Regular parallelisms on $\text{PG}(3, \mathbb{R})$ from generalized line stars: The oriented case

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\textit{Dedicated to the memory of Helmut Reiner Salzmann}

Abstract

Using the Klein correspondence, regular parallelisms of $\text{PG}(3, \mathbb{R})$ have been described by Betten and Riesinger in terms of a dual object, called a hyperflock determining ($hfd$) line set. In the special case where this set has a span of dimension 3, a second dualization leads to a more convenient object, called a generalized star of lines. Both constructions have later been simplified by the author.

Here we refine our simplified approach in order to obtain similar results for regular parallelisms of oriented lines. As a consequence, we can demonstrate that for oriented parallelisms, as we call them, there are distinctly more possibilities than in the non-oriented case. The proofs require a thorough analysis of orientation in projective spaces (as manifolds and as lattices) and in projective planes and, in particular, in translation planes. This is used in order to handle continuous families of oriented regular spreads in terms of the Klein model of $\text{PG}(3, \mathbb{R})$. This turns out to be quite subtle. Even the definition of suitable classes of dual objects modeling oriented parallelisms is not so obvious.

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1 Introduction

A parallelism in real projective 3-space is an equivalence relation on lines (always assumed to be continuous in a certain sense) such that every class is a spread (i.e., a partition of the point set into disjoint lines). The classical example is Clifford parallelism, but there are many more examples with varying amounts of symmetry, as was shown by Betten and Riesinger; see [1], [3] and other articles by these authors. From every parallelism one can construct an oriented parallelism (i.e., a similar equivalence relation on oriented lines), by a trivial process of ‘unfolding’. In [12], the author found that even among the most symmetric oriented parallelisms other than Clifford parallelism there are ‘non-foldable’ examples that do not arise in this trivial way. Our aim is to show that a vast number of non-foldable examples can be found among regular oriented parallelisms (where the spreads are all isomorphic to the complex spread), even with 2-torus symmetry.
In the present paper, we lay the foundations for this project by establishing powerful construction principles. In the case of non-oriented parallelisms, such principles were found by Betten and Riesinger and later simplified by the author. One works with the Klein model, which describes $\text{PG}(3, \mathbb{R})$ within $\text{PG}(5, \mathbb{R})$. The line space of $\text{PG}(3, \mathbb{R})$ corresponds to the Klein quadric $K$, and regular spreads become intersections of $K$ with certain 3-spaces.

First, there is a construction that works for all regular parallelisms; by dualizing with respect to the Klein quadric, the parallelism is turned into a set $\mathcal{H}$ of lines not meeting the quadric such that every tangent hyperplane of the quadric contains exactly one of them [3], [11]. Such sets are called hyperflock determining line sets, abbreviated $hfd$ line sets. In the special case where $\mathcal{H}$ spans (a plane or) a 3-space, one can dualize once more within the 3-space and one obtains a so-called generalized line star or $gl$ star [1], [11]. A $gl$ star is a set of lines in the 3-space such that every exterior point with respect to $K$ is on exactly one of them. This correspondence has been used to construct large sets of examples with torus symmetry [5], [13].

We shall prove similar results for parallelisms of oriented lines (briefly called oriented parallelisms). There are several obstacles that have to be overcome. This requires a careful analysis of orientation in various geometric contexts, see Section 2. A spread is a line set homeomorphic to the 2-sphere, and the section culminates in the proof that orienting a spread as a manifold amounts to the same as orienting all lines in the spread (Theorem 2.5). Regular spreads appear in the Klein quadric as intersections with certain 3-spaces. In Section 3, we show that this correspondence lifts to a continuous map from oriented 3-spaces to oriented spreads, where the topology on the set of oriented spreads is defined by a Hausdorff metric.

In Sections 4 and 5 we define the oriented analogs of $hfd$ sets and of $gl$ stars and obtain our main results. For example, given a 3-space $R$ that meets $K$ in an elliptic quadric $Q$, an oriented $gl$ star or $gl^+$ star with respect to $Q$ is a set of oriented secants of $Q$ such that every point $p$ of $R$, not in the interior of $Q$, is incident with exactly two of them and such that the set of these two oriented lines depends continuously on the point $p$. It is the latter condition which makes this approach work. Example 6.9 will show that compactness would not do as a surrogate. The main results of these two sections are Theorems 4.5 and 5.5. They may be summarized as follows.

**THEOREM 1.1.** a) There is a one-to-one correspondence between compact oriented regular parallelisms of $\text{PG}(3, \mathbb{R})$ and $hfd^+$ line sets in $\text{PG}(5, \mathbb{R})$.
b) Let $R$ be a 3-space of $\text{PG}(5, \mathbb{R})$ intersecting the Klein quadric $K$ in an elliptic quadric $Q$. There is a one-to-one correspondence between $hfd^+$ line sets in $R$ and $gl^+$ stars with respect to $Q$.

In the final Section 6 we give criteria that help to recognize oriented generalized line stars and to construct them, and we display non-foldable examples with and without rotational symmetry. A systematic study of regular oriented parallelisms with 2-torus action will be left to a future occasion.
2 Orientation

We consider various concepts of orientation arising in real projective geometry, and their relationships. Starting from fairly standard concepts, we proceed to develop specialized and not so obvious notions and results concerning orientation in projective planes, spreads and parallelisms. A useful reference to standard notions and constructions related to orientation is [7]. All notions, notation, and conventions introduced in this section shall be used tacitly in the sequel.

2.1 Oriented vector spaces and projective spaces

As usual, an orientation on the vector space $\mathbb{R}^k$ is given by an ordered basis $B = (v_1, ..., y_k)$, and two orientations given by $B_1$ and $B_2$ are considered as equal when the linear map sending $B_1$ to $B_2$ has positive determinant. A vector space $V$ with a fixed orientation will usually be denoted $(V, B)$ or simply $V^+$. Any basis defining the same orientation as $B$ will be called a positive basis of $(V, B)$. An oriented differential $k$-manifold is a differential manifold with compatible orientations on all its tangent spaces. An orientation in this sense defines also an orientation of the underlying topological manifold, i.e., preferred generators for all local homology groups in dimension $k$. An oriented vector space may be considered as an oriented differential manifold, since it coincides with each of its tangent spaces.

The projective space $P_k \mathbb{R}$ is defined as the quotient space of $\mathbb{R}^{k+1} \setminus \{0\}$ obtained by identifying a nonzero vector $v$ with every scalar multiple $rv, 0 \neq r \in \mathbb{R}$. By an orientation of $P = P_k \mathbb{R}$ we shall mean an orientation of $\mathbb{R}^{k+1}$ and denote the oriented projective space by $P^+$. We stress that this does not mean that we have oriented the differential or topological $k$-manifold $P$. However, if we restrict the quotient map $\rho : \mathbb{R}^{k+1} \setminus \{0\} \to P_k \mathbb{R}$ to the unit sphere $S_k \subseteq \mathbb{R}^{k+1}$, we obtain a two-sheeted covering map. The nontrivial deck transformation of this covering is the antipodal map $-id$. A given orientation of the differential manifold $\mathbb{R}^{k+1}$ induces an orientation on the unit ball (as a manifold with boundary), and this in turn yields an orientation on the boundary $S_k$ as follows, cp. [7]. A basis $B$ of the tangent space $T_s S_k$ at a point $s$ is positive if $B$, preceded by an outward pointing vector $v \in T_s \mathbb{R}^{k+1} = \mathbb{R}^{k+1}$, becomes a positive basis of $\mathbb{R}^{k+1}$. Now we try to transfer this orientation to $P_k \mathbb{R}$ via the map $\rho$ by insisting that the tangent map of $\rho$ should preserve orientation on tangent spaces. This works without conflict if and only if the deck transformation preserves the orientation of $S_k$, which is the case if and only if $k + 1$ is even, i.e., if $k$ is odd. So an odd-dimensional oriented projective space as defined here is in fact an oriented manifold, but an even-dimensional one is not.

Remark. The orientation on the manifold $P_k \mathbb{R}$ defined above does not depend on the choice of a quadratic form defining the unit sphere $S_k$. In other words, if $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$
is a linear automorphism with positive determinant, then replacing \( S_k \) with \( S'_k = \varphi S_k \) we get the same result. Indeed, \( \varphi \) is an orientation preserving diffeomorphism between the two spheres, and the quotient maps \( \rho \) and \( \rho' \) to the resulting oriented manifolds \( P_k \mathbb{R} \) and \( P'_k \mathbb{R} \) also preserve orientations. The induced map \( \varphi^\flat : P_k \mathbb{R} \to P'_k \mathbb{R} \) satisfies \( \rho' \circ \varphi = \varphi^\flat \circ \rho \), and therefore preserves orientations, as well.

We note that an orientation of a compact connected differential \( k \)-manifold corresponds to an orientation of the underlying topological manifold, which can be represented by a preferred generator of its top homology group, which is infinite cyclic. This amounts to the same as choosing a preferred generator for each local homology group in dimension \( k \) in a coherent way.

2.2 Grassmannians and orientation

In Incidence Geometry, a projective space is commonly considered as the subspace lattice of some vector space with the lattice operations \( \text{join} \ X \lor Y \) (the span of the union of two subspaces \( X \) and \( Y \)) and \( \text{intersection} \ X \land Y \). The \( k \)-dimensional real projective space, considered as the subspace lattice of \( \mathbb{R}^{k+1} \), will be denoted \( \text{PG}(k, \mathbb{R}) \). Its point set \( P_k \mathbb{R} \) is the set of 1-dimensional vector subspaces. The projective space \( P(X) \) associated with an \( (l+1) \)-dimensional subspace \( X \leq \mathbb{R}^{k+1} \) may be considered as a submanifold of \( P_k \mathbb{R} \), homeomorphic to \( P_l \mathbb{R} \). Such a subset is considered as an \( l \)-dimensional projective subspace of \( P_k \mathbb{R} \). In this way, \( \text{PG}(k, \mathbb{R}) \) becomes a lattice of subsets of \( P_k \mathbb{R} \).

The set of all \( l \)-dimensional subspaces of \( P_k \mathbb{R} \), or equivalently, the set of \( (l+1) \)-dimensional subspaces of \( \mathbb{R}^{k+1} \), is known as the Grassmann manifold \( G_{k+1,l+1} \). Indeed, the transitive action of the general linear group \( \Delta = \text{GL}_{k+1} \mathbb{R} \) turns this set into a compact, connected differential manifold, namely the homogeneous space of the group \( \Delta \) modulo the stabilizer \( \Delta_X \) of any fixed \( (l+1) \)-dimensional subspace \( X \). Specifically, this means that the subspace \( \delta X \) corresponds to the coset \( \delta \Delta_X \) for every \( \delta \in \Delta \). The differential manifold \( G_{k+1,l+1} \) equals the projective space \( P_k \mathbb{R} \) as defined earlier. When we think of the elements of a Grassmann manifold as projective subspaces, we prefer to write \( P_{k,l} \) rather than \( G_{k+1,l+1} \). Thus, we have \( P_k \mathbb{R} = P_{k,0} \). For the continuity properties of the lattice operations in \( \text{PG}(k, \mathbb{R}) \), a convenient reference is [9] or [6].

Now let us consider oriented subspaces. As before, the set of oriented \( (l+1) \)-dimensional vector subspaces of \( \mathbb{R}^{k+1} \) or, equivalently, the set of oriented \( l \)-dimensional projective subspaces of \( P_k \mathbb{R} \), becomes a compact differential manifold when considered as a coset space of \( \Delta = \text{GL}_{k+1} \mathbb{R} \). This time, the subgroup to be factored out is the stabilizer \( \Delta_X^+ \) of any oriented \( (l+1) \)-dimensional oriented vector subspace \( X^+ \). The elements of this stabilizer fix \( X^+ \) as a vector space and induce a linear map of positive determinant on \( X \). We denote this manifold by

\[
G_{k+1,l+1}^+ = P_{k,l}^+.
\]

We repeat that it is not possible to consider even-dimensional projective subspaces as
oriented manifolds. Note also that, in general, we do not have lattice operations for oriented subspaces. Clearly, the stabilizer $\Delta_{X^+}$ is an index 2 subgroup of $\Delta_X$, and therefore, the manifold $P_{k,l}^+$ is a 2-sheeted covering space of $P_{k,l}$. The covering maps will be denoted $\psi$. We could have used this fact as a definition of $P_{k,l}^+$, but this would not allow us to obtain a direct grip on the orientation carried by a subspace.

### 2.3 Polarities and orientation

A **polarity** is an antiautomorphism $\xi$ of order 2 of the lattice $\text{PG}(k, \mathbb{R})$. It is defined by a non-degenerate symmetric bilinear form $f$ on $\mathbb{R}^{k+1}$ and sends a subspace $X$ to $$\xi(X) = \{ v \in \mathbb{R}^{k+1} \mid f(v, X) = 0 \}.$$ We have $\dim \xi(X) + \dim X = k + 1$ for the vector space dimensions. In terms of projective dimensions, this means that $\xi$ sends $P_{k,l}$ to $P_{k,k-l-1}$. If $f$ restricts to a non-degenerate form on $X$, then $\xi(X)$ is indeed a vector space complement of $X$. In this case, we can define $\xi$ for oriented subspaces. We choose a fixed orientation for $\mathbb{R}^{k+1}$, and we define $\xi(X^+)$ by insisting that the sum decomposition

$$\mathbb{R}^{k+1} = X \oplus \xi(X)$$

is a sum of oriented vector spaces, i.e., that a positive ordered basis of $X$, followed by a positive ordered basis of $\xi(X)$, defines the given orientation of $\mathbb{R}^{k+1}$. Thus we have a partial lift of the polarity to the 2-sheeted covers, that is, a continuous partial map $\xi^+ : P_{k,l}^+ \to P_{k,k-l-1}^+$ that commutes with the covering maps $\psi$ in the sense that $\psi \circ \xi^+ = \xi \circ \psi$.

### 2.4 Oriented projective planes

Let $(P, \mathcal{L})$ be a compact, connected topological projective plane. This means first of all that $(P, \mathcal{L})$ is a projective plane, i.e., that the elements of $\mathcal{L}$ are subsets of the set $P$, called lines, and that two distinct points $p, q$ are joined by a unique line $p \lor q \in \mathcal{L}$ and two distinct lines $K, L$ meet in a unique point $K \land L \in P$. Moreover, we require that $P$ and $\mathcal{L}$ are compact, connected topological spaces and that the operations $\lor$ and $\land$ are continuous. The classical examples are the planes $\text{PG}(2, \mathbb{F})$, where $\mathbb{F}$ stands for the field of real or complex numbers, the skew field of quaternions, or the division algebra of octonions. The examples that we have in mind here are translation planes defined by spreads in $\text{PG}(3, \mathbb{R})$, see Section 2.5.

For simplicity, we shall assume that lines are topological manifolds, which implies that they are in fact spheres of dimension $l \in \{1, 2, 4, 8\}$, and that $P$ and $\mathcal{L}$ are $2l$-dimensional manifolds. If we only assume that $P$ has finite covering dimension $\dim P < \infty$, then lines are homology $l$-manifolds with the same possibilities for $l$ as before, and their homology
groups are those of an $l$-sphere. Proofs of these facts can be found in Chapter 5 of [16]. Background information without proofs is also given in [6].

Let $K, L$ be two lines and let $x$ be a point not contained in any of these lines. Then the central projection map

$$\omega(K, x, L) : y → (y \lor x) \land L$$

is a homeomorphism from $K$ to $L$, called a perspectivity. Compositions of perspectivities starting from a line $L$ and ending up on the same line are called projectivities. They form a group $\Omega(L)$. These groups have been studied extensively [10].

Every line $L \approx \mathbb{S}_l$ admits two possible orientations, and we denote the set of all oriented lines by $L^+$. There is a two to one surjective map

$$\psi : L^+ → L$$

that forgets orientations. Orientation forgetting maps will occur frequently, and they will always be denoted $\psi$. Our goal is to define a topology on $L^+$ such that $\psi$ becomes a two-sheeted covering map. The first such construction was given by Salzmann [15], p. 10, using the space of all embeddings $\mathbb{S}_l → P$ whose images are lines, equipped with the compact open topology. Here we use a different approach.

Let $L^+$ be an oriented line, and choose a point $x \notin L = \psi(L^+)$. Define a set of oriented lines

$$\mathcal{L}^+(L^+, x)$$

to be the set of all lines $M$ not containing $x$, endowed with the orientation transferred from $L^+$ via $\omega(L, x, M)$. The restriction of $\psi$ to this set is a bijection onto an open subset of $\mathcal{L}$ (the set of lines not containing $x$), and we define a topology on $\mathcal{L}^+(L^+, x)$ by insisting that this bijection be a homeomorphism.

Now consider two such sets $\mathcal{U}_1 = \mathcal{L}^+(L_1^+, x_1)$ and $\mathcal{U}_2 = \mathcal{L}^+(L_2^+, x_2)$, and let $\mathcal{X}^+$ be the set of oriented lines containing neither $x_1$ nor $x_2$. An oriented line $M^+ \in \mathcal{U}_1$ belongs to the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ if and only if it lies in $\mathcal{X}^+$ and the map $\omega(L_1, x_1, M, x_2, L_2)$ (to be read as a composition of perspectivities in the obvious manner) is orientation preserving as a map $L_1^+ → L_2^+$. If $M_t$ is a path in $\mathcal{X} = \psi\mathcal{X}^+$, then the corresponding maps $\omega_t$ are homotopic, hence they have the same effect on the top homology groups and are either all orientation preserving or all orientation reversing. Thus the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is mapped by the forgetful map $\psi$ onto a (possibly empty) union of some path connected components of $\mathcal{X}$. These components are open sets, so we see that $\mathcal{U}_1 \cap \mathcal{U}_2$ is open in $\mathcal{U}_1$ and in $\mathcal{U}_2$ and inherits the same topology from both sets.

Now we endow $\mathcal{L}^+$ with the topology generated by all the topologies on the various sets $\mathcal{L}^+(L^+, x)$, and it is still true that $\psi$ restricts to a homeomorphism on each of these sets with respect to this topology. Thus we have obtained
**Proposition 2.1.** If the set \( L^+ \) of oriented lines of a compact projective plane is equipped with the topology defined above, then the forgetful map \( \psi : L^+ \to L \) becomes a two-sheeted covering map.

From this, we obtain the next proposition almost as a corollary. By a *section* of the map \( \psi \) we mean a map \( \sigma \) in the opposite direction such that the composition \( \psi \circ \sigma \) is the identity map of \( L \).

**Proposition 2.2.**

a) Let \((P, L)\) be a projective plane with lines of dimension \( l \). The space \( L^+ \) of oriented lines is connected if \( l = 1 \) (in fact, it is then a 2-sphere) and is disconnected otherwise.

b) The forgetful map \( \psi : L^+ \to L \) admits a section if and only if \( l \geq 2 \).

**Proof.** If \( l \geq 2 \), then the point space \( P \) is simply connected, because the complement of a point \( P \setminus \{x\} \) deformation retracts onto every line not containing \( x \), cp. [16], 51.26. Exchanging the roles of points and lines, we see that \( L \) is simply connected as well. Therefore, a two-sheeted covering of \( L \) must be the topological sum of two copies of \( L \), each of which is mapped homeomorphically onto \( L \) by the covering map.

For \( l = 1 \), the space \( L^+ \) is connected. Indeed, the pencil \( L^+_x \) of oriented lines passing through a given point \( x \) is connected (in fact, homeomorphic to \( S_1 \)) since we have a continuous surjection from the boundary of a disc containing \( x \) in its interior onto \( L^+_x \), sending a point \( p \) to the line \( p \lor x \) oriented locally from \( p \) to \( x \). Furthermore, any two oriented lines belong to the pencil of oriented lines passing through their point of intersection. Now the space \( L \) is homeomorphic to \( P_2 \mathbb{R} \approx S_2 / \pm \text{id} \) for \( l = 1 \), see [16], 42.10. Thus, the 2-sphere is the only connected two-sheeted covering space of \( L \). A section to \( \psi \) would be a homeomorphism by domain invariance, a contradiction.

For our purposes, the preceding result is not enough. We need an explicit construction of sections in the case \( l \geq 2 \). This will be made possible with the aid of projectivity groups \( \Omega(L) \).

It is rather easy to see that projectivities do not preserve orientation of lines if \( l = 1 \). For example, in the real affine plane (which canonically extends to the projective plane) consider the lines \( X \) (the \( x \)-axis) and \( Y \) (the \( y \)-axis) and the points \( p = (1, -1) \) and \( q = (-1, -1) \). The projectivity \( \omega(X, p, Y, q, X) \) (to be read as a composition of projectivities in the obvious manner) reverses the orientation of the \( x \)-axis. However, for \( l \geq 2 \), we shall prove that all lines can be oriented in such a way that all projectivities and, hence, all projectivities preserve these orientations. By the definition of the topology on \( L^+ \), this then provides a section to the forgetful map \( \psi \).

As a preparation, we endow every group \( \Omega(L) \) of projectivities with the compact open topology, which turns it into a topological transformation group of the space \( L \). Next we note that a path \( \sigma : [0, 1] \to \Omega(L) \) corresponds to an isotopy on \( L \), and so the maps \( \sigma(0) \)
and $\sigma(1)$ are either both orientation preserving or both orientation reversing, because they have the same effect on the top homology group. Together with the above example, this shows that $\Omega(L)$ is not pathwise connected in the case of the real projective plane. However, we have the following well-known

**Lemma 2.3.** For $l \geq 2$, all groups $\Omega(L)$ of projectivities are pathwise connected.

**Proof.** We follow [10], Theorem 3.3. Consider a projectivity

$$\omega = \omega(L_1, x_1, L_2, x_2, ..., L_{n-1}, x_{n-1}, L_n)$$

with $L_n = L_1 = L$. We shall construct a path in $\Omega(L)$ joining $\omega$ to the identity. We choose a point $x$ not on any of the lines $L_i$ and join every $x_i$ to $x$ by a path $x_i(t)$ in the connected set $(x \vee x_i) \setminus (L_i \cup L_{i+1})$ (or by a constant path if $x = x_i$). Replacing every $x_i$ by $x_i(t)$ in the definition of $\omega$ we obtain a path $\omega(t)$ in $\Omega(L)$. Since all projection centers of $\omega(1)$ are identical, $\omega(1)$ is the identity, and $\omega(0) = \omega$. \hfill \qed

Now we obtain a geometric construction of sections of the forgetful map $\psi$.

**Theorem 2.4.** In a compact projective plane of dimension $2l \geq 4$ it is possible in exactly two ways to orient all lines in such a way that all perspectivities are orientation preserving.

**Proof.** Start with one line $L$ and orient it in one of the two possible ways. The condition on perspectivities then forces our way of orienting all other lines. If conflicts arise when we transfer orientations about, then it means that two projectivities from $L$ to some line $K$ take the orientation of $L$ to distinct orientations of $K$. Then the quotient of these projectivities is an orientation reversing projectivity of $K$ to itself. This is impossible, because $\Omega(K)$ is path connected, which implies that every projectivity of $K$ to itself is isotopic to the identity and hence induces the identity on the top homology group. \hfill \qed

Note that the orientations constructed in this proof depend continuously on the lines, by the very definition of the topology on $L^+$. Thus we have in fact obtained a section to the forgetful map.

### 2.5 Oriented spreads

We are now ready to study the orientation properties of spreads of PG(3,R), which are crucial to us since parallelisms are built from spreads. In fact, the results of this section, together with the continuity result Theorem 3.5 below, constitute the most subtle steps towards our final goals.

When we first introduced oriented parallelisms in [12], we were content with a very simple approach. A spread is a certain set $\mathcal{S}$ of lines of a projective 3-space, which is homeomorphic to the 2-sphere. Hence, the two-sheeted covering $\mathcal{S}^+$ is necessarily disconnected, and we defined an orientation of $\mathcal{S}$ to be a choice of one of the two connected components of
this cover. In the present situation, we need closer control of the orientations of the lines $L \in S$ involved here, so we need to refine our definition.

First we recall the definition of a spread; compare Section 64 of [16]. Consider the line space $P_{3,1}$ of $\text{PG}(3, \mathbb{R})$. A compact set $S \subseteq P_{3,1}$ is a (topological) spread if each point $x$ belongs to a unique element $S_x \in S$. By compactness, the map $x \to S_x$ is continuous, and $S$ is homeomorphic to the 2-sphere (this will become apparent later).

One reason for studying spreads is that a spread $S$ defines an affine translation plane $\mathcal{A}_S$. We may consider $S$ as a subset of the Grassmann manifold $G_{4,2}$, i.e., as a set of 2-dimensional vector subspaces of $\mathbb{R}^4$. The plane $\mathcal{A}_S$ has point set $\mathbb{R}^4$, and its lines are all translates $L + v$, where $L \in S$ and $v \in \mathbb{R}^4$. Thus, $S$ is the pencil of lines containing the origin $0 \in \mathbb{R}^4$ (which incidentally explains why $S \approx S_2$).

The projective plane associated with $\mathcal{A}_S$ is the projective translation plane $\mathcal{T}_S$ defined by $S$. There is a particularly nice description of this plane, which is somewhat hidden in the proof of Theorem 64.4 of [16]. (That book chapter contains a thorough introduction to translation planes.) The construction of $\mathcal{T}_S$ starts from $\text{PG}(4, \mathbb{R})$. Choose a hyperplane $H$ of that projective space. Then $H$ is isomorphic to $\text{PG}(3, \mathbb{R})$, and we may consider $S$ as a set of lines of $H$. We simply write $P_4$ for the point set $P_{4,0} = P_4 \mathbb{R}$ of $\text{PG}(4, \mathbb{R})$. Now the points of $\mathcal{T}_S$ are

- the points of $P_4$ not belonging to $H$ and
- the elements of $S$.

This partitions the point set $P_4$ into disjoint sets (most of them singletons), and the topology of the point set $T$ of $\mathcal{T}_S$ is the resulting quotient topology. The lines of $\mathcal{T}_S$ are

- the elements of $P_{4,2}$ (2-spaces in $\text{PG}(4, \mathbb{R})$) that meet $H$ in a line $S \in S$ and
- the set $S$ itself.

Incidence of points and lines is given by inclusion, and the topology of the line set is again a suitable quotient topology.

Now it is important to note that the topology of $S$ considered as a set of points of $\mathcal{T}_S$ is the same as the topology of $S$ inherited from $P_{3,1} = G_{4,2}$, the line space of $H$. By definition, $S$ considered as the point set of the line $S$ of $\mathcal{T}_S$ is the quotient of the point set $H_0$ of $H$ with respect to the map $H_0 \to S$ that sends a point to the unique spread line containing it. As we noted earlier, this map is continuous when $S$ is given the topology coming from the line space of $H$. It is also closed (by compactness) and surjective, hence it is a quotient map and our claim follows.
Remembering Theorem 2.4, we now conclude that orienting a spread \( S \) (the special line of \( T_S \)) as a manifold amounts to the same thing as orienting all lines of \( T_S \). The affine plane \( A_S \) is obtained from \( T_S \) by deleting the special line \( S \) and all its points, and so by orienting all lines of \( T_S \) we have in particular oriented all elements of \( S \), considered as 2-spaces in \( \mathbb{R}^4 \) (because they are lines of \( A_S \)).

In terms of the affine plane, this transfer of orientations is easy to visualize: \( S \) is the pencil of lines passing through the origin, and in order to orient \( S \in S \), apply to \( S \) a translation \( s \to s + v \) with \( v \notin S \). Then consider the bijection

\[ s + v \to (s + v) \lor 0 \]

of \( S + v \) onto \( S \setminus \{S\} \) and transfer the orientation via these two maps. This is, indeed, much simpler, but in order to prove consistency by applying Theorem 2.4 we prefer the projective version.

Summarizing these constructions, we obtain the following theorem.

**Theorem 2.5.** Let \( S \subseteq P_{3,1} \) be a compact spread of \( PG(3, \mathbb{R}) \). The construction given above defines a bijective correspondence between orientations of the manifold \( S \cong S_2 \) on the one hand and coherent orientations of all lines of the projective translation plane \( T_S \) associated with \( S \) on the other hand, i.e., sections to the orientation-forgetting map \( \psi \) of the latter plane.

In particular, orienting \( S \) as a manifold amounts to the same as orienting, in a coherent way, all the vector 2-spaces \( S \in S \) as manifolds (or as vector spaces).

### 2.6 Oriented parallelisms

A parallelism \( \Pi \) on \( PG(3, \mathbb{R}) \) is a set of pairwise disjoint spreads covering the line set \( P_{3,1} \). In other words, a parallelism partitions the line set and therefore is often thought of as an equivalence relation on the line set. Likewise, a parallelism of oriented lines or briefly, an oriented parallelism \( \Pi^+ \) is defined as a set of oriented spreads partitioning the set \( P_{3,1}^+ \) of oriented lines. If \( \Pi \) is a parallelism, then an oriented parallelism \( \Pi^+ \) can be obtained from it by taking all oriented spreads \( S^+ \) such that \( \psi S \in \Pi \), where as always \( \psi \) denotes the map that forgets orientations. This process will be called unfolding, and oriented parallelisms obtained in this way will be said to be foldable. Our investigation is motivated by the existence of non-foldable oriented parallelisms with nice properties, the discovery of which is described in [12], and by the desire to find more non-foldable examples worth looking at.

In a topology to be introduced shortly, an ordinary parallelism is a non-orientable 2-manifold (a projective plane) and an oriented parallelism is a 2-sphere. To avoid possible confusion, we stress here that the possible orientations of this sphere are irrelevant to us. So the term ‘oriented parallelism’ is merely a shorthand for ‘parallelism of oriented lines’. This contrasts with the situation for spreads, compare the preceding subsection.
For the convenience of the reader, we recall some basic facts obtained in [12], adding some details that require extra attention in the present situation.

We need a condition to ensure topological well-behavedness of a parallelism $\Pi$ or $\Pi^+$. A good choice for a topology on $\Pi$ or $\Pi^+$ is the topology defined by the Hausdorff metric [17], which we introduce next. Let $(X,d)$ be a compact metric space. The hyperspace $h(X)$ is the set of all compact subsets of $X$, endowed with the metric

$$d_h(A,B) = \max\{\max_{a \in A} d(a,B), \max_{b \in B} d(b,A)\},$$

where, as usual, $d(a,B) = \max_{b \in B} d(a,b)$. By the Hausdorff topology, we mean the topology on $h(X)$ induced by the Hausdorff metric. Two metric topologies agree if they induce the same notion of convergence. In the case of the Hausdorff metric, this notion is captured by the following Lemma, which follows easily from the definition, in view of compactness.

**Lemma 2.6.** Let $(X,d)$ be a compact metric space. A sequence $A_n \in h(X)$ converges to $A \in h(X)$ if and only if the following two conditions are satisfied.

i) If a sequence of points $a_n \in A_n$ converges to a point $a$ in $X$, then $a \in A$.

ii) Every point $a \in A$ is the limit of some sequence of points $a_n \in A_n$.

In what follows, we want to treat ordinary and oriented parallelisms simultaneously. We write $\Pi^*$ and $L^* \in P^*_3,1$ to indicate that we are thinking of both possibilities. By

$$\Pi^*(L^*)$$

we denote the unique spread in $\Pi^*$ that contains $L^*$. Similarly,

$$\Pi^*(x,L^*)$$

denotes the unique line that belongs to the same spread as $L^*$ and contains a given point $x$. In this way, we define two maps $P^*_3,1 \to \Pi^*$ and $P^*_3,1 \times P^*_3,1 \to P^*_3,1$; each of them contains the same information as $\Pi^*$ itself, which should justify the abuse of notation. We have the following

**Proposition 2.7.** Let $\Pi^*$ be an ordinary or oriented parallelism on $\text{PG}(3,\mathbb{K})$. The following conditions are equivalent.

1) $\Pi^*$ is compact with respect to the Hausdorff topology on $h(P^*_3,1)$.

2) The map $\Pi^*: P^*_3,1 \to h(P^*_3,1)$ defined above is continuous with respect to the Hausdorff topology on the hyperspace.

3) The map $\Pi^*: P^*_3,1 \times P^*_3,1 \to P^*_3,1$ defined above is continuous.
Proof. Assume (1). In order to show (3), we prove sequential continuity. If \((x_n, L_n^*) \to (x, L^*)\), we have to show that \(\Pi^*(x_n, L_n^*) \to \Pi^*(x, L^*)\). By compactness of \(\Pi^*\), we may assume that \(\Pi^*(L_n^*)\) converges to some spread \(S^* \in \Pi^*\). Then by Lemma 2.6 we have \(L^* \in S^*\), and so \(S^* = \Pi^*(L^*)\). Moreover, we may assume that \(\Pi^*(x_n, L_n^*)\) converges to some line \(K^* \in P^*_{3,1}\). Then \(x \in K^*\) since incidence is closed, and \(K^* \in \Pi^*(L^*)\) by Lemma 2.6 because \(\Pi^*(x_n, L_n^*) \in \Pi^*(L_n^*)\) and \(\Pi^*(L_n^*) \to \Pi^*(L^*)\). Thus, \(K^* = \Pi^*(x, L^*)\). This proves (3).

Now assume (3) and suppose that \(L_n^* \to L^*\). In order to prove (2), we have to show that \(\Pi^*(L_n^*) \to \Pi^*(L)\). Thus we have to verify conditions (i) and (ii) of Lemma 2.6. So let \(K_n^* \in \Pi^*(L_n^*)\) and assume that \(K_n^* \to K^* \in P^*_{3,1}\). We have to show that \(K^* \in \Pi^*(L)\). Choose points \(x_n \in K_n^*\). We may assume that \(x_n \to x \in P_{3,0}\). Then by (3), we have

\[K_n = \Pi^*(x_n, L_n^*) \to \Pi^*(x, L^*) \in \Pi^*(L^*).\]

This proves (i). For condition (ii), let \(K^* \in \Pi^*(L^*)\). We are looking for lines \(K_n^* \in \Pi^*(L_n^*)\) such that \(K_n^* \to K^*\). Choose any point \(x \in K^*\). Then \(K_n := \Pi^*(x, L_n^*)\) belongs to \(\Pi^*(L_n^*)\), and these lines converge to \(\Pi^*(x, L^*) = K^*\) by (3).

Finally, (2) implies (1) because the map considered in (2) restricts to a bijective map from the star of all lines or oriented lines containing any chosen point \(x\) to \(\Pi^*\). The star is compact, and (1) follows.

We shall say that \(\Pi^*\) is a topological (oriented) parallelism if it satisfies these equivalent conditions. Line stars are homeomorphic to \(P_2\mathbb{R}\) in the ordinary case and to \(S_2\) in the oriented case. Hence, the last step of the above proof shows:

**Lemma 2.8.** A topological parallelism \(\Pi\) is homeomorphic to \(P_2\mathbb{R}\), and a topological oriented parallelism \(\Pi^+\) is homeomorphic to \(S_2\). \(\square\)

### 3 Klein correspondence for oriented regular spreads

Within \(\text{PG}(5, \mathbb{R})\), the Klein correspondence sets up a model of \(\text{PG}(3, \mathbb{R})\) that is well suited for studying the line space \(P_{3,1}\). We summarize without proofs properties of the Klein correspondence that can be found in the literature, in particular in \([8, 5, 11]\). The Klein model arises from the index 3 bilinear form \(f(x, y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6\) on \(\mathbb{R}^6\). There are several kinds of special subspaces of \(\mathbb{R}^6\) with respect to this form:

- Totally isotropic one-dimensional subspaces. Viewed as points of \(\text{PG}(5, \mathbb{R})\), they constitute the Klein quadric \(K\), which represents the line set of \(\text{PG}(3, \mathbb{R})\).

- Two sorts of totally isotropic 3-dimensional subspaces (i.e., \(f\) induces the zero form on them). They represent the points and hyperplanes of \(\text{PG}(3, \mathbb{R})\). Incidence with lines is represented as reverse or direct inclusion, respectively.
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- Four-dimensional subspaces of signature (1, 3) or (3, 1), i.e., \( f \) induces a non-degenerate form of index one on them. Viewed as projective subspaces, they are 3-dimensional and intersect \( K \) in an elliptic quadric. They will be most important to us, and we call them (1,3)-spaces or (3,1)-spaces, respectively. As a vector space, a (1,3)-space contains a 3-dimensional negative definite subspace, and a (3,1)-space contains a 3-dimensional positive definite subspace.

We use the Klein correspondence mainly to describe regular spreads. A regular spread of \( \text{PG}(3, \mathbb{R}) \) is a spread isomorphic to the complex spread, which defines the complex affine plane. In other words, the latter spread consists of the one-dimensional complex subspaces of \( \mathbb{C}^2 = \mathbb{R}^4 \). We have the following Lemma, proved, e.g., in [3], Proposition 13.

**Lemma 3.1.** In the Klein model of \( \text{PG}(3, \mathbb{R}) \), the regular spreads are precisely the sets \( S = K \cap P \), where \( P \) is a (1,3)-space or a (3,1)-space.

Here, and frequently in what follows, we identify projective subspaces with their point sets, so that the lattice \( \text{PG}(5, \mathbb{R}) \) is viewed as a lattice of subsets of \( P_5 \mathbb{R} \). Our aim now is to obtain an oriented version of the above lemma, together with a continuity assertion. This will be achieved by the construction given below.

The space of oriented lines of \( \text{PG}(3, \mathbb{R}) \) is a two-sheeted covering of the line space, which we identify with the Klein quadric \( K \). Therefore, we shall write \( K^+ \) for the space of oriented lines and call it the oriented Klein quadric. The covering map \( K^+ \rightarrow K \) will be denoted \( \psi \), as usual. It is well-known that \( K^+ \cong S_2 \times S_2 \). An easy proof can be given using the oriented left and right Clifford parallelisms, see [12], Proposition 2.4.

**Construction.** Step 1. Let \( P^+ \in P^+_{5,3} \) be an oriented (3,1)-space, considered as projective space. The projective dimension of \( P^+ \) is odd, so, as explained in Section 2.4, we are given an orientation of \( P^+ \) as a differential manifold.

Step 2. Let \( P = \psi(P^+) \). The elliptic quadric \( S = P \cap K \), which represents a regular spread, separates \( P \) into two components, and is the boundary of both. The closure of the ‘interior’ component is a compact ball and inherits an orientation from \( P^+ \). This orientation induces an orientation on the boundary \( S \) as follows: Let \( B = (v_2, v_3) \) be an ordered basis of the tangent space \( T_s S \) at \( s \in S \) and let \( v \in T_s P \) be an outward pointing tangent vector. Then the orientation of \( S \) at \( s \) is defined by \( B \) if \( (v, v_2, v_3) \) is a positive basis of \( T_s P \).

Step 3. Finally, we apply Theorem 2.4 and from the orientation of the manifold \( S \) we obtain an orientation of all lines of the spread, such that we end up with one particular connected component \( S^+ \) of \( \psi^{-1}(S) \subseteq K^+ \). We shall denote this set by

\[
S^+ = S^+(P^+) \subseteq K^+.
\]

Our aim is to show now that the map \( P^+ \rightarrow S^+(P^+) \) from \( P^+_{5,3} \) to the hyperspace \( h(K^+) \) is continuous. This is the main step in the proof of Theorem 4.5 below, which describes a
way of constructing all oriented regular parallelisms. We need some preparation concerning group actions. For more details on the group actions discussed here, see [5].

We consider the group \( \Sigma = \text{SL}(4, \mathbb{R}) \).

By definition, this group acts on \( \mathbb{R}^4 \); it also acts (ineffectively) on \( \text{PG}(3, \mathbb{R}) \). Via the Klein correspondence, the latter action is translated to an \( f \)-orthogonal action on \( \mathbb{R}^6 \) and on \( \text{PG}(5, \mathbb{R}) \), where, as earlier, \( f \) denotes the form defining the Klein quadric. In fact, \( \Sigma \) induces an index 2 subgroup of the projective orthogonal group. In particular, the action of \( \Sigma \) leaves invariant all the sets of subspaces of special type enumerated at the beginning of this section. The actions on these sets of subspaces are transitive. The action also lifts to the sets \( P_{5,l}^+ \) of oriented subspaces.

A transitive action of a Lie group \( \Gamma \) on a manifold \( M \) always admits local sections. That is, given a point \( x \in M \), there exist a neighborhood \( U \) of \( x \) and a continuous map \( u \to \gamma_u \) from \( U \) into \( \Gamma \) such that \( \gamma_x \) is the identity and \( u = \gamma_u(x) \) for all \( u \in U \). This can be shown by proving that the map \( \Gamma \to M \) sending \( \gamma \) to \( \gamma(x) \) is a submersion. See [14] for a more general result.

**Lemma 3.2.** The actions of \( \Sigma \) on the sets of oriented and non-oriented special subspaces of \( \text{PG}(k, \mathbb{R}) \) listed at the beginning of this section admit local sections.

**Lemma 3.3.** Let \( \Gamma \) be a topological group acting on a metric space \( X \) and let \( Y \subseteq X \) be compact. If \( \gamma_\nu \) is a sequence in \( \Gamma \) converging to the unit element, then \( \gamma_\nu(Y) \to Y \) in the Hausdorff metric.

*Proof.* The continuity of the map \( \Gamma \times X \to X \) sending \((\gamma, x)\) to \( \gamma(x) \) implies that \( \gamma_\nu \to \text{id} \) uniformly on the compact set \( Y \). The claim follows easily.

**Lemma 3.4.** If \( \text{PG}(k, \mathbb{R}) \) is viewed as a lattice of subsets of \( P_k \mathbb{R} \), then the topology of the Grassmannians \( P_{k,l} \) is induced by the Hausdorff metric.

*Proof.* We use the continuity properties of the topological projective space \( \text{PG}(k, \mathbb{R}) \) given, e.g., in [2] or [5]. Suppose that \( P_\nu \to P \) in \( P_{k,l} \). Choose a subspace \( Q \) of dimension \( k-l-1 \) that is in general position to all these spaces. Then the projection map \( \pi_\nu : P \to P_\nu \) sending \( p \in P \) to \((p \lor Q) \land P_\nu \) is a homeomorphism, and \( \pi_\nu \) converges to the identity of \( P \). Hence for every point \( p \in P \), the sequence \( \pi_\nu p_\nu \in P_\nu \) converges to \( p \), and condition (ii) of Lemma 2.6 is satisfied. Condition (i) follows from the fact that the incidence relation is closed. This shows that \( P_\nu \to P \) with respect to the Hausdorff metric. Alternatively, this can be deduced from the preceding two lemmas.

Conversely, assume that \( P_\nu \to P \) in the Hausdorff sense. Let \( X \subseteq P \) be a set of \( l+2 \) points spanning \( P \) (a frame for \( P \)). Then for every \( \nu \) there is a set \( X_\nu \subseteq P_\nu \) of cardinality \( l+2 \) such that \( X_\nu \to X \) in the Hausdorff sense. It follows that for \( \nu \) large enough, \( X_\nu \) is a frame for \( P_\nu \), and the continuity of forming spans implies that \( P_\nu \to P \) in \( P_{k,l} \).
THEOREM 3.5. The above construction defines a continuous map \( P^+ \to S^+(P^+) \) from the set of oriented \((3,1)\)-spaces in \( P^+_{5,3} \) to the hyperspace \( h(K^+) \) of the oriented Klein quadric.

Proof. 1. Let \( P^+_\nu \to P^+ \) be a convergent sequence of \((3,1)\)-spaces in \( P^+_{5,3} \). According to Lemma 3.2, there is a sequence \( \sigma_\nu \in \Sigma \), converging to the identity, such that \( P^+ = \sigma_\nu P^+_\nu \) for all \( \nu \). Using Lemma 3.3, we infer that \( \psi P^+_\nu = P^+ \) converges to \( \psi P^+ = P^+ \) in the Hausdorff metric; compare also Lemma 3.4. Since \( K \) is invariant under \( \Sigma \), the same also holds for the intersections with \( K \), that is, \( S^\nu = P^+_\nu \cap K \to S = P \cap K \) in the hyperspace \( h(K) \).

2. The group \( \Sigma \) of isomorphisms respects all steps of the construction that turns an oriented projective subspace of odd dimension into an oriented manifold. Therefore, the oriented manifolds \( P^+_\nu \) converge to the oriented manifold \( P^+ \). By this we mean that there is a uniformly continuous sequence of orientation preserving embeddings of \( P^+_3 \to \mathbb{R} \) onto \( P^+_\nu \subseteq P^+ \) that converges to an orientation preserving embedding onto \( P^+ \). Indeed, the restrictions of the maps \( \sigma_\nu^{-1} \) form such a sequence.

3. The group \( \Sigma \) respects all structural features that were used in the construction of an orientation of the spreads \( S^\nu \) and \( S \) from the orientations of \( P^+_\nu \) and \( P^+ \), given in Step 2 of the construction following Lemma 3.1. As before, this implies that \( S^+_\nu \to S^+ \) as oriented manifolds.

4. The group \( \Sigma \) acts on both \( \mathbb{R}^4 \) and PG(3,\( \mathbb{R} \)). The group elements \( \sigma_\nu \) send the spread \( S^\nu \), considered as a subset of \( G^+_{4,2} \), to the spread \( S \). Consequently, \( \sigma_\nu \) is an isomorphism between the affine translation planes defined by these spreads, and extends to an isomorphism of the associated projective planes. Thus \( \sigma_\nu \) preserves all steps in the construction of orientations on the elements of \( S^\nu \) and of \( S \). Consequently, \( \sigma_\nu \) sends \( S^+(P^+_\nu) \to S^+(P^+) \). Moreover, \( \sigma_\nu \) converges to the identity on \( K^+ \), hence as before we may conclude that \( S^+(P^+_\nu) \to S^+(P^+) \) in the Hausdorff metric, as desired. \( \square \)

4 Oriented hfd line sets and first main result

We begin with a topological Lemma. It is probably known, but I do not remember seeing it in the literature. If \( X \) is a topological space, we let \( h_2(X) \) denote the set of subsets \( A \subseteq X \) with cardinality \#A = 2. We topologize this as the quotient

\[
h_2(X) = ((X \times X) \setminus \Delta_X) / \langle s \rangle,
\]

where \( \Delta_X \) denotes the diagonal \( \{(x,x) | x \in X\} \) and \( \langle s \rangle \) is the the group generated by the switching map \( s : (x,y) \to (y,x) \). If \( X \) is metric, then this topology is also induced by the Hausdorff metric, which is why we choose the symbol \( h_2 \).

LEMMA 4.1. Let \( q : \tilde{Y} \to Y \) be a two-sheeted covering map of connected Hausdorff spaces, and let \( X \) be a compact space. Let \( \tilde{g} : X \to \tilde{Y} \) be continuous and \( g = q \circ \tilde{g} \).
Suppose that all inverse images \( g^{-1}(y), y \in Y \), have cardinality 2 and that the resulting map \( g^{-1}: Y \to h_2(X) \) is continuous. Then the map \( \tilde{g} \) is bijective and, in fact, a homeomorphism.

Proof. Let \( U \subseteq Y \) be an open set which is evenly covered, that is, \( q^{-1}(U) \) is a union of two open subsets \( U_1, U_2 \) which are both mapped homeomorphically onto \( U \) by \( q \). If \( \tilde{g} \) maps the two \( g \)-inverse images of \( u \in U \) into the same sheet \( U_i \), then the images are in fact equal, and the same happens for nearby points \( u' \), by continuity of \( g^{-1} \). On the other hand, if those images are distinct, then the same holds in a neighborhood of \( u \). This shows that the cardinality of \( \tilde{g}(g^{-1}(y)), y \in Y \), is locally constant. By connectedness of \( Y \), this cardinality is always 1 or always 2. In the latter case, \( \tilde{g} \) is bijective and hence a homeomorphism by compactness. In the former case, looking at the sheets again one sees that the set \( \tilde{g}(X) \) and its complement are both open and nonempty, a contradiction to connectedness.

**Example 4.2.** The following shows that the assumption about continuity of the inverse is indispensable in the Lemma above. We take \( \tilde{Y} = X = S_2 \) and \( Y = P_5 \mathbb{R} \), the real projective plane. There is the two-sheeted covering \( q: \tilde{Y} \to Y \) sending \( y \) to \( \pm y \). Consider \( g = q \circ \tilde{g} \), where \( \tilde{g} \) is either the identity map or the folding map \( (x, y, z) \to (x, y, |z|) \). In the second case, \( g^{-1} \) is discontinuous at the equator \( (z = 0) \), and \( \tilde{g} \) is neither injective nor surjective.

We now return to the Klein model of \( \text{PG}(3, \mathbb{R}) \), and consider the polarity \( \pi_5 \) defined by the bilinear form \( f \) of signature \((3, 3)\). First we note that the polar \( \pi_5(x) \in P_{5,4} \) of a point \( x \in K \) (which represents a line of \( \text{PG}(3, \mathbb{R}) \)) is the tangent hyperplane of \( K \) at that point. This is not to be confused with the tangent vector space \( T_x K \) of the differential manifold \( K \), which is of no concern to us at the moment.

A line \( L \in P_{5,1} \) is called an exterior line with respect to \( K \) if \( L \cap K = \emptyset \). We note that this is the case if and only if \( L \) is a subspace of type \((2, 0)\) or \((0, 2)\), i.e., the form \( f \) is positive or negative definite on \( L \) considered as a two-dimensional vector space. Then \( \pi_5(L) \in P_{5,3} \) is a \((1,3)\)-space or a \((3,1)\)-space, respectively, and defines a regular spread \( \pi_5(L) \cap K \).

Betten and Riesinger \([3]\) defined a hyperflock determining line set or shortly, an hfd set to be a set \( \mathcal{H} \subseteq P_{5,1} \) of exterior lines such that every tangent hyperplane \( \pi_5(x), x \in K \), contains exactly one line from \( \mathcal{H} \). Since \( \pi_5 \) is an antiautomorphism of the lattice \( \text{PG}(5, \mathbb{R}) \) (i.e., it reverses inclusions), this implies that every \( x \in K \) is contained in exactly one element of \( \pi_5(\mathcal{H}) \). In other words, this set defines a regular parallelism \( \Pi(\mathcal{H}) \) by taking intersections with \( K \). Moreover, every regular parallelism arises in this way, and the parallelism \( \Pi(\mathcal{H}) \) is topological if and only if \( \mathcal{H} \) is compact; compare \([11]\). Now we imitate this in the oriented case, but we need to change in the pattern. For example, inclusion is only defined for non-oriented subspaces, so we cannot directly mimick the above definition of an hfd line set.
DEFINITION 4.3. a) An oriented hfd line set or briefly, an hfd$^+$ set is a set $\mathcal{H}^+ \subseteq P_{5,1}^+$ of oriented exterior lines such that

- for every $x \in K$, there are exactly two lines $H_i^+ \in \mathcal{H}^+$, $i = 1, 2$, such that the tangent hyperplane $\pi_5(x)$ contains $\psi H_1^+$ and $\psi H_2^+$, and

- the set $\{H_1^+, H_2^+\}$ of these oriented lines depends continuously on the point $x$.

b) If $\mathcal{H}^+$ is an hfd$^+$ set, we denote by $\Pi^+(\mathcal{H}^+) = S^+(\pi_5^+ \mathcal{H}^+)$ the set of all oriented spreads $S^+(\pi_5^+ H^+) \subseteq h(K^+)$, $H^+ \in \mathcal{H}^+$, as in Theorem 3.5.

In contrast to the non-oriented case, the definition of hfd$^+$ sets is quite useless without the condition on continuity of the inverse. This is because it works properly only in connection with Lemma 4.1; compare also Example 6.9. As a compensation, compactness can be deduced from this condition.

PROPOSITION 4.4. Let $\mathcal{H}^+$ be an hfd$^+$ set. Then

a) $\mathcal{H}^+$ is compact.

b) For each $H^+ \in \mathcal{H}^+$, there is a point $x \in K$ such that $\psi H^+ \subseteq \pi_5(x)$.

Proof. For assertion (b), note that exterior lines are of type $(2,0)$ or $(0,2)$. For every $x \in K$, the tangent hyperplane $\pi_5(x)$ contains subspaces of both those types, and the group $\Sigma$ is transitive on the subspaces of either type, whence (b) follows.

Now assertion (a) follows, because $K$ is compact. Indeed, from the continuity property of hfd$^+$ sets, we infer that the set of non-ordered pairs $\{H_1^+, H_2^+\}$ of oriented lines contained in some tangent hyperplane $\pi_5(x)$ is compact. By (b) it follows easily that $\mathcal{H}^+$ is compact, as well.

Here is our first main result. It describes all topological oriented regular parallelisms, whereas the final results of the next section only deal with the case that $\dim \text{span} \mathcal{H}^+ = 3$.

THEOREM 4.5. If $\mathcal{H}^+$ is an hfd$^+$ set, then the set $\Pi^+(\mathcal{H}^+) = S^+(\pi_5^+ \mathcal{H}^+)$ of oriented spreads is a topological oriented regular parallelism, and every topological oriented regular parallelism arises in this way.

COROLLARY 4.6. Every hfd$^+$ set $\mathcal{H}^+$ is homeomorphic to the 2-sphere $S_2$, and it consists entirely either of lines of type $(2,0)$ or of lines of type $(0,2)$.

Proof of Corollary. The polarity $\pi_5^+$ is continuous, and Theorem 3.5 asserts that the map $P^+ \to S^+(P^+)$ is continuous with respect to the Hausdorff metric. Since $\mathcal{H}^+$ is compact by Proposition 4.4, it follows that the oriented parallelism $\Pi^+(\mathcal{H}^+)$ is homeomorphic to $\mathcal{H}^+$, and we know that oriented parallelisms are homeomorphic to the 2-sphere. In particular, $\mathcal{H}^+$ is connected, and the second assertion follows.
Proof of Theorem 4.5. Using Theorem 3.5 and Proposition 4.4, we obtain that \( \Pi^+ = \Pi^+(\mathcal{H}^+) \) is a set of oriented spreads, and that this set is compact with respect to the Hausdorff metric. We want to use Lemma 4.1 in order to show that every oriented line belongs to exactly one of these oriented spreads. At first sight, there seems to be no mapping available to which the Lemma might be applied.

However, instead of the condition just stated, it suffices to consider the star \( \mathfrak{L}_p^+ \) of all oriented lines passing through some point \( p \) of \( \text{PG}(3, \mathbb{R}) \) and to prove that every oriented line in this star belongs to exactly one spread in \( \Pi^+ \). We view the star as a subset of \( \mathfrak{L}_x^+ \).

Now we have the two-sheeted covering map \( \psi : \mathfrak{L}_x^+ \to \mathfrak{L}_p^+ \), and we have a map \( \tilde{g} : \mathcal{H}^+ \to \mathfrak{L}_x^+ \) that sends an oriented line \( \mathcal{H}^+ \in \mathcal{H}^+ \) to the unique oriented line of the oriented spread \( S^+(\pi_5^+ \mathcal{H}^+) \) containing \( p \). This map is continuous by an argument similar to the proof of Theorem 2.7. By the properties of the \( hfd^+ \) set and those of the polarity, every line \( L \in \mathfrak{L}_x \) belongs to the spreads \( \psi S^+(\pi_5^+ \mathcal{H}^+) \) for exactly two lines \( \mathcal{H}^+_1, \mathcal{H}^+_2 \in \mathcal{H}^+ \) and, moreover, the set \( \{\mathcal{H}^+_1, \mathcal{H}^+_2\} \) depends continuously on \( L \). This means that the composite map \( g = \psi \circ \tilde{g} \) has inverse images \( g^{-1}(L) \) of cardinality 2, which depend continuously on \( L \). Now the Lemma tells us that \( \tilde{g} \) is bijective, which completes the proof that \( \Pi^+ \) is a compact oriented parallelism.

It remains to prove the converse, i.e., that every compact oriented parallelism comes from some \( hfd^+ \) set. This is obtained without difficulty by retracing all the steps. One special point is to show that the set of oriented lines from the \( hfd^+ \) set that are contained in a tangent hyperplane \( \pi_5^+(x), x \in K \), really depends on \( x \) continuously. The reason for this is the fact that the 2-valued inverse map of \( \psi : K^+ \to K \) is continuous because \( \psi \) is a covering map. Now the continuity property of the given oriented parallelism implies that the set of two oriented spreads containing the elements of \( \psi^{-1}(x) \) depends continuously on \( x \in K \), and this translates to the corresponding property of \( \mathcal{H}^+ \).

Every non-oriented \( hfd \) set yields an \( hfd^+ \) set by taking its inverse image with respect to \( \psi \), the forgetful map. \( hfd^+ \) sets obtained in this way will be called \textit{foldable}. Clearly we have the following.

**Proposition 4.7.** The oriented parallelism \( \Pi^+(\mathcal{H}^+) \) associated with an \( hfd^+ \) set \( \mathcal{H}^+ \) is foldable if and only if \( \mathcal{H}^+ \) is foldable.

5 Oriented generalized line stars and second main result

**Definition 5.1.** Let \( \Pi^* = \Pi^*(\mathcal{H}^*) \) be an oriented or non-oriented compact regular parallelism.

a) The space

\[ R = R(\Pi^*) := \text{span} \mathcal{H}^* \]

will be called the \textit{ruler} of the parallelism \( \Pi^* \).
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b) The number

$$\dim \Pi^* := \dim R = \dim \text{span} \mathcal{H}^*$$

will be called the dimension of $\Pi^*$. This terminology was introduced, in the non-oriented case, by Betten and Riesinger [3].

If $\dim \Pi^+ = 2$, then $\mathcal{H}^+ \approx S_2$ must consist of all oriented lines in $R$. According to [3], Lemma 2.7, $\psi(\Pi^+)$ is the ordinary Clifford parallelism. Hence $\Pi^+$ is its oriented unfolding, the oriented Clifford parallelism.

In this section, we shall deal with the case $\dim \Pi^* = 3$. In this case, passing to a dual object one obtains a very convenient description for $\mathcal{H}^*$. In the non-oriented case, this was shown by Betten and Riesinger [1]; see [11] for a simple proof. We shall see that this direct proof carries over to the oriented case almost verbatim. The only problem is to capture the right kind of dual object by a suitable definition.

Let an hfd set $\mathcal{H}^*$ with 3-dimensional span be given. We know that either all lines of $\mathcal{H}^*$ are (oriented) $(2,0)$-spaces, or all these lines are $(0,2)$-spaces. The two cases are interchanged by the map $(x_1, x_2, x_3, y_1, y_2, y_3) \rightarrow (y_1, y_2, y_3, x_1, x_2, x_3)$, so without loss of generality we may assume that the elements of $\mathcal{H}^*$ are of type $(0,2)$. The following result is essentially contained in the proof of [5], Theorem 23. We give an abbreviated proof for the sake of completeness.

**PROPOSITION 5.2.** Let $\Pi^* = \Pi^*(\mathcal{H}^*)$ be a 3-dimensional compact regular parallelism, and assume that the elements of $\mathcal{H}^*$ are of type $(0,2)$ (i.e., negative definite). Then the ruler $R(\Pi^*) \in P_5,3$ is of type $(1,3)$. In particular, it meets the Klein quadric $K$ in an elliptic quadric $Q = K \cap R$.

**Proof.** If $\mathcal{H}^*$ were a line star $\mathcal{L}_y^*$ of $R$, then the point $y$ would belong to all tangent hyperplanes $\pi_5(x)$, $x \in K$, a contradiction. Therefore, $R$ is spanned by two lines $H_1, H_2$ from $\psi \mathcal{H}^*$. Thus $\pi_5 R = \pi_5 H_1 \wedge \pi_5 H_2$ meets $K$ in the (empty) intersection of two spreads from $\Pi^*$ and is an exterior line. This implies our claim. 

Now we want to start with a potential ruler $R$ of type $(1,3)$ in $\text{PG}(5, \mathbb{R})$ and to find an hfd set $\mathcal{H}^*$ in $R$. We use the polarity $\pi_3$ of $R$ induced by $\pi_5$, in other words, the polarity defined by the restriction to $R$ of the form $f$. We say that a point of $R$ is non-interior with respect to $Q = R \cap K$ if it either belongs to $Q$ or is contained in a line that misses $Q$. Let $N_i Q$ be the set of non-interior points. By a 2-secant of $Q$ we mean a line meeting $Q$ in two distinct points.

In the non-oriented case, it is known from [1] and [11] that the compact hfd sets in $R$ are precisely the sets $\pi_3 G$, where $G$ is a compact set of 2-secants of $Q$ such that every point of $N_i Q$ belongs to exactly one line from $G$. A set $G$ of this kind is called a generalized line star, abbreviated gl star. The simplest example is an ordinary line star $G = \mathcal{L}_x$; then $\mathcal{H} = \pi_3 \mathcal{L}_x$ is the line set of the plane $\pi_3 x$, and we are in the Clifford case. We ask how to define the correct oriented analogue of a gl star. The answer is the following.
DEFINITION 5.3. Let \( Q \) be an elliptic quadric in a 3-dimensional real projective space. A set \( G^+ \) of oriented 2-secants of \( Q \) is called an \textit{oriented gl star} or just a \textit{gl} \( ^+ \) star if every point \( x \) of \( N \setminus Q \) is incident with exactly two oriented lines from \( G^+ \), and if the set of these two oriented lines depends continuously on the point \( x \).

As with \( hfd^+ \) sets, there is no version of the notion of \( gl^+ \) stars without the continuity condition, but compactness can be deduced from it.

**LEMMA 5.4.** Every \( gl^+ \) star \( G^+ \) is compact, but compactness cannot replace the continuity condition in the above definition.

**Proof.** Compactness of \( G^+ \) follows directly from compactness of \( Q \) via the continuity property. That compactness does not conversely imply the continuity condition will be demonstrated by Example 6.9.

**THEOREM 5.5.** Let \( R \) be a 3-space of type \((1,3)\) in \( \text{PG}(5,\mathbb{R}) \). The \( hfd^+ \) sets in \( R \) are precisely the sets \( \mathcal{H}^+ = \pi_3^+ G^+ \), where \( G^+ \) is a \( gl^+ \) star with respect to \( Q = R \cap K \). The space \( R \) is generated by \( \mathcal{H}^+ \) if and only if \( G^+ \) is not an ordinary star of oriented lines.

Combining this with Theorem 4.5 we obtain our main result for the 3-dimensional case:

**COROLLARY 5.6.** The compact oriented regular parallelisms of \( \text{PG}(3,\mathbb{R}) \) of dimension \( d \leq 3 \) are precisely the parallelisms

\[
\Pi^+(G^+) := \Pi^+(\pi_3^+ G^+),
\]

where \( G^+ \) is a \( gl^+ \) star in a 3-space \( R \) of \( \text{PG}(5,\mathbb{R}) \) that meets the Klein quadric \( K \) in an elliptic quadric of \( R \). The parallelism is Clifford if and only if \( G^+ \) is an ordinary star of oriented lines.

If the ruler \( R \) is of type \((1,3)\), as we have assumed previously, then the 3-spaces \( \pi_3^+ \pi_3^+ L^+ \), \( L^+ \in G^+ \), which define the oriented spreads of \( \Pi^+(G^+) \), are of type \((3,1)\).

The proof of Theorem 5.5 uses the following Lemma.

**LEMMA 5.7.** ([11], 2.4) Let \( U \in P_{3,1} \) be a subspace of type \((1,3)\) and consider the elliptic quadric \( Q = X \cap K \). For every \( x \in K \setminus Q \), the tangent hyperplane \( \pi_5(x) \) intersects \( Q \) in a non-degenerate conic, and every non-degenerate conic in \( Q \) arises in this way.

**Proof.** The quadric \( Q \) represents a spread in \( \text{PG}(3,\mathbb{R}) \), hence the line represented by \( x \notin Q \) intersects infinitely many lines in this spread. This means that \( x \) is \( f \)-orthogonal to infinitely many elements \( q \in Q \). These elements then belong to the plane \( \pi_5(x) \land R \). The converse follows using transitivity properties of the group \( \Sigma \).

**Proof of Theorem 5.3.** The proof is practically the same as the proof of [11], Theorem 2.3. Only the words ‘exactly one’ have to be replaced by ‘exactly two’ where appropriate.
For the sake of completeness, we give the proof that a $gl^+$ star $G^+$ yields an $hfd^+$ set; the converse direction is similar.

For $L^+ \in G^+$ and $x \in K$, we have that $\psi H^+ = \psi \pi_3^+(L^+) \in \psi H^+$ is contained in the tangent hyperplane $\pi_5(x)$ if and only if it is contained in $\pi_3(x) \cap R$, and this happens if and only if $\psi L^+$ contains the point $r(x) := \pi_3(\pi_5(x) \cap R)$. If $x \notin Q$, then $r(x) \in Ni Q$ by Lemma [5.7] If $x \in Q$, then again, $r(x) = x \in Ni Q$. There are exactly two lines $L_i^+ \in G^+$, $i = 1, 2$, containing $r(x)$. The set $\{L_1^+, L_2^+\}$ depends continuously on $r(x)$, which in turn is a continuous function of $x$.

Every compact non-oriented $gl$ star $G$ has the continuity property, that is, the line from $G$ passing through a non-interior point $p$ continuously depends on $p$, see [11], Theorem 3.2. This implies that taking the inverse image of $G$ with respect to $\psi$, we obtain a $gl^+$ star $G^+$. Such examples are said to be foldable. Clearly, we have the following.

**Proposition 5.8.** Let $G^+$ be a $gl^+$ star. Then $G^+$ and the associated $hfd^+$ set $\pi_3^+G^+$ as well as the associated oriented parallelism $\Pi^+(G^+)$ are either all foldable or all non-foldable.

**6 Examples**

With a non-oriented $gl$ star $G$, there is associated the involutory homeomorphism $\sigma : Q \to Q$ which sends a point $x \in Q$ to the second point of intersection of the line $G_x \in G$ containing $x$. This involution carries all information about $G$. In the oriented case, the intersection points of $L^+ \in G^+$ with $Q$ can be distinguished: at one point, $L^+$ enters the closed ball $B$ bounded by $Q$ (i.e., a positive tangent vector points inward), and at the other point, the line leaves $B$. Let us call these points $e(G^+)$ and $l(G^+)$, respectively.

**Lemma 6.1.** If $G^+$ is a $gl^+$ star with respect to $Q$, then at every point of $Q$ exactly one oriented line $L^+ \in G^+$ enters the closed ball $B$ bounded by $Q$, and exactly one leaves $B$.

**Proof.** As every oriented line $L^+ \in G^+$ has an entry point and a leave point, both the set of entry points and the set of leave points are nonempty. Lines entering at $x_n \to x$ cannot converge to a line leaving at $x$, hence both sets are closed. So the connected quadric $Q$ is a disjoint union of three closed sets: the set $EL$ of points where one line enters and one line leaves, the set $EE$ where two lines enter, and the set $LL$ where two lines leave. That these sets are closed follows from the continuity property in the definition of $gl^+$ stars. Only one of the three sets can be nonempty, and this set can only be $EL$. 

**Definition 6.2.** Given a $gl^+$ star $G^+$, define a map $\rho : Q \to Q$ by sending $x \in Q$ to the leave point of the unique line $G_x^+$ that enters the ball $B$ at $x$. We call this map the characteristic map of $G^+$.

The following lemma is now obvious.
**Lemma 6.3.** If $G^+$ is a $gl^+$ star with characteristic map $\rho$, then $\rho$ is a fixed point free homeomorphism, and $G^+$ is the set of all lines $G^+_x = x \vee \rho(x)$, oriented in such a way that the interval $G_x \cap B$ is traversed from $x$ to $\rho(x)$.

This opens up a huge set of candidates for $gl^+$ stars. Every fixed point free homeomorphism of the 2-sphere may be tested for its potential of defining a $gl^+$ star. The two oriented lines passing through $x \in Q$ are then $x \vee \rho(x)$ and $\rho^{-1}(x) \vee x$, so the continuity condition for the set of oriented lines of the $gl^+$ star passing through a given point is satisfied on $Q$ at least. In general, the test will fail nevertheless, but one may suspect that there are far more successes than with ordinary $gl$ stars, where the characteristic map is an involution. We pursue this a bit further.

**Definition 6.4.** An (oriented) $gl^+$ star is said to be foldable if by forgetting orientations it yields an ordinary $gl$ star. As before, we call an oriented parallelism $\Pi$ foldable if by forgetting orientations we get an ordinary parallelism.

Again we have two obvious facts:

**Proposition 6.5.**

a) A $gl^+$ star is foldable if and only if its characteristic map is an involution.

b) The oriented regular parallelism defined by a $gl^+$ star is foldable if and only if the $gl^+$ star is foldable.

**Example 6.6.** Here is a class of non-foldable $gl^+$ stars. Compare also Example 6.9. In [13], a large set of rotationally symmetric $gl$ stars were constructed. They are defined by their characteristic involutions $\sigma : S_2 \to S_2$. On the equator ($z = 0$) the involutions agree with the antipodal map, that is, $\sigma(x, y, 0) = -(x, y, 0)$. Now we take two different involutions $\sigma_1$ and $\sigma_2$ of this kind and define a homeomorphism $\rho S_2 \to S_2$ by sending $(x, y, z)$ to $\sigma_1(x, y, z)$ if $z \geq 0$, and to $\sigma_2(x, y, z)$ if $z \leq 0$. Then it is easily checked that this map $\rho$ is the characteristic map of a non-foldable parallelism.

**Theorem 6.7.** The examples described above define compact oriented regular parallelisms. These parallelisms are non-foldable, and they admit a 2-dimensional torus group of automorphisms.

The proof is almost automatic. For the automorphism group, compare [3], Theorem 31 or [13], Proposition 4.1. We note that a 2-torus is as much symmetry as an oriented regular parallelism can have without being Clifford, see [13], Theorem 2.1.

If a $gl^+$ star is to be constructed from its characteristic map, then the defining incidence condition can be relaxed and the continuity property can be replaced by an orientation rule. The analogous result for ordinary $gl$ stars is Proposition 5.1 of [13].

**Theorem 6.8.** Let $Q$ be an elliptic quadric in a real projective 3-space $R$ and let $\rho : Q \to Q$ be a fixed point free homeomorphism. Let $G^+ = G^+(\rho)$ be the set of all
oriented lines $G^+_x = x \lor \rho(x)$, oriented in such a way that the interval $\psi(G^+_x) \cap B$ is traversed from $x$ to $\rho(x)$.

If each point of $Ni Q$ is incident with at most two of these oriented lines, then $G^+$ is a gl$^+$ star. In particular, the continuity property of gl$^+$ stars comes for free in this situation.

**Proof.** 1. We may assume that $Q$ is the unit sphere in the affine space $\mathbb{R}^3$, of which $R$ is the projective closure. By construction, $G^+$ is compact. First we look at ‘affine’ points $a \in A := \mathbb{R}^3 \cap Ni Q$. The orientation of a line $L^+ \in G^+$ defines an order relation on the affine line $\psi L^+ \cap \mathbb{R}^3$. By the construction of $G^+$, the entry and leave points of $L^+$ satisfy $e(L^+) < l(L^+)$. We call $L^+$ a positive line with respect to $a \in L^+$ if $l(L^+) \leq a$. The only other possibility is $a \leq e(L^+)$, in which case we call $L^+$ a negative line with respect to $a$.

2. Let $L^+_n \in G^+$ be a sequence of oriented lines that are positive with respect to points $a_n$. If both sequences converge, then $\lim L^+_n$ is positive with respect to $\lim a_n$. Let $B$ be the set of points incident both with a positive line and with a negative line from $G^+$.

3. It suffices to show that every point of $A$ is on a negative line. For $x \in Q$, we have the oriented line $G^+_x \in G^+$. We define a ray $W_x$ as the connected component of $\psi(G^+_x) \cap \mathbb{A}$ that contains $x$. For $r \geq 1$, let $g_r(x)$ be the intersection point of $W_x$ with the sphere $rQ$ of radius $r$. Then $G^+_x$ is a negative line with respect to $g_r(x)$. Let $h : A \to Q$ be the map sending $y$ to $y \|x\|^{-1}$. Then $k_r := h \circ g_r$, $r \geq 1$, is a family of pairwise homotopic maps $Q \to Q$, with $k_1 = \text{id}$. So all of these maps have mapping degree one and, hence, are surjective. This proves our claim, and the continuity condition is proved on the set $A$.

4. It remains to prove continuity for points on the plane $I$ at infinity. Every point $y \in I$ has a neighborhood $U$ homeomorphic to $\mathbb{R}^2 \times [-1, 1]$ such that $U \cap I$ corresponds to $\mathbb{R}^2 \times \{0\}$ and that $U \cap rQ$ for some large number $r$ corresponds to $\mathbb{R}^2 \times \{-1, 1\}$. By compactness of $Q$, and for large enough $r$, there is a neighborhood $V \subseteq U$ of $y$ such that each oriented line meeting both $V$ and $Q$ intersects both the top layer $\mathbb{R}^2 \times \{1\}$ and the bottom layer $\mathbb{R}^2 \times \{-1\}$ of $U$. These oriented lines may therefore be divided into the set of upward lines, traversing $U$ from bottom to top, and downward lines. By the previous steps, each point of $V \setminus I$ is incident with both an upward line and a downward line from $G^+$. As before, we may conclude that the same is true for all points of $V$, and the continuity condition is guaranteed for these points.

We conclude with an example demonstrating the necessity of the condition on continuity of the inverse in the definitions of gl$^+$ stars and, hence, also in that of hfd$^+$ sets. It shows that compactness is not a possible surrogate condition.

**EXAMPLE 6.9.** We start by defining an ordinary gl star $G_1$ with respect to $Q = S_2$ in $\text{PG}(3, \mathbb{R})$. It contains all lines of the plane $z = 0$ that pass through the origin $o = (0, 0, 0) \in \mathbb{R}^3$, but no other lines containing $o$. This gl star does not have rotational symmetry. It has to be so, because rotationally symmetric gl stars inevitably contain the
rotation axis. Incidentally, this is the simplest known example with this property. For more such examples, see Section 7.2 of [4].

Consider the points

\[ p_t = (\sqrt{1-t^2}, 0, t) \in S_2 \quad \text{and} \quad q_t = (f(t), 0, 0), \]

where \( t \in [0, \frac{1}{2}] \) and where \( f : [0, \frac{1}{2}] \to [\frac{1}{2}, 0] \) is a strictly decreasing bijection. Let \( L_t \) be the line \( p_t \vee q_t \). Then the slope of \( L_t \) strictly decreases from 0 to \(-\infty\), and \( L_{\frac{1}{2}} \) is parallel to the \( z \)-axis \( Z \). Now rotate the line \( L_t \) about the axis \( A_t \) parallel to \( Z \) and passing through \( q_t \). These rotated lines fill a cone \( C_t \), and we define \( G_1 \) to be the set of all lines obtained by rotating \( L_t \) for all \( t \in [0, \frac{1}{2}] \). The non-interior part \( C_t \cap NiQ \) lies completely inside \( C_s \) for all \( s < t \). By continuity, every point of \( NiQ \) is incident with exactly one line from \( G_1 \), and we have defined a \( gl \) star.

Now consider the ordinary star of lines \( G_2 = \mathfrak{L}_o \), and observe that \( G_1 \cap G_2 \) consists precisely of the horizontal lines passing through \( o \). This gives us two possibilities to try forming a \( gl^+ \) star.

1. We take the horizontal lines with both orientations, the lines of \( G_1 \) with upward orientation, and the lines of \( G_2 \) with downward orientation. Like Example 6.6, this gives a nice non-foldable \( gl^+ \) star satisfying the continuity condition, but without rotational symmetry.

2. We take the elements of \( G_1 \cup G_2 \), all of them with upward orientation (and two orientations for the horizontal ones). This is a compact set \( A \approx S_2 \) of oriented lines such that every point \( p \in NiQ \) lies on precisely two distinct oriented lines in \( A \) (here we use the information about the intersection \( G_1 \cap G_2 \)). But the set of those two lines does not depend on \( p \) continuously when \( p \) is in the plane \( z = 0 \). So \( A \) is not a \( gl^+ \) star. By Corollary 5.6, it does not define an oriented parallelism.

The possibility of such examples quite puzzled the author until he understood the relevance of the continuity condition. What the example tells us is that, in contrast with non-oriented \( gl \) stars, compactness does not suffice to ensure the continuity property of a \( gl^+ \) star. Yet for oriented parallelisms themselves, compactness is enough, by Proposition 2.7.

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