Research Article
Positive Solutions of a Generalized Nonautonomous Fractional Differential System

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Received 7 April 2022; Accepted 12 July 2022; Published 8 September 2022

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In this work, we investigate the existence and uniqueness of positive solutions to a system of nonautonomous fractional differential equations. The fractional derivative of the system at hand is a Riemann–Liouville fractional derivative. Some significant properties of this derivative that assist in the analysis of our results substantial are proved. The analysis is based on Guo-Krasnoselskii’s and Banach’s fixed point theorem. Finally, we give two examples to verify our results.

1. Introduction

Fractional-order systems (FOSs) are viewed as more satisfactory than integer-order systems in some real-world issues, as fractional derivatives (FDs) supply a brilliant tool for the description of memory and hereditary properties of different processes and materials. Numerous applications of FOSs in various fields of engineering, chemistry [1], physics [2], aerodynamics [3, 4], and biological sciences [5] were presented. As a result, the topic of fractional differential equations (FDEs) has acquired great importance and unparalleled interest in recent times (see [6–8]). D-Gejji and his coauthor [9] have introduced a brief analysis of a system of FDEs. Once upon a time, Zhang [10] has discussed the existence of positive solutions for the next FDE:

\[ D^s_0 \omega(x) = f(x, \omega(x)), \quad 0 < s < 1. \]  

The existence result of positive solutions to the multi-term FDE is given by

\[ L(D) \omega(x) = f(x, \omega(x)), \]

where \( L(D) = D^s_0 - a_{m-1} D^{s_m-1} - \cdots - a_1 D^{s_1}, 0 < s_1 < \cdots < s_{m-1} < s_m < 1, a_i > 0 \) have been presented by Babakhani and Daftardar-Gejji [11]. In this regard, Daftardar-Gejji in [12] have studied the existence of positive solutions for the following system of FDE:

\[ D^{s_i}_0 \omega_i(x) = f_i(x, \omega(x)), \quad \omega_i(0) = 0, \quad i = 1, \ldots, m, \]

where \( \omega(x) = (\omega_1(x), \ldots, \omega_m(x)) \).

Since the beginning of 1695 fractional calculus was presented, over these years, there have been various local and nonlocal derivatives and amazing developments of FDs and integrals which have various kernels and applications, and
the most famous of these derivatives were introduced by Riemann–Liouville, Caputo, Hadamard, and Erdélyi–Kober (see [8]). Recently, Caputo and Fabrizio [13] suggested a new definition of FD without a singular kernel based on exponential law. Then, some properties of this operator are proved by Losada in [14]. In this regard, Atangana and Baleanu [15] contributed to siting a novel FD with nonlocal and nonsingular kernel based on Mittag–Leffler law. Some recent applications and results of these derivatives can be found in [16–18].

It was necessary to present a FD with respect to another function, by utilizing the FD in the concept of Riemann–Liouville given by [8].

\[
\mathcal{D}^{s\psi}_{a^+}\omega(\xi) = \left(\frac{1}{\psi(\xi)} \frac{d}{d\xi}\right)^m \mathcal{I}^{m-s\psi}_0 \omega(\xi),
\]

where \( m - 1 < s < m, m = [s] + 1 \) for \( s \notin \mathbb{N}. \) Like this definition is restricted to the possible FDs that contain the differentiation operator following up on the integral operator.

Very recently, Almeida [19], using the notion of the FD with respect to another function, proposed a new FD called the \( \psi \)-Caputo; in this framework, Sousa and Oliveira [20], based on previous ideas, proposed a derivative that generalizes a class of FDs, the so-called \( \psi \)-Hilfer. A perfect number

\[
\begin{align*}
\text{Definition 1 (see [8]). Let } s > 0, \text{ and } \omega : [a, b] \to \mathbb{R} \text{ such that } \\
\omega \in L^1 \left[ a, b \right]. \text{ Then, the } \psi \text{-Riemann–Liouville FI of order } s \text{ is defined by } \\
\mathcal{J}^{s\psi}_{a^+} \omega(\xi) = \frac{1}{\Gamma(s)} \int_a^\xi \frac{\psi'(\eta)}{(\psi(\eta) - \psi(\xi))^{s-1}} \omega(\eta) d\eta, \quad a < \xi < b. 
\end{align*}
\]

\[
\text{Definition 2 (see [8]). Let } m = [s] + 1 \in \mathbb{N}, \text{ and } \omega : [a, b] \to \mathbb{R}. \text{ Then, the } \psi \text{-Riemann–Liouville FD of order } s \text{ is given by }
\]

\[
\mathcal{D}^{s\psi}_{a^+} \omega(\xi) = \frac{1}{\Gamma(m-s)} \left( \frac{1}{\psi(\xi)} \frac{d}{d\xi}\right)^m \left( \frac{\psi'(\eta)}{(\psi(\eta) - \psi(\xi))^{m-s+1}} \omega(\eta) \frac{d\eta}{d\xi} \right),
\]

where \( m^n_\xi \psi \equiv (\psi(\eta) - \psi(\xi))^{(m-1)-n} \psi(\eta) \), \( a < \xi < b \),

2. Preliminaries

\[
\text{Proof.} \text{ Then, f from Definitions 1, 2, and Lemma 1, we have }
\]

\[
\mathcal{J}^{s\psi}_{a^+} \omega(\xi) = \mathcal{D}^{s\psi}_{a^+} \mathcal{J}^{s\psi}_{a^+} \omega(\xi) = \mathcal{D}^{s\psi}_{a^+} \omega(\xi)
\]

\[
= \frac{1}{\psi(\xi)} \frac{d}{d\xi} \int_a^\xi \psi'(\eta) \omega(\eta) d\eta,
\]

\[
\frac{d}{d\xi} \int_a^\xi \omega(\xi) d\eta = \omega(\xi).
\]

\[
\text{Lemma 2. Let } 0 < s < 1. \text{ If } \omega \in L^1 \left[ J_1, [a, b] \right], \text{ then }
\]

\[
\mathcal{D}^{s\psi}_{a^+} \omega(\xi) = \mathcal{J}^{s\psi}_{a^+} \mathcal{J}^{s\psi}_{a^+} \omega(\xi) = \mathcal{J}^{s\psi}_{a^+} \omega(\xi)
\]

\[
= \frac{1}{\psi(\xi)} \frac{d}{d\xi} \int_a^\xi \psi'(\eta) \omega(\eta) d\eta
\]

\[
= \frac{d}{d\xi} \int_a^\xi \omega(\xi) d\eta = \omega(\xi).
\]

\[
\text{Lemma 3. Let } 0 < s < 1, \text{ if } \mathcal{J}^{s\psi}_{a^+} \omega(\xi) \in L^1 \left[ a, b \right], \text{ then }
\]

\[
\mathcal{J}^{s\psi}_{a^+} \mathcal{J}^{s\psi}_{a^+} \omega(\xi) = \omega(\xi) - \mathcal{J}^{s\psi}_{a^+} \omega(\xi)\left|_{\xi=a}^{\xi = b} \right. \frac{\psi(\xi) - \psi(a)}{\Gamma(s)}.
\]

\[
\text{Proof.} \text{ The proof is deduced directly from Definitions 1 and 2, then the integration by part.}
\]

\[
\text{Lemma 4. Let } \mathcal{J}^{s\psi}_{a^+} \omega(\xi) \text{ is integrable, then}
\]
If \( \omega \) is continuous on \([a, b]\), then \( \mathcal{D}^\psi_{a, b} \omega (x) \) is integrable, \( \left[ \mathcal{F}^\psi_{a, b} \omega (x) \right]_{x=a} = 0 \) and (11) takes the type
\[
\mathcal{F}^\psi_{a, b} \left( \mathcal{D}^\psi_{a, b} \omega (x) \right) = \mathcal{F}^\psi_{a, b} \omega (x), \quad 0 < \rho \leq s < 1.
\] (12)

**Proof.** We first use Lemma 1 if \( \rho \leq s \), or the relation \( \mathcal{F}^\psi_{a, b} \mathcal{F}^\psi_{a, b} \omega (x) = \mathcal{D}^\psi_{a, b} \omega (x) \) if \( \rho \geq s \), and then property (10). This gives
\[
\mathcal{F}^\psi_{a, b} \left( \mathcal{D}^\psi_{a, b} \omega (x) \right) = \mathcal{D}^\psi_{a, b} \left[ \mathcal{F}^\psi_{a, b} \mathcal{D}^\psi_{a, b} \omega (x) \right]
\]
\[
= \mathcal{D}^\psi_{a, b} \left\{ \left( \mathcal{F}^\psi_{a, b} \mathcal{D}^\psi_{a, b} \omega (x) \right) \right\}
\]
\[
= \mathcal{D}^\psi_{a, b} \left( \mathcal{F}^\psi_{a, b} \omega (x) - \mathcal{F}^\psi_{a, b} \omega (x) \right)_{x=a} \frac{(\psi(x) - \psi(a))^{s-1}}{\Gamma(s)}
\]
\[
= \mathcal{F}^\psi_{a, b} \omega (x) - \mathcal{F}^\psi_{a, b} \omega (x)_{x=a} \frac{(\psi(x) - \psi(a))^{s-1}}{\Gamma(s)}
\]
(13)

where we used the next fact:
\[
\mathcal{F}^\psi_{a, b} \left( \psi(x) - \psi(a) \right)^{s-1} = \frac{\Gamma(s)}{\Gamma(s)}(\psi(x) - \psi(a))^{s-1}.
\] (14)

To prove the pair two, let \( M = \max |\omega (x)| \); then, using Definition 2, we get
\[
\int_a^x \mathcal{D}^\psi_{a, b} \omega (\zeta) \, d\zeta = \int_a^x \mathcal{F}^\psi_{a, b} \mathcal{D}^\psi_{a, b} \omega (\zeta) \, d\zeta
\]
\[
= \int_a^x \frac{1}{\psi'(\zeta)} \frac{d}{d\zeta} \mathcal{F}^\psi_{a, b} \omega (\zeta) \, d\zeta
\]
\[
\leq M \int_a^x \frac{1}{\psi'(\zeta)} \frac{d}{d\zeta} \frac{1}{\Gamma(1-\rho)} \, d\zeta
\]
\[
= M \frac{1}{\psi'(\zeta)} \frac{d}{d\zeta} \frac{1}{\Gamma(2-\rho)} |\psi(\zeta) - \psi(a)|^{1-\rho}
\]
\[
= M \frac{1}{\psi'(\zeta)} \frac{d}{d\zeta} \frac{1}{\Gamma(2-\rho)} |\psi(\zeta) - \psi(a)|^{1-\rho}
\]
(15)

So \( \mathcal{D}^\psi_{a, b} \omega (x) \) is integrable. Also,
\[
\mathcal{F}^\psi_{a, b} \omega (x)_{x=a} \leq \frac{M}{\Gamma(1-\rho)} \int_a^x \psi'(\zeta) \left( \psi(\zeta) - \psi(a) \right)^{s-1} \, d\zeta
\]
\[
= \frac{M}{\Gamma(2-\rho)} \left( \psi(x) - \psi(a) \right)^{1-\rho}
\]
(16)

Hence, (11) reduces to (12).

**Theorem 1** (see [26]). Let \( (\mathcal{X} \mathcal{H}) \) be a Banach space, and let \( \Omega_1, \Omega_2 \subset \mathcal{X} \) with \( 0 \in \Omega_1 \) and \( \Omega_1 \in \Omega_2 \). Assume \( Q: \mathcal{H} \cap \Omega_1 \longrightarrow \mathcal{H} \) be completely continuous s.t., either

(i) \( \| Q \| \leq \| \omega \| \) for \( \omega \in \mathcal{H} \cap \partial \Omega_1 \) and \( \| Q \| \geq \| \omega \| \) for \( \omega \in \mathcal{H} \cap \partial \Omega_2 \), or

(ii) \( \| Q \| \geq \| \omega \| \) for \( \omega \in \mathcal{H} \cap \partial \Omega_1 \) and \( \| Q \| \leq \| \omega \| \) for \( \omega \in \mathcal{H} \cap \partial \Omega_2 \).

Then, \( Q \) has a fixed point in \( \mathcal{H} \cap (\overline{\Omega_2}/\Omega_1) \).

**Theorem 2** (see [26]). Let \( \mathcal{X} \) be a Banach space and let the operator \( Q: \mathcal{X} \longrightarrow \mathcal{X} \) be a contraction. Then, \( Q \) admits a unique fixed point \( x \in \mathcal{X} \) with \( Qx = x \).

**3. Main Results**

Here, we demonstrate the existence of positive solutions to the following system of FDEs:
\[
\mathcal{D}^{s; \psi}_{0, b} \omega_i (x) = f_i (x, \overline{\omega}(x)), \quad \omega_i (0) = 0, \quad i = 1, \ldots, m,
\] (17)

where \( 0 < s_i < 1 \), and \( \mathcal{D}^{s; \psi}_{0, b} \) denote \( \psi \)-Riemann–Liouville FDEs of order \( s_i \) s, \( i \) \( \forall i \). Denote by \( \mathcal{X} = C(\mathcal{T}) \) the space of all continuous real functions on \( \mathcal{T} \). Let \( \mathcal{X}^m = \mathcal{X} \times \cdots \times \mathcal{X} \) denote the Banach space with the following norm:
\[
\| \overline{\omega} \| = \max_{1 \leq i \leq m} \left\{ \max_{x \in \mathcal{T}} |\omega_i (x)| \right\}, \quad \text{for} \omega_i \in \mathcal{X}, \overline{\omega} \in \mathcal{X}^m.
\] (18)

As per Lemma 4, the system (17) is equivalent to
\omega_i(x) = \mathcal{J}_{\psi}^{\omega_i} f_i(x, \varpi(\omega)) \quad i = 1, \ldots, m \text{ and } x \in \mathcal{O}. \quad (19)

Let \mathcal{H} \subset \mathcal{X}^m be a cone defined by
\mathcal{H} = \{ \varpi \in \mathcal{X}^m : \omega_i(x) \geq 0, \forall i = 1, \ldots, m, x \in \mathcal{O} \}, \quad (20)
and the pair (\mathcal{X}^m, \mathcal{H}) represents an ordered Banach space.

Let \( Q : \mathcal{X}^m \to \mathcal{X}^m \) be the operator defined as \( \omega \mapsto Q(\omega) = (Q_1(\omega), \ldots, Q_m(\omega)) \) such that
\[
Q_i(\omega)(x) = \mathcal{J}_{\psi}^{\omega_i} f_i(x, \varpi(\omega)), \quad i = 1, \ldots, m \text{ and } x \in \mathcal{O}. \quad (21)
\]

**Lemma 5.** It is assumed that \( f_i : \mathcal{O} \times \mathbb{R}_+^m \to \mathbb{R}_+ \) be continuous \( \forall i = 1, \ldots, m \). Then, \( Q : \mathcal{H} \to \mathcal{H} \) is completely continuous mapping.

**Proof.** Let \( \delta \in \mathcal{H} \) be bounded. Then, there exists \( \kappa > 0 \) such that \( \|\varpi\| \leq \kappa \), for all \( \varpi \in \delta \). Now, we prove that \( Q(\delta) \) is bounded. It is noted that \( Q(\mathcal{H}) \subset \mathcal{H} \), since \( f_i \) are non-negative for all \( i \).

Let \( M = \max_i \{1 + \|f_i(x, \varpi)\| : 0 < \|\varpi\| < \kappa\} \). For \( \varpi \in \mathcal{O} \) and \( \omega \in \delta \), we have
\[
\|Q\omega(\varpi)\| \leq \mathcal{J}_{\psi}^{\omega} f_i(x, \varpi(\omega))
\leq \frac{M}{\Gamma(s_i + 1)} (\psi(0) - \psi(0))^s_i
\leq \frac{M}{\Gamma(s_i + 1)} (\psi(0) - \psi(0))^s_i.
\]

Consequently, let \( \varepsilon > 0 \), there exists a \( 0 < \delta < r_1 \) such that \( \|\mathcal{J}(x, \varpi) - \mathcal{J}(x, \varpi)\| < (\varepsilon \Gamma(s + 1)/(\psi(0) - \psi(0))) \), where \( \varpi \in \mathcal{O} \), \( s = \max\{s_1, \ldots, s_m\} \), for \( \|\varpi - \varpi\| < \delta \). If \( \|\varpi - \varpi\| < \delta \), then \( \varpi \in \mathcal{P} \) and \( \|\varpi\| \leq r \). Since \( \varpi \in \mathcal{P} \subset \mathcal{H}, \|\omega\| \leq r \). Hence,

\[
\|Q\varpi - Q\varpi\| = \max_{1 \leq i \leq m} \left\{ \max_{x \in \mathcal{O}} \left| Q_i(\varpi)(x) - Q_i(\varpi)(x) \right| \right\}
\leq \max_{1 \leq i \leq m} \left\{ \max_{x \in \mathcal{O}} \left[ \frac{1}{\Gamma(s_i + 1)} \int_0^\infty \psi'((\psi(0) - \psi(0))) \mathcal{J}(x, \varpi(\omega)) f_i(x, \varpi(\omega)) d\zeta \right] \right\}
\leq \frac{\varepsilon \Gamma(s + 1)}{\Gamma(s + 1)} \leq \varepsilon.
\]

Thus, \( Q \) is continuous on \( \mathcal{H} \).

Let \( \omega \in \mathcal{O}, x_1, x_2 \in \mathcal{O} \) with \( x_1 < x_2 \). Then, for \( \varepsilon > 0 \) we can select \( \delta = \delta(\varepsilon) = (\varepsilon \Gamma(s)/(2M)^{1/s_i}) \) where \( s = \max\{s_1, \ldots, s_m\} \), and \( \rho = \min\{s_1, \ldots, s_m\} \), and \( 0 < \|\psi(x_2) - \psi(x_1)\| < \delta \), we have
\[ |Q_i\omega_i(x_1) - Q_i\omega_i dx_2| \]

\[
\leq \frac{1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x_1) - \psi(\zeta))^{s_i-1} f_i(\zeta, \omega(\zeta)) d\zeta - \frac{1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x_2) - \psi(\zeta))^{s_i-1} f_i(\zeta, \omega(\zeta)) d\zeta \\
\leq \frac{1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x_1) - \psi(\zeta))^{s_i-1} - (\psi(x_2) - \psi(\zeta))^{s_i-1} f_i(\zeta, \omega(\zeta)) d\zeta \\
= \frac{1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x_2) - \psi(\zeta))^{s_i-1} f_i(\zeta, \omega(\zeta)) d\zeta \\
\leq \frac{2M}{\Gamma(s_i)} \omega(x_2) - \omega(x_1) \leq \frac{2M}{\Gamma(s_i)} \delta_i \leq \frac{2M}{\rho \Gamma(s_i)} \delta_i = \epsilon. 
\]

As \(0 < s_i < s < 1\), \(\Gamma(s_i) < \Gamma(s_i)\). Therefore, \(|Q_i\omega_i(x_2) - Q_i\omega_i(x_1)| < \epsilon, \forall i\), which implies \(\|Q_i\omega_i(x_2) - Q_i\omega_i(x_1)\| < \epsilon\). Consequently, \(Q_i(s_i)\) is equicontinuous, and as per Arzela–Ascoli theorem, the operator \(Q: \mathcal{X} \rightarrow \mathcal{X}\) is completely continuous.

**Theorem 3.** Let \(f_i: \mathcal{O} \times \mathbb{R}^n_+ \rightarrow \mathbb{R}_+\) (\(i = 1, \ldots, m\)) be continuous. Suppose that there exist \(d_1, d_2 > 0, d_1 \neq d_2\) such that \(d_1 \leq f_i(x, \bar{\omega}) \leq d_2\) (\(i = 1, \ldots, m\)). Then, the system (17) has at least one positive solution.

**Proof.** We set

\[
V_1 = \left\{ \bar{\omega} \in \mathcal{X} : \|\bar{\omega}\| \leq \frac{d_1(\psi(b) - \psi(0))}{sl(s)} \right\},
\]

\[
V_2 = \left\{ \bar{\omega} \in \mathcal{X} : \|\bar{\omega}\| \leq \frac{d_2(\psi(b) - \psi(0))}{\rho \Gamma(s)} \right\},
\]

where \(s = \max\{s_1, \ldots, s_m\}\), and \(\rho = \min\{s_1, \ldots, s_m\}\). For \(\bar{\omega} \in \mathcal{X} \cap \partial V_2\), and \(\bar{x} \in \mathcal{O}\), we have

\[
0 \leq \omega_i(\bar{x}) \leq \frac{d_1(\psi(b) - \psi(0))}{\Gamma(s + 1)}, \quad \forall i.
\]

As \(f_i(x, \bar{\omega}) \leq d_2\), we get

\[
Q_i\omega_i(x) = \frac{1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x) - \psi(\zeta))^{s_i-1} f_i(\zeta, \omega(\zeta)) d\zeta \\
\leq \frac{d_2}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x) - \psi(\zeta))^{s_i-1} d\zeta \\
\leq \frac{d_2}{\Gamma(s_i + 1)} \left( \psi(b) - \psi(0) \right)^s.
\]

Hence,

\[
Q_i\omega_i(x) = \frac{1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x) - \psi(\zeta))^{s_i-1} f_i(\zeta, \omega(\zeta)) d\zeta \\
\leq \frac{d_2}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x) - \psi(\zeta))^{s_i-1} d\zeta \\
\leq \frac{d_2}{\Gamma(s_i + 1)} \left( \psi(b) - \psi(0) \right)^s.
\]

Hence,

\[
\|Q\omega\| = \max_{1 \leq i \leq m} \|Q_i\omega_i(x)\| = \max_{1 \leq i \leq m} \left\{ \max_{x \in \mathcal{O}} \|Q_i\omega_i(x)\| \right\} \\
\leq \frac{d_2(\psi(b) - \psi(0))}{\Gamma(s_i + 1)} \leq \frac{d_1(\psi(b) - \psi(0))}{\Gamma(s_i + 1)} = \|\omega\|.
\]

In this regard, for \(\bar{\omega} \in \mathcal{X} \cap \partial V_1\), we have

\[
0 \leq \omega_i(x) \leq \frac{d_1(\psi(b) - \psi(0))}{\Gamma(s) + 1}, \quad \forall i.
\]

As \(f_i(x, \bar{\omega}) \geq d_1\), we get

\[
Q_i\omega_i(x) = \frac{1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x) - \psi(\zeta))^{s_i-1} f_i(\zeta, \omega(\zeta)) d\zeta \\
\geq \frac{d_1}{\Gamma(s_i)} \int_0^{x_i} \omega'(\zeta) (\psi(x) - \psi(\zeta))^{s_i-1} d\zeta \\
\geq \frac{d_1(\psi(x) - \psi(0))}{\Gamma(s_i + 1)} \geq \frac{d_1(\psi(x) - \psi(0))}{\Gamma(s_i)}.
\]

Hence, \(|Q_i\omega_i(x)| \geq (d_1(\psi(b))^{s_i}/\Gamma(s_i))\), for \(i = 1, \ldots, m\). This means that

\[
\|Q\omega\| \geq \frac{d_1(\psi(b) - \psi(0))}{\Gamma(s + 1)} = \|\omega\|, \quad \bar{\omega} \in \mathcal{X} \cap \partial V_1.
\]

Therefore, Theorem 1 shows that \(Q\) has a fixed point in \(\mathcal{X} \cap (\mathcal{V}_2 \setminus V_1)\), which is consistent with the system’s positive solution (17).

**Theorem 4.** Let \(f_i: \mathcal{O} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+\) (\(i = 1, \ldots, m\)) be continuous. Suppose that \(\|f_i(x, \bar{\omega}) - f_i(x, \bar{\omega})\| \leq \lambda\|\bar{\omega} - \bar{\omega}\|, \quad \forall i, \bar{\omega}, \bar{\omega} \in \mathbb{R}_+^n\), and \(x \in \mathcal{O}\). If

\[
\lambda(\psi(b) - \psi(0))^s \leq \min\{\Gamma(s_i + 1), \ldots, \Gamma(s_m + 1)\} < 1.
\]

Then, the system (17) has unique positive solution.
Proof. Let \( \overline{\omega}, \overline{\omega} \in \mathcal{X} \). Then,

\[
\|Q\overline{\omega} - Q\overline{\omega}\| = \max_{1 \leq i \leq m} \left\{ \left| Q_i \omega_i - Q_i \overline{\omega}_i \right| \right\}
\]

\[
= \max_{1 \leq i \leq m} \left\{ \max_{\omega \in \mathcal{X}} \left| Q_i \omega_i(\omega) - Q_i \omega_i(\overline{\omega}) \right| \right\}
\]

\[
\leq \max_{1 \leq i \leq m} \left\{ \max_{\omega \in \mathcal{X}} \left| \frac{1}{\Gamma(s_i)} \int_0^\infty \psi'(t) (\psi(t) - \psi(\overline{\omega}))^{s_i-1} [f_i(t, \overline{\omega}) - f_i(t, \omega)] \, dt \right| \right\}
\]

\[
\leq \max_{1 \leq i \leq m} \left\{ \frac{1}{\Gamma(s_i)} \int_0^\infty \psi'(t) (\psi(t) - \psi(\overline{\omega}))^{s_i-1} \|f_i(t, \overline{\omega}) - f_i(t, \omega)\| \, dt \right\}
\]

\[
\leq \max_{1 \leq i \leq m} \left\{ \frac{\|\psi(\psi(t) - \psi(0))\|}{\Gamma(s_i + 1)} \right\}
\]

\[
\leq \frac{\lambda (\psi(\psi(t) - \psi(0)))^{s_i}}{\min_{1 \leq i \leq m} \{\Gamma(s_i + 1)\}} \|\overline{\omega} - \overline{\omega}\|.
\]

As per condition (33) along with Theorem 2, Q has unique fixed point in \( \mathcal{X} \), which is the unique positive solution. \( \square \)

4. Examples

Example 1. Consider the fractional model as

\[
\mathcal{D}^\alpha_0 \psi \omega_1(\omega) = c_1 + \frac{\omega_i^2}{1 + \omega_1 + \cdots + \omega_m^2},
\]

\[
\omega_1(0) = 0, \quad c_1 > 0,
\]

\[
\mathcal{D}^\alpha_0 \psi \omega_2(\omega) = c_2 + \frac{\omega_i^2}{1 + \omega_1 + \cdots + \omega_m^2},
\]

\[
\omega_2(0) = 0, \quad c_2 > 0,
\]

\[
\vdots
\]

\[
\mathcal{D}^\alpha_0 \psi \omega_m(\omega) = c_m + \frac{\omega_i^2}{1 + \omega_1 + \cdots + \omega_m^2},
\]

\[
\omega_m(0) = 0, \quad c_m > 0.
\]

Example 2. Let \( m = 2 \). Then, following coupled system

\[
\left\{ \begin{array}{l}
\mathcal{D}^\alpha_0 \psi \omega_1(\omega) = d_1 + [\psi(\omega) - \psi(0)]\tan^{-1} \omega_2(\omega), \quad d_1 > 0, \\
\mathcal{D}^\alpha_0 \psi \omega_2(\omega) = d_2 + [\psi(\omega) - \psi(0)]\tan^{-1} \omega_1(\omega), \quad d_2 > 0,
\end{array} \right.
\]

has positive solution in view of Theorem 1. To verify Theorem 4, let us choose \( s_1 = (1/2), s_2 = (1/3), \psi(\omega) = (\omega/3) \) for \( \omega \in [0, 1] \), and

\[
f_1(\omega_1, \omega_2) = d_1 + [\psi(\omega) - \psi(0)]\tan^{-1} \omega_2(\omega),
\]

\[
f_2(\omega_1, \omega_2) = d_2 + [\psi(\omega) - \psi(0)]\tan^{-1} \omega_1(\omega),
\]

where for \( i = 1, 2, \omega_1, \omega_2 \in \mathbb{R}^2_+ \) and \( \omega \in [0, 1] \), we obtain
\[\|f_1(\lambda, \omega_1, \omega_2) - f_1(\mu, \omega_1, \omega_2)\| \leq |\psi(\lambda) - \psi(\mu)| \|\tan^{-1}(\omega_2(\lambda)) - \tan^{-1}(\omega_2(\mu))\| \leq \frac{1}{3} \|\omega_2 - \omega_1\|,\]
\[\|f_2(\lambda, \omega_1, \omega_2) - f_2(\mu, \omega_1, \omega_2)\| \leq |\psi(\lambda) - \psi(\mu)| \|\tan^{-1}(\omega_1(\lambda)) - \tan^{-1}(\omega_1(\mu))\| \leq \frac{1}{3} \|\omega_1 - \omega_2\|,\]

Moreover, the condition (33) holds, i.e.,
\[\frac{\lambda(\psi(b) - \psi(0))}{\min\{\Gamma(s_1 + 1), \Gamma(s_2 + 1)\}} = \frac{\lambda(b^{s_1})}{\Gamma(s_1 + 1)} = \frac{2}{3\sqrt{3\pi}} < 1.\]  

Therefore, Theorem 4 can be applied to the problem (37).

5. Conclusions
In the theory of FDEs, the investigation and analysis of the existence and uniqueness of solutions and positive solutions are critical. In the past few years, many researchers have had great interest in qualitative analysis of FDEs, but the literature focused on Caputo FDs. In this work, utilizing new techniques, the existence and uniqueness of positive solutions for a system of FDEs under \(\psi\)-Riemann–Liouville derivatives have been successfully obtained. The analysis has been based on Guo-Krasnoselskii’s and Banach’s fixed point theorem. The results were consistent with Datta–Gerjji’s results [12] when \(\psi(\lambda) = \lambda\) and they have been new even for the special case: \(\psi(\lambda) = \log(\lambda)\) or \(\psi(\lambda) = \lambda^p\).
The suggested techniques can be extended to other \(\psi\)-Hilfer FDEs.

Data Availability
No real data were used to support this study. The data used in this study are hypothetical.

Conflicts of Interest
The authors declare no conflicts of interest related to this work.

Acknowledgments
The authors express their appreciation to “The Research Center for Advanced Materials Science (RCAMS)” at King Khalid University, Saudi Arabia, for funding this work under the grant number RCAMS/KKU/004-21. Taif University Researchers Supporting Project number (TURSP-2020/20), Taif University, Taif, Saudi Arabia.

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