Matrix $A_p$ Weights, Degenerate Sobolev Spaces, and Mappings of Finite Distortion

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Abstract We study degenerate Sobolev spaces where the degeneracy is controlled by a matrix $A_p$ weight. This class of weights was introduced by Nazarov, Treil and Volberg, and degenerate Sobolev spaces with matrix weights have been considered by several authors for their applications to PDEs. We prove that the classical Meyers–Serrin theorem, $H = W$, holds in this setting. As applications we prove partial regularity results for weak solutions of degenerate $p$-Laplacian equations, and in particular for mappings of finite distortion.

Keywords Matrix $A_p$ · Degenerate Sobolev spaces · Mappings of finite distortion

Mathematics Subject Classification 30C65 · 35B65 · 35J70 · 42B35 · 42B37 · 46E35

1 Introduction

In this paper we study matrix $A_p$ weights and their application to PDEs and mappings of finite distortion. Scalar Muckenhoupt $A_p$ weights have a long history: they were
introduced in the 1970s and are central to the study of weighted norm inequalities in harmonic analysis. They have extensive applications in PDEs and other areas. (For details and further references, see [10,13,16].) Matrix $A_p$ weights are more recent. They were introduced by Nazarov et al. [31,36,39] and arose from problems in stationary processes and operator theory. A matrix weight $W(x)$ is a $d \times d$ semi-definite matrix of measurable functions. It is used to define a weighted $L^p$ norm on vector-valued functions:

$$\|f\|_{L^p_W} = \left( \int_{\mathbb{R}^n} |W^{1/p}(x)f(x)|^p \, dx \right)^{1/p}.$$ 

The matrix $A_p$ condition is a natural generalization of the scalar Muckenhoupt $A_p$ condition and matrix $A_p$ weights also share many other analogous properties of their scalar counterparts. For instance, the Hilbert transform is bounded on $L^p_W(\mathbb{R})$ if and only if $W \in A_p$. Since their introduction these weights have been considered by a number of authors: see, for instance, [3,4,6,12,15,20,26,32,34].

In this paper we apply the theory of matrix $A_p$ weights to the study of degenerate Sobolev spaces. More precisely, we consider the space $W^{1,p}_W$ that consists of all functions in $W^{1,1}_{loc}$ such that

$$\|f\|_{W^{1,p}_W} = \|f\|_{L^p(v)} + \|\nabla f\|_{L^p_W} < \infty.$$ 

(The weight $v$ could in principle be arbitrary, but we will show that there exist scalar weights naturally associated with each matrix weight.) Such weighted Sobolev spaces are well known to play an important role in the study of degenerate elliptic equations: see [7,17,35,37]. Our main result extends the celebrated $H=W$ theorem of Meyers and Serrin [28] to Sobolev spaces $W^{1,p}_W(\Omega)$: we will show that if $W \in A_p$, then smooth functions are dense in $W^{1,p}_W(\Omega)$.

We give two applications of our results. First, we use them to prove partial regularity results for the degenerate $p$-Laplacian,

$$\mathcal{L}_{A,p}u = \text{div} \left( (A \nabla u, \nabla u)^{\frac{p-2}{2}} A \nabla u \right) = 0,$$

where $A$ is an $n \times n$ degenerate elliptic matrix. These results extend the work of the first two authors and Naibo [8]; in particular, assuming the matrix $A_p$ condition allows us to significantly weaken other hypotheses. Second, we apply these results for the degenerate $p$-Laplacian to the problem of partial regularity of mappings of finite distortion. Conditions guaranteeing the continuity of such mappings have been studied by many authors: see [18,21,25,27,38]. Our results approach the regularity problem from a significantly different direction. More precisely, we show that the mapping is continuous on a set determined by a maximal operator acting on the inner and outer distortion functions.

The remainder of this paper is organized as follows. In Sect. 2 we gather some preliminary material about scalar weights, particularly the Muckenhoupt $A_p$ weights. There is a close relationship between scalar $A_p$ and matrix $A_p$ and the scalar weights play a significant role in our work. In Sect. 3 we define matrix weighted spaces and
give some basic results. None of these ideas are new, but we have put them into a consistent framework and we give proofs for several results that are only implicit in the literature.

In Sect. 4 we define matrix $A_p$ weights and prove a number of new results, particularly for matrix $A_1$. The central theorem is that approximate identities converge in $L^p_W$, $1 \leq p < \infty$. We prove this without using the Hardy–Littlewood maximal operator, replacing it with a smaller averaging operator. This fact plays an important role in the proof of our main result, but it is of independent interest and should be useful in other settings.

In Sect. 5 we prove our main result, the generalization of the Meyers–Serrin $H = W$ theorem to matrix weighted Sobolev spaces. We prove several variations that correspond to well-known results in the scalar (unweighted) case.

The last three sections are applications. In Sects. 6 and 7 we apply our results to degenerate $p$-Laplacian equations. In Sect. 6 we reformulate and extend the results in [8] without using the matrix $A_p$ condition and instead give our hypotheses in terms of scalar weights. In Sect. 7 we show that the matrix $A_p$ condition yields a number of corollaries. Finally, in Sect. 8 we apply these results to prove partial regularity results for mappings of finite distortion. All of our results are based on assuming that the distortion tensor satisfies a matrix $A_p$ condition.

Throughout this paper we will use the following notation. The symbol $n$ will always denote the dimension of the Euclidean space $\mathbb{R}^n$. We will use $d$ to denote the dimension of matrix and vector-valued functions. In general $d$ can be any positive value, though in applications we will take $d = n$. We will take the domain of our functions to be an open, connected set $\Omega \subset \mathbb{R}^n$. The set $\Omega$ need not, a priori, be bounded. Given two values $A$ and $B$, we will write $A \lesssim B$ if there exists a constant $c$ such that $A \leq cB$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Constants $C$, $c$, etc., whether explicit or implicit, can change value at each appearance. Sometimes we will indicate the parameters constants depend on by writing, for instance, $C(n, p)$, etc. If the dependence is not indicated, the constant may depend on the dimension and other parameters that should be clear from context.

## 2 Scalar Weights

In this section we gather together, without proof, some basic definitions and results about scalar $A_p$ weights. Unless otherwise noted, these results can be found in [10, 13].

Given a domain $\Omega \subset \mathbb{R}^n$, we define a (scalar) weight $w$ to be a non-negative function in $L^1_{\text{loc}}(\Omega)$. The measure $w\, dx$ is a Borel measure and we define the weighted $L^p$ space, $L^p(w, \Omega)$, to be the Banach function space with norm

$$
\|f\|_{L^p(w,\Omega)} = \left( \int_{\Omega} |f(x)|^p w(x) \, dx \right)^{1/p}.
$$

Given a set $E$, let

$$
w(E) = \int_E w(x) \, dx, \quad \int_E w(x) \, dx = \frac{1}{|E|} \int_E w(x) \, dx.
$$
A weight $w$ is doubling if given any cube $Q$, $w(2Q) \leq Cw(Q)$, where $2Q$ is the cube with the same center as $Q$ and $\ell(2Q) = 2\ell(Q)$.

For $1 < p < \infty$, we say that $w \in A_p(\Omega)$ if

$$[w]_{A_p(\Omega)} = \sup_Q \left( \int_{Q \cap \Omega} w(x) \, dx \right)^{1/p} \left( \int_{Q \cap \Omega} w^{-p'/p}(x) \, dx \right)^{1/p'} < \infty,$$

where the supremum is taken over all cubes $Q$. When $p = 1$, we say $w \in A_1$ if for all cubes $Q$,

$$\int_{Q \cap \Omega} w(y) \, dy \leq [w]_{A_1} \inf_{x \in Q \cap \Omega} w(x).$$

Remark 2.1 Alternatively, we can define the doubling and $A_p$ conditions with respect to balls instead of cubes. If $\Omega = \mathbb{R}^n$, these two definitions are clearly equivalent; similarly, they are equivalent if $w$ is the restriction to $\Omega$ of a doubling or an $A_p$ weight defined on all of $\mathbb{R}^n$. However, depending on the geometry of $\Omega$ and its boundary these two definitions may not be equivalent. (For a characterization of the restriction problem for $A_p$ weights, see [13, Chapter IV.5].) Hereafter, given a domain $\Omega$ we will assume that our weights are defined on some unspecified set $\Omega'$ such that $\Omega \subset \Omega'$ and we assume that balls and cubes are interchangeable in the definition of doubling or $A_p$ on $\Omega$. Moreover, for simplicity, we will write $A_p$ instead of $A_p(\Omega)$: again, the precise domain will be implicit.

Define the class $A_\infty$ by

$$A_\infty = \bigcup_{p \geq 1} A_p.$$

If $w \in A_p \subset A_\infty$, then for every cube $Q$ and measurable set $E \subset Q$,

$$\frac{|E|}{|Q|} \leq [w]_{A_p}^{1/p} \left( \int_Q w(E) \, dx \right)^{1/p}. \quad (2.1)$$

A weight $w$ satisfies the reverse Hölder condition for some $s > 1$, denoted by $w \in RH_s$, if

$$[w]_{RH_s} = \sup_Q \left( \int_Q w(x)^s \, dx \right)^{1/s} \left( \int_Q w(x) \, dx \right)^{-1} < \infty.$$

We say that $w \in RH_\infty$ if for all cubes $Q$,

$$\inf_{x \in Q} w(x) \leq [w]_{RH_\infty} \int_Q w(y) \, dy.$$
Given a weight \( w \), \( w \in A_p \) for some \( p \) if and only if \( w \in RH_s \) for some \( s \): i.e.,

\[
\bigcup_{1 \leq p < \infty} A_p = A_\infty = \bigcup_{1 < s \leq \infty} RH_s.
\]

The reverse Hölder condition yields an estimate that is analogous to (2.1), exchanging
the roles of Lebesgue measure and the measure \( w \, dx \): if \( w \in RH_s \), then for every
cube \( Q \) and \( E \subset Q \),

\[
\frac{w(E)}{w(Q)} \leq [w]_{RH_s} \left( \frac{|E|}{|Q|} \right)^{1/s'}.
\]  

(2.2)

Below we will need a sharp estimate for the reverse Hölder exponent. The following
result is taken from Hytönen and Pérez [19]. If \( w \in A_\infty \), it satisfies the Fujii–Wilson
condition

\[
[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x) \, dx < \infty,
\]

where \( M \) is the Hardy–Littlewood maximal operator,

\[
Mf(x) = \sup_Q \int_Q |f(y)| \, dy \cdot \chi_Q(x).
\]

Then we have that \( w \in RH_s \) with \( [w]_{RH_s} \leq 2 \), where

\[
s = 1 + \frac{1}{2^{n+1}[w]_{A_\infty}}.
\]  

(2.3)

### 3 Matrix Weighted Spaces

In this section we define matrix weights and matrix weighted spaces, and prove some
basic properties. Recall that the symbol \( d \) denotes the dimension of vector functions
and matrices: in other words, we will consider vector-valued functions functions \( f : \Omega \to \mathbb{R}^d \), with

\[
f(x) = (f_1(x), \ldots, f_d(x)),
\]

and matrices \( A(x) = (a_{ij}(x))_{i,j=1}^d \). By \( Df \) we mean the \( n \times d \) matrix \( (\partial_i f_j) \).

Given a vector \( v = (v_1, \ldots, v_d) \), recall the vector \( \ell^p \) norms, \( 1 \leq p < \infty \),

\[
|v|_p = \left( \sum_{i=1}^d |v_i|^p \right)^{1/p},
\]
and let $|v|_\infty = \max(|v_1|, \ldots, |v_d|)$. When $p = 2$ we will often write $|v| = |v|_2$. We will frequently use the fact that given $1 \leq p < q \leq \infty$,

$$|v|_q \leq |v|_p \leq d^{1/p} |v|_\infty \leq d^{1/p} |v|_q.$$  

Let $\mathcal{M}_d$ denote the collection of all real-valued, $d \times d$ matrices. The norm of a matrix is the operator norm:

$$|A|_{\text{op}} = \sup_{v \in \mathbb{R}^d, |v|=1} |Av|.$$  

A matrix function is a map $W : \Omega \to \mathcal{M}_d$; we say that it is measurable if each component of $W$ is a measurable function.

Let $\mathcal{S}_d$ denote the collection of all those $A \in \mathcal{M}_d$ that are self-adjoint and positive semi-definite. If $A \in \mathcal{S}_d$, then it has $d$ non-negative eigenvalues, $\lambda_i, 1 \leq i \leq d$, and we have that

$$|A|_{\text{op}} = \max_i \lambda_i \leq \text{tr} A \leq d |A|_{\text{op}}.$$  

Moreover, there exists an orthogonal matrix $U$ such that $U^tAU$ is diagonal. We denote a diagonal matrix by $D(\lambda_1, \ldots, \lambda_d) = D(\lambda_i)$. If $W$ is a measurable matrix function with values in $\mathcal{S}_d$, then we can choose the matrices $U(x)$ to be measurable: the following result is from [33, Lemma 2.3.5]

**Lemma 3.1** Given a matrix function $W : \Omega \to \mathcal{S}_d$, there exists a $d \times d$ measurable matrix function $U$ defined on $\Omega$ such that $U^t(x)W(x)U(x)$ is diagonal.

If $A \in \mathcal{S}_d$ is diagonalized by an orthogonal matrix $U$ and has eigenvalues $\lambda_i$, for every $s > 0$ define $A^s = U D(\lambda_i^s)U^t$. By Lemma 3.1 we have that given any matrix function $W : \Omega \to \mathcal{S}_d$, $W^s$ is a measurable matrix function. For a fixed matrix function $W$ we will always implicitly assume that all of its powers are defined using the same orthogonal matrix $U$. Furthermore, if it is the case that $A$ is positive definite we can also define negative powers of $A$ through the orthogonal matrix $U$. Indeed, a simple calculation shows that $A^{-1} = U D(\lambda_i^{-1})U^t$ and for $s > 0$ we set $A^{-s} = U D(\lambda_i^{-s})U^t$.

By a matrix weight we mean a matrix function $W : \Omega \to \mathcal{S}_d$ such that $|W|_{\text{op}} \in L^1_{\text{loc}}(\Omega)$. Equivalently, we may assume that each eigenvalue $\lambda_i \in L^1_{\text{loc}}(\Omega), 1 \leq i \leq d$. We say that $W$ is an invertible matrix weight if $W$ is positive definite a.e.: equivalently, that det $W(x) \neq 0$ a.e. and so $W^{-1}$ exists. Hereafter, if $W$ is a matrix weight, we define $v(x) = |W(x)|_{\text{op}}$; if it is also invertible, we will always let $w(x) = |W^{-1}(x)|_{\text{op}}$.

**Proposition 3.2** Given an invertible matrix weight $W$, we have $0 < w(x) \leq v(x) < \infty$ for a.e. $x \in \Omega$. Furthermore, $W$ satisfies a two weight, degenerate ellipticity condition: for all $\xi \in \mathbb{R}^d$,

$$w(x)|\xi|^p \leq |W^{1/p}(x)\xi|^p \leq v(x)|\xi|^p.$$  

(3.1)
Proof First note that for a.e. \( x \in \Omega \), \( 1 = |I|_{\text{op}} \leq |W(x)|_{\text{op}}|W^{-1}(x)|_{\text{op}} \). Since \( W \) is a matrix weight, \( v \in L^1_{\text{loc}}(\Omega) \); since it is invertible, its eigenvalues are positive a.e. Hence, we must have that \( 0 < w(x) \leq v(x) < \infty \).

To prove the ellipticity conditions, we use the definition of matrix norm. The second inequality follows from it immediately:

\[
|W^{1/p}(x)\xi|^p \leq |W^{1/p}(x))|_{\text{op}}^p|\xi|^p \leq |W^{-1}(x)|_{\text{op}} |W^{1/p}(x)\xi|^p.
\]

The first follows similarly:

\[
|\xi|^p = |W^{-1/p}(x)W^{1/p}(x)\xi|^p \leq |W^{-1}(x)|_{\text{op}} |W^{1/p}(x)\xi|^p.
\]

\( \square \)

Remark 3.3 Note that if \( W \) is any matrix weight, the second inequality,

\[
|W^{1/p}(x)\xi|^p \leq v(x)|\xi|^p,
\]

still holds.

Given \( p, 1 \leq p < \infty \), and a matrix weight \( W : \Omega \rightarrow S_d \), define the weighted space \( L^p_W(\Omega) \) to be the set of all measurable, vector valued functions \( f : \Omega \rightarrow \mathbb{R}^d \) such that

\[
\|f\|_{L^p_W(\Omega)} = \left( \int_{\Omega} |W^{1/p}(x)f(x)|^p \, dx \right)^{1/p} < \infty.
\]

In this space, we identify two functions \( f, g \) as equivalent if \( \|f - g\|_{L^p_W(\Omega)} = 0 \). In the special case when \( p = 2 \), it is often useful to restate this norm in terms of the inner product on \( \mathbb{R}^d \):

\[
\|f\|_{L^2_W(\Omega)} = \left( \int_{\Omega} \langle W(x)f(x), f(x) \rangle \, dx \right)^{1/2}.
\]

The following lemma is proved in [30, 35].

Lemma 3.4 Given \( 1 \leq p < \infty \) and a matrix weight \( W : \Omega \rightarrow S_d \), the space \( L^p_W(\Omega) \) is a Banach space.

For a matrix weight that is non-invertible on a set of positive measure, the equivalence classes of functions can be quite large. However, if \( W \) is invertible, it is straightforward to identify them.

Lemma 3.5 Given \( 1 \leq p < \infty \), an invertible matrix weight \( W \), and \( f, g \in L^p_W(\Omega) \), then \( \|f - g\|_{L^p_W(\Omega)} = 0 \) if and only if \( f(x) = g(x) \) a.e.
Proof Clearly, if $f(x) = g(x)$ a.e., then $\|f - g\|_{L^p_W(\Omega)} = 0$. Since $W$ is an invertible matrix weight, we can apply Proposition 3.2 to prove the converse. By the ellipticity condition,

$$0 = \|f - g\|_{L^p_W(\Omega)} \geq \|f - g\|_{L^p(w, \Omega)},$$

and since $w(x) > 0$ a.e., it follows that $f(x) - g(x) = 0$ a.e. \[\Box\]

The set of bounded functions of compact support, $L^\infty_c(\Omega)$, and smooth functions of compact support, $C^\infty_c(\Omega)$, are both dense in $L^p_W(\Omega)$. These results seem to be known (cf. [15, Theorem 5.1]) but we have not found proofs in the literature. For completeness we include them here.

**Proposition 3.6** Given a matrix weight $W : \Omega \to S_d$, $L^\infty_c(\Omega)$ is dense in $L^p_W(\Omega)$.

**Proof** First assume that $W(x)$ is diagonal, that is $W(x) = D(\lambda_i(x))$. Fix $f \in L^p_W(\Omega)$. Then by the non-negativity of each $\lambda_i$ and the equivalence of norms,

$$\int_\Omega |W^{1/p}(x)f(x)|^p dx \approx \int_\Omega |W^{1/p}(x)f(x)|^p dx = \sum_{i=1}^d \int_\Omega |f_i(x)|^p \lambda_i(x) dx.$$

Therefore, we have that $f_i \in L^p(\lambda_i, \Omega)$. Since $\lambda_i \in L^1_{loc}(\Omega)$, $\lambda_i dx$ is a regular Borel measure, and so $L^\infty_c(\Omega)$ is dense in $L^p(\lambda_i, \Omega)$. Hence, given any $\epsilon > 0$, there exists $g_i \in L^\infty_c(\Omega)$ such that $\|f_i - g_i\|_{L^p(\lambda_i, \Omega)} < \epsilon$. Let $g = (g_1, \ldots, g_d)$. By our choice of the $g_i$’s we conclude that

$$\|f - g\|_{L^p_W(\Omega)} \lesssim \epsilon.$$

Now fix an arbitrary matrix weight $W$ and by Lemma 3.1 let $D = U^t W U$ be its diagonalization. Let $f \in L^p_W(\Omega)$ and set $h = U^t f$. Then by the orthogonality of $U$,

$$|D^{1/p}h| = |U^t W^{1/p}U U^t f| = |W^{1/p}f|.$$ 

Hence, $h \in L^p_D(\Omega)$ and by the previous argument, for any $\epsilon > 0$, there exists $g \in L^\infty_c(\Omega)$ such that $\|h - g\|_{L^p_D(\Omega)} < \epsilon$. Using orthogonality again, we have that

$$|D^{1/p}(h - g)| = |U^t W^{1/p}U(U^t f - g)| = |W^{1/p}(f - U g)|,$$

and since $|U g| \leq |U|_{op}|g|$, $U g \in L^\infty_c(\Omega)$. This completes the proof. \[\Box\]

As a consequence we have that smooth functions are dense in $L^p_W(\Omega)$.
**Proposition 3.7** Given a matrix weight \( W : \Omega \rightarrow S_d \), \( C^\infty_c(\Omega) \) is dense in \( L^p_W(\Omega) \).

**Proof** Fix \( f \in L^p_W(\Omega) \) and let \( \epsilon > 0 \). By Proposition 3.6, there exists \( g \in L^\infty_c(\Omega) \) such that \( \|f - g\|_{L^p_W(\Omega)} < \epsilon/2 \). Moreover, if we let \( v(x) = |W(x)|_{\text{op}} \) then \( v \in L^1_{\text{loc}}(\Omega) \) and

\[
\int_{\Omega} |g(x)|^p v(x) \, dx \leq \|g\|_\infty v(\text{supp}(g)) < \infty.
\]

Therefore, \( \|g - h\|_{L^p(v, \Omega)} < \epsilon/2 \). By Remark 3.3, \( |W^{1/p}(g - h)|^p \leq v |g - h|^p \), so we can conclude that \( \|f - h\|_{L^p_W(\Omega)} < \epsilon \).

\( \square \)

4 Matrix \( A_p \)

In this section we define matrix \( A_p \) weights and prove some of their properties. When \( p > 1 \) they are often defined in terms of norms on \( \mathbb{R}^d \), but here we take as our definition an equivalent condition due to Roudenko [34] that more closely resembles the definition of scalar \( A_p \) weights. Moreover, this approach also leads naturally to the definition of matrix \( A_1 \), which is due to Frazier and Roudenko [12].

**Definition 4.1** Given \( 1 < p < \infty \), an invertible matrix weight \( W : \Omega \rightarrow S_d \) is in matrix \( A_p(\Omega) \), denoted by \( W \in A_p(\Omega) \), if \( W^{-p/p'} \) is also a matrix weight and

\[
[W]_{A_p(\Omega)} = \sup_{Q} \left( \frac{\int_{Q \cap \Omega} |W^{1/p}(x)W^{-1/p}(y)|_{\text{op}}^p \, dy}{\int_{Q \cap \Omega} |W(x)|^p \, dx} \right)^{p/p'} < \infty,
\]

where the supremum is taken over all cubes in \( \mathbb{R}^n \) and where \( p' \) is the dual exponent to \( p \). When \( p = 1 \), we say that \( W \in A_1(\Omega) \) if \( W^{-1} \) is a matrix weight and

\[
[W]_{A_1} = \sup_{Q} \sup_{x \in Q} \int_{Q \cap \Omega} |W(y)W^{-1}(x)|_{\text{op}} \, dy < \infty.
\]

**Remark 4.2** As is the case for scalar weights (cf. Remark 2.1), if \( \Omega = \mathbb{R}^n \), then we get an equivalent definition if we replace cubes with balls. We will want to elide between balls and cubes on more general domains. Therefore, as in the scalar case, given any matrix weight \( W \) on a domain \( \Omega \), we will implicitly assume that it satisfies the matrix \( A_p \) condition on some larger domain \( \Omega' \) and we will suppress any reference to the domain, writing \( A_p \) instead of \( A_p(\Omega) \). We note in passing that the problem of characterizing those domains \( \Omega \) such that every \( W \in A_p(\Omega) \) is the restriction of a matrix in \( A_p(\mathbb{R}^n) \) is open.

**Remark 4.3** When \( d = 1 \) and \( W(x) = w(x) \) is a scalar valued weight, the matrix \( A_p \) condition becomes the \( A_p \) condition as defined in Sect. 2.
The matrix $A_p$ weights satisfy the same duality relationship as scalar $A_p$ weights. This is due to Roudenko [34, Corollary 3.3] when $Ω = \mathbb{R}^n$, but the proof given there extends without change to the more general setting.

**Lemma 4.4** Given $1 < p < \infty$ and a matrix weight $W$, $W \in A_p$ if and only if $W^{-p'/p} \in A_{p'}$.

By definition, if $W \in A_p$ it is an invertible matrix weight, so we have associated to it the scalar weights $v$ and $w$, and $W$ satisfies the degenerate ellipticity condition (3.1). Moreover, these weights are scalar $A_p$ weights.

**Lemma 4.5** Given $1 \leq p < \infty$, if $W \in A_p$, then $v(x) = |W(x)|_{op}$ and $w(x) = \|W^{-1}(x)\|_{op}^{-1}$ are scalar $A_p$ weights.

**Remark 4.6** The converse of this lemma is not true: for a counter-example, see Lauzon and Treil [26].

**Proof** First suppose that $p > 1$. The fact that $v \in A_p$ is due to Goldberg [15, Corollary 2.3]. (Again, his proof assumes $Ω = \mathbb{R}^n$, but it extends to the general case without change.) Further, by Lemma 4.4, $W^{-p'/p} \in A_{p'}$, so by the definition of the operator norm and what we just proved,

$$w^{-p'/p} = |W^{-1}|_{op}^{-p'/p} = |W^{-p'/p}|_{op} \in A_{p'}.$$ 

Therefore, by the duality of scalar $A_p$ weights (which follows at once from the definition), $w \in A_p$.

For the case $p = 1$ we modify an argument from Frazier and Roudenko [12, Lemma 2.1]. To prove that $v \in A_1$ we first construct a measurable vector function $v$ such that $|v(y)| = 1$ and $|W(y)v(y)| \leq |W(y)|_{op} \|v(y)\|_{op}$ a.e. If $W = D(\lambda_i)$ is diagonal, let $v$ be the constant vector $h = (d^{-1/2}, \ldots, d^{-1/2})$. Then

$$|D(y)h| \geq d^{-1/2} \max_i \lambda_i(y) = d^{-1/2}|D(y)|_{op}.$$ 

For a general $W$, let $D = UWU^1$ be the diagonalization of $W$ from Lemma 3.1 and let $v(y) = U^1(y)h$. Then $v$ is measurable and

$$|W(y)v(y)| = |U(y)W(y)U^1(y)h| = |D(y)h| \geq d^{-1/2}|D(y)|_{op} = d^{-1/2}|W(y)|_{op}.$$ 

Given such a vector function $v$, we can now estimate as follows. Fix a cube $Q$, let $x \in Q$, and set $w(y) = W(x)v(y)$. Then

$$\int_Q |W(y)|_{op} dy \leq \int_Q |W(y)v(y)| dy = \int_Q |W(y)W^{-1}(x)w(y)| dy \leq \int_Q |W(y)W^{-1}(x)|_{op}|w(y)| dy \leq [W]_{A_1|W(x)|_{op}}.$$
To prove that \( w \in A_1 \), we can argue similarly. Fix a cube \( Q \) and \( x \in Q \). Arguing as above, construct a vector \( \mathbf{w} = \mathbf{w}(x) \) so that \( |\mathbf{w}| = 1 \) and \( |W^{-1}(x)|_{\text{op}} \lesssim |W^{-1}(x)\mathbf{w}| \). Let \( \mathbf{v} = W^{-1}(\mathbf{x})\mathbf{w} \). Then for any \( y \in Q \),

\[
|W^{-1}(x)|_{\text{op}} \lesssim |\mathbf{v}| = |W^{-1}(y)W(y)\mathbf{v}| \leq |W^{-1}(y)|_{\text{op}}|W(y)\mathbf{v}|.
\]

Hence,

\[
|W^{-1}(x)|_{\text{op}} \int_Q |W^{-1}(y)|_{\text{op}}^{-1} dy \lesssim \int_Q |W(y)| dy
dy = \int_Q |W(y)W^{-1}(x)\mathbf{w}| dy
\leq \int_Q |W(y)W^{-1}(x)|_{\text{op}} dy \leq [W]_{A_1}.
\]

This completes the proof. \( \square \)

The matrix \( A_p \) condition characterizes the matrix weights \( W \) such that the averaging operators \( f \mapsto \int_Q f(x) dx \) are uniformly bounded on \( L^p_W(\Omega) \). (See \cite{15, Proposition 2.1} for the case \( p > 1 \).) This is also true for more general averaging operators.

**Proposition 4.7** Let \( Q \) be a collection of pairwise disjoint cubes in \( \mathbb{R}^n \). Given \( 1 \leq p < \infty \) and a matrix weight \( W \in A_p \), the averaging operator

\[
A_Q f(x) = \sum_{Q \in Q'} \int_Q f(y) dy \cdot \chi_Q(x)
\]

satisfies

\[
\|A_Q f\|_{L^p_W(\Omega)} \leq [W]_{A_p}^{1/p} \|f\|_{L^p_W(\Omega)}.
\]

**Proof** To begin, define \( f \equiv 0 \) on \( \mathbb{R}^n \setminus \Omega \). We first consider the case \( p > 1 \): since the cubes in \( Q \) are disjoint, by Hölder’s inequality and the definition of matrix \( A_p \),

\[
\int_\Omega |W^{1/p}(x)A_Q f(x)|^p dx \leq \int_{\mathbb{R}^n} |W^{1/p}(x)A_Q f(x)|^p dx
= \int_{\mathbb{R}^n} \left| \sum_{Q \in Q'} \int_Q \chi_Q(x)W^{1/p}(x)W^{-1/p}(y)W^{1/p}(y)f(y) dy \right|^p dx
\leq \int_{\mathbb{R}^n} \sum_{Q \in Q} \chi_Q(x) \left( \int_Q |W^{1/p}(x)W^{-1/p}(y)|_{\text{op}}^{p/p'} dy \right)^{p/p'} \left( \int_Q |W^{1/p}(y)f(y)| dy \right) dx
= \sum_{Q \in Q'} \int_Q \left( \int_Q |W^{1/p}(x)W^{-1/p}(y)|_{\text{op}}^{p/p'} dy \right)^{p/p'} dx \left( \int_Q |W^{1/p}(y)f(y)|^p dy \right)
\leq [W]_{A_p} \int_\Omega |W^{1/p}(y)f(y)|^p dy.
\]
When \( p = 1 \) the proof is almost identical, omitting Hölder’s inequality and using Fubini’s theorem and the definition of \( \mathcal{A}_1 \).

We now want to prove that for “nice” functions \( \phi \in C_c^\infty (B(0, 1)) \), the convolution operator \( f \mapsto \phi \ast f \) is bounded on \( L_W^p (\Omega) \) and that approximate identities defined using \( \phi \) converge. We first begin with a lemma.

**Lemma 4.8** Given \( 1 \leq p < \infty \) and \( W \in \mathcal{A}_p \), then for any cube \( Q \) and \( f \in L_W^p (\Omega) \)

\[
|||Q|^{-1} \chi_Q \ast f||_{L_W^p (\Omega)} \leq C(n, p) [W]_{\mathcal{A}_p} \| f \|_{L_W^p (\Omega)}.
\]

The same inequality is true if we replace the cube \( Q \) with any ball \( B \).

**Proof** Define the cubes \( \{ Q_k = Q + \ell (Q) k : k \in \mathbb{Z}^n \} \).

The cubes in \( Q_k \) form a partition of \( \mathbb{R}^n \). Further, we can then divide the cubes \( 3 Q_k \) into \( 3^n \) families \( Q_j \) of pairwise disjoint cubes. But then for every \( k \in \mathbb{Z}^n \) and \( x \in Q_k \), extending \( f \) by zero in \( \mathbb{R}^n \setminus \Omega \), we have that

\[
|W^{1/p}(x)| Q |^{-1} \chi_Q \ast f(x) | = \left| |Q|^{-1} \int_{\mathbb{R}^n} W^{1/p}(x)f(y)\chi_Q(x-y) dy \right|
\]

\[
\leq |Q|^{-1} \int_{\mathbb{R}^n} |W^{1/p}(x)f(y)\chi_Q(x-y)| dy
\]

\[
\leq C(n) \int_{3Q_k} |W^{1/p}(x)f(y)| dy.
\]

Therefore,

\[
\int_{\mathbb{R}^n} |W^{1/p}(x)| Q |^{-1} \chi_Q \ast f(x) |^p dx
\]

\[
\leq C(n, p) \sum_{k \in \mathbb{Z}^n} \int_{Q_k} \left( \int_{3Q_k} |W^{1/p}(x)f(y)| dy \right)^p dx
\]

\[
\leq C(n, p) \sum_{j=1}^{3^n} \sum_{Q \in Q_j} \int_Q \left( \int_{Q} |W^{1/p}(x)f(y)| dy \right)^p dx,
\]

and we can now argue exactly as in the proof of Proposition 4.7, starting at (4.1), to get the desired estimate for cubes.

To prove this for balls, fix a ball \( B \), and let \( Q \) be the smallest cube containing \( B \). Then \( |B| \approx |Q| \), and arguing as above, we get

\[
|W^{1/p}(x)| B |^{-1} \chi_B \ast f(x) | \leq C(n) \int_{3Q_k} |W^{1/p}(x)f(y)| dy,
\]

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and the proof continues as before. □

**Theorem 4.9** Given $1 \leq p < \infty$ and $W \in \mathcal{A}_p$, let $\phi \in C_c^\infty(B(0, 1))$ be a non-negative, radially symmetric and decreasing function with $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$, and for $t > 0$ let $\phi_t(x) = t^{-n/2} \phi(x/t)$. Then

$$
\sup_{t > 0} \|\phi_t * f\|_{L^p_W(\Omega)} \leq C(n, p)[W]^{1/p} \|f\|_{L^p_W(\Omega)}
$$

(4.2)

for every $f \in L^p_W(\Omega)$. As a consequence, we have that for every such $f$,

$$
\lim_{t \to 0} \|\phi_t * f - f\|_{L^p_W(\Omega)} = 0.
$$

(4.3)

**Proof** To prove (4.2), consider the function

$$
\Phi(x) = \sum_{k=1}^\infty a_k |B_k|^{-1} \chi_{B_k}(x),
$$

where the balls $B_k$ are centered at the origin, $B_{k+1} \subset B_k$ for all $k$, and the $a_k$ are non-negative with $\sum a_k = 1$. Extending $f$ by zero as before it will suffice to show that

$$
\|\Phi * f\|_{L^p_W(\mathbb{R}^n)} \leq C(n, p)[W]^{1/p} \|f\|_{L^p_W(\mathbb{R}^n)};
$$

inequality (4.2) follows by approximating $\phi_t$ from below by a sequence of such functions and applying Fatou’s lemma. But by Minkowski’s inequality and Lemma 4.8,

$$
\|\Phi * f\|_{L^p_W(\mathbb{R}^n)} \leq \sum_{k=1}^\infty a_k \|B_k|^{-1} \chi_{B_k} * f\|_{L^p_W(\mathbb{R}^n)} \leq C(n, p)[W]^{1/p} \sum_{k=1}^\infty a_k \|f\|_{L^p_W(\mathbb{R}^n)} = C(n, p)[W]^{1/p} \|f\|_{L^p_W(\mathbb{R}^n)}.
$$

To prove (4.3), fix $\epsilon > 0$. Given $f \in L^p_W(\Omega)$, by Proposition 3.7 there exists $g \in C_c(\Omega)$ such that $\|f - g\|_{L^p_W(\Omega)} < \epsilon$. By a classical result we have that $\phi_t * g \to g$ uniformly, and so by (3.1) for all $t$ sufficiently small,

$$
\|\phi_t * g - g\|_{L^p_W(\Omega)} \leq \|\phi_t * g - g\|_{L^p_\varnothing(\Omega)} < \epsilon.
$$

Therefore, by (4.2) we have that

$$
\|\phi_t * f - f\|_{L^p_W(\Omega)} \leq \|\phi_t * f - g\|_{L^p_W(\Omega)} + \|\phi_t * g - f\|_{L^p_W(\Omega)} + \|f - g\|_{L^p_W(\Omega)} < \epsilon + C\|f - g\|_{L^p_W(\Omega)} \lesssim \epsilon.
$$

□
Remark 4.10 In our proof of Theorem 4.9 the restrictions on $\phi$ seem artificial when compared to the scalar case, where any non-negative function $\phi \in C^\infty_c$ can be used. We need our restrictions to allow us to approximate $\phi$ by step functions like $\Phi_1$. It is also possible to prove inequality (4.2) by appealing to the bounds for singular integrals given in [15]. This approach only works for $p > 1$, but does allow for a larger class of functions $\phi$. Details are left to the interested reader. This was the approach we used in an early version of this paper; we want to thank S. Treil for suggesting the idea behind the proof we give above.

5 Degenerate Sobolev Spaces and $H = W$

In this section we define a family of degenerate Sobolev spaces using the matrix weighted spaces $L^p_W(\Omega)$. As we noted above, such spaces have been studied previously; here we consider them in the particular cases where $W$ is either an invertible matrix weight or a matrix $A_p$ weight.

Hereafter, let $W \in S_n$ be an invertible matrix weight and let $v(x) = |W(x)|_{op}$ and $w(x) = |W^{-1}(x)|_{op}^{-1}$. For $1 \leq p < \infty$, define the degenerate Sobolev space $\mathcal{W}^{1,p}_W(\Omega)$ to be the set of all $f \in \mathcal{W}^{1,1}_\text{loc}(\Omega)$ such that

$$\|f\|_{\mathcal{W}^{1,p}_W(\Omega)} = \|f\|_{L^p(v, \Omega)} + \|\nabla f\|_{L^p_W(\Omega)} < \infty.$$ 

Viewing this space as a collection of pairs of the form $(f, \nabla f)$, it is clear that we may consider $\mathcal{W}^{1,p}_W(\Omega)$ as a linear subspace of the Banach space $L^p_W(\Omega)$; since $v \in L^1_{\text{loc}}(\Omega)$, $L^p(v, \Omega)$ is a Banach space and by Lemma 3.4 so is $L^p_W(\Omega)$. Clearly, $\mathcal{W}^{1,p}_W(\Omega)$ is non-trivial: for instance, if $f \in C^\infty_c(\Omega)$, then $f \in \mathcal{W}^{1,p}_W(\Omega)$, since by Proposition 3.2,

$$\|f\|_{\mathcal{W}^{1,p}_W(\Omega)} \leq (\|f\|_\infty + \|\nabla f\|_\infty)v(\text{supp}(f)) < \infty.$$ 

Matrix weighted Sobolev spaces generalize the scalar weighted Sobolev spaces: that is, given a weight $u$, the space $\mathcal{W}^{1,p}(u, \Omega)$ of functions in $\mathcal{W}^{1,1}_{\text{loc}}(\Omega)$ such that

$$\|f\|_{\mathcal{W}^{1,p}(u, \Omega)} = \|f\|_{L^p(u, \Omega)} + \|\nabla f\|_{L^p(u, \Omega)} < \infty.$$ 

Every matrix weighted space $\mathcal{W}^{1,p}_W(\Omega)$ is nested between two scalar weighted spaces. By Proposition (3.2), we have that

$$\|f\|_{\mathcal{W}^{1,p}(w, \Omega)} \leq \|f\|_{\mathcal{W}^{1,p}_W(\Omega)} \leq \|f\|_{\mathcal{W}^{1,p}(v, \Omega)};$$

hence,

$$\mathcal{W}^{1,p}(v, \Omega) \subset \mathcal{W}^{1,p}_W(\Omega) \subset \mathcal{W}^{1,p}(w, \Omega).$$

In general, these inclusions are proper as the following example shows.
Example 5.1 Let $\Omega = (0, 1) \times (0, 1)$. Fix $1 < p < \infty$ and $\alpha \in (0, 1)$. Define the matrix weight

$$W(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & x^{-\alpha}y^{-\alpha} \end{bmatrix}.$$ 

Then a straightforward calculation shows that $W \in A_p$ since it is a diagonal matrix whose entries are the product of scalar $A_1$ weights in each independent variable. It is also easy to see that the weights $v$ and $w$ are given by

$$v(x, y) = x^{-\alpha}y^{-\alpha}, \quad w(x, y) = 1.$$ 

Clearly, $v, w \in L^1(\Omega)$. Now define two elements of $\mathcal{W}^{1, 1}_{loc}(\Omega)$: $f(x, y) = cx^{\frac{a-1}{p}}$ and $g(x, y) = cy^{\frac{a-1}{p}}$, where $c > 0$ is chosen so that

$$\nabla f(x, y) = \begin{bmatrix} \frac{a-1}{p} \\ 0 \end{bmatrix} \quad \text{and} \quad \nabla g(x, y) = \begin{bmatrix} 0 \\ \frac{a-1}{p} \end{bmatrix}.$$ 

Since $f, g$ are bounded, $f \in L^p(v, \Omega)$ and $g \in L^p(w, \Omega)$. The gradient of $f$ satisfies

$$\int_0^1 \int_0^1 |W^{1/p} \nabla f|^p dxdy = \int_0^1 \int_0^1 \frac{1}{x} dy dx = \frac{1}{\alpha} < \infty,$$

$$\int_0^1 \int_0^1 |\nabla f|^p v dxdy = \int_0^1 \int_0^1 y^{-\alpha} dy = \infty.$$ 

The opposite holds for $g$: that is $\|\nabla g\|_{L^p_W(\Omega)} = \infty$ and $\|\nabla g\|_{L^p_W(w, \Omega)} = \frac{1}{\alpha}$. Thus, $f$ belongs to $\mathcal{W}^{1, p}_W(\Omega) \setminus \mathcal{W}^{1, p}(v, \Omega)$, while $g$ belongs to $\mathcal{W}^{1, p}(w, \Omega) \setminus \mathcal{W}^{1, p}_W(\Omega)$.

Essential to our results is the requirement that $\mathcal{W}^{1, p}_W(\Omega)$ be a Banach space. This is achieved by imposing size conditions on $w^{-1}$ as the next theorem demonstrates.

Theorem 5.2 Given a domain $\Omega$, $1 \leq p < \infty$ and an invertible matrix weight $W$, suppose $w^{-p'}/p \in L^1_{loc}(\Omega)$ (if $p = 1$, $w^{-1} \in L^{\infty}_{loc}(\Omega)$). Then $\mathcal{W}^{1, p}_W(\Omega)$ is a Banach space. In particular, this is the case if $1 \leq p < \infty$ and $W \in A_p$.

Proof We need to show that $\mathcal{W}^{1, p}_W(\Omega)$ is a closed subspace of $L^p(v, \Omega) \oplus L^p_W(\Omega)$. Fix a Cauchy sequence $\{u_k\}$ in $\mathcal{W}^{1, p}_W(\Omega)$. Then there exists $u \in L^p(v, \Omega)$ and $U \in L^p_W(\Omega)$ such that $u_k \rightarrow u$ in $L^p(v, \Omega)$ and $\nabla u_k \rightarrow \nabla u$ in $L^p_W(\Omega)$. We will show that $u, U \in L^1_{loc}(\Omega)$ and that $U = \nabla u$ in the sense of distributional derivatives. Then $u \in W^{1, 1}_{loc}(\Omega)$ with $u \in L^p(v, \Omega)$ and $\nabla u \in L^p_W(\Omega)$. Thus $u$ belongs to $\mathcal{W}^{1, p}_W(\Omega)$.

Fix $\varphi \in C^\infty_c(\Omega)$; we need to show that

$$\int_\Omega U_j \varphi dx = -\int_\Omega u \partial_j \varphi dx, \quad 1 \leq j \leq n.$$
These integrals are finite since \( u \) and \( U \) are locally integrable. To see this suppose first that \( p > 1 \). Let \( K = \text{supp}(\phi) \subseteq \Omega \); then, since \( w^{-p'/p} \in L^1_{\text{loc}}(\mathbb{R}^n) \), we have that

\[
\int_{\Omega} |U_j\phi| \, dx \leq \|\phi\|_\infty \int_K |U_j| w^{1/p} w^{-1/p} \, dx \\
\leq w^{-p'/p}(K)^{1/p'} \|\phi\|_\infty \left( \int_{\Omega} |U_j|^p w \, dx \right)^{1/p} \\
\leq w^{-p'/p}(K)^{1/p'} \|\phi\|_\infty \|U\|_{L^p_w(\Omega)}.
\]

We can bound the other integral similarly: since \( w \leq v \) a.e., \( v^{-p'/p} \in L^1_{\text{loc}}(\Omega) \) and so

\[
\int_{\Omega} |u\partial_j \phi| \, dx \leq v^{-p'/p}(K)^{1/p'} \|\nabla \phi\|_\infty \|u\|_{L^p(v,\Omega)}.
\]

When \( p = 1 \), we can argue similarly, using the fact that \( v^{-1}, w^{-1} \in L^\infty_{\text{loc}}(\Omega) \).

We now show that these two integrals are equal. With \( \phi \) and \( K \) as before, by the weak differentiability of each \( u_k \) we have that

\[
\left| \int_{\Omega} U_j \phi + u \partial_j \phi \, dx \right| = \left| \int_{\Omega} (U_j - \partial_j u_k) \phi \, dx + \int_{\Omega} (u - u_k) \partial_j \phi \, dx \right| \\
\leq w^{-p'/p}(K)^{1/p'} \|\phi\|_\infty \|U - \nabla u_k\|_{L^p_w(\Omega)} \\
+ v^{-p'/p}(K)^{1/p'} \|\partial_j \phi\|_\infty \|u_k - u\|_{L^p(v,\Omega)}.
\]

Both terms on the right go to zero as \( k \to \infty \). Thus we have shown that \( U = \nabla u \) in the sense of distributional derivatives and so \( u \in W^{1,1}_{\text{loc}}(\Omega) \).

Finally, note that if \( p > 1 \) and \( W \in A_p \), then by Lemma 4.5, \( w \in A_p \) and so \( w^{-p'/p} \in A_{p'} \) and thus is locally integrable. When \( p = 1 \), it follows from the fact that \( w \in A_1 \) that \( w \) is locally bounded away from zero and so \( w^{-1} \) is locally bounded. This completes the proof. \( \square \)

The importance of the matrix \( A_p \) condition is that it lets us prove, as is the case in the classical Sobolev spaces, that smooth functions are dense in \( W^{1,1}_{\text{loc}}(\Omega) \). Define \( \mathcal{H}_{W^{1,p}}(\Omega) \) to be the closure of \( C^\infty(\Omega) \cap \mathcal{H}_{W^{1,p}}(\Omega) \) in \( \mathcal{H}_{W^{1,p}}(\Omega) \).

**Theorem 5.3** Given a domain \( \Omega \), if \( 1 \leq p < \infty \) and \( W \in A_p \), then

\[
\mathcal{H}_{W^{1,p}}(\Omega) = \mathcal{H}_{W^{1,p}}(\Omega).
\]

**Remark 5.4** The assumption that \( W \in A_p \) is not sharp for the conclusion of Theorem 5.3 to hold even when \( d = 1 \): see, for instance, the recent paper by Zhikov [40]. However, it appears to be very close to optimal: to illustrate this we sketch [29,
Example 3.9] for the case $p = 2$. There, the authors consider the matrix

$$A = \begin{pmatrix} |x|^{2\gamma} & 0 \\ 0 & 1 \end{pmatrix}$$

for $(x, y) \in \Omega = [-\frac{1}{3}, \frac{1}{3}] \times [-\frac{1}{3}, \frac{1}{3}]$ with $\gamma > 0$. It is clear that $A \in \mathcal{A}_2$ for $0 < \gamma < 1/2$ while $A \notin \mathcal{A}_2$ for $\gamma > 1/2$. In the latter case, the function $u(x, y) = |x|^{\alpha}$ with $\alpha = \max\{-\frac{1}{2}, \frac{1-2\gamma}{2}\}$, $0$ is shown to be a member of $\mathcal{H}^{1,2}_A(\Omega)$ while its gradient $\nabla u = (\alpha|x|^{\alpha-1}x, 0)$ is not an $L^1_{\text{loc}}(\Omega)$ function and hence $u \notin \mathcal{W}^{1,2}_A(\Omega)$.

**Proof** We will show that $\mathcal{W}^{1,p}_W(\Omega) \subset \mathcal{H}^{1,p}_W(\Omega)$ since the reverse inclusion holds by definition. The proof is an adaption of the classic proof that $H = W$: see [1, 28]. We will show that given any $f \in \mathcal{W}^{1,p}_W(\Omega)$ and any $\epsilon > 0$, there exists $g \in C^\infty(\Omega) \cap \mathcal{W}^{1,p}_W(\Omega)$ such that $\|f - g\|_{\mathcal{W}^{1,p}_W(\Omega)} < \epsilon$.

For each $j \in \mathbb{N}$, define the bounded sets

$$\Omega_j = \{x \in \Omega : |x| < j, \text{dist}(x, \partial \Omega) > 1/j\}.$$ 

Let $\Omega_0 = \Omega_{-1} = \emptyset$ and define the sets $A_j = \Omega_{j+1} \setminus \overline{\Omega}_{j-1}$. These sets are an open cover of $\Omega$, each $A_j$ is compact, and given $x \in \Omega$, $x \in A_j$ for only a finite number of indices $j$. We can therefore form a partition of unity subordinate to this cover: there exists $\psi_j \in C^\infty_c(A_j)$ such that for all $x \in \Omega$, $0 \leq \psi_j(x) \leq 1$ and

$$\sum_{j=1}^{\infty} \psi_j(x) = 1.$$ 

Since $f \in \mathcal{W}^{1,1}_{\text{loc}}(\Omega)$, $\psi_j f \in \mathcal{W}^{1,1}_{\text{loc}}(\Omega)$. Furthermore, since $\nabla(\psi_j f) = \psi_j \nabla f + f \nabla \psi_j$ a.e. in $\Omega$ (see [14, Sect. 7.3]), we have that

$$|W^{1/p} \nabla(\psi_j f)| \leq |\psi_j||W^{1/p} \nabla f| + |f||W^{1/p} \nabla \psi_j| \leq \|\psi_j\|_{L^\infty} |W^{1/p} \nabla f| + \|\nabla \psi_j\|_{L^\infty} |f| v^{1/p},$$

and so $\psi_j f \in \mathcal{W}^{1,p}_W(\Omega)$.

Fix a non-negative, radially symmetric and decreasing function $\phi \in C^\infty_c(B(0, 1))$ with $\int \phi \, dx = 1$. Then the convolution

$$\phi_t * (\psi_j f)(x) = \int_{A_j} \phi_t(x-y) \psi_j(y) f(y) \, dy$$

is only non-zero if for some $y \in A_j$, $|x-y| < t$.

Hence, for $j \geq 3$, if we fix $t = t_j$, $0 < t_j < (j+1)^{-1} - (j+2)^{-1}$, this will hold only if $(j+2)^{-1} < \text{dist}(x, \partial \Omega) \leq (j-2)^{-1}$. Therefore,

$$\text{supp}(\psi_j * (\psi_j f)) \subset \Omega_{j+2} \setminus \overline{\Omega}_{j-2} = B_j \subset \Omega.$$
We will fix the precise value of $t_j$ below.

Define
\[
g(x) = \sum_{j=1}^{\infty} \phi_{t_j} \ast (\psi_j f)(x).
\]

Since $\phi$ is smooth, each summand is in $C^\infty(\Omega)$. Further, given $x \in \Omega$, it is contained in a finite number of the $B_j$, so only a finite number of terms are non-zero. Thus the series converges locally uniformly and $g \in C^\infty(\Omega)$.

Finally, fix $\epsilon > 0$; we claim that for the appropriate choice of $t_j$ we have $\|f - g\|_{W^{1,p}(\Omega)} < \epsilon$. To prove this, we consider each part of the norm separately.

Since $v \in A_p$, the approximate identity $\{\phi_t\}_{t>0}$ converges in $L^p(v, \Omega)$. (See [37, Theorem 2.1.4].) Therefore, for each $j$ there exists $t_j$ such that
\[
\|f - g\|_{L^p(v, \Omega)} \leq \sum_{j=1}^{\infty} \|\psi_j f - \phi_{t_j} \ast (\psi_j f)\|_{L^p(v, \Omega)} \leq \sum_{j=1}^{\infty} \frac{\epsilon}{2j+1} = \frac{\epsilon}{2}.
\]

The argument for the second part of the norm is similar. Since $\psi_j f \in \mathcal{H}^{1,1}_\text{loc}(\Omega)$, $\phi_{t_j} \ast \nabla(\psi_j f) = \nabla(\phi_{t_j} \ast \psi_j f)$. Fix $j$; then by Theorem 4.9 there exists $t_j$ such that
\[
\|\nabla(\psi_j f - \phi_{t_j} \ast (\psi_j f))\|_{L^p_W(\Omega)} = \|\nabla(\psi_j f) - \phi_{t_j} \ast \nabla(\psi_j f)\|_{L^p_W(\Omega)} < \frac{\epsilon}{2j+1}.
\]

Therefore,
\[
\|\nabla(f - g)\|_{L^p_W(\Omega)} \leq \sum_{j=1}^{\infty} \|\nabla(\psi_j f - \phi_{t_j} \ast (\psi_j f))\|_{L^p_W(\Omega)} \leq \sum_{j=1}^{\infty} \frac{\epsilon}{2j+1} = \frac{\epsilon}{2}.
\]

Thus, we have shown that $\|f - g\|_{W^{1,p}(\Omega)} < \epsilon$ and our proof is complete. \(\square\)

As a corollary to Theorem 5.3 we can prove that when $\Omega = \mathbb{R}^n$, smooth functions of compact support are dense.

**Corollary 5.5** If $1 \leq p < \infty$ and $W \in A_p$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}^{1,p}_W(\mathbb{R}^n)$.

**Proof** Fix $\epsilon > 0$ and $f \in \mathcal{H}^{1,p}_W(\mathbb{R}^n)$. By Theorem 5.3 there exists $h \in C^\infty(\Omega) \cap \mathcal{H}^{1,p}_W(\mathbb{R}^n)$ such that
\[
\|f - h\|_{\mathcal{H}^{1,p}_W(\mathbb{R}^n)} < \epsilon/2.
\]

Therefore, to complete the proof we will construct $g \in C_c^\infty(\mathbb{R}^n)$ such that
\[
\|g - h\|_{\mathcal{H}^{1,p}_W(\mathbb{R}^n)} < \epsilon/2. \tag{5.1}
\]
For each $k \geq 2$, let $v_k \in C_c^\infty(\mathbb{R}^n)$ be such that $\text{supp}(v_k) \subset B(0, 2k), 0 \leq v_k \leq 1, v_k(x) = 1$ for $x \in B(0, k)$, and $|\nabla v_k| \leq 1/k$. Let $g_k = hv_k$. Then $g_k \in C_c^\infty(\mathbb{R}^n)$, $|g_k| \leq |h|$, and $g_k \to h$ pointwise as $k \to \infty$. Since $h \in L^p(v, \mathbb{R}^n)$, by the dominated convergence theorem,

$$\lim_{k \to \infty} \|g_k - h\|_{L^p(v, \mathbb{R}^n)} = 0.$$ 

Similarly, since $\nabla g_k = v_k \nabla h + h \nabla v_k$, $\nabla g_k \to \nabla h$ as $k \to \infty$. Furthermore, by (3.1),

$$|W^{1/p}(x)\nabla g_k(x)|^p \lesssim |v_k W^{1/p}(x)\nabla h(x)|^p + |h W^{1/p}(x)\nabla v_k(x)|^p$$

$$\lesssim |W^{1/p}(x)\nabla h(x)|^p + |\nabla v_k(x)|^p|h(x)|^p v(x)$$

$$\lesssim |W^{1/p}(x)\nabla h(x)|^p + |h(x)|^p v(x).$$

Since $h \in \mathcal{W}_W^{1,p}(\mathbb{R}^n)$, the final term is in $L^1(\mathbb{R}^n)$, so again by the dominated convergence theorem

$$\lim_{k \to \infty} \|\nabla g_k - \nabla h\|_{W^1_p(\mathbb{R}^n)} = 0.$$ 

Therefore, for $k$ sufficiently large, if we let $g = g_k$, we get inequality (5.1) as desired. 

By modifying the proof of Theorem 5.3 we can also show that functions that are smooth up the boundary are dense in $\mathcal{W}_W^{1,p}(\Omega)$ provided $\Omega$ has some boundary regularity. Given a bounded domain $\Omega$, let $S_W^{1,p}(\Omega)$ denote the closure of $C^\infty(\overline{\Omega})$ in $\mathcal{W}_W^{1,p}(\Omega)$.

**Theorem 5.6** Let $\Omega$ be a bounded domain such that $\partial \Omega$ is locally a Lipschitz graph. Then for $1 \leq p < \infty$ and $W \in \mathcal{A}_p$, $S_W^{1,p}(\Omega) = \mathcal{W}_W^{1,p}(\Omega)$.

**Proof** The proof of this result in the classical case (see, for instance, Evans and Gariepy [11, Sect. 4.2]) is an adaptation of the proof that $H = W$. In our setting, we can use the same modifications to adapt the proof of Theorem 5.3 and we leave the details to the reader. Here, we note that the heart of the changes is proving that, given a fixed vector $a \in \mathbb{R}^n$, $\phi_t * f(\cdot + at)$ converges to $f$ in $\mathcal{W}_W^{1,p}(\Omega)$. To modify the argument given above, it will suffice to prove that

$$\sup_{t > 0} \|\phi_t * f(\cdot + at)\|_{W^1_p(\Omega)} \leq C \|f\|_{W^1_p(\Omega)}.$$

But if we fix $t > 0$,

$$\phi_t * f(x + at) = \int_\Omega t^{-n} \phi \left( \frac{x - y}{t} + a \right) f(y) \, dy = \psi * f(x),$$
where $\psi$ is a positive, radially decreasing function centered at $a$. Such $\psi$ can be approximated by functions of the form

$$\Phi(x) = \sum_{k=1}^{\infty} a_k |B_k|^{-1} \chi_{B_k}(x),$$

where the balls $B_k$ are nested and centered at $a$. With such functions $\Phi$, the proof of Theorem 4.9 goes through without change. \hfill \square

### 6 Degenerate $p$-Laplacian Equations

We now consider the applications of matrix weighted Sobolev spaces to the study of degenerate elliptic equations. In this section we generalize some results from [8] for arbitrary matrix weights; in Sect. 7 we will apply these results in the special case when we assume the matrix $A_p$ condition. Throughout this section, let $\Omega$ be a bounded domain in $\mathbb{R}^n$.

In [8] the authors studied the partial regularity of solutions to the divergence form degenerate $p$-Laplacian

$$L_{A, p}u = \text{div}(\langle A \nabla u, \nabla u \rangle^{\frac{p-2}{2}} A \nabla u) = 0,$$  

(6.1)

where $1 < p < \infty$ and $A \in \mathcal{S}_n$ satisfies the ellipticity condition

$$w(x)^{2/p}|\xi|^2 \leq \langle A\xi, \xi \rangle \leq v(x)^{2/p}|\xi|^2,$$

where the weights $v, w$ are assumed to be locally integrable. In the terminology introduced above, we have that $A$ is a matrix weight. We want to recast this equation so that our results can be restated in terms of the degenerate Sobolev spaces defined in Sect. 5. If we define the matrix weight $W$ by $A^{1/2} = W^{1/p}$, then (6.1) becomes

$$L_{W, p}u = \text{div}(|W^{1/p} \nabla u|^{p-2} W^{2/p} \nabla u) = 0$$

(6.2)

with ellipticity condition

$$w(x)|\xi|^p \leq |W^{1/p}(x)\xi|^p \leq v(x)|\xi|^p.$$  

(6.3)

Hereafter, we will assume that $W$ is an invertible matrix weight and we will generally assume that $v = |W|_{\text{op}}$ and $w = |W^{-1}|_{\text{op}}^{-1}$. Since $v$ and $w$ are the largest and smallest eigenvalues of $W$, this choice is in some sense optimal.

As in [8], we introduce the notion of $p$-admissible pairs of weights on $\Omega$. Following the convention introduced above in Remarks 2.1 and 4.2, given a pair of scalar weights $(w, v)$ on $\Omega$, we will assume that they are in fact defined and locally integrable on a larger domain $\Omega'$ such that $\Omega \subseteq \Omega'$ and that balls and cubes are interchangeable in the definition of doubling and $A_p$ weights on $\Omega$. Note that as a consequence of this assumption, $v, w \in L^1(\Omega)$. 

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Definition 6.1 Given $1 < p < \infty$, a domain $\Omega$ and a pair of weights $(w, v)$, we say that the pair is $p$-admissible on $\Omega$ if:

1. $w \leq v$;
2. $w \in A_p$;
3. $v$ is doubling;
4. $(w, v)$ satisfies the balance condition: there exists $q > p$ such that for every ball $B \subset \Omega$ and $0 < r < 1$,

\[ r \left( \frac{v(rB)}{v(B)} \right)^{1/q} \lesssim \left( \frac{w(rB)}{w(B)} \right)^{1/p}. \]

(6.4)

If $W$ is an invertible matrix weight, then by Proposition 3.2 we have that (1) holds. Since $p > 1$, if we further assume $W \in A_p$, then (2) and (3) hold by Lemma 4.5. Therefore, the critical condition is the balance condition (4). We will consider this assumption more carefully in Sect. 7.

Given the assumption that $(w, v)$ are a $p$-admissible pair, then by Theorem 5.2 we have $W^{-1} \in A_p$. Therefore, we could define weak solutions $u$ to be functions in $W^{-1}$. However, to prove the results given below, we need the stronger assumption that $u \in S^1_{W,0}$. Indeed, if we apply the Cauchy–Schwarz inequality and then Hölder’s inequality, we get

\[ \int_{\Omega} \left| W^{1/p} \nabla u \right|^{p-2} \left( W^{1/p} \nabla u, W^{1/p} \nabla \varphi \right) dx = 0. \]

Note that if we only assume $u \in \mathcal{H}^{1,p}_W(\Omega)$, then the above integral is well defined even if $\varphi \in \mathcal{H}^{1,p}_W(\Omega)$. Indeed, if we apply the Cauchy–Schwarz inequality and then Hölder’s inequality, we get

\[ \int_{\Omega} \left| \left| W^{1/p} \nabla u \right|^{p-2} \left( W^{1/p} \nabla u, W^{1/p} \nabla \varphi \right) \right| dx \]

\[ \leq \int_{\Omega} \left| W^{1/p} \nabla u \right|^{p-1} \left| W^{1/p} \nabla \varphi \right| dx \leq \left\| \nabla u \right\|_{L^p_W(\Omega)}^{p-1} \left\| \nabla \varphi \right\|_{L^p_W(\Omega)}. \]

As a consequence, we could define weak solutions $u$ to be functions in $\mathcal{H}^{1,p}_W(\Omega)$. However, to prove the results given below, we need the stronger assumption that $u \in S^1_{W,0}$. However, if $W \in A_p$ and $\Omega$ has Lipschitz boundary (e.g., if $\Omega$ is a ball), then by Theorem 5.6 we have that $S^1_{W,0} = \mathcal{H}^{1,p}_W(\Omega) = \mathcal{H}^{1,p}_W(\Omega)$, so we can take our solution space to be either of these “larger” spaces. We will use this fact in Sect. 7 below.

The following results are from [8]. For brevity, in the next result let $S^1_{W,0}(\Omega)$ denote the closure of $C_c^{\infty}(\Omega)$ in $\mathcal{H}^{1,p}_W(\Omega)$.

Theorem 6.2 [8, Theorem 3.11] Let $1 < p < \infty$ and let $W \in S_n$ be an invertible matrix such that $(w, v)$ is a $p$-admissible pair. Then given any $\psi \in S^1_{W,0}(\Omega)$ there exists a weak solution $u \in S^1_{W,0}(\Omega)$ of $L^{\mathcal{W},p}u = 0$ such that $u - \psi \in S^1_{W,0}(\Omega)$.
Remark 6.3  In the original statement of this result there is an assumption that a global Sobolev inequality holds. This was necessary there because they were considering more general equations defined with respect to Hörmander vector fields. Since (6.2) is defined with respect to the gradient, this assumption always holds. See the discussion in [8, Sect. 3].

Theorem 6.4 [8, Theorem 3.16] If $B$ is a ball and $u \in S^{1,p}_{W}(2B)$ is a weak solution of $L_{W,p}u = 0,$ then $u$ is bounded on $B.$

Theorem 6.5 [8, Theorem 3.17] If $B$ is a ball and $u \in S^{1,p}_{W}(2B)$ is a non-negative weak solution of $L_{W,p}u = 0,$ then $u$ satisfies the following Harnack inequality:

$$\sup_{B} u \leq \exp \left( C\mu(B)^{1/p} \right) \inf_{B} u$$  \hspace{1cm} (6.5)

where $\mu(B) = \frac{v(B)}{w(B)}.$

To state our next result, we introduce an auxiliary operator: a weighted maximal operator. Given scalar weights $(w, v),$ for $x \in \Omega$ define the maximal operator

$$M_{\Omega}(w, v)(x) = \sup_{B} \frac{v(B)}{w(B)},$$

where the supremum is taken over all balls $B \subset \Omega$ centered at $x.$ Since $v \in L^{1}(\Omega),$ it follows from a Besicovitch covering lemma argument (cf. Journé [24, Chapter 1]) that

$$w(\{x \in \Omega : M_{\Omega}(w, v)(x) > \lambda\}) \leq \frac{C_{n}}{\lambda} v(\Omega).$$

In particular, the set $\{x \in \Omega : M_{\Omega}(w, v)(x) = \infty\}$ has measure zero.

Theorem 6.6 Given $1 < p < \infty$ and an invertible matrix weight $W \in S_{n},$ suppose that $(w, v)$ is a $p$-admissible pair. If $u \in S^{1,p}_{W}(\Omega)$ is a weak solution of $L_{W,p}u = 0,$ then $u$ is continuous on the set

$$F_{\Omega}(w, v) = \{x \in \Omega : M_{\Omega}(w, v)(x) < \infty\}.$$

In particular, $u$ is continuous almost everywhere in $\Omega.$

Proof Our proof follows closely the proofs in [8, Theorems 4.4 and 4.5], so here we only sketch the main ideas.

Note that by the above discussion we have that $|F_{\Omega}(w, v)| = |\Omega|.$ Fix $x \in F_{\Omega}(w, v)$ and let $B = B(x, r)$ be a ball such that $2B = B(x, 2r) \subset \Omega.$ Since $u \in S^{1,p}_{W}(\Omega)$ it is clearly in $S^{1,p}_{W}(2B)$ and is a solution to $L_{W,p}u = 0$ on $2B.$ So by Theorem 6.4 $u$ is
bounded on $B$. Therefore, if we let $M$ and $m$ be upper and lower bounds for $u$ on $B$, we can apply the Harnack inequality (6.5) to $M - u$ and $u - m$ to conclude that

$$\text{osc}_u(x, \frac{1}{2}B) \leq \frac{\exp \left( C \mu \left( \frac{1}{2}B \right)^{1/p} \right) - 1}{\exp \left( C \mu \left( \frac{1}{2}B \right)^{1/p} \right) + 1} \text{osc}_u(x, B),$$

where $\text{osc}_u(x, B) = \sup_B u - \inf_B u$ is the oscillation of $u$ on $B$.

Since

$$\mu \left( \frac{1}{2}B \right) = \frac{v(\frac{1}{2}B)}{w(\frac{1}{2}B)} \leq M_{\Omega}(w, v)(x),$$

we have that

$$\frac{\exp \left( C \mu \left( \frac{1}{2}B \right)^{1/p} \right) - 1}{\exp \left( C \mu \left( \frac{1}{2}B \right)^{1/p} \right) + 1} \leq \frac{\exp \left( CM_{\Omega}(w, v)(x)^{1/p} \right) - 1}{\exp \left( CM_{\Omega}(w, v)(x)^{1/p} \right) + 1} = \gamma(x).$$

Moreover, $F_{\Omega}(w, v) = \{ x \in \Omega : \gamma(x) < 1 \}$. Because $\gamma(x) < 1$ we may perform Moser iteration (see [14, Lemma 8.23]) to show there exists $0 < s(x) < \infty$ such that

$$\text{osc}_u(x, \alpha B) \leq c(x) s(x)^{\frac{1}{p}} \text{osc}_u(x, B), \quad 0 < \alpha < 1.$$}

It follows from this inequality that $u$ agrees a.e. with a function that is continuous on $F_{\Omega}(w, v)$. \hfill \Box

### 7 The Balance Condition

In this section we consider the partial regularity of solutions of the degenerate $p$-Laplacian equation $\mathcal{L}_W p u = 0$ with the additional assumption that $W \in A_p$. Since $W \in A_p$ implies (by Lemma 4.5) $w, v \in A_p$, we have that conditions (1), (2) and (3) of Definition 6.1 hold. However, the balance condition (6.4) does not follow automatically from the matrix $A_p$ condition, as the next example shows.

**Example 7.1** We modify Example 5.1. For ease of computation we will consider the balance condition for cubes instead of balls, but it is clear that they are interchangeable in this setting. Let $\Omega = Q = (0, 1) \times (0, 1)$. Fix $1 < p < 2$, $q > p$ and $p/2 < \alpha < 1$. We again define

$$W(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & x^{-\alpha} y^{-\alpha} \end{bmatrix}.$$}

Then as before $W \in A_p$ and $w(x, y) = 1$, $v(x, y) = x^{-\alpha} y^{-\alpha}$. Furthermore,

$$r \left( \frac{v(r Q)}{v(Q)} \right)^{1/q} \approx r^{\frac{2-2\alpha}{q} + 1}, \quad \left( \frac{w(r Q)}{w(Q)} \right)^{1/p} = r^{\frac{2}{p}}.$$
Therefore, the balance condition holds only if
\[
\frac{2 - 2\alpha}{q} + 1 \geq \frac{2}{p}.
\]
However, by our choice of \( p, q \) and \( \alpha \),
\[
\frac{2 - 2\alpha}{q} + 1 < \frac{2 - p}{p} + 1 = \frac{2}{p}.
\]

Given this example, we want to determine sufficient conditions on \( W \), or more precisely on \( v \) and \( w \), for the balance condition to hold. Intuitively, the above example fails because our choice of \( \alpha \) is too close to 1: the function \( x^{-\alpha} \) is in \( A_1 \), but it only satisfies the reverse Hölder inequality for small values of \( s > 1 \). Our main result, which is a generalization of [8, Theorems 4.8, 4.9], shows that a sufficiently large reverse Hölder exponent yields the balance condition.

**Theorem 7.2** Given \( 1 < p < \infty \), suppose \( 1 < s, t < \infty \), \( w \in A_t \) and \( v \in RH_s \) where
\[
0 < t - \frac{p}{n} = \frac{1}{s'}.
\]
Then \((w, v)\) satisfies the balance condition \((6.4)\).

**Proof** Since \( w \in A_t \) there exists \( \epsilon > 0 \) such that \( w \in A_{t-\epsilon} \) (see [10]). In particular, by \((7.1)\),
\[
0 < \frac{n}{p}(t - \epsilon) - 1 < \frac{n}{p s'}.
\]
Define
\[
q = \frac{n/s'}{(t - \epsilon)n/p - 1};
\]
then we have that \( q > p \). Fix \( 0 < r < 1 \) and a ball \( B \). By inequality \((2.1)\),
\[
r^{\frac{n}{p}(t-\epsilon)} = \left( \frac{|rB|}{|B|} \right)^{\frac{t-\epsilon}{p}} \leq C \left( \frac{w(rB)}{w(B)} \right)^{1/p}.
\]
Moreover, by inequality \((2.2)\),
\[
r \left( \frac{v(rB)}{v(B)} \right)^{1/q} \leq Cr \left( \frac{|rB|}{|B|} \right)^{1/(s'q)} = Cr^{\frac{n}{p}+1}.
\]
If we combine \((7.2)\) and \((7.3)\) we immediately get the balance inequality \((6.4)\). \( \square \)
Remark 7.3 A close examination of the proof shows that it is enough to assume that $1 < s, t < \infty$ satisfy

$$0 < t - \frac{p}{n} \leq \frac{1}{s'}$$

(7.4)

However, since the $A_t$ and $RH_s$ classes are nested (i.e., if $u < t$, then $A_u \subseteq A_t$, and if $q > s$, $RH_q \subseteq RH_s$), equality in (7.4) is the interesting case.

Theorem 7.2 seems to require a stronger condition on both $v$ and $w$. However, depending on the size of $p$ relative to the dimension $n$, we can shift the stronger condition to one weight or the other. We first consider $p$ small: in this case we require a stronger condition on $v$.

Corollary 7.4 Suppose $1 < p < n'$ and $W \in A_p$. If $v \in RH_{\frac{n'}{n'-p}}$, then $(w, v)$ satisfy the balance condition (6.4).

Proof Since $W \in A_p$, we have $w \in A_p$. Therefore, if we let $s = \frac{n'}{n'-p}$, then $r' = n'/p$ and so

$$\frac{n}{p} p - 1 = n - 1 = \frac{n}{n'} = \frac{n}{ps'}.$$

Therefore, by Theorem 7.2 the balance condition holds. \qed

When $p$ is large, we can shift the stronger hypothesis to $w$. The following two corollaries are immediate consequences of Theorem 7.2.

Corollary 7.5 Suppose $p \geq n$, $W \in A_p$ and $w \in A_t$, where

$$\frac{p}{n} < t \leq \frac{n}{n'} + \frac{1}{s'}$$

and $s > 1$ is such that $v \in RH_s$. Then the pair $(w, v)$ satisfies the balance condition (6.4).

Remark 7.6 Since $v \in A_p$ we know that $v \in RH_s$ for some $s > 1$, so there exists some $t > 1$ for which the hypotheses hold. Indeed, by (2.3), we can give a sharp estimate for $t$:

$$t \leq \frac{1}{2^{n+12}[v]_{A_\infty}} + \frac{p}{n}.$$

To state the next result, let $A_q^* = \bigcap_{p \geq q} A_p$. Note that this class is strictly larger than $A_q$.

Corollary 7.7 If $p \geq n$, $W \in A_p$ and $w \in A_q^*/n$, then the pair $(w, v)$ satisfies the balance condition (6.4).

As a consequence of Theorem 7.2 and its corollaries, we get the following partial regularity result.
Theorem 7.8 If $1 < p < \infty$, $W \in A_p$, and $w$, $v$ satisfy the hypotheses of any of the above results, and if $u$ is a weak solutions to $\mathcal{L}_{W,p}u = 0$, then $u$ is continuous on the set 

$$F_{\Omega}(w, v) = \{M_{\Omega}(w, v)(x) < \infty\}.$$ 

Remark 7.9 Theorem 7.8 is the best possible: that is, there exists $W$ satisfying the hypotheses such that a solution to $\mathcal{L}_{W,p}u = 0$ is discontinuous on the complement of $F_{\Omega}(w, v)$. See Example 8.9 below.

8 Mappings of Finite Distortion

In this section we apply our results on the partial regularity of solutions of the degenerate $p$-Laplacian to mappings of finite distortion. Hereafter, let $\Omega \subset \mathbb{R}^n$ be a domain that is not necessarily bounded. A vector function $f : \Omega \to \mathbb{R}^n$ is a mapping of finite distortion (MFD) if

1. $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$;
2. the Jacobian $J_f(x) = \det Df(x) > 0$ a.e., and $J_f \in L^1_{\text{loc}}(\Omega)$;
3. there exists $K(x) < \infty$ a.e. such that $|Df(x)|^{n}_{\text{op}} \leq K(x) J_f(x)$.

As we noted in the Introduction, the regularity of MFDs has been studied by a number of authors. A classical result due to Vodop’janov and Gol’dšteǐn [38] is that if $f \in W^{1,n}(\Omega)$, then $f$ is continuous. Generally speaking, most results in this area show that if $f \not\in W^{1,n}(\Omega)$, continuity follows if $\exp(K)$ satisfies some kind of integrability condition. Our results are quite different as we only prove partial regularity; they are similar in spirit, though not in detail, to the work of Manfredi [27].

To state our results, we first give some basic definitions and results on MFDs; for complete information, including proofs, see [17,22]. The smallest function $K$ such that the (3) holds is called the outer distortion of $f$ and is denoted $K_O$; i.e.,

$$|Df(x)|^{n}_{\text{op}} = K_O(x) J_f(x).$$

Since it is always the case that $|Df(x)|_{\text{op}} \geq J_f(x)$, we must have that $K_O(x) \geq 1$ a.e. Similarly, we define the inner distortion, denoted $K_I$, to be the smallest distortion function of the inverse differential matrix:

$$|Df^{-1}(x)|^{n}_{\text{op}} = K_I(x) J_{f^{-1}}(x) = K_I(x) J_f(x)^{-1}.$$ 

The inner and outer distortion functions are related by the inequalities

$$K_O \leq K_I^{n^{-1}} \quad \text{and} \quad K_I \leq K_O^{n^{-1}}.$$ 

Finally, if we define the maximal distortion $K_M$ by

$$K_M(x) = \max \left(K_I(x), K_O(x)\right),$$
then we have that
\[ K_I \leq K_M \leq K_I^{n-1} \quad \text{and} \quad K_O \leq K_M \leq K_O^{n-1}. \]

We now show that a mapping of finite distortion is a solution to a degenerate \( p \)-Laplacian equation. Define the distortion tensor of \( f \) to be the symmetric matrix
\[ G(x) = J_f(x)^{-2/n} Df(x)^t Df(x). \]

Hereafter, let \( W = G^{-n/2} \). Then we have that
\[ |W^{-1}(x)|_{op} = \frac{|Df(x)^t Df(x)|_{op}^{n/2}}{J_f(x)} = K_O(x) \quad (8.1) \]
and
\[ |W(x)|_{op} = |Df(x)^{-1} (Df(x)^{-1})^t|_{op}^{n/2} J_f(x) = K_I(x). \quad (8.2) \]

In particular, by inequality (3.1),
\[ K_O(x)^{-1} |\xi|^n \leq |W^{1/n} \xi|^n \leq K_I(x) |\xi|^n. \]

Let \( f = (f_1, \ldots, f_n) \). Then by definition, \( f_i \in W^{1,1}_{loc}(\Omega) \). Suppose \((K_I^{-1}, K_I)\) is an \( n \)-admissible pair. Then given any ball \( B \subseteq \Omega \), we have that \( f_i \in \mathcal{W}^{1,n}(B) \). We first show that \( \nabla f_i \in L^n_{W}(B) \). Given a matrix \( A \), let \( [A]_i \) denote its \( i \)-th column. Then, treating \( \nabla f_i \) as a column vector, we have that
\[ Df^{-1} (Df^{-1})^t \nabla f_i = Df^{-1} e_i = [Df^{-1}]_i \]
and \( [Df(x)^{-1}]_i \cdot \nabla f_i = 1 \). Therefore,
\[ \int_B |W^{1/n} \nabla f_i|^n \, dx = \int_B \langle G^{-1} \nabla f_i, \nabla f_i \rangle^{n/2} \, dx = \int_B J_f(x) \, dx < \infty. \]

To show that \( f_i \in L^n(K_I, B) \), we use the fact that since \((K_O^{-1}, K_I)\) is an \( n \)-admissible pair, we have a two-weight Poincaré inequality (see [5]):
\[
\frac{1}{K_I(B)} \int_B |f_i - (f_i)_B|^n K_I \, dx \lesssim \frac{r(B)^n}{K_I^{-1}(B)} \int_B |\nabla f_i|^n K_O^{-1} \, dx \\
\lesssim \frac{r(B)^n}{K_I^{-1}(B)} \int_B |W^{1/n} \nabla f_i|^n \, dx \\
= \frac{r(B)^n}{K_I^{-1}(B)} \int_B J_f(x) \, dx.
\]
It follows that
\[ \| f_i \|_{L^p_w(B)} \lesssim \| f_i \|_{L^1(B)} + \left( \int_B |f_i| \, dx \right)^{1/n}. \]

Finally, we have that if \( f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n) \), then the component functions, \( f_i \), are weak solutions of
\[ \mathcal{L}_{W,n} u = \text{div}(|W|^{1/n} \nabla u)^{n-2} W^{2/n} \nabla u) = 0. \]

See [22, Chapter 15] for details.

From these observations we see that given an MFD \( f \), we have that the component functions \( f_i \), \( 1 \leq i \leq n \), satisfy a degenerate \( p \)-Laplacian equation \( \mathcal{L}_{W,n} u = 0 \), where the matrix \( W \) satisfies the natural ellipticity conditions with bounds given by the distortion functions. In other words, these functions fall within the framework of our results in the previous two sections. This leads to the following partial regularity result for mappings of finite distortion.

**Theorem 8.1** Given an MFD, \( f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap S^{1,n}_W(\Omega) \), suppose \((K^{-1}_O, K_I)\) is an \( n \)-admissible pair. Then \( f \) is continuous almost everywhere on \( \Omega \). More precisely, given any ball \( B \subset \Omega \), then \( f \) is continuous on the set
\[ \{ x \in B : M_B(K^{-1}_O, K_I)(x) < \infty \}. \] (8.3)

**Proof** Fix a ball \( B \subset \Omega \). The component functions of \( f \) belong to \( S^{1,n}_W(B) \) and are weak solutions of \( \mathcal{L}_{W,n} u = 0 \). Therefore, by Theorem 6.6 each of the component functions is continuous on the set
\[ F_B(K_O, K_I) = \{ x \in B : M_B(K^{-1}_O, K_I)(x) < \infty \}. \]
Since \( |F_B(K_O, K_I)| = |B| \) and \( \Omega \) is the countable union of balls, we have that \( f \) is continuous almost everywhere on \( \Omega \). \( \square \)

Following our approach in Sect. 7, we now consider the hypothesis that \((K^{-1}_O, K_I)\) is an \( n \)-admissible pair given the additional assumption that \( W = G^{-n/2} \in \mathcal{A}_n \). (Equivalently, we may assume \( W^{-n'/n} = G^{n'/2} \in \mathcal{A}_n \). This is particularly useful when \( n = 2 \).) In this case, by Lemma 4.5 and the identities (8.1) and (8.2), we have that conditions (1) and (2) of Definition 6.1 hold, so the main problem is determining additional assumptions so that the balance condition (6.4) holds. Our first result is just a restatement of Theorem 7.2 in this setting.

**Corollary 8.2** Suppose \( f \) is an MFD and \( W = G^{-n/2} \in \mathcal{A}_n \). Suppose further that \( K^{-1}_O \in \mathcal{A}_t \) and \( K_I \in RH_s \), where
\[ 0 < t - 1 = \frac{1}{s}. \]
Then for each ball $B \subset \Omega$, $f$ is continuous on the set given by (8.3) and continuous a.e. on $\Omega$.

Since in our setting $p = n$, we can apply Corollaries 7.5 and 7.7. For brevity we will only consider the latter and leave the restatement of the former to the interested reader.

**Corollary 8.3** Suppose $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is an MFD, $W = G^{-n/2} \in \mathcal{A}_n$, and

$$1/K_O \in A^*_1 = \bigcap_{p > 1} A_p.$$  

Then for each ball $B \subset \Omega$, $f$ is continuous on the set given by (8.3) and continuous a.e. on $\Omega$.

As a consequence of Corollary 8.3 we give two results which implicitly require the outer distortion to be exponentially integrable. For the first result, note that a weight $w$ is such that $w, w^{-1} \in A^*_1$ if and only if $\log(w)$ is in the closure of $L^\infty$ in $BMO$; in particular, the latter inclusion holds if $\log(w) \in VMO$. (See [13, p. 474].)

**Corollary 8.4** Suppose $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $W = G^{-n/2} \in \mathcal{A}_n$, and $\log(K_O)$ is in the closure of $L^\infty$ in $BMO$. Then for each ball $B \subset \Omega$, $f$ is continuous on the set given by (8.3) and continuous a.e. on $\Omega$.

For the second result, we use the fact that if $b$ is a function such that $b, 1/b \in BMO$, then $b \in A^*_1$. (See [23].) Since $K_O \geq 1$, we always have that $K_O^{-1} \in L^\infty(\Omega) \subset BMO$.

**Corollary 8.5** Suppose $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $W = G^{-n/2} \in \mathcal{A}_n$, and $K_O \in BMO$. Then for each ball $B \subset \Omega$, $f$ is continuous on the set given by (8.3) and continuous a.e. on $\Omega$.

We now want to give some partial regularity theorems that are related to the results in [8]. The major improvement here is that by assuming that $W$ is in matrix $\mathcal{A}_n$ we no longer have to assume that a weak solution is in the closure of the smooth functions. To state our results we first note that in Theorem 6.6, while we implicitly assumed that $v = |W|_{\text{op}}$ and $w = |W^{-1}|_{\text{op}}$, we never used this in the proof. All we used was the fact that $(v, v)$ is a $p$-admissible pair, and the ellipticity condition (6.3) holds. Further, note that using the relationships relating them, we can give ellipticity conditions for $W = G^{-n/2}$ in terms of the distortion functions $K_M$, $K_O$ and $K_I$:

$$K_M(x)^{-1} |\xi|^n \leq |W^{1/n} \xi|^n \leq K_M(x) |\xi|^n$$  \quad (8.4)
$$K_O(x)^{-1} |\xi|^n \leq |W^{1/n} \xi|^n \leq K_O(x)^{n-1} |\xi|^n$$  \quad (8.5)
$$K_I(x)^{1-n} |\xi|^n \leq |W^{1/n} \xi|^n \leq K_I(x) |\xi|^n.$$  \quad (8.6)

To state our results we will need a local version of the Hardy–Littlewood maximal operator. Given a ball $B \subset \Omega$ and a locally integrable function $f$ define
$$M_B f(x) = \sup_{B'} \int_{B'} |f(y)| \, dy \cdot \chi_{B'}(x),$$

where the supremum is over all balls $B' \subset B$.

**Theorem 8.6** Given an MFD $f \in W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$, suppose $W = G^{-n/2} \in A_n$, and $K_M \in A_2 \cap RH_2$. Then $f$ is continuous almost everywhere on $\Omega$. More precisely, given any ball $B$, $f$ is continuous on the set

$$\{ x \in B : M_B(K_M)(x) < \infty \}.$$

**Proof** Since we have the ellipticity condition (8.4), to apply Theorem 6.6 we need to show that $(K^{-1}_M, K_M)$ is an $n$-admissible pair. Since $K_M \in A_2, K^{-1}_M \in A_2$, and so conditions (1) and (2) in Definition 6.1 hold. Since $K_M \in A_2 \cap RH_2$, we have that $K^2_M \in A_3$ (see [9, Theorem 2.2]), which by the duality of $A_p$ weights implies that $K^{-1}_M \in A_{3/2}$. We can therefore apply Theorem 7.2 with $t = 3/2$ and $s = 2$ to conclude that $(K^{-1}_M, K_M)$ satisfy the balance condition (6.4). Moreover, by Hölder’s inequality, we have that for any ball $B \subset \Omega$,

$$M_B(K^{-1}_M, K_M)(x) \leq M_B(K_M)(x)^2,$$

so

$$\{ x \in B : M_B(K_M)(x) < \infty \} \subset \{ x \in B : M_B(K^{-1}_M, K_M)(x) < \infty \}.$$

The desired conclusion now follows from Theorem 6.6. $\square$

**Theorem 8.7** Given an MFD $f \in W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$, suppose $W = G^{-n/2} \in A_n$, and $K_I \in A_{n'} \cap RH_n$. Then $f$ is continuous almost everywhere on $\Omega$: given any ball $B \subset \Omega$, $f$ is continuous on the set

$$\{ x \in B : M_B(K_I)(x) < \infty \}.$$

**Proof** We proceed as in the proof of Theorem 8.6: given the ellipticity condition (8.6), it will suffice to show that $(K^{1-n}_I, K_I)$ is an $n$ admissible pair. Since $K_I \in A_{n'} \cap RH_n$, we have that $K^n_I \in A_{n'+1}$, which in turn implies that

$$K^{1-n}_I = (K^n_I)^{\frac{1}{n'}} \in A_{1+\frac{1}{n'}}.$$

Hence conditions (1) and (2) hold. Moreover, if we take $t = 1 + \frac{1}{n'}$ and $s = n$ in Theorem 7.2, we see that the weights satisfy the balance condition. Finally,

$$M_B(K^{1-n}_I, K_I)(x) \leq M_B(K_I)(x)^n.$$

$\square$
Theorem 8.8 Given an MFD $f$, suppose $W = G^{-n/2} \in A_n$, and $K^{-1}_O \in A_n \cap RH_n'$. Then $f$ is continuous almost everywhere on $\Omega$: given any $B \subset \Omega$, $f$ is continuous on the set
\[ \{ x \in B : M_{B,n-1}(K_O)(x) < \infty \}, \]
where $M_{B,n-1}(K_I) = M_B(K_O^{n-1})^{1/(n-1)}$.

Proof First, by our assumption $K_O \in L^{n-1}_{loc}(\Omega)$, we do not need to assume a priori that $f \in \mathcal{W}^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. Indeed, since $J_f(x)$ is locally integrable, if $B \subset \Omega$, then
\[
\int_B |Df|^{n-1}_{op} dx = \int_B |Df| W^{1/n} W^{-1/n} |_{op}^{n-1} dx \\
\leq \int_B |Df| W^{1/n} |_{op}^{n-1} W^{-1/n} |_{op}^{n-1} dx \\
= \int_B |Df| W^{1/n} |_{op}^{n-1} K^{1/n}_O dx \\
\leq \left( \int_B |Df| W^{1/n} |_{op} dx \right)^{1/n'} \left( \int_B K^{n-1}_O dx \right)^{1/n} \\
\lesssim \left( \int_B J_f(x) dx \right)^{1/n'} \left( \int_B K^{n-1}_O dx \right)^{1/n} < \infty.
\]

For the last inequality we use the Frobenius norm, $|A|_F = \sqrt{\text{tr}(A^tA)} \approx |A|_{op}$, to get
\[
|Df| W^{1/n} |_{op} \leq |Df| G^{-1/2} |_F^2 = \text{tr}[(Df G^{-1/2})(Df G^{-1/2})]^n/2 = n^{n/2} J_f(x).
\]

We can now argue again as in the proof of Theorem 8.6 using the ellipticity condition (8.5). Since $K^{-1}_O \in A_n \cap RH_n'$ implies that $K^{-1}_O \in A_{n+1}$, by duality we have that $K^{-1}_O \in A_{1+1/n}$. This gives conditions (1) and (2). If we take $t = 1 + \frac{1}{n}$ and $s = n'$ in Theorem 7.2, then $(K^{-1}_O, K^{n-1}_O)$ satisfies the balance condition. Finally, again by Hölder’s inequality,
\[
M_B(K^{-1}_O, K^{n-1}_O)(x) \leq M_B(K^{n-1}_O)(x)^{1/(n-1)}.
\]

We conclude this section with an example to show that our results are sharp. Our example is adapted from an example due to Ball [2, Example 6.1]. As in all problems involving the matrix $A_p$ weights, the difficulty is in showing that the matrix is in this class. However, when $n = 2$, we can use a result due to Lauzon and Treil [26] to simplify the computations.

Example 8.9 Fix $n = 2$ and let $\Omega = B(0, 1)$. Define
\[
f(x) = (|x|^{-1} + |x|^{-1/2})x, \quad x \neq 0,
\]
and let \( f(0) = 0 \). Then \( f \) maps \( B(0, 1) \) to the annulus \( B(0, 2) \setminus B(0, 1) \); clearly no choice of value for \( f(0) \) will make \( f \) continuous there.

We will show that \( f \) satisfies the hypotheses and conclusions of Corollary 8.2. As shown in \([2]\), \( f \in W^{1,1}_{\text{loc}}(\Omega) \), and if we let \( x = (x_1, x_2) \), \( r = |x| \) and \( R = 1 + |x|^{1/2} \), then

\[
Df(x) = \begin{pmatrix}
\frac{R}{r} + \frac{(r'R-R)x_1^2}{r^3} & \frac{(r'R-R)x_1x_2}{r^3} \\
\frac{(r'R-R)x_1x_2}{r^3} & \frac{R}{r} + \frac{(r'R-R)x_2^2}{r^3}
\end{pmatrix}
\]

and

\[
J_f(x) = \det Df(x) = \frac{RR'}{r}.
\]

The eigenvalues of this matrix are (via Mathematica)

\[
\mu_1 = \frac{r^2R + rR'x_1^2 + rR'x_2^2 - Rx_1^2 - Rx_2^2}{r^3} = R' = \frac{1}{2|x|^{1/2}},
\]

\[
\mu_2 = \frac{R}{r} = \frac{1}{|x|} + \frac{1}{|x|^{1/2}}.
\]

Therefore (since \( n = 2 \)) we have that

\[
K_O(x) = K_I(x) = \frac{\mu_2}{\mu_1} = 2 + \frac{2}{|x|^{1/2}},
\]

and so

\[
K_I(x) \approx |x|^{-1/2}, \quad K_O(x)^{-1} \approx |x|^{1/2}.
\]

Thus \( K_O^{-1} \in A_t \) for \( t > 5/4 \), and \( K_I \in RH_s \) for \( s < 4 \). Therefore, we can take \( t = 3/2 \) and \( s = 2 \) and we satisfy the condition \( t - 1 = 1/s' \).

Therefore, it remains to show that \( W = G^{-1} \in A_2 \). In \([26, \text{Theorem 3.1}]\) they showed that this is the case if \( \langle W(x)v, v \rangle \) and \( \langle W^{-1}(x)v, v \rangle \) are uniformly in scalar \( A_2 \) for all unit vectors \( v \in \mathbb{R}^2 \). We first consider \( W^{-1}(x) = G(x) = J_f(x)^{-1} Df(x) Df(x) \).

This matrix has eigenvalues

\[
\lambda_1 = J_f^{-1} \mu_1^2 = \frac{|x|^{1/2}}{2(1 + |x|^{1/2})} \approx |x|^{1/2}, \quad \lambda_2 = J_f^{-1} \mu_2^2 = \frac{2(1 + |x|^{1/2})}{|x|^{1/2}} \approx |x|^{-1/2}.
\]

Therefore, \( \lambda_1, \lambda_2 \in A_2 \). Given any unit vector \( v \), we can write it as \( \alpha_1 \xi_1 + \alpha_2 \xi_2 \), where \( \xi_1, \xi_2 \) are an orthonormal basis of eigenvectors of \( W^{-1} \). Hence,

\[
\langle W^{-1}(x)v, v \rangle = \alpha_1^2 \lambda_1(x) + \alpha_2^2 \lambda_2(x).
\]
The linear combination of two scalar $A_2$ weights is again an $A_2$ weight, and its $A_2$ characteristic is dominated by $\alpha_1^2[\lambda_1]_2 + \alpha_2^2[\lambda_2]_2$. (See [16, p. 292].) Since $|v| = 1$, we get that (8.7) is uniformly in $A_2$. The argument for $W$ is exactly the same, using the fact that its eigenvalues are $\lambda_1^{-1}$ and $\lambda_2^{-2}$, and these are again in $A_2$. This completes our proof.

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