EXPANSIONS OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS FROM THE TAYLOR–STRATONOVICH EXPANSION BASED ON MULTIPLE TRIGONOMETRIC FOURIER SERIES. COMPARISON WITH THE MILSTEIN EXPANSION

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ABSTRACT. The article is devoted to comparison of the Milstein expansion of iterated Stratonovich stochastic integrals with the method of expansion of iterated stochastic integrals based on generalized multiple Fourier series. We consider the practical material connected with the expansions of iterated Stratonovich stochastic integrals from the Taylor–Stratonovich expansion based on multiple trigonometric Fourier series. The comparison of effectiveness of the Fourier–Legendre series as well as the trigonometric Fourier series for expansions of iterated Stratonovich stochastic integrals is considered.

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1. Introduction

Let $\left(\Omega, F, P\right)$ be a complete probability space, let $\left\{F_t, t \in [0, T]\right\}$ be a nondecreasing right-continuous family of $\sigma$-algebras of $F$, and let $f_t$ be a standard $m$-dimensional Wiener stochastic process, which is $F_t$-measurable for any $t \in [0, T]$. We assume that the components $f_t^{(i)} (i = 1, \ldots, m)$ of this process are independent. Consider an Itô stochastic differential equation (SDE) in the integral form
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\[ x_t = x_0 + \int_0^t a(x_\tau, \tau) d\tau + \int_0^t B(x_\tau, \tau) df_\tau, \quad x_0 = x(0, \omega). \]

Here \( x_t \) is some \( n \)-dimensional stochastic process satisfying the equation (1). The nonrandom functions \( a : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n \), \( B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m} \) guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let \( x_0 \) be an \( n \)-dimensional random variable, which is \( F_0 \)-measurable and \( \mathbb{M}\{ |x_0|^2 \} < \infty \) (\( \mathbb{M} \) denotes a mathematical expectation). We assume that \( x_0 \) and \( f_t - f_0 \) are independent when \( t > 0 \).

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]-[7]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

\[
J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \cdots \int_t^{t_1} \psi_1(t_1) dw_{i_1}^{(i_1)} \cdots dw_{i_k}^{(i_k)},
\]

\[
J^*[\psi^{(k)}]_{T,t} = \int_t^* \psi_k(t_k) \cdots \int_t^{t_1} \psi_1(t_1) dw_{i_1}^{(i_1)} \cdots dw_{i_k}^{(i_k)},
\]

where every \( \psi_l(\tau) \) \( (l = 1, \ldots, k) \) is a nonrandom function on \([t, T]\), \( w_{\tau}^{(i)} = f_{\tau}^{(i)} \) for \( i = 1, \ldots, m \) and \( w_\tau^{(0)} = \tau; i_1, \ldots, i_k = 0, 1, \ldots, m; \)

\[
\int \quad \text{and} \quad \int^*
\]

denote Ito and Stratonovich stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [3]).

Note that \( \psi_l(\tau) \equiv 1 \) \( (l = 1, \ldots, k) \) and \( i_1, \ldots, i_k = 0, 1, \ldots, m \) for the classical Taylor–Ito and Taylor–Stratonovich expansions [2]-[7] and \( \psi_l(\tau) \equiv (t - \tau)^q_l \) \( (l = 1, \ldots, k; q_1, \ldots, q_k = 0, 1, 2, \ldots) \) and \( i_1, \ldots, i_k = 1, \ldots, m \) for the unified Taylor–Ito and Taylor–Stratonovich expansions [8]-[24].

2. Milstein Expansion and Method of Generatized Multiple Fourier Series

Milestein G.N. proposed [2] (1988) an approach to the expansion of iterated stochastic integrals based on the trigonometric Fourier expansion of the Brownian bridge process (version of the so-called Karhunen–Loeve expansion).

Let us consider the Brownian bridge process [2]

\[
f_t = \frac{t}{\Delta} f_{\Delta}, \quad t \in [0, \Delta], \quad \Delta > 0,
\]

where \( f_t \) is a standard \( m \)-dimensional Wiener process with independent components \( f_{t}^{(i)} (i = 1, \ldots, m) \).
Consider the componentwise Karhunen–Loeve expansion of the process (11) [2]

\[
\begin{align*}
\eta_i^{(i)}(t) - \frac{t}{\Delta} \eta_i^{(i)}(s) &= \frac{1}{2} a_{i,0} + \sum_{r=1}^{\infty} \left( a_{i,r} \cos \frac{2\pi rt}{\Delta} + b_{i,r} \sin \frac{2\pi rt}{\Delta} \right)
\end{align*}
\]

converging in the mean-square sense, where

\[
\begin{align*}
a_{i,r} &= \frac{2}{\Delta} \int_{0}^{\Delta} \left( f_{s}^{(i)} - \frac{s}{\Delta} f_{\Delta}^{(i)} \right) \cos \frac{2\pi rs}{\Delta} ds, \\
b_{i,r} &= \frac{2}{\Delta} \int_{0}^{\Delta} \left( f_{s}^{(i)} - \frac{s}{\Delta} f_{\Delta}^{(i)} \right) \sin \frac{2\pi rs}{\Delta} ds,
\end{align*}
\]

where \( r = 0, 1, \ldots; i = 1, \ldots, m \).

It is easy to demonstrate [2] that the random variables \( a_{i,r}, b_{i,r} \) are Gaussian ones and they satisfy the following relations

\[
\begin{align*}
M \{ a_{i,r} b_{i,k} \} &= M \{ a_{i,r} a_{i,k} \} = M \{ b_{i,r} b_{i,k} \} = 0, \\
M \{ a_{i_1,r} a_{i_2,r} \} &= M \{ b_{i_1,r} b_{i_2,r} \} = 0, \\
M \{ a_{i_1,r}^2 \} &= M \{ b_{i_1,r}^2 \} = \frac{\Delta}{2\pi^2 r^2},
\end{align*}
\]

where \( i, i_1, i_2 = 1, \ldots, m; r \neq k; i_1 \neq i_2 \).

According to [5], we have

\[
\begin{align*}
\eta_i^{(i)}(t) &= \eta_{\Delta}^{(i)} \frac{t}{\Delta} + \frac{1}{2} a_{i,0} + \sum_{r=1}^{\infty} \left( a_{i,r} \cos \frac{2\pi rt}{\Delta} + b_{i,r} \sin \frac{2\pi rt}{\Delta} \right),
\end{align*}
\]

where the series converges in the mean-square sense.

Note that the trigonometric functions are the eigenfunctions of the covariance operator of the Brownian bridge process. That is why the basis functions are the trigonometric functions in the considered approach.

In [2] Milstein G.N. proposed to expand (2) or (3) (for the case \( k = 2 \) and \( \psi_1(s), \psi_2(s) \equiv 1 \)) into iterated series of products of standard Gaussian random variables by representing the Wiener process as the series (6). To obtain the Milstein expansion of (2) or (3), the truncated expansions (6) of components of the Wiener process \( f_{s} \) must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that obviously does not lead to a general expansion of (2) or (3) valid for an arbitrary multiplicity \( k \).

For this reason, only expansions of simplest single, double, and triple integrals (2), (3) were obtained (see [2]-[7]).

At that, in [2], [7] the case \( \psi_1(s), \psi_2(s) \equiv 1 \) and \( i_1, i_2 = 0, 1, \ldots, m \) is considered. In [3]-[6] the attempt to consider the case \( \psi_1(s), \psi_2(s), \psi_3(s) \equiv 1 \) and \( i_1, i_2, i_3 = 0, 1, \ldots, m \) is realized.

It should be noted that the authors of the works [3] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) use the Wong–Zakai approximation [38]-[40] (without rigorous proof) within the frames of the Milstein approach [2] based on the series expansion of the Brownian bridge process. See discussion in Sect. 7 of this paper for details.
Let us consider an another approach to the expansion of iterated stochastic integrals \( \text{[10]-[37]} \), which is referred to as the method of generalized multiple Fourier series.

Suppose that every \( \psi_l(\tau) (l = 1, \ldots, k) \) is a nonrandom function from the space \( L_2([t, T]) \). Define the following function on the hypercube \([t, T]_k\)

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\
0, & \text{otherwise}
\end{cases}
\]

where \( t_1, \ldots, t_k \in [t, T] (k \geq 2) \) and \( K(t_1) = \psi_1(t_1) \) for \( t_1 \in [t, T] \). Here \( 1_A \) denotes the indicator of the set \( A \).

Suppose that \( \{\phi_j(x)\}_{j=0}^\infty \) is a complete orthonormal system of functions in the space \( L_2([t, T]) \).

The function \( K(t_1, \ldots, t_k) \) belongs to the space \( L_2([t, T]_k) \). At this situation it is well known that the generalized multiple Fourier series of \( K(t_1, \ldots, t_k) \) converging to \( K(t_1, \ldots, t_k) \) in the hypercube \([t, T]_k\) in the mean-square sense, i.e.

\[
\lim_{k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \prod_{l=1}^{k} \phi_j(t_l) \right\|_{L_2([t, T]_k)} = 0,
\]

where

\[
C_{j_1 \ldots j_k} = \int_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_j(t_l) dt_1 \ldots dt_k
\]

is the Fourier coefficient and

\[
\|f\|_{L_2([t, T]_k)} = \left( \int_{[t, T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.
\]

Consider the partition \( \{\tau_j\}_{j=0}^N \) of \([t, T]\) such that

\[
t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N} \Delta \tau_j \to 0 \quad \text{if} \quad N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]

**Theorem 1** [10] (2006), [11]-[37]. Suppose that every \( \psi_l(\tau) (l = 1, \ldots, k) \) is a continuous nonrandom function on \([t, T]\) and \( \{\phi_j(x)\}_{j=0}^\infty \) is a complete orthonormal system of continuous functions in the space \( L_2([t, T]) \). Then

\[
J[\psi^{(k)}]_{T, t} = \lim\limits_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \prod_{l=1}^{k} \phi_j(t_l) - \\
- \lim\limits_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta w^{(i_k)}_{\tau_{l_1}} \ldots \phi_{j_k}(\tau_{l_k}) \Delta w^{(i_k)}_{\tau_{l_k}}.
\]
where
\[ G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1\}, \]
\[ L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1; \ l_g \neq l_r (g \neq r); \ g, r = 1, \ldots, k\}, \]

I.i.m. is a limit in the mean-square sense, \(i_1, \ldots, i_k = 0, 1, \ldots, m,\)

\[ (12) \quad \zeta_j^{(i)} = \int_T^t \phi_j(s) \, dw_s^{(i)} \]

are independent standard Gaussian random variables for various \(i\) or \(j\) (if \(i \neq 0\), \(C_{J_k \ldots J_1}\) is the Fourier coefficient \(\Delta w_j^{(i)} = w_j^{(i)} - w_{j-1}^{(i)} (i = 0, 1, \ldots, m)\), \(\{\tau_j\}_{j=0}^N\) is the partition of \([t, T]\), which satisfies the condition \(\Delta w_j^{(i)} = \zeta_j^{(i)} \cdot \Delta \tau_j\)).

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for \(k = 1, \ldots, 6\). \[10^{-37}\]

\[ (13) \quad J[\psi^{(1)}]_{T, t} = \text{l.i.m.}_{p_1 \to \infty} \sum_{j_1 = 0}^{p_1} C_{j_1} \zeta_j^{(1)} , \]

\[ (14) \quad J[\psi^{(2)}]_{T, t} = \text{l.i.m.}_{p_1, p_2 \to \infty} \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2 j_1} \left( \zeta_j^{(1)} \zeta_j^{(2)} - 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \right) , \]

\[ (15) \quad J[\psi^{(3)}]_{T, t} = \text{l.i.m.}_{p_1, p_2, p_3 \to \infty} \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} \sum_{j_3 = 0}^{p_3} C_{j_3 j_2 j_1} \left( \zeta_j^{(1)} \zeta_j^{(2)} \zeta_j^{(3)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(1)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} 1_{\{j_2 = j_3\}} \right) , \]

\[ (16) \quad J[\psi^{(4)}]_{T, t} = \text{l.i.m.}_{p_1, p_2, p_3, p_4 \to \infty} \sum_{j_1 = 0}^{p_1} \cdots \sum_{j_4 = 0}^{p_4} C_{j_4 \ldots j_1} \left( \prod_{i=1}^4 \zeta_j^{(i)} \right) - 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \zeta_j^{(1)} \zeta_j^{(2)} \zeta_j^{(3)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(1)} \zeta_j^{(2)} \zeta_j^{(4)} - 1_{\{i_1 = i_4 \neq 0\}} 1_{\{j_1 = j_4\}} \zeta_j^{(1)} \zeta_j^{(2)} \zeta_j^{(3)} - 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \zeta_j^{(1)} \zeta_j^{(2)} \zeta_j^{(4)} + 1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} 1_{\{i_3 = i_4 \neq 0\}} 1_{\{j_3 = j_4\}} + 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} 1_{\{i_2 = i_4 \neq 0\}} 1_{\{j_2 = j_4\}} + 1_{\{i_1 = i_4 \neq 0\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} , \]
\[ J[\psi^{(5)}]_{T,t} = \text{l.i.m.}_{p_1, \ldots, p_6 \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_6=0}^{p_6} C_{j_6 \ldots j_1} \left( \prod_{k=1}^{5} \xi_{j_k}^{(i_k)} - \right. \\
\left. -1_{\{i_1=i_2\neq 0\}} 1_{\{j_1=j_2\}} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} - 1_{\{i_1=i_3\neq 0\}} 1_{\{j_1=j_3\}} \xi_{j_2}^{(i_2)} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} - 1_{\{i_1=i_4\neq 0\}} 1_{\{j_1=j_4\}} \xi_{j_2}^{(i_2)} \xi_{j_3}^{(i_3)} \xi_{j_5}^{(i_5)} - 1_{\{i_1=i_5\neq 0\}} 1_{\{j_1=j_5\}} \xi_{j_2}^{(i_2)} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} - 1_{\{i_2=i_3\neq 0\}} 1_{\{j_2=j_3\}} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} - 1_{\{i_2=i_4\neq 0\}} 1_{\{j_2=j_4\}} \xi_{j_3}^{(i_3)} \xi_{j_5}^{(i_5)} - 1_{\{i_2=i_5\neq 0\}} 1_{\{j_2=j_5\}} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} - 1_{\{i_3=i_4\neq 0\}} 1_{\{j_3=j_4\}} \xi_{j_5}^{(i_5)} + \\
\left. 1_{\{i_3=i_5\neq 0\}} 1_{\{j_3=j_5\}} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} \right) + \\
\left( 1_{\{i_1=i_2\neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_3=i_4\neq 0\}} 1_{\{j_3=j_4\}} \xi_{j_5}^{(i_5)} \right) \right. \\
\left. + 1_{\{i_1=i_3\neq 0\}} 1_{\{j_1=j_3\}} 1_{\{i_4=i_5\neq 0\}} 1_{\{j_4=j_5\}} \xi_{j_6}^{(i_6)} + 1_{\{i_1=i_4\neq 0\}} 1_{\{j_1=j_4\}} 1_{\{i_2=i_5\neq 0\}} 1_{\{j_2=j_5\}} \xi_{j_6}^{(i_6)} + 1_{\{i_1=i_5\neq 0\}} 1_{\{j_1=j_5\}} 1_{\{i_2=i_4\neq 0\}} 1_{\{j_2=j_4\}} \xi_{j_6}^{(i_6)} + 1_{\{i_2=i_3\neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_4=i_5\neq 0\}} 1_{\{j_4=j_5\}} \xi_{j_6}^{(i_6)} + 1_{\{i_2=i_4\neq 0\}} 1_{\{j_2=j_4\}} 1_{\{i_3=i_5\neq 0\}} 1_{\{j_3=j_5\}} \xi_{j_6}^{(i_6)} + 1_{\{i_3=i_4\neq 0\}} 1_{\{j_3=j_4\}} 1_{\{i_2=i_5\neq 0\}} 1_{\{j_2=j_5\}} \xi_{j_6}^{(i_6)} + 1_{\{i_3=i_5\neq 0\}} 1_{\{j_3=j_5\}} 1_{\{i_2=i_4\neq 0\}} 1_{\{j_2=j_4\}} \xi_{j_6}^{(i_6)} + \\
\left. 1_{\{i_4=i_5\neq 0\}} 1_{\{j_4=j_5\}} \right) \]

\[ J[\psi^{(6)}]_{T,t} = \text{l.i.m.}_{p_1, \ldots, p_6 \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_6=0}^{p_6} C_{j_6 \ldots j_1} \left( \prod_{k=1}^{6} \xi_{j_k}^{(i_k)} - \right. \\
\left. -1_{\{i_1=i_2\neq 0\}} 1_{\{j_1=j_2\}} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} \xi_{j_6}^{(i_6)} - 1_{\{i_1=i_3\neq 0\}} 1_{\{j_1=j_3\}} \xi_{j_2}^{(i_2)} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} \xi_{j_6}^{(i_6)} - 1_{\{i_1=i_4\neq 0\}} 1_{\{j_1=j_4\}} \xi_{j_2}^{(i_2)} \xi_{j_3}^{(i_3)} \xi_{j_5}^{(i_5)} \xi_{j_6}^{(i_6)} - 1_{\{i_1=i_5\neq 0\}} 1_{\{j_1=j_5\}} \xi_{j_2}^{(i_2)} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} \xi_{j_6}^{(i_6)} - 1_{\{i_1=i_6\neq 0\}} 1_{\{j_1=j_6\}} \xi_{j_2}^{(i_2)} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} - 1_{\{i_2=i_3\neq 0\}} 1_{\{j_2=j_3\}} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} \xi_{j_6}^{(i_6)} - 1_{\{i_2=i_4\neq 0\}} 1_{\{j_2=j_4\}} \xi_{j_3}^{(i_3)} \xi_{j_5}^{(i_5)} \xi_{j_6}^{(i_6)} - 1_{\{i_2=i_5\neq 0\}} 1_{\{j_2=j_5\}} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} \xi_{j_6}^{(i_6)} - 1_{\{i_3=i_4\neq 0\}} 1_{\{j_3=j_4\}} \xi_{j_5}^{(i_5)} \xi_{j_6}^{(i_6)} - 1_{\{i_3=i_5\neq 0\}} 1_{\{j_3=j_5\}} \xi_{j_4}^{(i_4)} \xi_{j_6}^{(i_6)} - 1_{\{i_3=i_6\neq 0\}} 1_{\{j_3=j_6\}} \xi_{j_4}^{(i_4)} \xi_{j_5}^{(i_5)} \xi_{j_6}^{(i_6)} - 1_{\{i_4=i_5\neq 0\}} 1_{\{j_4=j_5\}} \xi_{j_6}^{(i_6)} + \\
\left. -1_{\{i_4=i_6\neq 0\}} 1_{\{j_4=j_6\}} \xi_{j_5}^{(i_5)} \right. \\
\left. -1_{\{i_5=i_6\neq 0\}} 1_{\{j_5=j_6\}} \right) \]
In order to do this, let us introduce some notations. Consider the unordered set

\[ \{i_1, i_2, \ldots, i_k\} \]

of the iterated Ito stochastic integral \[ \int_{t_0}^{t} \zeta(s)^j(s) \, ds \]

defined by (2).

For further consideration, let us consider the generalization of formulas (13)-(18) for the case of an arbitrary multiplicity \( k \in \mathbb{N} \) of the iterated Ito stochastic integral \[ J^{(\psi)}_{T, t} \] defined by (2). In order to do this, let us introduce some notations. Consider the unordered set \( \{1, 2, \ldots, k\} \) and

\[
\begin{align*}
+1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_4=\not{i}_5 \neq 0\}} 1_{\{j_4=j_5\}} 1_{\{i_1 \neq \not{i}_6 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + 1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_4=\not{i}_5 \neq 0\}} 1_{\{j_4=j_5\}} 1_{\{i_1 \neq i_6 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + \\
+1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_3=\not{i}_4 \neq 0\}} 1_{\{j_3=j_4\}} 1_{\{i_1 \neq \not{i}_5 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + 1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_3=\not{i}_4 \neq 0\}} 1_{\{j_3=j_4\}} 1_{\{i_1 \neq \not{i}_5 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + \\
+1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_3=\not{i}_4 \neq 0\}} 1_{\{j_3=j_4\}} 1_{\{i_1 \neq \not{i}_5 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + 1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_3=\not{i}_4 \neq 0\}} 1_{\{j_3=j_4\}} 1_{\{i_1 \neq \not{i}_5 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + \\
+1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_3=\not{i}_4 \neq 0\}} 1_{\{j_3=j_4\}} 1_{\{i_1 \neq \not{i}_5 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + 1_{\{i_2=\not{i}_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_3=\not{i}_4 \neq 0\}} 1_{\{j_3=j_4\}} 1_{\{i_1 \neq \not{i}_5 \}} \gamma_{j_1}(t_0) \gamma_{j_6} + \\
\end{align*}
\]

(18)

where \( 1_A \) is the indicator of the set \( A \).
separate it into two parts: the first part consists of \( r \) unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining \( k - 2r \) numbers. So, we have

\[
(19) \quad \left\{ g_{1,2}, \ldots, g_{2r-1,2r}, q_1, \ldots, q_{k-2r} \right\},
\]

where

\[
\{ g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r} \} = \{ 1, 2, \ldots, k \},
\]

braces mean an unordered set, and parentheses mean an ordered set.

We will say that (19) is a partition and consider the sum with respect to all possible partitions

\[
(20) \sum_{\left\{ \left\{ g_{1,2}, \ldots, g_{2r-1,2r}, q_1, \ldots, q_{k-2r} \right\} \right\}} a_{g_1 g_2 \ldots g_{2r-1} g_{2r} q_1 \ldots q_{k-2r}}.
\]

Below there are several examples of sums in the form (20)

\[
\sum_{\left\{ \{g_{1,2}\} \right\}} a_{g_1 g_2} = a_{12},
\]

\[
\sum_{\left\{ \{g_{1,2}\}, \{g_{3,4}\} \right\}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314},
\]

\[
\sum_{\left\{ \{g_{1,2}\}, \{q_1, q_2\} \right\}} a_{g_1 g_2 q_1 q_2} =
\]

\[
= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},
\]

\[
\sum_{\left\{ \{g_{1,2}\}, \{g_{1,2}, g_3\} \right\}} a_{g_1 g_2 g_3 g_2} =
\]

\[
= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +
\]

\[
+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},
\]

\[
\sum_{\left\{ \{g_{1,2}\}, \{g_{3,4}\}, \{q_1\} \right\}} a_{g_1 g_2 g_3 g_4 q_1} =
\]

\[
= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} +
\]

\[
+ a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} +
\]

\[
+ a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.
\]

Now we can write (11) as
where \([x]\) is an integer part of a real number \(x\); another notations are the same as in Theorem 1.

In particular, from (21) for \(k = 5\) we obtain

\[
J[\psi^{(5)}][T,t] = \lim_{p_1,\ldots, p_5 \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_5=0}^{p_5} C_{j_5 \ldots j_1} \left( \prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} - \sum_{\{(s_1, s_2) : (s_1, s_2, s_3, s_4) = (1,2,3,4,5)\}} \sum_{\{(i_1, i_2) : i_1 = i_2 \neq 0\}} \sum_{\{(i_3, i_4, i_5) : i_3 = i_4 \neq 0\}} \prod_{l=1}^{3} \zeta_{j_l}^{(i_l)} \right)
\]

The last equality obviously agrees with (17).

Let us consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space \(L_2([t, T])\) and \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])\).

**Theorem 2** [22] (Sect. 1.11), [29] (Sect. 15). Suppose that \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])\) and \(\{\psi_j(x)\}_{j=0}^{\infty}\) is an arbitrary complete orthonormal system of functions in the space \(L_2([t, T])\). Then the following expansion

\[
J[\psi^{(k)}][T,t] = \lim_{p_1,\ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]} (-1)^r \times \sum_{\{(i_1, i_2) : i_1 = i_2 \neq 0\}} \sum_{\{(i_3, \ldots, i_{k-2}) : i_3 = \ldots = i_{k-2} \neq 0\}} \prod_{l=1}^{k-2r} \zeta_{j_l}^{(i_l)} \right)
\]

converging in the mean-square sense is valid, where \([x]\) is an integer part of a real number \(x\); another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [41]. Note that we use another notations [22] (Sect. 1.11), [29] (Sect. 15) in comparison with [41]. Moreover, the proof of an analogue of Theorem 2 from [41] is somewhat different from the proof given in [22] (Sect. 1.11), [29] (Sect. 15).
3. Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 6

In a number of works of the author [15]-[24], Theorems 1, 2 have been adapted for the iterated Stratonovich stochastic integrals [30] of multiplicities 2 to 6. Let us first present some old results as the following theorem.

Theorem 3 [15]-[24], [30]. Suppose that \( \{ \phi_j(x) \}_{i=0}^\infty \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). At the same time \( \psi_2(t) \) is a continuously differentiable function on \([t, T]\) and \( \psi_1(t), \psi_3(t) \) are twice continuously differentiable functions on \([t, T]\). Then

\[
\begin{align*}
J^*[\psi^{(2)}]_{T,t} &= \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_1j_2} \psi_{(i_1)}^{(i_2)} \psi_{(i_2)}^{(i_2)} (i_1, i_2 = 1, \ldots, m), \\
J^*[\psi^{(3)}]_{T,t} &= \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_1j_2j_3} \psi_{(i_1)}^{(i_2)} \psi_{(i_2)}^{(i_3)} \psi_{(i_3)}^{(i_3)} (i_1, i_2, i_3 = 0, 1, \ldots, m), \\
J^*[\psi^{(4)}]_{T,t} &= \lim_{p \to \infty} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_1j_2j_3j_4} \psi_{(i_1)}^{(i_2)} \psi_{(i_2)}^{(i_3)} \psi_{(i_3)}^{(i_4)} (i_1, i_2, i_3, i_4 = 0, 1, \ldots, m),
\end{align*}
\]

where \( J^*[\psi^{(k)}]_{T,t} \) is defined by [34] and \( \psi_l(t) = 1 (l = 1, \ldots, 4) \) in [24], [26]; another notations are the same as in Theorems 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [22] (Sect. 2.10–2.16), [30] (Sect. 13–19), [33] (Sect. 7–13), [34] (Sect. 5–11), [54] (Sect. 4–9), [55]. Let us formulate four theorems that were obtained using this approach.

Theorem 4 [22], [30], [33], [34], [54]. Suppose that \( \{ \phi_j(x) \}_{i=0}^\infty \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). Furthermore, let \( \psi_1(t), \psi_2(t), \psi_3(t) \) are continuously differentiable nonrandom functions on \([t, T]\). Then, for the iterated Stratonovich stochastic integral of third multiplicity

\[
J^*[\psi^{(3)}]_{T,t} = \int_t^T \int_t^{t_2} \int_t^{t_3} \psi_1(t_1) d\omega_{t_1}^{(i_1)} d\omega_{t_2}^{(i_2)} d\omega_{t_3}^{(i_3)} (i_1, i_2, i_3 = 0, 1, \ldots, m)
\]

the following relations

\[
J^*[\psi^{(3)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3} \psi_{(i_1)}^{(i_2)} \psi_{(i_2)}^{(i_3)} \psi_{(i_3)}^{(i_3)},
\]
are fulfilled, where \(i_1, i_2, i_3 = 0, 1, \ldots, m\) in (28) and \(i_1, i_2, i_3 = 1, \ldots, m\) in (29), constant \(C\) is independent of \(p\),

\[
C_{j_3j_2j_1} = \int_t^T \psi_4(t_4) \int_t^{t_4} \psi_3(t_3) \int_t^{t_3} \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;
\]

are independent standard Gaussian random variables for various \(i\) or \(j\) (in the case when \(i \neq 0\)); another notations are the same as in Theorems 1, 2.

**Theorem 5** Let \(\{\phi_j(x)\}_{j=0}^\infty\) be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L^2(\mathbb{R})\). Furthermore, let \(\psi_1(\tau), \ldots, \psi_4(\tau)\) be continuously differentiable nonrandom functions on \([t,T]\). Then, for the iterated Stratonovich stochastic integral of fourth multiplicity

\[
J^*[\psi^{(4)}]_{T,t} = \int_t^T \psi_4(t_4) \int_t^{t_4} \psi_3(t_3) \int_t^{t_3} \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4
\]

the following relations

\[
J^*[\psi^{(4)}]_{T,t} = 1 \text{ i.m.} \quad \sum_{j_1, j_2, j_3, j_4=0}^p C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)};
\]

\[
M \left\{ \left( J^*[\psi^{(4)}]_{T,t} - \sum_{j_1, j_2, j_3, j_4=0}^p C_{j_3j_2j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}}
\]

are fulfilled, where \(i_1, \ldots, i_4 = 0, 1, \ldots, m\) in (28), (30) and \(i_1, \ldots, i_4 = 1, \ldots, m\) in (31), constant \(C\) does not depend on \(p, \varepsilon\) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \(L^2(\mathbb{R})\) and \(\varepsilon = 0\) for the case of complete orthonormal system of trigonometric functions in the space \(L^2(\mathbb{R})\),

\[
C_{j_3j_2j_1} = \int_t^T \int_t^{t_4} \int_t^{t_3} \int_t^{t_2} \int_t^{t_1} \psi_4(t_4) \phi_{j_4}(t_4) \psi_3(t_3) \phi_{j_3}(t_3) \psi_2(t_2) \phi_{j_2}(t_2) \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;
\]
another notations are the same as in Theorem 4.

Theorem 6 [22], [30], [33], [34], [54]. Assume that \( \{ \phi_j(x) \}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_5(\tau) \) are continuously differentiable nonrandom functions on \([t, T]\). Then, for the iterated Stratonovich stochastic integral of fifth multiplicity

\[
J^*[\psi^{(5)}]_{T,t} = \int_t^T \psi_5(t_5) \ldots \int_t^{t_2} \psi_1(t_1) dw_{t_1}^{(i_1)} \ldots dw_{t_5}^{(i_5)}
\]

the following relations

\[
J^*[\psi^{(5)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}(i_1) \ldots \zeta_{j_5}(i_5),
\]

\[
M \left\{ \left( J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1} \zeta_{j_1}(i_1) \ldots \zeta_{j_5}(i_5) \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}}
\]

are fulfilled, where \( i_1,\ldots,i_5 = 0,1,\ldots,m \) in (32), (33) and \( i_1,\ldots,i_5 = 1,\ldots,m \) in (34), constant \( C \) is independent of \( p, \varepsilon \) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \( L_2([t, T]) \) and \( \varepsilon = 0 \) for the case of complete orthonormal system of trigonometric functions in the space \( L_2([t, T]) \),

\[
C_{j_5\ldots j_1} = \int_t^T \psi_5(t_5) \phi_{j_5}(t_5) \ldots \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_5;
\]

another notations are the same as in Theorems 4, 5.

Theorem 7 [22], [30], [33], [34], [55]. Suppose that \( \{ \phi_j(x) \}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). Then, for the iterated Stratonovich stochastic integral of sixth multiplicity

\[
J^*[\psi^{(6)}]_{T,t} = \int_t^T \ldots \int_t^{t_2} dw_{t_1}^{(i_1)} \ldots dw_{t_6}^{(i_6)}
\]

the following expansion

\[
J^*[\psi^{(6)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1} \zeta_{j_1}(i_1) \ldots \zeta_{j_6}(i_6)
\]

that converges in the mean-square sense is valid, where \( i_1,\ldots,i_6 = 0,1,\ldots,m \),

\[
C_{j_6\ldots j_1} = \int_t^T \phi_{j_6}(t_6) \ldots \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_6;
\]
another notations are the same as in Theorems 4–6.

4. Exact Calculation of the Mean-Square Error in Theorems 1, 2

Theorems 1 and 2 allow us to accurately calculate the mean-square approximation error for iterated Ito stochastic integrals (see Theorem 8 below).

Assume that \( J_{T,t}^{[\psi(k)]p_1...p_k} \) is the approximation of (2), which is the expression on the right-hand side of (22) before passing to the limit

\[
J_{T,t}^{[\psi(k)]p_1...p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k...j_1} \left( \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} \right) + \sum_{r=1}^{[k/2]} (-1)^r \times
\]

\[
\sum_{s=1}^{r} \prod_{l=1}^{k} 1_{(i_{g_{2s-1}} = i_{g_{2s}} \neq 0)} 1_{(j_{g_{2s-1}} = j_{g_{2s}})} \prod_{l=1}^{k-2r} \zeta_{j_l}^{(i_l)},
\]

where \([x] \) is an integer part of a real number \( x \); another notations are the same as in Theorems 1, 2.

Let us denote

\[
E_{k}^{p_1...p_k} \overset{\text{def}}{=} M\left\{ \left( J_{T,t}^{[\psi(k)]} - J_{T,t}^{[\psi(k)]p_1...p_k} \right)^2 \right\},
\]

\[
E_{k}^{p_1...p_k} \overset{\text{def}}{=} E_{k}^{p} \quad \text{if} \quad p_1 = \ldots = p_k = p,
\]

\[
I_{k} \overset{\text{def}}{=} \| K \|_{L_2([t,T])}^2 = \int_{[t,T]^k} K^2(t_1, \ldots, t_k) dt_1 \ldots dt_k.
\]

In [10], [24], [29] it was shown that

\[
(36) \quad E_{k}^{p_1...p_k} \leq k! \left( I_{k} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k...j_1}^2 \right)
\]

if \( i_1, \ldots, i_k = 1, \ldots, m \) and \( 0 < T - t < \infty \) or \( i_1, \ldots, i_k = 0, 1, \ldots, m \) and \( 0 < T - t < 1 \).

Moreover, in [12], [24], [29] the following estimate

\[
M\left\{ \left( J_{T,t}^{[\psi(k)]} - J_{T,t}^{[\psi(k)]p_1...p_k} \right)^{2n} \right\} \leq
\]

\[
\leq (k!)^{2n} \left( n(2n-1) \right)^{n(k-1)} (2n-1)!! \left( I_{k} - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k...j_1}^2 \right)^n
\]

is obtained, where \( n \in \mathbb{N} \).
The value $E_k^p$ can be calculated exactly.

**Theorem 8** (Sect. 1.12), (Sect. 6). Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$, $i_1, \ldots, i_k = 1, \ldots, m$. Then

\[(38) \quad E_k^p = I_k - \sum_{j_1, \ldots, j_k=0}^p C_j \ldots \sum_{j_k} M \left\{ J[\psi^{(k)}]_{T,t} \sum_{(j_1, \ldots, j_k)} I_t^T \phi_{j_1}(t) \cdots \phi_{j_k}(t) \right\}, \]

where $i_1, \ldots, i_k = 1, \ldots, m$; the expression

\[\sum_{(j_1, \ldots, j_k)}\]

means the sum with respect to all possible permutations $(j_1, \ldots, j_k)$. At the same time if $j_r$ swapped with $j_q$ in the permutation $(j_1, \ldots, j_k)$, then $i_r$ swapped with $i_q$ in the permutation $(i_1, \ldots, i_k)$; another notations are the same as in Theorems 1, 2.

Note that

\[M \left\{ J[\psi^{(k)}]_{T,t} \sum_{(j_1, \ldots, j_k)} I_t^T \phi_{j_1}(t) \cdots \phi_{j_k}(t) \right\} = C_{j_k \ldots j_1}.\]

Then from Theorem 8 for pairwise different $i_1, \ldots, i_k$ and for $i_1 = \ldots = i_k$ we obtain

\[E_k^p = I_k - \sum_{j_1, \ldots, j_k=0}^p C_{j_k \ldots j_1}^2, \]

\[E_k^p = I_k - \sum_{j_1, \ldots, j_k=0}^p C_{j_k \ldots j_1} \left( \sum_{(j_1, \ldots, j_k)} C_{j_k \ldots j_1} \right).\]

Consider some examples of the application of Theorem 8 ($i_1, i_2, i_3 = 1, \ldots, m$)

\[E_2^p = I_2 - \sum_{j_1, j_2=0}^p C_{j_2 j_1}^2 - \sum_{j_1, j_2=0}^p C_{j_2 j_1} C_{j_1 j_2} \quad (i_1 = i_2), \]

\[E_3^p = I_3 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3), \]

\[E_3^p = I_3 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1} C_{j_3 j_2 j_1} \quad (i_1 \neq i_2 = i_3), \]

\[E_3^p = I_3 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_2 j_1} C_{j_3 j_2 j_1} \quad (i_1 = i_3 \neq i_2).\]
\begin{align*}
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_2 \neq i_3, i_4; \; i_3 \neq i_4), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_3 \neq i_2, i_4; \; i_2 \neq i_4), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_4)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_4 \neq i_2, i_3; \; i_2 \neq i_3), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_2, j_4)} C_{j_4 \ldots j_1} \right) \quad (i_2 = i_3 \neq i_1, i_4; \; i_1 \neq i_4), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_2, j_4)} C_{j_4 \ldots j_1} \right) \quad (i_2 = i_4 \neq i_1, i_3; \; i_1 \neq i_3), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_3, j_4)} C_{j_4 \ldots j_1} \right) \quad (i_3 = i_4 \neq i_1, i_2; \; i_1 \neq i_2), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_2 = i_3 \neq i_4), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_2, j_3, j_4)} C_{j_4 \ldots j_1} \right) \quad (i_2 = i_3 = i_4 \neq i_1), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2, j_4)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_2 = i_4 \neq i_3), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_3, j_4)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_3 = i_4 \neq i_2), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_2 \neq i_3 = i_4), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_3 \neq i_2 = i_4), \\
E_4^p &= I_4 - \sum_{j_1, \ldots, j_4=0}^{P} C_{j_4 \ldots j_1} \left( \sum_{(j_2, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_4 \neq i_2 = i_3),
\end{align*}
5. Some Technical Problems of the Milstein Approach

Let us denote

\[ I_{(i_1 \ldots i_k)}^{(0)}(T, t) = \int_t^T (T - s)^{3/2} \left( \frac{\zeta_0^{(i_1)}}{\pi} - \sqrt{\frac{2}{\pi}} \sum_{r=1}^{\infty} \frac{1}{r} \zeta_0^{(i_1) - 1} \right), \]

\[ I_{(i_1) (i_2)}^{(0)}(T, t) = \frac{1}{2} (T - t) \left( \zeta_0^{(i_1) \zeta_0^{(i_2)}} + \frac{1}{\pi} \sum_{r=1}^{\infty} \frac{1}{r} \zeta_2^{(i_1) \zeta_2^{(i_2)}} - \zeta_2^{(i_1) \zeta_2^{(i_2)}} + \sqrt{\frac{2}{\pi}} \left( \zeta_2^{(i_1) \zeta_2^{(i_2)}} - \zeta_0^{(i_1) \zeta_0^{(i_2)}} \right) \right), \]

\[ I_{(i_1) (i_2)}^{(2)}(T, t) = (T - t)^{5/2} \left( \frac{1}{3} \zeta_0^{(i_1)} + \frac{1}{\sqrt{2} \pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \zeta_2^{(i_1) \zeta_2^{(i_2)}} - \frac{1}{\sqrt{2} \pi} \sum_{r=1}^{\infty} \frac{1}{r^{1/2}} \zeta_2^{(i_1)} \right), \]

where \( i_1, i_2 = 1, \ldots, m; \]

\[ \zeta_j^{(i)} = \int_t^T \phi_j(s) \, ds \]

are independent standard Gaussian random variables for various \( i \) or \( j \), and

\[ \phi_j(s) = \begin{cases} 1 & \text{when } j = 0 \\ \frac{1}{\sqrt{T - t}} \sqrt{2} \sin(2\pi r(s - t)/(T - t)) & \text{when } j = 2r - 1 \\ \sqrt{2} \cos(2\pi r(s - t)/(T - t)) & \text{when } j = 2r \end{cases} \]
where \( r = 1, 2, \ldots \).

Obviously, that \( I_{(1)T,T}^{(i_1)}, I_{(2)T,T}^{(i_1)} \) have Gaussian distribution and the expansions (41), (43) are too complex for such simple stochastic integrals as \( I_{(1)T,T}^{(i_1)}, I_{(2)T,T}^{(i_1)} \).

Milstein G.N. proposed [2] the following mean-square approximations on the base of the expansions (45), (46).

\[
I_{(1)T,T}^{(i_1)q} = \frac{(T - t)^{3/2}}{2} \left( \phi_0^{(i_1)} - \frac{\sqrt{2}}{\pi} \left( \sum_{r=1}^{q} \frac{1}{r} \phi_{2r-1}^{(i_1)} + \sqrt{\alpha_q \sigma_q^{(i_1)}} \right) \right),
\]

\[
I_{(00)T,T}^{(i_1)q} = \frac{1}{2}(T - t) \left( \phi_0^{(i_1)} \phi_0^{(i_2)} + \frac{1}{\pi} \sum_{r=1}^{q} \frac{1}{r} \left( \phi_{2r-1}^{(i_1)} \phi_{2r-1}^{(i_2)} - \phi_{2r-1}^{(i_1)} \phi_{2r}^{(i_2)} + \phi_{2r-1}^{(i_2)} \phi_{2r}^{(i_1)} \right) \right) + \frac{\sqrt{2}}{\pi} \sqrt{\alpha_q} \left( \phi_0^{(i_1)} \phi_0^{(i_2)} - \phi_0^{(i_1)} \phi_0^{(i_2)} \right),
\]

where

\[
\phi_q^{(i)} = \frac{1}{\sqrt{\alpha_q}} \sum_{r=q+1}^{\infty} \frac{1}{r} \phi_{2r-1}^{(i)}, \quad \alpha_q = \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2},
\]

where \( \phi_0^{(i)}, \phi_2^{(i)}, \phi_{2r}^{(i)}, \phi_q^{(i)}; r = 1, \ldots, q; i = 1, \ldots, m \) are independent standard Gaussian random variables.

The approximation \( I_{(2)T,T}^{(i_1)q} \), which corresponds to (41), (43) has the form [3]

\[
I_{(2)T,T}^{(i_1)q} = (T - t)^{5/2} \left( \frac{1}{2} \phi_0^{(i_1)} + \frac{1}{\sqrt{2} \pi^2} \left( \sum_{r=1}^{q} \frac{1}{r^2} \phi_{2r}^{(i_1)} + \sqrt{\beta_q \mu_q^{(i_1)}} \right) \right) - \frac{1}{\sqrt{2} \pi} \left( \sum_{r=1}^{q} \frac{1}{r^2} \phi_{2r}^{(i_1)} + \sqrt{\alpha_q \phi_q^{(i_1)}} \right),
\]

where \( \phi_q^{(i)}, \beta_q \) has the form (17) and

\[
\mu_q^{(i)} = \frac{1}{\sqrt{\beta_q}} \sum_{r=q+1}^{\infty} \frac{1}{r^2} \phi_{2r}^{(i)}, \quad \beta_q = \frac{\pi^4}{90} - \sum_{r=1}^{q} \frac{1}{r^2},
\]

\( \phi_j(s) \) is defined by (13); \( \phi_0^{(i)}, \phi_2^{(i)}, \phi_{2r}^{(i)}, \phi_q^{(i)}, \mu_q^{(i)}; r = 1, \ldots, q; i = 1, \ldots, m \) are independent standard Gaussian random variables; \( i = 1, \ldots, m \).

Nevertheless, the expansions (41), (43) are too complex for the approximation of two Gaussian random variables \( I_{(1)T,T}^{(i_1)}, I_{(2)T,T}^{(i_1)} \).

Using Theorems 1–3 and complete orthonormal system of Legendre polynomials in the space \( L_2([0, T]) \), we obtain for \( i_1, i_2 = 1, \ldots, m \) [10], [37].
\[ I^{(i)}_{(0)T,t} = \sqrt{T - t} \zeta_0^{(i)} , \]

\[ I^{(i)}_{(1)T,t} = -\frac{(T - t)^{3/2}}{2} \left( \zeta_0^{(i)} + \frac{1}{\sqrt{3}} \zeta_1^{(i)} \right) , \]

\[ I^{(i)}_{(2)T,t} = -\frac{(T - t)^{5/2}}{3} \left( \zeta_0^{(i)} + \frac{\sqrt{15}}{2} \zeta_1^{(i)} + \frac{1}{2\sqrt{15}} \zeta_2^{(i)} \right) , \]

\[ I^{(i_1i_2)}_{(00)T,t} = \frac{T - t}{2} \left( \zeta_0^{(i_1i_2)} - \zeta_0^{(i_2)} \right) + \sum_{i=1}^{\infty} \frac{1}{\sqrt{4i^2 - 1}} \left( \zeta_0^{(i_1i_2)} - \zeta_0^{(i_1)} \zeta_0^{(i_2)} \right) , \]

\[ \zeta_j^{(i)} = \int_t^T \phi_j(s)dw_s^{(i)} \]

are independent standard Gaussian random variables for various \( i \) or \( j \), where

\[ \phi_j(x) = \sqrt{\frac{2j + 1}{T - t}} P_j \left( \left( x - \frac{T + t}{2} \right) \frac{2}{T - t} \right) ; \quad j = 0, 1, 2, \ldots , \]

where \( P_j(x) \) is the Legendre polynomial.

It is not difficult to see that the expansions (50), (51) are much simpler than the expansions (45), (48).

Obviously that the Milstein approach [2] leads to iterated series (iterated application of the operation of limit transitions) in contradiction to multiple series (the operation of limit transition is implemented only once) from Theorems 1–7.

For the case of simplest stochastic integral \( I^{(i_1i_2)}_{(00)T,t} \) of second multiplicity this problem was avoided as we saw earlier. However, the situation is not the same for the simplest iterated stochastic integral \( I^{(i_1i_2i_3)}_{(000)T,t} \) of third multiplicity.

Let us denote

\[ I^{(i_1\ldots i_k)}_{(\lambda_1\ldots \lambda_k)T,t} = \int_t^{t_2} \ldots \int_t^{t_k} dw_{t_1}^{(i_1)} \ldots dw_{t_k}^{(i_k)} , \]

where \( \lambda_l = 1 \) if \( i_l = 1, \ldots , m \) and \( \lambda_l = 0 \) if \( i_l = 0; \quad l = 1, \ldots , k \) (\( w_{\tau}^{(i)} = f^{(i)}_{\tau} \) for \( i = 1, \ldots , m \) and \( w_{\tau}^{(0)} = \tau \)).

Consider the expansion of iterated Stratonovich stochastic integral of third multiplicity obtained in [3–5] by the Milstein approach

\[ J^{(i_1i_2i_3)}_{(111)\Delta,0} = \frac{1}{\Delta} J^{(i_1)}_{(1)\Delta,0} r^{(0i_2i_3)}_{(011)\Delta,0} + \frac{1}{2} a_{i_1,0} J^{(i_2)}_{(1)\Delta,0} r^{(i_1i_3)}_{(1)\Delta,0} + \frac{1}{2\pi} b_{i_1} J^{(i_2)}_{(1)\Delta,0} r^{(i_3)}_{(1)\Delta,0} - \]

\[ - \Delta J^{(i_2)}_{(1)\Delta,0} B_{i_1i_3} + \Delta J^{(i_3)}_{(1)\Delta,0} \left( \frac{1}{2} A_{i_1i_2} - C_{i_2i_3} \right) + \Delta^{3/2} D_{i_1i_2i_3} , \]
where

\[
J_{(01)\Delta,0}^{(11)\Delta,0} = \frac{1}{6} J_{(1)\Delta,0}^{*(i_2)} J_{(1)\Delta,0}^{*(i_3)} - \frac{1}{\pi} \Delta J_{(1)\Delta,0}^{*(i_3)} b_{i_3} + \\
+ \Delta^2 B_{i_2 i_3} - \frac{1}{4} \Delta a_{i_3,0} J_{(1)\Delta,0}^{*(i_2)} + \frac{1}{2\pi} \Delta b_{i_3} J_{(1)\Delta,0}^{*(i_2)} + \Delta^2 C_{i_2 i_3} + \frac{1}{2} \Delta^2 A_{i_2 i_3},
\]

\[
A_{i_2 i_3} = \frac{\pi}{\Delta} \sum_{r=1}^{\infty} r (a_{i_2,r} b_{i_3,r} - b_{i_2,r} a_{i_3,r}),
\]

\[
C_{i_2 i_3} = -\frac{1}{\Delta} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \frac{r}{r^2 - l^2} (ra_{i_2,r} a_{i_3,l} + lb_{i_2,r} b_{i_3,l}),
\]

\[
B_{i_2 i_3} = \frac{1}{2\Delta} \sum_{r=1}^{\infty} (a_{i_2,r} a_{i_3,r} + b_{i_2,r} b_{i_3,r}),
\]

\[
b_1 = \sum_{r=1}^{\infty} \frac{1}{r} b_{i_1,r},
\]

\[
D_{i_1 i_2 i_3} = -\frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} l \left( a_{i_2,l} (a_{i_3,l+r} b_{i_1,r} - a_{i_1,r} b_{i_3,l+r}) + \\
+ b_{i_2,l} (a_{i_1,r} a_{i_3,r+l} + b_{i_1,r} b_{i_3,l+r}) \right) + \\
+ \frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{\infty} \sum_{r=1}^{l-1} l \left( a_{i_2,l} (a_{i_1,r} b_{i_3,l-r} + a_{i_3,l-r} b_{i_1,r}) - \\
- b_{i_2,l} (a_{i_1,r} a_{i_3,l-r} - b_{i_1,r} b_{i_3,l-r}) \right) + \\
+ \frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{\infty} \sum_{r=l}^{\infty} l \left( a_{i_2,l} (a_{i_3,r-l} b_{i_1,r} - a_{i_1,r} b_{i_3,r-l}) + \\
+ b_{i_2,l} (a_{i_1,r} a_{i_3,r-l} + b_{i_1,r} b_{i_3,r-l}) \right).
\]

From the form of expansion (61) and expansion of the stochastic integral \(J_{(01)\Delta,0}^{*(i_2 i_3)}\) we can conclude that they include iterated (double) series. Moreover, for approximation of the considered stochastic integral \(J_{(111)\Delta,0}^{*(i_1 i_2 i_3)}\) in the works [3] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) it is proposed to put upper limits of summation by equal \(q\) (on the base of the Wong–Zakai approximation [28-30], but without rigorous proof; also see discussion in Sect. 7).

For example, the value \(D_{i_1 i_2 i_3}\) is approximated in [3] (Sect. 5.8, pp. 202–204), [4] (pp. 82-84), [5] (pp. 438-439), [6] (pp. 263-264) by the double sums of the form
\begin{equation}
D^{(q)}_{i_1i_2i_3} = -\frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{q-1} \sum_{r=1}^{q} l \left( a_{i_2,l} \left( a_{i_3,l+r} b_{i_1,r} + a_{i_3,l-r} b_{i_1,r} \right) + b_{i_2,l} \left( a_{i_1,r} a_{i_3,r+l} + b_{i_1,r} b_{i_3,l+r} \right) + b_{i_2,l} \right)
+ \frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{q-1} \sum_{r=1}^{q} l \left( a_{i_2,l} \left( a_{i_1,r} b_{i_3,l-r} + a_{i_3,l-r} b_{i_1,r} \right) - b_{i_2,l} \left( a_{i_1,r} a_{i_3,l-r} - b_{i_1,r} b_{i_3,l-r} \right) + b_{i_2,l} \right)
+ \frac{\pi}{2\Delta^{3/2}} \sum_{l=1}^{q-1} \sum_{r=1}^{q} l \left( a_{i_2,l} \left( a_{i_3,r-l} b_{i_1,r} - a_{i_1,r} b_{i_3,r-l} \right) + b_{i_2,l} \left( a_{i_1,r} a_{i_3,r-l} + b_{i_1,r} b_{i_3,r-l} \right) \right).
\end{equation}

Obviously, we can avoid this problem (iterated application of the operation of limit transition) using the method based on Theorems 1–7.

If we prove that the terms of the expansion \((55)\) coincide with the terms of its analogue obtained using Theorems 1–3 (this fact is proved in \([10]-[24]\) for the simplest stochastic integrals \(I^{(1)}_{(1)T,t}, I^{(1)I_2}_{(0)T,t}\) of first and second multiplicity), then we can replace the iterated (double) series in \((54)\) by the multiple ones, as in Theorems 1–3 (as was made formally in \([3]-[6]\)). However, it requires a separate argumentation.

6. Approximation of Specific Iterated Stochastic Integrals of Multiplicities 1 to 3 Using Theorem 3 and Trigonometric System of Functions

In \([10]-[24]\) on the base of Theorems 1–3 the author of this paper obtained the following expansions of the iterated Stratonovich stochastic integrals \((59)\) (independently from the papers \([2]-[7]\) excepting the method in which additional random variables \(\xi^{(i)}_q, \mu^{(i)}_q\) are introduced)

\begin{equation}
I^{(i_1)}_{(0)T,t} = \sqrt{T-t} \xi^{(i_1)}_0,
\end{equation}

\begin{equation}
I^{(i_1)q}_{(1)T,t} = -\frac{(T-t)^{3/2}}{2} \left( \frac{\sqrt{2}}{\pi} \left( \sum_{r=1}^{q} \frac{\xi^{(i_1)}_0}{\xi^{(i_1)}_{2r-1}} + \sqrt{\alpha} \xi^{(i_1)}_q \right) \right),
\end{equation}

\begin{equation}
I^{(i_1i_2)q}_{(00)T,t} = \frac{1}{2}(T-t) \left( \xi^{(i_1)}_0 \xi^{(i_2)}_0 + \frac{1}{\pi} \sum_{r=1}^{q} \frac{1}{r} \left( \xi^{(i_1)}_{2r-1} \xi^{(i_2)}_{2r} - \xi^{(i_1)}_{2r-1} \xi^{(i_2)}_{2r-1} \right) \right)
+ \sqrt{\frac{2}{\pi}} \sqrt{\alpha} \left( \xi^{(i_1)}_q \xi^{(i_2)}_0 - \xi^{(i_1)}_0 \xi^{(i_2)}_q \right),
\end{equation}
\[ I^{(i_1 i_2 i_3)_q}_{(000)T,t} = (T-t)^{3/2} \left( \frac{1}{6} \xi^{(i_1)}_0 \xi^{(i_2)}_0 \xi^{(i_3)}_0 + \frac{\sqrt{3} \pi}{2 \sqrt{2} \pi} \left( \xi^{(i_1)}_q \xi^{(i_2)}_0 \xi^{(i_3)}_0 - \xi^{(i_1)}_0 \xi^{(i_2)}_q \xi^{(i_3)}_0 \right) \right) + \\
+ \frac{1}{2 \sqrt{2 \pi}} \sqrt{\pi} \left( \mu^{(i_1)}_q \xi^{(i_2)}_0 \xi^{(i_3)}_0 - 2 \mu^{(i_1)}_q \xi^{(i_1)}_0 \xi^{(i_2)}_0 + \mu^{(i_3)}_q \xi^{(i_1)}_0 \xi^{(i_2)}_0 \right) + \\
+ \frac{1}{2 \sqrt{2}} \sum_{r=1}^q \left( \frac{1}{\pi r} \left( \xi^{(i_1)}_r \xi^{(i_2)}_0 \xi^{(i_3)}_0 - \xi^{(i_2)}_r \xi^{(i_1)}_0 \xi^{(i_3)}_0 \right) \right) + \\
+ \frac{1}{2 \pi^2 r^2} \left( \xi^{(i_1)}_r \xi^{(i_2)}_r \xi^{(i_3)}_0 - \xi^{(i_2)}_r \xi^{(i_1)}_r \xi^{(i_3)}_0 - \xi^{(i_3)}_r \xi^{(i_1)}_r \xi^{(i_2)}_0 + \xi^{(i_2)}_r \xi^{(i_3)}_r \xi^{(i_1)}_0 \right) + \\
+ \sum_{r=1}^q \left( \frac{1}{4 \pi r} \left( \xi^{(i_1)}_r \xi^{(i_2)}_r \xi^{(i_3)}_r - \xi^{(i_2)}_r \xi^{(i_1)}_r \xi^{(i_3)}_r - \xi^{(i_3)}_r \xi^{(i_1)}_r \xi^{(i_2)}_r + \xi^{(i_2)}_r \xi^{(i_3)}_r \xi^{(i_1)}_r \right) \right) + D_{T,t}^{(i_1 i_2 i_3)_q}, \]
\[
I_{(10)T,t}^{*(i_{1}i_{2})q} = -(T-t)^2 \left( \frac{1}{6} \zeta_{(i_{1})}^{(i_{2})} - \frac{1}{2 \sqrt{2\pi}} \sqrt{\alpha q} \zeta_{q}^{(i_{1})} + \frac{1}{2 \sqrt{2\pi}} \sqrt{\beta q} \left( \mu_{q}^{(i_{2})} - 2 \mu_{q}^{(i_{1})} \right) + \frac{1}{2 \sqrt{2\pi}} \sum_{r=1}^{q} \left( \frac{1}{\pi r} \zeta_{(i_{1})}^{(i_{2})} + \frac{1}{\pi r^2} \left( \zeta_{2r}^{(i_{1})} - 2 \zeta_{2r}^{(i_{2})} \right) \right) - \frac{1}{2 \pi^2} \sum_{r=1}^{q} \frac{1}{r^2 - l^2} \left( \zeta_{2r}^{(i_{1})} \zeta_{2l}^{(i_{2})} + \zeta_{2r}^{(i_{2})} \zeta_{2l}^{(i_{1})} \right) \right) \right)
\]

\[
I_{(11)T,t}^{*(i_{1}i_{2})q} = -(T-t)^2 \left( \frac{1}{3 \zeta_{q}^{(i_{1})} - \frac{1}{2 \sqrt{2\pi}} \sqrt{\alpha q} \left( \zeta_{q}^{(i_{1})} - 2 \zeta_{q}^{(i_{2})} \right) + \frac{1}{2 \sqrt{2\pi}} \sqrt{\beta q} \left( \mu_{q}^{(i_{1})} - 2 \mu_{q}^{(i_{2})} \right) - \frac{1}{2 \sqrt{2\pi}} \sum_{r=1}^{q} \left( \frac{1}{\pi r} \zeta_{(i_{1})}^{(i_{2})} - 2 \zeta_{(i_{2})}^{(i_{1})} \right) - \frac{1}{\pi r^2} \left( \zeta_{2r}^{(i_{1})} - 2 \zeta_{2r}^{(i_{2})} \right) \right) + \frac{1}{2 \pi} \sum_{r=1}^{q} \frac{1}{r^2 - l^2} \left( \zeta_{2r}^{(i_{1})} \zeta_{2l}^{(i_{2})} + \zeta_{2r}^{(i_{2})} \zeta_{2l}^{(i_{1})} \right) \right)
\]

\[
I_{(2)T,t}^{*(i_{1})q} = -(T-t)^{5/2} \left( \frac{1}{3 \zeta_{q}^{(i_{1})} + \frac{1}{\sqrt{2\pi}} \sum_{r=1}^{q} \frac{1}{r^2} \zeta_{2r}^{(i_{1})} + \sqrt{\beta q} \mu_{q}^{(i_{1})} \right) - \frac{1}{\sqrt{2\pi}} \left( \sum_{r=1}^{q} \frac{1}{r^2} \zeta_{2r}^{(i_{1})} + \sqrt{\alpha q} \zeta_{q}^{(i_{1})} \right) \right)
\]

where

\[
\zeta_{q}^{(i)} = \frac{1}{\sqrt{\alpha q}} \sum_{r=q+1}^{\infty} \frac{1}{r} \zeta_{2r-1}^{(i)}, \quad \alpha q = \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2}, \quad \mu_{q}^{(i)} = \frac{1}{\sqrt{\beta q}} \sum_{r=q+1}^{\infty} \frac{1}{r^2} \zeta_{2r}^{(i)},
\]

\[
\beta q = \frac{\pi^4}{90} - \sum_{r=1}^{q} \frac{1}{r^4}, \quad \zeta_{j}^{(i)} = \int_{t}^{T} \phi_{j}(s) d\Phi_{s}^{(i)},
\]
where \(\phi_j(s)\) has the form (44); \(\zeta^{(i)}_0, k_2, \ldots, k_{2r-1}\), \(\xi^{(i)}_q, \mu_q\); \(r = 1, \ldots, q\); \(i = 1, \ldots, m\) are independent standard Gaussian random variables; \(i_1, i_2, i_3 = 1, \ldots, m\).

Note that from (59), (60) it follows that

\[
\sum_{j=0}^{\infty} C_{jj}^{10} = \sum_{j=0}^{\infty} C_{jj}^{01} = -\frac{(T - t)^2}{4},
\]

where

\[
C_{jj}^{10} = \int_{t}^{T} \phi_j(x) \int_{t}^{x} \phi_j(y)(t - y)dydx,
\]

\[
C_{jj}^{01} = \int_{t}^{T} \phi_j(x)(t - x) \int_{t}^{x} \phi_j(y)dydx.
\]

The formulas (62) are particular cases of the more general relation, which we applied for the proof of Theorem 3 for the case \(k = 2\) (see [15]-[24]).

Let us consider the mean-square errors of approximations (57)-(60). From the relations (57)-(60) when \(i_1 \neq i_2, i_2 \neq i_3, i_1 \neq i_3\) we obtain by direct calculation

\[
M \left\{ \left( I^{(i_1i_2)}(000)_{T,t} - I^{(i_1i_2q)}_{(000)T,t} \right)^2 \right\} = \frac{(T - t)^2}{2\pi^2} \left( \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right),
\]

\[
M \left\{ \left( I^{(i_1i_2i_3)}_{(000)T,t} - I^{(i_1i_2i_3q)}_{(000)T,t} \right)^2 \right\} = (T - t)^3 \left( \frac{1}{4\pi^2} \left( \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right) + \frac{55}{32\pi^4} \left( \frac{\pi^4}{90} - \sum_{r=1}^{q} \frac{1}{r^2} \right) + \frac{1}{4\pi^4} \left( \sum_{r=1}^{\infty} - \sum_{r=1}^{\infty} \frac{1}{r^2} \right) \frac{5l^4 + 4l^4 - 3l^2r^2}{l^2(l^2 - l^2)^2} \right),
\]

\[
M \left\{ \left( I^{(i_1i_2)}_{(01)T,t} - I^{(i_1i_2q)}_{(01)T,t} \right)^2 \right\} = (T - t)^4 \left( \frac{1}{8\pi^2} \left( \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right) + \frac{5}{32\pi^4} \left( \frac{\pi^4}{90} - \sum_{r=1}^{q} \frac{1}{r^2} \right) + \frac{1}{4\pi^4} \left( \sum_{k, l=1}^{\infty} - \sum_{k, l=1}^{\infty} \frac{k^2 + l^2}{k^2(l^2 - k^2)} \right) \right),
\]

\[
M \left\{ \left( I^{(i_1i_2i_3)}_{(10)T,t} - I^{(i_1i_2i_3q)}_{(10)T,t} \right)^2 \right\} = (T - t)^4 \left( \frac{1}{8\pi^2} \left( \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right) + \frac{5}{32\pi^4} \left( \frac{\pi^4}{90} - \sum_{r=1}^{q} \frac{1}{r^2} \right) + \frac{1}{4\pi^4} \left( \sum_{k, l=1}^{\infty} - \sum_{k, l=1}^{\infty} \frac{k^2 + l^2}{k^2(l^2 - k^2)} \right) \right).
\]
It is easy to demonstrate that the relations (64), (65), and (66) can be represented using Theorem 8 in the following form

\[
M \left\{ \left( I^{(i_1 i_2 i_3 \gamma)}_{(00)T,t} - I^{(i_1 i_2 i_3 \gamma) q}_{(00)T,t} \right)^2 \right\} = (T - t)^3 \left( \frac{4}{45} - \frac{1}{4 \pi^2} \sum_{r=1}^{q} \frac{1}{r^2} - \frac{55}{32 \pi^4} \sum_{r=1}^{q} \frac{1}{r^4} - \frac{1}{4 \pi^4} \sum_{r=1}^{q} \frac{5 l^4 + 4 r^4 - 3 r^2 l^2}{r^2 (r^2 - l^2)^2} \right),
\]

(67)

\[
M \left\{ \left( I^{(i_1 i_2 \gamma)}_{(10)T,t} - I^{(i_1 i_2 \gamma) q}_{(10)T,t} \right)^2 \right\} = \left( \frac{T - t}{4} \right)^4 \left( \frac{1}{9} - \frac{1}{2 \pi^2} \sum_{r=1}^{q} \frac{1}{r^2} - \frac{5}{8 \pi^4} \sum_{r=1}^{q} \frac{1}{r^4} - \frac{1}{8 \pi^4} \sum_{k,l=1 \atop k \neq l}^{q} \frac{k^2 + l^2}{l^2 (l^2 - k^2)^2} \right),
\]

(68)

\[
M \left\{ \left( I^{(i_1 i_2 \gamma)}_{(01)T,t} - I^{(i_1 i_2 \gamma) q}_{(01)T,t} \right)^2 \right\} = \left( \frac{T - t}{4} \right)^4 \left( \frac{1}{9} - \frac{1}{2 \pi^2} \sum_{r=1}^{q} \frac{1}{r^2} - \frac{5}{8 \pi^4} \sum_{r=1}^{q} \frac{1}{r^4} - \frac{1}{8 \pi^4} \sum_{k,l=1 \atop k \neq l}^{q} \frac{l^2 + k^2}{l^2 (l^2 - k^2)^2} \right).
\]

(69)

Comparing (67) – (69) and (64) – (66), we obtain

\[
\sum_{k,l=1 \atop k \neq l}^{\infty} \frac{l^2 + k^2}{k^2 (l^2 - k^2)^2} = \sum_{k,l=1 \atop k \neq l}^{\infty} \frac{l^2 + k^2}{l^2 (l^2 - k^2)^2} = \frac{\pi^4}{48},
\]

(70)

\[
\sum_{r=1 \atop r \neq 1}^{\infty} \frac{5 l^4 + 4 r^4 - 3 r^2 l^2}{r^2 l^2 (r^2 - l^2)^2} = \frac{9 \pi^4}{80}.
\]

(71)

Let us consider approximations of the stochastic integrals $I^{(i_1 i_2 \gamma)}_{(10)T,t}$, $I^{(i_1 i_2 \gamma)}_{(01)T,t}$ and conditions for selecting the number $q$ using the trigonometric system of functions

\[
I^{(i_1 i_2 \gamma) q}_{(10)T,t} = -(T - t)^2 \left( \frac{1}{6} \left( s_0^{(i_1)} \right)^2 - \frac{1}{2 \sqrt{2 \pi}} \sqrt{\alpha} \xi_0^{(i_1)} \right) - \frac{1}{2 \sqrt{2 \pi}} \sum_{r=1}^{q} \left( \frac{1}{\pi r^2} \phi^{(i_1)}_{2r-1} s_0^{(i_1)} + \frac{1}{\pi r^2} \phi^{(i_1)}_{2r} s_0^{(i_1)} \right) - \frac{1}{2 \sqrt{2 \pi}} \sum_{r=1}^{q} \left( \frac{1}{\pi r^2} \phi^{(i_1)}_{2r-1} s_0^{(i_1)} + \frac{1}{\pi r^2} \phi^{(i_1)}_{2r} s_0^{(i_1)} \right).
\]
Then, we obtain

\[
-\frac{1}{2\pi^2} \sum_{r=1}^{q} \frac{1}{r^2 - l^2} \left( \zeta_{2r}^{(i_1)} \zeta_{2l}^{(i_1)} + \frac{l}{r} \zeta_{2r-1}^{(i_1)} \zeta_{2l-1}^{(i_1)} \right) + \\
+ \frac{1}{8\pi^2} \sum_{r=1}^{q} \frac{1}{r^2} \left( 3 \left( \zeta_{2r-1}^{(i_1)} \right)^2 + \left( \zeta_{2r}^{(i_1)} \right)^2 \right),
\]

\[
I^{*\{i_1i_2\}}_{(01)T,t} = (T-t)^2 \left( -\frac{1}{3} \left( \zeta_{0}^{(i_1)} \right)^2 + \frac{1}{2\sqrt{2\pi}} \sqrt{\alpha_q} \zeta_{0}^{(i_1)} - \\
- \frac{1}{2\sqrt{2\pi}} \sqrt{\beta_q} \mu_{0}^{(i_1)} \zeta_{0}^{(i_1)} + \frac{1}{2\sqrt{2\pi}} \sum_{r=1}^{q} \left( \frac{1}{r} \zeta_{2r}^{(i_1)} \zeta_{0}^{(i_1)} - \frac{1}{r} \zeta_{2r}^{(i_1)} \zeta_{0}^{(i_1)} \right) + \\
+ \frac{1}{2\pi^2} \sum_{r=1}^{q} \frac{1}{r^2 - l^2} \left( \zeta_{2r}^{(i_1)} \zeta_{2l}^{(i_1)} + \frac{l}{r} \zeta_{2r-1}^{(i_1)} \zeta_{2l-1}^{(i_1)} \right) + \\
+ \frac{1}{8\pi^2} \sum_{r=1}^{q} \frac{1}{r^2} \left( 3 \left( \zeta_{2r-1}^{(i_1)} \right)^2 + \left( \zeta_{2r}^{(i_1)} \right)^2 \right) \right).
\]

Then, we obtain

\[
M \left\{ \left( I^{*\{i_1i_2\}}_{(01)T,t} - I^{*\{i_1i_2\}}_{(10)T,t} \right)^2 \right\} = M \left\{ \left( I^{*\{i_1i_2\}}_{(10)T,t} - I^{*\{i_1i_2\}}_{(10)T,t} \right)^2 \right\} = \\
= \frac{(T-t)^4}{4} \left( \frac{2}{\pi^4} \left( \frac{\pi^4}{90} + \sum_{r=1}^{q} \frac{1}{r^4} \right) + \frac{1}{\pi^4} \left( \frac{\pi^4}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right)^2 + \\
+ \frac{1}{\pi^4} \left( \sum_{k,l=1}^{\infty} - \sum_{k,l=1}^{q} \frac{l^2 + k^2}{k^2(l^2 - k^2)^2} \right) \right).
\]
Table 1. Confirmation of the formula (67)

| $\varepsilon/(T-t)^3$ | 0.0459 | 0.0072 | 7.5722·$10^{-4}$ | 7.5973·$10^{-5}$ | 7.5990·$10^{-6}$ |
|---|---|---|---|---|---|
| $q$ | 1 | 10 | 100 | 1000 | 10000 |

Table 2. Confirmation of the formulas (68), (69)

| $4\varepsilon/(T-t)^4$ | 0.0540 | 0.0082 | 8.4261·$10^{-4}$ | 8.4429·$10^{-5}$ | 8.4435·$10^{-6}$ |
|---|---|---|---|---|---|
| $q$ | 1 | 10 | 100 | 1000 | 10000 |

Table 3. Confirmation of the formula (73)

| $4\varepsilon/(T-t)^4$ | 0.0268 | 0.0034 | 3.3955·$10^{-4}$ | 3.3804·$10^{-5}$ | 3.3778·$10^{-6}$ |
|---|---|---|---|---|---|
| $q$ | 1 | 10 | 100 | 1000 | 10000 |

Table 4. Confirmation of the formula (70)

| $\varepsilon_q$ | 2.0294 | 0.3241 | 0.0330 | 0.0033 | 3.2902·$10^{-4}$ |
|---|---|---|---|---|---|
| $q$ | 1 | 10 | 100 | 1000 | 10000 |

In Tables 1–3, we confirm numerically the formulas (67)–(69), (73) for various values $q$. In Tables 1–3, the number $\varepsilon$ means the right-hand sides of the mentioned formulas.

The formulas (70), (71) appear to be interesting. Let us confirm numerically their correctness in Tables 4 and 5 (the number $\varepsilon_q$ is the absolute deviation of multiple partial sums with the upper limit of summation $q$ for the series (70), (71) from the right-hand sides of the formulas (70), (71); convergence of multiple series is regarded here when $p_1 = p_2 = q \to \infty$, which is acceptable according to Theorems 1, 2).

Using the trigonometric system of functions, let us consider the approximations of iterated stochastic integrals of the following form

$$J^{(i_1 \cdots i_k)}_{(\lambda_1 \cdots \lambda_k)T,t} = \int_t^T \cdots \int_t^{t_2} \, d\mathbf{w}_{t_1}^{(i_1)} \cdots d\mathbf{w}_{t_k}^{(i_k)},$$

where $\lambda_l = 1$ if $i_l = 1, \ldots, m$ and $\lambda_l = 0$ if $i_l = 0; l = 1, \ldots, k$ ($\mathbf{w}_{t}^{(i)} = \mathbf{f}_{t}^{(i)}$ for $i = 1, \ldots, m$ and $\mathbf{w}_{\tau}^{(0)} = \tau$).

It is easy to see that the approximations

$$J^{(i_1 i_2)q}_{(\lambda_1 \lambda_2)T,t}, J^{(i_1 i_2)q}_{(\lambda_1 \lambda_2 \lambda_3)T,t}$$

of the stochastic integrals

$$J^{(i_1 i_2)}_{(\lambda_1 \lambda_2)T,t}, J^{(i_1 i_2)}_{(\lambda_1 \lambda_2 \lambda_3)T,t}$$

are defined by the right-hand sides of the formulas (57), (58), where it is necessary to take
Table 5. Confirmation of the formula (71)

| $\varepsilon_q$ | 10.9585 | 1.8836 | 0.1968 | 0.0197 | 0.0020 |
|-----------------|---------|--------|--------|--------|--------|
| $q$             | 1       | 10     | 100    | 1000   | 10000  |

(74) \[ \zeta_j^{(i)} = \int_0^T \phi_j(s)d\omega_s^{(i)} \]

and $i_1, i_2, i_3 = 0, 1, \ldots, m$. Since

\[ \int_0^T \phi_j(s)d\omega_s^{(0)} = \begin{cases} \sqrt{T-t} & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}, \]

then it is easy to get from (57) and (58), considering that in these equalities $\zeta_j^{(i)}$ has the form (74) and $i_1, i_2, i_3 = 0, 1, \ldots, m$, the following family of formulas

\[
\begin{align*}
J_{(10)}^{(i_1)q} &= \frac{1}{2} (T-t)^{3/2} \left( \frac{\sqrt{T}}{\pi} \sum_{r=1}^{q} \frac{1}{r} \alpha_{r_{2r-1}}^{(i_1)} + \sqrt{\alpha} \xi_{r_{2r}}^{(i_1)} \right), \\
J_{(01)}^{(i_2)q} &= \frac{1}{2} (T-t)^{3/2} \left( \frac{\sqrt{T}}{\pi} \sum_{r=1}^{q} \frac{1}{r} \alpha_{r_{2r-1}}^{(i_2)} + \sqrt{\alpha} \xi_{r_{2r}}^{(i_2)} \right), \\
J_{(001)}^{(i_3)q} &= (T-t)^{5/2} \left( \frac{1}{6} \alpha_{s_0}^{(i_3)} + \frac{1}{2\sqrt{2\pi}} \left( \sum_{r=1}^{q} \frac{1}{r} \alpha_{r_{2r-1}}^{(i_3)} + \sqrt{\beta} \mu_{r_{2r}}^{(i_3)} \right) - \frac{1}{2\sqrt{2\pi}} \left( \sum_{r=1}^{q} \frac{1}{r} \alpha_{r_{2r-1}}^{(i_3)} + \sqrt{\alpha} \xi_{r_{2r}}^{(i_3)} \right) \right), \\
J_{(010)}^{(i_2)q} &= (T-t)^{5/2} \left( \frac{1}{6} \alpha_{s_0}^{(i_2)} - \frac{1}{2\sqrt{2\pi}} \left( \sum_{r=1}^{q} \frac{1}{r} \alpha_{r_{2r-1}}^{(i_2)} + \sqrt{\beta} \mu_{r_{2r}}^{(i_2)} \right) \right), \\
J_{(100)}^{(i_1)q} &= (T-t)^{5/2} \left( \frac{1}{6} \alpha_{s_0}^{(i_1)} + \frac{1}{2\sqrt{2\pi}} \left( \sum_{r=1}^{q} \frac{1}{r} \alpha_{r_{2r-1}}^{(i_1)} + \sqrt{\beta} \mu_{r_{2r}}^{(i_1)} \right) + \frac{1}{2\sqrt{2\pi}} \left( \sum_{r=1}^{q} \frac{1}{r} \alpha_{r_{2r-1}}^{(i_1)} + \sqrt{\alpha} \xi_{r_{2r}}^{(i_1)} \right) \right),
\end{align*}
\]
\[ J^{(0123q)}_{(011)T,t} = (T - t)^2 \left( \frac{1}{6} c_{(i_2)} c_{(i_3)} s_0 - \frac{1}{2 \sqrt{2\pi}} \alpha_{q} c_{(i_3)} c_{(i_2)} s_0 \right) + \]
\[ + \frac{1}{2 \sqrt{2\pi}^2} \beta_{q} \left( \mu_{q} c_{(i_3)} c_{(i_2)} s_0 - 2 \mu_{q} c_{(i_2)} c_{(i_3)} s_0 \right) + \]
\[ + \frac{1}{2 \sqrt{2\pi}^2} \sum_{r=1}^{q} \left( \frac{1}{\pi r^2} c_{(i_2)} c_{(i_3)} s_0 + \frac{1}{\pi^2 r^2} \left( c_{2r} c_{(i_2)} s_0 - 2 c_{2r} c_{(i_2)} s_0 \right) \right) - \]
\[ - \frac{1}{2 \sqrt{2\pi}^2} \sum_{r=1}^{q} \left( \frac{1}{\pi r^2} c_{2r} c_{(i_2)} s_0 + \frac{1}{\pi^2 r^2} \left( c_{2r} c_{(i_2)} s_0 - 2 c_{2r} c_{(i_2)} s_0 \right) \right) + \]
\[ + \sum_{r=1}^{q} \left( \frac{1}{4 \pi r} \left( c_{2r} c_{(i_2)} s_0 - c_{2r} c_{(i_2)} s_0 \right) + \frac{1}{8 \pi^2 r^2} \left( 3 c_{2r} c_{(i_2)} s_0 - c_{2r} c_{(i_2)} s_0 \right) \right) . \]

\[ J^{(i_1i_2q)}_{(110)T,t} = (T - t)^2 \left( \frac{1}{6} c_{(i_1)} c_{(i_2)} s_0 + \frac{1}{2 \sqrt{2\pi}^2} \alpha_{q} c_{(i_1)} c_{(i_2)} s_0 \right) + \]
\[ + \frac{1}{2 \sqrt{2\pi}^2} \beta_{q} \left( \mu_{q} c_{(i_1)} c_{(i_2)} s_0 - 2 \mu_{q} c_{(i_2)} c_{(i_1)} s_0 \right) + \]
\[ + \frac{1}{2 \sqrt{2\pi}^2} \sum_{r=1}^{q} \left( \frac{1}{\pi r^2} s_{r-1} c_{(i_2)} s_0 + \frac{1}{\pi^2 r^2} \left( s_{2r} c_{(i_2)} s_0 - 2 s_{2r} c_{(i_2)} s_0 \right) \right) + \]
\[ + \sum_{r=1}^{q} \left( \frac{1}{4 \pi r} \left( s_{2r-1} s_{2r} c_{(i_2)} s_0 + c_{2r} c_{(i_2)} s_0 \right) + \frac{1}{8 \pi^2 r^2} \left( 3 s_{2r-1} s_{2r} c_{(i_2)} s_0 + c_{2r} c_{(i_2)} s_0 \right) \right) , \]

\[ J^{(i_1q)}_{(101)T,t} = (T - t)^2 \left( \frac{1}{6} c_{(i_1)} c_{(i_1)} s_0 + \frac{1}{2 \sqrt{2\pi}} \alpha_{q} c_{(i_1)} c_{(i_1)} s_0 \right) + \]
\[ + \frac{1}{2 \sqrt{2\pi}^2} \beta_{q} \left( \mu_{q} c_{(i_1)} c_{(i_1)} s_0 + \mu_{q} c_{(i_1)} c_{(i_1)} s_0 \right) + \]
\[ + \frac{1}{2 \sqrt{2\pi}^2} \sum_{r=1}^{q} \left( \frac{1}{\pi r} \left( \zeta_{2r-1} c_{(i_1)} s_0 - \zeta_{2r-1} c_{(i_1)} s_0 \right) \right) + \]
$$+ \frac{1}{\pi r^2} \left( \hat{c}_{2r}^{(i_1)} \hat{c}_{2r}^{(i_3)} + \hat{c}_{2r}^{(i_2)} \hat{c}_{2r}^{(i_1)} \right) - \frac{1}{2\pi r^2} \sum_{r \neq s \neq 1} \frac{1}{r^2} \hat{c}_{2r-1}^{(i_1)} \hat{c}_{2r-1}^{(i_3)} -$$

$$- \sum_{p=1}^{q} \frac{1}{4\pi^2 r^2} \left( 3 \hat{c}_{2r-1}^{(i_1)} \hat{c}_{2r-1}^{(i_3)} + \hat{c}_{2r}^{(i_2)} \hat{c}_{2r}^{(i_1)} \right).$$

7. THEOREMS 1–7 FROM POINT OF VIEW OF THE WONG–ZAKAI APPROXIMATION

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components $f_s^{(i)}$, $i = 1, \ldots, m$ of the multidimensional Wiener process $f_s$, $s \in [0, T]$. Let $f_s^{(i)p}$, $p \in \mathbb{N}$ be some approximation of $f_s^{(i)}$, $i = 1, \ldots, m$. Suppose that $f_s^{(i)p}$ converges to $f_s^{(i)}$, $i = 1, \ldots, m$ if $p \to \infty$ in some sense and has differentiable sample trajectories.

A natural question arises: if we replace $f_s^{(i)}$ by $f_s^{(i)p}$, $i = 1, \ldots, m$ in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components $f_s^{(i)}$, $i = 1, \ldots, m$ of the multidimensional Wiener process $f_s$? The answer to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [38, 39], it was shown that under the special conditions and for some types of approximations of the Wiener process the answer is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [38–40] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let $f_s$, $s \in [0, T]$ be an $m$-dimensional standard Wiener process with independent components $f_s^{(i)}$, $i = 1, \ldots, m$. It is well known that the following representation takes place [42, 43]

$$f_{s,t}^{(i)} - f_t^{(i)} = \sum_{j=0}^{\tau} \int_{t}^{\tau} \phi_j(s) ds \, \zeta_j^{(i)}; \quad \zeta_j^{(i)} = \int_{t}^{\tau} \phi_j(s) df_s^{(i)},$$

where $\tau \in [t, T]$, $t \geq 0$, $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$, and $\zeta_j^{(i)}$ are independent standard Gaussian random variables for various $i$ or $j$. Moreover, the series (76) converges for any $\tau \in [t, T]$ in the mean-square sense.

Let $f_{s,t}^{(i)p} - f_t^{(i)p}$ be the mean-square approximation of the process $f_{s,t}^{(i)} - f_t^{(i)}$, which has the following form

$$f_{s,t}^{(i)p} - f_t^{(i)p} = \sum_{j=0}^{p} \int_{t}^{\tau} \phi_j(s) ds \, \zeta_j^{(i)}.$$

From (77) we obtain

$$df_{s,t}^{(i)p} = \sum_{j=0}^{p} \phi_j(\tau) \zeta_j^{(i)} \, d\tau.$$

Consider the following iterated Riemann–Stieltjes integral
\[
\int_{t}^{T} \psi_k(t_k) \ldots \int_{t}^{t_2} \psi_1(t_1) d\mathbf{w}_{i_1}^{(i_1)p_1} \ldots d\mathbf{w}_{i_k}^{(i_k)p_k},
\]

where \(p_1, \ldots, p_k \in \mathbb{N}, \ i_1, \ldots, i_k = 0, 1, \ldots, m,\)

\[
d\mathbf{w}^{(i)p}_{\tau} = \begin{cases} 
  d\mathbf{f}^{(i)p}_{\tau} & \text{for } i = 1, \ldots, m \\
  d\tau & \text{for } i = 0
\end{cases},
\]

and \(d\mathbf{f}^{(i)p}_{\tau}\) in defined by the relation (78).

Let us substitute (78) into (79) (81)

\[
\int_{t}^{T} \psi_k(t_k) \ldots \int_{t}^{t_2} \psi_1(t_1) d\mathbf{w}_{i_1}^{(i_1)p_1} \ldots d\mathbf{w}_{i_k}^{(i_k)p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} \sum_{j_{k-1}=0}^{C_{j_{k-1}} \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i)},
\]

where

\[
\zeta_{j_l}^{(i)} = \int_{t}^{T} \phi_{j_l}(s) d\mathbf{w}_{s}^{(i)}
\]

are independent standard Gaussian random variables for various \(i\) or \(j\) (in the case when \(i \neq 0\)), \(\mathbf{w}_{s}^{(i)} = \mathbf{f}_{s}^{(i)}\) for \(i = 1, \ldots, m\) and \(\mathbf{w}_{s}^{(0)} = s,\)

\[
C_{j_{k-1}} \ldots j_1 = \int_{t}^{T} \psi_k(t_k) \phi_{j_k}(t_k) \ldots \int_{t}^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_k
\]

is the Fourier coefficient.

To best of our knowledge [38]-[40] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [40] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (77) were not considered in [38], [39] (also see [40], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 [40] for approximations of the Wiener process based on its series expansion (76) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (81) to the iterated Stratonovich stochastic integral [3] does not follow from the results of the papers [38], [39] (also see [40], Theorems 7.1, 7.2).

From the other hand, Theorems 1–7 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals [40] of multiplicities 1 to 6 based on the approximation (77) of the Wiener process. At that, the iterated Riemann–Stieltjes integrals (79) converge (according to Theorems 1–7) to the appropriate iterated Stratonovich stochastic integrals [4]. Recall that \(\{\phi_{j}(x)\}_{j=0}^{\infty}\) (see (70), (77), and Theorems 3–7) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([t, T])\).

To illustrate the above reasoning, consider two examples for the case \(k = 2, \psi_1(s), \psi_2(s) \equiv 1; i_1, i_2 = 1, \ldots, m.\)
The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [38]-[40]).

Let \( b^{(i)}_\Delta(t), t \in [0,T] \) be the piecewise linear approximation of the \( i \)th component \( f^{(i)} \) of the multidimensional standard Wiener process \( f_t, t \in [0,T] \) with independent components \( f^{(i)}_t, i = 1, \ldots, m \), i.e.

\[
\Delta f^{(i)}_{k\Delta} = f^{(i)}_{(k+1)\Delta} - f^{(i)}_{k\Delta}, \quad t \in [k\Delta,(k+1)\Delta), \quad k = 0, 1, \ldots, N-1.
\]

Note that w. p. 1

\[
\frac{d b^{(i)}_\Delta}{dt}(t) = \frac{\Delta f^{(i)}_{k\Delta}}{\Delta}, \quad t \in [k\Delta,(k+1)\Delta), \quad k = 0, 1, \ldots, N-1.
\]

Consider the following iterated Riemann–Stieltjes integral

\[
\int_0^T \int_0^s \frac{db^{(i1)}_\Delta(\tau)db^{(i2)}_\Delta(s)}{ds} d\tau ds,
\]

Using (82) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

\[
\int_0^T \int_0^s \frac{db^{(i1)}_\Delta(\tau)db^{(i2)}_\Delta(s)}{ds} d\tau ds = \int_0^T \int_0^s \frac{db^{(i1)}_\Delta(\tau)db^{(i2)}_\Delta(s)}{ds} d\tau ds =
\]

\[
= \sum_{l=0}^{N-1} \int_{l\Delta}^{(l+1)\Delta} \left( \sum_{q=0}^{(l+1)\Delta} \frac{\Delta f^{(i1)}_{q\Delta}}{\Delta} d\tau + \int_{l\Delta}^{s} \frac{\Delta f^{(i1)}_{j\Delta}}{\Delta} d\tau \right) \frac{\Delta f^{(i2)}_{l\Delta}}{\Delta} ds =
\]

\[
= \sum_{l=0}^{N-1} \int_{l\Delta}^{(l+1)\Delta} \Delta f^{(i1)}_{q\Delta} \Delta f^{(i2)}_{l\Delta} + \frac{1}{\Delta} \sum_{l=0}^{N-1} \Delta f^{(i1)}_{l\Delta} \Delta f^{(i2)}_{l\Delta} \int_{l\Delta}^{s} d\tau ds =
\]

\[
\int_0^T \int_0^s \frac{db^{(i1)}_\Delta(\tau)db^{(i2)}_\Delta(s)}{ds} d\tau ds =
\]

\[
= \sum_{l=0}^{N-1} \sum_{q=0}^{(l+1)\Delta} \Delta f^{(i1)}_{q\Delta} \Delta f^{(i2)}_{l\Delta} + \frac{1}{\Delta} \sum_{l=0}^{N-1} \Delta f^{(i1)}_{l\Delta} \Delta f^{(i2)}_{l\Delta}. \quad \text{(83)}
\]

Using (83), it is not difficult to show that
\[
\text{l.i.m.}_{N \to \infty} \int_0^T \int_0^s \frac{d \mathbf{b}_{\Delta}^{(i_1)}(\tau) d \mathbf{b}_{\Delta}^{(i_2)}(s)}{s} = \int_0^T \int_0^s \frac{d f_{s}^{(i_1)} d f_s^{(i_2)}}{s} + \frac{1}{2} \mathbb{1}_{\{i_1=i_2\}} \int_0^T ds =
\]

\[
= \int_0^T \int_0^s \frac{d f_{s}^{(i_1)} d f_s^{(i_2)}}{s},
\]

where $\Delta \to 0$ if $N \to \infty$ ($N \Delta = T$).

Obviously, (84) agrees with Theorem 7.1 (see [40], p. 486).

The next example relates to the approximation (77) of the Wiener process based on its series expansion (76) for $t = 0$, where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([0, T])$.

Consider the following iterated Riemann–Stieltjes integral

\[
\int_0^T \int_0^s \frac{d f_{s}^{(i_1)} d f_s^{(i_2)}}{s},
\]

where $d f_{s}^{(i_1)p}$ is defined by the relation (78).

Let us substitute (78) into (85)

\[
\int_0^T \int_0^s \frac{d f_{s}^{(i_1)} d f_s^{(i_2)}}{s} = \sum_{j_1, j_2=0}^{p} C_{j_2 j_1} \phi_{j_2}^{(i_1)} \phi_{j_1}^{(i_2)},
\]

where

\[
C_{j_2 j_1} = \int_0^T \phi_{j_2}(s) \int_0^s \phi_{j_1}(\tau) d \tau ds
\]

is the Fourier coefficient; another notations are the same as in (81).

As we noted above, approximations of the Wiener process that are similar to (77) were not considered in [38], [39] (also see Theorems 7.1, 7.2 in [40]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [40] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [22]-[24]. More precisely, using Theorem 3, we obtain from (86) the desired result

\[
\text{l.i.m.}_{p \to \infty} \int_0^T \int_0^s \frac{d f_{s}^{(i_1)} d f_s^{(i_2)}}{s} = \text{l.i.m.}_{p \to \infty} \sum_{j_1, j_2=0}^{p} C_{j_2 j_1} \phi_{j_2}^{(i_1)} \phi_{j_1}^{(i_2)} =
\]

\[
= \int_0^T \int_0^s \frac{d f_{s}^{(i_1)} d f_s^{(i_2)}}{s},
\]

(87)
From the other hand, by Theorems 1, 2 (see (14)) for the case $k = 2$ we obtain from (86) the following relation

$$\lim_{p \to \infty} \int_0^T \int_0^s d\xi_1^{(i_1)} d\xi_2^{(i_2)} p = \lim_{p \to \infty} \sum_{j_1, j_2 = 0}^p C_{j_2 j_1} \xi_1^{(i_1)} \xi_2^{(i_2)} =$$

$$= \lim_{p \to \infty} \sum_{j_1, j_2 = 0}^p C_{j_2 j_1} \left( \xi_1^{(i_1)} \xi_2^{(i_2)} - 1_{\{i_1 = i_2\}} 1_{\{j_1 = j_2\}} \right) + 1_{\{i_1 = i_2\}} \sum_{j_1 = 0}^{\infty} C_{j_1 j_1} =$$

$$= \int_0^T \int_0^s d\xi_1^{(i_1)} d\xi_2^{(i_2)} + 1_{\{i_1 = i_2\}} \sum_{j_1 = 0}^{\infty} C_{j_1 j_1}.$$

Since

$$\sum_{j_1 = 0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1 = 0}^{\infty} \left( \int_0^T \phi_j(t) dt \right)^2 = \frac{1}{2} \left( \int_0^T \phi_0(t) dt \right)^2 = \frac{1}{2} \int_0^T ds,$$

then from (88) we obtain (87).

**References**

[1] Gihman I.I., Skorochod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982. 354 pp.

[2] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.

[3] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995. 632 pp.

[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994. 292 pp.

[5] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications, 10, 4 (1992), 431-441.

[6] Platen, E., Bruti-Liberati, N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg, 2010. 868 pp.

[7] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.

[8] Kulchitskii O.Yu., Kuznetsov D.F. The unified Taylor-Itô expansion. J. Math. Sci. (N. Y.), 99, 2 (2000), 1130-1140. DOI: [http://doi.org/10.1007/BF02673635](http://doi.org/10.1007/BF02673635)

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansion. J. Math. Sci. (N. Y.), 118, 6 (2003), 5586-5596. DOI: [http://doi.org/10.1023/A:1026138522239](http://doi.org/10.1023/A:1026138522239)

[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006. 764 pp. DOI: [http://doi.org/10.18720/SPBPU/2/s17-227](http://doi.org/10.18720/SPBPU/2/s17-227) Available at: [http://www.sde-kuznetsov.spb.ru/06.pdf](http://www.sde-kuznetsov.spb.ru/06.pdf) (ISBN 5-7422-1191-0)

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Program, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007. 778 pp. DOI: [http://doi.org/10.18720/SPBPU/2/s17-228](http://doi.org/10.18720/SPBPU/2/s17-228) Available at: [http://www.sde-kuznetsov.spb.ru/07b.pdf](http://www.sde-kuznetsov.spb.ru/07b.pdf) (ISBN 5-7422-1394-8)
[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian], Politechnical University Publishing House, Saint-Petersburg, 2007, XXXI+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 978-5-7422-1439-1)

[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian], Politechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[14] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian], Politechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-251 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[15] Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/z17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[16] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English], Politechnical University Publishing House, Saint-Petersburg, 2011, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-252 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[17] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English], Politechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-253 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[18] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. [In English], Politechnical University Publishing House, Saint-Petersburg, 2013, 382 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-234 Available at: http://www.sde-kuznetsov.spb.ru/13.pdf (ISBN 978-5-7422-3973-4)

[19] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals: Fourier-Legendre and Trigonometric Expansions. Approximations, Formulas. [In English], Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/z17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian], Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/s17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian], Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[22] Kuznetsov D.F. Multiple Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. [arXiv:2003.14184 [math.PR]]. 2022, 923 pp. [In English].

[23] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[24] Kuznetsov D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

[25] Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070096

[26] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070056

[27] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 2.5 order of strong convergence. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060
[28] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[29] Kuznetsov D.F. Expansion of iterated Itô stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746 [math.PR]. 2022, 111 pp. [In English].

[30] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516 [math.PR]. 2022, 204 pp. [In English].

[31] Kuznetsov D.F. Mean-square approximation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Itô and Taylor-Stratonovich expansions using Legendre polynomials. arXiv:1801.00231 [math.PR]. 2019, 106 pp. [In English].

[32] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Itô stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. arXiv:1807.02190 [math.PR]. 2018, 44 pp. [In English].

[33] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195 [math.PR]. 2022, 138 pp. [In English].

[34] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series. arXiv:1802.00643 [math.PR]. 2022, 129 pp. [In English].

[35] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2019, 68 pp.

[36] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. arXiv:1901.02545 [math.GM]. 2019, 40 pp. [In English].

[37] Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03723 [math.GM]. 2019, 41 pp. [In English].

[38] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[39] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[40] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

[41] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spb.ru/EN/numbers/2021.3/article.1.8.html

[42] Liptser R.Sh., Shiryaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.

[43] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[44] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs with non-commutative noise based on the unified Taylor-Itô and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR]. 2020, 443 pp. [In English].

[45] Kuznetsov D.F. Application of multiple Fourier–Legendre series to the implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs. [In English]. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry AMMAI-2020 (Crimea, Alushta, 6-13 September, 2020), MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[46] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2019), 18-62. Available at: http://diffjournal.spb.ru/EN/numbers/2019.3/article.1.2.html

[47] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spb.ru/RU/numbers/2020.2/article.1.6.html

[48] Kuznetsov D.F. Application of multiple Fourier–Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spb.ru/RU/numbers/2020.3/article.1.6.html
[49] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor–Ito expansion based on multiple Fourier–Legendre series [arXiv:2010.13564] [math.PR], 2020, 63 pp. [In English].

[50] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[51] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[52] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7-2

[53] Kuznetsov D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [arXiv:1912.02612] [math.PR], 2019, 32 pp. [In English].

[54] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[55] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html

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