Counting hexagonal lattice animals confined to a strip

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Abstract

We describe a bijection between hexagonal lattice animals and a special type of square lattice animals. Using this bijection we adopt Maple packages that automatically generates generating functions (and series expansions) for fixed square lattice lattice animals to that of fixed hexagonal animals on the two-dimensional hexagonal lattice confined to a strip $0 \leq y \leq k$, for arbitrary $k$.

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Polyominoes, Hexagonal Lattice, Square Lattice, Enumeration.

1. Background

Required reading: Symbol-Crunching with the Transfer-Matrix Method in Order to Count Skinny Physical Creatures [Z1].

In [Z1], Zeilberger used finite transfer matrix method and developed two Maple packages ANIMALS and FreeANIMALS to count square lattice animals confined to a strip. Here we define a bijection between hexagonal lattice animals and a special class of square lattice animals and adopt the packages HexANIMALS and HexaFreeANIMALS accompanying [Z1] to enumerate hexagonal lattice animals.

Definition 1.1 A hexagonal lattice animal (hexagonal polyominoes hence forth) on the hexagonal lattice is an edge-connected set of lattice cells on the hexagonal lattice. Two animals are equivalent if they are translations of each other.

\footnote{This article is accompanied by two Maple packages, HexANIMALS and HexaFreeANIMALS, that can be downloaded from \url{http://www.people.vcu.edu/~mapagodu/}}
Figure 1 shows the three hexagonal polyominoes comprised of two hexagons.

Figure 1: The three hexagonal polyominoes with two hexagons.

The number of nonequivalent hexagonal polyominoes with \( n \) cells, \( a(n) \), is given by Sloane's sequence **A001207** for \( n \leq 35 \) [NJAS]. For example, \( a(1) = 1 \), \( a(2) = 3 \), \( a(3) = 11 \), \( a(4) = 44 \), \( a(5) = 186 \), \( a(6) = 814 \), \( a(7) = 3652 \), \( a(8) = 16689 \), \( a(9) = 77,359 \), and \( a(10) = 362,671 \).

A hexagonal polyominoes in which each column contains at most \( k \) contiguous blocks of cells is referred to as a \( k \)-board polyominoes. Figure 2 shows examples of one and two-board polyominoes with 7 hexagons. The number of nonequivalent one-board polyominoes with \( n \) hexagonal cells, \( b(n) \), is given by Sloane's sequence **A059716** for \( n \leq 24 \) [GV]. For example, \( b(1) = 1 \), \( b(2) = 3 \), \( b(3) = 11 \), \( b(4) = 42 \), \( b(5) = 162 \), \( b(6) = 626 \), \( b(7) = 2419 \), \( b(8) = 9346 \), \( b(9) = 36,106 \), and \( b(10) = 139,483 \).

Figure 2: A one-board hexagonal polyominoes (a) and a two-board hexagonal polyominoes (b) with 7 hexagons.

The generating function that enumerates one-board hexagonal polyominoes is computed in [K], which can also be computed using the Maple package LEGO in [Z0] by taking \( p(a, b) = a + b \) (for definition of \( p(a, b) \) refer to [Z0]). To the best of the authors' knowledge, \( k \)-board hexagonal polyominoes have not been enumerated for \( k \geq 2 \). In this article, in addition to computing generating functions (and series expansions) for hexagonal lattice animals that fit into a prescribed but arbitrary height, we also compute the first 12 terms of the sequence that enumerate the number of board-pair-hexagonal polyominoes, the analog of square board-pair-polyominoes computed in [Z2]. First we recall some definitions and introduce the essential details of the Maple implementation of the transfer-matrix method [S] from [Z1].
2. The Transfer-Matrix Method

Let \( G(V, E) \) be a vertex weighted directed graph consisting a finite set of vertices, \( V \), and a finite set of directed edges, \( E \). A path \( P \) in \( G \) is a sequence \( v_1, e_1, v_2, e_2, \ldots, v_{m-1}, e_m, v_m \) where \( v_i \in V(1 \leq i \leq m) \) and \( e_i \in E(1 \leq i \leq m) \) is an edge from \( v_{i-1} \) to \( v_i \). The weight of a path \( P \) is the sum of the weights of all vertices participating in the path \( P \). In this case \( \text{Wt}(P) = \text{Wt}(v_1) + \text{Wt}(v_2) + \ldots + \text{Wt}(v_m) \).

For \( T \) a subset of \( V \), we want to compute the weight enumerator (generating function),

\[
F(z) = \sum_{j=0}^{\infty} a_j z^j
\]

where \( a_j \) is the number of paths with weight \( j \) that starts at any vertex in \( V \) and ends at a vertex in \( T \). For \( v \in V \), let \( F_v(z) \) be the generating function that enumerates all paths starting at vertex \( v \) and ends at a vertex in \( V \). Then, clearly,

\[
F_v(z) = \sum_{\text{paths } P} t^{\text{wt}(P)} \text{ init}(P)=v, \text{ fin}(P) \in T
\]

where \( \text{init}(P) \) is the initial vertex of \( P \) and \( \text{fin}(P) \) is the terminal vertex of \( P \). Let \( N(v) \) be the set of vertices adjacent to \( v \). Then, \( \{ F_v(z) | v \in V \} \) satisfies the following \( |V| \) (where \( |V| \) is the number of vertices) system of equations in \( |V| \) unknowns, namely \( \{ F_v(z) | v \in V \} \),

\[
F_v(z) = 1_T(v) z^{\text{wt}(v)} + z^{\text{wt}(v)} \sum_{u \in N(v)} n(v, u) F_u(z)
\]

where \( n(v, u) \) is the number of paths from \( v \) to \( u \) and

\[
1_T(z) = \begin{cases} 
1 & \text{if } z \in T \\
0 & \text{otherwise}
\end{cases}
\]

Since each \( F_u(z) \) exists as a power series in \( z \), our system has a solution. Once we solve for this system, the answer to our original problem is then

\[
F(z) = \sum_{v \in T} F_v(z)
\]
As an example consider the following vertex-weighted directed graph on four vertices with weights as shown next to the vertices.

![Example of vertex weighted directed graph on four vertices](image)

In this example, the generating function that enumerates all paths according to their weight is found by first finding the four generating functions \( \{F_u(z), F_v(z), F_w(z), F_t(z)\} \) for all paths starting at vertex \( u, v, w, \) and \( t \) respectively. These four generating functions are related by

\[
f_t(z) = z, f_w(z) = f_v(z) = zf_u(z) = \frac{z(1 + z + z^2 + z^3 + z^4)}{1 - z^5}.
\]

Solving these system and adding together we get the required generating function that enumerates all paths according to their weight as

\[
F(z) = \frac{2z + 2z^2 + z^3 + 3z^4 + 3z^5 + z^6}{1 - z^5}.
\]

For the hexagonal polyominoes case, we follow the structure of [Z1] and automate the problem and use Maple do the hard part. First, we recall the following definitions from [Z1]:

**Definition 2.1** A *Combinatorial Markov Process* is a six tuple \((V, E, \text{init}, \text{fin}, \text{Start}, \text{Finish})\), where \(V\) is a finite set of vertices, \(E\) is the set of directed edges, and \(\text{init}, \text{fin} : E \mapsto V\) are functions that assign the initial and terminal vertex to an edge, respectively.

**Definition 2.2** A vertex *Weighted Combinatorial Markov Process* is a seven tuple \((V, E, \text{init}, \text{fin}, \text{Start}, \text{Finish}, \text{wt})\), where \((V, E, \text{init}, \text{fin}, \text{Start}, \text{Finish})\) is a Combinatorial Markov Process and \(\text{wt} : V(E) \mapsto \mathbb{Z}^+\) is a function that assigns weights to vertices.

In Maple, we represent a Combinatorial Markov Process with vertices \(V = \{1, 2, \ldots, n\}\) and no multiple edges as a four tuple \([S,T,\text{ListOfOutGoingNeighbors},\text{ListOfWeights}]\) where \(S,T \subseteq V\), and \(\text{ListOfOutGoingNeighbors}\) and \(\text{ListOfWeights}\) are lists of length \(n\) whose \(i^{th}\) element is the set of vertices adjacent to vertex \(i\) and the weight of vertex \(i\), respectively.
Similarly, the Combinatorial Markov Process for a multiple edge graph is a four tuple \( [S,T,\text{ListOfOutGoingNeibors},\text{ListOfWeights}] \) where \( S, T \), and \( \text{ListOfWeights} \) are as previously defined and \( \text{ListOfOutGoingNeibors} \) is a list of length \( n \) whose \( i^{th} \) element is a multi-set \( j_1^{m_1} j_2^{m_2} \ldots j_k^{m_k} \) represented in the form \( \{[j_1, m_1], [j_2, m_2], \ldots, [j_k, m_k]\} \), which means that from vertex \( i \) there are \( m_1 \) edges to vertex \( j_1 \), \( m_2 \) edges to vertex \( j_2 \), and so on.

If we successfully model hexagonal polyominoes as a weighted Combinatorial Markov Process as it is done for the square lattice animals in [Z1] then we can employ the Maple package \textsc{MARKOV} given in [Z1] to automatically find the generating function and series expansion of the generating function. The next section describes how weighted Combinatorial Process for hexagonal polyominoes is created.

3. Maple Representation of hexagonal polyominoes

We could develop a grammar that describes height-restricted hexanimals from first principles and apply the same methods as Zeilberger [Z1] to encode the grammar as a Combinatorial Markov Process; however, this would result in rewriting much of the code that was already created for the square lattice animals. Our approach here is to define a bijection between hexagonal lattice animals and a special class of square lattice animals and adopt the Maple codes \textsc{ANIMALS} and \textsc{FreeANIMALS} in [Z1]. In the remaining sections, we describe this processes.

We overlay a hexagonal polyomino on the square lattice so that its leftmost vertex (vertices of the square contained inside the hexagonal) lie on the line \( x = 0 \) and its bottommost cell lie on the line \( y = 0 \) as shown in Fig.4(a). We then construct a polyominoes comprised of the square cells which have diagonally opposing corners falling on the same hexagonal cell as shown in Fig. 4(b).

As described in detail by Zeilberger [Z1], we represent a polyomino by the set of coordinates of the bottomleft corner of each of its cells. We further encode a polyomino as a word in the alphabet consisting of the non-empty subsets of \( \{0, 1, \ldots, k\} \) where \( k \) is the maximum \( y \) coordinate in the set representation of the polyomino. For example, the animal in Fig. 4 (b) is represented by the set

\[
\{(0,2), (0,3), (1,1), (1,2), (1,5), (1,6), (2,0), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2)\}
\]

and encoded as the word (in interval notation)

\[
\{[2,3]\}, \{[1,2],[5,6]\}, \{[0,5]\}, \{[1,2]\}.
\]

There are two important properties of the polyomino words that encode hexanimals. First, every interval has even size (i.e., it represents a contiguous block of an even number of cells); this is obvious from the mapping which associates each hexagonal cell with a
Figure 4: Mapping between a hexagonal lattice animals and a square lattice animals

pair of square cells. Second, within a letter, the starting position of every interval has the same parity (odd or even), and adjacent letters have different parity; this is again obvious from the mapping of each hexagonal cell to a pair of square cells and the vertical shift of adjacent hexagonal columns. Note, however, that the parity of the leftmost column may be either odd or even depending on the hexanimal.

We will call the class of square polyominoes to which hexagonal polyominoes are mapped *parity polyominoes*. It is clear that the mapping between hexagonal animals and parity polyominoes is unique and reversible, and thus defines a bijection. As a result of this bijection, we can count hexagonal animals by counting parity polyominoes. Zeilberger’s code counts general polyominoes (a much larger class than parity polyominoes), so in order to use it to count parity polyominoes, we need to restrict the alphabet. In the following sections of this paper, we describe how to modify Zeilberger’s code to count parity polyominoes while minimizing the amount of changes required; we assume that the reader is familiar with [Z1].

4. Counting Globally Skinny Hexagonal Animals

Zeilberger’s **ANIMALS** package uses the *transfer-matrix method* [S] to count polyominoes whose set representation has $y$ coordinates in the range $0 \leq y \leq k$ for some arbitrary $k$. Before the transfer-matrix method can be applied, a grammar needs to be constructed to describe how to combine letters into words that encode polyominoes. A key routine in the generation of this grammar is **PreLeftLetters**($a, b$) which returns the set of all possible subsets of the integers in the range $[a, b]$ written in interval notation. For example, **PreLeftLetters**($0, 2$) returns the following subsets:

$$\{\}, \{[0, 0]\}, \{[0, 1]\}, \{[1, 1]\}, \{[0, 2]\}, \{[1, 2]\}, \{[2, 2]\}, \{[0, 0], [2, 2]\}$$
For parity polyominoes, we are only interested in those subsets whose intervals have even size and the same parity for the starting position. Rather than rewriting \texttt{PreLeftLetters} from scratch, we can use a trick in Maple to subclass \texttt{PreLeftLetters} and alter its parameters without having to modify the original \texttt{PreLeftLetters} code:

```
read ANIMALS:

origPreLeftLetters := subs(PreLeftLetters=origPreLeftLetters,
 eval(PreLeftLetters)):

PreLeftLetters := proc(a, b)
    local halfS, S, i, PreLet, PreLet1:

    halfS := origPreLeftLetters(0, floor((b-a+1)/2)-1) minus {{}}:
    S := {{}}:
    for i from 1 to nops(halfS) do
        PreLet := {seq([a+2*halfS[i][j][1], a+2*halfS[i][j][2]+1],
                        j=1..nops(halfS[i]))}:
        PreLet1 := {seq([PreLet[j][1]+1, PreLet[j][2]+1],
                         j=1..nops(PreLet))}:
        S := S union {PreLet}:
        if max(seq(PreLet1[j][2], j=1..nops(PreLet1))) <= b then
            S := S union {PreLet1}:
        fi:
    od:
    S:
end:
```

The first line loads Zeilberger’s original \texttt{ANIMALS} package. The second line makes a copy of \texttt{PreLeftLetters}; the \texttt{subs} function is designed to modify \texttt{PreLeftLetters} so that any recursive calls are made to the copy rather than the original. The remaining lines redefine \texttt{PreLeftLetters} so that it produces the type of intervals that are required by parity polyominoes. The new \texttt{PreLeftLetters} uses the original \texttt{PreLeftLetters} to generate a set of subsets (in interval notation) for half the required range (stored in \texttt{halfS}). The \texttt{for} loop goes through each interval and doubles its size; this results in intervals that have even size and even parity (stored in \texttt{PreLet}). To create intervals with even size and odd parity, each interval in \texttt{PreLet} is shifted down by one, and the result is stored in \texttt{PreLet2}. The final condition checks that the shifted intervals do not go out of the range \([a, b]\).

For example, \texttt{PreLeftLetters(0,5)} would call \texttt{origPreLeftLetters(0,2)} and get the subsets shown above. The \texttt{for} loop would then modify these subsets to generate the following result which represents the parity polyomino preletters in the range \([0, 5]\\):
Once the preletters are generated, the grammar is extended by determining which of the preletters are valid extensions of existing polyomino words; the ANIMALS routine that performs this check is \texttt{PreLetToLet}. For parity polyominoes, not only must the preletter satisfy the same connectivity constraints as for general polyominoes, but the parity of the preletter’s interval starting positions must be different than the letter it follows. As we did with \texttt{PreLeftLetters}, a simple extension to the original code is that is required:

\begin{verbatim}
origPreLetToLet := subs(PreLetToLet=origPreLetToLet,
                      eval(PreLetToLet)):

PreLetToLet := proc(Let, PreLet)
    if PreLet <> {} and Let[1][1][1] mod 2 = PreLet[1][1] mod 2 then
        0:
    else
        origPreLetToLet(Let, PreLet):
    fi:
end:
\end{verbatim}

Since we know that every interval in a parity polyomino letter has the same starting position parity, it suffices to check whether or not the starting positions of the first intervals have differing parities; if that is the case, \texttt{PreLetToLet} calls the original ANIMALS routine to ensure that the connectivity constraints are satisfied.

In order to weight the transfer-matrix vertices so that polyominoes with a specific number of cells can be enumerated, the ANIMALS routines call \texttt{Weight(Let)} to determine the number of cells in the letter \texttt{Let}. For enumerating hexanimals, we want to weight the transfer-matrix vertices with the number of hexagonal cells. Since each hexagonal cell is represented by two polyomino cells, a letter’s weight in hexagonal cells is half its weight in polyomino cells:

\begin{verbatim}
origWeight := subs(Weight=origWeight,
                  eval(Weight)):

Weight := proc(Let)
    floor(origWeight(Let)/2):
end:
\end{verbatim}
One final change to ANIMALS is needed in order to enumerate hexanimals; the routine Khaya(L) that computes the number of polyominoes with $\leq L$ cells relies on symmetry that does not exist in the hexagonal lattice. Since a heximal with $L$ cells must map to a parity polyomino whose set representation has a maximum $y$ coordinate of $2L$, we can implement Khaya(L) by a call to GFseries:

```
Khaya := proc(L)
    [GFseries(2*L, L)]:
end:
```

5. A User’s Manual for HexANIMALS

The modifications to ANIMALS that are described in the previous section are contained in the package HexANIMALS. To run HexANIMALS, download it and ANIMALS from the first authors’ web site to a local directory and start Maple. Once in Maple, type: read HexANIMALS and follow the on-line help.

Excluding Khaya, all the generating function and series expansion routines in ANIMALS remain unchanged. GF(n,s) computes the generating function for hexanimals embedded in the square lattice of height $n$ (i.e., the height of the square lattice is comprised of $n$ cells). GFseries(n,L) computes the list of length $L$ whose $k^{th}$ term is the number of hexanimals with $k$ cells embedded in the square lattice of height $n$. Khaya(L) computes the list of length $L$ whose $k^{th}$ term is the number of hexanimals with $k$ cells. Gf and Gfseries are the analogs of GF and GFseries for hexanimals whose polyomino embeddings have height exactly $n$.

As an example of HexANIMALS usage, the call GF(1,s) returns 0 since no heximal can be embedded in a square lattice of height 1. GF(2,s) returns $s$ since there is exactly one heximal (the single cell) that can be embedded in a square lattice of height 2. GFseries(3,5) returns [1, 2, 2, 2, 2] since the heximals in the square lattice of height 3 with more than one hexagonal cell are chains, and each chain has two orientations depending on the parity of the starting cell. In Table 1 we compute the generating function for hexanimals embedded in a square lattice of height $n \leq 7$ along with the first few terms in the expansion by hexagonal cell count. Once $n \geq 13$, it takes too long to compute GF(n,s) exactly, but one can go much further with GFseries(n,L).

6. Counting Locally Skinny Hexagonal Animals

If we relax the condition that the entire hexagonal polyominoes must fit within a square lattice of fixed height, and only require that each column have height less than some ar-
The code for computing the generating function and series expansions for locally skinny polyominoes is contained in Zeilberger’s FreeANIMALS package. Since the changes to FreeANIMALS that are required in order to enumerate hexagonal lattice animals are virtually the same as those made to ANIMALS, we omit the details that were discussed in the previous sections and only discuss the major differences.

FreeANIMALS creates a Combinatorial Markov Process with multiple edges between letters. Each edge represents one of the possible vertical offsets of the letters. In contrast, ANIMALS creates a Combinatorial Markov Process with only one edge between letters; thus, each letter represents a particular arrangement of cells with an intrinsic offset. As a result, PreLeftLetters in HexANIMALS had to create the two possible parity offsets for each letter whereas PreLeftLetters in HexaFreeANIMALS needs to only create a single instance of the letter and the code that generates the Combinatorial Markov Process will create its own instances of the shifted letter. The resulting PreLeftLetters routine is similar to the PreLeftLetters routine in HexANIMALS except that the parity shifting code is omitted:

read FreeANIMALS:

origPreLeftLetters := subs(PreLeftLetters=origPreLeftLetters,
                             eval(PreLeftLetters)):

PreLeftLetters := proc(a, b)
    local halfS, S, i, PreLet:

    halfS := origPreLeftLetters(0, floor((b-a+1)/2)-1) minus {{}}:
    S := {{}}:
    for i from 1 to nops(halfS) do
        PreLet := proc():
            eval(PreLeftLetters):
        end proc:
        S := [op(S), PreLet(a)]:
    end do:
    return S:
end proc:

\[
\begin{array}{|c|c|c|}
\hline
n & F(z) & a_i \\
\hline
1 & 0 & 0, 1, 0, 0, \ldots \\
2 & z & 1, 0, 0, \ldots \\
3 & \frac{z(1+z)}{1-z} & 1, 2, 2, \ldots \\
4 & \frac{z(1+z)}{(z^2+z-1)(z-1)} & 1, 3, 6, 11, 19, 32, 53, 87 \\
5 & \frac{z^2+2z^2+z+1}{z^2+2z^2+z-1} & 1, 3, 10, 25, 61, 142, 323, 723 \\
6 & \frac{z(1+z^6+7z^5+4z^4+8z^3-3z^2-4z+3z^2+3z^2+z^2+4z^2)}{(3z^6+2z^2-1+2z^2+z+5z^2)(z^3+2z^2+z-1)} & 1, 3, 11, 37, 111, 320, 896 \\
\hline
\end{array}
\]

Table 1: Generating functions for hexanimals polyominoes embedded in the square lattice of height \( n \) and their series expansion by hexagonal cell count.
In addition, FreeANIMALS contains the routine \texttt{PreLeftLettersk}(a, b, k) which computes all possible subsets of the integers in the range \([a, b]\) that are represented by exactly \(k\) intervals. \texttt{PreLeftLettersk} is used to compute \(k\)-board polyominoes. For example, \texttt{PreLeftLettersk}(0, 4, 2) returns the following subsets:

\[
\{\{[0, 0], [2, 2]\}, \{[0, 0], [2, 3]\}, \{[0, 0], [3, 3]\}, \{[0, 1], [3, 3]\}\}
\]

The modifications to convert \texttt{PreLeftLettersk} for use with HexaFreeANIMALS are analogous to the changes made to \texttt{PreLeftLetters} as shown in the following code:

\begin{verbatim}
PreLeftLettersk := proc(a, b, k)
    local halfS, S, i, PreLet:
    halfS := origPreLeftLettersk(0, floor((b-a+1)/2)-1, k) minus {{}}:
    S := {{}}:
    for i from 1 to nops(halfS) do
        PreLet := {seq([a+2*halfS[i][j][1],
                         a+2*halfS[i][j][2]+1], j=1..nops(halfS[i]))}:
        S := S union {PreLet}:
    od:
    S:
end:
\end{verbatim}

The only other significant difference between HexANIMALS and HexFreeANIMALS is in the specification of height restrictions. When considering globally skinny hexanimals, the height restrictions were defined in terms of the square lattice in which the hexanimals were embedded; this was because the shifting of adjacent columns in hexanimals makes the definition of the height of a hexanimal in terms of hexagonal cells difficult. In the case of locally skinny hexanimals, the height restrictions apply to individual columns and are well-defined in terms of hexagonal cells. Since the underlying algorithms of HexaFreeANIMALS operate on parity polyominoes, the height restrictions for hexanimal columns are simply doubled when applied to parity polyominoes as shown in the following code:
7. A User’s Manual for the HexaFreeANIMALS

The modifications to FreeANIMALS are contained in the package HexaFreeANIMALS. \texttt{gf(n,s)} computes the generating function for hexagonal polyominoes whose columns span \( \leq n \) hexagonal cells. \texttt{gfSeries(n,L)} computes the list of length \( L \) whose \( k^{th} \) term is the number of hexagonal polyominoes with \( k \) cells whose columns span \( \leq n \) hexagonal cells. \texttt{gfList(List,s)} computes the generating function for polyominoes whose \( k \)-board columns span \( \leq List[k] \) hexagonal cells. \texttt{gfSeriesList(List,L)} computes the series expansion of \texttt{gfList(List,s)} up to \( L \) terms.

As an example of the usage of HexaFreeANIMALS, the call \texttt{gfList([7,5],s)} would compute the generating function for polyominoes whose 1-board columns span \( \leq 7 \) hexagonal cells and whose 2-board columns span \( \leq 5 \) hexagonal cells. \texttt{gfSeriesList([24],24)} computes Sloane’s sequence \texttt{A005971} [NJAS] of 1-board polyominoes.

The call \texttt{gfSeriesList([12,12],12)} computes the first 12 terms of the previously
unknown sequence

\[1, 3, 11, 44, 186, 814, 3648, 16611, 76437, 354112, 1647344, 7682237\]

that enumerates 2-board hexanimals with a given hexagonal cell count, the analog of Sloane’s sequence A001170 for board-pair-pile polyominoes.

8. Conclusion

We have described the Maple packages HexANIMALS and HexaFreeANIMALS. HexANIMALS enumerates globally skinny hexagonal polyominoes in which each polyominoes is embedded in a horizontal strip of prescribed height. HexaFreeANIMALS enumerates locally skinny polyominoes in which the total height of each hexagonal polyominoes is unbounded but whose columns are composed of a bounded number of hexagonal cells. In addition, we also demonstrated used HexaFreeANIMALS to compute the first few terms of the number of board-pair-hexagonal animals.

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