The Hamilton-Jacobi Equations for Strings and p-Branes

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Abstract

Simple derivation of the Hamilton-Jacobi equation for bosonic strings and p-branes is given. The motion of classical strings and p-branes is described by two and $p+1$ local fields, respectively. A variety of local field equations which reduce to the Hamilton-Jacobi equation in the classical limit are given. They are essentially nonlinear, having no linear term.

Dynamics of point particles can be described by the Hamilton-Jacobi (HJ) equation. It is a first-order partial differential equation, given by $(\partial S)^2 = m^2$ for a free relativistic particle. The HJ equation can be viewed as the classical limit of quantum field equations. The Klein-Gordon equation $(\hbar^2 \partial^2 + m^2)\Phi = 0$ or the Dirac equation $(i\hbar\gamma^\mu \partial_\mu - m)\Psi = 0$ reduces to the HJ equation with the ansatz $\Phi \sim \rho e^{iS/\hbar}$ or $\Psi \sim e^{iS/\hbar} \psi$.

The argument can be reversed for strings and p-branes. We attempt to first find the classical HJ equations for strings and p-branes, then use them as a guide to find quantum
string and p-branes equations. In particular, establishing the Hamilton-Jacobi equation for classical superstrings would serve as a big step for an ultimate formulation of quantum superstrings.

The Hamilton-Jacobi equation for strings has been discussed for many years. Rinke and Kastrup have given the HJ equation for strings, though their derivation being rather involved. For point particles the HJ function $S(x)$ can be identified with the action of trajectories ending at $x$. Such identification is not applicable to strings. The purpose of this Letter is to give simple derivation of the HJ equations for bosonic strings and p-branes, and to propose local field equations which reduce to the HJ equations in the classical limit. We believe that the simplicity and clarity of our derivation enhance the usefulness of the HJ equation.

We start with the Nambu-Goto action for strings:

$$I = -M^2 \int d\tau d\sigma \sqrt{-\frac{1}{2}v_{\mu\nu}^2}$$

where $v_{\mu\nu} = \partial(x^\mu, x^\nu)/\partial(\tau, \sigma)$. In terms of covariant momentum tensors

$$p_{\mu\nu} = M^2 \frac{v_{\mu\nu}}{\sqrt{-v^2/2}}, \quad -\frac{1}{2}p_{\mu\nu}p^{\mu\nu} = M^4$$

the equations of motions takes the form $\partial(p_{\mu\nu}, x^\nu)/\partial(\tau, \sigma) = 0$.

The key step for the HJ formulation is to consider a family of solutions to the equations such that they fill a $d$-dimensional domain in spacetime. There are $d-2$ parameters, $\phi_a$'s, to specify these solutions: $x^\mu = x^\mu(\tau, \sigma; \phi_1, \cdots, \phi_{d-2})$. They define a mapping from the parameter space $(\tau, \sigma, \phi_1, \cdots, \phi_{d-2})$ onto the spacetime domain $(x^\mu)$. In a region where the mapping is one-to-one, $p_{\mu\nu}(\tau, \sigma; \phi_a)$ can be viewed as a local field $p_{\mu\nu}(x)$. Then the equation of motion is tranformed to

$$p^{\mu\nu}\partial_\mu p_{\nu\lambda} = 0.$$  

(3)

Making use of the reparametrization invariance, one can choose $(\tau, \sigma)$ such that $v^2_{\mu\nu} = -2$, independent of $\tau, \sigma$ and $\phi_a$. Then $M^{-2}p^{\mu\nu}d\tau \wedge d\sigma = dx^\mu \wedge dx^\nu$ is the area element of the world sheet with fixed $\phi_a$'s. Further the two form $J(x) \equiv \frac{1}{2}p_{\mu\nu}dx^\mu \wedge dx^\nu = -M^2d\tau \wedge d\sigma$. With the mapping between \{\tau, \sigma, \phi_a\} and \{x^\mu\}, \tau and \sigma are regarded as local fields. Choosing $S_1 = M\tau(x)$ and $S_2 = -M\sigma(x)$, $p_{\mu\nu}(x)$ is expressed as

$$p_{\mu\nu} = \partial_\mu S_1 \partial_\nu S_2 - \partial_\nu S_1 \partial_\mu S_2.$$  

(4)
It is important to recognize that the equation of motion for strings is contained in the normalization condition. Indeed with (4), Eq. (3) becomes \( p^{\mu\nu} \partial_\mu P_{\nu\lambda} = \frac{1}{2} p^{\mu\nu} \partial_\mu P_{\nu\lambda} - \frac{1}{2} \partial_\lambda (p^{\mu\nu} P_{\mu\nu}) = -\frac{1}{2} \partial_\lambda (p^{\mu\nu} P_{\mu\nu}) = 0 \). Hence it is reduced to the normalization condition \( p^{\mu\nu} P_{\mu\nu} = \text{constant} \), or to

\[
(\partial S_1)^2 (\partial S_2)^2 - (\partial S_1 \partial S_2)^2 = -M^4 .
\] (5)

Conversely, if the \( S_1 \) and \( S_2 \) satisfy the condition (3), then \( p_{\mu\nu} \) given by (4) satisfies the equation of motion. From \( p_{\mu\nu}(x) \) a family of the Nambu-Goto string solutions are reconstructed. Eq. (2) is the Hamilton-Jacobi equation for bosonic strings, first derived by Rinke.\[1\] The usage of a family of solutions makes the proof significantly simple and transparent compared with those of Rinke\[1\] and of Kastrup\[6\]. The equivalence holds in any dimensions. Eqs. (1) and (5) naturally generalize \( p_\mu = \partial_\mu S \) and \( (\partial S)^2 = m^2 \) for point particles.

Kastrup, Nambu and Rinke proposed a relation \( p_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) a long time ago.\[4\] We see \( A_\mu = S_1 \partial_\mu S_2 \) up to a gauge transformation. The vector potential description is redundant. It was shown by one of the authors that string motion can be expressed in terms of \((d - 2)\) local scalar fields,\[5\] which is unnecessarily general and redundant.

The above argument applies to the point particle case as well. The equation of motion is \( \dot{p}^\mu = 0 \) where \( p^\mu = m \dot{x}^\mu / \sqrt{x^2} \). A family of solutions are parametrized by \( d - 1 \) \( \phi_a \)'s; \( x^\mu = x^\mu (\tau; \phi_1, \cdots, \phi_{d-1}) \). With this mapping the equation is converted to \( p^\mu \partial_\mu p_\lambda = 0 \). With the parametrization \( \dot{x}^2 = 1 \), \( p_\mu dx^\mu = md\tau \) so that the covariant momentum can be expressed as \( p_\mu = \partial_\mu S \). The equation of motion is reduced to \( p^\mu \partial_\mu p_\lambda = p^\mu \partial_\mu p_\mu = \frac{1}{2} \partial_\lambda p^2 = 0 \), i.e. \( (\partial S)^2 = m^2 \). If one considers a family of trajectories starting at one point \( x^\mu_0 \) at \( \tau = 0 \), \( \phi_a \)'s are just momenta at \( x^\mu_0 \). With this choice \( S(x) \) is the action evaluated at \( x \).

In the case of strings the meaning of \( S_1(x) \) and \( S_2(x) \) is yet to be found.

The generalization to p-branes is straightforward. The action for p-branes in the Nambu-Goto form is

\[
I = -M^{p+1} \int d\tau d^p\sigma \left[ \frac{(-p v^2)}{(p + 1)!} \right]^{1/2}
\] (6)

where \( v^{\mu_1 \cdots \mu_{p+1}} = \partial(x^{\mu_1}, \cdots, x^{\mu_{p+1}})/\partial(\tau, \sigma_1, \cdots, \sigma_p) \). The covariant momentum tensors are given by \( p^{\mu_1 \cdots \mu_{p+1}} = M^{p+1} \mu_1 \cdots \mu_{p+1} \) \( \left[ (-p v^2/(p + 1)! \right]^{-1/2} \). The equation of motion is given by \( \partial(p_{\mu_1 \cdots \mu_{p+1}}, x^{\mu_1}, \cdots, x^{\mu_p})/\partial(\tau, \sigma_1, \cdots, \sigma_p) = 0 \). Again by considering a family of solutions the equation is converted to

\[
p^{\mu_1 \cdots \mu_{p+1}} \partial_\mu_1 p_{\mu_2 \cdots \mu_{p+1}} = 0 .
\] (7)
With the parametrization $v^2 = (-1)^p(p + 1)!$, $p_{\mu_1 \cdots \mu_{p+1}}dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \propto d\tau \wedge d\sigma_1 \wedge \cdots \wedge d\sigma_p$. The covariant momentum tensor field is represented in terms of $(p + 1)$ local scalar fields:

$$p_{\mu_1 \cdots \mu_{p+1}} = \frac{\partial(S_1, \ldots, S_{p+1})}{\partial(x^{\mu_1}, \ldots, x^{\mu_{p+1})}}.$$  

(8)

In this representation $p_{\mu_1 \cdots \mu_{p+1}}\partial_{\mu_1}p_{\mu_2 \cdots \mu_{p+1}} = (-1)^p\partial_\nu(p_{\mu_1 \cdots \mu_{p+1}}p_{\mu_{p+1} \cdots \mu_{p+1}})/2(p+1)$ so that the equation of motion becomes equivalent to the Hamilton-Jacobi equation given by

$$\frac{1}{(p + 1)!} \left\{ \frac{\partial(S_1, \ldots, S_{p+1})}{\partial(x^{\mu_1}, \ldots, x^{\mu_{p+1})}} \right\}^2 = (-1)^p M^2(p+1).$$  

(9)

The equation for membranes ($p = 2$) has been discussed by Aurilia et al.\[11\]

Quantum equations which reduce to the Hamilton-Jacobi equation in the classical limit $\hbar \to 0$ can be easily found. There are several options. In the case of strings ($p$-branes) one may start with a functional of lines ($p$-dimensional surfaces). This approach leads to the string ($p$-brane) field theory. In this Letter we propose an alternative approach. We look for quantum equations in the form of local field equations, which turn out essentially nonlinear with no “free” part. Being local field equations, they are expected to describe only a part of dynamics of quantum strings. Yet it is interesting and noteworthy that they have connection to classical string dynamics through the HJ equation.

The first candidate is given by

$$\mathcal{L}_1 = -\frac{\hbar^4}{2} \Sigma_{\mu\nu}^\dagger \Sigma^{\mu\nu} - M^4\Phi_1^\dagger\Phi_1\Phi_2^\dagger\Phi_2,$$

where $\Phi_a$'s are complex scalar fields. The Euler equations reduce to the HJ equation \((5)\) in the $\hbar \to 0$ limit with the ansatz $\Phi_a = \rho_a e^{iS_a}/\hbar$. The Lagrangian \((10)\) contains no bilinear terms. The Hamiltonian density is positive semi-definite: $\mathcal{H}_1 = \hbar^4(\Sigma_{0k}^\dagger \Sigma_{0k} + \frac{1}{2} \Sigma_{jk}^\dagger \Sigma_{jk}) + M^4\Phi_1^\dagger\Phi_1\Phi_2^\dagger\Phi_2$. However, expressed in terms of conjugate momenta $\Pi_a = \hbar^4 \epsilon_{ab} \Sigma_{0k}^\dagger \partial_k \Phi_b$, it appears singular when $\nabla \Phi_1 \propto \nabla \Phi_2$:

$$\mathcal{H}_1 = \frac{1}{\hbar^4} \frac{\Pi_a^\dagger \partial_j \Phi^\dagger_j \Phi_a \Pi_b}{|\nabla \Phi_1|^2 |\nabla \Phi_2|^2 - |\nabla \Phi_1 | \nabla \Phi_2|^2} + \frac{\hbar^4}{2} \Sigma_{jk}^\dagger \Sigma_{jk} + M^4\Phi_1^\dagger\Phi_1\Phi_2^\dagger\Phi_2.$$  

(11)

This is an essentially nonlinear system. Although the canonical quantization can be carried out, the full consistency is yet to be examined.

The second candidate is obtained by generalizing Dirac’s approach. We prepare two multi-component fields $\Phi$ and $\Psi$ and write

$$\mathcal{L}_2 = \hbar^2 \bar{\Psi}_a \Phi_b \Gamma^{\mu\nu}_{ab,cd} \partial_\mu \Phi_d \partial_\nu \Psi_c - M^2 \bar{\Psi}_a \Psi_a \Phi_b \Phi_b.$$  

(12)
With the ansatz \( \Psi = e^{iS_1/\hbar} \psi \) and \( \Phi = e^{iS_2/\hbar} \phi \), the Euler equations reduce in the \( \hbar \to 0 \) limit to

\[
\Gamma^{\mu \nu}_{ab,cd} \partial_\mu S_1 \partial_\nu S_2 + M^2 \delta_{ac} \delta_{bd} = 0 ,
\]

or in short \( \Gamma^{\mu \nu}_{\mu} \partial_\mu S_1 \partial_\nu S_2 + M^2 = 0 \). This equation becomes identical to the HJ equation (13), provided

\[
\{ \Gamma^{\mu \nu}, \Gamma^{\rho \sigma} \}_{ab,cd} = -2(g^{\mu \rho} g^{\nu \sigma} - g^{\mu \sigma} g^{\nu \rho}) \delta_{ac} \delta_{bd} .
\]

\( \Gamma^{\mu \nu} \)'s act on the two fields. If \( \Psi \) and \( \Phi \) have \( n_1 \) and \( n_2 \) components, respectively, then \( \Gamma^{\mu \nu} \)'s are \( n_1 n_2 \) dimensional matrices. To find representations of the algebra (14), first consider the same algebra obeyed by \( n \)-dimensional matrices \( \gamma^{\mu \nu} \): \( \{ \gamma^{\mu \nu}, \gamma^{\rho \sigma} \} = -2(g^{\mu \rho} g^{\nu \sigma} - g^{\mu \sigma} g^{\nu \rho}) \). Since \( \{ \gamma^{\mu \nu} \} \) defines \( d(d-1)/2 \) dimensional Clifford algebra in \( d \) dimensions, \( n = (d-1)/2 \) in the minimal representation. This algebra for \( \gamma^{\mu \nu} \)'s has been previously investigated in the context of string and p-brane field equations in refs. [9] and [10].

Suppose that \( \Phi \) is \( m \)-component scalar field and \( \Psi_a \) is in a “spinor” representation of the \( \gamma^{\mu \nu} \) algebra. In other words we write

\[
\Gamma^{\mu \nu}_{ab,cd} = \gamma^{\mu \nu}_{ac} v_{bd} ,
\]

which satisfies (14) provided \( v_{ab} v_{cd} = \delta_{ac} \). The simplest choice is to consider a single component field \( \Phi (m = 1) \), which leads to

\[
\mathcal{L}_3 = \hbar^2 \overline{\Psi} \gamma^{\mu \nu} \partial_\mu \Psi \cdot \Phi^\dagger \partial_\nu \Phi - M^2 \overline{\Psi} \gamma^{\mu \nu} \partial_\mu \Phi .
\]

The Lorentz transformation properties of \( \Psi \) are clarified by constructing the corresponding generators. Under a Lorentz transformation \( \delta x^\mu = \epsilon^\mu_{\nu} x^\nu \), \( \delta \Psi = \frac{1}{2} \epsilon^{\mu \nu} s^{\mu \nu}_{ab} \Psi_b \).

As is easily confirmed, \( s^{\mu \nu} = -\frac{1}{4} [\gamma^{\mu \alpha}, \gamma^{\nu \alpha}] \) satisfies the desired Lorentz algebra. The Dirac conjugate is given by \( \overline{\Psi} = \Psi^\dagger \omega \) where \( \omega = \omega^\dagger = (i \text{ or } 1) \prod_{j=1}^{d-1} \gamma^{0j} \) for \( d = 4n \) or \( 4n + 2 \), respectively. Further \( -i [s^{\rho \sigma}, \gamma^{\mu \nu}] = g^{\mu \rho} \gamma^{\nu \sigma} + g^{\mu \sigma} \gamma^{\nu \rho} - g^{\mu \rho} \gamma^{\nu \sigma} - g^{\mu \sigma} \gamma^{\nu \rho} \) and \( \omega^{-1} s^{\mu \nu} \omega = s^{\mu \nu} \). These guarantee the invariance of the Lagrangian (16) under proper Lorentz transformations. In \( d = 4 \), for instance, \( \Psi \) has 8 components, which consists of two vectors as can be seen from \( s^{\mu \nu} \).

The discrete symmetry properties are subtle, however. Under parity \( P \), \( (x^0, \vec{x}) \to (x^0, -\vec{x}) \) and \( \Psi(x) \to \Psi^{\dagger}(x') = U_P \Psi(x) \). In order for the Lagrangian (16) to be invariant under \( P \), we need (i) \( U_P^\dagger \omega U_P = \omega \) and (ii) \( U_P^\dagger (\gamma^{0k}, \gamma^{jk}) U_P = (-\gamma^{0k}, \gamma^{jk}) \). These two are incompatible in even dimensions. The Lagrangian (16) is not invariant under \( P \).
Another choice is to consider $\Psi$ and $\Phi$ in the same $n$-plet representation of the $\gamma^{\mu\nu}$ algebra. Depending on the dimensionality, however, one needs two copies of either $\Psi$ or $\Phi$. In $d = 4 \ (\text{mod} \ 4)$, $\bar{\gamma} = \prod \gamma^{\mu\nu}$ (up to a phase) satisfies $\{\bar{\gamma}, \gamma^{\mu\nu}\} = 0$, $\bar{\gamma}\dagger = \bar{\gamma}$, and $\bar{\gamma}^2 = 1$. We observe

$$\Gamma_{ab,cd}^{\mu\nu} = \frac{1}{\sqrt{2}} \left( \gamma_{ac}^{\mu\nu} \delta_{bd} + \bar{\gamma}_{ac} \gamma_{bd}^{\mu\nu} \right),$$

or in short $\Gamma^{\mu\nu} = (\gamma^{\mu\nu} \otimes I + \bar{\gamma} \otimes \gamma^{\mu\nu})/\sqrt{2}$, satisfies the algebra (14). The corresponding Lagrangian is

$$L_4 = \frac{\hbar^2}{\sqrt{2}} \left\{ \bar{\Psi} \gamma^{\mu\nu} \partial_{\nu} \Psi \cdot \bar{\Phi} \partial_{\mu} \Phi - \bar{\Phi} \gamma^{\mu\nu} \partial_{\nu} \Phi \cdot \bar{\Psi} \gamma_{\mu\nu} \Psi \right\} - M^2 \bar{\Psi} \Psi \cdot \bar{\Phi} \Phi.$$  

The Lorentz generators are given by $S^{\mu\nu} = s^{\mu\nu} \otimes I + I \otimes s^{\mu\nu}$. As $[\bar{\gamma}, s^{\mu\nu}] = 0$, the Lagrangian $L_4$ is Lorentz invariant. The invariance of $L_4$ under $P$ is achieved if there exists $U_P$ such that (i) $U_P^\dagger \omega U_P = +\omega$ or $-\omega$, and (ii) $U_P^\dagger (\gamma^{0k}, \gamma^{jk}, \bar{\gamma}) U_P = (-\gamma^{0k}, \gamma^{jk}, \bar{\gamma})$. The conditions for $\gamma^{0k}$ and $\bar{\gamma}$ are incompatible. $L_4$ in the minimal representation is not invariant under $P$ in $d = 4 \ (\text{mod} \ 4)$. In $d = 2 \ (\text{mod} \ 4)$ the dimension of $\Psi$ must be doubled to have $\bar{\gamma}$. With the doubling the Lagrangian can be made $P$ invariant in even dimensions. Write $\Gamma^{\mu\nu} = (\hat{\gamma}^{\mu\nu} \otimes I + \bar{\gamma} \otimes \gamma^{\mu\nu})/\sqrt{2}$ in place of (17). The Dirac conjugate of $\Psi$ is written as $\Psi^\dagger = \Psi \hat{\omega}$. Under $P$, $\Psi \rightarrow \hat{U}_P \Psi$. The appropriate choice is $: \hat{\gamma}^{\mu\nu} = \gamma^{\mu\nu} \otimes \tau_1$, $\bar{\gamma} = I \otimes \tau_3$, $\hat{U}_P = i \omega \otimes \tau_2$. $\hat{\omega}$ can be either $\omega \otimes 1$ or $\omega \otimes \tau_2$.

In this paper we have given simple derivation of the Hamilton-Jacobi equations for bosonic strings and p-branes, Eqs. (5) and (9). We have then explored local field equations which reduce to the HJ equations in the classical limit. There are a variety of possibilities. In all cases the equations are essentially nonlinear, which makes the quantization formidable. We have not known whether the systems proposed are well defined at the quantum level. The critical dimensionality inherent in string theories must arise at the quantum level. The $\Gamma^{\mu\nu}$ algebra (14) is naturally associated with strings. Yet its implementation in field theories in (14) and (18) requires more elaboration.

As remarked before, the local field equations proposed above are expected to describe at best a part of string dynamics. Ultimate string field equations should be formulated in loop space with their projection yielding the local field equations proposed here. The extension of the current formalism to superstrings is yet to be achieved. We believe that the Hamilton-Jacobi equations should serve as a helpful guide in searching such ultimate superstring field equations.
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