Investigation on modal behaviour of FGM annular plate under hygrothermal effect

Pankaj Sharma¹, Rahul Singh²

¹ Assistant Professor, University Department of Mechanical Engineering, Rajasthan Technical University, Kota 324010, India
² Research Scholar, University Department of Mechanical Engineering, Rajasthan Technical University, Kota 324010, India

Email: ¹psharma@rtu.ac.in, ²drengineer92@gmail.com

Abstract. This paper investigates modal characteristics of FGM annular plate under hygrothermal effect. The mechanical properties are supposed to have a continuous variation along the thickness of the plate. The power law or sigmoid law functions are used to govern the resultant mechanical properties. The MUMPS eigen solver which is exists in COMSOL Multiphysics® (version 5.2) software is employed to obtain the eigenfrequencies and corresponding mode shapes. The accuracy of the method is validated by comparing the results with the results published work for particular case. Effects of functionally graded index, radius-to-thickness ratio, and temperature and moisture changes on the natural frequencies are discussed in detail. An excessive stress due to drastic moisture and temperature gradients make the engineering structures susceptible to failure. Due to the fact that, continuously identified natural frequencies of vibration under temperature and moisture effect can provide unique information during both initial fabrication and service life of different mechanical structures. Hence, this study can be helpful directly or indirectly in development of highly efficient environmental sensors that are very necessary in a structural health monitoring perspective. It can be observed that the fundamental frequency decreases with increase in functionally graded index and the moisture content rise under the power law model whereas under sigmoid law model the frequencies remain almost constant with increase in functionally graded index.

1. Introduction

Functionally graded materials (FGMs) are the new form of composite, which provides the facility of smooth material variation across any desired direction. Potential use of FGMs can be found in automotive structural components modelled as circular plates (e.g. thrust bearing plates, the driven plate of a friction clutch etc.) [1]. Annular plates are widely used parts in mechanical power transmission to minimize the weight of the whole structure [2]. Reduction of gear weight also reduce the natural frequency of the gears, thereby weakening the mesh noise [3]. In braking system also, FGM is a suitable material for automotive disc brake and pads [4].

Moisture absorption may be practically occurred during the fabrication of FGM structures (beam, plate and shell etc.) and subsequent variation in temperature and moisture reduces the elastic moduli and degrades the strength of material. Due to this reason there are many FGM structures which are failed because of vibrations under hygrothermal effect. Also, the Eigenfrequencies and corresponding mode shapes of any annular plate is affected by the presence of a hole in it.
2. Literature survey
There are many modelling techniques and solution methods exist in the literature to analyze the vibration characteristics of FGM plate under thermal effect [5] only. Also, it may be seen that in most of the cases the material properties were governed by the power law function.

Hygrothermal effect was first investigated by Whitney and Ashton [6] for laminated composite structures. Using Newton-Raphson method, Lee and Kim [7] calculated the Eigenfrequencies of simply supported FGM plates in hygrothermal environment. Using different plate theories, Zenkour [8,9] studied the hygrothermal effects on FGM plates.

In most of the above papers only power law has been considered to study the hygrothermal effects on modal behaviour of FGM plates. It has been stated in the literature [10] that the interfacial stresses arise which is the main cause of failure of composite structures is reduced up to an extent if the mechanical constants are varied according to sigmoid law distribution. To the author’s best knowledge, modal behaviour of FGM annular plate under hygrothermal effect where mechanical constants are varying according to sigmoid law has not been reported in the literature before. Herein, the results for an additional boundary condition clamped-free (C-F) is also reported. The finite element analysis software COMSOL Multiphysics® (version 5.2) is used for computation. The results are validated by comparing with the previous studies.

3. Problem formulation
A Functionally graded annular plate of thickness $h$ which is composed of 100% ceramic (ZrO$_2$) at the upper and 100% metal (Ti-6Al-4V) at the lower surface is shown in figure 1. It is assumed that the mechanical properties [11] are graded smoothly along $x_3$ direction from lower surface (metal rich) to upper surface (ceramic rich) according to power law or sigmoid law formulation. For meshing sequence in COMSOL®, physics-controlled mesh shown in figure 1 is used and extra fine element size (35 mm) is applied on the plate.

![FGM annular plate with physics-controlled mesh (35 mm)](image)

Figure 1. FGM annular plate with physics–controlled mesh (35 mm).

The formulation of power law function can be expressed as [12, 13]

$$P(x_3) = P_{ul} v(x_3) + P_l$$

(1)

The formulation of sigmoid law is given by combining the two power law functions as [12, 13]

$$P(x_3) = (P_{ul})v_1(x_3) + P_l$$

(2)

$$P(x_3) = (P_{ul})v_2(x_3) + P_l$$

(3)

here

$$v_1(x_3) = 1 - \frac{1}{2}\left(\frac{h}{x_3}\right)^n$$

for $0 \leq x_3 \leq \frac{h}{2}$
and
\[ \nu_2(x_3) = \frac{1}{2} \left( \frac{h_2 + h_3}{h} \right)^n \]
for \(-\frac{h}{2} \leq x_3 \leq 0\)

Where, \(P(x_3)\) is effective material property and \(\nu(x_3) = \left( \frac{2x_3 + h}{2h} \right)^n\) is volume fraction index. In which, \(P_{ul} = P_u - P_l\) and \(x_3\) is coordinate along the thickness of the plate. The \(P_u\) and \(P_l\) are effective mechanical properties at upper and lower part, \(h\) is the thickness and \(n\) is the volume fraction index of the FGM annular plate.

Here, uniform hygrothermal load along the plate thickness is considered. The present values of uniform moisture and temperature are:

\[ T = T_0 + \Delta T \]  
\[ C = C_0 + \Delta C \]

where \(T_0 = 300\) (k) and \(C_0 = 0\%\) are reference values, which are considered at the lower part of the plate. Where \(\Delta T\), \(\Delta C\) are the change in temperature and moisture respectively.

\[ W(T) = W_0(W_{-1}T^{-1} + 1 + W_1T + W_2T^2 + W_3T^3) \]

where \(W_0, W_{-1}, W_1, W_2, W_3\) are temperature dependent constants for ceramic and metal materials [11].

### 4. Convergence test

FGM annular plate with following dimension \((R_o = 1\ m; R_i = 0.1\ m; h = 0.1\ m)\) is taken for convergence test. All mechanical constants are supposed to be graded by simple power law from \(\text{ZrO}_2\) to \(\text{Ti-6Al-4V}\). Power law index \(n = 1\) and \(\Delta T = 100\) is considered and dimensionless fundamental frequencies for first three modes are obtained with different physics-controlled meshing for clamped - clamped (C-C) case. Results are obtained using COMSOL and presented in table 1. It may be notice that the results are agree well with the previous published work [11] with extra fine meshing (mesh size = 35 mm), so this meshing is used for further computation.

**TABLE 1.** First three dimensionless frequencies for FGM annular plate with CPU time and different physics-controlled mesh using COMSOL \((n = 1, \Delta T = 100)\).

| Mesh Size (mm) | Number of elements | CPU time (sec) | Mode (0,0) | Mode (1,0) | Mode (2,0) |
|---------------|--------------------|----------------|-------------|-------------|-------------|
| DQM [11]      |                    |                | 18.9389     | 19.9067     | 25.3866     |
| 20            | 42915              | 42             | 18.9386     | 19.9066     | 25.3865     |
| 35            | 8883               | 16             | 18.9389     | 19.9067     | 25.3866     |
| 55            | 2670               | 13             | 19.0968     | 20.0734     | 25.6044     |
| 80            | 1551               | 13             | 19.3278     | 20.3047     | 25.9683     |
| 100           | 993                | 13             | 19.5338     | 20.5236     | 26.2997     |
| 150           | 558                | 12             | 19.9712     | 20.9680     | 26.9376     |
| 190           | 387                | 12             | 20.9544     | 22.0710     | 28.6523     |
| 300           | 288                | 13             | 22.7261     | 24.1683     | 32.0253     |
| 500           | 135                | 12             | 27.1954     | 29.3258     | 38.7479     |

### 5. Validation study under thermal effect

The material constants can be refereed from [11]. The FGM annular plate is made of 100% ceramic \((\text{ZrO}_2)\) at the upper and 100% metal \((\text{Ti-6Al-4V})\) at the lower surface. In between these two, the mechanical properties are supposed to be graded with the power law formulation along the thickness of
the plate. The Eigenfrequencies are calculated by following relation: 
\[ \lambda R_o^2 \frac{D\lambda}{\rho_{oc}} \] 
in which \( D_{oc} = \frac{E_{oc}h^2}{12(1-\nu_{oc}^2)} \)
which is used for all computations in this study. Here, \( \lambda \) is the natural frequency, \( \rho_c, v_{oc}^2 \), and \( E_{oc} \) are the density, Poisson’s ratio and Young’s modulus for ceramic (ZrO\textsubscript{2}) material. Tables 2 and 3 display fundamental Eigenfrequencies for first three modes (0,0), (1,0) and (2,0) under thermal effect for clamped-clamped (C-C) end condition. Here, the first digit in the bracket shows number of nodal diameter while second digit shows number of nodal circles. It may be notice that, COMSOL which is used for current study gives near accurate result and also shows good agreement with the results obtained by differential quadrature method (DQM) \[ \text{[Reference]-(COMSOL)} \times 100. \]

Table 2. Accuracy of the present work (COMSOL), comparison of first dimensionless frequency (\( \lambda_1 \)) under clamped-clamped (C-C) boundary condition \( (h/R_o = 0.1, \Delta T = 800, n = 1) \) with DQM [11].

| Mode   | DQM [11] | COMSOL | Diff. (%) | DQM [11] | COMSOL | Diff. (%) |
|--------|----------|--------|-----------|----------|--------|-----------|
| (0,0)  | 18.438   | 18.9389| -2.71     | 30.006   | 30.5071| -1.66     |
| (1,0)  | 19.230   | 19.9067| -3.51     | 30.695   | 31.2401| -1.77     |
| (2,0)  | 24.528   | 25.3866| -3.05     | 33.331   | 33.9722| -1.92     |

Table 3. Accuracy of the present work (COMSOL), comparison of first dimensionless frequency (\( \lambda_1 \)) under clamped-clamped (C-C) boundary condition \( (h/R_o = 0.2, \Delta T = 800, n = 1) \) with DQM [11].

| Mode   | DQM [11] | COMSOL | Diff. (%) | DQM [11] | COMSOL | Diff. (%) |
|--------|----------|--------|-----------|----------|--------|-----------|
| (0,0)  | 15.470   | 15.3575| 0.72      | 23.603   | 23.3830| 0.93      |
| (1,0)  | 16.417   | 16.3527| 0.39      | 24.145   | 23.9287| 0.89      |
| (2,0)  | 21.215   | 21.1230| 0.43      | 26.333   | 26.1110| 0.84      |

6. Parametric study under hygrothermal effect

Modal analysis of FGM annular plate under hygrothermal effect where mechanical constants are varying along plate thickness with power law and sigmoid law for clamped-clamped (C-C) and clamped-free (C-F) end conditions is discussed briefly. Table 4 displays the dimensionless frequency for \( (\Delta C = 1\%, 2\% \text{ and } \Delta T = 100) \) with different volume fraction index using sigmoid law formulation.

Table 4. Dimensionless frequencies (\( \lambda \)) of FGM annular plate with hygrothermal effect for C - C & C - F end conditions under sigmoid law using COMSOL \( (h/R_o = 0.2, \Delta T = 0.3, \Delta T = 100) \).

| Mode | BCs     | \( \Delta C \) | n = 0 | n = 1 | n = 2 | n = 3 | n = 4 | n = 5 | n = 7 | n = 9 | n = 10 |
|------|---------|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (1,0) C-C | 1%   | 26.9433        | 26.9433| 26.9433| 26.9433| 26.9433| 26.9433| 26.9433| 26.9433| 26.9433| 26.9433|
|      | 2%   | 26.8889        | 26.8889| 26.8889| 26.8889| 26.8889| 26.8889| 26.8889| 26.8889| 26.8889| 26.8889|
| C-F | 1%   | 13.6069        | 13.6069| 13.6069| 13.6069| 13.6069| 13.6069| 13.6069| 13.6069| 13.6069| 13.6069|
|      | 2%   | 13.5524        | 13.5524| 13.5524| 13.5524| 13.5524| 13.5524| 13.5524| 13.5524| 13.5524| 13.5524|
| (2,0) C-C | 1%   | 20.026         | 20.026| 20.026| 20.026| 20.026| 20.026| 20.026| 20.026| 20.026| 20.026|
|      | 2%   | 20.0209        | 20.0209| 20.0209| 20.0209| 20.0209| 20.0209| 20.0209| 20.0209| 20.0209| 20.0209|
| C-F | 1%   | 16.5365        | 16.5365| 16.5365| 16.5365| 16.5365| 16.5365| 16.5365| 16.5365| 16.5365| 16.5365|
|      | 2%   | 16.5325        | 16.5325| 16.5325| 16.5325| 16.5325| 16.5325| 16.5325| 16.5325| 16.5325| 16.5325|
Figure 2. First dimensionless frequency with moisture content rise and volume fraction index using power law and sigmoid law formulation ($\Delta T = 100$) (a) clamped-clamped (b) clamped-free.

Eigenfrequency $= 480.38$ Hz

Mode (0,0)

Eigenfrequency $= 489.04$ Hz

Mode (1,0)

Eigenfrequency $= 625.02$ Hz

Mode (2,0)

Figure 3. First three mode shapes for clamped-clamped (C-C) boundary condition under hygrothermal effect using sigmoid law formulation ($\Delta T = 100, \Delta C = 2\%, R_i/R_o = 0.1, h/R_o = 0.1, n = 1$).

Eigenfrequency $= 82.539$ Hz

Mode (0,0)

Eigenfrequency $= 331.04$ Hz

Mode (1,0)

Eigenfrequency $= 589.54$ Hz

Mode (2,0)

Figure 4. First three mode shapes for clamped-free (C-F) boundary condition under hygrothermal effect using sigmoid law formulation ($\Delta T = 100, \Delta C = 2\%, R_i/R_o = 0.1, h/R_o = 0.1, n = 1$).

7. Results and discussions

7.1 Effect of moisture concentration rise

Table 4 presents the values of dimensionless frequency for $h/R_o = 0.2, R_i/R_o = 0.3$ with $\Delta T = 100$ using sigmoid law distribution. Figure 2 displays the first dimensionless frequency under hygrothermal ($\Delta C = 0\%, 1\%, 2\%$ and $\Delta T = 100$) effect with material properties graded using power law and sigmoid law distribution for modes (1,0) and (2,0). With the rise in moisture content, the fundamental frequency decreases with the increase in power law index and the moisture content rise. The frequency remains almost constant when properties are graded by sigmoid law with varying volume fraction index. The frequency of the FGM plate is smaller for high value of moisture content in comparison of FGM plate that without moisture content rise.
7.2 Effect of volume fraction index
It can be observed that, the value of dimensionless frequency decreases sharply as power law index increases from 0 to 1, and decreases smoothly for mid values and then it tends to remain constant for end values of power law index when the power law is used. While the frequencies remain almost constant when sigmoid law is used. Also, it is observed that with moisture content rise, the frequency decreases with the increase of volume fraction index. In case of clamped-free boundary condition, same phenomena can be observed besides of moisture effect because the effect of moisture content becomes insignificant. Figures 3 and 4 display the first three mode shapes (0,0), (1,0) and (2,0) for clamped-clamped (C-C) and clamped-free (C-F) end condition under hygrothermal effect.

8. Conclusion
The modal behaviour of FGM annular plate under hygrothermal effect is studied. Eigenfrequencies and corresponding mode shapes has been obtained for clamped-clamped and clamped-free boundary conditions. The power law and sigmoid law formulation is used to grade the mechanical constants along the plate thickness direction. The finite element analysis software COMSOL Multiphysics® (version 5.2) is used for computation. It can be observed that the frequency decreases with increase in functionally graded index and the moisture content rise under the power law formulation whereas under sigmoid law formulation the frequencies remain almost constant with increase in functionally graded index. Hence, this study can be helpful in making of hygrothermal sensors to provide unique information during both initial fabrication and service life of different FGM structures used in power transmission system.

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