On the asymptotic magnitude of subsets of Euclidean space

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Abstract Magnitude is a canonical invariant of finite metric spaces which has its origins in category theory; it is analogous to cardinality of finite sets. Here, by approximating certain compact subsets of Euclidean space with finite subsets, the magnitudes of line segments, circles and Cantor sets are defined and calculated. It is observed that asymptotically these satisfy the inclusion-exclusion principle, relating them to intrinsic volumes of polyconvex sets.

Keywords Magnitude · Metric space · Euler characteristic · Intrinsic volumes

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Introduction

In [10] one of us introduced the notion of the Euler characteristic of a finite category and showed how it linked together various notions of size in mathematics, including the cardinality of sets and the Euler characteristics of topological spaces, posets and graphs. In [11, 13] it was shown how to transfer this to a notion of ‘magnitude’ of a finite metric space, using the fact that a metric space can be viewed as an enriched category.

One way of viewing magnitude is as the ‘effective number of points’. Consider, for example, the $n$-point metric space in which any two points are a distance $d$ apart. When $d$ is very small, the magnitude is just greater than 1—there is ‘effectively only one point’. As $d$ increases, the magnitude increases, and when $d$ is very large, the magnitude is just less

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1 The terms ‘Euler characteristic’ and ‘cardinality’ could have been used here, as in [10] and [11], but we have decided to use a word with less mathematical ambiguity.

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than \( n \)—there are ‘effectively \( n \) points’. The magnitude of a finite metric space actually first appeared in the biodiversity literature [17], under the name ‘effective number of species’, although its mathematical properties were hardly explored. It should be noted that, contrary to the simple example given above, magnitude can display wild behaviour of various types [13]: when a space is scaled up, its magnitude can sometimes decrease, and there are some exceptional finite metric spaces for which the magnitude is not well-defined.

In this paper—which requires no category theory—we consider the notion of magnitude for certain non-finite metric spaces, in particular for certain compact subsets of Euclidean space. This is done by approximating such a subset \( A \) with a sequence of finite subsets of \( A \) and taking the limit of the corresponding sequence of magnitudes. In the cases we consider here—circles, line-segments and Cantor sets—as the subset is scaled up this answer behaves like a linear combination of ‘intrinsic volumes’, such as the length and Euler characteristic, which satisfy the inclusion-exclusion principle. This leads us to conjecture that for a subspace \( A \) the magnitude \( |A| \) decomposes as follows:

\[
|A| = P(A) + q(A)
\]

where \( P \) is a function, defined on some class of subsets of Euclidean space, which satisfies \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \) and \( q(A) \) tends to zero as \( A \) is scaled bigger and bigger. In other words, the magnitude of subsets of Euclidean space asymptotically satisfies the inclusion-exclusion principle. Whilst the proof of this for some examples requires only elementary analysis, the proof for circles requires more subtle asymptotic analysis. Empirical calculations for some subsets in two and three dimensions are consistent with this and appear elsewhere [19].

In an earlier version of this paper we were less definite in some of our assertions. We calculated the limiting value of approximations of each of the spaces and claimed that that should be the value of the magnitude of the space in question, and that it should be independent of the choice of approximating sequence. Motivated by our work, Meckes [15] proved that this was indeed the case and that our methods do indeed give unique results (see Sect. 1.3 below). Another development since the time of writing is the use of ‘weight measures’, this development means that several calculations presented here can be done more quickly, see [15] and [20]. However, the methods presented here seem to be more general as it is not known, and seems unlikely, that every compact metric space, or even every compact subset of Euclidean space, admits a weight measure.

There will now follow a more detailed description of the magnitude of metric spaces, the inclusion-exclusion principle and intrinsic volumes.

**Asymptotic conjectures.** Given a metric space \( X \) with \( n \) points one can try to associate to it an invariant called the magnitude; the definition is given in Sect. 1.2 and the definition is motivated by category theory in Sect. 1.1. If \( X \) is a metric subspace of some Euclidean space then it will have a well-defined magnitude (see Sect. 1.3). However, not every finite metric space has a well-defined magnitude, but every ‘sufficiently separated’ one does (Theorem 2). In particular, if for \( t > 0 \) we define \( tX \) to be \( X \) scaled by a factor of \( t \), so that it is a metric space with the same points as \( X \) but with the metric defined by \( d_{tX}(x, x') := td_X(x, x') \), then for \( t \) sufficiently large the magnitude \( |tX| \) is well-defined and \( |tX| \to n \) as \( t \to \infty \) (Theorem 3). So asymptotically, the magnitude is the number of points in the metric space.

Whilst the magnitude in the case of finite metric spaces is interesting, not least for its connections with biodiversity measures (see [12]), in this paper we consider extending this notion of magnitude to non-finite metric spaces, primarily in the form of compact subsets of Euclidean space with the subspace metric. In the cases we consider here one interesting...