Parametrizing Distinguished Varieties

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This paper is dedicated to Joseph Cima on the occasion of his 70th birthday.

Abstract

A distinguished variety is a variety that exits the bidisk through the distinguished boundary. We look at the moduli space for distinguished varieties of rank (2,2).

0 Introduction

In this paper, we shall be looking at a special class of bordered algebraic varieties that are contained in the bidisk $D^2$ in $\mathbb{C}^2$.

Definition 0.1 A non-empty set $V$ in $\mathbb{C}^2$ is a distinguished variety if there is a polynomial $p$ in $\mathbb{C}[z,w]$ such that

$$V = \{(z,w) \in D^2 : p(z,w) = 0\}$$

and such that

$$\overline{V} \cap \partial(D^2) = \overline{V} \cap (\partial D)^2.$$  (0.2)

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Condition \((0.2)\) means that the variety exits the bidisk through the distinguished boundary of the bidisk, the torus. We shall use \(\partial V\) to denote the set given by \((0.2)\): topologically, it is the boundary of \(V\) within the zero set of \(p\), rather than in all of \(C^2\).

In [?], the authors studied distinguished varieties, which we considered interesting because of the following two theorems:

**Theorem 0.3** Let \(T_1\) and \(T_2\) be commuting contractive matrices, neither of which has eigenvalues of modulus 1. Then there is a distinguished variety \(V\) such that, for any polynomial \(p\) in two variables, the inequality

\[
\|p(T_1, T_2)\| \leq \|p\|_V
\]

holds.

**Theorem 0.4** The uniqueness variety for a minimal extremal Pick problem on the bidisk contains a distinguished variety \(V\) that contains each of the nodes.

It is the goal of this paper to examine the geometry of distinguished varieties more closely, and in particular to parametrize the space of all distinguished varieties of rank \((2, 2)\) (see Definition 0.5 below).

Notice that if \(V\) is a distinguished variety, for each \(z\) in the unit disk \(D\), the number of points \(w\) satisfying \((z, w) \in V\) is constant (except perhaps at a finite number of multiple points, where the \(w\)’s must be counted with multiplicity). So the following definition makes sense:

**Definition 0.5** A distinguished variety is of rank \((m, n)\) if there are generically \(m\) sheets above every first coordinate and \(n\) above every second coordinate.

The principal result of this paper, Theorem 2.1, is a parametrization of distinguished varieties of rank \((2, 2)\).
1 Structure theory

For positive integers $m$ and $n$, let

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^m \oplus \mathbb{C}^n \to \mathbb{C}^m \oplus \mathbb{C}^n$$  \hspace{1cm} (1.1)$$

be an $(m+n)$-by-$(m+n)$ unitary matrix. Let

$$\Psi(z) = A + zB(I - zD)^{-1}C$$  \hspace{1cm} (1.2)$$

be the $m$-by-$m$ matrix valued function defined on the unit disk $\mathbb{D}$ by the entries of $U$. This is called the transfer function of $U$. Let

$$U' = \begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^m \to \mathbb{C}^n \oplus \mathbb{C}^m,$$

and let

$$\Psi'(w) = D^* + wB^*(I - wA^*)^{-1}C^*.$$  

Because $U^*U = I$, a calculation yields

$$I - \Psi(z)^*\Psi(z) = (1 - |z|^2) C^*(I - zD^*)^{-1}(I - zD)^{-1}C,$$  \hspace{1cm} (1.3)$$

so $\Psi(z)$ is a rational matrix-valued function that is unitary on the unit circle. Such functions are called rational matrix inner functions, and it is well-known that all rational matrix inner functions have the form (1.2) for some unitary matrix decomposed as in (1.1) — see e.g. [?] for a proof. The set

$$V = \{(z, w) \in \mathbb{D}^2 : \det(\Psi(z) - wI) = 0\}$$  \hspace{1cm} (1.4)$$

$$\quad = \{(z, w) \in \mathbb{D}^2 : \det(\Psi'(w) - zI) = 0\}$$  \hspace{1cm} (1.5)$$

$$\quad = \{(z, w) \in \mathbb{D}^2 : \det \begin{pmatrix} A - wI & zB \\ C & zD - I \end{pmatrix} = 0\}$$  \hspace{1cm} (1.6)$$

is a distinguished variety, because when $|z| = 1$, the eigenvalues of $\Psi(z)$ are unimodular (and a similar statement holds for $\Psi'$). The converse was proved in [?]: all distinguished varieties of rank $(m,n)$ can be represented in
this way. So the moduli space for distinguished varieties of rank \((m, n)\) is a quotient of the space of \((m + n)\)-by-\((m + n)\) unitaries. Let us write \(U^m_n\) to denote the set of \((m + n)\)-by-\((m + n)\) unitaries decomposed as in \((1.1)\). The following result is well-known.

**Proposition 1.7** Let \(U\) and \(U_1\) be in \(U^m_n\), with respective decompositions

\[
U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad U_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.
\]

Then they give rise to the same transfer function iff and only if there is an \(n\)-by-\(n\) unitary \(W\) such that

\[
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & W^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix}.
\]  (1.8)

**Proof:** By looking at the coefficients of powers of \(z\) in the transfer function, we see that \(U\) and \(U_1\) have the same transfer function if and only if

\[
A = A_1
\]

\[
BD^nC = B_1D_1^nC_1 \quad \forall \ n \in \mathbb{N}.
\]  (1.9)

Equation (1.8) is equivalent to

\[
A_1 = A
\]

\[
B_1 = BW
\]

\[
C_1 = W^*C
\]

\[
D_1 = W^*DW.
\]

Clearly these equations imply (1.9).

To see the converse, note that the fact that \(U\) and \(U_1\) are unitaries and \(A = A_1\) means \(BB^* = B_1B_1^*\). If \(B\) is invertible, define \(W\) to be \(B^{-1}B_1\). This is unitary since \(B\) and \(B_1\) have the same absolute values, and then the equations \(BC = B_1C_1\) and \(BDC = B_1D_1C_1\) yield (1.8).

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If $B$ is not invertible, then $A$ has norm one. Decompose
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
= \begin{pmatrix}
A' & 0 \\
0 & A''
\end{pmatrix}
\begin{pmatrix}
0 & B'' \\
0 & C''
\end{pmatrix},
\]
and apply the same argument to $B''$ and $C''$. 

Remark 1.10 Note that $W$ is unique unless $\|A\| = 1$.

2 Parametrizing distinguished varieties of rank $(2, 2)$

In this section, we address the question of when two different unitaries in $U_2^2$ give rise to the same distinguished variety. From the previous section we see that this is equivalent to asking when two rational matrix inner functions are isospectral.

Theorem 2.1 Let $U, \Psi$ and $V$ be as in (1.1), (1.2) and (1.4), with $U$ in $U_2^2$. Let
\[
U_0 = \begin{pmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{pmatrix}
\]
be another unitary in $U_2^2$. Then $U$ and $U_0$ give rise to the same distinguished variety iff
\begin{itemize}
  \item[(i)] $A$ and $A_0$ have the same eigenvalues.
  \item[(ii)] $D$ and $D_0$ have the same eigenvalues.
  \item[(iii)] $BC$ and $B_0C_0$ have the same trace.
\end{itemize}

Proof: For simplicity in the proof we will assume that $\det(A) \neq 0$ and that $A$ and $D$ both have two eigenvalues. (We can attain the remaining cases as a limit of these).

Let
\[
Q(z, w) = \det \begin{pmatrix}
A - wI & zB \\
C & zD - I
\end{pmatrix}
\]
(2.2)
\[
= \frac{\det D}{\det A^*} \det \begin{pmatrix} D^* - z & wB^* \\ C^* & wA^* - I \end{pmatrix}
\]  
(2.3)

\[
= p_2(z)w^2 + p_1(z)w + p_0(z)
\]  
(2.4)

\[
= q_2(w)z^2 + q_1(w)z + q_0(w),
\]  
(2.5)

where \( p_i \) and \( q_j \) are polynomials of degree at most 2. As \( V \) is the zero set of \( Q \), it is sufficient to prove that conditions (i) — (iii) completely determine \( Q \). Let \( \mu_1 \) and \( \mu_2 \) be the eigenvalues of \( D \) and \( l_1 \) and \( l_2 \) be the eigenvalues of \( A \).

We have

\[
p_2(z) = \det(zD - I)
\]

\[
= (z\mu_1 - 1)(z\mu_2 - 1),
\]

so is determined by (ii), the eigenvalues of \( D \). Similarly \( q_2(w) \) is determined by (i), the eigenvalues of \( A \).

From (2.3) we see that the coefficient of \( z^2 \) in \( Q \) is \( (\det D / \det A^*) \). Dividing (2.4) by \( p_2 \), we get

\[
\det(\Psi(z) - wI) = w^2 + \frac{p_1(z)}{p_2(z)}w + \frac{p_0(z)}{p_2(z)}.
\]  
(2.6)

As \( \Psi \) is a matrix inner function, we must have that the last term in (2.6), which is the product of the eigenvalues of \( \Psi \), is inner. Therefore

\[
p_0(z) = e^{i\theta}(z - \mu_1)(z - \mu_2).
\]

where

\[
e^{i\theta} = (\det D / \det A^*).
\]

It remains to determine \( p_1 \).

**Lemma 2.7** With notation as above, let

\[
\det(\Psi(z) - wI) = w^2 - a_1(z)w + a_0(z).
\]  
(2.8)

Then

\[
a_1(z) = a_0(z)a_1 \left( \frac{1}{z} \right).
\]  
(2.9)
Proof: For any fixed \( z \) in \( \mathbb{D} \), there are two \( w \)’s with \((z, w)\) in \( V \). The function \( a_0(z) \) is the product of these \( w \)’s, and \( a_1(z) \) is their sum. Labelling them (locally) as \( w_1(z) \) and \( w_2(z) \), the right-hand side of (2.9) is

\[
(w_1(z)w_2(z)) (w_1(\frac{1}{z}) + w_2(\frac{1}{z})).
\]

When the modulus of \( z \) is 1, because the variety is distinguished, the right-hand side of (2.9) equals the left-hand side. By analytic continuation, they must be equal everywhere.

Applying the lemma to \(-p_1/p_2\) and \(p_0/p_2\), we get

\[
p_1(z) = e^{i\theta} z^2 p_1(\frac{1}{z}).
\]

(2.10)

Writing

\[
p_1(z) = b_2 z^2 + b_1 z + b_0,
\]

(2.10) gives the two equations

\[
e^{i\theta} b_2 = b_0
\]

\[
e^{i\theta} b_1 = b_1.
\]

Comparing (2.4) and (2.5), the coefficient of \( z^2w \) gives us \( b_2 \) (since we know \( q_2 \)), and hence we also know \( b_0 \).

Finally, if we know the coefficient of \( zw \) in the power series expansion of (2.6), we will know

\[
b_1 p_2(0) - b_0 p_2'(0),
\]

and be done. But

\[
\Psi(z) - wI = A - wI + zBC + O(z^2),
\]

so the coefficient of \( zw \) is \(-\text{tr}(BC)\), which is given by (iii).
3 Open Problems

Two distinguished varieties are *geometrically equivalent* if there is a biholomorphic bijection between them.

**Question 3.1** When do two unitaries give rise to geometrically equivalent distinguished varieties?

Notice that all distinguished varieties of rank $(1, n)$ or $(m, 1)$ are geometrically equivalent, since they are all biholomorphic to the unit disk.

**Question 3.2** When are two distinguished varieties of rank $(2, 2)$ geometrically equivalent?

**Question 3.3** What is the generalization of Theorem 2.1 to distinguished varieties of rank $(2, 3)$ or $(3, 3)$?

W. Rudin showed that smoothly bounded planar domains are geometrically equivalent to distinguished varieties iff their connectivity is 0 or 1 [?].

**Question 3.4** Which distinguished varieties are geometrically equivalent to planar domains?

**Question 3.5** How can one read the topology of a distinguished variety from a unitary that determines it as in Section 1?