THE MARSDEN-WEINSTEIN REDUCTION STRUCTURE OF INTEGRABLE DYNAMICAL SYSTEMS AND A GENERALIZED EXACTLY SOLVABLE QUANTUM SUPERRADIANCE MODEL

BOGOLUBOV N.N (JR.)¹, PRYKARPATSKY Y.A².

Abstract. An approach to describing nonlinear Lax type integrable dynamical systems of modern mathematical and theoretical physics, based on the Marsden-Weinstein reduction method on canonically symplectic manifolds with group symmetry, is proposed. Its natural relationship with the well known Adler-Kostant-Souriau-Berezin-Kirillov method and the associated R-matrix approach is analyzed.

A new generalized exactly solvable spatially one-dimensional quantum superradiance model, describing a charged fermionic medium interacting with external electromagnetic field, is suggested. The Lax type operator spectral problem is presented, the related R-structure is calculated. The Hamilton operator renormalization procedure subject to a physically stable vacuum is described, the quantum excitations and quantum solitons, related with the thermodynamical equilibrity of the model, are discussed.

1. Introduction

As it is well known [3,1,23,12], the most popular canonically symplectic manifolds are supplied by cotangent spaces $M := T^*(P)$ to some ”coordinates” phase spaces $P$, which can often possess additional symmetry properties. If this symmetry can be identified with some Lie group $G$ action on the phase space $P$ and its natural extension on the whole manifold $M$ proves to be symplectic and even more, Hamiltonian, the Marsden-Weinstein reduction method [1,7] makes it possible to construct new Hamiltonian flows on the smaller invariant reduced phase space $\bar{M}_\xi := M_\xi/G_\xi$ subject to the group invariant constraint $p := \xi \in G^*$ for some specially chosen element $\xi \in G^*$, where $p : M \to G^*$ is the related momentum mapping on the symplectic manifold $M$ and $G^*$ is the adjoint space to the Lie algebra $G$ of the group Lie $G$.

As the corresponding Hamiltonian flows on the reduced phase space $\bar{M}_\xi$ possess often very interesting properties important for applications in many branches of mathematics and physics, their studies were topics of many researches during the past decades. Being interested in Lax type flows, we observed that their modern Lie algebraic descriptions by means of Hamiltonian group actions via the classical Lie—Poisson—Adler—Kostant—Souriau—Berezin—Kirillov (LPAKSBK) scheme is actually closely related to the Marsden—Weinstein reduction. In particular, the LPAKSBK on the adjoint space $\hat{G}^*$ to the Lie algebra $\hat{G}$ of a suitably chosen Lie group $G$ follows directly from an application of Marsden—Weinstein reduction to $M = T^*(P)$, where $P$ is chosen so that there is a naturally related Hamiltonian group $G$-action on $M$. Moreover, such classical integrability theory ingredients as the $R$-structures [27] and the related commutation properties of the related transfer matrices are also naturally retrieved from the Marsden-Weinstein reduction method within the scheme specified above. These and some related aspects of this reduction technique are topics of this investigation.

2. Loop groups, canonically symplectic manifold and Hamiltonian action

Consider a complex matrix Lie group $G = SL(\nu; \mathbb{C}), \nu \in \mathbb{Z}_+$, its Lie algebra $\mathfrak{g}$, and a related [12,18,23] formal loop group $\hat{G} \subset C^\infty(S^1; Hol(\mathbb{C}; G))$ of $G$-valued functions on the circle $S^1$, meromorphically depending on the complex parameter $\lambda \in \mathbb{C}$. Its Lie algebra $\hat{\mathfrak{g}}$ can be viewed as the completion

$$\hat{\mathfrak{g}} = \bigcup_{n \in \mathbb{Z}} \left\{ \sum_{j=-\infty}^{n} \hat{X}_j \lambda^j : \hat{X}_j \in C^\infty(S^1; \mathfrak{g}), \; j \leq n \right\}.$$
Using the standard procedure [12, 13] one can construct the centrally extended current algebra \( \hat{G} := \hat{G} \oplus \mathbb{C} \), on which the adjoint loop group \( \hat{G} \)-action is defined: for any \( g \in \hat{G} \),

\[
g : (T, c) \rightarrow (gT g^{-1}, c + (g^{-1} g_x, T)_{-1}).
\]

Here \( (T, c) \in \hat{G} \) and \( \langle \cdot, \cdot \rangle_{-1} : \hat{G} \times \hat{G} \rightarrow \mathbb{C} \) is the following nondegenerate symmetric scalar product on \( \hat{G} \):

\[
\langle A, B \rangle_{-1} := \text{res} \int_0^{2\pi} \text{tr}(A(x; \lambda)B(x; \lambda)) = \langle B, A \rangle_{-1},
\]

for any \( A, B \in \hat{G} \). The scalar product (2.3) is ad-invariant, that is

\[
(A, [B, C])_{-1} = ([A, B], C)_{-1}
\]

for any elements \( A, B \) and \( C \in \hat{G} \).

Define now the canonically symplectic phase space \( M := T^*(\hat{G}) \simeq (\hat{G}, \hat{G}^*) \) with the corresponding Liouville 1-form on \( M \):

\[
\alpha^{(1)}(T, c; l, k) = (l, dT)_{-1} + kdc,
\]

whose exterior derivative gives the symplectic structure on the functional manifold \( M \):

\[
\omega^{(2)}(T, c; l, k) := d\alpha^{(1)}(T, c; l, k) = (dl, \wedge dT)_{-1} + dk \wedge dc.
\]

Similarly to (2.2) one can naturally extend the group \( \hat{G} \)-action on the whole phase space \( M \), having

\[
g : (l, k) \rightarrow (glg^{-1} - kg_xg^{-1}, k)
\]

for any \( (l, k) \in \hat{G}^* \) and \( g \in \hat{G} \) as the corresponding co-adjoint action of the current group \( \hat{G} \) to the adjoint linear space \( \hat{G}^* \). The following lemma is almost evident.

**Lemma 2.1.** The \( \hat{G} \)-group action (2.2) and (2.7) on the symplectic phase space \( M \) is symplectic and Hamiltonian.

**Proof.** It is easy to check that the canonical Liouville 1-form (2.5) on the manifold \( M \) is \( \hat{G} \)-invariant:

\[
g^* \alpha^{(1)}(T, c; l, k) = (glg^{-1} - kg_xg^{-1}, gdT g^{-1})_{-1} + kdc + (g^{-1} g_x, dT)_{-1} =
\]

\[
\omega^{(2)}(T, c; l, k) := d\alpha^{(1)}(T, c; l, k) = (dl, \wedge dT)_{-1} + dk \wedge dc.
\]

From (2.8), owing to the expression (2.6), one obtains the symplectic form invariance

\[
g^* \omega^{(2)}(T, c; l, k) = \omega^{(2)}(T, c; l, k)
\]

for any element \( (T, c; l, k) \in M \).

To define the Hamiltonian \( \hat{G} \)-action on the symplectic manifold \( M \) we take the group flow \( g(t) := \exp(tX) \) for \( t \in \mathbb{R}, X \in \hat{G} \), and find the driven generated vector field \( K_X : M \rightarrow T(M) \) on the phase space \( M \):

\[
K_X(T, c; l, k) :=
\]

\[
\left. \frac{d}{dt} (g(t)T g^{-1}, c + (g(t)^{-1} g_x(t), T)_{-1}; g(t)lg(t)^{-1} - kg_x(t)g(t)^{-1}, k) \right|_{t=0} =
\]

\[
([X, T], [X_x, T]_{-1}; [X, l] - kX_x, 0),
\]

by a Hamiltonian function \( H_X : M \rightarrow \mathbb{C} \) owing to the canonical relationship \(-dH_X = iK_X \omega^{(2)} :\)

\[
-dH_X = - (\partial H / \partial l, dl)_{-1} - (\partial H / \partial T, dT)_{-1} +
\]

\[
+ \partial H / \partial k dk + \partial H / \partial c dc =
\]

\[
= ([X, l] - kX_x, dT)_{-1} - (dl, [X, T])_{-1} - (X_x, T)_{-1} dk.
\]
As a consequence of (2.11) one obtains
\begin{align}
\partial H_X / \partial l &= [X, T], & \partial H_X / \partial T &= kX_x - [X, l], \\
\partial H_X / \partial k &= (X_x, T)_{-1}, & \partial H_X / \partial c &= 0
\end{align}
for any point \((T, k; l, c) \in M\). From (2.12) it follows that
\begin{align}
H_X = ([T, l] - kT_x, X)_{-1} := (p(T, c; l, k), X)_{-1},
\end{align}
is linear with respect to the generator element \(X \in \hat{G}\). This means that the loop group \(\hat{G}^\ast\) action on the symplectic manifold \(M\) is Hamiltonian by definition [1] [22].

The corresponding mapping \(p : M \to \hat{G}^\ast\), where
\begin{align}
p(T, c; l, k) = [T, l] - kT_x,
\end{align}
is called the momentum mapping [1] [7] [22] which can be constrained to be fixed for further applications to the phase space \(M\) in the Marsden-Weinstein reduction procedure [1].

Let us describe in detail the related symplectic structure on the \(\xi\)-level submanifold
\begin{align}
M_\xi := \{ (T, c; l, k) \in M : [T, l] - kT_x = \xi \in \hat{G}^\ast \}
\end{align}
for a fixed element \(\xi \in \hat{G}^\ast\). As a more natural case we take that \(\xi = 0 \in \hat{G}^\ast\). The corresponding isotropy group \(\hat{G}_\xi = \hat{G}\), as \(\mathrm{Ad}_g\xi|_{\xi = 0} = 0\) holds for any element \(g \in \hat{G}\). To proceed further we need some additional properties of the submanifold \(M_\xi \subset M\), which we will describe in the next section.

3. Marsden-Weinstein Reduction, Commuting Vector Fields and Poisson Bracket

In this section we will be interested in describing the submanifold \(M_\xi \subset M\) parameterized by the points of the reduced phase space \(\tilde{M}_\xi := M_\xi / G_\xi\). It is known [1] [3], that this parametrization uniquely determines the points \((\tilde{T}, \tilde{c}; \tilde{l}, \tilde{k}) \in M_\xi \subset M\), which are invariant with respect to the appropriate loop group \(\hat{G}\) action (2.2) and (2.7). The last property makes it possible [1] [3] [7] [21] to define on the phase space \(\tilde{M}_\xi\) the reduced nondegenerate symplectic structure on the phase space \(\tilde{M}_\xi\) by means of the appropriate symplectic structure on the submanifold \(M_\xi\). Let us consider the point \((\tilde{T}, \tilde{c}; \tilde{l}, \tilde{k}) \in M_\xi\), where the elements \(\tilde{T} \in \hat{G}, \tilde{k} \in C\), according to the definition (2.15), satisfy the differential expressions:
\begin{align}
[\tilde{T}, \tilde{l}] - \tilde{k}\tilde{T}_x = 0, & \quad \tilde{k}_x = 0,
\end{align}
for all \(x \in S^1\). Consider now a Hamiltonian vector field \(\tilde{k}d/d\tau, \tau \in C\), on the submanifold \(M_\xi\), generated by the element \(X = l \in \hat{G}^\ast\) owing to the expressions
\begin{align}
-\tilde{k}\tilde{T}_\tau = [\tilde{l}, \tilde{T}] = -[\tilde{T}, \tilde{l}] = -\tilde{k}\tilde{T}_x, & \quad -\tilde{k}\tilde{l}_\tau = \tilde{k}\tilde{T}_x.
\end{align}
From (3.2) it follows that the equality \(\frac{d}{d\tau} = \frac{d}{dx}\) holds on the reduced phase space \(\tilde{M}_\xi\). Let us compute additionally the evolution of the element \(\tilde{c} \in C\) with respect to this vector field \(d/d\tau\) on \(\tilde{M}_\xi\):
\begin{align}
-\tilde{k}\tilde{c}_\tau = (\tilde{l}, \tilde{T}) = -\tilde{l}(\tilde{T}, \tilde{l}) - \tilde{k}\tilde{T}_x = \tilde{k}\tilde{T}_x = \tilde{k}\tilde{l}_\tau = 0.
\end{align}
coinciding with the \textit{a priori} assumed condition \(d\tilde{c}/dx = 0\) for any \(x \in S^1\).

Define similarly a vector field \(d/dt\), \(t \in C\), on the reduced phase space \(\tilde{M}_\xi\), generated by the Lie algebra element \(q(\tilde{l}) \in \hat{G}\), depending on the basis element \(\tilde{l} \in \hat{G}^\ast\) such that
\begin{align}
\tilde{T}_t = [q(\tilde{l}), \tilde{T}], & \quad \tilde{l}_t = [q(\tilde{l}), \tilde{l}] - \tilde{k}\tilde{T}_x, & \quad \tilde{c}_t = (q(\tilde{l}), \tilde{T})_{-1}, & \quad \tilde{k}_t = 0.
\end{align}
The latter, in particular, means that the flows \(d/dt\) and \(d/dx\) on the reduced phase space \(\tilde{M}_\xi\) possess the countable set \(\gamma_n(\tilde{l}) := \mathrm{tr}\tilde{T}^n(\tilde{l}), n \in \mathbb{Z}\), of conservation laws, where by definition, the element \(\tilde{T}(\tilde{l}) \in \hat{G}\) satisfies for a given element \(\tilde{l} \in \hat{G}^\ast\) the determining equation
\begin{align}
-\tilde{k}\tilde{T}_\tau(\tilde{l}) = [\tilde{l}, \tilde{T}(\tilde{l})]
\end{align}
for all \(x \in S^1\). From the equations (3.5) one easily finds that upon the reduced phase space \(\tilde{M}_\xi\)
\begin{align}
\tilde{c}_t = (q(\tilde{l}), \tilde{T})_{-1} = \tilde{k}^{-1}([q(\tilde{l}), \tilde{l}] - \tilde{l}, \tilde{T})_{-1} = \tilde{k}^{-1}([q(\tilde{l}), \tilde{l}])_{-1} - \tilde{k}^{-1}(\tilde{l}, \tilde{T})_{-1} = \tilde{k}^{-1}(\tilde{l}, \tilde{T})_{-1} - \tilde{k}^{-1}(\tilde{l}, \tilde{T})_{-1} = \tilde{k}^{-1}(\tilde{l}, \tilde{T})_{-1} - \tilde{k}^{-1}(\tilde{l}, \tilde{T})_{-1} =
\end{align}
(3.6)
Thus, from the $t$-evolution of the parameter $\bar{c} \in \mathbb{C}$ one finds that the constraint

$$\bar{c} = -\bar{k}^{-1}(\bar{l}, \bar{T})_{-1}$$

holds on the reduced phase space $\bar{M}_\xi$ subject to the vector field $d/dt$ generated by the element $q(\bar{l}) \in \bar{G}$. Moreover, as it is easy to observe, these two vector fields $d/d\tau$ and $d/dt$ on the reduced phase space $\bar{M}_\xi$ commute:

$$[d/dt, d/d\tau] = 0.$$  

The latter is very promising, since the condition (3.8) results in some differential relationships on the components of the reduced matrix $\bar{l} \in \bar{G}^*$, for which the related linear evolution equation

$$\bar{F}_x = i\bar{F},$$

augmented with the compatible differential equation

$$\bar{F}_t = q(\bar{l})\bar{F}$$

for the matrix $F \in \bar{G}$ are compatible. These equations (3.9) and (3.10) realize the well known [12, 19, 18, 23, 22, 17] generalized Lax type spectral problem, allowing to integrate the mentioned above differential relationships by means of either the inverse scattering or the spectral transform methods [12, 19, 18, 10] and algebraic geometry methods [19, 18], or their modern generalizations [23].

To make this aim more constructive, it is necessary to describe the evolution of the vector field $d/dt$ on the reduced phase space $\bar{M}_\xi$ in more detail subject to its dependence on the phase space element $\bar{l} \in \bar{G}^*$. Taking into account that the vector fields $d/dt$ and $d/dx$ satisfy the commutation condition (3.8) on the reduced manifold $\bar{M}_\xi$, we will apply the Marsden-Weinstein reduction theory to our symplectic manifold $M$ with the fixed value of the moment mapping $\xi = 0$ for computing the basic Poisson bracket

$$\{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\}_\xi$$

of the functions $\{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\}$ on the reduced phase space $\bar{M}_\xi$ for arbitrary $X, Y \in \bar{G}^*$. It can be shown [5, 21, 7] that this Poisson bracket on $\bar{M}_\xi$ in general is

$$\{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\}_\xi = \{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\} |_{\bar{M}_\xi} - \{\xi, [V_X, V_Y]_{-1}\} |_{\bar{M}_\xi},$$

where, by definition, the mappings $V_X, V_Y : \bar{M}_\xi \rightarrow \bar{G}$ denote the solutions to the following relationships:

$$\{\xi, Z, V_X\}_{-1} = K_Z(T, X)_{-1}, \{\xi, Z, V_Y\}_{-1} = K_Z(T, Y)_{-1},$$

which holds for all $Z \in \bar{G}$. The functions $\{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\} \in \mathcal{D}(\bar{M}_\xi)$ should be extended to those on the whole phase space $\bar{M}$ in such a way that their restrictions on the submanifold $\bar{M}_\xi \subset \bar{M}$ are $\bar{G}$-invariant.

To apply the Marsden-Weinstein reduction we will take into account that, by definition, there exists a group element $g(l) \in \bar{G}$ such that for arbitrarily chosen $l \in \bar{G}$ the expression

$$l = g(l)\bar{l}(l)g(l)^{-1} - \bar{k}g_x(l)g(l)^{-1}$$

holds and satisfies the normalization condition $g(\bar{l}) = \text{Id} \in \bar{G}$. By considering the function

$$f_X := (T, g(l)Xg(l)^{-1})_{-1},$$

one can observe that $f_X |_{\bar{M}_\xi} = (\bar{T}, X)_{-1}$ and, by construction, it is $\bar{G}$-invariant. The latter means that $f_X \in \mathcal{D}(\bar{M}_\xi)$ for any $l \in \bar{G}^*$. In fact, for any $a \in \bar{G}_\xi = \bar{G}$

$$a \circ f_X := \{(a \cdot T, g(a \circ l)Xg(a \circ l)^{-1})_{-1} =$$

$$= (aTa^{-1}, ag(l)Xg(l)^{-1} \cdot a^{-1}) = (T, g(l)Xg(l)^{-1})_{-1} = f_X,$$
where we made use of the property $g(a \circ l) = a \circ g(l)$, $l \in \tilde{G}^\ast$. The latter holds owing to the definitions (3.11) and (2.7):

$$a \circ l = a(l) a^{-1} - l a a^{-1} = a(g(l)) l g(l)^{-1} - l g(l)^{-1} g(l) a^{-1} - l a a^{-1} =$$

$$= a g(l) l (a g(l))^{-1} - l k g(l) g(l)^{-1} a^{-1} - l a a^{-1} =$$

$$= g(a \circ l) l (a \circ l)^{-1} - l k g(a \circ l) g(a \circ l)^{-1} =$$

$$= g(a \circ l) l (a \circ l)^{-1} - l k g(a \circ l) g(a \circ l)^{-1},$$

(3.17)



giving rise to relationship $g(a \circ l) = a \circ g(l)$ for any $a \in \tilde{G}^\ast$ and $l \in \tilde{G}^\ast$.

Returning to the Poisson bracket (3.12), we can replace the functions $(\bar{T}, X)_{-1}$ and $(\bar{T}, Y)_{-1} \in D(M_\xi)$ with their $\tilde{G}^\ast$-invariant extensions $f_X \in D(M_\xi)$. Before calculating the corresponding Poisson bracket

(3.18) \[ \{ f_X, f_Y \} \xi = \{ f_X, f_Y \} |_{\tilde{M}_\xi} - (\xi, [V_X, V_Y])_{-1} = \{ f_X, f_Y \} |_{\tilde{M}_\xi} - K_{V_X} f_Y |_{\tilde{M}_\xi}, \]

where $K_{V_X} : M \rightarrow T(M)$ is the vector field generated on $M$ by the element $V_X \in \tilde{G}$, we need to calculate the action $K_Z f_Y$ for any element $Z \in \tilde{G}$. Similarly to the calculations from [5], one finds that on the submanifold $M_\xi$ the general expression (3.19) implies

(3.20) \[ K_{V_X} f_Y |_{\tilde{M}_\xi} = (\bar{T}, [g'(l)]) \cdot ([V_X, l] - \bar{k} \frac{d}{dx} V_x) - V_X, Y)_{-1}. \]

Thus, the Poisson bracket (3.13), owing to the relationships $\{ f_X, f_Y \} = \omega(2)(K_{V_X}, K_{V_Y})$ and (3.20), becomes

(3.21) \[ \{(\bar{T}, X), (\bar{T}, Y)\} \xi = \]

$$= (\bar{T}, [g'(l)(Y), X] + [Y, g'(l)(X)])_{-1} - (\bar{T}, [g'(l)(Y), l] - \bar{k} \frac{d}{dx} V_x) - V_X, Y)_{-1}, \]

where we take into account that owing to (3.13) and (3.20), the expression

$$\left(\bar{T}, [g'(l)(Y), l] - \bar{k} \frac{d}{dx} V_x\right)_{-1} = K_{V_X} f_Y |_{\tilde{M}_\xi} = (\xi, [K_{V_X}, V_Y])_{-1} |_{\xi = 0} = 0. \]

Now one can rewrite the Poisson bracket (3.21) as

(3.22) \[ \{(\bar{T}, X), (\bar{T}, Y)\} \xi = (\bar{T}, [X, Y]_{D})_{-1}, \]

where, by definition, we have introduced the classical $D$-matrix structure in the Lie algebra $\tilde{G}^\ast$:

(3.23) \[ [X, Y]_{D} := [D(X), Y] + [X, D(Y)], \]

where $X, Y \in \tilde{G}^\ast$ and the linear homomorphism $D : \tilde{G}^\ast \rightarrow \tilde{G}^\ast$ is defined as

(3.24) \[ D(X) := -g'(l)(X). \]

The mapping (3.24) should satisfy [6] the well known condition

(3.25) \[ (\bar{T}, [X, D(Y), D(Z)] - D(Y, Z)_{D})_{-1} + (\bar{T}, [X, ((\bar{T}, Y), (\bar{T}, Z))])_{-1} + \text{cycles} = 0 \]

for any $X, Y \in \tilde{G}^\ast$ and $Z \in \tilde{G}$.

Now it is useful to recall that the mapping $g : \tilde{G}^\ast \rightarrow \tilde{G}$ satisfies the relationship (3.14), which implies [3] the following differential expression

(3.26) \[ g'(l)(X), l - \bar{k} \frac{d}{dx} g'(l)(X) = X. \]
for any \( X \in \hat{G}^* \), where \( g(^l): \hat{G}^* \to \hat{G}^* \) is the derivative mapping, depending from the chosen reduction \( \mathcal{G}^* \ni \tilde{l} \mapsto \tilde{l} \in G^* \).

The mapping (3.24) satisfies an additional relationship, which can be obtained from the group \( \hat{G} \)–action on the element \( \tilde{T}(\tilde{l}) \in \hat{G} \):

\[
T(l) = g(l)\tilde{T}(\tilde{l})g(l)^{-1},
\]

following naturally from (3.14). Differentiation of (3.27) with respect to \( \tilde{l} \in \hat{G}^* \) at the point \( l = \tilde{l} \), gives rise to the expression

\[
T'(\tilde{l})(X) = [g'(\tilde{l})(X), \tilde{T}(\tilde{l})]
\]

for an arbitrary \( X \in \hat{G}^* \). Moreover, since the matrix (3.27) satisfies the relationship (3.24), its differentiation with respect to \( \tilde{l} \in \hat{G}^* \) entails the differential expression:

\[
\tilde{l}\frac{d}{dx}T'(\tilde{l})(Y) + [\tilde{l}, T'(\tilde{l})(Y)] = [\tilde{T}(\tilde{l}), Y],
\]

which holds for any \( Y \in \hat{G}^* \). The above results can be formulated as the following proposition.

**Proposition 3.1.** The Poisson bracket (3.11) on the reduced phase space \( \hat{M}_\xi \) represented as a D-structure (3.22) on the linear space \( \hat{G}^* \), naturally generated by the gauge transformation (3.14), which reduces the arbitrary element \( \tilde{l} \in \hat{G}^* \) to the element \( \tilde{l} \in \hat{G}^* \), is uniquely defined on \( \hat{M}_\xi \).

As a consequence of representation (3.22) we find that there exists an infinite hierarchy of mutually commuting other functionals with respect to the Poisson bracket on the phase space \( \hat{M}_\xi \). The latter follows from the tensor form of the Poisson bracket (3.11) in the space \( \hat{G} \otimes \hat{G} \):

\[
\{ \tilde{T}(\tilde{l})(\lambda) \otimes \tilde{T}(\tilde{l})(\mu) \}_\zeta = [D(\lambda, \mu), \tilde{T}(\tilde{l})(\lambda) \otimes I + I \otimes \tilde{T}(\tilde{l})(\mu)]
\]

which holds for arbitrary \( \lambda, \mu \in \mathbb{C} \) and where \( D(\lambda, \mu): \hat{G}^* \to \hat{G}^* \) denotes the tensor form of the D-structure \( D: \hat{G}^* \to \hat{G}^* \). The trace operation in (3.30) causes the Poisson bracket vanish on the phase space \( \hat{M}_\xi \) for the functionals \( \text{tr}\tilde{T}(\tilde{l})(\lambda) \) and \( \text{tr}\tilde{T}(\tilde{l})(\mu) \) for arbitrary \( \lambda, \mu \in \mathbb{C} \).

4. Monodromy matrix, associated R-structure and Lie-Poisson bracket

Next we analyze possible forms of the D-mapping (3.24) as a function on the reduced phase space \( \hat{M}_\xi \). Since the parameter \( \tilde{k} \in \mathbb{C} \) is constant, its value for convenience is set at \( \tilde{k} = -1 \). Thus, taking into account the definition (3.24), the determining D-structure equation (3.26) takes the form:

\[
[D(\tilde{l})(Y), \tilde{l}] + \frac{d}{dx}D(\tilde{l})(Y) + Y = 0
\]

for any element \( Y \in \hat{G}^* \).

Let us consider the linear matrix equation

\[
\tilde{F}_x(x, s; \lambda) = \tilde{l}(x; \lambda)\tilde{F}(x, s; \lambda),
\]

where \( \tilde{l}(x; \lambda) = \tilde{G}^* \ni \tilde{F} \in \hat{G} \), with Cauchy data at a point \( x = s \in S^1 \):

\[
\tilde{F}(x, s; \lambda)|_{x=s} = I.
\]

The corresponding normalized monodromy matrix

\[
\tilde{T}(x; \lambda) := \tilde{F}(x + 2\pi, s; \lambda) - \nu^{-1}\text{tr}\tilde{F}(x + 2\pi, s; \lambda),
\]

for \( x \in S^1 \) and arbitrary \( \lambda \in \mathbb{C} \) satisfies the differential expression

\[
T_x - [\tilde{T}, \tilde{l}] = 0,
\]

exactly coinciding with (3.3). Thus, if by means of the co-adjoint transformation (2.7) this chosen matrix \( l \in \hat{G}^* \) will be transformed into the matrix \( \tilde{l} \in \hat{G}^* \), then the corresponding monodromy matrix of the equation (3.39) will transform into the monodromy matrix of the equation (4.2), which satisfies the expression (1.5).

Taking into account the differential relationships (4.2), (4.3) and (4.5), one can recalculate the Poisson bracket (3.22) by means of the identification

\[
\tilde{T}(\tilde{l})(z; \lambda) = \tilde{\tilde{T}}(z; \lambda)
\]
for arbitrary \( z \in S^1 \) and \( \lambda, \mu \in \mathbb{C} \). It yields the following tensor expression for the reduced phase space \( \tilde{M}_\xi \):

\[
\{\tilde{T}(\bar{l})(z; \lambda) \otimes \tilde{T}(\bar{l})(z; \mu)\}_{\xi} = \int_0^{2\pi} \frac{dx}{x} \int_0^{2\pi} \frac{dy}{y} \left\{ F(x + 2\pi, x; \lambda) \tilde{I}(x; \lambda) F(x, z; \lambda) \otimes F(y + 2\pi, y; \mu) \tilde{I}(y; \mu) F(y, z; \mu) \right\}_{\xi} = 0
\]

\[
= \int_0^{2\pi} \frac{dx}{x} \int_0^{2\pi} \frac{dy}{y} \left\{ F(z + 2\pi, x; \lambda) \tilde{I}(x; \lambda) \otimes \tilde{I}(y; \mu) \right\} F(x, z; \lambda) \otimes F(y, z; \mu)
\]

\[
= \int_0^{2\pi} \frac{dx}{x} \int_0^{2\pi} \frac{dy}{y} \left\{ F(z + 2\pi, x; \lambda) \tilde{I}(x; \lambda) \otimes \tilde{I}(y; \mu) \right\} F(x, z; \lambda) \otimes F(y, z; \mu)
\]

\[
(4.8)
\]

where \( z \in S^1 \), \( \lambda, \mu \in \mathbb{C} \) and, by definition,

\[
\{\tilde{l}(x; \lambda) \otimes \tilde{l}(y; \mu)\}_{\xi} := \tilde{\omega}(\lambda, \mu; x, y) = \sum_{i,k=0}^{N} \tilde{\omega}_{ik}(\lambda, \mu; x, y) \frac{\partial^i}{\partial x} \frac{\partial^k}{\partial y} \delta(x - y).
\]

Here the local functional matrices \( \tilde{\omega}_{ik}(\lambda, \mu; x, y) \in \tilde{G}^* \otimes \tilde{G}^* \) satisfy the antisymmetry property:

\[
P \tilde{\omega}_{ik}(\lambda, \mu; x, y) P = - \tilde{\omega}_{ki}(\lambda, \mu; x, y)
\]

for all \( i, k = 1, \ldots, N \), \( x, y \in S^1 \), \( \lambda, \mu \in \mathbb{C} \) and the permutation operator \( P : \tilde{G}^* \otimes \tilde{G}^* \), acts as \( PA \otimes BP := B \otimes A \) for any \( A, B \in \tilde{G}^* \). Just as in the calculation from [12, 29, 28] one obtains from (4.9) that

\[
(4.10)
\]

\[
\{\tilde{T}(z; \lambda) \otimes \tilde{T}(z; \mu)\}_{\xi} = \int_0^{2\pi} \frac{dx}{x} \int_0^{2\pi} \frac{dy}{y} \left\{ F(x + 2\pi, x; \lambda) \tilde{I}(x; \lambda) \tilde{I}(y; \mu) \right\} F(x, z; \lambda) \otimes F(y, z; \mu),
\]

where the matrix \( \tilde{\Omega}(\lambda, \mu; x) \in \tilde{G}^* \otimes \tilde{G}^* \) for all \( \lambda, \mu \in \mathbb{C} \), \( x \in S^1 \), depends only on \( \bar{l} \in \tilde{G}^* \).

The expression (4.10) allows the very compact representation

\[
(4.11)
\]

\[
\{\tilde{T}(z; \lambda) \otimes \tilde{T}(z; \mu)\}_{\xi} = \int_0^{2\pi} \frac{dx}{x} \frac{dy}{y} F(z + 2\pi, x; \lambda) \tilde{I}(x; \lambda) \tilde{I}(y; \mu) F(x, z; \lambda) \otimes F(y, z; \mu),
\]

if the tensor \( \mathcal{R} \)-matrix \( \mathcal{R} \in \tilde{G} \otimes \tilde{G}^* \) satisfies for \( x \in S^1 \) and \( \lambda, \mu \in \mathbb{C} \) the differential relationship

\[
(4.12)
\]

\[
\frac{d}{dx} \mathcal{R}(\lambda, \mu; x) + [\mathcal{R}(\lambda, \mu; x), l(x; \lambda) \otimes I + I \otimes l(x; \mu)] = \tilde{\Omega}(\lambda, \mu; x).
\]

If we define the mapping \( R : \tilde{G} \rightarrow \tilde{G} \) as

\[
R(Y) := \text{res}_{\mu=0}^{2\pi} \int_0^{2\pi} \frac{dy}{y} \mathcal{R}(\lambda, \mu; y) \delta(x - y) Y(y; \mu)
\]

for any \( Y \in \tilde{G}^* \), then the relationship (4.13) can be easily presented in the following operator form:

\[
(4.14)
\]

\[
-(X, \frac{d}{dx} Y)_{-1} + (\bar{l}, [X, Y]_R) = (X, R(Y))_{-1},
\]

which holds for any \( X, Y \in \tilde{G} \), where we denoted

\[
(4.15)
\]

\[
[X, Y]_R := [-R^*(x), Y] + [X, R(Y)].
\]
The result (4.15) can be used for rewriting the Poisson bracket (4.12) as

\[
\{ (X, \bar{T}(\bar{t}))_{-1}, (Y, \bar{T}(\bar{t}))_{-1} \}_\xi = \\
= \bar{\bar{t}} \left( \bar{\bar{t}}, [\bar{\bar{t}} \bar{X}, \bar{\bar{F}} \bar{F}_2 \bar{\bar{t}}, \bar{\bar{F}} \bar{Y} \bar{F}_2 \bar{\bar{t}} \bar{\bar{r}}]_{-1} - \left( \bar{\bar{t}}, \bar{\bar{F}} \bar{F}_2 \bar{\bar{t}}, \frac{\partial}{\partial \bar{\bar{t}}} (\bar{\bar{F}} \bar{Y} \bar{F}_2 \bar{\bar{t}}) \right)_{-1} \right) = \\
= \bar{\bar{t}} \left( \bar{\bar{t}}, \bar{\bar{F}} \bar{F}_2 \bar{\bar{t}}, \frac{\partial}{\partial \bar{\bar{t}}} (\bar{\bar{F}} \bar{Y} \bar{F}_2 \bar{\bar{t}}) \right)_{-1} - \\
- \left( \bar{\bar{t}}, \bar{\bar{F}} \bar{F}_2 \bar{\bar{t}}, \frac{\partial}{\partial \bar{\bar{t}}} (\bar{\bar{F}} \bar{Y} \bar{F}_2 \bar{\bar{t}}) \right)_{-1} ,
\]

where $\bar{\bar{F}} := \bar{\bar{F}}(\bar{\bar{t}}) (x, y; \lambda)$, $\bar{\bar{F}}_2 := \bar{\bar{F}}(\bar{\bar{t}}) (y + 2\pi, x; \lambda) \in \mathcal{G}$, $x, y \in \mathbb{S}^1, \lambda \in \mathbb{C}$, and we defined the gradients $\bar{\bar{\nabla}}(\bar{\bar{X}}, \bar{\bar{T}})(\bar{\bar{t}})$ and $\bar{\bar{\nabla}}(\bar{\bar{Y}}, \bar{\bar{T}})(\bar{\bar{t}}) \in \mathcal{G}$ in the standard way as

\[
(\nabla f(\bar{\bar{t}}), Z)_{-1} := \left. \frac{d}{d\bar{\bar{t}}} f(\bar{\bar{t}} + \varepsilon Z) \right|_{\varepsilon = 0}
\]

for any smooth functional $f \in \mathcal{D}(\mathcal{G}^*)$ and arbitrary $Z \in \mathcal{G}^*$.

It is easy to observe that under the antisymmetry condition $R^* = - R$ the right-hand side of (4.17) equals the Lie-Poisson bracket \cite{12, 25, 23, 18, 7} for the functionals $(\bar{\bar{X}}, \bar{\bar{T}})$ and $(\bar{\bar{Y}}, \bar{\bar{T}}) \in \mathcal{D}(\mathcal{G}^*)$. Here the adjoint space $\mathcal{G}^* = \mathcal{G}^* + \mathbb{C}$ is with respect to a new commutator structure $[\cdot, \cdot]_R$ on the centrally extended Lie algebra $\mathcal{G}$: for any $(\bar{\bar{X}}, c), (\bar{\bar{Y}}, r) \in \mathcal{G}$ with commutator

\[
[(\bar{\bar{X}}, c), (\bar{\bar{Y}}, r)]_R := \left( [\bar{\bar{X}}, \bar{\bar{Y}}], \frac{\partial}{\partial \bar{\bar{t}}} (\bar{\bar{X}}, \bar{\bar{Y}}) \right)_{-1} + \left( \frac{\partial}{\partial \bar{\bar{t}}} (\bar{\bar{X}}, \bar{\bar{Y}}) \right)_{-1} .
\]

In (4.19) the classical $R$-structure on the Lie algebra $\mathcal{G}$ $[[X, Y]]_R := [R(X), Y] + [X, R(Y)]$ under some conditions on the mapping $R : \mathcal{G} \to \mathcal{G}$ can generate on $\mathcal{G}$ a new Lie structure (which it must not).

The above results can be formulated as follows.

**Proposition 4.1.** The Marsden-Weinstein reduced canonical Poisson structure on the phase space $\mathcal{M}$ for the monodromy matrix $\bar{T}(\bar{t}) \in \mathcal{G}$ exactly coincides with the corresponding classical Lie-Poisson AKS-bracket on the centrally extended basis Lie algebra $\mathcal{G}$ subject to the $R$-structure (4.19) when it is antisymmetric.

If the antisymmetry property for the mapping $R : \mathcal{G} \to \mathcal{G}$ does not hold, the generated Lie-Poisson type bracket on the functional space $\mathcal{D}(\mathcal{G}^*)$ can be, owing to (4.17), defined as follows: for any $f, g \in \mathcal{D}(\mathcal{G}^*)$ the bracket

\[
\{ (\bar{\bar{f}}(\bar{\bar{t}}), (\bar{\bar{g}}(\bar{\bar{t}}))_R \} \xi = \left( \bar{\bar{t}}, \bar{\bar{f}}(\bar{\bar{t}}), \bar{\bar{g}}(\bar{\bar{t}}) \right)_{-1} + \left( \frac{\partial}{\partial \bar{\bar{t}}} \bar{\bar{f}}(\bar{\bar{t}}), R(\bar{\bar{g}}(\bar{\bar{t}})) \right)_{-1} + \left( \frac{\partial}{\partial \bar{\bar{t}}} R(\bar{\bar{g}}(\bar{\bar{t}})), \bar{\bar{g}}(\bar{\bar{t}}) \right)_{-1}
\]

where the generalized $R$-structure $[\cdot, \cdot]_R$ on $\mathcal{G}$ is given by the expression (4.16).

**5. D-STRUCTURE AND THE GENERALIZED R-STRUCTURE RELATIONSHIP ANALYSIS.**

As it was stated above, the reduced Poisson bracket on the phase space $\mathcal{M}_\xi$ is

\[
\{ (X, \bar{T}), (Y, T) \}_\xi = (\bar{T}, [X, Y]_{\mathcal{D}})_{-1},
\]

where for any $X, Y \in \mathcal{G}$ the corresponding $D$-structure on the Lie algebra $\mathcal{G}$ is defined by the classical expression (4.23) and the mapping (4.24). It is natural to assume that there exists a relationship between the $D$-structure $D : \mathcal{G} \to \mathcal{G}$ and the $R$-structure $R : \mathcal{G} \to \mathcal{G}$, described above in Section 3.

Assume, for brevity, that the $R$-structure (4.14) is antisymmetric, that is $R^* = - R$. Then it is easy to check that the following algebraic relationship

\[
D(X) := \frac{1}{2} R(\bar{T}X + XT)
\]
holds for any \( X \in \mathcal{G} \). In fact, the expression (4.12) is equivalent to the following

\[
(\mathcal{T}, X, (\mathcal{T}, Y)_{\xi} = (\mathcal{T}X, R(\mathcal{T}Y))_{-1} - (\mathcal{T}X, R(\mathcal{T}Y))_{-1}.
\]

Now, substituting the expression (5.2) into (3.22), one obtains that

\[
(\mathcal{T}, X, (\mathcal{T}, Y)_{\xi} = \frac{1}{2}(\mathcal{T}, [R(\mathcal{T}X + X\mathcal{T}), Y] + [X, R(\mathcal{T}Y + Y\mathcal{T})])_{-1}
\]

\[
= \frac{1}{2}([Y, \mathcal{T}], R(\mathcal{T}X))_{-1} + \frac{1}{2}([Y, \mathcal{T}], R(\mathcal{T}X))_{-1} +
\]

\[
+ \frac{1}{2}([\mathcal{T}, X], R(\mathcal{T}Y))_{-1} + \frac{1}{2}([\mathcal{T}, X], R(\mathcal{T}Y))_{-1} =
\]

\[
= \frac{1}{2}(Y\mathcal{T}, R(\mathcal{T}X))_{-1} - \frac{1}{2}(\mathcal{T}Y, R(\mathcal{T}X))_{-1} +
\]

\[
+ \frac{1}{2}(\mathcal{T}X, R(\mathcal{T}Y))_{-1} - \frac{1}{2}(\mathcal{T}X, R(\mathcal{T}Y))_{-1} +
\]

\[
= (\mathcal{T}X, R(\mathcal{T}Y))_{-1} - (\mathcal{T}X, R(\mathcal{T}Y))_{-1},
\]

which coincides exactly with (5.3).

Rewrite now for convenience the operator relationship (4.1) in the tensor form as

\[
(\mathcal{I} \otimes \mathbb{1})D - D(\mathbb{1} \otimes \mathcal{I}) - \frac{d}{dx}D = \mathbb{1},
\]

where the tensor \( D \in \mathcal{G} \otimes \mathcal{G}^* \), owing to the action (5.2), equals

\[
D = \frac{1}{2}(R(\mathbb{1} \otimes \mathcal{T}) + (\mathbb{1} \otimes \mathcal{T})R).
\]

Substituting the expression (5.6) into the equation (5.5) and taking into account the determining equation (4.13)

\[
[\mathcal{I} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{I}, R] - \frac{d}{dx}R = \Omega,
\]

one obtains the relationship for the tensor \( \Omega \in \mathcal{G} \rightarrow \mathcal{G}^*:

\[
2\mathbb{1} \otimes \mathcal{I} - \Omega = [R, \mathbb{1} \otimes \mathcal{T}] + (\mathbb{1} \otimes \mathcal{T})R(\mathbb{1} \otimes \mathcal{I}) - (\mathbb{1} \otimes \mathcal{I})R(\mathbb{1} \otimes \mathcal{T}).
\]

The latter makes two \( R \)- and \( D \)-structures on the Lie algebra \( \mathcal{G} \) compatible. Observe that the \( D \)-structure (5.2) is not antisymmetric even though the \( R \)-structure was assumed to be antisymmetric. Concerning the \( D \)-structure determining equation (5.7) one can anticipate that a study of its solutions would describe a set of nonlinear dynamical systems on the reduced phase space \( \hat{\mathcal{M}}_\xi \) possessing a priori an infinite hierarchy of mutually commuting conservation laws.

6. Example: A Fermionic Medium and a Related Quantum Exactly Solvable Superradiance Model

6.1. Model description. We shall demonstrate that the quantum superradiance properties of a generalized model of a one-dimensional many particle charged fermionic medium, interacting with an external electromagnetic field, can be completely described by means a quantum Lax type exactly solvable Hamiltonian system in a specially constructed Fock space.

Consider a Dirac type \( N \)-particle Hamiltonian operator of a quantum superradiance model which is expressed as

\[
H_N := i \sum_{j=1}^{N} \sigma_3^{(j)} \frac{\partial}{\partial x_j} \otimes \mathbb{1} - i \beta \mathbb{1} \otimes \int_{\mathbb{R}} \mathcal{E} \mathcal{E}^{*} \mathcal{E}_x + \alpha \sum_{j=1}^{N} \sigma_1^{(j)} \otimes \mathcal{E}(x_j),
\]
where $\sigma_3^{(j)}, \sigma_1^{(j)}, j = 1, \ldots, N$, are the usual Pauli matrices, $\alpha \in \mathbb{R}_+$ is an interaction constant, $0 < \beta < 1$ is the light speed in the linearly polarized fermionic medium, $\xi(x) := \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \varepsilon^+(x) \end{pmatrix}$ is the one-mode polarization matrix operator at particle location $x \in \mathbb{R}$ with quantized electric field boson operators $\varepsilon(x), \varepsilon^+(x) : \Phi_B \to \Phi_B$ acting in the corresponding Fock space $\Phi_B$ and satisfying the commutation relationships:

\begin{align}
[\varepsilon(x), \varepsilon^+(y)] &= \delta(x - y), \\
[\varepsilon(x), \varepsilon(y)] &= 0 = [\varepsilon^+(x), \varepsilon^+(y)]
\end{align}

for all $x, y \in \mathbb{R}$. We note that throughout the sequel we employ units for which the standard constants $\hbar = 1 = c$.

By construction, the $N$-particle Hamiltonian operator (6.1) acts in the Hilbert space $L_2^{(ns)}(\mathbb{R}^N; \mathbb{C}^2) \otimes \Phi_B$, where $L_2^{(ns)}(\mathbb{R}^N; \mathbb{C}^2)$ denotes the square-integrable antisymmetric vector functions on $\mathbb{R}^N, N \in \mathbb{Z}_+$. Correspondingly, the Fock space $\Phi_B$ allows the standard representation as the direct sum

\begin{equation}
\Phi_B := \oplus_{n \in \mathbb{Z}_+} L_2^{(s)}(\mathbb{R}^n; \mathbb{C})
\end{equation}

where $L_2^{(s)}(\mathbb{R}^n; \mathbb{C})$ denotes the space of symmetric square-integrable scalar functions on $\mathbb{R}^n, n \in \mathbb{Z}_+$.

Similarly, the fermionic Fock space

\begin{equation}
\Phi_F := \oplus_{n \in \mathbb{Z}_+} L_2^{(ns)}(\mathbb{R}^n; \mathbb{C}^2),
\end{equation}

can be used to represent [9, 13, 14, 2] the Hamiltonian operator (6.1) in the second quantized form

\begin{equation}
H := i \int_{\mathbb{R}} dx_1 \varepsilon_1^+\psi_1 - \varepsilon_2^+\psi_2 - \beta \varepsilon \varepsilon_2 + i\alpha(\varepsilon_1^+\psi_1 + \varepsilon_2^+\psi_2),
\end{equation}

which acts on the tensored Fock space $\Phi := \Phi_F \otimes \Phi_B$, where $\Phi_F$, defined by (6.4), can also be representes as

\begin{equation}
\Phi_F := \oplus_{n \in \mathbb{Z}_+} \text{span}\left\{ \int_{\mathbb{R}^n} dx_1 dx_2 \ldots dx_n \varphi_n^{(m)}(x_1, x_2, \ldots, x_n) \times \prod_{j=m+1}^n \psi_j^+(x_j) |0\rangle : 0 \leq m \leq n; \varphi_n^{(m)} \in L_2^{(as)}(\mathbb{R}^n; \mathbb{C}^2) \right\}
\end{equation}

and $|0\rangle \in \Phi_F$ is the corresponding vacuum state, satisfying the determining conditions

\begin{equation}
\psi_1(x) |0\rangle = 0 = \psi_2(x) |0\rangle, \varepsilon(x) |0\rangle = 0
\end{equation}

for all $x \in \mathbb{R}$. The creation and annihilation operators $\psi_j(x), \psi_j^+(y) : \Phi_F \to \Phi_F, j, k = 1, 2$, satisfy the anti-commuting

\begin{align}
\{\psi_j(x), \psi_k^+(y)\} &= \delta_{j,k}\delta(x - y), \\
\{\psi_j(x), \psi_k(y)\} &= 0 = \{\psi_j^+(x), \psi_k^+(y)\}
\end{align}

and commuting

\begin{align}
[\varepsilon(x), \psi_j(y)] &= 0 = [\varepsilon(x), \psi_j^+(y)], \\
[\varepsilon^+(x), \psi_j(y)] &= 0 = [\varepsilon^+(x), \psi_j^+(y)]
\end{align}

relationships for all $x, y \in \mathbb{R}$.

As we are interested in proving the exact integrability of our quantum superradiance model, it is necessary to find the corresponding Lax type representation of the Hamiltonian system

\begin{align}
\frac{d\psi_1}{dt} &= i[H, \psi_1] = \psi_1 + i\alpha \varepsilon^+ \psi_2, \\
\frac{d\psi_2}{dt} &= i[H, \psi_2] = -\psi_2 + i\alpha \varepsilon \psi_1, \\
\frac{d\varepsilon}{dt} &= i[H, \varepsilon] = -\beta \varepsilon + i\alpha \psi_1^+ \psi_2,
\end{align}

generated by the Hamiltonian operator (6.5) as a usual Heisenberg flow on the quantum operator manifold $M := \{(\psi_1, \psi_2, \varepsilon, \varepsilon^+, \psi_1^+, \psi_2^+) \in \text{End(}\Phi^6)\}$. The dynamical system (6.10) possesses also the following number operators as conservation laws:

\begin{equation}
N_F := \int_{\mathbb{R}} dx (\psi_1^+ \psi_1 + \psi_2^+ \psi_2), N_B := \int_{\mathbb{R}} dx (\varepsilon^+ \varepsilon + \psi_2^+ \psi_2),
\end{equation}
Proposition 6.1. The dynamical system (6.10) can be linearized by means of the quantum Lax type spectral problem

\[
\frac{df}{dx} = l(x; \lambda)f,
\]

where the operator matrix \(l(x; \lambda) \in \text{End}\Phi^3\) is

\[
l(x; \lambda) := \begin{pmatrix}
-\frac{i\lambda}{\beta} & i\xi_1 & i\xi_2 \\
i\xi_1 & -\frac{i\beta}{\beta} + i\xi_3 & i\xi_2 \\
i\xi_2 & i\xi_3 & -\frac{i\beta}{\beta} + i\xi_1
\end{pmatrix}
\]

for all \(x \in \mathbb{R}\), with \(\lambda \in \mathbb{C}\) an arbitrary time-independent spectral parameter, and

\[
\xi_1 := \xi_1(\alpha, \beta) = -18\alpha \left[ \frac{9 - 3\beta(\beta + 1)}{\beta + 3} \right]^{1/2} \times \\
\left( \frac{12\beta}{\beta + 3} \right)^{1/2} \frac{\beta + 3}{2\beta^2 + 3\beta + 3},
\]

\[
\xi_2 := \xi_2(\alpha, \beta) = 6\alpha(3 - 3\beta)^{1/2} \left( \frac{12\beta}{\beta + 3} \right)^{1/2} \frac{9 - 3\beta(\beta + 1)}{(\beta - 1)(2\beta^2 + 3\beta + 3)},
\]

\[
\xi_3 := \xi_3(\alpha, \beta) = 72\alpha\beta \frac{(3(1 - \beta))^{1/2}}{(\beta - 1)(2\beta^2 + 3\beta + 3)} \frac{9 - 3\beta(\beta + 1)}{\beta + 3}.
\]

are constants depending on the interaction parameter \(\alpha \in \mathbb{R}_+\) and the light speed in the polarized fermionic medium \(0 < \beta < 1\).

The quantum dynamical system (6.10) may be also regarded as an exactly solvable approximation of the three-level quantum model studied in [8] subject to its superradiance properties. Concerning the studies of such superradiance Dicke type one-dimensional models, it is necessary to mention the work [26] in which it was shown that the well-known quantum Bloch–Maxwell dynamical system

\[
d\psi_1/dt = i[\hat{H}, \psi_1] = i\alpha\varepsilon^+\psi_2,
\]

\[
d\psi_2/dt = i[\hat{H}, \psi_2] = i\alpha\varepsilon\psi_1,
\]

\[
d\varepsilon/dt = i[\hat{H}, \varepsilon] = -\beta\varepsilon_x + i\alpha\varepsilon^+\psi_2,
\]

generated by the reduced quantum Hamiltonian operator

\[
\hat{H} := -i\int_{\mathbb{R}} dx [\beta\varepsilon^+\varepsilon_x - i\alpha(\varepsilon^+\psi_1 + \psi_1^+\varepsilon)]
\]
in the strongly degenerate Fock space \(\Phi\) is also exactly solvable. Moreover, it possesses the corresponding Lax type operator whose spectral problem [12, 19, 7] is defined in the space \(\Phi^3\). But the important problem of constructing the stable physical vacuum for the Hamiltonian (6.17) was on the whole not discussed in [26], and neither was the problem of studying the related thermodynamics of quantum excitations over it. More interesting quantum one-dimensional models with the Hamiltonian similar to (6.5) describing the quantum interaction of just fermionic particles and only bosonic particles with an external electromagnetic field were studied, respectively, in [30] and [16]. In these investigations, the quantum localized Bethe states were constructed and analyzed in detail. The corresponding classical version of the quantum dynamical system (6.10), called the three-wave model, was studied in [19, 22] and elsewhere.

It is also worth mentioning here that the spectral operator problem (6.14) makes sense only if the light speed inside the polarized fermionic medium is less than the light speed in a vacuum. This is guaranteed by the dynamical stability of the quantum Hamiltonian system (6.10) following from the existence of an additional infinite hierarchy of conservation laws, suitably determined on the quantum operator phase space \(M\). Consequently, one can expect that the quantum dynamical system (6.10) also possesses the many-particle localized photonic states in the Fock space \(\Phi\), which
are called \textit{quantum solitons}, whose spatial range is inverse to the number of interior particles, and which can be interpreted as special Dicke type superradiance laser impulses. In particular, the quantum stability, solitonic formation aspects and construction of the physical ground state related with the unbounded a \textit{priori} from below Hamiltonian operator \((6.5)\) are of great importance for physical applications.

In the next subsection we will make use of use of the quantum spectral problem \((6.14)\) to prove that the quantum dynamical system \((6.10)\) allows the standard \(R\)-matrix description, which makes it possible to construct an infinite hierarchy of commuting conservation laws, thereby ensuring its complete quantum integrability.

6.2. The quantum \(R\)-matrix structure. The spectral problem \((6.14)\) belongs to exactly the same class whose Lie-algebraic properties were studied in Section 4. That means, in particular, that the system of dynamical equations

\[
d\psi_1/ dt = \psi_{1,x} + i\alpha \varepsilon^+ \psi_2, \\
d\psi_2/ dt = -\psi_{2,x} + i\alpha \varepsilon \psi_1, \\
d\varepsilon/ dt = -\beta \psi_x + i\alpha \psi_1^+ \psi_2
\]

(6.18)

jointly with their adjoint flows determine on an infinite-dimensional functional manifold \(M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^3)\) a completely Lax type integrable dynamical system and whose classical linear spectral problem \((6.13)\) entails the corresponding Poissonian relationships \((6.12)\) between the respectively defined monodromy matrix \(T(x; \lambda) \in \text{End } \mathbb{C}^3\) for any \(x \in \mathbb{R}\) and \(\lambda \in \mathbb{C}\). Taking into account the standard quantization rules \([11, 28]\) one can easily generalize these Poissonian relationships to the correspondingly defined operator relationship

\[
(6.19)
\]

\[\mathcal{R}(\lambda, \mu)T(x; \lambda) \otimes I = I \otimes T(x; \mu)\mathcal{R}(\lambda, \mu)\]

for some scalar \(\mathcal{R}\)-matrix \(\mathcal{R}(\lambda, \mu) \in \mathbb{C}^3 \otimes \mathbb{C}^3\) between quantum monodromy operators \(T(x; \lambda)\) and \(T(x; \mu)\in \text{End } \Phi^3\), acting already in the Fock space \(\Phi^3\). To realize this scheme, we first consider the following generalized quantum operator Cauchy problem for the spectral equation \((6.13)\) subject to the periodic conditions \(l(x + 2\pi; \lambda) = l(x; \lambda) \in \text{End } \Phi^3\) for all \(x \in \mathbb{R}\) and \(\lambda \in \mathbb{C}\):

\[
dF(x, y; \lambda)/ dx = \hat{\mathcal{L}}(x; \lambda)F(x, y; \lambda),
\]

(6.20)

where \(F(x, y; \lambda) \in \text{End } \Phi^3\) is the corresponding fundamental transition operator matrix satisfying

\[
F(x, y; \lambda)|_{y=x} = I,
\]

(6.21)

and the operation \(\hat{\cdot} : \cdot\) arranges operators \(\psi_j, \psi_j^+; j = 1, 2, \varepsilon\) and \(\varepsilon^+\), via the standard normal ordering \([28, 9]\) that does not change the position of any other operators; for instance, \(\hat{\psi}_1^+ \psi_2 \varepsilon^+ B\hat{\psi}_3 = \psi_1^+ \varepsilon^+ AB\psi_2\) for any \(A, B \in \text{End } \Phi\).

Construct now the operator products

\[
\hat{F}(x, y; \lambda, \mu) := \hat{F}(x, y; \lambda)F(x, y; \mu), \\
\hat{\mathcal{F}}(x, y; \lambda, \mu) := \hat{\mathcal{F}}(x, y; \mu)\hat{\mathcal{F}}(x, y; \lambda),
\]

(6.22)

where

\[
\hat{F}(x, y; \lambda) := F(x, y; \lambda) \otimes I, \\
\hat{\mathcal{F}}(x, y; \mu) := I \otimes F(x, y; \mu)
\]

(6.23)

are for all \(x, y \in \mathbb{R}\), \(\lambda, \mu \in \mathbb{C}\), the corresponding tensor products of operators acting in the space \(\Phi^3 \otimes \Phi^3\). The following proposition is crucial \([28, 2, 17]\) for the further analysis of integrability of the quantum dynamical system \((6.10)\) and is proved by a direct computation.

Proposition 6.2. The operator expressions \((6.23)\) satisfy the following differential relationships:

\[
\frac{\partial}{\partial x} \hat{F}(x, y; \lambda, \mu) = \hat{\mathcal{L}}(x; \lambda, \mu)\hat{F}(x, y; \lambda, \mu),
\]

\[
\frac{\partial}{\partial x} \hat{\mathcal{F}}(x, y; \lambda, \mu) = \hat{\mathcal{L}}(x; \lambda, \mu)\hat{\mathcal{F}}(x, y; \lambda, \mu),
\]

(6.24)
where the matrices
\[
\hat{\mathcal{L}}(x; \lambda, \mu) = \mathcal{I}(x; \lambda) + \mathcal{I}(x; \mu) - \alpha \hat{\Delta}(x; \lambda, \mu), \\
\hat{\Delta}(x; \lambda, \mu) = \mathcal{I}(x; \lambda) + \mathcal{I}(x; \mu) - \alpha \hat{\Delta}(x; \lambda, \mu),
\]
and \(\hat{\Delta}(x; \lambda, \mu), \hat{\Delta}(x; \lambda, \mu)\) satisfy the algebraic relationship \(P \hat{\Delta}(x; \lambda, \mu) P = \hat{\Delta}(x; \lambda, \mu)\) for all \(x \in \mathbb{R}, \lambda, \mu \in \mathbb{C}\), where \(P \in \text{End} \Phi^3 \otimes \Phi^3\) is the standard transmutation operator in the space \(\Phi^3 \otimes \Phi^3\), that is \(P(a \otimes b) := b \otimes a\) for any vectors \(a, b \in \Phi^3\).

Using Proposition 6.1 one can easily verify that there exists a scalar \(\mathcal{R}\)-matrix \(\mathcal{R}(\lambda, \mu), \mathcal{R} \in \text{End} \mathbb{C}^9\), such that
\[
\mathcal{R}(\lambda, \mu) \hat{\mathcal{L}}(x; \lambda, \mu) = \hat{\mathcal{L}}(x; \lambda, \mu) \mathcal{R}(\lambda, \mu)
\]
holds for all \(\lambda, \mu \in \mathbb{C}\) and \(x \in \mathbb{R}\). This, owing to the equations (6.24), implies the main functional Yang–Baxter type \([11, 28, 16, 17]\) operator relationship
\[
\mathcal{R}(\lambda, \mu) \mathcal{F}(x, y|\lambda, \mu) = \mathcal{F}(x, y|\lambda, \mu) \mathcal{R}(\lambda, \mu)
\]
is satisfied for any \(x, y \in \mathbb{R}\) and \(\lambda, \mu \in \mathbb{C}\), where
\[
\mathcal{R}(\lambda, \mu) := (\lambda - \mu) P - i \alpha \mathbb{I}
\]
is the corresponding quantum \(\mathcal{R}\)-operator. Recalling now that periodicity condition, from (6.27) one easily deduces by means of the trace-operation that the monodromy operator matrix \(T(x; \lambda) := F(x + 2\pi; x; \lambda)\) satisfies the algebraic expression
\[
\mathcal{R}(\lambda, \mu) T(x; \lambda) \otimes \mathbb{I} = \mathbb{I} \otimes T(x; \mu) \mathcal{R}(\lambda, \mu),
\]
giving rise, for all \(x \in \mathbb{R}\) and \(\lambda, \mu \in \mathbb{C}\), to the following commutation relationship
\[
[\text{tr}T(x; \lambda), \text{tr}T(x; \mu)] = 0.
\]
Actually, it follows from (6.27) that
\[
\text{tr}(T(x; \lambda) \otimes T(x; \mu)) = \text{tr}(\mathcal{R}^{-1}(x; \mu) \otimes T(x; \lambda)) = \text{tr}(T(x; \mu) \otimes T(x; \lambda)).
\]
Taking into account that \(\text{tr}(A \otimes B) = \text{tr}A \cdot \text{tr}B\) for any operators \(A, B \in \text{End} \Phi^3\), one easily obtains (6.30) from (6.31). Consequently, the \(\lambda\)-dependent operator functional
\[
\gamma(\lambda) := \text{tr}T(x, \lambda) \cong \sum_{j \in \mathbb{Z}^+} \gamma_j \lambda^{-j},
\]
as \(|\lambda| \to \infty\) generates an infinite hierarchy of commuting conservation laws \(\gamma_j : \Phi \to \Phi, j \in \mathbb{Z}^+ : \]
\[
[\gamma_j, \gamma_k] = 0
\]
for all \(j, k \in \mathbb{Z}^+\), where, in particular,
\[
\gamma_1 = N_F = \int_{\mathbb{R}} dx (\psi_1^+ \psi_1 + \psi_2^+ \psi_2), \quad \gamma_2 = N_B = \int_{\mathbb{R}} dx (\varepsilon^+ \varepsilon + \psi_2^+ \psi_2), \\
\gamma_3 = P = i \int_{\mathbb{R}} dx (\psi_1^+ \psi_{1, x} + \psi_2^+ \psi_{2, x} + \varepsilon^+ \varepsilon_x), \\
\gamma_4 = H = i \int_{\mathbb{R}} dx \left[ \psi_1^+ \psi_{1, x} - \psi_2^+ \psi_{2, x} - \varepsilon^+ \varepsilon_x + i \alpha (\varepsilon \psi_2^+ \psi_1 + \psi_2^+ \psi_2 \varepsilon^+) \right].
\]
Since the operator functional \(\gamma_4 = H\) is the Hamiltonian operator for the dynamical system \(6.10\), from (6.33) one obtains
\[
[H, \gamma_j] = 0
\]
for all \(j \in \mathbb{Z}^+\); that is, all of functionals \(\gamma_j : \Phi \rightarrow \Phi, j \in \mathbb{Z}^+\), are conservation laws.

Moreover, making use of the exact operator relationships (6.27) one can easily construct the physically stable quantum states \([N, M] > 0 \in \Phi\) for all \(N, M \in \mathbb{Z}^+\) upon redefining the Fock vacuum \(|0\rangle \in \Phi\), which is nonphysical for the dynamical system \(6.10\), governed by the unbounded from
below Hamiltonian operator (6.3). Following a renormalization scheme similar to those developed in [11, 31, 17], one can construct a new physically stable vacuum

\[(0)_{\text{phys}} := \prod_{q \leq \mu, \mu < Q} B^+(\mu) \langle 0 \rangle\]

by means of the new commuting to each other "creation" operators \(B^+(\mu) : \Phi \rightarrow \Phi, \mu \in \mathbb{C}\), generated by suitable components of the monodromy operator matrix \(T(x; \mu) : \Phi^3 \rightarrow \Phi^3, x \in \mathbb{R}\), whose commutation relationships with the Hamiltonian operator (6.5)

\[(\mathbf{H}, B^+(\mu)) = S(\mu; \alpha, \beta)B^+(\mu)\]

are parameterized by the two-particle scalar scattering factor \(S(\mu; \alpha, \beta), \mu \in \mathbb{C}\), and where values \(q < Q \in \mathbb{R}\) are to be determined [11] from the condition that quantum excitations over the physical vacuum (6.36) have positive energy. Since the physical vacuum (6.36) is an eigenstate of the Hamiltonian operator (6.3), the corresponding quantum eigenstates of the excitations can be represented as

\[\langle \mu \rangle := B^+(\mu) \langle 0 \rangle_{\text{phys}}\]

for some \(\mu \in \mathbb{R}\) and the new energy level can be taken into account in the renormalized Hamiltonian operator (6.5) by means of the chemical potentials \(a_F, a_B \in \mathbb{R}\):

\[\mathbf{H}_a := \mathbf{H} - a_F \mathbf{N}_F - a_B \mathbf{N}_B,\]

which should be determined from the conditions

\[\mathbf{H}_a \langle 0 \rangle_{\text{phys}} = 0, \langle \langle \mu \rangle | \mathbf{H}_a | \langle \mu \rangle \rangle > 0\]

for any \(\mu \in \mathbb{R}\). The physical vacuum state and quantum Hamiltonian renormalization construction described above make it possible to study the properties of superradiance quantum photonic impulse structures generated by interaction of the charged fermionic medium with an external electromagnetic field. Owing to the existence of quantum periodic eigenstates over the physically stable vacuum, one can also investigate the related thermodynamic properties of the model and analyze the generated superradiance photonic structures, which are important for explaining many existing experiments.

7. Conclusion

We have considered the standard canonically symplectic phase space \(M := T^*(\mathcal{G})\), generated by the centrally extended basis manifold to be an affine loop Lie algebra \(\mathcal{G}\) on the circle \(S^1\). Subject to the standard Hamiltonian Lie algebra \(\mathcal{G}\)-action on \(M\), with respect which the symplectic structure on \(M\) is invariant, constructed the corresponding momentum mapping and carried out the standard Marsden-Weinstein reduction of the manifold \(M\) upon the reduced phase space \(\tilde{M}_\xi\) endowed with the reduced Poisson bracket \(\langle \cdot, \cdot \rangle_\xi\). The latter allows to construct on the phase space \(\tilde{M}_\xi\) commuting to each other vector fields which are equivalent to some nonlinear dynamical systems possessing an infinite hierarchy of commuting conservation laws. Moreover, these mentioned commuting vector fields on \(\tilde{M}_\xi\) realize exactly their corresponding Lax type representations.

Presented detailed analysis of commutation properties for the related flows on the basis manifold makes it possible to define a suitable \(D\)-structure on the Lie algebra \(\mathcal{G}\), deeply related with the corresponding classical \(R\)-structure on \(\mathcal{G}\), generated by the reduced Poisson bracket on the phase space \(\tilde{M}_\xi\). As a bi-product of our analysis we stated that these \(\tilde{R}\) and \(D\)-structures are completely equivalent to a suitably generalized classical Lie-Poisson-Adler-Kostant-Symes-Kirillov-Berezin structure on the adjoint space \(\mathcal{G}^*\). We derived also the determining equation for the \(D\)-structure, classifying the generalized Lax type integrable nonlinear dynamical systems on the reduced phase space \(\tilde{M}_\xi\), whose respectively defined \(R\)-structures are not necessary both antisymmetric and local, as it was before described in [3] by means of an other approach. It is worth also to mention that the reduction scheme devised in this work can be applied also to the centrally extended algebra of pseudo-differential operators and affine loop algebras on the circle \(S^1\).

As an example of the physical Hamiltonian system whose exact solvability can be stated by means of a related \(R\)-structure, we have proposed a new generalized superradiance model, describing a one-dimensional many particle charged fermionic medium interacting with an external electromagnetic field. Its operator structure allows to calculate by means of the \(R\)-matrix approach
diverse superradiance effects, which are closely related to the formation of the bound quantum solitonic states and their stability. The existence of these states is established by suitable applying the physical vacuum renormalization subject to which all quantum excitations are of positive energy. This procedure, based on the determining operator relationships (6.27), enables one to describe the thermodynamic properties of the quantum dynamical system over the stable physical vacuum. In addition, it facilitates analysis of the corresponding thermodynamic states of the resulting quantum photonic system and its superradiance properties. Our work indicates that a more detailed investigation of these and related topics is in order, which we plan to undertake elsewhere.

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1) The V.A. Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia, and, the Abdus Salame International Center of Theoretical Physics, Trieste, Italy, 2) Department of Applied Mathematics, University of Agriculture, Krakow, Poland