ON SOME FAMILIES OF GUSHEL–MUKAI FOURFOLDS

GIOVANNI STAGLIANO

ABSTRACT. We give explicit descriptions of some Noether–Lefschetz divisors in the moduli space of Gushel–Mukai fourfolds. As a consequence we obtain that their Kodaira dimension is negative.

INTRODUCTION

In this paper, we consider (complex) ordinary Gushel–Mukai fourfolds, that is quadric hypersurfaces in a transverse intersection of the Grassmannian $G(1, 4) \subset \mathbb{P}^9$ in its Plücker embedding with a hyperplane. The interest in these fourfolds, especially for rationality questions, is classical (see e.g. [19]). They also appeared in the classification of Fano varieties of coindex 3 (see [16]).

There are strong similarities between these fourfolds and cubic fourfolds, i.e. smooth cubic hypersurfaces in $\mathbb{P}^5$. For instance, as in the case of cubic fourfolds, it is known that all Gushel–Mukai fourfolds are unirational. Some rational examples are classical and no examples have yet been proved to be irrational. The similarity with the case of cubic fourfolds became more evident thanks to the works [1, 2, 3, 4], where the authors, following Hassett’s study of cubic fourfolds (see [10, 11]), introduced, via Hodge theory and the period map, the analogous definition of the Noether–Lefschetz locus inside the moduli space of Gushel–Mukai fourfolds, which is a countably infinite union of irreducible divisors. They also gave explicit geometric descriptions for the first three components of this locus (see [1, Section 7] and Example 1.1 below).

The present paper is inspired by the work of Nuer [17], where he gave explicit descriptions for the first components of the Noether–Lefschetz locus in the moduli space of cubic fourfolds, deducing that they are unirational and hence of negative Kodaira dimension. Here we do basically the same in the case of Gushel–Mukai fourfolds. Our contribution is summarized in Theorem 2.1 and Table 1, and as a consequence we deduce that the first ten components of the Noether–Lefschetz locus in the moduli space of Gushel–Mukai fourfolds are uniruled and hence of negative Kodaira dimension (see Corollaries 2.3 and 2.5). Notice that in the case of cubic fourfolds, thanks to the work [30], there are only a few components of the Noether–Lefschetz locus for which we have no information about their Kodaira dimension.

This short paper is organized as follows. In Section 1 we recall some general facts about Gushel–Mukai fourfolds and del Pezzo fivefolds. In Section 2 we present our main results. In Section 3 we introduce a Macaulay2 package, which is essential to verify many of our claims.

2010 Mathematics Subject Classification. 14J35, 14J45, 14W30, 14Q10.
Acknowledgements. The author is grateful to F. Russo and M. Bolognesi for stimulating discussions, and to O. Debarre and A. Kuznetsov for pointing out Corollary 2.5.

1. Preliminaries

1.1. Parameter space. A Gushel–Mukai fourfold $X \subset \mathbb{P}^8$, GM fourfold for short, is a smooth Fano fourfold of degree 10 with $\text{Pic}(X) = \mathbb{Z}(\mathcal{O}_X(1))$ and $K_X \in |\mathcal{O}_X(-2)|$. Equivalently, $X$ is a quadratic section of a 5-dimensional linear section of the cone in $\mathbb{P}^{10}$ over the Grassmannian $G(1, 4) \subset \mathbb{P}^9$ of lines in $\mathbb{P}^4$ (see [16] and also [9]). The group of automorphisms of a GM fourfold is a finite subgroup of $\text{PGL}(9, \mathbb{C})$, and GM fourfolds are parameterized, up to isomorphism, by the points of a coarse moduli space $\mathcal{M}_{4}^{GM}$ of dimension 24. Every $[X] \in \mathcal{M}_{4}^{GM}$, outside a locus of codimension 2, can be realized as a quadratic section of a (uniquely determined) del Pezzo fivefold $Y \subset \mathbb{P}^8$, that is of a smooth hyperplane section of $G(1, 4) \subset \mathbb{P}^9$. Such fourfolds are called ordinary GM fourfolds, and they are the only ones in which we are interested in this paper. Recall also that all smooth hyperplane sections of $G(1, 4) \subset \mathbb{P}^9$ are projectively equivalent.

1.2. Noether–Lefschetz locus in $\mathcal{M}_{4}^{GM}$. We have a period map $p : \mathcal{M}_{4}^{GM} \rightarrow \mathcal{D}$ to a 20-dimensional quasi-projective variety $\mathcal{D}$, which is dominant with irreducible 4-dimensional fibers (see [4, Corollary 6.3]).

For a very general GM fourfold $[X] \in \mathcal{M}_{4}^{GM}$, the natural inclusion

$$H^4(G(1, 4), \mathbb{Z}) \cap H^{2,2}(G(1, 4)) \subseteq H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$$

of middle Hodge groups is an equality. A GM fourfold $X$ is said to be special if the inclusion (1.1) is strict. This means that the fourfold $X$ contains a surface whose cohomology class “does not come” from the Grassmannian $G(1, 4)$. The set of special GM fourfolds is called the Noether–Lefschetz locus.

Special GM fourfolds correspond to a countable union of hypersurfaces $\bigcup_d \mathcal{D}_d \subset \mathcal{D}$, labelled by the integers $d \geq 10$ with $d \equiv 0, 2, 4$ (mod 8) (see [1, Lemma 6.1]). The hypersurface $\mathcal{D}_d \subset \mathcal{D}$ is irreducible if $d \equiv 0$ (mod 4), and it has two irreducible components $\mathcal{D}_d'$ and $\mathcal{D}_d''$ if $d \equiv 2$ (mod 8) (see [1, Corollary 6.3]); the same holds true for the hypersurface $p^{-1}(\mathcal{D}_d) \subset \mathcal{M}_{4}^{GM}$.

1.3. Discriminants. Following [1, Section 7], suppose that an (ordinary) GM fourfold $X \subset Y = G(1, 4) \cap \mathbb{P}^8$ contains a smooth surface $S$. Let $a, b$ denote integers such that $[S] = a\sigma_{3,1} + b\sigma_{2,2}$ is the class of $S$ in the Chow ring of $G(1, 4)$. Then we have that $[X] \in p^{-1}(\mathcal{D}_d)$, where $d$ is the determinant (also called discriminant) of the intersection matrix with respect to $\{\sigma_{1,1|X}, \sigma_{2|X} - \sigma_{1,1|X}, [S]\}$, that is

$$d = \det \begin{pmatrix} 2 & 0 & b \\ 0 & 2 & a - b \\ b & a - b & (S^2_X) \end{pmatrix} = 4(S^2_X - 2(b^2 + (a - b)^2)).$$
The self-intersection of $S$ in $X$ is given by
\begin{equation}
(S^2)_{X} = 3a + 4b - 2 \deg(S) + 4g(S) - 12\chi(O_S) + 2K^2_S - 4,
\end{equation}
where $\deg(S)$ and $g(S)$ denote, respectively, the degree and the (sectional) genus of $S$.

When $d \equiv 2 \pmod{8}$, we have that $[X] \in p^{-1}(D'_d)$ if $a + b$ is even, and $[X] \in p^{-1}(D''_d)$ if $b$ is even.

1.4. Known explicit descriptions of Noether–Lefschetz divisors. The first irreducible components of the Noether–Lefschetz locus in $\mathcal{M}^{GM}_4$ are the following:
\begin{equation}
p^{-1}(D'_{10}), p^{-1}(D''_{10}), p^{-1}(D_{12}), p^{-1}(D_{16}), p^{-1}(D'_{18}),
p^{-1}(D'_{18}), p^{-1}(D_{20}), p^{-1}(D_{24}), p^{-1}(D'_{26}),
p^{-1}(D''_{26}), p^{-1}(D_{28}), p^{-1}(D_{32}), p^{-1}(D'_{34}), \ldots
\end{equation}
As far as the author knows, there are explicit descriptions only for the first three ones $p^{-1}(D'_{10}), p^{-1}(D''_{10}), p^{-1}(D_{12})$ (see [1]), and recently for $p^{-1}(D_{20})$ (see [13]).

Example 1.1. We recall here the explicit descriptions of the loci $p^{-1}(D'_{10}), p^{-1}(D''_{10}), p^{-1}(D_{12})$, which have been given in [1].
- A two-dimensional linear section of a Schubert variety $\Sigma_{1,1} \simeq G(1,3) \subset G(1,4)$ is a quadric surface; its class is $\sigma_1^2 \cdot \sigma_{1,1} = \sigma_{3,1} + \sigma_{2,2}$. The closure inside $\mathcal{M}^{GM}_4$ of the family of fourfolds containing such a surface coincides with $p^{-1}(D'_{10})$.
- A two-dimensional linear section of $G(1,4)$ is a quintic del Pezzo surface; its class is $\sigma_1^4 = 3\sigma_{3,1} + 2\sigma_{2,2}$. The closure inside $\mathcal{M}^{GM}_4$ of the family of fourfolds containing such a surface coincides with $p^{-1}(D''_{10})$.
- A two-dimensional linear section of a Schubert variety $\Sigma_2 \subset G(1,4)$ is a cubic scroll; its class is $\sigma_1^2 \cdot \sigma_2 = 2\sigma_{3,1} + \sigma_{2,2}$. The closure inside $\mathcal{M}^{GM}_4$ of the family of fourfolds containing such a cubic scroll coincides with $p^{-1}(D_{12})$.

The fourfolds in $p^{-1}(D''_{10})$, as well as some of those in $p^{-1}(D_{12})$, have been studied in [19].

Example 1.2. The description of $p^{-1}(D_{20})$ has been recently achieved in [13] as the closure inside $\mathcal{M}^{GM}_4$ of the family of fourfolds containing a suitable rational surface of degree 9 and sectional genus 2 with class $6\sigma_{3,1} + 3\sigma_{2,2}$. The family of such surfaces in a del Pezzo fivefold is irreducible and unirational of dimension 25. An explicit geometric construction of this family has been provided in [21].

In Section 2, we will provide explicit descriptions for all the components in the first two rows of (1.4), obtaining some alternative descriptions for those already known.

1.5. Count of parameters. Here we explain how to estimate the dimension of certain families of GM fourfolds just doing some calculations on a specific example. This leads to a computer-implementable algorithm which will be essential for our purposes. A similar argument has been given in [17] in the case of families of cubic fourfolds.

Suppose we have a specific example of a smooth surface $S$ embedded in a smooth del Pezzo fivefold $Y = G(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$, and that $X \in \mathbb{P}(\mathcal{H}^0(\mathcal{O}_Y(2)))$ is a smooth quadratic section of $Y$ which contains $S$. Let $\text{Hilb}_{\mathcal{Y}^{(\mathcal{O}_S(2))}}$ be the Hilbert scheme of subschemes of
Y with Hilbert polynomial $\chi(O_S(t))$, and assume that $\text{Hilb}_Y^\chi(O_S(t))$ is smooth at $[S]$. For instance, this condition is satisfied when we have

$$h^1(N_{S/Y}) = 0.$$  

Then there exists a unique irreducible component $S$ of $\text{Hilb}_Y^\chi(O_S(t))$ which contains $[S]$, and the dimension of this component is $h^0(N_{S/Y})$. Consider the incidence correspondence

$$Z = \{([S'], [X']) : S' \subset X' \} \subset S \times \mathbb{P}(H^0(O_Y(2)))$$

and let

$$p_2(Z) = \{\text{quadratic sections of } Y \text{ that contain some surface of } S\}$$

be the two projections. We are interested in estimating the codimension of the image

$$p_2(Z)$$

in the 39-dimensional projective space $\mathbb{P}(H^0(O_Y(2)))$ of quadratic sections of $Y$, which contains as an open set the locus of (ordinary) GM fourfolds contained in $Y$. By the semicontinuity theorem, the dimension of the fiber $p_1^{-1}([S']) \simeq \mathbb{P}(H^0(I_{S'/Y}(2)))$ of $p_1$ at a point $[S'] \in S$ achieves its minimum value on an open set of $S$. Therefore, if $h^0(I_{S/Y}(2))$ is minimal, then we deduce that $Z$ has a unique irreducible component $Z^o$ that dominates $S$, and the dimension of this component is

$$h^0(N_{S/Y}) + h^0(I_{S/Y}(2)) - 1.$$ 

Note that $h^0(I_{S/Y}(2))$ is automatically minimal if

$$h^1(O_S(2)) = 0 \text{ and } h^0(I_{S/Y}(2)) = h^0(O_Y(2)) - \chi(O_S(2)).$$

Now the value of $h^0(N_{S/X})$ is the dimension of the tangent space of $\text{Hilb}_X^\chi(O_S(t))$ at the special point $[S, X]$. Since $p_2|_{Z^o}([X])$ is a closed subscheme of $\text{Hilb}_X^\chi(O_S(t))$, again by semicontinuity, we deduce that the dimension of the general fiber of $p_2|_{Z^o}$ is bounded from above by $h^0(N_{S/X})$. It follows that the codimension of $p_2(Z^o)$ in $\mathbb{P}(H^0(O_Y(2)))$ satisfies

$$\text{codim}_{\mathbb{P}(H^0(O_Y(2)))}(p_2(Z^o)) \leq 39 - (h^0(N_{S/Y}) + h^0(I_{S/Y}(2)) - 1 - h^0(N_{S/X})).$$

Notice also that the inequality in (1.8) is actually an equality at least in the following case: the right-hand side of (1.8) equals 1 and the surface $S$ satisfies (1.2) and (1.3) for some value $d \neq 0$; when this happens, $p_2(Z^o)$ coincides with one of the (at most two) irreducible components of $p^{-1}(D_d)$, and $p_2(Z)$ and $p_2(Z^o)$ have the same support.

1.6. Birational representations on $\mathbb{P}^5$ of del Pezzo fivefolds. Let $Y = G(1, 4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ be a del Pezzo fivefold. We recall here the two well-known ways to obtain a rational parameterization of $Y$. 


1.6.1. Inside $Y$ there are two types of planes: a three-dimensional family of $\rho$-planes, having class $\sigma_{2,2}$ in $G(1,4)$; and a four-dimensional family of $\sigma$-planes, having class $\sigma_{3,1}$ in $G(1,4)$. The projection from a $\rho$-plane gives a birational map $Y \dasharrow \mathbb{P}^5$, whose inverse is defined by the linear system of quadrics through a rational normal cubic scroll $B = \mathbb{P}(O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)) \subset \mathbb{P}^4 \subset \mathbb{P}^5$. Conversely, the rational map defined by the quadrics through a rational normal cubic scroll $B$ is a birational map $\psi_B : \mathbb{P}^5 \dasharrow Y$, whose inverse is the projection from a $\rho$-plane.

The projection of $Y$ from a $\sigma$-plane gives a dominant map $Y \dasharrow Q \subset \mathbb{P}^5$ onto a smooth quadric hypersurface $Q \subset \mathbb{P}^5$.

1.6.2. The Hilbert scheme $\text{Hilb}^{2t+1}_{Y}$ of conics in $Y$ is irreducible of dimension 10. The plane $P$ obtained as the linear span of a general conic $[C] \in \text{Hilb}^{2t+1}_{Y}$ is not contained in $Y$. The projection from $P$ gives a birational map $Y \dasharrow \mathbb{P}^5$, whose inverse is defined by the linear system of cubics through a projection $B$ in $\mathbb{P}^5$ of a rational normal quartic scroll $\mathbb{P}(O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(1)) \subset \mathbb{P}^6$. Conversely, the rational map defined by the cubics through a general projection $B$ of a rational normal quartic scroll is a birational map $\psi_B : \mathbb{P}^5 \dasharrow Y$, whose inverse is the projection from the linear span of a conic in $Y$.

1.7. A way to construct examples of surfaces in a del Pezzo fivefold. Let $B \subset \mathbb{P}^5$ be either a rational normal cubic scroll or a rational quartic scroll as considered in Subsection 1.6. Then we have a birational map $\psi_B : \mathbb{P}^5 \dasharrow Y \subset \mathbb{P}^8$ whose base locus scheme is $B$ and where $Y$ is a del Pezzo fivefold. A practical way to construct “good” examples of surfaces in $Y$ consists in taking the image via the map $\psi_B$ of some surface $V \subset \mathbb{P}^5$ cutting $B$ along some curve $C$.

Example 1.3. Let us consider, for instance, the case in which we want to construct a triple $(B, V, C)$ such that $B \cap V = C$ and where $B \subset \mathbb{P}^4 \subset \mathbb{P}^5$ is a rational normal cubic scroll, $V \subset \mathbb{P}^5$ is a quintic del Pezzo surface, and $C \subset \mathbb{P}^3 \subset \mathbb{P}^5$ a twisted cubic curve.

Then a possible strategy is the following:

- take a twisted cubic curve $C \subset V$ (for example, since $V$ can be realized as the image of $\mathbb{P}^2$ via the linear system of cubic curves with 4 simple base points, we get $C \subset V$ by taking the image of a conic passing through 3 of the base points);
- take a twisted cubic curve $C' \subset B$ (for example, by intersecting $B$ with a general hyperplane);
- find a general automorphism $\eta : \mathbb{P}^5 \xrightarrow{\cong} \mathbb{P}^5$ such that $\eta(C') = C$ (for example, by taking standard parametrizations $\nu : \mathbb{P}^1 \to C' \subset \mathbb{P}^3$ and $\nu : \mathbb{P}^1 \to C \subset \mathbb{P}^3$, one sees that the composition $\nu \circ \nu^{-1} : C' \to C$ is defined by linear forms and can be extended to an automorphism of $\mathbb{P}^3 \subset \mathbb{P}^5$);
- replace $B$ by $\eta(B)$ and we are done.

Now one can verify that the restriction of $\psi_B$ induces an isomorphism between $V$ and a smooth surface $S = \overline{\psi_B(V)} \subset Y \subset \mathbb{P}^8$ cut out by 19 quadric hypersurfaces in $\mathbb{P}^8$ (see also Remark 2.6 and the 17th cases in Tables 1 and 2).

Example 1.4. As another example, we construct a triple $(B, V, C)$ such that $B \cap V = C$ and where $B \subset \mathbb{P}^4 \subset \mathbb{P}^5$ is again a rational normal cubic scroll, while $V \subset \mathbb{P}^5$ is a Veronese surface, and $C \subset \mathbb{P}^4 \subset \mathbb{P}^5$ is a rational normal quartic curve. Our strategy is the following:
• take a rational normal quartic curve $C \subset V$ (for example, by intersecting $V$ with a general hyperplane);
• take a rational normal quartic curve $C' \subset B$ (for example, since $B$ can be realized as the image of $\mathbb{P}^2$ via the linear system of conics with one base point, we get $C' \subset B$ by taking the image of a general conic);
• find a general automorphism $\eta: \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ such that $\eta(C') = C$ (for example, as before, we can take standard parametrizations $\nu': \mathbb{P}^1 \rightarrow C' \subset \mathbb{P}^4$ and $\nu: \mathbb{P}^1 \rightarrow C \subset \mathbb{P}^4$, and then extend $\nu \circ \nu'^{-1}: C' \rightarrow C$ to an automorphism of $\mathbb{P}^4 \subset \mathbb{P}^5$);
• replace $B$ by $\eta(B)$ and we are done.

One verifies that the restriction of $\psi_B$ induces an isomorphism between $V$ and another Veronese surface $S = \psi_B(V) \subset Y \subset \mathbb{P}^8$ (see also the 20th cases in Tables 1 and 2).

2. New descriptions of families of GM fourfolds

To construct surfaces $S$ embedded in a del Pezzo fivefold $Y \subset \mathbb{P}^8$ we proceed as described in Subsection 1.7, that is, we take the images of suitable surfaces $V \subset \mathbb{P}^5$ intersecting the base locus $B$ of the map along suitable curves $C$ via one of the two birational maps $\psi_B: \mathbb{P}^5 \dashrightarrow Y \subset \mathbb{P}^8$ described in Subsection 1.6.

More specifically, if we take the triple $(B,V,C)$ as in one of the 21 cases listed in Table 2, then the image $S = \psi_B(V) \subset \mathbb{P}^8$ of the surface $V$ via the map $\psi_B: \mathbb{P}^5 \dashrightarrow Y \subset \mathbb{P}^8$ is a smooth irreducible surface cut out scheme-theoretically by hypersurfaces of degree at most two, contained in smooth quadratic sections of $Y$, satisfying (1.5) and (1.7), and having all the invariants as reported in Table 1.

Everything can be verified from the explicit equations of the reducible scheme $B \cup V \subset \mathbb{P}^5$ using softwares as Macaulay2 [8] and Singular [5]. We give some more details in Section 3, by introducing a computational package [28] that can also produce random reducible schemes $B \cup V$ of the indicated type. As our results heavily depend on the existence of examples, we also provide selected examples, one for each case, on which we have verified all our claims.

Taking into account these examples and the facts mentioned in Section 1, we can now formulate the main result of this paper.

**Theorem 2.1.** Table 1 contains explicit descriptions for the first 9 components

\[ (2.1) \quad p^{-1}(D_{10} \cup D'_{10} \cup D_{12} \cup D_{16} \cup D'_{18} \cup D_{20} \cup D_{24} \cup D'_{26}) \]

of the Noether–Lefschetz locus in the moduli space of GM fourfolds.

We put in evidence the following immediate consequences.

**Corollary 2.2.** Each irreducible component of (2.1) can be described as the closure inside $M_4^{GM}$ of the family of fourfolds containing some surface of a family of smooth rational surfaces in a del Pezzo fivefold.

Since the general member $[S]$ of each of our families of surfaces in a del Pezzo fivefold $Y$ satisfies $h^0(I_S/Y(2)) > 1$, we deduce that the general fiber of the left projection in (1.6) is a positive-dimensional projective space. In particular, we have the following:

**Corollary 2.3.** Each irreducible component of (2.1) is uniruled, and hence of negative Kodaira dimension. (The 1st, 2nd, 3rd, and 7th are unirational, see Ex. 1.1 and 1.2.)
Remark 2.4. O. Debarre and A. Kuznetsov pointed out to the author that the divisors \( p^{-1}(\mathcal{D}'_d) \) and \( p^{-1}(\mathcal{D}''_d) \), with \( d \equiv 2 \pmod{8} \), are birationally isomorphic. This is because the moduli space \( \mathcal{M}_4^{GM} \) is a fiber space over the moduli space \( LG^0 \) of Lagrangians \( A \subset \wedge^3 V_6 \) with no decomposable vectors, with fiber over a point \([A]\) a dense open subset of the quotient of the EPW sextic \( Y^+_A \subset \mathbb{P}(V_6^\vee) \) by its automorphism group ([4, Corollary 6.3] and [2, Proposition 6.9]). One checks, using the many available results on automorphisms of double EPW sextics, that at a general point \([A]\) of any divisor \( D \subset LG^0 \) this automorphism group is trivial. Consider the duality involution \( \sigma: A \mapsto A^\perp \) on \( LG^0 \). Then, the EPW sextic hypersurfaces \( Y_A \) and \( Y_A^\perp \) are projectively dual (hence birationally isomorphic) by [2, Proposition 6.3], hence the divisors \( p^{-1}(D) \) and \( p^{-1}(\sigma(D)) \) of \( \mathcal{M}_4^{GM} \) are birationally isomorphic. This implies what we need since \( \mathcal{D}_d'' = \sigma(\mathcal{D}_d') \).

From the previous remark and Corollary 2.3, we immediately deduce the following:

**Corollary 2.5.** The divisor \( p^{-1}(\mathcal{D}''_{26}) \) is uniruled, and hence of negative Kodaira dimension.

Remark 2.6. All our descriptions of loci in \( \mathcal{M}_4^{GM} \), not only those of codimension one, are explicit in the sense that each of them is given as the closure of the family of fourfolds \( X \subset Y \) containing some smooth surface \( S \subset Y \) which varies on a specific irreducible component \( S \) of the Hilbert scheme of subschemes of a del Pezzo fivefold \( Y \). The existence of the family \( S \) is deduced from the existence of a single example \([S]\) \( \in S \), which can be constructed as indicated in Table 2. Nevertheless, by counting parameters, we see that our constructions yield only special examples that cannot describe a dense Zariski-open set of \( S \). This can be deduced from the last column of Table 2. For instance, in the 17th case, the family \( S \) of surfaces \( S \subset Y \) of degree 9 and sectional genus 2 has dimension 25. But the family of subschemes of \( \mathbb{P}^5 \) which are unions of a del Pezzo surface \( V \) and a cubic scroll \( B \) intersecting along a twisted cubic curve \( C \) has dimension 44. So, by sending the surfaces \( V \) via the maps \( \mathbb{P}^5 \rightarrow \rightarrow Y \) defined by the quadrics through \( B \), we get a family of surfaces \( S \subset Y \) of dimension just \( 44 - \dim(\text{Aut}(\mathbb{P}^5)) + \dim(\text{Aut}(Y)) = 44 - 35 + 15 = 24 \).

Remark 2.7. We are aware of several other examples besides those in Table 1, which can be constructed using the same methods. However, we have found no descriptions for the whole divisor \( p^{-1}(\mathcal{D}''_{26}) \) and \( p^{-1}(\mathcal{D}_d) \) with \( d > 26 \). See [27] for some sub-loci of \( p^{-1}(\mathcal{D}''_{26}) \).

### 2.1. Alternative constructions of surfaces in del Pezzo fivefolds.

Some of our examples of smooth surfaces \( S \subset Y = G(1, 4) \cap \mathbb{P}^8 \subset G(1, 4) \) admit an easy interpretation in terms of families of lines. We provide two of these examples below.

**Example 2.8.** Let \( X \subset \mathbb{P}^4 \) be a smooth cubic hypersurface. It is classically well-known that \( X \) is unirational. Indeed, let \( L \subset X \) be a general line, and consider the family \( \mathcal{F}_L \subset G(1, 4) \) of lines in \( \mathbb{P}^4 \) which are tangent to \( X \) at some point of \( L \). Then we get a \( 2:1 \) dominant rational map \( \mathcal{F}_L \rightarrow X \) by sending a line of \( \mathcal{F}_L \) to its residual intersection point with \( X \). It follows that \( X \) is unirational from the fact that \( \mathcal{F}_L \) is rational. Actually, \( \mathcal{F}_L \) is a cone of vertex \([L]\) over a smooth rational normal quartic scroll. By taking smooth hyperplane sections of \( \mathcal{F}_L \), we obtain surfaces in a del Pezzo fivefold as in case 2 of Table 1.
Example 2.9. As shown in [18], there are only four types of smooth threefolds in $\mathbb{P}^5$ that are scrolls over a surface. One of them is a scroll of degree 9 over a K3 surface of genus 8. Taking a general projection of such a scroll, we get a hypersurface of degree 9 in $\mathbb{P}^4$. The lines contained in this hypersurface are parameterized by the points of a smooth minimal K3 surface of genus 8 in $\mathbb{G}(1, 4) \cap \mathbb{P}^8 \subset \mathbb{G}(1, 4)$, which gives an example as in case 3 of Table 1 (see also [22]).

2.2. An application to cubic fourfolds. Recall that in the 20-dimensional moduli space $C$ of cubic fourfolds, the Noether–Lefschetz locus consists of a countable infinite union of irreducible divisors $C_d$, where the integer $d$, called the discriminant of the cubic fourfold, runs over all integers $d \geq 8$ with $d \equiv 0$ or 2 (mod 6); see [10, 11, 12] for details.

If a cubic fourfold $[X] \in C$ contains an irreducible surface $S$ of degree $\deg(S)$ and sectional genus $g(S)$, which has smooth normalization and only a finite number $\delta$ of nodes as singularities, then it is well-known that $[X] \in C_d$, where

$$d = 3(S)^2_X - \deg(S)^2,$$

and $(S)^2_X$, the self-intersection of $S$ on $X$, is given by

$$ (S)^2_X = 3 \deg(S) + 6 g(S) - 12 \chi(O_S) + 2 K^2_S + 2 \delta - 6. $$

Explicit geometric descriptions of the divisors $C_d$ are known only up to $d = 48$: the cases $d \leq 44$ with $d \neq 42$ are provided in [17]; the case $d = 42$ in [15, 7, 21]; and the case $d = 48$ in [21]. Moreover, as far as the author knows, there are no explicit examples of cubic fourfolds known to have discriminant $d > 48$.

We point out that the methods previously used can yield such examples. We just construct good surfaces in a del Pezzo fivefold $Y \subset \mathbb{P}^8$, and then we project them onto $\mathbb{P}^5$ from a general plane $P \subset Y$ of one of the two types of planes in $Y$. This idea has already been applied in [21] to get an explicit description of $C_{42}$ and deduce the rationality of its general member. Below is an example among many others that can be constructed in a similar way.

Example 2.10 (Some cubic fourfolds of discriminant 60). By taking the image of a Veronese surface $V \subset \mathbb{P}^5$ intersecting a cubic scroll $B$ along a conic $C$ via the birational map $\psi_B : \mathbb{P}^5 \dashrightarrow Y \subset \mathbb{P}^8$ defined by the quadrics through $B$, we obtain a surface $S \subset Y \subset \mathbb{P}^8$ which is a nodal projection of the 3-uple Veronese embedding $\nu_3(\mathbb{P}^2) \subset \mathbb{P}^9$. The projection of $S$ from a general $p$-plane yields a surface in $\mathbb{P}^5$ of degree 9 and sectional genus 1, cut out by 8 cubics, having 7 nodes as the only singularities, and $\nu_3(\mathbb{P}^2)$ as its normalization. A general cubic fourfold through such a surface is smooth, and formulas (2.2) and (2.3) tell us that its discriminant equals 60. A count of parameters shows that the family of these fourfolds gives a locus of codimension 3 in $C_{60}$.

3. Computations: the Macaulay2 package SpecialFanoFourfolds

In this section, we briefly illustrate some features of the Macaulay2 software package SpecialFanoFourfolds [28], with which the data in Table 1 can be verified. We refer to its documentation for further details.

This package provides two classes of objects, named SpecialCubicFourfold and SpecialGushelMukaiFourfold, which represent pairs $(S, X)$, with $S$ a surface contained in a fourfold $X$; in the former case $X$ is a cubic fourfold, and in the latter case it is a GM
fourfold. An object can be created by passing the equations of $S$ and $X$ (or just of $S$ if we want a random $X$ containing $S$). We now mention some of the functions available, which all take as input just such an object.

- The function `discriminant`, as the name suggests, calculates the discriminant of the fourfold. It applies the formulas (1.2), (1.3), (2.2), (2.3). The Euler-Poincaré characteristic of the surface, and hence the value of $K^2_S$, is calculated using tools from the packages `Cremona` [24] and `CharacteristicClasses` [14]. In the case of GM fourfolds, the class $a_3 + b_2$ of the surface is also obtained automatically through the calculation of an embedding into the Grassmannian $G(1, 4) \subset \mathbb{P}^9$, and then applying some tools from the package `Resultants` [25].

- The function `parameterCount` automates the count of parameters explained in Subsection 1.5, checking in particular (1.5) and (1.7). It gives the needed information on the cohomology of the normal bundles. We refer to [6, 29] for details on computations with sheaves and sheaf cohomology.

- The function `detectCongruence` may find eventual "congruences" of the surface, which were introduced in [20] (see also [21]). In the case of a special GM fourfold $(S, X)$ contained in a del Pezzo fivefold $Y$, the function counts for each $e \geq 1$ the number of curves of degree $e$ passing through a general point of $Y$ and that are contained in $Y$ and $(2e - 1)$-secants to the surface $S \subset Y$. If this number is 1 for some $e$, then a function is returned, which takes as input a point $p \in Y$ and returns the unique $(2e - 1)$-secant curve $C \subset Y$ of degree $e$ passing through $p$.

- Furthermore, after loading the package, the tools provided by the package `MultiprojectiveVarieties` [26] are also available. In particular, we have two functions that help to construct triples $(B, V, C)$ as the ones considered in Table 2 (see also Subsection 1.7):
  - the function `parametrize` tries to get a rational parametrization of a projective variety (in particular, it works with rational normal curves);
  - the function `===`, as an application of the function `parametrize`, tries to find a linear isomorphism between two projective varieties (in particular, it works with rational normal curves of the same degree). This function is also used internally (see the function `toGrass`) to calculate the embedding of a del Pezzo fivefold into $G(1, 4) \subset \mathbb{P}^9$.

We now show how to apply these functions in a specific example. For the convenience of the user, the package also includes a function called `GMtables`, which takes as input an integer $i$ between 1 and 21 together with a coefficient ring $K$, and it returns a random triple $(B, V, C)$ defined over $\mathbb{P}^5_K$ corresponding to the $i$-th case of Table 2. In the following code, we choose $i = 18$ and $K = \mathbb{Z}/65521$. (See also [23, 17] for the philosophy behind these calculations over finite fields, although we work here over a finite field only for speed reasons.)

```
$ M2 --no-preload
Macaulay2, version 1.18
i1 : needsPackage "SpecialFanoFourfolds";
i2 : (B,V,C) = GMtables(18,ZZ/65521);
```

Now we take the birational map $\psi : \mathbb{P}^5 \dashrightarrow Y \subset \mathbb{P}^8$ defined by the cubics through $B$, and then we construct a random GM fourfold $X \subset Y$ containing the image $S = \psi(V) \subset Y$ of the surface $V$ (the lack of error messages means that $X$ is smooth).
i3 : psi = rationalMap(ideal B,Dominant=>2);
o3 = RationalMap (cubic rational map from PP^5 to 5-dimensional subvariety of PP^8)
i4 : S = psi(ideal V);
i5 : X = specialGushelMukaiFourfold S;
o5 : ProjectiveVariety, GM fourfold containing a surface of degree 7 and sectional genus 0
The discriminant of X can be obtained with one of the two commands:
i6 : discriminant X
o6 = 20
i7 : describe X
o7 = Special Gushel-Mukai fourfold of discriminant 20
containing a surface in PP^8 of degree 7 and sectional genus 0
and with class in G(1,4) given by 4*s_(3,1)+3*s_(2,2)
The following code tells us that S corresponds to a smooth point of an irreducible family
S of dimension 21 of surfaces in Y, and that the family of GM fourfolds containing some
surface of S has codimension 3 in the space of all GM fourfolds (we omit a few lines of
output).
i8 : parameterCount X
-- h^1(N_{S,Y}) = 0
-- h^0(N_{S,Y}) = 21
-- h^1(0_{S(2)}) = 0 and h^0(I_{S,Y}(2)) = 16 = h^0(0_{Y}(2)) - \chi(0_{S(2)})
-- h^0(N_{S,X}) = 0
o8 = (3,(16,21,0))
Finally, the following code reveals a congruence of 3-secant conics of the surface S inside
Y. Indeed, the rational map \phi : Y \dashrightarrow \mathbb{P}^{15} defined by the quadrics through S is
birational onto a fivefold Z \subset \mathbb{P}^{15}, and through the general point \phi(p) of Z there pass 5
lines contained in Z. Of these 5 lines, only one comes from a 3-secant conic to S passing
through p. Note that this implies the rationality of X, a fact already known for the GM
fourfolds of discriminant 20 (see [13]).
i9 : detectCongruence X
-- phi: quadratic rational map from Y to PP^{15}, Z = phi(Y)
-- number lines contained in Z and passing through the point phi(p): 5
-- number 1-secant lines to S passing through p: 4
-- number 3-secant conics to S passing through p: 1
| Surface                                                                 | $K^3$ | Class in $\mathcal{M}_4^{GM}$ | Codim in $\mathcal{D}$ | Image in $\mathcal{D}$ | $h^0(I_{S/Y})$ | $h^0(N_{S/Y})$ | $h^0(N_{S/X})$ | Curves of degree $e$ in $Y$ passing through a general point of $Y$ and that are $(2e-1)$-secant to $S$ for $e = 1, 2, 3$ |
|------------------------------------------------------------------------|-------|--------------------------------|------------------------|------------------------|----------------|----------------|----------------|------------------------------------------------------|
| $\tau$-quadric surface [1]                                            | 6     | $\sigma_{3,1} + \sigma_{2,2}$ | 1                      | $P_{10}^\prime$         | 31             | 8              | 0              | 1, 0, 2, 3                                          |
| Quartic scroll                                                         | 8     | $3\sigma_{3,1} + \sigma_{2,2}$ | 1                      | $P_{10}^\prime$         | 25             | 14             | 0              | 3, 0, 2, 3                                          |
| K3 surface of degree 14 and genus 8 [22]                              | 0     | $5\sigma_{3,1} + 5\sigma_{2,2}$ | 1                      | $P_{10}^\prime$         | 10             | 39             | 10             | 9, 8, 1, 2, 3                                       |
| Quintic del Pezzo surface [19]                                        | 5     | $3\sigma_{3,1} + 2\sigma_{2,2}$ | 1                      | $P_{10}^\prime$         | 24             | 18             | 3              | 3, 0, 2, 3                                          |
| Rational surface of degree 9 and genus 3 [19]                         | 1     | $5\sigma_{3,1} + 4\sigma_{2,2}$ | 1                      | $P_{10}^\prime$         | 16             | 28             | 5              | 5, 0, 2, 3                                          |
| $\sigma$-plane [19]                                                   | 9     | $\sigma_{3,1}$                 | 2                      | $P_{10}^\prime$         | 34             | 4              | 0              | 1, 0, 2, 3                                          |
| Cubic scroll [1]                                                      | 8     | $2\sigma_{3,1} + \sigma_{2,2}$ | 1                      | $D_{12}$                | 28             | 11             | 0              | 2, 0, 2, 3                                          |
| Rational surface of degree 7 and genus 2 [19]                         | 3     | $4\sigma_{3,1} + 3\sigma_{2,2}$ | 1                      | $D_{12}$                | 20             | 23             | 4              | 4, 0, 2, 3                                          |
| $\rho$-plane [19]                                                     | 9     | $\sigma_{2,2}$                 | 3                      | $D_{12}$                | 34             | 3              | 0              | 0, 0, 2, 3                                          |
| Rational surface of degree 10 and genus 4 [3]                         | 0     | $6\sigma_{3,1} + 4\sigma_{2,2}$ | 1                      | $D_{12}$                | 15             | 29             | 5              | 6, 0, 2, 3                                          |
| Rational surface of degree 10 and genus 3                             | 2     | $6\sigma_{3,1} + 4\sigma_{2,2}$ | 1                      | $D_{12}$                | 13             | 29             | 3              | 6, 0, 2, 3                                          |
| K3 surface of degree 14 and genus 8                                    | 0     | $8\sigma_{3,1} + 6\sigma_{2,2}$ | 1                      | $D_{12}$                | 10             | 38             | 9              | 8, 0, 2, 3                                          |
| Rational surface of degree 12 and genus 5 [3]                         | $-1$  | $7\sigma_{3,1} + 5\sigma_{2,2}$ | 1                      | $D_{12}$                | 11             | 32             | 4              | 7, 0, 2, 3                                          |
| Rational surface of degree 8 and genus 2 [4]                          | 4     | $5\sigma_{3,1} + 3\sigma_{2,2}$ | $\leq 2$               | $D_{12}$                | 17             | 24             | 3              | 5, 0, 2, 3                                          |
| Rational surface of degree 9 and genus 3 [3]                          | 2     | $5\sigma_{3,1} + 4\sigma_{2,2}$ | 1                      | $D_{12}$                | 16             | 26             | 3              | 5, 0, 2, 3                                          |
| Rational surface of degree 11 and genus 5 [13]                        | $-1$  | $7\sigma_{3,1} + 4\sigma_{2,2}$ | 1                      | $D_{12}$                | 14             | 30             | 5              | 7, 0, 2, 3                                          |
| Rational surface of degree 9 and genus 2 [13]                         | 5     | $8\sigma_{3,1} + 3\sigma_{2,2}$ | $\leq 2$               | (really 1) $D_{20}$     | 14             | 25             | 1              | (but 0 when $S$ is general) 6, 1, 0                   |
| Septic scroll                                                         | 8     | $4\sigma_{3,1} + 3\sigma_{2,2}$ | 3                      | $D_{20}$                | 16             | 21             | 0              | 4, 1, 2, 3                                          |
| Rational surface of degree 10 and genus 3                             | 3     | $6\sigma_{3,1} + 4\sigma_{2,2}$ | 1                      | $D_{24}$                | 13             | 27             | 1              | 6, 0, 2, 3                                          |
| Veronese surface                                                      | 9     | $2\sigma_{3,1} + 2\sigma_{2,2}$ | 4                      | $D_{24}$                | 25             | 11             | 0              | 2, 0, 2, 3                                          |
| Rational surface of degree 12 and genus 5                             | 0     | $7\sigma_{3,1} + 5\sigma_{2,2}$ | 1                      | $D_{26}$                | 11             | 30             | 2              | 7, 0, 2, 3                                          |

Table 1. Families of GM fourfolds described as the closure of the locus of smooth quadric hypersurfaces in a del Pezzo fivefold $Y$ containing some smooth surface $S \subset Y$ varying in an irreducible component of $\text{Hilb}_{Y}^{\chi(O_{S}(t))}$. 
| $B \subset \mathbb{P}^5$ | $V \subset \mathbb{P}^5$ | $C \subset B \cap V$ | $B \cap V / C V \cong \psi_B(V)$ | $\dim \{C : B \supset C\}$ | $\dim \{V : V \supset C\}$ | $\dim \{(C, B, V) : C \subset B \cap V\}$ |
|----------------|----------------|----------------|-----------------|-----------------|-----------------|-----------------|
| 1 cubic scroll | quadric surface | irreducible conic curve | $\emptyset$ | yes | 11 | 6 | 31 |
| 2 cubic scroll | quadric surface | line of the ruling | $\emptyset$ | yes | 16 | 10 | 34 |
| 3 cubic scroll | $K_3$ surface of degree 8 and genus 5 | union of 3 lines of the ruling with the directrix line | $\emptyset$ | yes | 3 | 28 | 57 |
| 4 cubic scroll | del Pezzo surface of degree 4 | twisted cubic curve | $\emptyset$ | no | 7 | 13 | 40 |
| 5 cubic scroll threefold | cubic scroll surface | irreducible conic curve | 7 points | no | 22 | 11 | 47 |
| 6 cubic scroll | plane | line of the ruling | $\emptyset$ | yes | 16 | 3 | 27 |
| 7 cubic scroll | cubic scroll surface | twisted cubic curve | $\emptyset$ | yes | 7 | 7 | 34 |
| 8 quartic scroll | quadric surface | line | 5 points | no | 25 | 10 | 43 |
| 9 cubic scroll | plane | directrix line | $\emptyset$ | yes | 15 | 3 | 26 |
| 10 quartic scroll threefold | del Pezzo surface of degree 4 | rational normal quartic curve | 4 points | no | 14 | 8 | 48 |
| 11 quartic scroll threefold | quartic scroll surface | rational normal quartic curve | 6 points | no | 14 | 8 | 48 |
| 12 cubic scroll | $K_3$ surface of degree 8 and genus 5 | rational normal quartic curve | $\emptyset$ | yes | 2 | 28 | 56 |
| 13 cubic scroll | rational surface of degree 7 and genus 3, the image of the plane via the linear system of quintic curves with nine simple base points and one double point | rational normal quartic curve, the image of a general conic passing through three of the nine base points and through the triple point | $\emptyset$ | yes | 2 | 22 | 50 |
| 14 cubic scroll | del Pezzo surface of degree 4 | irreducible conic curve | $\emptyset$ | yes | 11 | 18 | 43 |
| 15 cubic scroll | rational surface of degree 6 and genus 2, the image of the plane via the linear system of quartic curves with six simple base points and one double point | rational normal quartic curve, the image of a general conic passing through two of the six base points and through the double point | $\emptyset$ | yes | 2 | 17 | 45 |
| 16 cubic scroll | rational surface of degree 6 and genus 3, the image of the plane via the linear system of quartic curves with ten simple base points | twisted cubic curve, the image of a general conic passing through five of the ten base points | $\emptyset$ | yes | 7 | 21 | 48 |
| 17 cubic scroll | del Pezzo surface of degree 5 | twisted cubic curve | $\emptyset$ | yes | 7 | 17 | 44 |
| 18 quartic scroll threefold | quartic scroll surface | rational normal quartic curve | 3 points | no | 11 | 2 | 45 |
| 19 quartic scroll threefold | Veronese surface | rational normal quartic curve | 6 points | no | 14 | 6 | 46 |
| 20 cubic scroll | Veronese surface | rational normal quartic curve | $\emptyset$ | yes | 2 | 6 | 34 |
| 21 cubic scroll | rational surface of degree 7 and genus 3, the image of the plane via the linear system of quartic curves with nine simple base points | rational normal quartic curve, the image of a general conic passing through four of the nine base points | $\emptyset$ | yes | 2 | 20 | 48 |

Table 2. Construction of surfaces $S \subset Y$ as in Table 1, obtained as images of surfaces $V \subset \mathbb{P}^5$ via the map $\psi_B : \mathbb{P}^5 \rightarrow Y$ associated to $B$. 
Table 3. Run-times to execute the function parameterCount in a code like the one in Section 3, for all the examples in Tables 1 and 2, on a laptop with an Intel Core i7-6500U processor, 16 GB of RAM.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 14s | 7m | 18m | 12s | 9.4h | 25s | 16s | 5h | 2s | 2h | 2.5h | 11.2h | 5.5h | 2h | 20m | 37m | 10m | 7m | 17m | 8m | 30m |

References

[1] O. Debarre, A. Iliev, and L. Manivel, *Special prime Fano fourfolds of degree 10 and index 2*, Recent Advances in Algebraic Geometry: A Volume in Honor of Rob Lazarsfeld’s 60th Birthday (C. Hacon, M. Mustaţă, and M. Popa, eds.), London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 2015, pp. 123–155.

[2] O. Debarre and A. Kuznetsov, *Gushel–Mukai varieties: Classification and birationalities*, Algebr. Geom. 5 (2018), 15–76.

[3] , *Gushel–Mukai varieties: Linear spaces and periods*, Kyoto J. Math. 59 (2019), no. 4, 897–953.

[4] , *Gushel–Mukai varieties: moduli*, Internat. J. Math. 31 (2020), no. 2.

[5] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönenmann, *SINGULAR — A computer algebra system for polynomial computations (version 4-1-2)*, home page: http://www.singular.uni-kl.de, 2018.

[6] D. Eisenbud, D. R. Grayson, M. Stillman, and B. Sturmfels (eds.), *Computations in algebraic geometry with Macaulay 2*, Algorithms Comput. Math, vol. 8, Springer Berlin Heidelberg, 2002.

[7] G. Farkas and A. Verra, *The unirationality of the moduli space of K3 surfaces of genus 22*, Math. Ann. 380 (2021), 953–973.

[8] D. R. Grayson and M. E. Stillman, *MACAULAY2 — A software system for research in algebraic geometry (version 1.18)*, home page: http://www.math.uiuc.edu/Macaulay2/, 2021.

[9] N. P. Gushel, *Fano varieties of genus 6* (in russian), Izv. Akad. Nauk USSR Ser. Mat. 46 (1982), no. 6, 1159–1174, English transl.: Math. USSR-Izv. 21 3 (1983), 445–459.

[10] B. Hassett, *Some rational cubic fourfolds*, J. Algebraic Geom. 8 (1999), no. 1, 103–114.

[11] , *Special cubic fourfolds*, Comp. Math. 120 (2000), no. 1, 1–23.

[12] , *Cubic fourfolds, K3 surfaces, and rationality questions*, Rationality Problems in Algebraic Geometry: Levico Terme, Italy 2015 (R. Pardini and G. P. Pirola, eds.), Springer International Publishing, Cham, 2016, pp. 29–66.

[13] M. Hoff and G. Staglianò, *New examples of rational Gushel-Mukai fourfolds*, Math. Z. 296 (2020), 1585–1591.

[14] C. Jost, *Computing characteristic classes and the topological Euler characteristic of complex projective schemes*, J. Softw. Algebra Geom. 7 (2015), no. 1, 31–39.

[15] K. Lai, *New cubic fourfolds with odd-degree unirational parametrizations*, Algebra & Number Theory 11 (2017), 1597–1626.

[16] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Natl. Acad. Sci. USA 86 (1989), no. 9, 3000–3002.

[17] H. Nuer, *Unirationality of moduli spaces of special cubic fourfolds and K3 surfaces*, Algebr. Geom. 4 (2015), 281–289.

[18] G. Ottaviani, *On 3-folds in $\mathbb{P}^5$ which are scrolls*, Ann. Sc. Norm. Super. Pisa 19 (1992), 451–471.

[19] L. Roth, *Algebraic varieties with canonical curve sections*, Ann. Mat. Pura Appl. 29 (1949), no. 1, 91–97.

[20] F. Russo and G. Staglianò, *Congruences of 5-secant conics and the rationality of some admissible cubic fourfolds*, Duke Math. J. 168 (2019), no. 5, 849–865.

[21] , *Trisecant Flops, their associated K3 surfaces and the rationality of some Fano fourfolds*, preprint: https://arxiv.org/abs/1909.01263, 2019.

[22] , *Explicit rationality of some cubic fourfolds*, to appear in the Proceeding of the Schiermonnikoog Conference Rationality of Algebraic Varieties, 2020.
[23] F.-O. Schreyer, *Computer aided unirationality proofs of moduli spaces*, Handbook of Moduli, Vol. III, Adv. Lect. Math., 26 (G. Farkas and I. Morrison, eds.), Int. Press, Somerville, 2013, pp. 257–280.

[24] G. Staglianò, *A Macaulay2 package for computations with rational maps*, J. Softw. Alg. Geom. 8 (2018), no. 1, 61–70.

[25] G. Staglianò, *A package for computations with classical resultants*, J. Softw. Alg. Geom. 8 (2018), no. 1, 21–30.

[26] G. Staglianò, *Computations with rational maps between multi-projective varieties*, to appear in J. Softw. Alg. Geom. (2021).

[27] G. Staglianò, *Some new rational Gushel fourfolds*, Mediterr. J. Math. 18 (2021), no. 5.

[28] G. Staglianò, *SpecialFanoFourfolds: a Macaulay2 package for working with special cubic fourfolds and special Gushel-Mukai fourfolds, version 2.3*, source code and documentation available at https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2/share/doc/Macaulay2/SpecialFanoFourfolds/html/index.html, 2021.

[29] M. Stillman, *Computing with sheaves and sheaf cohomology in algebraic geometry: preliminary version*, available at https://www.math.arizona.edu/~swc/aws/2006/06StillmanNotes.pdf, 2006.

[30] S. Tanimoto and A. Várilly-Alvarado, *Kodaira dimension of moduli of special cubic fourfolds*, J. Reine Angew. Math. 752 (2019), 265–300.

Dipartimento di Matematica e Informatica, Università degli Studi di Catania

Email address: giovannistagliano@gmail.com