The entropy of Hawking radiation and the generalized second law

Diego Pavón

1Departamento de Física, Facultad de Ciencias, Universidad Autónoma de Barcelona, 08193 Bellaterra (Barcelona), Spain

Abstract

We derive an approximate expression for the entropy of Hawking radiation filling a spherical box in stable thermodynamic equilibrium with the Schwarzschild black hole that produced the said radiation. The Bekenstein entropy bound is satisfied but the generalized second law might not be always guaranteed. We briefly discuss the possible origin of this unexpected result.

E-mail: diego.pavon@uab.es
I. INTRODUCTION

The stability of thermodynamic equilibrium between a Schwarzschild black hole and its own radiation was considered long ago [1]. A spherical, uncharged, classical black hole of mass $M$ can remain in equilibrium with its emitted radiation, enclosed in a spherical box of perfecting reflecting thin walls of radius $R$ ($R > 2M$), provided that total energy inside the box ($M + E_{rad}$) ($E_{rad}$ being the energy of Hawking’s radiation in the box) has a minimum. For this minimum to exist the total energy must be distributed between both components in such a way that the ratio $M/E_{rad}$ is not greater than a precise value that varies for every pair $(M, R)$. Actually, no stable equilibrium is possible if $E_{rad} > M/4$. This confirms the finding of Hawking [2] who (for mathematical simplicity) assumed the radiation to be purely thermal.

It is expedient to briefly recall the meaning of thermodynamic stable equilibrium in a macroscopic system. First, the temperature (as well other intensive thermodynamic parameters) must fulfill Tolman’s law [3] across the system and the macroscopic parameters (energy volume, number of particles, etc) are to stay constant. Second, the spontaneous microscopic fluctuations of temperature, pressure, etc, around the equilibrium state die away very quickly. In consequence, the macroscopic flows of energy, momentum, particles, etc, remain zero. By contrast, when the equilibrium is not stable the aforementioned fluctuations grow fastly and dissipative flows inevitably arise. In general, the latter take the system to a new stable equilibrium state of a larger entropy than the original.

Our aim is threefold: $(i)$ to obtain a well behaved, entropy function for the Hawking radiation enclosed in the aforesaid box when stable equilibrium between the Schwarzschild black hole and radiation prevails, $(ii)$ to see whether the Bekenstein bound [4] on the ratio entropy/energy is satisfied, and $(iii)$ to check whether the generalized second law (GSL) holds in the evaporation process of the black hole. This law (one should be mindful that actually rather than a law it is a reasonable conjecture) asserts that the Bekenstein-Hawking entropy of a black hole plus the entropy of its surroundings cannot decrease in time.

We shall use Page’s approximate stress-energy tensor of Hawking radiation [5] alongside Euler’s thermodynamic equation. A very reasonable expression for the entropy of Hawking radiation follows (Section I). In section II we check the Bekenstein bound, and in section III we explore whether the GSL is fulfilled. As it turns out, it may be violated when the size of
the system is not much larger than the wavelength of the typical Hawking radiation mode. In last Section we summarizes and discuss our findings.

We do not impose the evaporation process of the black hole to be unitary. As is well known, this assumption was adopted in several studies (see e.g., [6, 7] and references therein) of the evolution of the entropy of Hawking radiation in order to overcome the famous information loss paradox. Here we do not deal with this highly debatable issue given the lack of consensus on it.

We choose units such that $\hbar = c = G = k_B = 1$.

II. HAWKING RADIATION

Here we derive an approximate expression for the entropy of Hawking radiation filling a spherical thin box of radius $R$, of perfectly reflecting walls, in thermodynamic equilibrium with the black hole that produced the radiation. The formula we arrive to is not an exact one because: (i) we start from the expression for a conformally invariant scalar field in the Hartle-Hawking thermal state around a Schwarzschild black hole provided by Page, which, in spite of being very good, is only approximate. (ii) We ignore the vacuum polarization induced by the gravitational field of the black hole. Although the polarization dies away quickly with curvature (the latter is given by $48 M^2 r^{-6}$) is not negligible near the horizon. (iii) We also ignore the effect of the vacuum polarization of the wall of the box, and (iv) the back reaction of the radiation on the metric.

We begin by recalling the metric of a Schwarzschild black hole in spherical coordinates,

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

and Tolman’s law, $T = f^{-1/2}(r) T_{bh}$, where $f(r) = 1 - 2M/r$ and $T_{bh} = (8\pi M)^{-1}$.

Page’s stress-energy tensor reads

$$T^\mu_\nu = \frac{\pi^2}{90} \left( \frac{1}{8\pi M} \right)^4 \left\{ \frac{1 - (4 - 6/M)2(2M/r)^6}{(1 - 2M/r)^2} (\delta^\mu_\nu - 4\delta^0_\nu \delta^0_\mu) + 24 \left( \frac{2M}{r} \right)^6 (3\delta^0_\nu \delta^0_\mu + \delta^1_\nu \delta^1_\mu) \right\}.$$  

This quantity is conserved and has the right trace. From it the energy density ($-T^0_0$) and pressure ($T^k_k/3$)

$$\rho = \frac{\pi^2}{30} \left( \frac{1}{8\pi M} \right)^4 \left\{ \frac{1}{(1 - 2M/r)^2} \left[ 1 - \left( 4 - \frac{6M}{r} \right)^2 \left( \frac{2M}{r} \right)^6 \right] - 24 \left( \frac{2M}{r} \right)^6 \right\},$$

(2)
\[ P = \frac{\pi^2}{90} \left( \frac{1}{8\pi M} \right)^4 \left\{ \frac{1}{(1 - \frac{2M}{r})^2} \left[ 1 - \left( 4 - \frac{6M}{r} \right)^2 \left( \frac{2M}{r} \right)^6 \right] + 8 \left( \frac{2M}{r} \right)^6 \right\}, \]  

(3)
of Hawking radiation follow.

When \( r \gg M \) the equation of state is that of thermal radiation in flat space-time, \( P = \rho/3 \), as it should.

The entropy density is given by Euler’s equation, \( s_{rad} = T^{-1}(\rho + P) \) with the chemical potential set to zero because this quantity vanishes for thermal radiation. Thus,

\[ s_{rad} = \frac{2\pi^2}{45} \left( \frac{1}{8\pi M} \right)^3 \left( 1 - \frac{2M}{r} \right)^{1/2} \left\{ 1 - \left( 4 - \frac{6M}{r} \right)^2 \left( \frac{2M}{r} \right)^6 \right\} - 64 \left( \frac{2M}{r} \right)^6 \],

(4)
it goes to zero as \( r \to 2M \) and tends asymptotically to the thermal flat space-time value \((2\pi^2/45)T^3\) when \( r \gg M \).

The radiation entropy contained in a box of radius \( R \) is given by \( S_{rad} = \int_{2M}^{R} 4\pi r^2 f(r)^{-1/2} s_{rad} \, dr \).

Therefore,

\[ S_{rad} = \frac{2\pi^2}{45} \left( \frac{1}{8\pi M} \right)^3 \left\{ 32\pi M^3 \left[ \frac{1}{3} \left( \frac{R}{2M} \right) - 1 \right]^3 - 2 \left( \frac{R}{2M} - 1 \right)^2 + 6 \left( \frac{R}{2M} - 1 \right) + 4 \ln \left( \frac{R}{2M} \right) \right\} 
+1 - \left( \frac{2M}{R} \right)^2 + 3 \left( \frac{2M}{R} \right)^3 \right) + 5 \left( 1 - \frac{2M}{R} \right) + 64 \left( \frac{2M}{R} \right)^3 - 1 \right) \right\} \right\}. \]

(5)

This expression vanishes for \( R \to 2M \), it is bounded from below (at \((R/2M)_{\text{min}} \simeq 1.742\) it has a local minimum), and diverges when \( M \to 0 \). It is negative in the range \( 1 \leq R/2M \lesssim 4.912 \), which should not surprise us because the energy density \( (2) \) is negative when \( r \) is not much larger than \( 2M \). Thereby the GSL can be violated in that interval. As expected, for \( R \gg 2M \) the thermal flat space-time expression \( S_{rad} \big|_{R \gg 2M} = \frac{2\pi^2}{45} \frac{4\pi}{3} R^3 T_{bh}^3 \) is recovered.

It is seen that the entropy of Hawking radiation depends just on the ratio \( R/M \) and not on \( R \) or \( M \) separately. This is only natural after our using of Euler equation and integrating.

Reasonably, Euler’s equation (and thereby, Eqs. (4) and (5)) holds when the size of the box is at least three times larger than the wavelength of the typical Hawking radiation mode. The latter is of the order of the black hole radius \( r_s \), \( r_s = 2M \). Note also that for \( r \geq 3r_s \) the curvature is bounded from above by \( 1/972M^4 \); whence for \( M \geq 1 \) it results negligible.

For later convenience we rewrite (5) in terms of the dimensionless parameter \( x = R/2M \),
\[ S_{rad} = \frac{1}{360} \left[ \frac{1}{3}(x - 1)^3 - 2(x - 1)^2 + 6(x - 1) + 4 \ln x \right. \\
-5(x^{-1} - 1) - (x^{-2} - 1) + \left. \frac{73}{3}(x^{-3} - 1) \right]. \] (6)

A plot of this expression is shown in Fig. 1. The fact that \( S_{rad} \) is negative in some interval may be an un-physical consequence of: \((i)\) \( R \) being of the order of \( M \) in that interval, \((ii)\) having ignored the back reaction of the radiation on the metric and the vacuum polarization effects of the wall of the box.

![Figure 1. The entropy of Hawking radiation, Eq. (6), appears to be negative in the interval 1 \( \leq x \lesssim 4.912 \). However, see the text.](image)

**III. BEKENSTEIN’S ENTROPY BOUND**

In 1981 Bekenstein argued that the ratio between the entropy and total energy, \( E \), of a system cannot be arbitrarily large but subject to the constraint

\[ \frac{S}{E} \leq 2\pi R, \] (7)

where \( R \) is the radius of the circumference that circumscribes the system. This conjecture proved controversial, see e.g. [10, 11] and references therein; nevertheless, it bears some interest and we will explore it in the case at hand.
Upon defining $\eta = S/(2\pi RE)$ with $S = 4\pi M^2 + S_{\text{rad}}$, where $S_{\text{rad}}$ is given by (6) and $E = M + E_{\text{rad}}$ with

$$E_{\text{rad}} = \int_{2M}^{R} 4\pi r^2 \rho \, dr = \frac{1}{1920\pi R} \left[ \frac{x^4}{3} + x^3 + 3x^2 + 4x \ln x - \frac{22}{3} x - 5 - 3x^{-1} + 11x^{-2} \right],$$

(8)

we have

$$\eta = \frac{\pi R^2 x^{-2} + \frac{1}{360} \left[ \frac{1}{3} (x - 1)^3 - 2(x - 1)^2 + 6(x - 1) + 4 \ln(x) - 5(x^{-1} - 1) - (x^{-2} - 1) + \frac{73}{3} (x^{-3} - 1) \right]}{\pi R^2 x^{-1} + \frac{1}{960} \left[ \frac{x^4}{3} + x^3 + 3x^2 + 4x \ln x - \frac{22}{3} x - 5 - 3x^{-1} + 11x^{-2} \right]},$$

(9)

From last expression it is seen that $0 < \eta \leq 1$; more precisely, $\eta(x \to 1) = 1$ and $\eta(x > 1) \sim x^{-1}$. Altogether, the entropy’s Bekenstein bound is fulfilled. Figure 2 illustrates this for the case $R = 200$.

It is worth noting that because the entropy of any system is maximum at equilibrium, this bound is also satisfied when the black hole and radiation are out of mutual thermal equilibrium.

![Graph of $\eta(200, x)$ vs $x$. $\eta$ is upper-bounded by unity.](image)

**IV. THE GENERALIZED SECOND LAW**

As noted above, according to the GSL, the entropy of a black hole plus the entropy of its surroundings cannot decrease with time. On the one hand this seems rather natural and, to the best of our knowledge, no convincing counterexample has been provided thus far. On
the other hand, it has never been really proved. Here we examine whether our system (a Schawarschild black hole plus Hawking radiation in a spherical box) obeys this law.

Notice that Page’s stress-energy tensor \( T \) was obtained assuming a static space-time whereby \( M \) was necessarily constant. However, if \( M \) is large enough, the evaporation rate will be very small, namely: \( \dot{M} \simeq -\alpha(M) M^{-2} \), with \( \alpha(M) \) a small and nearly constant quantity for most of the black hole life-time. Therefore, in this situation temporal derivatives are admissible and equations (10) and (11), below, may be considered valid so long as \( M \gg 1 \).

The temporal derivative of the total entropy can be written as,

\[
\dot{S} = \dot{S}_{bh} + \dot{S}_{rad} = \lambda(R, x) \left| \dot{M} \right|
\]

with

\[
\lambda(R, x) = -\frac{4\pi R}{x} + \frac{1}{180R} \left[ x^4 - 6x^3 + 11x^2 + 4x + 5 + 2x^{-1} - 73x^{-2} \right].
\]

This equation is somewhat misleading as it seems to imply that the GSL is violated (\( \dot{S} < 0 \)) in the interval \( 1 < x \leq x_0 \), where \( x_0 \) is implicitly defined by setting \( \lambda(R, x) \) to zero, see Fig. 3.

In reality, it does appear violated only when, alongside \( \lambda \) being negative the system is out of
stable equilibrium. As shown in Ref. [1], for $R = 200$ in the interval $1 < x \leq x_{\text{min}} \simeq 38.558$ (see table 2 in the said reference) the black hole is in stable thermodynamic equilibrium with the radiation. As a consequence, $\dot{M} = 0$ and so $\dot{S} = 0$ as well. It is just in the subinterval $x_{\text{min}} < x < x_0$ (with $x_0 \simeq 40.24$) that $\lambda < 0$ and, simultaneously, the system is in an unstable thermodynamic equilibrium state whence $\dot{S} < 0$ and the GSL is violated. Naturally, when $x > x_0$ the GSL is satisfied since the equilibrium is unstable (whence $\dot{M} \neq 0$) and $\lambda > 0$. Fig. 4 illustrates this situation. As numerical calculation confirms, this pattern holds true regardless the radius of the box. Therefore though, formally, $\dot{S} < 0$ in the interval $(1, x_{\text{min}}]$, the entropy will remain constant in that interval (because stable thermodynamic equilibrium prevails in it). Then, $S$ will decrease and later on (from $x_0$ onwards) increase. Consider for instance the case shown in Fig. 4. Imagine that initially $x = 39$ (a value in the subinterval $(x_{\text{min}}, x_0)$); since there is no stable equilibrium, the black hole will lose mass to Hawking radiation continuously. This alongside the size of the box being a constant implies that $x$ will increase unbounded, and sooner or later $\dot{S}$ will become positive (right after $\lambda$ had crossed the horizontal axis). Thus even if the GSL is not fulfilled initially, it will eventually.

Clearly if $x_{\text{min}}$ could be greater than $x_0$, the region where $\lambda$ is negative would be non-existent and $\dot{S}$ would never become negative. However, as Fig. 5 depicts, this is not possible, i.e., the inequality $x_{\text{min}} > x_0$ is not to be satisfied ever. If it were, the $\lambda$ curve would undergo an un-physical discontinuity, $\Delta \lambda(R, x)$, at $x = x_{\text{min}}$. It would mean that right after the loss of stable thermal equilibrium between the black hole and Hawking radiation, the initial $\dot{S}$ instead of being close to zero (though positive), would be $\lambda(R, x_{\text{min}}) \mid \dot{M} \mid$. As it can be numerically verified, this result does not depend on the value of $R$. Therefore, the GSL won’t be violated whenever $x_{\text{min}}$ coincides with $x_0$.

### A. A Gedanken experiment

Consider a black hole of mass $M_1$ surrounded by spherical box, of radius $R_1$, of perfectly reflecting thin walls centered at the black hole, and further away a similar box of radius $R_2 > R_1$. Also assume that the black hole is in stable thermodynamic equilibrium with the Hawking radiation filling the inner box. Between both boxes there is a vacuum. Then, imagine that the smaller box is gently removed whence the Hawking radiation (originally
Figure 4. In the interval $1 < x \leq x_{\text{min}}$ ($x_{\text{min}} \simeq 38.558$, with $R = 200$) stable thermodynamic equilibrium between the black hole and Hawking radiation holds, hence $\dot{M} = \dot{S} = 0$. Beyond $x_{\text{min}}$ the equilibrium is unstable, therefore between $x_{\text{min}}$ and $x_0$ one has $\dot{M} \neq 0$ and $\dot{S} < 0$ as $\lambda$ is negative. For $x \geq x_0 \simeq 40.24$, $\lambda$ is positive and so is $\dot{S}$. The ascending line, starting at $x_{\text{min}}$, corresponds to the graph of $\lambda(200, x \geq x_{\text{min}})$.

Figure 5. In drawing the plot, for $R = 200$, it was artificially assumed that $x_{\text{min}} = 50$, larger than $x_0 \simeq 40.24$. As a result the $\lambda$ curve never lies below the horizontal axis but it experiences an un-physical discontinuity, namely $\Delta \lambda \simeq 103.28$ in our units.

filling just the smaller box) expands and fills the larger box while it gets colder and falls out
of equilibrium with the hole. By the Le Châtelier-Braun principle [9] the black hole reacts and gets hotter by emitting Hawking radiation. Now, two distinct possibilities, depending on whether the total heat capacity of the system is negative or positive, arise. In the first case, a new stable equilibrium, at a higher temperature, will be achieved. In the second, no stable equilibrium is feasible [1, 2, 13] and by a continuous evaporation process the black hole will shrink more and more while the radiation temperature rises enormously (the details of this process, especially his last stages, in which the black hole may even disappear altogether, are not well-known; actually, it is most likely a subject of quantum gravity). In any case, the Bekenstein-Hawking entropy of the black hole will decrease as $T_{bh}^{-2}$ whereas the radiation entropy will rise approximately as $T_{bh}^3$. Consequently, the final entropy will be larger than the entropy of the system before removing the inner box. We shall focus on the first possibility, namely, that of a final state of thermodynamic stable equilibrium.

In this thought experiment the total energy is conserved (i.e., $M_1 + E_{rad1} = M_2 + E_{rad2}$). If we fix the values of $R_1, x_1$ and $R_2$, then, $x_2$ will be set by the energy conservation equation (a constraint equation). Once $x_2$ is found, we can determine whether the total entropy $(S = S_{bh} + S_{rad})$ augments or decreases in the process $(M_1, R_1) \Rightarrow (M_2, R_2)$. Since the latter is irreversible, one should expect the final entropy to be larger than the initial one. But this is not ensured beforehand and it should be checked. Fig. 6 can be interpreted as follows. Let us focus first on the lower straight line. In this case, the system has a total energy $E = 4.2$ and it is in stable thermodynamic equilibrium. Then, the inner box, of radius $R_1 = 100$, is removed and the radiation expands up to the larger box, of radius $R_2 = 300$. The system attains a new stable equilibrium but at a higher temperature. In this process the quantity $x$ increases from about 11.946 up to about 41.608 and the black hole mass goes down by a factor close to 0.86. Next we focus on the upper straight line. In this instance the total energy is 8; everything parallels the previous case except that now $x_1 \simeq 8.339$, $x_2 \simeq 25.365$, and the mass of the black hole decreases by a factor of about 0.98.

We are now poised to calculate the variation of entropy in both cases. Clearly

$$\Delta S = \pi \left( \frac{R_2^2}{x_2} - \frac{R_1^2}{x_1} \right) + S_{rad2} - S_{rad1}, \tag{12}$$

where $S_{rad1}$ denotes $S_{rad}(M_1, R_1)$, and analogously so does $S_{rad2}$.

Upon inserting the numerical values quoted above ($R_1 = 100$, $R_2 = 300$, $x_1 = 11.946$ and $x_2 = 41.608$) for the case $E = 4.2$ in last expression, it yields $\Delta S \simeq -4.02$; this is to say,
Figure 6. Total energy vs $x$ assuming the radius of the inner and outer boxes being 100 and 300, respectively (from the left curve to the right one). The horizontal straight lines correspond to $E = 4.2$ and $E = 6$ in our units. The lower one first intersects the left and right curves at $x \simeq 11.946$ and $x \simeq 41.608$, respectively. The corresponding intersection points of the upper one are $x \simeq 8.339$ and $x \simeq 25.365$.

the GSL is not satisfied.

For the $E = 6$ case, we obtain $\Delta S \simeq -4.536$. Thus, the GSL is violated also in this instance. At any rate, inspection of Eq. (12) shows that for $x_2$ large enough, keeping $R_1$ and $x_1$ fixed, the GSL holds true.

One may think that the GSL could be easier analyzed by plotting the total entropy ($S = 4\pi M^2 + S_{rad}$) as a function of $x$. However, one should be mindful of the two only possibilities, namely: (i) The system, black hole plus Hawking radiation, is in state of stable thermodynamic equilibrium. In such a case $M$ as well $S_{rad}$ remain constant. (ii) The system is out of equilibrium. In this second case, the expression for $S_{rad}$, Eq. (6), does not apply.

V. SUMMARY AND DISCUSSION

By means of Page stress-energy tensor (11) and Euler’s thermodynamic equation we derived an approximate expression, Eq. (5), for the entropy of Hawking radiation filling a thin spherical box of perfectly reflecting walls. The said expression assumes thermal stable equi-
librium between the black hole and radiation. It has some nice features; among other things, it solely depends on the ratio between the radius of the box and the black hole mass but not on any of these two quantities by themselves.

While the Bekenstein bound \[4\] is fulfilled, our results suggest that the GSL might be violated not only when \(R\) is of the order of \(M\) but also when it is larger since the temporal derivative of the total entropy decreases with time. Also a Gedanken experiment that compares the total entropy after and before removing a intermediate box around the black hole shows violation of the GSL. The latter is recovered when the bigger box is much greater than the intermediate one.

The violation of the GSL is surprising. However, at this stage, it would be premature to claim that it is the case. This result may arise from the fact that Page’s stress-energy tensor, despite being a very good approximation, is not exact, especially near the black hole horizon where the energy density and entropy of the radiation become negatives. Further, we have ignored the back reaction of the radiation on the metric and the effect of the vacuum polarization of the wall of the box. Further, we might have overlooked some other source of entropy. Nevertheless, we believe our study points out the convenience to consider the subject more carefully. Actually, its solution may not be round the corner and, possibly, we are to wait for a reliable theory of quantum gravity to settle the matter.

Nonetheless, even if the GSL is not fulfilled in some phase of the evaporation it will be globally respected. Indeed, consider the entropy of a cloud of \(N\) particles, \(S_{\text{cloud}} \sim N\) (see e.g. \[14\]), just before undergoing a full gravitational collapse. The Bekenstein-Hawking entropy of the ensuing black hole will be \(S_{\text{bh}} \sim N^2\) and the GSL satisfied. However, the partial or total evaporation of this object results into a cloud of particles whose entropy will be again given by the above expression, \(S_{\text{new-cloud}} \sim N_{\text{new-cloud}}\) (in general both numbers will differ, and in the last phases of the evaporation \(N_{\text{new}}\) will attain a much bigger value than \(N\)). In any case, even though the GSL might be transcended in the evaporation process, the total process,

\[
\text{Initial cloud} \rightarrow \text{Black Hole} \rightarrow \text{Evaporation} \rightarrow \text{Final cloud},
\]

may well comply with the GSL on account of the inequality \(N_{\text{new-cloud}} > N\). If the GSL is fulfilled in the evaporation, then we would also have \(N_{\text{new-cloud}} > N^2\). Obviously, this is just a hand-waving argument whence it is desirable to verify its outcome by a more rigorous study.
Finally, it would be worthwhile to establish a relation between our expression of Hawking entropy and the expression for the entanglement entropy derived by Funai and Sugawara [15] on the principle of local field theory and the assumption of entanglement between the radiation and the black hole interior. However, this approach presents the drawback that for the process to be unitary the singularity at the center of the Schwarzschild hole must be replaced by a solid core. Maybe this difficulty will disappear when a better understanding of the process itself is achieved.

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