ON MINIMAL SINGULAR VALUES OF RANDOM MATRICES WITH CORRELATED ENTRIES

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Abstract. Let $X$ be a random matrix whose pairs of entries $X_{jk}$ and $X_{kj}$ are correlated and vectors $(X_{jk}, X_{kj})$, for $1 \leq j < k \leq n$, are mutually independent. Assume that the diagonal entries are independent from off-diagonal entries as well. We assume that $E X_{jk} = 0$, $E X_{jk}^2 = 1$, for any $j, k = 1, \ldots, n$ and $E X_{jk} X_{kj} = \rho$ for $1 \leq j < k \leq n$. Let $M_n$ be a non-random $n \times n$ matrix with $\|M_n\| \leq Kn^Q$, for some positive constants $K > 0$ and $Q \geq 0$. Let $s_n(X + M_n)$ denote the least singular value of the matrix $X + M_n$. It is shown that there exist positive constants $A$ and $B$ depending on $K, Q, \rho$ only such that

$$P(s_n(X + M_n) \leq n^{-A}) \leq n^{-B}.$$ 

As an application of this result we prove the elliptic law for this class of matrices with non identically distributed correlated entries.

1. Introduction

Let $M_n$ be an $n \times n$ matrix such that $\|M_n\| \leq Kn^Q$ with some positive constant $K > 0$ and a non-negative constant $Q \geq 0$. Consider an $n \times n$ matrix $W = X + M_n$, where $X$ denotes a random matrix with real entries $X_{j,k}$, $1 \leq j, k \leq n$, satisfying the following conditions (C0):

a) random vectors $(X_{jk}, X_{kj})$ are mutually independent for $1 \leq j < k \leq n$;

b) for any $j, k = 1, \ldots, n$

$$E X_{jk} = 0 \text{ and } E X_{jk}^2 = 1;$$

c) for any $1 \leq j < k \leq n$

$$E(X_{jk} X_{kj}) = \rho, |\rho| \leq 1;$$

In the case $\rho = 1$ $X$ is a.s. a symmetric matrix, if $\rho = 0$ and all random variables are Gaussian it is from the Ginibre ensemble.

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We say that the entries $X_{j,k}, 1 \leq j,k \leq n$, of the matrix $X$ satisfy condition (UI) if the squares of $X_{j,k}$’s are uniformly integrable, i.e.

$$\max_{j,k} \mathbb{E}|X_{j,k}|^2 \mathbb{P}\{|X_{j,k}| > M\} \to 0 \quad \text{as} \quad M \to \infty.$$  

Let us denote the least singular value of the perturbed matrix $W$ by $s_n := s_n(W)$. The main result of this note is

**Theorem 1.1.** Assume that $X_{j,k}, 1 \leq j,k \leq n$, satisfy the conditions (C0) and (UI). Let $X = \{X_{j,k}\}$ denote a $n \times n$ random matrix with entries $X_{j,k}$ and let $M_n$ denote a non-random matrix with $\|M_n\| \leq Kn^Q =: K_n$ for some $K > 0$ and $Q \geq 0$. Then there exist constants $C, A, B > 0$ depending on $K, Q$ such that

$$\mathbb{P}(s_n \leq n^{-B}) \leq Cn^{-A},$$  

**Remark.** Under the conditions of Theorem 1.1 we have, for $\gamma \geq \max\{Q, 1\} + \frac{1}{2}$

$$\mathbb{P}(\|X + M_n\| > n^\gamma) \leq \frac{\mathbb{E}\|X\|^2 + \|M_n\|^2}{n^{2\gamma}} \leq Cn^{-A}.$$

A similar result to Theorem 1.1 was obtained by Tao and Vu in [20] and by Götze and Tikhomirov in [11] for non-symmetric random matrices (that is $\rho = 0$ and $X_{j,k}$ and $X_{k,j}$ are independent). Under the condition that $X_{j,k}$, for $j,k = 1, \ldots, n$ are i.i.d. random variables with subgaussian distribution, Rudelson and Vershynin in [18] obtained an optimal bound for $s_n(X)$ (without non-random shift). In the case of symmetric matrices ($\rho = 1$) with i.i.d. entries which have a subgaussian distribution Vershynin proved

$$\mathbb{P}(s_n(X - zI) \leq \varepsilon n^{-1/2} \text{ and } \|X\| \leq K\sqrt{n}) \leq C\varepsilon^{\frac{1}{2}} + 2e^{-n\varepsilon}.$$

The result of Theorem 1.1 in the case of symmetric matrices ($M_n$ symmetric as well) was proved by H. Nguyen in [16] assuming a so-called anti-concentration condition. For the i.i.d. case, H. Nguyen used in his proof techniques which are very different from those used by Vershynin. Recently Nguyen and O’Rourke in [15] have proved the result of Theorem 1.1 for i.i.d. r.v.’s, assuming a finite second order moment. It seems though that their proof is rather involved. In the present note we consider the non-i.i.d. case. Our proof is short and based on the approach by Rudelson and Vershynin (see [18]) which divides the unit sphere into two classes of compressible and incompressible vectors.

Throughout this paper we assume that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By $C$ (with an index or without it) we shall denote generic absolute constants, whereas $C(\cdot, \cdot)$ will denote positive constants depending on various arguments. For any matrix $A$ we shall denote by $\|A\|_2$ the Frobenius norm of the matrix $A$ ($\|A\|_2^2 = \text{Tr}AA^*$) and by $\|A\|$ we shall denote the operator norm of the matrix $A$ ($\|A\| = \sup_{x: \|x\|=1} \|Ax\|$).
2. Proof of the main result

The proof is similar to the proof of Theorem 4.1 in [11]. We use ideas of Rudelson and Vershynin [18] to classify with high probability the vectors $x$ in the $(n-1)$-dimensional unit sphere $S^{(n-1)}$ such that $\|Wx\|_2$ is extremely small into two classes, called compressible and incompressible vectors. Note that

$$s_n = \inf_{x \in S^{(n-1)}} \|Wx\|_2.$$  \hfill (2.1)

First we note that without loss of generality we may assume that the matrix $W$ and all its principal minors are invertible. Otherwise we may consider the matrix $W + \exp\{-n\}rI$ where $r$ is a random variable which is uniformly distributed on the unit interval and independent of the matrix $W$. For such a matrix we have

$$\Pr\{\det(W + \exp\{-n\}rI) = 0\} = \mathbb{E}\Pr\{\prod_{j=1}^{n} (\lambda_j + r \exp\{-n\}) = 0 | W = 0\}. \hfill (2.2)$$

We denote here by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix $W$. Moreover, the last relation holds for any principal minor of the matrix $W$. We also have the inequality

$$s_n(W) \geq s_n(W + \exp\{-n\}rI) - \exp\{-n\}. \hfill (2.3)$$

Note that the matrix $X + \exp\{-n\}rI$ satisfies the conditions (C0).

We start now from the following lemma.

**Lemma 2.1.** Let $x^T = (x_1, \ldots, x_n)$ be a fixed unit vector and $X$ be a matrix as in Theorem [17]. Then there exist some positive absolute constants $c_0$ and $\tau_0$ such that for any $0 < \tau \leq \tau_0$ and any vector $u^T = (u_1, \ldots, u_n)$

$$\mathbb{P}(\|Wx - u\|_2/||x||_2 \leq \tau \sqrt{n}) \leq \exp\{-c_0n\}. \hfill (2.4)$$

**Proof.** Let $n_0 = \lfloor n/2 \rfloor$ and $n_1 = n - n_0$. Represent the matrix $X$ in the form

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \hfill (2.5)$$

and

$$M_n = \begin{bmatrix} M_n^{(11)} & M_n^{(12)} \\ M_n^{(21)} & M_n^{(22)} \end{bmatrix}, \hfill (2.6)$$

where $B, M_n^{(12)}$ are matrices of dimension $n_0 \times n_1$, $C, M_n^{(21)}$ are $n_1 \times n_0$ matrices, $A, M_n^{(11)}$ are $n_0 \times n_0$ matrices, and $D, M_n^{(22)}$ are $n_1 \times n_1$ matrices. Let $x^T = (x_0, x_1)^T \in S^{(n-1)}$ where $x_i \in \mathbb{R}^{n_i}$, $i = 0, 1$. Split the vector $u^T$ as $u^T = (u_0, u_1)^T$. Using these notations we have

$$\|Wx - u\|_2^2 = \|(Ax_0 + Bx_1 + M_n^{(11)}x_0 + M_n^{(12)}x_1 - u_0\|_2^2 \hfill (2.7)$$

$$+ \|(Cx_0 + Dx_1 + M_n^{(21)}x_0 + M_n^{(22)}x_1 - u_1\|_2^2$$
Note that max\{||x_0||_2, ||x_1||_2\} \geq \frac{||x||_2}{\sqrt{2}}. Without loss of generality we may assume that ||x_1||_2 \geq \frac{||x||_2}{\sqrt{2}}. We may write then

$$\|Wx - u\|_2^2 \geq \|Ax_0 + Bx_1 + M_n^{(11)}x_0 + M_n^{(12)}x_1 - u_0\|_2^2.$$ 

Denote by y^{(0)} = Ax_0 + M_n^{(11)}x_0 + M_n^{(12)}x_1 - u_0. Note that Bx_1 and y^{(0)} are independent. Furthermore, all entries of matrix B are independent and E[B]_{j,k} = 0, E[B]_{j,k}^2 = 1. We have

$$\|Bx_1 + y_0\|_2^2/||x_1||_2^2 = \sum_{j=1}^{n_0} (\sum_{k=1}^{n_1} [B]_{jk} \tilde{x}_{1k} + \tilde{y}_j)^2 =: \sum_{j=1}^{n_0} |\zeta_j + \tilde{y}_j|_2^2,$$

where \( \zeta_j = \sum_{k=n_0+1}^n x_{jk} \tilde{x}_{1k} \) and \( \tilde{x}_1 = x_1/||x_1||_2 = (\tilde{x}_{1n_0+1}, \ldots, \tilde{x}_{1n})^T, \tilde{y}^{(0)} = y^{(0)}/||x_1||_2 = (\tilde{y}_1^{(0)}, \ldots, \tilde{y}_n^{(0)})^T \). The remaining part of the proof is similar to the proof of Lemma 4.1 in [11]. By Chebyshev’s inequality

$$P(||Wx - u||_2/||x||_2 \leq \tau \sqrt{n}) \leq P(||Bx_1 + y^{(0)}||_2^2/||x_1||_2^2 \leq 2n\tau^2) \leq \exp\{nt^2\tau^2\} \prod_{j=1}^{n_0} E \exp\{-\frac{t^2}{2}(\zeta_j + \tilde{y}_j)^2\}. \tag{2.8}$$

Using \( e^{-t^2/2} = E \exp\{it\xi\} \), where \( \xi \) is a standard Gaussian random variable, we obtain

$$P(\sum_{j=1}^{n_0} (\zeta_j + \tilde{y}_j)^2 \leq 2n\tau^2) \leq \exp\{nt^2\tau^2\} \prod_{j=1}^{n_0} E_{\xi_j} \exp\{itn\tilde{y}_j\} \times \prod_{k=n_0+1}^n E_{X_{jk}} \exp\{it\xi_j \tilde{x}_{1k} X_{jk}\}, \tag{2.9}$$

where \( \xi_j, j = 1, \ldots, n_0 \) denote i.i.d. standard Gaussian r.v.’s and \( E_Z \) denotes expectation with respect to \( Z \) conditional on all other r.v.’s. Take \( \alpha = Pr\{|\xi| \leq C_1\} \) for some absolute positive constant \( C_1 \) which will be chosen later. Then it follows from (2.9)

$$P(\sum_{j=1}^{n_0} (\zeta_j + \tilde{y}_j)^2 \leq 2n\tau^2) \leq \exp\{nt^2\tau^2\} \times \prod_{j=1}^{n_0} \left( \alpha E_{\xi_j} \prod_{k=n_0+1}^n E_{X_{jk}} \left\{ \exp\{it\xi_j \tilde{x}_{1k} X_{jk}\} \right\} \right) + 1 - \alpha \right). \tag{2.10}$$

Furthermore, note that for any r.v. \( \xi \),

$$|E e^{it\xi}| \leq \exp\{-1 - |E e^{it\xi}|^2/2\}.$$ 

This implies

$$E_{X_{jk}} \left\{ \exp\{it\xi_j \tilde{x}_{1k} X_{jk}\} \right\} \leq \exp\{-1 - |f_{jk}(t\tilde{x}_{1k}\xi_j)^2/2\} \tag{2.11}$$
where $f_{jk}(u) = \mathbb{E} \exp \{iuX_{jk}\}$. Assuming (1.1), choose a constant $M > 0$ such that
\[
\sup_{j,k} \mathbb{E} |X_{jk}|^2 \mathbb{I}\{|X_{jk}| > M\} \leq \frac{1}{2}.
\]
Since $1 - \cos x \geq 11/24 \cdot x^2$, for $|x| \leq 1$, conditioning on the event $|\xi_j| \leq C_1$ we get for $0 < t \leq 1/(MC_1)$
\[
1 - |f_{jk}(t\tilde{x}_{1k}\xi_j)|^2 \geq \frac{11}{24} t^2 \tilde{x}_{1k}^2 \mathbb{E} |X_{jk}^{(q)}|^2 \mathbb{I}\{|X_{jk}| \leq M\} \geq 11/48 t^2 \tilde{x}_{1k}^2.
\]
It follows from (2.11) for $0 < t < 1/(MC_1)$ and for some constant $c > 0$
\[
\left|\mathbb{E}_{X_{jk}} \left\{ \exp \{it\xi_j\tilde{x}_{1k}X_{jk}\} \mid |\xi_j| \leq C_1 \right\} \right| \leq \exp \{-ct^2\tilde{x}_{1k}^2\}. \tag{2.12}
\]
This implies that conditionally on $|\xi_j| \leq C_1$ and for $0 < t < 1/(MC_1)$
\[
(2.12) \quad \prod_{k=1}^n \mathbb{E}_{X_{jk}} \left\{ \exp \{it\xi_j\tilde{x}_{1k}X_{jk}\} \mid |\xi_j| \leq C_1 \right\} \leq \exp \{-ct^2\xi_j^2\}
\]
Let $\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp \{-u^2/2\} \, du$. Then
\[
\mathbb{E}_{\xi_j} \left( \exp \{-ct^2\xi_j^2\} \mid |\xi_j| \leq C_1 \right) = \frac{1}{\sqrt{1+2ct^2}} \frac{\Phi_0(C_1 \sqrt{1+2ct^2})}{\Phi_0(C_1)}.
\]
We may choose $C_1$ large enough such that following inequality holds:
\[
\mathbb{E}_{\xi_j} \left( \exp \{-ct^2\xi_j^2\} \mid |\xi_j| \leq C_1 \right) \leq \exp \{-ct^2/24\},
\]
for all $0 < t \leq 1/(MC_1)$. Note that for every $\alpha, x \in [0, 1]$ and $\beta \in (0, 1)$ the following inequality holds
\[
\alpha x + 1 - \alpha \leq \max \{x^\beta, \left( \frac{\beta}{\alpha} \right)^{\frac{1}{1-\beta}} \}. \tag{2.13}
\]
Combining this inequality with inequalities (2.8), (2.10), (2.12), (2.13), we get
\[
\mathbb{P}(\sum_{j=1}^{n_0} (\zeta_j + y_j^{(0)})^2 \leq 2\tau^2 n) \leq \exp \{n \tau^2 t^2 \} \left( \exp \{-\beta cnt^2/24\} + \left( \frac{\beta}{\alpha} \right)^{\frac{1}{1-\beta}} \right).
\]
Without loss of generality we may take $C_1$ such that $\alpha \geq 4/5$ and choose $\beta = 2/5$. Then we obtain
\[
\mathbb{P}(\sum_{j=1}^{n_0} (\zeta_j + y_j^{(0)})^2 \leq 2\tau^2 n) \leq \exp \{n \tau^2 t^2 \} \left( \exp \{-cnt^2/60\} + \left( \frac{1}{2} \right)^{2n/3} \right).
\]
We conclude from here that there exists a constants $\tau_0 > 0$ and $c_0 > 0$ such that for every $0 < \tau \leq \tau_0$
\[
\mathbb{P}(\sum_{j=1}^{n_0} (\zeta_j + y_j^{(0)})^2 \leq 2\tau^2 n) \leq \exp \{-c_0 n\}.
\]
Thus Lemma 2.1 is proved. \qed
Following Rudelson and Vershynin [18] we shall partition the unit sphere $S^{(n-1)}$ into two sets of so-called compressible and incompressible vectors.

**Definition 2.2.** Let $\delta, r \in (0, 1)$. A vector $x \in \mathbb{R}^n$ is called $\delta$-sparse if $|\text{supp}(x)| \leq \delta n$. A vector $x \in S^{(n-1)}$ is called $(\delta, r)$-compressible if within Euclidean distance $r$ from the set of all $\delta$-sparse vectors. A vector $x \in S^{(n-1)}$ is called $(\delta, r)$-incompressible if it is not $(\delta, r)$-compressible.

For any fixed $k = 1, \ldots, n$ denote by $C_{k,n}$ the set of all sparse vectors $x \in S^{(n-1)}$ with $|\text{supp}(x)| \leq k$. Let $K_n := KnQ$ and $E_K := \{|W| \leq K_n\}$. Without loss of generality we may assume that $Q \geq 1/2$. We prove the following analogue of Proposition 4.6 in [11].

**Lemma 2.3.** Let $X$ be as in Theorem 1.1. Assume there exist an absolute constants $c_0 > 0$, $K \geq 1$, $Q \geq 1/2$, and values $\gamma_n, q_n > 0$ such that for any $x$ such that $x \parallel x \parallel_2 \in C \subset S^{(n-1)}$ and for any $u$ inequality

$$P(\text{||}Wx - u\text{||}_2/\text{||}x\text{||}_2 \leq \gamma_n \sqrt{n}, E_K) \leq \exp\{-c_0 q_n\}$$

holds. Then there exists a constant $\delta_0$ depending on $K, Q, \text{ and } c_0$ only such that, for $k = \lfloor \delta_0 nq_n/\ln n \rfloor$,

$$P\left(\inf_{\text{||}x\text{||}_2 \in C_{k,n} \cap C} \text{||}Wx - u\text{||}_2/\text{||}x\text{||}_2 \leq \gamma_n \sqrt{n}/2, E_K\right) \leq \exp\{-c_0 q_n/8\}.$$  

**Proof.** The proof is similar to the proof of Proposition 4.2 in [21]. Let $\eta > 0$ to be chosen later. There exists an $\eta$-net $N$ in $C_{k,n} \cap C$ of cardinality

$$|N| \leq \left(\frac{3 \ln n}{\delta_0 \eta q_n}\right)^{2k}$$

(see, e.g., [21] Lemma 3.7). Let $E$ denote the event in the left hand side of (2.15) whose probability we would like to bound. Assume that $E$ holds. Then there exist vectors $x_0 = \frac{x}{\text{||}x\text{||}_2} \in C_{k,n} \cap C$ and $u_0 = \frac{u}{\text{||}u\text{||}_2} \in \text{span}(u)$ such that

$$\text{||}Wx_0 - u_0\text{||}_2 \leq \gamma_n \sqrt{n}/2.$$  

By definition of $N$ there exists $y_0 \in N$ such that

$$\text{||}x_0 - y_0\text{||}_2 \leq \eta.$$  

On the other hand,

$$\text{||}Wy_0\text{||}_2 \leq \text{||}W\text{||} \leq K_n.$$  

Furthermore,

$$\text{||}Wy_0 - u_0\text{||}_2 \leq \text{||}W\text{||}\text{||}y_0 - x_0\text{||}_2 + \text{||}Wx_0 - u_0\text{||}_2 \leq K_n \eta + \gamma_n \sqrt{n}/2.$$  

We choose

$$\eta = \frac{\gamma_n \sqrt{n}}{4K_n}.$$  

Then we get

$$\text{||}u_0\text{||}_2 \leq K_n \eta + \gamma_n \sqrt{n}/2 + K_n \leq CK_n.$$
We see that
$$u_0 \in \text{span}(u) \cap CK_n C_2^n =: H.$$  

Here by $C_2^n$ we denote the unit ball in $C^n$. Let $\mathcal{M}$ be some fixed $(\gamma_n \sqrt{n})$-net on the interval $H$ such that

$$|\mathcal{M}| \leq \frac{4CK_n}{\gamma_n \sqrt{n}}.$$  

(2.18)

Let us choose a vector $v_0 \in \mathcal{M}$ such that $\|y_0 - v_0\| \leq \gamma_n \sqrt{n}$. It follows from (2.17) that

$$\|Wy_0 - v_0\|_2 \leq K_n \eta + \gamma_n \sqrt{n}/2 + \gamma_n \sqrt{n}/4 \leq \gamma_n \sqrt{n}.$$  

Summarizing, we have shown that the event $E$ implies the existence of vectors $y_0 \in \mathcal{N}$ and $v_0 \in \mathcal{M}$ such that

$$\|Wy_0 - v_0\|_2 \leq \gamma_n \sqrt{n}.$$  

Applying condition (2.14) and estimates (2.18), (2.16) on the cardinalities of the nets, we obtain

$$P(E) \leq \left( \frac{3 \ln n}{\delta_0 q_n} \right)^{2k} \frac{4CK_n}{\gamma_n \sqrt{n}} \exp\{-c_0 n q_n\}.$$  

Choosing $k = [\delta_0 n q_n / \ln n]$ for some small $\delta_0 > 0$, we get

$$P(E) \leq \exp\{-c_0 n q_n / 8\}.$$  

Thus Lemma 2.3 is proved completely.

Let $C(\delta)$ denote the set of all $\delta$-sparse vectors and $C(\delta, r)$ denote the set of $(\delta, r)$-compressible vectors.

**Lemma 2.4.** Let $X$ be a random matrix as described in Theorem 1.1. Assume that there exist an absolute constant $c_0 > 0$ and values $\gamma_n, q_n > 0$ such that (2.14) holds for any $x$ such that $\|x\| \in C$ and $u$. Then there exist $\delta_1, c_1 > 0$ that depend on $K, Q$ and $c_0$ only, such that

$$P \left( \inf_{x \in C(\delta_1 \hat{q}_n, r_n)} \frac{\|Wx - u\|_2}{\|x\|_2} \leq K_n \gamma_n \sqrt{n}/4, E_K \right) \leq \exp\{-c_1 n q_n\},$$  

(2.19)

where $\hat{q}_n = q_n / \ln n$ and $r_n = \gamma_n \sqrt{n}/(4K_n)$.

**Proof.** Choose now $r_n = \gamma_n \sqrt{n}/4K_n$. Let $V$ be the event on the left hand side of (2.19). Assume that the event $V$ occurs for some point $\frac{x}{\|x\|_2} \in C(\delta_1 \hat{q}_n, r_n)$. Choose a point $x_0 \in C(\delta_1 \hat{q}_n, r_n)$ such that $\|\frac{x}{\|x\|_2} - x_0\|_2 \leq r_n$. Then

$$\|Wx_0 - u\|_2/\|x\|_2 \leq \gamma_n \sqrt{n}/2.$$  

Put $x = \|y\|_2 x_0$. Then, $\frac{x}{\|x\|_2} = x_0 \in C(\delta_1 \hat{q}_n)$ and

$$\|Wx - u\|_2/\|x\|_2 \leq \gamma_n \sqrt{n}/2.$$  

(2.20)
Introduce the notation
\[ (2.24) \]
Furthermore, we have
\[
\text{Note that the vectors } y_i \text{ satisfies } P_i \text{ by } K_{n,0}. \]
By Lemma 2.5.

Thus Lemma 2.4 is proved.

Let \( IC(\delta, r) \) denote the set of \((\delta, r)\)-incompressible vectors in \( S^{(n-1)} \).

**Lemma 2.5.** Let \( \delta_n, r_n \in (0,1) \). Let \( X \) be a matrix as described in Theorem 1.1. Then there exist some positive constants \( c_1 \) and \( c_2 \) and \( \delta^{(1)} > 0 \) depending on \( K \) and \( Q \) such that for any \( 0 < \tau < \gamma_n \)

\[
\mathbb{P} \left( \inf_{\|x\|_2 \leq c \tau \sqrt{n}, x \in IC(\delta, r)} \|Wx - u\|_2 / \|x\|_2 \leq \tau \right) \leq \exp \left\{ -c_1 n \ln(n\delta_n) \right\}
\]

with \( \gamma_n = c_2 \left( \frac{\delta_n}{n} \right)^{1/4} \) and \( \tilde{q}_n = \frac{\ln(n\delta_n)}{\ln n} \).

**Proof.** Introduce representation of matrices \( X \) and \( M_n \) as in the proof of Lemma 2.1 with \( n_0 = \lceil n/2 \rceil \) and \( n_1 = n - n_0 \)

\[
X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad M_n = \begin{bmatrix} M_n^{(11)} & M_n^{(12)} \\ M_n^{(21)} & M_n^{(22)} \end{bmatrix}.
\]

Let \( \hat{x} \|x\|_2 \in IC(\delta, r_n) \). By Lemma 4.3 there exists a set \( \sigma(x) \subset \{1, \ldots, n\} \) of cardinality \( |\sigma(x)| \geq \frac{c_1}{2} n\delta_n \) such that

\[
\|P_{\sigma(x)}x\|_2^2 \geq \frac{r_n}{2} \|x\|_2
\]

and

\[
\frac{r_n}{\sqrt{2} \sqrt{n}} \|x\|_2 \leq |x_k| \leq \frac{1}{\sqrt{n\delta_n} / 2} \|x\|_2, \quad \text{for any } k \in \sigma(x).
\]

By \( P_{\sigma(x)} \) we denote the projection onto \( \mathbb{R}^{\sigma(x)} \) in \( \mathbb{R}^n \). Note that in the representation \( x = (x_0, x_1)^T \) at least one the sets \( \sigma(x_i) \) for the vectors \( x_i, i = 0, 1 \) satisfies \( |\sigma(x_i)| \geq \frac{1}{4} n\delta_n \) and \( \frac{r_n}{\sqrt{2} \sqrt{n}} \leq \frac{|x_k|}{\|x\|_2} \leq \frac{1}{\sqrt{n\delta_n} / 2} \) for \( k \in \sigma(x_i) \). Assume for definiteness that it holds for \( i = 1 \). Then

\[
\frac{r_n\sqrt{\delta_n}}{2\sqrt{2}} \|x\|_2 \leq \|x_1\|_2 \leq \|x\|_2.
\]

Furthermore, we have

\[
\|Wx - u\|_2 / \|x\|_2 \geq \|Ax_0 + M_n^{(11)}x_0 + M_n^{(12)}x_1 + Bx_1 - u_0\|_2 / \|x\|_2.
\]

Introduce the notation \( y = (y_1, \ldots, y_m)^T = Ax_0 + M_n^{(11)}x_0 + M_n^{(12)}x_1 - u_0 \). Note that the vectors \( y \) and \( Bx^{(0)} \) are independent and that all entries of the
matrix \( B \) are mutually independent. We may rewrite inequality (2.24) in the form

\[(2.25) \quad \|Wx - u\|^2_2 / \|x\|^2_2 \geq \|(Bx_1 + y)/\|x\|^2_2 = \sum_{j=1}^{n_0} (\zeta_j + \tilde{y}_j)^2,\]

where \( \zeta_j = \sum_{k=1}^{n_1} X_{j,k+n_0} \tilde{x}_{1k}, \) and \( \tilde{x}_1 = x_1/\|x\|_2 = (\tilde{x}_{11}, \ldots, \tilde{x}_{1n_1})^T, \) \( \tilde{y} = y/\|x\|_2 = (\tilde{y}_1, \ldots, \tilde{y}_{n_0})^T. \) Note that \( \zeta_j \) are mutually independent for \( j = 1, \ldots, n_0. \) We introduce now the maximal concentration function of weighted sums of the rows of the matrix \( B = (X_{jk})_{j=1, \ldots, n_0; k=n_0+1, \ldots, n} \) as,

\[p_x(\eta) = \max_{j=1, \ldots, n_0} \sup_{u \in \mathbb{R}} P \left( \left\| \sum_{k=1}^{n_1} X_{j,k+n_0} \tilde{x}_{1k} - u \right\| \leq \eta \right).\]

From Lemma 4.4 it follows that

\[(2.26) \quad p_x(\eta) \leq cn^{-1/4} \delta_n^{-3/8},\]

for all \( \eta \leq c' \left( \frac{\delta_n}{n} \right)^{1/4} r_n. \) Now we state an analogue of the tensorization Lemma in [11, Lemma 4.5].

**Lemma 2.6.** Let \( \zeta_1, \ldots, \zeta_n \) be independent non-negative random variables. Assume that

\[P(\zeta_j \leq \lambda) \leq b_n \]

for some \( b_n \in (0, 1/4) \) and \( \lambda > 0. \) Then

\[P \left( \sum_{j=1}^{n} \zeta_j^2 \leq \alpha_0 n \lambda^2 \right) \leq \exp \left\{ -\frac{1}{2} n \ln(1/2b_n) \right\},\]

where \( \alpha_0 = \frac{\sqrt{\pi} - 1}{2}. \)

**Proof of Lemma 2.6.** For any \( \tau > 0 \) we have

\[P \left( \sum_{j=1}^{n} \zeta_j^2 \leq nt^2 \right) \leq \exp \left\{ nt \prod_{j=1}^{n} E \exp \left( -\zeta_j^2 \tau / t^2 \right) \right\}.

Furthermore,

\[E \exp \left( -\tau \zeta_j^2 / t^2 \right) = \int_0^\infty P(\exp \left( -\tau \zeta_j^2 / t^2 \right) > s) ds = \int_0^1 P \left( \frac{1}{s} > \exp \left( \tau \zeta_j^2 / t^2 \right) \right) ds \leq \int_0^1 P(\zeta_j \leq \frac{t}{\sqrt{T}} \sqrt{\ln(1/s)}) ds \leq \left( \int_0^\infty \exp \left( -\frac{\lambda^2}{t^2} \right) ds + \int_0^1 \exp \left( -\frac{\lambda^2}{t^2} \right) \right) P(\zeta_j \leq \frac{t}{\sqrt{T}} \sqrt{\ln(1/s)}) ds \leq b_n + \exp \left\{ -\frac{\lambda^2 \tau}{t^2} \right\}.\]
Choose now \( \tau := \frac{1}{2} \ln(1/2b_n) \), \( t^2 := \frac{\ln^2 \tau}{\ln(1/b_n)} \). Then we get
\[
\mathbb{E} \exp\{-\tau \zeta_j^2/t^2\} \leq 2b_n
\]
and
\[
\mathbb{P}\left( \sum_{j=1}^n \zeta_j^2 \leq nt^2 \right) \leq \exp\{-\frac{1}{2} n \ln(1/2b_n)\}.
\]
To conclude the proof of Lemma we note that \( \frac{\ln(1/b_n)}{\ln(1/2b_n)} \geq \frac{\sqrt{2}-1}{2} \). Thus Lemma 2.6 is proved.

We continue now the proof of Lemma 2.5. By the inequality (2.26), we have, for every \( x \in \mathcal{IC}(\delta_n, r_n) \),
\[
\mathbb{P}(\|x\|_2 \leq \delta_n, r_n) \leq cn^{-1/4}\delta_n^{-3/8},
\]
for all \( \eta \leq c' \left( \frac{\delta_n}{n} \right)^{1/4} r_n \). By Lemma 2.6 and the inequality (2.25), we have
\[
\mathbb{P}(\|Wx - y\|_2/\|x\|_2 \leq \lambda \sqrt{n}) \leq \exp\{-c_1 n \ln(n\delta_n)\}.
\]
for all \( \lambda \leq c_2 \left( \frac{\delta_n}{n} \right)^{1/4} r_n \). The last inequality and Lemma 2.3 together conclude the proof of Lemma 2.5.

Now we continue with the proof of Theorem 1.1. Let \( \delta_n^{(1)} = \frac{\delta_0}{n^{1/2}} \) and \( r_n^{(1)} = \frac{\delta_0}{4K_n} \).
Consider the set
\[
\mathcal{C}_0 = \mathcal{C}(\delta_n^{(1)}, r_n^{(1)}).
\]
According to Lemma 2.1, for every \( x \) we have, for any \( 0 < \tau \leq \tau_0 \),
\[
\mathbb{P}(\|Wx - y\|_2/\|x\|_2 \leq \tau \sqrt{n}, \mathcal{E}_K) \leq \exp\{-c_0 n\}.
\]
This inequality and Lemmas 2.3 and 2.4 together imply, for any \( 0 < \tau \leq \tau_0/4 \),
\[
\mathbb{P}(\inf_{x \in \mathcal{C}_0} \|Wx - y\|_2/\|x\|_2 \leq \tau \sqrt{n}, \mathcal{E}_K) \leq \exp\{-c_0' n\}.
\]
By Lemma 2.5, the inequality (2.27), for every \( \frac{x}{\|x\|_2} \in \mathcal{IC}(\delta_n^{(1)}, r_n^{(1)}) \),
\[
\mathbb{P}(\|Wx - y\|_2/\|x\|_2 \leq \tau \sqrt{n}, \mathcal{E}_K) \leq \exp\{-c_1 n \ln(n\delta_n^{(1)})\}
\]
for all \( 0 < \tau < c_2 \left( \frac{\delta_n^{(1)}}{n} \right)^{1/4} r_n^{(1)} \). Furthermore, introduce
\[
\delta_n^{(2)} = \delta_1 \frac{\ln(n\delta_n^{(1)})}{\ln n} \quad \text{and} \quad r_n^{(2)} = c_3 \left( \frac{\delta_n^{(1)}}{n} \right)^{1/4} r_n^{(1)} \frac{1}{\ln n}.
\]
Let \( \mathcal{C}_1 = \mathcal{C}(\delta_n^{(2)}, r_n^{(2)}) \cap \mathcal{IC}(\delta_n^{(1)}, r_n^{(1)}) \). The inequality (2.28) and Lemmas 2.5 and 2.4 together imply, for any \( 0 < \tau \leq c_2' \left( \frac{\delta_n^{(1)}}{n} \right)^{1/4} r_n^{(1)} \),
\[
\mathbb{P}(\inf_{x \in \mathcal{C}_1} \|Wx - y\|_2/\|x\|_2 \leq \tau \sqrt{n}, \mathcal{E}_K) \leq \exp\{-c_1' n \ln(n\delta_n^{(1)})\}.
\]
Note that there exists an absolute constant \( \delta_2 > 0 \) such that \( \delta_n^{(2)} \geq \delta_2 \). This implies
\[
\mathcal{IC}(\delta_n^{(2)}, r_n^{(2)}) \subset \mathcal{IC}(\delta_2, r_n^{(2)}) =: \mathcal{C}_2.
\]
In what follows we shall bound the quantity
\[
P := \mathbb{P}(\inf_{x \in \mathbb{C}} \|Wx\|_2 \leq \tau \sqrt{n}, \mathcal{E}_K).
\]

We reformulate Lemma 4.9 from [11] (Lemma 3.5 in [18]) for our case.

**Lemma 2.7.** Let the matrix \(X\) be as described in Theorem 1.1. Let \(X_1, \ldots, X_n\) denote the columns of \(W\) and let \(H_k\) denote the span of all column vectors except the \(k\)th. Then for every \(\eta > 0\)
\[
\mathbb{P}(\inf_{x \in \mathbb{C}} \|Wx\|_2 \leq \eta (r_n^{(2)} / \sqrt{n})^2, \mathcal{E}_K) \leq \frac{1}{n \delta^2} \sum_{k=1}^{n} \mathbb{P}(\text{dist}(X_k, \mathcal{H}_k) < \eta (r_n^{(2)} / \sqrt{n}), \mathcal{E}_K).
\]

**Proof.** of Lemma 2.7. The proof of this lemma is given in [11]. Note that this proof doesn’t use the independence of the entries of the matrix \(X\). For the completeness we repeat this proof here. Recall that \(X_1, \ldots, X_n\) denote the column vector of the matrix \(W\). Writing \(Wx = \sum_{k=1}^{n} x_k X_k\), we have
\[
\|Wx\|_2 \geq \max_{k=1,\ldots,n} \text{dist}(x_k X_k, \mathcal{H}_k) = \max_{k=1,\ldots,n} |x_k| \text{dist}(X_k, \mathcal{H}_k)
\]

Put
\[
p_k = \mathbb{P}(\text{dist}(X_k, \mathcal{H}_k) < \eta (r_n^{(2)} / \sqrt{n})).
\]

Then
\[
\mathbb{E}(\{k : \text{dist}(X_k, \mathcal{H}_k) < \eta r_n^{(2)} \}) = \sum_{k=1}^{n} p_k.
\]

Denote by \(\mathcal{E}\) the event that the set \(\sigma_1 := \{k : \text{dist}(X_k, \mathcal{H}_k) \geq \eta r_n^{(2)} / \sqrt{n}\}\) contains more than \((1 - \delta^2)n\) elements. Then by Chebyshev inequality
\[
\mathbb{P}(\mathcal{E}^c) \leq \frac{1}{n \delta^2} \sum_{k=1}^{n} p_k
\]

On the other hand, for every incompressible vector \(x\), the set \(\sigma_2(x) = \{k : |x_k| \geq \eta r_n^{(2)} / \sqrt{n}\}\) contains at least \(\delta^2 n\) elements. (Otherwise, since \(\|P_{\sigma_2(x)} x\| \leq r_n^{(2)}\) we have \(\|x - y\| \leq r_n^{(2)}\) for the sparse vector \(y = P_{\sigma_2(x)} x\)).

Assume that the event \(\mathcal{E}\) occurs. Fix any \((\delta^2, r_n^{(2)})\)-incompressible vector \(x\). Then \(|\sigma_1| + |\sigma_2(x)| > (1 - \delta^2)n + \delta^2 n > n\), so the sets \(\sigma_1\) and \(\sigma_2(x)\) have non-empty intersection. Let \(k \in \sigma_1 \cap \sigma_2(x)\). Then by (2.30) and definitions of the sets \(\sigma_1\) and \(\sigma_2(x)\), we have
\[
\|Wx\|_2 \geq |x_k| \text{dist}(X_k, \mathcal{H}_k) \geq \eta (r_n^{(2)} / \sqrt{n})^2
\]

Summarizing, we have shown that
\[
\mathbb{P}(\inf_{x \in \mathbb{C}} \|Wx\|_2 \leq \eta (r_n^{(2)} / \sqrt{n})^2) \leq \mathbb{P}(\mathcal{E}^c) \leq \frac{1}{n \delta^2} \sum_{k=1}^{n} p_k.
\]

This completes the proof of Lemma 2.7. \(\Box\)
To conclude the proof of Theorem 1.1 it remains to bound the quantity
\[ \gamma_k := \mathbb{P}(\text{dist}(X_k, H_k) < \eta r_n^2 / \sqrt{n}, \mathcal{E}_k) \]
We shall use some modification of Vershynin’s approach. We reformulate the statement of Proposition 5.1 in [21] for matrices with correlated entries here.

**Statement 2.8.** Let \( A = (a_{jk}) \) be an arbitrary \( n \times n \) matrix. Let \( A_1 \) denote the first column of \( A \) and \( H_1 \) denote the span of the other columns. Furthermore, let \( B \) denote the \((n - 1) \times (n - 1)\) minor of \( A \) obtained by removing the first column and the first row from \( A \), and let \( u \in \mathbb{R}^{n-1} \) and \( v^T \in \mathbb{R}^{(n-1)} \) denote the first column and the first row of \( A \) respectively with first entry removed. Then
\[ \text{dist}(A_1, H_1) \geq \frac{|(B^{-T}v, u) - a_{11}|}{\sqrt{1 + \|B^{-T}v\|_2^2}} \]

**Proof of Statement 2.8.** The proof of this claim is given in [21]. We repeat this proof here. Represent the matrix \( A \) in the form
\[ (a_{11} \ v^T) \]
Let \( h \) be any unit vector orthogonal to \( A_2, ..., A_n \). It follows that
\[ 0 = \begin{pmatrix} V^T \\ B \end{pmatrix}^T h = h_1 v + B^T g, \]
where \( h = (h_1, g) \), and
\[ g = -h_1 B^{-T} v. \]
From the definition of \( h \)
\[ 1 = \|h\|_2^2 = |h_1|^2 + \|g\|_2^2 = |h_1|^2 + |h_1|^2 \|B^{-T}v\|_2^2 \]
Using this equations we get
\[ \text{dist}(A_1, H) \geq \|A_1, h\| = \frac{|a_{11} - (B^{-T}v, u)|}{\sqrt{1 + \|B^{-T}v\|_2^2}} \]
Thus Proposition 2.8 is proved. \( \square \)

In what follows we set \( A := W \). Next, we prove the following

**Lemma 2.9.** Let the matrix \( W \) denote a matrix as described in Theorem 1.1. Let \( u, v \) and \( B \) be determined by (2.32). Then
\[ \sup_{a \in \mathbb{R}} \mathbb{P} \left\{ \frac{|(B^{-T}v, u) - a|}{\sqrt{1 + \|B^{-T}v\|_2^2}} \leq \varepsilon, \text{ and } \|B\|_2 \leq K_n \right\} \leq C n^{-A} \]
with \( 0 < \varepsilon \leq n^{-B} \) for some constants \( A > 0 \) and \( B > 0 \).

To get this bound we need several statements. First we introduce the matrix and vector
\[ Q = \begin{pmatrix} O_{n-1} & B^{-T} \\ B^{-1} & O_{n-1} \end{pmatrix}, \quad w = \begin{pmatrix} u \\ v \end{pmatrix}, \]
where $O_{n-1}$ is $(n-1) \times (n-1)$ matrix with zero entries. Using the definition of $Q$ we may write

$$(B^{-T}v, u) = \frac{1}{2}(Qw, w).$$

Introduce vectors

$$(2.34) \quad w' = \begin{pmatrix} u' \\ v' \end{pmatrix}, \quad z = \begin{pmatrix} u \\ v' \end{pmatrix},$$

where $u', v'$ are independent copies of $u, v$ respectively. We need the following statement.

**Statement 2.10.**

$$\sup_{v \in \mathbb{R}} P_w (|(Qw, w) - v| \leq 2\varepsilon) \leq P_{w', w} (|(QP_{Jc}(w - w'), P_Jw) - u| \leq 2\varepsilon),$$

where $u$ doesn’t depend on $P_Jw = (P_Ju, P_Jv)^T$.

**Remark.** This Lemma was stated and proved in \cite[Statement 2]{14}. We repeat here the proof.

**Proof.** Let us fix $v$ and denote

$$p := P (|(Qw, w) - v| \leq 2\varepsilon).$$

We can decompose the set $[n]$ into the union $[n] = J \cup J^c$. We can take $u_1 = P_Ju, u_2 = P_{J^c}u, v_1 = P_Jv$ and $v_2 = P_{J^c}v$. In what follows we shall use a simple “decoupling” inequality. Let $X, Y$ be independent r.v.’s and denote by $X'$ an independent copy of $X$, which is also independent from $Y$. Let $\mathcal{E}(X, Y)$ be an event depending on $X$ and $Y$. Then

$$(2.35) \quad P^2\{\mathcal{E}(X, Y)\} \leq P\{\mathcal{E}(X, Y) \cap \mathcal{E}(X', Y)\}.$$

This inequality was stated in \cite[Lemma 8.5, and \cite[Lemma 14]{4}. It originated in \cite[Lemma 3.37]{8]. The proof is very simple. The claim follows from the inequality

$$P^2\{\mathcal{E}(X, Y)\} = (E P\{\mathcal{E}(X, Y)|Y\})^2 \leq E P^2\{\mathcal{E}(X, Y)|Y\} = E P\{\mathcal{E}(X, Y) \cap \mathcal{E}(X', Y)|Y\} = P\{\mathcal{E}(X, Y) \cap \mathcal{E}(X', Y)\}.$$

Applying inequality (2.35), we get

$$(2.36) \quad p^2 \leq P (|(Qw, w) - v| \leq 2\varepsilon, |(Qz, z) - v| \leq 2\varepsilon) \leq P (|(Qw, w) - (Qz, z)| \leq 4\varepsilon).$$

Let us rewrite $B^{-T}$ in the block form

$$B^{-T} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}.$$
We get
\[
(\mathbf{Q}_w, \mathbf{w}) = (\mathbf{E}_v, \mathbf{u}_1) + (\mathbf{F}_v, \mathbf{u}_2) + (\mathbf{G}_v, \mathbf{u}_1) + (\mathbf{H}_v, \mathbf{u}_2)
\]
\[
+ (\mathbf{E}^T \mathbf{u}_1, \mathbf{v}_1) + (\mathbf{F}^T \mathbf{u}_2, \mathbf{v}_2) + (\mathbf{G}^T \mathbf{u}_1, \mathbf{v}_1) + (\mathbf{H}^T \mathbf{u}_2, \mathbf{v}_2)
\]
\[
(\mathbf{Q}_z, \mathbf{z}) = (\mathbf{E}_v, \mathbf{u}_1) + (\mathbf{F}_v', \mathbf{u}_1) + (\mathbf{G}_v, \mathbf{u}_1') + (\mathbf{H}_v', \mathbf{u}_2')
\]
\[
+ (\mathbf{E}^T \mathbf{u}_1, \mathbf{v}_1) + (\mathbf{F}^T \mathbf{u}_2, \mathbf{v}_1) + (\mathbf{G}^T \mathbf{u}_1, \mathbf{v}_1) + (\mathbf{H}^T \mathbf{u}_2, \mathbf{v}_1)
\]
and
\[
(\mathbf{Q}_w, \mathbf{w}) - (\mathbf{Q}_z, \mathbf{z}) = 2(\mathbf{F}(\mathbf{v}_2 - \mathbf{v}_2'), \mathbf{u}_1) + 2(\mathbf{G}_T(\mathbf{u}_2 - \mathbf{u}_2'), \mathbf{v}_1)
\]
\[
+ 2(\mathbf{H}_v, \mathbf{u}_2) - 2(\mathbf{H}_v', \mathbf{u}_2')
\]
\[
(2.37)
\]

The last two terms in (2.37) depend only on \(\mathbf{u}_2, \mathbf{u}_2', \mathbf{v}_2, \mathbf{v}_2'\) and we conclude that
\[
\rho_1^2 \leq \mathbb{P} \left( \left( |\mathbf{Q}_P \mathbf{J}_c (\mathbf{w} - \mathbf{w}')| \right) \leq 2 \varepsilon \right),
\]
where \(\rho = \mathbb{P}(\mathbf{u}_2, \mathbf{v}_2, \mathbf{u}_2', \mathbf{v}_2, \mathbf{H})\).

Statement 2.11. For all \(\mathbf{u} \in \mathbb{R}^n\) there exists a constant \(c > 0\) such that
\[
\mathbb{P} \left( \frac{\mathbf{B}^{-T} \mathbf{u}}{||\mathbf{B}^{-T} \mathbf{u}||_2} \notin \mathcal{C}_2 \text{ and } ||\mathbf{B}|| \leq K_n \right) \leq e^{-cn}.
\]

Proof. Note that
\[
\mathcal{S}^{(n-1)} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2.
\]
Let \(\mathbf{x} = \mathbf{B}^{-T} \mathbf{u}\). It is easy to see that
\[
\left\{ \frac{\mathbf{B}^{-T} \mathbf{u}}{||\mathbf{B}^{-T} \mathbf{u}||_2} \notin \mathcal{C}_2 \right\} \subseteq \left\{ \exists \mathbf{x} : \frac{\mathbf{x}}{||\mathbf{x}||_2} \in \mathcal{C}_0 \cup \mathcal{C}_1 \text{ and } \mathbf{B}^T \mathbf{x} = \mathbf{u} \right\}
\]
This implies, for \(c > 0\),
\[
\mathbb{P} \left( \frac{\mathbf{B}^{-T} \mathbf{u}}{||\mathbf{B}^{-T} \mathbf{u}||_2} \notin \mathcal{C}_2 \text{ and } ||\mathbf{B}|| \leq K_n \right)
\]
\[
\leq \sum_{i=0}^{1} \mathbb{P} \left( \inf_{||\mathbf{x}||_2 \leq \mathcal{C}_i} ||\mathbf{B}^T \mathbf{x} - \mathbf{u}||_2/||\mathbf{x}||_2 \leq \tau \text{ and } ||\mathbf{B}|| \leq K_n \right)
\]
Now choose \(\tau\) small enough to apply Lemmas 2.3 and 2.4. \(\square\)

Remark. It is not very difficult to show that for all \(\mathbf{w} \in \mathbb{R}^{2n}\)
\[
\mathbb{P} \left( \frac{\mathbf{Q}_\mathbf{w}}{||\mathbf{Q}_\mathbf{w}||_2} \notin \mathcal{C}_2 \text{ and } ||\mathbf{B}|| \leq K_n \right) \leq e^{-cn}.
\]
To prove that one should observe that
\[
\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{O}_{n-1} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{O}_{n-1} \end{pmatrix}
\]
and repeat the proof of Lemmas 2.1–2.5 for the matrix \(\mathbf{W}\) replaced by \(\mathbf{Q}^{-1}\).
Statement 2.12. Let \( W \) satisfy condition (C0) and consider a matrix \( B \) and a vector \( v \) determined by the decomposition \( (2.32) \). Assume that \( \|B\| \leq K_n \).

Let \( \mathbf{x} = (X_1, \ldots, X_n) \), where \( X_1, \ldots, X_n \) are independent r.v.'s with \( \mathbb{E} X_j = 0 \) and \( \mathbb{E} X_j^2 = 1 \). We shall assume that the r.v.'s \( X_j \) are independent from the matrix \( B \). Then with probability at least \( 1 - e^{-cn} \) the matrix \( B \) has the following properties:

a) \( \mathbb{P}_v(\sqrt{1 + \|B^{-T}v\|^2_2} \leq \varepsilon^{-1/2}\|B^{-T}\|_2) \geq 1 - n^{-1/4}, \)

b) \( \mathbb{P}_x(\|B^{-T}\mathbf{x}\|_2 \geq \varepsilon\|B^{-1}\|_2) \geq 1 - cn^{-1/4}, \)

where \( \varepsilon < \min\left( \frac{n^{-1/4}}{2K_n}, c'n^{(2)} \right) \).

Proof. Let \( \{e_k\}_{k=1}^n \) be a standard basis in \( \mathbb{R}^n \). For all \( 1 \leq k \leq n \) define vectors by

\[ a_k := \frac{B^{-1}e_k}{\|B^{-1}e_k\|} \]

By Statement 2.11 the vector \( a_k \) is incompressible with probability \( 1 - e^{-cn} \).

Fix a matrix \( B \) with this property.

a) By Chebyshev inequality

\[ \mathbb{P}_v(\sqrt{1 + \|B^{-T}v\|^2_2} \geq \varepsilon^{-1/2}\|B^{-T}\|_2) \leq 2\varepsilon^{-2}K_n^2. \]

We may choose \( \varepsilon < \frac{n^{-1/4}}{2K_n^2} \).

b) Note that

\[ \|B^{-T}\mathbf{x}\|_2^2 = \sum_{k=1}^n \|B^{-1}e_k\|^2(a_k, \mathbf{x})^2. \]

We may conclude from Lemma 4.4 that

\[ \mathbb{P}(\|a_k, \mathbf{x}\| \leq \varepsilon) \leq cn^{-1/4}. \]

for all \( \varepsilon < c'n^{-1/4}n^{(2)} \). Applying now [21, Lemma 8.3] with \( p_k = \frac{\|B^{-1}e_k\|^2}{\|B^{-1}\|_2^2} \), we get

\[ \mathbb{P}_x(\|B^{-T}\mathbf{x}\|_2 \leq \varepsilon\|B^{-1}\|_2) = \mathbb{P}_x(\|B^{-T}\mathbf{x}\|_2^2 \leq \varepsilon^2\|B^{-1}\|_2^2) = \mathbb{P}_x(\sum_{k=1}^n p_k(a_k, \mathbf{x})^2 \leq \varepsilon^2) \leq 2\varepsilon \sum_{k=1}^n p_k \mathbb{P}_x(\|a_k, \mathbf{x}\| \leq \sqrt{2}\varepsilon) \leq cn^{-1/4}. \]

\[ \square \]

Proof of Lemma 2.7. Let \( \xi_1, \ldots, \xi_n \) be i.i.d. Bernoulli random variables with \( \mathbb{E} \xi_i = c_0/2 \), where \( c_0 \) is some constant which will be chosen later. Define the set \( J := \{i : \xi_i = 0\} \) and the event \( \mathcal{E}_0 := \{\|J^c\| \leq c_0n\} \). From a large deviation
inequality we may conclude that $P(\mathcal{E}_0) \geq 1 - 2 \exp(-c_0^2 n/2)$. Introduce the event 
\[ \mathcal{E}_1 := \{ \varepsilon_{\frac{1}{2}} \sqrt{1 + \|B^{-T}v\|^2_2} \leq \|B^{-1}\|_2 \leq \varepsilon_0^{-1} \|QP_{f}(w - w')\|_2 \}, \]
where $\varepsilon_0$ will be chosen later.

From Statement 2.12 we may conclude that
\[ P_B(\mathcal{E}_1 \cup \|B\| \geq K_n) \geq 1 - C'n^{-1/4} - 2e^{-c'n}. \]
Consider the random vector 
\[ w_0 = \frac{1}{\|QP_{f}(w - w')\|_2} \begin{pmatrix} B^{-T}P_{f}(v - v') \\ B^{-1}P_{f}(u - u') \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \]
By the remark after Statement 2.11 it follows that the event $\mathcal{E}_2 := \{ w_0 \in \mathcal{I}(\delta, r_n^{(2)}) \}$ holds with probability
\[ P_B(\mathcal{E}_2 \cup \|B\| \geq K_n|w, w', f) \geq 1 - 2 \exp(-c''n). \]
Combining these probabilities we have
\[ P_B, w', f(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \|B\| \geq K_n) \geq 1 - 2e^{-c_0^2 n/2} - C'n^{-1/4} - 2e^{-c'n} - 2e^{-c''n} := 1 - p_0. \]
We may fix $f$ that satisfies $|f^c| \leq c_0 n$ and
\[ P_B, w', f(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \|B\| \geq K_n) \geq 1 - p_0. \]
By Fubini’s theorem $B$ has the following property with probability at least $1 - \sqrt{p_0}$
\[ P_{w, w', f}(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \|B\| \geq K_n|B) \geq 1 - \sqrt{p_0}. \]
The event $\{\|B\| \geq K_n\}$ depends only on $B$. We may conclude that the random matrix $B$ has the following property with probability at least $1 - \sqrt{p_0}$: either $\|B\| \geq K_n$, or
\[ (2.38) \quad \|B\| \leq K_n \text{ and } P_{w, w', f}(\mathcal{E}_1 \cup \mathcal{E}_2 | B) \geq 1 - \sqrt{p_0}. \]
The event we are interested in is
\[ \Omega_0 := \left( \frac{\|(Qw, w - u)\|_2}{\sqrt{1 + \|B^{-T}v\|^2_2}} \leq 2 \varepsilon \right). \]
We need to estimate the probability
\[ P_B, w(\Omega_0 \cap \|B\| \leq K_n) \leq P_B, w(\Omega_0 \cap (2.38) \text{ holds}) + P_B, w(\|B\| \geq K_n \cap (2.38) \text{ fails}). \]
The last term is bounded by $\sqrt{p_0}$.
\[ P_B, w(\Omega_0 \cap \|B\| \leq K_n) \leq \sup_{B \text{ satisfies (2.38)}} P_w(\Omega_0|B) + \sqrt{p_0}. \]
We conclude that
\[ P_B, w(\Omega_0 \cap \|B\| \leq K_n) \leq \sup_{B \text{ satisfies (2.38)}} P_{w, w'}(\Omega_0, \mathcal{E}_1|B) + 2\sqrt{p_0}. \]
Let us fix \( B \) that satisfies (2.38) and denote \( p_1 := \mathbb{P}_{w, w'}(\Omega_0, \mathcal{E}_1 | B) \). By Statement 2.10 and the first inequality in \( \mathcal{E}_1 \) we have
\[
p_1^2 \leq \mathbb{P}_{w, w'} \left( |(Q \mathbf{P}_{\mathcal{J}}(w - w'), \mathbf{P}_{\mathcal{J}} w) - v| \leq \frac{2\varepsilon_0}{\sqrt{\varepsilon_0}} \|B^{-1}\|_2 \right) \tag{\Omega_1}
\]
and
\[
\mathbb{P}_{w, w'}(\Omega_1) \leq \mathbb{P}_{w, w'}(\Omega_1, \mathcal{E}_1, \mathcal{E}_2) + \sqrt{p_0}.
\]
Furthermore,
\[
p_1^2 \leq \mathbb{P}_{w, w'}(\|(w_0, \mathbf{P}_{\mathcal{J}} w) - v| \leq 2\varepsilon_0^{-3/2}, \mathcal{E}_2) + \sqrt{p_0}.
\]
By definition the random vector \( w_0 \) is determined by the random vector \( \mathbf{P}_{\mathcal{J}}(w - w') \), which is independent of the random vector \( \mathbf{P}_{\mathcal{J}} w \). Fix the vector \( \mathbf{P}_{\mathcal{J}}(w - w') \), obtaining
\[
p_1^2 \leq \sup_{w_0 \in \mathcal{I}(\delta r_n^{(2)})} \mathbb{P}_{\mathcal{J}} w \left( |(w_0, \mathbf{P}_{\mathcal{J}} w) - w| \leq 2\varepsilon_0^{-3/2} \right) + \sqrt{p_0}.
\]
Let us fix a vector \( w_0 \) and a number \( w \). We may rewrite
\[
(2.39) \quad (w_0, \mathbf{P}_{\mathcal{J}} w) = \sum_{i \in \mathcal{J}} (a_i X_i + b_i Y_i),
\]
where \( \|a\|_2^2 + \|b\|_2^2 = 1 \). For an arbitrary set \( I \) we introduce the notation \( S_I := \sum_{i \in I} (a_i X_i + b_i Y_i) \). Denote the concentration function of \( S_I \) by
\[
(2.40) \quad Q(S_I, \lambda) = \sup_{a \in \mathbb{R}} \mathbb{P}(|S_I - a| \leq \lambda).
\]
Our aim is to estimate \( Q(S_{\mathcal{J}}, \tilde{\varepsilon}) \). For this we would like to apply Lemma 4.1 but we can’t do it directly. We first need to get appropriate estimates for the coefficients \( a_i, b_i, i \in \mathcal{J} \).

From Lemma 4.3 we know that at least \( n\delta \) coordinates of vector \( w_0 \in \mathcal{I}(\delta, r_n^{(2)}) \) satisfy
\[
\frac{r_n^{(2)}}{2\sqrt{n}} \leq |w_{bi}| \leq \frac{1}{\sqrt{\delta n}}.
\]
Denote the spread set of the vectors \( a \) and \( b \) by \( \sigma(a) \) and \( \sigma(b) \) respectively. It is easy to see that \( \max(\sigma(a), \sigma(b)) \geq \delta n/2 \). Without loss of generality assume that \( \sigma(a) \geq \delta n/2 \). Let us denote by \( \mathcal{J}_0 := \mathcal{J} \cap \sigma(a) \). We may take \( c_0 = \delta/4 \) and conclude by construction of \( \mathcal{J} \) that \( |\mathcal{J}_0| \geq [\delta n/4] \). By Lemma 4.2 we get
\[
Q(S_{\mathcal{J}}, \tilde{\varepsilon}) \leq Q(S_{\mathcal{J}_0}, \tilde{\varepsilon}).
\]
The only information we know about the vector \( b \) is that
\[
0 \leq |b_i| \leq 1 \text{ for all } i \in \mathcal{J}_0.
\]
But from the inequality \( \|b\|_2^2 \leq 1 \) we may conclude that we can restrict the sum \( S_{\mathcal{J}_0} \) on the set
\[
\tilde{\mathcal{J}}_0 := \left\{ i \in \mathcal{J}_0 : |b_i| \leq \frac{c}{n^{1/4}} \right\}
\]
and $|\tilde{J}_0| \geq cn$. Now we split the set $\tilde{J}_0$ as a union of the disjoint sets $J_l$:

$$J_l = \left\{ i \in \tilde{J}_0 : \frac{2^{l-1}r_n^{(2)}}{2\sqrt{n}} \leq |a_i| \leq \frac{2l r_n^{(2)}}{2\sqrt{n}} \right\}, l = 1, ..., L,$$

$$J_{L+1} = \left\{ i \in \tilde{J}_0 : \frac{2L r_n^{(2)}}{2\sqrt{n}} \leq |a_i| \leq \frac{1}{\sqrt{\delta n}} \right\},$$

where $L = \lfloor c(Q) \ln n \rfloor$. It follows from the Dirichlet principle that there exists $l_0, 1 \leq l_0 \leq L + 1$, such that $|J_{l_0}| \geq cn^{-1} n$. Let’s set $I := J_{l_0}$. Again by Lemma 4.2 we may write $Q(S_{\tilde{J}_0}, \tilde{\varepsilon}) \leq Q(S_I, \tilde{\varepsilon})$. Set

$$\eta_n = \frac{2\sqrt{n}}{2^{l_0-1} r_n^{(2)}}.$$

It is easy to see that $\eta_n \geq c\sqrt{n}$.

Introduce the notations $\tilde{a}_i := \eta_n a_i$ and $\tilde{b}_i := \eta_n b_i$. It follows that

$$1 \leq |\tilde{a}_i| \leq 2, |\tilde{b}_i| \leq \eta_n n^{-1/4}.$$

Let $\tilde{S}_I = \sum_{i \in I} (\tilde{a}_i X_i + \tilde{b}_i Y_i)$, where $I$ is an arbitrary set. We decompose the set $I$ into the sum of two sets $I = I_1 \cup I_2$, where

$$I_1 := \{ i \in I : |\tilde{b}_i| \leq 2 \}$$

$$I_2 := \{ i \in I : 2 < |\tilde{b}_i| \leq \eta_n n^{-1/4} \}.$$

We have $\max(|I_1|, |I_2|) \geq cn^{-1} n/2$. From the properties of concentration functions it follows

$$Q(\tilde{S}_I, \varepsilon \eta_n) \leq \min(Q(\tilde{S}_{I_1}, \varepsilon \eta_n), Q(\tilde{S}_{I_2}, \varepsilon \eta_n)).$$

To finish the proof of the statement we shall consider two cases.

1) Suppose that $|I_1| \geq cn^{-1} n/2$. Denote by $\sigma^2 = \mathbb{E} \tilde{S}_{I_1}^2$. Elementary calculations show that on the set $I_1$ we have $\sigma^2 \geq (1 - \rho^2)|I_1|$. We may apply Lemma 4.1

$$Q(\tilde{S}_{I_1}, \varepsilon \eta_n) \leq \sqrt{\frac{\varepsilon \eta_n}{|I_1|}} \left[ 2(1 - \rho^2) - 8 \max_k \mathbb{E} Z_k^2 \mathbb{1}\{|Z_k| \geq \varepsilon \eta_n / 2 \} \right]^{1/2},$$

where $Z_i = \tilde{a}_i X_i + \tilde{b}_i Y_i$. We can estimate the maximum in the denominator in the following way

$$\max_k \mathbb{E} Z_k^2 \mathbb{1}\{|Z_k| \geq \varepsilon \eta_n / 2 \} \leq c \max_k \mathbb{E} X_k^2 \mathbb{1}\{|X_k| \geq \varepsilon \eta_n / 4 \}.$$

We take

$$\tilde{\varepsilon} = \varepsilon_1 := \frac{M_1 n^{1/4}}{\eta_n \ln n},$$

where $M_1$ is some constant. Note that $\tilde{\varepsilon} \leq c n^{-1/4} \ln^{-1} n$ and $\tilde{\varepsilon} \eta_n = M_1 n^{1/4} \ln^{-1} n$. Choosing $M_1$, the right hand side of (2.41) can be made as small as $1 - \rho^2$, assuming that $X_k$ has uniformly integrable second moment. We conclude that

$$Q(\tilde{S}_{I_1}, \tilde{\varepsilon} \eta_n) \leq n^{-A_1},$$

for some $A_1 > 0$. 
2) Suppose that $|I_2| \geq cn \ln^{-1} n/2$. For the set $I_2$ we repeat the above procedure and find the set $I_{k_0}$ such that

$$2^{k_0} \leq |b| \leq 2^{k_0+1}$$

for all $i \in I_{k_0}$ and $|I_{k_0}| \geq cn \ln^{-2} n$. We may reduce the sum $\tilde{S}_{I_2}$ to this set and $Q(\tilde{S}_{I_2}, \tilde{\eta}_n) \leq Q(\tilde{S}_{I_{k_0}}, \tilde{\eta}_n)$. On the set $I_{k_0}$ the variance $\sigma^2 = \mathbb{E} \tilde{S}_{I_2}^2$ is bounded below by

$$\sigma^2 \geq (1 - \rho^2)|b|^2 \geq (1 - \rho^2)|I_{k_0}|2^{2k_0}.$$

Applying Lemma 4.1 we get

$$Q(\tilde{S}_{I_{k_0}}, \tilde{\eta}_n) \leq \sqrt{\tilde{\eta}_n} \sqrt{|I_{k_0}|(2(1 - \rho^2)2^{2k_0} - 8 \max_k \mathbb{E} Z_k^2 \mathbb{I}\{|Z_k| \geq \tilde{\eta}_n/2\})^{1/2}}.$$

Note that

$$(2.42) \max_k \mathbb{E} Z_k^2 \mathbb{I}\{|Z_k| \geq \tilde{\eta}_n\} \leq c2^{2k_0} \max_k \mathbb{E} X_k^2 \mathbb{I}\{|X_k| \geq \tilde{\eta}_n2^{-k_0-1}/4\}$$

We take

$$\tilde{\epsilon} = \epsilon_2 := \frac{M_2 n^{1/8}2^{k_0}}{\eta_n \ln^2 n},$$

where $M_2$ is some constant. One may check that $\tilde{\epsilon} \leq \epsilon'' n^{-1/8} \ln^{-2} n$ and $\tilde{\epsilon}_n \leq M_2 n^{1/8} \ln^{-2} n$. Choosing $M_2$, the right hand side of (2.42) can be made as small as $(1 - \rho^2)2^{2k_0}$, assuming that $X_k$ has a uniformly integrable second moment. We conclude that

$$Q(\tilde{S}_{I_1}, \tilde{\eta}_n) \leq n^{-A_2},$$

for some $A_2 > 0$.

Now we take $\tilde{\epsilon} = \min(\epsilon_1, \epsilon_2)$ and conclude the statement.

\[ \square \]

3. Application to the elliptic law

In this section we briefly discuss the application of Theorem 1.1 to the elliptic law.

Denote by $\lambda_1, ..., \lambda_n$ the eigenvalues of the matrix $n^{-1/2}X$ and define the empirical spectral measure of eigenvalues by

$$\mu_n(B) = \frac{1}{n} \#\{1 \leq i \leq n : \lambda_i \in B\}, \quad B \in \mathcal{B}(\mathbb{C}),$$

where $\mathcal{B}(\mathbb{C})$ is a Borel $\sigma$-algebra of $\mathbb{C}$.

We say that a sequence of random probability measures $m_n(\cdot)$ converges weakly in probability to the probability measure $m(\cdot)$ if for all continuous and bounded functions $f : \mathbb{C} \to \mathbb{C}$ and all $\epsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}\left(\left|\int_{\mathbb{C}} f(x)m_n(dx) - \int_{\mathbb{C}} f(x)m(dx)\right| > \epsilon\right) = 0.$$
Figure 1. Eigenvalues of the matrix $n^{-1/2}X$ for $n = 3000$ and $\rho = 0.5$. On the left, each entry is an iid Gaussian normal random variable. On the right, each entry is an iid Bernoulli random variable, taking the values $+1$ and $-1$ each with probability $1/2$.

We denote weak convergence by the symbol $\xrightarrow{\text{weak}}$.

A fundamental problem in the theory of random matrices is to determine the limiting distribution of $\mu_n$ as the size of the random matrix tends to infinity. The following theorem gives the solution of this problem for matrices which satisfy the conditions (C0) and (UI).

**Theorem 3.1. (Elliptic Law)** Let the entries $X_{jk}, 1 \leq j, k \leq n$, of the matrix $X$ satisfy the conditions (C0) and (UI). Assume that $|\rho| < 1$. Then $\mu_n \xrightarrow{\text{weak}} \mu$ in probability, and $\mu$ has the density $g$:

$$g(x, y) = \begin{cases} \frac{1}{\pi(1-\rho^2)}, & x, y \in \mathcal{E}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\mathcal{E} := \left\{ u, v \in \mathbb{R} : \frac{u^2}{(1+\rho)^2} + \frac{v^2}{(1-\rho)^2} \leq 1 \right\}.$$  

Theorem 3.1 asserts that under assumptions (C0) and (UI) the empirical spectral measure weakly converges in probability to the uniform distribution on the ellipse. The axis of the ellipse are determined by the correlation $\mathbb{E}X_{jk}X_{kj} = \rho, 1 \leq j < k \leq n$. This result was called by Girko “Elliptic Law”. The limiting distribution doesn’t depend on the distribution of the matrix elements and in this sense the result is universal.

In 1985 Girko proved the elliptic law for rather general ensembles of random matrices under the assumption that the matrix elements have a density, see [6] and [7]. Girko used the method of characteristic functions. Using a so-called $V$-transform he reduced the problem to the problem for Hermitian matrices $(n^{-1/2}X - zI)^*(n^{-1/2}X - zI)$ and established the convergence of the empirical spectral distribution of singular values of $n^{-1/2}X - zI$ to some limit which
determines the elliptic law. Under the assumption of a finite fourth moment the elliptic law was recently proved by Naumov in \[13\]. Recently Nguyen and O’Rourke in \[15\] have proved the result of Theorem 3.1 for i.i.d. r.v.’s, assuming the second moment finite. Applying the result of Theorem 1.1 we may extend the elliptic law to the class of random matrices with non i.i.d. entries which satisfy the conditions \((\text{C0})\) and \((\text{UI})\).

Below we shall provide a short outline of the proof of Theorem 3.1.

3.1. **Gaussian case.** Assume that the elements of a real random matrix \(X\) have Gaussian distribution with zero mean and correlations

\[
\mathbb{E} X_{ij}^2 = 1 \quad \text{and} \quad \mathbb{E} X_{ij} X_{ij} = \rho, \quad 1 \leq i < j \leq n, \quad |\rho| < 1.
\]

In \[19\] it was shown that the ensemble of such matrices can be specified by the probability measure

\[
\mathbb{P}(dA) \sim \exp \left[ -\frac{n}{2(1-\rho^2)} \text{Tr}(AA^T - \rho A^2) \right],
\]

on the space of matrices \(A\). It was proved that \(\mu_n \xrightarrow{\text{weak}} \mu\), where \(\mu\) has a density from Theorem 1.1, see \[19\] for details. We will use this result to prove Theorem 3.1 in the general case.

For the discussion of the elliptic law in the Gaussian case see also \[5\], \[1\], Chapter 18 and \[12\].

3.2. **Proof of the elliptic law.** In the general case we have to consider elements of the matrix \(X\) with arbitrary distributions. To overcome this difficulty we shall use the method of logarithmic potentials. Let us denote by \(s_1 \geq s_2 \geq \ldots \geq s_n\) the singular values of \(n^{-1/2}X - zI\) and introduce the empirical spectral measure \(\nu_n(\cdot, z)\) of the singular values. The convergence in Theorem 3.1 will be proved via the convergence of the logarithmic potential of \(\mu_n\) to the logarithmic potential of \(\mu\). We can rewrite the logarithmic potential of \(\mu_n\) via the logarithmic moments of measure \(\nu_n\) by

\[
U_{\mu_n}(z) = -\int_{\mathbb{C}} \ln |z - w| \nu_n(dw) = -\frac{1}{n} \ln \left| \det \left( \frac{1}{\sqrt{n}} X - zI \right) \right|
\]

\[
= -\frac{1}{2n} \ln \det \left( \frac{1}{\sqrt{n}} X - zI \right)^* \left( \frac{1}{\sqrt{n}} X - zI \right) = -\int_0^\infty \ln x \nu_n(dx).
\]

This allows us to consider the Hermitian matrices \((n^{-1/2}X - zI)^*(n^{-1/2}X - zI)\) instead of \(n^{-1/2}X\). To prove Theorem 1.1 we need the following lemma.

**Lemma 3.2.** Suppose that for a.a. \(z \in \mathbb{C}\) there exists a probability measure \(\nu_z\) on \([0, \infty)\) such that

a) \(\nu_n \xrightarrow{\text{weak}} \nu_z\) as \(n \to \infty\) in probability

b) \(\ln x\) is uniformly integrable in probability with respect to \(\{\nu_n\}_{n \geq 1}\).
Then there exists a probability measure $\mu$ such that
\[ \mu \xrightarrow{\text{weak}} \mu \quad \text{as } n \to \infty \text{ in probability} \]
a) $\mu_n \xrightarrow{\text{weak}} \mu$ as $n \to \infty$ in probability
b) for a.a. $z \in \mathbb{C}$
\[ U_\mu(z) = -\int_0^\infty \ln x \nu_z(dx). \]

Proof. See [3] [Lemma 4.3] for the proof.

Suppose now that $X$ satisfies the conditions a) and b) of Lemma 3.2 and the measure $\nu_z$ is the same for all matrices which satisfy the conditions (C0) and (UI). Then there exist a probability measure $\hat{\mu}$ such that
\[ \mu_n \xrightarrow{\text{weak}} \hat{\mu} \]
and
\[ U_{\hat{\mu}}(z) = -\int_0^\infty \ln x \nu_z(dx). \]

We know that in the Gaussian case $\mu_n$ converges to the elliptic law $\mu$. Due to the assumption that $\nu_z$ is the same for all matrices which satisfy the conditions (C0), (UI) and Lemma 3.2 we have
\[ U_{\mu}(z) = -\int_0^\infty \ln x \nu_z(dx) \]
We get $U_{\hat{\mu}}(z) = U_{\mu}(z)$. From the uniqueness of the logarithmic potential we may conclude the statement of the Theorem.

It remains to check the assumptions we have made in the beginning of the proof.

The following lemma proves the condition a) of Lemma 3.2 and shows that $\nu_z$ is the same for all matrices which satisfy the conditions (C0) and (UI).

We say the entries $X_{j,k}$, $1 \leq j, k \leq n$, of the matrix $X$ satisfy Lindeberg’s condition (L) if
\[ \text{for all } \tau > 0 \quad \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E} X_{ij}^2 \mathbb{I}(|X_{ij}| \geq \tau \sqrt{n}) \to 0 \text{ as } n \to \infty. \]

It easy to see that (UI) $\Rightarrow$ (L)

Let $F_n(x, z)$ be the empirical distribution function of the singular values $s_1 \geq \ldots \geq s_n$ of the matrix $n^{-1/2}X - zI$ which corresponds to the measure $\nu_n(z, \cdot)$.

Lemma 3.3. Let $X_{jk}$, $1 \leq j, k \leq n$, satisfy the conditions (C0)and (L). Then there exists a non-random distribution function $F(x, z)$ such that for all continuous and bounded functions $f(x)$, a.a. $z \in \mathbb{C}$ and all $\varepsilon > 0$
\[ \mathbb{P} \left( \left| \int_{\mathbb{R}} f(x)dF_n(x, z) - \int_{\mathbb{R}} f(x)dF(x, z) \right| > \varepsilon \right) \to 0 \text{ as } n \to \infty, \]

Proof. Using the application of the general method of Bentkus from [2], see Theorem 1.3 in [9], one may reduce the problem to the Gaussian matrices and use the result of Theorem 5.1 from [13]. See also the recent paper of Götze and Tikhomirov [10].
Lemma 3.4. Let $X_{jk}$, $1 \leq j, k \leq n$, satisfy the conditions (C0) and (UI). Then $\ln(\cdot)$ is uniformly integrable in probability with respect to $\{\nu_n\}_{n \geq 1}$.

Proof of Lemma 3.4. It is enough to show that there exist $p, q > 0$ such that

\begin{equation}
\lim_{t \to \infty} \lim_{n \to \infty} \mathbb{P}\left( \int_0^\infty x^p \nu_n(dx) > t \right) = 0
\end{equation}

and

\begin{equation}
\lim_{t \to \infty} \lim_{n \to \infty} \mathbb{P}\left( \int_0^\infty x^{-q} \nu_n(dx) > t \right) = 0.
\end{equation}

From Kolmogorov’s strong law of large numbers it follows that

\[ \int_0^\infty x^2 dF(x, 0) \leq \frac{1}{n^2} \sum_{i,j=1}^n X_{ij}^2 \to 1 \text{ as } n \to \infty. \]

Applying this and the fact that $s_i(n^{-1/2}X - zI) \leq s_i(n^{-1/2}X) + |z|$ we may conclude (3.1) taking $p = 2$.

Using the uniform integrability of the second moment of $X_{jk}$ one may prove the extension of Lemma 4.3 from [13].

Lemma 3.5. If the conditions of Lemma 3.4 hold then there exist $c > 0$ and $0 < \gamma < 1$ such that a.s. for $n \gg 1$ and $n^{1-\gamma} \leq i \leq n - 1$

\[ s_{n-i}(n^{-1/2}X - zI) \geq c_i n. \]

We continue now the proof of Lemma 3.4. Denote the event $\Omega_1 := \Omega_{1,n} = \{\omega \in \Omega : s_{n-i} > c_i n^{1-\gamma} \leq i \leq n - 1\}$. Let us consider the event $\Omega_2 := \Omega_{2,n} = \Omega_1 \cap \{\omega : s_n \geq n^{-B-1/2}\}$, where $B > 0$ will be chosen later. We decompose the probability from (3.2) into two terms

\[ \mathbb{P}\left( \int_0^\infty x^{-q} \nu_n(dx) > t \right) = I_1 + I_2, \]

where

\[ I_1 := \mathbb{P}\left( \int_0^\infty x^{-q} \nu_n(dx) > t, \Omega_2 \right), \]

\[ I_2 := \mathbb{P}\left( \int_0^\infty x^{-q} \nu_n(dx) > t, \Omega_1 \right). \]

We may estimate $I_2$ by

\[ I_2 \leq \mathbb{P}(s_n(X - \sqrt{n}zI) \leq n^{-B}) + \mathbb{P}(\Omega_1^c). \]

From Theorem 1.1 it follows that there exist $A, B > 0$ such that

\begin{equation}
\mathbb{P}(s_n(X - \sqrt{n}zI) \leq n^{-B}) \leq Cn^{-A}.
\end{equation}

By Lemma 3.5

\begin{equation}
\lim_{n \to \infty} \mathbb{P}(\Omega_1^c) = 0.
\end{equation}
From (3.3) and (3.4) we conclude
\[ \lim_{n \to \infty} I_2 = 0. \]
To prove (3.2) it remains to bound \( I_1 \). From Markov’s inequality
\[ I_1 \leq \frac{1}{t} \mathbb{E} \left[ \int_0^\infty x^{-q} \nu_n(dx) I(\Omega_2) \right]. \]
By the definition of \( \Omega_2 \)
\[ \mathbb{E} \left[ \int_0^\infty x^{-q} \nu_n(dx) I(\Omega_2) \right] \leq \frac{1}{n} \sum_{i=1}^{n-\lceil n^{1-\gamma} \rceil} s_i^{-q} + \frac{1}{n} \sum_{i=n-\lceil n^{1-\gamma} \rceil + 1}^n s_i^{-q} \leq 2n^{q(B+1/2)-\gamma} + c^{-q} \int_0^1 s^{-q}ds. \]
If \( 0 < q < \min(1, \gamma/(B + 1/2)) \) then the last integral is finite and we conclude the proof of the Lemma.

4. Appendix

Let \( X_1, X_2, \ldots \) be independent random variables with
\[ \mathbb{E} X_k = 0 \quad \text{and} \quad \mathbb{E} X_k^2 = \sigma_k^2 > 0. \]
We denote \( \sigma^2 = \sum_{k=1}^n \sigma_k^2 \). Introduce the following quantity
\[ D(X, \lambda) = \frac{1}{\lambda^2} \int_{|X|<\lambda} x^2 dF(x) + \int_{|X|\geq\lambda} dF(x), \]
where \( F(x) \) is the distribution function of \( X \). Denote by
\[ Q(X, \lambda) = \sup_{a \in \mathbb{R}} \mathbb{P}(|X - a| \leq \lambda). \]
We shall prove here some simple results about the concentration function of sums of independent random variables.

**Lemma 4.1.** Assume that the condition (4.1) holds and let \( S_n = \sum_{k=1}^n X_k \).
Then
\[ Q(S_n, \lambda) \leq \frac{\sqrt{\lambda}}{(2\sigma^2 - 8 \sum_{i=1}^n \mathbb{E} X_i^2 \mathbb{P}(|X_i| \geq \lambda/2))^{1/2}}. \]

**Proof.** According to Theorem 3 in [17], Chapter II, §2, p. 43, we have
\[ Q(S_n, \lambda) \leq A\lambda \left( \sum_{k=1}^n \lambda_k^2 D(\tilde{X}_k, \lambda_k) \right)^{-\frac{1}{2}}, \]
where \( \tilde{X}_k = X_k - X'_k \), \( X'_k \) is independent copy of \( X_k \), and \( 0 < \lambda_k \leq \lambda \). Note that
\[ \lambda_k^2 D(\tilde{X}_k, \lambda_k) \geq \mathbb{E} \tilde{X}_k^2 \mathbb{I}(|\tilde{X}_k| < \lambda_k) = 2\sigma_k^2 - \mathbb{E} \tilde{X}_k^2 \mathbb{I}(|\tilde{X}_k| \geq \lambda_k). \]
Furthermore,
\[ \mathbb{E} \tilde{X}_k^2 (|\tilde{X}_k| \geq \lambda_k) \leq 2(\mathbb{E} (X_k^2) + (X_k')^2) \mathbb{I} (|\tilde{X}_k| \geq \lambda_k) \leq \\
\leq 2 \mathbb{E} X_k^2 \mathbb{I} (|X_k| \geq \lambda_k/2) + 2 \mathbb{E} X_k^2 \mathbb{I} (|\tilde{X}_k| \geq \lambda_k, |X_k| \leq \lambda_k/2) + \\
\leq 2 \mathbb{E} (X'_k)^2 \mathbb{I} (|X'_k| \geq \lambda_k/2) + 2 \mathbb{E} (X'_k)^2 \mathbb{I} (|\tilde{X}_k| \geq \lambda_k, |X'_k| \leq \lambda_k/2) + \\
\leq 2 \mathbb{E} (X'_k)^2 \mathbb{I} (|X'_k| \geq \lambda_k/2) + 2 \mathbb{E} (X'_k)^2 \mathbb{I} (|X_k| \geq \lambda_k/2, |X'_k| \leq \lambda_k/2) + \\
\leq 8 \mathbb{E} X_k^2 \mathbb{I} (|X_k| \geq \lambda_k/2).
\]
This implies that
\[ \lambda_k^2 D(\tilde{X}_k, \lambda_k) \geq 2 \sigma_k^2 - 8 \mathbb{E} X_k^2 \mathbb{I} (|X_k| \geq \lambda_k/2). \]
We take \( \lambda_k = \lambda \) and conclude the statement of the lemma.

\[ \text{Lemma 4.2. Let } S_J = \sum_{i \in J} \xi_i, \text{ where } J \subset [n], \text{ and } I \subset J \text{ then} \\
Q(S_J, \lambda) \leq Q(S_I, \lambda). \]

\[ \text{Proof. Let us fix an arbitrary } v. \text{ From the independence of } \xi_i \text{ we conclude} \\
\mathbb{P}(|S_J - v| \leq \lambda) \leq \mathbb{E} \mathbb{P}(|S_I + S_{J/I} - v| \leq \lambda | \{\xi_i\}_{i \in I}) \leq Q(S_I, \lambda). \]

\[ \text{Lemma 4.3. Let } \delta, \tau \in (0, 1). \text{ Let } x \in IC(\delta, \tau). \text{ Then there exists a set} \\
\sigma(x) \subset [n] \text{ of cardinality } |\sigma(x)| \geq \frac{1}{2} n \delta^2 \text{ such that} \\
\sum_{k \in \sigma(x)} |x_k|^2 \geq \frac{1}{2} \tau^2 \\
\text{and} \\
\frac{\tau}{\sqrt{2n}} \leq |x_k| \leq \frac{\sqrt{2}}{\sqrt{n} \delta} \text{ for any } k \in \sigma(x), \\
\text{which we shall call } \sigma(x) \text{ the "spread set of } x". \]

\[ \text{Proof. See in [18, Lemma 3.4] and [11, Lemma 4.3].} \]

\[ \text{Lemma 4.4. Assume that the condition (4.1) holds and let } S_n = \sum_{k=1}^n a_k X_k, \\
\text{where } a = (a_1, \ldots, a_n) \in IC(\delta_n, r_n). \text{ We additionally suppose that} \\
\max_{i=1, \ldots, n} \mathbb{E} |X_i|^2 \mathbb{I} (|X_i| > M) \to 0 \text{ as } M \to \infty \]

\[ \text{Then there exist constants } c \text{ and } c' \text{ such that} \\
Q(S_n, \varepsilon) \leq c n^{-1/4} \delta_n^{-3/8} \\
\text{for all } \varepsilon < c' \left( \frac{\delta_n}{n} \right)^{1/4} r_n. \]
Proof. For an arbitrary set $I \in [n]$ denote $S_I := \sum_{i \in I} a_i X_i$, where $a_i$ are the coordinates of $a$. We shall write $S_n$ instead of $S_I$ if $I = [n]$. From Lemma 4.3 there exist a set $\mathcal{I}$ such that $|\mathcal{I}| \geq \frac{1}{2} \delta_n n$ and 

$$
\frac{r_n}{\sqrt{2n}} \leq |a_i| \leq \sqrt{\frac{2}{\delta_n n}} \text{ for any } i \in \mathcal{I}.
$$

From Lemma 4.2

$$
Q(S_n, \varepsilon) \leq Q(S_{\mathcal{I}}, \varepsilon).
$$

Now we may split the set $\mathcal{I}$ into a union of the following disjoint sets $I_l$:

$$
I_l = \left\{ i \in \mathcal{I} : \frac{2^{l-1} r_n}{\sqrt{2n}} \leq |a_i| \leq \frac{2^l r_n}{\sqrt{2n}} \right\}, \quad l = 1, ..., L,
$$

$$
I_{L+1} = \left\{ i \in \mathcal{I} : \frac{2L r_n}{\sqrt{2n}} \leq |a_i| \leq \sqrt{\frac{2}{\delta_n n}} \right\},
$$

where $L = \lceil c(Q) \ln n \rceil$. It follows from Dirichlet’s principle that there exists $l_0, 1 \leq l_0 \leq L + 1$, such that $|I_{l_0}| \geq c \delta_n n \ln^{-1} n$. Let’s set $J := I_{l_0}$. Again by Lemma 4.2 we may write $Q(S_{\mathcal{I}}, \varepsilon) \leq Q(S_J, \varepsilon)$. Denote

$$
\eta_n = \frac{\sqrt{2n}}{2^{l_0-1} r_n}.
$$

One may check that $\eta_n \geq c \sqrt{\delta_n n} \geq c$. Let $\tilde{a}_i = \eta_n a_i$ and $\tilde{S}_J = \sum_{i \in J} \tilde{a}_i X_i$. Then on the set $J$

$$
1 \leq |\tilde{a}_i| \leq 2
$$

and the variance $\sigma^2$ of the sum $\tilde{S}_J$ is bounded below by $|J|$. Applying Lemma 4.4 we get

$$
Q(\tilde{S}_J, \varepsilon \eta_n) \leq \frac{\sqrt{\varepsilon \eta_n}}{\sqrt{|J|(2 - 8 \max_{i \in J} \mathbb{E}|\tilde{a}_i X_i|^2 \mathbb{I}\{|\tilde{a}_i X_i| \geq \varepsilon \eta_n / 2\})^{1/2}}}
$$

Note that

$$
\max_{i \in J} \mathbb{E}|\tilde{a}_i X_i|^2 \mathbb{I}\{|\tilde{a}_i X_i| \geq \varepsilon \eta_n / 2\} \leq 2 \max_{i=1,...,n} \mathbb{E}|X_i|^2 \mathbb{I}\{|X_i| \geq \varepsilon \eta_n / 4\}.
$$

We take

$$
\varepsilon = M \frac{(n \delta_n)^{1/4}}{\eta_n},
$$

where $M$ is some constant. It follows from (4.5) that we may choose $M$ such that

$$
\max_{i=1,...,n} \mathbb{E}|X_i|^2 \mathbb{I}\{|X_{jk}| \geq \varepsilon \eta_n / 4\} \leq 1/16.
$$

This concludes the statement of the Lemma from (4.6) and (4.7).
ON MINIMAL SINGULAR VALUES

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