Thermodynamic and dynamical stability of Freund–Rubin compactification

Shunichiro Kinoshita\textsuperscript{1,\ast} and Shinji Mukohyama\textsuperscript{2,\dagger}

\textsuperscript{1} Department of Physics, The University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan
\textsuperscript{2} Institute for the Physics and Mathematics of the Universe (IPMU), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8582, Japan

(Dated: June 18, 2009)

We investigate stability of two branches of Freund–Rubin compactification from thermodynamic and dynamical perspectives. Freund–Rubin compactification allows not only trivial solutions but also warped solutions describing warped product of external de Sitter space and internal deformed sphere. We study dynamical stability by analyzing linear perturbations around solutions in each branch. Also we study thermodynamic stability based on de Sitter entropy. We show complete agreement of thermodynamic and dynamical stabilities of this system. Finally, we interpret the results in terms of effective energy density in the four-dimensional Einstein frame and discuss cosmological implications.

PACS numbers: 04.20.-q, 04.50.+h, 11.25.Mj

I. INTRODUCTION AND SUMMARY

De Sitter or quasi-de Sitter spacetimes describe the inflationary epoch of the universe at its early stages, and also the present universe which has entered the period of accelerated expansion. One of the most intriguing issues today is to realize such de Sitter vacua in fundamental high-energy physics. Particularly, higher-dimensional spacetimes are required by string theory, which is a promising candidate for the fundamental theory. In order to obtain the effective four-dimensional theory in higher-dimensional spacetime we should usually compactify the extra dimensions and stabilize the compactified internal space. Thus we need to embed a four-dimensional de Sitter spacetime into higher-dimensional spacetimes together with stabilization of the extra dimensions.

The Freund–Rubin compactification \cite{1} is a simple model with a stabilization mechanism that the extra dimensions are dynamically compactified and stabilized by a flux of anti-symmetric tensor field or form field. In this model with $(p+q)$-dimensional spacetime, a $q$-form flux field is introduced to stabilize the $q$-dimensional compact space. Moreover, turning on a positive bulk cosmological constant allows an external de Sitter space and an internal manifold with positive curvature \cite{2, 3} (also, adding a dilaton field \cite{4}). Consequently, we obtain a $(p+q)$-dimensional product spacetime which consists of a $p$-dimensional de Sitter space $dS_p$ and a $q$-dimensional sphere $S^q$, that is the Freund–Rubin (FR) solution.

It has been known that the Freund–Rubin solution has two classes of dynamical instabilities \cite{2, 3}; one is attributed to homogeneous excitation ($l=0$ mode) of the internal space, corresponding to change of the radius of the extra dimensions; and the other is inhomogeneous excitation with quadrupole moment ($l=2$ mode) and higher multi-pole moments ($l \geq 3$ modes), representing deformation of the extra dimensions.

The $l=0$ mode is the so-called volume modulus or radion, and becomes tachyonic when the Hubble expansion rate of the external de Sitter space is too large (in other words, when the flux density on the internal space is small). This fact implies that if the energy scale of the inflationary external spacetime is sufficiently larger than the compactification scale of the internal space, the volume modulus will be destabilized. In order to avoid the emergence of this instability, configurations with small Hubble expansion rate is preferable.

While instability of the volume modulus exists already in $q=2$, instabilities arising from deformation of extra dimensions emerge only if the number of extra dimensions is larger than or equal to four. In the unstable region, in which at least one of the $l \geq 2$ modes are tachyonic, the external spacetime has small Hubble expansion rate including the Minkowski spacetime. It means that the flux densities are very large in the unstable region of this type. It should be noted that for more than four extra dimensions, the two unstable regions overlap so that stable configurations for the FR solution no longer exist.

\textsuperscript{\ast}Electronic address: kinoshita@utap.phys.s.u-tokyo.ac.jp
\textsuperscript{\dagger}Electronic address: shinji.mukohyama@ipmu.jp
Little has been known about the non-perturbative properties of the instability from higher multi-pole modes;\(^1\) how it turns out after the onset of this instability and whether any stable configuration exists as a possible end-state in this model, and so on. In the previous work [9], one of us has shown that in the Freund–Rubin compactification there is a new branch of solutions other than the FR solutions. Those solutions are described as the warped product of an external de Sitter space and an internal deformed sphere. It has been found that the branch of the FR solutions and that of the warped solutions intersect at the point where the FR solution becomes marginally stable for the \(l = 2\) mode. Although we have seen existence of non-trivial solutions other than the FR solutions, their stability remains unanswered.

In this paper we are particularly concerned with the close connection between dynamical stability and thermodynamic stability. The interesting relationship between dynamical and thermodynamic stability, which is well known as the correlated stability conjecture (or the Gubser–Mitra conjecture [7, 8]), has been suggested and confirmed for some black objects (strings, branes and so on) by many authors [9, 10, 11, 12, 13, 14, 15, 16, 17]. (See e.g. [18, 19] and references therein.) It is important to examine whether such connections really exist and whether they can be extended to systems other than black objects such as spacetimes with de Sitter horizons.

In fact, for the FR solutions we can simply reinterpret the instability from the \(l = 0\) mode based on thermodynamic arguments [6] as follows (and also see [20]). For a fixed total flux on the internal space, the branch of FR solutions are divided into two sub-branches in terms of entropy defined by the total area of the de Sitter horizon: a sub-branch of solutions with higher entropy and the other with lower entropy. Both sub-branches terminate at one critical point, where the \(l = 0\) mode becomes massless and the FR solution is marginally stable. Moreover, the solutions on the lower-entropy sub-branch, which are thermodynamically unfavorable, are dynamically unstable since the \(l = 0\) mode is tachyonic. Thus thermodynamic instability exactly coincides with dynamical instability for the \(l = 0\) mode of two sub-branches within the FR branch.

The aim of this paper is to examine the stability of the new branch of warped solutions from both dynamical and thermodynamic perspectives. This opens up new possibilities of the applicability of close connection between dynamical and thermodynamic stabilities.

We examine the dynamical stability by analyzing perturbative stability of the system in a straightforward way. For simplicity, we restrict our considerations to the sector which behaves as scalar with respect to the external de Sitter space since unstable perturbations of the FR solutions are in this sector.

In the case of four-dimensional external spacetime and four-dimensional internal space, we numerically obtain the Kaluza–Klein (KK) mass spectrum and show that the warped solutions are stable in the low Hubble regime, while the FR solutions are unstable due to the \(l = 2\) mode in the same regime within numerical accuracy.

In order to reveal the thermodynamic property we derive the first law of de Sitter thermodynamics for Freund–Rubin compactifications. Each branch of the FR solutions and the warped solutions obeys the first law in terms of entropy \(S\) and total flux \(\Phi\):

\[
dS = \frac{\Omega_p^{p-2b}}{4(p-1)h^p}d\Phi,
\]

where two parameters \(b\) and \(h\) characterize the flux density on the internal space and the Hubble expansion rate of the external de Sitter space, respectively. This fact means that for a fixed total flux, the branch with higher entropy should be thermodynamically favored. Comparison between the entropy of the FR branch and that of the warped branch for a given total flux tells us which branch is thermodynamically favored. The result is that the warped branch is entropically favored in the low Hubble regime while the FR branch is favored in the high Hubble regime.

The above results are briefly summarized as follows:

- For small Hubble expansion rate, the warped branch is thermodynamically favored and dynamically stable.
- For large Hubble expansion rate, the FR branch is thermodynamically favored and dynamically stable.

Thus, as we have expected, we see complete agreement of thermodynamic and dynamical stabilities for two branches of FR compactifications. This provides yet another example showing close connections between thermodynamic and dynamical properties of systems with horizons.

The rest of this paper is organized as follows. In Sec. II we review general Freund–Rubin compactifications and show the Freund–Rubin solutions describing \(dS_p \times S^q\) and the warped solutions describing a warped product of an external de Sitter space and an internal deformed sphere. In Sec. III we investigate dynamical stability of the warped solution by considering perturbations around the background solution. In Sec. IV we derive the first law of de Sitter

\(^1\) The time evolution of unstable solutions for the \(l = 0\) mode was studied in [9].
thermodynamics and discuss thermodynamic stability for the FR branch and the warped branch. In Sec. IV we interpret the above results in terms of effective energy density in the four-dimensional Einstein frame and discuss cosmological implications.

II. FREUND–RUBIN COMPACTIFICATION

In this section we review general Freund–Rubin flux compactifications, including a bulk cosmological constant. We consider the \((p + q)\)-dimensional action

\[
I = \frac{1}{16\pi} \int \! d^{p+q}x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{q!} F_{(q)}^2 \right),
\]

where \(\Lambda\) is a \((p + q)\)-dimensional bulk cosmological constant and \(F_{(q)}\) is a \(q\)-form field strength for stabilizing the \(q\)-dimensional internal manifold. (We use units in which \(G = 1\) unless otherwise noted.) The Einstein equation and Maxwell equation lead to

\[
G_{MN} = \frac{1}{(q-1)!} F_{ML_{1} \cdots L_{q-1}} F_{N}^{L_{1} \cdots L_{q-1}} - \frac{1}{2q!} F_{(q)MN} - \Lambda g_{MN},
\]

and

\[
\nabla_{M} F^{MN_{1} \cdots N_{q-1}} = 0,
\]

where the \(q\)-form field satisfies the Bianchi identity \(\nabla_{[M_{1}} F_{M_{2} \cdots M_{q+1}]} = 0\).

A. Freund–Rubin solution

These equations have well-known solutions originally found by Freund and Rubin \(\ll\). The metric and the \(q\)-form field strength in the Freund–Rubin solutions are given by

\[
ds^2 = -dt^2 + e^{2ht} dx_{p-1}^2 + \rho^2 d\Omega_{q}^2,
\]

and

\[
F_{(q)} = b \epsilon_{\mu_{1} \cdots \mu_{q}},
\]

where \(\epsilon_{\mu_{1} \cdots \mu_{q}}\) is the volume element of the \(q\)-sphere with a radius \(\rho\). These solutions describe the direct product of a \(p\)-dimensional external de Sitter space with Hubble expansion rate \(h\) and a \(q\)-sphere with radius \(\rho\). The internal \(q\)-sphere is supported by the \(q\)-form flux with a flux density \(b\). The Einstein equation and Maxwell equation yield relations among \(b\), \(h\) and \(\rho\):

\[
(p-1)(p+q-2)h^2 + (q-1)b^2 = 2\Lambda,
\]

\[
(q-1)^2 \rho^{-2} + (p-1)^2 h^2 = 2\Lambda.
\]

Evidently, these relations allow a one-parameter family of solutions.

B. Warped solution

There is another one parameter family of non-trivial solutions. Their geometry is a warped product of de Sitter space and a deformed sphere. Such warped solutions are described by the following ansatz for the metric

\[
ds^2 = e^{2\phi(r)} [-dt^2 + e^{2ht} dx_{p-1}^2] + e^{-\frac{2q}{q-1} \phi(r)} [dr^2 + a^2(r) d\Omega_{q}^2],
\]

and the \(q\)-form flux

\[
F_{(q)} = b e^{-\frac{2q}{q-1} \phi} a^{q-1} dr \wedge d\Omega_{q-1},
\]
where $b$ is a constant. Note that (10) automatically satisfies the Maxwell equation and the Bianchi identity. Then, from the Einstein equation we have the two equations,

$$
\frac{a''}{a} = \frac{b^2}{q-2} e^{-\frac{2p(q-1)}{q+2}\phi} + \frac{p(p-1)}{q-2} h^2 e^{-\frac{2(p+q-2)}{q+2}\phi} - \frac{2\Lambda}{q-2} e^{-\frac{2p}{q+2}\phi} - (q-2) \frac{a'^2 - 1}{a^2},
$$

and the constraint equation

$$
\frac{(q-1)(q-2)}{2} \left[ \left( \frac{a'}{a} \right)^2 - \frac{1}{a'^2} \right] = \frac{p(p+q-2)}{2(q-2)} \phi^2 + \frac{p(p-1)}{2} h^2 e^{\frac{2(p+q-2)}{q+2}\phi} + \frac{b^2}{2} e^{\frac{2p(q-1)}{q+2}\phi} - \Lambda e^{-\frac{2p}{q+2}\phi}.
$$

We are interested in the case where the internal space is compact. Thus, we consider an interval $r_- \leq r \leq r_+$, where $a(r)$ vanishes at the endpoints, $a(r_{\pm}) = 0$, and is positive between them. Then the regularity requires the following boundary conditions:

$$
|a'(r_{\pm})| = 1, \quad \phi'(r_{\pm}) = 0
$$

at the endpoints. These conditions ensure that the internal space with spherical topology is regular at the north and south poles.

A one-parameter family of solutions to the above equations and boundary conditions for $p = 4$ and $q = 4$ was found numerically by one of the authors [6]. This warped branch of solutions emanates from the marginally stable solution in the branch of Freund–Rubin solutions as shown in Fig. 1. In Fig. 1 two lines representing two branches intersect at one point. At the intersection, the solution in the warped branch is no more warped and reduces to a FR solution. For $b^2$ smaller than the value at the intersection, the internal space is prolate. On the other hand, for $b^2$ larger than the value at the intersection, the internal space is oblate. Therefore, while the numerically obtained value of $b^2$ at the intersection includes numerical errors, the statement that the warped solution reduces to the FR solution at the intersection is exact: there must be a boundary between the oblate and prolate cases; and at the boundary, the internal space is spherical and the warp factor is trivial.

In the following sections, we shall investigate stability of the FR and warped branches.

### III. DYNAMICAL STABILITY

In this section we investigate dynamical stability of FR solutions and warped solutions by considering linear perturbations around them. We will concentrate on scalar-type perturbation with respect to the external de Sitter space since in the case of the FR solution instability arises from perturbations of this type.
A. Background

In the previous section we have already shown the background metric and form field. In this subsection we rewrite them in a form which is convenient for the analysis of perturbation equations.

We suppose that the \((p+q)\)-dimensional metric is given by

\[
ds^2 = A^2(y_1)g_{\mu
u}(x)dx^\mu dx^\nu + dy^2 + B^2(y_1)\gamma_{ij}(z)dz^idz^j,
\]

where \(g_{\mu
u}\) is the metric of \(p\)-dimensional Lorentzian Einstein space and \(\gamma_{ij}\) is the metric of \((q-1)\)-dimensional Euclidean Einstein space. Note that the Riemann tensors with respect to \(g_{\mu
u}\) and \(\gamma_{ij}\) respectively satisfy \(\mathcal{R}_{\mu
u} = K(p-1)g_{\mu
u}\) and \(\mathcal{R}_{ij} = k(q-2)\gamma_{ij}\). The \(q\)-form field strength is

\[
F_q = b_1 B^{q-1}\sqrt{\gamma}dy_1^{q-2}\int dz^i,
\]

which satisfies the Maxwell equation and the Bianchi identity automatically.

Then, non-vanishing components of the \((p+q)\)-dimensional Einstein tensor \(G_{MN}\) become

\[
G_{\mu
u} = \left[ \frac{(p-1)(p-2)}{2} A^2 - K A^2 + (p-1) A'' A + \frac{(q-1)(q-2)}{2} B^{2} - k \right] g_{\mu
u} + (q-1) B'' A B + (p-1)(q-1) \frac{A'B'}{AB},
\]

\[
G_{yy} = \left[ \frac{p(p-1)}{2} A^2 - K A^2 + \frac{(q-1)(q-2)}{2} B^{2} - k \right] g_{yy} + p(q-1) \frac{A'B'}{AB},
\]

\[
G_{ij} = \left[ \frac{p(p-1)}{2} A^2 - K A^2 + \frac{A'' A}{A} + \frac{(q-2)(q-3)}{2} B^{2} - k \right] \gamma_{ij} + (q-2) B'' A B + p(q-2) \frac{A'B'}{AB}
\]

(here in this section the prime denotes the derivative with respect to \(y\)) and the energy-momentum tensor of the \(q\)-form flux field is given by

\[
T_{\mu
u} = -\frac{b_2}{2 A^{2(p-1)}} g_{\mu
u}, \quad T_{yy} = \frac{b_2}{2 A^{2p}}, \quad T_{ij} = \frac{b_2}{2 A^{2p}} B^2 \gamma_{ij},
\]

and the other components vanish.

Finally, we note that if we set \(A = e^\theta\), \(B = e^{-\frac{r}{\sqrt{2}}} a\), \(dy = e^{-\frac{\phi}{\sqrt{2}}} dr\), \(k = 1\) and \(K = h^2\) in the above equations, then all equations for the background ansatz in Sec. II are reproduced.

B. Perturbation

We now leave the subject of background and turn our attention to linear perturbations around the background. We suppose that the \(p\)-dimensional external spacetime and the \((q-1)\)-dimensional internal space are the Einstein manifolds with the metric \(g_{\mu\nu}\) and \(\gamma_{ij}\), respectively. In this case we can decompose tensors on the \((p+q)\)-dimensional spacetime into scalar-type, vector-type and tensor-type components with respect to \(g_{\mu
u}\) and \(\gamma_{ij}\). Hence, in our analysis we decompose perturbations of the metric and the form field into different types and obtain decoupled perturbation equations in each sector.

As we already mentioned before, we will concentrate on scalar-type perturbations. Especially, we suppose that the internal manifold is topologically a sphere and has \(SO(q)\)-isometry, namely \(\gamma_{ij}(z)dz^idz^j = d\Omega_q^{2}\) which is the metric of unit round \((q-1)\)-sphere. Then what we are interested in is perturbations which are scalar-type quantities with respect to not only \(p\)-dimensional de Sitter symmetry of the background external space but also \(SO(q)\) symmetry of the background internal space.

In this paper, for simplicity we assume that the perturbations preserve the \(SO(q)\) symmetry of the background internal space. By choosing an appropriate gauge (see Appendix A), we can write the perturbed metric and field strength as follows:

\[
ds^2 = (1 + \Pi Y) A^2(y_1)g_{\mu
u}(x)dx^\mu dx^\nu + [1 + (\Pi - \Omega) Y] dy^2 + \left[ 1 + \left( \frac{\Omega}{q-1} - \frac{p-1}{q-1}\Pi \right) Y \right] B^2(y_1)d\Omega_q^{2},
\]
and
\[ F_{(q)} = b \frac{B^{q-1}}{A^p} dy \wedge d\Omega_{q-1} + d(\varphi Y) \wedge d\Omega_{q-1} \] (19)

where \( Y(x) \) is scalar harmonics on the de Sitter space with the Hubble expansion rate \( h \) and variables \( \Pi, \Omega \) and \( \varphi \) depend on only \( y \)-coordinate due to the SO(q)-symmetry.

From the linearized Einstein equation and Maxwell equation, we obtain a set of two perturbation equations for two variables \( \Pi, \Omega \):
\[
(p + q - 2)\Pi'' + (q - 2)\Omega'' + (p + q - 2) \left[ \frac{A'}{A} - (q - 1) \frac{B'}{B} \right] \Pi' + (q - 2) \left[ p \frac{A'}{A} + (q - 1) \frac{B'}{B} \right] \Omega' + \left[ \mu^2 + \frac{2(q - 2)}{B^2} \right] [p + q - 2)\Pi - q\Omega] = 0,
\]
\[
\Omega'' + \left[ (3p - 2) \frac{A'}{A} + 3(q - 1) \frac{B'}{B} \right] \Omega' - \left[ \frac{2(p + q - 2)(q - 2)}{B^2} - 4\Lambda \right] \Pi + \left[ \mu^2 + 2h^2(p - 1)^2 \frac{1}{A^2} + 2\frac{q(q - 2)}{B^2} - 4\Lambda \right] \Omega = 0,
\] (20)

where \( \mu^2 \) is the KK mass squared which is defined by \( \nabla^a Y(x) = \mu^2 Y(x) \). Here \( \nabla_\mu \) denotes the covariant derivative with respect to \( g_{\mu\nu} \). Moreover we have an algebraic equation for \( \varphi \):
\[
\varphi = \frac{Ap^q-1}{2b} \left[ (p + q - 2) \frac{B'}{B} \Pi - \frac{1}{Ap^q-2B^q} \left( Ap^q-2B^q\Omega \right)^{2} \right]. \] (21)

Boundary conditions are specified by the regularity at the poles of the internal space of spherical topology, which are characterized by \( B = 0 \). They are given by
\[
(p + q - 2)\Pi - q\Omega = \Omega' = 0 \quad \text{at} \quad B = 0. \] (22)

Thus the perturbation equations are reduced to eigenvalue problems with eigenvalue \( \mu^2 \). If the spectrum of \( \mu^2 \) is non-negative, we can conclude that the background spacetime is dynamically stable.

Before we discuss stability of the warped solution, let us recall stability of the FR solution. For the FR solution, we set \( A = 1 \) and \( B = \rho \sin \frac{\mu y}{p} \). Then the perturbation equations reduce to
\[
(p + q - 2)\Pi'' + (q - 2)\Omega'' - (p + q - 2)(q - 1) \frac{B'}{B} \Pi' + (q - 2)(q - 1) \frac{B'}{B} \Omega' + \left[ \mu^2 + \frac{2(q - 2)}{B^2} \right] [p + q - 2)\Pi - q\Omega] = 0,
\]
\[
\Omega'' + 3(q - 1) \frac{B'}{B} \Omega' - \left[ \frac{2(p + q - 2)(q - 2)}{B^2} - 4\Lambda \right] \Pi + \left[ \mu^2 + 2h^2(p - 1)^2 \frac{1}{B^2} + \frac{2q(q - 2)}{B^2} - 4\Lambda \right] \Omega = 0.
\] (23)

Eliminating \( \Omega \) from these equations we obtain a single forth order differential equation for \( \Pi \):
\[
D^2 \cdot D^2 \Pi + 2 \left[ \mu^2 + (p - 1)h^2 + \frac{p(q - 1)}{p + q - 2}b^2 \right] D^2 \Pi + \mu^2 \left[ \mu^2 + 2(p - 1)h^2 - \frac{2(q - 1)(p - 2)b^2}{p + q - 2} \right] \Pi = 0,
\] (24)

where \( D^2 \) denotes Laplacian on \( S^q \) with a radius \( \rho \). We expand \( \Pi \) in terms of scalar harmonics \( Y(y) \) on \( S^q \), and then the mass eigenvalues \( \mu^2 \) are given by
\[
\mu^2 = \lambda + \frac{(q - 1)(p - 2)k}{p + q - 2}h^2 - (p - 1)h^2
\pm \sqrt{\left[ \frac{(q - 1)(p - 2)k}{p + q - 2}h^2 - (p - 1)h^2 \right]^2 + \frac{4q(q - 1)(p - 1)}{p + q - 2}b^2 \lambda},
\] (25)
where $D^2Y(y) = -\lambda Y(y)$ with $\lambda = l(l + q - 1)\rho^{-2}$. It is clear from this expression that the scalar perturbations for each multi-pole moment $l$ generically have two independent modes corresponding to the mass eigenvalues $\mu^2_+$ and $\mu^2_-$. However, for $l = 0$ and $l = 1$, only one of them is physical and the other is a gauge mode. For $l \geq 2$, we denote the modes with the mass squared $\mu^2_+$ and $\mu^2_-$, as $l = 2, 3, \ldots$ and $l = 2, 3, \ldots$, respectively. In the $l = 1$ case the physical mode has the mass eigenvalue $\mu^2_+(l = 1)$. On the other hand, in the $l = 0$ case we have a physical mode with $\mu^2(l = 0) = 2\frac{(q-1)(p-2)}{p+q-2}b^2 - 2(p-1)\Lambda^2$, corresponding to $\mu^2_+$ for $\frac{(q-1)(p-2)}{p+q-2}b^2 > (p-1)\Lambda^2$ and $\mu^2_-$ for $\frac{(q-1)(p-2)}{p+q-2}b^2 < (p-1)\Lambda^2$.

This mass spectrum leads to the following result for dynamical stability of the FR solutions; when $h^2 > h^2_{c(l=0)}$, where

$$h^2_{c(l=0)} = \frac{2\Lambda(p-2)}{(p-1)(p+q-2)},$$

the $l = 0$ mode is tachyonic and the FR solution is unstable arising from homogeneous excitation of the sphere. In addition, for $q \geq 4$, when $h^2$ becomes smaller than the critical value $h^2_{c(l=2)}$ given by

$$h^2_{c(l=2)} = \frac{2\Lambda[(p-1)q^2 - (3p-1)q + 2]}{q(q-3)(p-1)(p+q-2)},$$

the mass squared $\mu^2_-$ is negative for $l = 2$ and the FR solution is unstable arising from inhomogeneous excitations. As we have explained above, we call this mode $l = 2$... Some modes with $l \geq 3$ can be unstable when $h^2$ is even smaller. The mass spectrum for the scalar perturbations of the Freund–Rubin solution is shown as blue dashed lines in Fig. 2.

Now, let us consider the warped solutions. Especially we would like to examine dynamical stability of the warped solutions in the small Hubble regime in which the FR solutions suffer from instability of the inhomogeneous excitations. For the warped solution we numerically solve the eigenvalue problem for a set of differential equations (20) with the solutions in the small Hubble regime in which the FR solution suffers from instability of the inhomogeneous excitations. As we have explained above, we call this mode $l = 2$... mode since it is the first unstable mode for inhomogeneous perturbations on the FR branch. Note that “$l = 2$” means quadrupole moment with respect to the SO($q+1$)-symmetry of the internal space and that, rigorously speaking, this terminology is valid only for the FR branch. However, as explained in the end of Sec. II, there is a critical value of $b^2$ at which the solution in the warped branch reduces to an unwarped, FR solution. Since $h^2$ is determined by $b^2$ in each branch, this implies that there is a critical value of $h^2$ at which the solution in the warped branch and that in the FR branch represent the same solution. Therefore, it makes perfect sense to define the “$l = 2$ mode” for the warped branch as the mode which approaches the $l = 2$ mode of the FR branch as $(b^2, h^2)$ approaches the critical value. We can define $l = 3, 4, \ldots$ modes for the warped branch in a similar way. As one can see in the left panel of Fig. 2 for $h^2 < h^2_{c(l=2)} = \Lambda/18$, $\mu^2$ for the $l = 2$ mode of the warped branch is positive, while the $l = 2$ mode of the FR branch becomes tachyonic. Thus the warped branch is stable configuration in the low Hubble region, in which the FR branch is unstable. Actually, within numerical accuracy one can see that the red solid line and the blue dashed line for $l = 2$ intersect at $\mu^2 = 0$. This implies that the critical value of $h^2$ at which the warped solution reduces to the FR solution agrees with (27).

The mass squared for some other modes as well as the $l = 2$ mode is shown in the right panel of Fig. 2. As seen from the mass spectrum, the warped branch has no unstable mode in the low Hubble region. In addition we notice that $\mu^2$ of the warped branch (the red solid line) is larger than that of the FR branch (the blue dashed line) for $h^2$ smaller than $h^2_{c(l=2)}$ (the left hand side). This means that deformation of the internal space and warping tend to stabilize the shape modulus of the internal space in the low Hubble region. In other words, the tachyonic shape modulus is stabilized by the condensation of the modulus itself.

### IV. THERMODYNAMIC STABILITY

In the previous section we have investigated dynamical stability of two branches of Freund–Rubin compactification. In this section we shall investigate thermodynamic stability of the same system and compare the results with dynamical stability.
FIG. 2: The mass spectrum for scalar perturbations: the $l = 2$ mode (left) and some modes with $l = 0, 1, 2, \pm 3$ (right). Red solid lines and blue dashed lines indicate the warped branch and the FR branch, respectively. The green vertical line indicates the critical value $h^2 = h^2_{c(l=2)} = \Lambda/18$ at which two branches merge. In the low Hubble region where the FR branch is unstable, the warped branch is stable.

### A. Thermodynamic relations

In Freund–Rubin compactifications we can define various physical quantities characterizing thermodynamic properties of the system. One of the most important among them is the de Sitter entropy $S$, which is defined by one quarter of the total area $A$ of de Sitter horizon. For the metric (28) it is given by

$$S \equiv \frac{A}{4} = \frac{\Omega_{p-2}\Omega_{q-1}}{4h^{p-2}} \int_{r_-}^{r_+} dr e^{-\frac{2(p+q-2)}{q-2}\phi} a^{q-1}. \quad (28)$$

Also, we can define the total flux of the $q$-form field (10) as

$$\Phi \equiv \oint F_{(q)} = b\Omega_{q-1} \int_{r_-}^{r_+} dr e^{-\frac{2(p+q-1)}{q-2}\phi} a^{q-1}, \quad (29)$$

which is a conserved quantity for this system.

Before examining thermodynamic stability, let us derive the first law of de Sitter thermodynamics for Freund–Rubin compactifications.

For this purpose it is convenient to consider variations of the Euclidean action for the system since the on-shell Euclidean action is directly related to the de Sitter entropy as we shall see below. Assuming $\text{SO}(p+1) \times \text{SO}(q)$ isometry, we can take the metric ansatz as

$$d s^2_{\text{Euclid}} = e^{2\phi(r)} h^{-2} d\Omega_p^2 + e^{-2\frac{p}{q-2}\phi(r)}[dr^2 + a^2(r) d\Omega_{q-1}^2], \quad (30)$$

where $d\Omega_p^2$ and $d\Omega_{q-1}^2$ denote the metrics of the unit round $p$- and $(q-1)$-sphere, respectively. The $q$-form field strength is given by

$$F_{(q)} = \psi'(r) dr \wedge d\Omega_{q-1}. \quad (31)$$

The Euclidean action is given by

$$I_{\text{Euclid}} = -\frac{1}{16\pi} \int d^{p+q}x_{\text{E}} \sqrt{g_{\text{E}}} \left( R - 2\Lambda - \frac{1}{q!} F_{(q)}^2 \right). \quad (32)$$

The Ricci scalar and the field strength are

$$R = e^{\frac{2p}{q-2}\phi} \left[ (q-1)(q-2) \frac{a'^2 + 1}{a^2} - \frac{p(p+q-2)}{q-2} \phi'^2 + p(p-1)h^2 e^{-\frac{2(p+q-2)}{q-2}\phi} \right]$$

$$- 2e^{\frac{2p}{q-2}\phi} a^{q-1} \left\{ a^{q-1} \left[ (q-1) \frac{a'}{a} - \frac{p}{q-2} \phi' \right] \right\}', \quad (33)$$
and

$$- \frac{1}{q!} F_{(q)}^2 = \frac{2^{q-1}}{a^{2(q-1)}} \psi^2.$$  \hspace{1cm} (34)

Hence we have the following expression for the Euclidean action

$$I_{\text{Euclid}}[a, \phi, \psi] = - \frac{\Omega_p \Omega_{q-1}}{16\pi\hbar^p} \int_{r_-}^{r_+} dr \left[ (q-1)(q-2) a^{q+2} + \frac{p(p+q-2)}{q-2} \phi' + p(p-1) b e^{-\frac{2p(q-1)}{q-2} \phi} \right] a^{q-1},$$  \hspace{1cm} (35)

where the boundary terms have vanished since the boundary conditions require \(a(r_\pm) = 0\), \(a'(r_\pm) = 1\) and \(\phi'(r_\pm) = 0\) at the boundaries \(r = r_\pm\). Using the equations of motion, we evaluate the Euclidean action \(I_{\text{Euclid}}\) on shell and it turns out that

$$I_{\text{Euclid}} = -S,$$  \hspace{1cm} (36)

as explained in appendix [13]. Also, the equation of motion for the form field is given by

$$\left[ e^{\frac{2p(q-1)}{q-2} \phi} \psi' \right]' = 0,$$  \hspace{1cm} (37)

and it can be easily integrated as

$$e^{\frac{2p(q-1)}{q-2} \phi} \psi' = b,$$  \hspace{1cm} (38)

where \(b\) is an integration constant. The total flux \(\Phi\) is rewritten as

$$\Phi = \Omega_{q-1} [\psi(r_+) - \psi(r_-)] = b \Omega_{q-1} \int_{r_-}^{r_+} dr \ e^{-\frac{2p(q-1)}{q-2} \phi} a^{q-1}.$$  \hspace{1cm} (39)

We are now ready to derive the first law of de Sitter thermodynamics in our setup. We consider the first variation of the action \(I_{\text{Euclid}}[a, \phi, \psi]\) with respect to \(a, \phi\) and \(\psi\). Suppose both \(a, \phi, \psi\) and \(a + \delta a, \phi + \delta \phi, \psi + \delta \psi\) are different sets of solutions satisfying the equations of motion, the first variation of the action \(\delta I_{\text{Euclid}}\) is given by

$$\delta I_{\text{Euclid}} = \frac{\Omega_p \Omega_{q-1}}{8\pi\hbar^p} \left[ e^{\frac{2p(q-1)}{q-2} \phi} \delta \psi \right]^{r_+}_{r_-} + \int_{r_-}^{r_+} dr \ (\text{EOM for } a, \phi \text{ and } \psi).$$  \hspace{1cm} (40)

The integrand in the last term will vanish because of the equations of motion for \(a, \phi\) and \(\psi\). As a result, only the boundary term contributes to the first variation of the action. By using (38) and (39) the first law can be derived as

$$dS = -\frac{\Omega_p b}{4(p-1)\hbar^p} d\Phi.$$  \hspace{1cm} (41)

This implies that the entropy is described by a function of the total flux. Hence the entropy \(S\) is a thermodynamic potential with respect to the total flux \(\Phi\) as a natural thermodynamic variable. We shall call a sequence of solutions which satisfies the first law a “branch” of solutions.

So far, we have assumed that the bulk cosmological constant \(\Lambda\) is not a dynamical variable but a given constant. However it is probable that \(\Lambda\) is a dynamical variable induced by dynamical fields such as a scalar field. In this case we consider \(\Lambda\) as an additional thermodynamic variable so that the de Sitter entropy is now a function of \(\Phi\) and \(\Lambda\), \(S(\Phi, \Lambda)\). The first law (41) is easily generalized to include variation of \(\Lambda\) as follows. First, dimensional analysis leads to the following scaling relation

$$S(\lambda^{-(q-1)/2} \Phi, \lambda \Lambda) = \lambda^{-(p+q-2)/2} S(\Phi, \Lambda).$$  \hspace{1cm} (42)

Second, taking derivative with respect to \(\lambda\) and setting \(\lambda = 1\), we obtain

$$-\frac{q-1}{2} \frac{dS}{d\Phi} + \Lambda \frac{dS}{d\Lambda} = -\frac{p+q-2}{2} S.$$  \hspace{1cm} (43)

This is the first law of de Sitter thermodynamics for \(S(\Phi, \Lambda)\):

$$dS = -\frac{\Omega_p b}{4(p-1)\hbar^p} d\Phi - \frac{p+q-2}{2} S + \frac{q-1}{2} \int_{r_-}^{r_+} dr \frac{\Omega_p b}{4(p-1)\hbar^p} \frac{d\Lambda}{\Lambda}.$$  \hspace{1cm} (44)
B. Stability

In this subsection we discuss thermodynamic stability of the FR branch and the warped branch. As we have seen, the entropy $S$ is the thermodynamic potential when we choose the total flux $\Phi$ as a natural variable. Therefore, the second law of thermodynamics states that, for a fixed value of the total flux, a configuration with larger entropy is thermodynamically favored.

To begin with, let us examine thermodynamic property of the FR branch. We shall see that the FR branch has two sub-branches and one of them is thermodynamically preferred than the other. In the FR branch, the entropy $S$ and the total flux $\Phi$ are given by

$$S = \frac{\Omega_p - q \Omega_q \rho^q}{4 h_p - 2}, \quad \Phi = b \Omega_q \rho^q$$

where we have used Eqs. (28) and (39) with $a(r) = \rho \sin \frac{\pi}{p}$ and $\phi(r) = 0$. Note that we can explicitly check that these quantities satisfy the first law (11). We find that the entropy is written as a function of the total flux and splits the FR branch into two sub-branches: a lower-entropy sub-branch and a higher-entropy sub-branch, as shown in Fig. 3. (For example, see [20].) Therefore, for a given total flux the higher-entropy sub-branch is preferred than the lower-entropy sub-branch within the FR branch.

The critical point dividing the FR branch into two sub-branches is determined as follows. As seen before, the FR branch satisfies the first law. However, the entropy $S(\Phi)$ is a double-valued function of the total flux $\Phi$. Nonetheless, the FR solutions can be described as an one-parameter family of solutions, for example, in terms of the Hubble expansion rate $h$ of the external de Sitter space. Actually, the entropy is a single-valued function of $h$. These facts mean that a map from $\Phi$ to $h$ becomes singular at the critical value. Hence we can obtain the critical point by solving $d\Phi/dh = dS/dh = 0$, which yields the critical value

$$h_{c(l=0)}^2 = \frac{2\Lambda(p-2)}{(p-1)^2(p+q-2)}, \quad h_{c(l=0)}^2 = \frac{(p-1)(p+q-2)}{(p-2)(q-1)} h_{c(l=0)}^2.$$  

By comparing with (26), it is easy to see that this thermodynamical critical point agrees with the threshold at which the $l = 0$ mode becomes massless on the FR branch. In addition, the lower-entropy sub-branch is dynamically unstable against homogeneous ($l = 0$) excitation of the internal space. Therefore, we see complete agreement between thermodynamic and dynamical stability of the two sub-branches.

In the previous paragraphs we have compared entropies of two sub-branches within the FR branch. We now compare entropies of the FR branch and the warped branch. For simplicity we shall consider the case with $p = 4$ and $q = 4$ as an explicit example.

We denote the entropy of the FR branch and that of the warped branch as $S_{FR}$ and $S_w$, respectively. A difference between $S_w$ and $S_{FR}$ for various values of the total flux is shown in Fig. 4. It turns out that for $\Phi < 32\sqrt{3}\pi^2 \Lambda^{-3/2}$, or equivalently, for

$$h^2 < \frac{\Lambda}{18},$$

the warped branch has larger entropy and thus is thermodynamically stable. On the other hand, for $\Phi > 32\sqrt{3}\pi^2 \Lambda^{-3/2}$, or equivalently, for $h^2 > \frac{\Lambda}{18}$, the FR branch has larger entropy and is thermodynamically stable. At the critical point where two branches merge the solution is marginally stable.

It is worth noting that the thermodynamic stability investigated here agrees with the dynamical stability examined in Sec. 111. The FR branch has dynamical instability arising from inhomogeneous ($l = 2$) excitation when $h^2 < h_{c(l=2)}^2$, where $h_{c(l=2)}^2$ is given by (27). On the other hand, the warped branch is dynamically unstable for $h^2 > h_{c(l=2)}^2$. For $p = 4$ and $q = 4$, the critical value $h_{c(l=2)}^2$ for the dynamical stability agrees with the critical value for the thermodynamic stability given in the right hand side of (47). Therefore, we again see complete agreement between thermodynamic and dynamical stabilities.

V. COSMOLOGICAL IMPLICATIONS

In previous sections we have investigated stability of two branches of flux compactification from thermodynamic and dynamical perspectives. One branch has higher symmetry SO($q+1$), where $q$ is the number of extra dimensions, and corresponds to unwarped, Freund–Rubin solutions [1]. The other branch has lower symmetry SO($q$) and corresponds to warped solutions found recently by one of the authors [4]. By fixing or scaling out the higher-dimensional cosmological
FIG. 3: The entropy of the FR branch and the warped branch as functions of the total flux $\Phi$. Difference between entropies of the two branches is so small that two lines are indistinguishable in this figure. See Fig. 4 for the difference.

FIG. 4: The difference between the entropy of the warped branch $S_w$ and that of the FR branch $S_{FR}$.

constant, each branch is parameterized by one parameter, either total flux $\Phi$ of an antisymmetric field or the Hubble expansion rate $h$ of the 4-dimensional de Sitter metric. We have seen that the unwarped branch is dynamically stable for $h$ larger than a critical value $h_c(l=2)$ but unstable for smaller values. On the other hand, the warped branch is dynamically unstable for $h > h_c(l=2)$ and stable for $h < h_c(l=2)$. To investigate thermodynamic perspective, we have defined the total de Sitter entropy $S$ as the 4-dimensional de Sitter entropy integrated over extra dimensions. We have shown that the dynamically stable branch, i.e. the unwarped (or warped) branch for $h > h_c(l=2)$ (or $h < h_c(l=2)$, respectively), always has larger total de Sitter entropy than the dynamically unstable branch. Therefore, thermodynamic stability agrees with dynamical stability.

In this section we consider cosmological implications of these results. For this purpose we shall first define the 4-dimensional Einstein frame. For the $(4 + q)$-dimensional metric of the form

\[ G_{MN} dX^M dX^N = A^2(x,y) g_{\mu\nu}(x) dx^\mu dx^\nu + q_{mn}(y) dy^m dy^n, \]

the higher-dimensional Einstein-Hilbert action is

\[ I_{4+q} = \frac{(M_{4+q})^{2+q}}{2} \int d^{4+q} X \sqrt{-G} R[G] = \frac{M_4^2}{2} \int d^4 x \sqrt{-g} \Omega^2 R[g] + \cdots, \]

where $M_{4+q}$ and $M_4$ are $(4 + q)$- and 4-dimensional Planck scales (in this section we have temporarily restored the Planck scales), and

\[ \Omega^2 = \frac{(M_{4+q})^{2+q}}{M_4^2} \int d^q y \sqrt{q} A^2. \]
Since $\Omega$ in general depends on the 4-dimensional coordinates $x^\mu$, the resulting 4-dimensional effective theory describing $g_{\mu\nu}$ is not Einstein but a scalar-tensor theory. It is convenient to define the 4-dimensional Einstein frame $g_{\mu\nu}^{(E)}$ by

$$g_{\mu\nu}^{(E)} = \Omega^2 g_{\mu\nu},$$

in terms of which $I_{4+q}$ now includes the 4-dimensional Einstein-Hilbert term:

$$I_{4+q} = \frac{M_4^2}{2} \int d^4x \sqrt{-g^{(E)}} R[g^{(E)}] + \cdots.$$  \tag{51}

Suppose that the 4-dimensional metric $g_{\mu\nu}$ represents a de Sitter spacetime with the Hubble expansion rate $h$. The corresponding Einstein frame metric $g_{\mu\nu}^{(E)}$ has the Hubble expansion rate

$$h_E = \Omega^{-1} h,$$  \tag{52}

and the de Sitter entropy

$$S_E = 8\pi^2 M_4^2 h_E^{-2} = \frac{8\pi^2 (M_4)^2 + q}{h^2} \int d^4q \sqrt{qA^2}.$$  \tag{53}

Actually, this agrees with the total de Sitter entropy \cite{28}:

$$S_E = S.$$  \tag{54}

For cosmological considerations energy density is more convenient than de Sitter entropy since the former can easily be extended to a general FRW universe. The effective energy density in the Einstein frame is

$$\rho_E = 3M_4^2 h_E^{-2} = \frac{3M_4^2 h^2}{(M_4 + q)^2} \left( \int d^4q \sqrt{qA^2} \right)^{-1}.$$  \tag{55}

This is related to the total de Sitter entropy as

$$\frac{\rho_E}{3M_4^2} = \frac{8\pi^2}{S}.$$  \tag{56}

Therefore, the results of previous sections are restated in terms of the effective energy density $\rho_E$ in the Einstein frame as follows: dynamically stable branch, i.e. the unwarped (warped) branch for $h > h_{\xi(=2)}$ ($h < h_{\xi(=2)}$), always has lower $\rho_E$ than the dynamically unstable branch. This strongly suggests that a solution in the dynamically unstable branch should evolve to a solution in the dynamically stable branch. The latter solution is uniquely specified by the former since the flux $\Phi$ conserves. By this evolution, $\rho_E$ decreases.

Moreover, the results of the previous sections suggest a new type of phase transition. Suppose that $\Lambda$ is not the genuine constant but has a contribution from dynamical fields such as a scalar field. In this case $\Lambda$ is expected to decrease while the flux $\Phi$ stays constant. (See Fig. 5.) If we start with $h > h_{\xi(=2)}$ then the unwarped branch has lower energy density and is stable. Thus a solution in the unwarped branch should be realized initially. However, as $\Lambda$ decreases, $h$ also decreases and can reach the critical value $h_{\xi(=2)}$. At that point, the stable and unstable branches merge. After that, for $h < h_{\xi(=2)}$, solutions in the warped branch should be realized since this branch has lower energy density and is stable. This phase transition should be second-order since only one of the two branches is stable at a given time.

Second-order phase transitions play important roles in cosmology. For example, the end of hybrid inflation is due to second-order phase transition. Thus, this kind of phase transition may provide a new way of realizing hybrid inflation in higher-dimensional theories. This possibility will be investigated in future publications.

VI. DISCUSSION

In this paper we have investigated stability of two branches of Freund–Rubin compactification from two perspectives; one is thermodynamic stability based on de Sitter entropy, and the other is dynamical stability of linear perturbations around the background solutions.

We have analyzed linear perturbations around the warped solutions in order to examine dynamical stability of the solutions. The warped solutions are stable if the Hubble expansion rate of the external de Sitter spacetime is low
FIG. 5: The difference of the effective energy density between the warped branch and the FR branch with a fixed total flux.

enough. In the same regime of the Hubble expansion rate, the Freund–Rubin solutions have instability arising from the \( l = 2 \) mode. It follows from what has been said thus far that deformation of the internal space and warping will stabilize unstable configurations. This may be considered as spontaneous breaking of the symmetry of internal space in the sense that less symmetric configurations are dynamically chosen. Actually, for the reason explained below, this phenomenon is natural from gravitational viewpoints. In cosmology it is well known that configurations with high matter density suffer from gravitational instability due to long-wavelength modes, namely Jeans instability. Jeans instability develops inhomogeneities in the universe and results in structure formation. In the case of FR compactification, Einstein equation tells that the external space with small Hubble expansion rate corresponds to the internal space with large flux density. Therefore, it is natural to expect that when the Hubble expansion rate is low enough, the flux distribution may become inhomogeneous due to analogue of the Jeans instability.

In order to analyze thermodynamic properties, we have first derived the first law of de Sitter thermodynamics in terms of entropy and total flux. Sequence of solutions belonging to a branch obey this first law when those parameters characterizing the solutions change. Since the entropy is a natural thermodynamic potential for the total flux, for a given total flux, configuration with higher entropy should be favored thermodynamically. For \( p = 4 \) and \( q = 4 \) we have compared the entropy of the FR branch with that of the warped branch. There is a critical value of the Hubble expansion rate at which two branches merge and it is thus obvious that the entropies of the two branches agree at the critical value. For smaller Hubble expansion rate, the entropy of the warped branch is larger than that of the FR branch and thus the warped branch is thermodynamically favored. On the other hand, for larger Hubble expansion rate, the FR branch is thermodynamically favored. We found complete agreement of thermodynamic stability and dynamical stability.

It is intriguing to see that correlation between thermodynamic stability and dynamical stability exists for Freund–Rubin flux compactification. For a certain class of black objects the existence of such correlation has been known, that is the so-called Gubser–Mitra conjecture. This conjecture has been explicitly checked to hold for various black strings and branes. From what has been discussed above, it is probably natural to expect that the concept of correlated stability can be extended to a wider class of gravitating systems.

Acknowledgments

We would like to thank Jiro Soda, Masaru Shibata, Takahiro Tanaka and Tetsuya Shiromizu for valuable comments. We are deeply grateful to Katsuhiro Sato for his continuous support. The work of SK was in part supported by JSPS through a Grant-in-Aid. The work of SM was supported in part by MEXT through a Grant-in-Aid for Young Scientists (B) No. 17740134, by JSPS through a Grant-in-Aid for Creative Scientific Research No. 19GS0219 and through a Grant-in-Aid for Scientific Research (B) No. 19340054, and by the Mitsubishi Foundation. This work was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.
APPENDIX A: GAUGE

In the background spacetime given by

\[ g_{MN}dx^M dx^N = A^2(y)g_{\mu\nu}dx^\mu dx^\nu + dy^2 + B^2(y)\gamma_{ij}dz^i dz^j, \]  

where \( g_{\mu\nu} \) and \( \gamma_{ij} \) are respectively the metric of \( p \)-dimensional de Sitter space and \((q-1)\)-dimensional round sphere, we consider scalar-type perturbations,

\[
\delta g_{\mu\nu} = 2A^2 \left( \nabla_\mu \nabla_\nu - \frac{1}{p} g_{\mu\nu} \nabla^2 \right) H^{(\text{LL})} + A^2 g_{\mu\nu} H^{(Y)}, \\
\delta g_{\mu\nu} = \nabla_\mu H^{(L)}, \\
\delta g_{ij} = 2B^2 \left( D_i D_j - \frac{1}{q-4} \gamma_{ij} B^2 \right) h^{(\text{LL})} + B^2 \gamma_{ij} h^{(Y)}, \\
\delta g_{yy} = h_{yy}, \\
\delta g_{\mu\nu} = \nabla_\mu D_\nu \eta.
\]

Note that \( \nabla_\nu \) and \( D_i \) denote covariant derivatives associated with \( g_{\mu\nu} \) and \( \gamma_{ij} \), respectively.

These components transform as

\[
H^{(\text{LL})} \rightarrow H^{(\text{LL})} - A^{-2} \xi^{(L)}, \quad H^{(Y)} \rightarrow H^{(Y)} - A^{-2} \frac{2}{p} \nabla^2 \xi^{(L)} - 2 \frac{A'}{A} \xi_y, \quad H^{(L)} \rightarrow H^{(L)} - \xi_y - A^2 (A^{-2} \xi^{(L)})', \\
h^{(\text{LL})} \rightarrow h^{(\text{LL})} - B^{-2} \xi^{(L)}, \quad h^{(Y)} \rightarrow h^{(Y)} - B^{-2} \frac{2}{q-1} D^2 \xi^{(L)} - 2 \frac{B'}{B} \xi_y, \quad h^{(L)} \rightarrow h^{(L)} - \xi_y - B^2 (B^{-2} \xi^{(L)})', \\
h_{yy} \rightarrow h_{yy} - 2 \xi_y', \quad \eta \rightarrow \eta - \xi^{(L)} - \xi^{(L)},
\]

By setting

\[
\xi^{(L)} = A^2 H^{(\text{LL})}, \quad \xi_y = H^{(L)} - A^2 (H^{(\text{LL})})',
\]

and assuming that the perturbations does not depend on \( z^i \)-coordinates, we can simplify the form of metric perturbations so that non-vanishing components are

\[
\delta g_{\mu\nu} = A^2 g_{\mu\nu} H^{(Y)}, \quad \delta g_{ij} = B^2 \gamma_{ij} h^{(Y)}, \quad \delta g_{yy} = h_{yy},
\]

and the other components vanish. In addition, using parts of the linearized Einstein equations for scalar-type perturbations we have an algebraic relation

\[
(p-2)H^{(Y)} + h_{yy} + (q-1)h^{(Y)} = 0.
\]

Finally, we expand the perturbations in harmonics \( Y(x) \) on the \( p \)-dimensional de Sitter space \( g_{\mu\nu} \), and then we define two variables \( \Pi(y) \) and \( \Omega(y) \) as

\[
H^{(Y)} = \Pi Y(x), \quad h_{yy} = (\Pi - \Omega) Y(x), \quad h^{(Y)} = \left( \frac{\Omega}{q-1} - \frac{p-1}{q-1} \Pi \right) Y(x).
\]

APPENDIX B: EUCLIDEAN ACTION AND ENTROPY

The Euclidean action is given by

\[
I_{\text{Euclia}}[a,\phi,\psi] = - \frac{\Omega_p \Omega_{q-1}}{16\pi h^p} \int dr \left[ (q-1)(q-2)\frac{a'^2}{a^2} + \frac{p(p+q-2)}{q-2} \phi'^2 + p(p-1)h^2 e^{-\frac{2(q^2-2)}{q-2} \phi} \\
- 2\Lambda e^{-\frac{2(q-1)}{q-2} \phi} - \frac{e^{\frac{2(q-1)}{q-2} \phi}}{a^2(q-1)} \psi'^2 \right] a^{q-1} \\
= - \frac{\Omega_p \Omega_{q-1}}{16\pi h^p} \int dr \mathcal{L}(a, a', \phi, \phi', \psi').
\]
Here, since this action is invariant under the following transformation:

\[ h \rightarrow \lambda h, \quad \phi \rightarrow \phi + \ln \lambda, \quad a \rightarrow \lambda^{p/(q-2)} a, \quad r \rightarrow \lambda^{p/(q-2)} r, \quad \psi \rightarrow \psi; \]

(B2)

where \( \lambda \) is an arbitrary constant, we have an identity

\[
\left[ \frac{p}{q-2} \frac{\partial \mathcal{L}}{\partial a'} + \frac{\partial \mathcal{L}}{\partial \phi'} \right]' - p\mathcal{L} + 2p(p-1)h^2 e^{-2(e+p-2)/q-2} \phi a^{q-1} = 0,
\]

(B3)

provided that \( a(r), \phi(r) \) and \( \psi(r) \) satisfy the equations of motion. Hence the on-shell Euclidean action can be written as

\[
I_{\text{Euclid}} = -\frac{\Omega_p \Omega_{q-1}}{16\pi h^p} \int dr \mathcal{L}(a, a', \phi, \phi', \psi')
\]

(B4)

\[
= -\frac{\Omega_p \Omega_{q-1}}{8\pi h^{p-2}} \int dr (p-1)e^{-2(e+p-2)/q-2} \phi a^{q-1} \equiv -S,
\]

where the boundary terms vanish because of the boundary conditions. It is easy to show by using \( (p-1)\Omega_p = 2\pi \Omega_{p-2} \) that \( S \) agrees with the de Sitter entropy defined in (28).

[1] P. G. O. Freund and M. A. Rubin, Phys. Lett. B 97, 233 (1980).
[2] R. Bousso, O. DeWolfe and R. C. Myers, Found. Phys. 33, 297 (2003) [arXiv:hep-th/0205080].
[3] J. U. Martin, JCAP 0504, 010 (2005) [arXiv:hep-th/0412111].
[4] T. Torii and T. Shiromizu, Phys. Lett. B 551, 161 (2003) [arXiv:hep-th/0210002].
[5] C. Krishnan, S. Paban and M. Zanic, JHEP 0505, 045 (2005) [arXiv:hep-th/0503025].
[6] S. Kinoshita, Phys. Rev. D 76, 124003 (2007) [arXiv:0710.0707 [hep-th]].
[7] S. S. Gubser and I. Mitra, JHEP 0108, 018 (2001) [arXiv:hep-th/0011127].
[8] S. S. Gubser and I. Mitra, arXiv:hep-th/0009126.
[9] H. S. Reall, Phys. Rev. D 64, 044005 (2001) [arXiv:hep-th/0104071].
[10] T. Prestidge, Phys. Rev. D 61, 084002 (2000) [arXiv:hep-th/9907163].
[11] J. P. Gregory and S. F. Ross, Phys. Rev. D 64, 124006 (2001) [arXiv:hep-th/0106220].
[12] V. E. Hubeny and M. Rangamani, JHEP 0205, 027 (2002) [arXiv:hep-th/0202189].
[13] T. Hirayama, G. Kang and Y. Lee, Phys. Rev. D 67, 024007 (2003) [arXiv:hep-th/0209181].
[14] U. Miyamoto and H. Kudoh, JHEP 0612, 048 (2006) [arXiv:gr-qc/0609046].
[15] U. Miyamoto, Phys. Lett. B 659, 380 (2008) [arXiv:0709.1028 [hep-th]].
[16] Y. Brihaye, T. Delsate and E. Radu, Phys. Lett. B 662, 264 (2008) [arXiv:0710.4034 [hep-th]].
[17] S. Chen, K. Schleich and D. M. Witt, Phys. Rev. D 78, 126001 (2008) [arXiv:0809.1357 [hep-th]].
[18] B. Kol, Phys. Rept. 422, 119 (2006) [arXiv:hep-th/0411240].
[19] T. Harmark, V. Niarchos and N. A. Obers, Class. Quant. Grav. 24, R1 (2007) [arXiv:hep-th/0701022].
[20] S. Kinoshita, Y. Sendouda and S. Mukohyama, JCAP 0705, 018 (2007) [arXiv:hep-th/0703271].