Restoring the bulk-boundary correspondence in non-Hermitian Hamiltonians

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We address the breakdown of the bulk-boundary correspondence observed in non-Hermitian systems, where open and periodic systems can have distinct phase diagrams. The correspondence can be completely restored by considering the Hamiltonian’s singular value decomposition instead of its eigendecomposition. This leads to a natural topological description in terms of a flattened singular decomposition. This description is equivalent to the usual approach for Hermitian systems and coincides with a recent proposal for the classification of non-Hermitian systems. We generalize the notion of the entanglement spectrum to non-Hermitian systems, and show that the edge physics is indeed completely captured by the periodic bulk Hamiltonian. We exemplify our approach by considering the chiral non-Hermitian Su-Schrieffer-Heger and Chern insulator models. Our work advocates a different perspective on topological non-Hermitian Hamiltonians, paving the way to a better understanding of their entanglement structure.

Introduction.—Over the last decades, topology became a fundamental concept in condensed matter physics1–6. A comprehensive classification of different topological phases under various sets of symmetries led to vast advances in our understanding of electronic systems, in particular for closed systems described by Hermitian Hamiltonians7–11. Indeed, topology explains the existence and resilience of numerous physical properties (like unconventional edge or surface states), and provides a unified description of unconventional phases and phase transitions. One of the key ideas and guiding principles in topological matter is the bulk-boundary correspondence12,13: nontrivial topological invariants in the bulk of a system directly translate into gapless edge physics. This correspondence has been verified in a plethora of models, including even higher order topological phases.14,15

It is therefore not surprising that attempts have been made to extend those concepts to non-Hermitian models16,17. Non-Hermitian Hamiltonians provide a simple, albeit restricted, description of open systems. Instead of considering the full and overly complex problem of microscopically modeling a system coupled to its environment, or of working in a Lindblad formalism which governs the time evolution of density matrices, we can model dissipative memory-less environments by breaking the hermiticity of the Hamiltonian.18 This simplified approach has successfully described numerous experiments and phenomena, with applications ranging from mechanical and optical meta-materials19–23 to heavy-fermions systems24,25. The topological properties observed in these systems can significantly differ from their Hermitian counterparts, and several questions about their fundamental applicability remain open—the validity of the bulk-boundary correspondence being one of them.26–34 Indeed, it has been shown that in several models the phase diagram can strongly depend on the boundary conditions, where even the bulk spectra change depending on whether one considers periodic or open systems in sharp contrast with topological Hermitian systems. Similarly, the existence and stability of edge states in such models have been questioned29,32.

In this work, we provide a simple mathematical explanation for why the bulk-boundary correspondence breaks down in non-Hermitian systems, and propose a change of paradigm in the way we look at and define topology in these systems. For a topological classification of non-Hermitian matrices to make physical sense, with resilience to small perturbations, and for any bulk-boundary correspondence to stand, it is more fruitful to think in terms of singular values of the Hamiltonian than in terms of eigenvalues. Moreover, in contrast to the eigenvalues, the singular values are well-behaved in the thermodynamic limit. Below, through the example of the chiral non-Hermitian Su-Schrieffer-Heger (nH-SSH) model31,35,36, we illustrate concretely where the usual eigenvalue-based topology fails for non-Hermitian systems, and how standard results are phenomenologically recovered from the singular value decomposition (SVD). We then formalize a topological description for two-band non-Hermitian models, and give explicit examples of topological invariants in different symmetry classes. Finally, we introduce a generalization of the entanglement spectrum for the nH-SSH model, and show that the bulk-boundary correspondence is indeed recovered.

Notation.—We consider a matrix H. We denote with E and P its eigenvalue decomposition, while its singular value decomposition is the set of three matrices U, V and Λ such that:

\[ H = PEP^{-1} = UΛV^†. \]  

(1)

Λ is a diagonal and real positive matrix, whose eigenvalues are the so-called singular values, and U and V are two unitary matrices. The columns of U and V are
the left and right singular vectors. While the decomposition admits some gauge freedom, both $\Lambda$ (up to ordering) and $Q = UV^t$ are uniquely defined. The two decompositions are similar in a Hermitian setting: $\Lambda = |E|$ and one can choose $U = P$, $V^t = \text{sgn}(E)P^t$. This property breaks down for non-Hermitian matrices, with one crucial exception: each zero singular value corresponds to one Jordan block associated to a zero eigenvalue.

**Breakdown of the bulk boundary correspondence.**—The bulk-boundary correspondence breaks down in certain non-Hermitian models$^{26–32}$, an effect dubbed the non-Hermitian skin-effect. As an illustration, let us define the chiral nH-SSH model$^{31,35,36}$ as:

$$H_{\text{nH-SSH}} = -\sum_j t_1 (c_{j,1}^\dagger c_{j,2} + h.c.) + t_2 (c_{j,2}^\dagger c_{j+1,1} + h.c.) + \frac{\gamma}{2} \sum_j (c_{j,2}^\dagger c_{j,1} - c_{j,1}^\dagger c_{j,2}),$$  

(2)

where $c_{j,\alpha}^\dagger$ is the fermionic annihilation (creation) operator at site $j$ for the species or sublattices $\alpha = 1, 2$. $t_1$ and $t_2$ are usual hopping terms, while $\gamma$ is a dissipative chirality-preserving contribution to hopping. This model possesses the standard time-reversal, particle-hole and chiral symmetry, represented by:

$$K h_k K = h_{-k}, \quad K \sigma^z h_k \sigma^z K = -h_{-k}, \quad \{\sigma^z, h_k\} = 0,$$

(3)

where $K$ is the complex conjugation, $h_k$ is the Bloch Hamiltonian and $\sigma^z$ acts on the sublattice degree of freedom. For $\gamma = 0$, the model corresponds to the celebrated SSH model, which presents a topological phase for $|t_1| < |t_2|$ characterized by zero-energy edge states. Its phase diagram, using the criterion of energy gap closing, can be analytically computed both with open (OBC) and periodic (PBC) boundary conditions$^{32}$, and is shown in Fig. 1(a). For PBC, it presents four different phases: two topological phases with non-trivial winding numbers, and two trivial phases, one of each directly connected to the two phases of the Hermitian SSH model. For OBC on a finite system, there is no gap closing separating the two topological phases, and the boundaries separating the topological from the trivial phases differ from the periodic case. The topological phases are characterized by zero-energy edge states. As immediately visible in Fig. 1(b-d), there is no direct correspondence between the periodic bulk physics and the open edge physics. Gap closing points depend on the boundary conditions. Interestingly, computation of the energy spectrum directly in the thermodynamic limit on a semi-infinite chain$^{37}$ recovers the bulk phase diagram. The non-Hermitian topological phase is characterized by an infinite-dimensional zero-energy Jordan block. We therefore conclude that the phase diagram of the infinite open system differs from the infinite-limit of the phase diagram of the open system. Similar differences between finite and semi-infinite systems were observed in a study of Toeplitz matrices and operators$^{38}$.

We now turn to the stability of the edge states observed in the non-Hermitian topological phase with OBC. It was observed in Ref. 32 that these edge states are unstable to perturbations exponentially small in the total system size. Indeed, a simple weak link $\tilde{t}_2 \ll |t_2 - t_1|$ connecting the two edges of the nH-SSH model can lead to a change of order $|t_2 - t_1|$ in the energy of the edge states, as illustrated in Fig. 1(e-f). In the Hermitian limit, such perturbation only leads to a splitting of order $t_2$. Similar results are obtained with a domain-wall configuration: connecting the two edges by a strongly gapped segment of the SSH model can lead to the disappearance of the edge states$^{37}$. In contrast to what happens at the interface between different hermitian topological phases, the two topologically distinct segments are not interfaced by

**FIG. 1.** (a) Superposition of the phase diagrams of the non-Hermitian SSH model. Black lines mark gap closings for energies with PBC, and for singular values both with PBC and OBC. Orange lines mark gap closings for energies with OBC. In the PBC case, we observe four different phases: the Hermitian topological and trivial phase are simple extension of the phases of the Hermitian SSH model. (b-f) Singular values (blue lines) and absolute value of energies (orange dots) for different $\gamma$ and boundary conditions. (b-d) The mismatch between singular values and energies for OBC, and the breakdown of bulk-boundary correspondence for energies are apparent. (e) Hermitian and (f) Non-Hermitian SSH model with OBC and a weak link $10^{-12} t_2$ connecting the two edges, for $L = 50$. The edges states acquire a macroscopic energy in the non-Hermitian topological phase, while the Hermitian phase is essentially non-affected.
edge states. This is a direct consequence of the following inequalities: For Hermitian matrices $A$ and $B$, the Weyl inequalities guarantee that the variation of eigenvalues due to perturbations are well-behaved,

$$|E_j(A + B) - E_j(A)|_\infty \leq \|B\|$$  \hspace{1cm} (4)$$
where $E_j(A)$ (resp. $E_j(A + B)$) is the $j^{th}$ sorted eigenvalue of $A$ (resp. $A + B$) and the norm $\|\cdot\|$ is the spectral norm, that is to say the largest singular value of $B$ (largest absolute eigenvalue). On the other hand, for $n \times n$ non-Hermitian matrices $A$ and $B$, the following inequality\textsuperscript{39} holds:

$$d[E(A + B), E(A)] \leq c(n)(2M)^{-\frac{1}{2}}\|B\|^\frac{1}{2},$$  \hspace{1cm} (5)$$
where $c(n) = \frac{16}{3\sqrt{3}} 2^{-\frac{1}{3}} < 4$, $M = \max(\|A + B\|, \|A\|)$ and $d$ is the optimal matching distance:

$$d[E(A), E(B)] = \min_{\pi \in S_n} \max_{j} |E_j(A) - E_{\pi(j)}(B)|.$$  \hspace{1cm} (6)$$
$S_n$ is the group of all permutations. Physically, for a Hermitian system, a perturbation of energy smaller than $\varepsilon$, cannot change the system’s energy by more than $\varepsilon$, while an exponentially small perturbation of a non-Hermitian system can lead to macroscopic changes. This inequality is also the reason for the numerical noise often observed when diagonalizing non-Hermitian Hamiltonians.

Such sensitivity of energy eigenvalues to perturbations calls into question the use of the winning of energies around special points as a topological invariant for non-Hermitian systems. Working with a translation-invariant system with a finite number of bands, i.e., dealing with an effectively lower dimensional space, keeps under control the stability issue. Yet, instabilities immediately reappear when considering translation breaking perturbations, adding additional trivial bands, or considering interactions. Additionally, the sensitivity to perturbations potentially jeopardizes the validity of all the usual approximations assumed in condensed matter systems: longer range tunneling or interactions that quickly decay may still have macroscopic effects.

Restoring the correspondence.—Instead of considering eigenvalues, we can in the same way study non-Hermitian systems using the singular value decomposition. The SVD always satisfies the Weyl inequalities in Eq. (4) and is thus well-behaved in the thermodynamic limit. Phase transitions are marked by a gap closing, i.e., a continuum of singular values reaching 0. The singular spectrum for the nH-SSH model can be obtained analytically\textsuperscript{37}. In the Hermitian limit, it is doubly degenerate, due to the particle-hole symmetry. For nonzero $\gamma$, half of the singular spectrum corresponds to the spectrum of the Hermitian SSH model with renormalized hopping $\tilde{t}_1 = t_1 + \frac{\gamma}{2}$ (without particle-hole induced degeneracy) and the other half to the hermitian spectrum with $\tilde{t}_1 = t_1 - \frac{\gamma}{2}$. As illustrated in Fig. 1(a-d), the SVD phase diagram is identical for both OBC and PBC, and corresponds to the bulk energy phase diagram. The non-Hermitian topological phase is now characterized by a single zero-energy singular value, indicating the existence of a single zero-energy Jordan block (in contrast to two one-dimensional blocks in the Hermitian topological phase). Similarly, we can study the stability of the zero singular modes to perturbations. As expected, we recover the normal stability of Hermitian energy modes in Fig. 1(e-f). Domain walls between topologically distinct phases also translate into zero singular modes at the interfaces\textsuperscript{37}.

Topology in non-Hermitian systems.—Given the above observations, it is natural to shift language and reinterpret topology in terms of the SVD. By analogy with the Hermitian case, the actual value of the singular values should not matter, as long as they are nonzero. The natural object to consider is therefore the unitary matrix $Q = U V^\dagger$, as a generalized flattened singular decomposition. For a Hermitian system, $Q = P_+ - P_- = 2P_+ - 1 = 1 - 2P_-$, where $P_\pm$ are the projectors on the positive or negative energy bands. $Q$ is then also Hermitian and satisfies $Q^2 = 1$. The usual Hermitian topological invariants such as winding or Chern numbers can be rephrased in terms of $P_\pm$ and therefore also in terms of $Q$. In the non-Hermitian case, there is no longer a simple notion of occupied and unoccupied bands, and the eigenvalues of $Q$ are no longer limited to be $\pm 1$ but can take any value in $\mathbb{U}(1)$. Nevertheless, the notion of bands remains, and both topological invariants and classification can be achieved. Symmetries, such as particle-hole, time-reversal or chiral symmetry, play a similar role as in the Hermitian case. Moreover, the symmetries of $H$ are also symmetries of $Q$, since $H = Q \sqrt{H}$, with $\sqrt{H}$ taken to be positive-definite\textsuperscript{16}.

Topological invariants.—In noninteracting translation invariant systems, $Q$ can be written as the sum of $Q_k$, where $Q_k$ is obtained from the SVD of the Bloch Hamiltonian at momentum $\vec{k}$. Topological classification of $Q$ then simply corresponds to the classification of the mappings $\vec{k} \rightarrow Q_k$. We now present a few examples in one and two-dimensions for different symmetry classes. For pedagogical purposes, we focus on two-band models (i.e., $Q_k$ is a $2 \times 2$ matrix), the generalization being generally straightforward. For the non-Hermitian SSH model, the chiral symmetry implies

$$Q_\vec{k} = \begin{pmatrix} 0 & q_1(\vec{k}) \\ q_2(\vec{k}) & 0 \end{pmatrix},$$  \hspace{1cm} (7)$$
where $q_1$ and $q_2$ satisfy

$$\det Q_\vec{k} = -q_1 q_2 = \frac{\det H_\vec{k}}{|\det H_\vec{k}|}.$$  \hspace{1cm} (8)$$
The first homotopy group of the unitary matrices is $\mathbb{Z}$. The homotopy group for chiral-symmetric matrices is therefore $\mathbb{Z} \oplus \mathbb{Z}$. The topological invariants associated to such a decomposition are:

$$\nu_+ = \frac{i}{2\pi} \int_{\mathbb{BZ}} \text{Tr}(Q_k^\dagger \partial_\vec{k} Q_k), \quad \nu_- = \frac{i}{2\pi} \int_{\mathbb{BZ}} \text{Tr}(\sigma^z Q_k^\dagger \partial_\vec{k} Q_k),$$  \hspace{1cm} (9)$$
or equivalently:

\[ \nu_1 = \frac{\nu_+ - \nu_-}{2} \quad \text{and} \quad \nu_2 = \frac{\nu_+ + \nu_-}{2}. \tag{10} \]

These two topological invariants match those introduced in Ref. 36 and 41 previously to describe the nH-SSH model, while the definitions are different. Note that in the Hermitian case, \( q_1 = q_2 \), and therefore \( \nu_+ \) trivially vanishes while \( \nu_- \) is (twice) the usual winding number.

We similarly define the Chern number from \( Q \) in two dimensions. Though there is no direct simple link between the Berry curvature and \( Q \), one can write the Chern number as a winding of Wilson loops. We define the Wilson loop operator for a two-band system by:

\[ W_n(k_x) = \ln \left[ \text{Tr} \left( \prod_{k_y \in \text{BZ}} U_{k_x,k_y} |n\rangle \langle n| V_{k_x,k_y}^\dagger \right) \right], \tag{11} \]

where \( |n\rangle \langle n| \) is the projector on the \( n \)-th singular band (degeneracy in the Hermitian case can be taken care of by shifting by the identity matrix). The Chern number is then simply given by the winding of \( W_n \):

\[ C_n = \frac{1}{2\pi} \int dk_x W_n(k_x). \tag{12} \]

We have checked the validity of this definition in different non-Hermitian Chern insulator models.\textsuperscript{37}

**Entanglement spectrum.**—A striking signature of the bulk-boundary correspondence in Hermitian topological system is found in the entanglement spectrum\textsuperscript{42–45}. For Hermitian systems, the entanglement Hamiltonian is defined as \( H_{\text{ent}} = -\log \rho_A \), where \( \rho_A \) is the reduced density matrix of the groundstate in the subsystem \( A \)—its spectrum is the entanglement spectrum. In a topological closed system, \( H_{\text{ent}} \) contains a universal low-energy part that corresponds to the edge theory of \( H \) for OBC. In a noninteracting system, the entanglement spectrum can be obtained directly from the eigenvalues of the correlation matrix \( C_A = \langle c_{\vec{r}} c_{\vec{r}'}^\dagger \rangle \) where \( \vec{r}, \vec{r}' \) are restricted to \( A \).\textsuperscript{46} This matrix can be written as

\[ C_A = \frac{1 - Q_A^T}{2}, \tag{13} \]

where \( Q_A \) is the restriction of \( Q \) to \( A \). We therefore consider the matrix \( Q_A \) and its eigen and singular values as generalization of the entanglement spectrum (up to some rescaling)\textsuperscript{47}. Topological zero modes of the Hamiltonian should translate into zero modes of \( Q_A \). In Fig. 2, we show both the singular and the absolute eigenvalue spectra of \( Q_A \) for the nH-SSH model of size \( L \), computed from the periodic system for a segment \( A \) of length \( l \). The singular spectrum exactly matches the one obtained with open boundary: the Hermitian topological phase is associated to two zero singular values, while the non-Hermitian one has only a single zero singular value. Conversely, the energy spectrum exactly matches the one obtained for OBC, as long as \( A \) is not exactly half the system. In other words, by considering the singular flattened Hamiltonian, we have a true bulk-boundary correspondence: the physics of the open system is perfectly recovered for the periodic bulk Hamiltonian. We have obtained similar results\textsuperscript{37} for the two-dimensional models introduced in Ref. 32 and 48.

**Conclusions and discussions.**—In this work, we have shown how the bulk-boundary correspondence, considered to be broken in non-Hermitian systems, can actually be restored by shifting from eigenvalue considerations to a singular value decomposition. Our framework provides a path towards a natural classification of Hamiltonians in terms of their flattened singular decomposition \( Q = UV^\dagger \). We discussed how to recover topological invariants from \( Q \) and gave some concrete examples of topological non-Hermitian phases. Finally, we proposed a generalization of the entanglement spectrum to non-Hermitian systems. We showed that indeed, the bulk system contains complete information on the edge physics of both eigenvalues and singular values. Note that this approach for topology in non-Hermitian systems turns out to coincide with the one introduced in Ref. 16, while coming from a completely different perspective. Our construction is explicit for two-band Hamiltonians. A complete and full extension to multiband problems, and the full classification of topological non-Hermitian matrices taking into account all Bernard-LeClair symmetry classes\textsuperscript{49,50} are left to future work. Generalization of this approach to Lindblad systems would be an interesting subject of research. Similarly, comparison of these results to similar systems with a complete Hamiltonian description of the dissipative part would answer important issues. Indeed, non-Hermitian models are nothing but a simple approximation of complex Hermitian systems: whether this classification would carry through is an open question. Similarly, the instability of non-Hermitian systems to small perturbations could be addressed directly.

**FIG. 2.** Singular values (blue lines) and absolute value of the eigenspectrum (orange dots) of \( Q_A \) for the non-Hermitian SSH model. We consider a periodic wire with \( L = 200 \) cells and consider a subsystem with \( l = 25 \) unit-cells. The dashed (resp. dotted) vertical lines mark the OBC (resp. PBC) energy phase transitions. Both singular and energy entanglement spectrum match the behaviour of the open system.
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A shift by the identity of a non-Hermitian Hamiltonian is not trivial for its flattened representation. Let us look for a state $\psi_j \propto \langle j | \Psi_k \rangle$ and look for zero-energy left- and right- eigenstates of the form:

$$|\psi_{R,j}^0\rangle = \sum_{j \geq 0} r_j c_{j,1}^\dagger |0\rangle \quad \text{and} \quad |\psi_{L,j}^0\rangle = \sum_{j \geq 0} l_j c_{j,1}^\dagger |0\rangle .$$

Straightforward algebra leads to

$$r_j \propto \left(-\frac{t_1 + \gamma/2}{t_2}\right)^j \quad \text{and} \quad l_j \propto \left(-\frac{t_1 - \gamma/2}{t_2}\right)^j .$$

The normalization condition applied separately to each state leads to the PBC phase diagram, while the mutual normalization $\langle \psi_{R,j}^0 | \psi_{R,k}^0 \rangle = 1$ leads to the OBC phase diagram. The latter condition is not necessarily correct if these two states are the extremal states of a Jordan block. Let us look for a state $|\psi_R^0\rangle \propto \sum_{j \geq 0} r_j c_{j,2}^\dagger |0\rangle$ such that $H |\psi_R^0\rangle = |\psi_R^0\rangle$. If such a state exists, then $H$ is not diagonalizable and the zero eigenvalue is associated to a non-trivial Jordan block. The coefficients in $|\psi_R^0\rangle$ should satisfy the following set of equations:

$$-(t_1 - \frac{\gamma}{2}) r_{0,1} = 1 \quad \text{and} \quad -(t_1 - \frac{\gamma}{2}) r_{m,1} - t_2 r_{m-1,1} = \left(-\frac{t_1 + \gamma/2}{t_2}\right)^m .$$

**Appendix A: The chiral nH-SSH model**

1. **Phase diagram and boundary conditions**

The non-Hermitian chiral SSH model is a paradigmatic example of a chiral-symmetric topological model in one-dimension. Its real space Hamiltonian is given by Eq. (2), and the corresponding momentum Hamiltonian is:

$$H_{nH-SSH} = \sum_k \Psi_k^\dagger [\vec{n}(k) \vec{\sigma} + i \vec{d}(k) \vec{\sigma}] \Psi_k ,$$

with $\Psi_k^\dagger = (c_{k,1}^\dagger, c_{k,2}^\dagger)$, $\vec{n}(k) = (-t_1 - t_2 \cos k, -t_2 \sin k, 0)$ and $\vec{d}(k) = (0, \frac{\gamma}{2}, 0)$. The energies at momentum $k$ are given by:

$$E_k^2 = t_1^2 + t_2^2 + 2t_1t_2 \cos k - \frac{\gamma^2}{4} - i\gamma t_2 \sin k .$$

The gap-closing conditions for this two-band non-Hermitian model are:

$$||\vec{n}(k)|| = ||\vec{d}(k)|| \quad \text{and} \quad \vec{n}(k).\vec{d}(k) = 0 ,$$

which straightforwardly lead to phase boundaries for a periodic system given by $t_1 \pm t_2 = \pm \frac{\gamma}{2}$. Conversely, for a finite open system, the phase boundaries were computed analytically in Ref. 32 and correspond to $(t_1 \pm t_2)^2 = \frac{\gamma^2}{4}$. The mismatch between the two is illustrated in Fig. 1 in the main text.

Let us now consider the semi-infinite limit. We take a lattice defined for $j \geq 0$ and look for zero-energy left- and right- eigenstates of the form:

$$|\psi_{R,j}^0\rangle = \sum_{j \geq 0} r_j c_{j,1}^\dagger |0\rangle \quad \text{and} \quad |\psi_{L,j}^0\rangle = \sum_{j \geq 0} l_j c_{j,1}^\dagger |0\rangle .$$

Supplementary Material
which admit the following solution:

\[ r_{j,1} = \left( \frac{-t_1 + \gamma/2}{t_2} \right)^j \left( \frac{1}{\gamma/2 - t_1} - j \frac{t_1 + \gamma/2}{t_2^2} \right) \] (A8)

if \( t_2^2 - t_1^2 + \gamma^2/4 = 0 \), and

\[ r_{j,1} = \left( \frac{-t_1 + \gamma/2}{t_2} \right)^j \frac{t_1 + \gamma/2}{t_2^2 - t_1^2 + \gamma^2/4} + \left( \frac{t_2}{\gamma/2 - t_1} \right)^j \left( \frac{1}{\gamma/2 - t_1} - \frac{t_1 + \gamma/2}{t_2^2 - t_1^2 + \gamma^2/4} \right) \] (A9)

otherwise. Indeed, the matrix admits a Jordan block for the 0 eigenvalue, which imply that \( \langle \psi_0^L \psi_0^R \rangle = 1 \) is not the proper normalization condition. Our construction actually does not stop with only one such state, i.e., the Jordan block is not just a 2 \( \times \) 2 block. Similar calculation leads to an infinite tower of states satisfying \( H | \psi_{k+1} \rangle = | \psi_k \rangle \), with

\[ r_{m,2k} = \sum_{j=0}^{k} \alpha_k^e m^k \left( \frac{-t_1 + \gamma/2}{t_2} \right)^m \sum_{j=0}^{k-1} \beta_k^e m^k \left( \frac{t_2}{\gamma/2 - t_1} \right)^m, \] (A10)

\[ r_{m,2k+1} = \sum_{j=0}^{k} \alpha_k^o m^k \left( \frac{-t_1 + \gamma/2}{t_2} \right)^m \sum_{j=0}^{k} \beta_k^o m^k \left( \frac{t_2}{\gamma/2 - t_1} \right)^m. \] (A11)

The \( \alpha_k^e \) and \( \beta_k^e \) coefficients can be systematically computed by recurrence. The zero-energy space is actually infinite in the thermodynamic limit. This prevents us from properly determining the phase boundary by this approach, though the form of the coefficients that appear tend to imply that the PBC phase diagram is the correct one. To clarify this picture, let us turn to the computation of the singular values.

2. Evaluating the singular value decomposition

We first evaluate the singular values of the matrix \( h_k \). Computing singular values in general is a cumbersome task. Here we use the fact that they are the eigenvalues of \( h_k^\dagger h_k \). We then directly obtain

\[ \lambda_{\pm}(k)^2 = (t_1 + t_2 \cos k \pm \frac{\gamma}{2})^2 + t_2^2 \sin^2 k. \] (A12)

We see immediately that there is no simple link between eigenvalues and singular values, even in the case of a 2 \( \times \) 2 matrix. The singular values are zero when the eigenvalues of \( h(k) \) are also zero, and we recover the PBC phase diagram. We also see that \( \gamma \) acts as a translation of the hopping parameter \( t_1 \). Let us prove this result for arbitrary boundary conditions. One can rewrite the nH-SSH model as:

\[ H_{nH-SSH} = -t_1 \sigma^x \otimes \text{Id} - t_2 (\sigma^+ \otimes T^r + \sigma^- \otimes T^l) + i \frac{\gamma}{2} \sigma^y \otimes \text{Id} \] (A13)

where \( \sigma \) acts on the pseudo-spin subspace and the other operators on the unit-cell subspace. We use the convention \( 2\sigma^\pm = \sigma^x \pm i \sigma^y \). \( T^r \) is the translation operator to the right (left), taking into account the proper boundary conditions. We then obtain:

\[ H_{nH-SSH}^1 H_{nH-SSH} = (t_1 + \frac{\gamma}{2} \sigma^z)^2 + (t_1 + \frac{\gamma}{2} \sigma^z) t_2 (T^l + T^r) + t_2^2 (\sigma^- \sigma^+ \otimes T^r T^l + \sigma^+ \sigma^- \otimes T^r T^l) \] (A14)

This matrix is actually diagonal in the pseudo-spin space for all \( \gamma \). For \( \gamma = 0 \), the two pseudo-spin flavors have the same eigenvalues, and therefore lead to a double degeneracy of the singular spectrum, which is nothing but the \( \pm E \) particle-hole symmetry of the Hermitian model. On the other hand, for \( \gamma \neq 0 \), the two flavors correspond to two Hermitian models with different effective \( t_1 \). Half the non-Hermitian singular spectrum therefore corresponds to the positive energy spectrum of a Hermitian SSH (H-SSH) model with \( \gamma_{\text{eff}}^1 = t_1 + \frac{\gamma}{2} \) (without the particle-hole degeneracy) and half to the spectrum of a H-SSH model with \( \gamma_{\text{eff}}^2 = t_1 - \frac{\gamma}{2} \). This mapping immediately implies that the phase diagram given by the SVD with open or periodic boundary conditions or in the semi-infinite limit is identical to the phase diagram derived from the energies with PBC. It also tells us that the non-Hermitian topological phase corresponds to having only one copy of the SSH model in its topological phase, and therefore only a single zero singular value. Note that in the finite system, the two energy edge states are not necessarily in the same Jordan block, as they have an exponentially small residual energy.

3. Stability of the edge states

In the main text, we have studied the resilience of the edge states to the introduction of a direct coupling between the two edges. A fair critic could argue that directly coupling the edges should gap them out, albeit by a smaller amount. To show that this is not merely the gapping out of the edge modes, we propose a slightly different scheme based on a periodic wire whose one half is in the non-Hermitian topological phase and second half is in the Hermitian trivial phase. We choose parameters such that the trivial part is strongly gapped. This setup is a common setup for topological studies: at the interface between trivial and topological phases, we expect the appearance of the topological phase’s edge modes.

To be concrete, the Hamiltonian we study is given by Eq. (2), for \( L = L_1 + L_2 \) unit cells, but with the follow-
Finally, we demonstrate the validity of the winding number introduced in the main text to describe the periodic phase diagram. The phase diagram is presented in Fig. 4. Additionally, Fig. 5 represents the size dependence of the spectra of $Q_A$, our generalization of the entanglement spectrum. The half-system cut is anomalous and does not present the same physics as the others. While we do not have a full understanding of this feature and of its universality, the absence of continuity could be explained by the instability of non-Hermitian systems.
Appendix B: Two-dimensional models

1. Non-Hermitian Chern insulator

As a two-dimensional example for both the computation of the Chern number and the entanglement spectrum, we study a non-Hermitian generalization of the two-band Chern insulator introduced in Ref. 48. This model still exhibits the usual bulk-boundary correspondence, but its proximity to the standard Hermitian Chern insulator make it an ideal benchmark for our methods. We parametrize the Bloch Hamiltonian \( h(\vec{k}) \) as:

\[
h(\vec{k}) = \begin{bmatrix} \vec{n}(\vec{k}) + i \vec{d}(\vec{k}) \end{bmatrix} \vec{\sigma},
\]

with

\[
\vec{n}(\vec{k}) = (\Delta_x \sin k_x, \Delta_y \sin k_y, -\mu - t \cos k_x - t \cos k_y), \tag{B1}
\]

\[
\vec{d}(\vec{k}) = (\gamma_x, \gamma_y, \delta \mu). \tag{B2}
\]

If the two fermionic species are spin polarizations, \( \mu \) corresponds to a Zeeman field, \( t \) a hopping between lattice sites, \( \Delta_x \) and \( \Delta_y \) are spin orbit couplings, and \( \gamma_x \) and \( \gamma_y \) are constant dissipative spin-flip terms, while \( \delta \mu \) is a local source or drain coupled to the spin polarization. In the following, for simplification, we take \( t = \Delta_x = \Delta_y = 1 \).

In the Hermitian limit \( \vec{d} = \vec{0} \), for non-zero \( \Delta_x \) and \( \Delta_y \), the system is in a trivial gapped phase for \( |\mu| > 2|t| \), and in a gapped topological phase for \( |\mu| < 2|t| \). Two distinct topological phases exist, separated by a gapless point at \( \mu = 0 \). In both topological phases, each band is characterized by a Chern number \( \pm 1 \) and chiral edge states are present. Both topological phases host chiral edge states.

Obtaining the complete phase diagram analytically in presence of the non-Hermitian terms is a fairly involved computation, so we focus in this appendix on the phases adiabatically connected to the Hermitian phases. Using Weyl inequalities for singular values, it is straightforward to show that all three phases survive the presence of non-Hermitian perturbations smaller than their gap. Fig. 6 presents the winding number of the Wilson loops defined in Eq. (11) in the main text, for different sets of parameters. We have selected here the lowest singular band. The Chern number indeed survives the presence of non-Hermitian terms.

We also compute our generalization of the entanglement spectrum for this model. We consider a periodic system of length \( L_x = L_y = 80 \) unit-cells, and our subsystem \( \mathcal{A} \) is a strip of length \( l_x = L_x \) and \( l_y = 30 \) of the torus. Our results are summarized in Fig. 7. The spectra of \( Q_\mathcal{A} \) indeed shows the presence of the chiral modes even in the presence of the non-Hermitian dissipation, with an interesting caveat. In the open system, the chiral energy modes always have a finite dissipative part, i.e., the imaginary part of the energy does not vanish, with an effective low-energy dispersion relation given by:

\[
E_{\pm}(k_x) = \pm \left[ v_F(k_x - \pi) + i \epsilon_0 \right],
\]

where \( v_F \) is the Fermi velocity and \( \epsilon_0 \) a constant. Correspondingly, the singular modes do not exactly reach zero. Conversely, the spectra of \( Q_\mathcal{A} \) present clear low-energy modes which do vanish at \( k_x \approx \pi \), both for energies and singular values. We find that the effective low-energy modes in the spectrum of \( Q_\mathcal{A} \) are well fitted by:

\[
\xi_{\pm}(k_x) = \pm \left[ v_F(k_x - \pi) + i h_0 (k_x - \pi)^2 \right],
\]

where \( h_0 \) is a constant. The dissipative nature of the chiral boundary modes is therefore not fully captured by the entanglement spectrum here. Note that in this system, there are significant differences between singular and eigenspectra, even in the bulk.

2. Chern insulator with broken bulk-boundary correspondence

In the previous example, both energy and singular values led to similar phase diagrams: there was no breakdown of the bulk-boundary correspondence. We also check that our approach is valid in a toy model introduced in Ref. 32, where the broken bulk-boundary correspondence is restored in our generalization of the entanglement spectrum. The Bloch Hamiltonian is here given by:

\[
n_x = t_1 + \delta \cos k_x + (t_1 - \delta \cos k_x) \cos k_y, \\
n_y = (t_1 - \delta \cos k_x) \sin k_y, \\
z_z = t_1 - \Delta \sin k_x, \\
d_y = \frac{\gamma}{2}, \\
d_x = d_z = 0.
\]

FIG. 6. Wilson loop computed for the non-Hermitian Chern insulator, focusing on the lowest singular band. For numerical convenience, we shift the Hamiltonian's diagonal entries by 0.1 before computing the SVD. Black crosses correspond to the Hermitian limit \( \gamma_x = \gamma_y = \delta \mu = 0 \), orange pluses to \( \gamma_x = \gamma_y = 0.1 \), \( \delta \mu = 0 \) and blue stars to \( \gamma_x = \gamma_y = \delta \mu = 0.1 \). We considered a periodic system of \( L_x = 150 \) by \( L_y = 60 \) unit cells. The winding of the Wilson loops correctly capture the Chern number of the topological phase, even in the presence of non-Hermitian terms.
For pedagogical purposes, we focus on a single point of the phase diagram: \( t_1 = \Delta, \delta = 0.2t_1 \) and \( \gamma = 3 \). For PBC, the system is then gapped both for singular values and energies. If we now consider a system periodic in the \( x \) direction and open in the \( y \) direction, a zero-singular flat band appears and the energy spectrum remains gapped without zero modes. Computation of the entanglement spectrum for a strip of finite width in the \( y \) direction for the periodic system exactly matches what we observe in the open system. Results are summarized in Fig. 8. We can also compute the Chern number through the computation of the winding of the Wilson loops defined in Eq. (11). We find that the Hamiltonian is nontrivial, with a Chern number \(-1\) for the lowest singular band.