ASYMPTOTIC BEHAVIOR IN TIME OF SOLUTION TO THE NONLINEAR SCHRÖDINGER EQUATION WITH HIGHER ORDER ANISOTROPIC DISPERSION

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Abstract. We consider the asymptotic behavior in time of solutions to the nonlinear Schrödinger equation with fourth order anisotropic dispersion (4NLS) which describes the propagation of ultrashort laser pulses in a medium with anomalous time-dispersion in the presence of fourth-order time-dispersion. We prove existence of a solution to (4NLS) which scatters to a solution of the linearized equation of (4NLS) as \( t \to \infty \).

1. Introduction. We consider the asymptotic behavior in time of solution to the nonlinear Schrödinger equation with higher order anisotropic dispersion:

\[
    i \partial_t u + \alpha \Delta u + i \beta \partial_{x_1}^3 u + \gamma \partial_{x_1}^4 u = \lambda |u|^{p-1} u, \quad t > 0, x \in \mathbb{R}^d,
\]

where \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is an unknown function, \( \alpha, \beta, \gamma, \lambda \) are real constants and \( p > 1 \). Equation (1.1) arises in nonlinear optics to model the propagation of ultrashort laser pulses in a medium with anomalous time-dispersion in the presence of fourth-order time-dispersion (see [40, 12, 4] and the references therein). It also arises in models of propagation in fiber arrays (see [1, 11]).

To simplify (1.1), we introduce a new unknown function

\[
    v(t, x) = e^{-i \left( \frac{\alpha \beta^2}{16 \gamma^2} t + \frac{\beta^4}{256 \gamma^3} x_1 \right)} u(t, x_1 - \left( \frac{\alpha \beta}{2 \gamma} + \frac{3 \beta^2}{8 \gamma^2} \right) t, x_\perp),
\]

where \( x = (x_1, x_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1} \). Then the equation (1.1) can be rewritten as

\[
    i \partial_t v + \left\{ (\alpha + \frac{3 \beta^2}{8 \gamma}) \partial_{x_1}^2 + \alpha \Delta_{x_\perp} \right\} v + \gamma \partial_{x_1}^4 v = \lambda |v|^{p-1} v, \quad t > 0, x \in \mathbb{R}^d.
\]

Therefore if \( \alpha, \beta \) and \( \gamma \) satisfy \( (\alpha + \frac{3 \beta^2}{8 \gamma}) > 0 \) and \( \alpha \gamma < 0 \), then by using a suitable scaling transform, we can rewrite (1.1) into the Schrödinger equation with fourth order anisotropic dispersion:

\[
    i \partial_t u + \frac{1}{2} \Delta u - \frac{1}{4} \partial_{x_1}^4 u = \lambda |u|^{p-1} u, \quad t > 0, x \in \mathbb{R}^d.
\]

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In this paper, we study the asymptotic behavior in time of solutions to (1.2).

The Cauchy problem for the homogeneous fourth order nonlinear Schrödinger type equation

\[ i\partial_t u + \frac{1}{2} \Delta^2 u = \lambda |u|^{p-1} u, \quad t > 0, x \in \mathbb{R}^d \quad (1.3) \]

has been studied by many authors, most of the results holding also when lower dispersive terms are added. By using the Strichartz estimates in [3] one shows that the Cauchy problem is locally well-posed in the energy space \( H^2(\mathbb{R}^d) \) for the energy subcritical case \( 1 < p < 1 + 8/(d-4) \) when \( d \geq 5 \) and \( 1 < p < \infty \) when \( d \leq 4 \) and in \( L^2(\mathbb{R}^d) \) for the mass subcritical case \( 1 < p < 1 + 8/d \). We also refer to Bouchel [6] who studies the Cauchy problem and furthermore gives non-existence, existence and qualitative properties results of solitary wave solutions for (1.1). See also [12] for results on the Cauchy problem for slightly more general situations.

There are several results concerning the scattering and blow-up of solutions for (1.3). For the defocusing case \( \lambda < 0 \), the global well-posedness and scattering for (1.3) with the energy-critical nonlinearity (i.e., (1.3) with \( d \geq 5 \), and \( p = 1 + 8/(d-4) \)) was studied by Pausader [32] for radially symmetric initial data by combining the concentration-compactness argument by Kenig-Merle [25] and Morawetz-type estimate. Later, Miao, Xu and Zhao [28] proved a similar result for (1.3) in the energy-critical and higher dimensional case \( d \geq 9 \) without radial assumption on initial data. In [33], Pausader has shown the global well-posedness and scattering of (1.3) with cubic nonlinearity for the case \( 5 \leq d \leq 8 \). Pausader and Xia [34] proved the global well-posedness and scattering for (1.3) with mass super-critical nonlinearity (i.e., (1.3) with \( p > 1 + 8/d \)) for low dimensions \( 1 \leq d \leq 4 \) by using a virial-type estimate instead of the Morawetz-type estimates.

For the focusing case \( \lambda > 0 \), Pausader [31] and Miao, Xu and Zhao [27] independently showed the global well-posedness and scattering for (1.3) with the energy-critical nonlinearity for radially symmetric initial data with \( \dot{H}^2 \) and energy norms below that of the ground state. When the initial data is sufficiently small, Hayashi, Mendez-Navarro and Naumkin [14] proved the global existence and the scattering for (1.3) with \( d = 1 \) and \( p > 5 \) by using the factorization technique developed by the authors [16]. In [14] they also shown the small data global existence and the decay estimates for (1.3) with \( d = 1 \) and \( p > 4 \) under the assumption that the initial data is odd. In the subsequent paper [15], they proved that when \( d = 1, p = 5 \) and \( \lambda < 0 \), a solution to (1.3) has dissipative structure and gains additional logarithmic decay. Aoki, Hayashi and Naumkin [2] showed the global existence and scattering of (1.3) with \( d = 1, 2 \) and \( p > 1 + 4/d \). We refer also to the series of paper by Hayashi and Naumkin [18, 19, 21] and work by Hirayama and Okamoto [22] for interesting phenomena on the long time behavior of solutions to (1.3) with a derivative nonlinearity.

Recently, a blow-up result is proved by Boulenger and Lenzmann [7] for (1.3) with the mass critical and super critical focusing nonlinearity in the radial case which solves a long standing conjecture suggested by several numerical studies (see [11] for instance). Notice that most of their results hold also when lower dispersive term \( \mu \Delta u \) is added. See also the work by Bonheure, Casteras, Gou and Jeanjean [5] for the extension of the blow up results by [7].

We now return to the inhomogeneous case. Since the point-wise decay of the solution to the linear fourth order Schrödinger equation
\[ i\partial_t u + \frac{1}{2} \Delta u - \frac{1}{4} \partial^4_{x_1} u = 0, \quad t > 0, x \in \mathbb{R}^d \] (1.4)
is \( O(t^{-d/2}) \) as \( t \to \infty \) (see Ben-Artzi, Koch and Saut [3]), we expect that if \( p > 1 + 2/d \), then the (small) solution to (1.2) will scatter to the solution to the linearized equation (1.4). Compared to the homogeneous equation (1.3), there are few results on the long time behavior of solution for (1.2). For the one dimensional cubic equation (1.4). Compared to the homogeneous equation (1.3), there are few results. We shall show that for \( d = 1, p = 3 \) which behaves like a solution to the linearized equation (1.4) with a logarithmic phase correction. Furthermore, Hayashi and Naumkin [20] proved that for small initial data, there exists a global solution to (1.2) with \( d = 1, p = 3 \) which behaves like a solution to the linear equation (1.4) with a logarithmic phase correction.

In this paper, we consider the small data scatters of (1.2) for the higher dimensional case. Notice that the interesting case is for \( p \in (1 + 2/d, 1 + 4/d) \) since for the case \( p \geq 1 + 4/d \), the nonlinearity in (1.2) is weaker compared to the case \( p < 1 + 4/d \).

We shall show that for \( d = 2,3 \) and some range of \( p \) in \((1 + 2/d, 1 + 4/d)\), there exists a solution to (1.2) which scatters to the solution to the linearized equation (1.4). Let us consider the final state problem:

\[
\begin{align*}
&i\partial_t u + \frac{1}{2} \Delta u - \frac{1}{4} \partial^4_{x_1} u = \lambda |u|^{p-1} u, \quad t > 0, x \in \mathbb{R}^d, \\
&\lim_{t \to +\infty} (u(t) - W(t)\psi_+) = 0, \quad \text{in } L^2,
\end{align*}
\] (1.5)
2 \leq r \leq (2d)/(d - 2) \text{ for } d \geq 3, 2 \leq r < \infty \text{ for } d = 2 \text{ and } 2 \leq r \leq \infty \text{ for } d = 1. \text{ His proof is based on the pseudo-conformal identity which is not known for (1.2).}

To prove Theorems 1.1 and 1.2, we employ the argument by Ozawa [30], Hayashi and Naumkin [16, 17]. Let us introduce a modified asymptotic profile

$$W(t)F^{-1}[\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}],$$

where

$$S_+(t,\xi) = -\frac{\lambda}{1 - \frac{p-1}{2}d} \frac{\hat{\psi}_+(\xi)^{p-1}}{(3\xi_1^2 + 1)^{\frac{p-1}{2}}} t^{1 - \frac{p-1}{2}d}. \tag{1.7}$$

We first construct a solution $u$ to the final state problem

$$i\partial_t u + \frac{1}{2} \Delta u - \frac{1}{4} \partial_{\xi_1}^4 u = \lambda |u|^{p-1}u, \quad t > 0, x \in \mathbb{R}^d,$$

$$\lim_{t \to +\infty} (u(t) - W(t)F^{-1}w) = 0, \quad \text{in } L^2. \tag{1.8}$$

where $w(t,\xi) = \hat{\psi}_+(\xi)e^{iS_+(t,\xi)}$. To prove this, we first rewrite (1.8) as the integral equation

$$u(t) - W(t)F^{-1}w = i\lambda \int_t^{+\infty} W(t - \tau)[|u|^{p-1}u - |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w](\tau)d\tau$$

$$-i \int_t^{+\infty} W(t - \tau)R(\tau)d\tau, \tag{1.9}$$

where

$$R(t, x) = W(t)F^{-1}\left[\frac{t^{-\frac{p-1}{2}d}}{(3\xi_1^2 + 1)^{\frac{p-1}{2}}} \hat{\psi}_+(\xi)^{p-1}\hat{\psi}_+(\xi)e^{iS_+(t,\xi)}\right](x)$$

$$-\lambda|W(t)F^{-1}w|^{p-1}W(t)F^{-1}w.$$  

For the derivation of (1.9), see Section 4 below. Next, we apply the contraction mapping principle to the integral equation (1.9) in a suitable function space. In this step the asymptotic formula (Proposition 2.1) and the Strichartz estimate (Lemma 2.3) for the linear equation (1.4) play an important role. Finally, we show that the solutions of (1.8) converge to $W(t)\psi_+$ in $L^2$ as $t \to \infty$.

We introduce several notations and function spaces which are used throughout this paper. For $\psi \in \mathcal{S}'(\mathbb{R}^d)$, $\hat{\psi}(\xi) = \mathcal{F}[\psi](\xi)$ denote the Fourier transform of $\psi$. Let $\langle \xi \rangle = \sqrt{|\xi|^2 + 1}$. The differential operators $|\nabla|^s = (-\Delta)^s/2$ and $\langle \nabla \rangle^s = (1 - \Delta)^{s/2}$ denote the Riesz potential and Bessel potential of order $-s$, respectively. We define $\langle \partial_{\xi_1} \rangle^s = F^{-1}\langle \xi_1 \rangle^s F^{-1}$. For $1 \leq q, r \leq \infty$, $L^q(t, \infty; L^r_x(\mathbb{R}^d))$ is defined as follows:

$$L^q(t, \infty; L^r_x(\mathbb{R}^d)) = \{ u \in \mathcal{S}'(\mathbb{R}^{1+d}); \| u \|_{L^q(t, \infty; L^r_x)} < \infty \},$$

$$\| u \|_{L^q(t, \infty; L^r_x)} = \left( \int_t^{\infty} \| u(\tau) \|^q_{L^r_x} d\tau \right)^{1/q}.$$

We will use the Sobolev spaces

$$H^s(\mathbb{R}^d) = \{ \phi \in \mathcal{S}'(\mathbb{R}^d); \| \phi \|_{H^s} = \| \langle \nabla \rangle^s \phi \|_{L^2} < \infty \}$$

and their homogeneous version

$$\dot{H}^s(\mathbb{R}^d) = \{ \phi \in \mathcal{S}'(\mathbb{R}^d); \| \phi \|_{\dot{H}^s} = \| \langle \nabla \rangle^s \phi \|_{L^2} < \infty \}. $$
The weighted Sobolev spaces is defined by
\[ H^{m,\sigma}(\mathbb{R}^d) = \{ \phi \in \mathcal{S}(\mathbb{R}^d); \| \phi \|_{H^{m,\sigma}} = \| \langle x \rangle^s (\nabla)^m \phi \|_{L^2} < \infty \}. \]

We denote various constants by \( C \) and so forth. They may differ from line to line, when this does not cause any confusion.

The plan of the present paper is as follows. In Section 2, we prove several linear estimates for the fourth order Schrödinger type equation (1.4). In Section 3, we give several estimates for the asymptotic profile 1.6. In Section 4, we prove Theorems 1.1 and 1.2 by applying the contraction mapping principle to the integral equation (1.9). Finally in Section 5, we give several additional remarks.

2. Linear estimates. In this section, we derive several linear estimates for the fourth order Schrödinger type equation

\[
\begin{aligned}
&\left\{\begin{array}{ll}
  i\partial_t u + \frac{1}{2} \Delta u - \frac{1}{4} \partial_{x_1}^4 u = 0, & t > 0, x \in \mathbb{R}^d, \\
  u(0, x) = \psi(x), & x \in \mathbb{R}^d.
\end{array}\right.
\end{aligned}
\] (2.1)

Proposition 2.1. Let \( u \) be a solution to (2.1). Then we have

\[
u(t, x) = \frac{t^{-\frac{d}{4}}}{\sqrt{3\mu_1^4 + 1}} \hat{\psi}(\mu) e^{\frac{i}{4} t \mu_1^4 + \frac{1}{4} t \mu_1^2 - i \frac{d}{2} \pi} + R(t, x)
\]

for \( t \gg 1 \), where \( \mu = (\mu_1, \mu_\perp) \) is given by

\[
\begin{aligned}
\mu_1 &= \left\{ \frac{1}{2t} \left( x_1 + \sqrt{x_1^2 + \frac{4}{27} t^2} \right) \right\}^{1/3} + \left\{ \frac{1}{2t} \left( x_1 - \sqrt{x_1^2 + \frac{4}{27} t^2} \right) \right\}^{1/3}, \\
\mu_\perp &= \frac{x_\perp}{t}.
\end{aligned}
\] (2.2)

and \( R \) satisfies

\[
\| R(t) \|_{L^p_x} \leq C e^{-d\left(\frac{1}{4} - \frac{1}{p}\right) - \beta \| \psi \|_{H^{0,\sigma}}},
\]

for \( 2 \leq p \leq \infty \), \( 1/(4p) < \beta < 1/2 \) and \( s > d/2 - (d - 1)/p + 1 \).

To calculate the oscillatory integral effectively, we show the following elementary lemma.

Lemma 2.2. Let \( a, b, c \in \mathbb{R} \) and let \( \phi \in C^2([a, b], \mathbb{R}) \) and \( \psi \in C^1([a, b], \mathbb{C}) \). Then

\[
\int_a^b e^{-i\phi(\xi)} \psi(\xi) d\xi
= \left[ e^{-i\phi(\xi)} \frac{(\xi - c)\psi(\xi)}{1 - i(\xi - c)\phi'(\xi)} \right]_a^b - \int_a^b e^{-i\phi(\xi)} \frac{(\xi - c)\psi'(\xi)}{1 - i(\xi - c)\phi'(\xi)} d\xi
- i \int_a^b e^{-i\phi(\xi)} \frac{\{\phi'(\xi) + (\xi - c)\phi''(\xi)\} \psi(\xi)}{1 - i(\xi - c)\phi'(\xi)} d\xi.
\]

Proof of Lemma 2.2. Lemma 2.2 follows from the combination of the identity

\[
e^{-i\phi(\xi)} = \frac{e^{-i\phi(\xi)}(\xi - c)'}{1 - i(\xi - c)\phi'(\xi)}
\]

and integration by parts. \( \square \)
Proof of Proposition 2.1. Let $u$ be a solution to (2.1). Then we have

$$ u(t, x) = \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\xi \cdot (x - \frac{1}{2}t\xi^2 - \frac{1}{4}t\xi_1)} \psi(\xi) d\xi = \int_{\mathbb{R}^d} K(t, x - y) \psi(y) dy, $$

where

$$ K(t, z) = \left( \frac{1}{2\pi} \right)^{d} \int_{\mathbb{R}^d} e^{i\xi \cdot (-\frac{1}{2}t\xi^2 - \frac{1}{4}t\xi_1)} d\xi. $$

By the Fresnel integral formula

$$ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi_1 - \frac{1}{2}t\xi_1^2} d\xi_j = t^{-\frac{1}{2}} e^{i\frac{x^2}{4t} - i\frac{1}{4}x^2}, $$

for $j = 2, \ldots, d$, we have

$$ K(t, z) = \left( \frac{1}{2\pi} \right)^{\frac{d+1}{2}} t^{-\frac{d-1}{2}} e^{-i\frac{|\mu|^2}{4t} - i\frac{1}{4}x_1^2} \int_{\mathbb{R}} e^{i\xi \cdot \frac{x_1}{t}} \mathcal{F}[e^{-\frac{|\mu|^2}{4t} - 1} \psi](\xi_1, \frac{x_1}{t}) d\xi_1. $$

Therefore, we find

$$ u(t, x) = \frac{1}{\sqrt{2\pi}} t^{-\frac{d-1}{2}} e^{i\xi \cdot \frac{x_1}{t}} \int_{\mathbb{R}} e^{i\xi \cdot \frac{x_1}{t}} \mathcal{F}[e^{-\frac{|\mu|^2}{4t} - 1} \psi](\xi_1, \frac{x_1}{t}) d\xi_1. $$

We split $u$ into the following two pieces:

$$ u(t, x) = \frac{1}{\sqrt{2\pi}} t^{-\frac{d-1}{2}} e^{i\xi \cdot \frac{x_1}{t}} \int_{\mathbb{R}} e^{i\xi \cdot \frac{x_1}{t}} \mathcal{F}[\psi](\xi_1, \frac{x_1}{t}) d\xi_1 + \frac{1}{\sqrt{2\pi}} t^{-\frac{d-1}{2}} e^{i\xi \cdot \frac{x_1}{t}} \int_{\mathbb{R}} e^{i\xi \cdot \frac{x_1}{t}} \mathcal{F}[\psi - 1)(\xi_1, \frac{x_1}{t}) d\xi_1 $$

$$ =: L + R. \quad (2.3) $$

To evaluate $L$, we split $L$ into

$$ L = \frac{1}{\sqrt{2\pi}} t^{-\frac{d-1}{2}} e^{i\xi \cdot \frac{x_1}{t}} \mathcal{F}[\psi](\mu) \int_{\mathbb{R}} e^{i\xi \cdot \frac{x_1}{t}} \mathcal{F}[\psi]\xi_1 - \frac{1}{2}t\xi_1^2 - \frac{1}{4}t\xi_1, d\xi_1 $$

$$ + \frac{1}{\sqrt{2\pi}} t^{-\frac{d-1}{2}} e^{i\xi \cdot \frac{x_1}{t}} \mathcal{F}[\psi - 1)(\mu_1, \frac{x_1}{t}) d\xi_1 $$

$$ =: L_1(t, x) + L_2(t, x). \quad (2.4) $$

We rewrite $L_1$ as follows:

$$ L_1(t, x) = \frac{1}{\sqrt{2\pi}} t^{-\frac{d-1}{2}} e^{-\frac{1}{2}it\mu_1^2 + \frac{1}{2}it|\mu|^2 - i\frac{1}{4}x_1^2} \mathcal{F}[\psi](\mu) \int_{\mathbb{R}} e^{-itS(\mu_1, \xi_1)} d\xi_1, $$

where $S(\mu_1, \xi_1)$ is defined by

$$ S(\mu_1, \xi_1) = \frac{1}{4}\xi_1^4 + \frac{1}{2}\xi_1^2 - (\mu_1^3 + \mu_1)\xi_1 + \frac{3}{4}\mu_1^4 + \frac{1}{2}\mu_1^2. $$

Let

$$ \eta_1 = \mu_1 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}\mu_1^2 + 1} (\xi_1 - \mu_1) \sqrt{\xi_1^2 + 2\mu_1 \xi_1 + 3\mu_1^2 + 2}. $$
Furthermore, we easily see that

\[ L_1(t, x) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{i}{\hbar} S(\mu, \xi_1)} d\xi_1 + \int e^{-\frac{i}{\hbar} S(\mu, \xi_1)} (1 - \frac{d\eta}{d\xi_1}) d\xi_1 \]

\[ = \int e^{-\frac{i}{\hbar} S(\mu, \xi_1)} d\xi_1 + \int e^{-\frac{i}{\hbar} S(\mu, \xi_1)} (1 - \frac{d\eta}{d\xi_1}) d\xi_1 \]

\[ =: t^{-\frac{d-1}{2}} \mathcal{F}[\psi](\mu)e^{\frac{2}{d}\mu \xi^2 + \frac{2}{d}\mu |\xi|^2 - i\frac{d-1}{2}x^2} L_1(t, x) \]

\[ + t^{-\frac{d-1}{2}} \mathcal{F}[\psi](\mu)e^{\frac{2}{d}\mu \xi^2 + \frac{2}{d}\mu |\xi|^2 - i\frac{d-1}{2}x^2} L_1(t, x). \]  

(2.5)

For \( L_{1,1} \), changing the variable \( \xi_1 \rightarrow \eta_1 \), we have

\[ L_{1,1}(t, x) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}t(\eta_1^2 + 1)(\eta_1 - \mu_1)^2} d\eta_1. \]

In addition, changing the variable \( \xi_1 = (1/\sqrt{2})^{1/2} \sqrt{3\mu_1^2 + 1} (\eta_1 - \mu_1) \) \( (\eta_1 \rightarrow \zeta_1) \) and using the Fresnel integral formula, we obtain

\[ L_{1,1}(t, x) = \frac{\sqrt{2}}{\pi} \frac{t^{-1/2}}{\sqrt{3\mu_1^2 + 1}} \int e^{-i\zeta^2} d\zeta = \frac{t^{-1/2}}{\sqrt{3\mu_1^2 + 1}} e^{-i\zeta^2}. \]  

(2.6)

For \( L_{1,2} \), using Lemma 2.2, we have

\[ L_{1,2}(t, x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi_1 - \frac{i}{\hbar} t(\xi_1^2 + 1)} (1 - \frac{d\eta_1}{d\xi_1}) d\xi_1 \]

\[ = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi_1 - \frac{i}{\hbar} t(\xi_1^2 + 1)} (1 - \frac{d\eta_1}{d\xi_1}) \frac{(\xi_1 - \mu_1)}{1 - it(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1)} d\xi_1 \]

\[ - \frac{it}{\sqrt{2\pi}} \int e^{ix\xi_1 - \frac{i}{\hbar} t(\xi_1^2 + 1)} \frac{(\xi_1 - \mu_1)^2(4\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1)}{1 - it(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1)} d\xi_1 \]

\[ \times (1 - \frac{d\eta_1}{d\xi_1}) d\xi_1. \]

Since

\[ \frac{d\eta_1}{d\xi_1} = \sqrt{2} \frac{\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1}{\sqrt{3\mu_1^2 + 1} \sqrt{\xi_1^2 + 2\mu_1 \xi_1 + 3\mu_1^2 + 2}}, \]

\[ \frac{d^2\eta_1}{d\xi_1^2} = \sqrt{2} \frac{\xi_1^3 + 3\mu_1 \xi_1^2 + (6\mu_1^2 + 3)\xi_1 + (2\mu_1^3 + \mu_1)}{(\xi_1^2 + 2\mu_1 \xi_1 + 3\mu_1^2 + 2)^{3/2}}, \]

we see that

\[ \sup_{\xi_1 \in \mathbb{R}} \left| \frac{d^2\eta_1}{d\xi_1^2} \right| \leq \frac{C}{\sqrt{3\mu_1^2 + 1}}, \]

\[ \left| 1 - \frac{d\eta_1}{d\xi_1} \right| \leq \left| \frac{d\eta_1}{d\xi_1} \right|_{\xi_1 = \mu_1} \leq \sup_{\xi_1 \in \mathbb{R}} \left| \frac{d^2\eta_1}{d\xi_1^2} \right| \left| \xi_1 - \mu_1 \right| \leq C \frac{\left| \xi_1 - \mu_1 \right|}{\sqrt{3\mu_1^2 + 1}}. \]

Furthermore, we easily see that

\[ \sup_{\xi_1 \in \mathbb{R}} \left| t \frac{(\xi_1 - \mu_1)^2(4\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1)}{1 - it(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1)} \right| \leq C. \]  

(2.7)
Combining above three inequalities, we have
\[ |L_{1,2}(t, x)| \leq \frac{C}{\sqrt{3\mu_1^2 + 1}} \int_{\mathbb{R}} \frac{|\xi_1 - \mu_1|}{1 + t(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1^2 + 1)} d\xi_1. \]

We split the right hand side of above inequality into the following two pieces:
\[ |L_{1,2}(t, x)| \leq \frac{C}{\sqrt{3\mu_1^2 + 1}} \int_{|\xi_1 - \mu_1| \leq \sqrt{\mu_1^2 + 1}} \frac{|\xi_1 - \mu_1|}{1 + t(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1^2 + 1)} d\xi_1 + \frac{C}{\sqrt{3\mu_1^2 + 1}} \int_{|\xi_1 - \mu_1| > \sqrt{\mu_1^2 + 1}} \frac{|\xi_1 - \mu_1|}{1 + t(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1^2 + 1)} d\xi_1. \]

Using the inequalities
\[ \xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1 \geq \begin{cases} \frac{1}{2}(\mu_1^2 + 1), & \text{if } |\xi_1 - \mu_1| \leq \sqrt{\mu_1^2 + 1}, \\ \frac{1}{4}(\xi_1 - \mu_1)^2, & \text{if } |\xi_1 - \mu_1| \geq \sqrt{\mu_1^2 + 1}, \end{cases} \]
we have
\[ |L_{1,2}(t, x)| \leq \frac{C}{\sqrt{3\mu_1^2 + 1}} \int_{|\xi_1 - \mu_1| \leq \sqrt{\mu_1^2 + 1}} \frac{|\xi_1 - \mu_1|}{1 + t(\mu_1^2 + 1)(\xi_1 - \mu_1)^2} d\xi_1 + \frac{C}{\sqrt{3\mu_1^2 + 1}} \int_{|\xi_1 - \mu_1| > \sqrt{\mu_1^2 + 1}} \frac{|\xi_1 - \mu_1|}{1 + t|\xi_1 - \mu_1|^2} d\xi_1 \leq Ct^{-\beta - \frac{1}{2}}(\mu_1^2 + 1)^{-\beta - 1} \int_{|\xi_1 - \mu_1| \leq \sqrt{\mu_1^2 + 1}} |\xi_1 - \mu_1|^{-2\beta} d\xi_1 + Ct^{-\beta - \frac{1}{2}}(\mu_1^2 + 1)^{-\beta - \frac{1}{2}} \int_{|\xi_1 - \mu_1| > \sqrt{\mu_1^2 + 1}} |\xi_1 - \mu_1|^{-4\beta - 1} d\xi_1 \leq Ct^{-\beta - \frac{1}{2}}(\mu_1^2 + 1)^{-2\beta - \frac{1}{2}}, \quad (2.8) \]
where \( 0 < \beta < 1/2 \). By (2.5), (2.6) and (2.8), we have
\[ L_1(t, x) = \frac{t^{-\frac{d}{2}}}{\sqrt{3\mu_1^2 + 1}} \mathcal{F}[\psi](\mu)e^{\frac{1}{2}i\mu_1^2 + \frac{1}{2}t|\mu|^2 - \frac{1}{4}t} + R_1(t, x), \quad (2.9) \]
where \( R_1 \) satisfies
\[ |R_1(t, x)| \leq Ct^{-\frac{d}{2} - \beta}(\mu_1^2 + 1)^{-2\beta - \frac{1}{2}} |\mathcal{F}[\psi](\mu)|. \]

Hence the Hölder and Sobolev inequalities yield
\[ \|R_1(t)\|_{L^p_{\xi_1}} \leq Ct^{-\frac{d}{2} - \beta}(\mu_1^2 + 1)^{-2\beta - \frac{1}{2}} \|\mathcal{F}[\psi](\mu)\|_{L^p_{\xi_1}} \leq Ct^{-\frac{d}{2} - \beta}(\mu_1^2 + 1)^{-2\beta - \frac{1}{2}} \|\mathcal{F}[\psi](\cdot, \mu_{\perp})\|_{L^p_{\xi_1}} \leq Ct^{-\frac{d}{2} - \beta} \|\mathcal{F}[\psi](\cdot, \mu_{\perp})\|_{L^p_{\xi_1}} \leq Ct^{-\frac{d}{2} - \beta} \|\mathcal{F}[\psi](\cdot, \mu_{\perp})\|_{\mu_{\xi_1}}. \]
for $2 \leq p \leq \infty$ and $0 < \beta < 1/2$, where $s_1 > 1/2$. Combining the above inequality and the Minkowski inequality, we have

\[
\| R_1(t) \|_{L^p_{t,x}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \mathcal{F}[\psi](\xi_1, \mu_{\perp}) \|_{H^s_{\mu_{\perp}}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \mathcal{F}[\psi](\xi_1, \mu_{\perp}) \|_{L^p_{\xi_1}}
\]

Hence

\[
\| \mathcal{F}[\psi] \|_{L^p_{\xi_1}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \mathcal{F}[\psi] \|_{L^p_{\xi_1}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \mathcal{F}[\psi] \|_{H^{s_1}_{\mu_{\perp}}}
\]

where $s = s_1 + s_{\perp}$ and $s_{\perp} = (d - 1)(1/2 - 1/p)$ for $2 \leq p < \infty$ and $s > (d - 1)/2$ for $p = \infty$.

Next we evaluate $L_2$. We write

\[
L_2(t, x) = t^{-\frac{d-1}{2}} e^{-\frac{|x_\perp|^2}{4t} - i \frac{d}{2} - \frac{s_1}{2}} \tilde{L}_2(t, x).
\]

Using Lemma 2.2, we have

\[
\tilde{L}_2(t, x) = -\frac{1}{\sqrt{2\pi}} \int_R e^{it \xi_1 - \frac{t}{2} \xi_1^2 - \frac{t}{2} \xi_1^2} \frac{(\xi_1 - \mu_1)}{1 - it(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1)} \times \partial_{\xi_1} \mathcal{F}[\psi](\xi_1, \frac{x_\perp}{t}) d\xi_1
\]

\[
- \frac{it}{\sqrt{2\pi}} \int_R e^{it \xi_1 - \frac{t}{2} \xi_1^2 - \frac{t}{2} \xi_1^2} \frac{(\xi_1 - \mu_1)^2(4\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 2)}{(1 - it(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1))^2} \times (\mathcal{F}[\psi](\xi_1, \frac{x_\perp}{t}) - \mathcal{F}[\psi](\mu_1, \frac{x_\perp}{t})) d\xi_1.
\]

From (2.7), we have

\[
|\tilde{L}_2(t, x)| \leq C \| \partial_{\xi_1} \mathcal{F}[\psi](\cdot, \frac{x_\perp}{t}) \|_{L^\infty_{\xi_1}} \int_R \frac{|\xi_1 - \mu_1|}{1 + t(\xi_1 - \mu_1)^2(\xi_1^2 + \mu_1 \xi_1 + \mu_1^2 + 1)} d\xi_1.
\]

The same argument as that in (2.8) yields

\[
|\tilde{L}_2(t, x)| \leq C t^{-\beta - 1/2}(\mu_1^2 + 1)^{-2\beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\cdot, \frac{x_\perp}{t}) \|_{L^\infty_{\xi_1}}.
\]

Hence

\[
\| L_2(t) \|_{L^p_{t,x}} \leq C t^{\beta - \frac{d}{2} - \beta} \| (\mu_1^2 + 1)^{-2\beta} \|_{L^p_{\xi_1}} \| \partial_{\xi_1} \mathcal{F}[\psi](\cdot, \frac{x_\perp}{t}) \|_{L^\infty_{\xi_1}}
\]

\[
\leq C t^{\beta - \frac{d}{2} - \beta} \| (\mu_1^2 + 1)^{-2\beta} \|_{L^p_{\xi_1}} \| \partial_{\xi_1} \mathcal{F}[\psi](\cdot, \frac{x_\perp}{t}) \|_{L^\infty_{\xi_1}}
\]

\[
\leq C t^{\beta - \frac{d}{2} - \beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\cdot, \frac{x_\perp}{t}) \|_{H^s_{\mu_{\perp}}}
\]

for $2 \leq p \leq \infty$ and $1/(4p) < \beta < 1/2$. Combining the above inequality and the Minkowski inequality, we have

\[
\| L_2(t) \|_{L^p_{t,x}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\cdot, \frac{x_\perp}{t}) \|_{H^s_{\mu_{\perp}}}
\]

\[
\leq C t^{\beta - \frac{d}{2} - \beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\xi_1, \frac{x_\perp}{t}) \|_{L^p_{\xi_1}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\xi_1, \frac{x_\perp}{t}) \|_{H^s_{\mu_{\perp}}}
\]

\[
\| L_2(t) \|_{L^p_{t,x}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\cdot, \frac{x_\perp}{t}) \|_{H^s_{\mu_{\perp}}}
\]

\[
\leq C t^{\beta - \frac{d}{2} - \beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\xi_1, \frac{x_\perp}{t}) \|_{L^p_{\xi_1}} \leq C t^{\beta - \frac{d}{2} - \beta} \| \partial_{\xi_1} \mathcal{F}[\psi](\xi_1, \frac{x_\perp}{t}) \|_{H^s_{\mu_{\perp}}}
\]
Finally let us evaluate \( R \). \( R \) can be rewritten as
\[
R = t^{-\frac{d}{2} + \frac{1}{2}} e^{\frac{d}{4} t |\mu|^2 - i \frac{d}{4} \pi} W_{4LS}(t) \mathcal{F}_\perp [(e^{\frac{i |y|^2}{t^2}} - 1)\psi](x_1, \frac{x_\perp}{t}),
\]
where \( \{W_{4LS}(t)\}_{t \in \mathbb{R}} \) is a unitary group generated by the linear operator \((i/2)\partial_{x_1} - (i/4)\partial_{x_1}^4\);
\[
W_{4LS}(t)\phi = \frac{1}{\sqrt{2\pi}} \int e^{ix_1 \xi_1 - \frac{d}{4} t \xi_1^2 - \frac{i}{4} t \xi_1^4} \hat{\phi}(\xi_1) d\xi_1
\]
and \( \mathcal{F}_\perp \) is the Fourier transform in \( x_\perp \). By using the decay estimate (see [3, 35] for instance),
\[
\|W_{4LS}(t)\psi\|_{L_{x_1}^p} \leq C t^{-\frac{1}{2} - \frac{1}{p'}} \|\psi\|_{L_{x_1}^{p'}}.
\]
we obtain
\[
\|R(t)\|_{L_{x_1}^p} \leq C t^{-\frac{d}{2} + \frac{1}{2}} \|\mathcal{F}_\perp [(e^{\frac{i |y|^2}{t^2}} - 1)\psi](x_1, \frac{x_\perp}{t})\|_{L_{x_1}^p}.
\]
Combining the above inequality and the Minkowski inequality, we have
\[
\|R(t)\|_{L_{x_1}^p} \leq C t^{-\frac{d}{2} + \frac{1}{2}} \|\mathcal{F}_\perp [(e^{\frac{i |y|^2}{t^2}} - 1)\psi](x_1, \frac{x_\perp}{t})\|_{L_{x_1}^p} \leq C t^{-\frac{d}{2} + \frac{1}{2}} \|\mathcal{F}_\perp [(e^{\frac{i |y|^2}{t^2}} - 1)\psi](x_1, \frac{x_\perp}{t})\|_{L_{x_1}^p} \leq C t^{-\frac{d}{2} + \frac{1}{2}} \|\mathcal{F}_\perp [(e^{\frac{i |y|^2}{t^2}} - 1)\psi](x_1, \frac{x_\perp}{t})\|_{L_{x_1}^p} \leq C t^{-\frac{d}{2} + \frac{1}{2}} \|\psi\|_{L_{x_1}^{p'}}.
\]
where \( 0 < \beta < 1 \) and \( s > d/2 - d/p + 2\beta \). Collecting (2.3), (2.4), (2.9), (2.10), (2.11) and (2.12), we obtain the desired result. \( \square \)

To prove Theorems 1.1 and 1.2, we employ the Strichartz estimate for the linear fourth order Schrödinger equation (2.1).

**Lemma 2.3.** Let \( d \geq 2 \) and let \((q_j, r_j) \ (j = 1, 2)\) satisfy
\[
\frac{2}{q_j} + \frac{d}{r_j} = \frac{d}{2}, \quad 2 \leq r_j \leq \frac{2d}{d-2} \quad \text{and} \quad (q_j, r_j, d) \neq (2, \infty, 2).
\]
Then, the inequality
\[
\left\| \langle \partial_{x_1} \rangle \frac{2}{\pi^2} \int_{-\infty}^{+\infty} W(t - t')F(t') dt' \right\|_{L_{x_1}^{q_j}(t; \infty; L_{x_\perp}^{r_j})} \leq C \|\langle \partial_{x_1} \rangle \|_{L_{x_1}^{q_j}(t; \infty; L_{x_\perp}^{r_j})}^d
\]
holds.

**Proof of Lemma 2.3.** For the non-endpoint case (i.e., \( 2 \leq r_j < 2d/(d-2) \) for any \( j = 1, 2 \), see Kenig, Ponce and Vega [26, Theorem 3.1]. For the endpoint case (i.e., \( r_j = 2d/(d-2) \) for some \( j = 1, 2 \), see Keel and Tao [24, Theorem 10.2]. \( \square \)
Lemma 3.2. Assume that $s \geq 0$. Let $1 < p, p_1, p_4 < \infty$ and $1 < p_2, p_3 \leq \infty$. Then, we have
\[ \|\nabla^s (fg)\|_{L^p} \leq C \|\nabla^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|\nabla^s g\|_{L^{p_4}}, \]
provided that $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

Proof of Lemma 3.2. See [9, Proposition 3.3] for instance. \hfill \Box

Lemma 3.3. (Fractional chain rules). (i) Assume $G \in C^1(\mathbb{C}, \mathbb{C})$ and $0 < s \leq 1$. Let $1 < p, p_2 < \infty$ and $1 < p_1 \leq \infty$. Then we have
\[ \|\nabla^s G(u)\|_{L^p} \leq C \|G'(u)\|_{L^{p_1}} \|\nabla^s u\|_{L^{p_2}}, \]
provided $1/p = 1/p_1 + 1/p_2$.

(ii) Assume $G \in C^{0, \alpha}(\mathbb{C}, \mathbb{C})$ for some $0 < \alpha < 1$. Then for any $0 < s < \alpha, 1 < p < \infty$ and $s/\alpha < \sigma < 1$, we have
\[ \|\nabla^s G(u)\|_{L^p} \leq C \|u^{\alpha - \delta}\|_{L^{p_1}} \|\nabla^s u\|_{L^{p_2}}, \]
provided that $1/p = 1/p_1 + 1/p_2$ and $(1 - s/(\alpha \sigma))p_1 > 1$.

Proof of Lemma 3.3. For the proofs for (i) and (ii), see [9, Proposition 3.1] and [38, Proposition A.1], respectively. \hfill \Box

Lemma 3.4. Let $f(u) = |u|^{q-1}u$, $q = 1$ or $p$ and $S(u) = \mu |u|^{p-1}t^{-\delta}$. If $1 < s < 2$ and $s < p < 3$, then we have for $t \geq 1$,
\[ \left\{ \begin{array}{ll}
(1 + \|\sqrt{3} \xi_1 u\|_{L^\infty}^{2p-2}) \|\sqrt{3} \xi_1 u\|_{H^s} & \text{for } q = 1, \\
(1 + \|\sqrt{3} \xi_1 u\|_{L^\infty}^{2p-2}) \|\sqrt{3} \xi_1 u\|_{H^s}^{p-1} \|\sqrt{3} \xi_1 u\|_{H^s} & \text{for } q = p.
\end{array} \right. \]
If $2 \leq s < p < 3$, then we have for $t \geq 1$,
\[
\| \langle \sqrt{3}\xi_1 \rangle f(u) e^{iS(u)} \|_{H^s}
\leq C \begin{cases}
(1 + \| \langle \sqrt{3}\xi_1 \rangle u \|_{L^\infty}^{3p-3}) \| \langle \sqrt{3}\xi_1 \rangle u \|_{H^s} & \text{for } q = 1, \\
(1 + \| \langle \sqrt{3}\xi_1 \rangle u \|_{L^\infty}^{3p-3}) \| \langle \sqrt{3}\xi_1 \rangle u \|_{H^s}^{p-1} & \text{for } q = p.
\end{cases}
\]

Proof of Lemma 3.4. We consider the case $q = 1$ only since the case $q = p$ being similar. An $L^2$ estimate for $\langle \sqrt{3}\xi_1 \rangle f(u) e^{iS(u)}$ is trivial, so we consider $\dot{H}^s$ estimate.

We first consider the case $1 < s < p < 2$. Since
\[
\nabla((\sqrt{3}\xi_1)u e^{iS(u)}) = \nabla((\sqrt{3}\xi_1)u)(1 + iS'(u)u) e^{iS(u)} - iS'(u)u(\nabla, (\sqrt{3}\xi_1)u) e^{iS(u)},
\]
we obtain
\[
\| \nabla^s ((\sqrt{3}\xi_1)u e^{iS(u)}) \|_{L^2} \leq C(\| \nabla^s ((\sqrt{3}\xi_1)u) u \|_{L^2} + \| \nabla^{s-1} \nabla((\sqrt{3}\xi_1)u) S'(u) u e^{iS(u)} \|_{L^2} + C\| \nabla^s ((\sqrt{3}\xi_1)u) S'(u) u e^{iS(u)} \|_{L^2}).
\]

By Lemma 3.2, we have
\[
\| \nabla^s ((\sqrt{3}\xi_1)u e^{iS(u)}) \|_{L^2} \leq C(\| \nabla^s e^{iS(u)} \|_{L^{\frac{2s}{3}}} + \| \nabla^s (S'(u) u e^{iS(u)}) \|_{L^{\frac{2s}{3}}} + C(1 + \| S'(u) u \|_{L^\infty}) \| \nabla((\sqrt{3}\xi_1)u) \|_{H^s} + \| \nabla((\sqrt{3}\xi_1)u) \|_{H^{s-1}}).
\]

Lemma 3.2, Lemma 3.3 (ii) and the interpolation inequality yield
\[
\| \nabla^s e^{iS(u)} \|_{L^{\frac{2s}{3}}} \leq C\| u \|_{L^{\infty}}^{p-1} \| \nabla^s u \|_{L^{\infty}}^{\frac{s-1}{s}} \leq C\| u \|_{L^{\infty}}^{p+\frac{1}{2}} \| u \|_{H^s}^{1-\frac{1}{s}},
\]
where $\sigma$ satisfies $(s - 1)/(p - 1) < \sigma < 1$. Lemma 3.3 and (3.2) imply
\[
\| \nabla^s (S'(u) u e^{iS(u)}) \|_{L^{\frac{2s}{3}}} \leq C(1 + \| u \|_{L^{\infty}}^{p-1}) \| \nabla^s u \|_{L^{\infty}}^{p-1} \| u \|_{H^s}^{1-\frac{1}{s}} \leq C(1 + \| \nabla((\sqrt{3}\xi_1)u) \|_{L^{\infty}}^{p-1}) \| \nabla((\sqrt{3}\xi_1)u) \|_{H^s}^{1-\frac{1}{s}}.
\]

The interpolation inequality yields
\[
\| \nabla((\sqrt{3}\xi_1)u) \|_{L^{2s}} \leq C\| \nabla((\sqrt{3}\xi_1)u) \|_{L^{\infty}}^{1-\frac{1}{s}} \| \nabla((\sqrt{3}\xi_1)u) \|_{H^s}^{\frac{1}{s}}.
\]
Since $[\nabla, (\sqrt{3}\xi_1)] = 3\xi_1 (\sqrt{3}\xi_1)^{-1}$ is a pseudo-differential operator of order zero, by the $L^p$ boundedness of pseudo-differential operator (see [36, Chapter VI] for instance) and the interpolation inequality, we obtain
\[
\| [\nabla, (\sqrt{3}\xi_1)] u \|_{L^{2s}} \leq C\| u \|_{L^{2s}} \leq C\| u \|_{H^s}^{1-\frac{1}{s}} \| u \|_{H^s}^{\frac{1}{s}},
\]
\[
\| [\nabla, (\sqrt{3}\xi_1)] u \|_{H^{s-1}} \leq C\| u \|_{H^{s-1}} \leq C\| u \|_{H^s}.
\]
Combining (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6), we have
\[ \||\nabla||^s((\sqrt{3}\xi_1)ue^{iS(u)})||_{L^2} \leq C(1 + ||\langle 3\xi_1\rangle u||_{L^\infty}^{2p-2})||\langle 3\xi_1\rangle u||_{H^s}. \]

Next we consider the case \(1 < s < 2 \leq p\). As in the previous case, we obtain (3.1). Lemma 3.2, Lemma 3.3 (i) and the interpolation inequality yield
\[ \||\nabla||^{s-1}e^{iS(u)}||_{L^2} \leq C||u||_{L^\infty}^{p-2}||\nabla||^{s-1}u||_{L^2} \leq C||u||_{L^\infty}^{p-2+\frac{1}{2}}||u||_{H^s}^{1-\frac{1}{2}} \leq C||\langle 3\xi_1\rangle u||_{L^\infty}^{p-2+\frac{1}{2}}||\langle 3\xi_1\rangle u||_{H^s}^{1-\frac{1}{2}}. \] (3.7)

Lemma 3.3 and (3.7) imply
\[ \||\nabla||^{s-1}(S'(u)ue^{iS(u)})||_{L^2} \leq C(1 + ||\langle 3\xi_1\rangle u||_{L^\infty}^{2p-2})||\langle 3\xi_1\rangle u||_{H^s}. \]

Finally let us consider the case \(2 \leq s < p < 3\). Since
\[ \Delta((\sqrt{3}\xi_1)ue^{iS(u)}) \]
\[ = \Delta((\sqrt{3}\xi_1)u)(1 + iS'(u))e^{iS(u)} \]
\[ + \nabla u \cdot \nabla((\sqrt{3}\xi_1)u)(2iS'(u) + iS''(u)u - \Delta(u) S'(\xi_1))e^{iS(u)}, \]
\[ -iS'(u)u(\Delta, (\sqrt{3}\xi_1))u) e^{iS(u)} \]
\[ - \nabla u(\nabla, (\sqrt{3}\xi_1))u)(iS''(u)u - \Delta(u) S'(\xi_1))e^{iS(u)}, \]

we obtain
\[ \||\nabla||^s((\sqrt{3}\xi_1)ue^{iS(u)})||_{L^2} \]
\[ \simeq \||\nabla||^{s-2}\Delta((\sqrt{3}\xi_1)ue^{iS(u)})||_{L^2} \]
\[ \leq C(\||\nabla||^{s-2}\Delta((\sqrt{3}\xi_1)ue^{iS(u)})||_{L^2} + \||\nabla||^{s-2}\Delta((\sqrt{3}\xi_1)u)S'(u)ue^{iS(u)}||_{L^2} \]
\[ + C\||\nabla||^{s-2}\{\nabla u \cdot \nabla((\sqrt{3}\xi_1)u)S'(u)ue^{iS(u)}||_{L^2} \]
\[ + C\||\nabla||^{s-2}\{\nabla u \cdot \nabla((\sqrt{3}\xi_1)u)S'(u)ue^{iS(u)}||_{L^2} \]
\[ + C\||\nabla||^{s-2}\{\Delta, (\sqrt{3}\xi_1))u) e^{iS(u)}||_{L^2} \]
\[ + C\||\nabla||^{s-2}\{\nabla u((\nabla, (\sqrt{3}\xi_1))u)S'(u)ue^{iS(u)}||_{L^2} \]
\[ + C\||\nabla||^{s-2}\{\nabla u((\nabla, (\sqrt{3}\xi_1))u)S'(u)ue^{iS(u)}||_{L^2}. \]

Hence by Lemma 3.2, we have
\[ \||\nabla||^s((\sqrt{3}\xi_1)ue^{iS(u)})||_{L^2} \]
\[ \leq C(\||\nabla||^{s-2}e^{iS(u)}||_{L^\infty} + \||\nabla||^{s-2}(S'(u)ue^{iS(u)})||_{L^2} \]
\[ \times (||\Delta((\sqrt{3}\xi_1)u)||_{L^2} + ||\Delta, (\sqrt{3}\xi_1))u||_{L^s}) \]
By the interpolation inequality, we obtain
\begin{align}
&+C(1 + \|S'(u)u\|_{L^\infty})(\|\sqrt{\bar{c}_1}\|_{\dot{H}^s} + \|\|\sqrt{\bar{c}_1}\|_{\dot{H}^{s-2}}
+ C\|\|\nabla|^{-1}u\|_{L^{\frac{2}{s-1}}}\|\nabla((\sqrt{\bar{c}_1})u)\|_{L^{2s}} + \|\|\nabla|^{-1}u\|_{L^{\frac{2}{s-1}}}\|\nabla((\sqrt{\bar{c}_1})u)\|_{L^{2s}}
\end{align}
Proof of Proposition 3.1. The proof follows by applying Lemma 3.4 for $u = \langle \sqrt{3} \xi_1 \rangle^{-1} \hat{\psi}_+$. \hfill \Box

4. Proof of Theorems 1.1 and 1.2. In this section we prove Theorems 1.1 and 1.2. We first rewrite (1.8) as the integral equation. Let $L = i\partial_t + (1/2)\Delta - (1/4)\partial_x^4$, and let

$$w(t, \xi) = \hat{\psi}_+(\xi)e^{iS_+(t, \xi)},$$

where $S_+$ is given by (1.7). From (1.8) and (4.1), we obtain

$$i\partial_t (FW(-t)u) = FW(-t)Lu = \lambda FW(-t)|u|^{p-1}u,$$  \hspace{1cm} (4.2)

$$i\partial_t w = \lambda \frac{t^{-\frac{p+1}{2}d}}{(3\xi_1^2 + 1)^{\frac{p+1}{2}}} |\hat{\psi}_+(\xi)|^{p-1} \hat{\psi}_+(\xi)e^{iS_+(t, \xi)}.$$  \hspace{1cm} (4.3)

Subtracting (4.3) from (4.2), we have

$$i\partial_t (FW(-t)u - w) = \lambda FW(-t) \left[ |u|^{p-1}u - W(t)F^{-1} \left( \frac{t^{-\frac{p+1}{2}d}}{(3\xi_1^2 + 1)^{\frac{p+1}{2}}} |\hat{\psi}_+(\xi)|^{p-1} \hat{\psi}_+(\xi)e^{iS_+(t, \xi)} \right) \right].$$  \hspace{1cm} (4.4)

Let

$$u_+(t, x) := \frac{t^{-\frac{d}{2}}}{\sqrt{3\mu^2 + 1}} \hat{\psi}_+^t(\mu)e^{\frac{3}{2}it\mu^2 + \frac{1}{2}it|\mu|^4 + iS_+(t, \mu) - \frac{d}{2}\pi}.$$  \hspace{1cm}

Propositions 2.1 and 3.1 yield

$$W(t)F^{-1} \left( \frac{t^{-\frac{p+1}{2}d}}{(3\xi_1^2 + 1)^{\frac{p+1}{2}}} |\hat{\psi}_+(\xi)|^{p-1} \hat{\psi}_+(\xi)e^{iS_+(t, \xi)} \right) = |u_+|^{p-1}u_+ + R_1,$$

where $R_1(t)$ satisfies

$$\|R_1(t)\|_{L^2} \leq C t^{-\frac{p+1}{2}d - \beta} \|F^{-1} \left( \frac{1}{(3\xi_1^2 + 1)^{\frac{p+1}{2}}} |\hat{\psi}_+(\xi)|^{p-1} \hat{\psi}_+(\xi)e^{iS_+(t, \xi)} \right)\|_{H^{\beta}_{p, \omega}}.$$  \hspace{1cm}

Furthermore, by Propositions 2.1 and 3.1,

$$W(t)F^{-1} \left( \frac{t^{-\frac{p+1}{2}d}}{(3\xi_1^2 + 1)^{\frac{p+1}{2}}} |\hat{\psi}_+(\xi)|^{p-1} \hat{\psi}_+(\xi)e^{iS_+(t, \xi)} \right) = |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w + R_2(t),$$

where

$$\|R_2(t)\|_{L^2} \leq \|W(t)F^{-1}w|^{p-1}W(t)F^{-1}w - |u_+|^{p-1}u_+\|_{L^2} \leq (\|W(t)F^{-1}w\|_{L^2}^{p-1} + \|u_+\|^{p-1}_{L^2}) \|W(t)F^{-1}w - u_+\|_{L^2} \leq Ct^{-\frac{p+1}{2}d - \beta} \|\psi_+\|_{H^{\beta}_{p, \omega}}.$$  \hspace{1cm}

Combining the above inequalities, we obtain

$$W(t)F^{-1} \left( \frac{t^{-\frac{p+1}{2}d}}{(3\xi_1^2 + 1)^{\frac{p+1}{2}}} |\hat{\psi}_+(\xi)|^{p-1} \hat{\psi}_+(\xi)e^{iS_+(t, \xi)} \right) = |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w + R(t),$$  \hspace{1cm} (4.5)
where $R$ satisfies
\[
\|R\|_{L_1(t,\infty;L^2_x)} \leq C t^{1-\frac{1+\beta}{4}-\beta} p \|\psi_+\|_{H^{\alpha,p}_x},
\]  
(4.6)
where $1/8 < \beta < 1/2$. Substituting (4.5) into (4.4), we obtain
\[
i\partial_t(FW(-t)u - w) = \lambda FW(-t)[|u|^{p-1}u - |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w] - FW(-t)R.
\]
Integrating the above equation with respect to $t$ variable on $(t, \infty)$, we have
\[
u(t) - W(t)F^{-1}w
\]
\[
i \lambda \int_t^{+\infty} W(t-\tau)[|u|^{p-1}u - |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w](\tau)d\tau - i \int_t^{+\infty} W(t-\tau)R(\tau)d\tau.
\]
(4.7)
We prove the existence of a solution to (4.7). We first consider the case $d = 3$. Notice that in this case the end point Strichartz estimate is available.

**Case.** $d = 3$. To show the existence of $u$ satisfying (4.7), we shall prove that the map
\[
\Phi[u](t) = W(t)F^{-1}w + i\lambda \int_t^{+\infty} W(t-\tau)[|u|^{p-1}u - |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w](\tau)d\tau - i \int_t^{+\infty} W(t-\tau)R(\tau)d\tau
\]
(4.8)
is a contraction on
\[
X_{\rho,T} = \{ u \in C([T, \infty); L^2(\mathbb{R}^3)) \cap (\partial_{x_1})^{-\frac{2}{q}} L^q_{loc}(T, \infty; L^r(\mathbb{R}^3)), \quad \| u - W(t)F^{-1}w \|_{X_T} \leq \rho \},
\]
for some $T \geq 3$, where $(q, r) = (4/(3p - 5), 6/(8 - 3p))$.

Let $v(t) = u(t) - W(t)F^{-1}w$ and $v \in X_{\rho,T}$. Then
\[
\Phi[u](t) - W(t)F^{-1}w = i\lambda \int_t^{+\infty} W(t-\tau)[(|v| + W(t)F^{-1}w)^{p-1}(v + W(t)F^{-1}w)
\]
\[
- |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w](\tau)d\tau
\]
\[
- i \int_t^{+\infty} W(t-\tau)R(\tau)d\tau.
\]
Since
\[
\| v + W(t)F^{-1}w \|^{p-1}(v + W(t)F^{-1}w) - |W(t)F^{-1}w|^{p-1}W(t)F^{-1}w
\]
\[
\leq p \int_0^1 \| \theta v + W(t)F^{-1}w \|^{p-1} - |W(t)F^{-1}w|^{p-1}|d\theta| |v| + p |W(t)F^{-1}w|^{p-1} |v|
\]
\[
\leq C (|v|^{p-1} + |W(t)F^{-1}w|^{p-1}) |v|,
\]
the Strichartz estimate (Lemma 2.3) implies
\[
\|\Phi[u] - W(t)\mathcal{F}^{-1}w\|_{L^\infty(t,\infty;L_x^\infty)} + \|\langle \partial_x \rangle^{\frac{2}{p}}(\Phi[u] - W(t)\mathcal{F}^{-1}w)\|_{L^q(t,\infty;L_x^p)} \\
\leq C\left(\|v\|_{L^2(t,\infty;L_x^2)}^{p-1}\|v\|_{L_x^p} + \|W(t)\mathcal{F}^{-1}w\|_{L^1(t,\infty;L_x^2)}^{1/s}\frac{1}{s} \right) + \|R\|_{L^1(t,\infty;L_x^2)}.
\]
By the Hölder inequality, we find
\[
\|v\|_{L^2(t,\infty;L_x^2)}^{p-1}\|v\|_{L_x^p} \leq C\|v\|_{L^2(t,\infty)}^{p-1}\|v\|_{L_x^p}^{p},
\]
where $1/s = (7 - 3p)/4$. Substituting above two inequalities and (4.6) into (4.9), we have
\[
\|\Phi[u] - W(t)\mathcal{F}^{-1}w\|_{X_T} \leq C(\rho T^{-\alpha(p-1)+1+\frac{3}{4}(p-1)} + \rho T^{-\frac{2}{3}(p-1)+1} + T^{-\frac{2}{3}(p-1)+1+\alpha-\beta}).
\]
Choosing $1/(p-1) - 3/4 < \alpha < \beta < 1/2$ and $T$ large enough, we guarantee that $\Phi$ is a map onto $X_{p,T}$. In a similar way, we can conclude that $\Phi$ is a contraction map on $X_{p,T}$. Therefore, by Banach fixed point theorem one infers that $\Phi$ has a unique fixed point in $X_{p,T}$ which is the solution to the final state problem (1.8).

Next, we show that the solution to (1.8) with finite $X_T$ norm is unique. Let $u_1$ and $u_2$ be two solutions satisfying $\|u_1\|_{X_T} < \infty$ and $\|u_2\|_{X_T} < \infty$. We put $t_1 = \inf\{t \in [T, \infty) : u_1(s) = u_2(s) \text{ for any } s \in [t, \infty)\}$ and $\rho = \max\{|\|u_1\|_{X_T}, \|u_2\|_{X_T}\}$. If $t_1 = T$, then $u_1(t) = u_2(t)$ on $[T, \infty)$ which is desired result. If $T < t_1$, as in (4.9) by the Strichartz inequality (Lemma 2.3), we have
\[
\|\langle \partial_x \rangle^{\frac{2}{p}}(u_1 - u_2)\|_{L^q(t_0,t_1;L_x^p)} \leq C\rho^{p-1}(t_0^{1-s(p-1)\alpha} - t_1^{1-s(p-1)\alpha})^{1/s}\|\langle \partial_x \rangle^{\frac{2}{p}}(u_1 - u_2)\|_{L^q(t_0,t_1;L_x^p)},
\]
for $t_0 \in [T, t_1]$. Since $1 - s(p-1)\alpha < 0$, we can choose $t_0 \in [T, t_1]$ so that $C\rho^{p-1}(t_0^{1-s(p-1)\alpha} - t_1^{1-s(p-1)\alpha})^{1/s} < 1$. Then $\|\langle \partial_x \rangle^{\frac{2}{p}}(u_1 - u_2)\|_{L^q(t_0,t_1;L_x^p)} \leq 0$ which implies that $u_1(t) \equiv u_2(t)$ on $[t_0, t_1]$. This contradicts the assumption of $t_1$. Hence $u_1(t) = u_2(t)$ on $[T, \infty)$.

From (1.8), we obtain
\[
u(t) = W(t-T)u(T) - i\lambda \int_T^t W(\tau)u(\tau)^{p-1}u(\tau)d\tau.
\]
Since $u(T) \in L_x^2(\mathbb{R}^3)$, combining the argument by [37] with the Strichartz estimate (Lemma 2.3) and $L^2$ conservation law for (4.10), we can prove that (4.10) has a unique global solution in $C(\mathbb{R}; L_x^2(\mathbb{R}^3)) \cap \langle \partial_x \rangle^{-2/(3q)}L_{loc}^q(\mathbb{R}; L_x^p(\mathbb{R}^3))$. Therefore the solution $u$ of (1.8) can be extended to all times.
Finally we show that the solution to (1.8) converges to $W(t)\psi_+$ in $L^2$ as $t \to \infty$. By $u - W(t)\mathcal{F}^{-1}w \in \mathbb{X}_{p,T}$ and Proposition 2.1, we have
\[
\|u(t) - W(t)\psi_+\|_{L^2_x} \leq \|u(t) - W(t)\mathcal{F}^{-1}w\|_{L^2_x} + \|W(t)\mathcal{F}^{-1}w - \psi_+\|_{L^2_x}
\]
\[
= \|u(t) - W(t)\mathcal{F}^{-1}w\|_{L^2_x} + \|w - \psi_+\|_{L^2_x}
\]
\[
\leq Ct^{-\alpha} + \|S_+(t, \xi)\|_{L^{\infty}_x}\|\psi_+\|_{L^2_x}
\]
\[
\leq Ct^{-\alpha} + Ct^{1-\frac{2}{p}(p-1)}P(\|\psi_+\|_{H^0_x})
\]
\[
\leq t^{-\gamma}P(\|\psi_+\|_{H^0_x})
\]
where $1/(p-1) - 3/4 \leq \alpha < 1/2$ and $\min\{1/(p-1) - 3/4, 3(p-1)/2 - 1\} < \gamma$. This completes the proof of Theorem 1.2.

Case. $d = 2$. Next we show Theorem 1.1. In this case the end point Strichartz estimate is not available. Instead, we use the admissible pair which is close to the end point.

To show the existence of $u$ satisfying (4.7), we shall prove that the map $\Phi$ given by (4.8) is a contraction on
\[
\mathbb{X}_{p,T} = \{ u \in C([T, \infty); L^2(\mathbb{R}^2)) \cap \langle \partial_x \rangle^{-\frac{1}{4}} L^q_{x; t}(T, \infty; L^r(\mathbb{R}^2)); \}
\]
\[
\|u - W(t)\mathcal{F}^{-1}w\|_{\mathbb{X}_T} \leq \rho \},
\]
\[
\|v\|_{\mathbb{X}_T} = \sup_{t \geq T} (\|v\|_{L^q(t, \infty; L^r_x)} + \|\langle \partial_x \rangle^{\frac{1}{4}} v\|_{L^r(t, \infty; L^r_x)})
\]
for some $T \geq 3$, where $(q, r) = (2/(p - 2 + 2\varepsilon), 2/(3 - p - 2\varepsilon))$.

Let $v(t) = u(t) - W(t)\mathcal{F}^{-1}w$ and $v \in \mathbb{X}_{p,T}$. Then the Strichartz estimate (Lemma 2.3) implies
\[
\|\Phi[u] - W(t)\mathcal{F}^{-1}w\|_{L^q(t, \infty; L^r_x)} + \|\langle \partial_x \rangle^{\frac{1}{4}} (\Phi[u] - W(t)\mathcal{F}^{-1}w)\|_{L^r(t, \infty; L^r_x)}
\]
\[
\leq C(\|v\|^{p-1}_{L^r_x} + \|\langle \partial_x \rangle^{\frac{1}{4}} v\|_{L^r(t, \infty; L^r_x)})
\]
\[
+ \|R\|_{L^r(t, \infty; L^r_x)}).
\]

(4.11)

By the Hölder inequality,
\[
\|v\|^{p-1}_{L^r_x} \leq C\|v\|^{p-1}_{L^r_x} \|v\|_{L^{\frac{r'}{r}(t, \infty)}} \leq C\rho^{p-1}\|v\|^{p-1}_{L^r_x} \|v\|_{L^{\frac{r'}{r}(t, \infty)}}
\]
\[
\leq C\rho^{p-1}\|v\|^{p-1}_{L^r_x} \|v\|_{L^{\frac{r'}{r}(t, \infty)}}
\]
\[
\leq C\rho^{p-1}\|v\|^{p-1}_{L^r_x} \|v\|_{L^{\frac{r'}{r}(t, \infty)}}
\]
\[
\leq C\rho^{p-1}\|v\|^{p-1}_{L^r_x} \|v\|_{L^{\frac{r'}{r}(t, \infty)}}
\]
\[
\|w\|_{L^r_x} \|w\|_{L^r_x} \|w\|_{L^r_x} \|w\|_{L^r_x} \|w\|_{L^r_x}
\]
\[
\|\mathcal{F}[u] - W(t)\mathcal{F}^{-1}w\|_{\mathbb{X}_T} \leq C(p^{p-1}T^{-\alpha(p-1) + 1 - \frac{2}{p}(p-1)} + \rho T^{-(p-1) + 1 + \alpha - \beta}).
\]
Choosing $1/(p-1) - 1/2 < \alpha < \beta < 1/2$ and $T$ large enough, we guarantee that $\Phi$ is a map onto $X_{p,T}$. In a similar way we can conclude that $\Phi$ is a contraction map on $X_{p,T}$. Therefore, by Banach fixed point theorem one finds that $\Phi$ has a unique fixed point in $X_{p,T}$ which is the solution to the final state problem (1.8). Since the rest part of the proof is same as the proof of Theorem 1.2, we omit the detail. This completes the proof of Theorem 1.1.

5. Final comments. Finally, we give several comments.

1. As mentioned in the introduction, we restricted our study of (1.1) to the case where $\alpha$, $\beta$ and $\gamma$ satisfy $(\alpha + \frac{3\beta^2}{8\gamma}) \alpha > 0$ and $\alpha\gamma < 0$. If those conditions are violated, one should replace (1.2) by the “non-elliptic” NLS

$$i\partial_t u + \frac{1}{2} \Delta u - \frac{1}{2} \partial_{x_1}^2 u \pm \frac{1}{4} \partial_{x_1}^4 u = \lambda |u|^{p-1} u \quad t > 0, \quad x \in \mathbb{R}^d.$$ (5.1)

It is very likely that similar scattering results hold true in this case since the linear estimates should be essentially the same.

2. We were concerned in this paper with scattering issues for equation (1.1). On a different regime, one might look for a possible blow-up of solutions in the focusing case ($\lambda < 0$ in (1.2)). It is conjectured in [12] that a finite time blow-up should arise when $p \geq 1 + 8/(2d - 1)$. Proving this fact is an interesting open question.

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