IN Variant CURVES OF SMOOTH QUASI-PERIODIC MAPPINGS

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Abstract. In this paper we are concerned with the existence of invariant curves of planar mappings which are quasi-periodic in the spatial variable, satisfy the intersection property, $C^p$ smooth with $p > 2n + 1$, $n$ is the number of frequencies.

1. Introduction. In this paper we are concerned with the existence of invariant curves of the following planar quasi-periodic mapping

$$
\mathcal{M} : \begin{cases}
\theta_1 = \theta + r + f(\theta, r), \\
r_1 = r + g(\theta, r),
\end{cases} \quad (\theta, r) \in \mathbb{R} \times [a, b],
$$

where the perturbations $f(\theta, r)$ and $g(\theta, r)$ are quasi-periodic in $\theta$ with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, $C^p$ smooth in $\theta$ and $r$.

Moser [7] considered the twist mapping

$$
\mathcal{M}_0 : \begin{cases}
x_1 = x + \alpha(y) + \varphi_1(x, y), \\
y_1 = y + \varphi_2(x, y),
\end{cases}
$$

where the perturbations $\varphi_1, \varphi_2$ are assumed to be small and of period $2\pi$ in $x$. He obtained the existence of invariant closed curves of $\mathcal{M}_0$ which is of class $C^{333}$. About $\mathcal{M}_0$, an analytic version of the invariant curve theorem was presented in [13], a version in class $C^5$ in Rüssmann [10] and an optimal version in class $C^p$ with $p > 3$ in Herman [2, 3].

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When the perturbations \( f(\theta,r), g(\theta,r) \) in (1.1) are quasi-periodic in \( \theta \), there are some results about the existence of invariant curves of the following planar quasi-periodic mapping

\[
\mathfrak{M}_1 : \begin{cases}
\theta_1 = \theta + \beta + r + f(\theta,r), \\
r_1 = r + g(\theta,r),
\end{cases} \quad (\theta, r) \in \mathbb{R} \times [a, b],
\] (1.2)

where the functions \( f(\theta,r) \) and \( g(\theta,r) \) are quasi-periodic in \( \theta \) with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \), real analytic in \( \theta \) and \( r \), and \( \beta \) is a constant.

When the map \( \mathfrak{M}_1 \) in (1.2) is an exact symplectic map, \( \omega_1, \omega_2, \cdots, \omega_n, 2\pi \beta^{-1} \) are sufficiently incommensurable, Zharnitsky [15] proved the existence of invariant curves of the map \( \mathfrak{M}_1 \) and applied this result to present the boundedness of all solutions of Fermi-Ulam problem. His proof is based on the Lagrangian approach introduced by Moser [9] and used by Levi and Moser in [5] to show a proof of the twist theorem.

When the map \( \mathfrak{M}_1 \) in (1.2) is reversible with respect to the involution \( G : (\theta, r) \mapsto (-\theta, r) \), that is, \( G\mathfrak{M}_1 G = \mathfrak{M}_1^{-1}, \omega_1, \omega_2, \cdots, \omega_n, 2\pi \beta^{-1} \) satisfy the Diophantine condition

\[
\left| \left\langle k, \omega \right\rangle \right| \frac{\beta}{2\pi} - |j| \geq \frac{\gamma}{|k|^r}, \quad \forall \ k \in \mathbb{Z}^n \setminus \{0\}, \ j \in \mathbb{Z},
\]

Liu [6] stated some variants of the invariant curve theorem for quasi-periodic reversible mapping \( \mathfrak{M}_1 \).

In this paper, motivated by the above references, especially by Rüssmann [12], instead of the exact symplectic or reversibility assumption on \( \mathfrak{M} \), we assume that this mapping satisfies the intersection property, and obtain the invariant curve theorem for the quasi-periodic mapping \( \mathfrak{M} \) in the smooth case, other than analytic case.

Our efforts in this paper are same as Rüssmann [12], we are more interested in weak conditions for the perturbations \( f, g \) than in high differentiability properties of the constructed invariant curves, and the main line of the proofs is also similar to that of Rüssmann [12], which is proven by an iteration process.

Incidentally, in [4] we use this theorem to establish the existence of invariant curves of the planar quasi-periodic mapping

\[
\mathcal{M}_\delta : \begin{cases}
\theta_1 = \theta + \beta + \delta l(\theta,r) + \delta f(\theta,r,\delta), \\
r_1 = r + \delta m(\theta,r) + \delta g(\theta,r,\delta),
\end{cases} \quad (\theta, r) \in \mathbb{R} \times [a, b],
\]

where the functions \( l, m, f, g \) are quasi-periodic in \( \theta \) with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \), \( f(\theta,r,0) = g(\theta,r,0) = 0 \), \( \beta \) is a constant, \( 0 < \delta < 1 \) is a small parameter.

As an application, we also use them to study the existence of quasi-periodic solutions and the boundedness of all solutions for an asymmetric oscillation

\[
x'' + ax^+ - bx^- = f(t),
\]

where \( a, b \) are two different positive constants, \( x^+ = \max\{x,0\}, x^- = \max\{-x,0\}, f(t) \) is a smooth quasi-periodic function with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \).

Finally, we must point out that in order to obtain the existence of invariant curves for the quasi-periodic mapping \( \mathfrak{M} \), we need to assume that this mapping belongs to \( C^p \) with \( p > 2n+1 \) and \( n \) is the number of the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \). Meanwhile we note that when \( n = 1 \), quasi-periodic mappings are periodic mappings, and the optimal smoothness assumption is \( C^p \) with \( p > 3 \). Hence our smoothness assumption for quasi-periodic mappings agrees with that for periodic mappings, and is optimal in this sense.
The rest of the paper is organized as follows. In Section 2, we list some properties of quasi-periodic functions, and then state the main invariant curve theorem (Theorem 2.8) for the quasi-periodic mapping $\Psi$ which is given by (1.1). The proofs of Theorem 2.8 are given in Sections 3, 4, 5. In Section 6, we formulate the detailed proofs of Lemma 2.11 which has been used in the previous section.

2. Quasi-periodic functions and the main result.

2.1. The space of quasi-periodic functions. We first define quasi-periodic functions with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is rationally independent, that is, for all $k = (k_1, k_2, \cdots, k_n) \neq 0$, $\langle k, \omega \rangle = \sum_{j=1}^{n} k_j \omega_j \neq 0$.

**Definition 2.1.** $f(t)$ is called a continuous quasi-periodic function with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, if there is a continuous function $F(\theta_1, \theta_2, \cdots, \theta_n)$ which is $2\pi$-periodic in each $\theta_j$ ($1 \leq j \leq n$) such that

$$f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_n t).$$

Moreover, $f(t)$ is called a $C^p/\text{real analytic}$ quasi-periodic function, if $F$ is $C^p/\text{real analytic}$, meanwhile we say that $F$ is a shell function of $f$.

Denote by $Q(\omega)$ the space of real analytic quasi-periodic functions with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$. Given $f(t) \in Q(\omega)$, suppose that the corresponding shell function $F$ has the following Fourier expansion

$$F(\theta) = \sum_{k \in \mathbb{Z}^n} f_k e^{i\langle k, \theta \rangle},$$

which is $2\pi$-periodic in each variable, real analytic and bounded in a complex neighborhood $\Pi^n_\rho = \{ (\theta_1, \theta_2, \cdots, \theta_n) \in \mathbb{C}^n : |\text{Im } \theta_j| \leq r, j = 1, 2, \cdots, n \}$ of $\mathbb{R}^n$ for some $r > 0$. The function $f(t)$ is obtained from $F(\theta)$ by replacing $\theta$ by $\omega t$, and has the following expansion

$$f(t) = \sum_{k \in \mathbb{Z}^n} f_k e^{i\langle k, \omega \rangle t}.$$

**Definition 2.2.** For $r > 0$, let $Q_r(\omega) \subseteq Q(\omega)$ be the set of real analytic quasi-periodic functions $f$ such that the corresponding shell functions $F$ are bounded on the subset $\Pi^n_\rho$ with the supremum norm

$$|F|_r = \sup_{\theta \in \Pi^n_\rho} |F(\theta)| = \sup_{\theta \in \Pi^n_\rho} \left| \sum_{k \in \mathbb{Z}^n} f_k e^{i\langle k, \theta \rangle} \right| < +\infty.$$ 

Also we define the norm of $f$ as $|f|_r = |F|_r$.

The following properties of quasi-periodic functions can be found in [13, chapter 3].

**Lemma 2.3.** The following statements are true:

(i) Let $f(t), g(t) \in Q(\omega)$, then $g(t + f(t)) \in Q(\omega)$;

(ii) Suppose that

$$|\langle k, \omega \rangle| \geq \frac{c}{|k|^p \sigma}, \quad c, \sigma > 0$$

for all integer vectors $k \neq 0$. Let $h(t) \in Q(\omega)$ and $\tau = \beta t + h(t)$ ($\beta + h' > 0, \beta \neq 0$), then the inverse relation is given by $t = \beta^{-1} \tau + h_1(\tau)$ and $h_1 \in Q(\frac{\sigma}{\beta})$. In particular, if $\beta = 1$, then $h_1 \in Q(\omega)$. 

These statements provide a foundation for the further exploration of quasi-periodic mappings and their invariant curves.
Throughout this paper, we assume that the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) satisfies the Diophantine condition
\[
|\langle k, \omega \rangle| \geq \frac{c}{|k|^{\sigma_0}}, \quad c, \sigma_0 > 0
\]
for all integer vectors \( k \neq 0 \). It is not difficult to show that for \( \sigma_0 > n \), the Lebesgue measure of the set of \( \omega \) satisfying the above inequalities is positive for a suitably small \( c \).

2.2. The main result.

First we give the following definitions.

**Definition 2.4.** Let \( \mathcal{M} \) be a mapping given by (1.1). It is said that \( \mathcal{M} \) has the intersection property if
\[
\mathcal{M}(T) \cap T \neq \emptyset
\]
for every curve \( T : \theta = \xi + \varphi(\xi), \; r = \psi(\xi) \), where the continuous functions \( \varphi \) and \( \psi \) are quasi-periodic in \( \xi \) with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \).

**Definition 2.5.** Let \( \mathcal{M} \) be a mapping given by (1.1). We say that \( \mathcal{M} : \mathbb{R} \times [a, b] \to \mathbb{R}^2 \) is exact symplectic if \( \mathcal{M} \) is symplectic with respect to the usual symplectic structure \( dr \wedge d\theta \) and for every curve \( T : \theta = \xi + \varphi(\xi), \; r = \psi(\xi) \), where the continuous functions \( \varphi \) and \( \psi \) are quasi-periodic in \( \xi \) with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \), we have
\[
\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} r d\theta = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} r_1 d\theta_1.
\]

We claim that if the mapping \( \mathcal{M} \) is exact symplectic, then it has intersection property. In order to prove this result, we first give an useful lemma, and its proof is simple.

**Lemma 2.6.** If \( r = r(\theta) \) is quasi-periodic in \( \theta \) and \( F(\theta, r) \) is quasi-periodic in \( \theta \) with the same frequency, then \( F(\theta, r(\theta)) \) is also quasi-periodic in \( \theta \) with the same frequency.

Now we are going to prove the following lemma.

**Lemma 2.7.** If the mapping \( \mathcal{M} \) is exact symplectic, then it has intersection property.

**Proof.** Since the mapping \( \mathcal{M} \) is exact symplectic and it is also a monotonic twist mapping, according to the paper by Zharnitsky [15], there is a function \( H \) such that the mapping \( \mathcal{M} \) can be written by
\[
r = -\frac{\partial}{\partial \theta} H(\theta_1 - \theta, \theta), \quad r_1 = \frac{\partial}{\partial \theta_1} H(\theta_1 - \theta, \theta),
\]
where \( H \) is quasi-periodic in the second variable.

Now we prove the intersection property of the mapping \( \mathcal{M} \), that is, given any continuous quasi-periodic curve \( T : r = r(\theta) \), we need to prove that \( \mathcal{M}(T) \cap T \neq \emptyset \).

Define two sets \( \mathcal{B} \) and \( \mathcal{B}_1 : \) the set \( \mathcal{B} \) is bounded by four curves \( \{ (\theta, r) : \theta = T \} \); \( \{ (\theta, r) : \theta = T \} \); \( \{ (\theta, r) : r = r_* \} \) and \( \{ (\theta, r) : r = r(\theta) \} \), the set \( \mathcal{B}_1 \) is bounded by four curves \( \{ (\theta, r) : \theta = T \} \); \( \{ (\theta, r) : \theta = T \} \); \( \{ (\theta, r) : r = r_* \} \); and \( \{ (\theta, r) : r = r(\theta) \} \) and the image of \( T \) under \( \mathcal{M} \). Here we choose \( r_* < \min r(\theta) \). It is easy to show that the difference of the areas of \( \mathcal{B}_1 \) and \( \mathcal{B} \) is
\[
\Delta(t, T) = \int_{t}^{T} r_1 d\theta_1 - \int_{t}^{T} r d\theta = H(\theta_1(T) - T, T) - H(\theta_1(t) - t, t).
\]
From the definition of $\mathfrak{M}$ and Lemma 2.6, we know that $\theta_1(T) - T = r(T) + f(T, r(T))$ is quasi-periodic in $T$ and $\theta_1(t) - t = r(t) + f(t, r(t))$ is quasi-periodic in $t$. Hence using Lemma 2.6 again, it follows that $\Delta(t, T)$ is quasi-periodic in $t$ and $T$. Thus there are at least two pairs of $(t_1, T_1)$ and $(t_2, T_2)$ such that $\Delta(t_1, T_1) < 0, \Delta(t_2, T_2) > 0$. The intersection property of $\mathfrak{M}$ follows from this fact, which proves the lemma.

For the quasi-periodic mapping $\mathfrak{M}$ we assume that $f, g : \mathbb{R}^2 \to \mathbb{R}$ are of class $C^p$, and define

$$
|x| = \max (|\theta|, |r|) \quad \text{for} \quad x = (\theta, r) \in \mathbb{R}^2,
$$

$$
|h|_{\mathbb{R}^2} = \sup_{x \in \mathbb{R}^2} |h(x)|,
$$

$$
\|h\|_p = \sum_{|k| \leq p} \sup_{x \in \mathbb{R}^2} |D^k h(x)|
$$

if $p \geq 0$ is an integer, and

$$
\|h\|_p = \sup_{x \neq y \atop |k| = q} \frac{|D^k h(x) - D^k h(y)|}{|x - y|^s} + \sum_{|k| \leq l} \sup_{x \in \mathbb{R}^2} |D^k h(x)|
$$

if $p = l + s$, $l \geq 0$ is an integer, $s \in (0, 1)$, where

$$
D^k = \left( \frac{\partial}{\partial \theta} \right)^{k_1} \circ \left( \frac{\partial}{\partial r} \right)^{k_2}, \quad |k| = |k_1| + |k_2|, \quad k = (k_1, k_2).
$$

We choose a rotation number $\alpha$ satisfying the inequalities

$$
\begin{align*}
\alpha + 12^{-3}\gamma & \leq \alpha \leq b - 12^{-3}\gamma, \\
\langle k, \omega \rangle \frac{a}{2^k} - j & \geq \frac{\gamma}{|k|^\tau}, \quad \text{for all} \quad k \in \mathbb{N}^n \setminus \{0\}, \quad j \in \mathbb{Z}
\end{align*}
$$

with some constants $\gamma, \tau$ satisfying

$$
0 < \gamma < \frac{1}{2} \min \{1, 12^3(b - a)\}, \quad \tau > n. \quad (2.3)
$$

Now we are in a position to state our main result.

**Theorem 2.8.** Suppose that the quasi-periodic mapping $\mathfrak{M}$ given by (1.1) is of class $C^p$ ($p > 2\tau + 1$), and satisfies the intersection property, the functions $f(\theta, r), g(\theta, r)$ are quasi-periodic in $\theta$ with the frequency $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, and satisfy the following smallness conditions

$$
\begin{align*}
\|f\|_{\mathbb{R}^2} + |g|_{\mathbb{R}^2} & \leq \frac{6^{-(n+1)}}{3} \frac{a}{\sqrt{2}} \left( \frac{1}{c_0} \right)^\tau \left( \frac{\gamma}{\Gamma(\tau + 1)} \right)^2, \\
\|f\|_p + |g|_p & \leq \frac{6^{-(n+1)}}{3} \frac{a(1 - q)}{3600(3c_1 + c_2)} \left( \frac{1}{c_0} \right)^\tau \left( \frac{\gamma}{\Gamma(\tau + 1)} \right)^2,
\end{align*}
$$

where $\Gamma$ is the Gamma function, $\gamma, \tau$ satisfy (2.3), $c_0, c_1, c_2$ are positive constants depending only on $p$ and $\omega$, and $q$ is a number satisfying

$$
0 < q \leq \min \left\{ \frac{p - 2\tau - 1}{p + 1} \log 2, \ 10^{-2}4^{-\tau} \right\}. \quad (2.6)
$$

Then for any number $\alpha$ satisfying the inequalities (2.2), the quasi-periodic mapping $\mathfrak{M}$ has an invariant curve $\mathcal{T}_0$ with the form

$$
\begin{align*}
\theta = \theta' + \varphi(\theta'), \\
r = \psi(\theta'),
\end{align*}
$$
where \( \varphi, \psi \) are quasi-periodic with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \), and the invariant curve \( T_0 \) is continuous and quasi-periodic with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \). Moreover, the restriction of \( M \) onto \( T_0 \) is
\[
M|_{T_0} : \theta' = \theta' + \alpha.
\]

**Remark 2.9.** Here we assume that the mapping \( M \) is of class \( C^p \) with \( p > 2\tau + 1 > 2n + 1 \). \( n = 1 \) corresponds to the periodic case, in which \( p > 3 \) is the optimal smoothness condition. Hence our smoothness assumption for quasi-periodic mappings is optimal in this sense.

**Remark 2.10.** If all conditions of Theorem 2.8 hold, then the mapping \( M \) has many invariant curves \( T_0 \), which can be labeled by the form
\[
M|_{T_0} : \theta' = \theta' + \alpha.
\]
In fact, given any \( \alpha \) satisfying the inequalities (2.2), there exists an invariant curve \( T_0 \) of \( M \) which is quasi-periodic with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \), and the restriction of \( M \) onto \( T_0 \) has the form
\[
M|_{T_0} : \theta' = \theta' + \alpha.
\]
The existence of such \( \alpha \) can be found in Lemma 2.12.

The constants \( c_0, c_1, c_2 \) in the main result depend on how well functions of class \( C^p \) can be approximated by analytic ones.

**Lemma 2.11.** Let \( h(\cdot, y) \in C^p \) be a quasi-periodic function with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \), then for any \( \delta > 0 \), there exists a holomorphic function \( h_\delta : \mathbb{C}^2 \rightarrow \mathbb{C} \), \( h_\delta(\cdot, y) \in Q_\tau(\omega) \), \( h_\delta(\mathbb{R}^2) \subseteq \mathbb{R} \) such that the following inequalities
\[
\begin{align*}
|h_\delta|_{E_\delta} &\leq c_0 |h|_{\mathbb{R}^2}, \\
|h - h_\delta|_{\mathbb{R}^2} &\leq c_1 \|h\|_p \delta^p, \\
|h_\delta - h_\delta'|_{E_\delta} &\leq c_2 \|h\|_p \delta'^p
\end{align*}
\]
hold for \( 0 < \delta < \delta' \), where
\[
E_\delta = \{(x, y) \in \mathbb{C}^2 : |\text{Im } x| < \delta, \ |\text{Im } y| < \delta\}, \quad |\cdot|_{E_\delta} = \sup_{z \in E_\delta} |\cdot(z)|,
\]
c_0, c_1, c_2 are positive constants only depending on \( p, \omega \).

The detail proof of Lemma 2.11 is given in Appendix 6, which is similar to the periodic case. When \( h \in C^p \) is a periodic function, there are some detail proofs of Lemma 2.11 available in the literature, for example, see Moser [8, p. 528-529], Rüssmann [10, p. 74-78], Zehnder [14, p. 110-113].

### 2.3. The measure estimate.

**Lemma 2.12.** If \( \tau > n \), then for suitable small \( \gamma \), the set of \( \alpha \) satisfying (2.2) has positive measure.

**Proof.** Choose some \( n \)-dimensional frequency vector \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) satisfying (2.1) and let \( D_{\gamma, \tau}^\omega \) denote the set of all \( \alpha \in \mathbb{R} \) satisfying (2.2) with the fixed \( \gamma \) and
\[ \mathcal{R}_{\gamma, \tau}^\omega = \bigcup_{k \in \mathbb{Z}^n \setminus \{0\}} \bigcup_{j \in \mathbb{Z}} \mathcal{R}_{\omega, \gamma, \tau}^{k,j} \]
\[ = \bigcup_{k \in \mathbb{Z}^n \setminus \{0\}} \left\{ \alpha \in [a + 12^{-3} \gamma, b - 12^{-3} \gamma] : |\langle k, \omega \rangle \frac{\alpha}{2\pi} - j| < \frac{\gamma}{|k|^\tau} \right\}. \]

Now we estimate the measure of the set \( \mathcal{R}_{\omega, \gamma, \tau}^{k,j} \). Set \( |k_{\max}| = \max_{1 \leq i \leq n} |k_i| \), then there exists some \( 1 \leq m \leq n \) such that \( |k_m| = |k_{\max}| \), and \( 1 \leq \frac{|k|}{|k_{\max}|} \leq n \). Therefore, we have
\[ \mathcal{R}_{\omega, \gamma, \tau}^{k,j} = \left\{ \alpha \in [a + 12^{-3} \gamma, b - 12^{-3} \gamma] : |\langle k, \omega \rangle \frac{\alpha}{2\pi} - j| < \frac{\gamma}{|k|^\tau} \right\} \]
\[ = \left\{ \alpha \in [a + 12^{-3} \gamma, b - 12^{-3} \gamma] : |k_{\max} \omega_m \frac{\alpha}{2\pi} + \sum_{i \neq m} k_i \omega_i \frac{\alpha}{2\pi} - j| < \frac{\gamma}{|k|^\tau} \right\} \]
\[ = \left\{ \alpha \in [a + 12^{-3} \gamma, b - 12^{-3} \gamma] : |k_{\max} \omega_m| |\alpha + b_j| < \frac{2\pi \gamma}{|k|^\tau} \right\} \]
\[ = \left\{ \alpha \in [a + 12^{-3} \gamma, b - 12^{-3} \gamma] : -b_j - \delta_k < \alpha < -b_j + \delta_k \right\}, \]
where \( b_j = \frac{1}{k_{\max} \omega_m} \left\{ \sum_{i \neq m} k_i \omega_i - 2\pi j \right\} \) and \( \delta_k = \frac{2\pi \gamma}{|k|^\tau} \frac{1}{|k_{\max}| \omega_m}. \) Hence,
\[ \text{meas}(\mathcal{R}_{\omega, \gamma, \tau}^{k,j}) \leq 2\delta_k = \frac{4\pi \gamma}{|k|^\tau} \frac{1}{|k_{\max}| \omega_m} = \frac{4\pi \gamma}{|k|^\tau} \frac{1}{|k_{\max}| \omega_m}. \]

Since \( 1 \leq \frac{|k|}{|k_{\max}|} \leq n \), then we have the following measure estimate
\[ \text{meas}(\mathcal{R}_{\omega, \gamma, \tau}^{k,j}) \leq O\left( \frac{\gamma}{|k|^\tau+1} \right). \]

Next we estimate the measure of the set \( \mathcal{R}_{\gamma, \tau}^\omega \). Since for \( \alpha \in \mathcal{R}_{\omega, \gamma, \tau}^{k,j} \),
\[ |\langle k, \omega \rangle \frac{\alpha}{2\pi} - j| < \frac{\gamma}{|k|^\tau}, \]
then we have
\[ |j| \leq |\langle k, \omega \rangle | \frac{\alpha}{2\pi} + \frac{\gamma}{|k|^\tau} \leq c_0 |k|, \]
where \( c_0 \) is a constant independent of \( k \). Thus
\[ \text{meas}(\mathcal{R}_{\gamma, \tau}^\omega) \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \sum_{\substack{j \in \mathbb{Z} \ \text{s.t.} \ |j| \leq c_0 |k|}} \text{meas}(\mathcal{R}_{\omega, \gamma, \tau}^{k,j}) \]
\[ \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \sum_{\substack{j \in \mathbb{Z} \ \text{s.t.} \ |j| \leq c_0 |k|}} O\left( \frac{\gamma}{|k|^\tau+1} \right) \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} O\left( \frac{\gamma}{|k|^\tau} \right). \]

Also, if \( \tau > n \),
\[ \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|k|^\tau} \leq 2^n \sum_{m=1}^{\infty} \frac{1}{m^{\tau-n+1}} \leq 2^{2n-1} \sum_{m=1}^{\infty} \frac{1}{m^{\tau-n+1}} < +\infty. \]
Hence, for any \( \tau > n \),
\[
\text{meas}(R_{\gamma,\tau}^\omega) \leq O(\gamma)
\]
and
\[
\text{meas}(D_{\gamma,\tau}^\omega) \to b-a \quad \text{as} \quad \gamma \to 0.
\]
This completes the proof. \( \square \)

3. The iteration process. In this section we present an iteration process leading to the proof of Theorem 2.8.

Firstly, we introduce new variables by the linear transformation
\[
\begin{cases}
\theta = x, \\
\tau = \alpha + \varepsilon_0 y,
\end{cases}
\]
where \( \alpha \) is the chosen rotation number satisfying (2.2), \( \varepsilon_0 \) is defined by
\[
\varepsilon_0 = 6^{-(\tau + \frac{n+1}{2})} \frac{\gamma}{\Gamma(\tau+1)}. \tag{3.1}
\]
In the new coordinates the given mapping (1.1) having the intersection property in the strip \( S = \{(\theta, \tau) \in \mathbb{R}^2 : a < \tau < b \} \) gets the form
\[
A : \begin{cases}
x_1 = x + \alpha + \varepsilon_0 y + f(x, \alpha + \varepsilon_0 y), \\
y_1 = y + \varepsilon_0^{-1} g(x, \alpha + \varepsilon_0 y).
\end{cases}
\]
Clearly the intersection property is preserved and holds in the strip
\[
S^* = \{(x, \tau) \in \mathbb{R}^2 : |\tau| < 600^{-1} \}, \tag{3.2}
\]
where we have used (3.1) and \( \Gamma(\tau + 1) \geq 1 \) for \( \tau > n \).

Since the \( \zeta_p \) functions \( f(\cdot, \alpha + \varepsilon_0 y), g(\cdot, \alpha + \varepsilon_0 y) \) are quasi-periodic with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \), by the assumption we may apply Lemma 2.11 to obtain a family of holomorphic functions \( f_\delta(\cdot, \alpha + \varepsilon_0 y), g_\delta(\cdot, \alpha + \varepsilon_0 y) \in Q_r(\omega) (\delta > 0) \), with which we define the quasi-periodic mappings
\[
A_\delta : \begin{cases}
x_1 = x + \alpha + \varepsilon_0 y + f_\delta(x, \alpha + \varepsilon_0 y), \\
y_1 = y + \varepsilon_0^{-1} g_\delta(x, \alpha + \varepsilon_0 y).
\end{cases}
\]
Define
\[
E_\delta = \{(x, \tau) \in \mathbb{C}^2 : |\text{Im } x| < \delta, |\text{Im } y| < \delta\}
\]
and a sequence
\[
\delta_k = \left( \frac{1}{2} \right)^k, \quad k = 0, 1, 2, \cdots, \tag{3.3}
\]
where \( q \) is a real number satisfying (2.6), and set
\[
E_k = E_{\delta_k}, \quad A_k = A_{\delta_k}, \quad k = 0, 1, \cdots.
\]
Then the estimates of Lemma 2.11 can be written in the form
\[
\begin{cases}
|A_0 - \Omega_0|_{E_0} \leq \varepsilon_0^{-1} c_0 (|f|_{\zeta_2} + |g|_{\zeta_2}), \\
|A - A_k|_{\zeta_2} \leq \varepsilon_0^{-1} c_1 (\|f\|_p + \|g\|_p) \delta_k^p, \\
|A_k - A_{k+1}|_{E_{k+1}} \leq \varepsilon_0^{-1} c_2 (\|f\|_p + \|g\|_p) \delta_k^p,
\end{cases} \tag{3.4}
\]
where the mapping
\[
\Omega_0 : \begin{cases}
x_1 = x + \alpha + \varepsilon_0 y, \\
y_1 = y.
\end{cases}
\]
Before we describe the iteration process, we employ the definitions and notations in Definition 2 of Rüssmann [12], only in (iv), we need to define the mappings \( \Omega_k (k = 0, 1, \cdots) \) by

\[
\Omega_k : \begin{cases} 
x_1 = x + \alpha + \varepsilon_k y, \\
y_1 = y,
\end{cases} \quad (x, y) \in \mathbb{C}^2, \quad \varepsilon_k = 2^{-k\tau} \varepsilon_0.
\]

Now we are going back to the quasi-periodic mappings \( A_k : E_k \mapsto \mathbb{C}^2 \) defined above. We try to fix domains

\[
D_k = D(r_k, s_k), \quad D'_k = D(r'_k, s'_k),
\]

and to find mappings \( Z_k \in T(D'_k), H_k \in T(D_k), \) and \( Z_k - \text{id}, H_k - \Omega_k \) are quasi-periodic with the frequency \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) in the first variable, such that the diagrams

\[
\begin{array}{ccc}
D_k & \xrightarrow{Z_k|_{D_k}} & E_k \\
\downarrow H_k & & \downarrow A_k \\
D'_k & \xrightarrow{Z_k} & \mathbb{C}^2
\end{array}
\]

exist and commute for \( k = 0, 1, \cdots \).

A proper choice for the constants \( r_k, s_k, r'_k, s'_k \) is

\[
\begin{cases} 
    r_k = 2^{-k}, \quad s_k = 2^{-k}s_0, \quad s_0 = 300^{-1}2^{-\tau}, \\
    r'_k = \frac{4}{3}(r_k - s_k), \quad s'_k = \frac{4}{3}s_k, \quad k = 0, 1, \cdots.
\end{cases} \tag{3.6}
\]

Then obviously \( D_k \subseteq D'_k (k = 0, 1, \cdots) \). About the mappings \( Z_k, H_k \) the following relations are needed

\[
Z_k(D_k) \subseteq D_0, \quad \tag{3.7}_k
\]
\[
|Z_k(\zeta) - Z_k(\zeta')| \leq B_k|\zeta - \zeta'|, \quad \zeta, \zeta' \in D'_k, \quad \tag{3.8}_k
\]
\[
|Z_{k+1}(\zeta) - Z_k(\zeta)| \leq \frac{2}{3}qB ks_k, \quad \zeta = (\xi, 0) \in D_{k+1}, \quad \tag{3.9}_k
\]
\[

|H_k - \Omega_k|_{D_k} \leq M_k, \quad \tag{3.10}_k
\]

where \( b_k = 2^{-k\tau}(1 - q)^k, B_k = (1 + q)^k, M_k = 2^{-k(\tau + 1)}M_0, M_0 = \frac{1}{3}q\varepsilon_0s_0 \).

Finally define \( Z_0 = \text{id}, D'_0 \in T(D'_0), \) then by means of this iteration process, if it exists, the assertion of Theorem 2.8 can easily be proved.

In fact, from (3.6), (3.8)_k and (3.9)_k, the sequence \( Z_0, Z_1, \cdots \) converges uniformly on \( \mathbb{R} \times \{0\} \) and the limit \( Z_{\infty}(\xi) = \lim_{k \to \infty} Z_k(\xi, 0) \) is continuous on \( \mathbb{R} \).

Since \( Z_k \in T(D'_k) (k = 0, 1, \cdots) \),

\[
\sigma \circ Z_{\infty} = Z_{\infty} \circ \sigma |_{\mathbb{R}}, \quad \mathbb{Z}_{\infty}(\mathbb{R}) \subseteq \mathbb{R}^2,
\]

where \( \sigma \) is defined by

\[
(x, y) \mapsto \sigma(x, y) = (\bar{x}, \bar{y})
\]

for all \( (x, y) \in \mathbb{C}^2 \) with \( x = a + bi, \bar{x} = a - bi, a, b \in \mathbb{R} \). Now the commutativity of (3.5)_k yields

\[
Z_k \circ H_k = A_k \circ Z_k|_{D_k}.
\]

Hence

\[
A \circ Z_k(\xi, 0) - Z_k(\Omega_k(\xi, 0)) = (A - A_k)(Z_k(\xi, 0)) + Z_k(H_k(\xi, 0)) - Z_k(\Omega_k(\xi, 0))
\]
and by virtue of (3.8) \(_k\), (3.10) \(_k\) consequently
\[
|A \circ Z_k(\xi, 0) - Z_k(\alpha + \xi, 0)| \leq |A - A_k|_{\mathbb{R}^2} + B_k M_k, \quad \xi \in \mathbb{R}.
\]
Passing to the limit we get
\[
A \circ Z_\infty(\xi) = Z_\infty(\alpha + \xi), \quad \xi \in \mathbb{R}
\]
in view of (3.3), (3.4) and \(B_k M_k \to 0\).

Therefore, we can obtain the existence of invariant curves of the mapping \(A\), and from the relation between the mappings \(A\) and \(\mathcal{M}\), one can also get the existence of invariant curves of the mapping \(\mathcal{M}\).

From the above analysis, firstly we need to prove the assertion
\[
\begin{align*}
\{ & \text{The diagram (3.5) exists and commutes} \nonumber \\
& \text{with some } Z_k \in T(D_k), \quad H_k \in T(D_k) \nonumber \\
& \text{satisfying (3.7), (3.8), (3.10)} \nonumber \\
\} \tag{3.11}
\end{align*}
\]
and the estimate (3.9) for \(k = 0, 1, \ldots\).

The proofs of (3.9) \(_k\) and (3.11) \(_k\) \((k = 0, 1, \ldots)\) are done by the complete induction. The case (3.11) \(_0\) is same as the formula (26)\(_0\) in Rüssmann [12]. Now let us suppose that (3.11) \(_k\) is true for some \(k \geq 0\). We have to show (3.11) \(_{k+1}\) and (3.9) \(_k\). On this way the crucial result is the construction of the commuting diagram
\[
\begin{array}{ccc}
D_{k+1} & \xrightarrow{W_k} & D_k \\
\Phi_{k+1} \downarrow & & \downarrow H_k \\
D_{k+1}' & \xrightarrow{W_k} & D_k' \\
\end{array}
\tag{3.12}
\]
with
\[
D_{k+1}' = D(r_{k+1}' - \frac{1}{s} s_{k+1}' - \frac{1}{s} s_k)
\]
and mappings \(W_k \in T(D_{k+1}')\), \(\Phi_{k+1} \in T(D_{k+1})\), and \(W_k - \text{id}, \Phi_{k+1} - \Omega_{k+1}\) are quasi-periodic with the frequency \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)\) in the first variable.

The existence of the commuting diagram (3.12) is guaranteed by the inductive theorem (Theorem 5.1), which we will prove in Section 5. This theorem also gives the following estimates
\[
2^{-7}(1 - q)|\zeta - \zeta'| \leq |W_k(\zeta) - W_k(\zeta')| \leq (1 + q)|\zeta - \zeta'|, \quad \zeta, \zeta' \in D_{k+1}',
\tag{3.13}
\]
\[
|W_k(\zeta) - \zeta| \leq \frac{2}{3} q s_k, \quad \zeta = (\xi, 0) \in D_{k+1},
\tag{3.14}
\]
\[
|\Phi_{k+1} - \Omega_{k+1} - Q_k|_{D_{k+1}} \leq \frac{5}{24} M_{k+1},
\tag{3.15}
\]
where \(Q_k\) is a polynomial of degree 2 in the second variable only
\[
Q_k(\eta) = (0, a_{0k} + a_{1k} \eta + a_{2k} \eta^2), \quad a_{0k}, a_{1k}, a_{2k} \in \mathbb{R}.
\tag{3.16}
\]

With these assertions of the inductive theorem we can show (3.11) \(_{k+1}\) and (3.9) \(_k\). From the diagrams (3.5) \(_k\) and (3.12) we define
\[
Z_{k+1} = Z_k \circ W_k,
\]
of course, $Z_{k+1} \in T(D'_{k+1})$, $Z_{k+1} - \text{id}$ is quasi-periodic with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ in the first variable, and (3.7)$_{k+1}$ holds. The proof of (3.8)$_{k+1}$ by means of (3.8)$_{k}$ and (3.13) is obvious if we notice $W_k(D'_{k+1}) \subseteq D'_k$ in (3.12). The inequality (3.9)$_k$ follows from (3.8)$_k$, (3.14) and $W_k(D_{k+1}) \subseteq D'_k$ as a consequence of $D_{k+1} \subseteq D'_{k+1}$.

Combining the commuting diagrams (3.5)$_k$ and (3.12) yields the commuting diagram

\[
\begin{array}{ccc}
D_{k+1} & \xrightarrow{Z_{k+1}} & E_{k+1} \\
\downarrow & & \downarrow \\
D'_{k+1} & \xrightarrow{Z_{k+1}} & \mathbb{C}^2.
\end{array}
\]

of course we need to prove that $Z_{k+1}(D_{k+1}) \subseteq E_{k+1}$, which is also similar to that in [12], we omit it here.

Comparing the diagrams (3.5)$_{k+1}$ and (3.17), it remains to prove that we can replace $\Phi_{k+1}$ by $H_{k+1}$ if we replace $A_{k|E_{k+1}}$ by $A_{k+1}$. Moreover we have to show (3.10)$_{k+1}$, which is not possible without going back to the original quasi-periodic mapping $A$ in order to use the intersection property, and to estimate the polynomial (3.16) well enough such that (3.10)$_{k+1}$ follows from (3.15).

In the following, we will prove these assertions. First of all we give an useful lemma.

**Lemma 3.1** (Theorem 3 in [12]). Let $D$, $D'$, $E$ be open subsets of $\mathbb{F}^m$ belonging to $\Delta$ with $D \subseteq D'$ and $A'$, $Z$, $\Phi'$ be analytic mappings of class $\Sigma$ such that the diagram

\[
\begin{array}{ccc}
D & \xrightarrow{Z|_D} & E \\
\downarrow & & \downarrow \\
D' - d & \xrightarrow{A'} & \mathbb{F}^m \\
\downarrow & & \downarrow \\
D' & \xrightarrow{id} & \mathbb{F}^m
\end{array}
\]

exists and commutes with some $d > 0$, and the estimate

\[b|\zeta - \zeta'| \leq |Z(\zeta) - Z(\zeta')|\]

holds for all $\zeta, \zeta' \in D'$ with some $b > 0$. Then for any continuous mapping $A'' : E \mapsto \mathbb{F}^m$ of class $\Sigma$ satisfying the estimate

\[|A' - A''|_E \leq bd,
\]

(3.19)
there exists a continuous mapping $\Phi''$ of a class $\Sigma$ such that the diagram
\[
\begin{array}{ccc}
D & \xrightarrow{Z} & E \\
\Phi'' \downarrow & & \downarrow \Phi'' \\
D' & \xrightarrow{z} & E''
\end{array}
\]
exists and commutes, and the estimate
\[
|\Phi' - \Phi''|_D \leq b^{-1} |A' - A''|_E
\]
is valid. If $A''$ is analytic so is $\Phi''$.

We apply Lemma 3.1 to the diagram (3.17) in order to obtain (3.5)$_{k+1}$. Set
\[
A' = A_k|_{E_{k+1}}, \quad A'' = A_{k+1}, \quad \Phi' = \Phi_{k+1}, \quad Z = Z_{k+1}, \quad D = D_{k+1}, \quad D' = D'_{k+1}, \quad E = E_{k+1},
\]
\[
m = 2, \quad F = \mathbb{C}, \quad b = b_{k+1}, \quad d = \frac{1}{T} s_k.
\]
Moreover, $\Delta$ is the set of all subsets of $\mathbb{C}^2$ which are invariant under $\Lambda = \sigma$ such that $\Sigma$ is the class of all functions $F : D \mapsto \mathbb{C}^2$ with $D \in \Delta$ and $\sigma \circ F = F \circ \sigma$. Then $D_{k+1}$, $D'_{k+1}$, $E_{k+1}$ are open sets belonging to $\Delta$, and (3.21) represents analytic functions of class $\Sigma$. For $A'$ and $A''$ this follows from Lemma 2.11, for $\Phi'$ and $Z$ this is true because of $\Phi_{k+1} \in T(D_{k+1})$, $Z_{k+1} \in T(D'_{k+1})$. In addition (3.8)$_{k+1}$ is valid. Therefore Lemma 3.1 can be applied and gives a function $H_{k+1} = \Phi'' \in T(D_{k+1})$, $H_{k+1} - \Omega_{k+1}$ is quasi-periodic with the frequency $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ in the first variable, such that the diagram (3.5)$_{k+1}$ exists and commutes.

We also need to prove $H_{k+1}$ satisfies (3.10)$_{k+1}$. By (2.5), (3.1), (3.3), (3.4), we obtain
\[
|A_{k+1} - A_k|_{E_{k+1}} \leq \frac{1}{3} \varepsilon_0 c_2 \frac{q(1-q)}{600(3c_1 + c_2)} (1 + q)^k p 2^{3 - k \tau - k \tau p}.
\]
The necessary condition which we have to require is
\[
\frac{(1 + q)^p}{1 - q} \leq 2^{p - 1 - 2 \tau},
\]
then we get the estimate
\[
|A_{k+1} - A_k|_{E_{k+1}} \leq \frac{1}{3} \varepsilon_0 c_2 \frac{q(1-q)^{k+1}}{600(18c_1 + 6c_2)^2} 2^{3 - k \tau - 2k \tau}.
\]
By the definitions of $s_0$, $b_{k+1}$, $M_{k+1}$ in (3.6), (3.8)$_{k+1}$ and (3.10)$_{k+1}$, we have
\[
b_{k+1} = 2^{-(k+1)\tau} (1 - q)^{(k+1)}, \quad M_{k+1} = \frac{1}{3} q \varepsilon_0 300^{-1} 2^{-\tau} 2^{-(k+1)(\tau + 1)},
\]

hence
\[
|A_{k+1} - A_k|_{E_{k+1}} \leq \frac{c_2}{18c_1 + 6c_2} b_{k+1} M_{k+1}.
\]
Since $M_{k+1} s_k^{-1} \leq M_0 s_0^{-1} \leq 3^{-1}$, and put $d = \frac{1}{T} s_k$, we get
\[
|A_{k+1} - A_k|_{E_{k+1}} \leq b_{k+1} d.
\]
According to Lemma 3.1, we have
\[
|H_{k+1} - \Phi_{k+1}|_{D_{k+1}} \leq \frac{c_2}{18c_1 + 6c_2} M_{k+1}.
\]
4. Linear difference equations. In this section we will solve the difference equations

\[ u(x + \alpha, y) - u(x, y) = \varepsilon v(x, y) + f(x, y), \]
\[ v(x + \alpha, y) - v(x, y) = g(x, y) - \langle g \rangle, \] (4.1)

which play a central role in the proof of the inductive theorem. Here the mean value of the function \( g(x, y) \) over the variable \( x \) is defined by \( \langle g \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T g(x, y) dx \), and \( \alpha \) is a real number satisfying the Diophantine inequalities

\[
\begin{cases}
(k, \omega) \frac{\alpha}{2\pi} - j \geq \frac{2}{|k|^{\gamma}}, & \text{for all } k \in \mathbb{Z}^n \setminus \{0\}, \ j \in \mathbb{Z}, \\
0 < \gamma < \frac{1}{2}, \ \tau > n.
\end{cases}
\] (4.2)

The functions \( f(\cdot, y), g(\cdot, y) \in Q_\tau(\omega) \) are given holomorphic functions of the complex variables \( x, y \), and \( u, v \) are wanted holomorphic functions of the complex variables \( x, y \). \( \varepsilon \) is a positive constant to be determined in such a way that the functions \( u, v \) will be of the same size.

In order to get estimates for \( u, v \) which are good enough for the proof of Theorem 2.8, some technical preparations have to be made.

**Lemma 4.1** (Lemma 3.3 in [11]). Let \( \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_\ell) \in \mathbb{R}^\ell \) satisfying the inequalities \( D(k, \tilde{\omega}) \geq \psi(|k|) \), where \( D(k, \tilde{\omega}) = \min_{j \in \mathbb{Z}} \left| \langle (k, j), \tilde{\omega} \rangle \right|, k \in \mathbb{Z}^{\ell-1} \setminus \{0\} \), \( \psi \) is an approximation function. Then for \( m = 1, 2, \ldots \), we have

\[
\sum_{k \in \mathbb{Z}^\ell, 0 < |k| \leq m} \frac{1}{|\langle k, \tilde{\omega} \rangle|^2} \leq \pi^2 \frac{3^{\ell+2}}{8} \frac{\psi(m)^2}{\psi(m)},
\]
where \( \bar{k} = (k, j) \).

If we choose
\[
\bar{\omega} = (\omega_1 \frac{\alpha}{2\pi}, \omega_2 \frac{\alpha}{2\pi}, \cdots, \omega_n \frac{\alpha}{2\pi}, -1), \quad \ell = n + 1, \quad \psi(t) = \gamma t^{-\tau},
\]
\( k = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n \setminus \{0\}, \quad j \in \mathbb{Z}, \quad \bar{k} = (k, j) \),
then by Lemma 4.1 and the Diophantine inequalities (4.2), we obtain
\[
\sum_{k \in \mathbb{Z}^n \atop 0 < |k| \leq m} \frac{1}{\left| \langle k, \omega \rangle \frac{\alpha}{2\pi} - j \right|^2} \leq \frac{\pi^2}{8} \frac{n^2 + 3 \gamma^{-2} m^2}{2}.
\]
Meanwhile,
\[
\left| e^{i\langle k, \omega \rangle \alpha} - 1 \right| \geq \pi \left| \langle k, \omega \rangle \frac{\alpha}{2\pi} - j \right|.
\]
Therefore
\[
\sum_{k \in \mathbb{Z}^n \atop 0 < |k| \leq m} \frac{1}{\left| e^{i\langle k, \omega \rangle \alpha} - 1 \right|^2} \leq \frac{3n^2 + 3 \gamma^{-2} m^2}{8} \tag{4.3}
\]

**Lemma 4.2.** For \( r > 0 \), let \( f : \{x \in \mathbb{C} : |\text{Im } x| < r \} \to \mathbb{C} \) be a holomorphic function and \( f \in Q_r(\omega) \). Then we have the estimate
\[
\sum_{k \in \mathbb{Z}^n} |f_k|^2 |e^{2\pi r |k|}| \leq 2^n |f|^2_r,
\]
where \( f_k = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(\theta)e^{-i\langle k, \theta \rangle} d\theta, \quad k \in \mathbb{Z}^n \) are the Fourier coefficients of \( f \), \( F(\theta) \) is the shell function of \( f \) and
\[
|f|^2_r = \sup_{\theta \in \mathbb{H}^n} \left| \sum_k f_k e^{i\langle k, \theta \rangle} \right| = \sup_{\theta \in \mathbb{H}^n} |F(\theta)|.
\]

**Proof.** The Fourier coefficients \( f_k \) of \( f \) are given by
\[
f_k = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(\theta)e^{-i\langle k, \theta \rangle} d\theta, \quad k \in \mathbb{Z}^n
\]
For every \( \lambda \in \mathbb{R}^n \) with \( |\lambda| = \max_{1 \leq j \leq n} |\lambda_j| < r \), the domain of the function \( x \mapsto f(x+i\lambda) \) is \( r - |\lambda| \), its Fourier coefficients are
\[
f_k(\lambda) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(\theta+i\lambda)e^{-i\langle k, \theta \rangle} d\theta, \quad k \in \mathbb{Z}^n.
\]
By Bessel’s inequality,
\[
\sum_{k \in \mathbb{Z}^n} |f_k(\lambda)|^2 \leq \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |F(\theta+i\lambda)|^2 d\theta.
\]
Hence
\[
\sum_{k \in \mathbb{Z}^n} |f_k(\lambda)|^2 \leq |f|^2_r, \quad |\lambda| < r. \tag{4.4}
\]

Define a new function
\[
\lambda \mapsto f_k(\lambda)e^{i\langle k, \lambda \rangle} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(\theta+i\lambda)e^{-i\langle k, \theta + i\lambda \rangle} d\theta,
\]
then
\[
\frac{\partial}{\partial \lambda_j} \left( f_k(\lambda) e^{(k,\lambda)} \right) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} i \frac{\partial}{\partial \theta_j} F(\theta + i\lambda) e^{-i(k,\theta + i\lambda)} d\theta.
\]
Since \( F(\theta + i\lambda) e^{-i(k,\theta + i\lambda)} \) is \( 2\pi \)-periodic in \( \theta_j \) \((j = 1, 2, \cdots, n)\), then
\[
\frac{\partial}{\partial \lambda_j} \left( f_k(\lambda) e^{(k,\lambda)} \right) = 0.
\]
Hence the function \( f_k(\lambda) e^{(k,\lambda)} \) is independent of \( \lambda \), and
\[
f_k(\lambda) e^{(k,\lambda)} = f_k(0) = f_k,
\]
and consequently
\[
|f_k(\lambda)|^2 e^{2(k,\lambda)} = |f_k|^2.
\]
Finally, by (4.4), we have
\[
\sum_{k \in \mathbb{Z}^n} |f_k|^2 e^{2s(k,\lambda)} = \sum_{k \in \mathbb{Z}^n} |f_k(\lambda)|^2 \leq |f|^2, \quad |\lambda| < r. \tag{4.5}
\]
Define \( e_i \in \mathbb{R}^n \) \((i = 1, 2, \cdots, 2^n)\) which have components \( \pm 1 \), and
\[
Z_i = \left\{ k \in \mathbb{Z}^n : \langle k, e_i \rangle = -|k| \right\}.
\]
Then
\[
\bigcup_{i=1}^{2^n} Z_i = \mathbb{Z}^n. \tag{4.6}
\]
Let \( \lambda = se_i \) in (4.5), we obtain
\[
\sum_{k \in Z_i} |f_k|^2 e^{2s|k|} \leq |f|^2, \quad 0 < s < r, \quad i = 1, 2, \cdots, 2^n.
\]
Passing to the limit \( s \to r \) yields
\[
\sum_{k \in Z_i} |f_k|^2 e^{2r|k|} \leq |f|^2 = |f|^2.
\]
Adding these inequalities and by (4.6), we have
\[
\sum_{k \in \mathbb{Z}^n} |f_k|^2 e^{2r|k|} \leq 2^n |f|^2.
\]
The proof of this lemma is completed. \( \square \)

**Definition 4.3.** (i) For \( D = D(r, s) \), denote by \( P^m(r, s) = P^m(D) \) \((m = 1, 2, \cdots)\) the linear space of all holomorphic functions \( f : D \to \mathbb{C}^m \) satisfying
\[
f \circ \sigma |_D = f, \quad f(\mathbb{R}^2) \subseteq \mathbb{R}^m.
\]
Clearly \( f, g \in T(D) \) implies \( f - g \in P^2(D) \).
(ii) For a function \( f \in P^m(D) \), denote its mean value over the variable \( x \) by
\[
[f](y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x, y) dx.
\]
Theorem 4.4. Let $\alpha$ be a real number satisfying (4.2), and $f(\cdot, y) \in Q_r(\omega)$ be a function belonging to $P^1(r,s)$ for some positive constants $r, s$. Then the difference equation
\[
u(x + \alpha, y) - u(x, y) = f(x, y) - [f](y)
\]
has a unique solution $u \in Q(\omega)$, $u \in P^1(r,s)$ with $[u] = 0$. For this solution the estimate
\[
|u|_{r-\rho,s} \leq \varepsilon^{-1}|f|_{r,s}, \quad 0 < \rho < r
\]
holds, where $\varepsilon$ is defined by
\[
\varepsilon = \varepsilon(\rho) = 6 - \frac{n+1}{\Gamma(\tau + 1)} \rho^\tau.
\]

Proof. Since the restriction of $f(x, y)$ onto $\mathbb{R}^2$ is a continuously differentiable and quasi-periodic function in $x$, it can be expanded into its Fourier series
\[
f(x, y) = \sum_{k \in \mathbb{Z}^n} f_k(y) e^{i(k,\omega)x},
\]
where
\[
f_k(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(\theta, y) e^{-i(k,\theta)d\theta}, \quad k = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n
\]
amare the Fourier coefficients of $f(x, y)$. An application of Lemma 4.2 to the restriction of $f(x, y)$ onto $D(r,s)$ yields that
\[
\sum_{k \in \mathbb{Z}^n} |f_k(y)|^2 e^{2r|k|} \leq 2^n |f|_{r,s}^2.
\]
Let
\[
u(x, y) = \sum_k u_k(y) e^{i(k,\omega)x}.
\]
After straightforward calculations we obtain the relation between Fourier coefficients $f_k(y)$ and $u_k(y)$ as follows
\[
u_k(y) = \frac{f_k(y)}{e^{i(k,\omega)\alpha} - 1}, \quad k \neq 0,
\]
then
\[
u(x, y) = \sum_{k \neq 0} \frac{f_k(y)}{e^{i(k,\omega)\alpha} - 1} e^{i(k,\omega)x},
\]
which is the uniquely determined Fourier expansion of the wanted solution $u$ satisfying $u \in Q(\omega)$ with $[u] = 0$.

Firstly, we estimate the sum
\[
g_m(y) = \sum_{1 \leq |k| \leq m} \left| \frac{f_k(y)}{e^{i(k,\omega)\alpha} - 1} \right| e^{|k|r}.
\]
By Cauchy-Schwarz inequality, we have
\[
g_m(y) \leq \left( \sum_{k \in \mathbb{Z}^n} |f_k(y)|^2 e^{2|k|r} \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^n} \left| e^{i(k,\omega)\alpha} - 1 \right|^{-2} \right)^{\frac{1}{2}}.
\]
By (4.3), (4.9), we obtain

\[ \sqrt{\sum_{k \in \mathbb{Z}^n} |f_k(y)|^2 e^{2|k| r}} \leq 2^{\frac{3}{2}} |f|_{r,s}, \]

\[ \sqrt{\sum_{k \in \mathbb{Z}^n} \left| e^{i(k,\omega)\alpha} - 1 \right|^2} \leq \frac{3^{\frac{3}{2}} + 3}{2\sqrt{2}} m^\tau, \]

hence

\[ g_m(y) \leq \frac{3\sqrt{3}}{2\sqrt{2}} 6^{\frac{3}{2}} \frac{m^\tau}{\gamma} |f|_{r,s} \leq 6^{\frac{3}{2}} \frac{m^\tau}{\gamma} |f|_{r,s}. \]

Set \( g_0(y) = 0 \), we get

\[ \sum_{0 < |k| \leq N} \frac{f_k(y)}{e^{i(k,\omega)\alpha} - 1} e^{|k| (r - \rho)} = (1 - e^{-\rho}) \sum_{m=1}^{N} g_m(y)e^{-m\rho} + g_N(y) e^{-(N+1)\rho}. \]

Letting \( N \to \infty \), we have

\[ \sum_{k \neq 0} \frac{f_k(y)}{e^{i(k,\omega)\alpha} - 1} e^{|k| (r - \rho)} \leq (1 - e^{-\rho}) \sum_{m=1}^{\infty} g_m(y)e^{-m\rho} \]

\[ \leq \frac{6^{\frac{3}{2}}}{\gamma} |f|_{r,s} \sum_{m=1}^{\infty} m^\tau (e^{-m\rho} - e^{-(m+1)\rho}). \]

For the last series we get the estimate

\[ \sum_{m=1}^{\infty} m^\tau \left( e^{-m\rho} - e^{-(m+1)\rho} \right) = \rho \sum_{m=1}^{\infty} m^{\tau} \int_{m}^{m+1} e^{-x\rho} dx \]

\[ \leq \rho \int_{1}^{+\infty} x^\tau e^{-x\rho} dx = \rho \int_{1}^{+\infty} x^\tau e^{-x\rho} dx \]

\[ \leq \rho^{-\tau} \int_{1}^{+\infty} (xp)^\tau e^{-x\rho} d(xp) \leq \rho^{-\tau} \int_{0}^{+\infty} t^\tau e^{-t} dt \]

\[ = \rho^{-\tau} \Gamma(\tau + 1). \]

Hence

\[ |u|_{r - \rho, s} \leq \frac{6^{\frac{3}{2}}}{\gamma} \frac{\Gamma(\tau + 1)}{\rho^{-\tau}} |f|_{r,s}, \]

which completes the proof of the lemma. \( \Box \)

Now we are ready to solve equation (4.1).

**Theorem 4.5.** Let \( \alpha \) be a real number satisfying (4.2), and \( f(\cdot, y), g(\cdot, y) \in Q_r(\omega) \) be functions belonging to \( P^1(r, s) \) and satisfying the estimates

\[ |f|_{r,s} \leq M, \quad |g|_{r,s} \leq M \quad (4.10) \]

with some positive constants \( r, s, M \). Then the difference equations (4.1) with \( \varepsilon \) defined in (4.8) have a unique solution \( u, v \in Q(\omega), u, v \in P^1(r, s) \) with \( [u] = 0 \). For this solution the estimates

\[ |u|_{r-2\rho, s} \leq 2\varepsilon^{-1} M, \quad (4.11) \]
are valid for $0 < 2 \rho < r$.

Proof. In the first equation of (4.1) the mean value must vanish on both sides. Hence we get the condition

$$[v] = -\varepsilon^{-1} [f]$$

(4.13)

for the mean value of $v$. As a consequence, we have $[v] \in P^1(r,s)$ and

$$|[v]|_{r,s} \leq \varepsilon^{-1} M$$

(4.14)

in view of (4.10). Theorem 4.4 gives a unique solution $v = \tilde{v} \in P^1(r,s)$ of the second equation of (4.1) with $[\tilde{v}] = 0$. This solution has the estimate

$$|\tilde{v}|_{r-\rho,s} \leq \varepsilon^{-1} M$$

(4.15)

because of (4.10). Define $v = \tilde{v} + [v]$, we obtain the uniquely determined solution $v \in P^1(r,s)$ of the second equation of (4.1) satisfying (4.13). This solution has the estimate (4.12) as a consequence of (4.14) and (4.15).

Define $h = \varepsilon \tilde{v} + f$, note that $\tilde{v}$ is defined in $D(r-\rho,s)$, then $h$ is well defined in $D(r-\rho,s)$. As a consequence we have

$$|h|_{r-\rho,s} = |\varepsilon \tilde{v} + f|_{r-\rho,s} \leq 2M$$

(4.16)

and

$$h(x,y) - [h](y) = \varepsilon \tilde{v} + f - [\varepsilon \tilde{v} + f]$$

$$= \varepsilon \tilde{v} + f - \varepsilon \tilde{v} - [f]$$

$$= \varepsilon \tilde{v} + f + \varepsilon [v]$$

$$= \varepsilon v + f.$$

Hence, the first equation of (4.1) can be rewritten in the form

$$u(x + \alpha,y) - u(x,y) = h(x,y) - [h](y).$$

(4.17)

Thus Theorem 4.4 gives a uniquely determined solution $u \in P^1(r,s)$ of (4.17) with $[u] = 0$. For an estimate of $u$ we apply Theorem 4.4 to (4.17) restricted to $D(r-\rho,s)$ such that in (4.7) we have to replace $f$ by $h$ and $r$ by $r-\rho$. Then (4.11) follows by means of (4.16). The proof is finished. □

5. The inductive theorem. First of all we give together constants, domains, and mappings appearing in the formulation of the inductive theorem.

(I) Constants and their relations

We introduce the constants

$$\omega, \gamma, \tau, M, q, \varepsilon, \varepsilon_+, r, r_+, s, s_+, s', s'_{+}, s_{+}$$

(5.1)

and the auxiliary constants $\theta, \rho$ satisfying the relations

$$|\langle k, \omega \rangle\frac{\alpha}{2\pi} - j| \geq \frac{\gamma}{|k|^\tau}, \quad \text{for all} \quad k \in \mathbb{Z}^n \setminus \{0\}, \quad j \in \mathbb{Z},$$

$$0 < \gamma < \frac{1}{2}, \quad n \leq \tau, \quad 0 < r \leq 1,$$

$$0 < q \leq \left(\frac{\theta}{10}\right)^2, \quad s = \frac{\theta \rho}{50}, \quad \theta = 2^{-\tau}, \quad \rho = \frac{r}{6},$$

$$0 < \theta \rho \leq 1.$$
\[
\varepsilon = 6^{-\frac{q+s}{q+s+1}} \rho^\gamma, \quad M = \frac{1}{3} q \varepsilon s,
\]
\[
r' = \frac{2}{3} (r - s), \quad s' = \frac{4}{3} s,
\]
\[
r' = \frac{2}{3} s = \frac{s'}{r'} = s' = 1/2, \quad \varepsilon = \theta.
\]

(II) Domains and Mappings

Choose
\[
D = D(r, s), \quad D_+ = D(r_+, s_+), \quad D' = D(r', s'), \quad D'_+ = D(r'_+, s'_+),
\]
\[
D'_+ = D(r'_+ - \frac{1}{7} s, s'_+ - \frac{1}{7} s) = D'_+ - \frac{1}{7} s,
\]
and introduce the mappings
\[
\Omega(x, y) : \quad x_1 = x + \alpha + \varepsilon y, \quad y_1 = y.
\]  \tag{5.2}
\[
\Omega_+(x, y) : \quad x_1 = x + \alpha + \varepsilon_+ y, \quad y_1 = y.
\]  \tag{5.3}
\[
\Theta(x, y) : \quad x_1 = x, \quad y_1 = \theta y.
\]  \tag{5.4}

for all \((x, y) \in \mathbb{C}^2\), where we use the same symbol for the mappings \(\Omega, \Omega_+, \Theta\) as well as for their restrictions to subsets of \(\mathbb{C}^2\).

**Theorem 5.1** (Inductive Theorem). Let constants (5.1) and auxiliary constants \(\theta, \rho\) be given such that the relations in (I) are satisfied, and let domains \(D, D_+, D', D'_+\) and mappings \(\Omega, \Omega_+, \Theta\) be given as in (II). Then for any mapping
\[
H : D \rightarrow D', \quad H \in T(D),
\]
\(H - \Omega\) is quasi-periodic with the frequency \(\omega\) in the first variable and satisfying
\[
|H - \Omega|_D \leq M,
\]  \tag{5.5}
there are mappings \(W \in T(D'_+), \Phi_+ \in T(D_+)\) such that the diagram

\[
\begin{array}{ccc}
D_+ & \xrightarrow{W} & D \\
\downarrow{\Phi_+} & & \downarrow{H} \\
D'_+ & \xrightarrow{\text{id}} & D'
\end{array}
\]
exists and commutes. Moreover \(W - \Theta, \Phi_+ - \Omega_+ - Q\) are quasi-periodic with the frequency \(\omega\) in the first variable and the following estimates are satisfied
\[
|W - \Theta|_{D_+} \leq \frac{2}{3} q s,
\]  \tag{5.6}
\[
\theta(1 - q) |\zeta - \zeta'| \leq |W(\zeta) - W(\zeta')| \leq (1 + q) |\zeta - \zeta'|, \quad \zeta, \zeta' \in D'_+, \tag{5.7}
\]
\[
|\Phi_+ - \Omega_+ - Q|_{D_+} \leq \frac{5}{48} \theta M, \tag{5.8}
\]
where \(Q : D_+ \rightarrow \mathbb{C}^2\) is defined by
\[
(\xi, \eta) \mapsto Q(\eta) = (0, a_0 + a_1 \eta + a_2 \eta^2)
\]
with some constants \(a_0, a_1, a_2 \in \mathbb{R}\).
Proof. First of all, define
\[ h = \begin{pmatrix} f \\ g \end{pmatrix} := H - \Omega, \]
and by assumption, \( H - \Omega \) is quasi-periodic with the frequency \( \omega \) in the first variable and \( h \in P^2(D) \).

The results of section 4 enable us to solve the linear difference equations
\[
\begin{aligned}
&u(x + \alpha, y) - u(x, y) = \varepsilon v(x, y) + f(x, \theta y), \\
v(x + \alpha, y) - v(x, y) = g(x, \theta y) - [g](\theta y).
\end{aligned}
\tag{5.9}
\]

Let \( d\Omega \) be the differential of \( \Omega \) and define
\[
w = \begin{pmatrix} u \\ v \end{pmatrix},
\]
\[
\Omega^*(x, y) : x_1 = x + \alpha, \quad y_1 = y,
\]
\[
h^*(x, y) : x_1 = 0, \quad y_1 = [g](y).
\]

With these definitions and notations, we have
\[
w \circ \Omega^* = \begin{pmatrix} u(x + \alpha, y) \\ v(x + \alpha, y) \end{pmatrix},
\]
\[
(d\Omega)w = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} u(x, y) + \varepsilon v(x, y) \\ v(x, y) \end{pmatrix},
\]
\[
h \circ \Theta = \begin{pmatrix} f \\ g \end{pmatrix} \circ \Theta = \begin{pmatrix} f(x, \theta y) \\ g(x, \theta y) \end{pmatrix},
\]
\[
h^* \circ \Theta = \begin{pmatrix} 0 \\ [g] \end{pmatrix} \circ \Theta = \begin{pmatrix} 0 \\ [g](\theta y) \end{pmatrix}.
\]

Hence, the difference equations (5.9) can be written in the more compact form
\[
w \circ \Omega^* = (d\Omega)w + h \circ \Theta - h^* \circ \Theta. \tag{5.10}
\]

Define
\[
w = \begin{pmatrix} u \\ v \end{pmatrix} := W - \Theta, \quad \phi := \Phi_+ - \Omega_+,
\]
we ought to show \( w \in P^2(D) \), \( \phi \in P^2(D_+) \).

After having obtained a solution \( w \in P^2(D_+) \) of (5.10), we try to determine \( \phi \) from the equation
\[
H \circ W|_{D_+} = W \circ \Phi_+, \tag{5.11}
\]
which holds because the diagram in the inductive theorem can commute.

Now (5.11) can be rewritten in the form
\[
(h + \Omega) \circ (\Theta + w) = (\Theta + w) \circ (\phi + \Omega_+), \tag{5.12}
\]
which is
\[
h \circ (\Theta + w) + \Omega \circ (\Theta + w) = w \circ (\phi + \Omega_+) + \Theta \circ (\phi + \Omega_+).
\]

First,
\[ \Omega \circ (\Theta + w) = \left( x + u(x, y) + \alpha + \varepsilon(\theta y + v) \right) \]

\[ = \left( x + \alpha + \varepsilon\theta y \right) + \left( u + \alpha + \varepsilon v \right) - \left( \alpha \right) \]

\[ = \Omega \circ \Theta + \Omega \circ w - \left( \alpha \right), \]

Since \( \frac{\varepsilon}{\varepsilon} = \theta \), then

\[ \Theta \circ (\phi + \Omega_+) = \Theta \circ \phi + \Theta \circ \Omega_+. \]

Thus (5.11) is changed into the form

\[ \Theta \circ \phi = h \circ (\Theta + w) - w \circ (\phi + \Omega_+) + (d\Omega)w. \]

If we define

\[ \phi = \Theta^{-1}(z + h^* \circ \Theta), \]

which leads to

\[ \Theta \circ \phi = z + h^* \circ \Theta, \]

then

\[ z + h^* \circ \Theta = h \circ (\Theta + w) - w \circ (\phi + \Omega_+) + (d\Omega)w. \]

Now (5.11) gets by (5.10) the form

\[ \begin{cases} 
F_1 = w \circ \Omega_+ - w \circ (\Omega_+ + \phi), \\
F_2 = w \circ \Omega^* - w \circ \Omega_+, \\
F_3 = h \circ (\Theta + w) - h \circ \Theta, \\
z = F(z) := F_1 + F_2 + F_3.
\end{cases} \]

Careful estimates will lead to a solution \( z \in P^2(D') \) and \( z \) is quasi-periodic with the frequency \( \omega \) in the first variable, which implies that \( \phi \) is quasi-periodic with the frequency \( \omega \) in the first variable and \( \phi \in P^2(D') \) can be determined. Indeed, one can prove that \( F \) is a contraction, there is a fixed point \( F(z) = z \), which leads to the existence of a mapping \( \Phi_+ \) such that the diagram in the inductive theorem exists and commutes. The remaining proofs of the inductive theorem (Theorem 5.1) are similar to that in [12]. For a rigorous remaining proofs of the inductive theorem the reader is referred to Rüssmann [12], we omit it here. \( \square \)
Remark 5.2. Letting
\[ \varepsilon = \varepsilon_k, \quad r = r_k, \quad r' = r'_k, \quad s = s_k, \quad s' = s'_k, \quad M = M_k, \]
\[ D = D_k, \quad D' = D'_k, \quad W = W_k, \quad H = H_k, \]
and replacing the index + by \( k + 1 \), Theorem 5.1 confirms what we have asserted in section 3 concerning the construction of the commuting diagram (3.12) observing (3.13), (3.14), (3.15) and (3.16).

6. Appendix. In this appendix we give the detail proof of Lemma 2.11 which has been used in the previous section. For this purpose we need a well known and fundamental approximation result.

Lemma 6.1 (Lemma 2.1 in [1]). Let \( f \in C^p(\mathbb{R}^\ell) \) for some \( p > 0 \) with finite \( C^p \) norm over \( \mathbb{R}^\ell \). Let \( \phi \) be a radial-symmetric, \( C^\infty \) function, having as support the closure of the unit ball centered at the origin, also \( \phi \) is completely flat and takes value 1, let \( K = \hat{\phi} \) be its Fourier transform and for all \( \delta > 0 \) define
\[ f_\delta(x) := K_\delta * f(x) = \delta^{-\ell} \int_{\mathbb{R}^\ell} K\left(\frac{x - y}{\delta}\right) f(y) dy. \]
Then there exists a constant \( c \geq 1 \) depending only on \( p \) and \( \ell \) such that for any \( \delta > 0 \), the function \( f_\delta(x) \) is real analytic on \( \mathbb{C}^\ell \), and for all \( \beta \in \mathbb{N}^\ell \) with \( |\beta| \leq p \), one has
\[ \sup_{x \in \mathbb{C}^\ell} \left| \partial^\beta f_\delta(x) - \sum_{|\lambda| \leq p-|\beta|} \frac{\partial^{\lambda+\beta} f(\Re x)}{\lambda!} (\Im x)^\lambda \right| \leq c \|f\|_p \delta^{p-|\beta|}, \]
and, for all \( 0 < \delta < \delta' \),
\[ \sup_{x \in \mathbb{C}^\ell} \left| \partial^\beta f_{\delta'} - \partial^\beta f_{\delta} \right| \leq c \|f\|_p \delta^{p-|\beta|}. \]
Moreover, the Hölder norms of \( f_\delta \) satisfy, for all \( 0 \leq s \leq p \leq r \),
\[ \|f_\delta - f\|_s \leq c \|f\|_p \delta^{p-s}, \quad \|f_\delta\|_r \leq c \|f\|_p \delta^{p-r}. \]
The function \( f_\delta \) preserves periodicity, this is, if \( f \) is \( T \)-periodic in any of its variable \( x_j \), so is \( f_\delta \).

Now we are in a position to prove Lemma 2.11.

Proof of Lemma 2.11. Since the function \( h(x, y) \) is quasi-periodic in \( x \) with the frequency \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \), from Definition 2.1, there exists the corresponding shell function
\[ F(\theta, y) := F(\theta_1, \theta_2, \ldots, \theta_n, y), \quad \theta = (\theta_1, \theta_2, \ldots, \theta_n), \]
which is \( 2\pi \)-periodic in each \( \theta_j \), such that \( h(x, y) = F(\omega_1 x, \omega_2 x, \ldots, \omega_n x, y). \)
From the assumptions of Lemma 2.11, \( h \in C^p(\mathbb{R}^2) \), then \( F \in C^p(\mathbb{R}^{n+1}) \), and \( \|F\|_p \) is equivalent to \( \|h\|_p \). In fact, if \( p \geq 0 \) is an integer, then
\[ \|F\|_p = \sum_{|\beta| \leq p} \sup_{(\theta, y) \in \mathbb{R}^{n+1}} \left| \partial^\beta F(\theta, y) \right| = \sum_{|\beta| \leq p} \sup_{(\theta, y) \in \mathbb{R}^{n+1}} \frac{\partial^\beta F(\theta_1, \ldots, \theta_n, y)}{\partial \theta_1^{\beta_1} \cdots \partial \theta_n^{\beta_n} \partial y^{\beta_{n+1}}} \]
and for any \(0 < \delta < \delta'\),
\[
    \sup_{z \in \Pi_n} \left| \partial^\beta F_\delta(z) - \sum_{|\lambda| \leq p-|\beta|} \frac{\partial^{\lambda+\beta} F(\text{Re } z)}{\lambda!} (i \text{Im } z)^\lambda \right| \leq c_3 \| F \|_p \delta^{p-|\beta|},
\]
and for any \(0 < \delta < \delta'\),
\[
    \sup_{z \in \Pi_n} \left| \partial^\beta F_\delta - \partial^\beta F_\delta \right| \leq c_3 \| F \|_p \delta^{p-|\beta|},
\]
where \(c_3 \geq 1\) is a positive constant depending only on \(p\) and \(n\).

Specially, if \(\beta = 0\), one has
\[
    \sup_{z \in \Pi_n} \left| F_\delta(z) - \sum_{|\lambda| \leq p} \frac{\partial^\lambda F(\text{Re } z)}{\lambda!} (i \text{Im } z)^\lambda \right| \leq c_3 \| F \|_p \delta^p,
\]
\[
    \sup_{z \in \Pi_n} \left| F_\delta - F_\delta \right| \leq c_3 \| F \|_p \delta^p,
\]
where \(0 < \delta < \delta'\). Hence,
\[
    \sup_{(\theta, y) \in \Pi_n} \left| F_\delta(\theta, y) \right| \leq \sup_{z \in \Pi_n} \left| \sum_{|\lambda| \leq p} \frac{\partial^\lambda F(\text{Re } z)}{\lambda!} (i \text{Im } z)^\lambda \right| + c_3 \| F \|_p \delta^p
\]
\[
    \leq c_4 \| F \|_p + c_3 \| F \|_p \delta^p,
\]
where \(c_4\) is a positive constant depending only on \(p, n\).

Now we define the analytic approximation of \(h(x, y)\) in \(E_\delta\) as follows
\[
h_\delta(x, y) = F_\delta(\omega_1 x, \omega_2 x, \ldots, \omega_n x, y).
\]
Thus
\[
    \left| h_\delta \right|_{E_\delta} := \sup_{(\theta, y) \in \Pi_n} \left| F_\delta(\theta, y) \right| \leq c_4 \| F \|_p + c_3 \| F \|_p \delta^p.
\]
By (6.1), (6.2), we have
\[ |h_\delta|_{E_\delta} \leq c_4 \|F\|_p + c_3 \|F\|_p \delta^p \leq c_2 \|h\|_p. \]
Similarly, for any \(0 < \delta < \delta'\), one can obtain
\[ |h_{\delta'} - h_\delta|_{E_\delta} \leq \sup_{z \in \Pi_{n+1}} |F_{\delta'} - F_\delta| \leq c_2 \|h\|_p \delta'^p, \]
where \(c_2 > 0\) is a constant depending only on \(p, n, \omega\). Hence,
\[ |h_{\delta'} - h_\delta|_{E_\delta} \leq c_2 \|h\|_p \delta'^p. \]
The proof of this lemma is completed. \(\square\)

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