Stopping Criteria for, and Strong Convergence of, Stochastic Gradient Descent on Bottou-Curtis-Nocedal Functions

Vivak Patel

Abstract While Stochastic Gradient Descent (SGD) is a rather efficient algorithm for data-driven problems, it is an incomplete optimization algorithm as it lacks stopping criteria, which has limited its adoption in situations where such criteria are necessary. Unlike stopping criteria for deterministic methods, stopping criteria for SGD require a detailed understanding of (A) strong convergence, (B) whether the criteria will be triggered, (C) how false negatives are controlled, and (D) how false positives are controlled. In order to address these issues, we first prove strong global convergence (i.e., convergence with probability one) of SGD on a popular and general class of convex and nonconvex functions that are specified by, what we call, the Bottou-Curtis-Nocedal structure. Our proof of strong global convergence refines many techniques currently in the literature and employs new ones that are of independent interest. With strong convergence established, we then present several stopping criteria and rigorously explore whether they will be triggered in finite time and supply bounds on false negative probabilities. Ultimately, we lay a foundation for rigorously developing stopping criteria for SGD methods for a broad class of functions, in hopes of making SGD a more complete optimization algorithm with greater adoption for data-driven problems.

Keywords Stochastic Gradient Descent · Nonconvex · Stopping Criteria · Strong Convergence

This work is supported by the Wisconsin Alumni Research Foundation.

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1 Introduction

In data-driven and simulation-based disciplines, the optimization problem

\[
\min_{\theta} F(\theta) \tag{1}
\]

is frequently solved, where \( F(\theta) = \mathbb{E}[f(\theta, X)] \); \( f : \mathbb{R}^p \times \mathbb{R}^d \rightarrow \mathbb{R} \); \( X \) is a random variable with either finite, countable or uncountable support; and \( \mathbb{E} \) is the expectation operator. Depending on \( X \), the optimization problem’s objective function and its gradient may be impractical or impossible to evaluate directly \([2]\). Fortunately, when certain regularity conditions hold on \( f \) and \( X \) (e.g., \([4, 33]\)), the optimization problem’s structure is exploited to generate solvers that use the gradient of \( f \) with respect to the argument \( \theta \) for independent copies of \( X \) \([30, 7]\). Moreover, when the gradient of \( f \) is significantly cheaper to compute than the gradient of \( F \), the optimization problem can be efficiently solved using these so-called Stochastic Gradient Descent (SGD) methods \([2]\).

Despite the potential efficiency of SGD methods, they are incomplete algorithms as they lack of stopping criteria, which has limited their applicability to contexts where such criteria are necessary.\(^1\) Unfortunately, rigorously developing stopping criteria is particularly challenging for SGD methods as they require a more complex analysis in comparison to their deterministic counterparts. In particular, stopping criteria for SGD methods require understanding: (A) strong convergence (i.e., with probability one) to a stationary point, (B) detectability, (C) false negative control, and (D) false positive control. Below, these four points are discussed in detail as they relate to gradient-based stopping criteria.

(A) First, gradient-based stochastic stopping criteria require that the iterates converge with probability one to a stationary point—it is not enough to have convergence in probability. To illustrate, consider stopping a sequence of \( \{0, 1\} \)-valued, independent random variables when we observe the first zero. Moreover, to make this more realistic, this stopping criteria is only evaluated at periodic iterates, \( \{T_j\} \subset \mathbb{N} \), which we can let be random. To be more specific, consider the stopping criteria applied to an independent sequence, \( \{X_k\} \), with \( \mathbb{P}[X_k = 0] = 1 - k^{-\alpha} \) for some \( \alpha \in (0, 1] \). Then, \( \{X_k\} \) is converging to zero in probability, yet, by the second Borel-Cantelli lemma, \( \mathbb{P}[X_k = 1, \text{ i.o.}] = 1 \), where i.o. means infinitely often. As a result, it is entirely possible that \( X_{T_j} = 1 \) for all \( j \); that is, because strong convergence does not hold, a poor choice of \( \{T_j\} \) would prevent us from ever detecting that the random variables are zero. Thus, stopping criteria are predicated on establishing strong convergence, rendering convergence in probability insufficient.

Unfortunately, demonstrating strong convergence has been achieved with varying success. When \( F \) is convex, strong convergence can be readily established along with other convergence rate results \([7, 2, 4]\). Curiously, even when \( F \) is convex, strong convergence has not translated to useful stopping criteria

\(^1\) There are other reasons that SGD methods are incomplete. For example, see \([8]\).
with the exception of the limited stopping criteria demonstrated for quadratic problems [26, 5].

When $F$ is nonconvex, convergence in probability is already sufficiently challenging, even under the myriad of notions of nonconvexity that we catalog in §2. In fact, under the general Bottou-Curtis-Nocedal (BCN) structure (see §2.4), as popularized by [4], strong convergence has yet to be established. Despite this, there are results under more stringent structures:

1. In addition to the BCN structure, if twice differentiability is assumed and the Hessian-Gradient product is Lipschitz continuous, then convergence in probability has been demonstrated (see Corollary 4.12 of [4]).
2. If $f \geq 0$ with probability one and $\dot{f}$ is uniformly Lipschitz continuous in $\theta$ with probability one, then convergence in probability has also been demonstrated (see Theorem 2(c) of [20]).
3. In addition to the BCN structure, if $F$ is assumed to be Lipschitz continuous and $\mathbb{P} \left[ \bigcap_{\theta} \left\| \frac{\partial f}{\partial \theta}(\theta, X) - \frac{\partial F}{\partial \theta}(\theta) \right\|_2 \leq C \right] = 1$, then strong convergence has been demonstrated (see Theorem 1 of [21]).

This ultimate result is a rather important contribution, but the additional structural conditions exclude the simple linear regression problem, which is covered by the BCN structure. In summary, strong convergence for general, BCN nonconvex functions has yet to be established.

In order to address this gap, our first contribution is to prove the strong convergence (i.e., convergence with probability one) of SGD to a stationary point under the general BCN structure (see Corollary 1), which generalizes and strengthens the preceding results. Our proof employs several strategies that either refine current techniques or are atypical in the stochastic optimization literature; our general strategy and how it is distinct is discussed at the beginning of §3.2. Owing to this contribution, we have cleared the first barrier to developing stopping criteria for SGD methods.

(B) Owing to the fact that stopping criteria are evaluated at random times or depend on the values of random quantities, a stopping criterion may not be satisfied even though we have strong convergence. To illustrate, consider applying the independent-vote stopping criteria (see SC-4 below)—that is, stop if

$$\frac{1}{N} \sum_{i=1}^{N} \mathbf{1} \left[ \left| \frac{\partial f}{\partial \theta}(\theta, X_i) \right| \leq \epsilon \right] \geq \gamma,$$

where $\dot{f}(\theta, X_i)$ is the derivative of $f$ with respect to $\theta$; $X, X_i$ are independent and identically distributed; $\gamma, \epsilon \in (0, 1)$—to $f(\theta, X) = \theta X$ where $\theta \in \mathbb{R}$ and $X$ is a Rademacher random variable. Then, $\dot{F}(\theta) = 0$ for all $\theta$, yet the stopping criteria will never be triggered since $|\dot{f}(\theta, X_i)| = 1$.

To our knowledge, even the limited previous work on stopping criteria for convex functions, $F$, has not addressed the issue of detectability [26, 5]. In order to address the detectability issue, we will rigorously develop three stopping criteria by demonstrating that they will be triggered in finite time.
with probability one under common or feasible structural constraints. Thus, we will also address the detectability concerns that arise with stopping criteria for SGD methods.

\textbf{(C & D)} Owing to the fact that stopping criteria depend on the values of random quantities, a stopping criteria may incorrectly

1. (false negative) fail to stop the algorithm even though the norm of the iterate’s gradient is sufficiently small;\(^2\) or

2. (false positive) stop the algorithm even though the norm of the iterate’s gradient is too large.

Ideally, a stopping criteria is specified in order to control the probability of both types of falsehoods from occurring.

To our knowledge, there are no stopping criteria that have been specified that have rigorously addressed either of these concerns. Therefore, in this work, we will derive how three stopping criteria control the false negative probability. Unfortunately, we will not attempt to make conclusions about the false positive rate as this often depends on specifying probability bounds on the cumulative distribution function of \(\|f(\theta, X)\|_2\), which is an atypical consideration in the stochastic optimization literature. Thus, we will leave it to future work.

To summarize, in this work we

1. Prove strong convergence of SGD (§3.1) on general, nonconvex Bottou-Curtis-Nocedal functions (§2.4), which generalizes the existing results in the literature and is a prerequisite to rigorously developing stopping criteria; and

2. state several stopping criteria for SGD methods, and rigorously develop three of them by demonstrating their detectability in finite time with probability one and by deriving bound on the false negative probability.

The remainder of this paper is organized as follows. In §2, we review catalog and organize common structural conditions on \(F\) that specify its nonconvex structure, and, in particular, in §2.4, we specify the general Bottou-Curtis-Nocedal structure [4]. In §3, we precisely specify the stochastic gradient descent iterates (§3.2), and prove that these iterates converge strongly to a stationary point. In §4, we introduce and analyze several stopping criteria. In §5, we conclude this work.

2 Bottou-Curtis-Nocedal and Other Structural Conditions

There are many structural conditions in the literature and many of them are intimately related. Our goal here is to define these structural conditions as they appear in the literature, and demonstrate how they are related. At the end, we state the Bottou-Curtis-Nocedal (BCN) structure, which was popularized in [4] and which is the main structure under consideration in this work.

\(^2\) The notions of “sufficiently small” or “too large” are dependent on the application, just as they are in deterministic optimization.
Before enumerating these distinct conditions, we note that there is a core assumption that $F$ is bounded from below, both $\tilde{F}$ and $\tilde{f}$ exist, and that $\tilde{f}$ is unbiased. While it is possible to allow for bias in $\tilde{f}$ (see §4 of [4]), this is only useful when we are using a scalar step size; if we consider a matrix-based step size, as we will do here, then the bias in $\tilde{f}$ cannot be allowed without extra assumptions. Hence, we will take $\tilde{f}$ to be unbiased and we simply point out that the biased case with scalar step sizes is an easier analysis than what we will present below.

2.1 Control on the Iterate Space

There are three common ways in which the space of iterates of $\theta$ is controlled. The first condition assumes

(IS-1) The argument, $\theta$, is restricted to a compact, convex subset of $\mathbb{R}^p$, denoted by $\Theta$ [19].

Condition IS-1 is quite convenient in the analysis of SGD methods because it prevents the iterates from diverging out of a compact set—if an iterate is pushed outside of the convex, compact set then it is simply projected back into the set. As a result of this projection operation, the analysis of SGD methods where the iterates are restricted to a convex, compact is a special case of when the situation where $\Theta$ is not required to be bounded [2].

A less restrictive approach assumes

(IS-2) The argument, $\theta$, is restricted to a closed, convex subset of $\mathbb{R}^p$, denoted by $\Theta$ [13,31].

Condition IS-2 certainly contains Condition IS-1, and thus is more general. However, just as for Condition IS-1, Condition IS-2 is imposed by projecting the iterates into $\Theta$. Again, as a result of this projection operation, the analysis of SGD methods where the iterates are restricted to a convex, closed set is a special case of the analysis in which $\Theta$ is the whole space $\mathbb{R}^p$.

Thus, the final way in which the space of iterates of $\theta$ is controlled is to allow $\theta$ to be unrestricted. For uniformity, we define this as

(IS-3) The argument, $\theta$, is unrestricted. That is $\theta$ can take any value in $\mathbb{R}^p$.

2.2 Control on the Objective

There are several common ways in which the stochastic objective (SO) function, $f$, is directly controlled. The first one assumes
There exists a $C > 0$ such that
\[
\mathbb{P} \left[ \bigcap_{\theta} \{|f(\theta, X)| < C\} \right] = 1.
\]

Condition **SO-1** is assumed in [14,35]. In fact, both of these works require more stringent controls on $f$ and its derivatives. However, in [14], these added controls allow for the approximation of SGD methods by stochastic differential equations, which are then leveraged to characterize the escape times of an SGD method from a saddle point. It is worth noting that such a characterization supplies a more complete analysis than the more recent results of [10,16], which only supply conditions for finding an $\epsilon$-approximate stationary point with a nearly positive definite Hessian, but do not guarantee that such a point is stable (i.e., that the SGD method will not escape this point as well).

A natural relaxation of Condition **SO-1** is to assume that $F(\theta)$ is bounded from above and below. While bounding $F(\theta)$ from below is necessary when $\theta$ is not restricted to a compact subset, there were no works in our review that directly assumed that $F(\theta)$ is bounded from above. The closest condition in this regard come from [11], which assumes
\[
\text{There exists a } C > 0 \text{ such that for all } \theta \in \mathbb{R}^p,
\quad \mathbb{E} \left[ |f(\theta, X)|^2 \right] \leq C.
\]

Note, by Jensen’s inequality, Condition **SO-2** implies
\[
|F(\theta)|^2 = \mathbb{E} \left[ |f(\theta, X)|^2 \right] \leq \mathbb{E} \left[ |f(\theta, X)|^2 \right] \leq C;
\]
that is, Condition **SO-2** implies $F$ is bounded.

An alternative restriction to bounding the stochastic objective is to impose Lipschitz continuity, which is often more appropriate for a number of data-driven optimization problems. One specific form of using such continuity is assumed in [28,25], and specified by
\[
\text{There exists a } C > 0 \text{ such that}
\quad\mathbb{P} \left[ \bigcap_{\theta_1, \theta_2} \{|f(\theta_1, X) - f(\theta_2, X)| \leq C \|\theta_1 - \theta_2\|_2\} \right] = 1.
\]

Unfortunately, Condition **SO-3** excludes our litmus problem (i.e., the standard linear regression problem). Interestingly, Condition **SO-3** is assumed locally in [11] along with twice differentiability and constraints on the expected
Hessian of $F$ in order to establish local rates of convergence for nonconvex functions.

The natural relaxations of Conditions **SO-1** and **SO-3** to the deterministic objective (DO) function, $F$, are

**(DO-1)** There exists a $C > 0$ such that $|F(\theta)| \leq C$ for all $\theta$.

and

**(DO-2)** There exists a $C > 0$ such that for all $\theta_1$ and $\theta_2$,

\[ |F(\theta_1) - F(\theta_2)| \leq C \|\theta_1 - \theta_2\|_2. \]

By Jensen’s inequality, it follows that Condition **SO-1** implies Condition **DO-1**, and that Condition **SO-3** implies Condition **DO-2**. Note, Condition **DO-1** is used in [14] to establish an SDE approximation to the SGD iterates, and establish rather fine-resolutions results about the escape times of the SGD from saddle points. Condition **DO-2** was used in the early nonconvex results of [28,13], and was used recently by [25] to establish that SGD finds $\epsilon$-approximate second-order stationary points with high probability. Importantly, Condition **DO-2** is essential in Theorem 1 of [21] (i.e., the aforementioned convergence with probability one result): it is used three times in the proof of the result and done in such a way that it cannot be relaxed.

### 2.3 Control on the Gradient

There are many more conditions that are placed on the stochastic gradients (SG), $\dot{f}$, than the stochastic objective (SO), $f$, owing to the centrality of $\dot{f}$ in SGD methods. The first set of conditions will be analogous to those conditions on $f$. The most restrictive condition, integral to the results in [28,23,14,3], is

**(SG-1)** There exists a $C > 0$ such that

\[ \mathbb{P}\left[ \bigcap_\theta \left\{ \|\dot{f}(\theta,X)\|_2 \leq C \right\} \right] = 1. \]

A more applicable relative of this condition and an analogue of Condition **SO-3**, used in [29,23,15,35,22,1,10,16,3,25,31], is
Condition SG-2 is also used locally in the results of [11]. More importantly, Condition SG-2 is integral to the previously mentioned result, Theorem 2(c) of [20], which established convergence in probability. While we will not directly compare Condition SG-2 to the BCN structure, we will compare weaker structural conditions implied by Condition SG-2 to the BCN structure. In order to do so, we will first need to discuss common conditions placed on the noise model (NM).

The first common restriction on the noise model (NM), used in [6, 21], is specified by

\[(\text{NM-1})\quad \mathbb{P}\left[ \bigcap_{\theta_1, \theta_2} \left\{ \| \hat{f}(\theta_1, X) - \hat{f}(\theta_2, X) \|_2 \leq C \| \theta_1 - \theta_2 \|_2 \right\} \right] = 1.\]

The second and less restrictive noise model condition, used in [13, 22, 10, 16, 34, 25, 19], is specified by

\[(\text{NM-2})\quad \mathbb{E} \left[ \left\| \hat{f}(\theta, X) - \hat{F}(\theta) \right\|_2^2 \right] \leq C.\]

As we will see subsequently, we have the following noise model condition, which is motivated by Condition SG-2.

\[(\text{NM-3})\quad \mathbb{E} \left[ \left\| \hat{f}(\theta, X) - \hat{F}(\theta) \right\|_2^2 \right] \leq C_1 + C_2 F(\theta).\]

The following lemma relates Condition SG-2 to Condition NM-3. Note, in the following lemma, the parameter, \(\kappa\), is taken to be 0 in [20].
Lemma 1 Suppose there exists a $\kappa \in \mathbb{R}$ such that $\mathbb{P}[\forall \theta, \ f(\theta, X) \geq \kappa] = 1$ and suppose Condition $SG-2$ holds. Then, for all $\theta_1, \theta_2$,

$$\|\hat{F}(\theta_1) - \hat{F}(\theta_2)\|_2 \leq C\|\theta_1 - \theta_2\|_2,$$

and for some $C_1, C_2 \geq 0$, $NM-3$ holds.

Proof The first result follows by an application of Jensen’s inequality. For the second result, for any $\theta$, let 

$$\tilde{\theta} = \theta - \frac{1}{C} \hat{f}(\theta, X).$$

By Taylor’s theorem and Condition $SG-2$,

$$f(\tilde{\theta}, X) \leq f(\theta, X) + \hat{f}(\theta, X)'(\tilde{\theta} - \theta) + \frac{C}{2}\|\tilde{\theta} - \theta\|_2^2$$

$$\leq f(\theta, X) - \frac{1}{C}\|\hat{f}(\theta, X)\|_2^2 + \frac{1}{2C}\|\hat{f}(\theta, X)\|_2^2$$

$$\leq f(\theta, X) - \frac{1}{C}\|\hat{f}(\theta, X)\|_2^2.$$ (8)

Therefore, since $\kappa \leq f(\tilde{\theta}, X)$ with probability one,

$$\|\hat{f}(\theta, X)\|_2^2 \leq C[f(\theta, X) - f(\tilde{\theta}, X)] \leq C[f(\theta, X) - \kappa].$$ (9)

Moreover, this inequality with Jensen’s inequality implies that

$$\|\hat{F}(\theta)\|_2^2 \leq \mathbb{E}\left[\|\hat{f}(\theta, X)\|_2^2\right] \leq C[F(\theta) - \kappa]$$ (10)

Therefore,

$$\mathbb{E}\left[\|\hat{f}(\theta, X) - \hat{F}(\theta)\|_2^2\right] \leq 2\mathbb{E}\left[\|\hat{f}(\theta, X)\|_2^2\right] + 2\|\hat{F}(\theta)\|_2^2 \leq 4C(F(\theta) - \kappa).$$ (11)

If $\kappa$ is positive, then we can choose $C_1 = 0$ and $C_2 = 4C$. If $\kappa$ is negative, then we can choose $C_1 = -4C\kappa$ and $C_2 = 4C$. \hfill $\square$

We have not observed Condition $NM-3$ used in practice, but it is straightforward to prove convergence (in probability) of the objective function evaluated at the iterates to the optimal value under this noise model condition, Lipschitz continuity of $\hat{F}$, and the Polyak-Łojasiewicz (PL) condition (see [18] for an excellent overview of the PL condition and related conditions).

The final noise model condition is assumed in the BCN structure [4], and is specified by
There exists a $C_1, C_2 \geq 0$ such that, $\forall \theta \geq 0$,

\[ \mathbb{E} \left[ \left\| f(\theta, X) - \tilde{F}(\theta) \right\|^2 \right] \leq C_1 + C_2 \left\| \tilde{F}(\theta) \right\|^2. \]  

(NM-4)

In general, if $\dot{F}$ is Lipschitz continuous, then NM-4 implies NM-3 by a simple analogue of Lemma 1. On the other hand, under the PL condition, NM-3 implies NM-4. Specifically, if $\exists \mu > 0$ and an optimal objective function value $F^*$ such that

\[ \left\| \dot{F}(\theta) \right\|_2 \geq \mu [F(\theta) - F^*], \]  

then

\[ C_1 + C_2F(\theta) = (C_1 + C_2F^*) + C_2[F(\theta) - F^*] \leq (C_1 + C_2F^*) + \frac{C_2}{\mu} \left\| \dot{F}(\theta) \right\|^2. \]  

Therefore, given that the PL condition is a commonly used nonconvex structure to prove rates of convergence (e.g., [1]) and it is a relatively weak condition [18], we would argue that NM-4 and NM-3 are equivalent in practice, and, thus, NM-4 is more general than SG-2.

The final two conditions are the less restrictive implications that follow from Conditions SG-1 and SG-2 about the deterministic gradient (DG), $\dot{F}$.

The first condition, found in [15,12,6,32], is

\[ \text{There exists a } C > 0 \text{ such that, for all } \theta, \]  

(DG-1)

\[ \left\| \dot{F}(\theta) \right\|_2 \leq C. \]

The second condition, found in [21,36,34,19], is

\[ \text{There exists a } C > 0 \text{ such that for all } \theta_1, \theta_2, \]  

(DG-2)

\[ \left\| \dot{F}(\theta_1) - \dot{F}(\theta_2) \right\|_2 \leq C \left\| \theta_1 - \theta_2 \right\|_2. \]

2.4 Bottou-Curtis-Nocedal Structure

The Bottou-Curtis-Nocedal (BCN) structure, as popularized in [4], is a very general structure as it—arguably—takes the least restrictive conditions from those discussed above. That is, the BCN structure is given by

1. No restriction on the iterate space (IS-3).
2. No direct assumptions about the stochastic objective, $f$.
3. No direct control over the deterministic objective, $F$, except that it is bounded from below.
4. No direct control over the growth of the stochastic gradient, $\dot{f}$. We only assume that $\dot{F}(\theta) = \mathbb{E}[\dot{f}(\theta, X)]$, which can be relaxed (see remark below).
5. The variance of the stochastic gradients are bounded by a constant and a scaling of the norm-squared of the deterministic gradient with parameters $C_1$ and $C_2$ (NM-4).
6. The deterministic gradient, $\dot{F}$, is Lipschitz continuous with parameter $C$ (DG-2).

Remark 1 The BCN structure allows for biased stochastic gradients, whereas we have not allowed for this because we are considering (non-adaptive) second-order methods. Indeed, the BCN structure with biased stochastic gradients can readily used below if we use scalar step sizes.

3 Strong Global Convergence

Here, we establish one of our key results: that stochastic gradient descent (SGD) converges with probability one for Bottou-Curtis-Nocedal (BCN) non-convex functions. In order to do this, we will first need to specify the precise nature of the SGD iterates that we will consider. Then, we will establish global convergence of SGD with probability one for BCN functions, which includes a broad class of convex and nonconvex functions.

3.1 Stochastic Gradient Descent Method

Here, we will consider (non-adaptive) second-order SGD methods. In doing so, we will require the particular BCN structure described earlier; however, we emphasize that if the step size is scalar and the stochastic gradients are allowed to be bias, then the iterates can be analyzed with the same arguments below and with less difficulty.

Let $\{X_k : k \in \mathbb{N}\}$ be independent and identically distributed random variables that have the same distribution as $X$. Let $\beta_0$ be either a fixed or random quantity in $\mathbb{R}^p$. Let $\mathcal{F}_0 = \sigma(\beta_0)$ and $\mathcal{F}_k = \sigma(\beta_0, X_1, \ldots, X_k)$ denote the corresponding elements of the usual filtration. Define the Stochastic Gradient Descent iterates $\{\beta_k : k \in \mathbb{N}\}$ recursively by

$$\beta_{k+1} = \beta_k - M_k \dot{f}(\beta_k, X_{k+1}),$$

where $\{M_k\}$ are matrices that satisfy:

(P1) The matrices $\{M_k\}$ are symmetric and positive definite.
There exists an \( S > 0 \) such that
\[
\sum_{k=0}^{\infty} \lambda_{\max}(M_k)^2 < S,
\]
where \( \lambda_{\max}(\cdot) \) denotes the largest eigenvalue of the given matrix.

The sum,
\[
\sum_{k=0}^{\infty} \lambda_{\min}(M_k),
\]
diverges, where \( \lambda_{\min}(\cdot) \) is the smallest eigenvalue of the given matrix.

There are two remarks worth making at this point. First, if \( M_k \) are scalar multiples of the identity, we would recognize properties \( \text{P1} \) to \( \text{P3} \) as the Robbins-Monro conditions [30]. Second (and which we will demonstrate rigorously as needed), following directly from the independence \( \{X_k\} \), the iterates \( \{\beta_k\} \) enjoy an analogue of the strong Markov property; that is, for any stopping time \( \tau \), the iterates \( \{\beta_{\tau+k}\} \) are independent of \( F_{\tau} \) given \( \beta_{\tau} \) and \( M_{\tau} \) on the event \( \{\tau < \infty\} \). Moreover, since properties \( \text{P1} \) to \( \text{P3} \) still hold for \( \{M_{\tau+k}\} \), any property that holds for \( \{\beta_k\} \) must also hold for \( \{\beta_{\tau+k}\} \) given \( \beta_{\tau} \) and \( M_{\tau} \) on \( \{\tau < \infty\} \).

3.2 Strong Global Convergence

With the formulation of SGD in hand and the nature of the BCN structure specified, we are now ready to prove the strong global convergence of the iterates; that is, we will prove that
\[
P \left[ \lim_{k \to \infty} \| \hat{F}(\beta_k) \|_2 = 0 \right] = 1. \tag{15}
\]
The proof of this result will proceed by two steps. First, we will establish that, for any \( \delta > 0 \),
\[
P \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta, \ i.o. \right] = 1, \tag{16}
\]
where \( i.o. \) means infinitely often (see Theorem 1). Unfortunately, this alone will not imply convergence. Therefore, we will use this result to prove that, for any \( \delta > 0 \),
\[
P \left[ \| \hat{F}(\beta_k) \|_2 > \delta, \ i.o. \right] = 0, \tag{17}
\]
which implies that the iterates converge to a stationary point with probability one (see Theorem 2).

Before detailing the results, we would like to point out the general strategy that we use, and how it is similar to, or distinct from, previous efforts. First, we use coupling to relate the iterate sequence and a related sequence, for which we then establish an analogue of the strong Markov property mentioned previously. With these two pieces, a refinement of Zoutendijk’s global convergence strategy [37], and an induction argument, we are able to prove (16). Then, we leverage (16), the inclusion-exclusion principle, and a conditional version of the Borel-Cantelli lemma to conclude (17).

As mentioned, our approach adapts Zoutendijk’s global convergence strategy [37], which has been done previously in the stochastic optimization literature [28,29]. However, our approach refines this argument by establishing an analogue of the strong Markov property through coupling, which we have not observed in any of the stochastic optimization literature. This allows us to state a much stronger result than what has previously been established.

Another point of departure is that our approach avoids restating the behavior of the iterates as a martingale, which is the primary strategy when $F$ is convex (e.g., see [2]). Our approach also avoids restating any evaluation of the iterates with respect to the objective or the gradient as a martingale, which is the strategy that is used in Theorem 2(c) of [20] and which seems to require much stronger structural conditions than the general BCN structure. In fact, the use of martingales only appears in the proof of the conditional Borel-Cantelli lemma, which we do not derive but rather cite from source material.

**Remark 2** We also note that all of the inequalities and equalities below hold with probability one, even if this is not explicitly stated.

**Remark 3** We also point out that the probabilities and expectations below should be conditional on $\mathcal{F}_0$. However, to avoid the additional cumbersome notation, we will not explicitly state this.

### 3.2.1 Strong Markov Property and Coupling

Our first task will be to set the stage to the analogue of strong Markov property that will be relevant in analyzing the SGD method. Let $\tau$ be a finite stopping time with respect to $\{\mathcal{F}_k\}$; that is, $\mathbb{P}[\tau < \infty] = 1$. Moreover, as all of our arguments will be asymptotic in this section, we will assume also that $\mathbb{P}[\tau \geq K] = 1$ where $K \in \mathbb{N}$ such that $\forall k \geq K$,

$$\lambda_{\min}(M_k) > C\lambda_{\min}(M_k)^2 + CC_2\lambda_{\max}(M_k)^2,$$

(18)

where, we recall, $C$ is the Lipschitz constant in DG-2 and $C_2$ is the scaling parameter in NM-4. Note, if such a $K$ does not exist, then there is a subsequence of $\mathbb{N}$, $\{k_j\}$, such that

$$\frac{1}{C^2} \leq \lambda_{\min}(M_{k_j})^2 \leq \lambda_{\max}(M_{k_j})^2.$$ 

(19)
As a contradiction, the sum of the terms on the right most inequality over all \( j \) is bounded by \( S \) by \( \mathbf{P2} \), but the sum of the terms on the left most inequality diverges. Thus, we see that such a \( K \) exists.

Now, using \( \tau \), we will define a sequence of iterates \( \{ \psi_k \} \) that we will eventually couple with \( \{ \beta_k \} \). To define \( \{ \psi_k \} \), let

1. \( Z_k := X_{\tau + 1 + k} \) for all \( k \in \mathbb{N} \).
2. \( \psi_0 := \beta_{\tau + 1} \).
3. For all \( k \in \{0\} \cup \mathbb{N} \), \( P_k := M_{\tau + 1 + k} \) and

\[
\psi_{k+1} := \psi_k - P_k \dot{f}(\psi_k, Z_{k+1}) \mathbf{1} \left[ \left\| \dot{F}(\psi_k) \right\|_2^2 > \delta \right].
\]

There are several properties of these quantities that are worth noting: \( \{ Z_k \} \) are independent and identically distributed; and \( \{ P_k \} \) satisfy \( \mathbf{P1} \) to \( \mathbf{P3} \). The former is verified by Theorem 4.1.3 of [9], which states that \( \{ Z_k \} \) are mutually independent and independent of \( F_{\tau+1} \), and have the same distribution as \( X_1 \). The latter is verified by the following lemma.

**Lemma 2** With probability one, \( \{ P_k \} \) satisfy \( \mathbf{P1} \) to \( \mathbf{P3} \).

**Proof** The result follows from a standard divide and conquer argument. For \( \mathbf{P1} \),

\[
P \left[ P_k = P'_k, P_k > 0 \right] = P \left[ M'_{\tau + 1 + k} = M_{\tau + 1 + k}, M_{\tau + 1 + k} > 0 \right]
\]

\[
= \sum_{j=0}^{\infty} P \left[ M'_{j + 1 + k} = M_{j + 1 + k}, M_{j + 1 + k} > 0 \right] P \left[ \tau = j \right]
\]

\[
= \sum_{j=0}^{\infty} P \left[ \tau = j \right]
\]

\[
= P \left[ \tau < \infty \right] = 1.
\]

Similarly, for \( \mathbf{P2} \),

\[
P \left[ \sum_{k=0}^{\infty} \lambda_{\text{max}}(P_k)^2 < S \right]
\]

\[
= \sum_{j=0}^{\infty} P \left[ \sum_{k=0}^{\infty} \lambda_{\text{max}}(M_{j + 1 + k})^2 < S \right] P \left[ \tau = j \right]
\]

\[
= P \left[ \tau < \infty \right] = 1.
\]

The analogous argument will show that \( \mathbf{P3} \) holds with probability one. \( \square \)
From these properties, we see that if \((\beta_k)^{\ast}\) did not have the indicator term, then the only difference between \(\{\psi_k\}\) and \(\{\beta_k\}\) is the initialization—the fact that \(\{P_k\}\) and \(\{M_k\}\) are distinct is of little importance for our purposes as long as P1 to P3 are satisfied. Thus, we see that \(\{\beta_k\}\) exhibit an analogue of the strong Markov property.

Now, to couple these two iterate sequences, let \(G_0 = \mathcal{F}_{\tau+1}\) and \(G_k = \sigma(\mathcal{F}_{\tau+1}, Z_1, \ldots, Z_k)\), and, for \(\delta > 0\), define \(\tau_\delta\) to be a stopping time with respect to \(\{G_k\}\) such that

\[
\tau_\delta = \min \left\{ k \geq 0 : \left\| \hat{F}(\psi_k) \right\|_2^2 \leq \delta \right\}.
\]

Then, on the event \(\{k \geq \tau_\delta\}\), \(\psi_{k+1} = \psi_k\). Moreover, on the event \(\{k < \tau_\delta\}\),

\[
\psi_{k+1} = \psi_k - P_k \hat{f}(\psi_k, Z_{k+1}) = \beta_{\tau+1+k} - M_{\tau+1+k} \hat{f}(\beta_{\tau+1+k}, X_{\tau+2+k}) = \beta_{\tau+2+k}
\]

follows by induction. Therefore, for all \(0 \leq k \leq \tau_\delta\), \(\psi_k = \beta_{\tau+1+k}\); that is, the sequences are coupled in this interval. Owing to this coupling, we see that \(\tau_\delta + 1\) is number of iterates for \(\beta_k\) to be within a “\(\delta\)-region” of a stationary point after iterate \(\tau\). We now apply Zoutendijk’s global convergence approach to conclude that \(\tau_\delta\) is finite with probability one.

### 3.2.2 Zoutendijk’s Global Convergence Approach

We now apply Zoutendijk’s global convergence approach \([37]\) to \(\{\psi_k\}\) to conclude that \(\mathbb{P}[\tau_\delta < \infty] = 1\). First, by the fundamental theorem of calculus and DG-2 (recall, with constant \(C\)),

\[
F(\psi_{k+1}) \leq F(\psi_k) + \hat{F}(\psi_k)^{\prime}(\psi_{k+1} - \psi_k) + \frac{C}{2} \left\| \psi_{k+1} - \psi_k \right\|_2^2
\]

\[
= F(\psi_k) - \hat{F}(\psi_k)^{\prime} P_k \hat{f}(\psi_k, Z_{k+1}) \mathbf{1} \left[ \left\| \hat{F}(\psi_k) \right\|_2 > \delta \right]
\]

\[
+ \frac{C}{2} \left\| P_k \hat{f}(\psi_k, Z_{k+1}) \right\|_2^2 \mathbf{1} \left[ \left\| \hat{F}(\psi_k) \right\|_2 > \delta \right].
\]

We now take the conditional expectation of the resulting inequality with respect to \(G_k\). Note, since \(\psi_k, P_k\) are measurable with respect to \(G_k\) and \(Z_{k+1}\) is independent of \(G_k\), then \(\mathbb{E}[F(\psi_k) | G_k] = F(\psi_k)\) and

\[
\mathbb{E} \left[ \hat{F}(\psi_k)^{\prime} P_k \hat{f}(\psi_k, Z_{k+1}) | G_k \right] = \hat{F}(\psi_k)^{\prime} P_k \hat{F}(\psi_k).
\]
For the third term in (32), we will need to make use of NM-4 (with parameters $C_1, C_2 \geq 0$).

\[
\mathbb{E} \left[ \left\| P_k \left( \hat{f}(\psi_k, Z_{k+1}) - \hat{F}(\psi_k) \right) \right\|^2_{\lambda_k} \right]
= \mathbb{E} \left[ \left\| P_k \left( \hat{f}(\psi_k, Z_{k+1}) - \hat{F}(\psi_k) \right) \right\|^2_{\lambda_k} \right] + \hat{F}(\psi_k)' P_k^2 \hat{F}(\psi_k)
\leq C_1 \lambda_{\max}(P_k)^2 + C_2 \lambda_{\max}(P_k)^2 \left\| \hat{F}(\psi_k) \right\|^2_{\lambda_k} + \hat{F}(\psi_k)' P_k^2 \hat{F}(\psi_k)
\]  

(34)

Putting the calculation for these three terms together in (32), we conclude

\[
\mathbb{E} \left[ F(\psi_{k+1}) | G_k \right] \leq F(\psi_k) + \frac{CC_1}{2} \lambda_{\max}(P_k)^2 - \left( \left\| \hat{F}(\psi_k) \right\|^2_{\lambda_k} > \delta \right)
\times \left[ \hat{F}(\psi_k)' P_k \hat{F}(\psi_k) - \frac{C}{2} \hat{F}(\psi_k)' P_k^2 \hat{F}(\psi_k) - \frac{CC_2}{2} \lambda_{\max}(P_k)^2 \left\| \hat{F}(\psi_k) \right\|^2_{\lambda_k} \right].
\]

(36)

We can find an upper bound for (36) by finding a lower bound for the third term in the right hand side of the inequality. In particular, we will solve

\[
\min_{v \in \mathbb{S}^{p-1}} v' P_k v - \frac{C}{2} v' P_k^2 v - \frac{CC_2}{2} \lambda_{\max}(P_k)^2 \left\| v \right\|^2_{\lambda_k},
\]

(37)

where $\mathbb{S}^{p-1}$ is the unit sphere in $\mathbb{R}^p$. Using the Schur decomposition of $P_k$,\(^3\) we can transform (37) into the equivalent problem

\[
\min_{v \in \mathbb{S}^{p-1}} \sum_{i=1}^{p} \left[ \lambda_i - \frac{C}{2} \lambda_i^2 - \frac{CC_2}{2} \lambda_i^3 \right] v_i^2,
\]

(38)

where $\lambda_{\max}(P_k) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p = \lambda_{\min}(P_k)$; and $v_i$ are the components of $v$. Since $\tau \geq K$ and applying $K$ satisfying (18), the solution to (38) is

\[
\lambda_p - \frac{C}{2} \lambda_p^2 - \frac{CC_2}{2} \lambda_p^3 \geq \frac{1}{2} \lambda_p,
\]

(39)

where the last inequality follows by applying the requirement on $K$, (18), again (along with the divide-and-conquer argument used in Lemma 2 to show that $P_k$ satisfy the same condition as $M_k$ with probability one).

Plugging this lower bound into (36), we have that

\[
\mathbb{E} [F(\psi_{k+1}) | G_k] \leq F(\psi_k) + \frac{CC_1}{2} \lambda_{\max}(P_k)^2
- \frac{1}{2} \lambda_{\min}(P_k) \left\| \hat{F}(\psi_k) \right\|^2_{\lambda_k} \left[ \left\| \hat{F}(\psi_k) \right\|^2_{\lambda_k} > \delta \right].
\]

(40)

\(^3\) Since $P_k$ is random, its Schur decomposition is random.
Rearranging and applying the condition in the indicator,
\[
\frac{\delta}{2} \lambda_{\min}(P_k) 1 \left[ \left\| \dot{F}(\psi_k) \right\|_2^2 > \delta \right] 
\leq F(\psi_k) - \mathbb{E} \left[ F(\psi_{k+1}) \mid \mathcal{G}_k \right] + \frac{C C_1}{2} \lambda_{\max}(P_k)^2.
\]
(41)

Now, recall that \(P_k\) are measurable with respect to \(\mathcal{F}_{\tau+1}\) and recall that \(\mathcal{F}_{\tau+1} \subset \mathcal{G}_k\) for all \(k\). Therefore,
\[
\frac{\delta}{2} \lambda_{\min}(P_k) \mathbb{P} \left[ \left\| \dot{F}(\psi_k) \right\|_2^2 > \delta \right| \mathcal{F}_{\tau+1} 
\leq \mathbb{E} \left[ F(\psi_k) - F(\psi_{k+1}) \right| \mathcal{F}_{\tau+1} + \frac{C C_1}{2} \lambda_{\max}(P_k)^2.
\]
(42)

Moreover, by (a) summing both sides from \(k = 0\) to \(n \in \mathbb{N}\), (b) recalling that \(F(\psi_0)\) is finite with probability one given \(\mathcal{F}_{\tau+1}\), and (c) applying \(P2\) from Lemma 2, we conclude
\[
\frac{\delta}{2} \sum_{k=0}^{n} \lambda_{\min}(P_k) \mathbb{P} \left[ \left\| \dot{F}(\psi_k) \right\|_2^2 > \delta \right| \mathcal{F}_{\tau+1} 
\leq F(\psi_0) - \mathbb{E} \left[ F(\psi_{n+1}) \right| \mathcal{F}_{\tau+1}] + S.
\]
(43)

Recall that, we have assumed that \(F(\theta)\) is bounded from below by some constant \(F_{l.b.}\) as a core assumption. Using this, we see that
\[
\frac{\delta}{2} \sum_{k=0}^{n} \lambda_{\min}(P_k) \mathbb{P} \left[ \left\| \dot{F}(\psi_k) \right\|_2^2 > \delta \right| \mathcal{F}_{\tau+1} 
\leq F(\psi_0) - F_{l.b.} + S. \]
(44)

Moreover, note that, for all \(k \geq 0\),
\[
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \left\| \dot{F}(\psi_j) \right\|_2 > \delta \right\} \mid \mathcal{F}_{\tau+1} \right] \leq \mathbb{P} \left[ \left\| \dot{F}(\psi_k) \right\|_2^2 > \delta \right| \mathcal{F}_{\tau+1} \right].
\]
(45)

Therefore, for arbitrary \(n\),
\[
\mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \left\| \dot{F}(\psi_j) \right\|_2 > \delta \right\} \mid \mathcal{F}_{\tau+1} \right] \leq \frac{F(\psi_0) - F_{l.b.} - S}{(\delta/2) \sum_{k=0}^{n} \lambda_{\min}(P_k)}.
\]
(46)

By \(P3\) from Lemma 2, the right hand side of this inequality can be made arbitrarily small, which implies that the conditional probability on the left hand side is zero.

That is,
\[
0 = \mathbb{P} \left[ \bigcap_{j=0}^{\infty} \left\{ \left\| \dot{F}(\psi_j) \right\|_2 > \delta \right\} \mid \mathcal{F}_{\tau+1} \right] = \mathbb{P} \left[ \tau_\delta = \infty \mid \mathcal{F}_{\tau+1} \right].
\]
(47)
In other words, we have concluded that \( \mathbb{P} [ \tau_\delta < \infty \mid \mathcal{F}_{\tau + 1} ] = 1 \) with probability one for any finite stopping time \( \tau \). With this result, our last step is to use induction.

For now, we will say that the iterates are in a \( \delta \)-region of a stationary point if the squared-norm of the gradient of the iterate is no greater than \( \delta \). If we let \( \tau = -1 \), then \( \tau_\delta \) is the first time the iterates enter a \( \delta \)-region of a stationary point. Let \( T_1(\delta) = \tau_\delta \) when \( \tau = -1 \). Then, from the above argument, we have shown that \( T_1(\delta) \) is a finite stopping time. Now, define \( T_j(\delta) \) to be the \( j \)th time that the iterates enter a \( \delta \)-region of a stationary point. Suppose that \( T_j(\delta) \) is finite. Then, define \( \tau = T_j(\delta) \). Then, \( \tau_\delta \) for this \( \tau \) is the next time that the iterates enter a \( \delta \)-region of a stationary point. That is, \( T_{j+1}(\delta) = \tau_\delta + T_j(\delta) \). Since we have assumed that \( \tau = T_j(\delta) \) is finite, we conclude that \( \tau_\delta \) is finite, which implies that \( T_{j+1}(\delta) \) is finite. Therefore, by induction we have proven the following result.

**Theorem 1** Let \( F \) be a Bottou-Curtis-Nocedal function (§2.4) and let \( \{ \beta_k \} \) be the iterates generated by Stochastic Gradient Descent satisfying \( P1 \) to \( P3 \) (§3.1), then, for any \( \delta > 0 \),

\[
\mathbb{P} \left[ \left\| \dot{F}(\beta_k) \right\|_2^2 \leq \delta, \ \text{i.o.} \right\mid \mathcal{F}_0 \right] = 1 \text{ with probability } 1,
\]

where \( \mathcal{F}_0 = \sigma(\beta_0) \).

### 3.2.3 Inclusion-Exclusion and Markov’s Inequality

Our next step is to prove that

\[
\mathbb{P} \left[ \left\| \dot{F}(\beta_k) \right\|_2^2 > \delta, \ \text{i.o.} \right\mid \mathcal{F}_0 \right] = 0 \text{ with probability } 1.
\]

Again, we will temporarily drop the conditioning on \( \mathcal{F}_0 \) for simplicity of the notation. By Theorem 1 and the inclusion-exclusion principle,

\[
1 = \mathbb{P} \left[ \left\{ \left\| \dot{F}(\beta_k) \right\|_2 \leq \delta, \ \text{i.o.} \right\} \cup \left\{ \left\| \dot{F}(\beta_k) \right\|_2 > \delta, \ \text{i.o.} \right\} \right]
\]

\[
= \mathbb{P} \left[ \left\{ \left\| \dot{F}(\beta_k) \right\|_2 \leq \delta, \ \text{i.o.} \right\} \right] + \mathbb{P} \left[ \left\{ \left\| \dot{F}(\beta_k) \right\|_2 > \delta, \ \text{i.o.} \right\} \right]
\]

\[
- \mathbb{P} \left[ \left\{ \left\{ \left\| \dot{F}(\beta_k) \right\|_2 \leq \delta, \ \text{i.o.} \right\} \cap \left\{ \left\{ \left\| \dot{F}(\beta_k) \right\|_2 > \delta, \ \text{i.o.} \right\} \right\} \right\} \right].
\]

Applying Theorem 1 again, we conclude that

\[
\mathbb{P} \left[ \left\| \dot{F}(\beta_k) \right\|_2 > \delta, \ \text{i.o.} \right]\]

\[
= \mathbb{P} \left[ \left\{ \left\| \dot{F}(\beta_k) \right\|_2 \leq \delta, \ \text{i.o.} \right\} \cap \left\{ \left\| \dot{F}(\beta_k) \right\|_2 > \delta, \ \text{i.o.} \right\} \right].
\]

We will now show that the probability of the right hand side is zero. Note, for any outcome

\[
\omega \in \left\{ \left\| \dot{F}(\beta_k) \right\|_2 \leq \delta, \ \text{i.o.} \right\} \cap \left\{ \left\| \dot{F}(\beta_k) \right\|_2 > \delta, \ \text{i.o.} \right\},
\]
there must be an infinite subsequence of \( \mathbb{N} \) such that \( \beta_k \) is in a \( \delta^2 \)-region of a stationary point and then \( \beta_{k+1} \) exits this \( \delta^2 \)-region of a stationary point. Suppose this were not true. Then, there are two cases. In the first case, \( \beta_k \) enters a \( \delta^2 \)-region and then never leaves, in which case

\[
\omega \not\in \left\{ \| \hat{F}(\beta_k) \|_2 > \delta, \ i.o. \right\}. \tag{54}
\]

In the second case, we have that \( \beta_k \) exits a \( \delta^2 \)-region of a stationary point, and never enters again, which implies

\[
\omega \not\in \left\{ \| \hat{F}(\beta_k) \|_2 \leq \delta, \ i.o. \right\}. \tag{55}
\]

In both cases, we have a contradiction. Therefore, using just one of the cases, we conclude that

\[
\left\{ \| \hat{F}(\beta_k) \|_2 \leq \delta, \ i.o. \right\} \cap \left\{ \| \hat{F}(\beta_k) \|_2 > \delta, \ i.o. \right\}
\subset \left\{ \| \hat{F}(\beta_k) \|_2 \leq \delta, \ \| \hat{F}(\beta_{k+1}) \|_2 > \delta, \ i.o. \right\}. \tag{56}
\]

We can write this latter event as

\[
\left\{ \| \hat{F}(\beta_k) \|_2 \leq \delta, \ \| \hat{F}(\beta_{k+1}) \|_2 > \delta, \ i.o. \right\}
= \left\{ \| \hat{F}(\beta_{k+1}) \|_2 \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] > \delta, \ i.o. \right\}. \tag{57}
\]

We will now show that this ultimate event occurs with probability zero using Markov’s inequality and the Borel-Cantelli lemma. Let \( \epsilon > 0 \) and recall that \( C > 0 \) is the parameter in DG-2. Then,

\[
P \left[ \| \hat{F}(\beta_{k+1}) \|_2 \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] \geq \delta + C \epsilon \left| F_k \right| \right] \tag{58}
\]

\[
\leq P \left[ \left( \| \hat{F}(\beta_{k+1}) - \hat{F}(\beta_k) \|_2 + \| \hat{F}(\beta_k) \|_2 \right) \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] \geq \delta + C \epsilon \left| F_k \right| \right] \tag{59}
\]

\[
\leq P \left[ \| \hat{F}(\beta_{k+1}) - \hat{F}(\beta_k) \|_2 \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] \geq \delta + C \epsilon \left| F_k \right| \right] \tag{60}
\]

\[
\leq P \left[ C \| \beta_{k+1} - \beta_k \|_2 \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] \geq C \epsilon \left| F_k \right| \right] \tag{61}
\]

\[
\leq P \left[ \| M_k \hat{f}(\beta_k, X_{k+1}) \|_2 \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] \geq \epsilon \left| F_k \right| \right] \tag{62}
\]

Applying Markov’s inequality to the last conditional probability and using NM-4,

\[
P \left[ \| \hat{F}(\beta_{k+1}) \|_2 \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] \geq \delta + C \epsilon \left| F_k \right| \right] \tag{63}
\]

\[
\leq \frac{\lambda_{\max}(M_k)^2}{\epsilon^2} \left[ C_1 + (C_2 + 1) \left\| \hat{F}(\beta_k) \right\|_2^2 \mathbf{1} \left[ \| \hat{F}(\beta_k) \|_2 \leq \delta \right] \right. \tag{64}
\]

\[
\leq \frac{\lambda_{\max}(M_k)^2}{\epsilon^2} \left[ C_1 + (C_2 + 1) \delta^2 \right] \tag{65}
\]
By \( P2 \), the sum of the right hand side is bounded with probability one. Therefore, by the conditional second Borel-Cantelli lemma (Theorem 5.3.2 of [9]),

\[
P \left( \left\| \hat{F}(\beta_{k+1}) \right\|_2 \leq \delta \right) \geq \delta + C \epsilon, \text{ i.o.} = 0.
\]

(66)

Since \( \epsilon > 0 \) is arbitrary, then this conclusion holds for each element in the sequence \( \{\epsilon_m\} \) where \( \epsilon_m \downarrow 0 \). Therefore,

\[
P \left( \left\| \hat{F}(\beta_{k+1}) \right\|_2 \leq \delta \right) > \delta, \text{ i.o.}
\]

(67)

\[
= \sum_{m=1}^{\infty} P \left( \left\{ \left\| \hat{F}(\beta_{k+1}) \right\|_2 \leq \delta \right\} \cap \left\{ \left\| \hat{F}(\beta_k) \right\|_2 \geq \delta + C \epsilon_m \right\}, \text{ i.o.} \right)
\]

(68)

\[
= 0.
\]

(70)

Therefore, by using this result with (52), (56) and (57), we conclude the following result.

**Theorem 2** Let \( F \) be a Bottou-Curtis-Nocedal function (§2.4) and let \( \{\beta_k\} \) be the iterates generated by Stochastic Gradient Descent satisfying \( P1 \) to \( P3 \) (§3.1), then, for any \( \delta > 0 \),

\[
P \left[ \left\| \hat{F}(\beta_k) \right\|_2 > \delta, \text{ i.o.} \middle| \mathcal{F}_0 \right] = 0 \text{ with probability 1},
\]

(71)

where \( \mathcal{F}_0 = \sigma(\beta_0) \).

Theorem 2 supplies the following corollary.

**Corollary 1** Let \( F \) be a Bottou-Curtis-Nocedal function (§2.4) and let \( \{\beta_k\} \) be the iterates generated by Stochastic Gradient Descent satisfying \( P1 \) to \( P3 \) (§3.1), then

\[
P \left[ \lim_{k \to \infty} \left\| \hat{F}(\beta_k) \right\|_2 = 0 \middle| \mathcal{F}_0 \right] = 1, \text{ with probability 1},
\]

(72)

where \( \mathcal{F}_0 = \sigma(\beta_0) \).

**Proof** For any \( \delta > 0 \), by Theorem 2,

\[
1 = P \left( \left\{ \left\| \hat{F}(\beta_k) \right\|_2 > \delta \right\}, \text{ i.o.} \right| \mathcal{F}_0
\]

(73)

\[
= P \left[ \limsup_{k \to \infty} \left\| \hat{F}(\beta_k) \right\|_2 \leq \delta \middle| \mathcal{F}_0 \right].
\]

(74)
Since $\delta > 0$ is arbitrary, the preceding result applies to each element in the sequence $\{\delta_m\}$ where $\delta_m \downarrow 0$. Since the countable intersection of probability one events has probability one,

$$1 = P \left[ \bigcap_{m \in \mathbb{N}} \left\{ \limsup_{k \to \infty} \| F(\beta_k) \|_2 \leq \delta_m \right\} \mathcal{F}_0 \right],$$

which is the desired result. \hfill \square

### 4 Stopping Criteria

We have now resolved the first challenge of stopping criteria in a stochastic setting: we have demonstrated that the SGD iterates will converge with probability one to a stationary point of a BCN nonconvex function. We now turn our attention to whether the gradient-based stopping criteria, detailed below, will be detected and how they control false negatives.\(^4\) To be rigorous, we have the following notion of detectable.

**Definition 1** A stopping criteria is said to be detectable if the probability of satisfying the stopping criteria in finite time is equal to one.

With this definition, we will describe three structural scenarios under which we will rigorously develop the detectability and control over false negatives. Then, we will present five stopping criteria. For three of them, we will rigorously demonstrate that they are detectable and how they control false negatives. We summarize the results in Table 1 with an empty cell indicating that the property is not established for the given criteria under the given scenario.

**Table 1** A summary of detectability and false negative control results for the five stopping criteria and the three scenarios. An empty cell indicates that neither detectability nor false negative control for the stopping criteria was not established.

| Criteria | Scenario (a) | Scenario (b) | Scenario (c) |
|----------|--------------|--------------|--------------|
| SC-1     | Proposition 1| Proposition 1| Proposition 1|
| SC-2     | Proposition 2| Proposition 2| Proposition 2|
| SC-3     | Proposition 3| Proposition 3| |
| SC-4     |              |              |              |
| SC-5     |              |              |              |

\(^4\) Recall that a false negative occurs when the norm of the iterate’s gradient is sufficiently small, but the stopping criteria is not triggered.
4.1 Scenarios

For three stopping criteria proposed, we will consider the question of detectability and false negative control under three scenarios: (a) $F$ is a Bottou-Curtis-Nocedal function; (b) $F$ is a Bottou-Curtis-Nocedal function under homogeneity (i.e., $C_1 = 0$ in NM-4); and (c) $F$ is a Bottou-Curtis-Nocedal function satisfying

\[ \text{Let } \pi_1, \pi_2 \in (0, 1) \text{ and } \pi_3 \geq 1. \text{ For } \| \dot{F}(\theta) \|_2 \leq \pi_1 \text{ and for any } t \geq 0, } \]

\[ P \left( \| f(\theta, X) \|_2 \geq t \right) \leq \begin{cases} 1 & t \leq \pi_3 \| \dot{F}(\theta) \|_2 \\ \left( \frac{\pi_3 \| \dot{F}(\theta) \|_2}{t} \right)^{\frac{\pi_2}{\pi_3}} & t > \pi_3 \| \dot{F}(\theta) \|_2 \end{cases}. \]

These last two scenarios are worth elaboration. Scenario (b) induces homogeneity [27] of minima—a minimizer of $F$ is a minimizer of $f$ with probability one—which has become an important condition in overparametrized models in data science [1]. Scenario (c) induces the tail behavior of $f(\theta, X)$ to be at most a fat-tailed distribution (which includes heavy-tailed distributions), when $\dot{F}(\theta)$ is close to zero. Importantly, scenario (c)’s additional condition, T1, does not nullify or replace the other BCN conditions because a random variable satisfying T1 has infinite mean and an infinite variance. Moreover, scenario (c)’s additional condition is perhaps the weakest distributional requirement that we can place on $f$’s upper tail in order to preclude the non-detectability issue discussed in the introduction of this work.

4.2 Stopping Criteria by Gradient Estimation

The most natural extension from deterministic, gradient-based stopping criteria to the stochastic case is to apply deterministic stopping criteria to either a periodic evaluation of the deterministic gradient (when feasible, e.g., [17]), or to an estimate of the deterministic gradient over an independent sample. These two stopping criteria are stated below.

\[ \text{Let } \epsilon > 0. \text{ Let } \{ T_j \} \text{ be a sequence of positive-valued, strictly increasing, finite stopping times with respect to } \{ J_k \}. \text{ Then, the SGD iterates are stopped at iterate } T_j \text{ where} \]

\[ J = \min \left\{ j \geq 1 : \| \dot{F}(\beta_{T_j}) \|_2 \leq \epsilon \right\}. \]
Let \( \epsilon > 0 \). Let \( \{T_j\} \) be a sequence of positive-valued, strictly increasing, finite stopping times with respect to \( \{\mathcal{F}_k\} \). Let \( \{N_j\} \) be \( \mathbb{N} \)-valued random variables such that \( N_j \) is measurable with respect to \( \mathcal{F}_{T_j} \). Moreover, for each \( j \), let \( \{Z_{ij} : i = 1, \ldots, N_j\} \) be copies of \( X \) that are independent of each other and \( \{\mathcal{F}_k\} \). Then, the SGD iterates are stopped at iterate \( T_J \), where

\[
J = \min \left\{ j \geq 1 : \frac{1}{N_j} \left\| \sum_{i=1}^{N_j} f(\beta_{T_j}, Z_{ij}) \right\|_2 \leq \epsilon \right\}.
\]

**Remark 4** Letting \( \{T_j\} \) be deterministic and \( \{N_j\} \) be deterministic is allowed in both stopping criteria. By using stopping times, we allow for greater generality of these stopping criteria as they can be adaptive to the behavior of the sequence.

In both criteria, we evaluate the stopping criteria at strictly increasing stopping times, which ensures that we are not repeating the evaluation of the stopping criteria at the same iterate (as this can be represented by simply increasing the value of \( N_j \) and would be redundant). Moreover, our condition that \( \{T_j\} \) is increasing supplies the following instrumental result that is a direct consequence of Corollary 1.

**Corollary 2** Let \( F \) be a Bottou-Curtis-Nocedal nonconvex function (§2.4) and let \( \{\beta_k\} \) be the iterates generated by Stochastic Gradient Descent satisfying \( P1 \) to \( P3 \) (§3.1). If \( \{T_j\} \) are positive-valued, strictly increasing, finite stopping times with respect to \( \{\mathcal{F}_k\} \), then

\[
\mathbb{P} \left[ \lim_{j \to \infty} \left\| \hat{F}(\beta_{T_j}) \right\|_2 = 0 \bigg| \mathcal{F}_0 \right] = 1, \text{ with probability 1,} \tag{77}
\]

where \( \mathcal{F}_0 = \sigma(\beta_0) \).

**Proof** Let \( \{t_j\} \) be an increasing, deterministic sequence of integers. Then, \( \{\beta_{t_j}\} \subset \{\beta_k\} \) and

\[
\mathbb{P} \left[ \limsup_{j \to \infty} \left\| \hat{F}(\beta_{t_j}) \right\|_2 \leq \limsup_{k \to \infty} \left\| \hat{F}(\beta_k) \right\|_2 \bigg| \mathcal{F}_0 \right] = 1, \tag{78}
\]

with probability one. Therefore, by Corollary 1, the subsequence converges with probability one. Applying this with \( T = \sigma(\{T_j\}) \), we conclude that

\[
\begin{align*}
\mathbb{P} \left[ \limsup_{j \to \infty} \left\| \hat{F}(\beta_{T_j}) \right\|_2 = 0 \bigg| \mathcal{F}_0 \right] &= \mathbb{E} \left[ \mathbb{P} \left[ \limsup_{j \to \infty} \left\| \hat{F}(\beta_{T_j}) \right\|_2 = 0 \bigg| \mathcal{T} \right] \bigg| \mathcal{F}_0 \right] \\
&= \mathbb{E} \left[ 1 \bigg| \mathcal{F}_0 \right].
\end{align*}
\]
We see that Corollary 2 implies that \textbf{SC-1} is detectable; that is, it will be triggered in finite time under the BCN structure. Thus, it applies to all three scenarios.

**Proposition 1** Let $F$ be a Bottou-Curtis-Nocedal function (§2.4) and let $\{\beta_k\}$ be the iterates generated by Stochastic Gradient Descent satisfying \textbf{P1} to \textbf{P3} (§3.1). If the SGD iterates are subject to \textbf{SC-1} for some $\epsilon > 0$, then \textbf{SC-1} is detectable (i.e., $P[T_j < \infty | F_0] = 1$ with probability one.)

\textbf{Proof} We will drop the $F_0$ conditioning for simplification. Recall that $T_j$ are finite with probability one for all $j$. Therefore,

$$P[T_j < \infty] = \sum_{j=0}^\infty P[T_j < \infty | J = j] P[J = j]$$

$$= \sum_{j=0}^\infty P[J = j] = P[J < \infty]. \quad (81)$$

Hence, it is enough to prove that $P[J < \infty] = 1$. Note,

$$P[J < \infty] = P\left[ \bigcup_{j} \left\{ \|\dot{F}(\beta_{T_j})\|_2 \leq \epsilon \right\} \right] \quad (82)$$

$$\geq P\left[ \limsup_{j \to \infty} \|\dot{F}(\beta_{T_j})\|_2 \leq \epsilon \right] \quad (83)$$

By Corollary 2, the ultimate quantity has probability one. \hfill \Box

Given that we are directly evaluating $\|\dot{F}(\beta_{T_j})\|_2$, \textbf{SC-1} has no issues concerning false negatives or positives. Unfortunately, the story is not the same for \textbf{SC-2}. To understand \textbf{SC-2}, we will first need the following lemma.

**Lemma 3** Let $F$ be a Bottou-Curtis-Nocedal function (§2.4). For $N \in \mathbb{N}$, let $\{Z_1, \ldots, Z_N\}$ be independent copies of $X$. Then for $\rho > 0$,

$$P\left[ \frac{1}{N} \left\| \sum_{i=1}^N \dot{f}(\theta, Z_i) \right\|_2 \leq \rho \right] \geq 1 - \frac{C_1 + (C_2 + N) \|\dot{F}(\theta)\|_2^2}{N \rho^2}. \quad (84)$$

Moreover, if \textbf{T1} holds and $\|\dot{F}(\theta)\|_2 \leq \pi_1$, then

$$P\left[ \frac{1}{N} \left\| \sum_{i=1}^N \dot{f}(\theta, Z_i) \right\|_2 \leq \rho \right] \geq \begin{cases} 0 & \rho < \pi_3 \|\dot{F}(\theta)\|_2 \\ \pi_2 \rho \geq \pi_3 \|\dot{F}(\theta)\|_2 & \end{cases} \quad (85)$$
Proof In both cases, we will find the upper bound for the complement, from which a lower bound for the stated event is readily derived. For the first case, by Markov’s inequality and \textbf{NM-4},

\[ \mathbb{P} \left[ \frac{1}{N} \left\| \sum_{i=1}^{N} \dot{f}(\theta, Z_i) \right\|_2 > \rho \right] \leq \frac{\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^{N} \dot{f}(\theta, Z_i) \right\|_2^2 \right]}{\rho^2} \leq C_1 + \frac{C_2 + N}{N\rho^2} \left\| \dot{F}(\theta) \right\|_2^2. \]  

(86)

For the second case, if $\rho < \pi_3 \left\| \dot{F}(\theta) \right\|_2$, then the bound holds trivially. If $\rho \geq \pi_3 \left\| \dot{F}(\theta) \right\|_2$, then

\[ \mathbb{P} \left[ \frac{1}{N} \left\| \sum_{i=1}^{N} \dot{f}(\theta, Z_i) \right\|_2 > \rho \right] \leq N\mathbb{P} \left[ \left\| \dot{f}(\theta, Z_1) \right\|_2 > \rho \right], \]  

(87)

(88)

to which we apply \textbf{T1}. \qed

Remark 5 The fact that we have more samples does not help in the case where \textbf{T1} holds; in fact, it actually makes the situation worse. This is expected as such a fat-tailed distribution has a population mean of infinity, which, intuitively, would make a sample mean entirely useless.

With this lemma, the following result states when \textbf{SC-2} is detectable and how the false negative probability is controlled.

\textbf{Proposition 2} Let \( F \) be a Bottou-Curtis-Nocedal function (§2.4) and let \( \{\beta_k\} \) be the iterates generated by Stochastic Gradient Descent satisfying \textbf{P1} to \textbf{P3} (§3.1).

\textbf{Scenario (a):} Let $\rho \in (0, 1)$. If the SGD iterates are subject to \textbf{SC-2} for some $\epsilon > 0$ and for \( \{N_j\} \) such that

\[ \mathbb{P} \left[ \liminf_{j \to \infty} N_j > \frac{C_1}{\epsilon^2 \rho^2} + C_2 \right] = 1, \]  

(89)

then \textbf{SC-2} is detectable. Moreover, when $\|\dot{F}(\beta_{T_j})\|_2 \leq \rho \epsilon$ and $N_j > C_1/(\epsilon \rho)^2 + C_2$, then the probability of a false negative is controlled by $2\rho^2$.

\textbf{Scenario (b):} Suppose that $C_1 = 0$ in \textbf{NM-4}. Let $\rho \in (0, 1)$. If the SGD iterates are subject to \textbf{SC-2} for some $\epsilon > 0$, then \textbf{SC-2} is detectable. Moreover, when $\|\dot{F}(\beta_{T_j})\|_2 \leq (\rho C_2/N_j + 1)^{-1/2}$ then the probability of a false negative is controlled by $\rho^2$.

\textbf{Scenario (c):} Suppose \textbf{T1} holds. Let $\rho \in (0, 1)$. If the SGD iterates are subject to \textbf{SC-2} for some

\[ 0 < \epsilon \leq \frac{\pi_3 N_j^{1/\pi_2}}{\rho}, \]  

(90)
then SC-2 is detectable. Moreover, when \( \| \hat{F}(\beta_T) \|_2 \leq \epsilon \rho (\pi_3 N_j^{1/\pi_2})^{-1} \), then the probability of a false negative is controlled by \( \rho^{\pi_2} \).

**Proof** For each scenario, we first prove that, under the specified conditions, there exists a \( \bar{p} \in (0, 1) \), such that

\[
P \left[ \frac{1}{N_j} \left\| \sum_{i=1}^{N_j} \hat{f}(\beta_T, Z_{ij}) \right\|_2 \leq \epsilon \left| F_{T_j} \right| \right] \geq 1 - \bar{p}. \tag{91}
\]

Therefore, the probability of a false negative is controlled by \( \bar{p} \). Moreover, by Corollary 2, the upper bounds on \( \| \hat{F}(\beta_T) \|_2 \) specified in each scenario will hold for all \( j \) sufficiently large. Therefore, we will have a sequence of independent tests whose probability of failure is bounded from above by \( \bar{p} \); that is,

\[
P [ J = j | F_0 ] \leq \bar{p}^{j-1}. \tag{92}
\]

By comparing \( J \) to the negative binomial distribution, we conclude that \( J < \infty \) with probability one, which implies \( P [ T_J < \infty | F_0 ] = 1 \) as in the proof of Proposition 1. Therefore, we see that the stopping criteria is detectable and that \( \bar{p} \) is an upper bound (i.e., controls) the probability of a false negative. Therefore, we simply need to compute \( \bar{p} \) under each scenario to complete the proof.

**Scenario (a):** For \( j \) satisfying the hypotheses, Lemma 3 implies

\[
P \left[ \frac{1}{N_j} \left\| \sum_{i=1}^{N_j} \hat{f}(\beta_T, Z_{ij}) \right\|_2 \leq \epsilon \left| F_{T_j} \right| \right] \geq 1 - \frac{C_1 + (C_2 + N_j) \| \hat{F}(\beta_T) \|_2^2}{N_j \epsilon^2} \tag{93}
\]

\[
= 1 - \frac{C_1 + C_2 \rho^2 \epsilon^2}{N_j \epsilon^2} - \rho^2 \tag{94}
\]

\[
\geq 1 - \frac{C_2 + C_2 \rho^2 \epsilon^2}{C_1 / \rho^2 + C_2 \epsilon^2} - \rho^2 \tag{95}
\]

\[
= 1 - 2 \rho^2. \tag{96}
\]

**Scenario (b):** For \( j \) satisfying the hypotheses, Lemma 3 with \( C_1 = 0 \) implies

\[
P \left[ \frac{1}{N_j} \left\| \sum_{i=1}^{N_j} \hat{f}(\beta_T, Z_{ij}) \right\|_2 \leq \epsilon \left| F_{T_j} \right| \right] \geq 1 - \frac{(C_2 + N_j) \| \hat{F}(\beta_T) \|_2^2}{N_j \epsilon^2} \tag{97}
\]

\[
\geq 1 - \rho^2. \tag{98}
\]
**Scenario (c):** For \( j \) satisfying the hypotheses,
\[
\| \hat{F}(\beta_{T_j}) \|_2 \leq \frac{\epsilon\rho}{\pi_3 N_j^{1/\pi_2}} \leq \frac{\epsilon}{\pi_3}.
\] (99)

Therefore, Lemma 3 with \( T_1 \) implies
\[
P \left[ \frac{1}{N_j} \sum_{i=1}^{N_j} \| f(\beta_{T_j}, Z_{ij}) \|_2 \leq \epsilon \right| \mathcal{F}_{T_j} \right] \geq 1 - N_j \left( \frac{\pi_3 \| \hat{F}(\beta_{T_j}) \|_2}{\epsilon} \right)^{\pi_2}
\]
\[
\geq 1 - \rho^{\pi_2}.
\] (100)

\[
\geq 1 - \rho^{\pi_2}.
\] (101)

A criticism of both SC-1 and SC-2 is that the effort required to evaluate them is nearly identical to updating the parameter by the SGD method in order to achieve, presumably, an iterate that is closer to a stationary point. To address this criticism, we consider the following stopping criteria which makes use of information that is already computed in order to update the iterates. However, we will not address the detectability nor control over the false negative probability as it requires developing a maximal inequality that will be addressed in future work.

4.3 Stopping Criteria by Majority Vote

In the preceding discussion, we considered stopping criteria based on estimating the deterministic gradient, which are natural extensions of deterministic stopping criteria. However, we collecting this information is in some sense too much: we are simply trying to make an up-down decision about whether convergence has occurred, why should we not simply design a stopping criteria that exploits this structure directly? The following stopping criteria take this
exact approach. In effect, the following stopping criteria test each stochastic gradient against some condition and then use the majority decision (as determined by a threshold) to determine whether to stop or to continue the iterates. The first stopping criteria, **SC-4**, leverages independent samples, and would face the same aforementioned criticisms as **SC-2**. The second stopping criteria, **SC-5**, uses the samples that generate the iterate sequence and avoid these criticisms.

\[
(\text{SC-4}) \quad \text{Let } \epsilon > 0. \text{ Let } \{T_j\} \text{ be a sequence of positive-valued, strictly increasing, finite stopping times with respect to } \{F_k\}. \text{ Let } \{N_j\} \text{ be } \mathbb{N}\text{-valued random variables such that } N_j \text{ is measurable with respect to } F_{T_j}. \text{ Let } \bar{\gamma} \in (0,1) \text{ and let } \{\gamma_j\} \text{ be } (0, \bar{\gamma})\text{-valued random variables such that } \gamma_j \text{ is measurable with respect to } F_{T_j}. \text{ Moreover, for each } j, \text{ let } \{Z_{ij} : i = 1, \ldots, N_j\} \text{ be copies of } X \text{ that are independent of each other and } \{F_k\}. \text{ Then, the SGD iterates are stopped at iterate } T_J, \text{ where}
\]
\[
J = \min \left\{ j : \frac{1}{N_j} \sum_{i=1}^{N_j} 1 \left[ \| \hat{f}(\beta_{T_j}, Z_{ij}) \|_2 \leq \epsilon \right] \geq \gamma_j \right\}.
\]

\[
(\text{SC-5}) \quad \text{Let } \epsilon > 0. \text{ Let } \{T_j\} \text{ be a sequence of positive-valued, strictly increasing, finite stopping times with respect to } \{F_k\}. \text{ Let } \{N_j\} \text{ be } \mathbb{N}\text{-valued random variables such that } N_j \text{ is measurable with respect to } F_{T_j}. \text{ Let } \bar{\gamma} \in (0,1) \text{ and let } \{\gamma_j\} \text{ be } (0, \bar{\gamma})\text{-valued random variables such that } \gamma_j \text{ is measurable with respect to } F_{T_j}. \text{ The SGD iterates are stopped at iterate } T_J + N_j - 1, \text{ where}
\]
\[
J = \min \left\{ j : \frac{1}{N_j} \sum_{i \in I_j} 1 \left[ \| \hat{f}(\beta_i, X_{i+1}) \|_2 \leq \epsilon \right] \geq \gamma_j \right\},
\]
\[
\text{where } I_j = \{T_j, T_j + 1, \ldots, T_j + N_j - 1\}.
\]

The following supplies an analogue of Lemma 3 to the stopping criteria specified in **SC-4**. The result about detectability and control over the false negative probability follows.

**Lemma 4** Let \( F \) be a Bottou-Curtis-Nocedal nonconvex function (§2.4). For \( N \in \mathbb{N}, \) let \( \{Z_1, \ldots, Z_N\} \) be independent copies of \( X. \) Let \( \epsilon > 0 \) and \( \gamma \in (0,1) \) and define
\[
\Delta = \gamma - \mathbb{P} \left[ \| \hat{f}(\theta, X) \|_2 \leq \epsilon \right].
\]  

(102)
When $\Delta < 0$,
\[
\mathbb{P} \left[ \frac{1}{N} \sum_{i=1}^{N} I \left[ \left\| \dot{f}(\theta, Z_i) \right\|_2 \leq \epsilon \right] \geq \gamma \right] \geq 1 - \exp \left( -2N\Delta^2 \right). \tag{103}
\]

**Proof** The proof leverages McDiarmid’s inequality (see §3 of [24]). Let the range of $X$ be denoted by $X$. Let $z_1, ..., z_N \in X$ and define
\[
h(z_1, ..., z_N) = \frac{1}{N} \sum_{i=1}^{N} I \left[ \left\| \dot{f}(\theta, z_i) \right\|_2 \leq \epsilon \right]. \tag{104}
\]
Then, for any $j \in \{1, ..., N\}$ and $z_1, ..., z_N, z'_j \in X$,
\[
|h(z_1, ..., z_N) - h(z_1, ..., z'_j, ..., z_N)| \leq \frac{1}{N}. \tag{105}
\]
Since
\[
\mathbb{E} [h(Z_1, ..., Z_N)] = \mathbb{P} \left[ \left\| \dot{f}(\theta, Z_i) \right\|_2 \leq \epsilon \right], \tag{106}
\]
McDiarmid’s inequality implies
\[
\mathbb{P} [h(Z_1, ..., Z_N) < \gamma] = \mathbb{P} [h(Z_1, ..., Z_N) - \mathbb{E} [h(Z_1, ..., Z_N)] < \Delta] \tag{107}
\leq \exp(-2N\Delta^2). \tag{108}
\]
By computing the complement, the result follows. \qed

**Proposition 3** Let $F$ be a Bottou-Curtis-Nocedal function (§2.4) and let $\{\beta_k\}$ be the iterates generated by Stochastic Gradient Descent satisfying P1 to P3 (§3.1).

**Scenario (b):** Suppose that $C_1 = 0$ in NM-4. Let $\rho \in (0, 1)$. If the SGD iterates are subject to SC-4 for some $\epsilon > 0$, then SC-4 is detectable. Moreover, when
\[
\left\| \dot{F}(\beta_{T_j}) \right\|_2 \leq \epsilon \sqrt{\frac{\rho(1 - \bar{\gamma})}{C_2 + 1}}, \tag{109}
\]
then the probability of a false negative is controlled by $\exp(-2N \rho(1 - \bar{\gamma})^2 (1 - \bar{\gamma})^2)$.

**Scenario (c):** Suppose T1 holds. Let $\rho \in (0, 1)$. If the SGD iterates are subject to SC-2 for some
\[
0 < \epsilon \leq \pi\pi_3^3 [\rho(1 - \bar{\gamma})]^{1/\pi_2}, \tag{110}
\]
then SC-4 is detectable. Moreover, when
\[
\left\| \dot{F}(\beta_{T_j}) \right\|_2 \leq \frac{\epsilon}{\pi_3} [\rho(1 - \bar{\gamma})]^{1/\pi_2} \tag{111}
\]
then the probability of a false negative is controlled by $\exp(-2N \rho(1 - \bar{\gamma})^2 (1 - \bar{\gamma})^2)$. 

**Proof** The proof proceeds just as in Proposition 2. Therefore, it is sufficient to prove the lower bound on the true positive probability for each scenario.

**Scenario (b):** For \( j \) satisfying the hypotheses, Lemma 3 implies

\[
\mathbb{P} \left[ \| \dot{f}(\beta T_j, X) \|_2 \leq \epsilon \bigg| \mathcal{F}_{T_j} \right] \geq 1 - \rho(1 - \bar{\gamma}) > 0. \tag{112}
\]

Therefore,

\[
\Delta_j := \gamma_j - \mathbb{P} \left[ \| \dot{f}(\beta T_j, X) \|_2 \leq \epsilon \bigg| \mathcal{F}_{T_j} \right] \\
\leq \bar{\gamma} - 1 + \rho(1 - \bar{\gamma}) \tag{113}
\]

\[
= (1 - \rho)(\bar{\gamma} - 1) \tag{114}
\]

\[
< 0. \tag{115}
\]

Applying Lemma 4,

\[
\mathbb{P} \left[ \frac{1}{N_j} \sum_{i=1}^{N_j} 1 \left[ \| \dot{f}(\beta T_j, Z_{ij}) \|_2 \leq \epsilon \right] \geq \gamma_j \bigg| \mathcal{F}_{T_j} \right] \geq 1 - \exp \left( -2N_j(1 - \rho)^2(1 - \bar{\gamma})^2 \right). \tag{116}
\]

**Scenario (c):** For \( j \) satisfying the hypotheses, note that \( \pi_3 \| \dot{F}(\beta T_j) \|_2 \leq \epsilon_3 \). Therefore, applying Lemma 3 with \( T_1 \),

\[
\mathbb{P} \left[ \| \dot{f}(\beta T_j, X) \|_2 \leq \epsilon \bigg| \mathcal{F}_{T_j} \right] \geq 1 - \rho(1 - \bar{\gamma}) > 0. \tag{117}
\]

The remainder of the proof is identical to Scenario (b). \( \square \)

Just as for **SC-3**, we will not address the whether **SC-5** is detectable nor how we control its false negative probability. Again, such a result requires establishing a maximal inequality, which we intend to address in future efforts.

### 5 Conclusion

In this work, our goal was to lay a rigorous foundation for stopping criteria for stochastic gradient descent (SGD) as applied to Bottou-Curtis-Nocedal (BCN) functions, which includes a broad class of convex and nonconvex functions. We started by developing a strong global convergence result for SGD on BCN functions, which generalizes previous results on the convergence of SGD on nonconvex functions. Then, we stated five stopping criteria and rigorously analyzed three of them in regards to whether the stopping criteria would be triggered in finite time (i.e., detectability), and how they control the false negative probability.

This work has raised several questions that we enumerate below, and which we hope to address in future work.
1. Given strong global convergence, what is the local rate of convergence to a stationary point? Is this stationary point guaranteed to be a minimum? As evidenced by the many works cited, these issues are of great importance and can be answered more completely now that we have established strong global convergence.

2. Can something be said about SC-3 and SC-5? In some sense, these two stopping criteria are ideal as they are the least wasteful stopping criteria. However, to develop such results, we need a maximal inequality over the norms of the iterates.

3. For all of the stopping criteria, what are reasonable choices of \( \{T_j\} \) and \( \{N_j\} \)?

4. What are reasonable conditions to place on the lower tail probabilities (analogous to T1) and what are their implications for controlling the false positive probability of the stopping criteria studied in this work?

5. Finally, is there a context in which NM-3 is more appropriate than NM-4, and can the preceding results be developed in this context as well?

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