A framework of quantum spacetime reference frame is proposed and reviewed, in which the quantum spacetime at the Gaussian approximation is deformed by the Ricci flow. At sufficient large scale, the Ricci flow not only smooths out local small irregularities making the universe a homogeneous and isotropic Friedmann-Robertson-Walker metric, but also develops a local singularity at the physical-time origin. Due to the phenomenological suppression of the non-Gaussian primordial perturbations, we assume the validity of the Ricci flow applying to the high curvature region near the local singularity of the early universe. The no-local-collapsing theorem of Perelman ensures the existence of a canonical neighborhood around the large curvature pinching point, which resembles a gradient shrinking Ricci soliton (GSRS) solution of the Ricci flow. Without any inflaton field, the GSRS naturally reproduces an exact inflationary deSitter universe near the singularity at the leading order. Without any rolling-down behavior of inflaton, the deviation from exact deSitter described by the “slow roll parameters” can be calculated by a small deviation from the singular flow-time via the Ricci flow, and the primordial perturbations can also be studied on the GSRS background, the power spectrum of the scalar perturbation agrees with present observations, and the one of the tensor perturbation is predicted too small to be detectable than the standard inflation. The previous treatment of the cosmological constant and the effective gravity are also briefly reviewed in the framework. So we argue that the Ricci flow provides us a possible unified view and treatment of the late epoch accelerating expansion and early epoch inflation of the universe without introducing dark energy or inflaton (dark energy of the second kind).

I. INTRODUCTION

Issues concerning the origin of the universe and its primordial structure are among the deepest and most fundamental in physics. The textbook standard inflation theory, and a wide class of inflation paradigm or alternative theories are motivated by solving the fundamental problems of the big-bang theory of the universe at the early epoch, for instance, the homogeneous problem, the horizon problem and the flatness problem etc. On the one hand, the standard prediction from the simplest inflationary picture is extremely consistent with recent observation from the Cosmic Microwave Background (CMB) radiation [1], on the other hand, the inflationary paradigm is considered having its own conceptual problems (e.g. see [2, 3] and references therein), certainly the statement itself is also controversial [4]. At the fundamental level, in the absence of quantum description of gravity and spacetime, it is difficult to know which of the problems of inflation are more serious and correct to ask in order to get a better description of physics near the origin of the universe. However, it is no controversial that the basis of the inflation cosmology does not yet have the status of an established theory, the mechanism of inflation is based on the speculative inflaton in the early epoch (another unknown form of dark energy in the late epoch), the inflation theory (and other alternative theories) of the early universe are not yet fully quantum, it is based on a semi-classical combination of the classical general relativity and the quantum theory.

In the previous literature [5–12], a framework applying the quantum principle to a spacetime reference frame and gravity is proposed, in which the equivalence principle and the general covariance are generalized to the quantum level. It shows that the quantum behavior of the spacetime to a large extent is embodied in its Ricci flow. And the renormalizability of the theory is mathematically related to the uniformization of the spacetime to be proved by the Ricci flow approach. The Ricci flow was firstly introduced in 1980s by Friedan in $d = 2 + \epsilon$ non-linear sigma model [13, 14] and also independently invented by Hamilton in mathematics [15, 16]. From the mathematical side, the main motivation of the Ricci flow is to classify manifolds, especially, to prove the Poincare conjecture. Hamilton’s program is to use the Ricci flow as a useful tool to gradually deform a manifold into a “simpler and nicer” manifold whose topology can be readily recognized. But unfortunately, the program met some difficulties in treating generic initial conditions, because in general the flow may develop local singularities. A general realization of the program is achieved by Perelman at around 2003 [17, 18], who introduced several monotonic functionals to successfully deal with the local singularities. The monotonicity of the functionals are applied to prove the “no-local-collapsing theorem”, and

*Electronic address: mjluo@ujs.edu.cn
to prove the existence of a regular subset of the manifolds around the local singularity called “canonical neighborhood structure”. It is shown that the local singularity or the high curvature region around it belongs to finite number of “singularity models” resembling the “gradient shrinking Ricci soliton” (GSRS) configurations. A surgery can be performed in the canonical neighborhood around the singularity, and then the Ricci flow is able to continue, the breakthrough of Perelman finally removed the stumbling block in Hamilton’s program. The existence of the canonical neighborhood structure around a local singularity and the existence of the “singularity model” given in mathematics, make the studying of the initial singularity of the big bang universe feasible. The Ricci flow approach is not only powerful to study the compact geometry (as Hamilton’s and Perelman’s seminal works had shown) but also to the non-compact geometry [21][22]. In fact, the classification of a manifold in mathematics is nothing but equivalent to constructing the complete Hilbert space of the spacetime, and hence quantizing the spacetime in the language of physics. We first lay the physics foundation to the Ricci flow of spacetime based on the notion of quantum reference frame, and we consider the Ricci flow at least at the Gaussian approximation as a candidate quantum theory of spacetime. As long as the equivalence principle of Einstein is consistently generalized to the quantum level, the Ricci flow of the quantum spacetime can also be a candidate quantum theory for the gravity. The correctness of the framework depends on its mathematical consistency and the validity of applying it to explain and predict observations.

This framework of quantum spacetime is capable to discuss several fundamental issues of quantum gravity, for instance, the cosmological constant problem [5–8], the trace anomaly [9], the thermodynamics of the quantum spacetime [11] and the modified gravity [12]. Thus the main motivation and goal of the paper is trying to apply the framework to another important touchstone of a quantum gravity: the initial singularity of the universe and its possible early inflationary epoch.

The structure of the paper is as follows. In section II, we briefly review and introduce the background of the Ricci flow of quantum spacetime based on the notion of quantum reference frame. In section III, we focus on the early universe within framework of the Ricci flow limit of the quantum spacetime. The connection between a local singularity formation of the Ricci flow and the early universe is discussed in III-A; in III-B, we show that, the vicinity of the early epoch singularity can be modeled by a solution of a gradient shrinking Ricci soliton (GSRS) equation, the solution naturally gives rise to an inflationary universe without any speculative inflaton fields; the slightly deviation from the exact deSitter inflation metric is discussed in III-C where the slow roll parameters can be calculated by the Ricci flow; and how the inflation comes to an end is discussed in III-D; the power spectrums of the scalar and tensor primordial perturbations are calculated in III-E and be compared with the textbook standard inflation theory; in III-F, two important quantities: the manifolds density $u_*$ and Hubble rate $H$, during the early universe in the primordial perturbation power spectrum is estimated to give more practical predictions to the power spectrums. Finally, we summarize and conclude the paper in section IV.

II. RICCI FLOW OF QUANTUM SPACETIME

A. Quantum Reference System

The quantum reference frame (see e.g. [23][27] and references therein) is the conceptual foundation of the theory. In this theory, an under-study-system that is relative to the quantum reference system is described by a quantum state $|\psi\rangle$, and the quantum reference system is also described by a quantum state $|X\rangle$. Then the whole system is given by an entangled state

$$|\psi[X]\rangle = \sum_{ij} \alpha_{ij}|\psi\rangle_i \otimes |X\rangle_j$$

(1)

in the Hilbert space $\mathcal{H}_\psi \otimes \mathcal{H}_X$.

To explain the framework, let us first take a 1-dimensional quantum clock-time as a preliminary and simple example of a reference system, and then the example can be easily generalized to 4-dimensional quantum spacetime reference frame. In Newtonian mechanics, the under-study-system are the coordinates $X$ of a particle, as functions of the absolute time $t$, i.e. $X(t)$, called the equation of motion of the particle. To keep the speed of light a constant in any reference frame, Einstein pointed out that the coordinates of the particle in a moving frame are, instead, $X(\tau), T(\tau)$, including a physical clock-time $T(\tau)$ in each different moving frame, where $\tau$ is certain global parameter (e.g. proper time). An observer sees the equations of motion of the particle being reference to the relative time $T(\tau)$ is then given by functional $X[T(\tau)]$, which generalizes the functions $X(t)$ in Newtonian mechanics. The functionals describe the relation between the coordinate $X$ and the physical clock-time $T$. For example, the action of the coordinate $X$ of the particle w.r.t. the proper time $\tau$ (a global parameter) is given by a standard particle action with mass $m_X$ and
potential $V(X)$

$$S_X = \int d\tau \left[ \frac{1}{2} m_X \left( \frac{dX}{d\tau} \right)^2 - V(X) \right].$$

(2)

While time is imagined as an ideal and fiducial motion that other more complex motions are relative to. So we could also consider the clock-time $T$ as the coordinate of a pointer of a physical clock, which is considered uniformly moving and hence free. Thus the action of the coordinate $T$ of the pointer w.r.t. the proper time $\tau$ (global parameter) is given by a standard free particle action without potential

$$S_T = \int d\tau \left[ \frac{1}{2} m_T \left( \frac{dT}{d\tau} \right)^2 \right]$$

(3)

where $m_T$ is the mass of the clock pointer. The global parameter $\tau$ can be interpreted as the proper time of the lab. Since the under-study particle has no interaction with the clock after initial calibration between them, so the action of the whole system is a sum of them without their interaction,

$$S[X, T] = S_X + S_T.$$  

(4)

At the quantum level, the description of the particle is replaced by a quantum state $|X(\tau)\rangle$ given by the Hamiltonian of the action $S_X$, and the clock is also described by a quantum state $|T\rangle$ from the Hamiltonian of the action $S_T$. Then the evolution functional $X[T(\tau)]$ in classical physics, which describes a correspondence between functions $T(\tau)$ and $X(\tau)$, is now replaced by a quantum version of correspondence between the states $|T(\tau)\rangle$ and $|X(\tau)\rangle$, i.e. an entangled state

$$|X[T]\rangle = \sum_\tau C_\tau |X(\tau)\rangle \otimes |T(\tau)\rangle$$

(5)

which is a simple example of (1). The state predicts the output of the joint measurements of the clock time $|T(\tau)\rangle$ and the coordinate $|X(\tau)\rangle$ of the particle at the same “time” $\tau$, in this sense the equation of motion of the particle $X[T(\tau)]$ is measured at the quantum level. Different from that classical physics predicts a deterministic relation $X[T(\tau)]$, the quantum mechanics predicts a probabilistic state $|X[T]\rangle = \sum_\tau C_\tau |X(\tau)\rangle \otimes |T(\tau)\rangle$. Following the standard Copenhagen interpretation of the quantum state, $|C_\tau|^2$ is a joint probability of the particle state $|X(\tau)\rangle$ and the clock state $|T(\tau)\rangle$ happening at the same “time” $\tau$. The (conditional) probability of the particle at $|X(\tau)\rangle$ in the condition when the clock is at the state $|T(\tau)\rangle$ can also be given by $|C_\tau|^2/|C_\tau|^2$, where $|C_\tau|^2$ is the individual probability of the clock at the state $|T(\tau)\rangle$. And because the entangled state is inseparable, the joint probability $|C_\tau|^2$ will not be a direct product of the individual probability values of $|X(\tau)\rangle$ and $|T(\tau)\rangle$, so the conditional probability $|C_\tau|^2/|C_\tau|^2$ will not be an individual probability of the particle state $|X(\tau)\rangle$. In this sense, the entangled state (1) describes a “relational state” between the under-study-system (e.g. the coordinate of the particle) and the quantum reference system (e.g. the clock time), rather than an “absolute state” in textbook quantum mechanics.

In the semi-classical approximation, the quantum fluctuation $(T^2) - \langle T \rangle^2 = (\delta T)^2$ is ignored, so that the clock-time $T$ can be seen as a c-number parameter $T$, i.e. a delta wavefunction peaked at $T$, then the action is rewritten as

$$S[X, T] \approx S[X (\langle T \rangle)] = \int dT \left\{ \frac{1}{2} m_X \left( \frac{dX}{dT} \right)^2 + \frac{1}{2} m_T \right\} - V(X)$$

(6)

where (1) means the approximation is at the 1st order/semi-classical approximation compared with the 2nd order/Gaussian approximation in the coming discussions, $\| \frac{dT}{dT} \|$ is a Jacobian determinant, and $M_X = m_X \| \frac{dT}{dT} \| \left( \frac{dT}{dT} \right)^2 = m_X \frac{dT}{dT}$ is the effective mass of the particle. When the semi-classical clock time runs at exactly the same rate with the proper time $\tau$, then $M_X = m_X$, and hence the semi-classical action recovers the action (2) up to a constant, only the lab’s parameter time $\tau$ is replaced by the mean value of the physical clock time $\langle T \rangle$, and the derivative $\frac{dT}{d\tau}$ is replaced by the functional derivative $\frac{\delta}{\delta T}$. The entangled state $|X[T]\rangle$ recovers the textbook quantum absolute state $|X(\tau)\rangle$.

An important observation is that $T$ interpreted as the clock time is quadratic in the action (1), so its first functional derivative of $T$ vanishes, $\frac{\delta S[X, T]}{\delta T} = \langle E \rangle = 0$, and hence the Hamiltonian corresponding to the action is zero, that is
to say that the Schrödinger equation of the whole system is in fact a timeless wave equation with zero Hamiltonian (some literature call it Wheeler-DeWitt equation). There is no external time here, on the contrary, the equation is used to defined time $T$ which monitors the under-study particle system $X$. However, the second derivative of $T$ is not zero, so when the 2nd order fluctuation $\langle \delta T^2 \rangle \neq 0$ of the clock time is taken into account, the energy fluctuation $\sqrt{\langle \delta E^2 \rangle} \neq 0$ gives rise to a correct order of vacuum energy driving a late epoch acceleration expansion of the universe [4].

B. Non-Linear Sigma Model (NLSM) for the Quantum Spacetime Frame Fields

In the subsection, we generalize the quantum clock time to the quantum spacetime reference frame. To locate the coordinates of an event, the number of references must be generalized from 1 (one clock) to at least $D = 4$ (3 rods plus one clock), i.e. $X_{\mu} = (X_0, X_1, X_2, X_3)$. To contain these 4 frame fields in a lab, the number of the global parameter must be generalized from one (lab’s proper time $\tau$) to $d = 4 - \epsilon$ (lab’s Minkowskian/Euclidean background), i.e. $x = (x_0, x_1, x_2, x_3)$. From the mathematical point of view, the D-dimensional frame fields is a manifold $X_{\mu}$, which is a non-linear differentiable mapping $X(x)$ from a local coordinate patch $x \in \mathbb{R}^d$ to a D-manifolds $X \in M^D$. For the same logic of the previous quantum reference system, we can also consider the frame fields $X$ as the under-study-system w.r.t. the lab’s coordinates $x$ as the reference system, then the local mapping (i.e frame field) $X(x)$ at the quantum level is also given by an entangled state $\sum x \alpha_x |X \otimes \rangle$, rather than a direct product state, describing their local relation.

At the moment, the entanglement between the under-study-system (frame fields $X$) and a local reference system (local lab frame $x$) has not directly related to the inflation (the main subject of the paper), the inflation comes from the singularity developed by the local mapping and the entanglement in a indirect way, next we will introduce the dynamical theory and corresponding RG-flow of such local mapping and the entanglement.

The mapping in physics is usually realized by a kind of fields theory, the non-linear sigma model (NLSM) [13-14, 28-30]

$$S[X(x)] = \frac{1}{2} \lambda \int d^d x g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a}, \quad (7)$$

which is a generalization of the free clock time action [3]. $|X\rangle$ in (1) is the eigenstate of the Hamiltonian of this action. $\lambda$ is the only input constant with dimension of energy density $[L^{-d}]$ taking the value (69), which is the generalization of the clock mass. $x_a$ called the base space in NLSM, representing the lab’s wall and clock frame as the starting reference, which are the generalization of the single proper time $\tau$ in (3). In our common sense, the lab’s spacetime frame has dimension $d = 4 - \epsilon$, and is considered fiducial, classical, flat and external with infinite precision. Note that the NLSM is background signature independent, so without loss of generality, we consider the base space as an Euclidean one, i.e. $x \in \mathbb{R}^d$ which is better defined when we use the functional method to quantize the theory in latter section.

The differential mapping or the frame fields $X_{\mu}(x)$ with dimensional length $|L|$, now is the physical coordinates of a Riemannian or Lorentzian spacetime $M^D$ with generally curved metric $g_{\mu\nu}$, called the target space in NLSM. When quantizing the theory, we will promote the frame fields $X_{\mu}$ to D quantum fields, so in the language of quantum fields theory, $X_{\mu}(x)$ or their duals $X^\mu(x) = g^{\mu\nu} X_{\nu}(x)$ are the real defined scalar frame fields.

The under-study-system, without loss of generality, can be generalized from $X(\tau)$ (in previous subsection) to a standard scalar field $\psi(x)$, which shares the lab’s background $x$ with the frame fields $X(x)$. The action is generalized from (2) to

$$S[\psi(x)] = \int d^d x \left[ \frac{1}{2} \frac{\partial \psi}{\partial x_a} \frac{\partial \psi}{\partial x_a} - V(\psi) \right] \quad (8)$$

where $V(\psi)$ is some potential of the scalar field. $|\psi\rangle$ in (1) is the eigenstate of the Hamiltonian of the action.

Then the total action of the scalar field and the frame fields is a sum of each system without interaction between them

$$S[\psi, X] = \int d^d x \left[ \frac{1}{2} \frac{\partial \psi}{\partial x_a} \frac{\partial \psi}{\partial x_a} - V(\psi) + \frac{1}{2} \lambda g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right], \quad (9)$$

as the generalization of (4) or (5). The state of the whole state now gives the entangled state (1).

Because both $\psi$ field and the frame fields $X$ share the common lab’s background spacetime $x$, here they are described w.r.t. the lab’s background spacetime as the starting reference. If $\psi$ field is considered w.r.t. the physical frame fields
X, the action can be rewritten by transforming from x to X. At the semi-classical level, when the fluctuation of the frame fields \( \langle \delta X^2 \rangle \) can be ignored, it is simply a coordinates transformation x \( \rightarrow \langle X \rangle \),

\[
S[\psi, X] \approx S[\psi(\langle X \rangle)] = \int d^2X \sqrt{\text{det} g^{(1)}} \left[ \frac{1}{4} \left( g^{(1)\mu\nu} \frac{\partial \delta X^\mu}{\partial x_a} \frac{\partial \delta X^\nu}{\partial x_a} \right) \left( \frac{1}{2} g^{(1)\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} + 2\lambda \right) - V(\psi) \right]
\]

in which \( \frac{1}{4} \left( g^{(1)\mu\nu} \frac{\partial \delta X^\mu}{\partial x_a} \frac{\partial \delta X^\nu}{\partial x_a} \right) = \frac{1}{4} \left( g^{(1)\mu\nu} g^{(1)\mu\nu} \right) = \frac{1}{4} D = 1 \) has been used. The semi-classical action is a generalization of the \ref{eq:action}. It is easy to see, under the semi-classical treatment of frame fields X, the classical coordinates transformation reproduces the scalar field action \ref{eq:action} in general spacetime reference frame coordinates X up to a constant 2\( \lambda \), lab’s parameter background x is replaced by \( \langle X \rangle \), and the derivative \( \frac{\partial}{\partial x_a} \) is replaced by the functional derivative \( \frac{\delta}{\delta \psi} \).

\[
\sqrt{\text{det} g^{(1)}} = \left\| \frac{\delta}{\delta \psi} \right\| \text{ is the Jacobian determinant of the coordinate transformation, which is the generalization of } \| \frac{\delta}{\delta \psi} \| \text{ in previous subsection. Note that the coordinates transformation matrix must be a square matrix, so at semi-classical level } d \text{ should be close to } D = 4, \text{ which is seen obviously that the dimension of the lab is 4. However, it is for quantum and topological reasons that } d \text{ must not be exactly 4 but rather } d = 4 - \epsilon. \text{ In fact, the value of } d \text{ is very crucial for the renormalizability of the theory at the quantum level. It is well known that } d = 2 \text{ the NLSM is power counting and perturbative renormalizable. Although } 2 < d < 4 \text{ is not power counting and perturbative renormalizable, it is evident that the theory is non-perturbative renormalizable. From the topological point of view, NLSM is a fields model of mapping from the base space } x \text{ to the target space } X, \text{ whether NLSM is well-defined at the quantum or renormalization level depends on whether the mapping is free from intrinsic topological singularity. For simplicity, we consider the target space topologically a 4-sphere (after properly Wick rotated), } M^D = S^4, \text{ then the homotopy group of the mapping } X : \mathbb{R}^d \rightarrow S^4 \text{ is } \pi_d(S^4). \text{ The homotopy group is trivial when } d > 4 \text{, i.e. } \pi_d(S^4) = 0, \text{ which means that all possible (in the path integral sense) differentiable mapping } X(x) \text{ will not meet intrinsic topological obstacles and hence the mapping is always well-defined and free from intrinsic singularities. That is the reason we choose } d = 4 - \epsilon \text{ as the dimension of the base space, where } \epsilon \text{ can be considered as a small regularization parameter to avoid mathematical singularity at the quantum level. In fact, } d \text{ as an input parameter is not an observable of the theory, at the quantum level, } d \text{ can even be a fractal dimension due to the “dimension anomaly”. While at the classical or semi-classical level, it is no problem if one roughly considers } d = 4. \text{ When the location of an event is at a long distance scale far beyond the lab’s size, for instance, to the galaxy or cosmic scale, when the frame fields signal travels along such a long distance and be read by an observer, the broadening of the variance (2nd order moment) fluctuation of the frame fields } \langle \delta X^2 \rangle \text{ become unignorable. More precisely, the variance } \langle \delta X^2 \rangle \text{ inevitably modifies the quadratic form of distance of the Riemannian/Lorentzian spacetime}
\]

\[
\langle (\Delta X)^2 \rangle = \langle \Delta X \rangle^2 + \langle \delta X^2 \rangle.
\]

Since a local distance element is usual attributed to the local metric tensor at the point, so it is also convenient to think of the location point X being fixed, and the effect of the variance affects only the metric tensor \( g_{\mu\nu} \) at the point. As a consequence, the quantum expectation value of a metric tensor \( g_{\mu\nu} \) is modified by the 2nd moment quantum fluctuation of the frame fields

\[
\langle g^{\mu\nu} \rangle = \left\langle \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right\rangle = \left\langle \frac{\partial X^\mu}{\partial x_a} \right\rangle \left\langle \frac{\partial X^\nu}{\partial x_a} \right\rangle + \frac{1}{2} \frac{\partial^2}{\partial x_a^2} \langle \delta X^\mu \delta X^\nu \rangle = g^{(1)}_{\mu\nu}(X) + \delta g^{(2)}_{\mu\nu}(X),
\]

where

\[
\delta g^{(2)}_{\mu\nu}(X) = \left\langle \frac{\partial X^\mu}{\partial x_a} \right\rangle \left\langle \frac{\partial X^\nu}{\partial x_a} \right\rangle - \langle e^\mu_a \rangle \langle e^\nu_a \rangle
\]

is the 1st order moment (mean value) contribution, \( e^\mu_a \) is the vierbein. For correction from the 2nd order moment or variance \( \delta g^{(2)}_{\mu\nu} \) deforms the metric and the geometry of the physical spacetime, especially at long distance or cosmic scale. It is a renormalization or coarse-graining process of the quantum spacetime.

C. Metric deformation from Quantum Gaussian Fluctuations: Ricci Flow

To investigate the quantum correction \( \delta g^{(2)}_{\mu\nu}(X) \), we need to quantize the frame fields at the 2nd order moment or Gaussian approximation level, if the higher order moments (non-Gaussian fluctuations) are less important compared...
to the Gaussian fluctuation. When $\delta g^{\mu\nu}_{(2)}$ is relatively smaller than $g^{\mu\nu}_{(1)}$, it can be given by a perturbative one-loop calculation [29] of the NLSM

$$\delta g^{\mu\nu}_{(2)}(X) = 2 \frac{\partial^2}{\partial x_a^2} \langle \delta X^a \delta X^\nu \rangle = - \frac{R^{\mu\nu}_{(1)}(X)}{32\pi^2 \lambda} \delta k^2,$$

(14)

where $R^{\mu\nu}_{(1)}$ is the Ricci curvature given by the 1st order metric $g^{\mu\nu}_{(1)}$, $k^2$ is the cutoff energy scale of the Fourier components of the frame fields $X^\mu$. The validity of the perturbation calculation $R^{\mu\nu}_{(1)}(X) \ll \lambda$ is actually the validity of the Gaussian approximation. In fact, to recover the standard General Relativity, $\lambda$ must take the value of the critical density $\rho_c$ of the universe (shown in [30]), i.e. $\lambda \sim O(H_0^2/G)$, $H_0$ the current Hubble’s constant, $G$ the Newton’s constant. For the case when the curvature is of order of $H_0$, the condition $R^{\mu\nu}_{(1)}(X) \ll \lambda$ is equivalent to $\delta k^2 \ll 1/G$ which is reliable.

For the case when the curvature is large near a local singularity of a manifold, in principle, contributions from non-Gaussian fluctuations depicted by higher powers of the curvature $(R^{\mu\nu}/\lambda)^{n>1}$ may become important, the Gaussian approximation may be fail. In fact, depending on what is our interested physics and how close the period producing our interested physics is to the local singularity, the validity of the Gaussian approximation may be fail. In fact, describing what is our interested physics, the Gaussian approximation is a subtle issue in the early epoch of the universe which will be discussed in the next section. It is also worth stressing that here the metric fluctuation $\delta g^{\mu\nu}_{(2)}$ is not directly related to the observed tensor modes of metric perturbation in inflation. It will give a RG-flow to the spacetime, and finally gives rise to the curvature pinching near the singularity of the early universe (see section III).

The equation (14) is actually a RG equation of the target space, i.e. the physical spacetime, in mathematics it called the Ricci flow equation (some reviews see e.g. [31-33])

$$\frac{\partial g^{\mu\nu}}{\partial t} = 2R^{\mu\nu} \quad \text{or} \quad \frac{\partial g^{\mu\nu}}{\partial t} = -2R^{\mu\nu}.$$

(15)

For the same convention in mathematics literature, we often use the latter to describe a continuous deformation of the tangent spacetime metric driven by its Ricci curvature. The flow-time interval $\partial t = -\frac{1}{4\pi^2 \lambda} \delta k^d - 2$ has dimension of length square $[L^2]$ for any $d$, not restricted to Friedan’s original consideration of $d = 2 + \epsilon$.

For the Ricci curvature is non-linear in metric, the Ricci flow equation is in analogy to a non-linear “heat equation” for the metric, and flow along $t$ introduces a renormalizing or coarse-graining process to a spacetime and gravitational system which is highly non-trivial [34-38]. If it is free from local singularities during the flow, there exists a long flow-time solution in $t \in (-\infty, 0)$, which is often called an ancient solution in mathematics. In this situation, the range of the $t$-parameter corresponds to $k \in (0, \infty)$, from $t = -\infty$, i.e. a short distance (high energy) UV scale $k = \infty$ forwardly to $t = 0$ i.e. a long distance (low energy) IR scale $k = 0$. The coarse-grained metric at scale $t$ is given by being averaged out the fine-grain or short distance fine structure of the metric. So along $t$, the manifolds lose its fine-grain information, so that the flow is irreversible, that is it has no backwards solution. It is the underlying reason for the existence of an entropy of a spacetime discussed in [11] and in later subsections.

Note that in [11, 12], the variance of the metric modifies the local quadratic form of spacetime distance, thus the flow is essentially non-isometry. It is the underlying reason for the diffeomorphism anomaly of spacetime. The non-isometry does not affect its topology, so the flow preserves the topology of the spacetime. However, its local metric, shape and size (volume) changes during the flow. And there also exists a very special solution of the Ricci flow called the Ricci Soliton [39] in mathematics, which is a generalization of the notion of quasi-Einstein metric in physics [14]. The solution only deforms its local volume while keeps its local shape, the solution is self-similar. The Ricci Soliton, and its more general version, the Gradient Ricci Soliton, plays the role of the flow limit, are the generalization of the notion of fixed point in the RG flow and the deSitter metric. In the Gradient Ricci Soliton, the Gradient Shrinking Ricci Soliton (GSRS) is a particularly important model in understanding the gravity at cosmic scale and early epoch, we will see in the next section.

D. Density Matrix of the Frame Fields: induced Ricci-DeTurck Flow

Because the Ricci flow is a coarse-graining process of the spacetime, there exist irreversible entropy and diffeomorphism anomaly in the framework (see next subsection), so density matrix, rather pure state, is a more proper fundamental notion. By using the density matrix $u$, the previous (2nd order) results e.g. (12), (14) and hence the Ricci flow (15) can also be given by the expectation value

$$\langle O \rangle = \langle X | O | X \rangle = \lambda \int d^4 X \Psi^\dagger(X) O \Psi(X) = \lambda \int d^4 X u(X) O$$

(16)
via writing down the wavefunction $\Psi(X)$ or density matrix $u$ of the frame fields explicitly at the Gaussian approximation, where

$$d^4X = \sqrt{|\det g_{\mu\nu}|}dX^0dX^1dX^2dX^3$$

(17)

Remind that at the semi-classical approximation, the frame fields $X$ is a delta density peaking at its mean value. Thus at the Gaussian approximation level, finite Gaussian width/2nd moment fluctuation of $X$ must be introduced as the next order correction to the delta density. So at the Gaussian approximation, the fundamental solution of the wave function takes the Gaussian form

$$\Psi[X^\mu(x)] = \frac{|\det \sigma_{\mu\nu}|^{1/4}}{\sqrt{X(2\pi)^{D/2}|g|^{1/4}}} \exp \left[ -\frac{1}{4} (X^\mu(x) - x^\mu) \sigma_{\mu\nu} (X^\nu(x) - x^\nu) \right],$$

where the covariant matrix $\sigma_{\mu\nu}(x)$ measures the 2nd order moment fluctuations of the frame fields at the peaking point $x = \langle X \rangle$

$$\sigma_{\mu\nu}(x) = \frac{1}{\sigma_{\mu\nu}(x)} = \frac{1}{\langle \delta X^\mu(x)\delta X^\nu(x) \rangle}.$$  

(18)  

The absolute symbol in the exponential of the wavefunction is for keeping the Gaussian integral over $X$ positive as a probability, even in the Lorentzian (signature) spacetime.

The fundamental solution of the wavefunction gives rise to a dimensionless probability density matrix

$$u[X^\mu(x)] = \Psi^*(X)\Psi(X) = \frac{1}{X(2\pi)^{D/2}|\det g_{\mu\nu}|^{1/2}} \exp \left[ -\frac{1}{2} (X^\mu(x) - x^\mu) \sigma_{\mu\nu} (X^\nu(x) - x^\nu) \right],$$

(20)

in which $\sqrt{|\det \sigma_{\mu\nu}|}$ is given by the normalization condition

$$\lambda \int d^DX \Psi^*(X)\Psi(X) = \lambda \int d^DX u(X) = 1.$$  

(21)

There exists an arbitrariness in the density $u(X)$ for different choices of a diffeomorphism/gauge. Under a diffeomorphism $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu}$, $u(X)$ is transformed corresponding to a diffeomorphism of the covariant matrix

$$\sigma_{\mu\nu} \rightarrow \hat{\sigma}_{\mu\nu} = \sigma_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} h.$$  

(22)

where $h$ is certain transformation function.

Following the statistical interpretation of wavefunction with the normalization condition $\langle 21 \rangle$, density matrix $u(X^0, X^1, X^2, X^3)$ measures the probability density of finding the frame fields particles in the volume $d^DX$. During the Ricci flow along $t$, the volume $V_t$ in which the density is averaged also flows, so the density is coarse-grained in the volume $\Delta V_t$ at the scale $t$. If we consider the volume of the lab is rigid and fixed by $\lambda \int d^4x = 1$, so

$$u[X(x), t] = \frac{d^4x}{d^4X_t} = \lim_{\Delta V_t \rightarrow 0} \frac{1}{\Delta V_t} \int_{\Delta V} 1 \cdot d^4x = \langle 1 \rangle_{\Delta V_t \rightarrow 0}. $$

(23)

Thus the density $u(X, t)$ can be interpreted as a coarse-grained density at the scale $t$ w.r.t. a fine-grained unit density in the lab at UV $t \rightarrow -\infty$.

The coarse-grained density $u(X, t)$ not only has statistic meaning, playing a central role in analyzing the statistic physics $\langle 21 \rangle$ of the frame fields, but also has profound geometric meaning, generalizing the Riemannian/Lorentzian manifolds $(M^D, g)$ to a density manifolds $(M^D, g, u)$ $\langle 40 \rangle \langle 42 \rangle$, in which $u$ also called a manifold density in mathematics.

$u(X, t)$ associates a manifold density to each point $\langle X \rangle$ in a manifold. And it is worth stressing that $u$ is not equivalent to scaling the metric conformally by a factor, because in this case the integral measure of 4-volume or 3-volume in the expectation $\langle O \rangle = \lambda \int d^DX uO$ would scale by different powers. Since Gaussian $u$ density fuzzes the coordinates of the manifolds and hence deforms the metric and curvature in the density manifolds. Thus an important observation is that $u$ density generalizes the notion of curvature in the density manifolds. Indeed, there are various useful generalizations (e.g. see $\langle 43 \rangle$) of the Ricci curvature to the density manifolds, a widely accepted version is the Bakry-Emery generalization $\langle 44 \rangle$

$$R_{\mu\nu} \rightarrow R_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} \log u,$$

(24)
which is also used in Perelman’s seminal papers. Such generalization has many advantages in physics, for instance, the 2nd moment fluctuation eq. (22) of spacetime encoded in the $u$ density has more direct relation to curvature and gravity at the quantum level. We will use the definition throughout the paper. By using the generalized Ricci curvature, the Ricci flow for Riemannian manifolds $(M^D, g)$ is generalized to the Ricci-DeTurck flow [45] for the density manifolds $(M^D, g, u)$

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2 (R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u),$$

(25)

which is nothing but actually equivalent to the Ricci flow equation [115] up to a diffeomorphism. Different from the standard Ricci flow, the Ricci-DeTurck flow has the advantage that it turns out to be a gradient flow of some monotonic functionals introduced by Perelman, so the Ricci-DeTurck flow equation, instead of the Ricci flow equation, is a fundamental equation to the physical spacetime with density at the quantum level.

Remind that the eq. (23) also gives a volume constraint to the fiducial spacetime (the lab), the density $u(\mathbf{X}, t)$ in this sense cancels the flow of $\sqrt{\det g_{\mu\nu}}$, so we have

$$\frac{\partial}{\partial t} \left( u \sqrt{\det g_{\mu\nu}} \right) = 0.$$  

(26)

The relation, together with the Ricci-DeTurck flow equation (25), directly gives rise to the flow equation of the density

$$\frac{\partial u}{\partial t} = (R - \Delta) u,$$

(27)

in which $\Delta$ is the Laplacian of the 4-spacetime. And this equation is in analogy to the irreversible Boltzmann’s equation for his distribution function of dilute gas. However, note the minus sign in front of the Laplacian operator, it is a backwards heat-like equation of $u$. It seems that the solution of the backwards heat flow does not exist. But we also note that if one flows the manifolds to certain IR scale $t_s$, and at the scale $t_s$ one can certainly choose an appropriate $u(t_s) = u_0$ arbitrarily (up to a diffeomorphism gauge) and flows it backwards in $\tau = t_s - t$, and hence one obtains a solution $u(\tau)$ of the backwards equation. In the situation that if the flow is considered free from global singularities for the trivialness of the homotopy group $\pi_{d<4}(S^4) = 0$, we simply consider $t_s = 0$, so

$$\tau = -t = \frac{1}{64\pi^2 \lambda^2} k^2 \in (0, \infty).$$

(28)

As a consequence, the density satisfies a heat-like equation in terms of the backwards flow time $\tau$

$$\frac{\partial u}{\partial \tau} = (\Delta - R) u,$$

(29)

which indeed admit a solution along $\tau$, it is often called the conjugate heat equation in mathematics.

At this point, eq. (29) together with (25), the mathematical problem of the Ricci flow of a Riemannian/Lorentzian manifolds is generalized to a coupled equations

$$\begin{cases}
\frac{\partial g_{\mu\nu}}{\partial t} = -2 (R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u) \\
\frac{\partial u}{\partial t} = (\Delta - R) u \\
\frac{d\tau}{dt} = -1
\end{cases}$$

(30)

and hence the pure Riemannian/Lorentzian manifolds $(M^D, g)$ is generalized to a density manifolds $(M^D, g, u)$ with the constraint (21).

E. Diffeomorphism Anomaly

As is shown in previous subsection, an important feature of the Ricci flow or the Ricci-DeTurck flow to the manifolds is that the quantum fluctuation does not preserve the quadratic form of the distance of the manifolds. The non-isometry feature induces a breakdown of diffeomorphism or general coordinate transformation at the quantum level, namely the diffeomorphism anomaly [9].

We consider the functional quantization of the pure frame fields, the partition function is

$$Z(M^D) = \int [D\mathbf{X}] \exp (-S[\mathbf{X}]) = \int [D\mathbf{X}] \exp \left( -\frac{1}{2} \lambda \int d^4x g^{\mu\nu} \partial_\alpha X_\mu \partial_\alpha X_\nu \right),$$

(31)
in which, without loss of generality, the base spacetime is taken as Euclidean for it is better defined in path integral, and the final result is independent to the signature, i.e. the same for Minkowskian.

We have seen that the behavior of the Ricci flow (15) for the tangent and cotangent spacetime is opposite, for the same convention in mathematics, here we consider the general coordinate transformation of the tangent spacetime,

\[ X_\mu \to \hat{X}_\mu = \frac{\partial X_\mu}{\partial \hat{X}_\nu} \varepsilon^{\nu}. \] (32)

The coordinate transformation does not change the action \( S[X] = S[\hat{X}] \), but the measure of the functional integral changes

\[
\mathcal{D}\hat{X} = \prod_x \prod_{\mu=0}^{D-1} d\hat{X}_\mu(x) = \prod_x \varepsilon_{\mu\rho\sigma} \epsilon^{\mu}_\nu \epsilon^{\rho}_\sigma \epsilon^{\sigma}_\tau dX_0(x) dX_1(x) dX_2(x) dX_3(x)
\]

\[
\mathcal{D}\hat{X} = \prod_x |\det e(x)| \prod_{a=0}^{D-1} dX_a(x) = \left( \prod_x |\det e(x)| \right) \mathcal{D}X,
\] (33)

where

\[
\varepsilon_{\mu\rho\sigma} \epsilon^{\mu}_\nu \epsilon^{\rho}_\sigma \epsilon^{\sigma}_\tau = |\det e_\mu| = \sqrt{|\det g_{\mu\nu}|}
\] (34)

is the Jacobian of the coordinate transformation. In fact, the Jacobian is a local relative volume element \( d\mu(X_\mu) \) w.r.t. the fiducial one \( d\mu(X_\mu) \). Remind that the normalization condition (21) defines a fiducial volume element \( u d^4X \equiv u d\mu(\hat{X}_\mu) \), thus the Jacobian actually measures the frame fields density matrix

\[
u(\hat{X}) = \frac{d\mu(X_\mu)}{d\mu(\hat{X}_\mu)} = |\det e_\mu| = \frac{1}{|\det e_\mu|}.
\] (35)

The absolute value symbol used in the determinant is to keep \( u \) and hence the volume element positive defined, even in the Lorentzian signature spacetime. Otherwise, for the Lorentzian case, there must introduce extra imaginary factor \( i \) into (35) to preserve the normalization condition (21). It is a natural generalization from a 3-space density of Perelman to a 4-spacetime version with Lorentzian signature. It is such definition of the volume form for the Lorentzian 4-spacetime ensures the formalism of the framework formally identifies with Perelman’s standard formalism for the 3-space in Euclidean signature. From this observation, we can see that the manifolds density \( u \) encodes one of the most important information of a manifold, i.e. the local volume ratio, including not only the classical volume ratio (coming from the classical general coordinates transformation) but also the quantum counterpart (coming from the Ricci flow).

We could parameterize the solution \( u \) in terms of

\[
u(\hat{X}) = \frac{1}{\lambda^{(4\pi \tau)/2}} e^{-f(\hat{X},\tau)}.
\] (36)

Then by using it, the partition function \( Z(M^D) \) under the coordinate transformation gives

\[
Z(M^D) = \int [\mathcal{D}\hat{X}] \exp (-S[\hat{X}]) = \int \left( \prod_x |\det e| \right) [\mathcal{D}X] \exp (-S[X])
\]

\[
= \int \left( \prod_x e^{f + \frac{1}{2} \log(\lambda^{2/D} 4\pi \tau)} \right) [\mathcal{D}X] \exp (-S[X])
\]

\[
= \exp \left( \lambda \int d^4x \left[ f + \frac{D}{2} \log(\lambda^{2/D} 4\pi \tau) \right] \right) \int [\mathcal{D}X] \exp (-S[X])
\]

\[
= \exp \left( \lambda \int d^Dx u \left[ f + \frac{D}{2} \log(\lambda^{2/D} 4\pi \tau) \right] \right) \int [\mathcal{D}X] \exp (-S[X])
\] (37)

Finally, we can see that the change of the partition function, known as the anomaly, is

\[
Z(M^D) = e^{\lambda N(M^D)} Z(M^D),
\] (38)
in which \( N(\hat{M}^D) \) is nothing but a Shannon entropy in terms of the manifolds density \( u \)

\[
N(\hat{M}^D) = \int_{\hat{M}^D} d^D X u \left[ f + \frac{D}{2} \log(\lambda^{2/D} 4\pi\tau) \right] = -\int_{\hat{M}^D} d^D X u \log u. \tag{39}
\]

Because \( u \) in (35) is real defined, the Shannon entropy or the anomaly is real, even for the Lorentzian signature. It is a general result of the Ricci flow of spacetime.

Without loss of generality, if we simply consider the under-transformed coordinates \( X_a \) to be the coordinates of the fiducial lab \( x_a \), so they can be treated as classical parameter coordinates. In this situation the classical action of NLSM is simply \( \frac{D^2}{2} \), a topological invariant, i.e.

\[
\exp(-S_{cl}) = \exp\left( -\frac{1}{2} \lambda \int d^4 x g^{\mu\nu} \partial_{\mu} x_{\mu} \partial_{\nu} x_{\nu} \right) = \exp\left( -\frac{1}{2} \lambda \int d^4 x g^{\mu\nu} g_{\mu\nu} \right) = e^{-\frac{D^2}{2}}. \tag{40}
\]

Thus the total partition function (38) takes a simple form

\[
Z(\hat{M}^D) = e^{\lambda N(\hat{M}^D) - \frac{D^2}{2}}. \tag{41}
\]

In the Ricci flow limit, i.e. the Gradient Shrinking Ricci Soliton (GSRS) configuration

\[
R_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} f = \frac{1}{2\tau} g_{\mu\nu}, \tag{42}
\]

the covariance matrix \( \sigma^{\mu\nu} \) as the 2nd order moment of the frame fields with a IR cutoff \( k \) is simply proportional to the metric

\[
\frac{1}{2} \sigma_{*\mu\nu} = \frac{1}{2} (\delta X^\mu \delta X^\nu) = \frac{1}{2\lambda} g^{\mu\nu} \int_{|p|=k} d^4 p \frac{1}{(2\pi)^4} \frac{1}{p^2} = \frac{k^2}{64\pi^2 \lambda} g^{\mu\nu} = \tau g^{\mu\nu}, \tag{43}
\]

and then

\[
\sigma_{*\mu\nu} = (\sigma_{*}^{\mu\nu})^{-1} = \frac{1}{2\tau} g_{\mu\nu}, \tag{44}
\]

which means a uniform Gaussian broadening is achieved, i.e. its covariant gradient vanishes \( \nabla_{\rho} \sigma_{*\mu\nu} = 0 \). The subscript "\( * \)" represents the Ricci flow limit at which the Shannon entropy \( N \) approaches to its maximum value \( N_{*} \), and the density matrix

\[
u(X) = \frac{|\det \sigma_{\mu\nu}|^{1/2}}{\lambda(2\pi)^{D/2} |\det g_{\mu\nu}|^{1/2}} \exp\left( -\frac{1}{2} \frac{|X^{\mu} \sigma_{\mu\nu} X^{\nu}|}{\lambda |X^{\mu} X^{\nu}|} \right), \tag{45}\]

becomes a Maxwell-Boltzmann form of density

\[
u_{*}(X) = \frac{1}{\lambda(4\pi\tau)^{D/2}} \exp\left( -\frac{1}{4\tau} |g_{\mu\nu} X^{\mu} X^{\nu}| \right) \tag{46}\]

in the limit, in analogy to a “thermoequilibrium” state of spacetime [11]. We can also define a relative density \( \tilde{u} \) as the general density \( u(X) \) w.r.t. the “thermoequilibrium” density \( u_{*}(X) \) in the limit

\[
\tilde{u}(X) = \frac{u(X)}{u_{*}(X)}. \tag{47}\]

By using the relative density, a relative Shannon entropy \( \tilde{N} \) can also be defined by

\[
\tilde{N}(M^D) = -\int d^D X \log \tilde{u} = -\int d^D X u \log u + \int d^D X u_{*} \log u_{*} = N - N_{*} = -\log Z_P \leq 0, \tag{48}\]

where \( Z_P \) is Perelman’s partition function

\[
\log Z_P = \int_{M^D} d^D X u \left( \frac{D}{2} - f \right) \geq 0, \tag{49}\]
and $N_*$ is the maximum Shannon entropy

$$N_* = -\int d^D X u_* \log u_* = \int d^D X u_* \frac{D}{2} \left[1 + \log(\lambda^{2/D} 4\pi \tau)\right] = \frac{D}{2\lambda} \left[1 + \log(\lambda^{2/D} 4\pi \tau)\right].$$

(50)

For the reason that the relative Shannon entropy is real, the change of the partition function under diffeomorphism is in general non-unitary. Perelman defined the F-functional

$$\mathcal{F} = \frac{dN}{dt} = \int_{M^D} d^D X u \left(R + |\nabla f|^2\right)$$

(51)

with its maximum value at GSRS limit

$$\mathcal{F}_* \equiv \mathcal{F}(u_*) = \frac{dN_*}{dt} = \frac{D}{2\lambda \tau}.$$

(52)

It is easy to show that $\mathcal{F}$ is monotonic non-decreasing along $t$, because

$$\frac{d\mathcal{F}}{dt} = 2 \int d^D X u |R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u|^2 \geq \frac{2}{D} \int d^D X u \left|R - \Delta \log u\right|^2 \geq \frac{2}{D} \lambda F^2 \geq 0.$$

(53)

Thus the relative Shannon entropy is monotonic non-decreasing along the Ricci flow (along $t$),

$$\frac{d\tilde{N}(\hat{M}^D)}{dt} = -\tilde{\mathcal{F}} \geq 0,$$

(54)

where $\tilde{\mathcal{F}} = \mathcal{F} - \mathcal{F}_* \leq 0$ is the GSRS-normalized F-functional.

**F. Effective Action of Gravity and Non-Singular Ricci Flow**

If the equivalence principle is generalized to the quantum level, the quantum fluctuation of the frame fields should not only be a property of frame fields themselves but also be interpreted as universal property of the spacetime. In the interpretation, the mean value of the frame fields $\langle X \rangle$ measures the classical coordinate of the spacetime, the quantum fluctuation $\langle \delta X^2 \rangle$ of the frame fields measures the variance or fuzziness of the coordinates, and hence gravity as a relational phenomenon between different frames emerges from the frame fields system at the quantum level.

By using $\tilde{N}(M^D)$, the total partition function $\Pi$ now is written as

$$Z(M^D) = \frac{e^{\lambda N - \frac{D}{2}}} {e^{\lambda N_*}} = e^{\lambda \tilde{N}} e^{-\frac{D}{2}} = Z_\lambda e^{-\frac{D}{2}} = \exp \left[\lambda \int_{M^D} d^D X u (f - D)\right].$$

(55)

The relative Shannon entropy $\tilde{N}$ playing the role of the anomaly vanishes at GSRS limit at IR scale, but in general it is non-zero at lab scale up to UV. Since the lab’s volume is considered fiducial and fixed by $\lambda \int d^4 x = 1$, so the anomaly must be canceled at the lab scale up to UV. The physical requirement leads to the counter term $\nu(M^D_{\tau = \infty})$ which finally appears as the cosmological constant. The monotonicity of $\tilde{N}$ implies

$$\nu(M^D_{\tau = \infty}) = \lim_{\tau \to \infty} \lambda \tilde{N}(M^D, u, \tau) = \lambda \left[\tilde{N}_{UV}(\hat{M}^D_{UV}) - \tilde{N}_{IR}(\hat{M}^D_{IR})\right] < 0.$$

(56)

in which $\lim_{\tau \to 0} \tilde{N}(M^D) = \tilde{N}_{IR}(\hat{M}^D_{IR}) = 0$ is used.

The exponential of the counter term, $e^{\nu} < 1$, which is usually called the Gaussian density [46, 47] in mathematics, is a relative volume or the reduced volume $\hat{V}(M^D_{\tau = \infty})$ of the backwards limit manifolds introduced by Perelman. $e^{\nu}$ is also the inverse of the initial condition of the manifolds density $u_{-\frac{1}{2}}$. A finite value of the counter term $\nu(M^D_{\tau = \infty})$ makes an initial spacetime with unit fiducial volume at UV scale flow and converge to a finite $u_{\tau = 0}$ at IR. Thus the manifolds finally converge to a finite relative volume instead of shrinking to a singular point at IR $\tau = 0$.

As an example, i.e. a homogeneous and isotropic late epoch universe, with a positive curvature, its size in spatial and temporal parts are on an equal footing (with a “ball $B^4$” radius $a(\tau)$), i.e.

$$ds^2 = a^2(\tau) \left( dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 \right),$$

(57)

where $a(\tau)$ is the maximum size of the universe.
which is nothing but a (Lorentzian) Shrinking Ricci Soliton configuration. The solution tends to globally shrink $B^4$ to a singular point. Note that the metric satisfies the shrinking soliton equation $\frac{R_{\mu\nu}}{D} = \frac{1}{D} g_{\mu\nu}$, and its volume form are independent to the signature, the counter term can be approximately given by a 4-ball value $\nu(B^4_\infty) \approx -0.8$

Taken into account the counter term, the partition function $Z(M^D)$ changes to
\[ Z(M^D) = e^{\lambda \bar{N} - \frac{D}{2} \nu(B^4_\infty)}, \tag{58} \]
which is anomaly canceled at UV and hence having a fixed fiducial volume lab.

Considering $\lim_{\tau \to 0} \bar{N}(M^D) = 0$ and $\lambda \int d^D X \mu \tau |\nabla f_{\tau \to 0}|^2 = \frac{D}{2}$, at low energy or small $\tau$, $\bar{N}(M^D)$ can be expanded in powers of $\tau$
\[ \bar{N}(M^D) = \frac{\partial \bar{N}}{\partial \tau} \tau + O(\tau^2) = \tau \bar{F} + O(\tau^2) \]
\[ = \int_{M^D} d^D X u_0 \left[ \left( R_{\tau = 0} - \frac{D}{2\tau} \right) \tau - |\nabla f_{\tau = 0}|^2 \right] + O(\tau^2) \]
\[ = \int_{M^D} d^D X u_0 R_0 \tau + O(\tau^2). \tag{59} \]
In this expansion, the effective action of $Z(M^4)$ can be obtained for $D = 4$,
\[ -\log Z(M^4) = S_{eff} \approx \int_{M^4} d^4 X u_0 (2\lambda - \lambda R_0 \tau + \lambda \nu) \quad (\text{small } \tau) \tag{60} \]

Considering $u_0 d^4 X$ now recovers the classical invariant volume element \[ \sqrt{|g|} dV, \] and by using \[ \sqrt{|g|} \] to change the flow time $\tau$ to the cutoff $k$, we have
\[ S_{eff} = \int_{M^4} dV \sqrt{|g|} \left( 2\lambda - \frac{R_0}{64\pi^2} k^2 + \lambda \nu \right) \quad (\text{small } k). \tag{61} \]

As the cutoff scale $k$ ranges from the lab scale to at least the well-tested solar system scale ($k > 0$), the action must recover the Einstein-Hilbert (EH) action. However, at the cosmic scale ($k \to 0$), we know that the EH action deviates from observations and the cosmological constant becomes important. Following the requirement, as $k \to 0$, the action leaving $2\lambda + \lambda \nu$ should play the role of the standard EH action with a limit constant background scalar curvature $R_0$ plus the cosmological constant $\Lambda$, so
\[ 2\lambda + \lambda \nu = \frac{R_0 - 2\Lambda}{16\pi G}. \tag{62} \]

While at $k \to \infty$, $\lambda \bar{N} \to \nu$, the action leaving only the action $\frac{D}{2} \lambda = 2\lambda$ for the fiducial lab, when it should be interpreted as a constant EH action without the cosmological constant
\[ 2\lambda = \frac{R_0}{16\pi G}. \tag{63} \]
So we have the cosmological term
\[ \lambda \nu = -\frac{2\Lambda}{16\pi G} = -\rho_\Lambda. \tag{64} \]

As a result, the action can be rewritten as an effective EH action plus a cosmological term
\[ S_{eff} = \int_{M^4} dV \sqrt{|g|} \left( \frac{R_k}{16\pi G} + \lambda \nu \right) \quad (\text{small } k), \tag{65} \]
where
\[ \frac{R_k}{16\pi G} = 2\lambda - \frac{R_0}{64\pi^2} k^2, \tag{66} \]
which is in fact the solution of the flow equation of the scalar curvature $\frac{\partial R}{\partial \tau} = -\frac{2}{D} R^2$, i.e.
\[ R_k = \frac{R_0}{1 + \frac{1}{4\pi G} k^2}, \quad \text{or} \quad R_\tau = \frac{R_0}{1 + \frac{2}{D} R_0 \tau}. \tag{67} \]
The flow equation of the scalar curvature is a homogeneous and isotropic version of a more exact flow equation \( \frac{\partial R}{\partial \tau} = -\Delta R - 2R_{\mu\nu}R^{\mu\nu} \), if we consider the scalar curvature is nearly homogeneous and isotropic at IR i.e. \( \Delta R = 0 \) and \( R_{\mu\nu} = \frac{1}{D} R g_{\mu\nu} \).

Note that the effective scalar curvature is bounded by \( R_0 \) at the cosmic scale \( k \to 0 \), which can be measured by the “Hubble’s constant” \( H_0 \) at the cosmic scale,

\[
R_0 = D(D - 1)H_0^2 = 12H_0^2.
\]

As a consequence, to recover the standard Einstein’s gravity, \( \lambda \) must be identical with the critical density of the universe

\[
\lambda = \frac{3H_0^2}{8\pi G} = \rho_c,
\]

so the cosmological constant here is predicted of order of the critical density with a “dark energy” fraction

\[
\Omega = \frac{\rho\Lambda}{\rho_c} = -\nu \approx 0.8,
\]

which is close to the observations. The detail discussions about the cosmological constant problem and the cosmological effects, especially the modification of the distance-redshift relation at the second order by a deceleration parameter \( q_0 \approx -0.68 \), can be found in \[3, 9\].

Finally, if matter is incorporated in the pure gravity action. \( 2\lambda \) term in eq.(10) should be renormalized by the Ricci flow, by using eq.(61) and eq.(66), the effective action eq.(19) at the 2nd order moment or Gaussian approximation recovers the standard gravity+matter action

\[
S[\psi, X] \approx \frac{1}{2} \int dV \sqrt{|g|} \left[ \frac{1}{2} g^{\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + 2\lambda - \frac{R_0}{64\pi^2} k^2 + \lambda \nu \right]
\]

\[
= \int dV \sqrt{|g|} \left[ \frac{1}{2} g^{\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + \frac{R_k}{16\pi G} + \lambda \nu \right].
\]

III. THE EARLY UNIVERSE

A. Local Singularity Formation and Gradient Shrinking Ricci Soliton Model

In the previous section, by the cancellation effect of the cosmological constant in the effective action of gravity, we have a non-singular flow \( t_s = 0 \) starting from a special metric \( (57) \), i.e. an initial spacetime where space and time are on an equal footing at late epoch, homogeneous and with a positive curvature, however, when starting from a more general initial spacetime manifolds, (e.g. an early epoch Friedmann-Robertson-Walker metric \( (72) \) where time is treated different from space), the Ricci \( (18) \) or the generalized Ricci-DeTurck flow \( (29) \) may develop local curvature pinching or local singularities in some subset of spacetime at finite scale \( t_s \neq 0 \) \[18\], for instance, the early epoch singularity of the universe. The local singularity can not be canceled as the global cosmological constant did.

For physical consideration, near the local singularity, the curvature becomes larger and larger, the Ricci flow as the Gaussian approximation of the RG-flow of NLSM may not be valid, because non-Gaussian terms being composed of higher power of curvature may come into the flow equation and become more and more important. However, it in fact depends on how close the period producing our interested physics is to the local singularity, we note that the observed primordial perturbations produced from the early universe is highly Gaussian, while the non-Gaussian parts are suppressed \[1\], which is a hint that it is possible that the Ricci flow as a Gaussian approximation may still be a good approximation in that period when the primordial perturbations are produced, the non-Gaussian contributions to the RG-flow of the NLSM could be safely ignored at least at the phenomenological level.

Thus if we still assume the validity of the Ricci flow when dealing with the local curvature pinching, the no-local-collapsing theorem \[10, 17\] of Perelman indicates the local volume collapsing in the neighborhood of the singularity can not actually occur at scale \( O(\sqrt{t_s - t}) \) and hence existing a “canonical neighborhood” structure with a positive curvature around the large curvature point at finite scale, the blow-up limit of the canonical neighborhood around the local curvature pinching point should resemble an ancient and self-similar configurations \[49, 51\] satisfying the Gradient Shrinking Ricci Soliton (GSRS) equation. The mathematical theorem provides us a possible way to study the early universe where the curvature is highly pinched. In the following subsections we will study the canonical and regular metric of the curvature pinched region in the early universe and the primordial perturbations produced from it.
B. Curvature Pinching of the FRW Metric at early Epoch: Inflation

At sufficient large scale, if local irregularities (e.g. inhomogeneous, shear \[52\] etc) of the universe are not large, the Ricci flow gradually smooths out them making the universe seen spatially more and more uniform as the cosmological principle asserts. So we could always start from a spatially homogeneous isotropic Friedmann-Robertson-Walker (FRW) metric \(S^3 \times \mathbb{R}\)

\[
ds^2 = dT^2 - a^2(T, \tau)dX_idX_i,
\]

where the spatial metric, with \(K\) the spatial curvature, i.e. \(\text{Ric}^{(3)} = 2Kg^{(3)}\), has the form

\[
dX_idX_i = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2.
\]

The Ricci-DeTurck flow \[25\] not only smooths out the large scale universe to be FRW, but also tends to shrink the spatial part \(S^3\) rather than the temporal part \(\mathbb{R}\), so it develops an early epoch singularity at certain singular scale \(t_s\). We consider the FRW ansatz at the vicinity of physical-time origin \(T = 0\) (early epoch), and near the singular scale \(t_s\), i.e. \(\tau = t_s - t \rightarrow 0\), when the Ricci curvature pinches to very large value, the spacetime local region around the singularity is modeled by the GSRS equation \[12\] as the flow limit for the local spacetime with positive curvature \[17, 18\]

\[
\text{Ric}^{(4)} + \nabla\nabla f = \frac{1}{2\tau}g^{(4)},
\]

where \(\text{Ric}^{(4)}\) is the Ricci curvature of 4-spacetime

\[
\text{Ric}^{(4)} + \nabla\nabla f = \left(3\frac{\dot{a}}{a} + \dot{f}\right)dT^2 + \left(2K - 2\dot{a}^2 - a\ddot{a} + a\dot{a}\dot{f}\right)dX_idX_i,
\]

and dot denotes the physical-time derivative \(\dot{O} \equiv \frac{dO}{dT}\). At leading order, to extend \(a(T)\) and \(f(T)\) smoothly to the origin we naturally set the initial condition

\[
\lim_{T \rightarrow 0} a(T) \approx 0, \quad \lim_{T \rightarrow 0} \dot{f}(T) \approx 0.
\]

So we could first ignore the contribution from \(f\) at leading order, obtaining

\[
3\ddot{a} - \frac{1}{2\tau}a = 0,
\]

and

\[
2\dot{a}^2 + a\ddot{a} - 2K - \frac{1}{2\tau}a^2 = 0.
\]

The solution of the \[77\] is an exponential expanding scale factor, i.e. an inflation universe at early universe

\[
a(T) = a(0)e^{H_*T},
\]

where the Hubble rate is given by

\[
H_* \equiv \frac{\dot{a}}{a} = \sqrt{\frac{1}{6\tau_s}}.
\]

The subscript “\(s\)” in the following paper represents quantities related to the particular physical-time and scale of the inflation very near the singular scale \(t_s\). At small \(\tau_s \equiv t_s - t \rightarrow 0\), i.e. \(t\) slightly deviating from \(t_s\) (\(\tau = 0\)), we have a large expanding Hubble rate at early epoch. How close \(\tau_s\) is to the singular limit \(\tau = 0\) are effectively described by the input parameter: the e-folding number \(N\), see next subsection.

As \(a(T)\) is exponentially expanding, the spatial curvature term in \[78\] becomes more and more unimportant, the observed universe is seen more and more spatially flat, then the solution of \[78\] finally also tends to the exponential expanding profile \[77\] as the common inflation solution of \[14\] and \[78\].

In the framework, first, we see that a deSitter like configuration \[74\] (up to a diffeomorphism \[23\]) and hence a spatially inflationary universe is naturally obtained from the Ricci flow limit, so a Bunch-Davies vacuum as a common choice of vacuum state of inflation and nearly scale-invariant spectrum are naturally obtained at the leading order. Second, we note that the inflation behavior at early epoch is not driven by any speculative inflaton fields, but caused by a completely different mechanism, i.e. the shrinking behavior of the GSRS \[12\] or \[74\], which is also closely related to the conformal instability of the system. The above simple result shows, at the classical solution level, the early epoch inflation is a natural consequence when the spacetime is driven by the Ricci flow to form local singularity at the vicinity of the physical-time origin.
C. Small Deviation from Exact deSitter: Slow Roll Parameters

The small deviation of $\tau$ from zero not only gives a large and finite Hubble rate $H_*$ at early epoch, but also causes a small deviation from the exact deSitter configuration. Here we consider the scale factor $a(T, 0)$ at $\tau = 0$ is an exact deSitter configuration, i.e. the Hubble rate $H_{T=0}$ is exactly a constant independent to the physical-time $T$, and we calculate the physical-time dependence of the Hubble rate $H(T, \tau_*)$ when $\tau_*$ slightly deviates from exact 0. Since $\tau_*$ is a small, we can expand the scale factor in powers of $\tau_*$,

$$a(T, \tau_*) = a(T, 0) (1 + b_1 \tau_* + ...) , \quad (\tau_* \to 0, T \to 0).$$

(81)

Substituting it into the Ricci flow equation (15),

$$\frac{\partial \dot{a}^2(T, \tau_*)}{\partial \tau_*} = 2 \left( 2\dot{a}^2 + a\ddot{a} \right).$$

(82)

By using $\dot{a}_\tau = H_{\tau=0}$ and $\dot{H}_{\tau=0} \ll H_{\tau=0}^2$, we have the expansion coefficient

$$b_1 = 3H_{\tau=0}^2,$$

(83)

so the Ricci flow gives rise to a rescaling to the scale factor, which can also approximately interpreted as a physical-time varying of the Hubble rate, meaning a deviation from exact deSitter, i.e.

$$a(T, \tau_*) = a(T = 0)e^{H_{\tau=0} T} \left( 1 + 3H_{\tau=0}^2 \tau_* + ... \right) \approx a(T = 0)e^{H_* T},$$

(84)

where

$$H_*(T) \equiv H_*(T, \tau_*) \approx H_{\tau=0} + \frac{3H_{\tau=0}^2 \tau_*}{T}.$$  

(85)

In the standard terminology of inflation, the “slow roll parameter” as an approximate constant parameter is given by the physical-time derivative of the Hubble rate during the inflation, but in the case (85), the slow roll parameter is also change with $T$, so a typical-time $T_*$ is needed to approximately gives a frozen value or typical value of the slow roll parameter

$$\epsilon_* \equiv \epsilon(T_*) \equiv -\frac{\dot{H}_*(T_*)}{H_*^2(T_*)} = \frac{3H_{\tau=0}^2 \tau_*}{H_*^2 T_*^2},$$

(86)

where $T_* = \gamma^{-1}T_{end} < T_{end}$ is a typical-time of the inflation when the Hubble rate can be considered as constant $H_{\tau=0} \approx H_*$ and hence we could takes $\epsilon(T_*)$ as its typical value during the inflation. The typical-time $T_*$ is expected several times earlier than the end-time $T_{end}$ of the inflation, for instance,

$$\gamma = \frac{T_{end}}{T_*} > O(1).$$

(87)

Therefore, if we consider the Hubble rate $H_{\tau=0} \approx H_* = \frac{1}{\sqrt{6}\tau_*}$ and the typical-time $T_*$ are both constants during the inflation, then we have a constant slow roll parameter during the inflation

$$\epsilon_* \approx \frac{\gamma^2}{2N^2},$$

(88)

where the e-folding number $N$ is defined as

$$N = \ln \frac{a(T_{end}, \tau_*)}{a(0, 0)} \approx H_* T_{end} \gg 1.$$

(89)

The condition $|\epsilon_*| \ll 1$ is called the slow roll approximation, resulting $\sqrt{T_*} \ll T_* < T_{end}$, leading to a small deviation of the inflation background from deSitter.

In the framework, we also note that the term “slow roll parameters” is just for historical convention, it does not relate to any scalar fields rolling down a potential. The small but finite value of the parameter are completely due to a small deviation of $\tau_*$ from the singular flow-time $\tau = 0$ governed by short flow-time evolution of the Ricci flow.
D. End of the Inflation

The GSRS configuration mimics the local spacetime near the curvature pinching point \( T = 0 \). As the physical-time \( T \) evolves or varies away from the local point \( T = 0 \), the local GSRS part should smoothly connect to the non-de-Sitter rest part of the spacetime manifolds. These two parts are connected at about \( T_{\text{end}} \) when the inflation comes to an end, satisfying \( \varepsilon(T_{\text{end}}) \simeq 1 \). If we consider that \( H_{\tau=0} \) in the local GSRS part is much larger than \( H_{\text{end}} \) in the non-de-Sitter rest part, as \( H_{\tau} \) in (58) gradually decreases to \( H_{\text{end}} \), we have \( |\varepsilon| \ll 1 \) gradually increase to be of order one,

\[
\varepsilon(T_{\text{end}}) \simeq \frac{3H^2_{\tau=0}T_{\text{end}}^2}{H^2_{\text{end}}T_{\text{end}}^2} \simeq 1. \tag{90}
\]

It leads to the estimate that \( T_{\text{end}} \) is much larger than \( \sqrt{\tau_s} \sim O(H_s^{-1}) \),

\[
T_{\text{end}} \simeq \frac{H_{\tau=0}}{H_{\text{end}}} \sqrt{\tau_s}, \tag{91}
\]

which is much longer than \( H_s^{-1} \) consistent with general expectations in the standard inflation theories.

We note that since \( T_s \) and \( T_{\text{end}} \) are both much larger than the local non-collapsing scale \( O(\sqrt{\tau_s} - t_s = \sqrt{\tau_s}) \) of the Ricci flow. Thus the "canonical neighborhood" structure is indeed well-defined during the inflation, which is the self-consistent reason why we could model the inflation period in the early universe by a GSRS metric.

E. Primordial Perturbations

We have proved that at leading order the Ricci flow limit (GSRS) gives rise to an inflationary universe at early epoch. Important phenomenology of the early universe come from the primordial perturbations at the next leading order. In this section, we consider the scalar and tensor primordial perturbations produced during the inflation period.

Note that there are two kinds of scalar perturbations in the theory, first is the scalar (Newtonian potential) perturbation \( \varphi \) around the background, 

\[
ds^2 = (1 + 2\varphi) dT^2 - a^2(T, \tau_s) (1 - 2\varphi) dx_i dx_i, \quad (0 < \sqrt{\tau_s} < T < T_{\text{end}}) \tag{92}
\]

and the second is the scalar perturbation \( \delta u \) around the density \( u_s(T) \equiv u(T, \tau_s) \)

\[
u(k, T) = u_s(T) + \delta u(k, T), \tag{93}
\]

where \( k \) is the Fourier modes of the fluctuation. In fact, the additional scalar “field” \( u \) appearing in the effective action \( [55] \) is inevitable, and plays a fundamental role in the framework, already introduced in the subsection D of section II. It not only leads to the conformal instability and hence inflation due to the "wrong sign" in its kinetic term (see later), but also has direct statistic and geometric meanings \([14]\).

These two kinds of scalar fluctuations can be mixed up into a gauge invariant scalar perturbation

\[
\delta \varphi \equiv -\delta u + \frac{\dot{u}_s}{H_s} \varphi. \tag{94}
\]

By using the new variable \( \delta \varphi \), the fixed point action \( \bar{N} \) in (68) (up to constants) can be expanded to the quadratic order of \( \delta \varphi \),

\[
\bar{N} = \tau_s \bar{F} + \ldots = \lambda \tau_s \int_{M^D} d^D X \left[ uR + \frac{1}{u} |\nabla u|^2 - \frac{D}{2\tau} u \right] = \tau_s \left[ I_0(H_s, u_s) + I_2(\delta \phi) + \ldots \right], \quad (\tau_s \to 0). \tag{95}
\]

The action seems belong to a wide class of scalar-tensor theory of gravity \([53]\) with the density \( u \) playing the role of a “scalar field”. Note that the kinetic term of \( u \) has a “wrong sign” which gives rise to a conformal instability \([10]\) to the gravitational system. Such instability is the essential reason for the singularity formation and inflation. Since \( u \) can also be seen as a “conformal factor” of gravity, in a proper gauge, the instability of \( u \) can also be transformed to and interpreted as the instability of the metric leading to the inflation background eq. (79), leaving \( u_s \) does not directly feel the instability and mildly changes under the gauge. More precisely, note \( R_s = 6(2H^2_s + \dot{H}_s) \) being the scalar curvature in inflation, the lowest order fixed point action is given by

\[
I_0(H_s, u_s) = \lambda \int_{M^D} d^D X dT \frac{1}{u_s} \left\{ \dot{u}_s^2 + \left[ 6(2H^2_s + \dot{H}_s) - \frac{2}{\tau_s} \right] u_s^2 \right\}. \tag{96}
\]
and by using \( \frac{\lambda}{a_0} \) and \( 1/u_0 = a^3 \), its Euler-Lagrangian equation for \( u_* \) gives rise to the conjugate heat equation \( (29) \) taking the form

\[
\ddot{u}_* + 3H_* u_* - 6\dot{H}_* u_* = 0,
\]

in which \( u_* \) does not feel instability and mildly changes under the gauge. If \( \ddot{u}_* \) compared with other terms can be ignored then we have

\[
\frac{\dot{u}_*}{u_*} = \frac{2\dot{H}_*}{H_*} = -2\epsilon_* H_*.
\]

\( I_0 = 0 \) as the fixed point equation of the gradient flow of \( \hat{v} \) also gives rise to the classical GSRS equation \( (74) \) and hence gives the deSitter spacetime \( (79) \) near \( T = 0 \). By considering \( \dot{H}_* \sim O(\epsilon_* H_*^2) \) and \( \dot{u}_*/u_* \sim O(\epsilon_* H_*) \) proportional to \( \epsilon_* \) are both small, so at leading order, extremizing \( I_0(H_*, u_*) = 0 \) recovers the result \( (30) \).

At the next leading order, the fixed point action at the quadratic order of \( \delta \phi \) is

\[
I_2(\delta \phi) = \frac{1}{2} \lambda \int_{M^4} d^3 X dT Z \left\{ \delta \phi^2 - \frac{1}{a^2} |\nabla \delta \phi|^2 + \frac{1}{a^3 Z} \frac{H_*}{u_*} \left[ a^3 Z \left( \frac{\dot{u}_*}{H_*} \right)^4 \right] \delta \phi^2 \right\}
\]

with

\[
Z = \frac{1}{u_*} \left( 1 + \frac{\dot{u}_*}{a^3 H_*} \right)^2 \approx \frac{1}{u_*} (1 + 2\epsilon_*),
\]

where \( (98) \) has been used. The action is the starting point for studying the spectrum of the primordial scalar perturbation on the inflationary background given by \( I_0(H_*, u_*) \).

When \( u_* = 1 \) and \( \epsilon_* = 0 \), we have \( Z = 1 \), then \( I_2(\delta \phi) \) recovers the action of standard minimally coupled scalar inflation action that induces the standard Mukhanov-Sasaki equation. In this theory, the Euler-Lagrangian equation of \( I_2(\delta \phi) \) gives rise to a \( Z \)-modified Mukhanov-Sasaki equation

\[
\ddot{\delta \phi} + \left( \frac{a^3 Z}{a^3} \right) \frac{\dot{\delta \phi}}{\dot{\phi}} + \left\{ \frac{k^2}{a^2} - \frac{1}{a^3 Z} \frac{H_*}{u_*} \left[ a^3 Z \left( \frac{\dot{u}_*}{H_*} \right)^4 \right] \right\} \delta \phi = 0.
\]

Similar with the standard treatment, we introduce

\[
v(k, T) = \sqrt{Z} a \dot{\phi}, \quad z = \frac{a}{H_*} \sqrt{Z u_*^2},
\]

then the \( Z \)-modified Mukhanov-Sasaki equation becomes

\[
v'' + \left( k^2 - \frac{z''}{z} \right) v = 0,
\]

in which prime denotes the conformal physical-time \( d\eta = a^{-1} dT \) derivative.

By standard consideration, in the subhorizon limit \( k^2 \gg \frac{z''}{z} \), the \( v \) is fast oscillating inside the horizon, while in the superhorizon limit, \( k^2 \ll \frac{z''}{z} \), the solution can be written as a Hankel functions \( H^{(1)}_\nu \) in terms of the conformal physical-time \( \eta \),

\[
v_k(\eta) = \frac{\sqrt{\pi} \eta}{2} e^{i(1+2\nu)\pi/4 H^{(1)}_\nu(k|\eta)},
\]

where

\[
v^2 = \frac{1}{4} + \frac{z''}{z} \eta^2.
\]

If the slow roll parameters are considered constant (not vary with conformal physical-time), then we have

\[
v^2 = \frac{1}{4} + \frac{(1 + \epsilon_\delta + \alpha_\delta + \alpha_\epsilon)(2 + \delta_\delta + \alpha_\epsilon)}{(1 - \epsilon_\epsilon)^2},
\]
where the slow roll parameters are defined as

\[ \epsilon_* \equiv -\frac{\dot{H}_*}{H_*^2}, \quad \delta_* \equiv -\frac{\ddot{u}_*}{H_*u_*}, \quad \alpha_* \equiv -\frac{\dot{u}_*}{2H_*u_*} = \epsilon_* \] \tag{106}

in which (98) have been used in the 3rd parameter.

Thus the scalar power spectrum is

\[ P_{\delta \phi} \equiv 16\pi G \frac{k^3}{2\pi^2} |\delta \phi_k|^2, \] \tag{107}

where \( G \) is Newton’s constant. We finally have

\[ P_{\delta \phi} = \frac{16\pi G}{Q} \left[ (1 - \epsilon_*) \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{H_*}{2\pi} \right]^2 \frac{|k\eta|}{2}^{3-2\nu}, \] \tag{108}

where \( \Gamma \) is the Gamma function. The power spectrum of scalar perturbation is frozen when they cross to the outside of the horizon for which \( k = aH_* \), in the slow roll approximation the power spectrum of the scalar perturbation can be given by [56]

\[ P_{\delta \phi} = \frac{16\pi G}{Q} \left( \frac{H_*}{2\pi} \right)^2 \frac{k=aH_*}{\bigg| H_* \bigg|}^2 \approx 4\pi G \frac{H_*}{2\pi} \frac{\epsilon_*^2 u_*}{2\pi^2} \adjust{\bigg| \frac{k}{aH_*} \bigg|}^{3-2\nu}, \] \tag{109}

where

\[ Q = Z \left( \frac{\dot{u}_*}{H_*} \right)^2 \approx \frac{1}{u_*} (1 + 2\epsilon_*) \left( \frac{\dot{u}_*}{H_*} \right)^2 \approx 4\epsilon_*^2 u_* . \] \tag{110}

The spectral index for the scalar perturbation is given by

\[ n_s - 1 = \frac{d \ln P_{\delta \phi}}{d \ln k} \approx -4\epsilon_* - 2\delta_* - 2\alpha_* . \] \tag{111}

Tensor perturbations \( h_{ij} \) in \( g_{ij} = a^2(T, \tau)(\delta_{ij} + h_{ij}) \) have two polarization states \( h_p \) where \( p = +, \times \). By using the polarization tensors bases \( e^+_{ij} \) and \( e^\times_{ij} \), we have

\[ h_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij} . \] \tag{112}

For a similar consideration followed by the scalar perturbations, and in the slow roll approximation, the power spectrum of each polarization component \( h_p \) (\( p = +, \times \)) is given by [56]

\[ P_h \approx \frac{128\pi G}{u_*} \left( \frac{H_*}{2\pi} \right)^2 \frac{k=aH_*}{\bigg| H_* \bigg|}^2 , \] \tag{113}

with a much smaller spectral index beyond the order of slow roll approximation,

\[ n_t \equiv \frac{d \ln P_h}{d \ln k} \approx -2\epsilon_* + 2\alpha_* = -2\epsilon_* + 2\epsilon_* = 0 . \] \tag{114}

The power spectrum of the tensor perturbation is much smaller and more scale-invariant than the prediction of the standard inflation.

\[ \text{F. Estimate of } u_* \text{ and } H_* \]

To predict the power spectra of the scalar and tensor perturbations (109) (111) and (113) more precisely, several parameters are needed to further estimate in this framework, the most important ones are the \( u_* \equiv u(T, \tau_*) \) and \( H_* \equiv H(T_*, \tau_*) \).
Since \( u_\ast \) represents the volume ratio \(^{23}\) between the fiducial 3-volume (standard scale factor \( a = 1 \)) and the local 3-volume of early universe, by using \(^{26}\) and \(^{79}\), the physical-time derivative of \( u \) is volume changing rate during the inflation is

\[
\lim_{T \to 0} u_\ast = -3H_\ast.
\]  

(115)

By using \(^{88}\), so we have

\[
\lim_{T \to 0} u_\ast = \frac{3}{2\epsilon_\ast} - \frac{1}{a^3(T, \tau_\ast)},
\]  

(116)

so

\[
u_\ast(T) = \frac{3}{2\epsilon_\ast} - 3H_\ast T \approx \frac{3}{2\epsilon_\ast} e^{-2\epsilon_\ast H_\ast T}, \quad (T \to 0),
\]  

(117)

which is obviously also an approximate solution of \(^{97}\). The estimate of \( u_\ast \) make small corrections of order \( O(\epsilon_\ast) \) to the unperturbed conditions \(^{76}\) at the slow roll approximation.

From the estimate of \( u_\ast \), we can directly have the second slow roll parameter \( \delta_\ast \) in \(^{109}\)

\[
\delta_\ast \equiv \frac{\ddot{\nu}_\ast}{H_\ast \dot{u}_\ast} = -\frac{\dot{H}_\ast}{H_\ast^2} = \epsilon_\ast.
\]  

(118)

In this situation, if we take \( N \simeq 60 \), the spectral index of the scalar perturbation \(^{111}\) is in the range

\[
n_\ast = 1 - 8\epsilon_\ast \simeq 1 - \frac{4\gamma^2}{N^2} \approx (0.91 \sim 0.99),
\]  

(119)

if taking \( 1 < \gamma \leq 9 \), describing how close the typical-time \( T_\ast \) to the end-time \( T_{end} \) of inflation. The favored value of observations \( n_\ast \simeq 0.96 \) can be obtained if one takes \( \gamma \approx 6 \), which is consistent with the pre-assumption \(^{87}\).

In terms of Perelman’s seminal introduction of his monotonic functionals and reduced volume, the Ricci flow spacetime admits a remarkable comparison geometry picture. From the geometric point of view, one of the most crucial information of the geometry is coded in the local volume comparison. Remind in the subsection II-F that the fraction of the “dark energy” w.r.t. the critical density is related to the relative volume or reduced volume \( u_{\tau=0} = \tilde{V}(M_{\tau=0}) = e^\nu \leq 1 \) of order one, and the estimate of \( H_\ast \) in the local curvature pinching region also belongs to such kind of question in the framework. The value of \( H_\ast \equiv H(T_\ast, \tau_\ast) \) not only depends on the typical-time \( T_\ast \), but also on the scale \( \tau_\ast \), and its rough order of magnitude at fixed \( T_\ast \) is almost given by \( \tau_\ast \) and hence the e-folding \( N \).

The theory has the critical density \( \lambda = \rho_\ast \sim (10^{-3}eV)^4 \) \(^{89}\) as the only dimensional input of the theory, and together with the scale factor \( a(0) \sim e^{-N} \). So by dilating the current critical energy scale \( \lambda^{1/4} \) by the scale factor \( a(0) \) of the inflation period, we have a natural estimate to the energy scale and Hubble rate at the typical-time of the inflation if taking \( 60 \leq N \leq 70 \)

\[
H_\ast \simeq \lambda^{1/4} a^{-1}(0) \simeq (10^{14} \sim 10^{18}) \text{GeV},
\]  

(120)

which is within a generally accepted range of inflation scale. The estimate of \( H_\ast \) near the local singularity has direct relation to the estimate (e.g. Harnack estimate) of a local curvature and hence the local volume comparison in the Ricci flow. This estimate also suggests a possible picture for the mysterious large orders of magnitude between the energy scales of the early epoch accelerating inflation (of order of \( H_\ast \)) and the late epoch accelerating expansion (corresponding to the cosmological constant of order of \( \lambda^{1/4} \)); the high energy scale of \( H_\ast \) is because the local pinching curvature is almost unbounded given by the large e-folding number \( a^{-1}(0) \sim e^N \gg 1 \) at the early epoch, while the cosmological constant is of order of \( \lambda \) at late epoch, with fraction of order one, i.e. \(^{70}\).

By using \(^{116}\), \(^{120}\) and \(^{88}\) the scalar power spectrum \(^{109}\)

\[
P_\phi \approx \frac{4\pi G}{c^2 u_\ast} \left( \frac{H_\ast}{2\pi} \right)^2 \left[ \frac{8\pi G}{3\epsilon_\ast} \left( \frac{H_\ast}{2\pi} \right)^2 \right] \simeq 10^{-5}
\]  

(121)

is within the range of observations when taking \( N \simeq 60 \) with \( \gamma \simeq 6 \).

The tensor power spectrum \(^{113}\) is predicted as

\[
P_h \approx \frac{128\pi G}{u_\ast} \left( \frac{H_\ast}{2\pi} \right)^2 \left[ \frac{256\pi G}{3\epsilon_\ast} \left( \frac{H_\ast}{2\pi} \right)^2 \right] \simeq 10^{-8},
\]  

(122)
which is more difficult to be detected than the standard prediction. And the tensor-scalar ratio is given by

\[ r \equiv \frac{P_h}{P_{s\phi}} \simeq 32\epsilon^2 \simeq \frac{8\gamma^4}{N^4} \simeq 0.0008, \]  

(123)

when taking \( N \simeq 60 \) with \( \gamma \simeq 6 \), which compared with the standard prediction is more difficult to be observed and hence is certainly consistent with current observations.

And similar with standard consideration, (119) and (123) can be combined to give a relation,

\[ r \simeq \frac{1}{2}(1 - n_s)^2. \]  

(124)

This relation implies that the prediction in \( r - n_s \) plane is independent of the parameters \( N \) and \( \gamma \), which is inside the allowed range of the observations.

**IV. SUMMARY AND CONCLUSIONS**

The paper reviews the quantum fields theory of spacetime reference frame. The theory is described by a non-linear sigma model in \( d = 4 - \epsilon \) base space, and the target space is interpreted as the quantum reference frame fields. The 2nd order central moment quantum fluctuation of the field introduces the Ricci flow to the spacetime manifolds \((M, g)\) at Gaussian approximation. The normalized density matrix \( u \) of the theory can also be written explicitly at the Gaussian approximation, which induces the Ricci-DeTurck flow to the spacetime manifolds with density \((M, g, u)\). We use the functional integral method to deduce the diffeomorphism anomaly and the effective action of gravity based on the quantum spacetime theory.

When we apply the Ricci flow to the late epoch of the universe when the space and time are on an equal footing \((\tau, T)\), homogeneous and with positive curvature, the Ricci flow globally shrinks the spacetime \( B^4 \) isotropically. Since the base space interpreted as the fiducial lab is considered rigid, the diffeomorphism anomaly must be canceled in the fiducial lab up to UV scale, as a consequence, a cosmological constant emerging into the effective action can be calculated by the anomaly cancellation, which also normalizes the spacetime and prevents the Ricci flow shrinking the spacetime into a singular point but to a limit spacetime with finite relative volume (w.r.t. the fiducial lab volume).

When applying the Ricci flow to the early universe when time and space are treated differently \((\tau, T)\), \( S^3 \times \mathbb{R} \), the Ricci flow locally shrinks \( S^3 \) rather than \( \mathbb{R} \), and hence develops local curvature pinching near the origin of the physical-time. In this case, the local singularity can no longer be “normalized out” as the global cosmological constant does in the case of equal-footing-spacetime. The closeness of the early epoch producing the primordial perturbations to the singularity is effectively described by a finite e-folding number \( N \), due to the highly suppressed non-Gaussian primordial fluctuations in the cosmic observations, if we still assume the validity of the Ricci flow (being a Gaussian approximate) applying to the early epoch, the high curvature region of a manifold resembles a rescaled ancient non-collapse solution, the local non-collapsing theorem and the Gradient Shrinking Ricci Soliton (GSRS) model of the local singularity provide us possible approach to the quantum treatment of the early universe.

The paper shows that the Ricci flow limit configuration (GSRS) resembling the large curvature region of the spacetime is a promising model for the early universe because the following results are obtained: (i) the Ricci flow deforms a spatial homogeneous and isotropic FRW ansatz to the GSRS flow limit, which mimics a spatially inflationary (de-Sitter) universe at the leading order; (ii) the self-similarity of the GSRS configuration provides a natural explanation of nearly scale invariant and Gaussian spectrum of the primordial perturbations in cosmic observations; (iii) a finite scale \( \tau_\star \) slightly deviating from the singular flow-time \( \tau = 0 \) not only at leading order gives rise to a large inflation Hubble rate \( H_\star \) but also at the next leading order gives a small deviation from exact deSitter and hence gives rise to small slow roll parameters and small deviation of the primordial perturbation spectrum from exact scale invariant.

The slow roll parameters and the related inflation ending are also discussed within the framework. Because during the inflation, the geometric quantities such as the scale factor \( a(T, \tau) \), the Hubble rate \( H(T, \tau) \), and the manifolds density \( u(T, \tau) \) are functions of both the flow-time \( \tau \) and the physical-time \( T \), so we need to further estimate the scale \( \tau_\star \) and typical-time \( T_\star \) of the inflation, to evaluate their typical values during the inflation. It leads to 3 inputs in the calculating of the slow roll parameters and the power spectrum of the primordial perturbations: (a) the e-folding number \( N \) describing how close the scale \( \tau_\star \) of the inflation producing our interested physics is to the singular scale \( \tau = 0 \), (b) \( \gamma = T_{end}/\tau_\star \) describing how close between the end-time and the typical-time of the inflation, when the typical values of the slow roll parameters are taken and the Hubble rate \( H_\star \) can be considered as a constant, and beside those we also have (c) \( \lambda = \rho_c \simeq (10^{-3} eV)^4 \) being the fundamental dimensional input of the frame fields theory. \( \lambda \) is fixed from the first principle of the theory, while \( N \) and \( \gamma \) can be tuned in some ranges: e.g. the e-folding is taken within \( 60 \leq N \leq 70 \), and \( \gamma \) as the ratio between \( T_{end} \) and \( \tau_\star \) must only be several times larger than 1.
If we take $N \approx 60$ with $\gamma \approx 6$, first, the energy scale of the local curvature pinching of the early epoch is estimated consistent with generally accepted range of the standard inflation scale; and second, the predicted power spectrum and index of the primordial scalar perturbation could be consistent with observations, and the ones of the tensor perturbation are much smaller than the standard inflation theory, which are even more difficult to be detectable. Thus it seems that the tensor perturbation is too small to be used to distinguish this theory and the standard inflation theory.

The Ricci flow is a very powerful tool to study the geometry especially its short distance as well as long distance structure, and provide us a framework to ask meaningful questions to the geometry and physics. So if the approach really hits certain core of the geometry and physics in the early universe, we hope, beside the observables studied in the paper (taking the standard inflationary paradigm as the reference), our future investigations of the theory will provide us more possible predictions and insights to the early universe.

Acknowledgments

This work was supported in part by the National Science Foundation of China (NSFC) under Grant No.11205149, and the Scientific Research Foundation of Jiangsu University for Young Scholars under Grant No.15JDG153.

[1] N. Aghanim et al. (Planck), Astron. Astrophys. 641, A6 (2020), [Erratum: Astron.Astrophys. 652, C4 (2021)], 1807.06209.
[2] A. Ijjas, P. J. Steinhardt, and A. Loeb, Phys. Lett. B 736, 142 (2014), 1402.6980.
[3] R. Brandenberger, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 46, 109 (2014), ISSN 1355-2198.
[4] D. Chowdhury, J. Martin, C. Ringeval, and V. Vennin, Phys. Rev. D 100, 083537 (2019), 1902.03951.
[5] M. J. Luo, Nuclear Physics 884, 345 (2014), 1312.2759.
[6] M. J. Luo, Journal of High Energy Physics 06, 063 (2015), 1401.2488.
[7] M. J. Luo, Int. J. Mod. Phys. D 27, 1850081 (2018), 1507.08755.
[8] M. J. Luo, Found. Phys. 51, 2 (2021), 1907.05217.
[9] M. J. Luo, Class. Quant. Grav. 38, 155018 (2021), 2106.16150.
[10] M. J. Luo, Annals Phys. 441, 168861 (2022), 2201.10732.
[11] M. J. Luo, Int. J. Mod. Phys. D 32, 2350022 (2023), 2302.06851.
[12] M. J. Luo, Int. J. Theor. Phys. 62, 91 (2023), 2210.06082.
[13] D. Friedan, Physical Review Letters 45, 1057 (1980).
[14] D. Friedan, Annals of Physics 163, 318 (1980).
[15] R. S. Hamilton, Journal of Differential Geometry 17, 255 (1982).
[16] R. S. Hamilton et al., Journal of Differential Geometry 24, 153 (1986).
[17] G. Perelman, arXiv preprint math/0211159 (2002).
[18] G. Perelman, arXiv preprint math/0303109 (2003).
[19] G. Perelman, arXiv preprint math.DG/0307245 (2003).
[20] W. X. Shi, Journal of Differential Geometry 30, 303 (1989).
[21] W. X. Shi, J. Diff. Geom. 30, 223 (1989).
[22] B. L. Chen and X. P. Zhu, Journal of differential geometry 74, 119 (2005).
[23] Y. Aharonov and T. Kaufherr, Phys. Rev. D 30, 368 (1984).
[24] C. Rovelli, Classical and Quantum Gravity 8, 317 (1991).
[25] M. Dickson, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 35, 195 (2004).
[26] R. M. Angelo, N. Brunner, S. Popescu, A. J. Short, and P. Skrzypczyk, Journal of Physics A: Mathematical and Theoretical 44, 145304 (2011).
[27] Flaminia, Giacomini, Esteban, Castro-Ruiz, Caslav, and Brukner, Nature communications 10 (2019).
[28] M. Gell-Mann and M. Lévy, Il Nuovo Cimento 16, 705 (1960).
[29] A. Codello and R. Percacci, Physics Letters B 672, 280 (2009).
[30] C. De Rham, A. J. Tolley, and S. Y. Zhou, Physics Letters B 760 (2015).
[31] B. Chow and D. Knopf, The Ricci Flow: An Introduction; An Introduction, vol. 1 (American Mathematical Soc., 2004).
[32] B. Chow, P. Lu, and L. Ni, Hamilton’s Ricci flow, vol. 77 (American Mathematical Soc., 2006).
[33] P. Topping, Lectures on the Ricci flow, vol. 325 (Cambridge University Press, 2006).
[34] M. Carfora and K. Piotrzkowska, Physical Review D 52, 4393 (1995).
[35] K. Piotrzkowska, arXiv preprint gr-qc/9508047 (1995).
[36] M. Carfora and T. Buchert, in 14th International Conference on Waves and Stability in Continuous Media, eds. N. Mangana, R. Monaco, S. Rionero, World Scientific (2008), pp. 118–127.
[37] R. Zalaletdinov, Int. J. Mod. Phys. A23, 1173 (2008), 0801.3256.
[38] A. Paranjape, Ph.D. thesis, TIFR, Mumbai, Dept. Astron. Astrophys. (2009), 0906.3165.
[39] R. S. Hamilton, The Ricci flow on surfaces (The Ricci flow on surfaces, 1988).
[40] Morgan and Frank, American Mathematical Monthly (2009).
[41] W. Wylie and D. Yeroshkin, arXiv e-prints arXiv:1602.08000 (2016), 1602.08000.
[42] Corwin and Ivan, Rose Hulman Undergraduate Mathematics Journal (2017).
[43] M. M. Akbar and E. Woolgar, Classical and Quantum Gravity 26, 055015 (2009).
[44] D. Bakry and M. Emery, Diffusions hypercontractives (1985).
[45] D. M. DeTurck et al., Journal of Differential Geometry 18, 157 (1983).
[46] H.-D. Cao, R. S. Hamilton, and T. Ilmanen, arXiv preprint [math/0404165] (2004).
[47] H.-D. Cao, arXiv preprint arXiv:0908.2006 (2009).
[48] R. Hamilton, Surveys in Diff Geom 2 (1995).
[49] R. S. Hamilton, Journal of differential geometry 38 (1993).
[50] B. L. Chen and X. P. Zhu, Inventiones Mathematicae 140, 423 (2000).
[51] N. Sesum, Comm.anal.geom pp. 283–343 (2004).
[52] C. Pitrou, T. S. Pereira, and J.-P. Uzan, Journal of Cosmology and Astroparticle Physics 2008, 004 (2008).
[53] Y. Fujii and K. I. Maeda, Classical and Quantum Gravity 20, 4503 (2003).
[54] Jai-chan and Hwang, Physical Review D 53, 762 (1996).
[55] J.-c. Hwang, Class. Quant. Grav. 14, 3327 (1997), gr-qc/9607059.
[56] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010), 1002.4928.