CHARACTERISTIC CLASSES OF
FLAGS OF FOLIATIONS AND
LIE ALGEBRA COHOMOLOGY

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Abstract. We prove the conjecture by Feigin, Fuchs, and Gelfand describing the Lie
algebra cohomology of formal vector fields on an \( n \)-dimensional space with coefficients
in symmetric powers of the coadjoint representation. We also compute the cohomology
of the Lie algebra of formal vector fields that preserve a given flag at the origin. The
latter encodes characteristic classes of flags of foliations and was used in the formulation
of the local Riemann–Roch Theorem by Feigin and Tsygan.

Feigin, Fuchs, and Gelfand described the first symmetric power and to do this they
had to make use of a fearsomely complicated computation in invariant theory. By the
application of degeneration theorems of appropriate Hochschild–Serre spectral sequences,
we avoid the need to use the methods of FFG, and moreover, we are able to describe all
the symmetric powers at once.

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Introduction

1. Main results
The main result of this paper is a computation of the homology of the Lie algebra $W_n$ of formal vector fields on an $n$-dimensional space:

$$W_n := \left\{ \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \mid f_i \in \mathbb{k}[[x_1,\ldots,x_n]] \right\}.$$  

In the early 1970’s B. Feigin, D. Fuchs, and I. Gelfand stated a conjectural description of its cohomology with coefficients in symmetric powers of the coadjoint representation. They described applications of this conjecture to formal geometry, Gelfand–Fuchs cohomology, and the theory of foliations. This problem was formulated at the famous Gelfand seminar at Moscow State University. In [GFF] the same authors confirmed their conjecture for the particular case of the first symmetric power of the coadjoint representation. Later on, in 1989, this cohomological conjecture was offered as a proposition\(^2\) by B. Feigin and B. Tsygan in

\(^2\)without a proof