Analytical Solution of Cylindrical Wave Problem in the Frameworks of Micropolar Elasticity

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Abstract. Propagation problem for coupled harmonic waves of translational displacements and microrotations along the axis of a long cylindrical waveguide is discussed at present study. Microrotations modeling is carried out within the linear micropolar elasticity frameworks. The coupled system of vector differential equations of micropolar elasticity is presented. The translational displacements and microrotations in the coupled wave are decomposed into potential and vortex parts. The coupled differential equations are then reduced to uncoupled ones. The wave equations solutions for the translational and microrotational waves potentials are obtained for a high-frequency waves in the cylindrical domain.

1. Introduction

Modelling of abnormal mechanical and multi-physical behaviour of solids (metamaterials) is an actual problem of modern Continuum Mechanics [1–4]. The metamaterials exhibit physical properties not usually found in nature. For example, there are materials with negative Poissons ratio (auxetic materials), negative thermal expansion, negative electric permittivity and the magnetic permeability. These paradoxical physical phenomena cannot be described in the frameworks of conventional Continuum Mechanics. In this case, microstructure continuum theories based on the necessity of additional (extra) freedom degrees are required [5, 6]. The problems of propagating surfaces of discontinuities of displacements and microrotations are previously discussed in number of studies [1, 2, 6–13]. In some studies [10–13] the wave problems in circular micropolar waveguides are considered. The aim of the present paper is to investigate the translational and microrotation waves propagating along an infinite cylindrical waveguide. Microrotation waves in an elastic solid are modelled within the linear micropolar elasticity frameworks by microrotations introducing in the governing equations of the Continuum Mechanics. Such a modelling can be regularly carried out in terms of the classical field theory [2], starting from the action integral and the least action variational principle.

2. Reminder of the Governing Equations

Throughout the study we employ notations used in [7] while replacing \( \eta \rightarrow \alpha \). For micropolar elastic continuum, equations of motion are furnished in direct tensor representation for the case
of the absence of mass forces and mass moments
\[
\begin{align*}
\nabla \cdot \sigma &= \rho \ddot{u}, \\
\nabla \cdot m + \epsilon \cdot \sigma &= 3 \ddot{\phi},
\end{align*}
\]
(1)
where \(\nabla\) is the three-dimensional Hamiltonian differential operator (the nabla symbol), \(\rho\) is the mass density, \(3\) is a scalar dynamic characteristic of continuum (the rotational microinertia), dot over a symbol denotes partial differentiation with respect to time at fixed spatial coordinates, \(u\) is the translational displacement vector and \(\phi\) rotation vector, \(\epsilon\) is the three-dimensional Levi–Civita symbol (permutation symbol, antisymmetric symbol, or alternating symbol).

The constitute equations for the force stress tensor \(\sigma\) and the moment stress tensor \(\mu\) can be presented as follows
\[
\sigma = (\mu + \eta) \gamma + (\mu - \eta) \gamma^T + \lambda \text{tr} \gamma, \quad \mu = (\gamma + \varepsilon) \kappa + (\gamma - \varepsilon) \kappa^T + \beta \text{tr} \kappa.
\]
(2)

The equations for the strain tensor \(\gamma\) and the bending-torsion tensor \(\kappa\) are read by
\[
\kappa = \nabla \otimes \phi, \quad \gamma = \nabla \otimes u - \phi \cdot \epsilon,
\]
(3)
wherein \(I\) denotes the three dimensional unit tensor.

The coupled vector differential equations determining the coupled wavefields in a micropolar elastic continuum [7] can be derived after substituting constitutive equations (2) and (3) into equations of motion (1)
\[
\begin{align*}
(\lambda + 2\mu) \nabla \nabla \cdot u - (\mu + \eta) \nabla \times (\nabla \times u) + 2\eta \nabla \times \phi &= \rho \ddot{u}, \\
(\beta + 2\gamma) \nabla \nabla \cdot \phi - (\gamma + \varepsilon) \nabla \times (\nabla \times \phi) + 2\eta \nabla \times u - 4\eta \phi &= 3 \ddot{\phi},
\end{align*}
\]
(4)
where \(\lambda, \mu, \gamma, \beta, \varepsilon, \eta\) denote the constitutive constants of micropolar elastic continuum; \(\nabla\) is the three dimensional Hamilton operator (Hamilton nabla); \(u\) is the translational displacement vector; \(\phi\) is the microrotation vector; \(\rho\) is the mass density; \(3\) denotes the microelement inertia; dot denotes partial differentiation with respect to time.

3. Wave Problem Solution
The system (4) is coupled and can be uncoupled as follows. First, introduce the dynamic potentials of translational displacements and microrotations
\[
u = \nabla \Phi + \nabla \times \Psi, \quad \phi = \nabla \Sigma + \nabla \times H,
\]
(5)
where \(\Phi\) and \(\Sigma\) are the scalar potentials, and \(\Psi\) and \(H\) are the vector potentials of translational displacements and microrotations respectively.

Second, after substituting (5) in (4) it is seen that the system (4) is satisfied if the scalar and the vector potentials fulfill the calibration conditions
\[
\nabla \cdot \Psi = 0, \quad \nabla \cdot H = 0,
\]
(6)
and are solutions of coupled equations
\[
\begin{align*}
\Delta \Phi - \frac{1}{\varepsilon_1^2} \dddot{\Phi} &= 0, \\
\Delta \Sigma - 2 \frac{\Omega^2}{\mu_c^2} \dddot{\Sigma} - \frac{\Omega^2}{\mu_c^2} \Sigma &= 0, \\
\Delta \Psi - \frac{1}{\varepsilon_1^2} \dddot{\Psi} + 2d_1^2 \nabla \times H &= 0, \\
\Delta H - 2 \frac{\Omega^2}{\mu_c^2} \dddot{H} - \frac{\Omega^2}{\mu_c^2} H + \frac{\Omega^2}{2\mu_c^2} \nabla \times \Psi &= 0,
\end{align*}
\]
(7)
where the following notations are used

\[\Omega^2 = 4\eta \Omega^{-1}, \quad \rho c_0^2 = \lambda + 2\mu, \quad \Sigma_0 c_0^2 = \beta + 2\gamma, \]
\[\Sigma_0 c_1^2 = \mu + \eta, \quad \rho c_0^2 = \mu + \eta, \quad \rho c_1^2 = \eta, \quad w c_1^2 = \nu c_1^2. \tag{8}\]

Third, in present study only coupled high-frequency waves of translational displacements and microrotations are considered \((\omega > \Omega)\). In this case the following differential equations for the potentials can be obtained

\[(\Delta + \alpha_0^2)\Phi = 0, \quad (\Delta + \beta_0^2)\Sigma = 0, \quad (\Delta + \alpha_1^2)\Psi + 2i\beta_1^2 \nabla \times H = 0, \quad (\Delta + \beta_1^2)H + \frac{\Omega^2}{2\nu c_1^2} \nabla \times \Psi = 0. \tag{9}\]

wherein the following notations have been introduced

\[\alpha_0^2 = \omega^2 / \mu c_0^2, \quad \beta_0^2 = (\omega^2 - \Omega^2) / \mu c_0^2, \quad \alpha_1^2 = \omega^2 / \mu c_1^2, \quad \beta_1^2 = (\omega^2 - \Omega^2) / \nu c_1^2. \tag{10}\]

The equations (9) for vortex potentials \(\Psi\) and \(H\) are still coupled. Uncoupled equations for vector potentials can be derived by the increasing differentiation order.

At last, for potentials \(\Psi\) and \(H\) the separate equations can be furnished by

\[(\Delta + K_1^2)(\Delta + K_2^2)\Phi = 0, \quad (\Delta + K_1^2)(\Delta + K_2^2)\Psi = 0, \quad \sigma_1^2 = d_1^2 \Omega^2 / \mu c_1^2, \quad K_{1,2}^2 = -\Delta_{1,2}, \quad 2\Delta_{1,2} = -(\alpha_1^2 + \beta_1^2 + \sigma_1^2) \pm \sqrt{(\alpha_1^2 - \beta_1^2 + \sigma_1^2)^2 + 4\beta_1^2 \sigma_1^2}. \tag{11}\]

The coupled wavefields in a cylindrical domain are found in the cylindrical coordinates \(r, \varphi, z\) by separation of variables (see in details [8]). Thus, for the scalar potentials one can obtain

\[\Phi = C_1 I_n(p_1 r) \left\{ \cos n\varphi \begin{array}{c} \cos n\varphi \\ -\sin n\varphi \end{array} \right\} e^{\pm ikz}, \quad \sum = -C_2 I_n(p_2 r) \left\{ \sin n\varphi \begin{array}{c} \sin n\varphi \\ \cos n\varphi \end{array} \right\} e^{\pm ikz}, \tag{12}\]

wherein \(k\) denotes the wave number; \(C_1\) and \(C_2\) are arbitrary constants; \(I_n(\cdot)\) is the Bessel function of the first kind of an imaginary argument; \(p_1^2 = k^2 - \alpha_1^2, \quad p_2^2 = k^2 - \beta_1^2\).

The vortex potentials of translational displacements and microrotations are given by the following formulas

\[
\begin{align*}
\Psi_r &= [C'_3 I_{n-1}(q_1 r) + C'_4 I_{n+1}(q_1 r) + C'_3 I_{n-1}(q_2 r) + C'_4 I_{n+1}(q_2 r)] \left\{ \begin{array}{c} \sin n\varphi \\ \cos n\varphi \end{array} \right\} e^{\pm ikz}, \\
\Psi_\varphi &= [C'_3 I_{n-1}(q_1 r) - C'_4 I_{n+1}(q_1 r) + C'_3 I_{n-1}(q_2 r) - C'_4 I_{n+1}(q_2 r)] \left\{ \begin{array}{c} \cos n\varphi \\ -\sin n\varphi \end{array} \right\} e^{\pm ikz}, \\
\Psi_z &= [C'_5 I_n(q_1 r) + C'_5 I_n(q_2 r)] \left\{ \begin{array}{c} \sin n\varphi \\ \cos n\varphi \end{array} \right\} e^{\pm ikz}; \\
H_r &= [L'_3 I_{n-1}(q_1 r) + L'_4 I_{n+1}(q_1 r) + L'_3 I_{n-1}(q_2 r) + L'_4 I_{n+1}(q_2 r)] \left\{ \begin{array}{c} -\cos n\varphi \\ \sin n\varphi \end{array} \right\} e^{\pm ikz}, \\
H_\varphi &= [L'_3 I_{n-1}(q_1 r) - L'_4 I_{n+1}(q_1 r) + L'_3 I_{n-1}(q_2 r) - L'_4 I_{n+1}(q_2 r)] \left\{ \begin{array}{c} \sin n\varphi \\ \cos n\varphi \end{array} \right\} e^{\pm ikz}, \\
H_z &= [L'_5 I_n(q_1 r) + L'_5 I_n(q_2 r)] \left\{ \begin{array}{c} -\cos n\varphi \\ \sin n\varphi \end{array} \right\} e^{\pm ikz},
\end{align*}
\tag{13}\]

wherein \(C'_3 - C'_5, \quad C'_3 - C'_5, \quad L'_3 - L'_5\) and \(L'_3 - L'_5\) are arbitrary constants and \(q_1^2 = k^2 - K_1^2, \quad q_2^2 = k^2 - K_2^2\).
Conclusion

• The constitutive and governing equations of the linear micropolar elasticity have been furnished.

• Propagation problem for coupled harmonic waves of translational displacements and microrotations along the axis of a long circular waveguide has been solved.

• The translational displacements and microrotations in the coupled wave have been decomposed into potential and vortex parts and the coupled differential equations of motion has been reduced to uncoupled ones.

• The wave problem solution has been obtained for a high-frequency waves in the cylindrical domain.

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