An equivariant index formula for elliptic actions on contact manifolds

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Abstract

Given an elliptic action of a compact Lie group $G$ on a co-oriented contact manifold $(M, E)$ one obtains two naturally associated objects: A $G$-transversally elliptic operator $D_b$, and an equivariant differential form with generalised coefficients $J(E, X)$ defined in terms of a choice of contact form on $M$.

We explain how the form $J(E, X)$ is natural with respect to the contact structure, and give a formula for the equivariant index of $D_b$ involving $J(E, X)$. A key tool is the Chern character with compact support developed by Paradan-Vergne [11, 12].

1 Introduction

Let $(M, E)$ be a compact contact manifold; that is, a smooth manifold $M$, and $E \subset TM$ a contact hyperplane distribution. Suppose that $E$ is co-oriented, and let $\alpha \in \Omega^1(M)$ with $\ker(\alpha) = E$ be a contact form compatible with the co-orientation. The distribution $E$ is contact if and only if the restriction of $d\alpha$ to $E$ is symplectic.

A compatible complex structure $J$ on $E$ compatible with $d\alpha$ defines an almost-Cauchy-Riemann (CR) structure $E_{1,0} \subset TM \otimes \mathbb{C}$ on $M$ whose underlying real sub-bundle is $E$ [5]. We suppose $E^{0,1}$ is equipped with a compatible Hermitian metric $h$ and connection $\nabla$, and note the isomorphism $\psi : E^* \to E^{0,1}$ (7) given in Section 3 below.

Allowing a slight abuse of the definitions in [2], let $C(E)$ be the bundle of Clifford algebras over $M$ whose fibre at $x \in M$ is the Clifford algebra of $E^*_x$ with respect to the Riemannian metric on $E$ obtained from $h$. Then $\mathcal{E} = \bigwedge E^{0,1}$ is a spinor module for $C(E)$, with Clifford multiplication given by

$$c(\nu) = c(\psi(\nu)) - c(\psi(\nu)).$$  \hspace{1cm} (1)
Using the connection $\nabla$ and the Clifford multiplication $\mathbb{I}$, we can define a Spin$^c$-Dirac-like operator $\mathfrak{D}_b$ whose principal symbol is $\sigma_b(x, \xi) = -ic(q(\xi))$, where $(x, \xi) \in T^*M$, and $q : T^*M \to E^*$ denotes projection. Since $(\sigma_b)^2(x, \xi) = ||q(\xi)||^2$, the support of $\sigma_b$ is the anihilator bundle $E^0$ of $E$, whence $\mathfrak{D}_b$ is not elliptic. If the almost-CR structure associated to $E$ is integrable, then $M$ is a Cauchy-Riemann manifold, and $\mathfrak{D}_b$ is given in terms of the associated tangential Cauchy-Riemann operator $\overline{\nabla}_b$ by $\mathfrak{D}_b = \sqrt{2}(\overline{\nabla}_b + \overline{\nabla}_b^*)$.

Suppose now that a compact Lie group $G$ acts on $M$, such that the action preserves the contact distribution $E$, as well as its co-orientation, and choose $\alpha$ and $J$ to be $G$-invariant.

We require the action to be elliptic, meaning that $TM$ is spanned by $E$ and the vectors tangent to the $G$-orbits, or equivalently, that $E^0$ is transverse to the space $T_G^*M$ of covectors perpendicular to the $G$-orbits.

Thus, although $\mathfrak{D}_b$ is not elliptic, ellipticity gives $\text{Supp}(\sigma_b) \cap T^*_G M \subset M \times \{0\}$, which implies that $\mathfrak{D}_b$ is a $G$-transversally elliptic operator in the sense of Atiyah $\mathbb{I}$, and that the principal symbol $\sigma_b$ is a $G$-transversally elliptic symbol in the sense of Berline-Vergne $\mathbb{I}$.

Atiyah $\mathbb{I}$ has shown that the kernel and cokernel of any $G$-transversally elliptic operator $P$ will define trace-class representations of $G$, and that the principal symbol of $P$ defines an element in the equivariant $K$-theory $K_G(T^*_G M)$.

The $G$-equivariant index of $P$ is well-defined, but only as a generalised function on $G$, given by the formula $\mathbb{I} \mathbb{I}$:

$$\text{index}^G(P)(g) = \text{Tr}(g, \ker P) - \text{Tr}(g, \ker P^*) \quad (2)$$

Berline and Vergne $\mathbb{I} \mathbb{I}$ have given a character formula which gives the germ of (2) at $g \in G$ in terms of the integral over $T^*M(g)$ of certain equivariant differential forms, as follows:

For $\sigma$ a $G$-transversally elliptic symbol, we have, for $g \in G$ and $X \in \mathfrak{g}(g)$ sufficiently small,

$$\text{index}^G(\sigma)(ge^X) = (2i\pi)^{-\dim M(g)} \int_{T^*M(g)} \text{Ch}^2_{BV}(\sigma, X) \hat{A}^2(M(g), X) \frac{D_K(\mathcal{N}(g), X)}{D_K(\mathcal{N}(g), X)}, \quad (3)$$

where $\text{Ch}^2_{BV}(\sigma, X)$ is the Chern character of $\mathbb{I}$. For a $G$-transversally elliptic operator $P$ with principal symbol $\sigma(P)$, we have $\text{index}^G(P) = \text{index}^G(\sigma(P)) \mathbb{I}$.

Since $T^*M(g)$ is non-compact, this integral is in general defined only as a generalised function on $\mathfrak{g}(g)$, provided appropriate growth conditions are satisfied on the fibres of $T^*M(g) \to M(g)$.

More recent work of Paradan and Vergne $\mathbb{I} \mathbb{I}$ allows one to replace the non-compactly supported equivariant forms by ones with compact support, provided one passes to equivariant differential forms with generalised coefficients: these are $C^{-\infty}$ maps from $g$ to $\mathcal{A}(M)$, as in $\mathbb{I}$. The space of all such forms will be denoted by $\mathcal{A}^{-\infty}(g, M)$.
When one allows generalised coefficients, it is possible to define a natural differential form on $M$ adapted to the contact structure as follows:

Let $\alpha$ be a contact form on $M$, let $D = d - \iota(X)$ be the equivariant differential, and let $\delta_0$ be the Dirac delta distribution on $\mathbb{R}$. Then we may define the form

$$J(E, X) = \alpha \wedge \delta_0(D\alpha(X)),$$

which is well-defined as an element of $A^{-\infty}(g, M)$.

Moreover, using the properties of the delta distribution, one has that

1. $D J(E, X) = 0$, so that $J(E, X)$ defines a class in $H^{-\infty}(g, M)$, the equivariant cohomology of $M$ with generalised coefficients.

2. $J(E, X)$ is independent of the choice of contact form $\alpha$ and thus depends only on the contact structure $E$.

For a fixed $g \in G$, let $i: M(g) \to M$ denote the inclusion of the set of $g$-fixed points in $M$. In Proposition 2.7 we show that $(M(g), E(g))$ is again a contact manifold, with contact form $\alpha^g = i^*\alpha$, so that $J(E(g), X) = \alpha^g \wedge \delta_0(D\alpha^g(X))$ is again well defined, for $X \in \mathfrak{g}(g) \subset \mathfrak{g}$.

In this article, our interest in the form $J(E, X)$ is due to its appearance in our character formula for the equivariant index of the $G$-transversally elliptic operator $D_b/\partial$. The results of [11, 12] allow us to re-write the integrand of (3) in terms of a Chern character $\text{Ch}_M Q(\sigma, X)$ with “gaussian shape” along the fibres of $E^*$ in the sense of [10], and a differential form $P_\lambda(X)$ with generalised coefficients and compact support on $E^0$. We are then able to integrate over the fibres of $T^*M(g)$ to obtain:

**Theorem 1.1.** Let $(M, E)$ be a compact, co-oriented contact manifold of dimension $2n+1$, and let $G$ be a compact Lie group acting elliptically on $M$.

The $G$-equivariant index of $\mathcal{D}_b$ is the generalised function on $G$ whose germ at $g \in G$, is given, for $X \in \mathfrak{g}(g)$ sufficiently small, by

$$\text{index}^G(\mathcal{D}_b)(g e^X) = \frac{1}{(2\pi i)^k} \int_{M(g)} \frac{\text{Td}(E(g), X) J(E(g), X)}{D_C(N(g), X)},$$

where $\dim(M(g)) = 2k + 1$.

In particular, we have the following formula at the identity:

**Theorem 1.2.** For $X \in \mathfrak{g}$ sufficiently small,

$$\text{index}^G(\mathcal{D}_b)(e^X) = \frac{1}{(2\pi i)^n} \int_M \text{Td}(E, X) J(E, X).$$

In the case where $M$ is a principal $U(1)$-bundle over a Hamiltonian $G$-space $(B, \omega)$ with connection form $\tilde{\alpha}$, a contact form is given by $\alpha = i\tilde{\alpha}$, and we obtain the following special case, using the Poisson summation formula:
Corollary 1.3. Let \((B, \omega, G)\) be a Hamiltonian \(G\)-space of dimension \(2n\), with symplectic form \(\omega\), and let \(M\) be a principal \(U(1)\) bundle over \(B\), with associated pre-quantum line bundle \(L\) (on which \(G\) acts with weight 1). Then we have

\[
\text{index}^{G \times U(1)}(D_b)(e^X, e^{i\phi}) = \sum_{m \in \mathbb{Z}} \text{RR}^G(B, L^{\otimes m}, X) e^{im\phi},
\]

where \(\text{RR}^G(B, L^{\otimes m}, X)\) is the equivariant Riemann-Roch number of \(L^{\otimes m} \to B\), given by

\[
\text{RR}^G(B, L^{\otimes m}, X) = \frac{1}{(2\pi i)^n} \int_B \text{Td}(B, X) \text{Ch}(L^{\otimes m}, X).
\]

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2 Elliptic actions on contact manifolds

Let \(G\) be a compact Lie group, and let \(M\) be a \(G\)-manifold. We make use of the following notation:

Definition 2.1. The set \(\bigcup_{x \in M}(T_x(G.x))^0 \subset T^*M\) of covectors orthogonal to the \(G\)-orbits will be denoted \(T^*_G M\).

Definition 2.2. Let \(\eta \in \Omega^1(M)\) be an invariant 1-form on a \(G\)-manifold \(M\). We define the \(\eta\)-moment map to be the map \(f_\eta : M \to \mathfrak{g}^*\) given by the pairing

\[
< f_\eta(m), X > = \eta_m(X_M(m)),
\]

for any \(X \in \mathfrak{g}\), where \(X_M\) is the vector field on \(M\) corresponding to \(X\) via the infinitesimal action of \(\mathfrak{g}\) on \(M\).

We denote by \(C_\eta\) the zero-level set \(f_\eta^{-1}(0) \subset M\) of the \(\eta\)-moment map.

For any \(G\)-space \(V\), we will denote by \(V(g)\) the subset of \(V\) fixed by the action of an element \(g \in G\).

Remark 2.3. Let \(\theta\) be the canonical 1-form on \(T^*M\), and consider the lift of the action of \(G\) on \(M\) to \(T^*M\). This action is Hamiltonian, and \(f_{\theta} : T^*M \to \mathfrak{g}^*\) is the corresponding moment map. We may describe the space \(T^*_G M\) according to \(T^*_G M = C_\theta\).
Let \((M, E)\) denote a compact co-oriented contact manifold, and let \(G\) be a compact Lie group acting on \(M\) and preserving the contact structure \(E\), and the co-orientation. Choose a \(G\)-invariant contact form \(\alpha\) compatible with the co-orientation. That is, if we let \(E^0_+\) be the connected component of \(E^0\setminus 0\) that is positive with respect to the co-orientation, then \(\alpha(M) \subset E^0_+\). We will suppose such a choice of contact form has been fixed throughout the article.

**Remark 2.4.** Recall that the space \(E^0_+ \subset T^*M\) is a symplectic cone over the base \(M\), called the symplectization of \(M\). The symplectic form on \(E^0_+\) is the pullback under inclusion of the canonical symplectic form on \(T^*M\). The cotangent lift of an action of \(G\) on \(M\) preserving \(E\) restricts to a symplectic action of \(G\) on \(E^0_+\) commuting with the natural \(\mathbb{R}_+\) action.

**Definition 2.5.** The action of \(G\) on \((M, E)\) is said to be **elliptic** if and only if \(T^*_G M \cap E^0 = 0\).

For the remainder of this article, we will impose this stronger condition on the action of \(G\) on \(M\).

**Remark 2.6.** The action of \(G\) on \((M, E)\) is elliptic if the orbits of \(G\) in \(M\) are nowhere tangent to the contact distribution. Alternatively, if \(\Phi : E^0_+ \to g^*\) is the restriction of \(f_\theta\) to \(E^0_+\), then the action is elliptic if and only if \(\Phi^{-1}(0) = \emptyset\).

Associated to the chosen contact form \(\alpha\) is the Reeb vector field, which is the vector field \(Y \in \Gamma(TM)\) such that

\[\iota(Y)\alpha = 1\quad\text{and}\quad \iota(Y)d\alpha = 0.\]

Accordingly, we obtain a splitting \(TM = E \oplus \mathbb{R}Y\), dual to the splitting \(T^*M = E^* \oplus \mathbb{R}\alpha\) given by the choice of contact form.

The following proposition is a key lemma for our proof of the fixed-point formula (5).

**Proposition 2.7.** Let \((M, \alpha)\) be a co-oriented contact manifold, and suppose \(G\) is a compact group acting on \(M\) elliptically. For any \(g \in G\), let \(i : M(g) \to M\) denote inclusion of the \(g\)-fixed points. Then we have:

1. The submanifold \(M(g) \subset M\) is a contact manifold, and \(\alpha^g = i^\ast \alpha\) is a contact form.
2. The action of the centraliser \(G(g)\) of \(g\) in \(G\) on \(M(g)\) is elliptic.

**Proof.**

1. Let \(TM = E \oplus \mathbb{R}Y\), where \(Y\) is the Reeb vector field associated to the \(G\)-invariant contact form \(\alpha\), and thus \(G\)-invariant as well.

Denote by \(N(g)\) the normal bundle to \(M(g)\) in \(M\). Then we have that

\[TM|_{M(g)} = TM(g) \oplus N(g) = E(g) \oplus \mathbb{R}Y^g \oplus N(g)\]

by the invariance of \(Y\), where \(Y^g = Y|_{M(g)}\) and \(E(g)\) is the subset of \(E\) fixed by the action of \(g \in G\).
Let $\alpha^g = i^* \alpha$ be the restriction of the contact form $\alpha$ to $M(g)$. Choose any $m \in M(g)$. Then we know that $d\alpha|_{E_m}$ is symplectic. Moreover, the action of $G$ on the symplectic vector space $E_m$ preserves $\alpha$, and thus the symplectic structure, whence $E_m(g)$ is symplectic, with symplectic form $d\alpha^g$.

Finally, we observe that $\alpha(M(g)) \subset T^* M(g) = (TM(g))^*$, and since $\iota(Y^g)\alpha^g = 1$, we have $\ker(\alpha^g) \cap \mathbb{R}g^g = \emptyset$, whence $\ker(\alpha^g) \subset E(g)$, and a dimension count gives $\ker(\alpha^g) = E(g)$.

2. Recall that the action of $G$ on $M$ is elliptic if and only if zero is not in the image of the moment map $\Phi : E^0_+ \to g^*$ (Remark 2.6). Let $H = G(g)$, and let $\mathfrak{h}$ be the Lie algebra of $H$.

If $x \in E_0^0(g)$, we have by the equivariance of $\Phi$ that $\Phi(x) \in \mathfrak{h}^*$. Thus, the corresponding moment map $\Psi : E_0^0(g) \to \mathfrak{h}^*$ for the action of $G(g)$ on $E_0^0(g)$ is simply the restriction of $\Phi$ to $E_0^0$ (the projection from $g^*$ to $\mathfrak{h}^*$ being redundant).

Since $\Psi = \Phi|_{E^0_+}$, it follows that zero is not in the image of $\Psi$, and thus the action of $G(g)$ on $M(g)$ is elliptic. □

3 Definition of the operator $\mathfrak{D}_b$

Let $\alpha \in \Gamma(E^0 \setminus 0)$ be a given choice of contact form on $(M,E)$. Then $d\alpha|_E$ defines a symplectic structure on the fibres of $E$, so that $E$ is a symplectic vector bundle over $M$.

If $\beta$ is any other contact form, then $\beta = e^j \alpha$ for some $f \in C^\infty(M)$. Thus we have $d\beta = e^j df \wedge \alpha + e^j d\alpha$, whence $d\beta|_E = e^j d\alpha|_E$. Therefore, if $J$ is a complex structure on $E$ compatible with $d\alpha$, it is also compatible with $d\beta$, and thus depends only on the contact structure $E$. We may choose $J$ to be $G$-invariant.

The pair $(E,J)$ determines an almost-CR structure $E_{1,0}$ on $M$ whose underlying real bundle is the contact distribution. Thus $E_{1,0} \cap \overline{E_{1,0}} = \emptyset$, and so $E_{1,0} \oplus E_{0,1} = E \otimes \mathbb{C}$, where $E_{0,1} = \overline{E_{1,0}}$. Let $E^{1,0} = E_{1,0}$, giving the decomposition

$$E^* \otimes \mathbb{C} = E^{1,0} \oplus E^{0,1}$$

into $\pm i$-eigenspaces of $J$, where for $\eta \in T^* M$ and $\xi \in TM$ the induced almost-complex structure on $D^*$ is given by $J(\eta)(\xi) = \eta(J(\xi))$

Let $\psi : E^* \cong E^{0,1}$ be the isomorphism given by

$$\psi(\eta) = \eta + iJ(\eta).$$

(7)

Let $p : E^* \to M$, $q : T^* M \to E^*$, and $\pi_M : T^* M \to M$ denote projections. Let $h$ be the $G$-invariant Hermitian metric on $E^{0,1}$ determined by $J$ and $d\alpha$, and let $\nabla$ be a $G$-equivariant Hermitian connection on $E^{0,1}$.

The metric $h$ determines a Riemannian metric on $E$. Let $C(E) \to M$ be the bundle whose fibre over $x \in M$ is the Clifford algebra of $E_x$ with respect to this...
metric. For any \( \nu \in E \), we have the Clifford multiplication given by

\[
c(\nu) = \iota(\psi(\nu)) - \epsilon(\eta(\psi(\nu))),
\]
where \( \epsilon(\eta) \) denotes exterior multiplication by \( \eta \), and \( \iota(\eta) \) denotes contraction with respect to \( h: \iota(\eta, \xi) = h(\eta, \xi) \). The multiplication \( c \) makes \( E = \wedge E_{0,1} \) into a spinor module for \( C(E) \).

We may then use \( \nabla \) and \( c \) to define a \( G \)-equivariant differential operator \( \mathcal{D}_b : \Gamma(E) \to \Gamma(E) \) analogous to the \( \text{Spin}^c \)-Dirac operator by the composition

\[
\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{q} \Gamma(E^* \otimes E) \xrightarrow{c} \Gamma(E).
\]

In the case that \( E_{1,0} \) is integrable; that is, \( E_{1,0} \) defines a Cauchy-Riemann structure, we have

\[
\mathcal{D}_b = \sqrt{2}(\partial_b + \overline{\partial}_b),
\]
where \( \partial_b \) is the tangential Cauchy-Riemann operator defined with respect to \( E_{1,0} \).

Write \( E = E^+ \oplus E^- \) with respect to the \( \mathbb{Z}_2 \)-grading given by exterior degree, and let \( \sigma_b : \pi_M^*E^+ \to \pi_M^*E^- \) denote the morphism given for any \( (x, \xi) \in T^*M \) and \( \gamma \in E^+ \) by

\[
\sigma_b(x, \xi)(\gamma) = -ic(q(\xi))\gamma.
\]

For any morphism \( \sigma \) on \( \pi_M^*E \), define

\[
\text{Supp}(\sigma) = \{(x, \xi) \in T^*M | \sigma(x, \xi) \text{ is not invertible}\}.
\]

Since \( \sigma_b^2(x, \xi) = ||q(\xi)||^2 \), we have \( \text{Supp}(\sigma_b) = E^0 \). This implies that for an elliptic \( G \)-action on \( M \), \( \sigma_b \) is a \( G \)-transversally elliptic symbol on \( T^*M \) in the sense of Atiyah, since \( E^0 \cap T^*_G M = 0 \).

Therefore, \([\pi_M^*E, \sigma]\) defines an equivariant \( K \)-theory class in \( K_G(T^*_G M) \). This class is independent of the almost-CR structure (since any two such structures with underlying real bundle \( E \) are homotopic) and the Hermitian metric. A formula for the equivariant index of this class has been given by Berline and Vergne, but requires the integration of non-compactly supported forms on \( T^*M \).

Following Paradan and Vergne, we will instead pass to equivariant differential forms with generalised coefficients, which will allow us to construct a compactly supported form whose integral over \( T^*M \) agrees with that of the Berline-Vergne formula, and for which the integral over the fibres is easily carried out.

Before dealing with the technical details of this construction and the proof of our main theorem, we pause to consider two simple examples in which our index theorem may be applied.
4 Examples

The simplest examples of an elliptic group action on a contact manifold involve free circle actions. A particularly simple example is discussed in [14]: that of a circle acting on itself by multiplication.

**Example 1: \( S^1 \)**

Consider the circle \( S^1 = \{ e^{i\theta} | \theta \in \mathbb{R} \} \). The form \( d\theta \) is a contact form on \( S^1 \), with the zero section as the contact distribution. The group \( U(1) = \{ e^{i\phi} \} \) acts freely on \( S^1 \) by multiplication. The action is elliptic, since \( T^*G S^1 = 0 \) (while \( E^0 = T^*S^1 \)).

Here, our operator is \( D_b = 0 \), and since \( T^*G S^1 = 0 \), even the zero operator on \( S^1 \) is \( U(1) \)-transversally elliptic. The \( U(1) \)-equivariant index is given simply by

\[
\text{index}^G(0)(e^{i\phi}) = \int_{S^1} J(\phi) = 2\pi \delta_0(\phi) = \sum_{m \in \mathbb{Z}} e^{im\phi},
\]

where the last equality is valid for \( \phi \) sufficiently small, using the Poisson summation formula for \( \delta_0 \).

**Example 2: \( S^3 \)**

Let \( M = S^3 \) be the unit sphere in \( \mathbb{R}^4 \) with co-ordinates \((x_1, y_1, x_2, y_2)\), and consider the frame \( \{ X, Y, T \} \) for \( TS^3 \) given by

\[
X = x_2 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2},
Y = -y_2 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2},
T = y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_2}.
\]

A contact structure is given locally by \( E = TS^3 / RT \). If we let \( \{ \xi, \zeta, \alpha \} \) denote the corresponding co-frame, then \( \alpha \) is a contact form on \( S^3 \). In co-ordinates we have

\[
\alpha = y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2,
\]

and one readily sees that \( \alpha(T) = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 \), so that \( T \) is the Reeb field for \( \alpha \).

We let \( U(1) \) act on \( S^3 \), with action given in complex co-ordinates as follows: identify \( \mathbb{R}^4 \cong \mathbb{C}^2 \) via \( z_j = x_j + iy_j, j = 1, 2 \). The action of \( e^{i\phi} \in U(1) \) on \( \mathbb{C}^2 \) by \( e^{i\phi} (z_1, z_2) = (e^{i\phi} z_1, e^{i\phi} z_2) \) restricts to an action of \( U(1) \) on \( S^3 \). Let \( \mathfrak{g} = i\mathbb{R} \) denote the Lie algebra of \( G \), and note that the infinitesimal action of \( \mathfrak{g} \) on \( M \) is given by \( i\phi \mapsto -\phi T \). The orbits of the action are thus transverse to the contact distribution \( E \), making the action of \( U(1) \) on \( S^3 \) elliptic.

The almost-CR structure on \( M \) is given by taking \( E_{1,0} = \mathbb{C}Z \), where \( Z = \frac{1}{\sqrt{2}} (X + iY) \). The corresponding covector in \( E^* \otimes \mathbb{C} \) is \( \theta = \frac{1}{\sqrt{2}} (\xi - i\zeta) \), so that
$E^{1,0} = \mathbb{C} \theta$ and $E^{0,1} = \mathbb{C} \bar{\theta}$. The associated complex structure on $E$ comes from the complex structure on $\mathbb{C}^2$, and is given by $J(X) = -Y$ and $J(Y) = X$, so that $J(\xi) = \zeta$, and $J(\zeta) = -\xi$ on $E^*$. Since this structure is integrable, $M$ is a CR manifold, and $\mathcal{D}_b = \sqrt{2}(\partial_b + \partial_b^*)$.

Writing $\eta \in T_x^* S^3$ as $\eta = a\xi + b\zeta + c\alpha$, the symbol of $\mathcal{D}_b$ is given by

$$\sigma_b(x, \eta) = -i(\sqrt{2}(a + ib)\iota_Z - \sqrt{2}(a - ib)\epsilon(\bar{\theta})),$$

from which we see that $\sigma_b^2(x, \eta) = a^2 + b^2$.

Finally, the $U(1)$-equivariant index of $\mathcal{D}_b$ is given by

$$\text{index}^{U(1)}(\mathcal{D}_b)(e^{i\phi}) = \frac{1}{2\pi i} \int_{S^3} \text{Td}(E, \phi)J(E, \phi) = 2\pi(\delta_0(\phi) - i\delta'_0(\phi))$$

$$= \sum_{m \in \mathbb{Z}} (1 + m)e^{im\phi} = \sum_{m \in \mathbb{Z}} \text{RR}(S^2, \Lambda^m)e^{im\phi},$$

since

$$J(E, \phi) = \alpha \wedge \delta_0(d\alpha + \phi) = \alpha(\delta_0(\phi) + \bar{\alpha}'(\phi)d\alpha),$$

while, if $\pi : S^3 \to S^2$ denotes the projection onto the orbit space,

$$\text{Td}(E, \phi) = \text{Td}(\pi^*TS^2, \phi) = \pi^* \text{Td}(S^2)$$

$$= \pi^*(1 + i\omega) = 1 + i\omega.$$

The last equality is an instance of Corollary (6.6), where

$$\text{RR}(S^2, \Lambda^m) = \frac{1}{2\pi i} \int_{S^2} \text{Td}(S^2)e^{im\omega}.$$
supported test function $\phi \in C^\infty(g)$, $\int_g \alpha(X) \phi(X)$ is a smooth differential form on $N$.

The equivariant differential $D$ extends to $A^{-\infty}(g, N)$, with $D^2 = 0$, and so it is possible to define the space the space $\mathcal{H}^{-\infty}(g, N)$ of equivariant cohomology with generalised coefficients. See [8, 12, 14] for examples where this space is computed.

For non-compact $N$, we may consider as well the cohomology of (generalised) equivariant differential forms with compact support on $N$, denoted by $\mathcal{H}^{\pm\infty}(g, N)$.

### 5.1 Forms in $A^{-\infty}(g, N)$ derived from distributions on $\mathbb{R}$

It will be useful in our computation of the equivariant index to work in terms of the following generalised functions on $\mathbb{R}$:

$$\delta_+(x) = \frac{i}{2\pi} \lim_{\epsilon \to 0} \frac{1}{x + i\epsilon}, \quad \delta_-(x) = \frac{-i}{2\pi} \lim_{\epsilon \to 0} \frac{1}{x - i\epsilon}.$$  

Note that we have

$$\delta_+(x) + \delta_-(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \delta_0(x),$$

which we identify as the Dirac delta distribution $\delta_0(x)$, giving the first of the following identities:

$$\delta_+ + \delta_- = \delta_0, \tag{11}$$

and for any $a \in \mathbb{R} \setminus \{0\}$, we have

$$a \delta_0(ax) = \begin{cases} \delta_0(x), & \text{if } a > 0 \\ -\delta_0(x), & \text{if } a < 0 \end{cases}, \quad a \delta_\pm(ax) = \begin{cases} \delta_\pm(x), & \text{if } a > 0 \\ -\delta_\pm(x), & \text{if } a < 0 \end{cases}. \tag{13}$$

The integral representations of these generalised functions given by

$$\delta_+(x) = \frac{1}{2\pi} \int_0^\infty e^{ixt} dt, \quad \delta_-(x) = \frac{1}{2\pi} \int_{-\infty}^0 e^{ixt} dt$$

will be helpful in the computation of the index formulas to follow.

Distributions on $\mathbb{R}$, such as the above, can be used to define equivariant differential forms with generalised coefficients. For example, given an invariant 1-form $\beta$ on $N$, the form $\delta_0(D\beta)$, which we may view as the oscillatory integral $\int_0^\infty e^{ixt} dt$, is well-defined as a generalised equivariant form wherever the pairing $X \mapsto \beta(X_N)$ is non-zero.

We explain in general how such a form is well-defined: Let $u \in C^{-\infty}(\mathbb{R})$ is a distribution on $\mathbb{R}$. We may consider its pull-back by a smooth, proper map $h : g \to \mathbb{R}$, which will give a well-defined distribution $h^*u = u \circ h$ on $g$ provided

$$h^*(WF(u)) \cap (g \times \{0\}) = \emptyset \subset T^*g, \tag{14}$$

where $WF(u)$ denotes the wavefront set of $u \in g$.
Remark 5.1. Note that for the resulting distribution on $g$, we have $WF(h^*u) \subset h^*WF(u)$. Furthermore, for any derivative $u^{(j)}$ of $u$, we have $WF(u^{(j)}) \subset WF(u)$, so that if a map $h$ satisfies the condition above with respect to $u$, it does so for all the derivatives of $u$ as well.

Now, if $\beta$ is an invariant 1-form on $M$, then for a fixed point $m \in M$, $f_{\beta}(m)$ gives us a linear map from $g$ to $\mathbb{R}$. If $f_{\beta}(m)$ satisfies (14) for all $m \in M$, then we may set

$$u(D\beta)(X) = u(d\beta - f_{\beta}(X)) = \sum \frac{u^{(j)}(-f_{\beta}(X))}{j!} (d\beta)^j,$$

which is well-defined by Remark 5.1.

On a contact manifold $(M, E)$ on which a Lie group $G$ acts elliptically, consider the form

$$J(E, X) = \alpha \wedge \delta_0(D\alpha(X)),$$

where $\alpha$ is any contact form.

The ellipticity hypothesis ensures that the pairing $X \mapsto \alpha(X_M)$ is non-zero: We have $WF(\delta_0) = \{0\} \times (\mathbb{R} \setminus 0)$, while $f_{\alpha}^{-1}(0) = \emptyset$. Thus for all $m \in M$, $\eta = f_{\alpha}(m)$ is non-zero, and

$$\eta^*(WF(\delta_0)) = \{(X, t\eta) \in g \times g^*| (\eta(X), t) \in \{0\} \times (\mathbb{R} \setminus 0)\}.$$

Since $t\eta$ is never zero, (14) is satisfied.

Using the properties of the delta function given above, we obtain the following:

**Proposition 5.2.** Let $(M, \alpha)$ be a co-oriented contact manifold on which a Lie group $G$ acts elliptically. Then the form $J(E, X)$ is equivariantly closed, and independent of the choice of contact form.

**Proof.** We have:

$$D(\alpha \wedge \delta_0(D\alpha)) = D\alpha \wedge \delta_0(D\alpha) = 0 \quad \text{by (12)},$$

while if we change $\alpha$ to $e^f \alpha$ for some $f \in C^\infty(M)$ we have using (13) that

$$e^f \alpha \wedge \delta_0(D(e^f \alpha)) = e^f \alpha \wedge \delta_0(e^f (df \wedge \alpha + D\alpha)) = \alpha \wedge \delta_0(df \wedge \alpha + D\alpha) = \alpha \wedge \delta_0(D\alpha),$$

where in the last equality we have used (15), and the fact that $\alpha \wedge \alpha = 0$. \hfill $\square$

Note that since $J(E, X)$ is independent of the choice of contact form, it does not matter if $\alpha$ is $G$-invariant, as long as the action preserves $E$.

**Remark 5.3.** Recall [7] that given a symplectic manifold $(N, \omega)$ of dimension $2n$ and a Hamiltonian action of a Lie group $G$ on $N$ with moment map $\Psi$ we may define the Duistermaat-Heckman distribution $u_{DH}$ on $C_c^{\infty}(g^*)$ by

$$<\phi, u_{DH}> = \int_{g^*} \phi u_{DH} = \int_N (\Psi^* \phi) e^{-\omega}.$$
The Fourier transform of $u_{DH}$ is given, for $h \in C_c^{\infty}(g)$ by
\[
\langle \widehat{u_{DH}}, h \rangle = \int_{g^*} h(X)I(X)dX,
\]
so that $\widehat{u_{DH}} = I(X)dX$, where
\[
I(X) = \frac{1}{(2\pi i)^{n+1}} \int_{E^0} e^{i\omega(X)} = \frac{1}{(2\pi i)^n} \int_{M} \mathcal{J}(E, X).
\]

Now, given a co-oriented contact manifold $(M, E)$ of dimension $2n + 1$, consider the annihilator $E^0$ of $E$. Although not quite a symplectic manifold, since the form $\omega = d(t\alpha)$ is degenerate for $t = 0$, we have the moment map $\Psi = tf_\alpha$, and if we compute $I(X)$ in this case, we find
\[
I(X) = \frac{1}{(2\pi i)^{n+1}} \int_{E^0} e^{i\omega(X)} = \frac{1}{(2\pi i)^n} \int_{M} \mathcal{J}(E, X).
\]

**Remark 5.4.** By Proposition 2.7, for any $g \in G$ we know that $(M(g), E(g))$ is a contact manifold on which $G(g)$ acts elliptically, so that $\mathcal{J}(E(g), X)$ is well-defined as a $G(g)$-equivariant differential form with generalised coefficients. Furthermore, although in general the restriction of a distribution to a subspace is not well-defined, we have the following:

**Proposition 5.5.** Let $i : C^{-\infty}(g(g)) \otimes A^\infty(M(g)) \to C^{-\infty}(g) \otimes A^\infty(M)$ denote inclusion. Then $i^* \mathcal{J}(E, X)$ is well-defined, and therefore, we have $i^* \mathcal{J}(E, X) = \mathcal{J}(E(g), X)$.

**Proof.** Consider the form $\alpha \wedge \delta_0(\Phi(X))$, for $X \in g$. Let $f_\alpha : M \to g^*$ be the $\alpha$-moment map. Fix a point $m \in M$, and let $\phi = f_\alpha(m) : g \to \mathbb{R}$.

Then, as noted above, we have that
\[
WF(\delta_0(\Phi_m)) = WF(\phi^* \delta_0) \subset \phi^*WF(\delta_0)
\]
\[
= \{(x, \xi) \in g \times g^* | \phi(X) = 0 \text{ and } \xi = t\phi, \text{ for } t \in \mathbb{R} \setminus 0\}.
\]

The restriction of $\phi^* \delta_0$ to $h = g(g) \subset g$ is well-defined provided that
\[
WF(\phi^* \delta_0) \cap (h \times h^0) = \emptyset. \tag{17}
\]
(See [a].) Now, for $m \in M(g)$ the equivariance of $f_\alpha$ gives us that $\phi \in h*$:
\[
f_\alpha(m) = f_\alpha(g \cdot m) = g \cdot f_\alpha(m) \Rightarrow f_\alpha(m) \in g^*(g) = h^*.
\]

Thus, condition (17) holds for all $m \in M(g)$, so that $i^* \mathcal{J}(M, X)$ is indeed well-defined for $X \in g(g)$. 

12
5.2 Cohomology with support

We give here a quick summary of the material in [11] and [12] that is relevant to the proof of our index theorem.

Suppose \( F \) is a closed, \( G \)-invariant subset of \( N \). Then there are two cohomology spaces associated to \( F \) defined in [11]: the relative equivariant cohomology \( \mathcal{H}^\infty(g, N, N \setminus F) \) and the equivariant cohomology with compact support \( \mathcal{H}^\infty_c(g, N) \).

Cohomology classes in the former are represented by pairs \( (\eta, \xi) \), where \( \eta \in A^\infty(g, N) \) and \( \xi \in A^\infty(g, N \setminus F) \), that are closed under the relative equivariant differential \( D_{rel}(\eta, \xi) = (D\eta, \eta|_{N \setminus F} - D\xi) \), while cohomology classes in the latter are defined as follows:

Let \( U \subset N \) be any open, \( G \)-invariant subset containing \( F \). We may consider the spaces \( A^\infty_U(g, N) \) of equivariant differential forms with support contained in \( U \), and their corresponding cohomology spaces \( \mathcal{H}^\infty_U(g, N) \).

If we have two open subsets \( V \) and \( U \) with \( F \subset V \subset U \), the inclusion \( A^\infty_V(g, N) \hookrightarrow A^\infty_U(g, N) \) induces a map \( f_{U,V} : \mathcal{H}^\infty_V(g, N) \to \mathcal{H}^\infty_U(g, N) \), and so we obtain the inverse system \( (\mathcal{H}^\infty_U(g, N), f_{U,V}, U, V \in \mathcal{F}_F) \), where \( \mathcal{F}_F \) is the family of all open, \( G \)-invariant neighbourhoods of \( F \), letting us define the space of cohomology with support contained in \( F \) as the inverse limit of this system.

By [12], all of the above can be extended to equivariant cohomology with generalised coefficients, including the morphism

\[
p_F : \mathcal{H}^{\pm\infty}(g, N, N \setminus F) \to \mathcal{H}^{\pm\infty}_F(g, N)
\]

defined in [11] as follows:

Let \( U \) be any open, \( G \)-invariant neighbourhood of \( F \), and choose a cutoff function \( \chi \in C^\infty(N)^G \) with support contained in \( U \), such that \( \chi \equiv 1 \) on a smaller neighbourhood of \( F \). Let \( (\eta, \xi) \) represent a class in \( \mathcal{H}^{\pm\infty}(g, N, N \setminus F) \) (so that \( \eta|_{N \setminus F} = D\xi \)), and set

\[
p^\chi(\eta, \xi) = \chi\eta + d\chi\xi.
\]

By Proposition 3.14 in [11], \( p^\chi(\eta, \xi) \) is an equivariantly closed form with support in \( U \), whose class \( p_U(\eta, \xi) \) in \( \mathcal{H}^{\infty}_U(g, N) \) does not depend on \( \chi \). Moreover, \( f_{U,V} \circ p_V = p_U \), so that we may define \( p_F(\eta, \xi) \) to be the element defined by taking the inverse limit over invariant neighbourhoods of \( F \).

Remark 5.6. An element of \( \mathcal{H}^{\pm\infty}_F(g, N) \) in the image of \( p_F \) may be represented in calculations by one of the forms \( p^\chi(\eta, \xi) \).

If \( F \) is compact, there is a natural map

\[
\mathcal{H}^{\pm\infty}_F(g, N) \to \mathcal{H}^{\pm\infty}_c(g, N).
\]
The composition of \( p_F \) with \([20]\) defines a map denoted \( p_c \) in \([11]\).

In the case where \( N \) is a \( G \)-equivariant vector bundle we introduce two other complexes of differential forms: the complexes \( A^\pm_{rdm}(g, N) \) of differential forms that are rapidly decreasing in mean:

**Definition 5.7.** Suppose \( N \to B \) is a \( G \)-equivariant vector bundle over the compact base \( B \), and suppose \( \beta : g \to N \) is an equivariant differential form on \( N \) (possibly with generalised coefficients). We say that \( \beta \) is rapidly decreasing in mean if for any smooth, compactly supported density \( \rho \) on \( g \), the differential form \( \beta_\rho = \int g \beta(X) \rho(X) \, dX \) and all its derivatives are rapidly decreasing along the fibres of \( N \to B \).

The equivariant differential \( D \) is well-defined on \( A^-_{rdm}(g, N) \), and so we may define the cohomology space \( H^-_{rdm}(g, N) \).

Note that we have the inclusions

\[
A^\infty_{rdm}(g, N) \hookrightarrow A^-_{rdm}(g, N) \hookrightarrow A^\infty_{rdm}(g, N).
\]  

(21)

### 5.3 Chern characters

Suppose \( \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \) is a \( \mathbb{Z}_2 \)-graded \( G \)-equivariant vector bundle over \( N \), and let \( \mathbb{A} \) be a superconnection on \( \mathcal{E} \), in the sense of Mathai-Quillen \([10]\) or, more generally, Berline-Getzler-Vergne \([2]\). Thus \( \mathbb{A} \) is an odd operator on \( \mathcal{A}(N, \mathcal{E}) \) which preserves the \( \mathbb{Z}_2 \)-grading of \( \mathcal{E} \), and satisfies the derivation property

\[
\mathbb{A}(\omega s) = d \omega s + (-1)^{deg \omega} \omega \mathbb{A}s,
\]

for any form \( \omega \) on \( N \), and any section \( s \) of \( \mathcal{E} \). We suppose further that \( \mathbb{A} \) is invariant under the action of \( G \) on \( \mathcal{E} \). One example is the superconnection \( \mathbb{A} = \nabla + L \) on \( \mathcal{E} \) considered in \([10]\), where \( \nabla \) is a connection on \( \mathcal{E} \) in the usual sense, and \( L \) is an odd endomorphism of \( \mathcal{E} \).

**Definition 5.8** \([2][12]\). Given a superconnection \( \mathbb{A} \) on \( \mathcal{E} \), we define the moment of \( \mathbb{A} \) to be the map \( \mu^{\mathbb{A}} : g \to \mathcal{A}(N, \text{End}(\mathcal{E}))^+ \) given by

\[
\mu^{\mathbb{A}}(X) = \mathcal{L}(X) - [\iota_{X_N}, \mathbb{A}],
\]

where \( \mathcal{L}(X) = [d, \iota_{X_N}] \) is the Lie derivative in the direction of \( X_N \).

In the case \( \mathbb{A} = \nabla + L \) mentioned above, the moment of \( \mathbb{A} \) becomes simply \( \mu(X) = \mathcal{L}(X) - \nabla_X \).

The equivariant curvature of \( \mathbb{A} \) is the map \( \mathbb{F}(\mathbb{A}) : g \to \mathcal{A}(N, \text{End}(\mathcal{E}))^+ \) given by \( \mathbb{F}(\mathbb{A})(X) = \mathbb{A}^2 + \mu^{\mathbb{A}}(X) \), and the equivariant Chern character of \( (\mathcal{E}, \mathbb{A}) \) is the equivariant differential form \( \text{Ch}(\mathbb{A}, X) = \text{Str}(e^{\mathbb{F}(\mathbb{A})(X)}) \).

The equivariant Chern character is equivariantly closed, so that \( \text{Ch}(\mathbb{A}, X) \) defines a class in \( \mathcal{H}^\infty(g, N) \) equal to the Chern character of \( \mathcal{E} \).

Now, if we are given a smooth, \( G \)-equivariant morphism \( \sigma : \mathcal{E}^+ \to \mathcal{E}^- \), define \( \sigma^* \) using an invariant Hermitian metric on \( \mathcal{E} \). Then the map

\[
v_\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}
\]
defines an odd Hermitian endomorphism of $E$, and we can associate to it a differential form given by
\[ \text{Ch}(A^\sigma, X) \, \text{Ch}(A(\sigma, 1), X) = \text{Str}(e^{F(A, \sigma, 1)}(X)), \] (22)
where $A(\sigma, t) = A + itv_\sigma$, and $F(A, \sigma, t)(X)$ is the equivariant curvature of $A(\sigma, t)$. Explicitly, we have
\[ F(A, \sigma, t)(X) = -t^2 v_\sigma^2 + it[A, v_\sigma] + F(A)(X). \] (23)

We remark that in the non-equivariant setting, the Chern character (22) is essentially the form considered by Mathai-Quillen [10], in the case where $N$ is a vector bundle. We will denote by $\text{Ch}_{MQ}(\sigma, X)$ the corresponding equivariant Chern character studied in [11].

If we define as well the transgression form $\eta(A, \sigma, t) = -i \text{Str}(v_\sigma e^{F(A, \sigma, t)})$, then on $N \setminus \text{Supp}(\sigma)$, we have the well-defined equivariant differential form $\beta(A, \sigma) \in A^\infty(g, N \setminus \text{Supp}(\sigma))$ given by
\[ \beta(A, \sigma) = \int_0^\infty \eta(A, \sigma, t) dt. \]
Then $\text{Ch}(A)|_{N \setminus \text{Supp}(\sigma)} = D\beta(A, \sigma)$ [11], so that
\[ (\text{Ch}(A), \beta(A, \sigma)) \in \mathcal{H}^\infty(g, N, N \setminus \text{Supp}(\sigma)). \] (24)
This is the relative Chern character $\text{Ch}_{rel}(\sigma, X)$ of [11]. With $F = \text{Supp}(\sigma)$, we can use the map (15) to obtain a class
\[ \text{Ch}_{sup}(\sigma, X) = c_F(\text{Ch}_{rel}(\sigma, X)) \in \mathcal{H}^\infty_{\text{supp}(\sigma)}(g, N), \]
which is independent of the superconnection $\mathcal{A}$, and can be represented in computations by an equivariant form
\[ c(\sigma, \mathcal{A}, \chi) = \chi \text{Ch}(\mathcal{A}) + d\chi \beta(\mathcal{A}, \sigma), \]
where $\chi \in C^\infty(N)$ is a $G$-invariant cutoff function equal to 1 on a neighbourhood of $\text{Supp}(\sigma)$, with support contained in $U$, for some $G$-invariant neighbourhood of $\text{Supp}(\sigma)$.

Remark 5.9. If $\sigma$ is elliptic, so that $\text{Supp}(\sigma)$ is compact, then we obtain a class $\text{Ch}_c(\sigma) \in \mathcal{H}^\infty_c(g, N)$ under the the natural map (20) given in the previous subsection.

Furthermore, we have the following theorem [12]:

Theorem 5.10. Suppose $N$ is $G$-equivariant vector bundle over a manifold $B$. Then if $\sigma$ satisfies suitable growth conditions along the fibres of $N$ (see [12]), the form $\text{Ch}_{MQ}(\sigma, X)$ is an element of $A^\infty_{\text{adm}}(g, N)$ and represents the image of the class $\text{Ch}_{sup}(\sigma, X) \in \mathcal{H}^\infty_{\text{supp}(\sigma)}(g, N)$ in $\mathcal{H}^\infty_{\text{rdm}}(g, N)$.

Moreover, if the fibres of $\pi : N \to B$ are oriented, and the action of $G$ preserves the orientation, then we have $\pi_* \text{Ch}_{MQ}(\sigma, X) = \pi_* \text{Ch}_{sup}(\sigma, X)$ in $\mathcal{H}^\infty(g, B)$.
We now move from forms with smooth coefficients to those with generalised coefficients, which will allow us to shrink the support of our Chern character by using a $G$-invariant 1-form to modify the superconnection.

In [12] we see that the above results carry over to equivariant cohomology with generalised coefficients.

Let $\lambda \in \mathcal{A}^1(g, N)$ be an invariant 1-form. We use $\lambda$ to deform the part of our superconnection of exterior degree one, obtaining a new superconnection $\mathcal{A}^{\sigma, \lambda} = \mathcal{A}(\sigma, \lambda, 1)$, according to

$$\mathcal{A}(\sigma, \lambda, t) = \mathcal{A} + it(\sigma + \lambda), \quad \text{for } t \in \mathbb{R} \text{ and } v_\sigma + \lambda = \begin{pmatrix} \lambda & \sigma \\ \sigma^* & \lambda \end{pmatrix}.$$  

As before, we set $F(\mathcal{A}, \sigma, \lambda, t) = (\mathcal{A} + it(\sigma + \lambda))^2 + \mu$, so that $F(\mathcal{A}, \sigma, \lambda, 1)$ is the equivariant curvature of $\mathcal{A}^{\sigma, \lambda}$, $Ch(\mathcal{A}^{\sigma, \lambda}) = Str(e^{F(\mathcal{A}, \sigma, \lambda, 1)})$ is the associated character form, and

$$\eta(\sigma, \lambda, \mathcal{A}, t) = -i \text{Str}(v_\sigma + \lambda)e^{F(\mathcal{A}, \sigma, \lambda, t)}$$

the transgression form.

Then we may define $\beta(\mathcal{A}, \sigma, \lambda) = \int_{0}^{\infty} \eta(\mathcal{A}, \sigma, \lambda, t)dt$, which is now well-defined on $N \setminus (\text{Supp}(\sigma) \cap C_\lambda)$, but only as a differential form with generalised coefficients [12].

We thus obtain a class

$$Ch_{rel}(\sigma, \lambda) = (Ch(\mathcal{A}), \beta(\mathcal{A}, \sigma, \lambda)) \in \mathcal{H}^{-\infty}(g, N, N \setminus (\text{Supp}(\sigma) \cap C_\lambda)),$$

giving us

$$Ch_{\sup}(\sigma, \lambda) = c_F(Ch_{rel}(\sigma, \lambda)) \in \mathcal{H}^{-\infty}(g, N, N \setminus F),$$

where $F = \text{Supp}(\sigma) \cap C_\lambda$. The class $Ch_{\sup}(\sigma, \lambda)$ is independent of $\mathcal{A}$, and can be represented by a differential form

$$c(\sigma, \lambda, \mathcal{A}, \chi) = \chi Ch(\mathcal{A}) + d\chi \beta(\sigma, \lambda, \mathcal{A}),$$

where $\chi \in C^\infty(N)^G$ is equal to 1 on a neighbourhood of $\text{Supp}(\sigma) \cap C_\lambda$, and has support contained in a $G$-equivariant neighbourhood $U$ of $\text{Supp}(\sigma) \cap C_\lambda$.

As in the smooth case, we have the following [12]:

**Theorem 5.11.** Suppose that $N$ is a $G$-equivariant vector bundle over a $G$-manifold $B$. If $\sigma$ and $\lambda$ satisfy suitable growth conditions along the fibres of $N \to B$, then $Ch(\mathcal{A}^{\sigma, \lambda}) \in \mathcal{H}^{-\infty}_{rdm}(g, N)$ and $\beta(\mathcal{A}, \sigma, \lambda) \in \mathcal{A}^{-\infty}_{rdm}(g, N \setminus (\text{Supp}(\sigma) \cap C_\lambda))$, and $Ch(\mathcal{A}^{\sigma, \lambda})$ represents the image of $Ch_{\sup}(\sigma, \lambda)$ in $\mathcal{H}^{-\infty}_{rdm}(g, N)$.

Moreover, if the fibres of $\pi : N \to B$ are oriented, and the action of $G$ preserves the orientation, then the morphism $\pi_* : \mathcal{H}^{-\infty}_{rdm}(g, N) \to \mathcal{H}^{-\infty}(g, B)$ is well-defined, and $\pi_* Ch(\mathcal{A}^{\sigma, \lambda}) = \pi_* Ch_{\sup}(\sigma, \lambda)$ in $\mathcal{H}^{-\infty}(g, B)$.

In the case of a trivial bundle, we can define a class $P_\kappa \in \mathcal{H}^{-\infty}_C(g, N)$ via $P_\kappa = c_F(2\pi, \beta(\kappa))$, where $F = C_\kappa$ and $\beta(\kappa) = -i\kappa \int_{0}^{\infty} e^{itD_\kappa} dt = -2\pi i \delta_{\kappa}(D\kappa)$, so that $D\beta = 2\pi$ by [12]. If $U$ is a $G$-invariant neighbourhood of $C_\kappa$ and $\chi \in C^\infty(U)^G$ is equal to 1 on a neighbourhood of $C_\kappa$, then $P_\kappa$ can be represented by the form $P_U(\kappa, \chi) = 2\pi \chi + d\chi \beta(\kappa)$. 

16
5.4 The case of a contact manifold

We return now to the case of a compact, co-oriented contact manifold \((M, E)\). Consider the (almost) complex vector bundle \(p : E^{0,1} \to M\) obtained from an almost-CR structure on \(M\). Equip \(E^{0,1}\) with a \(G\)-invariant Hermitian metric \(h\) compatible with the symplectic structure on \(E\) and the almost-CR structure. Let \(\nabla\) be a \(G\)-equivariant Hermitian connection on \(E^{0,1}\) and let \(F(X)\) be its equivariant curvature.

The symbol \(\sigma_b\) \(^{[10]}\) on \(\pi_M^*\mathcal{E}\) is just the pullback by \(q : T^*M \to E^*\) of the equivariant morphism \(\sigma_{E^{0,1}}^G : p^*(\wedge E^{0,\text{even}}) \to p^*(\wedge E^{0,\text{odd}})\) defined in \(^{[11]}\). Furthermore, we have \(\sigma_0^G = \sigma_b\) with respect to the metric \(h\), so that \(\sigma_b^G = \sigma_b^G\) Id, giving \(\text{Ch}_{MQ}(\sigma_b, X)\) "Gaussian shape" along the fibres of \(E^*\) as in \(^{[10]}\).

If we define the equivariant Todd form of \((E^{0,1}, \nabla)\) for \(X \in \mathfrak{g}\) sufficiently small by

\[
\text{Td}(E^{0,1}, X) = \det_{\mathbb{C}} \left( \frac{F(X)}{e^{F(X)} - 1} \right),
\]

then we have \(^{[11]}\)

\[
\text{Ch}_{MQ}(\sigma_{E^{0,1}}, X) = (2\pi i)^n p^*(\text{Td}(E^{0,1}, X)^{-1}) \text{Th}_{MQ}(E^{0,1}) \text{ in } \mathcal{H}_{rdm}^{\infty}(\mathfrak{g}, E^{0,1}),
\]

where \(\text{Th}_{MQ}(E^{0,1})\) is an equivariant version of the Thom form defined in \(^{[10]}\), and \(n\) is the complex rank of \(E^{0,1}\).

If we pull back the above result to \(T^*M\), then we obtain

\[
\text{Ch}_{MQ}(\sigma_b, X) = (2\pi i)^n \pi_M^* (\text{Td}(E^{0,1}, X)^{-1}) q^* (\text{Th}_{MQ}(E^{0,1})),
\]

(26)

where \(\text{Ch}_{MQ}(\sigma_b, X) = \text{Ch}(\mathcal{A}_{\mathfrak{g}}^\infty, X)\).

Let \(\theta\) be the canonical 1-form on \(T^*M\). Since the action of \(G\) on \(M\) is assumed to be elliptic, we know that \(F = \text{Supp}(\sigma_b) \cap C_\theta = T^*_G M \cap E^0 = \{0\}\) is compact. Thus \(\text{Ch}_{sup}(\sigma_b, \theta)\) defines a class in \(\mathcal{H}_{c}^{\infty}(\mathfrak{g}, T^*M)\) under the mapping \(^{[24]}\).

Denote by \(\text{Ch}_{BV}(\sigma_b, X)\) the Chern character of \(^{[3]}\), given by

\[
\text{Ch}_{BV}(\sigma_b, X) = \text{Str}(e^{F(\cdot, \sigma_b, \theta, 1)}(X)),
\]

which is an element of \(\mathcal{A}_{rdm}^{\infty}(\mathfrak{g}, T^*M)\).

By Theorem 5.11 the image of \(\text{Ch}_{sup}(\sigma_b, \theta)\) in \(\mathcal{H}_{rdm}^{\infty}(\mathfrak{g}, T^*M)\), under the maps induced by the inclusions \(^{[24]}\), is represented by \(\text{Ch}_{BV}(\sigma_b, X)\).

Similarly, by Theorem 5.10 \(\text{Ch}_{MQ}(\sigma_b, X)\) is a representative of \(\text{Ch}_{sup}(\sigma_b)\) in \(\mathcal{H}_{rdm}^{\infty}(\mathfrak{g}, T^*M)\), and is related to \(\text{Ch}_{BV}(\sigma_b, X)\) by the following two lemmas from \(^{[12]}\).

**Lemma 5.12.** Let \(\kappa\) be a \(G\)-invariant 1-form on \(N\), and define \(P_\kappa\) as above. Then:

1. Under the natural map \(\mathcal{H}_{C_\kappa}^{-\infty}(\mathfrak{g}, N) \to \mathcal{H}^{-\infty}(\mathfrak{g}, N)\), the image of \(P_\kappa\) is equal to 1.
2. \( \text{Ch}(\sigma, \kappa) = P_\kappa \wedge \text{Ch}(\sigma) \) in \( \mathcal{H}_{\text{supp}(\sigma) \cap C_\kappa}(\mathfrak{g}, N) \).

**Lemma 5.13.** If \( \theta|_{\text{supp}(\sigma)} = \lambda|_{\text{supp}(\sigma)} \), then \( \text{supp}(\sigma) \cap C_\theta = \text{supp}(\sigma) \cap C_\lambda = T_{\overline{g}, M} \cap E^0 \), and \( \text{Ch}(\sigma, \theta) = \text{Ch}(\sigma, \lambda) \) in \( \mathcal{H}_{T_{\overline{g}, M} \cap E^0}(\mathfrak{g}, T^*M) \).

Together, the two above lemmas give:

**Proposition 5.14.** Let \( i : E^0 \hookrightarrow T^*M \) be the inclusion of \( E^0 \), and define \( \lambda = i^*\theta \). Using the splitting \( T^*M = E^* \oplus E^0 \), consider \( \lambda \) as a form on all of \( T^*M \), by taking \( \lambda|_{E^*} = 0 \), and \( \lambda|_{E^0} = i^*\theta \).

Then, \( \lambda \) and \( \theta \) agree on \( \text{supp}(\sigma_b) = E^0 \), and we have

\[
\text{Ch}_{BV}(\sigma_b, X) = P_\lambda(X) \wedge \text{Ch}_{MQ}(\sigma_b, X) \text{ in } \mathcal{H}_{rdm}(\mathfrak{g}, T^*M).
\] (27)

## 6 Calculation of the index

We now apply the results of the previous section, in the case of a compact, co-oriented contact manifold \((M, E)\) to the Berline-Vergne index formula \([3]\).

Recall that the equivariant \( \hat{A} \)-class is defined for any real \( G \)-equivariant vector bundle \( E \rightarrow M \), with \( G \)-equivariant connection \( \nabla \) and corresponding equivariant curvature \( F(X) \) by

\[
\hat{A}(E, X) = \det_{\mathbb{R}}^{1/2} \left( \frac{F(X)}{e^{F(X)/2} - e^{-F(X)/2}} \right),
\]

with the choice of square root depending on orientation. The equivariant \( \hat{A} \)-class of \( TM \rightarrow M \) is denoted by \( \hat{A}(M, X) \).

The form \( D_R(N(g), X) \) associated to the normal bundle is defined in \([\mathbb{M}]\) as follows:

**Definition 6.1.** For \( g \in G \), let \( F_N(X), X \in \mathfrak{g}(g) \), denote the equivariant curvature of \( N(g) \) with respect to a \( G(g) \)-equivariant connection. Then \( D_R(N(g), X) \) is the \( G(g) \)-equivariantly closed from on \( M(g) \) given for \( X \in \mathfrak{g}(g) \) by

\[
D_R(N(g), X) = \det_{\mathbb{R}}(1 - g^N e^{F_N(X)}),
\]

where \( g^N \) denotes the lifted action of \( g \in G \) on \( N(g) \).

We similarly define \( D_C(N(g), X) \) using the complex determinant in place of the real determinant used above. Note that using the canonical complex structure on \( N(g) \), we may write \( N(g) \otimes \mathbb{C} = N(g) \oplus \overline{N(g)} \) and obtain:

\[
D_R(N(g), X) = D_C(N(g) \otimes \mathbb{C}, X) = D_C(N(g), X)D_C(\overline{N(g)}, X).
\]

We are now ready to state the main theorem of this article:

**Theorem 6.2.** Let \((M, E)\) be a compact, co-oriented contact manifold of dimension \( 2n + 1 \), and let \( G \) be a compact Lie group acting elliptically on \( M \). Let \( g \in G \), and suppose \( \dim M(g) = 2k + 1 \).
The $G$-equivariant index of $\mathcal{D}_b$ is a generalised function on $G$ whose germ at $g \in G$, is given, for $X \in \mathfrak{g}(g)$ sufficiently small, by

$$\text{index}^G(\mathcal{D}_b)(ge^X) = \frac{1}{(2\pi i)^n} \int_{M(g)} \frac{\text{Td}(E(g), X) J(E(g), X)}{D_C(N(g), X)}.$$ (28)

### 6.1 The formula near the identity

We first consider the index formula for group elements $e^X$, for $X \in \mathfrak{g}$ sufficiently small. The calculation in this case is simpler, and employs the results of [12] directly. The general result will then follow an analogous approach.

**Theorem 6.3.** For $X \in \mathfrak{g}$ sufficiently small, we have

$$\text{index}^G(\mathcal{D}_b)(e^X) = \frac{1}{(2\pi i)^n} \int_M \text{Td}(E, X) J(M, X).$$ (29)

**Proof.** The formula of Berline-Vergne for the equivariant index of a transversally elliptic operator is given by

$$\text{index}^G(\mathcal{D}_b)(e^X) = (2\pi i)^{- (2n + 1)} \int_{T^* M} \pi_M^* (\hat{A}^2(M, X)) \text{Ch}_{BV}(\sigma, X).$$ (30)

Using the splitting $TM = E \oplus \mathbb{R}$, and the almost-complex structure on $E$, we have that

$$\hat{A}^2(M, X) = \hat{A}^2(E, X) = \text{Td}(E \otimes \mathbb{C}, X) = \text{Td}(E^{1,0}, X) \text{Td}(E^{0,1}, X).$$ (31)

By (26) and (27), we have

$$\text{Ch}_{BV}(\sigma_b, X) = P_\Lambda(X) \text{Ch}_{MQ}(\sigma_b, X)$$

$$= (2\pi i)^n P_\Lambda(X) \pi_M^* (\text{Td}(E^{0,1}, X)^{-1}) q^* (\text{Th}_{MQ}(E^{0,1})).$$ (32)

Combining (31) and (32), we find

$$\pi_M^* (\hat{A}^2(M, X)) \text{Ch}_{BV}(\sigma_b, X) = (2\pi i)^n \pi_M^* (\text{Td}(E^{1,0}, X)^{-1}) q^* (\text{Th}_{MQ}(E^{0,1})) P_\Lambda(X)$$ (33)

in $\mathcal{H}^{-\infty}(\mathfrak{q}, T^* M)$.

**Lemma 6.4.** In terms of the projections $q: T^* M \rightarrow E^*$ and $p: E^* \rightarrow M$ we have

$$q_* P_\Lambda(X) = 2\pi i p^* J(E, X).$$

That is, $(\pi_M)_* P_\Lambda = p_* q_* P_\Lambda = 2\pi i J(E, X)$ in $\mathcal{H}^{-\infty}(\mathfrak{q}, M)$. 

19
Proof. A representative of $P_\lambda$ is given by

$$P_\lambda^X = 2\pi \chi + d\chi \wedge \beta(\lambda) = 2\pi \chi - 2\pi i d\chi \wedge \lambda \delta_+(D\lambda),$$

where $\chi$ is a cutoff function with support in a neighbourhood of $E^*$, and $\chi \equiv 1$, on $E^*$.

Since $\lambda$ is a form on $E^0$, and $\chi$ is constant on $E^*$, $P_\lambda$ is independent of $E^*$, and so it remains to calculate the integral over the fibre of $E^0 = M \times \mathbb{R}$. Let $t$ be the co-ordinate along the fibre, and write $\chi = \chi(t)$. Then $\chi(t)$ is supported on a neighbourhood of $t = 0$, with $\chi(0) = 1$, and $\lambda$ may be written as $\lambda = t\alpha$, for $\alpha$ a contact form on $M$. Thus $D\lambda = D(t\alpha) = dt \wedge \alpha = tD\alpha$, and $P_\lambda$ becomes

$$P_\lambda = 2\pi \chi(t) - 2\pi i \chi'(t) dt \wedge \alpha t \delta_+(dt \wedge \alpha + tD\alpha)$$

$$= 2\pi \chi(t) - 2\pi i \chi'(t) \wedge \alpha t \delta_+(tD\alpha).$$

Thus, the integral over $\mathbb{R}$ becomes, with the help of the identities in Section 5.1,

$$\int_{-\infty}^{\infty} P_\lambda = -2\pi i \int_{-\infty}^{\infty} dt \chi'(t) \alpha t \delta_+(tD\alpha)$$

$$= -2\pi i \alpha \left[ \int_{0}^{\infty} dt \chi'(t) \delta_+(D\alpha) - \int_{-\infty}^{0} dt \chi'(t) \delta_-(D\alpha) \right]$$

$$= -2\pi i \alpha \left[ -\delta_+(D\alpha) - \delta_-(D\alpha) \right]$$

$$= 2\pi i \alpha \delta_0(D\alpha),$$

and we obtain our result. \qed

Let $\text{Td}(E, X)$ denote the cohomology class of the Todd form $\text{Td}(E^{1,0}, X)$, and write $\text{Th}_{MQ}(E^{0,1}) = \text{Th}_{MQ}(E^*)$ using the isomorphism (7). Then using Lemma 6.4 and (33) in the index formula (30), we obtain

$$\text{index}^G(\mathcal{D}_h(e^X)) = (2\pi i)^{-(2n+1)} \int_{T^*M} (2\pi i)^n \pi_M^*(\text{Td}(E, X)) q^* \text{Th}_{MQ}(E^*) P_\lambda(X)$$

$$= \frac{1}{(2\pi i)^n} \int_{E^*} p^*(\text{Td}(E, X) \mathcal{J}(E, X)) \text{Th}_{MQ}(E^*)$$

$$= \frac{1}{(2\pi i)^n} \int_M \text{Td}(E, X) \mathcal{J}(E, X).$$

\textbf{Remark 6.5.} Let $\pi : M \to B$ be a principal $U(1)$-bundle over a Hamiltonian $G$-space $(B, \omega, G)$ of dimension $2n$, with symplectic form $\omega$, such that $\frac{1}{2\pi} [\omega] \in H^2(B; \mathbb{Z})$. Then $M$ is a contact manifold, with contact form $\alpha$ such that $\tilde{\alpha} = -i\alpha$ is a connection form. The associated line bundle $L = M \times_{U(1)} \mathbb{C}$ can be equipped with a hermitian connection $\nabla$ with curvature $-i\omega$, so that $L$ is a pre-quantum line bundle over $B$, and $M$ can be viewed as the unit circle bundle inside $L$. 20
Corollary 6.6. Let $RR^G(B, L^\otimes m, X)$ denote the $G$-equivariant Riemann-Roch number of $L^\otimes m$, given by

$$RR^G(B, L^\otimes m, X) = \frac{1}{(2\pi i)^n} \int_B Td(B, X)e^{im\omega(X)}.$$ 

Then

$$\text{index}^{G\times U(1)}(\mathcal{P}_h)(e^X, e^{i\phi}) = \sum_{m \in \mathbb{Z}} RR^G(B, L^\otimes m, X)e^{im\phi}.$$ 

Proof. The group $G \times S^1$ acts freely on $M$, since $S^1$ does. Since $E = \pi^*TB$ and $d\alpha = \pi^*\omega$, we have $Td(E, (X, \phi)) = \pi^*Td(B, X)$, and $D\alpha(X, \phi) = \pi^*\omega(X) + \phi$. Thus,

$$\text{index}^{G\times U(1)}(\sigma)(e^X, e^{i\phi}) = \int_M Td(E, (X, \phi))J(E, (X, \phi))$$ 

$$= \frac{1}{(2\pi i)^n} \int_B Td(B, X)2\pi\delta_0(\omega(X) + \phi)$$ 

$$= \frac{1}{(2\pi i)^n} \int_B Td(B, X) \sum_{m \in \mathbb{Z}} e^{im(\omega(X) + \phi)}$$ 

$$= \sum_{m \in \mathbb{Z}} RR^G(B, L^\otimes m, X)e^{im\phi}. \quad \square$$

6.2 Fixed Point Formula

The general calculation of the pushforward of $[\mathfrak{g}]$ from $T^*M(g)$ to $M(g)$ is analogous to the proof given above, since by Proposition 2.7, we know that $(M(g), E(g))$ is again a contact manifold on which $G(g)$ acts elliptically. The primary added difficulty comes from the appearance of the action of $g \in G$ in the Chern character form of $[\mathfrak{g}]$.

Proof of Theorem 6.2. By Proposition 2.7 $(M(g), E(g))$ is again a contact manifold, and we have the splitting

$$T^*M(g) = E^*(g) \oplus \mathbb{R}\alpha^g.$$ 

Let $\dim M(g) = 2k+1$, so that $E(g)$ is a vector bundle over $M(g)$ of complex rank $k$.

Denote by $j : T^*M(g) \to T^*M$ the inclusion of the $g$-fixed point set in $T^*M$. Let $H = G(g)$, with Lie algebra $\mathfrak{h}$. Let $\mathfrak{h}^g = j^*\mathfrak{h}$, $\theta^g = j^*\theta$ and $\sigma^g = j^*\sigma$ denote the restrictions of the superconnection, canonical 1-form and symbol of the previous section to $T^*M(g)$. Let $p, q$ denote the projections $p : E^*(g) \to M(g)$ and $q : T^*M(g) \to E^*(g)$.
The Chern character $\text{Ch}_B^g(\sigma_b)(X)$ of Berline and Vergne is given by

$$\text{Ch}_B^g(\sigma_b)(X) = \text{Str}(g^E \cdot j^*(e^{iF(A_E \oplus \nabla_N)}(X))).$$

By [3], since $\sigma_b$ is $G$-transversally elliptic, $\sigma_b^g$ is $H$-transversally elliptic, and $\text{Ch}_B^g(\sigma_b)$ defines a class in $\mathcal{H}_{rdm}(\mathfrak{h}, T^*M(g))$.

Let $j^*\nabla = \nabla_E \oplus \nabla_N$ denote the decomposition of the restriction of the Hermitian connection $\nabla$ on $E^{0,1}$ into connections on $E^{0,1}(g)$ and $N(g)$, respectively. Since $\mathfrak{h} = q^*\nabla$, we have $\mathfrak{h}^g = \mathfrak{h}_E \oplus \nabla_N$, where $\mathfrak{h}_E = q^*\nabla_E$.

By Lemma 19 of [3], the canonical 1-form on $T^*M(g)$ is simply the restriction $\theta^g$ of the canonical 1-form $\theta$ on $T^*M$. Since $\theta^g$ is invariant under the action of $g$, we have

$$\text{Ch}_B^g(\sigma_b, X) = e^{i\theta^g} \text{Str}(g^E \cdot e^{F(A_E \oplus \nabla_N \cdot \sigma_b)}(X)).$$

**Lemma 6.7.** Let $V$ be a complex vector space of dimension $k$, and let a Lie group $G$ act on $V$, such that the action commutes with the natural $U(k)$ action on $V$. Let $\rho: U(k) \to \bigwedge V^*$ denote the representation of $U(k)$ on $\bigwedge V^*$ as in [70], and let $w \in \mathfrak{u}(k)$ be a skew-symmetric Hermitian matrix. Then for any $g \in G$, we have

$$\text{Str}(g \cdot e^{\rho(w)}) = \det_C(1 - g \cdot e^{-w}).$$

*Proof.* Since the action of $G$ on $\bigwedge V^*$ commutes with the representation $\rho$, the actions of $g$ and $w$ can be simultaneously diagonalised. $\square$

Using the above Lemma we may write

$$\text{Ch}_{BV}(\sigma_b, X) = e^{i\theta^g} \text{Ch}(\mathfrak{h}_E, \sigma_b^g, X)D_{\mathbb{C}}(N(g), X),$$

using

$$\det_{\mathbb{C}}(1 - g^E \cdot (j^*e^{-F(A_E \oplus \nabla)}(X))) = \det_{\mathbb{C}}(1 - e^{-F(A_E)}(X)) \det_{\mathbb{C}}(1 - g^N \cdot e^{-F(N)}(X)),$$

since $g$ acts trivially on $T^*M(g)$.

The form $\text{Ch}(\mathfrak{h}_E, \sigma_b^g, X)$ appearing in (34) is simply the Mathai-Quillen form $\text{Ch}_{MQ}(\sigma_b^g, X)$ on $E^{0,1}(g)$. We again use Theorem 5.11 and Lemmas 5.12 and 5.13 as follows:

A representative of the class $\text{Ch}_{\sup}(\sigma_b^g, \theta^g, X) \in \mathcal{H}_{\sup(\sigma_b^g) \cap C_{\theta^g}}(\mathfrak{h}, T^*M(g))$ in $\mathcal{H}_{\sup}(\mathfrak{h}, T^*M(g))$ is given by $e^{i\theta^g(X)} \text{Ch}_{MQ}(\sigma_b^g, X) = \text{Ch}(\mathfrak{h}_E, \sigma_b^g, \theta^g, X) \in \mathcal{A}_{rdm}(\mathfrak{h}, T^*M(g))$, and we have

$$\text{Ch}_{\sup}(\sigma^g, \theta^g, X) = \text{Ch}_{\sup}(\sigma^g, \lambda^g, X) = P_{\lambda^g}(X) \text{Ch}_{\sup}(\sigma^g, X).$$

Since a representative of $\text{Ch}_{\sup}(\sigma_b^g, X)$ is $\text{Ch}_{MQ}(\sigma_b^g, X)$, we have

$$\text{Ch}_{BV}(\sigma_b, X) = P_{\lambda^g}(X) \text{Ch}_{MQ}(\sigma_b^g, X)D_{\mathbb{C}}(N(g), X)$$

in $\mathcal{H}_{\sup(\sigma_b^g) \cap C_{\theta^g}}(\mathfrak{h}, T^*M(g))$. 

22
As before, $\text{Ch}^g_{MQ}(\sigma_b)$ is the pull-back of a form on $E^*(g)$, while $P_\alpha$ depends only on $R_\alpha$. Integrating $P_\alpha$ over $\mathbb{R}$ proceeds the same as in the proof of Theorem 6.3, giving $2\pi i \mathcal{J}(E(g), X)$ as the result.

Finally, we substitute (35) into (8) and use (26) to obtain:

\[
\text{index}^G(D_b/\mathbb{R})(g \cdot e^X) = \frac{1}{(2\pi i)^{2k+1}} \int_{T^*M(g)} \frac{\text{Ch}^g_{BV}(\sigma_b)(X) \hat{A}^2(M(g), X)}{D_\mathbb{R}(N(g), X)}
\]
\[
= \frac{1}{(2\pi i)^{k+1}} \int_{E^*(g) \oplus \mathbb{R}^k} \frac{P_{ab}(X) \pi^*_M(\text{Td}(E^{1,0}(g), X)q^* \text{Th}_{MQ}(E^{0,1}(g)))}{\pi^*_M(D_\mathcal{C}(N(g), X))}
\]
\[
= \frac{1}{(2\pi i)^k} \int_{E^*(g)} \frac{p^*(\mathcal{J}(E(g), X) \text{Td}(E^{1,0}(g), X)) \text{Th}_{MQ}(E^{0,1}(g))}{p^*(D_\mathcal{C}(N(g), X))}
\]
\[
= \frac{1}{(2\pi i)^k} \int_{M(g)} \text{Td}(E(g), X) \mathcal{J}(E(g), X) \frac{D_\mathcal{C}(N(g), X)}{D_\mathcal{C}(N(g), X)}.
\]

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