GHOST SYMMETRY OF THE DISCRETE KP HIERARCHY

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Abstract. In this paper, we define the $S$ function and ghost symmetry for the discrete KP hierarchy. By spectral representation of Baker-Akhiezer function and adjoint Baker-Akhiezer function, we derive ghost flow of eigenfunction and adjoint eigenfunction. From these observations above, some important distinctions between the discrete KP hierarchy and KP hierarchy are shown. Also we give ghost flow on tau function and another kind of proof of ASvM formula of the discrete KP hierarchy.

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Contents

§1. Introduction 1
2. The discrete KP hierarchy and $S$ function 3
3. The ghost symmetry of the discrete KP hierarchy 7
4. Properties and Spectral representation of discrete KP hierarchy 9
5. Ghost flow on tau function and ASvM formula 14
6. Conclusions and Discussions 15
References 17

§1. Introduction

The discrete KP(dKP) hierarchy is an interesting object in the research of the integrable system[1, 2, 3]. The discrete KP-hierarchy can be viewed as the classical KP-hierarchy [4, 5] with the continuous derivative $\frac{\partial}{\partial x}$ replaced formally by the discrete derivative $\Delta$ whose action on function $f(n)$ as

\begin{equation}
\Delta f(n) = f(n+1) - f(n).
\end{equation}

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The Hamiltonian structures and tau function for the discrete KP hierarchy was introduced in ([1]-[3]). In [6], the determinant representation of the gauge transformation for the discrete KP hierarchy was introduced. Sato Backlund transformation, additional symmetries and ASvM formula for the discrete KP hierarchy was considered in [7]. The fermionic approach to Darboux transformations was considered in [8] for the 1-component KP hierarchy and showed that any solution of the associated (adjoint) linear problems can always be expressed as a superposition of KP (adjoint) wave functions. It is shown how Darboux and binary Darboux transformations for a nonautonomous discrete KP equation can be obtained from fermion analysis in [9].

More recently, the extended discrete KP hierarchy and the algebraic structures of the non-isospectral flows of the discrete KP hierarchy were investigated in [10]. After discretization, many important integrable properties are inherited by the discrete KP hierarchy from the KP hierarchy([11], [12], [13]). For example: the existence of tau function, hamiltonian structure of the discrete KP hierarchy and the close relationship between tau function of the discrete KP hierarchy and the one of the KP hierarchy [3]. It is also interesting to further explore the new facts to show the difference between the discrete KP hierarchy and the KP hierarchy from the point of view of symmetry.

As we all know, symmetry is always an important research object in integrable systems. Many important characters of integrable systems have a close relation with symmetry, e.g. conserve laws, hamiltonian structure and so on. As one kind of additional symmetry, ghost symmetry was discovered by W. Oevel [14]. After that, it attracts a lot of research ([15] - [21]). H. Aratyn used the method squared eigenfunction potentials(SEP or $S$ function later) to construct ghost symmetry of KP hierarchy and connect this kind of symmetry with constrained KP hierarchy([16]-[18]). W. Oevel and S. Carillo used $S$ function to represent 2+1-dimensional hierarchies of the KP equation, the modified KP equation and the Dym equation[19, 20]. J. P. Cheng etc. gave a good construction of ghost symmetry of BKP hierarchy[21]. This paper will concentrate on the ghost symmetries of the discrete KP hierarchy. In order to study the action of ghost flow on the wave functions and tau function, it is necessary to introduce $S$ function and spectral representation of the discrete KP hierarchy. The spectral together with ghost symmetry will tell us an important difference of the discrete KP hierarchy from the KP hierarchy.

The paper is organized as follows. In Section 2, after recalling some basic facts of the discrete KP hierarchy ([1]-[3]), the squared eigenfunction potentials($S$ function)is introduced. In Section 3, we define the ghost symmetry of the discrete KP hierarchy with the help of $S$ function. In Section 4, the spectral representation of eigenfunctions for the discrete KP hierarchy help us deriving the ghost flow of eigenfunctions from the flow on Baker-Akhiezer wave functions, meanwhile we give some nice properties of these functions. Using these properties we give the ghost flow on tau function and another different proof of ASvM formula from[7] in Section 5. Section 6 is devoted to conclusions and discussions.
2. The discrete KP hierarchy and $S$ function

To save the space in this section, we would like to follow reference [3] to recall some basic known facts about the discrete KP hierarchy. Firstly we need to use a space $F$ on which the definition of the discrete KP hierarchy[3] is based

$$F = \{ f(n) = f(n, t_1, t_2, \cdots, t_j, \cdots); n \in \mathbb{Z}, t_i \in \mathbb{R} \},$$

and the shift operator is defined as

$$\Lambda f(n) = f(n+1).$$

Difference operator $\triangle$ which acts on function $f(n)$ is defined as following

$$\triangle f(n) = f(n+1) - f(n) = (\Lambda - I)f(n),$$

where $I$ is the identity operator.

For any $j \in \mathbb{Z}$, the following operation can be defined,

$$\Delta^j \circ f = \sum_{i=0}^{\infty} \left( \begin{array}{c} j \\ i \end{array} \right) (\Delta^i f)(n + j - i) \Delta^{j-i}, \quad \left( \begin{array}{c} j \\ i \end{array} \right) = \frac{j(j-1) \cdots (j-i+1)}{i!}.$$  

So we obtain an associative ring $F(\Delta)$ of formal pseudo difference operators, with the operation “$+$” and “$\circ$”

$$F(\Delta) = \left\{ R = \sum_{j=-\infty}^{d} f_j(n) \Delta^j, f_j(n) \in R, n \in \mathbb{Z} \right\},$$

and denote $R_+ := \sum_{j=0}^{d} f_j(n) \circ \Delta^j$ as the positive projection of $R$ and by $R_- := \sum_{j=-\infty}^{-1} f_j(n) \circ \Delta^j$, the negative projection of $R$. Also the adjoint operator to the $\Delta$ operator is defined by $\Delta^*$,

$$\Delta^* f(n) = (\Lambda^{-1} - I)f(n) = f(n-1) - f(n),$$

where $\Lambda^{-1} f(n) = f(n-1)$, and the corresponding “$\circ$” operation is

$$\Delta^* \circ f = \sum_{i=0}^{\infty} \left( \begin{array}{c} j \\ i \end{array} \right) (\Delta^i f)(n + i - j) \Delta^{j-i}.$$ 

Then we obtain the adjoint ring $F(\Delta^*)$ to the $F(\Delta)$, and define the formal adjoint to $R \in F(\Delta)$ is $R^* \in F(\Delta^*)$ as $R^* = \sum_{j=-\infty}^{d} \Delta^* \circ f_j(n)$. The $*$ operation satisfies rules as $(F \circ G)^* = G^* \circ F^*$ for two operators and $f(n)^* = f(n)$ for a function.

The discrete KP hierarchy [3] is a family of evolution equations depending on infinitely many variables $t = (t_1, t_2, \cdots)$

$$\frac{\partial L}{\partial t_i} = [B_i, L], \quad B_i := (L^i)_+,$$
where $L$ is a general first-order pseudo difference operator
\begin{equation}
L(n) = \triangle + \sum_{j=0}^{\infty} f_j(n) \triangle^{-j}.
\end{equation}

Similar to the KP hierarchy, $L$ can also be dressed by operator $W$
\begin{equation}
W(n; t) = 1 + \sum_{j=1}^{\infty} w_j(n; t) \triangle^{-j},
\end{equation}
by
\begin{equation}
L = W \circ \triangle \circ W^{-1}.
\end{equation}

There are Baker-Akhiezer wave function $\Phi_{BA}(n; t, z)$ and adjoint Baker-Akhiezer wave function $\Psi_{BA}(n-1; t, z)$ constructed by,
\begin{equation}
\Phi_{BA}(n; t, z) = W(n; t)(1 + z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)
\end{equation}
\begin{equation}
= \left(1 + \frac{w_1(n; t)}{z} + \frac{w_2(n; t)}{z^2} + \cdots \right)(1 + z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right),
\end{equation}
and
\begin{equation}
\Psi_{BA}(n; t, z) = (W^{-1}(n-1; t))^\ast(1 + z)^{n-1} \exp\left(\sum_{i=1}^{\infty} -t_i z^i\right)
\end{equation}
\begin{equation}
= \left(1 + \frac{w_1^\ast(n; t)}{z} + \frac{w_2^\ast(n; t)}{z^2} + \cdots \right)(1 + z)^{n-1} \exp\left(\sum_{i=1}^{\infty} -t_i z^i\right),
\end{equation}
which satisfy
\begin{equation}
L(n) \Phi_{BA}(n; t, z) = z \Phi_{BA}(n; t, z), \quad L^\ast(n) \Psi_{BA}(n; t, z) = z \Psi_{BA}(n; t, z),
\end{equation}
and
\begin{equation}
\partial_t \Phi_{BA}(n; t, z) = B_j(n) \Phi_{BA}(n; t, z), \quad \partial_t \Psi_{BA}(n; t, z) = -B_j^\ast(n-1) \Psi_{BA}(n; t, z).
\end{equation}
Also there exists a tau function $\tau_\triangle = \tau(n; t)$ for discrete KP hierarchy, which satisfies
\begin{equation}
\Phi_{BA}(n; t, z) = \frac{\tau(n; t - [z^{-1}])}{\tau(n; t)}(1 + z)^n \exp(\sum_{i=1}^{\infty} t_i z^i),
\end{equation}
and
\begin{equation}
\Psi_{BA}(n; t, z) = \frac{\tau(n; t + [z^{-1}])}{\tau(n; t)}(1 + z)^{-n} \exp(\sum_{i=1}^{\infty} -t_i z^i),
\end{equation}
where $[z^{-1}] = (\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \cdots)$.
Now we prove some useful properties for the operators which are used later.

**Lemma 2.1** For \( f \in F \) and \( \triangle, \Lambda \) as above, the following identities hold.

\[
\begin{align*}
(2.18) & \quad \triangle \circ \Lambda = \Lambda \circ \triangle, \\
(2.19) & \quad \triangle^* = -\triangle \circ \Lambda^{-1}, \\
(2.20) & \quad (\triangle^{-1})^* = (\triangle^*)^{-1} = -\Lambda \circ \triangle^{-1}, \\
(2.21) & \quad \triangle^{-1} \circ f \circ \triangle^{-1} = (\triangle^{-1} f) \circ \triangle^{-1} - \triangle^{-1} \circ \Lambda (\triangle^{-1} f).
\end{align*}
\]

**Proof.** The proof is standard and direct. We omit it here. \( \square \)

To define \( S \) function (called Squared Eigenfunction Potential for KP hierarchy), we need the following proposition firstly similar as [14] which is about constrained KP hierarchy.

**Proposition 2.1.** The following identities hold:

\[
\begin{align*}
(2.22) & \quad \text{res}(\triangle A)(j) = \text{res}(\triangle \circ A(j) - A(j + 1) \circ \triangle), \\
(2.23) & \quad P_{<0}(\triangle^{-1} A) = \Delta^{-1} P_{<0}(A) + \Delta^{-1} P_0(A^*), \\
(2.24) & \quad \text{res}(\triangle^{-1} A) = -\text{res}(\Lambda^{-1} P_0(A^*)),
\end{align*}
\]

where \( (\triangle A) \) denotes the action of \( \triangle \) on operator \( A \), \( (\Lambda^{-1} P_0(A^*)) \) means a backward shift of function \( P_0(A^*) \) (zero order term of operator \( A^* \) on \( \triangle ) \) on discrete parameter.

**Proof.** Firstly operator \( A(j) \) is supposed to have following form

\[
A(j) = A_n(j) \Delta^n + A_{n-1}(j) \Delta^{n-1} + \cdots + A_1(j) \Delta^{-1} + \cdots,
\]

where parameter \( j \) denotes the discrete parameter. Then it is easy to do following calculation

\[
\text{res}(\triangle A)(j) = A_{-1}(j + 1) - A_{-1}(j) = \text{res}(\triangle \circ A(j) - A(j + 1) \circ \triangle).
\]

If we rewrite operator \( A \) into \( \sum_{i \in \mathbb{Z}} \Delta^i \tilde{A}_i \), then (2.23) can be got as following

\[
P_{<0}(\triangle^{-1} A) = P_{<0}(\Delta^{-1} \sum_{i \in \mathbb{Z}} \Delta^i \tilde{A}_i) = \Delta^{-1} P_{<0}(A) + \Delta^{-1} P_0(A^*).
\]

Taking the residue of both sides of (2.23) will lead to (2.24). \( \square \)

Using above proposition, following proposition can be easily got similarly as [14].

**Proposition 2.2.** If \( \alpha \) and \( \beta \) are two local difference operators, then

\[
\text{res}(\Delta^{-1} \alpha \beta \Delta^{-1}) = \text{res}(\Delta^{-1} P_0(\alpha^*) \beta \Delta^{-1}) + \text{res}(\Delta^{-1} \alpha P_0(\beta) \Delta^{-1}).
\]

**Proof.** Using Proposition 2.1 following calculation will lead to the proposition.

\[
\text{res}(\Delta^{-1} \alpha \beta \Delta^{-1}) = \text{res}(\Delta^{-1} \alpha P_{21}(\beta) \Delta^{-1}) + \text{res}(\Delta^{-1} \alpha P_0(\beta) \Delta^{-1})
\]

\[
= \text{res}(P_{<0}(\Delta^{-1} \alpha) P_{21}(\beta) \Delta^{-1}) + \text{res}(\Delta^{-1} \alpha P_0(\beta) \Delta^{-1})
\]

\[
= \text{res}(\Delta^{-1} P_0(\alpha^*) P_{21}(\beta) \Delta^{-1}) + \text{res}(\Delta^{-1} \alpha P_0(\beta) \Delta^{-1})
\]

\[
= \text{res}(\Delta^{-1} P_0(\alpha^*) \beta \Delta^{-1}) + \text{res}(\Delta^{-1} \alpha P_0(\beta) \Delta^{-1}).
\]
This proposition will be used to prove the existence of $S$ function.

The functions $\phi, \psi$ which satisfy $\phi_n(j) = P_0(B_n(j)\phi(j))$ and $\psi_n(j) = -P_0(B_n^*(j)\psi(j))$ will be called the eigenfunction and adjoint eigenfunction of the discrete KP hierarchy respectively. By $S$ function, following proposition can be got.

**Proposition 2.3.** For eigenfunction $\phi$ and adjoint eigenfunction $\psi$ of the discrete KP hierarchy, there exists function $S(\psi, \phi)$, s.t.

(2.29) \[ S(\psi, \phi)_\Delta = \psi \phi, \]

(2.30) \[ S(\psi, \phi)_t_n = \text{res}(\Delta^{-1} \circ \psi \circ B_n \circ \phi \circ \Delta^{-1}). \]

**Proof.** In the following proof, we temporarily omit symbol “ $\circ$ ” in the operator multiplication. Eq. (2.29) and eq. (2.30) can be rewritten as

(2.31) \[ S(\psi(j), \phi(j))_\Delta = \psi(j)\phi(j), \]

(2.32) \[ S(\psi(j), \phi(j))_{t_n} = \text{res}(\Delta^{-1}\psi(j)B_n(j)\phi(j)\Delta^{-1}). \]

The commutativity of eq. (2.29) and eq. (2.30) can be proved as following

\[
S(\psi, \phi)_{t_n, \Delta} = \Delta \text{res}(\Delta^{-1}\psi(j)B_n(j)\phi(j)\Delta^{-1}) \\
= \text{res}(\psi B_n(j)\phi(j)\Delta^{-1} - \Delta^{-1}\psi(j+1)B_n(j+1)\phi(j+1)) \\
= \psi P_0(B_n(j)\phi(j)) + \phi^*(j)P_0(B_n^*(j)\psi^*(j)) \\
= \psi(j)\phi(j)_{t_n} + \phi(j)\psi(j)_{t_n} \\
= S(\psi, \phi)_{\Delta t_n} \\
S(\psi, \phi)_{t_n t_m} - S(\psi, \phi)_{t_m t_n} \\
= [\text{res}(\Delta^{-1}\psi_{t_n}(j)B_n(j)\phi(j)\Delta^{-1}) + \text{res}(\Delta^{-1}\psi(j)(B_n)_{t_m}(j)\phi(j)\Delta^{-1}) \\
+ \text{res}(\Delta^{-1}\psi(j)B_n(j)\phi(j)\Delta^{-1}) - \text{res}(\Delta^{-1}\psi_{t_m}(j)B_n(j)\phi(j)\Delta^{-1}) \\
- \text{res}(\Delta^{-1}\psi(j)B_m(j)\phi(j)\Delta^{-1}) - \text{res}(\Delta^{-1}\psi(j)B_n(j)\phi(j)\Delta^{-1})] \\
= [-\text{res}(\Delta^{-1}P_0(B_m^*(j)\psi(j))B_n(j)\phi(j)\Delta^{-1}) + \text{res}(\Delta^{-1}\psi(j)(B_n)_{t_m}(j)\phi(j)\Delta^{-1}) \\
+ \text{res}(\Delta^{-1}\psi B_n(j)P_0(B_m(j)\phi(j))\Delta^{-1}) + \text{res}(\Delta^{-1}P_0(B_n^*(j)\psi(j))B_m(j)\phi(j)\Delta^{-1}) \\
- \text{res}(\Delta^{-1}\psi(j)(B_m)_{t_m}(j)\phi(j)\Delta^{-1}) - \text{res}(\Delta^{-1}\psi(j)B_n(j)P_0(B_n(j)\phi(j))\Delta^{-1})] \\
= [\text{res}(\Delta^{-1}\psi(j)[(B_n)_{t_m} - (B_m)_{t_n} + [B_n, B_m]](j)\phi(j)\Delta^{-1})] \\
= 0.
\]

This is the end of the proof.

Because of the following formula

(2.33) \[ f_1 \Delta^{-1}g_1 \circ f_2 \Delta^{-1}g_2 = f_1S(g_1, f_2)\Delta^{-1}g_2 - f_1 \Delta^{-1} \circ \Lambda(S(g_1, f_2))g_2, \]

we can get some properties of the difference operator in the following proposition.
**Proposition 2.4.** The following identities hold

\[(2.34) \quad S(S(\Lambda(g_1), \Lambda(f_2))g_2, f_3) = S(S(g_1, f_2)^{-1}(g_2), \Lambda^{-1}(f_3)),\]

\[(2.35) \quad S(\Lambda(S(g_1, f_2))g_2, f_3) + \Lambda(S(g_1, f_2S(g_2, f_3))) = \Lambda(S(g_1, f_2))\Lambda(S(g_2, f_3)).\]

for arbitrary functions \(g_1, g_2, f_2, f_3\).

**Proof.** Direct calculation will lead to following two identities

\[
(f_1\Delta^{-1}g_1 \circ f_2\Delta^{-1}g_2) \circ f_3\Delta^{-1}g_3
= f_1S(g_1, f_2)S(g_2, f_3)\Delta^{-1}g_3 - f_1S(g_1, f_2)\Delta^{-1} \circ \Lambda(S(g_2, f_3))g_3
- f_1S(\Lambda(S(g_1, f_2))g_2, f_3)\Delta^{-1}g_3 + f_1\Delta^{-1} \circ \Lambda(S(g_1, f_2))g_2, f_3\Delta^{-1}g_3,

f_1\Delta^{-1}g_1 \circ (f_2\Delta^{-1}g_2 \circ f_3\Delta^{-1}g_3)
= f_1S(g_1, f_2S(g_2, f_3))\Delta^{-1}g_3 - f_1\Delta^{-1} \circ \Lambda(S(g_1, f_2S(g_2, f_3)))g_3
- f_1S(g_1, f_2)\Delta^{-1} \circ \Lambda(S(g_2, f_3))g_3 + f_1\Delta^{-1} \circ \Lambda(S(g_1, f_2))\Lambda(S(g_2, f_3))g_3.
\]

Let

\[(2.36) \quad (f_1\Delta^{-1}g_1 \circ f_2\Delta^{-1}g_2) \circ f_3\Delta^{-1}g_3 = f_1\Delta^{-1}g_1 \circ (f_2\Delta^{-1}g_2 \circ f_3\Delta^{-1}g_3)
\]

and compare the \(\Delta^{-1}\) term, we can get

\[(2.37) \quad S(S(\Lambda(g_1), \Lambda(f_2))g_2, f_3) = S(S(g_1, f_2)^{-1}(g_2), \Lambda^{-1}(f_3)).\]

Comparing the \(\Delta^{-2}\) terms of both sides of eq. (2.37), we can get

\[S(\Lambda(S(g_1, f_2))g_2, f_3) + \Lambda(S(g_1, f_2S(g_2, f_3))) = \Lambda(S(g_1, f_2))\Lambda(S(g_2, f_3)).\]

\(\square\)

Till now, it is time to derive the existence of \(S\) function which is contained in following proposition. Before that, we need following definition.

Using this \(S\) function, we will define the ghost symmetry in the next section.

### 3. The Ghost Symmetry of the Discrete KP Hierarchy

In this section, the ghost flows on Lax operator of discrete KP hierarchy will be introduced firstly. Then we will prove that they are symmetries of the discrete KP hierarchy. After this, we naturally further consider the action of ghost flow on wave function (Baker-Akhiezer wave function). In the following part, we always omit the same discrete parameter in one equation.

Inspired by the definition of ghost flow of the KP hierarchy [18], here we define the flow for ghost symmetry as following

\[(3.1) \quad \partial_\xi L = [\phi\Delta^{-1}\psi, L],\]
where functions $\phi, \psi$ are the eigenfunction and adjoint eigenfunction of the discrete KP hierarchy. They correspond to the same $B_n$. Therefore we leave the discrete parameter out in the following part.

The following theorem will tell you why we call it ghost symmetry.

**Theorem 3.1.** The additional flows $\partial Z$ commute with the discrete KP flows $\partial t_n$, i.e.,

$$[\partial Z, \partial t_n]L = 0,$$

(3.2)

**Proof.** The commutativity of ghost flows and discrete KP flows is in fact equivalent to the following Zero-Curvature equation which includes detailed proof

$$\partial Z B_n - \partial t_n (\phi \Delta^{-1} \psi) + [B_n, \phi \Delta^{-1} \psi] = \phi S(\psi, B_n^\phi) \Delta^{-1} \psi + [B_n, \phi \Delta^{-1} \psi] = 0.$$

Above proposition tells us that the ghost flows are the symmetries of the discrete KP hierarchy.

The ghost symmetry on wave operator $W$ can be got as following

$$\partial Z W = \phi \Delta^{-1} \psi W.$$

(3.3)

According to eq. (3.1), the ghost flows acting on Baker-Akhiezer function $\Phi_{BA}(n; t, z)$ and adjoint Baker-Akhiezer function $\Psi_{BA}^*(n; t, z)$ are in following proposition.

**Proposition 3.2.** Baker-Akhiezer function $\Phi_{BA}(n; t, z)$ and adjoint Baker-Akhiezer function $\Psi_{BA}^*(n; t, z)$ satisfy the following equations

$$\partial Z \Phi_{BA}(n; t, z) = \phi S(\psi, \Phi_{BA}(n; t, z)),$$

(3.4)

$$\partial Z \Psi_{BA}^*(n; t, z) = \psi S(\Lambda(\phi), \Psi_{BA}(n; t, z)).$$

(3.5)

**Proof.** The ghost symmetry on wave operator $W^{-1}$ and $W^{-1*}$ can be got as following

$$\partial Z W^{-1} = -W^{-1} \phi \Delta^{-1} \psi,$$

(3.6)

$$\partial Z W^{-1*} = \psi \Lambda \Delta^{-1} \phi W^{-1*}.$$

(3.7)

Considering eq. (2.12) and eq. (2.13) and taking derivative by $\partial Z$ will lead to following calculation which will finish the proof.

$$\partial Z \Phi_{BA}(n; t, z) = (\partial Z W(n; t))(1 + z)^n exp(\sum_{i=1}^{\infty} t_i z^i)$$
\[
\begin{align*}
&= \phi \Delta^{-1} \psi W(1 + z)^n \exp \left( \sum_{i=1}^{\infty} t_i z^i \right) \\
&= \phi S(\psi, \Phi_{BA}(n; t, z)),
\end{align*}
\]
\[
\partial_Z \Psi_{BA}(n; t, z) = (\partial_Z W^{-1*}(n - 1; t))(1 + z)^{-n} \exp \left( \sum_{i=1}^{\infty} -t_i z^i \right)
\]
\[
= \psi \Lambda \Delta^{-1} \phi(W^{-1}(n - 1; t))^*(1 + z)^{-n} \exp \left( \sum_{i=1}^{\infty} -t_i z^i \right)
\]
\[
= \psi \Delta^{-1} \Lambda(\phi)(W^{-1}(n; t))^*(1 + z)^{-n-1} \exp \left( \sum_{i=1}^{\infty} -t_i z^i \right)
\]
\[
= \psi S(\Lambda(\phi), \Psi_{BA}(n + 1; t, z)).
\]

From above, we find the \( S \) function\(^[18]\) is not used directly in the definition of ghost flow on Baker-Akhiezer function \( \Phi_{BA}(n; t, z) \) and adjoint Baker-Akhiezer function \( \Psi_{BA}(n; t, z) \).

Till now, the action of ghost flow on Baker-Akhiezer function \( \Phi_{BA}(n; t, z) \) and adjoint Baker-Akhiezer function \( \Psi_{BA}(n; t, z) \) is derived. Therefore it is natural to further consider the ghost flow on tau function. Before that we need some preparation which is contained in the next section.

4. Properties and Spectral representation of discrete KP hierarchy

This section will be about the spectral representation of eigenfunction for the discrete KP hierarchy. Before that, we firstly introduce some properties of the tau function and wave functions of the discrete KP hierarchy. These all give a good preparation to derive the ghost flow on tau function and new proof of ASvM formula which will be given in the next section.

**Lemma 4.1.** Hirota bilinear identities of the discrete KP hierarchy are

\[
\text{res}_z(\Delta^j \Phi_{BA}(n, t', z))\Psi_{BA}(n, t, z) = 0, \quad j \geq 0.
\]

Using this bilinear identity and relation between tau function and Baker-Akhiezer wave function and adjoint Baker-Akhiezer wave function, following proposition can be got\(^[7]\).

**Proposition 4.2.** The tau functions of discrete KP hierarchies satisfy following Fay identity\(^[7]\)

\[
(s_0 - s_1)(s_2 - s_3)\tau(n, t + [s_0] + [s_1])\tau(n, t + [s_2] + [s_3]) + c.p. = 0.
\]

Set \( s_0 = 0 \) and divide by \( s_1 s_2 s_3 \), then shift time variables by \([s_2] + [s_3]\) and we get following identity

\[
(s_2^{-1} - s_3^{-1})\tau(n, t + [s_1] - [s_2] - [s_3])\tau(n, t) + (s_1^{-1} - s_2^{-1})\tau(n, t - [s_2])\tau(n, t + [s_1] - [s_3])
\]
Proposition 4.3. Tau functions of discrete KP hierarchies satisfy

\[ (4.4) \quad (1 + s_3^{-1}) \Delta \left( \frac{\tau(n, t + [s_1] - [s_3])}{\tau(n, t)} \right) = (s_3^{-1} - s_1^{-1}) \left( \frac{\tau(n, t - [s_3]) \tau(n + 1, t + [s_1])}{\tau(n, t) \tau(n + 1, t)} \right) - (1 + s_1^{-1}) \left( \frac{\tau(n + 1, t + [s_1] - [s_3]) \tau(n + 1, t)}{\tau(n, t) \tau(n + 1, t)} \right). \]

Proposition 4.3 can be rewritten as following new proposition \[7\].

Proposition 4.4. The tau functions of discrete KP hierarchies satisfy following difference Fay identity

\[ (4.5) \quad (1 + s_3^{-1}) \Delta \left( \frac{\tau(n, t + [s_1] - [s_3])}{\tau(n, t)} \right) = (s_3^{-1} - s_1^{-1}) \left( \frac{\tau(n, t - [s_3]) \tau(n + 1, t + [s_1])}{\tau(n, t) \tau(n + 1, t)} \right) - (1 + s_1^{-1}) \left( \frac{\tau(n, t + [s_1] - [s_3]) \tau(n + 1, t)}{\tau(n, t) \tau(n + 1, t)} \right). \]

Similar as \[18\], we can get following property of Baker-Akhiezer wave function and adjoint Baker-Akhiezer wave function.

Proposition 4.5. Following identity holds

\[ (4.6) \quad \Delta(\Phi_{BA}(n, t + [\lambda^{-1}], \mu)\Psi_{BA}(n, t, \lambda)) = \lambda \Psi_{BA}(n + 1, t, \lambda)\Phi_{BA}(n, t, \mu), \]

\[ (4.7) \quad \Delta(\Phi_{BA}(n, t, \mu)\Psi_{BA}(n, t - [\mu^{-1}], \lambda)) = -\mu \Psi_{BA}(n + 1, t, \lambda)\Phi_{BA}(n, t, \mu). \]

Proof. By Proposition 4.4, we can get following identity

\[
\frac{(1 + \mu)^{n+1}}{(1 + \lambda)^n} e^{\xi(t, \mu) - \xi(t, \lambda)} \frac{1}{1 - \frac{\mu}{\lambda}} \frac{\tau(n + 1, t + [\lambda] - [\mu])}{\tau(n + 1, t)} \frac{\tau(n + 1, t)}{\tau(n, t)} \frac{1}{1 - \frac{\mu}{\lambda}} \frac{\tau(n, t + [\lambda] - [\mu])}{\tau(n, t)} \frac{\tau(n, t)}{\tau(n, t + [\lambda])} = \frac{(1 + \mu)^n}{(1 + \lambda)^{n+1}} e^{\xi(t, \mu) - \xi(t, \lambda)} \frac{\tau(n, t - [\mu]) \tau(n + 1, t + [\lambda])}{\tau(n, t) \tau(n + 1, t)}.
\]

Then it further leads to following identity

\[ (4.8) \quad \frac{1}{\lambda - \mu} \Delta \left( \frac{\lambda^n (1 + \mu)^n}{\mu^n (1 + \lambda)^n} e^{\xi(t, \mu) - \xi(t, \lambda)} \frac{\tau(n, t + [\lambda] - [\mu])}{\tau(n, t)} \frac{\tau(n, t)}{\tau(n + 1, t)} \frac{1}{1 - \frac{\mu}{\lambda}} \frac{\tau(n + 1, t)}{\tau(n + 1, t + [\lambda])} = \frac{\lambda^n (1 + \mu)^n}{\mu^n (1 + \lambda)^{n+1}} e^{\xi(t, \mu) - \xi(t, \lambda)} \frac{\tau(n, t - [\mu]) \tau(n + 1, t + [\lambda])}{\tau(n, t) \tau(n + 1, t)} \right). \]
Equation (4.8) above can have following form if we change $\lambda$ and $\mu$ to $\lambda^{-1}$ and $\mu^{-1}$

\[
\frac{1}{\lambda - \mu} \Delta \left( \frac{(1 + \mu)^n e^{\xi(t,\mu) - \xi(t,\lambda)} \tau(n, t + [\lambda^{-1}] - [\mu^{-1}])}{\tau(n, t)} \right) = (1 + \mu)^n e^{\xi(t,\mu) - \xi(t,\lambda)} \frac{\tau(n, t - [\mu^{-1}]) \tau(n + 1, t + [\lambda^{-1}])}{\tau(n, t) \tau(n + 1, t)}.
\]

Denoting

\[
X(n, \lambda, \mu) := \frac{(1 + \mu)^n e^{\xi(t,\mu) - \xi(t,\lambda)} e^{\sum_{i=1}^{\infty} \frac{1}{i}(\lambda^{-1} - \mu^{-1}) \frac{\partial}{\partial t}},
\]

we can get following result

\[
\frac{X(n, \lambda, \mu) \tau(n, t)}{\tau(n, t)} = (1 + \mu)^n e^{\xi(t,\mu) - \xi(t,\lambda)} \frac{\tau(n, t + [\lambda^{-1}] - [\mu^{-1}])}{\tau(n, t)} = (1 - \frac{\mu}{\lambda}) \Phi_{BA}(n, t + [\lambda^{-1}], \mu) \Psi_{BA}(n; t, \lambda).
\]

Similarly, we can also get

\[
\frac{X(n, \lambda, \mu) \tau(n, t)}{\tau(n, t)} = (1 - \frac{\lambda}{\mu}) \Phi_{BA}(n, t, \mu) \Psi_{BA}(n; t - [\mu^{-1}], \lambda),
\]

by considering another different convergence field.

Therefore we further get

\[
\frac{1}{\lambda - \mu} \Delta \left( \frac{X(n, \lambda, \mu) \tau(n, t)}{\tau(n, t)} \right) = \Psi_{BA}(n + 1, t, \lambda) \Phi_{BA}(n, t, \mu),
\]

which further leads to eq. (4.6) in the proposition. In the same way, eq. (4.7) can be easily proved.

Besides above proposition, following lemma can also be got easily whose continuous version can be found in [15].

**Lemma 4.6.** The following identity holds

\[
\frac{1}{\mu} \hat{\Delta}_z (\Phi_{BA}(n, t + [\mu^{-1}], \lambda) \Psi_{BA}(n; t, \mu)) = \frac{1}{z} \Phi_{BA}(n, t, \lambda) \Psi_{BA}(n; t - [z^{-1}], \mu)
\]

where

\[
\hat{\Delta}_z f(t) = f(t - [z^{-1}]) - f(t).
\]

**Proof.** According eq. (2.16) and eq. (2.17), we can get following identity

\[
\Phi_{BA}(n, t + [\mu^{-1}], \lambda) \Psi_{BA}(n; t, \mu) = \frac{(1 + \lambda)^n}{(1 + \mu)^n} \frac{1}{1 - \frac{\lambda}{\mu}} \frac{\tau(n, t + [\mu^{-1}] - [\lambda^{-1}])}{\tau(n, t)}.
\]

Then we can derive the following calculation which can finish the proof,
Proposition 4.7. Eigenfunction \( \phi(n; t) \) and adjoint eigenfunction \( \psi(n; t) \) have the following spectral representation using Baker-Akhiezer function \( \Phi_{BA}(n; t, z) \) and adjoint Baker-Akhiezer function \( \Psi_{BA}(n; t, z) \)

\[
\int dz \Phi_{BA}(m; t, z))\Psi_{BA}(n; t', z)) = 0, \quad m, n \in \mathbb{Z}_+.
\]

In this proof, we choose \( n = m + 1 \). Deriving the right side of the eq.(4.15) and using eq.(4.1), we can find \( t' \), i.e.

\[
\partial_{t'} \left( \int dz \Phi_{BA}(n; t, z))S(\phi(n, t'), \Psi_{BA}(n + 1; t', z)) \right) = 0.
\]

That means the right side of eq.(4.15) do not depend on \( t' \). Set \( t' = t \), then the right side of eq.(4.15) becomes

\[
- \int dz \Phi_{BA}(n; t, z))S(\phi(n, t), \Psi_{BA}(n + 1; t, z)) = \int dz \phi(n, t)(z^{-1} + o(z^{-2})) = \phi(n, t).
\]
So the eq.(4.15) is proved. To prove eq.(4.16), we need to choose \( m = n \) in eq.(4.18). The process is quite similar, therefore we omit it here. \( \square \)

From above spectral representation, one can see the difference between discrete KP hierarchy and KP hierarchy which shows the discrete effect.

The spectral representation will help us to get the ghost flow of eigenfunction \( \phi(n; t) \) and adjoint eigenfunction \( \psi(n; t) \).

Considering Proposition 4.5 and above spectral representation, this will lead to following proposition.

**Proposition 4.8.** Following identities hold

\[
\begin{align*}
S(\Psi_{BA}(n + 1, t, \lambda), \Phi_{BA}(n, t, \mu)) &= \lambda^{-1} \Phi_{BA}(n, t + [\lambda^{-1}], \mu) \Psi_{BA}(n, t, \lambda), \\
S(\Psi_{BA}(n + 1, t, \lambda), \Phi_{BA}(n, t, \mu)) &= -\mu^{-1} \Phi_{BA}(n, t, \mu) \Psi_{BA}(n, t - [\mu^{-1}], \lambda), \\
S(\Psi_{BA}(n + 1, t, \lambda), \phi(n, t)) &= \lambda^{-1} \phi(n, t + [\lambda^{-1}]) \Psi_{BA}(n, t, \lambda), \\
S(\psi(n, t), \Phi_{BA}(n, t, \lambda)) &= -\lambda^{-1} \Phi_{BA}(n, t, \lambda) \psi(n - 1; t - [\lambda^{-1}]), \\
S(\psi(n, t), \phi(n, t)) &= \int \int d\lambda d\mu \bar{\phi}(\mu) \bar{\psi}(\lambda) S(\Psi_{BA}(n + 1, t, \lambda), \Phi_{BA}(n, t, \mu)).
\end{align*}
\]

We need note that eq.(4.20) and eq.(4.21) are defined in two different field.

After the proposition above about spectral representation of the discrete KP hierarchy, it is easy to lead to the ghost flow of eigenfunction \( \phi(n; t) \) and adjoint eigenfunction \( \psi(n; t) \) as following proposition.

**Proposition 4.9.** Eigenfunction \( \phi(n; t) \) and adjoint eigenfunction \( \psi(n; t) \) satisfy the following equation

\[
\begin{align*}
\partial_Z \phi(n; t) &= \phi S(\psi, \phi(n; t)), \\
\partial_Z \psi(n; t) &= \psi S(\Lambda(\phi), \psi(n; t)).
\end{align*}
\]

**Proof.** Direct calculation will lead to following identities

\[
\begin{align*}
\partial_Z \phi(n; t) &= \int dz \bar{\phi}(z) \partial_Z \Phi_{BA}(n, t, z) \\
&= \phi \Delta^{-1} \psi \int dz \bar{\phi}(z) \Phi_{BA}(n, t, z) \\
&= \phi S(\psi, \phi(n; t)), \\
\partial_Z \psi(n; t) &= \int dz \bar{\psi}(z) \partial_Z \Psi_{BA}(n + 1; t, z) \\
&= \psi \Lambda \Delta^{-1} \phi \int dz \bar{\psi}(z) \Psi_{BA}(n + 1; t, z) \\
&= \psi S(\Lambda(\phi), \psi(n; t)).
\end{align*}
\]

Then we finish the proof. \( \square \)
Proposition 4.10. The following identity holds

\[
\hat{\Delta}_z(S(\phi(n, t), \psi(n; t)) = -\frac{1}{z}\phi(n, t)\psi(n - 1; t - [z^{-1}])
\]

where

\[
\hat{\Delta}_zf(t) = f(t - [z^{-1}]) - f(t).
\]

Proof. Considering Lemma 4.6 and eq.(4.20), eq.(4.21) can lead to equation as

\[
\hat{\Delta}_z(S(\Phi_{BA}(n, t, \lambda), \Psi_{BA}(n + 1; t, \mu))) = -\frac{1}{z}\Phi_{BA}(n, t, \lambda)\Psi_{BA}(n; t - [z^{-1}], \mu).
\]

Then using the spectral representation of eigenfunction and adjoint eigenfunction, we can derive the proposition directly.

\[\square\]

5. Ghost flow on tau function and ASvM formula

After the good preparation in the last section, this section will be devoted to derive the ghost flow on tau function and new proof of ASvM formula.

All ghost flows above is about

\[
\partial_Z := \phi \Delta^{-1} \psi
\]

where the choices of a pair \((\phi, \psi)\) are in the set \((E, E')\) of eigenfunctions and adjoint eigenfunctions of discrete KP hierarchy. Also \(\phi, \psi\) should correspond to the same \(B_n(j)\) in the definition of eigenfunctions and adjoint eigenfunctions of discrete KP hierarchy. That means the ghost symmetry of discrete KP hierarchy can be generalized to

\[
\partial_Z := \sum_{\phi \in E, \psi \in E'} \phi \Delta^{-1} \psi.
\]

Now we will think about one specific ghost flow denoted as

\[
\partial_{Z_n} := \phi(n) \Delta^{-1} \psi(n).
\]

After above preparation, it is time to derive the ghost flow acting on tau function of discrete KP hierarchy which is contained in the following proposition.

Proposition 5.1. The ghost flow of discrete KP hierarchy on its tau function is as following

\[
\partial_{Z_n} \tau(n; t) = -S(\phi(n, t), \psi(n; t)) \tau(n; t),
\]

Proof. Eq.(3.4), eq.(4.23) and eq.(4.27) will lead to following calculation

\[
\partial_{Z_n} \Phi_{BA}(n; t, z) = \phi(n) S(\psi(n), \Phi_{BA}(n; t, z))
\]

\[
= z^{-1} \phi(n) \psi(n; t - [z^{-1}]) \Phi_{BA}(n, t, z)
\]

\[
= -\hat{\Delta}_z(S(\phi(n, t), \psi(n; t))) \Phi_{BA}(n, t, z).
\]
Therefore we get
\[ \partial Z_n \log \tau(n; t) = -S(\phi(n, t), \psi(n; t)), \]
which further lead to eq. (5.4).

One can easily check that following identity holds using methods in \[5, 22\] which is about KP hierarchy
\[ X(n, \lambda, \mu) \Phi_{BA}(n, t, z) = \Phi_{BA}(n, t, z) \Delta^{-1} \Phi_{BA}(n + 1, t, \lambda), \]
where \( X(n, \lambda, \mu) \) is defined as (4.10) whose acting on \( \Phi_{BA}(n, t, z) \) is an infinitesimal acting on tau function.

Defining \( Y(n, \lambda, \mu) := (\lambda - \mu)\Phi_{BA}(n, t, \mu) \Delta^{-1} \Psi_{BA}(n + 1, t, \lambda) \) and using several propositions, lemmas mentioned above we will give a proof of the ASvM formula as following proposition in a different way from [7].

**Proposition 5.2.** Vertex operator \( X(n, \lambda, \mu) \) as an infinitesimal operator on Baker-Akhiezer function \( \Phi_{BA}(n, t, z) \) can be equivalently expressed by action of infinitesimal operator \( Y(n, \lambda, \mu) \)
\[ (5.7) \]
\[ X(n, \lambda, \mu) \Phi_{BA}(n, t, z) = Y(n, \lambda, \mu) \Phi_{BA}(n, t, z). \]

**Proof.** Using eq. (4.13) and eq. (5.6), the following equation can be got
\[ X(n, \lambda, \mu) \Phi_{BA}(n, t, z) = \Phi_{BA}(n, t, \mu) \Delta^{-1} \Phi_{BA}(n + 1, t, \lambda), \]
which further lead to following identity by considering eq. (4.29)
\[ X(n, \lambda, \mu) \Phi_{BA}(n, t, z) = (\lambda - \mu)\Phi_{BA}(n, t, \mu) \Delta^{-1} \Psi_{BA}(n + 1, t, \lambda) \Phi_{BA}(n, t, z - [z^{-1}], \lambda). \]

Then eq. (4.21) can help us deriving following identity
\[ X(n, \lambda, \mu) \Phi_{BA}(n, t, z) = (\lambda - \mu)\Phi_{BA}(n, t, \mu) \Delta^{-1} \Psi_{BA}(n + 1, t, \lambda) \Phi_{BA}(n, t, z), \]
which is exactly what we need to prove. This is the end of the simple proof for ASvM formula.

\[ \square \]

6. Conclusions and Discussions

In this paper, we define the ghost symmetry of the discrete KP hierarchy by acting on Lax operator \( \hat{L} \), give ghost flows on Baker-Akhiezer function \( \Phi_{BA}(n, t, z) \), adjoint Baker-Akhiezer function \( \Psi_{BA}(n, t, z) \) in Proposition 3.2. By spectral representation using Baker-Akhiezer function \( \Phi_{BA}(n, t, z) \) and adjoint Baker-Akhiezer function \( \Psi_{BA}(n, t, z) \) in Proposition 4.15 we derive ghost flows of eigenfunction \( \phi(n; t) \) and adjoint eigenfunction \( \psi(n; t) \) in Proposition 4.9. Meanwhile some nice properties of Baker-Akhiezer function \( \Phi_{BA}(n; t, z) \) and adjoint Baker-Akhiezer function \( \Psi_{BA}(n; t, z) \) are also got with the help of Fay-identities. By these properties, ghost flow on tau function is derived nicely in Proposition 5.1. Also we give a new proof of ASvM formula with the help of \( S \) function.

Our next step is to connect these ghost flows with the discrete constrained KP hierarchy. Another interesting problem is to consider ghost flows of sub-hierarchies of the discrete KP hierarchy.
hierarchy including discrete BKP hierarchy and discrete CKP hierarchy. The difficulty is identifying the discrete algebraic structure hidden in the discrete KP hierarchy such that we can find suitable reduction to define sub-hierarchies of the discrete KP hierarchy.

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