Bipartite post-quantum steering in generalised scenarios

Ana Belén Sainz,1,2 Matty J. Hoban,3 Paul Skrzypczyk,4 and Leandro Aolita5

1Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, Ontario, Canada, N2L 2Y5
2International Centre for Theory of Quantum Technologies, University of Gdańsk, 80-308 Gdańsk, Poland
3Department of Computing, Goldsmiths, University of London, London SE14 6NW, UK
4H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol, BS8 1TL, UK
5Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, 21941-972 Rio de Janeiro, RJ, Brazil

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In the traditional bipartite steering scenario, the assemblages (ensembles of ensembles) into which Alice remotely steers Bob’s system may always be explained via quantum theory. This is a renowned result by Gisin, Hughston, Jozsa and Wootters. Here we ask the question of whether, by suitably relaxing the traditional steering setup, it is possible to have steering that is incompatible with quantum theory in a bipartite setting. First we prove a new no-go result in the ‘sequential steering’ setup, where Alice performs a sequence of measurements, by demonstrating that such experiments can still always be explained in quantum theory. We then exhibit two scenarios where it is possible to find post-quantum steering: (i) where Bob also has an input, which informs the preparation of his local quantum system, and (ii) where the causal relations between the parties corresponds to the ‘instrumental steering’ scenario. We finally show that post-quantum steering in these two scenarios does not necessarily follow from post-quantum black-box correlations in the corresponding setups, rendering the phenomenon discovered a genuinely new type of post-quantum nonlocality.

Introduction.— Einstein-Podolsky-Rosen steering is a striking nonlocal feature of Nature present in quantum theory [1, 2]. It refers to the phenomenon where one party, Alice, by performing measurements on one half of a shared system, seemingly remotely ‘steers’ the states held by a distant party, Bob, in a way which has no classical explanation. Quantum steering was first discussed by Schrödinger [1], and recently reinterpreted with a quantum information approach [2]. This modern approach to steering describes it as a way to certify entanglement in situations where Alice’s devices are uncharacterised or untrusted. From this perspective, steering allows for a “one-sided device independent” implementation of several information-theoretic tasks, such as quantum key distribution [3], randomness certification [4, 5], measurement incompatibility certification [6–8], and self-testing of quantum states [9, 10].

Abstractly, we may view the steering scenario as one where Alice has a device that accepts a classical input, x, usually thought of as labelling the choice of measurement, and produces a classical outcome, α, usually thought of as the measurement result, while Bob has a device without an input, that produces a quantum outcome, ρα|x, correlated with the input and outcome of Alice, and usually thought of as the steered state. From this perspective, it becomes natural to ask the question of whether there could be steering beyond quantum theory. That is, could it be possible to find a pair of devices for Alice and Bob which could not be produced by Alice and Bob sharing a quantum state, upon which Alice performs quantum measurements labelled by x and with outcomes α? The only requirements that we wish to impose on the devices is that of relativistic causality – that the devices do not allow for signalling from Alice to Bob. In particular, if we conceive of hypothetical situations where the local structure of quantum theory is maintained, but the global structure is not – for instance having more general states or dynamics globally – a natural question is what observable nonlocal effects this might lead to, in the form of post-quantum steering.

A celebrated theorem by Gisin [11] and Hughston, Jozsa and Wootters [12] (GHJW) shows that in fact, post-quantum steering cannot occur. Namely, any pair of devices that satisfies the no-signalling constraints from Alice and Bob can always be realised by some carefully chosen set of measurements on a carefully chosen quantum state. This traditional bipartite steering scenario is however not the only interesting scenario where one can see the general effect of steering. We can think more generally that steering is about correlations between devices that have only classical inputs and outcomes, and separated devices that produce quantum states. In [13], using this perspective, post-quantum multi-partite steering was demonstrated. In particular, it was shown that in a tri-partite scenario, when Alice and Bob have devices where the inputs and outcomes are classical, and Charlie’s device produces a quantum state, there are correlations that are consistent with relativistic causality, but which provably cannot arise in quantum theory, i.e., it is impossible to find a tripartite entangled states and measurements for Alice and Bob that reproduce the observed correlations. Unified frameworks for the study of quantum and post-quantum steering in the multipartite setting have subsequently been found, providing a playground for exploration of this fascinating effect [14, 15].

A key question that however remained open is whether it is possible to have post-quantum steering in a suitable bipartite generalised scenario. In this work, in order to provide an answer to this question, we explore three natural bipartite generalisations of the traditional steering scenario: (i) One where Alice performs a sequence of measurements on her untrusted black-box device while Bob remains passive; (ii) another in which Alice performs a single measurement, as in the conventional scenario, but where Bob has a random input that is used to prepare a quantum system; (iii) and finally...
one similar to (ii) but where Bob’s preparations are now pos-
sibly conditioned on Alice’s measurement outcome instead.
Generalisation (iii) corresponds to a specific type of setup, known as instrumental causal networks, that are ubiquitous in causal inference [23, 24]. Here we show that the cele-
bra ted GHJW theorem extends to case (i), thus ruling out post-
quantum steering in an even more general setting than previ-
ously known. For the other two scenarios post-quantum steer-
ing is shown to be possible. Finally, we show that the types of post-quantum steering we discover constitute a genuinely
distinct effect from post-quantum device-independent corre-
lations in the corresponding generalised setups, and present examples of the former which do not feature the latter.

Preliminaries.— In the traditional bipartite quantum steer-
ing scenario (see Fig. 1 (a)) Alice and Bob share a system in a possibly entangled quantum state \( \rho \). Alice is al-
lowed to perform generalised measurements on her share of the system, which correspond to positive-operator val-
ued measures (POVM). Alice chooses one such measure-
ment \( \{ M_{a|x} \}_a \), labelled by \( x \), from a set of measure-
ments, and obtains an outcome \( a \) with probability \( p(a|x) = \text{tr} \left( (M_{a|x} \otimes I_B)\rho \right) \). After the measurement, Bob’s steered state is \( \rho_{a|x} = \text{tr}_A \left( (M_{a|x} \otimes I_B)\rho \right) / p(a|x) \). It is conve-
nient to work with the unnormalised steered states \( \sigma_{a|x} = p(a|x) \rho_{a|x} = \text{tr}_A \left( (M_{a|x} \otimes I_B)\rho \right) \), which contain both the information about both Alice’s conditional probabilities \( p(a|x) = \text{tr} \{ \sigma_{a|x} \} \), and Bob’s conditional states \( \rho_{a|x} \). The collection \( \{ \sigma_{a|x} \}_{a,x} \) of unnormalised states Bob is steered into is called an assemblage. Due to the completeness rela-
tion for Alice’s measurements, \( \sum_a M_{a|x} = \mathbb{I} \) for all \( x \), it fol-
lows that \( \sum_a \sigma_{a|x} = \text{tr}_A \{ \rho \} = \rho_2 \), independent of \( x \). This can be seen as a no-signalling condition from Alice to Bob, since Bob, without knowledge of the outcome of Alice, has no information about the choice of measurement she made.

Generalisation 1: sequential steering.— One natural general-
isation of the traditional steering scenario is to allow Alice to make a sequence of measurements on her share of the system, such that each measurement has the potential to steer the state of Bob. For clarity in the presentation, we will focus on the case depicted in Fig. 1 (b), where Alice makes two measurements, with general case of an arbitrary number of measurements following by induct-
ion. In this scenario, Alice first chooses to make a measure-
ment labelled by \( x_1 \), with outcomes labelled by \( a_1 \). Since we will need the post-measurement state, we must spec-
ify the Kraus operators \( \{ K_{a_1|x_1} \}_a \) such that \( M_{a_1|x_1} = K_{a_1|x_1}^\dagger K_{a_1|x_1} \), and not just the POVM elements \( M_{a_1|x_1} \). Alice then chooses to perform a second measurement, labelled by \( x_2 \), with outcome \( a_2 \), with corresponding POVM elements \( \{ M_{a_2|x_2} \}_a \) [16]. In this case, Bob’s assemblage has elements \( \sigma_{a_1,a_2|x_1,x_2} = \text{tr}_A \left( \{ K_{a_1|x_1} M_{a_2|x_2} K_{a_1|x_1} \otimes I_B \} \rho \right) \) such that the probabilities for Alice’s pair of measurement outcomes are \( p(a_1,a_2|x_1,x_2) = \text{tr} \{ \sigma_{a_1,a_2|x_1,x_2} \} \). This leads us to the fol-
lowing definition:

Definition 1. Quantum sequential assemblages.
An assemblage \( \{ \sigma_{a_1,a_2|x_1,x_2} \}_{a_1,a_2,x_1,x_2} \) has a quantum realisa-
tion in the sequential steering scenario (with two measure-
ments), if there exists a Hilbert space \( \mathcal{H}_A \) for Alice, Kraus op-
erators \( \{ K_{a_1|x_1} \}_a \) and POVMs \( \{ M_{a_2|x_2} \}_a \) for Alice, and a state \( \rho \) in \( \mathcal{H}_A \otimes \mathcal{H}_B \), such that

\[
\sigma_{a_1,a_2|x_1,x_2} = \text{tr}_A \left( \{ K_{a_1|x_1} M_{a_2|x_2} K_{a_1|x_1} \otimes I_B \} \rho \right).
\]

We denote this set of assemblages as \( \mathcal{Q}_S \).

We are then interested in thinking about this scenario more abstractly than from the point of view of quantum theory. That is, we would like to think about the most general type of corre-
lation that could arise in this sequential scenario, potentially beyond what quantum theory predicts. For this, thus, we need to find the corresponding no-signalling conditions that apply. Here, we still have no-signalling between Alice and Bob, just as before, but now taking into account all the information on
Alice’s side: that is, \( \sum_{a_1,a_2} \sigma_{a_1,a_2|x_1,x_2} \) is independent of \( x_1 \) and \( x_2 \). Moreover, we have an additional no-signalling con-
straint, from the future to the past within Alice’s lab: namely, with all the information available after the measurement \( x_1 \) (to Alice and Bob), it should be impossible to infer \( x_2 \). This means that \( \sum_{a_2} \sigma_{a_1,a_2|x_1,x_2} \) must be independent of \( x_2 \). We thus naturally arrive at the following definition for the most general non-signalling assemblage.

Definition 2. Non-signalling sequential assemblages.
An assemblage \( \{ \sigma_{a_1,a_2|x_1,x_2} \}_{a_1,a_2,x_1,x_2} \) is non-signalling in the steering scenario (with two sequential measure-
ments) iff \( \sigma_{a_1,a_2|x_1,x_2} \geq 0 \) for all \( a_1,a_2,x_1,x_2 \), and

\[
\sum_{a_1,x_2} \sigma_{a_1,a_2|x_1,x_2} = \sum_{a_1,a_2} \sigma_{a_1,a_2|x_1',x_2'} \quad \forall x_1, x_1', x_2, x_2', \quad \forall \sigma a_1 \sigma a_2 \sigma x_1 \sigma x_2 \sigma x_1' \sigma x_2' \quad (2)
\]

\[
\sum_{a_2} \sigma_{a_1,a_2|x_1,x_2} = \sum_{a_2} \sigma_{a_1,a_2|x_1,x_2} \quad \forall a_1, x_1, x_2, x_2'.
\]

We denote the set of such assemblages as \( \mathcal{G}_S \).
The choice of notation $G_S$ comes from thinking of these assemblages as the most general ones compatible with the no-signalling constraints of the setup.

The first question we consider is then whether every non-signalling sequential assemblage is also a quantum sequential assemblage. That is, whether the sequential relaxation of the traditional steering scenario allows for post-quantum steering. As a first main result, we prove that this is not the case:

**Theorem 1.** The set of all non-signalling sequential assemblages coincides with the set of quantum sequential assemblages, $Q_S = G_S$. That is, there is no post-quantum steering in the bipartite sequential steering scenario.

The proof of this theorem (in the generalised case, with $n$ measurements in sequence) is given in the Supplemental Material. The basic idea is to extend the standard GHJW construction working sequentially through the measurements of Alice in the order they are performed, using the Lüders rule to specify their Kraus operators. This demonstrates that the GHJW theorem generalises to the sequential scenario and rules out post-quantum steering there. We now move on to two other scenarios that do allow for post-quantum steering.

**Generalisation 2:** Bipartite steering when Bob has an input. We consider now the generalisation where Bob’s device also accepts an input before producing a quantum state. Intuitively, we can think that this input may determine the preparation of some quantum system, which could come about from a transformation on a quantum system inside Bob’s device. This situation is depicted in Fig. 1 (c), where $y$ denotes the input. In this generalised scenario, the members of the assemblage will be $\{\sigma_{a|xy}\}_{a,x,y}$.

In the context of quantum theory, we again assume that Alice and Bob share a quantum state $\rho$ and that Alice performs measurements labelled by $x$, as in the standard scenario. Given that Bob now has an input, the most general operation that he could apply is a Completely-Positive and Trace-Preserving (CPTP) channel onto their part of the quantum system. Thus, the quantum assemblages that can be generated are:

**Definition 3.** Quantum Bob-with-input assemblages. An assemblage $\{\sigma_{a|xy}\}_{a,x,y}$ has a has a quantum realisation in the steering scenario where Bob has an input if and only if there exists a Hilbert space $H_A$ and POVMs $\{M_{a|x}\}_{a,x}$ for Alice, a state $\rho$ in $H_A \otimes H_B$, and a collection of CPTP maps $\{E_y\}_y$ in $H_B$ for Bob, such that

$$\sigma_{a|xy} = E_y \left[ tr_A \{ (M_{a|x} \otimes I) \rho \} \right].$$

We denote this set of assemblages as $Q_{BI}$.

To go beyond quantum theory, we have to identify the most general constraints that apply here. Not only must we now ensure no-signalling from Alice to Bob, but since Bob has an input, we must also ensure no-signalling from Bob to Alice. These constraints are captured by the following definition:

**Definition 4.** Non-signalling Bob-with-input assemblages. An assemblage $\{\sigma_{a|xy}\}_{a,x,y}$ is non-signalling in the scenario where Bob has an input if $\sigma_{a|xy} \geq 0$ for all $a, x, y$, and

$$\sum_a \sigma_{a|xy} = \sum_a \sigma_{a|x'y} \quad \forall x, x', y,$$  

$$tr \{ \sigma_{a|xy} \} = p(a|x) \quad \forall a, x, y,$$  

$$tr \sum_a \sigma_{a|xy} = 1 \quad \forall x, y,$$

where $p(a|x)$ is the probability that Alice obtains outcome $a$ when performing measurement $x$ on her share of the system. We denote the set of such assemblages as $G_{BI}$.

We can now return to our central question of whether there can exist post-quantum steering in this scenario. Here we find that this is indeed the case, as stated in the following theorem:

**Theorem 2.** The set of all non-signalling Bob-with-input assemblages is strictly larger than the set of quantum Bob-with-input assemblages, $Q_{BI} \neq G_{BI}$. Hence, there is post-quantum steering in the Bob-with-input steering scenario.

**Proof.** We construct an explicit example of a assemblage in $G_{BI}$ which cannot be realised in quantum theory.

Consider the specific scenario where Alice has binary inputs and outcomes, $x \in \{0, 1\}$ and $a \in \{0, 1\}$. Bob has a binary input $y \in \{0, 1\}$, and the dimension of Bob’s Hilbert space is 2. Consider the following assemblage:

$$\sigma_{a|xy}^* := \frac{1}{2} (|a\rangle \langle a| + |a \oplus 1\rangle \langle a \oplus 1|) \delta_{xy}$$

Note that (i) $\sigma_{a|xy}^* \geq 0$ for all $a, x, y$; (ii) $\sum_a \sigma_{a|xy}^* = \frac{1}{2} (\delta_{xy} = 0 + \delta_{xy} = 1) I = \frac{1}{2} I$, which is independent of $x$ and $y$; (iii) $tr \{ \sigma_{a|xy}^* \} = \frac{1}{2} (\delta_{xy} = 0 + \delta_{xy} = 1) = \frac{1}{2}$ is independent of $y$ and (iv) $\sum_a \sigma_{a|xy}^* = 1$. This shows that $\sigma_{a|xy}^*$ is a valid no-signalling assemblage, i.e., $\{\sigma_{a|xy}^*\}_x \in G_{BI}$.

Now we show that this assemblage cannot arise in quantum theory, i.e., $\{\sigma_{a|xy}^*\}_x \not\in Q_{BI}$. We do so by first noting that for a quantum-realisable assemblage, since $(I_A \otimes E')[\rho]$ is a bipartite quantum state when $E'$ is a CPTP channel, Alice and Bob can only produce quantum Bell correlations, should Bob choose to measure his system. Namely, let Bob make an arbitrary measurement $\{N_b\}_b$ on his state in a quantum assemblage $\{\sigma_{a|xy}\}_{a,x,y}$. Then, the correlations obtained are

$$p(a, b|x, y) = tr_B \{ N_b \sigma_{a|xy} \} = tr_B \{ N_b E_y \left[ tr_A \{ (M_{a|x} \otimes I) \rho \} \right] \} = tr \{ (M_{a|x} \otimes E_y^b(N_b)) \rho \}.$$
Let $N_b = |b\rangle \langle b|$ be the computational basis measurement. The correlations that Alice and Bob obtain are

$$p(a, b|x, y) = \langle b|\sigma^*_{a|xy} |b\rangle = \begin{cases} \frac{1}{2} & \text{if } a \oplus b = xy \\ 0 & \text{otherwise} \end{cases}.$$ 

These are the correlations of the "Popescu-Rohrlich" box [17], which are not achievable within quantum theory. Hence, $\sigma^*_{a|xy} \not\in Q_{BI}$ and so $Q_{BI} \not\supseteq G_{BI}$. \hfill \Box

We see then that post-quantum steering can arise in a generalised bipartite steering scenario. This example given however relies on post-quantum nonlocality and hence the post-quantum steering found may be argued to be just another guise of the former effect. In the following theorem, we prove that the two phenomena are genuinely different:

**Theorem 3.** Post-quantum steering in the Bob-with-input steering scenario is independent of post-quantum nonlocality. Namely, there exist non-signalling assemblages $\{\sigma_{a|xy}\}$ that are not quantum realisable, but which can only lead to quantum correlations $p(a, b|x, y)$ in the Bell scenario.

The proof of this theorem is given in the Supplemental Material. The main idea is to use the method of [14] and consider a quantum setup with the relaxation that Bob can apply positive but not completely positive maps to his share of the joint quantum system. All such assemblages are shown not to lead to post-quantum nonlocality. An example of such an assemblage is the following:

$$\sigma_{a|xy} = \frac{1}{4}(\mathbb{I} + (-1)^a \sigma_x)^{T_y} \quad (9)$$

where $x \in \{1, 2, 3\}, (\sigma_1, \sigma_2, \sigma_3) = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli operators, and $T$ is transpose (w.r.t. the basis $|b\rangle$). That is, this is the assemblage that arises from Alice performing Pauli measurements on the maximally entangled state, with Bob applying the identity or transpose map depending on $y$. In the Supplemental Material we prove that this assemblage has no quantum realisation.

Hence, just as with multipartite post-quantum steering [13], the effect here is independent of the existence of post-quantum Bell nonlocality.

**Generalisation 3: Instrumental steering.**— We consider finally the Instrumental steering scenario [18]. In this case, Bob still has an input that can inform the preparation of a quantum system, however now this input can depend on Alice’s measurement outcome (see Fig. 1 (d)). For example, Bob’s input could just decide a transformation upon a quantum system. This scenario is closely related to the so-called ‘Instrumental setup’ [23, 24], only now one of the variables has become a quantum system.

In the Instrumental steering scenario then, an assemblage is given by the collection of subnormalised states $\{\sigma_{a|x}\}$, where $x$ denotes the choice of measurement by Alice, and $a$ denotes both Alice’s outcome and Bob’s input. Within quantum theory, the assemblages they can generate have the following form:

**Definition 5.** Quantum instrumental assemblages. An assemblage $\{\sigma_{a|x}\}$ has a has a quantum realisation in the instrumental steering scenario if and only if there exists a Hilbert space $\mathcal{H}_A$ and POVMs $\{M_{a|x}\}$ for Alice, a state $\rho$ in $\mathcal{H}_A \otimes \mathcal{H}_B$, and a collection of CPTP maps $\{\mathcal{E}_a\}$ in $\mathcal{H}_B$ for Bob, such that

$$\sigma_{a|x} = \mathcal{E}_a [\text{tr}_A \{ (M_{a|x} \otimes I) \rho \}] . \quad (10)$$

We denote this set of assemblages by $Q_{I}$.

The instrumental steering scenario has no straightforward non-signalling constraints. Hence, in order to define general assemblages here, we adopt the relation between non-signalling Bell correlations and generic instrumental correlations in the black-box scenario found in Ref. [20] (see also Supplementary Material of Ref. [19]):

**Definition 6.** General instrumental assemblages. An assemblage $\{\sigma_{a|x}\}$ is a general instrumental assemblage if there exists a non-signalling Bob-with-input assemblage $\{\omega_{a|xy}\} \in G_{BI}$ such that $\sigma_{a|x} = \omega_{a|x,y=a}$ for all $a$ and $x$. We denote the set of such general assemblages by $G_{I}$.

Returning one final time to our central question, we now show that there is post-quantum steering in the instrumental steering scenario. Moreover, we show that this does not follow from post-quantum instrumental black-box correlations, and it is hence another independent form of post-quantumness.

**Theorem 4.** The set of general instrumental assemblages strictly contains the set of quantum instrumental assemblages, $Q_{I} \not\supseteq G_{I}$. Hence, post-quantum instrumental steering exists.

**Theorem 5.** Post-quantum steering in the instrumental steering scenario is independent of post-quantum instrumental correlations. Namely, there exist general assemblages $\{\sigma_{a|x}\}$ that are not quantum realisable, but which can only lead to quantum correlations $p(a, b|x)$ in the Instrumental scenario.

These two theorems are proven together in the Supplementary Material, but their proof is very similar to that of Theorem 3. The general assemblage that is used here as an example is that which derives from (9) by setting $y = a$, which is both provably post-quantum in the instrumental scenario, and can only lead to quantum instrumental black-box correlations.

Thus, post-quantum steering is also possible within the instrumental scenario, and this is independent of the existence of correlations with no quantum explanation in the fully device-independent instrumental scenario. Hence, post-quantum instrumental steering is another genuinely new effect. Moreover, since instrumental assemblages can be seen as classically post-selected Bob-with-input assemblages, post-quantum instrumental steering is a stronger phenomenon than post-quantum Bob-with-input steering, in the sense of the former implying the latter. Finally, in terms of number of variables (inputs and outputs), the instrumental scenario is the simplest one where post-quantum steering can exist.
Discussion.— We have shown that steering beyond what quantum theory allows is possible in bipartite steering scenarios by considering suitable generalisations of the traditional steering scenario. Remarkably, our examples of post-quantum steering include cases with no post-quantum correlations arising in the corresponding Bell and Instrumental scenarios, rendering the phenomenon genuinely new.

On the other hand, we have shown that post-quantum steering is impossible in the sequential measurement generalisation of steering, where Alice steers Bob by performing a sequence on measurements. To do so, we have shown the GHJW theorem can be extended to this setting.

The instrumental causal structure is known to be the one with the fewest number of variables able to admit classical-versus-quantum gaps [22]. Such gaps have been explicitly found in Bell-type scenario [19–21] and the steering scenario [18]. Furthermore, quantum-versus-post-quantum gaps have also been found in Bell-type scenario [19, 20]; but the existence of post-quantum instrumental steering remained an open question. The discovery of the latter thus brings a crucial missing piece to both steering theory and generalised probabilistic theories. Our findings are relevant in the fields of quantum foundations and information, and causal inference.

Going forward, the most interesting questions which arise are now to understand the power of post-quantum steering. In particular, we would like to know if there are information theoretic or physical principles which can be violated by post-quantum steering, or whether they are particularly powerful for certain tasks. We believe this new approach to studying quantum theory ‘from the outside’ might lead to novel insights into the nonlocal structure of quantum theory.

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Supplemental material

Review of the Gisin’s and Hughston-Josza-Wootters’s theorem

In this section we review the proof of the theorem by Gisin [11] and Hughston, Josza and Wootters [12] (GHJW), which states that post-quantum steering in bipartite traditional scenarios compatible with the No Signalling (NS) principle does not exist.

The idea is to find a quantum realization of a generic NS assemblage. One could view this NS assemblage as a non-signalling sequential assemblage (as defined in the main body of the paper) with only one round of measurements. Given an NS assemblage \( \{ \sigma_a|x \} \), we want to find a Hilbert space for Alice, a state \( \rho \) shared by Alice and Bob, and measurement operators \( \{ M_{a|x} \} \) for Alice such that:

\[
\begin{align*}
\sum_a M_{a|x} &= I_A \quad \forall x, \\
\sigma_{a|x} &= \text{tr}_A \left( (M_{a|x} \otimes I_B) \rho \right) \quad \forall a, x.
\end{align*}
\]

GHJW find such a general construction as follows. First, notice that the reduced state \( \sigma_R = \sum_a \sigma_{a|x} \) satisfies \( \sigma_R \geq 0 \), hence it has a diagonal decomposition as:

\[
\sigma_R = \sum_{r \in S_R} \mu_r \left| r \right\rangle \left\langle r \right|,
\]

where \( \{ \left| r \right\rangle \} \) is an orthonormal basis, and \( S_R \) is the set of those basis’ vectors that \( \sigma_R \) has support on (that is, we can assume \( \mu_r > 0 \)). GHJW then define the following state and measurements for the quantum realisation of the given assemblage:

\[
\begin{align*}
|\psi\rangle &= \sum_{r \in S_R} \sqrt{\mu_r} \left| r \right\rangle \otimes \left| r \right\rangle, \\
M_{a|x} &= \begin{cases} 
\sqrt{\sigma_R^{-1}} \sigma_{a|x}^T \sqrt{\sigma_R^{-1}}, & \text{if } a > 0 \\
\sqrt{\sigma_R^{-1}} \sigma_{a|x}^T \sqrt{\sigma_R^{-1}} + I - I_R, & \text{if } a = 0
\end{cases},
\end{align*}
\]

where

\[
\sqrt{\sigma_R^{-1}} = \sum_{r \in S_R} \frac{1}{\sqrt{\mu_r}} \left| r \right\rangle \left\langle r \right|,
\]

and \( I_R = \sum_{r \in S_R} |r\rangle \langle r| \). These state and measurements are a valid quantum realisation of the assemblage, as we see next.

Let’s first show that the collection of operators \( \{ M_{a|x} \} \) forms well defined measurements. Since the operators are positive semidefinite by definition, we only need to check that the are suitably normalised:

\[
\begin{align*}
\sum_a M_{a|x} &= \sqrt{\sigma_R^{-1}} \sigma_{a|x}^T \sqrt{\sigma_R^{-1}} + I - I_R \\
&= \sum_{r,r',r'' \in S_R} \frac{\mu_{r''}}{\sqrt{\mu_r \mu_{r'}}} \left| r \right\rangle \left\langle r \right| \left| r' \right\rangle \left\langle r' \right| \left| r'' \right\rangle \left\langle r'' \right| + I - I_R \\
&= \sum_{r \in S_R} \left| r \right\rangle \left\langle r \right| + I - I_R \\
&= I.
\end{align*}
\]

Now let’s show that the model actually recovers the assemblage. When \( a > 0 \):

\[
\text{tr}_A \left\{ (M_{a|x} \otimes I_B) \rho \right\} = \text{tr}_A \left\{ \left( \sqrt{\sigma_R^{-1}} \sigma_{a|x}^T \sqrt{\sigma_R^{-1}} \otimes I_B \right) \left( \sum_{r,r' \in S_R} \sqrt{\mu_r \mu_{r'}} \left| rr \right\rangle \left\langle rr' \right| \right) \right\} \\
= \text{tr}_A \left\{ \sigma_{a|x}^T \otimes I_B \right\} \left( \sum_{r,r' \in S_R} \left| rr \right\rangle \left\langle rr' \right| \right) \\
= \sigma_{a|x}.
\]
where to see that these equalities follow we made use of some technical lemmas that you may find in the next section of our supplemental material: by Lemma 7, the support of $\sigma_{a|x}$ is not outside that of $\sigma_R$, and hence Lemma 8 applies.

Now, when $a = 0$:

$$\text{tr}_A \{ (M_{a|x} \otimes I_B) \rho \} = \text{tr}_A \left\{ \left( \sqrt{\sigma_R^{-1}} \sigma_{a|x}^T \sqrt{\sigma_R^{-1}} \otimes I_B \right) \left( \sum_{r,r' \in S_R} \sqrt{\mu_r} \sqrt{\mu_{r'}} \langle rr' \rangle \langle r'r' \rangle \right) \right\}$$

$$+ \text{tr}_A \{ (I - I_R) \otimes I_B \} \left( \sum_{r,r' \in S_R} \sqrt{\mu_r} \sqrt{\mu_{r'}} \langle rr' \rangle \langle r'r' \rangle \right)$$

$$= \text{tr}_A \left\{ \left( \sigma_{a|x}^T \otimes I_B \right) \left( \sum_{r,r' \in S_R} |rr\rangle \langle r'r'| \right) \right\}$$

$$= \sigma_{a|x}.$$

This completes the proof of the GHJW theorem for bipartite steering scenarios.

**Proof of Theorem 1:** Bipartite steering with sequential measurements does not admit post-quantum steering

One natural way to generalise the setup of a traditional steering scenario is to allow Alice to make a sequence of measurements on her share of the system. These scenarios, however, may not feature post-quantum steering compatible with the No Signalling principle, as we show here. We do so by proving how a generalisation of the GHJW theorem applies to these scenarios.

For clarity in the presentation, we will focus on the case depicted in Fig. 1(b), where Alice makes two consecutive measurements. The general case of an arbitrary number of measurements follows by induction. The question we want to answer is whether $G_s \equiv \mathbb{Q}_s$, i.e., whether any assemblage in the scenario has a quantum realization. This is answered in the affirmative by Theorem 1, which we recall here for completeness.

**Theorem 1.** The set of all non-signalling sequential assemblages coincides with the set of quantum sequential assemblages, $\mathbb{Q}_S = G_S$. That is, there is no post-quantum steering in the bipartite sequential steering scenario.

Before proceeding to the proof, we need three technical lemmata. The first lemma concerns a traditional bipartite steering scenario, for an assemblage of the form $\{\sigma_{a|x}\}_{a,x}$:

**Lemma 6.** The support of $\sigma_{a|x}$ is equal to or contained in that of $\sigma_R$, for all $a, x$.

**Proof.** Since $\sigma_R = \sum_{a} \sigma_{a|x}$, then $\sigma_R - \sigma_{a|x} = \sum_{a \neq a} \sigma_{a|x} \geq 0$.

Now let $I_R^a$ be the projector onto the orthogonal complement to the support of $\sigma_R$. Then, by definition $I_R^a \sigma_R I_R^a = 0$. Now let’s assume that the support of $\sigma_{a|x}$ is not contained in that of $\sigma_R$. That is, $I_R^a \sigma_{a|x} I_R^a > 0$. Then,

$$I_R^a \left( \sigma_R - \sigma_{a|x} \right) I_R^a = -I_R^a \sigma_{a|x} I_R^a < 0,$$

which is a contradiction. Hence, $\sigma_{a|x}$ must live in the support of $\sigma_R$. $\Box$

The second lemma concerns a sequential bipartite steering scenario, for an assemblage $\{\sigma_{a_1 a_2|x_1 x_2}\}_{a_1, a_2, x_1, x_2} \in G_s$.

**Lemma 7.** (i) The marginal state $\sigma_{a_1|x_1}$ lives in the subspace supported by $\sigma_R$, for all $a, x$.

(ii) The support of $\sigma_{a_1 a_2|x_1 x_2}$ is equal to or contained in that of $\sigma_{a_1|x_1}$, for all $a, a_2, x_1, x_2$.

**Proof.** The proof of (i) follows similarly to that of Lemma 6 once we see that when considering only the marginal states the situation is equivalent to the traditional bipartite steering scenario.

The proof of (ii) follows a similar logic than that of Lemma 6, by having $\sigma_{a_1 a_2|x_1 x_2}$ play the role of $\sigma_{a_1|x_1}$ and $\sigma_{a_1|x_1}$ that of $\sigma_R$. In the following we present the proof explicitly. For simplicity we will replace $a_1$ and $a_2$ with $a$ and $b$ respectively, and $x_1$ and $x_2$ with $x$ and $y$ respectively.

Since $\sigma_{a|x} = \sum_{a} \sigma_{a|y}$, then $\sigma_{a|x} - \sigma_{a|y} = \sum_{a \neq b} \sigma_{a|y} \geq 0$. Now let $I_{xy}$ be the projector onto the space not supported by $\sigma_{a|x}$. Then, by definition $I_{xy} \sigma_{a|y} I_{xy} = 0$. Now let’s assume that $\sigma_{a|y}$ has support outside the space where $\sigma_{a|x}$ lives. That is, $I_{xy} \sigma_{a|y} I_{xy} > 0$. Then,

$$I_{xy} \left( \sigma_{a|x} - \sigma_{a|y} \right) I_{xy} = -I_{xy} \sigma_{a|y} I_{xy} < 0,$$
which is a contradiction. Hence, \( \sigma_{ab|xy} \) must live in the support of \( \sigma_{a|x} \).

The third, and final, lemma is a special instance of the link product [26] in the Choi representation, and we make it explicit for simplicity in the before mentioned proofs.

**Lemma 8.**

\[
\text{tr}_1 \{(A^T \otimes I_2) |\psi\rangle \langle \psi|\} = A,
\]

where \( |\psi\rangle = \sum_k |k\rangle \otimes |k\rangle \), and \( \{|k\rangle\}_k \) spans the space where \( A \) has support on.

**Proof.** Let \( d \) be the dimension of the Hilbert space where \( A \) lives, and \( n \) the dimension of the space spanned by \( \{|k\rangle\}_k \).

\[
\text{tr}_1 \{(A^T \otimes I_2) |\psi\rangle \langle \psi|\} = \sum_{j=1}^d \sum_{k=1}^n \langle j| A^T |k\rangle \langle l| \langle l| = \sum_{j=1}^d \sum_{k=1}^n A^T_{jk} |k\rangle \langle j| = \sum_{j,k=1}^d A_{jk} |k\rangle \langle j| = A.
\]

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** Let us start from an assemblage \( \{\sigma_{ab|xy}\} \in G_a \), where for simplicity we change slightly the notation and denote by \((b, y)\) the labels of the second time step, and \((a, x)\) those of the first time step. Now define the following:

\[
|\psi\rangle = \sum_{r \in S_R} \sqrt{\mu_r} |r\rangle \otimes |r\rangle,
\]

\[
K_{a|x} = \begin{cases} \sqrt{\sigma_{a|x}} \sqrt{\sigma_{R}^{-1}} & \text{if } a > 0 \\ \sqrt{\sigma_{a|x}} \sqrt{\sigma_{R}^{-1}} + I - \mathbb{I}_{\mathbb{P}} & \text{if } a = 0 \end{cases},
\]

\[
M_{b|y}^{a,x} = \begin{cases} \left(\sqrt{\sigma_{a|x}}\right)^{-1} \sigma_{ab|xy}^{T} \left(\sqrt{\sigma_{a|x}}\right)^{-1} & \text{if } b > 0 \\ \left(\sqrt{\sigma_{a|x}}\right)^{-1} \sigma_{ab|xy}^{T} \left(\sqrt{\sigma_{a|x}}\right)^{-1} + I - \mathbb{I}_{\mathbb{P}} & \text{if } b = 0 \end{cases},
\]

where:

- \( \sigma_R = \sum_{r \in S_R} \mu_r |r\rangle \langle r| \), with \( \{|r\rangle\}_r \) an orthonormal basis, and \( S_R \) the set of those basis’ vectors that \( \sigma_R \) has support on.
- \( \sqrt{\sigma_{a|x}^{-1}} = \sum_{r \in S_R} \frac{1}{\sqrt{\mu_r}} |r\rangle \langle r|.\)
- \( \sigma_{a|x} = \sum_{k \in S_{a,x}} \nu_{k,a,x}^{a,x} |k\rangle_{a,x} \langle k|_{a,x} \), with \( \{|k\rangle_{a,x}\}_k \) an orthonormal basis, and \( S_{a,x} \) the set of those basis’ vectors where \( \sigma_{a|x} \) has support on.
- \( \sqrt{\sigma_{a|x}^{-1}} = \sum_{k \in S_{a,x}} \nu_{k,a,x}^{a,x} |k\rangle_{a,x} \langle k|_{a,x} \), with \( \{|k\rangle_{a,x}\}_k \) an orthonormal basis, and \( S_{a,x} \) the set of those basis’ vectors where \( \sigma_{a|x} \) has support on.
- \( \mathbb{I}_{\mathbb{P}} \), the projector onto the orthogonal complement of the support of \( \sigma_R \).
- \( \mathbb{I}_{\mathbb{P}} \), the projector onto the orthogonal complement of the support of \( \sigma_{a|x} \).

Now we see how the state, measurement operators and Kraus operators reproduce the assemblage and give well defined measurements for each time step.

Let us begin showing that the Kraus operators \( \{K_{a|x}\} \) correspond to well defined measurements for the first time step. First, \( \sum_a K_{a|x}^{\dagger} K_{a|x} \geq 0 \) by definition. Second,

\[
\sum_a K_{a|x}^{\dagger} K_{a|x} = \sum_{a \neq 0} \sqrt{\sigma_{a|x}^{-1}} \sigma_{a|x}^{T} \sqrt{\sigma_{R}^{-1}} + \left(\sqrt{\sigma_{R}^{-1}} \sqrt{\sigma_{a|x}^{T}} + \mathbb{I}_{\mathbb{P}}\right) \left(\sqrt{\sigma_{a|x}^{T}} \sqrt{\sigma_{R}^{-1}} + \mathbb{I}_{\mathbb{P}}\right)
\]

\[
= \sum_{a \neq 0} \sqrt{\sigma_{a|x}^{-1}} \sigma_{a|x}^{T} \sqrt{\sigma_{R}^{-1}} + \sqrt{\sigma_{R}^{-1}} \sigma_{a|x}^{T} \sqrt{\sigma_{R}^{-1}} + \mathbb{I}_{\mathbb{P}}
\]

\[
= \sqrt{\sigma_{R}^{-1}} \sigma_R \sqrt{\sigma_{R}^{-1}} + \mathbb{I}_{\mathbb{P}} = \mathbb{I}_R + \mathbb{I}_{\mathbb{P}}
\]

\[= \mathbb{I}.
\]
where we used the fact that $\mathbb{I}_x^T \sigma_{a|x}^T = 0 = \sigma_{a|x}^T \mathbb{I}_x$ according to Lemma 7. These two properties show that the Kraus operators correspond indeed to a well defined measurement.

Now let us show that the operators $\{M_{b|y}^{a,x}\}$ define a measurement for each $a, x, y$ in the second time step. First, notice that each $M_{b|y}^{a,x}$ is positive semidefinite by definition, hence we only need to check that they are properly normalised. This is shown as follows:

$$
\sum_b M_{b|y}^{a,x} = \sum_b \left( \sqrt{\sigma_{a|x}^T} \sigma_{a|x}^{-1} \right)^{-1} \sigma_{a|x}^{-1} \left( \sqrt{\sigma_{a|x}^T} \right)^{-1} + \mathbb{I} - \mathbb{I}_{ax}^T
$$

Finally, let us show that these state and measurements recover the assemblage. Here we will use some technical lemmas presented earlier. Let us begin with the case of $a > 0$ and $b > 0$:

$$
\text{tr}_A \left\{ K_{a|x}^T M_{b|y}^{a,x} K_{a|x} \otimes \mathbb{I}_B \rho \right\}
$$

where we use the fact that Lemma 7 implies $\mathbb{I}_a \sigma_{a|xy} \mathbb{I}_a = \sigma_{a|xy}$, and Lemmas 7 and 8 imply the last step.
Now consider the case where \( a > 0 \) and \( b = 0 \):

\[
\text{tr}_A \left\{ \begin{array}{l}
K_{a|x}^\dagger \left( M_{b|y}^{ax} + I_{\mathcal{M}} \right) K_{a|x} \otimes I_B \rho \\
= \text{tr}_A \left\{ K_{a|x}^\dagger M_{b|y}^{ax} K_{a|x} \otimes I_B \rho \right\} + \text{tr}_A \left\{ K_{a|x}^\dagger I_{\mathcal{M}} K_{a|x} \otimes I_B \rho \right\} \\
= \text{tr}_A \left\{ \left( \sqrt{\sigma_R^{-1}} \sqrt{\sigma_{a|x}^T} \left( \sqrt{\sigma_{a|x}^T} \right)^{-1} \sigma_{ab|xy}^{\dagger} \left( \sqrt{\sigma_{a|x}^T} \right)^{-1} \sqrt{\sigma_{a|x}^T} \sqrt{\sigma_R^{-1} \otimes I_B} \right) \rho \right\} \\
= \text{tr}_A \left\{ \left( \sqrt{\sigma_{a|x}^T} \left( \sqrt{\sigma_R^{-1}} \sigma_{ab|xy}^{\dagger} \left( \sqrt{\sigma_R^{-1}} \right)^{-1} \sqrt{\sigma_{a|x}^T} \otimes I_B \right) \sqrt{\sigma_R^{-1} \rho} \sqrt{\sigma_R^{-1}} \right) \right\} \\
= \text{tr}_A \left\{ \left( \sigma_{ab|xy}^{\dagger} \right) \left( \sum_{r,r' \in S_R} |rr \rangle \langle r' r'| \right) \right\} \\
= \sigma_{ab|xy}.
\]

where we use similar techniques to those in the previous case, together with the fact that Lemma 7 implies that \( K_{a|x}^\dagger I_{\mathcal{M}} K_{a|x} = 0 \).

Consider now the case \( a = 0 \) and \( b > 0 \). Here, the operator inside the trace is:

\[
\left( K_{a|x}^\dagger + I_{\mathcal{M}} \right) M_{b|y}^{ax} \left( K_{a|x} + I_{\mathcal{M}} \right) = K_{a|x}^\dagger M_{b|y}^{ax} K_{a|x} ,
\]

since Lemma 7 guarantees that \( I_{\mathcal{M}} M_{b|y}^{ax} = 0 = M_{b|y}^{ax} I_{\mathcal{M}} \). Hence,

\[
\text{tr}_A \left\{ \left( K_{a|x}^\dagger + I_{\mathcal{M}} \right) M_{b|y}^{ax} \left( K_{a|x} + I_{\mathcal{M}} \right) \otimes I_B \rho \right\} \\
= \text{tr}_A \left\{ K_{a|x}^\dagger M_{b|y}^{ax} K_{a|x} \otimes I_B \rho \right\} \\
= \text{tr}_A \left\{ \left( \sqrt{\sigma_R^{-1}} \sqrt{\sigma_{a|x}^T} \left( \sqrt{\sigma_{a|x}^T} \right)^{-1} \sigma_{ab|xy}^{\dagger} \left( \sqrt{\sigma_{a|x}^T} \right)^{-1} \sqrt{\sigma_{a|x}^T} \sqrt{\sigma_R^{-1} \otimes I_B} \right) \rho \right\} \\
= \text{tr}_A \left\{ \left( \sqrt{\sigma_{a|x}^T} \left( \sqrt{\sigma_R^{-1}} \sigma_{ab|xy}^{\dagger} \left( \sqrt{\sigma_R^{-1}} \right)^{-1} \sqrt{\sigma_{a|x}^T} \otimes I_B \right) \sqrt{\sigma_R^{-1} \rho} \sqrt{\sigma_R^{-1}} \right) \right\} \\
= \text{tr}_A \left\{ \left( \sigma_{ab|xy}^{\dagger} \right) \left( \sum_{r,r' \in S_R} |rr \rangle \langle r' r'| \right) \right\} \\
= \sigma_{ab|xy}.
\]

Finally, consider the case \( a = 0 \) and \( b = 0 \). Here

\[
\left( K_{a|x}^\dagger + I_{\mathcal{M}} \right) \left( M_{b|y}^{ax} + I_{\mathcal{M}} \right) = K_{a|x}^\dagger M_{b|y}^{ax} K_{a|x} + I_{\mathcal{M}} I_{\mathcal{M}} = K_{a|x}^\dagger M_{b|y}^{ax} K_{a|x} + I_{\mathcal{M}} I_{\mathcal{M}} ,
\]
where we used the fact that Lemma 7 implies $\|R M_{b|y}^{ax} = 0 = M_{b|y}^{ax} R$ and $K_{a|x}^{1} \|R = 0 = K_{a|x}$. Hence,

$$
\text{tr}_A \left\{ (K_{a|x}^{1} \|R + \|R) \left( M_{b|y}^{ax} + \|R \right) (K_{a|x} \|R + \|R) \otimes \mathbb{I}_B \rho \right\}
$$

$$
= \text{tr}_A \left\{ (K_{a|x}^{1} M_{b|y}^{ax} K_{a|x} + \|R \|R \otimes \mathbb{I}_B \rho + K_{a|x} M_{b|y}^{ax} K_{a|x} \otimes \mathbb{I}_B \rho \right\}
$$

$$
= \text{tr}_A \{ K_{a|x} M_{b|y}^{ax} K_{a|x} \otimes \mathbb{I}_B \rho \} + \text{tr}_A \{ \|R \|R \otimes \mathbb{I}_B \rho \}
$$

$$
= \text{tr}_A \{ K_{a|x} M_{b|y}^{ax} K_{a|x} \otimes \mathbb{I}_B \rho \} + \text{tr}_A \{ \|R \otimes \|B (\|R \otimes \|B) \rho (\|R \otimes \|B) \}
$$

$$
= \text{tr}_A \{ K_{a|x} M_{b|y}^{ax} K_{a|x} \otimes \mathbb{I}_B \rho \}
$$

$$
= \text{tr}_A \left\{ \left( \sqrt{\sigma_R} \sigma_{ab|xy} \sigma_{ab|x} \right) \left( \sqrt{\sigma_R} \right) \rho \right\}
$$

$$
= \sum_{k,k' \in S_{a,x}} \left| k \right\rangle_{a,x} \left| k' \right\rangle_{a,x} \sigma_{ab|xy} \left( \sqrt{\sigma_R} \right) \left| k' \right\rangle_{a,x} \left( \sqrt{\sigma_R} \right) \rho \left( \sqrt{\sigma_R} \right)
$$

$$
= \text{tr}_A \left\{ \sigma_{ab|xy} \left( \sum_{r,r' \in S_R} \left| r \right\rangle \left\langle r' \right| \right) \right\}
$$

$$
= \sigma_{ab|xy}
$$

where we used similar techniques to those in the previous cases, together with the fact that $\|R \otimes \|B \langle \psi \rangle = 0$.

We see then that the state defined in Eq. (13), together with the measurement operators from Eqs. (14) and (15) are indeed well defined and reproduce the assemblage. Hence, it follows that the GHJW theorem generalises to these sequential scenarios.

□

**Proof of Theorem 3:** post-quantum steering does not imply post-quantum non-locality in the Bob-with-input scenario

In this section we will prove Theorem 3, which says that post-quantum steering in the Bob-with-input setting does not imply post-quantum non-locality. To prove this we introduce a new set of Bob-with-input assemblages, which strictly contain the quantum Bob-with-input assemblages, but provably can never give rise to post-quantum correlations. We will call this set the set of **PTP assemblages**, since they are defined with respect to positive and trace-preserving (PTP) maps, which may not necessarily be completely positive. We will define these assemblages, but first it is instructive to recall the definition of quantum Bob-with-input assemblages.

**Definition 3. Quantum Bob-with-input assemblages.**

An assemblage $\{ \sigma_{a|xy} \}_{a,x,y}$ has a has a quantum realisation in the steering scenario where Bob has an input if and only if there exists a Hilbert space $\mathcal{H}_A$ and POVMs $\{ M_{a|x} \}_{a,x}$ for Alice, a state $\rho$ in $\mathcal{H}_A \otimes \mathcal{H}_B$, and a collection of CPTP maps $\{ \mathcal{E}_y \}_{y}$ in $\mathcal{H}_B$ for Bob, such that

$$\sigma_{a|xy} = \mathcal{E}_y \left[ \text{tr}_A \left\{ (M_{a|x} \otimes \mathbb{I}) \rho \right\} \right].$$

A relaxation of the quantum assemblages from the previous setup is that where $\mathcal{E}_y$ is a PTP map (but not necessarily completely positive) instead:

**Definition 7. PTP assemblage.**

An assemblage $\{ \sigma_{a|xy} \}_{a,x,y}$ is a PTP assemblage iff there exists a Hilbert space $\mathcal{H}_A$ for Alice, POVMs $\{ M_{a|x} \}_{a,x}$ for Alice, a state $\rho$ in $\mathcal{H}_A \otimes \mathcal{H}_B$, and a collection of PTP maps $\{ \mathcal{E} \}_{y}$ in $\mathcal{H}_B$ for Bob, such that $\sigma_{a|xy} = \mathcal{E}_y \left[ \text{tr}_A \{ M_{a|x} \otimes \mathbb{I} \} \right].$
When some PTP maps that realise the assemblage are not completely positive, they may allow the produced assemblage to be post-quantum. This method has been previously used in Ref. [14] to construct post-quantum assemblages that may only exhibit quantum correlations in Bell scenarios.

In the bipartite steering scenarios considered here, this method of constructing assemblages has a similar advantage to that in [14]: when Bob measures on his system, the correlations $p(ab|xy)$ are compatible with quantum theory. This claim is formalised below.

**Theorem 9.** Let $\rho$ be the state of a quantum system shared by Alice and Bob. Let $\{M_{a|x}\}_a$ be a POVM for Alice, for each $x$, and $\mathcal{E}^y$ a PTP map for Bob, for each $y$. These define the assemblage $\sigma_{a|x,y} = \text{tr}_A \{M_{a|x} \otimes \mathcal{E}^y \rho\}$.

Let $\{N_b\}_b$ be a POVM for Bob. Then, the correlations $p(ab|xy) = \text{tr} \{N_b \sigma_{a|x,y}\}$ have a quantum realisation in a Bell experiment. Moreover, let $\{N_{b|z}\}_b$ be a POVM for Bob, for each $z$. Then, the correlations $p(ab|xyz) = \text{tr} \{N_{b|z} \sigma_{a|x,y}\}$ also have a quantum realisation in a Bell experiment.

**Proof.** Let’s start from the correlations $p(ab|xy) = \text{tr} \{N_b \sigma_{a|x,y}\}$. For each PTP map $\mathcal{E}^y$, there exists the dual map $\mathcal{F}^y$, which is positive and unital, such that

$$p(ab|xy) = \text{tr} \{M_{a|x} \otimes F_{b|y} \rho\} = \text{tr} \{M_{a|x} \otimes F^y(N_b) \rho\}.$$ 

For each $y$, since the map $\mathcal{F}^y$ is positive and unital, it takes a general measurement $N_b$ to a general measurement with POVM elements $M_{b|y} := F^y(N_b)$. Therefore,

$$p(ab|xy) = \text{tr} \{M_{a|x} \otimes M_{b|y} \rho\},$$

which gives a quantum realisation for the correlations $p(ab|xy)$.

Now let’s move on to the second statement in the theorem. Let Bob choose from a set of general measurements indexed by the variable $z$ with POVM elements $N_{b|z}$. In this case the correlations are of the form

$$p(ab|xyz) = \text{tr} \{M_{a|x} \otimes N_{b|z} \otimes (1_A \otimes \mathcal{E}^y) \rho\} = \text{tr} \{M_{a|x} \otimes F^y(N_{b|z}) \rho\}.$$ 

By a similar argument as before, $M_{b|yz} := F^y(N_{b|z})$ defines a POVM for each choice of $(y, z)$. By reinterpreting $(y, z)$ as Bob’s total input, this provides a quantum realisation for the correlations $p(ab|xyz)$. \hfill \square

Theorem 9 shows that PTP assemblages cannot generate post-quantum correlations in a Bell scenario. In order to certify the post-quantumness of a PTP assemblage, thus, we need a new method that does not rely on Bell correlations. The one we develop here relies on the following observation:

**Lemma 10.** Consider the Bob-with-input scenario. Let $\{\sigma_{a|x,y}\}$ be a quantumly realisable assemblage, with the following property: $\sigma_{a|x,1}$ is (proportional to) a pure state for all $a, x$. Then, there exist CPTP maps $\{\mathcal{F}^y\}_{y>1}$ such that $\sigma_{a|x,y} = F^y[\sigma_{a|x,1}] \forall y > 1$.

In order to prove this Lemma, we first need to show a more general result, stated as a theorem below. The proofs will be presented in the diagrammatic notation of Ref. [25] for simplicity.

**Theorem 11.** Consider the Bob-with-input scenario. Let $\{\sigma_{a|x,y}\}$ be a quantum realisable assemblage. Then, there exist CPTP maps $\{\mathcal{F}^y\}_{y>1}$ and a dilation $\tilde{\sigma}_{a|x,1}$ of $\sigma_{a|x,1}$ for each $(a, x)$ through an auxiliary system $E$, such that:

$$\begin{cases}
\sigma_{a|x,y} = \mathcal{F}^y[\tilde{\sigma}_{a|x,1}] \forall y > 1, \\
\tilde{\sigma}_{a|x,1} = \text{tr}_{E'} \{\tilde{\sigma}_{a|x,1}\}.
\end{cases}$$

**Proof.** Let $\rho, \{M_{a|x}\}$ and $\{\mathcal{E}_y\}$ be the state, measurements and CPTP maps that provide a quantum realisation of $\{\sigma_{a|x,y}\}$. In diagrammatic notation:

$$\sigma_{a|x,y} = \begin{array}{c}
\xymatrix{A \ar@/^/[rr]^{\mathcal{E}_y} & \mathcal{E}_y \\
\rho & B \\
& B'}
\end{array}.$$
Let the auxiliary system $E$ on state $|\chi\rangle$ and the unitary operators $\{U_y\}$ provide a unitary dilation for the CPTP maps $\{E_y\}$, namely:

$$\sigma_{a|xy} = \begin{pmatrix} A \rightarrow B \rightarrow E \rightarrow E' \\ \rho \rightarrow U_y \rightarrow \chi \rightarrow \chi' \end{pmatrix}.$$ 

Let us now define the maps $\{F_y\}$ as follows:

$$F_y \begin{pmatrix} B \\ B' \end{pmatrix} := \begin{pmatrix} B \rightarrow E' \\ U_y \rightarrow U_y^{-1} \rightarrow B \rightarrow E' \end{pmatrix}.$$ 

Notice that each of these $F_y$ is a CPTP map acting on the joint system $BE'$.

Let us now show the first part of the claim, i.e. $\sigma_{a|xy} = F_y[\tilde{\sigma}_{a|x1}] \quad \forall y > 1$:

For the second part of the claim, notice that

$$\text{tr}_{E'} \{\tilde{\sigma}_{a|x1}\} = \begin{pmatrix} A \rightarrow B \rightarrow E \rightarrow E' \\ \rho \rightarrow U_1 \rightarrow \chi \rightarrow \chi' \end{pmatrix} = \begin{pmatrix} A \rightarrow B \rightarrow E \rightarrow E' \\ \rho \rightarrow \chi \rightarrow \chi' \end{pmatrix} = \sigma_{a|x1},$$

which concludes the proof of the theorem.

We can now prove Lemma 10 as a corollary of Theorem 11:

**Proof of Lemma 10.** From Theorem 11 we know there exists a purification $\tilde{\sigma}_{a|x1}$ of $\sigma_{a|x1}$ for each $(a, x)$, and CPTP maps $F_y$ for each $y > 1$, such that $\sigma_{a|xy} = F_y[\tilde{\sigma}_{a|x1}] \quad \forall y > 1$.

Since $\sigma_{a|x1}$ is pure for each $(a, x)$, then $\tilde{\sigma}_{a|x1} = \sigma_{a|x1} \otimes \omega$, where $\omega$ is a fixed state of the auxiliary system $E$.

Hence, $\sigma_{a|xy} = F_y[\sigma_{a|x1} \otimes \omega] = F_y[\tilde{\sigma}_{a|x1}]$, where the operators $F_y$ are CPTP since the $F_y$’s are. This concludes the proof of the Lemma.

Now we are in a position to present the proof of Theorem 3, recalled below for convenience.
Theorem 3. The following PTP assemblage has no quantum realisation:

\[
\begin{align*}
\sigma_{0|00} &= \frac{1}{2} |0\rangle \langle 0|, & \sigma_{1|00} &= \frac{1}{2} |1\rangle \langle 1|, \\
\sigma_{a|10} &= \frac{1}{2} ((0) + (-1)^a |1\rangle)(\langle 0| + (-1)^a |1\rangle), \\
\sigma_{a|20} &= \frac{1}{2} ((0) - (-1)^a i |1\rangle)(\langle 0| - (-1)^a i |1\rangle), \\
\sigma_{a|1x} &= \sigma_{a|x0}.
\end{align*}
\]

Proof. First notice that the assemblage \(\{\sigma_{a|x|y}\}\), where \(x \in \{0, 1, 2\}\), \(a \in \{0, 1\}\) and \(y \in \{0, 1\}\), may arise from Alice and Bob sharing a maximally entangled state of two qubits, Alice choosing between the three Pauli measurements to perform on her qubit, and Bob applying on his qubit the identity channel when \(y = 0\) and the transpose when \(y = 1\). Hence, \(\{\sigma_{a|x|y}\}\) is a indeed PTP assemblage, and by virtue of Theorem 9 may never yield post-quantum correlations in Bell experiments.

Now we will show that even though the assemblage \(\{\sigma^*_{a|x|y}\}\) is such that \(\{\sigma^*_{a|x|1}\}\) is a collection of pure quantum states, it does not comply with Lemma 10, and hence has no quantum realisation. For this, let \(X, Y\) and \(Z\) be the Pauli matrices, and notice that:

\[
\begin{align*}
\sigma^*_{0|00} - \sigma^*_{1|00} &= \frac{1}{2} Z, & \sigma^*_{0|01} - \sigma^*_{1|01} &= \frac{1}{2} Z, \\
\sigma^*_{0|10} - \sigma^*_{1|10} &= \frac{1}{2} X, & \sigma^*_{0|11} - \sigma^*_{1|11} &= \frac{1}{2} X, \\
\sigma^*_{0|20} - \sigma^*_{1|20} &= \frac{1}{2} Y, & \sigma^*_{0|21} - \sigma^*_{1|21} &= -\frac{1}{2} Y.
\end{align*}
\]

Should \(\{\sigma^*_{a|x|y}\}\) have a quantum realisation, then by Lemma 10 there exists a CPTP map \(\Lambda\) such that \(\Lambda[1] = \mathbb{I}, \Lambda[X] = X, \Lambda[Z] = Z, \Lambda[Y] = -Y\). However, if the map \(\Lambda\) is applied to one half of the maximally-entangled state \(\rho = \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} - X \otimes X + Y \otimes Y - Z \otimes Z)\) one gets a non-positive matrix, which shows that \(\Lambda\) is actually not CPTP. This contradiction shows that \(\{\sigma^*_{a|x|y}\}\) is indeed a post-quantum assemblage. \(\square\)

Proof of Theorems 4 and 5: post-quantum steering exists in the Instrumental steering scenario, and does not imply post-quantum instrumental correlations

Here we present the proof of Theorem 5, since Theorem 4 is implied by it. Here we recall the theorem below with its proof.

Theorem 5. The following assemblage in the instrumental steering scenario has no quantum realisation, and may only yield quantum correlations in the traditional instrumental setup.

\[
\begin{align*}
\sigma^*_{0|0} &= \frac{1}{2} |0\rangle \langle 0|, & \sigma^*_{1|0} &= \frac{1}{2} ((1) \langle 1|)\langle 1|, \\
\sigma^*_{a|1} &= \frac{1}{2} ((0) + |1\rangle)(\langle 0| + (1)) \\
\sigma^*_{0|2} &= \frac{1}{2} ((0) - i |1\rangle)(\langle 0| + i (1)), \\
\sigma^*_{1|1} &= \frac{1}{2} ((|0\rangle - |1\rangle)(\langle 0| - (1))\langle 1|) \\
\sigma^*_{1|2} &= \frac{1}{2} ((|0\rangle + i |1\rangle)(\langle 0| - i (1))\langle 1|)
\end{align*}
\]

Proof. First notice that the assemblage \(\{\sigma^*_{a|x}\}\), where \(x \in \{0, 1, 2\}\) and \(a \in \{0, 1\}\), may arise from Alice and Bob sharing a maximally entangled state of two qubits, Alice choosing between the three Pauli measurements to perform on her qubit, and Bob applying on his qubit the identity channel when \(a = 0\) and the transpose the when \(a = 1\). Hence, this assemblage may be mathematically obtained from that in Theorem 3 by setting \(\sigma^*_{a|x} = \sigma^*_{a|xa}\). This shows that \(\{\sigma^*_{a|x}\}\) is indeed a valid assemblage in the scenario, by Def. 6.

Let us now prove that the assemblage is post-quantum. Let us assume, for contradiction, that \(\{\sigma^*_{a|x}\}\) has a quantum realisation. That is, assume there is a state \(\rho\), POVMs \(\{M_{a|x}\}\) for Alice, and CPTP maps \(\{\mathcal{E}^a\}\) for Bob, such that \(\sigma^*_{a|x} = \mathcal{E}^a [\text{tr}_A \{M_{a|x} \otimes \mathbb{I} \rho\}]\). A self-testing argument, imposes that \(\{M_{a|x}\}\) be Pauli measurements and \(\rho\) a maximally entangled state, up to a local isometry. It hence follows that the assemblage \(\{\sigma^*_{a|x|y}\}\) of Theorem 3 may be expressed as \(\sigma^*_{a|x|y} \equiv \mathcal{E}^y [\text{tr}_A \{M_{a|x} \otimes \mathbb{I} \rho\}]\). This is, however, impossible, since \(\{\sigma^*_{a|x|y}\}\) has no quantum realisation. This proves that a quantum model for \(\{\sigma^*_{a|x}\}\) cannot exist.

Finally, let us show that the correlations \(p(ab|x) = \text{tr} \{N_0 \sigma_{a|x}\}\) have a quantum realisation in the instrumental scenario for all POVM \(\{N_0\}\). For a given choice of POVM \(\{N_0\}\), consider the correlations \(p(ab|x)\) in a Bell scenario given by \(p(ab|x) =\).
\[ \text{tr} \left\{ N_b \sigma_{a|xy}^s \right\}, \text{ where } \{ \sigma_{a|xy}^s \} \text{ is the assemblage of Theorem 3.} \]

On the one hand, notice that \( p(ab|x) \equiv p(ab|xa) \). On the other hand, \( \{ \sigma_{a|xy}^s \} \) is a PTP assemblage, and by Theorem 9 the correlations \( p(ab|xy) \) have a quantum realisation. This quantum model for \( p(ab|xy) \) then gives a quantum model for \( p(ab|x) \), and the claim follows.