On approximation properties related to unconditionally $p$-compact operators and Sinha–Karn $p$-compact operators

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Abstract
We establish new results on the $I$-approximation property for the Banach operator ideal $I = \mathcal{K}_{up}$ of the unconditionally $p$-compact operators in the case of $1 \leq p < 2$. As a consequence of our results, we provide a negative answer for the case $p = 1$ of a problem posed by Kim. Namely, the $\mathcal{K}_{u1}$-approximation property implies neither the $\mathcal{SK}_1$-approximation property nor the (classical) approximation property; and the $\mathcal{SK}_1$-approximation property implies neither the $\mathcal{K}_{u1}$-approximation property nor the approximation property. Here, $\mathcal{SK}_p$ denotes the $p$-compact operators of Sinha and Karn for $p \geq 1$. We also show for all $2 < p, q < \infty$ that there is a closed subspace $X \subset \ell^q$ that fails the $\mathcal{SK}_r$-approximation property for all $r \geq p$.

KEYWORDS
Banach operator ideals, approximation properties, unconditionally $p$-compact operators, Sinha–Karn $p$-compact operators

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1 | INTRODUCTION

For Banach spaces $X$ and $Y$ let $\mathcal{K}(X, Y)$ denote the space of all compact operators $X \to Y$, and let $\mathcal{A}(X, Y) := \overline{F(X, Y)}$ denote the uniform closure of the linear subspace $F(X, Y)$ of all bounded finite-rank operators $X \to Y$. A Banach space $X$ is said to have the approximation property (AP) if the identity operator on $X$ can be uniformly approximated by bounded finite-rank operators on compact subsets of $X$. By a classical characterization due to Grothendieck (see, e.g., [32, Theorem 1.e.4]), $X$ has the AP if and only if $\mathcal{K}(Y, X) = \mathcal{A}(Y, X)$ for every Banach space $Y$. This criterion was generalized by Lassalle and Turco [29] and Oja [35] for Banach operator ideals as follows. Let $I = (I, \| \cdot \|_I)$ be a Banach operator ideal. A Banach space $X$ is said to have the $I$-AP if

$$I(Y, X) = \overline{F(Y, X)}^{\| \cdot \|_I}$$

for every Banach space $Y$. Note that the AP is precisely the $\mathcal{K}$-AP for the Banach operator ideal $\mathcal{K} = (\mathcal{K}, \| \cdot \|)$ of the compact operators. The $I$-AP has recently been studied in several papers for various Banach operator ideals $I$, see, for example, [9, 25, 27, 28, 30, 35], and references therein. Moreover, results about special instances of the $I$-AP have been used in the recent studies [47, 49] of radical quotient algebras of Banach operator ideals.

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The study of the $\mathcal{K}_{up}$-AP was initiated in [22] for the Banach operator ideal $\mathcal{K}_{up}$ of the unconditionally $p$-compact operators [21], where $p \in [1, \infty)$. It is known that if the Banach space $X$ has the AP, then $X$ has the $\mathcal{K}_{up}$-AP for all $1 \leq p < \infty$. Due to results of Kim [22, 23] and Lassalle and Turco [31], it holds for all $1 \leq p < \infty$ that $X$ has the $\mathcal{K}_{up}$-AP whenever the dual space $X^*$ has the $SK_{p}$-AP, and interchangeably, $X$ has the $\mathcal{K}_{up}$-AP whenever $X^*$ has the $\mathcal{K}_{up}$-AP. Here, $SK_{p}$ denotes the Banach operator ideal of the Sinha–Karn $p$-compact operators introduced by Sinha and Karn [45]. These duality results connect the $\mathcal{K}_{up}$-AP to the theory of the Sinha–Karn $p$-compact operators, which is a class of operators that has received a lot of attention over the past 20 years, see, for example, [5, 10, 17, 38, 39, 45].

A general framework for the theory of the classes $\mathcal{K}_{up}$ and $SK_{p}$ as well as the corresponding approximation properties is provided by the class $\mathcal{K}_{I}$ of $I$-compact operators introduced by Carl and Stephani [4]. In fact, the identities $\mathcal{K}_{up} = \mathcal{K}_{K_{p}'}$ and $SK_{p} = \mathcal{K}_{N_{p}}$ hold isometrically [30, 31], where $\mathcal{K}_{p'}$ and $N_{p}$ are the respective Banach operator ideals of the classical $p'$-compact operators ($p'$ is the dual exponent of $p$) and the right $p$-nuclear operators. The $\mathcal{K}_{I}$-AP was introduced by Lassalle and Turco [30] and it has subsequently been studied for various classes of Banach operator ideals $I$ in [26, 28, 31].

The main purpose of this paper is to establish new results on the $\mathcal{K}_{up}$-AP in the case of $1 \leq p < 2$. We approach the $\mathcal{K}_{up}$-AP mainly through the characterization $\mathcal{K}_{up} = \mathcal{K}_{N_{p}}$ due to Muñoz, Oja and Piñeiro [34] (see also [15, 31]), where $\mathcal{K}_{N_{p}}$ denotes the surjective hull of $\mathcal{K}_{p'}$. By considering $\mathcal{K}_{up}$ through this characterization, many of the classical compact factorization results related to $p$-compact operators become accessible in a natural way. This allows for new insights and also to recover known facts on the $\mathcal{K}_{up}$-AP.

Our main results are established in Section 3. In particular, we establish the following monotonicity behavior of the $\mathcal{K}_{up}$-AP in the range $[1,2)$: if the Banach space $X$ has the $\mathcal{K}_{up}$-AP, then $X$ has the $\mathcal{K}_{up}$-AP whenever $1 \leq p < q < 2$. Note that this monotonicity behavior does not extend to the case of $1 \leq p < 2 < q < \infty$ since, in that case, there are Banach spaces with the $\mathcal{K}_{up}$-AP that fail the $\mathcal{K}_{up}$-AP (Example 3.5). Here, we have omitted the trivial cases $p = 2$ or $q = 2$ since every Banach space has the $\mathcal{K}_{u2}$-AP. Moreover, by applying a factorization result of Pisier [40] and a result of John [19] on the AP, we show that a Banach space $X$ has the $\mathcal{K}_{up}$-AP for all $1 \leq p < 2$ whenever $X$ has cotype 2.

As a consequence of our results, we show that there exist reflexive Banach spaces that have the $\mathcal{K}_{u1}$-AP but fail the $SK_{1}$-AP. For such a space $X$, the dual $X^*$ has the $SK_{1}$-AP but fails the $\mathcal{K}_{u1}$-AP. This answers the case $p = 1$ of questions posed by Kim [23, Problem 1] on the relationship between the AP, the $SK_{p}$-AP and the $\mathcal{K}_{up}$-AP. Namely, the $\mathcal{K}_{u1}$-AP implies neither the $SK_{1}$-AP nor the AP, and moreover, the $SK_{1}$-AP implies neither the $\mathcal{K}_{u1}$-AP nor the AP. In addition, we show that there exist Banach spaces that have the $W_{1}$-AP but fail the AP, where $W_{1}$ denotes the weakly 1-compact operators of Sinha and Karn [45]. This answers a query of Kim posed in [27].

In Section 4, we exhibit for all $2 < p, q < \infty$ a closed subspace $X \subseteq \ell^{q}$ that fails the $SK_{1}$-AP for all $r \geq p$. This is essentially the closed subspace $E \subseteq \ell^{q}$ constructed by Davie [6, 7] that fails the AP, and it complements a result of Choi and Kim [5] in which they established that a variant of the Davie space $E \subseteq \ell^{q}$ fails the $p$-AP whenever $p > 2$ and $q > 2p/(p - 2)$. We also draw attention to some remaining questions related to the connections between the AP, the $\mathcal{K}_{up}$-AP and the $SK_{p}$-AP for $2 < p < \infty$.

2 | BANACH OPERATOR IDEALS AND APPROXIMATION PROPERTIES

In this section, we recall relevant definitions, notation, and concepts related to Banach operator ideals and the corresponding approximation properties. Moreover, we establish a reverse monotonicity property for the classes $\mathcal{K}_{up}$ of unconditionally $p$-compact operators in the range $1 \leq p \leq 2$, which will be used in Section 3. We also provide a characterization of the $\mathcal{K}_{up}$-AP for dual Banach spaces which complements similar characterizations of the $SK_{p}$-AP and the uniform $SK_{p}$-AP obtained in [9, 11], respectively.

Throughout the paper, we consider Banach spaces over the same, either real or complex, scalar field $\mathbb{K}$. The closed unit ball of any Banach space $X$ is denoted by $B_{X}$ and the dual exponent of $p \in [1, \infty)$ is denoted by $p'$ (i.e., $1/p + 1/p' = 1$). We refer to the monographs [8, 37] and the survey [12] for the definition and the theory of Banach operator ideals.

2.1 | Banach operator ideals

Let $p \in [1, \infty)$ be fixed and let $X$ and $Y$ be arbitrary Banach spaces. We denote by $\ell^{p}_{w}(X)$ the space of all weakly $p$-summable sequences in $X$, and by $\ell^{p}_{S}(X)$ that of all strongly $p$-summable sequences in $X$. Recall that $\ell^{p}_{w}(X)$ and $\ell^{p}_{S}(X)$...
are Banach spaces when equipped with the following natural norms:

\[ ||(x_k)||_{p,w} = \sup_{x^* \in B_{X^*}} \left( \sum_{k=1}^{\infty} |x^*(x_k)|^p \right)^{1/p}, \quad (x_k) \in \ell^p_w(X), \]

\[ ||(x_k)||_p = \left( \sum_{k=1}^{\infty} ||x_k||^p \right)^{1/p}, \quad (x_k) \in \ell^p(X). \]

Following Sinha and Karn [45], the \( p \)-convex hull of a weakly \( p \)-summable sequence \((x_k) \in \ell^p_w(X)\) is defined as

\[ p\text{-co}\{x_k\} := \left\{ \sum_{k=1}^{\infty} \lambda_k x_k \mid (\lambda_k) \in B_{\ell^p_p} \right\} \quad \text{(if } p = 1, \text{ then } (\lambda_k) \in B_{c_0}) \]

We say that the (bounded linear) operator \( T \in \mathcal{L}(X, Y) \) is a Sinha–Karn \( p \)-compact operator, denoted by \( T \in \mathcal{SK}_p(X, Y) \), if there is a strongly \( p \)-summable sequence \((y_k) \in \ell^p_s(Y)\) such that

\[ T(B_X) \subset p\text{-co}\{y_k\}. \quad (2.1) \]

According to [45, Theorem 4.2] and [10, Proposition 3.15], the class \( \mathcal{SK}_p = (\mathcal{SK}_p, \mathcal{K}_p) \) of the Sinha–Karn \( p \)-compact operators is a Banach operator ideal, where the ideal norm \( \| \cdot \|_{\mathcal{SK}_p} \) is defined by

\[ ||T||_{\mathcal{SK}_p} = \text{inf} \left\{ ||(y_k)||_p \mid (2.1) \text{ holds for } (y_k) \in \ell^p_s(Y) \right\} \]

for all \( T \in \mathcal{SK}_p(X, Y) \).

We point out that Sinha–Karn \( p \)-compact operators were introduced by Sinha and Karn [45] as \( p \)-compact operators and with the notation \((K_p, \kappa_p)\). However, we have decided to adopt the terminology and notation from [47, 49] in order to obtain a clear distinction from the classical ideal of \( p \)-compact operators independently introduced by Fourie and Swart [16] and Pietsch [37], which are denoted by \( \mathcal{K}_p \) in this paper (see below).

For the definition of unconditionally \( p \)-compact operators, recall from, for example, [8, 8.2] that a weakly \( p \)-summable sequence \((x_k) \in \ell^p_w(X)\) is called \text{
th{it}unconditionally \( p \)-summable}, denoted by \((x_k) \in \ell^p_u(X)\), if

\[ \lim_{k \to \infty} ||(0, \ldots, 0, x_k, x_{k+1}, \ldots)||_{p,w} = 0. \]

Following Kim [21], the operator \( T \in \mathcal{L}(X, Y) \) is called \text{
th{it}unconditionally \( p \)-compact}, denoted by \( T \in \mathcal{K}_{up}(X, Y) \), if there is an unconditionally \( p \)-summable sequence \((y_k) \in \ell^p_u(Y)\) such that

\[ T(B_X) \subset p\text{-co}\{y_k\}. \quad (2.2) \]

The class \( \mathcal{K}_{up} = (\mathcal{K}_{up}, \mathcal{K}_{up}) \) is a Banach operator ideal [21, Theorem 2.1] where the ideal norm \( \| \cdot \|_{\mathcal{K}_{up}} \) is defined by

\[ ||T||_{\mathcal{K}_{up}} = \text{inf} \left\{ ||(y_k)||_{p,w} \mid (2.2) \text{ holds for } (y_k) \in \ell^p_u(Y) \right\} \]

for all \( T \in \mathcal{K}_{up}(X, Y) \).

Following Pietsch [37, 6.7.1], the inclusion \( I \subset J \) of two Banach operator ideals \( I = (I, || \cdot ||_I) \) and \( J = (J, || \cdot ||_J) \) means that \( I(X, Y) \subset J(X, Y) \) for all Banach spaces \( X \) and \( Y \), and \( || \cdot ||_J \leq || \cdot ||_I \). The identity \( I = J \) means that \( I \subset J \) and \( J \subset I \).

It is known that \( \mathcal{K}_{up} \subset \mathcal{K} \) for all \( 1 \leq p < \infty \), see, for example, [1, pp. 1574–1575]. Moreover, clearly \( \ell^p(Y) \subset \ell^p_u(Y) \) for any Banach space \( Y \) and \( ||(y_k)||_{p,w} \leq ||(y_k)||_p \) for any strongly \( p \)-summable sequence \((y_k) \in \ell^p_u(Y)\). Thus,

\[ \mathcal{SK}_p \subset \mathcal{K}_{up} \subset \mathcal{K} \quad (2.3) \]
for all $1 \leq p < \infty$. For the Sinha–Karn $p$-compact operators, we have the following monotonicity property according to [45, Proposition 4.3] (see also [35, p. 949]):

$$SK_p \subset SK_q$$  \hfill (2.4)

for all $1 \leq p < q < \infty$. We point out that the classes $K_{up}$ are not monotone for $p \in [1, \infty)$, see Remark 2.5.

In [34, pp. 2885–2886] (see also [15, Theorem 4.5] and [31, Remark 4.6]) it was established that

$$K_{up} = K_{p}^{\text{sur}}$$  \hfill (2.5)

for all $1 \leq p < \infty$. Our results on the $K_{up}$-AP mostly draw on this characterization and we will next recall the definitions associated with the surjective hull $K_{p}^{\text{sur}}$ for the convenience of the reader.

Let $1 \leq p \leq \infty$. Following [16] or [37, 18.3], the operator $T \in \mathcal{L}(X, Y)$ is called $p$-compact, denoted by $T \in \mathcal{K}_p(X, Y)$, if there are compact operators $A \in \mathcal{K}(X, \ell^P)$ and $B \in \mathcal{K}(\ell^P, Y)$ such that $T = BA$, where $\ell^P$ is replaced by $c_0$ in the case of $p = \infty$. The $p$-compact norm is defined by $\|T\|_{\mathcal{K}_p} = \inf \|A\| \cdot \|B\|$, where the infimum is taken over all such factorizations $T = BA$. According to [16, Theorem 2.1] or [37, 18.3.1], the class $\mathcal{K}_p = (\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p})$ is a Banach operator ideal for all $1 \leq p \leq \infty$.

The surjective hull $\mathcal{K}_p^{\text{sur}} = (\mathcal{K}_p^{\text{sur}}, \|\cdot\|_{\mathcal{K}_p^{\text{sur}}})$ of $\mathcal{K}_p$ is defined by the components

$$\mathcal{K}_p^{\text{sur}}(X, Y) = \{T \in \mathcal{L}(X, Y) \mid TQ \in \mathcal{K}_p(\ell^1(B_X), Y),$$

and the associated ideal norm is defined by $\|T\|_{\mathcal{K}_p^{\text{sur}}} = \|TQ\|_{\mathcal{K}_p}$ for all $T \in \mathcal{K}_p^{\text{sur}}(X, Y)$. Here, $Q : \ell^1(B_X) \to X$ is the canonical metric surjection.

Recall further that the injective hull $\mathcal{K}_p^{\text{inj}} = (\mathcal{K}_p^{\text{inj}}, \|\cdot\|_{\mathcal{K}_p^{\text{inj}}})$ of $\mathcal{K}_p$ is defined by the components

$$\mathcal{K}_p^{\text{inj}}(X, Y) = \{T \in \mathcal{L}(X, Y) \mid j_0T \in \mathcal{K}_p(X, \ell^\infty(B_{Y^*})),$$

where $j_0 : Y \to \ell^\infty(B_{Y^*})$ is the canonical isometric embedding. The associated ideal norm is defined by $\|T\|_{\mathcal{K}_p^{\text{inj}}} = \|j_0T\|_{\mathcal{K}_p}$ for all $T \in \mathcal{K}_p^{\text{inj}}(X, Y)$.

We proceed by revisiting a classical compact factorization result due to Johnson [20]. The novelty here is a careful verification of the equality of the respective operator ideal norms in (2.7), which will be used in the proof of Proposition 2.4. Note that Proposition 2.2 can also be deduced from [16, Corollary 2.4] for closed subspaces $Y \subset \ell^P$. However, we provide a proof for closed subspaces $Y \subset L^p(\mu)$ for an arbitrary measure space $(\Omega, \Sigma, \mu)$, since we will require the case $L^p[0,1]$ in Proposition 2.4 and the case $\ell^P$ later on. We will use the following lemma which essentially comes from the proof of [20, Theorem 2].

**Lemma 2.1.** Suppose that $1 \leq p < \infty$ and let $X$ be a Banach space and $(\Omega, \Sigma, \mu)$ a measure space. Then, $\|T\| = \|T\|_{\mathcal{K}_p}$ for all $T \in F(X, L^p(\mu))$.

**Proof.** Let $T \in F(X, L^p(\mu))$ and suppose that $\lambda > 1$. It is known that $L^p(\mu)$ is an $\ell^p,\lambda$-space (see e.g. [13, Theorem 3.2] or [48, Corollary 7]) and thus there is a finite-dimensional subspace $TX \subset F \subset L^p(\mu)$ and a linear isomorphism $\theta : F \to \ell^p$ such that $\|\theta\| \cdot \|\theta^{-1}\| < \lambda$, where $\ell^p = (\mathcal{K}_{\text{dim}}^p, \|\cdot\|_{\mathcal{K}_{\text{dim}}})$. Let $J : \ell^p \to \ell^p$ be the natural isometric embedding and $P : \ell^p \to \ell^p$ the corresponding natural projection. Note that $T = j\theta^{-1}Pj\theta \hat{T}$, where $\hat{T} : X \to F$ is defined by $\hat{T}x = Tx$ for all $x \in X$ and $J : F \to L^p(\mu)$ is the inclusion map. It follows that

$$\|T\|_{\mathcal{K}_p} \leq \|j\theta^{-1}P\| \cdot \|\hat{T}\| < \lambda \|\hat{T}\| = \lambda \|T\|,$$

which implies that $\|T\|_{\mathcal{K}_p} \leq \|T\|$. Consequently, (2.6) yields that $\|T\| = \|T\|_{\mathcal{K}_p}$. \qed
Above we used the known fact that
\[ ||T|| \leq ||T||_I \]  \hspace{1cm} (2.6)\]

for any Banach operator ideal \( I = (I, \| \cdot \|_I) \) and operator \( T \in I(X,Y) \), see, for example, [8, Proposition 9.3].

**Proposition 2.2.** Suppose that \( 1 \leq p < \infty \) and let \( X \) be a Banach space and \( (\Omega, \Sigma, \mu) \) a measure space. If \( Y \) is a closed subspace of \( L^p(\mu) \), then \( \mathcal{K}(X,Y) = \mathcal{K}^{\text{inj}}_p(X,Y) \) and

\[ ||T|| = ||T||_{\mathcal{K}^{\text{inj}}_p} \]  \hspace{1cm} (2.7)\]

for all \( T \in \mathcal{K}(X,Y) \).

**Proof.** Let \( T \in \mathcal{K}(X,Y) \) and let \( j : Y \hookrightarrow L^p(\mu) \) denote the inclusion map so that \( jT \in \mathcal{K}(X,L^p(\mu)) \). It is well known that \( L^p(\mu) \) has the AP, and thus there is a sequence \( (F_n) \subset \mathcal{K}(X,L^p(\mu)) \) such that

\[ ||F_n - jT|| \to 0 \]  \hspace{1cm} as \( n \to \infty \), see [32, Theorem 1.e.4]. As \( (\mathcal{K}_p, \| \cdot \|_{\mathcal{K}_p}) \) is complete, it follows from Lemma 2.1 and (2.6) that \( jT \in \mathcal{K}_p(X,L^p(\mu)) \) and

\[ ||F_n - jT||_{\mathcal{K}_p} \to 0 \]  \hspace{1cm} as \( n \to \infty \). Consequently,

\[ ||jT||_{\mathcal{K}_p} = \lim_{n \to \infty} ||F_n||_{\mathcal{K}_p} = \lim_{n \to \infty} ||F_n|| = ||jT|| = ||T||. \]  \hspace{1cm} (2.8)\]

Finally, since \( jT \in \mathcal{K}_p(X,L^p(\mu)) \) we have \( T \in \mathcal{K}^{\text{inj}}_p(X,Y) \) and \( ||T||_{\mathcal{K}^{\text{inj}}_p} \leq ||jT||_{\mathcal{K}_p} \), see [37, Proposition 8.4.4]. Thus, by applying (2.6) and (2.8) one gets that \( ||T|| = ||T||_{\mathcal{K}^{\text{inj}}_p} \). \( \square \)

We next collect from [15, Section 3] (see also [14, Section 2]) the following pertinent characterization of the injective and the surjective hull of the \( p \)-compact operators as well as the precise relationship between these two classes of operators for convenient reference. For part (iv) below recall that the dual ideal \( I^{\text{dual}} = (I^{\text{dual}}, \| \cdot \|_{I^{\text{dual}}}) \) of the Banach operator ideal \( I = (I, \| \cdot \|_I) \) is defined by the components

\[ I^{\text{dual}}(X,Y) = \{ T \in L(X,Y) \mid T^* \in I(Y^*,X^*) \}, \]

and the associated ideal norm is defined by \( ||T||_{I^{\text{dual}}} = ||T^*||_I \) for all \( T \in I^{\text{dual}}(X,Y) \).

**Fact 2.3.** Let \( X \) and \( Y \) be Banach spaces and let \( T \in L(X,Y) \). Suppose that \( 1 \leq p < \infty \) and \( 1 < q \leq \infty \). Here, \( \ell^q \) is replaced by \( c_0 \) in the case of \( q = \infty \) in part (ii) and part (iii).

(i) By [15, Proposition 3.4] the following holds: \( T \in \mathcal{K}^{\text{inj}}_p(X,Y) \) if and only if there is a closed subspace \( M \subset \ell^p \) together with compact operators \( A \in \mathcal{K}(X,M) \) and \( B \in \mathcal{K}(M,Y) \) such that \( T = BA \) and \( ||T||_{\mathcal{K}^{\text{inj}}_p} = \inf ||A|| \|B|| \),

where the infimum is taken over all such factorizations \( T = BA \).
(ii) By [15, Proposition 3.7] the following holds: \( T \in \mathcal{K}^{\text{sur}}_q (X, Y) \) if and only if there is a quotient space \( Z \) of \( \ell^q \) and compact operators \( A \in \mathcal{K}(X, Z) \) and \( B \in \mathcal{K}(Z, Y) \) such that
\[
T = BA \quad \text{and} \quad ||T||_{\mathcal{K}^{\text{sur}}_q} = \inf ||A|| ||B||.
\] (2.9)
where the infimum is taken over all such factorizations \( T = BA \).

(iii) By [15, Proposition 3.7.(c)] it suffices to assume that \( A \) in (2.9) is a bounded operator. In particular, if \( Z \) is a quotient space of \( \ell^q \) then (by choosing \( A \) in (2.9) to be the identity operator \( I : Z \to Z \)) one gets that
\[
\mathcal{K}(Z, Y) = \mathcal{K}^{\text{sur}}_q (Z, Y) \quad \text{and} \quad ||T|| = ||T||_{\mathcal{K}^{\text{sur}}_q}
\] (2.10)
for all \( T \in \mathcal{K}(Z, Y) \).

(iv) By [15, Theorem 3.13] the following holds:
\[
\mathcal{K}^{\text{inj}}_p = \left( \mathcal{K}^{\text{sur}}_{p'} \right)^{\text{dual}} \quad \text{and} \quad \mathcal{K}^{\text{sur}}_{p'} = \left( \mathcal{K}^{\text{inj}}_p \right)^{\text{dual}}.
\]

We proceed by establishing the following reverse monotonicity property for the classes \( \mathcal{K}^{\text{inj}}_p \) and \( \mathcal{K}_{uq} \) in the range \([1,2]\), which is used in the proof of Theorem 3.1. Here, the range cannot be extended to \([1,q]\) for any \( q > 2 \) as we note in Remark 2.5.

**Proposition 2.4.** Suppose that \( 1 \leq p < q \leq 2 \). Then, the following hold:

(i) \( \mathcal{K}^{\text{inj}}_q \subset \mathcal{K}^{\text{inj}}_p \).

(ii) \( \mathcal{K}_{uq} \subset \mathcal{K}_{uq} \).

Proof.

(i) Let \( \varepsilon > 0 \) and suppose that \( T \in \mathcal{K}^{\text{inj}}_q (X, Y) \), where \( X \) and \( Y \) are arbitrary Banach spaces. By applying Fact 2.3.(i), there exists a factorization \( T = VU \) through a closed subspace \( M \subset \ell^q \), where \( U \in \mathcal{K}(X, M) \) and \( V \in \mathcal{K}(M, Y) \) are compact operators such that
\[
||V|| = 1 \quad \text{and} \quad ||U|| < ||T||_{\mathcal{K}^{\text{inj}}_q} + \varepsilon.
\] (2.11)
Now, \( \ell^q \) embeds isometrically into \( L^p[0, 1] \) (see, e.g., [2, Theorem 6.4.18]), and so does \( M \). By Proposition 2.2, we have \( U \in \mathcal{K}^{\text{inj}}_p (X, M) \) and
\[
||U|| = ||U||_{\mathcal{K}^{\text{inj}}_p}.
\] (2.12)
Thus, \( T = VU \in \mathcal{K}^{\text{inj}}_p (X, Y) \) by the operator ideal property. Moreover, by applying (2.11) and (2.12) one gets that
\[
||T||_{\mathcal{K}^{\text{inj}}_p} = ||VU||_{\mathcal{K}^{\text{inj}}_p} \leq ||V|| ||U||_{\mathcal{K}^{\text{inj}}_p} = ||U|| = ||V|| < ||T||_{\mathcal{K}^{\text{inj}}_q} + \varepsilon.
\]
Consequently, \( ||T||_{\mathcal{K}^{\text{inj}}_p} \leq ||T||_{\mathcal{K}^{\text{inj}}_q} \). This yields the claim in part (i).

(ii) As \( \mathcal{I} \subset \mathcal{J} \) for Banach operator ideals \( \mathcal{I} \subset \mathcal{J} \), the following holds by (2.5), Fact 2.3.(iv) and part (i):
\[
\mathcal{K}_{uq} = \mathcal{K}^{\text{sur}}_q = \left( \mathcal{K}^{\text{inj}}_q \right)^{\text{dual}} \subset \left( \mathcal{K}^{\text{inj}}_p \right)^{\text{dual}} = \mathcal{K}^{\text{sur}}_p = \mathcal{K}_{uq}.
\]
This completes the proof. \( \square \)
Remark 2.5. The reverse monotonicity behavior of the classes $\mathcal{K}_{p}^{\text{inj}}$ and $\mathcal{K}_{up}$ in Proposition 2.4 does not extend to the case of $1 \leq p < 2 < q < \infty$. In fact, in that case there are, according to [47, Proposition 3.6], closed subspaces $X \subset \ell^{p}$ and $Y \subset \ell^{q}$ for which $\mathcal{K}_{p}^{\text{inj}}(Z,Z)$ and $\mathcal{K}_{q}^{\text{inj}}(Z,Z)$ are incomparable classes of operators for the direct sum $Z = X \oplus Y$. Consequently, also $\mathcal{K}_{up}(Z^{*},Z^{*})$ and $\mathcal{K}_{up}(Z^{*},Z^{*})$ are incomparable classes by (2.5) and the first identity in Fact 2.3.(iv). Note that the injective hull $\mathcal{K}_{p}^{\text{inj}}$ is denoted by $Q\mathcal{K}_{p}$ in [47], see the discussion after [47, Remarks 3.2].

Let $I = (I, \| \cdot \|_{I})$ and $J = (J, \| \cdot \|_{J})$ be Banach operator ideals. Recall that the quasi-Banach operator ideal $J \circ I = (J \circ I, \| \cdot \|_{J \circ I})$ is defined as follows: the operator $T \in J \circ I(X,Y)$ by definition if there is a Banach space $Z$ and operators $A \in I(X,Z)$ and $B \in J(Z,Y)$ such that $T = BA$. The components $J \circ I(X,Y)$ are equipped with the quasi-norm

$$||T||_{J \circ I} = \inf ||A||_{I} ||B||_{J},$$

where the infimum is taken over all such factorizations $T = BA$. We refer to [37, 6.1.1] or [8, 9.2] for the definition of a quasi-norm.

We will require the following useful characterization of the unconditionally $p$-compact operators obtained by Kim [24, Theorem 2.2]:

$$\mathcal{K}_{up} = \mathcal{K}_{up} \circ \mathcal{K}_{up} \quad \text{for all } 1 \leq p < \infty. \tag{2.13}$$

Remark 2.6. The argument in [24] uses a technical lemma on certain collections of summable sequences of positive real numbers. We note in passing that one can also establish the isometric identity in (2.13) by applying the characterization $\mathcal{K}_{up} = \mathcal{K}_{up}^{\text{sur}}$ from (2.5) and using the identities in (2.9) and (2.10). We leave the details to the reader.

2.2 Approximation properties

Following the terminology in [47], we say that the Banach space $X$ has the uniform $I$-AP if

$$I(Y,X) \subset A(Y,X)$$

for every Banach space $Y$. The uniform $I$-AP was considered by Lassalle and Turco [30] with a slightly different terminology. Observe that if $X$ has the $I$-AP, then $X$ has the uniform $I$-AP, since $|| \cdot || \leq || \cdot ||_{I}$ (see (2.6)). However, in general the converse fails, see [30, p. 2460].

We proceed with a simple lemma which exhibits some connections between the AP, the $I$-AP and the uniform $I$-AP in special situations that are relevant for us.

Lemma 2.7. Suppose that $I = (I, \| \cdot \|_{I})$ and $J = (J, \| \cdot \|_{J})$ are Banach operator ideals and let $X$ be a Banach space.

(i) Suppose that $I(Y,X) = J \circ I(Y,X)$ for every Banach space $Y$. If $X$ has the uniform $J$-AP, then $X$ has the $I$-AP.

(ii) Suppose that $I(Y,X) = \mathcal{K} \circ I(Y,X)$ for every Banach space $Y$. If $X$ has the $\mathcal{K}$-AP, then $X$ has the $I$-AP.

Proof.

(i) Suppose that $X$ has the uniform $J$-AP. Let $\varepsilon > 0$ and suppose that $T \in I(Y,X)$ for an arbitrary Banach space $Y$. By assumption, $T = BA$ for compatible operators $A \in I(Y,Z)$ and $B \in J(Z,X)$. Since $X$ has the uniform $J$-AP, there is a bounded finite-rank operator $F \in F(Z,X)$ such that $||B - F|| < \varepsilon/||A||_{I}$. It follows that

$$||T - FA||_{I} = ||BA - FA||_{I} \leq ||B - F|| \cdot ||A||_{I} < \varepsilon.$$

Consequently, $T \in \overline{F(Y,X)}$ which concludes the proof.
(ii) Suppose that \( X \) has the AP. Then, \( X \) has the uniform \( \mathcal{K} \)-AP by Grothendieck’s characterization of the AP [32, Theorem 1.e.4]. Thus part (i) (with \( J = \mathcal{K} \)) yields that \( X \) has the \( I \)-AP. □

Suppose that \( 1 \leq p < \infty \) and let \( X \) be a Banach space. It is known that if \( X \) has the AP, then \( X \) has the \( SK_\mathcal{K} \)-AP, see [17, Proposition 3.10]. Moreover, using an internal characterization of the \( \mathcal{K} \)-AP that involves a locally convex topology on \( \mathcal{L}(X, X) \), it was established in [22, Section 2] that if \( X \) has the AP, then \( X \) has the \( \mathcal{K} \)-AP. Note that this also follows from the more general result [30, Proposition 3.5] since \( \mathcal{K} = \mathcal{K}^{\mathcal{K}} \) and \( \mathcal{K}^{\mathcal{K}} \) is an accessible Banach operator ideal, see [8, 22.3 and 25.3].

We next observe that the above facts can also easily be verified by applying Lemma 2.7.(ii) with essentially the same method as in the proof of [17, Proposition 3.10].

**Fact 2.8.** Suppose that \( X \) is a Banach space with the AP. Then, \( X \) has both the \( SK_\mathcal{K} \)-AP and the \( \mathcal{K} \)-AP for all \( 1 \leq p < \infty \).

**Proof.** Let \( 1 \leq p < \infty \) and let \( Y \) be an arbitrary Banach space. It follows from [5, Theorem 3.1] and the operator ideal property that \( SK_\mathcal{K}(Y, X) = \mathcal{K} \circ SK_\mathcal{K}(Y, X) \). Moreover, since \( \mathcal{K} \subseteq \mathcal{K} \) (see (2.3)), the factorization result (2.13) of Kim yields that \( \mathcal{K}(Y, X) = \mathcal{K} \circ \mathcal{K}(Y, X) \). Thus, both claims follow from Lemma 2.7.(ii). □

**Remark 2.9.** Every Banach space has the \( SK_2 \)-AP by [9, Corollary 3.6] (see also Oja [35, p. 952]) and the \( \mathcal{K} \)-AP by [22, Corollary 1.2]. The argument for the \( \mathcal{K} \)-AP in [22] relies on a duality result between the \( SK_2 \)-AP and the \( \mathcal{K} \)-AP, see Equation (3.3). Moreover, it is known that \( \mathcal{K}(Y, X) = \mathcal{K}(Y, X) \) for all Banach spaces \( Y \) and \( X \), which follows from, for example, [37, 18.1.4].

It follows from [11, Theorem 2.1] that the uniform \( SK_\mathcal{K} \)-AP coincides with the \( p \)-AP of Sinha and Karn introduced in [45], which is a strictly weaker property than the \( SK_\mathcal{K} \)-AP at least for \( 1 \leq p < 2 \), see [9, Theorem 2.4]. We next note that the identity \( \mathcal{K} = \mathcal{K} \circ \mathcal{K} \) of Kim in (2.13) implies that the \( \mathcal{K} \)-AP is equivalent to the uniform \( \mathcal{K} \)-AP for all \( 1 \leq p < \infty \). This also follows by combining (2.5) and [25, Proposition 4.4], but we provide for convenience a short proof by applying Lemma 2.7.(i).

**Proposition 2.10.** Suppose that \( 1 \leq p < \infty \) and let \( X \) be a Banach space. Then, \( X \) has the \( \mathcal{K} \)-AP if and only if \( X \) has the uniform \( \mathcal{K} \)-AP.

**Proof.** The forward implication clearly holds. The converse follows from Lemma 2.7.(i) since \( \mathcal{K} = \mathcal{K} \circ \mathcal{K} \) by (2.13). □

We conclude this section with a characterization of the \( \mathcal{K} \)-AP for dual Banach spaces, which complements the characterizations of the \( SK_\mathcal{K} \)-AP and the uniform \( SK_\mathcal{K} \)-AP for dual spaces obtained in [9, Theorem 2.3] and [11, Theorem 2.8], respectively.

**Theorem 2.11.** Suppose that \( 1 \leq p < \infty \) and let \( X \) be a Banach space. Then, the following are equivalent:

(i) \( X^* \) has the \( \mathcal{K} \)-AP.

(ii) \( \mathcal{K}(Z, X^*) = F(Z, X^*) \) for every quotient space \( Z \) of \( \ell^p \) (respectively, of \( c_0 \) if \( p = 1 \)).

(iii) \( \mathcal{K}(Z, X^*) = A(Z, X^*) \) for every quotient space \( Z \) of \( \ell^p \) (respectively, of \( c_0 \) if \( p = 1 \)).

(iv) \( \mathcal{K}(X, M) = A(X, M) \) for every closed subspace \( M \subset \ell^p \).

**Proof.**

(i) \( \Rightarrow (ii) \) Suppose that \( Z \) is a quotient space of \( \ell^p \) (respectively, of \( c_0 \) if \( p = 1 \)) and let \( T \in \mathcal{K}(Z, X^*) \). By Fact 2.3.(iii) and (2.5), we have \( T \in \mathcal{K}(Z, X^*) \). It follows by the assumption that \( T \in F(Z, X^*) \).
(ii) \( \Rightarrow \) (iii) This is obvious since \( \| \cdot \| \leq \| \cdot \|_{K_{up}} \), see (2.6).

(iii) \( \Rightarrow \) (iv) Suppose that \( T \in \mathcal{K}(X, M) \), where \( M \) is an arbitrary closed subspace of \( \ell^p \). By Proposition 2.2 we have \( T \in \mathcal{K}^\text{fin}_{up}(X, M) \), and thus \( T^* \in \mathcal{K}^\text{ex}(M^*, X^*) \) by Fact 2.3.(iv). By Fact 2.3.(ii) there is a quotient space \( Z \) of \( \ell^p \) (respectively, of \( c_0 \) if \( p = 1 \)) and compact operators \( A \in \mathcal{K}(M^*, Z) \), \( B \in \mathcal{K}(Z, X^*) \) such that \( T^* = BA \). By assumption, \( B \in \mathcal{A}(Z, X^*) \) and thus \( T^* \in \mathcal{A}(M^*, X^*) \) by the operator ideal property. It follows that \( T \in \mathcal{A}(X, M) \), see, for example, [8, Ex. 9.12].

(iv) \( \Rightarrow \) (i) Suppose \( T \in \mathcal{K}_{up}(Y, X^*) \) for an arbitrary Banach space \( Y \). By (2.5) and Fact 2.3.(iv) we have \( T^* \in \mathcal{K}^\text{inj}_{up}(X^{**}, Y^*) \).

Thus, according to Fact 2.3.(i), there is a closed subspace \( M \subset \ell^p \) and compact operators \( A \in \mathcal{K}(X^{**}, M) \) and \( B \in \mathcal{K}(M, Y^*) \) such that \( T^* = BA \).

Next, let \( j_X : X \to X^{**} \) and \( j_Y : Y \to Y^{**} \) denote the canonical isometric embeddings. By assumption \( A j_X \in \mathcal{K}(X, M) = \mathcal{K}_{up}(X, M) \), and thus \( (A j_X)^* \in \mathcal{K}_{up}(M^*, X^*) \) by the operator ideal property. Consequently, \( X^* \) has the uniform \( \mathcal{K}_{up} \)-AP and thus the \( \mathcal{K}_{up} \)-AP by Proposition 2.10.

\[ \Box \]

3 THE \( \mathcal{K}_{up} \)-APPROXIMATION PROPERTY IN THE CASE OF \( 1 \leq p < 2 \)

In this section, we establish the main results of this paper concerning the \( \mathcal{K}_{up} \)-AP for \( 1 \leq p < 2 \). As an application, we provide a negative answer to the case \( p = 1 \) of questions of Kim [23, Problem 1]; namely, the \( \mathcal{K}_{u1} \)-AP implies neither the \( \mathcal{SK}_{u1} \)-AP nor the AP, and the \( \mathcal{SK}_{u1} \)-AP implies neither the \( \mathcal{K}_{u1} \)-AP nor the AP. We also provide an answer to a query of Kim posed in [27, Example 4.4]; namely, the \( \mathcal{W}_1 \)-AP does not imply the AP, where \( \mathcal{W}_1 = (\mathcal{W}_1, \| \cdot \|_{\mathcal{W}_1}) \) denotes the Banach operator ideal of the weakly 1-compact operators [45].

Our first main result follows from results established in Section 2.

**Theorem 3.1.** Suppose that \( 1 \leq p < q < 2 \) and let \( X \) be a Banach space. If \( X \) has the \( \mathcal{K}_{up} \)-AP, then \( X \) has the \( \mathcal{K}_{uj} \)-AP.

**Proof.** As for any Banach operator ideals \( I \subset J \), the uniform \( J \)-AP implies the uniform \( I \)-AP, the result is a straightforward combination of Propositions 2.10 and 2.4.(ii).

Recall from [42, 0.2] or [8, 31.5] that the bounded operator \( T \in \mathcal{L}(X, Y) \) is called compactly approximable, denoted \( T \in \mathcal{C}\mathcal{A}(X, Y) \), if for all \( \varepsilon > 0 \) and all compact subsets \( K \subset X \) there is a bounded finite-rank operator \( U \in \mathcal{F}(X, Y) \) such that

\[ \sup_{x \in K} ||Ux - Tx|| < \varepsilon. \]

The class \( \mathcal{C}\mathcal{A} = (\mathcal{C}\mathcal{A}, || \cdot ||) \) is a Banach operator ideal equipped with the uniform operator norm \( || \cdot || \). Furthermore, if \( X \) or \( Y \) has the AP, then

\[ \mathcal{C}\mathcal{A}(X, Y) = \mathcal{L}(X, Y). \]  

(3.1)

We refer to [47, Proposition 4.1] for a proof of these facts.

We proceed with a result which involves a factorization result of compactly approximable operators due to Pisier [40] and a lemma of John [19] on the AP for reflexive spaces. This complements a similar result obtained in [47, Theorem 2.2], which in turn combines a factorization result of Kwapien and Maurey with John’s lemma. See also [18, Theorem 2.2] and its subsequent comments as well as the remark on [13, p. 248] for other results in this direction. Recall that \( \ell^p \) and all closed subspaces \( M \subset \ell^p \) have cotype 2 whenever \( 1 \leq p \leq 2 \) and type 2 whenever \( 2 \leq p < \infty \), see, for example, [2, Theorem 6.2.14 and Remark 6.2.11.(f)].

**Theorem 3.2.** Let \( X \) and \( Y \) be Banach spaces. Suppose that \( X^* \) has cotype 2 and that \( Y \) has cotype 2 and the AP. Then

\[ \mathcal{K}(X, M) = \mathcal{A}(X, M) \]

for every closed subspace \( M \subset Y \).
Proof. Let $M$ be a closed subspace of $Y$ and suppose that $T \in \mathcal{K}(X, M)$. Since $Y$ has the AP, it follows from (3.1) that $jT \in CA(X, Y)$, where $j : M \hookrightarrow Y$ is the inclusion map. Furthermore, since $X^*$ and $Y$ have cotype 2, a factorization result of Pisier [40, Corollaire 2.16] (see also [42, Theorem 4.1]) yields a factorization

$$jT = VU,$$

where $U \in \mathcal{L}(X, H)$ and $V = \mathcal{L}(H, Y)$ are compatible bounded operators and $H$ is a Hilbert space. It follows that $T = \tilde{V} \tilde{U}$, where $\tilde{U} : X \rightarrow UX$ and $\tilde{V} : UX \rightarrow M$ are the bounded operators defined as follows:

$$\tilde{U} : x \mapsto Ux \quad \text{and} \quad \tilde{V} : h \mapsto Vh.$$

Now, since $UX \subset H$ is reflexive and has the AP, we have $T \in A(X, M)$ according to [19, Remarks 3] (see also the proof of [47, Theorem 2.2] for a more direct argument). Consequently, $\mathcal{K}(X, M) = A(X, M)$. □

By applying Theorem 3.2 we obtain the following result, which exhibits a large class of Banach spaces that have the $\mathcal{K}^{up}$-AP for all $1 \leq p < 2$. For an alternative proof using a different method, see Remark 3.4.

**Theorem 3.3.** Suppose that $X$ is a Banach space with cotype 2. Then, $X$ has the $\mathcal{K}^{up}$-AP for all $1 \leq p < 2$.

**Proof.** Suppose that $1 \leq p < 2$ and let $T \in \mathcal{K}^{up}(Y, X)$, where $Y$ is an arbitrary Banach space. By Proposition 2.10, it suffices to verify that $T \in A(Y, X)$.

To this end, note that $T^* \in \mathcal{K}^{up}(X^*, Y^*)$ by (2.5) and Fact 2.3.(iv). Thus, there is, according to Fact 2.3.(i), a closed subspace $M \subset \ell^p$ and compact operators $A \in \mathcal{K}(X^*, M)$ and $B \in \mathcal{K}(M, Y^*)$ such that

$$T^* = BA.$$

Since $X$ has cotype 2, the bidual $X^{**}$ has cotype 2, see, for example, [13, Corollary 11.9]. Thus, by applying Theorem 3.2, one gets that

$$A \in \mathcal{K}(X^*, M) = A(X^*, M).$$

It follows from the operator ideal property that $T^* \in A(X^*, Y^*)$, and thus $T \in A(Y, X)$, see, for example, [37, Theorem 11.7.4]. This completes the proof. □

**Remark 3.4.** Theorem 3.3 can also be established by an approach which involves the duality between the $\mathcal{K}^{up}$-AP and the $SK^{up}$-AP due to Kim [22] and Lassalle and Turco [31]. In fact, suppose that $1 \leq p < 2$ and let $X$ be a Banach space with cotype 2. By applying a classical result of Maurey on absolutely $p$-summing operators and a characterization of the $SK^{up}$- AP for dual spaces due to Delgado et al. [9], it is shown in [49, Proposition 3.6] that $X^*$ has the $SK^{up}$-AP. Consequently, $X$ has the $\mathcal{K}^{up}$-AP by [22, Theorem 1.1] if $1 < p < 2$ and by [31, Theorem 4.7] if $p = 1$.

The following example shows that the monotonicity property in Theorem 3.1 does not extend to the case of $1 \leq p < 2 < q < \infty$.

**Example 3.5.** Suppose that $1 \leq p < 2 < q < \infty$. Then, there is a reflexive Banach space that has the $\mathcal{K}^{up}$-AP but fails the $\mathcal{K}^{up}_q$-AP. In fact, let $X \subset \ell^q$ be a closed subspace that fails the AP, see [32, Theorem 2.d.6]. Then, $X$ fails the $SK^{up}_q$-AP according to [35, Theorem 1], and thus $X^*$ fails the $\mathcal{K}^{up}_q$-AP by [22, Theorem 1.1] (see (3.2)). Moreover, since $X$ has type 2, the dual space $X^*$ has cotype 2, see, for example, [13, Proposition 11.10]. Consequently, $X^*$ has the $\mathcal{K}^{up}_q$-AP by Theorem 3.3.

In [23, Section 5], Kim discusses the relationship between the AP, the $SK^{up}_p$-AP, and the $\mathcal{K}^{up}_p$-AP. In particular, the author demonstrates for all $1 < p < 2$ that the $\mathcal{K}^{up}_p$-AP implies neither the $SK^{up}_p$-AP nor the AP, and vice versa, the $SK^{up}_p$-AP implies neither the $\mathcal{K}^{up}_p$-AP nor the AP. Whether or not the same holds for $p = 1$ or for $2 < p < \infty$ was stated as a problem [23, Problem 1].
The reasoning above by Kim relies among others on results on the AP of order $p$, whose study was initiated by Saphar [44], and a duality result of Kim. In fact, according to [22, Theorem 1.1], the following claims hold for all $1 < p < \infty$ and all Banach spaces $X$:

If $X^*$ has the $\mathcal{K}_u^p$-AP, then $X$ has the $\mathcal{S}\mathcal{K}_p$-AP. \hfill (3.2)

If $X^*$ has the $\mathcal{S}\mathcal{K}_p$-AP, then $X$ has the $\mathcal{K}_u^p$-AP. \hfill (3.3)

Subsequently, it was shown by Kim [23, Theorem 1.1] that the claim (3.2) also holds for $p = 1$, and by Lassalle and Turco [31, Theorem 4.7] that also the claim (3.3) holds for $p = 1$. See also [28, Theorem 2.5] for a general result that includes both statements (3.2) and (3.3) for $p = 1$.

We will next demonstrate that Theorems 3.1 and 3.3 together with these duality results yield an answer to the case $p = 1$ of [23, Problem 1], where the following questions were posed by Kim (with our notation):

(Q1) If $X$ has the $\mathcal{K}_{u1}$-AP (respectively, the $\mathcal{S}\mathcal{K}_{1}$-AP), then does $X$ have the $\mathcal{S}\mathcal{K}_{1}$-AP (respectively, the $\mathcal{K}_{u1}$-AP) or the AP?

The analogous questions were also asked for $2 < p < \infty$ and we will in Section 5 draw attention to a few related questions and remarks that could potentially be of use for the case $2 < p < \infty$. We also obtain a negative answer to the case $p = 1$ of the follow-up questions [23, Problem 2] of Kim:

(Q2) If $X^*$ has the $\mathcal{K}_{u1}$-AP (respectively, the $\mathcal{S}\mathcal{K}_{1}$-AP), then does $X$ have the $\mathcal{K}_{u1}$-AP (respectively, the $\mathcal{S}\mathcal{K}_{1}$-AP)?

**Proposition 3.6.** Suppose that $1 < q < 2$ and let $X \subset \ell^q$ be a closed subspace that fails the AP. Then, $X$ has the $\mathcal{K}_{u1}$-AP but fails the $\mathcal{S}\mathcal{K}_{1}$-AP. Furthermore, the dual space $X^*$ has the $\mathcal{S}\mathcal{K}_{1}$-AP but fails both the AP and the $\mathcal{K}_{u1}$-AP.

In particular, the $\mathcal{K}_{u1}$-AP implies neither the $\mathcal{S}\mathcal{K}_{1}$-AP nor the AP, and the $\mathcal{S}\mathcal{K}_{1}$-AP implies neither the $\mathcal{K}_{u1}$-AP nor the AP. This answers (Q1) above in the negative. Moreover, note that the fact that the closed subspace $X \subset \ell^q$ in Proposition 3.6 is reflexive yields a negative answer to (Q2) above.

**Proof.** First, recall that such a closed subspace $X$ exists due to Szankowski, see [46] or [33, Theorem 1.g.4]. Since $X$ has cotype 2, it has the $\mathcal{K}_{u1}$-AP by Theorem 3.3. Consequently, since $X$ is reflexive, the dual space $X^*$ has the $\mathcal{S}\mathcal{K}_{1}$-AP according to [23, Theorem 1.1].

Next, we show that $X^*$ fails the $\mathcal{K}_{u1}$-AP. For this, assume toward a contradiction that $X^*$ has the $\mathcal{K}_{u1}$-AP. Then, $X^*$ has the $\mathcal{K}_{uq}$-AP by Theorem 3.1. Consequently, $X$ has the $\mathcal{S}\mathcal{K}_{q}$-AP by the duality (3.2). But since $X \subset \ell^q$ is a closed subspace, it follows from [35, Theorem 1] that $X$ has the AP, which is a contradiction. Thus, $X^*$ fails the $\mathcal{K}_{u1}$-AP.

Since $X^*$ is reflexive and fails the $\mathcal{K}_{u1}$-AP, the duality result [23, Theorem 1.3] or [31, Theorem 4.7] yields that $X$ fails the $\mathcal{S}\mathcal{K}_{1}$-AP. Moreover, $X^*$ fails the AP by Fact 2.8 or [32, Theorem 1.e.7].

It is also interesting to observe that even the combination of the $\mathcal{K}_{u1}$-AP and the $\mathcal{S}\mathcal{K}_{p}$-AP for all $1 \leq p < 2$ does not imply the AP as our following examples shows.

**Example 3.7.** Let $P$ denote the Banach space that fails the AP constructed by Pisier in [41] (see also [42, Section 10]), which has the following properties:

- $P \hat{\otimes} P = P \hat{\otimes} P$.
- Both $P$ and $P^*$ have cotype 2.

Now, $P$ has the $\mathcal{K}_{u1}$-AP for all $1 \leq p < 2$ by Theorem 3.3. The same is true for $P^*$ and thus the duality result (3.2) of Kim, that is, [22, Theorem 1.1] for $p > 1$ and [23, Theorem 1.1] for $p = 1$, yields that $P$ also has the $\mathcal{S}\mathcal{K}_{p}$-AP for all $1 \leq p < 2$.

As a final observation of this section, we note that Theorem 3.3 also yields an answer to a query posed in [27, Example 4.4] for $p = 1$; namely, the $\mathcal{W}_1$-AP does not imply the AP. Here, $\mathcal{W}_1 = (\mathcal{W}_1, \| \cdot \|_{\mathcal{W}_1})$ denotes the Banach operator ideal of the weakly 1-compact operators, which is a special case of the weakly $p$-compact operators $\mathcal{W}_p = (\mathcal{W}_p, \| \cdot \|_{\mathcal{W}_p})$ of Sinha and Karn [45]. Recall by definition that the operator $T \in \mathcal{W}_1(X, Y)$ if there is a weakly 1-summable sequence $(y_k) \in \ell^1_w(Y)$.
such that
\[ T(B_X) \subset 1\text{-co}(y_k) = \left\{ \sum_{k=1}^{\infty} \lambda_k y_k \mid (\lambda_k) \in B_{c_0} \right\}. \quad (3.4) \]

The ideal norm \( \| \cdot \|_{\mathcal{W}_1} \) is defined by
\[ \|T\|_{\mathcal{W}_1} = \inf \{ \|y_k\|_{1,w} \mid (3.4) \text{ holds for } (y_k) \in \ell_w^1(Y) \} \]
for all \( T \in \mathcal{W}_1(X,Y) \), see [27, p. 864]. Note that \( \mathcal{K}_{u_1} \subset \mathcal{W}_1 \) since \( \ell_1^u(X) \subset \ell_w^1(X) \) for any Banach space \( X \).

**Proposition 3.8.** Let \( X \) be a Banach space with cotype 2 that does not contain an isomorphic copy of \( c_0 \). Then, \( X \) has the \( \mathcal{W}_1 \)-AP.

In particular, the \( \mathcal{W}_1 \)-AP does not imply the AP, which answers the query posed in [27, Example 4.4].

**Proof.** Since \( X \) has cotype 2, it has the \( \mathcal{K}_{u_1} \)-AP by Theorem 3.3. Moreover, as \( X \) does not contain an isomorphic copy of \( c_0 \), the identity \( \mathcal{W}_1(Y,X) = \mathcal{K}_{u_1}(Y,X) \) hold isometrically for every Banach space \( Y \), see [27, Example 4.1]. Consequently, \( X \) has the \( \mathcal{W}_1 \)-AP. \( \square \)

4 \quad THE FAILURE OF THE \( \mathcal{K}_p \)-AP FOR \( p > 2 \)

Let \( 1 \leq p < \infty \) and let \( X \) be a Banach space. Recall that a subset \( K \subset X \) is called relatively \( p \)-compact if \( K \subset p\text{-co}\{x_k\} \) for a strongly \( p \)-summable sequence \( (x_k) \in \ell_p^s(X) \), see [45, pp. 19–20]. Following Sinha and Karn [45, Section 6], the Banach space \( X \) is said to have the \( p \)-AP if for all \( \varepsilon > 0 \) and all relatively \( p \)-compact subsets \( K \subset X \) there is a bounded finite-rank operator \( U \in \mathcal{L}(X,X) \) such that
\[ \sup_{x \in K} \|Ux - x\| < \varepsilon. \]
If \( X \) has the AP, then \( X \) has the \( p \)-AP for all \( 1 \leq p < \infty \) since relatively \( p \)-compact subsets are relatively compact, see [45, p. 20]. Moreover, it follows from [11, Theorem 2.1] that the Banach space \( X \) has the \( p \)-AP if and only if \( X \) has the uniform \( \mathcal{K}_p \)-AP.

Let \( 2 < p < q < \infty \) be such that \( q > 2p/(p - 2) \). In [5, Corollary 2.9] Choi and Kim established that a variant of the closed subspace \( E \subset \ell_q^p \) that fails the AP constructed by Davie [6] fails the \( p \)-AP, that is, the uniform \( \mathcal{K}_p \)-AP. It follows from the monotonicity property (2.4) that \( E \) fails the uniform \( \mathcal{K}_p \)-AP for all \( r \geq p \). In this section, we complement the result of Choi and Kim in the setting of the \( \mathcal{K}_p \)-AP by showing that for all \( 2 < p < q \) there is a closed subspace \( X \subset \ell_q^p \) that fails the uniform \( \mathcal{K}_r \)-AP for all \( r \geq p \). In particular, we do not require the condition \( q > 2p/(p - 2) \) in our example. We also point out here that it appears unknown whether the AP, the \( \mathcal{K}_p \)-AP, and the uniform \( \mathcal{K}_p \)-AP are different properties for \( 2 < p < \infty \), see Section 5.

The closed subspace \( X \subset \ell_q^p \) we uncover is essentially Davie’s subspace. However, our approach is based on a factorization argument of Reinov [43, Lemma 1.1] in contrast to the approach of Choi and Kim in [5]. For other similar constructions based on [43, Lemma 1.1], we refer to [47, Theorem 3.9] and [49, Proposition 4.3].

Suppose that \( 1 \leq p < \infty \) and let \( X \) and \( Y \) be arbitrary Banach spaces. Recall from [37, 18.1 and 18.2] or [13, pp. 111–112] that the operator \( T \in \mathcal{L}(X,Y) \) is called \( p \)-nuclear, denoted by \( T \in \mathcal{N}_p(X,Y) \), if there is a strongly \( p \)-summable sequence \( (x_k^*) \in \ell_p^s(X^*) \) and a weakly \( p' \)-summable sequence \( (y_k) \in \ell_w^{p'}(Y) \) such that
\[ Tx = \sum_{k=1}^{\infty} x_k^*(x)y_k, \quad x \in X. \quad (4.1) \]

The class \( \mathcal{N}_p = (\mathcal{N}_p, \| \cdot \|_{\mathcal{N}_p}) \) is a Banach operator ideal, where the \( p \)-nuclear norm \( \| \cdot \|_{\mathcal{N}_p} \) is defined by
\[ \|T\|_{\mathcal{N}_p} = \inf \{ \|x_k^*\|_p \|y_k\|_{p',w} \mid (4.1) \text{ holds} \} \]
for all \( T \in \mathcal{N}_p(X,Y) \).
Following Persson and Pietsch [36], the operator $T \in \mathcal{L}(X, Y)$ is called \textit{quasi $p$-nuclear}, denoted by $T \in Q\mathcal{N}_p(X, Y)$, if there is a strongly $p$-summable sequence $(x_k^*) \in \ell_p^p(X^*)$ such that

$$||Tx|| \leq \left( \sum_{k=1}^{\infty} |x_k^*(x)|^p \right)^{1/p}, \quad x \in X. \quad (4.2)$$

The class $Q\mathcal{N}_p = (Q\mathcal{N}_p, \| \cdot \|_{Q\mathcal{N}_p})$ is a Banach operator ideal, where the ideal norm $\| \cdot \|_{Q\mathcal{N}_p}$ is defined by

$$||T||_{Q\mathcal{N}_p} = \inf \{ ||(x_k^*)||_p \mid (4.2) \text{ holds for } (x_k^*) \in \ell_p^p(X^*) \}$$

for all $T \in Q\mathcal{N}_p(X, Y)$.

It is known that $Q\mathcal{N}_p$ coincides with the injective hull of $\mathcal{N}_p$, see [36, Satz 38 and Satz 39] or the comment before [38, Theorem 6]. Moreover, it follows from results in [10, Section 3] (see also the discussion preceding [17, Corollary 2.7]) that

$$Q\mathcal{N}_p = SK_p^{\text{dual}}$$

(4.3)

for all $1 \leq p < \infty$.

**Proposition 4.1.** Let $2 < p, q < \infty$. Then, there is a closed subspace $X \subset \ell^q$ that fails the $SK_r$-AP for all $r \geq p$.

**Proof.** Let $A = (a_{k,j})_{k,j \in \mathbb{N}}$ be an infinite matrix of scalars with the following properties:

(i) $A^2 = 0$,
(ii) $\text{tr} A := \sum_{k=1}^{\infty} a_{k,k} \neq 0$,
(iii) $\sum_{k=1}^{\infty} \lambda_k^b < \infty$ for all $b > 2/3$, where $\lambda_k := \sup_{j \in \mathbb{N}} |a_{k,j}| > 0$ for all $k \in \mathbb{N}$.

Recall that such a matrix exists due to Davie [6, 7] (see also [32, Theorem 2.d.3]). Since $2 < p, q < \infty$ we may pick a positive real number $c > 0$ such that

$$0 < c < \min \left\{ \frac{1}{3} - \frac{2}{3p}, \frac{1}{3} - \frac{2}{3q} \right\}.$$

Denote $s := \frac{2}{3q} + c$ and let $B = (b_{k,j})_{k,j \in \mathbb{N}}$ be the matrix with elements $b_{k,j} = (\frac{\lambda_j}{\lambda_k})^s a_{k,j}$ for all $k, j \in \mathbb{N}$. By (i) and (ii), the matrix $B$ has the following properties:

(iv) $B^2 = 0$,
(v) $\text{tr} B = \sum_{k=1}^{\infty} b_{k,k} \neq 0$.

(Not that $B$ is defined as the matrix in [32, Theorem 2.d.6], but with a different exponent $s$.)

Since $c > 0$ we have $sq > 2/3$. Consequently,

$$\left( \sum_{j=1}^{\infty} |b_{k,j}|^q \right)^{1/q} \leq \left( \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\lambda_k} \right)^{sq} |a_{k,j}|^q \right)^{1/q} \leq \left( \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\lambda_k} \right)^{sq} \right)^{1/q} = \lambda_k^{1-s} \left( \sum_{j=1}^{\infty} \lambda_j^q \right)^{1/q} < \infty$$

for all $k \in \mathbb{N}$, where the last expression is finite by (iii). It follows that $y_k := (b_{k,1}, b_{k,2}, \ldots) \in \ell^q$ for all $k \in \mathbb{N}$, and moreover,

$$||y_k||_q \leq L \cdot \lambda_k^{1-s}, \quad (k \in \mathbb{N}), \quad (4.4)$$
where \( L := \left( \sum_{j=1}^{\infty} \lambda_j^q \right)^{\frac{1}{2}} < \infty \) and \( t := 1 - s \).

Claim. The closed linear span

\[ X := \text{span}\{y_k \mid k \in \mathbb{N}\} \subset \ell^q \]

fails the \( SK_r \)-AP for all \( r \geq p \).

To this end, fix \( r \in \mathbb{R} \) such that \( r \geq p \). In order to establish the claim, we will next define relevant spaces and operators which are illustrated in the diagram (4.9). Since

\[ t = 1 - s = 1 - \frac{2}{3q} - c > 1 - \frac{2}{3q} - \left( \frac{1}{3} - \frac{2}{3q} \right) = \frac{2}{3}, \quad (4.5) \]

we have

\[ \sum_{k=1}^{\infty} ||y_k||_q \leq L \cdot \sum_{k=1}^{\infty} \lambda_k^t < \infty \quad (4.6) \]

by (4.4) and (iii). It follows that the matrix \( B \) defines a 1-nuclear operator \( B : \ell^q' \to \ell^q' \) defined by

\[ Bx = (\langle x, y_k \rangle)_{k=1}^{\infty}, \quad x = (x_k) \in \ell^q', \]

for which (iv) and (v) hold. This means that \( Bx = \sum_{k=1}^{\infty} y_k(x)e_k \) for all \( x \in \ell^q' \), where \( (e_k) \) denotes the unit vector basis in \( \ell^q' \).

Define the following operators:

\[ V : \ell^q' \to \ell^\infty, \quad Vx = (\lambda_k^{-1} \langle x, y_k \rangle)_{k=1}^{\infty}, \quad x = (x_k) \in \ell^q', \]

\[ \Delta : \ell^\infty \to \ell^q', \quad \Delta x = (\lambda_k^t x_k)_{k=1}^{\infty}, \quad x = (x_k) \in \ell^\infty. \]

By (4.4), the operator \( V \) is bounded (with \( ||V|| \leq L \)). Furthermore, \( \Delta \) is a 1-nuclear diagonal operator by (iii) since \( t > 2/3 \) by (4.5). Clearly \( B = \Delta V \).

Let \( E := \ell^q' / \ker V \). Note that \( \ker V = X^\perp \) since \( \lambda_k \neq 0 \) for all \( k \in \mathbb{N} \). Thus,

\[ X = X^\perp \cong (\ell^q'/X^\perp)^* = E^*. \quad (4.7) \]

Denote \( d := \min\{r', q'\} > 1 \) and pick \( a > 0 \) such that

\[ \frac{2}{3r} < a < t - \frac{2}{3d}. \quad (4.8) \]

This is possible since

\[ t - \frac{2}{3r'} > \frac{2}{3} - \frac{2}{3r'} = \frac{2}{3r} \]

by (4.5), and since

\[ t - \frac{2}{3q'} = 1 - s - \frac{2}{3q'} = 1 - \left( \frac{2}{3q} + c \right) - \frac{2}{3q'} = \frac{1}{3} - c > \frac{2}{3p} \geq \frac{2}{3r}, \]

where the last inequality holds since \( r \geq p \).

Next, define the following diagonal operators:

\[ \Delta_1 : \ell^\infty \to \ell^r, \quad \Delta_1 x = (\lambda_k^a x_k)_{k=1}^{\infty}, \quad x = (x_k) \in \ell^\infty, \]

\[ \Delta_2 : \ell^r \to \ell^q', \quad \Delta_2 x = (\lambda_k^{-a} x_k)_{k=1}^{\infty}, \quad x = (x_k) \in \ell^r. \]
It follows from (4.8) that \( ar > 2/3 \), and thus (\( \lambda^a \)) \( \in \ell^r \) by (iii). Hence, \( \Delta_1 \) is \( r \)-nuclear. Similarly, we have (\( \lambda^{r-a} \)) \( \in \ell^d \), and thus \( \Delta_2 \) is \( d \)-nuclear. Clearly \( \Delta = \Delta_1 \Delta_2 \).

Consider the closed subspace \( F_0 := \overline{V \ell^q} \subset \ell^\infty \) and let \( j_1 : F_0 \hookrightarrow \ell^\infty \) denote the inclusion map. Let \( Q : \ell^q' \to E \) be the canonical projection onto \( E = \ell^q / \ker V \) and let \( \overline{V} : E \to F_0 \) be the injective operator induced by \( V \) such that \( V = j_1 \overline{V} \).

Furthermore, define the closed subspace \( F := \Delta_1 j_1 F_0 \subset \ell^r \) and let \( j_2 : F \hookrightarrow \ell^r \) denote the inclusion map. Define also the operator

\[
\tilde{\Delta}_1 : F_0 \to F, \quad \tilde{\Delta}_1 x = \Delta_1 j_1 x, \quad x \in F_0.
\]

This means that \( j_2 \tilde{\Delta}_1 = \Delta_1 j_1 \). Finally, let \( U := \tilde{\Delta}_1 \overline{V} : E \to F \).

The above-defined spaces and operators are illustrated in the following commuting diagram:

\[
\begin{array}{cccc}
\ell^q & \overset{\nu}{\longrightarrow} & \ell^\infty & \overset{\Delta}{\longrightarrow} & \ell^q' \\
\downarrow{\phi} & & \downarrow{j_1} & & \downarrow{\Delta_1} \\
E & \overset{\overline{V}}{\longrightarrow} & F_0 & \overset{\Delta_1}{\longrightarrow} & F \\
\end{array}
\]

(4.9)

We proceed by verifying that

\[
U \in Q\mathcal{N}'(E, F) \setminus \overline{F(E, F)}
\]

(4.10)

Firstly, since \( \Delta_1 \) is \( r \)-nuclear, the operator \( \tilde{\Delta}_1 \) is quasi \( r \)-nuclear by [36, Satz 39]. Hence, \( U \in Q\mathcal{N}'(E, F) \) by the operator ideal property. We continue by showing that \( U \notin \overline{F(E, F)} \).

For this, define

\[
\phi(T) = \text{tr}(\Delta_2 j_2 T Q)
\]

(4.11)

for every \( T \in Q\mathcal{N}'(E, F) \). The mapping \( T \mapsto \phi(T) \) defines a continuous linear functional \( \phi \in Q\mathcal{N}'(E, F)^* \). In fact, since \( d = \min\{r', q'\} \leq r' \) we have

\[
\Delta_2 \in \mathcal{N}_d(\ell^r, \ell^q') \subset \mathcal{N}_{r'}(\ell^r, \ell^q')
\]

by monotonicity, see, for example, [13, Corollary 5.24]. Thus by the multiplication table in [36, Satz 48] the following hold:

\[
||\Delta_2 j_2 T Q||_{\mathcal{N}_1} \leq ||\Delta_2||_{\mathcal{N}_1} ||j_2 T Q||_{Q\mathcal{N}_r}
\]

(4.12)

for all \( T \in Q\mathcal{N}'(E, F) \). Since \( \ell^{q'} \) has the (metric) AP, it follows that the trace on the right-hand side in (4.11) is well-defined and

\[
|\phi(T)| = |\text{tr}(\Delta_2 j_2 T Q)| \leq ||\Delta_2||_{\mathcal{N}_1} ||j_2 T Q||_{Q\mathcal{N}_r}
\]

(4.13)

for all \( T \in Q\mathcal{N}'(E, F) \), see, for example, [37, 10.3.2]. By combining the norm estimates (4.12) and (4.13), one obtains that

\[
|\phi(T)| \leq ||\Delta_2||_{\mathcal{N}_1} ||j_2 T Q||_{Q\mathcal{N}_r} \leq ||\Delta_2||_{\mathcal{N}_1} ||T||_{Q\mathcal{N}_r}
\]

for all \( T \in Q\mathcal{N}'(E, F) \). Thus, the mapping \( T \mapsto \phi(T) \) defines a continuous linear functional \( \phi \in Q\mathcal{N}'(E, F)^* \).

Next, note that

\[
\phi(U) = \text{tr}(\Delta_2 j_2 U Q) = \text{tr}(\Delta V) = \text{tr}(B) \neq 0,
\]

(4.14)

where the final step holds by (v). However, we claim that

\[
\phi(x^* \otimes y) = 0
\]

(4.15)
for all bounded rank-one operators $x^* \otimes y \in F(E, F)$, which by linearity and continuity implies that $\phi(T) = 0$ for all $T \in F(E, F)$. This will show that $U \notin F(E, F)_q$, since $\phi(U) \neq 0$ by (4.14).

To show the claim (4.15), let $x^* \in E^*$ and $y \in F$. By construction $F = UQ\ell^q$ and we assume first that $y \in UQ\ell^q$. Thus, $y = UQx$ for some $x \in \ell^q$. It then follows that

$$\phi(x^* \otimes y) = \text{tr}(\Delta j_2(x^* \otimes y)Q) = \text{tr}(Qx^* \otimes \Delta j_2 y)$$

$$= \langle \Delta j_2 y, Qx^* \otimes \Delta j_2 Q \rangle = \langle Q\Delta j_2 y, x^* \rangle = \langle QBx, x^* \rangle = 0,$$

where the last equality above holds since $QB = 0$. In fact, $\Delta j_1 \tilde{V}QB = B^2 = 0$ by (iv), and since $\Delta j_1 \tilde{V}$ is injective, we have $QB = 0$.

For an arbitrary $y \in F$, pick a sequence $(z_n) \subset UQ\ell^q$ such that $||z_n - y|| \to 0$ as $n \to \infty$. Then

$$||x^* \otimes y - x^* \otimes z_n||_q = ||x^* \otimes (y - z_n)||_q = ||x^*|| ||z_n - y|| \to 0$$

as $n \to \infty$. Consequently, by continuity and (4.16), we have that

$$\phi(x^* \otimes y) = \lim_{n \to \infty} \phi(x^* \otimes z_n) = 0,$$

which then yields (4.15). Thus, $U \notin F(E, F)_q$.

We have thus established the claim in (4.10). It then follows by the duality (4.3) and reflexivity of the spaces $E$ and $F$ that

$$U^* \in SK_p(F^*, E^*) \setminus F(F^*, E^*)_{\mathcal{K}_p}.$$ 

Consequently, $E^*$ fails the $SK_p$-AP and thus $X$ fails the $SK_p$-AP since $X \cong E^*$ by (4.7).

\section{Final Remarks and Problems}

It is unknown whether the AP, the $\mathcal{K}_{up}$-AP and the $\mathcal{K}_p$-AP are distinct properties whenever $2 < p < \infty$. More precisely, in [23, Problem 1] Kim posed the following question (with our notation and excluding the case $p = 1$ which was answered in the negative in Proposition 3.6):

\textbf{Question 1} (Kim). Let $2 < p < \infty$. If $X$ has the $\mathcal{K}_{up}$-AP (respectively, the $\mathcal{K}_p$-AP), then does $X$ have the $\mathcal{K}_p$-AP (respectively, the $\mathcal{K}_{up}$-AP) or the AP?

In this section, we draw attention to a few related questions and remarks suggested by our results. The first of our questions is motivated by Theorem 3.2. For a related question, see [47, Question 5.2].

\textbf{Question 2}. Let $2 < p < \infty$. Is there a Banach space $X$ such that $X^*$ fails the AP and $\mathcal{K}(X, M) = \mathcal{A}(X, M)$ for every closed subspace $M \subset \ell^p$? We note that for such a Banach space $X$, the dual $X^*$ would have the $\mathcal{K}_{up}$-AP by Theorem 2.11.

It is known that there are Banach spaces that have the uniform $SK_p$-AP but fail the $SK_p$-AP whenever $1 \leq p < 2$, see [30, p. 2460]. In fact, every Banach space has the uniform $SK_p$-AP for $1 \leq p < 2$ (see [45, Theorem 6.4] or [11, Corollary 2.5]), and by [9, Theorem 2.4] there are Banach spaces that fail the $SK_p$-AP. However, it appears unknown whether there are Banach spaces that have the uniform $SK_p$-AP but fail the $SK_p$-AP for $2 < p < \infty$.

Recall the following result [35, Theorem 1] due to Oja which has been used in Example 3.5 and Proposition 3.6: if $X$ is a closed subspace of $L^p(\mu)$, then $X$ has the AP if and only if $X$ has the $SK_p$-AP.

Oja’s result suggests the following question on the relationship between the $SK_p$-AP and the uniform $SK_p$-AP.
**Question 3.** Let $2 < p < \infty$. Is there a significant class $C$ of Banach spaces for which the $\mathcal{K}_p$-AP is equivalent to the uniform $\mathcal{K}_p$-AP for all Banach spaces $X \in C$?

By a significant class above, we mean (somewhat vaguely) a class which is defined independently of any approximation properties and is large enough to contain spaces that have the AP and spaces that fail the uniform $\mathcal{K}_p$-AP. Note that the (uniform) $\mathcal{K}_{up}$-AP implies the uniform $\mathcal{K}_p$-AP since $\mathcal{K}_p \subset \mathcal{K}_{up}$, see (2.3). Consequently, the $\mathcal{K}_{up}$-AP would imply the $\mathcal{K}_p$-AP in such a class $C$ of Banach spaces.

Let $2 < p < \infty$ be fixed. Lemma 2.7.(i) points to the following class of Banach spaces which seems relevant for Question 3 at first sight: let $C'$ denote the class of all Banach spaces $X$ for which

$$\mathcal{S}\mathcal{K}_p(Y, X) = \mathcal{S}\mathcal{K}_p \circ \mathcal{S}\mathcal{K}_p(Y, X)$$

for every Banach space $Y$. In fact, Lemma 2.7.(i) implies that if $X \in C'$, then $X$ has the $\mathcal{K}_p$-AP if and only if $X$ has the uniform $\mathcal{K}_p$-AP. However, Example 5.1 shows that the class $C'$ only contains the finite-dimensional Banach spaces and is thus not a relevant class here.

**Example 5.1.** Suppose that $2 < p < \infty$ and let $X$ be an arbitrary infinite-dimensional Banach space. We claim that

$$\mathcal{S}\mathcal{K}_p \circ \mathcal{S}\mathcal{K}_p(\ell_p', X) \subset \mathcal{S}\mathcal{K}_p(\ell_p', X).$$

(5.1)

In fact, since $p > 2$ and $X$ is infinite-dimensional, there is a relatively $p$-compact subset $K \subset X$ that is not relatively $(p/2)$-compact according to [39, Proposition 3.5.(3)], see also [38, Proposition 20.(3)]. By the definition, we may assume that $K = p\text{-co}\{x_k\}$ for some strongly $p$-summable sequence $(x_k) \in \ell_p^s(X)$. Let $\theta : \ell_p' \to X$ be the bounded operator defined as follows:

$$\theta(a) = \sum_{k=1}^{\infty} a_k x_k, \quad a = (a_k) \in \ell_p'.$$

This means that $K = \theta(B_{\ell_p'})$, and thus $\theta \in \mathcal{S}\mathcal{K}_p(\ell_p', X)$. We next verify that $\theta \notin \mathcal{S}\mathcal{K}_p \circ \mathcal{S}\mathcal{K}_p(\ell_p', X)$.

Assume toward a contradiction that $\theta \in \mathcal{S}\mathcal{K}_p \circ \mathcal{S}\mathcal{K}_p(\ell_p', X)$. Let $\theta = UV$ be a factorization where $U, V \in \mathcal{S}\mathcal{K}_p$ are compatible Sinha–Karn $p$-compact operators. According to the duality result [10, Proposition 3.8], the adjoint operators $U^*, V^* \in QN_p'$ are quasi $p$-nuclear, and thus $\theta^* \in QN_p \circ QN_p'(X^*, \ell_p)$.

Next, according to the multiplication rule [36, Satz 48] we have $\theta^* \in QN_{p/2}(X^*, \ell_p)$, and thus $\theta \in \mathcal{S}\mathcal{K}_{p/2}(\ell_p', X)$ by [10, Proposition 3.8]. This means that $\theta(B_{\ell_p'}) = K \subset X$ is relatively $(p/2)$-compact, which is a contradiction. We have thus shown that

$$\theta \in \mathcal{S}\mathcal{K}_p(\ell_p', X) \setminus \mathcal{S}\mathcal{K}_p \circ \mathcal{S}\mathcal{K}_p(\ell_p', X),$$

which yields the claim (5.1).

As a final remark, we recall that it is a long standing open question whether the Hardy space $H^\infty$ of the bounded analytic functions on the open unit disc has the AP, see [32, Problem 1.e.10] or [8, 5.2(4)]. The connection here to Question 1 by Kim is the following observation:

$H^\infty$ has the $\mathcal{K}_{up}$-AP for all $1 < p < \infty$.

This follows from a similar argument as in [11, Corollary 2.10], where it was shown that all odd duals $(H^\infty)^*, (H^\infty)^{***}, \ldots$ have the uniform $\mathcal{S}\mathcal{K}_p$-AP for all $1 < p < \infty$. In fact, fix $1 < p < \infty$ for the argument. Recall from, for example, [11, Section 1] or [8, 21.7] that the Banach space $X$ has the AP of order $p$ (AP$_p$) if the natural mapping

$$\theta : Y^* \hat{\otimes}_{g_p} X \to \mathcal{N}_p(Y, X)$$

is injective for all Banach spaces $Y$. Here, $g_p$ denotes the Chevet–Saphar tensor norm and we refer to, for example, [8, 12.7] for a description of $g_p$. A result of Bourgain and Reinov [3, Theorem 1] implies that the bidual $(H^\infty)^{**}$ has the AP$_{p'}$. 


It then follows that $(H^\infty)^*$ has the $S\mathcal{K}_p$-AP by [9, Corollary 2.5]. Consequently, $H^\infty$ has the $\mathcal{K}_u\mathcal{K}_p$-AP by [22, Theorem 1.1] (see (3.3)).

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