Separable N-particle Hookean models

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Abstract. A model of $N$ particles, where interactions between disjoint pairs are described by two-particle potentials which depend only on the distance between the particles and the remaining interaction are harmonic is considered. If the coupling constants in the quadratic form describing the harmonic part of the interaction potential fulfill certain separability conditions then the model is separable. Several systems known in the literature as the Hookean two-electron atom (harmonium) and the Hookean hydrogen molecules, appear to be special cases of this model. Some properties of this model are discussed in detail.

1. Introduction

Theory of many-electron systems, in particular quantum chemistry and solid state physics, are based on approximations which impose separability to non-separable multi-dimensional Schrödinger equations. The most important examples include the one-electron model and the Born-Oppenheimer approximation. In the resulting models some features of non-separable effects related to interactions as, for example, the electron correlation or the non-adiabatic effects, are obscured by the enforced separability. Therefore, exactly separable models of interacting particles are of particular interest. Among them, particularly important are exactly and quasi-exactly solvable models.

A system of coupled harmonic oscillators can be decoupled by a linear transformation of the coordinates which diagonalizes the matrix of the coupling constants. The resulting Schrödinger equation is separable and exactly solvable. It is commonly used to model a large number of physical systems ranging from vibrations of polyatomic molecules and the electronic shells of atoms to the structure of atomic nuclei [1].

About 20 years ago Makarewicz, in a very interesting and largely overlooked work [2], formulated a separable and exactly solvable model of $N$ particles in which the attractive interactions are represented by the harmonic potentials and the repulsive ones by the potentials proportional to the inverse of the square of the distance between the particles. Several years ago separable models of three and four particle systems, in which one or two pairs of particles interact by arbitrary binary forces while the remaining interactions are harmonic, have been constructed and applied to a description of hydrogen molecules [3–7]. The separability of these models supplied a possibility of an exact solving a molecular problem without imposing the Born-Oppenheimer approximation.

It appears [8] that the separability is retained regardless of the number of particles if the interactions within disjoint pairs of particles are described by arbitrary two-particle potentials while the remaining interactions are harmonic and the coupling constants fulfill certain
conditions referred to as the separability conditions. With a proper choice of the two-particle potentials the resulting model may be exactly [2, 7, 8] or quasi-exactly solvable [8–10].

In this presentation some specific properties of the separable model of interacting particles as well as its applications in atomic and molecular physics are discussed. In particular, the parametric dependence of the solutions leading to the notion of the molecular shape (otherwise defined a priori on the ground of the Born-Oppenheimer model) [10, 11] and the structure of the separable N-particle potentials are analyzed.

2. Calogero-Sutherland model
A separable and exactly solvable model of interacting identical one-dimensional particles was introduced by Calogero [12] and by Sutherland [13, 14] nearly 40 years ago. In the original formulation the Calogero Hamiltonian contains square inverse interactions and reads:

$$H = \frac{1}{2} \sum_{j=1}^{N} \left( p_j^2 + \omega^2 x_j^2 \right) + \sum_{j<k}^{N} \frac{A}{(x_j - x_k)^2}, \quad (1)$$

The model of Sutherland is defined by a similar Hamiltonian:

$$H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \sum_{j<k}^{N} \frac{A}{\sin^2(x_j - x_k)}. \quad (2)$$

Both models are closely related and over the decades were modified and generalized in many ways. Two different modes of generalization have been recently constructed by Khare [15] and by Enciso et al. [16] who extended the model by introducing to the particles some internal degrees of freedom, referred to as “spin”. A review of the subject has been published recently by Polychronakos [17]. Among many applications of this model one should mention quantum field and string theory, theory of solitons, quantum Hall effect, random matrix theory, many-body problems, quantum chemistry.

3. Separable three-dimensional models
Hamiltonian of coupled harmonic oscillators

$$H_{osc}(r_1, r_2, \ldots, r_N) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} |r_i - r_j|^2 \equiv \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \mathbb{R}^\dagger A_R, \quad (3)$$

where $a_{ij} = a_{ji}$, $a_{ii} = 0$, $\mathbb{R}^\dagger = \{r_1, r_2, \ldots, r_N\}$ and

$$A_{ij} = \sum_{k=1}^{N} a_{ik} \delta_{ij} - a_{ij} (1 - \delta_{ij})$$

can be decoupled by a linear transformation of the coordinates:

$$q_i = \sum_{j=1}^{N} m_j r_j C_{ji}, \quad (4)$$

where $A\mathcal{C} = M\mathcal{C}A$, $M_{ij} = m_i \delta_{ij}$ and $A_{ij} = \lambda_{ij} \delta_{ij}$ [18]. The decoupled Hamiltonian reads

$$H_{osc}(q_1, q_2, \ldots, q_N) = \sum_{i=1}^{N} h_i(q_i), \quad (5)$$
where
\[ h_i(q_i) = \frac{1}{2} p_i^2(q_i) + \lambda_i q_i^2, \] (6)

The resulting Schrödinger equation is separable and exactly solvable. Numerous applications include modelling the vibrations of polyatomic molecules, the electronic shells of atoms (Moshinsky atom), the internal structure of atomic nuclei, and many more.

In the case of two interacting particles the Schrödinger equation may always be decoupled. Let
\[ H(r_1, r_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V_{12}(|r_1 - r_2|). \] (7)
The decoupling is performed by introducing the coordinates of the center of the mass:
\[ r_{12} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}, \] (8)
and the relative coordinates
\[ r_{12} = r_1 - r_2. \] (9)

In the new coordinates
\[ H(r_1, r_2) = h_0(r_{12}^+) + h_1(r_{12}^-), \] (10)

The first one-particle Hamiltonian
\[ h_0(r_{12}^+) = \frac{p_2^2(r_{12}^+)}{2M_{12}}, \] (11)
describes the free motion of the center of the mass and the second one,
\[ h_1(r_{12}^-) = \frac{p_2^2(r_{12}^-)}{2\mu_{12}} + V_{12}(|r_{12}^-|), \] (12)
describes the relative motion of the particles. Here and in the following we denote
\[ M_{ab} = m_a + m_b \] (13)
and
\[ \mu_{ab} = \frac{m_a m_b}{m_a + m_b}, \] (14)

Note, that the relative motion can also be interpreted as a motion of a quasi-particle of mass \( \mu_{12} \) confined in a spherically-symmetric external potential \( V_{12}(|r_{12}^-|) \).

In general, separability is not possible for more than two particles. However, several years ago, separable models describing three- and four-particle systems with specific forms of potentials have been constructed [2–6]. Very recently [8] the model has been generalized for the case of \( N \) particles. The generalization is based on the following theorem:

An \( N \)-particle Hamiltonian
\[ H = \sum_{i=1}^{K} H_{2i-1}(r_{2i-1}, r_{2i}) + \sum_{i=2K+1}^{N} \frac{p_i^2}{2m_i} + W(r_1, r_2, \ldots, r_N), \] (15)

where
\[ H_{2i-1}(r_{2i-1}, r_{2i}) = \frac{p_{2i-1}^2}{2m_{2i-1}} + \frac{p_{2i}^2}{2m_{2i}} + V_{2i-1,2i}(|r_{2i-1} - r_{2i}|), \] (16)
and
\[ W(r_1, r_2, \ldots, r_N) = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} |r_i - r_j|^2 \]  

may be decoupled to \( N \) one-particle Hamiltonians if the masses \( m_i \) and the coefficients \( a_{ij} \) fulfill separability conditions.

Out of these equations:
- One equation describes the free motion of the center of the mass,
- \( K \) equations describe the relative motion of the pairs of particles \( \{1, 2\}, \{3, 4\}, \ldots, \{2K - 1, 2K\} \), interacting by potentials
  \[ V_{2i-1,2i}(|r_{2i-1} - r_{2i}|) + c_{2i-1,2i} |r_{2i-1} - r_{2i}|^2, \quad i = 1, 2, \ldots, K, \]
  where \( c_{2i-1,2i} \) are constants which depend on the parameters of the Hamiltonian (15),
- \( N - K - 1 \) equations describe a set of spherical harmonic oscillators in properly constructed normal coordinates.

The decoupling procedure may be seen as a simple generalization of the one for two particles. Similarly as in the two-particle case, we introduce the center of the mass and the relative coordinates for the pairs of particles

\[ \{a, b\} = \{1, 2\}, \{3, 4\}, \ldots, \{2K - 1, 2K\} \]
\[ a = 2i - 1, \quad b = 2i, \quad i = 1, 2, \ldots, K, \quad K \leq N/2, \]
\[ r_{ab}^+ = \frac{m_a r_a + m_b r_b}{m_a + m_b}, \]
\[ r_{ab}^- = r_a - r_b. \]

As a consequence we can extract from the \( N \)-particle Hamiltonian (15) the one-particle relative-motion Hamiltonians

\[ h_a(r_{ab}) = \frac{p_{ab}^2}{2 \mu_{ab}} + V_{ab}(|r_{ab}|) + c_{ab} |r_{ab}|^2. \]

If the remaining part of the interaction potential (a quadratic form) is \( r_{ab}^- \)-independent, then the one-particle Hamiltonian (21) may be separated. As one can check, the separability condition reads:

\[ a_{pq} = \begin{cases} 
m_p m_q \alpha_{ab}, & \text{if } p, q \leq 2K, \\ m_p \alpha_{aq}, & \text{if } p \leq 2K, \quad q > 2K, \end{cases} \]

where \( \alpha_{ab}^{cd} \) and \( \alpha_{aq} \) are constants with

\[ b = 2 \left[ \frac{p + 1}{2} \right], \quad d = 2 \left[ \frac{q + 1}{2} \right], \quad a = b - 1, \quad c = d - 1, \quad \alpha_{ab} = 0, \quad \alpha_{ab}^{cd} = \alpha_{cd}^{ab} \]

For example, if \( N = 6, \ K = 3 \), then

\[ a_{13} = m_1 m_3 \alpha_{13}^{34}, \quad a_{14} = m_1 m_4 \alpha_{14}^{34}, \quad a_{23} = m_2 m_3 \alpha_{23}^{34}, \quad a_{24} = m_2 m_4 \alpha_{24}^{34}, \]
\[ a_{15} = m_1 m_5 \alpha_{15}^{34}, \quad a_{16} = m_1 m_6 \alpha_{16}^{34}, \quad a_{25} = m_2 m_5 \alpha_{25}^{34}, \quad a_{26} = m_2 m_6 \alpha_{26}^{34}, \]
\[ a_{35} = m_3 m_5 \alpha_{35}^{34}, \quad a_{36} = m_3 m_6 \alpha_{36}^{34}, \quad a_{45} = m_4 m_5 \alpha_{45}^{34}, \quad a_{46} = m_4 m_6 \alpha_{46}^{34}. \]

Let us assume, for simplicity, that \( N = 2K \) and \( a_{2i-1,2i} = 0, \ i = 1, 2, \ldots, K \). The last condition does not restrict the generality since the terms \( a_{2i-1,2i} |r_{2i-1} - r_{2i}|^2 \) may always be
Figure 1. Hookean three-particle system.

included to $V_{2i-1,2i}(|r_{2i-1} - r_{2i}|)$. Then, if the separability condition (21) is fulfilled, after some algebra, one gets

$$W(r_1, r_2, \ldots, r_N) = \sum_{\{ab\}, \{cd\}} \alpha_{ab}^{cd} M_{cd} \left[ \mu_{ab} |r_{ab}^-|^2 + \frac{1}{2} M_{ab} |r_{ab}^+ - r_{cd}^+|^2 \right],$$

(23)

where $\{a, b\} = \{2i - 1, 2i\}$, $i = 1, 2, \ldots, K$. Consequently, if the constant $c_{ab}$ in Eq. (21) is set equal to

$$c_{ab} = \mu_{ab} \sum_{\{cd\}} \alpha_{ab}^{cd} M_{cd}$$

(24)

then the residual part of $W$ does not depend on the relative coordinates of the decoupled pairs.

4. Examples

4.1. $N = 3$, $K = 1$

The case of $N = 3$ and $K = 1$ is represented in Fig 1. It may be used to modelling a large set of systems including the Hookean two-electron atom (if $m_3 >> m_1 = m_2$, the negative ion of positronium (if $m_1 = m_2 = m_3$), and the molecular hydrogen ion (if $m_3 << m_1 = m_2$) [6]. In this case

$$W(r_1, r_2, r_3) = \alpha \left[ m_1 m_3 |r_{13}^-|^2 + m_2 m_3 |r_{23}^-|^2 \right]$$

$$= \alpha \mu_{12} m_3 |r_{12}^-|^2 + \alpha M_{12} m_3 |r_{12}^+ - r_3|^2.$$ 

(25)

Consequently,

$$H = h_0(R_0) + h_1(r_{12}) + h_2(r_{12} + r_3),$$

(26)

with

$$R_0 = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3}{m_1 + m_2 + m_3},$$

(27)

$$h_0(R_0) = \frac{P^2(R_0)}{2M_{12,3}},$$

(28)

$$h_1(r_{12}) = \frac{1}{2J_{12}} + V(|r_{12}|) + \alpha \mu_{12} m_3 |r_{12}^-|^2,$$

(29)
4.2. $N = 4, K = 2$
This case is represented in Fig 2 and corresponds to the Hookean $H_2$ molecule, to $Ps_2$ and to many other four-particle systems [3]. In this case

$$W = \alpha [m_1 m_3 |r_{13}^-|^2 + m_1 m_4 |r_{14}^-|^2 + m_2 m_3 |r_{23}^-|^2 + m_2 m_4 |r_{24}^-|^2]$$

$$= \alpha M_{12} M_{34} |r_{12}^-|^2 + \alpha M_{34} M_{12} |r_{34}^-|^2 + \alpha M_{12} M_{34} |r_{12}^- - r_{34}^+|^2,$$  

(31)

where $\alpha \equiv \alpha_{12}^{34}$. 

Figure 2. Hookean four-particle system.

Figure 3. Hookean six-particle system. All interactions included.
4.3. \( N = 6 \), \( K = 3 \)

This case is represented in Fig 3. The quadratic part \( W \) of the interaction potential has the following form:

\[
W = \alpha_{12}^{34} (m_1 m_3 r_{13}^2 + m_1 m_4 r_{14}^2 + m_2 m_3 r_{23}^2 + m_2 m_4 r_{24}^2) \\
+ \alpha_{12}^{56} (m_1 m_5 r_{15}^2 + m_1 m_6 r_{16}^2 + m_2 m_5 r_{25}^2 + m_2 m_6 r_{26}^2) \\
+ \alpha_{34}^{56} (m_3 m_5 r_{35}^2 + m_3 m_6 r_{36}^2 + m_4 m_5 r_{45}^2 + m_4 m_6 r_{46}^2) \\
= \mu_{12} (\alpha_{12}^{34} M_{34} + \alpha_{12}^{56} M_{56}) |r_{12}^-|^2 + \alpha_{12}^{34} M_{12} M_{34} |r_{12}^- - r_{34}^+|^2 \\
+ \mu_{34} (\alpha_{34}^{12} M_{12} + \alpha_{34}^{56} M_{56}) |r_{34}^-|^2 + \alpha_{34}^{12} M_{12} M_{34} |r_{12}^- - r_{34}^+|^2 \\
+ \mu_{56} (\alpha_{56}^{12} M_{12} + \alpha_{56}^{34} M_{34}) |r_{56}^-|^2 + \alpha_{56}^{12} M_{12} M_{56} |r_{34}^- - r_{56}^+|^2
\]

If all masses, all interaction potentials and all parameters \( \alpha_{cd}^{ab} \) are equal then, with increasing \( N \), we get a series of “crystals” composed of pairs symmetrically distributed on a sphere.

4.4. A \( K \)-segment ladder

If the Hooke-type interactions are retained between the neighbouring pairs only, we get a ladder. The three-segment case is shown in Fig 4. If the ladder is composed of two kinds of particles with masses \( m \) and \( M \), and \( m_1 = m_2 = m_5 = m_6 = \cdots = m_{N-1} = m_N = m \) and \( m_3 = m_4 = m_7 = m_8 = \cdots = m_{N-3} = m_{N-2} = M \), then

\[
W = \alpha Mm \left[ |r_{12}^-|^2 + 2 |r_{34}^-|^2 + \cdots + 2 |r_{N-3,N-2}^-|^2 + |r_{N-1,N}^-|^2 \right] \\
+ 4 \alpha Mm \left[ |r_{12}^- - r_{34}^+|^2 + |r_{34}^- - r_{56}^+|^2 + \cdots + |r_{N-3,N-2}^- - r_{N-1,N}^+|^2 \right].
\]

5. Three-particle systems: general solution

One most interesting features of the separable model is a possibility of an exact analysis of the dependence of the properties of a Hookean molecule on the masses of the particles without imposing the Born-Oppenheimer approximation. Let particles \{1\} and \{2\} be identical fermions of mass \( m \) and let the mass of of the third particle be equal to \( m_3 \). Let us assume that the center of the mass is at the origin of our coordinate system, i.e. \( \mathbf{R}_0 = 0 \). Thus, Eq. (27) implies

\[
m(\mathbf{r}_1 + \mathbf{r}_2) + m_3 \mathbf{r}_3 = 0
\]
Figure 5. Shapes of the effective potential $V_{\text{eff}}(r)$ and the energy levels of confined positronium (A) and of harmonium (B). The values of $\omega$ correspond to analytically solvable cases with $l = 0, 1$. The analytical energies (thick lines) of harmonium and of confined positronium are the same.

In our case $\mu_{12} = m/2$, $M_{12} = 2m$, $M_{12,3} = 2m + m_3$,

$$\mu_{12,3} = \frac{2mm_3}{2m + m_3} \equiv \mu,$$

$$r_3 - r_{12}^+ = \frac{m_3}{\mu} r_3.$$

We select

$$V_{12}(|r_{12}|) = \frac{\zeta}{r_{12}},$$

and substitute $\omega^2 = 2\alpha mm_3$ and $r \equiv r_{12}^{-1}$. Since $\alpha$ is an arbitrary constant, we can assume that $\omega$ is a constant independent of the masses. Then Eqs. (26) – (30) yield

$$H = h_1 + h_2,$$

and

$$h_1 = \frac{1}{m} p^2(r) + \frac{\zeta}{r} + \frac{\omega^2}{4} r^2,$$

(35) and

(36)
\[ h_2 = \frac{1}{2\mu} p^2(\rho) + \omega^2 \rho^2, \]  
(37)

where \[ \rho = \frac{m_3 r_3}{\mu}. \]

The first Hamiltonian, \( h_1 \), describes the relative motion of particles \{1\} and \{2\}. It is the well known Hamiltonian of either harmonium (\( \zeta > 0 \)) or confined positronium (\( \zeta < 0 \)) [19] and the corresponding Schrödinger equation is quasi-exactly solvable. The second Hamiltonian, \( h_2 \), describes the relative motion of particle \{3\} and of the center of the mass of particles \{1\} and \{2\}. It is the Hamiltonian of a spherical harmonic oscillator and the corresponding Schrödinger equation is exactly solvable.

The total wavefunction may be expressed as

\[ \Psi(\mathbf{r}, r_3) = \Phi_{nlm_3}(\mathbf{r}) \Xi_{\nu\lambda\mu_3}(r_3) = (r \, r_3)^{-\frac{1}{2}} \phi_{nl}(r) \xi_{\nu\lambda}(r_3) Y_{lm_3}(\hat{\mathbf{r}}) Y_{\nu\lambda}(\hat{\mathbf{r}}_3), \]

(38)

where the radial functions are solutions of the following equations:

\[ -\frac{1}{m} \frac{d^2}{dr^2} + V_{\text{eff}}(r) - E_{nl}^{(1)} \phi_{nl}(r) = 0, \]

(39)

where

\[ V_{\text{eff}}(r) = \frac{l(l+1)}{mr^2} + \frac{\zeta}{r} + \frac{\omega^2 r^2}{4} \]

(40)

and

\[ -\frac{1}{2\mu} \frac{d^2}{dp^2} + \frac{\lambda(\lambda+1)}{2\mu r^2} + \omega^2 p^2 - E_{\nu\lambda}^{(2)} \xi_{\nu\lambda}(p) = 0. \]

(41)

Let us consider in more detail the ground state solutions of Eqs. (39) and (41). In the case of harmonium (39) the analytical solution exists if some specific relations between \( m, \zeta \) and \( \omega \) are fulfilled [10, 19]. In particular, for the ground state, the analytical solution exists if

\[ \omega = \frac{\zeta^2 m^{3/2}}{2(l+1)}. \]

(42)

The corresponding wavefunction reads:

\[ \phi(r)_{0l} \sim r^{l+1} \left[ 1 + \frac{m\zeta}{2(l+1)} r \right] e^{-\gamma r^2} \]

(43)

where

\[ \gamma = \frac{m^2 \zeta^2}{8(l+1)} = \frac{\omega \sqrt{m}}{4}. \]

As one can show [23], if for a given set of parameters an analytical solution exists for harmonium \( \zeta > 0 \), it also exists for the same set of parameters but with \( \zeta \) replaced by \(-\zeta\), i.e. for the confined positronium. The energies for these two cases appear to be the same. More to say, the energies of harmonium and of the confined positronium are the same only for the states for which analytical solutions exist (see Fig. 5). This property resembles the behaviour of spectra of supersymmetric partner potentials [24, 25] and reflects a specific symmetry related to the transformation \( r \rightarrow -r \).

An approximate solution, valid for arbitrary values of \( m, \zeta \) and \( \omega \), may be obtained by the replacement of \( V_{\text{eff}} \) by its harmonic approximation [20, 21]:

\[ V_{\text{eff}} = V_0 + \frac{3\omega^2}{4}(r - r_0)^2, \]

(44)
where
\[ r_0 = \left( \frac{2\zeta}{\omega^2} \right)^{1/3}. \] (45)

The effective potential (44) corresponds to a shifted harmonic oscillator. Its ground state wavefunction may be expressed as
\[ \phi_{00}(r) \approx \left( \frac{2D}{\pi} \right)^{1/4} e^{-D(r-r_0)^2}, \] (46)
where
\[ D = \frac{1}{4} \sqrt{3} m \omega. \] (47)

Solutions of Eq. (41) are well known. In particular,
\[ \xi_{0\lambda}(r_{12}^+) \sim |r_{12}^+|^{\lambda+1} e^{-c|r_{12}^+|^2}, \] (48)
where
\[ c = \omega \sqrt{m} \left( \frac{2m}{\mu} \right)^{3/2}. \] (49)

Now, using Eqs. (34), (42) and (43), we can get the exact wavefunction of the ground state of the three-particle problem:
\[ \psi(r,r_1^+) = \phi_{00}(r) \xi_{0\lambda}(r_{12}^+) (r_{1r_2^+}) \]
\[ \sim (r_1^2 - r_2^2)^{l} (r_{12}^+)^{\lambda-l} \left[ 1 + \frac{m\zeta}{2(l+1)} r \right] e^{-\gamma[a_+(r_1^2+r_2^2)+a_-r_1r_2]} \] (50)
where
\[ a_+ = (2m/\mu)^{3/2} + 1, \quad a_- = (2m/\mu)^{3/2} - 1. \]

As one can see, for \( m \to \infty \), i.e. for the one-center problem (Hookean helium atom with the infinite mass of the nucleus), \( a_+ \to 2 \) and \( a_- \to 0 \). Consequently, the wavefunction in the limit of two-electron one-center problem becomes
\[ \psi \sim (r_1^2 - r_2^2)^l (r_{12}^+)^{\lambda-l} \left[ 1 + \frac{m\zeta}{2(l+1)} r_{12} \right] e^{-2\gamma(r_1^2+r_2^2)} \] (51)

If \( m = 1 \) and \( \zeta = 1 \) (the Coulomb repulsion of two electrons) then for \( r_{12}^- \to 0 \) and \( l = 0 \) we get the cusp condition of Kato:
\[ \psi \sim 1 + \frac{1}{2} r_{12}. \] (52)

6. Three-particle system: the mass dependence

An analysis of the dependence of solutions of models describing systems of several particles on the particle masses leads to a better understanding of some specific features of the Born-Oppenheimer approximation. In particular, it allows to trace the process of the emergence of the molecular structure from the atomic shape [11]. In order to study this process, let us consider the system of three particles. The density of mass operator of particle \{3\} is defined as
\[ \rho(r; r_3) = m_3 \delta(r - r_3) \] (53)
and for particles \{1\} and \{2\} as
\[ \rho(r; r_1, r_2) = m [\delta(r - r_1) + \delta(r - r_2)], \] (54)
Figure 6. Left panels: Normalized to 1 density of mass of particle \{3\} versus \(z\); \(m_3\) changes from 1 (the upper panel) to 7356 (the lowest panel) while \(m = 1, \zeta = 1\) and \(\omega = 1/2\). Right panels: Normalized to 1 density of mass of particles \{1\} and \{2\}; mass \(m\) changes from 0.25 (the upper panel) to 5000 (the lowest panel) while \(m_3 = 1, \zeta = 1\) and \(\omega = 1/2\).

where

\[
\mathbf{r}_1 = \frac{1}{2} \mathbf{r}_{12} - \frac{m_3}{2m_2} \mathbf{r}_3, \quad \mathbf{r}_2 = \frac{1}{2} \mathbf{r}_{12} + \frac{m_3}{2m_2} \mathbf{r}_3.
\]  

(55)

From here one can calculate the normalized to 1 density of mass distribution for particle \{3\}:

\[
\rho(\mathbf{r})_{m_3} = \langle \Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) | \delta(\mathbf{r} - \mathbf{r}_3) | \Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \rangle = \langle \Phi(\mathbf{r}_{12}) \Xi(\mathbf{r}_3) | \delta(\mathbf{r} - \mathbf{r}_3) | \Phi(\mathbf{r}_{12}) \Xi(\mathbf{r}_3) \rangle = |\Xi(\mathbf{r})|^2
\]  

(56)

and similarly for particles \{1\} and \{2\}:

\[
\rho(\mathbf{r})_m = \frac{1}{2} \langle \Phi(\mathbf{r}_{12}) \Xi(\mathbf{r}_3) | \delta(\mathbf{r} - \mathbf{r}_1) + \delta(\mathbf{r} - \mathbf{r}_2) | \Phi(\mathbf{r}_{12}) \Xi(\mathbf{r}_3) \rangle.
\]  

(57)
Now, using Eqs. (46) and (48), one can perform the integration and express the densities in terms of the parameters of the pertinent wavefunctions [26]. Examples of the densities, as functions of the coordinate $z$, are shown in Fig 6. As one can see, with increasing masses the localized nuclei emerge. At the limit of the Born-Oppenheimer model, i.e. for either $m \rightarrow \infty$ or $m_3 \rightarrow \infty$, the density distributions of the heavy particles approach the Dirac deltas. Let us note, that in the present model the internuclear distance in the Hookean $H_2^+$ molecule is mass-independent and equal to $r_0$. It is in contrast to the Coulombic molecules where the bond length scales inversely proportional to the mass of the light particle. For example, the bond length in a muonic molecule is by factor 207 shorter than in an electronic one.

7. Final remarks
The Schrödinger equation for a systems of particles in which disjoint pairs interact by arbitrary two-particle potentials while the remaining interactions are described by harmonic oscillator potentials is separable. Special cases of this model, known as the Hookean atoms and molecules, are interesting in the context of applications in atomic and molecular physics.

All equations resulting from this model may be reduced to a set of spherically-symmetric one-particle equations. For a large set of physically interesting interaction potentials the radial parts of these equations are either exactly or quasi-exactly solvable. Their analysis, apart of the academic curiosity, supplies interesting data on the nature of various schemes of separability. In particular, it allows to analyze the electron correlation problems and the nature of the Born-Oppenheimer approximation from a new perspective.

Acknowledgement
Supported by the Polish Ministry of Science and Higher Education, project No. N202 041 32/1045.

References
[1] Moshinsky M and Smirnov Y F 1996 The harmonic oscillator in modern physics (Amsterdam: Harwood Academic Publishers)
[2] Makarewicz J 1986 Exact solvable three-dimensional models of many-body systems Am. J. Phys. 54 178-180
[3] Ludeña E V, Lopez X and Ugalde J M 2005 Non-Born-Oppenheimer treatment of the $H_2$ Hookean molecule J. Chem. Phys. 123 024102
[4] Lopez X, Ugalde J M and Ludeña E V 2005 Extracural densities of the non-Born-Oppenheimer Hookean $H_2$ molecule Chem. Phys. Letters 412 381-385
[5] Lopez X, Ugalde J M and Ludeña E V 2006 Exact non-Born-Oppenheimer wave function for the Hooke-Calogero model of the $H_2$ molecule Eur. Phys. J. D 37 351-359
[6] Lopez X, Ugalde J M, Echevarría L and Ludeña E V 2006 Exact non-Born-Oppenheimer wave functions for three-particle Hookean systems with arbitrary masses Phys. Rev. A 74 042504
[7] Howard I A, Amovilli C, Gidopoulos N and March N H 2007 Exactly solvable model mimicking $H_2$ molecule in the limit of large nuclear masses J. Math. Chem. 42 603-615
[8] Karwowski J 2008 A separable model of $N$ interacting particles Int. J. Quantum Chem. 108 2253-2260
[9] Karwowski J 2008 Few-particle systems: quasi-exactly solvable models J. Phys.: Conf. Series 104 012033
[10] Karwowski J and Szewc K 2008 Quasi-exactly solvable models in quantum chemistry Collect. Czech. Chem. Commun. 73 1372-1390
[11] Müller-Herold U 2006 On the emergence of molecular structure from atomic shape in the $1/r^2$ harmonium model J. Chem. Phys. 124 041105
[12] Calogero F 1971 Solution of the one-dimensional $N$-body problems with quadratic and/or inversely-quadratic pair potentials J. Math. Phys. 12 419-436
[13] Sutherland B 1971 Quantum many-body problem in one dimension J. Math. Phys. 12 246-250
[14] Sutherland B 1972 Exact results for a quantum many-body problem in one dimension Phys. Rev. A 5 1372-6
[15] Khare A 1996 Exact solution of an $N$-body problem in one dimension J. Phys. A: Math. Gen. 29 L45-L48
[16] Enciso A, Finkel F, González-Lopéa A and Rodríguez M A 2005 Solvable scalar and spin models with near-neighbors interactions Phys. Lett. B 605 214-22 (2005)
[17] Polychronakos A P 2006 The physics and mathematics of Calogero particles J. Phys. A: Math. Gen. 39 12793-12845
[18] Thouless D J 1972 The quantum mechanics of many-body systems, (New York: Academic Press)
[19] Karwowski J and Cyrnek L 2004 Harmonium Ann. Phys. (Leipzig) 13 181-193
[20] Taut M 1993 Two electrons in an external oscillator potential: Particular analytic solutions of a Coulomb correlation problem Phys. Rev. A 48 3561-3566
[21] Mandal S, Mukherjee P K and Diercksen G H F 2003 Two electrons in a harmonic potential: an approximate analytical solution J. Phys. B: At. Mol. Phys. 36 4483-4494
[22] Kato T 1957 On the eigenfunctions of many-particle systems in quantum mechanics Commun. Pure Appl. Math. 10 151-177
[23] Karwowski J and Cyrnek L 2005 A class of exactly solvable Schrödinger equations Collect. Czech. Chem. Commun. 70 864-880
[24] Dutt R, Khare A and Sukhatme U P 1988 Supersymmetry, shape-invariance, and exactly solvable potentials Am. J. Phys. 56 163-168.
[25] Cooper F, Khare A and Sukhatme U P 1995 Supersymmetry and quantum mechanics Physics Reports 251 267-385.
[26] Szewc K 2009 Quasi-exactly solvable models of atoms and molecules, (Thesis, Nicolaus Copernicus University, Toruń).