Non-chiral Intermediate Long Wave equation and inter-edge effects in narrow quantum Hall systems

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We present a non-chiral version of the Intermediate Long Wave (ILW) equation that can model nonlinear waves propagating on two opposite edges of a quantum Hall system, taking into account inter-edge interactions. We obtain exact soliton solutions governed by the hyperbolic Calogero-Moser-Sutherland (CMS) model, and we give a Lax pair, a Hirota form, and conservation laws for this new equation. We also present a periodic non-chiral ILW equation, together with its soliton solutions governed by the elliptic CMS model.

Introduction. One important feature of the Fractional Quantum Hall Effect (FQHE) is the strikingly high accuracy by which the Hall conductance, $\sigma_H$, is measured in units of the inverse von Klitzing constant, $e^2/h$. Therefore, satisfactory explanations of these FQHE measurements, $\sigma_H e^2/h = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots$, must be based on exact analytic arguments, and theories of the FQHE have close connections to integrable systems. Two important classes of integrable systems which are seemingly very different but which are both connected with the FQHE are (i) Calogero-Moser-Sutherland (CMS) models describing FQHE edge states, and (ii) soliton equations of Benjamin-Ono (BO) type describing the dynamics of nonlinear waves propagating along FQHE edges.

These systems are related by a fundamental correspondence between CMS systems and BO-type soliton equations, which provides the basis for a mathematically precise derivation of hydrodynamic descriptions of CMS systems. It is worth noting that this subject has recently received considerable attention in the context of non-equilibrium physics.

While the CMS-BO correspondence has been successfully used to understand FQHE physics, it is incomplete. Indeed, CMS systems come in four types: (I) rational, (II) trigonometric, (III) hyperbolic, and (IV) elliptic, and while the soliton equations related to the rational and trigonometric cases are well-understood since a long time, soliton equations related to the hyperbolic and elliptic cases were only recently identified as the Intermediate Long Wave (ILW) equation and the periodic ILW equation, respectively. However, as we will show in this paper, the latter two soliton equation are not unique: there are other equations which are more interesting in that they are of a different kind and describe new physics.

The correspondence between CMS and BO systems exists both at the classical and at the quantum level, and our discussion above applies to both cases. We first discovered the quantum elliptic version of the soliton equation presented in this letter from a second quantization of the quantum elliptic CMS model. However, when presenting results in the following, we restrict ourselves to the classical case for simplicity; corresponding quantum results will be presented elsewhere. We first give and prove our results in the hyperbolic case; the generalization to the elliptic case is surprisingly easy, as will be shown later on.

Outline. We start with some physics motivation which suggests that there exists a parity-invariant soliton equation accounting for nonlinear waves propagating on two opposite edges of a FQHE system boundary and taking into account interactions between different edges. Next, the non-chiral ILW equation is presented, together with its $N$-soliton solutions governed by the hyperbolic CMS model. To show integrability, a Lax pair, a Hirota form, and conservation laws for this equation are presented. Finally, we generalize of our results to the elliptic case. Some technical details are given in two appendices.

Motivation. The CMS models can be defined by Newton’s equations

$$\ddot{z}_j = -\sum_{k \neq j} V'(z_j - z_k) \quad (j = 1, \ldots, N) \quad (1)$$

where the two-body interaction potential is $V(r) = 4r^{-2}$ in the rational case and $V(r) = 4(\pi/L)^2 \sin^2(\pi r/L)$, $L > 0$, in the trigonometric case (the arguments in this paragraph apply to both cases). Eq. (1) describes an arbitrary number, $N$, of interacting particles with positions $z_j \equiv z_j(t)$ at time $t$. While one often restricts to real positions when interpreting the CMS model as a dynamical system, one has to allow for complex $z_j$ when studying the relation to the BO equation. This generalization preserves the integrability. The CMS model is invariant under the parity transformation $P : z_j \rightarrow -z_j$ for all $j$. However, the corresponding BO equation is not parity-invariant: it is given by $u_t + 2uu_x + Hu_{xx} = 0$, where $u \equiv u(x,t)$ and $H$ is the Hilbert transform (in the rational case, $(Hf)(x) = (1/\pi) \int_0^\infty (x' - x)^{-1} f(x') dx'$, and under the parity transformation $P : u(x,t) \rightarrow u(-x,t) \equiv v(x,t)$, it changes to $v_t - 2vv_x - HV_{xx} = 0$. This mismatch of symmetry is paradoxical at first sight, but the paradox is resolved by interpreting $u$ as a wave propagating on one edge of a FQHE system and noting that, in general, there is another edge far away carrying another wave $v$. Thus, actually, the rational CMS model corresponds to two uncoupled BO equations for $u$ and $v$. This system of equations is invariant under a parity
transformation interchanging $u$ and $v$, \[ P : [u(x,t), v(x,t)] \to [v(-x,t), u(-x,t)]. \] (2)

It is peculiar that these two BO equations are uncoupled, and it is for this reason that one can reduce the system to a single equation, ignoring the other. While this un-coupling is reasonable if the two edges are infinitely far apart, it is natural to ask what would happen if the two edges are parallel and close together; see Fig. 1. In this case, one would expect that the nonlinear waves propagating on the two edges interact. We now give a simple heuristic argument to suggest that the hyperbolic CMS model can describe this situation.

FIG. 1. Schematic picture of a narrow FQHE system with two edges carrying the two nonlinear waves $u(x,t)$ and $v(x,t)$.

The hyperbolic CMS model can be defined by Newton’s equations [1] with the interaction potential \[ V(r) = 4 \sum_{n \in Z} \frac{1}{(r + 2i\delta n)^2} = 4 \left( \frac{\pi}{2\delta} \right)^2 \sinh^{-2} \left( \frac{\pi}{2\delta} r \right), \] (3)

where $\delta > 0$ is an arbitrary length parameter. Dividing the particle positions $z_j$ into two groups and shifting the ones in the second group by the imaginary half-period, \[ w_k \equiv z_k - N_1 + i\delta \] for $k = 1, \ldots, N_2 = N - N_1$, with $1 < N_1 < N$, we can write these Newton’s equations as

\[ \begin{align*}
\ddot{z}_j &= -\sum_{j' \neq j}^N V'(z_j - z_{j'}) - \sum_{k=1}^{N_2} \tilde{V}'(z_j - w_k), \\
\ddot{w}_k &= -\sum_{k' \neq k}^{N_2} V'(w_k - w_{k'}) - \sum_{j=1}^{N_1} \tilde{V}'(w_k - z_j),
\end{align*} \] (4)

with $\tilde{V}(r) \equiv V(r - i\delta) = -4(\pi/2\delta)^2 \cos^{-2}(\pi r/2\delta)$, for $j = 1, \ldots, N_1$ and $k = 1, \ldots, N_2$. This can be interpreted as a model of two kinds of particles, $z_j$ and $w_k$, in which particles of the same kind interact via the singular repulsive two-body potential $V$, whereas particles of different kinds interact via the weakly attractive non-singular potential $\tilde{V}$. We interpret $\delta$ as the distance between the two edges of the FQHE system. In the rational limit $\delta \to \infty$, we have $V \to 0$, so particles of different types do not interact and the two corresponding soliton equations for $u$ and $v$ decouple; for finite $\delta$, the system is coupled.

**Non-chiral ILW equation.** In the hyperbolic case, the two-component generalization of the BO equation we propose in this letter is given by

\[ \begin{align*}
u_t + 2uu_x + T u_{xx} + \tilde{T} v_{xx} &= 0, \\
v_t - 2vv_x - T v_{xx} - \tilde{T} u_{xx} &= 0
\end{align*} \] (5)

for $u = u(x,t)$ and $v = v(x,t)$, with

\[ \begin{align*}
(Tf)(x) &= \frac{1}{2\delta} \int_{\mathbb{R}} \coth \left( \frac{\pi}{2\delta} (x' - x) \right) f(x') dx', \\
(\tilde{T}f)(x) &= \frac{1}{2\delta} \int_{\mathbb{R}} \tanh \left( \frac{\pi}{2\delta} (x' - x) \right) f(x') dx'.
\end{align*} \] (6)

The ILW equation is given by \[ u_t + 2uu_x + Tu_{xx} = 0; \] it reduces to the BO equation in the limit $\delta \to \infty$.\[23\text{-}27\]

Thus, if one drops the $\tilde{T}$-terms, \[ 5 \] corresponds to a system of uncoupled ILW equations generalizing the system of uncoupled BO equations discussed above. However, due to the presence of the $\tilde{T}$-terms, the nonlinear waves $u$ and $v$ interact. For this reason, and since equation \[ 6 \] is invariant under the parity transformation \[ 2 \], we call it the non-chiral ILW equation.\[28\] As shown below, it provides an integrable model of nonlinear waves on two edges of a FQHE system taking into account interaction effects between the edges.

**$N$-soliton solutions.** The following fundamental result shows that \[ 5 \] admits $N$-soliton solutions whose dynamics is described by the hyperbolic CMS model, thus generalizing a famous result for the rational case.\[29\] For arbitrary integers $N \geq 1$ and complex parameters $a_j$ with imaginary parts in the range $\delta/2 < \text{Im} a_j < 3\delta/2$ for $j = 1, \ldots, N$, the following is an exact solution of the non-chiral ILW equation \[ 5 \]:

\[ \begin{align*}
(u(x,t), v(x,t)) &= i \sum_{j=1}^N \left( \alpha(x - z_j(t) - i\delta/2) - \alpha(x - z_j(t) + i\delta/2) \right) + \text{c.c.}
\end{align*} \] (7)

where $\alpha(x) \equiv (\pi/2\delta) \coth(\pi x/2\delta)$ and the poles $z_j(t)$ are determined by Newton’s equations \[ 1 \] with $V(r)$ given by \[ 3 \] and with initial conditions $z_j(0) = a_j$ and

\[ z_j(0) = 2i \sum_{j' \neq j}^N \alpha(a_j - a_{j'}) - 2i \sum_{k=1}^N \alpha(a_j - \bar{a}_k + i\delta) \] (8)

(the bar denotes complex conjugation, c.c.). Thus, to obtain an exact solution of \[ 3 \], one chooses complex parameters $a_j$ satisfying $\delta/2 < \text{Im} a_j < 3\delta/2$; next, the time-evolution of $z_j(t)$ is obtained by solving the hyperbolic CMS model with initial conditions determined by the $a_j$; finally, the solution of \[ 5 \] is obtained from \[ 7 \].

Using the exact analytic solution of the hyperbolic CMS model obtained by the projection method,\[13\] the numerical effort to compute such an $N$-soliton solution at an arbitrary time, $t$, is reduced to diagonalizing an explicitly known $N \times N$ matrix. As elaborated in Appendix \[ 1 \], we tested this result by comparing with a numeric solution of \[ 5 \].
FIG. 2. Time evolution of a 2-soliton solution of the non-chiral ILW equation (5) with a $u$-channel dominated soliton (big blue and small red humps) colliding with a $v$-channel dominated soliton (big red and small blue humps), as explained in the main text. The plots show $u(x,t)$ (blue line) and $v(x,t)$ (red line) at successive times $t = (n-1)t_0$, $n = 1, \ldots, 5$; the parameters are $\delta = \pi$, $a_1 = -4 + 1.2i\delta$, $a_2 = 3 + 0.85i\delta$, and $t_0 = 2.25$.

FIG. 3. (a) Time evolution of the poles $z_j(t), j = 1, 2$, in the complex plane corresponding to the 2-soliton solution in Fig. 2. The times $t = (n-1)t_0$, $n = 1, \ldots, 5$, defined in the caption of Fig. 2 are indicated by circles; the arrows mark circles corresponding to $n = 1$. The dotted lines indicate the evolution of poles without interactions. (b) Time evolution of the centre-of-mass locations of the solitons given by $\text{Re} z_j(t)$.

Examples. The 1-soliton solution of (5) is given by

\[
\begin{pmatrix}
u(x,t) \\
\alpha(x-z(t)-i\delta/2)
\end{pmatrix} = i \left( \begin{array}{c}
\alpha(x-z(t)-i\delta/2) \\
-\alpha(x-z(t)+i\delta/2)
\end{array} \right) + \text{c.c.}
\]

(9)

where $z(t) = a + \dot{z}(0)t$, $\dot{z}(0) = -2i\alpha(a - \bar{a} + i\delta)$, with $a \in \mathbb{C}$ such that $\delta/2 < \text{Im}a \leq \delta/2$. It is important to note that $\dot{z}(0)$ is real, and therefore, $\text{Im}z(t) = \text{Im}a$ independent of $t$. Thus, the functions $u(x,t)$ and $v(x,t)$ both describe humps whose shapes do not change with time. These humps are centered at the same point and move with constant velocity, $\text{Re} z_j(t) = \text{Re}a + \dot{z}(0)t$, and their heights, max $u > 0$ and max $v > 0$, are determined by $\text{Im}a$. For $\text{Im}a$ close to $\delta/2$, max $u \gg \text{max} v$, and the solitons move to the right, $\dot{z}(0) > 0$. As $\text{Im}a$ decreases, max $u$ and $\dot{z}(0)$ decrease while max $v$ increases until, at $\text{Im}a = \delta$, max $u = \text{max} v$ and $\dot{z}(0) = 0$. Thus, if $\text{Im}a$ lies in the range $\delta < \text{Im}a < 3\delta/2$, then the 1-soliton is mainly in the $u$-channel and moves to the right; it is therefore similar to the 1-soliton solution of the standard ILW equation $u_t + 2nu_x + T u_{xx} = 0$. Similarly, when $\delta/2 < \text{Im}a < \delta$, the 1-soliton is mainly in the $v$-channel and moves to the left, similar to a 1-soliton solution of the $P$-transformed ILW equation $v_t - 2vv_x - T v_{xx} = 0$.

For parameters $a_j$ such that $\text{Re}(a_j - a_k) \gg \delta$ for all $j \neq k$, the $N$-soliton solution of (5) is well-approximated by a sum of $N$ 1-solitons of the form (8) where $\dot{z}_j(t) \approx -2i\alpha(a_j - \bar{a}_j + i\delta)$ is time-independent for times such that $\text{Re}(\dot{z}_j(t) - \dot{z}_k(t)) \gg \delta$; see Fig. 2(a) for a 2-soliton solution, with the corresponding motion of poles in Fig. 2(b).

However, when two solitons meet, they interact in a non-trivial way, and after the interaction they re-emerge with the same shape but with phase-shifts; see Fig. 3(b). Non-trivial such interactions between solitons can also be modeled by the system of decoupled ILW equations obtained from (5) by dropping the $T$-terms. A qualitatively new effect stemming from the $T$-terms is that $u$-channel dominated solitons ($u$-solitons) interact non-trivially with $v$-solitons, as clearly seen in our example in Figs. 2 and 3. It is interesting to note that the poles corresponding to the $u$- and $v$-solitons interchange their imaginary parts and directions during the collision and thus, in this sense, exchange their identities: while the first pole corresponds to the $u$-soliton and the second to the $v$-soliton before the collision, it is the other way round after the collision; see Figs. 2(a) and (b). We note that such an identity change of poles during soliton collisions is known for the BO equation but only for solitons moving in one direction.

Derivation of N-soliton solutions. We explain the key difference between the derivation of solitons for (5), and the corresponding derivation in the rational case; further details can be found in Appendix A 1.

The Hilbert transform, $H$, satisfies $H^2 = -I$, and this property is crucial for the existence of eigenfunctions of $H$ needed in the derivation of the CMS-related soliton solutions of the BO equation $u_t + 2nu_x + Hu_{xx} = 0$ However, while the trigonometric generalization of $H$ also has this property, the hyperbolic generalization of $H$ is the operator $T$ in (10), and $T^2 \neq -I$. This is the reason why the soliton solution of the BO equation straightforwardly generalizes to the trigonometric case but the naive generalization to the hyperbolic case fails. However, the non-chiral ILW equation can be written in vector form as

\[
u_t + (u \cdot u)_x + T u_{xx} = 0,
\]

\[
\mathbf{u} \equiv \left(\begin{array}{c}u \\
v\end{array}\right), \quad \mathbf{u} \cdot \mathbf{u} \equiv \left(\begin{array}{c}u^2 \\
v^2\end{array}\right), \quad T \equiv \left(\begin{array}{cc}T & \tilde{T} \\
-\tilde{T} & -T\end{array}\right)
\]

(10)

where the matrix operator, $T$, satisfies $T^2 = -I$. Moreover, $(\alpha(x+z+\pm i\delta/2), -\alpha(x+z\mp i\delta/2))$ are eigenfunctions of $T$ with eigenvalues $\pm 1$. The latter are the eigenfunctions needed to be able to use the method developed for
the rational case\textsuperscript{29} using well-known identities for the function $\alpha(z)$\textsuperscript{31} as well as a Bäcklund transformation for the hyperbolic CMS model\textsuperscript{32}; it is straightforward to adapt a known derivation of N-soliton solutions of the BO equation\textsuperscript{29} to the hyperbolic case.

Integrability. We found a Lax pair, a Hirota bilinear form, a Bäcklund transformation, and an infinite number of conservation laws for \textsuperscript{[10]}. Thus, the non-chiral ILW equation is a soliton equation that is integrable in the same strong sense as the standard ILW equation\textsuperscript{29}. Below we present some of these results that can be checked by straightforward computations.

The Lax pair we found is as follows: Let $\psi(z; t, k)$ be an analytic function on the union of the strips $0 < \Im z < \delta$ and $\delta < \Im z < 2\delta$ and extended to $\mathbb{C}$ by $2i\delta$-periodicity, $\psi_0^R(x; t, k)$ and $\psi_0^L(x; t, k)$ the boundary values of this function on $\mathbb{R}$ and $\mathbb{R} + i\delta$, respectively, and $\mu_1$, $\mu_2$, $\nu_1$, and $\nu_2$ arbitrary functions of the spectral parameter $k$.

Then the compatibility of the following linear equations yields\textsuperscript{[5]}:

\begin{align*}
(i\partial_x - u - \mu_1)\psi_0^- &= \nu_1 \psi_0^+,
(i\partial_x + v + \mu_1)\psi_0^+ &= \nu_2 \psi_0^-,
(i\partial_t - 2\mu_1i\partial_x - \partial_x^2 + Tu_x + \bar{T}v_x + iu_x + 2\mu_2)\psi_0^+ &= 0,
(i\partial_t - 2\mu_1i\partial_x - \partial_x^2 + Tv_x + \bar{T}u_x + iv_x + 2\mu_2)\psi_0^- &= 0.
\end{align*}

Inspired by known results for the BO equation\textsuperscript{13} we obtained the following Hirota bilinear form of (5):

\begin{align}
&(iD_t - D_x^2)F_- \cdot G_+ = (iD_t + D_x^2)G_- \cdot F_+ = 0
\end{align}

with $u = i\partial_x \log(F_-/G_+)$ and $v = i\partial_x \log(G_-/F_+)$, where $F_\pm(x, t) = F(x \pm i\delta/2, t)$ and similarly for $G$, using standard Hirota derivatives\textsuperscript{33}.

The first three of the conservation laws we found are

\begin{align}
I_1 &= \int_R (u + v)dx, \quad I_2 = \int_R (u^2 - v^2)dx, \\
I_3 &= \int_R \left[ \frac{u^3}{3} + \frac{uTu_x}{2} + \frac{uTv_x}{2} + (u \leftrightarrow v) \right]dx
\end{align}

with $(u \leftrightarrow v)$ short for the same three terms but with $u$ and $v$ interchanged.

Bäcklund transformations, other conservation laws, and detailed derivations will be given elsewhere.

Elliptic case. To generalize \textsuperscript{[6]} to the periodic setting, we use the Weierstrass functions $\wp(z)$ and $\zeta(z)$ with periods $(2\omega_1, 2\omega_2) \equiv (L, 2i\delta)$\textsuperscript{34} $L > 0$, and the related functions $\zeta_j(z) \equiv \zeta(z) - n_j \omega_1/\omega_1$, $n_j \equiv \zeta(x)$, $j = 1, 2$. The function $\zeta_1(z)$ is $L$-periodic, $\zeta_1(z + L) = \zeta_1(z)$, whereas the function $\zeta_2(z)$ is $2i\delta$-periodic, $\zeta_2(z + 2i\delta) = \zeta_2(z)$; recall that $\zeta(z)$ is neither $L$- nor $2i\delta$-periodic.

The periodic generalization of the non-chiral ILW equation we found can be written as in \textsuperscript{[3]} but changing the definitions of $T$, $\hat{T}$ in \textsuperscript{[10]} as follows: replace (1/2$\delta$) $\coth(\pi(x' - x)/2\delta)$ and (1/2$\delta$) $\tanh(\pi(x' - x)/2\delta)$ by $\zeta_1(x' - x)/\pi$ and $\zeta_2(x' - x + i\delta)/\pi$, respectively; see \textsuperscript{[A17]}. With that, $\hat{T}$ in \textsuperscript{[10]} still satisfies $\hat{T}^2 = -I$.

and the derivation of the $N$-soliton equation outlined above generalizes straightforwardly to the elliptic case provided $\alpha(z)$ in \textsuperscript{[A9]} is chosen as the $2i\delta$-periodic variant of $\zeta(z)$: The functions $u(x, t)$ and $v(x, t)$ given in \textsuperscript{[11]}, with $\alpha(x) = \zeta_2(z)$, satisfy the periodic non-chiral ILW equation provided that $z_j(t)$ satisfy Newton's equations \textsuperscript{[1]} with the elliptic CMS model potential $V(r) = 4\zeta(r)$, and with initial conditions $z_j(0) = a_j$ and $\dot{z}_j(0)$ in \textsuperscript{[10]}.

Final remarks. We presented the novel soliton equation \textsuperscript{[9]}. We call it the non-chiral ILW equation because it is parity invariant and can describe interacting solitons moving in both directions. We obtained exact $N$-soliton solutions determined by poles satisfying the equations of motion of the hyperbolic CMS model, and we gave a Lax pair, a Hirota form, and conservation laws. We also presented a periodic non-chiral ILW equation and its soliton solutions determined by the elliptic CMS model.

Many soliton equations containing only first-order derivatives in time are chiral, i.e., they can only describe solitons moving in one direction, left or right, and thus are not parity invariant. Examples include the Korteweg-de Vries equation, the BO equation and, more generally, the ILW equation. However, the fundamental equations in hydrodynamics from which these soliton equations are derived are parity invariant. This mismatch of symmetries is similar to the one discussed in this paper. Using the non-chiral ILW equation instead of the standard ILW equation would reconcile symmetries, and it therefore is tempting to speculate that the former is a better approximation to the fundamental equations than the latter.

We hope that our results open up a route to generalize recent results on a generalized hydrodynamic description of the Toda chain\textsuperscript{16,17} to the elliptic CMS model. This would be interesting since, in the elliptic CMS model, one can change the qualitative character of the interaction from long-range in the trigonometric case, to short-range in the hyperbolic case, to near-neighbor in the Toda limit.

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Appendix A: Derivation of soliton solutions

We give details on the derivation of the $N$-solution solutions presented in the main text, both in the hyperbolic and elliptic cases.

1. Hyperbolic case

We construct solutions of (10) with $T, \hat{T}$ defined in (6) by generalizing a known method for the BO equation.

a. Integral operators in Fourier space

We compute the Fourier space representation of the matrix operator $T$ in (10).

We start by transforming the operators $T, \hat{T}$ in (6) to Fourier space, using the following exact integral,

$$
\int_{\mathbb{R}} \frac{\pi}{2\delta} \coth\left(\frac{\pi}{2\delta} x + ia\right) e^{-ikx} dx = -\pi e^{\pm(ak - k\delta)} \sinh(k\delta) \quad (A1)
$$

for real parameters $a, k$ such that $0 < a < 2\delta$ and $k \neq 0$ (a derivation of this result can be found at the end of this section). This implies

$$
\int_{\mathbb{R}} \frac{1}{2\delta} \coth\left(\frac{\pi}{2\delta} x\right) e^{-ikx} dx = -i \coth(k\delta) \quad (A2)
$$

for real $k \neq 0$. Indeed, the first of these identities is equivalent to the average of the two integrals in (A1) in the limit $a \downarrow 0$, and the second is obtained from (A1) in the special case $a = \delta$. Observe that the integrals in (6) are convolutions. Using the following conventions for Fourier transformation, $\hat{u}(k) = \int_{\mathbb{R}} u(x) e^{-ikx} dx,$ the operators defined in (6) can therefore be expressed in Fourier space as follows,

$$
\left(\mathcal{T} u\right)(k) = i \coth(k\delta) \hat{u}(k),
\left(\mathcal{T} \hat{u}\right)(k) = \frac{1}{i \sinh(k\delta)} \hat{u}(k). \quad (A3)
$$

Thus, for the matrix operator $\mathcal{T}$ defined in (10), $\hat{T} u(k) = \mathcal{T}(k) \hat{u}(k)$ with

$$
\mathcal{T}(k) = i \begin{pmatrix}
\coth(k\delta) & 1/\sinh(k\delta)
-1/\sinh(k\delta) & -\coth(k\delta)
\end{pmatrix} \quad (A4)
$$

and $\hat{u}(k) = (\hat{u}(k), \hat{v}(k))^t$ for $u(x) = (u(x), v(x))^t$. Using this, it is easy to check that $\mathcal{T}(k)^2 = -I$, which is equivalent to $\mathcal{T}^2 = -I$.

b. Eigenfunctions

Since $\mathcal{T}^2 = -I$, the eigenvalues of $\mathcal{T}$ are $\pm i$. We now construct the corresponding eigenfunctions.

By straightforward computations we obtain the following eigenvectors of the matrix $\mathcal{T}(k)$ in (A4),

$$
\hat{g}(k) \begin{pmatrix}
e^{\pm ik\delta/2}
-e^{\mp ik\delta/2}
\end{pmatrix} \quad (A5)
$$

with corresponding eigenvalues $\pm i$, for an arbitrary function $g(k)$ of $k$. To get eigenfunctions of $\mathcal{T}$ with appropriate analyticity properties, we restrict ourselves to functions $\hat{g}(k)$ such that $\hat{g}(ke^{\alpha})$ has a well-defined inverse Fourier transform $g(x - ia)$ in a strip $-A < \alpha < A$ with $A > \delta/2$. For such functions,

$$
\int_{\mathbb{R}} \frac{dk}{2\pi} \hat{g}(k) e^{\pm ik\delta/2} e^{ikx} = g(x \mp i\delta/2), \quad (A6)
$$

and the eigenfunctions of the operator $\mathcal{T}$ are therefore as follows: For arbitrary complex valued functions $g(z)$ of $z \in \mathbb{C}$ analytic in a strip $-A < \text{Im}(z) < A$ with $A > \delta/2$, the vector valued functions

$$
v_{\pm}(x) \equiv \begin{pmatrix} g(x \mp i\delta/2) 
g(x \pm i\delta/2) \end{pmatrix} \quad (A7)
$$

satisfy

$$
\mathcal{T} v_{\pm}(x) = \pm i v_{\pm}(x). \quad (A8)
$$

c. Pole ansatz

Inspired by the CMS-related soliton solutions known for the BO equation, we make the following ansatz

**Derivation of (A1).** Suppose $0 < a < 2\delta$ and define the function $h(x)$ by

$$
h(x) = \frac{\pi}{2\delta} \coth\left(\frac{\pi}{2\delta} (x - ia)\right). \quad (A9)
$$

Even though $h(x)$ does not decay as $x \to \pm\infty$, the Fourier transform $\hat{h}$ of $h$ is well-defined as a tempered distribution. Indeed, the derivative

$$
h'(x) = -\left(\frac{\pi}{2\delta} \sinh\left(\frac{\pi(x - ia)}{2\delta}\right)\right)^2
$$

has exponential decay as $x \to \pm\infty$ and has a double pole at $x = ia + 2i\delta n$ for each integer $n$. Its Fourier transform $\hat{h}'$ can be computed by a residue computation. The Fourier transform $\hat{h}$ can then be obtained for $k \neq 0$ by $\hat{h}(k) = (\hat{h}')(k)/(ik)$. A similar computation applies if $-2\delta < a < 0$, and we arrive at (A1).
to solve (10),
\[
\mathbf{u}(x, t) = i \sum_{j=1}^{N} \left( \alpha(x - z_j(t) - i \delta/2) \right.
\left. - \alpha(x - z_j(t) + i \delta/2) \right)
- i \sum_{j=1}^{M} \left( \alpha(x - w_j(t) + i \delta/2) \right.
\left. - \alpha(x - w_j(t) - i \delta/2) \right)
\]  
(A9)

where \(\alpha(x) = (\pi/2\delta) \coth(\pi x/2\delta)\), \(N, M\) are arbitrary integers \(\geq 0\), and with poles \(z_j(t)\) and \(w_j(t)\) to be determined. We note that, to obtain real-valued solutions, one must restrict this ansatz to (7), i.e., \(M = N\) and \(w_j(t) = \bar{w}_j(t)\) for all \(j\), but we find it convenient to derive a more general result. In the following, we sometimes write \(z_j\) as shorthand for \(z_j(t)\), etc.

The function \(\alpha(z)\) is meromorphic with poles at \(z = 2i \delta n, n \text{ integer}\). Thus, if we restrict the imaginary parts of \(z_j\) and \(w_j\) as follows,
\[
\text{Im}(z_j \pm i \delta/2) \neq 2\delta n, \quad \text{Im}(w_j \pm i \delta/2) \neq 2\delta n \quad \text{(A10)}
\]
for all integers \(n\), then the result in (A7)–(A8) implies
\[
\mathcal{T} \mathbf{u}_{xx} = - \sum_{j=1}^{N} \left( \alpha''(x - z_j - i \delta/2) \right.
\left. - \alpha''(x - z_j + i \delta/2) \right)
- \sum_{j=1}^{M} \left( \alpha''(x - w_j + i \delta/2) \right.
\left. - \alpha''(x - w_j - i \delta/2) \right)
\]  
(A11)

with \(\alpha'(z) \equiv \partial_x \alpha(z)\) etc. We now use \(\alpha(-z) = -\alpha(z)\) and the well-known identity:
\[
\alpha''(z) = -\frac{1}{4} V'(z), \quad \partial_x [\alpha(z)^2] = \frac{1}{4} V'(z),
\]
\[
\alpha(z + 2i \delta) = \alpha(z), \quad \partial_x [\alpha(x-a)\alpha(x-b)] = \partial_x [\alpha(x-a) - \alpha(x-b)]\alpha(a-b),
\]  
(A12)

with \(V\) in (3), and for arbitrary \(a, b, \in \mathbb{C}\). Using this we compute
\[
\mathbf{u}_t + (\mathbf{u}, \mathbf{u})_x + \mathcal{T} \mathbf{u}_{xx} = \frac{1}{4} \sum_{j=1}^{N} \left( V'(x - z_j - i \delta/2) - V'(x - z_j + i \delta/2) \right)
\times \left( i \dot{z}_j + 2 \sum_{k \neq j} \alpha(z_j - z_k) - 2 \sum_{k=1}^{M} \alpha(z_j - w_k + i \delta) \right)
+ \frac{1}{4} \sum_{j=1}^{M} \left( V'(x - w_j + i \delta/2) - V'(x - w_j - i \delta/2) \right)
\times \left( i \dot{w}_j + 2 \sum_{k=1}^{N} \alpha(w_j - z_k + i \delta) - 2 \sum_{k \neq j} \alpha(w_j - w_k) \right)
\]

(the computations leading to this result are nearly the same as in the BO case (12) and thus omitted). This implies the following result: The function \(\mathbf{u}(x, t)\) in (A9) is a solution of the non-chiral ILW equation (10) provided the following system of equations is satisfied,
\[
\dot{z}_j = 2i \sum_{k \neq j}^N \alpha(z_j - z_k) - 2i \sum_{k=1}^{M} \alpha(z_j - w_k + i \delta),
\]  
(A13)
\[
\dot{w}_j = 2i \sum_{k \neq j}^N \alpha(w_j - z_k + i \delta) - 2i \sum_{k \neq j}^M \alpha(w_j - w_k),
\]

and the conditions in (A10) hold true.

The system in (A13) is known as a Bäcklund transformation for the hyperbolic CMS system. It implies two decoupled systems of Newton’s equations,
\[
\ddot{z}_j = - \sum_{k \neq j}^N V'(z_j - z_k) \quad (j = 1, \ldots, N),
\]  
(A14)
\[
\ddot{w}_j = - \sum_{k \neq j}^M V'(w_j - w_k) \quad (j = 1, \ldots, M)
\]

with \(V\) as in (3), see Ref. 33 for a recent alternative derivation of this result. We thus obtain the following generalization of the result stated in the main text: For arbitrary non-negative integers \(N, M\) and complex parameters \(a_j, j = 1, \ldots, N\), and \(b_j, j = 1, \ldots, M\), satisfying
\[
\text{Im}(a_j \pm i \delta/2) \neq 2\delta n, \quad \text{Im}(b_j \pm i \delta/2) \neq 2\delta n \quad \text{(A15)}
\]
for all integers \(n\), the function \(\mathbf{u}(x, t)\) in (A9) is a solution of the non-chiral ILW equation (10) provided the poles \(z_j(t)\) and \(w_j(t)\) satisfy Newton’s equations for the hyperbolic CMS model in (A14) with initial conditions
\[
z_j(0) = a_j, \quad w_j(0) = b_j,
\]
\[
\dot{z}_j(0) = 2i \sum_{k \neq j}^N \alpha(a_j - a_k) - 2i \sum_{k=1}^{M} \alpha(a_j - b_k + i \delta),
\]
\[
\dot{w}_j(0) = 2i \sum_{k \neq j}^M \alpha(b_j - a_k + i \delta) - 2i \sum_{k \neq j}^M \alpha(b_j - b_k).
\]

Restricting to \(M = N\) and \(b_j = \bar{a}_j\) for all \(j\), we obtain the result stated in the main text (note that, in this special case, the initial conditions imply \(w_j(t) = \bar{z}_j(t)\) for all \(t\)).

A technical remark is in order. Strictly speaking, we proved the result above only for times, \(t\), where the conditions in (A10) hold true. We did not point out this restriction before since we believe that, if the conditions in (A10) and (A13) hold true at time \(t = 0\), then the solutions \(z_j(t)\) and \(w_j(t)\) of (A14) satisfy the conditions in (A10) for all \(t > 0\). We checked this in several special cases by integrating (A14) numerically. We expect that this can be proved in general using the known explicit solution of the hyperbolic CMS model obtained with the projection method (13) this is left for future work.
2. Elliptic case

We give details on how the derivation in Appendix A1 generalizes to the $L$-periodic case.

a. Periodic non-chiral ILW equation

We recall that the function $\zeta_1(z)$ defined in the main text has the following representation:

$$\zeta_1(z) = \frac{\pi}{L} \lim_{M \to \infty} \sum_{n=-M}^{M} \cosh \left( \frac{\pi x}{2M} \right), \quad (A16)$$

and it therefore is the natural $L$-periodic analogue of

$$\frac{\pi}{2}\coth \left( \frac{\pi x}{2L} \right) = \lim_{M \to \infty} \sum_{n=-M}^{M} \frac{1}{z - 2i\delta n}.$$ 

This suggests that the $L$-periodic version of the non-chiral IILW equation is as in [3] but with

$$(Tf)(x) = \frac{1}{\pi} \int_{-L/2}^{L/2} \zeta_1(x' - x)f(x')dx', \quad (A17)$$

$$(\tilde{T}f)(x) = \frac{1}{\pi} \int_{-L/2}^{L/2} \zeta_1(x' - x + i\delta)f(x')dx'.$$

To see that this is the correct generalization, one can check that $[A3]$ still holds true but with Fourier modes, $k$, restricted to integer multiples of $(2\pi/L)$, and for $L$-periodic functions $f(x)$ that have zero mean, $\hat{f}(0) \equiv \int_{-L/2}^{L/2} f(x)dx = 0$. Thus, $T^2 = -I$, and the result in [A7] holds true as it stands provided the function $f(z)$ is $L$-periodic, has zero mean, and is analytic in a strip $-A < \text{Im}(z) < A$ for $A > \delta/2$. In particular,

$$T \partial_x^2 \left( \begin{array}{c} \zeta_2(x - z + i\delta/2) \\ -\zeta_2(x - z - i\delta/2) \end{array} \right) = \mp i \left( \begin{array}{c} \psi'(x - z + i\delta/2) \\ -\psi'(x - z - i\delta/2) \end{array} \right) \quad (A18)$$

using $\zeta_2'(z) = -\psi'(z)$. We can use this to construct soliton solutions related to the elliptic CMS model defined by Newton’s equations [1] with the potential

$$V(x) = 4\psi(x). \quad \text{(A19)}$$

b. Pole ansatz

The discussion above suggests to use the pole ansatz in [A3] with $\alpha(x)$ equal to $\zeta_1(x)$. However, this choice does not work since the third identity in [A12] is not satisfied. The choice that works is

$$\alpha(x) = \zeta_2(x) \quad \text{(A20)}$$

since $\zeta_2(z)$ is $2\delta$-periodic. However, $\zeta_2(z)$ is not $L$-periodic: $\zeta_2(z+L) = \zeta_2(z) + c$ for some non-zero constant $c$. Thus, $u(x + L, t) = u(x, t) + i(N - M)(c, -c)^t$, and, to get a $L$-periodic function $u(x, t)$, we must restrict to $M = N$.

We use [A18] to obtain

$$Tu_{xx} = \frac{1}{4} \sum_{j=1}^{N} \left( V'(x - z_j - \delta/2) - V'(x - z_j + \delta/2) \right)$$

$$+ \frac{1}{4} \sum_{j=1}^{N} \left( V'(x - w_j + \delta/2) - V'(x - w_j - \delta/2) \right) \quad (A21)$$

with $V$ in [A19]. We define $f_2(z) \equiv \partial_z[\zeta_2(z)^2 - \varphi(z)]$ and observe that the generalizations of the second and fourth identities in [A12] are

$$\partial_z \alpha(z)^2 = \frac{1}{4} V'(z) + f_2(z) \quad \text{(A22)}$$

and

$$\partial_x [\alpha(x - a)\alpha(x - b)] = \partial_x [\alpha(x - a) - \alpha(x - b)]$$

$$\times \alpha(a - b) + \frac{1}{2} [f_2(x - a) + f_2(x - b)], \quad \text{(A23)}$$

respectively (the latter follows from the following well-known functional equation satisfied by the Weierstrass functions [33]):

$$[\zeta(x) + \zeta(y) + \zeta(z)]^2 = \psi(x) + \psi(y) + \psi(z)$$

provided $x + y + z = 0$. The first and third identities in [A12] hold true as they stand.

While $f_2(z) = 0$ in the hyperbolic case, it is a non-trivial function in the elliptic case. However, going through the computations described in Appendix A1c one finds that they generalize straightforwardly to the elliptic case provided $M = N$ (that [A13] for $M = N$ implies [A14] even in the elliptic case has been known for a long time). One thus obtains the same result as in the hyperbolic case but with the restriction $M = N$.

Appendix B: Numerical method

We verified our soliton solutions numerically by adapting a method developed for solving the standard IILW equation [30] to the non-chiral IILW equation [3]. The numerical method applies to the periodic problem on the interval $[-L/2, L/2]$; for initial conditions and for times, $t$, such that $u(x, t)$ and $v(x, t)$ are significantly different from zero only in an interval $[-\ell/2, \ell/2]$ with $0 < \ell \ll L$, this is an excellent approximation for the non-periodic problem on $\mathbb{R}$. We thus checked numerically various 2- and 3-soliton solutions both for the periodic and non-periodic problem, and we found excellent agreement. For example, the 2-soliton solution in Fig. 2 computed with our numerical method cannot be distinguished with bare
eyes from the one obtained with our analytic result. We mention in passing that our numerical method is much more stable for initial conditions which give rise to soliton solutions than for generic initial conditions. In what follows, we describe our numeric method in more detail.

We employ the discrete Fourier transform

\[ u(x, t) \approx \sum_{n=-N}^{N-1} \hat{u}_n(t)e^{ik_nx}, \quad k_n = \frac{n2\pi}{L} \]  

(B1)

\[ \hat{u}_n(t) = \frac{1}{2N} \sum_{j=-N}^{N-1} u(x_j)e^{-ik_nx_j}, \quad x_j \equiv j\frac{L}{2N} \]

and the Fourier multiplier representations (A3) of the singular integral operators \( \overline{T_u} \),

\[ (\overline{T_u})_n(t) = i\coth(k_n\delta)\hat{u}_n(t), \]

\[ (\overline{T_u})_n(t) = i\frac{1}{\sinh(k_n\delta)}\hat{u}_n(t), \]  

(B2)

to obtain a system of ordinary differential equations for the time evolution of the Fourier coefficients via a semi-discrete collocation approximation (note that \( \hat{u}_n(t)/L \) can be identified with the Fourier transform \( \hat{u}(k_n,t) \)).

The numerical approximation for the nonlinear terms is

\[ (\overline{uur}_n(t) = ik_n(u^2)_n(t) \] with

\[ (\overline{uu}_n(t) = \sum_{m} \hat{u}_{n-m}(t)\hat{u}_m(t) \quad (-N \leq n \leq N-1) \]  

(B3)

where the sum on the right-hand side is over the integers \( m \) in the range \(-N \leq m \leq N-1\) such that \(-N \leq n-m \leq N-1\). In our tests we used \( L = 200 \) and \( N = 512 \).
One sometimes adds a term $\frac{1}{2}u_x$ to the left-hand side of the ILW equation to ensure that it reduces to the Korteweg-de Vries equation in the limit $\delta \to 0$. This term is trivial in that it can be removed by a Galilean transformation, and we find it convenient to ignore it. The known soliton equation related to the elliptic CMS model that we mentioned in the introduction is the periodic generalization of the ILW equation.

One motivation for this name originates in conformal field theory (CFT): while the quantum trigonometric BO equation can be obtained using a chiral CFT, the quantum analogue of (5) is obtained from a non-chiral CFT.

H.H. Chen, Y.C. Lee and N.R. Pereira, Algebraic internal wave solitons and the integrable Calogero-Moser-Sutherland N-body problem, Phys. Fluids 22, 187 (1979).

Y. Matsuno, Interaction of the Benjamin-Ono solitons, J. Phys. A: Math. Gen. 13, 1519 (1980).

F. Calogero, Exactly Solvable One-Dimensional Many-Body Problems, Lett. Nuovo Cim. 13, 411 (1975).

S. Wojciechowski, The analogue of the Backlund transformation for integrable many-body systems, J. Phys. A: Math. Gen. 15, L653 (1982).

E.T. Whittaker and G.N. Watson, A course of modern analysis, Fourth Edition, Cambridge University Press (1940).

M. Kulkarni and A.P. Polychronakos, Emergence of Calogero family of models in external potentials: Duality, Solitons and Hydrodynamics, J. Phys. A: Math. Theor. 50, 455202 (2017).

B. Pelloni and V.A. Dougalis, Numerical solution of some nonlocal, nonlinear dispersive wave equations, J. Nonlinear Sci. 10, 1 (2000).