The hollow Gaussian beam propagation on curved surface based on matrix optics method

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Abstract
In this paper, based on the Euler–Lagrange equation, an ABCD matrix is constructed out to study the paraxial transmission of light on a constant Gaussian curvature surface (CGCS), which is the first time to our knowledge. Then, by using the method of matrix optics, we extend the CGCS matrix to a general transfer matrix which is suitable for a gently varying curvature. As a beam propagation example, based on the Collins integral and the derived ABCD matrix elements, an analytical propagation formula for the hollow Gaussian beams (HGBs) on the CGCS is deduced. The propagation characteristics of HGBs on a CGCS are illustrated graphically in detail, mainly including the change of dark spot size and splitting rays. Besides its propagation periodicity and diffraction properties, a criterion for convergence and divergence of the spot size is proposed. The area of the dark region of the HGBs can easily be controlled by proper choice of the beam parameters and the shape of CGCS. In addition, we also study the special propagation properties of the hollow beam with fractional order. Compared with propagation characteristics of HGBs in flat space, these novel propagation characteristics of HGBs on curved surface may further expand the application range of hollow beams.

Keywords: curved surface, hollow Gaussian beam, matrix optics

(Some figures may appear in colour only in the online journal)

1. Introduction
We know that under the general relativity theory, the gravitational field is described as space-time curvature, therefore, it is obvious that the study of the light beam propagation on curved space is meaningful for cosmology, such as the measurement of the angular size of stars [1]. Moreover, the interaction between electromagnetic waves and curved space has many intriguing effects, such as Hawking radiation [2, 3] and Unruh effect [4]. On the other hand, it appears many analogical theories and experiments of the curved space since the work by Unruh [5]. For instance, use a Bose–Einstein condensation system to simulate the black hole Gibbons–Hawking effect [6, 7]. Recently, the moving dielectric medium [8, 9] and the nonlinear Schrödinger–Newton system [10, 11] are used to demonstrate the gravitational field caused by the curved space. Thus, the study of the transmission characteristics of light on curved space is not only a generalization of flat space but also a reference for a large number of analogical experiments.

Currently, the two-dimensional (2D) surface is a common simplification model in the study of curved space, such as the 2D Wolf effect of curved space [12], and other studies on geometric optics [13–15] and physical optics [16, 17] of 2D manifolds. But in most cases [18, 19], due to the complexity of curved surface algebra and differential geometry which need
a lot of numerical calculations, sometimes there are singularities, greatly reducing the generality of the analytical expression and thus reducing its physical significance. Moreover, most of the research work is based on the transmission along the generatrix, this paper mainly studies the equatorial transmission. Due to the effective generalization of Wenzel, Kramers, and Brillouin (WKB) approximation from Euclidean space to curved surface [20], we need to further study whether other optical transformation methods can be effectively generalized. Collins’ work [21] shows that ground on the content of simple geometric optics, we can study the properties of physical optics under complex optical transformations. Therefore, in this paper, we hope to utilize the tool—ABCD matrix and Collin’s formula to describe the curved surface optics, which is undoubtedly universal to explore the light propagation characteristics on the curved surface under the paraxial approximation.

Furthermore, since Lin’s work [22], the propagation properties of hollow Gaussian beams (HGBs) have been gradually revealed, such as its generation [23], far-field structure [24], and the propagation through misaligned optical systems [25]. HGB originates from simple Gaussian light and is a typical representative of all other hollow light. Founded on the HGB, a large number of other hollow beams [26–28], including vortex beams, are derived, which have a wide range of applications.

In general relativity, an isotropic and uniformly dense universe can be described using the Roberson–Walker metric (R–W metric), and mathematically, its geometry of the equatorial plane of the constant Gaussian curvature surface (CGCS) [16]. Therefore, we can simulate the optical transformation on the equatorial plane of the R–W metric by studying the beam transmission properties on CGCS.

In our previous work [29], the eikonal function of light rays on a CGCS was derived based on the wave equation. In this paper, we will obtain the CGCS transfer matrix from the Euler–Lagrange equation and further extend it to a general surface by using the language of matrix optics. We prove that this method has good universality, the transformation matrix is multiplicative, so we can easily extend it to the case of multiple devices. Additionally, we study the related propagation properties of a hollow beam on the CGCS. Due to the periodic nature of CGCS transmission, some properties of HGB can even be applied to the far-field. Here, we would like to mention that any light beam can be used to calculate its transmission on CGCS by this ABCD matrix method, the hollow beam is just one example. We hope that this research can provide ideas for beam propagation on other surfaces.

Figure 1. (a) Definition of the surface coordinates and parameters. The blue line represents a light ray on the surface. (b) HGB propagation diagram on the surface, which shows the simulation of the light intensity distribution when \( n = 3 \) and \( r/r_0 = 2 \). (c) A general surface can be thought of as a combination of CGCS.

2. The matrix optical method of curved surface

In this paper, we first consider the transfer matrix of the beam propagation on a CGCS and then generalize it to an ordinary curved surface in paraxial approximation. The CGCS model, with an assumption called surface of revolution, will be formed by the rotation of a certain radius parameter \( \rho(h) \), which is the distance between one point on the surface and the axis of symmetry.

For a 2D surface which is a sub-manifold embedded into three-dimensional space, we define the expression of CGCS as \( \rho(h) = r_0 \cos (h/r) \) where \( r_0 \) and \( r \) are transverse parameters. During the propagation of beam, there is an entrance point and an exit point on this surface, the position \( h_1, h_2 \) and transmission direction \( h_1', h_2' \) of the lights as shown in figure 1(a). According to Fermat’s principle on the surface [19], we know that light travels along geodesics, then we can calculate the propagation path by using Lagrange’s method. First of all, the line element of CGCS: \( ds^2 = dh^2 + \rho^2(h) d\theta^2 \) can be constructed as a d’Alembert action \( s = \int_{h_1}^{h_2} \sqrt{1 + \rho^2(h) \theta^2} dh \), the Euler–Lagrange equation is written as follows:

\[
\frac{d}{dh} \frac{\partial L}{\partial \theta} - \frac{\partial L}{\partial \theta} = 0
\]

where \( \theta \) is the angle of rotation around the axis of symmetry between the incident point and the exit point, \( h \) is the arc
length from the alternative point towards the maximum rotational circuit, $h' = dh/(r_d d\theta)$ represents the transmission direction and the Lagrangian $L = \sqrt{1 + \rho^2 (h)^2}$. According to equation (1), we can obtain the propagation path, which satisfies the following expression:

$$ \tan (h/r) = \sin \left( \frac{r_0}{r} \left( \theta - c_1 \right) \right) c_2. \quad (2) $$

Here $c_1$, $c_2$ are the integral constants. We take the derivative of both sides with respect to $\theta$,

$$ \frac{1}{r} \sec^2 (h/r) \frac{dh}{d\theta} = \frac{r_0}{r} \cos \left( \frac{r_0}{r} \left( \theta - c_1 \right) \right) c_2. \quad (3) $$

According to our coordinate definition, let us assume the initial conditions are $h(0) = h_1, h'(0) = h_1'$ and the final conditions are $h(\theta) = h_2, h'(\theta) = h_2'$, the relationship between the entrance point and an exit point can be derived:

$$ \begin{align*} h_2' &= h_1' \cos \frac{r_0}{r} \theta - \frac{1}{2} \sin \left( \frac{r_0}{r} \theta \right), \\ h_2 &= h_1 \sin \frac{r_0}{r} \theta + \frac{2}{2} \cos \frac{r_0}{r} \theta. \end{align*} \quad (4) $$

Perform the paraxial approximation and ignore the higher-order small terms, we can get the linear relationships: $\sin(h_{1,2}/r) \approx \tan(h_{1,2}/r) \approx h_{1,2}/r$, $\cos(h_{1,2}/r) \approx 1$. That is to say, the transfer function of the beam is the homogeneous expression of $h_1$ and $h_2$. So, in order to better describe the propagation characteristics of light, we introduce the method of matrix optics into the curved surface system in the following:

$$ \begin{pmatrix} h_2 \\ h_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} h_1 \\ h_1' \end{pmatrix} = \begin{pmatrix} \cos \frac{r_0}{r} \theta & \frac{r_0}{r} \theta \\ \frac{1}{r} \cos \frac{r_0}{r} \theta & \frac{r_0}{r} \theta \end{pmatrix} \begin{pmatrix} h_1 \\ h_1' \end{pmatrix}. \quad (5) $$

From equation (5), we can also deduce that the eikonal function of light rays between the incident point and the exit point had the following form as equation (6), which is consistent with our previous work [29]. The method we employ here is simpler, has fewer parameters, and avoids the tedious algebra of differential geometry

$$ s \approx s_0 + \frac{1}{2r} \left( h_1^2 \frac{r_0}{r} \theta - 2h_1 h_2 \frac{r_0}{r} \theta + h_2^2 \frac{r_0}{r} \theta \right). \quad (6) $$

The introduction of matrix optics can facilitate the calculation of paraxial optics. In particular, matrix optics greatly reduces the computational complexity in engineering, where the curved surface may be combined with several optical devices. The significant advantage is that the transformation matrix of the complex system can be simply acquired by multiplied the matrix of the optical devices in order.

The derivation of the $ABCD$ matrix is completely dependent on geometrical optics, which means that the $ABCD$ matrix can be derived by paraxial approximation of the geodesic equation on the surface in addition to using the definition of the ray equation. Here, our study on the surface of constant Gaussian curvature can be extended to an ordinary curved surface. To make the matrix more general, we can rewrite equation (5) by using the transmission distance $z = r_0 \theta$ and the Gaussian curvature $\kappa = r^{-2}$. The new optical transmission transformation matrix $M(z)$ will then be written in the following form as equation (7), which is a function of the transmission distance with constant Gaussian curvature

$$ M(z) = \begin{pmatrix} \cos (\sqrt{\kappa} z) & \frac{1}{\sqrt{\kappa}} \sin (\sqrt{\kappa} z) \\ -\sqrt{\kappa} \sin (\sqrt{\kappa} z) & \cos (\sqrt{\kappa} z) \end{pmatrix}. \quad (7) $$

$z$ axis is usually the geodesic of the surface. For a generally curved surface, $\kappa$ is not constant, but varies with the change of $z$, set as $\kappa(z)$. According to the idea of the element method, as shown in figure 1(c), a general curved surface is divided into more enough CGCS segments. The total equivalent matrix is going to be the product of all of these matrices of CGCS segments:

$$ M(\kappa(z), z) = \lim_{n \to \infty} \left[ \prod_{m=n}^{1} M(\kappa(z_m), \frac{z}{n}) \right] = \lim_{n \to \infty} \left[ \prod_{m=n}^{1} \begin{pmatrix} \cos (\sqrt{\kappa(z_m)} \frac{z}{n}) & \frac{1}{\sqrt{\kappa(z_m)}} \sin (\sqrt{\kappa(z_m)} \frac{z}{n}) \\ -\sqrt{\kappa(z_m)} \sin (\sqrt{\kappa(z_m)} \frac{z}{n}) & \cos (\sqrt{\kappa(z_m)} \frac{z}{n}) \end{pmatrix} \right]. \quad (8) $$

The symbol of $\prod_{m=n}^{1}$ means multiplying matrices in reverse order, $n$ is the number of segments. Using the trigonometric properties and gently varying curvature approximation ($k(z_m) \approx k(z_{m+1})$), equation (8) can be converted to the following form:
\[ M(\kappa(z), z) \approx \lim_{n\to\infty} \left( \cos \left( \frac{\kappa(z)z}{n} \right) \right) \left( \cos \left( \frac{\kappa(z)z}{n} \right) \right) \]

where \( \bar{\kappa} = \frac{\kappa(z)z}{z_2 - z_1} \) from and says the average Gaussian curvature. Equation (9) is worth noting that the matrix is valid only for the surface when the Gaussian curvature change is not too big.

As we have the transformation matrix of the CGCS system, we can study the beam propagation properties passing through this kind of optical system. Combining with the ABCD matrix and the Collins formula, the output light field \( E_2(h_2) \) can be obtained after the input light field \( E_1(h_1) \) passing through the CGCS system, as shown in equation (10):

\[
E_2(h_2) = \left( -\frac{ik}{2\pi B} \right)^{1/2} e^{ikL_0} \times \int_{-\infty}^{\infty} E_1(h_1) \exp \left[ \frac{ik}{2B} \left( Ah_1^2 - 2h_1h_2 + Dh_2^2 \right) \right] dh_1.
\]  

(10)

The above calculation is the light strictly on the curved surface without any thickness at all. Actually, in the experiment, it is common to use thin waveguides to simulate curved surfaces. The thickness of waveguides can indeed affect the transmission of light beams, but it is controllable [13]. When study the problem of thin waveguide propagation, in general, the effect of intrinsic bending of the waveguide on signal transmission is rarely considered in previous studies. As a potential information transmission signal, the hollow beam and its characteristic on curved surface are also valuable for waveguide propagation.

3. Propagation of HGBs on curved surfaces

As was mentioned at the beginning, in this paper, the derived \( ABCD \) matrix will be utilized to study a special case—the propagation of a HGB on CGCS. We can not only visually see the propagation properties of the split light, but also study its hollowness, a property common to many other hollow beams (including vortex beams). In general, we set the electric field of the initial HGB simply as follows [22]:

\[
E_1(h_1) = \left( \frac{h_1^2}{\sigma^2} \right) \text{e}^{-h_1^2/\sigma^2}.
\]  

(11)

In which, \( n \) represents the order of the HGB and the initial transverse spot size is \( w_0 = 2\sigma/\sqrt{n} \). Using the method depicted in the first part, the equation (11) is substituted into the Collins’ formula equation (8), then an analytic solution is deduced:

\[
E_2 = \frac{1}{\sqrt{2\pi i}} \text{e}^{\frac{iax^2}{B} + i\alpha n} \sqrt{\frac{k}{iB}} \left( \frac{1}{\sigma^2} - \frac{1A}{2B} \right)^{-\frac{1}{2} + n} \Gamma \left( \frac{1}{2} + n \right) \text{HF} \left( \frac{1}{2} + n, \frac{1}{2}, -\frac{h_2^2k^2\sigma^2}{4R^2 - 2iABk\sigma^2} \right)
\]  

(12)

where \( \Gamma(x) \) denotes the gamma function. \( \text{HF}(a,b,x) \) represents the Cumor conflunce hypergeometric function and the first few terms of the expansion are

\[
\text{HF}(a,b,x) = 1 + \frac{ax}{b} + \frac{a(1 + a)x^2}{2b(1 + b)} + \frac{a(1 + a)(2 + a)x^3}{6b(1 + b)(2 + b)} + O \left[ x^4 \right].
\]  

(13)

For purpose of facilitating the discussion of the propagation properties of the hollow beam, the curved surface is flattened into a plane surface and a two-period image can be drawn, as shown in figure 2(a). The horizontal coordinate of figure 2(a) is the angle \( \theta \) of revolution for transmission and its vertical coordinate is \( h \). Figure 2(a) is the light intensity flattened maps of the Gaussian beam and the HGB during the propagation on the curved surface. We explore that the HGB can keep the hollow characteristic in the near field, but becomes no longer ‘hollow’ in the far field, the area of the dark region depends on \( n \).
Figure 2. Propagation of the $n$-order HGBs on curved surfaces (a) and flat surfaces (b). On the curved surfaces, we calculated the propagation of HGBs of different order $n$ ($n = 0, 1, 4, 10$). Each plot holds two periods. As observed, with the increase of $n$, the maximum number of splitting rays increased in the process of transmission. The transverse comparison shows that $\gamma_d$ significantly affects the dark spot size. Here, the disappearance of ‘hollow’ is due to the beam interference. The hollow light on the CGCS can be regarded as two separated identical beams at $z = 0$ plane. During its propagation, interference will occur in the area where two beams of light are superimposed. On the axis, the eikonal difference is 0, obviously, the intensity is the main maximum. Due to the geometric symmetry and periodicity of the CGCS, we find that there will be multiple interference strengthening points on its axis. For contrast, the propagation of hollow beams in flat space is presented in figure 2(b). We can explore that the beam intensity on the propagation axis will also increase from zero. Actually, HGB is not a pure mode, but the superposition of a series of Laguerre–Gaussian modes. The different modes have different evolutions, overlapping and interfering during propagation, which brings the intriguing properties of HGBs [22].

In the following, we focus on the beam divergence and convergence properties during transmission. Before that, the study of Gaussian beam (i.e. $n = 0$) can serve as a reference for us, the transverse waist width of the Gaussian beam is calculated as follows:

$$w_{h, n=0} = \sigma \sqrt{\frac{4B^2}{k^2 \sigma^2} + A^2},$$

(14)

the change of waist width equals to

$$\frac{dw_{h, n=0}^2}{d\theta} = \frac{r_0 \sigma^2}{r} \sin \left( \frac{2r_0 \theta}{r} \right) \left[ \left( \frac{2r}{k \sigma^2} \right)^2 - 1 \right].$$

(15)

From equation (15), we explore that the derivative is a periodic function with period $\theta = \pi r/r_0$, and the first half of the period is positive or negative depending on the parameter $\gamma_d = 2r/(k \sigma^2)$, which is called the divergence coefficient in this paper. This coefficient is closely related to the original beam size and the shape of the surface. When $\gamma_d < 1$, the beam converges and then diverges, when $\gamma_d > 1$, the beam first diverges and then converges after emission, and in the case of $\gamma_d = 1$, the variation of Gaussian beam spot size vanishes. The last two transmission features are demonstrated in the first raw plot of figure 2(a). But as the order $n$ increases, it does not work that way. The second-order moment is used to describe the overall mean spot width:

$$w_{h, n}^2 = \frac{\int_{-\infty}^{\infty} h_2^2 |E_2|^2 dh_2}{\int_{-\infty}^{\infty} |E_2|^2 dh_2} = \frac{4n + 1}{4} \left[ 1 + \sin^2 \left( \frac{r_0 \theta}{r} \right) \left( \frac{8n - 1}{16n^2 - 1} \gamma_d^2 - 1 \right) \right].$$

(16)

So, we can use the change in average waist width as a criterion for overall convergence and divergence. According to equation (16), now the divergence coefficient changes from $\gamma_d \frac{\sqrt{(8n - 1)(16n^2 - 1)}}{(16n^2 - 1) \times \gamma_d}$. Compared with figures 2(a) and (b), we also can discover that the curved surface accelerates the near-field and the far-field conversion of light transmission, i.e. the fast conversion between the Fresnel diffraction region and the Fraunhofer diffraction region within half a transmission period on the curved surface. So, we think that the curved space of creating far-field conditions over short distances can be considered for the manufacture of optical elements.
For the Gaussian beam, the light intensity distribution along $h$ direction has only one single peak, while for the propagation process of HGB, the transverse light intensity develops from bimodal image to multiple peaks. The maximum number of peaks is equal to $2n + 1$, which can also be proved analytically by equation (12). To obtain the transverse positions of each peak, we can calculate the extreme value of light intensity during the half period based on equation (12), we demonstrate that their positions satisfy the following expression

$$h_2^2 \sum_{i=0}^{n} \sum_{j=0}^{n} \left( -h_2^2 k^2 \sigma^2 / r^2 \right)^{i+j} \frac{(n!)^2}{(n-i)!(n-j)!(2i)!(2j+1)!} = 0. \quad (17)$$

From equation (17), we can see that, except for the zero points at infinity, $h_2$ has $(4n + 1)$th order at most, that is, there are $4n + 1$ solutions corresponding to $4n + 1$ extremum points of light intensity, including $2n + 1$ peaks and $2n$ valleys, alternately.

Considering the Gaussian beam, when the light diverges, the intensity on the propagation axis becomes weaker, and vice versa, but for the hollow light, the situation is more complicated. The light intensity on the propagation axis is as follows:

$$|E_z|^2 |_{h_2=0} = \left( \frac{(2n - 1)!}{2^{2n-1} (n-1)!} \right)^2 \frac{\gamma_d^2 \left( \gamma_d^2 + \cot^2 \frac{\theta_0 \sqrt{22}}{r} \right)^{-n-1/2}}{\sin \frac{\theta_0 \sqrt{22}}{r}}. \quad (18)$$

According to equation (18), we divide the intensity shapes on the propagation axis into two types as revealed in figure 3: A type and B type. There is only one maximum intensity between two minimum intensity positions for A type, but there are two maximum intensity positions for B type. In fact, these two types distributions describe the axial intensity of the beams shown in the left (A type) and right columns (B type) of figure 2 when $n = 4$, respectively. For different $\gamma_d$, the convergence ability of the curvature surface is different. For type A, the beam converges, interferes then diverges during its propagation. For type B, the beam diverges, interferes, diverges again, then converges, interferes, converges again during the propagation. Surprisingly, the critical boundary conditions for these two types are very simple, that is, whether $\gamma_d$ is greater than $\sqrt{2n + 1}$. It means besides the curvature of the surface and the Rayleigh distance of the beam; the order of the hollow beam can directly affect its hollowness. It is shown that the maximum intensity of type A and the second minimum intensity of type B are both at the half period position with $\theta = r\pi / 2r_0$. The longitudinal length of the dark spot on the propagation axis $w_d$ is defined as the distance from the center of the dark spot to half of the maximum light intensity (type A), or to the first maximum intensity (type B), as shown in figure 3.

For type A, $w_d$ has no analytic solution and satisfies the following equation:

$$\frac{\left( \gamma_d^2 + \cot^2 \frac{2\alpha_0 w}{r} \right)^{-n-1/2}}{\sin \frac{2\alpha_0 w}{r}} = \frac{\gamma_d - 2n - 1}{2}, \quad \gamma_d \leq \sqrt{2n + 1}. \quad (19)$$

While based on equation (18), we can simply obtain the size of the dark spot of type B by calculating the maximum value of the light intensity as:

$$w_d = \frac{r}{r_0} \arcsin \left( \sqrt{\frac{2n}{\gamma_d^2 - 1}} \right), \quad \gamma_d > \sqrt{2n + 1}. \quad (20)$$

Here, we mainly focus on the beam propagation of type B. With the increase of $r$, the longitudinal size of the dark spot becomes smaller, but tends to a constant value $r_0w_d = \sqrt{2n\alpha_0^2 / 2}$, which is consistent with the result of flat space [22]. Compared to flat space, its hollow region will keep longer on curved surface, and it will grow faster than flat space as $n$ increases. However, the transverse size of our dark spot is consistent with that of the flat space, which indicates that our treatment of the curved surface follows such a phenomenon: the paraxial approximation ignores the surface property of light perpendicular to the propagation axis but focuses on the surface property along the propagation axis.

All the above work is based on the fact that the beam order $n$ is an integer, but $n$ can also be a fraction. Next, we focus on the optical transmission characteristics when $n$ is a fraction. Figure 4 displays the examples when the value of $n$ is fraction and half-integer. Interestingly, for a hollow beam with fractional order, we can also consider it as two beams, but the initial phases are inconsistent. As we can see, when $n$ is a half-integer, the phase difference of $\pi$ is introduced into the two beams, so the intensity on the axis will always cancel out then keep 'hollow', which is as same as the beam propagation on flat space. Besides, for other general fractions, the transverse intensity distribution is no longer axisymmetric. During its propagation, the beam converges or diverges, but after the focus point ($\theta = r\pi / r_0$), the transverse intensity distribution flips up and down, then the beam fringes in adjacent periods are distorted, causing periodic changes in the axial light intensity. Here, we can use the statistical skewness to describe the overall distortion of the light field distribution in two adjacent periods. For convenience, the half-period light field distribution is taken as the skew analysis in figure 5. Here, we take the position of half period within two adjacent periods as an index to describe the overall skewness. With the change of $\theta n$, the skewness oscillates and reaches zero when $n$ is an integer.
Figure 4. Fractional order beam transmission images on the curved surfaces (a) and in flat space (b). The intensity of half-integer-order beams on the axis remains dark during its propagation, other fractional orders can cause asymmetry of the transverse light intensity distribution.

Figure 5. Skewness changes with respect to \( n \). The parameter \( m \) denotes to an arbitrary integer. and a half-integer. Moreover, the variation of skewness is complementary for two adjacent periods, as shown by the red solid line and the black dotted line.

Figure 5 indicates that the centroid of the beam deviates from the transmission axis at most values of \( n \), showing the novel property of the self-bending of the fractional hollow beam transmitting along the equator on the curved surface. When \( n \) is an arbitrary number, we can also get a more general magnitude of the light intensity on the axis as:

\[
|E_{2}\rangle_{\phi_0=0}^2 = \frac{\cos^2(\gamma_0^2 \theta_0^2) \gamma^2 (1 + 2n - \gamma_0^2 \theta_0^2 \gamma^2 \cos^2 \theta_0^2)}{\sin \gamma_0^2 \theta_0^2} - \frac{n-1/2}{2}.
\]  

(21)

From equation (21), it can be seen that, due to the existence of the factor \( \cos^2 \theta_0 \), the light intensity on the propagation axis oscillates with \( n \). When \( n \) is half-integer, the light intensity on the propagation axis is zero everywhere, which is very interesting for optical trapping or optical guiding.

4. Conclusion

In conclusion, we introduce an \( ABCD \) matrix of the CGCS based on the Euler–Lagrange equation. Then, we extend the results to a general surface by using the properties of the \( ABCD \) matrix. As the focus of the whole manuscript, we derive an analytical propagation formula for HGBs on CGCS by using the Collins integral and plot the transmission properties of HGB. We find the periodic divergence and convergence of light and the light is split into some rays in the process of transmission. The longitudinal dark area depends on the order \( n \), the original beam size, and the shape of the curved surface, which is almost different from that of flat space. The transverse dark size is consistent with that of flat space. Furthermore, we discuss the optical transmission of HGB when \( n \) is a fraction. The \( ABCD \) matrix has been shown to be an ideal and convenient model with which to describe beam propagation on CGCS. In addition, our matrix processing method is not limited to the propagation on the selected axis. Since the misaligned optical system can be further described by using a \( 4 \times 4 \) tensor matrix, in the future, we can also deal with the cases of eccentricity or rotation of HGB on curved surface [25].

Here, we would like to discuss some possible applications. From the optical transformation matrix form and its periodic, CGCS can be a fractional Fourier transform system, which is useful in the information process [2]. Moreover, the research of hollow beam on curved surface is also valuable for wave-guide propagation [3]. In addition, as mentioned in the introduction, the CGCS system has the same geometry as a R–W black hole on the equatorial plane at a given time, so the research on CGCS may be used as an analog for laboratory simulations of general relativity [1, 4].

Data availability statement

No new data were created or analysed in this study.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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