We study the quasinormal modes (QNMs) of the 5D electrically charged Bardeen black holes by considering the scalar and the electromagnetic field perturbations. The black holes spacetime is an exact solution of the Einstein gravity coupled to the nonlinear electrodynamics in five dimensions which has nonsingular behavior. In order to calculate the QNMs, we use the WKB approximation method upto sixth order corrections. Due to the presence of electric charge $q_e > 0$, both the scalar field and the electromagnetic field perturbations decay more slowly in comparison to the neutral black holes. We find that the scalar field perturbations oscillate more rapidly as compare to the electromagnetic field perturbations. In terms of damping, the scalar ones damp more rapidly than the electromagnetic ones. We further show graphically that the transmission (reflection) coefficients decreases (increases) with an increase in the QNMs frequencies when we increase the value of electric charge $q_e$. The emission of gravitational waves allows the spacetime to undergo the damped oscillations due to nonzero value of the imaginary part which in always negative. The imaginary part of the QNMs is a monotonically decreasing function as we increase the value of electric charge $q_e$ for a given mode $(l, n)$. A connection between the QNMs and the black hole shadow as well as the geometric cross-section in the eiokonal limit is also described.
I. INTRODUCTION

The Einstein field equations when solved confirm the existence of the black hole solutions, describing stationary isolated black holes, such as the Schwarzschild/Reisner-Nordström and the Kerr/Kerr-Newman spacetime which contain an unavoidable curvature singularity in their interior [1]. General relativity cannot overcome the existence of singularity at very high curvature regime. To overcome such singular nature one considers regular or nonsingualr black holes as one of the possible viable models. The first model of a regular black hole was developed by the Bardeen who considered a collapse of a charged matter with a repulsive de Sitter core inside the black hole instead of its singularity [2]. The regularity of the solution means that for a fixed value of the nonlinear parameters the curvature of the spacetime is finite everywhere including at the origin, $r = 0$ with the assumption that the limiting curvature condition is satisfied [3]. After its first inception, the research on singularity-free models of the black holes been studied with significant efforts in last decades. Ayón-Beato and Garcia [4] obtained the first exact spherically symmetric regular black holes solution where general relativity is coupled to the nonlinear electrodynamics field. In the follow up papers, there has been a plenty of research motivated by the ideas of the regular black holes. Many exact solutions of the regular black holes has been obtained so far (see [5–10] and reference there in). The rotating regular black holes have been a test bed of the non-Kerr family of the black holes [11, 12]. They have been used to study several exotic properties ranging from the thermal phase transition [13] to the shadow properties [14, 15], the particle acceleration and particle collisions [16–18], geodesics completeness [19] etc. The regular black hole models have been extended to higher dimensions in modified theories of gravity including Einstein-Gauss-Bonnet [20, 21], Lovelock [22] and in others [23] to study their various properties.

When black holes are perturbed, they emit gravitational waves of the electromagnetic or scalar nature and are characterized by some complex frequencies called the quasinormal modes (QNMs). The real part of the frequencies maintains the oscillations of the gravitational waves while complex part being the damped or undamped in their nature. When one detects such waves emanating from the black holes, one could infer relevant information about the concerned black holes. Perturbations of the black holes are needed in order to have imaginations and to draw information about the nature of the astrophysical black holes. The first research in this direction was performed by Regge and Wheeler [24] and then due to Zerilli [25], who studied respectively, for the radial and polar perturbations of the Schwarzschild black holes. They added some small changes onto the unperturbed background, restricting the conditions that the energy-momentum tensor does not
affect with such perturbations. Following these works, later Vishveshwara [26, 27] and then Chandrasekhar in his monograph [28] explored the QNMs explicitly. The QNMs of the black holes have been studied for several spacetime in general relativity as well as in alternative theories of gravity [29–33], and also in modified theories of gravity including nonsingular black hole models [34–37] and many more. The QNMs have also been described in context of the AdS/CFT correspondence, as there may be a exciting correlation of the thermodynamic properties of the loop quantum black holes with the quasinormal ringing of the astrophysical black hole candidates [38, 39], also in the context of possible connection to the collapsing scenario [40].

A connection between the QNMs of the black holes to their shadows attracted much attentions at present, since the first image of the black hole has been already released. The Event Horizon Telescope group recently detected the black hole images using the shadow properties [41–43]. There exists a plenty of work available in the literature on shadow properties (see [44–53] and references there in). On the other hand, the detection of gravitational waves [54] relating the compact objects such as the black holes was one of the outstanding discoveries and motivates us to study the marriage between the QNMs and the shadow properties of the black holes. This is very fair and motivating to connect these two properties together as they open new arena to the black hole physics. The QNMs and the greybody factors are two important physical phenomena occurring in the background of curved spacetime. On the other hand, the Hawking radiation being a quantum mechanical effect experienced in curved spacetime has paramount importance in study of the black holes related phenomena. Whenever the particles emitted from the black holes, they face an effective potential barrier which forces the particles to go back into the black hole, a phenomena called the back-scattering [55]. The greybody factor is a frequency dependent quantity which measures the deviation from the ideal black body radiation and provides us valuable information about the horizon structure and related physics [56].

The main motivation of this work is to study the QNMs and their connection with the black hole shadows for 5D electrically charged Bardeen spacetime [57]. Along with this, we elaborate our discussion to find a connection between the QNMs and the greybody factor. Recently, the study connecting real part of the QNMs and the shadow radius has been explored for the static [58, 59] and the rotating spacetime [60]. Therefore, the discussion of the present work is twofold: in one hand, we find the QNMs and utilizing these modes, we find the greybody factors of the black holes. On the other hand, using the eikonal approximation, we find a connection between the QNMs and the shadows of the black holes. The electrically charged solutions are important for regular black hole solutions. There exists a no-go theorem regarding the electrical charge solutions
which says that there is no Lagrangian function, having a Maxwell weak field limit, which gives
a solution having regular center. It was shown that the Einstein gravity coupled to the nonlinear
electrodynamics with a Lagrangian having proper Maxwell weak field limit, gives a nontrivial
symmetric solutions with a globally regular spacetime with the proviso that the electric charge is
zero [61–64]. With our Lagrangian, we cannot get the Maxwell limit anywhere, but we allow to
have the regular black holes which are electrically charged as permitted by the no-go theorem. To
investigate the stability of the higher-dimensional black hole solutions which could exist in nature,
we need to derive the spectra of gravitational QNMs. The damped and undamped situations are
related, respectively, to the stable and unstable black holes. The study of the QNMs have been
performed by using many numerical techniques including the WKB method [65], Frobenius method
[39], method of continued fractions [66], the Mashhoon method [67, 68] etc. In our work, we follow
the WKB approximation to calculate the QNMs frequency initially developed by the Schutz and
Will [65]. Iyer and Will [69], in their original paper calculated the QNMs upto third order. The
higher order contribution was due to the Konoplya [38] to show that it allows us to compute the
QNMs frequently without taking account of the complicated numerical methods.

The paper is organized as follows. We give a short description of the 5D electrically charged
Bardeen black holes in Sec. II. The QNMs due to the scalar field perturbations and the electromag-
netic field perturbations are comprehensively discussed in Sec. III and Sec. IV, respectively. The
scattering and greybody factor is the subject of Sec. V. Section VI is devoted to study a connection
between the QNMs and the shadow, and the absorption cross-section is discussed in VII. We end
by concluding the key results in Sec. VIII.

II. 5D ELECTRICALLY CHARGED BARDEEN BLACK HOLES

We begin with a brief description of the 5D electrically charged Bardeen black holes. The
corresponding Einstein-Hilbert action coupled to the nonlinear electrodynamics can be expressed
[57] as follows

\[ S = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left[ R - 4\mathcal{L}(\mathcal{F}) \right], \]

where \( g \) represents the determinant of spacetime metric, \( R \) is the Ricci scalar, and \( \mathcal{L}(\mathcal{F}) \) is the
Lagrangian. Note that \( \mathcal{L}(\mathcal{F}) \) is a nonlinear function of the electromagnetic field strength \( \mathcal{F} =
F_{\mu\nu}F^{\mu\nu}/4 \) with \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The Einstein field equations and the Maxwell equations are
given by
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2 (\mathcal{L}_F F_{\mu\alpha} F^\alpha_{\nu} - g_{\mu\nu} \mathcal{L}) , \]
\[ \nabla_\mu (\mathcal{L}_F F^{\mu\nu}) = 0 , \] (2)

However, the nonlinear source equations for the spacetime (5) have the following forms [57]
\[ \mathcal{L}(r) = \frac{\mu q^3_e (3q^3_e - 4r^3)}{(r^3 + q^3_e)^{10/3}} , \quad \mathcal{L}_F(r) = \frac{(r^3 + q^3_e)^{10/3}}{7 \mu q_e r^9} , \] (3)

and the corresponding vector [57] potential is
\[ A^\mu = - \frac{\mu r^7}{q_e (r^3 + q^3_e)^{7/3}} \delta^\mu_t . \] (4)

The spherically symmetric 5D electrically charged Bardeen black hole spacetime [57] reads simply
\[ ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 + \cos^2 \theta \, d\psi^2 \right) , \] (5)

where the metric function \( f(r) \) has the form,
\[ f(r) = 1 - \frac{\mu r^2}{(r^3 + q^3_e)^{4/3}} . \] (6)

Here the parameter \( q_e \) represents the nonlinear electric charge and \( \mu \) is the black hole mass. This spacetime is asymptotically flat and there the effect of the nonlinear parameter becomes ineffective. A detailed discussion about the spacetime and its regular behavior can be found in [57]. Our next task is to study the quasinormal modes (QNMs) by considering the scalar and the electromagnetic field perturbations.

### III. QNMS OF SCALAR FIELD PERTURBATION

This section is devoted to a comprehensive discussion on the QNMs due to the scalar field perturbation in background of the 5D electrically charged Bardeen black holes. QNMs are basically solutions of the Schrödinger-like wave equation that satisfies the boundary conditions at the event horizon and far away from the black hole. Thus, we require a wave equation in order to compute QNMs. Let us start with the equation of motion for a massless scalar field which is the Klein-Gordon equation [70] and in background of curve spacetime can be written explicitly as follows
\[ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \, g^{\mu\nu} \partial_\nu \Phi \right) = 0 , \] (7)
where $\Phi$ represents the massless scalar field and it is a function of coordinates $(t, r, \theta, \phi, \psi)$. We now consider an ansatz of the scalar field

$$\Phi(t, r, \theta, \phi, \psi) = \sum_{lm} e^{-i\omega t} \frac{\Psi_l(r)}{r^{3/2}} Y_{lm}(r, \theta),$$

(8)

where $e^{-i\omega t}$ represents the time evolution of the field and $Y_{lm}(r, \theta)$ denotes the spherical harmonics function. Plunging the ansatz (8) into (7) and applying the separation of variables method, we obtain the standard Schrödinger-like wave equation

$$\frac{d^2 \Psi_l(r_*)}{dr_*^2} + (\omega^2 - V_s(r_*)) \Psi_l(r_*) = 0,$$

(9)

where $\omega$ is the frequency of the perturbation and $r_*$ represents the tortoise coordinates having the relation

$$dr_* = \frac{dr}{f(r)} \Rightarrow r_* = \int \frac{dr}{f(r)}.$$

(10)

Here the advantage of using the tortoise coordinates is to extend the range used in the survey of QNMs. Actually, the tortoise coordinate is being mapped the semi-infinite region from the horizon to infinity into $(-\infty, +\infty)$ region. The effective potential in (9) is given [71] as following

$$V_s(r_*) = \left(1 - \frac{\mu r^2}{(r^3 + q_e^3)^{4/3}}\right) \left[ l(l+2) - \frac{6\mu q_e^3 r^2}{(r^3 + q_e^3)^{7/3}} + \frac{3}{4r^2} \left(1 - \frac{\mu r^2}{(r^3 + q_e^3)^{4/3}}\right)\right],$$

(11)

where $l$ denotes the multipole number. It is noticeable that the effective potential (11) has the form of a potential barrier. We can now apply the WKB approach in order to compute the QNMs due to scalar field perturbation. The third order QNMs frequencies [69] are given by

$$\omega^2 = \left(V_0 + \sqrt{-2V_0'' \Lambda_2}\right) - i \left(n + \frac{1}{2}\right) \sqrt{-2V_0''(1 + \Lambda_3)},$$

(12)
where $\Lambda_2$ and $\Lambda_3$ are defined as follows

$$
\Lambda_2 = \frac{1}{\sqrt{-2V_0'}} \left[ \frac{1}{8} \left( \frac{V_0^{(4)}}{V_0''} \right) \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V_0^{(3)}}{V_0''} \right)^2 (7 + 60\alpha^2) \right],
$$

$$
\Lambda_3 = \frac{1}{\sqrt{-2V_0'}} \left[ \frac{5}{6912} \left( \frac{V_0^{(3)}}{V_0''} \right)^4 (77 + 188\alpha^2) - \frac{1}{384} \left( \frac{V_0''^2 V_0^{(4)}}{V_0'^{10}} \right) (51 + 100\alpha^2) 
+ \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0''} \right)^2 (67 + 68\alpha^2) - \frac{1}{288} \left( \frac{V_0''' V_0^{(5)}}{V_0''^2} \right) (19 + 28\alpha^2) 
- \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0''} \right) (5 + 4\alpha^2) \right].
$$

(13)

On the other hand, $\alpha$ and $V_0^{(m)}$ have the following definitions

$$
\alpha = n + \frac{1}{2}, \quad V_0^{(m)} = \left. \frac{e^m V}{dr^m} \right|_{r_*},
$$

(14)

where $n$ represents the overtone number. We are going to use the sixth order WKB correction method which is described in [38]. The corresponding formula can have the form as follows

$$
i \omega_n^2 - \frac{V_0}{\sqrt{-2V_0'}} - \sum_{i=2}^{6} \Lambda_i = n + \frac{1}{2},
$$

(15)

where the definitions of $\Lambda_4$, $\Lambda_5$, $\Lambda_6$ can be find in [38]. Note that here $V_0$ represents the height of the barrier and $V_0''$ denotes the second derivative with respect to tortoise coordinate of the potential at maximum. The typical behavior of the effective potential (11) for the scalar field perturbation can be seen in Fig. 1. It is found that the corrections depend on the value of potential and its higher derivatives at the maximum. We present our numerical results for the scalar field perturbations in Table I. The QNMs are depicted in Figs. 2 and 3. We see that when the charge $q_e$ increases

| $q_e$ | $l = 1, n = 0$ | $l = 2, n = 0$ | $l = 2, n = 1$ |
|------|----------------|----------------|----------------|
| 0    | 1.01444 - 0.36524 i | 1.51050 - 0.35770 i | 1.39249 - 1.10537 i |
| 0.1  | 1.01469 - 0.36512 i | 1.51087 - 0.35756 i | 1.39307 - 1.10491 i |
| 0.2  | 1.01641 - 0.36426 i | 1.51346 - 0.35656 i | 1.39717 - 1.10165 i |
| 0.3  | 1.02126 - 0.36167 i | 1.52062 - 0.35371 i | 1.40836 - 1.09229 i |
| 0.4  | 1.03129 - 0.35551 i | 1.53516 - 0.34755 i | 1.43036 - 1.07169 i |
| 0.5  | 1.04892 - 0.34209 i | 1.56114 - 0.33511 i | 1.46649 - 1.02934 i |

the real part of the QNMs increases. On the other hand, we find that the charge $q_e$ decreases the
FIG. 2. (Left panel) Plots showing the dependence of real part of the QNMs with electric charge \( q_e \) for scalar field perturbations. (Right panel) Plots showing the dependence of real part of the QNMs versus the imaginary part of the QNMs in absolute value. \((\mu = 1)\).

imaginary part of the QNMs in absolute value. This indicates that the scalar field perturbations in the presence of electric charge \( q_e > 0 \) decays more slowly compared to the neutral black holes. It is worth noting that we have not calculated the QNMs in case of fundamental mode \( l = n = 0 \) in Table I. This is related to the fact that the WKB method is applicable when \( l > n \) and does not give a satisfactory degree of precision for this fundamental mode \((l = n = 0)\). Nevertheless, one can use other methods such as for example the Frobenius method to include this fundamental mode (see for example [72]).
IV. QNMS OF ELECTROMAGNETIC FIELD PERTURBATION

In this section, we present the QNMs by considering the electromagnetic field perturbation in background of the 5D electrically charged Bardeen black holes. As the field strength of the electromagnetic tensor is given \[57\] by the following expression

\[
\mathcal{F} = -\frac{q_e^2}{2L_F^2(r)r^6}.
\]  

We can rewrite the Maxwell equations \(\nabla \mu (\mathcal{L}_\mathcal{F} F^{\mu \nu}) = 0\) as follows

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} \mathcal{L}_\mathcal{F} g^{\lambda \mu} g^{\sigma \nu} (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda) \right] = 0.
\]  

We substitute all related expressions into (17) and using the standard tortoise coordinate transformation \(dr_s = dr/f(r)\), which turns out to be the second-order differential equations of the following form \([73, 74]\):

\[
\frac{d^2 \Psi(r_s)}{dr_s^2} + \left[ \omega^2 - V_{1,2}(r_s) \right] \Psi(r_s) = 0.
\]  

This is interesting to notice that we have two categories of effective potential in case of the higher-dimensional black holes while discussing the electromagnetic perturbation. They are divided into the electromagnetic scalar perturbation and electromagnetic vector perturbation. The effective potential corresponding to the electromagnetic scalar perturbation is given \([73, 74]\) by

\[
V_1(r) = \left(1 - \frac{\mu r^2}{(r^3 + q_e^3)^{4/3}}\right) \left[ \frac{l(l+2)}{r^2} + \frac{3}{4r^2} \left(1 - \frac{\mu r^2}{(r^3 + q_e^3)^{4/3}}\right) + \frac{2\mu q_e^3 r^2}{(r^3 + q_e^3)^{7/3}} \right],
\]
and the effective potential corresponding to the electromagnetic vector perturbation has the following form [73, 74],

$$V_2(r) = \left( 1 - \frac{\mu r^2}{(r^3 + q_e^3)^4/3} \right) \left[ \frac{(l + 1)^2}{r^2} - \frac{3}{4r^2} \left( 1 - \frac{\mu r^2}{(r^3 + q_e^3)^4/3} \right) - \frac{2\mu q_e^2 r^2}{(r^3 + q_e^3)^7/3} \right]. \quad (20)$$

In order to see the nature, we plot these expressions of the effective potential functions. One can see the typical behavior of these potentials in Figs. 4 and 5. We see that an increase in the magnitude of charge $q_e$ changes the height of the potential barrier. Similarly, the height of potential barrier changes with an increase in multipole number.

We present numerical calculations of the QNMs in case of both scalar and vector types electromagnetic field perturbations (cf. Table II and III). We observe that by increasing the electric charge $q_e$, the real part of the QNMs increases while the imaginary part decreases in absolute value.

---

**FIG. 4.** Plot showing the behavior of effective potential function in case of electromagnetic scalar perturbation for different values of multipole number, electric charge $q_e$, and mass $\mu = 1$.

**FIG. 5.** Plot showing the behavior of effective potential function in case of electromagnetic vector perturbation for different values of multipole number $l$, electric charge $q_e$, and mass $\mu = 1$.
TABLE II. Real and imaginary parts of the QNMs in electromagnetic scalar field perturbations ($\mu = 1$).

| $q_e$ | $\omega (WKB)$ | $\omega (WKB)$ | $\omega (WKB)$ |
|-------|----------------|----------------|----------------|
| 0     | 0.73685 - 0.31537 i | 1.34123 - 0.33725 i | 1.21139 - 1.04385 i |
| 0.1   | 0.73771 - 0.31492 i | 1.34165 - 0.33712 i | 1.2121 - 1.04342 i |
| 0.2   | 0.74378 - 0.31183 i | 1.34453 - 0.33617 i | 1.21702 - 1.04039 i |
| 0.3   | 0.76020 - 0.30411 i | 1.35250 - 0.33352 i | 1.23035 - 1.03188 i |
| 0.4   | 0.79081 - 0.29230 i | 1.36856 - 0.32783 i | 1.25617 - 1.01378 i |
| 0.5   | 0.82919 - 0.28502 i | 1.39710 - 0.31630 i | 1.29788 - 0.97684 i |

TABLE III. Real and imaginary parts of the QNMs in electromagnetic vector field perturbations ($\mu = 1$).

| $q_e$ | $\omega (WKB)$ | $\omega (WKB)$ | $\omega (WKB)$ |
|-------|----------------|----------------|----------------|
| 0     | 0.95143 - 0.35304 i | 1.46852 - 0.35248 i | 1.34827 - 1.08979 i |
| 0.1   | 0.95174 - 0.35296 i | 1.46894 - 0.35234 i | 1.34892 - 1.08934 i |
| 0.2   | 0.95393 - 0.35240 i | 1.47186 - 0.35137 i | 1.35348 - 1.08614 i |
| 0.3   | 0.95996 - 0.35068 i | 1.47993 - 0.34858 i | 1.36596 - 1.07699 i |
| 0.4   | 0.97226 - 0.34634 i | 1.49638 - 0.34253 i | 1.39060 - 1.05692 i |
| 0.5   | 0.99438 - 0.33563 i | 1.52601 - 0.33011 i | 1.43181 - 1.01517 i |

as compared to the neutral black hole. This particular effect can be clearly seen from Figs. 6, 7, 8, and 9. We do a comparison of the numerical results between the scalar field perturbations and the electromagnetic field perturbations cases (cf. Table I, II and III). We find that the numerical values of real/imaginary part of the QNMs in absolute value are comparatively higher in scalar field perturbations. This means that the scalar field perturbations oscillate more rapidly compared to the electromagnetic field perturbations, but in terms of damping, scalar field ones damp more rapidly than electromagnetic field ones. Moreover, we compare the numerical results of the real/imaginary part in absolute value of the QNMs for scalar type and vector type perturbations of the electromagnetic field. It is found that the vector type perturbations oscillate and damp more rapidly in comparison to the scalar type perturbation (cf. Table II and III). In general, however, both the electromagnetic and scalar field perturbations in presence of the electric charge $q_e$ decay more slowly as compare to the neutral black hole.
FIG. 6. (Left panel) Plot showing the dependence of real part of the QNMs with electric charge $q_e$ for electromagnetic scalar field perturbations. (Right panel) Plot showing the dependence of real part of the QNMs versus the imaginary part of the QNMs in absolute value. ($\mu = 1$).

V. SCATTERING AND GREYBODY FACTORS

Now we are going to discuss the greybody factors for the 5D electrically charged Bardeen black holes. This study is important in order to determine, for instance, the amount of initial quantum radiation in vicinity of the event horizon of the black hole. It is reflected back to it by the potential barrier. Therefore, it is natural to interpret the greybody factor as the tunneling probability of the wave through the barrier determined by the effective potential in the given black hole spacetime.
We begin with the Schrödinger-like equation which describes the scattering of waves in the black holes. Since this equation cannot be solved analytically in general and even in order to find the greybody factor, it is not necessary to solve the equation exactly. Thus we consider the following asymptotic solutions of the equation

\[ \Psi = A e^{-i\omega r_*} + B e^{i\omega r_*}, \quad r_* \to -\infty, \]
\[ \Psi = C e^{-i\omega r_*} + D e^{i\omega r_*}, \quad r_* \to +\infty, \]  

Furthermore, we need to impose the conditions, \( B(\omega) = 0 \) for waves coming to the black hole from infinity and \( R(\omega) = D(\omega)/C(\omega) \) for the reflection amplitude. It is noticeable that due to the symmetry of the scattering properties this is identical to the scattering of a wave coming from the horizon. The transmission amplitude is given by \( T(\omega) = A(\omega)/C(\omega) \), thence

\[ \Psi = T e^{-i\omega r_*}, \quad r_* \to -\infty, \]
\[ \Psi = e^{-i\omega r_*} + R e^{i\omega r_*}, \quad r_* \to +\infty. \]

Our next task is to determine the square of the wave function’s amplitude. The probability of finding the wave satisfies the following condition

\[ |R|^2 + |T|^2 = 1. \]

While discussing the greybody factor, we consider \( \omega \simeq V(r_0) \), which is the most interesting case. In order to compute the reflection and transmission coefficients, we follow the WKB approximation.
FIG. 8. (Left panel) Plots showing the dependence of real part of the QNMs with electric charge $q_e$ for electromagnetic vector field perturbations. (Right panel) Plots showing the dependence of real part of the QNMs versus the imaginary part of the QNMs in absolute value. ($\mu = 1$).

Hence, one can get the reflection coefficient as follows

$$R = \frac{1}{\sqrt{1 + \exp(-2\pi i K)}},$$

(26)

where $K$ is given by

$$K = i \frac{\omega_n^2 - V(r_0)}{\sqrt{-2 V''(r_0)}} - \sum_{i=2}^{6} \Lambda_i,$$

(27)

On having the expression of the reflection coefficient, now we can use the condition (25) in order
FIG. 9. (Left panel) Plots showing the dependence of real part of the QNMs with electric charge $q_e$ for different values of $l$ and $n$ for electromagnetic vector perturbation. (Right panel) Plots showing the dependence of real part of the QNMs in absolute value with electric charge $q_e$ for different values of $l$ and $n$. ($\mu = 1$).

to obtain the transmission coefficient which takes the form

$$|T|^2 = 1 - \left| \frac{1}{\sqrt{1 + \exp(-2\pi iK)}} \right|^2.$$  \hspace{1cm} (28)

In Fig. 10, we show the effect of charge $q_e$ on the transmission and the reflection coefficients for the scalar field perturbations and the electromagnetic field perturbations. We see that by increasing the magnitude of electric charge $q_e$, the transmission coefficient decreases in all the cases. On the other hand, the reflection coefficients increases with an increase in the magnitude of the electric charge $q_e$.

VI. CONNECTION BETWEEN SHADOW RADIUS AND QNMS

In this section, we are going to demonstrate a relationship between the shadow and the QNMs of the 5D electrically charged Bardeen black holes. To achieve the goal, let us first start by discussing the shadow of the 5D electrically charged Bardeen black holes. As we know from the symmetry of the spacetime (5) that it admits three Killing vectors, namely $\partial_t$, $\partial_\phi$, and $\partial_\psi$. The presence of these Killing vectors can give rise to the associated conserved quantities, namely, the energy $E$, and the two angular momentum $L_\phi$ and $L_\psi$ in $\phi$ and $\psi$ directions, respectively. By using these conserved quantities and their relations with conjugate momenta, we can easily have the geodesic
FIG. 10. (Left panel): From top to bottom, plots showing the behavior of transmission coefficients for the scalar field perturbations, electromagnetic field perturbations of scalar and vector types, respectively. (Right panel): From top to bottom, plots showing the reflection coefficient for the scalar field perturbations, electromagnetic field perturbations of scalar and vector types, respectively. ($\mu = 1$ and $l = 1$).

equations

\[
\begin{align*}
\frac{dt}{d\sigma} &= \frac{E}{f(r)}', \\
\frac{d\phi}{d\sigma} &= \frac{L_{\phi}}{r^2 \sin^2 \theta}', \\
\frac{d\psi}{d\sigma} &= \frac{L_{\psi}}{r^2 \cos^2 \theta}'.
\end{align*}
\] (29)
where $\sigma$ is an affine parameter. Besides, the radial and the angular geodesic equations can be derived by using the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \sigma} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},$$  

(30)

where $S$ being the Jacobi action. The Jacobi action $S$ can be separated in the following form

$$S = \frac{1}{2} m_0^2 \sigma - Et + L_\phi \phi + L_\psi \psi + S_r(r) + S_\theta(\theta),$$  

(31)

where $S_r$ and $S_\theta$ are the functions of $r$ and $\theta$ only, respectively and $m_0$ denotes the mass of the test particle which vanishes in case of photon. We now substitute (31) into (30) and after some straightforward calculation, we obtain

$$r^2 \frac{dr}{d\sigma} = \pm \sqrt{R(r)},$$

$$r^2 \frac{d\theta}{d\sigma} = \pm \sqrt{\Theta(\theta)},$$  

(32)

where the function $R(r)$ and $\Theta(\theta)$ read simply

$$R(r) = E^2 r^4 - r^2 f(r) \left( K + L_\phi^2 + L_\psi^2 \right),$$

$$\Theta(\theta) = K - L_\phi^2 \cot^2 \theta - L_\psi^2 \tan^2 \theta.$$  

(33)

Here $K$ denotes the Carter constant which appears when we separate the coefficients of $r$ and $\theta$ during Hamilton-Jacobi formulation.

In order to describe the black hole shadow, we introduce the celestial coordinates which in case of the 5D black holes are given [75] as following:

$$x = - \lim_{r_0 \to \infty} r_0 \frac{p^\phi + p^\psi}{p^t},$$

$$y = \lim_{r_0 \to \infty} r_0 \frac{p^\theta}{p^t},$$  

(34)

where $p^i$ being the contravariant components of the momenta in new coordinate basis. These contravariant components of the momenta can be easily computed by using the orthonormal basis vectors for the local observer [75, 76], they become

$$p^\hat{t} = \frac{E}{f(r)}, \quad p^\hat{\phi} = \frac{L_\phi}{r \sin \theta}, \quad p^\hat{\psi} = \frac{L_\psi}{r \cos \theta},$$

$$p^\hat{r} = \pm \sqrt{f(r) R(r)}, \quad p^\hat{\theta} = \pm \sqrt{\Theta(\theta)} \frac{1}{r}.$$  

(35)
Plunging (35) into (34) and taking the limit $r_0 \to \infty$, provides

$$x = - (\xi_\phi \csc \theta + \xi_\psi \sec \theta),$$

$$y = \pm \sqrt{\eta - \xi_\phi^2 \cot^2 \theta - \xi_\psi^2 \tan^2 \theta}, \quad (36)$$

where we introduce the new quantities $\xi_\phi = L_\phi / E$, $\xi_\psi = L_\psi / E$, and $\eta = K / E^2$, also known as the impact parameters. If the observer is situated in equatorial plane ($\theta_0 = \pi/2$), then the angular momentum $L_\psi = 0$, which implies $\xi_\psi = 0$, thus we have

$$x = -\xi_\phi, \quad y = \pm \sqrt{\eta}. \quad (37)$$

On the other hand, when $\theta_0 = 0$, in this case $L_\phi = 0$, which implies $\xi_\phi = 0$, thereby

$$x = -\xi_\psi, \quad y = \pm \sqrt{\eta}. \quad (38)$$

The expressions (37) and (38) represent a connection between the celestial coordinates and the impact parameters when the observer is situated at inclination angles $\theta_0 = \pi/2$ and $\theta_0 = 0$, respectively. These expressions are very important in order to extract the informations regarding the shadows of the black holes.

We further derive a relationship between the impact parameters of the spacetime (5) by using the unstable spherical photon orbits conditions, $\mathcal{R} = 0$ and $d\mathcal{R}/dr = 0$, which turns out to be

$$\eta + \xi_\phi^2 + \xi_\psi^2 = \frac{r^2}{f(r)}, \quad r = \frac{2f(r)}{f'(r)}. \quad (39)$$

where prime ($'$) being the derivative with respect to $r$. Note that the metric function $f(r)$ is defined in (6) and the derivative of it reads simply

$$f'(r) = \frac{2\mu(r^4 - q_e^2 r)}{(r^3 + q_e^2)^{7/3}}. \quad (40)$$

On having all the related expressions with us, now we can portrait the shadows of the 5D electrically charged Bardeen black holes. The typical behavior of these shadows with electric charge $q_e$ can be seen in Fig. 11. We notice that the shape of black hole shadow is a perfect circle because of the spherical symmetry. We see a decrease in radius of the black hole shadow with increasing magnitude of the electric charge $q_e$. Therefore, we can conclude that the presence of electric charge $q_e$, in the black holes decreases the radius of the black hole shadow.

In seminal paper by the Cardoso et al. [77], it was argued that in the eikonal limit, the real part of the QNMs is related to the angular velocity of the unstable null geodesic while the imaginary
FIG. 11. Illustration of shadows of the 5D electrically charged Bardeen black holes for different values of electric charge $q_e$ and mass $\mu = 1$.

Part of the QNMs is related to the Lyapunov exponent that determines the instability time scale of the orbits. Thus, the relationship is given \cite{77} as follows

$$\omega_{QNM} = \Omega_c l - i \left( n + \frac{1}{2} \right) |\lambda|,$$

where $\Omega_c$ is the angular velocity at the unstable null geodesic and $\lambda$ denotes the Lyapunov exponent. Furthermore, this important result is expected to be valid not only for the static spacetimes but also for the stationary ones. Later on, Stefanov et al. \cite{78} pointed out a connection between the QNMs of the black holes in the eikonal limit and the strong lensing. Most recently, one of the authors of this paper pointed out that the real part of the QNMs and the shadow radius are related by the following relation (see for details \cite{58, 79})

$$\omega_R = \lim_{l \gg 1} \frac{l}{R_s},$$

which is precise only in the eikonal limit having large values of $l$. Here $R_s$ denotes the radius of the black hole shadow. Hence, we can easily write

$$\omega_{QNM} = \lim_{l \gg 1} \frac{l}{R_s} - i \left( n + \frac{1}{2} \right) |\lambda|.$$
FIG. 12. Shadow radius of the 5D electrically charged Bardeen black holes as a function of electric charge $q_e$ obtained via the relation (42) with mass $\mu = 1$.

In other words, instead of the angular velocity, it is more convenient to express the real part of the QNMs in terms of the black hole shadow radius. This close connection can be understood from the fact that the gravitational waves can be treated as massless particles propagating along the last null unstable orbit and out to infinity. At this point, we note that the correspondence is not guaranteed for the gravitational fields, as the link between the null geodesics and the quasinormal modes is shown to be violated in the context of the Einstein-Lovelock theory even in the eikonal limit [80]. Although the relation (42) is not accurate for small $l$, as we are going to see, it can provide important information regarding the effect of charge $q_e$ on the shadow radius, once we have calculated the real part of the QNMs or vice versa. The real part of the QNMs we show that $\omega_\Re$ increases with $q_e$, and subsequently we can make use of the inverse relation between $\omega_\Re$ and the shadow radius $R_S$,

$$\omega_\Re(q_e) \propto \frac{1}{R_s(q_e)}, \quad (44)$$

to show that the shadow radius decreases with increasing magnitude of charge $q_e$. This indeed is shown to be the case as depicted in Fig. 11, by using the geodesic method. It is quite amazing that we can deduce this information directly from the inverse relation between the real part of the QNMs and the shadow radius (42) even in case of small multipoles $l$, although the relation (42) is precise only in the eikonal regime having $l \gg 1$. We have used (42) to show the effect of electric charge $q_e$ on the shadow radius to be depicted in Fig. 12. We observe that the shadow radius of the 5D electrically charged Bardeen black holes is smaller than the shadow radius of the 5D neutral black holes.
VII. ABSORPTION CROSS-SECTION

We now determine the partial absorption cross-section for the 5D electrically charged Bardeen black holes. This is another important quantity which is defined in $D$-dimensions as following [81]:

$$\sigma_l = \frac{\pi^{D-2}}{\Gamma\left(\frac{D-2}{2}\right) \omega^{D-2}} \frac{(l + d - 4)! (2l + d - 3)}{l!} |T_l(\omega)|^2.$$  (45)

Therefore, in case of the 5D electrically charged Bardeen spacetime, the partial absorption cross-section reads

$$\sigma_l = \frac{4\pi (l + 1)^2}{\omega^3} |T_l(\omega)|^2.$$  (46)

In some cases, it is useful to define the so-called total absorption cross-section which is given as follows

$$\sigma = \frac{\pi^{D-2}}{\Gamma\left(\frac{D-2}{2}\right) \omega^{D-2}} \sum_{l=0}^{\infty} \frac{(l + d - 4)! (2l + d - 3)}{l!} |T_l(\omega)|^2.$$  (47)

One can confirm the Figs. 13 and 14 for more information where we show the dependence of partial absorption cross-section versus frequency for different values of the electric charge $q_e$. This can be observed that the partial absorption cross-section decreases by increasing the electric charge $q_e$ (cf. Figs. 13 and 14). Moreover, it can be seen that the partial absorption cross-section initially increases by increasing $\omega$, then it reaches to the maximum value at some critical value of $\omega$ and finally it falls off regardless of the black hole parameters due to the dependence on the term $\omega^{-3}$.

In the high-energy scale, we know that the wavelength is almost negligible relative to the horizon scale of the black hole, hence, we can use the classical capture cross-section of the light by the black holes as the geometric cross-section of the light rays. The geometric cross-section of the light rays is given [81, 82] by the expression

$$\sigma_{geom} = \frac{\pi^{D-2} b_{ps}^{D-2}}{\Gamma(D/2)},$$  (48)

where $b_{ps}$ is the critical impact parameter of the light defined by the ratio of the angular momentum and the energy of photon moving along the spherical photon orbits as $b_{ps} = J/E$. Now we can use the geometric-optics correspondence between the parameters of the QNMs, and the conserved quantities along geodesics. In particular, the energy of the particle can be identified with the real part of the QNMs and the azimuthal quantum number corresponds to the angular momentum [83], hence, we can easily get

$$E \rightarrow \omega_R, \quad \text{and} \quad J \rightarrow l.$$  (49)
FIG. 13. The plot shows the dependence of partial absorption cross-section versus frequency of the scalar field perturbations. ($\mu = 1$ and $l = 1$).

FIG. 14. (Left panel) Plot shows the dependence of partial absorption cross-section versus frequency of the electromagnetic field scalar perturbations. (Right panel) Plot shows the dependence of partial absorption cross-section versus frequency of the electromagnetic field vector perturbations. ($\mu = 1$ and $l = 1$).

This means that we can identify the impact parameter of the light with the shadow radius, i.e., $b_{ps} \to R_s$. We now use (42) to find the eikonal limit, the geometric cross-section, in terms of the real part of the QNMs

$$\sigma_{geom} = \lim_{l \gg 1} \pi \frac{D-2}{D-2} \frac{1}{\Gamma(D/2)\omega_{R}^{D-2}},$$

which is valid in the eikonal regime. As we already pointed out, due to the symmetry of the scattering properties, the greybody factors are also present in the emission spectrum of the black holes. In other words, if the black hole emits in the eikonal regime, the connection between the real part of the QNMs and the geometric cross-section given in (50) is to be expected. Given
FIG. 15. (Left panel) Geometric cross-section as a function of electric charge $q_e$. In both plots we have used the scalar field perturbation to calculate $\omega_R$ with $\mu = 1$. One can reach the same conclusion using the electromagnetic field perturbations, provided $l \gg 1$. (Right panel) Plot of the total absorption cross-section versus frequency of the field scalar perturbations. We have used $q_e = 0.1$ (red curve) and $q_e = 0.5$ (blue curve), respectively. ($l = 10^3$, $\lambda = 0.8$ and $\mu = 1$).

the fact that the geometric cross-section scales with the shadow radius, and one can calculate the geometric cross-section by means of the real part of the QNMs. To the best of our knowledge, the relation between the geometric cross-section and the real part of the QNMs has not been explored before. In a very interesting paper [81], it has been shown that there are fluctuations (of regular oscillations) of the high-energy (frequency) absorption cross-section around the limiting value of the geometric cross section. In particular, it was found that the oscillatory part of the absorption cross-section of the massless scalar waves is given [81] by

$$\sigma_{osc} = (-1)^{D-3} 4(D-2)\pi \lambda R_s e^{-\pi \lambda R_s} \sigma_{geom} \text{sinc}(2\pi R_s \omega),$$

where we have used the correspondence, $b_{ps} \rightarrow R_s$. Moreover, the function has been introduced $\text{sinc}(x) = \sin(x)/x$ and $\lambda$ is the Lyapunov exponent used for analysis of the instability of the null geodesics. The total absorption cross-section of the massless scalar waves is the sum of the geometric and oscillatory cross-sections

$$\sigma \approx \sigma_{geom} + \sigma_{osc}. \quad (52)$$

Eventually, in case of the 5D black holes, the geometric cross-section in terms of the real part of the QNMs is given as follows

$$\sigma_{geom} = \lim_{l \gg 1} \frac{4\pi l^3}{3 \omega^5_R}. \quad (53)$$
In Fig. 15 (left panel), we have used (53) to show that the geometric cross-section decreases with an increase in electric charge $q_e$ similar to that of the shadow radius. In other words, once we compute the real part of the QNMs in eikonal limit, we can easily find the geometric cross-section as it simply scales with the shadow radius. On the other hand, the oscillatory part in case of the 5D black holes reads simply

$$\sigma_{osc} = 12\pi \lambda R_e e^{-\pi \lambda R_e} \sigma_{geom} \text{sinc}(2\pi R_e \omega).$$

In Fig. 15 (right panel), we show the total absorption cross-section for the scalar field perturbations in the 5D electrically charged Bardeen black holes for different values of electric charge $q_e$. We see that by increasing the electric charge $q_e$, the total absorption cross-section decreases. In this sense, our result is consistent with the effect of electric charge in case of 4D electrically charged black hole and generalizes the results reported in Ref. [81].

VIII. CONCLUSION

In this work, we perform a comprehensive discussion on QNMs of the 5D electrically charged Bardeen black holes. The study provides us very interesting results in five dimensions while considering the general relativity coupled to the nonlinear electrodynamics. In order to compute the QNMs of the black holes, we have considered the scalar field and the electromagnetic field perturbations. We have used the WKB approach upto sixth order corrections to determine the numerical results on the QNMs. The effect of nonlinear electric charge $q_e$ on the QNMs has been investigated which provides a significant effect. To study the scalar field perturbations, it has been found that the charge $q_e$ increases the real part of the QNMs and decreases the imaginary part of the QNMs in absolute amount. While discussing the electromagnetic field perturbations, we discover that an increase in charge $q_e$ increases the real part of the QNMs and it decreases the imaginary part of the QNMs in both cases of scalar and vector types. We have noticed that the numerical values of the QNMs are higher in scalar field perturbation as compare to the electromagnetic field perturbations. This indicates that the scalar field perturbations oscillate more rapidly in comparison to the electromagnetic field perturbations. In terms of damping, the scalar field perturbations damp more rapidly than the electromagnetic field perturbations.

In further analysis, we have discussed the scattering and the greybody factors in case of the 5D electrically charged Bardeen black holes. We have calculated analytical expressions of the reflection and the transmission coefficients. It has been found that the reflection coefficient increases with
charge $q_e$ while the transmission coefficient decreases with charge $q_e$. Moreover, we have extended our analysis to determine a direct connection between the black hole shadow and the QNMs. The null geodesics and spherical photon orbits condition have been discussed in order to describe the shadow of the 5D electrically charged Bardeen black holes. The shadow of the black hole has been portrayed by varying the charge $q_e$ as well as the typical behavior is discussed. We have observed that the presence of charge $q_e$ affects the radius of the black hole shadow which decreases by an increase in the magnitude of charge $q_e$. We have shown that this result can be obtained by means of the real part of the QNMs valid in the eikonal limit. We have also discussed how the charge $q_e$ affects the partial absorption cross-section in case of the 5D electrically charged black holes. Finally, we have explored the total absorption cross-section in the high-energy scale which consists of the geometric cross-section part and oscillatory part. Importantly, using the geometric-optics correspondence, we have expressed the geometric cross-section, in terms of the real part of the QNMs. Our result shows that the total absorption cross-section decreases with an increase in magnitude of charge $q_e$.

Our work can be extended to the AdS spaces as they open a new avenue on the onset of the connection of the photon orbits to the thermodynamic phase transition of the AdS black holes. On the other hand, shadow and the QNMs of AdS black holes are connected with the stability, the thermodynamic phase transition and also in hydrodynamic region of strongly coupled field theories. Hence, the study of the QNMs in 5D electrically charged regular AdS black holes and its connection to the shadow properties will be a natural extension of our work in the future project.

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