A NEW EXPLICIT WAY OF OBTAINING SPECIAL GENERIC MAPS INTO THE 3-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. A special generic map is a smooth map regarded as a natural generalization of Morse functions with just 2 singular points on homotopy spheres. Canonical projections of unit spheres are simplest examples of such maps and manifolds admitting special generic maps into the plane are completely determined by Saeki in 1993 and ones admitting such maps into general Euclidean spaces are determined under appropriate conditions.

Moreover, if the difference of dimensions of source and target manifolds are not so large, then the diffeomorphism types of source manifolds are often limited; for example, homotopy spheres except standard spheres do not admit special generic maps into Euclidean spaces whose dimensions are not so low. As another example, simply connected manifolds admitting special generic maps are only manifolds represented as the connected sum of the total spaces of 2-dimensional sphere bundles over the 2-dimensional sphere (or the 4-dimensional standard sphere) and there are many manifolds homeomorphic and not diffeomorphic to these manifolds. These explicit facts make special generic maps attractive objects in the theory of Morse functions and higher dimensional versions and application to algebraic and differentiable topology of manifolds, which is an important study in both singularity theory of maps and algebraic and differential topology of manifolds.

In this paper, we demonstrate a way of constructing of special generic maps into the 3-dimensional Euclidean space. For this, first we prepare maps onto 2-dimensional polyhedra regarded as simplicial maps naturally called pseudo quotient maps, which are generalizations of the quotient maps to the spaces of all the connected components of inverse images, so-called Reeb spaces of original smooth maps, being fundamental and important tools in the studies. More precisely, we prepare specific pseudo quotient maps, locally construct maps onto 3-dimensional manifolds, glue them and realize the map as a special generic map into the 3-dimensional space. Most of the procedure for the construction is based on technique the author previously noticed and used in simpler situations.

The success of the construction explicitly shows that a class of maps which seems to cover a larger class of source manifolds may not be not so large and that the diffeomorphism types of source manifolds may be restricted as strongly as special generic maps. We also explain differential topological facts and problems related to this.

1. Introduction.

1.1. Backgrounds and fundamental tools. Morse functions and higher dimensional versions and application to algebraic and differentiable topology of manifolds
is an important study in both singularity theory of maps and algebraic and differential topology of manifolds.

A fold map is a smooth map such that each singular point is $p$ is of the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i(p)} x_k^2 - \sum_{k=m-n+i(p)+1}^{m} x_k^2)$$

for some integers $m, n, i(p)$. $i(p)$ is taken as a non-negative integer not larger than $\frac{m-n+1}{2}$ uniquely and we call $i(p)$ the index of $p$. The set of all the singular points of an index is a smooth submanifold of dimension $n-1$. Morse functions are regarded as fold maps. A fold map is stable or the $C^\infty$ equivalence classes of the maps are invariant under slight perturbations in the $C^\infty$ Whitney topology, if and only if the restriction to the set of all the singular points (of a fixed index), which are codimension 1 smooth immersions, are transversal. Note that for example, stable Morse functions or Morse functions such that at distinct singular points, the values are distinct exist densely. For Morse functions, fold maps and stable maps etc., see [3] for example. For algebraic and differential topological properties of fold maps, see [22] for example.

Special generic maps are fold maps indices of whose singular points are 0. They are regarded as simplest generalizations of Morse functions with just 2 singular points on homotopy spheres. Canonical projections of unit spheres are simplest examples of such maps and manifolds admitting special generic maps into the plane are completely determined by Saeki in 1993 [23] and ones admitting such maps into general Euclidean spaces are determined under appropriate conditions. More precise facts have been shown. For example, if the difference of dimensions of source and target manifolds are not so large, then the diffeomorphism types of source manifolds are often limited; homotopy spheres except standard spheres do not admit special generic maps into Euclidean spaces whose dimensions are not so low. As another example, 4-dimensional simply connected and closed manifolds admitting special generic maps into $\mathbb{R}^3$ are only manifolds represented as the connected sum of the total spaces of smooth $S^2$-bundles over the 2-dimensional sphere (or the 4-dimensional standard sphere) and there are many manifolds homeomorphic and not diffeomorphic to these manifolds; in [27] and [28], there are several 4-dimensional closed manifolds admitting fold maps into $\mathbb{R}^3$ and admitting no special generic maps into $\mathbb{R}^3$ homeomorphic to one admitting special generic maps into $\mathbb{R}^3$.

These facts make special generic maps attractive objects from the viewpoint of differential topology of smooth manifolds.

1.2. Contents of this paper. We define a normal spherical fold map.

Definition 1. A stable fold map from a closed manifold of dimension $m$ into $\mathbb{R}^n$ satisfying $m \geq n$ is said to be normal spherical (standard-spherical) if the inverse image of each regular value is a disjoint union of (resp. standard) spheres (or points) and the connected component containing a singular point of the inverse image of a small interval intersecting with the singular value set at once in its interior is either of the following.

(1) The $(m-n+1)$-dimensional standard closed disc.
(2) A manifold PL homeomorphic to an $(m-n+1)$-dimensional compact manifold obtained by removing the interior of three disjoint $(m-n+1)$-dimensional smoothly embedded closed discs from the $(m-n+1)$-dimensional standard sphere.
As a fundamental fact, this class includes special generic maps. Such fold maps are studied by Saeki and Suzuoka for example and for example, they can be obtained by projections of special generic maps by fundamental discussions of Saeki and Suzuoka [29]. Conversely, whether manifolds admitting such maps admit special generic maps into one dimensional higher Euclidean spaces is a natural and interesting problem, explicitly stated in the paper. Several answers have been given in [10] and [11] in the case of Morse functions. In this paper, we consider such maps and extended ones and apply technique based on ideas applied to the shown theorem for several cases of Morse functions, we obtain special generic maps. As additional remarks, we explain diffeomorphism types of source manifolds admitting such maps, including one stating that classes of source manifolds are not so large as conjectured and related topics on differential topology of manifolds.

This paper is organized as follows. In the next section, we review Reeb spaces and we define pseudo special generic maps and as general PL maps, pseudo quotient maps, which was first introduced in [12], pseudo special generic maps have been introduced in [9] and [11]. They are fundamental objects in the paper. After introducing these maps, we perform construction of special generic maps into $\mathbb{R}^3$; this is a main result. Last, we explain remarks on differential topological meanings of the obtained result and related facts and problems.

Throughout the paper, all the manifolds and maps between them, bundles over manifolds with fibers being manifolds etc. are smooth and of class $C^\infty$ unless otherwise stated. We also note that maps between polyhedra are PL unless otherwise stated in the paper.

Last, we call the set of all the singular points of a smooth map the singular set of the map and we call the image of the singular set the singular value set. We call the set of all the regular values the regular value set.

2. Reeb spaces, triangulable maps and two classes of maps on manifolds.

2.1. Reeb spaces and triangulable maps. We review Reeb spaces. For precise facts, see also [21] and introduced papers.

Definition 2. Let $X$ and $Y$ be topological spaces. For points $p_1, p_2 \in X$ and for a continuous map $c : X \to Y$, we define as $p_1 \sim_c p_2$ if and only if $p_1$ and $p_2$ are in the same connected component of $c^{-1}(p)$ for some $p \in Y$. The relation is an equivalence relation and we call the quotient space $W_c := X/\sim_c$ the Reeb space of $c$.

Example 1. (1) For a Morse function, the Reeb space is a graph.
(2) For a special generic map, the Reeb space is regarded as an immersed compact manifold with a non-empty boundary whose dimension is same as that of the target manifold. The Reeb space of a special generic map is known to be contractible if and only if the source manifold is a homotopy sphere.

It is also a fundamental and important property that the quotient map from the source manifold onto the Reeb space induces isomorphisms on homology groups whose degrees are not larger than the difference of the dimension of the source manifold and that of the target one. See [23] for example. Moreover, we mention remarks on special generic maps in some parts of the present paper.
The Reeb spaces of stable (fold) maps are polyhedra whose dimensions and those of the target manifolds coincide. For a normal spherical Morse function, it is also a fundamental and important property that the quotient map from the source manifold onto the Reeb space induces isomorphisms on homology groups whose degrees are smaller than the difference of the dimension of the source manifold and that of the target one. See [29] and see also [7] and [8] for example.

For a (proper) stable map, the Reeb space is a polyhedron ([31]).

We mention triangulable maps.

**Definition 3.** Let $X$ and $Y$ be polyhedra. A continuous map $c : X \rightarrow Y$ is said to be triangulable if there exists a pair of triangulations of $X$ and $Y$ and homeomorphisms $(\phi_X, \phi_Y)$ onto $X$ and $Y$ respectively such that the composition $\phi_Y^{-1} \circ c \circ \phi_X$ is a simplicial map with respect to the given triangulations. We also say that $c$ is triangulable with respect to $(\phi_X, \phi_Y)$.

**Fact 1 ([31]).** (Proper) stable maps are always triangulable (with respect to pairs of homeomorphisms giving the canonical triangulations of the smooth manifolds).

**Fact 2 ([4] and [5]).** For a triangulable map $c : X \rightarrow Y$ with respect to $(\phi_X, \phi_Y)$, the Reeb space $W_c$ is a polyhedron given by a homeomorphism $\phi_c$ from a polyhedron and two maps $q_c : X \rightarrow W_c$ and $c : W_c \rightarrow Y$ are triangulable maps with respect to the corresponding pairs of the homeomorphisms.

2.2. **Two classes of PL or smooth maps.** The following is introduced by the author [11].

**Definition 4.** Let $m > n$ be positive integers. A smooth map $f_p$ from a closed manifold $M$ of dimension $m$ onto a compact manifold $W_p$ of dimension $n$ satisfying $\partial W_p \neq \emptyset$ is said to be pseudo special generic if the following hold.

1. $f_p|_{f_p^{-1}(\partial W_p)} : f_p^{-1}(W_p - \partial W_p) \rightarrow W_p - \partial W_p$ gives a smooth $S^{m-n}$-bundle.

2. For a small collar neighborhood $N(\partial W_p)$ of $\partial W_p$ in $W_p$, regarded as a trivial bundle $\partial W_p \times [0, 1]$ where $\partial W_p \times \{0\}$ corresponds to the boundary $\partial W_p$, for each point $(p, 0) \in \partial W_p \times \{0\}$ and a small open neighborhood $U_p$, $f|_{f_p^{-1}(U_p \times [0,1])} : f^{-1}(U_p \times [0, 1]) \rightarrow U_p \times [-1,1]$ has the same local form as a singular point of index 0 of a fold map from an $m$-dimensional manifold into $\mathbb{R}^n$. From fundamental discussion of [22] for example, $f_p^{-1}(N(\partial W_p))$ is a linear $D^{m-n+1}$-bundle over $\partial W_p$ given by the composition $f_p$ and the canonical projection and the bundle is seen as a normal bundle of the submanifold $f^{-1}(\partial W_p) \subset M$.

All the quotient maps onto the Reeb spaces defined from special generic maps into Euclidean spaces are regarded as pseudo special generic. We also note that if the manifold $W_p$ can be immersed into the Euclidean space whose dimension is same as that of $W_p$, then by composing the immersion to the map, we have a special generic map. Special generic maps into Euclidean spaces are obtained like this, which is based on a fundamental discussion of [23].

We introduce pseudo spherical fold maps based on pseudo quotient maps. Pseudo quotient maps were first introduced in [12] and later defined again by [9].
Definition 5. Let $m > n$ be positive integers. Let $f_p$ be a triangulable map from a closed manifold $M$ of dimension $m$ onto a polyhedron $W_p$ of dimension $n$ with respect to the corresponding canonical homeomorphisms. If for each point $p \in W_q$ and the interior of a small connected and closed neighborhood $N_p$ being $n$-dimensional and satisfying $p \in \text{Int} N_p$, there exist a spherical fold map $f$, a point $p' \in W_f$, a small connected and closed neighborhood $N_{p'}$ with its interior containing the point, and a diffeomorphism $\Phi$ and a PL homeomorphism $\phi$ such that for the maps $q_{f,N_p} = f_q|f_q^{-1}(N_p) : f_q^{-1}(N_p) \to N_p$ and $q_{f,N_{p'}} : q_f^{-1}(N_{p'}) \to N_{p'}$, the relation $q_{f,N_{p'}} \circ \Phi = \phi \circ q_{f,N_p}$ holds, then $W_p$ is said to be a pseudo quotient space and $f_p$ is said to be a pseudo normal spherical fold map.

We can naturally define a singular point, a singular value, a regular value, the singular set, the singular value set, the regular value set etc. of a pseudo spherical fold map. There are some types of singular values of pseudo spherical fold maps and we introduce them in the following fundamental proposition. We can see the statements from fundamental explanations of articles on Turaev’s shadow such as [15] and [34] and ones on algebraic and differential topological properties of stable fold maps such as [12] [22], [23] and [29] for example: they are referenced later as important articles for fundamental and important facts related to such topics.

Proposition 1. Let $f_p$ be a pseudo normal spherical fold map from a closed manifold $M$ of dimension $m > 2$ onto a polyhedron $W_p$ of dimension 2.

1. For a singular value $p \in W_p$, its inverse image has one or two singular points. In the latter case, we call the singular value a double point. In the former case, we call the value a single point.

2. For a singular value $p \in W_p$, if its inverse image has one singular point, then for a small regular neighborhood of $p$, the intersection of the regular neighborhood and the regular value set consists of one or three connected components PL homeomorphic to the 2-dimensional closed disc. For the former case, we call $p$ a definite single point. Moreover, for the latter case, the following hold.

(a) Distinct connected components of the three connected components are in distinct connected components of the regular value set of $f_q$. In this case, we call the singular value a normal single point.

(b) Two of the three connected components are in a connected component of the regular value set of $f_p$ and the other is in another one.

(c) All the three connected components are in a connected component of the regular value set of $f_p$. If $f_p$ is obtained as the quotient map onto the Reeb space of a normal spherical fold map, then the case never happens.

Definition 6. For a pseudo normal spherical fold map, if each singular value set is a definite single point, normal single point or a double point, then the map is said to be standard.

Example 2. A pseudo special generic map is also a standard pseudo spherical fold map.
Figure 1. Around a double point (with the number of connected components of inverse images).

3. Construction of special generic maps.

**Theorem 1.** For a pseudo normal spherical fold map $f_p$ such that a small regular neighborhood of the singular value set contains no non-orientable surfaces on a 4-dimensional closed and connected manifold $M$ onto a 2-dimensional polyhedron $W_p$, there exists a 3-dimensional compact and connected orientable smooth manifold $W_P$ satisfying the following.

1. $\partial W_P \neq \emptyset$.
2. There exists a pseudo special generic map $f_P$ on $M$ onto $W_P$.
3. There exists a continuous map $g : W_P \to W_p$ satisfying $f_p = g \circ f_P$.

**Proof.** We construct $f_P$ and $g$ by constructing maps locally and glue them together so that the source manifold of $g$ is orientable through the following five steps. For some figures presented, see also the proof of Theorem 4.1 of [29]; same local models are depicted.

**STEP 1 Around a double point.**
As FIGURE 1 shows, we can construct a local map from a 4-dimensional compact smooth manifold onto a 3-dimensional compact and orientable manifold by piling two maps each of which is regarded as the product of a cobordism of special generic functions on $S^2$ and the identity map on $[-1, 1]$. For cobordisms of special generic functions which will be used later again, see [26] for example and see [16] for h-cobordisms including (h-)cobordisms of homotopy spheres. We also obtain the canonical projection from the 3-dimensional manifold to the small regular neighborhood of the double point.

**STEP 2 Around a normal single point.**
As FIGURE 2 or 3 shows, we can construct a local map from a 4-dimensional compact smooth manifold onto a 3-dimensional compact and orientable manifold by the product of a cobordism of special generic functions on $S^2$ and the identity map on $[-1, 1]$. By the assumption that a small regular neighborhood of the singular value set of the pseudo normal spherical fold map contains no non-orientable surface and the cobordism of special generic maps are invariant under a diffeomorphism on the whole space smoothly isotopic to a given diffeomorphism sending each connected component of the boundary onto the original component or a component different from the, original one we can attach obtained maps and manifolds together with the ones obtained in the previous step compatibly.

STEP 3 Around a definite single point.
As FIGURE 4 shows, we can construct a local map from a 4-dimensional compact smooth manifold onto a 3-dimensional compact and orientable manifold by the product of a cobordism between a special generic function on $S^2$ and a function on the empty set and the identity map on $[-1, 1]$. By a discussion similar to that of the previous step, we can attach the obtained objects to the constructed ones compatibly.

We have constructed maps on the singular value set and its small neighborhood. We construct maps on the remained part of the regular value set by extending the
STEP 4 Around a 1-dimensional skeleton on the regular value set. We extend the existing map to each 1-dimensional skeleton. To make the resulting 3-dimensional compact manifold orientable, we need to construct the local map as FIGURE 5 shows (see [11] for a precise explanation on this part); more precisely, we need to be careful in the case where the regular value set contains a non-orientable connected component.

STEP 5 Around a 2-dimensional skeleton on the regular value set. On each 2-dimensional cell $D$, we can construct a product of a special generic function on $S^2$ and the identity map on the cell $D$. Consider the attaching map on the boundary $\partial D \times S^2$. It is a bundle isomorphism on the trivial $S^2$-bundle over $\partial D^2$. Moreover, it is regarded as a bundle isomorphism preserving the Morse functions.
since the natural homomorphism $\pi_1(SO(2)) \rightarrow \pi_1(SO(3))$ is known to be surjective and the diffeomorphism groups of $S^1$ and $S^2$ are regarded as linear or homotopy equivalent to $O(2)$ and $O(3)$, respectively. The discussion enables us to extend the constructed maps and obtain a 3-dimensional compact and connected orientable manifold. See also FIGURE 6.

We can obtain the resulting maps and manifolds.

Remark 1. We note about Shibata’s work [30], in which essentially same figures as some figures in the proof of Theorem 1 appear. Shibata investigated the condition for a smooth map on a 3-dimensional compact and orientable smooth manifold with non-empty boundary into the plane to be represented as the composition of an immersion into $\mathbb{R}^3$ and the canonical projection onto the plane. In the proof, as a partially similar but essentially different work, we construct a 3-dimensional compact and orientable smooth manifold and a continuous map onto a given 2-dimensional polyhedron realized as the target space of a standard pseudo spherical fold map. For example, note that the 2-dimensional polyhedron may not be realized as the Reeb space of a spherical fold map into the plane; consider the case where the regular value set contains a non-orientable connected component.

Remark 2. In the present paper, as fold maps, stable fold maps are considered. Generally, a stable map is defined as a map such that by slight perturbations, the resulting maps are always $C^\infty$ equivalent to the original map; as another definition, it is defined as a smooth map such that there exists an open neighborhood of the
map and the maps there are always $C^\infty$ equivalent to the original one where the topology of spaces of smooth maps is the $C^\infty$ Whitney topology.

Two smooth maps $f_1$ and $f_2$ are said to be $C^\infty$ equivalent if there exists a pair $(\Phi, \phi)$ of diffeomorphisms of the source manifolds and of the target manifolds and the relation $\phi \circ f_1 = f_2 \circ \Phi$ holds.

If a stable map from a closed manifold of dimension larger than 1 into the plane is stable, then each singular point is of the same form as that of a fold map or a cusp. The set of all the singular points whose forms are same as that of a fold map and that of a singular point of a fixed index is a smooth submanifold of codimension 1 and the number of cusps is finite. Moreover, if the restriction of the map to the set of all the singular points whose form is same as that of a fold map is transversal and at distinct cusps, the values are different and the inverse image of the value of each cusp has only one singular point, then the map is stable.

For these elemental terminologies and facts, see [3] for example. For stable maps into the plane, see also [33] and [14] for example.

Moreover, we can define the extension of pseudo normal spherical fold map considering not only spherical fold maps but also stable maps such that inverse images satisfy same conditions and we can prove Theorem 1. We must consider about cusps; see FIGURE 7 and also Figure 6 of [29].

**Corollary 1.** A 4-dimensional closed and connected manifold $M$ admitting a standard pseudo normal spherical fold map onto a 2-dimensional polyhedron admits a special generic map into $\mathbb{R}^3$. Moreover, by a canonical projection, we have a normal spherical fold map and the resulting quotient map onto the Reeb space is standard.

**Proof.** As Theorem 1, we can obtain a pseudo special generic map onto a 3-dimensional orientable manifold with non-empty boundary. By a well known fact, we can immerse the target manifold into $\mathbb{R}^3$ and we have a special generic map. The latter follows from fundamental theory of [29] and [30]. To remove cusps mentioned in Remark 2, apply the theory of [14] and the fact that the Euler number of the source manifold is even.

**Proposition 2 ([29],[7],[8]).** A pseudo normal spherical fold map $f_p$ from a 4-dimensional closed and connected manifold $M$ onto a 2-dimensional polyhedron $W_p$ induces isomorphisms between the 1st homology and homotopy groups. Moreover, $H_2(M;\mathbb{Z})$ and $H_2(W_p;\mathbb{Z})$ are free and the rank of the former is that of the latter. If the dimension of the source manifold is $m > 3$, then a pseudo normal spherical
Figure 8. Two round fold maps.

Figure 9. Another example.

fold map \( f_p \) from an \( m \)-dimensional closed and connected manifold \( M \) onto a 2-dimensional polyhedron \( W_p \) induces isomorphisms between the \( k \)-th homology and homotopy groups for \( 0 \leq k \leq m - 3 \).

Remark 3. The proposition above is proven as a proposition for spherical fold maps or more generally, spherical stable maps explained also in Remark 2 from 4-dimensional closed and connected manifold into 2-dimensional manifold with no boundary in [29]. However, by the manner of the proof, we have the mentioned result.

This means that in the situation of Theorem 1 and Corollary 1, for the target Reeb spaces or the target pseudo quotient spaces for a given manifold, 1st homology and homotopy groups and the 2nd homology groups are invariant.

In Example 3 and Example 4, we introduce simple examples. The numbers in each connected component of the regular value set represents the numbers of the inverse images of the connected components.

Example 3. FIGURE 8 shows two kinds of fold maps. They are round fold maps, defined as fold maps whose singular value sets are embedded concentric spheres and introduced and systematically studied in [7] and [8] for example. The source manifolds are \( S^4 \) and the total space of an \( S^2 \)-bundle over \( S^2 \), respectively. Note that both trivial and non-trivial bundles admit fold maps like ones the right figure shows. FIGURE 9 show a fold map which is not round. The source manifold is represented as a connected sum of two total spaces of \( S^2 \)-bundles over \( S^2 \).

Example 4 ([32]). An example by Suzuoka depicted as FIGURE 10 is studied in [32]. The source manifold is \( S^2 \times S^2 \) or the total space of a non-trivial \( S^2 \)-bundle over \( S^2 \). We also note that these manifolds admit fold maps as depicted. We can apply Theorem 1 to this case.
4. DIFFERENTIAL TOPOLOGICAL MEANINGS OF THE RESULT AND RELATED FACTS AND PROBLEMS.

4.1. Manifolds admitting special generic maps. As mentioned in the abstract and the introduction or section 1, manifolds admitting special generic maps into Euclidean spaces are often restricted if the dimensions of the target spaces are not so low. For example, it is known that if a homotopy sphere of dimension $m > 3$ admits a special generic map into $\mathbb{R}^n$ satisfying $n = m - 1, m - 2, m - 3$, then it is the standard sphere. For the 4-dimensional case, first, a 4-dimensional closed and connected manifold admits a special generic map into the plane if the manifold is represented as a connected sum of the total spaces of $S^3$-bundles over $S^1$ ($S^4$ is also included in the trivial case). Moreover, a 4-dimensional closed and connected manifold whose fundamental group is represented as a free product of a finite number of $\mathbb{Z}$ is represented as a connected sum of the total spaces of $S^3$-bundles over $S^1$ and $S^2$-bundles over $S^2$ ($S^4$ is included). In a comment by Saeki in 2010, the following on the rigidity of diffeomorphism types of 4-dimensional manifolds admitting special generic maps is believed to be true.

**Conjecture 1.** If two 4-dimensional homeomorphic closed and connected manifolds admit special generic maps into an Euclidean space of dimension smaller than 4, then they are diffeomorphic.

Theorem 1 implies that the class of source manifolds of maps of the considered class, which seems to be wider than the class of 4-dimensional closed and connected manifolds admitting special generic maps, may not be so large. In fact, Saeki and Suzuoka [29] presented a problem on the diffeomorphism types of 4-dimensional closed manifolds admitting spherical fold maps into the plane, questioning whether a manifold homeomorphic and not diffeomorphic to a manifold represented as connected sums of the total spaces $S^2$-bundles over $S^2$ such as the Moishezon-Teicher surface with zero signature admits a spherical fold map into the plane: according to Theorem 1, it is false under the assumption that the resulting quotient map onto the Reeb space is standard. See also [27] and [28] for such topics on diffeomorphism types.

Note that this is false for pairs of general dimensions. Every homotopy sphere of dimension larger than 1 except 4-dimensional exotic homotopy spheres admits a special generic map into the plane ([23]).

Last, as a related result, recently, Wrazidlo [35] has shown that 7-dimensional oriented homotopy spheres whose oriented diffeomorphism types belong to a family of 14 types of 28 types do not admit special generic maps into $\mathbb{R}^3$.

4.2. 2-dimensional pseudo quotient spaces called simple polyhedra including the cases of Turaev’s shadows.
Definition 7. A simple polyhedron is a 2-dimensional polyhedron locally PL homeomorphic to a closed neighborhood of a point in the target space of a pseudo spherically fold map on 4-dimensional closed manifold onto the 2-dimensional polyhedron.

Remark 4. We can replace the dimension 4 by any other integer larger than 4. We cannot replace it by 2 or 3: for the 3-dimensional case, some shadows, explained later, are regarded as pseudo normal spherical fold map. We can know this from fundamental explanations of [24] and [25] and also from [2] and [6] for example.

We state the following fundamental fact as a proposition.

Proposition 3. Let $W_p$ be a simple polyhedron. For any integer $m > 2$, there exists a closed smooth manifold of dimension $m$ and a pseudo map onto the polyhedron $W_p$.

Some shadows, regarded as pseudo quotient maps, have been introduced by Turaev [34] and they explain for the case $m = 3$ with the source manifolds orientable. Costantino and Thurston have made use of the fact that Reeb spaces of stable fold maps from 3-dimensional closed orientable manifolds into surfaces without connected components of inverse images called II type, are regarded as shadows in [2]. Recently a related study [6] has been done. There are various related studies on shadows and manifolds admitting shadows whose singular value sets are disjoint unions of circles or whose singular value sets have intersections fixing the number of intersections. Martelli studied manifolds admitting shadows systematically in [15] and later in [13] with Koda and Naoe. In [17], [18] and [19], Naoe systematically and explicitly studied contractible shadows and 3-dimensional manifolds admitting such shadows. In these studies, there appear a lot of shadows for which we cannot apply Theorem 1.

As an explicit fact, the following is also known. It comes from a study of Kobayashi and Saeki [12] and that of Naoe [17], which are mutually independent studies, and some additional discussions.

Fact 3 ([12],[17] etc.). Let $M$ be a closed and connected manifold of dimension $m > 2$ admitting a pseudo spherical fold map into a 2-dimensional simple polyhedron $W_p$ such that the singular value set is a disjoint union of circles or contains no double points. If $m \geq 4$ holds, $M$ is simply-connected and $H_2(W_p;\mathbb{Z})$ is zero, then $W_p$ is contractible and can be collapsed to the 2-dimensional closed disc and $M$ is a homotopy sphere. Especially, for $m = 4, 5, 6$, $M$ is the standard sphere. If $m = 3$ holds and $W_p$ is simply-connected and $H_2(W_p;\mathbb{Z})$ is zero, then $W_p$ is contractible and can be collapsed to the 2-dimensional closed disc and $M$ is $S^3$.

We remark on [17].

Remark 5. Naoe [17] has shown that a 2-dimensional simple polyhedron $W_p$ without double points being simply-connected and satisfying $H_2(W_p;\mathbb{Z}) \cong \{0\}$ can be collapsed to the 2-dimensional closed disc based on the study [15], in which representations of shadows or simple polyhedra with the singular value sets being disjoint unions of circles or having no double points by labeled graphs have been introduced and presented. Note also that a main purpose of [17] is to show that a 4-dimensional compact manifold bounded by the source 3-dimensional manifold obtained by a canonical procedure on shadows is diffeomorphic to $D^4$ and that [18] is also on such studies of 4-dimensional manifolds including 3-dimensional manifolds appearing as the boundaries.
Figure 11. An example satisfying the assumption of Fact 3 which is not standard and for which we cannot apply Theorem 1 (see also section 7 of [12]).

Figure 12. A non-trivial example: arrows indicate two crossings in the singular value set, consisting of spheres and the right figure shows a realization by a fold map into the plane with the number representing the number of the connected components of the inverse images.

Note that in these cases the singular value set is a disjoint union of spheres and that we cannot always apply Theorem 1; for example, to the case of FIGURE 11, appearing in [12]. Last, related to Theorem 1, the source manifold is diffeomorphic to $S^4$ in the case where $m = 4$ and the pseudo normal spherical fold map is standard; it is true even if there exist double points. FIGURE 12 shows an explicit case. Moreover, we have the following for example.

**Theorem 2.** Bing's house is a contractible simple polyhedron and a pseudo normal spherical fold map onto this polyhedron is a homotopy sphere if the dimension of the source manifold is larger than 3. Moreover, the source manifold is a standard sphere if the dimension is 4, 5 or 6.

It is known that the singular value set has 2 double points.

**Proof.** We can know this from Proposition 2 except the 4-dimensional case or the fact that the source manifold is $S^4$. We can show this by seeing that we can apply Theorem 1 to this case. \qed

For the case where the source manifold is 3-dimensional in the theorem, see [18] for example; the source manifold may not be a homotopy sphere.

**Remark 6.** In [32], Suzuoka determined the diffeomorphism types of manifolds admitting spherical fold maps into the plane by considering the open book structures or bundle over the circle with fibers being inverse images of Morse functions
introduced from the structures of the fold maps and applying Kirby diagrams of 4-dimensional manifolds (FIGURE 13). He investigated simplest generalizations of spherical fold maps regarded as round fold maps. We can apply this technique in the cases FIGURE 8, FIGURE 9 and FIGURE 10. However, we cannot apply the method for the case of FIGURE 12.

5. Results and Remarks for Higher Dimensions.

**Theorem 3.** Theorem 1 and Corollary 1 hold also in the case where the source manifold is 5-dimensional.

**Sketch of the proof.** We can show the theorem for this case similarly. Through STEPS 1–4. In STEP 5, note that the diffeomorphism group of $S^3$ is homotopy equivalent to $O(4)$ and the corresponding homomorphism $\pi_1(SO(2)) \to \pi_1(SO(4))$ (consider a special generic Morse function on $S^3$) between the corresponding homotopy groups are surjective also in this case. □

**Example 5.** In [23], Saeki has shown that a 5-dimensional closed and simply connected manifold, completely classified in [1], admits a special generic map into $\mathbb{R}^3$ if and only if it is represented as a connected sum of total $S^3$-bundles over $S^2$. This condition is same as that for a such a manifold to admit a special generic map into $\mathbb{R}^4$ proven in [20]. Note that this condition on the manifold is equivalent to the assumption that the 2nd homology group is free.

We cannot extend Theorem 1 and 3 generally. As presented, according to Saeki, 7-dimensional homotopy spheres admitting a special generic map into $\mathbb{R}^4$ must be standard and Wrzidlo [35] has shown that there are at least 14 oriented diffeomorphism types of 7-dimensional homotopy spheres such that the homotopy spheres do not admit special generic maps into $\mathbb{R}^3$. However, in Theorem 3 the source manifold can be any homotopy sphere except exotic 4-dimensional spheres; consider a special generic map whose singular set is an embedded circle into the plane constructed in [23].

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