Random magnetic field and quasi-particle transports in the mixed state of high $T_c$ cuprates

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By a singular gauge transformation, the quasi-particle transport in the mixed state of high $T_c$ cuprates is mapped into charge-neutral composite Dirac fermion moving in short-range correlated random scalar and long-range correlated vector potential. A fully quantum mechanical approach to longitudinal and transverse thermal conductivities is presented. The semi-classical Volovik effect is presented in a quantum mechanical way. The quasi-particle scattering from the random magnetic field which was completely missed in all the previous semi-classical approaches is the dominant scattering mechanism at sufficient high magnetic field. The implications for experiments are discussed.

The general problem of quasi-particle transport in the mixed state of high $T_c$ cuprates is important, because simultaneous measurements of thermal conductivities $\kappa_{xx}$ and $\kappa_{xy}$ provide a lot of information on the new physics pertinent to $d$ wave superconductors. On the experimental side, Krishna et al observed that in superconducting BSCCO and YBCO, at temperature $T > 5K$, the longitudinal thermal conductivity $\kappa_{xx}(H)$ initially decreases with applied magnetic field $H$, then reaches a plateau $\bar{\kappa}_{xx}$. They also measured $\kappa_{xy}$ thermal Hall conductivity $\kappa_{xy}$ at $T > 10K$ and extracted the thermal Hall angle $\tan \theta = \frac{\kappa_{xy}}{\kappa_{xx}}$. On the theoretical side, employing semi-classical approximation, Volovik pointed out that the circulating supercurrents around vortices induce Doppler energy shift to the quasi-particle spectrum, which leads to a finite density of states at the nodes $\bar{\kappa}_{xy}$. This effect (Volovik effect) has been employed to explain the above experimental observations of $\kappa_{xx}$ by several authors $\bar{\kappa}_{xy}$. However, semi-classical method can not be used to calculate $\kappa_{xy}$. A fully quantum mechanical approach is needed to get any information on $\kappa_{xy}$.

Starting from BCS Hamiltonian, Wang and MacDonald performed a first numerical calculation on quasi-particle spectrum in vortex lattice state $\bar{\kappa}_{xy}$. By phenomenological scaling argument, Simon and Lee (SL) $\bar{\kappa}_{xy}$ proposed the approximate scaling forms for $\kappa_{xx}$ and $\kappa_{xy}$ for $d$ wave superconductors in the mixed state. Anderson $\bar{\kappa}_{xy}$ employed a singular gauge transformation to study quasi-particle dynamics in the mixed state. Franz and Tesanovic (FT) employed a different singular gauge transformation to map the quasi-particle in a square vortex lattice state to Dirac fermion moving in an effective periodic scalar and vector potential with zero average and studied the quasi-particle spectrum $\bar{\kappa}_{xy}$. Using FT transformation, Marinelli et al studied the spectrum in various kinds of vortex lattice states in great detail $\bar{\kappa}_{xy}$.

In this paper, by considering carefully the gauge invariance overlooked by previous authors $\bar{\kappa}_{xy}$, we extend FT singular gauge transformation to include the curvature term which is important to $\kappa_{xy}$. After clarifying some important subtle points of the singular gauge transfor-
\[ + \Delta(\vec{r})\{p_x + \frac{1}{2}\partial_x \phi, p_y + \frac{1}{2}\partial_y \phi\} \]  

(4)

where \( \phi \) is the phase of \( \Delta(\vec{r}) \) [12].

The gauge invariance in \( \Delta \) has not been taken into account in Refs. [13,14]. Although its correct treatment does not affect the linearized Hamiltonian (see Eq.6), it is crucial to the curvature term (see Eq.7) which is important to \( n_{xy} \). The similar gauge invariance was considered in Eq.2.12 of Ref. [13].

Following FT [14], we introduce composite fermion [15] \( d_c \) by performing a singular unitary transformation \( \tilde{U} = U_{imp} \):

\[ H_s = U^{-1}HU, \quad U = \begin{pmatrix} e^{i\phi_A(\vec{r})} & 0 \\ 0 & e^{-i\phi_B(\vec{r})} \end{pmatrix} \]  

(5)

where \( \phi_A \) is the phase from the vortices in sublattice \( A \) and \( \phi_B \) is the phase from the vortices in sublattice \( B \).

In this letter, we assume the thermal currents are sufficiently weak that the vortices remain pinned by non-magnetic impurities. Therefore, the transport properties of \( d \) is exactly the same as the composite fermions \( d_c \). It is easy to check that \( U \) commutes with \( \tau_z \), therefore the transformation leaves Eq.2 \( H_{imp} \) invariant.

Expanding \( H_s \) around the node 1 where \( \vec{p} = (p_F, 0) \), we obtain \( H_s = H_1 + H_c \) where the linearized Hamiltonian \( H_1 \) is given by:

\[ H_1 = v_f(p_x + a_z)\tau^3 + v_2(p_y + a_y)\tau^1 + v_f v_2(\vec{r}) + V(x)\tau^3 + (1 \rightarrow 2, x \rightarrow y) \]  

(6)

where \( \vec{v}_s = \frac{1}{2}(\vec{v}_s^A + \vec{v}_s^B) = \frac{1}{2}\vec{\nabla}\phi - \frac{1}{2}\vec{A} \) is the total superfluid momentum and \( a_{\alpha} = \frac{1}{2}(\vec{v}_s^A - \vec{v}_s^B) = \frac{1}{2}((\vec{\nabla}\phi_A - \vec{\nabla}\phi_B) \) is the internal gauge field. Anderson’s gauge choice is \( \phi_A = \phi, \phi_B = 0 \) or vice versa [13,14].

We get the corresponding expression at node 1 and 2 by changing \( v_f \rightarrow -v_f, v_2 \rightarrow -v_2 \) in the above Eq.

The curvature term \( H_c \) can be written as:

\[ H_c = \frac{1}{m}[[\Pi_{\alpha}, v_\alpha] + \frac{1}{2}\vec{\Pi}^2 + \vec{\sigma}^2 + \frac{\Delta_0}{2\epsilon_F} \{\Pi_x, \Pi_y\}\tau^1] \]  

(7)

Where \( \vec{\Pi} = \vec{p} + \vec{a} \) is the covariant derivative. \( H_c \) takes the same form for all the four nodes.

It is easy to see that \( v_\alpha(\vec{r}) \) acts as a scalar scattering potential, it respects time-reversal (T) symmetry, but breaks Particle-Hole (PH) symmetry [14]. There are two very different kinds of internal gauge fields in \( H_1; V(x) \) is due to non-magnetic impurity scattering at zero field, \( a_{\alpha} \) is completely due to Aharonov and Bohm (AB) phase scattering [13] from vortices generated by external magnetic field. They both respect P-H symmetry. In general, \( V(x) \) breaks T symmetry, but T symmetry is restored in the unitary limit [18]. For general flux quantum \( \alpha \), \( a_{\alpha} \) breaks T symmetry, but T symmetry is restored at \( \alpha = 1/2 \) [19] because \( \alpha = 1/2 \) flux quantum is equivalent to \( \alpha = -1/2 \) one due to the periodicity under \( \alpha \rightarrow \alpha + 1 \). This is also the underlying physical reason why we are able to choose the two sublattices \( A \) and \( B \) freely without changing any physics. Due to this exact T symmetry, there is no Landau level quantization as claimed in [8].

In \( H_c \), the only term which breaks both P-H and T symmetry [14] is \( \psi^\dagger(\vec{p}_a, v_\alpha)\psi = -i\psi(\nabla_\alpha \psi^\dagger-\psi^\dagger \nabla_\alpha \psi) \). This term will lead to thermal Hall conductivity to be discussed in the following [13].

From Eq.8, it is easy to identify the conserved charge currents at node 1:

\[ j_{1x} = \psi_1^\dagger(x)v_F \tau^3 \psi_1(x), \quad j_{1y} = \psi_1^\dagger(x)v_2 \tau^1 \psi_1(x) \]  

(8)

with the currents at node 2 differing from the above expressions by \( (1 \rightarrow 2, x \rightarrow y) \). It is known that the charge conductivity of \( d_c \) corresponds to the spin conductivity of \( c \) electrons. Because the spin \( \sigma \) and thermal conductivities are related by Wiedemann-Franz law [11,21].

The Hamiltonian \( H_1 + H_c \) enjoys gauge symmetry \( U_{\alpha}(1) \times U_{\alpha}(1) \), the first being uniform (or external) and the second being staggered (or internal) gauge symmetry. Although the composite fermion \( d_c \) is charge neutral to the external magnetic field, it carries charge 1 to the internal gauge field \( a_{\alpha} \).

We assume a randomly pinned vortex array with logarithmic interaction between vortices. In the hydrodynamic limit, after averaging over all the possible positions of the vortices \( R_{1i} \), we find:

\[ <v_\alpha> = <a_{\alpha}> = 0, \quad <v_\alpha(\vec{k}) a_{\beta}(\vec{k}'>) = 0 \]

\[ <v_\alpha(\vec{k}) v_\beta(-\vec{k})> = \pi^2(\delta_{\alpha\beta} - \frac{k_a k_b}{k^2}) \frac{n_v}{k^2 + n_v} \]

\[ <a_{\alpha}(\vec{k}) a_{\beta}(-\vec{k})> = \pi^2(\delta_{\alpha\beta} - \frac{k_a k_b}{k^2}) \frac{n_v}{k^2 + n_v} \]  

(9)

Where the vortex density is \( n_v = \frac{N}{L} = \frac{H}{2\epsilon_F} \).

The first line in Eq.9 is exact. The \( v - v \) and \( a - a \) correlators are the most general forms consistent with the incompressibility of the vortex system [22]. The pinning strength will only enter as prefactors in front of \( n_v \). For notational simplicity, we suppress these prefactors.

Because \( v \) and \( a \) are decoupled at quadratic order (see Eq.9), the long-range logarithmic interaction between vortices suppresses the fluctuation of superfluid velocity, but does not affect the fluctuation of the internal gauge field. Therefore the scalar field acquires a mass determined by the vortex density, but the gauge field remains massless. The gauge field is a pure quantum mechanical effect, it was completely missed in all the previous semi-classical approaches [13,14]. Here we explicitly demonstrate that being gapless, its fluctuation even dominates over the well-known Volovik effect in the low energy limit. It also dominates over that from
the non-magnetic scattering at sufficiently high field and low energy limit.

In fact, in the weakly type II limit $\xi < \lambda < d_v$, the superfluid velocity vanishes in the interior of superconductor, the long-range correlated gauge potential in Eq.9 becomes the only scattering mechanism [6].

From the Eq.9, it is easy to realize that the internode scattering $\sim k_F^{-2}$ is weaker than the intra-node scattering $\sim p_0^{-2}$ by a factor $\alpha^{-1} = \frac{2\pi}{\xi_0} \ll 1$. In the following we neglect the internode scattering.

Up to the order of Gaussian cumulants, the scalar potential and vector potential are uncorrelated. However, they are correlated by Non-Gaussian cumulants. The lowest order non-Gaussian cumulants are the skewness:

\[
< v_a(r_1)v_b(r_2)a_\gamma(r_3) > = < a_a(r_1)a_\beta(r_2)a_\gamma(r_3) > = 0
\]

\[
< v_a(k_1)v_b(k_2)v_\gamma(k_3) > = \pi^2 n_v \delta(k_1 + k_2 + k_3)
\]

\[
- \frac{\alpha \delta_{k_1 k_3}(k_2 \cdot \delta k_2 \delta k_3 - k_2 k_3)}{(k_1^2 + n_v)(k_2^2 + n_v)(k_3^2 + n_v)}
\]

\[
< v_a(k_1)a_\beta(k_2)a_\gamma(k_3) >= \pi^2 n_v \delta(k_1 + k_2 + k_3)
\]

\[
- \frac{\alpha \delta_{k_1 k_3}(k_2 \cdot \delta k_2 \delta k_3 - k_2 k_3)}{(k_1^2 + n_v)(k_2^2 + n_v)(k_3^2 + n_v)}
\]

In fact, because any distribution function satisfies $P[a_\alpha(x)] = P[-a_\alpha(x)]$, any correlators involving odd number of $a_\alpha$ vanish. After coarse graining, the exact $T$ symmetry of $H_i$ is approximated by the average one. This approximation will lead to correct behaviors of self-averaging physical quantities. But it does not apply to non-self-averaging quantity such as Hall conductance fluctuation.

The discussion on $\kappa_{xx}$: The scalar potential capture the essential physics of Volovik effect: the quasi-particles energies are shifted by superfluid flow. Following the RG analysis in Ref. [7], it can be shown that the random scalar potential is marginally relevant, therefore generates finite density of states at zero energy. Because $k_F l_{tr} \sim k_F d_v \gg k_F \xi \sim 5$, the SCBA in standard impurity scattering process can be applied to calculate the low energy scattering rate. In $k, \omega \rightarrow 0$ limit, it leads to

\[
1 = v_F^2 \pi^2 n_v \int \frac{d^2 p}{(2\pi)^2} \frac{p_0^2}{p^2 + n_v} \Gamma_0^2 + E_p^2
\]

\[
1/\tau_l \sim \Gamma_0 \sim \Delta_0 \sqrt{\frac{H}{H_{c2}}}
\]

where $E_p^2 = (v_F k_1)^2 + (v_2 k_2)^2$, the retarded self-energy $\Sigma^R(k, \omega) = \Sigma(\vec{k}, \omega \rightarrow \omega + i\delta)$, the zero energy scattering rate is $\Gamma_0 = -Im \Sigma^R(0, 0)$.

The above equation leads to the quasi-particle lifetime:

\[
1/\tau_l \sim \Gamma_0 \sim \Delta_0 \sqrt{\frac{H}{H_{c2}}}
\]

Now we look at the vertex correction to $\kappa_{xx}$ due to the ladder diagram shown in Fig.1 at $T = 0$.

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**Fig 1:** Vertex correction to longitudinal conductivity

Fig.1a is just the bubble diagram. It leads to the well known bubble conductivity $[1][2]$: $\sigma_{xx}^0 = \frac{v_F^2}{v_F^2 + v_2^2}$.

By checking the integral equation satisfied by the vertex function $\Gamma(p, \omega_i, i\omega_f + i\Omega) = \pi^3 (1 + \Lambda(p, \omega_i, \omega_f + i\Omega))$ depicted in Fig.1c, we find $\Lambda$ is at the order of 1 independent of $n_v$, therefore $\sigma_{xx}$ receives vertex correction of order 1 independent of $n_v$. This vertex correction was completely missed in the semiclassical treatments [3].

The random gauge field gives additional scattering mechanism, it has scaling dimension 2, therefore strongly relevant [24] and dominates over Volovik effect at low energy limit. Similar SCBA to Eq.11 leads to logarithmic divergent quasi-particle scattering rate $1/\tau_l$ which is not gauge-invariant anyway. But the vertex correction similar to Fig.1 removes the logarithmic divergence and leads to the finite gauge-invariant transport time $\tau_{tr} \sim \frac{H_{c2}}{H} \Delta_0$ [23].

The vertex correction to the bubble conductivity due to the non-magnetic impurity scattering Eq.6 among the four nodes was calculated in Ref. [11], it was found to be negligible. It is obvious that the two vertex corrections are different due to different scattering mechanisms, therefore the two conductivity values are different, although the bubble results are the same.

At finite temperature, the $T$ dependence comes solely from the Fermi function, $\sigma_{xx}$ should satisfy the following scaling ($T_c \sim \Delta_0$):

\[
\sigma_{xx}(H,T) = F_1(a T \tau_{tr}) = F_1(\frac{T}{T_c} \sqrt{\frac{H_{c2}}{H}})
\]

This scaling is consistent with Simon and Lee [3] using phenomenological scaling argument. Our derivation bring out explicitly the underlying physical process: the quasiparticle scattering due to the long-range correlated random gauge potential. Pushing further, we conclude that in the high field limit $H \gg a^2 H_{c2} \left(\frac{\Delta_0}{\tau_c}\right)^2$, $\sigma_{xx}$ should approach the $T = 0$ value $F_1(0) = \sigma_{xx}(0)$ at the order of 1. This value depends on not only the anisotropy parameter $\alpha = v_F/v_2$, but also the pinning strength appearing in Eq.1. This dependence could explain the different plateau values observed in the experiments [3].

Unfortunately, the value $\sigma_{xx}(0)$ is hard to be sort out experimentally due to the large background contributions from phonons [10].

The discussion on $\kappa_{xy}$: We start with $H_l$. In order to get a non-vanishing $\kappa_{xy}$, we must identify terms which break both $T$ and P-H symmetry. As shown previously, $H_l$ respects exact $T$ symmetry, therefore $\angle \sigma_{xy} = 0$ to
the linear order. We have to go to the curvature term Eq. 7 to see its contribution to $<\sigma_{xy}>$.

The first contribution comes from the skewness between Volovik term and the \{p_\alpha, v_\alpha\} term $\partial v_\alpha (\vec{r}_1) v_\beta (\vec{r}_2) v_\gamma (\vec{r}_3) >$ in Eq. 10 which breaks both T and P-H symmetry.

Just like in $\kappa_{xx}$, the skewness between scalar and random gauge field $\partial v_\alpha (\vec{r}_1) a_\beta (\vec{r}_2) a_\gamma (\vec{r}_3) >$ in Eq. 10 gives additional scattering mechanism. It even dominates over the pure scalar skewness at low energy limit.

Due to the antisymmetric tensor $\epsilon_{\alpha \delta}$, we find $\sigma_{2xy} = -\sigma_{1yx} = \sigma_{1xy}$. Because both skewnesses are even under $v_f \rightarrow -v_f, v_2 \rightarrow v_2$, it is easy to find that $\sigma_{1xy} = \sigma_{1yx}$, therefore $\sigma_{xy} = \sigma_{1xy} + \sigma_{2xy} + \sigma_{1yx} + \sigma_{2yx} = 4\sigma_{1xy}$.

Because the \{p_\alpha, v_\alpha\} term contains one more derivative, simple power counting leads to

$$<\sigma_{xy}(H,T)> = \frac{T_c}{T} \sqrt{\frac{H}{H_c}} F_2 (b \frac{T}{T_c}) \sqrt{\frac{H_c}{H}}$$

This scaling is consistent with Simon and Lee by phenomenological scaling argument. Our derivation bring out explicitly the leading contributions from the \{p_\alpha, v_\alpha\} term in the curvature term and also its small numerical factor $\frac{T_c}{T}$. Pushing further, we conclude that in high field limit $H \gg b^2 H_c a (\frac{T}{T_c})^2$, $\sigma_{xy}$ should increase with $H$ as $\frac{T_c}{T} \sqrt{\frac{H}{H_c}} F_2 (0)$. Taking $\alpha \sim 10$, $H \sim 10T, H_c \sim 150T$, we find the prefactor is about 1/40, so $\kappa_{xy}$ is smaller than $\kappa_{xx}$ by a factor of 1/40. The smallness of $\kappa_{xy}$ make it difficult to measure experimentally.

The most recent data at $10K < T < 30K$ for $\kappa_{xy}/T^2$ was shown to satisfy quite well the scaling $\kappa_{xy}/T^2 = F(b\sqrt{H}/T)$ which follows from Eq. 14, however, it continues to decrease up to $H = 14T$ instead of increasing linearly with $\sqrt{H}/T$ as follows from Eq. 14. The discrepancy may be due to the inelastic scattering at $T > 10K$ not considered in this paper. For technical reason, so far the data is not available at $T < 10K$.

In conclusion, we point out that the dominant scattering mechanism at sufficient high magnetic field is due to the long-range correlated random gauge potential instead of the well-known Volovik effect.

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