Quantum stabilization of 1/3-magnetization plateau in Cs$_2$CuBr$_4$

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We consider the phase diagram of a spatially anisotropic 2D triangular antiferromagnet in a magnetic field. Classically, the ground state is umbrella-like for all fields, but we show that the quantum phase diagram is much richer and contains a 1/3 magnetization plateau, two commensurate planar states, two incommensurate chiral umbrella phases, and, possibly, a planar state separating the two chiral phases. Our analysis sheds light on several recent experimental findings for the spin-1/2 system Cs$_2$CuBr$_4$.

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Introduction. A defining characteristic of frustrated quantum magnets is the appearance of numerous competing orders. This competition dramatically enhances quantum fluctuations, generating highly non-classical behavior as exemplified by, e.g., Cs$_2$CuCl$_4$ and Cs$_2$CuBr$_4$. These materials comprise quasi-2D spin-1/2 triangular antiferromagnets with spatially anisotropic exchange [see Fig. 1(a)] and weak Dzyaloshinskii-Moriya (DM) coupling. Absent the latter, both systems classically should realize a zero-field coplanar spiral, which evolves into non-coplanar “umbrella” states in a field as in Fig. 1(b) with smoothly increasing magnetization up to saturation [1]. Experiments, however, reveal decidedly different, non-classical behavior: in fields directed along the triangular layers Cs$_2$CuCl$_4$ realizes commensurate coplanar order in a wide field range with smoothly increasing magnetization $\theta$, while Cs$_2$CuBr$_4$ exhibits collinear “up-up-down” (UUD) order shown in Fig. 1(c) over a finite field interval, yielding a 1/3 magnetization plateau $\theta$. Neither observation is accounted for within a classical analysis [1]. NMR [8, 9] and neutron scattering [2] additionally find planar states adjacent to the UUD phase, with neutron and thermodynamic measurements [7] indicating that the transitions are first order. Additional experiments [2, 11] on Cs$_2$CuBr$_4$ also suggest the presence of a narrow 2/3-plateau and additional intervening collinear phases near particular fields.

While the existence of the UUD phase is well-established for the isotropic triangular antiferromagnet, much less is known about the stability of the plateau and the proximate quantum phases in the anisotropic case. The challenge here is illuminated by first observing that in the isotropic limit, the UUD state appears due to an “accidental” classical degeneracy between umbrella and planar states shown in Figs. 1(b) and (c), which quantum fluctuations lift in favor of the latter [11]. When $J' \neq J$, however, this degeneracy is lifted already at the classical level, but in favor of umbrella states for all fields. The planar phases can then only emerge if quantum effects overshadow those of spatial anisotropy. This in turn implies that the standard spin-wave expansion is not applicable since planar phases cease to be classical ground states.

Addressing the quantum phase diagram for the anisotropic system is particularly the phases near 1/3 magnetization, we introduce a modified approach here which is controlled by the smallness of 1/$S$ and spatial anisotropy, and yields results which are non-analytic in both parameters.

Figure 2 summarizes our results. We find that the physics is controlled by the parameter $\delta = (40/3)S(J - J')^2/J^2$. For $\delta < 1$ the stability of the UUD phase is, counter-intuitively, unaffected by anisotropy. Moreover, the spin order remains coplanar and commensurate at fields both below and above the UUD phase; incommensurate phases appear only at small

![Figure 1](image1.png)

**FIG. 1:** (a) Anisotropic triangular lattice with horizontal exchange $J$ and diagonal exchange $J'$. (b) Umbrella and (c) planar phases comprise competing classical ground states of the isotropic nearest-neighbor model.

![Figure 2](image2.png)

**FIG. 2:** Proposed phase diagram for the anisotropic nearest-neighbor Heisenberg model near 1/3 magnetization (full field range not shown). The horizontal axis is $\delta = (40/3)S(J - J')^2/J^2$. Planar states shown are commensurate, though they are expected to be incommensurate at small and large fields. The shaded area is where the UUD and adjacent phases are metastable, the energy being minimized by umbrella states of Fig. 1(b).
and high fields. For $1 < \delta < 4$, the UUD phase persists, but at the boundaries it becomes unstable towards non-coplanar, incommensurate phases which can be regarded as distorted umbrellas (this happens for $\delta > 1$ at the lower boundary and for $\delta > 3$ at the upper boundary). These two phases emerge as finite-$k$ instabilities of the two low-energy spin-wave branches of the UUD phase, and both have a non-zero Ising order parameter associated with chirality $K_{ABC} = \xi \cdot (S_A \times S_B + S_B \times S_C + S_C \times S_A)$ for each plaquette. For $\delta > 4$, the UUD state ceases to exist, and there is no magnetization plateau. Since the chiralities of the low- and high-field distorted umbrella phases are uncorrelated, the two must be separated by (at least) a first order transition in this region. At still larger $\delta$ (stronger anisotropy), 1D physics becomes important, and the system cannot be described by our semi-classical theory.

As a further complication, for $\delta > 2$ the energy of the UUD state becomes larger than that of the classical, undistorted umbrella, i.e., for $2 < \delta < 4$, the UUD state and neighboring distorted umbrellas are metastable. We represent this by shading the region $\delta > 2$ in Fig. 2. We expect that these metastable phases may be probed in pulsed field experiments [12].

**Model and UUD state in the Anisotropic System.** We consider a simple Heisenberg model with

$$H = \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} J_{r r'} \mathbf{S}_r \cdot \mathbf{S}_{r'} - h S \sum_r S_z^r,$$

(1)

where $\mathbf{S}_r$ are spin-$S$ operators, the exchanges $J_{r r'}$ are as shown in Fig. 1(a), and $h$ is the (scaled) magnetic field. The saturation field is $h_{\text{sat}} = (2J + J')^2 / J$. Since we wish to treat quantum effects and the effects of anisotropy on equal footing, we will organize our analysis by assuming that both $(J - J') / J$ and $1 / S$ are small.

With $J = J'$, the two competing classically degenerate states are commensurate (three-sublattice) umbrella and planar states shown in Fig. 1. Quantum fluctuations favor planarity, and spin re-arrangement in a field occurs as in Fig. 1(c). This process includes an intermediate UUD phase, which is classically stable only at $h_{\text{sat}} / 3$, but quantum fluctuations extend its stability to a finite field interval, $h_{c1} \leq h \leq h_{c2}$ [11], resulting in a 1/3-magnetization plateau. This is not surprising given that quantum fluctuations generally favor collinear states [13, 14]. To leading order in $1 / S$

$$h_{c1} = 3J - \frac{0.50J}{2S}, \quad h_{c2} = 3J + 1.3J / 2S,$$

(2)

which for $S = 1 / 2$ yields a plateau in a range $\Delta h^0 = h_{c2}^0 - h_{c1}^0 = 1.8J / (2S)$, in good agreement with exact diagonalization [15]. Inside this range, there are two low-energy spin-wave modes with gaps $\propto |h_{c1,2}^0 - h|$ at $k = 0$.

When $J \neq J'$, the umbrella state becomes incommensurate, and classically has lower energy than the planar phase for all fields. The naive expectation, then, is that the UUD phase must immediately shrink and disappear as $|J - J'|$ increases. We show, however, that the actual situation is much more complex, with new phases emerging when $J \neq J'$.

To study the stability of the classically unfavorable UUD state, we explore a modified large-$S$ approach to Eq. (1). First, we introduce a three-sublattice representation where spins on the A and B sublattices point up while those on the C sublattice point down, and use the standard Holstein-Primakoff mapping. The usual linear spin-wave Hamiltonian obtained in this fashion is not an appropriate starting point due to the classical instability of harmonic spin waves at $\delta \neq 0$. However, the interacting spin-wave Hamiltonian must support a stable UUD plateau over a finite anisotropy range, as exact diagonalization finds [18]. Therefore, we extend the linear spin-wave Hamiltonian of the UUD state to include the leading $1 / S$ self-energy corrections obtained by decoupling the quartic interactions using correlations from the isotropic system.

Diagonalizing this Hamiltonian, we obtain three spin-wave branches. One branch describes a precession of the total magnetization and has a high energy $\sim h_0 \equiv J + 2J'$. The energies of the other two branches are small near $k = 0$:

$$H_{\text{uud}} = S \sum_k [\omega_1 d_1^\dagger k d_1 k + \omega_2 d_2^\dagger k d_2 k],$$

(3)

where the leading expressions at small $k$ are

$$\omega_{1,2}(k) = \pm \left( h - h_0 - \frac{1}{5S} J - \frac{3}{4} J k^2 \right) + \frac{3JZ}{20S},$$

(4)

with $Z = \sqrt{9 + 10S(6k_x^2 + 10S k_y^2 - 3k_z^2(\delta - 2))]$. The critical fields obtained from these energies are

$$h_{c1,2} = h_{c1,2}^0 + 2(J' - J) \pm \frac{3J}{4} \min \left( \pm k_x^2 + \frac{Z - 3}{5S} \right),$$

(5)

where $h_{c1,2}^0$ is given by (2), and the minimum is taken with respect to $k_x$ ($k_y = 0$ at the minimum for all $\delta$). The UUD phase is stable for $h_{c1} < h < h_{c2}$.

These results, which are non-analytic in $1 / S$ and $J - J'$, encode the physics governing the local stability of the UUD state in the anisotropic system. One can verify by sending $S \rightarrow \infty$ above that the UUD state is indeed unstable for any non-zero anisotropy in the classical limit, due to an instability at finite $k_x$. Surprisingly, in the quantum system a finite amount of anisotropy is required to begin destabilizing the plateau. Specifically, for $\delta < 1$ both modes are minimized at $k = 0$, so it follows from Eq. (5) that the plateau width $\Delta h$ is unchanged from the isotropic system. The effect of anisotropy in this regime is only to shift the plateau’s location and soften the dispersion around $k = 0$.

For $\delta > 1$, the minimum of $\omega_1$ shifts to $k_{1,\pm} = (\pm k_1,0)$, where $k_1^2 = [3\delta - 6 + \sqrt{3}\delta(4 - \delta)] / (20S)$; the lower critical field then moves upward, reducing the width of the UUD plateau (see Fig. 2). Similarly, for $\delta > 3$ the minimum of $\omega_2$ shifts to $k_{2,\pm} = (\pm k_2,0)$, with $k_2^2 = [3\delta - 6 - \sqrt{3}\delta(4 - \delta)] / (20S)$. At this point the upper critical field moves to a smaller value, further reducing the UUD region. The plateau ceases to be locally stable at $\delta = 4$, when both spin-waves become gapless at $k_1^2 = k_2^2 = k_m^2 = 3 / (10S)$. 


Let us now explore the phases that emerge immediately away from the UUD state. At \( h_{c1} \) and \( h_{c2} \), magnons Bose condense, and one must determine the energetically favorable combination of operators \( d_{1,2} \) that condenses, and what this implies for the spin components \( \langle S^{x,y} \rangle \). For \( \delta < 1 \) this is straightforward: the minima of \( \omega_1(k) \) occur at \( k = 0 \), and the order parameters are simply \( \psi_{1,2} \propto \langle d_{1,2,0} \rangle \). One can easily verify that condensation of \( \psi_1(\psi_2) \) at \( h = h_{c1} (h_{c2}) \) leads to the \textit{commensurate} coplanar spin configurations displayed in Fig. 1(c). The prediction of commensurate order adjacent to the UUD state over a range of anisotropy is rather nontrivial, and could be tested in exact diagonalization studies.

The situation is subtler at the lowest critical field when \( \delta > 1 \), since here \( \omega_1(k) \) possesses two inequivalent minima at \( k_{\pm} \). There are then two order parameters, \( \psi_{\pm} = \sqrt{3/NS}(d_{1,\pm}k) \) (\( N \) is the number of spins), whose energy derived from the interacting spin wave Hamiltonian is

\[
\frac{2E}{JNS^2} = r(|\psi_+|^2 + |\psi_-|^2) + (|\psi_+|^2 + |\psi_-|^2) + u |\psi_+|^2 |\psi_-|^2.
\] (6)

Here \( r \propto h - h_{c1} \) and \( u = 2 \cos^2 2 \phi_{k_1} \), where

\[
\tanh(2\phi_{k_1}) = \frac{6(J - J')k_1}{\omega_1(k_1) + \omega_2(k_1)} = \sqrt{\frac{3\delta}{1 + 30S^2k_1}}.
\] (7)

Since \( u > 0 \), below the transition interactions favor \( \psi_+ \neq 0, \psi_- = 0 \) or vice versa. Choosing along the former, the spin configuration can be written as

\[
\langle S^\pm \rangle = -S\psi_+(\cos \phi_{k_1} + i \sin \phi_{k_1}) e^{-ik_1x},
\] (5a)

\[
\langle S^\mp \rangle = S\psi_+(\cos \phi_{k_1} + i \sin \phi_{k_1}) e^{-ik_1x},
\] (5b)

\[
\langle S^x \rangle = 2iS\psi_+(\cos \phi_{k_1} + i \sin \phi_{k_1}) e^{-ik_1x}. \]

This corresponds to \textit{non-coplanar}, incommensurate order that can be described as a distorted umbrella. Non-coplanarity of this state leads to a finite chirality \( K^{(1)} \), the sign of which is determined by that of the condensate momentum via \( K^{(1)}_{ABC} = \pm 3S^2|\psi_+|^2 \sinh 2\phi_{k_1} \).

The same consideration holds at the upper critical field when \( \delta > 3 \): \( \omega_2(k) \) again has two inequivalent minima at \( k_{2,\pm} \), and the energy has the same form as in (5), with the order parameter \( \psi_\pm = \sqrt{3/NS}(d_{2,\pm}k_\pm) \). The spin configuration above \( h_{c2} \) is another distorted umbrella with chirality \( K^{(2)}_{ABC} = \pm 3S^2|\psi_\pm|^2 \sinh 2\phi_{k_2} \).

At \( \delta = 4 \), the UUD plateau shrinks to a point at \( h_c = h_0 + 17J/(40S) \), and becomes unstable at larger \( \delta \). How the two distorted umbrellas merge in this regime presents an interesting issue. Since these states arise upon condensation of different spin-wave modes at \( h_{c1,2} \), their chiralities are uncorrelated. The two phases then cannot gradually transform into each other and must be separated either by a first order transition, or by an intermediate phase with no chirality.

To gain insight here we study the instability of the UUD phase at \( \delta = 4 \), \( h = h_c \). At this point, the two spin-wave branches become gapless at the same \( \pm k_{1,2} \), and the coherence factors \( \sinh \phi_{k_{1,2}} \) and \( \cosh \phi_{k_{1,2}} \) diverge as \( 1/\sqrt{4 - \delta} \), so that \( \tanh 2\phi_{k_{1,2}} \to 1 \). There are more choices for the order parameter at \( h = h_c \), compared to either \( h_{c1} \) or \( h_{c2} \) as both \( \psi_\pm \) and \( \psi_\pm \) condense at \( h_c \). The full expression for the ground state energy at \( \delta = 4 \), \( h = h_c \) to fourth order in \( \psi \) and \( \bar{\psi} \), and to leading order in \( 1/S \) is

\[
\frac{2E}{JNS^2} = \left( |\psi_+|^2 + |\psi_-|^2 - |\psi_+|^2 - |\psi_-|^2 \right)^2
+ 2|\psi_+|^2 |\psi_-|^2 + 2|\psi_+|^2 |\psi_-|^2.
\] (8)

subject to constraint \( |\psi_+|^2 - |\psi_-|^2 = i(|\psi_+|^2 + |\psi_-|^2) \) which eliminates infinitely large terms from the energy. Choosing just one of the four order parameters non-zero, we obtain the same distorted umbrella states as before, with \( E \propto |\psi|^4 \) and a finite chirality. However, we see from (8) that there is a better choice—taking \( |\psi_+|^2 = |\psi_-|^2 \) at the end-point of the UUD phase is unconstrained implying that \( |\psi_+|^2 = |\psi_-|^2 \) jumps to a finite value right at \( \delta = 4 \). The chirality \( K_{ABC} \) of such a state depends on the relative phase \( \theta \) of the two order parameters as \( K_{ABC} \propto \sinh \theta \) and vanishes when \( \theta = -\arcsin(\tanh 2\phi_{k_{1,2}}) \to -\pi/2 \). We verified that this particular \( \theta \) is the only choice at which the transverse magnetization given by \( \langle S^{x,y} \rangle = 0 \), \( \langle S^z \rangle = 0 \), \( \langle S^z \rangle \sim |\psi_+|^2 \sqrt{4 - \delta} \) does not diverge together with the coherence factors but rather remains zero at \( \delta = 4 \), even though \( |\psi_+|^2 \neq 0 \) there. It is tempting to speculate that such a zero-chirality state persists beyond \( \delta = 4 \) along a line in the \( h - \delta \) plane and continues to separate the two distorted umbrella phases (it cannot exist in a finite \( h \)-range since, unlike the UUD phase, it does not have two gapped low-energy modes). The spin structure along this line is either collinear, as at \( \delta = 4 \), or coplanar, with \( \langle S^{x,y} \rangle = 0 \), \( \langle S^z \rangle = 0 \); the difference can not be resolved within our formalism.

\textit{Energy considerations and the phase diagram.} So far we have analyzed the UUD phase’s local stability without addressing whether it globally minimizes the energy. There are three regimes where one can easily compare the umbrella and planar energies. First is the high field regime \( h \approx h_{sat} \). There, the umbrella state, which at arbitrary \( h \) is described by

\[
S_r = S\cos \theta \cos (Q \cdot r) \hat{x} + \sin (Q \cdot r) \hat{y} + \sin \theta \hat{z}
\] (9)

with \( Q = 2 \cos^{-1}(-J'/2J) \) and \( \sin \theta = h/h_{sat} \), wins for all \( J' \neq J \) simply because quantum effects vanish at \( h_{sat} \).

We have verified this explicitly by computing the analog of Eq. (5) at the saturation field to show that indeed interactions drive the system into the umbrella state for arbitrary \( J' \neq J \). As a result, the critical line which begins at \( \delta = 3 \), \( h = h_{c2} \) should end at \( \delta = 0, h = h_{sat} \).

The second regime occurs at small \( h \to 0 \). Here the lowest-energy planar configuration is \textit{incommensurate}, with the same \( Q \) as the umbrella state and

\[
S_r = S\cos (Q \cdot r + \varphi_r) \hat{x} + \sin (Q \cdot r + \varphi_r) \hat{y},
\] (10)

where \( \varphi_r = -(2h/u) \sin (Q \cdot r) + O(h^2) \) and \( u = h_{sat}[1 + (J - J')^2/J^2] \). At small \( h \), the energy difference between the
incommensurate umbrella and planar states of Eqs. (9) and (10) is \( \Delta E_{\text{h} \to 0} = (E_{\text{umb}} - E_{\text{pl}})/N S^2 = -(1/2)\hbar^2 \Delta \chi \), where \( \Delta \chi = \chi_{\text{umb}} - \chi_{\text{pl}} \) is the difference of susceptibilities. In the classical limit we find \( \chi_{\text{umb}} = 1/h_{\text{sat}}, \chi_{\text{pl}} = 1/\mu \), so that \( \Delta \chi = (J - J')^2/(9J^3) \) and the umbrella state has lower energy. The competition comes from quantum fluctuations: 1/S corrections to \( \chi_{\text{umb}} \) and \( \chi_{\text{pl}} \) are different already for \( J = J' \), and such that \( \Delta \chi_{\text{qu}} \approx -0.16/(18J^3) \) (Ref. (11)). Adding the two contributions, we find that \( \Delta E_{\text{h} \to 0} = [0.008h^2/(2JS)][(1.1 - \delta)], \) i.e., the incommensurate planar state has lower energy for \( \delta < 1.1 \). This implies that the commensurate planar state that we found immediately below \( h_{c1} \) should undergo either a second- or first-order transition into an incommensurate planar state at some \( h < h_{c1} \). We therefore expect the line separating planar and distorted umbrella states at low fields to depart at \( \delta = 1, h = h_{c1} \) and end up at \( \delta = 1.1, h = 0 \).

Finally, at \( h_{\text{sat}}/3 \) the energy difference between the umbrella and UUD phase is \( \Delta E_{1/3} = [0.067J/(2S)][(2.0 - \delta)], \) where the first and second terms, respectively, are the classical and quantum contributions; see Ref. (11). Consequently, the UUD phase and the neighboring distorted umbrella phases remain global minima only up to \( \delta = 2.0 \) and become metastable at larger \( \delta \). This suggests that for \( \delta > 2 \) the UUD state can be observed only via a transient magnetization plateau, similar to the situation in a kagomé system (12). Equilibrium measurements should reveal only umbrella-like states in that region of \( \delta \).

The resulting phase diagram near 1/3 magnetization is shown in Fig. 2. It contains an UUD phase; two commensurate planar states from Fig. 1(c); and two non-coplanar incommensurate distorted umbrella phases. The shaded region corresponds to the regime where the classical umbrella minimizes the energy globally. Additionally, incommensurate planar states are expected at small fields when \( \delta \leq 1 \), and near the saturation field for small \( \delta \). We also expect new phases at small \( J' / J \) (large \( \delta \)), where one-dimensional physics takes over the semi-classical analysis.

This phase diagram is in agreement with data for Cs\(_2\)CuBr\(_4\), where \( J' / J = 0.7 \) implies that \( \delta = 0.6 \) if we extrapolate to \( S = 1/2 \). For this \( \delta \), the UUD state is present, and the nearby phases are planar, in agreement with NMR (8, 9) and neutron (10) experiments. These experiments also observe that both transitions out of the UUD state are first order. Our calculations predict continuous transitions as a consequence of the U(1) spin symmetry exhibited by the Hamiltonian (1). However, when this U(1) symmetry is broken explicitly by spin-orbit coupling, cubic terms in the free energy are permissible, which generically render the transition first order. In particular, DM coupling of the form present in Cs\(_2\)CuBr\(_4\) breaks this symmetry when the field is directed along the triangular layers. In addition, a direct first order transition from UUD phase into the incommensurate planar phase is also a possibility, which should be investigated by numerical calculations similar to those in (17). For Cs\(_2\)CuCl\(_4\), the anisotropy is much higher (\( \delta \approx 2.9 \)), and the system very likely lies outside of the applicability region of our analysis, and should be approached from a 1D perspective (19). Still, even within our framework, \( \delta > 2 \) implies no UUD phase, and no plateau is seen in Cs\(_2\)CuCl\(_4\).

An intriguing question concerns the possible appearance of a 2/3-magnetization plateau at \( \delta < 1 \) and higher fields, as observed in Cs\(_2\)CuBr\(_4\) (5), which would correspond to (at least) a “5-up, 1-down” configuration. While such states are never ground states to order 1/S, we verified that their energy is reduced when \( J' \neq J \). We speculate that, due to a large degeneracy of 5-up, 1-down configurations, a 2/3-plateau may be entropically stabilized at finite temperature. Regarding this issue, the role of spin-phonon couplings should be seriously investigated (20).

Conclusions. Using a modified large-S approach, we studied the quantum phase diagram of an anisotropic triangular antiferromagnet, with particular emphasis on the classically unstable UUD state and proximate phases. Fig. 2 summarizes our findings. The UUD phase with 1/3 magnetization plateau survives a substantial range of anisotropy, and at its boundaries transforms either into commensurate planar phases, or into umbrella-like incommensurate chiral phases, depending on whether the spin-wave instabilities of the UUD phase are at zero or finite momenta. Our results explain a number of experimental findings for Cs\(_2\)CuBr\(_4\).

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References

[1] M. Y. Veillette, J. T. Chalker, and R. Coldea, Phys. Rev. B 71, 214426 (2005).
[2] Y. Tokiwa et al., Phys. Rev. B 73, 134414 (2006).
[3] M.Y. Veillette and J.T. Chalker, Phys. Rev. B 74, 052402 (2006).
[4] T. Ono et al., Phys. Rev. B 67, 104431 (2003).
[5] T. Ono et al., J. Phys.: Condens. Matter 16, S773 (2004).
[6] T. Ono et al., Prog. Theor. Phys. Suppl. 159, 217 (2005).
[7] H. Tsuji et al., Phys. Rev. B 76, 060406(R) (2007).
[8] Y. Fujii et al., Physica B 346-347, 45 (2004).
[9] Y. Fujii et al., J. Phys.: Condens. Matter 19, 145237 (2007).
[10] N. Fortune, S. Hannahs, Y. Yoshida, Y. Takano, T. Ono, and H. Tanaka, unpublished.
[11] A. V. Chubukov and D. I. Golosov, J. Phys.: Condens. Matter 3, 69 (1991). The lifting of the “accidental” degeneracy by classical, thermal fluctuations has been analyzed by H. Kawamura and S. Miyashita, J. Phys. Soc. Jpn. 54, 4530 (1985).
[12] Y. Narumi et al., Europhys. Lett. 65, 705 (2004).
[13] E.F. Shender, Sov. Phys. JETP 56, 178 (1982).
[14] C. L. Henley, Phys. Rev. Lett. 62, 2056 (1989).
[15] A. Honecker, J. Schulenburg, and J. Richter, J. Phys.: Condens. Matter 16, S749 (2004).
[16] T. Nikuni and S. Shiba, J. Phys. Soc. Jpn. 64, 3471 (1995).
[17] S. Yoshikawa et al., J. Phys. Soc. Jpn. 73, 1798 (2004).
[18] S. Miyahara, K. Ogino, and N. Furukawa, Physica B 378-380.
[19] O. A. Starykh and L. Balents, Phys. Rev. Lett. 98, 077205 (2007).

[20] F. Wang and A. Vishwanath, Phys. Rev. Lett. 100, 077201 (2008).