Gaussian free fields coupled with multiple SLEs driven by stochastic log-gases

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Abstract

Miller and Sheffield introduced the notion of an imaginary surface as an equivalence class of pairs of simply connected proper subdomains of $\mathbb{C}$ and Gaussian free fields (GFFs) on them under the conformal equivalence. They considered the situation in which the conformal maps are given by a chordal Schramm–Loewner evolution (SLE). In the present paper, we construct GFF-valued processes on $\mathbb{H}$ (the upper half-plane) and $\mathbb{D}$ (the first orthant of $\mathbb{C}$) by coupling a GFF with a multiple SLE evolving in time on each domain. We prove that a GFF on $\mathbb{H}$ and $\mathbb{D}$ is locally coupled with a multiple SLE if the multiple SLE is driven by the stochastic log-gas called the Dyson model defined on $\mathbb{R}$ and the Bru–Wishart process defined on $\mathbb{R}_+$, respectively. We obtain pairs of time-evolutionary domains and GFF-valued processes.

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1 Introduction

The present study is motivated by the recent work by Sheffield on the quantum gravity zipper and the AC geometry [32] and a series of papers by Miller and Sheffield on the imaginary geometry [25, 26, 27, 28]. In both of them, a Gaussian free field (GFF) on a simply connected proper subdomain $D$ of the complex plane $\mathbb{C}$ (see, for instance, [31]) is coupled with a Schramm–Loewner evolution (SLE) [30, 24, 23] driven by a Brownian motion moving on the boundary $\partial D$, or its variant called an SLE($\kappa, \rho$).

Consider a simply connected domain $D \subset \mathbb{C}$ and write $C_c^\infty(D)$ for the space of real smooth functions on $D$ with compact support. Assume $h \in C_c^\infty(D)$ and consider a smooth vector field $e^{\sqrt{-1} (h/\chi + \theta)}$ with parameters $\chi, \theta \in \mathbb{R}$. Then, the flow line along this vector field, $\eta: (0, \infty) \ni t \mapsto \star$
\( \eta(t) \in D \), starting from \( \lim_{t \to 0} \eta(t) =: \eta(0) = x \in \partial D \) is defined (if exists) as the solution of the ordinary differential equation (ODE)

\[
\frac{d\eta(t)}{dt} = e^{\sqrt{-1}(\chi(\eta(t))/\chi + \theta)}, \quad t \geq 0, \quad \eta(0) = x.
\]

Let \( \tilde{D} \subseteq \mathbb{C} \) be another simply connected domain and consider a conformal map \( \varphi : \tilde{D} \to D \). Then, we define the pull-back of the flow line \( \eta \) by \( \tilde{\eta}(t) = (\varphi^{-1} \circ \eta)(t) \). That is, \( \varphi(\tilde{\eta}(t)) = \eta(t) \), and the derivatives with respect to \( t \) of both sides of this equation gives \( \varphi'(\tilde{\eta}(t))d\tilde{\eta}(t)/dt = d\eta(t)/dt \) with \( \varphi'(z) := d\varphi(z)/dz \). We use the polar coordinates \( \varphi'(s) = |\varphi'(s)| e^{\sqrt{-1}\arg \varphi'(s)} \), where \( \arg \zeta \) of \( \zeta \in \mathbb{C} \) is a priori defined up to additive multiples of \( 2\pi \), and hence, we have

\[
\frac{d\tilde{\eta}(t)}{dt} = e^{\sqrt{-1}(\varphi^{-1}(\chi \arg \varphi')(\tilde{\eta}(t))/\chi + \theta)}, \quad t \geq 0.
\]

Since a time change preserves the image of a flow line, we can identify \( h \) on \( D \) and \( h \circ \varphi - \chi \arg \varphi' \) on \( \tilde{D} = \varphi^{-1}(D) \). In \cite{32, 25, 26, 27, 28}, such a flow line is considered also in the case that \( h \) is given by an instance of a GFF defined as follows.

**Definition 1.1** Let \( D \subseteq \mathbb{C} \) be a simply connected domain and \( H \) be a GFF on \( D \) with zero boundary condition (constructed in Section 4). A GFF on \( D \) is a random distribution \( h \) of the form \( h = H + u \), where \( u \) is a deterministic harmonic function on \( D \).

Since a GFF is not function-valued, but it is a *distribution-valued random field* (see Remark 4.1 in Section 4), the ODE in the form (1.1) no longer makes sense mathematically in the classical sense. Using the theory of SLE, however, the notion of flow lines was generalized as follows.

Consider the collection

\[
S := \left\{ (D, h) \Bigg| \begin{array}{c} D \subseteq \mathbb{C} : \text{simply connected} \\ h : \text{GFF on } D \end{array} \right\}.
\]

Fixing a parameter \( \chi \in \mathbb{R} \), we define the following equivalence relation in \( S \).

**Definition 1.2** Two pairs \( (D, h) \) and \( (\tilde{D}, \tilde{h}) \) \( \in S \) are equivalent if there exists a conformal map \( \varphi : \tilde{D} \to D \) and \( h = (\text{law}) h \circ \varphi - \chi \arg \varphi' \). In this case, we write \( (D, h) \sim (\tilde{D}, \tilde{h}) \).

We call each element belonging to \( S/ \sim \) an *imaginary surface* \cite{25} (or an *AC surface* \cite{32}). That is, in this equivalence class, a conformal map \( \varphi \) causes not only a coordinate change of a GFF as \( h \mapsto h \circ \varphi \) associated with changing the domain of definition of the field as \( D \mapsto \varphi^{-1}(D) \), but also an addition of a deterministic harmonic function \( -\chi \arg \varphi' \) to the field. Notice that this definition depends on one parameter \( \chi \in \mathbb{R} \).

As will be explained in Section 4, each instance \( H \) of a GFF with zero boundary condition depends on the choice of a complete orthonormal system (CONS) of a Hilbert space starting from which a GFF is constructed. The probability law of a zero-boundary GFF is, however, independent of such construction and uniquely determined.

Consider the case in which \( D \) is the upper half-plane \( \mathbb{H} := \{ z \in \mathbb{C} : \Im z > 0 \} \) with \( \partial \mathbb{H} = \mathbb{R} \cup \{ \infty \} \). Let \( (B(t))_{t \geq 0} \) be a one-dimensional standard Brownian motion starting from the origin
defined on a probability space \((\Omega^{1\text{SLE}}, \mathcal{F}^{1\text{SLE}}, \mathbb{P}^{1\text{SLE}})\) and adapted to a filtration \((\mathcal{F}_{t}^{1\text{SLE}})_{t \geq 0}\). We consider the chordal SLE(\(\kappa\)) driven by \((\sqrt{\kappa}B(t))_{t \geq 0}\) on \(S := \mathbb{R} \) \(\kappa > 0\) \([30, 24, 23]\), associated to which we obtain a random curve (called a chordal SLE(\(\kappa\)) curve) parameterized by time, \(\eta: (0, \infty) \ni t \mapsto \eta(t) \in \mathbb{H}\), such that \(\lim_{t \to 0} \eta(t) = \eta(0) = 0\), \(\lim_{t \to \infty} \eta(t) = \infty\). At each time \(t > 0\), let \(\eta(0, t) := \{\eta(s) : s \in (0, t]\}\) and write \(\mathbb{H}^\eta\) for the unbounded component of \(\mathbb{H} \setminus \eta(0, t]\). Then, the chordal SLE(\(\kappa\)) gives a conformal map from \(\mathbb{H}^\eta\) to \(\mathbb{H}\). It is also known that, if \(\kappa \in (0, 4]\), then \(\eta(0, t]\) is almost surely a simple curve at each \(t > 0\) and, hence, \(\mathbb{H}^\eta = \mathbb{H} \setminus \eta(0, t]\). In this paper, we will write the chordal SLE(\(\kappa\)) as \((g_{\mathbb{H}^\eta})_{t \geq 0}\). Let \(H(\cdot)\) be a GFF on \(\mathbb{H}\) with zero boundary condition on \(\mathbb{R}\) that is defined on a probability space \((\Omega^\text{GFF}, \mathcal{F}^\text{GFF}, \mathbb{P}^\text{GFF})\). To couple the SLE and the GFF, we introduce a probability space \((\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^\text{GFF} \times \Omega^{1\text{SLE}}, \mathcal{F}^\text{GFF} \vee \mathcal{F}^{1\text{SLE}}, \mathbb{P}^\text{GFF} \otimes \mathbb{P}^{1\text{SLE}})\) and extend the SLE and the GFF onto this probability space. Then, the SLE is adapted to the filtration \((\mathcal{F}_{t})_{t \geq 0}\) defined by \(\mathcal{F}_{t} = \emptyset, \mathbb{P}^{\text{GFF}}\} \vee \mathcal{F}_{t}^{1\text{SLE}}\). Instead of \(H(\cdot)\) itself, we consider the following GFF on \(\mathbb{H}\) by adding a deterministic harmonic function,

\[
h(\cdot) := H(\cdot) - \frac{2}{\sqrt{\kappa}} \arg(\cdot).
\]

Notice that arg(\(\cdot\)) = Im log(\(\cdot\)) and the real and imaginary parts of a complex analytic function are harmonic. Hence, the random distribution \([1,2]\) is in fact a GFF in the sense of Definition \([1,1]\). Given \(\kappa > 0\) for the SLE(\(\kappa\)), we fix the parameter \(\chi\) as \(\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2\). Note that the well-known relation between \(\kappa\) and the central charge \(c\) of conformal field theory is simply expressed using the present parameter \(\chi\) as \(c = 1 - 6\chi^{2}\) (see, for instance, \([3, \text{Eq.}(6)]\)). Let \(f_{\mathbb{H}^\eta} := g_{\mathbb{H}^\eta} - \sqrt{\kappa}B(t) = \sigma_{-\sqrt{\kappa}B(t)} \circ g_{\mathbb{H}^\eta}\), where \(\sigma_{s}\) denotes the translation by \(s \in \mathbb{R}\); \(\sigma_{s}(z) = z + s, z \in \mathbb{H}\). Let \(A \subset \mathbb{H}\) be an open set and take an \((\mathcal{F}_{t})_{t \geq 0}\)-stopping time

\[
\tau_{A} := \inf\left\{ t \geq 0 \mid \eta(0, t] \cap A \neq \emptyset \right\}.
\]

Let \(\tau\) be any \((\mathcal{F}_{t})_{t \geq 0}\)-stopping time such that \(\tau \leq \tau_{A}\) a.s. Then, we can prove the following equality in probability \([23, \text{Theorem 1.1, Lemma 3.11}]\) (see also \([10, \text{Lemma 6.1}]\)); for any \(f \in \mathcal{C}_{c}^{\infty}(\mathbb{H})\) such that \(\text{supp}(f) \subset A\),

\[
(h, f) \overset{\text{(law)}}{=} (h \circ f_{\mathbb{H}^\eta} - \chi \arg f_{\mathbb{H}^\eta}^{\prime}, f) \quad \text{under } \mathbb{P},
\]

where the pairing \((\cdot, \cdot)\) is defined by \([1,3]\) below. We comment that, due to the conformal invariance of a zero-boundary GFF (see Section \([4,2]\) below), for an instance of the SLE(\(\kappa\)), the random distribution \(h \circ f_{\mathbb{H}^\eta} - \chi \arg f_{\mathbb{H}^\eta}^{\prime}\) is a GFF on \(\mathbb{H}^\eta\) in the sense of Definition \([1,1]\). Notice that pairs \((\mathbb{H}, h)\) and \((\mathbb{H}^\eta, h \circ f_{\mathbb{H}^\eta} - \chi \arg f_{\mathbb{H}^\eta}^{\prime})\) with \([1,2]\) are equivalent in the sense of Definition \([1,2]\). In other words, an imaginary surface whose representative is given by \((\mathbb{H}, h)\) is constructed as a pair of a time-evolutionary domain, \(f_{\mathbb{H}^\eta}^{-1}(\mathbb{H}) = \mathbb{H}^\eta, t \geq 0\), and a GFF-valued process, \(h \circ f_{\mathbb{H}^\eta} - \chi \arg f_{\mathbb{H}^\eta}^{\prime}, t \geq 0\) defined on it. With the establishment of the equality \([1,3]\), we say that the local coupling between a GFF and an SLE is constructed (see \([10, 22, 25]\) for lifting the local coupling to the ‘global’ one). It was proved \([10, 25]\) that, under the coupling between a GFF and an SLE, the SLE-curve is a deterministic functional of the GFF. By virtue of it, in \([25]\), the authors referred to an SLE(\(\kappa\)) curve as a flow line of the GFF \(h\).
Here, first we consider the case in which the conformal maps are generated by a multiple Loewner equation associated with a multi-slit. Let $N \in \mathbb{N} := \{1, 2, \ldots \}$ and suppose that we have $N$ slits $\eta_i = \{\eta_i(t) : t \in (0, \infty)\} \subset \mathbb{H}$, $1 \leq i \leq N$, which are simple curves, disjoint with each other, $\eta_i \cap \eta_j = \emptyset$, $i \neq j$, starting from $N$ distinct points $\lim_{t \to 0} \eta_i(t) = \eta_i(0)$ on $\mathbb{R}$; $\eta_1(0) < \cdots < \eta_N(0)$, and all going to infinity; $\lim_{t \to \infty} \eta_i(t) = \infty$, $1 \leq i \leq N$. A multi-slit is defined as the union of them, $\bigcup_{i=1}^{N} \eta_i$, and

$$\mathbb{H}_t^\eta := \mathbb{H} \setminus \bigcup_{i=1}^{N} \eta_i(0, t) \text{ for each } t > 0 \text{ with } \mathbb{H}_0^\eta := \mathbb{H}.$$  

We write the time evolution of the conformal map which transforms $\mathbb{H}_t^\eta$ to $\mathbb{H}$ at each time $t \geq 0$ under the hydrodynamic normalization as $(g_{\mathbb{H}_t^\eta})_{t \geq 0}$ and call it a multiple SLE. The images of the tips of the multi-slit $g_{\mathbb{H}_t^\eta}(\eta_i(t))$, $1 \leq i \leq N$ exist as points on $\mathbb{R}$ for $t \geq 0$ and if we put $X_t^R := g_{\mathbb{H}_t^\eta}(\eta_i(t))$, the multiple SLE $(g_{\mathbb{H}_t^\eta})_{t \geq 0}$ is given as a unique solution of the following equation,

$$\frac{dg_{\mathbb{H}_t^\eta}(z)}{dt} = \frac{2}{\sum_{i=1}^{N} g_{\mathbb{H}_t^\eta}(z) - X_t^R(i)}, \quad t \geq 0,$$

under a proper parameterization of the multi-slit. Here

$$X_t^R = (X_1^R(t), \ldots, X_N^R(t)) \in \mathbb{R}^N, \quad t \geq 0$$

is called the driving process of the multiple SLE.

In the sequel, we will consider the case when $(X_t^R(t))_{t \geq 0}$ is a stochastic process defined on a probability space $(\Omega_{\text{NSLE}}, \mathcal{F}_{\text{NSLE}}, \mathbb{P}_{\text{NSLE}})$ and adapted to a filtration $(\mathcal{F}_t^\text{NSLE})_{t \geq 0}$. In this case, although it is not ensured that the solution generates a multi-slit depending on $\kappa$, we can still find a family of domains $\mathbb{H}_t^\eta \subset \mathbb{H}$, $t \geq 0$ so that $g_{\mathbb{H}_t^\eta} : \mathbb{H}_t^\eta \to \mathbb{H}$ is a conformal map at each $t \geq 0$. (See Remark 1.4 (1) in [19].)

We again consider a zero-boundary GFF $H$ on $\mathbb{H}$ defined on a probability space $(\Omega_{\text{GFF}}, \mathcal{F}_{\text{GFF}}, \mathbb{P}_{\text{GFF}})$ and introduce a coupled probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_{\text{GFF}} \times \Omega_{\text{NSLE}}, \mathcal{F}_{\text{GFF}} \vee \mathcal{F}_{\text{NSLE}}, \mathbb{P}_{\text{GFF}} \otimes \mathbb{P}_{\text{NSLE}}).$$

Then, the multiple SLE and the GFF are naturally extended to $(\Omega, \mathcal{F}, \mathbb{P})$, and the multiple SLE is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ defined by $\mathcal{F}_t = \{\emptyset, \Omega_{\text{GFF}}\} \vee \mathcal{F}_{\text{NSLE}}$.

Regarding (1.2) and (1.3), we see that $h \circ f_{\mathbb{H}_t^\eta}^\eta(\cdot) - \chi \arg f_{\mathbb{H}_t^\eta}^\eta(\cdot)$ is equal to

$$\left( H \circ \sigma_{-\sqrt{\kappa}B(t)} \right) \circ g_{\mathbb{H}_t^\eta}(\cdot) - \frac{2}{\sqrt{\kappa}} \arg \left( g_{\mathbb{H}_t^\eta}(\cdot) - \sqrt{\kappa}B(t) \right) - \chi \arg g_{\mathbb{H}_t^\eta}(\cdot)$$

$$= H \circ g_{\mathbb{H}_t^\eta}(\cdot) - \frac{2}{\sqrt{\kappa}} \arg \left( g_{\mathbb{H}_t^\eta}(\cdot) - g_{\mathbb{H}_t^\eta}(\eta(t)) \right) - \chi \arg g_{\mathbb{H}_t^\eta}(\cdot)$$

under $\mathbb{P}$.
t ≥ 0, where the translation invariance of H was used. Motivated by this observation, we study the GFF-valued process defined by

\[ H_\mathbb{H}(\cdot, t) := H \circ g_{\mathbb{H}}(\cdot) \]

\[ - \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg \left( g_{\mathbb{H}}(\cdot) - g_{\mathbb{H}}(\eta(t)) \right) - \chi \arg g'_{\mathbb{H}}(\cdot) \]

\[ = H \circ g_{\mathbb{H}}(\cdot) \]

\[ - \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg \left( g_{\mathbb{H}}(\cdot) - X_i^\mathbb{R}(t) \right) - \chi \arg g'_{\mathbb{H}}(\cdot) \]

on \( \mathbb{H}_i^\mathbb{R}, t ≥ 0 \). This process starts from

\[ H_\mathbb{H}(\cdot, 0) = H(\cdot) - \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg (\cdot - x_i^\mathbb{R}) , \]

where we assume that \( x_1^\mathbb{R} < \cdots < x_N^\mathbb{R} \). We let the boundary points evolve according to the stochastic process \( (X_i^\mathbb{R}(t))_{t≥0} \) starting from \( x_i^\mathbb{R} := (x_i^\mathbb{R})_{i=1}^{N} \). At each time \( t > 0 \), we consider the GFF \( H + u_t \) on \( \mathbb{H} \) where \( u_t(\cdot) = - (2/\sqrt{\kappa}) \sum_{i=1}^{N} \arg (\cdot - X_i^\mathbb{R}(t)) \). Then, the GFF \( H_\mathbb{H}(\cdot, t) \) on \( \mathbb{H}_i^\mathbb{R} \) is defined by the property that \( (\mathbb{H}, H + u_t) \sim (\mathbb{H}_i^\mathbb{R}, H_\mathbb{H}(\cdot, t)) \) in the sense of Definition 1.2.

A part of the main theorem in this paper (Theorem 5.4) is stated as follows.

**Theorem 1.3** Let \( A \subset \mathbb{H} \) be an open subset and take an \((\mathcal{F}_t)_{t≥0}\)-stopping time

\[ \tau_A := \inf \left\{ t ≥ 0 \mid A \not\subseteq \mathbb{H}_i^\mathbb{R} \right\} . \]

Let \( \tau \) be any \((\mathcal{F}_t)_{t≥0}\)-stopping time such that \( \tau ≤ \tau_A \) a.s. Then, for any \( f \in \mathcal{C}_c^{\infty}(\mathbb{H}) \) such that \( \text{supp}(f) \subset A \),

\[ (H_\mathbb{H}(\cdot, 0), f) \overset{(\text{law})}{=} (H_\mathbb{H}(\cdot, \tau), f) \quad \text{under } \mathbb{P}, \]

if the driving process \( (X_i^\mathbb{R}(t))_{t≥0} \) is equal to the time changed version \( Y_i^\mathbb{R}(t) = (Y_1^\mathbb{R}(t), \ldots, Y_N^\mathbb{R}(t)) \), \( t ≥ 0 \) of the Dyson model on \( \mathbb{R} \) which solves the following system of stochastic differential equations (SDEs) with \( \kappa > 0 \),

\[ dY_i^\mathbb{R}(t) = \sqrt{\kappa} dB_i(t) + 4 \sum_{1 ≤ j ≤ N, j \neq i} \frac{dt}{Y_i^\mathbb{R}(t) - Y_j^\mathbb{R}(t)}, \]

\( t ≥ 0, 1 ≤ i ≤ N, \) where \( (B_i(t))_{t≥0} \) are mutually independent one-dimensional standard Brownian motions starting from \( B_i(0) = Y_i^\mathbb{R}(0) =: y_i^\mathbb{R} = \eta_i(0), 1 ≤ i ≤ N, \) satisfying \( y_1^\mathbb{R} ≤ \cdots < y_N^\mathbb{R} \).

The Dyson model [12] is one of the most studied stochastic log-gases in one dimension, which is a dynamical version of the one-parameter \( (\beta = 8/\kappa) \) extension of the Gaussian unitary ensemble (GUE) of point processes studied in random matrix theory [13, 17]. It is also known that the multiple SLE driven by \( (Y_i^\mathbb{R}(t))_{t≥0} \) is an example of multiple SLEs that is defined in terms of an SLE partition function [4, 15].
It is possible that, for a certain choice of the open set \( A \subset \mathbb{H} \) in Theorem 1.3, the stopping time \( \tau \) has to be \( \tau = 0 \). Let us see that we can take \( \tau > 0 \) a.s. for a generic choice of \( A \). For each \( 1 \leq i \leq N \), let \( U_i \) be a neighborhood of \( y_i^R \) in \( \mathbb{H} \) and suppose that \( U_i \cap U_j = \emptyset \) if \( i \neq j \). We call such \( U_i, 1 \leq i \leq N \) localization neighborhoods. Then, we define the first exit time of the SLE from the union of localization neighborhoods, \( U := \bigcup_{i=1}^N U_i \), by

\[
\tau_U = \inf \left\{ t \geq 0 \mid (\mathbb{H} \setminus U) \not\subset \mathbb{H}_t^0 \right\}.
\]

Note that we have \( \tau_U > 0 \) a.s. since, otherwise, it contradicts the fact that \( (Y^R(t))_{t \geq 0} \) is continuous in \( t \geq 0 \) [8] [14]. Notice that we can take the stopping time \( \tau \) in Theorem 1.3 in such a way that \( \tau \geq \tau_U \) for some localization neighborhoods that are disjoint from \( A \), if the closure of \( A \) in \( \mathbb{H} \) does not contain any of \( y_i^R, 1 \leq i \leq N \). Therefore, we can take it so that \( \tau > 0 \) a.s. in such a generic case.

As we have already pointed out, it is not clear if a solution of the multiple Loewner equation generates a multi-slit. In our subsequent paper [19], we will prove under the assumption \( 1 \leq U \) the union of localization neighborhoods, that, if the driving process is the time changed version \( Y \) case.

\[\{\eta_i\}_{i=1}^N \] is a multiple SLE.

The process \( (H_H(\cdot,t))_{t \geq 0} \) is a generalization of \( (h \circ f_{H^R} - \chi \arg f_{H^R})_{t \geq 0} \) considered by Miller and Sheffield [32] [23] as explained above. The equality [1.3] [10] [32] [25] has been extended to the equality [1.7] in Theorem 1.3 which we think of as the local coupling between a GFF and a multiple SLE.

We will also construct another GFF-valued process in the first orthant in \( C; \mathbb{O} := \{ z \in \mathbb{C} : \Re z > 0, \Im z > 0 \} \). There, a GFF on \( \mathbb{O} \), denoted as \( H_{\mathbb{O}}(\cdot,0) \) is locally coupled with a multiple version of the quadrant SLE [33] defined on \( \mathbb{O} \), which is driven by a stochastic log-gas defined on \( S = \mathbb{R}_+ := \{ x \in \mathbb{R} : x \geq 0 \} \). This driving process is a dynamical version of the one-parameter \( (\beta = 8/\kappa) \) extension of the chiral GUE of point processes with parameter \( \nu \in [0,\infty) \) studied in random matrix theory [20] [13], and we call it the Bru–Wishart process in this paper [34] [6]. We note that \( (\mathbb{H},H_{\mathbb{H}}(\cdot,0)) \sim (\mathbb{O},H_{\mathbb{O}}(\cdot,0)) \) in the sense of Definition 1.2.

Construction of such GFF-valued processes will be meaningful for the study of multiple SLEs. The main problem in defining a multiple SLE correctly in \( D \subset \mathbb{C} \) may be how to find a correct principle to choose a driving process \( (X^S(t))_{t \geq 0} \) defined on a part of the boundary \( S \subset \partial D \) (e.g., conformal invariance, statistical mechanics consideration, reparameterization invariance, absolute continuity to the SLE with a single slit, commutation relations) [7] [4] [22] [15] [9]. In the present paper, we simply assume the form of SDEs for \( (X^S(t))_{t \geq 0} \) as

\[
dX^S_i(t) = \sqrt{rdB_i}(t) + F^S_i(X^S(t))dt, \quad t \geq 0, \quad 1 \leq i \leq N,
\]

where \( (B_i(t))_{t \geq 0}, 1 \leq i \leq N \) are mutually independent one-dimensional standard Brownian motions, \( \kappa > 0 \), and \( F^S_i(x) \in C^\infty(S^N \setminus \bigcup_{j \neq k} \{ x_j = x_k \}) \), \( 1 \leq i \leq N \), which do not explicitly depend on \( t \). Then, the equality [1.7] for a GFF-valued process on \( D = \mathbb{H} \) determines the driving process \( (X^R(t))_{t \geq 0} \) as (a time change of) the Dyson model \( (Y^R(t))_{t \geq 0} \). That is, the local coupling between a GFF and a multiple SLE provides a new scheme to choose a driving process for a multiple SLE.

Notice again that \( \arg z \) in [1.2] is the imaginary part of the complex analytic function \( \log z \).
Sheffield studied another type of distribution-valued random field on $\mathbb{H}$ given by [32]

$$\tilde{h}(\cdot) := \tilde{H}(\cdot) + \frac{2}{\sqrt{\kappa}} \Re \log(\cdot) = \tilde{H}(\cdot) + \frac{2}{\sqrt{\kappa}} \log | \cdot |,$$

where $\tilde{H}(\cdot)$ is a free boundary GFF on $\mathbb{H}$ and found that $\tilde{h}(\cdot)$ is coupled with a backward SLE in the context of quantum gravity [11]. This coupling was later generalized in [18,21] to the situations where backward multiple SLEs driven by stochastic log-gases play analogous roles as multiple SLEs did in the present work.

The present paper is organized as follows. We give brief reviews of stochastic log-gases in one dimension in Section 2 and the SLE both for a single-slit and a multi-slit in Section 3. In Section 4 we define a GFF with zero boundary condition on $D \subset \mathbb{C}$ based on the Bochner–Minlos theorem. The construction of GFF-valued processes by locally coupling GFFs with multiple SLEs driven by specified stochastic log-gases on $S$ are given in Section 5 for $(D,S) = (\mathbb{H},\mathbb{R})$ and $(\mathbb{O},\mathbb{R}_+)$.

2 One-dimensional Stochastic Log-Gases

2.1 Eigenvalue and singular-value processes

For $N \in \mathbb{N}$, let $H_N$ and $U_N$ be the space of $N \times N$ Hermitian matrices and the group of $N \times N$ unitary matrices, respectively. Consider complex-valued processes $(M_{ij}(t))_{t \geq 0}, 1 \leq i,j \leq N$ with the condition $M_{ji}(t) = M_{ij}(t)$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We consider an $H_N$-valued process by $M(t) = M_{ij}(t)_{1 \leq i,j \leq N}$. For $S = \mathbb{R}$ and $\mathbb{R}_+$, define the Weyl chambers as $W_N(S) := \{x = (x_1, \ldots, x_N) \in S^N : x_1 < \cdots < x_N\}$, and write their closures as $\overline{W}_N(S) = \{x \in S^N : x_1 \leq \cdots \leq x_N\}$. For each $t \geq 0$, there exists $U(t) = (U_{ij}(t))_{1 \leq i,j \leq N} \in U_N$ such that it diagonalizes $M(t)$ as $U^\dagger(t)M(t)U(t) = \text{diag}(\Lambda_1(t), \ldots, \Lambda_N(t))$ with the eigenvalues $\{\Lambda_i(t)\}_{i=1}^N$ of $M(t)$, where $U^\dagger(t)$ is the Hermitian conjugate of $U(t)$; $U_{ij}^\dagger(t) = \overline{U_{ji}(t)}, 1 \leq i,j \leq N$, and we assume $\Delta(t) := (\Lambda_1(t), \ldots, \Lambda_N(t)) \in \overline{W}_N(\mathbb{R}), t \geq 0$. For $dM(t) := (dM_{ij}(t))_{1 \leq i,j \leq N}$, define a set of quadratic variations,

$$\Gamma_{ij,kl}(t) := \left\langle (U^\dagger dMU)_{ij}, (U^\dagger dMU)_{kl} \right\rangle_t, \quad 1 \leq i,j,k,l \leq N, \quad t \geq 0.$$ 

We write $1_E$ for the indicator function of an event $E$; $1_E = 1$ if $E$ occurs, and $1_E = 0$ otherwise. The following is proved [5,20,17]. See Section 4.3 of [1] for details of proof.

**Proposition 2.1** Assume that $(M_{ij}(t))_{t \geq 0}, 1 \leq i,j \leq N$ are continuous semi-martingales. The eigenvalue process $(\Delta(t))_{t \geq 0}$ satisfies the following system of SDEs,

$$d\Lambda_i(t) = dM_i(t) + dJ_i(t), \quad t \geq 0, \quad 1 \leq i \leq N,$$

where $(\mathcal{M}_i(t))_{t \geq 0}, 1 \leq i \leq N$ are martingales with quadratic variations $(\mathcal{M}_i, \mathcal{M}_j)_t = \int_0^t \Gamma_{ii,jj}(s) ds$, and $(J_i(t))_{t \geq 0}, 1 \leq i \leq N$ are the processes with finite variations given by

$$dJ_i(t) = \sum_{j=1}^N \frac{1_{\Lambda_i(t) \neq \Lambda_j(t)}}{\Lambda_i(t) - \Lambda_j(t)} \Gamma_{ij,ii}(t) dt + d\Upsilon_i(t).$$

Here $d\Upsilon_i(t)$ denotes the finite-variation part of $(U^\dagger(t)dM(t)U(t))_{ii}, t \geq 0, 1 \leq i \leq N$. 

7
We will show two basic examples of \( M(t) \in \mathbb{H}_N, t \geq 0 \) and applications of Proposition 2.1. Let \( \nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \((B_{ij}(t))_{t \geq 0}, (\tilde{B}_{ij}(t))_{t \geq 0}, 1 \leq i \leq N + \nu, 1 \leq j \leq N \) be independent one-dimensional standard Brownian motions. For \( 1 \leq i \leq j \leq N \), put

\[
S_{ij}(t) = \begin{cases} 
B_{ij}(t)/\sqrt{2}, & (i < j), \\
B_{ii}(t), & (i = j), \\
\tilde{B}_{ij}(t)/\sqrt{2}, & (i = j), \\
0, & (i = j),
\end{cases}
\]

and let \( S_{ij}(t) = S_{ji}(t) \) and \( A_{ij}(t) = -A_{ji}(t), t \geq 0 \) for \( 1 \leq j < i \leq N \).

**Example 2.1** Put \( M_{ij}(t) = S_{ij}(t) + \sqrt{-1}A_{ij}(t), t \geq 0, 1 \leq i, j \leq N \). By definition, \((dM_{ij}, dM_{kl})_t = \delta_{il}\delta_{jk}dt, t \geq 0, 1 \leq i, j, k, l \leq N \). Hence, by unitarity of \( U(t), t \geq 0 \), we see that \( \Gamma_{ij,kl}(t) = \delta_{il}\delta_{jk} \), which gives \((dM_i, dM_j)_t = \Gamma_{ii, jj}(t)dt = \delta_{ij}dt \) and \( \Gamma_{ij, jj}(t) \equiv 1 \), \( t \geq 0, 1 \leq i, j \leq N \). Then, Proposition 2.1 proves that the eigenvalue process \((\Lambda(t))_{t \geq 0}\) satisfies the following system of SDEs with \( \beta = 2 \),

\[
d\Lambda_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{1 \leq j \leq N, j \neq i} \frac{dt}{\Lambda_i(t) - \Lambda_j(t)},
\]

\( t \geq 0, 1 \leq i \leq N \). Here, \((B_i(t))_{t \geq 0}, 1 \leq i \leq N \) are independent one-dimensional standard Brownian motions, which are different from \((B_{ij}(t))_{t \geq 0}\) and \((\tilde{B}_{ij}(t))_{t \geq 0}\) used to define \((S_{ij}(t))_{t \geq 0}\) and \((A_{ij}(t))_{t \geq 0}\), \( 1 \leq i, j \leq N \).

**Example 2.2** Consider an \((N + \nu) \times N\) rectangular-matrix-valued process given by \( K(t) = (B_{ij}(t) + \sqrt{-1}\tilde{B}_{ij}(t))_{1 \leq i \leq N + \nu, 1 \leq j \leq N}, t \geq 0 \), and define an \( \mathbb{H}_N\)-valued process by \( M(t) = K(t)K(t)\), \( t \geq 0 \). The matrix \( M(t) \) is positive semi-definite and hence the eigenvalues are non-negative; \( \Lambda_i(t) \in \mathbb{R}_+, t \geq 0, 1 \leq i \leq N \). We see that the finite-variation part of \( dM_{ij}(t) \) is equal to \( 2(N + \nu)\delta_{ij}dt \), \( t \geq 0 \), and

\[
\langle dM_i, dM_j \rangle_t = 2(M_{ii}(t)\delta_{jk} + M_{kj}(t)\delta_{ik})dt, t \geq 0, 1 \leq i, j, k, l \leq N,
\]

which implies that \( d\Gamma_i(t) = 2(N + \nu)dt \), \( \Gamma_{ij, jj}(t) = 2(\Lambda_i(t) + \Lambda_j(t)) \), and \( \langle dM_i, dM_j \rangle_t = \Gamma_{ii, jj}(t)dt = 4\Lambda_i(t)\delta_{ij}dt \), \( t \geq 0, 1 \leq i, j \leq N \). Then, we have the SDEs for eigenvalue processes

\[
d\Lambda_i(t) = 2\sqrt{\Lambda_i(t)}d\tilde{B}_i(t) + \beta \left( (\nu + 1) + 2\Lambda_i(t) \sum_{1 \leq j \leq N, j \neq i} \frac{1}{\Lambda_i(t) - \Lambda_j(t)} \right) dt,
\]

\( t \geq 0, 1 \leq i \leq N \) with \( \beta = 2 \), where \((\tilde{B}_i(t))_{t \geq 0}, 1 \leq i \leq N \) are independent one-dimensional standard Brownian motions, which are different from \((B_{ij}(t))_{t \geq 0}\) and \((\tilde{B}_{ij}(t))_{t \geq 0}, 1 \leq i, j \leq N \), used above to define the rectangular-matrix-valued process \((K(t))_{t \geq 0}\). The positive roots of eigenvalues of \( M(t) \) give the *singular values* of the rectangular matrix \( K(t) \), which are denoted by \( S_i(t) = \sqrt{\Lambda_i(t)}, t \geq 0, 1 \leq i \leq N \). The system of SDEs for them is readily obtained from (2.2) as

\[
dS_i(t) = d\tilde{B}_i(t) + \frac{\beta(\nu + 1) - 1}{2S_i(t)} dt + \frac{\beta}{2} \sum_{1 \leq j \leq N, j \neq i} \left( \frac{1}{S_i(t) - S_j(t)} + \frac{1}{S_i(t) + S_j(t)} \right) dt,
\]

\( t \geq 0, 1 \leq i \leq N \) with \( \beta = 2 \) and \( \nu \in \mathbb{N}_0 \).

Other examples of \( \mathbb{H}_N\)-valued processes \((M(t))_{t \geq 0}\) are shown in [20], in which the eigenvalue processes following the SDEs (2.1), (2.2), and (2.3) with \( \beta = 1 \) and 4 are also shown.
2.2 2D-Coulomb gases confined in 1D

In the next section, we will consider the SLE. Schramm used a parameter $\kappa > 0$ in order to parameterize time changes of a Brownian motion [30]. Accordingly, we relate the parameter $\beta$ to $\kappa$ by setting $\beta = 8/\kappa$, and perform a time change $t \rightarrow \kappa t$. Since $(B(\kappa t))_{t \geq 0} \overset{\text{(law)}}{=} (\sqrt{\kappa} B(t))_{t \geq 0}$, if we put $Y_i^R(t) := \Lambda_i(\kappa t), Y_i^{R+}(t) := \mathcal{S}_i(\kappa t), t \geq 0, 1 \leq i \leq N$, the system of SDEs (2.1) gives (1.8) and that of (2.3) gives

\begin{equation}
\begin{aligned}
dY_i^{R+}(t) &= \sqrt{\kappa d B_i(t)} + \frac{8(\nu + 1) - \kappa}{2Y_i^{R+}(t)} dt \\
 &+ 4 \sum_{1 \leq j \leq N, j \neq i} \left( \frac{1}{Y_i^{R+}(t) - Y_j^{R+}(t)} + \frac{1}{Y_i^{R+}(t) + Y_j^{R+}(t)} \right) dt,
\end{aligned}
\end{equation}

$t \geq 0, 1 \leq i \leq N$, where $\nu \geq 0$. In the present paper, we call $(Y_i^R(t))_{t \geq 0}$ the $(8/\kappa)$-Dyson model and $(Y_i^{R+}(t))_{t \geq 0}$ the $(8/\kappa, \nu)$-Bru–Wishart process, respectively. The above systems of SDEs for $(Y_i^S(t))_{t \geq 0}$ can be written as

\begin{equation}
\begin{aligned}
dY_i^S(t) &= \sqrt{\kappa d B_i(t)} + \left. \frac{\partial \phi^S(x)}{\partial x_i} \right|_{x = Y_i^S(t)} dt, \quad t \geq 0, \quad 1 \leq i \leq N,
\end{aligned}
\end{equation}

$S = \mathbb{R}$ or $\mathbb{R}_{+}$, when we introduce the following logarithmic potentials,

\begin{equation}
\begin{aligned}
\phi^S(x) := \begin{cases} 
4 \sum_{1 \leq i < j \leq N} \log(x_j - x_i), & \text{for } S = \mathbb{R}, \\
4 \sum_{1 \leq i < j \leq N} \left[ \log(x_j - x_i) + \log(x_j + x_i) \right] \\
+ \frac{8(\nu + 1) - \kappa}{2} \sum_{1 \leq i \leq N} \log x_i, & \text{for } S = \mathbb{R}_{+}.
\end{cases}
\end{aligned}
\end{equation}

In this sense, the $(8/\kappa)$-Dyson model and the $(8/\kappa, \nu)$-Bru–Wishart process are regarded as stochastic log-gases in one dimension [13]. Since the logarithmic potential describes the two-dimensional Coulomb law in electrostatics, the present processes are also considered as stochastic models of 2D-Coulomb gases confined in 1D.

3 Multiple Schramm–Loewner Evolution

3.1 Loewner equations for a single-slit and a multi-slit

Let $D$ be a simply connected domain $D \subset \mathbb{C}$ with boundary $\partial D$. We consider a slit in $D$, which is defined as a simple curve $\eta = \{\eta(t) : t \in (0, \infty)\} \subset D$; $\eta(s) \neq \eta(t)$ for $s \neq t$ and suppose that $\lim_{t \to 0} \eta(t) =: \eta(0) \in \partial D$. Let $\eta(0, t] := \{\eta(s) : s \in (0, t]\}$ and $D_t^\eta := D \setminus \eta(0, t], t \in (0, \infty)$ with $D_0^\eta := D$. The Loewner theory describes the slit $\eta$ by encoding it into a time-dependent analytic function $(g_{D_t^\eta})_{t \geq 0}$ such that

\[ g_{D_t^\eta} : \text{conformal map } D_t^\eta \to D, \quad t \in [0, \infty). \]
Proof

We parameterize each curve

For

Then, there exists a set of weight functions

where

This is called the hydrodynamic normalization and \(\text{hcap}(\eta(0, t])\) gives the half-plane capacity of \(\eta(0, t] \). The following can be proved (see, for instance, [23, Proposition 4.4]).

**Theorem 3.1** Let \(\eta\) be a slit in \(\mathbb{H}\) such that \(\text{hcap}(\eta(0, t]) = 2t, t > 0\). Then, the solution \((g_t)_{t \geq 0}\) of the differential equation (chordal Loewner equation)

\[
\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - V(t)}, \quad t \geq 0, \quad g_0(z) = z,
\]

where

\[
V(t) = g_{\mathbb{H}_t^g}(\eta(t)) := \lim_{z \to \eta(t), z \in \mathbb{H}_t^g} g_{\mathbb{H}_t^g}(z), \quad t \geq 0,
\]

coincides with \((g_{\mathbb{H}_t^g})_{t \geq 0}\).

Theorem 3.1 can be extended to the situation such that \(\eta\) is given by a multi-slit \(\bigcup_{i=1}^N \eta_i \subset \mathbb{H}\) and \(\mathbb{H}_t^g := \mathbb{H} \setminus \bigcup_{i=1}^N \eta_i(0, t], t > 0\) with \(\mathbb{H}_0^g := \mathbb{H}\).

**Theorem 3.2** For \(N \in \mathbb{N}\), let \(\bigcup_{i=1}^N \eta_i\) be a multi-slit in \(\mathbb{H}\) such that \(\text{hcap}(\bigcup_{i=1}^N \eta_i(0, t]) = 2t, t > 0\). Then, there exists a set of weight functions \((\lambda_i(t))_{t \geq 0}, 1 \leq i \leq N\) satisfying \(\lambda_i(t) \geq 0, 1 \leq i \leq N, \sum_{i=1}^N \lambda_i(t) = 1, t \geq 0\) such that the solution \((g_t)_{t \geq 0}\) of the differential equation (multiple chordal Loewner equation)

\[
\frac{dg_t(z)}{dt} = \sum_{i=1}^N \frac{2\lambda_i(t)}{g_t(z) - V_i(t)}, \quad t \geq 0, \quad g_0(z) = z,
\]

where

\[
V_i(t) = g_{\mathbb{H}_t^{\eta_i}}(\eta_i(t)) := \lim_{z \to \eta_i(t), z \in \mathbb{H}_t^{\eta_i}} g_{\mathbb{H}_t^{\eta_i}}(z), \quad t \geq 0, \quad 1 \leq i \leq N,
\]

coincides with \((g_{\mathbb{H}_t^{\eta_i}})_{t \geq 0}\).

**Proof** We parameterize each curve \(\eta_i\) separately by \(\eta_i : (0, \infty) \to \mathbb{H}; s_i \mapsto \eta_i(s_i)\) so that \(\text{hcap}(\bigcup_{j=1}^N \eta_j(0, s_j))\) is differentiable with respect to \(s_i, 1 \leq i \leq N\). For each \(s = (s_1, \ldots, s_N) \in [0, \infty)^N\), we set \(\mathbb{H}_s^g = \mathbb{H} \setminus \bigcup_{i=1}^N \eta_i(0, s_i)\). A similar argument as in [23, Proposition 4.4] shows that the family of conformal maps \(g_s = g_{\mathbb{H}_s^g}, s \in [0, \infty)^N\) satisfies partial differential equations

\[
\frac{\partial g_s(z)}{\partial s_i} = \frac{1}{g_s(z) - V_i(s)} \frac{\partial}{\partial s_i} \text{hcap}\left(\bigcup_{j=1}^N \eta_j(0, s_j)\right), \quad 1 \leq i \leq N,
\]

where

\[
V_i(s) = g_{\mathbb{H}_s^g}(\eta_i(s)) := \lim_{z \to \eta_i(s), z \in \mathbb{H}_s^g} g_{\mathbb{H}_s^g}(z).
\]

We have these parameters dependent on a single parameter so that \(s_i = s_i(t), t \geq 0, 1 \leq i \leq N\) are increasing and differentiable in \(t\), and write \(s = s(t), t \geq 0\). Then, we may understand \(\mathbb{H}_t^g = \mathbb{H}_{s(t)}^g\),
$g_t = g_{\eta(t)}$, and $V_i(t) = V_i(\eta(t))$, $1 \leq i \leq N$, $t \geq 0$. Furthermore, we impose a condition that $\text{hcapi}(\bigcup_{j=1}^{N} \eta_j(0, s_j(t))) = 2t$. Then, the family of conformal maps $(g_t)_{t \geq 0}$ satisfies the desired differential equation (3.2), where

$$\lambda_i(t) = \frac{1}{2} \frac{\partial}{\partial s_i} \text{hcapi} \left( \bigcup_{j=1}^{N} \eta_j(0, s_j) \right) \bigg|_{s_i = \tilde{g}(t)} \frac{ds_i(t)}{dt}, \quad 1 \leq i \leq N, \quad t \geq 0$$

are subject to the constraint $\sum_{i=1}^{N} \lambda_i(t) = 1, t \geq 0$.

The multiple chordal Loewner equation ($\tilde{\eta}$) for $D = \mathbb{H}$ can be mapped to other simply connected proper subdomains of $\mathbb{C}$ by conformal maps. Here, we consider a conformal map $\tilde{\varphi}(z) = z^2 : \Omega \rightarrow \mathbb{H}$. We set $\tilde{g}_t(z) = \sqrt{g_t(z^2)} + c(t), t \geq 0$ with a function of time $c(t), t \geq 0$. Then, we can see that (3.2) is transformed to

$$\tilde{g}_0(z) = z \in \Omega, \quad \tilde{V}_i(t) = \sqrt{V_i(t) + c(t)}, \quad 1 \leq i \leq N \quad \text{and} \quad 2 \sum_{i=1}^{N} \tilde{\lambda}_i(t) + \tilde{\lambda}_0(t) = (1/4) d\lambda c(d), t \geq 0.$$

The equation (3.3) can be regarded as a multi-slit version of the quadrant Loewner equation studied in [24]. The solution of (3.3) gives a conformal map $\tilde{g}_t = g_{\eta(t)} : \Omega_t \rightarrow \Omega$, where $\Omega_t^n := \Omega \setminus \bigcup_{i=1}^{N} \eta_i(0, t), t > 0, \Omega_t^n := \Omega$, and $g_{\eta(t)}(\eta(t)) = \tilde{V}_i(t) \in \mathbb{R}_+, t \geq 0, 1 \leq i \leq N$.

3.2 SLE

So far we have considered the problem in which given a single slit $\eta(0, t), t > 0$ or a multi-slit $\bigcup_{i=1}^{N} \eta_i(0, t), t > 0$ in $\mathbb{H}$, the time-evolution of the conformal map from $\mathbb{H}_t^n$ to $\mathbb{H}, t \geq 0$ is asked. The answers are given by Theorem 3.1 and Theorem 3.2. For $\mathbb{H}$ with a single slit, Schramm considered the inverse problem in a probabilistic setting [30]. He first asked a suitable family of driving stochastic processes $(X(t))_{t \geq 0}$ on $\mathbb{R}$. Then, he asked the probability law of the random slit $\eta$ in $\mathbb{H}$ that is determined by the solution $g_t = g_{\eta(t)}, t \geq 0$ of the Loewner equation (3.1) via $X(t) = g_{\eta(t)}(\eta(t)), t \geq 0$. Schramm argued that the conformal invariance and the domain Markov property of the law of the curve imply that the driving process $(X(t))_{t \geq 0}$ should be $(B(\kappa t))_{t \geq 0} \overset{\text{law}}{=} (\sqrt{\kappa B(t)})_{t \geq 0}$ with a parameter $\kappa > 0$. The solution of the chordal Loewner equation (3.1) driven by $X(t) = \sqrt{\kappa B(t)}, t \geq 0$ is called the chordal Schramm–Loewner evolution with parameter $\kappa > 0$ and is written as chordal SLE($\kappa$) for short.

The following was proved by Lawler, Schramm, and Werner [24] for $\kappa = 8$ and by Rohde and Schramm [29] for $\kappa \neq 8$.

Proposition 3.3 A chordal SLE($\kappa$) $(g_{\eta(t)})_{t \geq 0}$ determines a continuous curve $\eta = \{\eta(t) : t \in [0, \infty)\} \subset \mathbb{H}$ a.s.

In this inverse problem, the domain $\mathbb{H}_t^n$ is defined as the unbounded component of $\mathbb{H}\setminus\eta(0, t)$ so that the solution $g_t$ is a conformal map from $\mathbb{H}_t^n$ to $\mathbb{H}$ at each $t > 0$, which verifies writing the solution as $g_{\eta(t)} = g_t, t \geq 0$. 11
The continuous curve $\eta$ determined by an SLE($\kappa$) is called an SLE($\kappa$) curve. The probability law of an SLE($\kappa$) curve qualitatively depends on $\kappa$. When $\kappa \in (0, 4]$, the SLE($\kappa$) curve is a simple curve in $\mathbb{H}$. It becomes self-intersecting and can touch the real axis $\mathbb{R}$ when $\kappa > 4$, and becomes a space-filling curve when $\kappa \geq 8$ (see, for instance, [23, 17]).

3.3 Multiple SLE

For simplicity, we assume that $\lambda_i(t) \equiv 1/N, t \geq 0, 1 \leq i \leq N$ in (3.2) in Theorem 3.2. Then, by a simple time change $t/N \rightarrow t$ associated with a change of notation, $g_{N t} \rightarrow g_{O t}$, the multiple chordal Loewner equation is written as (1.3). Then, we ask what is a suitable family of driving stochastic processes of $N$ particles $(X_{\mathbb{R}}(t))_{t \geq 0}$ on $\mathbb{R}$.

Bauer, Bernard, and Kytölä [4] and Graham [15] argued that $X_{\mathbb{R}}^R(t), t \geq 0, 1 \leq i \leq N$ are semi-martingales and the quadratic variations should be given by $\langle dX_{i}^{\mathbb{R}}, dX_{j}^{\mathbb{R}} \rangle_t = \kappa \delta_{ij} dt, t \geq 0, 1 \leq i, j \leq N$ with $\kappa > 0$. Then, we can assume that the system of SDEs for $(X_{\mathbb{R}}^R(t))_{t \geq 0}$ is in the form (1.9).

In the orthant system (3.3), we put $\hat{\lambda}_i(t) \equiv r/(2N), t \geq 0, r \in (0, 1], 1 \leq i \leq N, dc(t)/dt \equiv 4, t \geq 0$, and perform a time change $rt/(2N) \rightarrow t$ associated with a change of notation $g_{2Nt/r} \rightarrow g_{O t}$. Then, the multiple Loewner equation in $\mathbb{O}$ is written as

$$
\frac{dg_{O t}(z)}{dt} = \sum_{i=1}^{N} \left( \frac{2}{g_{O i}^{+}(z) - X_{i}^{R}(t)} + \frac{2}{g_{O i}^{+}(z) + X_{i}^{R}(t)} \right) + \frac{4\delta}{g_{O}^{+}(z)},
$$

$t \geq 0$ with $g_{O i}^{+}(z) = z \in \mathbb{O}$, where $\delta := N(1-r)/r \geq 0$. We assume that the system of SDEs for $X^{R}_{i}(t) \in (\mathbb{R}_{+})^{N}, t \geq 0$ is in the form (1.9).

Analogously to the case of the SLE for a single slit, in both cases of $D = \mathbb{H}$ and $\mathbb{O}$, we find a family of domains $(D_{i}^{0} \subseteq D)_{t \geq 0}$ such that $g_{D_{i}^{0}}$ is a conformal map from $D_{i}^{0}$ to $D$ at each $t \geq 0$. (See Remark 1.4 (1) in [19].)

4 Gaussian Free Field with Zero Boundary Condition

4.1 Bochner–Minlos Theorem

Let $D \subseteq \mathbb{C}$ be a simply connected domain. Consider the real $L^2$ space with the inner product, $(f, g) := \int_{D} f(z) \overline{g(z)}d\mu(z), f, g \in L^2(D)$, where $\mu(z)$ is the Lebesgue measure on $\mathbb{C}$; $d\mu(z) = \sqrt{-1}dzd\bar{z}/2$. Let $\Delta$ be the Dirichlet Laplacian acting on $L^2(D)$. In the present subsection 4.1 we assume that $D$ is bounded. Then $-\Delta$ has positive discrete eigenvalues so that $-\Delta e_n = \lambda_n e_n, e_n \in L^2(D), n \in N$. We assume that the eigenvalues are labeled in the non-decreasing order; $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. The system of eigenfunctions $\{e_n\}_{n \in N}$ forms a CONS of $L^2(D)$. The asymptotic behavior of eigenvalues obeys Weyl’s formula; $\lim_{n \rightarrow \infty} \lambda_n/n = O(1)$.

For $f, g \in C_{0}^\infty(D)$, the Dirichlet inner product is defined by

$$
(f, g)_{\nabla} := \frac{1}{2\pi} \int_{D} (\nabla f)(z) \cdot (\nabla g)(z) d\mu(z).
$$

The Hilbert space completion of $C_{0}^\infty(D)$ with respect to $(\cdot, \cdot)_{\nabla}$ will be denoted by $W(D)$. We write $\|f\|_{\nabla} = \sqrt{(f, f)_{\nabla}}, f \in W(D)$. If we set $u_n = \sqrt{2\pi/\lambda_n} e_n, n \in N$, then, by integration by parts,
we have \((u_n, u_n)_\nabla = (u_n, (-\Delta)u_n)/(2\pi) = \delta_{nm}, n, m \in \mathbb{N}\). Therefore, \(\{u_n\}_{n \in \mathbb{N}}\) forms a CONS of W(D).

Let \(\hat{H}(D)\) be the space of formal infinite series in \(\{u_n\}_{n \in \mathbb{N}}\), which is obviously isomorphic to \(\mathbb{R}^N\) by setting \(\hat{H}(D) \ni \sum_{n \in \mathbb{N}} a_n u_n \mapsto (f_n)_{n \in \mathbb{N}} \in \mathbb{R}^N\). As a subspace of \(\hat{H}(D), W(D)\) is isomorphic to \(\ell^2(\mathbb{N}) \subset \mathbb{R}^N\). For two formal series \(f = \sum_{n \in \mathbb{N}} f_n u_n, g = \sum_{n \in \mathbb{N}} g_n u_n \in \hat{H}(D)\) such that \(\sum_{n \in \mathbb{N}} |f_n g_n| < \infty\), we define their pairing as \((f, g)_\nabla := \sum_{n \in \mathbb{N}} f_n g_n\). In the case when \(f, g \in W(D)\), their pairing, of course, coincides with the Dirichlet inner product (1.1).

Notice that, for any \(a \in \mathbb{R}\), the operator \((-\Delta)^a\) acts on \(\mathcal{H}(D)\) as \((-\Delta)^a \sum_{n \in \mathbb{N}} f_n u_n := \sum_{n \in \mathbb{N}} \lambda_n^a f_n u_n, (f_n)_{n \in \mathbb{N}} \in \mathbb{R}^N\). Using this fact, we define \(\mathcal{H}_a(D) := (-\Delta)^a W(D), a \in \mathbb{R}\), each of which is a Hilbert space with the inner product \((f, g)_a := ((-\Delta)^{-a} f, (-\Delta)^{-a} g)_\nabla, f, g \in \mathcal{H}_a(D)\). We can prove that \(\mathcal{H}_a(D) \subset \mathcal{H}_b(D)\) for \(a < b\) using Weyl’s formula for \(\{\lambda_n\}_{n \in \mathbb{N}}\), and that the dual Hilbert space of \(\mathcal{H}_a(D)\) is given by \(\mathcal{H}_a(D)\) (see [2]).

**Remark 4.1** Since
\[
(f, g)_{1/2} = (\langle (-\Delta)^{-1/2} f, (-\Delta)^{-1/2} g \rangle_\nabla = (f, g)/(2\pi), \quad f, g \in \mathcal{H}_{1/2}(D),
\]
\(\mathcal{H}_{1/2}(D) \simeq L^2(D)\). This implies that the members of \(\mathcal{H}_a(D)\) with \(a > 1/2\) cannot be functions, but are distributions.

Define \(\mathcal{E}(D) := \bigcup_{a \geq 1/2} \mathcal{H}_a(D)\). Then, its dual Hilbert space is identified with \(\mathcal{E}(D)^* := \bigcap_{a \leq -1/2} \mathcal{H}_a(D)\) and \(\mathcal{E}(D)^* \subset W(D) \subset \mathcal{E}(D)\) is established. Here \((\mathcal{E}(D)^*, W(D), \mathcal{E}(D))\) is called a Gel’fand triple. We set \(\Sigma_{\mathcal{E}(D)} = \sigma(\langle \cdot, f \rangle_\nabla : f \in \mathcal{E}(D)^*)\). On such a setting, the following is proved. This theorem is called the Bochner–Minlos theorem [16, 31, 2].

**Theorem 4.1 (Bochner–Minlos theorem)** Let \(\psi\) be a continuous function of positive type on \(W(D)\) such that \(\psi(0) = 1\). Then there exists a unique probability measure \(P\) on \((\mathcal{E}(D), \Sigma_{\mathcal{E}(D)})\) such that \(\psi(f) = \int_{\mathcal{E}(D)} e^{\sqrt{-1} \langle h, f \rangle_\nabla} P(\text{d}h)\) for \(f \in \mathcal{E}(D)^*\).

Under certain conditions on \(\psi\), the domain of the random functional \(h\) in the above formula can be extended from \(\mathcal{E}(D)^*\) to \(W(D)\). It is easy to verify that the functional \(\Psi(f) := e^{-\|f\|_{\nabla^*}^2/2}\) satisfies the conditions. Then, the following is established with the probability measure \(P\) on \((\mathcal{E}(D), \Sigma_{\mathcal{E}(D)})\),

\[
\int_{\mathcal{E}(D)} e^{\sqrt{-1} \langle h, f \rangle_\nabla} P(\text{d}h) = e^{-\|f\|_{\nabla^*}^2/2} \quad \text{for } f \in W(D).
\]

**Definition 4.2 (zero-boundary GFF)** A Gaussian free field (GFF) with zero boundary condition (zero-boundary GFF) is defined as a pair \(((\Omega^{GFF}, F^{GFF}, P^{GFF}), H)\) of a probability space \((\Omega^{GFF}, F^{GFF}, P^{GFF})\) and an isometry \(H : W(D) \to L^2(\Omega^{GFF}, F^{GFF}, P^{GFF})\) such that each \(H(f), f \in W(D)\) is a centered Gaussian random variable.

For each \(f \in W(D)\), we write \((H, f)_\nabla \in L^2(\mathcal{E}(D), \Sigma_{\mathcal{E}(D)}, P)\) for the random variable defined by \(h \mapsto \langle h, f \rangle_\nabla, h \in \mathcal{E}(D)\). Then (4.2) ensures that the pair of \(((\mathcal{E}(D), \Sigma_{\mathcal{E}(D)}, P), H)\) gives a GFF with zero boundary condition. We often just call \(H\) a zero-boundary GFF without referring to the probability space \((\Omega^{GFF}, F^{GFF}, P^{GFF}) = (\mathcal{E}(D), \Sigma_{\mathcal{E}(D)}, P)\).
4.2 Conformal invariance of a zero-boundary GFF

Assume that $D, \tilde{D} \subseteq \mathbb{C}$ are simply connected domains and let $\varphi : \tilde{D} \to D$ be a conformal map.

**Lemma 4.3** The Dirichlet inner product (4.1) is conformally invariant. That is, for $f, g \in C_c^\infty(D)$,

$$\int_D (\nabla f)(z) \cdot (\nabla g)(z) d\mu(z) = \int_{\tilde{D}} (\nabla (f \circ \varphi))(z) \cdot (\nabla (g \circ \varphi))(z) d\mu(z).$$

From the above lemma, we see that $\varphi^*: W(D) \ni f \mapsto f \circ \varphi \in W(\tilde{D})$ is an isomorphism. This allows one to consider a GFF on an unbounded domain. Namely, if $\tilde{D}$ is bounded on which a zero-boundary GFF is defined, but $D$ is unbounded, we can define a family $\{(\varphi^*H, f) : f \in W(D)\}$ by $(\varphi^*H, f) = (H, f \circ \varphi)$, $f \in W(D)$ so that we have the covariance structure,

$$E^{\text{GFF}}[(\varphi^*H, f) (\varphi^*H, g)] = (f, g), \quad f, g \in W(D),$$

where $E^{\text{GFF}}$ is the expectation value with respect to $P^{\text{GFF}}$. Relying on the formal computation,

$$(\varphi^*H, f) = (H, f \circ \varphi) = \frac{1}{2\pi} \int_D (\nabla H)(z) \cdot (\nabla f \circ \varphi)(z) d\mu(z)$$

we understand the equality $\varphi^*H = H \circ \varphi^{-1}$. By the fact shown above that the covariance structure does not change under a conformal map $\varphi$, we say a zero-boundary GFF is conformally invariant.

4.3 Green’s function of a zero-boundary GFF

Assume that $D \subseteq \mathbb{C}$ is a simply connected domain. In the previous subsections, we have constructed a family $\{(H, f) : f \in W(D)\}$ of random variables whose covariance structure is given by

$$E^{\text{GFF}}[(H, f)(H, g)] = (f, g), \quad f, g \in W(D).$$

By a formal integration by parts, we see that

$$(H, f) = \frac{1}{2\pi} \int_D (\nabla H)(z) \cdot (\nabla f)(z) d\mu(z) = \frac{1}{2\pi} \int_D H(z)(-\Delta f)(z) d\mu(z)$$

Motivated by this observation, we define

$$(4.3) \quad (H, f) := 2\pi (H, (-\Delta)^{-1} f) \quad \text{for } f \in D((-\Delta)^{-1}),$$

where $D((-\Delta)^{-1})$ denotes the domain of $(-\Delta)^{-1}$ in $W(D)$. The action of $(-\Delta)^{-1}$ is expressed as an integral operator as

$$((-\Delta)^{-1} f)(z) = \frac{1}{2\pi} \int_D G_D(z, w)f(w) d\mu(w) \quad \text{a.e. } z \in D.$$
\[ f \in \mathbb{D}((-\Delta)^{-1}) \], where the integral kernel \( G_D \) is known as the Green’s function of \( D \) under the Dirichlet boundary condition: \( G_D(z, w) = 0, w \in D \) if \( z \in \partial D \). Hence the covariance of \((H, f)\) and \((H, g)\) with \( f, g \in \mathbb{D}((-\Delta)^{-1}) \) is written as

\[ \mathbb{E}^{\text{GFF}}[(H, f)(H, g)] = \int_{D \times D} f(z)G_D(z, w)g(w)d\mu(z)d\mu(w). \]

When we symbolically write

\[ (H, f) = \int_D H(z)f(z)d\mu(z), \quad f \in \mathbb{D}((-\Delta)^{-1}), \]

the covariance structure can be expressed as

\[ \mathbb{E}^{\text{GFF}}[H(z)H(w)] = G_D(z, w), \quad z, w \in D, \quad z \neq w. \]

The conformal invariance of a zero-boundary GFF implies that for a conformal map \( \varphi : \tilde{D} \to D \), we have the equality,

\[ G_{\tilde{D}}(z, w) = G_D(\varphi(z), \varphi(w)), \quad z, w \in \tilde{D}, \quad z \neq w. \]

**Example 4.1** When \( D = \mathbb{H} \),

\[ G_{\mathbb{H}}(z, w) = \log \left| \frac{z - \overline{w}}{z - w} \right|, \quad z, w \in \mathbb{H}, \quad z \neq w. \]

**Example 4.2** When \( D = \mathbb{O} \),

\[ G_{\mathbb{O}}(z, w) = \log \left| \frac{(z - \overline{w})(z + \overline{w})}{(z - w)(z + w)} \right|, \quad z, w \in \mathbb{O}, \quad z \neq w. \]

From the formula \([4.4]\), we see that \( \mathcal{C}_c^\infty(D) \subset \mathbb{D}((-\Delta)^{-1}) \). In the following, we will consider the family of random variables \( \{(H, f) : f \in \mathcal{C}_c^\infty(D)\} \) to characterize a GFF \( H \).

## 5 Gaussian Free Fields Coupled with Multiple SLEs

In this section, we take a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which a zero-boundary GFF \( H \) and a multiple SLE \((g_D^\eta)_{t \geq 0}\) are defined in such a way that they are independent, and a filtration \((\mathcal{F}_t)_{t \geq 0}\) to which the multiple SLE is adapted (see Section \([1]\) for a precise setting). We fix an open set \( A \subset D \), where \( D = \mathbb{H} \) or \( \mathbb{O} \). Then the \((\mathcal{F}_t)_{t \geq 0}\)-stopping time \( \tau_A \) is defined in exactly the same expression as \([1.6]\). We also take an \((\mathcal{F}_t)_{t \geq 0}\)-stopping time \( \tau \) such that \( \tau \leq \tau_A \) a.s. as in Theorem \([I.3]\).

### 5.1 Zero-boundary GFF transformed by a multiple SLE

Here, we write the zero-boundary GFF defined on \( D = \mathbb{H} \) or \( \mathbb{O} \) as \( H_D \). Consider the transformation of \( H_D \) by the multiple SLE, \( H_D^\eta := H_D \circ g_D^\eta, 0 \leq t \leq \tau \) on \( \mathbb{D}^\eta_t \). By the conformal invariance, the Green’s function of \( H_D^\eta, 0 \leq t \leq \tau \) is given by \( G_{D^\eta}(z, w) = G_D(g_D^\eta(z), g_D^\eta(w)), z, w \in \mathbb{D}^\eta_t, z \neq w, 0 \leq t \leq \tau \). The following is obtained.
Lemma 5.1  For $D = \mathbb{H}$ and $\mathbb{O}$, the increments of $G_D^\tau(z,w)$, $z,w \in A$ in time $0 \leq t \leq \tau$ are given as

$$dG_{\mathbb{H}}(z,w) = -\sum_{i=1}^{N} \text{Im} \frac{2}{g_{\mathbb{H}}(z) - X_i^R(t)} \frac{2}{g_{\mathbb{H}}(w) - X_i^R(t)} dt,$$

$$dG_{\mathbb{O}}(z,w) = -\sum_{i=1}^{N} \text{Im} \left( \frac{2}{g_{\mathbb{O}}(z) - X_i^R(t)} - \frac{2}{g_{\mathbb{O}}(w) - X_i^R(t)} \right) dt.$$

**Proof**  Using the explicit expressions of the Green’s functions given in Examples 4.1 and 4.2 and the multiple Loewner equations (1.4) and (4.4), the increments of $(G_D^\tau)_0 \leq t \leq \tau$ with $D = \mathbb{H}$ and $\mathbb{O}$ are calculated. The above expressions are obtained using the equality $\text{Re} \zeta \omega - \text{Re} \zeta \omega = 2\text{Im} \zeta \text{Im} \omega$ for $\zeta, \omega \in \mathbb{C}$. $\blacksquare$

5.2  $\mathbb{C}$-valued logarithmic potentials and martingales

We have remarked in Section 2.2 that the Dyson model and the Bru–Wishart process studied in random matrix theory can be regarded as stochastic log-gasses defined on a line $S = \mathbb{R}$ and a half-line $S = \mathbb{R}_+$, respectively. There, the logarithmic potentials are given by (2.3). Here, we consider a complex-valued logarithmic potentials between a point $z$ in the two-dimensional domain $D \subseteq \mathbb{C}$ and $N$ points $x = (x_1, \ldots, x_N)$ on the boundary $S$. For $(D,S) = (\mathbb{H},\mathbb{R})$ and $(\mathbb{O},\mathbb{R}_+)$, put

$$\Phi_{\mathbb{H}}(z,x) = \sum_{i=1}^{N} \log(z - x_i),$$

$$\Phi_{\mathbb{O}}(z,x) = \Phi_{\mathbb{O}}(z,x;q) = \sum_{i=1}^{N} \{ \log(z - x_i) + \log(z + x_i) \} + q \log z,$$

where $z \in D$, $x \in S^N$, and $q \in \mathbb{R}$.

Now, we consider a time evolution of the $\mathbb{C}$-valued potential $\Phi_D$ by letting $x$ be the driving process $(X^S(t))_{t \geq 0}$ of the multiple SLE $(g_D^\tau)_{t \geq 0}$ and by transforming the function $\Phi_D(\cdot, X^S(t))$ by $(g_D^\tau)_{t \geq 0}$. We obtain the following.

Lemma 5.2  For $D = \mathbb{H}$ and $\mathbb{O}$, the increments of the $\mathbb{C}$-valued potentials are given as follows. For $z \in A$, $X^S(t) \in \mathbb{W}_N(S)$, $0 \leq t \leq \tau$,

$$(5.1) \quad d\Phi_{\mathbb{H}}(g_{\mathbb{H}}(z), X^R(t)) = -\sum_{i=1}^{N} \frac{\sqrt{d}B_i(t)}{g_{\mathbb{H}}(z) - X_i^R(t)} - \left( 1 - \frac{K}{4} \right) d\log g_{\mathbb{H}}(z)$$

$$-\sum_{i=1}^{N} \left( F_i^R(X^R(t)) - 4 \sum_{1 \leq j \leq N, \ j \neq i} \frac{1}{X_i^R(t) - X_j^R(t)} \right) \frac{dt}{g_{\mathbb{H}}(z) - X_i^R(t)}.$$
Proposition 5.3

Let $D$. Apply Itô’s formula and use the equalities such as

Proof

the third term in the RHS of (5.1) vanishes. Regarding (5.2), we first put

(5.2) vanishes.

The proof is given by direct calculation.

$$
\frac{d\Phi_D(g_{Q^T}(z), X^{R^+}(t); q)}{
= - \sum_{i=1}^{N} \left( \frac{1}{g_{Q^T}(z) - X^{R^+}_i(t)} - \frac{1}{g_{Q^T}(z) + X^{R^+}_i(t)} \right) \sqrt{\kappa} dB_i(t)
- \sum_{i=1}^{N} \left[ I_i^{R^+}(X^{R^+}(t)) \right.
- \left\{ 4 \sum_{1 \leq j \leq N, j \neq i} \left( \frac{1}{X^{R^+}_i(t) - X^{R^+}_j(t)} + \frac{1}{X^{R^+}_i(t) + X^{R^+}_j(t)} \right) \right.
\left. + 2(1 + 2\delta + q) \frac{1}{X^{R^+}_i(t)} \right\}]
\times \left( \frac{1}{g_{Q^T}(z) - X^{R^+}_i(t)} - \frac{1}{g_{Q^T}(z) + X^{R^+}_i(t)} \right) dt
- 4\delta \left( 1 - \frac{\kappa}{4} - q \right) \frac{dt}{(g_{Q^T}(z))^2} - \left( 1 - \frac{\kappa}{4} \right) d\log g_{Q^T}(z).
$$

If we assume that $(\mathbf{X}(t))_{t \geq 0}$ is given by the $(8/\kappa)$-Dyson model $(\mathbf{Y}(t))_{t \geq 0}$ satisfying (1.8), the third term in the RHS of (5.1) vanishes. Regarding (5.2), we first put $q = 1 - \kappa/4$ to make the third term in the RHS become zero. Then, if we assume that $\delta = \nu$ and $(\mathbf{X}^{R^+}(t))_{t \geq 0}$ is given by the $(8/\kappa, \nu)$-Bru–Wishart process $(\mathbf{Y}^{R^+}(t))_{t \geq 0}$ satisfying (2.4), the second term in the RHS of (5.2) vanishes.

Define

$$
\mathcal{M}_H(z, t) = -\Phi_H(g_{H^T}(z), Y^R(t)) - \left( 1 - \frac{\kappa}{4} \right) \log g_{H^T}(z),
$$

$$
\mathcal{M}_D(z, t) = -\Phi_D(g_{Q^T}(z), Y^{R^+}(t); 1 - \kappa/4) - \left( 1 - \frac{\kappa}{4} \right) \log g_{Q^T}(z).
$$

Proposition 5.3 Let $\kappa > 0$, $q = 1 - \kappa/4$, $\delta = \nu \geq 0$. Then, for each point $z \in A$, $(\mathcal{M}_D(z, t))_{0 \leq t \leq \tau}$, $D = \mathbb{H}$ and $\mathbb{O}$, provide local martingales with increments

$$
d\mathcal{M}_H(z, t) = \sum_{i=1}^{N} \frac{\sqrt{\kappa} dB_i(t)}{g_{H^T}(z) - Y_i^R(t)},
$$

$$
d\mathcal{M}_D(z, t) = \sum_{i=1}^{N} \left( \frac{1}{g_{Q^T}(z) - Y_i^{R^+}(t)} - \frac{1}{g_{Q^T}(z) + Y_i^{R^+}(t)} \right) \sqrt{\kappa} dB_i(t).
$$
5.3 GFF-valued Processes

Now, we consider the sum \( H_{D^q}(\cdot) + F[M_D(\cdot,t)] \), \( 0 \leq t \leq \tau \), where \( F[\cdot] \) denotes a functional. Comparing Lemma 5.1 and Proposition 5.3 we observe that

\[
\begin{align*}
\langle \Im M_D(z,\cdot), \Im M_D(w,\cdot) \rangle_t &= -\frac{\kappa}{4} dG^q_D(z,w),
\end{align*}
\]

\( z, w \in D^q, 0 \leq t \leq \tau \), for \((D, S) = (\mathbb{H}, \mathbb{R}) \) and \((\mathbb{D}, \mathbb{R}_+) \). Hence, we put \( F[\cdot] = (2/\sqrt{\kappa}) \Im [\cdot] \), and define the following GFF-valued processes for \((D, S) = (\mathbb{H}, \mathbb{R}) \) and \((\mathbb{D}, \mathbb{R}_+) \),

\[
H_D(\cdot, t) := H_{D^q}(\cdot) + \frac{2}{\sqrt{\kappa}} \Im M_D(\cdot, t), \quad 0 \leq t \leq \tau
\]

with \( \chi = \frac{2}{\sqrt{\kappa}}(1 - \kappa/4) = 2/\sqrt{\kappa} - \sqrt{\kappa}/2 \). The second term of (5.4) contains an imaginary part of the complex-valued logarithmic potential \(-\Phi_D(g_D^q(z), Y^S(t))\), \( t \geq 0 \). This is the unique harmonic function satisfying the boundary condition

\[
\frac{2}{\sqrt{\kappa}} \Im M_D(x, t) = \begin{cases} 
-\frac{2\pi}{\sqrt{\kappa}} N, & \text{if } x < Y^S(t), \\
-\frac{2\pi}{\sqrt{\kappa}}(N - i), & \text{if } x \in (Y^S_i(t), Y^S_{i+1}(t)), 1 \leq i \leq N,
\end{cases}
\]

with the convention \( Y^S_{N+1}(t) \equiv +\infty \). That is, it has discontinuity at \( Y^S(t) \) by \( 2\pi/\sqrt{\kappa} \) along \( S \), \( 1 \leq i \leq N, t \geq 0 \). We will think that the GFF \( H_D(\cdot, t) \) has the same boundary condition as \( (2/\sqrt{\kappa}) \Im M_D(\cdot, t), 0 \leq t \leq \tau \). For further arguments concerning the second term of (5.4), see Section V.C in [18].

Theorem 5.4 Let \( \kappa > 0, q = 1 - \kappa/4, \delta = \nu \geq 0 \). Assume that \((D, S) = (\mathbb{H}, \mathbb{R}) \) or \((\mathbb{D}, \mathbb{R}_+) \), and \((Y^S(t))_{t \geq 0} \) is the \((8/\kappa)\)-Dyson model if \( S = \mathbb{R} \) and the \((8/\kappa, \nu)\)-Bru–Wishart process if \( S = \mathbb{R}_+ \), starting from a configuration in \( \mathbb{W}_N(S) \). Then, for each \( f \in \mathcal{C}^\infty_c(D) \) such that \( \text{supp}(f) \subset A \), we have the following equality:

\[
(H_D(\cdot, 0), f) \overset{(\text{law})}{=} (H_D(\cdot, \tau), f) \quad \text{under } \mathbb{P}.
\]

Proof From (5.3) we have

\[
d \left( \langle \frac{2}{\sqrt{\kappa}} \Im M_D(\cdot, \cdot), f \rangle \right)_t = -dE_t(f), \quad 0 \leq t \leq \tau,
\]

where

\[
E_t(f) := \int_{A \times A} f(z)G_{D^q}(z,w)f(w)d\mu(z)d\mu(w)
\]

is called the Dirichlet energy of \( f \) in \( D^q \). We have \( \text{Var}[(H_{D^q}, f)] = E_t(f) \) due to the conformal invariance of a zero-boundary GFF. Introducing a parameter \( \theta \in \mathbb{R} \), we see

\[
\mathbb{E} \left[ e^{\sqrt{-1} \theta (H_D(\cdot, \tau), f)} \right] = \mathbb{E} \left[ e^{\sqrt{-1} \theta (H_{D^q}, f)} \right] e^{\sqrt{-1} \theta \frac{2}{\sqrt{\kappa}} \Im M_D(\cdot, \tau), f} \right],
\]

18
where $E$ is the expectation value with respect to the probability measure $\mathbb{P}$, since $(\text{Im} \mathcal{M}_D(\cdot, \tau), f)$ is $\mathcal{F}_\tau$-measurable. By definition of a zero-boundary GFF and the Dirichlet energy (5.6), we obtain

$$E \left[ e^{\sqrt{-1} \theta (H_D(\cdot, \tau), f)} \right] \bigg| \mathcal{F}_\tau = e^{-\frac{\theta^2}{2}} E_\tau(f).$$

Hence, by Proposition 5.3 and (5.5) we have

$$E \left[ e^{\sqrt{-1} \theta (H_D(\cdot, \tau), f)} \right] = E \left[ e^{\sqrt{-1} \theta (\text{Im} \mathcal{M}_D(\cdot, \tau), f) - \frac{\theta^2}{2} E_\tau(f)} \right] = E \left[ e^{\sqrt{-1} \theta (\text{Im} \mathcal{M}_D(\cdot, 0), f) - \frac{\theta^2}{2} E_0(f)} \right].$$

This implies the desired coincidence under the probability law $\mathbb{P}$.

**Remark 5.1** There are two local formulations of multiple SLE; one of them is based on commutation relation of Loewner chains each of which generates a single random curve [9] and the other one is a single Loewner chain driven by multiple driving processes [4]. Though it is expected that these two formulations are equivalent, the equivalence has not been proved and even its precise statement is not obvious. In the work by Miller and Sheffield [25], they studied coupling between a GFF and a variant of SLE called $\text{SLE}(\kappa, \rho)$, which reduces to a member of commuting Loewner chains at a specific setting of parameters. Hence, it can be said that they also considered coupling between a GFF and a multiple SLE in the former sense. This does not, however, imply that a multiple SLE in the latter sense can be coupled with the same GFF, which is exactly the result we presented in the current article.

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