EXISTENCE OF REGULAR SOLUTIONS TO THE FULL LIQUID CRYSTAL SYSTEM

MIMI DAI

Abstract. We study the general Ericksen-Leslie system with non-constant density, which describes the flow of nematic liquid crystal. In particular the model investigated here is associated with Parodi’s relation. We prove that: in two dimension, the solutions are globally regular with general data; in three dimension, the solutions are globally regular with small initial data, or for short time with large data. Moreover, a weak-strong type of uniqueness result is obtained.

Keywords: liquid crystals, Parodi’s relation, regularity.

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1. Introduction

The flows of nematic liquid crystals can be treated as slow moving particles where the fluid velocity and the alignment of the particles influence each other. The hydrodynamic theory of liquid crystals was established by Ericksen [13, 14] and Leslie [28, 29] in the 1960’s. As Leslie points out in his 1968 paper: “liquid crystals are states of matter which are capable of flow, and in which the molecular arrangements give rise to a preferred direction”. The full Ericksen-Leslie system consists of the following equations (cf. [13, 28, 29, 32]):

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho \dot{u} &= \rho F + \nabla \cdot \hat{\sigma}, \\
\rho_1 \dot{\omega} &= \rho_1 G + \hat{\theta} + \nabla \cdot \pi,
\end{align*}
\]

in \(\Omega \times (0, T)\), where \(\Omega\) is a domain in \(\mathbb{R}^n\) with \(n = 2, 3\). The three equations in system (1.1) describe the conservation of mass, linear momentum and angular momentum, respectively. The anisotropic feature of liquid crystal materials is exhibited in the third equation and the nonlinear coupling is represented in second equation. In the above equations, \(\rho : \Omega \times [0, T] \to \mathbb{R}\) is the fluid density, \(u : \Omega \times [0, T] \to \mathbb{R}^n\) is the fluid velocity, \(d : \Omega \times [0, T] \to \mathbb{R}^n\) is the director field representing the alignment of the molecules, \(\rho_1\) is the inertial constant, \(\hat{\theta}\) is the intrinsic force associated with \(d\), \(\pi\) is the director stress, \(F\) and \(G\) are external body force and external director body force, respectively. In this paper we consider the incompressible flow with \(\nabla \cdot u = 0\). The superposed dot denotes the material derivative \(\partial_t + u \cdot \nabla\). The notations

\[
\begin{align*}
A &= \frac{1}{2} (\nabla u + \nabla^T u), & \Omega &= \frac{1}{2} (\nabla u - \nabla^T u), \\
\omega &= \dot{d} = d_t + (u \cdot \nabla)d, & N &= \omega - \Omega d,
\end{align*}
\]

represent the rate of the strain tensor, the skew-symmetric part of the strain rate, the material derivative of \(d\) and the rigid rotation part of director changing rate by fluid vorticity, respectively.
We have the following constitutive relations for $\hat{\sigma}$, $\pi$ and $\hat{g}$ in (1.1):

$$\hat{\sigma}_{ij} = -P \delta_{ij} - \frac{\partial (\rho W)}{\partial d_{k,i}} d_{k,j} + \sigma_{ij},$$

$$\pi_{ij} = \beta_i d_j + \frac{\partial (\rho W)}{\partial d_{j,i}},$$

$$\hat{g}_i = \gamma d_i - \beta_j d_{i,j} - \frac{\partial (\rho W)}{\partial d_i} + g_i.$$  
(1.2)

Here the scalar function $P$ represents the pressure. The vector $\beta = (\beta_1, \beta_2, \beta_3)^T$ and the scalar function $\gamma$ are Lagrangian multipliers for the constraint $|d| = 1$. The term $\rho W$ denotes the Oseen-Frank energy functional for the equilibrium configuration of a unit director field. For simplicity, we consider the relaxation form of the elastic energy associated with $d$:

$$\rho W = \frac{1}{2} |\nabla d|^2 + \frac{1}{4\eta^2} (|d|^2 - 1)^2$$  
(1.3)

with constant $\eta > 0$. And

$$g_i = \lambda_1 N_i + \lambda_2 d_j A_{ji},$$

$$\sigma_{ij} = \mu_1 d_k A_{k,i} d_{k,j} + \mu_2 N_i d_j + \mu_3 d_i N_j$$

$$+ \mu_4 A_{i,j} + \mu_5 d_k A_{i,k} d_j + \mu_6 d_i A_{j,k} d_k,$$

$$N_i = \omega_i + \Omega_i d_k.$$  
(1.4)

In the above equations, the constants $\lambda_1, \lambda_2$ represent the molecular shape and $\mu_1, \ldots, \mu_6$ are Leslie coefficients which regard certain local correlations in the fluid (cf. [12]). They satisfy (cf. [28])

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6,$$

$$\mu_5 - \mu_6 = -(\mu_2 + \mu_3).$$  
(1.5)

The relations in (1.5) arise from the second law of thermodynamics. The relation (1.6) is called Parodi’s condition, which is derived from the Onsager’s reciprocal relation.

To further simplify the model, we take $\rho_1 = 0$, $\beta = 0$, $\gamma = 0$, and $F = G = 0$. Thus, the incompressible Ericksen-Leslie system (1.1) is reformulated as

$$\rho_t + (u \cdot \nabla) \rho = 0,$$

$$\rho u_t + \rho (u \cdot \nabla) u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d) + \nabla \cdot \sigma,$$

$$d_t + (u \cdot \nabla) d - \Omega d + \frac{\lambda_2}{\lambda_1} Ad = -\frac{1}{\lambda_1} (\Delta d - f(d)),$$

$$\nabla \cdot u = 0,$$

(1.7) (1.8) (1.9) (1.10)

with

$$\rho F(d) = \frac{1}{4\eta^2} (|d|^2 - 1)^2, \quad f(d) = \nabla_d (\rho F(d)) = \frac{1}{\eta^2} (|d|^2 - 1) d.$$  

The force term $\nabla d \otimes \nabla d$ in the equation of the conservation of momentum denotes the $3 \times 3$ matrix whose $ij$-th entry is given by “$\nabla_i d \cdot \nabla_j d$” for $1 \leq i, j \leq 3$.

There is a vast literature on the hydrodynamics of the liquid crystal system. For background we list a few names, with no intention to be complete: [3, 4, 5, 6, 7, 8, 11, 16, 20, 21, 22, 30, 31, 32, 36, 39, 43]. Particularly, in [43], the general Ericksen-Leslie system with constant density is studied. In the work, the global regularity and long-time behavior
of solutions are obtained with the assumption that the viscosity coefficient is sufficiently large (3D). With the Parodi’s relation, the authors established the global well-posedness and Lyapunov stability near local energy minimizers. The authors also discussed the connection between Parodi’s relation and the linear stability. On the other hand, considering the density is not constant, a relatively full model for the dynamic of Smectic-A liquid crystals is studied in [35]. In this model, several terms in \( \sigma \) are assumed to be zero, and the term \( -\Omega d + \frac{\lambda_2}{\lambda_1} Ad \) does not appear in equation (1.9). The author proved the existence of global classical solutions in both two and three dimensional cases. In 2D, no additional assumption is needed; while in 3D, the flow viscosity coefficient is assumed to be sufficiently large. Moreover, a “weak-strong" type of uniqueness result, a long-time behavior and stability of solutions are obtained. Also, with a non-constant density, a simplified Ericksen-Leslie model is studied in [11]. The authors indicated that a regular solution exists globally in 2D with general data, while exists globally in 3D with small initial data or for short time with general data.

In the present paper the consideration is given to the full Ericksen-Leslie model (1.7)-(1.10) with non-constant density and under the Parodi’s relation (1.6). We establish that: in 2D, there exists a global regular solution to the system with general data (no extra condition); in 3D, there exists a global regular solution with small initial data or a local (short time) regular solution with general data. We also show that a “weak-strong" type of uniqueness result holds with certain assumption on the weak solution. Namely, if there exist a regular solutions satisfying a higher order energy estimate and a weak solution satisfying the basic energy estimate and two auxiliary estimates, they must be identical.

For the simplified liquid crystal model with constant density (cf. [30]) or non-constant density (cf. [11]), and for the general Ericksen-Leslie system (1.7)-(1.10) with the artificial assumption \( \lambda_2 = 0 \) (cf. [32]), the transport equation of \( d \) satisfies a certain type of maximum principle. In the present paper, for the general model (1.7)-(1.10) with \( \lambda_2 \neq 0 \), the stretching effect causes the loss of maximum principle for \( d \) (cf. [43]). However, in the analysis of sequel, the estimate

\[
d \in L^{\infty}(0, T; L^{\infty})
\]

turns out to be essential in the derivation of higher-order energy estimates and hence to assure that the stress term \( \nabla \cdot \sigma \) can be handled successfully. Therefore, as in [43], we consider the periodic boundary conditions which help to avoid the difficulties from boundary terms when deriving the higher order energy estimates. As such, we restrict ourselves to the following boundary conditions:

\[
\begin{align*}
    u(x + e_i, t) &= u(x, t), & d(x + e_i, t) &= d(x, t), & \text{for } (x, t) \in \partial Q \times \mathbb{R}^+ \\
    \rho(x, 0) &= \rho_0(x), & 0 < M_1 \leq \rho_0(x) \leq M_2, \\
    u(x, 0) &= u_0(x), & \nabla \cdot u_0 &= 0, & \text{and } d(x, 0) &= d_0(x).
\end{align*}
\]

In the rest of the introduction we describe our main results:

**Theorem 1.1.** (2D) Suppose that \( Q \subset \mathbb{R}^2 \) is a unit square. Let \( \rho_0, u_0 \) and \( d_0 \) satisfy (1.12) and (1.13). Suppose that \( \rho_0 \in C^1 \), \( u_0 \in H^2_p \) and \( d_0 \in H^3_p \). Then, system (1.7)-(1.10) has
a global classical solution \((\rho, u, d)\), that is, for all \(T > 0\) and some \(\alpha \in (0, 1)\)

\[
\begin{align*}
  u &\in C^{1+\alpha/2, 2+\alpha}((0, T) \times Q), \\
  \nabla p &\in C^{\alpha/2, \alpha}((0, T) \times Q), \\
  d &\in C^{1+\alpha/2, 2+\alpha}((0, T) \times Q), \\
  \rho &\in C^1((0, T) \times Q).
\end{align*}
\]

(1.14)

The notations \(H^2_p\) and \(H^3_p\) will be introduced in Section 2.

Provided we have sufficiently small data or we work with sufficiently short time, we also obtain the regularity in three dimensional case.

**Theorem 1.2.** (3D) Suppose that \(Q \subset \mathbb{R}^3\) is a unit cube. Let \(\rho_0, u_0\) and \(d_0\) satisfy (1.12) and (1.13). Assume that \(\rho_0 \in C^1, u_0 \in H^2_p\) and \(d_0 \in H^3_p\). Then

1. There is a positive small number \(\epsilon_0\) such that if

\[
\rho_0 \|u_0\|_{H^1_p}^2 + \|d_0\|_{H^1_p}^2 + \|\Delta d_0 - f(d_0)\|_{L^2}^2 \leq \epsilon_0,
\]

then system (1.7)-(1.10) has a classical solution \((\rho, u, d)\) in the time period \((0, T)\), for all \(T > 0\). That is, (1.14) holds for some \(\alpha \in (0, 1)\).

2. For general data, there exists a positive number \(\delta_0 = \delta_0(\rho_0, u_0, d_0)\) such that (1.14) holds in the interval \((0, T)\) for some small \(T \leq \delta_0\).

**Remark 1.3.** It is pointed out in [30, 44] that the large viscosity assumption is not equivalent to the small initial data assumption for the Ericksen-Leslie system (1.7)-(1.10), due to its much more complicated structure. Thus it is particularly interesting to investigate the regularity of solutions to the Ericksen-Leslie system under the small initial data assumption.

**Remark 1.4.** In contrast to the initial condition in [11] (Theorem 1.3) where \((u_0, \nabla d_0) \in H^1 \times H^1\), we require a higher order condition on the initial data here, that is \((u_0, \nabla d_0) \in H^2_p \times H^3_p\). The reason is that, in the full system, the term \(\nabla \cdot \sigma\) in (1.3) contains the high order term \(\nabla \cdot (N d)\) which is not in the simplified system in [11]. To deal with the high order term \(\nabla \cdot (N d)\), we need a higher order energy estimate compared to the case of the simplified system. But to obtain the higher order energy estimate, the smallness assumption on \((u_0, \nabla d_0)\) in \(H^1_p \times H^1_p\) is sufficient.

The basic idea of the regularity proof is to get some high order energy estimates (Ladyzhenskaya method) as described in [30, 11]. To fully utilize the smallness assumption (1.15), we follow the principle in [11] by keeping the potentially small terms \(\|u\|_{L^2}\) and \(\|\nabla d\|_{L^2}\) instead of throwing them away. However, we point out that the situation for the full system in the present work is much more complicated than that of the simplified system in [11], because the velocity field equation (1.8) and director field equation (1.9) are coupled not only through \(\nabla \cdot (\nabla d \otimes \nabla d)\) but also through \(\nabla \cdot \sigma\) and \(-\Omega d + \frac{\Lambda}{\mu} Ad\), while \(\sigma\) consists of six non-zero terms.

In the following, we state the uniqueness of solutions.

**Theorem 1.5.** Let \((\rho, u, d)\) be a solution to system (1.7)-(1.10) and (1.11)-(1.13) obtained in Theorem 1.1 for two dimensional case or in Theorem 1.2 for three dimensional case. Let \((\bar{\rho}, \bar{u}, \bar{d})\) be a weak solution to system (1.7)-(1.10) with (1.11)-(1.13) satisfying the
following energy inequalities:

\begin{align*}
\int_Q |\bar{\rho}|^2 &\leq \int_Q |\rho_0|^2, \\
\int_Q \frac{1}{2} |\bar{\rho}|^2 + \frac{1}{2} |\nabla \bar{d}|^2 + \rho F(d) &\leq \int_Q \mu_4 |\nabla \bar{u}|^2 - \frac{1}{\lambda_1} |\Delta \bar{d} - f(\bar{d})|^2 \, dx dt \\
&\leq \int_Q \frac{1}{2} |\rho_0|^2 + \frac{1}{2} |\nabla d_0|^2 + \rho_0 F(d_0) \, dx.
\end{align*}

In addition, |\bar{d}| is bounded. Then (\rho, u, d) \equiv (\bar{\rho}, \bar{u}, \bar{d}).

The uniqueness result is achieved by a standard approach where one establishes a Gronwall’s inequality for the difference of the two solutions. A uniqueness proof for the simplified system was presented in [11]. For the full system the process will be similar. Since the computation work is huge and tedious, we give the proof of Theorem 1.5 in Appendix (Section 4).

The rest of the paper is organized as: in Section 2 we introduce some notations that shall be used throughout the paper and the basic energy law governing the full system (1.7)-(1.10); in Section 3 we prove Theorems 1.1 and 1.2 by several steps; in Appendix we devote to proving Theorem 1.5.

2. Preliminary

2.1. Notations. We adopt the standard functional settings and notations for periodic problems (cf. [42]) in the following:

\[
H^m_p(Q) = \{ u \in H^m(\mathbb{R}^n, \mathbb{R}^n) | u(x + e_i) = u(x) \} \\
\dot{H}^m_p = H^m_p(Q) \cap \left\{ \int_Q u(x) \, dx = 0 \right\} \\
H = \{ u \in L^2_p(Q) | \nabla \cdot u = 0 \} \quad \text{with} \quad L^2_p(Q) = H^0_p(Q) \\
V = \{ u \in \dot{H}^1_p(Q) | \nabla \cdot u = 0 \} \\
V' = \text{the dual space of } V.
\]

At certain places in the paper, we also denote the space of scalar functions by \( H^m_p(Q) \).

The inner product on \( L^2_p(Q) \) and \( H \) is denoted by \( \langle \cdot, \cdot \rangle \) and the associated norm by \( \| \cdot \| \). For simplicity, the space \( H^m_p(Q) \) is denoted by \( H^m_p \). The inner product on \( H^m_p \) is defined as \( \langle u, v \rangle_m = \Sigma_{|k|=0}^m (D^k u, D^k v) \) with \( k = (k_1, \ldots, k_n) \) being the multi-index of length \( |k| = \Sigma_{i=1}^n k_i \) and \( D^k = \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \).

2.2. Definition of weak solution. The weak formulation of the problem is given as follows.

**Definition 2.1.** The triplet \((\rho, u, d)\) is called a weak solution to the system (1.7)-(1.10) in \( Q_T = Q \times (0, T) \) subject to the boundary and initial conditions (1.11)-(1.13), if it satisfies

\[
0 < M_1 \leq \rho \leq M_2 \\
u \in L^\infty(0, T; H) \cap L^2(0, T; V) \\
d \in L^\infty(0, T; H^1_p \cap L^\infty_p) \cap L^2(0, T; H^2_p)
\]
and moreover, for any smooth function \( \psi(t) \) with \( \psi(T) = 0 \) and \( \phi(x) \in H^1_p \), the following integral equations hold:

\[
\begin{align*}
\int_0^T (\rho, \psi_t \phi) dt - \int_0^T (\rho u, \psi \nabla \phi) dt &= (\rho_0, \phi) \psi(0), \\
\int_0^T (u, \psi_t \phi) dt - \int_0^T (u \cdot \nabla u, \psi \phi) dt &= (u_0, \phi) \psi(0) - \int_0^T (\nabla d \otimes \nabla d, \psi \nabla \phi) dt + \int_0^T (\sigma, \psi \nabla \phi) dt, \\
\int_0^T (d, \psi_t \phi) dt - \int_0^T (u \cdot \nabla d, \psi \phi) dt + \int_0^T (\Omega d, \psi \phi) dt - \frac{\lambda_2}{\lambda_1} \int_0^T (Ad, \psi \phi) dt &= (d_0, \phi) \psi(0) + \frac{1}{\lambda_1} \int_0^T (\Delta d - f(d), \psi \phi) dt.
\end{align*}
\]

2.3. **Basic energy law.** The total energy of the full Ericksen-Leslie system (1.7)-(1.10) is given by

\[
E(t) = \int_Q \frac{1}{2} \rho |u|^2 + \frac{1}{2} |\nabla d|^2 + \rho F(d) dx
\]

which consists of kinetic and potential energies. Formally, a smooth solution \((\rho, u, d)\) satisfies (cf. \[32, 35\])

\[
\frac{d}{dt} E(t) = - \int_Q \mu_1 |d^T Ad|^2 + \mu_4 |\nabla u|^2 + (\mu_5 + \mu_6) |Ad|^2 dx \\
+ \lambda_1 \|N\|^2 + (\lambda_2 - \mu_2 - \mu_3) (N, Ad).
\]

To guarantee the dissipation of the director field, it is assumed that (cf. \[15, 29\])

\[
\begin{align*}
\lambda_1 &< 0 \\
\mu_1 &\geq 0, \quad \mu_4 > 0, \\
\mu_5 + \mu_6 &\geq 0.
\end{align*}
\]

With the Parodi’s relation (1.6) we have the following basic energy law.

**Lemma 2.2.** Suppose that (1.5), (1.6) and (2.20) are satisfied. In addition, we assume

\[
\begin{align*}
\frac{\lambda_2}{\lambda_1} \leq \mu_5 + \mu_6.
\end{align*}
\]

Then the total energy \(E(t)\) for smooth solution satisfies

\[
\begin{align*}
\frac{d}{dt} E(t) &= - \int_Q \mu_1 |d^T Ad|^2 + \mu_4 |\nabla u|^2 + \frac{1}{\lambda_1} \|\Delta d - f(d)\|^2 \\
- (\mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1}) \|Ad\|^2 \leq 0.
\end{align*}
\]

The proof of Lemma 2.2 is identical to the proof of Lemma 2.1 in \[43\] and thus it is omitted here.
3. Regular solutions to the full Ericksen-Lesie System

3.1. Galerkin Approximate Solutions. We construct a sequence of Galerkin approximating solutions that satisfy both of the basic energy estimate and a higher order energy estimate. The higher order energy estimate obtained through Ladyzhenskaya method yields a subsequence that will converge to the classical solution.

We first introduce the functional settings of Galerkin approximation method. Let
\[ \mathcal{H}(Q) = \text{closure of } \{ f \in C_0^\infty(Q, \mathbb{R}^n) : \nabla \cdot f = 0 \} \text{ in } H. \]

Let \( \{ \phi_i \}_{i=1}^\infty \) be the unit eigenvectors of the Stokes problem in the periodic case with zero mean:
\[ -\Delta \phi_i + \nabla \pi_i = k_i \phi_i \text{ in } Q, \quad \int_Q \phi_i(x) dx = 0 \]
with \( \pi_i \in L^2 \) and \( 0 < k_1 \leq k_2 \leq \ldots \) being eigenvalues. It is known that \( \phi_i \) are smooth and \( \{ \phi_i \}_{i=1}^\infty \) forms an orthogonal basis of \( \mathcal{H} \) (see [42]). Let
\[ P_m : \mathcal{H} \rightarrow \mathcal{H}_m = \text{span} \{ \phi_1, \ldots, \phi_m \} \]
be the orthonormal projection. We seek approximate solutions \( (\rho^m, u^m, d^m) \) with \( u^m \in \mathcal{H}_m \), satisfy the following equations:
\[
\begin{align*}
\rho^m_i + u^m \cdot \nabla \rho^m &= 0, \\
P_m(\rho^m \frac{\partial}{\partial t} u^m) &= P_m(\Delta u^m - \rho^m u^m \cdot \nabla u^m - \nabla \cdot (\nabla d^m \otimes \nabla d^m - \nabla \cdot \sigma^m),
\end{align*}
\]
\[
\begin{align*}
da^m_i + u^m \cdot \nabla d^m - \Omega^m d^m + \frac{\lambda_2}{\lambda_1} A^m d^m &= -\frac{1}{\lambda_1}(\Delta d^m - f(d^m))
\end{align*}
\]
with
\[
\begin{align*}
\Omega^m &= \frac{1}{2}(\nabla u^m - \nabla u^m)^T, \\
A^m &= \frac{1}{2}(\nabla u^m + \nabla u^m)^T, \\
N^m &= \partial_t d^m + (u^m \cdot \nabla) d^m + \Omega^m d^m, \\
\sigma^m &= \mu_1 ((d^m)^T A^m d^m) d^m \otimes d^m + \mu_2 N^m d^m + \mu_3 A^m d^m + \mu_4 A^m d^m \\
&+ \mu_5 A^m d^m \otimes d^m + \mu_6 A^m d^m,
\end{align*}
\]
and with the initial and boundary conditions
\[
\begin{align*}
\rho^m(x, 0) &= \rho_0(x), \quad u^m(x, 0) = P_m u_0(x), \quad d^m(x, 0) = d_0(x), \\
u^m(x + e_i, t) &= u^m(x, t), \quad d^m(x + e_i, t) = d^m(x, t).
\end{align*}
\]
Let
\[ u^m(x, t) = \sum_{i=1}^m g_i^m(t) \phi_i(x), \]
with \( g_i^m(t) \in C^1[0, T] \). Hence (3.24) is equivalent to the following system of ordinary differential equations:
\[
\sum_{i=1}^m A_i^{mj}(t) \frac{d}{dt} g_i^m(t) = -\sum_{i,k} B_{ik}^{mj}(t) g_i^m(t) g_k^m(t) - \sum_{i=1}^m C_i^j g_i^m(t) + D^{mj}(t),
\]
for \( j = 1, 2, \ldots, m \), where

\[
\begin{align*}
A_i^{mj}(t) &= \int_\Omega \rho^m(t)\phi_i(x)\phi_j(x)dx, \\
B_{ik}^{mj}(t) &= \int_\Omega \rho^m(t)(\phi_i(x)\cdot \nabla \phi_k(x))\phi_j(x)dx, \\
C_i^j &= \int_\Omega \nabla \phi_i(x) \cdot \nabla \phi_j(x)dx, \\
D^{mj}(t) &= \int_\Omega \sum_{k,l} \left( \frac{\partial^2}{\partial x_k \partial x_l}d^m + \sigma_{kl}^m \right) \phi_k^j(x)dx.
\end{align*}
\]

Here \( \phi^j_k(x) \) is the \( k \)-th component of the vector \( \phi_j(x) \). And

\[
u^m(\cdot, 0) = \sum_{i=1}^m g^{m}_i(0)\phi_i(x), \quad \text{where} \quad g^{m}_i(0) = \int_\Omega u_0(x)\phi_i(x)dx.
\]

**Lemma 3.1.** There exists a solution \((\rho^m, u^m, d^m)\) to the problem \([3.23]-[3.25]\) in \(Q_T = Q \times [0, T]\), for any \(T \in (0, \infty)\), satisfying

\[
M_1 \leq \rho^m \leq M_2
\]

\[
u^m \in L^\infty(0, T; H) \cap L^2(0, T; V)
\]

\[
d^m \in L^\infty(0, T; H^1_p \cap L^\infty_p) \cap L^2(0, T; H^2_p).
\]

Moreover, \((\rho^m, u^m, d^m)\) is smooth in the interior of \(Q_T\) and satisfies the basic energy equality,

\[
\frac{d}{dt}E^m(t) = -\int_\Omega \mu_1 |(d^m)^TA^m d^m|^2 + \mu_4 |\nabla u^m|^2 dx + \frac{1}{\lambda_1} \|\Delta d^m - f(d^m)\|^2
\]

\[
- \left( \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) \|A^m d^m\|^2 \leq 0,
\]

with

\[
E^m(t) = \int_Q \left( \frac{1}{2} \rho^m |u^m|^2 + \frac{1}{2} |\nabla d^m|^2 + \rho^m \mathcal{F}(d^m) \right) dx.
\]

**Proof:** The proof of existence of weak solutions is based on an application of the Leray-Schauder fixed point theorem. Let \(u^m = \sum_{i=1}^m h_i^m i \in C^1(0, T; H_m)\). For each \(m\) let \(\rho^m\) be a solution to

\[
\rho + v^m \cdot \nabla \rho = 0
\]

with initial condition \(\rho(\cdot, 0) = \rho_0\). Let \(d^m\) be a solution to

\[
d_t + v^m \cdot \nabla d - \tilde{\Omega}^m d + \frac{\lambda_2}{\lambda_1} \tilde{A}^m d = -\frac{1}{\lambda_1}(\Delta d - f(d))
\]

\[
\tilde{\Omega}^m = \frac{1}{2}(\nabla v^m - \nabla T v^m), \quad \tilde{A}^m = \frac{1}{2}(\nabla v^m + \nabla T v^m)
\]

with initial condition \(d(\cdot, 0) = d_0\) and boundary condition \(d(x + e_i, t) = d(x, t)\). The reason the transport equation \([3.23]\) is solvable for \(v^m \in C^1(0, T; H_m)\) is due to the regularity of the eigenfunctions of the Stokes operators (cf. \([41][26]\)). Let \(u^m = \sum_{i=1}^m g_i^m \phi_i \in C^1(0, T; H_m)\) be the solution of the system of linear equations

\[
\sum_{i=1}^m A_i^{mj}(t) \frac{d}{dt}g_i^m(t) = -\sum_{i,k} B_{ik}^{mj}(t)h_i^m(t)g_k^m(t) - \sum_{i=1}^m C_i^j g_i^m(t) + D^{mj}(t).
\]
This system of linear equations is solvable because the eigenvalues of matrix of the coefficients \( A_i^{m,j} (t) \) are bounded from below, since

\[
A_i^{m,j} \gamma_i \gamma_j = \int_{\Omega} \rho |\psi|^2 \, dx \geq M_1 \int_{\Omega} |\psi|^2 \, dx \quad \text{where} \quad \psi = \sum_{i=1}^{m} \gamma_i \phi_i.
\]

Thus we constructed a mapping \( M \) with \( M(u^m) = u^m \). The energy estimate (3.27) will be obtained similarly as in Lemma 2.2 and it plays the key to allow one to apply Leray-Schauder fixed point theorem for \( M \).

3.2. Higher order energy estimates. As a first step, we use the Ladyzhenskaya energy method [26] to show that \( u^m \in L^\infty(0, T; V) \cap L^2(0, T; H^2_0) \) and \( d^m \in L^\infty(0, T; H^2_0) \cap L^2(0, T; H^3_0) \), provided \( u_0 \in H^1_0 \) and \( d_0 \in H^2_0 \). In 2D, these estimates will be obtained with general initial data; while in 3D, they will be obtained under the assumption of small initial data. We then pass to the limit for the Galerkin approximating solutions \((\rho^m, u^m, d^m)\) to yield weak solutions for the system (1.7)-(1.10). The key inequalities used often in this paper are the following Gagliardo-Nirenberg inequality (cf. [17]) :

**Lemma 3.2. (Gagliardo-Nirenberg)** If \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \), then

\[
\|u\|_{L^4(\Omega)}^4 \leq C \|u\|_{L^2(\Omega)}^2 (\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2), \quad n = 2,
\]

\[
\|u\|_{L^4(\Omega)}^4 \leq C \|u\|_{L^2(\Omega)}^2 (\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2)^{\frac{3}{2}}, \quad n = 3.
\]

By a similar strategy as in [30, 11], we will establish uniform estimates for the following energy quantity

\[\Phi^2_m(t) = \|\sqrt{\rho^m} \nabla u^m\|_{L^2}^2 + \|\Delta d^m - f(d^m)\|_{L^2}^2.\]

**Lemma 3.3. (3D small data assumption.)** Let \( Q \subset \mathbb{R}^3 \) and the initial data \((\rho_0, u_0, d_0)\) satisfy (1.7), (1.12). Suppose that \( \Phi^2(0) = \|\sqrt{\rho_0} \nabla u_0\|_{L^2}^2 + \|\Delta d_0 - f(d_0)\|_{L^2}^2 < \infty. \) There is \( \epsilon_0 > 0 \) such that if

\[
(3.30) \quad \left( \frac{M_2}{\mu_4} + 1 \right) \rho_0 \|u_0\|_{H^1}^2 + \|d_0\|_{H^1}^2 + \|\Delta d_0 - f(d_0)\|_{L^2}^2 \leq \epsilon_0,
\]

then the approximating solutions \((\rho^m, u^m, d^m)\) obtained in Lemma 3.1 satisfy

\[
(3.31) \quad \|\sqrt{\rho^m} \nabla u^m\|_{L^2}^2 + \|\Delta d^m - f(d^m)\|_{L^2}^2 \leq C(\Phi^2(0) + C)
\]

for all \( t \in [0, T] \) and

\[
(3.32) \quad \int_0^T \|\Delta u^m\|_{L^2}^2 + \|\nabla (\Delta d^m - f(d^m))\|_{L^2}^2 + \mu_1 \|d^m T \nabla A^m d^m\|_{L^2}^2 \, dt \leq C
\]

where the constants \( C \) depend only on initial data, \( Q, M_1, M_2 \) and the physical coefficients in system (1.7) - (1.10).

**Proof:** Note that the approximating solutions \((\rho^m, u^m, d^m)\) obtained in Lemma 3.1 also satisfy equations (1.7) - (1.9) point wisely. To simplify the notation, through the proof we drop the approximating index \( m \) and denote \((\rho, u, d)\) as the Galerkin approximating
applying the periodic boundary conditions (1.11) and integration by parts yields
\[(3.33)\]
\[
\frac{1}{2} \frac{d}{dt} \Phi^2(t) = \int_Q \left( \frac{1}{2} \rho |\nabla u|^2 + \rho \nabla u \cdot \nabla u dx + \int_Q (\Delta d - f(d)) \cdot (\Delta d_t - f'(d) dt) dx \right)
\]
\[
= \int_Q \left( \frac{1}{2} \rho (u \cdot \nabla) |\nabla u|^2 - \rho \mu_4 \Delta u dx + \int_Q (\Delta d - f(d)) \Delta d_t dx 
- \int_Q (\Delta d - f(d)) f'(d) dt dx \right).
\]

It follows from equation (1.8)
\[(3.34)\]
\[
\int_Q \left( \frac{1}{2} \rho (u \cdot \nabla) |\nabla u|^2 - \rho \mu_4 \Delta u dx = \int_Q \frac{1}{2} \rho (u \cdot \nabla) |\nabla u|^2 + \rho (u \cdot \nabla) u \Delta u + \Delta u \nabla d \Delta d - \Delta u \nabla \cdot \sigma dx \right.
\]
\[
= - \mu_4 \int_Q |\Delta u|^2 dx + \int_Q 2 \rho (u \cdot \nabla) u \Delta u + \Delta u \nabla d \Delta d - \Delta u \nabla \cdot \bar{\sigma} dx
\]
\[
= - \mu_4 \int_Q |\Delta u|^2 dx + I_1 + I_2 + I_3
\]
with \( \bar{\sigma} = \sigma - \mu_4 A \). Acting Laplacian on equation (1.9) gives
\[
\Delta d_t + \Delta (u \cdot \nabla d) - \Delta (\Omega d) + \frac{\lambda_2}{\lambda_1} \Delta (Ad) = - \frac{1}{\lambda_1} \Delta (\Delta d - f(d)).
\]

Thus,
\[(3.35)\]
\[
\int_\Omega (\Delta d - f(d)) \Delta d_t dx
= \frac{1}{\lambda_1} \int_\Omega \left( |\nabla (\Delta d - f(d))|^2 dx - \int_\Omega (\Delta d - f(d)) \Delta (u \cdot \nabla d) dx \right.
\]
\[
+ \int_\Omega (\Delta d - f(d)) \Delta (\Omega d) dx - \frac{\lambda_2}{\lambda_1} \int_\Omega (\Delta d - f(d)) \Delta (Ad) dx
\]
\[
= \frac{1}{\lambda_1} \int_\Omega |\nabla (\Delta d - f(d))|^2 dx + I_4 + I_5 + I_6.
\]

We also have the following by equation (1.9)
\[(3.36)\]
\[
- \int_\Omega (\Delta d - f(d)) f'(d) dt dx = \frac{1}{\lambda_1} \int_\Omega f'(d) |\nabla d - f(d)|^2 dx
\]
\[
+ (\Delta d - f(d), f'(d) (u \cdot \nabla d) - (\Delta d - f(d), f'(d) (\Omega d - \frac{\lambda_2}{\lambda_1} Ad))
= I_7 + I_8 + I_9.
\]
Combining (3.33), (3.34), (3.35) and (3.36) yields
\[(3.37)\]
\[
\frac{1}{2} \frac{d}{dt} \Phi^2(t) = - \mu_4 \int_Q |\Delta u|^2 dx + \frac{1}{\lambda_1} \int_\Omega |\nabla (\Delta d - f(d))|^2 dx
\]
\[
+ I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9,
\]
with $I_k$ being defined as in the above equations for $k = 1, \ldots, 9$. In the following, we shall estimate these $I_k$ term by term.

By Hölder inequality and Gagliardo-Nirenberg inequality in Lemma 3.2 it follows that

\begin{equation}
|I_1| \leq M_2 \int_{\Omega} |u| |\nabla u| |\Delta u| dx
\leq \epsilon \|\Delta u\|_2^2 + C \|u\|_2^2 \|\nabla u\|_2^2
\leq \epsilon \|\Delta u\|_2^2 + C \|u\|_2^{1/2} \|\nabla u\|_2^{3/2}
\leq \epsilon \|\Delta u\|_2^2 + C \|u\|_2^2 \|\nabla u\|_8^4
\leq \epsilon \|\Delta u\|_2^2 + CM^{-4}_\epsilon \|u\|_2^2 \|\sqrt{\rho} \nabla u\|_6^3.
\end{equation}

By Agmon’s inequality in 3D, we have

\begin{equation}
\|d\|_\infty \leq C \|\nabla d\|_2^{1/2} \|\Delta d\|_2^{1/2}.
\end{equation}

Thus we infer that

\begin{equation}
\|\Delta d\|_2 \leq \|\Delta d - f(d)\|_2 + \|f(d)\|_2 \leq \|\Delta d - f(d)\|_2 + C,
\end{equation}

\begin{align}
\|\nabla d\|_2 &\leq \|\nabla (\Delta d - f(d))\|_2 + \|\nabla f(d)\|_2 \\
&\leq \|\nabla (\Delta d - f(d))\|_2 + \|\nabla f(d)\|_2 \\
&\leq \|\nabla (\Delta d - f(d))\|_2 + (1 + 1\|\nabla d\|_2) \|\nabla d\|_2 \\
&\leq \|\nabla (\Delta d - f(d))\|_2 + C (1 + 1\|\nabla d\|_2 \|\Delta d - f(d)\|_2) \|\nabla d\|_2.
\end{align}

Therefore, applying Hölder inequality, Gagliardo-Nirenberg inequality and (3.41) yields

\begin{align}
|I_2| \leq &\epsilon \|\Delta u\|_2^2 + C \|\nabla d\|_2 \|\Delta d\|_2^2 \\
\leq &\epsilon \|\Delta u\|_2^2 + C \|\nabla d\|_2^{1/2} \|\Delta d\|_2^{3/2} \|\nabla d\|_2^{3/2} \\
\leq &\epsilon \|\Delta u\|_2^2 + C \|\nabla d\|_2 \|\Delta d\|_2^2 \\
\leq &\epsilon \|\Delta u\|_2^2 + \epsilon \|\nabla (\Delta d - f(d))\|_2^2 + C \|\nabla d\|_2^2 (\|\Delta d - f(d)\|_2^2 + 1) \\
&+ C \|\nabla d\|_2 (1 + 1\|\nabla d\|_2 \|\Delta d - f(d)\|_2).
\end{align}

From the equation of $\sigma$ in (1.4) we have

\begin{align}
I_3 &= -\mu_1 \int_Q \nabla_j (d_k d_p A_{kp} d_j d_j) \nabla l \nabla l u_{i} dx - \mu_2 \int_Q \nabla_j (d_k N_i) \nabla l \nabla l u_{i} dx \\
&\quad - \mu_3 \int_Q \nabla_j (d_1 N_j) \nabla l \nabla l u_{i} dx - \mu_5 \int_Q \nabla_j (d_1 d_k A_{k2}) \nabla l \nabla l u_{i} dx \\
&\quad - \mu_6 \int_Q \nabla_j (d_1 d_k A_{kj}) \nabla l \nabla l u_{i} dx.
\end{align}

Integration by parts yields

\begin{align}
-I_3 &= -\mu_1 \int_Q \nabla_j (d_k d_p A_{kp} d_j d_j) \nabla l \nabla l u_{i} dx \\
&= \mu_1 \int_Q (d_k d_p A_{kp} d_j d_j) \nabla l \nabla l (A_{ij} + \Omega_{ij}) dx \\
&= \mu_1 \int_Q (d_k d_p A_{kp} d_j d_j) \nabla l \nabla l A_{ij} dx
\end{align}
where we used the fact that $\Omega$ is antisymmetric which implies

$$
\mu_1 \int_Q (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_l \Omega_{ij} dx = -\mu_1 \int_Q \nabla_l (d_k d_p A_{kp} d_i d_j) \nabla_l \Omega_{ij} dx
= \mu_1 \int_Q \nabla_l (d_k d_p A_{kp} d_i d_j) \nabla_l \Omega_{ji} dx
= \mu_1 \int_Q (d_k d_p A_{kp} d_i d_j) \nabla_l \Omega_{ij} dx
= -\mu_1 \int_Q (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_l \Omega_{ij} dx
$$

hence $\int_Q (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_l \Omega_{ij} dx = 0$. Therefore

$$
- \mu_1 \int_Q \nabla_j (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_l u_i dx
= - \mu_1 \int_Q |d^T \nabla A|^2 dx - \mu_1 \int_Q \nabla (d \otimes d) d \otimes d A \nabla A dx
= - \mu_1 \int_Q |d^T \nabla A|^2 dx + I_{31}.
$$

Integration by parts also yields

$$
- \mu_2 \int_Q \nabla_j (d_j N_i) \nabla_l \nabla_l u_i dx - \mu_3 \int_Q \nabla_j (d_i N_j) \nabla_l \nabla_l u_i dx
= \mu_2 \int_Q d_j N_i \Delta (A_{ij} + \Omega_{ij}) dx + \mu_3 \int_Q d_i N_j \Delta (A_{ij} + \Omega_{ij}) dx
= (\mu_2 + \mu_3) \int_Q d_j N_i \nabla A_{ij} dx + (\mu_2 - \mu_3) \int_Q d_j N_i \nabla \Omega_{ij} dx
= I_{32} + I_{33}
$$

$$
- \mu_5 \int_Q \nabla_j (d_j d_k A_{ki}) \nabla_l \nabla_l u_i dx - \mu_6 \int_Q \nabla_j (d_i d_k A_{ki}) \nabla_l \nabla_l u_i dx
= (\mu_5 + \mu_6) \int_Q d_j d_k A_{ki} \nabla A_{ij} dx + (\mu_5 - \mu_6) \int_Q d_j d_k A_{ki} \Delta \Omega_{ij} dx
= - 2(\mu_5 + \mu_6) \int_Q d \nabla d \nabla A dx + (\mu_5 - \mu_6) \int_Q d \nabla \Delta d \Omega dx
- (\mu_5 + \mu_6) \int_Q |d \nabla A|^2 dx = I_{34} + I_{35} + I_{36}.
$$
Using integration by parts and equation (1.9), we infer
\[
I_5 = \int_Q (\Delta d - f(d)) \Delta \Omega d \, dx + 2 \int_Q (\Delta d - f(d)) \nabla \Omega \nabla d \, dx \\
+ \int_Q (\Delta d - f(d)) \Omega \Delta d \, dx \\
= - \lambda_1 (N, \Delta \Omega d) - \lambda_2 (Ad, \Delta \Omega d) + 2 \int_Q (\Delta d - f(d)) \nabla \Omega \nabla d \, dx 
\]

Using integration by parts and equation (1.9), we infer
\[
I_6 = \lambda_2 \int_Q N \Delta (Ad) \, dx + \frac{\lambda_2^2}{\lambda_1} \int_Q Ad \Delta (Ad) \, dx \\
+ \int_Q N \nabla A \nabla d \, dx - \frac{\lambda_2^2}{\lambda_1} \int_Q (|\nabla A|^2) \, dx - \frac{\lambda_2^2}{\lambda_1} \int_Q |\nabla d|^2 \, dx 
\]

Note that there are cancelations among \( I_5, I_5 \), and \( I_6 \), due to the parameter relations (1.5) and (1.6), and the assumption (2.21). Indeed,
\[
I_{32} + I_{61} = 0, \quad I_{33} + I_{51} = 0, \quad I_{35} + I_{52} = 0, \quad I_{36} + I_{64} \leq 0. 
\]

In the following we estimate the rest terms in \( I_5, I_5 \), and \( I_6 \). Applying Hölder’s inequality, Gagliardo-Nirenberg’s inequality and Agmon’s inequality (3.39) yields
\[
|I_{31}| \leq \epsilon \|\Delta u\|^2 + C \|d\|^6 \|\nabla u\| \|\nabla d\|^2 \\
\leq \epsilon \|\Delta u\|^2 + C \|\nabla d\|^2 \|\Delta d\|^2 \|\nabla u\| \|\nabla d\|^2 \|\Delta d\|^2 \\
\leq \epsilon \|\Delta u\|^2 + C \|\nabla d\|^2 \|\Delta d\|^2 \|\nabla u\|^2 \\
\leq \epsilon \|\Delta u\|^2 + C \|\nabla d\|^2 \|\Delta d\|^2 \|\nabla u\|^2 \\
\leq \epsilon \|\Delta u\|^2 + C \|\nabla d\|^2 \|\Delta d\|^2 \|\nabla u\|^2 \\
\leq \epsilon \|\Delta u\|^2 + C \|\nabla d\|^2 \|\Delta d\|^2 \|\nabla u\|^2 \\
= |I_{34}| \leq \epsilon \|\Delta u\|^2 + C \|\nabla d\|^2 \|\Delta d - f(d)\|^2 \|\nabla u\|^2 \\
(3.48) 
\]

\[
|I_{53}| \leq \epsilon \|\Delta u\|^2 + \|\nabla (\Delta d - f(d))\|^2 + C \|\nabla d\|^2 \|\Delta d - f(d)\|^2 \|\nabla u\|^2 \\
(3.49) 
\]

and
\[
|I_{54}| \leq \epsilon \|\Delta u\|^2 + \|\nabla (\Delta d - f(d))\|^2 + C \|\nabla d\|^2 \|\Delta d - f(d)\|^2 \|\nabla u\|^2 \\
(3.50) 
\]

and
\[
|I_{62}| \leq \int_Q |Ad \Delta u \nabla d| \, dx + \int_Q |(\Delta d - f(d)) \Delta u \nabla d| \, dx \\
\leq \epsilon \|\Delta u\|^2 + \|\nabla (\Delta d - f(d))\|^2 + C \|\nabla d\|^2 \\
\cdot (\|\Delta d - f\|^2 + \|\Delta d - f\| + 1) \|\nabla u\|^2 \\
= (3.51) 
\]
since the two integrals are similar to $I_{34}$ and $I_{53}$ respectively. While, we have by integration by parts

$$I_{63} + I_{65} = -\lambda_2 \int_Q N \nabla A \nabla d \, dx - \lambda_2 \int_Q \nabla N A \nabla d \, dx - \frac{\lambda_2^2}{\lambda_1} \int_Q |A \nabla d|^2 \, dx$$

$$= -\lambda_2 \int_Q N \nabla A \nabla d \, dx + \frac{\lambda_2^2}{\lambda_1} \int_Q \nabla (Ad) A \nabla d \, dx$$

$$+ \frac{\lambda_2}{\lambda_1} \int_Q \nabla (\Delta d - f) A \nabla d \, dx - \frac{\lambda_2^3}{\lambda_1} \int_Q |A \nabla d|^2 \, dx$$

$$= -\lambda_2 \int_Q N \nabla A \nabla d \, dx + \frac{\lambda_2^2}{\lambda_1} \int_Q \nabla Ad A \nabla d \, dx$$

where the three integrals are similar to $I_{62}, I_{34}$ and $I_{54}$ respectively. Thus

$$|I_{63} + I_{65}| \leq \epsilon (\|\Delta u\|_2^2 + \|\nabla (\Delta d - f)\|_2^2) + C \|\nabla d\|_2^2 \cdot (\|\Delta d - f\|_2^2 + \|\nabla (\Delta d - f)\|_2^2).$$

The estimate for $I_4 + I_8$ is as follows. The facts $\nabla d \cdot f(d) = \nabla F(d)$ and $\nabla \cdot u = 0$ imply that $(f, \Delta u \cdot \nabla d) = 0$. Also by $\nabla \cdot u = 0$, it has $(\Delta d - f, u \cdot \nabla (\Delta d - f)) = 0)$. Thus, we have

$$I_4 = - (\Delta d - f, \Delta u \cdot \nabla d) - 2 \int_\Omega (\Delta d_i - f_i) \nabla_i v_j \nabla_j d_i \, dx - (\Delta d - f, u \cdot \nabla \Delta d)$$

$$= - (\Delta d \nabla d, \Delta u) + 2 \int_\Omega \nabla_j (\Delta d_i - f_i) \nabla_i v_j \nabla_i d_i \, dx - (\Delta d - f, u \cdot \nabla f(d))$$

$$= - (\Delta d \nabla d, \Delta u) + 2 \int_\Omega \nabla_j (\Delta d_i - f_i) \nabla_i v_j \nabla_i d_i \, dx - I_8.$$

Therefore,

$$I_4 + I_8 = - (\Delta d \nabla d, \Delta u) + 2 \int_\Omega \nabla_j (\Delta d_i - f_i) \nabla_i v_j \nabla_i d_i \, dx.$$

where the two terms are similar to $I_2, I_{54}$ respectively. Hence, from (3.51) and (3.50) we have

$$|I_4 + I_8| \leq \epsilon \left(\|\Delta u\|_2^2 + \|\nabla (\Delta d - f)\|_2^2\right) + C \|\nabla d\|_2^2 \left(\Phi^8 + \Phi^6 + 1\right).$$

By Agmon’s inequality, the terms $I_7$ and $I_9$ are estimated as

$$|I_7| \leq C \|f'(d)\|_\infty \|\Delta d - f(d)\|_2^2 \leq C \|d\|_\infty^2 \|\Delta d - f(d)\|_2^2$$

$$\leq C \|\nabla d\|_2^2 (\|\Delta d - f(d)\|_2^2 + 1) \|\Delta d - f(d)\|_2^2,$$

$$|I_9| \leq C \|f'(d)\|_\infty \|d\|_\infty (\|\Delta d - f(d)\|_2^2 + \|\nabla u\|_2^2)$$

$$\leq C \|\nabla d\|_2^{3/2} (\|\Delta d - f(d)\|_2^{3/2} + 1) \Phi^2.$$
Denote $D = \|\Delta f - f(d)\|_2$. Combining (3.37), (3.38) and (3.41)–(3.55) yields
\[
\frac{1}{2} \frac{d}{dt} \Phi^2(t) + (\mu_4 - \epsilon) \int_Q |\Delta u|^2 dx + \left( - \frac{1}{\lambda_1} - \epsilon \right) \int_{\Omega} |\nabla (\Delta d - f(d))|^2 dx + \mu_1 \int_Q |D\nabla Ad|^2 dx
\]
(3.56)
\[
\leq CM^{-1} \|u\|_2^2 \|\sqrt{\rho} \nabla u\|_2^2 + C \|
abla d\|_2^2 (D^8 + D^2 + 1) + C \|\nabla d\|_2^2 (D^{18} + D^{10} + D^6 + 1) (\|u\|_2^2 + 2)
\]
\[
+ C \|\nabla d\|_2 (D + 1) D^2 + C \|
abla d\|_2^{3/2} (D^{3/2} + 1) (\|u\|_2^2 + |\nabla d|_2)
\]
\[
\leq \epsilon \|u\|_2 + |\nabla d|_2 (\Phi^{18} + \Phi^{10} + \Phi^6 + \Phi^2 + \Phi + 1) \Phi^2,
\]
where we used the facts $\|\nabla u\|_2^2 \leq M^{-1} \|\sqrt{\rho} \nabla u\|_2^2$ and $\|u\|_2 + |\nabla d|_2 \leq C$ by the basic energy estimate.

Set
\[
\tilde{\Phi}^2 = \Phi^2 + \|u\|_2 + |\nabla d|_2
\]
and observe from (3.56) that
\[
\frac{d}{dt} \tilde{\Phi}^2 \leq C (1 + (\|u\|_2 + |\nabla d|_2) \Phi^{18}) \tilde{\Phi}^2.
\]
Recall that $\|u\|_2 + |\nabla d|_2$ is small by the basic energy estimate (3.27) and the smallness assumption (3.30). Thus following a similar argument as in [11] we infer that for a small enough $\epsilon_0 > 0$ in the assumption (3.30), the conclusions of the Lemma will be proved. Suppose that
\[
\left( \frac{M_2}{\mu_4} + 1 \right) \rho_0 \|u_0\|_H^2 + \|d_0\|_H^2 + \|\Delta d_0 - f(d_0)\|^2 = \epsilon_0.
\]
By the basic energy estimate we have
\[
\|u\|_2 + |\nabla d|_2 \leq \|u_0\|_2 + |\nabla d_0|_2 \leq \sqrt{2\epsilon_0}.
\]
We claim that, if $\epsilon_0$ is so small that
\[
\sqrt{2\epsilon_0} (4e^{2C} \sqrt{\epsilon_0})^9 < 1
\]
then
\[
\tilde{\Phi}^2 < 4e^{2C} \sqrt{\epsilon_0}, \text{ for all } t > 0.
\]
First we prove the claim for $t \in [0, 1]$. Assume otherwise, there must be $t_0 \in (0, 1)$ such that
\[
\tilde{\Phi}^2(t_0) = 4e^{2C} \sqrt{\epsilon_0} \quad \text{or} \quad \tilde{\Phi}^2(t) \leq 4e^{2C} \sqrt{\epsilon_0}, \text{ for all } t \in (0, t_0).
\]
(3.60)
Therefore, from (3.61), by the choice of $\epsilon_0$ in (3.59), we have
\[
\frac{d}{dt} \tilde{\Phi}^2 \leq 2C \tilde{\Phi}^2
\]
for all $t \in (0, t_0)$ and $\tilde{\Phi}^2(0) \leq \epsilon_0 + \sqrt{2\epsilon_0} < 3 \sqrt{\epsilon_0}$, which implies that
\[
\tilde{\Phi}^2(t_0) \leq e^{2C} \tilde{\Phi}^2(0) \leq 3e^{2C} \sqrt{\epsilon_0}
\]
and thus contradicts (3.60). For \( t > 1 \), we simply observe, that from the basic energy inequality (3.27) and the assumption (3.58)

\[
\int_{t-1}^{t} \tilde{\Phi}^2(t) dt \leq 3\sqrt{\epsilon_0}
\]

which implies that there is \( t_0 \in (t-1, t) \) such that

\[
\tilde{\Phi}^2(t_0) \leq 3\sqrt{\epsilon_0}.
\]

One may repeat the above argument to conclude that

\[
\tilde{\Phi}^2(t) \leq 4e^{2C} \sqrt{\epsilon_0}, \quad \text{for all } t > 0.
\]

The inequality (3.32) follows directly from the basic energy estimate (3.27) and (3.61). It completes the proof of the lemma.

Without the assumption of small initial data, the higher order energy estimate holds for short time.

Lemma 3.4. (3D short time.) Let \( Q \subset \mathbb{R}^3 \) and the initial data \( (\rho_0, u_0, d_0) \) satisfy (1.12)-(1.13). Suppose that \( \Phi(0)^2 = \|\sqrt{\rho_0}\nabla u_0\|^2 + \|\Delta d_0 - f(d_0)\|^2 < \infty \). Then the approximating solutions \( (\rho^m, u^m, d^m) \) obtained in Lemma 3.1 satisfy (3.31) and (3.32) on the time interval \( (0, T] \) for some \( T < \delta \).

Proof: Set

\[
\tilde{\Phi}^2_m = \Phi^2_m + \|u^m\|_{L^2} + \|\nabla d^m\|_{L^2} + 1.
\]

By (3.56) and the uniform estimate \( \|u^m\|_{L^2} + \|\nabla d^m\|_{L^2} \leq C \), we have

(3.61)

\[
\frac{d}{dt} \tilde{\Phi}^2_m \leq C\tilde{\Phi}^{20}_{m}.
\]

Hence,

\[
\int \frac{d\tilde{\Phi}^2_m}{\tilde{\Phi}^{18}_m} \leq C \int_{0}^{T} \tilde{\Phi}^2_m dt
\]

which produces for any \( t \in (0, T] \)

\[
\tilde{\Phi}^{16}_{16}(t) \leq \frac{\tilde{\Phi}^{16}_{16}(0)}{1 - C\tilde{\Phi}^{16}_{16}(0) \int_{0}^{T} \tilde{\Phi}^2_m (0) dt}.
\]

The basic energy estimate (3.27) indicates, there exists a small time \( T < \delta \) such that

\[
C\tilde{\Phi}^{16}_{16}(0) \int_{0}^{T} \tilde{\Phi}^2_m (0) dt \leq \frac{1}{2}
\]

which consequently implies

\[
\tilde{\Phi}^2_m(t) \leq C\tilde{\Phi}^2_m (0), \quad \text{for all } t \in (0, T].
\]

The conclusion of the lemma thus holds true.

Next we shall establish the higher order energy estimate with general data in two dimensional case. Note that in 2D, we usually assume in addition either

(i) \( \mu_1 \geq 0, \lambda_2 = 0 \), or
(ii) \( \mu_1 = 0, \lambda_2 \neq 0 \).

For the literature of the Ericksen-Leslie system with assumption (i), we refer the readers to [32]; while for the work with assumption (ii), we refer to [19, 39, 43].
Lemma 3.5. (2D.) Let \( Q \subset \mathbb{R}^2 \) and the initial data \((\rho_0, u_0, d_0)\) satisfy (1.2), (1.7). Suppose that \( \Phi(0)^2 = \|\sqrt{\rho_0}u_0\|_2^2 + \|\Delta d_0 - f(d_0)\|_2^2 < \infty \). Assume either (i) or (ii) holds. Then the approximating solutions \((\rho^n, u^n, d^n)\) obtained in Lemma 3.1 satisfy

\[
(3.62) \quad \|\sqrt{\rho^n}u^n\|_2^2 + \|\Delta d^n - f(d^n)\|_2^2 \leq \Phi^2(0) + C \|\sqrt{\rho_0}u_0\|_2^2 + \|\Delta d_0 - f(d_0)\|_2^2 + 1
\]

with \( t \in [0, T] \) and

\[
(3.63) \quad \int_0^T \|\Delta u^n\|_2^2 + \|\nabla(\Delta d^n - f(d^n))\|_2^2 + \mu_1 \|d^{nT} \nabla A^n d^n\|_2^2 dt \leq C
\]

for all \( T > 0 \). The constants \( C \) depend only on initial data, \( Q, M_1, M_2 \) and the physical coefficients in system (1.7), (1.10).

Proof: Case (i). Since \( \lambda_2 = 0 \), a maximum principle for \( d \) holds (cf. [32]). The Agmon’s inequality is not needed here. The highest nonlinear term \( I_{31} \) is estimated as

\[
|I_{31}| \leq C \|d^n\|_\infty^3 \int_Q |\nabla d^n \nabla u^n \Delta u^n| dx
\]

\[
\leq \epsilon \|\Delta u^n\|_2^2 + C \|\nabla u^n\|_2^2 \|\nabla d^n\|_2^2
\]

\[
\leq \epsilon \|\Delta u^n\|_2^2 + C \|\nabla u^n\|_2^2 (\| \Delta d^n - f(d^n) \|_2^2 + 1)
\]

where we used the 2D Gagliardo-Nirenberg inequality in Lemma [33]. Similarly, we apply Hölder’s equality, Gagliardo-Nirenberg inequality, the basic energy estimate and \( \|d^n\|_\infty \leq C \) to all the other terms in (3.33) and obtain that

\[
\frac{1}{2} \frac{d}{dt} \Phi^2_m(t) + (\mu_4 - \epsilon) \int_Q |\Delta u^n|^2 dx
\]

\[
+ \left( -\frac{1}{\lambda_1} - \epsilon \right) \int_\Omega |\nabla(\Delta d^n - f(d^n))|^2 dx + \mu_1 \int_Q |d^{nT} \nabla A^n d^n|^2 dx
\]

\[
\leq C(\Phi^4_m + \Phi^2_m + 1).
\]

Let \( \tilde{\Phi}^2_m = \Phi^2_m + 1 \). We have

\[
\frac{d}{dt} \tilde{\Phi}^2_m(t) + \mu_4 \int_Q |\Delta u^n|^2 dx - \frac{1}{\lambda_1} \int_\Omega |\nabla(\Delta d^n - f(d^n))|^2 dx
\]

\[
+ 2\mu_1 \int_Q |d^{nT} \nabla A^n d^n|^2 dx \leq C \tilde{\Phi}^4_m.
\]

Thus,

\[
\frac{1}{\Phi^2_m} \frac{d}{dt} \tilde{\Phi}^2_m \leq C \tilde{\Phi}^2_m.
\]

The conclusion of the lemma follows from the above inequality and the basic energy estimate (3.27) immediately.

Case (ii). Without loss of the generality, we assume \( |d^n| \) is large. Since otherwise we have \( \|d^n\|_\infty \leq C \) and hence it can be handled similarly as in case (i). Thus there exists a constant \( C_1 \) such that

\[
\int_Q |d^n|^2 dx \leq |Q|^{1/2} \left( \int_Q |d^n|^4 dx \right)^{1/2} \leq C_1 |Q|^{1/2} \left( \int_Q (|d^n|^2 - 1)^2 dx \right)^{1/2} \leq C
\]
where we used the fact \( \int_Q F(\xi)dx \) is bounded by the initial data from the basic energy estimate. It then follows from the Agmon’s inequality in 2D that

\[
\|d_m\|_\infty^2 \leq C\|d_m\|_2 \|\Delta d_m\|_2 \leq C\|\Delta d_m\|_2.
\]

Hence, slightly different from (3.41) we have

\[
\|\nabla \Delta d_m\|_2 \leq \|\nabla (\Delta d_m - f(d_m))\|_2 + C(1 + \|\Delta d_m - f(d_m)\|_2)\|\nabla d_m\|_2.
\]

Keep in mind that \( \mu_1 = 0 \) and thus the highest nonlinear term disappears here, and we have the Gagliardo-Nirenberg inequality in 2D. An analogous computation as in the proof of Lemma 3.3 produces that

\[
\frac{1}{2} \frac{d}{dt} \Phi_m^2(t) + (\mu_4 - \epsilon) \int_Q |\Delta u_m|^2 dx + \left(-\frac{1}{\lambda_1} - \epsilon\right) \int_\Omega |\nabla (\Delta d_m - f(d_m))|^2 dx \leq \epsilon \sqrt{\rho_m} \|\nabla u_m\|_2^2 |\Delta u_m|_2^2 + C(\Phi_m^4 + \Phi_m^2 + 1).
\]

Set \( \tilde{\Phi}_m^2 = \Phi_m^2 + 1 \). It follows

\[
\frac{d}{dt} \tilde{\Phi}_m^2(t) \leq -\left(\mu_4 - \epsilon \tilde{\Phi}_m^2\right) \int_Q |\Delta u_m|^2 dx + C \tilde{\Phi}_m^4.
\]

For a mall enough \( \epsilon \) such that

\[
\epsilon 2 \tilde{\Phi}_m^2(0)e^{C(\rho_0 u_0^2 + \|\nabla d_0\|^2_2 + 1)} \leq \frac{\mu_4}{2}
\]

then

\[
\tilde{\Phi}_m^2(t) < 2 \tilde{\Phi}_m^2(0)e^{C(\rho_0 u_0^2 + \|\nabla d_0\|^2_2 + 1)}, \quad \text{for all } t > 0.
\]

First we prove the claim for \( t \in [0, 1] \). Assume otherwise, there must be \( t_0 \in (0, 1) \) such that

\[
\begin{cases}
\tilde{\Phi}_m^2(t_0) = 2 \tilde{\Phi}_m^2(0)e^{C(\rho_0 u_0^2 + \|\nabla d_0\|^2_2 + 1)} \\
\tilde{\Phi}_m^2(t) \leq 2 \tilde{\Phi}_m^2(0)e^{C(\rho_0 u_0^2 + \|\nabla d_0\|^2_2 + 1)}, \quad \text{for all } t \in (0, t_0].
\end{cases}
\]

Therefore, from (3.65), by the choice of \( \epsilon \) in (3.66), we have

\[
\frac{1}{\Phi_m^2} \frac{d}{dt} \Phi_m^2 \leq C \Phi_m^2
\]

for all \( t \in (0, t_0) \) which implies that

\[
\tilde{\Phi}_m^2(t_0) \leq \tilde{\Phi}_m^2(0)e^{C \int_0^{t_0} \Phi_m^2(t) dt} \leq \tilde{\Phi}_m^2(0)e^{C(\rho_0 u_0^2 + \|\nabla d_0\|^2_2 + 1)}
\]

and thus contradicts (3.67). For \( t > 1 \), we apply a similar argument as in the proof of Lemma 3.3 to show that

\[
\tilde{\Phi}_m^2(t) \leq 2 \tilde{\Phi}_m^2(0)e^{C(\rho_0 u_0^2 + \|\nabla d_0\|^2_2 + 1)}.
\]

It completes the proof of the lemma.
3.3. Passage of limit to the weak solutions. In Subsection 3.2, the higher order energy estimates in Lemmas 3.3, 3.4, and 3.5 are all independent of the approximation index $m$ and time $t$. It implies the following uniform estimates hold

$$u^m \in L^\infty(0, T; V) \cap L^2(0, T; H^2_p),$$
$$d^m \in L^\infty(0, T; H^2_p) \cap L^2(0, T; H^3_p).$$

By the mass conservation equation (1.7), we also have

$$0 < M_1 \leq \rho^m \leq M_2.$$ 

Moreover, the positive lower bound of density $\rho^m$ combined with the uniform estimates for $u^m$ and $d^m$ indicates that

$$u^m_t \in L^2(0, T; L^2_p), \quad d^m_t \in L^2(0, T; H^1_p).$$

Therefore, with these higher order energy estimates, a standard procedure will show that the approximating solutions $(\rho^m, u^m, d^m)$ (a subsequence) converge to a limit $(\rho, u, d)$ such that the limit is a weak solution of system (1.7)-(1.10) and satisfies

(3.68) \quad 0 < M_1 \leq \rho \leq M_2,

(3.69) \quad u \in L^\infty(0, T; V) \cap L^2(0, T; H^2_p),

(3.70) \quad d \in L^\infty(0, T; H^2_p) \cap L^2(0, T; H^3_p).

Thus we obtained the existence of weak solutions.

**Theorem 3.6.** (3D) Let $Q \subset \mathbb{R}^3$ and the initial data $(\rho_0, u_0, d_0)$ satisfy the conditions (1.12), (1.13). Suppose that $\Phi^2(0) = \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\Delta d_0 - f(d_0)\|^2_{L^2} < \infty$.

(i) There is $\epsilon_0 > 0$ such that if

$$\left(\frac{M_2}{\mu_4} + 1\right)\rho_0\|u_0\|^2_{H^1} + \|d_0\|^2_{H^1} + \|\Delta d_0 - f(d_0)\|^2_{H^2} \leq \epsilon_0,$$

then system (1.7)-(1.10) with the periodic boundary condition (1.11) has a weak solution $(\rho, u, d)$ satisfying the basic energy estimate inequality

(3.71) \quad \mathcal{E}(t) + \int_{Q} \int_{1_{0}}^{t} |d| \mu_1 |d T| + \mu_4 |\nabla u|^2 - \frac{1}{\lambda_1} |\Delta d - f(d)|^2 dx

$$+ \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1^3}\right) \int_{1_{0}}^{t} \int_{Q} |d T|^2 dx \leq \mathcal{E}(t_0),$$

for all $t > t_0$ and almost every $t_0$. Moreover, the weak solution $(\rho, u, d)$ satisfies the higher order energy estimates (3.68)-(3.70) for all $T > 0$.

(ii) On the other hand, without the smallness assumption on initial data, there exists a weak solution to the system satisfying (3.71) for all $t > t_0$ a.e. $t_0$, but satisfying (3.68)-(3.70) only for a short time $T < \delta$, for some $\delta > 0$.

**Theorem 3.7.** (2D) Let $Q \subset \mathbb{R}^2$ and the initial data $(\rho_0, u_0, d_0)$ satisfy the conditions (1.12), (1.13). Suppose that $\Phi^2(0) = \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\Delta d_0 - f(d_0)\|^2_{L^2} < \infty$. Assume in addition that either

(i) $\mu_1 \geq 0$, $\lambda_2 = 0$, or (ii) $\mu_1 = 0$, $\lambda_2 \neq 0$.

The system (1.7)-(1.10) with the periodic boundary condition (1.11) has a weak solution $(\rho, u, d)$ satisfying the basic energy estimate inequality (3.71) with $t \in (0, T)$ and the higher order energy estimates (3.68)-(3.70) for all the time $T > 0$. 

**Theorem 3.8.** (1D) Let $Q \subset \mathbb{R}$ and the initial data $(\rho_0, u_0, d_0)$ satisfy the conditions (1.12), (1.13). Suppose that $\Phi^2(0) = \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\Delta d_0 - f(d_0)\|^2_{L^2} < \infty$. Assume in addition that either

(i) $\mu_1 \geq 0$, $\lambda_2 = 0$, or (ii) $\mu_1 = 0$, $\lambda_2 \neq 0$.

The system (1.7)-(1.10) with the periodic boundary condition (1.11) has a weak solution $(\rho, u, d)$ satisfying the basic energy estimate inequality (3.71) with $t \in (0, T)$ and the higher order energy estimates (3.68)-(3.70) for all the time $T > 0$. 

**Theorem 3.9.** (1D) Let $Q \subset \mathbb{R}$ and the initial data $(\rho_0, u_0, d_0)$ satisfy the conditions (1.12), (1.13). Suppose that $\Phi^2(0) = \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\Delta d_0 - f(d_0)\|^2_{L^2} < \infty$. Assume in addition that either

(i) $\mu_1 \geq 0$, $\lambda_2 = 0$, or (ii) $\mu_1 = 0$, $\lambda_2 \neq 0$.

The system (1.7)-(1.10) with the periodic boundary condition (1.11) has a weak solution $(\rho, u, d)$ satisfying the basic energy estimate inequality (3.71) with $t \in (0, T)$ and the higher order energy estimates (3.68)-(3.70) for all the time $T > 0$. 

**Theorem 3.10.** (1D) Let $Q \subset \mathbb{R}$ and the initial data $(\rho_0, u_0, d_0)$ satisfy the conditions (1.12), (1.13). Suppose that $\Phi^2(0) = \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\Delta d_0 - f(d_0)\|^2_{L^2} < \infty$. Assume in addition that either

(i) $\mu_1 \geq 0$, $\lambda_2 = 0$, or (ii) $\mu_1 = 0$, $\lambda_2 \neq 0$.

The system (1.7)-(1.10) with the periodic boundary condition (1.11) has a weak solution $(\rho, u, d)$ satisfying the basic energy estimate inequality (3.71) with $t \in (0, T)$ and the higher order energy estimates (3.68)-(3.70) for all the time $T > 0$. 

**Theorem 3.11.** (1D) Let $Q \subset \mathbb{R}$ and the initial data $(\rho_0, u_0, d_0)$ satisfy the conditions (1.12), (1.13). Suppose that $\Phi^2(0) = \|\sqrt{\rho_0} u_0\|^2_{L^2} + \|\Delta d_0 - f(d_0)\|^2_{L^2} < \infty$. Assume in addition that either

(i) $\mu_1 \geq 0$, $\lambda_2 = 0$, or (ii) $\mu_1 = 0$, $\lambda_2 \neq 0$.

The system (1.7)-(1.10) with the periodic boundary condition (1.11) has a weak solution $(\rho, u, d)$ satisfying the basic energy estimate inequality (3.71) with $t \in (0, T)$ and the higher order energy estimates (3.68)-(3.70) for all the time $T > 0$.
Applying a similar method to obtain the higher order energy estimates as in Subsection 3.2 the weak solution \((\rho, u, d)\) satisfies even higher order energy estimates provided more regular initial data.

**Corollary 3.8.** Let \(Q \subset \mathbb{R}^n\) with \(n = 2, 3\). The initial data \((\rho_0, u_0, d_0)\) satisfies the conditions in Theorem 3.6 and Theorem 3.7 respectively for \(n = 3, 2\). Assume, in addition, \(u_0 \in H_p^2\) and \(d_0 \in H_p^4\). Then there exists a weak solution \((\rho, u, d)\) to system (1.7)-(1.10) satisfying the basic energy estimate (3.71) and (3.68). Mean while,

\[
\begin{align*}
(3.72) & \quad u \in L^\infty(0, T; H_p^2) \cap L^2(0, T; H_p^3), \\
(3.73) & \quad d \in L^\infty(0, T; H_p^3) \cap L^2(0, T; H_p^4).
\end{align*}
\]

Indeed, compared to the simplified Ericksen-Leslie model studied in [11], the full model contains higher order derivative terms, for example, the term \(\nabla \cdot (Nd)\) in equation (1.8). When applying the \(L^p\) theory of parabolic equation on (1.8) to improve the regularity of velocity, the higher order energy estimates (3.72) and (3.73) are needed. Please see a detailed discussion in Subsection 3.5.

### 3.4. Auxiliary estimates on density

As a consequence of the higher order energy estimates obtained in Subsection 3.3 for the weak solution \((\rho, u, d)\), the regularity of the density can be improved in both of two and three dimensional cases.

**Lemma 3.9.** ([2]) Assume that \(\rho(0) \in C^1(Q)\) and that \(Q \subset \mathbb{R}^2\) is smooth and bounded. Suppose \(u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)\), and

\[
\rho_t + u \cdot \nabla \rho = 0
\]

in \(Q \times (0, T)\). Then \(\rho \in C^\alpha(\Omega \times [0, T])\) for some \(\alpha \in (0, 1)\) which depends only on the initial data, \(T\) and \(Q\).

**Lemma 3.10.** ([11]) Assume that \(\rho(0) \in C^1(Q)\) and that \(Q \subset \mathbb{R}^3\) is smooth and bounded. Suppose \(u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)\), and

\[
\rho_t + u \cdot \nabla \rho = 0
\]

in \(Q \times (0, T)\). Let \(t_1 \in (0, T)\) and \(p \in Q\), define

\[
\mathcal{A}(p, t_1) = (B_{r_0}(p) \cap Q) \times ([t_1 - r_0, t_1 + r_0] \cap [0, T]).
\]

Then, for any \(\epsilon > 0\), there exists \(r_0 > 0\) such that for \(p \in Q\) and all \(T > 0\),

\[
\sup_{(q, t_2) \in \mathcal{A}(p, t_1)} |\rho(q, t_2) - \rho(p, t_1)| \leq \epsilon.
\]

We refer the readers to [11] for a detailed proof of this lemma.

**Remark 3.11.** In 2D, the Hölder continuity for the fluid density guarantees that a frozen coefficient method (cf. [27]) can be applied to the Navier-Stokes equation (1.8) and hence the regularity of the velocity \(u\) will be improved through a standard \(L^p\) theory for the parabolic equation. In 3D, the density has the property of small oscillations over small balls in \(Q \times [0, T]\) provided that either the initial data is small or for short time. This turns out to be enough to carry out the frozen coefficient method to improve the regularity of the fluid velocity too. We refer the reader to [27] for a general idea of the frozen coefficient method. Also a more specific discussion relevant to the current model can be found in [11] (Appendix 6).
3.5. **Classical solution.** In this subsection, using the estimates (3.68)-(3.70), we apply the frozen coefficient method to improve regularities for \( \rho, p, u, d \) by a bootstrapping argument among the three equations (1.7)-(1.9). Hence Theorem 1.1 and 1.2 will be proved. The process of obtaining regular solutions in two dimensional case is much easier than the one in three dimensional case. And based on the previous work for the simplified Ericksen-Leslie model in [11], we just briefly show the steps to obtain regular solutions in 3D in the following.

We notice from (3.69) and (3.70)

\[ u \in L^\infty(0, T; L^6_p), \quad \nabla d \in L^\infty(0, T; L^3_p), \]

which implies

\[ u \cdot \nabla d \in L^\infty(0, T; L^3_p). \]

In the mean time, (3.69) and (3.70) indicate

\[ \Omega d, Ad \in L^\infty(0, T; L^q_p), \quad \forall q > 1. \]

By the standard parabolic estimates on equation (1.9) (cf. [25] and [1]), we have

\[ d \in W^{1, r}(W^{2, 3}_p), \quad \forall r > 1 \]

which implies that \( \nabla d \in L^\infty(0, T; L^q_p) \) for any \( q \in (1, \infty) \). Thus, we have

\[ u \cdot \nabla d \in L^\infty(0, T; L^q_p), \quad \forall q \in (1, 6). \]

Applying the same standard parabolic estimates on (1.9) again yields

\[ d \in W^{1, r}(W^{2, q}_p), \quad \forall r \in (1, \infty) \quad \text{and} \quad q \in (1, 6), \]

which implies that \( d \in C^{\alpha/2, 1+\alpha}([0, T] \times \bar{Q}) \) for some \( \alpha \in (0, 1) \) and

\[ \nabla d \Delta d \in L^\infty(0, T; L^3_p), \quad \forall q \in (1, 6). \]

In Navier-Stokes equation (1.8), the estimates for the conservation of momentum with constant density can be extended to the non-constant density case when Lemma 3.10 is available. This is done via the frozen coefficient method.

The estimates (3.69) and (3.70) implies

\[ u \cdot \nabla u \in L^\infty(0, T; L^{3/2}_p), \]

while (3.72) and (3.73) implies

\[ \nabla \cdot \sigma \in L^\infty(0, T; L^2_p). \]

Now we apply the frozen coefficient method using the oscillation estimates for the density as in Lemma 3.10 to yield

\[ u \in W^{1, q}(W^{2, 3/2}_p), \quad \forall q \in (1, \infty). \]

Thus \( u \in L^\infty(0, T; W^{1, 3}) \) and \( u \cdot \nabla u \in L^\infty(0, T; L^2_p) \). Repeating the above argument yields

\[ u \in W^{1, q}(W^{2, 2}_p), \quad \forall q \in (1, \infty), \]

from which it follows that \( u \in C^\alpha([0, T] \times Q) \).

Back to equation (1.9), we conclude that

\[ d \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, T) \times Q), \quad \text{for some} \ \alpha \in (0, 1). \]

Finally \( \rho \in C^1((0, T) \times Q) \) follows from the regularity of \( u \) and the regularity of the pressure \( p \) follows easily from the regularity of \( (\rho, u, d) \) similarly as in [2]. This completes the
4. APPENDIX: PROOF OF UNIQUENESS

In this appendix we sketch the proof of Theorem 1.5. For the full Ericksen-Leslie system with constant density, Wu, Xu and Liu [44] proved that the regular solution \((u, d)\) is unique in the sense: two regular solutions starting from the same initial data are identical. In present paper, for the full Ericksen-Leslie system with non-constant density, we show a weak-strong type of uniqueness result with certain conditions on the weak solution. The idea is to calculate the energy law satisfied by the difference of the regular and weak solutions and establish a Gronwall’s inequality. In our case, to calculate the energy law of the difference of the regular and the weak solutions it has some extra terms involving the density. We shall proceed an analogous computation as we did in [11] to achieve the goal. Similarly, the estimates are more involved requiring additional bounds on the strong solution \((\rho, u, d)\) to yield a Gronwall’s inequality. In 2D, we need

\[
\nabla \rho, \nabla u \in L^\infty((0,T) \times \Omega), \ u_t, u \cdot \nabla u \in L^\infty(0,T; L^q(\Omega)), q > 2.
\]

In 3D, we need

\[
\nabla \rho, \nabla u \in L^\infty((0,T) \times \Omega), \ u_t, u \cdot \nabla u \in L^\infty(0,T; L^3(\Omega)).
\]

With the assumption on data, \(\rho_0 \in C^1(\bar{\Omega}), u_0 \in C^{2+\alpha}(\bar{\Omega})\) and \(d_0 \in C^{2+\alpha}(\bar{\Omega})\), the solution \((\rho, u, d)\) from Theorem 1.1 or Theorem 1.2 satisfies (4.75) or (4.76), respectively.

Proof of Theorem 1.5: First recall that the regular solution \((\rho, u, d)\) from Theorem 1.1 or Theorem 1.2 satisfies the energy equality:

\[
\int_\Omega \frac{1}{2} \rho|u|^2 + \frac{1}{2} |\nabla d|^2 + \rho F(d) \, dx
\]

\[
+ \int_0^T \int_Q \mu_4 |\nabla u|^2 - \frac{1}{\lambda_1} |\Delta d - f(d)|^2 + \mu_1 |d^T Ad|^2 + (\mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1}) |Ad|^2 \, dx dt
\]

\[
= \int_Q \frac{1}{2} \rho_0 |u_0|^2 + \frac{1}{2} |\nabla d_0|^2 + \rho_0 F(d_0) \, dx.
\]

The density \(\rho\) is the strong solution of the transport equation, hence it satisfies that

\[
\int_Q \rho^2 \, dx = \int_\Omega \rho^2_0 \, dx.
\]

On the other hand, \(\bar{\rho}\) is a weak solution of the transport equation and \(M_1 \leq \bar{\rho} \leq M_2\). We have by hypothesis that

\[
\int_\Omega \bar{\rho}^2 \, dx \leq \int_\Omega \rho^2_0 \, dx.
\]

Thus,

\[
\frac{1}{2} \int_\Omega |\rho - \bar{\rho}|^2 \, dx = \frac{1}{2} \int_\Omega \rho^2 \, dx + \frac{1}{2} \int_\Omega \bar{\rho}^2 \, dx - \int_\Omega \rho \bar{\rho} \, dx
\]

\[
\leq \int_\Omega \rho^2_0 \, dx - \int_\Omega \rho \bar{\rho} \, dx.
\]
Since $\rho \in C^1([0, T] \times \bar{\Omega})$, we can take $\rho$ as a test function. Thus, multiplying
\[
\bar{\rho}_t + \bar{u} \cdot \nabla \bar{\rho} = 0
\]
by $\rho$ and integrating by parts yields
\[
(4.80) \quad \int_\Omega \rho \bar{\rho}_t \, dx - \int_\Omega \rho \bar{\rho} \, dx = -\int_0^t \int_\Omega \bar{\rho}_t \rho \, dx - \int_0^t \int_\Omega (\bar{u} \cdot \nabla \rho) \bar{\rho} \, dx = \int_0^t \int_\Omega (u \cdot \nabla \rho) \bar{\rho} \, dx - \int_0^t \int_\Omega (\bar{u} \cdot \nabla \rho) \bar{\rho} \, dx.
\]
Here we used again that $\rho$ is a classical solution of the transport equation. Substituting (4.80) in (4.79) gives
\[
(4.81) \quad \frac{1}{2} \int_\Omega |\rho - \bar{\rho}|^2 \, dx \leq \int_0^t \int_\Omega \rho(u - \bar{u}) \nabla \rho \, dx = \int_0^t \int_\Omega (\rho - \bar{\rho})(u - \bar{u}) \nabla \rho \, dx.
\]
Next, calculate the following term
\[
\frac{1}{2} \int_\Omega \bar{\rho}|u - \bar{u}|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla d - \nabla \bar{d}|^2 \, dx
\]
\[
= \frac{1}{2} \int_\Omega (\bar{\rho} - \rho)|u|^2 \, dx + \frac{1}{2} \int_\Omega |\bar{\rho}|^2 \, dx + \frac{1}{2} \int_\Omega \bar{\rho} \cdot \bar{u} \, dx + \frac{1}{2} \int_\Omega |\nabla d|^2 \, dx - \int_\Omega \nabla \otimes \nabla \bar{d} \, dx
\]
where $\nabla d \otimes \nabla \bar{d}$ denotes the $3 \times 3$ matrix whose $ij$-th entry is given by $\nabla i d \cdot \nabla j \bar{d}$ for $1 \leq i, j \leq 3$.
Using energy equality (4.77) for the regular solution $(\rho, u, d)$ and inequality (1.16) for the weak solution $(\bar{\rho}, \bar{u}, \bar{d})$ combined with the last equation yields
\[
\frac{1}{2} \int_\Omega \bar{\rho}|u - \bar{u}|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla d - \nabla \bar{d}|^2 \, dx
\leq - \int_0^t \int_\Omega \mu_4 |\nabla u - \nabla \bar{u}|^2 - \frac{1}{\lambda_1} |\Delta d - \Delta \bar{d}|^2 + \mu_1 |d^T Ad - \bar{d}^T \bar{A} \bar{d}|^2 \, dx dt
\]
\[
- \int_0^T \int_\Omega \left( \mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1} \right) |Ad - \bar{A}\bar{d}|^2 \, dx dt + \frac{1}{2} \int_\Omega (\bar{\rho} - \rho)|u|^2 \, dx
\]
\[
- \int_\Omega \bar{\rho} \otimes \bar{u} - \rho_0 |u_0|^2 \, dx - \int_\Omega \nabla d \otimes \nabla \bar{d} - |\nabla d_0|^2 \, dx
\]
\[
- \int_\Omega \rho F(d) + \bar{\rho} F(d) - 2\rho_0 F(d_0) \, dx - 2\mu_1 \int_0^t \int_\Omega d^T Ad \, dx dt
\]
\[
- 2\mu_4 \int_0^t \int_\Omega \nabla u \nabla \bar{u} \, dx dt + \frac{2}{\lambda_1} \int_0^t \int_\Omega (\Delta d) \, dx dt
\]
\[
- 2 \left( \mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1} \right) \int_0^t \int_\Omega Ad \bar{A} \, dx dt
\]
\[
+ \frac{1}{\lambda_1} \int_0^T \int_\Omega |f(d)|^2 + |f(\bar{d})|^2 \, dx dt - \frac{2}{\lambda_1} \int_0^t \int_\Omega \Delta f(d) + \Delta \bar{f}(\bar{d}) \, dx dt.
\]
Since $u, d \in C^{1+\alpha/2,2+\alpha}([0,T] \times \bar{Q})$, we can take $u, d$ as test functions for the weak solution $\bar{u}, \bar{d}$. Thus it follows that

$$
\int_Q \bar{\rho} \otimes \bar{\rho} - \rho_0 |u_0|^2 dx = \int_0^t \int_Q \bar{\rho} \bar{u} \bar{u} dxdt + \int_0^t \int_Q \bar{\rho} \bar{u} (\bar{u} \cdot \nabla u) dxdt
+ \int_0^t \int_Q \nabla \bar{d} \nabla \bar{d} \nabla u dxdt + \int_0^t \int_Q \nabla \cdot \bar{\sigma} u dxdt,
$$

(4.83)

$$
\int_Q \nabla d \otimes \nabla d - |\nabla d_0|^2 dx
= - \int_0^T \int_Q \Delta \bar{d}_d dxdt + \int_0^T \int_Q \Delta d (\bar{u} \cdot \nabla \bar{d}) dxdt - \int_0^T \int_Q \Delta \bar{d} \bar{d} dxdt
+ \frac{\lambda_2}{\lambda_1} \int_0^T \int_Q \Delta \bar{d} \bar{d} dxdt + \frac{1}{\lambda_1} \int_0^T \int_Q \Delta (\bar{d} - f(\bar{d})) dxdt.
$$

(4.84)

Substituting (4.83) and (4.84) in (4.82) and adding (4.81) yields

$$
\begin{align*}
\frac{1}{2} \int_Q |\rho - \bar{\rho}|^2 dx &+ \frac{1}{2} \int_Q |\bar{\rho} u - \bar{u} u|^2 dx + \frac{1}{2} \int_Q |\nabla d - \nabla d|^2 dx \\
&\leq - \int_0^t \int_Q \mu_4 |\nabla u - \nabla \bar{u}|^2 - \frac{1}{\lambda_1} |\Delta d - \Delta \bar{d}|^2 + \mu_4 |d^T A d - \bar{d}^T \bar{A} \bar{d}|^2 dxdt \\
&- \int_0^T \int_Q \left( \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) |\nabla d - \nabla \bar{d}|^2 dxdt + \frac{1}{2} \int_Q |\bar{\rho} - \rho| |u|^2 dx \\
&- \int_0^Q \rho F(d) + \bar{\rho} F(\bar{d}) - 2 \rho_0 F(d_0) dx + \int_0^t \int_Q (\bar{\rho} - \rho) (u - \bar{u}) \nabla \rho dxdt \\
&- \int_0^t \int_Q \bar{\rho} u u dxdt - \int_0^t \int_Q \bar{\rho} \bar{u} \bar{u} dxdt - \int_0^t \int_Q \nabla \bar{d} \nabla u dxdt \\
&- \int_0^t \int_Q u \nabla \cdot \bar{\sigma} dxdt + \mu_4 \int_0^t \int_Q \nabla \bar{u} \nabla u dxdt + \int_0^t \int_Q \Delta d_d dxdt \\
&- \int_0^t \int_Q u \nabla \bar{d} \nabla d dxdt + \int_0^t \int_Q \bar{d} \nabla d dxdt - \int_0^t \int_Q \nabla \bar{d} \nabla d dxdt \\
&- 2 \mu_4 \int_0^t \int_Q d^T \bar{A} d \bar{d} dxdt - 2 \mu_4 \int_0^t \int_Q \nabla u \nabla \bar{u} dxdt \\
&+ \frac{2}{\lambda_1} \int_0^t \int_Q \Delta d \Delta \bar{d} dxdt - 2 \left( \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) \int_0^t \int_Q A \bar{d} \bar{d} dxdt \\
&+ \frac{1}{\lambda_1} \int_0^t \int_Q |f(d)|^2 + |f(\bar{d})|^2 dxdt - \frac{2}{\lambda_1} \int_0^t \int_Q \Delta d f(d) + \Delta \bar{d} f(\bar{d}) dxdt.
\end{align*}
$$

(4.85)

Note that

$$
- \int_0^t \int_Q \bar{\rho} u u dxdt = - \int_0^t \int_Q (\bar{\rho} - \rho) (u - \bar{u}) u dxdt
- \int_0^t \int_Q (\bar{\rho} - \rho) u u dxdt - \int_0^t \int_Q \bar{\rho} u u dxdt,
$$
while

\[ -\int_0^t \int_Q (\bar{\rho} - \rho) uu_t \, dx \, dt = -\frac{d}{dt} \int_0^t \int_Q (\bar{\rho} - \rho) \frac{|u|^2}{2} \, dx \, dt + \int_0^t \int_Q (\bar{\rho} - \rho) \frac{|u|^2}{2} \, dx \, dt \]

\[ = -\int_0^t \int_Q (\bar{\rho} - \rho) \frac{|u|^2}{2} \, dx \, dt - \int_0^t \int_Q \text{div}(\bar{\rho} \bar{u} - \rho u) \frac{|u|^2}{2} \, dx \, dt \]

\[ = -\int_0^t \int_Q (\bar{\rho} - \rho) \frac{|u|^2}{2} \, dx \, dt + \int_0^t \int_Q \bar{\rho} uu \nabla u \, dx \, dt \]

\[ - \int_0^t \int_Q (\bar{\rho} - \rho) uu \nabla u \, dx \, dt - \int_0^t \int_Q \bar{\rho} uu \nabla u \, dx \, dt \]

and

\[ -\int_0^t \int_Q \bar{\rho} uu_t \, dx \, dt = \mu_4 \int_0^t \int_Q \nabla u \nabla \bar{u} \, dx \, dt + \int_0^t \int_Q \rho uu \nabla u \, dx \, dt \]

\[ + \int_0^t \int_Q \nabla d \Delta \bar{u} \, dx \, dt - \int_0^t \int_Q \bar{u} \nabla \cdot \sigma' \, dx \, dt. \]

On the other hand, we have

\[ \int_0^t \int_Q \Delta \bar{d} \, dx \, dt \]

\[ = -\int_0^t \int_Q \Delta \bar{d} \nabla \bar{d} \, dx \, dt + \int_0^t \int_Q \Delta \bar{d} \Omega d \, dx \, dt - \frac{\lambda_2}{\lambda_1} \int_0^t \int_Q \Delta \bar{d} A d \, dx \, dt \]

\[ - \frac{1}{\lambda_1} \int_0^t \int_Q \Delta \bar{d} \Delta d \, dx \, dt + \frac{1}{\lambda_1} \int_0^t \int_Q \Delta \bar{f}(d) \, dx \, dt. \]
Thus with some cancelation (4.85) becomes

\[
\frac{1}{2} \int_Q |\rho - \bar{\rho}|^2 \, dx + \frac{1}{2} \int_Q |\bar{\rho} - \bar{\bar{\rho}}|^2 \, dx + \frac{1}{2} \int_Q |\nabla d - \nabla \bar{d}|^2 \, dx \leq - \int_0^t \int_Q \mu_4 |\nabla u - \nabla \bar{u}|^2 - \frac{1}{\lambda_1} |\Delta d - \Delta \bar{d}|^2 + \mu_1 |d^T A d - \bar{d}^T \bar{A} \bar{d}|^2 \, dx \, dt \\
\quad - \int_0^t \int_Q \left( \mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1} \right) |A d - \bar{A} \bar{d}|^2 \, dx \, dt \\
\quad + \int_0^t \int_Q (\bar{\rho} - \rho) u \nabla \rho \, dx \, dt - \int_0^t \int_Q (\bar{\rho} - \rho) (\bar{u} - u) u \, dx \, dt \\
\quad - \int_0^t \int_Q \bar{\rho} \bar{u} \nabla u - \bar{\rho} \bar{u} \nabla u + \bar{\rho} u \nabla u - \bar{\rho} \bar{u} \nabla u \, dx \, dt \\
\quad - \int_0^t \int_Q (\rho - \bar{\rho}) u u \nabla u - (\rho - \bar{\rho}) \bar{u} \nabla u \, dx \, dt \\
\quad + \int_0^t \int_Q \nabla d \Delta \bar{d} u + \nabla d \Delta \bar{d} \bar{u} - \nabla \bar{d} \Delta d \bar{u} - \nabla \bar{d} \Delta \bar{d} \bar{u} \, dx \, dt \\
\quad - \int_0^t \int_Q u \nabla \cdot \bar{\sigma}' + \bar{u} \nabla \cdot \sigma' \, dx \, dt \\
\quad - \int_0^t \int_Q 2 \mu_1 d^T A d \bar{d} d + 2 \left( \mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1} \right) A d \bar{A} \bar{d} \, dx \, dt \\
\quad + \int_0^t \int_Q \Omega d \Delta \bar{d} + \Omega d \Delta d - \frac{\lambda_2}{\lambda_1} (A d \Delta d + A d \Delta \bar{d}) \, dx \, dt \\
\quad + \frac{1}{\lambda_1} \int_0^t \int_Q \Delta \bar{d} f (d) + \Delta df (\bar{d}) - 2 \Delta df (d) - 2 \Delta \bar{d} f (\bar{d}) \, dx \, dt \\
\quad - \int_Q \rho F (d) + \bar{\rho} F (\bar{d}) - 2 \rho_0 F (d_0) \, dx + \frac{1}{\lambda_1} \int_0^T \int_Q |f (d)|^2 + |f (\bar{d})|^2 \, dx \, dt \\
:= - G_1 - G_2 + I_1 + I_2 + \ldots + I_{11}.
\]

Note that

\[
I_3 = - \int_0^t \int_Q \bar{\rho} \nabla u |u - \bar{u}|^2 \, dx \, dt, \\
I_4 = \int_0^t \int_Q (\rho - \bar{\rho}) u \nabla (\bar{u} - u) \, dx \, dt, \\
I_5 = \int_0^t \int_Q u (\nabla d - \nabla \bar{d}) (\Delta d - \Delta \bar{d}) \, dx \, dt \\
\quad - \int_0^t \int_Q (\bar{u} - u) (\nabla \bar{d} - \nabla d) \Delta d \, dx \, dt.
\]
Using the relation $\lambda_2 = \mu_2 + \mu_3$, $I_6 + I_7$ can be reorganized as

$$I_6 + I_7 = \mu_1 \int_0^t \int_Q d^T A(d \otimes d - \tilde{d} \otimes \tilde{d})(\nabla \tilde{u} - \nabla u)dxdt$$

$$+ \mu_1 \int_0^t \int_Q (d^T Ad - d^T \tilde{A}d)(d \otimes d - \tilde{d} \otimes \tilde{d})\nabla udxdt$$

$$+ \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right) \int_0^t \int_Q d^T A(d - \tilde{d})(\nabla \tilde{u} - \nabla u)dxdt$$

Applying (1.9) and the fact $\int_Q u \cdot \nabla d f(d)dx = 0$, we derive that

$$I_8 + I_9 + I_{10} + I_{11}$$

$$= -\int_0^t \int_Q \left(\Omega - \frac{\lambda_2^2}{\lambda_1} A\right)(d - \tilde{d})(\Delta d - \Delta \tilde{d})dxdt$$

$$+ \int_0^t \int_Q \left(\left(\Omega - \frac{\lambda_2^2}{\lambda_1} A\right) - \left(\tilde{\Omega} - \frac{\lambda_2^2}{\lambda_1} \tilde{A}\right)\right)(d - \tilde{d})\Delta d dxdt$$

$$- \frac{1}{\lambda_1} \int_0^t \int_Q (\Delta d - \Delta \tilde{d})(f(d) - f(\tilde{d}))dxdt$$

(4.87)

Recall that the regular solution $(\rho, u, d)$ satisfies (4.75) in 2D and (4.76) in 3D. By Hölder and Gagliardo-Nirenberg inequalities on the terms of $I_1, \ldots, I_{11}$, it follows that

$$\frac{1}{2} \int_Q |\rho(t) - \bar{\rho}(t)|^2 + |\rho(t)|u(t) - \bar{u}(t)|^2 + |\nabla d(t) - \nabla \bar{d}(t)|^2 dx$$

$$\leq C \int_0^t \int_Q |\rho - \bar{\rho}|^2 + |\rho|u - \bar{u}|^2 + |\nabla d - \nabla \bar{d}|^2 dxdt. (4.88)$$

To handle the last integral in (4.87), we used the fact that $|d| \leq 1$ and the hypothesis $|\tilde{d}|$ is bounded, which imply $|f(d) - f(\tilde{d})| \leq C|d - \tilde{d}|$ by the definition of $f(d)$. Thus,

$$\int_0^t \int_Q |f(d) - f(\tilde{d})|^2 dxdt \leq C \int_0^t \int_Q |d - \tilde{d}|^2 dxdt \leq C(Q) \int_0^t \int_Q |\nabla d - \nabla \tilde{d}|^2 dxdt$$

where the constant $C$ depends on space domain $Q$ not on time $T$.

Thanks to the fact $\bar{\rho} \geq M_1 > 0$, applying Gronwall’s inequality to (4.88) we obtain

$$\frac{1}{2} \int_0^t \int_\Omega |\rho(t) - \bar{\rho}(t)|^2 + |\rho|u(t) - \bar{u}(t)|^2 + |\nabla d(t) - \nabla \tilde{d}(t)|^2 dx$$

$$\leq \int_\Omega |\rho(0) - \bar{\rho}(0)|^2 + |\rho(0)|u(0) - \bar{u}(0)|^2 + |\nabla d(0) - \nabla \tilde{d}(0)|^2 dx e^{Ct}$$

$$= 0$$

for all $t > 0$ which implies

$$\bar{\rho} - \rho = \bar{u} - u = \bar{d} - d \equiv 0.$$

This completes the proof of Theorem 1.5.
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DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF COLORADO BOULDER, BOULDER, CO, 80309, USA

E-mail address: mimi.dai@colorado.edu