A SYSTEMATIC STUDY OF THE FINAL MASSES OF GAS GIANT PLANETS

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ABSTRACT

We construct an analytic model for the gas accretion rate onto planets embedded in protoplanetary disks as a function of planetary mass, viscosity, scale height, and unperturbed surface density, and systematically study the long-term accretion and final masses of gas giant planets. We first derive an analytical formula for the surface density profile near the planetary orbit from considerations of the balance of force and dynamical stability. Using it in the empirical formula of normalized gas accretion rate that is derived based on hydrodynamic simulations, we then simulate the mass evolution of gas giant planets in viscously evolving disks. We finally determine the final mass as a function of semimajor axis of the planet. We find that the disk can be divided into three regions characterized by different processes by which the final mass is determined. In the inner region, the planet grows quickly and forms a deep gap to suppress the growth by itself before disk dissipation. The final mass shows the same trend as the mass determined by the viscous condition for gap opening, but is about 10 times larger than that. In the intermediate region, the disk’s viscous diffusion limits gas accretion onto planets before deep gap formation. The final mass can be up to the disk mass, when the disk’s viscous evolution occurs faster than disk evaporation. In the outer region, planets capture only tiny amounts of gas within the disk lifetime to form Neptune-like planets. We also derive analytic formulae for the final masses in the different regions and the locations of the boundaries, which are helpful to gain a systematic understanding of the masses of gas giant planets.

Subject headings: methods: analytical — methods: numerical — solar system: formation

1. INTRODUCTION

A fundamental but unresolved issue with planet formation is how the mass of a giant planet is fixed. In the solar system there are four giant planets, which are characterized by their massive hydrogen/helium envelopes. Jupiter and Saturn consist mostly of hydrogen/helium, while Uranus and Neptune consist mostly of ice but have significant amounts of hydrogen/helium. The giant planets are different in mass: Jupiter’s mass ($M_{\text{J}}$) is $1 \times 10^{-3} M_\odot$, Saturn’s mass is $\sim 0.3 M_{\text{J}}$, and Uranus’s and Neptune’s masses are $\sim 0.05 M_{\text{J}}$. Furthermore, the extrasolar planets detected so far also range in mass from about one Neptune mass to $\sim 10 M_{\text{J}}$; those planets are believed to have massive hydrogen/helium envelopes like the giant planets in the solar system.

There are two competing ideas on the formation of giant planets, the core accretion model (e.g., Mizuno 1980; Bodenheimer & Pollack 1986) and the disk instability model (e.g., Cameron 1978; Boss 1989). In the core accretion model, a rocky/icy core first forms through collisional aggregation of planetesimals, followed by envelope formation due to substantial accretion of gas from the circumstellar (protoplanetary) disk. In the disk instability model, density fluctuation of the disk gas grows to form a gaseous planet, followed by core formation due to sedimentation of heavy elements in its interior. The advantages and disadvantages of both models are discussed in the literature (e.g., Boss 2002) and are not repeated here. In this paper we consider giant planet formation in the context of the core accretion model.

In the core accretion model, the process of the accumulation of the envelopes is divided into two phases, the “subcritical accretion” and “supercritical accretion” phases, in terms of the dominant energy source. The transition from the former to the latter occurs when the mass of a core reaches a critical value. In the subcritical accretion phase, incoming planetesimals supply energy to the envelope so that the envelope is in hydrostatic equilibrium. The core accretion thus controls the gas accretion in this phase. Phase 2 found by Pollack et al. (1996) is a part of the subcritical accretion phase. In the supercritical accretion phase, the energy supplied by planetesimals is insufficient to keep the hydrostatic structure of the envelope, so the envelope substantially contracts and releases its gravitational energy, resulting in runaway accretion of the disk gas.

The supercritical accretion phase can be further divided into two subphases. In the former subphase, the gas accretion is controlled by contraction of the envelope and thus occurs on the Kelvin-Helmholtz timescale (Bodenheimer & Pollack 1986; Pollack et al. 1996; Ikoma et al. 2000; Ikoma & Genda 2006). However, because the gas accretion driven by the envelope contraction accelerates rapidly with time, the supply of the disk gas inevitably becomes unable to keep up with the demand of the contracting envelope. Thus, in the latter subphase, the disk gas supply limits the gas accretion (Tanigawa & Watanabe 2002, hereafter TW02).

In this paper we focus on the gas accretion in the latter subphase of the supercritical accretion phase, i.e., the phase in which the Kelvin-Helmholtz contraction of the envelope is under way but the gas accretion is limited by disk-gas supply. After the onset of the runaway gas accretion (i.e., in the supercritical accretion phase), the planet always experiences the limited gas supply (i.e., the latter subphase) by the end of its growth. Although there is an idea that the masses of Uranus and Neptune were fixed in the subcritical accretion phase because of dissipation of the disk gas (Pollack et al. 1996), we explore another possibility that the masses of those planets were fixed in the supercritical accretion phase.

One idea is that the mass of a giant planet is completely fixed when the planet opens a gap in the disk. From a physical point of

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1 See http://exoplanet.eu.
view, the gap opens when the gravitational scattering by the planet overpowers both viscous diffusion (the viscous condition) and push due to the pressure gradient (the thermal condition; e.g., Lin & Papaloizou 1993). In normal protoplanetary disks, the final mass of a giant planet is determined by the thermal condition (e.g., Ida & Lin 2004), which is equivalent to the condition that the Hill radius is equal to the disk scale height at the planet’s location. The final mass is thus determined only by temperature of the disk gas for given stellar mass and planet’s location. In protoplanetary disks similar to the minimum-mass solar nebula (Hayashi 1981), the thermal condition yields a final mass of approximately 1 M_\text{Jupiter} at 5 AU but predicts that the masses of giant planets increase with the semimajor axis, which is in clear contradiction to the current configuration of the solar system. Furthermore, because in reality the gap is not a vacuum but a low-density region, continuous gas accretion through the gap takes place, as suggested by several studies (e.g., Artymowicz & Lubow 1996). The effects of the subsequent accretion on the final masses of giant planets have been poorly understood.

Another idea is that gas accretion is truncated by depletion of the disk gas before gap opening (e.g., viscous dissipation and photoevaporation of the disk gas). This means that the final mass of a giant planet is the total mass of the disk gas that the planet has captured until the disk gas dissipates completely. This may be responsible for the current configuration of the solar system, since it takes more time for outer giant planets to accrete principally because of their long orbital periods. However, there have been no quantitative studies of this possibility.

The purpose of this paper is to gain a systematic understanding of how the final mass of a giant planet is determined. Specifically, we clarify (1) how much the mass of an accreting giant planet increases via slow, continuous accretion through the gap, (2) when and where disk dissipation dominates gap opening, and finally (3) how the final masses of giant planets depend on several disk parameters. To do so, we simulate the long-term accretion of a giant planet in a self-consistent fashion. The essential ingredients of the modeling are global viscous evolution of the disk and the flow pattern of the accreting gas, as described below; the latter effect in particular was completely neglected by previous studies. Furthermore, we derive analytical formulas for the final masses of giant planets and the boundaries between several different regimes.

During the viscous evolution of the disk, gas radially migrates from outer regions to the site of giant planet formation. The mass flux limits gas accretion onto giant planets in some situations. Because of the limited CPU speed of computers, azimuthally averaged one-dimensional simulations of the disk evolution are necessary in order to follow the long-term accretion. Several previous studies simulated the radial mass flux in viscously evolving disks and then calculated gas accretion rate onto giant planets. Lecar & Sasselov (2003) and Guillot & Hueso (2006) also simply assumed that gas accretion rate onto giant planets is always proportional to the viscous mass flux. Veras & Armitage (2004) and Alibert et al. (2004) used an approximate formula for gas accretion rate that fits the results of two-dimensional hydrodynamic simulations of disk gas with an embedded planet done by Lubow et al. (1999) and D’Angelo et al. (2002). However, the formula was derived based on a limited number of numerical simulations. Hence, whether the formula applies in a variety of situations is in question.

To understand the detailed pattern of accretion flow and to obtain the gas accretion rate onto the planet, several authors (e.g., Kley 1999; Lubow et al. 1999; D’Angelo et al. 2002, 2003; Tate et al. 2003) performed two- or three-dimensional hydrodynamic simulations of interaction between a planet and disk gas. Since they simulated global disks, the effect of gap formation on accretion rate was automatically included. However, because of the time-consuming simulations, they did not simulate the growth of giant planets on timescales of the viscous disk evolution.

This study is clearly different from the previous studies in that this study includes the effects of the flow pattern of the accreting gas in simulating long-term growth of planets embedded in globally evolving disks. To do so, we use the empirical relation between gas accretion rate and unperturbed surface density that was derived by TW02 (see § 3 for the details). They carried out detailed two-dimensional hydrodynamic simulations of gas accretion flow onto a planet and demonstrated the importance of the local flow pattern on the accretion rate.

In this paper, we first derive an analytical formula for radial surface density distribution of protoplanetary disks with an embedded planet in § 2. Next, in § 3 we describe how to include the effect of the flow pattern of accreting gas, following TW02, to calculate gas accretion rate onto giant planets. In § 4, we simulate the long-term accretion of giant planets, based on the prescription given in §§ 2 and 3, and obtain the final masses of giant planets for wide ranges of several disk parameters. In addition, we derive analytical approximate formulas for the final masses of giant planets and the boundaries between several regimes. Finally, we conclude this paper in § 5.

2. SURFACE DENSITY

We consider a planet embedded in a gas disk, both of which are assumed to rotate with the Keplerian velocity around a star. We derive an approximate formula for azimuthally averaged surface density as a function of radial distance to the central star, r. We adopt a local coordinate system in which all quantities except the surface density are independent of r. We assume steady states and no mass flow toward the central star. We do not consider migration of the planet. The validity of these assumptions is discussed in § 4.4.

We employ a local coordinate system corotating with the planet on a circular Keplerian orbit around the central star (a local shearing-sheet approximation). The origin of the coordinate system is the planet’s position. The x-axis is on the line from the star through the planet; the direction of the y-axis is the same as that of the velocity vector of the planet. The equation of motion of the disk gas in the y-direction in this coordinate system is given by

$$\frac{\partial}{\partial t} (\Sigma v_y) + \frac{\partial}{\partial x} (\Sigma v_x v_y) = \frac{\partial}{\partial x} \left( \nu \frac{\partial v_y}{\partial x} \right) + \Sigma \ddot{v}_y,$$

where \( \Sigma \) is the surface density, \( v_x \) and \( v_y \) are the velocities in the x- and y-directions, \( \nu \) is the viscosity coefficient, and \( \dot{v}_y \) is the force acting on the gas per unit mass. Here we have omitted the advection term in the y-direction because of axisymmetry. We also eliminate all the terms on the left-hand side of this equation on the assumptions of steady states (\( \partial / \partial t = 0 \)) and no mass flow (\( v_x = 0 \)). Hereafter we consider only \( x > 0 \) because of symmetry.

To derive an analytical formula for \( \Sigma(x) \), we assume that \( v_y \) is equal to the Keplerian velocity, \( v_y = -\frac{3}{2} \Omega_p x \), where \( \Omega_p \) is the Keplerian angular velocity at the planet’s position. This assumption is shown to be reasonable below. Equation (1) is thus transformed into

$$-\frac{3}{2} \Omega_p \ln \frac{\Sigma}{\sigma} + \ddot{v}_y = 0.$$

The gravity of the planet perturbs the motion of the disk gas. We follow the impulse prescription (e.g., Lin & Papaloizou 1979)
that approximates scattering of the disk gas as small-angle scattering of a particle by a point-mass object. On the prescription, $i_y$ is given by

$$i_y = \frac{4}{9\pi} \left( \frac{M_p}{M_*} \right)^2 r_p^2 \rho^2 x^{-4}, \quad (3)$$

where $r_p$ is the distance of the planet from the central star and $M_p$ and $M_*$ are the masses of the planet and the central star, respectively. This approximation is inappropriate for $x < \min\{2\rho H, r_p[(M_p/M_*)^{1/2}]\}$, where $2\rho H$ is the Hill radius, because this region is what is called the “horseshoe region,” in which gas is not pushed away from the planet. However, since the formula is unable to apply for wide ranges of the parameters, as they mentioned, we do not use their formula in this study.

Substituting equation (3) into equation (2), we obtain

$$\Sigma(x) = \Sigma_\infty \exp \left[ -\left( \frac{x}{l} \right)^{-3} \right] \equiv \Sigma_{vis}(x), \quad (4)$$

where $\Sigma_\infty$ is the surface density at infinity (i.e., far away from the planet’s orbital radius) and $l$ is defined as

$$l = \frac{\nu}{81\pi} \left( \frac{\rho}{r_p^2 \Omega_p} \right)^{1/3} \left( \frac{M_p}{M_*} \right)^2 r_p \quad (5)$$

$$= 0.146 \left( \frac{\nu}{10^{-5} r_p^2 \Omega_p} \right)^{1/3} \left( \frac{M_p}{10^{-3} M_*} \right)^2 r_p. \quad (6)$$

This surface density profile is basically the same as that derived by Lubow & D’Angelo (2006) except for the value of the coefficient.

The dynamical stability of the surface density profile given by equation (4) must be checked using the well-known Rayleigh criterion (e.g., Chandrasekhar 1961). For the surface density profile to be stable, in the local coordinate system,

$$\frac{\partial v_y}{\partial x} \geq -2\Omega_p \quad (7)$$

must be fulfilled. The velocity $v_y$ can be obtained by solving the equation of motion for the $x$-component,

$$3\Omega_p^2 x + 2\Omega_p v_y - c^2 \frac{\partial \ln \Sigma}{\partial x} = 0, \quad (8)$$

where $c$ is the sound speed. Substituting equation (4) into equation (8), we obtain

$$v_y = -\frac{3}{2} \Omega_p x \left[ 1 - \left( \frac{h}{7} \right)^2 \left( \frac{x}{7} \right)^{-5} \right], \quad (9)$$

where $h$ is the disk scale height defined as $h \equiv c/\Omega_p$. From equations (7) and (9), we find that condition (7) is not fulfilled for $x < x_m$, where

$$x_m = 12^{1/5} \left( \frac{h}{7} \right)^{2/5} l = 12^{1/5} \left( \frac{1}{7} \right)^{3/5} h$$

$$= 0.207 \left( \frac{h}{0.1 r_p} \right)^{2/5} \left( \frac{\nu}{10^{-5} r_p^2 \Omega_p} \right)^{-1/5} \left( \frac{M_p}{10^{-3} M_*} \right)^{2/5} r_p. \quad (10)$$

In the region $x \leq x_m$, the density gradient thus has to be small enough to satisfy the Rayleigh condition: the density profile would be relaxed to be marginally stable for the Rayleigh condition.

Integrating $\partial \Sigma/\partial x = -2\Omega_p$ inward from $x_m$ we obtain

$$v_y = v_{y,m} = -2\Omega_p(x - x_m) \quad (12)$$

in the region $x \leq x_m$. Substituting this into equation (8), we finally obtain

$$\Sigma(x) = \Sigma_\infty \exp \left[ -\frac{1}{2} \left( \frac{x}{h} - \frac{5 x_m}{h} \right)^2 + \frac{1}{32} \left( \frac{x_m}{h} \right)^3 \right] \equiv \Sigma_R, \quad (13)$$

for $x \leq x_m$.

In summary, the equilibrium profile of surface density is given as

$$\Sigma(x) = \begin{cases} \Sigma_R(x) & \text{for } x \leq x_m, \\ \Sigma_{vis}(x) & \text{for } x \geq x_m. \end{cases} \quad (14)$$

An example of the density profile is shown in Figure 1.

As mentioned above, we have assumed $v_y = -\frac{1}{2} \Omega_p x$ in deriving equation (4). However, based on equation (9), the velocity at $x = x_m$ is

$$v_{y,m} = -\frac{11}{8} \Omega_p x_m. \quad (15)$$
This means that the maximum deviation from the Keplerian shear velocity, $-\frac{1}{2} \Omega_p x$, is less than 10% of the absolute value. Hence, the assumption of the Keplerian shear velocity is reasonable for $x \geq x_m$.

Finally, we briefly describe the dependence of the surface density on the parameters. Figures 2a, 2b, and 2c show the surface density as a function of $x/r_p$, for several different values of planetary mass $M_p$, viscosity $\nu$, and scale height $h$, respectively. Figures 2a and 2b show that the gap becomes deeper and wider with increasing planetary mass and decreasing viscosity. The width and depth of the gap are determined by competition between gravitational scattering by the planet and viscous diffusion of the disk gas. The gap becomes wider and deeper when the planet’s gravity is stronger or viscous diffusion is less efficient. From a mathematical point of view, this is because both $\Sigma_{\text{vis}}$ and $\Sigma_{\text{R}}$ depend on $M_p$ and $\nu$ in the form of $(M_p^2/\nu)^{1/3}$ through $l$. As for the dependence of $\Sigma$ on $h$ (see Fig. 2c), all the curves are independent of $h$ for $x \geq x_m$, where $\Sigma = \Sigma_{\text{vis}}$, which is independent of $h$ (see eq. [4]). In other words, surface density in $x \geq x_m$ is determined by the balance between viscous torque and gravitational torque from the planet and has nothing to do with pressure. On the other hand, difference in surface density appears at $x < x_m$, where $\Sigma$ is determined by the Rayleigh condition (see eq. [13]) and thus depends on all of $h$, $\nu$, and $M_p$ through $x_m$. Symbols used in this paper are listed in Table 1.

3. GAS ACCRETION RATE ONTO PLANETS

In this study, we describe the gas accretion rate onto the planet, $\dot{M}_p$. Through a series of local high-resolution hydrodynamic simulations of the gas accretion flow onto giant planets in protoplanetary disks, TW02 obtained an empirical relation between accretion rate normalized by unperturbed surface density and $h/r_1$, which is the only parameter of the local system (see eq. [18] of TW02). Furthermore, TW02 demonstrated that only the gas in the two bands at $|x| \sim 2r_1$ accretes onto the planet. Thus, by writing equation (18) of TW02 in the explicit form, for $0.5 \leq h/r_1 \leq 1.8$, we have

$$\dot{M}_p = \dot{A} \Sigma_{\text{acc}},$$

(16)

where

$$\dot{A} \approx 0.29 \left( \frac{h}{r_p} \right)^{-2} \left( \frac{M_p}{M_\ast} \right)^{4/3} r_p^2 \Omega_p,$$

(17)

which is the area in which the gas is to be accreted onto the planet per unit time (hereafter the accretion area), and $\Sigma_{\text{acc}} = \Sigma(2r_1)$.

The reason why small $h$ or large $M_p$ results in large $\dot{M}_p$ is explained as follows. Disk gas loses its energy by passing through spiral shocks around the planet and consequently accretes onto the planet. Small $h$ (i.e., small sound velocity) or large $M_p$ (i.e., strong perturbation on gas motion) yields a large Mach number of the disk gas, which results in strong shocks. In addition, an increase in $M_p$ expands the planetary feeding zone, resulting in large $\dot{M}_p$.

Note that the position of the accretion band, $x_{\text{acc}}$, is located at $\sim 2r_1$ in the highest mass case of TW02 (i.e., $C_{\text{iso}} = 0.5$ in their notation; see Fig. 7 of TW02, where $x_{\text{acc}}$ is shown to depend slightly on $h/r_1$). When $C_{\text{iso}} < 0.5$, a gap exists around the planetary orbit. In such situations, appropriate choice of $x_{\text{acc}}$ is crucial in determining $\dot{M}_p$ because a small difference in $x_{\text{acc}}$ makes a large difference in $\Sigma_{\text{acc}}$ and thus $\dot{M}_p$, while the choice of $x_{\text{acc}}$ has little

![Fig. 2.—Surface density profiles as a function of distance from the planetary orbit. (a) Planetary mass dependence: $M_p/M_\ast = 10^{-4}$ (solid line), $10^{-3.5}$ (long-dashed line), $10^{-3}$ (short-dashed line), $10^{-2.5}$ (dotted line), and $10^{-2}$ (dotted line). (b) Viscosity parameter dependence: $\nu/(r_p^2 \Omega_p) = 10^{-1}$ (solid line), $10^{-2}$ (long-dashed line), $10^{-3}$ (short-dashed line), and $10^{-4}$ (dotted line). (c) Scale height dependence: $h/r_p = 10^{-2}$ (solid line), $10^{-1.5}$ (long-dashed line), $10^{-1}$ (short-dashed line), and $10^{-0.5}$ (dotted line). The standard values of the parameters are $M_p/M_\ast = 10^{-3}$, $\nu/(r_p^2 \Omega_p) = 10^{-3}$, and $h/r_p = 10^{-2}$. The plus signs indicate where $x = x_m$, and crosses in (a) and vertical lines in (b) and (c) indicate where $x = 2r_1$. $\Sigma_{\text{acc}}/\epsilon$ is also shown as a horizontal line in each panel.](image-url)
influence on \(M_p\) in low-mass cases. That is why we adopt the value of \(x_{\text{acc}} (=2r_H)\) for the highest mass case of TW02.

Note also that, as shown by D’Angelo et al. (2003), the gas accretion rate obtained by two-dimensional simulations is usually higher than that obtained by three-dimensional simulations. Thus, the accretion rate given by equation (16) could be overestimated. We discuss this issue in § 4.4.

### 4. EVOLUTION

In this section we show the evolution of the growth rate and mass of an accreting planet. To gain a proper understanding of the basic behavior of the planetary accretion, we first explore two simple cases with no disk dissipation (i.e., constant \(\Sigma_{\\infty}\)) and with exponentially decreasing \(\Sigma_{\\infty}\) in § 4.1 and 4.2, respectively. Then we investigate the accretion of a giant planet embedded in a viscously evolving protoplanetary disk in § 4.3.

The numerical procedure is as follows. Except in § 4.1, we first calculate the unperturbed surface density, \(\Sigma_{\\infty}\), from equation (21) in § 4.2 or from equation (26) in § 4.3. For given values of the parameters, \(h/r_p\) and \(\nu/(r_p^2\Omega_p)\), we then calculate \(\Sigma_{\\infty}\) and \(A\), using equation (4) if \(x_{\text{acc}} \geq x_m\) or equation (13) if \(x_{\text{acc}} \leq x_m\). Finally, we integrate \(M_p\) with respect to time using \(\Sigma_{\\infty}\) and \(\dot{A}\) (see eq. [16]) to obtain the time evolution of the planetary mass.

#### 4.1. Case without Disk Dissipation

We first show the evolution of the planetary mass without global disk dissipation. In this case the input parameters are \(\Sigma_{\\infty}r_p^2/M_*, h/r_p\), and \(\nu/(r_p^2\Omega_p)\).

| Variable | Meaning | Definition |
|----------|---------|------------|
| \(M_p\) | Planetary mass | ... |
| \(M_*\) | Mass of the central star | ... |
| \(M_{\text{disk}}\) | Disk mass at the initial condition | \(\int_0^\infty 2\pi r\Sigma_m (r, 0) \, dr\) Eq. (23) |
| \(M_{\text{final}}\) | Final mass of a planet | Eq. (36) |
| \(M_{\text{final, gap}}\) | Final mass of a planet in the gap-limiting region | Eq. (36) |
| \(M_{\text{final, diff}}\) | Final mass of a planet in the diffusion-limiting region | Eqs. (38) and (39) |
| \(M_{\text{trans}}\) | Planetary mass when \(\Sigma_{\\infty}\) is \((1/e)\Sigma_{\\infty}\) in the case of \(2r_H \geq x_m\) | Eq. (B2) |
| \(M_{\text{init}}\) | Initial planetary mass | ... |
| \(M_{\text{crit}}\) | Maximum planetary mass that does not violate a geometrical limit for eq. (17) | Eq. (48) |
| \(M_{\text{local}}\) | Final mass if \(M_p\) is assumed to be \(M_{\text{local}}\) | Eq. (35) |
| \(M_{\text{local, init}}\) | Final mass if \(M_p\) is assumed to be \(M_{\text{local, init}}\) | Eq. (35) |
| \(M_{\text{vis}}\) | Planetary mass when viscous condition for gap opening is fulfilled | Eq. (44) |
| \(M_{\text{vis, acc}}\) | Accretion rate onto a planet | Eqs. (16) or (32) |
| \(M_{\text{vis, local}}\) | Accretion rate onto a planet using eqs. (14), (16), and (26) | See § 4.3.1 |
| \(M_{\text{vis, disk}}\) | Radial mass flux of the disk due to viscous evolution | Eq. (30) |
| \(\Sigma_{\\infty}\) | Unperturbed surface density at planet’s orbit | Eqs. (4), (13), (21), and (26) |
| \(\Sigma_{\\infty, \text{vis}}\) | Surface density at the accretion band (\(x = 2r_H\)) | Eq. (16) |
| \(\Sigma_{\\infty, \text{vis, local}}\) | Surface density determined by the balance between viscous and gravitational torques | Eq. (4) |
| \(\Sigma_{\\infty, \text{vis, init}}\) | Surface density determined by marginally stable state of the Rayleigh condition | Eq. (13) |
| \(\Sigma_{\\infty, \text{vis, gap}}\) | Surface density of self-similar solution for viscous evolving disk | Eq. (27) |
| \(\Sigma_{\\infty, \text{vis, gap, dep}}\) | Initial unperturbed surface density at planet’s orbit | Eq. (21) |
| \(r_p\) | Semimajor axis of a planet | See § 4.3.3 |
| \(r_g\) | Boundary position between the gap-limiting region and the diffusion-limiting region | See § 4.3.3 |
| \(r_{\text{diff}}\) | Boundary position between the diffusion-limiting region and the no-growth region | See § 4.3.3 |
| \(l\) | Position where \(\Sigma_{\\infty}\) is \((1/e)\Sigma_{\\infty}\) | Eqs. (4) or (5) |
| \(x_{\text{acc}}\) | Position where \(\Sigma_{\\infty}\) is \((1/e)\Sigma_{\\infty}\) | Eq. (21) |
| \(\tau_{\text{diff}}\) | Time when the accretion rate is the maximum | Eq. (19) |
| \(\tau_{\text{dep}}\) | Exponential depletion timescale of a disk | Eq. (21) |
| \(\tau_{\text{vis}}\) | Viscous evolution timescale of a disk | Eq. (19) |
| \(\tau_{\text{atm}}\) | Effective disk lifetime (i.e., shorter one of \(\tau_{\text{dep}}\) and \(\tau_{\text{vis}}\)) | Eq. (34) |
| \(\zeta\) | Normalized parameter (surface density times disk depletion time) | Eq. (24) |

#### 4.1.1. General Properties

Figure 3 shows the evolution of the planetary mass and the gas accretion rate for several values of the three parameters. Without global disk depletion, the only way to suppress gas accretion is by opening a deep gap around the planetary orbit by the planet itself. As seen in Figure 3, the evolution can be divided into two phases. The first phase is growth without a gap (pregap phase), while the second phase is growth with a deep gap (postgap phase). Properties of the evolution in both phases are explained in analytical ways below.

In the pregap phase, \(M_p\) is almost proportional to \(M_p^{1/3}\) (see Figs. 3c, 3f, and 3i). In this phase, the planetary mass is insufficient to open a gap, so \(\Sigma_{\\infty} \approx \Sigma_{\\infty}\). Then we can easily integrate equation (16) with equation (17):

\[
M_p = S\Sigma_{\\infty}r_p^2\Omega_p\left(\frac{h}{r_p}\right)^{-2} \times \left[\left(\frac{M_{\text{p, init}}}{M_*}\right)^{-1/3} - \frac{S}{3} \left(\frac{\Sigma_{\\infty}}{M_*/r_p}\right)^{1/2} \left(\frac{h}{r_p}\right)^{-1/2} \left(\frac{\Omega_p}{\Omega_*}\right)^{-1}\right]^{-4},
\]

(18)

where \(S\) corresponds to 0.29 in equation (17) and \(M_{\text{p, init}}\) is the initial mass of the planet that corresponds to the mass beyond which the gas accretion is limited by disk-gas supply. This equation...
implies that if $\Sigma_{\text{acc}}$ is constant, the gas accretion rate diverges at a time

$$
\tau_{\text{div}} = \frac{3}{S} \left( \frac{M_{\text{p,init}}}{M_*} \right)^{-1/3} \left( \frac{\Sigma_{\infty}}{M_* / r_p^2} \right)^{-1} \left( \frac{h}{r_p} \right)^2 \Omega_p^{-1}
$$

$$
= 4.8 \times 10^4 \left( \frac{M_{\text{p,init}}}{10^{-3} M_*} \right)^{-1/3} \left( \frac{\Sigma_{\infty}}{10^{-3} M_* / r_p^2} \right)^{-1}
$$

$$
\times \left( \frac{h}{10^{-1.5} r_p} \right)^2 \Omega_p^{-1},
$$

(19)

$\tau_{\text{div}}$ corresponds to the end of the pregap phase.

In the postgap phase, $\dot{M}_p$ decreases with $M_p$ in an exponential fashion (see Figs. 3c, 3f, and 3i). Such a steep decrease in $\dot{M}_p$ is due to a steep decrease in $\Sigma_{\text{acc}}$ with respect to $M_p$ via $l$, $x_m$, and $r_H$ (see eqs. [4] and [13]). We can derive approximate expressions for the accretion rate in this phase, which shows $\dot{M}_p$ is inversely proportional to time as shown in Figures 3a–3c (see Appendix A for the details).

4.1.2. Dependence on the Parameters

We now see the dependence of the evolution of planetary mass on the unperturbed surface density, $\Sigma_{\infty}$ (Figs. 3a–3c). The evolution timescale is inversely proportional to $\Sigma_{\infty}$ (e.g., Fig. 3b). This is simply because $\Sigma_{\text{acc}}$ is proportional to $\Sigma_{\infty}$, whereas $A$ is independent of $\Sigma_{\infty}$. The peak value of $M_p$ is thus accordingly proportional to $\Sigma_{\infty}$ (e.g., Fig. 3c).

The dependence on scale height, $h$, is shown in Figures 3d–3f. In the pregap phase, $\dot{M}_p$ is simply $\propto h^{-2}$ and the evolution timescale is thus $\propto h^{-2}$. This is because the accretion area $A \propto h^{-2}$ (see Figs. 3a–3c).
In Appendix B, we have derived approximate solutions for $\dot{M}_p$ that are valid in the postgap phase. From equations (B2) and (B3), one finds $\dot{M}_p \propto \nu$ where $x_m$ is the peak value does not depend on $\nu$ either (see eq. [19]). In the postgap phase, $\dot{M}_p$ at a given $t$ increases with $\nu$.

In Appendix B, we have derived approximate solutions for $\dot{M}_p$ in the postgap phase. From equations (B2) and (B3), one finds that $\dot{M}_p \propto \nu$ in high-$\nu$ cases (i.e., $x_{\text{acc}} > x_m$). In low-$\nu$ cases (i.e., $x_{\text{acc}} \ll x_m$), $\dot{M}_p$ also increases with $\nu$, but the dependence is rather weak (see eqs. [B7] and [B8]). In addition, the accretion rate is inversely proportional to time (see eqs. [B3] or [B8]), so the maximum accretion rate can be roughly estimated by equations (B3) or (B8) at $t = \tau_{\text{div}}$.

### 4.2. Case with Exponential Disk Dissipation

Next we examine the evolution of the planetary mass in a simple case where the disk surface density decreases in an exponential fashion with a time constant of $\tau_{\text{div}}$.

$$\Sigma_{\infty} = \Sigma_{\infty,\text{init}} \exp \left( \frac{-t}{\tau_{\text{div}}} \right),$$

where $\Sigma_{\infty,\text{init}}$ is the initial surface density at infinity. Figure 4 shows the evolution of $\dot{M}_p$ and $\dot{M}_p$ for $\tau_{\text{div}} = 10^5 \Omega_p^{-1}$ and three values of $\Sigma_{\infty,\text{init}}$. In the high-$\Sigma_{\infty,\text{init}}$ case, $\dot{M}_p$ increases with time to reach a peak at $t \approx 1.5 \times 10^3 \Omega_p^{-1}$ and then decreases with time, which is similar to the evolution without disk dissipation shown in Fig. 4. This is because $\tau_{\text{div}} < \tau_{\text{div}}$ in this case. In the low-$\Sigma_{\infty,\text{init}}$ case, $\dot{M}_p$ decreases without experiencing a significant increase that occurs in the high-$\Sigma_{\infty,\text{init}}$ case. This is because $\tau_{\text{div}} > \tau_{\text{div}}$ in this case ($\tau_{\text{div}} \propto \Sigma_{\infty,\text{init}}^{-1}$; see eq. [19]). Because of such different evolution of $\dot{M}_p$, the mass evolution also differs between high- and low-$\Sigma_{\infty,\text{init}}$ cases (see Fig. 4, middle): The planet captures a significant amount of gas to be a Jupiter-like planet in the high-$\Sigma_{\infty,\text{init}}$ case, whereas the planet captures only a small amount of gas to be a Neptune-like planet in the low-$\Sigma_{\infty,\text{init}}$ case. The boundary between the two regimes is determined by the condition $\tau_{\text{div}} = \tau_{\text{div}}$, namely, from equation (19),

$$\sum_{\text{init}} / M_{\ast}/r_p^2 \cdot \left( \frac{\dot{M}_{\ast}}{M_{\ast}} \right)^{-1/3} \left( \frac{h}{r_p} \right) \left( \frac{\tau_{\text{div}}}{\Omega_p^{-1}} \right)^{-1} \cdot \left( \frac{\tau_{\text{div}}}{\Omega_p^{-1}} \right).$$

Figure 5 shows the final mass defined by

$$M_{\text{final}} \equiv \int_0^\infty \dot{M}_p \, dt$$

as a function of a quantity $\zeta$ defined by

$$\zeta \equiv \left( \frac{\Sigma_{\text{init}}}{M_{\ast}/r_p^2} \right) \left( \frac{\tau_{\text{div}}}{\Omega_p^{-1}} \right).$$

![Figure 4](image_url) Evolution of planetary mass and accretion rate with disk dissipation when $h/r = 0.1, \nu/(r^2 \Omega_p) = 10^{-5}$. The three panels have the same axes as Fig. 3. The thick lines show different initial surface densities $[\Sigma_{\text{init}}/(M_{\ast} r_p^2)] = 10^{-8.5}$ (solid line), $10^{-8.5}$ (long-dashed line), and $10^{-8.5}$ (short-dashed line), and evolutions without disk dissipation are shown as thin lines.

![Figure 5](image_url) Final mass of planets as a function of $\zeta$ (defined at eq. [24]) in the case with $h/r = 0.1, \nu/(r^2 \Omega_p) = 10^{-5}$. The three lines are for different initial planetary masses ($M_{\text{init}}/M_{\ast} = 1 \times 10^{-5}, 3 \times 10^{-5}$, and $1 \times 10^{-4}$).
The dependence of $M_{\text{final}}$ on $\tau_{\text{dep}}$ is the same as that on $\Sigma_{\text{in},\text{init}}$ because the ratio of $\tau_{\text{div}}$ ($\propto \Sigma_{\text{in},\text{init}}$) to $\tau_{\text{dep}}$ determines the evolution. Figure 5 illustrates that when $\zeta$ is small ($\leq 1$), the final mass is almost the same as the initial mass. In this regime, the planet captures only a tiny amount of gas and becomes a Neptune-like planet. When $\zeta \geq 10$, the final mass is almost equal to the initial mass and thus almost independent of viscosity. On the right side, the final mass increases with viscosity given by (Hartmann et al. 1998),

$$\Sigma_{\text{in}}(r, t) = \frac{M_{\text{disk}}}{2\pi R_{\text{out}}^2} \left(\frac{r}{R_{\text{out}}}\right)^{-1} \tilde{\tau}_{ss}^{-3/2} \exp\left(-\frac{r}{\tilde{\tau}_{ss} R_{\text{out}}}\right), \quad (27)$$

where $M_{\text{disk}}$ is the initial disk mass and $R_{\text{out}}$ is the initial disk size; $\tilde{\tau}_{ss}$ is defined as

$$\tilde{\tau}_{ss} = \frac{t}{\tau_{\text{vis}}} + 1, \quad (28)$$

with a typical timescale of global viscous evolution

$$\tau_{\text{vis}} \equiv \frac{R_{\text{out}}^2}{3
\nu_{\text{out}}} = 5.3 \times 10^5 \left(\frac{\alpha}{0.01}\right)^{-1} \left(\frac{h_1 \text{ AU}}{10^{-1.5} \text{ AU}}\right)^{-2} \left(\frac{R_{\text{out}} \text{ AU}}{100 \text{ AU}}\right) \text{ yr}, \quad (29)$$

where $\nu_{\text{out}}$ is the viscous coefficient of the disk gas at $r = R_{\text{out}}$. In deriving equation (27), we have assumed a temperature distribution, $T \propto r^{-1/2}$, which results in $\nu \propto r$.

Using the equations above, we calculate the local gas accretion rate onto the planet, denoted hereafter by $M_{\text{p,local}}$, in the same way as we did in § 4.2. However, when $M_{\text{p,local}}$ is larger than the radial mass flux due to viscous diffusion in the disk, the latter limits the planetary growth. The mass flux through a ring with radius $r$ is given by

$$\dot{M}_{\text{disk}}(r, t) = 2\pi r \Sigma_{\text{in}}(r, t) v_r,$$

instead of equation (21): $\Sigma_{\text{in}}$ represents a change in the surface density due to viscous diffusion of the disk gas and $e^{-t/\tau_{\text{dep}}}$ is introduced to mimic photoevaporation of the disk. For $\Sigma_{\text{in}}$, we adopt the self-similar solution with $\alpha$-prescription for disk viscosity given by (Hartmann et al. 1998),

$$\Sigma_{\text{in}}(r, t) = \frac{M_{\text{disk}}}{2\pi R_{\text{out}}^2} \left(\frac{r}{R_{\text{out}}}\right)^{-1} \tilde{\tau}_{ss}^{-3/2} \exp\left(-\frac{r}{\tilde{\tau}_{ss} R_{\text{out}}}\right), \quad (27)$$

where $M_{\text{disk}}$ is the initial disk mass and $R_{\text{out}}$ is the initial disk size; $\tilde{\tau}_{ss}$ is defined as

$$\tilde{\tau}_{ss} = \frac{t}{\tau_{\text{vis}}} + 1, \quad (28)$$

with a typical timescale of global viscous evolution

$$\tau_{\text{vis}} \equiv \frac{R_{\text{out}}^2}{3\nu_{\text{out}}} = 5.3 \times 10^5 \left(\frac{\alpha}{0.01}\right)^{-1} \left(\frac{h_1 \text{ AU}}{10^{-1.5} \text{ AU}}\right)^{-2} \left(\frac{R_{\text{out}} \text{ AU}}{100 \text{ AU}}\right) \text{ yr}, \quad (29)$$

where $\nu_{\text{out}}$ is the viscous coefficient of the disk gas at $r = R_{\text{out}}$. In deriving equation (27), we have assumed a temperature distribution, $T \propto r^{-1/2}$, which results in $\nu \propto r$.

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Instead of equation (21): $\Sigma_{\text{in}}$ represents a change in the surface density due to viscous diffusion of the disk gas and $e^{-t/\tau_{\text{dep}}}$ is introduced to mimic photoevaporation of the disk. For $\Sigma_{\text{in}}$, we adopt the self-similar solution with $\alpha$-prescription for disk viscosity given by (Hartmann et al. 1998),

$$\Sigma_{\text{in}}(r, t) = \frac{M_{\text{disk}}}{2\pi R_{\text{out}}^2} \left(\frac{r}{R_{\text{out}}}\right)^{-1} \tilde{\tau}_{ss}^{-3/2} \exp\left(-\frac{r}{\tilde{\tau}_{ss} R_{\text{out}}}\right), \quad (27)$$

where $M_{\text{disk}}$ is the initial disk mass and $R_{\text{out}}$ is the initial disk size; $\tilde{\tau}_{ss}$ is defined as

$$\tilde{\tau}_{ss} = \frac{t}{\tau_{\text{vis}}} + 1, \quad (28)$$

with a typical timescale of global viscous evolution

$$\tau_{\text{vis}} \equiv \frac{R_{\text{out}}^2}{3\nu_{\text{out}}} = 5.3 \times 10^5 \left(\frac{\alpha}{0.01}\right)^{-1} \left(\frac{h_1 \text{ AU}}{10^{-1.5} \text{ AU}}\right)^{-2} \left(\frac{R_{\text{out}} \text{ AU}}{100 \text{ AU}}\right) \text{ yr}, \quad (29)$$

where $\nu_{\text{out}}$ is the viscous coefficient of the disk gas at $r = R_{\text{out}}$. In deriving equation (27), we have assumed a temperature distribution, $T \propto r^{-1/2}$, which results in $\nu \propto r$.

Using the equations above, we calculate the local gas accretion rate onto the planet, denoted hereafter by $M_{\text{p,local}}$, in the same way as we did in § 4.2. However, when $M_{\text{p,local}}$ is larger than the radial mass flux due to viscous diffusion in the disk, the latter limits the planetary growth. The mass flux through a ring with radius $r$ is given by

$$\dot{M}_{\text{disk}}(r, t) = 2\pi r \Sigma_{\text{in}}(r, t) v_r,$$

Instead of equation (21): $\Sigma_{\text{in}}$ represents a change in the surface density due to viscous diffusion of the disk gas and $e^{-t/\tau_{\text{dep}}}$ is introduced to mimic photoevaporation of the disk. For $\Sigma_{\text{in}}$, we adopt the self-similar solution with $\alpha$-prescription for disk viscosity given by (Hartmann et al. 1998),

$$\Sigma_{\text{in}}(r, t) = \frac{M_{\text{disk}}}{2\pi R_{\text{out}}^2} \left(\frac{r}{R_{\text{out}}}\right)^{-1} \tilde{\tau}_{ss}^{-3/2} \exp\left(-\frac{r}{\tilde{\tau}_{ss} R_{\text{out}}}\right), \quad (27)$$

where $M_{\text{disk}}$ is the initial disk mass and $R_{\text{out}}$ is the initial disk size; $\tilde{\tau}_{ss}$ is defined as

$$\tilde{\tau}_{ss} = \frac{t}{\tau_{\text{vis}}} + 1, \quad (28)$$

with a typical timescale of global viscous evolution

$$\tau_{\text{vis}} \equiv \frac{R_{\text{out}}^2}{3\nu_{\text{out}}} = 5.3 \times 10^5 \left(\frac{\alpha}{0.01}\right)^{-1} \left(\frac{h_1 \text{ AU}}{10^{-1.5} \text{ AU}}\right)^{-2} \left(\frac{R_{\text{out}} \text{ AU}}{100 \text{ AU}}\right) \text{ yr}, \quad (29)$$

where $\nu_{\text{out}}$ is the viscous coefficient of the disk gas at $r = R_{\text{out}}$. In deriving equation (27), we have assumed a temperature distribution, $T \propto r^{-1/2}$, which results in $\nu \propto r$.

Using the equations above, we calculate the local gas accretion rate onto the planet, denoted hereafter by $M_{\text{p,local}}$, in the same way as we did in § 4.2. However, when $M_{\text{p,local}}$ is larger than the radial mass flux due to viscous diffusion in the disk, the latter limits the planetary growth. The mass flux through a ring with radius $r$ is given by

$$\dot{M}_{\text{disk}}(r, t) = 2\pi r \Sigma_{\text{in}}(r, t) v_r,$$
with the radial drift velocity of the diffusing gas in a Keplerian disk

\[ \nu_r = -\frac{3\nu}{r} \left[ \frac{\partial \ln(\Sigma_i)}{\partial \ln r} + \frac{1}{2} \right]. \] (31)

Thus, we calculate the gas accretion rate onto the planet in such a way that

\[ M_p = \begin{cases} M_{p,\text{local}} & \text{if } M_{p,\text{local}} < M_{\text{disk}}, \\ M_{\text{disk}} & \text{if } M_{p,\text{local}} > M_{\text{disk}}. \end{cases} \] (32)

There are several studies that modeled gas accretion rate onto giant planets. Guillot & Hueso (2006) used a formula in which the gas accretion rate onto the planet is 0.3 times the radial mass flux of the disk gas due to global viscous diffusion. However, their formula is an empirical one and applies in a limited situation. Lubow et al. (1999) carried out a series of hydrodynamic simulations to obtain gas accretion rate onto planets. Their simulations demonstrated that the planetary accretion rate can be larger than the diffusion flux because of gradients imposed by the gap. However, what they found is probably a transient phenomenon that occurs on a timescale much shorter than the viscous timescale. Veras & Armitage (2004) used an approximate formula based on hydrodynamic simulations done by Lubow et al. (1999) and D’Angelo et al. (2002).

4.3.2. Dependence on the Disk Parameters

Figure 7 is similar to Figure 6 and shows the final mass of the planet, \( M_{\text{final}} \), as a function of the initial surface density, \( \Sigma_{\text{ss}} \), and viscosity, \( \nu \), in the case where \( h/r = 0.1 \) (left) and 0.032 (right), \( R_{\text{out}} = 10r_p \), and \( \tau_{\text{dep}} = 10^6 \Omega_p^{-1} \). The value of \( \tau_{\text{dep}} \) corresponds to \( \zeta = 10^6 \frac{\Sigma_{\text{ss}}}{M_{\text{p,init}}(r_p^2)} \), so the ranges of \( \tau_{\text{dep}} \) are the same as those of Figure 6 for the horizontal axis as well as the vertical axis.

Since we have adopted the self-similar solution for the surface density evolution of the global disk, there appear two additional limits to the final mass. One arises from the appearance of the viscous timescale \( \tau_{\text{vis}} \), in addition to the timescale of exponential decay \( \tau_{\text{dep}} \). When \( \tau_{\text{vis}} < \tau_{\text{dep}} \), the effective disk lifetime is \( \tau_{\text{vis}} \). Thus, in order for planets to be massive, the growth timescale \( \tau_{\text{div}} \) should be shorter than \( \tau_{\text{vis}} \), so the additional limit is given by \( \tau_{\text{div}} = \tau_{\text{vis}} \):

\[ \frac{\nu}{r_p^2} \Omega_p \simeq \frac{1}{7} \left( \frac{M_{\text{p,init}}}{10^{-3}M_\odot} \right)^{1/3} \frac{1}{(0.1r_p)^2} \times \left( \frac{\Sigma_{\text{ss}}}{10^{-4}M_\odot r_p^2} \right) \left( \frac{R_{\text{out}}}{10r_p} \right), \] (33)

which runs from the lower left to the upper right in Figure 7 because \( \nu \) is linearly proportional to the surface density. As described above, the effective disk lifetime, \( \gamma_{\text{div}} \), is the shorter of the two timescales, so we here define

\[ \gamma_{\text{div}} = \begin{cases} \tau_{\text{dep}} & \text{if } \tau_{\text{dep}} < \tau_{\text{vis}}, \\ \tau_{\text{vis}} & \text{if } \tau_{\text{dep}} > \tau_{\text{vis}}. \end{cases} \] (34)

The other limit is the total mass of the disk. This limit does not appear in Figure 6 (i.e., the case without global viscous evolution). This can be clearly seen in the case of \( h/r = 0.032 \) on the right panel of Figure 7 where contour lines are vertical. Note that the steep boundary (being equivalent to \( \zeta = \zeta_0 \)) characterized by \( \tau_{\text{div}} = \tau_{\text{dep}} \) that is seen in Figure 6 is not found in Figure 7. This is simply because the two limits described above are more severe than the condition of \( \tau_{\text{div}} = \tau_{\text{dep}} \) in the cases shown in Figure 7. However, when \( \tau_{\text{dep}} \) is smaller, the boundary determined by \( \tau_{\text{div}} = \tau_{\text{dep}} \) moves rightward, so the vertical steep boundary determined by \( \tau_{\text{div}} = \tau_{\text{dep}} \) emerges even in the case with global disk evolution described in this subsection.

4.3.3. Classification by Semimajor Axis

In Figure 7, the dependence of the final mass was shown for the normalized disk parameters. However, one might want to
know its dependence on semimajor axis. We thus show the final mass as a function of semimajor axis in Figure 8 for a typical case with \( \alpha = 0.01, h/r = 0.032 \) at 1 AU, \( t_{\text{dep}} = 10^6 \) yr, \( M_{\text{disk}} = 1.3 \times 10^{-3} M_\odot \), and \( R_{\text{out}} = 100 \) AU. The dashed line shows \( M_{\text{p, disk}} \) (i.e., mass when all viscous-accreting gas is assumed to accrete onto planets), and the dotted line shows \( M_{\text{p, init}} \) (i.e., mass when the global viscous evolution with self-similar solution is not assumed). The initial mass of the planets is set as \( 3.2 \times 10^{-5} M_\odot \) which corresponds to 10 \( M_\oplus \) in the solar mass system.

In the intermediate region (1 AU \( \leq r_p \leq 100 \) AU), the final mass suddenly decreases to \( M_{\text{p, init}} \) which is consistent with the result shown in Figure 8.

The positions of the boundaries between the three regions are given as follows. At the boundary between the gap-limiting region and the diffusion-limiting region (its position being denoted by \( r_b \)), \( M_{\text{final, gap}} \approx M_{\text{final, diff}} \). When \( \tau_{\text{vis}} < \tau_{\text{dep}} \), from equations (36) and (38), we obtain

\[
r_b \approx 5 \left( \frac{\alpha}{10^{-2}} \right)^{-2} \left( \frac{h_1 \text{AU}}{10^{-1.5} \text{AU}} \right)^{-4} \left( \frac{M_{\text{disk}}}{10^{-2} M_\odot} \right)^2 \left( \frac{\log(\tau_{\text{vis}}/\tau_{\text{div}})}{5} \right)^{-2}.
\]

Here we have assumed the log factor in equation (36) is constant. When \( \tau_{\text{vis}} > \tau_{\text{dep}} \), from equations (36) and (39), we have

\[
r_b \approx \frac{1}{2} \left( \frac{\tau_{\text{dep}}}{\tau_{\text{vis}}} \right)^2 \left( \frac{M_{\text{disk}}}{10^{-2} M_\odot} \right)^2 \left( \frac{\log(\tau_{\text{vis}}/\tau_{\text{div}})}{5} \right)^{-2}.
\]
The location of the boundary between the diffusion-limiting region and the no-growth region is basically determined by \( \tau_{\text{div}} \sim \tau_{\text{life-time}} \) and is denoted by \( r_c \). When \( \tau_{\text{vis}} < \tau_{\text{dep}} \), from \( \tau_{\text{div}} = \tau_{\text{vis}} \), we obtain
\[
r_c \simeq 70 \left( \frac{M_{p,\text{init}}}{10^{-5} M_\odot} \right)^{1/3} \left( \frac{\Sigma_{\infty,1}}{10^{-5} M_\odot/\text{AU}^2} \right) \times \left( \frac{h_{1 \text{ AU}}}{10^{-1.5} \text{ AU}} \right)^{-4} \left( \frac{\alpha}{10^{-2}} \right)^{-1} \left( \frac{R_{\text{out}}}{100 \text{ AU}} \right) \text{AU.} \tag{42}
\]
When \( \tau_{\text{vis}} > \tau_{\text{dep}} \), from \( \tau_{\text{div}} = \tau_{\text{dep}} \), we obtain
\[
r_c \simeq 130 \left( \frac{M_{p,\text{init}}}{10^{-5} M_\odot} \right)^{1/3} \left( \frac{\Sigma_{\infty,1}}{10^{-5} M_\odot/\text{AU}^2} \right)^{-2} \left( \frac{\tau_{\text{dep}}}{10^6 \text{ yr}} \right) \text{AU.} \tag{43}
\]

Figure 9 shows the final mass for several different values of the disk parameters. The discussion above suffices to understand this figure. Note that there is a hollow at \( \sim 50 \text{ AU} \) in the case where \( \alpha \) is lower by a factor of 10 than its nominal value (thick short-dashed line). This is, however, artificial. In the self-similar solution we have adopted, the direction of the radial gas flow due to viscous diffusion changes at \( r = (R_{\text{out}}/2)^{\tau_{\text{vis}}} \), which means that the mass flux is zero at the point, and the accretion rate onto the planet is accordingly also zero (see eq. [32]). Since \( \tau_{\text{vis}} \) increases with time (see eq. [28]), the point moves outward with time. When \( \tau_{\text{dep}} > \tau_{\text{vis}} \), the point moves significantly with time and integration of mass flux at a particular point eliminates the effect. In contrast, when \( \tau_{\text{dep}} \ll \tau_{\text{vis}} \), the point hardly changes before the exponential depletion; the planet around \( r = R_{\text{out}}/2 \) thus cannot grow significantly.

Another important aspect is on the final mass in the gap-limiting region. We can see that the \( r_f \) dependence of \( M_{\text{final}} \) agrees with that of the mass given by the viscous condition for gap opening (Lin & Papaloizou 1979, 1993):
\[
M_{p,\text{vis}} = 40 \left( \frac{\nu}{r^2 \Omega} \right) M_\odot \tag{44}
\]
\[
= 4 \times 10^{-4} \left( \frac{\alpha}{10^{-2}} \right)^{1/2} \left( \frac{r}{1 \text{ AU}} \right) M_\odot. \tag{45}
\]

The dependence also agrees with that of \( M_{\text{trans}} \), which is the mass when \( \Sigma_{\text{acc}} \) is \((1/\epsilon)\Sigma_{\infty} \) in the case of \( 2r_{11} \geq x_m \) (see eq. [B2]), and rewritten as
\[
M_{\text{trans}} \simeq 9 \times 10^{-4} \left( \frac{\alpha}{10^{-2}} \right) \left( \frac{r}{1 \text{ AU}} \right)^{1/2} M_\odot. \tag{46}
\]

Both of the two masses indicate gap-forming masses, and they agree with each other within a factor of about 2. However, the final mass shown in our model is larger by a factor of 10 than \( M_{p,\text{vis}} \) or \( M_{\text{trans}} \). This is because the two masses correspond to the masses at which a gap is about to form and does not mean the masses at which growth stops. A further increase in \( M_\odot \) after reaching \( M_{p,\text{vis}} \) or \( M_{\text{trans}} \) is reflected in the log term in equation (36).

4.4. Remarks on the Uncertainties in the Model

We have assumed that the distribution of the disk gas around the planetary orbit (i.e., the surface density) is always in the equilibrium state that is determined by the balance between viscous stress and gravitational scattering by the planet. However, for the equilibrium state to be achieved, nonequilibrium distributions must be relaxed by diffusion. The diffusion timescale, \( \tau_{\text{vis,local}} \), is estimated by the gap width \( \sim 2 \times 2r_{11} \) divided by the viscosity as
\[
\tau_{\text{vis,local}} = 7.7 \times 10^3 \left( \frac{M_p}{10^{-3} M_\odot} \right)^{2/3} \left( \frac{\nu}{10^{-5} r_p^2 \Omega_p} \right) \left( \frac{r_p}{1 \text{ AU}} \right)^{-1} \Omega_p^{-1}. \tag{47}
\]

This timescale should be compared with the typical growth timescale (i.e., \( \tau_{\text{div}} \); see eq. [19]). The comparison indicates that \( \tau_{\text{vis,local}} < \tau_{\text{div}} \) in most of the cases presented in this paper. Hence, the assumption is appropriate unless quite low viscosities (yielding large \( \tau_{\text{vis,local}} \)) or high surface densities (yielding small \( \tau_{\text{div}} \)) are considered.

The empirical formula for the accretion rate based on local two-dimensional isothermal hydrodynamic simulations (TW02) can be different from that derived based on more realistic simulations including, for example, three-dimensional accretion flow (D’Angelo et al. 2003; Bate et al. 2003), a nonisothermal equation of state (TW02), and a magnetic field (Machida et al. 2006). The modification would change the form of \( \tau_{\text{div}} \), which would yield quantitatively different results. However, even in that case, by following our prescription given in this paper, one can easily calculate the mass evolution of planets in a similar way. The formula given by TW02 can be considered as the highest limit; thus, for example, \( r_c \) is expected to shift inward if more realistic models for the accretion rate are used. We compare the accretion rate (eq. [16]) with those obtained by global simulations (Kley 1999; Lubow & D’Angelo 2006). Although the value of our accretion rate is usually larger (by up to a factor of 10) than those given by the simulations, the dependence on viscosity and planetary mass is consistent.
We have not taken into account a geometry of the accretion flow, namely, one within which equation (17) is applicable. We can make a simple estimate of the maximum accretion rate if we assume all the gas within \(0 \leq x \leq 2\sqrt{3} r_{\text{H}}\) in \(y > 0\) and \(-2\sqrt{3} r_{\text{H}} \leq x \leq 0\) in \(y < 0\) accrete to the planet, where \(x = 2\sqrt{3} r_{\text{H}}\) at \(y \gg r_{\text{H}}\) is the point where potential energy on the rotating frame is the same as that at the Lagrange points L1 or L2. In this case, the maximum accretion rate (normalized by surface density) is given by \(3^{1/2} 6^{2/3} \rho_{\text{p}} \Omega_{\text{p}} (M_{\text{p}}/M) \), and thus \(A\) (eq. [17]) becomes larger than the maximum value when

\[
M_{\text{p}} > \frac{5.6 \times 10^{-3} M_{\odot}}{0.032 r_{\text{p}}} \equiv M_{\text{p, crit}}. \tag{48}
\]

Although planetary mass can be larger than \(M_{\text{p, crit}}\) depending on the parameters, such a massive planet should already have a deep gap, so the accretion rate is reduced greatly when the planetary mass reaches \(M_{\text{p, crit}}\). Consequently, the geometric effect has no significant influence on the final mass. Note that the discussion here is based on an approximation that streamlines can be well described by particle motions on the framework of the restricted three-body problem. Since this approximation is not valid when \(r_{\text{H}} < h\) (Masset et al. 2006), it is not applicable for the cases of small-mass planets. However, we consider the upper limit of applicable mass, where the planet mass is, in most cases, large enough to satisfy \(r_{\text{H}} < h\), so the approximation is valid for the situations that we consider here.

In this paper, we have put off the issue of planetary migration. Since we consider a phase in which a gap exists around the planet, we may have to consider the type II migration (e.g., Ward 1997; Ward & Harn 2000), especially in the gap-limiting region (e.g., \(r_{\text{H}} < 1\) AU in Fig. 8). The deep gap created by the planet blocks the accretion flow toward the central star. It follows that the planet is pushed inward by the gas exterior to the planet’s orbit. We will include the effect of planetary migration in our future work.

5. SUMMARY

To gain a systematic understanding of the final masses of gas giant planets, we have simulated the long-term accretion of gas giant planets after the onset of the supercritical gas accretion in a variety of situations, depending on four disk parameters such as disk mass, viscosity, scale height, and semimajor axis. To do so, we have made a semianalytical model to simulate the mass evolution, which enabled us to study the final mass of gas giant planets for extensive ranges of all the parameters.

We have first made a one-dimensional analytical model of the equilibrium surface density profile around a protoplanet, from consideration of the balance of torque and the dynamical stability (§ 2). Combining the surface density profile with an empirical formula for gas accretion rate that was obtained on the basis of hydrodynamic simulations by TW02, we have obtained a formula for gas accretion rate as a function of planetary mass, viscosity, scale height, and unperturbed surface density (§ 3). We have then integrated the gas accretion rate numerically with respect to time to simulate the long-term accretion of gas giant planets (§ 4). To understand the basic behavior of the planetary accretion, we have explored two simple cases with no disk dissipation (§ 4.1) and with exponentially decreasing surface density (§ 4.2). Finally, we have simulated the long-term accretion of gas giant planets embedded in a viscously evolving and evaporating disk to obtain the final mass of gas giant planets as a function of semimajor axis (§ 4.3). We have consequently found the following three different regions depending on limiting processes on the final mass.

In the inner region \((r_{\text{p}} \leq r_{\text{H}});\) see eqs. [40] and [41]), the planet grows quickly to form a deep gap to suppress the gas accretion from the disk by itself within the disk lifetime (“gap-limiting” region). We have found that the final mass in this region is roughly 10 times larger than that determined by the viscous condition for gap opening (Lin & Papaloizou 1993). This is because the condition for gap opening only expresses the condition when a gap begins to form, and is by no means equivalent to the condition that the growth is terminated.

In the intermediate region \((r_{\text{p}} > r_{\text{H}});\) see eqs. [42] and [43]), radial transfer of the disk gas toward the planetary orbit limits the gas accretion before the planet opens a deep gap; the final mass is thus limited by viscous diffusion of the disk (“diffusion-limiting” region). We have found that when the evaporation timescale \(\tau_{\text{dep}}\) is shorter than the viscous-diffusion timescale \(\tau_{\text{vis}}\), the relationship between the final mass \(M_{\text{final}}\) and the disk mass \(M_{\text{disk}}\) is given by \(M_{\text{final}} \sim \frac{3}{2} (\tau_{\text{dep}}/\tau_{\text{vis}}) M_{\text{disk}}\), whereas \(M_{\text{final}} \sim M_{\text{disk}}\) when \(\tau_{\text{dep}} > \tau_{\text{vis}}\).

In the outer region \((r \geq r_{\text{H}});\) the planet captures only a tiny amount of gas by the time the disk gas completely dissipates (“no-growth” region). Saturn and possibly Uranus/Neptune are likely to have experienced the situation.

In this study, we have gained a clear understanding of the final masses of gas giant planets, deriving analytical expressions for them in three characteristic regions (eqs. [36]–[39]) and the locations of the boundaries between the three regions (eqs. [40]–[43]). To understand the mass-period distribution of gas giant planets in extrasolar systems found by radial velocimetry, we need to take several additional processes into consideration. Planets in the gap-limiting region would be especially susceptible to the type II migration because the gas exterior to the planetary orbit blocked by the gap pushes the planet inward. Inclusion of planetary migration is our future work. In addition, inclusion of core accretion processes and the gas accretion process governed by the Kelvin-Helmholtz contraction of the envelope is needed especially to determine the initial mass and the origin time of our model.

Although the final masses of gas giant planets were focused on in this paper, the accretion process for reaching the final mass is also important to resolve issues relevant to planet formation. Growing giant planets dynamically affect other bodies in a planetary system. Satellites are likely to form in subdisks around accreting gas giant planets (e.g., Canup & Ward 2002, 2006). The long-term accretion of gas giant planets may affect the internal structure and evolution of isolated young gas giants (Marley et al. 2007). This would be important for future direct detection of young gas giants.

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DISK PROPERTIES

Typical values of the disk parameters in our model are summarized with their dependence on the semimajor axis of the planet. We assume that the disk is optically thin and in radiative equilibrium with the stellar radiation. The disk temperature is given by (Hayashi 1981)

\[ T = (280 \text{ K}) \left( \frac{r_p}{1 \text{ AU}} \right)^{-1/2} \left( \frac{L_*}{L_\odot} \right)^{1/4}, \]  

where \( L_* \) and \( L_\odot \) are the luminosities of the central star and the Sun, respectively. Hence, the normalized scale height (i.e., the aspect ratio of the disk) is

\[ \frac{h}{r_p} = 0.033 \left( \frac{r_p}{1 \text{ AU}} \right)^{1/4}. \]  

Note that the ratio is independent of the stellar mass because we assume the stellar luminosity is proportional to the fourth power of stellar mass, which is observationally known in the case of main-sequence stars.

On the \( \alpha \)-prescription \([\alpha \equiv \nu/(ch)]\), which is widely used to describe poorly known turbulent viscosity in the disks (Shakura & Sunyaev 1973), the normalized viscosity coefficient with the temperature distribution given by equation (A1) is

\[ \frac{\nu}{r_p^2 \Omega_p} = 1.1 \times 10^{-5} \left( \frac{\alpha}{10^{-2}} \right) \left( \frac{r_p}{1 \text{ AU}} \right)^{1/2}. \]  

If we assume that disk surface density is proportional to the mass of the central star, the normalized surface density is

\[ \frac{\Sigma}{M_\odot/r_p} = 1.9 \times 10^{-4} \left( \frac{r_p}{1 \text{ AU}} \right)^{1/2} \left( \frac{L_*}{L_\odot} \right), \]  

where \( f \) is a factor relative to the surface density of the minimum-mass solar nebula model when the mass of the central star equals to that of the Sun. The dissipation timescale of the disk is

\[ \frac{\tau_{\text{dep}}}{\Omega_p^{-1}} = 2\pi \times 10^6 \left( \frac{\tau_{\text{dep}}}{10^6 \text{ yr}} \right) \left( \frac{r_p}{1 \text{ AU}} \right)^{-3/2} \left( \frac{M_*}{M_\odot} \right)^{-1/2}, \]  

where \( M_\odot \) is the mass of the Sun. The accretion rate is

\[ \frac{\dot{M}_p}{M_* \Omega_p} = 4.8 \times 10^{-9} \left( \frac{\dot{M}_p}{M_\odot \text{ yr}^{-1}} \right) \left( \frac{M_*}{M_\odot} \right)^{-3/2} \left( \frac{r_p}{1 \text{ AU}} \right)^{3/2}. \]  

APPROXIMATE SOLUTIONS FOR THE ACCRETION RATE IN THE POSTGAP PHASE

The surface density of the gas in the accretion band at \( x \sim 2r_H \), \( \Sigma_{\text{acc}} \), is given by \( \Sigma_{\text{vis}} \), when the accretion band exists in the region with the dynamically stable profile of surface density determined by the balance between viscous diffusion and scattering due to the planet, namely, when \( 2r_H \geq x_m \). Substituting \( x = 2r_H \) into equation (4), one obtains

\[ \Sigma_{\text{acc}} = \Sigma_{\text{vis}}(2r_H) = \Sigma_\infty \exp \left( -\frac{\dot{M}_p}{M_{\text{trans}}} \right), \]  

where

\[ M_{\text{trans}} = 27\pi \left( \frac{\nu}{r_p^2 \Omega_p} \right) M_\odot. \]  

Using equation (B1), one can rewrite \( \dot{M}_p = \Sigma_{\text{acc}} \dot{A} \) as

\[ \dot{M}_p \approx \frac{M_{\text{trans}}}{t}, \]  

for large \( t \) and \( \dot{M}_p \), if one neglects the weak dependence of \( \dot{A} \) on \( M_p \).
When $2r_H \leq x_m$, $\dot{M}_p$ depends on $M_p$ in a more complicated manner. As seen in equation (13), $\Sigma_R$ is not a simple exponential function of $M_p$, unlike $\Sigma_{vis}$. Defining a function $f(M_p)$ as

$$f(M_p) \equiv -\frac{1}{2} \left( \frac{2r_H}{h} - \frac{5}{4} \frac{x_m}{h} \right)^2 + \frac{1}{32} \left( \frac{x_m}{l} \right)^2 - \left( \frac{x_m}{l} \right)^{-3},$$

one can write $\Sigma_{acc}$ in the form

$$\Sigma_{acc} = \Sigma_R(2r_H) = \Sigma_{\infty} \exp \left[ f(M_p) \right].$$

(B4)

(Since $M_p$ does not change so much in the postgap phase, one only has to integrate $\dot{M}_p$ in a limited range of $M_p$. Referring to $M_{p,0}$ as an increment from a mass, $M_p$; i.e., $M_p = M_{p,0} + \Delta M_p$), one expands $e^f$ in terms of the small quantity $\Delta M_p$ in such a way that

$$\Sigma_{acc} = \Sigma_R(M_{p,0} + \Delta M_p) = \Sigma_{\infty} \exp \left[ f(M_{p,0}) + \frac{df}{dM_p} \Delta M_p \right]$$

(B5)

$$= \Sigma_R(M_{p,0}) \exp \left[ -\frac{\Delta M_p}{(df/dM_p)^{-1}} \right],$$

where

$$\frac{df}{dM_p} = \frac{1}{M_p} \left[ -\frac{1}{3} \left( \frac{2r_H}{h} \right)^2 + \frac{11}{1245} \left( \frac{2r_H}{h} \right) \left( \frac{l}{h} \right)^{3/5} - \frac{8}{120} \left( \frac{l}{h} \right)^{6/5} \right].$$

(B6)

Thus, one obtains

$$\dot{M}_p \approx \frac{( - df/dM_p )^{-1}}{t},$$

for large $t$ and $M_p$.

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