Exact solution to sodium-iridate-BCS–Hubbard model along the symmetric line: non-trivial topology in the ferromagnetic order

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Abstract

We study the sodium-iridates model on the honeycomb lattice with both BCS pairing potential and Hubbard interaction term. It is shown that this model can be exactly solved with appropriate choices of amplitude of pairing gaps, where the interacting terms are transformed to external field terms. The band structures of these exact solutions on both torus and cylinder geometry are discussed in great details. It is found that the ground state assumes an anti-ferromagnetic configuration, which breaks the time reversal symmetry spontaneously and renders the superconductor topologically trivial. On the other hand, the nontrivial topology is preserved with ferromagnetic configuration and can be characterized by the isospin Chern number.

Keywords: topological superconductor, exact solutions, isospin Chern number

(Some figures may appear in colour only in the online journal)

1. Introduction

Topological quantum matters have been a central topic in the area of condensed matter physics for the last decade [1, 2]. The topology of gapped non-interacting fermionic systems has been classified according to three types of discrete symmetries, which leads to the famous ten-fold way classification scheme [3, 4]. The scheme has also been generalized and applied to systems with gapless dispersion and spatial symmetries [5]. On the other hand, the topological properties of interacting fermionic systems have always been intriguing topics, especially in the strongly correlated limit. For example, the pioneer works of Kitaev \textit{et al} showed that the Hubbard interaction can reduce the \(Z_N\) classification of topological superconductors to a finite group such as \(Z_N\) [6–8]. More general considerations of interacting fermionic systems lead to the concept of symmetry protected topological state [9], which is still under active study.

In some recent works [10–16], it is proposed that certain Bardeen-Cooper-Schrieffer (BCS) superconductor with Hubbard interactions can be exactly solved when the amplitude of pairing potentials are tuned to be equal to the hopping constants, i.e. along a symmetric line in the parameter space. This type of exact solution opens up a new approach to study the topological superconductors with arbitrary interaction strength. The mechanism behind these exact solutions strongly resembles the exact solution of Kitaev spin liquid model on the Honeycomb lattice [17]. When the BCS model is expressed in terms of Majorana fermions, half of them has zero kinetic term along the symmetric line. This gives rise to infinitely many conserved quantities and also transforms the Hubbard interacting terms into simple quadratic terms.

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of fermions. In reference [16], Ezawa carried out a detailed
study of the BCS superconductor based on the Kane–Mele
(KM) model with Hubbard interaction. It is known that KM
model is proposed for graphene, which does not have strong
eough spin orbital coupling to become topologically nontriv-
ial. In this paper, we propose the topological superconduc-
tor based on the sodium-iridate (SI) model [18, 19], which
possesses a stronger spin orbital coupling. This model can
also be exactly solved along the symmetric line when Hub-
bard interacting terms are considered. It will enrich the fam-
ily of such exactly solvable models and provide a valu-
able alternative for exploring the nontrivial topology in these
interacting systems.

This paper is organized as follows. In section 2, we intro-
duce the model Hamiltonian and discuss its symmetry and
topological classification. We also propose a possible experi-
mental construction of the system. Then in section 3, the non-
interacting limit is examined in detail. The band structures
are displayed in both torus and cylinder geometry. The Isospin
Chern number is computed to confirm the bulk-edge corre-
spondence in this model. With the help of isospin Chern num-
ber, we also obtain the phase diagram. Next we introduce the
on-site Hubbard interactions and reveal the exact solvability
in section 4. By expressing the fermions in terms of Majorana
fermions under the perfect flat band condition, we demon-
strate that one of the species of the Majorana fermions is decoupled,
which makes the interacting terms quadratic. We analyze the
interaction limits in detail. The band structures are
described in the rest of this paper. Finally we make a conclusion
in section 5. We will use the convention of \( h = 1 \) in the rest
of this paper.

2. Model Hamiltonian

The model Hamiltonian focused in this work consists of three
terms,

\[
H = H_{\text{SI}} + H_{\text{pair}} + H_{\text{int}},
\]

where \( H_{\text{SI}} \) describes the hopping terms with spin–orbital
coupling (SOC) of the SI type [18, 19], \( H_{\text{pair}} \) describes the BCS
pairing terms, and \( H_{\text{int}} \) contains the on-site Hubbard interac-
tions. Explicitly, we have

\[
H_{\text{SI}} = -t \sum_{(ij)x} c_{jx}^\dagger c_{ix} + i \lambda \sqrt{3} \sum_{(ij)y} v_{ij} \sigma_x c_{ix}^\dagger c_{jy}^\dagger,
\]

\[
H_{\text{pair}} = -\Delta_1 \sum_{(ij)} c_{jx}^\dagger c_{ix} + i \lambda \sqrt{3} \sum_{(ij)} v_{ij} \sigma_x c_{ix}^\dagger c_{jx}^\dagger + \text{h.c.},
\]

\[
H_{\text{int}} = U \sum_j \left( c_{jx}^\dagger c_{jx} - \frac{1}{2} \right) \left( c_{jx}^\dagger c_{jy} - \frac{1}{2} \right),
\]

where \( c_{jx}^\dagger \) and \( c_{jx} \) are electron creation and annihilation opera-
tors at site \( j \) with spin polarization \( x \), and \( (ij) \) and \( (ij) \) represent
the nearest and next-nearest-neighbor sites respectively.

The first term in \( H_{\text{SI}} \) is the usual nearest-neighbor hopping with

kinetic energy, and the second term is the anisotropic SOC,
which involves three different Pauli matrices \( \sigma^\alpha (\alpha = x, y, z) \)
for three inequivalent next-nearest-neighbor hopping (NNNH)
directions. See figure 1. The sign \( v_{ij} = +1 \) if the NNNH is anti-
clockwise with respect to the positive direction of \( z \) axis, and
\( v_{ij} = -1 \) if clockwise. \( \lambda \) is the SOC strength. For the super-
conducting pairing terms, we follow the convention in refer-
ence [16]. \( \Delta_{1,2} \) are pairing gaps, and \( U \) is the on-site Hubbard
interaction strength.

We make some remarks on the realization of the model
Hamiltonian. The prototype is the SI, a 5d transition metal
oxide proposed as a layered correlated Quantum Spin Hall
(QLSH) insulator [18]. When the electron–electron interac-
tion is not too strong, the SI becomes an interacting topo-
logical insulator [19, 20]. Recent study shows that spin-
triplet \( f \)-wave pairing can be induced when certain dop-
ing is made [21]. On the other hand, due to the super-
conducting proximity, artificial topological superconductivity
can be realized in superconductor-topological insulator het-
erostructure [22–27]. Therefore, we can consider a bilayer
van der Waals heterostructure with the upper layer being
the undoped SI with Hubbard interaction, and the lower
layer being the doped topological superconductor. This will
induce the BCS pairing term we want as a result of super-
conducting proximity. Then we could arrive at an effect-
ive Hamiltonian as shown in equation (1) for the upper
layer.

Let us now briefly study the symmetry of the model
Hamiltonian. Particle hole transformation \( C \) is a unitary trans-
formation that recombines creation and annihilation opera-
tors of fermion, and the operators transforms as \( Cc_{jx}^\dagger C^{-1} =
(-1)^y c_{jx}, Cc_{jy} C^{-1} = (-1)^y c_{jx}^\dagger \), where the sign \((-1)^y \)
is \(-1\) for sublattice \( A(B) \) of site \( j \). One can easily check

Figure 1. Upper panel: the plus (minus) signs of superconducting
pairing between the nearest neighbor sites are shown by the forward
(backward) direction of the arrows. Lower panels: the anisotropic
pairing term between the next nearest neighbor sites for sublattices
are shown by solid and dashed lines with labels.
that equation (1) is invariant under such a transformation, thus it has particle hole symmetry. Time reversal transformation $T$ is another discrete operation acting on the fermion operators. For spinful systems it reads $T = iσ^yK$, where $K$ denotes the complex conjugation, and $T^2 = -1$. It is known that the SI model Hamiltonian, as a bond-dependent generalization of KM model, is a time reversal invariant $Z_2$ topological insulator and belongs to the same universality class with the KM model [18]. When the pairing term is taken into consideration, the time reversal symmetry is unaffected. And we will show that the interaction will not break the time reversal symmetry in section 4. Then such a system falls into class DIII in the classification of topological quantum matter [5].

3. Non-interacting limit: SI-BCS model

First we investigate the non-interacting limit with $U = 0$. In this case, the model can be called SI-BCS model. For convenience, we will use the following notations. The basis vectors are

$$a_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad a_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

where $a$ is the lattice constant and will be set as a unit, $a = 1$. We label the vectors connecting the next-nearest neighbors as $d_1 = a_1, d_2 = -a_2$, and $d_3 = a_2 - a_1$, we have

$$d_1 = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad d_2 = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad d_1 = (-1, 0).$$

On the other hand, the vectors along the bonds are

$$e_1 = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad e_2 = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad e_3 = (0, 0, 1).$$

These two sets of vectors are related by

$$d_1 = e_2 - e_3, \quad d_2 = e_3 - e_1, \quad d_3 = e_1 - e_2.$$  

(4)

The unit cells are located at $n = ia_1 + ja_2$. Adopting the periodic-boundary condition (PBC), we can define the Fourier transformation as

$$c_n = \frac{1}{L_xL_y} \sum_k c_k \exp(i\mathbf{k} \cdot \mathbf{n}),$$  

(5)

where $L_x$ and $L_y$ are the lattice site numbers in the $x$ and $y$ directions, respectively, and the wave vector $k = (k_x, k_y)$, so as to obtain an equivalent Hamiltonian in momentum space. Here we would like to point out that the SI model has both spin and sublattice degrees of freedom. In the SI-BCS model, the pairing terms require us to put the particle and hole creation operators together to form a Nambu spinor as follows

$$\Psi_k = \begin{pmatrix} \psi_k^0 \\ \psi_k \end{pmatrix}^T, \quad \psi_k = (c_k A^\dagger, c_k A, c_k B^\dagger, c_k B)^T.$$  

(6)

Therefore the final Bogoliubov–de Gennes Hamiltonian of SI-BCS model is a $8 \times 8$ matrix which can be written as

$$H = \begin{pmatrix} \lambda G_i s_i & -tF^s & -\Delta_2 G_i s_i & -\Delta F^s s_i \\ -tF^s & -\lambda G_i s_i & \Delta_1 F^s & -\Delta_2 (s_i f_i) \\ -\Delta_2 G_i s_i & \Delta_1 F^s & \lambda G_i s_i & tF \\ -\Delta F^s s_i & -\Delta_2 G_i s_i & tF^s & -\lambda G_i s_i \end{pmatrix},$$  

(8)

where the repeated indices means summation over $i = 1, 2, 3$ and $s_i$ are Pauli matrices applying to the spin space and $s^0$ are $2 \times 2$ identity matrix. Here we have also introduced the following abbreviations,

$$F = \sum_{j=1}^3 \exp(i\mathbf{k} \cdot \mathbf{e}_j) = e^{-ik_y/\sqrt{3} + 2e^{ik_y/(2\sqrt{3})}} \cos k_x^2.$$  

(9)

$$G_1 = \frac{2}{\sqrt{3}} \sin k_x \cdot d_1 = \frac{2}{\sqrt{3}} \sin \left(\frac{k_x}{2} + \frac{3k_y}{2}\right),$$  

(10)

$$G_2 = \frac{2}{\sqrt{3}} \sin k_x \cdot d_2 = \frac{2}{\sqrt{3}} \sin \left(\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2}\right),$$  

(11)

$$G_3 = \frac{2}{\sqrt{3}} \sin k_x \cdot d_3 = \frac{2}{\sqrt{3}} \sin k_x.$$  

(12)

It is convenient to expand the above Hamiltonian in terms of the spinor basis. To this end, we define the following spinor basis

$$\Gamma_{abc} = \tau_a \otimes \sigma_b \otimes \sigma_c.$$  

(13)

Here the indices $a, b, c = 0, \ldots, 3$. For $i = 1, 2, 3$ and $\tau_i$ are also Pauli matrices applying to the Nambu pseudo-spin and sublattice space, respectively. $\tau_0$ and $\sigma_0$ are a $2 \times 2$ identity matrix. Then the Hamiltonian can be rewritten as

$$H = -t \text{Re}(F^t \Gamma_{310}) - t \text{Im}(F^t) \Gamma_{320} + \Delta_1 \text{Re}(F^t) \Gamma_{220} - \Delta_1 \text{Im}(F^t) \Gamma_{210} + \sum_{i=1}^3 \lambda G_i \Gamma_{10i} = \sum_{i=1}^3 \Delta_i G_i \Gamma_{10i}$$  

(14)

in the last line the repeated indices are summed, and we find the energy eigenvalues are

$$E(k) = \pm \sqrt{(t \pm \Delta_1) c^2 + (\lambda \pm \Delta_2)^2 + \sum_i \lambda_i^2}.$$  

(15)

The band gap closes at two inequivalent Dirac points $K$ and $K'$. In our basis choice, they are given by $K = (4\pi/3, 0), (-2\pi/3, \pm 2\pi/\sqrt{3})$ and $K' = (-4\pi/3, 0), (2\pi/3, \pm 2\pi/\sqrt{3})$. We notice that the perfect flat-band condition is satisfied when $t = \Delta_1$ and $\lambda = \Delta_2$. Later on, we will show that the perfect flat bands condition makes one half of the Majorana fermions decouple from the rest of the Hamiltonian and become localized in the bulk. This in turn leads to an exact solution even if the Hubbard interaction is also included.

In the following we mainly focus on the system with perfect flat band. If the SOC is turned off, the system reduces to
a Dirac-nodal superconductor, which can be described by the graphene-BCS model, and the energy gap closed linearly at the \( K \) and \( K' \) point. The energy spectrum is obtained in both torus and cylinder geometry, i.e., PBC is imposed in both \( x \) and \( y \) direction, or with open-boundary condition in \( y \) direction, as illustrated in figure 2. The upper panels shows the dispersions in the momentum space, while the lower panels displays the energy spectrum as a function of \( k_x \), with open boundary in \( y \) direction. In these momentum space plots, there are always four topologically trivial zero-energy flat bands lying in the middle. In the left two panels, the SOC and Hubbard interaction strength are all set to be thus. Thus, we obtain a graphene-BCS model. In the upper left panel, the two-fold degenerate graphene-like conduction bands and valence bands linearly touch with each other at \( K \) and \( K' \) points in the Brillouin zone. Correspondingly, the lower left plots with open boundary in \( y \) direction shows that the conduction band and valence band touch on the line connecting the two inequivalent Dirac points. Therefore it describes a nodal superconductor.

If the SOC is introduced, the Dirac fermions at \( K \) and \( K' \) point obtain a mass, meanwhile an energy gap opens up. In this case the model is described by SI-BCS Hamiltonian. In the upper right panel, one can see that the two-fold degenerate conduction bands and valence bands are fully gapped in the bulk. In the lower right panel with open boundary in \( y \) direction, there are gapless helical edge states connecting the valence and conduction bands, which indicates non-trivial topological property of the model.

We now investigate the topological properties of the system. The topological invariant describing class DIII topological superconductors is the \( \mathbb{Z}_2 \) index [3, 5, 28], or spin Chern number equivalently [29–31]. Making the Taylor expansion near the \( K \) point, one gets the effective Hamiltonian

\[
\mathcal{H}_k = -\eta v_F k_x \Gamma_{310} - v_F k_y \Gamma_{132} \\
+ \eta \Delta_k \Gamma_{220} - \Delta_k \Gamma_{210} \\
+ \lambda \eta \sum_j \Gamma_{03j} - \Delta_2 \eta \sum_j \Gamma_{10j}.
\]

Here \( \eta = \pm 1 \) for \( K \) and \( K' \) point respectively. For convenience, we define \( v_F = \sqrt{3}\tau/2 \) and \( \Delta_1 = \sqrt{3}\Delta_1/2 \). Then we can make the unitary transformation \( \tilde{H}(k) = U^{-1} H_k U \) with

\[
U = \exp(i \varphi \Gamma_{002} \Gamma_{003}) \quad \text{and} \quad \varphi_1 = \pi/4, \quad \varphi_2 = \arctan \sqrt{2}.
\]

The resulting \( \tilde{H}(k) \) is a block diagonal matrix in the form,

\[
\tilde{H}(k) = \begin{pmatrix} \tilde{H}_+(k) & 0 \\ 0 & \tilde{H}_-(k) \end{pmatrix}
\]

with \( s = \pm \) and

\[
\tilde{H}_s(k) = \begin{pmatrix} s \eta \lambda & v_F k_x & -s \eta \Delta_2 \\ v_F k_y & -s \eta \lambda & -s \eta \Delta_2 \\ -s \eta \Delta_2 & -s \eta \lambda & -s \eta \lambda \end{pmatrix}.
\]

The eigen-wavefunctions for the two valence bands are

\[
|\psi_{\pm}(k)\rangle = \left[ \sin \alpha_{\pm}(k) - \cos \alpha_{\pm}(k) e^{i\phi(k)} \right].
\]

where \( 2\alpha_{\pm}(k) = \arctan \frac{s \eta \Delta_2 k_y / (s \eta \Delta_2 k_x)}{\sqrt{1 + s^2 (\eta \lambda)^2 k_x^2}} \) and

\[
e^{i\phi(k)} = \frac{(\eta k_y - i k_x)}{k}.
\]

The Berry connection is defined as \( A_{\pm}(k) = -i \langle \psi_{\pm}(k) | \nabla_k | \psi_{\pm}(k) \rangle \) and the Berry curvature is

\[
\mathcal{F}_{\pm}(k) = \nabla_k \times A_{\pm}(k)
\]

\[
= \frac{s (\lambda \pm \Delta_2) (1 \pm \Delta_1/\tau) v_F^2}{2 \left[ (\lambda \pm \Delta_2)^2 v_F^2 + (\lambda \pm \Delta_2)^2 \right]^{3/2}}.
\]

Then the Chern number is the integral over the polar plane

\[
C_{\pm,\pm} = \int \mathcal{F}_{\pm}(k) d^3k/2\pi = \frac{1}{2} \text{sgn} [s (\lambda \pm \Delta_2)]
\]

with \( \text{sgn}(x) = \lim_{x \to 0} x / \sqrt{x^2 + \epsilon^2} \) bring the sign function. The total Chern number and isospin Chern number for each band is defined as

\[
C_{\pm} = C_{\pm,\pm} + C_{\pm,\mp}
\]

\[
C_{\pm,\sigma} = \frac{1}{2} \left( C_{\pm,\pm} - C_{\pm,\mp} \right)
\]
The phase diagram is shown in figure 3. As a result of time reversal symmetry, the total Chern number is always zero. If $|\lambda| > |\Delta_2|$, the isospin Chern number is 1 for both two bands, which indicates the system is a helical topological superconductor. While if $|\lambda| < |\Delta_2|$, the isospin Chern numbers of the two bands have opposite sign, and the system turns out to be another kind of helical topological superconductor. And we see that if the perfect flat band condition is satisfied, the Chern number of the flat band is zero, and system is always topological with nonzero $\lambda$.

4. Exact solution of interacting model along symmetric lines

In this section we show that the model above is exactly solvable even with the Hubbard interaction being taken into account. By introducing the Majorana fermion operators $\eta$ and $\gamma$ for each sublattice site,

$$c_{n,\alpha}^\dagger = \eta_{n,\alpha} + i \gamma_{n,\alpha}, \quad c_{n,\alpha} = \eta_{n,\alpha} - i \gamma_{n,\alpha},$$

we immediately find that the Majorana fermions disappear in $H_1$ and $H_2$. To make it more clear, we can introduce $D_j = 4i\eta_j\gamma_j$ since it commutes with $H_3$ for all sites $j$, i.e. it becomes a c-number. Then with the commutation relation in equation (26)

$$\{\eta_{m,\alpha}, \eta_{n,\beta}\} = \{\gamma_{m,\alpha}, \gamma_{n,\beta}\} = \frac{1}{2}\delta_{mn}\delta_{AB}\delta_{\alpha\beta},$$

the Hamiltonian in equation (1) can be rewritten as

$$H = H_1 + H_2 + H_3$$

with

$$H_1 = 2i\sum_{(i)\beta} (\Delta_1 + i) \eta_{i\alpha}\gamma_{i\beta} + (\Delta_1 - i) \eta_{i\beta}\gamma_{i\alpha},$$

$$H_2 = \frac{i}{2\sqrt{3}} \sum_{(i)\alpha\beta} v_{ij} (\lambda + \Delta_2) \eta_{i\alpha}\gamma_{j\beta} \gamma_{i\beta},$$

$$H_3 = U\sum_{ij} (2i\eta_{i\alpha}\gamma_{j\gamma}) (2i\eta_{j\gamma}\gamma_{i\alpha}).$$

Because the time-reversal transformation $T$ acts as $T^{-1}i\eta_j\gamma_j T = -i\eta_j\gamma_j$ and $T^{-1}\eta_j\gamma_j T = -i\eta_j\gamma_j$, the Hubbard interaction term does not break the time-reversal symmetry. Notice that the anti-commutation relations are
being taken into consideration, we find that \( D_j^2 = 1 \) and hence \( D_j = \pm 1 \). Finally, we arrive at the total Hamiltonian,

\[
H = 4it \sum_{\langle ij \rangle} \sum_{\sigma} \gamma_{ij} \sigma_{ij} + \frac{i\lambda}{\sqrt{3}} \sum_{\langle ij \rangle} \sum_{\nu>\alpha} \nu_{ij} \sigma_{\alpha\beta} \gamma_{ij} - iU \sum_j D_j (\gamma_{j\uparrow} \gamma_{j\downarrow}).
\] (27)

The decouple of \( \eta \) Majorana fermions makes the original interaction terms to be quadratic and thus equation (27) is exactly solvable for any fixed set of \( D_j \), which serves as a background \( \mathbb{Z}_2 \) gauge field. For a \( N \)-site system there are \( 2^N \) choices of the set \( D_j \), and the total Hilbert space is the direct product of that of one Hamiltonian with certain \( D_j \)'s configuration. Two uniform configurations are of most interest. One is the ferromagnetic (FM) configuration with uniform \( D_j = 1 \) for both of the two sublattices \( A \) and \( B \). Another is the antiferromagnetic (AFM) configuration, in which the signs of \( D_j \)'s are opposite for the two sublattices. In the basis of Nambu spinor, \( H_A \) can be written as

\[
H_{FM} = -\frac{U}{4} (\Gamma_{002} - \Gamma_{132}), \quad H_{AFM} = -\frac{U}{4} (\Gamma_{002} - \Gamma_{102})
\] (28)

for the two particular orders. And the eigenvalues are

\[
E_{FM,\pm}^2 = t^2|F|^2 + \lambda^2 \sum_{j=1}^3 G_j^2 + \left( \frac{U}{4} \right)^2 \pm \frac{U}{2} \sqrt{t^2|F|^2 + \lambda^2 G_2^2}.
\] (29a)

\[
E_{AFM}^2 = t^2|F|^2 + \lambda^2 \sum_{j=1}^3 G_j^2 + \left( \frac{U}{4} \right)^2 \pm \frac{\lambda U}{2} G_2.
\] (29b)

We see that for ferromagnetic order, the background \( D_j \) splits the energy bands for different isospin, while for the AFM order the bands remains to be two-fold degenerate. By using the arithmetic mean-root mean inequality

\[
\sum_{\pm} E_{FM,\pm} \leq \sqrt{2 \sum_{\pm} E_{FM,\pm}^2} < 2E_{AFM}
\]

we find that the ground state energy which is the sum of all negative energy states satisfies the following inequality

\[
-\sum_{\pm} \sum_k E_{FM,\pm} > -2E_{AFM}.
\] (30)

This explicitly shows that the AFM order has lower ground state energy than that of FM order (please see figure 4), then the ground state fall into the Hilbert space of the Hamiltonians with AFM configuration, while the time-reversal symmetry holds for the total Hamiltonian equations (1) and (24). Such a configuration does not satisfy the time reversal symmetry, so the time reversal symmetry is spontaneously broken.

We now study the band structure of the interacting topological superconductors with time reversal symmetry being broken. When the SOC strength \( \lambda = 0 \), the model reduces to the graphene-BCS–Hubbard model. The energy band is shown in figure 5. For the AFM case, the interactions can be treated as a staggered potential on isospins and thus the graphene-like partial flat bands are moved away from zero energy but the isospin degeneracy is not affected. Such a system is a full gapped superconductor. While for the FM case, the isospin degeneracy is lifted, meanwhile the conduction and valence bands touch with each other along a circle around the K points. Such a system is a loop-nodal superconductor. Because of the absence of SOC, both above mentioned two cases are topologically trivial.

Then we moved on to investigate the cases with SOC being turned on. The results are shown in figure 6. In the AFM case, the presence of SOC makes the spin-up and spin-down configurations in y direction inequivalent. And because of this, the Hubbard interaction plays a role of gate voltage in y direction, which opens up a band gap and makes the two K valleys imbalanced. Now the energy eigenvalues is not symmetric, i.e. \( E(k_x) \neq E(-k_x) \), which is the consequence of the time reversal symmetry breaking. While for the FM case the band is symmetric.

Although the time reversal symmetry breaking makes the \( \mathbb{Z}_2 \) index ill-defined, the isospin Chern number may be used to describe the non-trivial topological properties. For the superconductor with AFM order, the band gap will not close and re-open again. This is ascribed to a quantum phase transition into a trivial topological superconductor, just as described in references [32, 33]. On the other hand, topological non-trivial phase could survive in FM order [34, 35]. To explore whether there are some topologically nontrivial phases in the FM order, we first take a close look at the gap-closing condition.
necessary condition for gap-closing is

$$\sin k_x = 0, \quad \sin \left( \frac{1}{2} k_x \pm \frac{\sqrt{3}}{2} k_y \right) = 0$$

and we see that the SOC strength is not involved. At the momentum determined by the above equation, the requirement of gap-closing leads to $|F| \pm U = 0$. Therefore, we find that at two special interaction strength $U = 4t$ or $U = 12t$, the conduction and valence bands will touch with each other (figure 7).

To further check that whether the gap-closing could induce a topological transition, we numerically compute the Chern numbers for the valence bands, and the result is shown in figure 8. We see that the total Chern numbers are zero for $U/t < 4$ or $U/t > 12$, while the isospin Chern numbers are 1, indicating the quantum spin Hall phase. Surprisingly, we find that for $4 < U/t < 12$, the total Chern number becomes 1, indicating the quantum anomalous Hall phase. To clarify the mystery, we reinvestigate the band structures and localized edge states to see how the interaction term affects the topological properties. We choose three points $U/t = 0.8, 5, 13$, and the band structures together with corresponding localized edge states are plotted in figure 9. Both edge states are taken in gap from the valence bands with momentum $k_x = \pi$.

The three representative diagrams help us build a better understanding of the evolution of band structures under the interaction. Roughly speaking, the FM interaction term in equation (28) can be regarded as a chemical potential or gate voltage in the spin-$y$ direction. When $U/t < 4$ the interaction strength is weak and the degeneracy of two isospins are lifted. The two edge states crossing at $k_x = \pi$ connects two inequivalent valley of two bands with different isospins. The system is in a quantum spin Hall phase. With the interaction strength increasing to $4 < U/t < 12$, we can see that the bands are strongly inverted. The electrons become full...
polarized and results in the quantum anomalous Hall phase. It is noticed that such an interaction induced isospin-polarized quantum anomalous Hall phase is indexed by one Chern and one-half isospin Chern numbers, which is similar as that induced by the strong circular polarized light in silicene [36]. Finally, when the interaction strength is so strong that the conduction band and valence band with the same isospin are completed inverted. Therefore, the quantum spin Hall phase is restored.

5. Conclusion

In conclusion, we have proposed a SI model on the honeycomb lattice with both BCS pairing potential and Hubbard interaction term to explore the nontrivial topology in an interacting system. Due to the Hubbard interaction, the time-reversal symmetry is spontaneously broken. The ground state of this model is a solution with AFM configuration. For the solution with ferromagnetic configuration, nontrivial topology is found and characterized by the isospin Chern number.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix

To further confirm whether the AFM configuration holds the ground state, we make a comparison with the ground state energy of the Hamiltonian with randomly chosen $D_j$. We numerically diagonalized a $32 \times 32$ system with random on-site Hubbard interaction for 150 times and the results are shown in figure 10. We see that the AFM configuration always has the lowest ground state energy, and the FM configuration has the highest ground state energy. For the randomly chosen $D_j$'s cases the ground state energy always fall between those of the former two cases.

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