UNIFORM ATTRACTORS OF STOCHASTIC TWO-COMPARTMENT GRAY-SCOTT SYSTEM WITH MULTIPlicative NOISE

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Abstract. We first show that the stochastic two-compartment Gray-Scott system generates a non-autonomous random dynamical system. Then we establish some uniform estimates of solutions for stochastic two-compartment Gray-Scott system with multiplicative noise. Finally, the existence of uniform and cocycle attractors is proved.

1. Introduction. In this paper, we study the following stochastic two-compartment Gray-Scott system with multiplicative noise

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} - d_1 \Delta \bar{u} + (F + k)\bar{u} - \bar{u}^2 \bar{v} - D_1 (\bar{w} - \bar{u}) &= \sigma(x,t) + \alpha \bar{u} \circ d\omega, \\
\frac{\partial \bar{v}}{\partial t} - d_2 \Delta \bar{v} - F(1 - \bar{v}) + \bar{u}^2 \bar{v} - D_2 (\bar{y} - \bar{v}) &= \sigma(x,t) + \alpha \bar{v} \circ d\omega, \\
\frac{\partial \bar{w}}{\partial t} - d_1 \Delta \bar{w} + (F + k)\bar{w} - \bar{w}^2 \bar{y} - D_1 (\bar{u} - \bar{w}) &= \sigma(x,t) + \alpha \bar{w} \circ d\omega, \\
\frac{\partial \bar{y}}{\partial t} - d_2 \Delta \bar{y} - F(1 - \bar{y}) + \bar{w}^2 \bar{y} - D_2 (\bar{v} - \bar{y}) &= \sigma(x,t) + \alpha \bar{y} \circ d\omega,
\end{align*}
\]

where \( t > 0, x \in \mathcal{O} \subset \mathbb{R}^n (n \leq 3) \) is a bounded domain. \( \mathcal{O} \) has a locally Lipschitz continuous boundary. \( F, k, \alpha, D_i \) and \( d_i \) are positive constants, where \( i = 1, 2 \). \( \sigma \in \Sigma \), \( \circ \) denotes the Stratonovich sense of the stochastic term. Assume that (1) has the following homogeneous Neumann boundary condition

\[
\frac{\partial \bar{u}}{\partial \nu}(x,t) = \frac{\partial \bar{v}}{\partial \nu}(x,t) = \frac{\partial \bar{w}}{\partial \nu}(x,t) = \frac{\partial \bar{y}}{\partial \nu}(x,t) = 0, \quad t > 0, \ x \in \partial \mathcal{O},
\]

and has the following initial condition

\[
\bar{u}(x,0) = \bar{u}_0(x), \quad \bar{v}(x,0) = \bar{v}_0(x), \quad \bar{w}(x,0) = \bar{w}_0(x), \quad \bar{y}(x,0) = \bar{y}_0(x),
\]
where $\frac{\partial}{\partial \nu}$ is the outward normal derivative.

A non-autonomous random dynamical system (NRDS) is a measurable mapping $\phi: \mathbb{R}^+ \times \Omega \times \Sigma \times X \to X$, see [10, 11, 21]. $\phi$ has two base flows $\{\theta_t\}_{t \in \mathbb{R}}$ and $\{\vartheta_t\}_{t \in \mathbb{R}}$ working on $\Omega$ and $\Sigma$, respectively. $\mathbb{R}^+ = [0, \infty)$, $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space [1, 5], $\Sigma$ is a symbol space constructed by time-dependent terms, called the symbol of the system [4], and $X$ is a phase space.

In [8], Crauel, Kloeden and Yang firstly introduced the concept of cocycle attractors for NRDS. Then Wang explained cocycle attractors in detail in [21, 22]. It is noted that the attraction universes are non-autonomous in [21, 22]. For other different kinds of attractors, such as random attractors, pullback attractors, global attractors and so on, we can see [3, 6, 7, 14, 18, 20, 23, 24]. In [13], Flandoli and Schmalfuss studied random attractors of stochastic Navier-Stokes equation. Uniform attractors of non-autonomous three-component reversible Gray-Scott system were considered in [15]. In [16], Jia, Gao and Ding studied random attractors for stochastic two-compartment Gray-Scott equations with a multiplicative noise. Cui and Langa investigated uniform attractors for non-autonomous random dynamical systems in [11].

This paper is organized as follows. In Section 2, we will introduce some concepts about uniform and cocycle attractors. In Section 3, we will provide some basic settings of stochastic two-compartment Gray-Scott system. In Section 4, we give some uniform estimates of solutions. In Section 5, the existence of uniform and cocycle attractors is provided.

The following notations will be used throughout the paper. Denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and inner product in $L^2(\mathcal{O})$ or $[L^2(\mathcal{O})]^4$. We use $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^1}$ to denote the norm in $L^p(\mathcal{O})$ and $H^1(\mathcal{O})$.

Using Poincaré’s inequality, there exists a constant $\gamma > 0$ satisfying

$$\| \nabla \phi \|^2 \geq \gamma \| \phi \|^2, \quad \text{for} \ \phi \in H^1_0(\mathcal{O}) \ \text{or} \ [H^1_0(\mathcal{O})]^4. \quad (4)$$

We know that $H^1_0(\mathcal{O}) \hookrightarrow L^6(\mathcal{O})$ for $n \leq 3$. There is a constant $\eta > 0$ satisfying the following embedding inequality

$$\| f \|_{H^1(\mathcal{O})} \geq \eta \| f \|_{L^6(\mathcal{O})}, \quad \text{for} \ \phi \in H^1_0(\mathcal{O}) \ \text{or} \ [H^1_0(\mathcal{O})]^4. \quad (5)$$

2. Preliminaries. In this part, some concepts about uniform attractors and cocycle attractors are provided, see [9, 11, 4]. Let $\phi$ be a NRDS, let $\mathcal{D}$ be an autonomous universe and let $(X, d)$ be a Polish metric space. For non-empty sets in $X$, the Hausdorff semi-metric is defined by

$$\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} d(a, b), \quad A, B \in 2^X \setminus \emptyset.$$ 

Let $(\Xi, d_{\Xi})$ be the extended space of $\Sigma \times X$, i.e., $\chi \in \Xi$ if and only if $\chi = \{\sigma\} \times \{x\}$ for some $\sigma \in \Sigma$ and $x \in X$. For any metric space $M$, let $B(M)$ be the Borel sigma-algebra of $M$. Let $(\Sigma, d_{\Sigma})$ be a compact Polish space which is invariant

$$\theta_t \Sigma = \Sigma, \quad \forall t \in \mathbb{R}.$$ 

For each $\Xi \subset \Sigma$ and $\omega \in \Omega$, define the following omega limit set of any $B \in \mathcal{D}$,

$$W(\omega, \Xi, B) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \phi(t, \vartheta_{-t} \omega, \vartheta_{-t} \Xi, B(\vartheta_{-t} \omega)). \quad (6)$$
Normally, it is not difficult to find that $\mathcal{W}(\omega, \Xi, B) \neq \cup_{\sigma \in \Xi} \mathcal{W}(\omega, \sigma, B)$. In reality,
\[
\bigcup_{\sigma \in \Xi} \mathcal{W}(\omega, \sigma, B) = \mathcal{W}(\omega, \Xi, B).
\]

**Definition 2.1.** For any $s \in \mathbb{R}$, define an operator $A_{\sigma(s)}(\cdot) : G_1 \to G_0$, where $G_1$, $G_0$ are Banach spaces. $\sigma(s)$, $s \in \mathbb{R}$ reflects the dependence on time. The function $\sigma(s)$ is called the symbol of system (1).

**Definition 2.2.** The set $\Sigma$ is called the symbol space of system (1) if $\Sigma$ contains all $\sigma(\cdot)$ and
\[
\theta_t \sigma(\cdot) := \sigma(\cdot + t), \quad \forall t \in \mathbb{R}.
\]

**Property 2.3.** $\{\theta_t\}_{t \in \mathbb{R}}$ is the smooth translation operator satisfying
1. $\theta_0$ = identity operator on $\Sigma$;
2. $\theta_s \circ \theta_t = \theta_{t+s}$, $\forall t, s \in \mathbb{R}$;
3. $(t, \sigma) \mapsto \theta_t \sigma$ is continuous.

**Property 2.4.** $\{\vartheta_t\}_{t \in \mathbb{R}}$ is the base flow of $\phi$ satisfying
1. $\vartheta_0 = \text{identity operator on } \Omega$;
2. $\vartheta_t \Omega = \Omega$, $\forall t \in \mathbb{R}$;
3. $\vartheta_s \circ \vartheta_t = \vartheta_{t+s}$, $\forall t, s \in \mathbb{R}$;
4. $(t, \omega) \mapsto \vartheta_t \omega$ is $(B(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable;
5. $\mathcal{P}$-preserving: $\mathcal{P}(\vartheta_t F) = \mathcal{P}(F)$, $\forall t \leq 0, F \in \mathcal{F}$.

**Definition 2.5.** A mapping $\phi(t, \omega, \sigma, x) : \mathbb{R}^+ \times \Omega \times \Sigma \times X \to X$ is called a non-autonomous random dynamical system (NRDS) on $X$ if
1. $\phi$ is $(B(\mathbb{R}^+) \times \mathcal{F} \times B(\Sigma) \times B(X), B(X))$-measurable;
2. $\phi(0, \omega, \sigma, \cdot) =$ identity operator on $X$, $\forall \sigma \in \Sigma, \omega \in \Omega$;
3. for any fixed $\sigma \in \Sigma, x \in X$ and $\omega \in \Omega$, $\phi$ has the following cocycle property
\[
\phi(t + s, \omega, \sigma, x) = \phi(t, \vartheta_s \omega, \vartheta_s \sigma) \circ \phi(s, \omega, \sigma, x), \quad \forall t, s \in \mathbb{R}^+.
\]

**Definition 2.6.** For any fixed $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in X$, if the mapping $\sigma \mapsto \phi(t, \omega, \sigma, x)$ is continuous, then the NRDS $\phi$ is continuous in $\Sigma$.

Similarly, the continuity of $\phi$ in $X$ can be defined.

**Definition 2.7.** An (autonomous) random set $A \in \mathcal{D}$ is called the (random) $\mathcal{D}$-uniform attractor for a NRDS $\phi$ if
1. $A$ uniformly (pullback) attracts every $D \in \mathcal{D}$ under $\phi$, i.e.,
\[
\lim_{t \to \infty} \sup_{\sigma \in \Sigma} \text{dist}(\phi(t, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, D(\vartheta_{-t} \omega)), A(\sigma)) = 0, \quad \forall \omega \in \Omega;
\]
2. $A$ is the minimal compact (autonomous) random set satisfying (1).

**Definition 2.8.** A non-autonomous random set $A = \{A_{\sigma}(\omega)\}_{\sigma \in \Sigma, \omega \in \Omega}$ is said to be the (random) $\mathcal{D}$-cocycle attractor for an NRDS $\phi$ if
1. each $A_{\sigma}(\cdot)$ is a compact random set in $X$;
2. $A$ pullback attracts every random set $D$ of $\mathcal{D}$ under $\phi$, i.e.,
\[
\lim_{t \to \infty} \text{dist}(\phi(t, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, D(\vartheta_{-t} \omega)), A_{\sigma}(\omega)) = 0, \quad \forall \omega \in \Omega, \sigma \in \Sigma;
\]
3. $A$ is invariant under $\phi$, i.e.,
\[
\phi(t, \omega, \sigma, A_{\sigma}(\omega)) = A_{\theta_t \sigma (\vartheta_t \omega)}, \quad \forall t \in \mathbb{R}^+;
\]
(4) A is the minimal compact non-autonomous random set in $X$ satisfying (2).

**Theorem 2.9.** Suppose that NRDS $\phi$ is continuous in both $\Sigma$ and $X$, then there is the relation between uniform attractor $A$ and cocycle attractor $\Phi$ for $\phi$, that is,

$$A(\omega) = \cup_{\sigma \in \Sigma} A_\sigma(\omega), \ \forall \omega \in \Omega.$$ 

**Theorem 2.10.** For a NRDS $\Phi$, let $\pi : \mathbb{R}^+ \times \Omega \times X \to X$ be a mapping

$$\pi(t, \omega, \{\sigma\} \times \{x\}) = \{\theta_t \sigma\} \times \{\phi(t, \omega, \sigma, x)\}.$$ (8)

Then $\pi$ is a (random) cocycle satisfying

1. $\pi$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
2. $\pi(0, \omega, \chi) = \chi, \ \forall \omega \in \Omega, \chi \in X$;
3. $\pi(t + s, \omega, \chi) = \pi(t, \theta_s \chi, \pi(s, \omega, \chi))$, $\forall t, s \in \mathbb{R}^+, \omega \in \Omega, \chi \in X$.

**Proposition 2.11.** If a random set $A$ is uniformly $\mathcal{D}$-pullback attracting under NRDS $\phi$, then $A$ is forward uniformly attracting in probability, i.e.,

$$\lim_{t \to \infty} \mathcal{P}\{\omega \in \Omega : \sup_{\sigma \in \Sigma} \text{dist}(\phi(t, \omega, \sigma, B(\omega)), A(\theta_t \omega)) > \varepsilon\} = 0, \ \forall \varepsilon > 0, B \in \mathcal{D}.$$ 

**Lemma 2.12.** Assume that NRDS $\phi$ is continuous in $\Sigma$. If $\Xi \in \Sigma$ is dense, then

$$\mathcal{W}(\omega, \Xi, B) = \mathcal{W}(\omega, \Sigma, B), \ \forall \omega \in \Omega, B \in \mathcal{D}.$$ 

**Theorem 2.13.** Assume that NRDS $\phi$ is continuous in $\Sigma$ and $X$, and each $\Xi \in \Sigma$ is dense. If $\phi$ possesses a compact uniformly $\mathcal{D}$-attracting set $K$ and a closed uniformly $\mathcal{D}$-absorbing set $B \in \mathcal{D}$, then $\phi$ possesses a unique random uniform attractor $A \in \mathcal{D}$,

$$A(\omega) = \mathcal{W}(\omega, \Sigma, B) = \mathcal{W}(\omega, \Xi, B), \ \forall \omega \in \Omega.$$ 

Furthermore, $A$ is negatively semi-invariant

$$A(\theta_t \omega) \subseteq \phi(t, \omega, \Sigma, A(\omega)), \ \forall t \geq 0, \omega \in \Omega.$$ 

**Proposition 2.14.** Assume that NRDS $\phi$ is continuous in $\Sigma$ and $X$, and $\mathcal{U}$ is a random uniform attractor. If $\phi$ has a $\mathcal{D}$-random uniform attractor $A$, and $\mathcal{U}$ uniformly attracts deterministic compact sets, then

$$\mathcal{P}(A = \mathcal{U}) = 1.$$ 

3. NRDS generated by stochastic two-compartment Gray-Scott system.

In this part, we will provide some basic settings of system (1) from [11, 16, 25, 26, 27]. Moreover, we show that system (1) generates a NRDS.

Let $\vec{g} = (\vec{u}, \vec{v}, \vec{w}, \vec{y})^T$, system (1)-(3) can be rewritten as

$$\begin{align*}
\frac{\partial \vec{g}}{\partial t} - A \vec{g} + H(\vec{g}) &= \vec{\sigma}(x, t) + \alpha \vec{g} \odot \frac{d\omega}{dt}, \ \ t > 0, \\
\vec{g}(x, 0) &= \vec{g}_0(x), \ \ x \in \mathcal{O}, \\
\frac{\partial \vec{g}}{\partial \nu}(x, t) &= 0, \ \ x \in \partial \mathcal{O},
\end{align*}$$

where

$$\vec{\sigma}(x, t) = (\sigma(x, t), \sigma(x, t), \sigma(x, t), \sigma(x, t))^T,$$

$$A = \begin{pmatrix}
d_1 \Delta & 0 & 0 & 0 \\
0 & d_2 \Delta & 0 & 0 \\
0 & 0 & d_1 \Delta & 0 \\
0 & 0 & 0 & d_2 \Delta
\end{pmatrix},$$
After this, we will assume that $\tilde{\Omega}$ equals $\Omega$. Let

$$H(\tilde{g}) = \begin{pmatrix}
(F + k)\tilde{u} - \tilde{u}^2\tilde{v} - D_1(\tilde{w} - \tilde{u}) \\
-F(1 - \tilde{v}) + \tilde{u}^2\tilde{v} - D_2(\tilde{y} - \tilde{v}) \\
(F + k)\tilde{w} - \tilde{w}^2\tilde{y} - D_1(\tilde{u} - \tilde{w}) \\
-F(1 - \tilde{y}) + \tilde{w}^2\tilde{y} - D_2(\tilde{v} - \tilde{y})
\end{pmatrix},$$

here $T$ denotes the transposition.

Let $\Omega = \{\omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0\}$ be a probability space. Let $\mathcal{F}$ be a Borel sigma-algebra. Let $\mathbb{P}$ be the two-sided Wiener measure on $(\Omega, \mathcal{F})$. $\vartheta$ is a translation operator satisfying

$$\vartheta_t \omega = \omega(\cdot + t) - \omega(t), \quad \forall t \in \mathbb{R}, \; \omega \in \Omega.$$ 

Hence $\mathbb{P}$ is ergodic and invariant under $\vartheta$ [13, 17]. Denote by

$$z(\vartheta_t \omega) = -\int_{-\infty}^{0} e^\tau (\vartheta_t \omega)(\tau)\,d\tau, \quad \forall \omega \in \Omega.$$

It is not difficult to find that $z(\vartheta_t \omega)$ is the stationary solution of the following Ornstein-Uhlenbeck equation

$$dz(\vartheta_t \omega) + z(\vartheta_t \omega) dt = d\omega.$$ 

What’s more, $\Omega$ has a $\vartheta$-invariant subset $\tilde{\Omega}$. This ensures that $z(\vartheta_t \omega)$ is continuous. Meanwhile, $|z(\cdot)|$ is tempered satisfying the properties [1, 2, 12]

$$\lim_{t \to \pm \infty} \frac{|z(\vartheta_t \omega)|}{|t|} = 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t z(\vartheta_s \omega) ds = 0.$$

After this, we will assume that $\tilde{\Omega}$ equals $\Omega$. Let

$$u(t) = e^{-\alpha z(\vartheta_t \omega)}\bar{u}(t),$$
$$v(t) = e^{-\alpha z(\vartheta_t \omega)}\bar{v}(t),$$
$$w(t) = e^{-\alpha z(\vartheta_t \omega)}\bar{w}(t),$$
$$y(t) = e^{-\alpha z(\vartheta_t \omega)}\bar{y}(t),$$

system (1)-(3) can be written as

$$\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u + (F + k - \alpha z(\vartheta_t \omega))u - e^{2\alpha z(\vartheta_t \omega)}u^2v - D_1(u - w) &= e^{-\alpha z(\vartheta_t \omega)}\sigma, \\
\frac{\partial v}{\partial t} - d_2 \Delta v - Fe^{-\alpha z(\vartheta_t \omega)} + (F - \alpha z(\vartheta_t \omega))v + e^{2\alpha z(\vartheta_t \omega)}u^2v - D_2(y - v) &= e^{-\alpha z(\vartheta_t \omega)}\sigma, \\
\frac{\partial w}{\partial t} - d_1 \Delta w + (F + k - \alpha z(\vartheta_t \omega))w - e^{2\alpha z(\vartheta_t \omega)}w^2y - D_1(u - w) &= e^{-\alpha z(\vartheta_t \omega)}\sigma, \\
\frac{\partial y}{\partial t} - d_2 \Delta y - Fe^{-\alpha z(\vartheta_t \omega)} + (F - \alpha z(\vartheta_t \omega))y + e^{2\alpha z(\vartheta_t \omega)}w^2y - D_2(v - y) &= e^{-\alpha z(\vartheta_t \omega)}\sigma,
\end{align*}$$

with following conditions

$$u(x, 0) = u_0(x), \; v(x, 0) = v_0(x), \; w(x, 0) = w_0(x), \; y(x, 0) = y_0(x),$$
$$\frac{\partial u}{\partial \nu}(x, t) = \frac{\partial v}{\partial \nu}(x, t) = \frac{\partial w}{\partial \nu}(x, t) = \frac{\partial y}{\partial \nu}(x, t) = 0, \quad t > 0, \; x \in \partial \mathcal{O}. \quad (10)$$
Let $g = (u, v, w, y)^T$, system (9)-(10) can be rewritten as

\[
\begin{align*}
\frac{\partial g}{\partial t} - Ag + H(g, \omega) &= e^{-\alpha z(\omega)} \sigma(x, t), \quad t > 0, \\
g(x, 0) &= g_0(x), \quad x \in \mathcal{O}, \\
\frac{\partial g}{\partial \nu}(x, t) &= 0, \quad x \in \partial \mathcal{O},
\end{align*}
\]

where

\[
\tilde{\sigma}(x, t) = (\sigma(x, t), \sigma(x, t), \sigma(x, t), \sigma(x, t))^T,
\]

\[
A = \begin{pmatrix}
  d_1 \Delta & 0 & 0 & 0 \\
  0 & d_2 \Delta & 0 & 0 \\
  0 & 0 & d_1 \Delta & 0 \\
  0 & 0 & 0 & d_2 \Delta
\end{pmatrix},
\]

\[
H(g, \omega) = \begin{pmatrix}
(F + k - \alpha z(\partial_i \omega))u - e^{2\alpha z(\partial_i \omega)}u^2v - D_1(w - u) \\
-Fe^{-\alpha z(\partial_i \omega)} + (F - \alpha z(\partial_i \omega))v + e^{2\alpha z(\partial_i \omega)}u^2v - D_2(y - v) \\
(F + k - \alpha z(\partial_i \omega))w - e^{2\alpha z(\partial_i \omega)}u^2y - D_1(u - w) \\
-Fe^{-\alpha z(\partial_i \omega)} + (F - \alpha z(\partial_i \omega))y + e^{2\alpha z(\partial_i \omega)}u^2y - D_2(v - y)
\end{pmatrix},
\]

here $T$ denotes the transposition.

In [4, 19], we can find a standard method of solving system (9). Clearly, when $g_0 \in [L^2(\mathcal{O})]^4$, system (9) exists a unique solution $g(\cdot, \omega, \sigma, g_0) \in C([0, \infty); [L^2(\mathcal{O})]^4)$ and $L^2((0, \infty); [H^1(\mathcal{O})]^4)$, where $g(0, \omega, \sigma, g_0) = g_0$.

For every $t > 0, \omega \in \Omega, \sigma \in \Sigma$ and $\bar{g}_0 \in [L^2(\mathcal{O})]^4$, define

\[
\phi(t, \omega, \sigma, \bar{g}_0) = g(t, \omega, \sigma, e^{-\alpha z(\omega)}g_0)e^{\alpha z(\partial_i \omega)}.
\]

Clearly, $\phi(t, \omega, \sigma, \bar{g}_0)$ is the solution of (1) at time $t$ with initial data $\bar{g}_0$ (at time $t = 0$) satisfying Definition 2.5. Therefore, the NRDS defined by (8) is continuous in initial data and symbols.

We can investigate the tempered uniform and cocycle attractors of (1). Define an attraction universe $\mathcal{D}$,

\[
\mathcal{D} = \{ D : D \text{ is a bounded stochastic set in } [L^2(\mathcal{O})]^4 \text{ satisfying} \}
\]

\[
\lim_{t \to \infty} e^{-Ft} ||D(\partial_i \omega)||^2 = 0, \forall \omega \in \Omega.
\]

It is easy to find that $\mathcal{D}$ is inclusion-closed and neighborhood-closed.

4. Uniform estimates of solutions. In this part, we can estimate uniformly the solutions of (9). Based on [19], let

\[
\begin{align*}
Y_1(x, t) &= u(x, t) + v(x, t) + w(x, t) + y(x, t), \quad Y_{1,0} = u_0 + v_0 + w_0 + y_0, \\
Y_2(x, t) &= u(x, t) + w(x, t), \quad Y_{2,0} = u_0 + w_0, \\
Y_3(x, t) &= u(x, t) + v(x, t) - w(x, t) - y(x, t), \quad Y_{3,0} = u_0 + v_0 - w_0 - y_0, \\
Y_4(x, t) &= v(x, t) - y(x, t), \\
Y_5(x, t) &= u(x, t) - w(x, t).
\end{align*}
\]
Lemma 4.1. For any $D \in \mathcal{D}$ and $\omega \in \Omega$, we can find a time $T = T(D, \omega) > 1$ such that, for any $\sigma \in \Sigma$,

$$
\|g(t, \vartheta - t\omega, \theta - t\sigma, g_0)\|^2 \\
\leq \left(31 + \frac{4(d_1 - d_2)^2}{d_1d_2} + \frac{2(F + k + 2D_1) + 2k(2D_1 - 2D_2)^2}{k\gamma d_2(F + k + 2D_1)}\right) \\
\left(2F|\mathcal{O}| \int_{-\infty}^{0} e^{\int_{0}^{s} 2\sigma(\theta, \omega) d\tau - 2\alpha(\theta, \omega) ds} + \frac{2}{F} \int_{-\infty}^{0} e^{\int_{0}^{s} 2\sigma(\theta, \omega) d\tau - 2\alpha(\theta, \omega) ||\sigma(s)||^2 ds} + 1\right)
$$

holds uniformly in $g_0 \in D$ and $t \geq T$.

Proof. For (9), taking the inner products $((\partial v/\partial t), v)$ and $((\partial y/\partial t), y)$, and then summing up the results, we get

$$
\frac{1}{2} \frac{d}{dt}(\|v(t, \omega, \sigma, g_0)\|^2 + \|y(t, \omega, \sigma, g_0)\|^2) + d_2(\|\nabla v\|^2 + \|\nabla y\|^2) \\
= F \int_{\mathcal{O}} e^{-\alpha(\theta, \omega)}(v + y)dx + \int_{\mathcal{O}} (\alpha z(\theta, \omega) - F)(v^2 + y^2)dx \\
- \int_{\mathcal{O}} e^{2\alpha(\theta, \omega)}(u^2v^2 + w^2y^2)dx - \int_{\mathcal{O}} D_2(y - v)^2 dx \\
+ \int_{\mathcal{O}} e^{-\alpha(\theta, \omega)}\sigma v dx + \int_{\mathcal{O}} e^{-\alpha(\theta, \omega)}\sigma y dx
$$

$$
\leq F \int_{\mathcal{O}} e^{-\alpha(\theta, \omega)}(v + y)dx + \int_{\mathcal{O}} (\alpha z(\theta, \omega) - F)(v^2 + y^2)dx \\
+ \int_{\mathcal{O}} e^{-\alpha(\theta, \omega)}\sigma v dx + \int_{\mathcal{O}} e^{-\alpha(\theta, \omega)}\sigma y dx
$$

$$
\leq (\alpha z(\theta, \omega) - \frac{F}{2})(\|v\|^2 + \|y\|^2) - \frac{F}{2} \int_{\mathcal{O}} (v - e^{-\alpha(\theta, \omega)})^2 dx \\
- \frac{F}{2} \int_{\mathcal{O}} (y - e^{-\alpha(\theta, \omega)})^2 dx + F|\mathcal{O}|e^{-2\alpha(\theta, \omega)} + \frac{1}{F} e^{-2\alpha(\theta, \omega)} ||\sigma||^2 \\
+ \frac{F}{2}(\|v\|^2 + \|y\|^2)
$$

$$
\leq \alpha z(\theta, \omega)(\|v\|^2 + \|y\|^2) + F|\mathcal{O}|e^{-2\alpha(\theta, \omega)} + \frac{1}{F} e^{-2\alpha(\theta, \omega)} ||\sigma||^2. \quad (12)
$$

Therefore,

$$
\frac{d}{dt}(\|v(t, \omega, \sigma, g_0)\|^2 + \|y(t, \omega, \sigma, g_0)\|^2) + 2d_2(\|\nabla v\|^2 + \|\nabla y\|^2) \\
\leq 2\alpha z(\theta, \omega)(\|v\|^2 + \|y\|^2) + 2F|\mathcal{O}|e^{-2\alpha(\theta, \omega)} + \frac{2}{F} e^{-2\alpha(\theta, \omega)} ||\sigma||^2. \quad (13)
$$
Using Gronwall’s inequality to obtain
\[
\begin{align*}
&\|v(t, \omega, \sigma, g_0)\|^2 + \|y(t, \omega, \sigma, g_0)\|^2 \\
&\leq e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} (\|v_0\|^2 + \|y_0\|^2) \\
&+ 2F|\Omega| e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \|v\|^2 \, ds \\
&+ \frac{2}{F} e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \|\sigma(s)\|^2 \, ds \\
&- 2d_2 e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} (\|\nabla v\|^2 + \|\nabla y\|^2) \, ds. \quad (14)
\end{align*}
\]
Replacing \(\omega\) and \(\sigma\) with \(\vartheta - \omega\) and \(\vartheta - \sigma\) to get
\[
\begin{align*}
&\|v(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 + \|y(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
&\leq e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} (\|v_0\|^2 + \|y_0\|^2) \\
&+ 2F|\Omega| e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \|v\|^2 \, ds \\
&+ \frac{2}{F} e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \|\sigma(s)\|^2 \, ds \\
&- 2d_2 e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} (\|\nabla v\|^2 + \|\nabla y\|^2) \, ds. \quad (15)
\end{align*}
\]
Since \(g_0(\vartheta - \omega) \in D(\vartheta - \omega)\), by the tempered properties, we have
\[
\begin{align*}
&\lim_{t \to +\infty} e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} (\|v_0(\vartheta - \omega)\|^2 + \|y_0(\vartheta - \omega)\|^2) \\
&= \lim_{t \to +\infty} e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} (\|v_0(\vartheta - \omega)\|^2 + \|y_0(\vartheta - \omega)\|^2) = 0, \quad (16)
\end{align*}
\]
\[
\begin{align*}
2F|\Omega| e^{\int_0^t 2\alpha z(\theta, \omega) \, d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \, ds \\
&= 2F|\Omega| \int_0^t e^{\int_0^\tau 2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \, ds \\
&= 2F|\Omega| \int_{-t}^0 e^{\int_0^\tau 2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \, ds \\
&\leq 2F|\Omega| \int_{-\infty}^0 e^{\int_0^\tau 2\alpha z(\theta, \omega) \, d\tau -2\alpha z(\theta, \omega) \, d\tau} \, ds \\
&< +\infty, \quad (17)
\end{align*}
\]
Therefore,

\[
2 \int_{-\infty}^{0} e^{\int_{0}^{t} 2\alpha z(\theta \omega) \, d\tau} \int_{0}^{t} e^{\int_{0}^{\tau} 2\alpha z(\theta \omega) \, d\tau} - 2\alpha z(\theta \omega) \|\sigma(s-t)\|^2 \, ds
\]

\[
= 2 \int_{-\infty}^{0} e^{\int_{0}^{t} 2\alpha z(\theta \omega) \, d\tau} - 2\alpha z(\theta \omega) \|\sigma(s-t)\|^2 \, ds
\]

\[
= 2 \int_{-\infty}^{0} e^{\int_{0}^{t} 2\alpha z(\theta \omega) \, d\tau} - 2\alpha z(\theta \omega) \|\sigma(s)\|^2 \, ds
\]

\[
\leq 2 \int_{-\infty}^{0} e^{\int_{0}^{t} 2\alpha z(\theta \omega) \, d\tau} - 2\alpha z(\theta \omega) \|\sigma(s)\|^2 \, ds.
\]  

Thus, there is a \( T = T(\omega, D) > 1 \) such that, for any \( t \geq T \),

\[
\|v(t, \vartheta \omega, \theta \omega, k)\|^2 + \|y(t, \vartheta \omega, \theta \omega, g)\|^2
\]

\[
\leq 1 + 2F|\mathcal{O}| \int_{-\infty}^{0} e^{\int_{0}^{t} 2\alpha z(\theta \omega) \, d\tau} - 2\alpha z(\theta \omega) \|\sigma(s)\|^2 \, ds
\]

\[
+ 2 \int_{-\infty}^{0} e^{\int_{0}^{t} 2\alpha z(\theta \omega) \, d\tau} - 2\alpha z(\theta \omega) \|\sigma(s)\|^2 \, ds = p_0^2(\omega).
\]  

By (9), we conclude

\[
\frac{\partial Y_1}{\partial t} - d_1 \Delta Y_1 + (F + k - \alpha z(\theta \omega)) Y_1 + (d_1 - d_2) \Delta (v + y)
\]

\[
- k(v + y) - 2Fe^{-\alpha z(\theta \omega)} = 4e^{-\alpha z(\theta \omega)}
\]

Taking the inner product \((\partial Y_1/\partial t, Y_1)\), then using Hölder’s inequality, Young’s inequality and Poincaré’s inequality (4) to get

\[
\frac{1}{2} \frac{d}{dt} \|Y_1\|^2 + d_1 \|\nabla Y_1\|^2 + (F + k - \alpha z(\theta \omega)) \|Y_1\|^2
\]

\[
= \int_{\mathcal{O}} (d_2 - d_1) \Delta (v + y) Y_1 \, dx + k \int_{\mathcal{O}} (v + y) Y_1 \, dx
\]

\[
+ 2F \int_{\mathcal{O}} e^{-\alpha z(\theta \omega)} Y_1 \, dx + 4 \int_{\mathcal{O}} e^{-\alpha z(\theta \omega)} \sigma Y_1 \, dx
\]

\[
\leq |d_1 - d_2| \|\nabla (v + y)\| \|\nabla Y_1\| + k \|v + y\| \|Y_1\| + 2F|\mathcal{O}| \frac{1}{2} \|
\]

\[
\|Y_1\|^2 + 4 \int_{\mathcal{O}} e^{-\alpha z(\theta \omega)} \sigma Y_1 \, dx
\]

\[
\leq \frac{d_1}{2} \|\nabla Y_1\|^2 + \left(\frac{d_1 - d_2}{2d_1}\right) \|\nabla (v + y)\|^2 + \frac{k}{2} \|Y_1\|^2 + \frac{1}{2k \gamma} \|\nabla (v + y)\|^2
\]

\[
+ F \frac{1}{2} \|Y_1\|^2 + 2F|\mathcal{O}| e^{-2\alpha z(\theta \omega)} + \frac{8}{F} e^{-2\alpha z(\theta \omega)} \|\sigma\|^2.
\]  

Therefore,

\[
\frac{d}{dt} \|Y_1\|^2 + d_1 \|\nabla Y_1\|^2 \leq 2\alpha z(\theta \omega) \|Y_1\|^2 + \left(\frac{d_1 - d_2}{d_1} + \frac{1}{k \gamma}\right) \|\nabla (v + y)\|^2
\]

\[
+ 4F|\mathcal{O}| e^{-2\alpha z(\theta \omega)} + \frac{16}{F} e^{-2\alpha z(\theta \omega)} \|\sigma\|^2.
\]
By Gronwall’s inequality to get

$$\|Y_1(t, \omega, \sigma, g_0)\|^2 \leq e^{\int_0^t 2\alpha z(\vartheta(t))\frac{d\tau}{d\tau}}\|Y_{1,0}\|^2$$

$$+ \left( \frac{(d_1 - d_2)^2}{d_1} + \frac{1}{k\gamma} \right) e^{\int_0^t 2\alpha z(\vartheta(t))\frac{d\tau}{d\tau}} \int_0^t e^{\int_s^t -2\alpha z(\vartheta(t))\frac{d\tau}{d\tau}} \|\nabla (v + y)\|^2 ds$$

$$+ 4F|\mathcal{O}|e^{\int_0^t 2\alpha z(\vartheta(t))\frac{d\tau}{d\tau}} \int_0^t e^{\int_s^t -2\alpha z(\vartheta(t))\frac{d\tau}{d\tau} - 2\alpha z(\vartheta(s))\frac{d\tau}{d\tau}} \|\sigma(s)\|^2 ds$$

$$- d_1 e^{\int_0^t 2\alpha z(\vartheta(t))\frac{d\tau}{d\tau}} \int_0^t e^{\int_s^t -2\alpha z(\vartheta(t))\frac{d\tau}{d\tau}} \|\nabla Y_1\|^2 ds. \quad (23)$$

Replacing \( \omega \) and \( \sigma \) with \( \vartheta - \omega \) and \( \vartheta - \sigma \), we have

$$\|Y_1(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \leq e^{\int_0^t 2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}}\|Y_{1,0}\|^2$$

$$+ \left( \frac{(d_1 - d_2)^2}{d_1} + \frac{1}{k\gamma} \right) e^{\int_0^t 2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} \int_0^t e^{\int_s^t -2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} \|\nabla (v + y)\|^2 ds$$

$$+ 4F|\mathcal{O}|e^{\int_0^t 2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} \int_0^t e^{\int_s^t -2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau} - 2\alpha z(\vartheta - \omega(s))\frac{d\tau}{d\tau}} \|\sigma(s - t)\|^2 ds$$

$$- d_1 e^{\int_0^t 2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} \int_0^t e^{\int_s^t -2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} \|\nabla Y_1\|^2 ds. \quad (24)$$

Similar to (16), for all \( g_0(\vartheta - \omega) \in D(\vartheta - \omega) \),

$$\lim_{t \to +\infty} e^{\int_0^t 2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} \|Y_{1,0}(\vartheta - \omega)\|^2 = \lim_{t \to +\infty} e^{\int_t^0 2\alpha z(\vartheta(t))\frac{d\tau}{d\tau}} \|Y_{1,0}(\vartheta - \omega)\|^2 = 0. \quad (25)$$

By (15)-(19), for any \( t \geq T \),

$$2d_2 e^{\int_0^t 2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} \int_0^t e^{\int_s^t -2\alpha z(\vartheta - \omega(t))\frac{d\tau}{d\tau}} ||\nabla v||^2 ds \leq \rho_0^2(\omega). \quad (26)$$
Hence, for any $t \geq T$, setting $c_1 = \left(\frac{(d_1 - d_2)^2}{d_1} + \frac{1}{\kappa \gamma d_2}\right)$,

\[
\left(\frac{(d_1 - d_2)^2}{d_1} + \frac{1}{\kappa \gamma}\right) e^{\int_0^t \alpha z(\vartheta_{r-\tau} \omega) d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\vartheta_{r-\tau} \omega) d\tau} \\
\|\nabla (v + y)(s, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 ds \\
\leq 2 \left(\frac{(d_1 - d_2)^2}{d_1} + \frac{1}{\kappa \gamma}\right) e^{\int_0^t \alpha z(\vartheta_{r-\tau} \omega) d\tau} \int_0^t e^{\int_0^\tau -2\alpha z(\vartheta_{r-\tau} \omega) d\tau} \\
(\|\nabla v(s, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 + \|\nabla y(s, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2) ds \\
\leq c_1 \rho_0^2(\omega). \tag{27}
\]

By (17)-(18), (24) and (27), we obtain

\[
\|Y_1(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 \leq (c_1 + 10) \rho_0^2(\omega). \tag{28}
\]

Therefore, in together with (19) and (28), we have

\[
\|Y_2(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 \\
= \|Y_1(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0) - (v + y)(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 \\
\leq 2(\|Y_1(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 + \|(v + y)(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2) \\
\leq 2(\|Y_1(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 + \|v(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2 + \|y(t, \vartheta_{-r} \omega, \theta_{-r} \sigma, g_0)\|^2) \\
\leq (2c_1 + 24) \rho_0^2(\omega). \tag{29}
\]

By (9), we get

\[
\frac{\partial Y_3}{\partial t} = d_1 \Delta Y_3 - (F + k - \alpha z(\vartheta_{i} \omega))Y_3 + 2D_1(w - u) + 2D_2(y - v) + k(v - y) \\
+ (d_2 - d_1) \Delta (v - y) \\
= d_1 \Delta Y_3 - (F + k + 2D_1 - \alpha z(\vartheta_{i} \omega))Y_3 + (k + 2D_1 - 2D_2)(v - y) \\
+ (d_2 - d_1) \Delta (v - y). \tag{30}
\]

Taking the inner product $(\partial Y_3/\partial t, Y_3)$, then using Hölder’s inequality, Young’s inequality and Poincaré’s inequality (4) to obtain

\[
\frac{1}{2} \frac{d}{dt}\|Y_3\|^2 + d_1 \|\nabla Y_3\|^2 + (F + k + 2D_1 - \alpha z(\vartheta_{i} \omega))\|Y_3\|^2 \\
= \int_{\mathcal{O}} (k + 2D_1 - 2D_2)(v - y)Y_3 dx - \int_{\mathcal{O}} (d_2 - d_1) \nabla(v - y) \nabla Y_3 dx \\
\leq |k + 2D_1 - 2D_2| \|v - y\|\|Y_3\| + |d_2 - d_1| \|\nabla(v - y)\| \|\nabla Y_3\| \\
\leq \left(\frac{(d_1 - d_2)^2}{2d_1} + \frac{(k + 2D_1 - 2D_2)^2}{2\gamma (F + k + 2D_1)}\right) \|\nabla(v - y)\|^2 + \frac{d_1}{2} \|\nabla Y_3\|^2 \\
+ (F + k + 2D_1) \|Y_3\|^2 \\
\leq 2 \left(\frac{(d_1 - d_2)^2}{2d_1} + \frac{(k + 2D_1 - 2D_2)^2}{2\gamma (F + k + 2D_1)}\right) \|\nabla v\|^2 + \|\nabla y\|^2 + \frac{d_1}{2} \|\nabla Y_3\|^2 \\
+ (F + k + 2D_1) \|Y_3\|^2. \tag{31}
\]

Hence,

\[
\frac{d}{dt}\|Y_3\|^2 + d_1 \|\nabla Y_3\|^2 \leq 2\alpha z(\vartheta_{i} \omega)\|Y_3\|^2 + c_2(\|\nabla v\|^2 + \|\nabla y\|^2), \tag{32}
\]
where \( c_2 = \left( \frac{(2d_1 - d_2)^2}{d_1} + \frac{2(k + 2D_1 - 2D_2)^2}{g(F + k + 2D_1)} \right) \). Applying Gronwall’s inequality, we get

\[
\|Y_3(t, \omega, \sigma, \vartheta)\|^2 \leq e^{\int_0^t 2\alpha \omega(\vartheta, \omega) \, d\tau} \|Y_{3,0}\|^2 \\
+ c_2 e^{\int_0^t \frac{2\alpha \omega(\vartheta, \omega)}{d_1} \, d\tau} \int_0^t e^{\int_0^s -2\alpha(\vartheta, \omega) \, d\tau} (\|\nabla v\|^2 + \|\nabla y\|^2) \, ds \\
- d_1 e^{\int_0^t \frac{2\alpha \omega(\vartheta, \omega)}{d_1} \, d\tau} \int_0^t e^{\int_0^s -2\alpha(\vartheta, \omega) \, d\tau} \|\nabla Y_3\|^2 \, ds. \tag{33}
\]

Replacing \( \omega \) and \( \sigma \) with \( \vartheta - \omega \) and \( \vartheta - \sigma \), we conclude

\[
\|Y_3(t, \vartheta - \omega, \vartheta - \sigma, \vartheta, g_0)\|^2 \\
\leq e^{\int_0^t 2\alpha \omega(\vartheta - \omega) \, d\tau} \|Y_{3,0}\|^2 \\
+ c_2 e^{\int_0^t \frac{2\alpha \omega(\vartheta - \omega)}{d_1} \, d\tau} \int_0^t e^{\int_0^s -2\alpha(\vartheta - \omega) \, d\tau} (\|\nabla v(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
+ \|\nabla y(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2) \, ds \\
- d_1 e^{\int_0^t \frac{2\alpha \omega(\vartheta - \omega)}{d_1} \, d\tau} \int_0^t e^{\int_0^s -2\alpha(\vartheta - \omega) \, d\tau} \|\nabla Y_3(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \, ds. \tag{34}
\]

Similar to (16), for all \( g_0(\vartheta - \omega) \in D(\vartheta - \omega) \),

\[
\lim_{t \to +\infty} e^{\int_0^t 2\alpha \omega(\vartheta - \omega) \, d\tau} \|Y_{3,0}(\vartheta - \omega)\|^2 = \lim_{t \to +\infty} e^{\int_0^t 2\alpha \omega(\vartheta - \omega) \, d\tau} \|Y_{3,0}(\vartheta - \omega)\|^2 = 0, \tag{35}
\]

and

\[
c_2 e^{\int_0^t \frac{2\alpha \omega(\vartheta - \omega)}{d_1} \, d\tau} \int_0^t e^{\int_0^s -2\alpha(\vartheta - \omega) \, d\tau} (\|\nabla v(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
+ \|\nabla y(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2) \, ds \leq \frac{c_2}{2d_2} \rho_0^2(\omega). \tag{36}
\]

By (34)-(36), for any \( t \geq T \), setting \( c_3 = \frac{c_2}{2d_2} + 1 \), we obtain that

\[
\|Y_3(t, \vartheta - \omega, \vartheta - \sigma, \vartheta, g_0)\|^2 \leq c_3 \rho_0^2(\omega). \tag{37}
\]

It follows from (19) and (37) that,

\[
\|Y_3(t, \vartheta - \omega, \vartheta - \sigma, \vartheta, g_0)\|^2 \\
= \|Y_3(t, \vartheta - \omega, \vartheta - \sigma, g_0) - Y_4(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
\leq 2(\|Y_3(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 + \|Y_4(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2) \\
\leq 2\|Y_3(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 + 4(\|v(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 + \|y(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2) \\
\leq c_4 \rho_0^2(\omega). \tag{38}
\]

with \( c_4 = 2c_3 + 4 \). Thus, for any \( t \geq T \),

\[
\|u(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 + \|w(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
= \frac{1}{2} \|Y_2(t, \vartheta - \omega, \vartheta - \sigma, g_0) + Y_6(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
+ \frac{1}{2} \|Y_2(t, \vartheta - \omega, \vartheta - \sigma, g_0) - Y_6(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
\leq \|Y_2(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 + \|Y_6(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|^2 \\
\leq c_5 \rho_0^2(\omega), \tag{39}
\]
with \( c_5 = 2c_1 + 24 + c_4 \). Finally, we know that, for any \( t \geq T \),

\[
\|g(t, \theta, \sigma, y_0)\|^2 = \|u(t, \theta, \sigma, y_0)\|^2 + \|v(t, \theta, \sigma, y_0)\|^2 + \|w(t, \theta, \sigma, y_0)\|^2 + \|y(t, \theta, \sigma, y_0)\|^2 \\
\leq (c_5 + 1)\rho_0(Y)
\]

\[
= \left(31 + \frac{4(d_1 - d_2)^2}{d_1d_2} + \frac{2(F + k + 2D_1 + 2k(F + k + 2D_1 - 2D_2)^2)}{k\gamma d_2}\right)
\]

\[
\left(2F|\mathcal{O}| \int_{-\infty}^{0} e^{\int_0^s 2\alpha z(\theta, \omega)\text{d}t - 2\alpha z(\theta, \omega)}\text{d}s + \frac{2}{F} \int_{-\infty}^{0} e^{\int_0^s 2\alpha z(\theta, \omega)\text{d}t - 2\alpha z(\theta, \omega)}\|\sigma(s)\|^2 \text{d}s \right.
\]

\[
+ 1\). \quad (40)
\]

In this way, the proof is completed. \(\square\)

**Lemma 4.2.** For any \( D \in \mathcal{D} \) and \( \omega \in \Omega \), we can find a time \( T = T(D, \omega) > 1 \) such that, for any \( \sigma \in \Sigma \),

\[
\|v(t, \theta, \sigma, y_0)\|_{L^6}^6 + \|y(t, \theta, \sigma, y_0)\|_{L^6}^6 \\
\leq 1 + 2F|\mathcal{O}| \int_{-\infty}^{0} e^{\int_0^s 2\alpha z(\theta, \omega)\text{d}t - 6\alpha z(\theta, \omega)}\text{d}s \\
+ \frac{18}{F} \int_{-\infty}^{0} e^{\int_0^s 2\alpha z(\theta, \omega)\text{d}t - 6\alpha z(\theta, \omega)}\|\sigma(s)\|^6 \text{d}s
\]

holds uniformly in \( y_0 \in D \) and \( t \geq T \).

**Proof.** For (9), taking the inner products \((\partial v/\partial t), v^5\) and \((\partial y/\partial t), y^5\), and then summing up the results, we get

\[
\frac{1}{6} \frac{d}{dt} (\|v(t, \omega, \sigma, y_0)\|_{L^6}^6 + \|y(t, \omega, \sigma, y_0)\|_{L^6}^6) + 5d_2(\|v^2\nabla v\|^2 + \|y^2\nabla y\|^2)
\]

\[
= F \int_\Omega (e^{-\alpha z(\theta, \omega)}(v^5 + y^5) - (v^6 + y^6)) dx + \int_\Omega \alpha z(\theta, \omega)(v^6 + y^6) dx
\]

\[
- \int_\Omega e^{2\alpha z(\theta, \omega)}(u^2v^6 + w^2y^6) dx + D_2 \int_\Omega (y - v)(v^5 - y^5) dx
\]

\[
+ \int_\Omega e^{-\alpha z(\theta, \omega)}\sigma(v^5 + y^5) dx. \quad (41)
\]

Applying Young’s inequality to obtain

\[
F \int_\Omega (e^{-\alpha z(\theta, \omega)}(v^5 + y^5) - (v^6 + y^6)) dx
\]

\[
\leq F \int_\Omega \left( \frac{5}{6}(v^6 + y^6) + \frac{1}{3} e^{-6\alpha z(\theta, \omega)} - (v^6 + y^6) \right) dx
\]

\[
\leq -\frac{F}{6} (\|v\|_{L^6}^6 + \|y\|_{L^6}^6) + \frac{F}{3} |\mathcal{O}| e^{-6\alpha z(\theta, \omega)}, \quad (42)
\]
\[
D_2 \int_\mathcal{O} (v - y)(v^5 - y^5)dx \\
= D_2 \int_\mathcal{O} (-v^6 - y^6 + vy^5 + vy^5)dx \\
\leq D_2 \int_\mathcal{O} (-v^6 - y^6 + \frac{1}{6}y^6 + \frac{5}{6}v^6 + (\frac{1}{6}v^6 + \frac{5}{6}y^6))dx \\
= 0, \\
\] 

and

\[
\int_\mathcal{O} e^{-\alpha z(\vartheta, \omega)}(v^5 + y^5)dx \leq \frac{F}{6}(\|v\|_{L^6}^6 + \|y\|_{L^6}^6) + \frac{3}{F}e^{-6\alpha z(\vartheta, \omega)}\|\sigma\|_{L^6}^6. \\
\]

Therefore,

\[
d\frac{d}{dt}(\|v(t, \omega, \sigma, g_0)\|_{L^6}^6 + \|y(t, \omega, \sigma, g_0)\|_{L^6}^6) \leq 6\alpha z(\vartheta(t, \omega)) (\|v_0\|_{L^6}^6 + \|y_0\|_{L^6}^6) + 2F|\mathcal{O}|e^{-6\alpha z(\vartheta, \omega)} + \frac{18}{F}e^{-6\alpha z(\vartheta, \omega)}\|\sigma\|_{L^6}^6. \\
\]

By Gronwall’s inequality to get

\[
\|v(t, \omega, \sigma, g_0)\|_{L^6}^6 + \|y(t, \omega, \sigma, g_0)\|_{L^6}^6 \leq e^{\int_0^t 6\alpha z(\vartheta, \omega)d\tau}(\|v_0\|_{L^6}^6 + \|y_0\|_{L^6}^6) + 2F|\mathcal{O}|e^{\int_0^t 6\alpha z(\vartheta, \omega)d\tau - 6\alpha z(\vartheta, \omega)}\|\sigma\|_{L^6}^6 \\
+ \frac{18}{F}e^{\int_0^t 6\alpha z(\vartheta, \omega)d\tau - 6\alpha z(\vartheta, \omega)}\|\sigma\|_{L^6}^6ds. \\
\]

Replacing \( \omega \) and \( \sigma \) with \( \vartheta - \omega \) and \( \vartheta - \sigma \), we have

\[
\|v(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|_{L^6}^6 + \|y(t, \vartheta - \omega, \vartheta - \sigma, g_0)\|_{L^6}^6 \leq e^{\int_0^t 6\alpha z(\vartheta - \omega)d\tau}(\|v_0\|_{L^6}^6 + \|y_0\|_{L^6}^6) + 2F|\mathcal{O}|e^{\int_0^t 6\alpha z(\vartheta - \omega)d\tau - 6\alpha z(\vartheta - \omega)}\|\sigma\|_{L^6}^6 \\
+ \frac{18}{F}e^{\int_0^t 6\alpha z(\vartheta - \omega)d\tau - 6\alpha z(\vartheta - \omega)}\|\sigma\|_{L^6}^6ds. \\
\]

Since \( g_0(\vartheta - \omega) \in D(\vartheta - \omega), \) by the tempered properties, there is a \( T = T(\omega, D) > 1, \) such that, for any \( t \geq T, \)

\[
\lim_{t \to +\infty} e^{\int_0^t 6\alpha z(\vartheta - \omega)d\tau}(\|v_0(\vartheta - \omega)\|_{L^6}^6 + \|y_0(\vartheta - \omega)\|_{L^6}^6) \\
= \lim_{t \to +\infty} e^{\int_0^t 6\alpha z(\vartheta + \omega)d\tau}(\|v_0(\vartheta + \omega)\|_{L^6}^6 + \|y_0(\vartheta + \omega)\|_{L^6}^6) = 0, \\
\]
\[2F|O| e^{\int_t^0 6\alpha_z(\theta_{-1}\omega) d\tau} \int_0^t e^{\int_0^s 6\alpha_z(\theta_{-1}\omega) d\tau - 6\alpha_z(\theta_{-1}\omega)} ds = 2F|O| e^{\int_t^0 6\alpha_z(\theta_{-1}\omega) d\tau - 6\alpha_z(\theta_{-1}\omega)} ds = 2F|O| e^{\int_{-t}^0 6\alpha_z(\theta_{-2}\omega) d\tau - 6\alpha_z(\theta_{-2}\omega)} ds \leq 2F|O| e^{\int_{-\infty}^0 6\alpha_z(\theta_{-2}\omega) d\tau - 6\alpha_z(\theta_{-2}\omega)} ds < +\infty, \quad (49)\]

and
\[\frac{18}{F} e^{\int_t^0 6\alpha_z(\theta_{-1}\omega) d\tau} \int_0^t e^{\int_0^s 6\alpha_z(\theta_{-1}\omega) d\tau - 6\alpha_z(\theta_{-1}\omega)} ||\sigma(s - t)||_L^6 ds = 18 \int_0^t e^{\int_0^s 6\alpha_z(\theta_{-1}\omega) d\tau - 6\alpha_z(\theta_{-1}\omega)} ||\sigma(s - t)||_L^6 ds = 18 \int_{-t}^0 e^{\int_0^s 6\alpha_z(\theta_{-2}\omega) d\tau - 6\alpha_z(\theta_{-2}\omega)} ||\sigma(s)||_L^6 ds \leq 18 \int_{-\infty}^0 e^{\int_0^s 6\alpha_z(\theta_{-2}\omega) d\tau - 6\alpha_z(\theta_{-2}\omega)} ||\sigma(s)||_L^6 ds. \quad (50)\]

Thus, there is a \( T = T(\omega, D) > 1 \) such that, for any \( t \geq T \),
\[||v(t, \theta_{-1}\omega, \theta_{-1}\sigma, g_0)||_L^6 + ||g(t, \theta_{-1}\omega, \theta_{-1}\sigma, g_0)||_L^6 \leq \rho_1(\omega), \quad (51)\]

with
\[\rho_1(\omega) \equiv 1 + 2F|O| e^{\int_{-\infty}^0 6\alpha_z(\theta_{-2}\omega) d\tau - 6\alpha_z(\theta_{-2}\omega)} ds + 18 \int_{-\infty}^0 e^{\int_0^s 6\alpha_z(\theta_{-2}\omega) d\tau - 6\alpha_z(\theta_{-2}\omega)} ||\sigma(s)||_L^6 ds. \quad (52)\]

This completes the proof. \( \square \)

**Lemma 4.3.** For any \( D \in \mathcal{D} \) and \( \omega \in \Omega \), we can find a time \( T = T(D, \omega) > 1 \) such that, for any \( \sigma \in \Sigma \),
\[\int_{t-1}^t ||\nabla g(s, \theta_{-1}\omega, \theta_{-1}\sigma, g_0)||_L^2 ds \leq \left( \frac{22}{d_1} + \frac{9}{2d_2} + \frac{4(d_1 - d_2)^2}{d_1 d_2} + \frac{2}{k\gamma d_1 d_2} + \frac{2(k + 2D_1 - 2D_2)^2}{\gamma d_1 d_2(F + k + 2D_1)} \right) e^{-2\alpha \max_{1 \leq r \leq 0} |z(\theta, \omega)|} \left( 2F|O| \int_{-\infty}^0 e^{\int_0^s 2\alpha z(\theta_{-1}\omega) d\tau - 2\alpha z(\theta_{-1}\omega)} ds + \frac{2}{F} \int_{-\infty}^0 e^{\int_0^s 2\alpha z(\theta_{-2}\omega) d\tau - 2\alpha z(\theta_{-2}\omega)} ||\sigma(s)||_L^2 ds + 1 \right)\]

holds uniformly in \( g_0 \in D \) and \( t \geq T \).
Proof. By (15)-(19), we conclude

\[
2d_2 e^{\int_0^t 2\alpha z(\partial s, \iota_0)\, dt} \int_0^t e^{\int_0^t -2\alpha z(\partial s, \iota_0)\, dt} \left( \|\nabla v(s, \partial s, \theta, g_0)\|^2 + \|\nabla y(s, \partial s, \theta, g_0)\|^2 \right) ds \leq \rho_0^2(\omega).
\]

As

\[
2d_2 e^{\int_0^t 2\alpha z(\partial s, \iota_0)\, dt} \int_0^t e^{\int_0^t -2\alpha z(\partial s, \iota_0)\, dt} \left( \|\nabla v(s, \partial s, \theta, g_0)\|^2 + \|\nabla y(s, \partial s, \theta, g_0)\|^2 \right) ds
\]

\[
\geq 2d_2 e^{\int_0^t 2\alpha z(\partial s, \iota_0)\, dt} \int_0^t \left( \|\nabla v(s, \partial s, \theta, g_0)\|^2 + \|\nabla y(s, \partial s, \theta, g_0)\|^2 \right) ds
\]

\[
\geq 2d_2 e^{-2\alpha \max_{1 \leq s \leq 0} |z(\partial s)|} \int_0^t \left( \|\nabla v(s, \partial s, \theta, g_0)\|^2 + \|\nabla y(s, \partial s, \theta, g_0)\|^2 \right) ds,
\]

therefore, for any \( t \geq T = T(\omega, D) > 1 \),

\[
\int_{t-1}^t \left( \|\nabla v(s, \partial s, \theta, g_0)\|^2 + \|\nabla y(s, \partial s, \theta, g_0)\|^2 \right) ds \leq c_6 \rho_0^2(\omega),
\]

where \( c_6 = \frac{1}{2d_2} e^{-2\alpha \max_{1 \leq s \leq 0} |z(\partial s)|} \). Similarly, applying (24)-(28) and (34)-(37), we have

\[
\int_{t-1}^t \|\nabla Y_1(s, \partial s, \theta, g_0)\|^2 ds \leq c_7 \rho_0^2(\omega),
\]

\[
\int_{t-1}^t \|\nabla Y_2(s, \partial s, \theta, g_0)\|^2 ds \leq c_8 \rho_0^2(\omega),
\]

\[
\int_{t-1}^t \|\nabla Y_3(s, \partial s, \theta, g_0)\|^2 ds \leq c_9 \rho_0^2(\omega),
\]

with \( c_7 = \frac{2\alpha + 10}{4d_1} e^{-2\alpha \max_{1 \leq s \leq 0} |z(\partial s)|} \), \( c_8 = \frac{Q^2}{8d_1} e^{-2\alpha \max_{1 \leq s \leq 0} |z(\partial s)|} \). Let \( c_9 = 2c_7 + 4c_6 \), and \( c_{10} = 2c_8 + 4c_6 \), we get

\[
\int_{t-1}^t \|\nabla Y_3(s, \partial s, \theta, g_0)\|^2 ds
\]

\[
= \int_{t-1}^t \|\nabla Y_1(s, \partial s, \theta, g_0) - \nabla (v + y)(s, \partial s, \theta, g_0)\|^2 ds
\]

\[
\leq 2 \int_{t-1}^t \|\nabla Y_1(s, \partial s, \theta, g_0)\|^2 ds + 4 \int_{t-1}^t \|\nabla v(s, \partial s, \theta, g_0)\|^2 ds + 4 \int_{t-1}^t \|\nabla y(s, \partial s, \theta, g_0)\|^2 ds
\]

\[
\leq c_{10} \rho_0^2(\omega),
\]
Thus, for any $t_{1,2}$ and $\varphi_t$, this completes the proof.

By (58)-(59), we obtain

\[
\int_{t_{1,2}}^t (\|\nabla u(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2 + \|\nabla v(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2)\,ds = \frac{1}{4} \int_{t_{1,2}}^t (\|\nabla Y_2(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0) + \nabla Y_5(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2
\]

\[
+ \|\nabla Y_2(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0) - \nabla Y_5(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2)ds
\]

\[
\leq \int_{t_{1,2}}^t \|\nabla Y_2(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2 ds + \int_{t_{1,2}}^t \|\nabla Y_5(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2 ds
\]

\[
\leq (c_9 + c_{10}) \rho_0^2(\omega).
\]

Thus, for any $t \geq T = T(\omega, D) > 1$,

\[
\int_{t_{1,2}}^t \|\nabla g(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2 ds
\]

\[
= \int_{t_{1,2}}^t (\|\nabla u(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2 + \|\nabla v(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2
\]

\[
+ \|\nabla w(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2 + \|\nabla y(s, \partial_{-\tau} \omega, \theta_{-\tau} \sigma, g_0)\|^2)ds
\]

\[
\leq (c_9 + c_9 + c_{10})\rho_0^2(\omega)
\]

\[
= \left(\frac{22}{d_1} + \frac{9}{2d_2} + \frac{4(d_1 - d_2)^2}{d_1^2 d_2} + \frac{2}{k\gamma d_1 d_2} + \frac{2(k + 2D_1 - 2D_2)^2}{\gamma d_1 d_2(F + k + 2D_1)}\right) e^{2\alpha \max_{-1 \leq \tau \leq 0} \sigma(\varphi_t, \omega)}
\]

\[
\left(2F|\mathcal{O}| \int_{-\infty}^0 e^{\int_0^\tau 2\alpha(z, \omega) d\tau - 2\alpha(z, \omega)} ds + \frac{2}{F} \int_{-\infty}^0 e^{\int_0^\tau 2\alpha(z, \omega) d\tau - 2\alpha(z, \omega)} \sigma(s)^2 ds + 1\right)
\]

\[
= \rho_2(\omega).
\]

This completes the proof.
Lemma 4.4. There is a random variable $\rho_3(\omega)$ such that, for any $D \in D$ and $\omega \in \Omega$, we can find a time $T = T(D, \omega) > 1$ such that, for any $\sigma \in \Sigma$,
\[ \| \nabla u(t, \theta_{-1}\omega, \theta_{-1}\sigma, g_0) \|^2 + \| \nabla w(t, \theta_{-1}\omega, \theta_{-1}\sigma, g_0) \|^2 \leq \rho_3(\omega) \]
holds uniformly in $g_0 \in D$ and $t \geq T$.

Proof. For (9), taking the inner products $((\partial u/\partial t), -\Delta u)$ and $((\partial w/\partial t), -\Delta w)$, and then summing up the results, we get
\[
\frac{1}{2} \frac{d}{dt} (\| \nabla u \|^2 + \| \nabla w \|^2) + d_1 (\| \Delta u \|^2 + \| \Delta w \|^2) \\
+ (F + k - \alpha z(\theta_1\omega))(\| \nabla u \|^2 + \| \nabla w \|^2)
\]
\[ = - \int_D e^{2\alpha z(\theta_1\omega)}(w^2 v \Delta u + w^2 y \Delta w) dx + D_1 \int_D (w - u)(\Delta w - \Delta u) dx
\]
\[ - e^{-\alpha z(\theta_1\omega)} \int_D \sigma \Delta u dx - e^{-\alpha z(\theta_1\omega)} \int_D \sigma \Delta w dx. \]

As
\[ \int_D (w - u)(\Delta w - \Delta u) dx = - \int_D (\nabla w - \nabla u)^2 dx \leq 0, \]
by Hölder’s inequality, Young’s inequality and (5), we have
\[
- \int_D e^{2\alpha z(\theta_1\omega)}(u^2 v \Delta u + w^2 y \Delta w) dx
\]
\[ \leq d_1 (\| \Delta u \|^2 + \| \Delta w \|^2) + \frac{1}{4d_1} e^{4\alpha z(\theta_1\omega)} \int_D (u^4 v^2 + w^4 y^2) dx
\]
\[ \leq d_1 (\| \Delta u \|^2 + \| \Delta w \|^2) + \frac{1}{4d_1} e^{4\alpha z(\theta_1\omega)}(\| u \|^4 \| v \|^2_{L^6} + \| w \|^4 \| y \|^2_{L^6})
\]
\[ \leq d_1 (\| \Delta u \|^2 + \| \Delta w \|^2) + \frac{1}{2d_1 \eta^2} e^{4\alpha z(\theta_1\omega)}[\| u \|^4 + \| \nabla u \|^4] \| v \|^2_{L^6}
\]
\[ + (\| u \|^4 + \| \nabla u \|^4) \| y \|^2_{L^6}, \]

and
\[
- e^{-\alpha z(\theta_1\omega)} \int_D \sigma \Delta u dx - e^{-\alpha z(\theta_1\omega)} \int_D \sigma \Delta w dx
\]
\[ = e^{-\alpha z(\theta_1\omega)} \int_D (\nabla u \nabla \sigma + \nabla w \nabla \sigma) dx
\]
\[ \leq F (\| \nabla u \|^2 + \| \nabla w \|^2) + \frac{1}{2F} e^{-2\alpha z(\theta_1\omega)} \| \nabla \sigma \|^2. \]

It follows from (62)-(65) that
\[
\frac{d}{dt} (\| \nabla u \|^2 + \| \nabla w \|^2) - 2\alpha z(\theta_1\omega)(\| \nabla u \|^2 + \| \nabla w \|^2) - \frac{1}{F} e^{-2\alpha z(\theta_1\omega)} \| \nabla \sigma \|^2
\]
\[ \leq \frac{1}{d_1 \eta^2} e^{4\alpha z(\theta_1\omega)}[\| u \|^4 + \| \nabla u \|^4] \| v \|^2_{L^6} + (\| u \|^4 + \| \nabla w \|^4) \| y \|^2_{L^6}
\]
\[ \leq \frac{1}{d_1 \eta^2} e^{4\alpha z(\theta_1\omega)}(\| u \|^2 + \| w \|^2)^2 (\| v \|^2_{L^6} + \| y \|^2_{L^6})
\]
\[ + \frac{1}{d_1 \eta^2} e^{4\alpha z(\theta_1\omega)}(\| \nabla u \|^2 + \| \nabla w \|^2)^2 (\| v \|^2_{L^6} + \| y \|^2_{L^6}). \]
Replacing which can be rewritten as

where

(39) and (51), for all

Applying uniform Gronwall inequality to estimate

\[
(\beta \rho) \sigma_0 + \mu \sigma_0 \leq \beta(s) \rho(s) + \mu(s),
\]

where

\[
\begin{align*}
\rho(s) &= \|\nabla u(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2 + \|\nabla w(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2, \\
\beta(s) &= \frac{1}{d_1 \eta^2} e^{4\alpha z(\vartheta_{-t} \omega)} (\|u(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2 + \|\vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0\|^2), \\
\mu(s) &= \frac{1}{d_1 \eta^2} e^{4\alpha z(\vartheta_{-t} \omega)} (\|u(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2 + \|w(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2).
\end{align*}
\]

Applying uniform Gronwall inequality to estimate \( \int_{t-1}^{t} \beta(s)ds \) and \( \int_{t-1}^{t} \mu(s)ds \). By (39) and (51), for all \( t \geq T \), we have

\[
\|u(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2 + \|w(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2
\]

\[
= c_0 \|u(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2 + \|w(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, \vartheta_{-t}g_0)\|^2
\]

\[
\leq c_0 \rho_0^2(\vartheta_{-t}\omega),
\]
and
\[\|v(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|_{L_6}^2 + \|y(s, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|_{L_6}^2 = \|v(s, \vartheta_{-s}\vartheta_{-t}\omega), \vartheta_{-s}\vartheta_{-t}\sigma, g_0)\|_{L_6}^2 + \|y(s, \vartheta_{-s}\vartheta_{-t}\omega), \vartheta_{-s}\vartheta_{-t}\sigma, g_0)\|_{L_6}^2 \leq c_{11}\rho_1^{1/3}(\vartheta_{-s}\vartheta_{-t}\omega), \] (72)

where \(c_{11}\) is a positive constant. By (71) and (72), we conclude
\[
\int_{t-1}^{t} \mu(s) ds \leq \int_{t-1}^{t} \left( \frac{c_{11}^2}{\alpha^2} e^{2\alpha|z(\vartheta_{-t}\omega)|} \rho_0^4(\vartheta_{-t}\omega) \rho_1^{1/3}(\vartheta_{-t}\omega) \right) ds
\]
\[+ \frac{1}{F} e^{2\alpha|z(\vartheta_{-t}\omega)|} \|\nabla s(t)\|^2) ds
\]
\[
\leq \frac{c_{11}^2}{d_{11}\eta^2} \max_{-1 \leq \tau \leq 0} \left[ e^{2\alpha|z(\vartheta_{-t}\omega)|} \rho_0(\vartheta_{-t}\omega) \rho_1^{1/3}(\vartheta_{-t}\omega) \right]
\]
\[+ \frac{1}{F} \max_{-1 \leq \tau \leq 0} \left[ e^{2\alpha|z(\vartheta_{-t}\omega)|} \|\nabla s(\tau)\|^2 \right]
\]
\[\equiv M_1(\omega). \] (73)

By Lemma 4.3, we get
\[
\int_{t-1}^{t} \beta(s) ds \leq \frac{c_{11}}{d_{11}\eta^2} \max_{-1 \leq \tau \leq 0} \left[ e^{4\alpha|z(\vartheta_{-t}\omega)|} \rho_1^{1/3}(\vartheta_{-t}\omega) \rho_2(\vartheta_{-t}\omega) \right] + 2\alpha \max_{-1 \leq \tau \leq 0} |z(\vartheta_{-t}\omega)|
\]
\[\equiv M_2(\omega). \] (74)

Hence,
\[
\|\nabla u(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 + \|\nabla w(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 \leq (M_1(\omega) + \rho_2(\omega)) e^{M_2(\omega)}
\]
\[\equiv \rho_3(\omega). \] (75)

The proof is completed. □

Lemma 4.5. There is a random variable \(\rho_4(\omega)\) such that, for any \(D \in D\) and \(\omega \in \Omega\), we can find a time \(T = T(D, \omega) > 1\) such that, for any \(\sigma \in \Sigma\),
\[
\|\nabla u(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 + \|\nabla y(t, \vartheta_{-t}\omega, \vartheta_{-t}\sigma, g_0)\|^2 \leq \rho_4(\omega)
\]
holds uniformly in \(g_0 \in D\) and \(t \geq T\).

Proof. For (9), taking the inner products ((\(\partial v/\partial t\), \(-\Delta v\)) and ((\(\partial y/\partial t\), \(-\Delta y\)), and then summing up the results, we get
\[
\frac{1}{2} \frac{d}{dt}(\|\nabla v\|^2 + \|\nabla y\|^2) + d_2(\|\Delta v\|^2 + \|\Delta y\|^2)
\]
\[= - \int_{\Omega} F e^{-\alpha z(\vartheta_{t}\omega)}(\Delta v + \Delta y) dx + (\alpha z(\vartheta_{t}\omega) - F)(\|\nabla v\|^2 + \|\nabla y\|^2)
\]
\[+ e^{2\alpha z(\vartheta_{t}\omega)} \int_{\Omega} (u^2 v \Delta v + w^2 y \Delta y) dx - D_2 \int_{\Omega} (y - v) \Delta (v - y) dx
\]
\[- e^{-\alpha z(\vartheta_{t}\omega)} \int_{\Omega} \sigma \Delta v dx - e^{-\alpha z(\vartheta_{t}\omega)} \int_{\Omega} \sigma \Delta y dx. \] (76)
By Green’s formula, we have

\[\int_O F e^{-\alpha z(\theta, \omega)} (\Delta v + \Delta y) dx = 0,\]  
(77)

and

\[-D_2 \int_O (y - v) \Delta (v - y) dx = -D_2 \int_O \|\nabla (v - y)\|^2 dx \leq 0.\]  
(78)

Using Young’s inequality, we obtain

\[e^{2\alpha z(\theta, \omega)} \int_O (u^2 v \Delta v + w^2 y \Delta y) dx \leq d_2 (\|\Delta v\|^2 + \|\Delta y\|^2) + \frac{1}{4d_2} e^{4\alpha z(\theta, \omega)} \int_O (u^4 v^2 + w^4 y^2) dx,\]  
(79)

and

\[-e^{-\alpha z(\theta, \omega)} \int_O \sigma \Delta v dx - e^{-\alpha z(\theta, \omega)} \int_O \sigma \Delta y dx \]
\[= e^{-\alpha z(\theta, \omega)} \int_O (\nabla v \nabla \sigma + \nabla y \nabla \sigma) dx \]
\[\leq F (\|\nabla v\|^2 + \|\nabla y\|^2) + \frac{1}{2F} e^{-2\alpha z(\theta, \omega)} \|\nabla \sigma\|^2.\]  
(80)

It follows from (76)-(80) that

\[\frac{d}{dt}(\|\nabla v\|^2 + \|\nabla y\|^2) - 2\alpha z(\theta, \omega) (\|\nabla v\|^2 + \|\nabla y\|^2) - \frac{1}{F} e^{-2\alpha z(\theta, \omega)} \|\nabla \sigma\|^2 \]
\[\leq \frac{1}{2d_2} e^{4\alpha z(\theta, \omega)} (\|u\|_{L^\infty}^4 \|v\|_{L^6}^2 + \|w\|_{L^6}^4 \|y\|_{L^6}^2) \]
\[\leq \frac{1}{d_2 \eta^2} e^{4\alpha z(\theta, \omega)} \left[ (\|u\|^4 + \|\nabla u\|^4) \|v\|_{L^6}^2 + (\|w\|^4 + \|\nabla w\|^4) \|y\|_{L^6}^2 \right] \]
\[\leq \frac{1}{d_2 \eta^2} e^{4\alpha z(\theta, \omega)} \left[ (\|u\|^2 + \|w\|^2)^2 + (\|\nabla u\|^2 + \|\nabla w\|^2)^2 \right] (\|v\|_{L^6}^2 + \|y\|_{L^6}^2).\]  
(81)

Replacing t, ω and σ with s, θ−tω and θ−tσ, we conclude

\[\frac{d}{ds}(\|\nabla v(s, \theta-t\omega, \theta-t\sigma, \omega, 0)\|^2 + \|\nabla y(s, \theta-t\omega, \theta-t\sigma, 0)\|^2) \]
\[\leq 2\alpha z(\theta s-t\omega) (\|\nabla v(s, \theta-t\omega, \theta-t\sigma, 0)\|^2 + \|\nabla y(s, \theta-t\omega, \theta-t\sigma, 0)\|^2) \]
\[+ \frac{1}{d_2 \eta^2} e^{4\alpha z(\theta s-t\omega)} \left[ (\|u(s, \theta-t\omega, \theta-t\sigma, 0)\|^2 + \|w(s, \theta-t\omega, \theta-t\sigma, 0)\|^2)^2 \right] \]
\[+ \left( \|\nabla u(s, \theta-t\omega, \theta-t\sigma, 0)\|^2 + \|\nabla w(s, \theta-t\omega, \theta-t\sigma, 0)\|^2 \right)^2 \]
\[\left( \|v(s, \theta-t\omega, \theta-t\sigma, 0)\|_{L^6}^2 + \|y(s, \theta-t\omega, \theta-t\sigma, 0)\|_{L^6}^2 \right) \]
\[+ \frac{1}{F} e^{-2\alpha z(\theta s-t\omega)} \|\nabla \sigma(s-t)\|^2,\]  
(82)

which can be rewritten as

\[\frac{d\phi}{ds} \leq \beta(s)\rho(s) + \mu(s),\]  
(83)
where
\[
\begin{align*}
\rho(s) &= \| \nabla v(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 + \| \nabla g(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2, \\
\beta(s) &= 2\alpha z(\vartheta_t \omega) \\
\mu(s) &= \frac{1}{d_2 \eta^2} e^{4\alpha z(\vartheta_t \omega)} \left[ \left( \| u(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 + \| w(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 \right)^2 \\
&\quad + \left( \| \nabla u(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 + \| \nabla w(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 \right)^2 \right] \\
&\quad + \| v(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2_L + \| g(s, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2_{L^2} \\
&\quad + \frac{1}{F} e^{2\alpha z(\vartheta_t \omega)} \| \nabla \sigma(s - t) \|^2.
\end{align*}
\]

Similar to Lemma 4.4, we use uniform Gronwall inequality to estimate \( \int_{t-1}^t \beta(s) ds \) and \( \int_{t-1}^t \mu(s) ds \).

By (71), (72) and Lemma 4.4, we know that, for all \( t \geq T \),

\[
\int_{t-1}^t \mu(s) ds \leq \frac{C_{11}}{d_2 \eta^2} \max_{-1 \leq \tau \leq 0} \left[ e^{4\alpha z(\vartheta_{\tau} \omega)} \left( e^{2\alpha z(\vartheta_{\tau} \omega)} \| \nabla \sigma(\vartheta_{\tau} \omega) \|^2 \right) \right]
\]

\[
\leq M_4(\omega).
\]

Hence,
\[
\| \nabla v(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 + \| \nabla g(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 \\
\leq (M_4(\omega) + \rho_2(\omega)) e^{M_5(\omega)} \\
\equiv \rho_4(\omega).
\]

Finally,
\[
\| \nabla g(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 \\
= \| \nabla u(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 + \| \nabla v(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 \\
+ \| \nabla w(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 + \| \nabla g(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 \\
\leq \rho_3(\omega) + \rho_4(\omega) \\
= (M_1(\omega) + \rho_2(\omega)) e^{M_2(\omega)} + (M_4(\omega) + \rho_2(\omega)) e^{M_5(\omega)},
\]

where \( M_1(\omega), M_2(\omega), M_3(\omega), M_4(\omega) \) and \( \rho_2(\omega) \) are given in Lemma 4.3, Lemma 4.4 and Lemma 4.5.

Applying (8) and Lemma 4.5, the solutions of (1) can be estimated uniformly.

**Corollary 4.6.** For every \( D \in \mathcal{D} \) and \( \omega \in \Omega \), we can find a time \( T = T(D, \omega) > 1 \) such that, for any \( \sigma \in \Sigma \),

\[
\| \nabla \phi(t, \vartheta_t \omega, \theta_t \sigma, g_0) \|^2 \leq (M_1(\omega) + \rho_2(\omega)) e^{M_2(\omega)} + (M_4(\omega) + \rho_2(\omega)) e^{M_5(\omega)}
\]

holds uniformly in \( g_0 \in D \) and \( t \geq T \). \( M_1(\omega), M_2(\omega), M_3(\omega), M_4(\omega) \) and \( \rho_2(\omega) \) are given in Lemma 4.3, Lemma 4.4 and Lemma 4.5.
5. Uniform and cocycle attractors. In this part, the existence of uniform and cocycle attractors is proved [11].

For every $\omega \in \Omega$ and $\sigma \in \Sigma$, let
\[
E(\omega) = \left\{ \bar{g} \in [H^1(\mathcal{O})]^4 : \|\nabla \bar{g}\|^2 \leq (M_1(\omega) + \rho_2(\omega))e^{M_2(\omega)} + \rho_2(\omega)e^{M_3(\omega)} \right\},
\]
(89)

here $M_1(\omega)$, $M_2(\omega)$, $M_3(\omega)$ and $M_4(\omega)$ and $\rho_2(\omega)$ are given in Lemma 4.3, Lemma 4.4 and Lemma 4.5. Applying Sobolev compactness embeddings, $E$ is a compact random set in $[L^2(\mathcal{O})]^4$, and $E$ belongs to $\mathcal{D}$. Moreover, by Corollary 4.6, it is easy to find that $E$ ia a uniformly $\mathcal{D}$-pullback absorbing set for $\phi$.

Next, the existence of $\mathcal{D}$-cocycle and $\mathcal{D}$-uniform attractors is proved.

**Theorem 5.1.** The NRDS $\phi$ created by (1) possesses a unique $\mathcal{D}$-cocycle attractor $A = \{A_\sigma(\cdot)\}_\sigma \in \Sigma$ defined by
\[
A_\sigma(\omega) = W(\omega, \sigma, E), \quad \forall \sigma \in \Sigma, \omega \in \Omega,
\]
(90)
where $E$ is a random set from (89). Furthermore, $A$ possesses the following properties:

1. $A$ is upper semi-continuous, namely, for any $\omega \in \Omega$,
   \[
   \text{dist}_H(A_\sigma(\omega), A_{\sigma_0}(\omega)) \to 0, \quad \forall \sigma \to \sigma_0;
   \]
2. $A$ is uniformly compact, that is, for any $\omega \in \Omega$, $\cup_{\sigma \in \Sigma} A_\sigma(\omega)$ is compact in $H$;
3. $A$ is described by $\mathcal{D}$-complete trajectories of $\phi$, namely,
   \[
   A_\sigma(\omega) = \{ \xi(\omega, 0) : \xi \text{ is a } \mathcal{D}-\text{complete trajectory of } \phi \}, \quad \forall \sigma \in \Sigma, \omega \in \Omega.
   \]

By Theorem 2.9, Theorem 2.13, Proposition 2.11 and Proposition 2.14, Theorem 5.1 can be written as follows.

**Theorem 5.2.** The NRDS $\phi$ created by (1) possesses a $\mathcal{D}$-cocycle attractor $A$ and a $\mathcal{D}$-uniform attractor $A \in \mathcal{D}$ such that
\[
A(\omega) = W(\omega, \Sigma, E)
= \cup_{\sigma \in \Sigma} A_\sigma(\omega)
= \{ \xi(\omega, 0) : \xi \text{ is a } \mathcal{D}-\text{complete trajectory of } \phi \}, \quad \forall \omega \in \Omega.
\]
(91)
$A$ is forward-attracting in probability. Furthermore, $A$ is upper semi-continuous in symbols.

**REFERENCES**

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.
[2] P. W. Bates, H. Lisei and K. Lu, Attractors for stochastic lattice dynamical systems, *Stoch. Dyn.*, 6 (2006), 1–21.
[3] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, J. Differential Equations, 246 (2009), 845–869.
[4] V. V. Chepyzhov and M. I. Vishik, Attractors for equations of mathematical physics, in *American Mathematical Society Colloquium Publications*, Vol. 49, American Mathematical Society, Providence, RI, 2002.
[5] I. Chueshov, *Monotone Random Systems Theory and Applications*, Springer-Verlag, Berlin, 2002.
[6] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dynam. Differential Equations*, 9 (1997), 307–341.
[7] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Related Fields*, 100 (1994), 365–393.
[8] H. Crauel, P. E. Kloeden and M. Yang, Random attractors of stochastic reaction-diffusion equations on variable domains, *Stoch. Dyn.*, 11 (2011), 301–314.
[9] H. Cui, M. M. Freitas and J. A. Langa, On random cocycle attractors with autonomous attraction universes, *Discrete Contin. Dyn. Syst. Ser. B*. 22 (2017), 3379–3407.

[10] H. Cui and P. E. Kloeden, Invariant forward attractors of non-autonomous random dynamical systems, *J. Differential Equations*. 265 (2018), 6166–6186.

[11] H. Cui and J. A. Langa, Uniform attractors for non-autonomous random dynamical systems, *J. Differential Equations*. 263 (2017), 1225–1268.

[12] X. Fan, Attractors for a damped stochastic wave equation of sine-Gordon type with sublinear multiplicative noise, *Stoc. Anal. Appl.*. 24 (2006), 767–793.

[13] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise, *Stochastics Stochastics Rep.*. 59 (1996), 21–45.

[14] A. Gu and H. Xiang, Upper semicontinuity of random attractors for stochastic three-component reversible Gray-Scott system, *Appl. Math. Comput.*. 225 (2013), 387–400.

[15] A. Gu, S. Zhou and Z. Wang, Uniform attractor of non-autonomous three-component reversible Gray-Scott system, *Appl. Math. Comput.*. 219 (2013), 8718–8729.

[16] X. Jia, J. Gao and X. Ding, Random attractors for stochastic two-compartment Gray-Scott equations with a multiplicative noise, *Open Math.*. 14 (2016), 586–602.

[17] H. Liu and H. Gao, Ergodicity and dynamics for the stochastic 3D Navier-Stokes equations with damping, *Commun. Math. Sci.*. 16 (2018), 97–122.

[18] K. Lu and B. Wang, Global attractors for the Klein-Gordon-Schrödinger equation in unbounded domains, *J. Differential Equations*. 170 (2001), 281–316.

[19] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Second edition, Applied Mathematical Sciences, Vol. 68, Springer-Verlag, New York, 1997.

[20] B. Wang, Attractors for reaction-diffusion equations in unbounded domains, *Phys. D*. 128 (1999), 41–52.

[21] B. Wang, Sufficient and necessary criteria for exitence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*. 253 (2012), 1544–1583.

[22] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*. 34 (2014), 269–300.

[23] B. Wang, Pullback attractors for non-autonomous reaction-diffusion equations on $\mathbb{R}^n$, *Front. Math. China*. 4 (2009), 563–583.

[24] Z. Wang and S. Zhou, Random attractor for stochastic reaction-diffusion equation with multiplicative noise on unbounded domains, *J. Math. Anal. Appl.*. 384 (2011), 160–172.

[25] Y. You, Dynamics of two-compartment Gray-Scott equations, *Nonlinear Anal.*. 74 (2011), 1969–1986.

[26] Y. You, Dynamics of three-compartment reversible Gray-Scott model, *Discrete Contin. Dyn. Syst. Ser. B*. 14 (2010), 1671–1688.

[27] Y. You, Global attractor of the Gray-Scott equations, *Commun. Pure Appl. Anal.*. 7 (2008), 947–970.

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