Strongly connected orientation with minimum lexicographic order of indegrees *

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Abstract

Given a simple undirected graph $G$, an orientation of $G$ is to assign every edge of $G$ a direction. Borradaile et al gave a greedy algorithm SC-Path-Reversal (in polynomial time) which finds a strongly connected orientation that minimizes the maximum indegree, and conjectured that SC-Path-Reversal is indeed optimal for the "minimizing the lexicographic order" objective as well. In this note, we give a positive answer to the conjecture, that is we show that the algorithm SC-PATH-REVERSAL finds a strongly connected orientation that minimizes the lexicographic order of indegrees.

1 Introduction

Graph orientation has long been studied and is a rich field under different conditions. In this note we mainly concern about the strongly-connected orientation with minimum lexicographic order. This objective arises from a telecommunication network design problem \cite{2,4}. Let $G = (V,E)$ be an undirected simple graph. An orientation $\Lambda$ of $G$ is an assignment to each edge a direction. By a strongly-connected orientation, we mean the digraph that we obtain is strongly-connected. In a digraph $D = (V,\Lambda)$, the indegree of a vertex $v$ is the number of arcs that are directed to $v$, denoted by $d^-_\Lambda(v)$. The indegree sequence of a digraph (or an orientation) is defined as a non-increasing sequence of the indegrees of all the vertices, that is, we place the indegrees of vertices in a non-increasing order. To compare distinct indegree sequences of two orientations of an undirected graph, we apply the lexicographic order, i.e. let $s = (s_1, s_2, \ldots, s_n)$ and $t = (t_1, t_2, \ldots, t_n)$ be the

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indegree sequences of two distinct orientations of an undirected graph, respectively, we say $s$ is smaller than $t$ if there exists an integer $k$ with $1 \leq k \leq n$ such that $s_k < t_k$ and $s_i = t_i$ for all $i < k$, and vice versa.

Let $D = (V, \Lambda)$ be a digraph and $u, v \in V$, we say $u$ two-reaches to $v$ (or $v$ is two-reachable from $u$) if there are two arc-disjoint directed paths from $u$ to $v$ in $D$. A directed path from $u$ to $v$ is called reversible if $d^-(u) < d^-(v) - 1$, and is called strongly reversible if $d^-(u) < d^-(v) - 1$ and $u$ two-reaches $v$ in $D$.

The following greedy algorithm was first given by de Fraysseix and de Mendez [3]. It has been shown that the algorithm finds an orientation that minimizes the maximum indegree, which is proved by Venkateswaran [4], Asahiro et al. [1], and de Fraysseix and de Mendez [3], respectively. In fact, Path-Reversal can do something more, Borradaile et al. [2] showed that Path-Reversal indeed finds an orientation that minimizes the lexicographic order of the indegrees.

**Lemma 1** (Lemma 2 in [2]). *Reversing a directed path from $u$ to $v$ maintains the strong connectivity if and only if $u$ two-reaches $v$. Particularly, $v$ two-reaches $v$ itself.*

By Lemma [1] we know that reversing a strongly reversible direct path remains the resulting digraph strongly-connected. Based on PATH-REVERSAL, Borradaile et al. [2] gave a modified version of it as shown in the following.

![Algorithm 1.1: PATH-REVERSAL](image)

| Algorithm 1.1: PATH-REVERSAL |
|-------------------------------|
| **Input:** Undirected simple graph $G$ |
| **Output:** Orientation $\Lambda$ |
| 1 Arbitrarily orient every edge of $G$. |
| 2 While there is a reversible path, reverse it. |
| a Repeat step 2. |

Borradaile et al. [2] showed that the algorithm SC-Path-Reversal finds (in polynomial time) a strongly connected orientation that minimizes the maximum indegree, and conjectured that SC-Path-Reversal is indeed optimal for the the ”minimizing the lexicographic order” objective as well.

![Algorithm 1.2: SC-PATH-REVERSAL](image)

| Algorithm 1.2: SC-PATH-REVERSAL |
|-------------------------------|
| **Input:** Undirected simple graph $G$ admitting a strongly connected orientation |
| **Output:** Orientation $\Lambda$ |
| 1 Find an arbitrary strongly-connected orientation. |
| 2 If there exists a strongly reversible path, reverse it. |
| a Repeat step 2. |
**Conjecture 2** (Borradaile et al. [2]). The algorithm **SC-Path-Reversal** finds a strongly connected orientation that minimizes the lexicographic order of indegrees.

In this note, we give a positive answer to Conjecture 2. The following is our main theorem.

**Theorem 3.** The algorithm **SC-PATH-REVERSAL** finds an orientation that minimizes the lexicographic order of the indegrees.

In the rest of the note, we give the proof of Theorem 3.

## 2 Proof of Theorem 3

Before proving that the orientation output has the minimum lexicographic order, we introduce two lemmas given in [2].

**Lemma 4** (Lemma 2 in [2]). In a digraph, let $s$ and $t$ be two vertices (can be identical) that 2-reach a vertex $v$. If there exists a vertex $u$ such that one $u \to s$ and one $u \to t$ paths are arc-disjoint, then $u$ two-reaches $v$.

The following lemma is a variable version of Lemma 4 in [2], the proof is the same as the one of that lemma, so we omit it here.

**Lemma 5.** Let $v$ be a vertex in a strongly connected digraph $D = (V, \Lambda)$ and let $U$ be the set of vertices that 2-reach $v$. Then for any component $C$ of $D[V \setminus U]$, there is exactly one arc from $C$ to $U$.

**Remark 1.** By a component we mean a connected component instead of a strongly-connected component.

Now we are ready to give the proof of the main theorem.

**Proof of Theorem 3** Let $G = (V, E)$ be the underlying graph. Let $D_{pr} = (V, \Lambda_{pr})$ be the orientation founded by the SC-PATH-REVERSAL, and let $D_{lex} = (V, \Lambda_{lex})$ be a strongly connected orientation that minimizes the lexicographic order of indegrees among all strongly connected orientations of $G$. Write $d^-_{pr}(v)$ and $d^-_{lex}(v)$ for the indegree of $v$ in $D_{pr}$ and $D_{lex}$, respectively. Define

$$
\Delta := \sum_{v \in V} |d^-_{lex}(v) - d^-_{pr}(v)|
$$

and

$$
S := \{ v \mid d^-_{lex}(v) \neq d^-_{pr}(v) \}.
$$

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Choose $\Lambda_{\text{lex}}$ such that it minimizes $\Delta$. If $S = \emptyset$, then $\Lambda_{\text{pr}}$ is a strongly-connected orientation having the same lexicographic order of indegree as $\Lambda_{\text{lex}}$, we are done. So assume $S \neq \emptyset$. Denote $M_1 = \max \{ d_{\text{lex}}^{-}(v) \mid v \in S \}$ and $S_1 = \{ u \in S \mid d_{\text{lex}}^{-}(u) = M_1 \}$. Denote $M_2 = \max \{ d_{\text{pr}}^{-}(u) \mid u \in S_1 \}$. Choose $v \in S_1$ such that $d_{\text{pr}}^{-}(v) = M_2$.

**Case 1:** $d_{\text{lex}}^{-}(v) > d_{\text{pr}}^{-}(v)$.

Let $U$ be the set of vertices that two-reach $v$ in $D_{\text{lex}}$. By Lemma 5 there is exactly one arc from each component of $G[V \setminus U]$ to $U$. Thus, on the one hand,

$$\sum_{u \in U} d_{\text{lex}}^{-}(u) = |E(G[U])| + c(G[V \setminus U]),$$

where $c(G)$ denotes the number of components of a graph $G$.

While, on the other hand, since $D_{\text{pr}}$ is strongly connected, there is at least one arc from each component of $G[V \setminus U]$ to $U$. Thus

$$\sum_{u \in U} d_{\text{pr}}^{-}(u) \geq |E(G[U])| + c(G[V \setminus U]).$$

So we get $\sum_{u \in U} d_{\text{lex}}^{-}(u) \leq \sum_{u \in U} d_{\text{pr}}^{-}(u)$. Since $d_{\text{lex}}^{-}(v) > d_{\text{pr}}^{-}(v)$, there exists a vertex $w \in U$ such that $d_{\text{lex}}^{-}(w) < d_{\text{pr}}^{-}(w)$. Clearly, $w \in S$. By the choice of $v$, we have $d_{\text{lex}}^{-}(w) \leq d_{\text{lex}}^{-}(v)$. We claim that $d_{\text{lex}}^{-}(w) < d_{\text{lex}}^{-}(v)$. If not, $w \in S_1$. But $d_{\text{pr}}^{-}(w) > d_{\text{lex}}^{-}(w) = d_{\text{lex}}^{-}(v) > d_{\text{pr}}^{-}(v)$, a contradiction to the choice of $v$. If $d_{\text{lex}}^{-}(w) < d_{\text{lex}}^{-}(v) - 1$, then reversing a directed path from $w$ to $v$ remains strong connectivity by Lemma 11 but the resulting orientation has a smaller lexicographic order of indegree, a contradiction to the choice of $\Lambda_{\text{lex}}$. Thus $d_{\text{lex}}^{-}(w) = d_{\text{lex}}^{-}(v) - 1$. Now, reversing a directed path from $w$ to $v$ in $D_{\text{lex}}$, we get another orientation with minimum lexicographic order, and however, with $d_{\text{lex}}^{-}(v) > d_{\text{pr}}^{-}(v)$ and $d_{\text{lex}}^{-}(w) < d_{\text{pr}}^{-}(w)$ before the reverse, we get a smaller $\Delta$, which contradicts to the choice of $\Lambda_{\text{lex}}$, too.

**Case 2:** $d_{\text{lex}}^{-}(v) < d_{\text{pr}}^{-}(v)$.

Let $U$ be the set of vertices that 2-reach $v$ in $D_{\text{pr}}$. With a similar discussion as in Case 1, we get $\sum_{u \in U} d_{\text{lex}}^{-}(u) \geq \sum_{u \in U} d_{\text{pr}}^{-}(u)$. Since $d_{\text{lex}}^{-}(v) < d_{\text{pr}}^{-}(v)$, there exists a vertex $w \in U$ such that $d_{\text{lex}}^{-}(w) > d_{\text{pr}}^{-}(w)$. Then we have

$$d_{\text{pr}}^{-}(v) > d_{\text{lex}}^{-}(v) \geq d_{\text{lex}}^{-}(w) > d_{\text{pr}}^{-}(w).$$

This implies that $d_{\text{pr}}^{-}(w) < d_{\text{pr}}^{-}(v) - 1$. So there is a strongly reversible directed path in $D_{\text{pr}}$, which contradicts to the property $D_{\text{pr}}$ has no strongly reversible directed path.

We conclude that $S = \emptyset$. Therefore the orientation $\Lambda_{\text{pr}}$ found by the SC-PATH-REVERSAL algorithm has the minimum lexicographic order of indegree. \[\square\]
References

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A Proof of Lemma 5

Let $C$ be a component of $G[V \setminus U]$. Note that there is no arc between $C$ and the other components of $G[V \setminus U]$. Since $D$ is strongly connected, there is at least one arc from $C$ to $U$. Let $v_1, \ldots, v_p$ be the tails of the arcs from $C$ to $U$ and let $W_k$ be the set of vertices in $C$ that reach $v_k$ for $k = 1, \ldots, p$. Note that there is exactly one arc from $v_i$ to $U$ for $i = 1, \ldots, p$, otherwise $v_i$ 2-reaches $v$ by Lemma 4 which is a contradiction to the choice of $U$. So in the following it is sufficient to show that $p = 1$. Denote the head vertex of the arc from $v_i$ to $U$ by $u_i$, $i = 1, \ldots, p$. We first claim that $W_i \cap W_j \neq \emptyset$ for any pair of different $W_i$ and $W_j$. If not, suppose there is $x \in W_i \cap W_j$, then there exists a directed path $P_i$ from $x$ to $v_i$ and a directed path $P_j$ from $x$ to $v_j$ in $C$. Let $y$ be the last common vertex in $V(P_i) \cap V(P_j)$ along the direction of $P_i$. Then $y \in W_i \cap W_j$ and there are two arc-disjoint directed paths from $y$ to $v_i$ and $v_j$ in $C$ and hence from $y$ to $u_i$ and $u_j$ in $G$, respectively. By Lemma 4 $y$ 2-reaches $v$, a contradiction to $y \not\in U$. Thus $W_i \cap W_j = \emptyset$ for any $i \neq j$. The claim also implies that there is no arc between $W_i$ and $W_j$. However, $C$ is connected (not necessarily strongly connected). This forces that $p = 1$. The proof is complete.