Some extended hypergeometric matrix functions and their fractional calculus

Ashish Verma †, Ravi Dwivedi † and Vivek Sahai ‡

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Abstract

In this paper, we study some extended hypergeometric functions from matrix point of view. We have given the integral representations of these matrix functions. Finally, we obtain some generating function relations using fractional derivative operators.

Keywords: Matrix functional calculus, Appell functions, Lauricella functions, Fractional derivative operators

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1 Introduction

The Gauss hypergeometric function and Kummer hypergeometric function are amongst the most appeared special functions in mathematics as well as in physics. Due to a wide range of applications, these hypergeometric functions have been studied from different aspects viz. from matrix point of view [10, 12, 14]; extended Gauss and Kummer hypergeometric functions [4] and finite field analogue [9] to name a few. Recently, the matrix version of extended beta function, extended Gauss hypergeometric function and Kummer hypergeometric function have been studied by Abdalla and Bakhet in [1, 2]. The regions of convergence, integral representations and differential formulas satisfied by these matrix functions are determined. Verma discussed recursion formula, infinite summation formula for the Srivastava’s triple hypergeometric matrix functions $H_A$, $H_B$ and $H_C$ [18]. In [15, 16, 17, 19], authors introduced the incomplete first, second and fourth Appell hypergeometric matrix functions and incomplete Srivastava’s triple hypergeometric matrix functions and studied some basic properties; matrix differential equation, integral formula, recursion formula, recurrence relation and differentiation formula of these functions. In this paper, we study the matrix analogues of some extended hypergeometric functions of two and three variables. We obtain the integral representations for these matrix functions.

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*Department of Mathematics, Prof. Rajendra Singh (Rajju Bhaiya), Institute of Physical Sciences for Study and Research V.B.S. Purvanchal University, Jaunpur (U.P.)- 222003, India.
Email: vashish.hu@gmail.com (Corresponding author)

†Department of Basic Science and Humanities, PSIT College of Engineering, Kanpur 209305, India.
Email: dwivedir999@gmail.com

‡Department of Mathematics and Astronomy, Lucknow University, Lucknow 226007, India.
Email: sahai_vivek@hotmail.com
Finally, we use the extended Riemann-Liouville derivative operator and obtain certain generating function relations in terms of these matrix functions. The section-wise treatment is as follows:

In Section 2, we give the basic definitions related to special matrix functions that are needed in the sequel. In Section 3, we define the extended Appell matrix functions and extended Lauricella matrix function. We also discuss the regions of convergence as well as the integral representations satisfied by these matrix functions. In Section 4, we define the matrix analogue of extended Riemann-Liouville derivative operators and transform several matrix functions under these derivative operators. Finally, in Section 5, we obtain certain generating function relations which turn out in terms of extended Appell matrix functions and extended Lauricella matrix function.

2 Preliminaries

Throughout the paper, \( \mathbb{C}^{r \times r} \) is the vector space of \( r \)-square matrices with complex entries. For a matrix \( A \in \mathbb{C}^{r \times r} \) the spectrum, denoted by \( \sigma(A) \), is the set of eigenvalues of \( A \). If \( \Re(z) \) denotes the real part of a complex number \( z \), then a matrix \( A \in \mathbb{C}^{r \times r} \) is said to be positive stable if \( \Re(\lambda) > 0 \) for all \( \lambda \in \sigma(A) \).

If \( A \) is a positive stable matrix in \( \mathbb{C}^{r \times r} \), then \( \Gamma(A) \) can be expressed as \[ \Gamma(A) = \int_0^\infty e^{-t} t^{A-I} dt. \] (2.1)

Furthermore, if \( A + kI \) is invertible for all integers \( k \geq 0 \), then the reciprocal gamma matrix function is defined as \[ \Gamma^{-1}(A) = A(A + I) \ldots (A + (n - 1)I)\Gamma^{-1}(A + nI), \quad n \geq 1. \] (2.2)

If \( M \in \mathbb{C}^{r \times r} \) is a positive stable matrix and \( n \geq 1 \) is an integer, then the gamma matrix function can also be defined in the form of a limit as \[ \Gamma(M) = \lim_{n \to \infty} (n - 1)! (M_n^{-1} n^M). \] (2.3)

By application of the matrix functional calculus, the Pochhammer symbol for \( A \in \mathbb{C}^{r \times r} \) is given by \[ (A)_n = \begin{cases} I, & \text{if } n = 0, \\ A(A + I) \ldots (A + (n - 1)I), & \text{if } n \geq 1. \end{cases} \] (2.4)

This gives \[ (A)_n = \Gamma^{-1}(A) \Gamma(A + nI), \quad n \geq 1. \] (2.5)

If \( A \) and \( B \) are positive stable matrices in \( \mathbb{C}^{r \times r} \), then, for \( AB = BA \), the beta matrix function is defined as \[ \mathcal{B}(A, B) = \Gamma(A)\Gamma(B)\Gamma^{-1}(A + B) = \int_0^1 t^{A-I}(1 - t)^{B-I} dt \] (2.6)

\[ = \int_0^\infty u^{A-I}(1 + u)^{-(A+B)} du. \] (2.7)
Let $A$, $B$ and $X$ be positive stable and commuting matrices in $\mathbb{C}^{r \times r}$ such that $A + kI$, $B + kI$ and $X + kI$ are invertible for all integer $k \geq 0$. Then the extended beta matrix function $\mathfrak{B}(A, B; X)$ is defined by

$$\mathfrak{B}(A, B; X) = \int_0^1 t^{A-I}(1-t)^{B-I} \exp \left( -\frac{X}{t(1-t)} \right) dt. \quad (2.8)$$

Hence,

$$\mathfrak{B}(A, B; X) = \Gamma(A, X) \Gamma(B, X) \Gamma^{-1}(A + B, X). \quad (2.9)$$

It is obvious that $X = O$ gives the original beta matrix function $\mathfrak{B}$. Let $A$, $B$, $C$ be positive stable matrices in $\mathbb{C}^{r \times r}$ such that $C + kI$ is invertible for all integers $k \geq 0$. Then the Gauss hypergeometric matrix function is defined by

$$2F_1(A, B; C; z) = \sum_{n=0}^{\infty} (A)_n (B)_n (C)^{-1} \frac{z^n}{n!}. \quad (2.10)$$

The series (2.10) converges absolutely for $|z| < 1$ and for $z = 1$, if $\alpha(A) + \alpha(B) < \beta(C)$, where $\alpha(A) = \max\{ \Re(z) \mid z \in \sigma(A) \}$, $\beta(A) = \min\{ \Re(z) \mid z \in \sigma(A) \}$ and $\beta(A) = -\alpha(-A)$.

Furthermore, if $CB = BC$ and $C$, $B$ and $C - B$ are positive stable, then for $|z| < 1$ an integral representation of (2.10) is given by

$$2F_1(A, B; C; z) = \left( \int_0^1 (1 - zt)^{-A} t^{B-I}(1-t)^{C-B-I} dt \right) \times \Gamma^{-1}(B) \Gamma^{-1}(C-B) \Gamma(C). \quad (2.11)$$

Recently, Abdalla et.al generalized the Gauss and Kummer hypergeometric matrix function. Let $A$, $B$, $C$, $C - B$ and $X$ be positive stable matrices in $\mathbb{C}^{r \times r}$ such that $CB = BC$, $CX = XC$ and $BX = XB$. Then the extended Gauss hypergeometric matrix function (EGHMF) and extended Kummer hypergeometric matrix function (EKHMF) are defined by

$$F^{(X)}(A, B; C; z) = \left( \sum_{m\geq0} (A)_m \mathfrak{B}(B + mI, C - B; X) \frac{z^m}{m!} \right) \times \Gamma(C) \Gamma^{-1}(B) \Gamma^{-1}(C-B); \quad (2.12)$$

and

$$\phi^{(X)}(B; C; z) = \left( \sum_{m\geq0} \mathfrak{B}(B + mI, C - B; X) \frac{z^m}{m!} \right) \times \Gamma(C) \Gamma^{-1}(B) \Gamma^{-1}(C-B) \quad (2.13)$$

respectively.
Integral representation of extended matrix functions given in Equations (2.12) and (2.13) are as follows

\[
F(3)(A, B; C; z) = \left( \int_0^1 t^{B-I}(1-t)^{C-B-I} \exp \left( \frac{-X}{t(1-t)} \right) dt \right) \times \Gamma(C) \Gamma^{-1}(B) \Gamma^{-1}(C-B); \\
\phi(3)(B; C; z) \\
= \left( \int_0^1 t^{B-I}(1-t)^{C-B-I} \exp \left( \frac{-X}{t(1-t)} \right) dt \right) \times \Gamma(C) \Gamma^{-1}(B) \Gamma^{-1}(C-B).
\]

3 The extended hypergeometric matrix functions of two and three variables

In this section, we introduce extended Appell matrix functions (EAMF’s) and extended Lauricella’s matrix function (ELMF) of three variables. More explicitly, we give the extended form of Appell matrix functions \( F_1(A, B, B'; C; z, w) \), \( F_2(A, B, B'; C, C'; z, w) \) and Lauricella matrix function of three variables \( F_D^{(3)}(A, B, B''; C; z, w, v) \), \([5, 6, 7]\), in terms of the extended beta matrix function. We also give here the integral representations for these extended hypergeometric matrix functions.

Let \( A, B, B', C, C - A \) and \( X \) be positive stable matrices in \( \mathbb{C}^{r \times r} \) such that \( A, C, X \) commutes with each other and \( CB = BC \), \( CB' = B'C \). Then, we define extended Appell hypergeometric matrix function \( F_1(A, B, B'; C; z, w; X) \) as

\[
F_1(A, B, B'; C; z, w; X) = \Gamma \left( \frac{C}{A}, C - A \right) \sum_{m,n \geq 0} \mathfrak{B}(A + (m+n)I, C - A; X)(B)(B') n^m w^n \frac{z^m}{m! n!}, \\
\]

where \( \Gamma \left( \frac{C}{A}, C - A \right) = \Gamma(C) \Gamma^{-1}(A) \Gamma^{-1}(C-A) \).

For positive stable matrices \( A, B, B', C, C', C - B, C' - B' \) and \( X \) in \( \mathbb{C}^{r \times r} \) such that \( B, B', C, C' \) and \( X \) commutes with each other, we define the extended Appell hypergeometric matrix function \( F_2(A, B, B'; C, C'; z, w; X) \) as

\[
F_2(A, B, B'; C, C'; z, w; X) \\
= \sum_{m,n \geq 0} (A)_{m,n} \mathfrak{B}(B + mI, C - B; X) \mathfrak{B}(B' + nI, C' - B'; X) \frac{z^m w^n}{m! n!} \\
\times \Gamma \left( \frac{C}{B}, C - B, C' - B' \right).
\]

Suppose \( A, B, B', B'', C, C - A \) and \( X \) be positive stable matrices in \( \mathbb{C}^{r \times r} \) such that \( A, C, X \) commutes with each other and \( CB = BC \), \( CB' = B'C \), \( CB'' = B''C \). Then, we define the extended Lauricella hypergeometric matrix function \( F_D^{(3)}(A, B, B''; C; z, w, v) \) as

\[
F_D^{(3)}(A, B, B''; C; z, w, v)
\]
\[ (3.3) \]

We now turn our attention in finding the integral representations of extended Appell matrix functions (EAMF’s) and extended Lauricella matrix function (ELMF) of three variables. We start with the integral representation of \( F_1(A, B; B'; C; z, w; \mathbb{X}) \) determined in the next theorem.

**Theorem 3.1.** Let \( A, B, B', C, C - A \) and \( \mathbb{X} \) be positive stable matrices in \( \mathbb{C}^{r \times r} \) such that \( A, C, \mathbb{X} \) commute with each other and \( CB = BC, CB' = B'C \). Then the EAMF \( F_1(A, B, B'; C; z, w; \mathbb{X}) \) can be presented in the integral form as

\[
F_1(A, B, B'; C; z, w; \mathbb{X}) = \Gamma \left( \frac{C}{A, C - A} \right) \sum_{m,n,p \geq 0} \sum_{(m',n',p') \geq 0} \mathfrak{B}(A + (m + n + p)I, C - A; \mathbb{X})(B)_{m}(B')_{n}(B'')_{p} \frac{z^{m}w^{n}v^{p}}{m!n!p!}.
\]

(3.4)

**Proof.** Using the integral representation of extended beta matrix function from (2.8) in the definition of the EAMF \( F_1(A, B, B'; C; z, w; \mathbb{X}) \), we get

\[
F_1(A, B, B'; C; z, w; \mathbb{X}) = \Gamma \left( \frac{C}{A, C - A} \right) \sum_{m,n \geq 0} \int_{0}^{1} u^{A-I}(1-u)^{C-A-I}(1-zu)^{-B} \times (1-wu)^{-B'} \exp \left( \frac{-\mathbb{X}}{u(1-u)} \right) du.
\]

(3.5)

Using the process discussed in [5] we can show that the sequence of matrix functions in (3.5) is integrable and by dominated convergence theorem [8], the summation and the integral can be interchanged in (3.3). Now applying the matrix identity,

\[
(1 - z)^{-A} = \sum_{n=0}^{\infty} (A)_{n} \frac{z^{n}}{n!}.
\]

(3.6)

we can rewrite the Equation (3.5) as follows

\[
F_1(A, B, B'; C; z, w; \mathbb{X}) = \Gamma \left( \frac{C}{A, C - A} \right) \sum_{m,n \geq 0} \int_{0}^{1} u^{A-I}(1-u)^{C-A-I} \exp \left( \frac{-\mathbb{X}}{u(1-u)} \right) \times (1-zu)^{-B}(1-wu)^{-B'} du.
\]

(3.7)

This completes the proof of Theorem 3.1.

**Theorem 3.2.** Let \( A, B, B', C, C', C - B \) and \( \mathbb{X} \) be positive stable matrices in \( \mathbb{C}^{r \times r} \) such that \( B, B', C, C' \) and \( \mathbb{X} \) commute with each other. Then the EAMF \( F_2(A, B, B'; C, C'; z, w; \mathbb{X}) \) defined in (3.2) has following integral representation:

\[
F_2(A, B, B'; C, C'; z, w; \mathbb{X}) = \int_{0}^{1} \int_{0}^{1} (1 - zu - wv)^{-A}u^{B-I}(1-u)^{C-B-I}v^{B'-I}(1-v)^{C'-B'-I} \times \exp \left( \frac{-\mathbb{X}}{u(1-u)} - \frac{\mathbb{X}}{v(1-v)} \right) du dv \Gamma \left( \frac{C, C'}{B, B'; C - B, C' - B'} \right).
\]

(3.8)
Proof. Using extended beta matrix function and the EAMF $F_2(A, B, B'; C, C'; z, w; X)$ defined in (2.3) and (3.2) respectively, we have

$$F_2(A, B, B'; C, C'; z, w; X) = \sum_{m,n \geq 0} \int_0^1 \int_0^1 (A)_{m+n} (zu)^m (wu)^n u^{B-I} (1-u)^{C-B-I} v^{B'-I} (1-v)^{C'-B'-I} \times \exp \left(-\frac{X}{u(1-u)} \times \frac{X}{v(1-v)} \right) \, du \, dv \Gamma \left( B, B', C - B, C' - B' \right).$$

Summation and integral in (3.9) can be interchanged by using the dominated convergence theorem. Taking into account the summation formula $\sum_{m,n \geq 0} f(m+n) \frac{z^m w^n}{m! n!}$, we get

$$F_2(A, B, B'; C, C'; z, w; X) = \sum_{m,n \geq 0} \int_0^1 \int_0^1 (A)_{m+n} (zu)^m (wu)^n u^{B-I} (1-u)^{C-B-I} v^{B'-I} (1-v)^{C'-B'-I} \times \exp \left(-\frac{X}{u(1-u)} \times \frac{X}{v(1-v)} \right) \, du \, dv \Gamma \left( B, B', C - B, C' - B' \right).$$

From (3.3) and (3.11), we get (3.8).

**Theorem 3.3.** Suppose $A, B, B', B'', C, C - A$ and $X$ be positive stable matrices in $\mathbb{C}^{r \times r}$ such that $A, C, X$ commutes with each other and $CB = BC, CB' = B'C, CB'' = B''C$. Then the ELMF $F^{(3)}_{D,X}(A, B, B', B''; C; z, w, v)$ defined in (3.3) have the following integral representation:

$$F^{(3)}_{D,X}(A, B, B', B''; C; z, w, v) = \Gamma \left( \frac{C}{A, C - A} \right) \int_0^1 u^{A-I} (1-u)^{C-A-I} \exp \left(-\frac{X}{u(1-u)} \right) \times (1-zu)^{-B} (1-wu)^{-B'} (1-vu)^{-B''} \, du.$$

Proof. Equations (2.8) and (3.3) together yield

$$F^{(3)}_{D,X}(A, B, B', B''; C; z, w, v) = \Gamma \left( \frac{C}{A, C - A} \right) \sum_{m,n,p \geq 0} \int_0^1 u^{A-I} (1-u)^{C-A-I} \exp \left(-\frac{X}{u(1-u)} \right) (B)_m (B')_n \times (B'')_p (zu)^m (wu)^n (uv)^p \frac{m! n! p!}{m! n! p!}.$$

Now, using the matrix relation (3.6) and proceeding in the similar as in Theorem 3.1, we get the required result (3.12).
4 Fractional calculus of extended hypergeometric matrix function

The extended Riemann-Liouville fractional derivative of order $\mu$ is given by [13]

$$D_{\mu,p} z \{ f(z) \} = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1} \exp \left( \frac{-pt^2}{t(z-t)} \right) dt, \quad \Re(\mu) < 0, \Re(p) > 0 \quad (4.1)$$

and for $m-1 < \Re(\mu) < m \ (m=1,2,\ldots)$

$$D_{\mu,p} z \{ f(z) \} = d^m dz^m D_{\mu-m,p} z \{ f(z) \} = d^m dz^m \left[ \frac{1}{\Gamma(-\mu + m)} \int_0^z f(t)(z-t)^{-\mu+m-1} \exp \left( \frac{-pt^2}{t(z-t)} \right) dt \right] \quad (4.2)$$

where the path of integration is a line from 0 to $z$ in the complex t-plane. For the case $p = 0$, we obtain the classical Riemann-Liouville fractional derivative operator.

**Definition 1.** Let $X$ be a positive stable matrix in $\mathbb{C}^{r \times r}$ and $\mu \in \mathbb{C}$ such that $\Re(\mu) < 0$. Then, the extended Riemann-Liouville fractional derivative of order $\mu$ is defined as follows

$$D_{\mu,X} z \{ A \} = z^A_{\mu} \times \frac{\mathcal{B}(A + I, -\mu I; X)}{\Gamma(-\mu)} \quad (4.3)$$

and for $m-1 < \Re(\mu) < m \ (m=1,2,\ldots)$

$$D_{\mu,X} z \{ f(z) \} = d^m dz^m D_{\mu-m,X} z \{ f(t) \} = d^m dz^m \left[ \frac{1}{\Gamma(-\mu + m)} \int_0^z f(t)(z-t)^{-\mu+m-1} \exp \left( \frac{-Xz^2}{t(z-t)} \right) dt \right], \quad (4.4)$$

where the path of integration is a line from 0 to $z$ in the complex t-plane. For the case $X = O$ we obtain the classical Riemann-Liouville fractional derivative operator.

We start our investigation by calculating the extended fractional derivatives of some elementary functions.

**Theorem 4.1.** Let $A$ and $X$ be positive stable matrices in $\mathbb{C}^{r \times r}$ and $\mu \in \mathbb{C}$ such that $\Re(\mu) < 0$. Then,

$$D_{\mu,X} z^A = A^\mu \times \frac{\mathcal{B}(A + I, -\mu I; X)}{\Gamma(-\mu)} \quad (4.5)$$

**Proof.** From (2.8) and (4.3), we obtain

$$D_{\mu,X} z^A = \frac{1}{\Gamma(-\mu)} \int_0^z t^A(z-t)^{-\mu-1} \exp \left( \frac{-Xz^2}{t(z-t)} \right) dt. \quad (4.6)$$
\( \frac{1}{z} = u \) and \( \frac{d}{dt} = zdu \) in the equation (4.6), we get

\[
D_z^{\lambda-\mu X} \{z^{A}\} = \frac{z^{A-\mu I}}{\Gamma(\lambda - \mu)} \int_0^1 u^{A}(1-u)^{(-\mu-1)I} \exp \left( \frac{-X}{u(1-u)} \right) \, du \\
= z^{A-\mu I} \frac{\mathfrak{B}(A + I; \mu I; X)}{\Gamma(\lambda - \mu)}. 
\]

This completes the proof. \( \square \)

**Theorem 4.2.** Let \( A, B \) and \( X \) be positive stable matrices in \( \mathbb{C}^{r \times r} \) and \( \lambda, \mu \in \mathbb{C} \) such that \( \Re(\mu) > \Re(\lambda) > 0 \). Then,

\[
D_z^{\lambda-\mu X} \{z^{A-I}(1-z)^{-B} \} = \Gamma(A) \Gamma^{-1}(A + (\mu - \lambda)I) z^{A+(\mu-\lambda-1)I} F^{(X)}(B, A; A + (\mu - \lambda)I; z). \tag{4.7}
\]

**Proof.** Proceeding similarly as in Theorem 4.1, we get

\[
D_z^{\lambda-\mu X} \{z^{A-I}(1-z)^{-B} \} = \frac{z^{A+(\mu-\lambda-1)I}}{\Gamma(\mu - \lambda)} \int_0^1 u^{A-I}(1-uz)^{-B}(1-u)^{\mu-\lambda} \times \exp \left( \frac{-X}{u(1-u)} \right) \, du. \tag{4.8}
\]

Using (2.14), we can write

\[
D_z^{\lambda-\mu X} \{z^{A-I}(1-z)^{-B} \} = \frac{z^{A+(\mu-\lambda-1)I}}{\Gamma(\mu - \lambda)} \mathfrak{B}(A, (\mu - \lambda)I) F^{(X)}(B, A; A + (\mu - \lambda)I; z) \\
= z^{A+(\mu-\lambda-1)I} \Gamma(A) \Gamma^{-1}(A + (\mu - \lambda)I) F^{(X)}(B, A; A + (\mu - \lambda)I; z).
\]

Hence the proof is completed. \( \square \)

The transform of several other matrix functions can be obtained easily by using the extended Riemann-Liouville fractional derivative operators defined in (4.3) and (4.4). Since the proofs are similar to Theorems 4.1 and 4.2, we produce the results without proofs.

**Theorem 4.3.** Let \( A, B, B' \) and \( C \) be matrices in \( \mathbb{C}^{r \times r} \) such that \( AB = BA \), \( A B' = B' A \), \( AC = CA; A, C \) and \( C - A \) are positive stable and \( \lambda, \mu \in \mathbb{C} \) such that \( \Re(\mu) > \Re(\lambda) > 0 \); \( |az| < 1, |bz| < 1 \). Then

\[
D_z^{\lambda-\mu X} \{z^{A-I}(1-az)^{-B}(1-bz)^{-B'} \} = \Gamma(A) \Gamma^{-1}(A + (\mu - \lambda)I) z^{A+(\mu-\lambda-1)I} F_1(A, B, B'; A + (\mu - \lambda)I; az, bz; X). \tag{4.9}
\]

**Theorem 4.4.** Let \( A, B, B', B'' \) and \( C \) be matrices in \( \mathbb{C}^{r \times r} \) such that \( AC = CA \), \( A \) and \( C \) commute with all remain matrices; \( A, C \) and \( C - A \) are positive stable and \( \lambda, \mu \in \mathbb{C} \) such that \( \Re(\mu) > \Re(\lambda) > 0 \); \( |az| < 1, |bz| < 1, |cz| < 1 \). Then

\[
D_z^{\lambda-\mu X} \{z^{A-I}(1-az)^{-B}(1-bz)^{-B'}(1-cz)^{-B''} \} = \Gamma(A) \Gamma^{-1}(A + (\mu - \lambda)I) z^{A+(\mu-\lambda-1)I} F_{D,X}^{(3)}(A, B, B', B''; A + (\mu - \lambda)I; az, bz, cz). \tag{4.10}
\]
Theorem 4.5. Let $A$, $B$, $B'$ and $C$ be matrices in $\mathbb{C}^{r\times r}$ such that $B$, $B'$, $C$ commute with each other and $B$, $B'$, $C$, $C - B$ and $C' - B'$ are positive stable. Also let $\lambda, \mu \in \mathbb{C}$ such that $\Re(\mu) > \Re(\lambda) > 0$. Then, for $|\frac{z}{1-t}| < 1$, we have

$$D_2^{\lambda-\mu,\mathbb{X}}\{zB'-I(1-z)^{-A}F^{(\mathbb{X})}(A, B; C; \frac{x}{1-z})\} = \frac{zB' + (\mu - \lambda - 1)I}{\Gamma(\mu - \lambda)}F_2(A, B, B'; C, \mu I; x, z; \mathbb{X}) \times \Gamma\left(\begin{array}{c} C \\ B, C - B \end{array}\right).$$ (4.11)

5 Generating functions

Theorem 5.1. Let $A$, $B$ and $X$ be positive stable matrices in $\mathbb{C}^{r\times r}$ and $\lambda, \mu \in \mathbb{C}$ such that $\Re(\mu) > \Re(\lambda) > 0$. Then, for $|z| < |1-t|$, we have

$$\sum_{n=0}^{\infty} \frac{(A)_n}{n!} F^{(X)}(A + nI, B; B + (\mu - \lambda)I; z) t^n = (1 - t)^{-A} F^{(X)}(A, B; B + (\mu - \lambda)I; \frac{z}{1-t})$$ (5.1)

Proof. Consider the matrix identity

$$[(1 - z) - t]^{-A} = (1 - t)^{-A}\left[1 - \frac{z}{1-t}\right]^{-A},$$

which can be written as

$$\sum_{n=0}^{\infty} \frac{(A)_n}{n!} (1 - z)^{-A} \left(\frac{t}{1-z}\right)^n = (1 - t)^{-A}\left[1 - \frac{z}{1-t}\right]^{-A}.$$ (5.2)

Now, multiplying by $zB - I$ in Equation (5.2) and applying the extended fractional derivative operator $D^{\lambda-\mu,\mathbb{X}}$, we can write

$$\sum_{n=0}^{\infty} \frac{(A)_n}{n!} D^{\lambda-\mu,\mathbb{X}}\{zB-I(1-z)^{-A-nI}\} t^n = (1 - t)^{-A} D^{\lambda-\mu,\mathbb{X}}\{zB-I\left[1 - \frac{z}{1-t}\right]^{-A}\}.$$ 

Using Theorem 4.2, we get the desired result. \hfill \Box

Theorem 5.2. Let $A$, $B$, $B'$ and $X$ be positive stable matrices in $\mathbb{C}^{r\times r}$ and $\lambda, \mu \in \mathbb{C}$ such that $\Re(\mu) > \Re(\lambda) > 0$. Then, for $|t| < \frac{1}{1+|z|}$, we have the following generating relation

$$\sum_{n=0}^{\infty} \frac{(B')_n}{n!} F^{(X)}(B - nI, A; A + (\mu - \lambda)I; z) t^n = (1 - t)^{-B'} F_1(A, B, B'; A + (\mu - \lambda)I; z, \frac{-zt}{1-t}, \mathbb{X}).$$

Proof. Using (5.2), we have

$$\sum_{n=0}^{\infty} \frac{(B')_n}{n!} (1 - z)^n t^n = (1 - t)^{-B'}\left[1 - \frac{-zt}{1-t}\right]^{-B'}.$$ (5.3)
Now multiplying both sides of the Equation (5.3) by $z^{A-I}(1 - z)^{-B}$ and applying the extended fractional derivative operator $D_z^{\lambda-\mu,X}$, we get

$$
\sum_{n=0}^{\infty} \frac{(B')_n}{n!} D_z^{\lambda-\mu,X}\{z^{A-I}(1 - z)^{-(B-nI)}\} t^n \\
= (1 - t)^{-B'} D_z^{\lambda-\mu,X}\{z^{A-I}(1 - z)^{-B}\left[1 - \frac{-zt}{1-t}\right]^{-B'}\}.
$$

Using the Theorems 4.2 and 4.3 we get the desired result. \qed

6 Conclusion

In this paper, we studied the generalized version of some Appell matrix function and Lauricella matrix function. We obtained some generating relations in terms of matrix functions using the extended Riemann-Liouville fractional derivative of order $\mu$. The particular case of our results, i.e. if we take matrices from $\mathbb{C}^{1\times 1}$, coincides with the results obtained in [3].

References

[1] M. Abdalla, A. Bakhet, Extension of Beta matrix function, Asian J. Math. Comput. Res. 9 (2016), 253–264.

[2] M. Abdalla, A. Bakhet, Extended Gauss hypergeometric matrix functions, Iran. J. Sci. Technol. Trans. A Sci. 42 (2018), no. 3, 1465–1470.

[3] M. A. ¨Ozarslan, E. ¨Ozergin, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator, Math. Comput. Modelling, 52 (2010), no. 9–10, 1825–1833.

[4] M. A. Chaudhry, A. Qadir, H. M. Srivastava, R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, Appl. Math. Comput. 159 (2004), no. 2, 589–602.

[5] R. Dwivedi, V. Sahai, On the hypergeometric matrix functions of two variables, Linear Multilinear Algebra 66 (2018), no. 9, 1819–1837.

[6] R. Dwivedi, V. Sahai, On the hypergeometric matrix functions of several variables, J. Math. Phys. 59 (2018), no. 2, 023505, 15pp.

[7] R. Dwivedi, V. Sahai, A note on the Appell matrix functions, Quaest. Math. 43 (2020), No. 3, pp. 321-334.

[8] G. B. Folland, Fourier Analysis and Its Applications, Wadsworth and Brooks, pacific Grove, CA 1992.

[9] J. Greene, Hypergeometric functions over finite fields, Trans. Amer. Math. Soc. 301 (1987), no. 1, 77–101.
[10] L. Jódar, J.C. Cortés, *Some properties of gamma and beta matrix functions*. Appl. Math. Lett. 11 (1998), no. 1, 89–93.

[11] L. Jódar, J.C. Cortés, *On the hypergeometric matrix function*, Proceedings of the VI-IIth Symposium on Orthogonal Polynomials and Their Applications (Seville, 1997). J. Comput. Appl. Math. 99 (1998), no. 1-2, 205–217.

[12] A. M. Mathai, *Jacobians of matrix transformations and functions of matrix argument*, World Scientific Publishing Co., Inc., River Edge, NJ, 1997. xii+435 pp.

[13] H.M. Srivastava, H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[14] J. A. Tirao: *The matrix-valued hypergeometric equation*, Proc. Natl. Acad. Sci. USA, 100 (2003), 8138 –8141.

[15] A. Verma, On the incomplete first Appell hypergeometric matrix functions $\gamma_1$ and $\Gamma_1$, *Ramanujan J.*, Communicated (2019).

[16] A. Verma, On the incomplete fourth Appell hypergeometric matrix functions $\gamma_4$ and $\Gamma_4$, *Indian J. pure appl. Math.*, Communicated (2020).

[17] A. Verma, On the incomplete Srivastava’s triple hypergeometric matrix functions, *Quaest. Math.*, Accepted (2020) DOI: 10.2989/16073606.2020.1753123.

[18] A. Verma, Some results on the Srivastava’s triple hypergeometric matrix functions, *Asian-Eur. J. Math.*, Accepted (2020) DOI: 10.1142/S179355712150056X.

[19] A. Verma, S. Yadav, On the incomplete second Appell hypergeometric matrix functions, *Linear Multilinear Algebra*, Accepted (2019) DOI:10.1080/03081087.2019.1640178.