The Kähler Geometry of Bott Manifolds

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BON ANNIVERSAIRE JACQUES
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Background

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2. Bott Manifolds which are related to Bott-Samelson manifolds were anticipated by Raoul Bott and first studied in detail in Grossberg’s thesis and later used in representation theory by Grossberg and Karshon.
3. The topology of Bott manifolds was then studied by Choi, Masuda, Panov, Suh and others.
4. Bott Manifolds are smooth projective toric varieties; hence, they are integrable systems.
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4 Bott Manifolds are smooth projective toric varieties; hence, they are integrable systems.

5 They are best approached through the notion of a Bott Tower which we now describe.
Consider **Closed Complex Manifolds** $M_k$ for $k = 0, 1, \ldots, n$ with $M_0 = \{pt\}$ and $M_k$ the total space of the $\mathbb{CP}^1$-bundle $\pi_k: \mathbb{P}(1 \oplus L_k) \to M_{k-1}$ giving the sequence

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots M_2 \xrightarrow{\pi_2} M_1 = \mathbb{CP}^1 \to \{pt\}$$

where $L_k$ is a holomorphic line bundle on $M_{k-1}$.
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$M_k$ is called the stage $k$ **Bott manifold** of the **Bott tower** of height $n$. 

Stage 2 Bott manifolds are nothing but **Hirzebruch surfaces**.

A **Bott tower** is a collection $(M_k, \pi_k, \sigma_0^k, \sigma_\infty^k)_{k=1}^n$ where $\sigma_0^k$ and $\sigma_\infty^k$ are the zero and infinity sections of $L_k$, respectively.

The **Quotient Construction**: Any Bott tower is obtained from the complex torus action $(t_i)_{n_i=1}^n \in (\mathbb{C}^*)^n$ on $(z_j, w_j)_{n_j=1}^n \in (\mathbb{C}^2 \setminus \{0\})^n$ by

$$(t_i)_{n_i=1}^n \cdot (z_j, w_j)_{n_j=1}^n \mapsto (t_jz_j, (n \prod_{i=1}^n t_i A_{ij})w_j)_{n_j=1}^n$$

where $A_{ij}$ are the entries of a lower triangular unipotent integer-valued matrix $A$.

The **Cohomology Ring**: $H^*(M_n, \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \ldots, x_n]/I$ where $I$ is generated by $x_k y_k$ with $y_k = \sum_{n_j=1}^n A_{jk} x_j$.

Problem: When does the cohomology ring determine the diffeomorphism type? (Choi, Masuda, Panov, Suh)
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7. **Problem**: When does the cohomology ring determine the **diffeomorphism type**? (Choi, Masuda, Panov, Suh)
Bott towers form the objects $g^0_{BT}$ of a groupoid $g^{BT}$ (Bott tower groupoid) whose morphisms $g^1_{BT}$ are $\mathbb{T}^n$ equivariant biholomorphisms.
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2 Elements of $g_1^{BT}$ give equivalences of Bott towers.
Bott towers form the objects $\mathcal{G}^{BT}_0$ of a groupoid $\mathcal{G}^{BT}$ (Bott tower groupoid) whose morphisms $\mathcal{G}^{BT}_1$ are $\mathbb{T}^n$ equivariant biholomorphisms.

Elements of $\mathcal{G}^{BT}_1$ give equivalences of Bott towers.

The set of $n$ dimensional Bott towers $\mathcal{G}^{BT}_0$ can be identified with the set of $n \times n$ lower triangular unipotent matrices $A$ over the integers $\mathbb{Z}$, hence with $\mathbb{Z}^{\frac{n(n-1)}{2}}$. 
**The Bott Tower Groupoid**

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4. The isotropy subgroup $Iso(M_n(A)) \subset G_1^{BT}$ at $M_n(A) \in G_0^{BT}$ is $\text{Aut}(M_n(A))$. 
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Bott towers form the objects $G^0_{\text{BT}}$ of a groupoid $G^{\text{BT}}$ (Bott tower groupoid) whose morphisms $G^1_{\text{BT}}$ are $\mathbb{T}^n$ equivariant biholomorphisms.

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7 A Bott manifold has a $\mathbb{T}^n$ invariant compatible Kähler form $\omega$. In fact its Kähler cone $\mathcal{K}$ is $n$ dimensional.
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8. $\mathcal{K}$ is isomorphic to the ample cone $\mathcal{A}$ of $\mathbb{T}^n$ invariant ample divisors.
Symplectic Structures

- Given a **Bott tower** $M_n(A)$ choose a $\mathbb{T}^n$ invariant compatible symplectic form $\omega$. Then say that the **symplectic manifold** $(M^{2n}, \omega)$ is of **Bott type**
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- The number $N_B(M^{2n}, \omega)$ is finite (McDuff).
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**Theorem (1)**

Let $(M^{2n}, \omega)$ be a symplectic manifold of Bott type. Then the number of conjugacy classes of maximal tori of dimension $n$ in the symplectomorphism group $\text{Symp}(M^{2n}, \omega)$ equals $N_B(M^{2n}, \omega)$.
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To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional torus in $\text{Symp}(M^{2n}, \omega)$ and hence its conjugacy class.
Given a **Bott tower** $M_n(A)$ choose a $T^n$ invariant compatible symplectic form $\omega$. Then say that the **symplectic manifold** $(M^{2n}, \omega)$ is of **Bott type**. $N_B(M^{2n}, \omega)$ denotes the **number** of $T^n$ invariant **complex structures** that are compatible with $(M^{2n}, \omega)$ which is isomorphic to the **number** of compatible **Bott manifolds**. The number $N_B(M^{2n}, \omega)$ is **finite** (McDuff).

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- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional **torus** in $\text{Symp}(M^{2n}, \omega)$ and hence its **conjugacy class**.
- There is a **bijection** between the set of **Bott manifolds** compatible with $(M^{2n}, \omega)$ and the set of $G_1^{BT}$ **orbits** in $G_0^{BT}$ with an element compatible with $\omega$. 

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To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional torus in $\text{Symp}(M^{2n}, \omega)$ and hence its conjugacy class. There is a bijection between the set of Bott manifolds compatible with $(M^{2n}, \omega)$ and the set of $G_{BT}$ orbits in $G_{0BT}$ with an element compatible with $\omega$. Elements of $G_{0BT}/G_{1BT}$ have distinct complex structures.
Given a Bott tower $M_n(A)$ choose a $T^n$ invariant compatible symplectic form $\omega$. Then say that the symplectic manifold $(M^{2n}, \omega)$ is of Bott type. 

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**Ingredients of Proof:**

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There is a bijection between the set of Bott manifolds compatible with $(M^{2n}, \omega)$ and the set of $G_1^{BT}$ orbits in $G_0^{BT}$ with an element compatible with $\omega$. 

Elements of $G_0^{BT}/G_1^{BT}$ have distinct complex structures. 

Then a cohomological rigidity result of Choi-Suh and Masuda-Panov gives a bijection between the set of Bott manifolds compatible with $(M^{2n}, \omega)$ and the set of $T^n$ invariant integrable complex structures $J$ compatible with $(M^{2n}, \omega)$.
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- There is a **bijection** between the set of **Bott manifolds** compatible with $(M^{2n}, \omega)$ and the set of $\mathcal{G}^{BT}_1$ **orbits** in $\mathcal{G}^{BT}_0$ with an element compatible with $\omega$.
- Elements of $\mathcal{G}^{BT}_0/\mathcal{G}^{BT}_1$ have **distinct complex structures**.
- Then a **cohomological rigidity** result of Choi-Suh and Masuda-Panov gives a **bijection** between the set of **Bott manifolds compatible** with $(M^{2n}, \omega)$ and the set of $\mathbb{T}^n$ invariant integrable **complex structures** $J$ compatible with $(M^{2n}, \omega)$.
- Delzant’s Theorem $\Rightarrow (M^{2n}, \omega, J)$ is a smooth projective toric variety.
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- There is a **bijection** between the set of **Bott manifolds** compatible with $(M^{2n}, \omega)$ and the set of $G^{BT}_1$ **orbits** in $G^{BT}_0$ with an element compatible with $\omega$.
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- Then a **cohomological rigidity** result of Choi-Suh and Masuda-Panov gives a **bijection** between the set of **Bott manifolds compatible** with $(M^{2n}, \omega)$ and the set of $T^n$ invariant integrable **complex structures** $J$ compatible with $(M^{2n}, \omega)$.
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- The corresponding **Delzant polytope** $P$ is **combinatorially equivalent** to $n$ cube.
Symplectic Structures

- Given a **Bott tower** $M_n(A)$ choose a $\mathbb{T}^n$ invariant compatible symplectic form $\omega$. Then say that the **symplectic manifold** $(M^{2n}, \omega)$ is of **Bott type**
- $N_B(M^{2n}, \omega)$ denotes the **number** of $\mathbb{T}^n$ invariant **complex structures** that are compatible with $(M^{2n}, \omega)$ which is isomorphic to the **number** of compatible **Bott manifolds**.
- The number $N_B(M^{2n}, \omega)$ is **finite** (McDuff).

**Theorem (1)**

Let $(M^{2n}, \omega)$ be a symplectic manifold of Bott type. Then the **number of conjugacy classes of maximal tori** of dimension $n$ in the symplectomorphism group $\text{Symp}(M^{2n}, \omega)$ equals $N_B(M^{2n}, \omega)$.

**INGREDIENTS OF PROOF:**

- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional **torus** in $\text{Symp}(M^{2n}, \omega)$ and hence its **conjugacy class**.
- There is a **bijection** between the set of **Bott manifolds** compatible with $(M^{2n}, \omega)$ and the set of $G_1^{BT}$ **orbits** in $G_0^{BT}$ with an element compatible with $\omega$.
- Elements of $G_0^{BT}/G_1^{BT}$ have **distinct complex structures**.
- Then a **cohomological rigidity** result of Choi-Suh and Masuda-Panov gives a **bijection** between the set of **Bott manifolds compatible** with $(M^{2n}, \omega)$ and the set of $T^n$ invariant integrable **complex structures** $J$ compatible with $(M^{2n}, \omega)$.
- Delzant’s Theorem $\Rightarrow (M^{2n}, \omega, J)$ is a smooth projective toric variety.
- The corresponding **Delzant polytope** $P$ is **combinatorially equivalent** to $n$ cube.
- The corresponding **smooth projective toric varieties** $\{M_\mathcal{F}\} \simeq \{F_M\}$ **smooth normal fans** $\mathcal{F}$ over $\{P\}$.  

Charles Boyer (University of New Mexico)  
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Gauge Theories, Monopoles, Moduli Spaces and Integrable Systems  
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BON ANNIVERSAIRE JACQUES
Karshon proved the Theorem for Hirzebruch surface (stage 2 Bott manifolds) and gave a formula for $N_B(M^4, \omega)$.

Example:
Stage 3 Bott manifolds
diffeomorphic to $(S^2)^3 = S^2 \times S^2 \times S^2$ with symplectic form $\omega_{k_1, k_2, k_3}$ with $k_i \in \mathbb{R}^+$ and ordered $0 < k_3 \leq k_2 \leq k_1$.

$(S^2)^3, \omega_{k_1, k_2, k_3}$ is Kähler with respect to the Bott manifold $M_3(2a, 2b, 2c)$ if and only if one of the following two cases hold:
1. $c = 0$ with $k_1 - |a| k_2 - |b| k_3 > 0$, $k_2 > 0$, $k_3 > 0$.
2. $c \neq 0$ and $b = ac$ with $k_1 - |a| (k_2 - |c| k_3) > 0$, $k_2 - |c| k_3 > 0$, $k_3 > 0$.

Then $N_B(M_6, \omega_{k_1, k_2, k_3}) = \sum_{b \max j = 0} \left\lceil k_1 - jk_3 \right\rceil + \sum_{c \max j = 1} \left\lceil k_1 k_2 - jk_3 \right\rceil$ where $\left\lceil a b \right\rceil$ is least integer greater than or equal to $a b$. 
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Calabi’s Extremal Kähler Metrics

- Calabi **Energy functional** $E(g) = \int_M s_g^2 d\mu_g$, where $s_g$ is the scalar curvature of a Kähler metric $g$ with Kähler form $\omega$ on a compact complex manifold $M$. 
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- \( \text{Aut}(X_{\mathcal{F}})_0 \) is reductive if and only if and only if \( R(\mathcal{F}) = -R(\mathcal{F}) \).
The Generalized Calabi Construction

Ingredients

- Given a principal \( T^\ell \) bundle \( \mathcal{G} \to \mathbb{CP}^1 \) construct the associated fiber bundle \( M = \mathcal{G} \times_{T^\ell} V \) with fiber \( V \) where \( V \) is a compact toric Kähler manifold of complex dimension \( \ell \).
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A constant $\hat{c} \in \mathbb{R}$ such that the $(1,1)$-form $\hat{c} \omega_{\Sigma} + \langle v, \omega_{\Sigma} \otimes p \rangle$ is positive for $v \in P$.

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where $G = \text{Hess}(U) = H^{-1}$, $U$ is the symplectic potential of the chosen toric Kähler structure $g_V$ on $V$, and $\langle \cdot, \cdot, \cdot \rangle$ denotes the pointwise contraction $g_{t^*} \times S^2 t_\ell \times t^*_\ell \rightarrow \mathbb{R}$ or the dual contraction.
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**Lemma (Apostolov, Calderbank, Gauduchon, Tønnesen-Friedmann)**

*If $V$ admits an extremal toric Kähler metric $g_V$, then $M$ admits compatible extremal Kähler metrics (at least in some Kähler classes).*
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Any Bott manifold $M_n$ admits a toric extremal Kähler metric. Alternatively, the extremal Kähler cone $\mathcal{E}(M_n)$ is a non-empty open cone in the Kähler cone $\mathcal{K}(M_n)$.
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Theorem (3)

Let $M_n(A)$ be a Bott tower. If the elements below the diagonal of any row of the lower triangular unipotent matrix $A$ all have the same sign, then $M_n(A)$ does not admit a Kähler metric with constant scalar curvature. In particular, if $A_2^1 \neq 0$ then $M_n(A)$ does not admit a Kähler metric with constant scalar curvature.
Extremal Kähler Metrics on Bott Manifolds

Theorem (2)

Any Bott manifold $M_n$ admits a toric extremal Kähler metric. Alternatively, the extremal Kähler cone $\mathcal{E}(M_n)$ is a non-empty open cone in the Kähler cone $\mathcal{K}(M_n)$.

- The proof is induction using the Lemma of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedmann.
- The last statement uses the well known LeBrun-Simanca rigidity result.

Problem

Describe the extremal Kähler cone $\mathcal{E}(M_n)$. In particular, when is $\mathcal{E}(M_n) = \mathcal{K}(M_n)$?

- We can describe the Kähler cone $\mathcal{K}(M_n)$ of a Bott manifold $M_n$. It is often, but not always, the first orthant in $\mathbb{R}^n$.
- For a Bott tower $M_n(A)$ the connected component $\mathfrak{aut}(M_n(A))_0$ is the connected component of the isotropy subgroup of $\mathfrak{g}_1^{BT}$ at $M_n(A)$.

Theorem (3)

Let $M_n(A)$ be a Bott tower. If the elements below the diagonal of any row of the lower triangular unipotent matrix $A$ all have the same sign, then $M_n(A)$ does not admit a Kähler metric with constant scalar curvature. In particular, if $A_2^1 \neq 0$ then $M_n(A)$ does not admit a Kähler metric with constant scalar curvature.

- The proof essentially follows from Demazure’s Theorem by computing possible root vectors.
Following Choi-Suh we let $t$ denote the number of non-trivial topological fibrations in the defining sequence of a Bott tower $M_n(A)$. It is well defined and $t = 0, 1, \ldots, n - 1$. 

A $t$-twist Bott manifold is diffeomorphic to a bundle over $(S^2)^{n-t}$ with fiber a stage $t$ Bott manifold.
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The Twist of Bott Manifolds

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**Theorem**

Let $M_n(A)$ be a Bott tower with twist $t$ and matrix $A$ of the form

$$A = \begin{pmatrix}
\tilde{A} & 0 & \cdots & 0 \\
A_1 & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
A_n & A_{n-t+1} & \cdots & 1
\end{pmatrix}, \quad A_i^j \in \mathbb{Z},$$

where $\tilde{A} \neq \mathbb{I}_n$ has 0-twist. Then $M_n(A)$ does not admit a compatible Kähler metric with constant scalar curvature. In particular, if $t = 0$ and the Bott manifold $M_n(A)$ has a compatible Kähler metric with constant scalar curvature, then it is the product $(\mathbb{C}P^1)^n$. 
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The only 0 twist Fano Bott manifold is the product $(\mathbb{C}P^1)^n$. 
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Consider Bott manifolds $M_n(k)$ with $k = (k_1, \ldots, k_{n-1})$ satisfying $k_1 k_2 \cdots k_{n-1} \neq 0$ with $A$ matrix

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If not all $k_i$ have the same sign, then some of these metrics will have **constant scalar curvature**.
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The monotone Kähler class admits a Kähler-Ricci soliton which is Kähler-Einstein if and only if the number of $+1$ in $k$ equals the number of $-1$ in $k$. 
1 Twist Bott Manifolds

- A 1 twist **Bott manifold** is diffeomorphic to a non-trivial $\mathbb{C}P^1$ bundle over $(S^2)^{n-1}$.
- The **diffeomorphism type** of a 1 twist **Bott manifold** is determined by its **cohomology ring** (Choi-Suh).
- Consider Bott manifolds $M_n(k)$ with $k = (k_1, \ldots, k_{n-1})$ satisfying $k_1 k_2 \cdots k_{n-1} \neq 0$ with $A$ matrix
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- Then $M_n(k)$ admits an **extremal Kähler metric** in every Kähler class.
- If not all $k_i$ have the same sign, then some of these metrics will have **constant scalar curvature**.
- $M_n(k)$ is Fano if and only if $k_i = \pm 1$ for all $i$.
- The **monotone Kähler class** admits a **Kähler-Ricci soliton** which is **Kähler-Einstein** if and only if the number of $+1$ in $k$ equals the number of $-1$ in $k$.
- Much of this case recovers previous work of Koiso, Sakane, Guan, Hwang, and Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman.
Some Results for Stage 3 Bott Manifolds

For stage 3 Bott manifolds the cohomology ring determines its diffeomorphism type (Choi-Masuda-Suh).

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Note that types 2 and 3 can have non-trivial intersection.

A type 2 stage 3 Bott manifold $M_3(0, A_1^3, A_2^3)$ can be realized as the projectivization $\mathbb{P}(1 \oplus O(\Lambda_1^3, \Lambda_2^3))$. If $\Lambda_1^3 \Lambda_2^3 \neq 0$, it is a 1 twist Bott manifold. There is an infinite number of diffeomorphism types determined by $\Lambda_1^3 \Lambda_2^3$. The number of Bott manifolds in each diffeomorphism type is determined by the factorizations of $\Lambda_1^3 \Lambda_2^3$ with fixed parity of $(1 + \Lambda_1^3)(1 + \Lambda_2^3)$.

A 0 twist stage 3 Bott manifold is $M_3(2A_1^2, 2A_1^2, 0)$ or $M_3(2A_1^2, 2A_1^2A_2^3, 2A_2^3)$. The former is type 3 whereas generically the latter is type 1.

There are 5 stage 3 Fano Bott manifolds $M_3(\Lambda_1^2, \Lambda_1^3, \Lambda_2^3)$, up to equivalence, with representatives $M_3(0, 0, 0)$, $M_3(0, 1, -1)$, $M_3(0, 1, 1)$, $M_3(1, 0, 0)$, $M_3(-1, 0, 1)$. The first 2 admit constant scalar curvature Kähler metrics, the remaining 3 do not.

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THANK YOU FOR YOUR ATTENTION

and

HAPPY BIRTHDAY JACQUES