TORSION POINTS ON THE COHOMOLOGY JUMP LOCI OF COMPACT KÄHLER MANIFOLDS

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Abstract. We prove that each irreducible component of the cohomology jump loci of rank one local systems over a compact Kähler manifold contains at least one torsion point. This generalizes a theorem of Simpson for smooth complex projective varieties. An immediate consequence is the conjecture of Beauville and Catanese for compact Kähler manifolds. We also provide an example of a compact Kähler manifold, whose cohomology jump loci can not be realized by any smooth complex projective variety.

1. Introduction

Let $X$ be a connected CW complex of finite type. The cohomology jump loci capture the geometry of $X$. There is a series of results in the rank one case obtained by Beauville, Green-Lazarsfeld, Arapura, Simpson and many others. In this note, we generalized a result of Simpson about smooth complex projective varieties to compact Kähler manifolds.

We recall some definitions first. Let $X$ be a CW complex of finite type. Define $\text{Char}(X) = \text{Hom}(\pi_1(X), \mathbb{C}^*)$ to be the variety of rank one characters of $\pi_1(X)$. For each point $\rho \in \text{Char}(X)$, there exists a unique rank one local system $L_\rho$, whose monodromy representation is isomorphic to $\rho$. We can also regard $\text{Char}(X)$ as the moduli space of rank one local systems on $X$. $\text{Char}(X)$ is determined by the homology group $H_1(X, \mathbb{Z})$, and it is isomorphic to the direct product of $(\mathbb{C}^*)^{\text{b}_1(X)}$ and a finite abelian group. We use the following conventions as in [Sc1].

Definition 1.1. Let $\rho : X \to Y$ be a morphism of complex tori, and let $f^* : \text{Char}(Y) \to \text{Char}(X)$ be the map induced by $f_* : \pi_1(X) \to \pi_1(Y)$. Then $\text{Im}(f^*)$ is an algebraic subgroup in $\text{Char}(X)$. Then a linear subvariety of $\text{Char}(X)$ is of the form $\rho \cdot \text{Im}(f^*)$, for some $f$ as above and some $\rho \in \text{Char}(X)$. Such a linear subvariety of $\text{Char}(X)$ is called arithmetic, if $\rho$ can be chosen to be a torsion point in $\text{Char}(X)$.

The cohomology jump loci $\Sigma^i_k(X) = \{ \rho \in \text{Char}(X) \mid \dim H^i(X, L_{\rho}) \geq k \}$ are canonically defined Zariski closed subsets of $\text{Char}(X)$. It is known in many examples that these cohomology jump loci reflect the geometry of the topological space $X$. For instance, when $X$ is a smooth complex projective variety, each $\Sigma^i_k(X)$ is a finite union of arithmetic subvarieties of $\text{Char}(X)$ (see [Si2]), and this is recently generalized to smooth quasi-projective varieties (see [BW]). When $X$ is a compact Kähler manifold, it is known that each $\Sigma^i_k(X)$ is a finite union of linear subvarieties of $\text{Char}(X)$ ([GL], [A]). Moreover, Campana [C] proved that $\Sigma^1_k(X)$ is a finite union of arithmetic subvarieties.

The main result of this note is the following.

Theorem 1.2. Suppose $X$ is a compact Kähler manifold. Then for any $i, k \in \mathbb{N}$, $\Sigma^i_k(X)$ is a finite union of arithmetic subvarieties for any $i, k \in \mathbb{N}$.

An immediate consequence of the theorem is a conjecture of Beauville and Catanese. For a compact complex manifold $X$, the Dolbeault cohomology jump loci are defined to be $\Sigma^{pq}_k(X) = \{ E \in \text{Pic}^c(X) \mid \dim H^q(X, \Omega^p_X \otimes E) \geq k \}$. Green and Lazarsfeld [GL] showed that when $X$ is a compact Kähler manifold, each $\Sigma^{pq}_k(X)$ is a finite union of translates of subtori. Beauville and Catanese [B] conjectured that these translates are always by torsion points. When $X$ is a smooth complex projective variety, this is proved by Simpson [Si2]. As a corollary of Theorem 1.2 we know it is true for any compact Kähler manifold.

Corollary 1.3. When $X$ is a compact Kähler manifold, each $\Sigma^{pq}_k(X)$ is a finite union of torsion translates of subtori.
Proof. Hitchin-Kobayashi correspondence ([UY] for the compact Kähler manifold version) gives a natural real Lie group isomorphism between the torsion Picard group Pic^r(X) and the group of unitary rank one characters Char^u(X) of π_1(X). Now, the corollary follows from the Hodge decomposition for unitary local systems [SH] and the argument in the proof of Lemma 3.3. The same proof appeared in [SH1] and [Sc1]. We shall not give all the details here. 

Our approach is to extend a theorem of Schnell [Sc1] from abelian varieties to complex tori. Recall that the cohomology jump loci can be defined relative to a constructible complex on X. Let M be a constructible complex of C-modules on X, the cohomology jump loci of M can be defined relative to a constructible complex X, such that M ⊂ N ⊂ C.

Theorem 1.4. Let T be a compact complex torus, and let M be a perverse sheaf that underlies a polarizable Hodge module. Assume that M admits a Z-structure. Then for any i ∈ Z and k ∈ N, Σ_k(T, M) is a finite union of arithmetic subvarieties in Char(T).

Assuming Theorem 1.4, we can easily deduce Theorem 1.2. The proof of Theorem 1.4 will appear in Section 4.

Proof of Theorem 1.2. Let a : X → Alb(X) be the Albanese map. Denote the connected component of Char(X) containing the trivial character by Char^0(X). Then a induces an isomorphism between the character varieties a∗ : Char(Alb(X)) → Char^0(X). For any local system L on Alb(X),

H^i(Alb(X), Ra∗a(CX ⊗_C L)) ∼= H^i(X, a∗(L)).

Therefore,

a∗(Σ_k(Alb(X), Ra∗a(CX[dim X]))) = Σ_k^{i + dim X}(X) ∩ Char^0(X).

According to [Sh] and [PS],

Ra∗a(CX[dim X]) ∼= ⊕_{|i| ≤ δ} P_i[−i]

where δ is the defect of semismallness of a, and each P_i is a perverse sheaf that underlies a polarizable Hodge module. By [Sc1] Lemma 5.2, each P_i admits a Z-structure. Hence, by Theorem 1.4 and Lemma 3.3, Σ_k(Alb(X), Ra∗a(CX[dim X])) is a finite union of arithmetic subvarieties for any i ∈ Z, k ∈ N. Therefore, Σ_k(X) ∩ Char^0(X) is a finite union of arithmetic subvarieties for any i, k ∈ N. Suppose H^i(X, C) does not contain any torsion element. Then Char(X) = Char^0(X), and hence the theorem is proved.

If H^i(X, Z) has torsion elements, we can always find a finite cover f : X → X, such that the image of f∗ : H^i(X, Z) → H^i(X, Z) does not contain any torsion element. For such f, the image of the f∗ : Char(X) → Char( X) is contained in Char^0( X). By the computation in the proof of Lemma 3.2,

(1) (f∗)^−1 Σ_k( X, ⊕_{τ ∈ ker(f∗)} L_τ) = Σ_k(X, ⊕_{τ ∈ ker(f∗)} L_τ).

Since the finite cover X of X is also a compact Kähler manifold, Σ_k(X) ∩ Char^0(X) is a finite union of arithmetic subvarieties for any i, k ∈ N. Moreover, Im(f∗) ⊂ Char^0(X). Therefore, Σ_k(X) ∩ Im(f∗) is a finite union of arithmetic subvarieties. f being a finite covering map implies f∗ : H^i(X, C) → H^i(X, C) is surjective, and hence f∗ is a finite covering map from Char(X) to Im(f∗). Therefore, Σ_k(X, ⊕_{τ ∈ ker(f∗)} L_τ) is also a finite union of arithmetic subvarieties for any i, k ∈ N. By Lemma 3.4 for any τ ∈ ker(f∗), Σ_k(X, L_τ) is a finite union of arithmetic subvarieties. Since the trivial character 1 ∈ Char(X) is in ker(f∗), Σ_k(X, L_τ) = Σ_k(X) is a finite union of arithmetic subvarieties for any i, k ∈ N. 

□
Even though we have shown the same property of smooth complex projective varieties holds for compact Kähler manifolds too, there exists complex torus, whose cohomology jump loci can not be realized by any smooth complex projective variety. An example is given in the last section, based on the method of Voisin [V].

Convention. In contrast to [BW], here by a complex torus, we always mean a connected complex Lie group. Its underlying topological space is a real torus.

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2. Subvarieties of a complex torus

In this section, we prove some classification results of subvarieties of complex tori, and their divisors. These results will be useful to relate Hodge modules on a complex torus to Hodge modules on an abelian variety.

We first recall some notations and results from [U], with some modifications. Let $X$ be a compact analytic variety. A weak algebraic reduction of $X$ is a morphism $\psi : X' \rightarrow X_{\text{alg}}$ such that:

1. $X'$ is bimeromorphically equivalent to $X$;
2. $X_{\text{alg}}$ is a projective variety;
3. $\psi^*$ induces an isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(X_{\text{alg}})$, the field of meromorphic functions.

A weak algebraic reduction is called an algebraic reduction, if both $X'$ and $X_{\text{alg}}$ are smooth.

An algebraic reduction always exists and the birational class of $X_{\text{alg}}$ is uniquely determined. The following theorem is proved in [U] Section 12.

Theorem 2.1 ([U]). Let $\psi : X' \rightarrow X_{\text{alg}}$ be an algebraic reduction of an analytic variety $X$. For any Cartier divisor $D$ on $X'$, and for a very general point $u \in X_{\text{alg}}$, the fibre $X_u' = \psi^{-1}(u)$ is connected and smooth. Moreover, the Kodaira-Iitaka dimension $\kappa(D_u, X_u') \leq 0$, where $D_u$ is the restriction of $D$ to the fibre $X_u'$. In particular, for a very general point $u \in X_{\text{alg}}$, $\kappa(X_u') \leq 0$.

If $\psi : X' \rightarrow X_{\text{alg}}$ is only a weak algebraic reduction, and if a general fibre $X_u'$ of $\psi$ is smooth, then we can obtain the same conclusion as in the theorem using resolution of singularity.

Lemma 2.2. Let $B$ be a complex torus and let $X \subset B$ be an irreducible analytic subvariety. Suppose $X$ is a Moishezon space and $X$ generates $B$, i.e., there does not exist a proper subtorus of $B$ containing $\{x_1 = x_2 \mid x_1, x_2 \in X\}$. Then $B$ is an abelian variety. In particular, if a Moishezon space is isomorphic to an analytic subvariety of a complex torus, then it is a projective variety.

Proof. According to [LM], there exist a smooth projective variety $X'$ and a bimeromorphic morphism $X' \rightarrow X$. The composition $X' \rightarrow X \rightarrow B$ induces a map on the Albanese $\gamma : \text{Alb}(X') \rightarrow \text{Alb}(B) = B$. Since $X$ generates $B$, $\gamma$ is surjective. Since $X'$ is a smooth projective variety, $\text{Alb}(X')$ is an abelian variety, and hence $B$ is also an abelian variety.

In the rest of this section, we assume $T$ to be a complex torus, and $Y$ to be an irreducible analytic subvariety of $T$. The following proposition is an analog of [U] Theorem 10.9.

Proposition 2.3. There is a subtorus $S$ of $T$, such that the action of $S$ on $T$ preserves $Y$ and the quotient map $py : Y \rightarrow Y/S$ is a weak algebraic reduction of $Y$.

Proof. Let $\psi : Y' \rightarrow Y_{\text{alg}}$ be an algebraic reduction of $Y$. By Theorem 2.1 a very general fibre $Y_u'$ of $\psi : Y' \rightarrow Y_{\text{alg}}$ has $\kappa(Y_u') \leq 0$. It is also proved in [U] Section 10] that if $Z$ is a projective subvariety of $T$ with $\kappa(Z) \leq 0$, then $Z$ is a translate of a subtorus. Thus a very general fibre of $\psi : Y' \rightarrow Y_{\text{alg}}$ is bimeromorphically equivalent to a subtorus of $T$.

Let $U$ be the open subset of $Y_{\text{alg}}$, where $\psi$ is smooth. Since the Albanese map can be defined for a smooth family of analytic varieties, and since the Albanese dimension is a topological invariant (see [LP]), a small deformation of an analytic variety bimeromorphic to a complex torus is also bimeromorphic to a complex torus. Thus every fibre $Y_u'$ of $\psi : Y' \rightarrow Y_{\text{alg}}$ for $u \in U$ is bimeromorphic to a subtorus of $T$. Since there are only countably many subtori in $T$, all $Y_u'$ for $u \in U$ are isomorphic to some subtorus $S$ of $T$. Therefore, the algebraic reduction $\psi : Y' \rightarrow Y_{\text{alg}}$ is bimeromorphically equivalent to the quotient map $Y \rightarrow Y/S$. Hence $Y/S$ is a Moishezon space. Since $Y/S$ is an analytic subvariety of a complex torus $T/S$, by Lemma 2.2 $Y/S$ is a projective variety.
Proposition 2.4. Let $S$ be the subtorus of $T$ as in Proposition 2.3. Any Cartier divisor $D$ on $Y$ is the pullback of a Cartier divisor on $Y/S$ by $p_Y:Y \to Y/S$.

Proof. When $\dim S = 0$, the statement is trivial. So we can assume $\dim S \geq 1$. Without loss of generality, we can also assume $D$ is nontrivial and effective. Suppose $D$ is not the pullback of any divisor on $Y/S$. Then for a general point $u \in Y/S$, $D_u$ is a nontrivial effective Cartier divisor of $Y_u$, where $Y_u$ is the fibre of $p_Y:Y \to Y/S$ and $D_u$ is the restriction of $D$ to $Y_u$.

According to Theorem 2.21 it suffices to show $\kappa(D_u,S) \geq 1$ for a nontrivial effective Cartier divisor $D_u$ in $S$. To this end, we choose any irreducible component $D'$ of $D_u$. If $D'$ is a subtorus of $S$, then $S/D'$ is an elliptic curve. Now, $\kappa(D_u,S) \geq \kappa(D',S) = \kappa(0,S/D') = 1$. If $D'$ is a translate of a subtorus, we can translate $D'$ back to the subtorus, which does not change the Kodaira-Iitaka dimension. If $D'$ is not a translate of subtorus of $S$, by Proposition 2.2 there exists a subtorus $S'$ of $S$ such that $D'$ is preserved by the action of $S'$ and the quotient map $D' \to D'/S'$ is an algebraic reduction of $D'$. Now, $D'/S'$ is an algebraic variety and it is a divisor of $S'/S'$. Since $D'$ is not a subtorus of $S$, $D'/S'$ generates the whole torus $S'/S'$. By Lemma 2.2 $S'/S'$ is an abelian variety. By the succeeding lemma, $\kappa(D'/S',S/S') \geq 1$. Therefore, $\kappa(D_u,S) \geq \kappa(D',S) = \kappa(D'/S',S/S') \geq 1$. \hfill \Box

Lemma 2.5. Let $A$ be a complex abelian variety, and let $D$ be an irreducible divisor of $A$. Then $\kappa(D,A) \geq 1$.

Proof of Lemma 2.5. First, we assume there does not exist any morphism of abelian varieties $\phi: A \to A'$, such that $\dim A' < \dim A$ and $D = \phi^{-1}(D')$ for some divisor $D'$ on $A'$. In this case, it is proved in [3] that $D$ is ample. Hence, $\kappa(D,A) = \dim A \geq 1$.

Suppose there exist a morphism of abelian varieties $\phi: A \to A'$ with $\dim A' < \dim A$ and a divisor $D' \subset A$ such that $D = \phi^{-1}(D')$. We assume $D'$ has the smallest dimension among all such possibilities. Then as a divisor on $A'$, $D'$ is ample. Now, $\kappa(D, A) = \kappa(D', A') = \dim A' \geq 1$. \hfill \Box

3. Cohomology jump loci

In this section, we review some useful results about cohomology jump loci.

Lemma 3.1. Given any complex torus $T$ and a constructible complex $E$ on $T$, suppose $E = \bigoplus_{1 \leq j \leq t} E_j$. Then the following statements are equivalent for a fixed $i \in \mathbb{N}$,

$(1)$ $\Sigma_i^i(T, E)$ is a finite union of arithmetic subvarieties for every $k \in \mathbb{N}$;

$(2)$ $\Sigma_i^i(T, E_j)$ is a finite union of arithmetic subvarieties for every $k \in \mathbb{N}$ and $1 \leq j \leq t$.

Proof. Since for any rank one local system $L$ on $T$,

$$H^i(T, L \otimes E) \cong \bigoplus_{1 \leq j \leq t} H^i(T, L \otimes E_j)$$

we have

$$\Sigma_i^i(T, E) = \bigcup_{\mu} \bigcap_{1 \leq j \leq t} \Sigma_{i,\mu(j)}(T, E_j)$$

where the union is over all functions $\mu: \{1, 2, \ldots, t\} \to \mathbb{N}$ satisfying $\sum_{1 \leq j \leq t} \mu(j) = k$. Therefore, (2)$\Rightarrow$(1).

Conversely, assuming that (1) is true, without loss of generality, we only need to prove $\Sigma_i^i(T, E_1)$ is a finite union of arithmetic subvarieties. Let $C$ be an irreducible component of $\Sigma_i^i(T, E_1)$. Take a general point $\rho_0 \in C$. Here by $\rho_0$ being a general point, we require that

$$\dim H^i(T, L_{\rho_0} \otimes E_j) = \min\{\dim H^i(T, L_{\rho} \otimes E_j) \mid \rho \in C\}$$

for every $1 \leq j \leq t$. Then $C$ is an irreducible component of $\Sigma_{i,\rho_0}^i(T, E)$, where

$$k_0 = \sum_{1 \leq j \leq t} \dim H^i(T, L_{\rho_0} \otimes E_j).$$

Therefore, $C$ is an arithmetic subvariety, and hence $\Sigma_i^i(T, E_1)$ is a finite union of arithmetic subvarieties. \hfill \Box

Lemma 3.2. Let $\hat{T}$ and $T$ be two complex tori, and let $\alpha: \hat{T} \to T$ be an isogeny. We denote the induced map between the character varieties by $\alpha^*: \text{Char}(T) \to \text{Char}(\hat{T})$. Let $E$ be a constructible complex on $T$. The following statements are equivalent for a fixed $i \in \mathbb{N}$,
(1) $\Sigma_k^1(T, E)$ is a finite union of arithmetic subvarieties for every $k \in \mathbb{N}$.

(2) $\Sigma_k^1(\hat{T}, \alpha^*(E))$ is a finite union of arithmetic subvarieties for every $k \in \mathbb{N}$.

Proof. Since for any rank one local system $L$ on $T$, we have $H^i(\hat{T}, \alpha^*(L) \otimes_{\mathbb{C}} \alpha^*(E)) \cong H^i(T, L \otimes_{\mathbb{C}} \alpha_* \alpha^*(E))$.

Therefore,

$\Sigma_k^i(\hat{T}, \alpha^*(E)) = \alpha^* \left( \Sigma_k^i(T, \alpha_* \alpha^*(E)) \right)$.

By definition, ker$(\alpha^*)$ consists of all rank one local systems on $T$, whose pull-back to $\hat{T}$ are isomorphic to the trivial local system. Since $\alpha_* \alpha^*(\mathcal{C}_F) \cong \bigoplus_{\rho \in \ker(\alpha^*)} \rho$, by projection formula,

$\alpha_* \alpha^*(E) \cong \bigoplus_{\rho \in \ker(\alpha^*)} \rho \otimes_{\mathbb{C}} E$.

Moreover, since the cohomology of a local system can be computed using the derived push-forward to a point,

$H^i(\hat{T}, \alpha^*(E)) \cong H^i(T, \alpha_* \alpha^*(E))$.

Therefore, we have

$$(\alpha^*)^{-1} \left( \Sigma_k^i(\hat{T}, \alpha^*(E)) \right) = \Sigma_k^i \left( T, \bigoplus_{\rho \in \ker(\alpha^*)} \rho \otimes_{\mathbb{C}} E \right).$$

Now, $\Sigma_k^i(\hat{T}, \alpha^*(E))$ is a finite union of arithmetic subvarieties if and only if $\Sigma_k^i \left( T, \bigoplus_{\rho \in \ker(\alpha^*)} \rho \otimes_{\mathbb{C}} E \right)$ is a finite union of arithmetic subvarieties. Hence by Lemma 3.1 $\Sigma_k^1(\hat{T}, \alpha^*(E))$ is a finite union of arithmetic subvarieties if and only if $\Sigma_k^1(T, \alpha_* \alpha^*(E))$ is a finite union of arithmetic subvarieties for every $\rho \in \ker(\alpha^*)$. Thus, (2) $\Rightarrow$ (1). Conversely, notice that any $\rho \in \ker(\alpha^*)$ is a torsion point in $\text{Char}(T)$. Since $\Sigma_k^1(T, E)$ is a finite union of arithmetic subvarieties, $\Sigma_k^1(T, L \rho \otimes_{\mathbb{C}} E)$ is also a finite union of arithmetic subvarieties for any $\rho \in \ker(\alpha^*)$. Thus, (1) $\Rightarrow$ (2). $\square$

4. Polarizable Hodge module

Let $M$ be a pervasive sheaf on a complex torus $T$ that underlies a simple polarizable Hodge module. Let $Y$ be the support of $M$. Then $Y$ is an irreducible analytic subvariety of $T$. We can assume $M$ is obtained from the intermediate extension of a polarizable variation of Hodge structure $H$ on $U$, where $U$ is a Zariski open subset of the smooth locus of $Y$. In other words, $M|_U \cong H[\text{dim } Y]$. Moreover, we can also assume that every irreducible component of the complement of $U$ has codimension one. To prove Theorem 4.1 we can assume $Y$ generates $T$ without loss of generality.

By Proposition 2.2 there exists a subtorus $S$ of $T$ such that the action of $S$ on $T$ preserves $Y$ and the quotient map $p_T : Y \to Y/S$ is an algebraic reduction of $Y$. By Proposition 2.2 $U$ is the preimage of some open set of $Y/S$ under the quotient map $p_T : Y \to Y/S$. Thus, $U$ is preserved by the action of $S$. We denote the quotient maps by $p_U : U \to U/S$ and $p_T : T \to T/S$.

Lemma 4.1. Under the above notations, there exists a finite cover $\alpha : \hat{T} \to T$, satisfying the following:

(1) $\alpha^{-1}(S)$ is connected.

(2) the restriction of $\alpha_{\hat{T}}^*(H)$ to any fibre of $p_T \circ \alpha_U : \hat{T} \to U/S$ has trivial monodromy actions, where $\hat{U} = \alpha^{-1}(U)$, and $\alpha_U$ is the restriction of $\alpha$ to $\hat{U}$.

Proof. Let $F$ be any fibre of $p_U : U \to U/S$. Since $H$ is a polarizable variation of Hodge structure with coefficients in $\mathbb{Z}$, so is its restriction to $F$. It is proved in Lemma 4.1 of [13] that the underlying local system of a polarizable variation of Hodge structure with coefficients in $\mathbb{Z}$ on an abelian variety is torsion (the direct sum of rank one local systems whose monodromy actions are all torsion). When the abelian variety is replaced by a complex torus, the proof works the same. Thus, $H|_F$ is a direct sum of rank one local systems whose monodromy actions are torsion. Therefore, there exists some finite covering map $\beta : \hat{F} \to F$, such that $\beta^*(H|_F)$ is a trivial rank $n$ local system. Notice that by choosing an origin, $F$ is isomorphic to the complex torus $S$. Hence, after choosing an origin, $\hat{F}$
becomes a complex torus, which we denote by \( \hat{S} \). Moreover, by identifying \( F \) with \( S \) and \( \hat{F} \) with \( \hat{S} \), we can assume \( \beta : \hat{S} \to S \) is a morphism of complex tori.

As topological spaces or real Lie groups, \( T \cong S \times T/S \). The underlying topological space of \( \hat{T} \) is defined to be \( \hat{S} \times T/S \). The composition

\[
\hat{T} \cong \hat{S} \times T/S \xrightarrow{\beta \times \text{id}} S \times T/S \cong T
\]

is a covering map, which we set to be \( \alpha \). We define the complex structure on \( \hat{T} \) by setting the covering map to be holomorphic.

By Lemma 3.1, we can assume the perverse sheaf \( M \) underlies a simple polarizable Hodge module. Let \( \hat{T} \) and \( \alpha : \hat{T} \to T \) be defined as in Lemma 4.1. According to Lemma 3.2, it suffices to prove each \( \Sigma \) as in Lemma 4.1.

Proof of Theorem 1.4. By Lemma 5.1 [5,1] we can assume the perverse sheaf \( M \) underlies a simple polarizable Hodge module. Let \( \hat{T} \) and \( \alpha : \hat{T} \to T \) be defined as in Lemma 4.1. According to Lemma 3.2, it suffices to prove each \( \Sigma \) as in Lemma 4.1. According to

\[
(2) \quad \alpha^*(M) \cong \hat{\rho}_T^*(M_0).
\]

By projection formula,

\[
(3) \quad R\hat{p}_T\star(\hat{\rho}_T(M_0)) \cong M_0 \otimes_{\mathcal{C}_{T/S}} R\hat{p}_T\star(\hat{\mathcal{C}}_T).
\]

Since \( \hat{p}_T : \hat{T} \to T/S \) is a trivial topological fibration,

\[
R\hat{p}_T\star(\hat{\mathcal{C}}_T) \cong \bigoplus_{0 \leq i \leq 2 \dim S'} (\mathcal{C}_{T/S} \otimes _\mathbb{C} H^i(S', \mathbb{C})) [i].
\]

Thus,

\[
(4) \quad M_0 \otimes_{\mathcal{C}_{T/S}} R\hat{p}_T\star(\hat{\mathcal{C}}_T) \cong \bigoplus_{0 \leq i \leq 2 \dim S'} (M_0 \otimes _\mathbb{C} H^i(S', \mathbb{C})) [i].
\]

Now, combining (2), (3) and (4), we have

\[
(5) \quad R\hat{p}_T\star(\alpha^*(M)) \cong \bigoplus_{0 \leq i \leq 2 \dim S'} (M_0 \otimes _\mathbb{C} H^i(S', \mathbb{C})) [i].
\]

Recall that we have assumed \( M \) admits a \( \mathbb{Z} \)-structure. Since \( \mathbb{Z} \)-structure is preserved under the standard operations (see [Sc1]), \( R\hat{p}_T\star(\alpha^*(M)) \) also admits a \( \mathbb{Z} \)-structure. We have assumed that \( Y \) generates \( T \). Therefore, \( Y/S \) generate \( T/S \). By Lemma 2.2, \( T/S \) is an abelian variety. Now,
$R_\hat{\mathcal{T}}(\alpha^*(M))$ is a direct sum of twists of perverse sheaves that underlie polarizable Hodge modules on $T/S$ with $\mathbb{Z}$-structures. Theorem 2.2 of [SS1] and Lemma 3.1 implies that $\Sigma_k(T/S, R_\hat{\mathcal{T}}(\alpha^*(M)))$ is a finite union of arithmetic subvarieties. Thus, by (4),

$$\Sigma_k(\hat{T}, \alpha^*(M)) = \hat{\rho}_T(\Sigma_k(T/S, M_1)).$$

is a finite union of arithmetic subvarieties. By Lemma 3.2, $\Sigma_k(T/S, M_0)$ is a finite union of arithmetic subvarieties.

**Claim.** Let $\hat{\rho}_T : \text{Char}(T/S) \to \text{Char}(\hat{T})$ be the morphism on the character varieties induced by $\hat{\rho} : \hat{T} \to T/S$. Denote by $M_1$ the direct sum $\bigoplus_{0 \leq i \leq 2 \dim S'}(M_0 \otimes \mathbb{C} H^i(S', \mathbb{C}))[i]$. Then,

$$\Sigma_k(\hat{T}, \alpha^*(M)) = \hat{\rho}_T(\Sigma_k(T/S, M_1)).$$

**Proof of Claim.** Let $L$ be any rank one local system on $\hat{T}$. Since $H^i(\hat{T}, L \otimes_{\mathbb{C}_T} \alpha^*(M))$ can be computed as the $i$-th derived push forward of $L \otimes_{\mathbb{C}_T} \alpha^*(M)$ under the map $\hat{T} \to \text{point}$, there is a natural isomorphism

$$H^i(\hat{T}, L \otimes_{\mathbb{C}_T} \alpha^*(M)) \cong H^i(T/S, R_\hat{\mathcal{T}}(L \otimes_{\mathbb{C}_T} \alpha^*(M))).$$

On the other hand, $\alpha^*(M) \cong \hat{\rho}_T^*(M_0)$. By projection formula,

$$R_\hat{\mathcal{T}}(L \otimes_{\mathbb{C}_T} \hat{\rho}_T^*(M_0)) \cong R_\hat{\mathcal{T}}(L) \otimes_{\mathbb{C}_{T/S}} M_0.$$  \hspace{1cm} (6)

If the restriction of $L$ to the fibre of $\hat{\rho}_T : T \to T/S$ is a trivial local system, or equivalently if there exists a rank one local system $L_0$ on $T/S$ such that $L \cong \hat{\rho}_T(L_0)$, then by projection formula,

$$R_\hat{\mathcal{T}}(L) = \bigoplus_{0 \leq i \leq 2 \dim S'} L_0 \otimes \mathbb{C} H^i(S', \mathbb{C}))[i].$$  \hspace{1cm} (7)

If the restriction of $L$ to the fibre of $\hat{\rho}_T : T \to T/S$ is not trivial, then

$$R_\hat{\mathcal{T}}(L) \cong 0.$$  \hspace{1cm} (8)

Now a direct computation shows that the claim follows from (2), (6), (7) and (8). \hspace{1cm} \square

We have showed that $\Sigma_k(T/S, M_0)$ is a finite union of arithmetic subvarieties for any $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. Since $M_1$ is a direct sum of some twists of $M_0$, by Lemma 3.1, $\Sigma_k(T/S, M_1)$ is also a finite union of arithmetic subvarieties for any $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. Since $\hat{\rho}_T : \text{Char}(T/S) \to \text{Char}(\hat{T})$ is a morphism of abelian algebraic groups induced by the morphism of complex tori $\hat{\rho}_T$, $\hat{\rho}_T(\Sigma_k(T/S, M_1))$ is a finite union of arithmetic subvarieties. Hence, the claim implies that $\Sigma_k(\hat{T}, \alpha^*(M))$ is a finite union of arithmetic subvarieties. Thus, by Lemma 3.2, we have completed the proof. \hspace{1cm} \square

5. **Topology of compact Kähler manifolds and smooth complex projective varieties**

There exists compact Kähler manifold that is not of the (real) homotopy type of any smooth complex projective varieties (see [V]). Even though Theorem 1.2 shows the same property of smooth complex projective varieties holds for compact Kähler manifolds. We can still distinguish them by cohomology jump loci.

**Proposition 5.1.** There exists a compact Kähler manifold $X$, satisfying the following. If $X'$ is another compact Kähler manifold, and if there exists an isomorphism $f : \text{Char}(X) \to \text{Char}(X')$, which induces an isomorphism between $\Sigma^2(X) = \Sigma^2(X')$, then $X'$ is not projective.

The example of $X$ will be quite similar to the one of Voisin [V]. We recall a theorem there.

**Lemma 5.2** ([V]). Let $T$ be a complex torus of dimension at least two, and let $\phi$ be an endomorphism of $T$. Suppose the characteristic polynomial $f$ of $\phi^* : H^1(T, \mathbb{Z}) \to H^1(T, \mathbb{Z})$ satisfies the property that the Galois group of its splitting field acts as the symmetric group on the roots of $f$. Then $T$ is not an abelian variety.
Proof of Proposition 5.2. We first construct $X$. Let $T$ be the complex torus as in the lemma. Define $X_0 = T \times T \times \mathbf{P}^1_C$. Take four distinct points $P_i$, $1 \leq i \leq 4$, in $\mathbf{P}^1_C$. We denote four subvarieties of $T \times T$, $T \times \{0\}$, $\{0\} \times T$, and the graph of $\phi$ by $Z_i$, $1 \leq i \leq 4$ respectively. Let $X$ be the blow-up of $X_0$ along $\bigcup_{1 \leq i \leq 4} Z_i \times \{P_i\}$.

Clearly, there is a natural isomorphism $\gamma : \text{Alb}(X) \to T \times T$. The Albanese map $a : X \to \text{Alb}(X)$ is equal to the composition of the blowing up map $X \to X_0$ and the projection $X_0 \to T \times T$. Hence, $\text{Ra}_a(\mathcal{C}_X)$ is isomorphic to the direct sum of five constructible complexes, the first four have support equal to $\gamma^{-1}(Z_i)$, $1 \leq i \leq 4$ respectively, and the last one has support equal to the whole $\text{Alb}(X)$. Notice each $\gamma^{-1}(Z_i)$ is a torus of $T \times T$. Denote $T \times T / \gamma^{-1}(Z_i)$ by $B_i$, and denote the quotient maps by $p_i : T \times T \to B_i$ respectively. A direct computation shows

$$
\Sigma^2_1(X) = \bigcup_{1 \leq i \leq 4} p_i^* \text{Char}(B_i).
$$

Suppose $X'$ satisfies the assumption in the proposition. Then $\Sigma^2_1(X')$ is the union of four arithmetic subvarieties $C_i$, $1 \leq i \leq 4$. Notice $\text{Pic}^0(X') = \text{Pic}^0(X')$, since $\text{Char}(X) \cong \text{Char}(X')$ and $\text{Char}(X)$ is connected. Hitchin-Kobayashi correspondence allows us to identify the Picard group $\text{Pic}^0(X')$ with the group of unitary characters $\text{Char}^u(X')$. Under this identification, Hodge decomposition for unitary local systems implies

$$
\Sigma^{p,q}_k(X') = \Sigma^{1,1}_1(X') \cup \Sigma^{2,0}_1(X') = \Sigma^2_1(X') \cap \text{Char}^u(X').
$$

Since $\Sigma^{p,q}_k(X')$ is always an analytic subvariety of $\text{Pic}^0(X')$ for any $p, q, k \in \mathbb{N}$, $\text{Pic}^0(X')$ has four subvarieties corresponding to $C_i \cap \text{Char}^u(X')$, $1 \leq i \leq 4$. According to [4], the presence of these four subvarieties of $\text{Pic}^0(X')$ induces $\text{Pic}^0(X') \cong T^X \otimes T^X$ and $T^X$ has an endomorphism $\phi^*$ whose characteristic polynomial is equal to the characteristic polynomial of $f$. By Lemma 5.2, $T^X$ is not an abelian variety. Thus, $\text{Pic}^0(X')$ is not an abelian variety, and hence $X'$ is not projective. $\square$

Remark 5.3. Since cohomology jump loci are homotopy invariant, Proposition 5.1 implies any compact Kähler manifold that is of the homotopy type as $X$ is not projective. In fact, this is still true for any compact Kähler manifold that is of the same real homotopy type as $X$. This is because the germs of cohomology jump loci at origin are determined only by the real homotopy type (see [DP]). Knowing the germs at origin is sufficient to recover the endomorphism in Lemma 5.2 which induces $\text{Pic}^0(X')$ is not an abelian variety.

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