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Viscous evolution of point vortex equilibria: The collinear state

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We describe the viscous evolution of a collinear three-vortex structure that corresponds initially to an inviscid point vortex fixed equilibrium, with the goal of elucidating some of the main transient dynamical features of the flow. Using a multi-Gaussian “core-growth” type of model, we show that the system immediately begins to rotate unsteadily, a mechanism we attribute to a “viscously induced” instability. We then examine in detail the qualitative and quantitative evolution of the system as it evolves toward the long-time asymptotic Lamb–Oseen state, showing the sequence of topological bifurcations that occur both in a fixed reference frame and in an appropriately chosen rotating reference frame. The evolution of passive particles in this viscously evolving flow is shown and interpreted in relation to these evolving streamline patterns. © 2010 American Institute of Physics. [doi:10.1063/1.3516637]

I. INTRODUCTION

When point vortex equilibria of the two-dimensional (2D) Euler equations (inviscid) are used as initial conditions for the corresponding Navier–Stokes equations (viscous), typically an interesting and complex dynamical process unfolds at short and intermediate time scales, which depends crucially on the details of the initial configuration. For long enough times, Gallay and Wayne proved recently that the Lamb–Oseen solution is an asymptotically stable attracting solution for all (integrable) initial vorticity fields. While very powerful, this asymptotic result does not elucidate the intermediate dynamics that take place in finite time and allow a given initial vorticity field to reach the single-peaked Gaussian distribution of the Lamb–Oseen solution. Given the rather large (and growing) literature on point vortex equilibria of the Euler equations, (see for example, Refs. 2–4), we thought an analysis of how these equilibria evolve under the evolution of the full Navier–Stokes system would merit a systematic treatment. Hence, in this paper, we begin an investigation of the viscous evolution of a class of initial vorticity fields consisting of the superposition of N Dirac-delta functions or point vortices. Our initial configuration, shown in Fig. 1, is a collinear configuration of three point vortices, evenly spaced along a line (the x axis), with strengths 2Γ, −Γ, and 2Γ, respectively. Such a configuration, for the Euler equations, is known to be an unstable fixed equilibrium, as fleshed out most recently and comprehensively in Ref. 5, but earlier in Ref. 6. We point out that this configuration, because of the strengths chosen for each of the point vortices, is not what is commonly referred to as the “tripole” state7–10 in which the vortex strengths sum to zero. Our focus in this paper will be the dynamics that takes place at the short and intermediate time scales, using this initial state in the Navier–Stokes equations, before the long-time asymptotic Lamb–Oseen solution dominates. This includes the dynamics of the surrounding passive field and the corresponding background time-dependent streamline pattern in an appropriately chosen reference frame which we argue is very helpful as a diagnostic tool to interpret the resulting flowfield.

II. PROBLEM SETTING

Consider an incompressible fluid in an unbounded 2D domain \( \mathbb{R}^2 \). The fluid motion is governed by Navier–Stokes equations, written in terms of the vorticity field \( \omega(x,t) \), a scalar-valued function of position \( x \) and time \( t \), as follows:

\[
\frac{\partial \omega}{\partial t} = -\mathbf{u} \cdot \nabla \omega + \nu \Delta \omega. \tag{1}
\]

The kinematic viscosity \( \nu \) is assumed to be constant. The fluid velocity \( \mathbf{u}(x,t) \) is a vector-valued function of \( x \) and \( t \). Both \( x \) and \( \mathbf{u} \) are expressed in an inertial frame \( \{e_i\}_{i=1,2,3} \), where \( (e_1, e_2) \) span the plane of motion, that is to say, one has \( x = xe_1 + ye_2 \) and \( \mathbf{u} = u_x e_1 + u_y e_2 \), or, equivalently, \( x = (x,y) \) and \( \mathbf{u} = (u_x, u_y) \). By definition, the vorticity vector \( \omega = \nabla \times \mathbf{u} \) is always perpendicular to the plane of motion and can thus be expressed as \( \omega = \omega e_3 \). The velocity \( \mathbf{u} \) and vorticity \( \omega \) are related via the 2D Biot–Savart law

\[
\mathbf{u}(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - \tilde{x})^\perp}{|x - \tilde{x}|^3} \omega(\tilde{x},t) d\tilde{x}, \tag{2}
\]

where \( \tilde{x} \) is an integration variable and \( x^\perp = (-y,x) \). Note that for 2D flows, the stretching term \( \omega \cdot \nabla \mathbf{u} \) is identically zero thus does not appear in Eq. (1) while the continuity equation \( \text{div}(\mathbf{u}) = 0 \) is trivially satisfied when expressed in terms of vorticity.

The solution of the system of Eqs. (1) and (2) depends, of course, on the choice of initial conditions \( \omega(x,0) \). One solution of particular interest in this work is the well-known Lamb–Oseen solution corresponding to a Dirac-delta initial condition \( \omega(x,0) = \Gamma \delta(x) \), i.e., a point vortex placed at the origin with circulation or strength \( \Gamma \) (more generally, the circulation \( \Gamma \) around any closed curve \( C \) in the fluid domain is defined as \( \Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s} = \int_C \omega \, d\mathbf{a} \) and can be thought of as the flux of vorticity through the area \( A \) enclosed by the curve \( C \).
Traditionally, the problem can be expressed compactly in complex notation with position variable $z$, $z=x+iy$ and $i=\sqrt{-1}$. The Lamb–Oseen solution is given by (see, for example, Refs. 2 and 11 for more details)

$$\omega(z,t) = \frac{\Gamma}{4\pi |t|} \exp \left( -\frac{|z|^2}{4|t|} \right).$$

The corresponding velocity field $u$ is given by

$$\dot{z} = u_x - iu_y = \frac{\Gamma}{2\pi i z} \left[ 1 - \exp \left( -\frac{|z|^2}{4|t|} \right) \right],$$

where the dot notation ($\dot{\cdots}$)=$d(\cdots)/dt$ refers to the time derivative and the notation $\dot{z}=x-iy$ refers to the complex conjugate. According to Eq. (3), the evolution of a vorticity field that is initially concentrated at the origin is such that the vorticity diffuses axisymmetrically as a Gaussian distribution. The spreading of the vorticity concentration can be quantified by the vortex core or support defined as $\rho=\sqrt{4\pi t}$, where $r=vt$.

Despite the explicit, exact, and simple nature of solution (3), for more complicated initial data, explicit solutions of Eqs. (1) and (2) are not analytically available for a general initial vorticity field. We are particularly interested in the viscous evolution of a class of initial vorticity fields $\omega(z,0) = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(z-z_\alpha)$ consisting of the superposition of $N$ Dirac-delta functions or point vortices. Note that Gallay and Wayne proved recently that the Lamb–Oseen solution is an asymptotically stable attracting solution for all $N$ Dirac-delta functions or point vortices. For more complicated initial data, explicit solutions of the inviscid point vortex model (6) are not analytically available for a general initial vorticity field, see Ref. 1, of which $\omega(z,0) = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(z-z_\alpha)$ is a special case. It is the dynamics that unfolds as the system evolves toward this final state that we are interested in.

The dynamics of $N$ point vortices in the plane is extensively analyzed in the context of the inviscid fluid model ($v=0$), see for example Ref. 2 and references therein. The vorticity field remains then concentrated for all times at $N$ points whose position $z_\alpha(t)(\alpha=1, \ldots, N)$ is dictated by the local fluid velocity induced by the presence of the other vortices. The fluid velocity at an arbitrary point $z$ in the plane that does not coincide with a point vortex is obtained from the Biot–Savart law (2) which takes the form

$$\dot{z}_\alpha = \sum_{\alpha'=1}^{N} \frac{1}{2\pi i} \frac{\Gamma_{\alpha'}}{z - z_{\alpha'}},$$

whereas the velocity at a point vortex $z_\beta$ is given by subtracting the effect of that point vortex from Eq. (5), and replacing $z$ with $z_\beta$, namely,
the numerical algorithm devised in Ref. 14 that utilizes a
by presented here, we diffuse the initial Dirac-delta vorticity field
conditions of the vorticity field as a superposition of three
domain. This is a mesh-based method which poses a problem
nonreflecting boundary condition to emulate the infinite fluid
domain. This is a mesh-based method which poses a problem
in handling the Dirac-delta initial conditions because they
are not well-posed for discretization on a standard Euclidean
mesh. To overcome this problem, we consider the initial
conditions of the vorticity field as a superposition of three
slightly diffused Gaussian peaks. In all the simulations
presented here, we diffuse the initial Dirac-delta vorticity field
by ε such that ε/Re=2×10^{-3}, e.g., ε=0.2 when Re=1000.
This, of course, introduces a slight mismatch in the initial
conditions used for the numerical simulation with those used
in the model, an error we cannot completely eliminate, but
should be kept in mind when comparing the simulation with
the model. We compute the time evolution of the vorticity
field in the window [-1.5, 1.5]×[-1.5, 1.5] while the nonre-
fecting boundary conditions are imposed using the multidom-
ain technique with ten nested domains, the largest of which
is 2^{10} times the size of our result window. The spatial and
time steps are set to Δx=Δy=0.01, Δt=0.02.

Figure 2 depicts the time evolution of the vorticity con-
tours (top row) and streamlines (bottom row) of Navier–
Stokes solution for Re=1000 at four time instants: t=0, 2.8,
43.7, and 47.4. One notices that the vortex configuration be-
gins to rotate unsteadily for t>0; we refer to this motion as
viscosity-induced rotation. One also notices that the center
vortex stretches and diffuses out first, then the outer two
vortices begin to merge. Eventually the vortex configuration
approaches a single Gaussian vortex.

III. THE MULTI-GAUSSIAN MODEL

In this section, we use a simple, analytically tractable
model to describe the dynamic evolution of N point vortices
for nonzero (but small) viscosity (ν≠0). The model assumes
that the vorticity of each initial point vortex spreads axisym-
metrically as an isolated Lamb–Oseen vortex, thus modeling
the diffusion term νΔω in Eq. (1), while its center moves
according to the local velocity induced by the presence of the
other (diffusing) vortices, thus accounting for the convection
term u·Vω in Eq. (1). It is worth noting here the recent work
of Gallay who analyzes the inviscid limit ν→0 of the 2D
Navier–Stokes evolution of Dirac-delta initial conditions and
proves, under certain assumptions, that the solution of the
Navier–Stokes equation converges, as ν→0, to a superposi-
tion of Lamb–Oseen vortices. In this work, we show that, for
small yet finite ν, the multi-Gaussian model is able to capture
qualitatively, though not quantitatively, some of the main
features of the Navier–Stokes solution. Generally speaking,
this class of models has been most highly developed in the
numerical literature (see for example Refs. 16 and 17 and
subsequent analysis in Ref. 18) and is referred to as a “core-
growth” class of models. One can trace the “splitting” idea of
the advection and the diffusion terms of the 2D Navier–
Stokes equations, on which the core-growth model is based,
at least back to Chorin’s influential paper, also used by
Milinazzo and Saffman. In these papers, the diffusion was
handled by a random walk approach. Core-growth models
based on time-dependent solutions of the heat-equation were
developed and used mostly by the numerical/computational
vortex dynamics community and are discussed and devel-
oped explicitly in Refs. 21–24. In the context of numerical
simulations, focused studies can be found in the works of
Barba and Leonard and used in specific models in Refs.
27–29. We mention, of course, also the works in Refs. 30,
31, and 1 and the 2009 Ph.D. thesis of Uminsky which
develops an eigenfunction expansion method based on the form of the heat-kernel. Addition-
ally, we mention the body of work generated by Dritschel and co-workers, of which Refs. 34 and 35 would be two relevant examples, whose aim is to elucidate via Lagrangian type numerical simulations the host of complex processes associated with mixing and dynamics in viscously evolving two-dimensional flows.

The model assumes that the vorticity field at all times is a superposition of multiple Lamb–Oseen vortices,

$$\omega(z, t) = \sum_{\alpha=1}^{N} \frac{\Gamma_{\alpha}}{4\pi\nu t} \exp\left(-\frac{||z - z_{\alpha}||^2}{4\nu t}\right).$$  \hspace{1cm} (10)

The associated velocity field is computed by substituting Eq. (10) into Eq. (2),

$$\mathbf{z}^* = \sum_{\alpha=1}^{N} \frac{\Gamma_{\alpha}}{2\pi i(z - z_{\alpha})} \left[1 - \exp\left(-\frac{||z - z_{\alpha}||^2}{4\nu t}\right)\right].$$  \hspace{1cm} (11)

The velocity at the center $z_{\beta}$ of the $\beta$th vortex is given by subtracting the effect of that vortex and replacing $z$ by $z_{\beta}$ in Eq. (11),

$$z_{\beta}^* = \sum_{\alpha \neq \beta}^{N} \frac{\Gamma_{\alpha}}{2\pi i(z_{\beta} - z_{\alpha})} \left[1 - \exp\left(-\frac{||z_{\beta} - z_{\alpha}||^2}{4\nu t}\right)\right].$$  \hspace{1cm} (12)

The system of equations in Eqs. (10)–(12) is referred to as the multi-Gaussian model. We emphasize that we include in this model the Eq. (11) for the evolution of passive tracers in the field which is transported under the dynamics generated by Eqs. (10) and (12). This will be discussed more thoroughly in Sec. V and is relevant for comparisons of panels (a)–(d) of Fig. 2 with (a)–(d) of Fig. 3.

According to Eq. (10), the vorticity field associated with the initial three-vortex configuration shown in Fig. 1 is given by

$$\omega(z, t) = \frac{1}{4\pi\nu t} \left[2\Gamma \exp\left(-\frac{||z - z_{L}||^2}{4\nu t}\right) - \Gamma \exp\left(-\frac{||z - z_{C}||^2}{4\nu t}\right) + 2\Gamma \exp\left(-\frac{||z - z_{R}||^2}{4\nu t}\right)\right].$$  \hspace{1cm} (13)

The location of the centers of the vortices $z_{L}$, $z_{C}$, and $z_{R}$ is obtained by solving the set of six first-order, ordinary differential equations in Eq. (12). From symmetry, one can readily verify that $z_{C}^* = 0$ and that the centers of the vortices remain collinear and equally spaced with constant distances for all time. The vorticity contours of Eq. (10) are then plotted in Fig. 3 (top row). The streamlines associated with the velocity field in Eq. (12) is shown in Fig. 3 (bottom row). Similarly to the Navier–Stokes solution depicted in Fig. 2, the dynamic evolution of the multi-Gaussian model is characterized by: (i) an unsteady rotation of the whole vortex configuration for $t > 0$, (ii) a stretching of the middle vortex, and (iii) eventual merging of the outer two vortices to form one single-peaked Gaussian of strength $3\Gamma$ as shown in Fig. 3. However, here some care is in order, as clearly Figs. 2(a)–2(d) (DNS) and Figs. 3(b)–3(d) show some important differences. Not only are the timescales different, but Fig. 2(b) shows a convective “wrapping” and “stretching” of the middle vortex around the outer two before the diffusive effects kick in, whereas Fig. 3(b) shows the stretching, but not the wrapping. Here it is important to remember that the passively advected field, as shown in Fig. 13, is an important part of the model, and this field does show some of the same nonlinear wrapping features that appear in the DNS Fig. 2(b). One could say, in some respects, that the outer two vortices, being twice the strength of the inner one, are the primary drivers of the flowfield, which is perhaps why Figs. 2(e)–2(h) match relatively well with Figs. 3(e)–3(h). The “passively advected” inner vortex shown in Fig. 2(b) is better reflected in the passive particle field shown in Fig. 13 and discussed at length in Sec. V. In turn, because the passively advected field in our model is not affecting the vorticity evolution, whereas in the DNS it is, this helps explain why the timescales associated
with the two are different. The model is not an exact solution of the Navier–Stokes equations and this appears to be the main physical manifestation of this fact.

The unsteady rotation rate of the vortex structure is obtained analytically as follows. From Eq. (12), the velocity of one of the outer vortices, say the right vortex, takes the form

\[ \dot{z}_R = \frac{2\Gamma}{2\pi i(z_R - z_i)} \left[ 1 - \exp \left(-\frac{r_0^2}{4\nu t}\right) \right] \]

\[ + \frac{-\Gamma}{2\pi i(z_R - z_C)} \left[ 1 - \exp \left(-\frac{r_0^2}{4\nu t}\right) \right]. \tag{14} \]

Now, by symmetry one has \( z_i = -r_0 \hat{e}_i, z_C = 0, \) and \( z_R = r_0 \hat{e}_i, \) where \( \theta \) is the angle between the line traced by the vortex centers and the x axis, and \( z_R = i\nu_0 \dot{\theta} \hat{e}_i. \) One gets, upon substituting into Eq. (14) and simplifying, that the rotation rate \( \dot{\theta} \) is given by

\[ \dot{\theta} = \frac{\Gamma}{2\pi r_0^2} \exp \left(-\frac{r_0^2}{4\nu t}\right) - \exp \left(-\frac{r_0^2}{4\nu t}\right). \tag{15} \]

In Fig. 4 is a depiction of \( \dot{\theta} \) versus \( \tau = \nu t \) which shows that the rotation rate starts from zero, reaches a maximum value \( \dot{\theta}_{\text{max}} \) at an intermediate time \( \tau_{\text{max}} = \nu t_{\text{max}} \), and eventually decays to zero as \( \nu t \to \infty. \) The values of \( \dot{\theta}_{\text{max}} \) and \( \tau_{\text{max}} \) are given by

\[ \tau_{\text{max}} = \frac{3r_0^2}{8 \ln 2} \approx 0.1353, \]

\[ \dot{\theta}_{\text{max}} = \frac{\Gamma}{2\pi r_0^2} \exp \left(-\frac{2\ln 2}{3}\right) - \exp \left(-\frac{8\ln 2}{3}\right). \tag{16} \]

The orientation angle \( \theta \) can be readily obtained by integrating Eq. (15) in time

\[ \theta = \frac{\Gamma}{2\pi r_0^2} \left[ \exp \left(-\frac{r_0^2}{4\nu t}\right) - \exp \left(-\frac{r_0^2}{4\nu t}\right) \right] \]

\[ + \frac{\dot{\theta}_0}{4} \left[ \exp \left(-\frac{r_0^2}{4\nu t}\right) - \exp \left(-\frac{r_0^2}{4\nu t}\right) \right], \tag{17} \]

where the exponential integral is defined as \( \text{Ei}(x) = -\int_{-x}^{\infty} \exp(t)/t \, dt \) in the sense of principle value, which can be evaluated numerically to machine accuracy.

It is convenient for analyzing the flow to express the fluid velocity field \( \dot{z} \) in a frame corotating with the vortex configuration at the time-dependent rotation \( \dot{\theta} \). Let \( \xi = (\zeta, \eta) \) denote position of a point in the plane expressed in the rotating frame. The point transformation from the rotating to the inertial frame is given by

\[ \dot{z} = R \dot{\xi}, \quad R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \tag{18} \]

The fluid velocity transforms as \( \dot{\mathbf{v}} = R \dot{\mathbf{v}} \) where \( \dot{\mathbf{v}} = (v_\zeta, v_\eta) \) is the velocity field expressed in the rotating frame. In component form, Eq. (11) transforms as

\[ \frac{2\pi}{\Gamma} v_\zeta = \frac{\eta}{\zeta^2 + \eta^2} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \]

\[ - \frac{2\eta}{(\zeta + \rho_0)^2 + \eta^2} \left[ 1 - \exp \left( -\frac{(\zeta + \rho_0)^2 + \eta^2}{4\tau} \right) \right], \tag{19} \]

and

\[ \frac{2\pi}{\Gamma} v_\eta = -\frac{\zeta}{\zeta^2 + \eta^2} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \]

\[ + \frac{2(\zeta + \rho_0)}{(\zeta + \rho_0)^2 + \eta^2} \left[ 1 - \exp \left( -\frac{(\zeta + \rho_0)^2 + \eta^2}{4\tau} \right) \right] \]

\[ + \frac{2(\zeta - \rho_0)}{(\zeta - \rho_0)^2 + \eta^2} \left[ 1 - \exp \left( -\frac{(\zeta - \rho_0)^2 + \eta^2}{4\tau} \right) \right], \tag{20} \]

where we used \( \tau = \nu t. \)

The instantaneous stagnation points of the velocity field [obtained by setting the right-hand side of Eqs. (19) and (20) to zero] reveal important information about the instantaneous streamlines of the fluid velocity field. From symmetry of the velocity field, the instantaneous stagnation points must lie on the \( \zeta \) and \( \eta \) axes. One finds a total of five fixed points: one initially elliptic point at the origin, a pair of initially hyperbolic points at \((0, \pm \eta_0), \) and a pair of initially elliptic points at \((\pm \zeta_0, 0). \) The hyperbolic and elliptic character of these
stagnation points on the \( H \) axis, as shown in Fig. 5. Initially, the separatrix divides the fluid domain into four regions: three regions, one around each point vortex or elliptic point, and a fourth region bounded by the separatrix and the bound at infinity and void of point vortices. As time evolves, the location of the instantaneous stagnation points change as shown in Fig. 5 and the separatrix evolves accordingly. Note that the time-dependent separatrix does not constitute barriers to fluid motion and fluid particles typically move across this separatrix as time evolves as discussed in more details in Sec. V. Figure 6 shows the coordinates of the stagnation points \( \pm \eta_f \) and \( \pm \zeta_f \) as functions of time. The pair of initially hyperbolic points \( (0, \pm \eta_0) \) start from \( (0, \pm r_0/\sqrt{3}) \) then collide together with the elliptic point at the origin in finite time \( \tau_1 \approx 0.016 \) to transform the origin into a hyperbolic point. This collision of instantaneous stagnation points is accompanied by a change in the streamline topology where the region around the center vortex disappears, see Fig. 5. Time \( \tau_1 \) is referred to as the first bifurcation time. (Note that the first bifurcation time \( \tau_1 \) does not correspond to when the cores of the three Gaussian vortices touch for the first time which takes place at \( \tau = r_0/16 = 0.015625 \), nor does it correspond to when the cores of the two outer Gaussian vortices touch \( \tau = r_0^2/4 = 0.0625 \). Indeed, the definition of core size of a Gaussian function is rather ad hoc and bears little relevance to the dynamics of the multi-Gaussian model.) Similarly, \( (\pm \zeta_f, 0) \) starts from \( (\pm r_0, 0) \) and collides at the now hyperbolic point at the origin at time \( \tau_2 \approx 0.086 \). Time \( \tau_2 \) is referred to as the second bifurcation time. For \( \tau > \tau_2 \), one has one single elliptic point at the origin as expected from the asymptotic Lamb–Oseen solution.

IV. COMPARISON TO NAVIER–STOKES

The residual \( \sigma \) of the model is computed by substituting the solution of Eqs. (10) and (11) into the Navier–Stokes Eq. (9). If the solution of the model is also an exact solution of the Navier–Stokes equation for a given set of initial conditions, the residual \( \sigma \) is identically 0. In general, \( \sigma \) is not zero (see discussions of this in Ref. 21) and it can be viewed as an indication of the inaccuracy of the multi-Gaussian model. The \( L_2 \) norm of residual is plotted as a function of time \( \tau = \nu t \) in Fig. 7(a) for the collinear vortex configuration considered here. Figure 7(a) shows that as \( \tau \) increases, the \( L_2 \) norm of \( \sigma \) tends to zero, indicating that the multi-Gaussian model agrees with the Navier–Stokes solution for \( \tau \) large. From the result of Gallay and Wayne, and since the total circulation of the initial vorticity field is \( 3\Gamma \neq 0 \), we know as \( \tau \to \infty \) the Navier–Stokes solution approaches a single Gaussian vorticity distribution \( \omega_\nu = (3\Gamma/4\pi \nu t) \exp(-|z|^2/4\nu t) \) centered at the origin with circulation \( 3\Gamma \). We compute the difference

![Figure 5](image_url)  
**FIG. 5.** Evolution of the streamlines of the multi-Gaussian model. The separatrices are depicted in thick lines with arrows showing the direction of the flow. Instantaneous hyperbolic points are at intersections of separatrices while elliptic points are represented by circles.

![Figure 6](image_url)  
**FIG. 6.** Instantaneous stagnation points: (a) a pair of hyperbolic stagnation point located at \( (0, \pm \eta_0) \) for \( \Gamma = 1, r_0 = 0.5, \) and \( \nu = 10^{-3} \). This pair collides at (0.0) at bifurcation time \( \tau_1 \approx 0.016 \). (b) A pair of elliptic stagnation points located at \( (\pm \zeta_0, 0) \). This pair collides with the now hyperbolic origin at bifurcation time \( \tau_2 \approx 0.086 \).
Multi-Gaussian model and the single-peaked Lamb–Oseen vortex with circulation \( \Gamma \), and \( \nu=10^{-3} \). (b) \( L_2 \) norm of the difference in the vorticity field of the multi-Gaussian model and the single-peaked Lamb–Oseen vortex with circulation \( \Gamma \), and \( \nu=10^{-3} \). (c) shows the difference in velocity field. Clearly, for long time, the model approaches the single-peaked Gaussian but in short time, the multi-Gaussian, while not numerically accurate in comparison to the Navier–Stokes model as indicated in (a), its dynamics is richer than the single Gaussian as indicated in (b) and (c).

between the multi-Gaussian model and this asymptotic solution \( \omega_0 \). Figures 7(b) and 7(c) show the \( L_2 \) norm of the difference in both vorticity and velocity, respectively. These plots confirm that the multi-Gaussian model approaches the asymptotic Lamb–Oseen solution for large time but at the intermediate times, the multi-Gaussian model exhibits richer dynamics than the asymptotic Lamb–Oseen solution. While the dynamics of the multi-Gaussian model at these intermediate time scales does not faithfully track the Navier–Stokes solution [as seen from Fig. 7(a)], it does capture more details than the asymptotic Lamb–Oseen vortex and its evolution seems to exhibit the main qualitative features of the Navier–Stokes model as argued next.

It is evident from Figs. 2 and 3 that both the Navier–Stokes equations and the multi-Gaussian model exhibit a viscosity-induced rotation as \( t>0 \). In Fig. 8 we compare the qualitative trends of rotation angle \( \theta \) obtained from the numerical solution to the Navier–Stokes equation and the analytical solution of multi-Gaussian model. In the Navier–Stokes solution, the rotation angle \( \theta \) is obtained by computing the angle between the line traced by the vorticity peaks (see Fig. 2) and the \( x \) axis while it is given by Eq. (17) in the multi-Gaussian model. Clearly, both the Navier–Stokes solution and the model, although quantitatively distinct, exhibit similar qualitative trends in that the rotation angle \( \theta \) is smaller when \( \text{Re} \) increases (in Navier–Stokes) or equivalently when \( \Gamma/\nu \) increases (in the model). As cautioned earlier about comparing DNS Reynolds numbers with the model Reynolds number, if both are thought of strictly as \( \Gamma/\nu \), we can only claim qualitative overlap with the model and DNS. To obtain more quantitative overlap would require more effort on our part to obtain an accurate Lagrangian based DNS to get a more detailed handle on the effective numerical Reynolds number, along with a modified model system that does more to couple rotational effects with diffusive effects, neither of which are the immediate goals of the current work.

To quantify the difference between the Navier–Stokes solution and the multi-Gaussian model, we focus on comparing the first bifurcation time \( \tau_1 \) in the Navier–Stokes simulation for different \( \text{Re} \) to the first bifurcation time in the model. The result is plotted in log-log scale in Fig. 9 for \( \text{Re}=100, 500, 1000, 2000, 3000, 4000, \) and \( 5000 \) (plotted in squares). The dashed line is the best fitted straight line using the least squares distance rule. The fitted line can be expressed as \( \ln(\tau_1)=\ln(\text{Re})-5.356 \), which means in linear scale, the fitting is \( \tau_1=0.00472 \text{Re} \). The simulation results are compared to the first bifurcation time \( \tau_1=0.0161 \Gamma/\nu \) as predicted by the multi-Gaussian model. While the first bifurcation in the
When minded that the fluid velocity at a point is dependent core. By way of background, the reader is responding to the total circulation of the collinear vortex structure becomes analogous to that of a Rankine vortex at the origin with vorticity $3\Gamma$ and time-dependent core, as seen in Fig. 10. This analogy is especially evident in Fig. 10(b) where we superimpose the velocity field of the Rankine vortex on that induced by the viscously evolving collinear vortex structure at three different instances. Close to the origin, the velocity field of the collinear vortex structure looks like a rigid rotation and the rotation rate is given by $\dot{\theta}$ in Eq. (15). Since the rotation rate $\dot{\theta}$ is unsteady, the core size $R_{cr}$ of the Rankine vortex, obtained by equating $3\Gamma/2\pi R_{cr}^2 = \dot{\theta}$, is time-dependent and it increases with time $t$ as shown in Fig. 10(a). As the distance from the origin increases, the velocity field of the collinear vortex structure decays analogously to the inverse decay with vorticity $3\Gamma$.

Motivated by this analogy with the Rankine vortex, we examine the time evolution of the relative velocity field

$$\tilde{\xi} = \mathbf{v} - \dot{\theta}\mathbf{\xi}_r^\perp,$$  

(22)

obtained by subtracting a rigid body rotation from the fluid velocity field $\mathbf{v}$ expressed in the rotating frame [written in Eqs. (19) and (20) in component form]. Similarly to the analysis in Sec. III, we identify the instantaneous stagnation structure is perpendicular to the distance $r$ from the origin and its value is given by

$$v(r) = \begin{cases} 
3\Gamma & \text{for } r \leq R_{cr} \\
\frac{3\Gamma}{2\pi R_{cr}^2}r & \text{for } r > R_{cr}
\end{cases}$$  

(here $r^2 = \xi^2 + \eta^2$).

The value $R_{cr}$ is referred to as the core of the Rankine vortex. For $r \leq R_{cr}$, the fluid velocity corresponds to a rigid rotation while for $r > R_{cr}$, the velocity field decays proportionally to the inverse of the distance $r$. As time evolves, the velocity field induced by the viscously evolving collinear vortex structure becomes analogous to that of a Rankine vortex with vorticity $3\Gamma$ and time-dependent core, as seen in Fig. 10.

As time evolves, the vorticity field, initially concentrated at $z_c=0$ and $z_{L,R}$ = ±1/2, begins to spread spatially inducing a velocity field similar to that of a Rankine vortex with time-dependent core. By way of background, the reader is reminded that the fluid velocity at a point $(\zeta, \eta)$ associated with a Rankine vortex at the origin with vorticity $3\Gamma$ (corresponding to the total circulation of the collinear vortex structure) is perpendicular to the distance $r$ from the origin and its value is given by

$$v(r) = \begin{cases} 
3\Gamma & \text{for } r \leq R_{cr} \\
\frac{3\Gamma}{2\pi R_{cr}^2}r & \text{for } r > R_{cr}
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Motivated by this analogy with the Rankine vortex, we examine the time evolution of the relative velocity field

$$\tilde{\xi} = \mathbf{v} - \dot{\theta}\mathbf{\xi}_r^\perp,$$  

(22)

obtained by subtracting a rigid body rotation from the fluid velocity field $\mathbf{v}$ expressed in the rotating frame [written in Eqs. (19) and (20) in component form]. Similarly to the analysis in Sec. III, we identify the instantaneous stagnation
points associated with the relative velocity field \( \mathbf{v} \). Immediately as \( t \) increases from zero \( (t > 0) \), in addition to the elliptic stagnation points located at the origin \((0, \pm \eta_2)\) and the hyperbolic points at \((0, \pm \eta_2)\), one gets two new pairs of stagnation points appearing from infinity: one elliptic pair located at \((0, \pm \eta_2)\) and one hyperbolic pair located at \((\pm \zeta_2, 0)\). Figure 11(a) shows the values of \( \pm \eta_2 \) as functions of time. Clearly, \( \pm \eta_2 \) start from \( \pm \infty \) and eventually converge to \( 0, \pm \eta_2 \) as \( t \to \infty \), respectively.

\[
\eta_2(t) = \frac{1}{1 + \frac{t}{10}}
\]

\[
\zeta_2(t) = \frac{1}{1 + \frac{t}{10}}
\]

Figure 12. Evolution of the separatrices of the relative velocity field \( \mathbf{v} \). Instantaneous hyperbolic points are at intersections of separatrices and elliptic points are represented by circles. The outside separatrices and elliptic points in (b) and (c) are plotted out of scale.
FIG. 13. (Color) Colored passive tracers advected by the velocity field \( \mathbf{v} \) given in Eq. (11) and depicted in the frame rotating with the vortex structure. As time evolves, the passive tracers stretch and mix forming large lobes at a finite distance from the initial location of the vortex structure. The separatrices of the relative velocity field \( v - \dot{\mathbf{R}} \) are superimposed in black at various instants in time.

The first bifurcation in the streamline topology is due to the same mechanism explained in Sec. III and takes place at the same time \( \tau_1 = \tau_1 \), see Fig. 12(c). The second bifurcation does not coincide in time with the second bifurcation identified in Sec. III, that is, \( \tau_2 \neq \tau_2 \). It is associated with a change in the streamline topology caused by a collapse of the separatrices associated with the hyperbolic pair \( (\pm \xi_2, 0) \) onto the separatrices of the new hyperbolic point at the origin, see Fig. 12(e). The third bifurcation occurs at \( \tau_3 \) when the hyperbolic points at \( (\pm \xi_2, 0) \) collide with the elliptic points at \( (\pm \eta, \pm r_0, 0) \), respectively, causing them to change to hyperbolic points, see Fig. 12(g). After the third bifurcation, one still has two pairs \( (\pm \xi_2, 0) \) and \( (\pm \eta, 0) \) of stagnation points on the \( \zeta \)-axis but with exchanged hyperbolic/elliptic characters. The fourth bifurcation takes place at \( \tau_4 \) due yet to another collapse of the separatrices of the hyperbolic point at the origin with the separatrices at the new hyperbolic points at \( (\pm r_0, 0) \), see Fig. 12(i). The fifth bifurcation takes place at \( \tau_5 = 0.2045 \) when the now elliptic pair \( (\pm \xi_2, 0) \) collides with the hyperbolic origin causing it to turn into an elliptic point, see Fig. 12(k). This bifurcation sequence turns out to be crucial in dictating the time evolution of the vorticity field which we visualize using colored passive tracers as commonly done in experimental and computational fluid mechanics (see, for example, Ref. 9).

We seed the flow at time \( t=0 \) with passive tracers of four different colors as shown in Fig. 13(a) to distinguish the initial four fluid regions identified in Sec. III, namely, the three regions around the vortices bounded by the separatrix (seeded with red, blue, and green particles, respectively) and the fourth region (seeded with yellow particles) bounded by the separatrix and the bound at infinity. We let the passive tracers be advected by the fluid velocity field given in Eq. (11). Snapshots of the passive tracers at six distinct instants in time are depicted in Fig. 13. As time evolves, the location of the stagnation points and the associated separatrices change. Due to incompressibility, the particles initially in the region around the middle vortex (blue color) “leak” along the unstable branch of separatrices associated with the instantaneous hyperbolic points \( (0, \pm \eta_2) \). At \( \tau_5 \), Fig. 13(b) shows that all the particles are squeezed out of the middle region. Meanwhile as time progresses, the fluid particles in yellow begin to form lobes that stretch at a finite distance away from the initial location of the vortices, see Fig. 13(c). Qualitatively, the passive tracers in Fig. 13(c) indicate a vorticity field similar to that obtained from the Navier–Stokes simulation in Fig. 2(d) (modulo the rigid rotation of the whole structure). The formation of these lobes cannot be explained based on the analysis of the streamline patterns in Sec. III. Indeed, the formation of these lobes is initiated when the yellow passive tracers encounter the separatrices associated with the hyperbolic points of the relative velocity field (22) \( (\pm \xi_2, 0) \) that appear from infinity and move toward the ori-
origin along the $\zeta$-axis [see Fig. 11(b)]. The lobes then stretch and rotate around the elliptic points $(0, \pm \eta_2)$ that appear from infinity and converge to a finite distance away from the origin [see Fig. 11(a)]. Eventually, the passive particles initially placed in the regions around the point vortices, whose detailed evolution is also dictated by the sequence of bifurcations described in Fig. 12, join the large lobes as well and begin to stretch and rotate at a finite distance away from the initial vortex configuration, see Figs. 13(d)–13(f). After the last bifurcation in Fig. 12(k), all the passive particles continue to rotate as shown in Fig. 13(f). We emphasize that this interesting dynamics of the passive particles, which in turn indicates the evolution of the vorticity field, cannot be explained based solely on the analysis of the streamlines of the fluid velocity field of Sec. III. In addition, because of the detailed and delicate nature of the full series of topological bifurcations that occur, to capture all but the first of these in a DNS would require considerable further effort and is beyond the scope of the current manuscript.

VI. CONCLUSIONS

The redistribution (inviscid) and diffusion (viscous) of delta-function initial distributions of vorticity, although configuration independent for sufficiently long timescales, is highly dependent on the initial positions and strengths of the point vortices on short and intermediate timescales. These are typically the timescales in which much of the important mixing, transport, and redistribution of vorticity is achieved in many settings. Greengard’s 1985 paper notwithstanding21 pointing out that the types of models based on advection and core diffusion are not exact solutions of the Navier–Stokes equations, these ideas are exceptionally useful in getting a handle on some of the important dynamical mechanisms that occur during the evolution toward the ultimate Lamb–Oseen state. In fact, one contribution of the current manuscript is to further quantify and understand the limitations of core-growth type models as diagnostic tools for understanding more and more complex flows and to point out some of the delicate issues in comparing a DNS with these models. Not surprisingly, core-growth type models are also useful as starting points for more sophisticated numerical methods which systematically exploit some of the main features16,24 (also, see BLOBFLOW, an open source vortex method package developed by Rossi,36 available at http://www.math.udel.edu/~rossi/BlobFlow as of October 2010).

We summarize here with three main points associated with the viscous evolution of the three-vortex collinear state whose initial configuration corresponds to an unstable inviscid fixed equilibrium:

(i) The presence of viscosity immediately “triggers” the underlying instability of the equilibrium, causing the vortices to rotate unsteadily.

(ii) In a fixed frame of reference, as the system evolves toward the ultimate Lamb–Oseen solution, the streamline patterns associated with the velocity field undergo a clear sequence of topological bifurcations which we depict in Fig. 14. We show the “homotopic equivalence” of each of the distinct patterns in the panels: the time and quantitative values of the pattern are not depicted, just the sequence of distinct patterns that appear during the time sequence.

(iii) More interestingly, since the velocity field near the origin is of approximate solid-body (Rankine) form, if we subtract off this field and replot the homotopic sequence of patterns that emerges, shown in Fig. 15, a far richer and more instructive sequence of patterns is revealed, one that is far more relevant for the understanding of the evolution of passive particle transport, as shown clearly in Fig. 13.

We finish by mentioning connections of this work in two other contexts. First, there is by now a growing body of work on calculating “time-dependent separatrices” in developing flows that goes under the name of “Lagrangian coherent structures.”37,38 Certainly these tools are potentially useful for further elucidating the intermediate timescale dynamics associated with the evolution toward the Lamb–Oseen state, particularly for more complex initial patterns that perhaps start out as relative equilibria of the Euler equations. Second, if one regards the vorticity field as a probability density function associated, for example, with the positions of initial system of point vortices undergoing a random walk, there are meaningful interpretations of the models used in this paper that have been discussed most recently, for example, in Refs. 39–41. While this interpretation has not been the main focus of our work, we do find it potentially ripe for future development.

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