On the $I = 2$ channel $\pi - \pi$ interaction in the chiral limit 

H. R. Fiebig$^a$, H. Markum$^b$, A. Mihály$^c$, and K. Rabitsch$^b$

$^a$Physics Department, FIU-University Park, Miami, Florida 33199, USA

$^b$Institut für Kernphysik, Technische Universität Wien, A-1040 Vienna, Austria

$^c$Department of Theoretical Physics, Lajos Kossuth University, H-4010 Debrecen, Hungary

An approximate local potential for the residual $\pi^+ - \pi^+$ interaction is computed. We use an $O(a^2)$ improved action on a coarse $9^3 \times 13$ lattice with $a \approx 0.4$fm. The results present a continuation of previous work: Increasing the number of gauge configurations and quark propagators we attempt extrapolation of the $\pi^+ - \pi^+$ potential to the chiral limit.

1. INTRODUCTION

The use of improved lattice actions allows to work with lattice volumes large enough to accommodate systems of two hadrons, with manageable computational effort. We take advantage of this opportunity to study the residual effective interaction of two pseudo-scalar mesons on the lattice. This paper is a report on the current status of this project. An earlier exploratory study [1] was made with a smaller number of gauge configurations and two, somewhat large, quark masses. The new results presented here are based on 208 configurations and 6 values of quark masses, with otherwise unchanged lattice parameters. The current simulation allows extrapolation of the extracted $\pi^+ - \pi^+$ potential to the chiral limit and a comparison of lattice-based scattering phase shifts, computed with the potential, to experimental results [2,3] in the isospin $I = 2$ channel.

2. LATTICE PARAMETERS

An $L^3 \times T = 9^3 \times 13$ lattice was used with an $O(a^2)$ tree-level and tadpole improved action based on next-nearest neighbor couplings [4]. At $\beta = 6.2$, in the conventions of [4], the corresponding lattice constant is $a \approx 0.4$fm, or $a^{-1} \approx 500$MeV. We have used $N_U = 208$ quenched gauge configurations. The hopping parameters for Wilson fermions were set to $\kappa^{-1} = 5.720, 5.804, 5.888, 5.972, 6.056, 6.140$. The critical value for $\kappa^{-1}$ is $\approx 5.5$. Quark propagator matrix elements were computed using a random-source technique [7] with $N_R = 8$ Gaussian sources.

3. METHOD

An outline of the theoretical framework may be found in [1,7], we here mention only the essential points. Suitable operators for the $\pi^+ - \pi^+$ system, having isospin $I = 2$, are

$$\Phi_{\vec{p}}(t) = \phi_{-\vec{p}}(t) \phi_{\vec{p}}(t),$$

where

$$\phi_{\vec{p}}(t) = L^{-3} \sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \bar{\psi}(\vec{x}, t) \gamma_5 \nu(\vec{x}, t)$$

describes single mesons with lattice momenta $\vec{p}$. The correlation matrix for the $\pi^+ - \pi^+$ system

$$C_{\vec{p}, \vec{q}} = \langle \Phi_{\vec{p}}(t) \Phi_{\vec{q}}(t_0) \rangle = C_{\vec{p}, \vec{q}} + C_I_{\vec{p}, \vec{q}}$$

is a sum of a free, $\tilde{C}$, and a residual-interaction contribution, $C_I$. The free $\pi^+ - \pi^+$ correlator $\tilde{C}$ is diagonal in $p, q$. We also implement link variable fuzzing and operator smearing [8,9] at the sink.

We define an effective interaction through

$$H_1 = -\frac{\partial}{\partial t} \ln(\tilde{C}^{-1/2} C \tilde{C}^{-1/2})$$

Matrix elements of $H$ are obtained from linear fits to the logarithm of the eigenvalues of the correlators $\bar{C}$ and $C$. At this point, only the diagonal elements of $C$ were utilized. Momentum-space matrix elements $(\vec{p}|H|\vec{q})$ are computed in a truncated basis of small lattice momenta. The Fourier transform to coordinate space contains a local potential

$$V(\vec{r}) = \sum_\vec{q} e^{-2i\vec{q} \cdot \vec{r}} (-\vec{q}|H_1||\vec{q}) .$$

Its $s$-wave projection ($\ell = 0$) makes use of only the irreducible representation $A_1$ of the lattice symmetry group $O(3, Z)$. In terms of the corresponding reduced matrix elements we have

$$V(r) = \sum_q j_0(2qr)(q|H_1^{(A_1)}|q) ,$$

where $j_0$ is a spherical Bessel function.

4. RESULTS

Potentials according to (6) for one selected value of $\kappa$ are shown in Fig. 1. The sum over (on-axis) momenta $q = \frac{2\pi}{L} k$ was truncated at increasing $k_{\text{max}} = 0, 1, 2, 3, 4$, respectively. A detailed error analysis is in progress, however, errors appear to increase with $k$. The results presented below are for the truncation at $k_{\text{max}} = 3$.

Chirally extrapolation of the potential was done by linear fits of $V(r)$ versus $m^2$ for a fine (plot-grade) mesh of values $r$, fixed one at a time. Using sets of 3 through 6 data points, corresponding to the smallest available values of $m^2$, gives very similar results. The subsequent analysis was performed with 3 data points.

The extrapolated potential $V(r)$ is shown in Fig. 2 as a dashed line. The oscillations of $V(r)$ in the region $r > 2$ are due to the Fourier transform of the truncated momentum sums. The wave length is indicative of the lattice resolution at the current truncation.

A parametric fit to $V(r)$ with

$$V^{(\alpha)}(r) = \alpha_1 \frac{1 - \alpha_2 r^{\alpha_5}}{1 + \alpha_3 r^{\alpha_5} + \alpha_4 r} + \alpha_0 ,$$

at $\alpha_5 = 2$ fixed, was applied to the extrapolated potential. The result is shown in Fig. 2 as a solid line. It suggests attraction at short distances followed by a repulsive barrier.

We have used $V^{(\alpha)}(r)$ in a Schrödinger equation to calculate $s$-wave scattering phase shifts $\delta_{I=2}(p)$, see Fig. 3. The pion mass was set to multiples of the experimental value, corresponding to $m_{\pi} = 0.28$ in units of $a^{-1}$. The repulsive nature of the phase shifts is due to the hump of $V^{(\alpha)}(r)$ around, and extending beyond, $r \approx 1$, see Fig. 2. The data points in Fig. 3 are experimental results compiled from [2,3].

5. CONCLUSION

Scattering phase shifts for the $I = 2$ channel $\pi-\pi$ system were computed from lattice QCD by way of extracting a non-relativistic potential. Since at this point an error analysis is pending, particularly systematic, errors are unknown. The range of the extracted potential is short compared to the current spatial resolution. The latter is determined by the somewhat large value of the lattice constant, and by the limitations imposed by the momentum truncation for the correlator matrices. This situation makes it difficult to reli-
Figure 2. Result $V(r)$ of the chiral extrapolation shown by the dashed line. A parametric fit with $V^{(\alpha)}(r)$, see (7), gives the solid line.

ably extract details of $V(r)$. (The current lattice parameters should be better suited for studying interactions involving, larger sized, baryons.) In addition to $V(r)$ there is also present a nonlocal potential \[[1]\] which has not yet been computed.

The scattering phase shifts obtained from the underlying lattice study show repulsive behavior in the low-momentum region and, in this respect, compare favorably to experimental findings. Quantitatively, the computed phase shifts are too small by a sizeable factor. Relativistic corrections are at the 40% level at $p \approx 0.6a^{-1}$.

Acknowledgement: We would like to thank R.M. Woloshyn for a multiple-mass solver.

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