On a generalized pseudorelativistic Schrödinger equation with supercritical growth

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To Francesca, with love

We prove that the generalized pseudorelativistic equation
\[
\left(-c^2 \Delta + m^2 c^4 s^{\frac{s}{s-2}}\right)^s u - m^2 c^2 s^{\frac{s}{s-2}} u + \mu u = |u|^{p-1} u
\]
can be solved for large values of the “light speed” \(c\) even when \(p\) crosses the critical value for the fractional Sobolev embedding.

Keywords: Schrödinger equation; fractional Laplacian
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1 Introduction

The pseudorelativistic Schrödinger equation
\[
\frac{i}{\partial t} \psi = \sqrt{-c^2 \Delta + m^2 c^4 \psi - mc^2 \psi + f(|\psi|^2) \psi},
\]
in which \(c\) denotes the speed of the light, \(m > 0\) represents the particle mass and \(f: [0, \infty) \to \mathbb{R}\) is a nonlinear function, is one of the relativistic versions of the more familiar NLS
\[
\frac{i}{\partial t} \psi = -\frac{1}{2m} \Delta \psi + f(|\psi|^2) \psi.
\]

Equation (1) describes, from the physical viewpoint, the dynamics of systems consisting of identical spin-0 bosons whose motions are relativistic, like \textit{boson stars}. We refer to [8, 9, 10, 12, 14, 15] for the rigorous derivation of the equation and the study of its dynamical properties.

When \(f(t) = -t^{\frac{p-1}{2}}\), standing waves \(\psi(t, x) = \exp(i\mu t)u(x)\) must satisfy the stationary equation
\[
\sqrt{-c^2 \Delta + m^2 c^4 u - mc^2 u + \mu u} = |u|^{p-1} u.
\]

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Coti Zelati and Nolasco showed in [7] that for $c = 1$ and $1 < p < (N + 1)/(N - 1)$ there exists a radially symmetric positive solution to [2]. This result was extended later in [3]. Another variant of the above equation is the pseudorelativistic Hartree equation

$$\sqrt{-c^2\Delta + m^2c^4}u - mc^2u + \mu u = (I_\alpha \ast |u|^p)|u|^{p-2}u,$$

where $I_\alpha: \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a singular convolution kernel. If, formally, $I_\alpha$ degenerates to a Dirac delta, the Hartree equation reduces to [1]. The case in which $I_\alpha(x) = |x|^{N-\alpha}$ is particularly important. We refer to [18] for a survey of recent results.

As already noticed, the application of variational techniques to [2] requires a bound from above on the exponent $p$, since the natural Sobolev space in which [2] can be set is $H^{1/2}(\mathbb{R}^N)$ and this space is embedded into $L^q(\mathbb{R}^N)$ only if $q \leq 2N/(N-1)$. Local compactness of the embedding excludes the limiting exponent, and therefore it is customary to assume that $1 < p < (N + 1)/(N - 1)$.

On the other hand, if we observe that the pseudorelativistic operator $\sqrt{-c^2\Delta + m^2c^4} - mc^2$ converges to $-\frac{1}{2mc^2}\Delta$ as $c \to +\infty$, we may expect that solutions could exist for $c \gg 1$ as soon as $p < (N+2)/(N-2)$, namely below the critical Sobolev exponent for the operator $-\frac{1}{2mc^2}\Delta + 1$. This fact has been proved recently in [6].

In this paper we consider the generalized model

$$(-c^2\Delta + m^2c^{4s})^s u - m^{2s}c^{2s}u + \mu u = |u|^{p-1}u,$$

where $1/2 < s < 1$ and $p > 1$, which reduces to [2] for $s = 1/2$. For $c = 1$, this equation has been studied in [1, 13, 19, 20, 21]. Following the ideas of [6], we prove that (3) is actually solvable in the whole range $1 < p < (N + 2)/(N - 2)$ in the régime $c \gg 1$.

**Theorem 1.1.** Let $N \geq 3$ and $1/2 < s < 1$. For every $p \in (1, \frac{N+2}{N-2})$, equation (3) admits at least a nontrivial solution in $H^1_{\text{rad}} \cap L^\infty$ provided that $m^{2s}c^{2s}/\mu$ is sufficiently large.

**Remark 1.2.** The restriction $1/2 \leq s < 1$ is somehow natural, if we want bounded solutions. In the fractional framework, an exponent $s < 1/2$ does not ensure high regularity properties of solutions.

**Remark 1.3.** Although we have stated Theorem 1.1 for any dimension $N \geq 3$, it is rather easy to check that the same result holds also for $N = 1$ or $2$. In this case, however, the Sobolev critical exponent no longer exists.

Our approach is based on the reduction of equation (3) to a fixed-point problem. By means of some pseudo-differential calculus we can overcome the limitation of the variational setting.

We leave as an open problem the study of (3) in the régime $m^{2s}c^{2s}/\mu \ll 1$. Some results appear in [1] when $c = 1 = \mu$ and, consequently, $m \to 0$. Anyway, since the limit equation is in this case

$$(-\Delta)^s u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N,$$

we do not expect any improvement in the range of the exponent $p$.

## 2 Preliminaries and estimates

It will be useful to collect some notation that we are going to employ throughout the paper.

- An integral over the whole space $\mathbb{R}^N$ is denoted simply by $\int$ instead of $\int_{\mathbb{R}^N}$.
The Fourier transform of a (suitably regular) function $\varphi$ is denoted by
$$\hat{\varphi} : \xi \mapsto (2\pi)^{-N/2} \int e^{-i\xi \cdot x} \varphi(x) \, dx.$$  

Function spaces like $L^p(\mathbb{R}^N)$ or $W^{1,p}(\mathbb{R}^N)$ are denoted by $L^p$ and $W^{1,p}$.

If $X$ is some function space, we denote by $X_{\text{rad}}$ the subspace of $X$ consisting of radially symmetric functions.

The identity operator is denoted by $I$. Sometimes, however, we denote the multiplication operator against a given function $u$ by $uI$ instead of $uI$.

Derivatives are always denoted by the letter $D$ or by the symbol $\partial$. In particular, $\partial_{x_j}$ denotes the partial derivative with respect to the variable $x_j$. If $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index, we denote $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_j}^{\alpha_j}$.

Since we are interested in asymptotic estimates as $c \to +\infty$, we use the symbol $\lesssim$ to denote an inequality with a multiplicative constant independent of $c$. Therefore $a \lesssim b$ means that $a \leq Cb$ with some constant $C > 0$ that does not depend on $c$.

For two given Banach spaces, we denote by $L(X,Y)$ the space of continuous linear operators from $X$ to $Y$. The norm in $L(X,Y)$ is the usual one: $\|A\|_{L(X,Y)} = \sup_{\|x\|_X = 1} \|Ax\|_Y$.

If $A$ is an invertible linear operator, we sometimes use the “algebraic” piece of notation $1/A$ to denote the inverse $A^{-1}$.

We begin with an “almost necessary” condition so that solutions to (3) may exist. This is also a motivation for our attempt to construct a solution in a suitable supercritical setting.

**Theorem 2.1.** If $p \geq \frac{N+2s}{N-2s}$ but $m^2 c^{\frac{2s}{N-2s}} \leq \mu$, there is no non-trivial solution $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

**Proof.** Let us consider, for simplicity, the equation
$$\tag{4} (a^2 - b^2 \Delta)^s u = f(u) \quad \text{in } \mathbb{R}^N,$$
where $f$ is a suitable nonlinearity. For a smooth function $\varphi$, we recall that
$$\mathcal{F}(\partial_{x_k} \varphi) = i\xi_k \hat{\varphi}, \quad \mathcal{F}(x_k \varphi) = i\partial_{x_k} \hat{\varphi}.$$  

This implies that
$$\mathcal{F}(x \cdot \nabla \varphi) : \xi \mapsto -(\xi \cdot \nabla \hat{\varphi} + N \hat{\varphi})$$
and
$$\mathcal{F}((a^2 - b^2 \Delta)^s (x \cdot \nabla \varphi)) : \xi \mapsto -(a^2 + b^2 |\xi|^2)^s (\xi \cdot \nabla \hat{\varphi} + N \hat{\varphi}).$$

Hence
$$\mathcal{F} \left( x \cdot \nabla (a^2 - b^2 \Delta)^s \varphi \right) : \xi \mapsto -\left( \xi \cdot \nabla \mathcal{F}((a^2 - b^2 \Delta)^s \varphi) + N \mathcal{F}((a^2 - b^2 \Delta)^s \varphi) \right)$$
$$= -\left( \xi \cdot \nabla \left( (a^2 + b^2 |\xi|^2)^s \varphi \right) + N(a^2 + b^2 |\xi|^2)^s \varphi \right)$$
$$= -(a^2 + b^2 |\xi|^2)^s (\xi \cdot \nabla \hat{\varphi} + N \hat{\varphi}) - 2s b^2 (a^2 + b^2 |\xi|^2)^s - 1 |\xi|^2 \hat{\varphi}$$
$$= -(a^2 + b^2 |\xi|^2)^s (\xi \cdot \nabla \hat{\varphi} + N \hat{\varphi}) - 2s (a^2 + b^2 |\xi|^2)^s \hat{\varphi} + 2s a^2 (a^2 + b^2 |\xi|^2)^{s-1} \hat{\varphi}. $$
We have proved the following pointwise identity:

\[(a^2 - b^2 \Delta)^s (x \cdot \nabla \varphi) - x \cdot \nabla (a^2 - b^2 \Delta)^s \varphi = 2s(a^2 - b^2 \Delta)^s \varphi - 2a^2 s(a^2 - b^2 \Delta)^{s-1} \varphi.\]  

(5)

In the rest of the proof, we will be somehow sketchy. For a rigorous argument, we should replace \( u_\varepsilon = \rho_\varepsilon \ast u \) where \( \rho_\varepsilon \) is a mollifier, and then take the limit as \( \varepsilon \to 0 \). We omit the technical details. Using (5) we get

\[\int (a^2 - b^2 \Delta)^s u(x \cdot \nabla u) = \int u(a^2 - b^2 \Delta)^s (x \cdot \nabla u) = \int u (x \cdot \nabla f(u) + 2s f(u) - 2a^2 s(a^2 - b^2 \Delta)^{s-1} u)\]

\[= -N \int uf(u) + N \int F(u) + \int 2s \int f(u) - \int 2a^2 s u(a^2 - b^2 \Delta)^{s-1} u.\]

Since

\[\int f(u) (x \cdot \nabla u) = -N \int F(u),\]

we find the identity

\[(N - 2s) \int uf(u) - 2N \int F(u) + 2a^2 s \int u(a^2 - b^2 \Delta)^{s-1} u.\]

(6)

We observe that

\[\int u(a^2 - b^2 \Delta)^{s-1} u = \int (a^2 + b^2 |\xi|^2)^{s-1} |\hat{u}|^2 \geq 0.\]

We now choose

\[f(s) = |s|^{p-1} s - \kappa s\]

with \( \kappa \in \mathbb{R} \). Then \( F(s) = \frac{1}{p+1} |s|^{p+1} - \frac{\kappa}{2} s^2 \), and (6) yields

\[\left( \frac{1}{p+1} - \frac{N - 2s}{2N} \right) \int |u|^{p+1} = \frac{\kappa s}{N} \int |u|^2 + \frac{a^2 s}{N} \int u(a^2 - b^2 \Delta)^{s-1} u.\]

If \( p \geq \frac{N + 2s}{N - 2s} \) and \( \kappa \geq 0 \), then \( u = 0 \).

We apply this conclusion with

\[a^2 = m^2 c^{1-s}, \quad b^2 = c^2, \quad \kappa = \mu - m^2 c^{2s},\]

to prove our result.

On the other hand, we are going to construct a (non-trivial) solution to (3) when the quotient \( m^2 c^{2s} / \mu \) is sufficiently large. As a first step, we show that some parameters in (3) are not essential. It is anyway clear that the pseudorelativistic Schrödinger equation is not scale-invariant, but this obstacle is irrelevant in the limit \( c \to +\infty \).

**Proposition 2.2.** The equation

\[\left( \left( -\tilde{c}^2 \Delta + m^2 c^{2s} \right)^8 - m^2 c^{2s} \right) v + \mu v = |v|^{p-1} v\]

is equivalent to

\[\left( -c^2 \Delta + s^{1-s} c^{2s} \right)^8 u - s^{1-s} c^{2s} u + u = |u|^{p-1} u\]

(7)

for a suitable choice of \( c \) and \( \tilde{c} \).

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Proof. We suppose that \( v \) satisfies (at least formally) the equation
\[
\left( -\hat{c}^2 \Delta + m^2 \hat{c}^{-2} \right)^s v + \mu v = f
\]
with \( f = |v|^{p-1} v \). Let us set
\[
u(x) = \rho v(\sigma x)\]
for some constants \( \rho > 0, \sigma > 0 \), so that
\[
\hat{u}(\xi) = \frac{\rho}{\sigma^s} \hat{v} \left( \frac{\xi}{\sigma} \right).
\]
Then, by using Fourier variables,
\[
\left( \hat{c}^2 |\xi|^2 + m^2 \hat{c}^{-2} \right)^s \hat{v} - m^2 \hat{c}^{-1} \hat{v} + \mu \hat{v} = \hat{f}.
\]
Equivalently,
\[
\left( \hat{c}^2 \left( \frac{\xi}{\sigma} \right)^2 + m^2 \hat{c}^{-2} \right)^s \hat{v} \left( \frac{\xi}{\sigma} \right) - m^2 \hat{c}^{-1} \hat{v} \left( \frac{\xi}{\sigma} \right) + \mu \hat{v} \left( \frac{\xi}{\sigma} \right) = \hat{f} \left( \frac{\xi}{\sigma} \right).
\]
If we multiply through by \( \rho/\sigma \) and move back to Euclidean variables, recalling that \( f = |v|^{p-1} v \),
\[
\left( -\frac{\hat{c}^2}{\sigma^2} \Delta + m^2 \hat{c}^{-2} \right)^s u - m^2 \hat{c}^{-1} u + \mu u = \rho^{1-p} |u|^{p-1} u.
\]
Now we choose
\[
\rho = \mu^{\frac{1}{1-p}}, \quad \hat{c} = \sqrt{s \mu^{\frac{p-1}{m(1-s)}}} c, \quad \sigma^2 = \frac{s}{\mu m^{2(1-s)}}.
\]
We remark in particular that \( \hat{c} \to +\infty \) if and only if \( c \to +\infty \). After some elementary but lengthy computation we can show that (3) reduces to
\[
\left( -c^2 \Delta + \frac{1}{s^2-s} \hat{c} \right)^s u - \frac{1}{s^2-s} \hat{c} \hat{u} + u = |u|^{p-1} u,
\]
that is (7).

As a consequence, it is not restrictive to assume that \( m = s^{1/(2-2s)} \) and \( \mu = 1 \), and we will consider equation (7) in the rest of the paper.

We now introduce the (pseudodifferential) operators
\[
P_c(D) = \left( \left( c^2 |D|^2 + \frac{1}{s^2-s} c \right)^s - \frac{1}{s^2-s} c \right) + 1
\]
\[
P_{\infty}(D) = |D|^2 + 1,
\]
which are associated to the symbols
\[
P_c(\xi) = \left( \left( c^2 |\xi|^2 + \frac{1}{s^2-s} c \right)^s - \frac{1}{s^2-s} c \right) + 1
\]
\[
P_{\infty}(\xi) = |\xi|^2 + 1.
\]
Remark 2.3. In general, we recall that given a symbol \( m : \mathbb{R}^N \to \mathbb{R} \), the associated Fourier multiplier operator \( m(D) \) is defined (on smooth functions) by

\[
\widehat{m(D)f}(\xi) = m(\xi)\hat{f}(\xi).
\]

With this notation our problem is equivalent to

\[
P_c(D)u = |u|^{p-1}u
\]
as \( c \gg 1 \). The main idea is to begin with a solution \( u_\infty \) of

\[
P_\infty(D)(u_\infty) = |u_\infty|^{p-1}u_\infty
\]
and set \( w = u - u_\infty \). Hence

\[
P_c(D)w = P_c(D)u - P_c(D)u_\infty = P_c(D)u + P_\infty(D)u_\infty - P_c(D)u_\infty = P_c(D)u_\infty
\]
\[
= (P_\infty(D) - P_c(D))u_\infty + P_c(D)u - P_\infty(D)u_\infty
\]
\[
= (P_\infty(D) - P_c(D))u_\infty + |w + u_\infty|^{p-1}(w + u_\infty) - u_\infty^p.
\]

Equivalently,

\[
L_{c,\infty}w = (P_\infty(D) - P_c(D))u_\infty + Q(w),
\]
where

\[
L_{c,\infty} = P_c(D) - pu_\infty^{p-1}
\]
and

\[
Q(w) = |w + u_\infty|^{p-1}(w + u_\infty) - u_\infty^p - pu_\infty^{p-1}w.
\]

If we can invert \( L_{c,\infty} \) in a suitable space, then we may write the fixed-point equation

\[
w = (L_{c,\infty})^{-1}(P_\infty(D) - P_c(D))u_\infty + (L_{c,\infty})^{-1}Q(w).
\]

We will prove that the nonlinear operator

\[
\Phi_c(w) = (L_{c,\infty})^{-1}(P_\infty(D) - P_c(D))u_\infty + (L_{c,\infty})^{-1}Q(w)
\]
is contractive in a small ball of a suitable Sobolev space. To do this we need, as expected, some careful estimates.

**Lemma 2.4.** There results

\[
\frac{|\xi|^2 + 1}{2} \leq P_c(\xi) \leq |\xi|^2 \quad \text{if } |\xi| \leq c^{\frac{1}{s+1}} \sqrt{\frac{2^{1+\frac{1}{s+1}} - 1}{s^{\frac{1}{s+1}}}},
\]

and

\[
\left(1 + \frac{1}{2^{s+1} - 1}\right)^s - \frac{1}{(2^{s+1} - 1)^s} \right)^s c^{2s} |\xi|^{2s} + 1 \leq P_c(\xi) \leq c^{2s} |\xi|^{2s} + 1
\]
if \( |\xi| \geq c^{\frac{1}{s+1}} \sqrt{\frac{2^{1+\frac{1}{s+1}} - 1}{s^{\frac{1}{s+1}}}} \).
Proof. We observe that
\[
P_c(\xi) = \left(\left(c^2|\xi|^2 + \frac{1}{s^{\frac{1}{s-1}} c^{\frac{2}{s-1}} s^{1-s}}\right)^s - \frac{1}{s^{\frac{1}{s-1}} c^{\frac{2}{s-1}}}\right) + 1
\]
\[
= \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\left(\left(1 + \frac{1}{s^{\frac{1}{s-1}} c^{-\frac{2s}{s-1}} |\xi|^2 s^{1-s}}\right)^s - 1\right) + 1
\]
\[
= \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\left(f\left(\frac{s^{\frac{1}{s-1}} |\xi|^2}{c^{\frac{2s}{s-1}}}\right)\right) + 1,
\]
where
\[
f(t) = (1 + t)^s - 1 \quad \text{for every } t \geq 0.
\]
Clearly \(f(0) = 0\), and
\[
Df(t) = \frac{s}{(1+t)^{1-s}}.
\]
Plainly \(Df(t) \leq s\) for every \(t \geq 0\), while
\[
\frac{s}{(1+t)^{1-s}} \geq \frac{s}{2} \quad \text{if } t \leq 2^{1-s} - 1.
\]
Assume that
\[
|\xi| \leq \frac{c^{\frac{1}{s-1}}}{s^{\frac{1}{s-1}}} \sqrt{\frac{2^{1-s} - 1}{s^{\frac{1}{s-1}}}}
\]
so that, by Taylor's theorem
\[
P_c(\xi) - 1 = \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\left(\frac{s^{\frac{1}{s-1}} |\xi|^2}{c^{\frac{2s}{s-1}}}\right) \leq \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\left(f(0) + Df(0) s^{\frac{1}{s-1}} |\xi|^2\right)
\]
\[
\leq |\xi|^2.
\]
Similarly, (11) implies
\[
P_c(\xi) - 1 \geq \frac{1}{2} |\xi|^2.
\]
Hence (9) is proved. Assume on the contrary that
\[
|\xi| \geq \frac{c^{\frac{1}{s-1}}}{s^{\frac{1}{s-1}}} \sqrt{\frac{2^{1-s} - 1}{s^{\frac{1}{s-1}}}}.
\]
and write
\[
P_c(\xi) - 1 = \left(c^2|\xi|^2 + \frac{c^{\frac{2}{s-1}}}{s^{\frac{1}{s-1}}} s^{1-s}\right)^s - \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\]
\[
= c^2|\xi|^2 s^{1-s} \left(1 + \frac{c^{\frac{2}{s-1}}}{s^{\frac{1}{s-1}} c^2|\xi|^2 s^{1-s}}\right)^s - \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\]
\[
= c^2|\xi|^2 s^{1-s} \left(1 + \frac{c^{\frac{2}{s-1}}}{s^{\frac{1}{s-1}} c^2|\xi|^2} s^{1-s} |\xi|^2 s^{1-s}\right)^s - \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\]
\[
= c^2|\xi|^2 s^{1-s} \left(1 + \frac{c^{\frac{2}{s-1}}}{s^{\frac{1}{s-1}} |\xi|^2} s^{1-s} |\xi|^2 s^{1-s}\right)^s - \frac{c^{\frac{2s}{s-1}}}{s^{\frac{1}{s-1}}}
\]
The concavity of the function $t \mapsto t^s$ implies easily that
\[
\left( 1 + \frac{c^{1-s}}{s^{1-s}|\xi|^2} \right)^s - \frac{c^{2s}}{s^{2s-1}|\xi|^2} \leq 1.
\]
On the other hand, by monotonicity, $|\xi| \geq c^{1-s} \sqrt{2s^{1-s} - 1}$ implies
\[
\left( 1 + \frac{2 \zeta}{s^{1-s}} \right)^s - \frac{2 \zeta}{s^{1-s}} \geq \left( 1 + \frac{1}{s^{1-s} c^{1-s} \frac{2 \zeta}{s^{1-s} - 1}} \right)^s - \frac{2 \zeta}{s^{1-s} c^{1-s} \frac{2 \zeta}{s^{1-s} - 1}}.
\]
This proves (10).

\[\square\]

**Lemma 2.5.** There results
\[
|P_c(\xi) - P_\infty(\xi)| \leq \frac{s(s-1)}{2s^{2-s}} c^{2-s} |\xi|^4.
\]

**Proof.** By direct computation, and using again the same notation as in Lemma 2.4,
\[
P_c(\xi) - P_\infty(\xi) = \left( c^2 |\xi|^2 + \frac{c^{1-s}}{s^{1-s}} \right)^s - \frac{c^{2s}}{s^{2s-1}} - |\xi|^2 = \frac{c^{2s}}{s^{2s-1}} f \left( s^{1-s} \frac{|\xi|^2}{c^{1-s}} \right) - |\xi|^2 - \frac{2}{2} D^2 f(\zeta) \frac{s^{2-s}}{1 + \zeta} \frac{|\xi|^4}{c^{2-s}} \frac{c^{1-s}}{s^{1-s}} - |\xi|^2
\]
for some $0 < \zeta < s^{2-s} \frac{|\xi|^2}{c^{1-s}}$. Since $D^2 f(\zeta) = s(s-1)(\zeta + 1)^{s-2}$, we conclude that
\[
P_c(\xi) - P_\infty(\xi) = \frac{c^{1-s}}{s^{1-s}} \left( \frac{s^{2-s}}{c^{1-s}} |\xi|^2 + \frac{1}{2} s(s-1) \frac{s^{2-s}}{(\zeta + 1)^{2-s}} \frac{|\xi|^4}{c^{2-s}} \right) - |\xi|^2
\]
\[
= \frac{s(s-1)}{2s^{2-s}} \frac{|\xi|^4}{(1 + \zeta)^{2-s} c^{1-s}} \leq \frac{s(s-1)}{2s^{2-s}} \frac{|\xi|^4}{c^{2-s} c^{1-s}}.
\]

\[\square\]

We can now prove the core result for our estimates.

**Proposition 2.6.** (a) For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$ there exists a constant $C_\alpha > 0$ such that for all $c \geq 2$,
\[
|D^\alpha \left( \frac{1}{P_c(\xi)} - \frac{1}{P_\infty(\xi)} \right)| \leq C_\alpha \frac{1}{|\xi|^{|\alpha|}} \min \left\{ \frac{1}{c^{1-s}}, \frac{1}{c^{2-s}} (|\xi|^2 + 1)^{1-s} \right\}.
\]

(b) For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$ there exists a constant $C_\alpha > 0$ such that for all $c \geq 2$,
\[
|D^\alpha \left( \frac{P_c(\xi)}{P_\infty(\xi)} \right)| \leq C_\alpha \frac{1}{|\xi|^{|\alpha|}}.
\]
Proof. Let us denote
\[
a(\xi) = \frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)} = \frac{P_c(\xi) - P_\infty(\xi)}{P_c(\xi)P_\infty(\xi)}.
\]
According to the Leibniz rule for differentiation,
\[
\partial_\xi a(\xi) = \partial_\xi (P_c(\xi) - P_\infty(\xi)) \frac{1}{P_c(\xi)P_\infty(\xi)} + (P_c(\xi) - P_\infty(\xi)) \partial_\xi \left( \frac{1}{P_c(\xi)} \right) \frac{1}{P_\infty(\xi)}
\]
\[
+ (P_c(\xi) - P_\infty(\xi)) \frac{1}{P_c(\xi)} \partial_\xi \left( \frac{1}{P_\infty(\xi)} \right).
\]
(15)

Since
\[
\partial_\xi (P_c(\xi) - P_\infty(\xi)) = 2sc^2 \xi_j \left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{s-1} - 2\xi_j
\]
\[
\partial_\xi \left( \frac{1}{P_c(\xi)} \right) = -\frac{2sc^2 \xi_j \left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{s-1}}{P_c(\xi)}
\]
\[
\partial_\xi \left( \frac{1}{P_\infty(\xi)} \right) = -\frac{2\xi_j}{P_c(\xi)P_\infty(\xi)}
\]
\[
\partial_\xi \left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{s-1} = \frac{2sc^2 \xi_j}{\left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right) \left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{s-1}},
\]
we conclude from (15) that
\[
\partial_\xi a(\xi) = \left( 2sc^2 \xi_j \left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{s-1} - 2\xi_j \right) \frac{1}{P_c(\xi)} \frac{1}{P_\infty(\xi)}
\]
\[
- \frac{2sc^2 \xi_j \left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{s-1}}{P_c(\xi)} a(\xi) - \frac{2\xi_j}{P_\infty(\xi)} a(\xi),
\]
(17)

namely each partial derivative of \(a\) is the sum of products of the following factors: \(a\), \(1/P_c\), \(1/P_\infty\), \(\left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{s-1}\) and a polynomial of \(c, \xi_1, \ldots, \xi_N\). If we iterate this procedure, we conclude that \(D^\alpha a(\xi)\) can be expressed as a sum of products of:
\[
a(\xi), P_c(\xi)^{-\ell_1}, P_\infty(\xi)^{-\ell_2}, \left( c^2 |\xi|^2 + \frac{c^2}{s^{-1}} \right)^{-\ell_3},\]
and a polynomial of \(c, \xi_1, \ldots, \xi_N\),
where \(\ell_1, \ell_2\) and \(\ell_3\) are positive integers.
Furthermore, from Lemma 2.4 we deduce that

\[
\left| 2sc^2 \xi_j \left( c^2 |\xi|^2 + \frac{c^2 \xi_j^2}{s^2} \right) \right| P_\xi(\xi)^{s-1} \leq \begin{cases} \frac{1}{|\xi|} & \text{if } |\xi| \leq c^{\frac{2s-1}{2}} \sqrt{\frac{\xi^2}{s^{2s}}} \\ \frac{2s}{\left( \frac{1}{2} - \frac{2s-1}{s^{2s}} \right)^{s-1}} \frac{1}{|\xi|^s} \left( 1 + \frac{1}{2^{s-1}} \right)^s - \frac{1}{\left( \frac{1}{2} - \frac{2s-1}{s^{2s}} \right)^s} |\xi| & \text{if } |\xi| \geq c^{\frac{2s-1}{2}} \sqrt{\frac{\xi^2}{s^{2s}}} \end{cases}
\]

(18)

We now recall Lemma 2.5 and estimate for \( |\xi| \leq c^{\frac{2s-1}{2}} \sqrt{\frac{\xi^2}{s^{2s}}} \),

\[
|a(\xi)| \leq \frac{s(s-1) |\xi|^4}{2s^2 c^{2s} \xi^2} \leq \frac{s(s-1) |\xi|^4}{2s^2 c^{2s} \xi^2} \leq \frac{s(s-1)}{2s^2 c^{2s} \xi^2} \left( |\xi|^2 + 1 \right)^2
\]

\[
\leq \min \left\{ \frac{s(s-1)}{2s^2 c^{2s} \xi^2}, \frac{s(s-1)}{2s^2 c^{2s} \xi^2} \right\} \leq \frac{s(s-1)}{2s^2 c^{2s} \xi^2} \min \left\{ \frac{1}{c^{2s}}, \frac{1}{c^{2s}} \left( |\xi|^2 + 1 \right)^{1-s} \right\}
\]

\[
\leq \frac{s(s-1)}{2s^2 c^{2s} \xi^2} \min \left\{ \frac{1}{c^{2s}}, \frac{1}{c^{2s}} \left( \frac{2s-1}{s^{2s}} \right)^{1-s} \frac{1}{\left( |\xi|^2 + 1 \right)^{1-s}} \right\}.
\]

(19)

In the previous estimate we have used the fact that

\[
\frac{|\xi|^4}{\left( |\xi|^2 + 1 \right)^2} = \left( \frac{\left( |\xi|^2 + 1 \right)^{1-s}}{\left( |\xi|^2 + 1 \right)^{1-s}} \right)^{\frac{2s-1}{s}} \leq \frac{|\xi|^{2s}}{\left( |\xi|^2 + 1 \right)^{1-s}},
\]

which is true because \( \frac{|\xi|^2}{|\xi|^2 + 1} \leq 1 \). On the other hand, if \( |\xi| \geq c^{\frac{2s-1}{2}} \sqrt{\frac{\xi^2}{s^{2s}}} \),

\[
|a(\xi)| \leq \frac{1}{P_\xi(\xi)} + \frac{1}{P_\xi(\xi)} \leq \frac{1}{|\xi|^2 + 1} + \frac{1}{\left( \frac{1}{2^{s-1}} \right)^s - \frac{1}{\left( \frac{1}{2^{s-1}} \right)^s}} c^{2s} |\xi|^{2s} + 1
\]

In particular,

\[
|a(\xi)| \lesssim \frac{1}{c^{2s} |\xi|^2 + 1} + \frac{1}{c^{2s} |\xi|^2 + 1} \lesssim \frac{1}{c^{2s}}.
\]

(20)
We can also write

$$|a(\xi)| \leq \frac{1}{|\xi|^2 + 1} + \left(1 + \frac{1}{2^{1-s} - 1}\right) s - \frac{1}{2^{1-s} - 1} \right) c^{2s} |\xi|^{2s} + 1$$

Then

$$= \frac{(|\xi|^2 + 1)^{1-s}}{|\xi|^2 + 1} \frac{1}{(|\xi|^2 + 1)^{1-s}} + \left(1 + \frac{1}{2^{1-s} - 1}\right) s - \frac{1}{2^{1-s} - 1} \right) c^{2s} |\xi|^{2s} + 1$$

Putting together (20) and (21) we get also in this case

$$\approx \frac{1}{c^{2s} |\xi|^2 + 1)} \frac{1}{(|\xi|^2 + 1)^{1-s}} + \frac{|\xi|^{2(1-s)}}{c^{2s} |\xi|^2 + 1)} \frac{1}{(|\xi|^2 + 1)^{1-s}} \approx \frac{1}{c^{2s} |\xi|^2 + 1)} \frac{1}{(|\xi|^2 + 1)^{1-s}}.$$ (21)

Equations (19) and (22) prove that (13) holds for $|\alpha| = 0$.

If $|\alpha| = 1$ we turn back to (17). The first term can be estimated as

$$\left| 2sc^2 \varepsilon_j \left( c^2 |\xi|^2 + \frac{c^{2s}}{s^{s-1}} \right)^{s-1} - 2\varepsilon_j \right| \frac{1}{P_c(\xi)} \frac{1}{P_\infty(\xi)} \approx \frac{2|\xi|}{(|\xi|^2 + 1) c^2 |\xi|^2 + 1)} \frac{1}{(|\xi|^2 + 1)^{1-s}} \approx \frac{1}{c^{2s} |\xi|^2 + 1)} \frac{1}{(|\xi|^2 + 1)^{1-s}}$$

But

$$\left| 2sc^2 \varepsilon_j \left( c^2 |\xi|^2 + \frac{c^{2s}}{s^{s-1}} \right)^{s-1} - 2\varepsilon_j \right| \frac{1}{P_c(\xi)} \frac{1}{P_\infty(\xi)} \approx \frac{1}{c^{2s} |\xi|^2 + 1)} c^{2(1-s)} |\xi|^2 (|\xi|^2 + 1)^{1-s} \approx \frac{1}{c^{2s} |\xi|^2 + 1)} \frac{1}{(|\xi|^2 + 1)^{1-s}}$$

by the monotonicity of the map

$$t \mapsto \frac{c^{1-s}}{c^s + 1}$$ on the interval $\left[\frac{c^{1-s}}{c^{s-1} + 1}, +\infty\right)$.
for \(1/2 < s < 1\). Hence the first term can be estimated by
\[
\frac{1}{|\xi|} \min \left\{ \frac{1}{c^{2s}}, \frac{1}{c^{2s} (|\xi|^2 + 1)^{1-s}} \right\}.
\]
By using (18) and (22) it is easy to check that the other terms in (17) satisfy the same estimate.
We conclude by induction. Indeed, when we compute a term like
\[
\partial_{\xi_j} D^\alpha a,
\]
we just differentiate sum of products of terms as described above. If we differentiate a polynomial, the total degree is reduced by one. Otherwise, see (16) and (18), some extra terms appear that are estimated by \(1/|\xi|\).
This completes the induction step, and the proof of (13).

The proof of (14) is rather similar, and we only sketch the main steps. First of all, (12), which is valid for every \(\xi \in \mathbb{R}^N\) since \(Df(t) \leq s\) for every \(t \geq 0\), immediately implies that
\[
\left| \frac{P_c(\xi)}{P_\infty(\xi)} \right| \leq 1 \quad \text{for every } \xi \in \mathbb{R}^N.
\]
Then
\[
\partial_{\xi_j} \left( \frac{P_c(\xi)}{P_\infty(\xi)} \right) = \frac{1}{P_\infty(\xi)} \frac{2s c^2 \xi_j}{\left( c^2 |\xi|^2 + \frac{2 \xi_j}{s^{1-s}} \right)^{1-s}} \left( \frac{P_c(\xi)}{P_\infty(\xi)} \right) \frac{2 \xi_j}{P_\infty(\xi)}
\]
is dominated by some multiple of
\[
\frac{|\xi|}{P_\infty(\xi)} + \frac{|\xi|}{P_\infty(\xi)} 1 \leq \frac{1}{|\xi|},
\]
thanks to (23).
Then we check that \(D^\alpha (P_c/P_\infty)\) can be expressed as a sum of products of terms like a polynomial of \(c, \xi_1, \ldots, \xi_N, P_c/P_\infty, 1/P_\infty^{\ell_1}, \text{ and } (c^2 |\xi|^2 + \frac{e^{\xi_j}}{s^{1-s}})^{s-\ell_2}\), where \(\ell_1, \ell_2 \in \mathbb{N}\). Using again (18) and the induction hypothesis, we can prove that
\[
\left| \partial_{\xi_j} D^\alpha \left( \frac{P_c(\xi)}{P_\infty(\xi)} \right) \right| \lesssim \frac{1}{|\xi|^{(|\alpha|+1)}}
\]

We recall a celebrated result, see [16] for the original proof.

**Theorem 2.7 (Hörmander-Mikhlin).** Suppose that \(m: \mathbb{R}^N \rightarrow \mathbb{R}\) satisfies
\[
|D^\alpha m(\xi)| \leq \frac{B_\alpha}{|\xi|^{(|\alpha|+1)}}
\]
for all multi-indices \(\alpha \in (\mathbb{N} \cup \{0\})^N\) such that \(0 \leq |\alpha| \leq N/2 + 1\). Then for any \(1 < q < \infty\), there exists a constant \(C = C(q, N) > 0\) such that
\[
\|m(D)f\|_{L^q} \leq C \left( \sup_{0 \leq |\alpha| \leq N/2 + 1} B_\alpha \right) \|f\|_{L^q}.
\]
This fundamental result allows us to transform the analytic inequalities of Proposition 2.6 into useful operator inequalities.
Theorem 2.8. (1) For $1 < q < \infty$, there exists a constant $C = C(q, N) > 0$ such that
\[
\left\| \left( \frac{1}{P_\infty(D)} - \frac{1}{P_c(D)} \right) f \right\|_{L^q} \leq \frac{C}{c^{1/2}} \|f\|_{L^q}.
\] (24)

(2) For $1 < q < \infty$, there exists a constant $C = C(q, N) > 0$ such that
\[
\left\| \left( \frac{1}{P_\infty(D)} - \frac{1}{P_c(D)} \right) f \right\|_{L^q} \leq \frac{C}{c^{1/2}} \left\| \frac{1}{P_\infty(D)} f \right\|_{L^q}. \] (25)

Proof. Pick any multi-index $\alpha \in (\mathbb{N} \cup \{0\})^N$. From Proposition 2.6 we derive that
\[
\left| D^\alpha \left( \frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)} \right) \right| \lesssim \frac{1}{c^{1/2} |\xi|^{\|\alpha\|}}
\]
and
\[
\left| D^\alpha \left( \left( \frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)} \right) P_\infty(\xi)^{1-s} \right) \right|
\leq \sum_{\alpha_1 + \alpha_2 = \alpha} \left| D^{\alpha_1} \left( \frac{1}{P_\infty(\xi)} - \frac{1}{P_c(\xi)} \right) \right| \left| D^{\alpha_2} \left( P_\infty(\xi)^{1-s} \right) \right|
\lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{1}{c^{1/2} \xi^{\|\alpha_1\| + 2 - \alpha_2} |\xi|^{\|\alpha_2\| + 2 - 2s}} \lesssim \frac{1}{c^{1/2} \xi^{\|\alpha\|}}.
\]

We conclude by Theorem 2.7.

We conclude this Section with a statement that we will use to set up our fixed point argument.

Theorem 2.9. For $1 < q < \infty$ there exists a constant $C = C(q, N) > 0$ such that for $c \gg 1$,
\[
C^{-1} \|f\|_{W^{1,q}} \leq \|P_c(D)f\|_{L^q} \leq C \|f\|_{W^{2,q}}.
\]

Proof. By the triangle inequality and Theorem 2.8 we can write
\[
\left\| \frac{1}{P_c(D)} f \right\|_{L^q} \leq \left\| \frac{1}{P_\infty(\xi)} f \right\|_{L^q} + \left\| \left( \frac{1}{P_\infty(D)} - \frac{1}{P_c(D)} \right) f \right\|_{L^q}
\lesssim \left\| \frac{1}{P_\infty(\xi)} f \right\|_{L^q} + \frac{2}{c^{1/2}} \left\| \frac{1}{P_\infty(D)}^{1-s} f \right\|_{L^q}
\lesssim \|f\|_{W^{-2,q}} + \|f\|_{W^{2s-2,q}}
\lesssim \|f\|_{W^{-2,q}} \lesssim \|f\|_{W^{-1,q}}
\]
by the Sobolev embedding $W^{2-2s,q} \subset W^{2,q}$.

If we replace $f$ by $P_c(D) \sqrt{P_\infty(D)} f = \sqrt{P_\infty(D)} P_c(D) f$ we get
\[
\|f\|_{W^{1,q}} \lesssim \left\| \sqrt{P_\infty(D)} f \right\|_{L^q} \lesssim \left\| \sqrt{P_\infty(D)} P_c(D) f \right\|_{W^{-1,q}} \lesssim \|P_c(D)f\|_{L^q}.
\]
The other inequality follows from (14) and Theorem 2.7.
\[
\|P_c(D)f\|_{L^q} = \left\| \frac{P_c(D)}{P_\infty(D)} P_\infty(D) f \right\|_{L^q} \lesssim \|P_\infty(D)f\|_{L^q} \lesssim \|f\|_{W^{2,q}}.
\]
3 A fixed-point argument

We revert to the problem of finding a fixed point for the operator $\Phi_c$ defined in (3).

**Definition 3.1.** If $X$ and $Y$ are two Banach spaces, the norm of $X \cap Y$ is defined to be $\| \cdot \|_{X \cap Y} = \max \{ \| \cdot \|_X, \| \cdot \|_Y \}$. 

**Remark 3.2.** We will use the fact that the non-relativistic ground state $u_\infty$ is positive, radially symmetric (about the origin without loss of generality), and non-degenerate in the subspace of radially symmetric functions (see [4, 17]), namely

$$\ker \mathcal{L}_\infty \cap H^1_{rad} = \{ 0 \},$$

where

$$\mathcal{L}_\infty = -\Delta + 1 - pu_\infty^{p-1}: H^2 \to L^2.$$

**Lemma 3.3.** For any $2 \leq q < \infty$, the operator

$$\mathcal{A} = I - pu_\infty^{p-1} P_\infty(D)^{-1}$$

is invertible from $L^2_{rad} \cap L^q$ into $L^2_{rad} \cap L^q$.

**Proof.** Since the ground state $u_\infty$ decays exponentially fast at infinity (see [3]), it is easy to check that the operator $pu_\infty^{p-1} P_\infty(D)^{-1}$ is compact as the composition of a compact multiplication operator and a bounded operator. By the Fredholm alternative, it suffices to show that $\mathcal{A}$ is injective. But if $v \in \ker \mathcal{A}$, then $P_\infty(D)^{-1}v \in \ker \mathcal{L}_\infty$. It follows from Remark 3.2 that $P_\infty(D)^{-1}v = 0$, or $v = 0$. 

Now we can write

$$\mathcal{L}_{c, \infty} = \left( I - pu_\infty^{p-1} P_c(D)^{-1} \right) P_c(D)
= \left( \mathcal{A} + pu_\infty^{p-1} \left( P_\infty(D)^{-1} - P_c(D)^{-1} \right) \right) P_c(D)
= \left( I + pu_\infty^{p-1} \left( P_\infty(D)^{-1} - P_c(D)^{-1} \right) \mathcal{A}^{-1} \right) P_c(D),$$

where $\mathcal{A}$ was defined in (26).

**Lemma 3.4.** Suppose that $1 < p < \infty$ if $N = 1, 2$, and $1 < p < (N+2)/(N-2)$ if $N \geq 3$. Then there exists $c_0 > 0$ such that for every $c \geq c_0$ there results

$$\left\| pu_\infty^{p-1} \left( P_\infty(D)^{-1} - P_c(D)^{-1} \right) \mathcal{A}^{-1} \right\|_{L(L^2_{rad} \cap L^q)} \leq \frac{1}{2}.$$ 

**Proof.** We start with the simple estimate

$$\left\| pu_\infty^{p-1} \left( P_\infty(D)^{-1} - P_c(D)^{-1} \right) \mathcal{A}^{-1} \right\|_{L(L^2_{rad} \cap L^q)} \leq p \| u_\infty^{p-1} I \|_{L(L^2_{rad} \cap L^q)} \| P_\infty(D)^{-1} - P_c(D)^{-1} \|_{L(L^2_{rad} \cap L^q)} \| \mathcal{A}^{-1} \|_{L(L^2_{rad} \cap L^q)}. \quad (28)$$

Hölder’s inequality implies that $\| u_\infty^{p-1} I \|_{L(L^2_{rad} \cap L^q)} \| P_\infty(D)^{-1} - P_c(D)^{-1} \|_{L(L^2_{rad} \cap L^q)} \lesssim c^{-2}$. By (24),

$$\| P_\infty(D)^{-1} - P_c(D)^{-1} \|_{L(L^2_{rad} \cap L^q)} \lesssim c^{-2}.$$ 

By Lemma 3.3 $\| \mathcal{A}^{-1} \|_{L(L^2_{rad} \cap L^q)} \lesssim \infty$. We conclude the proof by inserting these estimates into (28). 

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We can now proceed with the proof of the invertibility of $\mathcal{L}_{c,\infty}$.

**Proposition 3.5.** Let $2 \leq q < \infty$. For every $c > 0$ sufficiently large, the operator

$$\mathcal{L}_{c,\infty} : H^1_{\text{rad}} \cap W^{1,q} \to L^2_{\text{rad}} \cap L^q$$

is invertible. Furthermore, its inverse is uniformly bounded in the sense that

$$\sup_{c>0} \left\| (\mathcal{L}_{c,\infty})^{-1} \right\|_{L(H^1_{\text{rad}} \cap W^{1,q}, L^2_{\text{rad}} \cap L^q)} < \infty,$$

where $L(H^1_{\text{rad}} \cap W^{1,q}, L^2_{\text{rad}} \cap L^q)$ is the Banach space of continuous linear operators with its standard norm.

**Proof.** By (27) and the previous Lemmas, the operator $\mathcal{L}_{c,\infty}$ is invertible for $c \geq c_0$, and

$$\mathcal{L}_{c,\infty}^{-1} = P_c(D)^{-1} A^{-1} \left( I + \nu_{\infty}^{-1} \left( P_{\infty}(D)^{-1} - P_c(D)^{-1} \right) A^{-1} \right)^{-1}.$$

Moreover, by Theorem 2.9 and Lemmas 3.3 and 3.4 we have

$$\left\| \mathcal{L}_{c,\infty}^{-1} \right\|_{L(L^2_{\text{rad}} \cap L^q, L^2_{\text{rad}} \cap W^{1,q})} \leq \left\| P_c(D)^{-1} \right\|_{L(L^2_{\text{rad}} \cap L^q, L^2_{\text{rad}} \cap W^{1,q})} \left\| \left( I + \nu_{\infty}^{-1} \left( P_{\infty}(D)^{-1} - P_c(D)^{-1} \right) A^{-1} \right)^{-1} \right\|_{L(L^2_{\text{rad}} \cap L^q)} \leq C,$$

where $C > 0$ is a constant independent of $c \geq c_0$.

To prove that $\Phi_c$ is a contraction (in some suitable space), we must provide bounds for its terms. Recalling that

$$\Phi_c(w) = (\mathcal{L}_{c,\infty})^{-1} (P_{\infty}(D) - P_c(D)) u_\infty + (\mathcal{L}_{c,\infty})^{-1} Q(w),$$

we first estimate for large $c$ the quantity

$$\mathcal{R}_c = (\mathcal{L}_{c,\infty})^{-1} (P_{\infty}(D) - P_c(D)) u_\infty.$$

**Lemma 3.6.** Let $2 \leq q < \infty$. Then we have

$$\left\| \mathcal{R}_c \right\|_{H^1_{\text{rad}} \cap W^{1,q}} = \begin{cases} O(c^{-\frac{2}{p-1}}) & \text{if } 1 < p \leq 2, \\ O(c^{-\frac{2}{p-1}}) & \text{if } p > 2. \end{cases}$$

**Proof.** It is well known that $u_\infty \in H^{|p|}_{\text{rad}} \cap W^{|p|,q}_{\text{rad}}$, where $\lfloor p \rfloor$ is the largest integer less than or equal to $p$.

If $p > 2$, by Proposition 3.5 (21) and Theorem 2.9 we deduce that

$$\left\| \mathcal{R}_c \right\|_{H^1_{\text{rad}} \cap W^{1,q}} \leq \left\| (\mathcal{L}_{c,\infty})^{-1} \right\|_{L(L^2_{\text{rad}} \cap L^q, H^1_{\text{rad}} \cap W^{1,q})} \left\| \frac{P_{\infty}(D) - P_c(D)}{P_\infty(D)P_c(D)} \right\|_{L(L^2_{\text{rad}} \cap L^q)} \left\| (P_\infty(D) - P_c(D)) u_\infty \right\|_{L^2_{\text{rad}} \cap L^q} \lesssim c^{-\frac{2}{p-1}} \left\| (P_\infty(D) - P_c(D)) u_\infty \right\|_{H^{|p|}_{\text{rad}} \cap W^{|p|,q}_{\text{rad}}} \lesssim c^{-\frac{2}{p-1}} \left\| u_\infty \right\|_{H^{|p|+1}_{\text{rad}} \cap W^{|p|+1,q}_{\text{rad}}}.$$}

Analogously, if $1 < p < 2$, again by Proposition 3.5 (25) and Theorem 2.9 we deduce that

$$\left\| \mathcal{R}_c \right\|_{H^1_{\text{rad}} \cap W^{1,q}} \lesssim c^{-\frac{2}{p-1}} \left\| (P_\infty(D) - P_c(D)) u_\infty \right\|_{H^{|p|}_{\text{rad}} \cap W^{|p|,q}_{\text{rad}}} \lesssim c^{-\frac{2}{p-1}} \left\| u_\infty \right\|_{H^{|p|+1}_{\text{rad}} \cap W^{|p|+1,q}_{\text{rad}}}.$$}

since $1/2 < s < 1$. 

\[ \square \]
We turn to the estimate of the second term in the formula of \( \Phi_c \).

**Lemma 3.7.** Fix \( q > N \) and suppose that \( 0 < \delta \leq \| u_\infty \|_{H^1} \). Then for \( c \geq c_0 \) we have

\[
\left\| L^{-1}_{c, \infty} Q(w) \right\|_{H^1_{rad} \cap W^{1,q}} \lesssim \delta^{\min(p,2)}
\]

and

\[
\left\| L^{-1}_{c, \infty} Q(w) - L^{-1}_{c, \infty} Q(\tilde{w}) \right\|_{H^1_{rad} \cap W^{1,q}} \lesssim \delta^{\min(p-1,1)} \left\| w - \tilde{w} \right\|_{H^1_{rad} \cap W^{1,q}}.
\]

**Proof.** Clearly (29) follows from (30) by choosing \( \tilde{w} = 0 \). Hence we focus on the second estimate. By definition of \( Q \), we write

\[
Q(w) - Q(\tilde{w}) = \left( |u_\infty + w|^{p-1}(u_\infty + w) - |u_\infty + \tilde{w}|^{p-1}(u_\infty + \tilde{w}) \right) - p\tilde{w}^{p-1}(w - \tilde{w})
\]

\[
= \int_0^1 \frac{d}{dt} \left( |u_\infty + (1 - t)\tilde{w} + tw|^{p-1}(u_\infty + (1 - t)\tilde{w} + tw) \right) dt - p\tilde{w}^{p-1}(w - \tilde{w})
\]

\[
= p \int_0^1 \left( |u_\infty + (1 - t)\tilde{w} + tw|^{p-1} \right) (w - \tilde{w}) dt.
\]

If \( 1 < p \leq 2 \), we conclude that \( |Q(w) - Q(\tilde{w})| \leq C(\| w \| + |\tilde{w}|)^{p-1}|w - \tilde{w}| \) by the elementary inequality \( |a|^{\ell} - |b|^{\ell} \leq |a| - |b| \leq |a - b|\) for \( 0 < \ell < 1 \). By Proposition 3.5 and the Sobolev embedding \( W^{1,q} \subset L^\infty \),

\[
\left\| L^{-1}_{c, \infty} Q(w) - L^{-1}_{c, \infty} Q(\tilde{w}) \right\|_{H^1_{rad} \cap W^{1,q}} \leq C \left\| Q(w) - Q(\tilde{w}) \right\|_{L^2_{rad} \cap L^q}
\]

\[
\leq C \left( \| w \|_{H^1_{rad} \cap W^{1,q}} + \| \tilde{w} \|_{H^1_{rad} \cap W^{1,q}} \right)^{p-1} \| w - \tilde{w} \|_{H^1_{rad} \cap W^{1,q}}.
\]

If \( p > 2 \), we proceed as before and conclude that

\[
|Q(w) - Q(\tilde{w})| \leq C (|u_\infty + | w | + | \tilde{w} |)^{p-2} (|w| + |\tilde{w}|)|w - \tilde{w}|.
\]

This yields as above

\[
\left\| L^{-1}_{c, \infty} Q(w) - L^{-1}_{c, \infty} Q(\tilde{w}) \right\|_{H^1_{rad} \cap W^{1,q}}
\]

\[
\leq C \left\| Q(w) - Q(\tilde{w}) \right\|_{L^2_{rad} \cap L^q} \leq C \left( \| u_\infty + | w | + | \tilde{w} | \right)^{p-2} (|w| + |\tilde{w}|)|w - \tilde{w}|_{L^2_{rad} \cap L^q}
\]

\[
\leq C \left( \| u_\infty \|_{H^1_{rad} \cap W^{1,q}} + \| w \|_{H^1_{rad} \cap W^{1,q}} + \| \tilde{w} \|_{H^1_{rad} \cap W^{1,q}} \right)^{p-2} \cdot \left( \| w \|_{H^1_{rad} \cap W^{1,q}} + \| \tilde{w} \|_{H^1_{rad} \cap W^{1,q}} \right) \| w - \tilde{w} \|_{H^1_{rad} \cap W^{1,q}}.
\]

The proof is complete. \(\square\)

**Proposition 3.8.** Let \( q > N \). For any \( \delta > 0 \) sufficiently small, there exists \( c_0 > 0 \) such that, if \( c \geq c_0 \), then \( \Phi_c \) has a unique fixed point in the (closed) ball

\[
B_\delta = \left\{ w \in H^1_{rad} \cap W^{1,q} \mid \| w \|_{H^1_{rad} \cap W^{1,q}} \leq \delta \right\}.
\]

**Proof.** By Lemma 3.6 we choose \( c_0 \) so large that \( \| R_c \|_{H^1_{rad} \cap W^{1,q}} \leq \delta/2 \) for \( c \geq c_0 \). By Lemma 3.7, \( w, \tilde{w} \in B_\delta \) implies \( \| \Phi_c(w) \|_{H^1_{rad} \cap W^{1,q}} \leq \delta/2 + C\delta^{\min(p,2)} \leq \delta \) and

\[
\| \Phi_c(w) - \Phi_c(\tilde{w}) \|_{H^1_{rad} \cap W^{1,q}} \leq C\delta^{\min(p-1,1)} \| w - \tilde{w} \|_{H^1_{rad} \cap W^{1,q}} \leq \frac{1}{2} \| w - \tilde{w} \|_{H^1_{rad} \cap W^{1,q}}
\]

as soon as \( C^{\min(p-1,1)} \delta \leq 1/2 \). A application of the Contraction Theorem (see [2]) yields the result. \(\square\)
Lemma 3.9. For $w$ the fixed point $w$ constructed in Proposition 3.8, the function $u = u_\infty + w$ is a solution of the equation $P_c(D)u = |u|^{p-1}u$.

Proof. Indeed, we already know that $w = \mathcal{R}_c + (\mathcal{L}_{c,\infty})^{-1}Q(w)$ in $H^1_{\text{rad}} \cap W^{1,q}$, and thus

$$0 = \mathcal{L}_{c,\infty}w - \mathcal{L}_{c,\infty}\mathcal{R}_c - Q(w)$$

$$= (P_c(D) - pu_\infty^{p-1})w - (P_\infty(D) - P_c(D))u_\infty - (|u|^{p-1}u - u_\infty^p - pu_\infty^{p-1}w)$$

$$= P_c(D)w - pu_\infty^{p-1}w - P_\infty(D)u_\infty + P_c(D)u_\infty - |u|^{p-1}u + u_\infty^p + pu_\infty^{p-1}w = P_c(D)u - |u|^{p-1}u.$$

The proof of Theorem 1.1 follows immediately from the previous Lemma and a rescaling, see (7).

Corollary 3.10. Let $u_c$ be a solution of (3) in $H^1_{\text{rad}} \cap L^\infty$ that converges to $u_\infty$ as $c \to +\infty$. Then for sufficiently large $c \geq 1$, $u_c$ is unique and moreover

$$\|u_c - u_\infty\|_{H^1 \cap W^{1,q}} = \begin{cases} O(c^{-\frac{q+1}{2s}}) & \text{if } 1 < p \leq 2 \\ O(c^{-\frac{q}{2s}}) & \text{if } p > 2. \end{cases}$$

(31)

Proof. First of all, we claim that there exists $\delta > 0$ so small that the solution to (3) is unique in $B_\delta(u_\infty) = \{u \in H^1_{\text{rad}} \cap L^\infty | \|u - u_\infty\|_{H^1 \cap L^\infty} < \delta\}$.

Indeed, by Lemma 3.9 $w = u_c - u_\infty$ is a fixed point of $\Phi_c$. Since we are assuming that $q > N$, the norms $\|\cdot\|_{H^1 \cap L^\infty}$ and $\|\cdot\|_{H^1 \cap W^{1,q}}$ are equivalent. The claim follows from the uniqueness of the fixed point of $\Phi_c$ in a small ball around zero.

Furthermore, by Lemma 3.6 we can take $\delta \sim c^{-\alpha}$ such that $\|\mathcal{R}_c\|_{H^1_{\text{rad}} \cap W^{1,q}} \leq \delta$, where $\alpha$ is either equal to $2s^2/(1-s)$ or $2s/(1-s)$. As before, we can prove that $\Phi_c$ is a contraction in a (closed) ball of radius $\sim c^{-\alpha}$, so that it admits a fixed point there. By uniqueness, taking $c$ larger if needed, this fixed point must be equal to the fixed point $w$ already constructed. Therefore the difference $u_c - u_\infty$ must satisfy (31).

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