Gauge transformations and quasitriangularity

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Abstract

Natural conditions on a Poisson/quantum group $G$ to implement Poisson/quantum gauge transformations on the lattice are investigated. In addition to our previous result that transformations on one lattice link require $G$ to be coboundary, it is shown that for a sequence of links one needs a quasitriangular $G$.

0 Introduction: one link results

Is it possible to formulate lattice (group valued) gauge fields with a Poisson (or quantum) group as the structure group, in such a way that the gauge transformations are Poisson or quantum actions? Which conditions should such a group satisfy?

In [1] we have answered the last question in case of gauge transformations performed on one link only. We recall these results.

1. Let $(G, \pi)$ be a Poisson Lie group. There exists a Poisson structure $\sigma$ on $G$ such that the map

$$G \times G \times G \ni (x, y, z) \mapsto xyz^{-1} \in G$$

is Poisson as a map from $(G, \pi) \times (G, \sigma) \times (G, \pi)$ to $(G, \sigma)$ if and only if $(G, \pi)$ is coboundary. In this case $\sigma$ is just the ‘plus’ structure

$$\sigma(g) = \pi_+(g) := rg + gr,$$

where $r \in \Lambda^2 g$ such that $\pi(g) = rg - gr$.

2. Let $(H, m, \Delta)$ be a Hopf algebra. Let $\Delta^\text{op}$ denote the comultiplication opposite to $\Delta$: $\Delta^\text{op} = P \circ \Delta$, where $P$ is the permutation in the tensor product.
There exists a (new) coalgebra structure $\Sigma: H \to H \otimes H$ such that the map
\[ m(m \otimes m): H \otimes H \otimes H \to H \] is a morphism from $(H, \Delta) \otimes (H, \Sigma) \otimes (H, \Delta^{\text{op}})$ to $(H, \Sigma)$ if and only if there exists $R \in H \otimes H$ such that
\[ \Delta(a)R = R\Delta^{\text{op}}(a) \quad \text{and} \quad [(\Delta \otimes \text{id}) R]_{12} = [(\text{id} \otimes \Delta) R]_{23}. \] (4)

In this case, $\Sigma$ is given by the ‘plus structure’
\[ \Sigma(a) = \Delta_+(a) := \Delta(a)R. \] (5)

3. For matrix quantum groups with basic relations given by
\[ \hat{R}(u \uplus u) = (u \uplus u)\hat{R}, \] (6)
the basic ‘plus’ commutation relations are given by
\[ \hat{R}(u \uplus u) = (u \uplus u)\hat{R}_{21} \quad (R_{21} \equiv P\hat{R}P). \] (7)

1 Two links — the results

In this paper we consider the case of gauge transformations on two consecutive links of the lattice. In the sequel, we concentrate mainly on the Poisson case. We assume already, that we have the ‘plus’ structure on links, and we investigate conditions under which the map
\[ G \times G \times G \ni (a, g, b) \mapsto (ag^{-1}, gb) \in G \times G \] (8)
is Poisson as a map from $(G, \pi_+) \times (G, \pi) \times (G, \pi_+) \to (G, \pi_+) \times (G, \pi_+)$. Due to appearance of the same $g$ in both components of map (8), the push-forward of $\pi$ by this map has necessarily a cross-term. It is therefore necessary to admit an additional braiding (nontrivial cross-relations) between links. The full Poisson structure on two links will be then given by
\[ \pi_{+++}(a, b) = \pi_+(a) \oplus \pi_+(b) \oplus \pi_{\text{st}}(a, b), \quad a, b \in G, \] (9)
according to the obvious decomposition of the tangent space $T_{(a, b)}(G \times G)$. The cross-term $\pi_{\text{st}}$ can be always written in the form
\[ \pi_{\text{st}}(a, b) = (a, e)\phi(a, b)(e, b), \] (10)
where
\[ \phi(a, b) \in \mathfrak{g}_{(1)} \bigwedge \mathfrak{g}_{(2)} \cong \mathfrak{g} \otimes \mathfrak{g} \] (11)
($e \in G$ is the group unit).
Let us mention that the braiding in the corresponding quantum case will be given by a cross-product coalgebra structure on $H \otimes H$:

$$\Delta_+ = (\text{id} \otimes Q \otimes \text{id})(\Delta_+ \otimes \Delta_+),$$  \hspace{1cm} (12)

where $Q: H \otimes H \rightarrow H \otimes H$ is a linear invertible map.

Now we state our results (their proof is given in the Appendix).

**Proposition 1.1** The map given by formula (8) is Poisson if and only if

$$\phi(a, b) = -r + \psi(a, b), \quad \text{where} \quad \psi(ag^{-1}, gb) = \text{Ad}_g \psi(a, b).$$  \hspace{1cm} (13)

**Remark 1.2** The above condition for $\psi$ has a general solution given by $\psi(a, b) = \text{Ad}_b f(ab)$, where $f$ is an arbitrary function on $G$ with values in $\mathfrak{g} \otimes \mathfrak{g}$.

**Proposition 1.3** (constant case). Assume that $\phi = \text{const} = -w$ satisfies (13). Then $w - r$ is invariant and we may assume that it is symmetric (one can modify $r$, if necessary). The bivector field $\pi_+$ is Poisson if and only if

$$[w, w] \equiv [w_{12}, w_{13}] + [w_{12}, w_{23}] + [w_{13}, w_{23}] = 0.$$  \hspace{1cm} (14)

The above Proposition clearly states that in the constant case, our condition is equivalent to the quasitriangularity.

## 2 Composability

Another natural requirement concerning two links is that the composition (of parallel transports) should be a Poisson map, more precisely, the multiplication

$$G \times G \ni (a, b) \mapsto ab \in G$$  \hspace{1cm} (15)

should be a Poisson map as a map from $(G \times G, \pi_+)$ to $(G, \pi_+)$. 

**Proposition 2.1** The multiplication is a Poisson map if and only if

$$\phi(a, b)_{\text{anti}} = -r$$  \hspace{1cm} (16)

('anti' refers to the antisymmetric part).

**Proof:** For $\phi(a, b) = X \otimes Y \cong X(1) \wedge Y(2)$ we have $\pi_M(a, b) = (aX)(1) \wedge (Yb)(2)$ which is mapped by the multiplication on

$$(aXb)\wedge(aYb) = a(X \wedge Y)b = a(\phi(a, b) - P\phi(a, b)b) = a(2\phi(a, b)_{\text{anti}})b.$$ 

The result being valid for arbitrary $\phi$ (by linearity), the map $(a, b) \mapsto ab$ is Poisson if and only if

$$\pi_+(a, b) = (ra + ar)b + a(rb + br) + 2a(\phi(a, b)_{\text{anti}})b = rab + abr,$$
which yields the required result.

The above result complements Proposition [1], the antisymmetric part must be constant.

Concerning the quantum case, we have the following proposition.

**Proposition 2.2** The multiplication map \( m: H \otimes H \to H \) is a coalgebra homomorphism as a map from \((H \otimes H, \Delta_{++})\) to \((H, \Delta_+)\),

\[
\Delta_+ m = (m \otimes m) \Delta_{++},
\]

(17)

if and only if \( Q \) is given by

\[
Q = P(m \otimes m)(id \otimes P R_D),
\]

(18)

where \( R_D := R^{-1} \) (which is the \( R \)-matrix of Drinfeld's type). \( \Delta_{++} \) is coassociative if and only if \( R_D \) satisfies

\[
(\Delta \otimes id)R_D = (R_D)_{13}(R_D)_{23}, \quad (id \otimes \Delta)R_D = (R_D)_{12}(R_D)_{13},
\]

(19)

i.e. the quasitriangularity property.

**Proof:** First we show that (18) satisfies (17). Using \( \Delta_+(a) = \Delta(a) R \), we have

\[
(m \otimes id)(id \otimes Q)(\Delta_+ \otimes id) = (m \otimes id)(\Delta \otimes id)
\]

(diagrams!), hence

\[
(m \otimes m)(id \otimes Q \otimes id)(\Delta_+ \otimes \Delta_+) = (m \otimes m)(id \otimes P \otimes id)(\Delta \otimes \Delta_+) = \Delta_+ m.
\]

We shall show that this solution is unique. It is sufficient to show that

\[
m(X \otimes id)\Delta_+ = 0
\]

(20)

implies \( X = 0 \) for each \( X \in \text{End}(H) \). Assuming (20), and using

\[
(id \otimes m)(\Delta_+ \otimes S)P \Delta = (m \otimes id)(id \otimes P)(R \otimes id)
\]

(21)

(\( S \) is the antipode), we have

\[
0 = m(m \otimes id)(X \otimes id \otimes S)(\Delta_+ \otimes id)P \Delta = m(X \otimes id)(id \otimes m)(\Delta_+ \otimes S)P \Delta
\]

\[
= m(X \otimes id)(m \otimes id)(id \otimes P)(R \otimes id).
\]

Denoting the last expression by \( F \), we have

\[
0 = m(F \otimes id)(m \otimes id)(id \otimes P)(R_D \otimes id) = X.
\]
For the second part of the proof, recall that $\Delta_{++}$ is coassociative if and only if
\[
(\Delta_{+} \otimes \text{id})Q = (\text{id} \otimes Q)(\Delta_{+} \otimes \text{id}), \quad (\text{id} \otimes \Delta_{+})Q = (Q \otimes \text{id})(\Delta_{+} \otimes \text{id}).
\]
(23)
We denote by $P_{k_1k_2...k_n}$ the permutation of the tensor product sending the $j$-th tensor factor to the $k_j$-th place. Let $L$ and $R$ denote the left and right hand side of the first condition in (19). This condition is equivalent to $A = B$, where
\[
A = (m \otimes m \otimes \text{id})P_{13524}(L \otimes \text{id} \otimes \text{id}), \quad B = (m \otimes m \otimes \text{id})P_{13524}(R \otimes \text{id} \otimes \text{id}).
\]
(24)
Now note that the first condition in (23) can be written as
\[
(\text{id} \otimes \text{id} \otimes m)P_{3124}(id \otimes A)(id \otimes \Delta_{+}) = (\text{id} \otimes \text{id} \otimes m)P_{3124}(id \otimes B)(id \otimes \Delta_{+}).
\]
(25)
or,
\[
A\Delta_{+} = B\Delta_{+}.
\]
(26)
Therefore, the first condition in (19) implies the first condition in (23). The converse implication follows by applying (26) to $I$, and using $L = A(I \otimes I), R = B(I \otimes I)$. 

Using (18), we can derive a formula for the braiding on the level of matrix quantum groups. Calculating the transposed $Q$,
\[
Q^T = (\text{id} \otimes R^T_{P}P \otimes \text{id})(\Delta \otimes \Delta)
\]
(27)
where $\Delta = m^T$ is the comultiplication in the ‘function algebra’: $\Delta v^a_l = v^a_j \otimes v^j_l$ on $w^a_l \otimes v^k_b$, we obtain
\[
Q^T(w^a_l \otimes v^k_b) = (\text{id} \otimes R^T_{P} \otimes \text{id})(v^k_j \otimes w^a_m \otimes v^j_b \otimes w^m_l) = R^{a'_{mb}}_{m'_{ab}}v^k_j \otimes w^m_l,
\]
(28)
where $R^{a'_{mb}}_{m'_{ab}} = \langle R_{D}^{T}, w^a_m \otimes v^j_b \rangle$ is the $R$-matrix of FRT type ($R = P \hat{R}$), hence the crossed multiplication rule in the tensor product of two copies of algebras with the ‘plus’ structure
\[
\hat{R}v^k \oplus v = v \oplus v \hat{R}_{21}, \quad \hat{R}w^k \oplus w = w \oplus w \hat{R}_{21},
\]
(29)
is given by
\[
w^a_l v^k_b = v^k_s R^{a'_{mb}}_{m'_{ab}} w^s_l.
\]
(30)
One can see directly that the composition is a homomorphism, by checking that $v^{q}_{a}w^a_l$ satisfy again the ‘plus’ relations:
\[
\hat{R}^{pq}_{jk} v^q_{a} w^a_l v^k_b w^b_m = \hat{R}^{pq}_{jk} v^q_{a} \hat{R}^{sa}_{t b} w^t_l w^b_m = v^{p}_{j} v^{q}_{k} \hat{R}^{kj}_{s a} w^s_{t} w^a_{b} \hat{R}^{b t}_{m l} = v^{p}_{j} v^{q}_{a} w^a_{b} \hat{R}^{b t}_{m l}.
\]
(One can also see that (28) and the transpose of (23) define $Q^T$ consistently on higher order polynomials, due to Yang-Baxter equation.)

5
3 Final remarks

We have seen that in order to implement gauge transformations (1), (8) as Poisson (or quantum) homomorphisms, the Poisson (quantum) gauge group should be quasitriangular, in the \textit{real sense}. This excludes in particular the standard deformations of compact simple groups, which are always quasitriangular in the \textit{imaginary sense} (cf. also [2]).

A simple imaginary quasitriangular case is provided by the standard deformation of $SU(2)$. Expanding the standard $R$-matrix for $SU(2)$ up to linear terms in $\varepsilon = q - 1$, we get

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-\frac{1}{2}} & 0 & 0 \\ 0 & q^{-\frac{1}{2}} - q^{-\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix} \sim I + \varepsilon \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 2 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$= I + \varepsilon (2X_+ \otimes X_+ + \frac{1}{2} \sigma_3 \otimes \sigma_3) = I + iw,$$

where

$$X_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$iw = \varepsilon X_- \wedge X_+ + \varepsilon (X_- \otimes X_+ + X_+ \otimes X_- + \frac{1}{2} \sigma_3 \otimes \sigma_3)$$

$$= \frac{i\varepsilon}{2} \sigma_1 \wedge \sigma_2 + \frac{\varepsilon}{2} (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) = i(r - is),$$

$$r := \frac{\varepsilon}{2} \sigma_1 \wedge \sigma_2 \in \bigwedge^2 su(2), \quad s := \frac{1}{2} \sum_{j=1}^{3} \sigma_j \otimes \sigma_j \in \bigotimes_{\text{symm}}^2 su(2).$$

(Here $\sigma_j$ are the Pauli matrices.) We see that the symmetric part of $w$ is imaginary.

Consider now the case of the (standard) quantum $SU(N)$. We assume therefore that the matrices $v$ and $w$ in (30) are unitary. It is easy to see that the conjugation of (30) gives again (30) (does not lead to new relations) if and only if $\hat{R}$ is unitary. This is in contradiction with the fact that it is actually self-adjoint (but not involutive).

Finally, we remark that having transformations (1), (8) implemented on the Poisson (quantum) level, they are automatically implemented on the whole one-dimensional lattice.

4 Appendix
4.1 Proof of Prop. 1.1

Let $\Phi$ be the map in (8) and let us denote isomorphism (11) explicitly by

$$g \otimes g \ni X \otimes Y \mapsto (X \otimes Y)_{(12)} := X_{(1)} \wedge Y_{(2)} \in g_{(1)} \wedge g_{(2)}.$$ 

For $X, Y \in \mathfrak{g}$, we have $\Phi^*(Xg) = (-ag^{-1}X)_{(1)} + (Xgb)_{(2)}$ and

$$\Phi^*(Xg) \wedge \Phi^*(Yg) = (ag^{-1}X)_{(1)} \wedge (ag^{-1}Y)_{(1)} + (Xgb)_{(2)} \wedge (Ygb)_{(2)} -
-(ag^{-1}X)_{(1)} \wedge (Ygb)_{(2)} - (Xgb)_{(2)} \wedge (ag^{-1}Y)_{(1)},$$

hence, for $r \in \mathfrak{g}$,

$$\Phi^* (rg) = (ag^{-1}r)_{(1)} + (rgb)_{(2)} - (ag^{-1})_{(1)} r_{12} (gb)_{(2)}. \quad (31)$$

Similarly, using $\Phi^*(gX) = (-aXg^{-1})_{(1)} + (gXb)_{(2)}$

we obtain

$$\Phi^* (gX) = (-aXg^{-1})_{(1)} + (gXb)_{(2)}$$

$$\Phi^* (gr) = (arg^{-1})_{(1)} + (grb)_{(2)} - (a, g) r_{12} (g^{-1}, b). \quad (32)$$

Since $a_{(1)} (X \otimes Y)_{12} b_{(2)} = (aX)_{(1)} \wedge (Yb)_{(2)}$, and

$$\Phi^* (aX)_{(1)} \wedge \Phi^* (Yb)_{(2)} = (aXg^{-1})_{(1)} \wedge (gYb)_{(2)} = (a, g) X_{(1)} \wedge Y_{(2)} (g^{-1}, b), \quad (33)$$

we have

$$\Phi^* \pi_\otimes (a, b) = (a, g) \phi(a, b)_{12} (g^{-1}, b). \quad (34)$$

Of course

$$\Phi^* \pi_+ (a) = ((ra + ar)g^{-1})_{(1)}, \quad \Phi^* \pi_+ (b) = (g(ra + ar))_{(2)}. \quad (35)$$

Inserting (31)-(35) into

$$\Phi^* (\pi_+ (a) + \pi_+ (b) + \pi_\otimes (a, b) + \pi(g)) = (rag^{-1} + ag^{-1}r)_{(1)} + (rgb + gbr)_{(2)} + \pi_\otimes (ag^{-1}, gb)$$

we obtain

$$\phi(ag^{-1}, gb) + r = \text{Ad}_g (\phi(a, b) + r). \quad (36)$$

4.2 Proof of Prop. 1.3

For $X \in \mathfrak{g}$ we set $X^L (g) := gX, X^R (g) := Xg$ and, similarly for higher rank tensors. In particular, $\pi_+ = r^L + r^R$. Expanding $w$ in some basis $X_k$ in $\mathfrak{g}$, we have

$$-\pi_\otimes (a, b) = a_{(1)} w_{12} b_{(2)} = a_{(1)} w^{kl} X_{k(1)} \wedge X_{l(2)} b_{(2)} = w^{kl} X^L_{k(1)} \wedge X^R_{l(2)} =: w^L_{12}.$$
Since $\pi_+ = r^{L}_{(1)} + r^{R}_{(1)} + r^{L}_{(2)} + r^{R}_{(2)} - w^{LR}_{12}$, we have
\[ [\pi_+, \pi_+] = -2[r^{L}_{(1)} + r^{L}_{(2)} + r^{R}_{(2)}, w^{LR}_{12}] + [w^{LR}_{12}, w^{LR}_{12}] \]
\[ = -2[r^{L}_{(1)} + r^{R}_{(2)}, w^{LR}_{12}] + [w^{LR}_{12}, w^{LR}_{12}]. \]

We set $r = \frac{1}{2} r^{ij} X_i \wedge X_j$ and calculate the first term:
\[ [r^{L}_{(1)}, w^{LR}_{12}] = \frac{1}{2} r^{ij} w^{kl} [X^{L}_{i(1)} \wedge X^{L}_{j(1)}, X^{L}_{k(1)} \wedge X^{R}_{l(2)}] = r^{ij} w^{kl} X^{L}_{i(1)} \wedge [X_{j}, X^{L}_{k(1)} \wedge X^{R}_{l(2)}] \]
\[ \cong r^{ij} w^{kl} X^{L}_{i(1)} \wedge [X_{j}, X^{L}_{k(1)} \otimes X^{R}_{l(2)}] \]
\[ = r^{ij} w^{kl} (X^{L}_{i(1)} \otimes [X_{j}, X^{L}_{k(1)}]) - [X_{j}, X^{L}_{k(1)} \otimes X^{R}_{l(2)}] \wedge X^{R}_{l(2)} = [r_{12}, w_{23}] + [r_{12}, w_{13}], \]
the last expression being interpreted as an element of $g_{[11]2} := (\bigwedge g^{L}_{(1)} \wedge g^{R}_{(1)} \otimes 2)$. Similarly,
\[ [r^{R}_{(2)}, w^{LR}_{12}] = [r_{23}, w_{13}] + [r_{23}, w_{12}], \]
(37)
where the right hand side is understood as an element of $g_{[12]}^{2} := g^{L}_{(1)} \wedge (\bigwedge g^{R}_{(2)})$, and also
\[ [w^{LR}_{12}, w^{LR}_{12}] = 2[w_{12}, w_{13}] - 2[w_{13}, w_{23}], \]
(38)
where the terms on the right hand side are elements of $g_{[12]}^{2}$ and $g_{[11]2}$, respectively. It follows that $\pi_+$ is Poisson if and only if
\[ [r_{12}, w_{13}] + [r_{12}, w_{23}] + [w_{13}, w_{23}] = 0 \]
(39)
and
\[ [w_{12}, w_{13}] + [w_{12}, r_{23}] + [w_{13}, r_{23}] = 0. \]
(40)
Due to the invariance of the symmetric part $w-r$ of $w$, both equations are equivalent to (14).

References

[1] S. Zakrzewski, A characterization of coboundary Poisson groups and Hopf algebras, Proceedings of the Banach Center Minisemester on Quantum Groups, Warsaw, November 1995, to appear. Also, q-alg/9602002.

[2] S. Zakrzewski, Phase spaces related to standard $r$-matrices, J. Phys. A: Math. Gen. 29 (1996) 1841–1857.