A HYPERGEOMETRIC BASIS FOR THE ALPERT 
MULTIRESOLUTION ANALYSIS

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Abstract. We construct an explicit orthonormal basis of piecewise \( \begin{pmatrix} i+1 \\ i \end{pmatrix} \) hypergeometric polynomials for the Alpert multiresolution analysis. The Fourier transform of each basis function is written in terms of \( \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) hypergeometric functions. Moreover, the entries in the matrix equation connecting the wavelets with the scaling functions are shown to be balanced \( \begin{pmatrix} 4 \\ 3 \end{pmatrix} \) hypergeometric functions evaluated at 1, which allows to compute them recursively via three-term recurrence relations.

The above results lead to a variety of new interesting identities and orthogonality relations reminiscent to classical identities of higher-order hypergeometric functions and orthogonality relations of Wigner 6j-symbols.

1. Introduction

Wavelet theory has had proved useful in many areas of mathematics and engineering such as functional analysis, Fourier analysis, and signal processing. The Alpert multiresolution analysis is associated with the spline spaces of piecewise polynomials of degree at most \( n \), discontinuous at the integers. The generators for this multiresolution analysis are the Legendre polynomials, restricted and scaled to \([0, 1] \) and set to zero otherwise. Using the symmetry inherent in this system, Alpert [1, 2] constructed a set of wavelet functions and then used them to analyze various integral operators (see also [3]). This basis has been used in [6] with the moment interpolating technique to construct smooth multiwavelets. An interesting problem from both computational and theoretical point of view is to find a wavelet basis which can be written in terms of explicit formulas.

In [12] an analysis was performed on the coefficients in the refinement equation satisfied by the modified Legendre polynomials and it was shown that the entries in these matrices could be written as multiples of certain generalized Jacobi polynomials evaluated at \( 1/2 \) and that these entries satisfy generalized eigenvalue equations. Moreover a new basis of wavelets was implicitly introduced through the matrix equation connecting the wavelets to the scaling functions by considering upper triangular matrices with positive diagonal entries.

In the present paper we provide explicit formulas for these wavelets, their Fourier transforms, and related matrix coefficients in terms of hypergeometric functions. We give a different construction and direct proofs of the characteristic properties of this new basis thus making the paper self-contained. Our results imply new
identities between higher-order hypergeometric functions and suggest interesting connections to representation theory.

The paper is organized as follows. In Section 2 we set the notation and review the elements of multiresolution analysis needed for the sequel. In Section 3 we postulate the orthogonality and symmetry conditions which characterize the wavelet functions. We show that these properties are satisfied by a sequence of piecewise polynomials supported on \([0, 1]\), which on \([0, 1/2]\) and \([1/2, 1]\) can be written as \(i+1\)F\(i\) hypergeometric functions. We also exhibit families of differential equations satisfied by these functions. In Section 4 we explain the differences between the wavelets constructed here and the ones introduced by Alpert. We also prove that the entries in the matrices relating our wavelets to the scaling functions can be written as balanced \(4F3\) hypergeometric functions evaluated at 1. In particular, these formulas imply that the matrices are upper triangular with positive diagonal entries, thus relating the formulas here to the implicit construction in [12]. We also indicate how these results are reminiscent to classical identities of higher-order hypergeometric functions and orthogonality relations of Wigner 6\(j\)-symbols. In Section 5 we derive a simple closed formula for the Fourier transform of these wavelets in terms of \(2F3\) hypergeometric functions and we write associated differential equations for them. Finally, in Section 6 we give recurrence formulas for the entries in the matrices of the wavelet equation.

2. Preliminaries

Let \(\phi_0, \ldots, \phi_r\) be compactly supported \(L^2\)-functions, and suppose that \(V_0 = \text{cl}_{L^2} \text{span}\{\phi_i(-j) : i = 0, 1, \ldots, r, j \in \mathbb{Z}\}\). Then \(V_0\) is called a finitely generated shift invariant (FSI) space. Let \((V_p)_{p \in \mathbb{Z}}\) be given by \(V_p = \{\phi(2^p \cdot) : \phi \in V_0\}\). Each space \(V_p\) may be thought of as approximating \(L^2\) at a different resolution depending on the value of \(p\). The sequence \((V_p)\) is called a multiresolution analysis (MRA) [5, 11, 13] generated by \(\phi_0, \ldots, \phi_r\) if (a) the spaces are nested, \(\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots\), and (b) the generators \(\phi_0, \ldots, \phi_r\) and their integer translates form a Riesz basis for \(V_0\). Because of (a) and (b) above, we can write

\[
V_{j+1} = V_j \oplus W_j, \quad \forall j \in \mathbb{Z}. \tag{1}
\]

The space \(W_0\) is called the wavelet space, and if \(\psi_0, \ldots, \psi_r\) generate a shift-invariant basis for \(W_0\), then these functions are called wavelet functions. If, in addition, \(\phi_0, \ldots, \phi_r\) and their integer translates form an orthogonal basis for \(V_0\), then \((V_p)\) is called an orthogonal MRA. Let \(S^n_{-1}\) be the space of polynomial splines of degree \(n\) continuous except perhaps at the integers, and set \(V^n_0 = S^n_{-1} \cap L^2(\mathbb{R})\). With \(V^n_p\) as above these spaces form a multiresolution analysis. If \(n = 0\) the multiresolution analysis obtained is associated with the Haar wavelet while for \(n > 0\) they were introduced by Alpert [11, 2]. Let

\[
\phi_j(t) = \hat{\phi}_j(2^j t - 1) \chi_{[0,1)}(t)
\]

where \(\hat{\phi}_j(t)\) is the Legendre polynomial [17] of degree \(j\) orthonormal on \([-1, 1]\) with positive leading coefficient i.e. \(\hat{\phi}_j(t) = k_j t^j + \text{lower degree terms with } k_j > 0\) and

\[
\int_{-1}^{1} \hat{\phi}_j(t)\hat{\phi}_k(t)dt = \delta_{k,j}.
\]
These polynomials have the following representation in terms of a \( _2F_1 \) hypergeometric function [17, p. 80],
\[
\hat{p}_n(t) = \frac{\sqrt{2n + 1}}{\sqrt{2}} _2F_1 \left( -n, n + 1; 1 - t; \frac{1}{2} \right),
\]
where formally,
\[
_pF_q \left( a_1, \ldots, a_p; b_1, \ldots, b_q; t \right) = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_p)_i}{(b_1)_i \cdots (b_q)_i} \frac{t^i}{i!}
\]
with \((a)_0 = 1\) and \((a)_i = a(a+1) \ldots (a+i-1)\) for \(i > 0\). Since one of the numerator parameters in the definition of \(\hat{p}_n\) is a negative integer, the series in equation (2) has only finitely many terms. With the normalization taken above \(\|\phi_j\|_{L^2(\mathbb{R})} = \frac{1}{2}\), and
\[
\Phi_n = \begin{bmatrix} \phi_0 & \cdots & \phi_n \end{bmatrix}^T
\]
and its integer translates form an orthogonal basis for \(V_0\). For the convenience in later computations we set
\[
P_n(t) = \begin{bmatrix} \hat{p}_0(t) \\
\vdots \\
\hat{p}_n(t) \end{bmatrix}.
\]
Equation (1) implies the existence of the refinement equation,
\[
\Phi_n \left( \frac{t+1}{2} \right) = C_{n-1}^n \Phi_n(t) + C_1^n \Phi_n(t - 1).
\]
In order to exploit the symmetry of the Legendre polynomials we shift \(t \to t + 1\) so that
\[
\Phi_n \left( \frac{t + 1}{2} \right) = P_n(t) \chi_{[-1,1]}(t) = C_{n-1}^n \Phi_n(t + 1) + C_1^n \Phi_n(t)
\]
\[
= C_{n-1}^n P_n(2t + 1) \chi_{[-1,0]}(t) + C_1^n P_n(2t - 1) \chi_{[0,1]}(t).
\]

3. Construction of the functions

We want to construct a set \(\{h_0^n, \ldots, h_n^n\}\) of functions such that,
\begin{enumerate}
\item On \([-1, 0)\) and \([0, 1)\), \(h_i^n\) is a polynomial of at most degree \(n\),
\item \(\int_{-1}^{1} t^i h^n_i(t) dt = 0\), \(0 \leq s \leq n\), \(0 \leq i \leq n\),
\item \(\int_{-1}^{1} t^i h^n_i(t) h^n_j(t) dt = 2 \delta_{i,j}\), \(0 \leq i, j \leq n\),
\item \(h^n_0(-t) = (-1)^{i+1} h_i^n(t)\) and,
\item \(\int_{0}^{1} t^i h^n_i(t) dt = 0\), \(s < i\).
\end{enumerate}

We will show,
\begin{align*}
\textbf{Theorem 1.} & \text{ For } 0 \leq t < 1, \text{ we have } \\
h_i^n(t) & = \sum_{k=0}^{n} d_{n,k} t^k, \quad \text{if } i = 0 \text{ or } i \text{ odd} \\
h_i^n(t) & = \sum_{k=0}^{n} d_{n-1,k} t^k, \quad \text{for } i > 0 \text{ even},
\end{align*}
where
\[
d_{n,k}^i = (-1)^n \frac{\sqrt{2n-2i+1}}{(n)_i} \frac{(-n)_k(n-i+1)_k}{(1)_k(1)_k} \\
\times \prod_{m=0}^{n} (n+k+1-2m) \prod_{m=0}^{n} (n-k-1-2m), \quad 0 \leq k \leq n, \quad i = 0, \text{ or } i \text{ odd.}
\]

We extend \( h_{n-i}^n(t) \) on \([-1,0)\) using iv.

Remark 2. From the above formula it follows that the polynomials \( \frac{(-n)_k h_{n-i}^n(t)}{\sqrt{2n-2i+1}} \) for \( i = 0 \) or \( i \text{ odd} \) and \( \frac{(-n+1)_k h_{n-i}^n(t)}{\sqrt{2n-2i+1}} \) for \( i > 0 \) even have integer coefficients.

We prove this theorem after developing a few lemmas. We begin with,

**Lemma 3.** For \( i \leq n \)
\[
\sum_{k=0}^{n} \frac{(-n)_k}{k!} \left( \frac{n-i+k}{n-i} \right) \frac{1}{x+k} = \frac{n!(-x+1)_{n-i}}{(n-i)!x_{n+1}}.
\]

In particular when \( x = n - i + 1 \) we have
\[
\sum_{k=0}^{n} \frac{(-n)_k}{k!} \left( \frac{n-i+k}{k} \right) \frac{1}{n-i+k+1} = (-1)^{n+i} \frac{n!(n-i)!}{(2n-i+1)!}.
\]

**Proof.** Consider a polynomial \( f(x) \) of degree at most \( n \). The Lagrange interpolation formula at the points \( x_k = -k, \ k = 0, 1, \ldots, n \) gives
\[
(x)_{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{f(-k)}{x+k} = f(x),
\]
or equivalently
\[
\sum_{k=0}^{n} \frac{(-n)_k f(-k)}{k!} \frac{1}{x+k} = \frac{n!f(x)}{(x)_{n+1}}.
\]

Applying the above formula with \( f(x) = \binom{n-i-x}{n-i} \) we find
\[
\sum_{k=0}^{n} \frac{(-n)_k}{k!} \left( \frac{n-i+k}{n-i} \right) \frac{1}{x+k} = \frac{n!(-x+1)_{n-i}}{(n-i)!x_{n+1}},
\]
which gives the result. \( \square \)

It is easy to see,

**Lemma 4.** For \( k \geq n - i + 2 \) and \( n - k \text{ odd} \) \( d_{n,k}^i = 0 \).

Also

**Lemma 5.** If \( i + l < n \) and \( l \geq -1 \) then for any polynomial \( p(k) \) with \( \deg p \leq i + l \),
\[
\sum_{k=0}^{n} \frac{(-n)_k(n-i+1)_k}{(1)_k(l+2)_k} p(k) = 0.
\]
Proof. By linearity, it is enough to show (12) for the polynomials \( p(k, r) = k(k-1) \cdots (k-r+1) \) with \( 0 \leq r \leq i + l \). If the common factor in the numerator and denominator are canceled and the change of variable \( k = r + m \) made, the above sum is equal to

\[
\sum_{k=0}^{n} \frac{(-n)_k(n-k+1)_k}{(1)_k(l+2)_k} p(k, r) = c_{n, r, i, l} \sum_{m=0}^{n-r} \frac{(-n+r)_m(n-i+r+1)_m}{(1)_m(r+l+2)_m} \times 2F1 \left( \frac{-n+r}{r+l+2}; 1 \right)
\]

\[
= c_{n, r, i, l} \frac{(-n+i+l+1)_{n-r}}{(r+l+2)_{n-r}} = 0,
\]

where \( c_{n, r, i, l} = \frac{(-n)_l(n-i+1)}{(l+2)_r} \). The Chu-Vandermonde identity was used to obtain the last line in the above equation.

Proof of Theorem[7]. Note first that by iv we have

\[
\int_{-1}^{1} t^s h_{n-i}^n(t) dt = (\frac{(-1)^{n-i+s+1} + 1}{2}) \int_{0}^{1} t^s h_{n-i}^n(t) dt. \tag{13}
\]

Thus, if the parity of \( s \) and \( n-i \) is the same, the above integral is equal to zero. From this it is easy to see that (3) follows from (7).

If \( i = 0 \), (7) shows that \( h_{n}^n(t) \) is the normalized Legendre polynomial on \([0, 1]\) which is orthogonal to \( t^j \) for \( 0 \leq j < n \). This combined with (13) establishes properties i-v for \( h_{n}^n(t) \) completing the proof in this case. It remains to show that for \( i \) odd the functions defined by (7) satisfy the required conditions, so we fix \( i \) odd below.

Formula (7) yields

\[
\int_{0}^{1} t^s h_{n-i}^n(t) dt = \sum_{k=0}^{n} d_{n,k} \frac{1}{k+s+1} = c_{n,i} \sum_{k=0}^{n} \frac{(-n)_k(n-i+1)_k}{(1)_k(s+2)_k} q(k), \tag{14}
\]

where \( c_{n,i} = (-1)^n \frac{n+2n-1}{(n)_1(n+1)} \) and

\[
q(k) = \prod_{j=1}^{s} (k+j) \prod_{m=0}^{i-1} (n+k+1-2m) \prod_{m=0}^{i-3} (n-k-1-2m).
\]

For \( s < n - i \) the degree of \( q(k) \) is \( s + i < n \) and we deduce from Lemma[5] that the right-hand side of (14) is equal to zero which gives v.

For ii, we need to show only that \( \int_{0}^{1} t^s h_{n-i}^n(t) dt = 0 \) when \( s > n - i \) and \( s \) and \( n-i \) have opposite parity (the rest follows from (13) and v). For such \( s \) we find

\[
\int_{0}^{1} t^s h_{n-i}^n(t) dt = \sum_{k=0}^{n} d_{n,k} \frac{1}{k+s+1}. \tag{15}
\]

Note that \( k+s+1 \) cancels one of the factors in \( \prod_{m=0}^{i-3} (n+k+1-2m) \) so the right-hand side of (15) is equal to zero from Lemma[5] with \( l = -1 \). This completes the proof of ii. For the orthogonality among the wavelet functions we begin with

\[
\int_{-1}^{1} h_{n-j}^n(t) h_{n-i}^n(t) dt = ((-1)^{i+j} + 1) \int_{0}^{1} h_{n-j}^n(t) h_{n-i}^n(t) dt.
\]
Thus we see that for \( i \) and \( j \) of opposite parity the above integral is equal to zero. Suppose that \( i \) and \( j \) are odd and \( i < j \). Then \( v \) shows that
\[
\int_0^1 h_{n-j}^n(t)h_{n-1}^n(t)dt = \sum_{k=n-i}^n d_{n,k}^j \int_0^1 t^k h_{n-i}^n(t)dt
\]
\[
= \sum_{n-k \text{ even}} d_{n,n-k}^j \int_0^1 t^k h_{n-i}^n(t)dt + \sum_{n-k \text{ odd}} d_{n,n-k}^j \int_0^1 t^k h_{n-i}^n(t)dt.
\]
In the sums on the last line of the above equation \( n-i \leq k \leq n \). Since in the second sum \( n-k \) is odd, Lemma 4 shows that \( d_{n,k}^j = 0 \) while in the first sum \( k > n-i \) and \( k \) and \( n-i \) are of opposite parity so the discussion after (15) shows that the integral is equal to zero. We now show that
\[
\int_0^1 h_{n-i}^n(t)^2 dt = 1.
\]
With the substitution of (19) we find from above that
\[
\int_0^1 h_{n-i}^n(t)^2 dt = d_{n,n-i}^j \int_0^1 t^{n-i} \psi_{n-i}(t)dt
\]
\[
= \frac{(-1)^n \sqrt{2n-2i} \Gamma(1)}{(-n)_i} \sum_{k=0}^n (-n)(n-i+1)_k \frac{\pi_i(k)(1)(1)_k}{n-i+k+1},
\]
where \( \pi_i(k) \) is the polynomial in \( k \) of degree \( i \) given by
\[
\pi_i(k) = \prod_{m=0}^{i-1} (n+k+1-2m) \prod_{m=0}^{i-3} (n-k-1-2m).
\]
If we add and subtract \( \pi_i(-n+i-1)/(n-i+k+1) \) to the summand then using Lemma 5 with \( s = -1 \) and the fact that \((\pi_i(k) - \pi_i(-n+i-1))/(n-i+k+1)\) is a polynomial of degree \( i-1 \) in \( k \) we find
\[
\int_0^1 h_{n-i}^n(t)^2 dt = (-1)^i (2n-2i+1)/n!(n-i)! \sum_{k=0}^n (-n)(n-i+1)_k \frac{\pi_i(k)(1)(1)_k}{n-i+k+1}.
\]
The proof now follows from equation (11) in Lemma 3. 

Remark 6. From the explicit formulas given in Theorem 1 it is clear that \( h_i^n(t) \) are hypergeometric functions although some care is needed in the hypergeometric representation since some of the coefficients vanish (see Lemma 4). If we set \( n_\epsilon = n + \epsilon \), then for \( \epsilon > 0 \) odd we can rewrite formula (17) as
\[
\frac{\lim_{\epsilon \to 0} \lim_{i+2} F_{i+1} \left( \begin{array}{c} -n, \alpha_1, \beta_1 \\ 1, \alpha_0, \beta_0 \end{array} ; t \right),}{(n+1) \prod_{m=0}^{i-3} (n-1-2m)^2}
\]
\[
\times \lim_{\epsilon \to 0} \lim_{i+2} F_{i+1} \left( \begin{array}{c} -n, \alpha_1, \beta_1 \\ 1, \alpha_0, \beta_0 \end{array} ; t \right),
\]
where \( \alpha_1 = \{ n_\epsilon - i + 1, n_\epsilon - i + 3, \ldots, n_\epsilon + 2 \}, \alpha_0 = \{ n_\epsilon - i + 2, n_\epsilon - i + 4, \ldots, n_\epsilon + 1 \}, \beta_1 = \{ -n_\epsilon + 2, -n_\epsilon + 4, \ldots, -n_\epsilon + i - 1 \}, \beta_0 = \{ -n_\epsilon + 1, -n_\epsilon + 3, \ldots, -n_\epsilon + i - 2 \} \). A similar formula can be written for \( i \) even using 8.
Remark 7. If we write \((-n, \alpha_1, \beta_1) = (a_1, \ldots, a_{i+2})\) and \((1, \alpha_0, \beta_0) = (b_1, \ldots, b_{i+1})\) it follows from the theory of generalized hypergeometric functions that \(i_{i+2}F_{i+1}\) satisfies the differential equation

\[
[D(D + b_1 - 1) \cdots (D + b_{i+1} - 1) - t(D + a_1) \cdots (D + a_{i+2})]_{i+2}F_{i+1} = 0
\]

where \(D = \frac{d}{dt}\). If the limit indicated above is taken we find that \(h^n_{-i}(t)\) satisfies the differential equation

\[
[t(1 - t)\frac{d^2}{dt^2} + (1 - (i + 2)t)\frac{d}{dt} + (n - i + 1)(n + 2)]L_i h^n_{-i}(t) = 0,
\]

where

\[
L_i = \prod_{m=0}^{n-i-1} (D - n + 2m) \prod_{m=0}^{n-i} (D + n - i + 1 + 2m).
\]

The use of equation (10) shows that

\[
L_i h^n_{-i}(t) = (-1)^{n+i+1}(-n - 1)i\sqrt{2n - 2i + 1}2F_1\left(\begin{array}{c}
-n+i-1, n+2; \\
1
\end{array}\right) (1 - 2t),
\]

where \(p_{n-i+1}(t)\) is the orthonormal Jacobi polynomial on \([-1, 1]\). This implies that for fixed \(i\), the functions \(L_i h^n_{-i}(t)\) are orthogonal on \([0, 1]\) for \(n \geq i\).

Using [15] and the specific form of the hypergeometric representation in (10), it is possible to derive even lower-order differential equations for \(h^n_{-i}(t)\), but they also correspond to generalized eigenvalue equations.

4. Wavelets for the Alpert Multiresolution

As mentioned in section 2 the functions \(\Phi_n\) generate a multiresolution analysis with \(V_0 = \text{span}\{\phi_i(-i), i \in \mathbb{Z}, j = 0, \ldots, n\}\) where the function \(\phi_j(-\frac{n-1}{2})\) when restricted to \([-1, 1]\) is the normalized Legendre polynomial \(\tilde{p}_j(t)\). In equation (11) the wavelet space \(W_0\) is the orthogonal complement of \(V_0\) in \(V_1\). We look for a set of \(L^2(\mathbb{R})\) functions \(\Psi_n = (\psi^n_0, \ldots, \psi^n_n)^T\) each supported on \([0, 1]\) whose integer translates provide a basis for \(W_0\). Using properties i-iv of the functions \(h^n_{-i}(t)\) and a standard argument (see for instance [2]) we see that if we choose

\[
\psi^n_j(t) = h^n_j(2t - 1)\chi_{[0,1]}(t),
\]

then we obtain a basis for \(L^2(\mathbb{R})\) by dilations and translations.

Theorem 8. The set of functions \(\{2^j\psi^n_j(2^j \cdot i), i, k \in \mathbb{Z}, j = 0, \ldots, n\}\) forms an orthonormal, compactly supported, piecewise polynomial basis for \(L^2(\mathbb{R})\).

Remark 9. We note that conditions i. through iv. are the same as those imposed by Alpert in his wavelet construction. To fix a basis he imposes more vanishing moments on \([-1, 1]\) (see [2] p. 248 condition 4). We were not able to find a hypergeometric representation for his functions.

On the interval \([0, 1]\) we have

Theorem 10. The set of functions \(\{\sqrt{2}\phi^n_j, 2^j\psi^n_j(2^j \cdot i), k \in \mathbb{Z}^+, i = 0, \ldots, 2^k - 1, j = 0, \ldots, n\}\) forms an orthonormal, piecewise polynomial basis for \(L^2([0, 1])\).
From the theory of multiresolution analysis, there are \((n+1) \times (n+1)\) matrices \(D^n_{-1}\) and \(D^n_1\) such that the following equations

\[
\Psi_n \left( \frac{t+1}{2} \right) = D^n_{-1} \Phi_n(t+1) + D^n_1 \Phi_n(t)
\]

\[
= D^n_{-1} P_n(2t+1) \chi_{[-1,0)}(t) + D^n_1 P_n(2t-1) \chi_{[0,1)}(t)
\]

hold, where

\[
D^n_1 = 2 \int_0^1 \Psi_n \left( \frac{t+1}{2} \right) P_n(2t-1) dt,
\]

and

\[
D^n_{-1} = 2 \int_{-1}^0 \Psi_n \left( \frac{t+1}{2} \right) P_n(2t+1) dt.
\]

Thus

\[
(D^n_1)_{i,j} = 2 \int_0^1 \psi^n_i \left( \frac{t+1}{2} \right) \hat{p}_j(2t-1) dt = 2 \int_0^1 h^n_i(t) \hat{p}_j(2t-1) dt, \quad 0 \leq i, j \leq n.
\]

These formulas and property v. show that \(D^n_i\) is an upper triangular matrix and as we will see below the diagonal entries are positive. From the orthogonality properties of the above functions we find

\[
4I_n = D^n_{-1}(D^n_{-1})^T + D^n_1(D^n_1)^T. \tag{21}
\]

We note that the right-hand side of the above equation differs by a factor of 2 from equation (5) in [12]. This is due to the fact that we have normalized the components of \(\Psi_n\) to be orthonormal. The symmetry properties of the wavelet functions give

\[
(D^n_{-1})_{i,j} = (-1)^{i+j+1}(D^n_1)_{i,j}, \tag{22}
\]

which combined with (21) leads to the orthogonality relations

\[
0 = ((-1)^{i+k} + 1) \sum_{j=i}^n (D^n_1)_{i,j}(D^n_1)_{k,j}, \quad k < i, \tag{23}
\]

and

\[
2 = \sum_{j=i}^n (D^n_1)_{i,j}(D^n_1)_{i,j}. \tag{24}
\]

Using the representation developed above for \(\psi^n_j\) we will show,

**Theorem 11.** The nonzero entries in \(D^n_1\) are given as follows:

\[
(D^n_1)_{n,n} = \sqrt{2}, \tag{25}
\]

for \(i\) odd and \(j \leq i,

\[
(D^n_1)_{n-i,n-j} = c_{n,i,j}(-1)^{n+j} \sqrt{2(n-i)} + 1 \sqrt{2(n-j)} + 1 \\
\times _4F_3 \left( \begin{array}{c} -i-j, i+j+1, n - i+j-1, n - i+j+1 \\ n - i+j, n - i+j, n - i+j \end{array} ; 1 \right), \tag{26}
\]

where

\[
c_{n,i,j} = \frac{(2n-i)! \Gamma(n-j+1)}{\Gamma(2n-j+1)}. \tag{27}
\]

while for \(i\) even and positive and \(j \leq i,

\[
(D^n_1)_{n-i,n-j} = (D^n_{-1})_{n-i,n-j}. \tag{28}
\]
Remark 12. Note that the $_4F_3$ hypergeometric functions are balanced and satisfy the orthogonality equations \cite{23,24}. From the explicit formulas above we see that the diagonal entries are positive. As shown in \cite[Theorem 4]{12} the orthogonality, upper triangularity, and positivity of the diagonal entries uniquely specify the matrix $D^l_n$.

Proof. We prove the result for $i = 0$ and $i$ odd since for $i > 0$ even the formula follows from the properties of the wavelets. For $i = 0$ or $i$ odd equation (2) in Theorem \cite[condition v.]{11} and condition v. give,

$$(D^l_n)_{n-i,n-j} = (-1)^{n-j} \sqrt{2} \sqrt{2(n-j)+1} \sum_{s=n-i}^{n-j} \sum_{k=0}^{n-s} \frac{(-n+j)_s(n-j+1)_s}{(1)_s(1)_s} \frac{d^l_{n,k}}{k+s+1}$$

Lemma \cite{3} yields

$$\sum_{k=0}^{n} \frac{(-n)_k(n-i+1)_k}{(1)_k(1)_k} \frac{1}{k+s+1} = \frac{n!(-s)^{n-i}}{(n-i)!(s+1)_{n+1}} = (-1)^i \frac{(-n)_i(-s)_{n-i}}{(s+1)_{n+1}}$$

which coupled with Lemma \cite{5} shows that

$$\sum_{k=0}^{n} \frac{d^l_{n,k}}{k+s+1} = (-1)^{n+i} \sqrt{2n-2i+1} \frac{(-s)_{n-i}}{(s+1)_{n+1}} \prod_{m=0}^{i-1} \frac{n-s+1}{n-s+2m} \prod_{m=0}^{i-2} \frac{n+s+1}{n+s+2m}.$$

Thus

$$(D^l_n)_{n-i,n-j} = (-1)^{n+j} \sqrt{2} \sqrt{2(n-i)+1} \sqrt{2(n-j)+1} \sum_{s=n-i}^{n-j} \frac{(-n+j)_s(n-j+1)_s}{(s+1)_{n+1}} \prod_{m=0}^{i-1} \frac{n-s+1}{n-s+2m} \prod_{m=0}^{i-3} \frac{n+s+1}{n+s+2m},$$

where the fact that $\frac{(-s)^{n-i}}{(1)_s(1)_s(s+1)_{n+1}} = \frac{(-1)^{s-i}}{(s+n+1)!(s-n+i)!}$ has been used. Making the change of variable $s = n-i+l$ yields,

$$(D^l_n)_{n-i,n-j} = (-1)^{n+j} \sqrt{2} \sqrt{2(n-i)+1} \sqrt{2(n-j)+1} \frac{(-n+j)_{n-i}(n-j+1)_{n-i}}{(2n-i+1)!} \prod_{m=0}^{i-1} \frac{j-i}{(2n-i+2)!} \prod_{m=0}^{i-3} \frac{n-s+1}{n-s+2m} \prod_{m=0}^{i-3} \frac{n+s+1}{n+s+2m}.$$

Note that the above sum on $l$ is zero if $l$ is odd (here $i$ is odd) since the first product starts from $i-l$ which is positive and even and ends up with $-l+1$ which is negative or zero. Using the identity

$$(a)_{2l} = 2^{2l} \left(\frac{a}{2}\right)_l \left(\frac{a+1}{2}\right)_l$$

for $(j-i)_{2l}, (2l)!, (2n-i+2)_{2l}$ and $(2n-i-j+1)_{2l}$ together with the identities,

$$\prod_{m=0}^{i-1} \frac{(i-2l-2m)}{\left(\frac{i}{2}\right)_l} = \frac{i!}{\left(\frac{i}{2}\right)_l}.$$
Proof. We only consider the case when formation of a balanced $4$ $(27)$ while for the remaining coefficients use equation $26$.

Because of the way that the indices enter in the above formulas it is not so simple to obtain recurrence relations. For this reason we obtain another representation.

Lemma 13. Suppose $j < i$. For $i$ odd and $j$ even,

$$(D_1^i)_{n-i,j} = (-1)^{i-j+1} (j+1)!((i-j-2)!!2^{2i-1}(n-i+\frac{3}{2})^{i+1}}{(n-i+\frac{3}{2})i} \sqrt{2(n-i)+1} \sqrt{2(n-j)+1}$$

\times 4F_3 \left( \begin{array}{c}
-j, i-j+1, n+\frac{3}{2}, 1 \\
1, \frac{3}{2}, \frac{3}{2}, 1
\end{array} \right), \tag{29}

for $i, j$ odd,

$$(D_1^i)_{n-i,j} = (-1)^{i-j+1} \frac{(j)!!(i-j-1)!!2^{2i-1}(n-i+\frac{3}{2})^{i+1}}{(n-i+\frac{3}{2})i} \sqrt{2(n-i)+1} \sqrt{2(n-j)+1}$$

\times (2n-j+2)(j+1) 4F_3 \left( \begin{array}{c}
-j-1, i-j+1, n-\frac{j+1}{2}+1, -n+\frac{i+j+1}{2} \\
2, \frac{3}{2}, \frac{3}{2}, 1
\end{array} \right), \tag{30}

while for the remaining coefficients use equation $(27)$.

Proof. We only consider the case when $i$ is odd and $j$ is even. The Whipple transformation of a balanced $4$ $4F_3$ hypergeometric function is the following

$$4F_3 \left( \begin{array}{c}
-n, x, y, z \\
u, v, w
\end{array} ; 1 \right) = \frac{(1-v+z-n)n(1-w+z-n)^n}{(v)_n(w)_n} 4F_3 \left( \begin{array}{c}
n, u-x, u-y, z \\
u, 1-v+z-n, 1-w+z-n
\end{array} ; 1 \right).$$

Thus with $i$ odd, $j$ even, $n = \frac{i-j-1}{2}$, $u = -i/2$, and $z = n - \frac{i+j+1}{2} + 1$, we find

$$4F_3 \left( \begin{array}{c}
-j, i-j+1, n-\frac{i+j-1}{2}, -n+\frac{i+j+1}{2}+1 \\
n-i+\frac{3}{2}, -\frac{i}{2}, n-i+\frac{3}{2}
\end{array} ; 1 \right)$$

$$= \frac{(1)_{i-j-1}(\frac{j+1}{2}+1)^{i-1}}{(n-i+\frac{3}{2})^{i-1}(n-i+\frac{3}{2})^{i-1}} 4F_3 \left( \begin{array}{c}
-i, \frac{j+i+1}{2}, -n+\frac{i-1}{2}, n+i+1 \\
1, -\frac{j}{2}, -\frac{j}{2}+1
\end{array} ; 1 \right).$$

An application of this formula to equation $(26)$ and the identity

$$\left( n-\frac{i+j-1}{2} \right)_{i-1} \left( n-\frac{i-3}{2} \right)_{i-1} = \left( n-\frac{i+j-1}{2} \right)_{i+1},$$

give the result. □
yields
\[(D^n_i)_{n-i,n-j} = (-1)^{j-i} \sqrt{2^{i-j-2\theta}} \cdot \left( \frac{n-1}{n-i} \frac{(i-j)!}{(n-i)!} \right) \cdot \sqrt{2(n-i)+1} \sqrt{2(n-j)+1} \times {}_4F_3 \left( \frac{-i, j-i, n-i+1, n-i+1}{1, -\frac{i}{2}, -\frac{j}{2} + 1} \right). \]

Another use of Whipple’s transformation with \(n = \frac{i}{2}\), \(z = \frac{j-i+1}{2}\), and \(u = 1\) gives the result.

**Remark 14.** Besides the orthogonality relations (23)-(24), equations (1) and (6) imply the relations,
\[2I_{n+1} = C_{n+1}^{-1} C_{n+1}^{T} + C_{1}^{n} C_{1}^{T},\]
and
\[C_{n+1}^{T} D_{n+1}^{-1} + C_{1}^{n} D_{1}^{T} = 0.\]

Using the symmetry property (22) and equation (15) of [12] we see that that the matrix \(A\) composed of the even rows of \(C^n\) and the odd rows of \(\frac{D^n}{\sqrt{2}}\) or vice versa is unitary which yields orthogonality relations among the entries of \(C^n\) and \(\frac{D^n}{\sqrt{2}}\). It is interesting to compare these orthogonality relations with other known orthogonality relations for \(4F_3\) series, and in particular with the orthogonality of the Wigner 6j-Symbols and Racah polynomials, see [18] or the book [16]. However, we could not relate the orthogonality relations above to this theory. Providing a Lie-theoretic interpretation of these new orthogonality equations is a very interesting problem. Another challenging problem is to connect the orthogonality relations here to an appropriate extension of the Fields and Wimp expansion formula [10].

5. **Fourier Transform**

An important tool in wavelet theory is the Fourier transform given by,
\[\hat{\psi}^n_k(\theta) = \int_0^1 \psi^k(t) e^{-i\theta t} dt.\]

In order to compute this Fourier transform we will begin with
\[\hat{h}^n_k(\theta) = \int_0^1 h^k(t) e^{-i\theta t} dt.\]

**Theorem 15.** For \(j\) odd \(\hat{h}^n_k(\theta)\) is given by
\[\hat{h}^n_{n-j}(\theta) = \frac{(-1)^{j+1} 2j+1!(j+1)!}{(j+2)!2(n+3)!(n+3/2-j)!} \sqrt{2n-2j+1} \theta^{n+2} \times {}_2F_3 \left( \frac{j+2}{2}, \frac{n+3}{2}, n+3/2; \frac{-\theta^2}{4} \right),\]
while for \(j\) even, \(\hat{h}^n_{n-j}(\theta) = \hat{h}^{n-1}_{n-j}(\theta).\)

**Remark 16.** It is remarkable that while the wavelet functions are limits of higher-order hypergeometric functions, their Fourier transforms have a simple closed formula in terms of \(\theta\) series. Since \(\hat{h}^n_k(\theta)\) is a polynomial of degree \(n\) on \([0, 1]\), Euler’s formula and integration by parts show that \(\hat{h}^n_{n-j}(\theta)\) can be written in terms of \(\sin \theta\)
and \( \cos \theta \) multiplied by polynomials in \( 1/\theta \) of degree \( n + 1 \). This is also true of the wavelets constructed by Alpert et al.

**Proof.** We prove the formula for \( j = 0 \) or \( j \) odd since for \( j \) even and positive the result follows from equation (33). From the symmetry properties of \( h_{n,j}^{n} \) we find

\[
\int_{-1}^{1} h_{n,j}^{n}(t)f(t)dt = \int_{0}^{1} [(-1)^{n-j+1}f(-t) + f(t)]h_{n-j}^{n}(t)dt,
\]

so that

\[
\hat{h}_{n-j}^{n}(\theta) = \int_{0}^{1} [(-1)^{n-j+1}e^{i\theta t} + e^{-i\theta t}]h_{n-j}^{n}(t)dt.
\] (37)

From property ii., \( \int_{-1}^{1} h_{k}^{n}(t)t^{s}dt = 0 \), for \( 0 \leq s \leq n \). Thus only the moments for \( s > n \) and \( s \equiv n - j + 1 \mod 2 \) are nonzero. For such \( s \) we find

\[
\int_{0}^{1} h_{n-j}^{n}(t)t^{s}dt = \sum_{k=0}^{n} d_{n,k} \frac{1}{k + s + 1}
\]

\[
= (-1)^{n} \frac{\sqrt{2n - 2j + 1}}{(-n)_{j}} \sum_{k=0}^{n} (-n)_{k}(n - j + 1)_{k} \prod_{m=0}^{i-1} (n + k + 1 - 2m) \prod_{m=0}^{i-3} (n - k - 1 - 2m).
\]

By Lemma 5, \( k \) can be replaced by \(-s + 1\) in the above products which gives

\[
\int_{0}^{1} h_{n-j}^{n}(t)t^{s}dt = (-1)^{n} \frac{\sqrt{2n - 2j + 1}}{(-n)_{j}} \prod_{m=0}^{i-1} (n - s - 2m) \prod_{m=0}^{i-3} (n + s - 2m)
\]

\[
\times \sum_{k=0}^{n} \frac{(-n)_{k}(n - j + 1)_{k}}{(k!)^{2}} \frac{1}{k + s + 1}.
\]

The use of Lemma 3 with \( x = s + 1 \) shows that the above sum is equal to

\[
\frac{(-n)_{j}(n + 1)_{n-j}}{(s + 1)_{n+1}!(s + j + 1)_{n-j}}
\]

so that

\[
\int_{0}^{1} h_{n-j}^{n}(t)t^{s}dt = (-1)^{n+j} \frac{\sqrt{2n - 2j + 1}}{(s + 1)_{n+1}} \prod_{m=0}^{i-1} (n - s - 2m) \prod_{m=0}^{i-3} (n + s - 2m).
\]

The identity \( \frac{(-s)_{n-j}(1)_{n+j}}{s!} = \frac{1}{(s-n+j)!} \) allows the above integral to be rewritten as

\[
\int_{0}^{1} h_{n-j}^{n}(t)t^{s}dt = \frac{\sqrt{2n - 2j + 1}}{(s - n + j)!((s + 1)_{n+1})} \prod_{m=0}^{i-1} (n - s - 2m) \prod_{m=0}^{i-3} (n + s - 2m).
\] (38)

For \( j > 0 \) the change of variables \( s = n + 2 + 2r, \ r = 0, 1, \ldots \) yields

\[
\int_{-1}^{1} h_{n-j}^{n}(t) \frac{t^{n+2+2r}}{(n + 2 + 2r)!}dt
\]

\[
= \frac{2^{j+1}(1)_{j+1}^{j+1} \sqrt{2n - 2j + 1}}{(2 + 2r + j)!(n + 3 + 2r)_{n+1}} \prod_{m=0}^{i-1} (m + r + 1) \prod_{m=0}^{i-3} (n + 1 + r - m).
\]
Thus
\[
\frac{1}{(n + 3 + 2r)_{n+1}} = \frac{(n + 2)!}{(2n + 3)! (n + 2)! (n + \frac{3}{2} - r)!},
\]
\[
\prod_{m=0}^{i-1} (m + r + 1) = \frac{(\frac{i-1}{2} + r)!}{r!} = \frac{(\frac{i-1}{2})! (\frac{i-3}{2})!}{r!},
\]
and
\[
\prod_{m=0}^{i-1} (n + r + 1 - m) = \frac{(n + 1 + r)!}{(n + \frac{3}{2} - \frac{7}{2} + r)!} = \frac{(n + 1)! (n + 2)!}{(n + \frac{3}{2} - \frac{7}{2} + r)!}.
\]
to obtain
\[
\int_{-1}^{1} h_{n-j}^{n}(t) \frac{t^{n+2+2r}}{(n + 2 + 2r)!} dt = \frac{(-1)^{\frac{n+1}{2} + 1} (\frac{i-1}{2})! (n + 1)! (n + 2)!}{(j + 2)! (2n + 3)! (n + \frac{3}{2} - \frac{7}{2})!} \sqrt{2n - 2 j + 1} \frac{(\frac{n+1}{2})! (\frac{n+3}{2})!}{r! (\frac{j+1}{2})! (\frac{j+3}{2})! (n + \frac{5}{2} - \frac{7}{2})!} \frac{1}{4^r}.
\]

Thus
\[
\hat{h}_{n-j}^{n}(\theta) = \sum_{s=0}^{\infty} \int_{-1}^{1} h_{n-j}^{n}(t)(-i\theta)^{s} \frac{t^{s}}{s!} dt
\]
\[
= (-i\theta)^{n+2} \sum_{r=0}^{\infty} (-i\theta)^{2r} \int_{-1}^{1} h_{n-j}^{n}(t) \frac{t^{n+2+2r}}{(n + 2 + 2r)!} dt
\]
and the result for \(j\) odd is obtained by the substitution of \(\theta\) in the above formula. For \(j = 0\) the substitution \(s = n + 1 + 2r\) in equation (38) shows that \(\hat{h}_{n}^{n} = \hat{h}_{n}^{n-1}\) which completes the proof. \(\square\)

Remark 17. As in the case of \(\psi_{n-j}^{n}\) the above formulas can be used to obtain differential equations for \(\hat{\psi}_{n-j}^{n}\). Since \(D_{\theta} = \frac{\theta}{2} \frac{d}{d\theta} = \theta^{2} \frac{d}{d\theta^{2}}\) and \(D_{\theta} \theta^{m} f(\theta) = \frac{n}{2} \theta^{m} f(\theta) + \theta^{m} D_{\theta} f(\theta)\) we find for \(j\) odd
\[
\left[ \left( D_{\theta} - \frac{n + 2}{2} \right) \left( D_{\theta} + \frac{n + 1}{2} \right) \left( D_{\theta} + \frac{j - n}{2} \right) \left( D_{\theta} + \frac{n + 1 - j}{2} \right) \right] \hat{\psi}_{n-j}^{n}(\theta) = 0,
\]
while for \(j\) even replace \(n\) by \(n - 1\) and \(j\) by \(j - 1\) in the above formula.

Remark 18. The formula for \(\hat{h}_{n-j}^{n}\) allows the development of asymptotic formulas for large \(n\) and \(j\). We will not systematically explore this here but be content to give a simple example in the case when \(j = tn\), \(0 < t \leq 1\). In this case for \(j\) odd
\[
\hat{h}_{(1-t)n}^{n}(\theta) = \frac{(-1)^{\frac{tn+1}{2}} 2^{tn+1} (\frac{tn+1}{2})! (n + 1)! (n + 2)! \sqrt{2n - 2tn + 1} (-i\theta)^{n+2}}{(tn + 2)! (2n + 3)! (n + \frac{5}{2} - \frac{7}{2})!} \times 2 F_{3} \left( \frac{tn+4}{2}, \frac{n+3}{2}, \frac{n+4}{2}, \frac{n + 5 - tn}{2} - \frac{\theta^{2}}{4} \right).
\]
The entries of the matrix

\[ D^n \]

are given by

\[
\begin{aligned}
&\frac{(tn+1)!}{(tn+2)!} \frac{(2n)!}{(2n+3)!} \frac{1}{(n+\frac{3}{2})!} \\
&= e^n (4n)^{-n} \left(1 - t \right)^{-n(1-\frac{1}{2})} \frac{(2t)^{-\frac{n}{2}}}{16\sqrt{2}t^{3/2}n^3} \left(1 - \frac{104 - 12t + 27t^2}{24(2-t)nt} + O \left(\frac{1}{n^2}\right)\right).
\end{aligned}
\]

Since the hypergeometric function can be expanded as

\[ 2F_3 \left( \frac{n+3}{2}, \frac{n+4}{2}, \frac{n+5}{2} ; n+5-tn ; -\frac{\theta^2}{4} \right) = 1 - \frac{\theta^2}{4nt(2-t)} + O \left(\frac{1}{n^2}\right), \]

we find

\[ \hat{h}_{n,(1-t)n}(\theta) = \frac{(-1)^{n/4}}{2^{n+3/2}t^{n+3}t^{2n}n^3} \left(1 - \frac{104 - 12t + 27t^2 + 6\theta^2}{24(2-t)tn} + O \left(\frac{1}{n^2}\right)\right). \]

6. Recurrence Formulas

The formulas for the entries in \( D^n \) given by Lemma 13 allow simple recurrences in \( n \) and \( i \). To this end we use formula (3.7.8) in [1] for balanced \( 4F_3 \) series:

\[ \frac{b(e-a)(f-a)(g-a)}{a-b-1}(F(a-,b+) - F) - \frac{a(e-b)(f-b)(g-b)}{b-a-1}(F(a+,b-) - F) + cd(a-b)F = 0, \]

where \( F = 4F_3 \left( \frac{a, b, e, f, g}{c, d} ; 1 \right) \). From this equation we find

**Theorem 19.** The entries of the matrix \( D^n \) satisfy the following recurrence relations

\[ (D^n)_{n-i+2,n-j} + k_{n,i,j}^1 (D^n)_{n-i,n-j} + k_{n,i,j}^2 (D^n)_{n-i-2,n-j} = 0, \]

where

\[ k_{n,i,j}^1 = \rho \left( \frac{(cd(a-b) - b(e-a)(f-a)(g-a))}{(a-b-1)} \right) \left( \frac{1 + a-b}{a(e-b)(f-b)(g-b)} - 1 \right), \]

\[ k_{n,i,j}^2 = \rho \left( \frac{b(e-a)(f-a)(g-a)(1 + a-b)}{(a-b-1)a(e-b)(f-b)(g-b)} \right), \]

and

- for \( i \) odd and \( j \) even, \( j < i - 1 \), we set \( a = \frac{i+1}{2}, b = -n + \frac{i+1}{2}, c = -\frac{i}{2}, d = n + \frac{3-i}{2}, e = 1, f = \frac{1}{2}, g = \frac{3}{2}, h = \frac{\sqrt{2(n+i)+5(-2n+i-j-1)}}{\sqrt{2(n+i)+3(i-j)(i-j-2)}}, \) and

\[ l = \frac{\sqrt{2(n+i)+5(2n-i-j-1)(2n-i-j+1)}}{\sqrt{2(n+i)-3(i-j)(i-j-1)}}. \]

- while for \( i \) and \( j \) odd, \( j < i - 2 \), we set \( a = \frac{i+j+2}{2}, b = -n + \frac{i+j+1}{2}, c = -\frac{j-1}{2}, d = n + \frac{3-j}{2}, e = 2, f = \frac{3}{2}, g = \frac{3}{2}, h = \frac{\sqrt{2(n+i)+5(-2n+i-j-2)}}{\sqrt{2(n+i)+3(i-j-1)(i-j-2)}}, \) and

\[ l = \frac{\sqrt{2(n+i)+5(2n-i-j)(2n-i-j+2)}}{\sqrt{2(n+i)-3(i-j+1)(i-j+2)}.} \]
Likewise we can obtain a recurrence relation in $n$.

**Theorem 20.** The entries of the matrix $D_n^n$ satisfy the following recurrence relations

$$(D_n^{n+1})_{n-i+1,n-j+1} + k_{n,i,j}^1(D_n^n)_{n-i,n-j} + k_{n,i,j}^2(D_n^{n-1})_{n-i-1,n-j-1} = 0,$$  

(46)

where $k^1$ and $k^2$ are given in equations (14)-(15), and

- for $i$ odd and $j$ even, $j < i < n$, we set $a = n + \frac{3-j}{2}$, $b = -n + \frac{i+j}{2}$, $c = -\frac{j}{2}$, $d = \frac{i-j+1}{2}$, $e = 1$, $f = \frac{1}{2}$, $g = \frac{3}{2}$, $h = \frac{\sqrt{2(n-i)+3}\sqrt{2(n-j)+3(2n-i-j)+1}}{2(n-i)+2\sqrt{2(n-j)+1}2(n-j+1)(2n-j+2)}$,

- while for $i$ and $j$ odd, $j < i < n$, we set $a = n + \frac{3-j}{2}$, $b = -n + \frac{i+j}{2}$, $c = -\frac{i-1}{2}$, $d = \frac{i-j+2}{2}$, $e = 2$, $f = \frac{3}{2}$, $g = \frac{3}{2}$, $h = \frac{\sqrt{2(n-i)+3}\sqrt{2(n-j)+3(2n-i-j)+1}(2n-j+2)}{2(n-i)-1\sqrt{2(n-j)+1}(2n-j+1)(2n-j+3)}$.

**References**

[1] B. K. Alpert, *Sparse representation of smooth linear operators*, Thesis, Yale University, 1990.

[2] B. K. Alpert, *A Class of bases in $L^2$ for the sparse representation of integral operators*, SIAM J. Math. Anal. 24 (1993), pp. 246–262.

[3] B. Alpert, G. Beylkin, R. Coifman, V. Rokhlin, *Wavelet-like bases for the fast solution of second-kind integral equations*, SIAM J. Math. Anal. 14 (1983), pp. 159–184.

[4] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyc. of Math. and its Appl 71, Cambridge University Press, Cambridge, 1999.

[5] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Series in Applied Math 61, SIAM, Philadelphia, 1992.

[6] D. L. Donoho, N. Dyn, D. Levin, and T. P. Y. Yu, *Smooth multiwavelets duals of Alpert bases by moment-interpolating refinement*, Appl. Comput. Harmon. Anal. 9 (2000), pp. 166–203.

[7] G. C. Donovan, J. S. Geronimo, and D. P. Hardin, *Intertwining multiresolution analyses and the construction of piecewise polynomial wavelets*, SIAM J. Math. Anal. 27 (1996), pp. 1791–1815.

[8] G. C. Donovan, J. S. Geronimo, and D. P. Hardin, *Orthogonal polynomials and the construction of piecewise polynomial smooth wavelets*, SIAM J. Math. Anal. 30 (1999), pp. 1029–1056.

[9] G. C. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Construction of orthogonal wavelets using fractal interpolation functions*, SIAM J. Math. Anal. 27 (1996), pp. 1158–1192.

[10] J. L. Fields and J. Wimp, *Expansions of hypergeometric functions in hypergeometric functions*, Math. Comp. 15 (1961), pp. 390–395.

[11] J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Fractal functions and wavelet expansions based on several scaling functions*, J. Approx. Theory 78 (1994), pp. 373–401.

[12] J. S. Geronimo and F. Marcellán, *On Alpert multiwavelets*, to appear in Proc. Amer. Math. Soc., arXiv:1309.6931.

[13] T. N. T. Goodman, and S. L. Lee, *Wavelets of multiplicity r*, Trans. Amer. Math. Soc. 342 (1994), pp. 307–324.

[14] I. S. Gradshteyn, and I. M. Ryzhik, *Tables of integrals, series, and products*, Academic Press, New York, 1965.

[15] J. Letessier, G. Valent, and J. Wimp, *Some differential equations satisfied by hypergeometric functions*, Approximation and computation, pp. 371–381, Internat. Ser. Numer. Math. 119, Birkhäuser Boston, Boston, MA, 1994.

[16] A. F. Nikiforov, S. K. Suslov and V. B. Uvarov, *Classical orthogonal polynomials of a discrete variable*, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991.

[17] G. Szegő, *Orthogonal polynomials*, AMS Colloq. Publ. 23, AMS, Providence, RI, 1939.
[18] J. A. Wilson, *Some hypergeometric orthogonal polynomials*, SIAM J. Math. Anal. **11** (1980), no. 4, 690–701.

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