ON THE 2-HEAD OF THE COLORED JONES POLYNOMIAL FOR
PRETZEL KNOTS

PAUL BEIRNE

Abstract. In this paper, we prove a formula for the 2-head of the colored Jones polynomial for an infinite family of pretzel knots. Following Hall, the proof utilizes skein-theoretic techniques and a careful examination of higher order stability properties for coefficients of the colored Jones polynomial.

1. Introduction

The colored Jones polynomial $J_{N,K}(q)$ of a knot $K$ is an important quantum knot invariant which, conjecturally, contains information about the geometry of $K$ [18]. Here, $N \in \mathbb{N}$ is the number of strands colored by the $N$-th Jones-Wenzl idempotent of the knot diagram of $K$. Following Armond and Dasbach [2], the tail $T_K(q)$ of the sequence $\{J_{N,K}(q)\}_{N \in \mathbb{N}}$ is a power series in $q$ such that its lowest $N$ coefficients match the lowest $N$ coefficients of $J_{N,K}(q)$ for all $N \geq 1$. Since its inception, there has been considerable interest in proving the existence (and non-existence) of the tail for various families of knots [1, 2, 3, 6, 7, 8, 10, 13, 14] and its connection to the volume conjecture [5], quantum spin networks [9] and Rogers-Ramanujan type identities [4, 12].

Similarly, the head $H_K(q)$ of the sequence $\{J_{N,K}(q)\}_{N \in \mathbb{N}}$ is the power series in $q$ formed by considering the highest $N$ coefficients of $J_{N,K}(q)$ for all $N \geq 1$. Note that the colored Jones polynomial of a knot $K$ is related to the colored Jones polynomial of $-K$, the mirror of $K$, via

$$J_{N,K}(q) = J_{N,-K}(q^{-1})$$

and so, in particular, $H_K(q) = T_{-K}(q)$. The objective of this paper is to examine the higher order stability for the coefficients of the colored Jones polynomial for an infinite family of pretzel knots which we now describe.

A negative twist region is a region of the knot with a positive number of negative half twists (see Figure 1).
Consider the family of pretzel knots obtained by connecting three negative twist regions with strands as in Figure 2.

The Tait graph of a knot is a planar graph found by labelling each region in the knot diagram either as an $A$-region or as a $B$-region, according to the rule in Figure 3.
The knots in Figure 2 are alternating and thus such a labelling is well-defined. The \( B \)-
checkerboard graph (or “Tait graph”) is formed by considering the \( B \)-regions of the knot diagram
as vertices and joining two vertices by an edge for each crossing that is simultaneously adjacent
to both of the corresponding regions. To construct the reduced Tait graph, replace every set of
multiple edges connecting two vertices with a single edge. For example, the reduced Tait graph
for the \( 4_1 \) knot is given in Figure 4.

Note that any knot in Figure 2 will have a triangle as its reduced Tait graph.

First order stability for the coefficients of the colored Jones polynomial of pretzel knots with
three negative twist regions was first studied by Hall in [11]. In [7], Elhamdadi, Hajij and
Saito proved that second order stability for this family of knots is ensured. To illustrate these
stabilities, consider the highest \( 3N + 1 \) coefficients of the \( N \) colored Jones polynomial for the
knot \(-9_{35}\).

\[
H_{-9_{35}}(q) = 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\]

| \( N \) | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| \( N = 2 \) | 1 | -1 | 3 | -4 | 3 | -5 | 4 | -3 | 2 | -1 | 0 | 0 | 0 | 0 | 0 |
| \( N = 3 \) | 1 | -1 | -1 | 4 | -1 | -6 | 7 | 0 | -11 | 8 | 4 | -13 | 7 | 9 | -13 | 4 |
| \( N = 4 \) | 1 | -1 | -1 | 0 | 4 | 0 | -4 | -5 | 7 | 6 | -4 | -13 | 4 | 10 | 3 | -14 |
| \( N = 5 \) | 1 | -1 | -1 | 0 | 0 | 5 | -1 | -3 | -3 | -6 | 11 | 5 | 2 | -6 | -20 | 8 |
| \( N = 6 \) | 1 | -1 | -1 | 0 | 0 | 1 | 4 | 0 | -4 | -3 | -3 | -2 | 9 | 9 | 2 | -4 |
| \( N = 7 \) | 1 | -1 | -1 | 0 | 0 | 1 | 0 | 5 | -1 | -4 | -3 | -3 | 0 | -3 | 14 | 7 |

Table 1. Highest coefficients of \( J_{N,-9_{35}}(q) \)

In Table 1, we see that the highest \( N \) coefficients of each polynomial stabilise to the highest
coefficients of

\[
\prod_{i=1}^{\infty}(1 - q^{-i}). \tag{1.1}
\]

Subtracting the head from each of the polynomials in Table 1 gives us the following in Table
2.
Table 2. Coefficients of \( J_{N,-935}(q) \) after subtracting \( H_{-935}(q) \)

| \( N = 2 \) | 1 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | −1 | 0 | 0 | −1 |
| \( N = 3 \) | 0 | 0 | 0 | 4 | −1 | 7 | 7 | −1 | −1 | −11 | 8 | 4 | −13 | 8 | 9 | −13 |
| \( N = 4 \) | 0 | 0 | 0 | 0 | 4 | −1 | 4 | −6 | 7 | 6 | −4 | −13 | 5 | 10 | 3 | −13 |
| \( N = 5 \) | 0 | 0 | 0 | 0 | 0 | 4 | −1 | −4 | −3 | −6 | 11 | 5 | 3 | −6 | −20 | 9 |
| \( N = 6 \) | 0 | 0 | 0 | 0 | 0 | 0 | 4 | −1 | −4 | −3 | −3 | −2 | 10 | 9 | 2 | −3 |
| \( N = 7 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | −1 | −4 | −3 | −3 | 1 | −3 | 14 | 8 |

Table 3. Stabilising coefficients of \( H_{1,-935}(q) \)

| \( H_{1,-935}(q) \) | 4 | −1 | −4 | −3 | −3 | 1 | 0 | 4 | 3 | 3 | 3 | 3 |
| \( N = 2 \) | 4 | −4 | 3 | −6 | 4 | −4 | 2 | −1 | 0 | 0 | 1 | 0 |
| \( N = 3 \) | 4 | −1 | −7 | 7 | −1 | −11 | 8 | 4 | −13 | 8 | 9 | −13 |
| \( N = 4 \) | 4 | −1 | −4 | −6 | 7 | 6 | −4 | −13 | 5 | 10 | 3 | −13 |
| \( N = 5 \) | 4 | −1 | −4 | −3 | −6 | 11 | 5 | 3 | −6 | −20 | 9 | 6 |
| \( N = 6 \) | 4 | −1 | −4 | −3 | −3 | −2 | 10 | 9 | 2 | −3 | −12 | −16 |
| \( N = 7 \) | 4 | −1 | −4 | −3 | −3 | 1 | −3 | 14 | 8 | 2 | −3 | −9 |

Table 4. Stabilising coefficients of \( H_{2,-935}(q) \)

| \( H_{2,-935}(q) \) | −3 | 10 | 5 | −1 | −6 | −12 |
| \( N = 2 \) | −3 | 7 | −3 | 7 | −5 | 2 |
| \( N = 3 \) | −3 | 10 | 2 | −12 | 8 | 0 |
| \( N = 4 \) | −3 | 10 | 5 | −4 | −17 | 2 |
| \( N = 5 \) | −3 | 10 | 5 | −1 | −9 | −23 |
| \( N = 6 \) | −3 | 10 | 5 | −1 | −6 | −15 |
| \( N = 7 \) | −3 | 10 | 5 | −1 | −6 | −12 |

We observe that the highest \( N − 1 \) coefficients of each polynomial in this new sequence stabilise to

\[
H_{1,-935}(q) = \prod_{i=1}^{\infty} \left( 1 - q^{-i} \right) \left( 1 + \frac{3}{1 - q^{-1}} \right),
\]

which was proven by Hall [11]. Repeating this process of subtracting \( H_{1,-935}(q) \) and left-justifying the coefficients, we notice that this stabilisation continues, thus forming the “2-head” \( H_{2,-935}(q) \) of \(-935\) as in Table 4.

Here, we observe that the highest \( N − 1 \) coefficients of each polynomial in this new sequence stabilise to
ON THE 2-HEAD OF THE COLORED JONES POLYNOMIAL FOR PRETZEL KNOTS

\[ H_{2,-935}(q) = \prod_{i=1}^{\infty} (1 - q^{-i}) \left( \frac{-3 + 10q^{-1} + 5q^{-2} - 4q^{-3} + q^{-4}}{(1 - q^{-1})(1 - q^{-2})} \right). \] (1.3)

Equation (1.3) is one instance of our general result which we now state.

**Theorem 1.1.** For any pretzel knot \( K \) with three negative twist regions where \( n \) is the number of twist regions with exactly two half-twists and \( m \) is the number of twist regions with at least three half-twists, we have

\[ H_{2,K}(q) = \prod_{i=1}^{\infty} (1 - q^{-i}) \left( q^{-1} + n \cdot \frac{1}{(1 - q^{-1})} - \frac{f_{n,m}(q)}{(1 - q^{-1})(1 - q^{-2})} \right) \] (1.4)

where

\[ f_{n,m}(q) = \begin{cases} 0 & \text{if } n + m = 0, \\ 1 - 3q^{-1} - q^{-2} + 2q^{-3} & \text{if } n + m = 1, \\ 2 - 6q^{-1} - 3q^{-2} + 3q^{-3} & \text{if } n + m = 2, \\ 3 - 9q^{-1} - 6q^{-2} + 3q^{-3} & \text{if } n + m = 3. \end{cases} \]

This paper is organised as follows. In Section 2, we provide definitions, notation and some preliminary results necessary to prove the main result. In Section 3, we prove Theorem 1.1 by calculating an expression for the first \( 3N+1 \) coefficients of \( J_{N+1,K}(q) \) in the case where \( K \) is a knot with three negative twist regions, each with at least three half-twists. We then consider cases defined by combinations of the number of crossings in each negative twist region with the previous knot as the base case.

2. Preliminaries

We first recall a formula for \( J_{N+1,K}(q) \) as given in [11]. For further details, see also [16] [17]. Fusion is given by

\[ \begin{array}{c} \text{a} \\ \text{b} \end{array} = \sum_c \frac{\Delta_c}{\theta(a,b,c)} \begin{array}{c} \text{a} \\ \text{b} \end{array} \begin{array}{c} \text{a} \\ \text{b} \end{array} \]

where

\[ \Delta_n = \left\langle \begin{array}{c} \text{a} \end{array} \right\rangle = (-1)^n[n + 1], \]

\[ [n] = \left\{ \frac{n}{1} \right\}, \quad \{n\} = A^{2n} - A^{-2n} \text{ and } A^{-4} = a^{-2} = q. \] (2.1)

Here, \([n]!\) and \(\{n\}!\) are naturally defined as

\[ [n]! = [n][n-1]\cdots[2][1], \quad \{n\}! = \{n\}\{n-1\}\cdots\{2\}\{1\}. \] (2.2)

Additionally, we set

\[ \Delta_n! = \Delta_n\Delta_{n-1}\cdots\Delta_1. \]

We also make use of the trihedron coefficient \( \theta(a,b,c) \) defined by
\[ \theta(a, b, c) = \left( \begin{array}{ccc} a \\ b \\ c \end{array} \right) = (-1)^{i+j+k} \frac{[i + j + k + 1][i][j][k]!}{[i + j][j + k][i + k]!} \]

where

\[ i = \frac{b + c - a}{2}, \quad j = \frac{a + c - b}{2}, \quad k = \frac{a + b - c}{2}. \]

Also, half-twists can be removed using

\[ \gamma(a, b, c) = (-1)^{\frac{a+b-c}{2}} A^{a+b-c} + \frac{a^2 + b^2 - c^2}{2} \]

is the negative half-twist coefficient.

A formula for the \( N + 1 \) colored Jones polynomial for pretzel knots with three negative twist regions is now given by (see (4.1) in [11])

\[ J_{N+1,K}(q) = \sum_{j_1, j_2, j_3=0}^{N} S_{j_1, j_2, j_3} = \sum_{j_1, j_2, j_3=0}^{N} \prod_{i=1}^{3} \gamma(N, N, 2j_i) m_i \frac{\Delta_{2j_i}}{\theta(N, N, 2j_i)} \Gamma_{N,(j_1, j_2, j_3)} \]

(2.3)

where \( m_i \) is the number of crossings in the \( i \)th twist region and \( \Gamma_{N,(j_1, j_2, j_3)} \) is the multiset of trivalent graphs resulting from the fusion of strands and the removal of half-twists from the knot diagram (see (4.2) in [11]).

Throughout this paper, we will only be concerned with a certain number of the highest coefficients of the various terms in an expression. To facilitate such calculations, we write

\[ f(q) \overset{\text{in}}{\Rightarrow} g(q) \]

if the coefficients of the \( n \) terms of highest degree in \( f(q) \) agree with the coefficients of the \( n \) terms of highest degree in \( g(q) \), up to a common sign change. We write \( f(q) \overset{\text{in}}{\Rightarrow} g(q) \) if \( f(q) = \pm q^s g(q) \) for \( s \in \mathbb{Z} \). Our first result is a routine extension of equation (4.15) in [11].

**Lemma 2.1.** For \( K \) a pretzel knot with three negative twist regions, each with at least three twists, we have

\[ J_{N+1,K}(q) \overset{\text{in}}{\Rightarrow} \frac{(-1)^N \{3N+1\}! \{N\}!^3}{\{2N\}!^3 \{1\}}. \]

(2.4)

We will also recall (4.22) in [11].

**Lemma 2.2.** We have

\[ (-1)^N \{N\}! \overset{\text{in}}{\Rightarrow} \prod_{i=1}^{N} (1 - q^{-i}). \]

(2.5)

---

1. We follow the convention in [11] for normalisation where \( J_{N,K}(q) \) is the colored Jones polynomial of \( K \) with each component colored by the \( N \)-dimensional irreducible representation of \( \mathfrak{sl}_2 \).
Lemma 2.3. We have

\[ (2N)! \cdot (-1)^N \{N\}! \left( 1 - \frac{q^{-N-1} - q^{-2N-1}}{1-q^{-1}} + \frac{q^{-2N-2}}{2(1-q^{-1})^2} - \frac{q^{-2N-2}}{2(1-q^{-2})} \right) \tag{2.6} \]

and

\[ (2N)!^2 \cdot (-1)^{3N+1} \{N\}!^2 \left( 1 - 2 \cdot \frac{q^{-N-1} - q^{-2N-1}}{1-q^{-1}} + \frac{2q^{-2N-2}}{(1-q^{-1})^2} - \frac{q^{-2N-2}}{1-q^{-2}} \right). \tag{2.7} \]

Proof of Lemma 2.3. By (2.2), dividing by an appropriate power of \( q \) and keeping track of terms that do not affect the highest \( 3N+1 \) terms (here this corresponds to powers of \( q \) of degree less than \(-3N \) as our maximal degree term is \( 1 \)), we obtain

\[ (2N)! = (2N)\{2N-1\} \ldots \{N+1\} \{N\}! \]

\[ = (q^{-2N} - q^{-1})^N (q^{-2N} - q^{-1}) \ldots (q^{-2N} - q^{-1}) \{N\}! \]

\[ = (-1)^N (q^{-2N} - q^{-2}) (q^{-2N} - q^{-1}) \ldots (q^{-2N} - q^{-2}) \{N\}! \]

\[ = (-1)^{3N+1} (-1)^N \{N\}! (1 - q^{-2N}) (1 - q^{-2N-1}) \ldots (1 - q^{-2N-1}) \]

\[ = (-1)^N \{N\}! \left( 1 - \sum_{i=1}^{N} q^{-N-i} + \sum_{k,l=1}^{N} q^{-2N-k-l} \right) \]

\[ = (-1)^N \{N\}! \left[ 1 - \frac{q^{-N-1} - q^{-2N-1}}{1-q^{-1}} + \frac{\sum_{k,l=1}^{N} q^{-2N-k-l}}{2} - \frac{\sum_{m=1}^{N} q^{-2N-2m}}{2} \right] \]

\[ = (-1)^N \{N\}! \left[ 1 - \frac{q^{-N-1} - q^{-2N-1}}{1-q^{-1}} + \frac{\sum_{l=1}^{N} q^{-2N-1-l} - q^{-3N-1-l}}{2(1-q^{-1})} \right. \]

\[ \left. - \frac{q^{-2N-2} - q^{-3N-2}}{2(1-q^{-2})} \right] \]

\[ = (-1)^{3N+1} (-1)^N \{N\}! \left( 1 - \frac{q^{-N-1} - q^{-2N-1}}{1-q^{-1}} + \frac{\sum_{l=1}^{N} q^{-2N-1-l}}{2(1-q^{-1})} - \frac{q^{-2N-2}}{2(1-q^{-2})} \right) \]

\[ = (-1)^N \{N\}! \left( 1 - \frac{q^{-N-1} - q^{-2N-1}}{1-q^{-1}} + \frac{q^{-2N-2} - q^{-3N-2}}{2(1-q^{-2})} - \frac{q^{-2N-2}}{2(1-q^{-2})} \right) \]

\[ = (-1)^{3N+1} (-1)^N \{N\}! \left( 1 - \frac{q^{-N-1} - q^{-2N-1}}{1-q^{-1}} + \frac{q^{-2N-2}}{2(1-q^{-1})^2} - \frac{q^{-2N-2}}{2(1-q^{-2})} \right). \tag{2.8} \]
Equation (2.7) follows upon squaring (2.6) and simplification.

Here, the fifth line of (2.7) is given by how we can take none, exactly one or exactly two of the negative powers of $q$ when multiplying the terms in the fourth line of (2.7) without affecting the highest $3N + 1$ terms. The sum involving $k$ and $l$ is realised as a sum allowing $k = l$ and $k < l$. Dividing by 2 removes the contributions of the $k < l$ terms and halves the contributions of the $k = l$ terms. Then the $m$-sum removes the other half of the $k = l$ terms.

Lemma 2.4. We have

$$\{3N + 1\}! = (-1)^{N+1}\{2N\}! \left(1 - \frac{q^{-2N-1}}{1-q^{-1}}\right)$$

and

$$\{3N\}! = (-1)^N\{2N\}! \left(1 - \frac{q^{-2N-1}}{1-q^{-1}}\right).$$

Proof of Lemma 2.4. We follow the proof of Lemma 2.3 to obtain

$$\{3N + 1\}! = \{3N + 1\}\{3N\} \ldots \{2N + 1\}\{2N\}!$$

$$= (q^{3N+1} - q^{3N})\ldots(q^{2N+1} - q^{2N})\{2N\}!$$

$$= (-1)^{N+1}(q^{3N+1} - q^{3N})\ldots(q^{2N+1} - q^{2N})\{2N\}!$$

$$\triangleq (-1)^{N+1}\{2N\}!(1-q^{-3N})\ldots(1-q^{-2N-1})$$

$$\triangleq (-1)^{N+1}\{2N\}! \left(1 - \sum_{i=1}^{N+1} q^{-2N-i}\right)$$

$$= (-1)^{N+1}\{2N\}! \left(1 - \frac{q^{-2N-1} - q^{-3N-2}}{1-q^{-1}}\right)$$

$$\triangleq (-1)^{N+1}\{2N\}! \left(1 - \frac{q^{-2N-1}}{1-q^{-1}}\right).$$

A similar calculation yields (2.10).

3. Proof of Theorem 1.1

The structure of the proof of Theorem 1.1 is as follows. We will first prove the formula for our base case $(3^+, 3^+, 3^+)$ in which all of the negative twist regions have at least three half-twists. In this case, after applying (2.4), (2.9) and (2.7), we obtain the highest $3N + 1$ coefficients of the normalised colored Jones polynomial

$$J'_{N+1,K}(q) = J_{N+1,K}(q) \cdot \frac{(-1)^N\{1\}}{\{N+1\}}.$$
We obtain the 2-head $H_{2,K}(q)$ after subtracting the highest $3N + 1$ coefficients of the head $H_K(q)$ and the shifted (by $q^{-N-1}$) 1-head $H_{1,K}(q)$ from the highest $3N + 1$ coefficients of $J'_{N+1,K}(q)$.

We will then split the remaining nine cases into four groups depending on the number of negative twist regions with exactly one half-twist. For simplicity, we start by choosing the case with maximal $m$ within each group. Once we have proven this case, we use the argument from Section 4.3 in [11] to prove cases within the same group. Namely, each negative twist region with exactly two half-twists will contribute a summand from (2.3) corresponding to a $j_i$ being $N-1$ and the corresponding $m_i$ equalling two. This summand does not contribute to the highest $2N + 1$ coefficients and thus contributes to $H_{2,K}(q)$ in the same way as (4.29) in [11]. We now prove Theorem 1.1 in the following five sections.

3.1. $(3^+, 3^+, 3^+)$. We proceed with the base case $(3^+, 3^+, 3^+)$, a pretzel knot $K$ with three negative twist regions, each with at least three half-twists.

By (2.1), Lemmas 2.1 and 2.4 and (3.1), we have

$$J_{N+1,K}^0(q)^{-3N+1} = \frac{3N+1}{2N} \frac{(N+1)!}{(N)!} \left( 1 - \frac{q^{-2N-1}}{1 - q^{-1}} \right) \left( 1 - 2 \cdot \frac{q^{-N-1} - q^{-2N-1}}{1 - q^{-1}} + \frac{2q^{-2N-2}}{(1-q^{-1})^2} - \frac{q^{-2N-2}}{1-q^{-1}} \right).$$

By Lemma 4.1 in [11], Lemma 2.3 and factoring out a power of $q$, we simplify (3.2) to obtain

$$J_{N+1,K}^0(q)^{-3N+1} = \frac{(-1)^N}{1-q^{-N-1}} \left( 1 - \frac{q^{-2N-1}}{1 - q^{-1}} \right) \left( 1 + \frac{2q^{-N-1} - q^{-2N-1}}{1 - q^{-1}} - \frac{2q^{-2N-2}}{(1-q^{-1})^2} + \frac{q^{-2N-2}}{1-q^{-1}} \right)$$

$$+ \left( 2 \cdot \frac{q^{-N-1} - q^{-2N-1}}{1 - q^{-1}} - \frac{2q^{-2N-2}}{(1-q^{-1})^2} + \frac{q^{-2N-2}}{1-q^{-2}} \right)^2 \left( 1 - \frac{q^{-2N-1}}{1 - q^{-1}} \right) \left( 1 + 2 \cdot \frac{q^{-N-1} - q^{-2N-1}}{1 - q^{-1}} + \frac{2q^{-2N-2}}{(1-q^{-1})^2} + \frac{q^{-2N-2}}{1-q^{-1}} \right).$$

We are set to subtract the head and 1-head. As the head is $\prod_{i=1}^{\infty} (1 - q^{-i}) = (-1)^N \{N\}$ and the 1-head is given as (4.30) in [11], we subtract the first $3N+1$ coefficients of these two terms (shifting the 1-head by $q^{-N-1}$) to get

$$H_{2,K}(q)^{-3N+1} = \frac{(-1)^N}{1-q^{-N-1}} \left( 1 - \frac{q^{-2N-1}}{1 - q^{-1}} \right) \left( 1 + 2 \cdot \frac{q^{-N-1} - q^{-2N-1}}{1 - q^{-1}} + \frac{2q^{-2N-2}}{(1-q^{-1})^2} + \frac{q^{-2N-2}}{1-q^{-1}} \right)$$

$$- (-1)^N \{3N\}! \cdot q^{-N-1} \left( (-1)^N \{3N\}! + \frac{3(-1)^N \{3N\}!}{1-q^{-1}} \right).$$
\[
\sum_{N=1}^{3N+1} \frac{(-1)^N\{N\}!}{(1 - q^{-N-1})(1 - q^{-1})^3(1 - q^{-2})} \left( 1 - 3q^{-1} + 2q^{-2} + 2q^{-3} - 3q^{-4} + q^{-5} + 2q^{-N-1} \\
- 4q^{-N-2} + 4q^{-N-2} - 2q^{-N-5} - 3q^{-2N-1} + 9q^{-2N-2} - 5q^{-2N-3} - 5q^{-2N-4} \\
+ 4q^{-2N-5} \right) - (-1)^N\{3N\}! - q^{-N-1} \left( (-1)^N\{3N\}! + \frac{3(-1)^N\{3N\}!}{1 - q^{-1}} \right). \tag{3.4}
\]

After applying (2.10) followed by (2.6) and some routine (but tedious) calculations, we obtain

\[
H_{2,K}(q) \sum_{N=1}^{3N+1} \frac{(-1)^N\{N\}!}{(1 - q^{-N-1})(1 - q^{-1})^4(1 - q^{-2})} (-3q^{-2N-1} + 19q^{-2N-2} - 34q^{-2N-3} + 14q^{-2N-4} \\
+ 18q^{-2N-5} - 20q^{-2N-6} + 7q^{-2N-7} - q^{-2N-8}) \\
\sum_{N=1}^{3N+1} \frac{(-1)^N\{N\}!q^{-2N-1}}{(1 - q^{-1})^4(1 - q^{-2})} (-3 + 19q^{-1} - 34q^{-2} + 14q^{-3} + 18q^{-4} - 20q^{-5} + 7q^{-6} - q^{-7}) \\
\sum_{N=1}^{3N+1} \sum_{i=1}^{\infty} (1 - q^{-i}) \left( \frac{-3 + 10q^{-1} + 5q^{-2} - 4q^{-3} + q^{-4}}{(1 - q^{-1})(1 - q^{-2})} \right), \tag{3.5}
\]
as required.

Here we see that when we subtract we get $2N + 1$ copies of zero in our list of coefficients and then the first $N$ coefficients of the 2-head are given by the first $N$ coefficients of the expression in the last line, which is independent of $N$.

We note that the proof of (3.5) begins with (2.4) which results from only considering the summand $S_{N,N,N}$ in (2.3). The remaining summands in (2.3) do not contribute to the first $3N + 1$ coefficients as each of the $m_i$ are at least three. For the remaining nine cases, more care is required. As lower values for one or more of the $m_i$'s occur, we consider the normalised summands

\[
\overline{S}_{j_1,j_2,j_3} = S_{j_1,j_2,j_3} \cdot \frac{(-1)^{N+1}}{\{N+1\}}
\]
in the following lemmas.

**Lemma 3.1.** We have

\[
S_{N-1,N,N} \overset{2N}{=} -\prod_{i=1}^{\infty} (1 - q^{-i}) \left( 1 + \frac{2q^{-(N+1)}}{1 - q^{-1}} - \frac{q^{-(2N-1)}}{1 - q^{-1}} \right).
\]

**Proof of Lemma 3.1.** By (2.5), (2.10) and Section 4.3 of [11] and after simplification, we have

\[
S_{N-1,N,N} \overset{\infty}{=} \frac{\{3N\}!\{N\}!^3}{\{1\}{2N-2}!\{2N\}!^2\{2N\}} \\
\overset{2N}{=} \frac{(-1)^{N+1}}{\{1\}{2N-2}!\{2N\}!\{2N\}} \\
\overset{2N}{=} \frac{(-1)^N\{N\}!}{1 - q^{-1}} \left( 1 + 2q^{-(N+1)} + 2q^{-(N+2)} + \cdots + 2q^{-(2N-2)} + q^{-(2N-1)} + 2q^{-2N} \right)
\]
We have
\[ S_{N-2,N,N} = \prod_{i=1}^{\infty} (1 - q^{-i}) \left( 1 + \frac{2q^{-2(N+1)}}{1 - q^{-1}} - \frac{q^{-2(2N-1)}}{1 - q^{-1}} \right). \]

\[ S_{N-2,N,N} \overset{\text{N}}{=} \frac{\prod_{i=1}^{\infty} (1 - q^{-i})}{(1 - q^{-2})(1 - q^{-1})}. \]

**Lemma 3.2.** We have
\[ \overline{S}_{N-3,N,N} = -\prod_{i=1}^{\infty} (1 - q^{-i}) \left( 1 - \frac{2q^{-2(N+1)}}{1 - q^{-1}} + \frac{q^{-2(2N-1)}}{1 - q^{-1}} \right). \]

\[ \overline{S}_{N-3,N,N} \overset{\text{N}}{=} -\prod_{i=1}^{\infty} (1 - q^{-i}) \left( 1 - \frac{2q^{-2(N+1)}}{1 - q^{-1}} + \frac{q^{-2(2N-1)}}{1 - q^{-1}} \right). \]

**Proof of Lemma 3.2.** By (2.3), (2.5), Appendix A in [11] and Lemma 14.5 in [15], we have
\[ \overline{S}_{N-2,N,N} = \frac{\prod_{i=1}^{\infty} (1 - q^{-i})}{(1 - q^{-2})(1 - q^{-1})}. \]

**Lemma 3.3.** We have
\[ \overline{S}_{N-3,N,N} = -\prod_{i=1}^{\infty} (1 - q^{-i}) \left( 1 - \frac{2q^{-2(N+1)}}{1 - q^{-1}} + \frac{q^{-2(2N-1)}}{1 - q^{-1}} \right). \]

\[ \overline{S}_{N-3,N,N} \overset{\text{N}}{=} -\prod_{i=1}^{\infty} (1 - q^{-i}) \left( 1 - \frac{2q^{-2(N+1)}}{1 - q^{-1}} + \frac{q^{-2(2N-1)}}{1 - q^{-1}} \right). \]

**Proof of Lemma 3.3.** By (2.3), (2.5), (A.1) in [11] and Lemma 14.5 in [15], we have
\[ \overline{S}_{N-3,N,N} = \frac{\prod_{i=1}^{\infty} (1 - q^{-i})}{(1 - q^{-2})(1 - q^{-1})}. \]
\[
\begin{align*}
\sum_{N} (-1)^N \{N\}! / \{3\}! \\
\sum_{N} \prod_{i=1}^{\infty} (1 - q^{-i}) \\
\prod_{i=1}^{\infty} (1 - q^{-3})(1 - q^{-2})(1 - q^{-1}).
\end{align*}
\]

**Lemma 3.4.** We have

\[
\text{Lemma 3.4. We have}
\]

\[
\sum_{N} \prod_{i=1}^{\infty} (1 - q^{-i}) \\
\prod_{i=1}^{\infty} (1 - q^{-3})(1 - q^{-2})(1 - q^{-1}).
\]

**Proof of Lemma 3.4.** Proceeding as in the proof of Lemma 3.3, we obtain

\[
\sum_{N} \prod_{i=1}^{\infty} (1 - q^{-i}) \\
\prod_{i=1}^{\infty} (1 - q^{-3})(1 - q^{-2})(1 - q^{-1}).
\]

3.2. \((3^+, 3^+, 2), (3^+, 2, 2), (2, 2, 2)\). For any pretzel knot \(K\) with three negative twist regions each with at least two half-twists, we have in (2.3) that \(m_i = 2\) if the \(i\)th negative twist region has two half-twists and \(m_i \geq 3\) otherwise. By Corollary 3.5 in (11), the maximal \(q\)-degree, say \(a\), arises from the \(j_1 = j_2 = j_3 = N\) term in (2.3). It follows from Lemmas 3.2–3.4 in (11) that for each \(m_i\) equal to 2, the contribution from the corresponding \(j_i = N - 1\) term has maximal \(q\)-degree \(a = 2N - 1\). No other term contributes to \(H_{2,K}(q)\) as decreasing any of the \(j_i\) corresponding to an \(m_i \geq 3\) decreases the maximum \(q\)-degree by at least \(3N + 1\). Decreasing any \(j_i\) corresponding to an \(m_i = 2\) from \(N - 1\) to \(N - 2\) decreases the maximum \(q\)-degree by either \(4N\) or \(4N + 1\), both of which are greater than or equal to \(3N + 1\) for all \(N \geq 1\). Any combinations of decreases in multiple \(j_i\)'s would further decrease the maximum \(q\)-degree and thus not contribute to \(H_{2,K}(q)\). In total, by (3.5)
\[ H_{2,K}(q) = H_{2,-9_{35}}(q) + nq^{2N+1}S_{N,N,N-1} \]

and thus
\[ H_{2,K}(q) = \prod_{i=1}^{\infty} (1 - q^{-i}) \left( \frac{-3 + 10q^{-1} + 5q^{-2} - 4q^{-3} + q^{-4}}{(1 - q^{-1})(1 - q^{-2})} \right) + n \prod_{i=1}^{\infty} (1 - q^{-i}) \left( \frac{1}{1 - q^{-i}} \right). \]

3.3. \((3^+, 3^+, 1), (3^+, 2, 1), (2, 2, 1)\). We first prove for \(K\) a knot in the family \((3^+, 3^+, 1)\) that \(H_{2,K}(q)\) is given by
\[ \prod_{i=1}^{\infty} (1 - q^{-i}) \left( \frac{-2 + 7q^{-1} + 2q^{-2} - 4q^{-3} + q^{-4}}{(1 - q^{-1})(1 - q^{-2})} \right). \]

To see this, first note that \(H_{2,K}(q)\) is the series that has its first \(N\) coefficients the \((2N+1)st\) to \((3N+1)st\) coefficients of
\[ J'_{N+1,K}(q) - H_K(q) - q^{N+1}H_{1,K}. \]

Without loss of generality, take \(K\) to be the \((-3, -3, -1)\) pretzel knot from the family \((3^+, 3^+, 1)\) and \(-9_{35}\) (which is the \((-3, -3, -3)\) pretzel knot) from the family \((3^+, 3^+, 3^+)\). By Corollary 3.5 in [3], the only summands in (2.3) that contribute to \(J'_{N+1,K}(q)\) yet do not contribute to the first \(3N+1\) coefficients of \(J'_{N+1,-9_{35}}(q)\) are those coming from decreasing the \(j_i\) term, say \(j_3\), corresponding to the twist region with only one half-twist. By considering the maximal \(q\)-degree in
\[ \gamma(N, N, 2j_3) \frac{\Delta_{2j_3}}{\Theta(N, N, 2j_3)} \Gamma_{N,N,N}, \]
where \(0 \leq j_3 \leq N\), we need only consider the terms corresponding to \(j_3 = N, N-1, N-2\) and \(N-3\), with respective maximal \(q\)-degrees \(a, a-N-1, a-2N-1, a-3N\).

Using (3.3), we express \(H_{2,K}(q)\) as
\[ H_{2,-9_{35}}(q) + q^{2N+1}H_{-9_{35}}(q) + q^N H_{1,-9_{35}}(q) + S_{N,N,N-1} + S_{N,N,N-2} + S_{N,N,N-3} - q^{2N+1}H_{K}(q) - q^{N}H_{1,K}(q). \]

Applying Lemmas 3.1–3.3, the fact that \(H_{-9_{35}}(q) = H_{K}(q)\) and (1.5) from [3] yields (after simplification)
\[ H_{2,K}(q) \overset{3N+1}{=} H_{2,-9_{35}}(q) + q^{N}(H_{1,-9_{35}}(q) - H_{1,K}(q)) + S_{N,N,N-1} + S_{N,N,N-2} + S_{N,N,N-3} \]
\[ \overset{N}{=} H_{2,-9_{35}}(q) + q^N \left( \prod_{i=1}^{\infty} (1 - q^{-i}) \right) - \frac{1}{1 - q^{-i}} - \frac{\prod_{i=1}^{\infty} (1 - q^{-i})(q^{-(N+1)})}{(1 - q^{-i})^2} \]
\[ - q^N \left( \prod_{i=1}^{\infty} (1 - q^{-i}) \right) \left( 1 + \frac{2q^{-N+1}}{1 - q^{-1}} - q^{-(2N-1)} \right) \]
\[ \left( 1 + \frac{2q^{-N+1}}{1 - q^{-1}} - q^{-(2N-1)} \right). \]
\[
\frac{1}{1-q} \prod_{i=1}^{\infty} (1-q^{-i}) - \frac{q^{-N} \prod_{i=1}^{\infty} (1-q^{-i})}{(1-q^{-1})(1-q^{-2})} (1-q^{-2})
\]  
(3.6)

Now, let \( K' \) be any knot in the family \((3^+, 2, 1)\) and consider the link \( L_{5a_1} \) which is \((2, 2, 1)\). If we do not pick up additional terms in \( J_{K_1,K_2}(q) \) due to a combination of decreases in the \( j_i \)'s, then we can apply the same argument as in Section 3.2. A decrease from a \( j_i \) corresponding to a twist region with two half-twists from \( N \) to \( N-1 \), coupled with a decrease from a \( j_i \) corresponding to a twist region with one half-twist from \( N \) to \( N-1 \) decreases the maximal \( q \)-degree of the corresponding summand by at least \( 3N+2 \). As this is the smallest possible decrease in \( q \)-degree due to combinations of decreases of the \( j_i \)'s, there is no further contribution to \( H_{2,K}(q) \) aside from those discussed in Section 3.2. Thus, we get

\[
H_{2,K}(q) = \prod_{i=1}^{\infty} (1-q^{-i}) \left( \frac{-1 + 7q^{-1} + q^{-2} - 4q^{-3} + q^{-4}}{(1-q^{-1})(1-q^{-2})} \right)
\]

and

\[
H_{2,L_{5a_1}}(q) = \prod_{i=1}^{\infty} (1-q^{-i}) \left( \frac{7q^{-1} - 4q^{-3} + q^{-4}}{(1-q^{-1})(1-q^{-2})} \right).
\]

3.4. \((3^+, 2, 1)\). Let \( K \) be a knot in the family \((3^+, 1, 1)\). Our first step is to prove that

\[
H_{2,K}(q) = \prod_{i=1}^{\infty} (1-q^{-i}) \left( \frac{-1 + 4q^{-1} - 3q^{-3} + q^{-4}}{(1-q^{-1})(1-q^{-2})} \right).
\]

Proceeding as above, we express \( H_{2,K}(q) - H_{2,-9_{35}}(q) \) as

\[
H_{-9_{35}}(q) - H_{K}(q) + q^{-(N+1)} (H_{1,-9_{35}}(q) - H_{1,K}(q)) + 2q^{-(2N+1)} \left[ H_{2,-9_{35}}(q) - H_{2,-7_4}(q) \right] + q^N \left( \prod_{i=1}^{\infty} (1-q^{-i}) - \frac{q^{-(N+1)} \prod_{i=1}^{\infty} (1-q^{-i})}{(1-q^{-1})^2} \right) + S_{N,N-1,N-1}.
\]

(3.7)

This is due to the fact that each twist region in \( K \) that has exactly one half-twist gives us a copy of the change in 2-head between \(-7_4 \) (or any knot in the family \((3^+, 3^+, 1)\)) and \(-9_{35} \), provided that we take into account the difference, shifted by an appropriate power of \( q \), between \( H_{1,K} - H_{1,-9_{35}} \) and \( H_{1,-7_4} - H_{1,-9_{35}} \).

We also need to consider the summand \( S_{N,N-1,N-1} \) as a result of decreasing the \( j_i \)'s corresponding to the twist regions with exactly one half-twist simultaneously. Any other combinations of decreases in the \( j_i \)'s leads to a decrease in the maximal \( q \)-degree of at least \( 3N+2 \) and thus will not contribute to \( H_{2,K}(q) \).

By \((3.6), (3.7)\), Lemmas 2.3, 3.1 and 3.4, the fact that \( H_{-9_{35}}(q) = H_{K}(q) \) and Theorem 1.2 in \( \Pi \) we have
3.5. (1, 1, 1). Consider $3_1$ which is the (1, 1, 1) pretzel knot. We claim that

$$H_{2,3_1} = \prod_{i=1}^{\infty} (1 - q^{-i}). \quad (3.9)$$

The difference $H_{2,3_1}(q) - H_{2, -9_{35}}(q)$ is given by

$$H_{-9_{35}}(q) - H_{3_1}(q) + q^{-(N+1)}(H_{1,-9_{35}}(q) - H_{1,3_1}(q)) + 3q^{-2(N+1)} \left[ H_{2,-9_{35}}(q) - H_{2,-7_4}(q) \right]$$
suggestions. for their financial support and his Ph.D. advisor Robert Osburn for helpful comments and suggestions.

\[ + q^N \left( \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{1 - q^{-1}} - q^{-(N+1)} \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{(1 - q^{-1})^2} \right) + 3S_{N,N-1,N-1}. \]

This is due to the fact that each twist region with exactly one half-twist gives us a copy of the change in 2-head between \(-74\) (or any knot in the family \((3^+, 3^+, 1)\)) and \(-9_{35}\), provided that we take into account the difference, shifted by an appropriate power of \(q\), between \(H_{1,K} - H_{1,-9_{35}}\) and \(H_{1,-74} - H_{1,-9_{35}}\).

We also need to consider the summand \(S_{N,N-1,N-1}\) as a result of decreasing the three possible pairs of \(j_i\)’s. Any other combinations of decreases in the \(j_i\)’s leads to a decrease in maximal \(q\)-degree of at least \(3N + 2\) and thus will not contribute to \(H_{2,3_1}(q)\).

Thus, along with (3.6), (3.7), Lemmas 2.3, 3.1 and 3.4, the fact that \(H_{-9_{35}}(q) = H_{3_1}(q)\) and Theorem 1.2 in [11], following the same argument as Section 3.4, we obtain

\[ H_{2,3_1}(q) = H_{2,-9_{35}} + q^{2N+1} \left[ H_{-9_{35}}(q) - H_{3_1}(q) + q^{-(N+1)} \left( H_{1,-9_{35}}(q) - H_{1,3_1}(q) \right) \right] \]

\[ + 3q^{-(2N+1)} \left[ H_{2,-9_{35}}(q) - H_{2,-74}(q) + q^N \left( \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{1 - q^{-1}} - q^{-(N+1)} \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{(1 - q^{-1})^2} \right) \right] + 3S_{N,N-1,N-1} \]

\[ \vdash H_{2,-9_{35}} + q^{2N+1} \left[ 3q^{-(N+1)} \left( \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{1 - q^{-1}} - q^{-(N+1)} \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{(1 - q^{-1})^2} \right) \right] \]

\[ - 3q^{-(N+1)} \left( \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{1 - q^{-1}} + 2q^{-(N+1)} \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{(1 - q^{-1})^2} \right) \]

\[ + 3q^{-(2N+1)} \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{1 - q^{-1}(1 - q^{-2})} - 3q^{3N} \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{(1 - q^{-1})(1 - q^{-2})(1 - q^{-3})} \]

\[ + 3q^{-(2N+3)} \prod_{i=1}^{\infty} \frac{(1 - q^{-i})}{1 - q^{-1}} \]

\[ \vdash \prod_{i=1}^{\infty} (1 - q^{-i}). \]

This completes the proof of Theorem 1.1.

**Acknowledgements**

The author would like to thank the Irish Research Council (Grant No. GOIPG/2018/2494) for their financial support and his Ph.D. advisor Robert Osburn for helpful comments and suggestions.
References

[1] C. Armond, *The head and tail conjecture for alternating knots*, Algebr. Geom. Topol. 13 (2013), no. 5, 2809–2826.

[2] C. Armond, O. T. Dasbach, *Rogers-Ramanujan type identities and the head and tail of the colored Jones polynomial*, preprint available at https://arxiv.org/abs/1106.3948

[3] C. Armond, O. T. Dasbach, *The head and tail of the colored Jones polynomial for adequate knots*, Proc. Amer. Math. Soc. 145 (2017), no. 3, 1357–1367.

[4] P. Beirne, R. Osburn, *q-series and tails of colored Jones polynomials*, Indag. Math. (N.S.) 28 (2017), no. 1, 247–260.

[5] O. T. Dasbach, X. Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math. 142 (2006), no. 5, 1332–1342.

[6] M. Elhamdadi, M. Hajij, *Foundations of the colored Jones polynomial of singular knots*, Bull. Korean Math. Soc. 55 (2018), no. 3, 937–956.

[7] M. Elhamdadi, M. Hajij and M. Saito, *Twist regions and coefficients stability of the colored Jones polynomial*, Trans. Amer. Math. Soc., 370 (2018), no. 7, 5155–5177.

[8] S. Garoufalidis, T. T. Q. Lê, *Nahm sums, stability and the colored Jones polynomial*, Res. Math. Sci. 2 (2015), Art. 1, 55pp.

[9] M. Hajij, *The tail of a quantum spin network*, Ramanujan J. 40 (2016), no. 1, 135–176.

[10] M. Hajij, *The colored Kauffman skein relation and the head and tail of the colored Jones polynomial*, J. Knot Theory Ramifications 26 (2017), no. 3, 1741002, 14pp.

[11] K. Hall, *Higher order stability in the coefficients of the colored Jones polynomial*, J. Knot Theory Ramifications 27 (2018), no. 3, 1840010, 26 pp.

[12] A. Keilthy, R. Osburn, *Rogers-Ramanujan type identities for alternating knots*, J. Number Theory 161 (2016), 255–280.

[13] C. Lee, R. van der Veen, *Slopes for pretzel knots*, New York J. Math. 22 (2016), 1339–1364.

[14] C. Lee, R. van der Veen, *Colored Jones polynomials without tails*, preprint available at https://arxiv.org/abs/1806.04565

[15] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, 175 (1997), Springer-Verlag, New York.

[16] G. Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, Algebr. Geom. Topol. 3 (2003), 537–556.

[17] G. Masbaum, P. Vogel, *3-valent graphs and the Kauffman bracket*, Pacific J. Math. 164 (1994), no. 2, 361–381.

[18] H. Murakami, *An introduction to the volume conjecture*, Contemp. Math. 541 (2011), 1–40.

School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland
E-mail address: paul.beirne@ucdconnect.ie