Learning in Multi-Stage Decentralized Matching Markets

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Abstract

Matching markets are often organized in a multi-stage and decentralized manner. Moreover, participants in real-world matching markets often have uncertain preferences. This article develops a framework for learning optimal strategies in such settings, based on a nonparametric statistical approach and variational analysis. We propose an efficient algorithm, built upon concepts of “lower uncertainty bound” and “calibrated decentralized matching,” for maximizing the participants’ expected payoff. We show that there exists a welfare-versus-fairness trade-off that is characterized by the uncertainty level of acceptance. Participants will strategically act in favor of a low uncertainty level to reduce competition and increase expected payoff. We prove that participants can be better off with multi-stage matching compared to single-stage matching. We demonstrate aspects of the theoretical predictions through simulations and an experiment using real data from college admissions.

1 Introduction

Two-sided matching markets have played an important role in microeconomics for several decades [34]. Matching markets are used to allocate indivisible “goods” to multiple decision-making agents based on mutual compatibility as assessed via sets of preferences. Such a market does not clear through prices. For example, a student applicant cannot simply demand the college she prefers but must also be chosen by the college. Matching markets are often organized in a decentralized way. Each agent makes their decision independently of others’ decisions, and each agent can have multiple stages of interactions with the other side of the market. College admissions with waiting lists and academic job markets are notable examples. We refer to such markets as multi-stage decentralized matching markets.

Uncertain preference is ubiquitous in multi-stage decentralized matching markets. For instance, colleges competing for students lack information on students’ preferences. An admitted student may receive offers from other colleges. She needs to accept one or reject all offers within a short period during each stage of early, regular, and waiting-list admissions [5]. This admission process provides little opportunity for colleges to learn students’ preferences, which are uncertain due to competition among colleges and variability in the relative popularity of colleges over time. Such uncertain preferences pose a challenge for colleges in their attempt to formulate an optimal admission strategy. Consequently, colleges may end up enrolling too many or too few students relative to their capacity or having enrolled students overly far from the attainable optimum in quality.

This paper addresses the following two research questions: (i) Given the uncertain preferences on one side of the market (e.g., students), how can agents (e.g., colleges) learn an optimal strategy that maximizes expected payoffs based on historical data? (ii) What are the fundamental implications of multi-stage decentralized matching on the welfare and fairness for both sides of the market? We study these two questions using nonparametric statistical methodology and variational analysis. We propose a new algorithm for maximizing agents’ expected payoffs that is based on learning.
We consider a simple timeline for multi-stage markets. At each stage, agents simultaneously pull sets with side payments. The model in [37] is also related to the maximum weighted bipartite matching whether we are maximizing the averaged or minimal expected payoff with respect to the uncertain welfare and fairness. We show that agents will favor arms with realistic and stable opportunities for matching instead of only targeting the top-ranked arms. Moreover, we show that agents are better off with multi-stage decentralized matching as compared to single-stage decentralized matching.

Adopting literature from the bandit literature, our model has a set of agents, each with limited capacity, and a set of arms. Each agent values two attributes of an arm: a “score” that is common to all agents and a “fit” that is agent-specific and independent across agents. Agents rank arms according to their scores and fits. An agent’s strategy consists of how many and which arms to pull at each stage. On the other hand, there is no restriction on the preferences of arms. The model allows uncertainty in the preferences, which is incorporated into the arms’ stage-wise acceptance probabilities. The acceptance probability depends on the unknown state of the world and the competition of agents at each stage. We consider a simple timeline for multi-stage markets. At each stage, agents simultaneously pull sets of arms. Each arm accepts at most one of the agents that pulled it. The arms have to make irreversible decisions at each stage without knowing which other agents might select them in later stages.

**Our contributions** There are two main contributions in this paper, which correspond to the two questions above. Our first contribution is to propose a new algorithm that maximizes the agent’s expected payoff in multi-stage decentralized matching markets. The algorithm sequentially learns the optimal strategy at each stage and is built upon notions of lower uncertainty bound (LUB) and calibrated decentralized matching (CDM). The key idea is to calibrate the state parameter in a data-driven approach and take the opportunity cost and penalty for exceeding the quota into account. The calibration can be performed under both average-case and worst-case metrics, depending on whether we are maximizing the averaged or minimal expected payoff with respect to the uncertain state. Given the calibrated state, the algorithm efficiently learns the optimal strategy using historical data via statistical machine learning methods.

The second contribution is providing an analytical framework for understanding the welfare and fairness implications. We show that agents favor arms with low uncertainty in levels of acceptance, suggesting that agents prefer arms with a realistic and stable chance for matching instead of only targeting the top-ranked arms. Such strategic behavior improves the agent’s expected payoff since otherwise, by the time that arms have rejected that agent, the next-best arms that the agent has in mind may already have accepted other agents. However, the strategic behavior leads to unfair outcomes for arms because some arms are not pulled by their favorite agents even though these agents pull arms ranked below them. We prove that agents are better off in multi-stage decentralized matching markets compared to single-stage decentralized matching markets.

**Related work** This paper is related to three strands of literature. The first line is on matching markets. Most theoretical work on matching markets traces back to [21] that formulated a model of two-sided matching without side payments, and [37] that formulated a model of two-sided matching with side payments. The model in [37] is also related to the maximum weighted bipartite matching and its to stochastic and online generalizations [27]. Our goal is to design algorithms for maximizing the agent’s welfare under the model of [21], given the uncertain preferences of arms. This is different from the goal of finding a matching with the largest size in maximum matching literature [37, 27]. The second strand of literature is on the decentralized interactions in matching markets [15–17, 30, 35] and search literature [28, 31]. Our paper contributes to this strand of literature via its analysis of multi-stage markets that allow uncertain preferences. We also study the economic implications for strategic behaviors in multi-stage decentralized markets. The third related body of literature is on algorithmic studies of college admissions. The celebrated work in [21] introduced the deferred acceptance algorithm implemented under central clearinghouses. Recent works have been focused on equilibrium admissions, students’ efforts, and students’ information acquisition costs in forming preferences; see, [7, 11–13, 18, 20, 22, 24]. In contrast, we emphasize students’ multidimensional abilities and multiple colleges competing for students. The students’ preferences are uncertain due to the competition among colleges and variability in the relative popularity of colleges over time. We develop a statistical model for learning the optimal strategies using historical data.
2 Problem Formulation

Multi-stage decentralized matching markets Let \( \mathcal{P} = \{P_1, P_2, \ldots, P_m\} \) be a set of \( m \) agents. Let \( A = \{A_1, A_2, \ldots, A_n\} \) the a set of \( n \) arms. Here \( \mathcal{P} \) and \( A \) are the sets of participants on the two sides of the matching market. Each agent \( P_i \) has a quota \( q_i \geq 1 \). We assume that \( q_1 + q_2 + \cdots + q_m \leq n \). There are total of \( K \geq 1 \) stages of the matching process. At each stage, an agent who has not used up its quota can pull available arms in the market. When multiple agents select the same arm, only one agent can successfully pull the arm according to the arm’s preference. We denote \([m] \equiv \{1, \ldots, m\}, [n] \equiv \{1, \ldots, n\}, \) and \([K] \equiv \{1, \ldots, K\}\). Decentralized matching markets require participants to make their decisions independently of others’ decisions \[33, 35\]. Notable examples of such markets include college admissions in the United States, Korea, and Japan, where \( \mathcal{P} \) and \( A \) represent the sets of colleges and students, respectively \[5, 6\]. Our goal is to learn the agent’s optimal strategy for maximizing the expected payoff. A strategy consists of deciding how many and which arms to pull at each stage. Agent’s decision-making in decentralized markets faces incomplete information about other agents’ decisions and arms’ preferences.

Participants’ preferences The agents’ preferences are based on the arms’ latent utilities. Consider the following latent utility model:

\[
U_i(A_j) = v_j + e_{ij}, \quad \forall i \in [m], j \in [n],
\]

where \( v_j \in [0, 1] \) is arm \( A_j \)'s systematic score considered by all agents, and \( e_{ij} \in [0, 1] \) is an agent-specific idiosyncratic fit considered only by agent \( P_i \), \( i \in [m] \). A utility model with a similar separable structure has been widely used in the matching market literature \[4, 13, 15\].

The arms’ preferences have no restrictions and can involve uncertainty. From an agent’s perspective, arms accept offers with probabilities dependent on opponents’ strategies and arms’ preferences. Let the parameter \( s_{i,k} \in [0, 1] \) be the state of the world \[36\] for agent \( P_i \), such that the probability that an arm \( A_j \) accepts \( P_i \) at stage \( k \) is \( \pi_{i,k}(s_{i,k}, v_j) \), \( \forall i \in [m], j \in [n], k \in [K] \). Since agents compete for arms with a higher score, the acceptance probability \( \pi_{i,k}(s_{i,k}, v_j) \) models the agents’ competition through the dependence on the score \( v_j \). Moreover, \( \pi_{i,k}(s_{i,k}, v_j) \) incorporates the arm’s uncertain preference into the state \( s_{i,k} \). It is known that there exists a valid probability mass function \( \pi_{i,k}(s_{i,k}, v_j) \) \[15\]. We assume that \( \pi_{i,k}(s_{i,k}, v_j) \) is strictly increasing and continuous in \( s_{i,k} \). Thus, a larger value of the state \( s_{i,k} \) corresponds to the case that agent \( P_i \) is more popular. In practice, the true state is unknown a priori to \( P_i \) and needs to be estimated from data. For instance, the yield in college admissions is defined as the rate at which a college’s admitted students accept the offers. However, the yield is unknown a priori to the college in the current year \[13\]. Colleges can only estimate the distribution of the yield from historical data. In this paper, we study a nonparametric model of \( \pi_{i,k}(\cdot, \cdot) \) by assuming it belongs to a reproducing kernel Hilbert space (RKHS) \[3, 39\]. Later, in Section 3.2, we propose an algorithm for calibrating \( s_{i,k} \) and efficiently estimating \( \pi_{i,k}(\cdot, \cdot) \) using historical data. Given the latent utility \( U_i(A_j) \) and the acceptance probability \( \pi_{i,k}(s_{i,k}, v_j) \), agent \( P_i \)'s expected utility of pulling arm \( A_j \) at stage \( k \) is \((v_j + e_{ij})\pi_{i,k}(s_{i,k}, v_j)\).

Timeline of the matching First, Nature draws a state such that arms’ preferences are realized. Denote by \( s_{i,k}^{*} \), the true state for agent \( P_i \) at stage \( k \). Next, arms display their interests to all agents. For example, students apply to colleges in a given period. Under the assumption that students incur negligible application costs, submitting applications to all colleges is the dominant strategy as students lack information on how colleges evaluate their academic ability or personal essays \[6, 13\]. Next, at each stage \( k \in [K] \), agents simultaneously pull available arms that have not previously rejected them. Each arm either accepts one of the agents that pulled it (if any) or rejects all. An arm exits the market once it accepts an agent, and agents are allowed to exit the market at any time. The arms act simultaneously at each stage. They cannot “hold” offers for accepting or rejecting at a later stage. Hence, agents make “exploding” offers, and arms have to make irreversible decisions without knowing what other offers are coming in later stages. Finally, this multi-stage matching process ends when all agents have exited or when a pre-specified number of stages has been reached. If there remain arms in the market when the matching has terminated, these arms are unmatched.

Agent’s expected payoff An agent’s goal is to maximize the expected payoff, which consists of two parts: the expected utilities and the penalty for exceeding the quota. Let \( A_k \) be the set of arms that are available in the market at stage \( k \in [K] \). Suppose that agent \( P_i \) pulls arms from the set
We consider a variational formulation of the optimal strategy in Section 3.1 and propose a two-step methodology for finding the optimal strategy. Third, we note that the multi-stage decentralized matching problem is different from the multi-armed bandit problem [10, 25, 26]. A bandit problem is a sequential algorithm using a statistical machine learning method in Section 3.2.

### 3 Statistical Learning of the Optimal Strategy

We consider a variational formulation of the optimal strategy in Section 3.1 and propose a two-step algorithm using a statistical machine learning method in Section 3.2.

#### 3.1 Variational formulation

The problem of finding the optimal set of arms, and the corresponding optimal value $\hat{U}_i$, can be described as follows:

$$
\hat{U}_i = \max_{B_{i,k} \subseteq (A_k \cup \cup_{l \leq k-1} B_{i,l})} \sum_{k \in [K]} U_{i,k} (B_{i,k}),
$$

(3)

where the expected payoff $U_{i,k}$ is defined in (2). Finding and checking an optimal solution to (3) is difficult. Suppose that an arm set $\cup_{k \in [K]} B_{i,k}$ is given and that it is claimed to be the optimal solution to (3). It is clear that the problem of verifying that $\cup_{k \in [K]} B_{i,k}$ is optimal is computationally intractable; because we need to individually check a significant fraction of the combinations of arms to determine which combination might give a larger expected payoff than the given arm set $\cup_{k \in [K]} B_{i,k}$. Since the number of combinations grows exponentially with the number of arms, the complexity of any systematic algorithm becomes impractically large. Moreover, the expected payoff $U_{i,k}$ depends on the unknown true state $s_{i,k}$, which creates yet another layer of difficulty for finding and checking an optimal solution.

**Variational problem** We introduce the following notation: $\delta_{i,k}(v) \equiv \frac{1}{2} [\max_{s_{i,k}} \pi_{i,k}(s_{i,k}, v) - \min_{s_{i,k}} \pi_{i,k}(s_{i,k}, v)]$, which measures the uncertainty of the acceptance probability with respect to the unknown state. Using this notation, we show that a variational formulation gives a practical methodology for finding the optimal strategy.

**Theorem 1.** There exist parameters $\eta_{i,k} > 0$, for $k \leq K - 1$, and $\eta_{i,K} = 0$ such that with high probability, the minimizer of the following variational loss, $\forall k \in [K]$,

$$
L_{i,k} (B_{i,k}) = \sum_{j \in B_{i,k}} (v_j + e_{ij}) \left[ \eta_{i,k} \delta_{i,k}(v_j) - \pi_{i,k}(s_{i,k}^*, v_j) \right] + \gamma_i \max\{N_{i,k}(B_{i,k}) - q_i, 0\},
$$

(4)

gives a maximizer of the total expected payoff $\sum_{k=1}^{K} U_{i,k} (B_{i,k})$. Here the expected payoff $U_{i,k} (B_{i,k})$ is given in (2), and $B_{i,k} \subseteq (A_k \cup \cup_{l \leq k-1} B_{i,l})$ for any $k \in [K]$.

We make four remarks regarding this theorem. First, the parameter $\eta_{i,k} \geq 0$ in (4) is induced by the hierarchical structure in the sense that the arms available at subsequent stages are worse than the current ones; see Appendix B.1. Hence, each agent prefers arms with a stable acceptance probability, and for which $\eta_{i,k}$ controls the penalty on the uncertainty. Second, $\eta_{i,k}$ serves as a regularization parameter in the optimization (4) for the uncertainty measure $\delta_{i,k}$. In practice, we may choose a large value of $\eta_{i,k}$ if the agents’ competition is tense, as the arms available at subsequent stages are much worse than the current ones. Third, we note that the multi-stage decentralized matching problem is different from the multi-armed bandit problem [10, 25, 26]. A bandit problem is a sequential...
We propose a greedy algorithm that gives an approximate solution to the optimization problem in Theorem 2.

Suppose the true state is fixed at \( r \). Then the algorithm ranks arms according to its associated value of \( \lambda \). Let \( \lambda \) be the arm set at stage \( k \). We refer to \( (v_j + e_{ij})[\pi_{i,k}(s_{i,k},v_j) - \eta_{i,k}\delta_{i,k}(v_j)] \) as arm \( A_j \)'s variational expected utility. For each \( A_j \in \{A_k \setminus \cup_{l \leq k-1} B_{i,l}\} \), the greedy algorithm computes the variational expected utility per unit of acceptance probability, that is,

\[
r(A_j) \equiv (v_j + e_{ij})[\pi_{i,k}(s_{i,k},v_j) - \eta_{i,k}\delta_{i,k}(v_j)]/\pi_{i,k}(s_{i,k},v_j).
\]

Then the algorithm ranks arms according to its associated value of \( r \) so that \( r(1) \geq r(2) \cdots \geq r(\text{card}(A_k \setminus \cup_{l \leq k-1} B_{i,l})). \) Starting with the first arm corresponding to \( r(1) \) and continuing in order, the algorithm selects the arm if its variational expected utility is larger than the expected penalty of exceeding the quota. This algorithm terminates when it arrives at a cutoff value of \( r \). Then only arms whose associated \( r \) value are better than or equal to the cutoff are selected for agent \( P_i \) to pull at stage \( k \in [K] \). We present the formalized cutoff \( r = r_\ast \) in Appendix B.2. Then using the greedy algorithm, agent \( P_i \) pulls arms from the following set,

\[
\hat{B}_{i,k}(s_{i,k}) = \{ j \mid A_j \in \{A_k \setminus \cup_{l \leq k-1} B_{i,l}\} \text{ satisfying } r(A_j) \geq r_\ast \}.
\]

**Theorem 2.** Suppose the true state is fixed at \( s_{i,k}^* = s_{i,k} \). The arm set \( \hat{B}_{i,k}(s_{i,k}) \) in (5) is near-optimal as its loss satisfies

\[
\min_{\hat{B}_{i,k} \subseteq \{A_k \setminus \cup_{l \leq k-1} B_{i,l}\}} \lambda_{i,k}^l[\hat{B}_{i,k}] \leq \lambda_{i,k}^l[\hat{B}_{i,k}(s_{i,k})] \leq \min_{\hat{B}_{i,k} \subseteq \{A_k \setminus \cup_{l \leq k-1} B_{i,l}\}} \lambda_{i,k}^l[\hat{B}_{i,k}] + UE^\dagger,
\]

where the loss function \( \lambda_{i,k}^l \) is defined in (4). The quantity \( UE^\dagger \geq 0 \) and it equals 0 if there is a continuum of arms and \( \pi_{i,k}(\cdot,v) \) is continuous in \( v \).

### 3.2 A two-step learning algorithm

Since the true state and the acceptance probability are unknown a priori in practice, the greedy strategy in (5) is unknown a priori to the agent \( P_i \). We propose a two-step algorithm to learning the greedy strategy by using historical data and statistical machine learning methods. The two-step algorithm is built upon the concepts of lower uncertainty bound (LUB) and calibrated decentralized matching (CDM) [15]. In the first step, we compute an estimated expected utility of each arm and its lower uncertainty bound. Many machine learning methods can be applied here for the modeling of historical data. In the second step, we calibrate the state parameter in a data-driven approach that takes the opportunity cost and penalty for exceeding the quota into account. Based on the calibrated state, an agent selects arms with the largest lower uncertainty bounds of the expected utility. The key idea is to select arms which have large expected utility or little uncertainty in the expected utility.

**Step 1: Lower uncertainty bound** Let \( A^l = \{A_1^l, A_2^l, \ldots, A_n^l\} \) be the arm set at \( t \in [T] \equiv \{1, \ldots, T\} \). Let \( s_{i,k}^t \) be the state of agent \( P_i \) at stage \( k \) and time \( t \). The state \( s_{i,k}^t \) is unknown until the next stage or the next time point, and the state \( s_{i,k}^t \) varies over time. For instance, the yield rate of a college may change over the years. For any arm \( A_j^l \in A^l \), there are an associated pair of the
score and fit values \((v^t_{ij}, e^t_{ij})\) obtained from (1), where \(i \in [m], j \in [n']\). Let \((v^t_{ij}, e^t_{ij})\) denote the attributes of arm \(A^t_j\). Define the set \(B^t_{i,k} = \{j \mid P_i\) pulls arm \(A^t_j\) at time \(t\) and step \(k\), \(1 \leq j \leq n'\), \(t \in [T]\}\). Let \(\text{card}(B^t_{i,k}) = n^t_{i,k} \leq n'\). For any \(j \in B^t_{i,k}\), the outcome that \(P_i\) observes is whether an arm \(A^t_j\) accepted \(P_i\), that is, \(y^t_{ij} = 1\) \(\{A^t_j\) accepts \(P_i\}\). We want to estimate \(\pi^*_i,k\) based on the historical data, \(D = \{(s^t_{i,k}, v^t_{i,k}, e^t_{i,k}, y^t_{ij}) \mid i \in [m], j \in \bigcup_{k=1}^{K} B^t_{i,k}, t \in [T]\}\).

A wide range of machine learning methods, e.g., reproducing kernel methods, random forests, or neural networks, can be applied here to learn \(\pi^*_i,k\) (cf. [23]). For concreteness, we consider a penalized estimator in RKHS. Let the log odds ratio \(\hat{f}_{i,k}(s^t_{i,k}, v) = \log\{\pi^*_i,k(s^t_{i,k}, v)/[1 - \pi^*_i,k(s^t_{i,k}, v)]\}\), which is assumed to reside in an RKHS \(\mathcal{H}_{\mathcal{K}, i,k}\) with the kernel \(\mathcal{K}_{i,k}\). Then we solve for \(\hat{f}_{i,k} \in \mathcal{H}_{\mathcal{K}, i,k}\) that minimizes the objective function:

\[
\frac{1}{n^t_{i,k}} \sum_{j \in B^t_{i,k}} \left[ -y^t_{ij} \hat{f}_{i,k}(s^t_{i,k}, v^t_{i,k}) + \log (1 + \exp \{ \hat{f}_{i,k}(s^t_{i,k}, v^t_{i,k}) \} ) \right] + \lambda_i,k \| \hat{f}_{i,k} \|^2_{\mathcal{H}_{\mathcal{K}, i,k}},
\]

where \(\lambda_i,k \geq 0\) is a tuning parameter. Consider the tensor product structure of \(\mathcal{H}_{\mathcal{K}, i,k}\), where \(\mathcal{K}_{i,k}(v, v') = \mathcal{K}^t_{i,k}(v, v') \mathcal{K}^*_{i,k}(v, v')\) with some kernel functions \(\mathcal{K}^t_{i,k}\) and \(\mathcal{K}^*_{i,k}\) [40]. It is known that \(\hat{f}_{i,k}\) is minimax rate-optimal and satisfies \(\mathbb{E}[|\hat{f}_{i,k} - f^{*}_{i,k}|^2] \leq c_f/2 \log T + c_f\) for any \(i \in [m]\) (cf. [15]). Here, \(c_f > 0\) is a constant independent of \(T\), and \(r \geq 1\) denotes the order of smoothness. The value of learning from historical data is particularly significant when a new arm is introduced into the problem. Let \(A^{T+1} = \{A_1, \ldots, A_n\}\) be the new arm set at time \(T + 1\), where \(A_j\) has attributes obtained from (1). Then the probability that \(A_j\) accepts \(P_i\) at stage \(k\) is estimated by \(\hat{\pi}_{i,k}(s^t_{i,k}, v_j) = (1 + \exp[-\hat{f}_{i,k}(s^t_{i,k}, v^t_{i,k})])^{-1}\). The expected utility of \(A_j\) is \(\hat{\pi}_{i,k}(s^t_{i,k}, v_j)(v^t_{i,k} + e_{ij})\) for any \(j \in [n]\). Finally, we construct a lower uncertainty bound for \(\pi^*_i,k(s^t_{i,k}, v_j)\) as

\[
\hat{\pi}^t_{i,k}(s^t_{i,k}, v_j) = \begin{cases} \hat{\pi}_{i,k}(s^t_{i,k}, v_j) - \eta_{i,k} \delta_{i,k}(v^t_{ij}), & \text{if } v^t_{ij} \in \min\{v^t_{ij} \mid j \in \bigcup_{t=1}^{T} B^t_{i,k}\} \max\{v^t_{ij} \mid j \in \bigcup_{t=1}^{T} B^t_{i,k}\}; \\ 1, & \text{o.w.,} \end{cases}
\]

where \(\delta_{i,k}(v^t_{ij}) = \frac{1}{2} \left[ \max_{s^t_{i,k}} \hat{\pi}_{i,k}(s^t_{i,k}, v^t_{ij}) - \min_{s^t_{i,k}} \hat{\pi}_{i,k}(s^t_{i,k}, v^t_{ij}) \right]\). The parameter \(\eta_{i,k} \geq 0\) is defined in (4). Note that (6) assigns probability one to arms with scores that agent \(P_i\) has never pulled. Hence it encourages the exploration of previously untried arms. A lower uncertainty bound for the expected utility is then given by \(\hat{\pi}^t_{i,k}(s^t_{i,k}, v_j)(v^t_{ij} + e_{ij})\) for any \(j \in [n]\).

The prediction of match compatibility is also possible in another direction that an arm \(A_j\) can also learn how much an agent \(P_i\) may like itself by predicting the probability that \(A_j\) can be pulled by \(P_i\). The arms would make the decisions based on the prediction that if they have a realistic potential of being pulled by a better agent. This feature also distinguishes the two-sided matching platform from a one-sided recommendation engine that only considers which arms an agent may like, but not which arms may also like the agent in return.

**Step 2: Calibrated decentralized matching** Since the true state \(s^*_i,k\) is unknown in practice, a natural question is how to calibrate the state parameter \(s^*_i,k\) in (5). Consider the average-case loss, \(\mathbb{E}_{s^*_i,k} \{L^*_i,k | \hat{B}_{i,k}(s^*_i,k)\}\), where the loss \(L^*_i,k\) is defined in (4). Define the marginal set as \(\partial\hat{B}_{i,k}(s^*_i,k) = \lim_{\delta \to 0^+} (\hat{B}_{i,k}(s^*_i,k) - \delta s) \setminus \hat{B}_{i,k}(s^*_i,k)\). Hence \(\partial\hat{B}_{i,k}(s^*_i,k)\) represents the change of \(\hat{B}_{i,k}(s^*_i,k)\) with a perturbation of \(s^*_i,k\).

**Theorem 3.** The average-case loss \(\mathbb{E}_{s^*_i,k} \{L^*_i,k | \hat{B}_{i,k}(s^*_i,k)\}\) is minimized if \(s^*_i,k \in (0, 1)\) is chosen as the solution to

\[
\mathbb{P}(s^*_i,k \neq s^*_i,k) \sum_{j \in \partial\hat{B}_{i,k}(s^*_i,k)} (v^t_{ij} + e_{ij}) \mathbb{E}_{s^*_i,k} \left[ \hat{\pi}_{i,k}(s^*_i,k, v^t_{ij}) - \eta_{i,k} \delta_{i,k}(v^t_{ij}) \mid s^*_i,k \neq s^*_i,k \right] = \gamma_t [1 - F_{s^*_i,k}(s^*_i,k)] \sum_{j \in \partial\hat{B}_{i,k}(s^*_i,k)} \mathbb{E}_{s^*_i,k} \left[ \hat{\pi}_{i,k}(s^*_i,k, v^t_{ij}) \mid s^*_i,k < s^*_i,k \leq 1 \right],
\]

where \(F_{s^*_i,k}\) is the cumulative distribution function of \(s^*_i,k \in [0, 1]\).
We show in Appendix B.2 that the cutoff agents in a multi-stage decentralized matching markets cannot observe other agents’ quotas or uncertainty level defined in (9).

Algorithm 1
The two-step algorithm for multi-stage decentralized matching

1: Inputs: Historical data for an agent $P_i$: \{$(s^i_{t,k}, v^i_j, e^i_{ij}, y^i_{ij}) : j \in \mathcal{B}_i^t : t = 1, 2, \ldots, T\}$; New arm set $\mathcal{A}^{T+1}$ at time $T + 1$, where the arms have attributes \{$(v_j, e_{ij}) : j \in [n]$\}; Penalty $\gamma_i$ for exceeding the quota. Regularization parameter $\eta_{i,k} \geq 0$.

2: for stage $k = 1, 2, \ldots, K$ do

3: Construct the lower uncertainty bound $\pi^L_{i,k}(s_{i,k}, v_j)$ by (6).

4: Estimate the distribution $P_{s^i_{k+1}}(\cdot)$ by the kernel density method [38].

5: Calibrate the state $s_{i,k}$ according to Theorem 3.

6: Determine the arm set $\hat{B}^L_{i,k}(s_{i,k})$ in (8).

7: Calculate the remaining quota: $q_i = \text{card}(\bigcup_{t \leq k-1} \mathcal{C}_{i,t})$ and the available arms.

8: end for

9: Outputs: The arm set $\hat{B}^L_{i,k}(s_{i,k})$ for agent $P_i$ at each stages.

The key idea of (7) is to balance the trade-off between opportunity cost and penalty for exceeding the quota. If (7) has more than one solution, then $s_{i,k}$ is chosen as the largest one. If the distribution $F^L_{s^i_{k+1}}$ has discrete support, the objective in Theorem 3 needs to be changed as follows: choosing the minimal $s_{i,k} \in [0,1]$ such that the left side of (7) is not less than the right side of (7), where the search of $s_{i,k}$ starts from the maximum value in the support and decreases to the minimal value. Moreover, instead of the average-case loss in Theorem 3, we can also perform the calibration under the worst-case loss, which is discussed in Appendix B.3.

Summary of the two-step algorithm Using (6) and (7), we can obtain the cutoff estimate $\hat{r}_s$ and calibrated state $s_{i,k}$, which suggests agent $P_i$ to pull arms from the following set at stage $k$:

$$\hat{B}^L_{i,k}(s_{i,k}) = \{ j \in \mathcal{A}^{T+1} \setminus \bigcup_{t \leq k-1} \mathcal{B}_i^t \} \text{ satisfying } r(A_j) \geq \hat{r}_s \}. \tag{8}$$

Here $\mathcal{A}^{T+1}_t$ is the set of arms that are available at stage $k$ of time $T + 1$. Due to the minimax optimality of $\hat{f}_{i,k}$, we have the consistency result that $\hat{B}^L_{i,k}(s_{i,k}) \rightarrow \hat{B}_{i,k}(s_{i,k})$ as $T \rightarrow \infty$, where the set $\hat{B}_{i,k}(s_{i,k})$ is defined in (5). We summarize the above two-step algorithm in Algorithm 1. We also remark that although the negligible application costs is assumed in Section 2, Algorithm 1 is applicable to non-negligible application costs, in which different agents (i.e., colleges) would have different sets of available arms (i.e., student applicants).

4 Strategic Behavior and Economic Implications

Agents in a multi-stage decentralized matching markets cannot observe other agents’ quotas or the choices of the arms that accept other agents. Each agent only observes the arms that are left in the market at each stage. Theorem 1 implies that agents prefer arms with stable acceptance probability. This preference lead to strategic behavior on the part of the agents as follows. Define the uncertainty level as the uncertainty measure $\delta_{i,k}(v)$ in Section 3.1 relative to the acceptance probability $\pi_{i,k}(s_{i,k}, v)$. That is,

$$\text{uncertainty level } \equiv \delta_{i,k}(v)/\pi_{i,k}(s_{i,k}, v). \tag{9}$$

We show in Appendix B.2 that the cutoff $r_s$ in (5) is strictly increasing in the uncertainty level for any $v \in [0,1]$ and $k \leq K - 1$, which implies that an agent favors arms with a low uncertainty level. Hence, an agent’s strategic behavior in this market is to strategically select arms with a low uncertainty level. We now study the implications of such strategic behavior on fairness and welfare.

No justified envy The fairness studied here is defined in terms of no justified envy [1, 8]. Specifically, an arm $A_j$ has justified envy if, at a stage $k \in [K]$, $A_j$ prefers an agent $P_{v'}$ to another agent $P_i$ that pulls $A_j$, even though $P_{v'}$ pulls an arm $A_{j'}$ which ranks below $A_j$ according to the true preference of $P_{v'}$. We define a multi-stage matching procedure to be fair if there is no arm having justified envy at any stage.

Proposition 1. The probability that an arm has justified envy is strictly increasing in the arm’s uncertainty level defined in (9).
The fairness issue has been noted in practical multi-stage matching markets. For example, candidates in job markets may “fall through the cracks”—an employer that values a candidate highly perceives that the candidate is unlikely to accept the job offer and hence declines to conduct an interview with the candidate; hence, candidates may have justified envy [14]. Besides our ex-ante definition of no justified envy, there are other choices of no justified envy, including ex-post definition, which could lead to a different set of technical results [19].

Fairness vs. welfare trade-off We note that by Theorem 1, an agent has increased expected payoff under \( \eta_{i,k} > 0 \) than under \( \eta_{i,k} = 0 \) for all stages \( k \leq K - 1 \). Define the number of arms with justified envy to be the level of justified envy of the matching outcome. Then if the level of justified envy is zero, the matching outcome is fair for arms.

Proposition 2. The level of justified envy is strictly increasing in \( \eta_{i,k} \geq 0 \).

This proposition implies a trade-off between welfare and fairness since both the level of justified envy and welfare increase when changing \( \eta_{i,k} = 0 \) to \( \eta_{i,k} > 0 \). We give an example of two-stage decentralized matching, that is, \( K = 2 \). Such two-stage matching is typical in college admissions, which may include regular admissions and waiting-list admissions. By Theorems 1 and 2, agents in the first stage would strategically pull arms with low uncertainty levels by taking \( \eta_{i,1} > 0 \). In this way, agents would reduce head-on competition. Next, agents in the second stage would act according to their true preferences and pull available arms with top latent utilities by taking \( \eta_{i,2} = 0 \). Theorem 1 shows that agents’ strategic behavior in the first stage increases the welfare compared to acting according to their true preferences, whereas in the second stage, agents acting according to their true preferences suffices. Proposition 2 shows that agents’ strategic behavior in the first stage results in increased welfare, but at the cost of arms’ fairness.

Comparison with single-stage matching markets Different from multi-stage matching markets, the optimal strategy in single-stage matching gives a fair outcome for arms [15]. However, we show that agents are better off in multi-stage markets compared to single-stage markets.

Proposition 3. Agents have improved welfare under multi-stage decentralized matching than under single-stage decentralized matching.

We provide an empirical example in Appendix A.4 to illustrate the gap between multi-stage welfare and single-stage welfare.

Comparison with centralized matching markets Many centralized matching markets are implemented by employing the celebrated deferred acceptance (DA) algorithm [21]; see examples in [1, 32]. In the arm-proposing version of DA (e.g., student-proposing in college admissions), agents and arms report their ordinal preferences to a clearinghouse, which simulates the following multi-stage procedure. Every arm shows its interest to the most preferred agent that has not yet rejected it at each stage. Every agent tentatively pulls the most preferred arms up to its quota limit and permanently rejects the remaining arms that have indicated their interest to the agent. Once the process terminates, each arm is assigned to the agent that has tentatively pulled it or otherwise remains unmatched. The multi-stage decentralized matching is different from DA in practice, mainly due to the acceptance is not tentative (i.e., non-deferrable) in decentralized matching. Moreover, there is usually a restriction on the number of stages in decentralized matching due to the time cost at each stage of multi-stage decentralized matching is not negligible. We show in a numerical example of Appendix A.3 that some agents are better off in decentralized markets than centralized markets. This finding gives a partial explanation of the prevalence of decentralized college admissions in many countries.

5 Numerical Studies

In this section we demonstrate aspects of the theoretical predictions through a simulation and a real data application in college admissions. We provide extensive numerical comparisons of Algorithm 1 with other methods in Appendix. We also give additional real data analysis in Appendix. The total computing hour is within one hour in personal laptop with Intel Core i5.
Simulated graduate school admissions Consider 50 graduate schools from three tiers of colleges: five top colleges \( \{P_1, \ldots, P_5\} \), ten good colleges \( \{P_6, \ldots, P_{15}\} \), and 35 other colleges \( \{P_{16}, \ldots, P_{50}\} \). Each has the same quota \( q = 5 \) and penalty \( \gamma = 2.5 \). The simulation generates students’ preferences with ten different states \( \{s_1, \ldots, s_{10}\} \subset [0, 1] \). For any state, students’ preferences for colleges from the same tier are random. However, students prefer top colleges to good colleges, and the other colleges are the least favorite. The random preferences depend on the state due to colleges’ uncertain reputation and popularity in the current year. We consider varying numbers of students \( \{250, 260, 270, 280, 290, 300\} \). For each size of students, there are ten students having score \( v_j \) chosen uniformly and i.i.d. from \([0.9, 1]\) and 100 students having score \( v_j \) i.i.d. uniformly chosen from \([0.7, 0.9]\). The rest of the students have score \( v_j \) randomly chosen from \([0, 0.7]\). The fits \( e_{ij} \) for all college-student pairs are drawn uniformly and i.i.d. from \([0, 1]\).

![Figure 1: Performance of the proposed Algorithm 1 (i.e., LUB-CDM) and the Simple Cutoff Strategies with varying numbers of students. The results are averaged over 500 data replications. (a): College \( P_1 \) from tier 1. (b): College \( P_6 \) from tier 2. (c): College \( P_{16} \) from tier 3.](image)

We compare the college’s expected payoff achieved by the proposed Algorithm 1 with the simple cutoff strategy, where the latter method has each college choosing the most preferred students up to the remaining quota at each stage. The training data are simulated from colleges’ random proposing by pulling a random number of arms according to the latent utilities. The training data consists of 20 times of random proposing under each of the arms’ preference structures with the two-stage admissions. This training data simulates the graduate school admissions over 20 years. The testing data draws a random state from \( \{s_1, \ldots, s_{10}\} \) which gives the corresponding arms’ preferences. Then we apply Algorithm 1 with \( \eta_{1,1} = 0.1, \eta_{1,2} = 0 \) and \( \eta_{1} = 2.5 \). Figure 1 reports the averaged payoffs of three colleges \( P_1, P_6, \) and \( P_{16} \) over 500 data replications. Here colleges \( P_1, P_6, \) and \( P_{16} \) belong to the three different tiers, respectively. In Figure 1, all colleges except \( P_1 \) use Algorithm 1 while \( P_1 \) uses one of the two methods: Algorithm 1 and the simple cutoff strategy. It is seen that Algorithm 1 gives the largest average payoffs for all of \( P_1, P_6 \) and \( P_{16} \). In particular, Algorithm 1 performs significantly better for \( P_6 \) and \( P_{16} \) compared to the simple cutoff strategy.

U.S. college admissions We study a public data on college admissions from the New York Times “The Choice” blog. In this dataset, 37 U.S. colleges reported their admission yields and waiting list offers for 2015–17 applicants without personally identifiable information. As we discussed in Section 2, a college’s yield is a proxy for the state \( s_{i,k} \) as it indicates the college’s popularity. The set of 37 colleges consists of liberal arts colleges, national universities, and other undergraduate programs.

We estimate the uncertainty level \( \delta_{i,k}(v)\pi_{i,k}^{-1}(s_{i,k}, v) \) defined in (9) and study colleges’ strategic responses. While conclusive evidence on the individual students’ acceptance probability is difficult to obtain, we estimate the college-wise uncertainty on the yield: \( \sqrt{\text{Var}(s_{i,k})} \). Since the choice set for admitted students differs across years, the yield’s uncertainty underestimates the uncertainty facing a college. Figure 2 shows that colleges’ uncertainty levels are much smaller than one, which, together with Theorem 1, implies that students face limited unfairness. In particular, the yield uncertainty is robust to the size of admitted students; see the left plot of Figure 2. On the other hand, top-ranked national universities may have higher uncertainty levels; see the right plot of Figure 2, where the
outlier is the University of Chicago at the .19 uncertainty level. We verify the higher uncertainty level for top universities using the waiting list data. We perform Fisher’s exact test for the rank data on the difference of rates of accepted waiting list students to total enrolled students over 2015–16. This statistic reflects the uncertainty on both the regular admission yield and the wait-listed students’ quality. We reject the null hypothesis that the uncertainty of acceptance is the same for all national universities at the .05 significance level. The higher uncertainty for top-ranked national universities may arise due to the intense competition. Those universities are better off by employing strategic admission to reduce the enrollment uncertainty. This result implies that students are more likely to experience unfairness when applying for top national universities.

6 Conclusion

This paper develops a nonparametric statistical model to learn optimal strategies in multi-stage decentralized matching markets. The model provides insight into the interplay between learning and economic objectives in decentralized matching markets. In the model, arms have uncertain preferences that depend on the unknown state of the world and competition among the agents. We propose an algorithm, built upon the concepts of lower uncertainty bound and calibrated decentralized matching, for learning optimal strategies using historical data. We find that agents act strategically in favor of arms with low uncertainty levels of acceptance. The strategic targeting improves an agent’s welfare but leads to unfairness for arms. Our theory allows analytical comparisons between single-stage decentralized markets and centralized markets.

For future directions, it is of interest to study algorithmic strategies when agents’ preferences show complementarities or indifference. These settings have important applications, as firms may demand workers that complement one another in terms of their skills and roles, or some applicants are indistinguishable to a firm. We leave these questions for future work.

The problem of machine learning in economics has become increasingly important in many application domains. In this work, we aim to deepen the understanding of decentralized matching markets from a learning perspective and propose an efficient and scalable algorithm to solve optimal strategies. We do not foresee any negative impact to society from our work.

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A Supplementary Numerical Results

A.1 Comparison with the straightforward strategy

In this example, we compare the proposed Algorithm 1 with the straightforward strategy, where the latter method pulls arms according to the latent utility defined in Eq. (1) and calibrates the state in the same way as Algorithm 1.

Suppose there are \( n \) arms \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \) and three agents \( \mathcal{P} = \{P_1, P_2, P_3\} \), where each agent has a quota \( q < n/3 \). There are two equally likely states: \( s_a \) and \( s_b \) with \( s_a = 1 - s_b > 1/2 \). All arms prefer \( P_1 \) and \( P_2 \) to \( P_3 \), but the arms prefer \( P_3 \) compared to being unmatched. Agents \( P_1 \) and \( P_2 \) evaluate each arm based on score \( v \) and with probability \( p^* \in (0, 1) \), each of \( P_3 \) and \( P_2 \) finds an arm unacceptable. Agent \( P_3 \) evaluates each arm only based on the score. For each state \( j \in \{a, b\} \), a fraction \( s_j \) of arms receives utility \( u_1 \) when matched to \( P_1 \) and utility \( u_2 \) when matched to \( P_2 \), where \( u_1 > u_2 \) and the remaining \( 1 - s_j \) of arms receive the opposite utilities. Hence, \( P_1 \) is more popular under the state \( s_a \) and \( P_2 \) is more popular under the state \( s_b \). In each state, an arm gets utility \( u_3 \) from \( P_3 \), where \( (1 - p^*)u_1 < u_3 < u_1 \). This condition implies that an arm is better off by accepting \( P_3 \) than waiting for \( P_1 \) or \( P_2 \). We consider a two-stage matching, where at the first stage, each agent pulls a set of arms and wait-lists other arms. An arm pulled by an agent must accept or reject the agent immediately.

**Proposition A.4.** Agent \( P_1 \) is better off by using Algorithm 1 than using the straightforward strategy, where the expected payoff is improved by \( O(\eta_{1,1}) \). Here \( \eta_{1,1} \) is the regularization parameter defined in Theorem 1.

![Figure 3: Comparison of the proposed Algorithm 1 (i.e., LUB-CDM) and the straightforward strategy. The results are averaged over 500 data replications. (a) The relative increase of \( P_1 \)'s payoffs when \( P_1 \) changes from the straightforward strategy to the LUB-CDM, where the improvement is \( O(\eta_{1,1}) \). (b) The relative decrease of \( P_2 \)'s payoffs when \( P_1 \) changes from the straightforward strategy to the LUB-CDM.](image)

To illustrate the improvement, we consider the states \( s_a = 0.6, s_b = 0.4 \), the number of arms \( n = 100 \), the quota \( q = 10 \), the utilities \( u_1 = 1, u_2 = 0.9, u_3 = 0.8 \), and the probability \( p^* = 0.3 \). Suppose that the score \( v \) follows a deterministic uniform design points \( \{1.05, 1.1, 1.15, \ldots, 2.95, 3\} \subset [1, 3] \). The penalties of exceeding the quota are \( \gamma_1 = \gamma_2 = \gamma_3 = 5 \). We compare the proposed Algorithm 1 (i.e., LUB-CDM) with the straightforward strategy (i.e., CDM). The latter method is a straightforward strategy as it pulls arms according to the latent utilities in Eq. (1) without strategic behaviors. Figure 3 reports \( P_1 \)'s and \( P_2 \)'s relative changes in payoffs, when \( P_1 \) changes from using the CDM to using the LUB-CDM. The results are averaged over 500 data replications. Here \( P_1 \) using the LUB-CDM and the CDM correspond to \( \eta_{1,1} > 0 \) and \( \eta_{1,1} = 0 \), respectively. The \( P_2 \) uses CDM. It is seen the LUB-CDM improves \( P_1 \)'s expected payoff, where the improvement is at the cost of \( P_2 \)'s payoff.
A.2 Comparison with the patient strategy

In this example, we compare the proposed Algorithm 1 with the patient strategy, where the latter method pulls arms according to the latent utility at the beginning stage but has more strategic behaviors as the matching proceeds. We consider a search model due to [2], which captures the search process in matching markets and builds a connection between the multi-stage decentralized matching markets and the centralized matching markets.

Suppose there are \( n \) arms \( A = \{A_1, A_2, \ldots, A_n\} \) and \( m \) agents \( P = \{P_1, P_2, \ldots, P_m\} \), where each agent has quota \( q = 1 \). At each stage, each agent comes across a randomly sampled arm. Let \( v_P(i) \) and \( v_A(j) \) be the reservation utilities of agent \( P_i \) and arm \( A_j \) from staying unmatched and continuing the search. Recall the latent utility \( \bar{U}_i(A_j) \) in Section 2. Similarly, we define \( U_j(P_i) \) as the utility that arm \( A_j \) receives when matched to \( P_i \). Let \( v_P(i) \) and \( v_A(j) \) be the reservation utilities of agent \( P_i \) and arm \( A_j \) from staying single and continuing the search for a match. Hence \( 1\{P_i \text{ pulls } A_j\} = 1\{U_i(A_j) \geq v_P(i)\} \), and \( 1\{A_j \text{ accepts } P_i\} = 1\{U_j(P_i) \geq v_A(j)\} \). The utility that agent \( P_i \) gets upon coming across arm \( A_j \) is

\[
\bar{U}_i(A_j) = U_i(A_j)1\{U_i(A_j) \geq v_P(i)\}1\{U_j(P_i) \geq v_A(j)\} + v_P(i)[1 - 1\{U_i(A_j) \geq v_P(i)\}1\{U_j(P_i) \geq v_A(j)\}],
\]

where the first term on the right-hand side is the utility from a successful match and the second term is the utility when no match occurs. Adachi’s model involves a stage discount factor \( \rho > 0 \), where the Bellman equations for the optimal reservation values and search rules are

\[
v_P(i) = \rho \int \bar{U}_i(A_j)dF_A(j) \quad \text{and} \quad v_A(j) = \rho \int U_j(P_i)dF_P(i), \quad (A.10)
\]

where \( F_A \) and \( F_P \) are the distributions that each agent and arm came across. In [2] the authors show that Bellman equations in Eq. (A.10) defines an iterative mapping that converges to the equilibrium reservation utilities \((v_P^*(i), v_A^*(j))\). Furthermore, as \( \rho \to 1 \), the Bellman equations lead to the matching outcomes that are stable in the sense of Gale and Shapley [21].

![Figure 4: Performance of the proposed Algorithm 1 (i.e., LUB-CDM) and the patient strategy. The results averaged over 500 data replications. (a) \( P_i \)’s reservation utility under the LUB-CDM given by 50 - 5log\((k)\), where the stage \( k = 1, \ldots, 500 \). (b) \( P_i \)’s reservation utility under the patient strategy given by 50 + 5log\((\frac{N+1-k}{N})\). (c) \( P_i \)’s payoffs with varying number of arms.](image-url)

Since the equilibrium reservation utilities \((v_P^*(i), v_A^*(j))\) are unknown in practice, agents need to learn an optimal strategy of choosing the reservation utility \( v_P(i) \) at different stages. We compare the proposed Algorithm 1 (i.e., LUB-CDM) with the patient strategy, where the latter is defined as the strategy with \( \rho = 1 \) at the beginning stage \( k = 1 \) and decreasing \( \rho \) as the matching proceeds in Eq. (A.10). Note that LUB-CDM has less strategic behaviors as the matching proceeds. Hence it corresponds to the case that \( v_P(i) \) is a convex function of the stages. On the other hand, the patient strategy has more strategic behaviors as the matching proceeds. Hence it corresponds to the case that \( v_P(i) \) is a concave function of the stages. Suppose that different arms receive the same utility for matching the same agent, that is, \( U_j(P_i) = U_j(P_i') \), \( \forall j \neq j' \), which utility is unknown to \( P_i \). Similarly, different agents receive the same utility for matching the same arm, that is, \( U_i(A_j) = U_i(A_j) \), \( \forall i \neq i' \), which utility is known to \( A_j \). Then \( P_i \) matches with \( A_j \) if the event \( \{U_j(P_i) \geq U_i(A_j) \geq v_P(i)\} \) holds. Suppose that agent \( P_i \)’s utility is \( U_i(P_i) = 40 \), and \( m = n \in \{100, 200, \ldots, 1000\} \). Let the reservation utility \( v_P(1) \) at the stage \( k \) be 50 - 5log\((k)\) and...
with the number indicating the arms’ ranking of agents. For example, which result corroborates Theorem 1.

Table 2: (a) Arm’s latent utilities for each agent. (b) Arms’ preferences with the number indicating the unique stable matching outcome. Here both $A_1$ and $P_4$’s payoff under two methods, where the LUB-CDM outperforms the patient strategy. Therefore, the strategic behavior at early stages improves the agent’s payoff in practice, which result corroborates Theorem 1.

A.3 Comparison of multi-stage matching and DA

In this example, we compare the multi-stage decentralized matching with the DA algorithm [21]. Suppose there are four arms $A = \{A_1, A_2, A_3, A_4\}$ and three agents $P = \{P_1, P_2, P_3\}$. Agents have varied quotas: $q_1 = 2$ and $q_2 = q_3 = 1$. Arms’ attributes are given by $v_1 = v_2 = v_3 = 2$, $v_4 = 1$, and $e_{13} = e_{23} = e_{32} = 0, e_{21} = e_{32} = 1$. Agents have equal $v_1 = e_{21} = e_{33} = 1, e_{14} = 0.2, e_{24} = 0.5, e_{34} = 0.8$. The latent utilities and arms’ true preferences are shown in Table 1. For the decentralized matching, suppose that at each stage, every agent uses the straightforward strategy by pulling its most preferred arms up to the quota. Arms accept their most preferred agent (if any) or wait until the next stage. Then the decentralized matching has the outcome $(A_1, P_1), (A_2, P_1), (A_3, P_3), (A_4, P_2)$. On the other hand, the DA algorithm gives the outcome $(A_1, P_2), (A_2, P_2), (A_3, P_1), (A_4, P_1)$, which the unique stable matching outcome. Here both $P_1$ and $P_3$ strictly prefer the decentralized matching outcome to DA outcome. This result corroborates the remark in Section 4 that some agents are better off under the decentralized matching.

| (a) Arm’s latent utility | (b) Arm’s preference |
|--------------------------|----------------------|
| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
| $P_1$ | 3 | 2.5 | 2 | 1.2 | $P_1$ | 2 | 2 | 1 |
| $P_2$ | 3 | 2.5 | 2 | 1.5 | $P_2$ | 3 | 1 | 3 | 2 |
| $P_3$ | 2.5 | 2 | 3 | 1.8 | $P_3$ | 1 | 3 | 2 | 3 |

Second, we study the incentive of agents in the multi-stage decentralized matching. We show that it is not a dominant strategy for each agent to use the straightforward strategy by pulling arms according to the latent utility. For example, consider the preferences in Table 1. If $P_2$ skips over $A_1$ and firstly pulls $A_2$, and other agents pull their most preferred arms up to their quotas. Then the decentralized matching has the outcome $(A_1, P_1), (A_2, P_2), (A_3, P_3), (A_4, P_1)$, where $P_2$ is strictly better off compared to the outcome when $P_2$ firstly pulls $A_1$.

Table 2: (a) Arm’s latent utilities for each agent. (b) Arms’ preferences with the number indicating the arms’ ranking of agents. For example, $A_1$ ranks $P_3$ first, $P_1$ second, $P_3$ third, $P_2$ fourth.

| (a) Arm’s latent utility | (b) Arm’s preference |
|--------------------------|----------------------|
| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_1$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ |
| $P_1$ | 3 | 2 | 2.6 | 2.3 | $A_1$ | 2 | 4 | 3 | 1 |
| $P_2$ | 2 | 2.6 | 3 | 2.3 | $A_2$ | 4 | 2 | 1 | 3 |
| $P_3$ | 2.3 | 2 | 3 | 2.6 | $A_3$ | 1 | 3 | 4 | 2 |
| $P_3$ | 2 | 2.3 | 2.6 | 3 | $A_4$ | 3 | 1 | 2 | 4 |

Finally, we show that arms can also be better off if they are strategic in multi-stage decentralized matching. Suppose there are four agents and four arms, and each agent has a quota one. The latent utilities and arms’ true preferences are given in Table 2. When agents and arms are not strategic, the decentralized matching has the outcome $(A_1, P_1), (A_2, P_3), (A_3, P_2), (A_4, P_2)$. However, suppose arms are strategic, where $A_4$ rejects $P_4$ as $P_4$ is $A_1$’s least favorite agent and $A_4$ believes the coming agent will not be worse. The outcome becomes $(A_1, P_1), (A_2, P_2), (A_3, P_2), (A_4, P_3)$. Hence $A_2$ is strictly better off. Besides, if $A_3$ also rejects $\{P_2, P_3\}$ as they are $A_3$’s two least favorite agents, the decentralized matching gives the outcome $(A_1, P_1), (A_2, P_2), (A_3, P_3), (A_4, P_3)$. Hence $A_3$ and $A_4$ are both strictly better off. Moreover, suppose there is a coordination mechanism among arms such
that each arm only accepts the most preferred agent. The decentralized matching gives the outcome \((A_1, P_4), (A_2, P_3), (A_3, P_1), (A_4, P_2)\), which is the arm-optimal stable matching.

A.4 Comparison of multi-stage and single-stage matching

In this example, we show the gap between multi-stage welfare and single-stage welfare. Suppose there are four arms \(A = \{A_1, A_2, A_3, A_4\}\) and three agents \(P = \{P_1, P_2, P_3\}\). Agents have varied quotas: \(q_1 = 2\) and \(q_2 = q_3 = 1\). Arms’ attributes are given by \(v_1 = v_2 = v_3 = 2\), \(v_4 = 1\), and \(e_{13} = e_{23} = e_{32} = 0\), \(e_{12} = e_{22} = e_{31} = 0.5\), \(e_{11} = e_{21} = e_{33} = 1\), \(e_{14} = 0.2\), \(e_{24} = 0.5\), \(e_{34} = 0.8\). The latent utilities and arms’ true preferences are shown in Table 1. Suppose each agent uses the straightforward strategy by pulling its most preferred arms up to the quota. Then the single-stage matching has the outcome \((A_1, P_1), (A_2, P_1), (A_3, P_3)\). The multi-stage matching gives the outcome \((A_1, P_1), (A_2, P_1), (A_3, P_3), (A_4, P_2)\). Hence \(P_2\) is strictly better off in multi-stage matching as \(P_2\)’s welfare increases from 0 to 1.5 by changing from single-stage matching to multi-stage matching. On the other hand, \(P_1\) and \(P_3\) have the same welfare in single-stage and multi-stage matching. This result corroborates Proposition 3.

A.5 Supplementary results for real application

We give supplementary results to the real data analysis, where the admission data is from the *New York Times* “The Choice” blog (available at https://thechoice.blogs.nytimes.com/category/admissions-data). Two colleges, Harvard and Yale, are excluded from the sample due to a significant proportion of missing values.

A.5.1 Chi-squared test with FDR control

We test if the yields of colleges changed over 2015–17. The null hypothesis is that the state is the same. We use a simultaneous chi-squared test for all colleges with the count data on accepted and enrolled students and under an FDR control at a .05 significance level [9]. Figure 5 shows that colleges with large numbers of admitted students are likely to have significantly varied yields. Moreover, top-ranked national universities and liberal arts colleges are likely to have significantly varied yields. This observation corroborates the uncertainty in applicants’ preferences facing colleges. Tables 3 and 4 report the 13 colleges with significant \(p\)-values and the 22 colleges with insignificant \(p\)-values, respectively.

![Figure 5: Regression of the yield on the size of admitted class and the ranking, respectively. We fit the dashed curves using smoothing splines with the tuning parameter chosen by GCV. The labels \{1, 2, \ldots, 35\} of each point indicates colleges’ ranking according to *U.S. News and World Report*, where two (or more) colleges might tie in the ranking, and liberal arts colleges, national universities, and other undergraduate programs are ranked separately within their categories. Gray and black points denote colleges with insignificant and significant \(p\)-values, respectively, in chi-squared tests under an FDR control.](image-url)

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Table 3: 13 chi-squared tests with significant p-value under the FDR control at the .05 significance level. Colleges’ ranking data are from *U.S. News and World Report*. The "Y/N" means the use of the waiting list varied during 2015–17.

| College                   | p-value  | Category            | Ranking | Waiting list |
|---------------------------|----------|---------------------|---------|--------------|
| Boston University         | .0013    | National University | 40      | Yes          |
| Brown University          | .0012    | National University | 14      | No           |
| Claremont McKenna College | .0003    | Liberal Arts College| 7       | Y/N          |
| College of Holy Cross     | 2.20E-16 | Liberal Arts College| 27      | Yes          |
| Emory University          | 2.20E-16 | National University | 21      | Yes          |
| Georgia Tech              | .0022    | National University | 29      | Yes          |
| Middlebury College        | .0065    | Liberal Arts College| 7       | Y/N          |
| Princeton University      | 8.31E-12 | National University | 1       | Yes          |
| Stanford University       | 2.50E-06 | National University | 6       | Y/N          |
| University of Chicago     | 2.20E-06 | National University | 6       | Y/N          |
| University of Rochester   | .0001    | National University | 29      | Y/N          |
| USC                       | 2.31E-11 | National University | 22      | No           |
| University of Wisconsin   | .0008    | National University | 46      | Y/N          |

Table 4: 22 chi-squared tests with insignificant p-value under the FDR control at the .05 significance level. Colleges’ ranking data are from *U.S. News and World Report*. The "Y/N" means the use of the waiting list varied during 2015–17.

| College                | p-value  | Category             | Ranking | Waiting list |
|------------------------|----------|----------------------|---------|--------------|
| Babson College         | .8994    | Other Program        | 31      | Yes          |
| Barnard College        | .6159    | Liberal Arts College | 25      | Yes          |
| Bates College          | .0798    | Liberal Arts College | 21      | Yes          |
| CalTech                | .0584    | National University  | 12      | Y/N          |
| Carnegie Mellon University | .4988   | National University  | 25      | Yes          |
| College of William & Mary | .2227  | National University  | 40      | Yes          |
| Cooper Union           | .9512    | Other Program        | 3       | Yes          |
| Dartmouth College      | .2217    | National University  | 12      | Y/N          |
| Dickinson College      | .4727    | Liberal Arts College | 46      | Y/N          |
| Elon University        | .6872    | National University  | 84      | Y/N          |
| George Washington University | .0309  | National University  | 70      | Yes          |
| Johns Hopkins University | .1799  | National University  | 10      | Yes          |
| Kenyon College         | .8012    | Liberal Arts College | 27      | Yes          |
| Lafayette College      | .8719    | Liberal Arts College | 39      | Yes          |
| Olin College of Engineering | .5317 | Other Program        | 5       | Y/N          |
| Rensselaer Polytech    | .0285    | National University  | 50      | Y/N          |
| Scripps College        | .6511    | Liberal Arts College | 33      | Y/N          |
| St. Lawrence University | .0587   | Liberal Arts College | 58      | Yes          |
| University of Maryland | .4438    | National University  | 64      | Y/N          |
| University of Michigan | .0277    | National University  | 25      | Y/N          |
| University of Pennsylvania | .3665 | National University  | 6       | Y/N          |
| Vanderbilt University  | .7576    | National University  | 15      | Y/N          |

A.5.2 Evidence on hierarchical structure

We present the evidence on the hierarchical structure in the sense that students who were invited to the waiting list and remain available at a later stage are likely to be far worse than the admitted
students at the regular admission stage. The report of National Association for College Admission Counseling [29] shows that the admission rate of the waiting list is significantly lower than that of regular admission. The top students in a college’s waiting list, uncertain about their rankings in the list and whether the college would admit them later, may have accepted offers from their less preferred colleges. We calculate the admission rate of the waiting list as follows:

\[
\text{admission rate} = \frac{\text{the number of offers sent to wait-listed students}}{\text{the total number of students invited to the waiting list}}.
\]

Figure 6 reports that the majority (> 77%) of admission rate of the waiting list are below 5%, which result corroborates the existence of the hierarchical structure in college admissions with waiting lists.

### B Proofs

#### B.1 Proof of Theorem 1

##### B.1.1 Hierarchical structure

We exploit the underlying hierarchical structure of the optimization problem in Eq. (3). For an arm set \(B_{i,k} \subseteq \{A_k \cup \bigcup_{l \leq k-1} B_{i,l}\}\), its loss can be formulated by comparing its expected payoff to the expected payoff of \(\bar{B}_{i,k}\), where we suppose that \(\bigcup_{k \in [K]} \bar{B}_{i,k}\) achieves the optimal value \(U_i\) in (3). Then the loss of \(B_{i,k}\) for any \(k \in [K]\) becomes

\[
L_{i,k}[\bar{B}_{i,k}] = 1\left\{ \sum_{j \in B_{i,k}} \pi_{i,k}(s^*_{i,k}, v_j) > q_i \right\} \text{OE}[\bar{B}_{i,k}] + 1\left\{ \sum_{j \in B_{i,k}} \pi_{i,k}(s^*_{i,k}, v_j) \leq q_i \right\} \text{UE}[\bar{B}_{i,k}],
\]

(B.11)

Here the over-enrollment (OE) loss in (B.11) is defined as

\[
\text{OE}[\bar{B}_{i,k}] = \gamma_i \left\{ \sum_{j \in \bar{B}_{i,k}} \pi_{i,k}(s^*_{i,k}, v_j) + \text{card}(\bigcup_{l \leq k-1} C_{i,l}) - q_i \right\}
\]

\[
- \left\{ \sum_{j \in B_{i,k}} (v_j + e_{ij})\pi_{i,k}(s^*_{i,k}, v_j) - \sum_{j \in B_{i,k}} (v_j + e_{ij})\pi_{i,k}(s^*_{i,k}, v_j) \right\}, \quad \forall k \in [K],
\]

where we recall that penalty parameter \(\gamma_i\) is defined in (2). The under-enrollment (UE) loss in (B.11) is given by

\[
\text{UE}[\bar{B}_{i,k}] = \begin{cases} 
\rho_{i,k}\left[ \sum_{j \in B_{i,k}} (v_j + e_{ij})\pi_{i,k}(s^*_{i,k}, v_j) - \sum_{j \in B_{i,k}} (v_j + e_{ij})\pi_{i,k}(s^*_{i,k}, v_j) \right], & k \leq K-1, \\
\sum_{j \in B_{i,k}} (v_j + e_{ij})\pi_{i,K}(s^*_{i,k}, v_j) - \sum_{j \in B_{i,k}} (v_j + e_{ij})\pi_{i,K}(s^*_{i,k}, v_j), & k = K,
\end{cases}
\]

(B.12)
where $\rho_{i,k} \in (0, 1)$ is a discount factor for $k \leq K - 1$. Note that $\rho_{i,k} < 1$ is because $P_i$ can fill the remaining quota (if any) in subsequent stages of the matching process. On the other hand, $\rho_{i,k} > 0$ is due to the observation that the arms available at subsequent stages are likely to be worse than the arms available at the current stage. Specifically, we refer to this observation as the hierarchical structure of the multi-stage matching and it is defined as follows: For any agent $P_i$, the $j$th best arm available at the subsequent stage has lower latent utility than the $j$th best arm available at the current stage, where $j \geq 1$. The hierarchical structure has been noted in college admissions with waiting lists [13]. Unlike the stages $k \leq K - 1$, the last stage $k = K$ has the discount factor equals to 1 since the agent cannot fill the remaining quota (if any) after the last stage.

The formulation in Eq. (B.11) allows one to study stage-wise optimal sets $B_{i,k}$ that minimize the loss $L_{i,k}$ for each $k \in [K]$. This makes the optimization problem easier compared to jointly finding $B_{i,k}$ for all $k \in [K]$ such that $\cup_{k \in [K]} B_{i,k}$ maximizes the expected payoff in (3).

### B.1.2 Main proof of Theorem 1

**Proof.** We introduce additional notations. Let $V_{i,k}(s^*_{i,k}, B_{i,k})$ be the expected utility of arms from $B_{i,k} \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}$ for agent $P_i$ at stage $k \in [K]$. That is,

$$V_{i,k}(s^*_{i,k}, B_{i,k}) = \sum_{j \in B_{i,k}} (v_j + e_{ij}) \pi_{i,k}(s^*_{i,k}, v_j).$$

Let $N_{i,k}(s^*_{i,k}, B_{i,k})$ be the expected number of arms in $B_{i,k}$ accepting $P_i$. That is,

$$N_{i,k}(s^*_{i,k}, B_{i,k}) = \sum_{j \in B_{i,k}} \pi_{i,k}(s^*_{i,k}, v_j).$$

By Lagrangian duality, the optimization of $L_{i,k}[B_{i,k}]$ in Eq. (B.11) can be reformulated to the constraint form:

$$\max_{\sum_{j \in B_{i,k}} \pi_{i,k}(s^*_{i,k}, v_j) \geq \eta'_{i,k}} \left\{ V_{i,k}(s^*_{i,k}, B_{i,k}) - \gamma_1 \max\{N_{i,k}(s^*_{i,k}, B_{i,k}) + \text{card}(\cup_{l \leq k-1} C_{i,l}) - q_i, 0\} \right\} ,$$

subject to $\sum_{j \in B_{i,k}} \pi_{i,k}(s^*_{i,k}, v_j) \geq \eta'_{i,k}$.

Here $\eta'_{i,k} > 0$ is an appropriately chosen tolerance parameter for $k \leq K - 1$, and $\eta'_{i,K} = 0$. The constraint $\sum_{j \in B_{i,k}} \pi_{i,k}(s^*_{i,k}, v_j) \geq \eta'_{i,k}$ can be written as

$$V_{i,k}(s^*_{i,k}, B_{i,k}) \leq V_{i,k}(s^*_{i,k}, B^*_{i,k}) - \eta'_{i,k}, \quad \forall s^*_{i,k} \in B_{i,k} \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}.$$  

where $B_{i,k} \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}$. Since $\pi_{i,k}(\cdot, \cdot)$ is assumed to belong to an RKHS, $\pi_{i,k}(\cdot, \cdot)$ is bounded [39]. By Hoeffding’s bound, with probability at least $1 - e^{-\epsilon}$, for all $\epsilon > 0$,

$$V_{i,k}(s^*_{i,k}, B_{i,k}) \leq \mathbb{E}_{s^*_{i,k}}[V_{i,k}(s^*_{i,k}, B_{i,k})] + \sqrt{\frac{2\epsilon}{|B_{i,k}|}} \sum_{j \in B_{i,k}} \delta_{i,k}(v_j)(v_j + e_{ij})^2$$

$$< \mathbb{E}_{s^*_{i,k}}[V_{i,k}(s^*_{i,k}, B_{i,k})] + \sqrt{2\epsilon} \sum_{j \in B_{i,k}} \delta_{i,k}(v_j)(v_j + e_{ij}).$$

Hence a sufficient condition for Eq. (B.13) is to control

$$\sum_{j \in B_{i,k}} \delta_{i,k}(v_j)(v_j + e_{ij}) < \eta''_{i,k}, \quad \text{for } B_{i,k} \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}.$$  

Here $\eta''_{i,k} > 0$ is a tolerance parameter for $k \leq K - 1$. Both the $I_1$ and Eq. (B.14) are convex, and so by Lagrangian duality, they can be reformulated in the penalized form that finding $B_{i,k} \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}$ to maximize

$$\sum_{j \in B_{i,k}} (v_j + e_{ij})[\pi_{i,k}(s_{i,k}, v_j) - \eta_{i,k} \delta_{i,k}(v_j)]$$

$$- \gamma_1 \max\{N_{i,k}(s^*_{i,k}, B_{i,k}) + \text{card}(\cup_{l \leq k-1} C_{i,l}) - q_i, 0\} ,$$

where $\eta_{i,k} > 0$ for $k \leq K - 1$ and $\eta_{i,K} = 0$. This completes the proof. \qed
To choose between $b_{i,k}$ be the value of $r$ of those arms on the cutoff. That is, arms on the cutoff satisfy $b_{i,k} = (v + e_i) \{1 - \eta_{i,k}\delta_{i,k}(v)\pi^{-1}_{i,k}(s_{i,k}, v)\} \geq 0$. Let $\Pi_{i,k}(b_{i,k})$ be the expected number of arms in $\hat{B}_{i,k}(s_{i,k})$ that would accept $P_i$. That is,

$$\Pi_{i,k}(b_{i,k}) = \sum_{j \in A} 1 \{ e_{ij} \geq \min \{ \max \{ b_{i,k}[1 - \eta_{i,k}\delta_{i,k}(v)\pi^{-1}_{i,k}(s_{i,k}, v)]^{-1} - v, 0 \}, 1 \} \} \pi_{i,k}(s_{i,k}, v_j).$$

If there exists some $b_{i,k} \geq 0$ such that $\Pi_{i,k}(b_{i,k}) = q_i - \text{card}(\cup_{l \leq k-1} C_{i,l})$, we let $\hat{b}_{i,k}(s_{i,k}) = b_{i,k}$ and the cutoff $\bar{e}_{i,k}(s_{i,k}, v) = \min \{ \max \{ \hat{b}_{i,k}(s_{i,k})[1 - \eta_{i,k}\delta_{i,k}(v)\pi^{-1}_{i,k}(s_{i,k}, v)]^{-1} - v, 0 \}, 1 \}$. However, if there is no solution to $\Pi_{i,k}(b_{i,k}) = q_i - \text{card}(\cup_{l \leq k-1} C_{i,l})$, we let

$$\hat{b}_{i,k}^+(s_{i,k}) = \arg \max \{ \Pi_{i,k}(b_{i,k}) > q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) \},$$

$$\hat{b}_{i,k}^-(s_{i,k}) = \arg \min \{ \Pi_{i,k}(b_{i,k}) < q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) \}.$$

To choose between $b_{i,k}^+$ and $b_{i,k}^-$, it is necessary to balance the expected utility and the expected penalty for exceeding the quota due to pulling arms on the boundary. Define two cutoffs $e_{i,k}^+(s_{i,k}, v) \equiv \min \{ \max \{ b_{i,k}[1 - \eta_{i,k}\delta_{i,k}(v)\pi^{-1}_{i,k}(s_{i,k}, v)]^{-1} - v, 0 \}, 1 \}$ and $e_{i,k}^-(s_{i,k}, v) \equiv \min \{ \max \{ \hat{b}_{i,k}(s_{i,k})[1 - \eta_{i,k}\delta_{i,k}(v)\pi^{-1}_{i,k}(s_{i,k}, v)]^{-1} - v, 0 \}, 1 \}$. The two cutoffs correspond to two sets, $\hat{B}_{i,k}^e(s_{i,k}) = \{ j \mid e_{ij} \geq e_{i,k}^+(s_{i,k}, v) \}$ and $\hat{B}_{i,k}^\ominus(s_{i,k}) = \{ j \mid e_{ij} \geq e_{i,k}^-(s_{i,k}, v) \}$, respectively.

Consider the following condition for the arms on the boundary $\{\hat{B}_{i,k}^e(s_{i,k}) \setminus \hat{B}_{i,k}^\ominus(s_{i,k})\}$. This condition formalizes the comparison of the variational expected utility and the expected penalty of exceeding the quota:

$$\sum_{j \in \hat{B}_{i,k}^e(s_{i,k}) \setminus \hat{B}_{i,k}^\ominus(s_{i,k})} (v_j + e_{ij})[\pi_{i,k}(s_{i,k}, v_j) - \eta_{i,k}\delta_{i,k}(v_j)] \geq \gamma_i \sum_{j \in \hat{B}_{i,k}^\ominus(s_{i,k})} \pi_{i,k}(s_{i,k}, v_j) - \gamma_i[q_i - \text{card}(\cup_{l \leq k-1} C_{i,l})]. \quad (B.15)$$

If (B.15) holds, let $\hat{b}_{i,k}(s_{i,k}) = \hat{b}_{i,k}^+(s_{i,k})$ and otherwise, let $\hat{b}_{i,k}(s_{i,k}) = \hat{b}_{i,k}^-(s_{i,k})$. Then the cutoff

$$\bar{e}_{i,k}(s_{i,k}, v) = \min \{ \max \{ \hat{b}_{i,k}(s_{i,k})[1 - \eta_{i,k}\delta_{i,k}(v)\pi^{-1}_{i,k}(s_{i,k}, v)]^{-1} - v, 0 \}, 1 \}. \quad (B.16)$$

Finally, using the greedy strategy, agent $P_i$ pulls arms from

$$\hat{B}_{i,k}(s_{i,k}) = \{ j \mid A_j \in \{ A_k \setminus \cup_{l \leq k-1} B_{i,l} \} \text{ with } (v_j, e_{ij}) \text{ satisfying } e_{ij} \geq \bar{e}_{i,k}(s_{i,k}, v_j) \} = \{ j \mid A_j \in \{ A_k \setminus \cup_{l \leq k-1} B_{i,l} \} \text{ satisfying } r(A_j) \geq r_* \},$$

where $r_*$ is the cutoff defined in Section 3.1.

### B.2.2 Main proof of Theorem 2

**Proof.** We define the function,

$$\text{UE}^+ \equiv \min_{j \in B_{i,k}^\ominus(s_{i,k})} (v_j + e_{ij})(1 - \eta_{i,k}\delta_{i,k}(v_j)\pi^{-1}_{i,k}(s_{i,k}, v_j)) \cdot \left[ q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) - \sum_{j \in B_{i,k}^\ominus(s_{i,k})} \pi_{i,k}(s_{i,k}, v_j) \right].$$

It is not hard to see that $\text{UE}^+ \geq 0$ and it equals 0 if there is a continuum of arms and $\pi_{i,k}(\cdot, v)$ is continuous in $v$. We divide the main proof of Theorem 2 into five steps.
Step 1. We show that the optimal strategy prefers an arm with higher fit given the same score. Suppose that arms $A_{j_1}, A_{j_2} \in \{A_k \cup \{v_{i,k} \leq v_{j,k} \} \}$ have the same score $v_{j_1} = v_{j_2}$, but $A_{j_1}$ has a worse fit than $A_{j_2}$ to agent $i$. Now assume that $A_{j_1}$ was pulled by $P_i$ at stage $k$ but $A_{j_2}$ was not, that is, $A_{j_1} \in \tilde{B}_{i,k}(s_{i,k}), A_{j_2} \notin \tilde{B}_{i,k}(s_{i,k})$. Then the expected number of arms accepting $P_i$ is unchanged if $P_i$ replaces $A_{j_1}$ with $A_{j_2}$ in $\tilde{B}_{i,k}(s_{i,k})$. On the other hand, since the loss function in Eq. (4) is strictly decreasing in fit $v_{i,k}$, $P_i$ should pull $A_{j_2}$ instead $A_{j_1}$, This argument holds regardless of strategies of other agents.

Step 2. We show that the cutoff curve $\bar{c}_{i,k}(s_{i,k}, v)$ in Eq. (B.16) is well-defined. If the boundary $\{B_{i,k}(s_{i,k}) \} \neq \emptyset$ is not empty, then $P_i$ pulling an arm $A_j$ on the boundary yields the loss

$$\mathcal{L}_{i,k}^+ [A_j] \leq 0,$$

which justifies the condition specified by Eq. (B.15). Since $\bar{c}_{i,k}(s_{i,k}, v) \in [0, 1]$, the cutoff curve is well-defined.

Step 3. We show that the cutoff strategy of pulling arms from the set $\tilde{B}_{i,k}(s_{i,k})$ is near-optimal. Let $\tilde{B}_{i,k}(s_{i,k})$ be any other arm set. Define the following mixed strategy:

$$\sigma_{i,k}(s_{i,k}, v, e_i; t) = t \cdot 1\{(v, e_i) \in \tilde{B}_{i,k}(s_{i,k})\} + (1 - t) \cdot 1\{(v, e_i) \in \tilde{B}_{i,k}(s_{i,k})\}, \text{ for } t \in [0, 1].$$

The corresponding loss of the mixed strategy $\sigma_i$ is

$$\bar{L}_{i,k}(t) = \sum_{j \in \{A_k \cup \{v_{i,k} \leq v_{j,k} \} \}} (v_j + e_{ij})[\pi_{i,k}(s_{i,k}, v_j) - \pi_{i,k}(s_{i,k}, v_j, e_{ij}; t)] + \gamma_i \max \left\{ \sum_{j \in \{A_k \cup \{v_{i,k} \leq v_{j,k} \} \}} \pi_{i,k}(s_{i,k}, v_j) \sigma_{i,k}(s_{i,k}, v_j, e_{ij}; t) + \text{ card}(\cup_{l \leq k-1} C_{i,l}) - q_i, 0 \right\}.$$

It is clear that $\bar{L}_{i,k}(t)$ is convex in $t$. We discuss the local change $d\bar{L}_{i,k}(t)/dt$ in three cases.

Case (I): Consider removing a single arm from $\tilde{B}_{i,k}(s_{i,k})$. If the arm is from the non-empty boundary $\{B_{i,k}(s_{i,k}) \} \neq \emptyset$, the condition specified by Eq. (B.15) implies that the loss $\bar{L}_{i,k}(t)$ increases if not pulling the arm. Moreover, by construction, any other arm $A_j$ in $\tilde{B}_{i,k}(s_{i,k})$ satisfies

$$(v_j + e_{ij})[\pi_{i,k}(s_{i,k}^*, v_j) - \pi_{i,k}(s_{i,k}, v_j)] > \tilde{B}_{i,k}(s_{i,k}) \pi_{i,k}(s_{i,k}^*, v_j)$$

$$\geq \gamma_i \sum_{j' \in \tilde{B}_{i,k}(s_{i,k})} \pi_{i,k}(s_{i,k}, v_{j'}) - \gamma_i [q_i - \text{ card}(\cup_{l \leq k-1} C_{i,l})].$$

Hence, removing $A_j$ from $\tilde{B}_{i,k}(s_{i,k})$ results in a strict increase in $\bar{L}_{i,k}(t)$. We have $d\bar{L}_{i,k}(t)/dt > 0$ in this case. By the convexity of $\bar{L}_{i,k}(t)$ in $t$, we obtain

$$\bar{L}_{i,k}(1) = \bar{L}_{i,k}(0) + \frac{d\bar{L}_{i,k}(0)}{dt} (1 - 0) > \bar{L}_{i,k}(0),$$

Case (II): Consider adding a new arm with attributes $\{v_{j'}, e_{ij'}\}$ to $\tilde{B}_{i,k}(s_{i,k})$, where the new arm is not from the set $B_{i,k}^+(s_{i,k})$. Denote by $B_{i,k}^+(s_{i,k})$ the new arm set with the added arm. Note that $P_i$ pulls a new arm only if the arm reduces the loss $\bar{L}_{i,k}(t)$, that is,

$$(v_{j'} + e_{ij'})[\pi_{i,k}(s_{i,k}, v_{j'}) - \pi_{i,k}(s_{i,k}, v_{j'})]$$

$$\geq \gamma_i \sum_{j \in B_{i,k}^+(s_{i,k})} \pi_{i,k}(s_{i,k}, v_{j'}) - \gamma_i [q_i - \text{ card}(\cup_{l \leq k-1} C_{i,l})].$$  \hspace{1cm} (B.17)
Since the added new arm is not in $B^+_{i,k}(s_{i,k})$ and $\sum_{j \in B^+_{i,k}(s_{i,k})} \pi_{i,k}(s_{i,k}, v_j) \geq q_i - \text{card}(\cup_{l \leq k-1} C_{i,l})$, we have

$$\sum_{j \in B^+_{i,k}(s_{i,k})} \pi_{i,k}(s_{i,k}, v_j) - [q_i - \text{card}(\cup_{l \leq k-1} C_{i,l})] \geq \sum_{j \in B^+_{i,k}(s_{i,k})} \pi_{i,k}(s_{i,k}, v_j) - \sum_{j \in B^+_{i,k}(s_{i,k})} \pi_{i,k}(s_{i,k}, v_j)$$

(B.18)

Therefore, exchanging an arm in $B_{i,k}(s_{i,k})$ results in an increase in the loss $\mathcal{L}_{i,k}(t)$. Hence, $d\mathcal{L}_{i,k}(0)/dt > 0$ in this case. By the convexity of $\mathcal{L}_{i,k}(t)$ in $t$, we obtain

$$\mathcal{L}_{i,k}(1) = \mathcal{L}_{i,k}(0) + \frac{d\mathcal{L}_{i,k}(0)}{dt}(1 - 0) > \mathcal{L}_{i,k}(0).$$

Case (III): Consider removing an arm with attributes $(v_j, e_{ij})$ from $B_{i,k}(s_{i,k})$ and simultaneously adding new arms to $B_{i,k}(s_{i,k})$. Suppose that the new arms have attributes $(v_{j''}, e_{ij''})$ and are from $B^+_{i,k}(s_{i,k})$. If $\hat{B}_{i,k}(s_{i,k}) = B_{i,k}(s_{i,k})$, then the new arms are not in $B^+_{i,k}(s_{i,k})$ and by definition,

$$(v_{j''} + e_{ij''})\left[\pi_{i,k}(s_{i,k}, v_{j''}) - \eta_{i,k} \delta_{i,k}(v_{j''})\right] \pi_{i,k}^{-1}(s_{i,k}, v_{j''}) \leq \min_{j \in B^+_{i,k}(s_{i,k})} \left\{(v_j + e_{ij})(\pi_{i,k}(s_{i,k}, v_j) - \eta_{i,k} \delta_{i,k}(v_j)) \pi_{i,k}^{-1}(s_{i,k}, v_j)\right\}.$$

Hence,

$$\mathcal{L}^*_{i,k}(B_{i,k}(s_{i,k})) - \mathcal{L}^*_{i,k}(1) \leq \sum_{j'' \in B^+_{i,k}(s_{i,k})} (v_{j''} + e_{ij''})\left[\pi_{i,k}(s_{i,k}, v_{j''}) - \eta_{i,k} \delta_{i,k}(v_{j''})\right] \pi_{i,k}^{-1}(s_{i,k}, v_{j''}) \cdot \pi_{i,k}(s_{i,k}, v_{j''})$$

(B.19)

$$\leq \left[\min_{j \in B^+_{i,k}(s_{i,k})} (v_j + e_{ij})\left(1 - \eta_{i,k} \pi_{i,k}^{-1}(s_{i,k}, v_j) \delta_{i,k}(v_j)\right)\right] \cdot \left[q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) - \sum_{j \in B^+_{i,k}(s_{i,k})} \pi_{i,k}(s_{i,k}, v_j)\right]$$

$$= \text{UE}^\dagger.$$

If $\hat{B}_{i,k}(s_{i,k}) = B_{i,k}(s_{i,k})$, then by definition of $B^+_{i,k}(s_{i,k})$

$$\mathcal{L}^*_{i,k}(B_{i,k}(s_{i,k})) - \mathcal{L}^*_{i,k}(1) \leq \mathcal{L}^*_{i,k}(B_{i,k}(s_{i,k})) - \mathcal{L}^*_{i,k}(1) \leq \text{UE}^\dagger,$$

where the last inequality is by Eq. (B.19). Hence,

$$\mathcal{L}_{i,k}(0) - \mathcal{L}_{i,k}(1) \leq \text{UE}^\dagger.$$

Therefore, exchanging an arm in $\hat{B}_{i,k}(s_{i,k})$ with arms not in $\hat{B}_{i,k}(s_{i,k})$ could result in an increase in the loss $\mathcal{L}_{i,k}(t)$ by at most $\text{UE}^\dagger$. Combining the cases (I), (II), (III), we obtain that

$$\mathcal{L}^*_{i,k}[\hat{B}_{i,k}(s_{i,k})] \leq \min_{B_{i,k} \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}} \mathcal{L}^*_{i,k}[B_{i,k}] + \text{UE}^\dagger.$$

**Step 4.** We prove the other direction of the inequality. Since $\hat{B}_{i,k}(s_{i,k}) \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}$,

$$\mathcal{L}^*_{i,k}[\hat{B}_{i,k}(s_{i,k})] \geq \min_{B_{i,k} \subseteq \{A_k \cup \cup_{l \leq k-1} B_{i,l}\}} \mathcal{L}^*_{i,k}[B_{i,k}].$$
Step 5. If there is a continuum of arms and \( \pi_i(\cdot) \) is continuous in \( v \), then there exists \( b_{i,k} \geq 0 \) such that \( \Pi_{i,k}(b_{i,k}) = q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) \), where \( \Pi_{i,k}(b_{i,k}) \) is defined in Section 3.1:

\[
\Pi_{i,k}(b_{i,k}) = \sum_{j \in A} 1 \left( e_{ij} \geq \min \left\{ \max \left\{ b_{i,k} | 1 - \eta_{i,k} \delta_i(k(v_j)) - \eta_{i,k} \delta_i(k(v_j)) \right\}^{-1} - v_j, 0 \right\}, 1 \right) \pi_i(s_{i,k}, v_j).
\]

Therefore, by definition, \( \hat{B}_{i,k}(s_{i,k}) = B^+_{i,k}(s_{i,k}) = B^-_{i,k}(s_{i,k}) \), and

\[
q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) - \sum_{j \in B^{-}_{i,k}(s_{i,k})} \pi_i(s_{i,k}, v_j) = 0.
\]

Hence \( \text{UE}_i^* = 0 \). This completes the proof.

B.3 Proof of Theorem 3

B.3.1 Main proof of Theorem 3

Proof. We follow the proof arguments for Theorem 4 in [15]. The only difference is that here we define the following penalized expected utility and the expected number of arms:

\[
V_{i,k}(s^*_i, \hat{B}_{i,k}) = \sum_{j \in \hat{B}_{i,k}} (v_j + e_{ij}) [\pi_i(s^*_i, v_j) - \eta_{i,k} \delta_i(k(v_j))],
\]

\[
N_{i,k}(s^*_i, \hat{B}_{i,k}) = \sum_{j \in \hat{B}_{i,k}} \pi_i(s^*_i, v_j).
\]

We omit the details for simplicity.

B.3.2 Calibration under the worst-case loss

Besides the average-case loss in Theorem 3, we also consider the worst-case loss with respect to the unknown \( s^*_i \). Theorem 4 gives minimax calibration, which calibrates \( s_{i,k} \) to minimize the maximum loss \( \max_{s^*_i} \{ C_{i,k}[\hat{B}_{i,k}(s_{i,k})] \} \) over the unknown \( s^*_i \).

Theorem 4. The worst-case loss \( \max_{s^*_i} \{ C_{i,k}[\hat{B}_{i,k}(s_{i,k})] \} \) is minimized if \( s_{i,k} \in [0, 1] \) is chosen as the solution to

\[
\sum_{j \in \hat{B}_{i,k}(s_{i,k})} 2(v_j + e_{ij}) \delta_i(k(v_j)) + \sum_{j \in \hat{B}_{i,k}(0)} (v_j + e_{ij}) [\pi_i(0, v_j) - \eta_{i,k} \delta_i(k(v_j))]
\]

\[
= \sum_{j \in \hat{B}_{i,k}(1)} (v_j + e_{ij}) [\pi_i(1, v_j) - \eta_{i,k} \delta_i(k(v_j))] + \gamma_i \sum_{j \in \hat{B}_{i,k}(s_{i,k})} \pi_i(1, v_j) - \gamma_i q_i.
\]

The proof follows from Theorem 5 in [15].

B.4 Proof of Proposition 1

Proof. Recall the cutoff parameter \( \hat{b}_{i,k}(s_{i,k}) \) defined in Eq. (B.16). Similarly, we define a cutoff parameter \( b'_{i,k}(s_{i,k}) \) for the linear cutoff: \( e'_{i,k}(s_{i,k}, v) = \min \{ \max \{ b'_{i,k}(s_{i,k}, v) - v, 0 \}, 1 \} \) following three steps. First, we define that

\[
\Pi'_{i,k}(b_{i,k}) \equiv \sum_{j \in A} 1(e_{ij} \geq \min \{ \max \{ b_{i,k} - v_j, 0 \}, 1 \}) \pi_i(s_{i,k}, v_j).
\]

If there exists \( b_{i,k} \geq 0 \) such that \( \Pi'_{i,k}(b_{i,k}) = q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) \), we let \( b'_{i,k}(s_{i,k}) = b_{i,k} \). Second, if there is no solution to \( \Pi'_{i,k}(b_{i,k}) = q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) \), we let

\[
b^+_{i,k}(s_{i,k}) = \arg \max_{b_{i,k} \geq 0} \left\{ \Pi'_{i,k}(b_{i,k}) > q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) \right\},
\]

\[
b^-_{i,k}(s_{i,k}) = \arg \min_{b_{i,k} \geq 0} \left\{ \Pi'_{i,k}(b_{i,k}) < q_i - \text{card}(\cup_{l \leq k-1} C_{i,l}) \right\}.
\]
The term in the bracket of Eq. (B.21), i.e.,
\[ \frac{\hat{b}_{i,k}(s_{i,k})}{1 - \eta_{i,k}\delta_{i,k}(v_j)\pi_{i,k}^{-1}(s_{i,k}, v_j)} - b'_{i,k}(s_{i,k}) \]

is strictly increasing in \( \eta_{i,k} \). Hence the number of arms having justified envy is strictly increasing in \( \eta_{i,k} \geq 0 \). This completes the proof.

**B.6 Proof of Proposition 3**

Proof. We show the improved welfare for agents by construction. Consider the strategy of an agent, for example, \( P_i \) with \( i \in [m] \). Suppose that \( P_i \) pulls arms at the first stage in multi-stage matching using the strategy that \( P_i \) would have used in single-stage matching. All arms that would have accepted \( P_i \) in single-stage matching accept \( P_i \). The reason is that arms have incomplete information on what other offers are coming in later stages. Hence, \( P_i \) can achieve at least as well as its payoff from single-stage matching. Therefore, agents benefit from multi-stage matching. \( \square \)
We then consider the matching outcome of the LUB-CDM algorithm. Suppose that \( \delta = \frac{\eta_{1,1}(v_j)}{s} \) where \( s \) satisfies

\[
\sum_{j \in A} 1(v_j \geq \bar{v}) \cdot s_a \cdot (1 - p^*) = q, \quad \text{and} \quad \sum_{j \in A} 1(v_j \geq \bar{v}) \cdot (1 - s_b) \cdot (1 - p^*) = q.
\] (B.22)

Here the boundary arm set is assumed to be empty in Eq. (B.22). Next, we consider \( P_1 \)'s strategy. Arms with the scores worse than \( \bar{v} \) will accept \( P_3 \) since if they accept \( P_3 \), they get \( u_3 \) for sure, but if they reject \( P_3 \), they will at best be pulled by \( P_1 \) or \( P_2 \) with probability \( (1 - p^*) \) and get the utility at most \( u_1 \), but \( u_3 > (1 - p^*)u_1 \). Suppose now \( P_3 \) pulls arms with the score \( v \geq \bar{v} \), where \( \bar{v} < \tilde{v} \). By Eq. (B.22), there are total \((p^*)^2 q [s_a(1 - p^*])^{-1}\) of arms with \( v \geq \bar{v} \) that are not pulled by \( P_1 \) or \( P_2 \) and they will accept \( P_3 \). Thus, we can quantify \( \tilde{v} \) by letting it satisfy

\[
\sum_{j \in A} 1(\tilde{v} \leq v_j < \bar{v}) = q \left[ 1 - \frac{(p^*)^2}{s_a(1 - p^*)} \right].
\]

See an illustration of the cutoffs in Figure 7. Then we analyze \( P_1 \)'s expected payoff by using the CDM. If the true state is \( s_b \), \( P_1 \) does not fill its capacity during the first stage and needs to pull more arms at the second stage. Suppose that \( P_1 \) pulls arms with \( v \in [\bar{v}, \tilde{v}] \) at the second stage, where \( \tilde{v} \) satisfies

\[
\sum_{j \in A} 1(\tilde{v} \leq v_j < \bar{v}) \cdot (1 - p^*) = q - \sum_{j \in A} 1(v_j \geq \bar{v}) \cdot s_b \cdot (1 - p^*).
\] (B.23)

Hence, \( P_1 \)'s expected payoff by using CDM is

\[
\mathcal{U}_{1}^{\text{CDM}} = \frac{1}{2} (1 - p^*) \left[ \sum_{v_j \geq \bar{v}} v_j + \sum_{\tilde{v} \leq v_j < \bar{v}} v_j \right].
\] (B.24)

We then consider the matching outcome of the LUB-CDM algorithm. Suppose that \( P_1 \) uses the LUB-CDM while \( P_2 \) still uses the CDM. By Theorem 2, \( P_1 \) pulls arms according to the ranking of the following quantity:

\[
v_j \left[ 1 - \eta_{1,1} \cdot \frac{s_a - s_b}{2s} \right] = v_j \left[ 1 - \eta_{1,1} \cdot \frac{s_a - s_b}{2s} \right],
\] (B.25)

where \( \eta_{1,1}(v) = \frac{1}{2}(s_a - s_b) \) in this example and \( \eta_{1,1} \geq 0 \) is the regularization parameter defined in Theorem 1. The calibration in Theorem 3 calibrates the state parameter as \( s = s_a \) for \( P_1 \). Then \( P_1 \) pulls the arms with the score \( v \in [\bar{v} - \kappa', \bar{v}] \cup \{ \bar{v} \geq \bar{v} + \kappa \} \) and rejects those with \( v \in [\bar{v} - \kappa', \bar{v}] \). Here the boundary arm set is assumed to be empty, and \( \kappa, \kappa' \) satisfy

\[
\sum_{j \in A} 1(\bar{v} - \kappa' \leq v_j < \bar{v}) = \sum_{j \in A} 1(\bar{v} \leq v_j < \bar{v} + \kappa) \cdot s_a.
\] (B.26)

By Eq. (B.25), \( \kappa \) and \( \kappa' \) also need to satisfy that

\[
\bar{v} - \kappa' = (\bar{v} + \kappa) \left[ 1 - \eta_{1,1} \cdot \frac{s_a - s_b}{2s_a} \right].
\] (B.27)

---

**Figure 7**: Cutoffs at the two stages.
Then we analyze $P_1$’s expected payoff by using the LUB-CDM. If the true state is $s_b$, $P_1$ needs to pull more arms at the second stage. Since the second stage is the last stage and by Theorem 1, it is optimal for $P_1$ to choose $\eta_{1,2} = 0$, where the LUB-CDM coincides with the CDM. Suppose that $P_1$ pulls arms with $v \in [\bar{v}, \tilde{v}]$ at the second stage, where $\tilde{v}$ satisfies

$$\sum_{j \in A} 1(\bar{v} \leq v_j < \tilde{v}) \cdot (1 - p^*) = q - \sum_{j \in A} 1(v_j \geq \bar{v} + \kappa) \cdot s_b \cdot (1 - p^*) - \sum_{j \in A} 1(\bar{v} < v_j < \tilde{v}) \cdot (1 - p^*). \tag{B.28}$$

Subtracting Eq. (B.28) from Eq. (B.23), we obtain that

$$\sum_{j \in A} 1(\bar{v} \leq v_j < \tilde{v}) = \sum_{j \in A} 1(\bar{v} - \kappa' \leq v_j < \tilde{v}) - \sum_{j \in A} 1(\bar{v} < v_j < \bar{v} + \kappa) \cdot s_b
$$

$$= \sum_{j \in A} 1(\bar{v} < v_j < \tilde{v} + \kappa) \cdot (s_a - s_b) > 0. \tag{B.29}$$

where the second equality is by Eq. (B.26). Thus, $\tilde{v} > \bar{v}$, and the $P_1$’s expected payoff by using the LUB-CDM is

$$U_1^{LUB-CDM} = (1 - p^*) \sum_{\bar{v} - \kappa' \leq v_j < \tilde{v}} v_j + \frac{1}{2} (1 - p^*) \left[ \sum_{\bar{v} \leq v_j < \tilde{v} + \kappa} v_j + \sum_{\tilde{v} \leq v_j < \bar{v}} v_j \right]. \tag{B.30}$$

We now comparing the two expected payoffs in Eqs. (B.30) and (B.24), respective. By taking the difference, we have

$$U_1^{LUB-CDM} - U_1^{CDM}
$$

$$= (1 - p^*) \sum_{\bar{v} - \kappa' \leq v_j < \tilde{v}} v_j - \frac{1}{2} (1 - p^*) \left[ \sum_{\bar{v} \leq v_j < \tilde{v} + \kappa} v_j + \sum_{\tilde{v} \leq v_j < \bar{v}} v_j \right]
$$

$$> (1 - p^*) (\tilde{v} - \kappa') \sum_{j \in A} 1(\bar{v} - \kappa' \leq v_j < \tilde{v})$$

$$- (\bar{v} + \kappa) \sum_{j \in A} 1(\bar{v} \leq v < \tilde{v} + \kappa) - \tilde{v} \sum_{j \in A} 1(\tilde{v} \leq v < \bar{v}) \tag{B.31}$$

$$= ([\tilde{v} - \bar{v}] (s_a - s_b) - (2\kappa' s_a + \kappa)) \sum_{j \in A} 1(\tilde{v} \leq v_j < \tilde{v} + \kappa)
$$

$$= \{(s_a - s_b) [1 - \eta_{1,1}] \tilde{v} - \bar{v}] + [2s_a - \eta_{1,1} (s_a - s_b) - 1] \kappa \} \sum_{j \in A} 1(\tilde{v} \leq v_j < \tilde{v} + \kappa),$$

where the second equality is due to Eqs. (B.26) and (B.29), and the last equality is by Eq. (B.27). For sufficiently small $\kappa$ and $\eta_{1,1}$, we have

$$U_1^{LUB-CDM} > U_1^{CDM}.$$ 

Last, we quantify the improvement of the expected payoff. From Eq. (B.26), $\kappa$ satisfies that

$$\sum_{j \in A} 1 \left(\bar{v} + \kappa\right) \left(1 - \eta_{1,1} \frac{s_a - s_b}{2s_a}\right) \leq v_j < \tilde{v} \right) = \sum_{j \in A} 1(\tilde{v} \leq v_j < \tilde{v} + \kappa) \cdot s_a.$$

Suppose that $v_j$ is uniformly distributed, we have a first-order approximation of the above equations:

$$\bar{v} - (\bar{v} + \kappa) \left(1 - \eta_{1,1} \frac{s_a - s_b}{2s_a}\right) = \kappa s_a,$$

which implies that

$$\kappa = \frac{\bar{v}(s_a - s_b)\eta_{1,1}}{2s_a(1 + s_a) - (s_a - s_b)\eta_{1,1}} = O(\eta_{1,1}).$$

26
Plugging it to Eq. (B.31) suggests that a sufficient condition for $U_{1}^{\text{UB-CDM}} > U_{1}^{\text{CDM}}$ is

$$\eta_{1,1} < \frac{2s_a}{s_a - s_b} \cdot \frac{(1 + s_a)(s_a - s_b)(\bar{v} - \tilde{v})}{(s_a - s_b)(\bar{v} - \tilde{v}) + (2s^2_a + 1)\bar{v}}.$$  \hspace{1cm} \text{(B.32)}

By the condition that $\kappa' > 0$, we have

$$\eta_{1,1} < \frac{2s_a}{s_a - s_b}(1 + s_a - \bar{v}).$$ \hspace{1cm} \text{(B.33)}

Under Eqs. (B.32) and (B.33), and noting that,

$$\sum_{j \in A} 1(\bar{v} \leq v_j < \bar{v} + \kappa) = O(\kappa) = O(\eta_{1,1}),$$

we have that,

$$U_{1}^{\text{UB-CDM}} - U_{1}^{\text{CDM}} = O(\eta_{1,1}).$$

This completes the proof. \hfill \Box