QCD sum rules in a Bayesian approach

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Abstract. A novel technique is developed, in which the Maximum Entropy Method is used to analyze QCD sum rules. The main advantage of this approach lies in its ability of directly generating the spectral function of a given operator. This is done without the need of making an assumption about the specific functional form of the spectral function, such as in the “pole + continuum” ansatz that is frequently used in QCD sum rule studies. Therefore, with this method it should in principle be possible to distinguish narrow pole structures from continuum states. To check whether meaningful results can be extracted within this approach, we have first investigated the vector meson channel, where QCD sum rules are traditionally known to provide a valid description of the spectral function. Our results exhibit a significant peak in the region of the experimentally observed ρ-meson mass, which agrees with earlier QCD sum rules studies and shows that the Maximum Entropy Method is a useful tool for analyzing QCD sum rules.

1. Introduction
QCD sum rules are a strong tool for investigating various properties of hadrons directly from QCD [1, 2]. By exploiting the analytic properties of two-point functions, this method connects perturbative and non-perturbative sectors of QCD, and therefore allows one to describe hadrons by the perturbative procedure of the operator product expansion (OPE). The higher-order terms of the OPE contain condensates of various operators and thus incorporate information on the QCD vacuum into the calculation.

While the success of QCD sum rules in describing many aspects of hadrons is quite impressive, this method also has its limitations. Firstly, the uncertainties that are involved in the calculation of the OPE are somewhat large, mainly originating from the possible range of values of the condensates and the unknown contributions from the higher-order terms that have been neglected due to the truncation of the OPE. Secondly, it has been necessary to introduce some simple ansatz for parametrizing the spectral function in all the QCD sum rule studies carried out so far. For instance, it is most common to assume the “pole + continuum” functional form, where the pole represents the hadron in question and the continuum stands for the excited and scattering states that contribute to the spectral function. While this ansatz may be justified in cases where the low-energy part of the spectral function is dominated by a single pole and the continuum states become important only at higher energies (the ρ-meson channel is such a case), it is not at all clear if it is also valid in other cases.

These features indicate that the Maximum Entropy Method (MEM) might be suitable for the investigation of QCD sum rules, because MEM makes it possible to obtain the most probable spectral function directly from the OPE data and their uncertainties, without having to use a priori assumptions about its explicit form. To test whether this is really true, we have...
investigated the vector meson channel as a first exploratory study. Our findings are summarized in these proceedings, while the details are given elsewhere [3].

2. Formalism

2.1. QCD sum rules

The QCD sum rule approach [1, 2] is based on the idea to make use of the analytic properties of the correlator of a general operator $J(x)$:

$$\Pi(q^2) = i \int d^4 x e^{iqx} \langle 0 | T[J(x)J^\dagger(0)] | 0 \rangle. \quad (1)$$

This expression is rewritten as a dispersion relation, which connects the imaginary part of $\Pi(q^2)$ with its values in the deep euclidean region, where it is possible to systematically carry out the operator product expansion. The dispersion relation can be given as:

$$\Pi(q^2) = \frac{1}{\pi} \int_0^\infty ds \frac{\text{Im}\Pi(s)}{s - q^2}. \quad (2)$$

After having calculated the OPE of $\Pi(q^2)$ in the deep euclidean region, the Borel transformation, symbolically denoted as $\hat{L}_M$, is applied to both sides of Eq.(2), giving the following expression for $G_{OPE}(M) \equiv \hat{L}_M[\Pi^{OPE}(q^2)]$, which depends only on $M$, the so-called Borel mass:

$$G_{OPE}(M) = \frac{2}{M^2} \int_0^\infty d\omega e^{-\omega^2/M^2} \omega \rho(\omega). \quad (3)$$

Here, $s = \omega^2$ and Im$\Pi(s) \equiv \pi \rho(\omega)$ have been used. As a next step, it is common to make some assumptions about the explicit from of the spectral function $\rho(\omega)$. The simplest and most popular choice is the “pole + continuum” ansatz, where one introduces a $\delta$-function for the state that dominates the spectral function in the low-energy region and parametrizes the higher-energy continuum states in terms of the expression obtained from the OPE. One of the main purposes of the approach introduced in this paper is to abandon this sort of assumption. Instead, we will start from the more general expression of Eq.(3) and employ the Maximum Entropy Method to directly extract the spectral function from this equation.

2.2. The Maximum Entropy Method (MEM)

In this subsection, the main ideas of the MEM method are briefly reviewed. For more details, see [3, 4, 5].

The kind of problem that one aims to solve with the help of MEM is the following. One is interested in some function $\rho(\omega)$, but has only information about an integral of $\rho(\omega)$:

$$G_{OPE}(M) = \int_0^\infty d\omega K(M, \omega) \rho(\omega), \quad (4)$$

where $K(M, \omega)$ is the kernel and corresponds to

$$K(M, \omega) = \frac{2\omega}{M^2} e^{-\omega^2/M^2} \quad (5)$$

in the case of QCD sum rules. If $G_{OPE}(M)$ is known only with limited accuracy or is only calculable in a limited range of the Borel mass $M$, the problem of obtaining $\rho(\omega)$ from $G_{OPE}(M)$ is ill-posed and will not be analytically solvable.
The MEM approach now uses Bayes’ theorem, by which additional information about \( \rho(\omega) \) such as positivity and/or its asymptotic behavior at small or large energies can be included into the analysis in a systematic way and by which one can finally deduce the most probable from of \( \rho(\omega) \). Bayes’ theorem can be expressed as

\[
P[\rho|GH] = \frac{P[G|\rho H]P[\rho|H]}{P[G|H]},
\]

where the prior knowledge about \( \rho(\omega) \) is denoted as \( H \) and \( P[\rho|GH] \) stands for the conditional probability of \( \rho(\omega) \) given \( G_{OPE}(M) \) and \( H \). Dropping the constant term \( P[G|H] \) in the denominator because it does not depend on \( \rho(\omega) \) and maximizing the remaining functional will give the most probable \( \rho(\omega) \). \( P[G|\rho H] \) is the “likelihood function” and can be constructed as

\[
P[G|\rho H] = e^{-L[\rho]},
\]

\[
L[\rho] = \frac{1}{2(M_{\text{max}} - M_{\text{min}})} \int_{M_{\text{min}}}^{M_{\text{max}}} dM \frac{[G_{OPE}(M) - G_{\rho}(M)]^2}{\sigma^2(M)}.
\]

\( G_{OPE}(M) \) is obtained from the OPE of the two-point function and \( G_{\rho}(M) \) is defined as on the right hand side of Eq.(4), and therefore implicitly depends on \( \rho(\omega) \). The function \( \sigma(M) \) describes the uncertainty of \( G_{OPE}(M) \) at the corresponding value of the Borel mass.

On the other hand, \( P[\rho|H] \) is called the “prior probability” and is given as

\[
P[\rho|H] = e^{\alpha S[\rho]},
\]

\[
S[\rho] = \int_0^\infty d\omega \left[ \rho(\omega) - m(\omega) - \rho(\omega) \log \left( \frac{\rho(\omega)}{m(\omega)} \right) \right],
\]

where \( S[\rho] \) is called the Shannon-Jaynes entropy. The function \( m(\omega) \), introduced in Eq.(8) is the “default model”. In the case of no available data \( G_{OPE}(M) \), the MEM procedure will just give \( m(\omega) \) for \( \rho(\omega) \) because this function maximizes \( P[\rho|H] \). The default model can be used to incorporate known information about \( \rho(\omega) \) into the calculation.

Assembling the functions given above, we get the final form for the probability \( P[\rho|GH] \):

\[
P[\rho|GH] \propto P[G|\rho H]P[\rho|H] = e^Q[\rho],
\]

\[
Q[\rho] \equiv \alpha S[\rho] - L[\rho].
\]

At this point, it is only a numerical problem to obtain the form of \( \rho(\omega) \) that maximizes \( Q[\rho] \) and is the most probable \( \rho(\omega) \) given \( G_{OPE}(M) \) and \( H \). For this, we use the Bryan algorithm [6], which is widely used in MEM studies.

Once we have found \( \rho_\alpha(\omega) \) which maximizes \( Q[\rho] \) for a fixed value of \( \alpha \), this parameter is integrated out by averaging \( \rho(\omega) \) over a range of values of \( \alpha \). The explicit procedure for this step is explained in [3]. The result this last integration then leads to our final result \( \rho_{out}(\omega) \).

3. Obtaining a suitable default model

The default model \( m(\omega) \) enters the MEM analysis through the Shannon-Jaynes entropy of Eq.(8). There is in principle no general criterion, according to which the default model has to be chosen and we can thus employ a \( m(\omega) \) that serves best for our purposes of accurately describing the
Figure 1. The results of the MEM investigation of mock data with various default models. The solid lines stand for the output of the analysis \( \rho(\omega)_{\text{out}} \), the long-dashed lines for the default model with the parameters shown in the figure, and the short-dashed lines for the input function \( \rho(\omega)_{\text{in}} \) of Eq. (10). The horizontal bars show the values of the spectral function, averaged over the peaks and the corresponding ranges. For figures c), d) and e), the lower error bars of the second peak are not shown because they lie below \( \rho(\omega) = 0 \).

The spectral function in the low-energy region. To find such a default model, we analyze mock data with realistic errors, obtained from a phenomenological model spectral function as shown below:

\[
\rho_{\text{in}}(\omega) = \frac{2F^2}{\pi} \frac{\Gamma_\rho m_\rho}{(\omega^2 - m_\rho^2)^2 + \Gamma_\rho^2 m_\rho^2} + \frac{1}{4\pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) \frac{1}{1 + e^{(\omega_0 - \omega)/\delta}},
\]

\[
\Gamma_\rho(\omega) = \frac{g_{\rho\pi\pi}}{48\pi} m_\rho \left( 1 - \frac{4m_\pi^2}{\omega^2} \right)^{3/2} \theta(\omega - 2m_\pi).
\]

(10)

The concrete values used for the various parameters are given in [3].

As a result of this analysis, we have found that it is important to choose a default model that has a value close to zero at low energy and at the same time approaches the asymptotic value \( 1 + \frac{\alpha_s}{\pi} \) at high energies. For this purpose, we have chosen the following form for the default model:

\[
m(\omega) = \frac{1}{4\pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) \frac{1}{1 + e^{(\omega_0 - \omega)/\delta}}.
\]

(11)

We have tried various combinations of \( \omega_0 \) and \( \delta \) and found that the values of \( \omega_0 = 2.0 \) GeV and \( \delta = 0.1 \) GeV give the most satisfying results. This is illustrated in Fig. 1.

It is important to understand the meaning of the default model \( m(\omega) \) of Eq. (11) in the present calculation. As mentioned above, our results show that the default model should behave according to the asymptotic values of the spectral function at low and high energy. In fact,
Here, \( \alpha \) is the (averaged) quark mass of the u- and d-quark, and \( \langle \bar{q}q \rangle \) represents the corresponding quark condensate. The gluon condensate \( \langle \frac{\alpha_s}{\pi}G^2 \rangle \) is an abbreviated expression for \( \langle \frac{\alpha_s}{\pi}G_{\mu \nu}G^{\mu \nu} \rangle \) and \( \kappa \) parametrizes the breaking of the vacuum saturation approximation, which was employed to calculate the above result for \( \kappa = 1 \). Various estimates of the values of the condensates and their ranges exist, and we use the ones given in three recent publications: [7, 8, 9]. The explicit values are given in Table 1. Concerning value of \( m(\bar{q}q) \), which does not play an important role in the present sum rule, we make use of the Gell-Mann-Oakes-Renner relation, which gives \( m(\bar{q}q) = -\frac{1}{2}m^2 + \frac{1}{\pi}f^2_\pi \) and take the experimental values of \( m_\pi \) and \( f_\pi \) for all three cases, leading to

\[
m(\bar{q}q) = -8.5 \times 10^{-5} \text{ GeV}^4.
\]

Using the three different parameter sets of Table 1, we have carried out the MEM analysis of \( G_{\text{OPE}}(M) \) with the default model of Eq.(11). The results are shown in Fig. 2. It clearly seen that all three data sets give a significant lowest peak, corresponding to the \( \rho \)-meson resonance. Next, fitting the spectral functions of Fig. 2 to a relativistic Breit-Wigner peak plus a second order polynomial background, we determine the coupling strength \( F_\rho \) from our obtained spectral function. All these results are summarized in Table 2.
Figure 2. The results of the MEM analysis using the OPE data. The dashed lines show the default model, which we have used and the horizontal bars stand for the values of the spectral function, averaged over the peaks and the corresponding ranges. For the two figures on the right, the lower error bars of the second peak are not shown because they lie below \( \rho(\omega) = 0 \).

Table 2. The final results for the three parameter sets. The corresponding errors were determined from our analysis of mock data.

|                  | Colangelo et al. [7] | Narison [8] | Ioffe [9] | Experiment |
|------------------|----------------------|-------------|-----------|------------|
| \( m_\rho \)     | 0.76 ± 0.07 GeV      | 0.84 ± 0.12 GeV | 0.84 ± 0.10 GeV | 0.77 GeV |
| \( F_\rho \)     | 0.174 ± 0.039 GeV    | 0.190 ± 0.053 GeV | 0.187 ± 0.050 GeV | 0.141 GeV |

5. Summary and Conclusion
The MEM technique has been applied to QCD sum rules and we have presented first results of our exploratory study of the \( \rho \)-meson channel in the present proceedings. The advantage of the MEM approach is that one is not forced to make assumptions about the explicit form of the spectral function, which has always been a necessity in former studies of QCD sum rules.

Fixing the limiting values of the spectral function at low and high energy with the help of the default model and carrying out the MEM analysis, we obtain the most probable form of the spectral function, given the OPE data from \( G_{OPE}(M) \). The results as shown in Fig. 2 and in Table 2 imply that the method works well in this channel and that we are able to reproduce the experimental \( \rho \)-meson mass \( m_\rho \) with a precision of about 10 % and the coupling strength \( F_\rho \) with a precision of about 30 %.

These promising findings encourage us to expand this method to other channels and to the investigation of the properties of hadrons at finite temperature or density. Such calculations are currently in progress.

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