Temperature dependence of the anomalous effective action of fermions in two and four dimensions

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Abstract

The temperature dependence of the anomalous sector of the effective action of fermions coupled to external gauge and pseudo-scalar fields is computed at leading order in an expansion in the number of Lorentz indices in two and four dimensions. The calculation preserves chiral symmetry and confirms that a temperature dependence is compatible with axial anomaly saturation. The result checks soft-pions theorems at zero temperature as well as recent results in the literature for the pionic decay amplitude into static photons in the chirally symmetric phase. The case of chiral fermions is also considered.

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I. INTRODUCTION

It is well established in the literature that the axial anomaly has a temperature independent form [1–13]. This result is consistent with our present understanding of anomalies, since they are induced by the ultraviolet divergences present in the theory whereas the finite temperature modifies the infrared sector only, namely, by imposing periodic or antiperiodic boundary conditions in the Euclidean time direction [4–14]. In particular for Weyl fermions, the known topological origin of the anomaly guarantees its independence under changes on the space-time manifold [17]. Likewise, the axial anomaly is mass independent and also density independent [13]. Analogous statements hold for the parity anomaly in odd-dimensional theories [18].

Recently, it has been found in [19] that the anomalous amplitudes, such as the neutral pion decay into two photons, are temperature dependent. At first, this would seem surprising since the numerical value of the pion width is very well accounted for by the axial anomaly prediction and furthermore, such decay is considered the standard proof that chiral anomalies are not just mathematical artifacts. In order to clarify this point it should be noted that even at zero temperature the axial anomaly and the neutral pion decay are different entities; Adler’s theorem [20], relates the pion decay to the axial anomaly in the soft-pion limit only. Translated to the language of effective actions, this means that, at leading order in a gradient expansion, the pseudo-parity odd component of the effective action [21] (or equivalently, the component that is imaginary in Euclidean time or also the component containing the Levi-Civita pseudo-tensor) is accounted for by the gauged Wess-Zumino-Witten (WZW) action [22,21]. In other words, all other contributions are of higher order. They contain more gradients and are therefore subdominant in the soft-pion limit. Such higher order terms will necessarily be chirally invariant since the gauged WZW action saturates the anomaly. If one tries to write down Euclidean (Lorentz) and chiral invariant terms of the same order as the WZW action, it is immediately clear that they vanish identically. At finite temperature the conditions are less restrictive and the pion decay is no longer determined by the axial anomaly [23,24]. Indeed, at finite temperature the time direction is privileged and Euclidean invariance is partially broken. Moreover, the effective action is not expected to admit a gradient expansion with local terms in general [16,23–25]. This lower symmetry allows to write down new chiral invariant terms which can compete with those coming from the anomaly rendering the anomalous (or better, pseudo-parity odd) amplitudes temperature dependent even at leading order.

Ref. [19] makes use of a linear sigma model with constituent quarks to study the mesonic decay amplitudes at temperatures near the chirally symmetric phase through direct computation of the relevant Feynman diagrams. As already noted, a non trivial temperature dependence is found and furthermore the neutral pion decay turns out to be suppressed in the chirally symmetric phase. This conclusion is confirmed in [26] using functional methods. Both calculations use the imaginary time formalism to introduce the finite temperature. A similar calculation is carried out in [27] using the real time formalism. There it is found that the pion decay amplitude is indeed temperature dependent although the suppression in the chirally symmetric phase is not reproduced. In [28] the problem is studied in full generality regarding the kinematical conditions of the pion and photons within the real time formulation. The analysis there indicates that the discrepancy comes from the different kinematical
configurations assumed, namely, static photons in [19] versus on-shell photons in [27]. It is also noteworthy that, according to [28], the vanishing of the pionic decay amplitude in the chirally symmetric phase will presumably be recovered when quantum pionic fluctuations are properly taken into account since they regulate the infrared sector. Another set of calculations of anomalous (and non anomalous) mesonic amplitudes should be mentioned here [29,23–25]. They correspond to the low temperature limit and thus well within the phase were chiral symmetry is broken. In this case the spirit of chiral perturbation theory applies. These calculations make use of effective Lagrangians of pions. In this approach, the temperature dependence comes through pionic loop corrections (instead of quark loops as in the constituent quark models cited above) and the pionic Lagrangian itself is assumed to be temperature independent.

In the present work we carry out a calculation of the pseudo-parity odd component of the effective action of fermions in two and four dimensions in the presence of external non-Abelian vector, axial and pseudo-scalar fields on the chiral circle at finite temperature, and at leading order in a suitable expansion. We will use the ζ-function prescription and will emphasize the chiral symmetry preservation aspects such as the anomaly and explicit vector gauge invariance. Some implications for chiral fermions are also presented.

II. GENERAL CONSIDERATIONS

The Euclidean action describing fermions in the presence of external bosonic fields is

\[ \int \bar{\psi} D \psi, \]

where \( D = \partial^\mu + A^\mu + MU^{\gamma_5}, \) (1)

\( D^\mu = \partial^\mu + V^\mu \) is the covariant derivative, \( U^{\gamma_5} \) stand for \( U \) in the subspace \( \gamma_5 = +1 \) and \( U^{-1} \) when \( \gamma_5 = -1 \). \( M \) is a constant c-number mass term, the constituent mass of the quarks, \( V^\mu(x), A^\mu(x) \) and \( U(x) \) are matrices in flavor (and color) space. \( M \) is real and positive, \( V^\mu(x) \) and \( A^\mu(x) \) are anti-Hermitian, and \( U(x) \) is unitary. The matrices \( \gamma^\mu \) and \( \gamma^5 \) are Hermitian and satisfy

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 \delta^\mu_\nu, \quad \gamma^5 = \eta^2 \gamma_0 \cdots \gamma_{d-1}, \] (2)

d = 2, 4 being the space-time dimension and \( \eta_2 = i, \eta_4 = 1 \). We will use the imaginary time formalism to implement the finite temperature condition, namely, through periodic boundary conditions in the Euclidean time direction for the bosonic fields and antiperiodic conditions for the fermionic fields with period \( \beta = 1/T, T \) being the temperature [14–16]. We will assume a space-time topology of the form \( \mathbb{R}^{d-1} \times S^1 \). As a consequence Euclidean (Lorentz) invariance is partially broken due to the anisotropic boundary conditions.

Chiral symmetry corresponds to the transformation

\[ D \rightarrow \Omega_R^{-P_L} \Omega_L^{-P_R} D \Omega_R^{P_R} \Omega_L^{P_L}, \] (3)

where \( P_{R,L} = \frac{1}{2}(1 \pm \gamma_5) \) are the projectors on the subspaces \( \gamma_5 = \pm 1 \) and \( \Omega_{R,L}(x) \) are independent unitary flavor matrices. In terms of the chiral fields \( V^{R,L}_\mu(x) = V^\mu \pm A^\mu \), the Dirac operator can be written as
where \( D_{\mu}^{R,L} = \partial_\mu + V_{\mu}^{R,L} \) are the chiral covariant derivatives, and the chiral transformation takes the form

\[
D_{\mu}^{R,L} \rightarrow \Omega_{R,L}^{-1} D_{\mu}^{R,L} \Omega_{R,L} , \quad U \rightarrow \Omega_L^{-1} U \Omega_R .
\]  

Vector gauge transformations correspond to the diagonal subgroup \( \Omega_R = \Omega_L \), i.e. \( D \rightarrow \Omega^{-1} D \Omega \). For finite rotations, \( \Omega_{R,L} = \exp(\alpha_{R,L}) \) the matrices \( \alpha_{R,L} \) being anti-Hermitian. Infinitesimal vector and axial rotations are defined by

\[
\delta \alpha_{R,L} = \delta \alpha_V \pm \delta \alpha_A .
\]

In the Abelian case finite axial transformations form a group.

The fermionic effective action is formally defined as \( W = -\text{Tr} \log(D) \). To renormalize the ultraviolet divergent trace we will adopt the \( \zeta \)-function prescription [30,32], that is, \( W = -\sum_n \lambda_n^s \log \lambda_n |_{s=0} \). Here, \( \lambda_n \) are the eigenvalues of \( D \) and \( s = 0 \) is understood as an analytical continuation from sufficiently negative \( s \). This method is mathematically well founded [30] and it is particularly convenient in the context of finite temperature since it preserves vector gauge invariance under both small and large gauge transformations [33,18]. This is because, within the \( \zeta \)-function prescription, the effective action depends only on the spectrum of the Dirac operator which is invariant under gauge transformations. Note, however, that in general the effective action is invariant modulo \( 2\pi i \), due to the multivaluation of the logarithm.

It is convenient to introduce the concept of pseudo-parity transformation [21], namely, \( V_\mu \rightarrow V_\mu , M \rightarrow M , A_\mu \rightarrow -A_\mu \) and \( U \rightarrow U^{-1} \). Due to parity invariance, which involves and additional \( (x_0 , x) \rightarrow (x_0 , -x) \) in the fields, the pseudo-parity odd component of the effective action is that containing the Levi-Civita pseudo-tensor. Also, it corresponds to the imaginary part of the effective action. As it is well-known, the real part of the effective action may be regularized preserving chiral symmetry and the chiral anomaly is only essential (i.e., not removable by counterterms) in the pseudo-parity odd component [17]. Here we will concentrate on this latter component, which will be denoted \( W^- \).

The consistent chiral anomaly, defined as the variation of the effective action under an infinitesimal chiral rotation, is given by

\[
\delta W_{d=2}^- = \frac{i}{\pi} \int \text{tr} \left( F - A^2 \right) \delta \alpha_A , \\
\delta W_{d=4}^- = \frac{-1}{12\pi^2} \int \text{tr} \left( 3F^2 + F_A^2 - 4AF_A - \{F,A^2\} - A^4 \right) \delta \alpha_A ,
\]

where we have adopted a standard differential geometry notation: \( D = D_\mu dx_\mu , V = V_\mu dx_\mu , A = A_\mu dx_\mu , F = D^2 = dV + V^2 , F_A = \{D, A\} \), the \( dx_\mu \) anticommute and \( dx_0 dx_1 \cdots dx_{d-1} = \epsilon_{01\cdots d-1} dx \). It is noteworthy that the anomaly depends on the gauge fields only and not on \( M \) or \( U(x) \) and also that there is no anomaly associated to purely vector gauge transformations.

The anomaly can be integrated to yield the gauged WZW action [22,21,34,36], which in two dimensions takes the form

\[
\Gamma(V, A, U) = \frac{1}{12\pi} \int \text{tr}(R^3) + \frac{i}{4\pi} \int \text{tr} \left( V^R R - V^L L + V^R U^{-1} V^L U - V^R V^L \right) ,
\]

where \( R = U^{-1} dU \) and \( L = U dU^{-1} \). In the Wess-Zumino term, \( \int \text{tr}(R^3) \), the integral refers to a three-dimensional manifold on the gauge group which interpolates between the
original $U(x)$ configuration and a fixed configuration belonging to the same homotopy class. The latter can be taken as a constant at zero temperature. The finite temperature case is discussed in great detail in Ref. 37. There it is shown, in particular, that $\Gamma$ does not depend on small variations (in the topological sense) of the interpolating path and also that the variation of $\Gamma$ is independent of the homotopy class representative chosen. For this invariance it is essential that the Wess-Zumino integrand is a closed form.

Alternatively, the same gauged WZW action can also be written as 17, 38–40, 23

$$\Gamma(V,A,U) = -\frac{i}{4\pi} \int \text{tr} \left( \frac{1}{6} R_c^3 - R_c F_R - 2AF_R + \frac{4}{3} A^3 \right) - \text{p.p.c}, \quad (8)$$

where $R_c = R - V^R + U^{-1} V^LU$, $F_R = D_R^2 = dV_R + V^2_R$ transform covariantly under $\Omega_R$ and are invariant under $\Omega_L$, and p.p.c stands for pseudo-parity conjugate, i.e., $A \rightarrow -A$, $U \rightarrow U^{-1}$ (and thus exchanging the labels $R$ with $L$ everywhere). This version can be obtained directly by starting from the Wess-Zumino action and applying minimal coupling, $R \rightarrow R_c$. The result is then chiral invariant but it is no longer a closed 3-form, i.e., it has a spurious dependence on the particular interpolating path taken. This is cured by adding new terms involving the gauge fields and preserving vector gauge invariance. An advantage of this version is that the chiral breaking terms are manifestly polynomial thus yielding a polynomial anomaly. These terms cannot be removed by local polynomial counterterms, however, since they do not form an exact three-form by themselves. The corresponding expression in four dimensions is

$$\Gamma(V,A,U) = -\frac{1}{48\pi^2} \int \text{tr} \left( \frac{1}{10} R_c^5 + 2R_c F^2_R - R_c^3 F_R + R_c F_R U^{-1} F_L U + 2AF_R F_L \
+ 4AF^2_R - 8A^3 F_R + \frac{16}{5} A^5 \right) - \text{p.p.c}. \quad (9)$$

(The LR versions are given in section VI.)

We remark that the mathematical statement is that the infinitesimal variation of $\Gamma(V,A,U)$ equals the consistent anomaly given in the right-hand side of eqs. (6), provided that $V$, $A$ and $U$ transform under chiral rotations as in eq. (5). Therefore, $U$ can actually stand for any field taking values on the gauge group and transforming as $\Omega_R^{-1} U \Omega_R$. This is true when $U$ is the pseudo-scalar field appearing in the Dirac operator, but would apply also for a $U$ suitably constructed out of $V$ and $A$, for instance. This remark is specially relevant when $M = 0$, since then there is no pseudo-scalar field in the Dirac operator. In any case, by construction, the gauged WZW term $\Gamma(V,A,U)$ and the effective action $W^-(V,A,U;M,T)$ differ by chiral covariant terms only.

In order to carry out the ulterior calculation we will fix the chiral gauge: by taking $\Omega_R = \Omega$ and $\Omega_L = U \Omega$ a new field configuration is obtained such that $U = 1$. Let us denote the new gauge fields by $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$. The field $\Omega(x)$ represents a vector gauge freedom. It will be convenient to partially fix this gauge by imposing stationarity of the $\mathcal{V}_0$ field. As shown in Ref. 18, the remaining vector gauge freedom consists of two kinds of transformations, namely, stationary gauge transformations and discrete transformations of the form $\Omega = \exp(x_0 \Lambda(x))$, where $\Lambda(x)$ is any anti-Hermitian matrix commuting with $\mathcal{V}_0(x)$ and with eigenvalues of the form $2\pi in/\beta$, for integer $n$. The quantization of $\Lambda$ ensures the preservation of the periodic boundary conditions. In the Abelian case, the discrete transformations are large in the topological sense.
The difference between the effective actions of the original and rotated configurations is accounted for by the gauged WZW term, which vanishes identically when \( U(x) = 1 \), therefore
\[
W^-(V, A, U; M, T) = W^-(V, A, 1; M, T) + \Gamma(V, A, U).
\] (10)

By construction, since the chiral gauge has been fixed, the effective action of the \((V, A, U = 1)\) configuration defines a chiral invariant action when re-expressed in terms of the original fields. In the actual calculation, which necessarily truncates the exact result to some order, chiral invariance will be maintained provided that the calculation in terms of the rotated fields preserves vector gauge invariance.

### III. Calculation of the Effective Action

In this section we will explain our calculation of the effective action at finite temperature and the expansion used. The result of the calculation is discussed in the next section. Let us introduce the \(\zeta\)-function
\[
\Omega_s(D) = \text{Tr}(D^s) = \sum_n \lambda_n^s,
\] (11)
which is ultraviolet finite if \(\text{Re}(s) < -d\) and a meromorphic function of \(s\) with simple poles at \(s = -d, -d + 1, \ldots, -1\). Then, the \(\zeta\)-function prescription is
\[
W(D) = -\frac{d}{ds} \Omega_s(D)\big|_{s=0}.
\] (12)

Applying Cauchy theorem
\[
\Omega_s = -\text{Tr} \int \frac{dz}{2\pi i} \frac{z^s}{D - z},
\] (13)
where the integration path in the complex plane starts at infinity following for instance the negative real axis, encircles the origin clockwise and goes back to infinity along the same ray. In this regularization, the mass-like term \(z\) is responsible for the explicit breaking of chiral symmetry needed to allow for the chiral anomaly.

To deal with the trace on the space-time degrees of freedom, we apply the Wigner transformation method:
\[
\Omega_s = -\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\beta} \sum_n \int \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | \frac{1}{\not\!p + D - z} | 0 \rangle.
\] (14)

Here, \(p = ik\), \(p_0 = \omega_n\) with \(\omega_n = 2\pi i (n + 1/2)/\beta\) and the sum on \(n\) refers to all integers. \(\text{tr}\) refers to flavor and Dirac spaces. \(|0\rangle\) is the state of zero momentum and zero energy in the space-time Hilbert space, normalized as \(\langle x|0\rangle = 1\). In particular, this implies \(\partial_\mu |0\rangle = \langle 0|\partial_\mu = 0\) whenever a derivative inside \(D\) reaches any of the ends of the matrix element. Also, \(\langle 0|0\rangle = \int d^dx\).

In order to proceed we will work with a chiral gauge fixed configuration, i.e., with the Dirac operator
\[
D = \gamma_\mu (D_\mu + A_\mu \gamma_5) + M, \quad \partial_0 V_0 = 0. \quad (15)
\]

(For simplicity, in this section we will not use a special notation for the rotated fields. Also, for the correct interpretation of the formulas, we remark that in this section we are not using a differential geometry notation.) The main idea of the calculation, in order to be able to carry out the sums and integrations indicated in eq. (14), is to perform a series expansion on some of the pieces contained in the Dirac operator. Depending on the precise choice made one can obtain perturbation theory, gradient expansions, inverse mass expansions, etc. We want to organize the calculation so that operators of higher dimension are also of higher order, however, it is essential to do this preserving gauge invariance. In the Wigner transformation formulation, the integral over \(k\) projects out the contribution which is invariant under time-independent gauge transformations whereas the sum over frequencies does the same thing for the discrete transformations \([40,18]\). (Loosely speaking, a time-independent gauge transformation can be compensated by a shift in \(k\) and a discrete gauge transformation can be compensated by a shift in \(\omega_n\).) Therefore, gauge invariance is preserved by expanding in powers of \(\gamma D + A / \gamma 5\) while keeping \(p / \gamma 0 + D_0 + M - z\) in the denominator. At finite temperature, making a further expansion in powers of \(D_0\) would break gauge invariance because \(D_0\) transforms discretely under discrete gauge transformations. The deep reason for this is that counting powers of \(\partial_0\) effectively means to study the change of the functional under dilatations in the time coordinate, and such dilatations are not supported by the boundary conditions.

Treating \(D_0\) fully non perturbatively in the way just described is possible since, due to our choice of chiral gauge fixing, the quantities \(p_\mu, \partial_0, V_0\) and \(M - z\) appearing in the denominator are all commuting. (See \([18]\) for such a non-perturbative treatment in three dimensions.) However, in the spirit of retaining only lower dimensional operators, a further natural expansion is that in powers of the quantity \(\hat{D}_0\) defined by \(\hat{D}_0(X) = [D_0, X]\) which preserves gauges invariance.

Let us illustrate this kind of calculations by detailing the procedure in the two dimensional case. Of course, due to the identity \(\gamma_\mu \gamma_5 = i \epsilon_{\mu \nu} \gamma_\nu\), in this case it is algebraically simpler to use a single non anti-Hermitian vector field \(W_\mu = V_\mu - i \epsilon_{\mu \nu} A_\mu\) in the calculation. Nevertheless, for greater similarity with the four dimensional case, we will keep the vector and axial fields as independent variables. Since the leading order terms are in principle of the form \(V_0 A\) and \(A_0 V\), we have to retain terms of first and second order in the expansion of the \(\zeta\)-function,

\[
\Omega_{n,1+2} = - \int \frac{dk}{2\pi i} \sum_n \int \frac{dz}{2\pi i} z^n \text{tr}[0] \left( - \frac{1}{\gamma_0 Q + \gamma p + \mu} \left( \gamma D + A_\mu \gamma_5 \right) \frac{1}{\gamma_0 Q + \gamma p + \mu} \right) |0\rangle, \quad (16)
\]

where \(Q = \omega_n + D_0\) and \(\mu = M - z\). Since \(Q, p\) and \(\mu\) commute with each other we can use the identity

\[
\frac{1}{\gamma_0 Q + \gamma p + \mu} = \frac{\mu - \gamma_0 Q - \gamma p}{\Delta}, \quad \Delta = \mu^2 - Q^2 + k^2. \quad (17)
\]

The Dirac trace can then be evaluated. Retaining only the pseudo-parity odd terms one finds
\[\Omega^0_{s,1+2} = 2i \int \frac{dk}{2\pi} \frac{1}{\beta} \sum_n \int \frac{dz}{2\pi i} z^s \mu \text{tr}(0) \frac{1}{\Delta} \left( A Q + Q A + (3k^2 - \mu^2) \right) \left( A_0 \frac{1}{\Delta} D + D \frac{1}{\Delta} A_0 \right) \]
\[-Q A_0 \frac{1}{\Delta} [D, Q] + [D, Q] \frac{1}{\Delta} A_0 Q + A_0 Q \frac{1}{\Delta} D Q + Q D \frac{1}{\Delta} Q A_0 \right) \frac{1}{\Delta} |0\rangle. \tag{18}\]

Next, all the “naked” \( D \), i.e., those which are not inside commutators, are brought to the right of the matrix element. For instance
\[D \frac{1}{\Delta} = \frac{1}{\Delta} D + \frac{1}{\Delta} \{Q, [D, Q]\} \frac{1}{\Delta}. \tag{19}\]

This produces commutator terms of the form \([D, X]\) which are gauge invariant under time-independent gauge transformations. There are also non-covariant terms, of the form \(\text{tr}(0)XD|0\rangle\), which can be replaced by \(\text{tr}(0) XV|0\rangle\), since \(\partial|0\rangle = 0\). Thus at the end all appearing operators are multiplicative in \(x\)-space, that is, \(x\) becomes just a parameter and the operators effectively act on a Hilbert space which is the tensor product of flavor and time spaces only.

The problem now is that the sum over the variables \(k\), \(\omega_n\), and \(z\) cannot be done in a straightforward manner because they appear in different operators which do not commute. At this point, one can insist on obtaining a strict expansion in powers of \(D\) and \(A_\mu\) (i.e., without further expanding in powers of \(D_0\) \[18\]. Instead of doing so, we will simplify the problem by retaining only terms which are of lowest dimension in fields and derivatives. The counting is defined as follows: \(M\) will be taken of order zero, \(V_\mu\), \(A_\mu\) and \(\partial_\mu\) count as first order each. This is equivalent to count the number of Lorentz indices. The treatment of \(D_0\) requires some care: the quantity \(Q\) is in principle of zeroth order due to \(\omega_n\), however, inside commutators \(\omega_n\) does not contribute, so \(Q\) becomes \(D_0\) and counts as first order. Then (recalling that expanding in powers of \(D_0\) is forbidden) the strategy to follow is to bring all naked \(Q\) (the explicit ones as well as those inside \(\Delta\)) together to the left of the matrix element. There they commute and the indicated sums and integrals can be easily carried out. In doing this we keep only leading order terms. Moving explicit the \(Q\) operators to the left can be done systematically, generating commutator terms. On the other hand, moving \(\Delta\) operators to the left is more subtle in the sense that it cannot be done exactly in closed form. It can only be done up to some given order in the expansion. This comes from the identity
\[X \frac{1}{\Delta} = \frac{1}{\Delta} X - 2Q \frac{1}{\Delta} [D_0, X] \frac{1}{\Delta} + \frac{1}{\Delta} [D_0, [D_0, X]] \frac{1}{\Delta}, \tag{20}\]

which shows that new terms with a \(\Delta^{-1}\) factor at the right are generated. Nevertheless, such terms always come with commutators and thus they are of higher order than the original one. Therefore repeating the commutation operation a sufficient number of times, all naked \(Q\) will end up at the left, modulo higher order terms. Once this has been achieved, all \(Q\) not at the left has been replaced by \(D_0\) and they appear inside commutators. The next thing to observe is that in all the naked \(Q\) (explicit and implicit) which are now at the left, \(D_0\) can be replaced by \(V_0\) since \(\langle 0|\partial|0\rangle\) vanishes and \(V_0\) is stationary. So finally we have a multiplicative operator both in \(x\)- and \(x_0\)-spaces and hence \(\langle 0|X|0\rangle\) is just \(\int d^4x X\). Note that there is no difficulty of principle in computing the expansion at any given order. In practice, it is simpler to bring the naked \(Q\) to the left and the naked \(D\) to the right doing the truncation at the same time.
Applying this method to eq. (18), the pseudo-parity odd $\zeta$-function in two dimensions becomes, at leading order,

$$\Omega_{s,\text{lead.}} = 2i \int \frac{dk}{2\pi \beta} \sum_n \int \frac{dz}{2\pi i} \zeta_n^s \mu \text{tr} \langle 0 | \left[ \frac{2}{\Delta^2} A + \left( \frac{1}{\Delta^2} + 4 \frac{Q^2}{\Delta^3} \right) [A, D_0] ight. $$

$$+ \left\{ \frac{4 k^2}{\Delta^3} - \frac{1}{\Delta^2} \right\} ([D, A_0] + 2 A_0 V) \rangle | 0 \rangle,$$

(21)

where effectively $Q = \omega_n + V_0$. In this expression, the contributions from $[D, A_0]$ and $A_0 V$ vanish after the integration over $k$. The cancellation of non-covariant terms of the form $\text{tr} \langle 0 | X V | 0 \rangle$ is a non-trivial check of the calculation. Likewise, the contribution from $[A, D_0]$ vanishes using integration by parts and cyclic property. (The cyclic property does not apply directly here since the operation $\langle X \rangle = \sum_n \text{tr} \langle 0 | X | 0 \rangle$ is not a trace, however, it can be shown that $\langle XY \rangle = \langle YX \rangle$ whenever $X$ is multiplicative in $x$-space and $Y$ is a function of $D_0$ [18].)

The only remaining term $Q \Delta^{-2} A$ presents a further subtlety. In principle, this term would be of order one but it is actually of second order. This is because, after summing over frequencies, the factor $Q \Delta^{-2}$ yields an odd function of $V_0$ (recall that $\partial_0$ is no longer present here) and thus this factor is of first order. In general, the Levi-Civita pseudo-tensor requires the saturation of $d$ Lorentz indices in a $d$-dimensional space-time, thus the leading order term will be of order $d$.

In four dimensions the calculation of the leading contribution is algebraically more involved, but it can be done along the same lines. For instance, there appear terms of the form $\langle \Delta^{-2} [A, D_0], F \rangle$, which in principle would count as fourth order, however, using again the cyclic property, it is clear that the non-trivial contribution from $\Delta$ starts at second order in $V_0$, so this term is actually of sixth order. (This was to be expected in this and similar formally ultraviolet divergent terms, since otherwise there would be an spurious contribution to the scale anomaly which is absent in the pseudo-parity odd sector.)

The calculation can alternatively be done using a $(d + 2)$-dimensional formalism, which relates the pseudo-parity odd part of the effective action in $d$ dimensions with the baryon number in $d + 2$ dimensions [11][13][10] (for instance, the two dimensional Wess-Zumino term $(12\pi)^{-1} \int \text{tr} R^3$ is the correctly normalized baryon number in four dimensions). Such formalism gives directly the effective action as the integral of a $(d + 1)$-form, as in eqs. (8,9). One advantage of this procedure is that enforcement of vector gauge invariance is sufficient to fix the action without introducing a $\zeta$-function regularization, but working in more space-time dimensions is also more involved. The results found with this formalism (at least in two dimensions) are consistent with those found in the $\zeta$-function prescription.

Let us remark that fixing the chiral gauge by $U = 1$ and $\partial_0 V_0 = 0$ is essential to carry out the calculation. Fixing $V_0$ to be stationary allows to set $\partial_0 = 0$ in the operators $Q$ moved to the left. Eliminating $U$ allows to use the identity in eq. (17). An exception is the case of Abelian and stationary fields, since in this case $U$, $V_0$ and $\partial_0$ commute and this allows to carry out the calculation without fixing $U = 1$. Such calculation can be done most conveniently using the $(d + 2)$-dimensional formalism and in particular it checks, once more, that the anomaly is temperature independent.
IV. THE EFFECTIVE ACTION AT FINITE TEMPERATURE

Recalling the exact formula eq. (10), we define the leading order of the pseudo-parity odd component of the effective action as

\[ W_{\text{lead.}}(V, A, U; M, T) = W_{\text{lead.}}(V, A, 1; M, T) + \Gamma(V, A, U), \]  

(22)

where \( V \) and \( A \) refer to the rotated fields. The chiral invariant term, \( W_{\text{lead.}}(V, A, 1; M, T) \), has been computed with the method described in the previous section and is given by

\[ W_{\text{2,lead.,c.i.}}^- = -2i \int \text{tr} \left( \varphi_{1,1}^{(2)} A \right), \]  

(23)

\[ W_{\text{4,lead.,c.i.}}^- = \int \text{tr} \left( 2\varphi_{2,1}^{(4)}(A, F) + \left( \frac{2}{3}\varphi_{2,0}^{(4)} + \frac{8}{3}\varphi_{3,2}^{(4)} \right) A_0(A, F) - \frac{16}{3}\varphi_{3,3}^{(4)} A^3 \right), \]

for two and four dimensions, respectively. Here, a differential geometry notation has been used with \( A = A_i dx_i, \) \( F = \frac{1}{2}[D_i, D_j]dx_i dx_j \) and \( F_A = [D_i, A_j]dx_i dx_j, \) and \( A_0 \) actually stands for \( A_0 dx_0. \) Further, the coefficients \( \varphi_{r,m}^{(d)} \) are functions of \( V_0 \) (and \( M \) and \( T \)) given by

\[ \varphi_{r,m}^{(d)} = \partial_s \left( 2r \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\beta} \sum_n \int \frac{dz}{2\pi i} z^s(M - z) \frac{Q^m}{\Delta r + 1} \right)_{s=0}. \]  

(24)

where \( Q = \omega_n + \Im \) (with \( \omega_n = 2\pi i(n + \frac{1}{2})/\beta \)) and \( \Delta = (M - z)^2 - Q^2 + k^2. \) Besides, a factor \( dx_0 \) is implicit when \( m \) is odd so that in this case \( \varphi_{r,m}^{(d)} \) is a 1-form. When \( m \) is even, \( \varphi_{r,m}^{(d)} \) is an even function of \( V_0 \) and is of zeroth order, on the other hand for odd \( m, \) \( \varphi_{r,m}^{(d)} \) is odd in \( V_0 \) and it counts as first order. As a consequence all the terms in \( W_{\text{d,lead.}}^- \) are of dimension \( d. \)

The following simpler formulas are equivalent whenever they are convergent:

\[ \varphi_{r,m}^{(d)} = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\beta} \sum_n \frac{Q^m}{\Delta r}, \]  

(25)

where \( \Delta = M^2 - Q^2 + k^2 \) and again a factor \( dx_0 \) should be added when \( m \) is odd. Explicitly convergent formulas for \( \varphi_{1,1}^{(2)} \) are

\[ \varphi_{1,1}^{(2)} = -\frac{1}{4} \int \frac{dk}{2\pi} \left[ \tanh \left( \frac{\beta}{2} \sqrt{k^2 + M^2 + \Im} \right) - \text{h.c.} \right] dx_0 \]

\[ = -\frac{1}{2} \int \frac{dk}{2\pi} \frac{\sinh(\beta \Im)}{\cosh(\beta \sqrt{k^2 + M^2}) + \cosh(\beta \Im)} dx_0. \]

(26)

These functions are not all independent. The following relations are useful

\[ \varphi_{2,1}^{(4)} = \frac{1}{4\pi}\varphi_{1,1}^{(2)}, \]

\[ \varphi_{3,2}^{(4)} = -\frac{1}{4}\varphi_{2,0}^{(4)} + M^2\varphi_{3,0}^{(4)} - \frac{1}{32\pi^2}, \]

\[ \varphi_{3,3}^{(4)} = -\frac{1}{4}\varphi_{2,1}^{(4)} + M^2\varphi_{3,1}^{(4)}. \]

(27)

So the four-dimensional action can also be written as
\[ W_{4, \text{lead,c.i.}}^\perp = \int \text{tr} \left( 2\varphi_{2,1}^{(4)} \{ \mathcal{A}, \mathcal{F} \} + \left( \frac{8}{3} M^2 \varphi_{3,0}^{(4)} - \frac{1}{12\pi^2} \right) \mathcal{A}_0 \{ \mathcal{A}, \mathcal{F}_A \} + \frac{16}{3} \left( \frac{1}{4} \varphi_{2,1}^{(4)} - M^2 \varphi_{3,1}^{(4)} \right) \mathcal{A}^2 \right). \]  

(28)

In two dimensions, \( W_{\text{lead,c.i.}}^\perp \) contains only a term with the structure \( \mathcal{V}_0 \mathcal{A} \). The absence of \( \mathcal{A}_0 \mathcal{V} \) can be understood as a two dimensional peculiarity. In this case the Dirac operator depends only on the two combinations \( \mathcal{V}_0 - i\mathcal{A}_1 \) and \( \mathcal{V}_1 + i\mathcal{A}_0 \) and not on the four fields independently. Because gauge invariance requires \( \mathcal{D}_1 + i\mathcal{A}_0 \) to appear only inside commutators, the possible terms of dimension two of the form \( \mathcal{A}_0 \mathcal{V} \) would have to come from \( [\mathcal{D}_1 + i\mathcal{A}_0, \mathcal{D}_1 + i\mathcal{A}_0] \) which is identically zero. On the other hand, in four dimensions it is not clear why \( W_{\text{lead,c.i.}}^\perp \) contains no terms with the structure \( \mathcal{A}_0 \mathcal{V} \mathcal{V} \mathcal{V} \), and why no time derivative should appear.

The gauge invariance of \( W_{\text{lead,c.i.}}^\perp \) is obvious under time-independent gauge transformations. Under discrete gauge transformations \( \mathcal{V}_0 \) transforms to \( \mathcal{V}_0 + \Lambda \), so its eigenvalues are shifted by an integer multiple of \( 2\pi i/\beta \). Due to the sum over frequencies, \( \varphi_{r,m}^{(d)} \) are periodic functions of \( \mathcal{V}_0 \) with period \( 2\pi i/\beta \) and the result is invariant.

Another important test is the zero temperature limit of the result. As noted before, soft-pion theorems imply that at zero temperature, the only pieces present at leading order in the pseudo-parity odd sector are those coming from the gauged WZW term, which already saturates the anomaly. Indeed, the new contributions would be chirally invariant, and thus gauge invariant in terms of the rotated fields (i.e., in the chiral gauge \( U = 1 \)). The possible leading order terms consistent with Euclidean and gauge invariance are \( \langle F_A \rangle \) in two dimensions and \( \langle F_A F_A \rangle \) and \( \langle F_A A^2 \rangle \) in four dimensions, all of which vanish identically. Of course, there are new chiral invariant terms at zero temperature at sub-leading orders, for instance, in two dimensions, the fourth order term is (e.g. using the formulas in Ref. [40])

\[ -\frac{i}{12\pi M^2} \int d^2x \epsilon_{\mu\nu} \text{tr} \left( (F_{\mu\nu} - [A_\mu,A_\nu])(D_\lambda,A_\lambda) \right). \]  

(29)

Therefore, restoration of Euclidean invariance requires \( W_{\text{lead,c.i.}}^\perp \) to vanish in the zero temperature limit. In this limit \( \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\beta} \sum_n \) becomes \( \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \). All terms containing \( \varphi_{r,m}^{(d)} \) with odd \( m \) vanish: since \( \omega_n \) is now a continuous variable, it can be shifted to eliminate \( \mathcal{V}_0 \). The resulting integral vanish due to antisymmetry of the integrand. This cancellation is illustrated by eq. (27). Likewise, the coefficient of the term \( \mathcal{A}_0 \{ \mathcal{A}, F_A \} \) also vanishes at zero temperature. This can be seen using the form in eq. (28) and computing \( \varphi_{3,0}^{(4)} \) with eq. (25).

V. ANOMALOUS AMPLITUDES

In this section we will obtain the mesonic amplitudes derived from the effective action at finite temperature. We will consider only the Abelian and stationary case, since it allows to compare with previous calculations in the literature [13,26].

In the Abelian and stationary case, \( \mathcal{V} = V \) and \( A = A - \frac{1}{2}d\phi \), where \( U = \exp(\phi) \), thus the chiral invariant part of the action at leading order reads

\[ W_{2,\text{lead,c.i.}}^\perp = -2i \int \varphi_{1,1}^{(2)} \left( A - \frac{1}{2}d\phi \right), \]  

(30)

\[ W_{4,\text{lead,c.i.}}^\perp = \int \left( 4\varphi_{2,1}^{(4)} F + \left( \frac{16}{3} M^2 \varphi_{3,0}^{(4)} - \frac{1}{6\pi^2} \right) A_0 F_A \right) \left( A - \frac{1}{2}d\phi \right). \]
In order to find the amplitudes, let us retain the leading order in an expansion in powers of $V_0$. A simple calculation, gives

\[\varphi_{1,1}^{(2)} = -\frac{1}{2\pi} (1 - f)V_0 + O(V_0^3),\]
\[\varphi_{2,1}^{(4)} = -\frac{1}{8\pi^2} (1 - f)V_0 + O(V_0^3),\]
\[\varphi_{3,0}^{(4)} = \frac{1}{32\pi^2 M^2} f + O(V_0^2),\]

where we have introduced the dimensionless function

\[f = \sum_{n \in \mathbb{Z}} \frac{\pi (\beta M)^2}{((\beta M)^2 + \pi^2 (2n + 1)^2)^3/2},\]

which has limits $f \to 1$ when $T \to 0$ and $f \to 0$ when $M \to 0$. On the other hand, in the Abelian and stationary case the gauged WZW term reduces to

\[\Gamma_2(V, A, U) = \frac{i}{2\pi} \int V_0 d\phi\]
\[\Gamma_4(V, A, U) = -\frac{1}{12\pi^2} \int (3V_0 F + A_0 F_A) d\phi.\]

Therefore the leading anomalous amplitudes can be read from the following effective actions

\[W_2 = \frac{i}{2\pi} \int V_0 (f d\phi + 2(1 - f)A),\]
\[W_4 = -\frac{1}{12\pi^2} \int (3V_0 F + A_0 F_A) (f d\phi + 2(1 - f)A).\]

These formulas suggest that the temperature dependence of the amplitudes are independent of the space-time dimension. In the two cases considered, introducing a finite temperature amounts to make the substitution $d\phi \to f d\phi + 2(1 - f)A$ in the anomalous amplitudes at zero temperature. In particular, the term $V_0 F d\phi$, related to the process $\pi^0 \to \gamma\gamma$, gets an extra factor of $f$ as compared with the zero temperature amplitude. For $M \ll T$, this factor behaves as

\[f = \frac{7}{4\pi^2} \zeta(3) \frac{M^2}{T^2} + O\left(\frac{M^4}{T^4}\right).\]

A result which is in agreement with those in Refs. [19,26].

In the zero temperature limit $f = 1$ and the amplitudes reduce to the gauged WZW term, as they should. Thus in this case there is a tight relation between the anomalous amplitudes and the anomaly. On the other hand, at finite temperature the amplitudes are modified, yet the anomaly is preserved since the terms added to the gauged WZW action are chirally invariant.

The fact that $f$ vanishes as $M \to 0$ implies that the amplitudes involving $\phi$ cancel in the chirally symmetric phase. In principle, such a cancellation is to be expected on general grounds (and thus it is a non trivial test of the calculation): since the variable $U$ no longer appears in the Dirac operator when $M = 0$, the effective action should also be independent
of $U$. More generally, the effective action should depend analytically on the external fields $V_\mu$, $A_\mu$ and the combination $MU$. There are, however, two considerations which affect this conclusion. First, the argument may be spoiled by infrared divergences. For instance, at zero temperature the dominant anomalous term is the WZW action which depends on $U$ rather than $MU$. At finite temperature, $\omega_n$ is always different from zero so no infrared divergences should appear in the fermionic effective action. Second, even if the argument holds for the exact effective action, it needs not apply when only the leading order is retained. The truncation might introduce a spurious $\phi$ dependence at $M = 0$. The situation at $M = 0$ beyond the Abelian and stationary restrictions is further studied in the next section.

Another important remark is that the fact that $f \to 0$ as $M \to 0$ does not directly imply that the amplitude for $\pi \gamma \gamma$ vanish in the chirally symmetric phase. For instance, in a linear sigma model, the relevant piece of the action is $g \bar{\psi} (M + \sigma + i \gamma_5 \vec{r} \vec{p}) \psi$ hence the pion field $\pi(x)$ is related to $\phi$ as $\pi \sim M \phi$, as a consequence the relevant amplitude goes as $f/MV_0F d\pi$ and the renormalization factor due to the temperature is $f/M$. In our calculation, in agreement with [19,26], $f = O(M^2/T^2)$ and $f/M \to 0$ as $M \to 0$. However, in Refs. [27,28], $f = O(M/T)$. Whereas $f$ still vanishes as $M \to 0$, the $\pi \gamma \gamma$ amplitude does not vanish, presumably due to the different kinematical conditions assumed [28].

### VI. MASSLESS FERMIONS

In the case of massless fermions, the gauge fixing condition $U = 1$ is meaningless because there is no field $U$ in the Dirac operator and no chiral rotation is necessary. Thus the calculation of the effective action described in section III can be applied to any gauge field configuration $(V,A)$ provided only that the requirement $\partial_0 V_0 = 0$ is satisfied. In order not to unnecessarily complicate the notation, we will use the symbols $V$ and $A$ to denote, not the original fields, but the fields after a vector transformation so that $V_0$ is stationary. That is, the Dirac operator is

$$D = \gamma_\mu (D_\mu + A_\mu \gamma_5), \quad \partial_0 V_0 = 0,$$

and the effective action at leading order is given by $W_{\text{lead}, \text{c.i.}}$ in eqs. (35) but using $V$ and $A$ instead of $V$ and $A$ and with $M = 0$. The required massless functions $\varphi^{(d)}_{r,m}$ can be computed in closed form and are given by

$$\varphi^{(d)}_{1,1} = -\frac{1}{2\pi} V_0, \quad \varphi^{(d)}_{2,1} = -\frac{1}{8\pi^2} V_0,$$

where $V_0$ actually stands for the 1-form $V_0 dx_0$. (A detailed analysis shows that in this formula $V_0$ is to be understood modulo $2\pi i/\beta$ so that periodicity is preserved.) Note that these functions no longer vanish modulo $2\pi i/\beta$ so that periodicity is preserved.) Note that these functions no longer vanish as $T \to 0$ in the massless case; the two limits $M \to 0$ and $T \to 0$ do not commute. The leading order actions become

$$W_{\text{lead}, \text{c.i.}} (V,A) = \frac{i}{\pi} \int \text{tr} (V_0 A),$$

$$W_{\text{lead}, \text{c.i.}} (V,A) = -\frac{1}{12\pi^2} \int \text{tr} \left( 3V_0 \{A,F\} + A_0 \{A,F_A\} + 2V_0 A^3 \right).$$
Consistently with dimensional arguments, these actions are temperature independent (except, of course, through the boundary conditions). It should be noted that they are not really local polynomial actions due to the gauge fixing: when expressed in terms of the original fields, these actions are in fact non-local.

In the massive case, the correct chiral transformation of the action was not an issue, since the chiral gauge had been fixed. In the present case, however, the correct transformation is not evident by construction. If these formulas are applied, not directly to \((V,A)\) but to a chirally rotated configuration \((\mathcal{V},\mathcal{A})\) (with \(\mathcal{V}_0\) stationary) chiral symmetry would require

\[
W_{\text{lead}}^{-}(V,A) = W_{\text{lead}}^{-}(\mathcal{V},\mathcal{A}) + \Gamma(V,A,U),
\]

where \(U\) denotes the rotation. This equality holds for the exact effective action, what is not evident is that it should hold for its leading order too. Note that an equivalent statement is that the right-hand side is independent of \(U\), that is, there is no spurious dependence on \(U\). The discussion of the previous section shows that this is true in the Abelian and stationary case. (In fact, the eqs. (31) which give \(\varphi^{(2)}_{1,1}\) and \(\varphi^{(4)}_{2,1}\) at lowest order in \(V_0\), turn out to coincide with the exact ones when \(M = 0\), eqs. (37).) For Abelian but time dependent configurations, still \(\mathcal{A} = A - \frac{1}{2}d\phi\), where \(U = \exp(\phi)\), and \(\mathcal{V} = V\) (there is no vector gauge transformation involved since \(\partial_0 V_0 = \partial_0 \mathcal{V}_0 = 0\)). Then, in two dimensions, the right-hand side of eq. (39) yields

\[
W_{\text{lead}}^{-}(V,A) + \frac{i}{2\pi} \int \mathcal{V} d\phi
\]

(where \(d\) stands for \(\partial_0 dx_0\)). This formula implies that in general there is a spurious dependence on \(U\).

In order to consider the general non-Abelian case, note that eq. (39) is equivalent to say that \(W_{\text{lead}}^{-}(V,A)\) displays the correct chiral anomaly. A detailed calculation shows that these actions do not reproduce the correct chiral anomaly in general, but they do so for stationary configurations. It is rather remarkable that, even if restricted to the stationary case, these actions are able to saturate the non-Abelian anomaly in two and four dimensions. This is more so since the calculation of \(W_{\text{lead}}^{-}\), detailed in section [14,10], knows nothing of the anomaly or the WZW term. From this point of view, it constitutes a non-trivial check of the calculation.

The fact that the anomaly is not always reproduced means that, in the massless case, the truncation of the effective action to its leading order violates chiral symmetry. This is because we have done an expansion in powers of the operator \(\gamma D + A \gamma_5\), which is not chiral covariant; under chiral rotations it mixes with the other piece of the Dirac operator, \(\gamma_0 D_0\). This implies that the correct anomaly is only recovered through cancellations among the axial variation of terms of different order. (A different matter is directly computing the axial anomaly use our expansion. The axial anomaly within the \(\zeta\)-function renormalization is given by \(\text{Tr}(-2\alpha A \gamma_5 D^s)_{s=0}\ [14,10]\) which, in the massless case, can computed with the same technique described in section [11] and gives the correct result in closed form.)

It is interesting to compare our result in the two-dimensional case with the exact effective action for the massless Dirac operator (the Weyl determinant) known in closed form [13]. The exact result follows from the observation that, upon complex analytical extension of the chiral group parameters, chiral transformations are sufficient to bring any gauge field
configuration to a space-time constant configuration. Therefore, the exact effective action is given by two terms. First, the gauged WZW term associated to the required analytically extended chiral rotation and second, the effective action corresponding to the constant configuration. This latter term will depend on the temperature, as will also depend higher orders in our expansion. In the Abelian case (and concentrating from now on on the temperature independent term only) the corresponding exact effective action for the pseudo-parity odd sector is

\[ W^-(V, A) = \frac{i}{\pi} \int V_t A_\ell. \quad (41) \]

The labels \( \ell \) and \( t \) stand for longitudinal and transverse, respectively, that is,

\[ V_\ell = d \left( (\partial_\mu)^{-1} \partial_\nu V_\nu \right), \quad V_t = V - V_\ell. \quad (42) \]

This action is non-local thus it cannot be removed by a local polynomial counterterm, i.e. by a suitable choice of the renormalization prescription. In the stationary case, \( V_t = V_0 \) and \( A_\ell = A \), so the exact effective action coincides with that given in eq. (38). That is, in this particular case, the leading order is exact (in the temperature independent sector of the effective action). On the other hand, for non stationary configurations, an expansion of the exact result to extract its leading order is not well-defined due to the presence of \( \partial^{-2} \) in the definition of \( V_\ell \), i.e., due to infrared divergences in the expansion.

The exact Weyl determinant is not known in four dimensions. The trick of analytical extension of the chiral group only doubles the dimension of the chiral group and thus it is insufficient to rotate to zero all the components of the gauge configuration. Nevertheless, in the Abelian case it is easy to write down a pseudo-parity odd action which saturates the correct anomaly, namely,

\[ W^-(V, A) = \frac{1}{12\pi^2} \int (2FAV_t - FVA_\ell - F_A A_\ell). \quad (43) \]

(A version of it exists for any even number of dimensions.) Unlike the two dimensional case, this action does not reduce directly to that given in eq. (38) for stationary configurations. This does not necessarily imply that they are inconsistent with each other, since the difference are terms which are subject to infrared ambiguities if one insists on a gradient expansion.

Of course, for massless fermions it is much more natural to work with chiral fields. Since right and left fields are completely decoupled, the effective action must satisfy

\[ W^-(V, A) = W^-(V^R) - W^-(V^L) + P(V^R, V^L), \quad (44) \]

where \( P \) is a local polynomial introduced by the renormalization prescription. In terms of the chiral field \( v \) (say \( V^R \)) the consistent anomalies take the well-known form

\[ \delta W_2^- = \frac{i}{4\pi} \int \text{tr} \left( F - v^2 \right) \delta \alpha, \]

\[ = \frac{i}{4\pi} \int \text{tr} \, v d \delta \alpha, \]

\[ \delta W_4^- = \frac{1}{48\pi^2} \int \text{tr} \left( -2F^2 + Fv^2 + vFv + v^2F - v^4 \right) \delta \alpha, \]
\[ \Gamma_2(v, U) = -\frac{i}{12\pi} \int \text{tr}(R^3) + \frac{i}{4\pi} \int \text{tr}(vR) \]

\[ \Gamma_4(v, U) = -\frac{1}{240\pi^2} \int \text{tr}(R^5) + \frac{1}{48\pi^2} \int \text{tr}\left(v^3R - 2FvR + v^2R^2 - \frac{1}{2}vR_vR + vR^2\right) \]

\[ \Gamma_4(v, U) = -\frac{1}{240\pi^2} \int \text{tr}\left(R^5 + v^5 - 5(R_c^3 + v^3)F + 10(R_c + v)F^2\right). \]  

Here, \( R = U^{-1}dU \) and \( R_c = R - v \), and \( U \) transforms as \( U\Omega \) when \( D \) transforms as \( \Omega^{-1}D\Omega \). In the \((d + 1)\)-dimensional version of formulas, the polynomial term is the normalized \( d + 1 \) Chern-Simons term and thus it yields the correct anomaly \([17]\).

The Abelian actions in Eqs. (41) and (43) satisfy the decoupling condition, eq. (44), with

\[ W^-(v) = -\frac{i}{4\pi} \int v_\ell v, \quad P = \frac{i}{2\pi} \int VA, \]

\[ W^-(v) = \frac{1}{24\pi^2} \int v_\ell vd\ell, \quad P = \frac{1}{6\pi^2} \int AAvdV, \quad (47) \]

in two and four dimensions respectively. In each case the polynomial \( P \) is uniquely determined by vector gauge invariance. These actions are constructed so that they saturate the anomaly, that is, they correspond to the gauged WZW term associated to the chiral rotation which brings the field \( v \) to the gauge \( v_\ell = 0 \). (This is achieved by taking \( U = \exp(\phi) \) where \( \phi \) is defined by \( v_\ell = d\phi \).) Of course, besides the gauged WZW term the exact effective action has another contribution (namely, the exact effective action of the purely transverse field) but it is chiral invariant by construction and, at least in two dimensions, it can be shown to be pseudo-parity even. The same procedure can be followed in the \( VA \) version of the theory, that is, taking the gauged WZW term associated to axially rotate to the gauge \( A_\ell = 0 \). In two dimensions this procedure reproduces eq. (41) and thus it gives nothing new, however, in four dimensions it yields

\[ W^-(V, A) = -\frac{1}{12\pi^2} \int (3FV + FA_A)A_\ell. \]

This action differs from that in eq. (43) by a chiral invariant term of the form \( \langle FV_\ell A_\ell \rangle \). Since such term is not a local polynomial (and as far as I can see, is not identically zero) both actions are not related by a change in the renormalization prescription. In principle only the action in eq. (14), which satisfies the decoupling formula (44), could come from integration of the fermions.

The chiral version of our leading order action corresponds to take \( V^R = v \) and \( V^L = 0 \) in the \( VA \) version, eqs. (38). This gives

\[ W^-_{2,\text{lead.}}(v) = \frac{i}{4\pi} \int \text{tr}(v_0v), \quad (49) \]

\[ W^-_{4,\text{lead.}}(v) = \frac{1}{24\pi^2} \int \text{tr}\left(-v_0\{v, F\} + \frac{1}{2}v_0v^3\right), \]
which refer to the gauge $\partial_0 v_0 = 0$.

In the two dimensional case there is an important difference with the $VA$ form of the action: due to the gauge condition, $\partial_0 v_0 = 0$, the only allowed infinitesimal gauge transformations are time-independent ones. Under this restriction it can be immediately verified that the chiral two dimensional action at leading order always yields the correct anomaly; the configuration no longer has to be stationary as in the $VA$ form. (In the $VA$ form of the theory, the analogous statement holds if the formula is applied only in the chiral gauge $\partial_0 V_0 = \partial_0 A_0 = 0$. Eq. (40) confirms this observation since in such a gauge $\phi$ can only be stationary and the spurious term $V d_0 \phi$ cancels.) This means that for any configuration, the effective action can be computed by integrating the anomaly up to a $v_0$-stationary gauge and applying our leading order formula. The result will be independent of the particular $v_0$-stationary gauge chosen. Unfortunately, the action so constructed differs from the correct one by gauge invariant terms, unless both $v_0$ and $v_1$ are stationary. What would be needed is to “rotate” the original configuration to the “gauge” $\partial_0 v_0 = \partial_0 v_1 = 0$. This can be done using an analytical extension of the chiral group, in the same spirit as Rothe’s calculation [3].

Not unexpectedly, in four dimensions the situation is worse. The anomaly is reproduced by $W_{4,\text{lead}}(v)$ if the configuration is stationary, but not otherwise. Therefore it is not possible to give a well-defined effective action in this case; the precise value would depend on the particular $v_0$-stationary gauge taken. It would be necessary to bring the configuration to the stationary case, but analytical extension of the chiral group is not sufficient to do this.

**VII. SUMMARY AND CONCLUSIONS**

We have studied the temperature dependence of the effective action of fermions in the presence of external bosonic fields. Due to the complexity of the problem, only the leading order terms, that is, those which can compete with the anomalous ones, have been retained. However, the calculation could in principle be carried out to higher orders as well. The temperature dependence of the anomalous amplitudes is shown to be fully consistent with chiral symmetry including the known temperature independence of the axial anomaly. The known zero temperature limit, given by soft-pions theorems, is verified. Also, for temperatures near chiral symmetry restoration, previous results in the literature obtained with the imaginary time formalism for static final states are reproduced. Finally, some non trivial tests are also verified in the case of massless fermion, but full $U$-independence of the effective action as $M \to 0$ is only reproduced in the stationary case, due to the truncation at leading order.

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