Role of topology on the work distribution function of a quenched Haldane model of graphene

Sourav Bhattacharjee, Utso Bhattacharya and Amit Dutta
Department of Physics, Indian Institute of Technology Kanpur-208016, India
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We investigate the effect of equilibrium topology on the statistics of non-equilibrium work performed during the subsequent unitary evolution, following a sudden quench of the Semenoff mass of the Haldane model. We show that the resulting work distribution function for quenches performed on the Haldane Hamiltonian with broken time reversal symmetry (TRS) exhibits richer universal characteristics as compared to those performed on the time-reversal symmetric massive graphene limit whose work distribution function we have also evaluated for comparison. Our results show that the work statistics of non-equilibrium work may therefore be used as an effective tool to detect the presence of equilibrium topology of a given system.

I. INTRODUCTION

Studying the probability distribution of work in a driven quantum system is an interesting area of recent research. We recall that in quantum mechanics, work (W) is not an observable, rather it acquires a stochastic behavior due to the inherent probabilistic nature of quantum measurement. Naturally, the object of interest therefore is no longer W itself but rather a distribution function P(W) which encodes its fluctuating behavior. Remarkably, it has been shown that the work distribution function attains a universal behavior following a gap in the small W(W → 0) limit with a power-law behavior with W and the associated exponent depends on the initial and final value of the quench parameter (with respect to the critical point) and the spatial dimensionality. This universal behavior has been probed extensively in free bosonic as well as free fermionic models for both global and local quenches.

To elaborate, let us assume that a closed d-dimensional quantum many body system is initially prepared in the ground state |ψ0⟩ of an initial Hamiltonian H0; a certain parameter of the Hamiltonian is then quenched at time t = 0 using some protocol following which the system is allowed to evolve unitarily. The work distribution function P(W) characterising the probability that W amount of work has been done in the system after a time τ is

\[
P(W) = \sum_n \delta(W - |E_n^0 - E_n^f|) |\langle \phi_n^f | \psi_\tau \rangle|^2 \tag{1}
\]

where |ψ_\tau⟩ is the evolved state of the system at time τ. |ϕ_n^f⟩ and E_n^0 denote the n-th instantaneous energy eigenstate and its eigen energy respectively while E_n^f is the (ground state) energy of |ψ_0⟩. If the quench is performed suddenly, the subsequent time evolution of |ψ_0⟩ is dictated by the final time independent Hamiltonian H_f with the final value of the quench parameter, i.e.

\[
|\psi_\tau⟩ = e^{-iH_\tau t} |\psi_0⟩. \text{ One immediately finds,}
\]

\[
|\langle \phi_n^f | \psi_\tau \rangle|^2 = |\langle \phi_n^f | e^{-iH_f \tau} |\psi_0⟩|^2 = |\langle \phi_n^f | \psi_0⟩|^2 \tag{2}
\]

where |\phi_n^f⟩ are the instantaneous energy eigenstates of H_f. It is now straightforward to show that

\[
P(W) = \int_{-\infty}^{\infty} e^{iW \tau} G(\tau) d\tau \tag{3}
\]

where G(τ) is the characteristic function of P(W) and is given as

\[
G(\tau) = e^{-i\Delta E_0 \tau} \langle \psi_0 | e^{i(E_f^0 - H_f)\tau} |\psi_0⟩. \tag{4}
\]

Here \(\Delta E_0 = E_f^0 - E_0^0\) is the difference in ground state energies of the final and initial Hamiltonians and hence is the minimum threshold of possible work. We also note in passing that upon rescaling \(E_f^0\) to zero, the inner product term in Eq. (4) reduces to the conventional Loschmidt overlap amplitude [22].

An analytic continuation to imaginary time \(\tau = -iS\) enables us to rewrite Eq. (4) in the following way [3]

\[
G(S) = e^{-S\Delta E_0} Z(S) \tag{5a}
\]

\[
Z(S) = \langle \psi_0 | (e^{i(E_f^0 - H_f)S}) |\psi_0⟩ \tag{5b}
\]

where Z(S), in accordance with the quantum to classical correspondence principle, can be interpreted as the partition function of a \((d+1)\)-dimensional classical system defined on a strip geometry of width S with boundary states |ψ_0⟩. The associated free energy (F) can be decoupled into three contributions as follows:

\[
F = - \log G(S) = L^d (f_b + 2f_s + f_c(S)) \tag{6}
\]

where \(f_b = \Delta E_0/L^d\) is the bulk free energy density, \(f_s\) is the surface free energy due to the two boundaries of the strip and hence is independent of its thickness S while \(f_c(S)\) is the contribution due to the Casimir interaction between the boundaries which decays to zero for large S [23].
Close to a critical point, the response of the system is characterized by a diverging correlation length \( \xi \), thus there is a slower non-exponential decay of the two point correlation functions of fluctuations of the order parameter. In such a scenario, the existence of the boundary states impose effective boundary conditions on the order parameter which leads to a Casimir like force between the boundaries. This results in contribution of an additional parameter which leads to a Casimir like force between the boundaries. This results in contribution of an additional part \( f_c(S) \) (Eq. (6)) to the free energy of the system, which assumes the scaling form

\[
F_c(S) = S^{-d}F(S/\xi); \quad (7)
\]

here, \( F(S/\xi) \) is a universal scaling function which is independent of microscopic details and only depends on the surface and bulk universality classes. This is the source of the emergence of universal behaviour of \( P(W) \) close to criticality, where the scaling function \( F(S/\xi) \) and hence \( f_c(S) \) can be asymptotically expanded for \( S/\xi \to 1 \). Therefore, the universality in the behavior of \( P(W) \) for small \( W \) can be extracted from the large \( S \) behavior \( f_c(S) \). For the rest of the paper, we will only focus on this low work regime of \( P(W) \).

Let us now briefly recapitulate some of the generic aspects of the universal behavior of \( P(W) \) valid for a wide class of free fermionic models. Especially, focusing on the 1-D transverse field Ising model with the transverse field close to its critical value \( g_c \), \( P(W) \) depends solely on the relative value of the initial field \( g_i \) and the final field \( g_f \) (after a sudden quench) with respect to \( g_c \). In other words, it depends on whether the quench is carried out within the same quantum phase \( (g_i, g_f \gtrless g_c) \), or across the quantum phases \( (g_i > g_c, g_f < g_c) \) or \( (g_i < g_c, g_f > g_c) \), or from (to) the critical point \( (g_i(f) = g_c) \). However, a few characteristics are common in all the cases; there is a delta function peak at the origin with a weight factor given by the ground state fidelity \( \langle \phi_0 | \psi_0 \rangle^2 \). This corresponds to the reversible work which is the difference of the initial and final ground state energies as discussed above. In addition, there also exists an edge at a lower cutoff of \( W \) below which \( P(W) \) is zero.

In this paper, we explore the effect of equilibrium topology on the non-equilibrium work statistics following a sudden quench of a parameter of the system Hamiltonian. This is relevant in the light of a growing number of recent studies which explore connections between equilibrium topology and dynamics, both in the context of periodic\(^{29-31}\) and quenches\(^{32-40}\) dynamics. We study the non-equilibrium dynamics of the paradigmatic Haldane model\(^{11}\) which is an integrable two dimensional model of spin-less electrons; the phase diagram of the model (Fig. 1a) hosts topological as well as trivial phases. This model is based on an infinite graphene like honeycomb lattice (Fig. 1b) with broken sub-lattice symmetry (SLS) and time-reversal symmetry (TRS) manifested in nearest neighbor (NN) and complex next-nearest neighbor (NNN) hoppings. The Hamiltonian of the Haldane model can be decomposed as a sum of Hamiltonians of decoupled two-level systems,

\[
H = \sum_{\vec{k}} H(\vec{k}) = \sum_{\vec{k}} \tilde{h}(\vec{k}) \cdot \vec{\sigma} + h_0(\vec{k}) I \quad (8)
\]

where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices, \( I \) is the \( (2 \times 2) \) identity matrix and

\[
h_z(\vec{k}) = -t \left( \cos (\vec{k} \cdot e_1) + \cos (\vec{k} \cdot e_2) + \cos (\vec{k} \cdot e_3) \right) \quad (9a)
\]
\[ h_y(\vec{k}) = -t \left( \sin (\vec{k} \cdot \vec{e}_1^1) + \sin (\vec{k} \cdot \vec{e}_2^1) + \sin (\vec{k} \cdot \vec{e}_3^1) \right) \]  
\[ h_z(\vec{k}) = M - 2t' \sin \phi \left( \sin (\vec{k} \cdot \vec{v}_1) + \sin (\vec{k} \cdot \vec{v}_2) + \sin (\vec{k} \cdot \vec{v}_3) \right) \]  
\[ h_0(\vec{k}) = -2t' \cos \phi \left( \cos (\vec{k} \cdot \vec{v}_1) + \cos (\vec{k} \cdot \vec{v}_2) + \cos (\vec{k} \cdot \vec{v}_3) \right). \]

Here, for a given lattice site, the vectors \{\vec{e}_i\} and \{\vec{v}_i\} (i = 1, 2, 3) are the locations of NN and NNN sites respectively. Further, t is the amplitude of NN hopping in the graphene honeycomb lattice, \(t'\) is the absolute part of the complex NNN hopping and \(\phi\) is its argument; \(M\), on the other hand, denotes the staggered on-site potential at the lattice sites, also known as the Semenoff mass.

II. CASIMIR FREE ENERGY

As discussed already, the universality in the behavior of \(P(W)\) is directly linked to large \(S\) behavior of \(f_c(S)\). To present the outline of the procedure to extract \(f_c(S)\) from the total free energy, we first substitute Eq. (5a) in Eq. (6) so that

\[ \log Z(S) = -L^2 (2f_s + f_c(S)). \]  

where we have set \(d = 2\) for the 2D Haldane model. The fact that the quasi-momentum modes are conserved and independent of each other, allows one to construct the initial state as

\[ |\psi_0\rangle = \bigotimes_{\vec{k}} |\psi_0(\vec{k})\rangle \]  

where \(|\psi_0(\vec{k})\rangle\) is the energy eigenstate of \(H_I(\vec{k})\) and the direct product is taken over the first Brillouin zone (BZ) of the lattice. This simplification, together with Eq. (9) immediately implies that Eq. (5b) can be rewritten as

\[ Z(S) = e^{SE_0^f} \prod_{\vec{k}} \langle |\psi_0(\vec{k})\rangle |e^{-H_f(\vec{k})S} |\psi_0(\vec{k})\rangle. \]  

where \(E_0^f = -\sum_{\vec{k}} \epsilon_f(\vec{k})\) and \(-\epsilon_f(\vec{k})\) is the ground state energy of final Hamiltonian \(H_f(\vec{k})\).

Further, Eq. (8) also suggests that the Hilbert space of the decoupled two-level systems can be mapped to the surface of a Bloch sphere of radius \(|\vec{h}|\). Let us assume that the initial state lies at a point \((\theta_i, \Phi_i)\) on this Bloch sphere where \(\theta\) and \(\Phi\) are the azimuthal and polar angles, respectively. It can be easily checked from Eq. (9a), (9b) and (9c) that the quench which is performed on \(\vec{M}\) only effects the \(h_z(\vec{k}) = |\vec{h}| \cos \theta\) component of \(\vec{h}(\vec{k})\), thereby limiting the subsequent dynamics of the state to a great circle passing through the poles on the surface of the Bloch sphere. Finally, expanding \(\psi_0(\vec{k})\) in the eigenbasis of \(H_f(\vec{k})\), we obtain

\[ Z(S) = \prod_{\vec{k}} \cos^2 (\varphi(\vec{k})) \left( 1 + \tan^2 (\varphi(\vec{k})) e^{-2S\epsilon_f(\vec{k})} \right) \]  

where \(\varphi(\vec{k}) = \frac{\theta_f(\vec{k}) - \theta_i(\vec{k})}{2}\) and \(\theta_i(f) = \cos^{-1} \frac{h_z(I(f))}{|\vec{h}_z(I(f))|}.\)
Substituting this expression for $Z(S)$ in Eq. (10), we have

$$-L^2(2f_s + f_c(S)) = \sum_K 2\log\left(\cos(\varphi(\vec{k}))\right) + \sum_K \log\left(1 + \tan^2(\varphi(\vec{k}))e^{-2\epsilon_f(\vec{k})}\right).$$

(14)

Now, assuming the continuum limit, we can identify the surface and Casimir free energy contributions as

$$f_s = -\frac{1}{L^2 A_B} \int_{BZ} \log\left(\cos(\varphi(\vec{k}))\right) d\vec{k}$$

(15a)

$$f_c(S) = -\frac{1}{L^2 A_B} \int_{BZ} \log\left(1 + \tan^2(\varphi(\vec{k}))e^{-2\epsilon_f(\vec{k})}\right) d\vec{k}$$

(15b)

where $A_B = \int_{BZ} d\vec{k}$ is the area of the Brillouin zone.

III. WORK STATISTICS IN TOPOLOGICALLY TRIVIAL GRAPHENE

When $\phi = 0$, the amplitude of NNN hoppings are real and their only effect is to rescale the energy spectrum of the massive graphene Hamiltonian with NN hoppings by $h_0(\vec{k})$. We analyze the large $S$ behavior of $f_c(S)$ for quenches close to the gap-less graphene point ($M = 0, \phi = 0$ in Fig. 1a) as follows:

It is clear from Eq. (15b) that in the large $S$ limit, the contributions to $f_c(S)$ from the quasi-momentum modes $\vec{k}$ fall off exponentially as we move away from the two Dirac points $\vec{K}_1$ and $\vec{K}_2$ which are time-reversed partners of each other, thus $\epsilon(\vec{K}_1) = \epsilon(\vec{K}_2) \approx 0$ as $M$ tends to zero. Thus, the dominant contribution to the integral in Eq. (15b) comes from the lowest energy continuum around each of the Dirac points which contribute equally and identically to $f_c(S)$. Expanding the energy spectrum around $\vec{K}_1$ to leading non-trivial order in $k = |\vec{k} - \vec{K}_1|$, we have

$$\epsilon(k) = \sqrt{M^2 + k^2}. \quad \text{(16)}$$

In the continuum limit, the limits of the integration in Eq. (15b) extend to infinity to yield,

$$f_c(S) = -\frac{1}{L^2 A_B} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \log\left(1 + \tan^2(\varphi(\vec{k}))e^{-2\epsilon_f(\vec{k})}\right).$$

(17)

Further simplification requires the explicit form of $\tan(\varphi(\vec{k}))$ for small $k$ which we now evaluate for several cases as elaborated below:

A. Quench without crossing QCP ($M_i, M_f \gtrless 0$)

In this case, the initial and the final Semenoff masses are either both positive or negative (Fig. 2a) and we have (see Appendix A)

$$\tan(\varphi(\vec{k})) = C(M_i, M_f)k$$

(18)

to the leading order in $k$ where $C(M_i, M_f) = (M_i - M_f)/2M_i M_f$ depends only on $M_i$ and $M_f$. Substituting in Eq. (17), we get

$$f_c^1(S) = -\frac{2\pi}{L^2 A_B} \int_0^{\infty} \log\left(1 + C^2(M_i, M_f)k^2\right) dk.$$ \quad (19)

where the superscript 1 in $f_c^1(S)$ refers to the fact that we are considering contribution from the lowest energy continuum from only around $\vec{K}_1$. Following few steps of algebra (see Appendix B for detail), we eventually obtain

$$f_c(S) = -2 \times \frac{\pi(1 - M_f/M_i)^2}{4L^2 A_B} \left(\frac{e^{-2S|M_f|}}{S^2}\right)$$

(20)

where the multiplicating factor 2 accounts for the fact that each Dirac point contributes identically. The characteristic function defined in Eq. (5a) takes the form

$$G(S) = e^{-\Delta E_0 S} e^{-2L^2 f_s} e^{-L^2 f_c(S)}$$

$$= e^{-\Delta E_0 S} e^{-2L^2 f_s} \left(1 - L^2 f_c(S) + \ldots\right)$$

(21)
where we have expanded the third exponential to leading order in \( f_c(S) \) exploiting the fact that \( f_c(S) \) decays exponentially with \( S \). Substituting the form of \( f_c(S) \) from Eq. (20) and performing an inverse Laplace transform on \( G(S) \) finally gives us the small \( W \) behavior of \( P(W) \) as

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - | \Delta E_0 |) + \Theta(W - | \Delta E_0 - 2| M_f |) \right] \\
\times \left\{ \frac{\pi (1 - M_f/M_i)^2}{2 A_B} (W - | \Delta E_0 - 2| M_f |) \right\}. \tag{22}
\]

\( P(W) \) therefore has a delta function peak at \( W = | \Delta E_0 | \) and the presence of the Heavyside theta function in the second term implies the existence of an edge singularity. Note that the quench amplitudes and other microscopic details only appear in the coefficient of the edge-singularity while the exponent of \( (W - | \Delta E_0 - 2| M_f |) \) is independent of such details and is thus universal.

### B. Quench across the QCP \((M_i \gtrless 0 \gtrless M_f)\)

When \( M_i \) and \( M_f \) are on either side of the gapless graphene point (Fig. 2b), the leading order term in the expansion of \( \tan(\varphi(\vec{k})) \) takes the form (again, referring to Appendix A)

\[
\tan(\varphi(\vec{k})) = -\frac{1}{C(M_i, M_f) \vec{k}}. \tag{23}
\]

Proceeding similarly as in Case IIIA we obtain the Casimir interaction term as

\[
f_c(S) = -\frac{16 \pi (1 - \gamma) M_i^2 M_f^2}{L^2 A_B (M_i - M_f)^2} e^{-2S|M_f|}. \tag{24}
\]

where \( \gamma \) is the Euler-Mascheroni constant. The work distribution function is thus

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - | \Delta E_0 |) + \Theta(W - | \Delta E_0 - 2| M_f |) \right] \\
\times \left\{ \frac{16 \pi (1 - \gamma) M_i^2 M_f^2}{A_B (M_i - M_f)^2} \right\}. \tag{25}
\]

which interestingly has two delta function peaks at \( W = \Delta E_0 \) and \( W = \Delta E_0 + 2| M_f | \) and contains no continuum.

### C. Quench from the QCP \((M_i = 0)\)

If the quench originates from the critical (graphene) point (Fig. 2c), \( \tan(\varphi(\vec{k})) \) depends only on the relative position of \( M_f \) and is independent of its absolute value,\n
\[
\tan(\varphi(\vec{k})) = -\text{sgn}(M_f). \tag{26}
\]

The Casimir interaction term assumes the simple form

\[
f_c(S) = -\frac{2 \pi M_i}{L^2 A_B} \left( e^{-2S|M_f|} \right). \tag{27}
\]

and the work distribution is

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - | \Delta E_0 |) + \Theta(W - | \Delta E_0 - 2| M_f |) \right] \\
\times \left\{ \frac{2 \pi M_i}{A_B} \Theta(W - | \Delta E_0 - 2| M_f |) \right\} \tag{28}
\]

Thus, the continuum begins with a finite discontinuity at \( W = |\Delta E_0 + 2| M_f | \).

### D. Quench ending at the QCP \((M_f = 0)\)

In this case (Fig. 2d), \( \tan(\varphi(\vec{k})) \) once again is independent of the absolute value of \( M_f \) and depends only on its relative position to the QCP:

\[
\tan(\varphi(\vec{k})) = \text{sgn}(M_i). \tag{29}
\]

However, \( f_c(S) \) now undergoes a power law decay with \( S \),

\[
f_c(S) = -\frac{\pi}{L^2 A_B} \left( \frac{1}{S^2} \right). \tag{30}
\]

This is expected from the fact that the correlation length diverges at the gapless critical point and the two-point correlations exhibit a power law decay. \( P(W) \) thus assumes the form

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - | \Delta E_0 |) + \Theta(W - | \Delta E_0 |) \right] \\
\times \left\{ \frac{\pi}{L^2 A_B} (W - | \Delta E_0 |) \right\}. \tag{31}
\]

which shows that there is no gap in the low energy regime of \( P(W) \) and the continuum starts from \( W = | \Delta E_0 | \). In summary, when \( \phi = 0 \), the behavior of \( P(W) \) is identical to that of the one dimensional transverse Ising model. However, in the present case, two gapless Dirac points contribute.

### IV. WORK STATISTICS IN TOPOLOGICAL HALDANE MODEL

In this section, we shall set \( \phi \neq 0 \) and probe the non-trivial influence of the equilibrium topology on the distribution function \( P(W) \). Let us recall that when complex NNN hoppings are introduced into the graphene Hamiltonian, the two Dirac points are no longer connected through TRS. The resulting asymmetry in the spectrum prohibits simultaneous gap-closings at the two
FIG. 3: Schematic of the quenches performed in the topological Haldane model with a small \((\phi \neq 0)\) and near the QCLs. The cases analyzed are quench (a) within same phase, (b) across one QCL, (c) starting from the QCL and (d) across the topological phase.

Dirac points. The system now has two quantum critical lines (QCLs) (Fig. 1a), \(M_{c1} = 3\sqrt{3}t'\sin \phi\) and \(M_{c2} = -3\sqrt{3}t'\sin \phi\) for vanishing of the two Dirac points, respectively. However if \(\phi\) is small, the two QCLs are very close to each other as

\[ |M_{c1} - M_{c2}| \approx 6\sqrt{3}t'\phi. \]  

(32)

Importantly, the spectrum at the two Dirac points, though non-identical, are still of the same orders of magnitude. For our purpose, this means that the lowest energy continuum around both the Dirac points still make dominant contributions to the Casimir interaction term \(f_c(S)\) in the large \(S\) limit. The spectrum around the Dirac points can now be expanded to leading non-trivial order in \(k_1 = |\vec{k} - \vec{K}_1|\) and \(k_2 = |\vec{k} - \vec{K}_2|\) in the form

\[ \epsilon(k_{1(2)}) = \sqrt{m_1^2(k_{1(2)}) + k_{1(2)}^2} \]  

(33)

where \(m_1 = M - 3\sqrt{3}t'\phi\) and \(m_2 = M + 3\sqrt{3}t'\phi\). A quench in the Semenoff mass from \(M_i\) to \(M_f\) is therefore equivalent to simultaneous quenches in \(m_1\) and \(m_2\) from \((m_{i1}, m_{i2})\) to \((m_{f1}, m_{f2})\). In view of the above situations, we now proceed to evaluate \(f_c(S)\) and \(P(W)\) for the following cases:

### A. Quench within trivial phase

\((M_i, M_f \geq 3\sqrt{3}t'\sin \phi)\) or within topological phase

\((3\sqrt{3}t'\sin \phi > M_i, M_f > -3\sqrt{3}t'\sin \phi)\)

The situation here (Fig. 3a) is similar to the quenches carried out without crossing QCP in the massive graphene model. We observe that

\[ \tan(\varphi(k_1)) = C(m_{i1}, m_{f1})k_1 \]  

(34a)

\[ \tan(\varphi(k_2)) = C(m_{i2}, m_{f2})k_2. \]  

(34b)

The Casimir interaction term assumes the form

\[ f_c(S) = -\frac{\pi}{4L^2A_B}\left[ (1 - m_{f1}/m_{i1})^2 e^{-2S|m_{f1}|/S^2} + (1 - m_{f2}/m_{i2})^2 e^{-2S|m_{f2}|/S^2}\right]. \]  

(35)

and the work distribution function is obtained as

\[ P(W) = e^{-2L^2f_s}\left[ \delta(W - \Delta E_0) + \Theta(W - \Delta E_0 - 2|m_{f1}|) \times \left\{ \frac{\pi(1 - m_{f1}/m_{i1})^2}{4A_B} (W - \Delta E_0 - 2|m_{f1}|) \right\} \right] \]

\[ + \Theta(W - \Delta E_0 - 2|m_{f2}|) \times \left\{ \frac{\pi(1 - m_{f2}/m_{i2})^2}{4A_B} (W - \Delta E_0 - 2|m_{f2}|) \right\} \].  

(36)

Comparing with Eq. (22), we see that \(P(W)\) now has two Heavyside theta functions due to unequal contributions from the two Dirac points.

### B. Quench from trivial to topological phase

\((M_i \geq 3\sqrt{3}t'\sin \phi \geq M_f \geq -3\sqrt{3}t'\sin \phi)\) or vice-versa

In this case (Fig. 3b), the quench is performed across one of the two QCLs. One finds:

\[ \tan(\varphi(k_1)) = -\frac{1}{C(m_{i1}, m_{f1})k_1} \]  

(37a)

\[ \tan(\varphi(k_2)) = C(m_{i2}, m_{f2})k_2. \]  

(37b)

It should be noted that unlike the previous case, \(\tan(\varphi(k))\) has a pole at \(k_1 = 0\) while it is analytic for \(k_2\). Therefore, the two Dirac points contribute differently to the Casimir interaction term and we obtain:

\[ f_c(S) = -\frac{\pi}{L^2A_B}\left[ \frac{8(1 - \gamma)m_{i1}^2m_{f1}^2e^{-2S|m_{f1}|}}{(m_{i1} - m_{f1})^2} + (1 - m_{f2}/m_{i2})^2 e^{-2S|m_{f2}|/4S^2}\right]. \]  

(38)

This is also reflected in the work distribution as

\[ P(W) = e^{-2L^2f_s}\left[ \delta(W - \Delta E_0) + \Theta(W - \Delta E_0 - 2|m_{f1}|) \times \left\{ \frac{\pi(1 - m_{f1}/m_{i1})^2}{4A_B} (W - \Delta E_0 - 2|m_{f1}|) \right\} \right] \]

\[ + \Theta(W - \Delta E_0 - 2|m_{f2}|) \times \left\{ \frac{8\pi(1 - \gamma)m_{i1}^2m_{f1}^2}{A_B(m_{i1} - m_{f1})^2} \delta(W - \Delta E_0 - 2|m_{f1}|) \right\} \].  

(39)
| Quench | Additional delta-function peak position(s) at \( W = \) | Theta function discontinuity position(s) at \( W = \) | Scaling exponent of \( W \) associated with Theta function edge | Overall nature of \( P(W) \) for small \( W \) |
|--------|-------------------------------------------------|--------------------------------|---------------------------------|--------------------------------|
| A. within trivial or within topological phase | - | i. 2\(|m_1f|\) ii. 2\(|m_2f|\) | i. 1 ii. 1 | Continuum starts from \( \min\{2|m_1f|, 2|m_2f|\} \) and the slope changes sharply at \( \max\{2|m_1f|, 2|m_2f|\} \). |
| B. from trivial to topological phase or vice-versa | 2\(|m_1f|\) | 2\(|m_2f|\) | 1 | Continuum starts from 2\(|m_2f|\) and a delta function peak exist at 2\(|m_1f|\), which may either lie prior to the continuum or be superimposed on it. |
| C. away from one QCL and ending: | a. - | a.i. 2\(|m_1f|\) a.ii. 2\(|m_2f|\) | a.i. 0 a.ii.1 | a. If \(|m_1f| \leq |m_2f|\), continuum starts with a non-zero finite value at 2\(|m_1f|\) and the slope changes sharply at 2\(|m_2f|\); if \(|m_1f| > |m_2f|\), continuum begins at 2\(|m_2f|\) with a finite discontinuity at 2\(|m_1f|\). |
| | b. - | b.i. 0 b.ii. 2\(|m_1f|\) | b.i. 1 b.ii. 0 | b. Continuum starts from the origin and the slope changes sharply at 2\(|m_1f|\). |
| | c. across the other QCL | c. 2\(|m_2f|\) | c. 0 | c. Continuum starts from 2\(|m_2f|\) with a non-zero finite value and a delta function peak exist at 2\(|m_2f|\), which may either lie prior to the continuum or be superimposed on it. |
| D. across the topological phase | i. 2\(|m_1f|\) ii. 2\(|m_2f|\) | - - | | Delta function peaks at 2\(|m_1f|\) and 2\(|m_2f|\). |

TABLE I: Summary of the universal characteristics of \( P(W) \) for quenches performed in the topological Haldane model. \( W \) has been rescaled to \( W = W - \Delta E_0 \). In all the cases, there is a delta function at \( W = 0 \) which has not been reported separately here.

where \( P(W) \) now consists of an additional delta function which may appear in the gap or be superimposed on the continuum depending on which among \(|m_1f|\) or \(|m_2f|\) is greater.

\[
f_c(S) = -\frac{\pi}{L^2 AB} \left[ \frac{m_{1f} e^{-2S|m_{1f}|}}{S} + \left( 1 - \frac{m_{2f}}{6\sqrt{3}' \sin \phi} \right)^2 e^{-2S|m_{2f}|} \right] \]  
\( (40a) \)

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - \Delta E_0) + \frac{\pi m_{1f}}{AB} \Theta (W - \Delta E_0 - 2|m_{1f}|) \right. \\
\left. + \Theta (W - \Delta E_0 - 2|m_{2f}|) \right] \times \left\{ \frac{\pi}{4AB} \left( 1 - \frac{m_{2f}}{6\sqrt{3}' \sin \phi} \right)^2 (W - \Delta E_0 - 2|m_{2f}|) \right\} \]  
\( (40b) \)

**C. Quench starting from the QCLs** \( M_i = \pm 3\sqrt{3}' \sin \phi \)

For quenches originating from one of the QCLs, there are three possible scenarios (Fig. 3c) depending on relative position of \( M_f \) with respect to the other QCL. For example, if the quench originates from \( M_i = 3\sqrt{3}' \phi \), the Casimir interaction term and the work distribution function for each of the three scenarios are listed below:
Here, \( P(W) \) consists two Heavyside theta functions and there exists a finite discontinuity at \( W = 2|m_{1f}| \).

b. \( M_f = -3\sqrt{3} \sin \phi \)

\[
f_c(S) = -\frac{\pi}{L^2 A_B} \left[ m_{1f} e^{-2S|m_{1f}|} + \frac{1}{2S^2} \right]
\]

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - \Delta E_0) + \Theta(W - \Delta E_0) \right.
\]

\[
\times \frac{\pi}{L^2 A_B} \left( W - \Delta E_0 \right) + \frac{\pi m_{1f}}{A_B} \Theta(W - \Delta E_0 - 2|m_{1f}|) \]

(41b)

Once again, we obtain two Heavyside theta functions and the continuum begins from \( W = \Delta E_0 \) with no gapped region.

c. \( M_f < -3\sqrt{3} \sin \phi \)

\[
f_c(S) = -\frac{\pi}{L^2 A_B} \left[ m_{1f} e^{-2S|m_{1f}|} + \frac{8(1 - \gamma)m_{1f}^2 m_{2f}^2 e^{-2S|m_{2f}|}}{(m_{2f} - m_{1f})^2} \right]
\]

(42a)

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - \Delta E_0) \right.
\]

\[
\left. + \frac{\pi m_{1f}}{A_B} \Theta(W - \Delta E_0 - 2|m_{1f}|) + \delta(W - \Delta E_0 - 2|m_{2f}|) \right]
\]

\[
\times \frac{8\pi(1 - \gamma)m_{1f}^2 m_{2f}^2}{A_B(m_{2f} - m_{1f})^2} \right] \]

(42b)

Here, unlike the previous cases, \( P(W) \) has a delta function peak at \( W = 2m_{2f} \) and the continuum begins with a finite discontinuity.

D. Quench across the topological phase

\((M_i > 3\sqrt{3} \sin \phi, M_f < -3\sqrt{3} \sin \phi)\)

The quench in this case is performed across the topological phase from one trivial phase to other as indicated in Fig. 34. Proceeding as before, the work distribution function evaluates to

\[
P(W) = e^{-2L^2 f_c} \left[ \delta(W - \Delta E_0) + \right.
\]

\[
\left. + \frac{8\pi(1 - \gamma)m_{1f}^2 m_{2f}^2}{(m_{1f} - m_{2f})^2} \delta(W - \Delta E_0 - 2|m_{1f}|) \right]
\]

\[
\left. + \frac{m_{2f}^2}{(m_{2f} - m_{1f})^2} \delta(W - \Delta E_0 - 2|m_{2f}|) \right] \]

(43)

Therefore, there exist two additional delta function peaks and no continuum.

V. DISCUSSIONS AND CONCLUSIONS

In this article, we have investigated the effect of topology on the emerging universality of the work distribution function for small work when the system is close to criticality. For this purpose, we first analyzed the \( P(W) \) following sudden quenches on the Semenoff mass, in the topologically trivial phase (\( \phi = 0 \)) of the Haldane model with intact TRS. In addition to a delta function peak at \( W = \Delta E_0 \), the nature of \( P(W) \) was found to be solely dependent on the relative position of the initial and final value of the Semenoff mass with respect to the QCP, i.e. \( M = 0 \). To elucidate the effect of topology on \( P(W) \), we then allowed an infinitesimal non-zero value for \( \phi \) which broke the TRS and allowed us to perform quenches to and from the topological phases of the Haldane model. Depending on the initial and final Semenoff masses with respect to the QCLs, the \( P(W) \) now displayed far richer universal characteristics as summarized in Table. I. This observation therefore indicates that the statistics of nonequilibrium work may serve as a probe to detect the equilibrium topology of a given system.

However, we would like to point out that for quenches performed in \( M \) at a large constant value of \( \phi \), the ap proximate equality in Eq. (42) is no longer satisfied. The spectrum at the two Dirac points are of different orders of magnitude and therefore only one of them contributes dominantly to the Casimir interaction term for any given quench. In this scenario, the \( P(W) \) reduces to a form similar to that obtained in the topologically trivial case where the two Dirac points contributed identically. However, it is to be noted that the two Dirac points can contribute together, even at a large value of \( \phi \), if the quench is performed such that \( M_f \neq 0 \), as the spectrum then becomes similar but significantly gapped at the two Dirac points. The large gap on the other hand signals departure from universality and therefore we conclude that the effect of topology on the universality of \( P(W) \) is only manifested at small values of \( \phi \).

VI. EXPERIMENTAL POSSIBILITIES

The experimental reconstruction of the nonequilibrium work probability distribution in a closed quantum system alongwith the experimental study of the corresponding quantum fluctuation relations has been carried out by Batalhao et al.\(^{10}\) The experiment used a liquid-state nuclear magnetic resonance platform that offered complete control on the preparation of the system under study and thereby, in turn enabling the characterization of the out-of-equilibrium dynamics of a quantum spin from the viewpoint of finite-time thermodynamics. Moreover, the experimental realization of the paradigmatic Haldane model and the characterization of its topological band structure, using ultracold fermionic atoms in a periodically modulated optical honeycomb lattice has also been achieved by Jotzu et al.\(^{13}\) Considering the tremendous...
experimental progress observed in recent years in realizing such topological systems, and the immense experimental control achieved in generating out of equilibrium dynamics through quantum quenches in quantum many-body systems, it would be possible to experimentally probe the effect of equilibrium topology on the statistics of work performed, in accordance with our results.

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Appendix A: Evaluation of $\tan (\varphi(\vec{k}))$ for small $\vec{k}$ for trivially gapped graphene

We have,

$$\cos \theta(\vec{k}) = \frac{h_x(\vec{k})}{\epsilon(k)} = \frac{M}{\epsilon(k)} \quad (A1a)$$

where $h_x(\vec{k})$ is actually independent of $\vec{k}$ for each Dirac point in all the cases we consider throughout.

$$\sin \theta(\vec{k}) = \frac{\sqrt{h_x^2(\vec{k}) + h_y^2(\vec{k})}}{\epsilon(k)} \approx \frac{k}{\epsilon(k)} \quad (A1b)$$

to leading order in $k$. A simple trigonometric manipulation allows one to write

$$\tan (\varphi(\vec{k})) = \frac{1 - \cos (\theta_f(\vec{k}) - \theta_t(\vec{k}))}{\sin (\theta_f(\vec{k}) - \theta_t(\vec{k}))} \quad (A2)$$

Substituting Eq. (A1) in the above equation, we obtain

$$\tan (\varphi(\vec{k})) = \frac{1}{\sin (\theta_f(\vec{k}) - \theta_t(\vec{k}))} \quad (A3)$$

Expanding binomially and retaining terms upto $O(k^2)$,

$$\tan (\varphi(\vec{k})) = \frac{|M_i|M_f| - M_iM_f + k^2(M_iM_f)^2}{(M_i - M_f)k} \quad (A4)$$

Hence if $M_i, M_f \geq 0$, we have

$$\tan (\varphi(\vec{k})) = \frac{(M_i - M_f)}{2M_iM_f}k \quad (A5)$$

where we have retained only the leading order term in $k$. Similarly, if $M_i \geq 0 \geq M_f$, the leading order term is

$$\tan (\varphi(\vec{k})) = -\frac{2M_iM_f}{(M_i - M_f)k}. \quad (A6)$$

Finally, it is straightforward to see from Eq. (A3) that if $M_i = 0(M_f = 0)$, we have $\tan (\varphi(\vec{k})) = -\text{sgn}(M_f)(\text{sgn}(M_i))$ respectively.

Appendix B: Evaluation of the integral form of the Casimir term

We choose $\tan (\varphi(\vec{k})) = C(M_i, M_f)k$ to outline the procedure for evaluating $f_c(S)$. Other forms of $\tan (\varphi(\vec{k}))$ can be likewise evaluated. First we recall the following inverse Mellin transformation,

$$\log (1 + x) = \frac{1}{2\pi i} \int_{a-\infty}^{a+\infty} \frac{\pi}{u \sin \pi u} x^{-u}du \quad (B1)$$

where $u \in \mathbb{C}$ and $-1 < a < 0$. Substituting in Eq. (17),

$$f_c^1(S) = \frac{i}{L^2AB} \int_0^\infty kdk \times \int_{a-i\infty}^{a+i\infty} \frac{\pi}{u \sin \pi u} C(M_i, M_f)^{-2u} k^{-2u} e^{2uS|M_f|} e^{uSk^2/|M_f|}du \quad (B2)$$

where we have expanded $\epsilon_f(k)$ to order $O(k^2)$ as

$$\epsilon_f(k) = \sqrt{M_f^2 + k^2} = |M_f|(1 + k^2/2M_f^2) \quad (B3)$$

The integral in $k$ can be evaluated as $Re[u] = a < 0$, and therefore the Eq. (B2) assumes the form

$$f_c^1(S) = \frac{i}{L^2AB} \int_0^{a+i\infty} \frac{\pi(-u)^n \Gamma(-u)}{2u \sin \pi u} \left( \frac{S}{M_f} \right)^{u-1} C(M_i, M_f)^{-2u} e^{2uS|M_f|}du \quad (B4)$$

Further,

$$\int_{a-i\infty}^{a+i\infty} g(u)du = \int_{b-i\infty}^{b+i\infty} g(u)du + \sum_{b < Re[u] < a} \text{res}[g(u)] \quad (B5)$$

where $b < a$ and the residues are summed up over all the poles that lie within the strip $b < Re[u] < a$. The integrand $g(u)$ has poles on the real axis, which can be easily seen if we notice that

$$\frac{\pi}{\sin \pi u} = \Gamma(u)\Gamma(1-u), \quad (B6)$$

and the gamma function has simple poles at $u = -n$ where $n \in \mathbb{N}^+$. The residue at the $n^{th}$ pole is $(-1)^n/n!$. On choosing $b = -\infty$, the integral in the R.H.S. of Eq. (B5) reduces to zero and the summation is now over all $n \in \mathbb{N}^+$. However, since $S$ is large, we consider only the contribution from the pole at $u = -1$ and therefore we obtain,

$$f_c^1(S) = \frac{i}{L^2AB} \frac{\Gamma(1)\Gamma(2)}{-2} \left( \frac{S}{M_f} \right)^{-2} C(M_i, M_f)^2 e^{-2S|M_f|} \times (-2\pi i) \quad (B7)$$
where the last term within braces is the residue of $\Gamma(-1)$, i.e. $-1$, multiplied by $2\pi i$. The final expression is therefore,

$$f_c^e(S) = -\frac{\pi (1 - M_f/M_i)^2}{4L^2 A_B} \left( \frac{e^{-2S|M_f|}}{S^2} \right).$$

(B8)