Lorenz’s electromagnetic theory of light

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Abstract. The Lorenz electromagnetic theory of light, published two years after the Maxwell theory, starts by postulating that both scalar and vector potentials are retarded. We show that in spite of this postulate, Lorenz’s theory gives a longitudinal electric field in vacuum that remains the instantaneous action at a distance it is in the Maxwell theory. There is in fact no difference between the two theories for electromagnetic phenomena in vacuum.

1 Introduction

The retarded scalar potential found in almost every textbook of electromagnetic theory today was first presented by Riemann to the Göttingen Academy in 1858, but the short paper was subsequently withdrawn (footnote 2, [1]). It was finally published posthumously in 1867 [2], a year after Riemann’s death. With this retarded potential, Riemann proposed that the electrical action was not instantaneous, but propagated with light speed $c$.

In 1867, Lorenz [3] independently proposed that Kirchhoff’s equations of electrical currents in conductors be slightly modified by requiring that these currents propagated with light speed in order to obtain an electrical theory of light. Drawing upon his previous results on wave propagation in elastic bodies [4], he proposed an electromagnetic theory of light even in a non-conducting space such as the vacuum by postulating that both scalar and vector potentials were really retarded and satisfied wave equations where light speed $c$ appeared.

In this way, Lorenz introduced what is known today as the Lorenz gauge. For many years, the gauge was mistakenly called the Lorentz gauge, after Lorentz who also used retarded potentials in a long paper in 1892 [5]. Fortunately, the story of the misattribution has recently been told in detail by Jackson and Okun [6].

In this paper, we are interested rather in the differences between the Maxwell and Lorenz theories of electromagnetism in vacuum. In his 1873 Treatise, Maxwell acknowledged that Lorenz’s theory of light was ‘similar to’ his own 1865 theory [7] of the electromagnetic field, but considered Lorenz’s work too little and too late ([8], p.183 [9]). To this day, there are Maxwellians who take the same point of view. However, there were [9] and still are [10] Lorenzians who take Lorenz’s retarded potentials literally and consider the Lorenz theory as the more correct description.

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2 Riemann was reluctant to publish incomplete work. On his untimely death at age 40, he left much work unpublished.
The scientific proposals of Maxwell and Lorenz to be discussed in this paper were made almost 150 years ago when the mechanistic nature of the vacuum was a scientific question of great interest. The term Maxwellian today can have a meaning very different from its true historical meaning. Our view of the physical world has changed very substantially a century after J. J. Thompson’s discovery of the electron in 1897 and Einstein’s 1905 invention of the special theory of relativity and analysis of the electrodynamics of moving bodies. Maxwell’s vacuum equations have stood the test of time, however. In our modern view, Maxwell’s displacement current is no longer the artificial and awkward device found so distasteful by the anti-Maxwellians. It appears instead as a natural partner to Faraday’s induction in the electric-magnetic duality structure of electromagnetism. It is conceptually very satisfying that electromagnetic wave motion is another manifestation of the experimentally well established and technologically highly important phenomenon of Faraday induction.

In this paper, we mean by a Maxwellian approach the formulation of the basic laws of electromagnetism in terms of Faraday induction and Maxwell displacement. This approach is in stark contrast to Lorenz’s proposal to start instead with a postulated retarded form of the scalar and vector potentials. Lorenz’s justification was that the light speed $c$ though large was known experimentally to be finite. So he believed that the classical concept of action at a distance must be an idealization and an approximation.

The purpose of this paper is to determine how similar Lorenz’s electromagnetic (EM) theory is to Maxwell’s. We shall do this with the method of Fourier transform used previously in our study of the Maxwell theory [12,13,14]. In this method, partial differential equations whose derivative terms carry constant coefficients are transformed into simple algebraic equations in Fourier space. It then becomes clear that the two theories agree in their treatments of the causal transverse EM fields and transverse vector potential. The apparent difference appears only in their scalar and longitudinal vector potentials. We then show that Lorenz’s retarded scalar/vector potentials appear in the longitudinal electric field in a certain combination that makes their retardations cancel completely. What is left is the same action at a distance described by the time-dependent Coulomb/Gauss law of the Maxwell theory. Hence the two theories describe exactly the same phenomena from two different starting points. Neither Riemann nor Lorenz was correct in supposing that instantaneous action at a distance can be eliminated completely from electromagnetic phenomena.

2 Lorenz’s theory in Fourier space

In the Lorenz theory ([3, pp. 267–270, [1], chap. VI, pp. 181–202 [9]), one begins by postulating that the scalar and vector potentials satisfy the wave equations.

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi(r, t) = -\frac{\rho(r, t)}{\varepsilon_0},$$

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A(r, t) = -\mu_0 J(r, t),$$

(1)
in SI units in the notation of Jackson [15]. Here $\partial_t = \partial / \partial t$. We shall work in the Fourier space $(k, \omega)$ where all differential equations with constant coefficients become algebraic equations that can be manipulated transparently.

Suppose these potentials and their first to third derivatives in space-time have Fourier representations of the type

$$\Phi(r, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot r - i \omega t} \tilde{\Phi}(k, \omega),$$

(2)

$$\nabla \Phi(r, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot r - i \omega t} i k \tilde{\Phi}(k, \omega),$$

(3)

eq etc. Note that the differential operator $\nabla$ simply operates on the Fourier basis functions $\exp(i k \cdot r - i \omega t)$. The wave Eqs. (1) then simplify to the algebraic equations in Fourier space

$$\left( k^2 - \frac{\omega^2}{c^2} \right) \tilde{\Phi} = \frac{\tilde{\rho}}{\varepsilon_0},$$

$$\left( k^2 - \frac{\omega^2}{c^2} \right) \tilde{A} = \mu_0 \tilde{J}.$$  \hspace{1cm} (4)

Since the operations $\nabla \cdot$ and $\nabla \times$ appear in Fourier space as the simple vector operations $ik \cdot$ and $ik \times$, the Helmholtz decomposition [16] of a vector field in Fourier space simplifies to the BAC identity

$$\tilde{A} = \tilde{A}_\parallel + \tilde{A}_\perp = e_k (e_k \cdot \tilde{A}) - e_k \times (e_k \times \tilde{A}),$$

(5)

where $e_k = k/k$. It gives a unique separation of the longitudinal and transverse (L/T) parts of the vector field $\tilde{A}$. We can now write down all the remaining equations of the Lorenz theory in Fourier space without further ado.

The electric field is defined to be

$$\tilde{E} = -i k \tilde{\Phi} + i \omega \tilde{A}.$$  \hspace{1cm} (6)

Its L/T parts are found by inspection:

$$\tilde{E}_\parallel = -i k \tilde{\Phi} + i \omega \tilde{A}_\parallel,$$

(7)

$$\tilde{E}_\perp = i \omega \tilde{A}_\perp.$$  \hspace{1cm} (8)

The magnetic induction, defined to be

$$\tilde{B} = i k \times \tilde{A} = i k \times \tilde{A}_\perp,$$  \hspace{1cm} (9)

turns out to be purely transverse. Its vanishing longitudinal part $\tilde{B}_\parallel = 0$ describes the fact that no magnetic charges or monopoles are present. We thus see that the same transverse causal vector potential $\tilde{A}_\perp$ determines both transverse causal EM fields uniquely. These results are the same as the Maxwell results. Indeed it had been realized by Young in 1817 ([17], pp. 114-5 [1]) and by Frenel in 1821 ([18], pp. 115-22, [1]) that the two independent polarization states seen in light passing through crystalline solids could be understood only if light propagation was caused by transverse vibrations and not longitudinal vibrations.

It is now useful to describe the consequence of the continuity equation for a conserved charge density:

$$\frac{d\rho}{dt} = \partial_t \rho + \nabla \cdot J = 0$$  \hspace{1cm} (10)
in spacetime, and
\[ \tilde{J}_\parallel = \frac{\omega}{k} \tilde{\rho}. \]
In Fourier space. Used with the wave Eqs. (4), it yields the expression
\[ \left( k^2 - \frac{\omega^2}{c^2} \right) \left( \epsilon_0 \omega \tilde{\Phi} - \frac{k}{\mu_0} \tilde{A} \right) = \omega \tilde{\rho} - k \tilde{J}_\parallel = 0. \]
(12)
This means that the Lorenz gauge condition
\[ \epsilon_0 \omega \tilde{\Phi} - \frac{k}{\mu_0} \tilde{A} = 0 \]
holds if \( k^2 \neq \omega^2/c^2 \). The inequality for \( k^2 \) appears because the Lorenz condition actually involves the solutions of two inhomogeneous wave equations whose sources when thus combined cancel everywhere in Fourier space or in spacetime. Thus the Lorenz theory seems to have the nice feature that the Lorenz condition comes out naturally from the postulated retarded scalar/vector potentials. However, its deeper significance is that this postulated retardation actually disappears completely from \( \tilde{E}_\parallel \), as we shall show in the following.

For completeness, we should mention that the Lorenz condition is violated for the solution of the homogeneous wave Eq. (12) satisfying the condition \( k^2 = \omega^2/c^2 \), but the resulting complementary function is only concerned with a change of the boundary/initial conditions satisfied by a solution of an inhomogeneous wave equation.

We turn finally to the single scalar field in \( \tilde{E}_\parallel \). It can be expressed in terms of two of the solutions \( \tilde{\rho} \) and \( \tilde{A}_\parallel \) of the wave Eqs. (4):
\[ \tilde{E}_\parallel = -i \epsilon_0 k \left( \frac{\tilde{\rho} k^2 - \tilde{J}_\parallel k \omega^2/c^2}{k^2 - \omega^2/c^2} \right). \]
(14)
Further simplification obtains on using the continuity Eq. (11):
\[ \tilde{E}_\parallel = -\frac{i \tilde{\rho}}{\epsilon_0 k} \left( \frac{k^2 - \omega^2/c^2}{k^2 - \omega^2/c^2} \right) = -\frac{i \tilde{\rho}}{\epsilon_0 k}. \]
(15)
The final step is justified because \( k^2 \neq \omega^2/c^2 \). Hence the Lorenz \( \tilde{E}_\parallel \) is no longer retarded, because it does not depend on the light speed \( c \) anymore. In fact, it is just the Coulomb/Gauss law in the Maxwell equation
\[ \nabla \cdot \mathbf{E}(\mathbf{r}, t) = \rho(\mathbf{r}, t)/\epsilon_0 \]
(16)
in spacetime. The div operator on the left introduces a spatial nonlocality, but the action occurs at the same time \( t \) on both sides of the equation. \( \mathbf{E}_\parallel \) is therefore also an instantaneous action at a distance in the Lorenz theory.

In conclusion, we have shown that the Lorentz theory gives exactly the same EM fields as the Maxwell theory. In particular, \( \mathbf{E}_\parallel \) remains an instantaneous action at a distance because the retarded contributions to \( \mathbf{E}_\parallel \) from \( \rho \) and \( \mathbf{J}_\parallel \) cancel completely.

The Maxwell and Lorenz theories are two examples of a common gauge-invariant theory with the universal characteristics that all causal EM fields are transverse and that \( \mathbf{E}_\parallel \) is an instantaneous action at a distance \( [12,14] \). The Maxwell and Lorenz theories are special cases of the common universal theory realized in the Coulomb and Lorenz gauges, respectively. More specifically, the result that \( \mathbf{E}_\parallel \) is non-causal and not associated with light propagation is obtained in Lorenz theory with the help of the continuity equation for the charge density and in the Maxwell theory directly from the Coulomb/Gauss law.
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