LOW MODES REGULARITY CRITERION FOR A CHEMOTAXIS-NAVIER-STOKES SYSTEM

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ABSTRACT. In this paper we study the regularity problem of a three dimensional chemotaxis-Navier-Stokes system. A new regularity criterion in terms of only low modes of the oxygen concentration and the fluid velocity is obtained via a wavenumber splitting approach. The result improves certain existing criteria in the literature.

1. Introduction. We consider the following chemotaxis-Navier-Stokes system

\[
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(c) \nabla c), \\
    c_t + u \cdot \nabla c &= \Delta c - nf(c), \\
    u_t + (u \cdot \nabla)u + \nabla P &= \Delta u + n\nabla \Phi, \\
    \nabla \cdot u &= 0,
\end{aligned}
\]

\( (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3. \) \tag{1}

This coupled system arises from modelling aerobic bacteria, e.g. \textit{Bacillus subtilis}, suspended into sessile drops of water. It describes a scenario in which both the bacteria, whose population density is denoted by \( n = n(t,x) \), and oxygen, whose concentration is denoted by \( c = c(t,x) \), are transported by the fluid and at the same time diffuse randomly. In addition, the bacteria, which have chemotactic sensitivity \( \chi(c) \), tend to swim towards their nutrient oxygen and consume it at a per-capita rate \( f(c) \). Meanwhile, since the bacteria are heavier than water, their chemotactic swimming induces buoyant forces which affects the fluid motion. This buoyancy-driven effect is reflected in the third equation in system (1), represented by an extra term \( n\nabla \Phi \) added to the Navier-Stokes equation. In this extra term, \( \Phi \) denotes the gravitational potential, whereas the Navier-Stokes equation is conventionally written with \( u = u(t,x) \) denoting the fluid velocity, and \( P = P(t,x) \) the pressure.

In this paper, we study a simple yet prototypical case in which

\[
\nabla \Phi \equiv \text{const.}, \quad \chi(c) \equiv \text{const.}, \quad f(c) \equiv c. \tag{2}
\]

We note that in this case, solutions to system (1) satisfy the following scaling property:

\[
n_{\lambda}(t,x) = \lambda^2 n(\lambda^2 t, \lambda x), \quad u_{\lambda}(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad c_{\lambda}(t,x) = c(\lambda^2 t, \lambda x),
\]

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solve (1) with initial data
\[ n_{\lambda,0} = \lambda^2 n(\lambda x), \quad u_{\lambda,0} = \lambda u(\lambda x), \quad c_{\lambda,0} = c(\lambda x), \]
if \((n(t, x), u(t, x), c(t, x))\) solves (1) with initial data \((n_0(x), u_0(x), c_0(x))\). It is obvious that in 3D the Sobolev space \(H^{-\frac{1}{2}} \times H^{\frac{1}{2}} \times H^\frac{3}{2}\) is scaling invariant (aka critical) for \((n, c, u)\) under the above natural scaling of the system.

Experiments showed that under the chemotaxis-fluid interaction of system (1), even almost homogeneous initial bacteria distribution can evolve and exhibit quite intricate spatial patterns (see [10, 31, 25]). In [25] Lorz proved the existence of a local-in-time weak solution to the 3D chemotaxis-Navier-Stokes system on bounded domains. In a recent work by Winkler [34], the existence of global weak solutions was proved under more general assumptions via entropy-energy estimates. We refer readers to the works of Winkler [32, 33, 34], Liu and Lorz [24], Duan, Lorz and Markowich [11], Chae, Kang and Lee [3, 4], Jiang, Wu and Zheng [16] as well as He and Zhang [14] for more details about the well-posedness results for the chemotaxis-Navier-Stokes system.

As of now, the global regularity of the 3D Navier-Stokes equations remains an outstanding unresolved problem, which is a fundamental reason why a mathematical theory for system (1) is yet to be completed. A classical result due to Prodi [27], Serrin [30] and Ladyzhenskaya [25] states that if a Leray-Hopf solution \(u\) to the 3D Navier-Stokes equations satisfies
\[
\|u\|_{L^{q}(0,T;L^p)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p \leq \infty,
\]
then \(u\) is in fact smooth on \([0, T]\). It is also well-known that a smooth solution to the Navier-Stokes equations on \([0, T]\) can be extended beyond time \(T\) if
\[
\int_0^T \|\nabla \times u\|_{L^\infty} \, dt < \infty,
\]
which is the Beale-Kato-Majda regularity criterion (see [2]). Among a myriad of refined or generalized criteria for the Navier-Stokes equations, we list the ones from [8, 12, 19, 26]. Particularly relevant to this paper is the regularity criterion due to Cheskidov and Shvydkoy (see [8]). It was discovered that by devising the concept of the critical wavenumber \(2^Q\), the condition in Beale-Kato-Majda regularity criterion can be weakened into
\[
\int_0^T \|(\nabla \times u) \leq Q\|_{B^0_{\infty, \infty}} \, dt < \infty,
\]
with \(B^0_{\infty, \infty}\) being a Besov space and \((\nabla \times u) \leq Q\) the low frequency part of \(\nabla \times u\) below the wavenumber \(2^Q\), both of which can be clearly defined within the framework of Littlewood-Paley theory in the upcoming sections. Intuitively speaking, condition (5) implies that the integrability of a certain lower frequency part of a solution alone can ensure the regularity of the solution. The idea of separating high and low frequency parts by a wavenumber originates in Kolmogorov’s theory of turbulence, which predicts the existence of a critical wavenumber above which the dissipation term is dominant. Recently, this wavenumber splitting mechanism has been applied to various fluid models, e.g., the liquid crystal model with Q-tensor configuration (see [9]), leading to a series of refined regularity criteria.

Concerning the three dimensional chemotaxis-Navier-Stokes system, we are aware of several regularity criteria. In [3], Chae, Kang and Lee obtained local-in-time
classical solutions and Prodi-Serrin type regularity criteria. In particular, suppose that
\[ \|u\|_{L^q(0,T;L^p)} + \|\nabla c\|_{L^2(0,T;L^\infty)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p \leq \infty, \] (6)
then the corresponding classical solution can be extended beyond time \( T \). In [4], Chae, Kang and Lee also obtained regularity criteria in terms of the \( L^p \) norms of \( u \) and \( n \). Jiang, Wu and Zheng, in their recent paper [17], proved that a classical solution to the initial boundary value problem of the Keller-Segel model i.e. the fluid free version of system (1), can be extended beyond time \( T \) if
\[ \|\nabla c\|_{L^2(0,T;L^\infty)} < \infty, \] or \[ \|n\|_{L^q(0,T;L^p)} < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 2, \quad \frac{3}{2} < p \leq \infty. \]

In this paper, we aim to establish a regularity condition only imposed on the low frequency part of the concentration function \( c \) and the velocity field \( u \). For a given solution \( (n,u,c) \) without knowing its regularity, we shall define the wavenumber \( \Lambda_u = 2^{Q_u} \) and \( \Lambda_c = 2^{Q_c} \) for \( u \) and \( c \) as in (11), respectively, according to the structure of the equations. Let \( u_{\leq Q_u} \) and \( c_{\leq Q_c} \) denote the low modes of the velocity and oxygen concentration below wavenumber \( \Lambda_u \) and \( \Lambda_c \), respectively. The main result is stated as follows.

**Theorem 1.1.** Let \( (n(t), u(t), c(t)) \) be a weak solution to (1) on \([0,T]\) on \( \mathbb{R}^3 \). Assume that \( (n(t), u(t), c(t)) \) is regular on \([0,T)\) and
\[ \int_0^T \|\nabla c_{\leq Q_c(t)}(t)\|_{L^\infty}^2 + \|u_{\leq Q_u(t)}(t)\|_{H^s_{\infty,\infty}} dt < \infty, \] (7)
then \( (n(t), u(t), c(t)) \) is regular on \([0,T]\).

**Remark 1.** We note that the quantity in (7) is invariant with respect to the scaling of system (1). It is obvious that the condition on the oxygen concentration \( c \) in (7) is weaker than that of (6). It was also shown that the condition on velocity \( u \) in (7) is weaker than that of (6) (see [7]).

**Remark 2.** The same result as in Theorem 1.1 holds on torus \( \mathbb{T}^3 \) as well. However, on bounded domain with boundary, such result may not be achieved since the analysis to obtain Theorem 1.1 relies heavily on harmonic analysis techniques and Littlewood-Paley theory on bounded domain is more involved.

The upcoming sections are organized as follows – in Section 2, we shall give concise introductions to the mathematical tools used in this paper and various notions of solutions to system (1), then in Section 3 we shall formulate the wavenumbers \( \Lambda_u \) and \( \Lambda_c \) via Littlewood-Paley theory and proceed to prove Theorem 1.1.

2. Preliminaries.

2.1. Notation. The symbol \( A \lesssim B \) denotes an estimate of the form \( A \leq CB \) with some absolute constant \( C \), and \( A \sim B \) denotes an estimate of the form \( C_1 B \leq A \leq C_2 B \) with absolute constants \( C_1, C_2 \). The Sobolev norm \( \|\cdot\|_{L^p} \) is shortened as \( \|\cdot\|_p \) without no confusion. The symbols \( W^{k,p} \) and \( H^s \) represent the standard Sobolev spaces and \( L^2 \)-based Sobolev spaces, respectively.
2.2. Littlewood-Paley decomposition. The main analysis tools are the frequency localization method and a wavenumber splitting approach based on the Littlewood-Paley theory, which we briefly recall here. For a complete description of the theory and applications, the readers are referred to the books [1] and [13].

We construct a family of smooth functions \( \{ \varphi_q \}_{q = -1}^{\infty} \) with annular support that forms a dyadic partition of unity in the frequency space, defined as

\[
\varphi_q(\xi) = \begin{cases} 
\varphi(\lambda_q^{-1} \xi) & \text{for } q \geq 0, \\
\chi(\xi) & \text{for } q = -1,
\end{cases}
\]

where \( \lambda_q = 2^q \), \( \varphi(\xi) = \chi(\xi/2) - \chi(\xi) \) and \( \chi \in C_0^\infty(\mathbb{R}^d) \) is a nonnegative radial function chosen in a way such that

\[
\chi(\xi) = \begin{cases} 
1, & \text{for } |\xi| \leq \frac{3}{4} \\
0, & \text{for } |\xi| \geq 1.
\end{cases}
\]

Introducing \( \tilde{h} := \mathcal{F}^{-1} \chi \) and \( h := \mathcal{F}^{-1} \varphi \), we define the Littlewood-Paley projections for a function \( u \in S' \) as

\[
\begin{aligned}
u_{-1} &= \mathcal{F}^{-1}(\chi(\xi) \mathcal{F} u) = \int \tilde{h}(y) u(x - y) dy, \\
u_q &= \Delta_q u = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1} \xi) \mathcal{F} u) = \lambda_d^q \int h(\lambda_q y) u(x - y) dy, \text{ for } q \geq 0.
\end{aligned}
\]

Then the identity

\[
u = \sum_{q = -1}^{\infty} u_q
\]

holds in the sense of distributions. To simplify the notation, we denote

\[
\tilde{u}_q = u_{q-1} + u_q + u_{q+1}, \quad u_{\leq Q} = \sum_{q = -1}^{Q} u_q, \quad u_{(P,Q]} = \sum_{q = P}^{Q} u_q.
\]

We note that

\[
\|u\|_{H^s} \sim \left( \sum_{q = -1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{1/2},
\]

for each \( u \in H^s \) and \( s \in \mathbb{R} \). Using the Littlewood-Paley projections, we can define the Besov space \( B_{p,\infty}^s \) as follows.

**Definition 2.1.** Let \( s \in \mathbb{R}, \) and \( 1 \leq p \leq \infty \). The Besov space \( B_{p,\infty}^s \) is the space of tempered distributions \( u \) whose Besov norm \( \|u\|_{B_{p,\infty}^s} < \infty \), where

\[
\|u\|_{B_{p,\infty}^s} := \sup_{q \geq -1} \lambda_q^s \|u_q\|_p.
\]

Moreover, we recall Bernstein’s inequality, whose proof can be found in [1].

**Lemma 2.2.** Let \( d \) be the space dimension and \( 1 \leq s \leq r \leq \infty \). Then for all tempered distributions \( u \),

\[
\|u_q\|_r \lesssim \lambda_q^{(d-1)/r} \|u_q\|_s.
\]

Throughout the paper, we will also utilize Bony’s paraproduct decomposition

\[
\Delta_q(u \cdot v) = \sum_{|q-p| \leq 2} \Delta_q(u_{\leq p-2} \cdot v_p) + \sum_{|q-p| \leq 2} \Delta_q(u_p \cdot v_{\leq p-2}) + \sum_{p \geq q-2} \Delta_q(u_p \cdot \tilde{v}_p),
\]
Lemma 2.3. Let $\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1}$, we have the estimate
$$\| [\Delta_q, u_{\leq p-2} \cdot \nabla] v_q \|_{r_1} \lesssim \| v_q \|_{r_3} \sum_{p' \leq p-2} \lambda_{p'} \| u_{p'} \|_{r_2}.$$  

2.3. Weak solution and regular solution to system (1). From [34], we know that on bounded, smooth and convex domain $\Omega$ in three dimension, system (1) has a global weak solution $(n, c, u)$ which satisfies the equations in (1) in the distributional sense, provided that the initial data $(n_0, c_0, u_0)$ satisfy $n_0 > 0$, $c_0 > 0$, and
$$n_0 \in L^1 \cap L \log L, \quad c_0 \in L^\infty, \quad \sqrt{n_0} \in H^1, \quad u_0 \in L^2, \quad \nabla \cdot u_0 = 0.$$ (9)

Global existence of weak solutions on the full space $\mathbb{R}^d$ with $d = 2, 3$ was established with initial data satisfying (9) in [14]. Adapting the same argument therein on a periodic domain shows that initial data satisfying (9) also generate at least one global weak solution $(n, c, u)$.

We highlight the following properties of the weak solution $(n, c, u)$ in particular
$$n \in L^\infty(0, \infty; L^1(\Omega)), \quad c \in L^\infty(0, \infty; L^\infty(\Omega)), \quad u \in L^{\infty}_{\text{loc}}(0, \infty; L^2(\Omega)) \cap L^2_{\text{loc}}(0, \infty; H^1_0(\Omega)).$$

A regular solution of (1) is understood in the way that the solution has enough regularity to satisfy the equations of the system in a point-wise manner. Typically, a solution in a space with higher regularity than its critical space can be shown regular via bootstrap arguments. The local-in-time existence of regular solutions to system (1) was shown in [3].

2.4. Parabolic regularity theory. We consider the heat equation on $\mathbb{R}^d, d \geq 2$
$$u_t - \Delta u = f$$ (10)
with initial data $u_0$. Classical parabolic regularity theory (see [20] and [28]) asserts that the solution $u$ turns out to be smoother than the source term $f$. In what follows, we shall prove a parabolic smoothing lemma via Littlewood-Paley theory.

Lemma 2.4. Let $u$ be a solution to (10) with $u_0 \in H^{\alpha+1}$ and $f \in L^2(0, T; H^\alpha)$ for $\alpha \in \mathbb{R}$. Then we have $u \in L^2(0, T; H^{\alpha+2}) \cap H^1(0, T; H^\alpha).$

Proof. Projecting equation (10) by $\Delta_q$ and taking inner product of the resulted equation with $\lambda_q^{2\alpha+4} u_q$ leads to
$$\frac{1}{2} \frac{d}{dt} \lambda_q^{2\alpha+4} \| u_q \|_2^2 + \lambda_q^{2\alpha+4} \| \nabla u_q \|_2^2 = \lambda_q^{2\alpha+4} \int f_q u_q \, dx.$$ Applying Hölder’s and Young’s inequalities to the right hand side yields
$$\frac{d}{dt} \lambda_q^{2\alpha+4} \| u_q \|_2^2 + \lambda_q^{2\alpha+4} \| \nabla u_q \|_2^2 \leq 4 \lambda_q^{2\alpha+2} \| f_q \|_2^2.$$ As a consequence of Duhamel’s formula, summation in $q$ and integration over $[0, T]$, we obtain
$$\int_0^T \sum_{q \geq -1} \lambda_q^{2\alpha+4} \| u_q(t) \|_2^2 \, dt \leq \int_0^T \sum_{q \geq -1} \lambda_q^{2\alpha+4} \| u_q(0) \|_2^2 e^{-\lambda_q^{2\alpha+2}} \, dt$$
3. Proof of Theorem 1.1. This section is devoted to the proof of the main result. Recall \( \lambda_q = 2^q \) for any integer \( q \). We start by introducing the dissipation wavenumber \( \Lambda_u(t) \) for \( u \) and \( \Lambda_c(t) \) for \( c \), respectively

\[
\Lambda_u(t) = \min \left\{ \lambda_q : \lambda_p^{-1} \|u_p(t)\|_\infty < C_0, \forall p > q, q \in \mathbb{N} \right\},
\]

\[
\Lambda_c(t) = \min \left\{ \lambda_q : \lambda_p^2 \|c_p(t)\|_r < C_0, \forall p > q, q \in \mathbb{N} \right\}, \quad r \in \left( 3, \frac{3}{1 - \varepsilon} \right),
\]

(11)

where \( \varepsilon > 0 \) is a fixed arbitrarily small constant, and \( C_0 \) is a small constant to be determined later. Throughout this section, we use \( C \) for various absolute constants which can be different from line to line. In addition, we let \( Q_u(t) \) and \( Q_c(t) \) be the integers such that

\[
\Lambda_u(t) = \lambda_{Q_u(t)} \quad \text{and} \quad \Lambda_c(t) = \lambda_{Q_c(t)}.
\]

The constraint on the low modes is then defined as

\[
f(t) := \|\nabla c_{\leq Q_u(t)}(t)\|_{L_\infty}^2 + \|u_{\leq Q_u(t)}(t)\|_{B^0_{\infty, \infty}}\]

The first integral on the right hand side is handled as

\[
\int_0^T \sum_{q \geq -1} \lambda_q^{2n+2} \int_0^t e^{-\lambda_q^2 (t-s)} \|f_q(s)\|_2^2 \, ds \, dt.
\]

The constraint on the low modes is then defined as

\[
\sum_{q \geq -1} \lambda_q^{2n+2} \|u_q(0)\|_2^2 e^{-\lambda_q^2 t} \leq \sum_{q \geq -1} \lambda_q^{2n+2} \|u_q(0)\|_2^2 \left( 1 - e^{-\lambda_q^2 T} \right) \lesssim \|u_0\|_{H^{\alpha+1}}^2.
\]

In order to estimate the second integral, we exchange the order of integration to obtain

\[
\int_0^T \sum_{q \geq -1} \lambda_q^{2n+2} \int_0^t e^{-\lambda_q^2 (t-s)} \|f_q(s)\|_2^2 \, ds \, dt
\]

\[
\lesssim \int_0^T \int_s^T \sum_{q \geq -1} \lambda_q^{2n+2} e^{-\lambda_q^2 (t-s)} \|f_q(s)\|_2^2 \, ds \, ds
\]

\[
\lesssim \int_0^T \sum_{q \geq -1} \lambda_q^{2n} \|f_q(s)\|_2^2 \left( 1 - e^{-\lambda_q^2 (T-s)} \right) \, ds
\]

\[
\lesssim \|f\|_{L^2(0,T;H^\alpha)}^2.
\]

Combining the estimates above, we conclude that \( u \in L^2(0,T;H^{\alpha+2}) \) for \( \alpha \in \mathbb{R} \).

To prove \( u \in H^1(0,T;H^\alpha) \), we first project equation (10) to the \( q \)-th dyadic shell

\[
(u_k)_q = \Delta u_q + f_q.
\]

It follows

\[
\|(u_k)_q\|_2^2 \leq 2\|\Delta u_q\|_2^2 + 2\|f_q\|_2^2.
\]

Thus we deduce that

\[
\int_0^T \sum_{q \geq -1} \lambda_q^{2n} \|(u_k)_q\|_2^2 \, dt \lesssim \int_0^T \sum_{q \geq -1} \lambda_q^{2n} \|\Delta u_q\|_2^2 \, dt + \int_0^T \sum_{q \geq -1} \lambda_q^{2n} \|f_q\|_2^2 \, dt
\]

\[
\lesssim \|u\|_{L^2(0,T;H^{\alpha+2})}^2 + \|f\|_{L^2(0,T;H^\alpha)}^2.
\]

It is then clear that \( u \in H^1(0,T;H^\alpha) \), which completes the proof of the lemma. \( \square \)
Notice that the wavenumber $\Lambda_u$ separates the inertial range from the dissipation range where the viscous term $\Delta u$ dominates; and $\Lambda_c$ has the analogous property. Precisely, we have

$$\|u(t)Q_u\|_\infty \geq C_0 \Lambda_u(t), \quad \Lambda_c^\frac{3}{2}(t)\|c(t)Q_c\|_r \geq C_0;$$

$$\lambda_q\|u(t)\|_q \geq C_0, \quad \forall q > Q_u; \quad \lambda_c^\frac{3}{2}(t)\|c(t)\|_r < C_0, \quad \forall q > Q_c.$$ 

The crucial part of the proof is to establish a uniform (in time) bound for each of the unknowns $n, u$ and $c$ in a space with higher regularity than the critical Sobolev space. In fact, it is sufficient to prove that

$$(n, u, c) \in L^\infty(0, T; \dot{H}^{s_1}) \times L^\infty(0, T; \dot{H}^{s_2}) \times L^\infty(0, T; \dot{H}^{s_3})$$

for some $s_1 > \frac{1}{2}, s_2 > \frac{1}{2}$ and $s_3 > \frac{3}{2}$. Due to the complicated interactions among the three equations in (1), the aforementioned goal will be achieved in two steps. The first step is to show that

$$(n, u, c) \in L^\infty(0, T; \dot{H}^s) \times L^\infty(0, T; \dot{H}^{s+1}) \times L^\infty(0, T; \dot{H}^{s+1})$$

for some $s \in (-\frac{1}{2}, 0)$. The second step consists of applying bootstrap arguments, the $L^p-L^q$ theory for parabolic equations and a mixed derivative theorem to the equation of oxygen concentration $c$ to improve the regularity of $c$.

To start, we multiply the equations in (1) by $\lambda_q^\frac{3}{2} \Delta_q^\frac{3}{2} n, \lambda_q^\frac{3}{2} \Delta_q^\frac{3}{2} c$ and $\lambda_q^\frac{3}{2} \Delta_q^\frac{3}{2} u$, respectively. Integrating and summing lead to

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^2 \|n_q\|^2_2 \leq - \sum_{q \geq -1} \lambda_q^2 \|\nabla n_q\|^2_2 - \sum_{q \geq -1} \lambda_q^2 \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla n) n_q dx$$

$$- \sum_{q \geq -1} \lambda_q^2 \int_{\mathbb{R}^3} \Delta_q(\nabla \cdot (n \chi(c) \nabla c)) n_q dx;$$

(12)

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2+2} \|c_q\|^2_2 \leq - \sum_{q \geq -1} \lambda_q^{2+2} \|\nabla c_q\|^2_2 - \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla c) c_q dx$$

$$- \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(n f(c)) c_q dx;$$

(13)

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2+2} \|u_q\|^2_2 \leq - \sum_{q \geq -1} \lambda_q^{2+2} \|\nabla u_q\|^2_2 - \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla u) u_q dx$$

$$- \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(n \nabla \Phi) u_q dx.$$  

(14)

For simplicity we label the terms

$$I := - \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla u) u_q dx, \quad II := - \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla c) c_q dx,$$

$$III := - \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla n) n_q dx, \quad IV := - \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(n \nabla \Phi) u_q dx,$$

$$V := - \sum_{q \geq -1} \lambda_q^{2+2} \int_{\mathbb{R}^3} \Delta_q(n f(c)) c_q dx, \quad VI := - \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(\nabla \cdot (n \chi(c) \nabla c)) n_q dx.$$
Moreover we have

\[ |I| \lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+4} \| u_q \|_2^2 + Q_u f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \| u_q \|_2^2. \]

**Proof.** Applying Bony’s paraproduct decomposition to \( I \) leads to

\[
I = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(u_{p-2} \cdot \nabla u_p) u_q \, dx \\
- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla u_{p-2}) u_q \, dx \\
- \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla u_p) u_q \, dx \\
= I_1 + I_2 + I_3.
\]

Using the fact \( \sum_{|q-p| \leq 2} \Delta_q u_p = u_q \) and the commutator notation, we have

\[
I_1 = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} [\Delta_q, u_{p-2} \cdot \nabla] u_p u_q \, dx \\
- \sum_{q \geq -1} \lambda_q^{2s+3} \int_{\mathbb{R}^3} u_{p-2} \cdot \nabla u_q u_q \, dx \\
- \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} (u_{p-2} - u_{q-2}) \cdot \nabla \Delta_q u_p u_q \, dx \\
=: I_{11} + I_{12} + I_{13}.
\]

Moreover we have \( I_{12} = 0 \) due to that \( \nabla \cdot u_{q-2} = 0 \).

We then split \( I_{11} \) based on definition of \( \Lambda_u(t) \)

\[
|I_{11}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \| [\Delta_q, u_{p-2} \cdot \nabla] u_p u_q \| \, dx \\
\leq \sum_{p \leq Q_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \| [\Delta_q, u_{p-2} \cdot \nabla] u_p u_q \| \, dx \\
+ \sum_{p > Q_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \| [\Delta_q, u_{p-2} \cdot \nabla] u_p u_q \| \, dx \\
+ \sum_{p > Q_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \| [\Delta_q, u_{Q_u+2} \cdot \nabla] u_p u_q \| \, dx \\
=: I_{111} + I_{112} + I_{113}.
\]

Using (2.3), Hölder’s inequality, and definition of \( f(t) \), we obtain

\[
I_{111} \lesssim \sum_{1 \leq p \leq Q_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \| \nabla u_{p-2} \|_\infty \| u_p \|_2 \| u_q \|_2 \\
\lesssim f(t) \sum_{1 \leq p \leq Q_u+2} \| u_p \|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \| u_q \|_2 \sum_{p' \leq p-2} 1
\]
Using H"older's inequality and definition of \( I \)

\[
\sum_{1 \leq p \leq Q_u + 2} \lambda_p^{s+1} \||u_p||_2 \sum_{|q-p| \leq 2} \lambda_q^{s+1} \||u_q||_2
\]

and similarly

\[
\sum_{q \geq -1} \lambda_q^{2s+2} \||u_q||_2^2;
\]

and similarly

\[
I_{112} \leq \sum_{p > Q_u + 2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \||\nabla u|_{L_\infty} \||u_p||_2 \||u_q||_2
\]

\[
\sum_{p > Q_u + 2} \||u_p||_2 \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \||u_q||_2 \sum_{Q_u < q' \leq p-2} \lambda_{q'} \||u_{q'}||_\infty
\]

\[
\leq C_0 \sum_{p > Q_u + 2} \||u_p||_2 \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \||u_q||_2 \sum_{Q_u < q' \leq p-2} \lambda_{q'}^2
\]

\[
\leq C_0 \sum_{p > Q_u + 2} \lambda_p^{2s+4} \||u_p||_2^2 \sum_{Q_u < q' \leq p-2} \lambda_{q'}^2
\]

We estimate \( I_{113} \) with the help of Hölder’s inequality and Lemma (2.3)

\[
I_{113} \leq \sum_{p > Q_u + 2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\|u\|_p \leq 2} \|\Delta_q \cdot \nabla u\|_{2 \cdot \|u_q\|_2}
\]

\[
\leq \sum_{-1 \leq q \leq Q_u} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\|u\|_p \leq 2} \|\Delta_q \cdot \nabla u\|_{2 \cdot \|u_q\|_2}
\]

Using Hölder’s inequality and definition of \( f(t) \) we can bound \( I_{131} \).

\[
|I_{131}| \leq \sum_{-1 \leq q \leq Q_u} \lambda_q^{2s+3} \|u_q\|_\infty \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_2 \|u_p\|_2
\]

\[
\sum_{-1 \leq q \leq Q_u} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \||u_{p-2} - u_{q-2}\|_2 \|u_p\|_2
\]

\[
\sum_{-1 \leq q \leq Q_u} \lambda_q^{2s+2} \||u_q||_2^2.
\]
And we estimate $I_{132}$ using Hölder’s inequality and the definition of $\Lambda_u(t)$,

$$|I_{132}| \leq \sum_{q \geq Q_u} \lambda_q^{2s+2} \|u_q\|_\infty \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_2 \|u_p\|_2$$

$$\lesssim C_0 \sum_{q > Q_u} \lambda_q^{2s+4} \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_2 \|u_p\|_2$$

$$\lesssim C_0 \sum_{q > Q_u} \lambda_q^{2s+4} \|u_q\|_2^2.$$

We omit the detailed estimation of $I_2$ as it is similar to that of $I_{11}$. Meanwhile, for $I_3$ we have for $s > -\frac{1}{2}$

$$|I_3| \leq \sum_{q \geq -1} \lambda_q^{2s+2} \sum_{p \geq q-2} \|\Delta_q(u_p \otimes \tilde{u}_p) \nabla u_q\|_2 dx$$

$$\leq \sum_{q \geq Q_u} \lambda_q^{2s+3} \|u_q\|_\infty \sum_{p \geq q-2} \|u_p\|_2^2 + \sum_{-1 \leq q \leq Q_u} \lambda_q^{2s+3} \|u_q\|_\infty \sum_{p \geq q-2} \|u_p\|_2^2$$

$$\leq C_0 \sum_{q > Q_u} \lambda_q^{2s+4} \sum_{p \geq q-2} \|u_p\|_2^2 + f(t) \sum_{-1 \leq q \leq Q_u} \lambda_q^{2s+2} \sum_{p \geq q-2} \lambda_p^{2s+2}$$

$$\lesssim C_0 \sum_{q > Q_u} \lambda_q^{2s+4} \|u_q\|_2^2 + f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2.$$

We combine the above estimates to conclude that

$$|I| \lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+4} \|u_q\|_2^2 + Q_u f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|c_q\|_2^2.$$

\[
\square
\]

3.2. Estimate for $II$.

**Lemma 3.2.** Let $s > -\frac{1}{2}$. We have

$$|II| \leq \left(C C_0 + \frac{1}{32}\right) \sum_{q \geq -1} \lambda_q^{2s+4} \|c_q\|_2^2 + \lambda_q^{2s+4} \|u_q\|_2^2 + C Q_u f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|c_q\|_2^2.$$

**Proof.** We start with Bony’s para-product decomposition for $II$,

$$II = \sum_{q \geq -1} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(u_{\leq p-2} \cdot \nabla c_p) c_q dx$$

$$+ \sum_{q \geq -1} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla c_{\leq p-2}) c_q dx$$

$$+ \sum_{q \geq -1} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla \tilde{c}_p) c_q dx$$

$$= II_1 + II_2 + II_3.$$

Using the commutator notation, we rewrite $II_1$ as

$$II_1 = \sum_{q \geq -1} \lambda_q^{2s+2} \int_{\mathbb{R}^3} [\Delta_q, u_{\leq p-2} \cdot \nabla] c_q dx$$
+ \sum_{q \geq -1} |\Delta_q, u_{p-2} \cdot \nabla|c_q| dx \\
+ \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} (u_{p-2} - u_{q-2}) \cdot \nabla \Delta_q c_p c_q dx \\
=: II_{11} + II_{12} + II_{13}.

Here, just as in proposition (3.1), we used \sum_{|q-p| \leq 2} \Delta_p c_q = c_q to obtain II_{12}, which vanishes as \nabla \cdot u_{q-2} = 0.

One can see that II_{11} enjoys the same estimates that I_{11} does. Splitting the summation by \mathcal{Q}_u(t) yields

\begin{align*}
|II_{11}| &\leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} |[\Delta_q, u_{p-2} \cdot \nabla]|c_q| dx \\
&\leq \sum_{p \leq \mathcal{Q}_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} |[\Delta_q, u_{p-2} \cdot \nabla]|c_q| dx \\
&\quad + \sum_{p > \mathcal{Q}_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} |[\Delta_q, u_{\mathcal{Q}_u-2} \cdot \nabla]|c_q| dx \\
&\quad + \sum_{p > \mathcal{Q}_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} |[\Delta_q, u_{\mathcal{Q}_u-p-2} \cdot \nabla]|c_q| dx \\
&=: II_{111} + II_{112} + II_{113}.
\end{align*}

We estimate the first two of the above three terms by Hölder’s inequality, Lemma (2.3) and the definition of f(t).

\begin{align*}
II_{111} &\leq \sum_{1 \leq p \leq \mathcal{Q}_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \|\nabla u_{p-2}\|_\infty \|c_p\|_2 \|c_q\|_2 \\
&\leq \mathcal{Q}_u f(t) \sum_{1 \leq p \leq \mathcal{Q}_u+2} \lambda_p^{s+1} \|c_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{s+1} \|c_q\|_2 \\
&\leq \mathcal{Q}_u f(t) \sum_{q \geq -1} \lambda_q^{2s+2} \|c_q\|_2^2,
\end{align*}

\begin{align*}
II_{112} &\leq \sum_{p > \mathcal{Q}_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \|\nabla u_{\mathcal{Q}_u}\|_\infty \|c_p\|_2 \|c_q\|_2 \\
&\leq \mathcal{Q}_u f(t) \sum_{p > \mathcal{Q}_u+2} \lambda_p^{s+1} \|c_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{s+1} \|c_q\|_2 \\
&\leq \mathcal{Q}_u f(t) \sum_{q > \mathcal{Q}_u} \lambda_q^{2s+2} \|c_q\|_2^2.
\end{align*}

And with the help of Hölder’s inequality, Lemma (2.3) and the definition of \Lambda_u(t), we estimate II_{113} as

\begin{align*}
II_{113} &\leq \sum_{p > \mathcal{Q}_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \|\nabla u_{\mathcal{Q}_u-p-2} \cdot \nabla|c_p\|_2 \|c_q\|_2 \\
&\leq \sum_{p > \mathcal{Q}_u+2} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \|c_q\|_2 \sum_{Q_u < p' < p-2} \lambda_{p'} \|u_{p'}\|_\infty \|c_p\|_2.
\end{align*}
\[ \begin{align*}
&\leq \sum_{p > Q_u + 2} \|c_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \|c_q\|_2 \sum_{Q_u < q' \leq p-2} \lambda_q^{2s+2} \|c_q\|_2 \sum_{Q_u < q' \leq p-2} \lambda_{q'}\|u_{p'}\|_\infty \\
&\lesssim C_0 \sum_{p > Q_u + 2} \|c_p\|_2 \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \|c_q\|_2 \sum_{Q_u < q' \leq p-2} \lambda_{q'} \\
&\lesssim C_0 \sum_{p > Q_u + 2} \lambda_q^{2s+4} \|c_p\|_2^2 \sum_{Q_u < q' \leq p-2} \lambda_{q'}^2 \\
&\lesssim C_0 \sum_{p > Q_u + 2} \lambda_q^{2s+4} \|c_p\|_2^2 \\
\end{align*} \]

\(I_{13}\) can be estimated in the same fashion as \(I_{13}\). Splitting the sum by the wavenumber \(Q_u(t)\), we have

\[ |I_{13}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} |(u_{p-2} - u_{q-2}) \cdot \nabla \Delta q c_q| dx \]

\[ \leq \sum_{-1 \leq q \leq Q_u} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} |(u_{p-2} - u_{q-2}) \cdot \nabla \Delta q c_q| dx \]

\[ + \sum_{q > Q_u} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} |(u_{p-2} - u_{q-2}) \cdot \nabla \Delta q c_q| dx \]

\[= : I_{131} + I_{132}, \]

which, using Hölder’s inequality along with the definition of \(f(t)\) and \(\Lambda_u(t)\), we estimate as

\[ |I_{131}| \leq \sum_{-1 \leq q \leq Q_u} \lambda_q^{2s+3} \|c_q\|_2 \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_\infty \|c_p\|_2 \]

\[ \lesssim f(t) \sum_{-1 \leq q \leq Q_u} \lambda_q^{2s+2} \|c_q\|_2 \sum_{|q-p| \leq 2} \|c_p\|_2 \]

\[ \lesssim f(t) \sum_{-1 \leq q \leq Q_u} \lambda_q^{2s+2} \|c_q\|_2^2, \]

\[ |I_{132}| \leq \sum_{q > Q_u} \lambda_q^{2s+3} \|c_q\|_2 \sum_{|q-p| \leq 2} \|u_{p-2} - u_{q-2}\|_\infty \|c_p\|_2 \]

\[ \lesssim C_0 \sum_{q > Q_u} \lambda_q^{2s+3} \|c_q\|_2 \sum_{|q-p| \leq 2} \lambda_p \|c_p\|_2 \]

\[ \lesssim C_0 \sum_{q > Q_u} \lambda_q^{2s+4} \|c_q\|_2^2. \]

To estimate \(I_2\), we make use of the wavenumber \(Q_c(t)\) instead of \(Q_u(t)\). Splitting the summation by \(Q_c(t)\) yields

\[ I_2 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta q(u_{p} \cdot \nabla c_{q \leq p-2}) c_q dx \]

\[ = \sum_{q \geq -1} \sum_{|q-p| \leq 2, p \leq Q_c + 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta q(u_{p} \cdot \nabla c_{q \leq p-2}) c_q dx \]

\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2, p > Q_c + 2} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta q(u_{p} \cdot \nabla c_{q \leq p-2}) c_q dx \]
Proceeding with the help of Hölder’s, Young’s and Jensen’s inequalities, we have

\[ II_{21} + II_{22} + II_{23}. \]

It follows from Hölder’s and Young’s inequalities that

\[
|II_{21}| \leq \sum_{q \geq 1} \sum_{|q-p| \leq 2, p \geq Q_c + 2} \lambda_q^{2s+2} \|u_p\|_2 \|c_q\|_2 \|\nabla c_{p}\|_2 \|\nabla c_{Q_c + p - 2}\|_\infty \leq \sum_{q \geq 1} \lambda_q^{2s+2} \|u_q\|_2 \|c_q\|_2 \|\nabla c_{Q_c + p - 2}\|_\infty \leq \sum_{q \geq 1} \lambda_q^{2s+2} \|u_q\|_2 \|c_q\|_2 \sum_{|q-p| \leq 2, p \geq Q_c + 2} \lambda_p^{s+1} \|c_p\|_r \sum_{|q-p| \leq 2, p \geq Q_c + 2} \lambda_p \lambda_{p-q} \sum_{q \geq 1} \lambda_q^{2s+4} \|u_q\|_2 \|c_q\|_2 + C_0 \sum_{q > Q_c} \lambda_q^{2s+4} \|c_q\|_2^2. \]

While the term II_{22} can be treated the same way as II_{21}, the third term is estimated as follows, by utilizing the definition of \( \Lambda_c(t) \)

\[
|II_{23}| \leq \sum_{q \geq 1} \sum_{|q-p| \leq 2, p \geq Q_c + 2} \lambda_q^{2s+2} \|u_p\|_2 \|c_q\|_2 \|\nabla c_{Q_c - p}\|_\infty \leq \sum_{q \geq 1} \lambda_q^{2s+2} \|u_q\|_2 \|c_q\|_2 \|\nabla c_{Q_c - p}\|_\infty \leq \sum_{q \geq 1} \lambda_q^{2s+2} \|u_q\|_2 \|c_q\|_2 \sum_{Q_c < p \leq q} \lambda_{p} \sum_{Q_c < p \leq q} \lambda_{p-q} \sum_{q \geq 1} \lambda_q^{2s+4} \|u_q\|_2 \|c_q\|_2 + C_0 \sum_{q > Q_c} \lambda_q^{2s+4} \|c_q\|_2^2. \]

As for II_3, we first integrate by parts, then split by the wavenumber \( Q_u(t) \),

\[
|II_3| \leq \left| \sum_{p \geq 1} \lambda_q^{2s+2} \sum_{1 \leq q \leq p + 2} \int_{\mathbb{R}^3} \Delta_q(u_p \otimes \hat{c}_p) \nabla c_q \, dx \right| \leq \sum_{p \geq Q_u} \lambda_q^{2s+2} \sum_{1 \leq q \leq p + 2} \int_{\mathbb{R}^3} \Delta_q(u_p \otimes \hat{c}_p) \nabla c_q \, dx \right| \leq \sum_{1 \leq q \leq p + 2} \int_{\mathbb{R}^3} \Delta_q(u_p \otimes \hat{c}_p) \nabla c_q \, dx \right| =: II_{31} + II_{32}. \]

Proceeding with the help of Hölder’s, Young’s and Jensen’s inequalities, we have for \( s > -\frac{1}{2} \)

\[
|II_{31}| \lesssim \sum_{p > Q_u} \|u_p\|_\infty \|c_p\|_2 \sum_{1 \leq q \leq p + 2} \lambda_q^{2s+3} \|c_q\|_2. \]
We combine the above estimates to conclude that

\[ |II| \lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} ||q||_2^2 + \sum_{q \geq -1} \lambda_q^{2s+3} ||c_q||_2 \]

\[ \lesssim C_0 \sum_{p > Q_u} \lambda_p ||c_p||_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+3} ||c_q||_2 \]

\[ \lesssim C_0 \sum_{p > Q_u} \lambda_p ||c_p||_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+2} ||c_q||_2 ||2\lambda_q^{-1} - 1|| \]

\[ \lesssim C_0 \sum_{p > Q_u} \left( \lambda_p^{2s+4} ||c_p||_2^2 + \left( \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+2} ||c_q||_2 ||2\lambda_q^{-1} - 1|| \right)^2 \right) \]

\[ \lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+4} ||c_q||_2^2, \]

\[ |II|_3 \lesssim \sum_{-1 \leq p \leq Q_u} \lambda_p ||c_p||_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+3} ||c_q||_2 \]

\[ \lesssim f(t) \sum_{-1 \leq p \leq Q_u} \lambda_p ||c_p||_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+3} ||c_q||_2 \]

\[ \lesssim f(t) \sum_{p > Q_u} \lambda_p ||c_p||_2 \sum_{-1 \leq q \leq p+2} \lambda_q^{2s+3} ||c_q||_2 ||2\lambda_q^{-1} - 1|| \]

\[ \lesssim f(t) \sum_{q \geq -1} \lambda_q^{2s+2} ||c_q||_2^2. \]

We combine the above estimates to conclude that

\[ |II| \lesssim C_0 \sum_{q \geq -1} (\lambda_q^{2s+4} ||c_q||_2^2 + \lambda_q^{2s+4} ||u_q||_2^2) + Q_u f(t) \sum_{q \geq -1} \lambda_q^{2s+2} ||c_q||_2^2. \]

\[ \square \]

### 3.3. Estimate for II.

**Lemma 3.3.** Let \( s < 0 \). We have

\[ |III| \lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} ||n_q||_2^2 + Q_u f(t) \sum_{q \geq -1} \lambda_q^{2s} ||n_q||_2^2. \]

**Proof.** We start with Bony’s paraproduct decomposition for both of the terms.

\[ III = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_{p-2} \cdot \nabla n_q) n_q \ dx \]

\[ \quad + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla n_{p-2}) n_q \ dx \]

\[ \quad + \sum_{q \geq -1} \sum_{p \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla n_p) n_q \ dx \]

\[ = III_1 + III_2 + III_3. \]

Since we can estimate \( III_1 \) and \( III_3 \) in the same manner as \( II_1 \) and \( II_3 \), respectively, the details of computation are omitted for simplicity. We claim

\[ |III_1| + |III_3| \lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} ||n_q||_2^2 + Q_u f(t) \sum_{q \geq -1} \lambda_q^{2s} ||n_q||_2^2. \]
We are then left with the estimation of $III_2$ to complete the conclusion. Splitting the term by the wavenumber $Q_u(t)$, we have

\[
III_2 = \sum_{-1 \leq p \leq Q_u} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla n_{\leq p-2}) n_q \, dx \\
+ \sum_{p > Q_u} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla n_{\leq p-2}) n_q \, dx
\]

\[= III_{21} + III_{22}.
\]

Applying Hölder’s, Young’s and Jensen’s inequalities, we have for $s < 1$

\[
|III_{21}| \leq \sum_{-1 \leq p \leq Q_u} \| u_p \|_\infty \sum_{|q-p| \leq 2} \lambda_q^{2s} \| n_q \|_2 \sum_{p' \leq p-2} \lambda_{p'} \| n_{p'} \|_2
\]

\[\leq \sum_{-1 \leq p \leq Q_u} \lambda_p \| u_p \|_\infty \sum_{|q-p| \leq 2} \lambda_q^{2s-1} \| n_q \|_2 \sum_{p' \leq p-2} \lambda_{p'} \| n_{p'} \|_2
\]

\[\leq \sum_{-1 \leq p \leq Q_u} \lambda_p \| u_p \|_\infty \sum_{|q-p| \leq 2} \lambda_q^s \| n_q \|_2 \sum_{p' \leq p-2} \lambda_{p'} \| n_{p'} \|_{2 \lambda_{q-p}^{s-1}}
\]

\[\lesssim f(t) \sum_{q \geq -1} \lambda_q^{2s} \| n_q \|_2^2,
\]

and for $s < 0$

\[
|III_{22}| \leq \sum_{p > Q_u} \| u_p \|_\infty \sum_{|q-p| \leq 2} \lambda_q^{s+1} \| n_q \|_2 \sum_{p' \leq p-2} \lambda_{p'} \| n_{p'} \|_2
\]

\[\leq \sum_{p > Q_u} \lambda_p^{-1} \| u_p \|_\infty \sum_{|q-p| \leq 2} \lambda_q^{s+1} \| n_q \|_2 \sum_{p' \leq p-2} \lambda_{p'} \| n_{p'} \|_2
\]

\[\leq C_0 \sum_{q > Q_u} \lambda_q^{s+1} \| n_q \|_2 \sum_{p' \leq q} \lambda_{p'}^{s+1} \| n_{p'} \|_{2 \lambda_{q-p}^s}
\]

\[\lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} \| n_q \|_2^2.
\]

3.4. **Estimate for IV.** Here we use Hölder’s and Young’s inequalities to obtain

\[
|IV| \leq \sum_{q \geq -1} \lambda_q^{s+1} \| n_q \|_2 \lambda_q^{s+1} \| u_q \|_2
\]

\[\leq \frac{1}{32} \sum_{q \geq -1} \lambda_q^{2s+2} \| n_q \|_2^2 + C \sum_{q \geq -1} \lambda_q^{2s+2} \| u_q \|_2^2
\]

for an absolute constant $C$.

3.5. **Estimate for V.**

**Lemma 3.4.** If $r > 3$ and $s < 0$, we have

\[
|V| \leq \left( \frac{1}{32} + C C_0 \right) \sum_{q \geq -1} \left( \lambda_q^{2s+4} \| c_q \|_2^2 + \lambda_q^{2s+2} \| n_q \|_2^2 \right)
\]
Proof. Bony’s paraproduct decomposition leads to

\[ V = \sum_{q \geq -1} \sum_{|q-p| \leq 3} \lambda_q^{2s+2} \left( \Delta_q (n_{p-2} c_p) c_q \right) d \mathbf{x} \]

We further split

\[ V = V_1 + V_2 + V_3. \]

We further split \( V_1 \) by the wavenumber \( \lambda_c(t) \)

\[ V_1 = \sum_{q \leq Q_c} \sum_{|q-p| \leq 3} \lambda_q^{2s+2} \left( \Delta_q (n_{p-2} c_p) c_q \right) d \mathbf{x} \]

To estimate \( V_{11} \), Hölder’s inequality gives

\[ |V_{11}| \leq \sum_{q \leq Q_c} \lambda_q^{2s+2} \| c_q \|_\infty \sum_{|q-p| \leq 3} \| c_p \|_2 \sum_{p' \leq p-2} \| n_{p'} \|_2 \]

\[ \lesssim \sum_{q \leq Q_c} \lambda_q \| c_q \|_\infty \sum_{|q-p| \leq 3} \lambda_p^{s+2} \| c_p \|_2 \sum_{p' \leq p-2} \lambda_{p'} \| n_{p'} \|_2 \| \lambda_{q-p}' \|_\infty \lambda_{q-p}^{-1} \lambda_{p'}^{-1}. \]

Since \( s - 1 < 0 \) for \( s < 0 \), we apply Young’s and Jensen’s inequalities to obtain

\[ |V_{11}| \leq \frac{1}{32} \sum_{q \leq Q_c + 2} \lambda_q^{2s+4} \| c_q \|_2^2 + C \sum_{q \leq Q_c + 2} \lambda_q^2 \| c_q \|_\infty \left( \sum_{p' \leq q} \lambda_{p'}^s \| n_{p'} \|_2 \lambda_{q-p}' \right) \]

\[ \leq \frac{1}{32} \sum_{q \leq Q_c + 2} \lambda_q^{2s+4} \| c_q \|_2^2 + C f(t) \sum_{q \leq Q_c + 2} \sum_{p' \leq q} \lambda_{p'}^{2s} \| n_{p'} \|_2^2 \lambda_{q-p}' \]

\[ \leq \frac{1}{32} \sum_{q \leq Q_c + 2} \lambda_q^{2s+4} \| c_q \|_2^2 + C f(t) \sum_{q \leq Q_c} \lambda_q^{2s} \| n_q \|_2^2 \]

Regarding \( V_{12} \), we have

\[ |V_{12}| \leq \sum_{q > Q_c} \lambda_q^{2s+2} \| c_q \|_\infty \sum_{|q-p| \leq 3} \| c_p \|_2 \sum_{p' \leq p-2} \| n_{p'} \|_2 \]

\[ \lesssim \sum_{q > Q_c} \lambda_q^{2s+2} \| c_q \|_r \sum_{|q-p| \leq 3} \| c_p \|_2 \sum_{p' \leq p-2} \| n_{p'} \|_2 \]

\[ \lesssim C_0 \sum_{q > Q_c - 2} \lambda_q^{2s+2} \| c_q \|_2 \sum_{p' \leq q} \lambda_{p'}^s \| n_{p'} \|_2 \lambda_{q-p}' \]

\[ \lesssim C_0 \sum_{q > Q_c - 2} \lambda_q^{2s+2} \| c_q \|_2 \sum_{p' \leq q} \lambda_{p'}^s \| n_{p'} \|_2 \lambda_{q-p}' \].
Again, since $s < 0$, we can apply Young’s and Jensen’s inequalities

$$|V_{12}| \lesssim C_0 \sum_{q > Q_c - 2} \lambda^2_q \|c_q\|^2 + \sum_{q > Q_c} \left( \sum_{q' \leq q} \lambda^s_{q'} \|n_{q'}\| \|2\lambda^s_{q-q'}\right)^2 \lesssim C_0 \sum_{q > Q_c - 2} \lambda^2_q \|c_q\|^2 + \sum_{q \geq -1} \lambda^2_{q'} \|n_{q'}\|^2$$

We also split $V_2$ by the wavenumber $\Lambda_c(t)$ as

$$V_2 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2_q \int_{\mathbb{R}^3} \Delta_q(n_p c_{p-2}) c_q \, dx$$

Hölder’s and Young’s inequalities yield the following estimate on $V_{21}$,

$$|V_{21}| \lesssim \sum_{q \geq -1} \lambda^{2+2} q \|c_q\|_2 \sum_{|q-p| \leq 2} \|n_p\|_2 \sum_{p' \leq p-2 \leq Q_c} \|c_{p'}\|_\infty$$

$$\lesssim \sum_{q \geq -1} \lambda^{s+1} q \|c_q\|_2 \sum_{|q-p| \leq 2} \lambda^{s+1} p \|n_p\|_2 \sum_{p' \leq p-2 \leq Q_c} \lambda_{p'} \|c_{p'}\|_\infty \lambda^{-1}_{p'}$$

$$\lesssim \frac{1}{32} \sum_{q \geq -1} \lambda^{2+2} q \|n_q\|_2^2 + C \sum_{p \geq -1} \lambda^{2+2} p \|c_p\|_2 \left( \sum_{p' \leq p-2 \leq Q_c} \lambda_{p'} \|c_{p'}\|_\infty \lambda^{-1}_{p'} \right)^2$$

$$\lesssim \frac{1}{32} \sum_{q \geq -1} \lambda^{2+2} q \|n_q\|_2^2 + C \sum_{p \geq -1} \lambda^{2+2} p \|c_p\|_2 \sum_{p' \leq p-2 \leq Q_c} \lambda_{p'} \|c_{p'}\|_\infty \lambda^{-1}_{p'}$$

Note that $V_{22}$ can be estimated in the same way. Provided $3 \leq r \leq 1$, the last term $V_{23}$ is estimated as follows,

$$|V_{23}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq p+2} \lambda^{2+2} q \|n_p\|_2 \|c_q\|_r \sum_{q > Q_c, Q_c < p' \leq q} \|c_{p'}\|_r$$

$$\lesssim \sum_{q \geq -1} \lambda^{2+2} q \|n_q\|_2 \|c_q\|_2 \sum_{q > Q_c, Q_c < p' \leq q} \|c_{p'}\|_r$$

$$\lesssim C_0 \sum_{q \geq -1} \lambda^{2+2} q \|n_q\|_2 \|c_q\|_2 \sum_{q > Q_c, Q_c < p' \leq q} \lambda^{-\frac{3}{2}}_{p'}$$
Lemma 3.5. Let $\lambda_q^{2s+2} = \lambda_q^{2s+2} ||n_q||_2 \lambda_q^{3/2-1} \sum_{q > Q_c, q < p' \leq q} \lambda_p^{3/2}$

\[
\lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} ||n_q||_2 \lambda_q^{3/2-1} \sum_{q > Q_c, q < p' \leq q} \lambda_p^{3/2}
\]

\[
\lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} ||n_q||_2 \lambda_q^{3/2} \|c_q\|_2
\]

\[
\lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} ||n_q||_2^2 + C_0 \sum_{q \geq -1} \lambda_q^{2s+4} \|c_q\|_2^2.
\]

The term $V_3$ can be dealt with in a similar way as $V_1$. We first split the sum,

\[
V_3 = \sum_{q \leq Q_c} \sum_{p \geq q^{-2}} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(\tilde{n}_p c_p) c_q dx
\]

\[
+ \sum_{q<Q_c} \sum_{p \geq q^{-2}} \lambda_q^{2s+2} \int_{\mathbb{R}^3} \Delta_q(\tilde{n}_p c_p) c_q dx.
\]

Without giving details, we claim that

\[
|V_3| \leq \left(\frac{1}{32} + CC_0\right) \sum_{q \geq -1} \lambda_q^{2s+4} \|n_q\|_2^2 + C(f(t) + 1) \sum_{q \geq -1} \lambda_q^{2s} ||n_q||_2^2.
\]

3.6. Estimate for $V_1$.

Lemma 3.5. Let $-\frac{1}{2} < s < 0$ and $3 < r < \frac{3}{1+s}$. We have

\[
|V_1| \leq \left(\frac{1}{32} + CC_0\right) \sum_{q \geq -1} \lambda_q^{2s+2} ||n_q||_2^2 + C(f(t) \sum_{q \geq -1} \lambda_q^{2s} ||n_q||_2^2.
\]

Proof. Utilizing Bony’s paraproduct, $VI$ can be decomposed as

\[
VI = -\sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_{p-q} \nabla c_p) \nabla n_q dx
\]

\[
-\sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_p \nabla c_p) \nabla n_q dx
\]

\[
-\sum_{q \geq -1} \sum_{p \geq q^{-2}} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(\tilde{n}_p c_p) \nabla n_q dx
\]

\[
=: VI_1 + VI_2 + VI_3.
\]

We continue to decompose $VI_1$ by $Q_c$,

\[
VI_1 = -\sum_{q \geq -1} \sum_{|q-p| \leq 2, p \leq Q_c} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_{p-q} \nabla c_p) \nabla n_q dx
\]

\[
-\sum_{q \geq -1} \sum_{|q-p| \leq 2, p > Q_c} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(n_{p-q} \nabla c_p) \nabla n_q dx
\]

\[
=: VI_{11} + VI_{12}.
\]

To estimate $VI_{11}$, we apply Hölder’s inequality first,

\[
|VI_{11}| \lesssim \sum_{q \geq -1} \lambda_q^{2s+1} ||n_q||_2 \sum_{|q-p| \leq 2, p \leq Q_c} \|\nabla c_p\|_\infty \sum_{p' \leq p-2} ||n_{p'}||_2
\]
Following Young’s and Jensen’s inequalities, we obtain for $s$

$$\lambda_{q-1}^{s+1} \|n_q\|_2 \sum_{|q-p| \leq 2, p \leq Q_c} \|\nabla c_p\|_\infty \sum_{p' \leq p-2} \lambda_{p'}^{s} \|n_{p'}\|_2 \lambda_{p-p'}^{s}.$$  

We then apply Young’s and Jensen’s inequalities,

$$|V_1| \leq \frac{1}{32} \sum_{q \geq -1} \lambda_{q-1}^{2s+2} \|n_q\|_2^2 + Cf(t) \sum_{q \geq -1} \left( \sum_{p' \leq q} \lambda_{p'}^{s} \|n_{p'}\|_2 \lambda_{q-p'}^{s} \right)^2$$

$$\leq \frac{1}{32} \sum_{q \geq -1} \lambda_{q-1}^{2s+2} \|n_q\|_2^2 + Cf(t) \sum_{q \geq -1} \lambda_{q}^{2s} \|n_q\|_2^2,$$

where we require $s < 0$ to ensure $\sum_{p' \leq p-2} \lambda_{p-p'}^{s} < \infty$.

Again, applying Hölder’s inequality first to $V_{I21}$ yields

$$|V_{I21}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_{q}^{2s} \|\nabla n_q\|_2 \|\nabla c_p\|_r \|n_{p-q}\|_\infty \sum_{p' \leq q} \lambda_{p'}^{s+1} \|n_{p'}\|_2 \lambda_{q-p}^{s+\frac{1}{2}}$$

$$\leq C_0 \sum_{q \geq -1} \lambda_{q-1}^{2s+2} \|n_q\|_2 \sum_{p' \leq q} \lambda_{p'}^{s+1} \|n_{p'}\|_2 \lambda_{q-p}^{s+\frac{1}{2}}.$$  

Following Young’s and Jensen’s inequalities, we obtain for $s + 1 < \frac{3}{r}$

$$|V_{I21}| \leq C_0 \sum_{q \geq -1} \lambda_{q-1}^{2s+2} \|n_q\|_2^2 + C_0 \sum_{q \geq -1} \left( \sum_{p' \leq q} \lambda_{p'}^{s+1} \|n_{p'}\|_2 \lambda_{q-p}^{s+\frac{1}{2}} \right)^2$$

$$\leq C_0 \sum_{q \geq -1} \lambda_{q-1}^{2s+2} \|n_q\|_2^2 + C_0 \sum_{q \geq -1} \sum_{p' \leq q} \lambda_{p'}^{2s+2} \|n_{p'}\|_2 \lambda_{q-p}^{s+\frac{1}{2}}$$

$$\leq C_0 \sum_{q \geq -1} \lambda_{q-1}^{2s+2} \|n_q\|_2^2.$$  

The first step of dealing with $V_{I2}$ is to split it by $Q_c$ as well,

$$V_{I2} = - \sum_{q \geq -1} \sum_{|q-p| \leq 2, p \leq Q_c+2} \lambda_{q}^{2s} \int_{\mathbb{R}^3} \Delta_q(n_p \nabla c_{p-2}) \nabla n_q \, dx$$

$$- \sum_{q \geq -1} \sum_{|q-p| \leq 2, p > Q_c+2} \lambda_{q}^{2s} \int_{\mathbb{R}^3} \Delta_q(n_p \nabla c_{Q_c}) \nabla n_q \, dx$$

$$- \sum_{q \geq -1} \sum_{|q-p| \leq 2, p > Q_c+2} \lambda_{q}^{2s} \int_{\mathbb{R}^3} \Delta_q(n_p \nabla c_{Q_c-p-2}) \nabla n_q \, dx$$

$$=: V_{I21} + V_{I22} + V_{I23}.$$  

Using Hölder’s and Young’s inequalities, and the fact that $\lambda_{q-p} \sim C$ as $p, q$ are close to each other, we have

$$|V_{I21}| \leq \sum_{q \geq -1} \lambda_{q-1}^{2s+1} \|n_q\|_2 \sum_{|q-p| \leq 2, p \leq Q_c+2} \|n_p\|_2 \|\nabla c_{p-2}\|_\infty$$
\[
\begin{align*}
&\leq \sum_{q \geq -1} \lambda_q^{s+1} \|n_q\|_2 \sum_{|q-p| \leq 2, p \leq Q_{-2}+2} \lambda_p \|n_p\|_2 \|\nabla c_{p-2}\|_{\infty} \lambda_{q-p}^s \\
&\leq \frac{1}{32} \sum_{q \geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 + C f(t) \sum_{q \geq -1} \lambda_q^{2s} \|n_q\|_2^2.
\end{align*}
\]

We skip the computation for \(VI_{22}\) which can be estimated in a likely way as \(VI_{21}\). We proceed to estimate \(VI_{23}\), provided \(r > 3\).

\[
|VI_{23}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2, p > Q_{-2}+2} \lambda_q^{2s} \|n_q\|_2 \|\nabla c_{(Q_{-2}-2)}\|_r \|\nabla n_q\|_{\frac{r}{r-2}} \\
\lesssim \sum_{q \geq -1} \lambda_q^{2s+1+\frac{2}{r}} \|n_q\|_2^2 \sum_{Q_{c} < p' \leq q} \lambda_{p'} \|c_{p'}\|_r \\
\lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+1+\frac{2}{r}} \|n_q\|_2^2 \sum_{Q_{c} < p' \leq q} \lambda_{p'}^{-\frac{2}{r}} \\
\lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} \|n_q\|_2^2 \sum_{Q_{c} < p' \leq q} \lambda_{p'}^{-\frac{2}{r}} \\
\lesssim C_0 \sum_{q \geq -1} \lambda_q^{2s+2} \|n_q\|_2^2.
\]

Finally, we notice that \(VI_3\) can be handled in an analogous way of \(VI_1\); thus the details of computation are omitted. It completes the proof of the lemma. \(\square\)

Summing inequalities in Lemma 3.1- Lemma 3.5 and (12)-(15) produces the following Grönwall type of inequality,

\[
\frac{d}{dt} \sum_{q \geq -1} (\lambda_q^{2s} \|n_q\|_2^2 + \lambda_q^{2s+2} \|c_q\|_2^2 + \lambda_q^{2s+2} \|u_q\|_2^2) \\
\leq (-2 + CC_0) \sum_{q \geq -1} (\lambda_q^{2s} \|n_q\|_2^2 + \lambda_q^{2s+4} \|c_q\|_2^2 + \lambda_q^{2s+4} \|u_q\|_2^2) \\
+ CQ_u f(t) \sum_{q \geq -1} (\lambda_q^{2s} \|n_q\|_2^2 + \lambda_q^{2s+2} \|c_q\|_2^2 + \lambda_q^{2s+2} \|u_q\|_2^2).
\]

(Note that \(Q_u\) appears due to Lemma 3.1, Lemma 3.2 and Lemma 3.3.) Thus, \(C_0\) can be chosen small enough such that \(CC_0 < \frac{1}{4}\). On the other hand, combining the definition of \(\Lambda_u(t)\) and Bernstein’s inequality, one can deduce

\[
1 \leq C_0^{-1} \Lambda_u^{-1} \|uQ_u\|_{\infty} \leq C_0^{-1} \Lambda_u^{-\frac{1}{2}} \|uQ_u\|_2 \leq C_0^{-1} \Lambda_u^{-\frac{s}{2}} \|u\|_{H^{s+1}},
\]

which implies that for \(s > -\frac{1}{2}\),

\[
Q_u = \log \Lambda_u \lesssim 1 + \log \|u\|_{H^{s+1}}.
\]

Hence the energy inequality becomes

\[
\frac{d}{dt} \sum_{q \geq -1} (\lambda_q^{2s} \|n_q\|_2^2 + \lambda_q^{2s+2} \|c_q\|_2^2 + \lambda_q^{2s+2} \|u_q\|_2^2) \\
\leq - \sum_{q \geq -1} (\lambda_q^{2s+2} \|n_q\|_2^2 + \lambda_q^{2s+4} \|c_q\|_2^2 + \lambda_q^{2s+4} \|u_q\|_2^2) \\
+ C f(t) (1 + \log \|u\|_{H^{s+1}}) \sum_{q \geq -1} (\lambda_q^{2s} \|n_q\|_2^2 + \lambda_q^{2s+2} \|c_q\|_2^2 + \lambda_q^{2s+2} \|u_q\|_2^2).
\]
By the hypothesis (7) and Grönwall’s inequality, we can conclude that
\[ n \in L^\infty(0, T; H^3) \cap L^2(0, T; H^{s+1}), \]
\[ c \in L^\infty(0, T; H^{s+1}) \cap L^2(0, T; H^{s+2}), \]
\[ u \in L^\infty(0, T; H^{s+1}) \cap L^2(0, T; H^{s+2}). \]

We consider a particular case \( s = -\varepsilon \) for small enough \( \varepsilon > 0 \). We realize that our solution is regular via bootstrapping arguments. In fact, we have
\[ n \in L^\infty(0, T; H^{-\varepsilon}), \quad u \in L^\infty(0, T; H^{1-\varepsilon}), \quad c \in L^\infty(0, T; H^{1-\varepsilon}) \cap L^\infty(0, T; L^\infty). \]

By scaling, it is known that \( H^{-\frac{1}{2}}, H^{\frac{1}{2}}, \) and \( H^\frac{3}{2} \) are critical for \( n, u \) and \( c \), respectively. For small enough \( \varepsilon > 0 \), \( H^{-\varepsilon} \) is subcritical for \( n \); so is \( H^{1-\varepsilon} \) for \( u \). Thus, it suffices to bootstrap the equation of \( c \) to obtain higher regularity for \( c \). We recall
\[ c_t - \Delta c = -u \cdot \nabla c - nc. \]

Since \( u \in L^\infty(0, T; H^{1-\varepsilon}) \) and \( \nabla c \in L^2(0, T; H^{1-\varepsilon}) \), by Sobolev embedding \( H^{1-\varepsilon} \hookrightarrow L^{\frac{3}{1-\varepsilon}} \), we have \( u \cdot \nabla c \in L^2(0, T; L^{\frac{3}{1-\varepsilon}}) \). Similarly, the fact of \( c \in L^\infty(0, T; H^{1-\varepsilon}) \) and \( n \in L^2(0, T; H^{1-\varepsilon}) \) implies \( nc \in L^2(0, T; L^{\frac{3}{1-\varepsilon}}) \). Then the standard maximal regularity theory of heat equation yields
\[ c \in H^1(0, T; L^{\frac{3}{1-\varepsilon}}) \cap L^2(0, T; W^2, L^{\frac{3}{1-\varepsilon}}). \]

We need to bootstrap one more time. Now we have \( \nabla c \in L^2(0, T; W^{1, \frac{3}{1-\varepsilon}}) \) which, along with \( u \in L^\infty(0, T; H^{1-\varepsilon}) \), implies \( u \cdot \nabla c \in L^2(0, T; L^{\frac{3}{1-\varepsilon}}) \). On the other hand, we have \( nc \in L^2(0, T; L^{\frac{3}{1-\varepsilon}}) \) from the estimate \( n \in L^2(0, T; H^{1-\varepsilon}) \) and the maximal principle \( c \in L^\infty(0, T; L^{\infty}) \). Again, the maximal regularity theory of heat equation produces that
\[ c \in H^1(0, T; L^{\frac{3}{1-\varepsilon}}) \cap L^2(0, T; W^2, L^{\frac{3}{1-\varepsilon}}). \]

As a consequence of the mixed derivative theorem (see [28]), we have
\[ c \in W^{1-\theta, 2}(0, T; W^{2\theta, \frac{3}{1-\varepsilon}}) \]
for any \( \theta \in [0, 1] \). In fact, if we take \( \theta = (\frac{16\varepsilon}{1-\varepsilon}, \frac{1}{2}) \), Sobolev embedding theorem shows that
\[ c \in W^{1-\theta, 2}(0, T; W^{2\theta, \frac{3}{1-\varepsilon}}) \hookrightarrow L^\infty(0, T; H^{\frac{3}{2}+\varepsilon_0}) \]
for an small enough constant \( \varepsilon_0 > 0 \). Notice that \( H^{\frac{3}{2}} \) is critical for \( c \). Thus we can stop the bootstrapping for \( c \) equation. Regarding the density function \( n \), although the obtained estimates are in subcritical space already, we would like to further improve the estimates to reach spaces with even higher regularity (i.e. Sobolev spaces with positive smoothness index). Indeed, we look at the \( n \) equation again,
\[ n_t - \Delta n = -u \cdot \nabla n - \nabla \cdot (n \nabla c). \]

Due to the fact \( \nabla c \in L^\infty(0, T; H^{\frac{3}{2}+\varepsilon_0}) \) and \( n \in L^2(0, T; H^{1-\varepsilon}) \), Sobolev embedding theorem yields \( n \nabla c \in L^3(0, T; L^2) \) and hence \( \nabla \cdot (n \nabla c) \in L^2(0, T; H^{-1}) \). While \( u \in L^\infty(0, T; H^{1-\varepsilon}) \) and \( n \in L^2(0, T; H^{1-\varepsilon}) \) together imply \( un \in L^2(0, T; L^{\frac{3}{1-\varepsilon}}) \) which is embedded in \( L^2(0, T; L^2) \). Thus \( u \cdot \nabla n = \nabla \cdot (un) \in L^2(0, T; H^{-1}) \). Applying Lemma 2.4 with \( \alpha = -1 \) to the \( n \) equation, we claim
\[ n \in H^1(0, T; H^{-1}) \cap L^2(0, T; H^1). \]

Summarizing the analysis above gives us
\[ n \in L^\infty(0, T; H^{-\varepsilon}) \cap L^2(0, T; H^3) \cap H^1(0, T; H^{-1}), \]
\[ u \in L^\infty(0, T; H^{1-\varepsilon}) \cap L^2(0, T; H^{2-\varepsilon}), \]
\[ c \in L^\infty(0, T; H^{3/2+\varepsilon_0}) \cap L^2(0, T; W^{2, 6+6\varepsilon}). \]

Since each of the three functions \( n, u \) and \( c \) is in higher regularity space than its critical Sobolev space, further bootstrapping procedures for parabolic equations and standard argument of extending regularity can be applied to infer that the solution \((n, u, c)\) is regular up to time \( T \).

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**REFERENCES**

[1] H. Bahouri, J. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehrender Mathematischen Wissenschaften, Vol. 343, Springer, Heidelberg, 2011.

[2] J. Beale, T. Kato and A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Commun. Math. Phys.*, 94 (1984), 61–66.

[3] M. Chae, K. Kang and J. Lee, Existence of smooth solutions to coupled chemotaxis-fluid equations, *Discrete Contin. Dyn. Syst.*, 33 (2013), 2271–2297.

[4] M. Chae, K. Kang and J. Lee, Global existence and temporal decay in Keller-Segel models coupled to fluid equations, *Commun. Partial Differ. Equ.*, 39 (2014), 1205–1235.

[5] A. Cheskidov and M. Dai, Determining modes for the surface quasi-geostrophic equation, *Physica D*, 376/377 (2018), 204–215.

[6] A. Cheskidov, M. Dai, and L. Kavlie, Determining modes for the 3D Navier-Stokes equations, *Physica D*, 374/375 (2018), 1–9.

[7] A. Cheskidov and M. Dai, Regularity criteria for the 3D Navier-Stokes and MHD equations, *Proc. Edinb. Math. Soc.*, (2019), To appear.

[8] A. Cheskidov and R. Shvydkoy, A unified approach to regularity problems for the 3D Navier-Stokes and Euler equations: the use of Kolmogorov’s dissipation range, *J. Math. Fluid Mech.*, 16 (2014), 263–273.

[9] M. Dai, Regularity problem for the nematic LCD system with Q-tensor in \( \mathbb{R}^3 \), *SIAM J. Math. Anal.*, 49 (2017), 5007–5030.

[10] C. Dombrowski, L. Cisneros, S. Chatkaew, R. Goldstein and J. Kessler, Self-concentration and large-scale coherence in bacterial dynamics, *Phys. Rev. Lett.*, 93.098103 (2004), 1–4.

[11] R. Duan, A. Lorz and P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, *Commun. Partial Differ. Equ.*, 35 (2010), 1635–1673.

[12] L. Esquivel, G. A. Serégin and V. Sverak, \( L^3, \infty \)-solutions of Navier-Stokes equations and backward uniqueness, *Uspekhi Mat. Nauk*, 58 (2003), 3–44.

[13] L. Grafakos, *Modern Fourier Analysis*, Graduate Texts in Mathematics, Vol. 250, 2nd edition, Springer, New York, 2009.

[14] H. He and Q. Zhang, Global existence of weak solutions for the 3D chemotaxis-Navier-Stokes equations, *Nonlinear Anal. Real World Appl.*, 35 (2017), 336–349.

[15] T. Hillen and K. Painter, A user’s guide to PDE models for chemotaxis, *J. Math. Biol.*, 58 (2009), 183–217.

[16] J. Jiang, H. Wu and S. Zheng, Global existence and asymptotic behavior of solutions to a chemotaxis-fluid system on general bounded domains, *Asymptotic Anal.*, 92 (2015), 249–258.

[17] J. Jiang, H. Wu and S. Zheng, Blow-up for a three dimensional Keller-Segel model with consumption of chemoattractant, *J. Differ. Equ.*, 264 (2018), 5432–5464.

[18] E. Keller and L. Segel, Travelling bands of chemotactic bacteria: a theoretical analysis, *J. Theor. Biol.*, 30 (1971), 235–248.

[19] H. Kozono, T. Ogawa, and Y. Tanibuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semilinear evolution equations, *Math. Z.*, 242 (2002), 251–278.

[20] N. Krylov, *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, Graduate Studies in Mathematics, Vol. 96, American Mathematical Society, Providence, RI, 2008.
[21] O. A. Ladyzhenskaya, Uniqueness and smoothness of generalized solutions to the Navier-Stokes equations, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 5 (1967), 169–185; English transl., Sem. Math. V. A. Steklov Math. Inst. Leningrad, 5 (1969), 60–66.

[22] O. Ladyzhenskaya, V. A. Solonnikov and N. Ural’tseva, Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, RI, 1968.

[23] P. G. Lemarié-Rieusset, Recent Developments in the Navier-Stokes Problem, Chapman and Hall/CRC Research Notes in Mathematics, Vol. 431, Chapman and Hall/CRC, Boca Raton, FL, 2002.

[24] J. Liu and A. Lorz, A coupled chemotaxis fluid model: global existence, Ann. Inst. Henri Poincare - Anal. Non Lineaire, 28 (2011), 643–652.

[25] A. Lorz, Coupled chemotaxis fluid model, Math. Mod. Meth. Appl. Sci., 20 (2010), 987–1004.

[26] F. Planchon, An extension of the Beale-Kato-Majda criterion for the Euler Equations, Commun. Math. Phys., 232 (2003), 319–326.

[27] G. Prodi, Teorema di unicita per le equazioni di Navier-Stokes, Ann. Mat. Pura Appl., 48 (1959), 173–182.

[28] J. Prüss and G. Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations, Monographs in Mathematics, Birkhäuser, Basel, 2016.

[29] G. Rosen, Steady-state distribution of bacteria chemotactic towards oxygen, Bull. Math. Biol., 40 (1978), 641–674.

[30] J. Serrin, The initial value problem for the Navier-Stokes equations, in Nonlinear Problems (Proc. Sympos., Madison, Wis.), Univ. of Wisconsin Press, Madison, Wis. (1963), 69–98.

[31] I. Tuval, L. Cisneros, C. Dombrowski, C. Wolgemuth, J. Kessler and R. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. U. S. A., 102 (2005), 2277–2282.

[32] M. Winkler, Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, Commun. Partial Differ. Equ., 37 (2012), 319–351.

[33] M. Winkler, Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity, Calc. Var. Partial Differ. Equ., 54 (2015), 3789–3828.

[34] M. Winkler, Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system, Ann. Inst. Henri Poincare - Anal. Non Lineaire, 33 (2016), 1329–1352.

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