Low-Density Parity-Check Codes for Nonergodic Block-Fading Channels

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Abstract

We solve the problem of designing powerful low-density parity-check (LDPC) codes with iterative decoding for the block-fading channel. We first study the case of maximum-likelihood decoding, and show that the design criterion is rather straightforward. Unfortunately, optimal constructions for maximum-likelihood decoding do not perform well under iterative decoding. To overcome this limitation, we then introduce a new family of full-diversity LDPC codes that exhibit near-outage-limit performance under iterative decoding for all block-lengths. This family competes with multiplexed parallel turbo codes suitable for nonergodic channels and recently reported in the literature.
I. INTRODUCTION

The block-fading (BF) channel model was first introduced in [16], and further elaborated upon in [2] (see also [1, p. 98 ff.]). This is a realistic and convenient model for a number of channels affected by slowly varying fading, and, as observed for example in [6], is especially relevant in wireless communications involving slow time–frequency hopping (e.g., cellular networks and wireless Ethernet) or multicarrier modulation using orthogonal frequency division multiplexing (OFDM). The design of error-control codes for BF channels offers a challenging problem, which differs greatly from its counterparts referred to additive white Gaussian noise (AWGN) or independent-fading channels (see [6] for a summary of recent results). The main reason for this unlikeness stems from the fact that in BF channels the random channel gains remain constant during a block of symbols (see below for additional details and definitions), and take independent values from block to block. As a result, while the word-error probability in independent-fading channels depends on the Hamming distances between code words, in BF channels it depends on a new parameter, the blockwise Hamming distance. Since codes exhibiting a large minimum Hamming distance may not have a large blockwise Hamming distance, codes that are good when used on the independent-fading channel may not be as good for a BF channel. In addition, over independently faded channels permutations of the symbols cause no variation of the code performance, but this property does not hold on the BF channel. Thus, if an off-the-shelf code, designed for the independent-fading channel, is used for transmission over the BF channel, it is important to carefully select the best permutation of its symbols. Finally, one must consider that the BF channel is nonergodic. As a consequence, to determine the information-theoretical rate limit which cannot be surpassed by the word error probability of any coding scheme, one cannot use channel capacity, but rather outage probability [1], [2], [16]. Classical random-like codes, designed to approach ergodic capacity, cannot generally approach the ideal performance limits of BF channels, and hence code designs suited to the nonergodic nature of the channel are called for. This paper is devoted to this design problem.

Two main parameters that determine the error rate of coded BF channels for high signal-to-noise (SNR) ratios are the diversity order and the coding gain. The former determines the
slope of the error-rate curve as a function of the SNR on a log-log scale. Since the error probability of any coding scheme is lower-bounded by the outage probability, the diversity order is upper-bounded by the *intrinsic diversity* of the channel, which reflects the slope of the outage limit. When maximum diversity is achieved by a code, the coding gain yields a measure of SNR proximity to the outage limit. The maximum achievable diversity order with discrete input constellations is given by the Singleton bound [6], [11], [14], and codes achieving the Singleton bound are termed blockwise maximum-distance separable (MDS). Blockwise MDS codes are outage-achieving over the (noiseless) block-erasure channel [7], but may not achieve the outage-probability limit on noisy BF channels. As a matter of fact, as shown in [6], blockwise MDS codes are necessary, but not sufficient to approach the outage probability of the channel.

Recent code designs for BF channels include near-outage schemes based on a suitable permutation of parallel turbo codes [3]–[5]. Multiplexers for convolutional, turbo and repeat-accumulate codes [3], [6], [11] appeared one decade after the analysis of random and periodic interleaving of convolutional codes on the block-erasure channel [13]. Random ensembles of low-density parity-check codes (LDPC) designed for ergodic AWGN channels [9], [19], in spite of the excellent decoding threshold of their irregular structures, do not have full-diversity, and hence exhibit a poor performance over a BF channel. Decoding thresholds of LDPC code ensembles over ergodic BF channels have been studied [10]. Unfortunately, these codes are not designed to be blockwise MDS, and therefore fail to achieve the outage limit in the nonergodic setup.

In this work, we introduce a new family of blockwise MDS LDPC codes, the *root LDPC codes*, based on a special type of checknode that we call *rootchecks*. Under iterative message-passing decoding, they achieve the outage-probability limit on block-erasure channels, and they perform close to that limit on Rayleigh BF channels. This paper is organized as follows. Section II introduces the channel model and the relevant notations. LDPC codes with full diversity under Maximum Likelihood (ML) decoding are discussed in section III. Our new family of LDPC codes suited for iterative decoding is further described. Section IV analyzes their density evolution in the presence of block fading. Conclusions are finally drawn in Section V. Complementary support material is shown in the Appendix.

1The diversity order is exactly the asymptotic slope for Rayleigh fading, while for other fading distributions it is only proportional to the slope. See [15], [22] for details. In this paper we shall restrict our attention to Rayleigh fading.
II. CHANNEL MODEL AND NOTATION

We consider codewords of $N$ binary digits transmitted on a BF channel, where $n_c$ independent fading gains (whose values form the channel state) affect each codeword. The length $N$ is a multiple of $n_c$, with $\ell \triangleq N/n_c$ denoting the number of bits per fading block. The received signal when symbol $x_i$ is transmitted is given by

$$y_i = \alpha_j x_i + z_i$$

where $y_i \in \mathbb{R}$, $i = 1 \ldots N$, and $j = 1 + [(i - 1)/\ell]$, with $[r]$ denoting the integer part of a real number $r$. The nonnegative real number $\alpha_j$ is the fading gain at block $j$, $j = 1 \ldots n_c$. The symbols $x_i$ are chosen from a BPSK alphabet, $x_i = \pm \sqrt{E_s}$, where $E_s$ is the average energy per symbol. The noise samples are i.i.d. with $z_i \sim N(0, \sigma^2)$, $\sigma^2 = N_0/2$. We assume perfect channel state information (CSI) at the receiver, and channel gains which are i.i.d. Rayleigh-distributed from block to block and from codeword to codeword. Thus, when the information rate is $R$ bits per channel use, the average SNR per symbol is given by $\gamma = E_s/N_0$, and the average SNR per bit is $E_b/N_0 = \gamma/R$. Fig. 1 illustrates the channel model for $n_c = 2$ and $\ell = N/2$.

In this work, we focus on linear binary codes $C(N, K)_{2}$ with block length $N$, dimension $K$, and rate $R = K/N \leq 1/n_c \leq 1/2$. The code $C$ is defined by an $L \times N$ parity-check matrix $H$ (Fig. 2), or, equivalently, by the corresponding Tanner graph [1]. This has $L$ single-parity checknodes. It is assumed that $H$ has full rank $L$, so that $R = 1 - L/N$.

For a given nonzero codeword $c \in C$, we define the blockwise Hamming weight vector...
Fig. 2. Parity-check matrix notations for a block-fading channel with $n_c = 2$. The $L - N/2$ extra rows are added in order to enhance the coding gain of a full-diversity code.

$(\omega_1, \ldots, \omega_{n_c})$, where $\omega_j$ is the Hamming weight of the coded bits affected by fading $\alpha_j$. Following [6], [11] we define the block diversity of $C$ as

$$d = \min_{c \in C \setminus \{0\}} |\{\omega_j \neq 0\}|.$$

In words, the block diversity is the minimum number of blocks that have non-zero Hamming weight, or the blockwise Hamming distance. Qualitatively, this implies that an ML decoder of $C$ will be able to decode correctly in presence of $d - 1$ deep fades, which one can think of as block erasures.

We also define the minimum blockwise Hamming weight as

$$\omega^* = \min_{c \in C \setminus \{0\}} (\omega_1, \ldots, \omega_{n_c}).$$

**Definition 1** An error-correcting code is said to have full diversity if $\omega^* > 0$.

Having $\omega^* > 0$ implies that $d = n_c$, having nonzero weight in all blocks.

Now, observe that the blockwise Singleton bound [1], [6], [11], [14]

$$d \leq 1 + \lceil n_c(1 - R) \rceil$$

determines $R = 1/n_c$ as the highest achievable rate for a full-diversity code. Furthermore, the word error probability of a code with diversity $n_c$ decreases as $1/\gamma^{n_c}$ at high SNR [1], [17],
The block-fading channel is not information stable [21], and therefore its Shannon capacity is zero since there is a non-vanishing probability that the decoder makes a word error. In the limit of large block length, this probability is the information outage probability, defined as [2], [16]

\[ P_{\text{out}}(\gamma, R) \triangleq \mathbb{P}\{I(\gamma, \alpha) < R\} \tag{2} \]

where \( I(\gamma, \alpha) \) is the instantaneous input–output mutual information between the input and output of the channel, defined as

\[ I(\gamma, \alpha) \triangleq \frac{1}{n_c} \sum_{i=1}^{n_c} I_{\text{AWGN}}(\gamma \alpha_i^2), \tag{3} \]

with \( I_{\text{AWGN}}(s) \) the input–output mutual information of an AWGN channel with SNR per symbol equal to \( s \). The BF channel is also commonly referred to as nonergodic since, for finite values of \( n_c \), \( I(\gamma, \alpha) \) is a non-constant random variable.

The information outage probability \( P_{\text{out}}(\gamma, R) \) is a fundamental lower bound on the word error rate for sufficiently large word length. Therefore, any code approaching \( P_{\text{out}}(\gamma, R) \) should have a word-error probability that, as \( N \) increases, becomes independent of the code length [4], [6].

Unless stated otherwise, we shall focus our study on a coding rate \( R = \frac{1}{2} \) (or just slightly smaller than \( \frac{1}{2} \)) and a nonergodic Rayleigh fading channel with \( n_c = 2 \) blocks per codeword, as depicted in Figs. 1 and 2. However, most of our results can be easily generalized to \( R = \frac{1}{n_c} \).

III. FULL-DIVERSITY LDPC CODES UNDER ML DECODING

In this section, we study LDPC codes in the presence of BF under ML decoding. As we shall see, the design of full-diversity LDPC codes under ML decoding is rather straightforward. We recognize that ML decoding is unfeasible in practice; however, it yields valuable insight into code structures suitable for nonergodic channels. The main result of this section is somewhat negative: under iterative decoding, ML-designed full-diversity codes fail to guarantee diversity, due to badly located pseudo-codewords.

Following the notations defined in the previous section, the \( L \times N \) parity-check matrix \( H \) is written in the form \( H = [H_1 \mid H_2] \), where the left and right parts \( H_1, H_2 \) are \( L \times N/2 \). The vector space generated by the \( N/2 \) left columns is denoted \( S_1 \). Similarly, \( S_2 \) is the vector space generated by the \( N/2 \) right columns.
Proposition 1 A binary code \( C \) with rate \( R \leq \frac{1}{2} \), i.e. \( L \geq N/2 \), has full diversity if and only if \( H_1 \) and \( H_2 \) are both full-rank.

Proof: If \( \dim S_1 = N/2 \), then a nonzero codeword cannot have its support on \( H_1 \), because all columns in \( H_1 \) are independent. Hence, \( \omega_2 > 0 \) for all nonzero codewords. Similarly, \( \omega_1 > 0 \) when \( \dim S_2 = N/2 \). Finally, \( \omega_1 > 0 \) and \( \omega_2 > 0 \) for all nonzero codewords, which yields \( \omega^* > 0 \).

The full-rank property of the above proposition was first observed in [8]. Its extension to coding rate \( 1/3 \) with \( H = [H_1 \mid H_2 \mid H_3] \) can be obtained by imposing that the matrices \([H_1 \mid H_2], [H_1 \mid H_3], \) and \([H_2 \mid H_3] \) all have full rank. Generalization to any rate \( R = \frac{1}{n_c} \) is straightforward\(^2\).

Proposition 2 Consider a binary code \( C \) with rate \( R = 1/2 \), and hence with \( L = K = N/2 \). If \( C \) has full diversity, then \( \omega^* = 1 \).

Proof: If \( C \) has full diversity, then \( \dim S_1 = \dim S_2 = N/2 \). Any column from \( H_1 \) can then be written as a linear combination of columns from \( H_2 \). This is also valid for any column belonging to \( H_2 \). Hence, nonzero codewords with \( \omega_i = 1 \) exist for both \( i = 1 \) and \( i = 2 \) if the coding rate is exactly equal to 1/2.

The minimum blockwise Hamming weight must be increased in order to improve the coding gain of \( C \). Proposition 2 states that to achieve this, one must decrease the coding rate. The next proposition shows that adding just one extra row is enough to move from \( \omega^* = 1 \) to \( \omega^* = 2 \) under ML decoding.

Proposition 3 There exists a binary code \( C \) of rate \( R = 1/2 - 1/N \) that has full diversity with \( \omega^* = 2 \).

Proof: The proof is based on the special parity-check matrix structure shown in Fig. 3 where \( H_2 \) is a full-rank matrix whose columns have odd Hamming weight (the identity matrix, for example). Let now \( H_1 \) be such that its first column is the all zero vector, and the remaining \( N/2 - 1 \) columns are all even-weight and full-rank.

\(^2\)Two interesting combinatorial problems arising from this Proposition are the following: (1) What is the probability of a random binary matrix to be full-rank? And, (2) What is the probability of a random binary sparse matrix to be full-rank?
Next, we show that the $\omega^*$ corresponding to this construction is 2. Clearly the first (leftmost) $N/2$ columns of $H$ and the last (rightmost) $N/2$ columns of $H$ have full rank, so that we have $\omega^* \geq 1$.

None of the first $N/2$ columns of $H$ can be a linear combination of the last $N/2$ columns of $H$, due to the 1 in the last position of each of the first columns. None of the last $N/2$ columns of $H$ can be a linear combination of the first $N/2$ columns of $H$, because columns of $H_2$ have odd weight and any linear combination of columns of $H_1$ has even weight.

These last statements imply that $\omega^* \geq 2$.

![Fig. 3. ML-designed full-diversity LDPC code with $\omega^* = 2$.](image)

The rate reduction necessary to achieve $\omega^* = 2$ is negligible for large code length $N$. If we now require $\omega^* = 3$, the following result holds:

**Proposition 4** Consider a binary code $C$ with rate $R \leq 1/2$. The code has $\omega^* = 3$ only if $R \leq 1/2 - (1/N) \log_2(1 + N/2)$.

**Proof:** Denote by $H_2^{\text{col}}$ the set of columns of $H_2$. Consider the $1 + N/2$ sets consisting of $H_2^{\text{col}}$ together with its translates $h_1 + H_2^{\text{col}}$ for all columns $h_1$ of $H_1$. No two of these sets can intersect, otherwise either a column of $H_1$, or a sum of two columns of $H_1$, equals a sum of columns of $H_2$, which would imply the existence of a codeword of weight at most 2 on the first $N/2$ positions. Therefore we must have $2^L \geq (1 + N/2)^{2N/2}$.
Proposition 5 There exists a full-diversity binary code with \( \omega^* \geq 3 \) and \( R = 1/2 - (1/N)2\log_2(N/2 + 1) \).

Proof: The code has the parity-check matrix of Fig. 4. The presence of a Hamming code whose minimum distance is 3 rules out a blockwise Hamming weight equal to 2.

![Diagram of parity-check matrix](image)

Fig. 4. ML-designed full-diversity LDPC code with \( \omega^* \geq 3 \).

We are now in a position to examine the word-error rate of full-diversity LDPC codes designed for ML decoding, and compare it to the outage capacity limit. The results are illustrated in Fig. 5 for \( n_c = 2 \) and the \((3, 6)\) ensemble using the constructions outlined above. With iterative decoding, an ML-designed LDPC code has diversity one. This effect is caused by the pseudo-codewords [12] whose support is restricted to \( H_1 \) or \( H_2 \), and hence have a minimum blockwise pseudo-weight equal to zero when iterative belief propagation decoding is applied. Even a random LDPC code (not shown in the figure) performs as poorly as an ML-designed code with \( \omega^* = 1 \). On the other hand, full diversity is guaranteed when a “genie-aided” ML decoder is used which knows whether errors occur in positions corresponding to \( H_1 \) or to \( H_2 \).

IV. FULL-DIVERSITY LDPC CODES FOR ITERATIVE BELIEF PROPAGATION DECODING

The results presented at the end of Section III show that, if iterative decoding is used, the design criteria derived under the assumption of ML decoding are irrelevant. In this section, we proceed to design LDPC codes with the stipulation of iterative decoding. Our design is based...
Fig. 5. Rate $1/2$ ML-designed LDPC codes with iterative decoding on a Rayleigh block-fading channel with $n_c = 2$. The thick solid line corresponds to the outage probability with BPSK inputs, the dotted lines with * markers corresponds to the ML-designed code with iterative decoding, the dotted lines with □ markers corresponds to the ML-designed code with $\omega^* = 1$ using a genie ML decoder and the dotted lines with + markers corresponds to the ML-designed code with $\omega^* = 3$ using the genie ML decoder. The genie ML curves show the performance of a decoder that knows whether errors occur in positions corresponding to $H_1$ or $H_2$.

on a graphical representation [1], [20], which is then translated into a matrix description. We then analyze the construction by means of log-ratio probability-density evolution.

A. A limiting case: block-erasure channels

We illustrate our solution to the design problem by referring to a limiting case. Specifically, observe that, if the fading coefficients $\alpha_i$ belong to the set $\{0, +\infty\}$, the BF channel becomes a block-erasure channel [7], [13]. This corresponds to the large SNR regime. The reader is referred
to Fig. 6, where the outage boundaries are illustrated (see [4] for more details).

In our approach, we need to find a graph whose topology yields full diversity. For simplicity, we illustrate the case of the (3, 6) LDPC ensemble with $n_c = 2$ (generalizations to other degree distributions and rates will be treated infra). Fig. 7 shows the notation employed in this section. Two examples of local graphs whose diversity is not guaranteed are shown in Fig. 8. The checknodes defining an LDPC code are single-parity check codes, and hence they cannot tolerate more than one erased bit. For example, if $\alpha_1 = 0$ then the checknodes in Fig. 8 are not able to recover the erased bit, because it is connected to bitnodes which are also erased, because they are subject to the same fading coefficient. Notice also that the design must be symmetric, i.e., any analysis with respect to $\alpha_1$ is valid for $\alpha_2$, and hence permuting the order of the two fading gains should yield an equivalent design.

The two unique local graphs that guarantee full diversity in the presence of block erasures are exhibited in Fig. 9. The immediate consequence is the definition of rootchecks. We start by building a regular (3, 6) structure where bitnodes have degree 3 and checknodes have degree 6, next we generalize to any $(\lambda(x), \rho(x))$ degree distribution [19]. A checknode $\Phi$ connected to

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**Fig. 6.** Outage boundaries in the fading plane for a BF channel with $n_c = 2$. To approach the outage limit, one should: (a) Reduce the gap on the ergodic line, which requires an excellent decoding threshold, and (b) Reduce the gap at infinity, which requires a full-diversity code (MDS) on a block-erasure channel.
bits \( \vartheta_1, \vartheta_2, \ldots, \vartheta_6 \) is written as \( \Phi(\vartheta_1, \vartheta_2, \ldots, \vartheta_6) \).

**Definition 2** Let \( \vartheta \) be a binary element transmitted on fading \( \alpha_1 \). A type-1 rootcheck for \( \vartheta \) is a checknode \( \Phi(\vartheta, \vartheta_1, \ldots, \vartheta_5) \) where all bits \( \vartheta_1, \ldots, \vartheta_5 \) are transmitted on fading \( \alpha_2 \).

Type-2 rootchecks are defined similarly.

![Diagram](image)

**Fig. 7.** Notations for graph representation.

Using Definition 2, consider a length-\( N \), rate-1/2 LDPC code. Information bits are split into two classes: \( N/4 \) bits (tagged 1i) are transmitted on \( \alpha_1 \), while \( N/4 \) bits (tagged 2i) are transmitted on \( \alpha_2 \). Parity bits are also partitioned into two sets, say 1p and 2p. Finally, we connect all information bits to rootchecks in order to guarantee full diversity when word error probability is measured on those bits. The protection of parity bits is abandoned. This design produces the bipartite Tanner graph drawn in Fig. 10(a). Its extension to rate 1/3 is portrayed in Fig. 11. Integers labeling edges indicate the degree of a node along those edges. The structure of \( H \) for a root-LDPC code is directly derived from its Tanner graph, and is shown in Fig. 10(b). The \( N/4 \times N/4 \) identity matrix is written twice in connections 1i \( \leftrightarrow \) 1c and 2i \( \leftrightarrow \) 2c. Two
Fig. 8. Two examples of bad configurations under belief propagation decoding on a block-fading channel.

Fig. 9. The two unique good configurations (rootchecks) under belief propagation decoding on a block-fading channel.

all-zero $N/4 \times N/4$ submatrices prohibit any edge of type $1p \leftrightarrow 1c$ and $2p \leftrightarrow 2c$. The other 4 submatrices are all sparse, $H_{1i}$ and $H_{2i}$ are random sparse matrices of Hamming weight 2 per row and per column. Similarly, $H_{1p}$ and $H_{2p}$ are random sparse matrices of Hamming weight 3 per row and per column.

An irregular version of a root-LDPC code can be built from a left degree distribution $\lambda(x)$ and a right degree distribution $\rho(x)$ by appropriately modifying the weight distribution of the 4 submatrices $H_{1i}$, $H_{2i}$, $H_{1p}$, and $H_{2p}$. Equivalently, the degree distribution changes the distribution of edges connected to non-rootchecks in the Tanner graph. Irregularity has no influence on the diversity order because rootchecks are maintained. Irregularity should enhance the coding gain by pushing the code boundary near the outage capacity limit on the ergodic line.
Fig. 10. Tanner graph and parity-check matrix for a regular (3,6) root-LDPC code of rate 1/2. An irregular structure \((\lambda(x), \rho(x))\) can be easily plugged on edges connected to non-root checknodes.

**Proposition 6** Consider a rate-\(R = 1/2\) root-LDPC code with degree distribution \((\lambda(x), \rho(x))\) transmitted on a block-erasure channel with \(n_c = 2\). Then, under iterative message passing decoding, the root-LDPC code has full-diversity.

**Proof:** The two fading coefficients \(\alpha_1\) and \(\alpha_2\) are independent and take two possible values \(\{0, +\infty\}\). Examining the Tanner graph of Fig. 10(a) we observe that the only outage event occurs when \(\alpha_1 = \alpha_2 = 0\) (both blocks erased). Indeed, when \(\alpha_1 = 0\) and \(\alpha_2 = +\infty\), it is
Fig. 11. Tanner graph for a regular (4,6) root-LDPC code of rate $1/3$. The introduction of any $(\lambda(x), \rho(x))$ irregularity is always possible on edges connected to non-root checknodes.

It is straightforward to see that information bits $1i$ are determined using rootchecks $1c$. Similarly, when $\alpha_1 = +\infty$ and $\alpha_2 = 0$, information bits $2i$ are determined using rootchecks $2c$.

On a block-erasure channel, let $\epsilon$ be the probability that $\alpha_i$ be equal to 0. From the proof of Proposition 6 above, we find that the word error probability of a root-LDPC code is $\epsilon^2$. As shown in [7], this is precisely the outage probability of the channel, and therefore, full-diversity blockwise MDS codes are outage achieving in the block-erasure channel. As remarked in [7], blockwise MDS codes are necessary, but not sufficient to achieve the outage limit in noisy channels. In the following, we study the behavior of root-LDPC over general Rayleigh BF AWGN channels.
B. The general case

Now we study the general case of Rayleigh BF. Some simple facts about 4th-order $\chi^2$ distributions are reviewed in the Appendix. In the sequel, we use the notations of the Appendix to analyze the diversity metric in log-ratio messages.

**Proposition 7** Consider a rate-1/2, $(\lambda(x), \rho(x))$ root-LDPC code transmitted on a Rayleigh block-fading channel with $n_c = 2$. Then, under iterative belief propagation decoding, the root-LDPC code has full-diversity.

*Proof:* As indicated in the design of a root-LDPC code before Proposition 6, the diversity order of a root-LDPC code does not depend on its left or right degree distribution. This can also be proved via the evolution trees in the next section. Thus, we restrict this proof to a regular $(3, 6)$ LDPC. The extension to the irregular case is straightforward.

Let $\Lambda^a_i$, $i = 1 \ldots \delta - 1$, denote the input log-ratio probabilistic messages to a checknode $\Phi$ of degree $\delta$. The output message $\Lambda^e$ for belief propagation is

$$\Lambda^e = 2 \th^{-1} \left( \prod_{i=1}^{\delta-1} \th \left( \frac{\Lambda^a_i}{2} \right) \right)$$

where $\th(x)$ denotes the hyperbolic-tangent function. Superscripts $a$ and $e$ stand for a priori and extrinsic, respectively. In order to simplify the proof, we will show that a suboptimal belief propagation decoder is able to achieve diversity order 2. Therefore, if a suboptimal decoder achieves full diversity, the optimal decoder also achieves full diversity. Consider the min–sum decoder. The output message produced by a checknode $\Phi$ is now approximated by

$$\Lambda^e = \min(|\Lambda^a_i|) \prod_{i=1}^{\delta-1} \sign(\Lambda^a_i)$$

*a) First decoding iteration:* We first study the output after one decoding iteration. We assume that the all-zero codeword has been transmitted. The channel crossover probability associated with fading $\alpha_j$, $j = 1, 2$, is

$$\epsilon_j = Q \left( \sqrt{2\gamma \alpha_j^2} \right)$$
The channel message for a bit \( \vartheta \) transmitted over fading coefficient \( \alpha \) is

\[
\Lambda_0 = \log \left( \frac{p(y|\vartheta = 0, \alpha)}{p(y|\vartheta = 1, \alpha)} \right) = \frac{2\alpha y}{\sigma^2} = \frac{2}{\sigma^2} (\alpha^2 + \alpha z)
\]  

(6)

where \( y = \alpha + z \) and \( z \sim \mathcal{N}(0, \sigma^2) \) (assuming \( E_s = 1 \)). At the first decoding iteration, all input messages \( \Lambda_i^0 \) in (5) have an expression identical to (6).

An information bit \( \vartheta \) of class \( i \) has \( \Lambda_0 = \frac{2}{\sigma^2} (\alpha_i^2 + \alpha_i z_0) \). It also receives 3 messages \( \Lambda_i^e \), \( i = 1 \ldots 3 \) from its 3 neighboring checknodes. The total \textit{a posteriori} message corresponding to \( \vartheta \) is \( \Lambda = \Lambda_0 + \Lambda_i^e + \Lambda_2^e + \Lambda_3^e \). Let \( \Lambda_i^e \) be the extrinsic message generated by the rootcheck of class \( 1c \) connected to \( \vartheta \). The error rate \( P_e(1i) \) on class \( 1i \) is given by the negative tail of the density of \( \Lambda \) messages. The addition of \( \Lambda_2^e + \Lambda_3^e \) to \( \Lambda_0 + \Lambda_i^e \) cannot degrade \( P_e(1i) \) because the convolution with the density of messages from non-rootchecks can only physically upgrade the resulting density. Thus, it is sufficient to prove that message \( \Lambda_0 + \Lambda_i^e \) brings full diversity. The expression of \( \Lambda_i^e \) is found by applying (5). Input messages to the rootcheck are negative with probability \( \epsilon_2 \). Then

\[
\Lambda_i^e = S_1 \frac{2}{\sigma^2} (\alpha_i^2 + \alpha_2 z_1)
\]

where

\[
S_1 = \sum_{i \text{ even}} \binom{4}{i} \epsilon_2^i (1 - \epsilon_2)^{4-i} - \sum_{i \text{ odd}} \binom{4}{i} \epsilon_2^i (1 - \epsilon_2)^{4-i}
\]

We obtain

\[
\Lambda_i^e = (1 - 2\epsilon_2)^4 \frac{2}{\sigma^2} (\alpha_i^2 + \alpha_2 z_1)
\]

The partial \textit{a posteriori} log-ratio message becomes

\[
\Lambda_0 + \Lambda_i^e = \frac{2}{\sigma^2} (\alpha_i^2 + (1 - 2\epsilon_2)^4 \alpha_2^2) + \alpha_1 z_0 + (1 - 2\epsilon_2)^4 \alpha_2 z_1
\]

The embedded metric \( Y = \alpha_i^2 + (1 - 2\epsilon_2)^4 \alpha_2^2 \) guarantees full diversity. At high SNR (i.e., when \( E_b/N_0 \to +\infty \)), \( Y \) behaves exactly as \( \alpha_i^2 + \alpha_2^2 \).

\textit{b) Further decoding iterations:} As can be seen from the decoding tree of a bitnode \( 1i \) in Fig. 14, the diversity order 2 is maintained after the first iteration. Indeed, at the input of the rootcheck, information bits of class \( 2i \) have already full diversity and parity bits \( 2p \) bring always a term proportional to \( \alpha_2^2 \). Due to the particular structure of root-LDPC codes, the density of
message $\Lambda_0 + \Lambda_1^e$ can only be improved with respect to the first iteration. Hence, full diversity is preserved.

The proof of the previous proposition is based on showing that the information bits have diversity 2. In the following, we examine the diversity of the parity bits. A parity bit $\vartheta$ of class $1p$ has $\Lambda_0 = \frac{2}{\sigma^2} (\alpha_1^2 + \alpha_1 z)$. It also receives 3 messages $\Lambda_i^e$, $i = 1 \ldots 3$ from its 3 neighboring checknodes all of class $2c$. The total a posteriori message of $\vartheta$ is $\Lambda = \Lambda_0 + \Lambda_1^e + \Lambda_2^e + \Lambda_3^e$. Now let us determine the nature of $\Lambda_i^e$ based on input messages to a checknode $\Phi$ of class $2c$ as illustrated in Figures 10(a) and 15. The node $\Phi$ is not a rootcheck. We need to determine the metric $Y$ embedded in its output message. In the case $\alpha_2 \leq \alpha_1$ (this happens with probability $1/2$), it can be shown that, after one decoding iteration, the extrinsic message produced by $\Phi$ satisfies

$$
\Lambda_i^e = \begin{cases} 
S \sigma^2 (\alpha_2^2 + \alpha_2 z) & \text{with probability } G^4 \geq \frac{1}{16} \\
S \sigma^2 (\alpha_1^2 + \alpha_1 z) & \text{with probability } 1 - G^4 \leq \frac{15}{16}
\end{cases}
$$

where the function $G$ is defined in the Appendix. On the contrary, when $\alpha_2 \geq \alpha_1$, it can be shown that

$$
\Lambda_i^e = \begin{cases} 
S \sigma^2 (\alpha_2^2 + \alpha_2 z) & \text{with probability } G^4 \leq \frac{1}{16} \\
S \sigma^2 (\alpha_1^2 + \alpha_1 z) & \text{with probability } 1 - G^4 \geq \frac{15}{16}
\end{cases}
$$

We conclude that, for parity bits, with a probability greater than $\frac{1}{2} \times \frac{15}{16}$, the output message has diversity order one. In spite of the presence (with a nonzero probability) of diversity-2 messages, the error probability of parity bits will be dominated by weak messages with diversity 1. The above arguments are still valid for further decoding iterations.

Finally, we look at the minimum partial Hamming weight $\omega^*$ under belief propagation decoding.

**Corollary 1** A root-LDPC code with $R = 1/2$ has full-diversity under iterative belief propagation decoding.

**Proof:** Consider an information bit $\vartheta$ of class $1i$. Let $\delta_b \geq 2$ be the degree of $\vartheta$. At high SNR, the log-ratio message produced by its rootcheck has an embedded metric $\alpha_1^2 + \alpha_2^2$. Consider the $\delta_b - 1$ non-root checknodes connected to $\vartheta$. Since parity bits of class $1p$ dominate the error
probability at the input of $2c$ checknodes, then its metric will be $\alpha_1^2$. Finally, the \textit{a posteriori} log-ratio message associated to $\vartheta$ will contain a metric of the type $\delta_0 \alpha_1^2 + \alpha_2^2$ which has diversity 2.

In Fig. 12, we illustrate the performance of the $(3, 6)$ root-LDPC ensemble. As we observe, the performance is similar for all ranges of $N$, and it is also close to the outage probability of the channel. This effect was first observed with blockwise-con catenated codes and repeat-accumulate codes in [6], and then in [3]–[5] for parallel turbo codes. This effect is due to the threshold behavior of \textit{good} codes, i.e., for a given channel realization, the code has a SNR threshold (independent of $N$) below which the decoder cannot decode successfully. Hence, whenever this threshold is larger than the SNR $\gamma$, the decoder will make an error for sufficiently large word length [6]. This is considered in more detail in the following section, where the analysis of the word error probability under iterative decoding for large $N$ is done using density evolution.

V. DENSITY EVOLUTION IN PRESENCE OF BLOCK FADEING

The evolution of message densities [18], [20] under iterative decoding is described through six evolution trees for a binary LDPC root-code. The evolution trees represent the local neighborhood of a bitnode in an infinite-length code whose graph has no cycles. Figs. 13, 14, and 15 show the local neighborhoods of classes $1i$ and $1p$. Similar evolution trees can be drawn for classes $2i$ and $2p$. Full diversity in the presence of fading is guaranteed, thanks to messages $1c \to 1i$ (respectively, $2c \to 2i$) as indicated in the proof of Proposition 7. Irregularity is defined in the standard way [19] through the polynomials $\lambda(x)$ and $\rho(x)$. The polynomial $\lambda(x)$ is replaced by $\tilde{\lambda}(x) = \lambda(x)/x$ each time an edge is isolated at the input of a bitnode. In addition, the polynomial $\rho(x)$ is replaced by $\tilde{\rho}(x) = \rho(x)/x$ each time an edge is isolated at the input of a checknode. The following notations are used, where the superscript $m$ is an integer denoting the decoding iteration order:

- $q_1^m(x)$ and $q_2^m(x)$: Probability density functions of log-ratio messages on the edges $1i \to 1c$ and $2i \to 2c$ respectively. See Fig. 13.
- $f_1^m(x)$ and $f_2^m(x)$: Probability density functions of log-ratio messages on the edges $1i \to 2c$ and $2i \to 1c$ respectively. See Fig. 14.
- $g_1^m(x)$ and $g_2^m(x)$: Probability density functions of log-ratio messages on the edges $1p \to 2c$
Fig. 12. Regular (3,6) root-LDPC codes with iterative decoding on a Rayleigh block-fading channel with $n_c = 2$. Word-error rate is measured on information bits. The thick solid line corresponds to the outage probability with BPSK, the dotted lines with $\times$ markers correspond to $N = 200$, the dotted lines with $\square$ markers correspond to $N = 2000$ and the dotted lines with markers $\ast$ correspond to $N = 20000$.

• Let $X_1 \sim p_1(x)$ and $X_2 \sim p_2(x)$ be two independent real random variables. The density function of $X_1 + X_2$ obtained by convolving the two original densities is written as $p_1(x) \otimes p_2(x)$. The notation $p(x) \otimes^n$ denotes the convolution of $p(x)$ with itself $n$ times. The expression $\lambda(p(x))$ represents the density function $\sum_i \lambda_i p(x)^{\otimes i-1}$.

• Let $X_1 \sim p_1(x)$ and $X_2 \sim p_2(x)$ be two independent real random variables. The density function $p(y)$ of the variable $Y = 2 \text{th}^{-1}(\text{th}(\frac{X_1}{2}) \text{th}(\frac{X_2}{2}))$ obtained through a checknode is written as $p_1(x) \odot p_2(x)$ and is called $R$-convolution [20]. The notation $p(x) \odot^n$ denotes the
R-convolution of $p(x)$ with itself $n$ times. The expression $\rho(p(x))$ represents the density function $\sum_i \rho_i p(x)^{\circ i-1}$.

**Proposition 8** Consider an (ergodic) AWGN channel (i.e., assume $\alpha_1 = \alpha_2 = 1$). Under iterative decoding, a $(\lambda(x), \rho(x))$ root-LDPC code has the same decoding threshold as a random $(\lambda(x), \rho(x))$ LDPC code.

**Proof:** With the two fading gains equal to unity, the six evolution trees degenerate into a single tree, and all densities become identical: $q_1^m(x) = q_2^m(x) = f_1^m(x) = f_2^m(x) = g_1^m(x) = g_2^m(x)$ for any decoding iteration $m$. Thus, density evolution of a root-LDPC code reduces to a classical density evolution of a random code given by $p^{m+1}(x) = \lambda(\rho(p^m(x)))$. 

**Proposition 9** Consider a nonergodic BF channel with $n_c = 2$. For fixed fading coefficients
(α₁, α₂), the density evolution equations of a (λ(x), ρ(x)) root-LDPC code are, for all m,

\[
\begin{align*}
q_{11}^{m+1}(x) &= \mu_1(x) \otimes \lambda(q_{21}^m(x) \circ \tilde{\rho}(f_e f_1^m(x) + g_e g_1^m(x))) \\
f_{11}^{m+1}(x) &= \mu_1(x) \otimes \tilde{\lambda}(q_{21}^m(x) \circ \tilde{\rho}(f_e f_1^m(x) + g_e g_1^m(x))) \otimes \rho(f_e f_1^m(x) + g_e g_1^m(x)) \\
g_{11}^m(x) &= q_{11}^m(x)
\end{align*}
\]

where the multi-edge type fraction is

\[
f_e = 1 - g_e = \frac{\sum_i \lambda_i/i}{\sum_i \lambda_i/(i-1) + \sum_i \lambda_i/i}
\]

and \(\mu_1(x)\) is the Gaussian density at the output of the channel with fading \(\alpha_1\). Similar density evolution equations are obtained by permuting the two fading gains.

**Proof:** The above equations are directly derived from local neighborhoods of bitnodes in the graphical representation of the LDPC code, following standard density evolution analysis of multi-edge type LDPC codes [20].
Fig. 15. Local neighborhood of bitnode $1p$. This tree is used to determine the evolution of messages $1p \rightarrow 2c$.

To evaluate the performance of LDPC codes via density evolution in presence of nonergodic fading, we illustrate the results obtained by applying Proposition 9 to the calculation of asymptotic error probability of the code, in a similar way to what is done in [4]. Three codes are shown in Fig. 16: a random $(3, 6)$ regular code, a root $(3, 6)$ regular code, and an LDPC irregular root code with left and right degree distributions given by the polynomials [19]

$$\lambda(x) = 0.24426x + 0.25907x^2 + 0.01054x^3 + 0.05510x^4 + 0.01455x^7 + 0.01275x^9 + 0.40373x^{11}$$

and

$$\rho(x) = 0.25475x^6 + 0.73438x^7 + 0.01087x^8.$$ 

Refer again to the outage boundary representation in the fading plane of Fig. 6. Let $\alpha_0$ be the fading value defined by the intersection of the BPSK outage boundary and the ergodic line. For rate $1/2$, this intersection point satisfies $I_b(\alpha_0^2 E_b/N_0) = 1/2$, where $I_b(x) \triangleq I_{\text{AWGN}}(Rx)$ is the average mutual information on an AWGN channel with a binary input and an SNR per bit.
equal to \( x \).

Let \( \alpha_{th} \) denote the fading value defined by the intersection of the LDPC code outage boundary and the ergodic line. Then we have

\[
\alpha_{th}^2 = \frac{E_b}{N_0}_{th},
\]
where \( \frac{E_b}{N_0}_{\text{th}} \) is the decoding threshold of the LDPC code over the ergodic AWGN channel. Finally, we obtain

\[
\alpha_{\text{th}} = \alpha_0 \sqrt{\frac{E_b}{N_0}_{\text{th}}} = \alpha_0 \sqrt{\Delta}
\]

where \( \Delta \) is the signal-to-noise ratio gap separating the decoding threshold and the capacity limit on the Gaussian channel. To better understand the gain due to irregularity illustrated in Fig. 16, we evaluate the ratio \( \alpha_{\text{th}}/\alpha_0 \).

- For the regular (3,6) LDPC code, the threshold is 1.09 dB above the Gaussian channel. Hence, \( \alpha_{\text{th}}/\alpha_0 = 1.107 \).
- For the irregular LDPC code given above, the threshold is 0.37 dB above the Gaussian channel. Hence, \( \alpha_{\text{th}}/\alpha_0 = 1.045 \).

Using the best irregular code proposed in [19] with a threshold of 0.25 dB, we obtain \( \alpha_{\text{th}}/\alpha_0 = 1.007 \). Hence, with \( \alpha_c/\alpha_0 \) close to 1, the area between the outage capacity boundary and the code outage boundary is decreased in the neighborhood of the ergodic line. However, this does not ensure that, the code outage boundary would be close to the outage capacity boundary in the critical region between the ergodic line and the block-erasure channel. Therefore, in order to approach the outage probability limit, a full-diversity capacity-achieving code is necessary, but may not be sufficient.

**VI. Conclusions**

We have studied LDPC codes in the block-fading channel under both ML and iterative decoding. We have shown that constructions designed for ML decoders fail to guarantee diversity under iterative decoding. Driven by this restriction, we have introduced the new family of root-LDPC codes, which achieve full diversity under iterative decoding. We have shown both finite- and infinite-length performance, and we have illustrated how the error-rate performance of root-LDPC is close to the outage probability limit and almost insensitive to the block-length. This makes root-LDPC codes attractive for slowly-varying wireless communications scenarios.
Appendix I
Coding Gain of a 4th-Order Unbalanced $\chi^2$ Distribution

Here we limit our description to a diversity order of 2, but all results are easily extendable to rate-1/$n_c$ coding on a channel with diversity order $n_c$. In the context of ML decoding, the Euclidean distance between two codewords is proportional to $\omega_1 \alpha_1^2 + \omega_2 \alpha_2^2$. As fading $\alpha_i$ have a Rayleigh density, their squares are exponentially distributed, i.e., $p_{\alpha_i^2}(x) = e^{-x}$. The latter is a central $\chi^2$ distribution of order 2 with parameter $\sigma^2 = 1/2$ [17]. Diversity 2 is achieved with a $\chi^2$ distribution of order 4. Hence, a full-diversity code must satisfy $\omega_1 > 0$ and $\omega_1 > 0$ in order to get the order-4, $\chi^2$ distributed, metric $\omega_1 \alpha_1^2 + \omega_2 \alpha_2^2$. Once maximum diversity is guaranteed, the maximization of the product $\omega_1 \omega_2$ increases the coding gain.

The above simple facts are still valid in the context of iterative probabilistic decoding. Let $\Lambda$ be the a posteriori probability log-ratio of a binary element $b$. Achieving full diversity under iterative decoding is equivalent to letting $\Lambda$ behave as the metric $Y = a\alpha_1^2 + b\alpha_2^2$, where $a$ and $b$ are two positive real numbers. The energy of $Y$ is normalized, $a + b = 1$. The exact mathematical expression relating $\Lambda$ to $Y$ depends on the type of iterative algorithm used for decoding, e.g., $\Lambda \propto Y + \nu$ where $\nu$ is an additive noise. To understand the influence of the product $ab$ on the performance, one should study the error probability associated with $Y$, i.e. $P(Y < T) = F(a,b,T)$. When $a = b = 1/2$, the order-4 $\chi^2$ distribution is balanced, and its probability density function is

$$p_Y(y) = 4ye^{-2y}$$

When $a \neq b = 1 - a$, the order-4 $\chi^2$ distribution is unbalanced, and its probability density function is

$$p_Y(y) = \frac{(e^{-y/a} - e^{-y/b})}{2a - 1}$$

The expression of $P(Y < T) = F(a,b,T)$ is obtained after integrating $p_Y(y)$. The diversity order and the coding gain embedded in $Y$ appear when $T \ll 1$. For a balanced $\chi^2$ distribution, we have

$$F(a,b,T) = 1 - e^{-2T}(1 + 2T) = 2T^2 + o(T^2)$$
For an unbalanced $\chi^2$ distribution, we obtain

$$F(a, b, T) = 1 - \frac{ae^{-T/a} - be^{-T/b}}{2a - 1} = \frac{T^2}{2ab} + o(T^2)$$  \hspace{1cm} (10)

In Fig. 17 the performance function $F(a, b, T)$ is plotted versus $\gamma = 1/T$ on a double logarithmic scale for different values of $a$ and $b$. The slope is always 2 (i.e., $F(a, b, T) \propto 1/\gamma^2$) for all positive values of $a$ and $b$. The function $F$ degenerates to $T + o(T)$ when $b = 0$ (diversity order equal to 1 instead of 2). Notice also that an unbalanced $\chi^2$ distribution with $a = 3/4$ and $b = 1/4$ generates a coding loss about 0.65 dB. This loss is slightly higher (about 0.75 dB) when considering $P(\Lambda < 0)$ for $\Lambda \propto Y + \nu$ since additive noise depends on the fading coefficients as shown in Section IV.

Fig. 17. Coding gain and diversity order of $Y = a\alpha_1^2 + b\alpha_2^2$ ($\chi^2$ of 4th order) where $\alpha_1$ and $\alpha_2$ are Rayleigh distributed.
Consider two real independent Gaussian random variables $X_1 \sim \mathcal{N}(\alpha_1^2, \alpha_1^2\sigma^2)$ and $X_2 \sim \mathcal{N}(\alpha_2^2, \alpha_2^2\sigma^2)$. We define the multivariate function $G(\alpha_1, \alpha_2, \sigma^2) \triangleq \mathbb{P}(|X_2| < |X_1|)$. The $G$ function is given by the integral expression

\[
G = 1 - \int_0^{\infty} \frac{dt}{\sqrt{2\pi\alpha_1^2\sigma^2}} \left( e^{-\frac{(t-\alpha_1^2)^2}{2\alpha_1^2\sigma^2}} + e^{-\frac{(t+\alpha_1^2)^2}{2\alpha_1^2\sigma^2}} \right) \left( Q\left(\frac{t - \alpha_2}{\alpha_2\sigma}\right) + Q\left(\frac{t + \alpha_2}{\alpha_2\sigma}\right) \right) \tag{11}
\]

where $Q(x)$ is the Gaussian tail function. A 3D plot of $G$ is illustrated in Fig. 18. The main properties of $G$ are:

**APPENDIX II**

**THE BIDIMENSIONAL CUMULATIVE DENSITY FUNCTION** $G = \mathbb{P}(|X_2| < |X_1|)$

Fig. 18. A 3D plot of $G = \mathbb{P}(|X_2| < |X_1|)$ versus $\alpha_1$ and $\alpha_2$ for a variance $\sigma^2 = 1/10$. 
• $G(\alpha, \alpha, \sigma^2) = 1/2$ for all $\sigma^2 > 0$.

• $G$ is a non-decreasing function of $\alpha_1$ and a decreasing function of $\alpha_2$. Hence, $G \leq 1/2$ if $\alpha_1 \leq \alpha_2$ and $G \geq 1/2$ if $\alpha_2 \leq \alpha_1$.

• For fixed $\sigma^2$ and $\alpha_2$, $G \rightarrow 1$ as $\alpha_1 \rightarrow +\infty$.

• For fixed $\sigma^2$ and $\alpha_1$, $G \rightarrow 0$ as $\alpha_2 \rightarrow +\infty$. 
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