BOOTSTRAP REGULARITY
FOR INTEGRO-DIFFERENTIAL OPERATORS
AND ITS APPLICATION
TO NONLOCAL MINIMAL SURFACES

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ABSTRACT. We prove that \(\mathcal{C}^{1,\alpha}_s\)-minimal surfaces are automatically \(\mathcal{C}^\infty\).
For this, we develop a new bootstrap regularity theory for solutions of integro-
differential equations of very general type, which we believe is of independent
interest.

1. Introduction

Motivated by the structure of interphases arising in phase transition models with
long range interactions, in [4] the authors introduced a nonlocal version of minimal
surfaces. These objects are obtained by minimizing a “nonlocal perimeter” inside
a fixed domain \(\Omega\): fix \(s \in (0, 1)\), and given two sets \(A, B \subset \mathbb{R}^n\), let us define the
interaction term

\[
L(A, B) := \int_A \int_B \frac{dx
dy}{|x - y|^{n+s}}.
\]

The nonlocal perimeter of \(E\) inside \(\Omega\) is defined as

\[
\text{Per}(E, \Omega, s) := L(E \cap \Omega, (\mathcal{E}E) \cap \Omega) + L(E \cap \Omega, (\mathcal{E}E) \cap (\mathcal{E}\Omega)) + L((\mathcal{E}E) \cap \Omega, E \cap (\mathcal{E}\Omega)),
\]

References
where \( \mathcal{C} E := \mathbb{R}^n \setminus E \) denotes the complement of \( E \). Then nonlocal \((s)\)-minimal surfaces correspond to minimizers of the above functional with the “boundary condition” that \( E \cap (\mathcal{C} \Omega) \) is prescribed.

It is proved in [4] that “flat \( s \)-minimal surface” are \( C^{1, \alpha} \) for all \( \alpha < s \), and in [3] that, as \( s \to 1^- \), the \( s \)-minimal surfaces approach the classical ones, both in a geometric sense and in a \( \Gamma \)-convergence framework, with uniform estimates as \( s \to 1^- \). In particular, when \( s \) is sufficiently close to 1, they inherit some nice regularity properties from the classical minimal surfaces (see also [8, 13, 14] for the relation between \( s \)-minimal surfaces and the interfaces of some phase transition equations driven by the fractional Laplacian).

On the other hand, all the previous literature only focused on the \( C^{1, \alpha} \) regularity, and higher regularity was left as an open problem. In this paper we address this issue, and we prove that \( C^{1, \alpha} \) \( s \)-minimal surfaces are indeed \( C^\infty \), according to the following result.

**Theorem 1.** Let \( s \in (0, 1) \), and \( \partial E \) be a \( s \)-minimal surface in \( K_R \) for some \( R > 0 \). Assume that

\[
\partial E \cap K_R = \{(x', x_n) : x' \in B_R^{n-1} \text{ and } x_n = u(x')\}
\]

for some \( u \in C^{1, \alpha}(B_R^{n-1}) \) for any \( \alpha < s \), with \( u(0) = 0 \). Then

\[
u \in C^\infty(B_R^{n-1}) \quad \forall \rho \in (0, R),
\]

and for any \( k \in \mathbb{N} \) we have

\[
\|u\|_{C^k(B_R^{n-1})} \leq C(s, n, k, \rho, R).
\]

The regularity result of Theorem 1 combined with [4, Theorem 6.1] and [10, Theorems 1, 3, 4, 5], implies also the following results (here and in the sequel, \( \{e_1, e_2, \ldots, e_n\} \) denotes the standard Euclidean basis):

**Corollary 2.** Fix \( s_0 \in (0, 1) \). Let \( s \in (s_0, 1) \) and \( \partial E \) be a \( s \)-minimal surface in \( B_R \) for some \( R > 0 \). There exists \( \varepsilon > 0 \), possibly depending on \( n \), \( s_0 \) and \( \alpha \), but independent of \( s \) and \( R \), such that if

\[
\partial E \cap B_R \subseteq \{|x \cdot e_n| \leq \varepsilon R\}
\]

then \( \partial E \cap B_{R/2} \) is a \( C^\infty \)-graph in the \( e_n \)-direction.

**Corollary 3.** There exists \( \varepsilon_0 \in (0, 1) \) such that if \( s \in (1 - \varepsilon_0, 1) \), then:

- If \( n \leq 7 \), any \( s \)-minimal surface is of class \( C^\infty \);
- If \( n = 8 \), any \( s \)-minimal surface is of class \( C^\infty \) except, at most, at countably many isolated points.

More generally, in any dimension \( n \) there exists \( \varepsilon_0 \in (0, 1) \) such that if \( s \in (1 - \varepsilon_0, 1) \) then any \( s \)-minimal surface is of class \( C^\infty \) outside a closed set \( \Sigma \) of Hausdorff dimension \( n - 8 \).

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\(^1\) Here and in the sequel, we write \( x \in \mathbb{R}^n \) as \( x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \). Moreover, given \( r > 0 \) and \( p \in \mathbb{R}^n \), we define

\[
K_r(p) := \{x \in \mathbb{R}^n : |x' - p'| < r \text{ and } |x_n - p_n| < r\}.
\]

As usual, \( B_r(p) \) denotes the Euclidean ball of radius \( r \) centered at \( p \). Given \( p' \in \mathbb{R}^{n-1} \), we set

\[
B_r^{n-1}(p') := \{x' \in \mathbb{R}^{n-1} : |x' - p'| < r\}.
\]

We also use the notation \( K_r := K_r(0) \), \( B_r := B_r(0) \), \( B_r^{n-1} := B_r^{n-1}(0) \).
Also, Theorem 1 here combined with Corollary 1 in [15] gives the following regularity result in the plane:

**Corollary 4.** Let \( n = 2 \). Then, for any \( s \in (0,1) \), any \( s \)-minimal surface is a smooth curve of class \( C^{\infty} \).

In order to prove Theorem 1 we establish in fact a very general result about the regularity of integro-differential equations, which we believe is of independent interest.

For this, we consider a kernel \( K = K(x,w) : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to (0,+\infty) \) satisfying some general structural assumptions. In the following, \( \sigma \in (1,2) \).

First of all, we suppose that \( K \) is close to an autonomous kernel of fractional Laplacian type, namely

\[
\begin{align*}
\text{there exist } a_0 \in L^\infty(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}), c_0, C_0, r_0 > 0, \text{ and } \eta \in (0, c_0/4), & \quad \text{such that} \\
& \quad \frac{|w|^{n+\sigma}K(x,w)}{2-\sigma} - a_0(w) \leq \eta \quad \forall x \in B_1, w \in B_{r_0} \setminus \{0\}, \\
& \quad |\nabla a_0(w)| \leq \frac{C_0}{|w|} \quad \forall w \in \mathbb{R}^n \setminus \{0\}. \\
\end{align*}
\]

Moreover, we assume that\(^2\)

\[
\begin{align*}
\text{there exist } k \in \mathbb{N} \cup \{0\} \text{ and } C_k > 0 \text{ such that} & \quad K \in C^{k+1}(B_1 \times (\mathbb{R}^n \setminus \{0\})) \\
& \quad \|
\partial^\mu_x \partial^\theta_w K(\cdot,w)\|_{L^\infty(B_1)} \leq \frac{C_k}{|w|^{n+\sigma+|\theta|}} \\
& \quad \forall \mu, \theta \in \mathbb{N}^n, |\mu| + |\theta| \leq k + 1, w \in \mathbb{R}^n \setminus \{0\}. \\
\end{align*}
\]

Our main result is a “Schauder regularity theory” for solutions\(^3\) of an integro-differential equation. Here and in the sequel we use the notation

\[
\delta u(x,w) := u(x+w) + u(x-w) - 2u(x). \quad (5)
\]

**Theorem 5.** Let \( \sigma \in (1,2), k \in \mathbb{N} \cup \{0\}, \text{ and } u \in L^\infty(\mathbb{R}^n) \) be a viscosity solution of the equation

\[
\int_{\mathbb{R}^n} K(x,w) \delta u(x,w) dw = f(x,u(x)) \quad \text{inside } B_1, \quad (6)
\]

with \( f \in C^{k+1}(B_1 \times \mathbb{R}) \). Assume that \( K : B_1 \times (\mathbb{R}^n \setminus \{0\}) \to (0, +\infty) \) satisfies assumptions (3) and (1) for the same value of \( k \).

Then, if \( \eta \) in (3) is sufficiently small (the smallness being independent of \( k \)), we have \( u \in C^{k+\sigma+\alpha}(B_{1/2}) \) for any \( \alpha < 1 \), and

\[
\|u\|_{C^{k+\sigma+\alpha}(B_{1/2})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}), \quad (7)
\]

\(^2\)Observe that we use \( |\cdot| \) both to denote the Euclidean norm of a vector and, for a multi-index case \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), to denote \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). However, the meaning of \( |\cdot| \) will always be clear from the context.

\(^3\)We adopt the notion of viscosity solution used in [5, 6, 7].
Let us notice that, since the right hand side in (6) depends on \( \sigma \) and \( |\theta| \), it makes the proofs longer and more tedious. Hence, since the proof of Theorem 5 already contains all the main ideas, we do not introduce any major additional difficulties.

Theorem 6. Let \( \sigma \in (1, 2) \), \( k \in \mathbb{N} \cup \{0\} \), and \( u \in L^\infty(\mathbb{R}^n) \) be a viscosity solution of equation (3) with \( f \in C^{k,\beta}(B_1 \times \mathbb{R}) \). Assume that \( K : B_1 \times (\mathbb{R}^n \setminus \{0\}) \to (0, +\infty) \) satisfies assumptions (4) and (8) for the same value of \( k \).

Then, if \( \eta \) in (3) is sufficiently small (the smallness being independent of \( k \)), we have \( u \in C^{k+\sigma+\alpha}(B_{1/2}) \) for any \( \alpha < \beta \), and

\[
\|u\|_{C^{k+\sigma+\alpha}(B_{1/2})} \leq C (1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}),
\]

where \( C > 0 \) depends only on \( n, \sigma, k, C_k \) and \( \|f\|_{C^{k+1}(B_1 \times \mathbb{R})} \).

The proof of Theorem 6 is essentially the same as the one of Theorem 5, the only difference being that instead of differentiating the equations (see for instance the argument in Section 2.4) one should use incremental quotients. Although this does not introduce any major additional difficulties, it makes the proofs longer and more tedious. Hence, since the proof of Theorem 5 already contains all the main ideas to prove also Theorem 6, we will show the details of the proof only for Theorem 5.

The paper is organized as follows: in the next section we prove Theorem 5 and then in Section 3 we write the fractional minimal surface equation in a suitable form so that we can apply Theorems 5 and 6 to prove Theorem 1.

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2.1.1. Scaled Hölder norms and coverings. Given $m \in \mathbb{N}$, $\alpha \in (0,1)$, $x \in \mathbb{R}^n$, and $r > 0$, we define the $C^{m,\alpha}$-norm of a function $u$ in $B_r(x)$ as

$$
\|u\|_{C^{m,\alpha}(B_r(x))} := \sum_{|\gamma| \leq m} \|D^\gamma u\|_{L^\infty(B_r(x))} + \sum_{|\gamma|=m} \sup_{y \neq z \in B_r(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y - z|^\alpha}.
$$

For our purposes it is also convenient to look at the following classical rescaled version of the norm:

$$
\|u\|_{C^{m,\alpha}(B_r(x))}^\ast := \sum_{j=0}^m \sum_{|\gamma|=j} r^j \|D^\gamma u\|_{L^\infty(B_r(x))}
+ \sum_{|\gamma|=m} r^{m+\alpha} \sup_{y \neq z \in B_r(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y - z|^\alpha}.
$$

This scaled norm behaves nicely under covering, as the next observation points out:

**Lemma 7.** Let $m \in \mathbb{N}$, $\alpha \in (0,1)$, $\rho > 0$, and $x \in \mathbb{R}^n$. Suppose that $B_\rho(x)$ is covered by finitely many balls $\{B_{\rho/10}(x_k)\}_{k=1}^N$. Then, there exists $C_\alpha > 0$, depending only on $m$, such that

$$
\|u\|_{C^{m,\alpha}(B_\rho(x))}^\ast \leq C_\alpha \sum_{k=1}^N \|u\|_{C^{m,\alpha}(B_{\rho/10}(x_k))}^\ast.
$$

**Proof.** We first observe that, if $j \in \{0, \ldots, m\}$ and $|\gamma| = j$,

$$
\rho^j \|D^\gamma u\|_{L^\infty(B_\rho(x))} \leq 10^j (\rho/10)^j \max_{k=1,\ldots,N} \|D^\gamma u\|_{L^\infty(B_{\rho/10}(x_k))} \leq 10^m \sum_{k=1}^N (\rho/10)^j \|D^\gamma u\|_{L^\infty(B_{\rho/10}(x_k))} \leq 10^m \sum_{k=1}^N \|u\|_{C^{m,\alpha}(B_{\rho/10}(x_k))}^\ast.
$$

Now, let $|\gamma| = m$: we claim that

$$
\rho^{m+\alpha} \sup_{y \neq z \in B_\rho(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y - z|^\alpha} \leq 200 \cdot 10^m \sum_{k=1}^N \|u\|_{C^{m,\alpha}(B_{\rho/10}(x_k))}^\ast.
$$

To check this, we take $y, z \in B_\rho(x)$ with $y \neq z$ and we distinguish two cases. If $|y - z| \leq \rho/100$ we choose $k_0 \in \{1, \ldots, N\}$ such that $y \in B_{\rho/100}(x_{k_0})$. Then $|z - x_{k_0}| \leq |z - y| + |y - x_{k_0}| \leq \rho/50$, which implies $y, z \in B_{\rho/10}(x_{k_0})$, therefore

$$
\rho^{m+\alpha} \sup_{y \neq z \in B_{\rho/10}(x_{k_0})} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y - z|^\alpha} \leq \rho^{m+\alpha} \sup_{\tilde{y} \neq \tilde{z} \in B_{\rho/10}(x_{k_0})} \frac{|D^\gamma u(\tilde{y}) - D^\gamma u(\tilde{z})|}{|\tilde{y} - \tilde{z}|^\alpha} \leq 10^{m+\alpha} \|u\|_{C^{m,\alpha}(B_{\rho/10}(x_{k_0}))}^\ast.
$$
Conversely, if $|y - z| > r/100$, recalling that $\alpha \in (0, 1)$ we have
\[
\frac{\rho^{m+\alpha} |D^\gamma u(y) - D^\gamma u(z)|}{|y - z|^{\alpha}} \leq 2 \cdot 100^\alpha \rho^m \|D^\gamma u\|_{L^\infty(B_{\rho}(x))}
\]
\[
\leq 2 \cdot 100^\alpha \rho^m \sum_{k=1}^N \|D^\gamma u\|_{L^\infty(B_{\rho/10}(x_k))}
\]
\[
\leq 200 \cdot 10^m \sum_{k=1}^N \|u\|^{*}_{C^{m,\alpha}(B_{\rho/10}(x_k))}.
\]
This proves the claim and concludes the proof. \qed

Scaled norms behave also nicely in order to go from local to global bounds, as the next result shows:

**Lemma 8.** Let $m \in \mathbb{N}$, $\alpha \in (0, 1)$, and $u \in C^{m,\alpha}(B_1)$. Suppose that for any $\epsilon > 0$ there exists $\Lambda_\epsilon > 0$ such that, for any $x \in B_1$ and any $r \in (0, 1 - |x|)$, we have
\[
\|u\|^{*}_{C^{m,\alpha}(B_{r/8}(x))} \leq \Lambda_\epsilon + \epsilon \|u\|^{*}_{C^{m,\alpha}(B_{r/6}(x))}.
\]
Then there exist constants $\epsilon_\alpha$, $C > 0$, depending only on $n$, $m$, and $\alpha$, such that
\[
\|u\|_{C^{m,\alpha}(B_{r/8}(x))} \leq C \Lambda_\epsilon.
\]

**Proof.** First of all we observe that
\[
\|u\|^{*}_{C^{m,\alpha}(B_{r/8}(x))} \leq \|u\|_{C^{m,\alpha}(B_{r/8}(x))} \leq \|u\|^{*}_{C^{m,\alpha}(B_1)}
\]
because $r \in (0, 1)$, which implies that
\[
Q := \sup_{x \in B_1} \|u\|^{*}_{C^{m,\alpha}(B_{r/8}(x))} < +\infty.
\]
We now use a covering argument: fixed any $x \in B_1$ and any $r \in (0, 1 - |x|)$, we cover $B_{r/8}(x)$ with finitely many balls $\{B_{r/80}(x_k)\}_{k=1}^N$, with $x_k \in B_{r/8}(x)$, for some $N$ depending only on the dimension $n$. We now observe that
\[
|x_k| + \frac{r}{8} \leq |x_k - x| + |x| + \frac{r}{8} \leq \frac{r}{8} + |x| + \frac{r}{8} < r + |x| \leq 1.
\]
Hence we can use (9) (with $x = x_k$ and $r$ scaled to $r/10$) to obtain
\[
\|u\|^{*}_{C^{m,\alpha}(B_{r/80}(x_k))} \leq \Lambda_\epsilon + \epsilon \|u\|^{*}_{C^{m,\alpha}(B_{r/12}(x_k))}.
\]
Then, using Lemma 7 with $\rho := r/8$, we get
\[
\|u\|^{*}_{C^{m,\alpha}(B_{r/8}(x))} \leq C_\alpha \sum_{k=1}^N \|u\|^{*}_{C^{m,\alpha}(B_{r/80}(x_k))}
\]
\[
\leq C_\alpha \Lambda_\epsilon + C_\alpha \epsilon \sum_{k=1}^N \|u\|^{*}_{C^{m,\alpha}(B_{r/12}(x_k))}
\]
\[
\leq C_\alpha \Lambda_\epsilon + \epsilon C_\alpha N Q.
\]
Recalling the definition of $Q$ this implies
\[
Q \leq C_\alpha \Lambda_\epsilon + \epsilon C_\alpha N Q,
\]
so that, by choosing $\epsilon_\alpha := 1/(2C_\alpha N)$,
\[
Q \leq 2C_\alpha \Lambda_\epsilon.
\]
Thus we have proved that
\[ \|u\|_{C^{\alpha,\alpha}(B_{r/8}(x))} \leq 2C_\alpha \|\nabla \alpha\| \quad \forall x \in B_1, \ r \in (0,1-|x|), \]
and the desired result follows setting \( x = 0 \) and \( r = 1 \). \( \square \)

2.1.2. Differentiating integral functions. In the proof of Theorem 5 we will need to differentiate, under the integral sign, smooth functions that are either supported near the origin or far from it. This purpose will be accomplished in Lemmata 11 and 12 after some technical bounds that are needed to use the Dominated Convergence Theorem.

Recall the notation in (5).

**Lemma 9.** Let \( r > r' > 0 \), \( v \in C^3(B_r) \), \( x \in B_{r'} \), \( h \in \mathbb{R} \) with \( |h| < (r-r')/2 \). Then, for any \( w \in \mathbb{R}^n \) with \( |w| < (r-r')/2 \), we have
\[ |\delta v(x + he_1, w) - \delta v(x, w)| \leq |h| |w|^2 \|v\|_{C^3(B_r)}. \]

**Proof.** Fixed \( x \in B_{r'} \) and \( |w| < (r-r')/2 \), for any \( h \in (r-r')/2, (r-r')/2 \) we set \( g(h) := v(x + he_1 + w) + v(x + he_1 - w) - 2v(x + he_1) \). Then
\[ |g(h) - g(0)| \leq |h| \sup_{|\xi| \leq |h|} |\partial_1 v(x + \xi e_1 + w) + \partial_1 v(x + \xi e_1 - w) - 2\partial_1 v(x + \xi e_1)|. \]

Noticing that \( |x + \xi e_1 \pm w| \leq r' + |h| + |w| < r \), a second order Taylor expansion of \( \partial_1 v \) with respect to the variable \( w \) gives
\[ |\partial_1 v(x + \xi e_1 + w) + \partial_1 v(x + \xi e_1 - w) - 2\partial_1 v(x + \xi e_1)| \leq |w|^2 \|\partial_1 v\|_{C^2(B_r)}. \]

Therefore
\[ |\delta v(x + he_1, w) - \delta v(x, w)| = |g(h) - g(0)| \leq |h| |w|^2 \|v\|_{C^3(B_r)}, \]
as desired. \( \square \)

**Lemma 10.** Let \( r > r' > 0 \), \( v \in W^{1,\infty}(\mathbb{R}^n) \), \( h \in \mathbb{R} \). Then, for any \( w \in \mathbb{R}^n \),
\[ |\delta v(x + he_1, w) - \delta v(x, w)| \leq 4|h| \|\nabla v\|_{L^\infty(\mathbb{R}^n)}. \]

**Proof.** It sufficed to proceed as in the proof of Lemma 10 but replacing (10) with the following estimate:
\[ |\partial_1 v(x + \xi e_1 + w) + \partial_1 v(x + \xi e_1 - w) - 2\partial_1 v(x + \xi e_1)| \leq 4\|\partial_1 v\|_{L^\infty(\mathbb{R}^n)}. \]
\( \square \)

**Lemma 11.** Let \( \ell \in \mathbb{N} \), \( r \in (0,2) \), \( K \) satisfy (4), and \( U \in C^{l+2}_0(B_r) \). Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n \) with \( |\gamma| \leq \ell \leq k + 1 \). Then
\[ \partial^\gamma_x \int_{\mathbb{R}^n} K(x, w) \delta U(x, w) \, dw = \int_{\mathbb{R}^n} \partial^\gamma_x \left( K(x, w) \delta U(x, w) \right) \, dw \\
= \sum_{1 \leq \ell_1 \leq n} \sum_{\ell \leq \ell_1} \left( \frac{\gamma_1}{\ell_1} \right) \ldots \left( \frac{\gamma_n}{\ell_n} \right) \int_{\mathbb{R}^n} \partial^\ell_x K(x, w) \delta (\partial^\ell x U)(x, w) \, dw \]
for any \( x \in B_r \).
and, by the Dominated Convergence Theorem, we get

By (4), if \(0 < \gamma\) result with \(\gamma \leq \ell - 1\) and, by inductive hypothesis, we know that

\[
g_{\gamma}(x) := \partial_x^\gamma \int_{\mathbb{R}^n} K(x, w) \delta U(x, w) \, dw = \int_{\mathbb{R}^n} \theta(x, w) \, dw
\]

with

\[
\theta(x, w) := \sum_{1 \leq i \leq n} \left( \frac{\gamma_1}{\lambda_1} \right) \cdots \left( \frac{\gamma_n}{\lambda_n} \right) \partial_{x_i}^\lambda K(x, w) \delta(\partial_{x}^{\gamma} - \lambda U)(x, w) \, dw.
\]

By [1], if \(0 < |h| < (r - r')/2\) then

\[
|\partial_x^\gamma K(x + he_1, w) - \partial_x^\gamma K(x, w)| \leq C_{|\lambda|+1} |h| |w|^{-n-\sigma}.
\] (12)

Moreover, if \(|w| < (r - r')/2\), we can apply Lemma [9] with \(v := \partial_x^{\gamma} - \lambda U\) and obtain

\[
|\delta(\partial_x^{\gamma} - \lambda U)(x + he_1, w) - \delta(\partial_x^{\gamma} - \lambda U)(x, w)| \leq |h| |w|^{2} \|U\|_{C^{\gamma + \lambda + 2}(B_{r})}.
\] (13)

On the other hand, by Lemma [10] we obtain

\[
|\delta(\partial_x^{\gamma} - \lambda U)(x + he_1, w) - \delta(\partial_x^{\gamma} - \lambda U)(x, w)| \leq 4 |h| \|\partial_x^{\gamma} - \lambda U\|_{C^1(\mathbb{R}^n)}.
\]

All in all,

\[
|\delta(\partial_x^{\gamma} - \lambda U)(x + he_1, w) - \delta(\partial_x^{\gamma} - \lambda U)(x, w)| \leq |h| \|U\|_{C^{\gamma + \lambda + 2}(\mathbb{R}^n)} \min\{4, |w|^{2}\}.
\] (14)

Analogously, a simple Taylor expansion provides also the bound

\[
|\delta(\partial_x^{\gamma} - \lambda U)(x, w)| \leq \|U\|_{C^{\gamma + \lambda + 2}(\mathbb{R}^n)} \min\{4, |w|^{2}\}.
\] (15)

Hence, (11), (12), (13), and (15) give

\[
|\partial_x^\gamma K(x + he_1, w) \delta(\partial_x^{\gamma} - \lambda U)(x + he_1, w) - \partial_x^\gamma K(x, w) \delta(\partial_x^{\gamma} - \lambda U)(x, w)|
\]

\[
\leq \|\partial_x^\gamma K(x + he_1, w) \delta(\partial_x^{\gamma} - \lambda U)(x + he_1, w) - \delta(\partial_x^{\gamma} - \lambda U)(x, w)|
\]

\[
+ \left| \left[ \partial_x^\gamma K(x + he_1, w) - \partial_x^\gamma K(x, w) \right] \delta(\partial_x^{\gamma} - \lambda U)(x, w) \right|
\]

\[
\leq C_1 |h| \min\{|w|^{-n-\sigma}, |w|^{2-n-\sigma}\},
\]

with \(C_1 > 0\) depending only on \(\ell, C_\ell\) and \(\|U\|_{C^{\ell+2}(\mathbb{R}^n)}\). As a consequence,

\[
|\theta(x + he_1, w) - \theta(x, w)| \leq C_2 |h| \min\{|w|^{-n-\sigma}, |w|^{2-n-\sigma}\},
\]

and, by the Dominated Convergence Theorem, we get

\[
\int_{\mathbb{R}^n} \partial_x \theta(x, w) \, dw = \lim_{h \to 0} \int_{\mathbb{R}^n} \frac{\theta(x + he_1, w) - \theta(x, w)}{h} \, dw
\]

\[
= \lim_{h \to 0} \frac{g_\gamma(x + he_1) - g_\gamma(x)}{h}
\]

\[
= \partial_{x_1} g_\gamma(x),
\]

which proves (14) with \(\gamma\) replaced by \(\gamma + e_1\). Analogously one could prove the same result with \(\gamma\) replaced by \(\gamma + e_1\), concluding the inductive step.

The differentiation under the integral sign in (11) may also be obtained under slightly different assumptions, as next result points out:
Lemma 12. Let $\ell \in \mathbb{N}$, $R > r > 0$. Let $U \in C^{\ell+1}(\mathbb{R}^n)$ with $U = 0$ in $B_R$. Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ with $|\gamma| \leq \ell$. Then (11) holds true for any $x \in B_r$.

Proof. If $x \in B_r, w \in B(R-r)/2$ and $|h| \leq (R-r)/2$, we have that $|x + w + he_1| < R$ and so $\delta U(x + he_1, w) = 0$. In particular

$$\delta U(x + he_1, w) - \delta U(x, w) = 0$$

for small $h$ when $w \in B(R-r)/2$. This formula replaces (13), and the rest of the proof goes on as the one of Lemma 11

2.1.3. Integral computations. Here we collect some integral computations which will be used in the proof of Theorem 5.

Lemma 13. Let $v : \mathbb{R}^n \to \mathbb{R}$ be smooth and with all its derivatives bounded. Let $x \in B_{1/4}$, and $\gamma, \lambda \in \mathbb{N}^n$, with $\gamma_i \geq \lambda_i$ for any $i \in \{1, \ldots, n\}$. Then there exists a constant $C' > 0$, depending only on $n$ and $\sigma$, such that

$$\left| \int_{\mathbb{R}^n} \partial_x^\lambda K(x, w) \delta(\partial_x^{\gamma-\lambda}v)(x, w) \, dw \right| \leq C' C_{|\gamma|} \|v\|_{C^{\gamma-\lambda+2}(\mathbb{R}^n)}. \quad (16)$$

Furthermore, if

$$v = 0 \text{ in } B_{1/2}$$

we have

$$\left| \int_{\mathbb{R}^n} \partial_x^\lambda K(x, w) \delta(\partial_x^{\gamma-\lambda}v)(x, w) \, dw \right| \leq C' C_{|\gamma|} \|v\|_{L^\infty(\mathbb{R}^n)}. \quad (17)$$

Proof. By (14) and (15) (with $U = v$),

$$\int_{\mathbb{R}^n} \left| \partial_x^\lambda K(x, w) \right| \delta(\partial_x^{\gamma-\lambda}v)(x, w) \, dw \leq C_{|\lambda|} \left( \|v\|_{C^{\gamma-\lambda+2}(\mathbb{R}^n)} \int_{B_2} |w|^{-n-\sigma+2} \, dw + 4\|v\|_{C^{\gamma-\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_2} |w|^{-n-\sigma} \, dw \right),$$

which proves (16).

We now prove (18). For this we notice that, thanks to (17), $v(x+w)$ and $v(x-w)$ (and also their derivatives) are equal to zero if $x$ and $w$ lie in $B_{1/4}$. Hence, by an integration by parts we see that

$$\int_{\mathbb{R}^n} \partial_x^\lambda K(x, w) \delta(\partial_x^{\gamma-\lambda}v)(x, w) \, dw$$

$$= \int_{\mathbb{R}^n} \partial_x^\lambda K(x, w) \partial_w^{\gamma-\lambda} [v(x+w) - v(x-w)] \, dw$$

$$= \int_{\mathbb{R}^n \setminus B_{1/4}} \partial_x^\lambda K(x, w) \partial_w^{\gamma-\lambda} [v(x+w) - v(x-w)] \, dw$$

$$= (-1)^{|\gamma-\lambda|} \int_{\mathbb{R}^n \setminus B_{1/4}} \partial_x^\lambda \partial_w^{\gamma-\lambda} K(x, w) [v(x+w) - v(x-w)] \, dw.$$
Finally, we define
\[ u = 1 \quad \text{in } B_{1/2}, \]
\[ 0 \quad \text{in } \mathbb{R}^n \setminus B_{3/4}, \]
and for any \( \varepsilon, \delta > 0 \) set \( \eta_\varepsilon(w) := \eta(\varepsilon^{-1} w) \) for any \( \varepsilon > 0 \), \( \bar{\eta}_\delta(x) := \delta^{-n} \eta(x/\delta) \). Then we define
\[ K_\varepsilon(x, w) := \eta_\varepsilon(w) \frac{2 - \sigma}{|w|^{n+\sigma}} + (1 - \eta_\varepsilon(w)) \hat{K}_\varepsilon(x, w), \]
(19)
where
\[ \hat{K}_\varepsilon(x, w) := K(x, w) * \left( \hat{\eta}_\varepsilon(x) \hat{\eta}_\varepsilon(w) \right), \]
(20)
and
\[ L_\varepsilon v(x) := \int_{\mathbb{R}^n} K_\varepsilon(x, w) \delta v(x, w) dw. \]
(21)
We also define
\[ f_\varepsilon(x) := f(x, u(x)) * \hat{\eta}_\varepsilon(x). \]
(22)
Note that we get a family \( f_\varepsilon \in C^\infty(B_1) \) such that
\[ \lim_{\varepsilon \to 0^+} f_\varepsilon = f \text{ uniformly in } B_{3/4}. \]
Finally, we define \( u_\varepsilon \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) as the unique solution to the following linear problem:
\[ \begin{cases} L_\varepsilon u_\varepsilon = f_\varepsilon(x) & \text{in } B_{3/4}, \\ u_\varepsilon = u & \text{in } \mathbb{R}^n \setminus B_{3/4}. \end{cases} \]
(23)
We observe that, since \( K \) satisfies assumption (ii) with \( k = 0 \), and the convolution parameter \( \varepsilon^2 \) in (19) is much smaller than \( \varepsilon \) the operators \( L_\varepsilon \) converge to the operator associated to \( K \) in the weak sense introduced in Definition 22. Indeed, let \( v \) a smooth function that satisfies
\[ |v(w) - v(x) - (w - x) \cdot \nabla v(x)| \leq M |x - w|^2, \quad w \in B_1(x), \quad M > 0 \quad \text{and} \quad |v| \leq M \quad \text{in } \mathbb{R}^n. \]
(24)
Then, from (4) and (24), it follows that
\[ \int_{\mathbb{R}^n} \left| \eta_\varepsilon(w) \frac{2 - \sigma}{|w|^{n+\sigma}} + (1 - \eta_\varepsilon(\omega))(K(x, w) * \hat{\eta}_\varepsilon(x) \hat{\eta}_\varepsilon(w)) - K(x, w) \right| \delta v(x, w) dw \]
\[ \leq \int_{\mathbb{R}^n} \left( \eta_\varepsilon(w) \left| \frac{2 - \sigma}{|w|^{n+\sigma}} - K(x, w) \right| + (1 - \eta_\varepsilon(\omega)) \left| K(x, w) * \hat{\eta}_\varepsilon(x) \hat{\eta}_\varepsilon(w) - K(x, w) \right| \right) \delta v(x, w) dw \]
\[ \times |\delta v(x, w)| dw \]
\[ \leq \int_{B_r} C|w|^{2-n-\sigma} + \int_{B_r} |K(x, w) * \hat{\eta}_\varepsilon(x) \hat{\eta}_\varepsilon(w) - K(x, w)| \delta v(x, w) dw \]
\[ \leq C\varepsilon^{2-\sigma} + I, \]
(25)
for some kernel satisfying $\eta \leq C \delta v \leq \lambda K(x, u + (2 - \sigma)\frac{a}{|w|^n}) \delta v(x, w) dw$ for all $w \neq 0$, while in our case the kernel $K$ and so also $K_0$ satisfies

$$\frac{(2 - \sigma)\Lambda}{|w|^n} \leq K(x, w) \leq \frac{(2 - \sigma)\Lambda}{|w|^n} \quad \forall |w| \leq r_0$$

with $\lambda := c_0 - \eta$, $\Lambda := \|a\|_{L^\infty(\mathbb{R}^n)} + \eta$, and $r_0 > 0$ (observe that, by our assumptions in [3], $\lambda \geq 3c_0/4$).

However this is not a big problem: if $v \in L^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} K_0(x, w) \delta v(x, w) dw = f(x) \quad \text{in } B_{3/4}$$

for some kernel satisfying $\frac{(2 - \sigma)\Lambda}{|w|^n} \leq K_0(x, w) \leq \frac{(2 - \sigma)\Lambda}{|w|^n}$ only for $|w| \leq r_0$, we define $K'(x, w) := \zeta(w)K_0(x, w) + (2 - \sigma)\frac{\zeta(w)}{|w|^n}$, with $\zeta$ a smooth cut-off function supported inside $B_{r_0}$, to get

$$\int_{\mathbb{R}^n} K'(x, w) \delta v(x, w) dw = f(x) + \int_{\mathbb{R}^n} [1 - \zeta(w)] \left( K_0(x, w) + \frac{2 - \sigma}{|w|^n} \right) \delta v(x, w) dw.$$  

Since $1 - \zeta(w) = 0$ near the origin, the second integral is uniformly bounded as a function of $x$, so [4] Lemma 3 applied to $K'$ gives the desired equicontinuity.

Finally, the uniqueness for the boundary problem

$$\begin{cases}
\int_{\mathbb{R}^n} K(x, u) \delta v(x, w) dw = f(x, u(x)) & \text{in } B_{3/4},
\int_{\mathbb{R}^n} K(x, w) \delta v(x, w) dw = f(x, u(x)) & \text{in } \mathbb{R}^n \setminus B_{3/4}.
\end{cases}$$

follows by a standard comparison principle argument (see for instance the argument used in the proof of [2] Theorem 3.2)).
as \(\varepsilon \to 0\).

2.3. **Smoothness of the approximate solutions.** We prove now that the functions \(u_\varepsilon\) defined in the previous section are of class \(C^\infty\) inside a small ball (whose size is uniform with respect to \(\varepsilon\)): namely, there exists \(r \in (0, 1/4)\) such that, for any \(m \in \mathbb{N}\)

\[
\|D^m u_\varepsilon\|_{L^\infty(B_r)} \leq C
\]  

(28)

for some positive constant \(C = C(m, \sigma, \varepsilon, \|u\|_{L^\infty(\mathbb{R}^n)}, \|f\|_{L^\infty(B_1 \times \mathbb{R})})\).

For this, given \(x \in B_1/4\) we observe that by (19)

\[
\frac{2 - \sigma}{|w|^{n+\sigma}} = K_\varepsilon(x, w) - (1 - \eta_\varepsilon(w))\hat{K}_\varepsilon(x, w) + (1 - \eta_\varepsilon(w)) \frac{2 - \sigma}{|w|^{n+\sigma}},
\]

so

\[
\frac{2 - \sigma}{2c_{n, \sigma}} (\Delta)^{\sigma/2} u_\varepsilon(x) = \int_{\mathbb{R}^n} \frac{2 - \sigma}{|w|^{n+\sigma}} \delta u_\varepsilon(x, w) dw
\]

\[
= f_\varepsilon(x) - \int_{\mathbb{R}^n} (1 - \eta_\varepsilon(w))\hat{K}_\varepsilon(x, w) \delta u_\varepsilon(x, w) dw
\]

\[
+ \int_{\mathbb{R}^n} (1 - \eta_\varepsilon(w)) \frac{2 - \sigma}{|w|^{n+\sigma}} \delta u_\varepsilon(x, w) dw
\]

(here \(c_{n, \sigma}\) is the positive constant that appears in the definition of the fractional Laplacian, see e.g. [11]). Then, for any \(x \in B_1/4\) it follows that

\[
(\Delta)^{\sigma/2} u_\varepsilon(x) = d_{n, \sigma} f_\varepsilon(x) + \int_{\mathbb{R}^n} (1 - \eta_\varepsilon(w))(\frac{2 - \sigma}{|w|^{n+\sigma}} - \hat{K}_\varepsilon(x, w)) \delta u_\varepsilon(x, w) dw
\]

(29)

\[
= d_{n, \sigma} f_\varepsilon(x) + \int_{\mathbb{R}^n} (1 - \eta_\varepsilon(w)) \frac{2 - \sigma}{|w|^{n+\sigma}} \delta u_\varepsilon(x, w) dw
\]

(30)

We now notice that “the function \(h_\varepsilon\) is locally as smooth as \(u_\varepsilon\),” is the sense that for any \(m \in \mathbb{N}\) and \(U \subset B_1/4\) open we have

\[
\|h_\varepsilon\|_{C^m(U)} \leq C_m (1 + \|u_\varepsilon\|_{C^m(U)})
\]

(31)

for some \(C_m > 0\). To see this observe that, in the first two integrals, the variable \(x\) appears only inside \(\eta\) and in the kernel \(K\), and \(\eta\) is equal to 1 near the origin. Hence the first two integrals are smooth functions of \(x\) (recall that \(\hat{K}\) is smooth, see (20)). The third term is clearly as regular as \(u_\varepsilon\) because the third integral is smooth by the same reason as before. This proves (31).

We are now going to prove that the functions \(u_\varepsilon\) belong to \(C^\infty(B_{1/5})\), with

\[
\|u_\varepsilon\|_{C^\infty(B_{1/4-r_m})} \leq C(r_1, m, s, \varepsilon, \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}, \|f\|_{L^\infty(B_1 \times \mathbb{R})})
\]

(32)
for any $m \in \mathbb{N}$, where $r_1 := 1/100$ and $r_m := 2r_{m-1} + r_{m-1}^2$.

To show this, we begin by observing that, since $\sigma \in (1, 2)$, by (29), (31), and [6, Theorem 61], we have that $u_\varepsilon \in L^\infty(\mathbb{R}^n) \cap C^{1,\beta}(B_{1/4-r_1})$ for any $\beta < \sigma - 1$ ($r_1 = 1/100$), and

$$
\|u_\varepsilon\|_{C^{1,\beta}(B_{1/4-r_1})} \leq C\left(\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} + \|f_\varepsilon\|_{L^\infty(B_{1/2} \times \mathbb{R})}\right). \tag{33}
$$

Now, to get a bound on higher derivatives, the idea would be to differentiate (29) and use again (31) and [6, Theorem 61]. However we do not have $C^1$ bounds on the function $u_\varepsilon$ outside $B_{1/4-r_1}$, and therefore we can not apply directly this strategy to obtain the $C^{2,\alpha}$ regularity of the function $u_\varepsilon$.

To avoid this problem we follow the localization argument in [5, Theorem 13.1]: we consider a smooth cut-off function

$$
\vartheta := \begin{cases} 1 & \text{in } B_{1/4-3r_1/2}, \\
0 & \text{on } \mathbb{R}^n \setminus B_{1/4-r_1},
\end{cases}
$$

and for fixed $c \in S^{n-1}$ and $|h| < r_1/16$ we define

$$
v(x) := \frac{u_\varepsilon(x+eh) - u_\varepsilon(x)}{|h|}, \quad x \in B_{1/4-7r_1/16}. \tag{34}
$$

We write $v(x) = v_1(x) + v_2(x)$, being

$$
v_1(x) := \frac{\partial u_\varepsilon(x+eh) - \partial u_\varepsilon(x)}{|h|} \quad \text{and} \quad v_2(x) := \frac{(1-\vartheta)u_\varepsilon(x+eh) - (1-\vartheta)u_\varepsilon(x)}{|h|}.
$$

By (33) it is clear that

$$
v_1 \in L^\infty(\mathbb{R}^n)
$$

and that (recall that $|h| < r_1/16$)

$$
v_1 = v \quad \text{inside } B_{1/4-7r_1/4}. \tag{35}
$$

Moreover, for $x \in B_{1/4-7r_1/4}$, using (29), (31) and (33) we get

$$
\left|(-\Delta)^{\sigma/2}v_1(x)\right| = \left|(-\Delta)^{\sigma/2}v(x) - (-\Delta)^{\sigma/2}v_2(x)\right|
= \left|\frac{g_\varepsilon(x+eh) - g_\varepsilon(x)}{|h|} - (-\Delta)^{\sigma/2}v_2(x)\right|
\leq C(1 + \|u_\varepsilon\|_{C^1(B_{1/4-r_1})}) + \left|(-\Delta)^{\sigma/2}v_2(x)\right|. \tag{36}
$$

Now, let us denote by $K_\sigma(y) := \frac{c_n}{|y|^{n+\sigma}}$ the kernel of the fractional Laplacian. Since for $x \in B_{1/4-7r_1/4}$ and $|\xi| < r_1/16$ we have that $(1-\vartheta)u_\varepsilon(x \pm \xi) = 0$, it follows
from a change of variable that

\[ |(-\Delta)^{\alpha/2} v_2(x)| \leq \left| \int_{\mathbb{R}^n} (v_2(x+y) + v_2(x-y) - 2v_2(x)) K_\varepsilon(y) dy \right| \]

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\[ \leq \left| \int_{\mathbb{R}^n} \frac{(1 - \vartheta) u_\varepsilon(x+y + e\vartheta) - (1 - \vartheta) u_\varepsilon(x+y)}{|h|} K_\varepsilon(y) dy \right| + \int_{\mathbb{R}^n} \frac{(1 - \vartheta) u_\varepsilon(x-y + e\vartheta) - (1 - \vartheta) u_\varepsilon(x-y)}{|h|} K_\varepsilon(y) dy \]

\[ \leq \left| \int_{\mathbb{R}^n} (1 - \vartheta) |u_\varepsilon|(x+y) \frac{K_\varepsilon(y - e\vartheta) - K_\varepsilon(y)}{|h|} dy \right| + \int_{\mathbb{R}^n} (1 - \vartheta) |u_\varepsilon|(x-y) \frac{K_\varepsilon(y - e\vartheta) - K_\varepsilon(y)}{|h|} dy \]

\[ \leq \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_{1/4}} \frac{1}{|y|^{n+\sigma+1}} dy \leq C\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}. \]

Therefore, by \(^6\) we obtain

\[ |(-\Delta)^{\alpha/2} v_1(x)| \leq C \left( 1 + \|u_\varepsilon\|_{C^1(B_1/4 - r_1)} + \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \right), \quad x \in B_{1/4} - 7r_1/4, \]

and we can apply \(^6\) Theorem 61\] to get that \( v_1 \in C^{1,\beta}(B_{1/4 - r_1}) \) for any \( \beta < \sigma - 1 \), with

\[ \|v_1\|_{C^{1,\beta}(B_{1/4 - r_1})} \leq C \left( 1 + \|v_1\|_{L^\infty(\mathbb{R}^n)} + \|u_\varepsilon\|_{C^1(B_1/4 - r_1)} + \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \right), \]

for \( r_2 := 2/100 + (1/100)^2 > 7r_1/4. \) By \(^6\), \(^6\), and \(^6\), this implies that \( u_\varepsilon \in C^{2,\beta}(B_{1/4 - r_2}) \), with

\[ \|u_\varepsilon\|_{C^{2,\beta}(B_{1/4 - r_2})} \leq C \left( 1 + \|u_\varepsilon\|_{C^1(B_1/4 - r_1)} + \|f_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \right) \]

\[ \leq C \left( 1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})} \right). \]

Iterating this argument we finally obtain \(^6\) as desired.

2.4. Uniform estimates and conclusion of the proof for \( k = 0 \). Knowing now that the functions \( u_\varepsilon \) defined by \(^6\) are smooth, our goal is to obtain a-priori bounds independent of \( \varepsilon \).

By \(^6\) Theorem 61\] applied\(^6\) to \( u \), we have that \( u \in C^{1,\beta}(B_{1-R_1}) \) for any \( \beta < \sigma - 1 \) and \( R_1 > 0 \), with

\[ \|u\|_{C^{1,\beta}(B_{1-R_1})} \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})} \right). \]  \((37)\)

Let \( r_2 \) be as in \(^6\). Then, for any \( \varepsilon \) sufficiently small, recalling \((37)\), we observe that \( f_\varepsilon \in C^1(B_{5/8}+r_2) \) with

\[ \|f_\varepsilon\|_{C^1(B_{5/8}+r_2)} \leq C' \left( 1 + \|u\|_{C^1(B_{1-R_1})} \right) \]

\[ \leq C' C \left( 1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})} \right), \]

where \( C' > 0 \) depends on \( \|f\|_{C^{1,\beta}(B_{1,\mathbb{R}})} \) only.

\(^6\)As already observed in the footnote on page\(^6\) the fact that the kernel satisfies \(^6\) only for \( w \) small is not a problem, and one can easily check that \(^6\) Theorem 61\] still holds in our setting.
Then, recalling [23], we write the equation satisfied by \( u_\varepsilon \) as
\[
 f_\varepsilon(x) = \int_{\mathbb{R}^n} K_\varepsilon(x, w) \delta(\eta u_\varepsilon)(x, w) dw + \int_{\mathbb{R}^n} K_\varepsilon(x, w) \delta((1 - \eta)u_\varepsilon)(x, w) dw,
\]
and by differentiating it, say in direction \( e_1 \), we obtain (recall Lemmata [11] and [12])
\[
 \partial_{x_1} f_\varepsilon(x) = \int_{\mathbb{R}^n} K_\varepsilon(x, w) \delta(\partial_{x_1} (\eta u_\varepsilon))(x, w) dw \\
+ \int_{\mathbb{R}^n} \partial_{x_1} [K_\varepsilon(x, w) \delta((1 - \eta)u_\varepsilon)](x, w) dw \\
+ \int_{\mathbb{R}^n} \partial_{x_1} K_\varepsilon(x, w) \delta(\eta u_\varepsilon)(x, w) dw
\]
for any \( x \in B_{(5/8) + r_2} \). It is convenient to rewrite this equation as
\[
 \int_{\mathbb{R}^n} K_\varepsilon(x, w) \delta(\partial_{x_1} (\eta u_\varepsilon))(x, w) dw = A_1 - A_2 - A_3,
\]
with
\[
 A_1 := \partial_{x_1} f_\varepsilon(x), \\
 A_2 := \int_{\mathbb{R}^n} \partial_{x_1} K_\varepsilon(x, w) \delta(\eta u_\varepsilon)(x, w) dw \\
 A_3 := \int_{\mathbb{R}^n} \partial_{x_1} [K_\varepsilon(x, w) \delta((1 - \eta)u_\varepsilon)](x, w) dw.
\]
We claim that
\[
 \|A_1 - A_2 - A_3\|_{L^\infty(B_{(5/8) + r_2})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|u_\varepsilon\|_{C^2(B_{(1/8) - r_2})}) \tag{39}
\]
with \( C \) depending on \( \|f\|_{C^1_1(\mathbb{R}^n \times \mathbb{R})} \). Indeed, by [35]
\[
 \|A_1\|_{L^\infty(B_{(5/8) + r_2})} \leq C \left(1 + \|u\|_{L^\infty(\mathbb{R}^n)}\right),
\]
and by [10] (used with \( \gamma = \lambda := (1, 0, \ldots, 0) \) and \( v := \eta u_\varepsilon \))
\[
 \|A_2\|_{L^\infty(B_{(5/8) + r_2})} \leq C \|u_\varepsilon\|_{C^2(B_{(1/8) - r_2})}.
\]
Moreover, since \( (1 - \eta)u_\varepsilon = 0 \) inside \( B_{1/2} \), we can use [13] with \( v := (1 - \eta)u_\varepsilon \) to obtain
\[
 \left| \int_{\mathbb{R}^n} \partial_{x_1} K_\varepsilon(x, w) \delta((1 - \eta)u_\varepsilon)(x, w) dw \right| \\
+ \left| \int_{\mathbb{R}^n} K_\varepsilon(x, w) \partial_{x_1} \delta((1 - \eta)u_\varepsilon)(x, w) dw \right| \\
\leq C C_\varepsilon \|1 - \eta\|_{L^\infty(\mathbb{R}^n)} (1 + \|u\|_{L^\infty(\mathbb{R}^n)})
\]
which gives (note that, by an easy comparison principle, \( \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)}) \))
\[
 \|A_3\|_{L^\infty(B_{(5/8) + r_2})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)}).
\]
The above estimates imply [39].

Since \( \partial_{x_1}(\eta u_\varepsilon) \) is uniformly bounded on the whole of \( \mathbb{R}^n \) we can apply [29], and [10] Theorem 61] to obtain that \( \partial_{x_1} (\eta u_\varepsilon) \in C^{1,\beta}(B_{(5/8) + r_2 - R_2}) \) for any \( R_2 > 0 \), with
\[
 \|\partial_{x_1}(\eta u_\varepsilon)\|_{C^{1,\beta}(B_{(5/8) + r_2 - R_2})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|u_\varepsilon\|_{C^2(B_{(1/8) - r_2})}),
\]
which implies
\[ \|u_\varepsilon\|_{C^{2,\beta}(B_{1/8})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|u_\varepsilon\|_{C^2(B_{(1/8)-r_2})}). \] (40)
To end the proof we need to reabsorb the $C^2$-norm on the right hand side. To do this, we observe that by standard interpolation inequalities (see for instance [12 Lemma 6.35]), for any $\delta \in (0, 1)$ there exists $C_\delta > 0$ such that
\[ \|u_\varepsilon\|_{C^2(B_{(1/8)-r_2})} \leq \delta \|u_\varepsilon\|_{C^{2,\beta}(B_{(1/4)-r_2})} + C_\delta \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}. \] (41)
Hence, by (40) and (41) we obtain
\[ \|u_\varepsilon\|_{C^{2,\beta}(B_{1/8})} \leq C_\delta (1 + \|u\|_{L^\infty(\mathbb{R}^n)}) + C_\delta \|u_\varepsilon\|_{C^{2,\beta}(B_{(1/4)-r_2})}. \] (42)
To conclude, one needs to apply the above estimates at every point inside $B_1$ at every scale: for any $x \in B_1$, let $r > 0$ be any radius such that $B_{r}(x) \subset B_1$. Then we consider
\[ \tilde{v}_{\varepsilon,r}(y) := u_\varepsilon(x + ry), \] (43)
and we observe that $\tilde{v}_{\varepsilon,r}$ solves an analogous equation as the one solved by $u_\varepsilon$ with the kernel given by
\[ K_{\varepsilon,r}(y,z) := r^{n+\sigma} K_{\varepsilon}(x + ry, rz) \]
and with right hand side
\[ F_{\varepsilon,r}(y) := r^\sigma \int_{\mathbb{R}^n} f(x + ry - \tilde{x}, u(x + ry - \tilde{x})) \eta_\varepsilon(\tilde{x}) d\tilde{x}. \]
We now observe that the kernels $K_{\varepsilon,r}$ satisfy the assumptions \[ and \[ with respect to $\varepsilon$, $r$, and $x$. Moreover, for $|x| + r \leq 1/2$ and $\varepsilon$ small enough, we have
\[ \|F_{\varepsilon,r}\|_{C^1(B_{(5/8)+r_2})} \leq r^\sigma C(1 + \|u\|_{C^1(B_{1/4})}), \]
with $C > 0$ depending on $\|f\|_{C^1(B_1 \times \mathbb{R})}$ only. Hence, by (37) this implies
\[ \|F_{\varepsilon,r}\|_{C^1(B_{(5/8)+r_2})} \leq r^\sigma C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}). \]
Thus, applying (42) to $\tilde{v}_{\varepsilon,r}$ (by the discussion we just made, the constants are all independent of $\varepsilon$, $r$, and $x$) and scaling back, we get
\[ \|u_\varepsilon\|_{C^{2,\beta}(B_{1/16})} \leq C_\delta (1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})} + \|f\|_{L^\infty(B_{(5/8)+r_2})}), \]
Using now Lemma \[ inside $B_{1/2}$ (with $m = 2$ and $\Lambda_\delta = C_\delta (1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})})$) we conclude
\[ \|u_\varepsilon\|_{C^{2,\beta}(B_{1/16})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}), \]
which implies
\[ \|u\|_{C^{2,\beta}(B_{1/16})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}) \]
by letting $\varepsilon \to 0$ (see (27)). Since $\beta < \sigma - 1$, this is equivalent to
\[ \|u\|_{C^{\sigma+\alpha}(B_{1/16})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}), \quad \text{for any } \alpha < 1. \]
A standard covering/rescaling argument completes the proof of Theorem 5 in the case $k = 0$. 
2.5. **The induction argument.** We already proved Theorem 5 in the case \( k = 0 \).

We now show by induction that

\[
\|u\|_{C^{k+\sigma+\alpha}(B_{1/2^{3k+4})}} \leq C_k (1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_{1/2})}),
\]  

(44)

for some constant \( C_k > 0 \): by a standard covering/rescaling argument, this proves (7) and so Theorem 5.

Assume (44) holds for \( k \), and let us prove it for \( k + 1 \).

Define \( g(x) := f(x, u(x)) \), and consider a cut-off function \( \eta \) which is 1 inside \( B_{1/2^{3k+5}} \) and 0 outside \( B_{1/2^{3k+4}} \).

By Lemmata 11 and 12 we differentiate the equation \( k \) times according to the following computation: first we observe that \( g \in C^{k+1}(B_{1/2^{3k+4}}) \) with

\[
\|g\|_{C^{k+1}(B_{1/2^{3k+4}})} \leq C \left( 1 + \|u\|_{C^{k+1}(B_{1/2^{3k+4}})} \right) \leq C \left( 1 + \|u\|_{C^k(B_{1/2^{3k+4}})} \right),
\]  

(45)

with \( C > 0 \) depending on \( \|f\|_{C^{k+1}(B_{1/2}\times\mathbb{R})} \) only. Now we take \( \gamma \in \mathbb{N}^n \) with \( |\gamma| = k + 1 \) and we differentiate the equation to obtain

\[
\partial^\gamma g(x) = \sum_{1 \leq |\lambda| \leq n \atop 0 \leq \lambda < \gamma} \binom{\gamma}{\lambda} \int_{\mathbb{R}^n} \partial^\lambda K(x, w) \delta(\partial^\gamma - \lambda(\eta u))(x, w) \, dw
\]

\[+ \sum_{1 \leq |\lambda| \leq n \atop 0 \leq \lambda < \gamma} \binom{\gamma}{\lambda} \int_{\mathbb{R}^n} \partial^\lambda K(x, w) \delta(\partial^\gamma - \lambda(1 - \eta)u)(x, w) \, dw.
\]

Then, we isolate the term with \( \lambda = 0 \) in the first sum:

\[
\int_{\mathbb{R}^n} K(x, w) \delta(\partial^\gamma - \lambda(\eta u))(x, w) \, dw = A_1 - A_2 - A_3
\]

with

\[
A_1 := \partial^\gamma g(x),
\]

\[
A_2 := \sum_{1 \leq |\lambda| \leq n \atop 0 \leq \lambda < \gamma} \binom{\gamma}{\lambda} \int_{\mathbb{R}^n} \partial^\lambda K(x, w) \delta(\partial^\gamma - \lambda(\eta u))(x, w) \, dw
\]

\[
A_3 := \sum_{1 \leq |\lambda| \leq n \atop 0 \leq \lambda < \gamma} \binom{\gamma}{\lambda} \int_{\mathbb{R}^n} \partial^\lambda K(x, w) \delta(\partial^\gamma - \lambda(1 - \eta)u)(x, w) \, dw
\]

We claim that

\[
\|A_1 - A_2 - A_3\|_{L^\infty(B_{1/2^{3k+6}})} \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{C^{k+\sigma+\alpha}(B_{1/2^{3k+4}})}),
\]  

(46)

Indeed, by the fact that \( |\gamma - \lambda| \leq k \) we see that

\[
\|A_2\|_{L^\infty(B_{1/2^{3k+6}})} \leq C_k \|\eta u\|_{C^{k+\sigma+\alpha}(\mathbb{R}^n)} \leq C_k \|u\|_{C^{k+\sigma+\alpha}(B_{1/2^{3k+4}})}.
\]  

(47)

Furthermore, since \( (1 - \eta)u = 0 \) inside \( B_{1/2^{3k+5}} \), we can use (13) with \( v := (1 - \eta)u \) to obtain that

\[
\|A_3\|_{L^\infty(B_{1/2^{3k+6}})} \leq C \|u\|_{L^\infty(\mathbb{R}^n)}.
\]
This last estimate, (45), and (47) allow us to conclude the validity of (46).

Now, by (46) and the case \( k = 0 \) applied to \( \partial_\gamma x(\eta u) \) we get

\[
\|u\|_{C^{s+k+1+\alpha}(B_{1/2^{k+7}})} \lesssim C(1 + \|u\|_{C^{k+\alpha}(B_{1/2^{2k+4}})} + \|u\|_{L^\infty(\mathbb{R}^n)}).
\]

Hence, by (44) we conclude that

\[
\|u\|_{C^{s+k+1+\alpha}(B_{1/2^{2(k+1)+4}})} \lesssim C(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}),
\]

completing the proof of Theorem 5.

3. Proof of Theorem 1

The idea of the proof is to write the fractional minimal surface equation in a suitable form so that we can apply Theorem 5.

3.1. Writing the operator on the graph of \( u \). The first step of our proof consists in writing the \( s \)-minimal surface functional in terms of the function \( u \) which (locally) parameterizes the boundary of a set \( E \). More precisely, we assume that \( u \) parameterizes \( \partial E \cap K_R \) and that (without loss of generality) \( E \cap K_R \) is contained in the ipograph of \( u \). Moreover, since by assumption \( u(0) = 0 \) and is of class \( C^{1,\alpha} \), up to rotating the system of coordinates (so that \( \nabla u(0) = 0 \)) and reducing the size of \( R \), we can also assume that

\[
\partial E \cap K_R \subset B_R^{n-1} \times [-R/8, R/8].
\]

Let \( \varphi \in C^\infty(\mathbb{R}) \) be an even function satisfying

\[
\varphi(t) = \begin{cases} 
1 & \text{if } |t| \leq 1/4, \\
0 & \text{if } |t| \geq 1/2,
\end{cases}
\]

and define the smooth cut-off functions

\[
\zeta_R(x') := \varphi(|x'|/R) \quad \eta_R(x) := \varphi(|x'|/R)\varphi(|x_n|/R).
\]

Observe that

\[
\zeta_R = 1 \quad \text{in } B_{R/4}^{n-1}, \quad \zeta_R = 0 \quad \text{outside } B_{R/2}^{n-1};
\]

\[
\eta_R = 1 \quad \text{in } K_{R/4}, \quad \eta_R = 0 \quad \text{outside } K_{R/2}.
\]

We claim that, for any \( x \in \partial E \cap B_{R/2}^{n-1} \times [-R/8, R/8] \),

\[
\int_{\mathbb{R}^n} \eta_R(y - x) \chi_{E}(y) - \chi_{E}(y) dy = 2 \int_{\mathbb{R}^{n-1}} F \left( \frac{u(x' - w') - u(x')}{|w'|} \right) \frac{\zeta_R(w')}{|w'|^{n-1+\alpha}} dw',
\]

where

\[
F(t) := \int_0^t \frac{d\tau}{(1 + \tau^2)^{(n+\alpha)/2}}.
\]
Indeed, writing \( y = x - w \) we have (observe that \( \eta_R \) is even)
\[
\int_{\mathbb{R}^n} \eta_R(y-x) \chi_E(y) \frac{\chi \varphi E(y)}{|x-y|^{n+s}} \, dy
\]
\[
= \int_{\mathbb{R}^n} \eta_R(w) \chi_E(x-w) - \chi \varphi E(x-w) \frac{dw}{|w|^{n+s}}
\]  
(50)
where the last equality follows from the fact that \( \varphi(|w_n|/R) = 1 \) for \( |w_n| \leq R/4 \), and that by (48) and by symmetry the contributions of \( \chi_E(x-w) \) and \( \chi \varphi E(x-w) \) outside \( \{|w_n| \leq R/4\} \) cancel each other.

We now compute the inner integral: using the change variable \( t := w_n/|w'| \) we have
\[
\int_{-R/4}^{R/4} \frac{\chi_E(x-w)}{1 + (w_n/|w'|)^2} \frac{1}{(n+s)/2} \, dw_n
\]
\[
= \left| w' \right| \frac{R}{4|w'|} - \frac{w(x') - u(x' - w')}{|w'|} \bigg[ F \left( \frac{R}{4|w'|} \right) - F \left( \frac{w(x') - u(x' - w')}{|w'|} \right) \bigg].
\]
In the same way,
\[
\int_{-R/4}^{R/4} \frac{\chi \varphi E(x-w)}{1 + (w_n/|w'|)^2} \frac{1}{(n+s)/2} \, dw_n
\]
\[
= \left| w' \right| \bigg[ F \left( \frac{w(x') - u(x' - w')}{|w'|} \right) - F \left( - \frac{R}{4|w'|} \right) \bigg].
\]
Therefore, since \( F \) is odd, we immediately get that
\[
\int_{-R/4}^{R/4} \frac{\chi_E(x-w) - \chi \varphi E(x-w)}{1 + (w_n/|w'|)^2} \frac{dw_n}{(n+s)/2} = 2|w'| F \left( \frac{w(x' - w') - u(x')}{|w'|} \right),
\]
which together with (50) proves (49).

Let us point out that to justify these computations in a pointwise fashion one would need \( u \in C^{1,1}(x) \) (in the sense of [3, Definition 3.1]). However, by using the viscosity definition it is immediate to check that (49) holds in the viscosity sense (since one only needs to verify it at points where the graph of \( u \) can be touched with paraboloids).

3.2. The right hand side of the equation. Let us define the function
\[
\Psi_R(x) := \int_{\mathbb{R}^n} \left[ 1 - \eta_R(y-x) \right] \frac{\chi_E(y) - \chi \varphi E(y)}{|x-y|^{n+s}} \, dy.
\]  
(51)
Since $1 - \eta_R(y - x)$ vanishes in a neighborhood of $\{x = y\}$, it is immediate to check that the function $\psi_R(z) := \frac{1 - \eta_R(z)}{|z|^{n+s}}$ is of class $C^\infty$, with

$$|\partial^\alpha \psi_R(z)| \leq \frac{C_{|\alpha|}}{1 + |z|^{n+s}} \quad \forall \alpha \in \mathbb{N}^n.$$ 

Hence, since $1/(1 + |z|^{n+s}) \in L^1(\mathbb{R}^n)$ we deduce that

$$\Psi_R \in C^\infty(\mathbb{R}^n),$$

with all its derivatives uniformly bounded. \hfill (52)

3.3. An equation for $u$ and conclusion. By \cite{4} Theorem 5.1] we have that the equation

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\partial E}(y)}{|x - y|^{n+s}} dy = 0$$

holds in viscosity sense for any $x \in (\partial E) \cap K_R$. Consequently, by (49) and (51) we deduce that $u$ is a viscosity solution of

$$\int_{\mathbb{R}^{n-1}} F \left( \frac{u(x' - w') - u(x')}{|w'|} \right) \frac{\zeta_R(w')}{|w'|^{n-1+s}} dw' = -\frac{\Psi_R(x', u(x'))}{2}$$

inside $B_{R/2}^{n-1}$. We would like to apply the regularity result from Theorem 6 (and then of Theorem 3), by exploiting (52) to bound the right hand side of (53). To this aim, using the Fundamental Theorem of Calculus, we rewrite the left hand side in (53) as

$$\int_{\mathbb{R}^{n-1}} (u(x' - w') - u(x')) \frac{a(x', -w') \zeta_R(w')}{|w'|^{n+s}} dw',$$ \hfill (54)

where

$$a(x', -w') := \int_0^1 \left( 1 + t^2 \left( \frac{u(x' - w') - u(x')}{|w'|} \right)^2 \right)^{-\frac{n+s}{2}} dt.$$ 

Now, we claim that

$$\int_{\mathbb{R}^{n-1}} \delta u(x', w') K_R(x', w') dw' = -\Psi_R(x', u(x')) + A_R(x')$$ \hfill (55)

where

$$K_R(x', w') := \frac{[a(x', w') + a(x', -w')] \zeta_R(w')}{2|w'|^{(n-1)+(1+s)}}$$

and

$$A_R(x') := \int_{\mathbb{R}^{n-1}} [u(x' - w') - u(x') + \nabla u(x') \cdot w'] \frac{[a(x', w') - a(x', -w')] \zeta_R(w')}{|w'|^{n+s}} dw'.$$

To prove (55) we introduce a short-hand notation: we define

$$u^\pm(x', w') := u(x' \pm w') - u(x'), \quad a^\pm(x', w') := a(x', \pm w') \zeta_R(w')/|w'|^{n+s},$$

while the integration over $\mathbb{R}^{n-1}$, possibly in the principal value sense, will be denoted by $I[\cdot]$. With this notation, and recalling (54), it follows that (55) can be written

$$-\frac{\Psi_R}{2} = I[u^- a^-].$$ \hfill (56)

Then, by changing $w'$ with $-w'$ in the integral given by $I$, we see that

$$I[u^+ a^+] = I[u^- a^-].$$
Consequently, (56) can be rewritten as
\[-\Psi_R^R = I[u^+a^+].\] (57)

Notice also that
\[u^+ + u^- = \delta u\]
and so, by adding (56) and (57), we conclude that
\[-\Psi_R = I[u^+a^+] + I[u^-a^-] = I[(u^+ + u^-)a^+] - I[u^-a^+] + I[u^-a^-] = I[\delta u a^+] - I[u^-a^+] + I[u^-a^-] = I[\delta u a^+] - I[u^-a^+]\]
and this proves (55) (notice that in the last integral we can add \(\nabla u(x') \cdot w' \) to \(u^-\), since it integrates to zero).

Now, to conclude the proof of Theorem 1 it suffices to apply Theorem 6 iteratively: more precisely, let us start by assuming that \(u \in C^{1, \beta}(B_{2r}^{-})\) for some \(r \leq R/2\) and any \(\beta < s\). Then, by the discussion above we get that \(u\) solves
\[\int_{\mathbb{R}^{n-1}} \delta u(x', w') K_r(x', w') \, dw' = -\Psi_r(x', u(x')) + A_r(x) \quad \text{in } B_{r}^{-}.\]

Moreover, one can easily check that the regularity of \(u\) implies that the assumptions of Theorem 6 are satisfied with \(\sigma := 1 + s\) and \(a_0(w) := 1/(1-s)\). (Observe that (58) holds since \(\|u\|_{C^{1, \beta}(B_{2r}^{-})}\). Furthermore, it is not difficult to check that, for \(|w'| \leq 1,\]
\[|u(x' - w') - u(x') + \nabla u(x') \cdot w'| \leq C|w'|^\beta + 1,\]
which implies that the integral is convergent by choosing \(\beta > s/2\). Furthermore, a tedious computation (which we postpone to Subsection 3.4 below) shows that
\[A_r \in C^{2\beta-s}(B_{r}^{-}).\] (58)

Hence, by Theorem 6 with \(k = 0\) we deduce that \(u \in C^{1, 2\beta}(B_{r/4}^{-})\). But then this implies that \(A_{r/4} \in C^{4\beta-s}(B_{r/8}^{-})\) and so by Theorem 6 again \(u \in C^{1, 4\beta}(B_{r/8}^{-})\) for all \(\beta < s\). Iterating this argument infinitely many times\(^7\) and by a simple covering argument, we obtain that \(u\) is of class \(C^\infty\) inside \(B_{\rho}\) for any \(\rho < R\). This completes the proof of Theorem 1.

\(^7\)Note that, once we know that \(\|u\|_{C^{k, \beta}(B_{2r}^{-})}\) is bounded for some \(k \geq 2\) and \(\beta \in (0, 1]\), for any \(|\gamma| \leq k - 1\) we get
\[\partial_\gamma^\gamma A_r(x) \int_{\mathbb{R}^{n-1}} \partial_\gamma^\gamma \left[u(x' - w') - u(x') + \nabla u(x') \cdot w' \right] \left[a(x', w') - a(x', -w')\right] \frac{\zeta(x')}{|w'|^{n+\beta}} \, dw',\]
and exactly as in the case \(k = 0\) one shows that
\[|\partial_\gamma^\gamma \left[u(x' - w') - u(x') + \nabla u(x') \cdot w' \right] \left[a(x', w') - a(x', -w')\right] | \leq C|w'|^{2\beta + 1} \quad \forall |w'| \leq 1,\]
and that \(A_r \in C^{k, 2\beta-s}(B_{r}^{-}).\)
3.4. Hölder regularity of $A_R$. We now prove (58), i.e., if $u \in C^{1,\beta}(B_{2r}^{n-1})$ then $A_r \in C^{2\beta-4(B_r^{n-1})}$ ($r \leq R/2$). For this we introduce the following notation:

$$U(x',w') := u(x' - w') - u(x') + \nabla u(x') \cdot w'$$

and

$$p(t) := \frac{1}{(1 + t^2)^{\frac{n-1}{2}}}.$$ 

In this way we can write

$$a(x',w') = \int_0^1 p\left(t \frac{u(x' - w') - u(x')}{|w'|}\right) dt. \quad (59)$$

Let us define

$$\mathcal{A}(x',w') := a(x',w') - a(x',-w').$$

Then we have

$$A_r(x') = \int_{\mathbb{R}^{n-1}} U(x',w') \frac{\mathcal{A}(x',w')}{|w'|^{n+s}} \zeta_r(w') dw'.$$

To prove the desired Hölder condition of the function $A_r(x')$, we first note that

$$U(x',w') = \int_0^1 [\nabla u(x') - \nabla u(x' - tw')] dt \cdot w.$$

Let $r$ be such that $2r \leq R$. Using that $u \in C^{1,\beta}(B_{2r}^{n-1})$ we get

$$|U(x',w') - U(y',w')| \leq C \min\{|x' - y'|^{\beta}|w'|, |w'|^{\beta+1}\}, \quad \text{for } y \in B_r^{n-1} \quad (60)$$

and

$$|U(x',w')| \leq C|w'|^{\beta+1}. \quad (61)$$

Therefore, from (60) and (61) it follows that, for any $y \in B_r^{n-1},$

$$|A_r(x') - A_r(y')| = \left| \int_{\mathbb{R}^{n-1}} \left(U(x',w') \mathcal{A}(x',w') - U(y',w') \mathcal{A}(y',w')\right) \frac{\zeta_r(w')}{|w'|^{n+s}} dw' \right|$$

$$\leq C \int_{\mathbb{R}^{n-1}} \min\{|x' - y'|^{\beta}|w'|, |w'|^{\beta+1}\} \frac{|\mathcal{A}(x',w') - \mathcal{A}(y',w')|}{|w'|^{n+s}} \zeta_r(w') dw'$$

$$+ C \int_{\mathbb{R}^{n-1}} |w'|^{\beta+1} \frac{|\mathcal{A}(x',w')|}{|w'|^{n+s}} \zeta_r(w') dw'$$

$$=: I_1(x',y') + I_2(x',y'). \quad (62)$$

To estimate the last two integrals we define

$$\mathcal{A}_s(x',w') := a(x',w') - \int_0^1 p\left(t \frac{u(x')}{|w'|}\right) dt.$$

With this notation

$$\mathcal{A}(x',w') = \mathcal{A}_s(x',w') - \mathcal{A}_s(x',-w'). \quad (63)$$

By (59) and (61), since $|p'(t)| \leq C$ and $p$ is even, it follows that

$$|\mathcal{A}_s(x',w')| \leq \int_0^1 \int_0^1 \left|\frac{d}{d\lambda} \left[\lambda t \frac{u(x' - w') - u(x')}{|w'|} + (\lambda - 1)t \nabla u(x') \cdot \frac{w'}{|w'|}\right]\right| d\lambda dt$$

$$\leq \int_0^1 \frac{|U(x',w')|}{|w'|} \left( \int_0^1 \left|p\left(t \frac{u(x')}{|w'|}\right) - t \nabla u(x') \cdot \frac{w'}{|w'|}\right| d\lambda \right) dt$$

$$\leq C|w'|^{\beta} \quad (64)$$
for all $|w'| \leq r.$

Estimating $\mathcal{A}_s(x',-w')$ in the same way, by (63) and (64), we get, for any $\beta > s/2,$

$$I_1(x',y') \leq C \int_{r_n-1} \min \{ |x' - y'|^{\beta}, |w'|^{\beta} \} \int |w'|^{\beta-n-\varepsilon} \xi_r(w') \, dw'$$

$$\leq C |x' - y'|^{\beta} \int r^{\beta-s-1} dt + \int \int |x' - y'| \, t^{2\beta-s-1} dt$$

$$\leq C |x' - y'|^{2\beta-s}.$$

(65)

On the other hand, to estimate $I_2$ we note that

$$|\mathcal{A}(x',w') - \mathcal{A}(y',w')| \leq |\mathcal{A}(x',w') - \mathcal{A}(y',w')|$$

$$\quad + |\mathcal{A}(y',-w') - \mathcal{A}(x',-w')|.$$  (66)

Hence, arguing as in (64) we have

$$|\mathcal{A}_s(x',w') - \mathcal{A}_s(y',w')|$$

$$\leq \int_0^1 \left[ \frac{U(x',w')}{|w'|} \right] \int_0^1 \left[ p'( \lambda \frac{U(x',w')}{|w'|} + (\lambda-1)t \nabla u(x') \cdot \frac{w'}{|w'|} ) \right]$$

$$\quad - p'( \lambda \frac{U(y',w')}{|w'|} + (\lambda-1)t \nabla u(y') \cdot \frac{w'}{|w'|} ) \right] \, d\lambda \, dt$$

$$\quad + \int_0^1 \left[ \frac{U(x',w') - U(y',w')}{|w'|} \right] \int_0^1 \left[ p'( \lambda \frac{U(y',w')}{|w'|} + (\lambda-1)t \nabla u(y') \cdot \frac{w'}{|w'|} ) \right] \, d\lambda \, dt$$

$$=: I_{2,1}(x',y') + I_{2,2}(x',y').$$  (67)

We bound each of these integrals separately. First, since $|p'(t)| \leq C$, it follows immediately from (60) that

$$I_{2,2}(x',y') \leq C \min \{ |x' - y'|^{\beta}, |w'|^{\beta} \}.$$  (68)

On the other hand, by (61), (60), and the fact that $u \in C^{1,\beta}(B^n_{R-1})$ and $p'$ is Lipschitz, we get

$$I_{2,1}(x',y') \leq C |w'|^{\beta} \left( \frac{U(x',w') - U(y',w')}{|w'|} + |\nabla u(x') - \nabla u(y')| \right)$$

$$\leq C |w'|^{\beta} \left( \min \{ |x' - y'|^{\beta}, |w'|^{\beta} \} + |x' - y'|^{\beta} \right)$$

$$\leq C |w'|^{\beta} |x' - y'|^{\beta}.$$  (69)

Then, assuming without loss of generality $r \leq 1$ (so that also $|x' - y'| \leq 1$), by (67), (68), and (69) it follows that

$$|\mathcal{A}_s(x',w') - \mathcal{A}_s(y',w')| \leq C \min \{ |x' - y'|^{\beta}, |w'|^{\beta} \} |x' - y'|^{\beta}.$$  (70)

As $|\mathcal{A}_s(y',-w') - \mathcal{A}_s(x',-w')|$ is bounded in the same way, by (66), we have

$$|\mathcal{A}(x',w') - \mathcal{A}(y',w')| \leq C \min \{ |x' - y'|^{\beta}, |w'|^{\beta} \}.$$
By arguing as in (65), we get that, for any \( s/2 < \beta < s \),
\[
I_2(x', y') \leq C \int_{\mathbb{R}^{n-1}} \frac{|w'|^{\beta+1} \min\{|x'-y'|^{\beta}, |w'|^{\beta}\}}{|w'|^{n+s}} \zeta_s(w')dw' \\
\leq C|x' - y'|^{2\beta-s}.
\]
(71)
Finally, by (62), (65) and (71), we conclude that
\[
|A_r(x') - A_r(y')| \leq C|x' - y'|^{2\beta-s}, \quad y' \in B^{n-1}_r,
\]
as desired.

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