Involution words: counting problems and connections to Schubert calculus for symmetric orbit closures

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Abstract

Involution words are variations of reduced words for involutions in Coxeter groups, first studied under the name of “admissible sequences” by Richardson and Springer. They are maximal chains in Richardson and Springer’s weak order on involutions. In this paper, we initiate the study of involution words as an object of independent interest. To investigate their enumerative properties, we define involution analogues of several objects associated to permutations, including Rothe diagrams, the essential set, Schubert polynomials, and Stanley symmetric functions. These objects have geometric interpretations for certain intervals in the weak order on involutions, which we refer to as the geometric cases. In these cases, our definition for “involution Schubert polynomials” can be viewed as a Billey-Jockusch-Stanley type formula for cohomology class representatives of $O_n$- and $Sp_{2n}$-orbit closures in the flag variety, defined inductively in recent work of Wyser and Yong. As a special case of a more general theorem, we show that the involution Stanley symmetric function for the longest element is a product of staircase-shaped Schur functions in both geometric cases. We prove, as an application, that the number of involution words for the longest element in the symmetric group is the dimension of a certain irreducible representation of a Weyl group of type $B$.

This paper is the first in a series of three papers on involution words. It focuses primarily on algebraic tools related to involution words. In the second paper, we study the combinatorics of objects we call “atoms,” which are central to the study of involution words. The third paper focuses on the bijective combinatorics of involution words, including a new insertion algorithm. All three papers can be read independently.

Contents

1 Introduction

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1 Introduction

Let \((W, S)\) be a Coxeter system and define \(I = I(W) = \{x \in W : x = x^{-1}\}\) as the set of involutions in \(W\). A reduced word for an element \(w \in W\) is a sequence \((s_1, s_2, \ldots, s_k)\) with \(s_i \in S\) of shortest possible length such that \(w = s_1s_2\cdots s_k\). An involution word for an element \(z \in I\) is a sequence \((s_1, s_2, \ldots, s_k)\) with \(s_i \in S\) of shortest possible length such that

\[
z = (\cdots ((1 \times s_1) \times s_2) \times \cdots) \times s_k
\]

(1.1)

where for \(g \in W\) and \(s \in S\) we let \(g \times s\) be either \(gs\) (if \(s\) and \(g\) commute) or \(sgs\) (if \(sg \neq gs\)). Note when \(g \in I\) that \(g \times s\) is also an involution. Less obviously, every \(z \in I\) has at least one involution word; for example, the empty sequence \(\emptyset\) is the unique involution word of \(1 \in I\). We write \(\mathcal{R}(w)\) for the set of reduced words of \(w \in W\) and \(\mathcal{R}(z)\) for the set of involution words of \(z \in I\). More generally, given any pair of involutions \(y, z \in I\), we define \(\mathcal{R}(y, z)\) as the set of sequences in \(S\) which, when appended to involution words for \(y\), produce involution words for \(z\). The set \(\mathcal{R}(y, z)\) may be empty, and we refer to its elements as involution words from \(y\) to \(z\).

Involution words display many properties analogous to those of ordinary reduced words, which accounts for our terminology. In particular, reduced words correspond to maximal chains in \(W\) under the right weak order, while involution words correspond to maximal chains in \(I\) under the involution weak order defined by Richardson and Springer in [47, §3.17]. For initial intervals (that is, intervals starting at \(y = 1\)), involution words are the same as what Hultman calls “\(S\)-expressions” in [27, 28] and are the right-handed versions of “admissible sequences” in [47, 48] and “\(I\)-expressions” in [43, 44]. For permutations, the involution weak order can be identified with the weak order on the set of \(B\)-orbit closures in certain spherical varieties, and involution words are studied in this form by Can, Joyce, and Wyser in [6, 7]. Specifically, the orbits induced by the actions of the orthogonal and symplectic groups on the flag variety have weak orders whose chains correspond to involution words in the intervals starting at \(1\) and \(v_n = [2, 1, 4, 3, \ldots, 2n, 2n-1] \in S_{2n}\), respectively.
These geometric cases are of particular interest, and lead us to define, alongside \( \hat{\mathcal{R}}(y) \), the set
\[
\hat{\mathcal{R}}_{\text{FPF}}(z) \overset{\text{def}}{=} \hat{\mathcal{R}}(v_n, z) \quad \text{for } z \in \mathcal{I}(S_{2n}).
\] (1.2)
Abusing notation, elements of this set will be called fixed-point-free involution words, since \( v_n \) is the unique minimal involution with no fixed points in \( S_{2n} \). The set \( \hat{\mathcal{R}}_{\text{FPF}}(z) \) will be non-empty if and only if \( z \in \mathcal{I}(S_{2n}) \) has no fixed points, in which case every involution in the interval between \( v_n \) and \( z \) in weak order will also be fixed-point-free.

Before describing our work on involution words, we provide a brief overview of the geometry underlying the geometric cases. Let \( B \) be the Borel subgroup of lower triangular matrices in \( \text{GL}_n(\mathbb{C}) \) and denote by \( \text{Fl}(n) = B \backslash \text{GL}_n(\mathbb{C}) \) the flag variety. The right orbits of the opposite Borel subgroup \( B^+ \) in \( \text{Fl}(n) \) decompose into Schubert cells, whose Zariski closures are the Schubert varieties \( \mathcal{X}_w \), which are indexed by permutations \( w \in S_n \), and which can also be defined explicitly using a fixed reference flag and rank conditions determined by \( w \). By instead considering the right actions on \( \text{Fl}(n) \) of another group \( K \), such as the orthogonal group \( O_n(\mathbb{C}) \) or (when \( n \) is even) the symplectic group \( \text{Sp}_n(\mathbb{C}) \), one obtains different orbit decompositions. These \( K \)-actions decompose \( \text{Fl}(n) \) into \( K \)-orbits \( Y^K_y \) indexed by arbitrary involutions \( y \in \mathcal{I}(S_n) \) when \( K = O_n(\mathbb{C}) \) and by fixed-point-free involutions in \( S_n \) when \( K = \text{Sp}_n(\mathbb{C}) \). Again, \( Y^K_y \) can be defined using a fixed reference flag and explicit rank conditions determined by \( y \).

Each Schubert variety determines a class \([\mathcal{X}_w] \) in the cohomology ring \( H^\ast(\text{Fl}(n), \mathbb{Z}) \), which can be identified with the quotient of \( \mathbb{Z}[x_1, \ldots, x_n] \) by the ideal generated by the symmetric polynomials of positive degree via the Borel isomorphism (see (2.9)). The Schubert polynomial \( \mathcal{S}_w \), defined by Lascoux and Schützenberger [34], is a particularly nice choice of representative under this map for the cohomology classes of the Schubert variety \( \mathcal{X}_w \). For \( K \)-actions where \( K = O_n(\mathbb{C}) \) or \( \text{Sp}_n(\mathbb{C}) \), the cohomology classes of \( Y^K_y \) in \( H^\ast(\text{Fl}(n), \mathbb{Z}) \) map to the same quotient of \( \mathbb{Z}[x_1, \ldots, x_n] \). For these classes \([Y^K_y] \), Wyser and Yong have defined similarly nice polynomial representatives \( \Upsilon^K_y \), which they call \( \Upsilon \)-polynomials [55]. The construction of \( \Upsilon^K_y \) in [55] relies on first choosing a representative for the class of a point, and then showing compatibility with certain compositions of divided difference operators.

In this paper, we initiate the study of involution words from an enumerative perspective. To do so, we develop many analogues of combinatorial objects used in the study of reduced words. Most notably, we introduce analogues of Rothe diagrams and Fulton’s essential set for the geometric cases and analogues of Schubert polynomials and Stanley symmetric functions for all involution words. Many of these definitions are simple extensions of the ordinary versions, especially in light of the following result, which is a consequence of [47 Lemma 3.16].

**Theorem-Definition 1.1.** For each \( y, z \in \mathcal{I}(W) \), there exists a finite subset \( A(y, z) \subset W \) such that \( \hat{\mathcal{R}}(y, z) = \bigcup_{w \in A(y, z)} \mathcal{R}(w) \). Equivalently, every involution word from \( y \) to \( z \) is a reduced word for some element of \( W \) and the set \( \hat{\mathcal{R}}(y, z) \) is closed under the braid relations for \((W, S)\). For \( y, z \in \mathcal{I} \) and \( w \in A(y, z) \), we say \( w \) is a relative atom from \( y \) to \( z \).

In the geometric cases, we define \( A(y) \overset{\text{def}}{=} A(1, y) \) and \( A_{\text{FPF}}(z) \overset{\text{def}}{=} A(v_n, z) \).

**Remark.** The theorem-definition follows from results of Richardson and Springer in [47 §3]. A more direct, general proof using our present notation appears in our complementary work [23].
In both geometric cases, we introduce the *involution Rothe diagrams* \( \hat{D}(y) \) and \( \hat{D}_{FPF}(y) \) of \( y \in \mathcal{I} \) as certain restrictions of the usual Rothe diagram \( D(y) \). We then define the essential sets \( \text{Ess}(\hat{D}(y)) \) and \( \text{Ess}(\hat{D}_{FPF}(y)) \) as sets of southeast corners in the corresponding involution diagram. This closely mirrors the definition of Fulton’s essential set \( \text{Ess}(D(w)) \) for \( w \in S_n \). In Proposition 3.16 we show that the involution essential sets determine a subset of the rank conditions sufficient to define \( Y^K \) when \( K = O_n(\mathbb{C}) \) or \( K = Sp_n(\mathbb{C}) \), respectively. The proof is largely a consequence of the analogous result for the \( B^+ \)-action, with some subtleties in the fixed-point-free case. These objects prove to be a key tool in our study of involution Schubert polynomials and involution Stanley symmetric functions.

As discussed previously, Schubert polynomials were originally defined using divided difference operators. However, they can also be viewed as a sort of generating function over reduced words. More specifically, Billey, Jockusch and Stanley [4] and Fomin and Stanley [13] showed the following explicit combinatorial formula. Let \( s_i \) denote the simple transposition \((i, i+1)\), so that \( S_n \) is a Coxeter group relative to the generating set \( \{s_1, s_2, \ldots, s_{n-1}\} \). Fix \( w \in S_n \), and for each \( a = (s_{a_1}, s_{a_2}, \ldots, s_{a_k}) \in \mathcal{R}(w) \), let \( C(a) \) be the set of sequences of positive integers \( I = (i_1, i_2, \ldots, i_k) \) satisfying

\[ i_1 \leq i_2 \leq \cdots \leq i_k \quad \text{and} \quad i_j < i_{j+1} \quad \text{whenever} \quad a_j < a_{j+1}. \]

We write \( I \leq a \) to indicate that \( i_j \leq a_j \) for all \( j \) and define \( x_I = x_{i_1} x_{i_2} \cdots x_{i_k} \). The Schubert polynomial corresponding to \( w \in S_n \) is then given by

\[
\mathcal{S}_w = \sum_{a \in \mathcal{R}(w)} \sum_{I \in C(a)} x_I \in \mathbb{Z}[x_1, \ldots, x_n]. \tag{1.3}
\]

This formula makes clear that \( \mathcal{S}_w \) is homogeneous with degree equal to the length of \( w \). Similarly, the *Stanley symmetric function* of \( w \) is

\[
F_w = \sum_{a \in \mathcal{R}(w)} \sum_{I \in C(a)} x_I \in \mathbb{Z}[[x_1, x_2, \ldots]] \tag{1.4}
\]

(this definition is actually \( F_{w^{-1}} \) in [50]). These were introduced by Stanley to help enumerate reduced words, since the coefficient of \( x_1 x_2 \cdots x_{|w|} \) in \( F_w \) is \( |\mathcal{R}(w)| \). Note \( F_w = \lim_{N \to \infty} \mathcal{S}_{1_N \times w} \) where \( 1_N \times w \) denotes the image of \( w \) under the natural embedding \( S_n \hookrightarrow S_N \times S_n \subset S_{N+n} \) and the limit is taken in the sense of formal power series. This limit is called *stabilization*, and Stanley symmetric functions are sometimes referred to as stable Schubert polynomials.

For \( y, z \in \mathcal{I}(S_n) \), we define *involution Schubert polynomials* and *involution Stanley symmetric functions* analogously as

\[
\hat{\mathcal{S}}_{y,z} = \sum_{a \in \mathcal{R}(y,z)} \sum_{I \in C(a)} x_I \quad \text{and} \quad \hat{F}_{y,z} = \lim_{N \to \infty} \hat{\mathcal{S}}_{1_N \times y, 1_N \times z} = \sum_{a \in \mathcal{R}(y,z)} \sum_{I \in C(a)} x_I,
\]

respectively. Note by Theorem-Definition 1.1 that \( \hat{\mathcal{S}}_{y,z} = \sum_{w \in \mathcal{A}(y,z)} \mathcal{S}_w \) and \( \hat{F}_{y,z} = \sum_{w \in \mathcal{A}(y,z)} F_w \).

For the geometric cases, we define

\[
\hat{\mathcal{S}}_y \overset{\text{def}}{=} \hat{\mathcal{S}}_{1,y}, \quad \hat{F}_y \overset{\text{def}}{=} \hat{F}_{1,y}, \quad \hat{\mathcal{S}}_{FPF} \overset{\text{def}}{=} \hat{\mathcal{S}}_{v_n,y}, \quad \text{and} \quad \hat{F}_{FPF} \overset{\text{def}}{=} \hat{F}_{v_n,y}.
\]

As one would hope, these involution Schubert polynomials are the same (up to scaling factor) as Wyser and Yong’s representatives for \([Y^K_y]\). Let \( \kappa(y) \) be the number of two-cycles in \( y \in \mathcal{I}(S_n) \).
\textbf{Theorem 1.2.} For each \( y \in \mathcal{I}(S_n) \) and each fixed-point-free \( z \in \mathcal{I}(S_{2n}) \), it holds that
\[
2^{\kappa(y)} \hat{\mathcal{S}}_y = \mathcal{Y}^O_y \quad \text{and} \quad \hat{\mathcal{S}}^{\text{FFP}} = \mathcal{Y}^{\text{SP}}._{2n}.
\]

In either geometric case, the longest permutation \( w_n = [n, n-1, \ldots, 1] \in S_n \) indexes the orbit of the fixed reference flag, i.e., the class of a point. For this class, Wyser and Yong’s polynomial representatives are
\[
\mathcal{Y}^O_{w_n} = 2^{1+\frac{n}{2}} \hat{\mathcal{S}}_{w_n} = \prod_{1 \leq i < j \leq n-1} (x_i + x_j) \quad \text{and} \quad \mathcal{Y}^{\text{SP}}_{w_{2n}} = \hat{\mathcal{S}}^{\text{FFP}}_{w_{2n}} = \prod_{1 \leq i < j \leq 2n-1} (x_i + x_j).
\]

A permutation is \textit{dominant} if it is 132-avoiding. We extend these product formulas to dominant involutions as follows.

\textbf{Theorem 1.3.} Let \( y \in \mathcal{I}(S_n) \) and let \( z \in \mathcal{I}(S_{2n}) \) be fixed-point-free. If \( y \) and \( z \) are dominant, then
\[
\hat{\mathcal{S}}_y = 2^{-\kappa(y)} \prod_{(i,j) \in \hat{D}(y)} (x_i + x_j) \quad \text{and} \quad \hat{\mathcal{S}}^{\text{FFP}}_z = \prod_{(i,j) \in \hat{D}^{\text{FFP}}(z)} (x_i + x_j)
\]
where \( \hat{D}(y) \) and \( \hat{D}^{\text{FFP}}(z) \) are defined as in Section 3.2.

Theorem 1.3 is restated as Theorem 3.26 and is a special case of Theorem 3.27, which describes a simple formula for the involution Schubert polynomials of a more general class of involutions that we call \textit{weakly dominant}. This formula is the product of \( \mathcal{S}^v_{w_k} \) and a specialization of the double Schubert polynomial of some \( v \in S_n \), where \( k \) and \( v \) are determined by the weakly dominant involution.

In \cite{Stanley1984}, Stanley showed that Stanley symmetric functions \( F_w \) are symmetric, and computed several functions explicitly, notably showing that \( F_{w_n} \) is the Schur function \( s_{\delta_n} \) indexed by the staircase shape partition \( \delta_n = (n - 1, n - 2, \ldots, 1) \). This implies that \( |\mathcal{R}(w_n)| \) is equal to \( f^{\delta_n} \), the number of standard Young tableaux of shape \( \delta_n \). As a consequence of work by Lascoux and Schützenberger \cite{Lascoux1987}, and as proven bijectively by Edelman and Greene \cite{Edelman1987}, Stanley symmetric functions are Schur positive, i.e., can be expressed as positive integer sums of Schur functions. Since involution Stanley symmetric functions are sums of Stanley symmetric functions, they inherit this property. In the geometric cases, we characterize the involutions whose involution Stanley symmetric function is a single Schur function. Moreover, by carefully studying the stabilization of certain weakly dominant involution Schubert polynomials, we obtain expressions for the corresponding involution Stanley symmetric functions. Most notably we derive the following, which was conjectured in 2006 in unpublished work of Cooley and Williams \cite{Cooley2006}.

\textbf{Theorem 1.4.} Let \( p = \left\lceil \frac{n+1}{2} \right\rceil \) and \( q = \left\lceil \frac{n+1}{2} \right\rceil \), and set \( P = \left( \begin{array}{c} \frac{p}{2} \\ \frac{q}{2} \end{array} \right) \), \( Q = \left( \begin{array}{c} \frac{q}{2} \\ \frac{q}{2} \end{array} \right) \), and \( N = \left( \begin{array}{c} \frac{n}{2} \\ \frac{n}{2} \end{array} \right) \). Then
\[
\hat{F}_{w_n} = s_{\delta_p} s_{\delta_q} \quad \text{and} \quad \hat{F}^{\text{FFP}}_{w_{2n}} = (s_{\delta_n})^2.
\]
Consequently, \( |\mathcal{R}(w_n)| = \left( \begin{array}{c} p+q \\ p \end{array} \right) f^{\delta_p} f^{\delta_q} \) and \( |\mathcal{R}^{\text{FFP}}(w_{2n})| = \left( \begin{array}{c} 2N \\ N \end{array} \right) (f^{\delta_n})^2 \).

Theorem 1.4 is a special case of Theorem 3.43 which provides product formulae for a certain family of weakly dominant involutions. Moreover, every involution Stanley symmetric function computed in Theorem 3.43 is Schur-\( P \) positive. In forthcoming work \cite{FutureWork}, we present a bijective proof that \( \hat{F}_y = \hat{F}_{1,y} \) is Schur-\( P \) positive for all \( y \in \mathcal{I}(S_n) \). We do not yet have a good understanding of when the symmetric function \( \hat{F}_{y,z} \) is Schur-\( P \) positive for arbitrary involutions \( y, z \in \mathcal{I}(S_n) \). It can happen that an involution Stanley symmetric function is not expressable using Schur-\( P \) functions. For example \( \hat{F}_{[2,1,3,4],[3,4,1,2]} = s_{(1,1)} \), which is not in the ring generated by Schur-\( P \) functions.
**Question 1.5.** For which $y, z \in \mathcal{I}(S_n)$ is $F_{y,z}$ Schur-P positive?

**Conjecture 1.6.** For $y \in \mathcal{I}(S_n)$ fixed-point-free, $\hat{R}^{\text{FPF}}_y$ is Schur-P positive.

Although our enumerative results are restricted to the geometric cases for the symmetric group, the objects we study have natural analogues in other Coxeter groups. Many tantalizing questions remain in this direction. For example, Haiman showed in [19] that $|\mathcal{R}(w_n^B)| = f(n^n)$ where $w_n^B$ is the longest element in the Weyl group $B_n$ and $n^n = (n, n, \ldots, n)$. Computations suggest the following version of this theorem for involution words.

**Conjecture 1.7.** The set $\hat{R}(w_n^B)$ has size $f^{\delta_n+1}$.

Additionally, there is a notion of *twisted involution words* for which Schubert polynomial and Stanley symmetric function analogues are readily defined. We do not explore these objects in the present paper, but it remains a question of great interest to find geometric interpretations for twisted involution Schubert polynomials.

Before outlining the structure of the paper, we provide a brief discussion of the relationship between our work and that of Wyser and Yong [55]. Our proof of Theorem 1.2 proceeds by generalizing a characterization of Schubert polynomials to the involution setting. This allows us to show our formula coincides with Wyser and Yong’s formula for the class of a point and behaves in a similar way with respect to divided difference operators. However, a general formula of Brion [5, Section 5] note that their representative for $[Y_y^K]$ is a linear combination of Schubert polynomials, and therefore is equal to $\sum_{w \in \mathcal{A}^K(y)} X_w$. Wyser and Yong [55, Section 5] note that their representative for $[Y_y^K]$ is a linear combination of Schubert polynomials, and therefore is equal to $\sum_{w \in \mathcal{A}^K(y)} X_w$ (again up to a power of 2). One then immediately gets an analogue of (1.3) for Wyser and Yong’s representatives by replacing $\mathcal{R}(w)$ with $\bigcup_{w \in \mathcal{A}(y)} \mathcal{R}(w)$ (in the $K = O_n(\mathbb{C})$ case) or $\bigcup_{w \in \mathcal{A}_{\text{FPF}}(y)} \mathcal{R}(w)$ (in the $K = \text{Sp}_n(\mathbb{C})$ case). From this point of view, the main contribution of Theorem 1.2 is combinatorial: we identify $\bigcup_{w \in \mathcal{A}^K(y)} \mathcal{R}(w)$ as $\hat{R}(y)$ (or $\hat{R}_{\text{FPF}}(y)$) and $\mathcal{A}^K(y)$ as the set $\mathcal{A}(y)$ (or $\mathcal{A}_{\text{FPF}}(y)$) defined in Theorem-Definition 1.1. While the latter is also done in [7], the connection is mostly left implicit.

As mentioned earlier, Theorem 1.3 is a special case of Theorem 3.27 which describes certain involution Stanley symmetric functions as the product of a Schubert polynomial and the specialization of a double Schubert polynomial. In Section 3, we present a proof relying on carefully studying the image of $\hat{S}_{w_n}$ and $\hat{S}_{w_n}^{\text{FPF}}$ under certain divided difference operators. However, by considering the intersection of appropriate varieties corresponding to these polynomials, one obtains a different characterization of the $[Y_y^K]$’s. In Appendix A, we present an alternate proof of Theorem 3.27 based on these geometric considerations. This approach is quite different from that of Wyser and Yong. Indeed, it provides a new proof of their product formulae (1.5) for $\hat{S}_{w_n}$ and $\hat{S}_{w_n}^{\text{FPF}}$ and that the involution Schubert polynomials are cohomology representatives, which is independent both of their work and Brion’s. However, this approach does not recover the fact, proved by Wyser and Yong, that these polynomials are equivariant cohomology representatives.

The remainder of the paper is structured as follows. Section 2 describes the previous understanding of involution words necessary for this paper. It then recalls definitions and known results for Rothe diagrams, Schubert polynomials, and Stanley symmetric function required to prove our results, as well as a more detailed discussion of the geometric connections. In Section 3 we prove the main results of this paper. This requires defining many objects, including involution diagrams, and developing significant background on techniques such as stabilization. Finally, Appendix 2.1 presents a geometric proof of Theorem 3.27.
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2 Preliminaries

Write \( \mathbb{P} = \{1, 2, 3, \ldots \} \) for the positive integers and define \( \mathbb{N} = \{0\} \cup \mathbb{P} \) and \( \{n\} = \{i \in \mathbb{P} : i \leq n\} \). If \((W,S)\) is a Coxeter system, then we write \( \ell : W \to \mathbb{N} \) for its length function, and denote by

\[
\text{Des}_L(w) = \{ s \in S : \ell(sw) < \ell(w) \} \quad \text{and} \quad \text{Des}_R(w) = \{ s \in S : \ell(ws) < \ell(w) \}
\]

the left and right descent sets of an element \( w \in W \).

2.1 General properties of involution words

Here we review the basic properties of involution words for an arbitrary Coxeter group. Most of this material appears in some form in \([47, 48, 49]\) or the more recent papers \([26, 27, 28]\).

Remark. Our definition of involution words has a straightforward generalization to twisted involutions in Coxeter groups, by which we mean elements \( w \in W \) satisfying \( w^{-1} = w^* \) for some fixed \( S \)-preserving automorphism * of \( W \) of order two. This more flexible setup is the point of view of our references, but our present applications will not require this generality.

Let \((W,S)\) be any Coxeter system and write \( \mathcal{I} = \mathcal{I}(W) = \{ w \in W : w^{-1} = w \} \).

Remark. Recall that for \( y \in \mathcal{I} \) and \( s \in S \) we define

\[
y \times s = \begin{cases} 
sys & \text{if } ys \neq sy \\
ys & \text{otherwise.}
\end{cases}
\]

Although \((y \times s) \times s = y\) for \( s \in S \), the operation \( \times : \mathcal{I} \times S \to \mathcal{I} \) usually does not extend to a right \( W \)-action. For example, if \( s, t \in S \) are such that \( sts = tst \) then \(((1 \times s) \times t) \times s = t \) but \(((1 \times t) \times s) \times t = s \). Nevertheless, we usually omit all parentheses in expressions like \((1,1)\).

Define \( \mathcal{R}(w) \) for \( w \in W \) and \( \hat{\mathcal{R}}(z) \) for \( z \in \mathcal{I} \) as in the introduction. Recall that for \( y, z \in \mathcal{I} \), the set \( \hat{\mathcal{R}}(y,z) \) consists of all words \( (s_1, \ldots, s_k) \) with \( s_i \in S \) such that for some (equivalently, every) word \( (r_1, \ldots, r_j) \in \hat{\mathcal{R}}(y) \) it holds that

\[
(r_1, \ldots, r_j, s_1, \ldots, s_k) \in \hat{\mathcal{R}}(z).
\]

Call \( \mathcal{R}(y,z) \) the set of (relative) involution words from \( y \) to \( z \). Note that \( \hat{\mathcal{R}}(y,y) = \{\emptyset\} \) where \( \emptyset \) denotes the empty sequence, and that \( \hat{\mathcal{R}}(y,z) \) may be empty, for example if \( y \) exceeds \( z \) in length.

Fix \( y \in \mathcal{I} \) and \( s \in S \). It is a consequence of the exchange principle that \( \ell(sys) = \ell(y) \) if and only if \( sys = y \) [27, Lemma 3.4], and so if \( s \in \text{Des}_R(y) \) then

\[
\ell(y \times s) = \begin{cases} 
\ell(y) - 2 & \text{if } y \times s = sys \\
\ell(y) - 1 & \text{if } y \times s = ys.
\end{cases}
\]
From this property, it follows by induction on length that \( \hat{\mathcal{R}}(y) \neq \emptyset \) for all \( y \in \mathcal{I} \), so we may set

\[
\hat{\ell}(y) \overset{\text{def}}{=} \text{the common length of all involution words for } y \in \mathcal{I}.
\] (2.3)

We abbreviate by writing \( \hat{\ell}(y, z) \) to denote the difference \( \hat{\ell}(z) - \hat{\ell}(y) \).

**Remark.** The map \( \hat{\ell} : W \to \mathbb{N} \) is denoted \( L \) in [17, §3] and \( \rho \) in [26, 27, 28]. Incitti [29, 30] has derived useful combinatorial formulas for \( \hat{\ell} \) when \( W \) is a classical Weyl group. In the case when \( W = S_n \) is a symmetric group, one has

\[
\hat{\ell}(y) = \frac{1}{2}(\ell(y) + \kappa(y)) \quad \text{for } y \in \mathcal{I}(S_n)
\]

where \( \ell(y) \) is the usual length and \( \kappa(y) \) is the number of 2-cycles of an involution \( y \).

Write \( \leq \) for the (strong) Bruhat order on \( W \). Recall that this is the partial order in which \( u \leq v \) if and only if in each reduced expression for \( v \) one can omit a certain number of factors to obtain a reduced expression for \( u \). Thus \( u < v \) implies \( \ell(u) < \ell(v) \), and it follows from (2.2) that if \( y \in \mathcal{I} \) and \( s \in \text{Des}_R(y) \) then \( y \times s \leq ys < y \). There is a close relationship between the the Bruhat order on \( \mathcal{I} \) and involution words, which was one motivation for their study originally in [17, 18]. For example, \( (\mathcal{I}, \leq) \) is a graded poset with rank function \( \hat{\ell} : \mathcal{I} \to \mathbb{N} \) [26, Theorem 4.8], and this poset inherits the subword characterization of \( (W, \leq) \) given above, but with the role of reduced words replaced by involution words [28, Theorem 2.8]. From these results, it is clear that if \( y \in \mathcal{I} \) and \( s \in S \) then the following are equivalent:

\[
y \times s < y \iff \ell(ys) = \ell(y) - 1 \iff \ell(y \times s) < \ell(y) \iff \hat{\ell}(y \times s) = \hat{\ell}(y) - 1.
\]

These properties imply the following useful alternative definition of the set \( \hat{\mathcal{R}}(y, z) \):

**Lemma 2.1.** If \( y, z \in \mathcal{I} \), then a word \((s_1, s_2, \ldots, s_k)\) with \( s_i \in S \) belongs to \( \hat{\mathcal{R}}(y, z) \) if and only if

\[
y < w_1 < w_2 < \cdots < w_k = z \quad \text{where } w_i = y \times s_1 \times s_2 \times \cdots \times s_i.
\]

Recall that \( \hat{\mathcal{R}}_{\text{FFF}}(y) = \hat{\mathcal{R}}(v_n, y) \) where \( v_n = s_1 s_3 \cdots s_{2n+1} \).

**Corollary 2.2.** If \( y \in \mathcal{I}(S_{2n}) \) then \( \hat{\mathcal{R}}_{\text{FFF}}(y) \) is non-empty if and only if \( y \) is fixed point free.

**Proof.** Note if \( y \in S_{2n} \) is a fixed-point-free involution and \( s \) is a simple transposition that \( sys \) is also fixed-point-free while \( y \times s = ys \) only if \( s \in \text{Des}_R(w) \); then invoke the preceding lemma. \( \Box \)

Recall the definition of \( \mathcal{A}(y, z) \) for \( y, z \in \mathcal{I}(W) \) from Theorem-Definition 1.1. The properties of \( \mathcal{A}(y, z) \) are one of the main subjects of our work [23]. Call \( \mathcal{A}(y, z) \) the set of (relative) atoms from \( y \) to \( z \), and recall \( \mathcal{A}(y) = \mathcal{A}(1, y) \). While \( \mathcal{A}(y, z) \) may be empty, for example when \( \ell(z) < \ell(y) \), the set \( \mathcal{A}(y) \) is always nonempty, and we always have \( \mathcal{A}(1) = \mathcal{A}(y, y) = \{1\} \). It is clear that

\[
\mathcal{A}(y, z) = \{w \in W : \ell(w) = \hat{\ell}(y, z) \text{ and } vw \in \mathcal{A}(z) \text{ for some } v \in \mathcal{A}(y)\},
\]

so \( \mathcal{A}(y, z) \) can be computed from \( \mathcal{A}(y) \) and \( \mathcal{A}(z) \).

**Example 2.3.** For the involutions \( v_2 = [2, 1, 4, 3] \) and \( w_4 = [4, 3, 2, 1] \) in \( S_4 \) we have

\[
\mathcal{A}(w_4) = \{[2, 4, 3, 1], [3, 4, 1, 2], [4, 2, 1, 3]\} \quad \text{and} \quad \mathcal{A}(v_2, w_4) = \{[1, 3, 4, 2], [3, 1, 2, 4]\}.
\]

In general, the sets \( \mathcal{A}(w_n) \) and \( \mathcal{A}(v_n, w_{2n}) \) have cardinality \((n - 1)!!\) and \( n! \) and are given by a simple recursive construction due to Can, Joyce, and Wyser [6, 7].
The essential properties of atoms are summarized by the following statement, which is proved as [23, Proposition 2.8].

**Proposition 2.4** (See [23]). Let \( y, z \in \mathcal{I} \) and \( s \in S \).

(a) If \( s \notin \text{Des}_R(z) \) then \( \mathcal{A}(y, z) = \{ws : w \in \mathcal{A}(y, z \times s) \text{ and } s \in \text{Des}_R(w)\} \).

(b) If \( s \in \text{Des}_R(y) \) then \( \mathcal{A}(y, z) = \{sw : w \in \mathcal{A}(y \times s, z) \text{ and } s \in \text{Des}_L(w)\} \).

Consequently, if \( u \in \mathcal{A}(y, z) \) then \( \text{Des}_R(u) \subset \text{Des}_R(z) \) and \( \text{Des}_L(u) \subset S \setminus \text{Des}_R(y) \).

We mention another order on \( \mathcal{I} \) which will be of relevance. Recall that the left and right weak orders \( \leq_L \) and \( \leq_R \) on \( W \) are the transitive closures of the relations \( w <_L sw \) and \( w <_R wt \) for \( w \in W \) and \( s, t \in S \) such that \( \ell(sw) > \ell(w) \) and \( \ell(wt) > \ell(w) \). Following [27, Section 5], we define the (two-sided) weak order \( \leq_T \) on \( \mathcal{I} \) as the transitive closure of the relations

\[
  w <_T w \times s \quad \text{for } w \in \mathcal{I} \text{ and } s \in S \text{ such that } \ell(w) < \ell(w \times s).
\]

Evidently \( \mathcal{A}(y, z) \) is nonempty if and only if \( y \leq_T z \), and each element of \( \mathcal{R}(y, z) \) corresponds to a maximal chain from \( y \) to \( z \) in the poset \( (\mathcal{I}, \leq_T) \). If \( y, z \in \mathcal{I} \) then \( y \leq_T z \) implies \( y \leq z \), but the reverse implication does not hold in general.

### 2.2 Diagrams and codes for permutations

We write \( S_{\infty} \) for the group of permutations \( w \) of \( \mathbb{P} \) whose support \( \text{supp}(w) = \{i \in \mathbb{P} : w(i) \neq i\} \) is finite, and identify \( S_n \) for \( n \in \mathbb{P} \) as the subgroup of permutations \( w \in S_{\infty} \) with \( \text{supp}(w) \subset [n] \). Recall that the right descent set of \( w \in S_{\infty} \), defined as in [24] with respect to the generating set of simple transpositions \( \{s_i = (i, i + 1) : i \in \mathbb{P}\} \), is given more explicitly by

\[
  \text{Des}_R(w) = \{s_i : i \in \mathbb{P} \text{ and } w(i) > w(i + 1)\}. \tag{2.4}
\]

We say that \( i \) is a *descent* of \( w \in S_{\infty} \) if \( w(i) > w(i + 1) \), so that \( s_i \in \text{Des}_R(w) \).

Recall (e.g., from [41, §2.1.1]) that the *Rothe diagram* of \( w \in S_{\infty} \) is the set of positions

\[
  D(w) = \{(i, j) \in \mathbb{P} \times \mathbb{P} : j < w(i) \text{ and } i < w^{-1}(j)\}.
\]

Note that \( D(w) \) is obtained by applying the map \((i, j) \mapsto (i, w(j))\) to the inversion set of \( w \). Consequently \( D(w^{-1}) = D(w)^T \) where \( T \) denotes the transpose map \((i, j) \mapsto (j, i)\), and if \( w \in S_n \) and has largest descent \( k \), then \( D(w) \subset [k] \times [n] \).

**Example 2.5.** We have \( D(v_n) = \{(2i - 1, 2i - 1) : i \in [n]\} \) and \( D(w_n) = \{(i, j) \in \mathbb{P}^2 : i + j \leq n\} \).

The *diagram* of an integer partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \) is the set \( \{(i, j) \in \mathbb{P} \times \mathbb{P} : j \leq \lambda_i\} \). We often identify partitions with their diagrams, and write \((i, j) \in \lambda \) to indicate that \((i, j)\) belongs to the diagram of \( \lambda \). If \( \lambda \) and \( \mu \) are partitions with \( \mu \subset \lambda \) then the *skew shape* \( \lambda/\mu \) is the complement of the diagram of \( \mu \) in the diagram of \( \lambda \). The *shifted shape* of a strict partition \( \lambda \) (i.e., a partition with distinct parts) is the set \( \{(i, j + i - 1) : (i, j) \in \lambda\} \). Two finite subsets of \( \mathbb{P} \times \mathbb{P} \) (in particular, Rothe diagrams or diagrams of partitions or skew shapes or shifted shapes) are *equivalent* if one can be transformed to the other by permuting its rows and then its columns.
Example 2.6. If \( \lambda = (2, 2, 1) \) and \( \mu = (1) \) then \( \{ (1, 1), (1, 3), (2, 3), (3, 1) \} \) is equivalent to \( \lambda/\mu \).

The code of \( w \in S_n \) is the sequence \( c(w) = (c_1(w), c_2(w), \ldots, c_n(w)) \in \mathbb{N}^n \) where
\[
c_i(w) = |\{ j \in [n] : i < j \text{ and } w(i) > w(j) \}|.
\] (2.5)
Observe that \( c_i(w) \) is the number of cells in the \( i \)th row of \( D(w) \). The shape \( \lambda(w) \) of \( w \in S_n \) is the partition of \( \ell(w) \) whose parts are the nonzero entries of \( c(w) \).

Example 2.7. If \( w = [3, 7, 4, 1, 6, 5, 2] \) then \( c(w) = (2, 5, 2, 0, 2, 1, 0) \) and \( \lambda(w) = (5, 2, 2, 2, 1) \).

2.3 Schubert polynomials

We sketch here the fundamental properties of the Schubert polynomials \( \mathcal{G}_w \) as defined in the introduction; our main references are the texts \cite{10,11} and papers \cite{2,4,10,32,34}. We write
\[
\mathcal{P}_n = \mathbb{Z}[x_1, x_2, \ldots, x_n] \quad \text{and} \quad \mathcal{P}_\infty = \mathbb{Z}[x_1, x_2, \ldots]
\]
for the rings of polynomials in finite and countable sets of commuting variables \( \{x_1, x_2, \ldots \} \). The group \( S_n \) (respectively, \( S_\infty \)) acts on \( \mathcal{P}_n \) (respectively \( \mathcal{P}_\infty \)) by permuting variables. With respect to this action, the divided difference operator \( \partial_i \) for \( i \in \mathbb{P} \) is defined by
\[
\partial_i f = (f - s_i f)/(x_i - x_{i+1}) \quad \text{for } f \in \mathcal{P}_\infty.
\]
For example, \( \partial_i (x_i^3) = x_i^2 + x_i x_{i+1} + x_{i+1}^2 \). It is a standard exercise to check that this formula in fact gives a linear map \( \partial_i : \mathcal{P}_\infty \rightarrow \mathcal{P}_\infty \), and that \( \partial_i(fg) = f \cdot \partial_i g \) whenever \( s_i f = f \).

We first observe, following the notes of Knutson \cite{32}, how one may characterize the Schubert polynomials without explicitly constructing them using the divided difference operators.

Theorem 2.8 (See \cite{32}). The Schubert polynomials \( \{ \mathcal{G}_w \}_{w \in S_\infty} \) are the unique family of homogeneous polynomials indexed by the elements of \( S_\infty \) such that
\[
\mathcal{G}_1 = 1 \quad \text{and} \quad \partial_i \mathcal{G}_w = \begin{cases} \mathcal{G}_{ws_i} & \text{if } s_i \in \Des_R(w) \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } i \in \mathbb{P}.
\]

The divided difference operators satisfy \( \partial_i^2 = 0 \) as well as the Coxeter relations for \( S_\infty \) given by
\[
\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \quad \text{and} \quad \partial_i \partial_j = \partial_j \partial_i \quad \text{for } i, j \in \mathbb{P} \text{ with } |i - j| > 1.
\] (2.6)
For \( w \in S_\infty \), we may thus define \( \partial_w = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} \) for any reduced word \( (s_{i_1}, s_{i_2}, \ldots, s_{i_k}) \in \mathcal{R}(w) \).

Recall that \( w_n \) denotes the longest element of \( S_n \).

Theorem 2.9 (See \cite{31}). If \( n \in \mathbb{P} \) and \( v \in S_n \) then
\[
\mathcal{G}_{w_n} = x_1^{n-1} x_2^{n-2} x_3^{n-3} \cdots x_{n-1} \quad \text{and} \quad \mathcal{G}_v = \partial_{v^{-1} w_n} \mathcal{G}_{w_n}.
\]

Let \( y = \{ y_1, y_2, \ldots \} \) be another countable set of commuting variables, which commute also with \( x = \{ x_1, x_2, \ldots \} \). If \( f \in \mathcal{P}_\infty \) then we write \( f(y) \) to denote the polynomial given by evaluating \( f \) at \( x_i = y_i \), and for emphasis we sometimes write \( f = f(x) \). We let \( \mathcal{P}_\infty(x; y) = \mathbb{Z}[x_1, y_1, x_2, y_2, \ldots] \) be the polynomial ring in \( x \) and \( y \) together, and take \cite{31} Proposition 2.4.7 for the following definition:
**Definition 2.10.** The double Schubert polynomial corresponding to \( w \in S_\infty \) is
\[
\mathcal{S}_w(x; y) = \sum_{\ell(w) = \ell(u) + \ell(v)} \mathcal{S}_u(x)\mathcal{S}_v(-y) \in \mathcal{P}_\infty(x; y).
\]

Let \( S_\infty \) act on \( \mathcal{P}_\infty(x; y) \) by permuting only the \( x_i \) variables, and extend the formula for \( \partial_i \) to an operator \( \mathcal{P}_\infty(x; y) \to \mathcal{P}_\infty(x; y) \) with respect to this action. The following then holds:

**Theorem 2.11** (See [41]). If \( n \in \mathbb{P} \) and \( v \in S_n \) then
\[
\mathcal{S}_{w_n}(x; y) = \prod_{i+j \leq n} (x_i - y_j) \quad \text{and} \quad \mathcal{S}_v(x; y) = \partial_{w^{-1}w_n} \mathcal{S}_{w_n}(x; y)
\]
where the product on the left is over \( i, j \in \mathbb{P} \). In particular, \( \mathcal{S}_v = \mathcal{S}_v(x; 0) \).

If \( w \in S_n \) then \( \mathcal{S}_w \) is a polynomial in at most \( n - 1 \) variables, though often fewer. Explicitly, we note the following useful fact, which appears as [41, Proposition 2.5.4].

**Proposition 2.12** (See [41]). The set of Schubert polynomials \( \mathcal{S}_w \) with \( w \in S_\infty \) ranging over all permutations with largest descent at most \( n \) forms a basis for \( \mathcal{P}_n \) over \( \mathbb{Z} \).

A permutation \( w \in S_n \) is *dominant* if it is 132-avoiding, or equivalently if its Rothe diagram is the diagram of a partition (see [41, Exercise 2.2.2]). A permutation \( w \in S_n \) is *Grassmannian* if it has at most one right descent, or equivalently if for some \( r \geq 0 \) it holds that \( c_1(w) \leq \cdots \leq c_r(w) \) and \( c_i(w) = 0 \) for all \( i > r \). For permutations of these types, the corresponding Schubert polynomials have the following formulas, which appear [41, Propositions 2.6.7 and 2.6.8]. Here, recall that \( s_\lambda \) denotes the Schur function indexed by a partition \( \lambda \).

**Proposition 2.13** (See [41]). Let \( w \in S_\infty \).

(a) If \( w \) is dominant then \( \mathcal{S}_w(x; y) = \prod_{(i,j) \in D(w)} (x_i - y_j) \).

(b) If \( w \neq 1 \) is Grassmannian with unique descent \( r \), then \( \mathcal{S}_w = s_{\lambda(w)}(x_1, \ldots, x_r) \).

### 2.4 Cohomology of flag varieties

We review the geometric context that lead to the consideration of Schubert polynomials. Let \( \text{Fl}(n) \) denote the set of complete flags \( F_\bullet = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^n) \), where each \( F_i \) is a subspace of dimension \( i \), given the structure of a projective algebraic variety via the Plücker embedding as in [41, §3.6.1]. We identify \( \text{Fl}(n) \) with the right coset space \( B \setminus \text{GL}_n(\mathbb{C}) \), where \( B \) is the Borel subgroup of lower triangular matrices in \( \text{GL}_n(\mathbb{C}) \).

The general linear group \( \text{GL}_n(\mathbb{C}) \) acts on the right on \( \text{Fl}(n) \) by multiplication. Let \( B^+ = w_n \cdot B \cdot w_n \) denote the Borel subgroup opposite to \( B \), consisting of the upper triangular matrices in \( \text{GL}_n(\mathbb{C}) \). It follows from the Bruhat decomposition of \( \text{GL}_n(\mathbb{C}) \) that the distinct orbits of \( B^+ \) on \( \text{Fl}(n) \) identified with \( B \setminus \text{GL}_n(\mathbb{C}) \) are given by \( B \setminus BwB^+ \) for \( w \in S_n \), where \( S_n \) is embedded as the subgroup of permutation matrices in \( \text{GL}_n(\mathbb{C}) \). Define
\[
X_w = B \setminus BwB^+ \quad \text{and} \quad \hat{X}_w = B \setminus BwB^+ \quad \text{for} \ w \in S_n
\]
where on the right the bar denotes the Zariski closure. We call \( X_w \) the Schubert cell attached to \( w \in S_n \) and \( \hat{X}_w \) the corresponding Schubert variety.
Remark. Because we identify \(\text{Fl}(n)\) with \(B\backslash GL_n(\mathbb{C})\) rather than \(GL_n(\mathbb{C})/B\) and define Schubert cells as right \(B^+\)-orbits rather than left \(B\)-orbits, our definitions differ from those in [41] \S 3.6] by a transformation of indices. Explicitly, the sets \(\Omega_w\) for \(w \in S_n\) which Manivel refers to as Schubert cells are given in our notation by \(\Omega_w = \tilde{X}_{w,w} \cdot w_n\) whence \(\tilde{X}_w = w_n \cdot \Omega_{w,w}\). What we call \(X_w\) is related to Manivel’s definition of the Schubert variety of \(w \in S_n\) by the same transformations. It thus follows from [41] \S 3.6.2] that \(X_w\) is an irreducible variety of codimension \(\ell(w)\) in \(\text{Fl}(n)\).

It will be useful to recall the following concrete description of Schubert cells and varieties. Choose a basis \(e_1, e_2, \ldots, e_n\) of \(\mathbb{C}^n\) for each \(j \in [n]\) define \(V_j = \mathbb{C}\cdot\{e_1, e_2, \ldots, e_j\}\). Given a vector space \(U \subset \mathbb{C}^n\), we write \(\text{proj} : U \to V_j\) for the restriction to \(U\) of the usual linear projection \(\mathbb{C}^n \to V_j\) mapping \(e_i \mapsto 0\) for \(i > j\). Also define

\[
\text{rk}_w(i,j) = |\{t \in [i] : w(t) \in [j]\}| \quad \text{for } w \in S_n \text{ and } i,j \in [n].
\]

By [41] Proposition 3.6.4 (noting the remark above), we then have

\[
\tilde{X}_w = \{F_\bullet \in \text{Fl}(n) : \dim(\text{proj} : F_i \to V_j) = \text{rk}_w(i,j) \text{ for each } i,j \in [n]\},
\]

\[
X_w = \{F_\bullet \in \text{Fl}(n) : \dim(\text{proj} : F_i \to V_j) \leq \text{rk}_w(i,j) \text{ for each } i,j \in [n]\}.
\]

These conditions say that \(F_\bullet\) belongs to \(\tilde{X}_w\) (respectively, \(X_w\)) if and only if for each \(i,j \in [n]\), the upper left \(i \times j\) submatrix of a matrix representing \(F_\bullet\) has rank equal to (respectively, at most) the number of 1’s in the upper left \(i \times j\) submatrix of the permutation matrix of \(w\).

If \(X\) is a smooth complex algebraic variety and \(V\) is a closed subvariety, then there is a corresponding cohomology class \([V] \in H^\ast(X,\mathbb{Z})\), with the important property that \([V \cap W] = [V][W]\) when \(V\) and \(W\) intersect transversely on an open subset of \(V \cap W\). When \(X\) is compact one defines \([V]\) by first triangulating \(V\) to obtain a homology class and then taking its Poincaré dual. In general, one can view \([V]\) as the image of the class of \(V\) in the Chow ring of \(X\) under an appropriate map to \(H^\ast(X)\); see [16] or [41] Appendix A] or [14] Chapter 19.

For each Schubert variety \(X_w \subset \text{Fl}(n)\) one obtains in this way a corresponding Schubert class \([X_w] \in H^\ast(\text{Fl}(n),\mathbb{Z})\) which is denoted \(\sigma_w\) in [41] \S 3.6.3. As in the introduction, we identify the Schubert classes with elements of the coinvariant algebra of the symmetric group via the Borel isomorphism (see [41] \S 3.6.4])

\[
H^\ast(\text{Fl}(n),\mathbb{Z}) \xrightarrow{\sim} \mathcal{P}_n/(\Lambda_n^+)\]

with \((\Lambda_n^+)\) denoting the ideal in \(\mathcal{P}_n\) generated by the symmetric polynomials of positive degree. Via these identifications, the divided differences \(\partial_w\) for \(w \in S_n\) make sense as an operators on \(H^\ast(\text{Fl}(n),\mathbb{Z})\), since \(\partial_i\) maps \((\Lambda_n^+)\) into itself. Bernstein, Gelfand, and Gelfand [3] show that

\[
\partial_s[X_w] = \begin{cases} [X_{ws}] & \text{if } s \in \text{Des}_R(w) \\ 0 & \text{otherwise} \end{cases} \quad \text{for } w \in S_n \text{ and } s \in \{s_i : i \in \mathbb{P}\}.
\]

Consequently, once one fixes a polynomial representing \([X_{w_n}]\) (the class of a point), representatives for all \([X_w]\) are determined by induction. Lascoux and Schützenberger [35] have shown that the Schubert polynomials are representatives of the Schubert classes formed in precisely this way:

**Theorem 2.14** (Lascoux and Schützenberger [35]). For all \(w \in S_n\) it holds that \(\mathcal{S}_w \equiv [X_w]\).
2.5 Stanley symmetric functions

Let \( \Lambda = \Lambda(x) \) be the algebra of symmetric functions over \( \mathbb{Z} \) in the variables \( x = \{ x_i : i \in \mathbb{P} \} \).

We follow the standard conventions from [11] for referring to the various well-known bases of this algebra.

Recall the definition of the Stanley symmetric function \( F_w \) from [14]. Stanley [50] was the first to consider this power series and prove that it belongs to \( \Lambda \). In this section we review an alternate definition due to Edelman and Greene [11] which makes this fact more transparent and explains the connection between \( F_w \) and the problem of counting reduced words.

**Remark.** Following Lam [37, 38], our conventions for \( F_w \) differ from Stanley’s original definition by the transformation \( w \leftrightarrow w^{-1}; \) [37, Corollary 2.2] is helpful for understanding these transformations.

Let \( T \) be a (Young) tableau, i.e., an assignment of positive integers to the cells of the diagram of a partition (or, more generally, to the cells of some sequence of partitions or skew shapes or shifted shapes), called the shape of \( T \). Say that \( T \) is strict if its entries are strictly increasing both from left to right in each row and from top to bottom in each column. A strict tableau is standard if its entries comprise the set \( [n] \) for some \( n \in \mathbb{N} \). The reverse reading word of \( T \), denoted \( rrw(T) \), is the word obtained by reading the rows of \( T \) from right to left, starting with the top row. For example,

\[
T = \begin{array}{ccc}
1 & 2 & 3 \\
2 & & 3
\end{array}
\]

has \( rrw(T) = (3, 2, 1, 3, 2) \). A tableau is reduced for \( w \in S_\infty \) if it is strict and its reverse reading word is a reduced word for \( w \), where we identify a sequence of positive integers \( (i_1, i_2, \ldots, i_k) \) with the word \( (s_{i_1}, s_{i_2}, \ldots, s_{i_k}) \). The tableau \( T \) above is reduced for \( w = s_3 s_2 s_1 s_3 s_2 = [4, 3, 1, 2] \). Results of [11] show that we may alternatively define the Stanley symmetric function \( F_w \) as follows:

**Theorem 2.15** (Edelman and Greene [11]). If \( w \in S_\infty \) then \( F_w = \sum_\lambda \alpha_{w,\lambda} s_\lambda \in \Lambda \) where the sum is over partitions \( \lambda \) and \( \alpha_{w,\lambda} \) is the number of reduced tableaux for \( w \) of shape \( \lambda \).

Recall that \( f^\lambda \) is the number of standard tableaux of shape \( \lambda \).

**Theorem 2.16** (Edelman and Greene [11]). If \( w \in S_\infty \) then \( |\mathcal{R}(w)| = \sum_\lambda \alpha_{w,\lambda} f^\lambda \).

Edelman and Greene [11] provide bijective proofs of these identities using a variant of the RSK correspondence, now referred to as Edelman-Greene insertion. The latter map gives an algorithm for calculating \( F_w \) for any \( w \in S_\infty \); other, more efficient methods of computation are described in [17], [22], [36], [39]. For our purposes, it will suffice to recall one exact formula [41, Proposition 2.4].

**Proposition 2.17** (Billey, Jockusch, Stanley [4]). If the Rothe diagram of \( w \in S_n \) is equivalent to a skew shape \( \lambda/\mu \), then \( F_w = s_{\lambda/\mu} \) is the corresponding skew Schur function.

A permutation \( w \in S_n \) is vexillary if it is 2143-avoiding or, equivalently, if its Rothe diagram is equivalent to the diagram of a partition [11, Proposition 2.2.7]. We also recall the following result, which derives from either [40, Eq. (7.24)(iii)] or [50, Theorem 4.1].

**Theorem 2.18** (Macdonald [40]; Stanley [50]). The Stanley symmetric function \( F_w \) is a Schur function if and only if \( w \) is vexillary, in which case \( F_w = s_{\lambda(w)} \).

**Example 2.19.** The reverse permutation \( w_n \in S_n \) is vexillary with \( \lambda(w_n) = \delta_n \), so \( F_{w_n} = s_{\delta_n} \) and via Theorem 2.16 we recover the result of Stanley [50] that \( |\mathcal{R}(w_n)| = f^{\delta_n} \).

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3 Involution words for symmetric groups

As in Section 2.1 we write $\mathcal{I}(S_{\infty})$ and $\mathcal{I}(S_n)$ for the sets of involutions in $S_{\infty}$ and $S_n$. In addition, we let $\mathcal{I}_{\text{FPF}}(S_{2n})$ be the set of fixed-point-free involutions in $S_{2n}$, and set $\mathcal{I}_{\text{FPF}}(S_{\infty}) = \bigcup_{n \in \mathbb{P}} \mathcal{I}_{\text{FPF}}(S_{2n})$. Throughout this section, we write $g_n$ for the Grassmannian involution

$$g_n = (1, n)(2, n + 1) \cdots (n, 2n) = [n + 1, n + 2, \ldots, 2n, 1, 2, \ldots, n] \in \mathcal{I}_{\text{FPF}}(S_{2n}). \quad (3.1)$$

Recall that $\kappa(y)$ is the number of 2-cycles in an involution $y$. We have

$$\hat{\ell}(y) = \frac{1}{2} (\ell(y) + \kappa(y)) \quad \text{and} \quad \hat{\ell}_{\text{FPF}}(z) = \frac{1}{2} (\ell(z) - \kappa(z))$$

for $y \in \mathcal{I}(S_n)$ and $z \in \mathcal{I}_{\text{FPF}}(S_{2n})$, where $\hat{\ell}(y)$ is as in (2.3) and $\hat{\ell}_{\text{FPF}}(z) \overset{\text{def}}{=} \hat{\ell}(v_n, z) = \hat{\ell}(z) - n$.

3.1 Atoms for permutations

Recall the sets $\mathcal{A}(y, z)$ from Theorem-Definition 1.1. As in the introduction, we write

$$\mathcal{A}_{\text{FPF}}(z) = \mathcal{A}(v_n, z) \quad \text{for} \quad z \in \mathcal{I}_{\text{FPF}}(S_{2n}),$$

where as usual $v_n = (1, 2)(3, 4) \cdots (2n-1, 2n)$, so that $\hat{\mathcal{R}}_{\text{FPF}}(z) = \bigcup_{u \in \mathcal{A}_{\text{FPF}}(z)} \hat{\mathcal{R}}(u)$.

As noted earlier, Can, Joyce, and Wyser [3, 7] have recently studied the sets $\mathcal{A}(y)$ and $\mathcal{A}_{\text{FPF}}(y)$ for $y \in \mathcal{I}(S_{\infty})$; they provide a useful set of conditions, involving only the one-line representations of permutations, which classify their elements. (Several left/right-handed conventions in [6, 7] are the mirror images of the ones we adopt here, so the elements in the sets described by the main results [7, Theorem 2.5 and Corollary 2.16] are actually the inverses of what we call atoms.) The sets $\mathcal{A}(y)$ and $\mathcal{A}_{\text{FPF}}(y)$ may also be viewed as special cases of the sets $W(Y)$ that Brion defines geometrically in [5, §1.1]. These sets of atoms have several special properties which do not generalize to other Coxeter groups, which we discuss in the complementary paper [23]. We require some results from that work, which we quote as follows. The following combines [23, Corollaries 6.11 and 6.23].

Theorem 3.1 (See [23]). Let $y \in \mathcal{I}(S_{\infty})$. Then $|\mathcal{A}(y)| = 1$ if and only if $y$ is 321-avoiding. Likewise, if $y$ is fixed-point-free, then $|\mathcal{A}_{\text{FPF}}(y)| = 1$ if and only if $y$ is 321-avoiding.

We can identify one atom of any involution $y$ by the following construction. Given a list $[c_1, c_2, \ldots]$, we write $[[c_1, c_2, \ldots]]$ for the sublist formed by omitting repeated entries after their initial occurrence. For example, $[[1, 1, 3, 4, 2, 4, 3, 2, 4]] = [1, 3, 4, 2]$. If the set of distinct elements in $[c_1, c_2, \ldots]$ is $\{1, 2, \ldots, n\}$, then interpret $[[c_1, c_2, \ldots]]$ as the one-line representation of a permutation in $S_n$. Now, for $y \in \mathcal{I}(S_n)$, we define $\alpha_{\text{min}}(y)$ and $\beta_{\text{min}}(y)$ as the permutations in $S_n$ given by

$$\alpha_{\text{min}}(y) = [[b_1, a_1, b_2, a_2, \ldots, b_k, a_k]]^{-1} \quad \text{and} \quad \beta_{\text{min}}(y) = [[a_1, b_1, a_2, b_2, \ldots, a_k, b_k]]^{-1}$$

where $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ are the elements of the set $\{(a, b) \in [n] \times [n] : a \leq b = w(a)\}$, indexed such that $a_1 < a_2 < \cdots < a_k$.

Example 3.2. If $y = [4, 7, 3, 1, 6, 5, 2] = (1, 4)(2, 7)(5, 6)$ then

$$\alpha_{\text{min}}(y) = [[4, 1, 7, 2, 3, 3, 6, 5]]^{-1} = [2, 4, 5, 1, 7, 6, 3] \quad \text{and} \quad \beta_{\text{min}}(y) = [1, 3, 5, 2, 6, 7, 4].$$

In turn, one computes that
Proposition 2.4], and we have

\[ \alpha_{\min}(g_k) = [2, 4, \ldots, 2k, 1, 3, \ldots, 2k - 1] \text{ and } \beta_{\min}(g_k) = [1, 3, \ldots, 2k - 1, 2, 4, \ldots, 2k]. \]

(b) \( \alpha_{\min}(w_{2k}) = [2, 4, \ldots, 2k, 2k - 1, \ldots, 3, 1] \) and \( \beta_{\min}(w_{2k}) = [1, 3, \ldots, 2k - 1, 2k, \ldots, 4, 2]. \)

Fix \( y \in \mathcal{I}(S_\infty) \) and \( z \in \mathcal{I}_{\FPF}(S_\infty). \) The following is a corollary of results in [7], and is included explicitly in [23] Theorems 6.10 and 6.22.

**Proposition 3.3** (See [23]). The permutations \( \alpha_{\min}(y) \) and \( \beta_{\min}(z) \) are the unique lexicographically minimal elements of \( A(y) \) and \( A_{\FPF}(z) \), respectively.

**Corollary 3.4.** If \( y \) (respectively, \( z \)) is 321-avoiding, then so is \( \alpha_{\min}(y) \) (respectively, \( \beta_{\min}(z) \)).

**Proof.** The set of 321-avoiding permutations is an order ideal under the left weak order (see [54] Proposition 2.4), and we have \( u <_L \pi \) whenever \( u \in A(\sigma, \pi) \) by inspection.

### 3.2 Diagrams and codes for involutions

For involutions \( y \in \mathcal{I}(S_\infty) \), we define

\[
\hat{D}(y) = \{(i, j) \in \mathbb{P} \times \mathbb{P} : j < y(i) \text{ and } i < y(j) \text{ and } j \leq i\},
\]

\[
\hat{D}_{\FPF}(y) = \{(i, j) : \hat{D}(y) : j < i\}.
\]

Call these sets *involution Rothe diagrams*. Observe that \( \hat{D}(y) \) and \( \hat{D}_{\FPF}(y) \) are the subsets of positions in the usual Rothe diagram \( D(y) \) that are weakly and strictly below the diagonal. Since \( D(y) \) is invariant under transpose as \( y^2 = 1 \), the diagram \( \hat{D}(y) \) uniquely determines \( D(y) \).

**Example 3.5.** If \( y = [4, 7, 3, 1, 6, 5, 2] \) then

\[
D(y) = \begin{pmatrix}
\circ & \circ & \circ & \circ & . & . & . \\
\circ & \circ & \circ & . & . & . & . \\
\circ & \circ & . & . & . & . & . \\
\circ & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & .
\end{pmatrix}
\]

and

\[
\hat{D}(y) = \begin{pmatrix}
\circ & . & . & . & . & . & . \\
\circ & . & . & . & . & . & . \\
\circ & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & .
\end{pmatrix}
\]

One similarly computes that

\[
\hat{D}(g_n) = \{(i, j) : 1 \leq j \leq i \leq n\} \quad \text{and} \quad \hat{D}(w_n) = \{(i, j) \in \mathbb{P} \times \mathbb{P} : i + j \leq n \text{ and } j \leq i\},
\]

which are the transposes of the shifted shapes of \((n, n - 1, \ldots, 2, 1)\) and \((n - 1, n - 3, n - 5, \ldots)\).

We discuss a few results that indicate why \( \hat{D}(y) \) and \( \hat{D}_{\FPF}(z) \) are the appropriate notions of diagrams for involutions. Recall that the cardinality of \( D(w) \) is the number of inversions of \( w \in S_\infty \), and so \( \ell(w) = |D(w)| \). An analogous fact holds for involution Rothe diagrams:

**Proposition 3.6.** If \( y \in \mathcal{I}(S_n) \) and \( z \in \mathcal{I}_{\FPF}(S_{2n}) \) then \( \hat{\ell}(y) = |\hat{D}(y)| \) and \( \hat{\ell}_{\FPF}(z) = |\hat{D}_{\FPF}(z)| \).

**Proof.** When \( y \) is an involution, \( D(y) \) is transpose-invariant and the number of diagonal positions \( (i, i) \in D(y) \) is precisely \( \kappa(w) \), so it follows that \( |\hat{D}(y)| = \frac{1}{2}(|D(y)| - \kappa(y)) \) plus \( \kappa(y) = \hat{\ell}(y) \). If \( z \) is a fixed-point-free involution, then \( |\hat{D}_{\FPF}(z)| = |\hat{D}(z)| - \kappa(z) = \hat{\ell}(z) - \kappa(z) = \hat{\ell}_{\FPF}(z) \).
Recall the definition of the code \( c(w) \in \mathbb{Z}^n \) from \([2,5]\). We define the \emph{involution codes}
\[
\hat{c}(y) = (\hat{c}_1(y), \hat{c}_2(y), \ldots, \hat{c}_n(y)) \quad \text{and} \quad \hat{c}_{FPF}(y) = (\hat{c}_{FPF,1}(y), \hat{c}_{FPF,2}(y), \ldots, \hat{c}_{FPF,n}(y))
\]
for \( y \in I(S_n) \) as the integer sequences with
\[
\hat{c}_i(y) = |\{ j \in [n] : y(j) \leq i < j \text{ and } y(i) > y(j) \}|,
\]
\[
\hat{c}_{FPF,i}(y) = |\{ j \in [n] : y(j) < i < j \text{ and } y(i) > y(j) \}|.
\]
The \( i \)th entries in these sequences count the number of cells in the \( i \)th rows of \( \hat{D}(y) \) and \( \hat{D}_{FPF}(y) \), respectively. These sequences do not depend in any serious way on \( n \): if \( y \) is viewed as belonging to a larger symmetric group \( S_N \supset S_n \), then the resulting codes are the same, extended by zeros.

**Example 3.7.** If \( y = [4,7,3,1,6,5,2] = (1,4)(2,7)(5,6) \in I(S_7) \) then
\[
\hat{c}(y) = (1,2,0,2,1,0) \quad \text{and} \quad \hat{c}_{FPF}(y) = (0,1,2,0,1,1,0).
\]

For the involutions \( g_n, w_{2n} \in I_{FPF}(S_{2n}) \) we have:
(a) \( \hat{c}(g_n) = (1,2,3,\ldots,n,0,\ldots,0) \) and \( \hat{c}_{FPF}(g_n) = (0,1,2,\ldots,n-1,0,\ldots,0) \).
(b) \( \hat{c}(w_{2n}) = (1,2,\ldots,n,2,1,0) \) and \( \hat{c}_{FPF}(w_{2n}) = (0,1,2,\ldots,n-1,2,1,0,0) \).

Recall the minimal atoms \( \alpha_{\min}(y) \in A(w) \) and \( \beta_{\min}(y) \in A_{FPF}(y) \) defined in Section 3.1.

**Lemma 3.8.** If \( y \in I(S_n) \) then \( \hat{c}(y) = c(\alpha_{\min}(y)) \) and \( \hat{c}_{FPF}(y) = c(\beta_{\min}(y)) \).

**Proof.** Define \( (a_i, b_i) \) relative to \( y \) as before Example 3.2 and fix \( i \in [k] \), where \( k \) is the largest descent in \( y \). It is straightforward to check from the definitions that
\[
c_{a_i}(\beta_{\min}(y)) = |\{ j \in [i - 1] : b_j > a_i \}| \quad \text{and} \quad c_{b_i}(\beta_{\min}(y)) = |\{ j \in [i - 1] : b_j > b_i \}|.
\]
On the other hand, since \( y(t) \leq t \) implies \( y(t) = a_j \) and \( t = b_j \) for some \( j \), and \( a_j < a_i \) if and only if \( j < i \), it follows that
\[
\hat{c}_{FPF,a_i}(y) = |\{ j \in [n] : y(j) < a_i < j \text{ and } y(a_i) > y(j) \}|
\]
\[
= |\{ j \in [k] : a_j < a_i < b_j \text{ and } b_i > a_j \}| = |\{ j \in [i - 1] : b_j > a_i \}| = c_{a_i}(\beta_{\min}(y))
\]
and likewise
\[
\hat{c}_{FPF,b_i}(y) = |\{ j \in [n] : y(j) < b_i < j \text{ and } y(b_i) > y(j) \}|
\]
\[
= |\{ j \in [k] : a_j < b_i < b_j \text{ and } a_i > a_j \}| = |\{ j \in [i - 1] : b_j > b_i \}| = c_{b_i}(\beta_{\min}(y)).
\]
As every \( t \in [n] \) is given by \( a_j \) or \( b_j \) for some \( j \in [k] \), we conclude that \( \hat{c}_{FPF}(y) = c(\beta_{\min}(y)) \). The proof that \( \hat{c}(y) = c(\alpha_{\min}(y)) \) is similar; we omit the details. \( \square \)

One can read off an involution word for \( y \in I(S_n) \) from its involution code in the following way. For any sequence \( c = (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n \), define
\[
\Theta(c) = (\underbrace{c_1, \ldots, 2, 1}_{c_1 \text{ terms}}, \underbrace{c_2 + 1, \ldots, 3, 2}_{c_2 \text{ terms}}, \underbrace{c_3 + 2, \ldots, 4, 3}_{c_3 \text{ terms}}, \ldots, \underbrace{c_n + n - 1, \ldots, n, n, n}_{c_n \text{ terms}}).
\]
In what follows, identify integer sequences \((i_1, \ldots, i_k)\) with words \((s_{i_1}, \ldots, s_{i_k})\).

**Proposition 3.9.** If \( y \in I(S_n) \) and \( z \in I_{FPF}(S_{2n}) \) then \( \Theta(\hat{c}(y)) \in \hat{R}(y) \) and \( \Theta(\hat{c}_{FPF}(z)) \in \hat{R}_{FPF}(z) \).

**Proof.** By \([41]\) Remark 2.1.9, if \( u \) is any permutation then \( \Theta(c(u)) \) gives a reduced word for \( u \), so this result follows from Lemma 3.8. \( \square \)
3.3 Involution Schubert polynomials

Recall the definition of $\mathcal{S}_w$ for $w \in S_\infty$ from Section 2.3. In this section we turn to the involution Schubert polynomials $\hat{\mathcal{S}}_y$ and $\hat{\mathcal{S}}^{FPF}_y$ defined in the introduction. Recall for $y, z \in \mathcal{I}(S_\infty)$ we define the involution Schubert polynomial

$$\hat{\mathcal{S}}_{y,z} = \sum_{u \in \mathcal{A}(y,z)} \mathcal{S}_u \in \mathcal{P}_\infty.$$  

We abbreviate by setting $\hat{\mathcal{S}}_y = \hat{\mathcal{S}}_{1,y}$ and $\hat{\mathcal{S}}^{FPF}_z = \hat{\mathcal{S}}_{v_{n-z}}$ for $z \in \mathcal{I}^{FPF}(S_{2n})$.

It follows immediately from Theorem-Definition 1.1 that these definitions of $\hat{\mathcal{S}}_w$ and $\hat{\mathcal{S}}^{FPF}_w$ are consistent with our earlier definitions in the introduction. Write $\Lambda_n$ for the ring of symmetric polynomials in $\mathcal{P}_n = \mathbb{Z}[x_1, \ldots, x_n]$. For the Grassmannian involution $g_k$ given by (3.1) we can compute $\hat{\mathcal{S}}_{g_k}$ and $\hat{\mathcal{S}}^{FPF}_{g_k}$ directly:

**Proposition 3.10.** For each $k \in \mathbb{N}$ it holds that

$$\hat{\mathcal{S}}_{g_k} = 2^{-k} \prod_{1 \leq i < j \leq k} (x_i + x_j) \in \Lambda_k \quad \text{and} \quad \hat{\mathcal{S}}^{FPF}_{g_k} = \prod_{1 \leq i < j \leq k} (x_i + x_j) \in \Lambda_k.$$  

**Proof.** By Proposition 2.13 and the discussion in Section 3.1 we have $\hat{\mathcal{S}}_{g_k} = s_{\delta_{k+1}}(x_1, \ldots, x_k)$ and $\hat{\mathcal{S}}^{FPF}_{g_k} = s_{\delta_k}(x_1, \ldots, x_k)$. Checking that these Schur polynomials have the given product formulas is a simple exercise from Jacobi’s determinantal definition of $s_\lambda$; see [41, Exercise 1.2.4]. \(\square\)

Involution Schubert polynomials may also be characterized along the lines of Theorem 2.8 as follows. We write $\delta_{v,w}$ to denote the Kronecker delta function, equal to 1 if $v = w$ and 0 otherwise. Recall the definition of the two-sided weak order $<_T$ from Section 2.1.

**Theorem 3.11.** Fix $y \in \mathcal{I}(S_\infty)$. Then $\{\hat{\mathcal{S}}_{y,z}\}_{z \in \mathcal{I}(S_\infty)}$ is the unique family of homogeneous polynomials indexed by $\mathcal{I}(S_\infty)$ such that

$$\hat{\mathcal{S}}_{y,z} = \delta_{y,z} \quad \text{if} \quad y \not<_T z \quad \text{and} \quad \partial_i \hat{\mathcal{S}}_{y,z} = \begin{cases} \hat{\mathcal{S}}_{y,z \times s_i} & \text{if} \ s_i \in \text{Des}_R(z) \\ 0 & \text{otherwise} \end{cases} \quad \text{for all} \ i \in \mathbb{P}. $$

**Proof.** We first claim that $\hat{\mathcal{S}}_{y,z}$ has these properties. The polynomials $\hat{\mathcal{S}}_{y,z}$ are homogeneous since the Schubert polynomial $\mathcal{S}_u$ is homogeneous of degree $\ell(u)$. That $\hat{\mathcal{S}}_{y,z} = \delta_{y,z}$ if $y \not<_T z$ follows from the definition of $\mathcal{A}(y, z)$. The given formula for $\partial_i \hat{\mathcal{S}}_{y,z}$, finally, is straightforward to check from our original definition of $\hat{\mathcal{S}}_{y,z}$ using Proposition 2.4 and Theorem 2.8.

For the uniqueness assertion, suppose $\{f_z\}_{z \in \mathcal{I}(S_\infty)}$ is another family of homogeneous polynomials with the properties of $\hat{\mathcal{S}}_{y,z}$ described in the theorem. We proceed as in the proof of [32, Theorem 2.3]. By hypothesis $f_z = \hat{\mathcal{S}}_{y,z} = \delta_{y,z}$ if $y \not<_T z$, so assume that $y <_T z$ and that $f_u = \hat{\mathcal{S}}_{u,y}$ if $u \in \mathcal{I}(S_\infty)$ is such that $\ell(u) < \ell(z)$. Then $\partial_i f_z = \partial_i \hat{\mathcal{S}}_{y,z}$ for all $i \in \mathbb{P}$, so we deduce that $f_z = \hat{\mathcal{S}}_{y,z} + a_y$ for some $a_y \in \mathbb{Z} = \bigcap_{i \in \mathbb{P}} \ker \partial_i$. Since $f_z$ and $\hat{\mathcal{S}}_{y,z}$ are both homogeneous and since $\hat{\mathcal{S}}_{y,z}$ has degree $\ell(y, z) > 0$, the constant $a_z$ must be zero so $f_z = \hat{\mathcal{S}}_{y,z}$ as desired. \(\square\)

By induction, we may express $\hat{\mathcal{S}}_y$ in terms of divided differences in the following way.

**Corollary 3.12.** Let $y, z \in \mathcal{I}(S_n)$. Then $\hat{\mathcal{S}}_{y,z} = \partial_u \hat{\mathcal{S}}_{y,u}$ for any $u \in \mathcal{A}(z, w_n)$.  

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For technical reasons we need a slightly different version of Theorem 3.11 to characterize the fixed-point-free involution Schubert polynomials \( \hat{S}_z^{\text{FPF}} \). Define \( \hat{I}_{\text{FPF}} \) as the set of fixed-point-free involutions \( w : \mathbb{P} \to \mathbb{P} \) with the property that, for some sufficiently large positive integer \( N \), it holds that \( w(2i - 1) = 2i \) and \( w(2i) = 2i - 1 \) for all \( i > N \). Note that each such permutation has infinite support. If \( n \) is finite and \( z \in \hat{I}_{\text{FPF}}(S_{2n}) \) then let \( z_\infty \) be the infinite product

\[
\tag{3.2}
z_\infty = z \cdot s_{2n+1} \cdot s_{2n+3} \cdot s_{2n+5} \cdots \in \hat{I}_{\text{FPF}}.
\]

We also let \( 1_{\text{FPF}} = 1_\infty \in \hat{I}_{\text{FPF}} \) be the map \( \mathbb{P} \to \mathbb{P} \) with \( 2i - 1 \mapsto 2i \) and \( 2i \mapsto 2i - 1 \) for all \( i \). Although no elements of \( \hat{I}_{\text{FPF}} \) belong to \( S_\infty \), we define \( w \times s_i \) for \( w \in \hat{I}_{\text{FPF}} \) exactly as if \( w \) were in \( S_\infty \), and we define the (now infinite) set \( \text{Des}_R(w) \) again by (2.4).

If \( z \in \hat{I}_{\text{FPF}}(S_{2n}) \), so that \( zs_{2n+1} \in \hat{I}_{\text{FPF}}(S_{2n+2}) \), then it follows either as a straightforward exercise or from the more general statement [23, Lemma 3.2] that \( A(v_n, z) = A(v_{n+1}, z s_{2n+1}) \) and therefore \( \hat{S}_z^{\text{FPF}} = \hat{S}_{zs_{2n+1}}^{\text{FPF}} \). By induction, if \( y \in \hat{I}_{\text{FPF}}(S_{2n}) \) and \( z \in \hat{I}_{\text{FPF}}(S_{2n}) \), then \( \hat{S}_y^{\text{FPF}} = \hat{S}_z^{\text{FPF}} \) whenever \( y_\infty = z_\infty \). It is therefore well-defined to set

\[
\hat{S}_z^{\text{FPF}} = \hat{S}_z = \hat{S}_{v_n, z} \quad \text{for any } z \in \hat{I}_{\text{FPF}}(S_{2n}) \text{ and } n \in \mathbb{P}.
\]

Since \( \hat{I}_{\text{FPF}} = \bigcup_{n \in \mathbb{P}} \{ z_\infty : z \in \hat{I}_{\text{FPF}}(S_{2n}) \} \), this defines \( \hat{S}_w^{\text{FPF}} \) for every \( w \in \hat{I}_{\text{FPF}} \).

**Corollary 3.13.** It holds that \( \{ \hat{S}_z^{\text{FPF}} \}_{z \in \hat{I}_{\text{FPF}}} \) is the unique family of homogeneous polynomials indexed by \( \hat{I}_{\text{FPF}} \) such that

\[
\hat{S}_1^{\text{FPF}} = 1 \quad \text{and} \quad \partial_i \hat{S}_z^{\text{FPF}} = \begin{cases} \hat{S}_z \times s_i & \text{if } s_i \in \text{Des}_R(z) \text{ and } z \times s_i \in \hat{I}_{\text{FPF}} \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } i \in \mathbb{P}.
\]

**Proof.** Note that if \( z \in \hat{I}(S_{2n}) \) is not fixed-point-free then it cannot hold that \( v_n \leq_T z \) (since \( y \leq_T z \) implies that \( y \) has at least as many fixed points as \( z \)) so \( \hat{S}_{v_n, z} = 0 \). From this observation, the assertion that the polynomials \( \{ \hat{S}_z^{\text{FPF}} \}_{z \in \hat{I}_{\text{FPF}}} \) have the given properties is an easy exercise from Theorem 3.11. The uniqueness assertion follows by the same argument as the one in the proof of Theorem 3.11 *mutatis mutandis*, after replacing the involution length \( \ell \) which is not well-defined on \( \hat{I}_{\text{FPF}} \) by the function \( \hat{\ell}_{\text{FPF}} : \hat{I}_{\text{FPF}} \to \mathbb{N} \) given by \( \hat{\ell}_{\text{FPF}}(z_\infty) = \hat{\ell}(v_n, z) \) for \( z \in \hat{I}_{\text{FPF}}(S_{2n}) \).

Given a sequence \( i = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n \), we let \( x^i = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \) and write \( x^i \prec_{\text{lex}} x^j \) when \( i <_{\text{lex}} j \in \mathbb{N}^n \), where \( \prec_{\text{lex}} \) denotes the lexicographic order on sequences. A different convention is sometimes taken to define lexicographic order on monomials, as we explain in the following remark.

**Remark.** Viewing \( x^i \) as a word in the alphabet \( \{ x_1, x_2, \ldots \} \) defines a sequence

\[
\Psi(x^i) = (1, \underbrace{2, \ldots, 2}_{i_1 \text{ times}}, \underbrace{\ldots, \ldots, \ldots}_{i_2 \text{ times}}, \underbrace{\ldots, \ldots, \ldots}_{i_n \text{ times}}).
\]

For example, \( \Psi(x^{(1,1,1)}) = \Psi(x_1 x_2 x_3) = (1, 2, 3) \) and \( \Psi(x^{(0,2,1)}) = \Psi(x_2 x_3) = (2, 2, 3) \). One checks that on the set of monomials of any fixed degree, the map \( x^i \mapsto \Psi(x^i) \) reverses lexicographic order; e.g., we have \( x_2^2 x_3 \prec_{\text{lex}} x_1 x_2 x_3 \) but \( (1, 2, 3) \prec_{\text{lex}} (2, 2, 3) \). For this reason, the order \( \prec_{\text{lex}} \) that we have defined on monomials is sometimes (e.g., in [2]) referred to as the reverse lexicographic order, though for us this order is the usual lexicographic total order.
We now show how to read the involution codes $\hat{c}(y)$ and $\hat{c}_{\text{FPF}}(z)$ from the polynomials $\hat{S}_y$ and $\hat{S}_z^{\text{FPF}}$. As these codes determine $y \in \mathcal{I}(S_{\infty})$ and $z \in \mathcal{I}_{\text{FPF}}(S_{\infty})$, one can reconstruct the index of an involution Schubert polynomial from the polynomial itself.

**Proposition 3.14.** Let $y \in \mathcal{I}(S_{\infty})$ and $z \in \mathcal{I}_{\text{FPF}}(S_{\infty})$. The lexicographically least monomials in the involution Schubert polynomials $\hat{S}_y$ and $\hat{S}_z^{\text{FPF}}$ are $x^{\hat{c}(y)}$ and $x^{\hat{c}_{\text{FPF}}(z)}$ respectively.

**Proof.** One checks that if $u, u' \in S_{\infty}$ with $u <_{\text{lex}} u'$ (interpreting $u$ and $u'$ in one-line notation), then $c(u) <_{\text{lex}} c(u')$. It therefore suffices by Proposition 3.3 and Lemma 3.8 to show that $x^{c(u)}$ is the lexicographically least monomial in $\hat{S}_u$. This property is clear from the proof of [2 Corollary 3.9], on noting that $x^{c(u)} = x_{D_{\text{bot}}(u)}$ in the notation of [2 §3], and that $<_{\text{lex}}$ is what the authors of [2] call the reverse lexicographic order on monomials. \[\square\]

### 3.4 Cohomology of flag varieties revisited

Recall the notation of Section 2.4. Throughout this section, we let $\beta$ be a non-degenerate bilinear form on $\mathbb{C}^n$ which is symmetric or skew-symmetric, and define $K \subset \text{GL}_n(\mathbb{C})$ as the subgroup of matrices preserving $\beta$. The group $K$ is given by the orthogonal group $O(n)$ when $\beta$ is symmetric, and by the symplectic group $\text{Sp}(n)$ when $\beta$ is skew-symmetric (which can only occur if $n$ is even).

As explained in [17 §10], the orbits of the symmetric subgroups $O(n)$ and $\text{Sp}(n)$ on $\text{Fl}(n)$ are naturally indexed by $\mathcal{I}(S_n)$ and $\mathcal{I}_{\text{FPF}}(n)$, respectively. To refer to these indexing sets, we define

$$\mathcal{I}_{O(n)} = \mathcal{I}(S_n) \quad \text{and} \quad \mathcal{I}_{\text{Sp}(n)} = \mathcal{I}_{\text{FPF}}(S_n).$$

The corresponding $K$-orbits may then be described by rank conditions analogous to the ones in Section 2.8 for Schubert varieties. Explicitly, we may define a $K$-orbit associated to an involution $y \in \mathcal{I}_K$ by

$$\hat{Y}_y^K = \left\{ F_\bullet : \text{rank} \left( \beta|_{F_i \times F_j} \right) = \text{rk}_y(i, j) \text{ for each } i, j \in [n] \right\},$$

where $\beta|_{F_i \times F_j}$ denotes the linear map $F_i \to F_j^*$ given by $v \mapsto \beta(v, \cdot)$, and $\text{rk}_y(i, j)$ is as in (2.7). It is not hard to see that $\hat{Y}_y^K$ is $K$-stable, and that $\text{Fl}(n)$ is the disjoint union $\bigcup_{y \in \mathcal{I}_K} \hat{Y}_y$. Wyser discusses why $\hat{Y}_y^K$ is actually a single $K$-orbit in [54 §2.1.2].

Let $Y_y^K$ denote the Zariski closure of $\hat{Y}_y^K$ as in Section 2.4. Then $Y_y^K$ is again defined by rank conditions; namely (see [54 Proposition 2.4])

$$Y_y^K = \left\{ F_\bullet : \text{rank} \left( \beta|_{F_i \times F_j} \right) \leq \text{rk}_y(i, j) \text{ for each } i, j \in [n] \right\}. \quad (3.3)$$

Recall from (2.8) that the Schubert variety $X_w$ for $w \in S_n$ is the set of complete flags $F_\bullet$ such that $\dim (\text{proj } F_i \to V_j) \leq \text{rk}_w(i, j)$ for each $(i, j) \in [n] \times [n]$. Fulton shows in [15] that a proper subset of these rank conditions actually imply all of the rest. Specifically, to determine $X_w$ one only needs the conditions corresponding to pairs $(i, j)$ in the *essential set* of the Rothe diagram $D(w)$. In general, the essential set of a diagram $D \subset \mathbb{P} \times \mathbb{P}$ is

$$\text{Ess}(D) = \{(i, j) \in D : (i + 1, j), (i, j + 1), (i + 1, j + 1) \notin D\}.$$ 

Observe that $\text{Ess}(D)$ is the set of southeast corners of the connected components of $D$.

**Example 3.15.** If $y = [4, 7, 3, 1, 6, 5, 2]$ as in Example 3.5 then

$$\text{Ess}(D(y)) = \{(2, 3), (2, 6), (3, 2), (5, 6), (6, 2)\} \quad \text{and} \quad \text{Ess}(\hat{D}(y)) = \{(2, 3), (2, 6), (5, 6)\}.$$
The rank conditions \((3.3)\) giving \(Y^K_y\) admit the following analogous simplification.

**Proposition 3.16.** The \(K\)-orbit closure associated to an involution \(y \in I_K\) satisfies

\[
Y^K_y = \{ F_S : \text{rank} (\beta|_{F_S \times F_S}) \leq \text{rk}_y(i,j) \text{ for each } (i,j) \in \text{Ess}(D) \}
\]

where \(D = \hat{D}(y)\) when \(K = O(n)\) and \(D = \hat{D}_{	ext{PPF}}(y)\) when \(K = \text{Sp}(n)\).

**Proof.** Fix \(y \in I_K\) and let \(C_{ij}\) denote the set of complete flags \(F_S \in \text{Fl}(n)\) satisfying the condition \(\text{rank} (\beta|_{F_S \times F_S}) \leq \text{rk}_y(i,j)\). Since \(y\) is an involution, we have \(C_{ij} = C_{ji}\), so \((3.3)\) implies that \(Y^K_y\) is the intersection of the sets \(C_{ij}\) for \(1 \leq j < i \leq n\). Define the implication graph of \(y\) as the directed graph on \(\{(i,j) : 1 \leq j \leq i \leq n\}\) with an edge from \((i,j)\) to \((k,l)\) if the two cells are adjacent in the same row or column and \(C_{ij} \subset C_{kl}\).

First assume \(K = O(n)\) so that \(D = \hat{D}(y)\). If \((i,j) \in D\), then \(\text{rk}_y(i,j) = \text{rk}_y(i,j - 1) = \text{rk}_y(i,j)\) so \(C_{ij}\) is contained in \(C_{i-1,j}\) and \(C_{1,j-1}\). If \((i,j) \notin D\), then either \(j \geq y(i)\) or \(i \leq y^{-1}(j)\). In the first case, \(\text{rk}_y(i,j) = \text{rk}_y(i,j - 1)\), \(\text{so } C_{i-1,j} \subset C_{ij}\); in the second case, we deduce \(C_{j-1,i} \subset C_{ij}\) by an analogous argument. Thus, each cell in \(D\) is the source of edges in the implication graph going both north and west, while each cell not in \(D\) is the target of an edge going either south or east. It follows that every \((i,j) \in D\) can be reached by a directed path in the implication graph starting at some cell in \(\text{Ess}(D)\), and every \((i,j) \notin D\) can be reached by a directed path starting at either \((1,1)\) or a cell in \(D\). Since \((1,1) \notin D\) only if \(w(1) = 1\) in which case \(C_{1,1} = \text{Fl}(n)\), we conclude that each \(C_{ij}\) contains \(C_{pq}\) for some \((p,q) \in \text{Ess}(D)\), so \(Y^K_y\) is the intersection of the sets \(C_{pq}\) for \((p,q) \in \text{Ess}(D)\), as desired.

Now suppose \(K = \text{Sp}(n)\) so that \(D = \hat{D}_{	ext{PPF}}(y)\). Since in this case \(\beta\) is skew-symmetric and \(y\) is fixed-point-free, the numbers \(\text{rank} (\beta|_{F_S \times F_S})\) and \(\text{rk}_y(i,i)\) are always even and bounded above by \(i\). Hence \(C_{1,1} = \text{Fl}(n)\). We claim, moreover, that \(C_{i,i-1} \subset C_{ii}\) for all \(1 < i \leq n\). To show this, suppose \(F_S \in C_{i,i-1}\). If \(\text{rank} (\beta|_{F_S \times F_S}) = \text{rank} (\beta|_{F_S \times F_S})\) or \(\text{rk}_y(i,i-1) < \text{rk}_y(i,i)\) then clearly \(F_S \in C_{ii}\); otherwise, it must happen that \(\text{rank} (\beta|_{F_S \times F_S}) = \text{rank} (\beta|_{F_S \times F_S}) - 1\) is odd and \(\text{rk}_y(i,i-1) = \text{rk}_y(i,i)\) even, so the strict inequality \(\text{rank} (\beta|_{F_S \times F_S}) < \text{rk}_y(i,i-1)\) holds, which again implies \(F_S \in C_{ii}\). From the claim just shown and \((3.3)\), we deduce that \(Y^K_y\) is the intersection of \(C_{ij}\) for \(1 \leq j < i \leq n\). The proposition thus follows as in the orthogonal case, by considering the implication graph on the set of cells \(\{(i,j) : 1 \leq j < i \leq n\}\) strictly below the diagonal.

As for Schubert varieties, to each orbit closure \(Y^K_y\) there is an associated cohomology class

\[
[Y^K_y] \in H^*(\text{Fl}(n), \mathbb{Z}) \quad \text{for } y \in I_K.
\]

(3.4)

The Borel isomorphism \((2.9)\) identifies these cohomology classes with elements of the quotient \(\mathcal{P}_n / (\Lambda^+ / \mathbb{Z}^n)\). Combining \([54]\) Propositions 2.1 and 2.7 with the discussion in \([55]\) §1.3] gives the following analogue of \((2.10)\) describing the action of the divided difference operators on \([Y^K_y]\).

**Theorem 3.17** (Wyser and Yong \([54, 55]\)). Let \(i \in [n - 1]\) and \(y \in I_K\) and set \(s = s_i\). Then

\[
\partial_i [Y^K_y] = \begin{cases} [Y^K_{y^s}] & \text{if } y \times s = sy < y \\ 2[Y^K_{y^s}] & \text{if } y \times s = ys < y \text{ and } ys \in I_K \\ 0 & \text{if } s \notin \text{Des}_R(y) \text{ or } y \times s \notin I_K. \end{cases}
\]

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Remark. We include a sketch of the proof of this theorem since it does not immediately follow from Wyser and Yong’s results in [54, 55] that \( \partial_t [Y^K_y] = 0 \) in the case when \( y \times s \notin \mathcal{T}_K \). Note that this occurs only if \( K = \text{Sp}(n) \) and \( y \times s = ys \), so that \( ys \) is not fixed-point-free.

Proof sketch. Fix \( i \in \mathbb{N} \). Suppose \( X \) is a subvariety of \( \text{Fl}(n) \), and consider the set (variety, in fact) \( X' \) obtained by replacing each flag \( F_i \in X \) by all flags \( E_i \) with \( E_j = F_j \) for \( j \neq i \) (informally, remove any conditions defining \( X \) that restrict \( F_i \)). If \( \dim X' = \dim X + 1 \), then there will be an integer \( d \) such that, generically, each flag \( E_i \in X' \) arises from \( d \) distinct flags in \( X \), in which case \( \partial_t [X] = d[X'] \). If, alternatively, \( \dim X' \neq \dim X + 1 \), then \( \partial_t [X] = 0 \). For a more detailed justification of these assertions, see [16, Chapter 10].

One can use the rank conditions on \( Y^K_y \) to understand \( (Y^K_y)' \) and the integers \( d \). For example, suppose \( K = \text{Sp}(n) \) and let \( y \in \mathcal{T}_K \) such that \( y \times s \) is not fixed-point-free. Then \( y(i) = i + 1 \). We claim that row \( i \) and column \( i \) of \( \text{Ess}(\mathcal{D}_{\text{FFF}}(y)) \) are both empty. Indeed, row \( i \) of the involution Rothe diagram \( \mathcal{D}_{\text{FFF}}(y) \) is empty by definition, while if \( (k, i) \in \mathcal{D}_{\text{FFF}}(y) \) for some \( k < i \), then we also have \( (k + 1, i) \in \mathcal{D}_{\text{FFF}}(y) \) (for otherwise \( y(k) = i + 1 \)), in which case \( (k, i) \notin \text{Ess}(\mathcal{D}_{\text{FFF}}(y)) \). Thus, any of the rank conditions defining \( Y^K_y \) in Proposition 3.16 that involve \( F_i \) are implied by others that do not, so \( (Y^K_y)' = Y^K_y \) and therefore \( \partial_t [Y^K_y] = 0 \). The other cases follow similarly.

The theorem shows that one may compute polynomial representatives for the cohomology classes \( Y^K_y \) just as for Schubert classes, i.e., by applying divided difference operators to suitable representatives for the longest element \( w_n \in S_n \). Wyser and Yong identify such representatives in [55]. To state their result, recall from the introduction that

\[
\gamma_{w_n}^{O(n)} = \prod_{1 \leq i+j \leq n \atop i < j} (x_i + x_j) \quad \text{and} \quad \gamma_{w_{2n}}^{\text{Sp}(n)} = \prod_{1 \leq i<j \leq 2n \atop i < j} (x_i + x_j).
\]

Wyser and Yong prove the following as [55, Theorem 1.1]:

**Theorem 3.18** (Wyser and Yong [55]). Let \( y \in \mathcal{T}_K \) and choose any \( u, v \in A(y, w_n) \). Then

\[
\gamma_{w_n}^{K} = [Y^K_{w_n}] \quad \text{and} \quad \partial_u \gamma_{w_n}^{K} = \partial_v \gamma_{w_n}^{K}.
\]

Let \( y \in \mathcal{T}_K \) and \( u, v \in A(y, w_n) \). If \( K = \text{O}(n) \) then define \( \gamma_y^{O(n)} = 2^{\kappa(y) - \kappa(w_n)} \partial_u \gamma^{O(n)}_{w_n} \) and if \( K = \text{Sp}(n) \) then define \( \gamma_y^{\text{Sp}(n)} = \partial_u \gamma^{\text{Sp}(n)}_{w_n} \). Theorems 3.17 and 3.18 show that \( \gamma_y^{K} \) is then a representative of \([Y^K_y]\); moreover, these polynomials do not depend on the choice of \( u \), and so are unambiguously indexed by \( \mathcal{T}_K \subset S_n \). Wyser and Yong note in [55, Theorem 1.1] that these representatives are nonnegative integer linear combinations of ordinary Schubert polynomials. We can identify this decomposition explicitly by showing that Wyser and Yong’s polynomials are actually scalar multiples of the involution Schubert polynomials defined in the previous section.

**Theorem 3.19.** Let \( y \in \mathcal{T}(S_n) \) and \( z \in \mathcal{T}_{\text{FFF}}(S_{2n}) \). Choose \( u, v \in A(y, w_n) \) and \( v \in A(z, w_{2n}) \). Then

\[
\hat{\gamma}_y = 2^{-[n/2]} \partial_u \gamma^{O(n)}_{w_n} \quad \text{and} \quad \hat{\gamma}^{\text{FFF}}_z = \partial_v \gamma^{\text{Sp}(2n)}_{w_{2n}}.
\]

Remark. In the following proof, we need to suppress \( n \), so we define \( \gamma_y = 2^{-\kappa(y)} \gamma^{O(n)}_{w_n} \) for \( y \in \mathcal{T}(S_n) \) and \( \gamma^{\text{FFF}}_z = \gamma^{\text{Sp}(2n)}_{w_{2n}} \) for \( z \in \mathcal{T}_{\text{FFF}}(S_{2n}) \), with \( z \) denoting the permutation of \( \mathbb{P} \) defined from \( z \) by (3.2). This gives us a family of polynomials \( \gamma_y \) indexed by elements of \( \mathcal{T}(S_{\infty}) \), and a family
of polynomials \( \Upsilon_z^{\text{FFP}} \) indexed by elements of the set \( \tilde{I}_{\text{FFP}} \) defined in the previous section. The fact that the polynomials \( \Upsilon_y \) and \( \Upsilon_z^{\text{FFP}} \) given in this way are well-defined, independent of our choices of \( n \), is proved by Wyser and Yong as [55] Theorem 1.4.

**Proof of Theorem 3.19.** Note that \( \Upsilon_y = 2^{-(n/2)} \partial_y \Upsilon_{w_n}(n) \) and \( \Upsilon_z^{\text{FFP}} = \partial_z \Upsilon_{w_{2n}}(2n) \) for any \( u \in \mathcal{A}(y, w_n) \) and \( v \in \mathcal{A}(z, w_{2n}) \) by Theorem 3.18. It therefore suffices to argue that \( \{ \Upsilon_y \}_{y \in I(S_\infty)} \) and \( \{ \Upsilon_z^{\text{FFP}} \}_{z \in \tilde{I}_{\text{FFP}}} \) have the properties in Theorem 3.11 and Corollary 3.13 that uniquely characterize \( \{ \hat{\Upsilon}_y \}_{y \in I(S_\infty)} \) and \( \{ \hat{\Upsilon}_z^{\text{FFP}} \}_{z \in \tilde{I}_{\text{FFP}}} \). For this, we first claim for all \( y \in I(S_\infty) \) and \( z \in \tilde{I}_{\text{FFP}} \) and \( i, j \in \mathbb{P} \) that

\[
\partial_i \Upsilon_y = \begin{cases} \Upsilon_{y \times s_i} & \text{if } i \in \text{Des}_R(y) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \partial_j \Upsilon_z^{\text{FFP}} = \begin{cases} \Upsilon_z^{\text{FFP}}(z \times s_j) & \text{if } j \in \text{Des}_R(z) \text{ and } z \times s_j \in \tilde{I}_{\text{FFP}} \\ 0 & \text{otherwise.} \end{cases}
\]

Choose \( n \in \mathbb{P} \) such that \( y \in I(S_n) \) and \( z = w_\infty \) for some \( w \in \tilde{I}_{\text{FFP}}(S_{2n}) \). Then \( \Upsilon_y \in \mathcal{P}_{n-1} \) and \( \Upsilon_z^{\text{FFP}} \in \mathcal{P}_{2n-1} \), so the claim holds automatically when \( i \geq n \) and \( j \geq 2n \) since both sides of the two equations are zero. When \( s_i \in \text{Des}_R(y) \) and \( s_j \in \text{Des}_R(z) \) and \( z \times s_j \in \tilde{I}_{\text{FFP}} \), the desired identities follow directly from the definitions and Lemma 2.4. Finally, suppose \( i \in [n-1] \) is not a descent of \( y \) and \( j \in [2n-1] \) is not a descent of \( z \). Then by the preceding case \( \Upsilon_y = \partial_i \Upsilon_{y \times s_i} \) and \( \Upsilon_z^{\text{FFP}} = \partial_j \Upsilon_z^{\text{FFP}} \), so as the divided difference operators square to zero, it follows that \( \partial_i \Upsilon_y = \partial_j \Upsilon_z^{\text{FFP}} = 0 \).

It remains to show that \( \Upsilon_1 = \Upsilon_{\text{FFP}}^{\text{FFP}} = 1 \) and that if \( z \times s \notin \tilde{I}_{\text{FFP}} \) then \( \partial_y \Upsilon^{\text{FFP}} = 0 \). Recall from the discussion above that \( 2^{|e(y)|} \Upsilon_y = [\Upsilon_y^{\text{FFP}}] \) if \( y \in I(S_n) \) and \( \Upsilon_z^{\text{FFP}} = [\Upsilon_{w_{2n}}^{\text{FFP}}] \) if \( z = w_\infty \) where \( w \in \tilde{I}_{\text{FFP}}(S_{2n}) \). The rank conditions in Proposition 3.16 show that \( \Upsilon_1 = \text{FI}(n) \) and \( \Upsilon_{w_{2n}}^{\text{FFP}} = \text{FI}(2n) \), so these orbit closures correspond to the identity elements in their corresponding cohomology rings.

As the Borel isomorphism (2.9) is an isomorphism of rings and since \( \Upsilon_y \) and \( \Upsilon_z^{\text{FFP}} \) are evidently homogeneous polynomials, it is immediate that \( \Upsilon_1 = \Upsilon_{\text{FFP}}^{\text{FFP}} = 1 \).

Finally suppose \( z \in \tilde{I}_{\text{FFP}} \) and \( j \in [2n-1] \) are such that \( z \times s_j \notin \tilde{I}_{\text{FFP}} \). As before, we then have \( z = w_\infty \) for some \( w \in \tilde{I}_{\text{FFP}}(S_{2n}) \), and necessarily \( j \in \text{Des}_R(w) \) and \( w \times s_j \notin \tilde{I}_{\text{FFP}}(S_{2n}) \). Theorem 3.17 implies \( \partial_j \Upsilon_{w_{2n}}^{\text{FFP}} = 0 \), so as \( \Upsilon_z^{\text{FFP}} = [\Upsilon_{w_{2n}}^{\text{FFP}}] \) it follows that \( \partial_j \Upsilon_z^{\text{FFP}} = 0 \). We conclude by Theorem 3.11 and Corollary 3.13 that \( \Upsilon_y = \hat{\Upsilon}_y \) and \( \Upsilon_z^{\text{FFP}} = \hat{\Upsilon}_z^{\text{FFP}} \) for all \( y \in I(S_\infty) \) and \( z \in \tilde{I}_{\text{FFP}} \), and so our result follows from Theorem 3.18.

3.5 Product formulas

Theorem 3.19 establishes an explicit product formula for \( \hat{\Upsilon}_{w_n} \) and \( \hat{\Upsilon}_{w_{2n}}^{\text{FFP}} \), and in this section we generalize that result to the following class of involutions:

**Definition 3.20.** An involution \( y \in I(S_\infty) \) is weakly dominant if it has the form

\[
y = (1, b_1)(2, b_2) \cdots (k, b_k)
\]

for some \( k \in \mathbb{N} \) and some integers \( b_1, b_2, \ldots, b_k \) with \( b_i > k \) for all \( i \in [k] \).

If \( y = (1, b_1)(2, b_2) \cdots (k, b_k) \in I(S_n) \) is weakly dominant then we define \( r(y) \in S_{n-k} \) as the permutation given in one-line notation by

\[
r(y) = [b_1 - k, b_2 - k, \ldots, b_k - k, c_1, c_2, \ldots, c_{n-2k}]
\]

where \( c_1 < c_2 < \cdots < c_{n-2k} \) are the elements of \( \{1, 2, \ldots, n-k\} \setminus \{b_1 - k, b_2 - k, \ldots, b_k - k\} \).

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Example 3.21. We have \( r ((1,6)(2,5)(3,8)) = [3, 2, 5, 1, 4] \) and \( r(g_k) = 1 \) and \( r(w_k) = w_{[k/2]} \). More generally, if \( y = u^{-1}g_ku \) for any \( u \in S_k \), then \( y \) is weakly dominant with \( r(y) = u \).

It follows from Section 2.2 that the permutation \( r(y) \) has these basic properties:

Observation 3.22. If \( y \in \mathcal{I}(S_n) \) is weakly dominant with \( k = \kappa(y) \) distinct 2-cycles, then \( r(y) \) belongs to \( S_{n-k} \) with largest descent at most \( k \), so \( D(r(y)) \subset [k] \times [n-k] \).

Recall that \( \hat{D}(g_k) \) is the transpose of the shifted shape of \( \delta_{k+1} \). For any permutation \( u \), define \( E_k(u) = \{(j+k, i) : (i, j) \in D(u)\} \) as the transpose of \( D(u) \), shifted down by \( k \) rows.

Lemma 3.23. Suppose \( y \in \mathcal{I}(S_{\infty}) \) is weakly dominant and \( k = \kappa(y) \). Then

\[
\hat{D}(y) = \hat{D}(g_k) \cup E_k(r(y)) \quad \text{and} \quad \hat{D}_{\text{FF}}(y) = \hat{D}_{\text{FF}}(g_k) \cup E_k(r(y)).
\]

Example 3.24. Before commencing the proof of this lemma, it is helpful to give an example illustrating the result. Let \( y = (1,6)(2,5)(3,8) = [6, 5, 8, 4, 2, 1, 7, 3] \) as in Example 3.21. Then

\[
\hat{D}(y) = \left\{ \begin{array}{c}
\circ \ldots \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array} \right\}, \quad \hat{D}(g_3) = \left\{ \begin{array}{c}
\circ \ldots \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array} \right\}, \quad D(r(y)) = \left\{ \begin{array}{c}
\circ \ldots \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array} \right\},
\]

and the lemma’s claim that \( \hat{D}(y) = \hat{D}(g_3) \cup E_3(r(y)) \) is evident.

Proof of Lemma 3.23. By definition, \( y = (1,b_1)(2,b_2) \cdots (k,b_k) \) for distinct integers \( b_i > k \). It is clear by construction that \( \hat{D}(g_k) \subset \hat{D}(y) \). Fix \( (i, j) \in \mathcal{P} \times \mathcal{P} \). We claim that \( (j+k, i) \in D(y) \) if and only if \( (i, j) \in D(r(y)) \). Recall that \( (j+k, i) \in D(y) \) if and only if

\[
j + k < y(i) \quad \text{and} \quad i < y(j+k).
\]

The claim holds when \( i \in [k] \) since then the first of these conditions is equivalent to \( j < r(y)(i) \), while the second may be rewritten as

\[
i < y(j+k) \iff j + k \notin \{b_1, b_2, \ldots, b_i\} \iff j \notin r(y)([i]) \iff i < r(y)^{-1}(j).
\]

On the other hand, if \( i > k \) then \( (i, j) \notin D(r(y)) \) by Observation 3.22 and \( (j+k, i) \notin D(y) \) since one checks that the conditions (3.5) never simultaneously hold. Hence the set of positions in \( D(y) \) below the \( k^{\text{th}} \) row is precisely \( E_k(r(y)) \). The latter set is contained entirely below the diagonal since \( D(r(y)) \subset [k] \times [n-k] \), so the lemma follows. \(\square\)

Recall the definition of dominant from before Proposition 2.13. We see by the following proposition that every dominant involution is weakly dominant.

Proposition 3.25. Let \( y \in \mathcal{I}(S_{\infty}) \). The following are equivalent:

(a) \( y \) is dominant.
(b) $\hat{D}(y)$ is the transpose of a shifted shape.

(c) $y$ is weakly dominant and $r(y)$ is dominant.

Proof. The equivalence of (a) and (b) is immediate from the definitions of dominant permutations and involution Rothe diagrams. By Lemma 3.23, it is clear that if $y$ is weakly dominant then the transpose of $\hat{D}(y)$ is a shifted shape if and only if $D(r(y))$ is the diagram of a partition, i.e., $r(y)$ is dominant. It is a straightforward exercise to check that a 132-avoiding (i.e., dominant) involution is weakly dominant, so it follows that (a) and (c) are equivalent. \qed

We now prove Theorem 1.3, which is the involution analogue of Proposition 2.13(a).

**Theorem 3.26.** Suppose $y \in \mathcal{I}(S_\infty)$ and $z \in \mathcal{I}_{\text{FFF}}(S_\infty)$ are dominant. Then

$$\hat{\mathcal{S}}_y = 2^{-\kappa(y)} \prod_{(i,j) \in \hat{D}(y)} (x_i + x_j) \quad \text{and} \quad \hat{\mathcal{S}}_{\text{FFF}}^z = \prod_{(i,j) \in D_{\text{FFF}}(z)} (x_i + x_j).$$

Proof. We prove the first identity, since the formula for $\hat{\mathcal{S}}_{\text{FFF}}^z$ follows by essentially the same argument. To begin, it is helpful to note (see [41, §2.1]) that the Rothe diagram of a permutation $w$ is the complement in $\mathbb{P} \times \mathbb{P}$ of the hooks through the points $(i, w(i))$ for $i \in \mathbb{P}$, where the hook through a cell $(i, j)$ is the set of positions of the form $(i + t, j)$ or $(i, j + t)$ for $t \in \mathbb{N}$. It follows that if $w$ is dominant, so that $D(w)$ is a partition, then the northwest corners of the complement of $D(w)$ are all of the form $(i, w(i))$, and that if $(i, w(i))$ is such a corner then $i$ is a descent of $w$ if and only if the $(i + 1)$th row of $D(w)$ is shorter than the $i$th row.

Let $y \in \mathcal{I}(S_n)$ be a dominant involution. The desired formula holds when $y = w_n$ by Theorem 3.19 so assume $y < w_n$ and that the product formula is valid for all dominant involutions $z \in \mathcal{I}(S_n)$ with $\ell(y) < \ell(z)$. The Rothe diagram of $D(y)$ is strictly contained in $D(w_n) = (n-1, n-2, \ldots, 2, 1)$, so we may define $j \in [n]$ to be minimal such that $(n - j, j) \notin D(y)$, and then define $i \in [n - j]$ to be minimal such that $(i, j) \notin D(y)$. The cell $(i, j)$ is then a northwest corner of the complement of $D(y)$, so $j = y(i)$. Moreover, rows $i, i + 1, \ldots, n - j + 1$ of $D(y)$ all have length $j - 1$, so we must have $s_i \notin \text{Des}_R(y)$ and also $j \leq i$, since $D(y)$ is symmetric under transpose.

Using these facts and the interpretation of the Rothe diagram as the complement of the hooks through the points of a permutation, it is a straightforward exercise to check that

$$\hat{D}(y \circ s_i) = \hat{D}(y) \cup \{(i, j)\}. \quad (3.6)$$

We omit the details, since the argument is easier to visualize than to transcribe and is similar to the proof of [41, Proposition 2.6.7]. By Proposition 3.25 the identity (3.6) implies that $y \circ s_i$ is itself a dominant involution of greater length than $y$, so by induction and Theorem 3.11 we obtain

$$\hat{\mathcal{S}}_y = \partial_i \hat{\mathcal{S}}_{y \circ s_i} = \partial_i \left[ 2^{-\kappa(y \circ s_i)} (x_i + x_j) \prod_{(k,l) \in \hat{D}(y)} (x_k + x_l) \right]. \quad (3.7)$$

The product $\prod_{(k,l) \in \hat{D}(y)} (x_k + x_l)$ is $s_i$-invariant since the map $(k, l) \mapsto (s_i(k), s_i(l))$ preserves the transposed shifted shape $\hat{D}(y)$, which has the same number of cells in rows $i$ and $i + 1$ and no cells in columns $i$ and $i + 1$. Since $\partial_i(fg) = f\partial_i(g)$ when $s_i f = f$, equation (3.7) becomes

$$\hat{\mathcal{S}}_y = 2^{-\kappa(y \circ s_i)} \prod_{(k,l) \in \hat{D}(y)} (x_k + x_l) \cdot \partial_i(x_i + x_j).$$

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This transforms to the desired formula for $\hat{S}_y$ on checking that $\partial_i(x_i + x_j)$ is 1 if $i \neq j$ and 2 otherwise, and then noting, since $\kappa(\sigma)$ is the number of diagonal cells in $D(\sigma)$ for any $\sigma \in \mathcal{I}(S_\infty)$, that $\kappa(y \times s_i) - \kappa(y)$ is likewise 0 if $i \neq j$ and 1 otherwise.

For $p, q \in \mathbb{N}$ define $\Phi_{p,q}$ as the map $\mathcal{P}_\infty(x;y) \to \mathcal{P}_{p+q}(x)$ with

$$\Phi_{p,q} : f(x;y) \mapsto f(x_1, x_2, \ldots, x_p, 0, 0, \ldots; -x_{p+1}, -x_{p+2}, \ldots, -x_{p+q}, 0, 0, \ldots).$$

In other words, $\Phi_{p,q}$ is the ring homomorphism which maps $x_i \mapsto x_i$ and $y_j \mapsto -x_{p+j}$ for $i \in [p]$ and $j \in [q]$, while mapping all other variables to zero. Suppose $z \in \mathcal{I}(S_n)$ is weakly dominant and $k = \kappa(z)$. If $E_k(u)$ is defined as before Lemma 3.23 then it follows from Observation 3.22 that

$$\prod_{(i,j) \in E_k(r(z))} (x_i + x_j) = \prod_{(i,j) \in D(r(z))} (x_i + x_{j+k}) = \Phi_{k,n-k} \left( \prod_{(i,j) \in D(r(z))} (x_i - y_j) \right). \quad (3.8)$$

This fact leads to the following result, generalizing the previous theorem.

**Theorem 3.27.** Suppose $z \in \mathcal{I}(S_\infty)$ is a weakly dominant involution. Let $p = \kappa(z)$, define $n$ as the smallest integer such that $z \in S_n$, and set $q = n - p$. Then

$$\hat{S}_z = \hat{S}_{gp} \cdot \Phi_{p,q} \left( \mathcal{S}_{r(z)}(x;y) \right) \quad \text{and} \quad \hat{S}_z^{FPF} = \hat{S}_{gp}^{FPF} \cdot \Phi_{p,q} \left( \mathcal{S}_{r(z)}(x;y) \right)$$

where the second identity applies only in the case when $z$ is fixed-point-free.

The discussion in Section 3.4 makes it natural try to interpret this statement geometrically, and we outline a geometric proof of the theorem along these lines in Appendix A.

**Proof.** Since $z$ is weakly dominant we may write $z = (1, b_1)(2, b_2) \cdots (p, b_p)$ for distinct integers $b_i$ all greater than $p$. First suppose $b_1 > b_2 > \cdots > b_p$. One checks that $z$ is then dominant, so it follows from Proposition 2.13(a) that the rightmost expression in (3.8) is precisely $\Phi_{p,q} \left( \mathcal{S}_{r(w)}(x;y) \right)$. By Proposition 3.10, noting Example 3.5 it always holds that $\hat{S}_{gp} = \prod_{(i,j) \in D(gp)} (x_i + x_j)$, so the desired formula for $\hat{S}_w$ follows by Lemma 3.23 and Theorem 3.26.

Suppose alternatively that there exists an index $i \in [p-1]$ such that $b_i < b_{i+1}$. Then $i$ is a (right) ascent of both $z$ and $r(z)$, and evidently $z \times s_i = s_i z s_i$ is also weakly dominant, so we may assume by induction that $\hat{S}_{z \times s_i} = \hat{S}_{gp} \cdot \Phi_{p,q} \left( \mathcal{S}_{r(z \times s_i)}(x;y) \right)$. As $\hat{S}_{gp} \in \Lambda_p$ by Proposition 3.10 it follows by our inductive hypothesis and Theorem 3.11 that

$$\hat{S}_z = \partial_i \hat{S}_{z \times s_i} = \partial_i \left[ \hat{S}_{gp} \cdot \Phi_{p,q} \left( \mathcal{S}_{r(z \times s_i)}(x;y) \right) \right] = \hat{S}_{gp} \cdot \partial_i \Phi_{p,q} \left( \mathcal{S}_{r(z \times s_i)}(x;y) \right).$$

Since $r(z \times s_i) = r(z)s_i > r(w)$, and since $\partial_i$ acts only on the $x_i$ variables when applied to an element of $\mathcal{P}_\infty(x;y)$, we have $\partial_i \Phi_{p,q} \left( \mathcal{S}_{r(z \times s_i)}(x;y) \right) = \Phi_{p,q} \left( \mathcal{S}_{r(z)}(x;y) \right)$. Substituting this into the preceding equation gives the desired formula for $\hat{S}_z$. When $z$ is fixed-point-free, the analogous identity for $\hat{S}_z^{FPF}$ follows by a similar argument. \qed
3.6 Involution Stanley symmetric functions

For \( y, z \in I(S_\infty) \), we recall from the introduction the involution Stanley symmetric function

\[
\hat{F}_{y,z} = \sum_{u \in A(y,z)} F_u \in \Lambda.
\]

We abbreviate as usual by setting \( \hat{F}_y = \hat{F}_{1,y} \) and \( \hat{F}_{y,z}^{\text{FPF}} = \hat{F}_{v^y_n,z} \) for \( z \in I_{\text{FPF}}(S_{2n}) \).

Observe that \( \hat{F}_{y,z} = 0 \) if \( y \not\leq_T z \), with \( \leq_T \) as in Section 2.1. The following slight modification to Theorem 2.15 holds by Theorem-Definition 1.1:

**Observation 3.28.** If \( y, z \in I(S_\infty) \) then \( \hat{F}_{y,z} = \sum_{\lambda} \beta_{y,z,\lambda} s_\lambda \) where the sum is over partitions \( \lambda \) and \( \beta_{y,z,\lambda} \) is the number of strict tableaux \( T \) of shape \( \lambda \) with \( \text{rrw}(T) \in \hat{R}(y,z) \).

Note that \( \beta_{y,z,\lambda} = 0 \) if \( \lambda \) is not a partition of \( \hat{\ell}(y,z) \), so the sum appearing in the observation’s formula for \( \hat{F}_{y,z} \) is finite. From Theorem 2.10 we obtain the following corollary:

**Corollary 3.29.** If \( y, z \in I(S_\infty) \) then \( |\hat{R}(y,z)| = \sum_{\lambda} \beta_{y,z,\lambda} f^\lambda \).

Thus, to count the number of elements in the sets \( \hat{R}(y,z) \), we need only determine the Schur decomposition of the symmetric functions \( \hat{F}_{y,z} \). The rest of this section is spent proving a few facts about such decompositions that follow directly from properties of Stanley symmetric functions and involution words.

Just as for ordinary Stanley symmetric functions, \( \hat{F}_y \) and \( \hat{F}_y^{\text{FPF}} \) are skew Schur functions when indexed by 321-avoiding permutations. In detail, given a sequence of nonnegative integers \( c = (c_1, c_2, \ldots, c_n) \) whose nonzero entries occur in positions \( k_1 < \cdots < k_l \), let \( \text{skew}(c) \) be the set of cells \( (i,j) \in \mathbb{P} \times \mathbb{P} \), with \( 1 \leq i \leq l \), such that

\[
i - k_i - c_{k_i} < j + L \leq i - k_i \quad \text{where} \quad L = l - k_l - c_{k_l}.
\]

By [41, §2.2.2], it follows that if \( c = c(w) \) is the code of a 321-avoiding permutation \( w \), then \( \text{skew}(c) \) is a skew shape. Recall that \( \delta_n = (n-1, n-2, \ldots, 2, 1) \).

**Example 3.30.** If \( y = v_n \in S_{2n} \) then \( c(y) = (1,0,1,0,\ldots,1,0) \) and \( \text{skew}(c(y)) = \delta_{n+1}/\delta_n \).

If \( y \in S_\infty \) is 321-avoiding then Proposition 2.17 asserts that \( F_y = s_{\text{skew}(c)} \). The following parallel statement holds for involution Stanley symmetric functions:

**Proposition 3.31.** Suppose \( y \in I(S_\infty) \) is 321-avoiding. Then \( \text{skew}(\hat{c}(y)) = \lambda/\mu \) is a skew shape and \( \hat{F}_y = s_{\lambda/\mu} \). If \( y \) is fixed-point-free, then \( \text{skew}(\hat{c}_{\text{FPF}}(y)) = \gamma/\nu \) is a skew shape and \( \hat{F}_y^{\text{FPF}} = s_{\gamma/\nu} \).

**Proof.** By Theorem 3.1 and Proposition 3.3, \( A(y) = \{\alpha_{\text{min}}(y)\} \) and (in the fixed-point-free case) \( \mathcal{A}_{\text{FPF}}(y) = \{\beta_{\text{min}}(y)\} \). By Corollary 3.4, \( \alpha_{\text{min}}(y) \) and \( \beta_{\text{min}}(y) \) are themselves 321-avoiding, so the result follows from its analogue for ordinary Stanley symmetric functions by Lemma 3.8.

We next classify the involutions \( y \in S_\infty \) for which \( \hat{F}_y \) and \( \hat{F}_y^{\text{FPF}} \) are Schur functions. To begin, we note the following lemma which derives from the discussion after [12] Proposition 5.4:

**Lemma 3.32** (Eriksson and Linusson [12]). A permutation \( w \in S_\infty \) is both 321-avoiding and 2143-avoiding if and only if either \( w \) or \( w^{-1} \) is Grassmannian.
If \( w \in S_k \) and \( m \in \mathbb{P} \) then we define the shifted permutation

\[
1_m \times w = [1, 2, \ldots, m, w(1) + m, w(2) + m, \ldots w(k) + m] \in S_{k+m}. \tag{3.9}
\]

For any \( v \in S_m \) we similarly define (with slight abuse of notation) \( v \times w = v \cdot (1_m \times w) \in S_{k+m} \).

With this convention, if \( z \in S_{2k} \) is a fixed-point-free involution, then \( v_m \times z \times v_n \) is as well.

**Example 3.33.** We have \( 1_4 \times w_4 = (5, 8)(6, 7) \) and \( v_2 \times w_4 \times v_5 = (1, 2)(3, 4)(5, 8)(6, 7)(9, 10) \).

Recall that we define \( g_k = (1, k + 1)(2, k + 2), \ldots (k, 2k) \in S_{2k} \).

**Proposition 3.34.** If \( y \in S_\infty \) is a Grassmannian involution then \( y = 1_m \times g_k \) for some \( m, k \in \mathbb{N} \).

**Proof.** Let \( y \in \mathcal{I}(S_\infty) \) be Grassmannian. It suffices to show that if \( k = y(1) - 1 \) is positive then \( y = g_k \). For this, observe that the Rothe diagram \( D(y) \) contains the cells \((1, i)\) for all \( i \in [k] \), and therefore also the cells \((i, 1)\) for \( i \in [k] \) since \( D(y) = D(y^{-1}) = D(y)^T \). As \( k \) is evidently the unique descent of \( y \), \( D(y) \) has no cells below the \( k^{\text{th}} \) row or (by symmetry) to the right of the \( k^{\text{th}} \) column; hence \( D(y) \subset [k] \times [k] \). On the other hand, by the definition of a Grassmannian permutation before Proposition 2.13, it holds that each nonempty row in \( D(y) \) contains at least as many cells as the row above it. Since the first row of \( D(y) \) already has \( k \) cells, it follows that in fact \( D(y) = [k] \times [k] = D(g_k) \), so \( y = g_k \) as desired.

The permutations \( y \in \mathcal{I}(S_\infty) \) whose involution Stanley symmetric functions are single Schur functions turn out to have a very restricted form, which we now describe.

**Proposition 3.35.** Suppose \( y = 1_m \times g_{k-1} \) and \( z = v_m \times g_k \times v_n \) for some \( m, n, k \). Then

\[
\hat{F}_y = \hat{F}_z^{\text{FPP}} = s_{\delta_k} \quad \text{and} \quad |\hat{R}(y)| = |\hat{R}^{\text{FPP}}(z)| = f^{|\delta_k|}.
\]

**Proof.** These identities follow directly from Proposition 3.31 and Corollary 3.20.

**Theorem 3.36.** Let \( y \in \mathcal{I}(S_\infty) \) and \( z \in \mathcal{I}^{\text{FPP}}(S_\infty) \).

(a) \( \hat{F}_y \) is a Schur function if and only if \( y = 1_m \times g_k \) for some \( m, k \in \mathbb{N} \).

(b) \( \hat{F}_z^{\text{FPP}} \) is a Schur function if and only if \( z = v_m \times g_k \times v_n \) for some \( m, n \in \mathbb{N} \) and \( k \in \mathbb{P} \).

**Proof.** Suppose \( \hat{F}_y \) is a Schur function. Combining Theorems 2.18 and 3.1 with Proposition 3.3 and its corollary shows that \( y \) must be 321-avoiding and that \( \alpha_{\text{min}}(y) \) must be both 321-avoiding and 2143-avoiding. By Lemma 3.32, either \( \alpha_{\text{min}}(y) \) or its inverse is therefore Grassmannian. It is apparent from the definition that \( \alpha_{\text{min}}(y)^{-1} \) is Grassmannian only if \( y \) is the identity or a simple transposition (as the only 321-avoiding transpositions are simple), in which case \( \alpha_{\text{min}}(y) \) is equal to its inverse. We conclude therefore that \( \alpha_{\text{min}}(y) \) must be Grassmannian. We now argue that \( y \) must be some shift of \( g_k \). To this end, suppose \( (a, b) \) and \( (a', b') \) are pairs of positive integers with \( a \leq b = y(a) \) and \( a' \leq b' = y(a') \). One checks that it cannot occur that \( a < a' \leq b' < b \) (as then \( y \) would contain the pattern 321) or that \( a < b < a' < b' \) (as then \( \alpha_{\text{min}}(y) \) would contain the pattern 2143), so if \( (a, b) \) and \( (a', b') \) are distinct with \( a < a' \) then \( a < a' < b < b' \). Using this property, it is straightforward to deduce that \( y \) must have the form \( 1_m \times g_k \) for some \( m, k \in \mathbb{N} \). We leave this exercise to the reader.
Suppose next that \( \hat{F}_{z}^{\mathcal{FF}} \) is a Schur function. By the same set of results as cited in the previous paragraph, it follows that \( z \) must be 321-avoiding and either \( \beta_{\min}(z) \) or its inverse must be Grassmannian. Let \( (a_{i}, b_{i}) \) for \( i = 1, 2, 3, 4 \) be elements of the set \( \{(i, j) \in \mathbb{P} \times \mathbb{P} : i < j = z(i)\} \). One checks that it cannot occur that \( a_{1} < a_{2} < b_{2} < b_{1} \) (as then \( z \) would contain the pattern 321) or that \( a_{1} < a_{2} < b_{1} < a_{3} < b_{2} < b_{3} \) (as then \( \beta_{\min}(z) \) and its inverse would both contain the pattern 132546) or that \( a_{1} < a_{2} < b_{1} < b_{2} < a_{3} < a_{4} < b_{3} < b_{4} \) (as then \( \beta_{\min}(z) \) and its inverse would both contain the pattern 13245678). Using these properties, it is a similarly elementary exercise to deduce that \( z \) must have the form \( v_{m} \times g_{k} \times v_{n} \) for some \( m, n, k \in \mathbb{N} \). We again leave this to the reader.

This proves one half of the theorem, and the converse holds by Proposition 3.35.

A permutation is antivexillary if it is both 321-avoiding and 351624-avoiding. For an explanation of this terminology, see [12, Proposition 5.1].

**Corollary 3.37.** Let \( y \in \mathcal{I}(S_{\infty}) \) and \( z \in \mathcal{I}_{\mathcal{FF}}(S_{2n}) \).

(a) \( \hat{F}_{y} \) is a Schur function if and only if \( y \) is vexillary and 321-avoiding.

(b) \( \hat{F}_{z}^{\mathcal{FF}} \) is a Schur function if and only if \( z \) is antivexillary and 231564-avoiding.

**Proof.** Given the theorem, part (a) is immediate from Lemma 3.32 and Proposition 3.34. To prove part (b), one must show that a fixed-point-free involution \( z \) is antivexillary and 231564-avoiding if and only if \( z = v_{m} \times g_{k} \times v_{n} \) for some \( m, k \in \mathbb{N} \). We leave this exercise to the reader.

### 3.7 Stabilization

To prove stronger statements about involution Stanley symmetric functions, we must leverage the results in Section 3.5; we discuss methods for this here. If \( f \) is a power series in the variables \( x = \{x_{1}, x_{2}, \ldots\} \), then we write \( r_{n}(f) \) or \( f(x_{1}, \ldots, x_{n}) \) for the power series formed by setting the variables \( x_{n+1} = x_{n+2} = \cdots = 0 \). Observe that if \( f \in \mathcal{P}_{\infty} \) then \( r_{n}(f) \in \mathcal{P}_{n} \) and that if \( f \in \Lambda \) then \( r_{n}(f) \in \Lambda_{n} \), where \( \Lambda_{n} \) denotes the ring of polynomials in \( \mathcal{P}_{n} \) fixed by the action of \( S_{n} \).

Following [40], we define the **stabilization (in degree \( n \in \mathbb{N} \))** of \( \mathcal{G}_{w} \) for \( w \in S_{\infty} \) as

\[
stb_{n}(\mathcal{G}_{w}) = F_{w}(x_{1}, \ldots, x_{n}) \in \Lambda_{n}.
\]

By Proposition 2.12, this formula extends by linearity to a map \( stb_{n} : \mathcal{P}_{\infty} \rightarrow \Lambda_{n} \). We then have

\[
stb_{n}(\mathcal{G}_{y,z}) = \hat{F}_{y,z}(x_{1}, \ldots, x_{n}) \quad \text{for } y, z \in \mathcal{I}(S_{\infty})
\]

and \( \lim_{n \to \infty} stb_{n}(\mathcal{G}_{y,z}) = \hat{F}_{y,z} \), where the limit is interpreted in the sense of formal power series, with a sequence of power series defined to be convergent if the sequence of coefficients of any fixed monomial is eventually constant.

By applying \( stb_{n} \) to both sides of the identities in Theorem 3.27, one might hope to show that similar factorizations hold for \( \hat{F}_{y} \) and \( \hat{F}_{z}^{\mathcal{FF}} \). This strategy cannot work in general, since stabilization is not a ring homomorphism and may fail to preserve products. However, we will find that in certain cases of interest the maps \( stb_{n} \) do behave as we would wish, but to prove this we will require several preliminaries about these operations. For \( w \in S_{k} \), recall from (3.9) the definition of \( 1_{m} \times w \in S_{m+k} \) and note the following corollary of (1.3), which appears as [41, Proposition 2.8.1]:
Corollary 3.38. If \( w \in S_\infty \) then \( \text{stb}_n(\mathcal{G}_w) = \mathcal{G}_{1_{N} \times w}(x_1, x_2, \ldots, x_n) \) for all \( N \geq n \).

More usefully, we can express \( \text{stb}_n \) in terms of certain modified divided differences. Following [40], we define the isobaric divided difference operator \( \pi_i : \mathcal{P}_\infty \to \mathcal{P}_\infty \) for \( i \in \mathbb{P} \) by \( \pi_i f = \partial_i(x_i f) \).

For example, \( \pi_i(x_i x_{i+1}) = \partial_i(x_i^2 x_{i+1}) = x_i x_{i+1} \). One checks that \( \pi_i^2 = \pi_i \) and that

\[
\pi_i(fg) = f \cdot \pi_i g \quad \text{whenever } f, g \in \mathcal{P}_\infty \text{ such that } s_i f = f.
\]

(3.11)

In particular \( \pi_i(f) = f \cdot \pi_i(1) = f \) if it holds that \( s_i f = f \). These operators, like the ordinary divided differences \( \partial_i \), satisfy the Coxeter relations (2.6), so for \( w \in S_\infty \) we may define

\[
\pi_w = \pi_{i_1} \cdots \pi_{i_k} \quad \text{for any reduced word } (i_1, \ldots, i_k) \in R(w).
\]

(3.12)

We note as the following lemma one less standard property of these operators.

Lemma 3.39. If \( i, n \in \mathbb{P} \) and \( f \in \mathcal{P}_\infty \) then \( r_n(\pi_i f) = \begin{cases} \pi_i r_n(f) & \text{if } i < n \\ r_n(f) & \text{if } i \geq n. \end{cases} \)

Proof. Checking the lemma is a simple exercise in algebra which we leave to the reader.

The next theorem appears as [40, Eq. (4.25)]. We include a proof for completeness, since this result is central to what follows and since [40] is currently out of print and difficult to obtain.

Theorem 3.40 (Macdonald [40]). For all \( f \in \mathcal{P}_n \) it holds that \( \text{stb}_n(f) = \pi_w f \).

Proof. Define the operator \( \tau_n = \pi_1 \cdots \pi_n \). Since \( x_i(\pi_j f) = \pi_j(x_i f) \) for \( i < j \), it follows that we may also write \( \tau_n f = \partial_1 \cdots \partial_n(x_1 \cdots x_n f) \) for \( f \in \mathcal{P}_\infty \). Suppose \( u \in S_\infty \) has largest descent at most \( n \).

We claim that \( \tau_n \mathcal{G}_u = \mathcal{G}_{\pi(u)} \) where \( \pi(u) \) denotes the permutation

\[
i(u) = 1_1 \times u = [1, u(1) + 1, u(2) + 1, \ldots] \in S_\infty.
\]

To show this, first assume \( u \in S_n \) and let \( v = w_{n+1} w_n u = [u(1) + 1, u(2) + 1, \ldots, u(n) + 1, 1] \in S_{n+1} \). Since the product \( x_1 \cdots x_n \) is invariant under \( S_n \), it holds by Theorem (2.9) that

\[
x_1 \cdots x_n \mathcal{G}_u = x_1 \cdots x_n \partial_{u^{-1} w_n} \mathcal{G}_{w_n} = \partial_{u^{-1} w_n}(x_1 \cdots x_n \mathcal{G}_{w_n}) = \partial_{u^{-1} w_n} \mathcal{G}_{w_n} = \mathcal{G}_v
\]

so \( \tau_n \mathcal{G}_u = \partial_1 \cdots \partial_n(x_1 \cdots x_n \mathcal{G}_u) = \partial_1 \cdots \partial_n \mathcal{G}_v \). It is clear that we have a descending chain

\[
v > v s_n > v s_n s_{n-1} > \cdots > v s_n s_{n-1} \cdots s_1 = \pi(u)
\]

so we conclude by Theorem (2.8) that \( \tau_n \mathcal{G}_u = \mathcal{G}_{\pi(u)} \) when \( u \in S_n \). To prove the claim in general, observe that \( \mathcal{G}_u \in \mathcal{P}_n \) by Proposition (2.12) so \( \pi_i \mathcal{G}_u = \mathcal{G}_u \) for all \( i > n \). Therefore if \( u \in S_N \) for some \( N \geq n \), then \( \tau_N \mathcal{G}_u = \tau_n \mathcal{G}_u = \mathcal{G}_{\pi(u)} \) by the part of the claim already shown.

Fix \( f \in \mathcal{P}_n \). Given our claim, it follows by Proposition (2.12) and Corollary (3.38) that \( \text{stb}_n(f) = r_n(\tau_{2n-1} \cdots \tau_{n+1} \tau_n f) \). One checks using Lemma (3.39) that if \( N \geq n \) then \( r_n(\tau_N g) = \tau_{n-1} r_n(g) \) for all \( g \in \mathcal{P}_\infty \). Using this property, we deduce that

\[
\text{stb}_n(f) = r_n(\tau_{2n-1} \cdots \tau_{n+1} \tau_n f) = \tau_{n-1}^n r_n(f) = \pi_{n-1}^n f.
\]

Recall that \( \pi_i^2 = \pi_i \) for all \( i \in \mathbb{P} \). Therefore, if \( w \in W \) and \( s \in S \), then \( \pi_w \pi_s \) is equal to \( \pi_w \) when \( s \in \text{Des}_R(w) \) and to \( \pi_{us} \) when \( s \notin \text{Des}_R(w) \). Using this property, it is a simple exercise to check that \( \tau_{n-1}^n = \pi_{w_n} \) which suffices to complete the proof.
We may now begin to say something about the “stability” of the formulas in Theorem 3.27. In view of the preceding theorem and (3.11), it follows that \( \text{stb}_m(fg) = f \text{stb}_m(g) \) if \( f \in \mathcal{P}_\infty \) is invariant under the action of \( S_n \). We would like to apply something like this identity to Theorem 3.27 but, problematically, the involution Schubert polynomials \( \hat{S}_k \) and \( \hat{S}_{k\ell} \) appearing in that result are symmetric only under the action of the subgroup \( S_k \times S_{n-k} \), not all of \( S_n \). To get around this difficulty, we factor \( \text{stb}_m \) into two operators, one of which respects the partial symmetry which we encounter, in the following way.

Fix nonnegative integers \( p, q \) with \( n = p + q \), and write \( \Lambda_{p \times q} \) for the subring of polynomials in \( \mathcal{P}_n \) which are fixed by the action of \( S_p \times S_q \subset S_n \). Thus \( \Lambda_n = \Lambda_0 \otimes \Lambda_n \subset \Lambda_{p \times q} \). Let

\[
g_{p,q} = w_n \cdot (w_p \times w_q) = [q + 1, q + 2, \ldots , n, 1, 2, \ldots , q] \in S_n
\]

and define \( \text{stb}_{p,q} : \mathcal{P}_n \to \Lambda_{p \times q} \) and \( \text{stb}_{n/p,q} : \Lambda_{p \times q} \to \Lambda_n \) by

\[
\text{stb}_{p,q} = \pi_{w_p \times w_q} \quad \text{and} \quad \text{stb}_{n/p,q} = \pi_{g_{p,q}}.
\]

Note that it is clear from Theorem 3.40 that \( \text{stb}_n(f) = \text{stb}_{n/p,q}\text{stb}_{p,q}(f) \) for all \( f \in \mathcal{P}_n \).

**Lemma 3.41.** Let \( p, q \in \mathbb{N} \) and \( n = p + q \). If \( f \in \Lambda_{p \times q} \) and \( g \in \mathcal{P}_\infty \) such that \( \text{stb}_{p,q}(g) \in \Lambda_n \), then

\[
\text{stb}_n(fg) = \text{stb}_n(f)\text{stb}_{p,q}(g).
\]

**Proof.** Our hypotheses together with (3.11) imply that \( \text{stb}_{p,q}(f) = f \) and

\[
\text{stb}_{p,q}(fg) = f \cdot \text{stb}_{p,q}(g) \quad \text{and} \quad \text{stb}_{n/p,q}(f \cdot \text{stb}_{p,q}(g)) = \text{stb}_{n/p,q}(f)\text{stb}_{p,q}(g).
\]

Hence \( \text{stb}_n(fg) = \text{stb}_{n/p,q}\text{stb}_{p,q}(fg) = \text{stb}_{n/p,q}(f)\text{stb}_{p,q}(g) = \text{stb}_n(f)\text{stb}_{p,q}(g) \). \( \square \)

The algebra of symmetric functions \( \Lambda \) may be identified with its graded dual and so given the structure of a graded, self-dual Hopf algebra; see [18, 56] for the details of this standard construction. The coproduct \( \Delta : \Lambda \to \Lambda \otimes \Lambda \) of this Hopf algebra is defined as the linear map satisfying \( \Delta(f) = \sum_i g_i \otimes h_i \) for \( f \in \Lambda \), where \( g_i, h_i \in \Lambda \) are symmetric functions such that, when written as a function of two countable, commuting sets of variables, \( f \) decomposes as

\[
f(x_1, x_2, \ldots , y_1, y_2, \ldots ,) = \sum_i g_i(x_1, x_2, \ldots )h_i(y_1, y_2, \ldots ).
\]

For any partition \( \nu \), it holds that \( \Delta(s_\nu) = \sum_{\lambda, \mu} c_{\lambda, \mu}^\nu s_\lambda \otimes s_\mu \) where \( c_{\lambda, \mu}^\nu \) are the Littlewood-Richardson coefficients. Write \( \omega : \Lambda \to \Lambda \) for the linear map with \( \omega(s_\nu) = s_{\nu^T} \) for all partitions \( \nu \).

**Lemma 3.42.** Let \( p, q \in \mathbb{N} \) and \( n = p + q \). Suppose \( w \in S_q \) has largest descent at most \( p \). If

\[
(id \otimes \omega) \circ \Delta(F_w) = \Delta(F_w)
\]

then \( \text{stb}_{p,q}(\Phi_{p,q}(G_w(x; y))) = F_w(x_1, \ldots , x_n) \).

**Proof.** First note using Definition 2.10 and Proposition 2.12 that

\[
\Phi_{p,q}(G_w(x; y)) = \sum_{u=\nu^{-1}} \sum_{l(u)=\ell(u)+\ell(\nu)} G_u(x_1, \ldots , x_p)G_v(x_{p+1}, \ldots , x_n) \in \mathcal{P}_n
\]

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On transposing the variables \( x \) expression) and then setting \( \omega \) acts on \( \Phi_{p,q} \) (\( \mathcal{S}_w(x; y) \)) as the operator \( \pi_{w_p \times w_q} \); it follows via Theorem 3.40 that
\[
\text{stb}_{p,q}(\Phi_{p,q}(\mathcal{S}_w(x; y))) = \sum_{w=v^{-1}u, \ell(w)=\ell(u)+\ell(v)} F_u(x_1, \ldots, x_p)F_v(x_{p+1}, \ldots, x_n). \tag{3.13}
\]

Now, we have from [38, Proposition 5 and Theorem 12] that \( \Delta(F_w) = \sum F_u \otimes F_v \) where the sum is over all \( u, v \in S_q \) with \( w = vu \) and \( \ell(w) = \ell(v) + \ell(u) \). On the other hand, Macdonald [40, Corollary 7.22] proves that \( \omega(F_w) = F_{w^{-1}} \). Hence, if \( (id \otimes \omega) \circ \Delta(F_w) = \Delta(F_w) \), then
\[
F_w(x_1, x_2, \ldots, y_1, y_2, \ldots) = \sum_{w=v^{-1}u, \ell(w)=\ell(v)+\ell(u)} F_u(x_1, x_2, \ldots)F_v(y_1, y_2, \ldots).
\]

On transposing the variables \( y_i \) and \( x_{p+i} \) for \( i \in \{q\} \) (which by symmetry does not affect either expression) and then setting \( x_{n+i} = y_i = 0 \) for \( i \in \mathbb{P} \), the left side of this identity becomes \( F_u(x_1, \ldots, x_n) \) while the right side becomes the formula \( [3.13] \) for \( \text{stb}_{p,q}(\Phi_{p,q}(\mathcal{S}_w(x; y))) \); these expressions are therefore equal when \( id \otimes \omega \) fixes \( \Delta(F_w) \).

Let \( p_n = x_1^n + x_2^n + \cdots \in \Lambda \) for \( n \in \mathbb{N} \) denote the usual power sum symmetric function. Since \( \Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n \) and \( \omega(p_n) = (-1)^n-1p_n \) (see [18, Proposition 2.23 and §2.4]), and since \( \Delta \) and \( \omega \) are algebra homomorphisms, it follows that \( (1 \otimes \omega) \circ \Delta(f) = \Delta(f) \) whenever \( f \) belongs to the Hopf subalgebra
\[
\Lambda^{\text{odd}} = \mathbb{Z}[p_1, p_3, p_5, \ldots] \subset \Lambda
\]
generated by the odd-indexed power sum symmetric functions. This subalgebra is studied in a few places (see, e.g., [11, 25, 52]), but does not seem to have an established name. The following theorem is the main result of this section, and will imply the results described in the introduction.

**Theorem 3.43.** Let \( y \in \mathcal{I}(S_\infty) \) be weakly dominant with \( k = \kappa(y) \). If \( F_{r(y)} \in \Lambda^{\text{odd}} \) then
\[
\hat{F}_y = \hat{F}_{g_k} F_{r(y)} \quad \text{and} \quad \hat{F}^{\text{FFP}}_y = \hat{F}^{\text{FFP}}_{g_k} F_{r(y)}
\]
where the second identity applies only in the case when \( y \) is fixed-point-free.

**Proof.** Let \( n \) be the smallest integer such that \( y \in S_n \), so that \( n = 2k \) when \( y \) is fixed-point-free. As noted in the preceding discussion, the operator \( id \otimes \omega \) preserves \( \Delta(F_{r(y)}) \) and so applying Lemmas 3.41 and 3.42 to Theorem 3.27 shows that \( \hat{F}_y(x_1, \ldots, x_n) = \hat{F}_{g_k}(x_1, \ldots, x_n)F_{r(y)}(x_1, \ldots, x_n) \) and, when \( y \) is fixed-point-free, that \( \hat{F}^{\text{FFP}}_y(x_1, \ldots, x_n) = \hat{F}^{\text{FFP}}_{g_k}(x_1, \ldots, x_n)F_{r(y)}(x_1, \ldots, x_n) \). It remains only to argue that these identities in \( \Lambda_n \) lift to identities in \( \Lambda \).

Write \( \ell(\lambda) \) for the number of parts in a partition \( \lambda \). Let \( \Lambda_{\infty,k} \) be the subspace of \( \Lambda_n \) spanned by the Schur polynomials \( s_\lambda(x_1, \ldots, x_n) \) for partitions \( \lambda \) with \( \ell(\lambda) \leq k \), and likewise define \( \Lambda_{\infty,k} = \mathbb{Z}\text{-span}\{s_\lambda : \ell(\lambda) \leq k\} \). It is well-known that the restriction map \( r_n \) defines a bijection \( \Lambda_{\infty,k} \to \Lambda_{n,k} \) whenever \( k \leq n \) (see, e.g., [41 Proposition 1.2.1]) and that \( fg \in \Lambda_{\infty,j+k} \) whenever \( f \in \Lambda_{\infty,j} \) and \( g \in \Lambda_{\infty,k} \) (see [41] §1.5.4]). Hence, if we have \( (f, g, h) \in \Lambda_{\infty,j} \times \Lambda_{\infty,k} \times \Lambda_{\infty,j+k} \) and \( j + k \leq n \), then
\[
h(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)g(x_1, \ldots, x_n) \in \Lambda_n \quad \Rightarrow \quad h = fg \in \Lambda \quad \tag{3.14}
\]
since we may obtain the right identity by applying the inverse of the bijection \( r_n : \Lambda_{\infty,j+k} \to \Lambda_{n,j+k} \) to both sides of the equation on the left.

It follows from [50, Theorem 4.1] that \( F_u \in \Lambda_{\infty,k} \) if \( u \in S_\infty \) has largest descent at most \( k \). In view of Proposition 2.4 we thus have \( \hat{F}_u \in \Lambda_{\infty,k} \) (respectively, \( \hat{F}_{\text{FPF}} \in \Lambda_{\infty,k} \)) whenever \( u \) is an involution (respectively, fixed-point-free involution) with largest descent at most \( k \). Since \( 2k \leq n \) and \( y \in S_n \), and since \( g_k \) and \( r(y) \) both have largest descent at most \( k \), we may apply (3.14) to deduce the desired identities from the formulas in the first paragraph.

Recall that \( \delta_n = (n - 1, n - 2, \ldots, 2, 1) \). The subalgebra \( \Lambda^\text{odd} \subset \Lambda \) has a distinguished basis \( \{ P_\lambda \} \) indexed by strict partitions (see [11, 52]), called the Schur \( P \)-functions. An element \( f \in \Lambda^\text{odd} \) is Schur \( P \)-positive if it is a nonnegative linear combination of Schur \( P \)-functions.

**Theorem 3.44.** Let \( y \in I(S_\infty) \) be weakly dominant with \( k = \kappa(y) \). Suppose \( D(r(y)) \) is equivalent to a skew shape of the form \( \delta_m/\lambda \) for some \( m \in \mathbb{P} \) and partition \( \lambda \subset \delta_m \). Then

\[
\hat{F}_y = s_{\delta_{k+1}} s_{\delta_m}/\lambda \quad \text{and} \quad \hat{F}_{\text{FPF}} = s_{\delta_k} s_{\delta_m}/\lambda
\]

where the second identity applies only in the case when \( y \) is fixed-point-free. Moreover, in this case the symmetric functions \( \hat{F}_y \) and (when defined) \( \hat{F}_{\text{FPF}} \) are Schur \( P \)-positive.

**Remark.** It will be shown in [24] that \( \hat{F}_y \) is Schur \( P \)-positive for all \( y \in I(S_\infty) \), but only in the special case just described does this follow directly from our present methods. It is still an open question whether \( \hat{F}_{\text{FPF}} \) for \( z \in I_{\text{FPF}}(S_\infty) \) is similarly Schur \( P \)-positive.

**Proof.** By Proposition 2.17 we have \( F_{r(y)} = s_{\delta_m}/\mu \), and it is well-known that this skew Schur function belongs to \( \Lambda^\text{odd} \), see Proof 2 of [46, Corollary 7.32], or just adapt the argument in [51, Proposition 7.17.7]. From this, the formulas for \( \hat{F}_y \) and \( \hat{F}_{\text{FPF}} \) are immediate by Proposition 3.35. For the last assertion, we note that skew Schur functions of the form \( s_{\delta_m}/\mu \) are Schur \( P \)-positive (see [1]), and that Schur \( P \)-positivity is closed under products (see [52, §8]).

The most important special case of the preceding result is Theorem 1.4 from the introduction, whose proof we now give.

**Proof of Theorem 1.4.** One checks that \( \kappa(w_n) = \lfloor n/2 \rfloor \) and \( r(w_n) = w_{\lfloor n/2 \rfloor} \). We have seen that \( D(w_n) \) is the shape of \( \delta_n \), so by Theorem 3.44 \( \hat{F}_{r_n} = s_{\delta_{k+1}} s_{\delta_m} \) for \( k = \lfloor \frac{n}{2} \rfloor \) and \( m = \lfloor \frac{n}{2} \rfloor \) and \( \hat{F}_{\text{FPF}} = (s_{\delta_n})^2 \). The theorem follows as \( \{ k+1, m \} = \{ p, q \} \) for \( p = \lfloor \frac{n-1}{2} \rfloor \) and \( q = \lfloor \frac{n+1}{2} \rfloor \).

### A Geometric complements

Throughout this appendix, we define \( \beta \) and \( K \subset \text{GL}_n(\mathbb{C}) \) as in Section 3.4, and fix a weakly dominant involution \( z \in I_K \) with \( p = \kappa(z) \) and set \( q = n - p \). It is convenient to define

\[
\mathcal{G}' = \begin{cases} 2^p \hat{G}_{\gamma_p} & \text{if } \beta \text{ is symmetric} \\ \hat{G}_{\gamma_p} & \text{if } \beta \text{ is skew-symmetric} \end{cases} \quad \text{and} \quad \mathcal{G}'' = \Phi_{p,q}(\mathcal{G}_{r(z)}(x;y)).
\]

We outline here a geometric proof of Theorem 8.27.

That statement asserts \( [Y^K_z] = \mathcal{G}' \cdot \mathcal{G}'' \), so it is natural to ask whether it is possible to write \( Y^K_z \) as a reasonable intersection of two subvarieties of \( \text{Fl}(n) \) with cohomology classes \( \mathcal{G}' \) and \( \mathcal{G}'' \). As
stated this question cannot have an affirmative answer, since the elements of the cohomology ring $H^*(\text{Fl}(n), \mathbb{Z})$ which are classes of subvarieties are exactly the nonnegative integer combinations of Schubert classes (cf. [33, Remark 2.1.2]), which include $\mathcal{G}'$ but not necessarily $\mathcal{G}''$.

Instead, define $\text{Fl}(k, n)$ as the variety of partial flags $F_\bullet = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq \mathbb{C}^n)$, with each $F_i$ being an $i$-dimensional subspace of $\mathbb{C}^n$, and consider the split flag variety

$$\text{Fl}^{sp}(p, q) = \{(F_\bullet, G_\bullet) \in \text{Fl}(p, n) \times \text{Fl}(q, n) : F_p \cap G_q = 0\}.$$

Let $\pi$ be the projection $\text{Fl}^{sp}(p, q) \to \text{Fl}(n)$ defined by

$$\pi(F_\bullet, G_\bullet) = (F_1 \subseteq \cdots \subseteq F_p \subseteq F_p \oplus G_1 \subseteq \cdots \subseteq F_p \oplus G_q).$$

It suffices by the following lemma to study the question posed in the previous paragraph in $\text{Fl}^{sp}(p, q)$ rather than $\text{Fl}(n)$.

**Lemma A.1.** The induced map $\pi^* : H^*(\text{Fl}(n), \mathbb{Z}) \to H^*(\text{Fl}^{sp}(p, q), \mathbb{Z})$ is an isomorphism.

**Proof.** The proof is similar to arguments discussed in [9 §3.6.13]. Let $\text{Gr}(q, \mathbb{C}^n)$ denote the Grassmannian of $q$-planes in $\mathbb{C}^n$. Then $\pi$ is the projection of a fiber bundle, whose fiber over $H_\bullet$ is

$$\{V \in \text{Gr}(q, \mathbb{C}^n) : V \cap H_p = 0\} \simeq \text{rowspan}\left[ \begin{array}{cccccc} * & \cdots & * & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ * & \cdots & * & 0 & \cdots & 1 \end{array} \right] \simeq \mathbb{C}^{pq}.$$

These fibers are contractible, and we claim that this makes $\pi$ a homotopy equivalence. One way to see this is to use the long exact sequence of a fibration to first conclude that $\pi$ is a weak homotopy equivalence. Both $\text{Fl}(n)$ and $\text{Fl}^{sp}(p, q)$ are smooth manifolds (the latter because it is an open subset of the smooth manifold $\text{Fl}(p, n) \times \text{Fl}(q, n)$) and every smooth manifold has the homotopy type of a CW-complex via Morse theory [32 Corollary 6.7]. By Whitehead’s theorem [21, Theorem 4.5], $\pi$ is a homotopy equivalence.

To prove Theorem 3.27 geometrically, we will exhibit subvarieties $Z$, $Z'$, and $Z''$ of $\text{Fl}^{sp}(p, q)$ whose cohomology classes correspond to $[Y^K_z]$, $\mathcal{G}'$, and $\mathcal{G}''$ under the isomorphism $\pi^*$, and which are such that the intersection $Z' \cap Z''$ is generically transverse and equal to $Z$. These hypotheses imply $[Z] = [Z'][Z'']$, and by passing back to $\text{Fl}(n)$, we will be able to conclude that Theorem 3.27 holds in the cohomology ring $H^*(\text{Fl}(n), \mathbb{Z})$.

To this end, define $Z'$ and $Z''$ as the subvarieties of $\text{Fl}^{sp}(p, q)$ given by

$$Z' = \{(F_\bullet, G_\bullet) \in \text{Fl}^{sp}(p, q) : \beta(F_p, F_p) = 0\}$$

$$Z'' = \{(F_\bullet, G_\bullet) \in \text{Fl}^{sp}(p, q) : \text{rank}(\beta|_{F_i \times G_j}) \leq \text{rank}(z)(i, j) \text{ for } (i, j) \in [p] \times [q]\}.$$

We also define an open subset $Z''$ of the second variety by replacing the inequality $\leq$ by $=$ in each rank condition. The following alternative description of $Z''$ will be useful.

**Observation A.2.** The rank of $\beta|_{F_i \times G_j}$ is equal to the rank of the Gram matrix $[\beta(f_i, g_j)]_{(i, j) \in [p] \times [q]}$.

For the rest of this section we write $\hat{\ell}_K(z)$ for the length given by $\hat{\ell}(z)$ when $K = O(n)$ and by $\hat{\ell}_{ppp}(z)$ when $K = \text{Sp}(n)$. Note that $\hat{\ell}_K(z)$ is then the codimension of $Y^K_z$ in $\text{Fl}(n)$.
Lemma A.3. Set-theoretically, we have $Z' \cap Z'' = \pi^{-1}(Y_z^K)$ and $Z' \cap \hat{Z}'' = \pi^{-1}(\hat{Y}_z^K)$.

Proof. We consider only the case when $K = O(n)$, since the symplectic case follows in the same way. Let $C_{ij}$ denote the set of $F_\bullet \in \text{Fl}(n)$ such that $\text{rank}(\beta|_{F_i \times F_j}) \leq \text{rk}_z(i,j)$. It follows by Proposition 3.16 and Lemma 3.23 that $\pi^{-1}(Y_z^K) = Y' \cap Y''$ where

$$Y' = \bigcap_{(i,j) \in \hat{D}(g_p)} \pi^{-1}(C_{ij}) \quad \text{and} \quad Y'' = \bigcap_{(i,j) \in E_p(r(z))} \pi^{-1}(C_{ij}).$$

If $(i,j) \in \hat{D}(g_p)$, so that $i, j \leq p$, then

$$\pi^{-1}(C_{ij}) = \{(F_\bullet, G_\bullet) \in \text{Fl}^{pp}(p,q) : \text{rank}(\beta|_{F_i \times F_j}) \leq \text{rk}_z(i,j)\}.$$

Since $\text{rk}_z(i,j) = 0$ whenever $i, j \leq p$, it follows that $Y' = Z'$. Suppose on the other hand that $(i,j) \in E_p(r(z))$, so that $j \leq p < i$. Then

$$\pi^{-1}(C_{ij}) = \{(F_\bullet, G_\bullet) \in \text{Fl}^{pp}(p,q) : \text{rank}(\beta|_{F_i \times F_j}) \leq \text{rk}_z(i,j)\}.$$

If $(F_\bullet, G_\bullet) \in Z'$ then $\text{rank}(\beta|_{F_j \times (G_{i-p} \times F_j)}) = \text{rank}(\beta|_{(G_{i-p} \times F_j)})$ since $\beta(F_p, F_p) = 0$. As $\text{rk}_z(i,j) = \text{rk}_{r(z)}(j, i-p)$ when $i > p$ by Lemma 3.23, it follows that

$$Z' \cap \pi^{-1}(C_{ij}) = \{(F_\bullet, G_\bullet) \in Z' : \text{rank}(\beta|_{G_{i-p} \times F_j}) \leq \text{rk}_z(i,j)\} = \{(F_\bullet, G_\bullet) \in Z' : \text{rank}(\beta|_{F_j \times G_{i-p}}) \leq \text{rk}_{r(z)}(j, i-p)\}.$$

Combining these observations, we see that $\pi^{-1}(Y_z^K) = \bigcap_{(i,j) \in E_p(r(z))} Z' \cap \pi^{-1}(C_{ij}) = Z' \cap Z''$. These arguments go through just as well if we replace the inequalities $\leq$ by $=$ in each rank condition, so we conclude that $\pi^{-1}(Y_z^K) = Z' \cap \hat{Z}''$ also. \qed

Lemma A.4. The varieties $Z'$ and $Z''$ intersect generically transversely; that is, they intersect transversely on a Zariski-open dense subset of $Z' \cap Z''$.

Proof sketch. It will be convenient to work at first in the full product $X = \text{Fl}(p,n) \times \text{Fl}(q,n)$ rather than $\text{Fl}^{pp}(p,q)$. To that end, we define subvarieties $\Gamma'$, $\Gamma''$, and $\hat{\Gamma}''$ of $X$ using the same rank conditions as $Z'$, $Z''$, and $\hat{Z}''$ but without the restriction that $F_p \cap G_q = 0$. Given $G_\bullet \in \text{Fl}(q,n)$, define

$$V'_{G_\bullet} = \{F_\bullet \in \text{Fl}(p,n) : (F_\bullet, G_\bullet) \in \Gamma'\} \quad \text{and} \quad \hat{V}''_{G_\bullet} = \{F_\bullet \in \text{Fl}(p,n) : (F_\bullet, G_\bullet) \in \hat{\Gamma}''\}.$$

It is not hard to show from the definition of transversality that if $V'_{G_\bullet}$ and $\hat{V}''_{G_\bullet}$ are transverse at $(F_\bullet, G_\bullet)$, and so are the analogous varieties where the first coordinate is fixed at $F_\bullet$, then $\Gamma'$ and $\hat{\Gamma}''$ are transverse at $(F_\bullet, G_\bullet)$. The second condition is easy to check, so we focus on transversality of $V'_{G_\bullet}$ and $\hat{V}''_{G_\bullet}$.

Let $h^*$ denote the adjoint of $h \in \text{GL}_n(\mathbb{C})$ with respect to the form $\beta$. It is a simple consequence of the relevant rank conditions that $\hat{V}''_{G_\bullet} = \hat{V}''_{G_\bullet}(h^*)^{-1}$. The varieties $V'_{G_\bullet}$ and $\hat{V}''_{G_\bullet}$ are smooth because they are homogeneous spaces for $K$ and a Borel subgroup depending on $G_\bullet$ respectively. Thus we can apply Kleiman’s transversality theorem [31] and conclude that, having fixed $G_\bullet$, there is a Zariski-open set of $h$ for which $V'_{G_\bullet}$ and $\hat{V}''_{G_\bullet}$ intersect transversely in the expected dimension.

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This leads to an open set $U \subseteq \text{Fl}^{p}(p,q)$ on which $\hat{Z}'$ and $Z''$ intersect transversely, and calculating the expected dimension of intersection shows that they do in fact intersect on $U$.

By Lemma A.3, $Z' \cap Z'' = \pi^{-1}(\hat{Y}_z K)$. Because $V'_{U}$ is $K$-stable, we can assume that $U$ is $K$-stable. Since $U \cap \pi^{-1}(\hat{Y}_z K) \neq \emptyset$ and $K$ acts transitively on $\pi^{-1}(\hat{Y}_z K)$, we see that $U$ intersects every component of $\pi^{-1}(\hat{Y}_z K)$, and therefore every component of its closure $\pi^{-1}(\tilde{Y}_z K)$. Thus $U \cap \pi^{-1}(\tilde{Y}_z K)$ is an open dense subset of $\pi^{-1}(\tilde{Y}_z K)$.

**Lemma A.5.** Relative to $\text{Fl}^{p}(p,q)$, it holds that $\text{codim} Z' = \hat{\ell}_K(g_p)$ and $\text{codim} Z'' = \ell(r(z))$.

**Proof.** Under the map $\text{Fl}^{p}(p,q) \to \text{Gr}(p, C^n)$ defined by $(F_{\bullet}, G_{\bullet}) \mapsto F_p$, the variety $Z'$ is the inverse image of the orthogonal (if $\beta$ is symmetric) or isotropic (if $\beta$ is skew-symmetric) Grassmannian

$$\{V \in \text{Gr}(p, C^n) : \beta(V, V) = 0\},$$

which one checks has codimension $\hat{\ell}_K(g_p)$ in $\text{Gr}(p, C^n)$. Since $(F_{\bullet}, G_{\bullet}) \mapsto F_p$ is the projection of a fiber bundle, $Z'$ has the same codimension.

Likewise, since $\pi$ is the projection of a fiber bundle, $\text{codim} \pi^{-1}(\tilde{Y}_z K) = \text{codim} Y_z K = \hat{\ell}_K(z)$. By Lemmas A.3 and A.4, the intersection $Z' \cap Z''$ is generically transverse and equal to $\pi^{-1}(\tilde{Y}_z K)$, so

$$\hat{\ell}_K(z) = \text{codim} \pi^{-1}(\tilde{Y}_z K) = \text{codim} Z' + \text{codim} Z'' = \hat{\ell}_K(g_p) + \text{codim} Z''.$$ 

Hence, $\text{codim} Z'' = \hat{\ell}_K(z) - \hat{\ell}_K(g_p) = \ell(r(z))$.

Next, we compute the cohomology classes of $Z'$ and $Z''$ by interpreting them as degeneracy loci, according to the following general setup. Suppose we have two vector bundles $P$ and $Q$ over a smooth complex variety $X$ and a bundle map $f : P \to Q$. We are interested in the locus $L$ of points $x \in X$ where $f_x : P_x \to Q_x$ satisfies some kind of rank conditions, and $P_x$ denotes the fiber of $P$ over $x$. We will compute the classes of $Z'$ and $Z''$ by realizing them as such loci $L$ and appealing to theorems which give universal formulas for $[L]$ in terms of Chern classes of $P$ and $Q$.

Let $0 \subseteq P_1 \subseteq \cdots \subseteq P_n \to X$ be a complete flag of vector bundles over a smooth complex variety $X$, so that rank $P_i = i$. Dually, let $X \leftarrow Q_n \to Q_{n-1} \to \cdots \to Q_1 \to 0$ be a sequence of surjections of vector bundles with rank $Q_i = i$. Suppose $f : P_n \to Q_n$ is a bundle map. Given $v \in S_n$, let $L_v$ be the locus where $\text{rank}(f : P_i \to Q_j) \leq \text{rk}_v(i, j)$; this construction covers all the rank conditions one could impose on $f$ which give non-empty loci, cf. [15] Lemma 3.1. Write $c_i(S)$ for the $i^{th}$ Chern class of $S$, an element of $H^{2i}(X, \mathbb{Z})$. Finally, define $x_i = c_1(\ker T_i \to T_{i-1})$ and $y_i = c_1(P_i/P_{i-1})$.

Fulton [15] gives this formula for $[L_v]$ in terms of double Schubert polynomials:

**Theorem A.6 (See [15]).** If $\text{codim} L_v = \ell(v)$, then $[L_v] = \mathcal{G}_v(x; y) \in H^{2\ell(v)}(X)$.

Say that a polynomial $f \in \mathcal{P}_n$ represents an element of $H^*(\text{Fl}^{p}(p,q), \mathbb{Z})$ if it represents the image of this element under the map given by composing the inverse of $\pi^*$ with the Borel isomorphism.

**Lemma A.7.** $\mathcal{G}''$ represents the cohomology class $[Z''] \in H^*(\text{Fl}^{p}(p,q), \mathbb{Z})$.

**Proof.** Let $T_{\bullet}$ be the tautological flag of vector bundles on $\text{Fl}(n)$, so that the fiber of $T_i$ over $F_{\bullet}$ is $F_i$. Upon pulling back to $\text{Fl}^{p}(p,q)$ via $\pi$, $T$ splits into two flags of bundles, $0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{p} = \pi^* T_p$ and $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{q}$, where the respective fibers of $U_i$ and $V_i$ over $(F_{\bullet}, G_{\bullet})$ are
Under the Borel isomorphism discussed in Section 2.4, \( x_i \in H^*(\text{Fl}(n), \mathbb{Z}) \) corresponds to \( c_1((T_i/\mathcal{T}_{i-1})^*) = -c_1(T_i/\mathcal{T}_{i-1}) \). Since
\[
\pi^*(T_i/\mathcal{T}_{i-1}) = \begin{cases} 
U_i/U_{i-1} & \text{if } i \leq p \\
V_{i-p}/V_{i-p-1} & \text{if } i > p,
\end{cases}
\]
the functoriality of Chern classes shows that \( \pi^*(x_i) = -c_1(U_i/U_{i-1}) \) or \( -c_1(V_{i-p}/V_{i-p-1}) \) as appropriate. By Lemma A.1, \( \pi^* \) is an isomorphism, and we identify \( x_i \) with \( \pi^*(x_i) \in H^*(\text{Fl}^p(p,q), \mathbb{Z}) \) in the rest of this section.

Define a bundle map \( f : V_q \to U_p^* \) on the fiber over \( (F_*, \mathcal{G}_*) \) by \( f(v) = \beta(v, \cdot)|_{F_p} \). Then \( Z'' \) is the locus where \( \text{rank}(f : V_i \to U_j) \leq \text{rk}_{r(z)}(i,j) \). Since \( \text{codim} Z'' \leq \ell(r(z)) \) by Lemma A.5, Theorem A.6 gives \( [Z''] = G_{r(z)}(u;v) \) where
\[
u_i = c_1(\ker(U_i^* \to U_{i-1}^*)) = c_1(U_i/U_{i-1}^*) = x_i \quad \text{and} \quad \nu_i = c_1(V_i/V_{i-1}) = -x_{k+i}
\]
which means precisely that \( [Z''] = \mathcal{O}' \). \( \square \)

To compute \( [Z'] \) we use a degeneracy locus result for bundle maps with some symmetries. Say that a linear map \( g : V \to V^* \) from a vector space to its dual is symmetric or skew-symmetric if the associated bilinear form \( (v, w) \mapsto g(v)(w) \) is symmetric or skew-symmetric.

**Theorem A.8** (See [20]). Suppose \( P \) is a rank \( n \) vector bundle over a smooth complex variety \( X \), and \( g : P \to P^* \) is a bundle map. Let \( L_r \subseteq X \) be the locus where \( g \) has rank at most \( r \).

(a) If \( g \) is symmetric and \( L_r \) has codimension \( \binom{n-r+1}{2} \), then
\[
[L_r] = 2^{n-r} \det[c_{n-r-2i+j+1}(P^*)]_{1 \leq i, j \leq n-r}.
\]

(b) If \( g \) is skew-symmetric and \( r \) is even and \( L_r \) has codimension \( \binom{n-r}{2} \), then
\[
[L_r] = \det[c_{n-r-2i+j}(P^*)]_{1 \leq i, j \leq n-r-1}.
\]

**Lemma A.9.** \( \mathcal{O}' \) represents the cohomology class \( [Z'] \in H^*(\text{Fl}^p(p,q), \mathbb{Z}) \).

**Proof.** Let \( U, V, \) and \( x_1, \ldots, x_n \) be as in the proof of Lemma A.7. Define \( g : U_p \to U_p^* \) on the fiber over \( (F_*, \mathcal{G}_*) \) by \( g(v) = \beta(v, \cdot)|_{F_p} \). Then \( g \) is symmetric (if \( K = O(n) \)) or skew-symmetric (if \( K = \text{Sp}(n) \)), and \( Z' \) is the locus where \( g \) has rank zero. By Lemma A.5, codim \( Z' \) is the expected codimension required for Theorem A.8, so
\[
[Z'] = \begin{cases} 
2^n \det[c_{p-2i+j+1}(U_p^*)]_{1 \leq i, j \leq p} & \text{if } K = O(n) \\
\det[c_{p-2i+j}(U_p^*)]_{1 \leq i, j \leq p-1} & \text{if } K = \text{Sp}(n).
\end{cases}
\]
We have \( U_p^* \simeq \bigoplus_{i=1}^n (U_i/U_{i-1})^* \), so the Whitney sum formula gives \( c_i(U_p^*) = c_i(x_1, \ldots, x_p) \). The Jacobi-Trudi identity alongside Proposition 3.10 now shows that \( [Z'] = \mathcal{O}' \). \( \square \)

**Remark.** As discussed in [31], involution Schubert polynomials also arise from degeneracy locus formulas. Suppose \( 0 \subseteq P_1 \subseteq \cdots \subseteq P_n \) is a complete flag of vector bundles over a smooth complex variety \( X \). Assume \( P_n \) is equipped with a non-degenerate bilinear form \( \beta \) which is either symmetric
or skew-symmetric. Given $y \in I_K$ where $K$ is the group of automorphisms of $\beta$ on a fiber of $P_n$, define the locus

$$L_y = \left\{ x \in X : \text{rank} \left( \beta|_{(P_i)_x \times (P_j)_x} \right) \leq \text{rk}_y(i, j) \right\} \text{ for all } 1 \leq i, j \leq n.$$ 

Let $x_i$ be the Chern class $c_1(P_i/P_{i-1})$. It is shown in [54] that if $\beta$ is suitably generic, then

$$[L_y] = \begin{cases} 2^{\kappa(y)} \hat{G}_y(x_1, \ldots, x_n) & \text{if } \beta \text{ is symmetric} \\ \hat{G}_{FPF}(x_1, \ldots, x_n) & \text{if } \beta \text{ is skew-symmetric.} \end{cases}$$

We can now reprove Theorem 3.27. Since $\pi$ is the projection of a fiber bundle, $\pi^*[Y^K_z] = [\pi^{-1}(Y^K_z)]$ (cf. [16, §B.1]), and so by Lemmas A.3 and A.4, we have

$$\pi^*[Y^K_z] = [Z'][Z''].$$

By Lemmas A.7, and A.9, passing back to $H^*(\text{Fl}(n), \mathbb{Z})$ via $(\pi^*)^{-1}$, while recalling our identification of $\pi^*(x_i)$ with $x_i$, gives

$$[Y^K_z] \equiv \mathcal{G}' \cdot \mathcal{G}''. \quad (A.1)$$

This proves Theorem 3.27 in $H^*(\text{Fl}(n), \mathbb{Z})$, which, however, is weaker than the equality of polynomials given by Theorem 3.27. To deduce the latter, it suffices to show that $\mathcal{G}' \cdot \mathcal{G}''$ is in $S = \mathbb{Z}\text{-span}\{\mathcal{G}_w : w \in S_n\}$, since these Schubert polynomials remain linearly independent in $H^*(\text{Fl}(n), \mathbb{Z})$, and $\hat{G}_z$ is also in $S$. As per [16, §10.4],

$$S = \mathbb{Z}\text{-span}\{x_1^{i_1} \cdots x_n^{i_n} : 0 \leq i_j \leq n - j \text{ for each } j\}.$$ 

Thus, if $x_1^{i_1} \cdots x_n^{i_n}$ is a monomial in $\mathcal{G}'$, then $i_j \leq p - j$ for $j \leq p$ while $i_j = 0$ for $j > p$. Similarly, one checks using Definition 2.10 that for a monomial in $\mathcal{G}''$, $i_j \leq n - p - j$ for $j \leq p$ while $i_j \leq n - j$ for $j > p$. Combining these, for a monomial in $\mathcal{G}' \cdot \mathcal{G}''$ one has $i_j \leq n - 2j$ for $j \leq p$ while $i_j \leq n - j$ for $j > p$. In particular, $\mathcal{G}' \cdot \mathcal{G}'' \in S$.

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