NON-ABELIAN CONGRUENCES BETWEEN SPECIAL VALUES OF
L-FUNCTIONS OF ELLIPTIC CURVES; THE CM CASE

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ABSTRACT. In this work we prove congruences between special values of elliptic curves with
CM that seem to play a central role in the analytic side of the non-commutative Iwasawa theory.
These congruences are the analogue for elliptic curves with CM of those proved by Kato, Ritter
and Weiss for the Tate motive. The proof is based on the fact that the critical values of elliptic
curves with CM, or what amounts to the same, the critical values of Grössencharacters, can be
expressed as values of Hilbert-Eisenstein series at CM points. We believe that our strategy can
be generalized to provide congruences for a large class of L-values.

1. INTRODUCTION

In [7,14] a vast generalization of the Main Conjecture of the classical (abelian) Iwasawa-theory
to a non-abelian setting is proposed. As in the classical theory, the non-abelian Main Conjecture predicts a deep relation between an analytic object (a non-abelian p-adic L-function) and an algebraic object (a Selmer group or complex over a non-abelian p-adic Lie extension). However the evidences for this non-abelian Main Conjecture are still very modest. One of the central difficulties of the theory seems to be the construction of non-abelian p-adic L-functions. Actually the only known results in this direction are mainly restricted to the Tate motive over particular p-adic Lie extensions as for example in [25][21][20][15]. We should also mention here that for elliptic curves there are some evidences for the existence of such non-abelian p-adic L-functions offered in [4][10] and also some computational evidences offered in [13][11].

The main aim of the present work is to address the issue of the existence of the non-abelian p-adic L-function for an elliptic curve with complex multiplication (but see also the remark later in the introduction) with respect specific p-adic Lie extension as for example, the so-called false Tate curve extension or Heisenberg type Lie extensions. Actually we prove congruences, under some assumptions, that are the analogue for elliptic curves with CM of those proved by Ritter and Weiss in [25] for the Tate motive. We remark that such congruences can be used to prove the existence of the non-abelian p-adic L-function as done for example in [21] or in [20] for the Tate motive. We start by making our setting concrete.

Let E be an elliptic curve defined over Q with CM by the ring of integers \( \mathcal{O}_0 \) of a quadratic imaginary field \( K_0 \). We fix an isomorphism \( \mathcal{O}_0 \cong \text{End}(E) \) and we write \( \Sigma_0 \) for the implicit CM type. Let us write \( \psi_{K_0} \) for the attached Grössencharacter to \( E \), that is \( \psi_{K_0} \) is a Hecke character of \( K \) of (ideal) type \((1,0)\) with respect to the CM type \( \Sigma_0 \) and satisfy \( L(E,s) = L(\psi_{K_0},s) \). We fix an odd prime \( p \) where the elliptic curve has good ordinary reduction. We fix an embedding \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p \) and using the selected CM type we fix an embedding \( K \hookrightarrow \overline{\mathbb{Q}} \). The ordinary assumption implies that \( p \) splits in \( K_0 \), say to \( p \) and \( \overline{p} \) where we write \( p \) for the...
prime ideal that corresponds to the $p$-adic embedding $K \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. We write $N_F$ for the conductor of $E$ and $j$ for the conductor of $\psi_{K_0}$.

We consider a finite totally real extension $F$ of $\mathbb{Q}$ which we assume unramified at the primes of $\mathbb{Q}$ that ramify in $K_0$ and at $p$. We write $r$ for its ring of integers and we fix an integral ideal $n$ of $\mathfrak{r}$ that is relative prime to $p$ and to $N_F$. Let now $F'$ be a totally real Galois extension of $F$, cyclic of order $p$ that is ramified only at primes of $F$ lying above $p$ or at primes of $F$ that divide $n$. We make the additional assumptions that the non-$p$ part of the conductor of $F'/F$ divides $n$, that is $F'$ is a subfield of $F(p^\infty n)$, the ray class field of conductor $p^\infty n$ and that the primes that ramify in $F'/F$ are split in $K$. That is if we write $\theta_{F'/F}$ for the relative different of $F'/F$ then $\theta_{F'/F} = \mathfrak{p}\mathfrak{p}'$ in $K'$. We write $\tau'$ for the ring of integers in $F'$. We write $K$ for the CM field $K_0$ and $K'$ for the CM field $F'K_0 = F'K$ and $\mathfrak{r}$ and $\mathfrak{r}'$ for their ring of integers respectively. We also write $\Gamma$ for the Galois group $\text{Gal}(F'/F) \cong \text{Gal}(K'/K)$. Note that in both $F$ and $F'$ all primes above $p$ split in $K$ and $K'$ respectively. Finally we write $\tau$ for the nontrivial element (complex conjugation) of $\text{Gal}(K'/F) \cong \text{Gal}(K'/F')$ and we set $g := [F : \mathbb{Q}]$.

We now consider the base changed elliptic curves $E/F$ over $F$ and $E/F'$ over $F'$. We note that the above setting gives the following equalities between the $L$ functions,

$$L(E/F, s) = L(\psi_K, s), \quad L(E/F', s) = L(\psi_{K'}, s)$$

where $\psi_K := \psi_{K_0} \circ N_{K/K_0}$ and $\psi_{K'} := \psi_K \circ N_{K'/K} = \psi_{K_0} \circ N_{K'/K_0}$, that is the base-changed characters of $\psi_{K_0}$ to $K$ and $K'$.

We write $G_{F'}$ for the Galois group $\text{Gal}(F(p^\infty n)/F)$ where $F(p^\infty n)$ denotes the ray class field modulo $p^\infty n$ over $F$, and also $G_{F'} := \text{Gal}(F'(p^\infty n)/F')$ for the analogue for $F'$. Note that the above setting introduce a transfer map $\text{ver} : G_F \to G_{F'}$. Moreover we have an action by conjugation of $\Gamma = \text{Gal}(F'/F)$ on $G_{F'}$. We consider the measures $\mu_{E/F}$ of $G_F$ and $\mu_{E/F'}$ of $G_{F'}$ that interpolate the critical values of the elliptic curve $E/F$ and $E/F'$ respectively twisted by finite order characters of conductor dividing $p^\infty n$. The precise interpolation properties is a delicate issue in our setting that we will discuss in the next section. However we can state now the main theorem of our work. We write $\mathfrak{j}$ for the smallest ideal of $\mathfrak{r}$ which contains $n\mathfrak{r} \cap F$ and its prime factors inert or ramify in $K$. If we write $3 := j\mathfrak{r}$ then we denote by $\text{Cl}_K(3)$ the ray class group of the ray class field $K(3)$. We define $\text{Cl}_{K'}(3)$ as the quotient of $\text{Cl}_K(3)$ by the natural image of $(\gamma/j)^\infty$. Similarly we make the analogous definitions for $K'$.

**Theorem 1.1.** We make the assumptions

1. The natural map $\text{Cl}_K(3) \to \text{Cl}_{K'}(3)^\Gamma$ is an isomorphism,
2. The natural map $\text{Cl}_F(1) \to \text{Cl}_{F'}(1)$ is an injection,
3. The relative different $\theta_{F'/F}$ of $F'$ over $F$ is trivial in $\text{Cl}_{F'}^+$, the strict ideal class group of $F'$. That is, there is $\xi \in F'$, totally positive so that $\theta_{F'/F} = (\xi)$.

Then,

$$\int_{G_F} \epsilon \circ \text{ver} \ d\mu_{E/F} \equiv \int_{G_{F'}} \epsilon \ d\mu_{E/F'} \mod p\mathbb{Z}_p$$

for all $\epsilon$ locally constant $\mathbb{Z}_p$-valued functions on $G_{F'}$, such that $\epsilon^\gamma = \epsilon$ for all $\gamma \in \Gamma$ where $\epsilon^\gamma(g) := (\epsilon^\gamma g \epsilon^{-1})$ for all $g \in G_{F'}$ and for some lift $\hat{\gamma} \in \text{Gal}(F'(p^\infty n)/F)$ of $\gamma$. More generally if we relax the assumption (1) and assume only that $\gamma : \text{Cl}_K(3) \to \text{Cl}_{K'}(3)^\Gamma$ is
injective then equation (1) reads

\[ \int_{G_F} \epsilon \circ \text{ver} \ d\mu_{E/F} \equiv \int_{G_{F'}} \epsilon \ d\mu_{E/F'} + \Delta(\epsilon) \mod p\mathbb{Z}_p \]

where \( \Delta(\epsilon) \) is an “error term” that depends on the cokernel of the map \( \iota \).

Remarks:

(1) It can be shown, see for example [25], that the above congruences imply the following congruences between measures. If we write \( f_{E/F} \) for the element in the Iwasawa algebra \( \mathbb{Z}_p[[G_F]] \) that corresponds to the measure \( \mu_{E/F} \) and similarly \( f_{E/F'} \) for that in \( \mathbb{Z}_p[[G'_{F'}]] \) then we obtain the congruences

\[ \text{ver}(f_{E/F}) \equiv f_{E/F'} \mod T \]

where \( T \) is the trace ideal in \( \mathbb{Z}_p[[G'_{F'}]] \) generated by elements \( \Sigma_{i=0}^{p-1} a_i \gamma^i \) for \( a \in \mathbb{Z}_p[[G'_{F'}]] \). Note that \( f_{E/F'} \) is in \( \mathbb{Z}_p[[G'_{F'}]] \) as it comes from base change from \( F \).

It is exactly this implication that motivates our work. The aim is to use this kind of congruences to establish the existence of non-commutative \( p \)-adic \( L \)-functions for our elliptic curve with respect to specific \( p \)-adic Lie groups, as for example Heisenberg type Lie groups, very much in the same spirit done by Kato for the Tate motive \( \mathbb{Z}_p(1) \) in [21] and by Kakde for false Tate curve extensions also for the Tate motive in [20].

(2) Our assumption that the elliptic curve is defined over \( \mathbb{Q} \) is made mainly for simplicity reasons. Our considerations could be applied in a more general setting. One can consider as starting “object” a Hilbert-modular form over \( F \) with CM by \( K \). The delicate issue however is the understanding of the various motivic periods that are associated to it. However the “philosophy” of our proof applies also in this setting.

(3) We believe that the term \( \Delta(\epsilon) \) always vanishes but we cannot prove it yet.

(4) The assumption that \( \epsilon \) is \( \mathbb{Z}_p \)-valued can be relaxed and consider any integrally-valued locally constant function. Then simply one obtains the congruences

\[ \int_{G_F} \epsilon \circ \text{ver} \ d\mu_{E/F} \equiv \int_{G_{F'}} \epsilon \ d\mu_{E/F'} \mod p\mathbb{Z}_p[\epsilon] \]

where \( \mathbb{Z}_p[\epsilon] \) is the ring of integers of the smallest extension of \( \mathbb{Q}_p \) that contains the values of \( \epsilon \).

On the strategy of the proof: Let us finish the introduction by explaining briefly the main idea of the proof. As we will explain shortly we are going to construct the measure \( \mu_{E/F} \) and \( \mu_{E/F'} \) by using the so-called Katz measure for Hecke characters of CM fields. The reason for this should be intuitively clear from the equation of \( L \) functions above. These measures are constructed by using the fact (going back to Damerell’s theorem) that the special values of the \( L \) of Grössencharacters can be expressed as values of Hilbert-Eisenstein series on CM points. The modular meaning of these CM points is that they correspond to Hilbert-Blumenthal abelian varieties (HBAV) with CM of the same type as the character under consideration. In our relative setting we have that the Grössencharacter \( \psi_K \) is the base change of \( \psi_{K'} \) in particular as we will explain in the next section if we write \( (K, \Sigma) \) for the CM type of \( \psi_K \) then the CM type of \( \psi_{K'} \) is \( (K', \Sigma') \) where \( \Sigma' \) is the lift of \( \Sigma \) to \( K' \). But now the key observation is that the HBAV with CM of type \( (K', \Sigma') \) are isogenous to \( [K': K'] \)-copies of HBAV with CM \( (K, \Sigma) \). In particular this says that the CM points that we need to evaluate our Eisenstein series over \( F' \)
are in some sense coming from $F$ through the natural diagonal embedding $\Delta : \mathbb{H}_F \hookrightarrow \mathbb{H}_{F'}$ of the Hilbert upper half planes. Note here the importance of $\Sigma'$ being lifted from $\Sigma$. Hence we can pull-back the Hilbert-Eisenstein that is used over $F'$ to obtain a Hilbert-Eisenstein series over $F$, so that its values on CM points of $\mathbb{H}_F$ are the same with those of the one over $F'$ evaluated on the image of the CM points with respect to $\Delta$. It is mainly this idea that we will use to prove the above congruences. We note here that a similar strategy was used by Kato [21] and Ritter and Weiss [25] for the cyclotomic character but in their works the $L$ values appeared as the constant term of Hilbert-Eisenstein series (or as “values” at the cusp at infinity). We believe that this strategy is more general. We have applied similar considerations in [3] for other $L$-values that can be understood either as values at CM points or at infinity of Eisenstein series of the unitary group. Actually what we are doing here could be rephrased in the unitary group setting, but we defer this discussion for our forthcoming work [3].

2. THE MEASURES ATTACHED TO THE ELLIPTIC CURVES $E/F$ AND $E/F'$

The statement of our main theorem involves measures on $G_F$ (resp $G_{F'}$) with the property that integrating these measures against finite characters of $G_F$ (resp $G_{F'}$) we obtain critical values of $E/F$ (resp $E/F'$) twisted with these characters up to some modification. Now we proceed in explaining the construction of these measures and their interpolation properties. We point right away that there are various construction of these measures; the modular symbol construction which we will not discuss at all, the construction of Katz, Hida and Tilouine on which we will use in the present work and finally in our specific setting the construction of Colmez and Schneps which we also discuss shortly below. In order to explain the definition of the above-mentioned measures we need to introduce some more notation.

**Archimedean and $p$-adic periods:** Since the elliptic curve $E$ is defined over $\mathbb{Q}$, we have that the class number of $K_0$ is one. In particular we can fix a well-defined complex period for $E$ as follows. We write $\Lambda$ for the lattice of $E$, that is $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$. Then we define $\Omega_\infty(E) \in \mathbb{C}^\times$ uniquely up to elements in $\mathfrak{R}_0^\times$ as $\Lambda = \Omega_\infty(E)\sigma_0(\mathfrak{R}_0)$, where $\sigma_0 : K \hookrightarrow \mathbb{C}$ the selected embedding. Moreover we define a $p$-adic period $\Omega_p(E) \in \mathcal{J}_\infty^\times$, where $\mathcal{J}_\infty$ denotes the ring of integers of the $p$-adic completion of the maximal abelian unramified extension of $\mathbb{Q}_p$. If we write $\Phi$ for the extension of Frobenius that operates on it, then it is well-known that this period is uniquely determined up to elements in $\mathbb{Z}_p^\times$ by the property

$$\frac{\Omega_p(E)^\Phi}{\Omega_p(E)} = u \in \mathbb{Z}_p^\times$$

where $u$ is the $p$-adic number determined by the equation (as $p$ is good ordinary for $E$)

$$1 - a_p(E)X + pX^2 = (1 - uX)(1 - wX)$$

**CM-types:** We fix some CM-types for the CM fields $K_0, K, K'$. We have fixed already an embedding of $K_0 \hookrightarrow \mathbb{C}$, say $\sigma_0$ and defined the CM type of $K_0$ by $\Sigma_0 = \{\sigma_0\}$. We normalized things so that the character $\psi_{K_0}$ is of infinite type $1 \Sigma_0$. Now we fix a CM type $\Sigma$ of $K$ by taking the lift of $\Sigma_0$ to $K$. That is, we pick embeddings that restrict to $\sigma_0$ in $K_0$. We also define $\Sigma'$ to be the lift of $\Sigma$ in $K'$. We note two things for these CM-types. First the characters $\psi_K$ and $\psi_K'$ are of type $1 \Sigma$ and $1 \Sigma'$. Second the types just picked are also $p$-ordinary in the terminology of Katz, that simply amounts of picking the primes of $K$ and $K'$ that are above $p$ and not $p$. We denote these sets of primes as $\Sigma_p$ and $\Sigma'_p$ respectively. Of course we set also $\Sigma_{0,p} = \{p\}$. Finally we note that all abelian varieties of dimension $[F : \mathbb{Q}]$ with CM by $K$
We have fixed above a Grössencharacter \( \psi \) and that \( \psi_K \) and type \( \Sigma \) are isogenous to the product of \( [F : \mathbb{Q}] \) (resp. \( [F' : \mathbb{Q}] \)) copies of the elliptic curve \( E \).

**The \( \infty \)-types of the Grössencharacters:** For the Grössencharacter \( \psi_K \) we have that \( \psi_K \psi = N_{K/Q} \) and that \( \psi_K(q) = \psi_K(q) \). In particular we have that

\[
L(\psi^{-1}K,0) = L(\psi_K N_{K/Q}^{-1},0) = L(\psi_K,1) = L(\psi_K,1)
\]

Moreover if we consider twists by finite cyclotomic characters, that is characters of the form \( \chi = \chi' \circ N_{K_0/q} \) for \( \chi' \) a finite Dirichlet character of \( \mathbb{Q} \), we have that \( L(\psi_K \chi,1) = L(\psi_K,1) \).

So from now on we are going to consider characters of infinite type \( \psi \), \( k \Sigma_0 \), \( -k \Sigma \) and \( -k \Sigma' \) for the various CM-types and \( k \geq 1 \) and the critical values that we study are at \( s = 0 \). The above equation explains why these are the values that we are interested in.

We now recall the interpolation properties of a slight modification of a \( p \)-adic measure \( \mu_{\delta}^{KHT} \) for Hecke characters constructed by Katz \([23]\) and later extended by Hida and Tilouine in \([16]\). Let \( \mathfrak{C} \) be some integral ideal of \( K \) relative prime to \( p \). Then for a Hecke character \( \chi \) of \( G_K := Gal(K(\mathfrak{C}^p)/K) \) of infinite type \( -k \Sigma \) we have

\[
\frac{\int_{G_K} \chi(g) \mu_{p}^{KHT}(g)}{\Omega_p^{\Sigma}} = (\Re^\times: \tau^\times) Local(\Sigma, \chi, \delta) \frac{(-1)^k \Gamma(k)^g}{\sqrt{D_p} \Omega_p^{\Sigma}} \prod_{q | \delta} (1 - \chi(q)) \prod_{\eta \notin \delta} (1 - \chi(\eta)) \prod_{\eta \notin \delta} (1 - \chi(\eta))(1 - \chi(\eta))L(0,\chi)
\]

where the ideals \( \mathfrak{g}, \mathfrak{g} \) are some factors of \( \mathfrak{C} \) and will be defined in the next section. Also in the next section we will explain in details the construction of the above measure but for the time being we just want to indicate three points:

1. The measure depends on a choice of an element \( \delta \in K \), totally imaginary with respect to \( \Sigma \) and such that its valuation at \( p \in \Sigma_p \) is equal with the valuation of the absolute different of \( K \).
2. The periods (archimedean and \( p \)-adic) that appear above depend only on the CM type \( \Sigma \) and not at all on the finite part of the Hecke character \( \chi \).
3. The factor \( Local(\chi, \Sigma, \delta) \) is similar to some epsilon factor of \( \chi \) but not equal. We will explain more on that shortly.

We have fixed above a Grössencharacter \( \psi \) (note that \( k = 1 \) for this character). We set, with notation as in the introduction, \( \mathfrak{C} := \mathfrak{nfR} \) and we consider the measure of \( G_K \) defined for every finite character \( \chi \) of \( G_K \) by

\[
\int_{G_K} \chi(g) \mu_{\psi, \delta} \mu_{\psi, \delta}(g) := \int_{G_K} \chi(g) \mu_{\psi, \delta}(g) \mu_{\psi, \delta}(g)
\]

where \( \psi_K \) is the \( p \)-adic avatar of \( \psi_K \) constructed by Weil. We will show later that in this case we can set \( \Omega_p^{\Sigma} = \Omega_p(E)^g \) and \( \Omega_p^{\Sigma} = \Omega_p(E)^g \). Then we define the measure \( \mu_{E/F} \) discussed above by (recall that \( G_F := Gal(F(\mathfrak{p}^{p}/F)) \)) by

\[
\int_{G_F} \chi(g) \mu_{E/F}(g) := \int_{G_K} \chi(g) \mu_{\psi, \delta}(g) \frac{\Omega_p(E)^g}{\Omega_p(E)^g}
\]

where \( \chi \) is the base change of \( \chi \) from \( F \) to \( K \). Then from our remarks on the critical value \( L(E/F,1) \) we see that this measure interpolates twists of this critical value of \( E/F \). The same considerations apply also for the datum \( (K', F', \psi_{K'}, G_{F'}, G_{K'}) \). We now observe that our
main theorem above amounts to prove the following congruences, under of course the same assumptions as in the theorem above,
\[
\frac{\int_{G_K} \epsilon \circ \text{ver} \ d\mu_{\psi_K, \delta}^{KHT}}{\Omega_p(E)^g} \equiv \frac{\int_{G_{K'}} \epsilon \ d\mu_{\psi_{K'}, \delta'}^{KHT}}{\Omega_p(E)^{pg}} \mod p\mathbb{Z}_p
\]
for all \(\epsilon\) locally constant \(\mathbb{Z}_p\)-valued functions on \(G_{K'}\) with \(\epsilon' = \epsilon\) and belong to the cyclotomic part of it, i.e., when it is written as a sum of finite order characters it is of the form \(\epsilon = \sum c_{\chi} \chi\) with \(\chi' = \chi\).

However these congruences do not hold when the extension \(F' / F\) is ramified. In order to overcome this difficulty we will need to modify (twist) the measure of Katz-Hida-Tilouine over \(K'\). The key question is whether the factor \(Local(\chi, \Sigma, \delta)\) is the "right" one. We believe that this is not so when the extension \(F' / F\) is ramified (in the appendix we offer evidences for this) and actually with our modification we aim to overcome this problem. In short, we will define for the datum \((K', F', \psi_{K'}, G_{F'}, G_{K'})\) the measure \(\mu_{E/F'}\) as
\[
\int_{G_{F'}} \chi(g) \mu_{E/F'}(g) := \frac{\int_{G_{K'}} \chi(g) \mu_{\psi_{K'}, \delta, \xi}^{KHT, tw}(g)}{\Omega_p(E)^{pg}} := \frac{\int_{G_{K'}} \chi(g) \psi_{K'}^{-1}(g) \mu_{\psi_{K'}, \delta, \xi}^{KHT, tw}(g)}{\Omega_p(E)^{pg}}
\]
where the measure \(\mu_{\psi_{K'}, \delta, \xi}^{KHT, tw}\), called in this work the twisted Katz-Hida-Tilouine measure, will be defined in section 4. Then we will show that
\[
\frac{\int_{G_K} \epsilon \circ \text{ver} \ d\mu_{\psi_K, \delta}^{KHT}}{\Omega_p(E)^g} \equiv \frac{\int_{G_K} \epsilon \ d\mu_{\psi_{K'}, \delta, \xi}^{KHT, tw}}{\Omega_p(E)^{pg}} \mod p\mathbb{Z}_p
\]
for all \(\epsilon\) locally constant \(\mathbb{Z}_p\)-valued functions on \(G_{K'}\) with \(\epsilon' = \epsilon\) and belong to the cyclotomic part of it.

The measure of Colmez and Schneps: We close this section by making a few more observations. In the setting that we consider we can apply the construction of [8]. Indeed in this work Colmez and Schneps construct a measure of \(G_K := \text{Gal}(K(\mathbb{Q}_p^\infty) / K)\) such that for every Grössencharacter \(\chi\) of \(K\) of infinite type \(\chi((a)) = N_{K/K_0}(a)^{-k}\) for \(a \equiv 1\) modulo the conductor of \(\chi\) has the interpolation property
\[
\int_G \chi(g) \mu^{CS}(g) = (-1)^k \Gamma(k)^g \prod_{p \in \Sigma_p} c_p(\chi, \psi, dx_1) \prod_{q | \mathfrak{c}} (1 - \chi(q)) \prod_{p | \mathfrak{c}} (1 - \chi(\mathfrak{p})) (1 - \bar{\chi}(\mathfrak{p})) L(0, \chi)
\]
Although Colmez and Schneps do not work the algebraicity of the measure we see here that their measure is normalized differently from that of Katz-Hida-Tilouine with respect to the local factors. Here one gets the epsilon factors of Deligne as local factors. It is exactly this construction that we explore in a common work with Filippo Nuccio [5] where we try to obtain a different proof of the congruences hoping also to relax some of the assumptions of the current work.

3. The Eisenstein measure of Katz-Hida-Tilouine

We start by recalling some Eisenstein series appearing in the work of Katz [23] and Hida and Tilouine [16]. We follow the notations of Hida and Tilouine and introduce the setting described in their paper. We consider a totally real field \(F\) with ring of integers \(\mathfrak{r}\) and write \(\theta\) for the different of \(F / \mathbb{Q}\). We also fix an odd prime \(p\). For an ideal \(\mathfrak{a}\) of \(F\) we write \(\mathfrak{a}^* = \mathfrak{a}^{-1}\theta^{-1}\). We fix a fractional ideal \(\mathfrak{c}\) and take two fractional ideals \(\mathfrak{a}\) and \(\mathfrak{b}\) such that \(\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}\). Let
Following Hida and Tilouine we then define the partial Tate module $F_L \otimes \mathbb{C}$ be a locally constant function such that $\phi(e^{-1}x, ey) = N(e)^k \psi(x, y)$, for all $e \in \mathbb{R}^k$, $k$ some positive integer and $f'$ and $f''$ integral ideals relative prime to $p$. We put $\mathfrak{f} := f' \cap f''$ and we assume that the ideals $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are prime to $\mathfrak{f}$. Moreover we assume that the ideal $\mathfrak{a}$ is prime to $p$ and that $p$ does not divide $\mathfrak{b}$. However we allow the case $(p, b^{-1}) \neq 1$. We consider the natural projection $T := \{\mathfrak{r}_p \times (\mathfrak{f}/f)\} \times \{\mathfrak{r}_p \times (\mathfrak{f}/f')\} \rightarrow \{\mathfrak{r}_p \times (\mathfrak{f})\}$ and consider $\phi$ as a locally constant function on $T$.

We define the partial Fourier transform of the first variable of $\phi$ and write

$$P\phi : \{(F_p/\theta_p^{-1} \times \mathfrak{f}^*/\theta^{-1}) \times (\mathfrak{r}_p \times (\mathfrak{f}/f))\} \rightarrow \mathbb{C}$$

as

$$P\phi(x, y) = \alpha|F: \mathbb{Q}| \sum_{a \in \mathbb{R}_\alpha} \phi(a, y)e_{F}(ax)$$

for $\phi$ factoring through $X_\alpha \times \mathfrak{r}_p \times (\mathfrak{f}/f)$ with $X_\alpha := \mathfrak{r}_p / \alpha \mathfrak{r}_p \times (\mathfrak{f}/f)$ with $\alpha \in \mathbb{N}$.

We attach an Eisenstein series to $\phi$. This is realized as a rule on triples $(\mathcal{L}, \lambda, \iota)$ where $\iota$ a $p^\infty \mathfrak{f}^2$ level structure.

The partial Tate module: From the $p^\infty \mathfrak{f}^2$ structure after restriction we obtain a short exact sequence of $\mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}$-modules

$$0 \rightarrow \theta^{-1} \otimes (\mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}) \rightarrow \mathcal{L} \otimes (\mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}) \rightarrow ? \rightarrow 0$$

From the given polarization after we obtain an isomorphism

$$\bigwedge^2_{\mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}} (\mathcal{L} \otimes (\mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r})) \cong \theta^{-1} c^{-1} \otimes (\mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r})$$

From where we conclude that

$$? \cong c^{-1} \otimes (\mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}) \cong c_p^{-1} \mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}$$

We obtain the projection $\pi'$

$$\pi' : (\mathcal{L} \otimes \mathfrak{r}_p) \times \mathcal{L}/\mathfrak{f} \mathcal{L} \rightarrow c_p^{-1} \mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}$$

Following Hida and Tilouine we then define the partial Tate module $PV(\mathcal{L})$ as a submodule of $\mathcal{L} \otimes F_{\mathfrak{f}^2}$ that contains $\mathcal{L} \otimes \mathfrak{r}_{\mathfrak{f}^2}$ such that

$$PV(\mathcal{L})/\mathcal{L} \otimes \mathfrak{r}_{\mathfrak{f}^2} \cong \text{Im}(F_p/\theta_p^{-1} \times \mathfrak{f}^*/\theta^{-1} \rightarrow p^{-\infty} \mathcal{L} \otimes \mathfrak{f}^{-1} \mathcal{L}/\mathcal{L})$$

Then as explained in [16] one obtains the projections

$$\pi' : PV(\mathcal{L}) \rightarrow c_p^{-1} \mathfrak{r}_p \times \mathfrak{f}/\mathfrak{f} \mathcal{r}, \quad \text{and,} \quad \pi : PV(\mathcal{L}) \rightarrow F_p/\theta_p^{-1} \times \mathfrak{f}^*/\theta^{-1}$$

We set $\mathcal{L}(\mathfrak{f}_p) := \mathfrak{f}^{-1} p^{-\infty} \mathcal{L} \cap PV(\mathcal{L})$ and for $w \in \mathcal{L}(\mathfrak{f}_p)$ we define $P\phi(w) := P\phi(\pi(w), \pi'(w))$.

For an integer $k \geq 1$ we define the $c$-polarized HMF $E_k(\phi, c)$ by

$$E_k(\phi, c)(\mathcal{L}, \lambda, \iota) := \frac{(-1)^k \Gamma(k + s)^\theta}{\sqrt(D \mathcal{F})} \sum_{w \in \mathcal{L}(\mathfrak{f}_p)/\mathfrak{f}^s} \frac{P\phi(w)}{N(w)^k |N(w)^{2s}|} |_{s=0}$$

Then from [23, 16] we have the following proposition,
Proposition 3.1. There exists a $\epsilon$-HMF $E_k(\phi, \epsilon)$ of level $p^\infty q^2$ and weight $k$ such that if $k \geq 2$ or $\phi(a, 0) = 0$ for all $a$ then its $q$-expansion is given by

$$E_k(\phi, \epsilon)(\text{Tate}_{a,b}(q), \lambda_{\text{can}}, \omega_{\text{can}}, \iota_{\text{can}}) = N(a)\{2^{-g}L(1 - k, \phi, a) + \sum_{a_{\epsilon}b_{\epsilon} \in \mathbb{Z}} \phi(a, b)\text{sgn}(N(a))N(a)^{k-1}q^\epsilon\}$$

where $L(s; \phi, a) = \sum_{x \in (a-0)/\epsilon} \phi(x, 0)\text{sgn}(N(x))^{k}|N(x)|^{-s}$.

Remark: The following remarks are in order

1. In the case that the locally constant function $\phi$ is supported on $T^\times := \{\tau^\times_p \times (\tau/f)^\times\} \times \{\tau^\times_p \times (\tau/f)^\times\}$ then the Eisenstein series has constant term equal to zero at the cusp $(a, b)$.

2. Note that the $p$-integrality of the $q$-expansion follows from the values of the function $\phi$ and from the fact that $(a, p) = 1$.

The Eisenstein Measure of Katz-Hida-Tilouine: Hida and Tilouine extended the work of Katz to obtain measures of the Galois group $\text{Gal}(K(\mathcal{O}_p^\infty)/K)$ for $K$ a CM field and $\mathcal{C}$ an integral ideal of $K$. We describe briefly the construction and the interpolation properties of these measures. We start with the decomposition $\mathcal{C} = \mathcal{F} \mathcal{G} \mathcal{J}$ such that

$$\mathcal{F} + \mathcal{G} = \mathcal{R}, \quad \mathcal{F} + \mathcal{G}^c = \mathcal{R}, \quad \mathcal{F} \mathcal{G} = \mathcal{R}, \quad \mathcal{F} \mathcal{G}^c = \mathcal{R}, \quad \mathcal{G} \supset \mathcal{G}^c$$

and $\mathcal{J}$ consists of ideals that inert or ramify in $K/F$. We set $\mathcal{J} := \mathcal{J}_c \cap \mathcal{F}$ and $\mathcal{J}^c := \mathcal{J}_c \cap \mathcal{F}$, $\mathcal{F} := \mathcal{J}^c \cap \mathcal{F}$ and $\mathcal{J} := \mathcal{J} \cap \mathcal{J}$, $\delta := \mathcal{J} \cap \mathcal{J}$ and $\iota := \mathcal{J} \cap \mathcal{J}$. As in Hida and Tilouine, we consider the homomorphism obtained from class field theory

$$i : \{(\tau^\times_p \times (\tau/f)^\times \times \tau^\times_p \times (\tau/s)^\times)/\bar{\mathcal{F}}^\times\} \to Cl_K(\mathcal{C})$$

We write $Cl_K(\mathcal{J})$ for the quotient of $Cl_K(\mathcal{J})$ by the natural image of $(\tau/j)^\times$. If $\{\mathcal{J}_j\}_j$ are representatives of $Cl_K(\mathcal{J})$, which we pick relative prime to $p\mathcal{C}^c$, then we have that $Cl_K(\mathcal{C}) = \prod_j \text{Im}(i)\mathcal{J}_j^{-1}$ where $\mathcal{J}_j$ the image of $\mathcal{J}_j$ in $Cl_K(\mathcal{C})$. We use the surjection $(\tau/f)^\times \to (\tau/s)^\times$ to obtain a projection

$$T := \{(\tau^\times_p \times (\tau/f)^\times \times \tau^\times_p \times (\tau/s)^\times)/\bar{\mathcal{F}}^\times\} \to \{(\tau^\times_p \times (\tau/f)^\times \times \tau^\times_p \times (\tau/s)^\times)/\bar{\mathcal{F}}^\times\}$$

Given a continuous function $\phi$ of $Cl_K(\mathcal{C}) \cong \text{Gal}(K(\mathcal{C}^\infty)/K)$, we define a function $\phi$ on $\text{Im}(i)\mathcal{J}_j$ by $\phi_j(x) := \phi(x[\mathcal{J}_j^{-1}])$ and through the above projection we view $\phi_j$ as function on $T$. Moreover we write $N$ for the function

$$N : (\tau^\times_p \times (\tau/f)^\times \times \tau^\times_p \times (\tau/f)^\times) \to \mathbb{Z}^\times_p$$

given by $N_k(x, a, y, b) := \prod_{\sigma \in \Sigma_p} x_{\sigma}$. The we define functions $\phi_j$ on $(\tau^\times_p \times (\tau/f)^\times \times \tau^\times_p \times (\tau/f)^\times)$ by $\phi_j(x, a, y, b) := N(x^{-1}\phi_j(x, x^{-1}a^{-1}, y, b))$.

In order to define the measure of Katz, Hida and Tilouine we need to pick polarization of HBAV with complex multiplication by $\mathcal{R}$ and CM type $\Sigma$. We pick an element $\delta \in K$ such that

1. $\delta^c = -\delta$ and $\text{Im}(\delta^c) > 0$ for all $\sigma \in \Sigma$.
2. The polarization $< u, v > := \langle u, v \rangle_{\Sigma_p} / 2\delta$ on $\mathcal{R}$ induces the isomorphism $\mathcal{R} \otimes \mathcal{R} \cong \theta^{-1}c^{-1}$ for $c$ relative prime to $p$. 

After the above choice of $\delta$ we can attach (see [16] page 211 for details) to the fractional ideals $\mathfrak{U}_j$ of $K$ a datum $(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \iota(\mathfrak{U}_j))$ consisting of a HBAV $X(\mathfrak{U}_j)$ with CM of type $(K, \Sigma)$, of polarization $c_{\mathfrak{U}_j}\mathfrak{U}_j$ and level structure $\iota(\mathfrak{U}_j)$ of type $p^\infty\Sigma^2$.

We define the measure, see [23, pages 260-261] as

$$\int_G \phi(g)\mu^{KHT}(g) := \sum_j \int_T \bar{\phi}_j dE_j := \sum_j E_1(\phi_j, c_j)(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \iota(\mathfrak{U}_j))$$

where $c_j := (\mathfrak{U}_j\mathfrak{U}_j)\lambda \Sigma$. We note here that when $\phi$ is a character of infinite type $-k\Sigma$ then we have that

$$E_1(\phi_j, c_j)(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \iota(\mathfrak{U}_j)) = E_k(\phi_{finite,j}, c_j)(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \iota(\mathfrak{U}_j))$$

where $\phi_{finite,j}$ is as in [23] page 277 and the above equation is explained in (5.5.7) of (loc. cit.). Here we note an important difference of our construction from the construction of Hida and Tilouine. We do not use the function $\phi^0$ in Hida-Tilouine’s notation (page 209). This is the reason why the following measure has slightly different interpolation properties from theirs.

The reason for doing that is related with the values of the measures $\mu_{E/F}$ and $\mu_{E/F'}$ that we will define later. If we want these measures to take $\mathbb{Z}_p$ values then we have to make sure that we put the right epsilon factors (viewed as periods) also away from $p$.

**Theorem 3.2 ((Interpolation Properties)).** For a character $\chi$ of $G := \text{Gal}(K(\mathbb{C}_p\infty)/K)$ of infinite type $-k\Sigma$ we have

$$\frac{\int_G \chi(g)\mu^\delta_{KHT}(g)}{\Omega^\delta_k} = (2\pi : \tau^\chi)\text{Local}(\Sigma, \chi, \delta) \left(\frac{(-1)^k\Gamma(k)\tau^g}{\sqrt{D_F}\Omega^\delta_k}\right) \times \prod_{q|\delta} (1 - \bar{\chi}(q)) \prod_{q|\delta} (1 - \chi(q)) \prod_{p|\Sigma, p} (1 - \chi(p))(1 - \bar{\chi}(p))L(0, \chi)$$

**Proof.** This is in principle the measure constructed by Katz and Hida-Tilouine in [16, 23]. The main difference of the above formula with the one in Theorem 4.1 of [16] is that we do also the partial Fourier transform for the primes that divide $\delta\Sigma$ (this is why in our definition we used $\phi$ and not $\phi^0$ as Hida and Tilouine do (page 209). Note that the computations in their work are local so what we do amounts simply moving some of the epsilon factors away from $p$ to the other part of the functional equation (compare with theorem 4.2 in Hida and Tilouine).

We now explain the local factor $\text{Local}(\chi, \Sigma, \delta)$ that shows up in the interpolation formula above. So we let $\chi$ be a Grössencharacter of a CM field $K$ of infinite type (after fixing $\text{incl}(\infty) : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$)

$$\chi_{\infty} : K^\times \to \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

given by

$$\chi_{\infty}(a) = \prod_{\sigma \in \Sigma} \frac{1}{\sigma(a)^k} \left(\frac{\sigma(\overline{a})}{\sigma(a)}\right)^{d(\sigma)}$$

We write $c : A_K^\times / K^\times \to \mathbb{C}^\times$ for the corresponding adelic character and we decompose it to $c = \prod_{\sigma \in \Sigma, \sigma} c_{\sigma} \prod_{\nu} e_{\nu}$. The infinite type of the character can be read from the parts at infinite $c_{\sigma} : \mathbb{C}^\times \to \mathbb{C}^\times$. These are given by

$$c_{\sigma}(re^{i\theta}) = c_{\sigma}(z) = z^{k+d(\sigma)} = r^k e^{i\theta(k+2d(\sigma))}$$
Let as pick \( q \), a prime ideal of \( K \) which we also take relative prime to 2. Then we define

\[
Local(\chi, \delta)_q := \frac{\hat{F}_{q,1}\left(\frac{-1}{20q}\right)}{c_q(a)}
\]

where \( a \in K \) such that \( ord_q(a) = ord_q(cond(\chi)) \). Here

\[
\hat{F}_{q,1}(x) := \frac{1}{N(q)^{ord_q(cond(\chi))}} \sum_{u \in (\mathfrak{O}/q)^{\times}} c_q(y)exp(-2\pi i Tr_q(ux))
\]

Then in the formula we have

\[
Local(\chi, \Sigma, \delta) := \prod_{q \mid \delta} Local(\chi, \delta)_q \prod_{p \in \Sigma \setminus \delta} Local(\chi, \delta)_p
\]

The discrepancy of the \( \epsilon \)-factors: Our next goal is to understand the relation of the local factor \( Local(\Sigma, \chi, \delta) \) appearing in the interpolation properties of the Hida-Katz-Tilouine measure and the standard epsilon factors of Tate-Deligne. We start by normalizing properly the epsilon factors. We follow Tate’s article [27] for the definition and properties of the epsilon factors of Deligne. We denote Deligne’s factor with \( \epsilon_p(\chi, \psi, dx) \) as is defined in Tate’s article [27] where as \( \psi \) we pick the additive character of \( K_p \) given by \( exp \circ (-Tr_p) \) (as above in the Gauss sum appearing in Katz’s work) and \( dx \) we pick the Haar measure that gives measure 1 to the units of \( \mathfrak{O}_p \). From the formula (3.6.11) in Tate (there is a typo there!) we have that

\[
\epsilon_p(\chi^{-1}, \psi, dx) = \epsilon_p^{-1}(\alpha)N(\theta_K(p)) \sum_{u \in (\mathfrak{O}/p)^{\times}} \epsilon_p(y)exp(-2\pi i Tr_p(\frac{u}{\alpha}))
\]

where \( \alpha \) is an element with \( ord_p(\alpha) = n(\chi) + n(\psi) \). In particular we conclude that

\[
\epsilon_p(\chi^{-1}, \psi, dx) = N(p)^{ord_p(cond(\chi))} \epsilon_p^{-1}(\delta)N(\theta_K(p))Local(\chi, \Sigma, \delta)_p
\]

We conclude

**Lemma 3.3.** The relation between Katz and Deligne’s epsilon factors is given by

\[
\epsilon_p(\chi^{-1}, \psi, dx) = N(p)^{ord_p(cond(\chi))} \epsilon_p^{-1}(\delta)N(\theta_K(p))Local(\chi, \Sigma, \delta)_p
\]

No we consider the take in the lemma above \( \chi \) equal to \( \chi_p^{-1} \) for \( \chi \) a finite character of \( K \). Then we have that

\[
\epsilon_p(\chi^{-1}\psi_{K^{-1}}, \psi, dx) = \epsilon_p(\chi^{-1}, \psi, dx)\psi_{K}(\pi_p^{n(\chi)+n(\psi)})
\]

In particular that implies

\[
Local(\chi_p^{\psi_{K^{-1}}}, \Sigma, \delta)_p = N(p)^{-n(\chi)}\epsilon_p(\delta)N(\theta_K(p))^{-1}\epsilon_p(\chi_p^{-1}, \psi, dx) =
\]

\[
= N(p)^{-n(\chi)}\epsilon_p(\delta)N(\theta_K(p))^{-1}\epsilon_p(\chi^{-1}, \psi, dx)\psi_{K}(\pi_p^{n(\chi)+n(\psi)}) =
\]

\[
= \epsilon_p(\delta)\epsilon_p(\chi^{-1}, \psi, dx)\psi_{K}(\pi_p^{n(\chi)})\pi_p^{n(\psi)} \frac{N(p)^{n(\chi)}}{N(p)^{n(\psi)}}
\]

where \( \epsilon_p(\delta) \) is the value of the adelic counterpart of \( \chi_p^{\psi_{K^{-1}}} \) at \( \delta \). But as \( \psi_K \) is unramified at \( p \) we have that \( \epsilon_p(\delta) = \psi_K(\pi_p^{-n(\psi)}) \chi_p(\delta) \). So we conclude that

\[
Local(\chi_p^{\psi_{K^{-1}}}, \Sigma, \delta)_p = \chi_p(\delta)\epsilon_p(\chi^{-1}, \psi, dx)\psi_{K}(\pi_p^{n(\chi)})\pi_p^{n(\psi)} \frac{N(p)^{n(\chi)}}{N(p)^{n(\psi)}}
\]
Remarks on the values of the measure of Katz-Hida-Tilouine and the periods: In order to determine where the measures $\mu_{E/F}$ and $\mu_{E/F}^\delta$ defined in section 2 above take their values we need first to explain where the measures $\mu_{K,\delta}^{KHT}$ and $\mu_{K'}^{KHT}$ of Hida-Katz-Tilouine take their values. The key point is to understand how the interpolation formulas of these measures are related to the period conjectures of Deligne that were proved by Blasius in our setting. As mentioned above in Theorem 3.2 the interpolation properties of the Katz-Hida-Tilouine measure for a character $\chi$ of $G := Gal(K(mp^{\infty})/K)$ of infinite type $k\Sigma$ are

$$\frac{\int_G \chi(g) \psi_{K,\delta}^{KHT}(g)}{\Omega_{K,\delta}^{KHT}} = (\mathfrak{R}^\chi : r^\chi) \text{Local}(\Sigma, \chi, \delta) \frac{(-1)^k \Gamma(k)^g}{\sqrt{DF} \Omega_{K,\delta}^{KHT}} \times$$

$$\prod_{q|\delta} (1 - \chi(q)) \prod_{\delta} (1 - \chi(q)) \prod_{p \in \Sigma_p} (1 - \chi(p))(1 - \chi(p)) L(0, \chi)$$

and we have fixed a Grössencharacter $\psi_K$ associated to $E/F$, unramified above $p$ and considered the measure of $G$ defined for every locally constant function $\chi$ of $G$ by

$$\int_G \chi(g) \mu_{K,\delta}^{KHT}(g) := \int_G \chi(g) \hat{\psi}_K^{-1}(g) \mu_{K,\delta}^{KHT}(g)$$

where $\hat{\psi}_K$ is the $p$-adic avatar of $\psi_K$ constructed by Weil. Then we consider the question in which field the algebraic elements $\int_G \chi(g) \psi_{K,\delta}^{KHT}(g)$ belong which is equivalent to addressing the question where the values

$$\text{Local}(\Sigma, \chi \psi_{K,\delta}^{-1}, \delta) \frac{L(0, \chi \psi_{K,\delta}^{-1})}{\sqrt{DF} \Omega_{K,\delta}^{KHT}}$$

exactly belong. As we will see later we can replace $\text{Local}(\Sigma, \chi \psi_{K,\delta}^{-1}, \delta)$ with $\text{Local}(\Sigma, \chi, \delta)$ as the two differ by an element in $K^\times$. Now we note that the element $\Omega_{\infty}$ defined by Katz depends only on the infinite type of $\psi_K$. However we will assume that $\Omega_{\infty}$ is so selected such that $\sqrt{DF} \Omega_{K,\delta}^{KHT}$ is equal to Deligne’s period $c^+(\psi_{K,\delta}^{-1})$. We note that this is not always possible in Katz’s construction as one is restricted to pick abelian varieties with CM by $K$ that arise from fractional ideals of $K$. However in our setting, as everything will be “coming” from an elliptic curve $E/\mathbb{Q}$, we are allowed this assumption and actually we will prove later that we are allowed to take $\Omega_{\infty}^{\Sigma} = \Omega(E)^\delta$ and $\Omega_{\infty}^{\Sigma} = \Omega(E)^g$ where we recall $g = [F : \mathbb{Q}]$. So we may assume that $\frac{L(0, \chi \psi_{K,\delta}^{-1})}{\sqrt{DF} \Omega_{K,\delta}^{KHT}} \in K$. As we have mentioned above, Blasius has proved in [1] Deligne’s conjecture for Hecke characters of CM fields, in particular we know that

$$\frac{L(0, \chi \psi_{K,\delta}^{-1})}{c^+(\chi \psi_{K,\delta}^{-1})} \in K(\chi)$$

where $c^+(\chi \psi_{K,\delta})$ is Deligne’s period for the Hecke character $\chi \psi_{K,\delta}^{-1}$. In general one has that $c^+(\chi \psi_{K,\delta}^{-1}) \neq c^+(\chi) c^+(\psi_{K,\delta}^{-1})$. Indeed it is shown in [26] (page 107 formula 3.3.1) that

$$\frac{c^+(\chi \psi_{K,\delta}^{-1})}{c^+(\psi_{K,\delta}^{-1})} = c(\Sigma, \chi) \mod K(\chi)^{\times}$$

Here $c(\Sigma, \chi) \in (K(\chi) \otimes \mathbb{Q})^\times$ is a period associated to the finite character $\chi$ and depending on the CM-type of the Grössencharacter $\psi_K$. Actually it can be determined, up to elements in $K(\chi)^{\times}$, from the following reciprocity law. If we write $F := K^+$ for the maximal totally real
subfield of $K$ then one can associate to the CM type $\Sigma$ the so-called half-transfer map of Tate (see [26] page 106)

$$Vern: \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{Gal}(\overline{\mathbb{Q}}/K)$$

Then one has that

$$(1 \otimes \tau) c(\Sigma, \chi) = (\chi \circ Vern)(\tau)c(\Sigma, \chi), \quad \tau \in \text{Gal}(\overline{\mathbb{Q}}/F)$$

So for our considerations we need to consider the question if $Local(\chi, \Sigma, \delta)$ is equal to $c(\Sigma, \chi)$ up to elements in $K(\chi)^\times$. This is in general not the case. Indeed as it is explained by Blasius in [2] (page 66) if we denote by $E$ the reflex field of $(K, \Sigma)$, this is a CM field itself, then the extension $E_\Sigma := E(c(\Sigma, \chi), \chi)$, where we adjoin to $E$ the values $c(\Sigma, \chi)$ for finite order characters $\chi$ over $K$, is the field extension of $E$ generated by values of arithmetic Hilbert modular functions on CM points of $\mathbb{H}[F: \mathbb{Q}]$ of type $(K, \Sigma)$, i.e. correspond to Hilbert-Blumenthal abelian varieties of dimension $[F : \mathbb{Q}]$ with CM of type $(K, \Sigma)$. This extension of $E$ is not included in $E\mathbb{Q}^{ab}$. However we will see later that the elements $Local(\chi_\psi, \Sigma, \delta)$ are almost equal to Gauss sums. In particular that implies that they can generate over $E$ only extensions that are included in $E\mathbb{Q}^{ab}$ (see also the comment in [26] page 109). Hence in general the two “periods” of $\chi$ are not equal up to elements in $K(\chi)^\times$. That implies, that in general the measures $\frac{1}{\Omega_p(E)^g} \mu^{KHT}_{\psi_K, \delta}$ and $\frac{1}{\Omega_p(E)^g} \mu^{KHT}_{\psi_K^*, \delta}$ are not elements of the Iwasawa algebras $\mathbb{Z}_p[[G_K]]$ and $\mathbb{Z}_p[[G_{K^*}]]$ respectively. However if $\chi$ is cyclotomic i.e. $\chi(\tau g \tau^{-1}) = \chi(g)$ for all $g \in G_K$ then we have the following

**Lemma 3.4.** For $\chi$ cyclotomic we have

$$\frac{\int_{G_K} \chi(g) \mu^{KHT}_{\psi_K^*, \delta}(g)}{\Omega_p(E)^g} \in \mathbb{Z}_p[\chi]$$

**Proof.** From the interpolation properties of the measure $\mu^{KHT}_{\psi_K^*, \delta}$ we have

$$\frac{\int_{G_K} \chi(g) \mu^{KHT}_{\psi_K^*, \delta}(g)}{\Omega_p(E)^g} = (\mathcal{R}^\times : \mathcal{R}^\times) Local(\Sigma, \chi_{\psi_K}^{-1}, \delta) \frac{(-1)^k \Gamma(k)^g}{\sqrt{D_F \Omega_\infty(E)^{p\Sigma}}} \times \prod_{q|\delta} (1 - \chi_{\psi_K}^{-1}(q)) \prod_{q|\delta} (1 - \chi_{\psi_K}^{-1}(q)) \prod_{p \in \Sigma_p} (1 - \chi_{\psi_K}^{-1}(p))(1 - \tilde{\chi}_{\psi_K}^{-1}(p)) L(0, \chi_{\psi_K}^{-1})$$

As the measure is integral valued we have only to show that

$$\frac{L(0, \chi_{\psi_K}^{-1})}{\sqrt{D_F \Omega_\infty(E)^{p\Sigma}}} Local(\Sigma, \chi_{\psi_K}^{-1}, \delta) \in \mathbb{Q}_p(\chi)$$

From the discussion above we have that $Local(\Sigma, \chi_{\psi_K}^{-1}, \delta)$ is equal to $\prod_{p \in \Sigma_p} e_p(\chi_{\psi_K}^{-1}) \prod_{q|\delta} e_q(\chi_{\psi_K}^{-1})$ up to elements in $K(\chi)$. But then if we write $f_{\psi_K}$ for the conductor of $\psi_K$ we have that $\prod_{q|\delta} e_q(\psi_K) = \pm 1$ as this is the sign of the functional equation of $E/F$. In particular up to elements in $K(\chi)$ (as $\psi_K$ is unramified above $p$ and $(\text{cond}(\chi), \text{cond}(\psi_K)) = 1$) we have that $\prod_{p \in \Sigma} e_p(\chi_{\psi_K}) \prod_{q|\delta} e_q(\chi_{\psi_K}) = \prod_{p \in \Sigma} e_p(\chi^{-1}) \prod_{q|\delta} e_q(\chi^{-1})$. We write now $f_{\psi_K}$ for the Hilbert modular form over $F$ that is induced by automorphic induction from $\psi_K$ (i.e. the one that corresponds to the modular elliptic curve $E/F$) and $\tilde{\chi}$ for the finite character over $F$ whom $\chi$ is the base change of from $F$ to $K$. Then we that up to elements in $K(\chi)$, $\prod_{p \in \Sigma} e_p(\chi^{-1}) \prod_{q|\delta} e_q(\chi^{-1}) = e(\tilde{\chi}^{-1})$ where $e(\tilde{\chi}^{-1})$ the global epsilon factor of
Moreover we have that $L(\chi \psi_{\tilde{K}}, 0) = L(f_{\psi_{\tilde{K}}}, \tilde{\chi}^{-1}, 1)$ (here is crucial that $\chi$ is cyclotomic). But it is known as for example is proved in [19] (page 435 Theorem I) that

$$\frac{L(f_{\psi_{\tilde{K}}}, \tilde{\chi}^{-1}, 1)}{\sqrt{D_F \Omega_{\infty}(E) \rho^\Sigma}} e(\tilde{\chi}^{-1}) \in \mathbb{Q}_p(\chi)$$

which allows us to conclude the proof of the lemma. \qed

Actually using the full force of the results in [19] we have that

$$\left( \frac{L(f_{\psi_{\tilde{K}}}, \tilde{\chi}^{-1}, 1)}{\sqrt{D_F \Omega_{\infty}(E) \rho^\Sigma}} e(\tilde{\chi}^{-1}) \right)^\sigma = \frac{L(f_{\psi_{\tilde{K}}}, \tilde{\chi}^{-\sigma}, 1)}{\sqrt{D_F \Omega_{\infty}(E) \rho^\Sigma}} e(\tilde{\chi}^{-\sigma})$$

for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which can be easily seen to imply that

$$\left( \frac{\int_{G_K} \chi(g) \mu_{\psi_{\tilde{K}}}^{\text{KHT}}(g)}{\Omega_p(E)^g} \right)^\sigma = \frac{\int_{G_K} (\chi(g))^\sigma \mu_{\psi_{\tilde{K}}}^{\text{KHT}}(g)}{\Omega_p(E)^g}$$

for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

4. The twisted Katz-Hida-Tilouine measure

In this section we modify the KHT-measure in the case where the relative different is principal. The interpolation properties of the twisted measure are going to be different with respect with the “epsilon” factors and with the modification of the Euler factors at $p$. We explain now this modification. We follow the construction that we presented above. We still consider the relative situation $F'/F$ and the corresponding $K'/K$ extension. Under our assumption we have that $(\xi) = \theta_{F'/F}$ where $\xi$ is a totally positive element in $F'$. Moreover our assumptions on $F'/F$ imply that $\theta_{F'/F}$ splits in $K'$ to $\mathfrak{P}'\overline{\mathfrak{P}}$.

Over $K'$ we define the $KHT$-measure by picking instead of $\delta'$ the element $\delta \in K \to K'$. Note that since the CM type $(K', \Sigma')$ is a lift of $(K, \Sigma)$ this is a valid choice. The polarization that the element $\delta$ induces to the lattice $\mathcal{R}'$ is

$$\bigwedge_2^2(\mathcal{R}') \cong \theta_{F'}^{-1} \epsilon^{-1} \mathfrak{c}'$$

if the same element, seeing as an element in $K$ induces the polarization

$$\bigwedge_2^2(\mathcal{R}) \cong \theta_{F}^{-1} \epsilon^{-1}$$

Indeed, under our assumptions about the ramification of $F'$ and $F$ and $K_0$ we have that $\mathcal{R}' = \mathfrak{c}'\mathfrak{R}_0$ and similarly $\mathcal{R} = \mathfrak{c}\mathfrak{R}_0$, from which we obtain $\mathcal{R}' = \mathcal{R}\mathfrak{c}'$ and the above claim follows.

With respect to this polarization we have for fractional ideals of $K'$ of the form $\mathfrak{U} \otimes \xi^{-1} = \mathfrak{U} \otimes \theta_{F'/F}^{-1}$ the polarization

$$\bigwedge_2^2(\mathfrak{U} \otimes \xi^{-1}) \cong \theta_{F}^{-1} \epsilon^{-1} \mathfrak{U} \theta_{F'/F}^{-2} = \theta_{F}^{-1} \epsilon^{-1} \mathfrak{U} \theta_{F'/F}^{-1}$$

The twisted triples: Our twisted measure is going to be defined again by evaluating Eisenstein series on the very CM abelian varieties as the measure of Katz-Hida-Tilouine but we will twist
them by $\xi^{-1}$ and use the above mentioned polarization. In particular the triples that we consider are

1. The abelian varieties are $X(\mathcal{U}_j^\xi) := X(\mathcal{U}_j \otimes \theta^{-1}_{F'/F})$.
2. The polarization $\lambda_\delta^\xi(\mathcal{U}_j \otimes (\theta^{-1}_{F'/F}))$ i.e. the one defined above and
3. The $p^{\infty}$-arithmetic structure is obtained from an isomorphism $X(\mathcal{U}_j) \cong X(\mathcal{U}_j \otimes \xi^{-1})$. We will amplify on this below.

We then define the twisted measure as follows

$$\int_{G'} \phi(g)\mu_{KHT,tw}^{G}(g) := \sum_{j} \int_{T} \phi_j dE_j := \sum_{j} E_1(\phi_j, \epsilon_j)(X(\mathcal{U}_j^\xi), \lambda_\delta^\xi(\mathcal{U}_j \otimes \theta^{-1}_{F'/F}), \tau^\xi(\mathcal{U}_j \otimes \theta^{-1}_{F'/F}))$$

with $\epsilon_j := \epsilon(\mathcal{U}^{-1})\theta_{F'/F}$. We next explore the interpolation properties of the twisted measure. Let us write $cond(\chi)_p = \prod_{p_j \in \Sigma'}^\chi_{p_j}^{a_j} p_j^{b_j}$ for the $p$-part of the conductor of $\chi$. We define $e_j := ord_p \epsilon_j$ for all $p_j \in \Sigma'_p$. We have already described a decomposition $\mathcal{C} = \mathfrak{F} \mathfrak{D} \mathfrak{E}$.

**Proposition 4.1 (Interpolation Properties of the “twisted” Katz-Hida-Tilouine measure).** For a character $\chi$ of $G' := Gal(K'(\mathfrak{C}p^{\infty})/K')$ of infinite type $-k\Sigma'$ we have

$$\int_{G'} \phi(\chi)(\mu_{KHT,tw}^{G})(g) \bigg/ \Omega_{\mathfrak{D}^{\Sigma'}}^{G} \bigg/ = (\mathfrak{R}'^\chi : \mathfrak{r}'^\chi) Local(\Sigma', \chi, \delta, \xi) \prod_{a_j = 0} \lambda(\mathfrak{p}_j)^{-e_j} \prod_{\ell_j = 0} \lambda(\mathfrak{q}_j)^{-d_j} \times$$

$$\prod_{\mathfrak{q}_j | \mathfrak{D}} (1 - \chi(\mathfrak{q}_j)) \left( \prod_{\mathfrak{q}_j | \mathfrak{D}} (1 - \chi(\mathfrak{q}_j))(1 - \chi(\mathfrak{q}_j)) \right) \left( \prod_{\mathfrak{p}_j \in \Sigma'_p} (1 - \chi(\mathfrak{p}_j)) (1 - \chi(\mathfrak{p}_j)) \right)$$

$$\frac{(-1)^{k\delta} \Gamma(k)^{d'}}{\sqrt{D_{F'}} \Omega_{\mathfrak{D}^{\Sigma'}}^{G}} \times L(0, \chi)$$

Here the factor $Local(\Sigma', \chi, \delta, \xi)$ is a modification of the local factor of the measure of Katz-Hida-Tilouine and it will be defined in the proof of the proposition. But before we proceed to the proof of the above proposition we must explain a little bit more the $p^{\infty}$-part of the given arithmetic structure of twisted HBAV used in the above proposition. As in Katz we use the ordinary type $\Sigma_p$ to obtain an isomorphism

$$\mathfrak{R}' \otimes \mathfrak{Z}_p \cong \prod_{p \in \Sigma_p} \mathfrak{R}'_p \times \prod_{p \in \Sigma_p} \mathfrak{R}'_p \cong \mathfrak{r}' \times \mathfrak{r}'$$

And similarly for any fractional ideal $\mathfrak{U}$ of $\mathfrak{R}'$ relative prime to $p$ we can identify $\mathfrak{U} \otimes \mathfrak{Z}_p = \mathfrak{R}' \otimes \mathfrak{Z}_p$ in $\mathfrak{K}' \otimes \mathfrak{Z}_p$. In particular we have an isomorphism for such ideals

$$\mathfrak{U} \otimes \mathfrak{Z}_p \cong \prod_{p \in \Sigma_p} \mathfrak{R}'_p \times \prod_{p \in \Sigma_p} \mathfrak{R}'_p \cong \mathfrak{r}' \times \mathfrak{r}'$$

Then as Katz explains (see [23] page 265 and lemma 5.7.5) the $p^{\infty}$ structure of $X(\mathfrak{U})$ is defined by picking the isomorphism

$$\mathfrak{r}'_p := \theta^{-1}_{F'} \otimes \mathfrak{Z}_p$$
In particular the computations of Katz for the twisted values now read, where \( \delta_0 \) is the image of \((2\delta')^{-1}\) in \( K'_p \) and using it to define the injection

\[
\theta_{F'}^{-1} \otimes \mathbb{Z}_p \hookrightarrow \mathcal{U} \otimes \mathbb{Z}_p \cong \mathfrak{t}'_p \times \mathfrak{t}'_p
\]

using the isomorphism in the first component. Now the \( p^{\infty} \) structure of the twisted varieties \( \mathcal{U} \otimes \xi^{-1} \) is defined using the isomorphisms

\[
(\mathcal{U} \otimes \xi^{-1}) \otimes \mathbb{Z}_p \cong \prod_{\mathfrak{p} \in \Sigma_p} \mathbb{Z} \mathfrak{t}'_p \times \prod_{\mathfrak{p} \in \Sigma_p} \mathbb{Z} \mathfrak{t}'_p \cong \prod_{\mathfrak{p} \in \Sigma_p} \mathbb{Z} \mathfrak{t}'_p \times \prod_{\mathfrak{p} \in \Sigma_p} \mathbb{Z} \mathfrak{t}'_p
\]

and picking the isomorphism

\[
\frac{1}{\mathfrak{t}'_p} = \theta_{F'/F}^{-1} \otimes \mathfrak{t}'_p \cong \theta_{F'}^{-1} \otimes \mathbb{Z}_p
\]

given by \( x \mapsto x\delta_0^{-1} \) where \( \delta_0 \) is the image of \( \delta \) in \( \prod_{\mathfrak{p} \in \Sigma_p} K'_p \cong \prod_{\mathfrak{p} \in \Sigma_p} F'_p \). Now we proceed to the proof of the proposition on the interpolation properties of the twisted Katz-Hida-Tilouine measure.

**Proof.** We will follow closely the proof of Katz in [23]. Actually we will mainly indicate the differences of our setting from his setting. We start with the following observation. As the computations are local in nature (see also the remark of Hida and Tilouine in [16] page 214) it is enough to prove the theorem for characters \( \chi \) of \( \mathcal{G}' \) that ramify only at \( p \).

Now we split the proof in two cases. We first consider the case where the character \( \chi \) is ramified in all primes \( \mathfrak{p} \in \Sigma_p \) and then we generalize.

**Special Case:** \( \chi \) ramified at all \( \mathfrak{p} \) in \( \Sigma_p \). We follow Katz [23] as in page 279 and use his notation. We write the conductor of the character \( \chi \), \( \text{cond}(\chi) = \prod_i \mathfrak{p}_i^{b_i} \), Moreover we decompose \( (\xi) = \mathfrak{P} \mathfrak{B} \) as ideals in \( K' \). We also write \( \mathfrak{P} \prod \mathfrak{p}_i^{a_i} = (\alpha) \mathfrak{B} \) for \( \alpha \in K'^\times \) and \( \mathfrak{B} \) prime to \( p \). In the case that we considered we have \( a_i \geq 1 \) for all \( i \). From the definition of the \( p^{\infty} \)-structure we have that the function \( P_\delta \tilde{F} \) is supported in

\[
(\prod_i \mathfrak{p}_i^{-a_i}) \mathcal{U}_j \mathcal{B}^{-1} = (\alpha^{-1}) \mathfrak{B}^{-1} \mathcal{U}_j
\]

In particular the computations of Katz for the twisted values now read,

\[
\sum_{j=1}^h \chi(\mathcal{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1} \mathcal{U}_j} \frac{P_\delta \tilde{F}(\alpha a)}{\prod_{\sigma} \sigma(a)^k |N_Q^/K'(\alpha a)|^s} = \sum_{j=1}^h \chi(\mathcal{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1} \mathcal{U}_j} \frac{P_\delta \tilde{F}(\alpha a)}{\prod_{\sigma} \sigma(\alpha^{-1} a)^k |N_Q^/K'(\alpha^{-1} a)|^s} = \sum_{j=1}^h \chi(\mathcal{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1} \mathcal{U}_j} \frac{P_\delta \tilde{F}(\alpha a)}{\prod_{\sigma} \sigma(\alpha^{-1} a)^k |N_Q^/K'(\alpha^{-1} a)|^s} = \left( P_\delta \tilde{F}(\alpha^{-1}) |N_Q^/K'(\alpha)|^s \prod_{\sigma} \sigma(a)^k \right) \sum_{j=1}^h \chi(\mathcal{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1} \mathcal{U}_j} \frac{\chi_{\text{finite}}(a)}{\prod_{\sigma} \sigma(a)^k |N_Q^/K'(\alpha)|^s}
\]

There is a special case where it is easy to see the difference of the new factors with those of Katz. Let us assume that for the decomposition \( \theta_{F'/F} = \mathfrak{P} \mathfrak{B} \) there exists \( \zeta \in K' \) so that
where \( \psi \). We define \( \alpha' \in K'^{\times} \) as in Katz by \( \prod_i p_i^{a_i} = (\alpha')\mathcal{B}' \) for \( \mathcal{B}' \) prime to \( p \) and we compare

\[
Local(\Sigma', \chi, \delta, \xi)_p := \frac{P_{\delta} \hat{F}(\alpha^{-1})}{\chi(\mathcal{B})} \prod_{\sigma} \chi(\alpha)^k
\]

against the local factor of Katz

\[
\frac{P_{\delta} \hat{F}(\alpha'^{-1})}{\chi(\mathcal{B})} \prod_{\sigma} \chi(\alpha'^{k})
\]

We consider

\[
\frac{P_{\delta} \hat{F}(\alpha^{-1})}{\chi(\mathcal{B})} \prod_{\sigma} \chi(\alpha)^k = \frac{P_{\delta} \hat{F}(\alpha'^{-1})}{\chi(\mathcal{B})} \times \chi(\mathcal{B}^{\prime-1}) \times \prod_{\sigma} \chi(\alpha'^{k})
\]

Note that from our assumptions \( \xi = \zeta \zeta^{\prime} \) hence we have \( \alpha = \alpha' \zeta \). This implies

\[
\frac{P_{\delta} \hat{F}(\alpha^{-1})}{\chi(\mathcal{B})} \prod_{\sigma} \chi(\alpha)^k = \prod_{p \in \Sigma_p} \hat{F}_{p, \delta}(\alpha^{-1}) \hat{F}_{p}(\alpha^{-1}) = \prod_{p \in \Sigma_p} \chi_p(\bar{\zeta}) \chi_p(\zeta^{\prime-1})
\]

and \( \mathcal{B} = \mathcal{B}' \) and \( \prod_{\sigma} (\alpha \sigma^{k}) = \prod_{\sigma} (\zeta^{\prime k}) \).

The general case: Now we consider the case where some of the \( a_i \)'s in \( cond(\chi) = \prod_i p_i^{a_i} \) are zero. We start by stating the following (see [23] page 282 or [16] page 209),

\[
\int_{\mathcal{N}_{\mathcal{B}}} \psi_{\beta'}(xy)dy = I_{\mathcal{N}_{\mathcal{B}}}(x) - \frac{1}{N_p} I_{p^{-1}\mathcal{N}_{\mathcal{B}}}(x)
\]

where \( \psi_{\beta'} \) is the additive character of \( K_{\mathcal{B}} \) given by

\[
\psi_{\beta'}(x) := exp \circ \text{Tr}_{p} \left( \frac{x}{\delta'} \right)
\]

In particular if we denote by \( \psi_{\delta} \) the additive character

\[
\psi_{\delta}(x) := exp \circ \text{Tr}_{p} \left( \frac{x}{\delta} \right)
\]

we have

\[
\int_{\mathcal{N}_{\mathcal{B}}} \psi_{\delta}(xy)dy = I_{\mathcal{N}_{\mathcal{B}}}(x\xi) - \frac{1}{N_p} I_{p^{-1}\mathcal{N}_{\mathcal{B}}}(x\xi)
\]

where we recall \( \xi = \frac{d}{\delta} \) up to elements in \( \mathcal{N}_{\mathcal{B}}^{\times} \). Now we follow the computations of Katz as in ([23] page 281-282). We use the same notation as in Katz. In our setting after the observation above we have that the function \( P\hat{F} \) is supported in

\[
\prod_{a_i \geq 1} p_i^{-a_i} (\prod_{a_j \geq 1} p_j^{-e_j}) (\prod_{a_j = 0} p_j^{-1-e_j}) \mathcal{U} = (\alpha^{-1}) \mathcal{B}^{-1} (\prod_{a_j = 0} p_j^{-1-e_j}) \mathcal{U}
\]

where \( \alpha \) relative prime to the \( p_i \)'s with \( a_i \geq 1 \), \( \mathcal{B} \) prime to \( p \) and \( e_j := ord_{p_j} \xi \). From the observation above we have that for \( a \in \mathcal{B}^{-1}(\prod_{a_j = 0} p_j^{-1-e_j}) \mathcal{U} \) we have

\[
P\hat{F}(\alpha^{-1} a) = P_{\delta} F(\alpha^{-1}) \chi_{2, \text{finite}}(a) \prod_{a_j = 0} \text{char}(p_j^{1+e_j})(a)
\]
where
\[ \text{char}(p_j^{1+e_j})(a) = \begin{cases} 1 - \frac{1}{Np_j}, & \text{if } ord_{p_j}(a) \geq -e_j; \\ -\frac{1}{Np_j}, & \text{if } ord_{p_j}(a) = -e_j - 1. \end{cases} \]

Following Katz (note a typo in Katz’s definition! compare 5.5.31 with 5.5.35) we extend the above function to the set \( \mathfrak{P} \) of fractional ideals \( I \) of \( K' \) of the form
\[ I = (\prod_{j=0}^{h} p_j^{-1-e_j}) \mathfrak{P} \]
where \( \mathfrak{P} \) is an integral ideal, prime to those \( p_i \) with \( a_i \neq 0 \) and to all \( p_k \) by
\[ \text{char}(p_j^{1+e_j})(I) = \begin{cases} 1 - \frac{1}{Np_j}, & \text{if } Ip_j^{e_j} \text{ is integral;} \\ -\frac{1}{Np_j}, & \text{if not.} \end{cases} \]

Following Katz’s computations we have that the values that we are interested in are
\[ \sum_{j=1}^{h} \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1}(\prod_{j=0}^{h} (p_j^{-1-e_j})) \mathfrak{U}_j} \frac{P_a \hat{F}(\alpha^{-1}a)}{\prod_{\sigma} \sigma(\alpha^{-1}a)^k |N_{Q'}(\alpha^{-1}a)|^s} = \]
\[ \left( \frac{P_a \hat{F}(\alpha^{-1})}{\chi(\mathfrak{B})} \right) \prod_{\sigma} \sigma(\alpha)^k \sum_{I_0 \in \mathbf{I}(p)} \frac{\chi(I_0)}{N(I_0)^s} \prod_{a_j=0} \sum_{n \geq -1-e_j} \frac{\chi_2(p_j)^n}{N(p_j)^{ns}} \text{char}(p_j^{1+e_j})(p_j^n) \]

As in Katz we compute the inner sum
\[ \sum_{n=-1-e_j}^{\infty} \frac{\chi_2(p_j)^n}{N(p_j)^{ns}} \text{char}(p_j^{1+e_j})(p_j^n) = \frac{-1}{N(p_j)} \frac{\chi_2(p_j)^{-1-e_j}}{N(p_j)^{(-1-e_j)s}} + \left( 1 - \frac{1}{N(p_j)} \right) \sum_{n=-e_j}^{\infty} \frac{\chi_2(p_j)^n}{N(p_j)^{ns}} \]
\[ = \sum_{n=-e_j}^{\infty} \frac{\chi_2(p_j)^n}{N(p_j)^{ns}} - \frac{1}{N(p_j)} \left( \sum_{n=-e_j}^{\infty} \frac{\chi_2(p_j)^n}{N(p_j)^{ns}} \right) \]
\[ = \left( 1 - \frac{1}{N(p_j)} \right) \sum_{n=-e_j}^{\infty} \frac{\chi_2(p_j)^n}{N(p_j)^{ns}} \]
\[ = \left( 1 - \frac{N(p_j)^s}{\chi_2(p_j)N(p_j)} \right) \sum_{n=0}^{\infty} \frac{\chi_2(p_j)^n}{N(p_j)^{ns}} \]
\[ = \left( 1 - \frac{N(p_j)^s}{\chi_2(p_j)N(p_j)} \right) \frac{\chi_2(p_j)^{-e_j}}{N(p_j)^{-e_j s}} \left( 1 - \chi_2(p_j)N(p_j)^{-s} \right)^{-1} \]
\[ = \left( 1 - \frac{N(p_j)^s}{\chi_2(p_j)N(p_j)} \right) \frac{\chi_2(p_j)^{-e_j}}{N(p_j)^{-e_j s}} \left( 1 - \chi_2(p_j)N(p_j)^{-s} \right)^{-1} \]

So we conclude,
\[ \sum_{j=1}^{h} \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1}(\prod_{j=0}^{h} (p_j^{-1-e_j})) \mathfrak{U}_j} \frac{P_a \hat{F}(\alpha^{-1}a)}{\prod_{\sigma} \sigma(\alpha^{-1}a)^k |N_{Q'}(\alpha^{-1}a)|^s} = \]
whose value at $s = 0$ is equal to
\[
\left( \frac{P_b \hat{F}(\alpha^{-1})}{\chi(2)} \prod_{\sigma} \sigma(\alpha)^k \right) L(0, \chi_1) \prod_{a_j=0} \left( \frac{1 - \chi(\tilde{p}_j)}{(1 - \chi_2(\tilde{p}_j))} \chi_2(\tilde{p}_j)^{-\epsilon_j} \right)
\]

But $L(s, \chi_1) = L(s, \chi) \prod_{\tilde{p}_j} (1 - \chi(\tilde{p}_j)N(\tilde{p}_j)^{-s}) (1 - \chi(\tilde{p})N(\tilde{p})^{-s})$ which allow us to conclude that the values are equal to
\[
\left( \frac{P_b \hat{F}(\alpha^{-1})}{\chi(2)} \prod_{\sigma} \sigma(\alpha)^k \right) L(0, \chi) \prod_{\tilde{p}_j \in \Sigma_p} (1 - \chi(\tilde{p}_j))(1 - \chi(\tilde{p})) \prod_{a_j=0} \chi(\tilde{p}_j)^{-\epsilon_j}
\]

\[\square\]

5. The relative setting: Congruences between Eisenstein series

Now we consider the following relative setting. We consider as in the introduction a totally real field galois extension $F'$ of $F$ of degree $p$ ramified only at $p$ and write $\Gamma = Gal(F'/F)$. We fix ideals $a, b, c$ and $f$ of $F$ and consider also the corresponding ideals in $F'$, that is their natural image under $F \hookrightarrow F'$. We write $T'$ and $T^{\times}$ for the corresponding spaces in the $F'$ setting that we have introduced for the $F$ setting. We note that $\Gamma$ operates naturally on the spaces $T'$ and $T^{\times}$. Moreover the embedding $F \hookrightarrow F'$ induces a natural diagonal embedding $\mathbb{H}[F:Q] \hookrightarrow \mathbb{H}[F':Q]$ with the property that the pull back of a Hilbert modular form of $F'$ is a Hilbert modular form of $F$. We need to make this last remark a little bit more explicit.

The Tate-Abelian Scheme and the modular interpretation of the diagonal embedding: We would like now to describe the geometric meaning of the diagonal embedding. We follow the book of Hida [13] as in chapter 4 (and especially section 4.1.5) and the notation there.

For fractional ideals $a$ and $b$ of the totally real field $F$ and a ring $R$ we define the ring $R[[((ab)_+)]]$ with $(ab)_+ := ab \cap F_+$ to be the ring of formal series
\[
R[[((ab)_+)]] := \{ a_0 \sum_{\xi \in (ab)_+} a_\xi q^\xi | a_\xi \in R \}
\]

We pick the multiplicative set $q^{((ab)_+)} := \{ q^\xi | \xi \in (ab)_+ \}$ and define $R\{ab\}$ as the localization of $R[[((ab)_+)]]$ to this multiplicative set. Then as explained in Hida the Tate semi-abelian scheme $\text{Tate}_{a,b}(q)$ is defined over the ring $R\{ab\}$ (with $R$ depending on the extra level structure that we impose) by the algebraization of the rigid analytic variety
\[
(G_m \otimes a^{-1} \theta_{F^{-1}})/q^b
\]

Let $X$ be a HBAV over a ring $R$ with real multiplication by $\tau$. We may define a HBAV $X'$ over $R$ with real multiplication by $\tau'$ by considering the functor from schemes $S$ over $R$ to $\tau'$ modules defined by
\[
S \mapsto X'(S) := X(S) \otimes_{\tau} \theta_{F'/F}^{-1}
\]

We let $\epsilon := ab^{-1}$ and consider the effect of our map on the Tate curve $\text{Tate}_{a,b}(q)$. That is we consider the HBAV with real multiplication by $\tau'$ defined by $\text{Tate}_{a,b}(q) \otimes_{\tau} \theta_{F'/F}^{-1} = \boxed{\text{\epsilon}}$
We consider the map \( tr_{F'/F} : R\{ab\theta_{F'/F}^{-1}\} \to R\{ab\} \) given by 
\[ q^\alpha \mapsto q^{tr_{F'/F}(\alpha)}. \]
Then we have,

**Lemma 5.1.**

\[ \text{Tate}_{\alpha',b\theta_{F'/F}^{-1}}(q) \times_{R\{b\theta_{F'/F}^{-1}\}} R\{ab\} \cong \text{Tate}_{a,b}(q) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} \]

**Proof.** Even though the lemma holds in general we are going to use it while working over number fields. Hence after fixing embeddings in the complex numbers we may just prove it over \( \mathbb{C} \). Over the complex numbers this follows easily by observing that \( \text{Tate}_{a,b}(q) \) corresponds to that lattice \( 2\pi i(bz + a^{-1}\theta_{F'}) \) for \( z \in \mathbb{H}_F \) and hence \( \text{Tate}_{a,b}(q) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} \) to the lattice

\[ 2\pi i(bz + a^{-1}\theta_{F'}) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} = 2\pi i(b\theta_{F'/F}z' + a^{-1}\theta_{F'}) \]

with \( z' \in \mathbb{H}_{F'} \) the image of \( z \) under the diagonal embedding \( \mathbb{H}_F \hookrightarrow \mathbb{H}_{F'} \) induced from \( F' \hookrightarrow F' \). Moreover in this case the map \( tr_{F'/F} : R\{ab\theta_{F'/F}^{-1}\} \to R\{ab\} \) given by \( q^\alpha \mapsto q^{tr_{F'/F}(\alpha)} \) corresponds to setting the indeterminate \( q := \exp(T_{r_{F'}}(z')) := \exp(\sum_{\sigma \in \Sigma'} z'_{\sigma}) \)

where \( \sigma \in \Sigma' \) the embeddings \( \sigma : F' \to \mathbb{C} \) and \( z' = (z'_{\sigma}) \in \mathbb{H}_{F';\mathbb{Q}} \) equal to the indeterminate \( q = \exp(T_{r_{F'}}(\Delta(z))) \) for \( \Delta : \mathbb{H}_{F';\mathbb{Q}} \hookrightarrow \mathbb{H}_{F';\mathbb{Q}} \), the diagonal map. In particular that implies that the complex points of \( \text{Tate}_{\alpha',b\theta_{F'/F}^{-1}}(q) \times_{R\{b\theta_{F'/F}^{-1}\}} R\{ab\} \) correspond to the lattice \( 2\pi i(b\theta_{F'/F}z' + a^{-1}\theta_{F'}) \) for \( z' = \Delta(z) \).

\[ \square \]

We can use the above lemma to study the effect of the diagonal embedding to the \( q \)-expansion, that is to the values of Hilbert modular forms on the Tate abelian scheme. For a \( \mathcal{C} \theta_{F'/F} \)-HMF \( \phi \) of \( F' \) we have that

\[ \phi(\text{Tate}_{a,b}(q) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1}) = \phi(\text{Tate}_{a',b\theta_{F'/F}^{-1}}(q) \times_{R\{b\theta_{F'/F}^{-1}\}} R\{ab\}) = \]

\[ = \phi(\text{Tate}_{a',b\theta_{F'/F}^{-1}}(q)) \times_{R\{b\theta_{F'/F}^{-1}\}} R\{ab\} \]

The next question that we need to clarify is what is happening under this diagonal map for an HBAV with real multiplication by \( \tau \) that has CM by \( \mathfrak{A} \), the ring of integers of a totally imaginary quadratic extension \( K \) of \( F \). It is well known that up to isomorphism these are given by the fractional ideals of \( K \). Let us write \( \mathfrak{M} \) for one of these and \( X(\mathfrak{M}) \) for the corresponding HBAV with CM by \( \mathfrak{M} \). We see that the above map gives us the HBAV \( X(\mathfrak{M}) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} \) with real multiplication by \( \tau' \). We set \( K' = F' \) and write \( \mathfrak{M}' \) for its ring of integers. Then we have,

**Lemma 5.2.** Assume that \( \mathfrak{M}' = \mathfrak{M} \tau' \). Then the HBAV \( X(\mathfrak{M}) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} \) has CM by \( \mathfrak{M}' \) and it corresponds to the fractional ideal \( \mathfrak{M} \mathfrak{D}^{-1} \) with \( \mathfrak{D} = \theta_{F'/F}^{\mathfrak{M}'} \).

**Proof.** We write \( K = F(d) \) and then \( K' = F'(d) \). In particular since \( X(\mathfrak{M}) \) has CM by \( K \) we conclude that \( X(\mathfrak{M}) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} \) has CM by \( K' \) as we have \( d \in End(X(\mathfrak{M})) \hookrightarrow End(X(\mathfrak{M})) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} \). Moreover we have

\[ X(\mathfrak{M}) \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} = X(\mathfrak{M}) \otimes_{\mathbb{Q}} \tau' \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} = X(\mathfrak{M}\mathfrak{D}') \otimes_{\mathbb{Q}} \theta_{F'/F}^{-1} = X(\mathfrak{M}\mathfrak{D}') / (X(\mathfrak{M}\mathfrak{D}'))[\theta_{F'/F}] \]
Similarly we have Proposition 5.3.

The key proposition is now is the following which later will allow us to compare the measures $cusp\frac{\text{the congruences of Eisenstein series}}{\text{that}}\frac{\text{We remark that the condition of the lemma,}}{\text{where}}\frac{\text{We consider the cusp,}}{\text{with}}\frac{\text{Eisenstein}}{\text{where}}\frac{\text{We have that}}{\text{that}}\frac{\text{Proposition 3.1 we know that the}}{\text{Proof.}}\frac{\text{frac}}{\text{We have that}}\frac{\text{but then we have}}{\text{The key proposition is now is the following which later will allow us to compare the measures of Katz-Hida-Tilouine over K and K'}}{\text{Proposition 5.3. (Congruences) Let c be a fractional ideal of F relative prime to p. We have the congruences of Eisenstein series}}\frac{\text{res}_\Delta(E_k(\phi', c\theta_{F'/F})) \equiv Frob_p(E_{pk}(\phi, c)) \mod p}{\text{where} \phi := \phi' \text{ over and} \phi' \text{ a locally constant } \mathbb{Z}_p\text{-valued function on } \mathfrak{r}'^\times \times (\mathfrak{c}'/f)^\times \times (\mathfrak{r}'/f)^\times}{\text{with } \phi' = \phi \text{ for all } \gamma \in \Gamma}{\text{Proof. We consider the cusp } (\mathfrak{r}', b\theta^{-1}_{F'/F}) \text{ for } b \text{ a fractional ideal of } F \text{ equal to } c^{-1}.}{\text{From Proposition 3.1 we know that the } q\text{-expansion of the Eisenstein series } E_k(\phi', c\theta_{F'/F}) \text{ at the cusp } (\mathfrak{r}', b\theta^{-1}_{F'/F}) \text{ is given by}}{E_k(\phi', c\theta_{F'/F})(Tate_{\mathfrak{r}', b\theta^{-1}_{F'/F}}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 < \xi \in b\theta^{-1}_{F'/F}} a(\xi, \phi', k)q^\xi}{\text{with } a(\xi, \phi', k) = \sum_{(a,b) \in (\mathfrak{r}' \times b\theta^{-1}_{F'/F})/\mathfrak{r}' \times ,ab=\xi} \phi'(a,b)sgn(N(a))N(a)^{k-1}}{\text{As the function } \phi' \text{ is supported on the units of } \mathfrak{r}'^\times \text{ with respect to the second variable (i.e. the } b\text{'s above)} \text{ we have that the above } q\text{-expansion with respect the selected cusp is given by}}{E_k(\phi', c\theta_{F'/F})(Tate_{\mathfrak{r}', b\theta^{-1}_{F'/F}}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 < \xi \in b} a(\xi, \phi', k)q^\xi}{\text{with } a(\xi, \phi', k) = \sum_{(a,b) \in (\mathfrak{r}' \times b)/\mathfrak{r}' \times ,ab=\xi} \phi'(a,b)sgn(N(a))N(a)^{k-1}}{\text{From Lemma 5.1 and the discussion after that it follows that the } q\text{-expansion of the restricted Eisenstein } res_\Delta E_k(\phi', c\theta_{F'/F}) \text{ series at the cusp } (r, b) \text{ is given by}}{res_\Delta E_k(\phi', c\theta_{F'/F})(Tate_{r,b}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 < \xi \in b} a(\xi, \phi', k)q^\xi}{\text{where } a(\xi, \phi', k) = \sum_{\xi' \in b, Tr_{F'/F}(\xi')=\xi} a(\xi', \phi', k)}{\text{The } q\text{-expansion of the Eisenstein series } E_{pk}(\phi, c) \text{ at the cusp } (r, b) \text{ is given by}}{E_{pk}(\phi, c)(Tate_{r,b}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 < \xi \in b} a(\xi, \phi, pk)q^\xi}
with
\[ a(\xi, \phi, pk) = \sum_{(a,b) \in (r \times b)/\mathfrak{r}^*} \phi(a, b) \text{sgn}(N(a)) N(a)^{pk-1} \]

and hence that of \( \text{Frob}_p(E_{pk}(\phi, c)) \) is given by
\[ \text{Frob}_p(E_{pk}(\phi, c))(\text{Tate}_\mathfrak{r}, b(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 \leq \xi \in b} a(\xi, \phi, pk)c^{pk} \]

In order to establish the congruences of the Eisenstein series it is enough, thanks to the \( q \)-expansion principle to establish the congruences between the \( q \)-expansions at the selected cusp \((r, b)\).

We start by observing that the Eisenstein series \( \text{Frob}_p(E_{pk}(\phi, c)) \) has non-zero terms only at terms divisible by \( p \) as we assume that the ideal \( b \) is prime to \( p \). We consider the \( \xi^{th} \)-term of \( \text{res}_E \mathfrak{E}_b(\phi', c) \). It is equal to
\[ a(\xi, \phi', k) = \sum_{\xi' \in b, Tr_{F'/F}(\xi') = \xi} \sum_{(a,b) \in (r \times b)/\mathfrak{r}^*} \phi'(a, b) \text{sgn}(N(a)) N(a)^{k-1} \]

We observe that the group \( \Gamma = \text{Gal}(F'/F) \) acts on the triples \((\xi', a, b)\) of the summation above by \((\xi', a, b)\gamma := (\xi'^{\gamma}, a^{\gamma}, b^{\gamma})\) as \( b \) is an ideal of \( F \) hence is preserved by \( \Gamma \), where the action on \( a \) and \( b \) is modulo the units in \( \mathfrak{r}' \) to understand. We write \( \gamma \) for a generator of \( \Gamma \). We consider two cases, the case where \((\xi, a, b)\) is fixed by \( \gamma \) and the case where it is not. In the first case we notice that as \( \phi' \) is fixed under \( \Gamma \) we have that \( \phi'(a^{\gamma}, b^{\gamma}) = \phi'(a, b) \). Hence we have
\[ \sum_{i=0}^{p^{-1}} \phi'(a^{\gamma^i}, b^{\gamma^i}) \text{sgn}(N(a^{\gamma^i})) N(a^{\gamma^i})^{k-1} = p \phi'(a, b) \text{sgn}(N(a)) N(a)^{k-1} \equiv 0 \pmod{p} \]

If \((\xi', a, b)\) is fixed by \( \gamma \) then that implies that (i) \( \xi' \in F \) and (ii) the ideals generated by \( a \) and \( b \) in \( \mathfrak{r}' \) are coming from ideals in \( r \) as they are relative prime to \( \theta_{F'/F} \) i.e. to the primes where the extension is ramified. Moreover as we assume that \( \text{Cl}_F \hookrightarrow \text{Cl}_{F'} \) we have that actually the elements themselves are (up to units) equal to elements from \( F \). In this case we first notice that \( \xi = Tr_{F'/F}(\xi') = p\xi' \) and as \( \xi' \in b\mathfrak{r}' \) with \( b \) prime to \( p \) we have that \( \xi \) is also divisible by \( p \) in the sense that is of the form \( p\xi' \) for \( \xi' \in b \). Further we have the congruences modulo \( p \)
\[ \phi'(a, b) \text{sgn}(N_{F'}(a)) N_{F'}(a)^{k-1} \equiv \phi(a, b) \text{sgn}(N_F(a)p) N_F(a)^{p(k-1)} \]

In particular we conclude that \( a(\xi, \phi', k) \equiv 0 \pmod{p} \) if \( \xi \) is not of the form \( p\xi' \) for \( \xi' \in b \subset F \). In the case where \( \xi \) is of the form \( p\xi' \) we have seen that
\[ a(p\xi', \phi', k) \equiv \sum_{(a,b) \in (r \times b)/\mathfrak{r}^*, ab = \xi'} \phi(a, b) \text{sgn}(N_F(a)) N_F(a)^{pk-1} = a(\xi, \phi, pk) \pmod{p} \]

But \( a(\xi, \phi, pk) \) is the \( p\xi'^{th} \) Fourier term of \( \text{Frob}_p(E_{pk}(\phi, c)) \) which allow us to conclude the proof of the proposition.
Before we prove our main theorem we need to make some preparation. In this section we explain how we can use the theory of complex multiplication to understand how Frobenious operates on values of Eisenstein series of CM points. We recall that we consider the CM types $(K_0, \Sigma_0)$ and its lift $(K, \Sigma)$. Moreover by our setting we have that the reflex field for both of these CM types is simply $(K_0, \Sigma_0)$. We first note that since we assume that $p$ is unramified in $F$ then the triples $(X(\Omega), \lambda(\Omega), i(\Omega))$ are defined over the ring of integers of $W = \mathbb{F}_p$ (see [17] page 69). We write $\Phi$ for the extension of the Frobenious in $\text{Gal}(\mathbb{Q}^{nr}/\mathbb{Q}_p)$ to $W$. In this section we prove the following proposition which is just a reformulation of what is done in [22] (page 539) in the case of quadratic imaginary fields.

**Proposition 6.1. (Reciprocity law on CM points)** For every fractional ideal $\Omega$ of the CM field $K$ and $\phi$ a $\mathbb{Z}_p$ valued locally constant function we have the reciprocity law

$$\text{Frob}_p(E_{pk}(\phi, c)(X(\Omega), \lambda(\Omega), i(\Omega))) = (E_{pk}(\phi, c)(X(\Omega), \lambda(\Omega), i(\Omega)))^\Phi$$

**Proof.** Let us write $\mathcal{R}$ for the ring of integers of $W$. As we are assuming that $\phi$ is $\mathbb{Z}_p$ valued and we know from above that the triple $(X(\Omega), \lambda(\Omega), i(\Omega))$ is defined over $\mathcal{R}$ we have that the value of the Eisenstein series is in $\mathcal{R}$. From the compatibility of $p$-adic modular forms with ring extensions and the fact that the Eisenstein series is defined over $\mathbb{Z}_p$ we have that

$$(E_{pk}(\phi, c)(X(\Omega), \lambda(\Omega), i(\Omega)))^\Phi = (E_{pk}(\phi, c)(X(\Omega), \lambda(\Omega), i(\Omega) \otimes_{\mathcal{R}, \Phi} \mathcal{R}))$$

where the tensor product is with respect to the map $\Phi : \mathcal{R} \rightarrow \mathcal{R}$, i.e. the base change of the triple $(X(\Omega), \lambda(\Omega), i(\Omega))$ with respect to the frobenious map. But then from the theory of complex multiplication see [24] (Lemma 3.1 in page 61 and Theorem 3.4 in page 66), the fact that the reflex field of $(K, \Sigma)$ is $(K_0, \Sigma_0)$ and that $p$ is ordinary we have that

$$(X(\Omega), \lambda(\Omega), i(\Omega)) \otimes_{\mathcal{R}, \Phi} \mathcal{R} \cong (X'(\Omega), \lambda'(\Omega), i'(\Omega))$$

where $(X'(\Omega), \lambda'(\Omega), i'(\Omega))$ is the quotient obtained by $X/H_{can}$ with $H_{can} := i(\theta_F \otimes \mu_p)$ as explained in Katz [23] page 223. Moreover as in Katz we have that the Tate HBAV $(\text{Tate}_{a,b}'(q), \lambda'_{can}, i'_{can})$ is obtained from $(\text{Tate}_{a,b}(q), \lambda_{can}, i_{can})$ by the map $q \mapsto q^p$ from which we conclude the proposition.

**7. Complex and $p$-adic periods.**

In this section we study the various periods (archimedean and $p$-adic) that appear in the interpolation properties of the $KHT$-measure. We also consider the relative situation and we focus especially in the case of interest with $(K_0, \Sigma_0) < (K, \Sigma) < (K', \Sigma')$.

**The periods of Katz:** We start by recalling the periods defined by Katz and then showing that in the case of the twisted measure the periods used remain unchanged. We follow Katz (see [23] page 268) and fix a nowhere vanishing differential over $A := \{a \in \mathbb{Q} : \text{incl}(p)(a) \in D_p\}$

$$\omega : \text{Lie}(X(\mathfrak{R})) \cong \theta_F^{-1} \otimes A$$

Then for any fractional ideal $\Omega$ of $K$ that is relative prime to the place induced by $\text{incl}(p)$ we have an identification $\text{Lie}(X((\Omega))) = \text{Lie}(X(\mathfrak{R}))$ and hence one may use the very same $\omega$ to fix a nowhere differential of $X(\Omega)$ by

$$\omega(\Omega) : \text{Lie}(X((\Omega))) = \text{Lie}(X(\mathfrak{R})) \cong \theta_F^{-1} \otimes A$$
We use \( \text{incl}(\infty) : A \mapsto \mathbb{C} \) to define the standard complex nowhere vanishing differential \( \omega_{\text{trans}}(X(\mathfrak{m})) \) associated to the torus \( \mathbb{C}^2 / \Sigma \mathfrak{m} \). Then as in Katz ([23], Lemma 5.1.45) we have an element \( \Omega_{K}^{\text{atz}} = (\ldots, \Omega(\sigma), \ldots) \in (\mathbb{C}^\times)^\Sigma \) such that for all fractional ideals \( \mathfrak{m} \) of \( K \) relative prime to \( p \) we have

\[
\omega(\mathfrak{m}) = \Omega_{K}^{\text{atz}} \omega_{\text{trans}}(\mathfrak{m})
\]

Of course the same considerations hold for \( K_0 \) and \( K' \). Especially for \( K' \) we want to compute also the periods for the twisted HBAV \( X(\mathfrak{m} \otimes \xi) \). From the isomorphism \( X(\mathfrak{m}) \cong X(\mathfrak{m} \otimes \xi^{-1}) \) we have that we can pick the invariant differentials \( \omega(\mathfrak{m} \otimes \xi^{-1}) \) and \( \omega_{\text{trans}}(\mathfrak{m} \otimes \xi^{-1}) \) as \( \xi \cdot \omega(\mathfrak{m}) \) and \( \xi \cdot \omega_{\text{trans}}(\mathfrak{m}) \) respectively. In particular we have that the selected periods are equal to \( \Omega_{K}^{\text{atz}} \). Similarly Katz ([23] Lemma 5.1.47) defines \( p \)-adic periods in \( (D^\times_p)^\Sigma \) relating the invariant differential \( \omega(\mathfrak{m}) \) to the invariant differential \( \omega_{\text{can}}(\mathfrak{m}) \) obtained from the \( p^\infty \)-structure. As above we obtain that the \( p \)-adic periods for the twisted HBAV are the same.

**Picking the periods compatible:**(See also [12] page 195 on the properties of the periods defined by Katz). Now we consider the more specific setting where \( (K, \Sigma) \) and \( (K', \Sigma') \) are lifted from the type \( (K_0, \Sigma_0) \). Moreover as we assume that \( K_0 \) is the CM field of an elliptic curve defined over \( \mathbb{Q} \), we have that \( \mathfrak{R}_0 \) has class number one, i.e. it is a P.I.D. That means that the ring of integers \( \mathfrak{R} \) and \( \mathfrak{R}' \) are free over \( \mathfrak{R}_0 \). That means that we have

\[
\text{Lie}(X(\mathfrak{R})) = \oplus_{j=1}^{q} \text{Lie}(X(\mathfrak{R}_0))
\]

and similarly

\[
\text{Lie}(X(\mathfrak{R}')) = \oplus_{j=1}^{q'} \text{Lie}(X(\mathfrak{R}_0))
\]

In particular that implies that

\[
\Omega_{K}^{\text{atz}} = (\ldots, \Omega(E), \ldots), \quad \text{and} \quad \Omega_{K'}^{\text{atz}} = (\ldots, \Omega(E), \ldots)
\]

Similarly for the \( p \)-adic periods we observe that \( X(\mathfrak{R}) \cong E \times \ldots \times E \) and hence \( X(\mathfrak{R})[p^\infty] \cong E[p^\infty] \times \ldots \times E[p^\infty] \) where \( E \) is the elliptic curve defined over \( \mathbb{Q} \) that corresponds to the ideal \( \mathfrak{R}_0 \) with respect to the CM type \( (K_0, \Sigma_0) \). These considerations imply that

\[
\Omega_{p,K}^{\text{atz}} = (\ldots, \Omega_p(E), \ldots), \quad \text{and} \quad \Omega_{p,K'}^{\text{atz}} = (\ldots, \Omega_p(E), \ldots)
\]

We note that the definition of the periods of Katz in general are independent of the Grössencharacter in general since they depend only on its infinite type. This is why it is important to pick the differentials \( \omega(\mathfrak{R}) \) and \( \omega(\mathfrak{R}') \) properly. And actually in our setting we have a very natural choice by considering the elliptic curve \( E / \mathbb{Q} \) to whom the Grössencharacter \( \psi_0 \) is attached (recall that \( \psi_K = \psi_0 \circ N_{K / \mathbb{Q}} \) and \( \psi_{K'} = \psi_0 \circ N_{K' / \mathbb{Q}} \).

8. **Congruences of measures**

We are now ready to prove our main theorem. We recall that this amounts to proving the following

**Theorem 8.1.** If (i) \( Cl_{K'}(3) \cong Cl_{K'\cdot}(3)^\Gamma \) (ii) \( Cl_F(1) \hookrightarrow Cl_{F'/F} \) and (iii) \( \theta_{F'/F} = (\xi) \) with \( \xi \gg 0 \) and \( \xi = \zeta \xi_0 \) for \( \zeta \in K' \) then we have the congruences

\[
\int_{G_{K'}} \epsilon \circ \text{ver} \frac{d\mu_K^{\text{KHT}}}{\Omega_p(E)^g} \equiv \int_{G_{K'}} \epsilon \frac{d\mu_p^{\text{KHT,tw}}}{\Omega_p(E)^g} \mod p\mathbb{Z}_p
\]


for all $\epsilon$ locally constant $\mathbb{Z}_p$-valued functions on $G_{K'}$, with $\epsilon^\gamma = \epsilon$ and belong to the cyclotomic part of it, i.e. when it is written as a sum of finite order characters it is of the form $\epsilon = \sum c_\chi \chi$ with $\chi^T = \chi$.

The strategy for proving the above theorem is as follows. By definition we have that the twisted $KHT$-measure is given as

$$\int_{G'} \phi(g) \mu^K_{H,T,w}(g) := \sum_{j} \int_T \tilde{\phi}_j dE_j := \sum_{j} E_1(\phi_j, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1}))$$

We consider the set of representatives $\{\Omega_j\}$ of $CL^{-\gamma}_{K'}(\mathcal{O})$. If we consider the map

$$\rho : CL^{-\gamma}_{K'}(\mathcal{O}) \rightarrow CL^{-\gamma}_{K'}(\mathcal{O})$$

We may pick representatives of $Im(\rho)$ to be fractional ideals $\Omega_j$ with the property $\Omega_j^\gamma = \Omega_j$ for all $\gamma \in \Gamma$. Moreover we may pick the other representatives of $CL^{-\gamma}_{K'}(\mathcal{O})$ such that if $\Omega_j$ is a representative then if $\Omega_j^\gamma$ is not in the same equivalence class as $\Omega_j$ then it is also a representative (and this must hold for all $\gamma \in \Gamma$). We may split the twisted measure as follows,

$$\int_{G'} \phi(g) \mu^K_{H,T}(g) = \sum_{\Omega_j \in Im(\rho)} E_1(\phi_j, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1})) + \sum_{\Omega_j \not\in Im(\rho)} E_1(\phi_j, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1}))$$

Our strategy is to compare the first summand (i.e those CM points that are coming from $K$) with the $KHT$-measure of $K$ through the diagonal embedding that we have worked above. For the other part we will prove directly that under the assumptions of our theorem is in $p\mathbb{Z}_p$.

We start with the following proposition

**Proposition 8.2.** Let $\Omega_j$ be a fractional ideal of $K'$. Then for $\phi$ a locally constant function invariant under $\Gamma$ we have,

$$E_k(\phi, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1})) = E_k(\phi, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1}))$$

for $\gamma \in \Gamma$.

**Proof.** The first thing that we note is that for $\phi$ with $\phi^\gamma = \phi$ the following equality holds

$$E_k(\phi, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1})) = E_k(\phi, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1}))$$

for all $\gamma \in \Gamma$. Indeed it is enough to observe that $\frac{\xi_{\Omega_j}}{\xi_{\Omega_j}} \in \mathbb{R}_+$ and hence we have the equality of ideals $\Omega_j \otimes (\xi) = \Omega_j \otimes (\xi)$.

We now have from the definition of the Eisenstein series

$$E_k(\phi, \gamma_j)(X(\Omega_j^\xi), \lambda_5^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}), \delta(\Omega_j \otimes \theta_{F'/F}^{-1})) = \frac{(-1)^{k'g} \Gamma(k + s)^{g'}}{\sqrt{(D_{F'})}} \sum_{w \in (\Omega_j \otimes (\xi))(f_{(p)})^2 |N(w)|} P\phi(w) \left| \frac{N(w)^k |N(w)|^{2s}}{s=0} \right.$$
for $\phi$ factoring through $X_\alpha \times \mathcal{T}_p \times (\mathcal{T}/J)$ with $X_\alpha := \mathcal{T}_p/\alpha \mathcal{T}_p \times (\mathcal{T}/J)$ with $\alpha \in \mathbb{N}$. But then

$$P\phi(x^\gamma, y^\gamma) = p^{\alpha[F:F']|N_f|^{-1}} \sum_{a \in X_\alpha} \phi(a, y^\gamma)e_{F'}(ax^\gamma)$$

As $\gamma$ permutes $X_\alpha$ we have

$$\sum_{a \in X_\alpha} \phi(a, y^\gamma)e_{F'}(ax^\gamma) = \sum_{a \in X_\alpha} \phi(a^\gamma, y^\gamma)e_{F'}(a^\gamma x^\gamma) = \sum_{a \in X_\alpha} \phi(a, y)e_{F'}(ax)$$

which concludes our claim.

Back to our considerations we have that

$$\sum_{w \in (\mathcal{U}_j \otimes (\xi))(\mathcal{T}_p)} \frac{P\phi(w)}{N(w)^k |N(w)^2|^s} |_{s=0} = \sum_{w \in (\mathcal{U}_j \otimes (\xi))(\mathcal{T}_p)} \frac{P\phi(w)}{N(w)^k |N(w)^2|^s} |_{s=0}$$

But the last sum is equal to $\sum_{w \in (\mathcal{U}_j \otimes (\xi))(\mathcal{T}_p)} \frac{P\phi(w)}{N(w)^k |N(w)^2|^s} |_{s=0}$ which concludes the proof. 

We know consider the measure $\mu_{KHT}^{\psi, \delta, \xi}$. We recall that $\psi'$ is a Grössencharacter of type $1\Sigma$. We write $\psi_{finite}'$ for its finite part. Then we define introduce the notation for a locally constant function $\phi_j$.

$$E_{\psi'}(\phi_j, \epsilon_j) \left(X(\mathcal{U}_j^\xi), \lambda_\delta^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}), \epsilon^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}) \right) := E_1(\phi_j \psi_{finite,j}', \epsilon_j)(X(\mathcal{U}_j^\xi), \lambda_\delta^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}), \epsilon^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}))$$

Moreover we define the subset $S$ of the selected representatives of $Cl_{K'}^{-}(\mathfrak{m})$ as the set of ideals that represent classes in $Cl_{K'}^{-}(\mathfrak{m})$ but not in $Im(\rho)$.

**Corollary 8.3.** For the twisted $KHT$-measure we have the congruences

$$\int_{G'} \phi(g) \mu_{KHT}^{\psi, \delta, \xi}(g) = \sum_{\mathcal{U}_j \in Im(\rho)} E_{\psi'}(\phi_j, \epsilon_j)(X(\mathcal{U}_j^\xi), \lambda_\delta^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}), \epsilon^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}))$$

$$+ \sum_{\mathcal{U}_j \in S} E_{\psi'}(\phi_j, \epsilon_j)(X(\mathcal{U}_j^\xi), \lambda_\delta^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}), \epsilon^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1})) \mod p$$

for all $\mathbb{Z}_p$-valued locally constant functions $\phi$ of $G'$ such that $\phi^\gamma = \phi$ for all $\gamma \in \Gamma$.

**Proof.** It follows directly from the fact that $|\Gamma| = p$ and that $\phi^\gamma = \phi$ for all $\gamma \in \Gamma$.

Our next aim is to prove the following proposition

**Proposition 8.4.** Under our assumption, for all $\mathbb{Z}_p$-valued locally constant $\phi$ with $\phi^\gamma = \phi$ for all $\gamma \in \Gamma$, we have the congruences

$$\Phi \left( \int_{G} (\phi \circ \sigma)(g) \mu_{\psi, \delta}^{KHT}(g) \right) = \sum_{\mathcal{U}_j \in Im(\rho)} E_{\psi'}(\phi_j, \epsilon_j)(X(\mathcal{U}_j^\xi), \lambda_\delta^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1}), \epsilon^\xi (\mathcal{U}_j \otimes \theta_{F'/F}^{-1})) \mod p$$

where $\Phi$ was the extension of the Frobenious element from its action on $\mathbb{Q}_p^\infty$ to its $p$-adic completion $J_\infty$. 

Proof. By definition we have that
\[\int_G (\phi \circ \text{ver})(g) \mu_{\psi, \delta}^{KHT}(g) = \sum_j E_{\psi}(\phi \circ \text{ver}_j, c_j)(X(\mathfrak{U}_j), \lambda_\delta(\mathfrak{U}_j), \iota(\mathfrak{U}_j), \sigma(\mathfrak{U}_j))\]
where the sum runs over a set of representatives of \(\text{Cl}_K^{-}(\mathfrak{3})\) and
\[E_{\psi}(\phi \circ \text{ver}_j, c_j)(X(\mathfrak{U}_j), \lambda_\delta(\mathfrak{U}_j), \iota(\mathfrak{U}_j), \sigma(\mathfrak{U}_j)) := E_p(\phi_{\psi'_\text{finite}} \circ \text{ver}_j, c_j)(X(\mathfrak{U}_j), \lambda_\delta(\mathfrak{U}_j), \iota(\mathfrak{U}_j), \sigma(\mathfrak{U}_j))\]
where we note that \(\psi' \circ \text{ver} = \psi^p\) as \(\psi' = \psi \circ N_{K'/K}\). From the congruences between the Eisenstein series that we have proved in Proposition 5.3 we have that
\[F_{\rho}(E_{\psi}(\phi \circ \text{ver}_j, c_j)(X(\mathfrak{U}_j), \lambda_\delta(\mathfrak{U}_j), \iota(\mathfrak{U}_j), \sigma(\mathfrak{U}_j))) = E_{\psi'}(\phi, c_j)(X(\mathfrak{U}_j^\xi), \lambda_\delta(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \iota(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}))\]
where of course in the right hand side \(\mathfrak{U}_j\) is understood as \(\mathfrak{U}_j^\mathfrak{3}\). We sum over all representatives of \(\text{Cl}_K^{-}(\mathfrak{3})\) and after using the Main Theorem of Complex Multiplication and our assumption that \(\rho\) is injective we obtain
\[\Phi(\int_G (\phi \circ \text{ver})(g) \mu_{\psi, \delta}^{KHT}(g)) \equiv \sum_{\mathfrak{U}_j \in \text{Im}(\rho)} E_{\psi}(\phi, c_j)(X(\mathfrak{U}_j^\xi), \lambda_\delta(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \iota(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1})) \mod p\]

Lemma 8.5. Let \(\phi\) be a locally constant \(\mathbb{Z}_p\)-valued function of \(G_K\) that is cyclotomic i.e. \(\phi\) is the restriction to \(G_K\) of a locally constant function on \(G_F\). Then we have that
\[\frac{\int_G \phi(g) \mu_{\psi, \delta}^{KHT}(g)}{\Omega_p(E)^g_k} \in \mathbb{Z}_p\]
for all \(k \in \mathbb{N}\)

Proof. This follows almost directly Lemma 3.4 and the discussion after it. Indeed we may write \(\phi = \sum \chi_c \chi\) where \(\chi\) are cyclotomic i.e. \(\chi \circ c = \chi\). For such characters it is known that for all \(\sigma \in \text{Gal}(\mathbb{Q}_p/\mathbb{Q})\) we have
\[\left(\frac{\int_G \chi(g) \mu_{\psi, \delta}^{KHT}(g)}{\Omega_p(E)^g_k}\right)^\sigma = \frac{\int_G (\chi(g))^\sigma \mu_{\psi, \delta}^{KHT}(g)}{\Omega_p(E)^g_k}\]
For all \(\sigma \in G_{\mathbb{Q}_p}\) and \(\phi\)'s cyclotomic we have
\[\left(\frac{\int_G \phi(g) \mu_{\psi, \delta}^{KHT}(g)}{\Omega_p(E)^g_k}\right)^\sigma = \sum_{\chi} c^\sigma_{\chi} \left(\frac{\int_G \chi(g) \mu_{\psi, \delta}^{KHT}(g)}{\Omega_p(E)^g_k}\right)^\sigma = \sum_{\chi} c^\sigma_{\chi} \frac{\int_G (\chi(g))^\sigma \mu_{\psi, \delta}^{KHT}(g)}{\Omega_p(E)^g_k}\]
But then as \(\phi(g) = (\phi(g))^\sigma = \sum \chi c^\sigma_{\chi} \chi(g)^\sigma\) the last sum is equal to \(\frac{\int_G \phi(g) \mu_{\psi, \delta}^{KHT}(g)}{\Omega_p(E)^g_k}\) which finishes the proof. \(\square\)

Note that a direct corollary of the proposition is
Corollary 8.6. If $\phi$ is cyclotomic then,
\[
\int_G \phi(g) \mu_{V,\delta}^{KHT}(g) - u^g \int_G (\phi \circ \text{ver})(g) \mu_{V,\delta}^{KHT}(g) \equiv 0
\]
\[
\equiv \sum_{\Omega_j \in S} E_\psi(\phi_j, \epsilon) (X(\Omega_j^0), \lambda_0^0(\Omega_j \otimes \theta_{F'/F}^{-1}), r^\xi(\Omega_j \otimes \theta_{F'/F}^{-1})) \mod p
\]

Proof. We have
\[
\Phi \left( \int_G (\phi \circ \text{ver})(g) \mu_{V,\delta}^{KHT}(g) \right) = \Phi \left( \frac{\int_G (\phi \circ \text{ver})(g) \mu_{V,\delta}^{KHT}(g)}{\int_G \mu_{V,\delta}^{KHT}(g)} \right) = \frac{u^g}{u^g} \int_G (\phi \circ \text{ver})(g) \mu_{V,\delta}^{KHT}(g)
\]
as $\frac{\Omega_p(E)^g}{\Omega_p(E)} = u$ and from the assumption on $\phi$ we have that $\int_G (\phi \circ \text{ver})(g) \mu_{V,\delta}^{KHT}(g) \mod p$. But as $u := \psi(\pi) \in \mathbb{Z}_p$, we have $u^g \equiv u \mod p$. \qed

Lemma 8.7. We have the congruences
\[
u^g \int_G \phi(g) \mu_{V,\delta}^{KHT}(g) \equiv \frac{\int_G \phi(g) \mu_{V,\delta}^{KHT}(g)}{\Omega_p(E)^g} \mod p
\]
for all locally constant $\mathbb{Z}_p$-valued functions $\phi$ of $G$.

Proof. As $\psi^g \equiv \psi \mod p$ we have that
\[
\int_G \phi(g) \mu_{V,\delta}^{KHT}(g) = \int_G (\phi(g) \psi) \mu_{\delta}^{KHT}(g) \equiv \int_G (\phi(g) \mu_{\delta}^{KHT}(g) = \int_G (\phi(g) \mu_{V,\delta}^{KHT}(g) \mod p
\]
Dividing by the unit $\Omega_p(E)^g$ and observing that $u = \frac{\Omega_p(E)^g}{\Omega_p(E)^g} \equiv \frac{\Omega_p(E)^g}{\Omega_p(E)^g} \mod p$ we have
\[
\frac{\int_G \phi(g) \mu_{V,\delta}^{KHT}(g)}{\Omega_p(E)^g} \equiv \frac{\int_G \phi(g) \mu_{V,\delta}^{KHT}(g)}{\Omega_p(E)^g} \times \frac{\Omega_p(E)^g}{\Omega_p(E)^g} \mod p
\]
which concludes the proof. \qed

Now our assumptions of the main theorem imply that $S = \emptyset$. Then the last two statements conclude the proof of the main theorem. Note that if we do not assume that $S = \emptyset$ then we obtain the congruences
\[
\int_{G_F} \epsilon \circ \text{ver} d\mu_{E/F} \equiv \int_{G_{F'}} \epsilon d\mu_{E/F'} + \Delta(\epsilon) \mod p\mathbb{Z}_p
\]
where
\[
\Delta(\epsilon) := \frac{1}{\Omega_p(E)^g} \sum_{\Omega_j \in S} E_\psi(\phi_j, \epsilon) (X(\Omega_j^0), \lambda_0^0(\Omega_j \otimes \theta_{F'/F}^{-1}), r^\xi(\Omega_j \otimes \theta_{F'/F}^{-1}))
\]

The Fukaya-Kato conjecture and the measure of Katz: We would like to finish this work by stating the question of whether the $p$-adic interpolation properties of the Katz-Hida-Tilouine measure are canonical. In [14] (page 67, theorem 4.2.22) Fukaya and Kato conjecture a general formula for $p$-adic $L$ functions for motives over any field. Does this formula agree with Katz-Hida-Tilouine’s formula in the case where the motive consider is the one attached to a Grössencharacter over a CM field? We remark that our question is more concerning the $p$-adic and archimedean periods that appear in the two formulas.
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9. APPENDIX

There is an easy way to see that there must be a modification in the interpolation properties of the measures in order for the congruences to hold. We assume for simplicity that $F'/F$ ramifies only above $p$. Moreover we assume that the character $\psi_K$ is unramified (we just divide out the finite part of it which has conductor $f$) and we pick with notation as in the introduction $n = r$.

Let us pick as the locally constant function $\phi$ that appear in the congruences the character $\phi := \phi \circ N_{K'/K}$ for some finite $\mathbb{Z}_p$-valued character of $G_K$, which we assume cyclotomic (for example $\phi := 1$ or some of the $p-1$ order characters factoring through the torsion of $G_F$ base changed to $G_K$). Then by the interpolation properties of the measure we have

$$\frac{\int_{G_{K'}} \tilde{\phi} \ d\mu_{\psi_{K'}}}{\Omega_p(E)^g} = \prod_{p \in \Sigma'_p} \frac{\text{Local}_p(\tilde{\phi}\psi_{K'}, \Sigma', \delta')(1 - \tilde{\phi}\psi_{K'}(\tilde{p}))(1 - \tilde{\phi}\psi_{K'}(\tilde{p}))}{\sqrt{|D_{F'}|}\Omega(E)^g} = \frac{\prod_{p \in \Sigma'_p} \text{Local}_p(\tilde{\phi}\psi_{K'}, \Sigma', \delta')} {\sqrt{|D_{F'}|}} \prod_{\chi} \prod_{p \in \Sigma_p} (1 - \phi \psi_K\chi(\tilde{p}))(1 - \phi \psi_K\chi(\tilde{p})).$$

where $\chi$ runs over the characters of the extension $K'/K$. Now we note that $\chi \equiv 1 \mod (\zeta_p - 1)$ and hence as $\text{Gal}(K'/K)$ is a quotient of $G_K$ we have that

$$\frac{\int_{G_K} \phi \chi \ d\mu_{\psi_K}}{\Omega_p(E)^g} \equiv \frac{\int_{G_K} \phi \ d\mu_{\psi_K}}{\Omega_p(E)^g} \mod (\zeta_p - 1)$$

or equivalently

$$\prod_{p \in \Sigma_p} \text{Local}_p(\phi \psi_K, \Sigma, \delta) \prod_{p \in \Sigma_p} (1 - \phi \psi_K\chi(\tilde{p}))(1 - \phi \psi_K\chi(\tilde{p})).$$

Taking the product over all $\chi$’s we obtain

$$\sqrt{|D_{F'}|} \prod_{\chi} \left( \prod_{p \in \Sigma_p} \text{Local}_p(\phi \psi_K, \Sigma, \delta) \right) \frac{\int_{G_{K'}} \tilde{\phi} \ d\mu_{\psi_{K'}}}{\Omega_p(E)^g} \equiv \left( \frac{\int_{G_K} \phi \ d\mu_{\psi_K}}{\Omega_p(E)^g} \right)^p \mod (\zeta_p - 1)$$

Now we note that

$$\left( \frac{\int_{G_K} \phi \ d\mu_{\psi_K}}{\Omega_p(E)^g} \right)^p \equiv \frac{\int_{G_K} \phi^p \ d\mu_{\psi_K}}{\Omega_p(E)^g} \mod p$$

as the values of the integrals are in $\mathbb{Z}_p$ as we assume that $\phi$ is cyclotomic. Hence we need to understand the factor $\frac{\sqrt{|D_{F'}|} \prod_{\chi} \left( \prod_{p \in \Sigma_p} \text{Local}_p(\phi \psi_K, \Sigma, \delta) \right)}{\sqrt{|D_F|} \prod_{p \in \Sigma'_p} \text{Local}_p(\phi \psi_{K'}, \Sigma', \delta')} \frac{\int_{G_{K'}} \tilde{\phi} \ d\mu_{\psi_{K'}}}{\Omega_p(E)^g}$ and where the quantity $\frac{\int_{G_{K'}} \tilde{\phi} \ d\mu_{\psi_{K'}}}{\Omega_p(E)^g}$
lies. We start with the local factors. From Lemma 3.3 we have that

\[ \text{Local}(\phi\chi\psi_K, \Sigma, \delta)_p = c_p^{(\chi)}(\delta)e_p(\phi^{-1}\chi^{-1}, \psi, dx_1) \left( \frac{\psi_{K'}^{-1}(\pi_p)}{N(p)} \right)^{n_p(\phi\chi) + n_p(\psi)} \]

and

\[ \text{Local}(\tilde{\phi}\psi_K, \Sigma', \delta')_p = c'_p(\delta')e_p(\tilde{\phi}^{-1}, \psi', dx_1) \left( \frac{\psi_{K'}^{-1}(\pi_p)}{N(p)} \right)^{n_p(\tilde{\phi}) + n_p(\psi')} \]

where \( c_p^{(\chi)} \) is the local part of \( \phi\chi\psi_K \) and \( dx_1 \) is the Haar measure that assigns measure 1 to the ring of integers of \( K_p \) (with similar notations for the second expression). Now we note that (as easily seen from the functional equation and the fact that \( \chi_p \)) we have that

\[ \prod_{\tilde{p}\in\Sigma'} e_p(\tilde{\phi}, \psi', dx'_\psi) = \prod_{\tilde{p}\in\Sigma'} e_p(\tilde{\phi}, \psi, dx_\psi) \]

where we follow Tate’s notation as in [27] for the Tamagawa measures \( dx_\psi \) and \( dx_\psi' \). The relation between the Tamagawa measure \( dx_\psi \) and the normalized measure \( dx_1 \) of a place \( p \) is given by \( dx_\psi = N(p)^{-n_p(\psi)/2} dx_1 \) (There is a typo in Tate’s [27] p.17, but see the same article in page 18 or Lang’s Algebraic Number Theory page 277). That implies,

\[ \prod_{\tilde{p}\in\Sigma'} e_p(\tilde{\phi}, \psi', dx'_\psi) = \prod_{\tilde{p}\in\Sigma'} e_p(\tilde{\phi}, \psi', dx_1)N(p)^{-n_p(\psi')/2} \]

and

\[ \prod_{\chi} \prod_{p\in\Sigma} e_p(\phi\chi, \psi, dx_\psi) = \prod_{\chi} \prod_{p\in\Sigma} e_p(\phi\chi, \psi, dx_1)N(p)^{-n_p(\psi)/2} = \prod_{p\in\Sigma} N(p)^{-n_p(\psi)/2} \prod_{\chi} e_p(\phi\chi, \psi, dx_1) \]

So we conclude the equation

\[ \prod_{\tilde{p}\in\Sigma'} e_p(\tilde{\phi}, \psi', dx_1)N(p)^{-n_p(\psi')/2} = \prod_{p\in\Sigma} N(p)^{-n_p(\psi)/2} \prod_{\chi} e_p(\phi\chi, \psi, dx_1) \]

or equivalently

\[ \prod_{\chi} \prod_{p\in\Sigma} e_p(\phi\chi, \psi, dx_1) = \frac{\prod_{p\in\Sigma'} N(p)^{-n_p(\psi')/2}}{\prod_{p\in\Sigma} N(p)^{-n_p(\psi)/2}} \prod_{\tilde{p}\in\Sigma'} e_p(\tilde{\phi}, \psi', dx_1) \]

As we assume that \( \Sigma \) and \( \Sigma' \) are ordinary and for simplicity we take the extension to be ramified only at \( p \) we have that

\[ \prod_{p\in\Sigma'} N(p)^{n_p(\psi')/2} = \frac{\sqrt{|D_{F'}|}}{|D_F|^p}. \]

Putting everything together we see that the discrepancy factor in the congruences

\[ \text{Diff} := \frac{\sqrt{|D_{F'}|}}{|D_F|^p} \times \prod_{\chi} \prod_{p\in\Sigma_p} \text{Local}_p(\phi\chi\psi_K, \Sigma, \delta) \prod_{p\in\Sigma'_p} \text{Local}_p(\tilde{\phi}\psi_K', \Sigma', \delta') \]

is equal to

\[ \text{Diff} = \frac{\prod_{\chi} \prod_{p\in\Sigma_p} c_p^{(\chi)}(\delta) \left( \frac{\psi_{K'}^{-1}(\pi_p)}{N(p)} \right)^{n_p(\phi\chi) + n_p(\psi)}}{\prod_{p\in\Sigma'_p} c'_p(\delta') \left( \frac{\psi_{K'}^{-1}(\pi_p)}{N(p)} \right)^{n_p(\tilde{\phi}) + n_p(\psi')}} \]
Now we claim that the factor
\[ \prod_{\chi} \prod_{p \in \Sigma_p} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi \chi) + n_p(\psi)} \prod_{p \in \Sigma_p} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi) + n_p(\psi')} = 1. \]

Indeed we have
\[ \prod_{p \in \Sigma_p} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi) + n_p(\psi')} = \prod_{p \in \Sigma_p} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi') + n_p(\psi)} \]

For those \( p' \in \Sigma_p' \) that are not ramified we have \( n_{p'}(\psi') = n_p(\psi) \) for \( p \in \Sigma_p \) the prime below \( p' \). Similarly \( n_{p'}(\phi) = n_p(\phi \chi) = n_p(\phi) \) for all \( \chi \) as these are ramified only at the primes that ramify in \( K'/K \). Then we have
\[ \prod_{p \in \Sigma_p, \text{ unram.}} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi) + n_p(\psi')} = \prod_{p \in \Sigma_p, \text{ unram.}} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi) + n_p(\psi)} \]
\[ = \prod_{\chi} \prod_{p \in \Sigma_p, \text{ unram.}} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi \chi) + n_p(\psi)} \]

Now we consider the ramified primes. We have
\[ \prod_{p \in \Sigma_p', \text{ ram.}} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi) + n_p(\psi')} = \prod_{p \in \Sigma_p} \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi) + n_p(\psi')} \]

For every \( p' \in \Sigma_p' \) that is ramified (totally as we consider a \( p \)-order extension) we have from the conductor-discriminant formula that
\[ n_{p'}(\psi') = \sum_{\chi} n_p(\chi) + pm_p(\psi) \]

for the prime \( p \in \Sigma_p \) below \( p' \). Moreover as the conductor-function \( n_p(\cdot) \) is additive and inductive in degree zero we have that
\[ n_{p'}(\phi) = n_{p'}(\text{Res}(\phi)) = n_{p'}(\text{Res}(\phi)) - n_{p'}(\text{Ind}(\text{Res}(\phi) \circ \text{Ind}(\text{1})) = n_p((\text{Ind}\text{Res}(\phi)) \circ \text{Ind}(\text{1})) = \]
\[ = n_p(\text{IndRes}(\phi)) - n_p(\text{Ind}(\text{1})) = n_p(\phi \chi) - n_p(\chi \phi) = \sum_{\chi} n_p(\phi \chi) - \sum_{\chi} n_p(\chi) \]

Putting all together we conclude our claim. Hence we have that
\[ \text{Diff} = \frac{\prod_{\chi} \prod_{p \in \Sigma_p} c_p^{(\chi)}(\delta) \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi \chi) + n_p(\psi)}}{\prod_{p \in \Sigma_p} c_p^{(\delta')}(\delta) \left( \frac{\psi_{K_p}^{-1}(n_p)}{N(p)} \right)^{n_p(\phi) + n_p(\psi')}} = \prod_{\chi} \prod_{p \in \Sigma_p} c_p^{(\chi)}(\delta) = \prod_{p \in \Sigma_p} \left( \phi \psi_K \right)_p(\delta) \prod_{\chi} c_p^{(\delta')} \]

Now we observe that
\[ \prod_{\chi} \prod_{p \in \Sigma_p} c_p^{(\chi)}(\delta) = \prod_{\chi} \prod_{p \in \Sigma_p} (\phi \chi \psi_K)_p(\delta) = \prod_{p \in \Sigma_p} (\phi \psi_K)_p(\delta) \prod_{\chi} c_p^{(\delta')} \]
\[
\prod_{p \in \Sigma_p} (\phi \psi_K)_p(\delta) \prod_{\chi} \chi_p(\delta) = \prod_{p \in \Sigma_p} (\phi \psi_K)_p(\delta^p)
\]

since \(\prod_{\chi} \chi_p(\delta) = 1\) because we multiply over all elements of the multiplicative group of characters of \(\text{Gal}(K'/K)\) and we know that \(\chi \neq \chi^{-1}\) for all \(\chi \neq 1\) as these are \(p\)-order characters. Also we have that

\[
\prod_{p \in \Sigma_p} c'_p(\delta') = \prod_{p \in \Sigma_p} (\phi \circ N_{K'/K})_p(\psi_K \circ N_{K'/K})_p(\delta') = \prod_{p \in \Sigma_p} (\phi \psi_K)_p(N_{K'/K} \delta')
\]

In particular we observe that in general we have that

\[
\prod_{\chi} \prod_{p \in \Sigma_p} c'_p(\delta) \neq \prod_{p \in \Sigma_p} c'_p(\delta').
\]

as \(N_{K'/K}(\delta') \neq \delta^p\) when the extension \(K'/K\) is ramified at \(p\). Actually the two expressions may not even have the same valuation.

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