Entanglement Teleportation Through 1D Heisenberg Chain

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Abstract

Information transmission of two qubits through two independent 1D Heisenberg chains as a quantum channel is analyzed. It is found that the entanglement of two spin-$\frac{1}{2}$ quantum systems is decreased during teleportation via the thermal mixed state in 1D Heisenberg chain. The entanglement teleportation will be realized if the minimal entanglement of the thermal mixed state is provided in such quantum channel. High average fidelity of teleportation with values larger than $2/3$ is obtained when the temperature $T$ is very low. The mutual information $I$ of the quantum channel declines with the increase of the temperature and the external magnetic field. The entanglement quality of input signal states cannot enhance mutual information of the quantum channel.

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1 Introduction

Quantum teleportation originally proposed by Bennett et. al. [1] has developed rapidly in recent years. The maximally entangled state shared between Alice and Bob plays a crucial role in the standard teleportation method. The entanglement is essential in quantum information because of its unique advantages that can be applied to many fields, such as quantum teleportation [1], quantum cryptography [2], etc. However, the decoherence from environment always impacts on the degree of entanglement, so the resource of maximally entangled states is hard to prepare in a real experiment. Certainly, a mixed entangled state as the resource is approximately near to the real circumstances. For experimental investigations of quantum computation and quantum communication, the thermal equilibrium state in Heisenberg model is one fundamental sort of mixed and entangled states [3, 4]. Recently, Bowen et. al. [5] and Horodeckies [6] have suggested that teleportation as the resource of mixed states via the standard method, i.e., Bell-type measurements and Pauli rotations, is equivalent to a general depolarising channel [5-7]. In some schemes of teleportation using Heisenberg model, half of two qubits can be teleported by the sole two-qubit mixed state [5]. In other schemes, entanglement of a special mixed state, i.e., Werner state, can be transferred through a pair of two-qubit mixed states [8, 9]. However, the entanglement teleportation of the whole two-qubit system as the resource of the thermal mixed state needs to be investigated. Meanwhile, to see more clearly the possible applications of Heisenberg model in quantum teleportation, mutual information of the quantum channel also needs to be analyzed.

In this paper, the information transmission by a pair of thermal mixed states in 1D Heisenberg XXX chains is investigated. The minimal entanglement in the quantum channel is needed to transfer entanglement information. The high average fidelity of teleportation is obtained when the temperature is very low and the magnetic field is weak. The entanglement quality of input signal states and the mutual information are analyzed. In section 2, entanglement of the thermal mixed state in a 1D Heisenberg
XXX chain is presented. The entanglement teleportation of two-qubit pure states and fidelity are derived. In section 3, the mutual information in such quantum channel is analyzed. A discussion concludes the paper.

2 Entanglement teleportation and fidelity

In an isotropic Heisenberg model [3], there are \( N \) spin-\( \frac{1}{2} \) units in the quantum system with coupling constant \( J \). For the case of \( N = 2 \), the Hamiltonian of the system in the magnetic field \( B \) can be written as

\[
H = \frac{1}{2} B (\sigma^1_z + \sigma^2_z) + \frac{J}{2} (\sigma^1_x \otimes \sigma^2_x + \sigma^1_y \otimes \sigma^2_y + \sigma^1_z \otimes \sigma^2_z),
\]

where the periodic boundary condition \( N = N + 1 \) is adopted, \( \sigma^i_x, \sigma^i_y, \sigma^i_z \) are Pauli rotation operators for the \( i \)th qubit. The density matrix of the thermal entangled state at the equilibrium temperature \( T \) can be expressed as

\[
\rho^c = \frac{1}{z} \begin{pmatrix}
  e^{-\frac{B}{T}} & 0 & 0 & 0 \\
  0 & \frac{1}{2} (e^{-\frac{J}{2T}} + e^{\frac{3J}{2T}}) & \frac{1}{2} (e^{-\frac{J}{2T}} - e^{\frac{3J}{2T}}) & 0 \\
  0 & \frac{1}{2} (e^{-\frac{J}{2T}} - e^{\frac{3J}{2T}}) & \frac{1}{2} (e^{-\frac{J}{2T}} + e^{\frac{3J}{2T}}) & 0 \\
  0 & 0 & 0 & e^{-\frac{B}{T}}
\end{pmatrix},
\]

where \( z = \exp\left(\frac{B - J}{T}\right) + \exp\left(-\frac{B - J}{T}\right) + \exp\left(-\frac{J}{2T}\right) + \exp\left(\frac{3J}{2T}\right) \). By the method introduced in Refs. [8-11], the amount of entanglement of the thermal state is

\[
E(\rho^c) = \max\{-2 \sum_i \lambda^-_i, 0\},
\]

where \( \lambda^-_i \) is the \( i \)th negative eigenvalue of \((\rho^c)^T\) which is the partial transposition matrix of \( \rho^c \). In a 1D Heisenberg XXX chain, the entanglement of \( \rho^c \) is

\[
E = \max\left\{ \frac{2 \exp\left(-\frac{J}{2T}\right) \cosh\left(\frac{B}{T}\right)}{z} \left[ \sqrt{1 + \frac{(\exp(\frac{3J}{2T}) - 1)^2 - 4}{4 \cosh^2\left(\frac{B}{T}\right)}} - 1 \right], 0 \right\}.
\]
It is found that there is a critical temperature \( T_c = \frac{2J}{3k_B} \). Below the critical temperature \( T_c \), the entanglement is increased when the magnetic field \( B \) is decreased. Above the critical temperature \( T_c \), the entanglement is always close to zero no matter the magnetic field \( B \) is increased or decreased.

For the entanglement teleportation of the whole two-qubit system as the resource of the thermal mixed state in a 1D Heisenberg XXX chain, the standard teleportation through mixed states can be regarded as a general depolarising channel \([5, 6]\). Similar to the standard teleportation, the entanglement teleportation for the mixed channel of an input entangled state is destroyed and its replica state appears at the remote place after applying local measurement in the form of linear operators. When two-qubit state \( \rho_{in} \) is teleported via the channel, the output state \( \rho_{out} \) is [11]

\[
\rho_{out} = \sum_{ij} p_{ij} (\sigma_i \otimes \sigma_j) \rho_{in} (\sigma_i \otimes \sigma_j) (\sum_{ij} p_{ij} = 1).
\] (5)

In the above equation, \( \sigma_i (i = 0, x, y, z) \) signify unit matrix \( I \) and are three components of Pauli matrix \( \vec{\sigma} \) correspondingly, \( p_{ij} = p_i p_j = \text{tr}(E_i \rho^c) \cdot \text{tr}(E_j \rho^c) \), \( p_{00} \) stands for the maximal possibility of successful teleportation, and \( p_i = \text{tr}(E_i \rho^c) \) is the ratio of maximally entangled state to the thermal mixed states \( \rho^c \) at the equilibrium temperature. Here \( \{E_i\} = \{|\psi^+\rangle\langle\psi^-|, |\phi^-\rangle\langle\phi^-|, |\phi^+\rangle\langle\phi^+|, |\psi^+\rangle\langle\psi^+|\} \).

To see more clearly the entanglement teleportation of two qubits, one kind of pure entangled state can be chosen as an example

\[
|\psi\rangle_{in} = c_1 |u_1\rangle |v_1\rangle + c_2 |u_2\rangle |v_2\rangle,
\] (6)

where \( \{|u_1\rangle, |u_2\rangle\} \) and \( \{|v_1\rangle, |v_2\rangle\} \) are two sets of basis vectors of two qubits. Without losing the generality, it can be assumed

\[
c_1 = \cos \frac{\theta}{2} \quad \text{and} \quad c_2 = \sin \frac{\theta}{2} \exp(i\phi), \quad \text{when} \quad \theta \in [0, \pi] \quad \text{and} \quad \phi \in [0, 2\pi].
\] (7)

Here different values of \( \theta \) describe all states with different amplitudes, and \( \phi \) stands for the phase of these states. From Eqs. (3)-(7), the amount of entanglement of \( \rho_{out} \) can be expressed as,

\[
E_{out} = \max \left\{ \frac{E_{in}(\exp(\frac{2J}{T}) - 1)^2 - 4 \cosh(\frac{B}{T})(\exp(\frac{2J}{T}) + 1)}{[2 \cosh(\frac{B}{T}) + \exp(\frac{2J}{T}) + 1]^2}, 0 \right\},
\] (8)
where $E_{\text{out}}$ is the entanglement of teleported state and $E_{\text{in}} = 2|c_1c_2|$ is the input entanglement.

The quantity $E_{\text{out}}$ as a function of $E_{\text{in}}$ is plotted in Fig. 1 when the magnetic field $B$, the temperature $T$ and the coupling coefficient $J$ are changed. Fig. 1(a) is a plot of $E_{\text{out}}$ as functions of $E_{\text{in}}$ and $B$. When the input entanglement $E_{\text{in}} = 0$, $E_{\text{out}}$ is always zero no matter $B$ is increased or not. The value of $E_{\text{out}}$ is decreased with increasing the value of $B$. This is due to the fact that the entanglement of 1D Heisenberg chain decreases when the magnetic field is increased. It is seen that the entanglement of the teleported state is closely related to that of the channel. While $E_{\text{out}}$ is increased with increasing value of $E_{\text{in}}$. This is due to their linear relationship in Eq. (8). Fig. 1(b) is a plot of $E_{\text{out}}$ as functions of $E_{\text{in}}$ and $T$. The value of $E_{\text{out}}$ is decreased with increasing value of $T$ since the entanglement of the mixed channel decreases with $T$. While $E_{\text{out}}$ is increased with increasing value of $E_{\text{in}}$. Fig. 1(c) is a plot of $E_{\text{out}}$ as functions of $E_{\text{in}}$ and $J$. It is seen that $E_{\text{out}}$ is increased with increasing values of both $J$ and $E_{\text{in}}$. This means that $E_{\text{out}}$ depends on the coupling strength $J$. The higher value of $E_{\text{out}}$ is due to the stronger coupling of $J$. It is found that $E_{\text{out}}$ is always zero for $J \leq (\ln 3/2)T$ when there is no entanglement in the mixed channel.

From Figs. 1(a) to 1(c), it is noted that $E_{\text{out}}$ is increased linearly with increasing value of $E_{\text{in}}$. While $E_{\text{out}}$ is decreased with increasing values of both $B$ and $T$. Under the general circumstances, the output entanglement of two-qubit state $|\psi\rangle_{\text{in}}$ will decrease via the quantum channel. Since the entanglement cannot be increased under local operations, Eq. (5) provides an upper bound to the output entanglement. That is, the output entangled mixed state has the entanglement smaller than that of the channel [5]. The entanglement after the teleportation is always less than that of the input state before the teleportation, i.e., $E_{\text{out}} < E_{\text{in}}$. To realize the entanglement teleportation of $E_{\text{out}} > 0$, the quantum channel needs the minimal entanglement. Fig. 1(d) is a plot of critical values of the magnetic field $B_c$, the temperature $T_c$, and the coupling coefficient $J_c$ for $E_{\text{out}} = 0$ when $E_{\text{in}} = 1$. The critical values of $B_c$, $T_c$, and $J_c$ are located on the plane. When the values of $B$, $T$, and $J$ are above the plane, the output
entanglement is always zero. That is, above these critical values, the entanglement teleportation cannot be achieved. If the values of $B$, $T$, and $J$ are located below the plane, $E_{out} > 0$. That is, the entanglement teleportation for the mixed channel can be realized.

The average fidelity $F_A$ of teleportation can be formulated by

$$F_A = \frac{\int_0^{2\pi} d\phi \int_0^\pi F \sin \theta d\theta}{4\pi}.$$  

(9)

If a 1D Heisenberg $XXX$ chain is used as quantum channel, $F_A$ can be expressed as

$$F_A = \frac{2}{3} \left\{ 1 + \frac{5}{2} \exp\left(\frac{4J}{T}\right) + 3 \exp\left(\frac{2J}{T}\right) + \frac{5}{2} - 2 \left[ \exp\left(\frac{2J}{T}\right) + \cosh\left(\frac{B}{T}\right) + 1 \right]^2 \right\}.$$  

(10)

In Eq. (10), the average fidelity $F_A$ is larger than $\frac{2}{3}$ when $T < \frac{2J}{8 \ln 11}$ and $B$ is not very large. This shows that the teleportation through 1D Heisenberg mixed states is better than the classical communication since $F_A = \frac{2}{3}$ is the limited value in classical communication.

The average fidelity $F_A$ is plotted as functions of the magnetic field $B$ and the coupling coefficient $J$ in Fig. 2 when the temperature $T = \frac{1}{2 \ln 3}$. From Fig. 2(a), it is seen that $F_A$ is increased when $J$ is increased. When $J$ is small, $F_A$ is gradually decreased and then increased with increasing value of $B$. If the coupling is weak, $J \to 0$, the average fidelity can be approximated by

$$F_A = \left( \frac{1}{\cosh\frac{B}{T} + 1} - \frac{1}{3} \right)^2 + \frac{2}{9}.$$  

(11)

It is easily seen that there is a minimum fidelity of $F_A = \frac{2}{3}$ when $B = B_m = T \cosh^{-1} 2$. The average fidelity $F_A$ can be increased if the magnetic field $B \neq B_m$. Therefore, the average fidelity $F_A$ in Fig. 2(a) is like a parabolic curve when $B$ is increased from zero. If the interaction in the Heisenberg chain is weak, the fidelity of teleportation may be improved to some degree by increasing magnetic field. Fig. 2(b) is a plot of the critical
values of the magnetic field $B_{cf}$, the temperature $T_{cf}$, and the coupling coefficient $J_{cf}$ for $F_A = \frac{2}{3}$. From Fig. 2(b), it is seen that the critical values of $B_{cf}$, $T_{cf}$, and $J_{cf}$ are located on the plane. If the values of $B$, $T$, and $J$ are located on the plane or below the plane, $F_A \geq \frac{2}{3}$. This means that the entanglement teleportation of the mixed channel is superior to the classical communication. If the values of $B$, $T$, and $J$ are located above the plane, $F_A < \frac{2}{3}$. That is, above these critical values, the fidelity $F_A$ will be less than $\frac{2}{3}$. The entanglement teleportation will be worse.

3 Mutual information of the quantum channel

Compared with the classical information theory, the mutual information $\mathcal{I}$ in the quantum communication theory can imply the classical capacity of a quantum channel. For two-qubit signal states, the value 2.0 of mutual information means that the classical information carried by signal states can be totally transmitted via the quantum channel. If the value of $\mathcal{I}$ is 0.0, it means that the original classical information coded in signal states is totally destroyed after quantum teleportation. If the value of $\mathcal{I}$ is $0.0 < \mathcal{I} < 2.0$, it means that the original classical information coded in signal states is destroyed partially after quantum teleportation. Given a set of input signal states $\{q_i, \pi_i\}$, the output states are $\{q_i, \chi_i\}$. Assuming $\chi = \sum_i q_i \chi_i$, the mutual information can be written as [12-15],

$$\mathcal{I}_n = S(\chi) - \sum_i q_i S(\chi_i) \left( \sum_i q_i = 1 \right),$$

where the subscript $n$ denotes the number of the channel used. In Eq. (12), the Von Neumann entropy is $S(\rho) = -tr(\rho \log_2 \rho)$. There are two independent 1D Heisenberg chains as the quantum channel with $n = 2$. The input signal states can be written as

$$|\pi_1\rangle = \cos \gamma |00\rangle + \sin \gamma |11\rangle,$$

$$|\pi_2\rangle = \sin \gamma |00\rangle - \cos \gamma |11\rangle,$$

$$|\pi_3\rangle = \cos \beta |01\rangle + \sin \beta |10\rangle,$$

$$|\pi_4\rangle = \sin \beta |01\rangle - \cos \beta |10\rangle.$$
Here, the possibility of each state is the same, i.e., $q_i = \frac{1}{4}$. The parameters $\gamma$ and $\beta$ describe two sets of entangled states with different amplitudes. By means of Eqs. (5), (12), and (13), the mutual information is

$$ I = I_2 = 2 - \frac{1}{4} \sum_{ij} \xi_{ij} \log_2 \xi_{ij}, $$

where $\xi_{ij}$ is the $j$th eigenvalue of $\chi_i$.

The mutual information $I$ is illustrated in Fig. 3. Fig. 3(a) is a plot of the mutual information $I$ as functions of signal states $(\gamma, \beta)$. From Fig. 3(a), it is seen that $I$ is a periodic function of both $\gamma$ and $\beta$. There is a minimum mutual information of $I = 1.70$ when the input signal states are maximally entangled with $\beta = \gamma = (n + \frac{1}{4})\pi$ or $(n + \frac{3}{4})\pi, (n = 0, 1, 2, 3, \ldots)$. When the input states are not entangled with $\beta = \gamma = n\pi$ or $(n + \frac{1}{2})\pi, (n = 0, 1, 2, 3, \ldots)$, the maximum mutual information of $I = 1.80$ can be obtained. It is interesting to note that the entanglement quality of the input states cannot enhance the mutual information in entanglement teleportation. The difference of mutual information $I$ between maximally entangled and non-entangled states is less than 6.0%. Fig. 3(b) is a plot of the mutual information $I$ as functions of the magnetic field $B$ and temperature $T$ when the input signal states are maximally entangled with $\beta = \gamma = (n + \frac{1}{4})\pi$ or $(n + \frac{3}{4})\pi, (n = 0, 1, 2, 3, \ldots)$. From Fig. 3(b), it is seen that high mutual information can be obtained when input states are entangled with small values of magnetic field $B$ and temperature $T$.

The mutual information $I$ is plotted in Fig. 4 when the input states $\beta$ and $\gamma$ are varied. From Fig. 4, it is seen that $I$ monotonously decreases with increasing values of both $B$ and $T$. Fig. 4(a) is a plot of $I$ as a function of $B$. In Fig. 4(a) of very strong magnetic field $B$, the mutual information $I$ of non-entangled input states (solid line) is decreased more rapidly than that of maximally entangled ones (dashed line). That is, the mutual information of non-entangled input states is more sensitive to the magnetic field $B$ than that of maximally entangled ones. When the magnetic field is stronger than 1.3, the mutual information of maximally entangled states decreases
slower than that of non-entangled states. Fig. 4(b) is a plot of $I$ as a function of $T$. From Fig. 4(b), it is seen that there is almost no difference in $I$ between maximally entangled input state and non-entangled input state when $T < 0.2$. When $T > 0.2$, the mutual information of non-entangled input states (solid line) is always higher than that of maximally entangled ones (dashed line).

4 Discussion

The entanglement teleportation of two-qubit mixed states via two independent 1D Heisenberg $XXX$ chains is analyzed. For suitable values of the magnetic field $B$, the temperature $T$, and the coupling coefficient $J$, the minimal entanglement of the thermal state in 1D Heisenberg chain is needed to realize the entanglement teleportation. There exist critical values of $B$, $T$, and $J$ when the minimal entanglement is obtained. Above the critical values, the entanglement teleportation cannot be achieved. When $T < \frac{2J}{\ln 11}$ and $B$ is small, the average fidelity is larger than $2/3$ which is better than the classical communication. To keep high value of the mutual information, the magnetic field and the temperature should be very small. The quality of the entanglement of input states cannot enhance the mutual information. For maximally entangled input states, minimal value of mutual information is obtained. While for non-entangled input states, maximal mutual information is achieved.

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Figure Captions

Fig. 1.

The teleported entanglement $E_{out}$ as a function of the input entanglement $E_{in}$ if
(a). the magnetic field $B$ is changed when $J = 1$ and $T = \frac{1}{2 \ln 3}$;
(b). the temperature $T$ is changed when $J = 1$ and $B \to 0$;
(c). the coupling coefficient $J$ is changed when $T = \frac{1}{2 \ln 3}$ and $B \to 0$.
(d). The critical values of $B_c$, $T_c$, and $J_c$ for $E_{out} = 0$ when $E_{in} = 1$.

Fig. 2.

(a). The average fidelity $F_A$ as functions of the magnetic field $B$ and the coupling coefficient $J$ when the temperature $T = \frac{1}{2 \ln 3}$.
(b). The critical values of $B_{cf}$, $T_{cf}$, and $J_{cf}$ when $F_A = \frac{2}{3}$.

Fig. 3.

The mutual information $I$ as functions of the input signal entangled states $(\beta, \gamma)$ and parameters $(B,T)$.
(a). $I$ as functions of $(\beta, \gamma)$ when $J = 1$, $T = \frac{1}{2 \ln 3}$, and $B \to 0$.
(b). $I$ as functions of $(B,T)$ when $J = 1$ and $\beta = \gamma = (n + \frac{1}{4})\pi$ or $(n + \frac{3}{4})\pi$, $(n = 0, 1, 2, 3, \ldots)$.

Fig. 4.

The mutual information $I$ as a function of the magnetic field $B$ and the temperature $T$.
(a). $I$ as a function of $B$ when $J = 1$ and $T = \frac{1}{2 \ln 3}$.
(b). $I$ as a function of $T$ when $J = 1$ and $B \to 0$.

The dash line labels maximally entangled input states when $\beta = \gamma = (n + \frac{1}{4})\pi$ or $(n + \frac{3}{4})\pi$, $(n = 0, 1, 2, 3, \ldots)$. The solid line labels non-entangled input states when $\beta = \gamma = n\pi$ or $(n + \frac{1}{2})\pi$, $(n = 0, 1, 2, 3, \ldots)$.
Fig. 1
Fig. 2
Fig. 3
Fig. 4