The Cardy-Verlinde formula and entropy of Topological Kerr-Newman black holes in de Sitter spaces

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Abstract

In this paper we show that the entropy of cosmological horizon in 4-dimensional Topological Kerr-Newman-de Sitter spaces can be described by the Cardy-Verlinde formula, which is supposed to be an entropy formula of conformal field theory in any dimension. Furthermore, we find that the entropy of black hole horizon can also be rewritten in terms of the Cardy-Verlinde formula for these black holes in de Sitter spaces, if we use the definition due to Abbott and Deser for conserved charges in asymptotically de Sitter spaces. Such result presume a well-defined dS/CFT correspondence, which has not yet attained the credibility of its AdS analogue.

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1 Introduction

Holography is believed to be one of the fundamental principles of the true quantum theory of gravity\cite{1, 2}. An explicitly calculable example of holography is the much-studied AdS/CFT correspondence. Unfortunately, it seems that we live in a universe with a positive cosmological constant which will look like de Sitter space-time in the far future. Therefore, we should try to understand quantum gravity or string theory in de Sitter space preferably in a holographic way. Of course, physics in de Sitter space is interesting even without its connection to the real world; de Sitter entropy and temperature have always been mysterious aspects of quantum gravity\cite{3}.

While string theory successfully has addressed the problem of entropy for black holes, dS entropy remains a mystery. One reason is that the finite entropy seems to suggest that the Hilbert space of quantum gravity for asymptotically de Sitter space is finite dimensional, \cite{4, 5}. Another, related, reason is that the horizon and entropy in de Sitter space have an obvious observer dependence. For a black hole in flat space (or even in AdS) we can take the point of view of an outside observer who can assign a unique entropy to the black hole. The problem of what an observer venturing inside the black hole experiences, is much more tricky and has not been given a satisfactory answer within string theory. While the idea of black hole complementarity provides useful clues, \cite{6}, rigorous calculations are still limited to the perspective of the outside observer. In de Sitter space there is no way to escape the problem of the observer dependent entropy. This contributes to the difficulty of de Sitter space.

More recently, it has been proposed that defined in a manner analogous to the AdS/CFT correspondence, quantum gravity in a de Sitter (dS) space is dual to a certain Euclidean CFT living on a spacelike boundary of the dS space \cite{7} (see also earlier works \cite{8}-\cite{10}). Following the proposal, some investigations on the dS space have been carried out recently \cite{9}-\cite{30}. According to the dS/CFT correspondence, it might be expected that as the case of AdS black holes \cite{31}, the thermodynamics of cosmological horizon in asymptotically dS spaces can be identified with that of a certain Euclidean CFT residing on a spacelike boundary of the asymptotically dS spaces.

One of the remarkable outcomes of the AdS/CFT and dS/CFT correspondence has been the generalization of Cardy’s formula (Cardy-Verlinde formula) for arbitrary dimensionality, as well as a variety AdS and dS backgrounds. In this paper, we will show that the entropy of cosmological horizon in the 4-dimensional Topological Kerr-Newman-de Sitter spaces (TKNdS) can also be rewritten in the form of Cardy-Verlinde formula. We then show that if one uses the Abbott and Deser (AD) prescription \cite{32}, the entropy of black hole horizons in dS spaces can also be expressed by the Cardy-Verlinde formula \cite{33}. In a previous paper \cite{34}, we have shown that the entropy of cosmological horizon in Topological Reissner-Nordström- de Sitter spaces in arbitrary dimension can be described by the Cardy-Verlinde formula. Each of these cases is found to have interesting implications in the context of the proposed correspondence. The Cardy-Verlinde formula in 4-dimensional Kerr-Newman-de Sitter has been studied previously in \cite{35} (the dS/CFT correspondence have considered for the three-dimensional Kerr-de Sitter space already at \cite{36}). In the KNdS case the cosmological horizon geometry is spherical, also the black hole have two event horizon, but in the TKNdS case the cosmological horizon geometry is spherical, flat and hyperbolic for k=1,0,-1, respectively, also for the case k=0,-1 the black hole have not event horizon. In the other hand the entropy of such spaces come
from both cosmological and event horizon, then absence of event horizon or existance of extra cosmological horizon change the result for the total entropy.

2 Topological Kerr-Newman-de Sitter Black Holes

The line element of TKNdS black holes in 4-dimension is given by

\[
d s^2 = -\frac{\Delta_r}{\rho^2} \left( d t - \frac{a}{\Xi} \sin^2 \theta d \phi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d \theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left[ d t - \frac{(r^2 + a^2)}{\Xi} d \phi \right]^2,
\]

where

\[
\Delta_r = (r^2 + a^2) \left( k - \frac{r^2}{l^2} \right) - 2Mr + q^2,
\]
\[
\Delta_\theta = 1 + \frac{a^2 \cos^2 \theta}{l^2},
\]
\[
\Xi = 1 + \frac{a^2}{l^2},
\]
\[
\rho^2 = r^2 + a^2 \cos^2 \theta.
\]

here the parameters \(M, a,\) and \(q\) are associated with the mass, angular momentum, and electric charge parameters of the space-time, respectively. The topological metric Eq.(1) will only solve the Einstein equations if \(k=1\), which is the spherical topology. In fact when \(k = 1\), the metric Eq.(1) is just the Kerr-Newman -de Sitter solution. Three real roots of the equation \(\Delta_r = 0\), are the location of three horizon, the largest on is the cosmological horizon \(r_c\), the smallest is the inner horizon of black hole, the one in between is the outer horizon \(r_b\) of the black hole.

If we want the \(k = 0, -1\) cases solve the Einstein equations, then we must set \(\sin \theta \rightarrow \theta\), and \(\sin \theta \rightarrow \sinh \theta\) respectively [37]-[40]. When \(k = 0\) or \(k = -1\), there is only one positive real root of \(\Delta_r\), and this locates the position of cosmological horizon \(r_c\). When \(q = 0\), \(a = 0\) and \(M \rightarrow -M\) the metric Eq.(1)is the TdS (Topological de Sitter) solution [43, 44], which have a cosmological horizon and a naked singularity, for this type of solution, the Cardy-Verlinde formula also work well.

Here we review the BBM prescription [23] for computing the conserved quantities of asymptotically de Sitter spacetimes briefly. In a theory of gravity, mass is a measure of how much a metric deviates near infinity from its natural vacuum behavior; i.e, mass measures the warping of space. Inspired by the analogous reasoning in AdS space [45, 46] one can construct a divergence-free Euclidean quasilocal stress tensor in de Sitter space by the response of the action to variation of the boundary metric, in 4-dimensional spacetime we have

\[
T^{\mu \nu} = \frac{2}{\sqrt{\h} \partial_{\h} \h_{\mu \nu}} = \frac{1}{8\pi G} \left[ K^{\mu \nu} - \h^{\mu \nu} + \frac{2}{l} \h^{\mu \nu} + l \h^{\mu \nu} \right],
\]

where \(h^{\mu \nu}\) is the metric induced on surfaces of fixed time, \(K_{\mu \nu}, K\) are respectively extrinsic curvature and its trace, \(G^{\mu \nu}\) is the Einstein tensor of the boundary geometry. To compute
the mass and other conserved quantities, one can write the metric $h_{\mu\nu}$ in the following form

$$h_{\mu\nu} dx^\mu dx^\nu = N_\rho^2 d\rho^2 + \sigma_{ab} (d\phi^a + N^a_\Sigma d\rho) (d\phi^b + N^b_\Sigma d\rho)$$

(4)

where the $\phi^a$ are angular variables parametrizing closed surfaces around the origin. When there is a Killing vector field $\xi^\mu$ on the boundary, then the conserved charge associated to $\xi^\mu$ can be written as \[45, 46\]

$$Q = \oint_\Sigma d^2 \phi \sqrt{\sigma} n^\rho \xi^\rho T_{\mu\nu}$$

(5)

where $n^\rho$ is the unit normal vector on the boundary, $\sigma$ is the determinant of the metric $\sigma_{ab}$. Therefore the mass of an asymptotically de Sitter space in 4-dimensions is as

$$M = \oint_\Sigma d^2 \phi \sqrt{\sigma} N_\rho \epsilon ; \quad \epsilon \equiv n^\mu n^\nu T_{\mu\nu},$$

(6)

where Killing vector normalized as $\xi^\mu = N_\rho n^\mu$. Using this prescription \[23\], the gravitational mass, subtracted the anomalous Casimir energy, of the 4-dimensional TKNdS solution is

$$E = -\frac{M}{\Xi}.$$  

(7)

Where the parameter $M$ can be obtain from equation $\Delta r = 0$. On this basis, the following relation for the gravitational mass can be obtained

$$E = -\frac{M}{\Xi} = \frac{(r_c^2 + a^2)(r_c^2 - kl^2) - q^2 l^2}{2\Xi r_c l^2}.$$  

(8)

The Hawking temperature of the cosmological horizon given by

$$T_c = -\frac{1}{4\pi} \frac{\Delta' (r_c)}{(r_c^2 + a^2)} = \frac{3r_c^4 + r_c^2(a^2 - kl^2) + (ka^2 + q^2)l^2}{4\pi r_c l^2(r_c^2 + a^2)^2}.$$  

(9)

The entropy associated with the cosmological horizon can be calculated as

$$S_c = \frac{\pi(r_c^2 + a^2)}{\Xi}.$$  

(10)

The angular velocity of the cosmological horizon is given by

$$\Omega_c = \frac{-a\Xi}{(r_c^2 + a^2)}.$$  

(11)

The angular momentum $J_c$, the electric charge $Q$, and the electric potentials $\phi_{qc}$ and $\phi_{qc0}$ are given by

$$J_c = \frac{Ma}{\Xi^2},$$

$$Q = \frac{q}{\Xi},$$

$$\Phi_{qc} = -\frac{qr_c}{r_c^2 + a^2},$$

$$\Phi_{qc0} = -\frac{q}{r_c},$$

(12)
The obtained above quantities of the cosmological horizon satisfy the first law of thermodynamics
\[ dE = T_c dS_c + \Omega_c dJ_c + (\Phi_{qc} + \Phi_{qc0}) dQ. \] (13)

Using the Eqs.(10,12) for cosmological horizon entropy, angular momentum and charge, also equation \( \Delta_r(r_c) = 0 \), we can obtain the metric parameters \( M, a, q \) as a function of \( S_c, J_c \) and \( Q \), after that we can write \( E \) as a function of these thermodynamical quantities \( E(S_c, J_c, Q) \) (see [47, 48]). Then one can define the quantities conjugate to \( S_c, J_c \) and \( Q \), as
\[ T_c = \left( \frac{\partial E}{\partial S_c} \right)_{J_c, Q}, \quad \Omega_c = \left( \frac{\partial E}{\partial J_c} \right)_{S_c, Q}, \quad \Phi_{qc} = \left( \frac{\partial E}{\partial Q} \right)_{S_c, J_c}, \quad \Phi_{qc0} = \lim_{a \to 0} \left( \frac{\partial E}{\partial Q} \right)_{S_c, J_c}. \] (14)

Making use of the fact that the metric for the boundary CFT can be determined only up to a conformal factor, we rescale the boundary metric for the CFT to be the following form
\[ ds_{CFT}^2 = \lim_{r \to \infty} \frac{R^2}{r^2} ds^2, \] (15)

Then the thermodynamic relations between the boundary CFT and the bulk TKNdS are given by
\[ E_{CFT} = \frac{l}{R} E, \quad T_{CFT} = \frac{l}{R} T, \quad J_{CFT} = \frac{l}{R} J, \quad \phi_{CFT} = \frac{l}{R} \phi, \quad \phi_{0CFT} = \frac{l}{R} \phi_0. \] (16)

The Casimir energy \( E_c \), defined as \( E_c = (n + 1) E - n(T_c S_c + \Omega_c Q + Q/2\phi_{qc} + Q/2\phi_{qc0}) \), and \( n = 2 \) in this case, is found to be
\[ E_c = -\frac{k(r_c^2 + a^2)}{R \Xi r_c}, \] (17)

in KNdS space case [35] the Casimir energy \( E_c \) is always negative, but in TKNdS space case Casimir energy can be positive, negative or vanishing depending on the choice of \( k \). Thus we can see that the entropy Eq.(10)of the cosmological horizon can be rewritten as
\[ S = \frac{2\pi R}{n} \sqrt{\frac{E_c}{k}} \left| \left(2(E - E_q) - E_c \right) \right|, \] (18)

where
\[ E_q = \frac{1}{2} \phi_{cd} Q. \] (19)

We note that the entropy expression (18) has a similar form as the case of TRNdS black holes [34].

For the black hole horizon, which there is only for the case \( k = 1 \) associated thermodynamic quantities are
\[ T_b = \frac{1}{4\pi} \frac{\Delta'_r(r_b)}{r_b^2 + a^2} = -\frac{3r_b^4 + r_b^2(a^2 - l^2) + (a^2 + q^2)l^2}{4\pi r_b l^2(r_b^2 + a^2)}, \] (20)

\[ S_b = \frac{\pi (r_b^2 + a^2)}{\Xi}. \] (21)
\[ \Omega_b = \frac{aE}{(r_b^2 + a^2)}. \]  
(22)

\[ J_b = \frac{Ma}{E^2}, \]  
(23)

\[ Q = \frac{q}{E}. \]  
(24)

\[ \Phi_{qb} = \frac{q r_b}{r_b^2 + a^2}, \]  
(25)

\[ \Phi_{qb0} = \frac{q}{r_b}. \]  
(26)

Now if we use the BBM mass Ee.(7) the black hole horizon entropy cannot be expressed by a form like Cardy-Verlinde formula [43]. The other way for computing conserved quantities of asymptotically de Sitter space is Abbott and Deser (AD) prescription [32]. According to this prescription, the gravitational mass of asymptotically de Sitter space coincides with the ADM mass in asymptotically flat space, when the cosmological constant goes to zero. Using the AD prescription for calculating conserved quantities the black hole horizon entropy of TKNdS space can be expressed in term of Cardy-Verlinde formula [33]. The AD mass of TKNdS solution can be expressed in terms of black hole horizon radius \( r_b \), \( a \) and charge \( q \),

\[ E' = \frac{M}{E} = \frac{(r_b^2 + a^2)(r_b^2 - l^2) - q^2 l^2}{2E r_b l^2}. \]  
(27)

The obtained above quantities of the black hole horizon also satisfy the first law of thermodynamics as

\[ dE' = T_b dS_b + \Omega_b dJ_b + (\Phi_{qb} + \Phi_{qb0})dQ. \]  
(28)

The thermodynamics quantities of the CFT must be rescaled by a factor \( \frac{l}{R} \) similar to the previous case. In this case, the Casimir energy, defined as \( E'_C = (n + 1)E' - n(T_b S_b + J_b \Omega_b + Q/2\phi_{qb} + Q/2\phi_{qb0}) \), is

\[ E'_C = \frac{(r_b^2 + a^2)l}{R E r_b}, \]  
(29)

and the black hole entropy \( S_b \) can be rewritten as

\[ S_b = \frac{2\pi R}{n} \sqrt{E'_C [(2E' - E'_q) - E'_C]}, \]  
(30)

where

\[ E'_q = \frac{1}{2} \phi_{qb0} Q. \]  
(31)

which is the energy of electromagnetic field outside the black hole horizon. Thus we demonstrate that the black hole horizon entropy of TKNdS solution can be expressed in a form as the Cardy-Verlinde formula. However, if one uses the BBM mass Eq.(8) the black hole horizon entropy \( S_b \) cannot be expressed by a form like the Cardy-Verlinde formula. Our result is in favour of the dS/CFT correspondence.
3 Conclusion

The Cardy-Verlinde formula recently proposed by Verlinde [42], relates the entropy of a certain CFT to its total energy and Casimir energy in arbitrary dimensions. In the spirit of dS/CFT correspondence, this formula has been shown to hold exactly for the cases of dS Schwarzschild, dS topological, dS Reissner-Nordström, dS Kerr, and dS Kerr-Newman black holes. In this paper we have further checked the Cardy-Verlinde formula with topological Kerr-Newman de Sitter black hole.

It is well-known that there is no black hole solution whose event horizon is not sphere, in de Sitter background although there are such solutions in anti-de Sitter background, then in TKNdS space for the case k=0,-1 the black hole have not event horizon, however the cosmological horizon geometry is spherical, flat and hyperbolic for k=1,0,-1, respectively. As we have shown there exist two different temperature and entropy associated with the cosmological horizon and black hole horizon, in TKNdS spacetimes. If the temperatures of the black hole and cosmological horizon are equal, then the entropy of system is the sum of entropies of cosmological and black hole horizons. The geometric features of black hole temperature and entropy seem to imply that the black hole thermodynamics is closely related to nontrivial topological structure of spacetime. In [49] Cai, et al in order to relate the entropy with Euler characteristic \( \chi \) of the corresponding Euclidean manifolds have been presented the following relation

\[
S = \frac{\chi_1 A_{BH}}{8} + \frac{\chi_2 A_{CH}}{8},
\]

in which the Euler number of the manifolds divided into two parts, one first part comes from the black hole horizon and the second part come from the cosmological horizon (see also [50, 51, 52]). If one uses the BBM mass of the asymptotically dS spaces, the black hole horizon entropy cannot be expressed by a form like the Cardy-Verlinde formula[43].

In this paper, we have found that if one uses the AD prescription to calculate conserved charges of asymptotically dS spaces, the TKNdS black hole horizon entropy can also be rewritten in a form of Cardy-Verlinde formula, which indicates that the thermodynamics of black hole horizon in dS spaces can be also described by a certain CFT. Our result is also reminiscent of the Carlip’s claim [53](to see new formulation which is free of inconsistencies encountered in Carlip’s ref.[54]) that for black holes in any dimension the Bekenstein-Hawking entropy can be reproduced using the Cardy formula [55]. Also we have shown that the Casimir energy for cosmological horizon in TKNdS space case can be positive, negative or vanishing depending on the choice of \( k \), by contrast the Casimir energy for cosmological horizon in KNdS space is always negative [35].

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