$ with “sufficiently irrational” flux and at least one homologically essential periodic point has infinitely many periodic orbits. Note
that here, in contrast with many other Conley conjecture type results (see, e.g., [FH, Gi, Hi2] and references therein), the assumption that the periodic point is homologically essential is necessary as the following example due to John Franks indicates.

**Example 1.10.** Consider the one-form \( \alpha_0 = -\theta_2 dq_1 + \theta_1 dq_2 \) on the two-torus \( \mathbb{T}^2 \) with angular coordinates \( (q_1, q_2) \) and symplectic form \( dq_1 \wedge dq_2 \). The time-one map of the symplectic flow generated by \( \alpha_0 \) is \( R_{\theta^0} \), where \( \theta^0 = (\theta_1, \theta_2) \). This map has no periodic points when at least one of the components of \( \theta^0 \) is irrational. Fix a small closed flow box \( U \) in \( \mathbb{T}^2 \), and let \( H \) be a smooth function on \( U \) such that \( H \) has only one critical point \( x \) in \( U \) and \( dH = \alpha_0 \) near \( \partial U \) and, finally, no Hamiltonian flow line of \( H \) other than \( x \) is entirely contained in \( U \). Denote by \( \alpha \) the closed one-form obtained from \( \alpha_0 \) by replacing \( \alpha_0 \) by \( dH \) in \( U \). It is easy to see that \( x \) is the only periodic point of the time-one map \( \varphi \) generated by \( \alpha \). (By construction, the flow of \( \alpha \) has no closed orbits other than the fixed point \( x \).) Furthermore, although the flux \( \theta \) of \( \varphi \) can be different from \( \theta^0 \), at least one component of \( \theta \) is irrational by the Arnold conjecture. Thus, \( \varphi \) satisfies all hypotheses of Theorem 1.7 but the non-degeneracy. (As has been pointed out above, \( \varphi \) is automatically a Hamiltonian perturbation \( R_{\theta^0} \); see [Ba].) Note that the fixed point \( x \) of \( \varphi \) is degenerate and, moreover, \( x \) can be destroyed by an arbitrarily small perturbation. In particular, the local Floer homology of \( x \) is trivial; see [Gi, GG1].

1.4. Periodic orbits of Reeb flows. In this section we state the analogues of Theorems 1.2 and 1.4 for Reeb flows. Similarly to the results from Section 1.2 that are based on the existence of resonance relations for mean indices and actions of Hamiltonian diffeomorphisms, the results of this section follow from the resonance relations for Reeb flows, [GK].

To state these results, consider a closed contact manifold \((M, \xi)\). Throughout this section we assume that \( c_1(\xi) = 0 \) although this condition may in some instances be relaxed. We refer the reader to [Ge] for a general introduction to contact topology; see also [Bo, BO, El]. Having the contact manifold \((M, \xi)\) fixed, we only consider contact forms \( \alpha \) on \( M \) with \( \ker \alpha = \xi \). We say that the **Weinstein conjecture holds for** \((M, \xi)\) **if the Reeb flow for every contact form** \( \alpha \) **has a periodic orbit**; see, e.g., [Ge]. We call a contact form **non-degenerate** when all periodic orbits of its Reeb flow are non-degenerate. (This notion is similar to strong non-degeneracy for Hamiltonian diffeomorphisms.) For a non-degenerate closed Reeb orbit \( x \), we set its degree to be \( |x| = \mu_{cz}(x) + n - 3 \), where \( \mu_{cz}(x) \) stands for the Conley–Zehnder index of \( x \) (with its standard normalization), and denote by \( \Delta(x) \) the mean index of \( x \).

Our generic existence results concern sets of contact structures satisfying some additional conditions. Namely, let \( \mathcal{C} \) be a set of contact forms \( \alpha \) (such that \( \ker \alpha = \xi \)), equipped with the \( C^\infty \)-topology, meeting the following two requirements:

1. The intersection of the set of non-degenerate contact forms with \( \mathcal{C} \) is a residual set in \( \mathcal{C} \).
2. For every non-degenerate form \( \alpha \in \mathcal{C} \), a periodic orbit \( x \) of the Reeb flow of \( \alpha \) and a small neighborhood \( U \) of \( x \), there exists a sequence of perturbations \( \alpha_k \to \alpha \) in \( \mathcal{C} \) such that \( \alpha_k - \alpha \) is supported in \( U \) and \( \Delta(x_k) \neq \Delta(x) \) for all \( k \), where \( x_k \) is the periodic orbit of the Reeb flow of \( \alpha_k \) arising as a perturbation of \( x \).
Note that the orbit \( x_k \) in (C2) is unique and close to \( x \) once \( U \) and \( \alpha - \alpha_k \) are sufficiently small and that \( \Delta(x_k) \to \Delta(x) \) due to continuity of \( \Delta \); see [SZ]. Thus, condition (C2) is the requirement that \( \Delta(x) \) can be varied by varying \( \alpha \) in \( \mathcal{C} \).

To state the first result, assume that \( M \) is the boundary of a compact manifold \( W \). We say that \((M, \xi)\) is symplectically fillable by \( W \) if \( d\alpha \to 0 \) for some contact form \( \alpha_0 \) on \((M, \xi)\), extends to a symplectically aspherical symplectic structure \( \omega_0 \) on \( W \). Then, as is not hard to see, the same is true for every contact form \( \alpha \) on \((M, \xi)\). The extension \( \omega \) of \( d\alpha \) is obtained from \( \omega_0 \) by modifying the latter near \( M = \partial W \), and \((M, \alpha_0)\) and \((M, \alpha)\) have the same linearized contact homology with respect to these fillings.

**Theorem 1.11.** Assume that \((M, \xi)\) is symplectically fillable, the Weinstein conjecture holds for \( \xi \), and that \( \mathcal{C} \) satisfies (C1) and (C2). Then non-degenerate contact forms with infinitely many periodic orbits form a residual subset in \( \mathcal{C} \).

**Example 1.12.** Assume that \((M, \xi)\) is symplectically fillable. Then requirements (C1) and (C2) are obviously satisfied for the set of all contact forms \( \alpha \) on \((M, \xi)\).

The second result concerns the situation where the contact manifold \((M, \xi)\) is not required to be fillable.

**Theorem 1.13.** Assume that the Weinstein conjecture holds for \( \xi \), the set \( \mathcal{C} \) meets requirements (C1), (C2) and also the following:

(C3) the Reeb flow of any non-degenerate contact form in \( \mathcal{C} \) has no contractible periodic orbits \( x \) with \( |x| = 0 \) or \( \pm 1 \).

Then non-degenerate contact forms with infinitely many periodic orbits form a residual subset in \( \mathcal{C} \).

**Remark 1.14.** Requirement (C3) is quite restrictive: there exist contact structures which admit no non-degenerate contact forms satisfying (C3). For instance, this is the case for an overtwisted contact structure on \( S^3 \); see, e.g., [Ya].

**Example 1.15.** Let \( M \) be the unit cotangent bundle \( ST^*P \) of a closed manifold \( P \), equipped with the standard contact structure and let \( \mathcal{C} \) be the set of contact forms on \( M \) associated with Riemannian metrics on \( P \). Then \((M, \xi)\) is fillable and \( \mathcal{C} \) satisfies requirements (C1) and (C2); condition (C3) is met whenever \( \dim P > 3 \). (See [Ab, An] for the proof of (C1) and [AS, Lo, Vi2] and references therein and, in particular [Du], for a discussion of indices; (C2) can be established similarly to the argument from [KT].) Likewise, let \( M = S^{2n-1} \) be equipped with the standard contact structure and let \( \mathcal{C} \) be the class of contact forms arising from embeddings of \( M \) into \( \mathbb{R}^{2n} \) as a strictly convex hypersurface enclosing the origin. Then (C1) is satisfied and it is not hard to see that \( \mathcal{C} \) also meets requirements (C2) and (C3); see, e.g., [Lo] and references therein. Furthermore, any contact form \( \alpha \) on \( S^{2n-1} \) giving rise to the standard contact structure is symplectically fillable by \( W = B^{2n} \). (Indeed, \( \alpha \) is the restriction of the form \( \sum (x_i dy_i - y_i dx_i)/2 \) on \( \mathbb{R}^{2n} \) to an embedding \( S^{2n-1} \hookrightarrow \mathbb{R}^{2n} \) bounding a starshaped domain.) Thus, Theorems 1.11 and 1.13 generalize the \( C^\infty \)-generic existence of infinitely many closed characteristics on convex hypersurfaces in \( \mathbb{R}^{2n} \) (see [Ek]) and the \( C^\infty \)-generic existence of infinitely many closed geodesics (see [Hi1, Ra1, Ra2]). Similar considerations apply, of course, to Finsler metrics, symmetric as well as asymmetric.
Remark 1.16. In Theorems 1.11 and 1.13 one can also require the homotopy classes of closed orbits to lie in a fixed set of free homotopy classes of loops in $M$, closed under iterations. (For instance, one can require the orbits to be contractible.) However, in this case the Weinstein conjecture must also hold for such orbits, which is a non-trivial restriction on the contact structure and the set of homotopy classes. (The argument from Section 2.3 readily proves this generalization of the theorems, cf. the discussion in [GK, Section 1.3].)

1.5. Acknowledgments. We are grateful to Alberto Abbondandolo, Christian Bonatti, Yasha Eliashberg, John Franks, Anatole Katok, Leonid Polterovich, and Marcelo Viana for useful comments and remarks.

2. Proofs and remarks

2.1. Hamiltonian diffeomorphisms: Proofs of Theorems 1.2 and 1.4.

Proof of Theorem 1.2. Recall that to a contractible periodic orbit $x$ of $H$ we can associate the mean index $\Delta(x) \in \mathbb{R}/2\mathbb{N}$ as in [SZ]. Strictly speaking, the mean index, viewed as a real number, depends on the choice of the capping of $x$. However, $\Delta(x)$ is well defined once it is regarded as an element of the circle $\mathbb{R}/2\mathbb{N}$. Consider the set $\Delta^\infty$ formed by the indices of simple contractible periodic orbits of $H$ for all periods. (Here we treat $\Delta^\infty$ as a genuine set: if two orbits have equal indices, their index enters the collection only once.) Denote by $\Delta^k$ the subset of $\Delta^\infty$ formed by the mean indices of periodic orbits with period less than $k$. Note that the set $\Delta^k$ is necessarily finite when $H$ is strongly non-degenerate.

Recall also that, as readily follows from [GK, Theorem 1.1 and Remark 1.6], whenever $\varphi_H$ has finitely many periodic orbits, the set $\Delta^\infty = (\Delta_1, \ldots, \Delta_m)$ satisfies a resonance relation of the form

$$a_1\Delta_1 + \ldots + a_m\Delta_m = 0 \mod 2N$$

for some non-zero vector $\vec{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$.

Let $\mathcal{N}_k(\vec{a})$ be the set of strongly non-degenerate Hamiltonian diffeomorphisms $\varphi$ that do not satisfy the resonance relation (2.1) up to period $k$. (Here we do not require $\varphi$ to have finitely many periodic points.) More precisely, a non-degenerate Hamiltonian diffeomorphism is in $\mathcal{N}_k(\vec{a})$ if either the number of non-zero components in the vector $\vec{a}$ exceeds the cardinality of $\Delta^k$ or (2.1) fails for any choice of a subset $\Delta = (\Delta_1, \ldots, \Delta_m)$ in $\Delta^k$ and any ordering of this subset.

It is routine to show that $\mathcal{N}_k(\vec{a})$ is an open, dense subset in Ham($M, \omega$) in the $C^\infty$-topology. Indeed, it is clear that $\mathcal{N}_k(\vec{a})$ is open. To prove that it is dense, let us consider a non-degenerate Hamiltonian diffeomorphism satisfying the resonance relation (2.1) for some subset $\Delta \subset \Delta^k$ formed by $\Delta_i = \Delta(x_i)$ where all orbits $x_i$ have period less than or equal to $k$ and, say, $a_1 \neq 0$. (We will assume for the sake of simplicity that $\Delta$ is the only ordered collection in $\Delta^k$ satisfying (2.1) and that $x_1$ is the only orbit with mean index $\Delta_1$ and period less than $k$. The general case can be dealt with in a similar fashion.) By applying a sufficiently $C^\infty$-small perturbation of $H$ localized near $x_1$ in the “space-time” $S^1 \times M$, we can change the value of $\Delta(x_1)$ without affecting other periodic orbits of period up to $k$. This change will destroy the resonance relation (2.1). It will also create no periodic orbits of period less than $k$ (since $x_1$ is non-degenerate), and hence no new resonance relations for such orbits. Thus, $\mathcal{N}_k(\vec{a})$ is dense.
Taking the intersection of these sets for all \( k \) and \( \bar{a} \), we obtain a \( C^\infty \)-residual subset \( \mathcal{N} \) of \( \text{Ham}(M,\omega) \), which, by the result from [GK] quoted above, contains no Hamiltonian diffeomorphisms with finitely many periodic orbits. \( \square \)

Proof of Theorem 1.4. The argument is similar to the proof of Theorem 1.2, but instead of the resonance relations (2.1) from [GK] we utilize a relation involving both the mean indices and actions and proved in [GG2].

Let, as above, \( H \) be a one-periodic in time Hamiltonian on \( M \) and let \( x \) be a \( k \)-periodic orbit of \( H \). The normalized augmented action of \( H \) on \( x \) is defined as

\[
\tilde{A}_H(x) = \left( A_H(\bar{x}) - \lambda \Delta_H(\bar{x}) \right)/k,
\]

where \( \bar{x} \) is the orbit \( x \) equipped with an arbitrary capping. Here \( A_H(\bar{x}) \) stands for the ordinary action of \( H \) on \( \bar{x} \) and \( \Delta_H(\bar{x}) \) is mean index of \( H \) on \( \bar{x} \). It is clear that \( \tilde{A}_H(x) \) is independent of the capping. This definition, borrowed from [GG2], is inspired by the considerations in [Sa, Section 1.6] and [EP, Section 1.4], where the Conley–Zehnder index is utilized in place of the mean index. For us, the main advantage of using the mean index is that iterating an orbit does not change the normalized augmented action, i.e., \( \tilde{A}_H(x^i) = \tilde{A}_H(x) \). The reason is that the ordinary action and the mean index are both homogeneous: \( \tilde{A}_H(\bar{x}^i) = l \tilde{A}_H(\bar{x}) \) and \( \Delta_H(\bar{x}^i) = l \Delta_H(\bar{x}) \). Moreover, geometrically identical orbits have equal normalized augmented action. (Two periodic orbits of \( H \) are said to be geometrically identical if the corresponding periodic orbits of \( \varphi_H \) coincide as subsets of \( M \); see [GG2, Section 1.3].)

By [GG2, Corollary 1.11], under the hypotheses of the theorem, there exist two geometrically distinct periodic orbits \( x \) and \( y \) of \( H \) with \( \tilde{A}_H(x) = \tilde{A}_H(y) \) whenever \( \varphi_H \) has finitely many periodic orbits. In case (ii), let \( \mathcal{N}_k \) be the set of strongly non-degenerate Hamiltonian diffeomorphisms \( \varphi_H \) such that all geometrically distinct periodic orbits of \( \varphi_H \) up to period \( k \) have different normalized augmented actions. As in the proof of Theorem 1.2, it is easy to show that \( \mathcal{N}_k \) is open and dense in \( \text{Ham}(M,\omega) \) in the \( C^\infty \)-topology. (In case (i), \( \mathcal{N}_k \) is defined similarly, but only as a subset of \( \mathcal{B} \). Then \( \mathcal{N}_k \) is open and dense in \( \mathcal{B} \).) The set \( \mathcal{N} = \bigcap_k \mathcal{N}_k \) is residual and contains no Hamiltonian diffeomorphisms with finitely many periodic orbits. \( \square \)

2.2. Symplectomorphisms of \( \mathbb{T}^2 \): Proof of Theorem 1.7. The proof of the theorem relies on the machinery of Floer–Novikov homology developed in [LO, On] and throughout the proof we use the conventions and notation from these works.

Proof of Theorem 1.7. Let us assume first that \( \varphi^k \) is a non-degenerate iteration of \( \varphi \) and denote by \( \text{CF}_*(\varphi^k) \) the Floer–Novikov complex of \( \varphi^k \). (Strictly speaking, to define this complex, we need to pick a path \( \psi_1 \) connecting \( \varphi^k \) to the identity. To this end, we fix a path \( \varphi_t \) from \( \varphi \) to \( \text{id} \) and set \( \psi_1 = \varphi^k_t \). The choice of the path \( \varphi_t \) is immaterial.) The complex \( \text{CF}_*(\varphi^k) \) is generated by the fixed points of \( \varphi^k \), corresponding to the contractible \( k \)-periodic orbits of \( \varphi_t \), over the Novikov ring \( \Lambda_{k\theta} \); see [LO]. It is essential for what follows that the elements of \( \Lambda_{k\theta} \) have zero degree. (To be quite precise, one has to consider the lifts of the fixed points to the universal covering \( \mathbb{R}^2 \) of \( \mathbb{T}^2 \). Note also that the Novikov ring \( \Lambda_{k\theta} \) depends on \( k \) if one of the components of \( \theta \) is rational.)

The complex \( \text{CF}_*(\varphi^k) \) is acyclic. Indeed, \( \varphi^k \) is a Hamiltonian deformation of \( R_{k\theta} \), and hence the two symplectomorphisms have the same Floer–Novikov homology.
Furthermore, since at least one of the components of $\theta$ is irrational $R_{k\theta}$ has no fixed points, and hence its Floer--Novikov homology vanishes.

Arguing by contradiction, assume now that $\varphi$ has no simple $k$-periodic points for a (large) prime $k$. Then the fixed points of $\varphi^k$ are the $k$th iterations of the fixed points of $\varphi$. (Here and in what follows, we consider only the fixed points corresponding to the contractible orbits of $\varphi_t$.) Furthermore, once $k$ is greater than the degree of any root of unity among the Floquet multipliers of the one-periodic orbits of $\varphi_t$, the iteration $\varphi^k$ is non-degenerate and the complex $\CF_*(\varphi^k)$ is generated by the $k$th iterations of the fixed points of $\varphi$. Let us group these fixed points according to their mean indices, placing all fixed points with the same index into one group. Thus, we have $r$ groups corresponding to different real numbers $\Delta_1, \ldots, \Delta_r$ occurring as the mean indices for $\varphi$. This grouping is carried over to the $k$th iteration $\varphi^k$. When $k$ is so large that $k|\Delta_i - \Delta_j| > 3$ (if $i \neq j$), the complex $\CF_*(\varphi^k)$ breaks down into a direct sum of complexes each of which is generated by the $k$th iteration of the fixed points from one group. Each of these complexes is acyclic. Our goal is to show that this is impossible.

To this end, consider the complex generated by the orbits in one group, say, $x^k_1, \ldots, x^k_m$. (Since $\varphi$ has a fixed point, there is at least one non-empty group.) Since all orbits $x^k_i$ are non-degenerate, we have

$$|\mu_{\text{CZ}}(x^k_i) - k\Delta| < 1,$$

where $\Delta$ is the mean index of the group; see [CZ, SZ].

Then $\mu_{\text{CZ}}(x^k_i) = k\Delta$ if $k\Delta \in \mathbb{Z}$. Thus, in this case all generators $x^k_i$ have the same degree, which is impossible, for the complex is acyclic. When $k\Delta \notin \mathbb{Z}$, every orbit $x^k_i$ is necessarily elliptic: its Floquet multipliers $\lambda, \bar{\lambda}$ are on the unit circle and different from $\pm 1$. For such an orbit, in dimension two, the mean index completely determines the Conley–Zehnder index. Hence again, we arrive at a contradiction with acyclicity, for all generators $x^k_i$ have the same degree. \hfill $\Box$

### 2.3. Reeb flows: Proofs of Theorems 1.11 and 1.13

The proof of both of these results is quite similar to the proofs of Theorems 1.2 and 1.4 except that now a different resonance relation is used.

**Proof of Theorems 1.11 and 1.13.** Let $\mathcal{C}$ be as in either of the theorems and let $\alpha \in \mathcal{C}$ be non-degenerate. Following [GK], we call a simple periodic orbit $x$ of the Reeb flow of $\alpha$ bad if the linearized Poincaré return map along $x$ has an odd number of real eigenvalues strictly smaller than $-1$. Otherwise, the orbit is said to be good. (This terminology differs slightly from the standard usage, cf. [Bo, BO].) When the orbit $x$ is good, the parity of the Conley--Zehnder indices $\mu_{\text{CZ}}(x^k)$ is independent of $k$: if $x$ is bad, the parity of $\mu_{\text{CZ}}(x^k)$ depends on the parity of $k$. We denote the mean index of an orbit $x$ by $\Delta(x)$ and set $\sigma(x) = (-1)^{|x|} = (-1)^{n+1}(-1)^{\mu_{\text{CZ}}(x)}$. In other words, $\sigma(x)$ is, up to the factor $(-1)^{n+1}$, the topological index of the orbit $x$ or, more precisely, of the Poincaré return map along $x$.

Furthermore, under the hypotheses of either of the theorems, the contact homology $\text{HC}_*(M, \xi)$ of $(M, \xi)$ is defined and independent of $\alpha \in \mathcal{C}$; see [Bo, BO]. (In the case of Theorem 1.11, we use the fact, mentioned in Section 1.4, that the linearized contact homology is independent of $\alpha$ as long as the fillings are adjusted...
Accordingly. As in [GK], set
\[
\chi^{\pm}(W, \xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{l=\pm}^{N} (-1)^{l} \dim \text{HC}_{\pm l}(W, \xi),
\]
where \( l_- = -2 \) and \( l_+ = 2n - 4 \), provided that all terms are finite and the limits exist, and let
\[
\chi(W, \xi) := \frac{\chi^+(W, \xi) + \chi^-(W, \xi)}{2}.
\]
We call \( \chi(W, \xi) \) the mean Euler characteristic of \( \xi \). (This invariant is also considered in [VK, Section 11.1.3].)

Then, whenever the Reeb flow of \( \alpha \) has finitely many simple periodic orbits, \( \dim \text{HC}_{\pm l}(W, \xi) < \infty \) when \( \pm l > l_\pm \), the limits in (2.2) exist, and the mean indices of the orbits satisfy the resonance relation
\[
\sum \sigma(x_i) \Delta(x_i) + \frac{1}{2} \sum \sigma(y_i) \Delta(y_i) = \chi(W, \xi),
\]
where the first sum is over all good simple periodic orbits \( x_i \), the second sum is over all bad simple periodic orbits \( y_i \), and in both cases the orbits with zero mean index are excluded. (See [GK] for a proof; this result generalizes the resonance relations from [EH, Vi].)

Denote by \( \mathcal{N}_k \) the set of non-degenerate forms \( \alpha \in \mathcal{C} \) such that (2.3) fails when the summation on the left hand side is taken over all simple periodic orbits of period less than \( k \). Note that, even though now we are not assuming that the Reeb flow has finitely many periodic orbits, the number of orbits with period bounded from above by \( k \) is finite as a consequence of non-degeneracy.

We claim that, when \( k \) is sufficiently large, the set \( \mathcal{N}_k \) is \( C^\infty \)-open and dense in \( \mathcal{C} \). Indeed, it is clear that \( \mathcal{N}_k \) is open. To show that it is dense, it suffices, by (C1), to prove that any neighborhood of a non-degenerate form \( \alpha_0 \in \mathcal{C} \) contains a form \( \alpha \in \mathcal{N}_k \). This is clear due to (C2): by varying \( \alpha_0 \) in \( \mathcal{C} \), one can make (2.3) fail for orbits with period bounded by \( k \). (At this point we use the assumption that the Weinstein conjecture holds for \( \xi \) to make sure that the left hand side of (2.3) contains at least one term when \( k \) is large.)

Finally, taking the intersection of the sets \( \mathcal{N}_k \), for all large \( k \), we obtain a \( C^\infty \)-residual set \( \mathcal{N} \subset \mathcal{C} \) of non-degenerate forms and, by (2.3), the Reeb flow of any form in \( \mathcal{N} \) has infinitely many periodic orbits. \( \square \)

2.4. Discussion. One common requirement in the majority of generic existence results for infinitely many periodic orbits is the existence of one such orbit, which in some instances must satisfy certain additional conditions.

To illustrate this point, recall that the Birkhoff--Moser fixed point theorem (see [Mo]) asserts that when \( x \) is a non-degenerate, non-hyperbolic periodic point of a Hamiltonian diffeomorphism \( \varphi \), a \( C^\infty \)-generic perturbation of \( \varphi \) (which can be taken to be strongly non-degenerate) has infinitely many periodic points in a neighborhood of \( x \). (A \( k \)-periodic point \( x \) is called non-hyperbolic if at least one of the eigenvalues of the linearized map \( d\varphi^k : T_x M \to T_x M \) is on the unit circle.) Thus, the Birkhoff--Moser fixed point theorem implies the \( C^\infty \)-generic existence of infinitely many periodic orbits in a neighborhood (in the \( C^\infty \)-topology) of a Hamiltonian diffeomorphism with a non-degenerate, non-hyperbolic periodic point. Then one may try to deal with the existence problem in the case where all periodic points
are hyperbolic by some different, usually \textit{ad hoc}, method. For instance, this is the approach used in [Hi1] to establish the generic existence of infinitely many geodesics on simply connected, compact symmetric spaces of rank one (e.g., spheres). In the case of Hamiltonian diffeomorphisms, arguing along these lines we obtain Proposition 1.6.

\textbf{Proof of Proposition 1.6.} By the Birkhoff–Moser fixed point theorem it suffices to show that a strongly non-degenerate Hamiltonian diffeomorphism $\varphi$ has a non-hyperbolic periodic point or $\varphi$ has infinitely many periodic orbits. Assume that all periodic points of $\varphi$ are hyperbolic. Then all periodic points of $\varphi$ have even Conley–Zehnder indices, which is impossible by Floer’s theory since $H_{\text{odd}}(M; \mathbb{Z}) \neq 0$ (see, e.g., [Sa, SZ]), or every iteration of $\varphi$ has a hyperbolic point with negative real eigenvalues. It follows then that every iteration of the form $\varphi^{2k}$ must have a simple periodic orbit and thus the number of periodic orbits is infinite. \hfill $\square$

In a similar vein, the existence of a periodic point (hyperbolic) plays an important role in the argument from [We], mentioned in Section 1.2, concerning Hamiltonian diffeomorphisms of $S^2$.

The method employed in this paper and utilizing the resonance relations is no exception in that it also relies, implicitly or explicitly, on the existence of one periodic point, even though we use global symplectic-topological rather than dynamical systems arguments. In Theorems 1.2 and 1.4, the existence of one fixed point is implicit, for this requirement is automatic due to the Arnold conjecture. In Theorem 1.7, the requirement is explicit and essential; see Section 1.3. In Theorems 1.11 and 1.13, the requirement is also explicit (the Weinstein conjecture), but is conjecturally always met.

\textbf{Remark 2.1.} We conclude this discussion by pointing out an aspect of the generic existence problem for periodic orbits that is not touched upon in this paper (except in Theorem 1.7). This is the question of the (generic) growth of the number of periodic orbits, which is of interest even when infinitely many periodic orbits exist unconditionally. For instance, when the manifold is symplectically aspherical, a generic Hamiltonian diffeomorphism has a simple periodic orbit for every sufficiently large prime period; see [SZ]. Thus, the number of geometrically distinct periodic orbits of period less than or equal to $k$ generically grows at least as $k/\log k$. A similar lower bound holds, up to a factor, for closed geodesics (cf. Example 1.15) on a Riemannian manifold with a finite, but non-trivial, fundamental group, [BTZ]. However, to the best of the authors’ knowledge no such results for generic Hamiltonian diffeomorphisms of, say, $\mathbb{CP}^n$ or under the hypotheses of Proposition 1.6 have been obtained.

\begin{thebibliography}{99}

[Ab] R. Abraham, Bumpy metrics, in 1970 Global Analysis (ProcSympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) pp. 1–3, Amer. Math. Soc., Providence, R.I.

[AS] A. Abbondandolo, M. Schwarz, On the Floer homology of cotangent bundles, \textit{Comm. Pure Appl. Math.}, 59 (2006), no. 2, 254–316.

[An] D.V. Anosov, Generic properties of closed geodesics, (in Russian) Izv. Akad. Nauk SSSR Ser. Mat., 46 (1982), no. 4, 675–709, 896; English translation: Math. USSR, Izv., 21 (1983), 1-29.

[AK] D.V. Anosov, A.B. Katok, New examples in smooth ergodic theory. Ergodic diffeomorphisms, (in Russian), \textit{Trudy Moskov. Mat. Obšč.}, 23 (1970), 3–36.

\end{thebibliography}
GENERIC EXISTENCE OF PERIODIC ORBITS

[BTZ] W. Ballmann, G. Thorbergsson, W. Ziller, Closed geodesics and the fundamental group, *Duke Math. J.*, 48 (1981), 585–588.

[Ba] A. Banyaga, Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comment. Math. Helv.* 53 (1978), 174–227.

[Bo] F. Bourgeois, Introduction to contact homology. Lecture notes available at [http://homepages.vub.ac.be/~fbourgeo/](http://homepages.vub.ac.be/~fbourgeo/).

[BO] F. Bourgeois, A. Oancea, An exact sequence for contact- and symplectic homology, *Invent. Math.*, 175 (2009), 611–680.

[CZ] C. Conley, E. Zehnder, A global fixed point theorem for symplectic maps and subharmonic solutions of Hamiltonian equations on tori, *Proc. Sympos. Pure Math.*, 45 (1986), 283–299.

[Du] J.J. Duistermaat, On the Morse index in variational calculus, *Advances in Math.*, 21 (1976), 173–195.

[Ek] I. Ekeland, Une théorie de Morse pour les systèmes hamiltoniens convexes, *Ann. Inst. H. Poncaré Anal. Non Linéaire*, 1 (1984), 19–78.

[EH] I. Ekeland, H. Hofer, Convex Hamiltonian energy surfaces and their periodic trajectories, *Comm. Math. Phys.*, 113 (1987), 419–469.

[El] Y. Eliashberg, Symplectic field theory and its applications, *International Congress of Mathematicians*, Vol. I, 217–246, Eur. Math. Soc., Zürich, 2007.

[EP] M. Entov, L. Polterovich, Rigid subsets of symplectic manifolds, Preprint 2007, arXiv:0704.0105.

[FK] B. Fayad, A. Katok, Constructions in elliptic dynamics, *Ergodic Theory Dynam. Systems*, 24 (2004), 1477–1520.

[FH] J. Franks, M. Handel, Periodic points of Hamiltonian surface diffeomorphisms, *Geom. Topol.*, 7 (2003), 713–756.

[Ge] H. Geiges, *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics, vol. 109; Cambridge University Press, Cambridge, 2008.

[Gi] V.L. Ginzburg, The Conley conjecture, Preprint 2006, math.SG/0610956.

[GG1] V.L. Ginzburg, B.Z. Gürel, Local Floer homology and the action gap, Preprint 2007, arXiv:0709.4077.

[GG2] V.L. Ginzburg, B.Z. Gürel, Action and index spectra and periodic orbits in Hamiltonian dynamics, *Geom. Topol.*, 13 (2009), 2745–2805.

[GK] V.L. Ginzburg, E. Kerman, Homological resonances for Hamiltonian diffeomorphisms and Reeb flows, *Int. Math. Res. Not. IMRN* 2009; doi: 10.1093/imrn/rnp120.

[He1] M.-R. Herman, Exemples de flots hamiltoniens dont aucune perturbation en topologie $C^\infty$ n’a d’orbites périodiques sur ouvert de surfaces d’énergies, *C. R. Acad. Sci. Paris Sér. I Math.*, 312 (1991), no. 13, 989–994.

[He2] M.-R. Herman, Differantialité optimale et contre-exemples à la fermeture en topologie $C^\infty$ des orbites récurrentes de flots hamiltoniens, *C. R. Acad. Sci. Paris Sér. I Math.*, 313 (1991), 49–51.

[Hi1] N. Hingston, Equivariant Morse theory and closed geodesics. *J. Differential Geom.*, 19 (1984), 85–116.

[Hi2] N. Hingston, Subharmonic solutions of Hamiltonian equations on tori, Preprint 2004; to appear in *Ann. of Math.*, available at [http://comet.lehman.cuny.edu/sormani/others/hingston.html](http://comet.lehman.cuny.edu/sormani/others/hingston.html).

[HS] H. Hofer, D. Salamon, Floer homology and Novikov rings, in *The Floer Memorial Volume*, 483–524; Progr. Math., 133, Birkhäuser, Basel, 1995.

[KT] W. Klingenberg, F. Takens, Generic properties of geodesic flows, *Math. Ann.*, 197 (1972), 323–334.

[LO] H.-V. Lê, K. Ono, Symplectic fixed points, the Calabi invariant and Novikov homology, *Topology*, 34 (1995), no. 1, 155–176.

[Lo] Y. Long, *Index Theory for Symplectic Paths with Applications*, Progress in Mathematics, 207. Birkhäuser Verlag, Basel, 2002.

[MS] D. McDuff, D. Salamon, *J-holomorphic Curves and Symplectic Topology*, American Mathematical Society Colloquium Publications, vol. 52. American Mathematical Society, Providence, RI, 2004.

[Mo] J. Moser, Proof of a generalized form of a fixed point theorem due to G. D. Birkhoff, *Lecture Notes in Math.*, Vol. 597, Springer, Berlin and New York, 1977, 464-494.
[On] K. Ono, Floer–Novikov cohomology and symplectic fixed points, *J. Symplectic Geom.*, 3 (2005), 545–563.

[Pi] D. Pixton, Planar homoclinic points, *J. Differential Equations*, 44 (1982), 365–382.

[Po] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*. Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2001.

[PR] C.C. Pugh, C. Robinson, The $C^1$ closing lemma, including Hamiltonians, *Ergodic Theory Dynam. Systems*, 3 (1983), 261–313.

[Ra1] H.B. Rademacher, On the average indices of closed geodesics, *J. Differential Geom.*, 29 (1989), 65–83.

[Ra2] H.B. Rademacher, On a generic property of geodesic flows, *Math. Ann.*, 298 (1994), 101–116.

[Sa] D.A. Salamon, Lectures on Floer homology, in *Symplectic Geometry and Topology (Park City, UT, 1997)*, 143–229, IAS/Park City Math. Ser., 7, Amer. Math. Soc., Providence, RI, 1999.

[SZ] D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.*, 45 (1992), 1303–1360.

[VK] O. van Koert, Open books for contact five-manifolds and applications of contact homology, Inaugural Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln, 2005; available at http://www.math.sci.hokudai.ac.jp/~okoert/.

[Vi1] C. Viterbo, Equivariant Morse theory for starshaped Hamiltonian systems, *Trans. Amer. Math. Soc.*, 311 (1989), 621–655.

[Vi2] C. Viterbo, A new obstruction to embedding Lagrangian tori, *Invent. Math.*, 100 (1990), no. 2, 301–320.

[We] H. Weiss, Genericity of symplectic diffeomorphisms of $S^2$ with positive topological entropy. A remark on: “Planar homoclinic points” [J. Differential Equations 44 (1982), no. 3, 365–382; MR0661158 (83h:58077)] by D. Pixton and “On the generic existence of homoclinic points” [Ergodic Theory Dynam. Systems 7 (1987), no. 4, 567–595; MR0922366 (89j:58104)] by F. Oliveira, *J. Statist. Phys.*, 80 (1995), 481–485.

[Ya] M.-L., Yau, Vanishing of the contact homology of overtwisted contact 3-manifolds. With an appendix by Yakov Eliashberg, *Bull. Inst. Math. Acad. Sin. (N.S.)*, 1 (2006), 211–229.

VG: DEPARTMENT OF MATHEMATICS, UC SANTA CRUZ, SANTA CRUZ, CA 95064, USA
E-mail address: ginzburg@math.ucsc.edu

BG: DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TN 37240, USA
E-mail address: basak.gurel@vanderbilt.edu