A spin-wave (SW) approach for hard-core bosons is presented to treat the problem of two dimensional boson localization in a random potential. After a short review of the method to compute 1/S-corrected observables, the case of random on-site energy is discussed. Whereas the mean-field solution does not display a Bose glass (BG) phase, 1/S corrections do capture BG physics. In particular, the localization of SW excitations is discussed through the inverse participation ratio.

I. INTRODUCTION

The problem of disordered superfluids (SF) and superconductors has attracted an increasing interest over the past decades [1–14]. Numerous theoretical techniques, such as mean-field, scaling theory, renormalization group, quantum Monte Carlo simulations, and cavity mean-field have been used to investigate the localization of interacting bosons in presence of disorder. Contrary to fermions where the non-interacting case is a good starting point to understand the physics of localization [15], non-interacting bosons are pathological since all bosons will condense into the lowest single particle state. The opposite limit is achieved when a hard-core constraint is imposed, such that no more than one boson can live on each site. This hard-core condition has been shown to be physically relevant in various situations: lattice model for Helium II [16], preformed cooper pairs in localized superconductors [1], spin-gapped antiferromagnets in an external field [17].

In this paper, we will discuss two dimensional hard-core bosons models using a spin wave approximation, as recently discussed for clean [18] and disordered lattices [19]. It is indeed quite appealing to ask whether boson localization can be captured when the first quantum corrections (namely 1/S corrections using linear spin wave theory) are included above the mean-field solution where Bose glass (BG) physics is absent. The Hamiltonian we will study is the following:

\[ H = -t \sum_{\langle ij \rangle} (a_i^\dagger a_j + a_j^\dagger a_i) - \sum_i \mu_i n_i, \]

(1)

where \( t \) is the hopping between neighboring sites, \( \mu_i \) the (possibly random) chemical potential and \( a_i^\dagger \) \( (a_i) \) denotes the operator creating (destroying) a hard-core boson at site \( i \). An interesting property of hard-core bosons is that they can be exactly mapped onto spins \( \frac{1}{2} \) using the Matsuda-Matsubara mapping [16]: \( n_i = S^z_i + 1/2, \)

\[ a_i^\dagger = S^+_i \quad \text{and} \quad a_i = S^-_i. \]

The equivalent \( S = \frac{1}{2} \) model is simply an XY model in a transverse magnetic field \( \mu_i \):

\[ H = -2t \sum_{\langle ij \rangle} \left( S^x_i S^x_j + S^y_i S^y_j \right) - \sum_i \mu_i \left( S^z_i + \frac{1}{2} \right). \]

(2)

Based on such a mapping, a semi-classical approximation can be developed starting from the large \( S \) limit of this magnetic Hamiltonian. This approach has been developed in a series of papers [18, 20–22]. Below, we review the key steps of this semi-classical treatment, and we discuss the SF - BG transition for bosons in a random potential.

II. SEMI-CLASSICAL TREATMENT OF HARD-CORE BOSONS

A. Spin wave approximation

Having rewritten the hard-core bosonic model (1) as a spin Hamiltonian (2), one first performs a classical treatment, replacing spin operators by 3D vectors: \( \vec{S}_i = S (\sin \theta_i \cos \varphi_i, \sin \theta_i \sin \varphi_i, \cos \theta_i) \). The classical energy then reads

\[ E_0 = -2tS^2 \sum_{\langle ij \rangle} \sin \theta_i \sin \theta_j \cos (\varphi_i - \varphi_j) - \sum_i \mu_i S \cos \theta_i. \]

(3)

In the absence of a twist at the boundary (useful to compute the response [23], see below), the energy is minimized for \( \varphi_i = \text{constant} \), and

\[ \mu_i \sin \theta_i = 2tS \cos \theta_i \sum_j n_{n_i} \sin \theta_j, \]

(4)
for all sites $i$, where $j \text{nn } i$ are the nearest neighbors of $i$. In the clean case, translational invariance simplifies the problem, yielding for all sites

$$\cos \theta = \frac{\mu}{8tS}. \tag{5}$$

For random potentials $\mu_i$, the minimization of the classical energy (4) generally cannot be done analytically, except for the special bimodal case where $\mu_i = \pm W$ with probability $1/2$. Indeed, for such a disorder distribution, the classical angles satisfy

$$\cos \theta_i = \pm \frac{W}{8tS} \quad \text{and} \quad \sin \theta_i = \sqrt{1 - \left(\frac{W}{8tS}\right)^2}. \tag{6}$$

For more general dense distributions of the $\mu_i$'s, for instance square box distribution ($P(\mu) = (2W)^{-1}$ if $|\mu| \leq W$, and $= 0$ otherwise), we have to numerically solve (4) using an iterative process. Once the classical angles are determined, we perform a rotation of the spin operators such that the new quantization axis $z'$ is aligned with the classical vector:

$$S_i^x = (\cos \theta_i) S_i^x + (\sin \theta_i) S_i^y$$
$$S_i^y = S_i^y$$
$$S_i^z = (\cos \theta_i) S_i^z - (\sin \theta_i) S_i^x. \tag{7}$$

The new spin operators can be expressed in terms of Holstein-Primakoff bosons [24]:

$$S_i^x' = S - b_i^\dagger b_i, \quad S_i^y' = \frac{\sqrt{2S}}{2}(b_i + b_i^\dagger) + \ldots, \quad S_i^z' = \frac{\sqrt{2S}}{2i}(b_i - b_i^\dagger) + \ldots \tag{8}$$

Combining the rotations (7) and the Holstein-Primakoff representation (8), when only linear corrections are kept, the original hard-core bosonic Hamiltonian (1) reads:

$$H = E_0 - \sum_{\langle ij \rangle} \left[ t_{ij} \left( b_i^\dagger b_j + b_j^\dagger b_i \right) + \bar{t}_{ij} \left( b_i b_j + b_j b_i^\dagger \right) \right] + \sum_i \epsilon_i n_i + \ldots, \tag{9}$$

where $E_0$ is the classical energy, $t_{ij} = tS(1 + \cos \theta_i \cos \theta_j)$, $\bar{t}_{ij} = tS(\cos \theta_i \cos \theta_j - 1)$, $\epsilon_i = \mu_i \cos \theta_i + 2tS \sin \theta_i \sum_{j \text{nn } i} \cos \theta_j$, and the ellipses denotes higher order terms. Hamiltonian (9) is then straightforwardly diagonalized using a generalized Bogoliubov transformation. New bosonic operators $\alpha_i$ diagonalize the quadratic Hamiltonian, such that (9) simplifies to

$$H = E_0 + 2 \sum_{p=1}^N \Omega_p \left( \frac{1}{2} + \alpha_p^\dagger \alpha_p \right) - \sum_{i=1}^N \epsilon_i / 2. \tag{10}$$

The $1/S$-corrected ground-state energy can be easily evaluated since the new ground-state corresponds to the vacuum of Bogoliubov quasi-particles $\langle \alpha_p^\dagger \alpha_p \rangle = 0 \forall p$, yielding

$$E_{1/S} = E_0 + \sum_p \Omega_p - \sum_i \frac{\epsilon_i}{2}. \tag{11}$$

**B. $1/S$ computation of physical observables**

The question of the correct determination of $1/S$-corrected expectation value of a physical observable $\hat{O}$ has been clarified recently in Ref. [18]. There, we have shown that if it is possible to express $\hat{O}$ as a $n^{th}$ derivative of the Hamiltonian $H$ with respect to an external field $\Gamma \to 0$, then it is very convenient to compute $1/S$ corrections (and higher order corrections as well) using

$$\langle \hat{O} \rangle_{1/S} = \left. \frac{\partial^n E_{1/S}}{\partial \Gamma^n} \right|_{\Gamma=0}. \tag{12}$$
FIG. 1: SF density $\rho_{sf}$ of two-dimensional hard-core bosons plotted against disorder strength $W/t$ for bimodal disorder, i.e. $\mu_i = \pm W$, for mean-field and $1/S$ spin-wave approximations. While at the mean-field level there is no intervening BG between SF at $W/t \leq 4$ and gapped insulator for $W/t > 4$, $1/S$ corrections can stabilize a BG for $3.74 \leq W/t \leq 4$. SW results are thermodynamic limit extrapolations (see Ref. [19]) and disorder average has been performed over several hundreds of independent samples. The inset shows the bimodal distribution of chemical potentials $P(\mu)$.

Let’s illustrate this on a simple example, assuming we want to compute the $T = 0$ SF density of our original hard-core bosons $\rho_{sf}$. The SF density can be obtained by imposing a phase gradient $\Phi_{i+e} - \Phi_i = \varphi$ to the system ($\vec{e}$ being the unit vector along one axis $x$ or $y$ of the lattice). Following Fisher, Barber and Jasnow [23], the density of kinetic energy of a SF of density $\rho_{sf}$ flowing at velocity $v_{sf} = (\hbar/m_s)\varphi$ in one direction is given by

$$E(\varphi) - E(0) = \frac{1}{2}m^*\rho_{sf}(v_{sf})^2$$  

(13)

thus yielding a SF density

$$\rho_{sf} = \frac{m^*}{\hbar^2} \frac{\partial^2 E(\varphi)}{\partial \varphi^2}|_{\varphi=0}$$  

(14)

where the effective mass is given by $2m^*/\hbar^2 = 1/(2t)$. In order to evaluate the SW corrections to the SF density, one needs to compute the SW-corrected energy with a small twist angle at the boundaries. While in the disorder-free situation, the global twist $\Phi$ will be uniformly distributed along the $L$ bonds $\varphi = \Phi/L$, this will not be necessary the case for general non-translationally invariant problems [25, 26]. We therefore introduce different local twist angles directly on the bosonic operators: $b_i^\dagger \rightarrow b_i^\dagger e^{i\Phi_i}$ and $b_i \rightarrow b_i e^{-i\Phi_i}$, leading to new local rotations of spin operators Eq. (7):

$$S_i^x = (\cos \theta_i S_i^u + \sin \theta_i S_i^w) \cos \Phi_i - S_i^y \sin \Phi_i$$
$$S_i^y = (\cos \theta_i S_i^u + \sin \theta_i S_i^w) \sin \Phi_i + S_i^x \cos \Phi_i$$
$$S_i^z = -\sin \theta_i S_i^u + \cos \theta_i S_i^w.$$  

(15)

Therefore, in the presence of a global twist angle $\Phi$, the $1/S$ SW Hamiltonian reads

$$\mathcal{H}(\Phi) = E_0(\Phi) - \sum_{(ij)} [t_{ij}(\varphi_i, \varphi_j) (b_i b_j^\dagger + b_j b_i^\dagger) + \bar{t}_{ij}(\varphi_i, \varphi_j) (b_i^\dagger b_j + b_j^\dagger b_i)] + \sum_i \epsilon_i(\varphi_i, \varphi_j n n i) n_i,$$  

(16)

where $E_0(\Phi)$ is the mean-field energy in the presence of $\Phi$, $t_{ij}(\varphi_i, \varphi_j) = t S \cos(\varphi_i - \varphi_j) (1 + \cos \theta_i \cos \theta_j)$, $\bar{t}_{ij}(\varphi_i, \varphi_j) = t S \cos(\varphi_i - \varphi_j) (\cos \theta_i \cos \theta_j - 1)$, and $\epsilon_i(\varphi_i, \varphi_j n n i) = \mu_i \cos \theta_i + 2t S \sin \theta_i \sum_{j n n} \sin \theta_j \cos(\varphi_i - \varphi_j)$.

The $1/S$-corrected SF density

$$\rho_{sf, (1/S)} = \frac{1}{4t} \frac{\partial^2 E_{(1/S)}(\Phi)}{\partial \varphi^2}\bigg|_{\Phi=0},$$  

(17)

only requires to compute the $1/S$ correction to the energy in the presence of $\Phi$. Mean-field and SW results for $\rho_{sf}$ are displayed in Fig. 1 versus the disorder strength $W/t$ for the case of bimodal disorder $\mu_i = \pm W$ with probability
At the mean-field level, the SF density is simply given by $\frac{1}{4} \sin^2 \theta = 1/4 - (W/8t)^2$ (black curve in Fig. 1) for $W/t \leq 4$, and zero for $W/t > 4$ where the system becomes a gapped insulator. In the SF regime, SW fluctuations lead to interesting corrections (red curve in Fig. 1): for small disorder, superfluidity is enhanced, as compared to mean-field [27], whereas for stronger disorder $W/t > 3$ quantum fluctuations and disorder start to cooperate to destroy superfluidity which is found to vanish at $W_c/t \approx 3.74$, before the mean-field transition point. Therefore a small but finite Bose-glass window, intervening between and gapped insulator, is found at $1/\sqrt{N}$ order, while absent in mean-field.

III. EXCITATION SPECTRUM

An interesting question concerns the spin-wave (bosonic) excitation spectrum in the presence of disorder, a topic only addressed in a few works [28–31]. Here we want to address this question using the inverse participation ratio (IPR) for single particle Bogoliubov excitations, usually defined as $\text{IPR}_p = \sum_j |\psi_p(j)|^4$ for normalized states $|\Psi_p\rangle = \sum_j \psi_p(j)|j\rangle$, with $j = 1, \ldots, N = L^d$ the lattice sites. For delocalized states, $\text{IPR}_p \sim 1/L^d$, it saturates to a finite value for localized states: $\text{IPR}_p \sim (1/\xi_p)^d$, and an anomalous scaling is expected at the localization-delocalization transition: $\text{IPR}_p \sim 1/L^{D_2}$, with $D_2 \leq 2$ the fractal dimension. Starting from the bosonic Bogoliubov transformation which diagonalized the quadratic spin-wave Hamiltonian $b_i^\dagger = \sum_p (u_p \alpha_p + v_p \alpha_p^\dagger)$, we use the following definition for the IPRs [28, 29]

$$\text{IPR}_p = \frac{\sum_j |v_p(j)|^4}{(\sum_j |v_p(j)|^2)^2} \quad (18)$$

for each eigenmode $p$. Then, in order to make a frequency-dependent study, we average over finite slices of frequencies centered around $\Omega$:

$$\text{IPR}(\Omega) = \frac{\sum_p \Theta(\Omega_p, \Omega \pm \delta \Omega) \text{IPR}_p}{\sum_p \Theta(\Omega_p, \Omega \pm \delta \Omega)}, \quad (19)$$

where $\Theta(\Omega_p, \Omega \pm \delta \Omega) = 1$ if $\Omega - \delta \Omega \leq \Omega_p \leq \Omega + \delta \Omega$, and 0 otherwise, with $\delta \Omega/v_0 = 1/20$ in the following. In Fig. III we present two representative results in the SF phase: (a) small disorder $W/t = 1$ where $1/\sqrt{\text{IPR}}$ keeps increasing with $N$ for all frequencies, signaling that all eigenmodes are extended, in agreement with what is expected for Goldstone modes [32]; (b) strong disorder $W/t = 3.1$ where a qualitative difference is clearly visible between low and high energy states. Indeed, for such a disorder, whereas low energy modes are clearly delocalized, again in agreement with Ref. [32], a transition occurs at finite frequency in the spectrum. In order to precisely locate this transition, we exploit

![Representative results for IPRs in the phase. (a) For weak disorder $W/t = 1$, $1/\sqrt{\text{IPR}}$ keeps increasing with $N$ for all frequencies, meaning that all modes are delocalized. (b) For stronger disorder (but still in the SF phase), a qualitative difference is visible in the spectrum between low and high energy states. A quantitative analysis is proposed in Fig. 3 where a localization transition is observed at $\Omega_c \approx 1.5$, as shown here by the dashed vertical line.](image-url)
FIG. 3: Best crossing of $IPR \times N^{D_2/2}$ obtained with $N = 256, \cdots, 4096$ at a mobility edge $\Omega_c \simeq 1.5$ for $W/t = 3.1$ (left) and $\Omega_c \simeq 0.9$ for $W/t = 3.6$ (right), both using the fractal dimension $D_2 = 1.48$. Results obtained for bimodal disordered hard-core bosons on the square lattice.

the fractal scaling at the transition $IPR \sim N^{-D_2/2}$. Plotting in Fig. 3 $IPR \times N^{D_2/2}$ for $N = 256, \cdots, 4096$, we get the best crossing using $D_2 = 1.48$ which works not only for this particular value of the disorder $W/t = 3.1$ but also for $W/t = 3.6$, as shown in the right panel of Fig. 3, and for other disorders [19]. These crossings signal a mobility edge at finite frequency $\Omega_c$, separating delocalized low energy excitations from localized ones at higher energies. As discussed in Refs. [19, 33], this mobility edge is expected to vanish in the BG phase where all excited modes are localized.

IV. CONCLUSION

To conclude, we have shown that $1/S$ spin wave corrections provide very interesting information about the superfluid - Bose glass transition for two-dimensional hard-core bosons in a random potential. For sufficiently strong randomness, quantum fluctuations cooperate with disorder such that the superfluid density vanishes at a critical point, leaving room for a stable gapless localized phase (the Bose glass) before entering in a gapped insulator, a phenomenology absent from a classical (mean-field) treatment. The study of the excitation spectrum above the superfluid ground-state gives non-trivial results, namely the existence of a mobility edge at finite frequency, separating delocalized modes at low energy from localized ones at higher energy.

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This fact is also known for clean hard-core bosons [18, 22] where zero point fluctuations increase the energy difference in the presence of a twist.

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