A Behavioral Approach to Data-Driven Control With Noisy Input–Output Data

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Abstract—This article deals with data-driven stability analysis and feedback stabilization of linear input–output systems in autoregressive (AR) form. We assume that noisy input–output data on a finite time-interval have been obtained from some unknown AR system. Data-based tests are then developed to analyze whether the unknown system is stable, or to verify whether a stabilizing dynamic feedback controller exists. If so, stabilizing controllers are computed using the data. In order to do this, we employ the behavioral approach to systems and control, meaning a departure from existing methods in data driven control. Our results heavily rely on a characterization of asymptotic stability of systems in AR form using the notion of quadratic difference form as a natural framework for Lyapunov functions of autonomous AR systems. We introduce the concepts of informative data for quadratic stability and quadratic stabilization in the context of input–output AR systems and establish necessary and sufficient conditions for these properties to hold. In addition, this article will build on results on quadratic matrix inequalities and a matrix version of Yakubovich’s $S$-lemma.

Index Terms—Behavioral approach, data-driven control, quadratic matrix inequalities (QMI), robust control, $S$-procedure.

I. INTRODUCTION

DATA-DRIVEN analysis and control is a research topic that has received a lot of attention in the past few years. The idea that lies at the core of this research area is to use data obtained from an unknown dynamical system to verify certain system properties and to design control laws for that system. The main challenge is to do the analysis and design without the usual first step of establishing a mathematical model of the system (for example, by using first principles modeling or system identification), but work directly with the data instead. This has been the subject of many recent publications in the area, for a large part in the context of input–state–output systems and under the assumption that the system’s state is measured, see, e.g., [1], [2], [3], [4], [5], [6], [7].

There are different contributions that extend these results to input–output measurements [8], [9], [10], [11]. A general strategy, adopted by all of these papers, is to rely on an artificial state-space representation of the system with a state comprised of shifts of the inputs and outputs. This leads to an input–state–output system to which techniques for state data are applicable. A potential downside of this approach is that the obtained state-space systems are nonminimal and of high dimension, thus requiring a large amount of data to control (see, e.g., [8, Sec. VIC]). In addition, the system matrices of the state-space representation are structured and consist of a combination of known and unknown blocks. Often, this structure is not taken fully into account, which leads to rather conservative conditions for data-driven control design. Exploiting this prior knowledge of the system matrices is an important problem, which has recently been studied in [11].

Motivated by the limitations of state space, the main purpose of this article is to develop a theory for the data-driven design of feedback controllers on the basis of input–output data, without relying on state construction. We will, thus, abandon the paradigm of systems in state-space form, and will, instead, work directly with the model class of all input–output systems described by higher order difference equations, also called autoregressive (AR) systems. The unknown dynamical system that we want to analyze or control is assumed to be a member of this model class of AR systems. We will assume that noisy input–output data on a given finite time-interval have been obtained from this unknown AR system. These data are employed to check stability or to verify whether a dynamic feedback controller exists that stabilizes the unknown system and, if so, to compute a stabilizing controller.

Essential to our development is a method that allows the verification of asymptotic stability of systems described by higher order difference equations. For this, we will heavily rely on the behavioral approach to systems and control. In particular, we will adopt the notion of quadratic difference form (QDF) as a natural framework for Lyapunov functions of autonomous AR systems, see [12], [13], [14] for the origins of this theory in continuous time and [15], [16] for the discrete-time counterpart.
A. Contributions of This Article

The contributions of this article are summarized as follows. 
1) Following the general framework of [17], we introduce the concepts of informative data for quadratic stability and quadratic stabilization in the context of input–output AR systems. 
2) We provide necessary and sufficient conditions under which the data are informative for quadratic stability and quadratic stabilization. These conditions are formulated in terms of data-based linear matrix inequalities (LMIs). If the LMI for quadratic stabilization is feasible, a controller can be extracted from one of its solutions. 
3) Using projection results in [18], we separate the computation of the controller and Lyapunov function, which leads to lower dimensional LMIs for the Lyapunov function, and an explicit formula for a dynamic controller.

B. Related Work

We note that behavioral theory has been popularized before in the context of data-driven control. Based on Willems’ fundamental lemma [19] and its various extensions [20], [21], [22], [23], a number of control problems were solved, such as output matching [24] and predictive control [25], [26], [27], [28]. The control design typically involves the computation of (open loop) sequences of control inputs that live in the image of data Hankel matrices. The results of this article complement these methods by providing behavioral results for the data-based computation of dynamic feedback controllers. Other relevant results at the intersection of behaviors and data-driven control include [29] that considers control by interconnection, and [30] in which data-driven dissipativity analysis is performed using QDFs, both in the exact data setting. In the state-space setting, dissipativity was investigated using QDFs in [31].

Informativity for stability and stabilization have been studied before in the context of state-space systems with noiseless input-state data in [17] and for noisy input-state data in [3]. We stress that in this article, we deal with noisy input–output data. The behavioral approach followed in this article is a radical departure from existing work on input–output systems [8], [9], [10], [11], which enables, among others, the formulation of necessary and sufficient conditions for data-driven analysis and control problems.

In order to cope with noise-corrupted measurements, this article will build on results on quadratic matrix inequalities (QMI) and a matrix version of Yakubovich’s S-lemma that were established in [3], [32], and [18].

C. Notation

The set of nonnegative integers will be denoted by \( \mathbb{Z}_+ \). We will denote by \( \mathbb{R}^n \) the \( n \)-dimensional Euclidean space. For given positive integers \( m \) and \( n \) the linear space of all real \( m \times n \) matrices will be denoted by \( \mathbb{R}^{m \times n} \). The subset of \( \mathbb{R}^{n \times n} \) consisting of all symmetric matrices will be denoted by \( \mathbb{S}^n \). For vectors \( x \) and \( y \) we will denote \( [x^\top \ y^\top]^\top \) by \( \text{col}(x, y) \). For given integer \( n \), we denote by \( I_n \) the \( n \times n \) identity matrix and \( 0_n \) the \( n \times n \) zero matrix. In order to enhance readability, we sometimes denote the \( n \times m \) zero matrix by \( 0_{n \times m} \). Given a real matrix \( M \), we will denote its Moore–Penrose pseudoinverse by \( M^\dagger \). For given \( T > 0 \), the discrete-time interval \( \{0, 1, \ldots, T\} \) is denoted by \( [0, T] \).

II. Quadratic Matrix Inequalities

An important role in this article is played by solution sets of QMIs and the so-called matrix S-lemma. For an extensive treatment of these, we refer to [3], [32], [33], and more recently [18]. In particular, for proofs of the propositions that are collected in this section, we refer to Section III and [18, Appendix A].

We consider symmetric partitioned matrices of the form
\[
\Pi = \begin{bmatrix}
\Pi_{11} & \Pi_{12} \\
\Pi_{21} & \Pi_{22}
\end{bmatrix} \in \mathbb{S}^{q+r}
\]
where \( \Pi_{11} \in \mathbb{S}^q \) and \( \Pi_{22} \in \mathbb{S}^r \) and, obviously, \( \Pi_{21} = \Pi_{12}^\top \). We define the generalized Schur complement of \( \Pi \) with respect to \( \Pi_{22} \) as
\[
\Pi|\Pi_{22} := \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21}.
\]
Define the following subset of all partitioned matrices in \( \mathbb{S}^{q+r} \) of the form
\[
\Pi_{q,r} := \left\{ \begin{bmatrix}
\Pi_{11} & \Pi_{12} \\
\Pi_{21} & \Pi_{22}
\end{bmatrix} \in \mathbb{S}^{q+r} \mid \Pi_{22} \preceq 0, \Pi|\Pi_{22} \succeq 0 \right\}.
\]
We will be interested in the solution sets of QMIs associated with matrices \( \Pi \in \Pi_{q,r} \). In particular, consider the sets
\[
Z_r(\Pi) := \left\{ Z \in \mathbb{R}^{r \times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} \succeq 0 \right\}
\]
\[
Z_+^r(\Pi) := \left\{ Z \in \mathbb{R}^{r \times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} > 0 \right\}.
\]

Proposition 1 (see [18, Theorems 3.2, 3.3]): Let \( \Pi \in \Pi_{q,r} \). Then, the following holds:
1) \( Z_r(\Pi) \) is nonempty if and only if \( \Pi|\Pi_{22} \succeq 0 \);
2) \( Z_+^r(\Pi) \) is nonempty if and only if \( \Pi|\Pi_{22} > 0 \). In that case, \( Z = Z_+^r(\Pi) \) if and only if \( Z = -\Pi_{22}^{-1}\Pi_{21} + ((-\Pi_{22})^\dagger)^\dagger S(\Pi|\Pi_{22})^\dagger + (I - \Pi_{22}\Pi_{22}^\dagger)T \) for some \( S, T \in \mathbb{R}^{r \times (q+r)} \) with \( S^2 < I \).

The following result gives necessary and sufficient conditions under which the solution set of one QMI is contained in that of second, strict, QMI. These conditions are in terms of feasibility of an LMI.

Proposition 2 (Strict matrix S-lemma [18, Theorem 4.10]): Let \( M, N \in \mathbb{S}^{q+r} \). If there exists a real scalar \( \alpha \geq 0 \) such that \( M - \alpha N > 0 \) then \( Z_r(N) \subseteq Z_+^r(M) \). Next, assume that \( N \in \Pi_{q,r} \) and \( N_{22} < 0 \). Then, \( Z_r(N) \subseteq Z_+^r(M) \) if and only if there exists a real scalar \( \alpha \geq 0 \) such that \( M - \alpha N > 0 \).
Another result that will be instrumental in this article is the following.

Let $W \in \mathbb{R}^{q \times p}$ and for $S \subseteq \mathbb{R}^{r \times q}$ define $SW := \{SW \mid S \in S\}$. Also, for $\Pi \in \mathbb{S}^{p+r}$ define $\Pi W \in \mathbb{S}^{p+r}$ by

$$
\Pi W := \begin{bmatrix} W^T & 0 \\ 0 & I_r \end{bmatrix} \Pi \begin{bmatrix} W & 0 \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} W^T \Pi_{11} W & W^T \Pi_{12} \\ \Pi_{21} W & \Pi_{22} \end{bmatrix}.
$$

(5)

Note that $\Pi W \in \Pi r$, if $\Pi \in \Pi r$. The sets $Z_r(\Pi)$ and $Z_r(\Pi W)$ and their strict analogues are related as follows.

**Proposition 3** (see [18, Theorems 3.4, 3.5]): Let $\Pi \in \Pi r$ and $W \in \mathbb{R}^{q \times p}$. Then, the inclusion $Z_r(\Pi)W \subseteq Z_r(\Pi W)$ holds. Moreover:

1) if either $\Pi_{22}$ is nonsingular or $W$ has full column rank, then $Z_r(\Pi)W = Z_r(\Pi W)$;

2) if $W$ has full column rank and $\Pi | \Pi_{22} > 2$ then $Z_r^+(\Pi)W = Z_r^+(\Pi W)$.

### III. Systems Represented by AR Models

In this article, we consider input–output systems with noise represented by AR models of the form

$$
y(t + L) + P_{L-1} y(t + L - 1) + \cdots + P_0 y(t) = Q_{L-1} u(t + L - 1) + \cdots + Q_0 u(t) + v(t). \tag{6}
$$

Here, $L$ is a positive integer, called the order of the system. The input $u(t)$ and output $y(t)$ are assumed to take their values in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. The term $v(t)$ represents unknown noise. The parameters of the model are real $p \times p$ matrices $P_0, P_1, \ldots, P_{L-1}$ and $p \times m$ matrices $Q_0, Q_1, \ldots, Q_{L-1}$. Using the shift operator $(\sigma f)(t) = f(t + 1)$, (6) can be written as

$$
P(\sigma) y = Q(\sigma) u + v \tag{7}
$$

where $P(\xi)$ and $Q(\xi)$ are the real $p \times p$ and $p \times m$ polynomial matrices defined by

$$
P(\xi) = I \xi^L + P_{L-1} \xi^{L-1} + \cdots + P_1 \xi + P_0 \tag{8}
$$

$$
Q(\xi) = Q_{L-1} \xi^{L-1} + \cdots + Q_1 \xi + Q_0.
$$

Since the leading coefficient matrix of $P(\xi)$ is the $p \times p$ identity matrix, $P(\xi)$ is nonsingular and $P^{-1}(\xi)Q(\xi)$ is proper. Thus, indeed, (7) represents a causal input–output system with control input $u$, noise input $v$ and output $y$.

Note that we consider the case that $Q(\xi)$ has degree at most $L - 1$. We do this to obtain well-posed feedback interconnections later on. This means that $P(\xi)^{-1}Q(\xi)$ is, in fact, strictly proper.

In this article, we will freely use terminology and notation originating from the behavioral approach, see, e.g., [34], [35]. In particular, we denote

$$
R(\xi) := \begin{bmatrix} -Q(\xi) & P(\xi) \end{bmatrix} \quad \text{and} \quad w := \text{col}(u, y). \tag{9}
$$

Clearly, $R(\xi)$ is a real $p \times q$ polynomial matrix with $q := p + m$, (7) can be written as

$$
R(\sigma) w = v. \tag{10}
$$

The homogeneous (i.e., noise free) system associated with (10) is given by $R(\sigma)w = 0$. Within the behavioral approach, this is called a kernel representation of its space of solutions $w : \mathbb{Z}_+ \to \mathbb{R}^q$. This space of solutions is called the behavior of the system, and is denoted by $B(R)$. The variable $w$ is called the manifest variable of the behavior. In the special case that $m = 0$, i.e., the system has no control inputs, the polynomial matrix $Q(\xi)$ is void and $R(\xi) = P(\xi)$. In that case $R(\sigma)w = 0$ reduces to the autonomous system represented by $P(\sigma)y = 0$. Its associated behavior, denoted by $B(P)$, is then a finite dimensional linear space.

This article deals with analysis and control design for systems of the form (7), where the polynomial matrices $P(\xi)$ and $Q(\xi)$ are unknown. We do assume that the order $L$ and the dimensions $m$ and $p$ are known. We assume that we have noisy input–output data on a given finite-time interval. These data are assumed to be obtained from an underlying true (but unknown) system of form (7). In particular, in case this unknown system has no control inputs, we want to use the output data to check whether it is stable, in the sense that if the noise $v = 0$ then all solutions $w$ tend to zero as time tends to infinity. On the other hand, in case that control inputs are present we want to use the input–output data to check whether there exists a stabilizing feedback controller and, if so, determine such controller using only the data.

So far, we have not specified the type of noisy data that we will be dealing with in this article. We will consider measurements in the form of a finite-length trajectory of the inputs and outputs. To be precise, we assume that we have noisy input–output data

$$
u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T) \tag{11}
$$

on a given time interval $[0, T]$ with $T \geq L$. These noisy data are obtained from the true system. Assume that this true system is represented by (unknown) polynomial matrices $P_s(\xi)$ and $Q_s(\xi)$ of the form (8). In other words, the true system is represented by the equation

$$
P_s(\sigma) y = Q_s(\sigma) u + v \tag{12}
$$

with $v$ unknown noise.

More concretely, we assume that $u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T)$ are samples on the interval $[0, T]$ of $u$ and $y$ that satisfy (12). We make the following assumption on the noise $v$ during the interval on which we collect data.

**Assumption 4:** The noise samples $v(0), v(1), \ldots, v(T - L)$, collected in the real $p \times (T - L + 1)$ matrix

$$
V := \begin{bmatrix} v(0) & v(1) & \cdots & v(T - L) \end{bmatrix}
$$

satisfy the QMI

$$
\begin{bmatrix} I \\ V^T \end{bmatrix} \Pi \begin{bmatrix} I \\ V^T \end{bmatrix} \geq 0 \tag{13}
$$

where $\Pi \in \mathbb{S}^{p+T-L+1}$ is a known partitioned matrix

$$
\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}
$$
with \( \Pi_{11} \in \mathbb{S}^p \), \( \Pi_{12} \in \mathbb{R}^{p \times (T-L+1)} \), \( \Pi_{21} = \Pi_{12}^\top \), and \( \Pi_{22} \in \mathbb{S}^{T-L+1} \). We assume that \( \Pi_{22} \leq 0 \) and \( \Pi_{22} \geq 0 \). In particular, this implies that \( \Pi \in \Pi_{p,T-L+1} \) and that the set \( \mathcal{Z}_{T-L+1}(\Pi) \) of matrices \( V \) that satisfy (13) is nonempty (see Proposition 1).

Assumption (4) on the noise samples \( v(0), \ldots, v(T-L) \) captures, for instance, the following.

1) **Energy bounds:** \( \Pi_{22} = -I \) and \( \Pi_{12} = 0 \) imply \( VV^\top = \sum_{t=0}^{T-L} \|v(t)\|^2 \leq \Pi_{11} \), which means that the energy of \( v \) on the time interval \([0, T-L]\) is bounded by \( \Pi_{11} \).

2) **Individual noise sample bounds:** If \( \|v(t)\|^2 < \epsilon \) for all individual noise samples at every time instant are bounded in norm, then (13) holds with \( \Pi_{22} = -I \), \( \Pi_{12} = 0 \) and \( \Pi_{11} = c(T-L+1)I \). Note that this introduces some conservatism.

3) **Sample covariance bounds:** \( \Pi_{22} = \frac{1}{T-L+1} \Pi \Pi^\top - I \), \( \Pi_{11} = (T-L+1)M \) with \( M \in \mathbb{S}^p \), and \( \Pi_{12} = 0 \). Defining the average \( \mu := \frac{1}{T-L+1} \sum_{t=0}^{T-L} v(t) \), this leads to \( \frac{1}{T-L+1} \sum_{t=0}^{T-L} (v(t) - \mu)(v(t) - \mu)^\top = \frac{1}{T-L+1}(V-I)^\top \Pi^\top \Pi V \leq M \) where \( \Pi \) denotes the \( T-L+1 \)-vector of ones: the sample covariance matrix of \( v \) is bounded by \( M \).

4) **Exact measurements:** \( \Pi_{11} = 0 \), \( \Pi_{12} = 0 \), and \( \Pi_{22} = -I \) leads to \( V = 0 \), i.e., the noise is zero.

Additional examples of special cases of noise models satisfying Assumption (4) can be found in [18].

Again denote \( q := p + m \), \( R(\xi) = [-Q(\xi) \; P(\xi)] \) and \( w = \text{col}(u, y) \) and recall that (7) can be written as

\[
R(\sigma)w = v. \tag{14}
\]

We collect the (unknown) coefficient matrices of \( R(\xi) \) in the \( p \times (ql) \) matrix

\[
R := \begin{bmatrix}
-Q_0 & P_0 & -Q_1 & P_1 & \cdots & -Q_{L-1} & P_{L-1}
\end{bmatrix}. \tag{15}
\]

Note that, with a slight abuse of notation, we denote both the polynomial matrix and its coefficient matrix by \( R \). With the definition of the signal \( w \), we also arrange the data \( u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T) \) into the vectors

\[
w(t) = \begin{bmatrix}
u(t) \\
y(t)
\end{bmatrix}, \quad (t = 0, 1, \ldots, T)
\]

and define the slightly adapted Hankel matrix \( H(w) \) defined by

\[
H(w) := \begin{bmatrix}
w(0) & w(1) & \cdots & w(T-L) \\
w(1) & w(2) & \cdots & w(T-L+1) \\
\vdots & \vdots & & \vdots \\
w(L-1) & w(L) & \cdots & w(T-1) \\
y(L) & y(L+1) & \cdots & y(T)
\end{bmatrix}. \tag{16}
\]

Furthermore, we partition

\[
H(w) = \begin{bmatrix}
H_1(w) \\
H_2(w)
\end{bmatrix}. \tag{17}
\]

where \( H_1(w) \) contains the first \( qL \) rows and \( H_2(w) \) the last \( p \) rows.

It is then easily verified that any input–output system (14) for which the coefficient matrix \( R \) defined in (15) satisfies

\[
\begin{bmatrix}
R \\
I
\end{bmatrix} \begin{bmatrix}
H_1(w) \\
H_2(w)
\end{bmatrix} = V \tag{18}
\]

for some \( V \in \mathcal{Z}_{T-L+1}(\Pi) \), could have generated the noisy input–output data (11). In other words, \( w(0), w(1), \ldots, w(T) \) are also samples on the interval \([0, T]\) of a \( w \) that satisfies \( R(w)v = v \) for some \( v \) satisfying Assumption 4. Therefore, if \( R \) satisfies (18) for some \( V \in \mathcal{Z}_{T-L+1}(\Pi) \), we call the AR system corresponding to the matrix \( R \) compatible with the data. Recall that, in particular, the true system is compatible with the data. Now define

\[
N := \begin{bmatrix}
I & H_2(w) \\
0 & H_1(w)
\end{bmatrix} \begin{bmatrix}
I & H_2(w) \\
0 & H_1(w)
\end{bmatrix}^\top. \tag{19}
\]

Then, by combining (13) and (18) we see that

\[
\begin{bmatrix}
I \\
V^\top
\end{bmatrix} = \begin{bmatrix}
I & H_2(w) \\
0 & H_1(w)
\end{bmatrix} \begin{bmatrix}
I \\
R^\top
\end{bmatrix}. \tag{20}
\]

Therefore, the system corresponding to the coefficient matrix \( R \) is compatible with the data if and only if \( R^\top \) satisfies the QMI

\[
\begin{bmatrix}
I \\
R^\top
\end{bmatrix} N \begin{bmatrix}
I \\
R^\top
\end{bmatrix} \succeq 0
\]

equivalently

\[
R^\top \in \mathcal{Z}_{qL}(N).
\]

Since the true system is compatible with the data, the set \( \mathcal{Z}_{qL}(N) \) is nonempty.

Recall that we are interested in finding a controller that stabilizes the unknown true system. Moreover, on the basis of the measurements, we cannot distinguish this true system from any of those corresponding to matrices in \( \mathcal{Z}_{qL}(N) \). Therefore, our goal requires us to find a stabilizing controller for all systems corresponding to matrices in this set.

**IV. QUADRATIC DIFFERENCE FORMS**

Studying stability and stabilization in the context of input–output AR systems requires the notion of Lyapunov functions given by QDFs. In this section, we will review the basic material and establish some useful preliminary results. For more details, we refer to [12], [13], [15], [16].

Let \( K \) and \( q \) be positive integers and for \( i, j = 0, 1, \ldots, K \) let \( \Phi_{i,j} \in \mathbb{R}^{q \times q} \) be such that \( \Phi_{i,i} \in \mathcal{S}^q \) and \( \Phi_{i,j} = \Phi_{j,i}^\top \) for all \( i \neq j \).

Arrange these matrices into the partitioned matrix \( \Phi \in \mathcal{S}^{(K+1)q} \) given by

\[
\Phi := \begin{bmatrix}
\Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,K} \\
\Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,K} \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{K,0} & \Phi_{K,1} & \cdots & \Phi_{K,K}
\end{bmatrix}.
\]
Then, the QDF associated with $\Phi$ is the operator $Q_\theta$ that maps $\mathbb{R}^q$-valued functions $w$ on $\mathbb{Z}_+$ to $\mathbb{R}$-valued functions $Q_\theta(w)$ on $\mathbb{Z}_+$ defined by

$$Q_\theta(w)(t) := \sum_{k,\ell=0}^K w(t+k)^\top \Phi_{k,\ell} w(t+\ell). \quad (21)$$

In terms of the matrix $\Phi$ this can be written as

$$Q_\theta(w)(t) = \begin{bmatrix} w(t) \\ w(t+1) \\ \vdots \\ w(t+K) \end{bmatrix}^\top \Phi \begin{bmatrix} w(t) \\ w(t+1) \\ \vdots \\ w(t+K) \end{bmatrix}.$$

We define the degree of the QDF (21) as the smallest integer $d$ such that $\deg(\Phi_{1,j}) = 0$ for all $i > d$ or $j > d$. This degree is denoted by $\deg(Q_\theta)$. The matrix $\Phi$ is called a coefficient matrix of the QDF. Note that a given QDF does not determine the coefficient matrix uniquely. However, if the degree of the QDF is $d$, it allows a coefficient matrix $\Phi \in S^{(d+1)d}\mathbb{R}^q$.

The QDF $Q_\theta$ is called nonnegative if $Q_\theta(w) \geq 0$ for all $w : \mathbb{Z}_+ \to \mathbb{R}^q$. We denote this as $Q_\theta \geq 0$. Clearly, this holds if and only if $\Phi \geq 0$. The QDF is called positive if it is nonnegative and, in addition, $Q_\theta(w) = 0$ if and only if $w = 0$. This is denoted as $Q_\theta > 0$. We define nonpositivity and negativity analogously.

For a given QDF $Q_\theta$, its rate of change along a given $w : \mathbb{Z}_+ \to \mathbb{R}^q$ is given by $Q_\theta(w)(t+1) - Q_\theta(w)(t)$. It turns out that the rate of change defines a QDF itself. Indeed, by defining the matrix $\nabla \Phi := [0_{q\times q} \quad 0 \quad \Phi \quad 0_{q\times q}]$ by

$$\nabla \Phi := \begin{bmatrix} 0_{q\times q} & 0 & \Phi & 0_{q\times q} \\ 0 & 0 & \Phi & 0_{q\times q} \end{bmatrix} \quad \text{(22)}$$

it is easily verified that

$$Q_{\nabla \Phi}(w)(t) = Q_\theta(w)(t+1) - Q_\theta(w)(t)$$

for all $w : \mathbb{Z}_+ \to \mathbb{R}^q$ and $t \in \mathbb{Z}_+$.

QDFs are particularly relevant in combination with behaviors defined by AR systems. Let $R(\xi)$ be a real $p \times q$ polynomial matrix and consider the AR system represented by $R(\sigma)w = 0$. Let $B(R)$ be the behavior of this system. The QDF $Q_\theta$ is called nonnegative on $B(R)$ if $Q_\theta(w) \geq 0$ for all $w \in B(R)$. It is called positive on $B(R)$ if, in addition, $Q_\theta(w) = 0$ if and only if $w = 0$. We denote this as $Q_\theta \geq 0$ on $B(R)$ and $Q_\theta > 0$ on $B(R)$, respectively. Likewise we define nonpositivity and negativity on $B(R)$.

Two given QDFs $Q_{\theta_1}$ and $Q_{\theta_2}$ are called $B(R)$-equivalent if they coincide on solutions of $R(\sigma)w = 0$, i.e. $Q_{\theta_1}(w) = Q_{\theta_2}(w)$ for all $w \in B(R)$.

We now return to the setup of AR systems introduced in Section III, and in particular consider QDFs for autonomous systems. In that case every QDF turns out to be equivalent to a QDF with degree at most the order of the system. Indeed, let $P(\xi)$ be a square polynomial matrix as given in (8) with corresponding autonomous system $P(\sigma)y = 0$ of order $L$. Let $B(P)$ be its behavior. Define the restricted behavior on the interval $[0, L-1]$ by

$$B(P)[0, L-1] := \left\{ \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(L-1) \end{bmatrix} \in \mathbb{R}^L \mid y \in B(P) \right\}.$$  

It is easily seen that $B(P)[0, L-1] = \mathbb{R}^L$. Using this fact, we obtain the following lemma (see also [12], Prop 4.9).

Lemma 5: For any QDF $Q_{\Phi}(y)$ there exists a $B(P)$-equivalent QDF $Q_{\Phi}(y)$ with degree at most $L-1$. In addition, if $Q_{\Phi} \geq 0$ on $B(P)$ then $Q_{\Phi} \geq 0$, equivalently, $\Phi \geq 0$.

Proof: Let $d = \deg(Q_{\Phi})$ and let $y \in B(P)$. Then, we have

$$Q_{\Phi}(y)(t) = \sum_{k=0}^d \sum_{l=0}^d y^{t+k}) \Phi_{k,\ell} y(t+l). \quad (23)$$

First consider the case that $d \leq L-1$. Since $B(P)[0, L-1] = \mathbb{R}^L$, we readily have that if $Q_{\Phi} \geq 0$ on $B(P)$ then $\Phi \geq 0$.

Next, suppose that $d \geq L$. Let $k$ be such that $L \leq k \leq d$. Then, we have

$$y(t+k) = -P_{L-1,y}(t+k-1) - \cdots - P_{0,y}(t+k-L).$$

That is, we can rewrite the term $y(t+k)$ as a linear combination of elements $y(\tau)$, where $\tau < t+k$. Performing this substitution recursively allows us to rewrite any term $y(t+k)$ in terms of $y(\tau)$, where $\tau < k + L$. In other words, we can define suitable $p \times p$ matrices $\Phi_{i,j}$ such that

$$Q_{\Phi}(y)(t) := \sum_{k=0}^d \sum_{l=0}^d y^{t+k}) \Phi_{k,\ell} y(t+l). \quad (24)$$

If we now define

$$\Phi := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,L-1} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{L-1,0} & \Phi_{L-1,1} & \cdots & \Phi_{L-1,L-1} \end{bmatrix}$$

it follows from (24) that $Q_{\Phi}(y) = Q_{\Phi}(y)$ for all $y \in B(P)$. Moreover, again by the fact that $B(P)[0, L-1] = \mathbb{R}^L$, we see that if $Q_{\Phi} \geq 0$ on $B(P)$ we have $\Phi \geq 0$, equivalently, $Q_{\Phi} \geq 0$.

A. Stability of autonomous AR systems

As noted, we require QDFs in order to study stability and Lyapunov theory in the context of autonomous systems represented by AR models. For this, we first define stability.

Definition 6: Let $P(\xi)$ be a nonsingular polynomial matrix. The corresponding autonomous system $P(\sigma)y = 0$ is called stable if $y(t) \to 0$ as $t \to \infty$ for all solutions $y$ on $\mathbb{Z}_+$.

Stability of autonomous AR systems can be characterized in terms of QDFs. In fact, the following proposition holds (see [12], [15]).

Proposition 7 (see [15, Theorem 1]): Let $P(\xi)$ be a nonsingular polynomial matrix. The corresponding autonomous system...
$P(\sigma)y = 0$ is stable if and only if there exists a QDF $Q_\Psi(y)$ such that $Q_\Psi \geq 0$ on $B(P)$ and $Q_{\mathcal{V}\Psi} < 0$ on $B(P)$.

For obvious reasons, we refer to $Q_\Psi$ as a Lyapunov function. In principle, the abovementioned theorem does not specify the degree of $Q_\Psi$. However, it turns out that if $P(\xi)$ is of the form as in (8) (with leading coefficient matrix the identity matrix) and the corresponding system $P(\sigma)y = 0$ of order $L$ is stable, there exists a Lyapunov function of degree at most $L - 1$.

Lemma 8: Let $P(\xi)$ be a polynomial matrix of the form (8). The corresponding autonomous system $P(\sigma)y = 0$ of order $L$ is stable if and only if there exists a QDF $Q_\Psi(y)$ of degree at most $L - 1$ such that $Q_\Psi \geq 0$ and $Q_{\mathcal{V}\Psi} < 0$ on $B(P)$.

Proof: We only need to prove the “only if” direction. By Proposition 7, there exists a QDF $Q_\Psi$ such that $Q_\Psi \geq 0$ on $B(P)$ and $Q_{\mathcal{V}\Psi} < 0$ on $B(P)$. By Lemma 5, there exists a QDF $Q_\Psi$ of degree at most $L - 1$ that is $B(P)$-equivalent to $Q_\Psi$ and $Q_\Psi \geq 0$. Finally, $Q_{\mathcal{V}\Psi}$ and $Q_{\mathcal{V}\Psi}$ are also $B(P)$-equivalent and therefore $Q_{\mathcal{V}\Psi} < 0$ on $B(P)$. 

V. DATA-DRIVEN STABILITY ANALYSIS OF AUTONOMOUS AR SYSTEMS

In order to provide a solid foundation for resolving the above-described stabilization problem, we first investigate the corresponding stability analysis problem. As such, in this section, we will take a more detailed look at the case that there are no control inputs, i.e., $m = 0$. In that case (6) reduces to

$$y(t + L) + P_{L-1}y(t + L - 1) + \cdots + P_1y(t + 1) + P_0y(t) = v(t)$$

and (7) to

$$P(\sigma)y = v$$

with $P(\xi)$ a nonsingular polynomial matrix. In this section, we will briefly discuss the notion of noisy data for this special case. In fact, in this case, we have only output data

$$y(0), y(1), \ldots, y(T)$$

on a time-interval $[0, T]$ with $T \geq L$. We assume that these data come from an unknown true system. Suppose this true system is represented by the unknown polynomial matrix $P_1(\xi)$, with $P_1(\xi)$ of the form (8). The true system is then represented by $P_1(\sigma)y = v$. Again we assume that the noise $v$ is unknown, but on the time interval $[0, T - L]$ its samples satisfy Assumption 4.

Any system in the model class of systems of the form (26) with fixed dimension $p$ and order $L$ is parametrized by its coefficient matrices $P_0, P_1, \ldots, P_{L-1}$. We collect these matrices in the $p \times pL$ matrix

$$P := \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}.$$  \hspace{1cm} (28)

Recalling that there are no control inputs, we have $w = y$. Therefore, we denote the Hankel matrix associated with the data as given by (16) by $H(y)$ and as before partition this matrix as

$$H(y) = \begin{bmatrix} H_1(y) \\ H_2(y) \end{bmatrix}$$

where $H_1(y)$ contains the first $pL$ rows and $H_2(y)$ the last $p$ rows. Also define

$$N := \begin{bmatrix} I & H_2(y) \\ 0 & H_1(y) \end{bmatrix} \Pi \begin{bmatrix} I & H_2(y) \\ 0 & H_1(y) \end{bmatrix}^\top.$$  \hspace{1cm} (29)

Then, as discussed in Section III, the autonomous system with coefficient matrices collected in the matrix $P$ is compatible with the data if and only if

$$\begin{bmatrix} I \\ P^\top \end{bmatrix} \begin{bmatrix} N \\ P^\top \end{bmatrix} \geq 0$$

equivalently $P^\top \in \mathcal{Z}_{pL}(N)$. Since the true system is assumed to be compatible with the data, the set $\mathcal{Z}_{pL}(N)$ is nonempty.

Our aim is to develop a test on the output data $y(0), y(1), \ldots, y(T)$ that determines whether the true system is stable (in the sense that if the noise $v = 0$ then all solutions $y$ tend to zero as time tends to infinity). As we saw in Section III, the data do not necessarily determine the true system uniquely.

Thus, we are forced to test stability for all systems that are compatible with the data, that is, for all systems for which the corresponding matrix $P$ [see (28)] is in $\mathcal{Z}_{pL}(N)$, where $N$ is given by (29).

In order to proceed, we will first express the existence of a Lyapunov function $Q_\Psi$ for the autonomous system $P(\sigma)y = 0$ in terms of a QMI. This QMI involves a symmetric matrix $\Psi$ of dimensions $pL \times pL$ leading to a Lyapunov function $Q_\Psi$, and the matrix $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$. Again, for ease of notation, we denote both the polynomial matrix and its coefficient matrix by $P$. Then, we have the following.

Theorem 9: Let $P(\xi) = I_{pL}^\xi + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0$ and let $P(\sigma)y = 0$ be the corresponding autonomous system. This system is stable if and only if there exists $\Psi \in \mathfrak{S}_{pL}$ such that $\Psi \geq 0$ and

$$\begin{bmatrix} I \\ -P \end{bmatrix}^\top \begin{bmatrix} 0_p & 0 \\ 0 & \Psi \end{bmatrix} \begin{bmatrix} 0_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ -P \end{bmatrix} < 0.$$  \hspace{1cm} (31)

Any such $\Psi$ defines a Lyapunov function $Q_\Psi$.

Proof: We first prove the “if” part by showing that the QDF $Q_\Psi$ associated with the matrix $\Psi$ is a Lyapunov function. Since $\Psi \geq 0$ we have $Q_\Psi \geq 0$ so by Proposition 7, it suffices to show that $Q_{\mathcal{V}\Psi} < 0$ on $B(P)$. As shown in (22), denote the matrix in the middle of (31) by $\nabla \Psi$. Let $y \in B(P)$. Then, for all $t \in \mathbb{Z}_+$ we have

$$y(t + L) + P_{L-1}y(t + L - 1) + \cdots + P_1y(t + 1) + P_0y(t) = 0.$$  \hspace{1cm} (30)

This implies

$$\begin{bmatrix} y(t) \\ \vdots \\ y(t + L) \end{bmatrix} = \begin{bmatrix} I \\ -P \end{bmatrix} \begin{bmatrix} y(t) \\ \vdots \\ y(t + L - 1) \end{bmatrix}$$
for all $t \in \mathbb{Z}_+$. Thus, we compute

$$Q_\psi(y)(t) = \begin{bmatrix} y(t) \\ \vdots \\ y(t+L) \\ y(t+L) \\ \vdots \end{bmatrix}^\top \nabla \Psi \begin{bmatrix} y(t) \\ \vdots \\ y(t+L) \\ y(t+L) \\ \vdots \end{bmatrix}.$$

This implies $Q_\psi(y)(t) \leq 0$ for all $t \in \mathbb{Z}_+$ and $Q_\psi(y)(t) = 0$ for all $t \in \mathbb{Z}_+$ if and only if $y(t) = 0$ for all $t \in \mathbb{Z}_+$, which shows that $Q_\psi < 0$ on $B(P)$.

Next, we turn to proving the “only if” part. Suppose the system is stable. According to Lemma 8, there exists a Lyapunov function $\Psi$ of degree at most $L - 1$ such that $Q_\psi \geq 0$. This QDF allows a coefficient matrix $\Psi \in \mathbb{S}^{pL}$, $\Psi \geq 0$. We claim that $\Psi$ satisfies (31). Indeed, take any $y_0, y_1, \ldots, y_{L-1}$ not all equal to zero. Clearly, since $B(P)|_{[0,L-1]} = \mathbb{R}^{pL}$ there exists $y \in B(P)$ such that $y(t) = y_t$, $t = 0, 1, \ldots, L - 1$. Finally

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{L-1} \\ y_0 \\ \vdots \\ y_{L-1} \end{bmatrix}^\top \begin{bmatrix} I \\ -P \\ \nabla \Psi \\ \vdots \\ \nabla \Psi \end{bmatrix} = Q_\psi(y)(0) < 0.$$

We now return to our problem of verifying stability on the basis of the output data. To this end, we give the following definition of informativity for quadratic stability.

**Definition 10:** The noisy output data $y(0), y(1), \ldots, y(T)$ are called informative for quadratic stability if there exists a matrix $\Psi \in \mathbb{S}^{pL}$ such that $\Psi \geq 0$ and the QMI (31) holds for all $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$ that satisfy the QMI (30), with $N$ defined by (29).

Informativity for quadratic stability, thus, implies that there exists a matrix $\Psi \in \mathbb{S}^{pL}$ such that the QDF $Q_\psi$ is a Lyapunov function for all systems that are compatible with the data, i.e., all systems in $Z_{pL}(N)$ are stable with a common Lyapunov function.

In the sequel, our aim is to establish necessary and sufficient conditions on the data $y(0), y(1), \ldots, y(T)$ to be informative in this manner. The idea is to apply the strict matrix $S$-lemma in Proposition 2 to obtain such conditions in the form of feasibility of an LMI. Note, however, that the QMI (30) is in terms of the matrix $P^\top$ whereas (31) is in terms of $P$. Therefore, immediate application of the matrix $S$-lemma is not possible. In this article, we will resolve this issue by reformulating the QMI (31) in terms of the variable $P^\top$. We first formulate the following instrumental lemma.

**Lemma 11:** Let $P(0) = I_{\xi^L} + P_{L-1} \xi^{L-1} + \cdots + P_1 \xi + P_0$ and, as before, let $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$. Define the $p(L-1) \times pL$ matrix $J$ by

$$J := \begin{bmatrix} 0_{p(L-1) \times p} & I_{p(L-1)} \end{bmatrix}.$$

Then, $\Psi$ satisfies (31) if and only if it satisfies the standard Lyapunov inequality

$$\begin{bmatrix} J \end{bmatrix}^\top \Psi \begin{bmatrix} J \end{bmatrix} - \Psi < 0.$$

Moreover, if $\Psi \geq 0$ satisfies (31) then $\Psi > 0$.

**Proof:** By inspection, it can be seen that (31) can equivalently be reformulated as (33). Suppose $\Psi \geq 0$ satisfies (31). It then immediately follows that

$$\Psi \geq \Psi - \begin{bmatrix} J \end{bmatrix}^\top \Psi \begin{bmatrix} J \end{bmatrix} > 0.$$

Using a Schur complement argument twice, the strict Lyapunov inequality (33) can be seen to be equivalent to

$$\Psi^{-1} - \begin{bmatrix} J \end{bmatrix}^\top \Psi^{-1} \begin{bmatrix} J \end{bmatrix} > 0, \quad \Psi > 0.$$

Using as an intermediate step that, obviously

$$\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} J \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \end{bmatrix}$$

it can be seen that (34) holds if and only if $\Psi > 0$ and

$$\begin{bmatrix} I_{pL} \\ 0 \end{bmatrix}^\top M \begin{bmatrix} I_{pL} \\ 0 \end{bmatrix} > 0$$

where the $2pL \times 2pL$ matrix $M$ is defined by

$$M := \begin{bmatrix} \Psi^{-1} - \begin{bmatrix} J \end{bmatrix}^\top \Psi^{-1} \begin{bmatrix} J \end{bmatrix} \\ \Psi^{-1} \begin{bmatrix} J \end{bmatrix} \end{bmatrix} - \Psi^{-1}.$$

From the above, we see that informativity for quadratic stability is equivalent to the existence of $\Psi > 0$ such that the QMI (35) holds for all coefficient matrices $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$ that satisfy the QMI (30). In terms of solutions sets of QMIs as discussed in Section II this can now be restated as

$$P^\top \in Z_{pL}(N) \implies P^\top \begin{bmatrix} 0 & -I_p \end{bmatrix} \in Z^+_{pL}(M)$$

or equivalently,

$$Z_{pL}(N) \begin{bmatrix} 0 & -I_p \end{bmatrix} \subseteq Z^+_{pL}(M).$$
In order to be able to apply the strict matrix S-lemma formulated in Proposition 2, we want to express the (projected) set on the left in (37) as the solution set of a QMI. To this end, define

\[
\bar{N} := \begin{bmatrix}
0 & -I_p \\
-I_p & 0 \\
0 & I_p L
\end{bmatrix}^T N \begin{bmatrix}
0 & -I_p \\
-I_p & 0 \\
0 & I_p L
\end{bmatrix}. \quad (38)
\]

Then, indeed, we have the following lemma.

**Lemma 12:** Assume that the Hankel matrix \( H_1(y) \) of depth \( L \) has full row rank. Then, \( Z_{pL}(N) \left[ \begin{array}{c} 0 \\ -I_p \\ 0 \\ I_p L \end{array} \right] = Z_{pL}(\bar{N}). \)

**Proof:** Note that \( N \) is partitioned as

\[
N = \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}
\]

with \( N_{22} = H_1(y) \Pi_{22} H_1(y)^T \). By Assumption 4, we have \( \Pi_{22} < 0 \) and, therefore, \( N_{22} < 0 \). The true system is compatible with the data, and therefore, \( Z_{pL}(N) \) is nonempty. By Proposition 1, we, thus, have that \( N \mid \bar{N} \geq 0 \). The result then follows from Proposition 3.

Summarizing our findings up to now, we see that under the assumption that \( H_1(y) \) has full row rank, informativity for quadratic stability is equivalent to the existence of \( \Psi > 0 \) such that the inclusion \( Z_{pL}(\bar{N}) \subseteq Z_{pL}^+ (M) \), holds. This inclusion is dealt with by Proposition 2.

**Lemma 13:** Let \( \Psi > 0 \) and let \( M \) be given by (36). Assume that \( H_1(y) \) has full row rank. Then, \( Z_{pL}(\bar{N}) \subseteq Z_{pL}^+ (M) \) if and only if there exist \( \alpha \geq 0 \) such that

\[
M - \alpha \bar{N} > 0. \quad (39)
\]

**Proof:** We check the conditions of Proposition 2 on \( \bar{N} \). Note that

\[
\bar{N} = \begin{bmatrix}
\bar{N}_{11} & \bar{N}_{12} \\
\bar{N}_{21} & \bar{N}_{22}
\end{bmatrix}
\]

\[
\bar{N}_{11} = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix} N_{11} \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}, \quad \bar{N}_{12} = \begin{bmatrix} 0 & -I_p \end{bmatrix} N_{12}.
\]

We have \( \bar{N}_{22} = H_1(y) \Pi_{22} H_1(y)^T < 0 \). Finally, the Schur complement \( N \mid \bar{N}_{22} \geq 0 \) since \( N \mid \bar{N}_{22} \geq 0 \).

In order to combine the above, recall that informativity for quadratic stability is equivalent to (37). Applying, in turn, Lemmas 12 and 13, we obtain an equivalent characterization of informativity for quadratic stability is equivalent as the existence of a scalar \( \alpha \geq 0 \) and a matrix \( \Psi > 0 \) such that (39) holds. Note that due to the negative definite lower right block in \( M \), the scalar \( \alpha \) is necessarily positive. By scaling the inequality (39), we can, therefore, take \( \alpha = 1 \). Putting \( \Phi := \Psi^{-1} \) we then finally obtain the following necessary and sufficient condition in terms of feasibility of an LMI. Recall the definition (32) of the matrix \( J \).

**Theorem 14:** Let \( \bar{N} \) be given by (38), where \( N \) is defined by (29). Assume that \( H_1(y) \) has full row rank. Then, the output data \( y(0), y(1), \ldots, y(T) \) are informative for quadratic stability if and only if there exists \( \Phi \in \mathbb{S}^{pL} \) such that \( \Phi > 0 \) and

\[
\begin{bmatrix}
\Phi - [J] & [J] \\
[0] & [0]
\end{bmatrix} - \begin{bmatrix}
\bar{N} - [0] & [0] \\
[0] & [0]
\end{bmatrix} > 0. \quad (41)
\]

In that case the QDF \( Q_{\Phi} \) with \( \Psi := \Phi^{-1} \) is a Lyapunov function for all systems of the form (26) compatible with the data.

**Remark 15:** Note that the size of the LMI (41) is \( 2pL \) whereas the number of unknowns is \( \frac{1}{2} pL(pL + 1) \). These are independent of the length \( T + 1 \) of the interval on which the input–output data are collected, and only depend on the order of the system and the number of outputs.

**VI. DATA-DRIVEN STABILIZATION OF INPUT–OUTPUT AR SYSTEMS**

In this section, we will discuss data-driven stabilization of input–output systems in AR form. We will work in the setup of Section III, with systems of the form (6), or equivalently (7), with polynomial matrices as in (8) of given degree \( L \). We assume that we have noisy input–output data \( u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T) \) on the interval \([0, T]\) with \( T \geq L \). These are samples of \( u \) and \( y \) obtained from the unknown true system (12).

The noise \( v \) is unknown, but its samples are assumed to satisfy Assumption 4. Recall that we can identify systems in the model class by their corresponding matrix \( R \) as shown in (15). Moreover, the system with coefficient matrices collected in \( R \) is compatible with the data if and only if

\[
R^T \in Z_{q L}(N).
\]

As discussed, we are interested in finding a stabilizing controller for all systems corresponding to elements in this set. In particular, we will consider controllers of essentially the same form as the plant. Thus, linear systems in AR form of degree \( L \), with leading coefficient \( I \). Of course, we will consider controllers such that the inputs of the controller are the outputs of the plant and vice-versa. We will assume that the controller is also strictly proper. To be precise, the controller will be taken to be of the form

\[
G(\sigma)u = F(\sigma)y \quad (42)
\]

with

\[
G(\xi) = I L + G_{L-1} L^{-1} + \ldots + G_1 L + G_0 \\
F(\xi) = F_{L-1} L^{-1} + \ldots + F_1 L + F_0.
\]

Here, the leading coefficient matrix of \( G(\xi) \) is assumed to be the \( m \times m \) identity matrix and \( G_i \in \mathbb{R}^{m \times m} \), and \( F_i \in \mathbb{R}^{m \times p} \) for \( i = 0, 1, \ldots, L - 1 \). The closed-loop system obtained by interconnecting a system of the form (7) and the controller is represented by

\[
\begin{bmatrix}
G(\sigma) - F(\sigma) \\
-Q(\sigma) & P(\sigma)
\end{bmatrix} \begin{bmatrix}
u \\
I_p
\end{bmatrix} = \begin{bmatrix}
0 \\
[0]
\end{bmatrix}. \quad (43)
\]
Since the leading coefficient matrix is the \( q \times q \) identity matrix, the controlled system with noise equal to zero is autonomous. We call controller (42) a stabilizing controller if the controlled system (43) is stable, in the sense that if \( v = 0 \), then all solutions \( u \) and \( y \) tend to zero as time tends to infinity. Now define

\[
C(\xi) := \begin{bmatrix} G(\xi) & -F(\xi) \end{bmatrix}
\]

and recall that \( \mathbf{w} = \text{col}(u, y) \). Then, (43) can equivalently be written as

\[
\begin{bmatrix} C(\sigma) \\ R(\sigma) \end{bmatrix} \mathbf{w} = \begin{bmatrix} 0 \\ \mathbf{I}_p \end{bmatrix} \mathbf{v}.
\]  

(44)

Collect the coefficient matrices of \( F(\xi) \) and \( G(\xi) \) in the matrix \( C \) defined by

\[
C := \begin{bmatrix} G_0 & -F_0 & G_1 & -F_1 & \cdots & G_{L-1} & -F_{L-1} \end{bmatrix}
\]

(45) and recall definition (15) of the matrix \( R(\xi) \). Recall that the leading coefficient matrix of \( \begin{bmatrix} C(\xi)^\top & R(\xi)^\top \end{bmatrix} \) is the \( q \times q \) identity matrix. Furthermore, the matrix \( \begin{bmatrix} C^\top & R^\top \end{bmatrix} \) collects the remaining coefficient matrices.

An immediate application of Theorem 9 then yields the following.

**Lemma 16:** The controlled system (44) is stable if and only if there exists \( \Psi \in \mathbb{S}^{qL} \) such that \( \Psi \geq 0 \) and

\[
\begin{bmatrix} I_{qL}^\top \\ C \\ -R \end{bmatrix} \begin{bmatrix} 0_q & 0 \\ 0 & \Psi \\ 0 & 0_{q_L} \end{bmatrix} \begin{bmatrix} I_{qL} \\ -C \\ -R \end{bmatrix} < 0.
\]  

(46)

Moreover, if \( \Psi \geq 0 \) satisfies (46), then \( \Psi > 0 \).

This leads to the following definition of informativity for quadratic stabilization.

**Definition 17:** The noisy input–output data \( u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T) \) are called informative for quadratic stabilization if there exist \( C \in \mathbb{R}^{m \times qL} \) and \( \Psi \in \mathbb{S}^{qL} \) with \( \Psi \geq 0 \) such that the QMI (46) holds for all \( R \) that satisfy the QMI (20), with \( N \) defined by (19).

Informativity for quadratic stabilization thus means that there exist a controller \( C(\sigma) \mathbf{w} = 0 \) (equivalently, \( G(\sigma)u = F(\sigma)y \)) and a matrix \( \Psi \in \mathbb{S}^{qL} \) such that the QDF \( \mathbf{G} \) is a common Lyapunov function for all closed-loop systems obtained by interconnecting the controller with an arbitrary system that is compatible with the data.

In this article, we will derive necessary and sufficient conditions for informativity for quadratic stabilization. Similar to Section V, the QMI (20) is in terms of the matrix \( R^\top \) whereas (46) is in terms of \( R \). We will, therefore, first reformulate the QMI (46) in terms of the variable \( R^\top \).

Define the \( q(L-1) \times qL \) matrix \( J \) by

\[
J := \begin{bmatrix} 0_{q(L-1) \times q} & I_{q(L-1)} \end{bmatrix}.
\]  

(47)

By Lemma 11, \( \Psi \in \mathbb{S}^{qL}, \Psi \geq 0 \) satisfies (46) if and only if \( \Psi > 0 \) and satisfies the strict Lyapunov inequality

\[
\begin{bmatrix} J \\ -C \\ -R \end{bmatrix} \Psi \begin{bmatrix} J \\ -C \\ -R \end{bmatrix}^\top < 0
\]

which is equivalent to

\[
\Psi^{-1} - \begin{bmatrix} J \\ -C \\ -R \end{bmatrix} \Psi^{-1} \begin{bmatrix} J \\ -C \\ -R \end{bmatrix}^\top > 0, \quad \Psi > 0.
\]  

(48)

By writing

\[
\begin{bmatrix} J \\ -C \\ -R \end{bmatrix} = \begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ R \end{bmatrix}
\]

it can be seen that (48) holds if and only if \( \Psi > 0 \) and

\[
R^\top \begin{bmatrix} I_{qL} \\ 0 \\ 0 \end{bmatrix} M R^\top \begin{bmatrix} I_{qL} \\ 0 \\ 0 \end{bmatrix} > 0
\]

(49)

where the \( 2qL \times 2qL \) matrix \( M \) is defined by

\[
M := \begin{bmatrix} \Psi^{-1} - \begin{bmatrix} J \\ -C \end{bmatrix} \Psi^{-1} \begin{bmatrix} J \\ -C \end{bmatrix}^\top & \Psi^{-1} \\ \begin{bmatrix} J \\ -C \end{bmatrix} \Psi^{-1} & -\Psi^{-1} \end{bmatrix}.
\]  

(50)

Thus, we see that informativity for quadratic stabilization is equivalent to the existence of an \( m \times qL \) matrix \( C \) and a matrix \( \Psi \in \mathbb{S}^{qL}, \Psi > 0 \) such that the QMI (49) holds for all coefficient matrices \( R \) that satisfy the QMI (20). The matrix \( C \) is then the coefficient matrix of a suitable controller. In terms of solutions sets of QMIs this can be restated as

\[
R^\top \in \mathbb{Z}_{qL}(N) \implies R^\top \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} \in \mathbb{Z}_{qL}^+(M)
\]

or equivalently

\[
\mathbb{Z}_{qL}(N) \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} \subseteq \mathbb{Z}_{qL}^+(M).
\]  

(51)

As before, in order to be able to apply the strict matrix S-lemma in Proposition 2, we want to express the set on the left in (51) as the solution set of a QMI. Define the \( 2qL \times 2qL \) matrix \( \bar{N} \) by

\[
\bar{N} := \begin{bmatrix} 0 & 0 & -I_p \\ 0 & I_{qL} \end{bmatrix} N \begin{bmatrix} 0 & 0 & -I_p \\ 0 & I_{qL} \end{bmatrix}.
\]  

(52)

Then, we have the following lemma.

**Lemma 18:** Assume that the Hankel matrix \( H_1(\mathbf{w}) \) has full row rank. Then, \( \mathbb{Z}_{qL}(N) \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} = \mathbb{Z}_{qL}(\bar{N}) \).

**Proof:** The proof is similar to that of Lemma 12.
From the above, we see that, under the assumption that $H_1(w)$ has full row rank, informativity for quadratic stabilization requires the existence of $C$ and $\Psi > 0$ such that the inclusion $Z_{qL}(\tilde{N}) \subseteq Z_{qL}^+(M)$. This inclusion is dealt with by Proposition 2.

Lemma 19: Let $\Psi > 0$, $C \in \mathbb{R}^{m \times qL}$ and let $M$ be given by (50). Assume that $H_1(w)$ has full row rank. Then $Z_{qL}(\tilde{N}) \subseteq Z_{qL}^+(M)$ if and only if there exists a scalar $\alpha \geq 0$ such that

$$M - \alpha \tilde{N} > 0.$$  

Proof: The proof is similar to that of Lemma 13.

Note that the unknowns $C$ and $\Psi$ appear in the matrix $M$ in a nonlinear way, and even in the form of an inverse. By putting $\Phi := \Psi^{-1}$ we get rid of the inverse, and rewrite the condition $M - \alpha \tilde{N} > 0$ as

$$\Phi - \begin{bmatrix} J & -C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J & -C \\ 0 & 0 \end{bmatrix}^\top \Phi - \begin{bmatrix} J & -C \\ 0 & 0 \end{bmatrix} - \alpha \tilde{N} > 0.$$  

Thus, informativity for quadratic stabilization holds if and only if there exists $\tilde{N} > 0$, a matrix $C$, and a scalar $\alpha \geq 0$ such that (54) holds. Note that $\alpha$ must be positive due to the negative definite lower right block in (54). By scaling $\Phi$, we can, therefore, take $\alpha = 1$. By introducing the new variable $D := -C\Phi$ and taking a suitable Schur complement, (54) can then be reformulated as the following LMI in the unknowns $\Phi$ and $D$:

This then immediately leads to the following characterization of informativity for quadratic stabilization and a method to compute a suitable feedback controller together with a common Lyapunov function.

Theorem 20: Assume that $H_1(w)$ has full row rank. Let the matrix $\tilde{N}$ be given by (52), with $N$ defined by (19). Then, the input–output data $u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T)$ are informative for quadratic stabilization if and only if there exist matrices $D \in \mathbb{R}^{m \times qL}$ and $\Phi \in \mathbb{S}^{qL}$ such that $\Phi > 0$ and the LMI (55) holds. Moreover, if this holds, the feedback controller with coefficient matrix $C := -D\Phi^{-1}$ stabilizes all systems of the form (7) that are compatible with the input–output data.

Moreover, the QDF $Q_\Phi$ with $\Psi := \Phi^{-1}$ is a common Lyapunov function for all closed-loop systems.

Remark 21: Thus, in order to compute an AR representation of the controller that stabilizes all systems compatible with the data.

1. Compute the matrix $\tilde{N}$ using the Hankel matrix associated with the data.
2. Check feasibility of the LMI (55), and if it is feasible, compute $D$ and $\Phi$.
3. Let $C = -D\Phi^{-1}$, and partition

$$C = \begin{bmatrix} G_0 & -F_0 & G_1 & -F_1 & \cdots & G_{L-1} & -F_{L-1} \end{bmatrix}$$

with $F_i \in \mathbb{R}^{m \times p}$ and $G_i \in \mathbb{R}^{m \times m}$.

4. Define $F(\xi) := F_{L-1}\xi^{L-1} + \cdots + F_0$ and $G(\xi) := I\xi^L + G_{L-1}\xi^{L-1} + \cdots + G_0$.

Now, the AR representation of the desirable controller is:

$$G(\sigma)u = F(\sigma)y.$$  

VII. REDUCTION OF COMPUTATIONAL COMPLEXITY

In this section, we will again take a look at the data-driven stabilization problem. In Section VI, we showed that finding a controller that stabilizes all systems that are compatible with the data requires checking feasibility of the LMI (55). The size of this LMI is $3qL$, while the number of unknowns is $\frac{1}{2}qL(qL + 2m + 1)$, both independent of the time horizon $T$. The unknowns in the LMI (55) are the matrices $\Phi$ and $D$ that together lead to a controller and a common Lyapunov function. In this section, we will decouple the computation of the common Lyapunov function from that of the controller. This will lead to checking feasibility of an LMI of smaller size and with a smaller number of unknowns.

In order to proceed, we will need the following lemma.

Lemma 22: Let $\Pi \in \Pi_{q,n}$ and let $W \in \mathbb{R}^{q \times p}$ have full column rank. Let $Y \in \mathbb{R}^{r \times p}$, then, there exists a matrix $Z \in \mathbb{R}^{r \times q}$ such that

1) $Z \in Z_{qL}^+(\Pi)$;
2) $ZW = Y$;

if and only if $\Pi \Pi_{22} > 0$ and $Y \in Z_{qL}^+(\Pi_W)$. If these two conditions hold and, in addition, $\Pi_{22} < 0$ then the matrix

$$Z := -\Pi_{22}^{-1} \Pi_{21} + (Y + \Pi_{22}^{-1} \Pi_{21} W)(\Pi_{22} W)^{-1}(\Pi_{22})^\top$$

(56)
satisfies 1) and 2).

Proof: For the "only if" part, note that 1) implies nonemptiness of $Z_{qL}^+(\Pi)$ and, hence, $\Pi \Pi_{22} > 0$. Moreover, $Z \in Z_{qL}^+(\Pi)$ and $ZW = Y$ clearly imply that $Y \in Z_{qL}^+(\Pi_W)$. Conversely, if $\Pi \Pi_{22} > 0$ and $W$ has full column rank, then by Proposition 3, we have $Z_{qL}^+(\Pi)W = Z_{qL}^+(\Pi_W)$, which proves the claim. Next, under the assumption that $\Pi_{22} < 0$, by Proposition 1 any $Z \in Z_{qL}^+(\Pi)$ can be written as

$$Z = -\Pi_{22}^{-1} \Pi_{21} + (-\Pi_{22})^{-\frac{1}{2}} S (\Pi \Pi_{22})^{-\frac{1}{2}} S^\top S < I.$$  

Thus, there exist $S$ with $S^\top S < I$ such that

$$Y = -\Pi_{22}^{-1} \Pi_{21} W + (-\Pi_{22})^{-\frac{1}{2}} S (\Pi \Pi_{22})^{-\frac{1}{2}} W.$$  

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equivalently
\[
(-\Pi_{22})^\frac{1}{2} Y - (-\Pi_{22})^{-\frac{1}{2}} \Pi_{21} W = S (\Pi | \Pi_{22})^\frac{1}{2} W.
\]
Hence, by [18, Lemma A.1], the matrix
\[
S = \left( (-\Pi_{22})^\frac{1}{2} Y - (-\Pi_{22})^{-\frac{1}{2}} \Pi_{21} W \right) (\Pi | \Pi_{22})^{-\frac{1}{2}} W ^\dagger
\]
does the job. It can be shown that
\[
(\Pi | \Pi_{22})^{-\frac{1}{2}} W ^\dagger = (\Pi W | \Pi_{22})^\dagger W ^\top (\Pi | \Pi_{22})^{-\frac{1}{2}}.
\]
Therefore, \(S\) is equal to
\[
\left( (-\Pi_{22})^\frac{1}{2} Y - (-\Pi_{22})^{-\frac{1}{2}} \Pi_{21} W \right) (\Pi W | \Pi_{22})^\dagger W ^\top (\Pi | \Pi_{22})^{-\frac{1}{2}}.
\]
Plugging this expression for \(S\) into the formula of \(Z\) then finally yields (56).

Now consider inequality (54) and recall that the existence of \(\Phi > 0\) and \(C\) satisfying this inequality with \(\alpha = 1\) is equivalent to informativity for quadratic stabilization. We can reformulate (54) as
\[
\begin{bmatrix}
I_{qL} & 0 & 0 & 0 & 0 \\
0 & I_{qL} & \Phi & 0 & -\bar{N} \\
J & -C & I_{qL} & 0 & 0 \\
0 & 0 & 0 & I_{qL} & 0 \\
0 & 0 & 0 & -\Phi & I_{qL}
\end{bmatrix} > 0.
\]
Then, by applying Lemma 22, we now obtain necessary and sufficient conditions for informativity for quadractic stabilization, together with a formula for a stabilizing controller. Define the \(2qL \times (2qL - m)\) matrix \(W\) by
\[
W := \begin{bmatrix}
I_{q(L-1)} & 0 & 0 & 0 \\
0 & 0 & m \times p & 0 \\
0 & I_{p} & 0 & 0 \\
0 & 0 & 0 & I_{qL}
\end{bmatrix}.
\]
In addition, partition the matrix \(\bar{N}\) as in (40), where \(\bar{N}_{11}\) and \(\bar{N}_{22}\) are in \(\mathbb{S}^{qL}\) and \(\bar{N}_{12} = \bar{N}_{21} = \bar{N}_{12}^\top \in \mathbb{R}^{qL \times qL}\).

**Theorem 23:** Assume that \(H_1^L(w)\) has full rank. Let the matrix \(\bar{N}\) be given by (52), with \(N\) defined by (19). Then, the input–output data \(u(0), u(1), \ldots, u(T), y(0), y(1), \ldots, y(T)\) are informative for quadratic stabilization if and only if there exists \(\Phi \in \mathbb{S}^{qL}\) such that
\[
\Phi > \bar{N} | \bar{N}_{22}
\]
and
\[
\begin{bmatrix}
J^\top & -C^\top & 0 & I_{qL}
\end{bmatrix} \Delta W ^\top \begin{bmatrix}
\Phi & 0 & 0 \\
0 & 0 & -\bar{N}
\end{bmatrix} \begin{bmatrix}
0_{q(L-1) \times m} \\
I_{m}
\end{bmatrix} = 0.
\]
Moreover, if \(\Phi\) satisfies these two LMIs, then the controller with coefficient matrix \(C\) defined by (CONT) shown at the bottom of the next page, satisfies (57). As a consequence, this controller stabilizes all systems compatible with the data, and the resulting closed-loop systems have common Lyapunov function \(Q_\phi\) with \(\Psi := \Phi^{-1}\).

**Proof:** We first prove the “only if” statement. Since \(\Phi > 0\), it follows immediately from (57) that
\[
\begin{bmatrix}
\Phi & 0 \\
0 & 0
\end{bmatrix} - \bar{N} > 0.
\]
In turn, this implies \(\Phi > \bar{N} | \bar{N}_{22}\). By multiplying (57) from the right by \(W\) and from the left by its transpose, we obtain inequality (59).

To prove the converse implication, recall that \(\bar{N} | \bar{N}_{22} > 0\). Hence, it follows from (58) that \(\Phi > 0\), and, using the fact that \(\bar{N}_{22} < 0\), that (60) holds. From this it follows that the matrix \(\Pi\) defined by
\[
\Pi := \begin{bmatrix}
\Phi & 0 \\
0 & 0 \\
0 & -\Phi
\end{bmatrix}
\]
is in the set \(\Pi_{2qL,qL}\). Then, applying Lemma 22 to \(\Pi, W\) and \(Y := [J^\top & 0 & I_{qL}]\) shows that there exists a matrix \(C\) such that (57) is satisfied. In other words, the data are informative for quadratic stabilization.

Finally, we will prove the formula (CONT) for \(C^\top\). To this end, again apply Lemma 22 to \(\Pi, W,\) and \(Y = [J^\top & 0 & I_{qL}]\). Introduce the shorthand notation
\[
\Delta := \left( W ^\top \begin{bmatrix}
\Phi & 0 \\
0 & 0 \\
0 & -\bar{N}
\end{bmatrix} W \right)^{-1}.
\]
By (56), a “structured” element \([J^\top & -C^\top & 0 & I_{qL}]\) in the set \(\mathcal{Z}^{qL}_{qL}(\Pi)\) is given by
\[
[J^\top & -C^\top & 0 & I_{qL}] = [J^\top & 0 & I_{qL}] \Delta W ^\top \begin{bmatrix}
\Phi & 0 \\
0 & 0
\end{bmatrix} - \bar{N}.
\]
As such, a controller is given by
\[
C^\top = [J^\top & 0 & I_{qL}] \Delta W ^\top \begin{bmatrix}
\Phi & 0 & 0 \\
0 & 0 & -\bar{N}
\end{bmatrix} \begin{bmatrix}
0_{q(L-1) \times m} \\
I_{m}
\end{bmatrix}.
\]
It is easily verified that
\[
\bar{N} \begin{bmatrix}
0_{q(L-1) \times m} \\
I_{m}
\end{bmatrix} = 0.
\]
Thus, we conclude that \(C^\top\) is given by (CONT) as claimed.

**Remark 24:** Note that we have indeed managed to reduce the size and the number of unknowns. The total size of the LMIs (58) and (59) is equal to \(3qL - m\), whereas the number of unknowns has been reduced to \(\frac{1}{2}qL(qL + 1)\). The computation of the controller has been decoupled from that of \(\Phi\). Indeed, a stabilizing controller is now computed using (CONT) in terms of \(\Phi\).
In this example, we will obtain a stabilizing controller for an inverted pendulum on a cart from collected measurements.

We consider a standard inverted pendulum on a cart as depicted in Fig. 1. Here, \( m \) and \( \ell \) denote the mass and length of the pendulum. The mass and coefficient of friction of the cart are denoted by \( M \) and \( b \). Lastly, we consider the following variables: the horizontal displacement of the cart is given by \( x \), the angle of the pendulum from the (unstable) equilibrium is \( \phi \), and the force applied to the cart is denoted by \( u \). Assuming that \( M, m, \) and \( \ell \) are nonzero, it is straightforward to derive the following equations of motion:

\[
(M + m)\ddot{x} + b\dot{x} - m\ell\ddot{\phi}\cos(\phi) + m\ell^2\dot{\phi}^2\sin(\phi) = u
\]

\[
\ell\ddot{\phi} - g\sin(\phi) = \ddot{x}\cos(\phi).
\]

To linearize, assume that \( \phi \) and \( \dot{\phi} \) are small, which implies that \( \cos(\phi) \approx 1, \sin(\phi) \approx \phi, \) and \( \dot{\phi}^2 \approx 0 \). Then

\[
(M + m)\ddot{x} + b\dot{x} - m\ell\ddot{\phi} - m\ell\delta = u, \quad (I + m\ell^2)\ddot{\phi} + mg\ell\dot{\phi} - m\ell\ddot{x} = 0.
\]

Assuming that \( I(M + m) + Mm\ell^2 \neq 0 \) and employing the standard discretization

\[
\dot{x}(t) \approx \frac{x(t + \delta) - x(t)}{\delta}, \quad \dot{\phi}(t) \approx \frac{x(t + 2\delta) - 2x(t + \delta) + x(t)}{\delta^2}
\]

and performing some algebraic manipulation allows us to obtain the linear discrete time model in (61) shown at the bottom of this page. After incorporating an additive noise term \( v(t) = (v_1(t) v_2(t))^T \) in (61), we obtain an input–output system in AR form as in (7), where \( L = 2 \) and

\[
y = \begin{pmatrix} x \\ \phi \end{pmatrix}.
\]

For this example, we let the parameters take the following values:

\[
M = 1 \text{ kg, } \quad m = 0.7 \text{ kg, } \quad b = 0.1 \frac{N}{m}, \quad g = 9.8 \frac{m}{s^2}, \quad l = 0.5 \text{ m, } \quad \delta = 0.01 \text{ s}.
\]

The resulting system (61) with additive unknown noise term \( v \) will now be considered as the “true,” unknown system.

### A. Measurements From the Linearization

In the first simulation example, we collect measurements from the noisy linearized system, i.e., system (61) with additive noise. We take \( T = 20 \), provide 2 initial conditions, and generate random inputs from the interval \([-1, 1]\).

As for the matrix of noise samples \( V \), we will assume a noise model of the form (13) by considering \( VV^T \leq \epsilon I_2 \). Note that, in order to discretize the system and make the leading coefficient equal to \( I_2 \), the dynamics were multiplied by a factor of \( \delta^2 \).

Indeed, it is seen in (61) that the effect of the input \( u \) on the dynamics is proportional to \( \delta^2 \). Therefore, it is reasonable to assume that the same holds for the noise signal \( v \). Consequently, \( \epsilon \) can be assumed to be proportional to \( \delta^2 \). In the present example, we, therefore, take \( \epsilon = 10^{-2}\delta^2 \).

We now generate a random noise signal that satisfies the noise model and apply the initial conditions, inputs and noise to the linearized system (61) with added noise. The measurements resulting from this are shown in (62) shown at the bottom of next page.

We will use Theorem 20 to show that these measurements are informative for quadratic stabilization. For this, we first form the matrices \( H_1', H_2' \), and \( \bar{N} \). It is straightforward to see that \( H_1' \) has full row rank. We now use Yalmip with Mosek as a solver in order to find matrices \( D \in \mathbb{R}^{1 \times 6} \), and \( \Phi \in \mathbb{S}^6 \), such that \( \Phi > 0 \) and the LMI (55) holds. Indeed, such matrices exist, and therefore, the data are informative for quadratic stabilization. We can find a stabilizing controller by taking \( C = -D\Phi^{-1} \), which results in

\[
[0.76 \quad 29168.72 \quad -18360.21 \quad 0.68 \quad -29515.03 \quad 19264.40].
\]

This corresponds to the controller of the form (42) given by

\[
u(t+2) = 0.68u(t+1) + 0.76u(t) = 29515.03x(t+1) - 19264.40\phi(t+1) - 29168.72x(t) + 18360.21\phi(t).
\]
The large difference in magnitude of the gains corresponding to $x$ and $\phi$ and those corresponding to $u$ is caused by the discretization.

In Fig. 2, we can see the results of applying this controller to the linear discretized model, with noise $v = 0$. To be precise, we plot both $x$ [see Fig. 2(a)] and $\phi$ [see Fig. 2(b)] for 200 steps originating from a given initial condition. This illustrates that the controller stabilizes the linearized system, as was guaranteed by Theorem 20.

B. Measurements From the Nonlinear System

In this example, instead of measuring the linear system (61) with a bounded noise term, we will perform measurements on the (discretized) nonlinear system directly. This means that we interpret the noise term $v(t)$ of the linear system as the effect of the nonlinearities. Again, we provide 2 initial conditions and take $T = 20$. We will generate measurements close to the equilibrium, in order to keep the effect of the nonlinearities relatively small. As such, we will assume that $VV^T \leq 10^{-4} \delta^4 I_2$, which we will validate experimentally.

We now generate random inputs from the interval $[-1, 1]$ and apply them to the nonlinear system with the given initial conditions. The measurements resulting from applying this can be seen in (63) shown at the top of next page. For the sake of simulations, we note that the effects of the nonlinearities for these initial conditions and inputs, as captured in the matrix $V$, indeed satisfy the assumed noise model.

Similar to earlier, we note that $H_1^T$ has full row rank, and we can find $\Phi$ and $D$ such that (55) holds. This means that

$$
\begin{bmatrix}
y(0) & \cdots & y(10)
\end{bmatrix} = 
\begin{bmatrix}
0.1000 & 0.1010 & 0.1020 & 0.1029 & 0.1039 & 0.1050 & 0.1061 & 0.1072 & 0.1084 & 0.1096 & 0.1108 \\
0.1000 & 0.0990 & 0.0981 & 0.0974 & 0.0969 & 0.0969 & 0.0970 & 0.0974 & 0.0982 & 0.0991 & 0.1003
\end{bmatrix}
$$

$$
\begin{bmatrix}
y(11) & \cdots & y(20)
\end{bmatrix} = 
\begin{bmatrix}
0.1121 & 0.1134 & 0.1149 & 0.1165 & 0.1182 & 0.1200 & 0.1219 & 0.1238 & 0.1258 & 0.1277 \\
0.1017 & 0.1035 & 0.1058 & 0.1085 & 0.1116 & 0.1153 & 0.1192 & 0.1235 & 0.1279 & 0.1327
\end{bmatrix}
$$

$$
\begin{bmatrix}
u(0) & \cdots & u(9)
\end{bmatrix} = 
\begin{bmatrix}
-0.9960 & -0.7388 & -0.6322 & 0.6612 & -0.8090 & -0.2520 & 0.1023 & -0.7179 & -0.0428 & -0.8528
\end{bmatrix}
$$

$$
\begin{bmatrix}
u(10) & \cdots & u(19)
\end{bmatrix} = 
\begin{bmatrix}
0.4309 & 0.6413 & 0.6225 & 0.2019 & 0.7475 & -0.1559 & -0.5855 & -0.7585 & -0.3562 & -0.6643
\end{bmatrix}
$$

(62)
the data are informative for quadratic stabilization. By taking $C = -D\Phi^{-1}$, we obtain
\[
\begin{bmatrix}
y(0) & \cdots & y(10)
\end{bmatrix} = \begin{bmatrix}
0.1000 & 0.1010 & 0.1020 & 0.1029 & 0.1040 & 0.1050 & 0.1061 & 0.1070 & 0.1081 & 0.1090 & 0.1100 \\
0.0400 & 0.0390 & 0.0380 & 0.0371 & 0.0364 & 0.0358 & 0.0353 & 0.0347 & 0.0342 & 0.0337 & 0.0333
\end{bmatrix}
\]
\[
\begin{bmatrix}
y(11) & \cdots & y(20)
\end{bmatrix} = \begin{bmatrix}
0.1111 & 0.1121 & 0.1132 & 0.1143 & 0.1155 & 0.1167 & 0.1180 & 0.1192 & 0.1205 & 0.1218 \\
0.0332 & 0.0331 & 0.0330 & 0.0332 & 0.0336 & 0.0340 & 0.0346 & 0.0352 & 0.0363 & 0.0370
\end{bmatrix}
\]
\[
\begin{bmatrix}
u(0) & \cdots & u(9)
\end{bmatrix} = \begin{bmatrix}
-0.6358 & -0.2516 & 0.1650 & -0.1941 & -0.4534 & -0.8523 & 0.1926 & -0.6554 & -0.0237 & 0.3687
\end{bmatrix}
\]
\[
\begin{bmatrix}
u(10) & \cdots & u(19)
\end{bmatrix} = \begin{bmatrix}
-0.0440 & -0.2577 & 0.3739 & 0.5910 & -0.1870 & 0.2488 & -0.6610 & 0.7050 & -0.3692 & 0.1016
\end{bmatrix}.
\]
(63)

As before, we apply the resulting controller to both the discretization of the nonlinear model and its linearization (61) without noise. For both models and a given initial condition the values of the position of the cart for 200 steps are shown in Fig. 2(c). In Fig. 2(d), we show the corresponding angles of the pendulum for the same interval of time.

**IX. CONCLUSION**

In this article, we have studied data-driven stability analysis and feedback stabilization of linear input–output systems in AR form. On the basis of noisy input–output data obtained from some unknown “true” AR system, it is in general not possible to identify this system uniquely. Indeed, we have shown that a given set of data gives rise to a whole set of systems that are compatible with these data, and this set is equal to the solution set of a certain QMI that is given in terms of the data and the noise model that we use. Next, in order to study stability and feedback stabilization, we have given a characterization of asymptotic stability of systems in AR form using QDFs as a framework for Lyapunov functions of autonomous AR systems. This has led to necessary and sufficient conditions for stability in terms of a second, strict, QMI. We have then defined informativity for quadratic stabilization as the property that all systems whose coefficient matrix satisfy the first QMI above also satisfy the second, strict, QMI. This has the interpretation that all systems that are compatible with the given data are stable with a common Lyapunov function. Using a version of the so-called strict matrix $S$-lemma, this set inclusion has been characterized in terms of feasibility of a strict LMI, again given in terms of the data. As shown in Theorem 9, feasibility of this LMI is then equivalent to the fact that all system compatible with the data are stable with a common Lyapunov function, so in particular the unknown “true” system is stable. Subsequently, we have used this framework to study data-driven stabilization. We have defined informativity of the given data for quadratic stabilization as the property that there exists a single feedback controller that stabilizes all systems that are compatible with the data, while leading to a common Lyapunov function for all closed-loop systems. We have shown in Theorem 14 that, again, this property can be characterized in terms of feasibility of a strict LMI that is given in terms of the data. Solutions of this LMI then immediately yield a controller together with a common Lyapunov function. In order to reduce the size of the LMIs and the number of variables, we have also deduced Theorem 23. This provides an alternative characterization of informativity in terms of feasibility of an LMI of reduced size. Finally, our results have been illustrated using an example in which noisy input–output data are used to compute a stabilizing controller for an inverted pendulum set-up.

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