On tidal capture of primordial black holes by neutron stars

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The fraction of primordial black holes (PBHs) of masses $10^{17} - 10^{26}$ g in the total amount of dark matter may be constrained by considering their capture by neutron stars (NSs), which leads to the rapid destruction of the latter. The constraints depend crucially on the capture rate which, in turn, is determined by the energy loss by a PBH passing through a NS. Two alternative approaches to estimate the energy loss have been used in the literature: the one based on the dynamical friction mechanism, and another on tidal deformations of the NS by the PBH. The second mechanism was claimed to be more efficient by several orders of magnitude due to the excitation of particular oscillation modes reminiscent of the surface waves. We address this disagreement by considering a simple analytically solvable model that consists of a flat incompressible fluid in an external gravitational field. In this model, we calculate the energy loss by a PBH traversing the fluid surface. We find that the excitation of modes with the propagation velocity smaller than that of PBH is suppressed, which implies that in a realistic situation of a supersonic PBH the large contributions from the surface waves are absent and the above two approaches lead to consistent expressions for the energy loss.

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I. INTRODUCTION

One of the prominent candidates to the role of the dark matter (DM) of the Universe is primordial black holes (PBH) [1, 3]. PBH could have been created in the early Universe by a variety of mechanisms [4–6] and could have survived till present epoch comprising all or a fraction of the cold DM. An attractive feature of this scenario is that it does not require the existence of a new stable fundamental particle.

PBH are characterized by a single parameter — the mass. Depending on the production mechanism, this parameter can take essentially any value ranging from the Planck mass to many solar masses. In some range of masses the fraction of PBH in the total amount of DM is constrained by observations. Barring the possibility of stable remnants of a Planckian mass, the PBH lighter than $10^{14}$ g cannot survive till present time due to Hawking evaporation [7]. The PBH of slightly higher masses, up to $\sim 10^{16}$ g, evaporate too efficiently and overproduce the $\gamma$-ray background radiation [8]. In the mass range $10^{26} - 10^{34}$ g the fraction of PBH is constrained by lensing [9, 10], while at even larger masses the DM completely consisting of PBH is excluded by the microwave observations [11].

In the intermediate mass range $10^{17} - 10^{26}$ g the PBH are of a microscopic size (from nuclear size to a fraction of a millimeter), and are very difficult to observe. Constraints on the fraction of PBH in this mass range may be imposed from observations of compact stars [12, 14] — neutron stars (NS) and white dwarfs. These constraints rely on the fact that if a compact star captures even a single PBH it gets destroyed in a short time. If an old compact star is observed, the probability that it captures a PBH in its lifetime must be much smaller than 1. Thus, a mere observation of an old neutron star in an environment with a high DM density and low velocity dispersion of DM, where the capture rate is the highest, may imply constraints on the fraction of PBH in the total amount of dark matter.

There are two ways to capture a PBH: during the formation of a main sequence star, with its subsequent evolution into a compact star [12, 14], or through direct capture by an already existing neutron star [13]. This is the second case that is relevant for the present paper.

In order to be captured by a compact star, a PBH has to lose its kinetic energy and become gravitationally bound to the star. Due to a microscopic size of the PBH and large mass, the energy loss in a single passage through the star is very small, so that only PBHs with very small asymptotic velocity can be captured. The energy loss, therefore, becomes one of the key parameters that determines the resulting constraints, as the amount of captured DM is directly proportional to it.

Two different approaches have been used in the literature to calculate the energy loss by a PBH passing through a NS. One is based on the dynamical friction mechanism [13]. In this approach the star is approximated as a collection of free particles which interact individually with the gravitational field of a passing PBH and absorb part of its momentum causing the PBH to slow down. The motivation for this approximation is that the speed with which the PBH passed through a NS after being accelerated in the gravitational field of the latter is at least a few times larger than the sound speed in the NS, so that, by causality, the collective properties of the matter do not have time to manifest themselves. The energy loss in this approach has been calculated in
Ref. [13] and is, parametrically,
\[ E_{\text{eff}} \sim \frac{Gm^2}{R} \ln \Lambda, \]
where \( G \) is Newton’s constant, \( R \) is the star radius, \( m \) the PBH mass and the subscript stands for dynamical friction. The factor \( \ln \Lambda \) is the Coulomb logarithm [15] whose presence is due to the long-range character of the Newtonian potential. This factor is estimated as \( \ln \Lambda \sim \ln(R/Gm) \) for an ordinary star and is somewhat reduced in the case of the neutron star because of the Fermi-degeneracy of the nuclear matter [13]. The constraints that are obtained with this energy loss are 1-2 orders of magnitude weaker than those resulting from the PBH capture at the stage of the star formation [14].

An alternative approach to the energy loss has been used in Ref. [16], where the NS matter has been treated as a medium rather than a collection of individual particles. A passing PBH excites oscillations of the medium by its gravitational field and thus loses the energy. In Ref. [17] it has been shown that as far as the excitation of the sound waves is concerned, the two approaches give identical answer in the supersonic case, in accord with the above causality arguments.

However, it was argued in Ref. [16] that in the case of a PBH crossing a NS, the dominant energy loss comes not from the normal sound waves, but from the surface or gravity waves excited on the surface of the NS core. When the energy is represented as a sum over partial waves with harmonic number \( l \), it was found that this sum diverges at large \( l \), giving the energy loss of the form
\[ E_{\text{sw}} \sim \frac{Gm^2}{R} \sum_{l=1}^{l_{\text{max}}} \frac{1}{l^n}, \]
where \( n \) depends on the matter equation of state (according to [16], \( n \approx 0.5 \) for the NS core and \( n = 0 \) for an incompressible fluid). When summed up to some large \( l_{\text{max}} \), Eq. (2) gives a much larger energy loss than the dynamical friction, Eq. (1), the dominant contribution coming from the shortest wavelengths, i.e., from the vicinity of the PBH impact point. This result is in disagreement with the causality arguments because the surface waves propagate in any case not faster than the sound, and thus the approximation of the dynamical friction approach should be valid.

The aim of the present paper is to resolve this apparent paradox (see also [18]). Keeping in mind that the enhancement factor in Eq. (2) comes from the vicinity of the PBH and is maximum for stiff fluids, we developed a simple analytically solvable model where the claims of Ref. [16] can be easily tested. Namely, we consider a semi-infinite incompressible fluid in the uniform external gravitational filed normal to the fluid surface. This model possesses surface waves fully analogous to the waves, say, on the surface of a lake. A PBH passing through the fluid surface excites an outgoing circular wave that propagates from the impact point. We calculated analytically the fluid perturbation and the amount of energy it carries to infinity which, by energy conservation, equals the energy loss by the PBH.

We found that this energy is convergent even in the idealized case of a point-like PBH, and parametrically coincides with Eq. (1), up to the factor \( \ln \Lambda \), after the appropriate identification of parameters. Moreover, when represented in momentum space — that is, in the form analogous to Eq. (2) — the contribution of high momenta is cut off. The cutoff occurs precisely at the value of the momentum beyond which the surface waves propagate slower than the PBH speed (note that, as will be discussed below, the higher is the momentum, the slower is the propagation speed of a surface wave). Thus, as expected, we recover the causality arguments in the case of the surface waves as well. We also identified problems in the calculations of Ref. [16] which we believe have lead to an erroneous answer.

The rest of this paper is organized as follows. In section II we set up the basic equations and consider the gravitational interaction of a point-like mass (the black hole) with a semi-infinite incompressible fluid in a uniform gravitational field (a flat neutron star). From the explicit expression of the surface deformation as a function of time, we compute the energy transfer from the black hole to the fluid and show that it is of order of Eq. (2). In section III we briefly consider the case of a sphere of incompressible fluid and recover, for large momenta, the behavior corresponding to the planar limit. In the same section we compare in more details our result with those of [16] and point to calculational issues that could explain the differences between our results. We then draw our conclusions.

II. ENERGY LOSS FOR A FLAT STAR

We begin by setting up the basic equations, which may be found in standard reference books (see, for instance, [19]). Since the fluid is incompressible \( \rho = \text{const} \), and so is the pressure at the surface, \( p_0 = \text{const} \). We set \( p_0 = 0 \) in what follows. The equation of continuity applied to an incompressible fluid gives \( \text{div} \vec{v} = 0 \). Since we focus on gravitational effects, which derive from a gradient, we consider that the flow is irrotational. Then the velocity field of the fluid takes the from a gradient, \( \vec{v} = \text{grad} \varphi \), with \( \varphi \) satisfying
\[ \Delta \varphi = 0. \]
The Euler equation in the vicinity of the boundary reduces to
\[ \partial_t \varphi + \frac{1}{2} (\text{grad} \varphi)^2 = -p/\rho - g z, \]
$z = \eta(x, y, t)$ for the deformation of the surface. To make progress we begin by neglecting the non-linear terms. This amounts to assuming that $\eta$ is always smaller than the characteristic wavelength of a given deformation $\lambda$, $|\eta| \ll \lambda$. At the surface we then have

$$(\partial_t \eta)_{z=\eta} + g\eta = 0.$$  

To eliminate $\eta$ from this equation, we use the fact that $v_z = \partial_x \phi \approx \partial_t \eta$, which is valid for small deformations, and get

$$(\partial_t^2 \phi + g \partial_x \phi)_{z=\eta} = 0. \quad (5)$$

Eqs. (5) together with Eq. (3) dictate the evolution of small surface deformations in presence of gravity, or gravity waves. Notice that they do not involve the density of the fluid, which is a manifestation of the equivalence principle.

For the problem at hand, we need the eigenfunctions of Eq. (3) with cylindrical symmetry. To simplify the subsequent expressions, we consider a fluid with infinite depth. We have checked that considering finite depth does not change our conclusions. In this limit, the velocity potential takes a simple form in terms of the Bessel functions $J_0$, $J_1$, and $1$.

$$\varphi_k(\vec{x}, t) = e^{-i\omega_k t} e^{kz} J_0(kr), \quad (6)$$

where $z < 0$. For further reference, we also present the expression for the velocity itself,

$$\vec{v}_k(\vec{x}, t) = e^{-i\omega_k t} e^{kz} \left( J_1(kr) \vec{I}_x - J_0(kr) \vec{I}_r \right) \quad (7)$$

$$\sim e^{-i\omega_k t} \vec{s}_k(\vec{x}),$$

where $\vec{s}_k(\vec{x})$ are the eigenmodes of the fluid displacement vector field, see Sect. 11.

From this and Eq. (5) we have

$$\omega^2_k = gk, \quad (8)$$

which is the familiar dispersion relation of surface waves in deep water (i.e. $\lambda \ll h$, with $h$ the depth of the fluid layer). The group velocity is

$$v_g = \frac{\partial \omega_k}{\partial k} = \frac{1}{2} \frac{\sqrt{g}}{k} \quad (9)$$

which implies that short wavelength modes travel more slowly.

We may now consider the effects of the PBH, which we will treat as a perturbation with gravitational potential

$$\Phi(t, r, z) = -\frac{Gm}{\sqrt{r^2 + (z + \vec{v}t)}^2}$$

where $m$ is the mass of the PBH and $\vec{v}$ is its velocity near the surface. We have approximated the gravitational field of the PBH by its Newtonian expression valid at distances $r \gg 2Gm$; the justification will be given later. For simplicity, we also neglected the PBH acceleration due to the gravitational field and thus consider $\vec{v}$ to be constant.

With the gravitational potential included, Eq. (5) becomes

$$\partial_t^2 \varphi + g \partial_x \varphi = -\partial_t \Phi \quad (10)$$

at $z = 0$ and we search for a solution for the velocity potential $\varphi$ that is zero at $t \to -\infty$. By representing $\Phi$ as a sum of the Bessel functions in a manner similar to Eq. (6), Eq. (10) can be diagonalized and solved. The result reads

$$\varphi(t, r, z) = \frac{Gm\bar{v}}{g} \int_0^\infty dk e^{kz} J_0(kr) \left[ -\epsilon(t) e^{-k\bar{v}t} \right.$$  

$$+ \theta(t) 2 \cos(\omega_k t) \right], \quad (11)$$

where $\epsilon(t)$ and $\theta(t)$ are the sign and Heaviside functions, respectively. One easily checks that this function and its time derivative are continuous at $t = 0$, and that Eq. (10) is indeed satisfied.

Using $\partial_x \varphi = \partial_t \eta$, we get that the deformation of the surface is given by

$$\eta(t, r) = \frac{Gm}{g} \int_0^\infty dk \int_0^\infty d\omega \frac{1}{1 + k^2 \bar{v}^2 / g}$$

$$\times \left[ e^{-k\bar{v}t} + 2\theta(t) \bar{v} \sqrt{\frac{k}{g}} \sin(\omega_k t) \right] J_0(kr). \quad (12)$$

For $t < 0$, the deformation has the form of a bump, with a height that is increasing as the BH comes closer to the surface. For $t > 0$, while the bump fades away, the perturbation evolves into an outgoing wave train, as described by the second (oscillating) term in Eq. (12). Because the group velocity (9) is decreasing for large $k$, at late times this surface wave has a characteristic shape — typical of water surface disturbances — with a front ahead and an oscillatory pattern with shorter wavelengths behind.

Using Eq. (12), we may readily calculate the energy transfer. To this end we compute the energy $E$ carried away at late times by the wave train, that is we consider only the contribution at $t \gg 0$ of the second term in Eq. (12). This energy is given by the sum of the kinetic and potential energy of the disturbance,

$$E = \int \frac{1}{2} \rho v^2_\text{w} d^3x + \frac{1}{2} \rho g \int \eta^2_\text{w} d^2x,$$

1 The profile of the deformation is diverging at the origin at $t = 0$. The divergence is mild, being only logarithmic, and has no — or little — incidence on the energy loss. The emergence of this divergence of is related to the point-like approximation of the gravitational potential of the PBH, and could be regulated — at the cost of substantial complication — by taking into account the Schwarzschild radius.
where the subscript ‘wt’ stands for the part of the solution describing the outgoing wave. In the second term that corresponds to the potential energy, we have performed the integration over $z$ and subtracted the contribution of the unperturbed fluid. From Eqs. (11) and (12) we get

$$E = 4\pi \rho \frac{G^2 m^2 \bar{v}^2}{g^2} \int_0^\infty \frac{1}{(1 + k\bar{v}/g)^2} \, dk$$

$$= 4\pi \frac{G^2 m^2}{g}.$$  

In Appendix A we recalculate the energy deposited into the fluid perturbations at arbitrary time, using a different method and retaining all the terms in the solution (12). We then show that the resulting expression reproduces Eq. (13) at $t \to \infty$.

The expression of Eq. (13) is our main result. First and foremost we notice that Eq. (13) is UV finite, as we advertised in the Introduction. Although this equation refers to the flat case, the disagreement with Eq. (2) is already evident from the large- $k$ behavior: should Eq. (2) be recovered, the integral over $k$ in Eq. (13) would have to linearly diverge, which is not the case.

Second, only sufficiently low-$k$ modes contribute substantially to the energy loss. The integral in Eq. (13) is cut at $k \lesssim g/\bar{v}^2$. Making use of Eq. (9), this condition translates into

$$v_g \gtrsim \bar{v}.$$  

We see that only the waves that propagate faster than the PBH are efficiently excited, in agreement with the causality arguments.

Finally, by identifying the acceleration $g$ with the acceleration at the surface of the neutron star and the velocity $\bar{v}$ with the velocity of the BH falling onto it,

$$g \sim \frac{GM}{R^2}, \quad \bar{v} = \sqrt{\frac{GM}{R}},$$

where $R$ is the radius of the star and $M$ its mass, one can cast Eq. (13) in the form

$$E \sim \frac{G m^2}{R},$$

which is parametrically the same, apart from the logarithmic factor, as the energy loss Eq. (11) calculated in the dynamical friction approach. A more detailed comparison with the spherical case is presented in the next Section.

To conclude this section, let us check the validity of the approximations used to obtain Eq. (13). It follows from Eq. (13) that the dominant contribution to the energy loss comes from the excitation of the waves with the wavelengths

$$\lambda \sim 1/k \sim \bar{v}^2/g.$$  

The Newtonian approximation for the gravitational potential of the PBH is, therefore, valid when this wavelength is much larger than the horizon size of the PBH, $\bar{v}^2/g \gg R_s = 2GM$, which is satisfied for sufficiently small $g$ and/or large $\bar{v}$. In the realistic case of a neutron star we have from Eqs. (14) that $\bar{v}^2/g \sim R \gg R_s$, so in practice the Newtonian approximation is satisfied for PBH lighter than about solar mass.

Consider now the variation of the PBH velocity. At the distance of order $\lambda$, the fractional change of the PBH velocity due to gravitational acceleration is $\delta \bar{v} = g\lambda/\bar{v}$. From Eq. (16) we conclude that $\delta \bar{v} \sim \bar{v}$, so the constant velocity approximation is only marginally satisfied — enough to establish the absence of divergency and make an order-of-magnitude estimate of the energy loss, but not enough for a precise answer.

### III. SPHERICAL STAR AND COMPARISON OF RESULTS

Armed with our understanding of the planar limit, we consider now the case of a sphere of incompressible fluid of radius $R$. In this we will follow the approach of Ref. [10] and consider the fluid displacement vector field $\vec{s}(\vec{x}, t)$, which reduces to the surface deformation $\eta$ for $\vec{x}$ at the surface. As the issue at hand is the large-momenta behavior of the energy deposition, we will study the limit of the solution for the sphere in the large $l$ limit, where $l$ is the multipole index. Our motivation is twofold. First, we want to provide and independent check of the planar solutions that we have derived. This will imply right away that the large-$l$ modes give the same contribution to the energy loss as in the planar case, as one should expect. In particular, the energy loss is UV finite (see the appendix for the expression of the energy loss for all $l$, Eqs. (A2) and (A3)). Second, we want to trace the origin of the difference between our conclusions and those of Ref. [10]. For this we compare our solution with that of Ref. [16] in the large-$l$ limit.

As mentioned in Ref. [16], the eigenproblem for a sphere of an incompressible gravitating fluid has been solved by Kelvin. Expressing the eigenfunctions in terms of spherical harmonics, for the sphere of radius $R$ and mass $M$, the fluid displacement eigenmodes $s_l(\vec{x})$ are given by

$$s_l(\vec{x}) = \sqrt{\frac{4\pi l}{3M}} \left( \frac{\tilde{r}}{R} \right)^{l-1} \left( Y_{l0}(\theta) \hat{t} + \frac{1}{\tilde{r}} \partial_\theta Y_{l0}(\theta) \hat{\theta} \right)$$

where $\tilde{r}$ is the radial coordinate $0 < \tilde{r} < R$. We will assume that the PBH trajectory is passing through the center of the sphere, so the only relevant modes correspond to $m = 0$. Their normalization is as in Ref. [16]. The frequencies of the eigenmodes are given by

$$\omega_l^2 = \frac{2l(l-1) GM}{2l + 1} \frac{1}{R^3}$$

The frequency depends only on the multipole index $l \geq 2$. 

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The content seems to be a continuation of the discussion on the energy loss of a primordial black hole (PBH) due to gravitational interaction with a neutron star. It involves a detailed analysis of the equations governing the fluid dynamics, comparing the planar and spherical cases, and discussing the validity of the Newtonian approximation. The content is dense and mathematical, focusing on the derivation of equations and the implications of their solutions.
because there are no excited radial modes for an incompressible sphere. Using \( g = GM/R^2 \), the dispersion relation becomes Eq. (8) for large \( l \approx kR \). Similarly, using

\[
Y_{l,0}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta) \approx \sqrt{\frac{kR}{2\pi}} J_0(kr)
\]

valid for large \( l \approx kR \), we obtain the expression of the eigenmodes in the planar limit

\[
\vec{s}_l(x) \sim \frac{k}{\sqrt{2\pi\rho R}} e^{kz} \left( J_0(kr) \vec{I}_z - J_1(kr) \vec{I}_r \right).
\]

(19)

Up to an arbitrary normalization factor, this agrees with Eq. (7).

Now we claim that the solution for the fluid displacement for all time in the presence of the PBH is given by

\[
\vec{s}(\vec{x}, t) = \sum_{l=2}^{\infty} \vec{s}_l(\vec{x}) \int_{-\infty}^{t} dt' A_l(t') \frac{\sin \omega_l(t-t')}{\omega_l}
\]

(20)

where

\[
A_l(t) = \int_{\text{star}} dV \vec{f} \cdot \vec{s}_l(x),
\]

(21)

\( \vec{f}(x) \) being the gravitational force of the PBH acting on a fluid element in the point \( \vec{x} \). This is pretty clear from the fact that, in the sector with given \( l \),

\[
G^+(t-t') = \theta(t-t') \frac{\sin \omega_l(t-t')}{\omega_l}
\]

is the retarded Green’s function for the problem at hand, with \( \omega_l \) given by Eq. (18). This solution, Eq. (20), differs from the expression used in [16], a point to which we will come back later on.

First we consider the large \( l \approx kR \) limit of the solutions. Using the force decomposition in spherical modes in the large-\( l \) limit

\[
\vec{f} = -Gm\rho \int_0^{\infty} dk k e^{-k|z+\vec{v}\cdot \vec{x}|} \\
\times \left( \epsilon(z+\vec{v}t)J_0(kr)\vec{I}_z + J_1(kr)\vec{I}_r \right)
\]

(22)

and Eq. (17) we have

\[
A_l(t) \sim \sqrt{\frac{2\pi\rho}{R}} Gm e^{-k|\vec{v}|t}
\]

(23)

from which it is straightforward to get

\[
\vec{s}(\vec{x}, t < 0) = Gm \int dk \frac{e^{k(z+\vec{v}t)}}{g + k\vec{v}^2} \left( J_0(kr)\vec{I}_z - J_1(kr)\vec{I}_r \right)
\]

(24)

and

\[
\vec{s}(\vec{x}, t > 0) = Gm \int dk \left( \frac{e^{-k\vec{v}t}}{g + k\vec{v}^2} - \frac{2\vec{v}k - \sin \omega_k t}{g + k\vec{v}^2 + \omega_k} \right) \\
\times e^{kz} \left( J_0(kr)\vec{I}_z - J_1(kr)\vec{I}_r \right)
\]

(25)

This solution gives precisely the surface deformation given by Eq. (12). Alternatively one may check that it corresponds to the velocity field resulting from Eq. (11), with \( \vec{v} = \nabla \varphi = ds/dt \).

Our first conclusion is thus that the two approaches — the direct resolution of the planar problem and the large \( l \) limit of the spherical problem — lead to precisely the same solutions, as they should. This not only provides an independent check of our approach, but also shows that the large \( l \) behavior on the sphere is mundane, the same as in the planar limit. From this we also conclude that the energy loss is finite. Indeed, the only divergence we have encountered in the planar limit is for small momenta, but this is regulated by the radius of the sphere \( R \).

Unfortunately, full calculations beyond the large-\( l \) limit, in particular that of the energy loss, are not as straightforward as in the planar case. Nevertheless, in the light of the discussion of the previous section, the final result should be similar to that of Eq. (15).

We do not reach the conclusions of [16], so what is the source of the disagreement? We believe that our results is sound, essentially it matches the intuition that there should be little quantitative difference between energy loss through tidal deformation and dynamical friction. Indeed, as we have argued above, the underlying mechanism is the same, and both approaches should match for supersonic propagation, as in [17]. Regarding the calculations, the comparison of the solutions in the large-\( l \) limit reveals several calculational issues that may explain the difference between our results and those of [16].

For the sake of simplicity we focus on Eq. (2.8) in [16], which together with their equations (2.2) and (2.3) should in principle give the fluid displacement for \( t < 0 \). In the large \( l \), corresponding to the planar limit, we get from their result (reinstating \( G \) and setting \( \bar{v} \) as the velocity near the surface)

\[
\vec{s}(\vec{x}, t < 0)|_{\text{planar}} = Gm \int dk \sqrt{\frac{Rk}{2\pi}} \frac{\bar{v}^2k}{g + k\bar{v}^2} \int e^{k(z+\bar{v}t)} \left( J_0(kr)\vec{I}_z - J_1(kr)\vec{I}_r \right)
\]

(26)

This solution, obtained from the intermediate expressions in Ref. [16], differs from our solution, Eq. (24). In particular, it has a much worse large-\( k \) behavior.

The first issue is the presence of the factor of \( \sqrt{kR/2\pi} \) which we believe should be absent, since the planar limit (if taken appropriately) should contain no explicit factors of \( R \). Re-doing the steps in [16] that lead to their Eqs. (2.8) and (2.9) one indeed realizes that a factor of \( \sqrt{4\pi/(2l+1)} \sim \sqrt{2\pi/kR} \) is missing. We do not know whether this a mere misprint, or if this missing factor has propagated in the calculations of the energy loss.

Another problem is the presence of the factor \( \bar{v}^2k/g \) compared to our Eq. (24), which leads to the an extra divergence in \( k \) of the energy loss obtained in [16]. We believe that the resolution to this specific issue is as follows. First it is clear Eqs. (20) and (21) are undoubtedly correct. However, the solution in [16] is obtained using
a distinct starting point, given by their Eqs. (2.2) and (2.3). To see the relation, we integrate by part Eq. (20) to get
\[
\tilde{s}(\vec{x}, t) = \text{Re} \sum_{l=2}^{\infty} \tilde{s}_l(\vec{x}) \int_{-\infty}^{t} dt' \partial_{t'} A_l(t') \left(1 - e^{i\omega_l(t-t')}\right).
\]
(27)

The first term in the brackets is the boundary term. Going from this expression to that of Eqs. (2.2) and (2.3) in [16] involves two further steps [21]. First, one must neglect the boundary term. This may be true if the source (here the PBH gravitational field) is far enough from the star, but leads to a wrong expression for the fluid deformation at any finite time. Second, one must assume that the derivative over time and the integration over space may be exchanged. These are subtle effects, which are difficult to spot in the full-fledged spherical system, so it is again useful to consider the planar limit.

Let us deal first with the boundary term. Using Eqs. (27) and (28), we get
\[
\tilde{s}(\vec{x}, t < 0) = Gm \text{Re} \int dk \frac{k}{\omega^2_k} \left(1 - \frac{\vec{v}k}{k\vec{u} - i\omega_k}\right) e^{k^2 + k\vec{v}t} \left(J_0(kr)\hat{I}_z - J_1(kr)\hat{I}_r\right),
\]
where the 1 in the first brackets comes from the boundary term in Eq. (27). If taken alone, the term $\propto \vec{v}k$ under the brackets gives the result of [16] (without the factor of $\sqrt{kR/2\pi}$ discussed in the previous paragraph). Including the boundary term (the 1 under the bracket) gives instead our result, Eq. (21), which has a much milder large $k$ behaviour. We believe that this settles this specific problem.

We have considered so far the case $t < 0$, which is most transparent, and found an extra factor of $k^{3/2}$ in the solution of Ref. [16] as compared to our result. This factor is also present in their solution at $t > 0$. However there is one extra issue, which is that in calculating $\partial_t A_l(t)$ it is assumed in [16] (and in [21]) that the time derivative and the volume integration commute,
\[
\int_{\text{star}} dV \vec{f} \cdot \vec{s}_l(\vec{x}) = \int_{\text{star}} dV \partial_t \vec{f} \cdot \vec{s}_l(\vec{x}).
\]
To see that this is not the case for the problem at hand it is again useful to take the large-$l$ limit. From Eq. (23) we have
\[
\int_{\text{star}} dV \vec{f} \cdot \vec{s}_l(\vec{x}) \asymp \epsilon(t) e^{-k|\vec{v}|t},
\]
where calculating the derivative and the integral in a different order — the one of Ref. [16] — gives instead
\[
\int_{\text{star}} dV \partial_t \vec{f} \cdot \vec{s}_l(\vec{x}) \propto e^{-k|\vec{v}|t}.
\]
(30)

The appearance of the sign function $\epsilon(t)$ in the correct expression (29) is crucial to get our result for the fluid displacement, Eq. (25). One may check that failing to take into account this change of sign leads to an expression for the fluid displacement for $t > 0$ that diverges linearly for large $k$. Together with the factor of $\sqrt{kR/2\pi}$ discussed above, one would get a result consistent with the large $l$ limit behaviour of the $t > 0$ solution in [10], which we think is incorrect and do not reproduce here to avoid further cluttering of equations.

To summarize, the fluid displacement in the spherical case should have a large-$l$ limit given by Eqs. (24)–(25) which is consistent with the planar case. However, the solution of Ref. [16] has a different (more singular) behaviour in $k$, which may explain the appearance of divergency in the resulting energy loss at large $l$.

IV. CONCLUSIONS

Motivated by the problem of energy loss by a PBH passing through a neutron star and the controversy existing in the literature, we have considered a simple model where the crucial questions concerning this phenomenon can be addressed analytically. The model consists of an infinite incompressible fluid in a uniform external gravitational field normal to the fluid surface, so that the fluid is in equilibrium. When perturbed, this system possesses surface waves reminiscent of those on the surface of a lake.

We have calculated the excitation of these waves by a gravitational field of a moving mass (say, a PBH) crossing normally the fluid surface. From the explicit solution for the fluid perturbations we have calculated the energy transferred to the fluid and, therefore, lost by the passing mass. We have found that:

- The transferred energy Eq. (13) is convergent at high momenta even in the limit of a point-like mass. Our result is thus in disagreement with that of Ref. [10], Eq. (2).

- The contributions into the energy transfer of individual waves with a given momentum $k$ is cut at high values of momentum. The cutoff corresponds to the suppression of modes that propagate (much) slower than the velocity of the passing mass. This supports the causality arguments in the case when surface waves are present.

- Upon the appropriate identification of the parameters, our resulting energy transfer [15] is parametrically the same as the dynamical friction result, apart from the logarithmic factor. The absence of the Coulomb logarithm in the flat infinite case suggests that this factor would also be absent in the realistic spherical case.

- The generalization of our model to the case of a spherical ball of an incompressible gravitating fluid is difficult to completely solve analytically. Nevertheless, the detailed comparison of the flat and
spherical cases allows one to identify a problem in the calculation of Ref. [16].

Even though a real neutron star is neither flat nor incompressible, our calculation can still shed some light on the relative importance of the surface wave contribution into the energy loss. First, the absence of the Coulomb logarithm suggest that this contribution is subdominant compared to the dynamical friction one, at least by the logarithmic factor. Second and more important, we see from Eq. (13) that contributions of slow (high- \( k \)) modes into the transferred energy are suppressed, because these modes are not excited efficiently. We think this is a general phenomenon not related to such peculiar features of our model as incompressibility of the fluid. The suppression factor is easy to estimate from Eq. (13). The modes propagating with velocities \( v \ll \bar{v} \) contribute to the transferred energy [13] the fraction of order \((v/\bar{v})^2\).

The PBH falling onto a neutron star attains a velocity which is about 0.5c. The speed of the surface waves at the boundary of the NS core depends on their wavelength, but in any case cannot exceed the sound speed, which is at least an order of magnitude smaller than the speed of the PBH. We thus expect that the energy transfer into the excitation of the surface waves in the case of a realistic NS is suppressed by a factor of \((v/\bar{v})^2 \lesssim 10^{-2}\) as compared to the dynamical friction result. A full calculation in the case of a spherical ball of a compressible fluid is needed to verify this expectation, which goes beyond the scope of this paper.

Appendix A: Alternative derivation of the energy loss

It is instructive to calculate the energy loss by the BH as a function of time in the planar limit. For this, following [22], we calculate the work done by the BH on the fluid elements,

\[
d\bar{E} = \int d^3\bar{x} \cdot \bar{f}_{BH}\quad (A1)
\]

where \( \bar{v} = \text{grad} \, \varphi \) is the velocity field of the fluid and \( \bar{f}_{BH} = -\text{grad} \, \Phi \). Integrating over \( t \), we get the energy transferred \( \Delta E \) as a function of time:

\[
\Delta E(t < 0) = 2\pi G^2 m^2 \rho \int_{k_{co}}^{\infty} \frac{dk}{2k(g + k\bar{v}^2)} e^{2k\bar{v}t} \quad (A2)
\]

\[
\Delta E(t > 0) = 2\pi G^2 m^2 \bar{v} \rho \int_{k_{co}}^{\infty} \frac{dk}{g + k\bar{v}^2} \left\{ \frac{e^{-2k\bar{v}t}}{2k\bar{v}} + \frac{g}{k\bar{v}^2 + \omega_k^2} \right\} \quad (A3)
\]

where \( k \rightarrow \infty \). It is straightforward to see that at positive times the term that does not contain the exponential factor exactly reproduces Eq. (13). The other terms are logarithmically divergent at small \( k \), which reflects the long-range character of the Newtonian potential. When cut at \( k_{co} \sim 1/R \), \( R \) being the star radius, they decay with time starting from the value parametrically given by Eq. (13). Thus, at \( t \rightarrow \infty \) the work performed by the BH on the fluid tends to the energy stored in the outgoing waves.

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