Derivation of effective parameters of magnetic metamaterials composed of passive resonant LC inclusions

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We determine rigorously the effective permeability of magnetic metamaterials composed of passive particles exhibiting magnetic resonances. The effective permeability is expressed in terms of circuit parameters characteristic to these particles (L, C, and R) and particle geometry. The derivation takes into account the magnetic coupling between particles. We show that a bigger concentration of particles leads to an improvement in the metamaterial performances (such as bandwidth or loss tangent), but this effect saturates as the particles are packed more tightly. The theory is validated through numerical simulations of physically realizable metamaterials.

Since the development of the first electromagnetic metamaterials based on split-ring-resonators (SRRs) and wires, it has been acknowledged that the narrow bandwidth, high dispersion, and significant loss inside them have been the main factors that limited their practical use in commercial applications. Theoretical analysis of the individual particles that compose such metamaterials have been performed before [1, 2], and gave valuable insight on their physics. However, a complete and reliable theoretical derivation of the effective material parameters of these metamaterials in terms of their geometry has not been developed yet.

In this paper we focus on metamaterials having negative permeability, which are commonly designed and built using periodic arrays of unit cells composed of inclusions such as SRRs [3], omega particles [4], and other types of similar resonant particles [5]. Our purpose is to derive rigorously the effective permeability of such metamaterials in terms of the circuit parameters characteristic to the composing particles (i.e. inductance, L, capacitance, C, and resistance, R) and the particle geometry represented by the geometry factor F. We will show that larger F factors lead to an increase in the performance in terms of bandwidth or loss tangent. However, we will show that F cannot be made arbitrarily large and reaches an upper bound as the particles are packed closer together. Numerical simulations will be performed to validate the theoretical results.

Consider the unit cell depicted in Figure 1. All the resonant particles commonly used to generate negative permeability (such as SRRs, omega particles) are essentially LC resonant circuits [1, 2] that can be represented in general by a planar, capacitively loaded loop as depicted in Figure 1. Losses are taken into account by using a resistor, R, in series with the capacitor, C, and the loop. For now, the loop can have an arbitrary shape. A metamaterial is generated by using periodic arrays of this unit cell. Consider a plane wave propagating through this metamaterial in the +z direction, such that the magnetic field, \( B_i = \hat{y} B_0 e^{-jkz} (e^{j\omega t} \text{ time dependence assumed}) \), is perpendicular on the plane of the loop. Writing Faraday’s Law in integral form for an arbitrary loop inside the metamaterial we obtain:

\[
\int_{C_{\text{loop}}} E \cdot dl = -j\omega \int_{A_{\text{loop}}} (B_i + B_{\text{loop}} + \sum_n B_n) \cdot ds \quad (1)
\]

where \( B_{\text{loop}} \) is the magnetic field produced by the current induced in the loop by the incident \( B_i \); \( B_n \) is the field produced by the current induced in the \( n \)-th loop. From the definition of auto inductance, \( \int_{A_{\text{loop}}} B_{\text{loop}} \cdot ds = L_{\text{loop}} I \), where \( L_{\text{loop}} \) is the auto-inductance of the loop, and I is the current induced in the loop. Similarly, \( \int_{A_{\text{loop}}} B_n \cdot ds = M_n I_n \), where \( M_n \) and \( I_n \) are the mutual inductance between the \( n \)-th loop and the loop for which Eq. (1) is written, and, respectively, the current through the \( n \)-th loop. Note that \( M_n \) can be positive or negative depending on the relative position between

FIG. 1: Unit cell containing a capacitively loaded loop. Periodic arrangements of this unit cell generate the metamaterial. Inset: The capacitively loaded loop used to generate the magnetic response of the metamaterial. The curve \( C_{\text{loop}} \) follows the contour of the loop; \( A_{\text{loop}} \) is the area delimited by \( C_{\text{loop}} \)
the two loops. In a bulk material, far from the edges, because of the symmetry of the metamaterial structure, we can consider that \( I = I_n \) for every \( n \). Moreover, the voltage across the resistor in series with the capacitor is \( I[R + 1/(j\omega C)] = \int_{C_{loop}} B_1 \cdot ds \). Therefore, Eq. \( \text{(7)} \) becomes:

\[
I \left( R + \frac{1}{j\omega C} \right) = -j\omega \left( \int_{A_{loop}} B_1 \cdot ds + IL_{eq} \right) \tag{2}
\]

where \( L_{eq} \equiv L + \sum_n M_n \). Letting \( \omega_0 \equiv (CL_{eff})^{-1/2} \) be the resonant frequency of the loop, the above equation gives the current induced in the loop:

\[
I = -\frac{\int_{A_{loop}} B_1 \cdot ds}{L_{eff}} \frac{\omega^2}{\omega^2 - \omega_0^2 - j\omega R/L_{eff}} \tag{3}
\]

In general, the magnetic flux through the loop, \( \int_{A_{loop}} B_1 \cdot ds \), is a function of the geometry of the loop. However, it is well known that in order for the metamaterial to be described in terms of effective material properties, the unit cell has to be much smaller than the wavelength. Therefore, for frequencies around \( \omega_0 \) where the permeability of the metamaterial becomes negative, we can make the approximation \( \int_{A_{loop}} B_1 \cdot ds \approx B_i A_{loop} = \mu_0 H_i A_{loop} \). Note that \( B_i \) and \( H_i \) represent the local incident B and H fields related through \( B_i = \mu_0 H_i \) since the loops are placed in vacuum. For example, for square loops (i.e. a geometry widely used for building SRRs)

\[
\int_{A_{loop}} B_1 \cdot ds = B_i A_{loop} \sin(k_0 l_{loop}/2),
\]

where \( l_{loop} \) is the length of the loop. If we require \( \lambda/l_{loop} = 10 \), as it is generally desired, then \( 1 > \sin(k_0 l_{loop}/2) > 0.99 \), which allows us to neglect the sinc term. Under this approximation, it follows from \( \text{(3)} \) that the magnetic dipole moment per unit volume is given by

\[
M = \frac{IA_{loop}}{V_{uc}} = H_i \frac{\mu_0 A_{loop}^2}{L_{eff} V_{uc}} \frac{\omega^2}{\omega^2 - \omega_0^2 - j\omega R/L_{eff}} \tag{4}
\]

where \( V_{uc} \) is the volume of the unit cell.

Note that \( M/\omega^2 \) follows a purely lorentzian model, with the quality factor of the material given by

\[
Q \equiv R/(L_{eff} \omega_0) = 1/(\omega_0 RC) \tag{5}
\]

For weak magnetically coupled loops, \( L_{eff} \approx L \), and \( Q \) is approximately the quality factor of the individual loop. Also, note that the term:

\[
F \equiv \frac{\mu_0 A_{loop}^2}{L_{eff} V_{uc}} \tag{6}
\]

only depends on the geometry of the unit cell and that of the loop, and, as recognized in Ref. \( \text{[3]} \), has an important contribution to the performances of the metamaterial.

The relative effective permeability of the metamaterial, given by \( \mu_r = 1 + M/H_i \), can be expressed in terms of

![FIG. 2: The effective permeability retrieved from the S parameters (solid line), and the permeability computed using (7) (dashed line). This excellent match was obtained for \( F = 0.16 \) and \( Q = 71 \).](image.png)
meability can be controlled to some extent by controlling the three parameters that enters \( \mu_0 \). Thus, \( \omega \mu_0 \) is relatively easy to control by loading the loop with lumped capacitors as suggested in Ref. 8. The quality factor, \( Q \), directly responsible for the losses inside the metamaterial, is harder to control since it depends on the dielectric and ohmic losses 10. Out of the three parameters, \( F \) is the easiest to control since it depends only on the geometry of the unit cell. Moreover, a closer look to Eq. 7 reveals that a bigger \( F \) factor leads to an increase in the negative permeability bandwidth and a decrease in the magnetic loss tangent, in agreement with Ref. 1. Consequently, the next step is to determine how large \( F \) can be made. Note that the parameters that enter Eq. 6 that gives \( F \) are dependent on each other. The area of the loop is strongly related to its inductance, and the unit cell volume. If we reduce the unit cell volume by packing the loops closer together in an attempt to increase \( F \), the effective inductance of the loops would increase as a result of an increase between the mutual inductance between the loops. This suggests that \( F \) cannot be made arbitrarily large.

Since it is commonly used in literature, we will focus next on metamaterials made of cubic unit cells. For the commonly used square loop \( A_{\text{loop}}^2/V_{uc} = l_{\text{loop}}(l_{\text{loop}}/l_{uc})^3 \), where \( l_{\text{loop}} \) is the diameter of the loop, and \( l_{uc} \) is the diameter of the unit cell. Obviously, the ratio \( l_{\text{loop}}/l_{uc} \) is always less than unity, and typically is less than 0.8 to reduce the coupling between adjacent loops. For example, in Ref. 10 this ratio is 0.79, while in Ref. 8 it is as low as 0.52. On the other hand the inductance is proportional to \( \mu_0 l_{\text{loop}} \). Since the unit cell is cubic we can neglect the inductive coupling between loops. Under this approximation, a more exact equation written for square loops made of cylindrical wires gives 8

\[
L_{\text{loop}} = \frac{\mu_0 l_{\text{loop}}}{2\pi} \left( 2.303 \log_{10} \frac{32l_{\text{loop}}}{w} - 2.853 \right) \tag{8}
\]

where \( w \) is the diameter of the wire. Typically, the term \( l_{\text{loop}}/w \) is of the order of hundreds, and has a moderate influence on the inductance \( L_{\text{loop}} \) because the logarithm is a slowly varying function for large arguments. Assuming \( l_{\text{loop}}/w > 100 \) it follows that \( L_{\text{loop}} > 3.33\mu_0 l_{\text{loop}} \), which combined to the constraint on \( A_{\text{loop}}^2/V_{uc} \) gives (see Eq. 6)

\[
F \approx \frac{0.83\mu_0 l_{\text{loop}}}{3.33\mu_0 l_{\text{loop}}} = 0.15 \tag{9}
\]

The above value was obtained for cubic unit cells and square loops. We expect loops of different shapes to give slightly lower values of \( F \) because the ratio \( A_{\text{loop}}^2/V_{uc} \) is smaller than for square loops while the inductance remains proportional to \( \mu_0 \sqrt{A_{\text{loop}}} \). For example, for circular loops it can be shown in a similar manner that \( F \approx 0.12 \). It follows from the above analysis that for cubic unit cells and arbitrary loop shapes \( F < 0.2 \).

This bound on \( F \) is determined directly from Eq. 6, and is supported by both the parameters retrieved from the HFSS simulation (shown in Figure 2), and the results obtained by others using numerical methods 11. Note that this analytically derived upper bound on \( F \) is significantly smaller than that predicted by other analytical models 1.

These results were obtained for cubic unit cells which are commonly used in the design of magnetic metamaterials. To increase \( F \) over the limit determined above, we can decrease the unit cell volume by reducing the cell axial dimension, or, equivalently, by increasing the concentration of loops. However, besides a decrease in \( V_{uc} \), we also expect a substantial increase in the effective inductance, \( L_{\text{eff}} \), due to the magnetic coupling between the loops (recall that \( F \propto 1/L_{\text{eff}} \)). The \( F \) factor can, thus, be accurately computed either analytically, from Eq. 6 by computing the mutual inductances that give \( L_{\text{eff}} \), or numerically using Ansoft HFSS and the procedure described above. For simplicity, we used the latter method to find \( F \) versus the concentration of loops (expressed as the number of loops per cubic unit cell). The results are presented in Figure 3 and show the geometry factor asymptotically approaching an upper bound. Thus, even for tightly packed loops (i.e. 20 loops per cubic unit cell), \( F \) remains below 0.7.

In conclusion, we derived the analytical equations that allow an accurate prediction of the effective permeability of magnetic metamaterials composed of passive resonant inclusions commonly used in literature, such as the splitting-resonator, or the omega particle. The analysis took into consideration the magnetic coupling between these particles. We showed that, for the commonly used cubic unit cell, the geometry factor \( F \) is typically 0.15. An increase in the \( F \) factor can be achieved by increasing the concentration of particles. However, even for tightly packed particles \( F \) reaches an upper bound significantly lower than 1. We validated these results through numerical simulations performed in Ansoft HFSS.
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