Transition probabilities and dynamic structure factor in the ASEP conditioned on strong flux

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We consider the asymmetric simple exclusion processes (ASEP) on a ring constrained to produce an atypically large flux, or an extreme activity. Using quantum free fermion techniques we find the time-dependent conditional transition probabilities and the exact dynamical structure factor under such conditioned dynamics. In the thermodynamic limit we obtain the explicit scaling form. This gives a direct proof that the dynamical exponent in the extreme current regime is $z = 1$ rather than the KPZ exponent $z = 3/2$ which characterizes the ASEP in the regime of typical currents. Some of our results extend to the activity in the partially asymmetric simple exclusion process, including the symmetric case.

\section{I. INTRODUCTION}

The Asymmetric Simple Exclusion Process (ASEP) is a minimal model for many traffic and queueing processes \cite{1-3}. It describes a non-equilibrium system of many driven particles, interacting via hard core repulsion. This Markov process is defined on a lattice, each site of which can be empty or occupied by one particle. Particles jump independently after an exponentially distributed random time with mean $1/(p + q)$ to a nearest neighbor site on the right or on the left, provided that the target site is empty (hard core exclusion rule). The probability to choose the right neighbour is $p/(p + q)$ while the probability of choosing the left neighbour is $q/(p + q)$. In spite of its simple formulation, and very simple product
stationary state, the time-dependent characteristics are very nontrivial to obtain, even in the simplest periodic one-dimensional case (closed ring of \(L\) sites). This is due to the fact that e.g. the time-dependent operators of type \(e^{Ht}\hat{f}(0)e^{-Ht}\) are complicated objects because the generator of the stochastic dynamics \(H\) is equivalent to an interacting many-body quantum Hamiltonian.

Recently, it has been found that the ASEP, under the restriction to produce an extreme flux or extreme activity is governed by effective long range interactions \([4]\). In this setting one considers realizations of the process for a duration \(T\) which for a long interval between large times \(t\) and \(T-t\) (where \(T-t\) is itself also large) have carried an atypically large flux. This extreme event quite surprisingly makes the conditioned process intrinsically related to a much simpler system than the original one, namely to a system of non-interacting fermions. This fact suggests to use field theoretic free fermion techniques to compute time-dependent correlations in this particular large deviation limit of the classical stochastic dynamics of the conditioned ASEP.

In the present contribution we shall employ these techniques to calculate two particular time-dependent correlation functions: the conditional probabilities \(P(\eta, t; \eta_0, 0)\) for the transition from a microscopic \(N\)-particle initial configuration \(\eta_0\) to a final configuration \(\eta\) at time \(t\) and the dynamical structure factor which is the Fourier transform of the time-dependent density-density correlation function \(\langle n_k(t)n_0(0)\rangle - \rho^2\) in the stationary ASEP in the two above mentioned limits: under the restriction to produce a very large flux and under the restriction to produce a very large activity. We note that for the totally asymmetric version of the ASEP (TASEP) on an infinite lattice and on a ring the conditional probabilities were found in \([5]\) and \([6]\), respectively using the Bethe ansatz. The dynamical structure for the infinite system was obtained by Prähofer and Spohn using random matrix techniques \([7]\). We remark that that the time-dependent correlations for atypical stochastic trajectories (in the large deviation limit) are practically unaccessible by numerical methods, even though some promising approaches have been developed recently \([8]\).

The paper is organized as follows. In Sec. \(\text{II}\) we introduce the ASEP conditioned to carry a certain nontypical flux or to exhibit a nontypical activity. In the first two subsections we review for self-containedness the our previous work \([4]\) which among other things clarifies how conditioning on large flux and activity is related to free fermions. In the third subsection of Sec. \(\text{II}\) we present a new result, viz. the form of temporal correlations in the large current
or large activity regime. In the following main sections we focus on large flux and derive the conditional transition probabilities (Sec. III) and dynamical structure factor (Sec. IV). We conclude with a brief summary and some remarks on the large limit activity in the ASEP and symmetric exclusion.

II. ASEP CONDITIONED ON A LARGE FLUX

A. Definitions and relation to an exclusion process with long-range interaction

The flux (or synonymously the current) in the ASEP is the mean net number of jumps in some time interval. Conditioning the ASEP to carry an typical flux is achieved by first ascribing to the ASEP space-time trajectories weighting factors $e^{sJ}$ where $s$ is a generalized chemical potential, conjugated to the total current $J$, registered along the trajectory up to time $t$ (see e.g. [9] for an elaborate treatment). Fixing $s$, one picks up an ensemble of space-time trajectories characterized by any desired average current $j(s)$. In particular, fixing the desired current to be very large (in the sense described in the introduction), one obtains a particular set of trajectories, where several tendencies are clearly seen [4] when taking the limit $s \to \infty$: (a) the backward hoppings are totally suppressed (b) formation of clusters of particles, potentially leading to the flux reduction, is strongly suppressed (c) long-range interactions between the particles are generated.

In more details, it was noted that the ASEP under the restriction to produce extreme flux after proper rescaling of time is described by a Master equation

$$\frac{\partial |P\rangle}{\partial t} = -H_{\text{eff}} |P\rangle$$

(1)

with effective stochastic Hamiltonian

$$H_{\text{eff}} = \Delta H \Delta^{-1} - \mu,$$

(2)

where $\Delta$ is a diagonal matrix with positive entries given further below, and $\mu$ is the lowest eigenvalue of the Hamiltonian $H$

$$H = -\sum_{n=1}^{L} \sigma_n^+ \sigma_{n+1}^-.$$

(3)

with the spin 1/2 lowering and raising operators of the Lie algebra $SU(2)$. We point out that in the limit of very large flux the same stochastic process describes the partially asymmetric
simple exclusion process (ASEP) or even the symmetric case, since in this limit any backward hopping does not contribute.

Similar considerations can be carried out for the activity where space-time trajectories are weighted by a factor $e^{sA}$ where $A$ is the total number of jumps along the trajectory, irrespective of the direction of the jump. In the case of ASEP restricted to have an extremely large activity (counting hoppings, irrespective of the direction), the effective Hamiltonian has the form (2) where $H$ is substituted by

$$H_{II} = -\sum_{n=1}^{L} \left( p\sigma_{n}^{+}\sigma_{n+1}^{-} + q\sigma_{n+1}^{+}\sigma_{n}^{-} \right),$$

(4)

and $\mu$ is substituted by the lowest eigenvalue of $H_{II}$ and $p$ is the hopping rate to the right (clockwise) while $q$ is the hopping rate to the left (anti-clockwise) of the ASEP. Both Hamiltonians (3), (4) are free fermion Hamiltonians, and in particular for the case of symmetric hopping $p = q = 1$, the so called Symmetric Simple Exclusion Process, or SSEP, the dynamics is governed by the effective Hamiltonian (2) with $H$ substituted by the physical $XX0$ Hamiltonian describing quantum fermions on a ring of $L$ sites,

$$H_{XX0} = -\sum_{n=1}^{L} \left( \sigma_{n}^{+}\sigma_{n+1}^{-} + \sigma_{n+1}^{+}\sigma_{n}^{-} \right),$$

(5)

and $\mu$ substituted by the lowest eigenvalue of $H_{XX0}$. All the above mentioned Hamiltonians commute with each other and share the same ground state $|\mu\rangle$. The components of $|\mu\rangle$ in the natural basis constitute the respective entries of the diagonal matrix $\Delta$ from (2) $\langle \eta | \mu \rangle = \Delta_{\eta\eta}$. By the Perron-Frobenius theorem the ground state is nondegenerate, all its components are positive $\langle \eta | \mu \rangle$, so the matrix $\Delta^{-1}$ exists. It is straightforward to verify that the stationary state (the state with zero eigenvalue) of the stochastic process (1) is given by $\Delta |\mu\rangle$. The stationary state is thus common for all Hamiltonians mentioned above, for details see [4], [10].

It may be worthwhile noting that the effective process is an interesting object of study in its own right, i.e., without reference to atypically large currents in the usual ASEP. This process is a totally asymmetric exclusion process with long-range interaction. It describes nearest neighbour hopping, but the hopping rate depends on the positions of the other particles. Specifically, the move of the $k$-th particle, located at position $n_k$ in a configuration $\eta$, to the consecutive position $n_k + 1$ in the new configuration $\eta'$ has the rate [4]

$$W_{\eta'\eta} = \prod_{l \neq k} \frac{\sin(\pi(n_k + 1 - n_l)/L)}{\sin(\pi(n_k - n_l)/L)}$$

(6)
where the product is over \( l \) and \( k \) from 1 through \( N \). If the target site \( n_k + 1 \) is occupied, the hopping rate is zero, in agreement with the exclusion rule.

For the case of extreme activity one obtains in the same fashion a partially asymmetric or symmetric exclusion process with long range interaction. The symmetric case has been studied some time ago by Spohn \cite{15} who derived a hydrodynamic fluctuation theory for the large scale dynamics of the process. Noticing that fluctuations are driven by a Gaussian field he presented a general form of the dynamical structure function which represents a universality different from the Edwards Wilkinson equation. He also noted the link of the hopping rates \( \text{(6)} \) to Dyson’s Brownian motion, which was pointed out independently in \cite{4} for the totally asymmetric case. Thus our discrete model has a natural interpretation as a discrete, biased random walk version of Dyson’s Brownian motion driven by an external field.

### B. Free Fermion states and spectrum

The complete set of eigenstates of \( H, H_{II} \) and \( H_{XX0} \) in a sector with \( N \) particles is characterized by a combination of \( N \) plane waves where each plane has a quasimomentum \( \alpha \) satisfying \( e^{i\alpha L} = (-1)^{N+1} \) because of the periodic boundary conditions \cite{11}. Therefore each \( \alpha \) takes one of the quantized values

\[
\alpha_k = \frac{2\pi}{L}(k - \frac{N + 1}{2}), \quad k \in \{1, \ldots, L\}
\]

and thus each eigenstate is defined by an \( N \)-tuple of integers \( \{k\} \equiv \{k_1, \ldots, k_N\} \) where each \( k_j \in \{1, \ldots, L\} \). With these definitions we can write the eigenvectors as

\[
|\mu\{k\}\rangle = \sum_{m_1m_2\ldots m_N} \chi_{m_1m_2\ldots m_N}(\{k\})|m_1m_2\ldots m_N\rangle
\]

where the vector \( |m_1m_2\ldots m_N\rangle \) denotes a state with particles at respective positions \( m_j \) (not necessarily ordered) and the sum is over all possible particle positions \( m_j \in \{1, 2, \ldots, L\} \).

The free fermion wave function in coordinate representation has the form

\[
\chi_{m_1m_2\ldots m_N}(\{k\}) = \frac{1}{N!L^{N/2}} T_{\{m\}} \sum_Q (-1)^Q e^{i \sum_{j=1}^N m_{Q_j} \alpha_k} \]

where \( Q \) is a permutation of indexes \( Q(1, 2, \ldots, N) = (Q_1, Q_2, \ldots, Q_N) \) and \( (-1)^Q \) denotes the sign of the permutation. The factor \( T_{\{m\}} \) is zero if in the set \( \{m_1m_2\ldots m_N\} \) some \( m_j \)
coincide, and otherwise is equal to 1 or $-1$, depending on whether the set $\{m_1 m_2 \ldots m_N\}$ was obtained from the ordered set by an even or odd number of pair permutations. Any exchange of particle positions or quasimomenta results in the eigenfunction $|\mu_k\rangle$ changing sign, which reflects the Pauli principle. The states $|\mu_k\rangle$ with all possible sets $\{k\}$ are normalized, orthogonal and form an orthonormal basis in the respective sector of Hilbert space with $N$ number of particles. The eigenvalues $E^{(k)}$ corresponding to the $|\mu_k\rangle$ are sums of those for each quasiparticle

$$E^Z_{\{k\}} = \sum_{j=1}^{N} \varepsilon^Z(\alpha_k)$$

(10)

where the quasiparticle energies are

$$\varepsilon^I(\alpha_k) = -e^{-i\alpha_k}$$

(11)

$$\varepsilon^{II}(\alpha_k) = -(p e^{-i\alpha_k} + r e^{i\alpha_k})$$

(12)

$$\varepsilon^{XX0}(\alpha_k) = -2 \cos \alpha_k$$

(13)

for the Hamiltonians (3), (4) and (5), respectively.

The ground state $|\mu_0\rangle$ which minimizes the energies $E^Z_{\{k\}}$ is characterized by the choice of quasimomenta $\alpha_k$ where $k_j = j$, i.e. for the set

$$\{k\}_0 = \{1, 2, \ldots N\}.$$  

(14)

For this choice the ground state energies $E^Z_0$ for all three cases are real. In the following by default we treat the Hamiltonian (3) with quasiparticle "energies" (11) and drop the superscript $I$.

C. Correlation Functions

Consider a general time-dependent correlation function $\langle G(t)F(0)\rangle_{eff}$ in the stationary distribution of the effective stochastic process defined by (2). Here $G$ and $F$ are functions of the occupation numbers and hence represented in the quantum Hamiltonian formalism by diagonal operators and $G(t) = \exp(H_{eff} t)G\exp(-H_{eff} t)$. The basis of our present work is the following simple, but fundamental property of conditioned processes of the form (2).

**Theorem.** Let the operators $G, F$ be diagonal in the natural basis defined by $H_{eff}$. Then

$$\langle G(t)F(0)\rangle_{eff} = \langle \mu | \tilde{G}(t)F|\mu \rangle$$

(15)
where $\tilde{G}(t) = \exp(\mathcal{H}t)G \exp(-\mathcal{H}t)$ and $\mathcal{H}$ is the respective free fermion Hamiltonian (3), (4) or (5).

Proof. The proof is simple, but for readers not familiar with the machinery we provide here the details. By definition,

$$\langle G(t) F(0) \rangle_{\text{eff}} = \langle s| e^{H_{\text{eff}} t} G e^{-H_{\text{eff}} t} F |P^* \rangle \quad (16)$$

where the summation vector $\langle s|$ and stationary distribution vector $|P^* \rangle$ are left and right lowest eigenvectors of $H_{\text{eff}}$ with eigenvalue 0. We recall that $|P^* \rangle = \Delta |\mu \rangle$ and that the summation vector $\langle s| = \langle 111...1|$ is a vector with all unit components. The property $\langle s| H_{\text{eff}} = 0$ is simply a stochasticity condition (conservation of probability) for the Hamiltonian $H_{\text{eff}}$. Using the latter condition and the fact that $[\Delta, F] = 0$ when both $\Delta$ and $F$ diagonal operators we obtain

$$\langle G(t) F(0) \rangle_{\text{eff}} = \langle s| G e^{-H_{\text{eff}} t} \Delta F |\mu \rangle. \quad (17)$$

Now we note that $e^{-H_{\text{eff}} t} \Delta = \Delta e^{-(H-\mu)t}$, which is verified by expanding the exponent on both sides in Taylor series and comparing the series term by term, using (2). Inserting this equality into (17), and using $[\Delta, G] = 0$, we obtain

$$\langle G(t) F(0) \rangle_{\text{eff}} = \langle s| \Delta G e^{-(H-\mu)t} F |\mu \rangle. \quad (18)$$

Finally, noting $\langle s| \Delta = \langle \mu |$ and $\langle \mu | e^{\mu t} = \langle \mu | e^{Ht}$, we obtain the right hand side of (15).

Some remarks are in order. Theorem (15) is straightforwardly generalized to multipoint space-time correlation functions and one obtains $\langle G(t_1) F(t_2) ... Q(t_n) \rangle_{\text{eff}} = \langle \mu | \tilde{G}(t_1) \tilde{F}(t_2) ... \tilde{Q}(t_n) |\mu \rangle$, provided that all operators $G, F, ..., Q$ are diagonal in the natural basis, i.e., represent observables of the classical stochastic process generated by $H_{\text{eff}}$.

We also point out that the theorem is phrased for our specific case of the current or activity in the ASEP. However, a similar results applies for any conditioned stochastic dynamics in which case $H \equiv H(s)$ is the weighted generator of the unconditioned process.

III. CONDITIONAL PROBABILITIES

We pick two arbitrary configurations $\eta_0$ and $\eta$ of our system containing $N$ particles at the positions $n_i$ and $m_i$ respectively, i.e., $|\eta_0 \rangle = |n_1 n_2 ... n_N \rangle$, $|\eta \rangle = |m_1 m_2 ... m_N \rangle$. We assume
both sets to be ordered, \( n_i < n_{i+1} \) and \( m_i < m_{i+1} \). The probability to find the system in configuration \( \eta \) at time \( t \), provided it has been in configuration \( \eta_0 \) at time \( t = 0 \) is given by

\[
P_L(\eta, t; \eta_0, 0) = \frac{\langle \hat{\eta}(t) \hat{\eta}_0 \rangle_{\text{eff}}}{\langle \hat{\eta}_0 \rangle_{\text{eff}}}
\]  

(19)

where the subscript denotes that the average is computed with respect to the stationary state defined by the effective dynamics (11). The operators \( \hat{\eta}(0) = |\eta\rangle \langle \eta| \) and \( \hat{\eta}_0 = |\eta_0\rangle \langle \eta_0| \) are diagonal operators, which allows to use the Theorem (15). The denominator in (19) can be obtained by using (15) with \( G = I \) and \( F = \hat{\eta}_0 \). Using the results for the stationary probabilities of ref. [4] we find

\[
\langle \hat{\eta}_0 \rangle_{\text{eff}} = \langle \mu|\eta_0\rangle \langle \eta_0|\mu \rangle = (N!)^2 \chi_\eta^* \chi_\eta = \frac{2^{N(N-1)}}{L^N} \prod_{i<j} \sin^2 \frac{\pi n_i - n_j}{L} \quad (20)
\]

The right hand side of (20) are (modulo squares) the components of a Slater determinant with quasimomenta (14) chosen so as to fill the Fermi sea. For details, see e.g. [11, 12].

For the numerator we have, using (15),

\[
\langle \hat{\eta}(t) \hat{\eta}_0 \rangle_{\text{eff}} = \langle \mu|e^{Ht}\eta e^{-Ht}\eta_0|\mu \rangle.
\]  

(21)

Since the eigenvectors (8) form an orthonormal basis the sum of projectors \((N!)^{-1} \sum_{\{k\}} |\mu_{\{k\}}\rangle \langle \mu_{\{k\}}| = I\) over all \( N \)-tuples \( \{k\} \) is a unit operator. The factor \((N!)^{-1}\) appears because in the sum \( \sum_{\{k\}} f(\{\alpha_{k}\}) = \sum_{j_1=1}^{L} \cdots \sum_{j_N=1}^{L} f(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_N}) \) each different set of \( \alpha \)-s occurs \( N! \) times. Inserting it in (21), we obtain

\[
\frac{1}{N!} \sum_{\{k\}} |\mu_{\{k\}}\rangle \langle \mu_{\{k\}}| \eta_0 \mu \rangle = \frac{1}{N!} \sum_{\{k\}} e^{(E_0 - E_{\{k\}})t} Z(\eta) Z(\eta_0),
\]

(22)

where \( E_0 \) is the ground state energy corresponding to the specific set of quasimomenta \( \{k\}_0 \) given by (14), and \( Z(\eta) = \langle \mu_{\{k\}}|\eta|\mu \rangle = (N!)^2 \chi_\eta^* \chi_\eta(\{k\}_0) \) are found using (8). Substituting \( Z(\eta) \) into (22), one obtains

\[
\langle \hat{\eta}(t) \hat{\eta}_0 \rangle_{\text{eff}} = (N!)^3 \chi_\eta^* (\{k\}_0) \chi_\eta (\{k\}_0) e^{Et} \sum_{\{k\}} e^{-E_{\{k\}}t} \chi_\eta (\{k\}) \chi_\eta (\{k\}_0)
\]

(23)

Using the explicit form of \( \chi (2) \), the sum over \( \{k\} \) in (23) can be rewritten as

\[
\frac{L^{-N}}{(N!)^2} \sum_{\{k\}} \sum_{Q, Q'} (-1)^{Q+Q'} e^{i \sum_{j=1}^{N} (mq_j - n_{Q}) \alpha_{Q} - t \sum_{j=1}^{N} \epsilon(\alpha_{Q})}.
\]

(24)
In the sum over the permutations \( \sum_{Q,Q'} \) there are \((N!)^2\) terms; however, under the summation over \( k_1, k_2, \ldots k_N \) and reshuffling of \( k \)-s only \( N! \) terms are independent, namely, 
\[
\sum_{\{k\}} \sum_{Q,Q'} (-1)^{Q+Q'} e^{i \sum (m_{Q_j} - n_{Q'_j}) \alpha_{k_j} - t \sum \varepsilon(\alpha_{k_j})} = N! \sum_{\{k\}} \sum_Q (-1)^{Q} e^{i \sum (m_{Q_j} - n_j) \alpha_{k_j} - t \sum \varepsilon(\alpha_{k_j})}.
\]
Substituting this into (24), we get 
\[
\frac{L^{-N}}{N!} \sum_{\{k\}} \sum_Q (-1)^Q e^{i \sum_{j=1}^{N} (m_{Q_j} - n_j) \alpha_{k_j} - t \sum_{j=1}^{N} \varepsilon(\alpha_{k_j})}.
\] (25)

So far the discussion has been general and applicable to all three Hamiltonians (3), (4) and (5). Focussing now on the default case (3) we use (10) and (11) for further simplification. We expand the exponents 
\[ e^{-te^\alpha_k} = \sum_{s=0}^{\infty} t^s e^{s \alpha_k} / s! \]
and collect the terms with the same \( \alpha_k \). Under the summation over \( \{k\} \), only the terms with \( m_{Q_j} - n_j - s = -\kappa L \) contribute to the sum, where \( \kappa = 0, 1, 2, \ldots \) if \( m_{Q_j} \geq n_j \) and \( \kappa = 1, 2, \ldots \) otherwise. Since each \( \alpha_k \) satisfies 
\[ e^{i \alpha_k L} = (-1)^{N+1} \]
for odd number of particles \( N \) each contributing term in (25) is equal to 
\[ \sum_{\{k\}} 1 = L^N. \]
For even \( N \), the signs of the terms with different \( \kappa \) will alternate. Now we introduce the function 
\[
g_L(d, t) = \sum_{\kappa=0}^{\infty} \left[ (-1)^{\kappa} \text{sign}(d) \right]^{N+1} \frac{t^{d_{L} + \kappa L}}{(d_{L} + \kappa L)!},
\] (26)
where \( d \) is an integer ranging from \(-L+1\) to \(L-1\) and \( d_{L} = d \) for \( d > 0 \) and \( d_{L} = d + L \) for \( d < 0 \). The function \( \text{sign}(d) = 1 \) for \( d \geq 0 \) and \( \text{sign}(d) = -1 \) for \( d < 0 \). With this function the expression (25) can be rewritten in compact determinantal form as
\[
\frac{L^{-N} L^N}{N! N!} \sum_Q (-1)^Q \prod_{k=1}^{N} g_L(m_{Q_k} - n_k, t) = \frac{1}{(N!)^2} \text{det}[g_L(m_j - n_i, t)]
\] (27)
where we used Leibnitz formula for the determinant. Finally, using (20), (23) and (27), the conditional probabilities (19) can be brought after some algebra into the final form
\[
P_L(\eta; t; \eta_0, 0) = e^{E_0 t} \sqrt{\langle \eta \rangle_{\text{eff}} / \langle \eta_0 \rangle_{\text{eff}}} \text{det}[g_L(m_j - n_i, t)]
\] (28)
The \( \langle \eta_0 \rangle_{\text{eff}}, \langle \eta \rangle_{\text{eff}} \) are stationary probabilities of the initial and the final state respectively, given by (20). At the last step of the calculation we have used the fact that all the components \( \chi_\eta(\{k\}_0) \) of the ground state eigenvector can be made positive, see the discussion after Eq. (13). Consequently 
\[ \chi_\eta(\{k\}_0)/\chi_\eta^\ast(\{k\}_0) = \chi_\eta(\{k\}_0)/\chi_\eta(\{k\}_0) = \sqrt{\langle \eta \rangle_{\text{eff}} / \langle \eta_0 \rangle_{\text{eff}}}. \]
Eq. (28) is the main result of this section. The determinantal structure of the conditional
probabilities was first noticed by Spohn [15] for the symmetric case. In contrast to the symmetric case, here the matrix elements of the determinant are the propagators of a totally asymmetric random walk. Extending the link of the symmetric model to Dyson’s Brownian motion we may regard our model as a totally asymmetric Dyson random walk.

Let us discuss some limiting cases of (28). For \( t = 0 \) one has \( \det[g_L(d_{ij}, t)] = \delta_{\eta \eta_0} \), yielding the correct normalization \( P(\eta, 0; \eta_0, 0) = \delta_{\eta \eta_0} \). In the simplest case of one particle \( N = 1 \), \( E_0 = -1 \), \( \langle \eta \rangle_{\text{eff}} = \langle \eta_0 \rangle_{\text{eff}} = 1/L \), and we obtain

\[
P_L(m, t; n, 0) = e^{-t} g_L(d, t) = \sum_{\kappa=0}^{\infty} e^{-t} \frac{t^{d+\kappa L}}{(d + \kappa L)!}
\]  

(29)

where \( d = m - n \) for \( m > n \) and \( d = L - n + m \) otherwise, and \( n, m \) are initial and final particle positions. Each term \( e^{-t} t^{d+\kappa L}/(d + \kappa L)! \) in the sum is a contribution of a Poisson process \( e^{-\lambda \kappa}/\kappa! \) where an event (hopping of a particle to the right with rate 1) has happened \( d + \kappa L \) times during time \( t \). Along this trajectory a particle starts from site \( n \), and arrives at site \( m \) after making \( \kappa \) complete circles on the ring of size \( L \). Thus the parameter \( \kappa \) in (26) is the winding number.

The formula (28) has been obtained for the continuous time Markov process (1). For the discrete time update [13] we expect the Poisson distribution terms in (26) to be substituted with the binomial distribution terms

\[
g_L^{\text{DISCRETE}}(d, t) = \sum_{\kappa=0}^{\infty} \left[(-1)^\kappa \text{sign}(d)\right]^{N+1} \left(t^{d+\kappa L}/(d + \kappa L)\right) p^{d_L+\kappa L}(1-p)^{t-d_L-\kappa L},
\]  

(30)

where \( p \) is the probability of a TASEP particle to hop. Unlike the function (26), the sum (30) is truncated for any finite \( t \).

**IV. DYNAMIC STRUCTURE FACTOR**

The dynamic structure factor in the large current regime of a periodic chain with \( L \) sites is defined as the Fourier transform of the stationary correlation function \( h_L(n_1, t_1; n_2, t_2; \rho) = \rho^2 = \langle \hat{n}_{n_1}(t_1)\hat{n}_{n_2}(t_2) \rangle_{\text{eff}} - \rho^2 \), where \( \hat{n}_k(t) = e^{H_{\text{eff}}t} \hat{n}_k e^{-H_{\text{eff}}t} \) are particle number operators and \( \rho = N/L \) is the stationary particle density. Without losing generality, we assume \( n_2 > n_1 \). Because of the translational symmetry and time independence of the Hamiltonian \( H_{\text{eff}} \), the correlation function \( h \) depends only on the differences \( n_2 - n_1 = n, t_2 - t_1 = t \), i.e. \( h(n_1, n_2, t_1, t_2) \equiv h_L(n, \rho, t) \). Thus we can write the real-space representation of the dynamic
structure factor as \( S_L(n, \rho, t) = h_L(n, \rho, t) - \rho^2 \). Moreover, by particle-hole symmetry we have that \( S_L(n, 1 - \rho, t) = S_L(-n, \rho, t) \) and trivially \( S_L(n, 0, t) = S_L(n, 1, t) = 0 \). Hence we can limit our discussion to the range \( 0 < \rho \leq 1/2 \). In order to simplify notation we drop the dependence on \( \rho \) in the structure function and the correlation function and simply write \( S_L(n, t) \) and \( h_L(n, t) \).

The number operator is diagonal and therefore the formula (15) is applicable. \( h_L(n, t) \) can then be calculated in similar manner as the conditional probabilities in the previous section. We shall present only the final result (the derivation proceeds analogously to the respective quantum mechanical calculation of \( \langle \sigma_{n_2}^z(t)\sigma_{n_1}^z(0) \rangle \) in [12]),

\[
S_L(n, t) = \frac{1}{L^2} \sum_{k=1}^{N} e^{-i\alpha_k n + \varepsilon(\alpha_k) t} \sum_{l=1}^{L} e^{i\alpha_l n - \varepsilon(\alpha_l) t} - \frac{1}{L^2} \sum_{k=1}^{N} e^{i\alpha_k n - \varepsilon(\alpha_k) t} \sum_{l=1}^{N} e^{-i\alpha_l n + \varepsilon(\alpha_l) t}. \tag{31}
\]

where the \( \alpha_k \) are of the form (7) with the ground state choice \( k_j = j \), and \( \varepsilon(\alpha_k) \) is given by one of expressions (11-13).

Notice that even though the expression (31) formally looks like the corresponding quantum formula in imaginary time, its analytical properties and limits are crucially different. All the sums in (31) are strictly real which can be verified straightforwardly, using symmetry of the ground state set (14) of pseudomomenta \( \alpha_k \). We remark that for our default choice \( \varepsilon(\alpha_k) = -e^{-i\alpha_k} \), the summation from 1 through \( L \) in Eq.(31) attains a simple form by expanding the exponent. We obtain then

\[
\frac{1}{L} \sum_{l=1}^{L} e^{i\alpha_l n - \varepsilon(\alpha_l) t} = g_L(n, t), \tag{32}
\]

where \( g_L(n, t) \) is given by (26).

From the two-point correlation function (31) we compute the dynamic structure factor

\[
\hat{S}_L(p, t) = \sum_{n=1}^{L} e^{-2\pi i p n / L} S_L(n, t) \tag{33}
\]

with the integer momentum variable \( p \in \{1, 2, \ldots, L\} \). Obviously \( \hat{S}_L(p, t) = \hat{S}_L(p + nL, t) \) for any integer \( n \) and \( \hat{S}_L(0, t) = 0 \) which allows us to restrict the subsequent study of the dynamic structure factor to the range \( p \in \{1, 2, \ldots, L - 1\} \). To evaluate (33) in this range we first observe that the Fourier transformation turns the exponentials of the summation variables \( \alpha_k, \alpha_l \) into the Kronecker-delta \( \delta_{p,k-l} \). Then we write the second sum in (31)
(which runs up to \(N\)) as a sum from 1 to \(L\) and subtract the part from \(N + 1\) to \(L\). This yields as an intermediate expression

\[
\hat{S}_L(p, t) = \frac{1}{L} \sum_{k=1}^{N} \left[ e^{(\varepsilon(\alpha_k) - \varepsilon(\alpha_{k+p}))t} - e^{-(\varepsilon(\alpha_k) - \varepsilon(\alpha_{k-p}))t} \right] \\
+ \frac{1}{L} \sum_{k=1}^{N} \sum_{l=N+1}^{L} e^{-(\varepsilon(\alpha_k) - \varepsilon(\alpha_l))t} \delta_{p,k-l} 
\]

\[\text{:=} \hat{S}_L^{(1)}(p, t) + \hat{S}_L^{(2)}(p, t)\] (34)

for which we analyse next the double sum \(\hat{S}_L^{(2)}(p, t)\). We focus on the default case (11).

For the default case the difference of relaxation times in the exponential takes the simple form

\[\varepsilon(\alpha_k) - \varepsilon(\alpha_l) = -(1 - e^{2\pi ip/L})\varepsilon(\alpha_k)\] (36)

and for notational convenience we introduce

\[t_p := (1 - e^{2\pi ip/L})t.\] (37)

Bearing in mind the range of definition of the momentum variable \(p \in \{1, 2, \ldots, L - 1\}\), the Kronecker-delta in conjunction with the summation limits of the double sum gives rise to three distinct regimes for \(p\). Careful analysis yields

\[
\hat{S}_L^{(2)}(p, t) = \begin{cases} 
\frac{1}{L} \sum_{k=1}^{p} e^{tp e^{-i\alpha_k}} & p = 1, \ldots, N - 1 \\
\frac{1}{L} \sum_{k=1}^{N} e^{tp e^{-i\alpha_k}} & p = N, \ldots, L - N \\
\frac{1}{L} \sum_{k=N+1-L+p}^{N} e^{tp e^{-i\alpha_k}} & p = L - N + 1, \ldots, L - 1.
\end{cases} \] (38)

In the thermodynamic limit \(L \gg 1\) the sums over \(k\) and \(j\) in (31) turn into integrals,

\[R(n, \rho, t) := \lim_{L \to \infty} \frac{1}{L} \sum_{k=1}^{N} e^{i\alpha_k n - \varepsilon(\alpha_k)t} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ipn - \varepsilon(p)t} dp.\]

Notice that here \(p\) is real-valued and we define it to be in the interval \([-\pi, \pi]\). This yields

\[S(n, t) := \lim_{L \to \infty} S_L(n, t) = R(n, 1, t)R(-n, \rho, -t) - R(n, \rho, t)R(-n, \rho, -t).\] (39)

where \(R(n, 1, t) = t^n/n!\) corresponds to the limit of the function \(g(n, t)\), where only the first term in (26) appears (no winding condition). With a view in large, but still finite systems we point out that this observation imposes obvious validity limitations on the integral expression
Figure 1: Dynamic structure factors $\langle n_m(t)n_0(0) \rangle - \rho^2$ for $\rho = 0.5$ as function of time for $0 \leq n < 8$, computed from Eq. (39). Panel (a): connected correlation function $h(n,t) - \rho^2$ for odd $n = 0, 2, 4, 6$, represented by bold, thin, dashed, and dotted lines respectively. Panel (b): connected correlation function $h(n,t) - \rho^2$ for odd $n = 1, 3, 5, 7$ (bold, thin, dashed, and dotted lines respectively).

as an approximation for large system size: The limiting behaviour cannot be used as an approximation for $t \gtrsim L$ and $n \gtrsim L$. The integrals $R(n, \rho, t)$, apart from obvious special cases $R(n, 1, t) = t^n/n!$, and $R(n, \rho, 0) = (n\pi)^{-1} \sin n\pi\rho$ are not expressed in elementary functions and must be evaluated numerically. In Fig. 1 we show the function $S(n,t)$ for even and odd $n$ for the particular case of half-filling $N/L = 1/2$ at different times $t$. For $t = 0$ the difference in the expression $\varepsilon_q$ for energies the quasiparticles become irrelevant and, using $R(n, \rho, 0) = (n\pi)^{-1} \sin n\pi\rho$, we obtain from (39) the static density-density correlation function

$$S(n, 0) = -\frac{\sin^2 n\pi\rho}{n^2\pi^2}$$

first derived in [14].

In order to explore the large-scale behaviour of the dynamic structure factor we study the behaviour for small momentum $p$ and large times $t$. To this end we return to the definition (33) and first observe that in the thermodynamic limit

$$\hat{S}^{(1)}(p, t) = \frac{1}{2\pi} \int_{-\rho\pi}^{\rho\pi} dx \left[ e^{-e^{-ix}t - p} - e^{-e^{-ix}t_p} \right]$$

(41)

where now $t_p = (1 - e^{ip})t$. Using $t_{-p} = -e^{-ip}t_p$ allows us to rewrite this expression as a difference of integrals over the same function $\exp(e^{-ix}t_p)$ with integration intervals $[-\rho\pi +
\[ p, \rho \pi + p \] (positive term) and \([-\rho \pi, \rho \pi]\) (negative term) respectively. On the other hand

\[
\hat{S}^{(2)}(p, t) = \begin{cases} 
\frac{1}{2\pi} \int_{-\rho \pi}^{\rho \pi+p} dx \ e^{tx - ix} & p \in [0, 2\rho \pi] \\
\frac{1}{2\pi} \int_{-\rho \pi}^{\rho \pi+p} dx \ e^{tx - ix} & p \in [-\pi, \ldots, -2\rho \pi] \cup [2\rho \pi, \ldots, \pi] \\
\frac{1}{2\pi} \int_{\rho \pi+p}^{\rho \pi} dx \ e^{tx - ix} & p \in [-2\rho \pi, 0] 
\end{cases}
\] (42)

Putting everything together we finally obtain the dynamic structure factor for \(\rho \leq 1/2\)

\[
\hat{S}(p, t) = \begin{cases} 
\frac{1}{2\pi} \int_{\rho \pi}^{\rho \pi+p} dx \ e^{tx - ix} & p \in [0, 2\rho \pi] \\
\frac{1}{2\pi} \int_{-\rho \pi}^{\rho \pi+p} dx \ e^{tx - ix} & p \in [-\pi, \ldots, -2\rho \pi] \cup [2\rho \pi, \ldots, \pi] \\
\frac{1}{2\pi} \int_{-\rho \pi}^{-\rho \pi+p} dx \ e^{tx - ix} & p \in [-2\rho \pi, 0] 
\end{cases}
\] (43)

This, along with the symmetry relation \(\hat{S}(1-\rho, p, t) = \hat{S}(\rho, -p, t)\) provides an exact integral presentation valid for all densities \(\rho\), momenta \(p\) and times \(t\). The static structure factor takes the simple form (\(\rho \leq 1/2\))

\[
\hat{S}(p, 0) = \begin{cases} 
|p| \frac{2\pi}{2\pi} & p \in [-2\rho \pi, 2\rho \pi] \\
\rho & p \in [-\pi, \ldots, -2\rho \pi] \cup [2\rho \pi, \ldots, \pi], 
\end{cases}
\] (44)

cf. the real-space result [40].

We are particularly interested in the large scale behaviour as expressed in the scaling limit of small \(p\) and large \(t\) of the form \(p^z t = u\) where \(z\) is the dynamical exponent and \(u\) is the scaling variable. In the limit \(p \to 0\) only the first and third expression in (43) are relevant. From the occurrence of the factor \(t_p = (1 - e^{ip})t\) we conclude that there is non-trivial scaling behaviour for \(z = 1\), i.e. for \(u = pt\). In this scaling we have \(t_p = -iut\) which yields the desired result

\[
\hat{S}(u) = \frac{|u|}{2\pi t} e^{-iu \cos \rho \pi - |u| \sin \rho \pi}
\] (45)

which is valid for all \(\rho \in [0, 1]\) and in agreement with the universal form the dynamic structure factor derived in [15] for the symmetric case. The presence of the particle drift does not change the universality class as it does for the usual unconditioned exclusion process where the undriven model is in the universality class of the Edwards-Wilkinson equation with dynamical exponent \(z = 2\) while the driven model is in the KPZ universality class with \(z = 3/2\). This is in agreement with an earlier observation that for stochastic dynamics which
have an underlying free-fermion structure an external drift can be absorbed into a Galilei transformation [16].

V. FINAL REMARKS

We obtained analytically conditional probabilities and the two point time-dependent density correlation functions for the ASEP conditioned to carry a very large average current. The conditional probabilities have determinantal form and can be expressed through elementary functions. The density correlation functions are obtained both for a finite system and in the thermodynamic limit. By Fourier transformation we have computed the exact dynamical structure factor and derived its large-scale behaviour. The natural scaling variable turns out to be $u = pt$ which proves that the dynamical exponent of the conditioned ASEP is $z = 1$. From the explicit scaling form we read off the collective velocity $v_c = \cos \rho \pi$ of density fluctuations which is in contrast to $v_c = (p - q)(1 - 2\rho)$ of the usual ASEP in the regime of typical currents. The relaxation part is symmetric in $u$ and very different from the corresponding quantity in the usual ASEP [7] which has dynamical exponent $z = 3/2$ for the driven case and $z = 2$ for the symmetric case. In our model the presence of a drift does not change the universality class.

Our results have a natural generalization to the study of large activity, i.e. to choosing the Hamiltonians (11) and (15). One has to replace the energies $\varepsilon(\alpha_k)$ in (31) by the respective expressions $-(pe^{-i\alpha_k} + re^{i\alpha_k})$ and $-2\cos\alpha_k$. It will be interesting to study not only the hydrodynamic limit, but also the microscopic structure of shocks for the general case. It would also be interesting to extend the analysis of the effective dynamics of driven systems under large deviation constraints to the non-diagonal case, e.g. to compute current-current time-dependent averages.

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