TANNAKIAN DUALITY OVER DEDEKIND RING AND APPLICATIONS

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Dedicated to Hélène Esnault, with admiration and affection

Abstract. We review the Tannakian duality for group schemes over a Dedekind ring, obtained by Saavedra, and recent related results of Wedhorn. We use it to study homomorphisms of flat group schemes over Dedekind rings. Applications to the study of the fundamental group schemes of algebraic schemes are discussed.

1. Introduction

Tannakian duality for algebraic groups over a field was studied by Saavedra [19]. The duality in the neutral case, as shown by Saavedra, is a dictionary between $k$-linear abelian rigid tensor categories equipped with a fiber functor to the category of vector spaces an affine group schemes over $k$. The duality consists of two parts:

- The reconstruction theorem which recover a group scheme from a neutral Tannakian category $(\mathcal{T}, \omega : \mathcal{T} \rightarrow \text{Vect}(k))$, as the group of automorphisms of $\omega$, which preserves the tensor product, the Tannakian group of $(\mathcal{T}, \omega)$.
- The presentation (or description) theorem which claims the equivalence between the original category $\mathcal{T}$ and the representation category of the Tannakian group of $\mathcal{T}$.

Saavedra also extended this result to the non-neutral case - when the fiber functor goes to a more general category of coherent sheaves over a $k$-scheme. The complete proof of this theorem was given by Deligne in [6].

The main applications of Tannakian duality are the following. Let $X$ be a scheme over an algebraically closed field $k$. There are various tensor abelian categories associated to $X$. For example, if $X$ is reduced and connected, M. Nori defined the category of essentially finite bundles. If $X$ is smooth and $k$
has characteristic zero, one has the category of flat connections, if \( k \) has positive characteristic, one has the category of stratified bundles (i.e. \( \mathcal{O}_X \)-coherent modules over the sheaf of algebras of differential operators). Fix a \( k \)-rational point \( x \) of \( X \) and consider the functor taking the fibers at \( x \), we see that the above categories become Tannakian category and Tannakian duality applies to yield a corresponding affine group scheme, which is usually called the fundamental group scheme of \( X \).

Let now \( X \) be a smooth scheme over a Dedekind ring \( R \). We are interested in the category of modules over \( \mathcal{D}(X/R) \), which are coherent as \( \mathcal{O}_X \)-modules, where \( \mathcal{D}(X/R) \) denotes the sheaf of algebras of differential operators on \( X/R \). Such a sheaf will be called a stratified sheaf over \( X \). The category \( \text{str}(X/R) \) of stratified sheaves on \( X/R \) is an abelian tensor category, in which an object is rigid if it is locally free as an \( \mathcal{O}_X \)-module. Assume that \( X \) admits an \( R \)-rational point \( \xi \). Then the functor taking fiber at \( \xi \) provides us a fiber functor for \( \text{str}(X/R) \). It is then natural to ask, if there exists a generalization of Tannakian duality to this case.

Recently there have been attempts to extend Tannakian duality to a more general settings. Most important is the case of flat group schemes over Dedekind rings, see, e.g., [26, 3]. It seems, a result of Saavedra has been ignored. In [19, II.2], Saavedra gave a condition for an abelian category equipped with an exact faithful functor to the module category (over a given Noetherian ring) to be equivalent to the comodule category over the coalgebra reconstructed from this functor. This duality can be developed to a Tannakian duality for flat affine group schemes over a Dedekind ring. The discussion of Saavedra, although very conceptual, was not easy to follow. This might be one reason for the ignorance of his result. It is therefore one of our aims to give a direct and self-contained proof of Saavedra’s result. We also relate Saavedra’s construction of the Tannakian group to the more common construction of coend originated from the work of MacLane. Some relationships to the recent work of Wedhorn [26] are also given. This is the contents of Section 2.

The Tannakian duality is applied in Section 3 to study the property of homomorphisms of flat coalgebras over a Dedekind ring. The main results (Propositions 3.9, 3.13) provide conditions for such a homomorphism to be (special) injective or surjective. In Section 4 these results are used to study homomorphisms of flat group schemes: to characterize closed immersion and faithful flatness. We introduce the notion of local finiteness. For coalgebras over fields, local finiteness is an important property: each coalgebra is the union of its finite dimensional subcoalgebras. The situation becomes more complicated for flat coalgebras over Dedekind ring. A flat coalgebra over a Dedekind ring is still the union of its finite subcoalgebras. However one generally cannot chose the subcoalgebras to be special (see Definition 3.1). Consequently, it is not true in general that a flat
group scheme is pro-algebraic in the sense that it is the limit of a pro-system of

It is even not known, which flat group scheme of finite type is locally finite (some
classes are known, see Proposition 3.7). It is thus natural to pose the question:

when the Tannakian group of a Tannakian category over a Dedekind ring is pro-
algebraic (see 4.4). Another question is to know, in terms of Tannakian duality,
when a flat group scheme is smooth.

In the last part of Section 4, using the Tannakian duality we give a criterion for
the exactness of sequences of homomorphisms of flat affine group schemes over
Dedekind rings.

In Section 5 we apply the Tannakian duality to the case of stratified sheaves
over a smooth scheme $X/R$, where $R$ is a Dedekind ring. When $R$ is complete
local discrete valuation ring with equal characteristic, we show that $\text{str}(X/R)$ is
a Tannakian category over $R$. The general case is open. However, there exists
an appropriate subcategory of $\text{str}(X/R)$ which is Tannakian and thus defines the
relative differential fundamental group. Not much is known about this group
scheme. For instance, we are interested in the question, whether this group
scheme is locally finite, pro-algebraic and smooth.

2. Tannakian duality for flat coalgebras over Dedekind rings

Let $R$ be a Dedekind ring. In [19], Saavedra showed that a flat $R$-coalgebra can
be reconstructed from the category of its comodules, which are finitely generated
over $R$. He also gave a criterion for a category to be equivalent to $\text{Comod}_f(L)$.
This result is then used to prove the Tannakian duality for flat group scheme
over Dedekind rings.

In this section we give a quick, complete and self-contained proof of this result.
We also relate this result to the current result of Wedhorn.

First we will recall the notion ind-category of an abelian category. The two
equivalent descriptions of the ind-category will play a crucial role in Saavedra’s
proof. A category $\mathcal{I}$ is called a filtering category if to every pair $i, j$ of objects in
$\mathcal{I}$ there exists an object $k$ such that $\text{Hom}(i, k)$ and $\text{Hom}(j, k)$ are both not empty,
and for every pair $u, v : i \to j$, there exists a morphism $w : j \to k$ such that
$wu = wv$.

Definition 2.1. Ind-categories. Let $\mathcal{C}$ be an abelian category. The category
$\text{Ind}(\mathcal{C})$ consists of functors $X : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is a filtering category. We
usually denote $X_i$ for $X(i), i \in \mathcal{I}$, an write

$$X = \lim_{i \in \mathcal{I}} X_i.$$
For two objects $X = \lim_{i \in I} X_i$ and $Y = \lim_{j \in J} Y_j$ their hom-set is defined to be
\[ \text{Hom}(X, Y) := \lim_{i \in I} \lim_{j \in J} \text{Hom}(X_i, Y_j). \]
\[ \square \]

Let $\omega : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The extension of $\omega$, $\text{Ind}(\omega) : \text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\mathcal{D})$ is defined by
\[ \text{Ind}(\omega)(\lim_{i} X_i) := \lim_{i} \omega(X_i). \]

There is an alternative description of $\text{Ind}(\mathcal{C})$. Denote $\text{Lex}(\mathcal{C}^{\text{op}}, \text{Sets})$ the category of left exact functors from $\mathcal{C}^{\text{op}}$ to the category of sets. For $X = \lim_{i} X_i$ we define functor
\[ \lim_{i} h_{X_i}(-) := \lim_{i} \text{Hom}(-, X_i) \in \text{Lex}(\mathcal{C}^{\text{op}}, \text{Sets}). \]

This yields a functor $\text{Ind}(\mathcal{C}) \rightarrow \text{Lex}(\mathcal{C}^{\text{op}}, \text{Sets})$ which is an equivalence (cf. [2], I.8.3.3). Recall that the Hom-sets for objects of $\text{Lex}(\mathcal{C}^{\text{op}}, \text{Sets})$ are by definition the sets of natural transformations. For simplicity, we shall use the notation $\text{Hom}(F, G)$ instead of $\text{Nat}(F, G)$ for objects of this category.

Suppose that $\mathcal{C}$ is an $R$-linear Noetherian abelian category. Let $\text{Lex}_R(\mathcal{C}^{\text{op}}, \text{Mod}(R))$ be category of $R$-linear left exact functors from $\mathcal{C}^{\text{op}}$ to the category of modules $\text{Mod}(R)$. Then the natural functor
\[ \text{Lex}_R(\mathcal{C}^{\text{op}}, \text{Mod}(R)) \cong \text{Lex}(\mathcal{C}^{\text{op}}, \text{Sets}) \]
is an equivalence (cf. Gabriel [10, II]). Thus, for an $R$-linear Noetherian abelian category we have an equivalence
\[ \text{Ind}(\mathcal{C}) \cong \text{Lex}_R(\mathcal{C}^{\text{op}}, \text{Mod}(R)), \quad X = \lim_{i} X_i \mapsto \lim_{i} h_{X_i}(-). \]

Further the category $\text{Ind}(\mathcal{C})$ is locally Noetherian and the inclusion $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$ identifies $\mathcal{C}$ with the full subcategory of Noetherian objects in $\text{Ind}(\mathcal{C})$, [10, II, 4, Thm.1].

The following are our main examples.

**Example 2.2.** The category $\text{Mod}_f(R)$ of finitely generated $R$-modules, where $R$ is a Noetherian ring, is a Noetherian category. Its Ind category is precisely the category $\text{Mod}(R)$ of all $R$-modules. This is obvious.

**Example 2.3.** Denote by $\text{Comod}(L)$ the category of right $L$-comodules and by $\text{Comod}_f(L)$ the subcategory of comodules which are finitely generated as $R$-module. Then:

(i) If $L$ is flat over $R$ then $\text{Comod}(L)$ is an abelian category. In fact, the flatness of $L$ implies that the kernel of a homomorphism of $L$-comodules is equipped with a natural coaction of $L$. In particular, the forgetful
functor from $\text{Comod}(L)$ to $\text{Mod}(R)$ is exact. The converse is also true: if the forgetful functor preserves kernels then $L$ is flat over $R$.

(ii) Assume that $L$ is flat over $R$. A theorem of Serre \[23\] states that each $L$-comodule is the union of its subcomodules, which are finitely generated over $R$. Consequently, if $R$ is Noetherian, $\text{Comod}(L)$ is locally Noetherian and $\text{Comod}_f(L)$ is the full subcategory of Noetherian objects.

Objects of $\text{Comod}_f(L)$ will usually be called finite $L$-comodules. They will be the main target of our study.

In what follows $R$ will be assumed to be a Dedekind ring, the tensor product, when not indicated, is understood as the tensor product over $R$. We notice that over such a ring, a torsion free module is flat and a finitely generated flat module is projective.

**Definition 2.4** (Subcategory of definition, cf. \[19\] II.2.2]). Let $\mathcal{C}$ be an $R$-linear abelian category, and $\omega : \mathcal{C} \to \text{Mod}_f(R)$ be an $R$-linear exact faithful functor. Suppose that $\mathcal{C}^o$ is a full subcategory of $\mathcal{C}$ such that:

(i) for any object $X \in \mathcal{C}^o$, $\omega(X)$ is a finitely generated projective $R$-module;

(ii) every object of $\mathcal{C}$ is quotient of an object of $\mathcal{C}^o$.

Then $\mathcal{C}^o$ is called a subcategory of definition of $\mathcal{C}$ with respect to $\omega$.

This definition is motivated by the following fact, due to Serre (see \[23\] Prop.3). For each $L$-comodule $E$ there exists a short exact sequence of $L$-comodules

$$0 \to F' \to F \to E \to 0$$

in which $F', F$ are flat as $R$-modules. Thus, the subcategory $\text{Comod}^o(L)$ of $L$-comodules which are flat over $R$ is a subcategory of definition in $\text{Comod}_f(L)$. In particular, the subcategory of projective modules in $\text{Mod}_f(R)$ is a subcategory of definition in $\text{Mod}_f(R)$.

Let $\mathcal{C}$ be an $R$-linear abelian category, and $\omega : \mathcal{C} \to \text{Mod}_f(R)$ be an $R$-linear exact faithful functor. Suppose that there exists a full subcategory of definition $\mathcal{C}^o$ in $\mathcal{C}$. Our aim is to show that there exists a flat $R$-coalgebra $L$ such that $\omega$ induces an equivalence between $\text{Comod}_f(L)$ and $\mathcal{C}$, and between $\text{Comod}(L)$ and $\text{Ind}(\mathcal{C})$.

The functor $\omega$ induces a functor $\text{Ind}(\mathcal{C}) \to \text{Mod}(R)$, which we, by abuse of language, will denote simply by $\omega$. Recall that we identify $\text{Ind}(\mathcal{C})$ with $\text{Lex}(\mathcal{C}^{op}, \text{Mod}(R))$, the category of left exact functors on $\mathcal{C}^{op}$ with values in $\text{Mod}(R)$. The key technique is to use alternatively these two equivalent descriptions of one category.

For any $R$-algebra $A$, we define functor

$$F^A : \mathcal{C}^{op} \to \text{Mod}(A), \quad X \mapsto \text{Hom}(\omega(X), A).$$
Then $F^A$ is an object of $\text{Lex}(\mathcal{C}^{\text{op}}, \text{Mod}(R))$. Set $F := F^R$. There is a natural $A$-linear transformation $A \otimes F \to F^A$.

$$\theta_X : A \otimes \text{Hom}(\omega(X), R) \to \text{Hom}(\omega(X), A), \quad a \otimes f \mapsto af.$$ 

**Lemma 2.5.** The $A$-linear transformation $\theta : A \otimes F \to F^A$ given above is an isomorphism.

**Proof.** For any $K, G \in \text{Lex}(\mathcal{C}^{\text{op}}, \text{Mod}(R))$ we denote $K^o, G^o$ their restrictions to $(\mathcal{C}^o)^{\text{op}}$, respectively. From the definition of $\mathcal{C}^o$ we have

(1) \hspace{1cm} \text{Hom}(K, G) := \text{Nat}(K, G) \simeq \text{Nat}(K^o, G^o).

For $X \in \mathcal{C}^o$, $\omega(X)$ is finitely generated projective over $R$, hence

$$F^A(X) = \text{Hom}(\omega(X), A) \simeq \text{Hom}(\omega(X), R) \otimes A = A \otimes F(X).$$

Therefore, for any $G \in \text{Lex}(\mathcal{C}^{\text{op}}, \text{Mod}(R))$, we have

(2) \hspace{1cm} \text{Nat}((F^A)^o, G^o) = \text{Nat}((A \otimes F)^o, G^o)

and (1) yields

(3) \hspace{1cm} \text{Nat}(F^A, G) = \text{Nat}(A \otimes F, G).

So we have $F^A \simeq A \otimes F$. \hfill \Box

We will show that $L := \omega(F)$ is the coalgebra to be found. To show this, first we will need

**Lemma 2.6.** For any $X \in \text{Lex}(\mathcal{C}^{\text{op}}, \text{Mod}(R))$ and $R$-algebra $A$ we have the following $A$-linear isomorphism:

(4) \hspace{1cm} \text{Hom}(X, F^A) \simeq \text{Hom}_A(A \otimes \omega(X), A) = \text{Hom}_R(\omega(X), A).

**Proof.** Every $X \in \text{Lex}(\mathcal{C}^{\text{op}}, \text{Mod}(R))$ can be represented as $X = \varinjt \lim h_{X_i}(X_i \in \mathcal{C})$, where $h_{X_i}$ is a functor over $\mathcal{C}$, defined by $h_{X_i}(-) := \text{Hom}_\mathcal{C}(-, X_i)$. Hence we have

$$\text{Hom}(X, F^A) = \text{Hom}(\varinjt \lim h_{X_i}, F^A) = \varinjlim \text{Hom}(h_{X_i}, F^A) = \varinjlim F^A(X_i) = \text{Hom}_R(\varinjlim \omega(X_i), A) = \text{Hom}_R(\omega(\varinjlim h_{X_i}), A) = \text{Hom}_R(\omega(X), A).$$

It is easy to see that all isomorphisms are $A$-linear. \hfill \Box
Isomorphism (4) for $A = R$ and $X = R$ reads $\text{Hom}(F, F) \simeq \text{Hom}_R(\omega(F), R)$. We denote $L := \omega(F)$ and let $\varepsilon : L \to R$ be the map on the right hand side that corresponds to the identity transformation on the left hand side of this isomorphism. One can replace the algebra $A$ in (4) by any $R$-module $M$. This time we have $R$-linear isomorphisms.

**Lemma 2.7.** There exists a natural $R$-linear isomorphism

\[(5) \Phi_{X,M} : \text{Hom}(X, M \otimes F) \simeq \text{Hom}_R(\omega(X), M).\]

which is given explicitly by

\[\Phi_{X,M}(f) = (\text{id}_M \otimes \varepsilon) \circ \omega(f).\]

**Proof.** For any $A$-module $M$, we can make $A \oplus M$ into an $A$-algebra by letting $M$ be an ideal with square null. Hence the isomorphism (5) is a direct consequence of (4). By definition $\Phi_{F,R}$ is given by

\[\Phi_{F,R}(f) = \varepsilon \circ \omega(f).\]

Each homomorphism $\iota : R \to M$ induces by functoriality the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(F, F) & \xrightarrow{\varepsilon \circ \omega(-)} & \text{Hom}_R(\omega(F), R) \\
(\iota \otimes \text{id}_F) \circ - & \downarrow & \iota \circ - \\
\text{Hom}(F, M \otimes F) & \xrightarrow{\Phi_{F,M}} & \text{Hom}_R(\omega(F), M)
\end{array}
\]

Now, the identity on $F$ yields the equality:

\[\iota \circ \varepsilon = \Phi_{F,M}(\iota \otimes \text{id}_{\omega(F)}) : \omega(F) \to M.\]

Hence, for $m = \iota(1)$, we have $\Phi_{F,M}(l) = \varepsilon(l)m$, $l \in \omega(F)$. Thus the claim holds for $X = F$. Since the $\omega$ and Hom-functor in the first variant commute with direct limits we conclude that the claim hold of $X = N \otimes F$ for any $R$-module $N$. Now the general case follows from the following diagram

\[
\begin{array}{ccc}
\text{Hom}(M \otimes F, M \otimes F) & \xrightarrow{\Phi_{F,M \otimes F}} & \text{Hom}(M \otimes \omega(F), M) \\
(-) \circ f & \downarrow & (-) \circ \omega(f) \\
\text{Hom}(X, M \otimes F) & \xrightarrow{\Phi_{X,M}} & \text{Hom}(\omega(X), M)
\end{array}
\]

applied for the identity of $M \otimes F$:

\[\Phi_{X,M}(f) = \Phi_{F,M}(\text{id}) \circ \omega(f) = (\text{id}_M \otimes \varepsilon) \circ \omega(f).\]

□
Proposition 2.8. Let $L := \omega(F)$. Then it is a coalgebra with $\varepsilon$ being the counit and $\omega$ factors though a functor
\[ \text{Ind}(C) \rightarrow \text{Comod}(L). \]

Proof. Choose $M = \omega(X)$ in \( \text{(5)} \) we have a morphism $\sigma_X : X \rightarrow \omega(X) \otimes F$ which corresponds to the identity element $\text{id}_{\omega(X)}$ under the isomorphism $\Phi_{X,\omega(X)}$ of Lemma 2.7, thus we have
\[ (\text{id}_{\omega(X)} \otimes \varepsilon) \circ \omega(\sigma_X) = \text{id}_{\omega(X)}. \]

For any morphism $\lambda : X \rightarrow Y$ in $\text{Ind}(C)$, according to \( \text{(2.7)} \) we have the following equalities:
\[ \Phi_{X,\omega(Y)}((\omega(\lambda) \otimes \text{id}_F) \circ \sigma_X) = \omega(\lambda), \]
\[ \Phi_{X,\omega(Y)}(\sigma_Y \circ \lambda) = \omega(\lambda). \]
Thus $((\omega(\lambda) \otimes \text{id}_F) \circ \sigma_X = \sigma_Y \circ \lambda$, i.e., the following diagram commutes:
\[ X \xrightarrow{\lambda} Y \]
\[ \sigma_X \downarrow \quad \sigma_Y \]
\[ \omega(X) \otimes F \xrightarrow{\omega(\lambda) \otimes \text{id}_F} \omega(Y) \otimes F. \]

For $Y = \omega(X) \otimes F$ and $\lambda = \sigma_X$, we get
\[ X \xrightarrow{\sigma_X} \omega(X) \otimes F \]
\[ \sigma_X \downarrow \quad \sigma_X \]
\[ \omega(X) \otimes F \xrightarrow{\omega(\sigma_X) \otimes \text{id}_F} \omega(X) \otimes L \otimes F. \]

Applying $\omega$ on this diagram we obtain a commutative diagram in $\text{Mod}(R)$:
\[ \omega(X) \xrightarrow{\omega(\sigma_X)} \omega(X) \otimes L \]
\[ \omega(\sigma_X) \downarrow \quad \omega(\sigma_X) \]
\[ \omega(X) \otimes L \xrightarrow{\omega(\sigma_X) \otimes \text{id}_L} \omega(X) \otimes L \otimes L, \]
where $\Delta := \omega(\sigma_F)$. Together with \( \text{(6)} \), this diagram for $X = F$ gives a coalgebra structure on $L$ with $\Delta$ being the coproduct and hence, for any $X$, it gives a comodule structure of $L$ on $\omega(X)$. \( \square \)

Theorem 2.9. Let $R$ be a Dedekind ring and let $C$ be an $R$-linear abelian category. Assume that there exist an $R$-linear exact faithful functor $\omega : C \rightarrow \text{Mod}_t(R)$ and a subcategory of definition $C^\circ$ with respect to $\omega$. Then $\omega$ factors though an equivalence $C \simeq \text{Comod}_t(L)$ and the forgetful functor, for some $R$-coalgebra $L$. 

Proof. Let $L$ be defined as in Proposition 2.8. We consider $\omega$ as a functor $C \to \text{Comod}_f(L)$. It is to show that $\omega$ is an equivalence of category. By definition it is faithful. To see the fullness, suppose $X, Y \in C'$ and $\alpha : \omega(X) \to \omega(Y)$ is a homomorphism of $L$-comodules, i.e., we have

$$(\alpha \otimes \text{id}) \circ \omega(\sigma_X) = \omega(\sigma_Y) \circ \alpha : \omega(X) \to \omega(Y) \otimes L.$$

Then $\omega(X) \xrightarrow{\alpha} \omega(Y) \xrightarrow{\omega(\sigma_Y)} \omega(Y) \otimes L$ is the image under $\omega$ of the morphism $X \xrightarrow{\sigma_X} \omega(X) \otimes F \xrightarrow{\alpha \otimes \text{id}} \omega(Y) \otimes F$.

Notice that (9) (for $X$ replaced by $Y$) yields a split exact sequence

$$0 \to \omega(Y) \to \omega(Y) \otimes L \xrightarrow{\delta} \omega(Y) \otimes L \otimes L,$$

where the second homomorphism is $\delta = \text{id} \otimes \Delta - \omega(\sigma_X) \otimes \text{id}$, and the splitting is given by $\text{id} \otimes \varepsilon : \omega(Y) \otimes L \to \omega(Y)$. This sequence is the similar image under $\omega$ of the sequence coming from (8):

$$0 \to Y \to \omega(Y) \otimes F \to \omega(Y) \otimes L \otimes F.$$

Hence the latter sequence is also exact. On the other hand, it follows from the faithfulness of $\omega$ that the composed map

$$X \xrightarrow{\sigma_X} \omega(X) \otimes F \xrightarrow{\alpha \otimes \text{id}} \omega(Y) \otimes F \xrightarrow{\omega(\sigma_Y)} \omega(Y) \otimes L \otimes F$$

is the zero morphism (since its image under $\omega$ is zero by means of (9) and the fact that $\alpha$ is a homomorphism of $L$-comodules). Consequently, the morphism $X \xrightarrow{\sigma_X} \omega(X) \otimes F \xrightarrow{\alpha \otimes \text{id}} \omega(Y) \otimes F$ factor through a morphism $f : X \to Y$ and the morphism $\sigma_Y$. Applying $\omega$ on the composition of these maps we conclude $\omega(f) = \alpha$, as $\omega(\sigma_Y)$ is injective. Thus $\omega$ is full.

It remains to show that $\varphi$ is essentially surjective. For any $L$-comodule $(E, \rho_E)$ let $E^\alpha \in C$ be such that the sequence

$$0 \to E^\alpha \to E \otimes F \xrightarrow{\delta} E \otimes L \otimes F$$

is exact, where $\delta = \rho_E \otimes \text{id} - \text{id} \otimes \sigma_F$. Applying $\omega$ to this sequence and comparing with (10) we conclude that $\omega(E^\alpha) = E$.

Thus $\omega : C \to \text{Comod}_f(L)$ is an equivalence of categories. Thus the forgetful functor $\text{Comod}_f(L) \to \text{Mod}(R)$ is exact, hence $L$ is flat over $R$. □

Remarks 2.10. (i) Under the equivalence of Theorem 2.9, $L$, with the right coaction of itself given by the coproduct, corresponds to $F$. Indeed, this follows from the natural isomorphism

$$\text{Hom}^L(E, L) \simeq \text{Hom}_R(E, R), \quad f \mapsto \varepsilon \circ f.$$
(ii) There is another way to determine $L$ from the category of its comodules as follows. We claim that there is a natural isomorphism
\begin{equation} \label{eq:11} \text{Nat}(\omega, \omega \otimes M) \simeq \text{Hom}_R(L, M), \end{equation}
for any $R$-module $M$. Indeed, we have
\begin{equation} \text{Hom}_R(L, M) \simeq \text{Hom}(F, F \otimes M) \simeq \text{Hom}(\text{Hom}(\omega, R), \text{Hom}(\omega, R) \otimes M). \end{equation}
By means of \eqref{eq:11}, it suffices to show the isomorphism
\begin{equation} \text{Nat}(\omega(X), \omega(X) \otimes M) \simeq \text{Hom}(\text{Hom}(\omega(X), R), \text{Hom}(\omega(X), R) \otimes M) \end{equation}
for any $X \in C^o$. Since for such $X$, $\omega(X)$ is finitely generated projective over $R$, the last isomorphism is obvious. $L$ is usually referred to as the Coend of $\omega$, denoted $\text{Coend}(\omega)$.

(iii) If $C = \text{Comod}_f(L)$ and $\omega$ is the forgetful functor from $C$ to $\text{Mod}(R)$, then the isomorphism \eqref{eq:11} implies that $\text{Coend}(\omega) \simeq L$. Thus a flat coalgebra over $R$ can be reconstructed from the category of its comodules. \hfill $\Box$

Remarks 2.11. Let $(C, \omega)$ and $(C', \omega')$ be two categories satisfying the condition of Theorem 2.9 and let $\eta : C \rightarrow C'$ be an $R$-linear functor such that $\omega' \eta = \omega$. Then $\eta$ induces a coalgebra homomorphism $f : L \rightarrow L'$. This can be seen from \eqref{eq:11} as follows. The coaction of $L'$ on $\omega'(X')$ defines a natural transformation $\delta' : \omega' \rightarrow \omega' \otimes L'$. Combine this with $\eta$ we obtain a natural transformation $\delta : \omega \rightarrow \omega \otimes L'$. Thus \eqref{eq:11} yields a linear map $L \rightarrow L'$, which satisfies the following commutative diagram:

\begin{equation*}
\begin{tikzpicture}
  \node (A) {$\omega(X)$};
  \node (B) {$\omega(X) \otimes L$};
  \node (C) {$\omega(X) \otimes L'$};
  \node (D) {$\omega'(X)$};
  \node (E) {$\omega'(X) \otimes L'$};

  \draw[->] (A) to node {$\delta$} (B);
  \draw[->] (B) to node [swap] {$\delta'$} (C);
  \draw[->] (A) to node [right] {$\text{id} \otimes f$} (D);
  \draw[->] (C) to node [right] {$\text{id} \otimes f$} (E);
\end{tikzpicture}
\end{equation*}

\hfill $\Box$

Using Theorem 2.9, Saavedra established the Tannakian duality for flat group schemes over a Dedekind ring, cf. \cite[II.4]{10}. We relate this result with a more recent result of Wedhorn on base change. We fix $R$ a Dedekind ring with field of fraction $K$.

Definition 2.12. (i) An abelian tensor category over a ring $R$ is an $R$-linear abelian tensor category, in which the endomorphism ring of the unit object is canonically isomorphic to $R$.

(ii) Let $\mathcal{T}$ be an abelian tensor category over $R$. $\mathcal{T}$ is said to be dominated by rigid objects if each rigid object is a quotient of a rigid object.

(iii) A (neutral) Tannakian category over a Dedekind ring $R$ is an abelian tensor category $\mathcal{T}$ over $R$, dominated by rigid objects, together with an exact faithful functor $\omega : \mathcal{T} \rightarrow \text{Mod}(R)$. \hfill $\Box$
The main example is of course the representation category of an affine flat group scheme over $R$. Let $G$ be such a group scheme. Let $L := R[G]$, the coordinate ring of $G$. Thus $L$ is an $R$-Hopf algebra, flat over $R$. Usually, for a Hopf algebra, we will denote the coproduct by $\Delta$, the counit by $\varepsilon$ and the antipode by $S$.

The category $\text{Rep}(G)$ of $R$-linear representations of $G$ (or $G$-modules) is identified with the category of right $L$-comodules. Similarly, the category of $G$-modules which are finitely generated over $R$, denoted $\text{Rep}^f(G)$, is identified with $\text{Comod}^f(L)$. This is an abelian tensor category over $R$. An object of $\text{Comod}^f(L)$ is rigid if the underlying $R$-module is projective (since $R$ is a Dedekind ring, this is equivalent to being flat over $R$). Then Serre’s theorem say that $\text{Rep}^f(G)$ is dominated by rigid objects. Hence it is a Tannakian category over $R$ with fiber functor being the forgetful functor. Denote the subcategory of rigid objects by $\text{Rep}^o(G)$ (resp. $\text{Comod}^o(L)$).

**Definition 2.13** (Base change of category, cf. [26, Sect. 3]). Let $C$ be an $R$-linear category. The category $C_K$ has the same as the objects of $C$ and for two objects $X$ and $Y$ in $C$ define

$$\text{Hom}_{C_K}(X, Y) := \text{Hom}_C(X, Y) \otimes_R K.$$ 

This is a $K$-linear category.

If $f$ is monomorphism (resp. an epimorphism) in $C$ then its image in $C_K$ is monomorphism (resp. an epimorphism) in $C_K$.

Let $D$ be a second $R$-linear category and let $\omega : C \rightarrow D$ be an $R$-linear functor. Then $\omega$ induces a functor $\omega_K : C_K \rightarrow D_K$. If $\omega$ is (fully) faithful, so is $\omega_K$.

Assume that $T$ is a monoidal $R$-linear category. The $R$-bilinear functor $\otimes : T \times T \rightarrow T$ extends to a $K$-bilinear functor $\otimes : T_K \times T_K \rightarrow T_K$. This way $T_K$ is monoidal $K$-linear category. It is symmetric (resp. rigid) if $T$ is symmetric (resp. rigid).

The main example is provided by comodule categories.

**Proposition 2.14.** [26, 6.4] (i) Let $L$ be an $R$-flat coalgebra. Set $L_K = L \otimes_R K$, this is a coalgebra over $K$. The tensor product with $K$ defines a functor from the category $\text{Comod}^o(L)_K$ to the category $\text{Comod}^o(L_K)$. This functor is an equivalence.

(ii) Let $G$ be a flat affine group scheme over $R$. It follows from (i) that we have an equivalence between $\text{Rep}^o(G)_K$ and $\text{Rep}^o(G_K)$ given by tensoring with $K$. This is in fact an equivalence of tensor categories.

Given a Tannakian category $(T, \omega)$. Let $T^o$ be the subcategory of rigid objects and denote by $\omega^o$ the restriction of this functor to $T^o$. One defines the group functor $\text{Aut}^\otimes(\omega^o) : \text{Alg}(R) \rightarrow \text{Grs}$:

$$\text{Aut}^\otimes(\omega^o)(A) := \text{Aut}^\otimes(\omega^o \otimes A).$$
It is shown in [26, Sect. 5] that this functor is representable by $\text{Coend}(\omega^\otimes)$, i.e. it is an affine group scheme over $R$. This group scheme is called the Tannakian group of $(\mathcal{T}, \omega)$.

**Theorem 2.15.** Let $(\mathcal{T}, \omega)$ be a Tannakian category over a Dedekind ring $R$. Then

(i) $\omega$ factors through an equivalence between $\mathcal{T}$ and $\text{Rep}_f(G)$, where $G := \text{Aut}^\otimes(\omega)$, cf. [19, II.3.4].

(ii) $\omega$ induces an equivalence between $\mathcal{T}^\circ K$ and $\text{Rep}_f(G_K)$, cf. [26, Sect. 5].

**Proof.** By 2.9, the natural functor $\mathcal{T} \to \text{Rep}_f(G)$ is an equivalence of abelian categories. Moreover, this functor is compatible with the tensor product. Hence it is an equivalence of tensor categories.

(ii) follows from (i) and Proposition 2.14.

**Remarks 2.16.** For an abelian tensor category $C$ over $R$, we define the special fiber $C_s$ at a closed point $s$ of $S := \text{Spec}(R)$ to be the full subcategory of objects which satisfy $m_sX = 0$, where $m_s$ is the maximum ideal of $R$ determining $s$. If $C$ is Tannakian then $C_s$ is abelian and closed under the tensor product. Moreover, the fiber functor $\omega$ yields an equivalence between $C_s$ and $\text{Rep}_f(G_s)$ where $G$ is the Tannakian group of $C$ and $G_s$ is the fiber of $G$ at $s$, cf. [14, Chapt. 10].

3. **Locally finite $R$-coalgebras**

An important property of coalgebras over a field is the local finiteness: a coalgebra is the union of its finite dimensional subcoalgebras. This property formally generalizes to flat coalgebras over a Dedekind ring. Indeed, as mentioned in Example 2.3 (ii), a flat coalgebra $L$ is the union of its finite subcomodules. Let $M \subset L$ be such a subcomodule. Since $M$ is finite and torsion free, it is projective. Then the coaction $\rho : M \to M \otimes L$ yields a map $M^\vee \otimes M \to M^\vee \otimes M \otimes L \to L$, the image of which is called the coefficient space of $M$, denoted by $\text{cf}(M)$. It is a subcoalgebra of $L$ and contains $M$. In fact, if follows from the coassociativity of $\rho$ that the image of $\varepsilon_M \otimes m$ is just $m$, for any $m \in M$, where $\varepsilon_M$ is the restriction of the counit $\varepsilon$ on $M$.

However this is not fully reflected in the Tannakian duality. For a subcoalgebra $C$ of coalgebra $L$ over a field $k$, the category $\text{Comod}_f(C)$ can be identified with a full, exact subcategory of $\text{Comod}_f(L)$, which is closed under taking subobjects. This is no more the case for flat coalgebras over a Dedekind ring as the example below shows. The reason is that the quotient module $L/C$ may be non-flat over the base ring. If we try to enlarge $C$ so that the quotient becomes flat, we cannot guarantee that it remains finite over $R$. In view of Tannakian duality, this reflects the fact that the full abelian subcategory generated by a single object in a comodule category may become very large.
This phenomenon is one of the main obstructions to the study of flat coalgebras and flat group schemes over a Dedekind ring. Below we will show that a coalgebra which is projective as a module over the base ring is locally finite. However we don’t know if finitely generated flat Hopf algebras over a Dedekind ring also behave well like this.

Following dos Santos [21], we make the following definition.

**Definition 3.1.** Let $L$ be a flat $R$-coalgebra.

(i) A submodule $M$ of an $L$-comodule $N$ is called special if $N/M$ is flat over $R$.

(ii) A special subquotient $M$ of an $L$-comodule $N$ is a special submodule of a quotient of $N$, or, equivalently, a quotient of a special submodule of $N$.

(iii) A comodule $N$ is called locally finite if any finite submodule of $N$ is contained in a finite special submodule.

(iv) A subcoalgebra $C$ of $L$ is called special if $L/C$ is flat. A homomorphism of flat coalgebras $f : L' \rightarrow L$ is called special if $f(L')$ is a special subcoalgebra of $L$.

(v) $L$ is called a locally finite coalgebra if for any finite submodule $C$, there exists a finite special subcoalgebra containing $C$.

Let $N$ be an $L$-comodule. Then $N_{\text{tor}}$, the $R$-torsion submodule of $N$ is an $L$-subcomodule. Hence for any $L$-subcomodule $M$, the preimage of $(N/M)_{\text{tor}}$ in $N$, denoted $M_{\text{sat}}$, is an $L$-comodule. Since $R$ is a Dedekind ring, the quotient $N/M_{\text{sat}}$ is flat, being torsion free. Thus $M_{\text{sat}}$ is the smallest special subcomodule of $N$, containing $M$. It is called the saturation of $M$ in $N$.

**Lemma 3.2.** An $R$-flat coalgebra $L$ is locally finite if and only if it is locally finite as a comodule on itself.

**Proof.** Assume that $L$ is locally finite as a right comodule on itself. Let $C$ be a finite subcoalgebra of $L$ and $C_{\text{sat}}$ the saturation. It is to show that $C_{\text{sat}}$ is a subcoalgebra.

We have the filtration $C_{\text{sat}} \otimes C_{\text{sat}} \subset C_{\text{sat}} \otimes L \subset L \otimes L$, the successive quotients of which are flat, hence $L \otimes L/C_{\text{sat}} \otimes C_{\text{sat}}$ is also flat. Thus

$$(C \otimes C)_{\text{sat}} \subset C_{\text{sat}} \otimes C_{\text{sat}}.$$

Hence, by the definition of $C_{\text{sat}}$, we have

$$\Delta(C_{\text{sat}}) \subset (C \otimes C)_{\text{sat}} \subset C_{\text{sat}} \otimes C_{\text{sat}}.$$

For the converse statement we use the well-known fact that each finite subcomodule of $L$ is contained in a finite subcoalgebra of $L$. Indeed, given $M \subset L$ finite, then it is projective and the coaction $M \rightarrow M \otimes L$ induces a map

$$M' \otimes M \rightarrow L, \quad \varphi \otimes m \mapsto \sum \varphi(m_i)m'_i, \quad \varphi \in M', m \in M,$$
where $\Delta(m) = \sum_i m_i \otimes m'_i$. This is a coalgebra homomorphism. Its image is called the coefficient space of $M$, denoted $\text{Cf}(M)$. It is a subcoalgebra of $L$ and contains $M$. In fact, let $\varepsilon_M$ be the restriction of $\varepsilon$ to $M$, then $\varepsilon_M \otimes m \mapsto \sum_i \varepsilon(m_i)m'_i = m$. □

The next example shows that there exist coalgebras which are not locally finite.

**Example 3.3** ([7], Remarque 11.10.1). Let $R$ be a Dedekind ring, $K$ be its quotient field and let $x \in R$ be a non-invertible element. Let $G$ the affine group scheme over $R$ determined by the Hopf subalgebra of $K[G_a] = K[T]$:

$$R[G] := \{ P \in K[T] \mid P(0) \in R \}$$

that is, $R[G]$ consists of polynomials in $K[T]$ with the free coefficients belonging to $R$. Let $C_i$ be the subcoalgebra spanned (over $R$) by $1$ and $x^{-i}T$. Then we have

$$C_i \subsetneq C_{i+1}, \quad xC_{i+1} \subsetneq C_i.$$  

Thus the saturation of $C_0$ is not finite.

In what follows we will need some standard facts on tensor the product and flat modules, cf. [1].

**Lemma 3.4.** Let $A \subset B$ be flat $R$-modules and $M, M_1, M_2 \subset N$ be arbitrary $R$-modules. Then

(i) $M_1 \otimes A \cap M_2 \otimes A = (M_1 \cap M_2) \otimes A$, $M_1 \otimes A + M_2 \otimes A = (M_1 + M_2) \otimes A$, as subsets of $N \otimes A$;

(ii) if $B/A$ is also flat, we have $N \otimes A \cap M \otimes B = M \otimes A$ as submodules in $N \otimes B$.

The following result is proved in more generality in [13, I.3.11], we recall it here for completeness.

**Proposition 3.5.** Let $R$ be a Dedekind ring and $L$ be an $R$-flat coalgebra.

(i) If $L$ is locally finite then $L$ is Mittag-Leffler as an $R$-module. Hence, if $L$ is moreover countably generated over $R$, it is $R$-projetive.

(ii) If $L$ is $R$-projetive, then it is locally finite as an $R$-coalgebra.

**Proof.** (i) Let $\{C_\alpha\}$ be the directed system of finite special subcoalgebra. Then for any finite $R$-module $N$, the system $\text{Hom}_R(C_\alpha, N)$ is Mittag-Leffler. In fact, each inclusion $C_\alpha \rightarrow C_\beta$ splits, as $C_\beta/C_\alpha$ is torsion free and finite, hence projective. Consequently the map

$$\text{Hom}_R(C_\beta, N) \rightarrow \text{Hom}_R(C_\alpha, N)$$

is surjective. Thus by definition, $L$ is Mittag-Leffler. It is well-known that a flat, countably generated, Mittag-Leffler module is projective.
(ii) We show that any $R$-projective comodule is locally finite. If $N \subset M$ is a subcomodule then $N^{\text{sat}}$ is the preimage of $(M/N)_{\text{tor}}$ under the quotient map $M \twoheadrightarrow M/N$. Thus we have to show that $N^{\text{sat}}$ is finite provided that $M$ is projective and $N$ is finite. This is a pure question of $R$-modules. Embed $M$ as a free $R$-module $F$ as a direct summand. Replacing $M$ by $F$ will only enlarge $N^{\text{sat}}$, thus we can assume that $M$ is free over $R$. Then, as $N$ is finite, we can find a free direct submodule of $F_0$ which contains $N$. Now $F_0 = F_0^{\text{sat}}$ implying $N^{\text{sat}} \subset F_0$, hence it is finite.

\[ \square \]

Questions 3.6. It is not known if any locally finite $R$-flat coalgebra is $R$-projective. Another interesting question is: which affine flat $R$-group scheme of finite type is locally finite.

Proposition 3.7. Let $G$ be a flat group scheme of finite type over a Dedekind ring $R$. Assume that the generic fiber $G_K$ is smooth and connected. Then $\mathbb{R}[G]$ is locally finite as an $R$-coalgebra.

Proof. Let $I$ be the augmented ideal of $\mathbb{R}[G]$, that is $I = \ker(\varepsilon)$. Since the $\mathbb{R}[G]$ is flat over $R$, the map $\mathbb{R}[G] \rightarrow \mathbb{R}[G] \otimes_R K = K[G_K]$ is injective and as $K$ is flat over $R$, the augmentation ideal of $K[G_K]$ is $I_K = I \otimes_R K$. With the assumption that $G_K$ is smooth and connected, $K[G_K]$ is an integral domain. By Krull’s intersection theorem $\bigcap_m (I_K)^m = 0$.

Let $M \subset \mathbb{R}[G]$ be a finite $R$-submodule. Then there exists $m$ such that $M \otimes K \cap (I_K)^m = 0$. We have $(I_K)^m = I^m \otimes K$. Hence $(M \cap I^m) \otimes K = 0$ implying $M \cap I^m = 0$. It follows that $M^{\text{sat}} \cap I^m = 0$. Indeed, if $0 \neq a \in M^{\text{sat}} \cap I^m$ then there exists $0 \neq r \in R$ such that $ra \in M \cap I^m$, it forces $ra = 0$, consequently $a = 0$ as $\mathbb{R}[G]$ is torsion free. Thus, the map $M^{\text{sat}} \rightarrow \mathbb{R}[G]/I^m$ is injective. But the module $\mathbb{R}[G]/I^m$ is finite, hence so is $M^{\text{sat}}$.

\[ \square \]

On the other hand, the situation for group schemes with finite fibers turns out to be more complicated, as in the following example, communicated to us by dos Santos.

Example 3.8. Let $R$ be a DVR of equal positive characteristic $p$, with uniformizer $\pi$. Let $G$ be the group scheme determined by the Hopf algebra

\[ \mathbb{R}[G] := R[T]/(\pi T^p - T). \]

Then the fibers of $G$ are étale group scheme but $\mathbb{R}[G]$ is not locally finite. Indeed, the saturation of the finite subcomodule spanned by $1$ and $T$ contains $T^p k, k \geq 1$, and is not finite.

In the rest of this paragraph we will use the Tannakian duality to characterize (special) injective and surjective homomorphisms of flat coalgebras.

Proposition 3.9. Let $f : L' \rightarrow L$ be homomorphism of flat coalgebras over a Dedekind ring $R$. Then
(i) \( f \) is injective if and only if the functor \( \omega_f \) is fully faithfull and its image in \( \text{Comod}_f(L) \) is closed under taking special subobjects.

(ii) \( f \) is injective and special if and only if the natural functor \( \omega_f \) is fully faithful and its image in \( \text{Comod}_f(L) \) is closed under taking subobjects. In this case, the functor \( \omega_f \) is also fully faithful and its image is closed under taking subobjects.

\textbf{Proof.} We shall proof the “only if” implication for both claims at once. Thus, assume that \( f \) is injective (resp. special).

The functor \( \omega_f : \text{Comod}_f(L') \to \text{Comod}_f(L) \) can be considered as the identity functor on the underlying module category, it changes only the coaction (from that of \( L' \) to that of \( L \)): for \((M, \rho') \in \text{Comod}_f(L')\) we have \( \omega_f : (M, \rho') \mapsto (M, \rho) \)

\[
\begin{array}{ccc}
M & \xrightarrow{\rho'} & M \otimes L' \\
\downarrow{\rho} & & \downarrow{id \otimes f} \\
M \otimes L & & M \otimes L
\end{array}
\]

Thus \( \omega_f \) is obviously faithful (and so is \( \omega_f^\circ \)).

The condition for a map \( \varphi : M \to N \) to be \( L' \)-comodules reads as follows: \( \rho_N' \varphi - (\varphi \otimes \text{id}) \rho_M' : M \to N \otimes L' \) is the zero map:

\[
\begin{array}{ccc}
M & \xrightarrow{\rho'_M} & M \otimes L' \xrightarrow{id \otimes f} M \otimes L \\
\downarrow{\varphi} & & \downarrow{\varphi \otimes \text{id}} \\
N & \xrightarrow{\rho'_N} & N \otimes L' \xrightarrow{id \otimes f} N \otimes L
\end{array}
\]

If either \( N \) or \( L/L' \) is flat over \( R \), the horizontal map \( \text{id} \otimes f : N \otimes L' \to N \otimes L \) in the above diagram is injective. Hence the map \( \rho'_N \varphi - (\varphi \otimes \text{id}) \rho'_M \) is zero if and only if its composition with \( \text{id} \otimes f \), which is \( \rho_N \varphi - (\varphi \otimes \text{id}) \rho_M : M \to N \otimes L \) is zero. Thus \( \omega_f \) is full.

We show the closedness under taking (special) subquotients. For \((M, \rho') \in \text{Comod}_f(L')\), its image under \( \omega_f \) is denoted by \((M, \rho)\). Let \((N, \rho_N)\) be a sub \( L \)-comodule of \((M, \rho)\). Thus we have commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\rho'} & M \otimes L' \\
\downarrow{\rho} & & \downarrow{id \otimes f} \\
N & \xrightarrow{\rho_N} & M \otimes L \\
\downarrow{\rho_N} & & \downarrow{\rho_N} \\
N \otimes L & & N \otimes L
\end{array}
\]
To show that $\rho_N$ comes from a coaction of $L'$ on $Y$ amounts to showing that $\rho_N(N) \subset N \otimes f(L')$. The above diagram shows that $\rho_N(N) \subset N \otimes L \cap M \otimes f(L')$. If either $M, N, M/N$ or $L/f(L')$ are flat over $R$, according to Lemma 3.4, one has equality

$$N \otimes L \cap M \otimes f(L') = N \otimes f(L').$$

Thus $\rho_N(N) \subset N \otimes f(L')$, that is, $\rho'$ restricts to a coaction of $L'$ on $N$.

Conversely, assume that the functor $\omega_f^o : \text{Comod}(L') \rightarrow \text{Comod}(L)$ is fully faithful and its image is closed under taking special subobjects. By the flatness of $K$ over $R$, the functor $\text{Comod}(L')_K \rightarrow \text{Comod}(L)_K$ is fully faithful and preserves mono- and epimorphisms. Therefore, by the equivalence in Proposition 2.14 the functor $\text{Comod}(L' \otimes K) \rightarrow \text{Comod}(L \otimes K)$ satisfies conditions of [5, Thm 2.21], hence $L' \otimes K \rightarrow L \otimes K$ is injective. Consequently the map $f : L' \rightarrow L$ is injective, (i) is proved.

Now assume that the image of $\omega_f^o$ is closed under taking subobjects. Thus the discussion above shows that $f$ is injective. We will identify $L'$ with a subcoalgebra of $L$. For a non-invertible $\pi \in R$, denote $C := \pi L \cap L' \supset \pi L'$. By means of Lemma 3.4 we see that $C$ is also an $L$-subcomodule of $L'$. Since an $L$-comodule is the union of its finite subcomodule, the assumption on $\omega_f^o$ implies that $C$ is in fact an $L'$-subcomodule of $L'$:

$$\Delta(C) \subset C \otimes L' \subset \pi L \otimes L'.$$

Thus for any $c \in C$ we have

$$\Delta(c) = \sum a_i \otimes b_i, \quad a_i \in L, \ b_i \in L'.$$

Hence $c = \sum \varepsilon(a_i)b_i \in \pi L'$. That is $C \subset \pi L'$, consequently $\pi L \cap L' = \pi L'$. The last equation holds for any $\pi \in R$, it follows that $L/L'$ is torsion free over $R$, hence flat, as $R$ is a Dedekind ring.

We now express the local finiteness in terms of Tannakian duality. First we have the following characterization of finite coalgebras. For an object $X$ in an abelian category $C$, let $\langle X \rangle$ denote the full subcategory of $C$ consisting of subquotients of direct sums of copies of $X$.

**Proposition 3.10.** Let $L$ be a flat coalgebra over a Dedekind ring $R$. Then $L$ is finite over $R$ if and only if $\text{Comod}(L)$ has a projective generator.

**Proof.** Assume that $L$ is finite. Then $L$ is projective over $R$ and there is an equivalence between $\text{Comod}(L)$ and $\text{Mod}(L^\vee)$, where $L^\vee$ is the dual $R$-module to $L$, and hence is an $R$-algebra: for any $L$-comodule $M$ with coaction $\delta$, $L^\vee$ acts on $M$ as follows:

$$\varphi \cdot m := \sum_i \varphi(l_i)m_i, \quad \text{where } \delta(m) = \sum_i m_i \otimes l_i.$$
Conversely, if $M$ is an $L^\vee$-comodule, let $(l_i)$ and $(\varphi^i)$ be dual bases on $L$ and $L^\vee$. We define the coaction of $L$ on $M$ as follows:

$$\delta(m) = \sum_i \varphi^i \cdot m \otimes l_i.$$ 

To check that these constructions are inverse of each other, if suffices to note that, by definition, the dual bases $(l_i)$ and $(\varphi^i)$ satisfies:

$$\sum_i \varphi^i(l) l_i = l \text{ and } \sum_i \varphi(l_i) \varphi^i = \varphi, \text{ for all } l \in L, \varphi \in L^\vee.$$ 

Now $L^\vee$ is a projective generator in $\text{Comod}_L(L)$.

Conversely, assume that $\text{Comod}_L(L)$ has a projective generator, say $P$. Then the category is equivalent to $\text{Mod}(A)$, where $A = \text{End}_L(P)$. We claim that $P$ is projective over $R$. The exact functor $\text{Hom}^L(P, -)$ on $\text{Comod}_L(L)$ extends to its ind-category $\text{Comod}(L)$. Since $L$ is flat, the functor $\text{Hom}^L(P, - \otimes L)$ is also exact. But

$$\text{Hom}^L(P, - \otimes L) \cong \text{Hom}_R(P, -).$$

Thus $P$ is projective as an $R$-module. It follows that $\text{End}_R(P)$ is also a projective module over $R$. Now $A$ is an $R$-submodule of $\text{End}_R(P)$, hence is also projective. Then $A^\vee$ is a finite flat $R$-coalgebra and the correspondence mentioned above gives an equivalence between $\text{Mod}(A)$ and $\text{Comod}(A^\vee)$. By Tannakian duality we conclude that $A^\vee \cong L$, whence $L$ is finite. \hfill \Box

**Proposition 3.11.** Let $L$ be a flat coalgebra over a Dedekind ring $R$ and let $\omega$ denote the forgetful functor $\text{Comod}_L(L) \rightarrow \text{Mod}(R)$.

(i) For each projective comodule $X$ of $L$ the natural map $\text{Coend}(\omega|\langle X \rangle) \rightarrow \text{Coend}(\omega) = L$ is special. If $L$ is locally finite, then the coalgebra $\text{Coend}(\omega|\langle X \rangle)$ is finite over $R$.

(ii) If for any finite projective comodule $X$, $\langle X \rangle$ is finite, then $L$ is locally finite.

**Proof.** (i) The first claim follows from Tannakian duality Theorem 2.9 and Proposition 3.9.

Assume now that $L$ is locally finite. Let $M$ be a projective comodule of $L$. Then there exists a special subcoalgebra $C$ of $L$ such that the coaction of $L$ on $M$ factors though that of $C$. According to (i), $\text{Comod}(C) \subset \text{Comod}(L)$ is a full subcategory of $\text{Comod}(L)$ which is closed under taking subobjects. Since $M \in \text{Comod}(C)$, we have $\langle M \rangle \subset \text{Comod}(C)$. Hence we have a factorization

$$\text{Coend}(\omega|\langle M \rangle) \rightarrow C \rightarrow L.$$

The first map is also injective, according to (i). But $C$ is finite by the assumption on $L$. Whence the claim.
(ii) According to (i), we can cover $L$ by its finite special subcoalgebras $L = \bigsqcup L_i$, where $\{L_i\}$ is a cofinal directed system, i.e. any two coalgebras $L_i, L_j$ are contained in some $L_k$. Hence, if $C$ is a finite subcoalgebra of $L$, then $C$ is contained in some $L_i$. Since $L_i$ is special, it is saturated, consequently $C^{\text{sat}} \subset L_i$, hence is finite.

Remarks 3.12. The proof of Proposition 3.9 is based on the fact that, when $R = K$ is a field, the map $\text{Coend}(\omega|\langle X \rangle) \rightarrow L$ is injective. The reader is referred to [24, Thm 6.4.4] for the detailed proof of this fact. In Saavedra’s proof of this fact [19, II.2.6.2.1], implication d) $\Rightarrow$ a), is incomplete.

Finally we provide a condition for the surjectivity of a colagebra homomorphism.

**Proposition 3.13.** Let $f : L' \rightarrow L$ be a morphism of flat coalgebras over $R$. Then $f$ is surjective if and only if the induced functor $\omega_f : \text{Comod}_f(L') \rightarrow \text{Comod}_f(L)$ satisfies the following condition: each $M \in \text{Comod}_f(L)$ is a special subquotient of $\omega_f(N)$ for some $N \in \text{Comod}_f(L')$.

**Proof.** Assume that $f$ is surjective. Let $M$ be a finite projective $L$-comodule. The coaction $\delta : M \rightarrow M \otimes L$ can be consider as a homomorphism of $L$-comodule. Since the composition of $\delta$ with the map $\text{id}_M \otimes \varepsilon$ is the identity on $M$, $M$ is a direct summand of $M \otimes L$ as $R$-modules. Hence $M$ is a special subcomodule of $M \otimes L$. Recall that $L$ coacts on $M \otimes L$ by means of the coaction on $L$. As $M$ is projective, we see that $M \otimes L$ is a direct summand of $L^\oplus r$ for some $r > 0$. Thus $M$ is a special subobject of $L^\oplus r$. We identify $M$ with a subcomodule of $L^\oplus r$. Choose a generating set $\{m_i\}$ of $M$. Let $m'_i \in L'^\oplus r$ be such that $f(m'_i) = m_i$. The set $\{m'_i\}$ is contained in some finite module $N$ of $L'^\oplus r$. The holomorphic image of $N$ in $L^\oplus r$ is denoted by $N_1$, this is an $L$-subcomodule of $L^\oplus r$ which contains $M$. Since $M$ is special in $L^\oplus r$, it is also special in $N_1$.

Conversely, assume that $\omega_f$ has the stated property. Let $C$ be a finite subcoalgebra of $L$. By assumption, there exists a finite $L'$-comodule $N$, such that $C$ is a special subquotient of $N$, as $L$-comodules. The torsion part of $N$, which is an $L'$-subcomodule, is irrelevant to $C$ as it is flat, so that we can assume that $N$ is torsion free, hence projective.

Denote by $\text{Cf}(N)$ the coefficient space of $N$, i.e. the image of $N^\vee \otimes N \rightarrow L$. This is a subcoalgebra of $L$. We claim that $C \subset \text{Cf}(N)$. This follows from Lemma 3.14 below and the fact that $C = \text{Cf}(C)$.

On the other hand, it follows from the construction that $\text{Cf}(N)$ is the image of $\text{Cf}_{L'}(N)$ - the coefficient space of $N$ considered as $L'$-comodule. Thus the map $L' \rightarrow L$ is surjective. $\square$
Lemma 3.14 ([21, Lemma 9]). Let $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ be an exact sequence of finite $R$-projective $L$-comodules. Then $\text{Cf}(N)$ contains both $\text{Cf}(M)$ and $\text{Cf}(P)$.

4. Tannakian description of group scheme homomorphisms

Let $R$ be a Dedekind ring with fraction field $K$. Let $f : G \rightarrow G'$ be a homomorphism of flat affine group schemes over $R$. We say $f$ is surjective or a quotient homomorphism if it is faithfully flat. In the first part of this chapter we give a necessary and sufficient condition for the faithful flatness of $f$, in terms of Tannakian duality. Then we will give a condition for $f$ to be a closed immersion. Finally we give a criterion for the exactness of a sequence of group homomorphisms.

The coordinate ring $R[G]$ of $G$ is a (commutative) Hopf algebra, which is flat over $R$ and $G$-modules are the same as $R[G]$-comodules. We recall some important $G$-modules. The right regular action of $G$ on $R[G]$, $(g, h) \mapsto gh : (gh)(x) = h(xg)$, $g \in G$, $h \in R[G]$ corresponds to the right coaction of $R[G]$ on itself by the coproduct $\Delta$. The left regular action of $G$ on $R[G]$, $(g, h) \mapsto gh : (gh)(x) = h(g^{-1}x)$ corresponds to the following (right) coaction of $R[G]$ on itself:

$$a \mapsto \sum_i a'_i \otimes S(a_i), \quad \text{where} \quad \Delta(a) = \sum_i a_i \otimes a'_i.$$  

The combination of these two actions is the conjugation action.

The following theorem is a generalization of the well-known faithful flatness theorem for Hopf algebras: a (commutative) Hopf algebra is faithfully flat of a Hopf subalgebra, see, e.g., [25, Thm 14.1]. The proof is developed from an idea of J.C. Moore [17].

**Theorem 4.1.** Let $L$ be a flat Hopf algebra over $R$ and $L'$ be a Hopf subalgebra. Then $L$ is faithfully flat over $L$ if and only if $L/L'$ is $R$-flat, i.e. $L'$ is a special subcoalgebra of $L$.

**Proof.** Assume that $L$ is faithfully flat over $L'$. Consider the tensor product of the exact sequence

$$0 \rightarrow L' \rightarrow L \rightarrow Q \rightarrow 0 \quad (13)$$

with $L$ over $L'$ we get an exact sequence

$$0 \rightarrow L \rightarrow L \otimes_{L'} L \rightarrow L \otimes_{L'} Q \rightarrow 0 \quad (14)$$

The multiplication $L \otimes_{L'} L \rightarrow L$ splits this sequence, hence $L \otimes_{L'} Q$ is flat over $L'$. Since $L$ is faithfully flat over $L'$, $Q$ is flat over $L'$ and therefore it is flat over $R$.  


Conversely, assume that \( L/L' \) is \( R \)-flat. We first show that \( L \) is flat over \( L' \), i.e., for any \( L' \)-module \( M \), \( \text{Tor}^R_1(M, L) = 0 \). The claim holds if \( R \) is a field.

First assume that \( M \) is \( R \)-flat. Choose \( P_* \) a projective resolution of \( L \) over \( L' \). Then \( P_* \otimes k \) is a projective resolution of \( L \otimes k \) over \( L' \otimes k \), where \( k \) is a residue field or the fraction field of \( R \). We have

\[
(M \otimes k) \otimes (L \otimes k) \cong (M \otimes_{L'} P_*) \otimes k,
\]

implying

\[
H_i((M \otimes_{L'} P_*) \otimes k) \cong \text{Tor}^R_{i+1}(M \otimes k, L \otimes k), \quad \text{for all } i \geq 0.
\]

Since \( M \otimes_{L'} P_* \) is flat over \( R \), a Dedekind ring, the universal coefficient theorem, (see, e.g., [27] Thm 3.1.6), applies. Thus, for each \( i \geq 1 \), we have an exact sequence

\[
0 \to H_i(M \otimes_{L'} P_*) \otimes k \to H_i((M \otimes_{L'} P_*) \otimes k) \to \text{Tor}^R_i(H_{i-1}(M \otimes_{L'} P_*), k) \to 0.
\]

That is, for all \( i \geq 1 \),

\[
0 \to \text{Tor}^R_i(M, L) \otimes k \to \text{Tor}^R_{i+1}(M \otimes k, L \otimes k) \to \text{Tor}^R_i(\text{Tor}^R_{i-1}(M, L), k) \to 0.
\]

As \( L/L' \) is flat, the map \( L' \otimes k \to L \otimes k \) is injective, hence flat. Therefore \( \text{Tor}^R_{i+1}(M \otimes k, L \otimes k) = 0 \), for all \( i \geq 1 \). Consequently

\[
\text{Tor}^R_i(\text{Tor}^R_{i-1}(M, L), k) = \text{Tor}^R_i(M, L) \otimes k = 0, \quad \text{for all } i \geq 1.
\]

This holds for any residue field and the fraction field of \( R \), hence \( \text{Tor}^R_i(M, L) \) is flat over \( R \) and \( \text{Tor}^R_i(M, L) = 0 \) for all \( i \geq 1 \).

Let now \( M \) be an arbitrary \( L' \)-module. Then the \( R \)-torsion submodule \( M_\tau \) of \( M \) is also an \( L' \)-submodule. The quotient module \( M/M_\tau \) is then \( R \)-flat. As we have the exact sequence

\[
\text{Tor}^R_i(M_\tau, L) \to \text{Tor}^R_i(M, L) \to \text{Tor}^R_i(M/M_\tau, L) \to \ldots
\]

it suffices to show \( \text{Tor}^R_1(M, L) \) for \( M \) being \( R \)-torsion.

For each non-zero ideal \( p \in R \), the submodule \( M_p \) of elements annihilated by elements of \( p \), is also an \( L' \)-submodule. As \( M \) is torsion, it is the direct limit of \( M_p \). Since the Tor-functor commutes with direct limits, one can replace \( M \) by some \( M_p \). Since \( R \) is a Dedekind ring, each non-zero ideal \( p \) is a product of finitely many prime ideals. Therefore each \( M_p \) has a filtration, each grade module of which is annihilated by a certain non-zero prime ideal. Thus using induction we can reduce to the case \( M \) is annihilated by a prime ideal \( p \). In this case \( M = M \otimes k_p \) is an \( L' \otimes k_p \)-module, where \( k_p := R/p \) and we have

\[
M \otimes_{L'} P_* = M \otimes_{L' \otimes k_p} (P_* \otimes k).
\]

Since \( P_* \otimes k_p \) is an \( L' \otimes k_p \)-projective resolution of \( L \otimes k_p \), we see that

\[
\text{Tor}^R_i(M, L) = \text{Tor}^R_{i+1}(M, L \otimes k_p) = 0,
\]
as $L \otimes k_p$ is flat over $L' \otimes k_p$.

Finally, we show that $L$ is faithfully flat over $L'$.

Let $M$ be an $L'$-module, such that $M \otimes_{L'} L = 0$. Then we have

$$M_k \otimes_{L'_k} L_k \cong (M \otimes_{L'} L) \otimes_R k = 0$$

for any residue field of the fraction field $k$ of $R$. Since $L'_k \rightarrow L_k$ is faithfully flat, we have $M_k = 0$ and $M_K = 0$. If $M$ is finite over $L'$, this implies that $M = 0$, according to [16, Thm 4.9]. In the general case, $M$ always contains a non-zero finite submodule and since $L$ is flat over $L'$, we see that $M = 0$ if $M \otimes_{L'} L = 0$. Thus $L$ is faithfully flat over $L'$.

As a corollary of Theorem 4.1 and Propositions 3.9, 3.13 we have the following theorem.

**Theorem 4.2.** Let $f : G \rightarrow G'$ be a homomorphism of affine flat groups over $R$, and $\omega_f$ be the corresponding functor $\text{Rep}_f(G') \rightarrow \text{Rep}_f(G)$.

(i) $f$ is faithfully flat if and only if $\omega_f^\circ : \text{Rep}^\circ(G') \rightarrow \text{Rep}^\circ(G)$ is fully faithful and its image is closed under taking subobjects.

(ii) $f$ is a closed immersion if and only if every object of $\text{Rep}^\circ(G)$ is isomorphic to a special subquotient of an object of the form $\omega_f(X')$, $X' \in \text{Rep}^\circ(G')$.

**Remarks 4.3.** (i) For (commutative) Hopf algebras over a field, any injective homomorphism $L' \rightarrow L$ is automatically faithfully flat. This is not the case for Hopf algebras over a Dedekind ring. Take for example $R = k[x]$ and $L = R[T] = k[x, T]$ with the $R$-coalgebra structure given by $\Delta(T) = T \otimes 1 + 1 \otimes T$, $\varepsilon(T) = 0$ $S(T) = -T$ (i.e. $L = R[G_a]$), and consider the map of Hopf $R$-algebras

$$f : L \rightarrow L, \quad T \mapsto xT.$$  

The quotient $L/f(L)$ is not flat over $R$, hence Theorem 4.2 implies that $L$ is not faithfully flat over $f(L)$.

(ii) Theorem 4.2 shows that if $f : L' \rightarrow L$ is a homomorphism of flat, finitely generated Hopf $R$-algebras, such that $L/f(L')$ is flat over $R$, then $L$ is faithfully flat over $f(L')$.

(iii) The claim (ii) of Theorem 4.2 generalizes a result of dos Santos, [21, Prop. 12].

**Questions 4.4.** Theorem 4.2 suggests the following question: find a criterion of a Tannakian category $\mathcal{C}$ such that its Tannakian group $G$ is pro-algebraic in the sense that

$$G = \lim_{\alpha} G_{\alpha}$$
where each group scheme $G_\alpha$ is of finite type and each structure map $G_\beta \to G_\alpha$ is faithfully flat. Similarly, find a criterion such that $G$ is smooth in the sense that the $G_\alpha$ above are smooth. It seems that the local finiteness mentioned in the previous section closely relates to these problems.

Let $G \to A$ be homomorphism of affine groups schemes over $R$. Let $I_A$ be the kernel of counit $\epsilon : R[A] \to R$, i.e. the augmentation ideal of $R[A]$, and let $I_A R[G]$ be the ideal generated by the image of $I_A$ in $R[G]$. Then the kernel of $G \to A$ is the closed subscheme of $G$ with coordinate ring $R[G]/I_A R[G]$. A sequence

$$1 \to H \xrightarrow{q} G \xrightarrow{p} A \to 1$$

is said to be exact if $p$ is a quotient map with kernel $H$. We will provide a criterion for the exactness in terms of the functors

$$\text{Rep}^o(A) \xrightarrow{p^*} \text{Rep}^o(G) \xrightarrow{q^*} \text{Rep}^o(H).$$

Lemma 4.5. Let $G \to A$ be a quotient map with kernel $H$.

(i) If $M$ is a $G$-module, then $M^H$ is a $G$-submodule of $M$.
(ii) $R[A]$ is equal to $R[G]^H$ as $G$-modules.

Proof. (i) See [14, I.3.2].

(ii) The proof is based on the fact that $R[G]$ is faithfully flat over $R[A]$ and follows closely the proof for group schemes over fields, cf. [25, Sect. 15.4].

There is an isomorphism $H \times G \xrightarrow{\sim} G \times A G; \quad (h, g) \mapsto (hg, g)$, which precisely means that $G \to A$ is a principal bundle under $H$. In terms of the coordinate rings this isomorphism has the form

$$\varphi : R[G] \otimes_{R[A]} R[G] \cong R[H] \otimes_R R[G], \quad a \otimes b \mapsto \sum_i q(a_i) \otimes a'_i b,$$

where $q : R[G] \to R[H]$ denotes the quotient map and $\Delta(a) = \sum_i a_i \otimes a'_i$. The inverse map is given by

$$q(a) \otimes b \mapsto \sum_i a_i \otimes S(a'_i) b$$

where $S$ denotes the antipode of $R[G]$, one checks that this assignment depends on $q(a) \in R[H]$ but does not depend on the choice of $a \in R[G]$. Now consider the following diagram:

$$\begin{array}{ccc}
R[G]^H & \xrightarrow{d} & R[G] \\
\downarrow{\phi} & & \downarrow{q^*} \\
R[H] \otimes_R R[G]
\end{array}$$
where \( d(a) := a \otimes 1 - 1 \otimes a \). Then \( d' \) is computed as follows:
\[
d'(a) = \sum_i q(a_i) \otimes a_i' - 1 \otimes a, \quad \text{where} \quad \Delta(a) = \sum_i a_i \otimes a_i'.
\]

Thus \( R[G]^H \) is precisely the kernel of \( d' \), hence is also the kernel of \( d \) as \( \varphi \) is an isomorphism. On the other hand, as \( R[G] \) is faithfully flat over \( R[A] \), \( R[A] \) is the kernel of \( d \). We conclude that \( R[A] = R[G]^H \).

**Theorem 4.6.** Let us be given a sequence
\[
H \xrightarrow{q} G \xrightarrow{p} A
\]
with \( q \) a closed immersion and \( p \) faithfully flat. Then this sequence is exact if and only if the following conditions are fulfilled:

(a) For an object \( V \in \text{Rep}^o(G) \), \( q^*(V) \) in \( \text{Rep}^o(H) \) is trivial if and only if \( V \cong p^*U \) for some \( U \in \text{Rep}^o(A) \).

(b) Let \( W_0 \) be the maximal trivial subobject of \( q^*(V) \) in \( \text{Rep}^o(H) \). Then there exists \( V_0 \subset V \in \text{Rep}^o(G) \), such that \( q^*(V_0) \cong W_0 \).

(c) Any \( W \in \text{Rep}^o(H) \) is a quotient in (hence, by taking duals, a subobject of) \( q^*(V) \) for some \( V \in \text{Rep}^o(G) \).

**Proof.** Assume that \( q : H \rightarrow G \) is the kernel of \( p : G \rightarrow A \). Then (a) and (b) follow from Lemmas (i) and (ii). We prove (c).

Let \( \text{Ind} : \text{Rep}(H) \rightarrow \text{Rep}(G) \) be the induced representation functor, it is the right adjoint functor to the restriction functor \( \text{Res} : \text{Rep}(G) \rightarrow \text{Rep}(H) \) that is
\[
\text{Hom}_G(V, \text{Ind}(W)) \cong \text{Hom}_H(\text{Res}(V), W).
\]

One has \( \text{Ind}(W) \cong (W \otimes_R R[G])^H \), where \( H \) acts on \( R[G] \) by the left regular action, [14, I.3.4]. Notice that the subspace \( (W \otimes_R R[G])^H \subset W \otimes_R R[G] \) is invariant under the action of \( R[A] \), i.e. it is an \( R[A] \) submodule.

Notice that the isomorphism [10]
\[
R[G] \otimes_{R[A]} R[G] \xrightarrow{\sim} R[H] \otimes_R R[G]
\]
is a map of \( G \)-modules where \( G \) acts on \( R[G] \) by the second tensor terms by the right regular action. Since \( R[G] \) is faithfully flat over its subalgebra \( R[A] \), taking the tensor product with \( R[G] \) over \( R[A] \) commutes with taking \( H \)-invariants, hence we have
\[
\text{Ind}(W) \otimes_{R[A]} R[G] \cong (W \otimes_R R[G])^H \otimes_{R[A]} R[G] \cong W \otimes_R R[G].
\]
This in turns implies that \( \text{Ind}(W) \) is faithful exact.

Setting \( V = \text{Ind}(W) \) in (17), one obtains a canonical map \( u_W : \text{Ind}(W) \rightarrow W \) in \( \text{Rep}(H) \) which gives back the isomorphism in (17) as follows:
\[
\text{Hom}_G(V, \text{Ind}(W)) \ni h \mapsto u_W \circ h \in \text{Hom}_H(\text{Res}(V), W).
\]
The map $u_W$ is non-zero whenever $W$ is non-zero. Indeed, since $\text{Ind}$ is exact and faithful, $\text{Ind}(W)$ is non-zero whenever $W$ is non-zero. Thus if were $u_W = 0$, then were the zero map for any $V$. On the other hand, for $V = \text{Ind}(W)$, the right hand side contains the identity map. A contradiction, which shows that $u_W$ cannot vanish.

We show now that $u_W$ is always surjective. Let $U = \text{im}(u_W)$ and $T = W/U \in \text{Rep}_f(H)$. We have the following diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Ind}(U) & \rightarrow & \text{Ind}(W) & \rightarrow & \text{Ind}(T) & \rightarrow & 0 \\
\downarrow{u_U} & & \downarrow{u_W} & & \downarrow{u_T} & & \downarrow & \\
0 & \rightarrow & U & \rightarrow & W & \rightarrow & T & \rightarrow & 0
\end{array}
$$

By assumption, the composition $\text{Ind}(W) \rightarrow \text{Ind}(T) \rightarrow T$ is 0, therefore $\text{Ind}(T) \rightarrow T$ is a zero map, implying $T = 0$.

Assume now $W$ is finitely generated projective over $R$. Then $\text{Ind}(W)$ is torsion free. Hence $\text{Ind}(W)$ is the union of its finitely generated projective modules, we can find a finitely generated $G$-submodule $W_0(W)$ of $\text{Ind}(W)$ which still maps surjectively on $W$. In order to obtain the statement on the embedding of $W$, we dualize the map $W_0(W^\vee) \rightarrow W^\vee$.

Assume now that (a), (b), (c) are satisfied. Then it follows from (a) that for $U \in \text{Rep}^\circ(A), q^*p^*(U) \in \text{Rep}^\circ(H)$ is trivial. Hence $pq : H \rightarrow A$ is the trivial homomorphism. Recall that by assumption, $q$ is injective, $p$ is surjective. Let $\bar{q} : \bar{H} \rightarrow G$ be the kernel of $p$. Then we have commutative diagram

$$
\begin{array}{cccccc}
H & \xrightarrow{i} & \bar{H} & \leftrightarrow & \text{Rep}^\circ(H) & \xleftarrow{i^*} & \text{Rep}^\circ(\bar{H}) \\
\downarrow{q} & & \downarrow{\bar{q}} & & \downarrow{q^*} & & \downarrow{q^*} \\
G & \xrightarrow{q} & \text{Rep}^\circ(G).
\end{array}
$$

It remains to show that $i$ is surjective. We use Proposition [3.9]. We first show the functor $i^*$ is fully faithful. The faithfulness is obvious. Let $W_0, W_1$ be objects in $\text{Rep}^\circ(\bar{H})$, and $\varphi : W_0 = i^*(\bar{W}_0) \rightarrow W_1 = i^*(\bar{W}_1)$ be a morphism of $H$-modules. Since $\bar{H}$ is the kernel of $p$, the first part of this proof shows that there exist surjection $q^*(V_0) \rightarrow \bar{W}_0$ and injection $\bar{W}_1 \hookrightarrow q^*(V_1)$ where $V_0, V_1$ are objects of $\text{Rep}^\circ(G)$. Thus $\varphi$ combined with these maps yields a map $\tilde{\varphi} : q^*(V_0) \rightarrow q^*(V_1)$. $\tilde{\varphi}$ corresponds to an element of $(q^*(V_1) \otimes q^*(V_0)^\vee)^H$. By conditions (a), (b) for $H$ and by the fact that $\bar{H}$ also satisfies (a), (b), there exists $\psi : q^*(V_0) \rightarrow q^*(V_1)$ such that

$$
\tilde{\varphi} = i^*(\psi) : q^*(V_0) \rightarrow \bar{W}_0 \xrightarrow{\varphi} \bar{W}_1 \hookrightarrow q^*(V_1).
$$

This implies that $\varphi = i^*(\tilde{\varphi})$ for some $\tilde{\varphi} : \bar{W}_0 \rightarrow \bar{W}_1$. Thus $i^*$ is full.
For any $W \in \text{Rep}^o(H)$, by (c) there exist $V_0, V_1$ in $\text{Rep}^o(G)$ and $\varphi : q^*(V_0) \rightarrow q^*(V_1)$ such that $W = \text{im} \varphi$. Since $i^*$ is full, $\varphi = i^* \bar{\varphi}$, hence $W \cong i^*(\text{im} \bar{\varphi})$. Thus we have proved that any object in $\text{Rep}^o(H)$ is isomorphic to the image under $i^*$ of an object in $\text{Rep}^o(\bar{H})$. Together with the discussion above this implies that $\bar{H} \cong H$. \[\square\]

5. Stratified sheaves on a smooth scheme over a Dedekind ring

Let $S$ be a smooth, connected affine curve over an algebraically closed field $k$. Thus $S = \text{Spec}(R)$, where $R$ is a Dedekind ring containing $k$. Let $f : X \rightarrow S$ be a smooth morphism with connected fibers. We shall assume that $f$ admits a section $\xi : S \rightarrow X$.

Consider the category $\text{str}(X/S)$ of $O_X$-coherent modules over the sheaf $\mathcal{D}(X/S)$ of algebras of differential operators on $X/S$. This is an abelian tensor category with the unit object being $O_S$. In characteristic 0, an object of $\text{str}(X/S)$ is nothing but a flat connection. In positive characteristic, there is a big difference between the two concepts. For convenience we will refer to objects of $\text{str}(X/S)$ as stratified sheaves.

The pull-back by $\xi$ provides a functor $\xi^* : \text{str}(X/S) \rightarrow \text{Mod}(R)$. The following results are similar to those of dos Santos \[21, \text{Sect 4.2}\].

**Proposition 5.1.** The following claims hold:

(i) An object of $\text{str}(X/S)$, which is torsion-free, is flat as an $O_X$-module and hence it is locally free. Consequently, the subcategory of $R$-torsion free objects is closed under taking subobjects.

(ii) The functor $\xi^* : \text{str}(X/S) \rightarrow \text{Mod}(R)$ is faithful and exact.

**Proof.** (i). Since this is a local property, we can assume that $R$ is a discrete valuation ring and $X$ is affine over $R$. Then the proof of \[21, \text{Lem. 19}\] can be used. Namely, by the local flatness criterion, an object $M$ of $\text{str}(X/S)$ is flat over $R$ if $\text{Tor}_1^R(M, O_{X_0}) = 0$ and $M_0 := M/tM$ is flat over $X_0$. The second condition is trivially satisfied as $X_0$ is a scheme over a field. The first condition just means $M$ is $t$-torsion free.

(ii) We show that $\xi^*$ is left exact. We can assume that $X$ is affine, $X = \text{Spec}(A)$. Let $m_\xi$ be the kernel of $\xi : A \rightarrow R$ then $\xi^*$ is the functor tensoring with $A/m_\xi$. Thus if suffices to check that $\text{Tor}_1^A(M, A/m_\xi)$ vanishes for any $M \in \text{str}(X/S)$. Let $M_t$ be the $R$-torsion part of $M$ and $M_f := M/M_t$ then $M_f$ is $R$-torsion free hence is flat by (i). Thus one is led to check the claim for those $M$, which are $R$-torsion. Then such an $M$ has a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M$$
where $M_i/M_{i-1}$ is killed by some maximum ideal $p_i$ of $R$. Thus $M_i$ is supported in $X_{s_i}$, where $s_i$ is the point determined by $p_i$ and we can replaced $R$ by the local ring $R_{(p_i)}$, which is a discrete valuation ring. Then the argument of dos Santos [21, Prop. 18] shows that $\text{Tor}^R_1(M, A/m_{\xi}) = 0$. □

We don’t know if $\text{str}(X/S)$ is a Tannakian category over $R$. However we can define the maximal Tannakian subcategory as follows. Let $\mathcal{C}(X/S)$ be the full subcategory of $\text{str}(X/S)$ consisting of objects which can be represented as a quotient of locally free objects of $\text{str}(X/S)$. According to the proposition above, $\mathcal{C}$ is an abelian tensor category over $R$ (by the connectedness of the fibers of $f : X \to S$, the endomorphism ring of the unit object in $\text{str}(X/S)$ is equal to $R$) and $\mathcal{C}$ contains the subcategory of definition $\text{str}^{\pi}(X/S)$ consisting of stratified bundles.

**Definition 5.2.** The fiber functor $\xi^*$ makes $\mathcal{C}$ a Tannakian category, its Tannakian group, denoted $\pi(X/S, \xi)$, is called the relative fundamental group scheme of $X/S$.

Let $s$ be a closed point of $S$ and let $X_s$ denote the fiber of $f$ at $s$. Consider the category $\text{str}(X_s/k)$. Its objects are automatically locally free as $O_{X_s}$-modules, i.e. they are vector bundles on $X_s$ (for the proof of this fact in characteristic 0 see [15] and in positive characteristic see [20]). Thus $\text{str}(X_s/k)$ is an abelian rigid tensor category over $k$.

The fiber of $\text{str}(X/S)$ at $s \in S$ is defined as in Remark 2.16 and is denoted by $\text{str}(X/S)_s$. This full subcategory of $\text{str}(X/S)$ is identified with the category of stratified bundle on $X_s$. Indeed, the restriction functor $\text{str}(X/S) \to \text{str}(X_s/k)$ (given by pulling-back along the closed immersion $X_s \to X$) can be identified with the functor which associates to each object $M$ of $\text{str}(X/S)$ the quotient $M/p_s M$, where $p_s$ the maximal ideal of $R$ determining $s$. In particular, $\text{str}(X_s/k)$ is naturally a subcategory of $\text{str}(X/S)$, consisting of those objects which are $p_s$-torsion.

There is a natural “inflation” functor from $\text{str}(X/k)$ to $\text{str}^{\pi}(X/S)$: each stratified bundle on $X/k$ can be considered as a stratified bundle on $X/S$. Finally we have the functor $\omega_f = f^*: \text{str}(S/k) \to \text{str}(X/k)$. Thus, to summarize, we have the following sequence of tensor functors:

$$\text{str}(S/k) \xrightarrow{\omega_f} \text{str}(X/k) \xrightarrow{\text{infl}} \mathcal{C}(X/S) \xrightarrow{\text{res}} \mathcal{C}(X/S)_s \xrightarrow{\text{res}} \text{str}(X_s/k)$$

**Lemma 5.3.** $\mathcal{C}(X/S)_s$ is a full subcategory of $\text{str}(X_s/k)$, closed under taking subquotients.

**Proof.** This is obvious. An object $M_0$ of $\mathcal{C}(X/S)_s$ is a $\pi$-torsion object which can be presented as a quotient of a locally free object of $\text{str}(X/S)$, say $M$. Thus any
quotient of $M_0$ is again in $\mathcal{C}(X/S)_s$. On the other hand, if $N_0$ is a subobject of $M_0$ then taking the pull-back of $N_0$ along the projection map $M \rightarrow M_0$ we get a subobject $N$ of $M$ which surjects onto $N_0$. But $N$ itself is locally free. \hfill \square 

Remarks 5.4. In general it is not necessarily the case that $M_0 = M/\pi M$. In other words, the kernel $M \rightarrow M_0$ maybe large than $\pi M$.

The fiber functor at a closed point $x \in X$ makes $\text{str}(X/k)$ a neutral Tannakian category. Its Tannakian group is denoted by $\pi(X, x)$ and called the fundamental group scheme of $X$ at $x$. Let $s = f(x)$. The functor $\omega_f$ is compatible with the fiber functors at $s$ and $x$. Thus, according to 2.11, we have a homomorphism of fundamental group schemes $f_*: \pi(X, x) \rightarrow \pi(S, s)$. The restriction functor $\text{str}(X/k) \rightarrow \text{str}(X_s/k)$ is also a tensor functor and is compatible with the fiber functors at $x$, hence induces a homomorphism $\pi(X_s, x) \rightarrow \pi(X, x)$.

Assume now that $x = \xi(s)$:

Theorem 5.5. The homomorphism $\pi(X_s/k, x) \rightarrow \pi(X/S, \xi)_s$ is surjective.

Proof. Here the group schemes are defined over a field. Hence we can use the criterion for surjectivity of Deligne-Milne [5, Thm. 2.21]. We show that for each object $M \in \text{str}(X/S)_s$, when considered as object in $\text{str}(X_s/k, x)$ all its subobjects will be an object in $\text{str}(X/S)_s$. This is obvious from the fact that the category $\text{str}^{\alpha}(X/S)$ is closed under taking subobjects. There exists by assumption an
$X \in \text{str}^o(X/S)$ which surjects on $M$. We have the following pull-back diagram

$$
\begin{array}{ccc}
X & \longrightarrow & M \\
\downarrow & & \downarrow \\
Y & \longrightarrow & N
\end{array}
$$

Since $Y$ is a subobject of $X$, it is itself locally free.

In \cite{22} dos Santos proved that the following homotopy sequence is exact:

$$
\pi(X_s, x) \longrightarrow \pi(X, x) \longrightarrow \pi(S, s) \longrightarrow 1,
$$

provided that $f$ is a proper map. Hence in this case we also have an exact sequence

$$
\pi(X/S, \xi)_s \longrightarrow \pi(X/k, x) \longrightarrow \pi(S/k, s) \longrightarrow 1.
$$

We do not know whether it is also left exact.

The general Tannakian duality applied to $\xi^*$ and the categories $\text{str}(X/k)$ and $\text{str}(S/k)$ yields the fundamental groupoid schemes $\Pi(X/k, \xi)$ and $\Pi(S/k, \xi)$, and the functor $\omega_f : \text{str}(S/k) \longrightarrow \text{str}(X/k)$ yields a surjective homomorphism $f_* : \Pi(X, \xi) \longrightarrow \Pi(S, \xi)$. The kernel of this groupoid homomorphism, is by definition

$$
L := S \times_{\Pi(S, \xi)} \Pi(X, \xi),
$$

where the map $S \longrightarrow \Pi(X, \xi)$ is given by the unit element. This is a flat group scheme over $S$. On other hand, the inflation functor $\text{str}(X/k) \longrightarrow \text{str}(X/S)$ induces a homomorphism $\pi(X/S, \xi) \longrightarrow \Pi(X/k, \xi)$. The following question is motivated by dos Santos’ result mentioned above.

**Questions 5.6.** Assume that $f : X \longrightarrow S$ is a smooth, proper map with connected fibers. Is the following sequence exact

$$
\pi(X/S, \xi) \longrightarrow \Pi(X/k, \xi) \longrightarrow \Pi(S/k, \xi) \longrightarrow 1?
$$

**The case of a complete discrete valuation ring.** Let $A = k[[t]]$ where $k$ is algebraically closed field, with quotient field $K = k((t))$. Let $\mathfrak{X}$ be a smooth connected formal affine scheme over $\text{Spf}A$. Let $X_0$ be the special fiber of $\mathfrak{X}$ and let $X$ be the generic fiber. Assume that $\mathfrak{X}$ admits an $A$-rational point $\xi$. Our aim is to show that the category $\text{str}(\mathfrak{X}/A)$ is Tannakian.

Recall the following results of dos Santos \cite{21} Sect 4.2].

**Lemma 5.7.** A stratified sheaf in $\text{str}(\mathfrak{X}/A)$ is in $\text{str}^o(\mathfrak{X}/A)$ iff it is $t$-torsion free. Consequently, the subcategory $\text{str}^o(\mathfrak{X}/A)$ is closed under taking subobjects. Further the fiber functor at $\xi$, $\text{str}(\mathfrak{X}/A) \longrightarrow \text{Mod}(A)$ is faithful and exact.

Let $\text{str}^o(\mathfrak{X}/A)$ denote the full subcategory of $\text{str}(\mathfrak{X}/A)$ consisting of stratified bundles on $\mathfrak{X}/A$. We will show that $\text{str}^o(\mathfrak{X}/A)$ is subcategory of definition in
Thus it is to show that any stratified sheaf in \( \text{str}(\mathcal{X}/A) \) is quotient of a stratified bundle. First we need the following.

**Proposition 5.8.** The restriction functor from \( \text{str}(\mathcal{X}/k) \) to \( \text{str}(X_0/k) \) is an equivalence. In particular, any exact sequence in \( \text{str}(X_0/k) \) can be lifted to an exact sequence in \( \text{str}(\mathcal{X}/k) \).

**Proof.** If \( k \) is of positive characteristic, this is a result of Gieseker \([11, \text{Lemma 1.5}]\). He constructed an explicit lift of a stratified bundle on \( X_0/k \) to a stratified bundle on \( \mathcal{X}/k \) and showed that this lift yields a functor which is quasi-inverse to the restriction functor, thus giving the equivalence.

The case of zero characteristic can be proved using the method of Katz in the proof of \([15, \text{Prop. 8.8}]\). Let \( M \) be a stratified module over \( D(\mathcal{X}/k) \). The action of this algebra on \( M \) will be denoted as usual by \( \nabla \). Since \( O_{\mathcal{X}} \) contains the field \( k \), \( D(\mathcal{X}/k) \) is generated by the derivations, that is a stratification is nothing but a flat connection. We first show that it is locally free. By means of Lemma 5.7 it suffices to show that \( M \) is \( t \)-torsion free. This is a local property on \( \mathcal{X} \).

Let \((x_1, \ldots, x_r, t)\) be local coordinates on \( \mathcal{X} \). Thus \( \partial_{x_i} \)'s commute each other and commute with \( \partial_t \), and we have

\[
\partial_{x_i}(x_j) = \delta_{ij}; \quad \partial_{x_i}(t) = \partial_i(x_i) = 0; \quad \partial_t(t) = 1.
\]

One considers the \( k \)-linear operator

\[
P := \sum_{i=0}^{\infty} \frac{(-t)^i}{i!} \partial_t.
\]

It has the following properties, cf. \([15, \text{Sect. 8}]\):

\[
P^2 = P; \quad P(m) = m \mod tM; \quad P(fm) = f(0)P(m), \quad f \in A, m \in M.
\]

Hence, setting \( M^\nabla := \ker \nabla(\partial_t) \), we have

\[
M^\nabla = \text{im}P; \quad M = M^\nabla \oplus tM; \quad M^\nabla \cong M_0 := M/tM.
\]

Further the map \( A \otimes M^\nabla \rightarrow M, f \otimes m \mapsto fm \) is injective, in particular, \( M^\nabla \) is \( t \)-torsion free.

Assume now that \( tm = 0 \) for some \( m \in M \). If \( m \neq 0 \) it has a unique presentation \( m = t^{k-1}(tm_1 + m_0) \), where \( k > 0 \) maximal, \( m_0 \in M^\nabla \) (this is due to the completeness of the \( t \)-adic topology on \( O_{\mathcal{X}} \)). Then we have

\[
0 = \nabla(\partial_t)^k(tm) = \nabla(\partial_t)^k(t^{k+1}m_1) + k!m_0,
\]

(since \( \nabla(\partial_t)(m) = 0 \)). Hence \( m_0 \in tM \), which implies \( m_0 = 0 \), contradiction. Hence \( m = 0 \). Thus \( M \) is \( t \)-torsion free, hence locally free over \( O_{\mathcal{X}} \) and consequently the restriction functor \( M \mapsto M_0 = M/tM \) is exact. It also implies that \( \text{str}(\mathcal{X}/k) \) is an abelian rigid tensor category.
A section of $M$ (as an object of $\text{str}(\mathfrak{X}/k)$) is horizontal iff (locally) it lies in $M\nabla t$, and is annihilated by $\nabla(\partial_{x_i})$, and hence iff its image in $M_0$ is a horizontal section of $M_0$ as an object of $\text{str}(X_0/k)$. We conclude that the restriction functor $\text{str}(\mathfrak{X}/k) \to \text{str}(X_0/k)$ is faithful.

Conversely, the third isomorphism in Eq. 19 shows that on an open $U$ of $X_0$ (which topologically homeomorphic to $X$), small enough so that local coordinates on it exist, a horizontal section of $M_0|_U$ can be uniquely lifted to a horizontal section of $M|_U$. Let now $s_0$ be a horizontal section of $M_0 \in \text{str}(X_0/k)$. Consider an open covering $(U_\alpha)$ of $X_0$ such that on each $U_\alpha$ there exist local coordinates $(x_i,t)$. Let $s_{0,\alpha}$ be the restriction of $s_0$ on $U_\alpha$. Lift $s_{0,\alpha}$ to a horizontal section $s_\alpha$ of $M$ on $U_\alpha$. The restrictions of $s_\alpha$ and $s_\beta$ on $U_\alpha \cap U_\beta$ agree as they are liftings of the same section. Hence the $s_\alpha$’s glue together to give a horizontal section of $M$ on $X$. Thus the restriction functor is full.

It remains to show that this functor is essentially surjective, that is, each stratified bundle $M_0$ on $X_0/k$ can be lifted to a stratified bundle on $\mathfrak{X}/k$. We first assume that on $X_0$ there exist global coordinates $(x_1,\ldots,x_n,t)$ and that $M_0$ is free over $O_{X_0}$ with basis $(e_0^0)$ and show that a flat connection on $M_0$ can be lifted to $M = \langle e_i \rangle_{O_X}$.

Consider the action of the operator $P$ defined above on the algebra $O_X$. We have $(D := \nabla(\partial_t))$

$$P(ab) = \sum_i \frac{(-t)^i}{i!} D^i(ab)$$

$$= \sum_i \frac{(-t)^i}{i!} \sum_j \binom{i}{j} D^j(a) D^{i-j}(b)$$

$$= \sum_j \frac{(-t)^j}{j!} D^j(a) \sum_{i \geq j} \frac{(-t)^{i-j}}{i!} D^{i-j}(b)$$

$$= P(a)P(b).$$

Hence the isomorphism $\varphi : O_{\mathfrak{X}}^\nabla \to O_{X_0}$, induced by $P$, is an isomorphism of algebras. Notice that, since $[\partial_{x_i}, \partial_t] = 0$ we have $[\partial_{x_i}, D] = 0$, for all $i$. Thus $\varphi$ commutes with the action of $\partial_{x_i}$.

Let $\psi$ be the inverse of $\varphi$. Assume that the actions of $\nabla(\partial_{x_i})$ on the basis $(e_0^0)$ is given by the matrices $a_{ij}^k$.

$$\nabla(\partial_{x_i})(e_0^0) = \sum_k a_{ij}^k e_0^0.$$ 

The flatness of $\nabla$ is expressed in terms of the Maurer-Cartan equation involving the matrices $(a_{ij}^k)$ and their partial derivatives in $x_i$’s. This equation is preserved by
ψ, which means we can lift them to a set of matrices $A^k_{ij}$ such that the equations:

$$\nabla(\partial x_i)(e_j) = \sum_k A^k_{ij}e_k$$

defines a flat connection on $\mathfrak{X}/A$.

Finally we simply set $\nabla(\partial t)(e_i) = 0$. It is straightforward to check that $\nabla(\partial t)$ commutes with $\nabla(\partial x_i)$ using the fact that $\nabla(\partial t)(A^k_{ij}) = 0$ as these elements lie in $A^{\nabla t}$. Thus, we have constructed a flat connection on $M$.

In the general case we consider an open covering of $X_0$, such that on each open, the connection $M_0$ is free and local coordinates exist. Then on each open we can lift $M_0$. As the lift on each open is unique, they glue together to give a lift of $M_0$ on the whole $\mathfrak{X}$.

Notice that if a stratified sheaf $E_0$ on $\mathfrak{X}/A$ is annihilated by $t$ then it can be considered as a sheaf on $X_0/k$ and hence can be lifted to a stratified bundle on $\mathfrak{X}/k$, say $E$ and we have exact sequence

$$0 \rightarrow E \rightarrow E \rightarrow E_0 \rightarrow 0$$

where $[t]$ denotes the map multiplication by $t$. Thus in this case $E_0$ is a quotient of $E$ (considered as stratified sheaf on $\mathfrak{X}/A$).

**Proposition 5.9.** Each object of $\text{str}(\mathfrak{X}/A)$ is a quotient of an object of $\text{str}(\mathfrak{X}/A)^0$. Consequently, $\text{str}(\mathfrak{X}/A)$ is a Tannakian category.

**Proof.** Let $E$ be an object of $\text{str}(\mathfrak{X}/A)$. Then the subsheaf $E_t$ consisting of sections annihilated by some power of $t$ is invariant under the stratification. We have an exact sequence

$$0 \rightarrow E_t \rightarrow E \rightarrow E_{t^r} \rightarrow 0$$

with $E_{t^r}$ a $t$-torsion free stratified sheaf, hence is locally free by the lemma above. There exists a least integer $r$ such that $E_t$ is annihilated by $t^r$. We will use induction on $r$.

For $r = 1$, the subsheaf $tE \subset E$ is $t$-torsion free. Indeed, if a section $ts$ in $tE$ is torsion then $s$ is itself torsion, hence is annihilated by $t$. Consider the exact sequence

$$0 \rightarrow tE \rightarrow E \rightarrow E/tE \rightarrow 0.$$

The sheaf $E/tE$ is in $\text{str}(X_0/k)$, hence can be lifted to a stratified bundle $F$ on $\mathfrak{X}/k$: $F \rightarrow E/tE$. Pull back the above sequence along this map (considered as
morphism in \( \text{str}(\mathcal{X}/A) \), we get the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & tE & \to & E & \to & E/tE & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & tE & \to & \hat{E} & \to & F & \to & 0
\end{array}
\]

In particular, the map \( \hat{E} \to E \) is surjective. But \( \hat{E} \) is torsion free as the two sheaves \( tE \) and \( F \) are locally free. Thus \( \hat{E} \) is the needed stratified bundle on \( \mathcal{X}/A \).

Let now \( E \) be such that \( E_t \), the subsheaf of torsion sections, is annihilated by \( t^n, n > 1 \). Let \( E_0 \) be the subsheaf of \( E \) of section annihilated by \( t \). Then we have exact sequence

\[
0 \to E_0 \to E \to E' \to 0
\]

where for \( E' \) its torsion part \( E'_t \) is annihilated by \( t^{n-1} \). By induction we can lift \( E' \) to \( F' \) and hence, by pulling-back, we can lift \( E \):

\[
\begin{array}{ccccccccc}
0 & \to & E_0 & \to & E & \to & E' & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & E_0 & \to & F & \to & F' & \to & 0
\end{array}
\]

Since \( F' \) is locally free, \( E_0 \) is the torsion subsheaf of \( F \) and we can lift \( F \). \( \square \)

**Remarks 5.10.** Y. Andre has given in [1, 3.2.1.5] an example showing that the Tannakian group scheme of a single connection in \( \text{str}(\mathcal{X}/A)^0 \) may be not of finite type over \( A \).

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