Phase transition in the Integrated Density of States of the Anderson model arising from a supersymmetric sigma model

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Abstract

We study the Integrated Density of States (IDS) of the random Schrödinger operator appearing in the study of certain reinforced random processes in connection with a supersymmetric sigma-model. We rely on previous results on the supersymmetric sigma-model to obtain lower and upper bounds on the asymptotic behavior of the IDS near the bottom of the spectrum in all dimension. We show a phase transition for the IDS between weak and strong disorder regime in dimension larger or equal to three, that follows from a phase transition in the corresponding random process and supersymmetric sigma-model. In particular, we show that the IDS does not exhibit Lifshitz tails in the strong disorder regime, thereby proving a conjecture by Sabot and Zeng up to a logarithmic factor. This is in stark contrast with other disordered systems, like the Anderson model. A Wegner type estimate is also derived, giving an upper bound on the IDS and showing the regularity of the function.

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1 Introduction and main results

In recent years, linearly reinforced random processes have attracted much interest from the mathematics community due to the discovery of connections to other models studied in the literature that have allowed for rigorous proofs of phase transitions in these systems. In \cite{25, 26, 27} the authors prove a connection between the edge reinforced random walk, the vertex reinforced jump process and the supersymmetric hyperbolic sigma model $H^{2|2}$ introduced in \cite{28} and \cite{8}. The latter was studied in \cite{7, 6} where it was shown to exhibit dimension-dependent phase transitions. This was a key ingredient in the work of Sabot, Tarrès and Zeng to prove phase transitions for the linearly reinforced random processes they studied. As shown in \cite{26}, a common factor in these models is the appearance of a random Schrödinger operator, denoted by $H_\beta$, to be defined below. The spectral properties of this operator were studied in \cite{26, 27} and more recently in \cite{4}. Sabot and Zeng conjecture that the phase transition in linearly reinforced random processes between recurrent and transient regimes is related to the dynamical localization and delocalization transition of $H_\beta$. This motivates the study of this random operator, as the localization-delocalization transition is a long-standing open problem in the theory of disordered systems, going back to the seminal work of P.W. Anderson \cite{2}.

Our aim is to study the asymptotics of the so called Integrated Density of States (IDS) of the operator $H_\beta$ for energies near the bottom of the spectrum. The IDS is a function on the spectrum of the operator that computes the average number of eigenvalues per unit volume. In disordered systems like the Anderson model with independent random variables, the IDS exhibits an exponential decay near the spectral edges at arbitrary dimension, known as Lifshitz tails. This is in stark contrast with the behavior of the IDS in periodic systems. The Lifshitz behavior of the IDS is a key ingredient to prove localization for random operators, although it is not a necessary condition (see e.g. Delone-Anderson models for which the IDS might not even exist but localization still holds \cite{24, 9}). The connection between the IDS behavior at the bottom of the spectrum and localization explains the important role played by the IDS in the spectral and dynamical study of random Schrödinger operators.

In \cite{27}, the authors conjecture that the asymptotic behavior of the IDS of the random Schrödinger operator $H_\beta$ appearing in connection to reinforced random processes does not exhibit Lifshitz tails. This is due to dependencies in the random variables, that imply that the bottom of the spectrum is not attained by extreme values of the random variables, but can be attained by
several configurations of the potential.

In this article we show that the IDS $N(E, H_\beta)$ of the operator $H_\beta$ does not exhibit Lifshitz tails, and undergoes a phase transition in its behavior as a function of $E$, depending on the dimension and the strength of the disorder. This follows from a phase transition in the associated reinforced random process and supersymmetric sigma-model. Namely, we prove that in dimension one, for any value of the disorder strength, the IDS behaves roughly as $\sqrt{E}$ as $E \downarrow 0$, while in dimension two and above, this behavior holds for large disorder. On the contrary, in dimension three and above the decay rate is bounded above by $E$ at strong disorder.

To the best of our knowledge, the operator $H_\beta$ is the first Anderson-type model for which the IDS is known to undergo a phase transition, whose dependence on the disorder strength and dimension is similar to the one in the metal-insulator transition conjectured for the Anderson model. Note that the transitions appearing in the literature for the IDS of Anderson-type models (the so-called classical-quantum transitions) are transitions in the exponents of the Lifshitz tails depending on the decay of the single site potential [18][11]. A phase transition which does not involve Lifshitz tails has been observed in the IDS for certain random spin models [10]. The operator $H_\beta$ provides a first physically motivated example where Lifshitz tails break down. The latter contributes to the family of very specific models for which the violation of Lifshitz tails is known [15][21][16][3].

We proceed to define the random Schrödinger operator $H_\beta$ on $\mathbb{Z}^d$ appearing in connection with the hyperbolic $H^{2|2}$ sigma-model and reinforced random processes. Let $\mathbb{Z}^d$ be the undirected square lattice, with vertex set $V(\mathbb{Z}^d)$ and edge set $E(\mathbb{Z}^d)$. By abuse of notation we will often identify the set in $\mathbb{Z}^d$ with its vertex set and, in particular, write $\mathbb{Z}^d$ instead of $V(\mathbb{Z}^d)$. The operator $H_\beta$ is defined as follows: let $W = W_{i,j} > 0$ be the edge weight of $e = \{i,j\}$ on $\mathbb{Z}^d$, and $P_W$ be the associated adjacency operator of $\mathbb{Z}^d$, or equivalently, $P_W$ is the operator on $\ell^2(\mathbb{Z}^d)$ defined by

$$P_W f(i) = \sum_{j:j \sim i} W_{i,j} f(j), \forall f \in \ell^2(\mathbb{Z}^d), \quad (1)$$

where $j \sim i$ means that $\{i,j\}$ is an edge of the lattice $\mathbb{Z}^d$. We consider $H_\beta \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$, the infinite symmetric matrix defined by

$$H_\beta := 2\beta - P_W, \quad (2)$$

where $\beta$ is a diagonal matrix and $\beta = (\beta_i)_{i \in \mathbb{Z}^d}$ is a family of positive random variables defined as follows (c.f. Theorem 1 of [26] and Proposition 1 of
∀ i ∈ Z^d, β_i > 0 a.s. and for all sub lattice Λ ⊂ Z^d finite, the Laplace transform of (β_i)_{i ∈ Λ} equals

$$E^W(e^{-⟨\lambda, β⟩_Λ})$$

(3)
$$= e^{-\sum_{i,j \in Λ, i \sim j} W_{i,j} \left(\sqrt{(1+\lambda_i)(1+\lambda_j)} - 1\right) - \sum_{i \in Λ, j \notin Λ, i \sim j} W_{i,j} \left(\sqrt{1+\lambda_i} - 1\right) \prod_{i \in Λ} \sqrt{1+\lambda_i}}.$$

The law of this random field β is characterized by the above Laplace transform, it will be denoted by ν^W or ν^W,1, the associated expectation is denoted by E^W. The operator \(H_β\) defined by

$$H_β f(i) = 2β_i f(i) - \sum_{j \sim i} W_{i,j} f(j), \quad ∀ i \in Z^d,$$

(4)
maps \(D → ℓ^2(Z^d)\) almost surely, where \(D ⊂ ℓ^2(Z^d)\) is the set of sequences with finite support, which is dense.

In this paper, we will set all \(W_{i,j}\) equal, and in an abuse of notation denote this common value \(W\) too (as it will not cause any ambiguity in the sequel).

Then, the operator \(H_β\) is ergodic w.r.t. the translations in \(Z^d\). In the distributions we still write \(W_{i,j}\), to insist on the generality of the probability \(ν^W,η,Λ\). In particular, as shown in Theorem 2.(i) of [27], the random operator \(H_β\) has deterministic spectrum \(σ(H_β) ⊂ [0, +∞]\).

By Proposition 1 of [26] or Lemma 4 of [27], any finite marginal \((β_i)_{i ∈ Λ}\) (i.e. \(Λ ⊂ Z^d\) is a finite subset) has the following explicit probability density w.r.t. the product Lebesgue measure \(dβ = \prod_{i \in Λ} dβ_i:\)

$$ν^W_Λ(dβ) = 1_{H_β,Λ > 0} e^{-\frac{1}{2} \left(⟨1, H_β,Λ 1⟩ + ⟨η^w_Λ, H_β,Λ^{-1}η^w_Λ⟩ - 2⟨1, η^w_Λ⟩\right)} \frac{1}{\sqrt{\det H_β,Λ}} \left(\frac{2}{π}\right)^{|Λ|/2} dβ,$$

(5)
where \(η^w_Λ\) is a vector denoting a wired boundary condition on Λ, defined by

$$η^w_Λ(i) := \sum_{j \notin Λ, j \sim i} W_{i,j}, \quad ∀ i ∈ Λ,$$

(6)
and

$$H_{β,Λ} := (2β - P_W)|_Λ = 1_{Λ_L} H_β 1_{Λ_L}$$

(7)
is the operator \(H_β\) restricted on the set \(Λ\) with simple boundary condition, i.e. a finite matrix defined by

$$H_{β,Λ} f(i) := 2β_i f(i) - \sum_{j ∈ Λ, j \sim i} W_{i,j} f(j), \quad ∀ f ∈ ℜ^Λ.$$

(8)
Here 1_{Λ_L} is the projection operator on Λ_L. The probability distribution of β has not been changed, but to insist on the restriction, we denote it by E^W_Λ. Note that even if we replace η^w_Λ by an arbitrary η ∈ ℝ^A_{≥0}, (3) is still a probability density. In this case the probability will be denoted by ν^W_Λ,η and the expectation by E^W_Λ,η, to stress the η dependence. A more general finite volume density is given in Theorem 4 below.

Sometimes we will write H^S_{β,Λ} := H_{β,Λ} to insist on the type of boundary conditions considered, that we call simple boundary condition. We will also consider the operator with Dirichlet boundary condition, which will be denoted H^D_{β,Λ} and is defined by

\[ H^D_{β,Λ} := (2β - P^W)_Λ + WM_{2d-n} = H_{β,Λ} + WM_{2d-n}, \]  

where M_{2d-n} is the multiplicative operator by 2d - n, where ψ(i) = n_iψ(i) for all ψ ∈ ℓ²(ℤ^d), and n_i := deg(i) in Λ, i.e. n_i = \sum_{j ∈ Λ, j ∼ i} 1.

In the usual Anderson model, the random Schrödinger operator H = −Δ + λV with a bounded potential, the edge weight equals 1 (in the discrete Laplacian Δ, entries are 0 or 1), and the disorder parameter λ > 0 modulating the intensity of the random potential allows for two well-defined regimes, that of strong disorder (λ ≫ 1) and that of weak disorder (small λ). In H_{β}, however, the edge weight equals W, and the law of the random potential depends also on W, hence the disorder parameter does not appear as a coupling constant but is encoded in the law of β. To have an expression that resembles the Anderson model we consider the rescaled operator H_{β} defined by

\[ H_{β} := \frac{1}{W}H_{β} = \frac{2β}{W} - P = \frac{2β}{W} - 2d + (-Δ), \quad P_{ij} := 1_{i ∼ j}. \]  

The corresponding finite volume operator with Dirichlet boundary condition is then

\[ H^D_{β,Λ} := \frac{1}{W}H^D_{β,Λ} = \left(\frac{2β}{W} - P\right)_Λ + M_{2d-n} = H_{β,Λ} + M_{2d-n}, \]  

where H_{β,Λ} = H^S_{β,Λ} is the operator with simple boundary condition.

Note that, the one point marginal of the random potential is known to be a reciprocal inverse Gaussian distribution, by the explicit Laplace transform [3] (c.f. [4, Theorem C]), precisely

\[ \mathbb{E}^W[e^{-λβ}] = \frac{e^{-2dW(\sqrt{1 + λ} - 1)}}{\sqrt{1 + λ}}. \]
It follows that the mean of $2\beta_i$ is $2dW + 1$, and its variance is $2dW + 2$. The corresponding rescaled potential $V_i := \frac{2\beta_i}{W}$ has mean $\mathbb{E}^W[V_i] = 2d + \frac{1}{W}$ and variance $Var[V_i] = \frac{2d}{W} + \frac{2}{W^2}$. Therefore, analogously to the case of the Anderson model, for $H_\beta$ we can identify two regimes: $W$ small corresponds to a strong disorder regime and $W$ large, to a weak disorder regime. Indeed, for large $W$ we have $\mathbb{E}^W[V_i] \simeq 2d$, and $Var[V_i] = O(1/W)$, hence $H_\beta$ is a small perturbation of $2d - P = -\Delta$. On the contrary, for small $W$ both mean and variance are large, $\mathbb{E}^W[V_j] = O(1/W)$, and $Var[V_j] = O(1/W^2)$, hence $H_\beta$ is dominated by the diagonal disorder.

Our main object of study is the Integrated Density of States (IDS) $N(E) = N(E, H_\beta)$ for $H_\beta$ at an energy $E \in \mathbb{R}$, defined by

$$N(E) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \sum_{\lambda \in \sigma(H_\beta^L)} 1 = \frac{1}{|\Lambda_L|} \text{tr} \left( 1_{(-\infty,E]}(H_\beta^L) \right).$$

(12)

Here $\# \in \{D, S\}$ indicates if we have Dirichlet (see (11)) or simple boundary conditions. Note that the usual definition of the IDS (see e.g. [1, Corollary 3.16]) does not contain the expectation in the right-hand side of the equation, but this follows from, e.g., [1, Lemma 4.12]). Also, the limiting function $N$ does not depend on the boundary conditions in the finite-volume restriction of $H_\beta$ to the box, so we can replace the simple boundary conditions with Dirichlet boundary conditions and the result still holds [1, Lemma 4.12]).

We are interested in the asymptotics of $N(E)$ for $E \searrow 0$, that is, at the bottom of the spectrum of $H_\beta$. For a random Schrödinger operator with i.i.d. random potential, the Lifshitz tails estimate (e.g. [17]) claims that, near the bottom of the spectrum (assuming it is 0), the integrated density of state behaves like

$$N(E) = c e^{-E^{-d/2 + o(1)}}$$

in $d$ dimensions, contrary to Weyl’s law of the free Laplacian, which is $N(E) \asymp E^{d/2}$. Lifshitz tails also appear in models exhibiting correlations in the potential, for example in potentials given by a linear combination of i.i.d. random variables [14].

For the $H_\beta$ operator we will prove the following three results, which show that in this case the IDS does not exhibit Lifshitz tails.
Theorem 1 (lower bound on the IDS) We define
\[ W_{cr} = W_{cr}(d) := \max\{W_c, W'_c\}, \tag{14} \]
where \( W_c > 0 \) (resp. \( W'_c > 0 \)) is the (dimensional dependent) parameter introduced in Theorem 6 (resp. Theorem 8). In particular \( W_{cr} = \infty \) for \( d = 1 \).

Then, for each \( 0 < W < W_{cr} \) there exist constants \( c = c(W, d) > 0 \) \( E_0 = E_0(W, d, c) > 0 \) such that
\[ N(E, H_{\beta}) \geq c(- \log E)^{-d} \sqrt{E}, \quad \forall 0 < E < E_0. \tag{15} \]

The next result concerns the regularity of the finite volume IDS with simple/Dirichlet boundary condition defined in (13) above.

Theorem 2 (Wegner type estimate) For all \( W > 0 \) the finite volume IDS \( N(E, H_{\beta, \Lambda_L}^\#) \), with \( \# \in \{D, S\} \) satisfies the bound
\[ E_{\Lambda_L} \left[ N(E + \varepsilon, H_{\beta, \Lambda_L}^\#) - N(E - \varepsilon, H_{\beta, \Lambda_L}^\#) \right] \leq 4 \sqrt{\frac{W}{2\pi}} \sqrt{\varepsilon} \tag{16} \]
uniformly in \( \Lambda_L, E \in \mathbb{R} \) and \( \varepsilon > 0 \).

Moreover, for \( d \geq 3 \) there exists a \( W_0 > 1 \) such that for all \( W \geq W_0 \), the following improved estimate holds
\[ E_{\Lambda_L} \left[ N(E + \varepsilon, H_{\beta, \Lambda_L}^\#) - N(E - \varepsilon, H_{\beta, \Lambda_L}^\#) \right] \leq C \sqrt{W} \varepsilon \tag{17} \]
uniformly in \( \Lambda_L, E \in \mathbb{R} \) and \( \varepsilon > 0 \), where \( C > 0 \) is some constant.

Theorem 3 (upper bound and regularity for the IDS) For all \( W > 0 \) the function \( E \mapsto N(E) \) is Hölder continuous with exponent \( \frac{1}{2} \) and Hölder seminorm \( |N|_{C^{0, \frac{1}{2}}} \leq 2 \sqrt{W/\pi} \). In particular, it satisfies the bound
\[ N(E) \leq 2 \sqrt{\frac{W}{\pi}} \sqrt{E} \quad \forall E > 0. \tag{18} \]

Moreover, for \( d \geq 3 \) there exists a \( W_0 > 1 \) such that for all \( W \geq W_0 \), the function \( E \mapsto N(E) \) is Lipschitz continuous with Lipschitz constant \( \text{Lip}(N) \leq C \sqrt{W}/2 \), where \( C \) is the constant introduced in Theorem 2. In addition, it satisfies the bound
\[ N(E) \leq C'E \quad \forall E > 0, \tag{19} \]
for some constant \( C' > 0 \) independent of \( W \).
Discussion on the results. Theorems 1 and 3 imply, in the strong disorder regime \( W < W_{cr} \),

\[
\frac{1}{c |\ln E|^d} \sqrt{E} \leq N(E) \leq 2 \sqrt{\frac{W}{\pi}} \sqrt{E}.
\]

as \( E \downarrow 0 \). Our result shows that, for a physically motivated ergodic and 1-dependent random potential, Lifshitz tails do not appear. Note that it is conjectured in [27] that the asymptotic behavior of the IDS of the operator \( H_\beta \) is \( \sqrt{E} \), in particular, there is no Lifshitz tail. Therefore we prove this conjecture in the strong disorder regime up to a logarithmic correction. Note that, for strong disorder \( W \ll 1 \) the random variables \( \beta \) are approximately iid with Gamma distribution (cf. the explicit Laplace transform in (3)). Therefore one expects that

\[
N(E, H_\beta) = N(EW, H_\beta) \simeq \mathbb{P}(2\beta_0 < EW) \propto \sqrt{EW}
\]

for \( EW < 1 \), which is indeed what we obtained. For weak disorder \( W \gg 1 \) the random variables approach the constant value \( 2d \), hence one expects convergence to the IDS of \( 2d - P = -\Delta \) as \( W \to \infty \). Note that the improved bound (19) is compatible with this expectation and moreover shows the IDS undergoes a phase transition at \( d \geq 3 \). A more precise comparison with \( -\Delta \) would require also a lower bound for \( N(E, H_\beta) \) at weak disorder, which is still missing.

Theorem 1: strategy of the proof. We argue in three steps.
Step 1. By standard arguments (see Section 3.1) we have, for all \( L \geq 1 \),

\[
N(E, H_\beta) \geq \mathbb{E}^W_{\lambda L} \left[ N(E, H^D_{\beta, \Lambda_L}) \right] \geq \frac{1}{|\lambda L|} \nu^{W, \eta^w}_{\lambda L} \left( (H^D_{\beta, \Lambda_L})^{-1}(0,0) \geq \frac{1}{E} \right),
\]

where \( H^D_{\beta, \Lambda_L} \) was defined in (11) and \( N(E, H^D_{\beta, \Lambda_L}) \) is the finite volume IDS with Dirichlet boundary condition defined in (13).

Step 2. We have no direct information on the probability density of \( (H^D_{\beta, \Lambda_L})^{-1}(0,0) \), but we do have detailed information on the distribution of \( (H^S_{\beta, \Lambda_L})^{-1}(0,0) = H^{-1}_{\beta, \Lambda_L}(0,0) \). In Section 3.2 we show that, for \( E \leq \frac{1}{2} \), \( W < W_{cr} \), and \( L > 1 \) large enough we have

\[
\left\{ (H^S_{\beta, \Lambda_L})^{-1}(0,0) > \frac{1}{E} \right\} \Rightarrow \left\{ (H^D_{\beta, \Lambda_L})^{-1}(0,0) > \frac{1}{2E} \right\},
\]

as \( W \downarrow 0 \).
on a configuration set \( \Omega_{\text{loc}} \) of probability close to one, precisely \( 1 - e^{-\kappa L} \), for some positive constant \( \kappa > 0 \). Hence

\[
N(E, H_\beta) \geq \frac{1}{|\Lambda_L|} \nu^{W,\eta}_{\Lambda_L} \left( \Omega_{\text{loc}} \cap \left\{ H^{-1}_{\beta,\Lambda_L}(0,0) \geq \frac{1}{2E} \right\} \right).
\]

**Step 3.** The conditional density of \( y := 1/H^{-1}_{\beta,\Lambda_L}(0,0) \), knowing \( \beta_0^c = (\beta_j)_{j \in \Lambda_L \setminus \{0\}} \), is denoted by \( d\rho_{a_0} \) and explicitly given in (69). All dependence on \( \beta_0^c \) is contained in the parameter \( a_0 \), defined in (70), which contains the two-point Green’s function of the ground state of \( H_{\beta,\Lambda \setminus \{0\}} \). Therefore

\[
E_{\Lambda_L}^{W,\eta^w} \left[ 1_{\Omega_{\text{loc}}} 1\{H^{-1}_{\beta,\Lambda_L}(0,0) \geq \frac{1}{2E}\} \right] = E_{\Lambda_L}^{W,\eta^w} \left[ \int 1_{\Omega_{\text{loc}}} 1\{y \leq 2E\} d\rho_{a_0}(\beta_0^c)(y) \right].
\]

A subtle point is to show that we can choose \( \Omega_{\text{loc}} \) as intersection of two events \( \Omega_{\text{loc}} = \Omega_{\text{loc},0} \cap \Omega_{\text{loc},1} \) where \( \Omega_{\text{loc},0} \) is measurable wrt \( \beta_0^c \) and \( \Omega_{\text{loc},1} = \{ y \leq e^{-\kappa L} \} \) is measurable wrt \( y \). As a result we can write

\[
E_{\Lambda_L}^{W,\eta^w} \left[ \int 1_{\Omega_{\text{loc}}} 1\{y \leq 2E\} d\rho_{a_0}(\beta_0^c)(y) \right] = E_{\Lambda_L}^{W,\eta^w} \left[ \int_{e^{-\kappa L}}^{2E} d\rho_{a_0}(\beta_0^c)(y) \right].
\]

The set \( \Omega_{\text{loc},0} \) guarantees that \( a_0(\beta_0^c) \leq e^{-\kappa L/2} \) for all \( \beta_0^c \in \Omega_{\text{loc},0} \). Then, using the explicit form of \( \rho_a \) and the fact that \( \Omega_{\text{loc},0} \) has probability close to one we get

\[
E_{\Lambda_L}^{W,\eta^w} \left[ \int_{e^{-\kappa L}}^{2E} d\rho_{a_0}(\beta_0^c)(y) \right] \geq (1 - ce^{-\kappa L/2}) (e^{-\kappa L} \leq y \leq 2E).
\]

The result now follows from a direct analysis of the one-dimensional measure \( \rho_a \). The details are explained in Section 4.

**Theorem 2** strategy of the proof. Note that \( H_{\beta,\Lambda_L} \pm 2\varepsilon = H_{\beta \pm \varepsilon,\Lambda_L} \). Following a standard argument we construct a sequence of potentials interpolating between \( \beta + \varepsilon \) and \( \beta - \varepsilon \), by switching \( \varepsilon \) one site at the time. As a result we obtain the following estimate (cf. Lemma 15)

\[
E_{\Lambda_L}^{W,\eta^w} \left[ N(E + \varepsilon, H_{\beta,\Lambda_L}^\#) - N(E - \varepsilon, H_{\beta,\Lambda_L}^\#) \right] \leq \frac{4}{|\Lambda_L|} \sum_{j \in \Lambda} E_{\Lambda_L}^{W,1,\eta^w} \left[ \mathcal{L}_{\rho_a_j(\beta_0^c)}(4\varepsilon) \right], \quad \forall L > 1,
\]
where $\mathcal{L}_{\rho_{aj}}$ denotes the Lévy concentration (defined in (79)) of the conditional measure $\rho_{aj}$. Using the explicit formula for $\rho_{a}$ we then show that $\mathcal{L}_{\rho_{a}}(\varepsilon) \leq c\sqrt{\varepsilon}$ for some constant $c > 0$ independent of $a$ (cf. Lemma (81)), which gives the first result. For $d \geq 3$ we bound the conditional density pointwise by

$$\rho_{a}(y) \leq \frac{1}{\sqrt{2\pi}} \left( \frac{1}{a} + \frac{1}{\sqrt{a}} \right) \quad \forall y > 0.$$ 

The result now follows from the following bound (cf. Lemma (17)).

$$\mathbb{E}_{\Lambda_{L}}^{W,1,\eta_{w}}[1/a] = \mathbb{E}_{\Lambda_{L}}^{W,1,\eta_{w}}[e^{-u_{0}H^{-1}_{\beta_{\Lambda_{L}}}(0,0)}] \leq C_{d}/W,$$

where $C_{d} > 0$ is a constant depending only on the dimension.

**Theorem 3: strategy of the proof.** The regularity bounds follow directly from the Wegner estimate and (12) by replacing $E$ and $\varepsilon$ by $E/2$. On the contrary, the improved upper bound (19) is proved in Proposition 20 using properties of the infinite volume distribution of $\beta$.

**Organization of this paper.** In Section 2 we review some definitions and known results, and derive the modifications of these results that will be needed in the rest of the paper. A few technical results that we will also use in this section are summarized in the Appendix. Section 3 covers the first two steps of the proof of Theorem 1. The final step is worked out in Section 4. Section 5 contains the proof of Theorem 2. Note that all the these proofs involve only properties of the finite volume marginal distribution but some of the above results can be recovered by exploiting properties of the infinite volume distribution. The main ideas are sketched in Section 6, while the detailed construction can be found in [23]. This alternative approach also provides the improved bound (19) in Theorem 3. All corresponding details are given in Section 6.

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2 Some previous results on the $\mathcal{H}_\beta$ operator

As mentioned in the introduction, the $\mathcal{H}_\beta$ operator has been studied in the literature in connection with linearly reinforced random processes and the $H^{2|2}$ sigma-model. In this section we collect some tools and results that we will use in the next sections. The following theorem can be found in [26, 27, 19].

**Theorem 4 (The multivariate inverse Gaussian distribution)** Let $\mathcal{G} = (V, E)$ be a finite graph. For any $W \in \mathbb{R}_>^E$, $\theta \in \mathbb{R}_V^>$ and $\eta \in \mathbb{R}_V^\geq$, the following holds

$$
\int_{H_{\beta,V} > 0} e^{-\frac{1}{2} \left(\langle \theta, H_{\beta,V} \theta \rangle + \langle \eta, H_{\beta,V}^{-1} \eta \rangle - 2\langle \theta, \eta \rangle \right)} \prod_i \theta_i \sqrt{\det H_{\beta,V}} \left(\frac{2}{\pi}\right)^{|V|/2} d\beta = 1,
$$

(22)

where $H_{\beta,V} := 2\beta - P_W \in \mathbb{R}_V^E \times \mathbb{R}_V^E$. We denote by $\nu_{\mathcal{G}, W, \theta, \eta}$ the probability defined by the above integral, in particular, $\nu_{\mathcal{G}, W, \eta} = \nu_{\mathcal{G}, W, \theta, \eta}$ with $\theta \equiv 1$. The associated expectations are denoted by $E_{\mathcal{G}, W, \theta, \eta}$ and the Laplace transform is given by

$$
E_{\mathcal{G}, W, \theta, \eta} \left( e^{-\langle \lambda, \beta \rangle} \right) = e^{-\sum_{i,j \in V, i \sim j} W_{i,j} \left( \sqrt{(\theta_i^2 + \lambda_i)(\theta_j^2 + \lambda_j) - \theta_i \theta_j} \right) - \sum_{i \in V} \eta_i \left( \sqrt{\theta_i^2 + \lambda_i} \right)} \prod_{i \in V} \frac{\theta_i}{\sqrt{\theta_i^2 + \lambda_i}}.
$$

(23)

Moreover, if $(\beta_i)_{i \in V}$ is distributed according to $\nu_{\mathcal{G}, W, \theta, \eta} \equiv 0$, and $\mathcal{G}' = (V', E')$ is the subgraph obtained by taking $V' \subset V$ and $E' := \{ \{i, j\} \in E \mid i, j \in V' \}$, then the marginal law of $(\beta_i)_{i \in V'}$ is $\nu_{\mathcal{G}', W, \theta, \eta}$ where $W', \theta'$ equal $W, \theta$ restricted on $\Lambda'$, and $\eta$ is defined by $\eta_i = \sum_{j \in V \setminus V'} W_{i,j} \theta_j$. Note that $\eta$ is a generalization of $\eta_{V'}$, defined in (6).

**Remarks on various marginals.** Note that, (see e.g. Remark 3.5 of [4]), in the case $\eta \equiv 0$ and $\theta \equiv 1$, for any $i \in V$,

$$
\gamma := \frac{1}{2H_{\beta,V}^{-1}(i, i)} \text{ has density } 1_{\gamma > 0} \frac{1}{\sqrt{\pi \gamma}} e^{-\gamma},
$$

(24)

i.e. it’s a Gamma random variable of parameter $1/2$. This holds for any $W$ and any finite $\mathcal{G}$.

If we consider a box $\Lambda_L$ of side $2L + 1$ in $\mathbb{Z}^d$ and $\theta \equiv 1$,

$$
\nu_{\Lambda_L, W, \eta} = \text{marginal of } \nu_{\Lambda_{L+1}, W, 1, 0},
$$

(25)
where $\eta^w$ is given in (6). Therefore, $\eta$ can be seen as the boundary condition of the law of the random potential. The case $\eta \equiv 0$ is called zero boundary condition. Note that $\nu^{W,1,\eta^w}_{\Lambda}$ also corresponds to the marginal of $\beta = (\beta_j)_{j \in \mathbb{Z}^d}$ on $\Lambda_L$. For general $\eta$,

$$
\nu^{W,1,\eta}_{\Lambda_L} = \text{marginal of } \nu^{W,1,0}_{\Lambda_L \cup \delta} \tag{26}
$$

where the graph $\Lambda \cup \delta$ has vertex set $\Lambda \cup \{\delta\}$ and edge set $E(\Lambda) \cup \{\{i, \delta\} | i \in \Lambda\}$, and we defined $W_{i\delta} = \eta_i \forall i \in \Lambda$.

**Connection with the $H^{2|2}$ model.** Let $\mathcal{G}$ be the graph associated to a box $\Lambda$ of $\mathbb{Z}^d$, $\theta \equiv 1$, and $\eta \in [0, \infty)^\Lambda$ with at least one strictly positive component. The following expression defines a probability measure for $u \in \mathbb{R}^\Lambda$ (cf. [6])

$$
\mu^{W,1,\beta}_{\Lambda}(u) = e^{-\sum_{i \sim j \in \Lambda} W_{ij}(\cosh(u_i - u_j) - 1)} e^{-\sum_{j \in \Lambda} \eta_j(\cosh u_j - 1)} \sqrt{\det H_{\beta(u),\Lambda}} \frac{1}{\sqrt{2\pi^n}} du_{\Lambda},
$$

where we defined

$$
2\beta_i(u) = \sum_{j \in \Lambda} W_{ij} e^{u_j - u_i} + \eta_i e^{-u_i} \quad \forall i \in \Lambda \tag{28}
$$

The corresponding average is denoted by $\mathbb{E}^{W,1,\eta}_{u,\Lambda}$. Note that the measure $\mu^{W,1,\eta}_{\Lambda}(u)$ is also the effective bosonic field measure in Section 2.3 of [7].

The next lemma connects $\mu^{W,1,\eta}_{\Lambda}(u)$ with $\nu^{W,1,\eta}_{\Lambda}(\beta)$ and can be found in Proposition 2 and Theorem 3 in [26].

**Lemma 5 (connection to $H^{2|2}$)** Let $\mathcal{G}$ be the graph associated to a box $\Lambda$ of $\mathbb{Z}^d$, $\theta \equiv 1$, and $\eta \in [0, \infty)^\Lambda$ with at least one strictly positive component. It holds

$$
\mathbb{E}^{W,1,\eta}_{\Lambda}[f(\beta_{\Lambda})] = \mathbb{E}^{W,1,\eta}_{u,\Lambda}[f(\beta_{\Lambda}(u))] \tag{29}
$$

for any function $f$ integrable with respect to the measure $\nu^{W,1,\eta}_{\Lambda}$. Moreover, remembering that $\nu^{W,1,\eta}_{\Lambda}$ corresponds to the marginal of $\nu^{W,1,0}_{\Lambda \cup \delta}$ with $W_{j,\delta} = \eta_j \forall j \in \Lambda$, (cf. eq. (26)) it holds

$$
e^{u_i} = \frac{H_{\beta_{\Lambda},\delta}(i, \delta)}{H_{\beta_{\Lambda},\delta}(\delta, \delta)}, \quad \forall i \in \Lambda, \tag{30}
$$

where the above fraction is independent of $\beta_{\delta}$. Hence we have

$$
\mathbb{E}^{W,1,\eta}_{u,\Lambda}[f(u)] = \mathbb{E}^{W,1,0}_{\Lambda \cup \delta}[f((u(\beta)))] = \mathbb{E}^{W,1,\eta}_{\Lambda}[f((u(\beta_{\Lambda})))]. \tag{31}
$$
Note that, by the resolvent identity we have, for all \( j \in \Lambda \),
\[
\mathcal{H}_{\beta, \Lambda \cup \delta}(j, \delta) = 0 + \sum_{k \in \Lambda} \mathcal{H}_{\beta, \Lambda}(j, k) W_{k, \delta} \mathcal{H}_{\beta, \Lambda \cup \delta}(\delta, \delta)
\]
\[
= \mathcal{H}_{\beta, \Lambda \cup \delta}(\delta, \delta) \sum_{k \in \Lambda} \mathcal{H}_{\beta, \Lambda}(j, k) \eta_k = \mathcal{H}_{\beta, \Lambda \cup \delta}(\delta, \delta) (\mathcal{H}_{\beta, \Lambda}^{-1} \eta_{\Lambda})(j)
\]
and hence
\[
e^{u_j} = \frac{\mathcal{H}_{\beta, \Lambda \cup \delta}(i, \delta)}{\mathcal{H}_{\beta, \Lambda \cup \delta}(\delta, \delta)} = (\mathcal{H}_{\beta, \Lambda}^{-1} \eta_{\Lambda})(j).
\]

It follows
\[
\mathbb{E}^{W,1,\eta}_{u,\Lambda} \left[ f \left( e^{u_j - u_{j'}} \right) \right] = \mathbb{E}^{W,1,\eta}_{\Lambda} \left[ f \left( \frac{(\mathcal{H}_{\beta, \Lambda}^{-1} \eta_{\Lambda})(j)}{(\mathcal{H}_{\beta, \Lambda}^{-1} \eta_{\Lambda})(j')} \right) \right].
\]
for any function \( f \), as long as the left and right hand side are well defined. In the special case \( \eta_j = \eta \delta_{jj_0} \) (pinning at one point) the formula simplifies to
\[
e^{u_j - u_{j_0}} = \frac{\mathcal{H}_{\beta, \Lambda}^{-1}(j, j_0)}{\mathcal{H}_{\beta, \Lambda}^{-1}(j_0, j_0)}.
\]

Therefore, we can translate Theorem 1 and 2 of \([6]\) into the following.

**Theorem 6 (decay of the ground state Green’s function (1))** Let \( \Lambda_L \) be a finite box of side \( 2L + 1 \) in \( \mathbb{Z}^d \), \( \theta = 1 \), and \( \eta \in [0, \infty)^\Lambda \) with at least one strictly positive component. We define
\[
I_W := \sqrt{W} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-W(\cosh t - 1)}.
\]
We define \( W_c > 0 \) as the unique solution of \( I_{W_c} e^{W_c(2d-2)(2d-1)} = 1 \). In particular, when \( d = 1 \), \( W_c = +\infty \). Finally set \( C_0 = 2e^{2W} \).

Then for all \( 0 < W < W_c \), \( I_W e^{W(2d-2)(2d-1)} < 1 \) and the following hold.

(i) Fix \( i, j \in \Lambda \), \( W_{ij} = W \ \forall i \sim j \in \Lambda \) and assume \( \eta_i > 0, \eta_j > 0 \). Then we have
\[
\mathbb{E}^{W,1,\eta}_{\Lambda_L} [\mathcal{H}_{\beta, \Lambda_L}^{-1}(i, j)] \leq C_0 e^{\sum_{k \in \Lambda_L} \eta_k} \left( \eta_i^{-1} + \eta_j^{-1} \right) \left[ I_W e^{W(2d-2)(2d-1)} \right]^{\mid i - j \mid},
\]
where \( \mid i - j \mid \) is the graph distance between \( i, j \).
(ii) Assume there is only one pinning at \( j_0 \in \Lambda_L \), i.e. \( \eta_j = \eta \delta_{j,j_0} \). Then 
\forall j \in \Lambda_L,
\begin{equation}
\mathbb{E}_{\Lambda_L}^{W,1,\eta} \left[ \sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}(j_0,j_0)}} \right] \leq C_0 \left( I_W e^{W(2d-2)(2d-1)} \right)^{|j-j_0|}. \tag{38}
\end{equation}

**Remark 1.** The function \( W \mapsto I_W \) is monotone increasing (cf. Remark 1 after Theorem 1 in [6]). Therefore the function \( W \mapsto F_d(W) := I_W e^{W(2d-2)(2d-1)} \) is also monotone increasing and \( W_c \) is well defined.

**Remark 2.** Note that in [6] we assumed in addition \( \sum_{k \in \Lambda} \eta_k = 1 \) and hence the term \( e^{\sum_{k \in \Lambda} \eta_k} \) is replaced by \( e^1 \).

In this article we will use the following extension of Theorem 6.

**Theorem 7 (decay with wired bc (1))** Let \( \Lambda_L \) be a finite box of side \( 2L+1 \) in \( \mathbb{Z}^d \). We consider \( (\beta_i)_{i \in \Lambda_L} \sim \nu_{\Lambda_L}^{W,1,\eta_w} \) where \( \eta_w^{\Lambda_L} \) is the wired boundary condition introduced in (6).

Let \( W_c \) be as in Theorem 6 above.

Then, there is a constant \( \kappa = \kappa(W,d) > 0 \), and \( C_0 = C_0(W,d) > 0 \) as before s.t.
\begin{equation}
\mathbb{E}_{\Lambda_L}^{W,1,\eta_w} \left[ \sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}(j_0,j_0)}} \right] \leq C_0 e^{-\kappa |j-j_0|}. \tag{39}
\end{equation}

holds for all \( 0 < W < W_c \), \( j,j_0 \in \Lambda \).

In particular, in \( d = 1 \) \( W_c = \infty \), hence the bound holds \( \forall W > 0 \).

**Proof.** By (25) we have
\begin{equation}
\mathbb{E}_{\Lambda_L}^{W,1,\eta_w} \left[ \sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}(j_0,j_0)}} \right] = \mathbb{E}_{\Lambda_L}^{W,1,0} \left[ \sqrt{\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}(j_0,j_0)}} \right].
\end{equation}

By a random walk representation (cf. Proposition 6 of [27] and notations therein)
\begin{equation}
\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}(j_0,j_0)} = \sum_{\sigma \in \mathcal{P}_L^{j_0 j}} \frac{W_{\sigma}}{(2\beta)^{1/2}},
\end{equation}

where \( \mathcal{P}_L^{j_0 j} \) is the set of nearest neighbor paths from \( j_0 \) to \( j \) in \( \Lambda_L \) that visit \( j_0 \) only once. It follows, for all \( j,j_0 \in \Lambda_L \),
\begin{equation}
\frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}(j_0,j_0)} \leq \frac{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)}{\mathcal{H}_{\beta,\Lambda_L}(j_0,j_0)}.
\end{equation}
since in the term in the right-hand side contains more paths. Hence
\[
E_{\Lambda_L}^{W,1,\eta}\left[ \frac{1}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)} \right] = E_{\Lambda_{L+1}}^{W,1,0}\left[ \frac{1}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(j_0,j)} \right] \leq E_{\Lambda_{L+1}}^{W,1,0}\left[ \frac{1}{\mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(j_0,j)} \right].
\]

By the monotonicity result [22, Theorem 6] (cf. Corollary [23] in the appendix) we have, setting \( \eta_j = W\delta_{j,j_0} \forall j \in \Lambda_{L+1} \),
\[
E_{\Lambda_{L+1}}^{W,1,0}\left[ \frac{1}{\mathcal{H}_{\beta,\Lambda_{L+1}}^{-1}(j_0,j)} \right] \leq E_{\Lambda_{L+1}}^{W,1,0,\eta}\left[ \sqrt{e^{u_j-u_{j_0}}} \right] \leq C_0 e^{-\kappa |j-j_0|},
\]
with \( \kappa = -\log(I_W e^{W(2d-2)(2d-1)}) > 0 \), and \( C_0 = 2e^{2W} \). In the last step we used Lemma [12] and Theorem [6(ii)].

Another useful result on the decay of the ground state Green’s function is Theorem 2.1 of [11], more precisely Equation (5.4) therein. We state this result for our applications

**Theorem 8 (decay of the ground state Green’s function (2))** Let \( \Lambda_L \) be a finite box of side \( 2L + 1 \) in \( \mathbb{Z}^d \), and define \( W'_c = \frac{\sqrt{\pi}}{\Gamma(1/4)4^{1/4}d} \).

Let \( (\beta_i)_{i \in \Lambda_L} \sim \nu_{\Lambda_L}^{W,1,0} \). Then for all \( 0 < W < W'_c \), there are constants \( \kappa = \kappa(d,W) \), and \( C'_0(d,W) \) s.t. for any \( i, j \in \Lambda_L \),
\[
E^{W,1,0}_{\Lambda_L}\left[ \frac{1}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j)^{1/4}} \right] \leq C'_0 e^{-\kappa |i-j|}. \tag{41}
\]

In this paper we will use the following corollary of the above result.

**Corollary 9 (decay with wired b.c. (2))** Let \( \Lambda_L \) be a finite box of side \( 2L + 1 \) in \( \mathbb{Z}^d \), and let \( (\beta_i)_{i \in \Lambda_L} \sim \nu_{\Lambda_L}^{W,1,\eta} \) where \( \eta_{\Lambda_L}^{\nu} \) is the wired boundary condition introduced in [9].

Remember the definition of \( W'_c \) in Theorem 8 above.

Then, for all \( 0 < W < W'_c \), there are constants \( \kappa = \kappa(W,d) > 0 \) and \( C'_0(W,d) > 0 \) s.t. s.t. for any \( i, j \in \Lambda_L \),
\[
E^{W,1,\eta}_{\Lambda_L}\left[ \frac{1}{\mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j)^{1/4}} \right] \leq C'_0 e^{-\kappa |i-j|}. \tag{42}
\]

In an abuse of notation, from now on we will write \( C_0 \) for the constant in both decay results in Theorem [6(ii)] and Theorem [8].
Proof of Corollary 9. By (25) we have
\[ E_{\Lambda_L}^{W,1,\eta} \left[ H_{\beta,\Lambda_L}^{-1}(i,j) \right]^{1/4} = E_{\Lambda_{L+1}}^{W,1,0} \left[ H_{\beta,\Lambda_L}^{-1}(i,j) \right]^{1/4}. \]

By random walk representation (cf. Proposition 6 of [27] and notations therein)
\[ H_{\beta,\Lambda_L}^{-1}(i,j) = \sum_{\sigma \in \mathcal{P}_{ij}^{\Lambda_L}} W_{\sigma} / (2\beta)^{\sigma} \]
where \( \mathcal{P}_{ij}^{\Lambda_L} \) is the set of nearest neighbor paths from \( j_0 \) to \( j \) in \( \Lambda_L \). It follows, for all \( i,j \in \Lambda_L \),
\[ H_{\beta,\Lambda_L}^{-1}(i,j) \leq H_{\beta,\Lambda_{L+1}}^{-1}(i,j) \] (43)
since in the second term we have more paths. Therefore
\[ E_{\Lambda_L}^{W,1,\eta} \left[ H_{\beta,\Lambda_L}^{-1}(i,j) \right]^{1/4} \leq E_{\Lambda_{L+1}}^{W,1,0} \left[ H_{\beta,\Lambda_L}^{-1}(i,j) \right]^{1/4} \leq C_0 e^{-\kappa|i-j|}, \]
where in the last step we applied Theorem 8.

Comparing \( W_c \) and \( W'_c \). In Section 3.2 we will construct two sets of measure close to one \( \Omega_1 \) (resp \( \Omega_2 \)) using Theorem 7 (resp. Corollary 9). Both sets can be used to construct the same lower bound on the IDS (cf. Section 4), which will be valid \( \forall W < W_c = W_c(d) \) (resp. \( \forall W < W'_c = W'_c(d) \)) if we use \( \Omega_1 \) (resp. \( \Omega_2 \)).

It is then reasonable to ask which result works for a larger set of parameters \( W \), i.e. which of the two critical values is larger. While we have an explicit numeric expression for \( W'_c(d) \), \( W_c(d) \) is only indirectly determined as the unique solution of \( F_d(W) = I_We^{W(2d-2)(2d-1)} = 1 \) (cf. Remark 1. after Theorem 7). For \( d = 1 \),
\[ W_c(1) = \infty > W'_c(1). \] (44)

For \( d = 2 \) we compute numerically \( F_2(W'_c(2)) \approx 2.908 \), hence
\[ W_c(2) < W'_c(2). \] (45)

We claim that the map \( d \mapsto f(d) := F_d(W'_c(d)) \) is monotone increasing and hence \( F_d(W'_c(d)) \geq F_2(W'_c(2)) \geq 2.9 \), which implies \( W'_c(d) > W_c(d) \ \forall d \geq 2. \)
Indeed, recalling the definition of modified Bessel function of the second kind \(K_\alpha(x) := \int_0^\infty \cosh(\alpha t)e^{-x \cosh t}dt\), we have \(I_W = e^W \sqrt{\frac{W}{2\pi}}K_0(W)\) and

\[F_d(W) = \sqrt{\frac{W}{2\pi}}K_0(W)e^{W(2d-1)(2d-1)}\]

Note that \(W'_d(d) = C/d\), where \(C\) is a constant independent of \(d\), and hence \(f(d) = F_d(C/d)\). Moreover

\[\partial W F_d(W) = \left(\frac{1}{2W} + (2d - 1) - \frac{K_0'(W)}{K_0(W)}\right) F_d(W)\]
\[\partial_d F_d(W) = \left(2W + \frac{2}{2d - 1}\right) F_d(W).\]

It follows

\[f'(d) = -\frac{C}{2} \partial W F_d(C/d) + \partial_d F_d(C/d)\]
\[= F_d(C/d) \left[\frac{2d + 1}{2d(2d-1)} + \frac{C}{2d} + \frac{C K_1(C/d)}{2\pi K_0(C/d)}\right] > 0,\]

which proves that \(f\) is monotone increasing.

Before going to the proof of the lower bound we list an additional useful corollary on the probability distribution of \(H^{-1}_{\beta,A}\).

**Corollary 10** Let \(A_L\) be a finite box of side \(2L+1\) in \(\mathbb{Z}^d\), and let \((\beta_i)_{i \in A_L} \sim \nu_{A_L}^{W,1,\eta^w}\) where \(\eta^w_{A_L}\) is the wired boundary condition introduced in (6).

Then, for any \(\delta > 0\) we have

\[\nu_{A_L}^{W,1,\eta^w}(H^{-1}_{\beta,A}(0,0) > \delta) \leq \int_0^{\frac{1}{2\delta}} \frac{1}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma.\] (46)

**Proof.** We argue

\[\nu_{A_L}^{W,1,\eta^w}(H^{-1}_{\beta,A}(0,0) > \delta) = \nu_{A_{L+1}}^{W,1,0}(H^{-1}_{\beta,A}(0,0) > \delta).\]

By (13) \(H^{-1}_{\beta,A}(0,0) \geq H^{-1}_{\beta,A+1}(0,0)\) and hence

\[H^{-1}_{\beta,A}(0,0) > \delta \Rightarrow H^{-1}_{\beta,A+1}(0,0) > \delta.\]

It follows

\[\nu_{A_{L+1}}^{W,1,0}(H^{-1}_{\beta,A}(0,0) > \delta) \leq \nu_{A_{L+1}}^{W,1,0}(H^{-1}_{\beta,A+1}(0,0) > \delta)\]
\[= \nu_{A_{L+1}}^{W,1,0}\left(\frac{1}{2H^{-1}_{\beta,A+1}(0,0)} < \frac{1}{2\delta}\right) = \int_0^{\frac{1}{2\delta}} \frac{1}{\sqrt{\pi \gamma}} e^{-\gamma} d\gamma,\]

where in the last step we used that \(1/2H^{-1}_{\beta,A}(0,0)\) is Gamma distributed (cf. Equation (24)).
3 Preliminary results

3.1 Connection between \( N(E) \) and the Green’s function with Dirichlet b.c.

To obtain a lower bound on \( N(E, H_\beta) \) we use the following classical argument (see, e.g. [13])

**Lemma 11** For any finite box \( \Lambda_L \) of side \( 2L + 1 \) we have

\[
N(E, H_\beta) \geq E_{\Lambda_L}^{W,1,\eta^w} [N(E, H_{\beta,\Lambda_L}^D)],
\]

where \( N(E, H_{\beta,\Lambda_L}^D) \) is the finite volume IDS with Dirichlet boundary condition defined in [13], \( E_{\Lambda_L}^{W,1,\eta^w} \) denotes the expectation with respect to the finite marginal \( \nu_{\Lambda_L}^{W,1,\eta^w} \) of \( (\beta_i)_{i \in \Lambda_L} \) given in (5), and \( \eta^w = \eta_{\Lambda_L}^w \).

**Proof.** Recalling the definition of the integrated density of states \( N \), by (12) we have

\[
N(E) = \lim_{K \to \infty} \frac{1}{|\Lambda_K|} E_{\Lambda_K}^{W,1,\eta^w} \left[ \text{tr}(1_{(-\infty,E]}(H_{\beta,\Lambda_K})) \right],
\]

where \( \Lambda_K \) is a finite box of side \( 2K + 1 \). We split the large box \( \Lambda_K \) into a tiling of smaller boxes of side \( 2L + 1 \) with \( L < K, \Lambda_K = \bigcup_{j=1}^{N_K} \Lambda_{L,j} \). Using \( (v_i - v_j)^2 \leq 2(v_i^2 + v_j^2) \) (Dirichlet–Neumann bracketing) we obtain

\[
H_{\beta,\Lambda_K} \leq \bigoplus_{j=1}^{N_K} H_{\beta,\Lambda_{L,j}}^D,
\]

as a quadratic form. Note that by min-max principle, if \( A > B \) then \( \lambda_{A,j} > \lambda_{B,j} \), where \( \lambda_{A,j} \) are ordered eigenvalues of \( A \). This, together with translation invariance and the relation \( |\Lambda_L|N_K = |\Lambda_K| \), yields

\[
\frac{1}{|\Lambda_K|} E_{\Lambda_K}^{W,1,\eta^w_K} \left[ \text{tr}(1_{(-\infty,E]}(H_{\beta,\Lambda_K})) \right] \geq \sum_{j=1}^{N_K} \frac{1}{|\Lambda_K|} E_{\Lambda_{L,j}}^{W,1,\eta^w_{\Lambda_K}} \left[ \text{tr}(1_{(-\infty,E]}(H_{\beta,\Lambda_{L,j}}^D)) \right]
\]

\[
= \sum_{j=1}^{N_K} \frac{1}{|\Lambda_K|} E_{\Lambda_{L,j}}^{W,1,\eta^w_{\Lambda_K}} \left[ \text{tr}(1_{(-\infty,E]}(H_{\beta,\Lambda_{L,j}}^D)) \right]
\]

\[
= \frac{N_L}{|\Lambda_K|} E_{\Lambda_{L,j}}^{W,1,\eta^w_{\Lambda_K}} \left[ \text{tr}(1_{(-\infty,E]}(H_{\beta,\Lambda_{L,j}}^D)) \right] = E_{\Lambda_L}^{W,1,\eta^w} [N(E, H_{\beta,\Lambda_L}^D)],
\]

(49)
for any finite box \( \Lambda_L \) in the family \( \{ \Lambda_{L,j} \}_{j=1}^{NK} \). Taking the limit \( K \to \infty \), keeping \( L \) fixed gives the desired result.

As we are looking for a lower bound of \( N(E,H) \), we can consider any finite box \( \Lambda_L \) (usually a larger \( L \) gives a better bound). We will fix \( \Lambda_L = [-L,L]^d \cap \mathbb{Z}^d \) in the sequel. At the end we will choose \( L \) depending on the energy \( E \).

**Lemma 12** Let \( \Lambda_L = [-L,L]^d \cap \mathbb{Z}^d \). It holds

\[
\mathbb{E}_{\Lambda_L}^{W,1,\eta}[N(E,H_{\beta,\Lambda_L})] \geq \frac{1}{|\Lambda_L|} \mathbb{E}_{\Lambda_L}^{W,1,\eta}[\nu_{\Lambda_L}^{W,1,\eta}\left((H_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E}\right)]
\]

(50)

**Proof.** \( H_{\beta,\Lambda_L}^D \) is a self adjoint finite random matrix, and by definition it is a.s. positive definite. As a consequence, its smallest eigenvalue \( \lambda_1 \) satisfies

\[
\lambda_1 > 0 \quad \text{and} \quad \frac{1}{\lambda_1} = \|(H_{\beta,\Lambda_L}^D)^{-1}\|_{op}
\]

where \( \| \cdot \|_{op} \) stands for the operator norm. It follows that

\[
\mathbb{E}_{\Lambda_L}^{W,1,\eta}[N(E,H_{\beta,\Lambda_L})] \geq \frac{1}{|\Lambda_L|} \mathbb{E}_{\Lambda_L}^{W,1,\eta}[\nu_{\Lambda_L}^{W,1,\eta}\left(\lambda_1 \leq E\right)] \geq \frac{1}{|\Lambda_L|} \mathbb{E}_{\Lambda_L}^{W,1,\eta}\left(\|(H_{\beta,\Lambda_L}^D)^{-1}\|_{op} \geq \frac{1}{E}\right).
\]

Note that

\[
\|(H_{\beta,\Lambda_L}^D)^{-1}\|_{op} = \sup_{\psi: \|\psi\|=1} \|(H_{\beta,\Lambda_L}^D)^{-1}\psi\| \geq \|(H_{\beta,\Lambda_L}^D)^{-1}(0,0)\| = (H_{\beta,\Lambda_L}^D)^{-1}(0,0),
\]

(52)

where \( e_0 = (\delta_{j0})_{j \in \mathbb{Z}^d} \). In the last step we used that, since the matrix is a.s. an M-matrix, the entries of its Green’s function are all positive. Therefore,

\[
\mathbb{E}_{\Lambda_L}^{W,1,\eta}[N(E,H_{\beta,\Lambda_L})] \geq \frac{1}{|\Lambda_L|} \mathbb{E}_{\Lambda_L}^{W,1,\eta}\left(\|(H_{\beta,\Lambda_L}^D)^{-1}\|_{op} \geq \frac{1}{E}\right)
\]

(53)

This concludes the proof of the lemma.  \( \square \)
3.2 From Dirichlet to simple boundary conditions.

Lemma 13 (Dirichlet versus simple bc (1)) Let $\Lambda_L$ be the finite box in $\mathbb{Z}^d$ of side $2L + 1$ centered at $0$. We consider $(\beta_i)_{i \in \Lambda_L} \sim \nu_{\Lambda_L}^{W,1,\eta^w}$ where $\eta^w_{\Lambda_L}$ is the wired boundary condition introduced in (6). Define $\Omega_1 = \Omega_{1,0} \cap \Omega_{1,1}$, with

$$
\begin{align*}
\Omega_{1,0} := \left\{ \left( \frac{H^{-1}_{\beta,\Lambda_L}(0,i)}{H_{\beta,\Lambda_L}(0,0)} \right) \leq e^{-\kappa|i|/2} \forall i \in \partial \Lambda_L \right\}, \\
\Omega_{1,1} := \left\{ H^{-1}_{\beta,\Lambda_L}(0,0) < e^{\kappa L} \right\},
\end{align*}
$$

(54)

where $\kappa$ is the constant introduced in Theorem 7 and remember that $H_{\beta,\Lambda_L} = W H_{\beta,\Lambda_L}$. Let $W_c$ be as in Theorem 6. Then there is a constant $L_0 = L_0(W,d) > 1$ and $C_1 = C_1(d,W_c)$ such that $\forall L \geq L_0$ and $\forall 0 < W < W_c$ we have

$$
\nu_{\Lambda_L}^{W,1,\eta^w}(\Omega_{1,j}) \geq 1 - C_1 e^{-\kappa L/4}
$$

for $j = 0, 1$, (55) and hence $\nu_{\Lambda_L}^{W,1,\eta^w}(\Omega_1) \geq 1 - 2C_1 e^{-\kappa L/4}$. Moreover on the set $\Omega_1$, for any $E > 0$ it holds

$$
\left\{ (H^{-1}_{\beta,\Lambda_L}(0,0) > \frac{1}{E} \right\} \Rightarrow \left\{ (H^{-1}_{\beta,\Lambda_L}(0,0) > \frac{1}{2E} \right\},
$$

(56)

In particular, when $d = 1$ this result holds for all $W > 0$, since $W_c = \infty$.

Note that the set $\Omega_{1,0}$ is measurable wrt $\{\beta_j\}_{j \in \Lambda \setminus \{0\}}$ while $\Omega_{1,1}$ is measurable wrt $(H^{-1}_{\beta,\Lambda_L}(0,0))$. This fact will be important in the proof of the lower bound for the IDS.

**Proof of Lemma 13.** The decay estimate (59) together with the Markov inequality entails that

$$
\nu_{\Lambda_L}^{W,1,\eta^w}(\Omega^c_{1,0}) \leq \sum_{i \in \partial \Lambda_L} \nu_{\Lambda_L}^{W,1,\eta^w}\left( \left( \frac{H^{-1}_{\beta,\Lambda_L}(0,i)}{H_{\beta,\Lambda_L}(0,0)} \right) > e^{-\kappa|i|/2} \right)
$$

$$
\leq C_0 |\partial \Lambda_L| e^{-\kappa L/2} \leq C_1 e^{-\kappa L/4}
$$

(57)

for some constants $C_1, L$ large enough depending on $W, d$ and $\forall 0 < W < W_c$. Lemma 10 with $\delta = e^{\kappa L/2} / W$ gives

$$
\nu_{\Lambda_L}^{W,1,\eta^w}(\Omega^c_{1,1}) = \nu_{\Lambda_L}^{W,1,\eta^w}(H^{-1}_{\beta,\Lambda_L}(0,0) > e^{\kappa L})
$$

$$
= \nu_{\Lambda_L}^{W,1,\eta^w}(H^{-1}_{\beta,\Lambda_L}(0,0) > e^{\kappa L} / W) \leq C_1 e^{-\kappa L/4}.
$$

(58)

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Therefore (55) holds.

Assume now we are in $\Omega_1$. By resolvent identity we have

$$
(H_{S,\Lambda_L}^\beta)^{-1}(0, 0) - (H_{D,\Lambda_L}^\beta)^{-1}(0, 0) = \sum_{j \in \partial \Lambda_L} (H_{S,\Lambda_L}^\beta)^{-1}(0, j)(2d - n_j)(H_{D,\Lambda_L}^\beta)^{-1}(j, 0). 
$$

(59)

By random walk representation we have, setting $P_{j_0}^{j}$ the set of nearest neighbor paths from $j_0$ to $j$ in $\Lambda_L$, (cf. Proposition 6 of [27] and notations therein)

$$
H_{\Lambda_L}^{-1}(0, i) = \sum_{\sigma \in P_{j_0,i}} W_{\sigma} \geq (H_{D,\Lambda_L}^\beta)^{-1}(0, i),
$$

(60)

because the denominator in the path expansion is increased in the one with Dirichlet boundary condition. Therefore, on the set $\Omega_1$ we have

$$
\left| \frac{(H_{D,\Lambda_L}^\beta)^{-1}(0, 0)}{(H_{S,\Lambda_L}^\beta)^{-1}(0, 0)} - 1 \right| = 1 - \frac{(H_{D,\Lambda_L}^\beta)^{-1}(0, 0)}{(H_{S,\Lambda_L}^\beta)^{-1}(0, 0)} 
$$

$$
= \sum_{j \in \partial \Lambda_L} (H_{S,\Lambda_L}^\beta)^{-1}(0, j)\left(2d - n_j\right)\frac{(H_{D,\Lambda_L}^\beta)^{-1}(j, 0)}{(H_{S,\Lambda_L}^\beta)^{-1}(0, 0)} 
$$

$$
\leq (H_{\Lambda_L}^{-1}(0, 0) \sum_{j \in \partial \Lambda_L} (H_{S,\Lambda_L}^\beta)^{-1}(0, j)(H_{S,\Lambda_L}^\beta)^{-1}(j, 0)(H_{S,\Lambda_L}^\beta)^{-1}(0, 0) 
$$

$$
\leq e^{\kappa L}2d \sum_{j \in \partial \Lambda_L} e^{-2\kappa |j|} \leq 2d|\partial \Lambda_L|e^{+\kappa L}e^{-2\kappa L} \leq e^{-\kappa L/4} \leq \frac{1}{2},
$$

for $L \geq L_0$, where $L_0$ depends on $W, d$. Thus, on $\Omega_1$, it holds

$$
(H_{D,\Lambda_L}^\beta)^{-1}(0, 0) = (H_{S,\Lambda_L}^\beta)^{-1}(0, 0) \left[ 1 - \left(1 - \frac{(H_{D,\Lambda_L}^\beta)^{-1}(0, 0)}{(H_{S,\Lambda_L}^\beta)^{-1}(0, 0)} \right) \right] 
$$

(61)

$$
\geq \frac{1}{2}(H_{S,\Lambda_L}^\beta)^{-1}(0, 0),
$$

and hence

$$
(H_{S,\Lambda_L}^\beta)^{-1}(0, 0) > 1/E \Rightarrow (H_{D,\Lambda_L}^\beta)^{-1}(0, 0) \geq \frac{1}{2E}.
$$
Lemma 14 (Dirichlet versus simple bc (2)) Let $\Lambda_L$ be the finite box in $\mathbb{Z}^d$ of side $2L+1$ centered at 0. We consider $(\beta_i)_{i\in\Lambda_L} \sim \nu_{\Lambda_L}^{W,1,\eta^w}$ where $\eta^w_{\Lambda_L}$ is the wired boundary condition introduced in (6). Define $\Omega_2 = \Omega_{2,0} \cap \Omega_{2,1}$, with

\[
\Omega_{2,0} := \left\{ \max_{j \in \partial \Lambda_L, i \sim 0} \left( \mathcal{H}_{\beta,\Lambda_L \setminus \{0\}}^{-1}(i,j) \right) < e^{-\frac{3}{2} \kappa L} \right\}, \quad \Omega_{2,1} := \left\{ H_{\beta,\Lambda_L}^{-1}(0,0) < e^{\kappa L} \right\},
\]

where $\kappa$ is the constant introduced in Theorem 8 and remember that $\mathcal{H}_{\beta,\Lambda_L} = WH_{\beta,\Lambda_L}$. Let $W'_c$ be as in Theorem 8. Then there is $L_0(W,d) > 1$ such that $\forall L \geq L_0$ and $\forall 0 < W < W'_c$ we have $\kappa L > 1$, and there is a constant $C_1 = C_1(W,d) > 0$ such that

\[
\nu_{\Lambda_L}^{W,1,\eta^w}(\Omega_{2,j}) \geq 1 - C_1 e^{-\kappa L/4} \quad \text{for } j = 0, 1,
\]

and hence $\nu_{\Lambda_L}^{W,1,\eta^w}(\Omega_2) \geq 1 - 2C_1 e^{-\kappa L/4}$. Moreover on the set $\Omega_2$ it holds

\[
\left\{ (H_{\beta,\Lambda_L}^S)^{-1}(0,0) > \frac{1}{E} \right\} \Rightarrow \left\{ (H_{\beta,\Lambda_L}^D)^{-1}(0,0) > \frac{1}{2E} \right\},
\]

for all energy $0 < E < \frac{1}{2}$.

Note that also here the set $\Omega_{2,0}$ is measurable wrt $\{\beta_j\}_{j \in \Lambda \setminus \{0\}}$ while $\Omega_{2,1}$ is measurable wrt $(H_{\beta,\Lambda_L})^{-1}(0,0)$. This fact will be important in the proof of the lower bound on the IDS.

Proof of Lemma 14. By the random path representation $\mathcal{H}_{\beta,\Lambda_L \setminus \{0\}}^{-1}(i,j) \leq \mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j)$. Then, the decay estimate (42) together with the Markov inequality entails, $\forall 0 < W < W'_c$,

\[
\nu_{\Lambda_L}^{W,1,\eta^w}(\Omega_{2,0}) \leq \sum_{j \in \partial \Lambda_L, i \sim 0} \nu_{\Lambda_L}^{W,1,\eta^w}(\mathcal{H}_{\beta,\Lambda_L \setminus \{0\}}^{-1}(i,j) > e^{-\frac{3}{2} \kappa L}) \leq \sum_{j \in \partial \Lambda_L, i \sim 0} e^{\kappa L/8} \nu_{\Lambda_L}^{W,1,\eta^w}(\mathcal{H}_{\beta,\Lambda_L \setminus \{0\}}^{-1}(i,j))^{1/4} \leq \sum_{j \in \partial \Lambda_L, i \sim 0} e^{\kappa L/8} \nu_{\Lambda_L}^{W,1,\eta^w}(\mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j))^{1/4} \leq C_0 \sum_{j \in \partial \Lambda_L, i \sim 0} e^{\kappa L/8} e^{-\kappa |i-j|} \leq C_0 |\partial \Lambda_L| e^{-\kappa L}/8 \leq C_1 e^{-\kappa L/4}
\]

for some constant $C_1 = C_1(W,d)$. The bound for $\Omega_{2,1}$ works exactly as the one for $\Omega_{1,1}$ in Lemma 13. Therefore (63) holds.
Assume now we are on $\Omega_2$. We have, for all $j \in \partial \Lambda_L$

$$H_{\beta,\Lambda_L}^{-1}(0,j) = \sum_{i \sim 0} H_{\beta,\Lambda_L}^{-1}(0,0) W H_{\beta,\Lambda_L}^{-1}(0,j).$$

Therefore

$$H_{\beta,\Lambda_L}^{-1}(0,j) \leq W^2 e^\kappa L e^{-\frac{3}{2} \kappa L} \leq W e^{-\frac{1}{4} \kappa L}.\quad (65)$$

By (59), and (60)

$$|H_{\beta,\Lambda_L}^D(0,0) - (H_{\beta,\Lambda_L}^S(0,0)| = \sum_{j \in \partial \Lambda_L} (H_{\beta,\Lambda_L}^S)^{-1}(0,j)(2d - n_j)(H_{\beta,\Lambda_L}^D)^{-1}(j,0) \leq \sum_{j \in \partial \Lambda_L} (H_{\beta,\Lambda_L}^S)^{-1}(0,j)(2d - n_j)(H_{\beta,\Lambda_L}^S)^{-1}(j,0) \leq W^2 |\partial \Lambda_L| 2de^{-\frac{3}{2} \kappa L} \leq e^{-\frac{1}{4} \kappa L}.\quad (66)$$

for $L$ large enough, depending on $W,d$. It follows that, if $E \leq 1/2$

$$\begin{align*}
(H_{\beta,\Lambda_L}^D)^{-1}(0,0) > \frac{1}{E} & \Rightarrow (H_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq (H_{\beta,\Lambda_L}^S)^{-1}(0,0) - e^{-\frac{1}{4} \kappa L} \geq \frac{1}{E} - e^{-\frac{1}{4} \kappa L} \geq \frac{1}{2E}
\end{align*}$$

for $L \geq L_0 = L_0(W,d)$. □

4 Lower bound on the IDS

We are now ready to prove Theorem 1. By Lemma 11 and Lemma 12 in Section 3.1 we have

$$N(E, H_{\beta}) \geq \frac{1}{|\Lambda_L|^w_{\Lambda_L}} W_{\eta}^{w \Lambda} \left( (H_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E} \right).$$

Remember the definition of $W_{cr}$ in (14) and the configuration sets $\Omega_{1,0}, \Omega_{1,1}, \Omega_{2,0}, \Omega_{2,1}$ introduced in (54) and (62). We define, for $j = 0, 1$,

$$\Omega_{loc,j} := \begin{cases} 
\Omega_{1,j} & \text{if } W_{cr} = W_c, \\
\Omega_{2,j} & \text{if } W_{cr} = W_c'.
\end{cases} \quad (67)$$

and $\Omega_{loc} := \Omega_{loc,0} \cap \Omega_{loc,1}$. Then, by Lemma 13 and 14 we have

$$N(E, H_{\beta}) \geq \frac{1}{|\Lambda_L|^w_{\Lambda_L}} \mathbb{E}^{W_{\eta}^{w \Lambda}} \left[ 1_{\Omega_{loc}} \left\{ H_{\beta,\Lambda_L}^{-1}(0,0) \geq \frac{1}{2E} \right\} \right].$$

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for all $0 < W < W_{cr}$, $E < \frac{1}{2}$ and $L$ large. Remember that $H_{\beta,L} = \frac{1}{W} H_{\beta,L}$, and set $0^c = \Lambda_L \setminus \{0\}$. By Schur decomposition,  

$$
\frac{1}{W} H_{\beta,L}^{-1}(0,0) = H_{\beta,L}^{-1}(0,0) = \frac{1}{2\beta_0 - P_{0,0}^W H_{\beta,L}^{-1}\{0\} P_{0,0}^W} = \frac{1}{y}. 
$$

(68)

The conditional density of $y$, given $\beta_{0^c} = (\beta_j)_{j \in 0^c}$ is given by

$$
d\rho_{ao}(y) = \rho_{ao}(y)dy = \frac{e^{a_0}}{\sqrt{2\pi}} \left( y + \frac{a_0^2}{y} \right) \frac{1}{\sqrt{y}} 1_{y > 0}dy,
$$

(69)

where

$$
a_0 = a_0(\beta_{0^c}) = \sum_{j \in 0_L} \frac{H_{\Lambda,L}^{-1}(0,j)}{H_{\Lambda,L}^{-1}(0,0)} \eta_j^w = W \sum_{i \sim 0, j \in 0_L} \frac{H_{\beta,L}^{-1}(0,0)}{H_{\beta,L}^{-1}(0,0)} \eta_j^w. 
$$

(70)

The corresponding average will be denoted by $E^{a_0}$. This decomposition follows from the factorization in Equation (5.14) of [27] with $U = \Lambda \setminus \{0\}$ and $U^c = \{0\}$. It also follows directly from the following relations:

$$
\langle 1, H_{\beta,L} \rangle = 2\beta_0 + \langle 1_{0^c}, H_{\beta,L} \{0 \} \rangle 1_{0^c} - 2 \sum_{i \sim 0} W = y + F(\beta_{0^c})\quad \quad (71)
$$

$$
\langle \eta^w, H_{\beta,L}^{-1} \eta^w \rangle = \langle \eta^w, H_{\beta,L}^{-1}(0,0) \eta^w \rangle + a_0^2 H_{\beta,L}^{-1}(0,0) = G(\beta_{0^c}) + \frac{a_0^2}{y} \quad \quad (72)
$$

$$
\det H_{\beta,L} = y \det H_{\beta,L} \{0 \} \quad \quad (73)
$$

where, $F, G$ are functions of $\beta_{0^c}$ and in second line we combined $\eta_0^w = 0$ and the resolvent identity $A^{-1} = B^{-1} + B^{-1}(B - A)B^{-1} + B^{-1}(B - A)A^{-1}(B - A)B^{-1}$ with $A = H_{\beta,L}$ and $B = 2\beta_0 \oplus H_{\beta,L} \{0 \}$. Therefore

$$
E_{\Lambda_L}^{W,\eta^w} \left[ 1_{\Omega_{loc}} \frac{1}{H_{\beta,L}^{-1}(0,0) > \frac{1}{2\pi}} \right] = E_{\Lambda_L}^{W,\eta^w} \left[ 1_{\Omega_{loc}} \frac{1}{H_{\beta,L}^{-1}(0,0) > \frac{1}{2\pi}} \right] = E_{\Lambda_L}^{W,\eta^w} \left[ 1_{\Omega_{loc},0} \int_1 \frac{1}{y} e^{-\frac{1}{2} \left( y + \frac{a_0^2}{y} \right)} \frac{1}{\sqrt{y}} 1_{y > 0}dy \right] \quad \quad (74)
$$

$$
= E_{\Lambda_L}^{W,\eta^w} \left[ 1_{\Omega_{loc},0} \int_1 \frac{1}{\frac{1}{W} e^{\frac{1}{2} \left( \frac{\eta^w}{\sqrt{2\pi}} - \frac{a_0^2}{y} \frac{1}{\sqrt{y}} 1_{y > 0}dy \right) \right] \right] \quad \quad (75)
$$

$$
= E_{\Lambda_L}^{W,\eta^w} \left[ 1_{\Omega_{loc},0} \int_1 \frac{1}{\frac{1}{W} e^{\frac{1}{2} \left( \frac{\eta^w}{\sqrt{2\pi}} - \frac{a_0^2}{y} \frac{1}{\sqrt{y}} 1_{y > 0}dy \right) \right] \right] \quad \quad (76)
$$

$$
= E_{\Lambda_L}^{W,\eta^w} \left[ 1_{\Omega_{loc},0} \int_1 \frac{1}{\frac{1}{W} e^{\frac{1}{2} \left( \frac{\eta^w}{\sqrt{2\pi}} - \frac{a_0^2}{y} \frac{1}{\sqrt{y}} 1_{y > 0}dy \right) \right] \right] \quad \quad (77)
$$

where we used that $\Omega_{loc,0}$ is measurable wrt $\beta_{0^c}$ and

$$
\Omega_{loc,1} = \left\{ H_{\beta,L}^{-1}(0,0) < e^{\kappa L} \right\} = \left\{ \frac{Y}{W} > e^{-\kappa L} \right\}. 
$$
We argue

\[ a_0(\beta_0) = \sum_{j \in \partial \Lambda_L} \frac{H_{\beta_0}(0, j)}{H_{\beta, \Lambda_L}(0, 0)} \eta_j^w \leq |\partial \Lambda_L| e^{-\kappa L} W(2d - 1) \leq W e^{-\kappa L/2} \quad \forall \beta_0 \in \Omega_{1,0} \]

\[ a_0(\beta_0) = W \sum_{i, j \in \partial \Lambda_L} \frac{H_{\beta, \Lambda_L}(0, 0)}{H_{\beta, \Lambda_L}(0, 0)} \eta_j \leq 2dW |\partial \Lambda_L| e^{-\frac{3}{2} \kappa L} \leq W e^{-\kappa L/2} \quad \forall \beta_0 \in \Omega_{2,0}. \]

for \( L \) large enough. Setting \( \bar{\pi}_0 := W e^{-\kappa L/2} \), and remarking that \( a_0 \geq 0 \), we argue

\[ e^{a_0(\beta_0)} e^{-\frac{(a_0(\beta_0))^2}{\gamma_0}} \geq e^{-\bar{\pi}_0} e^{-\frac{\bar{\pi}_0^2}{\gamma_0}} e^{-\bar{\pi}_0} = \rho_{\bar{\pi}_0}(y) e^{-\bar{\pi}_0} \quad \forall \beta_0 \in \Omega_{\text{loc}, 1}. \]

Therefore we obtain

\[ E_{\Lambda_L} W, \eta^w \left[ 1_{\Omega_{\text{loc}, 0}} E_{\Lambda_L}^{a_0(\beta_0)} \left[ 1_{e^{-\kappa L} \leq y \leq 2EW} \right] \right] \]

\[ \geq e^{-\bar{\pi}_0} \nu_{\Lambda_L} W, \eta^w (\Omega_{\text{loc}, 0}) \rho_{\bar{\pi}_0} (W e^{-\kappa L} \leq y \leq 2EW) \]

\[ \geq (1 - C'_1 e^{-\kappa L/A}) \rho_{\bar{\pi}_0} (W e^{-\kappa L} \leq y \leq 2EW) \]

for some constant \( C'_1 \), where we used (55), (63) and the bound \( e^{-\bar{\pi}_0} = e^{-W e^{-\frac{1}{2} \kappa L}} \geq (1 - ce^{-\kappa L/4}) \) for some constant \( c > 0 \).

It remains to extract a lower bound on \( \rho_{\bar{\pi}_0} (e^{-\kappa L} \leq y \leq 2EW) \). Set \( L = \frac{1}{2} \ln \frac{1}{E} \). For \( E \) small, \( L_E \) is large enough for all our results to hold. Then \( 2EW = 2W e^{-\kappa L} \) and hence \( W e^{-\kappa L} < 2EW < \bar{\pi}_0 \). Moreover

\[ \frac{(\bar{\pi}_0 - y)^2}{y} \leq W \quad \forall W e^{-\kappa L} \leq y \leq 2W e^{-\kappa L} \]

It follows

\[ \rho_{\bar{\pi}_0} (W e^{-\kappa L} \leq y \leq 2EW) = \frac{W}{\sqrt{2\pi}} \int_{W e^{-\kappa L}}^{2EW} e^{-\frac{(\bar{\pi}_0 - y)^2}{2y}} dy \]

\[ \geq \frac{W}{\sqrt{2\pi}} \int_{W e^{-\kappa L}}^{2EW} \frac{1}{\sqrt{y}} dy = c e^{-W/2} \sqrt{W E}, \]

for some constant \( c > 0 \) independent of \( E, \kappa, L, W \). Putting all these results together we obtain

\[ N(E, H_\beta) \geq \frac{1}{|\Lambda_L|} \int_{\Lambda_L} W, \eta^w \left[ (H_{\beta, \Lambda_L}^D)^{-1}(0, 0) \geq \frac{1}{E} \right] \]

\[ \geq \frac{1}{|\Lambda_L|} (1 - C'_1 e^{-\kappa LE/A}) c e^{-W/2} \sqrt{W \sqrt{E}} \geq c' \frac{1}{|\log E|^d} \sqrt{E} \]

(75)

for some constant \( c' > 0 \) depending on \( W, d, \kappa \). This concludes the proof of the lower bound.

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5 Wegner estimate

In this section we prove Theorem 2. We work out in detail the proof only for the case of simple boundary conditions. The proof in the case of Dirichlet boundary conditions works exactly in the same way. Note that, since $H = \mathcal{H}/W$, we have

$$N(E, H_{\beta,\Lambda_L}) = N(W_E, H_{\beta,\Lambda_L}) = \frac{1}{|\Lambda_L|} \text{tr} 1_{(-\infty,0]}(H_{\beta,\Lambda_L} - W_E).$$

We will smooth out the discontinous function $1_{(-\infty,0]}(x)$ as follows. Let $\rho$ be a smooth non-decreasing function satisfying $\rho = 0$ on $(-\infty,-\varepsilon)$ and $\rho = 1$ on $(\varepsilon,\infty)$. Then

$$1_{[-\varepsilon,\varepsilon]}(x) \leq \rho(x + 2\varepsilon) - \rho(x - 2\varepsilon) \quad \forall x \in \mathbb{R}.$$

Setting

$$\delta_L(E, \varepsilon) := \mathbb{E}_{\Lambda_L}^{W,1,\eta_w} \left[ \text{tr} (\rho(H_{\beta,\Lambda_L} - E + 2\varepsilon) - \rho(H_{\beta,\Lambda_L} - E - 2\varepsilon)) \right]$$

we have

$$0 \leq \mathbb{E}_{\Lambda_L}^{W,1,\eta_w} [N(E + \varepsilon, H_{\beta,\Lambda_L}) - N(E - \varepsilon, H_{\beta,\Lambda_L})] \leq \frac{1}{|\Lambda_L|} \delta_L(W_E, W\varepsilon).$$

(77)

Remember that the conditional measure for $2\beta_j - P_{j,j}^{W,\mathcal{H}_{\beta,\Lambda_L}(\{j\})} P_{j^*,j}^{W}$, given $\beta_{j^*} = (\beta_j)_{j \in \Lambda_L \setminus \{j\}}$, is $\rho_{a_j}$ defined in (69), where, for general $j \in \Lambda_L$,

$$a_j = a_j(\beta_{j^*}) := \eta_j^w + W \sum_{i \sim j, k \in \Lambda_L \setminus \{j\}} \mathcal{H}^{-1}_{\beta,\Lambda_L \setminus \{j\}}(i,k) \eta_k^w = \sum_{k \in \partial \Lambda_L} \mathcal{H}^{-1}_{\beta,\Lambda_L}(j,k) \eta_k^w \frac{\mathcal{H}^{-1}_{\beta,\Lambda_L}(j,j)}{\mathcal{H}^{-1}_{\beta,\Lambda_L}(j,j)}$$

(78)

(cf. Equation (5.14) of [27]). We also recall that the Lévy concentration of a measure $\mu$ on $\mathbb{R}$ is defined by

$$\mathcal{L}_\mu(\varepsilon) = \sup_x \mu([x,x + \varepsilon]).$$

(79)

The proof of Theorem 2 follows directly from the next two lemmas.

**Lemma 15** For all $E > 0$ and $0 < \varepsilon < E$ it holds

$$\delta_L(E, \varepsilon) \leq \sum_{j \in \Lambda} \mathbb{E}_{\Lambda_L}^{W,1,\eta_w} \left[ \mathcal{L}_{\rho_{a_j}(\beta_{j^*})}(4\varepsilon) \right].$$

(80)
Lemma 16 It holds, for all \( \epsilon > 0 \), and \( a > 0 \)

\[
\mathcal{L}_{p_a}(\epsilon) \leq \frac{1}{\sqrt{2\pi}}2\sqrt{\epsilon}.
\] (81)

Moreover, for \( d \geq 3 \) there exists a \( W_0 > 1 \) such that for all \( W \geq W_0 \), the following improved estimate holds for all \( \epsilon > 0 \)

\[
E_{\Lambda_L}^{W,1,\eta_w}[\mathcal{L}_{p_{\beta,j}(\beta_c)}(\epsilon)] \leq \frac{C_1}{\sqrt{W}}\epsilon, \quad \forall j \in \Lambda_L,
\] (82)

where \( C_1 > 0 \) is a constant depending only on the dimension.

Indeed, (80) implies

\[
E_{\Lambda_L}^{W,1,\eta_w}[N(E + \epsilon, H_{\beta,\Lambda_L}) - N(E - \epsilon, H_{\beta,\Lambda_L})] \leq \frac{1}{|\Lambda_L|}\delta_L(WE, W\epsilon) \leq \frac{1}{|\Lambda_L|} \sum_{j \in \Lambda_L} E_{\Lambda_L}^{W,1,\eta_w}[\mathcal{L}_{p_{\beta,j}(\beta_c)}(4W\epsilon)].
\]

Inserting (81) and (82) we obtain the result. This concludes the proof of the theorem.

Proof of Lemma 15 Note that

\[
\mathcal{H}_{\beta,\Lambda_L} \pm 2\epsilon = 2(\beta \pm \epsilon) - P = \mathcal{H}_{\beta \pm \epsilon,\Lambda_L}.
\]

Order the vertices in \( \Lambda_L \) as \( \{1, 2, \ldots, |\Lambda_L|\} \). For each \( 1 \leq k \leq |\Lambda| \), we define

\[
\tilde{\beta}_k = \tilde{\beta}_k(\epsilon) := (\beta_1 + \epsilon, \ldots, \beta_k - \epsilon, \beta_{|\Lambda|} - \epsilon).
\] (83)

With this convention we have

\[
\mathcal{H}_{\tilde{\beta}_{k+1},\Lambda_L}(i, j) = \mathcal{H}_{\tilde{\beta}_k,\Lambda_L}(i, j) + 1_{i=j=k}4\epsilon.
\]

Expanding in a telescopic sum we get

\[
\delta_L(E, \epsilon) = \sum_{k=1}^{|\Lambda|} E_{\Lambda_L}^{W,1,\eta_w}\left[\text{tr}\left(\rho(\mathcal{H}_{\tilde{\beta}_{k+1},\Lambda_L} - E) - \rho(\mathcal{H}_{\tilde{\beta}_k,\Lambda_L} - E)\right)\right].
\]

We concentrate now on the \( k \)-th term in the sum. For a fixed configuration \( \beta_{k^c} \), we define \( y_k := 2\beta_k - P_{k^c} W \mathcal{H}_{\beta,\Lambda_L}^{-1} \{k\} P_{k^c} W \). Note that \( \tilde{\beta}_k = \tilde{\beta_k}(y_k, \beta_{k^c}) \)
is a function of $y_k$ and $\beta_k$. We consider the function $y_k \mapsto F_k(y_k) := \text{tr} \rho(\mathcal{H}_{\beta_k(y_k, \beta_k)} - E)$. Then we can write

$$\mathbb{E}_{\Lambda_L}^{W, 1, \eta} \left[ \text{tr} \left( \rho(\mathcal{H}_{\beta_{k+1}} - E) - \rho(\mathcal{H}_{\beta_k} - E) \right) \right]$$

$$= \mathbb{E}_{\Lambda_L}^{W, 1, \eta} \left[ \int (F_k(y_k + 4\varepsilon) - F_k(y_k)) \rho_{\alpha_k}(y_k) dy_k \right].$$

We take the following primitive of $\rho_{\alpha_k}$:

$$G_k(y) := \int_0^y \rho_{\alpha_k}(t) dt.$$  

This function is differentiable and satisfies $G(\infty) = 1$ and $G(0) = 0$. Moreover we can write

$$\int_0^\infty \rho_{\alpha_k}(y) [F_k(y + 4\varepsilon) - F_k(y)] dy$$

$$= \lim_{M \to \infty} \int_0^M \rho_{\alpha_k}(y) [F_k(y + 4\varepsilon) - F_k(y_k)] dy =: \lim_{M \to \infty} I^k_M.$$  

Performing integration by parts, we argue

$$I^k_M = G_k(M)(F_k(M + 4\varepsilon) - F_k(M)) - \int_0^M G_k(y)(F'_k(y + 4\varepsilon) - F'_k(y)) dy$$

$$= \int_M^{M+4\varepsilon} G_k(y)F'_k(y) dy - \int_0^M G_k(y)F'_k(y + 4\varepsilon) dy + \int_0^M G_k(y)F'_k(y) dy.$$  

Now write

$$\int_0^M G_k(y)F'_k(y + 4\varepsilon) = \int_0^{4\varepsilon} G_k(y - 4\varepsilon)F'_k(y) dy + \int_M^{M+4\varepsilon} G_k(y - 4\varepsilon)F'_k(y) dy$$

and

$$\int_0^M G_k(y)F'_k(y) dy = \int_0^{4\varepsilon} G(y)F'_k(y) dy + \int_0^{4\varepsilon} G_k(y)F'_k(y) dy.$$  

Putting everything together we get

$$I^k_M = \int_M^{M+4\varepsilon} (G_k(M) - G_k(y - 4\varepsilon))F'_k(y) dy$$

$$+ \int_0^{4\varepsilon} (G_k(y) - G_k(y - 4\varepsilon))F'_k(y) dy + \int_0^{4\varepsilon} G_k(y)F'_k(y) dy.$$  

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Now
\[
G_k(y) - G_k(y - 4\varepsilon) = \int_{y-4\varepsilon}^{y} \rho_{ak}(t)dt \leq \mathcal{L}_{\rho_{ak}}(4\varepsilon) \quad \forall y \in [\varepsilon, M].
\]
The same bound holds for \(G_k(M) - G_k(y - 4\varepsilon)\) and \(G_k(y) = G_k(y) - G_k(0)\). Therefore
\[
I_k^M \leq \mathcal{L}_{\rho_{ak}}(4\varepsilon) \int_{0}^{M+4\varepsilon} F'_k(y)dy = \mathcal{L}_{\rho_{ak}}(4\varepsilon)(F_k(M + 4\varepsilon) - F_k(0)).
\]
Finally, using a standard argument of rank-one perturbation (see e.g. [13, Lemma 5.25]) \((F_k(M + 4\varepsilon) - F_k(0)) \leq 1\) uniformly in \(M\). The result follows.

**Proof of Lemma 16**  Note that, for all \(y > 0\) we have
\[
\rho_a(y) = \sqrt{\frac{1}{2\pi}} e^{-\frac{1}{2y}(y-a)^2} \frac{1}{\sqrt{y}} \leq \frac{1}{\sqrt{2\pi y}},
\]
and therefore
\[
\mathcal{L}_{\rho_a}(\varepsilon) = \sup_{x \geq 0} \rho_a([x, x + \varepsilon]) \leq \sup_{x \geq 0} \int_{x}^{x+\varepsilon} \frac{1}{\sqrt{2\pi y}} dy = \int_{0}^{\varepsilon} \frac{1}{\sqrt{2\pi y}} dy = \sqrt{\frac{2\varepsilon}{\pi}}.
\]
This gives the first bound (81). To obtain the improved bound (82) note that, by (78) \(a = a_j(\beta_{jc}) > 0\) almost surely, hence the function \(y \mapsto \rho_a(y)\) takes its maximum value in
\[
y_a := \frac{1}{2} \left(-1 + \sqrt{1+4a^2}\right).
\]
Therefore we have \(\mathcal{L}_{\rho_a}(\varepsilon) \leq \rho(y_a)\varepsilon\). Now, using
\[
\frac{1}{2y_a} = \frac{1 + \sqrt{1+4a^2}}{4a^2} \leq \frac{2 + 2a}{4a^2} = \frac{1}{2} \left(\frac{1}{a^2} + \frac{1}{a}\right),
\]
we obtain
\[
\rho(y_a) = \frac{1}{\sqrt{2\pi y_a}} e^{-\frac{1}{2y_a}(y_a-a)^2} \leq \frac{1}{\sqrt{2\pi y_a}} \leq \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a^2} + \frac{1}{a}\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a} + \frac{1}{\sqrt{a}}\right).
\]
It follows
\[
\mathbb{E}_{\Lambda_L}^{W,1,\eta,w}[\mathcal{L}_{\rho_{a_j}(\beta_{jc})}(\varepsilon)] \leq \frac{\varepsilon}{\sqrt{2\pi}} \left(\mathbb{E}_{\Lambda_L}^{W,1,\eta,w}\left[\frac{1}{a_j}\right] + \mathbb{E}_{\Lambda_L}^{W,1,\eta,w}\left[\frac{1}{\sqrt{a_j}}\right]\right)
\]
\[
\leq \frac{\varepsilon}{\sqrt{2\pi}} \left(\mathbb{E}_{\Lambda_L}^{W,1,\eta,w}\left[\frac{1}{a_j}\right] + \mathbb{E}_{\Lambda_L}^{W,1,\eta,w}\left[\frac{1}{a_j}\right]^{1/2}\right).
\]
The result now follows from Lemma 17 below setting \(C_1 := \sqrt{2C_2/\pi}\). ■
Lemma 17 For $d \geq 3$ there exists a $W_0 > 1$ such that $\forall W \geq W_0$ it holds
\[
\mathbb{E}_{\Lambda_L}^{W,1,\eta_\omega}\left[ \frac{1}{a_j(\beta_i)} \right] \leq \frac{C_2}{W} \quad j \in \Lambda,
\]
where $C_2 > 0$ is a constant depending only on the dimension.

Proof. In the case $j \in \partial \Lambda_L$ we have, by (T8), $a_j \geq \eta_j^w \geq W$ a.s. and hence (S4) holds with $C_2 = 1$.

Assume now $j \in \Lambda_L \setminus \partial \Lambda_L$. Using (S3) and (T8) we have
\[
a_j = \frac{\sum_{k \in \partial \Lambda_L} \mathcal{H}_{\beta,\Lambda}^{-1}(j,k) \eta^w_k}{\mathcal{H}_{\beta,\Lambda}^{-1}(j,j)} = \frac{e^{u_j}}{\mathcal{H}_{\beta,\Lambda}^{-1}(j,j)},
\]
therefore
\[
\mathbb{E}_{\Lambda_L}^{W,1,\eta_\omega}\left[ \frac{1}{a_j} \right] = \mathbb{E}_{u,\Lambda_L}^{W,1,\eta_\omega}\left[ \mathcal{H}_{\beta,\Lambda}^{-1}(j,j)e^{-u_j} \right] = \frac{1}{W} \mathbb{E}_{u,\Lambda_L}^{W,1,\eta_\omega}\left[ D^{-1}(j,j)e^{u_j} \right],
\]
where we used (T8) and the matrix $D = D(u) := e^u H_{\beta(u),\Lambda_L} e^u$ can be characterized via the quadratic form
\[
(v, D(u)v) = \sum_{k \sim k' \in \Lambda_L} e^{u_j + u_k}(\nabla_{kk'}v)^2 + \sum_{k \in \Lambda_L} \tilde{\eta}_k^w u_k v_k^2,
\]
where we defined $\tilde{\eta}_k^w := \eta_k^w / W$ and $\nabla_{kk'}v := v_k - v_{k'}$. To estimate the average of $D^{-1}(j,j)e^{u_j}$ we use the same strategy as in the proof of Theorem 3 of [7].

We can write $D^{-1}(j,j)e^{u_j} = (f, D^{-1}f)$, where $f_k := \delta_k e^{u_j/2} = e^{u_j/2}(\delta_j)(k)$. Setting $D_0 := D(0) = -\Delta + \tilde{\eta}_k^w$ we argue
\[
(f, D^{-1}f) = (D_0 D_0^{-1}f, D^{-1}f)
\]
\[
= \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1}f)(\nabla_{kk'} D^{-1}f) + \sum_k \tilde{\eta}_k^w (D_0^{-1}f)_k (D^{-1}f)_k
\]
\[
= \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1}f) \frac{1}{e^{(u_k + u_{k'})/2}} (\nabla_{kk'} D^{-1}f) \frac{1}{e^{-(u_k + u_{k'})/2}} + \sum_k \tilde{\eta}_k^w (D_0^{-1}f)_k (D^{-1}f)_k \frac{1}{e^{u_k/2}} \frac{1}{e^{-u_k/2}}
\]
\[
\leq \left( \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1}f)_k^2 \frac{1}{e^{(u_k + u_{k'})/2}} + \sum_k \tilde{\eta}_k^w (D_0^{-1}f)_k^2 \frac{1}{e^{u_k}} \right)^{1/2} (f, D^{-1}f)^{1/2},
\]
where in the last step we used Cauchy-Schwarz inequality. It follows
\[
(f, D^{-1}f) \leq \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1}f)_k^2 + \sum_k \tilde{\eta}_k^w (D_0^{-1}f)_k^2 \frac{1}{e^{u_k}}
\]
\[
= \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1}f)_k^2 e^{u_j - (u_k + u_{k'})} + \sum_k \tilde{\eta}_k^w (D_0^{-1}f)_k^2 e^{u_j - u_k}
\]
where we used the explicit form of $f$. Therefore

\[
\mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(f, D^{-1} f)] \leq \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1} \delta_j)^2 \mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[e^{u_{j-}-(u_k+u_{k'})}]
\]

\[
+ \sum_k \tilde{\eta}_k^w (D_0^{-1} \delta_j)^2 \mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[e^{u_{j-}-u_k}] .
\]

Note that

\[
\mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[e^{u_{j-}-(u_k+u_{k'})}] \leq 4 \mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(\cosh u_j - u_k)^2]^{\frac{1}{2}} \mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(\cosh u_{k'})^2]^{\frac{1}{2}}
\]

\[
\mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[e^{u_{j-}-u_k}] \leq 2 \mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(\cosh u_j - u_k)].
\]

For $d \geq 3$, $W \geq W_0$, Theorem 1 in [7] ensures that

\[
\mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(\cosh u_j - u_k)^2] \leq 2 \quad \forall j, k, m \leq W_0^{\frac{1}{s}}.
\] (86)

Note that, although the model considered in [7] has uniform pinning $\eta_j = \varepsilon > 0 \forall j \in \Lambda_L$ the proof of Theorem 1 is completely independent from the pinning choice.

Also, while in [7] only $d = 3$ is considered, the same proof works for any $d \geq 3$. Indeed the key dimension-dependent result (Lemma 5) is proved for general dimension $d \geq 3$. The same argument was used in a slightly different setting in [3].

Moreover Lemma 18 below ensures that $\mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(\cosh u_k)^2] \leq 8 \forall k \in \Lambda_L$.

Note that in [7] the analog bound is proved in Theorem 2 and requires quite some work due to the presence of uniform small pinning $\varepsilon \sim 1/|\Lambda_L|^{1-s}$, $0 < s \ll 1$. Here the same bound follows easily from (86) and the fact that we have large pinning at the boundary $\eta^w_j \geq W \gg 1 \forall j \in \partial \Lambda_L$.

Putting all these bounds together we obtain

\[
\mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(f, D^{-1} f)] \leq 16 \left( \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1} \delta_j)^2 + \sum_k \tilde{\eta}_k^w (D_0^{-1} \delta_j)^2 \right)
\]

\[
= 16(\delta_j, D_0^{-1} \delta_j) = 16(-\Delta_{\Lambda_L} + \tilde{\eta}_{jj}^{-1}) \leq C_2,
\]

for some constant $C_2$ independent of $j$ and $L$, since we are in dimension $d \geq 3$. This concludes the proof of the lemma. ■

**Lemma 18** Let $d \geq 3$. There exists $W_0 > 1$ such that $\forall W \geq W_0$ we have

\[
\mathbb{E}^{W,1,\eta^w}_{u,\Lambda_L}[(\cosh u_k)^2] \leq 8 \quad \forall k \in \Lambda_L.
\] (88)
Proof. For any $W > 1$, $j \in \partial \Lambda$ and $m \leq W/2$ we have

$$E_{u,\Lambda_L}^{W,1,\eta^w} [(\cosh u_j)^m] \leq \frac{1}{1 - \frac{m}{W}} \leq 2. \quad (89)$$

This inequality follows by a supersymmetric Ward identity analog to the one in section 5.1 of [7]. We sketch here the argument (we refer to the notation in [7])

$$1 = \langle (B_j + \bar{\psi}_j \psi_j e^{u_j})^m \rangle = E_{u,\Lambda_L}^{W,1,\eta^w} \left[ B_j^m \left( 1 - \frac{m}{B_j} \mathcal{H}_{\beta,\Lambda_L}^{-1}(j,j) \right) \right]$$

The bound now follows from $B_j \geq 1$ and $\mathcal{H}_{\beta,\Lambda}^{-1}(j,j) \leq \frac{1}{\eta^w} \leq \frac{1}{W}$ $\forall j \in \partial \Lambda$. Fix now some $k \in \Lambda_L \setminus \partial \Lambda_L$ and let $j$ be some vertex on $\partial \Lambda_L$. We have $\cosh u_k \leq 2 \cosh u_j \cosh(u_k - u_j)$ and hence

$$E_{u,\Lambda_L}^{W,1,\eta^w} [(\cosh u_k)^2] \leq 4 E_{u,\Lambda_L}^{W,1,\eta^w} [(\cosh u_j)^4]^{\frac{1}{2}} E_{u,\Lambda_L}^{W,1,\eta^w} [(\cosh u_j - u_k)^4]^{\frac{1}{2}}.$$

The result now follows from (86) and (89).

6 An alternative approach

Some of the above results can also be obtained by using the properties of the infinite volume measure $\nu^W$. This alternative approach also provides the improved bound (19) in Theorem 3. In this section we highlight the main differences with the finite volume approach and give the proof (19). For more details see [23].

To explain the strategy we need to introduce a few preliminary notions and results. Recall that $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$. We define, for every $i \in \mathbb{Z}^d$ and $L \in \mathbb{N}^*$,

$$\psi_L(i) := 1 \quad \text{if } i \notin \Lambda_L,$$

$$\psi_L(i) := \sum_{k \in \partial \Lambda_L} \mathcal{H}_{\beta,\Lambda_L}^{-1}(i,k) \eta_{\Lambda_L}^w(k) = e^{u_{\beta,i}}(\beta_L) \quad \text{if } i \in \Lambda_L.$$

The following result is an extract of Theorem 1 in [27].

Proposition 19

1. For every $(i,j) \in \mathbb{Z}^d$, $\left( \mathcal{H}_{\beta,\Lambda_L}^{-1}(i,j) \right)_{L \in \mathbb{N}^*}$ is increasing $\nu^W$-a.s. Moreover, it converges toward some almost surely finite random variable which is denoted by $G(i,j)$.
2. For every $i \in \mathbb{Z}^d$, $(\psi_L(i))_{L \in \mathbb{N}^*}$ is a positive martingale with respect to
the filtration $(\sigma(\beta, i \in \Lambda_L), L \in \mathbb{N}^*)$.

3. For every $i \in \mathbb{Z}^d$, the bracket of $(\psi_L(i))_{L \in \mathbb{N}^*}$ equals
$\left( \mathcal{H}_{\beta, L}^{-1}(i, i) \right)_{L \in \mathbb{N}^*}$.

In particular, $(\psi_L(i))^2 - H^{-1}\beta, \Lambda_L(0, 0)$ is a martingale for every
$i \in \mathbb{Z}^d$.

By Theorem 2 in [27], $\hat{G}$ is the inverse of the infinite volume operator $\mathcal{H}_\beta$
in the following sense $\hat{G}(i, j) = H^{-1}(i, j) := \lim_{\epsilon \to 0}(\mathcal{H}_\beta + \epsilon)^{-1}(i, j) \nu^W$-a.s.. These facts are the key for the construction of the infinite volume
environment of the related vertex reinforced jump process. A first application is
the improved bound (19).

Proposition 20 (Upper bound on the IDS for large $W$)

For $d \geq 3$ there exists a $W_0 > 1$ such that for all $W \geq W_0$, the function
$E \mapsto N(E)$ satisfies the bound

$$N(E) \leq C' E \quad \forall E > 0,$$

for some constant $C' > 0$ independent of $W$.

**Proof.** Note that $N(E, H_\beta) = N(W E, H_\beta) =: \tilde{N}(E)$, In the rest of the
proof we will work with $\tilde{N}$. For every $\epsilon > 0$, it holds that $\nu^W$-a.s,

$$\int_0^{+\infty} \frac{1}{u + \epsilon} d\mu_\delta_0(u) = (H_\beta + \epsilon)^{-1}(0, 0)$$

(90)

where $\mu_\delta_0$ is the random spectral measure of the operator $\mathcal{H}_\beta$ at point 0. Re-
member that $(H_\beta + \epsilon)^{-1}(0, 0) \rightarrow \hat{G}(0, 0), \nu^W$-a.s. Moreover, by monotone
convergence theorem,

$$\int_0^{+\infty} \frac{1}{u + \epsilon} d\mu_\delta_0(u) \rightarrow \int_0^{+\infty} \frac{1}{u} d\mu_\delta_0(u).$$

Together with (90), this implies that, $\nu^W_d$-a.s,

$$\int_0^{+\infty} \frac{1}{u} d\mu_\delta_0(u) = \hat{G}(0, 0).$$

Taking the expectation we obtain, using Fatou’s lemma,

$$\int_0^{+\infty} \frac{1}{u} d\tilde{N}(u) du = \mathbb{E}^W \left[ \hat{G}(0, 0) \right] \leq \liminf_{L \to +\infty} \mathbb{E}^W \left[ H_{\beta, L}^{-1}(0, 0) \right]$$

$$= \liminf_{L \to +\infty} \frac{1}{W} \mathbb{E}^{W, 1, y^w}_{u, \Lambda_L} \left[ D^{-1}(0, 0) e^{2 \nu_0} \right] = \liminf_{L \to +\infty} \frac{1}{W} \mathbb{E}^{W, 1, y^w}_{u, \Lambda_L} \left[ (f, D^{-1}f) \right],$$

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where the matrix $D$ was defined in (85) and $f_k := \delta_k e^{u_0}$ (instead of $f_k := \delta_{kj} e^{u_0}$) in the proof of Lemma 17. Repeating the same arguments as in the proof of Lemma 17 we obtain

$$
E_{u,\Lambda_{L}}^{W,1,\eta^w} [(f, D^{-1} f)] \leq \sum_{k \sim k'} (\nabla_{kk'} D_0^{-1} \delta_0)^2 E_{u,\Lambda_{L}}^{W,1,\eta^w} [e^{2u_0 - (u_k + u_{k'})}]
$$

$$
+ \sum_k \eta_k (D_0^{-1} \delta_0)^2 E_{u,\Lambda_{L}}^{W,1,\eta^w} [e^{2u_0 - u_k}]
$$

where remember that $D_0 := D(0) = -\Delta_{\Lambda_L} + \bar{\eta}$. Note that

$$
E_{u,\Lambda_{L}}^{W,1,\eta^w} [e^{2u_0 - (u_k + u_{k'})}] \leq 4 E_{u,\Lambda_{L}}^{W,1,\eta^w} [(\cosh u_0 - u_k)^2]^{1 \over 4} E_{u,\Lambda_{L}}^{W,1,\eta^w} [(\cosh u_0 - u_{k'})^2]^{1 \over 4},
$$

$$
E_{u,\Lambda_{L}}^{W,1,\eta^w} [e^{2u_0 - u_k}] \leq 4 E_{u,\Lambda_{L}}^{W,1,\eta^w} [(\cosh u_0 - u_k)^2]^{1 \over 2} E_{u,\Lambda_{L}}^{W,1,\eta^w} [(\cosh u_0)^2]^{1 \over 2}.
$$

Using (86) and Lemma 18 we obtain (cf. (87))

$$
E_{u,\Lambda_{L}}^{W,1,\eta^w} [(f, D^{-1} f)] \leq 16 (\delta_0, D_0^{-1} \delta_0) = 16 (-\Delta_{\Lambda_L} + \bar{\eta})_{00}^{-1} \leq C_2,
$$

where $C_2 > 0$ is the same constant we obtained in (87) and we used that we are in dimension $d \geq 3$. Hence $\int_0^{+\infty} \frac{1}{u} d\tilde{N}(u) du \leq C_2/W$. It follows

$$
N(E, H_{\beta}) = \tilde{N}(WE) = \int_0^{WE} \frac{1}{u} d\tilde{N}(u) \leq WE \int_0^{+\infty} \frac{1}{u} d\tilde{N}(u) \leq C_2 E.
$$

This concludes the proof setting $C'' := C_2$. $\blacksquare$

**Remark.** Note that $(\mathcal{H}_{\beta,\Lambda_{L}}^{-1}(0,0))_{L \in \mathbb{N}}$ is the quadratic variation of the martingale $(\psi_L(0))_{L \in \mathbb{N}}$ (cf Proposition 19). This observation gives the slightly weaker estimate

$$
\int_0^{+\infty} \frac{1}{u} d\tilde{N}(u) du = E^{W} [\tilde{G}(0,0)] \leq \liminf_{L \to +\infty} E^{W} [\mathcal{H}_{\beta,\Lambda_{L}}^{-1}(0,0)] = \liminf_{L \to +\infty} E^{W} [\psi_L(0)^2] \leq 16,
$$

where in the last inequality we used Lemma 18 together with

$$
E^{W} [\psi_L(0)^2] = E_{u,\Lambda_{L}}^{W,1,\eta^w} [e^{2u_0}] \leq 4 E_{u,\Lambda_{L}}^{W,1,\eta^w} [\cosh u_0^2] \leq 16.
$$

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The infinite volume measure approach also gives an alternative proof of the lower bound for \( N(E) \). For this we need some more definitions. Setting for \( i \in \mathbb{Z}^d \) we define
\[
\tilde{\beta}_i := \beta_i - \delta_{i,0} \frac{1}{2G(0,0)}
\] (91)
we have the following result.

**Proposition 21 (Proposition 2.4 in [12])** Recall the definition of \( W_{cr} \) in (14). Then, for all \( W < W_{cr} \), \( \frac{1}{2G(0,0)} \) has density \( L > 0 \sqrt{\gamma} e^{-\gamma} \). Moreover, \( \tilde{\beta} \) and \( \frac{1}{2G(0,0)} \) are independent random variables.

Note that this proposition works for any \( W \) such that the corresponding reinforced jump process is recurrent. This is true in particular for \( W < W_{cr} \).

The variable \( \tilde{\beta}_i \) arises naturally as the jump rate of the vertex reinforced jump process at vertex \( i \) (See [27] Theorem 1.(iii)). In the following we will consider \( \tilde{\mathcal{H}}_\beta := 2\tilde{\beta} - P^W \) and its Dirichlet restriction on the finite box \( \Lambda_L \)
\( \tilde{\mathcal{H}}^D_{\beta,\Lambda_L} \) (cf (11)).

Finally, recall that the graph \( \Lambda_L \cup \delta \) has vertex set \( \Lambda_L \cup \{\delta\} \) and edge set \( E(\Lambda_L) \cup \{i, \delta\} \mid i \in \Lambda_L \}, \) and we defined \( W_{i\delta} = \eta_{i\delta} = \sum_{j \sim i, j \notin \Lambda_L} W \forall i \in \Lambda_L \).

Now consider an electrical network on \( \Lambda_L \cup \delta \) with conductances
\[
c(i, j) := W \frac{\hat{G}(0,i) \hat{G}(0,j)}{\hat{G}(0,0)^2} \quad \forall i, j \in \Lambda_L,
\]
\[
c(i, \delta_L) := \sum_{\substack{j \sim i \\ j \notin \Lambda_L}} W \left( \frac{\hat{G}(0,i) \hat{G}(0,j)}{\hat{G}(0,0)^2} \right) \quad \forall i \in \Lambda_L,
\]
and let \( R_L(0 \leftarrow \delta) \) be the effective resistance of the random walk associated with this network. The following proposition is proved in [23].

**Proposition 22** Let \( W < W_{cr} \). Then, for every \( L \in \mathbb{N}^* \),
\[
(\tilde{\mathcal{H}}^D_{\beta,\Lambda_L})^{-1}(0,0) = R_L(0 \leftarrow \delta).
\]

Thanks to this result, we can use some tricks from the theory of electrical networks (e.g. Chap 2 of [20]) to construct an alternative proof of Theorem 1. We sketch below the argument. For details see [23].

**Proof of Theorem 1 (II).** As in the proof given in section 4, we start from
\[
N(E, \mathcal{H}_\beta) \geq \frac{1}{|\Lambda_L|} \nu^W \left( (\mathcal{H}^D_{\beta,\Lambda_L})^{-1}(0,0) \geq \frac{1}{E} \right),
\]
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Note that
\[
\nu^W \left( \frac{1}{2\hat{G}(0,0)} \leq \frac{E}{4} \right) = \nu^W \left( \hat{G}(0,0) \geq \frac{2}{E} \right)
\leq \nu^W \left( \hat{G}(0,0) - (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E} \right) + \nu^W \left( (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E} \right).
\]
Consequently,
\[
N(E, \mathcal{H}_\beta) \geq \frac{1}{|\Lambda_L|} \left[ \nu^W \left( \frac{1}{2\hat{G}(0,0)} \leq \frac{E}{4} \right) - \nu^W \left( \hat{G}(0,0) - (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E} \right) \right]
\geq \frac{1}{|\Lambda_L|} \left[ C\sqrt{E} - \nu^W \left( \hat{G}(0,0) - (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E} \right) \right],
\]
where \( C > 0 \) is some constant and we used that \( \frac{1}{2\hat{G}(0,0)} \) is a \( \Gamma(1/2,1) \) random variable (cf Prop. 21). We claim that
\[
\nu^W \left( \hat{G}(0,0) - (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E} \right) \ll \sqrt{E},
\]
which implies the result. To prove this asymptotic domination, note that
\[
(\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) = \frac{(\hat{\mathcal{H}}_{\beta,\Lambda_L}^D)^{-1}(0,0)}{1 + (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0)/\hat{G}(0,0)}
\]
and thus, using Proposition 22
\[
\hat{G}(0,0) - (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) \leq \frac{\hat{G}(0,0)^2}{(\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0)} = \frac{\hat{G}(0,0)^2}{\mathcal{R}_L(0 \leftrightarrow \delta)}.
\]
Then
\[
\nu^W \left( \hat{G}(0,0) - (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) \geq \frac{1}{E} \right) \leq \nu^W \left( \frac{\hat{G}(0,0)^2}{\mathcal{R}_L(0 \leftrightarrow \delta)} \geq \frac{1}{E} \right)
= \int_0^{+\infty} \frac{e^{-\gamma}}{\sqrt{\pi\gamma}} \nu^W \left( \frac{1}{\mathcal{R}_L(0 \leftrightarrow \delta)} \geq \frac{4\gamma^2}{E} \right) d\gamma \quad (92)
\]
where in the last equality of (92), we used the Proposition 21 and the measurability of \( (\mathcal{H}_{\beta,\Lambda_L}^D)^{-1}(0,0) = \mathcal{R}_L(0 \leftrightarrow \delta) \) with respect to \( \beta \). Then, one can use classical results in electrical networks (The Nash-Williams inequality) to control the inverse of \( \mathcal{R}_L(0 \leftrightarrow \delta) \), using the local conductances on the boundary of \( \Lambda_L \). Finally, we can control these local conductances thanks to Corollary 9 if we choose a "good" \( L \) as a function of \( E \).
A Monotonicity

The following monotonicity result can be found in [22, Theorem 6]

**Theorem 23** Let $V$ be a finite set, $W^+, W^- \in \mathbb{R}_{\geq 0}^{V \times V}$ two families of non-negative weights satisfying $W_{jj}^\pm = 0 \ \forall j \in V$ and

$$W_{ji}^- = W_{ij}^- \leq W_{ij}^+ = W_{ji}^+ \ \forall i \neq j.$$  

Let $E^+$ (resp. $E^-$) be the set of pairs with positive weight $W_{ij}^+ > 0$ (resp $W_{ij}^- > 0$) and denote by $G^\pm = (V, E^\pm)$ the corresponding graphs.

If $i, j \in V$ are connected by $G^-$, then it holds

$$\mathbb{E}_{G^-}^{W_{i,j}^- (1)} \left[ f \left( \frac{\mathcal{H}_{\beta,V,W^-}^{-1}(j,k)}{\mathcal{H}_{\beta,V,W^-}^{-1}(j,j)} \right) \right] \leq \mathbb{E}_{G^+}^{W_{i,j}^+ (1)} \left[ f \left( \frac{\mathcal{H}_{\beta,V,W^+}^{-1}(j,k)}{\mathcal{H}_{\beta,V,W^+}^{-1}(j,j)} \right) \right]$$ \hspace{1cm} (93)

for any concave function $f$. Here we write $\mathcal{H}_{\beta,V,W^\pm}$ instead of $\mathcal{H}_{\beta,V}$ to emphasize the dependence of $W^\pm$.

In this paper we use the following corollary.

**Corollary 24** Let $G = (V, E)$ be a connected finite graph and $W \in \mathbb{R}_{> 0}^E$ a given set of weights. Fix a vertex $j_0 \in V$ and set $\eta_j = \eta_{j,j_0}$ with $\eta > 0$ (one pinning at $j_0$). It holds, for all $j \in V$,

$$\mathbb{E}_{G}^{W_{i,j} (1)} \left[ \sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}} \right] \geq \mathbb{E}_{G}^{W_{i,j_0} (1)} \left[ \sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}} \right].$$

**Proof.** The measure $\nu_{G}^{W_{i,j} (1), \eta}$ is the marginal of $\nu_{G^\delta}^{W_{i,j} (1), \eta}$, where the graph $G^\delta$ has vertex set $V \cup \{\delta\}$ and edge set $E \cup \{j_0, \delta\}$, and we defined $W_{j_0, \delta} = \eta$. Moreover, by resolvent expansion, we have

$$\mathcal{H}_{\beta,V \cup \{\delta\}}^{-1}(j_0, j) = \mathcal{H}_{\beta,V}^{-1}(j_0, j) \left[ 1 + \eta^2 \mathcal{H}_{\beta,V}^{-1}(j_0, j_0) \mathcal{H}_{\beta,V \cup \{\delta\}}^{-1}(\delta, \delta) \right],$$

$$\mathcal{H}_{\beta,V \cup \{\delta\}}^{-1}(j_0, j_0) = \mathcal{H}_{\beta,V}^{-1}(j_0, j_0) \left[ 1 + \eta^2 \mathcal{H}_{\beta,V}^{-1}(j_0, j_0) \mathcal{H}_{\beta,V \cup \{\delta\}}^{-1}(\delta, \delta) \right],$$

and hence

$$\mathbb{E}_{G}^{W_{i,j} (1)} \left[ \sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}} \right] = \mathbb{E}_{G^\delta}^{W_{i,j} (1)} \left[ \sqrt{\frac{\mathcal{H}_{\beta,V}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V}^{-1}(j_0,j_0)}} \right] = \mathbb{E}_{G^\delta}^{W_{i,j_0} (1)} \left[ \sqrt{\frac{\mathcal{H}_{\beta,V \cup \{\delta\}}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V \cup \{\delta\}}^{-1}(j_0,j_0)}} \right].$$
Define \( \tilde{W}_{ij} = W_{ij} \) \( \forall i \sim j \in V \) and \( \tilde{W}_{j_0\delta} = 0 \). Then \( W_{ij} \geq \tilde{W}_{ij} \) \( \forall i \sim j \in G^\delta \) and the graph generated by \( \tilde{W} \) connects \( j_0 \) to \( j \) \( \forall j \in V \). Since \( f(x) = \sqrt{x} \) is a concave function, by Theorem 23 we have

\[
\mathbb{E}_{G^\delta}^{\tilde{W},1,0}\left[ \frac{\mathcal{H}_{\beta,V\cup\{\delta\},\tilde{W}}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V\cup\{\delta\},W}(j_0,j_0)} \right] \geq \mathbb{E}_{G^\delta}^{\tilde{W},1,0}\left[ \frac{\mathcal{H}_{\beta,V\cup\{\delta\},\tilde{W}}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V\cup\{\delta\},\tilde{W}}^{-1}(j_0,j_0)} \right].
\]

Since \( \tilde{W}_{j_0\delta} = 0 \) and \( \tilde{W} = W \) on \( V \), \( 2\beta_\delta \) is independent of the other random variables and we have \( \mathcal{H}_{\beta,V\cup\{\delta\},\tilde{W}} = 2\beta_\delta \oplus \mathcal{H}_{\beta,V} \), where we abbreviated \( \mathcal{H}_{\beta,V} = \mathcal{H}_{\beta,V,W} \). Therefore

\[
\mathcal{H}_{\beta,V\cup\{\delta\},\tilde{W}}^{-1}(j_0,j) = \mathcal{H}_{\beta,V}(j_0,j), \quad \mathcal{H}_{\beta,V\cup\{\delta\},\tilde{W}}^{-1}(j_0,j_0) = \mathcal{H}_{\beta,V}(j_0,j_0).
\]

It follows

\[
\mathbb{E}_{G^\delta}^{\tilde{W},1,0}\left[ \frac{\mathcal{H}_{\beta,V\cup\{\delta\},\tilde{W}}^{-1}(j_0,j)}{\mathcal{H}_{\beta,V\cup\{\delta\},W}(j_0,j_0)} \right] = \mathbb{E}_{G^\delta}^{\tilde{W},1,0}\left[ \frac{\mathcal{H}_{\beta,V}(j_0,j)}{\mathcal{H}_{\beta,V}(j_0,j_0)} \right] = \mathbb{E}_{G}^{W,1,0}\left[ \frac{\mathcal{H}_{\beta,V}(j_0,j)}{\mathcal{H}_{\beta,V}(j_0,j_0)} \right],
\]

where in the last step we used that \( \beta_\delta \) is independent of the other variables. This concludes the proof.

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