A quasi-solution approach to nonlinear problems—the case of the Blasius similarity solution

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Abstract

Using the simple case of the Blasius similarity solution, we illustrate a recently developed general method (Costin et al 2012 Nonlinearity 25 125–64; Costin et al 2012 arXiv:1209.1009) that reduces a strongly nonlinear problem into a weakly nonlinear analysis. The basic idea is to find a quasi-solution $F_0$ that satisfies the nonlinear problem and boundary conditions to within small errors. Then, by decomposing the true solution $F = F_0 + E$, a weakly nonlinear analysis of $E$, using the contraction mapping theorem in a suitable space of functions provides the existence of the solution as well as bounds on the error $E$. The quasi-solution construction relies on a combination of exponential asymptotics and standard orthogonal polynomial representations in a finite domain.

1. Introduction and main results

Nonlinear mathematical problems abound in vortex dynamics as they do in all areas of the sciences. For the most part, available mathematical tools are limited to numerical computations. While providing for valuable insights, computations do not usually address the associated existence and uniqueness questions. In problems involving an infinite domain for which numerical computations are usually truncated to finite subdomains, these questions are more than just of theoretical interest. For instance, determination of heteroclinic and homoclinic orbits of a dynamical system play an important role in Lagrangian chaos (see for instance the review paper, Aref (1990)). Yet, numerical computations of such two-point boundary value problems cannot by themselves resolve the question whether or not such orbits exist in the first place. Also, in many problems, such as in hydrodynamic stability, a clear understanding of the associated spectral problem is facilitated greatly by analytical...
representation of steady state solutions. This explains at least in part why analytical expressions for solutions to nonlinear problems remain an important area of research. Closed-form solutions, however, exist only for a small sub-class of problems (essentially for integrable models). On the other hand, if a problem involves some small parameter $\varepsilon$ (or a large parameter) and the limiting problem is exactly solvable, then there exist quite general asymptotic methods to obtain convenient expansions for the perturbed problem.

Indeed, consider for instance the question of finding the solution to $\mathcal{N}[u, \varepsilon] = 0$, where $\mathcal{N}$ is a (possibly nonlinear) differential operator in some space of functions satisfying boundary/initial conditions and that $u_0$ is the solution at $\varepsilon = 0$. Existence and uniqueness of a solution $u$ as well as bounds on the error $E = u - u_0$ may be found as follows. We write

$$ LE = -\delta - \mathcal{N}(E), $$

where $L = \frac{d\mathcal{N}}{du}ig|_{u=u_0}$ is the Fréchet derivative of $\mathcal{N}$, $\delta = \mathcal{N}[u_0]$ is the residual and $\mathcal{N}(E) = \mathcal{N}(u_0 + E) - LE = O(E^2)$. Assuming $L$ to be invertible in a suitable space of functions subject to appropriate initial/boundary conditions, and using the fixed point or contractive mapping theorems in an adapted norm, the small nonlinearity $\mathcal{N}(E)$ can be controlled.

Recently, a relatively general strategy has been employed (Costin et al. 2012a, 2012b) in problems without explicit small or large parameters. The approach uses exponential asymptotic methods and classical orthogonal polynomial techniques to find a function $u_0$; we call it a quasi-solution, which is a very accurate global approximation of the sought solution $u$, in the sense that $\delta = \mathcal{N}(u_0)$ is small in a suitable norm and the boundary conditions are satisfied to within small errors. Once this is accomplished, a perturbative approach similar to the one above applies with the role of $\varepsilon$ played by the norm of $\delta$ and one obtains an actual solution $u$ by controlling the equation satisfied by $E = u - u_0$. The method has been generalized to integro-differential equations arising in steady 2D deep water waves (Tanveer 2013); therefore, it might be expected that this should be generalizable to vortex patches as well. Indeed, the quasi-solution approach is more general and can be generalized to PDEs as well, though the details in higher dimensions are computationally challenging. The only crucial conceptual barrier is the ability to determine suitably appropriate bounds on $L^{-1}$.

Here, we explain the strategy for the relatively simple but well-known Blasius similarity solution (Blasius 1908) arising in boundary layer fluid-flow past a flat plate. Since the audience is mostly non-mathematicians, we limit ourselves to presenting the theorems and explaining the implications while omitting technical proofs given elsewhere (Costin and Tanveer 2013). We also elucidate the construction of a quasi-solution and give some indication on how error estimates are obtained. In the last section, we present new results when the boundary layer similarity solution is required to satisfy a more general boundary condition than the usual no-slip condition. The point is to briefly explain how parameters can be incorporated in a quasi-solution formulation. The detailed proofs will appear elsewhere (Kim 2014).

The classic Blasius similarity solution to boundary layer equations past a semi-infinite plate satisfies

$$ f''(x) + f(x)f'(x) = 0 \text{ for } x \in (0, \infty). \quad (1) $$

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with no-slip boundary conditions:

\[ f(0) = 0, \quad f'(0) = 0, \lim_{x \to +\infty} f'(x) = 1. \tag{2} \]

A generalization of (2) is also of interest (see, e.g., Hussaini and Laikin 1986 and Brighi and Hoernel 2005) and involves modification of the no-slip boundary conditions:

\[ f(0) = \alpha, f'(0) = \gamma, \lim_{x \to +\infty} f'(x) = 1. \tag{3} \]

The Blasius similarity solution have garnered much attention since Blasius (1908) derived it as an exact solution to the Prandtl boundary layer equations. Existence and uniqueness were first proved by Weyl (1942). Issues of existence and uniqueness for this and related equations have been considered as well by many authors (see, e.g., Hussaini and Laikin 1986 and Brighi and Hoernel 2005, the latter being a review paper). Hodograph transformations (Callegari and Friedman 1968) allow a convergent power series representation in the entire domain, but the convergence is slow at the edge of the domain and the representation is not quite convenient in finding an approximation to \( f \) directly. Empirically, there has been quite a bit of interest in obtaining simple expressions for Blasius and related similarity solutions. Liao (1999) for instance introduced a formal method for an empirically accurate approximation; the theoretical basis for this procedure and its limitations remain however unclear. We are unaware of any rigorous error control for this or any other efficient approximation in terms of simple functions. Also, we are unaware of any systematic procedure that allows for analytical representation of solution to any desired accuracy. Weyl (1942), using a transformation introduced by Töpfer (1912), proved that \( f \) in (1) and (2) can be expressed as

\[ f(x) = a^{-1/2} F(a^{-1/2}x), \tag{4} \]

where \( F \) satisfies the initial value problem

\[ F''(x) + F(x)F''(x) = 0 \quad \text{for} \quad x \in (0, \infty), \tag{5} \]

with initial conditions

\[ F(0) = 0, \quad F'(0) = 0, \quad F''(0) = 1. \tag{6} \]

In (4), \( \lim_{x \to +\infty} F'(x) = a \in \mathbb{R}^+ \) (cf. Weyl 1942, Topfer 1912). More general boundary conditions (3) on \( f \) translate into the following initial conditions on \( F \):

\[ F(0) = \alpha, \quad F'(0) = \gamma, \quad F''(0) = 1, \tag{7} \]

where \( \alpha = a^{1/2} \hat{a} \) and \( \gamma = \hat{a} \). Note that the solution \( f \) to the original problem is obtained through the transformation (4); the appropriately non-dimensionalized wall stress is given by

\[ f''(0) = a^{-3/2}. \tag{8} \]

It is to be emphasized that this transformation, though convenient, is by no means necessary to construct a quasi-solution, since a quasi-solution only needs to satisfy initial/boundary conditions approximately. This will be clearer in the error analysis where it will be seen that non-homogeneous initial/boundary conditions are allowable as long as they are small. For this reason, the methodology outlined here can be extended to more general two-point boundary value problems.

\[ ^1 \text{The equation in the original Blasius' paper has a coefficient } \frac{1}{2} \text{ for } f''; \text{ however the change of variable } x \to x/\sqrt{2}, \]

\[ f \to f/\sqrt{2} \text{ transforms (1) into Blasius' original equation. Thus, } f''(0) = 0.469600 \pm 0.0000022 \text{ transforms to } f''(0) = 0.3320574 \pm 0.0000016 \text{ in the original variables.} \]
First, for some given value of \((\alpha, \gamma)\), determining an accurate quasi-solution \(F_0\) satisfying (5) and (7) over a finite interval \(I\) (chosen to be \([0, \frac{5}{2}]\)), to within small errors is fairly general and straightforward. Projecting the empirically determined solution \(F''\) through numerical calculations of \(-FF''\) onto a truncated Chebyshev basis \(\{T_j(\frac{2}{\pi}x - 1)\}_{j=0}^{N+1}\) provides for an accurate approximation of \(F''\). We re-express this in terms of a polynomial of degree \(N + 1\), and use rational number approximation for the coefficients in order to avoid round-off errors. Using the boundary condition at \(x = 0\), integration gives rise to an expression for \(F_0\) in theorem 1 for \(N = 12\), when we also enforce \(F''(0) = 0\), which follows from (5) and (6). Comparison of numerically computed \(F''\) with \(F''_0\) over the interval \(I\) gives an error of at most \(3.5 \times 10^{-7}\), which is a factor of 10 smaller than the bounds proved in theorem 1. For \(N = 8\), similar comparison gives an empirical accuracy of \(2 \times 10^{-5}\); the corresponding \(F_0\) on \(I\) in the form (13) (see also (9)) is given in (19). Mathematical theory guarantees that larger \(N\) will give rise to a more accurate solution. However, in order to avoid round-off errors, it is useful to avoid polynomials with large coefficients. If large order polynomials arise for accurate representation, it is better to break up the interval \(I\) into smaller subintervals and use lower order polynomials in each subinterval to obtain the \(F''_0\) approximation and use the continuity of \(F_0\) to determine a piecewise polynomial quasi-solution \(F_0\) in the entire interval \(I\).

In addition to variable \(x\), if parameters are involved such as \(\alpha\) ranging over some interval \(\alpha\), we determine a polynomial representation in \(x\) for a set of \(\alpha \in I\) and fit a polynomial in \(\alpha\) to each \(x\)-polynomial coefficient. Alternately, numerically computed data can be projected into a Chebyshev polynomial representation in two variables \(\alpha, x \in I\) to obtain the quasi-solution \(F_0\) reported in theorem 2.

For the complimentary \(x\)-interval \([\frac{5}{2}, \infty)\), we use rigorously controlled exponential asymptotic theory (Blasius 1908) to obtain a suitable representation for \(F_0\).

2. Main results for \((\alpha, \gamma) = (0, 0)\)

Let

\[
P(y) = \sum_{j=0}^{12} \frac{2}{5(j + 2)(j + 3)(j + 4)} p_j y^j,
\]

where \([p_0, ..., p_{12}]\) are given by

\[
\begin{bmatrix}
-510 & -18.523 & -4.2598 & -11.344 & -6.5173 & -0.390 & -0.0236 & 0.169 \\
10445 & 149 & 441 & 811 & 22093 & 6016 & 0.9858 & 1.000 \\
4134 & 879 & 1928001 & 20880 & 1572554 & 1546782 & 1315241 & 32239
\end{bmatrix}
\]

(10)

Define

\[
t(x) = \frac{a}{2} \left(x + \frac{b}{a}\right)^2, \quad I_0(t) = 1 - \sqrt{\pi} t e^{-t} \text{erfc} \left(\sqrt{t}\right), \quad J_0(t) = 1 - \sqrt{2\pi} t e^{\frac{b^2}{a}} \text{erfc} \left(\sqrt{2t}\right),
\]

(11)
where \( \text{erfc} \) denotes the complementary error function and let
\[
q_0(t) = 2c\sqrt{t}e^{-t}I_0 + c^2e^{-t}(2I_0 - I_0^2).
\]
The theorem below provides an accurate representation of solution \( F \) to (5)–(6).

**Theorem 1.** Let \( F_0 \) be defined by
\[
F_0(x) = \begin{cases} 
\frac{x^2}{2} + x^2P\left(\frac{2}{5}\right) & \text{for } x \in \left[0, \frac{5}{2}\right] \\
ax + b + \frac{a}{2t(x)}q_0(t(x)) & \text{for } x > \frac{5}{2}.
\end{cases}
\]

Then, there is a unique triple \((a, b, c)\) close to \((a_0, b_0, c_0) = \left(\frac{3211}{1762}, -\frac{2761}{1762}, \frac{377}{1762}\right)\) in the sense that \((a, b, c) \in S\) where
\[
S = \left\{ (a, b, c) \in \mathbb{R}^3 : \sqrt{(a - a_0)^2 + \frac{1}{4}(b - b_0)^2 + \frac{1}{4}(c - c_0)^2} \leq \rho_5 \right\}
\]
with the property that \( F_0 \) is a representation of the actual solution \( F \) to the initial value problem (5)–(6) within small errors. More precisely,
\[
F(x) = F_0(x) + E(x),
\]
where the error term \( E \) satisfies on \([0, \frac{5}{2}]\)
\[
\|E\|_\infty \leq 3.5 \times 10^{-3}, \quad \|E\|_1 \leq 4.5 \times 10^{-6}, \quad \|E\|_\omega \leq 4 \times 10^{-6},
\]
and for \( x \geq \frac{5}{2} \)
\[
|E| \leq 1.69 \times 10^{-3}t^{-3/4}, \quad \left|\frac{d}{dx}E\right| \leq 9.20 \times 10^{-3}t^{-3/2}e^{-3t},
\]
\[
\left|\frac{d^2}{dx^2}E\right| \leq 5.02 \times 10^{-4}t^{-1}e^{-3t}.
\]

**Remark 1.** Certainly, \( F \) is smooth since it is an actual solution of (5)–(6), which exists on \([0, \infty)\) and is unique; see Weyl (1942). However, the particular choice \((a, b, c) \in S\) in theorem 1 needed in order for \( F = F_0 + E \) to solve (5)–(6) does not ensure continuity of the approximate solution \( F_0 \) at \( x = \frac{5}{2} \). Nonetheless, if \( F_0, F_0', F_0'' \) are needed to be continuous, this can be achieved by a slightly different choice of \((a, b, c) \in S\) (see remark 7), namely
\[
(a, b, c) = \left(1.655 190 456 1499..., -1.565 439 826 457..., 0.233 728 727 537...ight).
\]
Note also that (17) implies not only small absolute errors (that, in the far field hold even for the approximation of \( F^* \) by zero) but also very small relative errors on \([\frac{5}{2}, \infty)\).

**Definition 1.** Let \( a_i = a_0 - \rho_i, a_1 = a_0 + \rho_1, b_i = b_0 - 2\rho_i, b_1 = b_0 + 2\rho_1, c_i = c_0 - 2\rho_i\), and \( c_1 = c_0 + 2\rho_1 \). We see that \((a, b, c) \in S\) implies that \( a \in [a_i, a_i], b \in [b_i, b_i], c \in (c_i, c_i)\).
We define \( t_m = \frac{1}{2} \left( \frac{1}{2} + \frac{b}{a} \right)^2 \) and note that \( x \in \left[ \frac{1}{2}, \infty \right) \) corresponds to \( t \in [t_m, \infty) \) and \( t_m \in \left( \frac{1}{2} \left( \frac{1}{2} + \frac{b}{a} \right)^2, \frac{1}{2} \left( \frac{1}{2} + \frac{b}{a} \right)^2 \right) = (1.998859\ldots, 1.999438\ldots) \).

**Remark 2.** The error bounds proved for \( E \) in theorem 1 are likely a 10-fold over-estimate. Comparison with the numerically calculated \( F \) suggests that \( \| F - F_0 \|, \| F' - F'_0 \| \) and \( \| F'' - F''_0 \| \) in \([0, \frac{5}{2}]\) are at most \( 2 \times 10^{-7}, 2 \times 10^{-7} \) and \( 5 \times 10^{-7} \) respectively. Using the non-rigorous bounds on \( E \) and its derivatives reduces the \( \rho_0 \) in the definition of \( \Sigma \) from \( \times - 5 \times 10^{-5} \) to \( \times - 1.4 \times 10^{-5} \). It is thus likely that \((a, b, c) \approx (a_0, b_0, c_0)\) with five (rather than the proven four) digit accuracy. Further, there is no theoretical limitation in the accuracy in this approach. Higher accuracy will require a higher order or piecewise polynomial expressions in \([0, \frac{5}{2}]\) and using a higher order truncation of the series (56) for \( q_{0r} \), as explained in the ensuing.

**Remark 3.** If we choose to use \( N = 8 \) instead of \( N = 12 \) in the empirical polynomial approximation of \( F^{0r} \) on \( I \) explained earlier, the quasi-solution \( F_0 \) on \( I \) in the form (13) (see (9) as well) now has coefficients \( \left\{ p_j \right\}_{j=0}^{8} \) which take the values

\[
\begin{align*}
-749 & \quad -79\,496 \\
729 & \quad 151 \\
-25\,889 & \quad -21\,549 \\
21\,880 & \quad 154\,757 \\
18\,383 & \quad -581\,201 \\
19\,331 & \quad 877\,116 \\
847\,199 & \quad -282\,687 \\
10\,433 & \quad 714\,562 \\
11\,372 & \quad 4603 \\
\end{align*}
\]

(19)

**Remark 4.** The choice of the break point \( x_0 = \frac{5}{2} \) in the piecewise definition of \( F_0 \) in (13) is not special; other values will ensure a comparable degree of accuracy. For instance, if \( x_0 < \frac{5}{2} \), \( F_0 \) will be replaced by a lower degree polynomial on \([0, x_0]\). However, this would require a higher order truncation of the series (56) for \( q_{0r} \) for the far-field representation.

The proof of theorem 1 rests on the following three propositions; we will discuss the idea behind the proofs of these propositions in later sections.

**Proposition 2.** The error term \( E(x) = F(x) - F_0(x) \) verifies the equation

\[
\mathcal{L} \left[ E \right] = E'' + F'_0 E'' + F_0 E = -F'_0 E F''_0 - EE' + EE'' ,
\]

(20)

\[
E(0) = 0 = E'(0) = E''(0),
\]

(21)

and satisfies the bounds (16) on \( I = \left[ 0, \frac{5}{2} \right] \).

**Proposition 3.** For given \((a, b, c) \) with \( a > 0, |c| < \frac{5}{2}, \) in the domain \( x \geq \frac{a}{2} + \sqrt{\frac{a}{2}}, \) for \( T \geq 1.99, \) which corresponds to \( t \geq T \geq 1.99, \) there exists unique solution to (5) in the form

\[
F(x) = ax + b + \sqrt{\frac{a}{2t} \frac{x}{q(t)}} \left( q(t(x)) \right),
\]

(22)
that satisfies the condition \( \lim_{t \to \infty} \frac{q(t)}{t} = 0 \). Furthermore,
\[
q(t) = q_0(t) + \mathcal{E}(t),
\]
where \( \mathcal{E} \) is small and satisfies the following error bounds:
\[
\left| \mathcal{E}(t) \right| \leq 1.6667 \times 10^{-4} \frac{e^{-3t}}{9t^{1/2}},
\]
\[
\left| \mathcal{E}'(t) - \frac{1}{2t} \mathcal{E}(t) \right| \leq 1.6667 \times 10^{-4} \frac{e^{-3t}}{3t^{1/2}},
\]
\[
\left| \sqrt{t} \mathcal{E}''(t) - \frac{1}{\sqrt{t}} \mathcal{E}'(t) + \frac{1}{2t^{1/2}} \mathcal{E}(t) \right| \leq 1.6667 \times 10^{-4} t^{-1} e^{-3t}.\]

**Proposition 4.** There exists a unique triple \((a, b, c) \in S\) so that the functions in the previous two propositions: \( F_0(x) + E(x) \) for \( x \leq \frac{5}{2} \) and \( ax + b + \frac{a}{\sqrt{2t(\gamma)}} q(t(x)) \) for \( x \geq \frac{5}{2} \) and their first two derivatives agree at \( x = \frac{5}{2} \).

The proof of theorem 1 follows from propositions 2–4 in the following manner: proposition 2 implies that \( F(x) = F_0(x) + E(x) \) satisfies (5)–(6) for \( x \in I \); we note that \( F_0(0) = 0 = F_0(0) \) and \( F_0''(0) = 1 \).

Proposition 3 implies \( F(x) = ax + b + \frac{a}{\sqrt{2t(\gamma)}} \left[ q_0(t(x)) + \mathcal{E}(t(x)) \right] \) satisfies (5) in a range of \( x \) that includes \( \left[ \frac{5}{2}, \infty \right) \) when \( (a, b, c) \in S \). Further, proposition 4 ensures that this is the same solution of the ODE (5) as the one in proposition 2. Identifying \( F_0(x) \) and \( E(x) \) in theorem 1 in this range of \( x \) with \( ax + b + \frac{a}{\sqrt{2t(\gamma)}} q_0(t(x)) \) and \( \frac{a}{\sqrt{2t(\gamma)}} E(t(x)) \), respectively, and relating \( x \)-derivatives to \( t \)-derivatives, the error bounds for \( E, E' \) and \( E'' \) follow from the ones given for \( \mathcal{E} \) in proposition 3 for \((a, b, c) \in S \). We discuss these propositions in later sections.

3. Solution in the interval \( I = [0, \frac{5}{2}] \) for \((\alpha, \gamma) = (0, 0)\) and proof of proposition 2

As mentioned earlier, the quasi-solution \( F_0 \) in the compact set \( I = [0, \frac{5}{2}] \) is obtained simply by projecting a numerical solution on the subspace spanned by the first few Chebyshev polynomials \( \{ T_j \left( \frac{2x}{5} - 1 \right) \}_{j=0}^N \}. To avoid estimating derivatives of an approximation, which are not well controlled, we project instead the approximate third derivative \( F'' = -F^2 F \) on the interval \( I \). The rigorous control of the errors of the integrals of \( F'' \) is a much simpler task. For a given polynomial degree, a Chebyshev polynomial approximation of a function is known to be, typically, close to the most accurate polynomial approximation, in the sense of \( L^2 \). A power series is less efficient since it is constrained by complex plane behavior.

We seek to control the error term \( E \) in (15) by first estimating the remainder
\[
R(x) = F''_0(x) + F_0(x) F_0''(x),
\]
which will be shown to be small \((\leq 0.673 \times 10^{-6})\). Then, we invert the principal part of the linear part of the equation for the error term \(E\) by using initial conditions to obtain a nonlinear integral equation. The smallness of \(R\) and careful bounds on the resolvent \(L^{-1}\) help prove proposition 2.

3.1. Estimating the size of remainder \(R(x)\) for \(x \in I\)

Since \(P\) is a polynomial of degree 12, \(R(x)\) is a polynomial of degree 30. We estimate \(R\) in \(I\) in the following manner. We break up the interval into subintervals \(\left\{ \left[ x_{j-1}, x_j \right] \right\}_{j=1}^{14}\) with \(x_0 = 0\) and \(x_{14} = \frac{5}{7}\), while \(\{ x_j \}_{j=1}^{14}\) is given by \(\{ 0.0625, 0.125, 0.25, 0.375, 0.50, 0.75, 1.0, x_i, 1.5, 1.75, 2.0, 2.25, 2.40 \}\), where \(x_i = 1.322 040.\) The intervals were chosen based on how rapidly the polynomial \(R(x)\) varies locally.

We re-expand \(R(x)\) as a polynomial in the scaled variable \(\tau\), where

\[
x = \frac{1}{7} (x_j + x_{j-1}) + \frac{1}{7} (x_j - x_{j-1}) \tau
\]

and determine the maximum \(M_j\) and minimum \(m_j\) of the third degree polynomial \(P_3^{(j)}(\tau)\) for \(\tau \in \left[ -1, 1 \right]\) (using simple calculus). We bound the contribution of the remaining terms:

\[
E_k^{(j)} = \sum_{k=4}^{30} d_k^{(j)} \leq E_k^{(j)}.
\]

It follows that in the \(j\)-subinterval we have

\[
m_j - E_k^{(j)} \leq R(x) \leq M_j + E_k^{(j)}.
\]

The maximum and minimum over any union of subintervals is found simply taking min and max of \(m_j - E_k^{(j)}\) and \(M_j + E_k^{(j)}\) over the the indices \(j\) for subintervals involved. This elementary though tedious calculation\(^3\) yields

\[
-3.22 \times 10^{-7} \leq R(x) \leq 2.505 \times 10^{-7} \quad \text{for } x \in \left[ 0, x_i \right],
\]

\[
4.6 \times 10^{-8} \leq R(x) \leq 4.06 \times 10^{-7} \quad \text{for } x \in \left[ x_i, 2.0 \right],
\]

\[
2.78 \times 10^{-7} \leq R(x) \leq 6.73 \times 10^{-7} \quad \text{for } x \in \left[ 2.0, 2.5 \right].
\]

(28)

We note that the remainder is at most \(6.73 \times 10^{-7}\) in absolute value in the interval \(I\). In the same way, we find bounds for the polynomials \(F'_0(x), F'_0, F''_0(x)\). For \(x \in \left[ 0, 1 \frac{1}{8} \right]\),

\[\text{As described elsewhere (Blasius 1908), it is convenient to choose one of the subdivision points } x_i \text{ to be approximately, to the number of digits quoted, the value of } x \text{ where } F''_0(x) - 2F'_0(x) + 1 \text{ changes sign.}\]

\[\text{The maximum and minimum found through the analysis described here is found to be consistent with a numerical plot of the graph of } R(x), \text{ as must be the case. The calculations can be conveniently done with a computer algebra program, as they only involve operations with rational numbers.}\]
\[-5 \times 10^{-10} \leq F_0'(x) \leq 0.008, -8 \times 10^{-12} \leq F_0''(x) \leq 0.13, \]
\[0.99 \leq F_0^*(x) \leq 1 + 2 \times 10^{-9}, \quad (29)\]

while for \(x \in \left[\frac{1}{z}, \frac{2}{z}\right]\)
\[0.03 \leq F_0(x) \leq 2.59, 0.12 \leq F_0'(x) \leq 1.7, 0.09 \leq F_0''(x) \leq 1. \quad (30)\]

A less unwieldy strategy for residual error estimation is to find \(A_j\) so that
\[R(x) = \sum_{j=0}^{30} A_j T_j \left(\frac{4x}{5} - 1\right). \quad (31)\]

Since for \(y \in [-1, 1]\), \(T_j(y) = \cos\left(j \cos^{-1} y\right)\) is less than 1 in absolute value
\[
\| R \|_{\infty,y} \leq \sum_{j=0}^{30} |A_j| \leq 9.74 \times 10^{-7}. \quad (32)\]

Projecting \(R\) instead to Chebyshev polynomials in each of the sub intervals \([0, x_1], [x_1, 2]\) and \([2, 2.5]\) gives somewhat better bounds. In both cases, the bounds are not as sharp as the ones estimated through local Taylor series expansion. Nonetheless, this method is simpler and more easily adapted to multi-variables.

### 3.2. Error estimate on a sub interval \([x_l, x_r] \subset I\)

Consider the decomposition
\[F(x) = F_0(x) + E(x). \quad (33)\]

We seek to find error estimates for \(E(x)\) and its first two derivatives for \(x \in I\). On \([x_l, x_r] \subset I\), where \(E(x_l), E'(x_l)\) and \(E''(x_l)\) are considered known, \(E\) satisfies
\[L[E] = E''(x) + F_0(x)E''(x) + F_0'(x)E(x) = -E(x)E''(x) - R(x). \quad (34)\]

Using a variation of parameter approach, where \(\{\Phi_j\}_{j=1}^{3}\) are fundamental solutions to \(L\Phi = 0\), we may invert the operator \(L\) by using the boundary condition at \(x = 0\) to obtain an integral equation in the form
\[E'(x) = \sum_{j=1}^{3} E^{(j-1)}(x_j) \Phi_j'(x) - G[R](x) - G[EE'](x) = : N[E'](x), \quad (35)\]

and where \(E\) is given in terms of \(E'\):
\[E(x) = E(x_l) + (x - x_l)E'(x_l) + \int_{x_l}^{x} (x - t)E''(t)dt, \quad (36)\]

Note that (36) allows control of \(\|\|_{\infty}\) (sup)-norm of \(E\) and \(E'\) in terms of \(E''\). Error estimates follow by showing that integral equation (34) written abstractly \(E'' = N[E']\) has a unique solution in the space of continuous functions in a small ball in the \(\|\|_{\infty}\) norm by showing that \(N\) is contractive (see Costin and Tanveer (2013) for details). This is possible without explicit knowledge of \(\{\Phi_j\}_{j=1}^{3}\) or the resolvent operator \(G\), provided that the bounds are not too large.

In the following subsection we detail how energy methods can be used for that purpose.
3.3. Green’s function estimate

Consider now the problem of solving the linear generally inhomogeneous equation

$$L[\phi] = \phi^\nu(x) + F_0(x) \phi^\nu (x) + F_0^\nu (x) \phi (x) = r(x),$$

(37)

over a typical subinterval $[x_i, x_j] \subset I$, with initial conditions $\phi(x_i), \phi'(x_i)$ and $\phi''(x_i)$ known. The solution of this inhomogeneous equation is given by the standard variation of parameter formula

$$\phi(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_j) \Phi_j(x) + \sum_{j=1}^{3} \Phi_j(x) \int_{x_j}^{x} \Psi_j(t) r(t) dt,$$

(38)

where $\{\Phi_j\}_{j=1}^{3}$ form a fundamental set of solutions to $L\phi = 0$ and $\{\Psi_j(x)\}_{j=1}^{3}$ are elements of the inverse of the fundamental matrix constructed from the $\Phi_j$ and their derivatives. The precise expressions are unimportant in the ensuing: we only need their smoothness in $x$. It also follows from the properties of $\Phi_j$ and $\psi_j$ that

$$\phi''(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_j) \Phi_j''(x) + \sum_{j=1}^{3} \Phi_j''(x) \int_{x_j}^{x} \Psi_j(t) r(t) dt.$$

(39)

It is useful to write (39) in the following abstract form:

$$\phi''(x) = \sum_{j=1}^{3} \phi^{(j-1)}(x_j) \Phi_j''(x) + \mathcal{G}[r](x),$$

(40)

where from the general properties of the fundamental matrix and its inverse for the linear ODEs with smooth (in this case polynomial) coefficients $\mathcal{G}$ is a bounded linear operator on $C([x_i, x_j])$; denote its norm by $M$,

$$M = \| \mathcal{G} \|.$$

(41)

Then, on the interval $[x_i, x_j]$ we have,

$$\| \phi'' \|_{\infty} \leq \sum_{j=1}^{3} M_j \left| \phi^{(j-1)}(x_j) \right| + M \| r \|_{\infty} ; \quad M_j = \sup_{x \in [x_i, x_j]} \left| \Phi_j''(x) \right|.$$

(42)

We will outline how estimates of $M_j$ for $j = 1, \ldots, 3$ and $M$ may be obtained indirectly, using ‘energy’ bounds. Because of linearity of the problem, for the purposes of determining these bounds, it is useful to separately consider the cases (i)–(iii), when $r = 0$, $\phi^{(k-1)}(x_j) = 0$ for $1 \leq k \neq j \leq 3$, $\phi^{(l-1)}(x_i) = 1$ respectively, and, finally, (iv) when $\phi^{(k-1)}(x_j) = 0$ for $k = 1, \ldots, 3$ and $r (i) \neq 0$. For all cases (i)–(iv), we return to the ODE

$$\phi'' + F_0 \phi'' + F_0^\nu \phi = r.$$

(43)

Multiplying by $2\phi''$ and integration gives

$$\left( \phi''(x) \right)^2 = \left( \phi''(x_i) \right)^2 - \int_{x_i}^{x} \left\{ 2F_0(y) \left( \phi''(y) \right)^2 + 2F_0^\nu(y) \phi''(y) \phi(y) - 2\phi''(y) r(y) \right\} dy.$$

(44)

4 In particular, $\sum_{j=1}^{3} \phi_j(x) \psi_j(x) = 0, \sum_{j=1}^{3} \phi_j'(x) \psi_j'(x) = 0$. 
Further, for given $\phi(x)$ and $\phi'(x)$, $\phi(x)$ is determined from $\phi''(x)$ from
\[
\hat{\phi}(x) := \phi(x) - \phi(x_l) - (x - x_l)\phi'(x_l) = \int_{x_l}^{x} (x - y)\phi''(y)\,dy.
\] (45)

Using (45) in (44), it follows that
\[
(\phi''(x))^2 = (\phi''(x_l))^2 - \int_{x_l}^{x} 2F_0''(y)[\phi(x_l) + (y - x_l)\phi'(x_l)]\phi''(y)\,dy
- \int_{x_l}^{x} \left[2F_0''(y)(\phi''(y))^2 + 2F_0''(y)\phi''(y)\hat{\phi}(y) - 2\phi''(y)\hat{r}(y)\right]\,dy.
\] (46)

Using the Cauchy–Schwartz inequalities, the relation between $\phi$ and $\phi''$ and Gronwall’s inequality, it is not difficult to prove by considering separately cases (i)–(iv) that
\[
M_1 \leq (F_0''(x_l) - F_0''(x))^{1/2} \exp\left[\frac{1}{2} \int_{x_l}^{x} Q_1(y)\,dy\right],
\] (47)
\[
M_2 \leq \left(\int_{x_l}^{x} (y - x_l)^2F_0''(y)\,dy\right)^{1/2} \exp\left[\frac{1}{2} \int_{x_l}^{x} Q_1(y)\,dy\right],
\] (48)
\[
M_3 \leq \exp\left[\frac{1}{2} \int_{x_l}^{x} Q_1(y)\,dy\right],
\] (49)
\[
M \leq (x_l - x)^{1/2} \exp\left[\frac{1}{2} \int_{x_l}^{x} Q(y)\,dy\right],
\] (50)

where
\[
Q_1(x) = \begin{cases}
F_0''(x)\left(2 + \frac{(x - x_l)^2}{4}\right) - 2F_0(x) & \text{if } (2F_0''(x) - 2F_0(x) > 0) \\
\frac{(x - x_l)^4}{4}F_0''(x) & \text{if } (2F_0''(x) - 2F_0(x) \leq 0)
\end{cases}
\] (51)
\[
Q_2(x) = \begin{cases}
\left(1 + \frac{(x - x_l)^2}{4}\right)F_0''(x) - 2F_0(x) & \text{if } (F_0''(x) - 2F_0(x) > 0) \\
\frac{(x - x_l)^4}{4}F_0''(x) & \text{if } (F_0''(x) - 2F_0(x) \leq 0)
\end{cases}
\] (52)
\[
Q(x) = \begin{cases}
F_0''(x) - 2F_0(x) + 1 + \frac{(x - x_l)^2}{4}F_0''(x) & \text{if } (F_0'' - 2F_0 + 1 \geq 0) \\
\frac{(x - x_l)^4}{4}F_0''(x) & \text{if } (F_0'' - 2F_0 + 1 < 0)
\end{cases}
\] (53)

It is possible (Costin and Tanveer 2013) to estimate $M$, $M_1$, $M_2$, $M_3$ in three subintervals $[0, x_l]$, $[x_l, 2]$ and $[2, 2.5]$ and use them to get small error bounds for $E$, $E'$ and $E''$ to complete the proof of proposition 2.
4. Solution in $t \geq T \geq 1.99$ for $|c| < \frac{1}{2}$, $a > 0$ and proof of proposition 3

The construction of quasi-solution $F_0$ for $x \in \left[ \frac{5}{2}, \infty \right)$ relies on precise large $x$ asymptotics, which as it turns out, gives a desirably accurate solution representation in the entire interval. For the Blasius solution, it is known that any solution with $\lim_{x \to \infty} F'(x) = a > 0$ must have the representation

$$F(x) = ax + b + G(x),$$

(54)

where $G(x)$ is exponentially small in $x$ for large $x$. Indeed, through change of variable $t = t(x)$ given by (11) and $G = \sqrt{\frac{2}{\pi}} q(t)$, $q$ satisfies

$$\frac{d^3}{dt^3}q + \left(1 + \frac{q}{2t}\right) \frac{d^2}{dt^2}q + \left(-\frac{1}{2t} + \frac{3}{4t^2} - \frac{q}{4t^3}\right) \frac{dq}{dt} + \left(\frac{1}{2t^2} - \frac{3}{4t^3}\right)q + \frac{q^2}{4t} = 0,$$

(55)

and from a general theory (Costin 1998)\(^5\) it may be deduced that small solutions $q$ must have the convergent series representation

$$q(t) = \sum_{n=1}^{\infty} \xi^n Q_n(t) \quad \text{where } \xi = \frac{ce^{-t}}{\sqrt{t}},$$

(56)

where the equations for $Q_n$ may be deduced by plugging in (56) into (55) and equating different powers of $\xi$. With appropriate matching at $\infty$, one obtains $Q_0(t) = 2tI_0(t)$ and $Q_1(t) = -tI_0 - tI_0^2 + 2tJ_0$, where

$$I_0 = 1 - \sqrt{\pi} e^{\frac{1}{2}} \text{erfc} \left(\sqrt{\frac{1}{2}}\right),$$

(57)

$$J_0 = 1 - \sqrt{2\pi} e^{\frac{1}{2}} \text{erfc} \left(\sqrt{2}\right).$$

(58)

The two-term truncation of (56) proved adequate to determine an accurate quasi-solution in an $x$-domain that corresponds to $t \geq 1.99$ if $|c| < \frac{1}{2}$ to within the quoted accuracy. Note that the solution is only complete after determining $(a, b, c)$ through matching of $F$, $F'$ and $F''$ at $x = \frac{5}{2}$. Since $(a, b, c)$ only needs to be restricted to some small neighborhood of $(a_0, b_0, c_0)$ to accomplish matching (see proposition 4), the restriction $t \geq 1.99$ is seen to include $x \geq \frac{5}{2}$.

Furthermore, the restriction $|c| < \frac{1}{2}$ in proposition 4 is appropriate for the quoted error estimates in $x \geq \frac{5}{2}$ in theorem 1.

We decompose

$$q(t) = q_0(t) + E(t),$$

(59)

\(^5\) Though the non-degeneracy condition stated in Costin (1998) does not hold, a small modification leads to the same result.
where

\[ q_0(t) = \frac{ce^{-t}}{\sqrt{t}}Q_0(t) + \frac{c^2e^{-2t}}{t}Q_2(t), \] (60)

where

\[ Q_1(t) = 2tI_0(t), \quad \text{where } I_0(t) = 1 - \sqrt{\pi t}e^t \text{erfc} \left( \sqrt{t} \right) = \frac{1}{2} \int_0^\infty \frac{e^{-s}}{(1 + s^2)^{1/2}} ds, \] (61)

\[ Q_3(t) = -tI_0(t) + tI_0(t) \quad \text{where } I_0(t) = 1 - \sqrt{2\pi t}e^t \text{erfc} \left( \sqrt{2t} \right) \]
\[ = \frac{1}{4} \int_0^\infty \frac{e^{-s}}{(1 + s/2)^{1/2}} ds. \] (62)

We obtain a nonlinear integral equation for \( h \), which is related to \( \mathcal{E} \) as follows:

\[ \mathcal{E}(t) = \sqrt{t} \int_0^\infty \frac{dx}{\sqrt{x}} \int_0^x \frac{e^{-s}}{\sqrt{s}} h(\tau) d\tau. \] (63)

A contraction mapping argument in a small ball is possible by exploiting the smallness of the residual \( R = R(t) \) given by

\[ R = \frac{d^2}{dx^2}q_0 + \left( 1 + \frac{q_0}{2t} \right) \frac{d^2}{dt^2}q_0 + \left( -\frac{1}{2t} + \frac{3}{4t^2} - \frac{q_0}{4t^3} \right) \frac{dq_0}{dt} + \left( \frac{1}{2t^2} - \frac{3}{4t^3} \right) q_0 + \frac{q_0}{4t^4}. \] (64)

This leads to the proof of proposition 3 (see Costin and Tanveer (2013) for details).

**Remark 5.** The triple \((a_0, b_0, c_0)\) was chosen from the observation of this section. Indeed, let \( \tilde{F}(x) \) be the numerical solution of the given IVP (5)–(6) on the interval \( I \). From (54) and its asymptotic behavior at infinity, we expect \( \tilde{F}'(x) \) to be close to \( a \) for all sufficiently large \( x \). In fact, the numerical plot of \( \tilde{F}' \) is observed to asymptote to some value, which is taken to be \( a_0 \).

Similarly, we found that \( \tilde{F}(x) - a_0x \) approached some constant, call it \( b_0 \), for large\(^6 \) \( x \). To determine \( c_0 \), we look at the second derivative \( F''(x) \)

\[ F''(x) = \frac{d^2}{dx^2} \left( \sqrt{\frac{a}{2t(x)}} q \left( t(x) \right) \right). \]

Truncating the series \( q(t) \) at the first term, the right-hand side of the above becomes

\[ \frac{d^2}{dx^2} \left( \sqrt{\frac{a}{2t(x)}} e^{-t(x)} Q_0 \left( t(x) \right) \right) = \sqrt{2} a^{3/2} e^{-\frac{(a+b)^2}{2x}}. \]

We then set \( F''(x) = \sqrt{2} a_0^{3/2} e^{-\frac{(a+b)^2}{2x_0}} \) for a sufficiently large \( x \) and solve for \( c \); this is our \( c_0 \).

\(^6\) If \( x \) is taken too large, the small error in \( a_0 \) contaminates this calculation. So, some care is needed in the choice of large \( x \).
5. Matching for \((\alpha, \gamma) = (0, 0)\) and proof of proposition 4

In order for the two representations (33) and \(F(x) = ax + b + \sqrt{\frac{a}{\beta(x)}} \left( \frac{q_0(t(x)) + E(t(x))}{2m} \right) \) to coincide at \(x = \frac{5}{2} \), we match \(F\) and its two derivatives; from (60), (59) and (63) we get

\[
a = F' \left( \frac{5}{2} \right) - a \left( \frac{q_0(t_m; c)}{2m} \right) - a \int_{-\infty}^{\infty} e^{-t} h(t; c) \, dt = : N_1(a, b, c),
\]

(65)

\[
b = F' \left( \frac{5}{2} \right) - \frac{5}{2} N_1 - \frac{a}{2m} q_0(t_m; c) - \frac{a}{2} \int_{-\infty}^{\infty} e^{-\tau} \int_{-\infty}^{\tau} s^{-\frac{1}{2}} e^{-s} h(s; c) \, ds \]

\[
\quad : = N_2(a, b, c),
\]

(66)

\[
c = \frac{1}{\sqrt{2a^{3/2}}} \left[ V(t_m; c) + \frac{1}{c} h(t_m; c) \right]^{1/4} e^{\sigma F''} \left( \frac{5}{2} \right) = : N_3(a, b, c),
\]

(67)

where

\[
V(t; c) = -\frac{2}{c} teB(t; c) \quad \text{and} \quad B(t; c) = -\frac{q_0''(t)}{2t^{3/2}} + \frac{q_0(t)}{4t^{3/2}}.
\]

**Definition 5.** We define \(A = (a, b, c)\), \(A_0 = (a_0, b_0, c_0)\) and \(N(A) = (N_1, \frac{1}{4}N_2, \frac{1}{4}N_3)\). Define also

\[
S_A = \left\{ \| A - A_0 \|_2 \leq \rho_0 := 5 \times 10^{-5} \right\},
\]

where \(\| . \|_2\) is the Euclidean norm and let

\[
J = \begin{pmatrix}
\frac{1}{2} \partial N_1 & 2 \partial N_1 & 2 \partial N_1 \\
1 \partial N_2 & \partial N_2 & \partial N_2 \\
\frac{1}{2} \partial N_3 & \partial N_3 & \partial N_3
\end{pmatrix}.
\]

(68)

**Note 6.** We see that \(A \in S_A\) implies \((a, b, c) \in S\). The system of equations (65)–(67) is written as

\[
A = N[A].
\]

(69)

We define \(J = \frac{dN}{dA}\) to be the Jacobian and \(\|J\|_2\) denotes the \(l^2\)-norm of the matrix. We note that

\[
\| J \|_2^2 = (\partial N_1)^2 + 4 (\partial N_1)^2 + 4 (\partial N_1)^2 + \frac{1}{4} (\partial N_1)^2
\]

\[
+ (\partial N_2)^2 + (\partial N_2)^2 + \frac{1}{4} (\partial N_2)^2 + (\partial N_2)^2 + (\partial N_3)^2 + (\partial N_3)^2.
\]

(70)
Lemma 6. The inequalities
\[ \| A_0 - N[A_0] \|_2 \leq (1 - \alpha)\rho_0, \tag{71} \]
\[ \sup_{A \in S_0} \| J \|_2 \leq \alpha < 1, \tag{72} \]
for some \( \alpha \in (0, 1) \) imply that \( A = N[A] \) has a unique solution for \( A \in S_0 \).

Proof. The mean value theorem implies
\[ \| N[A] - A_0 \|_2 \leq \| N[A_3] - A_0 \|_2 + \| N[A] - N[A_3] \|_2 \leq \rho_0 (1 - \alpha) \]
\[ + \| J \|_2 \rho_0 \leq \rho_0 \tag{73} \]
and also, if \( A_1, A_2 \in S_0 \),
\[ \| N[A_1] - N[A_2] \|_2 \leq \| J \|_2 \| A_1 - A_2 \|_2 \leq \alpha \| A_1 - A_2 \|_2. \]
Thus, (71) and (72) imply that \( N: S_0 \rightarrow S_0 \) and that it is contractive there; the result follows
from the contractive mapping theorem.

5.1. Proof of proposition 4

Proposition 4 follows from lemma 6 once we verify that (71) and (72) hold. This has been
shown (Costin and Tanveer 2013) for \( \alpha \leq 0.764 \) and that
\[ \| A_0 - N[A_3] \|_2 \leq 1.16 \times 10^{-5} \leq (1 - \alpha)\rho_0 \] thereby completing the proof of proposition 4.

Remark 7. Note that the proof of proposition 4 only requires smallness of the norms of \( h \)
and \( E \) (we recall that \( F = F_0 + E \) and on no further details about them. If in some application
\( F_0 \) needs to be made \( C^2 \), then this can be ensured by iterating \( N \) with \( h = E = 0 \); the first
thirteen digits obtained in this way are given in (18).

6. Generalization for \( (\alpha, \gamma) \neq (0, 0) \)

Here we consider for simplicity the special case \( \gamma = 0, \alpha \in \left[ -\frac{3}{65}, \frac{1}{65} \right] \). Through piecewise
polynomial representatons, other intervals in \( \alpha \) can similarly be incorporated; it is to be noted
that the non-existence of a globally acceptable solution for some ranges of \( (\alpha, \gamma) \) is manifest
in the present approach by lack of matching at \( x = \frac{3}{2} \).

Let
\[ P(y; \beta) = \sum_{i=0}^{13} \sum_{j=0}^{3} \frac{P_{ij}}{(i+1)(i+2)(i+3)} \beta^i y^j, \tag{74} \]
where $p_{ij}$ is the $(i + 1, j + 1)$-entry of the following matrix

\[
\begin{bmatrix}
29589 & -9845 & -274 & 241 & -422 & 308 \\
29589 & -9845 & -274 & 241 & -422 & 308 \\
1493 & 185 & 36 & 25 & 221 & 308 \\
1706 & 376 & 476 & 686 & 948 & 9263 \\
203 & 116 & 15 & 44 & 121 & 296 \\
65155 & 970153 & 235 & 863 & 890 & 5983 \\
72804 & 239 & 497 & 213 & 995 & 79 \\
75433 & 147 & 253 & 192 & 583 & -29 \\
106800 & 112 & 122 & 155 & 285 & 525 \\
43663 & 86 & 717 & 19 & 732 & -17 \\
387344 & 77 & 473 & 304 & 475 & 304 \\
32609 & 4402 & 15 & 867 & 26 & 568 \\
3084825 & 1006 & 711 & 171 & 511 & 1097 \\
27611 & 9319 & 4286 & 24 & 721 & 2915 \\
2254258 & 3595 & 213 & 1049 & 674 & 2081 \\
5883 & 9561 & 2379 & 4399 & 19943 & 5459 \\
1915077 & 3165 & 632 & 5196 & 992 & 38399 \\
2126 & 3527 & 3543 & 1327 & 2153 & 9363 \\
2860297 & 3706 & 169 & 5245 & 388 & 6522 \\
1927 & 2627 & 1929 & 317 & 1366 & 33 \\
281944 & 3174 & 435 & 5003 & 871 & 6098 \\
179 & 2257 & 1621 & 1117 & 958 & 4606 \\
2506157 & 2704 & 059 & 8285 & 683 & 4186 \\
2481 & 3157 & 3873 & 1295 & 863 & 463 \\
2072736 & 1425 & 478 & 3778 & 762 & 3100 \\
5813 & 4881 & 4529 & 486 & 1537 & 821 \\
1051227 & 745 & 495 & 1839 & 247 & 2241 \\
19699 & 17357 & 13071 & 5276 & 6290 & 30274 \\
\end{bmatrix}
\]

(75)

**Theorem 2.** For any $\alpha \in \left[-\frac{3}{50}, \frac{3}{50}\right]$, let $F_\alpha$ be defined by

\[
F_\alpha(x; \alpha) = \begin{cases} 
\frac{\alpha + x^2}{2} + x^3\left(\frac{2}{5}; \frac{25}{3} \alpha + \frac{1}{2}\right) & \text{for } x \in \left[0, \frac{5}{2}\right] \\
ax + b + \frac{a}{2\sqrt{t(x)}}q_\alpha(t(x)) & \text{for } x \in \left(\frac{5}{2}, \infty\right).
\end{cases}
\]

(76)

where $t(x)$ and $q_\alpha(t)$ are as defined in (11) and (12). Let $a_\alpha(\alpha)$, $b_\alpha(\alpha)$, and $c_\alpha(\alpha)$ be defined by

\[
a_\alpha(\alpha) = \frac{3221}{1946} - \frac{797}{603} \alpha + \frac{176}{289} \alpha^2, \tag{77}
\]

\[
b_\alpha(\alpha) = -\frac{2763}{1765} + \frac{761}{284} \alpha - \frac{194}{237} \alpha^2, \tag{78}
\]
\[ c_1(\alpha) = \frac{377}{1613} + \frac{174}{1387} \alpha + \frac{937}{6822} \alpha^2. \]  

(79)

Then, for each \( \alpha \in \left[ -\frac{1}{50}, \frac{1}{350} \right] \), there exists a unique triple \((a, b, c)\) close to \((a_0(\alpha), b_0(\alpha), c_0(\alpha))\) in the sense that \((a, b, c) \in \mathcal{S}_\alpha\) where

\[
\mathcal{S}_\alpha = \left\{ (a, b, c) \in \mathbb{R}^3 : \sqrt{(a - a_0(\alpha))^2 + \frac{1}{4}(b - b_0(\alpha))^2 + \frac{1}{4}(c - c_0(\alpha))^2} \leq \rho_\alpha = 5 \times 10^{-1} \right\},
\]

(80)

with the property that \( \mathcal{F}_0 \) is a representation of the actual solution \( F \) to the initial value problem (5) and (7) with \( \gamma = 0 \) and \( \alpha \in \left[ -\frac{1}{50}, \frac{1}{350} \right] \) within small errors. More precisely,

\[
F(x; \alpha) = \mathcal{F}_0(x; \alpha) + E(x; \alpha),
\]

(81)

where the error term \( E \) satisfies on \( [0, \frac{5}{7}] \)

\[
\| E \|_\infty \leq 9.01 \times 10^{-5}, \quad \| E \|_\infty^2 \leq 4.51 \times 10^{-6}, \quad \| E \|_\infty^4 \leq 2.16 \times 10^{-6}, \quad (82)
\]

where \( \| E \|_\infty := \sup \{ |E(x; \alpha)| : x \in [0, \frac{5}{7}], \alpha \in \left[ -\frac{1}{50}, \frac{1}{350} \right] \} \), and for \( x \geq \frac{5}{7} \)

\[
|E(x; \alpha)| \leq 1.75 \times 10^{-5} r^{-2} e^{-3}, \quad \left| \frac{\partial}{\partial x} E(x; \alpha) \right| \leq 9.78 \times 10^{-5} r^{-3} e^{-3},
\]

\[
\left| \frac{\partial^2}{\partial x^2} E(x; \alpha) \right| \leq 5.47 \times 10^{-4} r^{-1} e^{-3}. \quad (83)
\]

6.1. Results in the interval \( \mathcal{I} = [0, \frac{5}{7}] \)

Let \( \mathcal{I} := [0, \frac{5}{7}] \) and \( \mathcal{J} := \left[ -\frac{1}{50}, \frac{1}{350} \right] \). For any \( \alpha \in \mathcal{J} \), the residual \( R(x; \alpha) \) on the interval \( x \in \mathcal{I} \) is given by

\[
R(x; \alpha) = \frac{\partial^3}{\partial x^3} \mathcal{F}_0(x; \alpha) + \frac{\partial^2}{\partial x^2} \mathcal{F}_0(x; \alpha). \quad (84)
\]

Let \( \{ x_k \}_{k=1}^4 = \{ 0, 1.2, 1.4, 2, 2.5 \} \). Then \( \mathcal{I}_k := [x_{k-1}, x_k], k = 1, \ldots, 4 \), provide a partition of \( \mathcal{I} \).

6.1.1. Estimating size of remainder of \( R(x; \alpha) \) on \( \mathcal{I}_k \). Let the sup-norms of \( R \) over different subintervals be distinguished as follows:

\[
\| R \|_{\infty, \mathcal{I}_k} := \sup \{ |R(x; \alpha)| : x \in \mathcal{I}_k, \alpha \in \mathcal{J} \} \quad \text{for} \quad k = 1, \ldots, 4. \quad (85)
\]

The following estimates are obtained from the \( l^1 \)-norm of the coefficients of Chebyshev expansion of \( R \) on appropriate intervals of \( x \) and \( \alpha \) as described in section 3.1.

\[
\| R \|_{\infty, \mathcal{I}_1} \leq 9.58 \times 10^{-7}, \quad \| R \|_{\infty, \mathcal{I}_2} \leq 1.03 \times 10^{-8},
\]

\[
\| R \|_{\infty, \mathcal{I}_3} \leq 1.82 \times 10^{-6}, \quad \| R \|_{\infty, \mathcal{I}_4} \leq 3.11 \times 10^{-6}. \quad (86)
\]

As a result, \( \| R \|_{\infty, \mathcal{J}} := \sup \{ |R(x; \alpha)| : x \in \mathcal{I}, \alpha \in \mathcal{J} \} \) is smaller than \( 3.11 \times 10^{-6} \).
6.1.2. Various suprema. On each subinterval $I_k$, the bounds of $M_1$, $M_2$, $M_3$ and $M_4$ can be obtained using the ‘energy’ method as shown in section 3.3, in particular, by using the inequalities (47)–(50). The results are summarized in table 1. The modified construction of functions, $Q_1$, $Q_2$, and $Q$ appearing in (47)–(50) is found in Kim 2014.

6.1.3. Estimates of errors on $I_k$. Using the contractive mapping principle (Costin and Tanveer 2013, Kim 2014) and the above result, we obtain the error bounds on the subintervals presented in table 2. Compared with the errors against the numerically calculated solution, these estimates turned out to be 10- to 20-fold over-estimates.

6.2. Results in the interval $t \geq T \geq 1.96$

Let $a_i(\alpha) = a_0(\alpha) - \rho_0$, $b_i(\alpha) = b_0(\alpha) - 2\rho_0$, $c_i(\alpha) = c_0(\alpha) - 2\rho_0$, $c_i(\alpha) = c_0(\alpha) + 2\rho_0$. Define $t_m = \frac{T}{2} \left( \frac{1}{2} + \frac{1}{2} \right)$ so that $x \in [t_m, \infty)$. Note that, for each $a \in J$, $(a, b, c) \in S$ implies that $a \in [a_i(\alpha), a_i(\alpha)]$, $b \in [b_i(\alpha), b_i(\alpha)]$, and $c \in [c_i(\alpha), c_i(\alpha)]$. Since $a_i$ and $b_i$ are quadratic functions in $\alpha$, simple calculations show that $0 < a_i(\alpha)$ and $b_i(\alpha) < 0$ on the interval $J$, and so

$$\frac{a_i(\alpha)}{2} \left( \frac{5}{2} + \frac{b_i(\alpha)}{a_i(\alpha)} \right) < t_m < \frac{a_i(\alpha)}{2} \left( \frac{5}{2} + \frac{b_i(\alpha)}{a_i(\alpha)} \right)$$

(87)

Thus it must be the case that

$$t_m \in \left[ \inf_{\alpha \in J} \left\{ \frac{a_i(\alpha)}{2} \left( \frac{5}{2} + \frac{b_i(\alpha)}{a_i(\alpha)} \right)^{\frac{3}{2}} \right\}, \sup_{\alpha \in J} \left\{ \frac{a_i(\alpha)}{2} \left( \frac{5}{2} + \frac{b_i(\alpha)}{a_i(\alpha)} \right)^{\frac{3}{2}} \right\} \right]$$

(88)

So, provided that $a > 0$, the domain $t \geq T \geq 1.96$ corresponds to the domain $x \geq -\frac{b}{a} + \sqrt{\frac{2T}{a}}$, with $T \geq 1.96$. On this domain, the inequalities (24)–(26) are modified to

| $I_1$ | $I_2$ | $I_3$ | $I_4$ |
|-------|-------|-------|-------|
| $M_1$ | 3.124 | 3.082 | 2.085 | 1.677 |
| $M_2$ | 0.470 | 0.423 | 0.048 | 1.001 |
| $M_3$ | 0.776 | 0.554 | 0.173 | 1.001 |
| $M_4$ | 0.708 | 0.316 | 0.079 | 1.001 |

Table 2. Bounds of $E$ and its derivatives on subintervals $I_k$.

| $\|E\|_{x,j}$ | $\|E'\|_{x,j}$ | $\|E''\|_{x,j}$ |
|----------------|----------------|------------------|
| $I_1$ | $2.16 \times 10^{-6}$ | $3.60 \times 10^{-6}$ | $3.00 \times 10^{-6}$ |
| $I_2$ | $9.11 \times 10^{-8}$ | $9.11 \times 10^{-7}$ | $4.56 \times 10^{-6}$ |
| $I_3$ | $1.12 \times 10^{-6}$ | $3.71 \times 10^{-6}$ | $6.17 \times 10^{-6}$ |
| $I_4$ | $1.13 \times 10^{-6}$ | $4.51 \times 10^{-6}$ | $9.01 \times 10^{-6}$ |
have the constant $1.6873 \times 10^{-4}$ in place of $1.667 \times 10^{-4}$. With these modified inequalities, we obtain (83). For details, see Kim (2014).

6.3. Matching of the two solutions

For each $\alpha \in \mathcal{J}$, define $A_\alpha = (a_\alpha, b_\alpha, c_\alpha)$. Let $A$, $N[A]$, and $J$ be defined as in definition 5. In addition, for each $\alpha \in \mathcal{J}$, define $S_{\alpha, \mathcal{A}} = \{ A \in \mathbb{R}^3 : \| A - A_\alpha \|_2 \leq \rho_\alpha^2 \leq 5 \times 10^{-4} \}$.

where $\| \cdot \|_2$ is the Euclidean norm. Costin and Tanveer (2013) show that for each $\alpha \in \mathcal{J}$

$\| A_\alpha (\alpha) - N[A_\alpha (\alpha)] \|_2 \leq 4.27 \times 10^{-4}, \quad (90)$

$\sup_{A \in S_{\alpha, \mathcal{A}}} \| J(\alpha) \|_2 \leq 0.822. \quad (91)$

(90) and (91) satisfy the conditions in lemma 6 for each $\alpha \in \mathcal{J}$ and it follows that the two solutions match at $x = \frac{1}{2}$ for any value of $\alpha \in \mathcal{J}$.

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