ON THE STOCHASTIC FLOWS ON 
\((m + n + 1)\)-DIMENSIONAL EXOTIC SPHERES

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ABSTRACT

Stochastic flows of Stratonovich stochastic differential equations on exotic spheres have been studied. The consequences of the choice of exotic differential structure on stochastic processes taking place on the topological space \(S^{m+n+1}\) as state space of the processes have been investigated. More precisely, we have investigated the properties of stochastic processes where the state spaces of the stochastic processes under consideration are \((m + n + 1)\)-dimensional differentiable manifolds which are homeomorphic but not necessarily diffeomorphic to standard \((m + n + 1)\)-dimensional sphere. The differentiable manifolds have been constructed from disjoint union \(\mathbb{R}^{m+1} \times S^n \sqcup S^m \times \mathbb{R}^{n+1}\) by identifying every pair of its points using a map \(u : \mathbb{R}^{m+1} \times S^n \to S^m \times \mathbb{R}^{n+1}\) which is constructed from a diffeomorphism \(h_1 \times h_2 : S^m \times S^n \to S^m \times S^n\). The diffeomorphisms \(h_1\) and \(h_2\), therefore, can be regarded as the carriers of the "exoticism" of the constructed manifolds. For all of the above purposes, homeomorphisms \(h\) from the above-constructed manifolds onto the standard sphere explicitly in term of the diffeomorphisms \(h_1\) and \(h_2\) have been constructed. Using the homeomorphisms \(h\) and all their associated maps derived from them and expressed in terms of \(h_1\) and \(h_2\) as well as their derivatives, we construct the stochastic processes or flows on the above-constructed manifolds corresponding to stochastics processes on the standard sphere \(S^{n} \times \mathbb{R}^{n+1}\). The stochastic processes yielded from the above construction on the constructed manifolds can be regarded as the same stochastic processes on \(S^{m+n+1}\) but described in exotic differential structures on \(S^{m+n+1}\).

Keywords: Stochastic flow; stratonovich stochastic differential equation; exotic spheres; exotic differential structure; homeomorphism; Sobolev regularities.

Mathematics Subject Classification (2010): 60H10; 57S15; 51H25.


1 Introduction

Stochastic processes appear in many physical phenomena, such as the well-known diffusion process, heat conduction, Brownian motion, the formation of river meander, the fluctuation current due to thermal noise, etc. Mathematically, a stochastic process is a set of random variables parametrized by another set (discrete or continuous) playing the role of time. A stochastic process is a solution of so-called stochastic differential equations. Since there exists a distribution function for every random variable, there is an associated distribution function depending on the time for every stochastic process. The distribution function associated with a stochastic process satisfies a deterministic partial differential equation called the Fokker-Planck equation associated with the process which is obtained from the stochastic differential equation governing the process via Itô equation. A Fokker-Planck equation is used by Einstein to define the time evolution of the probability density of particles whereas Langevin rewrites an explicit deterministic momentum equation for a particle augmented by a Gaussian white noise force which perturbs the particle trajectory.

The approach of Langevin now constitutes a canonical stochastic differential equation for a full scope of systems, but since its introduction, the mathematical and physical interpretation of the white noise term has been discussed and debated. The white noise interpretation of stochastic differential equations naturally leads to stochastic differential equations in Stratonovich sense. This is because discrete-time and smooth approximations of white noise drive stochastic differential equations to converge to stochastic differential equations in Stratonovich sense, not in Itô sense.

The microscopic description of the dynamics of a diffusion process on a connected compact differentiable manifold $M$, for instance, is typically represented by a so-called Stratonovich formulation of stochastic differential equations [6]. Likewise, the dynamic descriptions of the other stochastic processes on differentiable manifolds are also represented in the formulation. The advantage of this formulation is that Itô’s formula appears in the same form as the fundamental theorem of calculus; therefore, stochastic calculus in this formulation takes a more familiar form. The other advantage of Stratonovich formulation, which is more important and relevant to our investigation is that it allows one to construct more than one space-time with different differential structures. If the yielded space-times are not equivalent to each other, we may find more than one differential structures. Therefore, from a single topological space of all events, we may construct more than one space-time with different differential structures. If the yielded space-times are not distinguishable in the sense that they are not diffeomorphic, they may lead to inequivalent formulations of the law of physics. The totality of inequivalent differential structures on a topological space is called the exotica of differential structures on the topological space, and a differential structure that is not equivalent to the standard one is called an exotic differential structure on the topological space under consideration [1, 2]. Unfortunately, among the exotic differential structures whose existences are already known on various topological spaces, only the exotic differential structures on spheres can be constructed explicitly. Therefore, the exotic spheres can be used as toy models serving as configuration or phase spaces of physical systems as well as the base manifolds for various field theories. Various problems of real interest can be answered by studying the models and the results they are compared to those obtained from the standard spheres [3].
In mathematics, spheres are topological spaces which are interesting to investigate. They serve, for instance, as models for configuration spaces of some mechanical systems. In physics, for example, the standard 7-dimensional sphere \( S^7 \) is particularly interesting in relation to supersymmetry breaking [3] and to the work of Witten [10] in which he used it to cancel the global gravitational anomalies in 1985. The standard seven-dimensional sphere is also regarded as the total space of a principal \( SU(2) \) bundle of Yang-Mills theory [11][7]. Two differential structures on 7-dimensional sphere \( S^7 \) are said to be equivalent if there is a diffeomorphism pulling back the differentiable maximal atlas from the second to the first. The connected compact topological space \( S^7 \) has more than one distinct differential structure that are not equivalent each other in this sense or more precisely there are connected compact 7-dimensional differentiable manifolds which are homeomorphic but not diffeomorphic to the standard seven-sphere \( S^7 \) [8][9]. The differential structure on the standard sphere \( S^7 \) is called a standard differential structure, while the differential structures that are not equivalent to the standard one are called exotic differential structures. The topological space \( S^7 \) equipped with an exotic differential structure or a seven-dimensional differentiable manifold which is homeomorphic but not diffeomorphic to standard seven-dimensional sphere \( S^7 \) is called exotic 7-sphere.

The main purpose of this work is to understand the consequences of the choice of exotic differential structure on stochastic processes taking place on the topological space \( S^{m+n+1} \) as state space of the processes. More precisely, we investigate the properties of stochastic processes where the state spaces of the stochastic processes under consideration are \((m+n+1)\)-dimensional differentiable manifolds \( M^{m+n+1} \) which are homeomorphic but not necessarily diffeomorphic to standard \((m+n+1)\)-dimensional sphere \( S^{m+n+1} \). A manifolds \( M^{m+n+1} \) is constructed (see [8]) from disjoint union \( \mathbb{R}^{m+1} \times S^m \cup S^m \times \mathbb{R}^{n+1} \) by identifying every pair of its points using a map \( u : \mathbb{R}^{m+1} \times S^m \to S^m \times \mathbb{R}^{n+1} \) which is constructed from a diffeomorphism \( h_1 \times h_2 : S^m \times S^n \to S^m \times S^n \). Whether a manifold \( M^{m+n+1} \) is homeomorphic or even diffeomorphic to the standard sphere \( S^{m+n+1} \) depends on the diffeomorphism \( h_1 : S^m \to S^m \) and \( h_2 : S^n \to S^n \). We consider here only the cases where the diffeomorphisms \( h_1 \) and \( h_2 \) lead to the manifolds \( M^{m+n+1} \) which are at least homeomorphic to standard sphere. The diffeomorphisms \( h_1 \) and \( h_2 \) therefore can be regarded as the carriers of the "exoticism" of \( M^{m+n+1} \). For all of the above purposes, we firstly construct from the diffeomorphisms \( h_1 \) and \( h_2 \) a homeomorphism \( h : M^{m+n+1} \to S^{m+n+1} \) explicitly in term of the diffeomorphisms \( h_1 \) and \( h_2 \). The homeomorphism \( h \) topologically identifies the topological space \( M^{m+n+1} \) with \( S^{m+n+1} \). For instance, if the diffeomorphisms \( h_1 \) and \( h_2 \) determining \( M^{m+n+1} \) are the identity maps, then the corresponding homeomorphism \( h \) is a diffeomorphism. Assuming a stochastic differential equation which satisfies certain regularities on the standard spheres, then associated to the equation there exists a unique stochastic flow on the standard sphere in the sense of Zhang [12]. Using the identification \( h \), a stochastic flow can be constructed on \( M^{m+n+1} \). The stochastic flow yielded from the above construction on \( M^{m+n+1} \) can be regarded as the same stochastic flow on \( S^{m+n+1} \) but described in exotic differential structures on \( S^{m+n+1} \). In turn, by making use of the homeomorphism \( h \) and associated maps derived from it (expressed in terms of \( h_1 \) and \( h_2 \) as well as their derivatives), we construct a stochastic differential equation on \( M^{m+n+1} \) corresponding to the stochastic differential equation on the standard sphere \( S^{m+n+1} \). We investigate then the stochastic flow yielded in the above mentioned construction on the exotic spheres. The main result of our investigation is depicted than in Theorem 3.

2 Milnor’s Construction of Exotic Spheres and their Homeomorphism into Standard Sphere

Let \( h_1 \times h_2 : S^m \times S^n \to S^m \times S^n \) be a diffeomorphism from \( S^m \times S^n \) onto itself so that every \((x, y) \in S^m \times S^n\) is mapped into \((h_1(x), h_2(y)) \in S^m \times S^n\). Furthermore, define a map \( u : \mathbb{R}^{m+1} \times S^n \to S^m \times \mathbb{R}^{n+1} \) defined by \( u(tx, y) = (h_1(x), t^{-1}h_2(y)) \) for every \( t \in (0, \infty) \) and \((x, y) \in S^m \times S^n\). The manifold \( M^{m+n+1} \) is obtained from \( \mathbb{R}^{m+1} \times S^n \cup S^m \times \mathbb{R}^{n+1} \) by gluing or matching every \((tx, y) \in \mathbb{R}^{m+1} \times S^n\) with its image \( u(tx, y) = (h_1(x), t^{-1}h_2(y)) \) under the map \( u \). It is an \((m+n+1)\)-dimensional manifold. Wether the obtained manifold \( M^{m+n+1} \) is homeomorphic or even diffeomorphic to \( S^{m+n+1} \) depends merely on the maps \( h_1 \) and \( h_2 \).

Now consider two maps defined by

\[
\mathbb{R}^{m+1} \times S^n \ni (x, y) \mapsto \left( \frac{x}{\sqrt{1 + \|x\|^2}}, \frac{y}{\sqrt{1 + \|y\|^2}} \right) \in S^{m+n+1},
\]

(1)
and
\[ S^m \times \mathbb{R}^{n+1} \ni (\tilde{x}, \tilde{y}) \mapsto \left( \frac{h_1^{-1}(\tilde{x})}{\sqrt{1 + ||\tilde{y}||^2}}, \frac{||\tilde{y}||h_2^{-1}(\frac{\tilde{y}}{||\tilde{y}||})}{\sqrt{1 + ||\tilde{y}||^2}} \right) \in S^{m+n+1}. \quad (2) \]

If
\[ \tilde{x} = h_1\left( \frac{x}{||x||} \right) \quad \text{and} \quad \tilde{y} = h_2\left( \frac{y}{||y||} \right), \quad (3) \]
i.e. \((\tilde{x}, \tilde{y})\) and \((\tilde{x}, \tilde{y})\) represent the same point in \(M^{m+n+1}_{(h_1, h_2)}\), then it is clear that \(||\tilde{y}|| = ||\tilde{x}||^{-1}\) and
\[ \left( \frac{\tilde{x}}{\sqrt{1 + ||\tilde{x}||^2}}, \frac{\tilde{y}}{\sqrt{1 + ||\tilde{x}||^2}} \right) = \left( \frac{h_1^{-1}(\tilde{x})}{\sqrt{1 + ||\tilde{y}||^2}}, \frac{||\tilde{y}||h_2^{-1}(\frac{\tilde{y}}{||\tilde{y}||})}{\sqrt{1 + ||\tilde{y}||^2}} \right). \quad (4) \]

Therefore the maps \(h : M^{m+n+1}_{(h_1, h_2)} \to S^{m+n+1}_{\mathbb{R}}\) defined by Eq. (1) and Eq. (3) is well defined.

Now let \((\tilde{x}, \tilde{y})\) and \((\bar{x}, \bar{y})\) be any two elements of \(\mathbb{R}^{m+1} \times \mathbb{R}^n\) with \((\tilde{x}, \tilde{y}) \neq (\bar{x}, \bar{y})\). If
\[ \left( \frac{\tilde{x}}{\sqrt{1 + ||\tilde{x}||^2}}, \frac{\tilde{y}}{\sqrt{1 + ||\tilde{x}||^2}} \right) = \left( \frac{\bar{x}}{\sqrt{1 + ||\bar{x}||^2}}, \frac{\bar{y}}{\sqrt{1 + ||\bar{x}||^2}} \right), \quad (5) \]
then
\[ \frac{\tilde{x}}{\sqrt{1 + ||\tilde{x}||^2}} = \frac{\bar{x}}{\sqrt{1 + ||\bar{x}||^2}} \quad \text{and} \quad \frac{\tilde{y}}{\sqrt{1 + ||\tilde{x}||^2}} = \frac{\bar{y}}{\sqrt{1 + ||\bar{x}||^2}}. \quad (6) \]

Since \(\tilde{y}\) and \(\bar{y}\) are elements of \(S^n\), we have
\[ \sqrt{1 + ||\tilde{x}||^2} = \sqrt{1 + ||\bar{x}||^2}. \quad (7) \]

From Eq. (6) and Eq. (7) we have \((\tilde{x}, \tilde{y}) = (\bar{x}, \bar{y})\). It contradicts the above assumption. Therefore, the map \(h\) is an injection.

Let \((\gamma, \kappa) \in S^{m+n+1}_{\mathbb{R}}\). Furthermore let \((x, y) \in S^m \times S^n\) and \(t \in [0, \infty)\) such that
\[ \gamma = \frac{tx}{\sqrt{1 + t^2}} \quad \text{and} \quad \kappa = \frac{y}{\sqrt{1 + t^2}}. \quad (8) \]

From Eq. (8) we have
\[ ||\gamma||^2(1 + t^2) = t^2||x||^2 = t^2, \quad (9) \]
and
\[ ||\kappa||^2(1 + t^2) = ||y||^2 = 1, \quad (10) \]

since \(x \in S^m\) and \(y \in S^n\). From Eq. (9) and Eq. (10) we obtain then
\[ t = \sqrt{\frac{||\gamma||^2}{||\kappa||^2}} = \frac{||\gamma||}{||\kappa||} \quad (11) \]
\[ tx = \frac{||\gamma||}{||\kappa||} \sqrt{\frac{||\kappa||^2}{||\gamma||^2} + 1} = \gamma \sqrt{1 + \frac{||\gamma||^2}{||\kappa||^2}}, \quad (12) \]
and
\[ y = \kappa \sqrt{1 + \frac{||\gamma||^2}{||\kappa||^2}}. \quad (13) \]

This means the map \(h\) is surjective and therefore it is a bijection. Let \(f\) denote the invers of \(h\). Thus, \(f : S^{m+n+1} \rightarrow M^{m+n+1}_{(h_1, h_2)}\) is given by
\[ S^{m+n+1} \ni (\gamma, \kappa) \mapsto \begin{cases} \left( \frac{||\gamma||}{||\kappa||} \sqrt{\frac{||\kappa||^2}{||\gamma||^2} + 1}, \kappa \sqrt{1 + \frac{||\gamma||^2}{||\kappa||^2}} \right) \in \mathbb{R}^{m+1} \times \mathbb{R}^n \quad (14) \\ (h_1 \left[ \gamma \sqrt{\frac{||\kappa||^2}{||\gamma||^2}} + 1 \right], \frac{||\gamma||}{||\kappa||} h_2 \left[ \kappa \sqrt{1 + \frac{||\gamma||^2}{||\kappa||^2}} \right]) \in S^m \times \mathbb{R}^{n+1}. \end{cases} \]
3 Stochastic Flows on $M_{(h_1,h_2)}^{m+n+1}$

Let $(W_t)_{t \geq 0}$ denote the $d$-dimensional standard Brownian motion on the classical Wiener space $(\Omega, \mathcal{F}, \mathcal{P}; (\mathcal{F}_t)_{t \geq 0})$, where $\Omega$ is the space of all continuous function from $\mathbb{R}_+$ to $\mathbb{R}^d$ with locally uniform convergence topology, $\mathcal{F}$ is the Borel $\sigma$-field on $\Omega$ generated by the topology, $P$ is the Wiener measure on $\mathcal{F}$, and $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by $W_t(\omega) = \omega$. Now consider the Stratonovich’s stochastic differential equation

$$
\begin{align*}
 dq_t &= X_0(q_t)dt + \sum_{k=1}^{d} X_k(q_t) \circ dW^k_t, \quad q_0 = q
\end{align*}
$$

(15)
on a compact differentiable Riemannian manifold $(M, g)$, where $X_i$, $i = 0, \ldots, d$ are $d+1$ vector fields on $M$. There are at least two ways to solve the differential equation. In the first way, we solve it locally and then patch them up. In the second one, we obtain the solution of the equation in a suitable embedding space $\mathbb{R}^N$ and prove that the solution stays in $M$ whenever the starting point $q \in M$. It is well-known that the two ways as mentioned above work well only if the drift vector field $X_0$ and the other vectors $X_k$ ($k = 1, \ldots, d$) appearing in the above differential equation are smooth (at least $C^2$). For the case of $X_0$ having only Sobolev regularity and bounded divergence and $X_k$ ($k = 1, \ldots, d$) are $C^2$, Zhang [12] has shown the existence and uniqueness of the so-called $\nu$-almost everywhere stochastic invertible flows of SDE (15), where $\nu$ is the Riemannian measure on $M$.

**Definition 1 (Zhang [12])** Let $q_t(\omega, q)$ be an $M$-valued measurable stochastic field on $\mathbb{R}_+ \times \Omega \times M$. The flow $q_t(q)$ is called a $\nu$-almost everywhere stochastic flow of (15) corresponding to vector fields $X_i (i = 0, \ldots, d)$ if

1. For $\nu$-almost all $q \in M$, $t \mapsto q_t(q)$ is continuous and $(\mathcal{F}_t)$-adapted stochastic process and, satisfies that for any $t > 0$ and $\zeta \in C^\infty(M)$

$$
\zeta(q_t(q)) = \zeta(q) + \int_0^t X_0(\zeta(q_s(q)))ds + \int_0^t \nabla_\tau \zeta(q_s(q)) \circ dW_s^\tau \quad \forall t \geq 0.
$$

(16)

2. For arbitrary $t \geq 0$ and $P$-almost all $\omega \in \Omega$, $(\nu \circ q_t)(\omega, \cdot) \ll \nu$. For any $T > 0$, there is a positive constant $K_{T, X_0, X_k}$ such that for all non-negative measurable function $\zeta$ on $M$

$$
\sup_{t \in [0, T]} \mathbb{E} \int_M \zeta(q_t(q)) \nu(dq) \leq K_{T, X_0, X_k} \int_M \zeta(q) \nu(dq).
$$

(17)

Furthermore, the $\nu$-almost everywhere stochastic flow $q_t(q)$ of (15) is said to be invertible if for all $t \geq 0$ and $P$-almost all $\omega \in \Omega$, there exists a measurable invers $q_t^{-1}(\omega, \cdot)$ of $q_t(\omega, \cdot)$ so that $\nu \circ q_t^{-1}(\omega, \cdot) = \nu_t(\omega, \cdot) \nu$, where the density $\nu_t(q)$ is given by

$$
\nu_t(q) = \exp \left[ \int_0^t \text{div} X_0(q_s(q))ds + \int_0^t \text{div} X_k(q_s(q)) \circ dW_s^k \right].
$$

(18)

Let $C^k(TM)$ be the set of all $C^k$-differentiable vector fields on $M$, for every $k \in \mathbb{N} \cup \{\infty\}$. For every $p \geq 1$, we define

$$
\|X\|_p := \left[ \int_M |X(q)|^p \nu(dq) \right]^{1/p},
$$

(19)

and

$$
\|X\|_{1,p} := \|X\|_p + \left[ \int_M |\nabla X(q)|^p \nu(dq) \right]^{1/p},
$$

(20)

for every $X \in C^\infty(TM)$, where $\nabla$ is the Levi-Civita connection associated to the metric tensor $g$ on $M$. The completions of $C^\infty(TM)$ with respect to $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ will be denoted by $L^p(TM)$ and $\mathbb{H}^p(TM)$ respectively. Then Zhang found the following fact.

**Theorem 2 (Zhang [12])** Let $X_0 \in L^\infty(TM) \cap \mathbb{H}^1_1(TM)$ for some $p \geq 1$, $\text{div} X_0 \in L^\infty(M)$, and for each $k = 1, \cdots, d, X_k \in C^2(TM)$. Then there exists a unique $\nu$-almost everywhere stochastic invertible flow $q_t(q)$ of (15) in the sense of Definition[1]

Now consider the case where $M$ is the standard sphere $S_{m+n+1}$ provided by the natural Riemannian metric tensor $g$ induced from Euclidean metric in $\mathbb{R}^{m+n+2}$. Let $V_0 \in L^\infty(TS_{m+n+1}) \cap \mathbb{H}^p_1(TS_{m+n+1})$ and $V_k (k = 1, \cdots, d)$ be $C^2$-vector...
fields on $S^{m+n+1}$. Then according to Theorem 2, there exists a unique $\nu$-almost everywhere stochastic invertible flow $q_t(q)$ of the following stochastic differential equation

$$ dq_t = V_0(q_t)dt + \sum_{k=1}^d V_k(q_t) \circ dW^k_t, \quad q_0 = q \quad (21) $$

on $S^{m+n+1}$. With the identicivation $h$ and its invers $f$ constructed in the previous section, then $(f \circ q_t \circ h)(\tilde{q})$ is a stochastic flow on $M_{h_1,h_2}^{m+n+1}$, where $\tilde{q}$ is the element of $M_{h_1,h_2}^{m+n+1}$ which is identified with $q$, i.e., $h(\tilde{q}) = q$. The stochastic flow $(f \circ q_t \circ h)(\tilde{q})$ and the stochastic flow $q_t(q)$ is actually the same flow on the topological space $S^{m+n+1}$. The flow $q_t(q)$ is the stochastic flow on $S^{m+n+1}$ described using the standard differential structure on $S^{m+n+1}$. Whereas, the flow $(f \circ q_t \circ h)(\tilde{q})$ can be regarded as the same stochastic flow on $S^{m+n+1}$ described using another differential structure which may be exotic (depending on the maps $h_1$ and $h_2$). If the maps $h_1$ and $h_2$ leads to manifold $M_{h_1,h_2}^{m+n+1}$ which is diffeomorphic to $S^{m+n+1}$ so that $h$ is a diffeomorphism, then $(f \circ q_t \circ h)(\tilde{q})$ and $q_t(q)$ clearly are the same $\nu$-almost everywhere stochastic invertible flow on $S^{m+n+1}$. Now we will investigate the properties of the flow $(f \circ q_t \circ h)(\tilde{q})$ for the general case in which $M_{h_1,h_2}^{m+n+1}$ may not be diffeomorphic but homeomorphic to $S^{m+n+1}$.

The vector fields $V_0, V_1, \ldots, V_d$ on $S^{m+n+1}$ appearing in equation (21) are vector fields on $S^{m+n+1}$ when they are described using the standard differential structure on $S^{m+n+1}$. The vector fields are seen as $f_0V_0, f_1V_1, \ldots, f_dV_d$ on $M_{h_1,h_2}^{m+n+1}$ if they are described by using another (exotic) differential structure on $S^{m+n+1}$. The stochastic differential equation (21) is seen as

$$ d(f \circ q_t \circ h) = f_0V_0(f \circ q_t \circ h)dt + \sum_{k=1}^d f_kV_k(f \circ q_t \circ h) \circ dW^k_t, \quad f \circ q_0 \circ h = \tilde{q} \quad (22) $$

on $M_{h_1,h_2}^{m+n+1}$. The properties of the flow $f \circ q_t \circ h$ can be understood from the equation (22), especially from the vector fields appearing in the equation. At point $(\tilde{x}, \tilde{y}) \in M_{h_1,h_2}^{m+n+1}$ the value of vector field $f_iV_k$ is given by

$$ (f_iV_k)(\tilde{x}, \tilde{y}) = (f_i)_{(\gamma, \kappa)}[V_k(\gamma, \kappa)], \quad (23) $$

where $(\gamma, \kappa) = h(\tilde{x}, \tilde{y})$. If $V^\nu_k (\mu = 1, 2, \ldots, m + n + 2)$ is the components of $V_k$ in the ambient space $\mathbb{R}^{m+n+1}$, then for $\nu = 1, 2, \ldots, m + n + 2$

$$ (f_i)_{(\gamma, \kappa)}[V_k(\gamma, \kappa)]^\nu = \sum_{j=1}^{m+1} \frac{\partial f_i^\nu}{\partial \gamma_j}V_j^\nu(\gamma, \kappa) + \sum_{r=1}^{n+1} \frac{\partial f_i^\nu}{\partial \kappa_r}V^\nu_r(\gamma, \kappa). \quad (24) $$

If $(\tilde{x}, \tilde{y})$ is in the region $S^m \times \mathbb{R}^{n+1}$ of $M_{h_1,h_2}^{m+n+1}$, the differentials of $f$ are given by

$$ \frac{\partial f_i^\nu}{\partial \gamma_j} = \sum_{j=1}^{m+1} \frac{\partial h_j^\nu(\tilde{y})}{\partial \gamma_j} \left( \frac{\delta_i^j}{||\gamma||} - \frac{||\kappa||^2}{||\gamma||^2} \tilde{\gamma}_j \right) $$

$$ = \frac{\partial h_j^\nu}{\partial \gamma^j} \sqrt{1 + ||\gamma||^2} - \sum_{j=1}^{m+1} \frac{\partial h_j^\nu}{\partial \gamma^j} \frac{||\gamma||^2}{\sqrt{1 + ||\gamma||^2}} [h_1^{-1}(\tilde{x})]_j^i [h_2^{-1}(\tilde{x})]^j_i \quad (v, j = 1, \ldots, m+1), \quad (25) $$

$$ \frac{\partial f_i^\nu}{\partial \kappa_r} = \sum_{j=1}^{m+1} \frac{\partial h_j^\nu(\tilde{y})}{\partial \kappa_r} \tilde{\gamma}_j \kappa_r $$

$$ = \sum_{j=1}^{m+1} \frac{\partial h_j^\nu(\tilde{y})}{\partial \kappa_r} \frac{||\tilde{y}||}{\sqrt{1 + ||\gamma||^2}} [h_1^{-1}(\tilde{x})]_j^i [h_2^{-1}(\tilde{x})]_j^i \quad (v = 1, \ldots, m + 1, r = 1, \ldots, n + 1), \quad (26) $$

$$ \frac{\partial f_i^\nu}{\partial \gamma^j} = - \frac{||\gamma||}{||\gamma||} h_2^{-m-1}(\tilde{y}) \frac{\partial h_j^\nu}{\partial \gamma^j} \left( \frac{\delta_i^j}{||\gamma||} - \frac{||\kappa||^2}{||\gamma||^2} \tilde{\gamma}_j \right) $$

$$ = -\tilde{\gamma}_j^i \left( \frac{\delta_i^j}{||\gamma||} - \frac{||\kappa||^2}{||\gamma||^2} \tilde{\gamma}_j \right) + \sum_{j=1}^{n+1} \frac{\partial h_j^\nu}{\partial \kappa_r} \frac{||\gamma||^2}{\sqrt{1 + ||\gamma||^2}} [h_1^{-1}(\tilde{x})]_j^i [h_2^{-1}(\tilde{x})]_j^i \times \frac{\delta_i^j}{||\gamma||^2}, \quad (27) $$
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\( (j = 1, \ldots, m + 1, v = m + 2, \ldots, m + n + 2) \), and

\[
\frac{\partial f^v}{\partial \kappa^r} = h_2^{v-m-1}(\hat{k}) \frac{\kappa'}{||\gamma||} + \sum_{s=1}^{n} \frac{\partial h_2^{v-m-1}(\hat{k})}{\partial \kappa^s} \left( \frac{\delta_r'}{||\gamma||} - \frac{||\gamma||}{||\kappa||^3} \kappa'^r \right)
\]

\[
= \tilde{y}^{v-m} \sqrt{1 + ||\tilde{y}||^2} \frac{(h_2^{-1}(\tilde{y}/||\tilde{y}||))^r}{\sqrt{1 + ||\tilde{y}||^2}} + \sum_{s=1}^{n} \frac{\partial h_2^{v-m-1}(\hat{k})}{\partial \kappa^s} \left( \delta_r' \sqrt{1 + ||\tilde{y}||^2} - \frac{(h_2^{-1}(\tilde{y}/||\tilde{y}||))^r}{\sqrt{1 + ||\tilde{y}||^2}} \right),
\]

\eqref{28}

\((r = 1, \ldots, n + 1, v = m + 2, \ldots, m + n + 2)\), since

\[
h_2^{v-m-1}(\hat{k}) = \tilde{y}^{v-m-1}(h_2^{-1}(\tilde{y}/||\tilde{y}||)) = \tilde{y}^{v-m-1} \frac{||\tilde{y}||}{||\tilde{y}||}.
\]

If \((\hat{x}, \tilde{y})\) is in the region \(\mathbb{R}^{m+1} \times S^n\) of \(M_{\hat{k}, k_2}^{m+n+1}\), the differentials of \(f\) are given by

\[
\frac{\partial f^v}{\partial \gamma^j} = \frac{\gamma^v \gamma^j}{||\kappa||} + \frac{\delta^v}{||\kappa||}, \quad \gamma^v = \frac{\tilde{y}^v}{\sqrt{1 + ||\tilde{y}||^2}} + \delta^v \sqrt{1 + ||\tilde{y}||^2} \quad (v, j = 1, \ldots, m + 1),
\]

\eqref{30}

\[
\frac{\partial f^v}{\partial \kappa^r} = -\frac{||\gamma||^2 \gamma^v \kappa'^r}{||\kappa||^3},
\]

\eqref{31}

\[
\frac{\partial f^v}{\partial \gamma^j} = \frac{\delta^v y^{v-m-1}}{||\kappa||} \quad (v = m + 2, \ldots, m + n + 2, j = 1, \ldots, m + 1),
\]

\eqref{32}

\[
\frac{\partial f^v}{\partial \kappa^r} = \frac{\delta^v y^{v-m-1} ||\gamma||^2 \kappa'^r}{||\kappa||^3} - \frac{||\tilde{y}||^2 y^{v-m-1} \delta^v \kappa'^r}{\sqrt{1 + ||\tilde{y}||^2}},
\]

\eqref{33}

\((v = m + 2, \ldots, m + n + 2, r = 1, \ldots, n + 1)\). In the above expressions, \(\gamma := ||\gamma||^{-1} = ||\gamma^1, \ldots, \gamma^{m+1}||^{-1}\) and \(\hat{k} := \kappa||\kappa||^{-1} = (\kappa^{1}, \ldots, \kappa^{n+1})||\kappa||^{-1}\). Then, the vector fields \((f, V_k) (k = 0, 1, \ldots, d)\) at \((\hat{x}, \tilde{y})\) in \(S^m \times \mathbb{R}^{n+1}\) are given by

\[
[(f, V_k)(\hat{x}, \tilde{y})]^v = \sum_{j=1}^{m+1} \left[ \sum_{r=1}^{n+1} \frac{\partial h_1^r}{\partial \gamma} \sqrt{1 + ||\tilde{y}||^2} - \sum_{s=1}^{m+1} \frac{\partial h_1^r}{\partial \kappa^s} \frac{||\tilde{y}||^2}{\sqrt{1 + ||\tilde{y}||^2}} [h_1^{-1}(\tilde{y})][h_1^{-1}(\tilde{y})]^r \right] V_k^j (h(\hat{x}, \tilde{y}))
\]

\[
+ \sum_{r=1}^{n+1} \left[ \sum_{j=1}^{m+1} \frac{\partial h_2}{\partial \gamma} \frac{||\tilde{y}||}{\sqrt{1 + ||\tilde{y}||^2}} [h_2^{-1}(\tilde{y}/||\tilde{y}||)][h_2^{-1}(\tilde{y}/||\tilde{y}||)]^r \right] V_k^r (h(\hat{x}, \tilde{y})),
\]

\eqref{34}

for \(v = 1, \ldots, m + 1\), and

\[
[(f, V_k)(\hat{x}, \tilde{y})]^v = \sum_{j=1}^{m+1} \left[ \tilde{y}^{v-m-1} \sqrt{1 + ||\tilde{y}||^2} [h_1^{-1}(\tilde{y})]^j \right] V_k^j (h(\hat{x}, \tilde{y}))
\]

\[
+ \sum_{j=1}^{m+1} \left[ \sum_{r=1}^{n+1} \frac{\partial h_2^r}{\partial \gamma} [h_2^{-1}(\tilde{y}/||\tilde{y}||)][h_2^{-1}(\tilde{y}/||\tilde{y}||)]^r \right] V_k^r (h(\hat{x}, \tilde{y}))
\]

\[
+ \sum_{r=1}^{n+1} \left[ \tilde{y}^{v-m-1} \sqrt{1 + ||\tilde{y}||^2} [h_2^{-1}(\tilde{y}/||\tilde{y}||)][h_2^{-1}(\tilde{y}/||\tilde{y}||)]^r \right] V_k^r (h(\hat{x}, \tilde{y}))
\]

\[
+ \sum_{r=1}^{n+1} \sum_{s=1}^{m+1} \frac{\partial h_2^r}{\partial \kappa^s} \left( \delta_r' \sqrt{1 + ||\tilde{y}||^2} - \frac{[h_2^{-1}(\tilde{y}/||\tilde{y}||)][h_2^{-1}(\tilde{y}/||\tilde{y}||)]^r}{\sqrt{1 + ||\tilde{y}||^2}} \right) V_k^r (h(\hat{x}, \tilde{y})),
\]

\eqref{35}
for $\nu = m + 2, \ldots, m + n + 2$. Whereas, at $(\tilde{x}, \tilde{y}) \in \mathbb{R}^{m+1} \times S^n$ the value of vector fields $f,V_k$ $(k = 0, 1, \ldots, d)$ are given by

$$
[(f,V_k)(\tilde{x}, \tilde{y})]^{\nu} = \sum_{j=1}^{m+1} \left[ \frac{\tilde{x}^{\nu} \tilde{y}}{1 + \|\tilde{x}\|^2} + \delta^y \sqrt{1 + \|\tilde{x}\|^2} \right] V_k^i(h(\tilde{x}, \tilde{y}))
+ \sum_{r=1}^{n+1} \left[ - \frac{\|\tilde{x}\|^2 \tilde{x}^{\nu} \tilde{y}^r}{1 + \|\tilde{x}\|^2} \right] V_k^r(h(\tilde{x}, \tilde{y})), \quad \text{for } \nu = 1, \ldots, m + 1
$$

(36)

$$
[(f,V_k)(\tilde{x}, \tilde{y})]^{\nu} = \sum_{j=1}^{m+1} \left[ \frac{\tilde{x}^{\nu} \tilde{y}^{\nu-m-1}}{1 + \|\tilde{x}\|^2} \right] V_k^j(h(\tilde{x}, \tilde{y}))
+ \sum_{r=1}^{n+1} \left[ \frac{1 + \|\tilde{x}\|^2 \delta^y \tilde{x}^{\nu-m-1}}{1 + \|\tilde{x}\|^2} \right] V_k^r(h(\tilde{x}, \tilde{y})), \quad \text{for } \nu = m + 2, \ldots, m + n + 2.
$$

From the expressions of the components $[(f,V_k)(\tilde{x}, \tilde{y})]^{\nu}$ of the vector fields $f_V$ on the region $\mathbb{R}^{m+1} \times S^n$, i.e. Equation (36) and (37), it is clear that the vector fields $f,V_k$ $(k = 1, \ldots, d)$ are $C^2$-differentiable on that region.

Now we consider the expression of the components $[(f,V_k)(\tilde{x}, \tilde{y})]^{\nu}$ (i.e. Equation (34) and (35)) of the vector fields $f,V_k$ on the other region of $M_{h_1}^{m+n+1}$. From Equation (34) and (35), the differentiability of the components $[(f,V_k)(\tilde{x}, \tilde{y})]^{\nu}$ clearly depend on the differentiability of the components $V^\nu_k \circ h$ as well as on the differentiability of their coefficients appearing in Equation (34) and (35). Taking second partial derivative of $V^\nu_k \circ h$ with respect to $\tilde{x}^i$ and $\tilde{x}^j$ yields

$$
\frac{\partial^2 V^\nu_k(h(\tilde{x}, \tilde{y}))}{\partial \tilde{x}^i \partial \tilde{x}^j} = \sum_{i,a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^a \partial \tilde{x}^i \partial \tilde{x}^j} + \sum_{i=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^i \partial \tilde{x}^j} + \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^a \partial \tilde{x}^j} + \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^i \partial \tilde{x}^a}
$$

$$
= \frac{1}{\|\tilde{y}\|^{2} + 1} \sum_{i,a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^a \partial \tilde{x}^i \partial \tilde{x}^j} + \sum_{i=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^i \partial \tilde{x}^j} + \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^a \partial \tilde{x}^j} + \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{x}^i \partial \tilde{x}^a}.
$$

(38)

It is clear that the differentiability of $V^\nu_k \circ h$ with respect to $\tilde{x}$ depends only on the differentiability of $V^\nu_k$ with respect to $\gamma$, since $h_1$ is a diffeomorphism. Now taking second partial derivative of $V^\nu_k \circ h$ with respect to $\tilde{y}^i$ and $\tilde{y}^j$ yields

$$
\frac{\partial^2 V^\nu_k(h(\tilde{x}, \tilde{y}))}{\partial \tilde{y}^i \partial \tilde{y}^j} = \sum_{i,a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^a \partial \tilde{y}^i \partial \tilde{y}^j} + \sum_{i=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^i \partial \tilde{y}^j} + \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^a \partial \tilde{y}^j} + \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^i \partial \tilde{y}^a}
$$

$$
= - \frac{1}{\|\tilde{y}\|^{2} + 1} \sum_{i,a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^a \partial \tilde{y}^i \partial \tilde{y}^j} - \sum_{i=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^i \partial \tilde{y}^j} - \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^a \partial \tilde{y}^j} - \sum_{a=1}^{m+1} \frac{\partial^2 V^\nu_k}{\partial \tilde{y}^i \partial \tilde{y}^a}.
$$

(39)

It is also clear from the last equation that the second partial derivative of $V^\nu_k \circ h$ with respect to $\tilde{x}$ and $\tilde{y}$ is continuous only if the function $V^\nu_k$ is at least $C^2$. The second derivative of $V^\nu_k \circ h$ with respect to $\tilde{y}$ is given by
\[
\frac{\partial^2 V^V_\nu(h(\bar{x}, \bar{y}))}{\partial \bar{y} \partial \bar{y}^d} = \sum_{i,j=1}^{m+1} \frac{\partial^2 V^V_k}{\partial \bar{y}^i \partial \bar{y}^j} \frac{\partial \gamma_i}{\partial \bar{y}^d} + \sum_{i=1}^{m+1} \frac{\partial V^V_k}{\partial \bar{y}^i} \frac{\partial^2 \gamma_i}{\partial \bar{y}^d} + \sum_{j=1}^{m+1} \sum_{i=1}^{m+1} \frac{\partial^2 V^V_k}{\partial \gamma_i \partial \gamma_j} \frac{\partial \gamma_i}{\partial \bar{y}^d} + \sum_{j=1}^{m+1} \sum_{i=1}^{m+1} \frac{\partial^2 V^V_k}{\partial \gamma_i \partial \gamma_j} \frac{\partial \gamma_j}{\partial \bar{y}^d} + \sum_{i=1}^{m+1} \frac{\partial^2 V^V_k}{\partial \gamma_i \partial \gamma_j} \frac{\partial \gamma_i}{\partial \bar{y}^d} \frac{\partial \gamma_j}{\partial \bar{y}^d},
\]

where

\[
\frac{\partial \gamma_i}{\partial \bar{y}^d} \frac{\partial \gamma_j}{\partial \bar{y}^d} = \frac{[h_1^{-1}(\bar{x})]^i [h_1^{-1}(\bar{x})]^j \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^3},
\]

\[
\frac{\partial^2 \gamma_i}{\partial \bar{y}^d \partial \bar{y}^d} = \frac{3[h_1^{-1}(\bar{x})]^i \bar{y}^d}{(\|\bar{y}\|^2 + 1)^{5/2}} - \frac{[h_1^{-1}(\bar{x})]^i \delta^d_1}{(\|\bar{y}\|^2 + 1)^{3/2}},
\]

\[
\frac{\partial \gamma_i}{\partial \bar{y}^d} \frac{\partial \gamma_j}{\partial \bar{y}^d} = -\frac{\|\bar{y}\| \|[h_1^{-1}(\bar{x})]^i [h_2^{-1}(\bar{y})]^j \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^3} + \frac{[h_1^{-1}(\bar{x})]^i [h_2^{-1}(\bar{y})]^j \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|} - \sum_{s=1}^{n+1} \frac{[h_1^{-1}(\bar{x})]^j \frac{\partial [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{\partial \delta^s_1}}{\|\bar{y}\|^2 + 1} + \frac{[h_1^{-1}(\bar{x})]^j \frac{\partial [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{\partial \delta^s_1}}{\|\bar{y}\|^2 + 1} \frac{\|\bar{y}\|}{\|\bar{y}\|},
\]

\[
\frac{\partial \gamma_i}{\partial \bar{y}^d} \frac{\partial \gamma_j}{\partial \bar{y}^d} = \frac{\|\bar{y}\|^2 [h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^3} + \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|} + \frac{1}{\|\bar{y}\|^2 + 1} \frac{\|\bar{y}\|^2 [h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^3} - \frac{2[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} + \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} - \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} + \frac{1}{\|\bar{y}\|^2 + 1} \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^3} - \frac{2[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} + \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} - \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} + \frac{1}{\|\bar{y}\|^2 + 1} \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^3} - \frac{2[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} + \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} - \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^2} \frac{\|\bar{y}\|}{\|\bar{y}\|^2} + \frac{1}{\|\bar{y}\|^2 + 1} \frac{[h_2^{-1}(\bar{y})]^s [h_2^{-1}(\bar{y})]^s \bar{y}^d \bar{y}^d}{(\|\bar{y}\|^2 + 1)^3},
\]
and

\[
\frac{\partial^2 \kappa^c}{\partial \bar{y}^c \partial \bar{y}^d} = \frac{3\|y\| [h_2^{-1}(\bar{y})]^r \bar{y}^c \bar{y}^d}{(\|y\|^2 + 1)^{3/2}} - \frac{\bar{y}^c}{\|y\|^2} \frac{\partial [h_2^{-1}(\bar{y})]^r}{\partial \bar{y}^d} - \frac{\|y\| [h_2^{-1}(\bar{y})]^r \bar{y}^c \bar{y}^d}{\|y\|^2 (\|y\|^2 + 1)^{3/2}} - \frac{\|y\| \|h_2^{-1}(\bar{y})\|^r \delta^c_d}{\|y\|^2 \|y\| + 1} + \frac{1}{\|y\|^2 + 1} \|y\|^2 \|y\|^2 + 1 \\
\frac{\partial [h_2^{-1}(\bar{y})]^r}{\partial \bar{y}^d} - \frac{\|y\| \|h_2^{-1}(\bar{y})\|^r \delta^c_d}{\|y\|^2 \|y\| + 1} + \frac{1}{\|y\|^2 + 1} \|y\|^2 \|y\|^2 + 1 \\
\frac{\partial \|h_2^{-1}(\bar{y})\|^r}{\partial \bar{y}^d} - \frac{\|y\|^2 \|y\|^2 + 1} + \frac{1}{\|y\|^2 + 1} \|y\|^2 \|y\|^2 + 1 \\
\frac{\partial^2 [h_2^{-1}(\bar{y})]^r}{\partial \bar{y}^d \partial \bar{y}^d} \frac{\partial [h_2^{-1}(\bar{y})]^r}{\partial \bar{y}^d} - \frac{\|y\|^2 \|y\|^2 + 1} + \frac{1}{\|y\|^2 + 1} \|y\|^2 \|y\|^2 + 1 \\
\frac{\partial \|h_2^{-1}(\bar{y})\|^r}{\partial \bar{y}^d} - \frac{\|y\|^2 \|y\|^2 + 1} + \frac{1}{\|y\|^2 + 1} \|y\|^2 \|y\|^2 + 1 \\
\frac{\partial [h_2^{-1}(\bar{y})]^r}{\partial \bar{y}^d} - \frac{\|y\|^2 \|y\|^2 + 1} + \frac{1}{\|y\|^2 + 1} \|y\|^2 \|y\|^2 + 1.
\] (46)

All expressions appearing in Equation (41, 42, 43, 44, and 45) are continuous. However, the continuity of the expression appearing in Equation (46) very depends on the form of the diffeomorphism $h_2$. By calculating the second derivative of $\|y\| [h_2^{-1}(\bar{y}^r)]^r \bar{y}^c \bar{y}^d$ with respect to $\bar{y}^c$ and $\bar{y}^d$ ($c, d = 1, 2, \ldots, n + 1$) it is clear that $\|y\| [h_2^{-1}(\bar{y}^r)]^r \bar{y}^c \bar{y}^d$ is $C^2$-differentiable with respect to $\bar{y}$ on the region $S^m \times \mathbb{R}^{n+1}$, then the expression appearing in Equation (46) is $C^2$-differentiable. By inspecting the coefficients of the functions $V^c_k \circ h$ in Equation (34), it easy to see that the components $((f, V_k))^r$ ($v = 1, 2, \ldots, m + 1$) of the vector fields $((f, V_k))$ are $C^2$-differentiable on $S^m \times \mathbb{R}^{n+1}$ if $\|y\| [h_2^{-1}(\bar{y}^r)]^r (r = 1, 2, \ldots, n + 1)$ are $C^2$-differentiable with respect to $\bar{y}$ on the region. By direct calculation, it can be concluded that if the factors $[h_2^{-1}(\bar{y}^r)]^r \bar{y}^c \bar{y}^d$ are $C^2$-differentiable with respect to $\bar{y}$ for all $r = 1, 2, \ldots, n + 1$, then $\|y\| [h_2^{-1}(\bar{y}^r)]^r \bar{y}^c \bar{y}^d$ and $\|h_2^{-1}(\bar{y}^r)]^r \bar{y}^c \bar{y}^d$, for all $r, s = 1, 2, \ldots, n + 1$, are also $C^2$-differentiable with respect to $\bar{y}$. Furthermore, by inspecting the coefficients of $V^c_k \circ h$ in Equation (35), the components $((f, V_k))^r$ ($v = m + 2, \ldots, m + n + 2$) of the vector fields $((f, V_k))$ are $C^2$-differentiable on the region $S^m \times \mathbb{R}^{n+1}$ if the factors $[h_2^{-1}(\bar{y}^r)]^r \bar{y}^c \bar{y}^d$ are $C^2$-differentiable on the region for all $r = 1, 2, \ldots, n + 1$.

Now, we can extend the map $f : S^{m+n+1} \rightarrow M^{m+n+1}$ defined in Equation (14) to a homeomorphism $f_{ext}$ from an open subset $U$ containing $S^{m+n+1}_{h_2}$ of $\mathbb{R}^{m+n+2}$ onto an open subsets $U'$ containing $M^{m+n+1}_{h_1, h_2}$ of $\mathbb{R}^{m+n+2}$. Let $t_3$ be the natural inclusion of $S^{m+n+1}_{h_2}$ into $\mathbb{R}^{m+n+2}$ and $G$ a Riemannian metric tensor in the ambient space $\mathbb{R}^{m+n+2}$. Furthermore, let $g$ denote the Riemannian metric tensor on $S^{m+n+1}_{h_2}$ inherited from $G$, i.e. $g = t_3^* G$. By using the map $h_{ext}$, i.e. the invers of $f_{ext}$, we can pull back the metric tensor field $G$ yielding the Riemannian metric tensor $t_3^* G$ in $U'$. Let $t_{3y}$ be the natural inclusion of $M^{m+n+1}_{h_1, h_2}$ into $\mathbb{R}^{m+n+2}$. Then, the pull back $t_{3y}^* h_{ext}^* G$ of $h_{ext}^* G$ by $t_{3y}$ is a Riemannian tensor metric on $M^{m+n+1}_{h_1, h_2}$. It is easy to show that the metric tensor $t_{3y}^* h_{ext}^* G$ is the metric tensor $h^* g = h^* (t_3^* G)$. The metric tensor
\[ t_s^{h_s^*}G \] can be regarded as the metric tensor \( g \) on \( S_{s}^{m+n+1} \) when it is described with the exotic differential structure. The metric tensor field \( g \) gives rise to the Riemannian measures \( v \) on \( S_{s}^{m+n+1} \) and the metric tensor field \( h^*g \) to the Riemannian measure \( h^*v \) on \( M_{h_1,h_2}^{m+n+1} \) respectively. Now define
\[
|f,X((\tilde{\eta}))|_{h^*g} := (h^*g)(\tilde{\eta})((f,X)(\tilde{\eta}),(f,X)(\tilde{\eta})),
\]
for every vector field \( X \) defined on \( M_{h_1,h_2}^{m+n+1} \). It is straightforward to show that
\[
|f_s,X((\tilde{\eta}))|_{h^*g} = |X|_g(h(\tilde{\eta})).
\]
Since all the elements of the matrix representing the pull-back map \( h^* \) as well as of the push-forward map \( f_s \) are continuous then
\[
\int_{M_{h_1,h_2}^{m+n+1}} |f_s,V_0((\tilde{\eta}))|^p(h^*v)(d\tilde{\eta}) < \infty,
\]
whenever
\[
\int_{S_{s}^{m+n+1}} |V_0(q)|^p v(dq) < \infty,
\]
for \( 1 \geq p \), i.e. \( f_s,V_0 \in L^p(TM_{h_1,h_2}^{m+n+1}) \).

Since \( V_0 \in \mathbb{H}_1^p(TS_{s}^{m+n+1}) \), then there exist \( m+n+1 \) vector fields \( Y_1,\ldots,Y_{m+n+1} \) on \( S_{s}^{m+n+1} \) so that for \( \sigma = 1,\ldots,m+n+1 \) we have
\[
\int_{S_{s}^{m+n+1}} \frac{\partial \varphi}{\partial q^\sigma} V_0(q)v(dq) = -\int_{S_{s}^{m+n+1}} \varphi(q) Y_\sigma(q)v(dq),
\]
(the integrations appearing in Equation \( 51 \) are in the Bochner sense) for every \( \varphi \in C_c(S_{s}^{m+n+1}) \) and
\[
\int_{S_{s}^{m+n+1}} \nabla V_0(q)^*_g v(dq) < \infty,
\]
where
\[
|\nabla V_0(q)|_g = \left( \sum_{\eta=1}^{m+n+1} |Y_\eta(q)|_g^2 \right)^{1/2} = \left( \sum_{\eta=1}^{m+n+1} g(q)(Y_\eta(q),Y_\eta(q)) \right)^{1/2}.
\]
From Equation \( 53 \) we obtain
\[
|f_s,\nabla V_0(\tilde{\eta})|_{h^*g} = \left( \sum_{\eta=1}^{m+n+1} (h^*g)(\tilde{\eta})(f_s,Y_\eta(\tilde{\eta}),f_s,Y_\eta(\tilde{\eta})) \right)^{1/2} = |\nabla V_0(\tilde{\eta})|_g(h(\tilde{\eta})).
\]
Since all the elements of the matrix representing the pull-back map \( h^* \) are continuous, then
\[
\int_{M_{h_1,h_2}^{m+n+1}} |f_s,\nabla V_0(\tilde{\eta})|^p(h^*v)(d\tilde{\eta}) < \infty,
\]
and therefore \( f_s,V_0 \) is contained in the Sobolev space \( \mathbb{H}_1^p(TM_{h_1,h_2}^{m+n+1}) \).

Furthermore, if \( V_0 \) is in the space \( L^\infty(TM_{h_1,h_2}^{m+n+1}) \), then \( f_s,V_0 \) belongs to the space \( L^\infty(TM_{h_1,h_2}^{m+n+1}) \) of all bounded measurable vector fields on \( M_{h_1,h_2}^{m+n+1} \), since the map \( f_s \) is continuous. Now because \( S_{s}^{m+n+1} \) as well as \( M_{h_1,h_2}^{m+n+1} \) is compact and \( V_0 \in L^\infty(TM_{h_1,h_2}^{m+n+1}) \), it is clear that \( V_0 \) is in \( L^1(TM_{h_1,h_2}^{m+n+1}) \) and \( f_s,V_0 \) is in \( L^1(TM_{h_1,h_2}^{m+n+1}) \). If \( V_0 \) has (weak) divergence with \( \text{div } V_0 \in L^\infty(S_{s}^{m+n+1}) \), then so is \( f_s,V_0 \) with \( \text{div } (f_s,V_0) \in L^\infty(M_{h_1,h_2}^{m+n+1}) \), since the maps \( f_s \) and \( h^* \) are continuous.

As a conclusion we have

**Theorem 3** Let \( M_{h_1,h_2}^{m+n+1} \) be an exotic sphere, where \( h_1^{-1}(\tilde{\eta}/||\tilde{\eta}||) \) is \( C^2 \)-differentiable with respect to \( \tilde{\eta} \). Furthermore, let \( V_0 \in L^\infty(TM_{h_1,h_2}^{m+n+1}) \) for some \( p \geq 1 \), \( \text{div } V_0 \in L^\infty(S_{s}^{m+n+1}) \), and for each \( k = 1,\ldots,d \), \( V_k \in C^2(TM_{h_1,h_2}^{m+n+1}) \) so that \( q_k(\tilde{\eta}) \) is the unique \( \nu \)-almost everywhere stochastic invertible flow of \( 21 \). Then, the flow \( (f \circ q_k \circ h)(\tilde{\eta}) \) on \( M_{h_1,h_2}^{m+n+1} \) is the unique \( h^*\nu \)-almost everywhere stochastic invertible flow of \( 22 \) in the sense of Definition \( 7 \).
With the same conditions on the vector fields $V_k$ ($k = 0, 1, \cdots, d$), Theorem 3 means that whenever $h^{-1}_2(\vec{y}/||\vec{y}||)$ is $C^2$-differentiable with respect to $\vec{y}$, both stochastic flows $q_t(q)$ and $(f \circ q_t \circ h)(\vec{q})$ have the same regularities. It also means, both description (using the standard differential structure as well as using the choosen exotic differential structure) of the stochastic flow on the sphere $S^{m+n+1}$ look the same or indistinguishable. If both description of the stochastic flow on the sphere $S^{m+n+1}$ are distinguishable, then the diffeomorphism $h^{-1}_2(\vec{y}/||\vec{y}||)$ is not $C^2$-differentiable with respect to $\vec{y}$. Now if the map $h^{-1}_2(\vec{y}/||\vec{y}||)$ is not $C^2$-differentiable with respect to $\vec{y}$, then both description of the stochastic flow on the sphere $S^{m+n+1}$ may or may not indistinguishable.

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