A derivability criterion based on the existence of adjunctions.

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Abstract

In this paper we introduce a derivability criterion of functors based on the existence of adjunctions rather than on the existence of resolutions. It constitutes a converse of Quillen-Maltsiniotis derived adjunction theorem. We present two applications of our derivability criterion. On the one hand, we prove that the two notions for homotopy colimits corresponding to Grothendieck derivators and Quillen model categories are equivalent. On the other hand, we deduce that the internal hom for derived Morita theory constructed by B. Toën is indeed the right derived functor of the internal hom of dg-categories.

1. Introduction.

In homological algebra, the classical criteria of existence of derived functors are always based on the existence of resolutions: for instance, it is a basic fact that if $\mathcal{A}$ is an abelian category with enough injectives then there exists the right derived functor $\mathbb{R}F : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ of any left exact functor $F : \mathcal{A} \to \mathcal{B}$. More recently, the existence of the right derived functor $\mathbb{R}F : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ of $F$ for unbounded complexes over a Grothendieck abelian category $\mathcal{A}$ is deduced in [AJS] and [S] from the existence of K-injective resolutions. In the context of model categories, fibrant replacements ensure the existence of the right derived functor of any right Quillen functor ([Q]). And there are other kinds of resolutions, for instance the right deformations of [DHKS], and the fibrant models of [GNPR]. A general notion of resolution that includes the preceding examples consists of the ‘structures de dérivabilité’ developed in [KM], where a derivability criterion based on their existence is given.

In this paper we obtain a derivability criterion that uses a different approach: instead of assuming the existence of resolutions, we assume the existence of adjunctions. It is motivated by the Quillen derived adjunction theorem ([Q]), as generalized by G. Maltsiniotis ([M]):

**Derived adjunction theorem.** Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors. Assume that there exists $\mathbb{L}F$, the absolute left derived functor of $F$, and $\mathbb{R}G$, the absolute right derived functor of $G$. Then $(\mathbb{L}F, \mathbb{R}G)$ is again a pair of adjoint functors.

In other words, the derived adjunction theorem states that adjunctions are preserved by taking absolute derived functors. Our derivability criterion is somehow a converse of this...
fact: it states that the right adjoint of the absolute left derived functor of the left adjoint of $G$ is indeed the absolute right derived functor of $G$. More precisely,

**Theorem 3.3.** Let $F: C \rightleftarrows D : G$ be a pair of adjoint functors. Assume that there exists $\mathbb{L}F$, the absolute left derived functor of $F$. If $\mathbb{L}F$ admits a right adjoint $G'$

$$\mathbb{L}F : C[W^{-1}] \rightleftarrows D[W^{-1}] : G'$$

then the absolute right derived functor $\mathbb{R}G$ of $G$ exists, and it agrees with $G'$.

The proof relies on a result concerning iterated Kan extensions due to E. Dubuc ([D]). For an adjunction $F: C \rightleftarrows D : G$ in which $G$ is absolutely right derivable, the dual result guarantees the existence of $\mathbb{L}F$ provided that $\mathbb{R}G$ has a left adjoint.

A consequence of previous results in the context of homotopy colimits is that Grothendieck homotopy colimits (defined as left adjoints of the localized constant diagram functor) are indeed equivalent to Quillen homotopy colimits (defined as absolute left derived functors of the colimit). More generally, homotopy left Kan extensions may be equivalently defined as left adjoints of the localized inverse image functor, or as absolute left derived functors of the inverse image functor. In particular, we deduce that homotopy left Kan extensions are always composable.

In the model category framework, given a general model category $\mathcal{M}$ and a general small category $I$, there is no known model structure on $\mathcal{M}^I$ suitable to obtain $\mathbb{L}\text{colim}$, the left derived functor of the colimit, as $\text{colim}$ composed with a cofibrant resolution. This is the reason why one has to work harder to prove that a general model category is homotopically cocomplete, that is, it possesses all homotopy colimits (see [CS], [DHKS] or [C]). In [R] we prove that the corrected Bousfield-Kan construction of homotopy colimits, made through the choice of a cosimplicial frame as in [H], gives a left adjoint of the localized constant functor $\mathcal{M}^I[W^{-1}] \rightarrow \mathcal{M}[W^{-1}]$. By the equivalence between Grothendieck and Quillen homotopy colimits obtained here, this implies that the local (Bousfield-Kan construction) and global (left derived functor approach) homotopy colimits coincide for any model category.

Our second application concerns B. Toën’s internal hom for derived Morita theory, introduced in [T]. Derived Morita theory is developed in loc. cit. using a suitable homotopy theory of dg-categories. An essential point to do this was to show that the homotopy category of dg-categories, $\text{Ho}(\text{dgcat})$, which is easily seen to be symmetric monoidal, is indeed a closed symmetric monoidal category. This is proved directly in [T], providing an explicit construction for the internal hom, $\mathcal{R}\text{Hom}(-,-)$, in $\text{Ho}(\text{dgcat})$. But since $\mathcal{R}\text{Hom}(B,C)$ is not constructed as $\text{Hom}(-,-)$ applied to a cofibrant resolution of $B$ and to a fibrant resolution of $C$, it is not clear whether $\mathcal{R}\text{Hom}(-,-)$ is the right derived functor of $\text{Hom}(-,-)$ or not (see [Ta2]). We settle this question proving the

**Theorem 5.6.** Toën’s internal hom $\mathcal{R}\text{Hom}(-,-) : \text{Ho}(\text{dgcat})^\circ \times \text{Ho}(\text{dgcat}) \rightarrow \text{Ho}(\text{dgcat})$ of derived Morita theory is the absolute right derived functor $\mathcal{R}\text{Hom}(-,-)$ of the internal hom $\text{Hom}(-,-)$ of dg-categories.

The paper is organized as follows. In section 2 we develop the preliminaries on absolute Kan extensions and absolute derived functors needed later. In the third section we
give the derivability criterion, and a corollary regarding the composition of derived functors. Section 4 contains an application to the setting of homotopy colimits, where we prove the equivalence of Quillen and Grothendieck notions for the homotopy colimit. We also deduce that homotopy left Kan extensions are composable. Finally, in the last section we apply our derivability criterion to derived internal homs, deducing that Toën’s internal hom of derived Morita theory is indeed the absolute right derived functor of the internal hom of dg-categories.

ACKNOWLEDGEMENTS: I would like to thank V. Navarro Aznar, F. Guillén Santos and A. Roig Maranges for useful discussions and comments, specially to V. Navarro Aznar for pointed out a more general form of the initial derivability criterion conceived by the author.

2. Preliminaries.

In this section we introduce some preliminaries regarding absolute Kan extensions and derived functors. We assume the reader is familiar with the basics on these topics, that may be found at [ML], or sections 2 and 3 of [KM], for instance.

2.1 Absolute Kan extensions.

Given functors \( T : \mathcal{M} \to \mathcal{A} \) and \( K : \mathcal{M} \to \mathcal{C} \), we denote by \((\text{Ran}_K T, \epsilon)\) the right Kan extension of \( T \) along \( K \) (if it exists). It consists of a functor \( \text{Ran}_K T : \mathcal{C} \to \mathcal{A} \) and a natural transformation \( \epsilon : \text{Ran}_K T \circ K \to T \), called the unit of the Kan extension

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{T} & \mathcal{A} \\
\downarrow K & & \uparrow \epsilon \\
\mathcal{C} & \xrightarrow{\text{Ran}_K T} & \\
\end{array}
\]

that satisfy the usual universal property. Recall that a right Kan extension \((\text{Ran}_K T, \epsilon)\) is characterized by the existence for each \( L : \mathcal{C} \to \mathcal{A} \) of a natural bijection

\[
\tau : \mathrm{Nat}(L, \text{Ran}_K T) \longrightarrow \mathrm{Nat}(L \circ K, T)
\]

where \( \mathrm{Nat}(F, G) = \{ \text{natural transformations from } F \text{ to } G \} \). Indeed, given \((\text{Ran}_K T, \epsilon)\), \( \tau \) is defined as the map that sends \( \lambda : L \to \text{Ran}_K T \) to \( \epsilon \circ (\lambda \circ K) : L \circ K \to T \). Conversely, given \( \tau \) and setting \( L = \text{Ran}_K T \), \( \epsilon = \tau(1_{\text{Ran}_K T} : \text{Ran}_K T \to \text{Ran}_K T) \). A left Kan extension of \( T \) along \( K \) is defined dually by the existence of natural bijections

\[
\mathrm{Nat}(\text{Lan}_K T, L) \longrightarrow \mathrm{Nat}(T, L \circ K)
\]

Definition 2.1. A right Kan extension \((\text{Ran}_K T, \epsilon)\) is said to be absolute if it is preserved by any functor. More concretely, given \( S : \mathcal{A} \to \mathcal{B} \) then the right Kan extension of \( S \circ T \)
along $K$ exists and $\text{Ran}_K(S\circ T) = S\circ \text{Ran}_KT$. In addition, the unit $\text{Ran}_K(S\circ T)\circ K \to S\circ T$ is required to agree with $S\circ \epsilon$. In this case, for each $L : C \to A$ there are bijections

$$\text{Nat}(L, S\circ \text{Ran}_KT) \to \text{Nat}(L\circ K, S\circ T)$$

natural on $S$ and $L$. An absolute left Kan extension of $T$ along $K$ is defined dually, so there are natural bijections $\text{Nat}(S\circ \text{Lan}_K T, L) \to \text{Nat}(S\circ T, L\circ K)$.

Many of the constructions occurring in category theory may be expressed in terms of Kan extensions. An example is the case of adjunctions.

**Proposition 2.2.** Given a functor $F : C \to D$, the following are equivalent:

1. $F$ admits a right adjoint $G : D \to C$.
2. There exists an absolute left Kan extension $G = \text{Lan}_F 1_C$.
3. There exists a left Kan extension $G = \text{Lan}_F 1_C$ and it is preserved by $F$.

If these equivalent conditions hold, the units can be chosen in such a way that $\epsilon : 1_C \to \text{Lan}_F 1_C \circ F$ is both the unit of the adjunction and of the left Kan extension.

**Proof.** The equivalence between 1 and 3 is [ML, theorem X.7.2]. On the other hand 2 implies 3 is obvious, while 1 implies 2 is [ML, proposition X.7.3].

The dual result states in particular that $G$ admits a left adjoint $F$ if and only if there exists an absolute right Kan extension $F = \text{Ran}_G 1_D$ of $1_D : D \to D$ along $G$.

We will use the following result about iterated Kan extensions due to E. Dubuc. Its proof may be found in [D, Proposition I.4.1]

**Proposition 2.3.** Consider functors $K : M \to C$, $L : C \to D$ and $T : M \to A$. Assume that there exists $\text{Ran}_K T$, the right Kan extension of $T$ along $K$. Then there exists $\text{Ran}_{L\circ K} T$ if and only if there exists $\text{Ran}_L \text{Ran}_K T$. In this case, both agree and the units may be chosen in such a way that

$$\epsilon_{\text{Ran}_{L\circ K} T} = \epsilon_{\text{Ran}_K T}\circ (\epsilon_{\text{Ran}_L \text{Ran}_K T} \circ K)$$

In addition, if $\text{Ran}_K T$ is an absolute right Kan extension, then $\text{Ran}_L \text{Ran}_K T$ is absolute if and only if $\text{Ran}_{L\circ K} T$ is.

### 2.2 Absolute derived functors.

Given a class $W$ of morphisms in a category $C$, the localization of $C$ with respect to $W$ is the result of formally inverting the morphisms of $W$ in $C$. This gives a (possibly big category) $C[W^{-1}]$ plus a localization functor $\gamma_C : C \to C[W^{-1}]$ sending the elements of $W$ to isomorphisms, and inducing for each category $D$ an equivalence of categories

$$-\circ \gamma_C : \text{Cat}(C[W^{-1}], D) \to \text{Cat}_W(C, D)$$

Here $\text{Cat}(C[W^{-1}], D)$ is the category of functors $G' : C[W^{-1}] \to E$ and $\text{Cat}_W(C, D)$ is the full subcategory of $\text{Cat}(C, D)$ formed by those $G : C \to E$ that send the elements in $W$ to
isomorphisms.

If \( F : \mathcal{C} \to \mathcal{D} \), the left derived functor of \( F \) (if it exists), is the right Kan extension \((\mathbb{L}F = \text{Ran}_{\gamma_C} F, \epsilon : \mathbb{L}F \circ \gamma_C \to F)\). In case \( \mathbb{L}F \) is in addition an absolute right Kan extension of \( F \) along \( \gamma_C \), then \( \mathbb{L}F \) is called the absolute left derived functor of \( F \). In particular, for each \( S : \mathcal{D} \to \mathcal{E} \), \( \mathbb{L}(S \circ F) = S \circ \mathbb{L}F \).

If \( \mathcal{D} \) is also equipped with a distinguished class of morphisms, which we also write as \( \mathcal{W} \), then the (absolute) total left derived functor of \( F : \mathcal{C} \to \mathcal{D} \) with respect to the classes \( \mathcal{W} \) of \( \mathcal{C} \) and \( \mathcal{D} \) is the (absolute) right Kan extension of \( \gamma_{\mathcal{D}^o}F \) along \( \gamma_C \). It is also denoted by \( \mathbb{L}F \), and this time

\[
\mathbb{L}F = \text{Ran}_{\gamma_C} \gamma_{\mathcal{D}^o}F
\]

In the absolute case, for each functor \( S : \mathcal{D}[\mathcal{W}^{-1}] \to \mathcal{E} \) it holds that

\[
S \circ \mathbb{L}F = \text{Ran}_{\gamma_C} S \circ \gamma_{\mathcal{D}^o}F
\]

In the rest of the paper we will restrict ourselves to the total derived functors case. Then, since there is no risk of confusion about the notation \( \mathbb{L}F \), we drop the ‘total’ adjective for brevity. So we just say that \( F \) has an (absolute) left derived functor \( \mathbb{L}F : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}[\mathcal{W}^{-1}] \).

Dually, \( F \) is said to admit an (absolute) right derived functor if there exists the absolute left Kan extension \((\mathbb{R}F = \text{Lan}_{\gamma_C} \gamma_{\mathcal{D}^o}F, \epsilon : \gamma_{\mathcal{D}^o}F \to \mathbb{R}F \circ \gamma_C)\).

From now on we fix a class \( \mathcal{W} \) of morphisms, called weak equivalences, in the categories considered, and left or right derived functors are always defined with respect to these classes.

In practice, (absolute) derived functors are usually obtained through the existence of some kind of resolutions in \( \mathcal{C} \). An example is the Quillen theorem of existence of derived functors, in the context of model categories.

**Theorem 2.4.** ([Q]) Let \( F : \mathcal{M} \to \mathcal{D} \) be a functor from a Quillen model category \( \mathcal{M} \) to a category \( \mathcal{D} \). If \( F \) sends weak equivalences between cofibrant objects to weak equivalences, then the left derived functor of \( F \) exists and may be computed composing \( F \) with a cofibrant replacement. Dually, if \( F \) sends weak equivalences between fibrant objects to weak equivalences, then the right derived functor of \( F \) exists and may be computed composing \( F \) with a fibrant replacement.

**Remark 2.5.** Although no mention to absoluteness is made in the original form of the above theorem, those derived functors obtained through cofibrant or fibrant resolutions are easily seen to be absolute derived functors (see [M, p.2]). Also, cofibrant or fibrant resolutions induce a derivability structure in the sense of [KM] on the model category. Therefore the theorem above is a particular case of the derivability criterion given in loc. cit., where the derived functors obtained are absolute.
Remark 2.6. Previous theorem shows that a sufficient condition on a functor defined on a model category for being absolutely right derivable is that it preserves weak equivalences between fibrant objects. Indeed, some authors directly define the (absolute) right derived functor of such a functor as its composition with a fibrant replacement, instead as its (absolute) right Kan extension. Both definitions agree in this case by the above theorem. But note however that this is not a necessary condition on a functor to be absolutely right derivable: there are functors that do not preserve weak equivalences between fibrant objects but still admit an absolute right derived functor. An example is the internal hom of dg-categories, that will be studied in section 5.1.

3. Derivability criterion.

The derivability criterion introduced here is motivated by the Quillen derived adjunction theorem ([Q]), as generalized by G. Maltsiniotis ([M]):

**Derived adjunction theorem.** Let $F : C \rightleftarrows D : G$ be a pair of adjoint functors. Assume that there exists $\mathbb{L}F$, the absolute left derived functor of $F$, and $\mathbb{R}G$, the absolute right derived functor of $G$. Then

$$\mathbb{L}F : C[W^{-1}] \rightleftarrows D[W^{-1}] : \mathbb{R}G$$

is a pair of adjoint functors. In addition, the adjunction morphisms $a' : \mathbb{L}F \circ \mathbb{R}G \to 1$ and $b' : 1 \to \mathbb{R}G \circ \mathbb{L}F$ may be chosen in such a way that the two squares below commute

$$\begin{array}{ccc}
\mathbb{L}F \circ \gamma_C \circ G & \xrightarrow{\mathbb{L}F \circ \varepsilon} & \mathbb{L}F \circ \mathbb{R}G \circ \gamma_D \\
\gamma_D \circ F \circ G & \xrightarrow{\gamma_D \circ \varepsilon} & \gamma_D \\
\end{array}$$

$$\begin{array}{ccc}
\mathbb{R}G \circ \gamma_D \circ F & \xleftarrow{\mathbb{R}G \circ \varepsilon} & \mathbb{R}G \circ \mathbb{L}F \circ \gamma_C \\
\gamma_C \circ G \circ F & \xleftarrow{\gamma_C \circ \varepsilon} & \gamma_C \\
\end{array}
(1)

Here, $\varepsilon$ and $\varepsilon$ denote the respective units of $\mathbb{L}F$ and $\mathbb{R}G$, while $a$ and $b$ are the adjunction morphisms of $(F, G)$.

The following result is a converse of the previous derived adjunction theorem, and constitutes a derivability criterion of functors based on the existence of adjunctions rather than on the existence of resolutions.

**Theorem 3.1.** Let $F : C \rightleftarrows D : G$ be a pair of adjoint functors. Assume that there exists $\mathbb{R}G$, the absolute right derived functor of $G$. If $\mathbb{R}G$ admits a left adjoint $F'$

$$F' : C[W^{-1}] \rightleftarrows D[W^{-1}] : \mathbb{R}G$$

then the absolute left derived functor $\mathbb{L}F$ of $F$ exists, and it agrees with $F'$. In addition, the unit $\varepsilon : F' \circ \gamma_C \to \gamma_D \circ F$ may be chosen in such a way that the two squares in (1) commute.

**Proof.** We must prove that $F'$ is the absolute right Kan extension $\text{Ran}_{\gamma_C} \gamma_D \circ F$ of $\gamma_D \circ F$ along $\gamma_C$. Consider $S : D[W^{-1}] \to \mathcal{E}$. Since $(F', \mathbb{R}G)$ is a pair of adjoint functors, it follows from
Then, we deduce that \( F' \) have the bijections

\[
\text{Nat}(L, S\circ F') \xrightarrow{\lambda} \text{Nat}(L\circ \mathbb{R}G, S) \xrightarrow{\gamma} \text{Nat}(L\circ \mathbb{C}G, S\circ \gamma_D)
\]

Then, we deduce that \( F' \) is the absolute right Kan extension \( \text{Ran}_{\mathbb{R}G}\gamma_D \) with unit the composition

\[
F'\circ \mathbb{C}G \xrightarrow{\epsilon G} \gamma_D\circ F\circ G \xrightarrow{a'\circ \gamma_D} \gamma_D
\]

On the other hand, since \((F, G)\) is a pair of adjoint functors then \( F \) is the absolute right Kan extension \( \text{Ran}_G\gamma_D \) with unit \( a : FG \rightarrow 1_D \), by proposition 2.2. In particular, \( \gamma_D\circ F \) is the absolute right Kan extension \( \text{Ran}_G\gamma_D \) with unit \( \gamma_D\circ a : \gamma_D\circ F \rightarrow \gamma_D \). Summarizing all we have proved the existence of

\[
\text{Ran}_G\gamma_D = \gamma_D\circ F \quad \text{and} \quad \text{Ran}_{\mathbb{R}G}\gamma_D = F'
\]

By Dubuc’s result 2.3 we deduce that \( \text{Ran}_{\mathbb{R}G}\gamma_D = \text{Ran}_{\mathbb{R}C}\gamma_D\circ F \) exists, it is absolute, and it agrees with \( F' \). In addition, we may choose as unit \( \epsilon : (\text{Ran}_{\mathbb{R}G}\gamma_D\circ F)\circ \gamma_C \rightarrow \gamma_C\circ F \) in such a way such that

\[
\begin{align*}
F'\circ \mathbb{C}G \xrightarrow{\epsilon G} \gamma_D\circ F\circ G \\
\downarrow (a'\circ \gamma_D)\circ (F'\circ \gamma_C) & \quad \quad \downarrow \gamma_D\circ a \\
\gamma_D & \quad \quad \gamma_D
\end{align*}
\]

commutes. This means that \( F' \) is an absolute left derived functor of \( F \), and the left square in (1) is commutative. Finally, the commutativity of the right square in (1) means that the unit \( \epsilon : F'\circ \gamma_C \rightarrow \gamma_D\circ F \) of the absolute left Kan extension is the adjoint natural transformation through \((F', \mathbb{R}G)\) of

\[
\begin{align*}
\gamma_C \xrightarrow{\gamma_C\circ \gamma_D\circ F} \gamma_C\circ G\circ F \xrightarrow{\epsilon \circ F} \mathbb{R}G\circ \gamma_D\circ F
\end{align*}
\]

Using the naturality of \( \epsilon \) and the left square in (1) we deduce the commutative diagram

\[
\begin{align*}
F'\circ \gamma_C \xrightarrow{(F'\circ \gamma_C)\circ \epsilon} F'\circ \gamma_C\circ G\circ F \xrightarrow{\epsilon \circ (G\circ F)} F'\circ \mathbb{R}G\circ \gamma_D\circ F \xrightarrow{a'\circ (\gamma_D\circ F)} \gamma_D\circ F \xrightarrow{(\gamma_D\circ F)\circ \epsilon} \gamma_D\circ F \circ G\circ F
\end{align*}
\]

Since \((a\circ F)\circ (F\circ b)\) is the identity, it follows that the adjoint of (2) is \( \epsilon \) as required.

Together with Quillen-Maltsiniotis derived adjunction theorem, previous theorem implies the
Let \( F : \mathcal{C} \rightleftharpoons \mathcal{D} : G \) be a pair of adjoint functors. Assume that there exists \( \mathbb{R}G \), the absolute right derived functor of \( G \). Then, the following are equivalent:

1. \( \mathbb{R}G \) admits a left adjoint \( F' : \mathcal{C}[W^{-1}] \to \mathcal{D}[W^{-1}] \).
2. \( F \) admits an absolute left derived functor \( \mathbb{L}F : \mathcal{C}[W^{-1}] \to \mathcal{D}[W^{-1}] \).

In this case, \( F' \) and \( \mathbb{L}F \) agree and their corresponding unit and adjunction morphisms may be chosen in such a way that the two squares in (1) commute.

We will also use the duals of theorem 3.1 and corollary 3.2, given below.

**Theorem 3.3.** Let \( F : \mathcal{C} \rightleftharpoons \mathcal{D} : G \) be a pair of adjoint functors. Assume that there exists \( \mathbb{L}F \), the absolute left derived functor of \( F \). If \( \mathbb{L}F \) admits a right adjoint \( G' \)

\[
\mathbb{L}F : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : G'
\]

then the absolute right derived functor \( \mathbb{R}G \) of \( G \) exists, and it agrees with \( G' \). In addition, the unit \( \varepsilon : \gamma_C \circ G \to G' \circ \gamma_D \) may be chosen in such a way that the two squares in (1) commute.

**Corollary 3.4.** Let \( F : \mathcal{C} \rightleftharpoons \mathcal{D} : G \) be a pair of adjoint functors. Assume that there exists \( \mathbb{L}F \), the absolute left derived functor of \( F \). Then, the following are equivalent:

1. \( \mathbb{L}F \) admits a right adjoint \( G' : \mathcal{D}[W^{-1}] \to \mathcal{C}[W^{-1}] \).
2. \( G \) admits an absolute right derived functor \( \mathbb{R}G : \mathcal{D}[W^{-1}] \to \mathcal{C}[W^{-1}] \).

In this case, \( G' \) and \( \mathbb{R}G \) agree and their corresponding unit and adjunction morphisms may be chosen in such a way that the two squares in (1) commute.

Next we study the composition of derived functors obtained in this way. Given functors \( F_1 : \mathcal{C} \to \mathcal{D} \) and \( F_2 : \mathcal{D} \to \mathcal{E} \) such that \( \mathbb{L}F_1 \), \( \mathbb{L}F_2 \) and \( \mathbb{L}(F_2 \circ F_1) \) exist, there is a natural morphism

\[
\mathbb{L}F_2 \circ \mathbb{L}F_1 \longrightarrow \mathbb{L}(F_2 \circ F_1)
\]

(3)

It is obtained from \((\varepsilon_{\mathbb{L}F_2} \circ F_1) \circ (\mathbb{L}F_2 \circ F_1) : \mathbb{L}F_2 \circ \mathbb{L}F_1 \circ \varepsilon_C \to \varepsilon_E \circ F_2 \circ F_1 \) through the bijection

\[
\text{Nat}(\mathbb{L}F_2 \circ \mathbb{L}F_1, \mathbb{L}(F_2 \circ F_1)) \longrightarrow \text{Nat}(\mathbb{L}F_2 \circ \mathbb{L}F_1 \circ \varepsilon_C, \varepsilon_E \circ F_2 \circ F_1)
\]

Analogously, given \( G_1 : \mathcal{D} \to \mathcal{C} \) and \( G_2 : \mathcal{E} \to \mathcal{D} \) such that there exist \( \mathbb{R}G_1 \), \( \mathbb{R}G_2 \) and \( \mathbb{R}(G_1 \circ G_2) \), there is a natural morphism

\[
\mathbb{R}(G_1 \circ G_2) \longrightarrow \mathbb{R}G_1 \circ \mathbb{R}G_2
\]

(4)

Note that neither (3) nor (4) need to be an isomorphism for general \( F_1 \), \( F_2 \), \( G_1 \) and \( G_2 \). In our case we have the

**Proposition 3.5.** Consider two pairs of adjoint functors \((F_1, G_1)\) and \((F_2, G_2)\)

\[
\mathcal{C} \xrightarrow{F_1} \mathcal{D} \xleftarrow{F_2} \mathcal{E}
\]

such that the absolute derived functors \( \mathbb{L}F_1 \), \( \mathbb{L}F_2 \), \( \mathbb{R}G_1 \), \( \mathbb{R}G_2 \) and \( \mathbb{R}(G_1 \circ G_2) \) exist. If (4) is an isomorphism, then there exists the absolute left derived functor \( \mathbb{L}(F_2 \circ F_1) \) and (3) is an isomorphism as well.

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Proof. By corollary 3.2 we have the adjoint pairs of functors \((\mathbb{L}F_1, \mathbb{R}G_1)\) and \((\mathbb{L}F_2, \mathbb{R}G_2)\). Since the composition of adjunctions is again an adjunction, we have also the adjoint pair \((\mathbb{L}F_2, \mathbb{R}G_1, \mathbb{R}G_2)\). If \((\mathbb{H})\) is an isomorphism, it turns out that \(\mathbb{L}F_2 \circ \mathbb{L}F_1\) is left adjoint to \(\mathbb{R}(G_1 \circ G_2)\). But then, by corollary 3.2 applied to \((F_2 \circ F_1, G_1 \circ G_2)\) we have that \(\mathbb{L}F_2 \circ \mathbb{L}F_1\) is the absolute left derived functor of \(F_2 \circ F_1\).

It is left to the reader to establish the dual of previous proposition, concerning the composition of absolute right derived functors.

4. Equivalence between Grothendieck and Quillen homotopy colimits.

A nice particular case of corollary 3.2 occurs when in an adjunction \(F : \mathcal{C} \rightleftarrows \mathcal{D} : G\) the right adjoint preserves weak equivalences. In this case we deduce that \(F\) admits an absolute left derived functor if and only if \(G : \mathcal{D}[\mathcal{W}^{-1}] \rightarrow \mathcal{C}[\mathcal{W}^{-1}]\) admits a left adjoint, and in this case both agree. In the setting of homotopy colimits, a consequence of this fact is that the notions of homotopy colimit corresponding to Grothendieck derivators and model categories are indeed equivalent.

We begin by recalling how these two notions are defined.

Given a small category \(I\), the class \(\mathcal{W}\) of weak equivalences of \(\mathcal{C}\) induces pointwise a class of morphisms in the category of functors from \(I\) to \(\mathcal{C}\), \(\mathcal{C}^I\). This new class, which we also denote by \(\mathcal{W}\), is called the class of pointwise weak equivalences. More concretely, a pointwise weak equivalence of \(\mathcal{C}^I\) is a natural transformation \(\lambda : X \rightarrow Y\) such that \(\lambda_i \in \mathcal{W}\) for each \(i \in I\). Note that the constant diagram functor \(c_I : \mathcal{C} \rightarrow \mathcal{C}^I\), defined as \((c_I(x))(i) = x\) for all \(i \in I\), is then weak equivalence preserving. We also denote by \(c_I\) the induced functor \(c_I : \mathcal{C}[\mathcal{W}^{-1}] \rightarrow \mathcal{C}^I[\mathcal{W}^{-1}]\) on localized categories, and call it the localized constant diagram functor.

Definition 4.1. A Grothendieck homotopy colimit is defined as the left adjoint \(\text{hocolim}_I : \mathcal{C}^I[\mathcal{W}^{-1}] \rightarrow \mathcal{C}[\mathcal{W}^{-1}]\) of the localized constant diagram functor \(c_I : \mathcal{C}[\mathcal{W}^{-1}] \rightarrow \mathcal{C}^I[\mathcal{W}^{-1}]\) (if it exists).

On the other hand, if there exists the colimit \(\text{colim}_I : \mathcal{C}^I \rightarrow \mathcal{C}\), a Quillen homotopy colimit is defined as the absolute left derived functor \(\mathbb{L}\text{colim}_I : \mathcal{C}^I[\mathcal{W}^{-1}] \rightarrow \mathcal{C}[\mathcal{W}^{-1}]\) of the colimit \(\text{colim}_I\) (if it exists).

Proposition 4.2. Assume that there exists \(\text{colim}_I : \mathcal{C}^I \rightarrow \mathcal{C}\). Then a functor \(\mathcal{C}^I[\mathcal{W}^{-1}] \rightarrow \mathcal{C}[\mathcal{W}^{-1}]\) is a Grothendieck homotopy colimit if and only if it is a Quillen homotopy colimit.

Proof. Since by hypothesis \(\text{colim}_I : \mathcal{C}^I \rightarrow \mathcal{C}\) exists, we have an adjunction \(\text{colim}_I : \mathcal{C}^I \rightleftarrows \mathcal{C} : c_I\). Since \(c_I\) is weak equivalence preserving, it has in particular an absolute right derived functor and \(\mathbb{R}c_I = c_I : \mathcal{C}^I[\mathcal{W}^{-1}] \rightarrow \mathcal{C}[\mathcal{W}^{-1}]\). Hence the result follows directly from corollary 3.2. 

\[9\]
Recall that the inverse image of a functor \( f : I \to J \) of small categories is \( f^* : C^J \to C^I \) defined as \( (f^*Y)(i) = Y(f(i)) \). Again, it clearly preserves pointwise weak equivalences.

**Definition 4.3.** A Grothendieck homotopy left Kan extension along \( f \), \( \text{Hof}_f : C^I \to C^J \) \( W^{-1} \), is defined as the left adjoint of \( f^* : C^J \to C^I \) \( W^{-1} \) (if it exists).

On the other hand, if there exists the left Kan extension \( f^* : C^I \to C^J \), a Quillen homotopy left Kan extension along \( f \) is defined as the absolute left derived functor \( \mathbb{L}f^* : C^I \to C^J \) \( W^{-1} \) of \( f^* \) (if it exists).

Analogously, we deduce from corollary 3.2 the

**Proposition 4.4.** Assume that there exists \( f^* : C^I \to C^J \). Then a functor \( C^I \to C^J \) \( W^{-1} \) is a Grothendieck homotopy left Kan extension if and only if it is a Quillen homotopy left Kan extension.

In light of this equivalence, we deduce in next corollary that Quillen homotopy left Kan extensions are always composable.

**Corollary 4.5.** Assume given functors \( f : I \to J \) and \( g : J \to K \) such that there exist the absolute left derived functors \( \mathbb{L}f^* : C^I \to C^J \) \( W^{-1} \) and \( \mathbb{L}g^* : C^J \to C^K \) \( W^{-1} \). Then there exists the absolute left derived functor \( \mathbb{L}(g \circ f)^* \) and it agrees with \( \mathbb{L}g^* \mathbb{L}f^* \).

**Proof.** Since \( f^* \), \( g^* \) and \( f^* g^* = (g \circ f)^* \) are weak equivalence preserving functors, in particular \( \mathbb{R}(f^* g^*) = f^* g^* = \mathbb{R}f^* \mathbb{R}g^* \). Hence the corresponding composition morphism \( \mathbb{H} \) is an isomorphism, and the result follows from proposition 3.5 applied to

\[
\begin{array}{ccc}
C^I & \xrightarrow{f^*} & C^J \\
V & \xrightarrow{g^*} & C^K
\end{array}
\]

**Remark 4.6.** In [CS, section 16], it is proved that if \((\mathcal{C}, \mathcal{W})\) admits a so called ‘left model approximation’ then it has all Quillen homotopy left Kan extensions. Previous corollary imply in particular that these are composable.

The dual results obtained using corollary 3.4 state that Grothendieck homotopy limits (defined as right adjoints of the localized constant diagram functors) are equivalent to Quillen homotopy limits (defined as absolute right derived functors of the constant diagram functors). Also, Grothendieck homotopy right Kan extensions are equivalent to Quillen homotopy right Kan extensions, defined dually, and consequently Quillen homotopy right Kan extensions are always composable.
5. Derived internal homs.

Recall that a symmetric monoidal category \((\mathcal{C}, \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C})\) is said to be \textit{closed} if for each object \(B\) of \(\mathcal{C}\) the functor \(- \otimes B : \mathcal{C} \to \mathcal{C}\) has a right adjoint \(\text{Hom}_\mathcal{C}(B, -)\). This means that there are natural bijections

\[
\text{Hom}_\mathcal{C}(A \otimes B, C) \leftrightarrow \text{Hom}_\mathcal{C}(A, \text{Hom}_\mathcal{C}(B, C))
\]

In this case, \(\text{Hom}_\mathcal{C}(B, -)\) is also natural on \(B\) producing a bifunctor \(\text{Hom}_\mathcal{C}(-, -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\).

If \(\mathcal{W}\) is a class of morphisms in \(\mathcal{C}\), a closed symmetric monoidal structure on \(\mathcal{C}\) does not necessarily induce one on \(\mathcal{C}[\mathcal{W}^{-1}]\). In case \(\otimes\) passes to the localized category \(\mathcal{C}[\mathcal{W}^{-1}]\), we deduce the following result.

**Proposition 5.1.** Let \((\mathcal{C}, \otimes, \text{Hom})\) be a closed symmetric monoidal category. Assume that \((\mathcal{C}[\mathcal{W}^{-1}], \otimes^L)\) is a symmetric monoidal category in which \(- \otimes^L B\) is the absolute left derived functor of \(- \otimes B\), for each object \(B\) of \(\mathcal{C}\). Then, the following are equivalent:

1. \((\mathcal{C}[\mathcal{W}^{-1}], \otimes^L)\) is a closed symmetric monoidal category, with internal hom

\[
\text{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(-, -) : \mathcal{C}[\mathcal{W}^{-1}] \times \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{W}^{-1}]
\]

2. For each object \(B\) of \(\mathcal{C}\), the internal hom \(\text{Hom}_\mathcal{C}(B, -) : \mathcal{C} \to \mathcal{C}\) has an absolute right derived functor \(\mathbb{R}\text{Hom}_\mathcal{C}(B, -)\).

In addition, if these equivalent conditions hold, then \(\text{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(B, -)\) and \(\mathbb{R}\text{Hom}_\mathcal{C}(B, -)\) agree for each object \(B\) of \(\mathcal{C}\).

**Proof.** For a fixed object \(B\) of \(\mathcal{C}\), the left adjoint in the adjunction \(- \otimes B : \mathcal{C} \rightleftarrows \mathcal{C} : \text{Hom}_\mathcal{C}(B, -)\) admits by assumption an absolute left derived functor \(\mathbb{L}(- \otimes B) = - \otimes^L B\). By corollary 3.4, it follows that \(- \otimes^L B : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{W}^{-1}]\) has a right adjoint \(\text{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(B, -)\) if and only \(\text{Hom}_\mathcal{C}(B, -)\) has an absolute right derived functor \(\mathbb{R}\text{Hom}_\mathcal{C}(B, -)\), and in this case they agree. \(\square\)

5.1 The internal hom of derived Morita theory.

In case that \((\mathcal{C}, \mathcal{W})\) is a \textit{closed symmetric monoidal model category}, \(\mathcal{C}[\mathcal{W}^{-1}]\) does inherit a closed symmetric monoidal structure from \(\mathcal{C}\). Indeed, it is given by the (absolute) left derived functor of \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) and the (absolute) right derived functor of \(\text{Hom} : \mathcal{C}^\circ \times \mathcal{C} \to \mathcal{C}\) (see [Ho, Theorem 4.3.2]).

However, in some interesting situations one encounters a model category \((\mathcal{C}, \mathcal{W})\) such that \(\mathcal{C}\) is also closed symmetric monoidal, but the two structures are not compatible and consequently \((\mathcal{C}, \mathcal{W})\) is not a symmetric monoidal model category. This is precisely the case of derived Morita theory, which is developed in [T] in terms of differential graded categories (or dg-categories for short). Let us briefly recall how it is defined.
Definition 5.2. A dg-category $\mathcal{A}$ is a category enriched over the category $\mathcal{C}(k)$ of complexes of $k$-modules, where $k$ is some fixed ring. More concretely, $\mathcal{A}$ consists of a class (or set) $\text{Ob} \mathcal{A}$ of objects, and a hom-object $\text{Hom}_\mathcal{A}(x, y)$ of $\mathcal{C}(k)$ for any two objects $x$ and $y$. In addition, there are morphisms of complexes (composition)

$$\tau_{x, y, z} : \text{Hom}_\mathcal{A}(x, y) \otimes_k \text{Hom}_\mathcal{A}(y, z) \to \text{Hom}_\mathcal{A}(x, z)$$

subject to the usual associativity and unit axioms. From a dg-category $\mathcal{A}$ one can construct a usual category $[\mathcal{A}]$ with same objects as $\mathcal{A}$, and morphisms $\text{Hom}_{[\mathcal{A}]}(x, y) = \text{H}^0\text{Hom}_\mathcal{A}(x, y)$.

Definition 5.3. A dg-functor $f : \mathcal{A} \to \mathcal{B}$ between two dg-categories $\mathcal{A}$ and $\mathcal{B}$ is a functor of enriched categories over $\mathcal{C}(k)$. That is, $\text{Hom}(f) : \text{Hom}_\mathcal{A}(x, y) \to \text{Hom}_\mathcal{B}(f(x), f(y))$ is a morphism of complexes compatible with composition. Note that $f$ induces in particular a functor $[f] : [\mathcal{A}] \to [\mathcal{B}]$.

From the fact that $\mathcal{C}(k)$ is closed symmetric monoidal readily follows that the category $\mathcal{dgcat}$ of dg-categories and dg-functors is again closed symmetric monoidal. Indeed, $\mathcal{A} \otimes \mathcal{B}$ has $\text{Ob} \mathcal{A} \times \text{Ob} \mathcal{B}$ as objects, and $\text{Hom}_{\mathcal{A}}(\cdot, \cdot) \otimes_k \text{Hom}_{\mathcal{B}}(\cdot, \cdot)$ as morphisms.

As explained in [T], derived Morita theory may be established through a suitable homotopy theory of dg-categories. To do this, a weak equivalence of dg-categories is defined as a dg-functor $f : \mathcal{A} \to \mathcal{B}$ such that $\text{Hom}(f)$ is a quasi-isomorphism of complexes, and $[f]$ is essentially surjective (hence an equivalence of categories). We write $\text{Ho}(\mathcal{dgcat}) = \mathcal{dgcat}[W^{-1}]$.

With this notion of weak equivalences, $(\mathcal{dgcat}, W)$ is indeed a model category (see [1a]). However, with this model structure $\otimes : \mathcal{dgcat} \times \mathcal{dgcat} \to \mathcal{dgcat}$ does not preserve cofibrant objects, so $(\mathcal{dgcat}, W)$ is not a closed symmetric monoidal model category.

The situation is not as bad though, since for a cofibrant dg-category $\mathcal{A}$ it holds that $\mathcal{A} \otimes \cdot : \mathcal{dgcat} \to \mathcal{dgcat}$ still preserves weak equivalences. This readily implies that for an arbitrary dg-category $\mathcal{B}$, $\cdot \otimes \mathcal{B}$ preserves weak equivalences between cofibrant objects. Then, it follows from theorem 2.4 and remark 2.5 the

**Proposition 5.4.** Given a dg-category $\mathcal{B}$, $\cdot \otimes \mathcal{B} : \mathcal{dgcat} \to \mathcal{dgcat}$ admits an absolute left derived functor $\cdot \otimes^{L} \mathcal{B} : \text{Ho}(\mathcal{dgcat}) \to \text{Ho}(\mathcal{dgcat})$.

With this derived tensor product $\text{Ho}(\mathcal{dgcat})$ becomes a symmetric monoidal category (see [T]). An essential point to establish derived Morita theory was to prove that $\text{Ho}(\mathcal{dgcat})$ is in addition closed symmetric monoidal. This is done in loc. cit. directly, that is, constructing explicitly a right adjoint $\mathcal{R}\text{Hom}(\mathcal{B}, \cdot)$ to $\cdot \otimes^{L} \mathcal{B} : \text{Ho}(\mathcal{dgcat}) \to \text{Ho}(\mathcal{dgcat})$ for each dg-category $\mathcal{B}$.

The internal hom $\mathcal{R}\text{Hom}(\mathcal{B}, \mathcal{C})$ is constructed in two steps. First, one considers the dg-category $\text{Int}((\mathcal{B} \otimes^{L} \mathcal{C}^-) - \text{mod})$ with objects the fibrant and cofibrant $(\mathcal{B} \otimes^{L} \mathcal{C})^\circ$-modules and

$$\text{Hom}_{\text{Int}((\mathcal{B} \otimes^{L} \mathcal{C}^-) - \text{mod})}(E, F) = \text{Hom}(E, F)$$
where the right hand side is the $C(k)$-valued hom of $(B \otimes^L C) - mod$. Then, $\mathcal{RHom}(B, C)$ is the full sub-dg-category $\text{Int}((B \otimes^L C) - mod)^{qr}$ of $\text{Int}((B \otimes^L C) - mod)$ formed by right quasi-representable modules.

Since $\mathcal{RHom}(B, -)$ is not constructed as $\text{Hom}(B, -)$ composed with a resolution, it was not clear whether $\mathcal{RHom}(B, -)$ was or not the right derived functor of $\text{Hom}(B, -)$ (see [1a2]). But, a direct consequence of proposition [5.1] is the

**Proposition 5.5.** Given a dg-category $B$, the internal hom $\mathcal{RHom}(B, -) : \text{Ho}(\text{dgcat}) \to \text{Ho}(\text{dgcat})$ of derived Morita theory is the absolute right derived functor $\mathcal{RHom}(B, -)$ of the internal hom $\text{Hom}(B, -) : \text{dgcat} \to \text{dgcat}$ of dg-categories.

Finally, we conclude that $\mathcal{RHom}$ is the right derived functor of the internal hom of dg-categories in the two variables.

**Theorem 5.6.** The internal hom $\mathcal{RHom}(-, -) : \text{Ho}(\text{dgcat})^\times \times \text{Ho}(\text{dgcat}) \to \text{Ho}(\text{dgcat})$ of derived Morita theory is the absolute right derived functor $\mathcal{RHom}(-, -)$ of the internal hom $\text{Hom}(-, -) : \text{dgcat} \times \text{dgcat} \to \text{dgcat}$ of dg-categories.

**Proof.** Given a dg-category $B$, denote by $\varepsilon^B : \gamma^B \text{Hom}(B, -) \to \mathcal{RHom}(B, -) \gamma$ the unit of the absolute left Kan extension $\mathcal{RHom}(B, -) = \mathcal{RHom}(B, -)$, and by $\varepsilon^B : (- \otimes^L B) \gamma \to \gamma (\otimes B)$ the unit of the absolute right Kan extension $\otimes B$. Note that $\varepsilon^B$ is in this case natural on $B$: if $f : B \to B'$ is a morphism of dg-categories, the square

$$(\otimes^L B) \gamma \xrightarrow{\varepsilon^B} \gamma (\otimes B)$$

commutes. This follows from the fact that, given a cofibrant replacement $(Q : \text{dgcat} \to \text{dgcat}, \rho : Q \to 1_{\text{dgcat}})$ which acts by the identity on the sets of objects, then $(- \otimes^L B) \gamma = \gamma (Q (-) \otimes B)$ and $\varepsilon^B = \gamma (\rho \otimes B)$ (see [1] p. 631).

On the other hand, the commutativity of the left square of (1) implies that $\varepsilon^B$ is obtained by adjunction from $\varepsilon^B$ through the adjoint pairs $(- \otimes^L B, \mathcal{RHom}(B, -))$ and $(- \otimes B, \text{Hom}(B, -))$. Since $(\text{dgcat}, \otimes, \text{Hom})$ and $(\text{Ho}(\text{dgcat}), \otimes^L, \mathcal{RHom})$ are closed symmetric monoidal categories, the adjunction morphisms of $(- \otimes B, \text{Hom}(B, -))$ and $(- \otimes^L B, \mathcal{RHom}(B, -))$ are both natural on $B$. Then, we deduce that $\varepsilon^B : \gamma^B \text{Hom}(B, -) \to \mathcal{RHom}(B, -) \gamma$ is also natural on $B$, so it defines a natural transformation

$$\varepsilon : \gamma^B \text{Hom}(-, -) \to \mathcal{RHom}(-, -) \gamma$$

The universal property satisfied by each $\varepsilon_{(B, -)} = \varepsilon^B$ readily implies that $\mathcal{RHom}(-, -)$ is the absolute left Kan extension $\text{Lan}_\gamma \gamma^B \text{Hom}(-, -)$ with unit $\varepsilon$. □
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