THE HÖLDER QUASICONTINUITY FOR RIESZ-MORREY POTENTIALS AND LANE-EMDEN EQUATIONS

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Abstract. This note is devoted to exploring Hölder’s quasicontinuity for the Riesz-Morrey potentials, and its application to the corresponding nature of some nonnegative weak solutions of the quasilinear Lane-Emden equations for the $p$-Laplacian.

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1. Introduction

Let $(\alpha, p, \lambda) \in (0, n) \times [1, \infty) \times (0, n] \text{ and } \Omega \text{ be a bounded domain in the } 2 \leq n \text{-dimensional Euclidean space } \mathbb{R}^n. \text{ The main ideas in } \cite{5, 6, 7, 8, 9, 32} \text{ suggest us to deal with two basic concepts in the theory of Morrey spaces and their potentials.}

The first one is the so-called Riesz-Morrey potential – the $\alpha$-order Riesz singular integral

$$I_\alpha f(x) = \int_{\mathbb{R}^n} f(y)|y - x|^{\alpha - n} \, dy = \int_{\Omega} f(y)|y - x|^{\alpha - n} \, dy$$

of $f$ (whose value on $\mathbb{R}^n \setminus \Omega$ is defined to be 0) in the Morrey space

$$L^{p,\lambda}(\Omega) = \left\{ g \in L^p(\Omega) : \|g\|_{L^{p,\lambda}(\Omega)} = \sup_{x \in \Omega, 0 < r < \text{diam}(\Omega)} \left( r^{\lambda - n} \int_{B(x, r) \cap \Omega} |g|^p \right)^{\frac{1}{p}} < \infty \right\},$$

where diam$(\Omega)$ is the diameter of $\Omega$, $B(x, r)$ is the open ball with center $x$ and radius $r$, and the integral is taken with respect to the $n$-dimensional Lebesgue measure $dy$.

The second one is the Riesz-Morrey capacity of a set $E \subseteq \Omega$:

$$C_\alpha(E; L^{p,\lambda}(\Omega)) = \inf_{E \subseteq \text{open} \ O \subseteq \Omega} C_\alpha(O; L^{p,\lambda}(\Omega)) = \inf_{E \subseteq \text{open} \ O \subseteq \text{compact} \ K \subseteq \text{open} \ O} \sup_{C_\alpha(K; L^{p,\lambda}(\Omega))}$$

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where
\[ C_\alpha(K; L^{p,\lambda}(\Omega)) = \inf \{ \| h \|_{L^{p,\lambda}(\Omega)}^p : 0 \leq h \in L^{p,\lambda}(\Omega) \ \& \ I_\alpha h \geq 1_K \} \]
for which \(1_K\) stands for the characteristic function of the compact \(K \subseteq \Omega\).

In this note, through using the Riesz-Morrey capacity we study the quasicontinuous representative and the Hölder quasicontinuity of each Riesz-Morrey potential – see Theorems 2.2 & 2.4. Certainly, the discovered properties show their worth in connection with investigating Hölderian quasicontinuity of some nonnegative weak solutions \(u\) of the quasilinear Lane-Emden equations for \(p\)-Laplacian:
\[-\Delta_p u = -\text{div}(\nabla u |^{p-2} \nabla u) = u^{q-1} \text{ or } e^u \text{ in } \Omega,\]
where \((p, q) \in (1, n) \times (0, \infty)\) – see Theorems 3.3 & 3.4.

**Notation.** In what follows, \(\Omega\) is always assumed to be a bounded domain in \(\mathbb{R}^n\). For \(E \subseteq \Omega\) define \(\bar{E}\) to be the integral mean over \(\Omega\) with respect to the Lebesgue measure \(dy\). And, \(X \leq Y\) stands for \(X \leq cY\) for a constant \(c > 0\). Moreover, \(X \approx Y\) means both \(X \leq Y\) and \(Y \leq X\).

2. **Riesz-Morrey potentials**

2.1. **Quasicontinuous representation for \(I_\alpha L^{p,\lambda}\).** A function \(g\) on \(\Omega\) is said to be \(C_\alpha(\cdot; L^{p,\lambda})\)-quasicontinuous provided that for any \(\epsilon > 0\) there is a continuous function \(\tilde{g}\) on \(\Omega\) such that
\[ C_\alpha(\{ x \in \Omega : \tilde{g}(x) \neq g(x) \}; L^{p,\lambda}(\Omega)) < \epsilon. \]

Naturally, \(\tilde{g}\) is called a \((\cdot; L^{p,\lambda}(\Omega))\)-quasicontinuous representative of \(g\). The forthcoming Theorem 2.2 is an extension of [4 Theorem 6.2.1] from \(L^p\) to \(L^{p,\lambda}\).

**Lemma 2.1.** For \(1 < p < \infty\) and \(0 < \gamma \leq n\), let \(L^{p,\gamma}(\Omega)\) be the Zorko space (cf. [39]) of all \(f \in L^{p,\gamma}(\Omega)\) that can be approximated by \(C^1\) functions with compact support in \(\Omega\) under the norm \(\| \cdot \|_{L^{p,\gamma}(\Omega)}\). Then \(L^{p,\lambda}(\Omega) \subseteq L^{p,\gamma}(\Omega) \forall \lambda \in (0, \gamma).

**Proof.** If \(\gamma = n\), then \(L^{p,\gamma}(\Omega) = L^p(\Omega)\), and hence
\[ \int_{\Omega \cap B(x,r)} |f|^p \leq \| f \|_{L^{p,\lambda}(\Omega)}^p (\text{diam}(\Omega))^{n-1} \ \forall \ (\lambda, x, r) \in (0, n) \times \Omega \times (0, \text{diam}(\Omega)), \]
thereby deriving the desired inclusion.

If \(\gamma < n\), then the desired inclusion follows from [9 Lemma 3.4]. \(\square\)

**Theorem 2.2.** Let \(g = I_\alpha f, f \in L^{p,\lambda}(\Omega)\), and \(1 < p < \lambda/\alpha < \mu/\alpha \leq n/\alpha\). Then there is a set \(\Sigma \subseteq \Omega\) such that:
\(i\) \(C_\alpha(\Sigma; L^{p,\mu}(\Omega)) = 0;\)
\(ii\) \(\lim_{r \to 0} \int_{B(x,r)} g = \tilde{g}(x) \ \forall \ x \in \Omega \setminus \Sigma;\)
\(iii\) \(\lim_{r \to 0} \int_{B(x,r)} |g - \tilde{g}(x)| = 0 \ \forall \ x \in \Omega \setminus \Sigma.\)
Moreover, one has:
\(iv\) For any \(\epsilon > 0\) there is an open set \(O \subseteq \Omega\) such that \(C_\alpha(O; L^{p,\mu}(\Omega)) < \epsilon\) and the convergence in (ii)-(iii) is uniform on \(\Omega \setminus O;\)
\(v\) \(\tilde{g}\) is a \(C_\alpha(\cdot; L^{p,\mu}(\Omega))\)-quasicontinuous representative of \(g;\)
\(vi\) \(\tilde{g}(x) = g(x) \ \forall \ x \in \Omega \setminus O.\)

**Proof.** Given \(r \in (0, \infty),\) let
\[ \chi(x) = 1_{\mathbb{R}_+}(x) \omega_n^{-1} \ \& \ \chi_r(x) = r^n \chi(x/r), \]
where
where \( \omega_n \) is the volume of the unit ball \( \mathbb{B}^n \) of \( \mathbb{R}^n \). For \( f \in L^{p,\lambda}(\Omega) \), \( \epsilon > 0 \) and \( \mu \in (\lambda, n] \), we use Lemma 2.1 to find a Schwartz function \( f_0 \) on \( \mathbb{R}^n \) such that \( f_0 = 0 \) in \( \mathbb{R}^n \setminus \Omega \) and \( \| f - f_0 \|_{L^{p,\mu}(\Omega)} < \epsilon \). Consequently, \( g_0 = I_\alpha f_0 \) is a Schwartz function and \( \chi_r \ast g_0 \) converges to \( g_0 \) on \( \Omega \) as \( r \to 0 \). Note that

\[
\int_{B(x,r)} g = \chi_r \ast g(x) \quad \text{and} \quad \int_{B(x,r)} g_0 = \chi_r \ast g_0(x).
\]

Thus, for \( \delta > 0 \) letting

\[
J_\delta g(x) = \sup_{0 < r < \delta} (\chi_r \ast g)(x) - \inf_{0 < r < \delta} (\chi_r \ast g)(x),
\]

we have

\[
J_\delta g(x) \leq J_\delta (g - g_0)(x) + J_\delta g_0(x).
\]

By the previously-stated convergence, for any given \( \epsilon > 0 \) there exists \( \delta > 0 \) so small that \( \sup_{x \in \Omega} J_\delta g_0(x) < \epsilon \). If \( \mathcal{M} \) stands for the Hardy-Littlewood maximal operator, then

\[
|\chi_r \ast (g - g_0)(x)| \leq \mathcal{M}(g - g_0)(x) \quad \forall \quad x \in \Omega,
\]

and hence

\[
J_\delta g(x) \leq \mathcal{M}(g - g_0)(x) + \epsilon \quad \forall \quad x \in \Omega.
\]

Upon choosing \( \omega/2 > \epsilon > 0 \), the last estimate gives

\[
E_\omega := \{ x \in \Omega : J_\delta g(x) > \omega \} \subseteq \{ x \in \Omega : J_\delta (g - g_0)(x) > \omega/2 \} =: F_\omega.
\]

In view of the definition of \( C_\alpha(\cdot; L^{p,\mu}(\Omega)) \) and the boundedness of \( \mathcal{M} \) acting on \( L^{p,\mu}(\Omega) \), we find

\[
C_\alpha(E_\omega; L^{p,\mu}(\Omega)) \leq C_\alpha(F_\omega; L^{p,\mu}(\Omega)) \leq \omega^{-p} \| f - f_0 \|_{L^{p,\mu}(\Omega)} \leq (\epsilon \omega^{-1})^p.
\]

For each natural number \( j \) let \( \omega = 2^{-j}, \epsilon = 4^{-j} \), and \( \delta_j \) be their induced number. If

\[
G_j = \{ x \in \Omega : J_\delta g(x) > 2^{-j} \},
\]

then

\[
C_\alpha(G_j; L^{p,\mu}(\Omega)) \leq 2^{-jp}.
\]

Furthermore,

\[
O_k = \bigcup_{j=k}^{\infty} G_j \Rightarrow C_\alpha(O_k; L^{p,\mu}(\Omega)) \leq \sum_{j=k}^{\infty} 2^{-jp} \to 0 \quad \text{as} \quad k \to \infty.
\]

Consequently, under

\[
\begin{cases}
1 < p < \mu/\alpha; \\
\mu - \alpha p < d \leq n; \\
0 < q < dp/(\mu - \alpha p),
\end{cases}
\]

one has

\[
C_\alpha(\cap_{k=1}^{\infty} O_k; L^{p,\mu}(\Omega)) = 0.
\]

Note that

\[
x \notin O_k \Rightarrow J_\delta g(x) \leq 2^{-j} \quad \forall \quad \delta \leq \delta_j \quad \& \quad j \geq k.
\]

So,

\[
\lim_{r \to 0} \chi_r \ast g(x) = \tilde{g}(x) \quad \text{exists for any} \quad x \notin \bigcap_{k=1}^{\infty} O_k.
\]

Clearly, this convergence is uniform on \( \Omega \setminus O_k \) with sufficiently small \( C_\alpha(O_k; L^{p,\mu}(\Omega)) \). This proves the results of Theorem 2.2 with \( \Sigma = \cap_{k=1}^{\infty} O_k \) except the part (iii).
However, the demonstration of the part (iii) follows from a slight modification of the above argument plus defining

\[ J_\delta(g - \tilde{g})(x) = \sup_{0 < r \leq \delta} (\chi_r * |g - \tilde{g}(x)|)(x) \]

and so establishing

\[ J_\delta(g - \tilde{g})(x) \leq M(g - g_0)(x) + |(\tilde{g} - g_0)(x)| + \epsilon \]

under

\[ J_\delta(g - g_0(x))(x) < \epsilon. \]

\[ \square \]

2.2. Hölderian quasicontinuity for \( I_\alpha L^{p,1} \). Given \( \beta \in (0, 1] \). We say that \( g \in Lip_\beta(\Omega) \) provided that \( g \) satisfies

\[ \sup \left\{ \frac{|g(x) - g(y)|}{|x - y|^\beta} : x, y \in \Omega, x \neq y \right\} < \infty. \]

In particular, if \( \beta \in (0, 1) \) or \( \beta = 1 \) then \( g \) is called \( \beta \)-Hölder continuous or Lipschitz continuous. Moreover, a function \( g \) defined on \( \Omega \) is called Hölder quasicontinuous if for any \( \epsilon > 0 \) there is a set \( E \subset \Omega \) of a given capacity smaller than \( \epsilon \) such that \( g \) is of the Hölder continuity on \( \Omega \setminus E \). The forthcoming Theorem 2.4 shows that any function in \( I_\alpha L^{p,1} \) is of Hölder quasicontinuity. To be more precise, let us recall the Sobolev-Morrey type imbedding (cf. [1], [2]):

\[ I_\alpha : L^{p,1}(\Omega) \mapsto \begin{cases} L^{\frac{\lambda}{\alpha},p}(\Omega) \cap L^{p,1-\alpha p}(\Omega), & 1 < p < \lambda/\alpha; \\ BMO(\Omega), & 1 < p = \lambda/\alpha, \end{cases} \]

where

\[ f \in BMO(\Omega) \iff \sup_{x\in\Omega, 0<r<\text{diam} \Omega} \left\{ \fint_{B(x,r)\cap\Omega} \left| f - \fint_{B(x,r)\cap\Omega} f \right| \right\} < \infty. \]

Interestingly, the above imbedding can be extended from \( p \leq \lambda/\alpha \) to \( p > \lambda/\alpha \).

Lemma 2.3. Let \( g = I_\alpha f, f \in L^{p,1}(\Omega) \), and \( (\alpha, p, \lambda) \in (0, n) \times (1, \infty) \times (0, n] \). (i) If \( 0 < \delta = \alpha - \lambda/p < 1 \), then \( g \in Lip_\delta(\Omega) \).

(ii) If

\[ \begin{cases} 1 < p < \lambda/\alpha; \\ 1 < q < \min\{p, \lambda/\alpha\}; \\ \mu = n - (n - \lambda)q/p; \\ 0 < \beta < \min \left\{1, \alpha(1 - q/p), \lambda(1 - q/p)/(\lambda + (1 - \alpha)q)\right\}, \end{cases} \]

then for any \( r \in (0, 1) \) there exist \( f_r \in L^{p,1}(\Omega) \) and \( g_r = I_\alpha f_r \) such that

\[ \begin{cases} \|f - f_r\|_{L^{\mu,p}(\Omega)} \leq \rho^\beta; \\ |g_r(x) - g_r(y)| \leq |x - y|^{\beta} \quad \forall \quad y \in B(x, r) \subseteq \Omega. \end{cases} \]

Proof. (i) Since \( \alpha = \delta + \lambda/p \), an application of [1] Corollary (iii) and [10] page 91 gives

\[ I_\alpha L^{p,1}(\Omega) = I_\delta I_{\lambda/p} L^{p,1}(\Omega) \subseteq I_\delta BMO(\Omega) \subseteq Lip_\delta(\Omega), \]

whence implying \( g \in Lip_\delta(\Omega) \).
(ii) Without loss of generality, we may assume \( \|f\|_{L^{p,1}(\Omega)} \leq 1 \) and \( f\|_{\mathbb{R}^n,\Omega} = 0 \). Since \( \Omega \) is bounded, there is a big ball \( B(x, r) \) with center \( x \in \Omega \) and radius \( r \leq \text{diam}(\Omega) \) such that \( \Omega \subseteq B(x, r) \), and consequently,

\[
\|f\|_{L^p(\Omega)}^p = \int_{\Omega} |f|^p \leq (\text{diam}(\Omega))^{n-\lambda}.
\]

For \( r \in (0, 1) \) let \( O_r = \{ x \in \Omega : |f(x)| > s_r \} \), \( s_r = r^{\beta q/(q-p)} \), and

\[
f_r = \begin{cases} f & \text{on } \Omega \setminus O_r; \\ 0 & \text{on } O_r. \end{cases}
\]

Evidently,

\[
\int_{O_r} 1_{O_r} \leq s_r^{-\lambda}(\text{diam}(\Omega))^{n-\lambda}
\]

and \( g_r = I_\alpha f_r \) is bounded. Moreover, by Hölder’s inequality and the definition of \( \| \cdot \|_{L^{p,1}(\Omega)} \), one gets

\[
\|f - f_r\|_{L^{p,1}(\Omega)}^p \leq \|f\|_{L^{p,1}(\Omega)}^p \left( \int_{O_r} 1_{O_r} \right)^{\frac{\mu-1}{\mu n}} \leq (\text{diam}(\Omega))^{\frac{\mu-1}{\mu n}} s_r^{-\alpha/p} \leq (\text{diam}(\Omega))^{(n-\lambda)(1-q/p)} r^{\alpha \beta}. 
\]

Meanwhile, thanks to \( f_r \leq s_r \), we can use (i) above to get that if

\[
p < \bar{p} = \frac{\lambda(p-q) - \beta pq}{\alpha(p-q) - \beta p} \quad \text{and} \quad 0 < \bar{\beta} = \alpha - \lambda/\bar{p} < 1,
\]

then

\[
|g_r(x) - g_r(y)| = |I_\alpha f_r(x) - I_\alpha f_r(y)| \leq \|f_r\|_{L^{p,1}(\Omega)} |x - y|^\bar{\beta} \quad \forall \ y \in B(x, r).
\]

Another application of the Hölder inequality gives

\[
\|f_r\|_{L^{p,1}(\Omega)}^p \leq s_r^{-p} \|f\|_{L^{p,1}(\Omega)}^p \leq s_r^{-p}. 
\]

Thus, \( |g_r(x) - g_r(y)| \leq r^{\bar{\beta}} \) holds for any \( y \in B(x, r) \).

\[\Box\]

Below is the Hölder quasicontinuity for the Riesz-Morrey potentials which actually gives a nontrivial generalization of [23, Theorem 7] (see [19] for a further development of [23]).

**Theorem 2.4.** Let \( g = I_\alpha f, f \in L^{p,\lambda}(\Omega) \), and \( 1 < p < \lambda/\alpha \leq n/\alpha \). If

\[
\begin{cases}
1 < q < \min\{p, \lambda/\alpha\} = p; \\
\mu = n - (n - \lambda)q/p; \\
0 < \gamma < \min\{1, \alpha(1 - q/p), \lambda(1 - q/p)/(\lambda + (1 - \alpha)q)\}, \\
\end{cases}
\]

then for any \( \epsilon > 0 \) there exists an open set \( O \subseteq \Omega \) and a \( \gamma \)-Hölder continuous function \( h \) on \( \Omega \) such that

\[
C_\alpha(O; L^{q,\mu}(\Omega)) < \epsilon \quad \text{and} \quad g = h \quad \text{in} \quad \Omega \setminus O.
\]
Proof. The notations introduced in Lemma 2.3 and its proof will be used in the sequel. Given \( \gamma \in (0, \beta) \) with \( \beta \) as in Lemma 2.3. Now, for each natural number \( j \) let \( r_j \) be chosen so that

\[
(2.1) \quad r_0 = 1 \quad \& \quad \left( \frac{r_{j+1}}{r_j} \right)^\gamma \leq 1/2.
\]

For simplicity, set \( h_j = g_{r_j} \) and then \( f_j \) be the corresponding \( f_{r_j} \) and

\[
\sum_{j=1}^\infty \| f_{j+1} - f_j \|_{L^{p,\beta}(\Omega)} < \infty.
\]

Choosing

\[
\begin{cases}
&w_j = \max \{ -r_j^\gamma, \min\{r_j^\gamma, h_{j+1} - h_j\} \}; \\
&O_j = \{ x \in \Omega : |h_{j+1}(x) - h_j(x)| > r_j^\gamma \},
\end{cases}
\]

we use the already-established estimate

\[
\| f - f_r \|_{L^{p,q}(\Omega)} \leq \left( \text{diam}(\Omega) \right)^{(n-\lambda)(1/q - 1/p)} r^\rho
\]

and the definition of \( C_\alpha(\cdot; L^{q,\mu}(\Omega)) \) to obtain

\[
C_\alpha(O_j; L^{q,\mu}(\Omega)) \leq r_j^{-\gamma q} \| f_{j+1} - f_j \|_{L^{q,\mu}(\Omega)} \leq r_j^{(\beta - \gamma)q},
\]

Consequently, for any \( \epsilon > 0 \) there is a big integer \( J \) such that

\[
\sum_{j=J}^\infty C_\alpha(O_j; L^{q,\mu}(\Omega)) \leq \sum_{j=J}^\infty r_j^{(\beta - \gamma)} < \epsilon.
\]

Putting

\[
O = \bigcup_{j=J}^\infty E_j \quad \& \quad h = h_J + \sum_{j=J}^\infty w_j,
\]

we find that \( O \) is an open subset of \( \Omega \) and

\[
C_\alpha(O; L^{q,\mu}(\Omega)) < \epsilon \quad \& \quad h = g \quad \text{on} \quad \Omega \setminus O.
\]

It remains to check that \( h \) is \( \beta \)-Hölder continuous on \( \Omega \). Of course, it is enough to verify

\[
|h(x) - h(y)| \leq |x - y|^\beta \quad \forall \quad x, y \in \Omega \quad \text{with} \quad |x - y| \leq r_j.
\]

Obviously, \( h_j \) is \( \beta \)-Hölder continuous. To show the similar property for \( \sum_{j=J}^\infty w_j \), we may assume

\[
x, y \in \Omega; \quad 0 < |x - y| \leq r_j; \quad r_{k+1} < |x - y| \leq r_k.
\]

From (2.1) it follows that

\[
(2.2) \quad k \leq \left( \frac{\gamma}{\ln 2} \right) \ln \frac{1}{r_k} \leq \left( \frac{\gamma}{(\beta - \gamma) \ln 2} \right) r_k^{\gamma - \beta} \leq \left( \frac{\gamma}{(\beta - \gamma) \ln 2} \right) |x - y|^{\gamma - \beta}
\]

When \( 1 \leq j \leq k \), an application of the last estimate in Lemma 2.3 gives

\[
|w_j(x) - w_j(y)| \leq |x - y|^\beta.
\]

When \( j > k \), another application of (2.1) yields

\[
|w_j(x) - w_j(y)| \leq 2r_j^\gamma \leq 2^{k-j+2} r_{k+1}^\gamma \leq 2^{k-j+2} |x - y|^\gamma.
\]

This, together with (2.2) and \( h = h_J + \sum_{j=J}^\infty w_j \), derives

\[
|h(x) - h(y)| \leq |x - y|^\gamma + k|x - y|^\beta \leq |x - y|^\gamma.
\]

\[ \Box \]
3. Lane-Emden Equations

3.1. Hölderian quasicontinuity for \(-\Delta_p u = u^{r+1}\). Recall that for \(1 \leq p < \infty\) the Sobolev space \(W^{1,p}(\Omega)\) consists of all functions \(f\) with

\[
\|f\|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} |f|^p \right)^{\frac{1}{p}} + \left( \int_{\Omega} |\nabla f|^p \right)^{\frac{1}{p}} < \infty
\]

and the Sobolev space \(W_0^{1,p}(\Omega)\) is the completeness of \(C^1_0(\Omega)\) (all \(C^1\) functions \(f\) with compact support in \(\Omega\)) under \(\|\cdot\|_{W^{1,p}(\Omega)}\). According to [18, Lemma 7.14], any \(f \in W_0^{1,1}(\Omega)\) can be represented via

\[
f(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \int_{\Omega} |x-y|^{-n}(x-y) \cdot \nabla f(y) \, dy \quad |f(x)| \leq \left( \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \right) (I_{\Omega}) |\nabla f|(x) \quad \text{a.e. } x \in \Omega,
\]

where \(\Gamma(\cdot)\) is the usual gamma function. As a variant of \(C_1(K; L^{p,a}(\Omega))\) the variational \(p\)-capacity of a compact \(K \subseteq \Omega\) is defined by

\[
C(K; W_0^{1,p}(\Omega)) = \inf \left\{ \int_{\Omega} |\nabla f|^p : f \in W_0^{1,p}(\Omega) \land f \geq 1_K \right\}.
\]

Clearly, this definition is extendable to an arbitrary set \(E \subseteq \Omega\) through (cf. [21, p.27])

\[
C(E; W_0^{1,p}(\Omega)) = \inf_{E \subseteq \text{open } O \subseteq \Omega} C(O; W_0^{1,p}(\Omega)) = \inf_{E \subseteq \text{open } O \subseteq \Omega} \sup_{C \subseteq \text{compact } K \subseteq \text{open } O} C(K; W_0^{1,p}(\Omega)).
\]

Importantly, such a capacity can used to establish the following relatively independent Sobolev embedding whose (v) is indeed a sort of motivation to investigate the quasilinear Lane-Emden equations.

**Proposition 3.1.** Given \(1 < p < \min\{n, q\}\) and \(0 < r < q(1-p^{-1})\), let \(\nu\) be a nonnegative Radon measure on \(\Omega\). Then the following properties are mutually equivalent:

(i) \(I_1\) is a continuous operator from \(L^q(\Omega)\) into \(L^q(\Omega, \nu)\);

(ii) \(W_0^{1,p}(\Omega)\) continuously embeds into \(L^q(\Omega, \nu)\);

(iii) Isocapacitary inequality \(\nu(K) \leq C(K; W_0^{1,p}(\Omega))^{\frac{q}{p}}\) holds for all compact sets \(K \subseteq \Omega\);

(iv) Isocapacitary inequality \(\nu(B(x, r)) \leq r^{-\frac{q}{p}}\) holds for all \(B(x, r) \subseteq \Omega\);

(v) Faber-Krahn’s inequality \(\nu(O)^{\frac{q}{p}-1} \leq \lambda_{p,\nu}(O)\) holds for all bounded open sets \(O \subseteq \Omega\), where

\[
\lambda_{p,\nu}(O) = \inf \left\{ \int_O |\nabla f|^p : f \in C_0^1(O) \land f \neq 0 \text{ on } O \right\}.
\]

**Proof.** (ii)\(\Leftrightarrow\)(iii)\(\Leftrightarrow\)(iv)\(\Leftrightarrow\)(i) is essentially known – see, for example, [24, 25] and [4, Theorem 7.2.2].

So, it remains to prove (ii)\(\Leftrightarrow\)(v). If (ii) is valid, then the Hölder inequality yields that for any open set \(O \subseteq \Omega\) and \(f \in C_0^1(O)\),

\[
\int_O |f|^p \, d\nu \leq \left( \int_O |f|^p \, d\nu \right)^{\frac{q}{p}} \nu(O)^{1-\frac{p}{q}} \leq \left( \int_O |\nabla f|^p \nu(O)^{1-\frac{p}{q}} \right)^{\frac{q}{p}}
\]
holds, whence giving (v). For the converse, we use the argument methods in [12] pp. 159-161 and [14] to proceed. Suppose (v) is true. Then for any $f \in W_0^{1,p}(\Omega)$ and any $t > 0$,

$$
\int_{\Omega} |f|^p \, dv \leq \int_{\{y \in \Omega : |f(y)| > t\}} |f|^p \, dv + t^{p-1} \int_{\{y \in \Omega : |f(y)| \leq t\}} |f| \, dv
$$

$$
\leq \frac{\int_{\{y \in \Omega : |f(y)| > t\}} \nabla f|^p \, dv}{tn(|y \in \Omega : |f(y)| > t\})^\frac{p}{p-1} + t^{p-1} \int_{\{y \in \Omega : |f(y)| \leq t\}} |f| \, dv.
$$

Choosing

$$
t = \left( \frac{\int_{\Omega} \nabla f|^p \, dv}{\int_{\Omega} |f| \, dv} \right)^{\frac{1}{p-1}},
$$

we get a constant $c > 0$ such that

$$
\int_{\Omega} |f|^p \, dv \leq 2c \left( \int_{\Omega} |\nabla f|^p \right)^{\frac{p(q-1)}{p+q-1}} \left( \int_{\Omega} |f| \, dv \right)^{\frac{q-p}{p+q-1}}.
$$

Replacing this $f$ by

$$
f_k = \min \{ \max\{f - 2^k, 0\}, 2^k\}, \quad k = 0, \pm 1, \pm 2, \ldots,
$$

we have

$$
\left( \int_{\Omega} f_k|^p \, dv \right)^{\frac{p(q-1)}{pq-1}} \leq (2c)^{\frac{p(q-1)}{pq-1}} \left( \int_{\Omega} |\nabla f|^p \right)^{\frac{p(q-1)}{pq-1}} \left( \int_{\Omega} f_k \, dv \right)^{\frac{q-p}{pq-1}}.
$$

This implies

$$
\left( 2^k v(|y \in \Omega : f(y) \geq 2^k) \right)^{\frac{p(q-1)}{pq-1}} \leq (2c)^{\frac{p(q-1)}{pq-1}} \left( \int_{\{y \in \Omega : 2^k \leq f(y) < 2^{k+1}\}} |\nabla f|^p \right)^{\frac{p(q-1)}{pq-1}} \left( 2^k v(|y \in \Omega : f(y) \geq 2^k) \right)^{\frac{q-p}{pq-1}}.
$$

Setting

$$
\begin{align*}
  a_k &= 2^k v(|y \in \Omega : f(y) \geq 2^k)); \\
  b_k &= \int_{\{y \in \Omega : 2^k \leq f(y) < 2^{k+1}\}} |\nabla f|^p; \\
  \theta &= \frac{p(p-1)}{p+q-1},
\end{align*}
$$

one has $a_{k+1} \leq 2^{1+q} cb_k^\theta a_k^{(1-\theta)}$. This, together with Hölder’s inequality, derives

$$
\sum_k a_k \leq 2^{1+q} c \sum_k b_k^\theta a_k^{(1-\theta)}
$$

$$
\leq 2^{1+q} c \left( \sum_k b_k \right)^\theta \left( \sum_k a_k \right)^{(1-\theta)}
$$

$$
\leq 2^{1+q} c \left( \int_{\Omega} |\nabla f(y)|^p \, dy \right)^\theta \left( \sum_k a_k \right)^{(1-\theta)}.
$$

A simplification of these estimates yields (ii).
\textbf{Remark 3.2.} The part on Faber-Krahn’s inequality under \((p, q, dv) = (2, 2n/(n - 2), dy)\) of Proposition 3.1 appeared in [11, 20, 36, 37, 38]. In particular, if
\[ dv = \omega \, dy \quad \& \quad 1 < p < q < pn/(n - p), \]
then condition (iv) above says that \(0 \leq \omega\) belongs to the Morrey space \(L^{1,n-(a-p)q/p}(\Omega)\) – in other words – the Sobolev imbedding under this circumstance is fully controlled by this Morrey space; see [26] for a similar treatment on the Schrödinger operator \(-\Delta + V\). Furthermore, when the last \(\omega\) equals identically 1, there is a nonnegative function \(u \in W^{1,p}_0(\Omega)\) such that the Euler-Lagrange (or Lane-Emden type) equation
\[ -\Delta_p u = \lambda p,\nu(\Omega)u^{p-1} \quad \text{in} \quad \Omega \]
holds in the weak sense:
\[ \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = \lambda p,\nu(\Omega) \int_{\Omega} u^{p-1} \phi \quad \forall \ \phi \in W^{1,p}_0(\Omega); \]
see e.g. [22] and its related references.

In view of Proposition 3.1, Remark 3.2, and the research of the Lane-Emden equations in [27, 28, 29, 30, 31, 33, 34, 38, 17, 13], we consider the nonnegative weak solutions of the quasilinear Lane-Emden equation with index \((p, q) \in (1, n) \times (0, \infty)\):
\[ -\Delta_p u = u^{q+1} \quad \text{in} \quad \Omega, \]
and utilize Theorem 2.4 to get the following result.

\textbf{Theorem 3.3.} Let
\[ \begin{cases}  (p, q) \in (1, n) \times (0, \infty); \\ \tilde{q} \geq \max\{p, q + 2\}; \\ n \geq \lambda \geq \max\left\{ \frac{n(q+2)}{q}, \frac{p(q+1)}{q} \right\}. \end{cases} \]
If \(u \in L^{\tilde{q}}(\Omega)\) is a nonnegative weak solution of (3.1), then for any \( \varepsilon > 0 \) there is an open set \(O \subseteq \Omega\) such that \(C_1(O; L^{\tilde{q}+2}(\Omega)) < \varepsilon\) and \(I_1|\nabla u|\) is \(\tilde{q}\)-Hölder continuous in \(\Omega \setminus O\) where
\[ \begin{cases}  1 < \tilde{q} < p \leq \lambda < \tilde{\mu} = n - (n - \lambda)\tilde{q}/p; \\ 0 < \tilde{q} < 1 - \tilde{q}/p. \end{cases} \]

\textbf{Proof.} Suppose \(u \in L^{\tilde{q}}(\Omega)\) is a nonnegative weak solution of (3.1). Then
\[ \int_{\Omega} u^{q+1} \phi = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \quad \forall \ \phi \in W^{1,p}_0(\Omega). \]
Given \(x_0 \in \Omega\) and \(0 < r < \text{diam}(\Omega)\). Upon taking a test function \(\phi = \eta^2\) such that
\[ \begin{cases}  \eta(x) = 1 \quad \text{for} \quad x \in B(x_0, r/3); \\ \eta(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus B(x_0, r/2); \\ |\nabla \eta(x)| \leq r^{-1} \quad \text{for} \quad x \in B(x_0, r), \end{cases} \]
we utilize (3.2) to get
\[ \int_{\Omega} u^{q+2} \eta^2 = \int_{\Omega} |\nabla u|^{p-2} |\nabla u|^2 \eta^2 + 2^{-1} \int_{\Omega} |\nabla u|^{p-2} (\nabla(u^2)) \cdot (\nabla(\eta^2)). \]
Through the properties of \( \eta \), Young’s inequality

\[
ab \leq \frac{a^\theta}{\theta} + \frac{b^\theta}{\theta'} \quad \forall \quad a, b, \epsilon, \theta - 1 > 0 \quad \& \quad \theta' = \frac{\theta}{\theta - 1},
\]

(applied to the last integral), and Hölder’s inequality, we find

\[
\int_{B(x_0,r/3) \cap \Omega} |\nabla u|^p \leq \int_{B(x_0,r/3) \cap \Omega} u^{2+q} + r^{-p} \int_{B(x_0,r/3) \cap \Omega} u^p
\]

\[
\leq \left( \int_{B(x_0,r/3) \cap \Omega} u^\hat{q} \right)^{\frac{2+q}{\hat{q}}} r^p(1-\frac{2+q}{\hat{q}}) + \left( \int_{B(x_0,r/3) \cap \Omega} u^{\hat{q}} \right)^{\frac{p}{\hat{q}}} r^p(1-\frac{\hat{q}}{q} - p)
\]

where the assumption on \( p, q, \hat{q}, \lambda \) and the following definition

\[
\hat{\lambda} = \lambda - \max \left\{ \frac{n(q+2)}{q}, \frac{p(n+\hat{q})}{\hat{q}} \right\} \geq 0
\]

have been used. Therefore, \( |\nabla u| \in L^{p,\hat{\lambda}}(\Omega) \) and desired assertion follows from applying Theorem 2.4 to the Riesz-Morrey potential \( I_1|\nabla u| \).

\[\square\]

3.2. Hölderian quasicontinuity for \( -\Delta_p u = e^u \). The recent works [34, 35, 15, 16], along with Theorem 3.3, have driven us to consider the nonnegative weak solutions to the quasilinear Lane equation for the \( 1 < p < n \) Laplacian:

\[
-\Delta_p u = e^u \quad \text{in} \quad \Omega,
\]

thereby discovering the following fact.

**Theorem 3.4.** Let \( 1 < p < n \). If \( u \) with \( \int_{\Omega} u e^u < \infty \) is a nonnegative weak solution of (3.5), then for any \( \epsilon > 0 \) there is an open set \( O \subseteq \Omega \) such that \( C_1(O; L^{\hat{q}}(\Omega)) < \epsilon \) and \( I_1|\nabla u| \) is \( \hat{q} \)-Hölder continuous in \( \Omega \setminus O \) where

\[\begin{aligned}
1 < \hat{q} < p; \\
0 < \hat{\gamma} < 1 - \hat{q}/p.
\end{aligned}\]

**Proof.** Suppose \( u \geq 0 \) is a weak solution of (3.5) with the integrability \( \int_{\Omega} u e^u < \infty \). Then

\[
\int_{\Omega} e^u \phi = \int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \quad \forall \quad \phi \in W_0^{1,p}(\Omega).
\]

Given \((x_0, r) \in \Omega \times (0, \text{diam}(\Omega))\). Choosing \( \phi = u \eta^2 \) with (3.3) we obtain via (3.6):

\[
\int_{\Omega} u e^u \eta^2 = \int_{\Omega} |\nabla u|^{p-2}\nabla u^2 \eta^2 + 2^{-1} \int_{\Omega} |\nabla u|^{p-2}(\nabla(u^2)) \cdot (\nabla(\eta^2)).
\]
An application of the Young inequality (3.4), the Hölder inequality and the assumption $p \in (1, n)$ yields

$$\int_{B(x_0, r/3) \cap \Omega} |\nabla u|^p \lesssim \int_{B(x_0, r/3) \cap \Omega} ue^{u} + \frac{r^p}{p-1} \int_{B(x_0, r/3) \cap \Omega} u^{p-1} + \frac{r}{p-1} \int_{B(x_0, r/3) \cap \Omega} \left( \int_{\Omega} ue^{u} \right)^{1-\frac{p}{n}} \left( \int_{\Omega} u \right)^{\frac{1-p}{n}}.$$ 

Thus, $|\nabla u| \in L^{p,n}(\Omega)$. This, together with Theorem 2.4 for the Riesz-Morrey potential $I_1|\nabla u|$, derives the desired assertion. □

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