Stratification of spaces of locally convex curves by itineraries

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Abstract

Locally convex (or nondegenerate) curves in the sphere $\mathbb{S}^n$ (or the projective space) have been studied for several reasons, including the study of linear ordinary differential equations of order $n + 1$. Taking Frenet frames allows us to translate such curves $\gamma$ into corresponding curves $\Gamma$ in the flag space, the orthogonal group $SO_{n+1}$ or its double cover $Spin_{n+1}$. Determining the homotopy type of the space of such closed curves or, more generally, of spaces of such curves with prescribed initial and final jets appears to be a hard problem, which has been solved for $n = 2$ but otherwise remains open. This paper is a step towards solving the problem for larger values of $n$. In the process, we prove a related conjecture of B. Shapiro and M. Shapiro regarding the behavior of fundamental systems of solutions to linear ordinary differential equations.

We define the itinerary of a locally convex curve $\Gamma : [0, 1] \to Spin_{n+1}$ as a (finite) word $w$ in the alphabet $S_{n+1} \setminus \{e\}$ of non-trivial permutations. This word encodes the succession of non-open Bruhat cells of $Spin_{n+1}$ pierced by $\Gamma(t)$ as $t$ ranges from 0 to 1. We prove that, for each word $w$, the subspace of curves of itinerary $w$ is an embedded contractible (globally collared topological) submanifold of finite codimension, thus defining a stratification of the space of curves. We show how to obtain explicit (topologically) transversal sections for each of these submanifolds. We study both a space of curves with minimum regularity hypotheses, where only topological transversality applies, and spaces of sufficiently regular curves, where transversality has the usual meaning. In both cases we also study the adjacency relation between strata.

This is an important step in the construction of CW cell complexes mapped into the original space of curves by weak homotopy equivalences. Our stratification is not as nice as might be desired, lacking for instance the Whitney property. Somewhat surprisingly, the differentiability class of the curves affects some properties of the stratification. The necessary ingredients for the construction of a dual CW complex are proved.
1 Introduction

For a fixed integer \( n \geq 2 \), let \( \text{Spin}_{n+1} \) be the universal covering of the orthogonal group \( \text{SO}_{n+1} \). For \( j \in \{1, 2, \ldots, n\} \), consider the skew-symmetric tridiagonal matrices \( a_j = e_{j+1}^\top e_j^\top - e_j e_{j+1}^\top \in \mathfrak{so}_{n+1} \) as elements of the Lie algebra \( \mathfrak{spin}_{n+1} \), via the isomorphism of Lie algebras induced by the covering map \( \Pi \).

A map \( \Gamma : J \to \text{Spin}_{n+1} \) defined on an interval \( J \subseteq \mathbb{R} \) is called a locally convex curve if it is absolutely continuous (hence differentiable almost everywhere) and its logarithmic derivative is almost everywhere of the form

\[
(\Gamma(t))^{-1}\Gamma'(t) = \sum_{j \in [n]} \kappa_j(t)a_j,
\]

for positive functions \( \kappa_1, \ldots, \kappa_n : J \to (0, +\infty) \).

Given a smooth locally convex curve \( \Gamma \), the smooth curve \( \gamma : J \to S^n \) defined by \( \gamma(t) = \Pi(\Gamma(t))e_1 \) satisfies \( \det(\gamma(t), \gamma'(t), \ldots, \gamma^{(n)}(t)) > 0 \) for all \( t \in J \). A parametric curve \( \gamma : J \to \mathbb{R}^{n+1} \) of class \( C^n \) satisfying the inequality above is also called (positive) locally convex \([2, 30, 31]\) or (positive) nondegenerate \([13, 20, 22, 26]\). Such a curve \( \gamma \) can be lifted to a locally convex curve \( \tilde{\gamma} \) in \( \text{SO}_{n+1} \) (and therefore in \( \text{Spin}_{n+1} \)) of class \( C^1 \) by taking the orthogonal matrix \( \mathbb{F}_{\tilde{\gamma}}(t) \) whose column-vectors are the result of applying the Gram-Schmidt algorithm to the ordered basis \( (\gamma(t), \gamma'(t), \ldots, \gamma^{(n)}(t)) \) of \( \mathbb{R}^{n+1} \). The orthogonal basis of \( \mathbb{R}^{n+1} \) thus obtained is the (generalized) Frenet frame of the space curve \( \gamma \). By the classical Frenet-Serret formulæ, the coefficients \( \kappa_1, \ldots, \kappa_n \) of the logarithmic derivative of \( \mathbb{F}_{\gamma} \) admit geometric interpretations: \( \kappa_1 = v_\gamma = |\gamma'| \) is the velocity of \( \gamma \); \( \kappa_2 = v_\gamma x_1 \), where \( x_1 \) is the geodesic curvature of \( \gamma \); \( \kappa_3 = v_\gamma x_2 \), where \( x_2 \) is the geodesic torsion of \( \gamma \), and so on \([23, 27]\). The term locally convex comes from the fact that a nondegenerate curve \( \gamma : J \to \mathbb{R}^{n+1} \) can be partitioned into finitely many convex arcs, i.e., arcs that intersect any \( n \)-dimensional subspace of \( \mathbb{R}^{n+1} \) at most \( n \) times (with multiplicities taken into account); see Subsection 2.3.

Given \( r \in \mathbb{N}^n \) and \( z_0, z_1 \in \text{Spin}_{n+1} \), let \( \mathcal{L}_n^{[C^r]}(z_0; z_1) \) denote the space of locally convex curves \( \Gamma : [0, 1] \to \text{Spin}_{n+1} \) of differentiability class \( C^r \) with endpoints \( \Gamma(0) = z_0 \) and \( \Gamma(1) = z_1 \). We endow this space with the usual \( C^r \) topology and consider the problem of describing its homotopy type. This is equivalent to the problem of studying the homotopy type of the space \( \mathcal{L}_{S^n}(z_0; z_1) \) of nondegenerate spherical curves \( \gamma : [0, 1] \to S^n \) satisfying \( \mathbb{F}_\gamma(0) = z_0 \) and \( \mathbb{F}_\gamma(1) = z_1 \) with the subspace topology inherited from \( C^\infty([0, 1], \mathbb{R}^{n+1}) \) (see Subsection 2.2). Some historical motivation for this problem is given at the end of this introduction. Of course, we have the natural homeomorphism \( \mathcal{L}_n^{[C^r]}(z_0; z_1) \approx \mathcal{L}_n^{[C^r]}(1; z_0^{-1}z_1) \). The present paper provides an important preliminary step for the construction of an abstract cell complex \( \mathcal{D}_n(z) \) weak homotopy equivalent to \( \mathcal{L}_n^{[C^r]}(1; z) \).

The existence of \( \mathcal{D}_n(z) \) and the construction of its lowest dimensional skeletons
We call the complement $\text{Sing}(\Gamma) \setminus \{t_0, t_1\}$ the singular set group; this is a singular variety of codimension one. Accordingly, we define $\eta$ are open, and their union $\text{Bru}$ is the lift to $\text{Spin}_{n+1}$ of the subgroup $\text{SO}_{n+1}$ of signed permutation matrices with positive determinant. The acute map $\sigma \in S_{n+1} \mapsto \sigma \in \tilde{B}_{n+1}^+$, which is not a homomorphism, is a right inverse to the natural projection $\tilde{B}_{n+1}^+ \to S_{n+1}$ and is defined in Equation (8) in Subsection 2.1 (also, Equation (2) of [15]). We can then write each element $z \in \tilde{B}_{n+1}^+$ as $z = q\sigma$ for unique $\sigma \in S_{n+1}$ and $q \in \text{Quat}_{n+1}$. Here, $\text{Quat}_{n+1} \subset \tilde{B}_{n+1}^+$ is the lift to $\text{Spin}_{n+1}$ of the subgroup $\text{Diag}_{n+1}^+ \subset \tilde{B}_{n+1}^+$ of diagonal matrices.

Let $\eta \in S_{n+1}$ have the maximum number of inversions, $\text{inv}(\eta) = n(n+1)/2$, i.e., let $\eta : j \mapsto n+2-j$. The element $\eta$ is called the top permutation of $S_{n+1}$ and is often denoted in the literature by $w_0$. The cells $\text{Bru}_{q\eta}$, $q \in \text{Quat}_{n+1}$, are open, and their union $\text{Bru}_n$ is a dense open subspace of the spin group. We call the complement $\text{Sing}_{n+1} = \text{Spin}_{n+1} \setminus \text{Bru}_n$ the singular set of the spin group; this is a singular variety of codimension one. Accordingly, we define the singular set of a locally convex curve $\Gamma : [t_0, t_1] \to \text{Spin}_{n+1}$ as $\text{sing}(\Gamma) = \Gamma^{-1}[\text{Sing}_{n+1}] \setminus \{t_0, t_1\} \subset (t_0, t_1)$; the elements of $\text{sing}(\Gamma)$ are sometimes called...
the moments of non-transversality between the osculating flag of \( \gamma = (\Pi \circ \Gamma)e_1 \) and the standard complete flag of \( \mathbb{R}^{n+1} \) \cite{38}. Theorem 3 of \cite{13} implies that nondegenerate curves \( \Gamma \in \mathcal{L}_n(0; 1) \) have finite singular sets \( \text{sing}(\Gamma) \subset (0, 1) \).

Recall that the Hausdorff distance \cite{12} between two nonempty compact sets \( X, Y \subset [0, 1] \) is:

\[
d_H(X, Y) = \max \left\{ \left( \sup_{x \in X} \inf_{y \in Y} |x - y| \right), \left( \sup_{y \in Y} \inf_{x \in X} |x - y| \right) \right\};
\]

we also define \( d_H(\emptyset, X) = 1 \) for \( X \neq \emptyset \) and \( d_H(\emptyset, \emptyset) = 0 \). Let \( \mathcal{H}([0, 1]) \subset 2^{[0,1]} \) be the set of compact subsets of \([0, 1]\); this is a complete metric space with the Hausdorff distance where the empty set is an isolated point.

**Theorem 1.** Given \( z_0, z_1 \in \text{Spin}_{n+1} \), the map \( \text{sing} : \mathcal{L}_{n}(0; 1) \rightarrow \mathcal{H}([0, 1]) \) defined by \( \Gamma \mapsto \text{sing}(\Gamma) \) is continuous.

This is obtained as an immediate consequence of Lemma \cite{32}. These results imply in particular that when a locally convex curve is deformed (while remaining in \( \mathcal{L}_n(0; 1) \)), points in the singular set may join or split but never vanish or appear out of nowhere. This is closely related to the known fact \cite{4, 36, 39} that the Hausdorff distance \cite{12} between two nonempty compact sets \( X, Y \subset [0, 1] \) is:

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\]

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Given a locally convex curve \( \Gamma : [t_0, t_1] \rightarrow \text{Spin}_{n+1} \), write \( \text{sing}(\Gamma) = \{ \tau_1 < \cdots < \tau_\ell \} \subset (t_0, t_1) \) and, for each \( j \in \mathbb{Z} \), let \( \Gamma(\tau_j) \in \text{Bru}_{\sigma_j} \), \( \sigma_j \in S_{n+1} \setminus \{ e \} \). Let \( \mathcal{W}_n \) be the set of finite words in the alphabet \( S_{n+1} \setminus \{ e \} \). We define the *itinerary* of \( \Gamma \) by \( \text{iti}(\Gamma) = (\sigma_1, \ldots, \sigma_\ell) \in \mathcal{W}_n \).

We define our working space of locally convex curves as

\[
\mathcal{L}_n = \bigcup_{q \in \text{Quat}_{n+1}} \mathcal{L}_{n}(1; q).
\]

Given \( z_0, z_1 \in \text{Spin}_{n+1} \), we can determine explicitly an element \( q \in \text{Quat}_{n+1} \) such that the spaces \( \mathcal{L}_n(z_0; 1) \) and \( \mathcal{L}_n(1; q) \) are homeomorphic. Therefore, in order to understand all the spaces \( \mathcal{L}_n(z_0; 1) \), one may restrict attention to the disjoint union of \( 2^{n+1} \) spaces in Equation \( 2 \). The problem of determining whether the spaces \( \mathcal{L}_n(q_0) \) and \( \mathcal{L}_n(q_1) \) are homeomorphic (where \( q_0, q_1 \in \text{Quat}_{n+1} \), \( q_0 \neq q_1 \)) has been considered in \cite{2, 31}; Corollary 1.1 in \cite{14} gives partial results. For \( n = 3 \), our methods allow for a rather complete discussion in \cite{I}, culminating
in the computation of the homotopy type of $L_3(z)$ for all $z \in \text{Spin}_4$: see Corollary 10.1 and Theorem 5 in the Final Remarks.

For $w = (\sigma_1, \ldots, \sigma_\ell) \in W_n$, set

$$\hat{w} = \hat{\sigma}_1 \cdots \hat{\sigma}_\ell \in \text{Quat}_{n+1}, \quad \dim(w) = \dim(\sigma_1) + \cdots + \dim(\sigma_\ell),$$

where $\sigma \in S_{n+1} \mapsto \hat{\sigma} \in \text{Quat}_{n+1}$ is the hat map defined in Equation (8) and $\dim(\sigma) = \text{inv}(\sigma) - 1$, for all $\sigma \in S_{n+1}$. Notice that $\eta\hat{w}^2 \in \text{Quat}_{n+1}$ for all words $w \in W_n$. Let $L_n[w] \subset L_n$ be the subset of curves with itinerary $w$. 

\begin{remark}
We recall the concepts of tubular neighborhood and collared topological submanifold. Let $M_0$ be a (finite or infinite dimensional) manifold and $M_1 \subseteq M_0$: the subset $M_1$ is called a \textit{(globally) collared topological submanifold of codimension $d$} if and only if there exists an open set $\hat{A}_0$, $M_1 \subseteq \hat{A}_0 \subseteq M_0$, which is a \textit{tubular neighborhood of $M_1$} (based on \cite{[6]}). We say that $\hat{A}_0$ as above is a tubular neighborhood if there exist an open ball $B \subseteq \mathbb{R}^d$, $0 \in B$, a continuous projection $\Pi : \hat{A}_0 \to M_1 \subseteq \hat{A}_0$ and a continuous map $\hat{F} : \hat{A}_0 \to B$ such that the map $(\Pi, \hat{F}) : \hat{A}_0 \to M_1 \times B$ is a homeomorphism. Embedded $C^2$ submanifolds of Hilbert spaces with finite codimension are collared topological submanifolds: in this case $\Pi$ can be taken to be the normal projection. \hfill \Diamond
\end{remark}

We are mostly interested in the cases $H^1$ (essentially no regularity hypotheses) and $H^r$ for large $r$.

\begin{theorem}
For $r \in \mathbb{N}^*$ and $w \in W_n$, we have $L_n^{[H^r]}[w] \subseteq L_n^{[H^r]}(\eta\hat{w}^2)$; also:

\begin{enumerate}
  \item The set $L_n^{[H^1]}[w]$ is a contractible globally collared topological submanifold of $L_n^{[H^1]}(\eta\hat{w}^2)$ of codimension $\dim(w)$.
  \item If $r \geq 3$, then $L_n^{[H^r]}[w]$ is an embedded $C^{r-1}$ submanifold of $L_n^{[H^r]}(\eta\hat{w}^2)$ of codimension $\dim(w)$ (with tubular neighborhood fibred by normal balls).
\end{enumerate}

In particular, all words $w \in W_n$ are realizable as itineraries of locally convex curves, the empty word $(\ ) \in W_n$ being the itinerary of the convex curves. In fact, it follows from Lemma \cite{[8]} that $L_{n,\text{convex}} = L_n(\ )$ is a contractible connected component of $L_n$ contained in $L_n(\hat{\eta})$, where $\hat{\eta} = \eta^2$, consistently with known results \cite{[4], [36], [39]}. The proof of Theorem 2 is presented in Section 6. Some preliminary steps are covered in Sections 4 and 5.

We have thus defined the \textit{itinerary stratification} that gives this paper its title:

$$L_n = \bigsqcup_{w \in W_n} L_n[w]; \quad L_n[w] = \{\Gamma \in L_n \mid \text{iti}(\Gamma) = w\}. \quad (4)$$

We devote the last part of this paper to investigate how these strata fit together. Explicit parameterizations of transversal sections of $L_n[w]$ are constructed
in Section 7 with this goal in mind. Unlike the homotopy type of the spaces $L_n(z_0; z_1)$, this turns out to be sensitive to the regularity class, i.e., on which version $L^*_n$ we are actually using (see Section 9).

We produce a simple, visual example below. For $n \leq 4$, we use the simplified notation $a = a_1$, $b = a_2$, $c = a_3$, $d = a_4$ for the Coxeter generators of $S_{n+1}$. We also write a word in $W_n$ as a string of letters, as in, say, $ab[ab]abb[aba][ab] = (a, b, ab, a, b, b, aba, ab)$. Square brackets are used to avoid confusion between, say, $a[ba] = (a, ba)$, $[aba] = (aba)$ and $aba = (a, b, a)$, of respective lengths 2, 1 and 3.

Figure 1: A family of curves in $L_2$. The equator is dashed and the fat dot indicates $e_1$. The vector $e_2$ is pointing to the right.

**Example 1.2.** Let $n = 2$. In Figure 1, we draw the nondegenerate curve $\gamma : [t_0, t_1] \to \mathbb{S}^2$, $\gamma(t) = \Pi(\Gamma(t))e_1$, as a visual representation of the corresponding locally convex curve $\Gamma = F_\gamma : [t_0, t_1] \to \text{Spin}_3$. A letter $a = a_1$ in $\text{iti}(\Gamma)$ corresponds to the curve $\gamma$ transversally crossing the equator (i.e., the great circle $x_3 = 0$) at a point different from $\pm e_1$. A letter $b = a_2$ occurs when the tangent geodesic (great circle) to $\gamma$ at $t$ includes the points $\pm e_1$ but the $x_3$-coordinate of $\gamma(t)$ is non-zero. A letter $[ab]$ indicates that the curve is tangent to the equator, but not at $\pm e_1$. A letter $[ba]$ declares that the curve crosses the equator transversally at $\pm e_1$. Finally, $[aba]$ proclaims that the curve is tangent to the equator at $\pm e_1$. Figure 1 shows a two-parameter family of (arcs of) curves in $L_2$ illustrating all these cases. The reader may want to compare this with the explicit parameterization of a transversal section of $L_2[[aba]]$ obtained in Example 7.2.
We define a partial order in $W_n$ by

$$w_0 \preceq w_1 \iff \mathcal{L}_n^{[H^1]}[w_1] \subseteq \overline{\mathcal{L}_n^{[H^1]}[w_0]}.$$  (5)

The Hasse diagram in Figure 2 represents the above partial order restricted to \{w ∈ W_2 | w ≤ [aba]\} = \{[aba], a[ba], [ba]a, b[ab], [ab]b, aa, abab, baba, bb\}.

Equation (5) defines a poset structure in $W_n$ that inherits (so to speak) some features from the strong Bruhat order $\leq$ in $S_{n+1}$; recall that $\sigma_0 \leq \sigma_1$ in $S_{n+1}$ if and only if $\text{Bru}_{\sigma_0} \subseteq \text{Bru}_{\sigma_1}$ in $\text{Spin}_{n+1}$ (see, for instance, Corollary 1.1 of [15]; notice the reversion of the indices in relation to Equation (5)).

**Theorem 3.** For $w_0, w_1 = (\sigma_1, \ldots, \sigma_\ell) ∈ W_n$, $w_0 \preceq w_1$ is equivalent to each one of the following conditions:

(i) $\mathcal{L}_n^{[H^1]}[w_1] \subseteq \overline{\mathcal{L}_n^{[H^1]}[w_0]}$;

(ii) $\mathcal{L}_n^{[H^1]}[w_1] \cap \overline{\mathcal{L}_n^{[H^1]}[w_0]} \neq \emptyset$;

(iii) given $\Gamma_1 ∈ \mathcal{L}_n^{[H^1]}[w_1]$, $\epsilon > 0$, $\text{sing}(\Gamma_1) = \{t_1 < \cdots < t_\ell\}$ and an open neighborhood $U ⊂ \mathcal{L}_n^{[H^1]}$ of $\Gamma_1$ there exists $\Gamma ∈ U \cap \mathcal{L}_n^{[H^1]}[w_0]$ with $\Gamma$ and $\Gamma_1$ coinciding outside $\cup_{i∈[\ell]}(t_i - \epsilon, t_i + \epsilon)$;

(iv) there exist nonempty words $\tilde{w}_1, \ldots, \tilde{w}_\ell ∈ W_n$ such that $w_0 = \tilde{w}_1 \cdots \tilde{w}_\ell$ and, for all $i ∈ [\ell]$, $\tilde{w}_i ≤ (\sigma_i)$.

The empty word ( ) ∈ $W_n$ is an isolated point. We prove Theorem 3 in Section 8. The corresponding statement is false for $* = [H^r]$, $r$ large; indeed, we shall see in Section 9 that the Whitney condition fails:

$$\mathcal{L}_3^{[H^1]}[[acb]] \subset \overline{\mathcal{L}_3^{[H^1]}[cabca]},$$

$$\mathcal{L}_3^{[H^1]}[[acb]] \not\subset \overline{\mathcal{L}_3^{[H^1]}[cabca]}.$$  (6)

Thus, in the equivalent statement to Theorem 3 for $r ≥ 3$, the equivalence between conditions (i) and (ii) does not hold; likewise, condition (ii) and (iii) are not
equivalent in the $r \geq 3$ case. In Lemma 9.1 we state and prove the equivalence between (ii) and a version of (iv) for arbitrary large $r$.

As we write this paper, some natural and rather basic questions concerning the partial order $\preceq$ are still open; Conjecture 1.3 below is essentially equivalent to Conjecture 2.4 in [38]. For $\sigma \in S_{n+1}$, set $\text{mult}(\sigma) = (\text{mult}_1(\sigma), \ldots, \text{mult}_n(\sigma)) \in \mathbb{N}^n$, where, for each $j \in [n]$, we have

$$\text{mult}_j(\sigma) = (1^{\sigma} + \cdots + j^{\sigma}) - (1 + \cdots + j) \in \mathbb{N} = \{0, 1, 2, \ldots\},$$

as in Theorem 4 of [15]. For $w = (\sigma_1, \ldots, \sigma_\ell) \in W_n$, define

$$\text{mult}(w) = \text{mult}(\sigma_1) + \cdots + \text{mult}(\sigma_\ell) \in \mathbb{N}^n.$$ 

For $u, v \in \mathbb{N}^n$, we write $u \leq v$ if and only if $u_j \leq v_j$ for all $j \in [n]$.

**Conjecture 1.3.** Given $w_0, w_1 \in W_n$, if $w_0 \preceq w_1$ then $\text{mult}(w_0) \leq \text{mult}(w_1)$.

Given $n \in \mathbb{N}$, $n \geq 2$, let

$$r_\bullet(n) = \left(\frac{n+1}{2}\right)^2 = \max\{\text{mult}_j(\sigma); \sigma \in S_{n+1}, j \in [n]\}.$$ 

**Theorem 4.** For $w_0, w_1 \in W_n$, if $r > r_\bullet(n)$ and $L_n^{[H^r]}[w_1] \cap \overline{L_n^{[H^r]}[w_0]} \neq \emptyset$, then $\text{mult}(w_0) \leq \text{mult}(w_1)$.

Notice that these conditions imply $w_0 \preceq w_1$. We prove Theorem 4 in Section 9. In [14] we apply an argument similar to Poincaré duality to obtain from the itinerary stratification (Equation (4)) a CW complex $D_n$ with a cell of dimension $\dim(w)$ for each $w \in W_n$ and a weak homotopy equivalence $c : D_n \to L_n$ (see also [13] for an older version of this construction). A similar finite dimensional construction is presented in [3] with the aim of computing the homotopy type of the intersections of two real Bruhat cells. In both cases, the Whitney condition is violated (see Equation (6)). Theorem 3 allows us to circumvent, in the construction of $D_n$, the inconvenient fact that Conjecture 1.3 remains an open problem (for $n > 4$; but see [32] and [33] for some recent partial results).

The space $\mathcal{L}S^2(I) \approx \mathcal{L}_2(-1) \sqcup \mathcal{L}_2(1)$ of closed nondegenerate curves in $S^2$ was originally studied by J. Little in the seventies [26], and shown to have three connected components. These are, in our notation: $\mathcal{L}_2(+1)$, containing curves with an odd number of self-intersections (counted with multiplicity); $\mathcal{L}_2,\text{convex}(-1)$, the subspace of simple curves; and $\mathcal{L}_2,\text{non-convex}(-1)$, containing curves with positive even number of self-intersections (again with multiplicity). The works of B. Khesin, B. Shapiro and M. Shapiro in the nineties [22, 36, 39] extended this result for $n$ and $z \in \text{Spin}_{n+1}$ arbitrary, showing that $\mathcal{L}_n(z)$ has one or two connected components: one if and only if it does not contain convex curves and
two otherwise, one of them being the contractible subspace $L_{n,\text{convex}}(z)$. In [30] the spaces $L_3(z)$ were completely classified into three homotopy types explicitly described. Our approach via the CW complex $D_n$ has already allowed further progress on the problem of describing the homotopy types of the spaces $L_n(z)$ for $n > 2$. We expect to present our new results, in particular solving the problem for $n = 3$. For more information, see our final remarks in Section 10.

Our problem is related to the study of linear ordinary differential operators. This point of view was the original motivation of V. Arnold, B. Khesin, V. Ovsienko, B. Shapiro and M. Shapiro for considering this class of questions in the early nineties [20, 21, 22, 38]. Conjectures 2.4 and 2.6 of [38] (mentioned earlier in this introduction) are related to an attempt at a generalized (multiplicative) Sturm Theory for linear ordinary differential equations of order $n + 1 > 2$, the case $n = 1$ standing for the classical (additive) one. The first of these conjectures has been proved for $n \leq 4$ in [32, 37], but the general case remains open; the second one is essentially our Lemma 3.1. The second author was first led to consider this subject while studying the critical sets of nonlinear differential operators with periodic coefficients, in a series of works with D. Burghelea and C. Tomei [8, 9, 10, 34].

In Section 2 we review notation and results about Bruhat cells and related topics (Subsection 2.1), Hilbert and Banach manifolds of curves (Subsection 2.2), and convex curves (Subsection 2.3). Section 3 is dedicated to the proof of Theorem 1. The concept of accessibility is discussed in Section 4 for the lower triangular group and in Section 5 for the spin group. Section 6 is dedicated to the proof of Theorem 2. In Section 7 we present transversal sections to the strata. Section 8 is dedicated to the proof of Theorem 3. Section 9 begins by proving Theorem 4, we then discuss in detail the neighborhood of the stratum defined by the permutation $acb = [3142] \in S_4$. Some final remarks are given in Section 10. Theorem 5 (which is proved in [1]) describes the homotopy type of the spaces $L_3(q)$, $q \in \text{Quat}_4$ (locally convex curves in $S^3$).

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2 Notations and facts

2.1 Bruhat cells

We briefly recall some definitions from [15]. For \( j \in [n] \), set \( \alpha_j : \mathbb{R} \to Spin_{n+1}, \)

\[
\alpha_j(\theta) = \exp(\theta a_j), \quad a_j = e_{j+1}e_j^\top - e_je_{j+1}^\top \in so_{n+1} = spin_{n+1}.
\]

The spin group \( Spin_{n+1} \) is the universal double cover of \( SO_{n+1} \) and is contained in the Clifford algebra \( Cl^n_{n+1} \). Also, set \( \dot{a}_j = \alpha_j(\pi/2), \dot{a}_j = (\dot{a}_j)^{-1} \in \widetilde{B}_{n+1}^+ \subset Spin_{n+1} \).

Recall that the group \( \widetilde{B}_{n+1}^+ \) is the lift to the spin group of the group \( B_{n+1}^+ \subset SO_{n+1} \) of signed permutation matrices with positive determinant; the elements \( \dot{a}_j \) are generators of \( \widetilde{B}_{n+1}^+ \).

A reduced word for a permutation \( \sigma \in S_{n+1} \) is an expression \( \sigma = a_{i_1} \cdots a_{i_k} \) of minimum length \( k = \text{inv}(\sigma) \); here, \( a_j = (j, j+1) \in S_{n+1}, j \in [n], \) are the Coxeter generators of the symmetric group. Given a reduced word as above, set

\[
\acute{\sigma} = \text{acute}(\sigma) = \dot{a}_{i_1} \cdots \dot{a}_{i_k}, \quad \grave{\sigma} = \text{grave}(\sigma) = \dot{a}_{i_k} \cdots \dot{a}_{i_1} \in \widetilde{B}_{n+1}^+,
\]

\[
\hat{\sigma} = \hat{\sigma}(\sigma) = \acute{\sigma}(\grave{\sigma})^{-1} \in Quat_{n+1} \subset B_{n+1}^+.
\]

Let \( \prec \) be the covering relation for the Bruhat order; thus, \( \sigma_0 \prec \sigma_1 \) implies \( \text{inv}(\sigma_1) = 1 + \text{inv}(\sigma_0) \). An important special case is when \( \sigma_{j-1} \prec \sigma_j = \sigma_{j-1}a_{i_j} : \) in this case, there exists reduced words \( \sigma_{j-1} = a_{i_1} \cdots a_{i_{j-1}} \) and \( \sigma_j = a_{i_1} \cdots a_{i_{j-1}}a_{i_j} \).

Let \( Lo_{n+1}^1 \) be the group of lower triangular matrices with unit diagonal entries and \( lo_{n+1}^1 \) be its Lie algebra. Notice that \( Lo_{n+1}^1 \) is nilpotent and contractible. Let \( U_1 \subset SO_{n+1} \) be the open contractible set of orthogonal matrices \( Q \in SO_{n+1} \) that admit an \( LU \) decomposition \( Q = LU \) with \( L \in Lo_{n+1}^1 \) and \( U \in Up_{n+1}^+ \).

Here, \( Up_{n+1}^+ \) is the group of upper triangular matrices with positive diagonal entries. Also, let \( U_1 \subset Spin_{n+1} \) be the connected component of the identity in the subset \( \Pi^{-1}[U_1] \subset Spin_{n+1} \). The diffeomorphism \( L : U_1 \to Lo_{n+1}^1 \) is defined by taking the \( L \)-part \( L(z) \) in the \( LU \) decomposition of the matrix \( \Pi(z) \). Each \( z_0 \in Spin_{n+1} \) has an open neighborhood \( \mathcal{U}_{z_0} = z_0 U_1 \subset Spin_{n+1} \) diffeomorphic to \( Lo_{n+1}^1 \). We also consider the set of matrices \( z_0 Lo_{n+1}^1 = \Pi(z_0) Lo_{n+1}^1 \subset GL_{n+1}^+ \) and the diffeomorphism \( L_{z_0} : \mathcal{U}_{z_0} \to z_0 Lo_{n+1}^1, L_{z_0}(z) = z_0 L(z_0^{-1}z), \) with inverse \( Q_{z_0} = L_{z_0}^{-1} : z_0 Lo_{n+1}^1 \to \mathcal{U}_{z_0} \), where \( Q_{z_0}(M) \in \mathcal{U}_{z_0} \) is the lift to \( Spin_{n+1} \) of the \( Q \)-part of the \( QR \) decomposition of \( M \). We are particularly interested in the case \( z_0 \in \widetilde{B}_{n+1}^+ \). In this situation, the matrices in \( z_0 Lo_{n+1}^1 \) are, up to signs, triangular matrices with rows shuffled by the underlying permutation of \( z_0 \). We call \( (U_{z_0}, L_{z_0}) \) a triangular system of coordinates. When \( z_0 = 1 \), we write simply \( Q = Q_1 \), in accordance with \( L = L_1 \).

For each \( j \in [n] \), the Lie algebra element \( a_j \in spin_{n+1} \) is taken to a positive multiple of \( l_j = e_{j+1}e_j^\top \in lo_{n+1}^1 \) by the derivative of the map \( L \) at the identity
For a reduced word $\eta$ useful. The first two are convex and the third one is locally convex.

Recall that

for arbitrary elements $\Gamma : J \to Spin_{n+1}$ is a locally convex curve. Moreover, the arcs of the curves $\alpha_j$ contained in $U_1$ are taken by $L$ into (orientation-preserving) reparameterizations of $\lambda_j(t) = \exp(t\eta_j)$, $t \in \mathbb{R}$. We say that an absolutely continuous map $\Gamma : J \to Lo^1_{n+1}$, defined on an interval $J \subseteq \mathbb{R}$, is a convex curve if and only if its logarithmic derivative is given almost everywhere by

$$(\Gamma(t))^{-1}\Gamma'(t) = \sum_{j \in [n]} \beta_j(t) t_j,$$

for positive functions $\beta_1, \ldots, \beta_n : J \to (0, +\infty)$. In other words, $\Gamma : J \to Lo^1_{n+1}$ is a convex curve if and only if $Q \circ \Gamma : J \to Spin_{n+1}$ is a locally convex curve. Notice that $(Q \circ \Gamma)[J] \subseteq U_1$; in Subsection 2.3 we review the fact that a locally convex curve $\Gamma : J \to Spin_{n+1}$ is strictly convex if and only if $\Gamma[J] \subseteq U_2$ for some $z_0 \in Spin_{n+1}$. Notice that if $\Gamma : [t_0, t_1] \to Spin_{n+1}$ is strictly convex then $\Gamma$ is globally convex, i.e., $\text{sing}((\Gamma(t_0))^{-1}\Gamma) = \emptyset$; the reciprocal is not quite true.

Some distinguished Lie algebra elements are

$$n = \sum_{j \in [n]} l_j, \quad h_L = \sum_{j \in [n]} \sqrt{j(n+1-j)} l_j \in Lo^1_{n+1},$$

$$h = \sum_{j \in [n]} \sqrt{j(n+1-j)} a_j \in spin^1_{n+1}. \quad (9)$$

For arbitrary elements $g_0 \in G$, $v \in g$ of a Lie group and its Lie algebra, denote by $\Gamma_{g_0,v} : \mathbb{R} \to G$ the smooth parametric curve $\Gamma_{g_0,v}(t) = g_0 \exp(tv)$. The smooth curves $\Gamma_{g_0,v}$, $\Gamma_{g_0,h_L}$, and $\Gamma_{g_0,h_L}$, studied in Example 4.2 of [15], are particularly useful. The first two are convex and the third one is locally convex.

We denote by $\text{Pos}_\eta \subset Lo^1_{n+1}$, the open subset of totally positive matrices [5]. For a reduced word $\eta = a_{i_1} \cdots a_{i_m}$ ($m = n(n+1)/2$) for the Coxeter element of $S_{n+1}$, the map $(0, +\infty)^m \to \text{Pos}_\eta$, $(t_1, \ldots, t_m) \mapsto \lambda_{i_1}(t_1) \cdots \lambda_{i_m}(t_m)$, is a diffeomorphism. More generally, there are embedded submanifolds $\text{Pos}_\sigma, \text{Neg}_\sigma \subset Lo^1_{n+1}, \sigma \in S_{n+1}$, such that, given a reduced word $\sigma = a_{i_1} \cdots a_{i_k}$, $k = \text{inv}(\sigma)$, the maps $(0, +\infty)^k \to \text{Pos}_\sigma$, $(t_1, \ldots, t_k) \mapsto \lambda_{i_1}(t_1) \cdots \lambda_{i_k}(t_k)$, and $(-\infty, 0)^k \to \text{Neg}_\sigma$, $(t_1, \ldots, t_k) \mapsto \lambda_{i_1}(t_1) \cdots \lambda_{i_k}(t_k)$, are diffeomorphisms. We have

$$\overline{\text{Pos}_\eta} = \bigcup_{\sigma \in S_{n+1}} \text{Pos}_\sigma, \quad \overline{\text{Neg}_\eta} = \bigcup_{\sigma \in S_{n+1}} \text{Neg}_\sigma, \quad \overline{\text{Pos}_\eta} \cap \overline{\text{Neg}_\eta} = \{1\} = \text{Pos}_e = \text{Neg}_e.$$

This is closely related to the Bruhat stratifications:

$$\text{Spin}_{n+1} = \bigcup_{\sigma \in S_{n+1}} \text{Bru}_\sigma, \quad \text{Bru}_\sigma = \bigcup_{q \in \text{Quat}_{n+1}} \text{Bru}_{q\delta}.$$

Recall that $z \in \text{Bru}_\sigma$ if and only if there exist upper triangular matrices $U_0, U_1$ such that $\Pi(z) = U_0 \Pi(\delta) U_1$. We have $\text{Bru}_{q\delta} = U_{q\delta} \cap \text{Bru}_\sigma$. Also, given a reduced word $\sigma = a_{i_1} \cdots a_{i_k}$, $k = \text{inv}(\sigma)$, the map $(0, \pi)^k \to \text{Bru}_{q\delta}$, $(\theta_1, \ldots, \theta_k) \mapsto$
\(q\alpha_{i_1}(\theta_1)\cdots\alpha_{i_k}(\theta_k)\), is a diffeomorphism (Corollary 1.2 of [15]). For all \(\sigma \in S_{n+1}\) and \(q \in \text{Quat}_{n+1}\), the set \(q\mathcal{Q}[\text{Pos}_\sigma]\) is a contractible connected component of the submanifold \(\mathcal{U}_q \cap \text{Bru}_{q\hat{\sigma}}\). Similarly, \(q\mathcal{Q}[\text{Neg}_\sigma]\) is a contractible connected component of \(\mathcal{U}_q \cap \text{Bru}_{q\hat{\sigma}}\). We have \(q\hat{\sigma} = \tilde{q}\hat{\sigma} \in \tilde{B}_{n+1}^+, \tilde{q} = q\hat{\sigma}^{-1} \in \text{Quat}_{n+1}\).

For \(L_0, L_1 \in \text{Lo}^1_{n+1}\), we write \(L_0 \ll L_1\) if and only if \(L_0^{-1}L_1 \in \text{Pos}_\eta\) (equivalently, \(L_1^1L_0 \in \text{Neg}_\eta\)) and \(L_0 \leq L_1\) if and only if \(L_0^{-1}L_1 \in \text{Pos}_\eta\) (equiv., \(L_1^{-1}L_0 \in \text{Neg}_\eta\)). These are partial orders in \(\text{Lo}^1_{n+1}\) (Lemma 5.2 of [15]). We have \(L_0 \ll L_1\) if and only if there is a convex curve \(\Gamma : [0,1] \to \text{Lo}^1_{n+1}\) satisfying \(\Gamma(0) = L_0\) and \(\Gamma(1) = L_1\) (Lemma 5.3 of [15]). Convex curves \(\Gamma : J \to \text{Lo}^1_{n+1}\) are such that, for \(t_0 < t < t_1\) in \(J\), we have \(\Gamma(t) \in (\Gamma(t_0)\text{Pos}_\eta) \cap (\Gamma(t_1)\text{Neg}_\eta)\) (Lemma 5.7 of [15]).

**Projective transformations** are 1-1 correspondences between (locally) convex curves that preserve itineraries and singular sets. We consider two types of them:

1. Given an upper triangular matrix \(U\) with positive diagonal entries, we assign to each locally convex curve \(\Gamma : [t_0, t_1] \to \text{Spin}_{n+1}\) its projective transform \(\Gamma^U : [t_0, t_1] \to \text{Spin}_{n+1}\) given by \(\Gamma^U(t) = \mathcal{Q}(U^{-1}\Gamma(t))\);

2. Given \(\lambda > 0\), consider the diagonal matrix \(E_\lambda = \text{diag}(1, \lambda, \ldots, \lambda^n)\). We assign to each convex curve \(\Gamma : [t_0, t_1] \to \text{Lo}^1_{n+1}\) its projective transform \(\Gamma^\lambda : [t_0, t_1] \to \text{Lo}^1_{n+1}\) given by \(\Gamma^\lambda(t) = E_\lambda^{-1}\Gamma(t)E_\lambda\).

Projective transformations come from the smooth actions of Lie groups:

\[
\begin{align*}
\text{Spin}_{n+1} \times \text{Up}^+_{n+1} & \to \text{Spin}_{n+1}, \quad (z, U) \mapsto z^U = \mathcal{Q}(U^{-1}z), \\
\text{Lo}^1_{n+1} \times (0, +\infty) & \to \text{Lo}^1_{n+1}, \quad (L, \lambda) \mapsto L^\lambda = E_\lambda^{-1}LE_\lambda,
\end{align*}
\]

We abuse the distinction between \(z \in \text{Spin}_{n+1}\) and \(\Pi(z) \in \text{SO}_{n+1}\) in the first formula, so that \(\mathcal{Q}(U^{-1}z)\) is the lift to the spin group of the \(QR\) factorization of the invertible matrix \(U^{-1}\Pi(z)\). Both these actions preserve signed Bruhat cells \(\text{Bru}_{q\hat{\sigma}}\) (we consider \(L[\mathcal{U}_q \cap \text{Bru}_{q\hat{\sigma}}]\) as the corresponding signed Bruhat cell in \(\text{Lo}^1_{n+1}\)); the subgroup \(\text{Up}^+_{n+1} \subset \text{Up}^+_{n+1}\) of matrices with unit diagonal entries acts transitively on each signed Bruhat cell. See Section 6 in [15].

In projective transformations of type 1, the lift to \(\text{Spin}_{n+1}\) is made in such a way that, for each \(t\), \(\Gamma(t)\) and \(\Gamma^U(t)\) are in the same signed Bruhat cell \(\text{Bru}_{q\hat{\sigma}}\). Also notice that if \(\Gamma(t) \in \text{Bru}_{q\hat{\sigma}}\), then, for each \(z \in \text{Bru}_{q\hat{\sigma}}\), there is a projective transformation of type 1 such that \(\Gamma^U(t) = z\). Moreover, the matrix \(U\) can always be taken in the subgroup \(\text{Up}^1_{n+1} \subset \text{Up}^+_{n+1}\) of upper triangular matrices with unit diagonal entries; in type 2, notice that, for all \(t\), we have \(\lim_{t \to +\infty} \Gamma^\lambda(t) = I\).

The maps \(\text{chop}, \text{adv} : \text{Spin}_{n+1} \to \hat{\eta}\text{Quat}_{n+1} \subset \tilde{B}_{n+1}^+\) are defined by

\[
\text{adv}(z) = q_\alpha\hat{\eta}, \quad \text{chop}(z) = q_\alpha\hat{\eta}, \quad z \in \text{Bru}_{q_\alpha} \subset \text{Bru}_{q_0}, \quad z_0 = q_\alpha\hat{\sigma}_0 = q_\alpha\hat{\sigma}_0, \quad (10)
\]
where \( z_0 \in \mathbb{P}^+ \), \( \sigma_0 \in S_{n+1} \) and \( q_a, q_c \in \text{Quat}_{n+1} \). For \( \rho_0 = \eta \sigma_0 \), we have \( \text{adv}(z) = \text{chop}(z) \rho_0 \). Given a locally convex curve \( \Gamma : J \to \text{Spin}_{n+1} \), for each \( t \in J \), there is \( \epsilon > 0 \) such that \( \Gamma[[t, t + \epsilon]] \in \text{Bru}_{\text{chop}(\Gamma(t))} \) and \( \Gamma[[t - \epsilon, t]] \in \text{Bru}_{\text{adv}(\Gamma(t))} \) (Theorem 3 of [13]); notice that these are open signed Bruhat cells.

### 2.2 Hilbert and Banach manifolds of curves

There are many possible choices for the exact definition and topology of our spaces of locally convex curves. Notice that there are sections with a similar purpose in [30, 31, 35]. We also plan to discuss this subject in greater detail and generality in [16].

Given \( r \in \mathbb{N} = \{0, 1, 2, \ldots \} \) and a finite dimensional real vector space \( V \), we are interested in the Banach spaces \( C^r(V) = C^r([0, 1]; V) \) and the Hilbert space \( H^r(V) = H^r([0, 1]; V) \). The space \( H^r(V) \) consists of functions \( f : [0, 1] \to V \) of Sobolev class \( H^r \). In more detail, for \( r \geq 1 \), we have \( f \in H^r(V) \) if \( f : [0, 1] \to V \) is of class \( C^{(r-1)} \), its \( (r-1) \)th derivative \( f^{(r-1)} \) is absolutely continuous and its \( r \)th derivative \( f^{(r)} \) (defined a.e.) is a function of class \( L^2 \). We follow the convention that \( H^0(V) = L^2([0, 1], V) \).

Consider a compact Lie group \( G \) contained in a finite dimensional associative algebra \( A \) (such as \( G = \text{SO}_{n+1} \subset A = \mathbb{R}^{(n+1)\times(n+1)} \) or \( G = \text{Spin}_{n+1} \subset A = \text{Cl}^0_{n+1} \)). Define \( C^r(A) \) and \( H^r(A) \) as above. The spaces \( C^r(G) = C^r([0, 1]; G) \subset C^r(A) \) and \( H^r(G) = H^r([0, 1]; G) \subset H^r(A) \) of functions whose images are contained in \( G \) are Banach and Hilbert submanifolds, respectively. Notice that the maps \( \Gamma \mapsto \Gamma(t_0) \) are smooth surjective submersions onto \( G \).

Let \( T \subset \mathfrak{g} \) be a vector subspace which generates \( \mathfrak{g} \) (as a Lie algebra). An absolutely continuous curve \( \Gamma : [0, 1] \to G \) is \( T \)-holonomic if \( (\Gamma(t))^{-1}\Gamma'(t) \in T \) whenever \( \Gamma'(t) \) is defined (which is almost always). For \( r \geq 1 \), the subsets \( C^r(G; T) \subset C^r(G) \) and \( H^r(G; T) \subset H^r(G) \) of \( T \)-holonomic curves are smooth Banach and Hilbert submanifolds, respectively. In our example, \( G = \text{Spin}_{n+1} \) and \( T \) is spanned by \( a_1, \ldots, a_n \). Let \( J \subset T \) be the open cone of linear combinations of \( a_1, \ldots, a_n \) with positive coefficients. A holonomic curve is locally convex if \( (\Gamma(t))^{-1}\Gamma'(t) \in J \) whenever defined. This defines open subsets \( \mathcal{L}_{n}^{C^r}(\cdot; \cdot) \subset C^r(G; T) \) for \( r \geq 1 \) and \( \mathcal{L}_{n}^{H^r}(\cdot; \cdot) \subset H^r(G; T) \) for \( r \geq 2 \) of locally convex curves. These are Banach and Hilbert manifolds, respectively. The case \( H^1 \) requires a more delicate discussion and is postponed.

If we fix the initial point, we have submanifolds \( \mathcal{L}_{n}^{C^r}(z_0; \cdot) \subset \mathcal{L}_{n}^{C^r}(\cdot; \cdot) \) and \( \mathcal{L}_{n}^{H^r}(z_0; \cdot) \subset \mathcal{L}_{n}^{H^r}(\cdot; \cdot) \). Indeed, multiplication by \( z \in \text{Spin}_{n+1} \) shows that \( \Gamma(0) \) as a function of \( \Gamma \in \mathcal{L}_{n}(\cdot; \cdot) \) is a submersion. In these submanifolds the map \( \mu, \mu(\Gamma) = \Gamma(1) \), is called monodromy. Lemma [2.1] below shows that likewise \( \mathcal{L}_{n}(z_0; z_1) \subset \mathcal{L}_{n}(z_0; \cdot) \) is a submanifold.
Recall from Equation (1) that a locally convex curve is characterized by its initial point \( z_0 = \Gamma(0) \) and by the positive functions \( \kappa_j, j \in [n] \). If \( \Gamma \) is of class \( H^r, r > 1 \), the functions \( \kappa_j \) are in \( H^{r-1}([0,1]; \mathbb{R}) \). We could use this system of coordinates to produce an alternative description of the Hilbert manifold structure of the space \( \mathcal{L}_n^{[H^r]}(\cdot ; \cdot) \). We follow a similar method to define \( \mathcal{L}_n^{[H^1]}(\cdot ; \cdot) \). The functions \( \kappa_j \) would then be in \( H^0 = L^2([0,1]; \mathbb{R}) \). The difficulty is that the set of positive functions is not open in \( L^2([0,1]; \mathbb{R}) \). We circumvent this difficulty by defining functions \( \xi_j \in H^0 \) by

\[
\xi_j = \kappa_j - \frac{1}{\kappa_j}, \quad \kappa_j = \frac{\xi_j^2 + 4}{2}. \tag{11}
\]

By definition, a locally convex curve \( \Gamma \) (assumed to be absolutely continuous) is in \( \mathcal{L}_n^{[H^1]}(\cdot ; \cdot) \) if the corresponding functions \( \xi_j \) (through Equations (1) and (11)) are in \( H^0 = L^2([0,1]) \). Conversely, given functions \( \xi_j \in H^0 \), Equation (11) above yields positive functions \( \kappa_j \): Equation (1) is then an ODE, defining \( \Gamma \) (see [16] for details). Thus, the initial point \( z_0 \) and the functions \( \xi_j \) give a smooth Hilbert manifold structure to the space \( \mathcal{L}_n^{[H^1]}(\cdot ; \cdot) \). As in the case \( r \geq 2 \), \( \mathcal{L}_n^{[H^1]}(z_0; \cdot) \subset \mathcal{L}_n^{[H^1]}(\cdot ; \cdot) \) is clearly a submanifold for each \( z_0 \in \text{Spin}_{n+1} \).

**Lemma 2.1.** Consider a topology \( C^r, r > 1, H^1 \) or \( H^r, r > 2 \), and the monodromy map \( \mu : \mathcal{L}_n(z_0; \cdot) \to \text{Spin}_{n+1}, \mu(\Gamma) = \Gamma(1) \), for a fixed \( z_0 \in \text{Spin}_{n+1} \). The map \( \mu \) is a surjective submersion. In particular, \( \mathcal{L}_n(z_0; z_1) \subset \mathcal{L}_n(z_0; \cdot) \) is a nonempty submanifold for each \( z_1 \in \text{Spin}_{n+1} \).

The case \( n = 2 \) is discussed in [30] and that proof can be adapted by using some basic facts about total positivity.

**Proof.** We use spaces of convex curves in the triangular group \( \text{Lo}^1_{n+1} \). If the image of a locally convex arc \( \Gamma \) is contained in \( \mathcal{U}_{z_1}, z_1 \in \text{Spin}_{n+1} \), then \( t \mapsto \mathcal{L}(z_1^{-1} \Gamma(t)) \) is a convex curve.

Convex curves if \( \text{Lo}^1_{n+1} \) follow explicit formulae. In particular, it follows from Equation (10) in [15] that, given a convex curve in \( \text{Lo}^1_{n+1} \), there exist perturbations keeping the first half of the arc fixed and moving the end point in any prescribed direction.

Given \( \Gamma \in \mathcal{L}_n(z_0; \cdot) \), set \( z_1 = \Gamma(1) \). Set \( t_2 \in (0,1) \) such that the image of the arc \( \Gamma_{[t_2,1]} \) is contained in \( \mathcal{U}_{z_1} \). Identify \( \mathcal{U}_{z_1} \) with \( \text{Lo}^1_{n+1} \), as above. The perturbation in \( \text{Lo}^1_{n+1} \) can be brought back to \( \Gamma \), proving that \( \mu \) is a submersion. Surjectivity follows by adding loops, as in [31].
The inclusion $L_n^{[H^{r+1}]}(z_0;\cdot) \hookrightarrow L_n^{[H^r]}(z_0;\cdot)$ is continuous with dense image for all $r \geq 1$. The same happens for $L_n^{[C^{r+1}]}(z_0;\cdot) \hookrightarrow L_n^{[C^r]}(z_0;\cdot)$ and $L_n^{[C^r]}(z_0;\cdot) \hookrightarrow L_n^{[H^r]}(z_0;\cdot)$. The following facts imply that these are also homotopy equivalences.

**Fact 2.2** (Theorem 2 of [8]). Let $B_1$ and $B_2$ be infinite dimensional separable Banach spaces. Suppose $i : B_1 \rightarrow B_2$ is a bounded, injective linear map with dense image and $M_2 \subset B_2$ is a smooth closed Banach submanifold of finite codimension. Then, $M_1 = i^{-1}[M_2]$ is a smooth closed Banach submanifold of $B_1$ and $i : (B_1, M_1) \rightarrow (B_2, M_2)$ is a homotopy equivalence of pairs.

**Fact 2.3** (from Theorem 0.1 of [7] and Corollary 3 of [18]). Let $M_1$ and $M_2$ be topological (respectively, smooth) manifolds modeled on infinite dimensional separable Banach (resp. Hilbert) spaces. Any homotopy equivalence $i : M_1 \rightarrow M_2$ is homotopic to a homeomorphism (resp., diffeomorphism).

**Remark 2.4.** A natural question at this point would be whether $\mu$ qualifies as some sort of fibration. The reader of course knows that the spaces $L_n(z)$ exhibit different homotopy types as $z$ ranges over $\text{Spin}_{n+1}$ [26, 22, 30, 31, 36, 39]. In fact, $\mu$ is not even a Serre fibration, since it lacks the homotopy lifting property for polyhedra (see [22, 29]).

**Lemma 2.5.** For all $z_0, z_1 \in \text{Spin}_{n+1}$, we have that:

1. for all $r, r' \in \mathbb{N}^*$, $r \neq 2$, $r' \neq 1$, the subspaces $L_n^{[H^r]}(z_0; z_1)$ and $L_n^{[C^{r'}]}(z_0; z_1)$ are closed embedded smooth submanifolds of codimension $m = n(n+1)/2$ of $L_n^{[H^r]}(z_0; \cdot)$ and $L_n^{[C^{r'}]}(z_0; \cdot)$, respectively;

2. for all $r, \tilde{r} \in \mathbb{N}$, $r \geq 1$, the natural inclusion maps

\[ i_{r,\tilde{r}} : (L_n^{[H^{r+1}]}(z_0;\cdot), L_n^{[H^{r+1}]}(z_0; z_1)) \hookrightarrow (L_n^{[H^r]}(z_0;\cdot), L_n^{[H^r]}(z_0; z_1)), \]

\[ j_{r,\tilde{r}} : (L_n^{[C^{r+1}]}(z_0;\cdot), L_n^{[C^{r+1}]}(z_0; z_1)) \hookrightarrow (L_n^{[H^r]}(z_0;\cdot), L_n^{[H^r]}(z_0; z_1)), \]

\[ \ell_{r,\tilde{r}} : (L_n^{[C^r]}(z_0;\cdot), L_n^{[C^r]}(z_0; z_1)) \hookrightarrow (L_n^{[C^r]}(z_0;\cdot), L_n^{[C^r]}(z_0; z_1)) \]

are homotopy equivalences of pairs.

3. each of the natural inclusions $i_{r,\tilde{r}}$, $j_{r,\tilde{r}}$, $\ell_{r,\tilde{r}}$ of item 2 is homotopic to a homeomorphism between the respective pairs (a diffeomorphism for $i_{r,\tilde{r}}$).

**Proof.** Item 1 follows directly from Lemma 2.1 for $r, n \in \mathbb{N}^*$, consider the Banach spaces $H^{r,n} = (H^{r-1}([0,1]); \mathbb{R})^n$ and $C^{r,n} = (C^{r-1}([0,1]); \mathbb{R})^n$. Regard $L_n^{[H^r]}(z_0; z_1) \subset L_n^{[H^r]}(z_0; \cdot)$ as submanifolds of $H^{r,n}$ and $L_n^{[C^r]}(z_0; z_1) \subset L_n^{[C^r]}(z_0; \cdot)$ as submanifolds of $C^{r,n}$, given by the functions $\xi_1, \ldots, \xi_n$ of Equation (11). Item 2 now follows from Fact 2.2 by choosing $B_1$ and $B_2$ amongst the Banach spaces above. Item 3 follows from Item 2 and Fact 2.3. $\square$
In particular, we see that, given \( z_0, z_1 \in \text{Spin}_{n+1} \), all spaces \( L^*_n(z_0; z_1) \) are homeomorphic. In some situations, this warrants us the right to drop the superscripts altogether and to adopt a definition of \( L_n(z_0; z_1) \) that is well-suited to the purpose at hand. Throughout this paper, the spaces of locally convex curves of class \( H^r \) take precedence over their \( C^r \) counterparts for being Hilbert manifolds. We are particularly interested in \( L^{[H^1]}_n(z_0; z_1) \) and in \( L^{[H^r]}_n(z_0; z_1) \) for large \( r \).

2.3 Convex curves

A smooth parametric curve \( \gamma : J \to S^n \) defined on a compact interval \( J \subset \mathbb{R} \) is said to be strictly convex if for each nonzero linear functional \( \omega \in (\mathbb{R}^{n+1})^* \setminus \{0\} \) the function \( \omega \gamma : J \to \mathbb{R}, (\omega \gamma)(t) = \omega(\gamma(t)) \), has at most \( n \) zeroes counted with multiplicities (zeroes at endpoints taken into account). It is said to be convex if its restriction to any proper compact subinterval of \( J \) is strictly convex.

In other words, a convex curve is one that (possibly neglecting one endpoint at a time) intersects each \( n \)-dimensional vector subspace \( V \subset \mathbb{R}^{n+1} \) at most \( n \) times with multiplicities taken into account. Thus, for instance, a transversal intersection counts as 1; a generic tangency counts as 2; a generic osculation counts as 3. Other terms used for the same or closely related concepts are non-oscillatory curves [27, 39] and disconjugate curves [22, 36, 38].

In Appendix A in [13] we show that a smooth nondegenerate curve \( \gamma : [0, 1] \to S^n \) with initial frame \( \mathfrak{F}_\gamma(0) = 1 \) is convex if and only if its itinerary is the empty word, i.e., that the notion of convexity introduced in Section 2.3 and given in terms of the singular set of \( \mathfrak{F}_\gamma \) coincides with this geometric definition. These results are essentially present in [39].

Clearly, convexity implies nondegeneracy. Conversely, as we shall see in Lemma 2.6 (smooth) nondegeneracy implies local convexity: this is why the terms nondegenerate and locally convex are used interchangeably.

We quote below the main result of Appendix A of [13]. For \( J \) a compact interval, we say that a locally convex curve \( \Gamma : J \to \text{Spin}_{n+1} \) is short if there exists \( z \in \text{Spin}_{n+1} \) such that \( \Gamma[J] \subset U_z \). Recall that \( U_z \subset \text{Spin}_{n+1} \) is the domain of a triangular system of coordinates (see Subsection 2.1 or Section 4 of [15]).

**Lemma 2.6.** Let \( \gamma : J \to S^n \) be a smooth nondegenerate curve defined on a compact interval \( J \subset \mathbb{R} \). The following conditions are equivalent:

1. \( \gamma \) is strictly convex;
2. \( \mathfrak{F}_\gamma \) is short;
3. \( \forall t_0, t_+ \in J \ ( (t_0 < t_+) \rightarrow (\mathfrak{F}_\gamma(t_+) \in \mathfrak{F}_\gamma(t_0) \text{Bru}_d) ) \);
4. \( \forall t_0, t_- \in J ((t_- < t_0) \to (\mathfrak{F}_\gamma(t_-) \subseteq \mathfrak{F}_\gamma(t_0) \text{Bru}_q)) \).

**Proof.** See Appendix A in \[13\]; closely related sufficient conditions for convexity may be found in \[27, 39\]. \( \square \)

**Example 2.7.** Given \( z \in \text{Bru}_q \), consider the curve \( \Gamma = \Gamma_z : [0, 1] \to \text{Spin}_{n+1} \) passing through \( \Gamma_z(\tfrac{1}{2}) = z \) given by Lemma 6.1 of \[15\]. It follows immediately from Lemma 2.6 that \( \gamma_z = \Gamma_z e_1 \) is a convex curve (though not strictly convex). For an alternative proof, recall that \( \Gamma_z \) is obtained from \( \Gamma_q(t) = \exp(\pi t \eta) \) through a projective transformation and see Lemma 2.2 of \[31\] for a direct proof of the convexity of \( \Gamma_q \). For \( n = 2 \), \( \gamma_q(t) = \frac{1}{2}(1 + \cos(2\pi t), \sqrt{2}\sin(2\pi t), 1 - \cos(2\pi t)) \) is the circle of diameter \( e_1 e_3 \) in \( S^2 \). Notice that \( \gamma_q \) is closed if and only if \( n \) is even (as usual, a curve \( \gamma : [0, 1] \to S^n \) is closed if \( \mathfrak{F}_\gamma(0) = \mathfrak{F}_\gamma(1) \)). \( \diamond \)

### 3 Proof of Theorem \[1\]

Before proving Theorem \[1\] we state and prove two related results. The first lemma is essentially equivalent to Conjecture 2.6 in \[38\]. We thank B. Shapiro and M. Shapiro for insightful conversations on this subject.

**Lemma 3.1.** Let \( \Gamma \in \mathcal{L}_n(z_0; z_1) \) be a locally convex curve. Let \( t_* \in \text{sing}(\Gamma) \subset (0, 1) \), \( z_* = \Gamma(t_*) \in \text{Sing}_{n+1} \). Then there exists an open set \( U_* \subset \text{Spin}_{n+1} \), \( z_* \in U_* \) with the following properties. There exists \( \epsilon > 0 \) such that \( [t_* - \epsilon, t_* + \epsilon] \subset (0, 1) \), \( [t_* - \epsilon, t_* + \epsilon] \cap \text{sing}(\Gamma) = \{t_*\} \) and \( \Gamma([t_* - \epsilon, t_* + \epsilon]) \subset U_* \). There exist distinct open connected components \( U_*^- \) and \( U_*^+ \) of \( U_* \setminus \text{Sing}_{n+1} \) such that \( \Gamma([t_* - \epsilon, t_*]) \subset U_*^- \) and \( \Gamma([t_* - \epsilon, t_*]) \subset U_*^+ \).

**Proof.** Assume that \( z_* \in \text{Bru}_\rho, \rho \neq \eta \). After applying a projective transformation, we may assume that \( z_* \in q_* \text{Pos}_\rho \subset U_{q_*}, q_* \in \text{Quat}_{n+1} \). We shall take \( U_* = U_{q_*} \) and \( U_*^+ = q_* \text{Pos}_\eta \), which is a connected component of \( U_* \setminus \text{Sing}_{n+1} \). The number \( \epsilon > 0 \) can easily be chosen so as to satisfy the conditions in the statement. It follows from Lemma 5.7 of \[15\] that \( \Gamma([t_* - \epsilon, t_*]) \subset U_*^+ \) and that \( \Gamma([t_* - \epsilon, t_*]) \) is disjoint from \( U_*^+ \): let \( U_*^- \) be the connected component of \( U_* \setminus \text{Sing}_{n+1} \) containing \( \Gamma([t_* - \epsilon, t_*]) \). \( \square \)

**Lemma 3.2.** Let \( z_0, z_1 \in \text{Spin}_{n+1} \). Let \( K \) be a compact set and \( H : K \to \mathcal{L}_n(z_0; z_1) \) be a continuous function. Let

\[
K_1 = \bigcup_{s \in K} (\{s\} \times \text{sing}(H(s))) = \{(s, t) \in K \times (0, 1) \mid H(s)(t) \in \text{Sing}_{n+1}\}.
\]

Then \( K_1 \) is a compact set and satisfies the following condition:

\[
\forall (s_0, t_0) \in K_1, \forall \epsilon > 0, \exists \delta > 0, \forall s \in K, \quad |s - s_0| < \delta \quad \Rightarrow \quad (\exists t \in (0, 1), (s, t) \in K_1, |t - t_0| < \epsilon).
\]
Proof. Write \( \tilde{H}(s,t) = H(s)(t) \in \text{Spin}_{n+1} \) so that \( \tilde{H} : K \times [0,1] \to \text{Spin}_{n+1} \) is continuous. Notice that \( K_2 \subseteq K \times [0,1] \) defined by

\[
K_2 = \tilde{H}^{-1}[\text{Sing}_{n+1}] \subseteq K_1 \cup (K \times \{0,1\}), \quad \text{Sing}_{n+1} = \sqcup_{\eta \neq \eta} \text{Bru}_\eta,
\]
is closed and therefore compact. Furthermore, the sets \( A_0 = \tilde{H}^{-1}[\text{Bru}_{\text{adv}(z_0)}] \) and \( A_1 = \tilde{H}^{-1}[\text{Bru}_{\text{chop}(z_1)}] \) are open and disjoint from \( K_2 \). From Theorem 3 of [15], for each \( s \in K \) there exists \( \epsilon_s > 0 \) such that \( \{s\} \times (0, \epsilon_s) \subset A_0 \) and \( \{s\} \times (1 - \epsilon_s, 1) \subset A_1 \). By compactness of \( K \) there exists \( \epsilon_s > 0 \) such that \( K \times (0, \epsilon_s) \subset A_0 \) and \( K \times (1 - \epsilon_s, 1) \subset A_1 \), implying the compactness of \( K_1 = K_2 \setminus (K \times \{0,1\}) \).

The remaining claim follows from Lemma 3.1. \( \square \)

Proof of Theorem 2. It follows from the condition in Lemma 3.2 that the composite map \( \text{sing} \circ \tilde{H} \) is continuous. Since \( \mathcal{L}_n(z_0; z_1) \) is metrizable and \( K \) is arbitrary, this implies the continuity of the map \( \text{sing} : \mathcal{L}_n(z_0; z_1) \to \mathcal{H}([0,1]) \). \( \square \)

Recall from the introduction that \( \Gamma \in \mathcal{L}_{n,\text{convex}}(z) \) if and only if \( \Gamma \in \mathcal{L}_n(1; z) \) and \( \text{sing}(\Gamma) = \emptyset \). The following result is well known [11, 16, 39] and is presented here for completeness and as an example of an application.

Lemma 3.3. If \( z \in \tilde{B}_{n+1}^+ \) then the subset \( \mathcal{L}_{n,\text{convex}}(z) \subset \mathcal{L}_n(z) \) is either empty or a contractible connected component. It is nonempty if and only if \( \text{chop}(z) = \eta \).

Proof. From Theorem 1 and the fact that \( \emptyset \in \mathcal{H}([0,1]) \) is an isolated point it follows that \( \mathcal{L}_{n,\text{convex}}(z) \) is a union of connected components. By item 2 of Lemma 2.5 it suffices to show that \( \mathcal{L}_{n,\text{convex}}(z) \) is contractible.

Consider first the case \( z \in \text{Bru}_\eta \). By applying a projective transformation we may assume \( z = \eta = \exp(\frac{\pi}{2} h) \). Take \( \Gamma_0 \in \mathcal{L}_{n,\text{convex}}(\eta) \), \( \Gamma_0(t) = \exp(\frac{\pi}{2} t h) \). Notice that Equation (9) of [15] implies that \( \Gamma_0(t) \in \text{Bru}_\eta \) for \( 0 < t \leq 1 \), since \( \Gamma_0(t) = U_0(t) \Pi(\eta) U_1(t) \), where

\[
U_1(t) = \exp \left( -\log \left( \cos (\pi t/2) [h_L, h_L^T] \right) \right) \exp \left( -\tan (\pi t/2) h_L^T \right),
\]

\[
U_0(t) = \Pi(\eta) \exp(\tan(\pi t/2) h_L) \Pi(\eta)^T \in \text{Up}_{n+1}^h
\]

(recall that the commutator \([h_L, h_L^T] = \sum_{k=0}^{n}(2k-n)e_{k+1}e_{k+1}^T \) is a diagonal matrix).

For \( s \in (0, 1] \), let \( U_s \in \text{Up}_{n+1}^h \) be such that \( \eta^U_s = \Gamma_0(s) \). For \( \Gamma_1 \in \mathcal{L}_{n,\text{convex}}(\eta) \) and \( s \in (0, 1] \) define \( \Gamma_s \in \mathcal{L}_{n,\text{convex}}(\eta) \) by:

\[
\Gamma_s(t) = \begin{cases} 
(\Gamma_1(t))^U_s, & t \in [0, s], \\
\Gamma_0(t), & t \in [s, 1]. 
\end{cases}
\]

The map \([0,1] \to \mathcal{L}_{n,\text{convex}}(\eta), s \mapsto \Gamma_s \) is continuous (even at \( s = 0 \)).

The general case follows from Proposition 6.4 of [31]; see also Remark 5.6. \( \square \)
Remark 3.4. It follows from Lemma 2.6 that $\text{sing}(\Gamma) = \emptyset$ implies that $\Gamma$ is convex. The reciprocal is not true. Indeed, for any $\sigma \in S_{n+1}$, $\sigma \neq \eta$, take $\Gamma : [0, 1] \to \text{Spin}_{n+1}$, $\Gamma(t) = \hat{\sigma} \exp(v(t - \frac{1}{2})h)$. For small $v > 0$, $\Gamma$ is convex (short) but $\frac{1}{2} \in \text{sing}(\Gamma)$ (see also Example 5.2 below).

4 Accessibility in triangular coordinates

For $L_x \in \text{Pos}_{\eta} \subset L^1_{n+1}$, we shall be interested in the interval

$$[I, L_x) = \text{Pos}_{\eta} \cap (L_x \text{ Neg}_{\eta}) = \{L \in L^1_{n+1} \mid I \leq L \ll L_x\} = \bigcup_{\sigma \in S_{n+1}} \text{Ac}_\sigma(L_x)$$

where the strata $\text{Ac}_\sigma(L_x) \subset \text{Pos}_\sigma$ are given by

$$\text{Ac}_\sigma(L_x) = [I, L_x) \cap \text{Pos}_\sigma = \{L \in \text{Pos}_\sigma \mid L \ll L_x\}.$$

The sets $\text{Ac}_\sigma(L_x)$ will be called accessibility sets, suggesting that for $L \in \text{Pos}_\sigma$, $L \in \text{Ac}_\sigma(L_x)$ if and only if there exists a convex curve $\Gamma : [0, 1] \to L^1_{n+1}$ with $\Gamma(0) = L$ and $\Gamma(1) = L_x$.

Example 4.1. Take $n = 2$ and, for $x, y, z \in \mathbb{R}$, write

$$L_x = L(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} = \lambda_1(c_1)\lambda_2(c_2)\lambda_3(c_3) = \lambda_2(\tilde{c}_1)\lambda_1(\tilde{c}_2)\lambda_2(\tilde{c}_3),$$

$$c_1 = x - \frac{z}{y}, \quad c_2 = y, \quad c_3 = \frac{z}{y}, \quad \tilde{c}_1 = y - \frac{z}{x}, \quad \tilde{c}_2 = x, \quad \tilde{c}_3 = \frac{z}{x};$$

as in Section 2, $\lambda_j(t) = \exp(tI_j)$. We describe the strata $\text{Ac}_\sigma = \text{Ac}_\sigma(L_x)$. The first stratum is a point: $\text{Ac}_e = \{I\}$. Next we have line segments:

$$\text{Ac}_a = \{\lambda_1(t_1) \mid t_1 \in (0, c_1)\}, \quad \text{Ac}_b = \{\lambda_2(\tilde{t}_1) \mid \tilde{t}_1 \in (0, \tilde{c}_1)\}.$$

The next strata are surfaces:

$$\text{Ac}_{ab} = \{\lambda_1(t_1)\lambda_2(t_2) \mid t_1 \in (0, c_1), t_2 \in (0, g_2(t_1))\}, \quad g_2(t_1) = \frac{c_2c_3}{c_1 + c_3 - t_1},$$

$$\text{Ac}_{ba} = \{\lambda_2(\tilde{t}_1)\lambda_1(\tilde{t}_2) \mid \tilde{t}_1 \in (0, \tilde{c}_1), \tilde{t}_2 \in (0, \tilde{g}_2(\tilde{t}_1))\}, \quad \tilde{g}_2(\tilde{t}_1) = \frac{\tilde{c}_2\tilde{c}_3}{\tilde{c}_1 + \tilde{c}_3 - \tilde{t}_1}.$$

Translating this parametrization back to $(x, y, z)$ coordinates shows that $\text{Ac}_{ab}$ is contained in the plane $z = 0$ and $\text{Ac}_{ba}$ is contained in the hyperbolic paraboloid $z = xy$. Finally, the open stratum $\text{Ac}_{aba}$ can be described as

$$\text{Ac}_{aba} = \{\lambda_1(t_1)\lambda_2(t_2)\lambda_3(t_3) \mid t_1 \in (0, c_1), t_2 \in (0, g_2(t_1)), t_3 \in (0, g_3(t_1, t_2))\}$$

$$= \{\lambda_2(\tilde{t}_1)\lambda_1(\tilde{t}_2)\lambda_3(\tilde{t}_3) \mid \tilde{t}_1 \in (0, \tilde{c}_1), \tilde{t}_2 \in (0, \tilde{g}_2(\tilde{t}_1)), \tilde{t}_3 \in (0, \tilde{g}_3(\tilde{t}_1, \tilde{t}_2))\},$$

$$g_3(t_1, t_2) = \frac{c_2(c_1 - t_1)}{c_2 - t_2}, \quad \tilde{g}_3(\tilde{t}_1, \tilde{t}_2) = \frac{\tilde{c}_2(\tilde{c}_1 - \tilde{t}_1)}{\tilde{c}_2 - \tilde{t}_2}.$$

$\diamond$
A quasiprodct is a finite sequence \((X_j)_{1\leq j\leq k}\) of open sets \(X_j \subset (0, +\infty)^j\) such that there exist a constant \(c_1 \in (0, +\infty)\) and continuous functions \(g_j : X_{j-1} \rightarrow (0, +\infty)\) for \(2 \leq j \leq k\) such that \(X_1 = (0, c_1)\) and

\[
X_j = \{(t_1, \ldots, t_{j-1}, t_j) \in X_{j-1} \times (0, +\infty) \mid t_j < g_j(t_1, \ldots, t_{j-1})\}, \quad 2 \leq j \leq k.
\]

Notice that \(X_k\) is homeomorphic to \(\mathbb{R}^k\).

**Lemma 4.2.** If \(L_x \in \text{Pos}_\eta\), each stratum \(\text{Ac}_\sigma(L_x)\) is an open, bounded and contractible subset of \(\text{Pos}_\sigma\). Moreover, if \(\sigma = \sigma_k = a_{i_1} \cdots a_{i_k}\) is a reduced word and \(\sigma_j = a_{i_1} \cdots a_{i_j},\ j \leq k\), then

\[
\text{Ac}_{\sigma_j}(L_x) = \{\lambda_{i_1}(t_1) \cdots \lambda_{i_j}(t_j) \mid (t_1, \ldots, t_j) \in X_j\}
\]

where the sequence \((X_j)_{1\leq j\leq k}\) is a quasiprodct. Furthermore, the functions \(g_j : X_{j-1} \rightarrow (0, +\infty)\) are rational and bounded.

Example 4.1 above illustrates this claim for \(n = 2\).

**Proof.** Notice that \(I \leq L \ll L_x\) implies that \(L \in \overline{\text{Pos}_\eta}\) and that there exists \(\tilde{L} \in \text{Pos}_\eta\) with \(L \tilde{L} = L_x\). Computing \((L_x)_{ij}\) in this product yields \(0 \leq (L)_{ij} \leq (L_x)_{ij}\); it follows that the interval \([I, L_x]\) is bounded.

The proof is by induction on \(k = \text{inv}(\sigma)\); the case \(k = 1\) is easy. Write

\[
X_j = \{(t_1, \ldots, t_j) \in (0, +\infty)^j \mid \lambda_{i_1}(t_1) \cdots \lambda_{i_j}(t_j) \ll L_x\}.
\]

We assume by induction that \((X_j)_{1\leq j\leq k-1}\) is a quasiprodct; we need to construct the function \(g_k : X_{k-1} \rightarrow (0, +\infty)\) that obtains \(X_k\).

Let \(\eta = a_{j_1} \cdots a_{j_m}\) be a reduced word with \(j_1 = i_k\). Given \((t_1, \ldots, t_{k-1}) \in X_{k-1}\), let

\[
L_{\sigma_{k-1}} = \lambda_{i_1}(t_1) \cdots \lambda_{i_{k-1}}(t_{k-1}) \in \text{Ac}_{\sigma_{k-1}}(L_x) \subset \text{Pos}_{\sigma_{k-1}}
\]

and write

\[
L_{\sigma_{k-1}}^{-1} L_x = \lambda_{j_1}(\tau_1) \cdots \lambda_{j_m}(\tau_m) \in \text{Pos}_\eta
\]

so that \(\tau_1 > 0\) is a function of \((t_1, \ldots, t_{k-1})\): define \(g_k(t_1, \ldots, t_{k-1}) = \tau_1\). That \(g_k\) is a rational function follows from the fact that, for all \(i \in [n-1]\), \(s_1, s_2, s_3 \in \mathbb{R}\),

\[
\lambda_i(s_1)\lambda_{i+1}(s_2)\lambda_i(s_3) = \lambda_{i+1}\left(\frac{s_2s_3}{s_1 + s_3}\right)\lambda_i(s_1 + s_3)\lambda_{i+1}\left(\frac{s_1s_2}{s_1 + s_3}\right).
\]

As in Lemma 5.8 of [15], \(\lambda_{ik}(t) \ll L_{\sigma_{k-1}}^{-1} L_x\) if and only if \(t < g_k(t_1, \ldots, t_{k-1})\).
5 Accessibility in the spin group

For \( z_\chi \in Q[\text{Pos}_\eta] \subseteq U_1 \cap \text{Bru}_\eta \) and \( \sigma \in S_{n+1} \), we define
\[
\text{Ac}_\sigma(z_\chi) = Q[\text{Ac}_\sigma(L(z_\chi))] \subseteq Q[\text{Pos}\sigma] \subseteq \text{Bru}_\sigma.
\]

For each \( z \in \text{Ac}_\sigma(z_\chi) \), there exists a locally convex curve \( \Gamma : [0, 1] \to U_1 \) with \( \Gamma(0) = z \) and \( \Gamma(1) = z_\chi \). Indeed, just take a convex curve \( \Gamma_L : [0, 1] \to L_{n+1} \) with \( \Gamma_L(0) = L(z) \) and \( \Gamma_L(1) = L(z_\chi) \) and define \( \Gamma = Q \circ \Gamma_L \). Similarly, for \( z \in Q[\text{Pos}\sigma] \setminus \text{Ac}_\sigma(z_\chi) \), no such curve exists.

For \( z_\chi \in \text{Bru}_\eta \), choose \( U \in U_{n+1}^+ \) such that \( z_\chi = z_U \), \( z_0 \in Q[\text{Pos}_\eta] \). For \( \sigma \in S_{n+1} \), define \( \text{Ac}_\sigma(z_\chi) = (\text{Ac}_\sigma(z_0))^U \); this turns out to be well-defined and the properties above still hold. We want to define \( \text{Ac}_\sigma(z_\chi) \) for any \( z_\chi \in \text{chop}^{-1}[\{\eta\}] \).

This will require a certain detour. We shall first present a topological construction (using curves), then an algebraic one (using coordinates) and then finally prove their equivalence.

For \( q \in \text{Quat}_{n+1} \), set
\[
\text{Bru}^0_{\eta q} = \text{adv}^{-1}[\{q\eta\}] = \bigcup_{\sigma \in S_{n+1}} \text{Bru}_{\eta q\sigma},
\]
\[
\text{Bru}^1_{\eta q} = \text{chop}^{-1}[\{q\eta\}] = \bigcup_{\sigma \in S_{n+1}} \text{Bru}_{q\sigma}.
\]

A locally convex curve \( \Gamma : [0, 1] \to \text{Spin}_{n+1} \) satisfying \( \Gamma(t) \in \text{Bru}_\eta \) for \( t \in (0, 1) \) will necessarily satisfy \( \Gamma(0) \in \text{Bru}^0_{\eta q} \) and \( \Gamma(1) \in \text{Bru}^1_{\eta q} \). Notice that \( \text{Bru}_\eta \subseteq \text{Bru}^0_{\eta q} \cap \text{Bru}^1_{\eta q} \). Given \( \sigma \in S_{n+1} \), we have
\[
\text{Bru}_\sigma \cap \text{Bru}^0_{\eta q} = \text{Bru}_\sigma,
\]
\[
\text{Bru}_\sigma \cap \text{Bru}^1_{\eta q} = \text{Bru}_{\eta(\sigma^{-1})q}.
\]

and therefore
\[
\text{Bru}^0_{\eta q} \cap \text{Bru}^1_{\eta q} = \bigcup_{\sigma \in S_{n+1}, \sigma = \eta q} \text{Bru}_\sigma.
\]

It follows from Remark 3.8 of [15] that \( \text{Bru}^0_{\eta q} \cap \text{Bru}^1_{\eta q} = \text{Bru}_\eta \) precisely for \( n \leq 3 \).

In order to extend locally convex curves in \( \text{Bru}_\eta \) to the boundary and not mix up entry points with exit points we define a new larger space:
\[
\text{Bru}^0_\eta = ((\text{adv}^{-1}[\{\eta\}] \times \{0\}) \cup (\text{chop}^{-1}[\{\eta\}] \times \{1\})) / \sim
\]

where \((z, 0) \sim (z, 1)\) for \( z \in \text{Bru}_\eta \) (and only there). We abuse notation by writing
\[
\text{Bru}^0_{\eta q} \subseteq \text{Bru}^0_{\eta q q}, \quad \text{Bru}^1_{\eta q} \subseteq \text{Bru}^1_{\eta q}.
\]
in this context, \( \text{Bru}^0_{\eta q} \cap \text{Bru}^1_{\eta q} = \text{Bru}_\eta \). A locally convex curve \( \Gamma : [0, 1] \to \text{Bru}^0_{\eta q} \) corresponds to a locally convex curve \( \Gamma_1 : [0, 1] \to \text{Spin}_{n+1} \) satisfying \( \Gamma_1(t) \in \text{Bru}_\eta \) for \( t \in (0, 1) \) with \( \Gamma(0) = (\Gamma_1(0), 0), \Gamma(1) = (\Gamma_1(1), 1) \).
For \( z_0 \in \text{Bru}_q^0 \) and \( z_1 \in \text{Bru}_q^1 \), let \( \mathcal{L}_n^0(z_0; z_1) \subset \mathcal{L}_n^{[H]}(z_0; z_1) \) be the set of locally convex curves \( \Gamma : [0, 1] \to \text{Spin}_{n+1} \) such that \( \Gamma(0) = z_0, \Gamma(1) = z_1 \) and \( \Gamma(t) \in \text{Bru}_q \) for all \( t \in (0, 1) \). For \( z_0, z_1 \in \text{Bru}_q^0 \), write \( z_0 \ll z_1 \) if and only if \( z_0 \in \text{Bru}_q^0, z_1 \in \text{Bru}_q^1 \), and \( \mathcal{L}_n^0(z_0; z_1) \neq \emptyset \) (compare with Lemma 5.3 of [15]).

**Lemma 5.1.** Consider \( z_0 \in \text{Bru}_q^0 \) and \( z_1 \in \text{Bru}_q^1 \). The set \( \mathcal{L}_n^0(z_0; z_1) \) is either empty or contractible. If \( z_0 \ll z_1 \) then \( z_0^{-1}z_1 \in \text{Bru}_q^1 \) and \( \mathcal{L}_n^0(z_0; z_1) = \mathcal{L}_{n,\text{convex}}(z_0; z_1) \).

**Example 5.2.** Recall from Lemma 3.3 that \( \mathcal{L}_{n,\text{convex}}(z_0; z_1) \subset \mathcal{L}_n(z_0; z_1) \) is a contractible connected component if \( z_0^{-1}z_1 \in \text{Bru}_q^1 \) and is empty otherwise. It is entirely possible to have \( z_0 \in \text{Bru}_q^0, z_1 \in \text{Bru}_q^1, z_0^{-1}z_1 \in \text{Bru}_q^1 \) and \( z_0 \ll z_1 \) so that \( \mathcal{L}_n^0(z_0; z_1) = \emptyset \). In this case, there are convex curves in \( \mathcal{L}(z_0; z_1) \) but they never belong to \( \mathcal{L}_n^0(z_0; z_1) \). A simple case is \( n = 6 \), \( z_0 = \exp(3/4 \hat{h}) \) and \( z_1 = \exp(1/2 \hat{h}) \). Recall from Example 3.7 of [15] that, for \( n = 6 \), we have \( \hat{h} = \exp(7/2 \hbar) = 1 \). We have \( z_0^{-1}z_1 = \exp(1/2 \hbar) = \hat{h} \) and the curve \( \Gamma(t) = z_0 \exp(5/2 \hbar t) \) is convex.

**Proof of Lemma 5.1.** By definition, \( \mathcal{L}_n^0(z_0; z_1) = \text{sing}^{-1}([\emptyset]) \). Since \( \emptyset \) is an isolated point in \( \mathcal{H}([0, 1]) \), the set \( \mathcal{L}_n^0(z_0; z_1) \) is a union of connected components of \( \mathcal{L}_n(z_0; z_1) \), by Theorem 3.4.

We know from [31] that if \( \Gamma \) is not convex then \( \Gamma \) is in the same connected component as \( \Gamma \) with added loops, which clearly does not have empty singular set. Thus the only connected component of \( \mathcal{L}_n(z_0; z_1) \) which may be contained in \( \mathcal{L}_n^0(z_0; z_1) \) is \( \mathcal{L}_{n,\text{convex}}(z_0; z_1) \).

Given \( z_x \in \text{Bru}_q^1 \) and \( \sigma \in S_{n+1} \), consider \( \text{Bru}_\sigma \subset \text{Bru}_q^0 \); let

\[
\text{Ac}_\sigma(z_x) = \{ z \in \text{Bru}_\sigma \subset \text{Bru}_q^0 \mid z \ll z_x \}.
\]

**Lemma 5.3.** Consider \( z_x \in \text{Bru}_q^1 \). Consider \( \sigma_{k-1} \ll \sigma_k = \sigma_{k-1} \alpha_{i_k} \in S_{n+1} \), \( \text{inv}(\sigma_k) = k \). Consider \( z_{k-1} \in \text{Bru}_{\sigma_{k-1}} \) and \( z_k = z_{k-1} \alpha_{i_k}(\theta_k) \in \text{Bru}_{\sigma_k}, \theta_k \in (0, \pi) \). If \( z_k \ll z_x \) then \( z_{k-1} \alpha_{i_k}(\theta) \ll z_x \) for all \( \theta \in [0, \theta_k] \).

As in Section 2, \( \ll \) denotes the covering relation for the Bruhat order in \( S_{n+1} \).

**Proof.** From Corollary 6.4 of [15], if \( z_k \in \text{Q}[\text{Pos}_{\sigma_k}] \) then

\[
z_{k-1} \alpha_{i_k}(\theta) \in \text{Q}[\text{Pos}_{\sigma_{k-1}} \cup \text{Pos}_{\sigma_k}] .
\]

In this case, take \( L_k = \text{L}(z_k) \) and \( L_\theta = \text{L}(z_{k-1} \alpha_{i_k}(\theta)) \). Consider a locally convex curve \( \Gamma \in \mathcal{L}_n^0(z_k; z_x) \). As in the proof of Theorem 3 of [15], take \( \Gamma_L(t) = \text{L}(\Gamma(t)) \) so that \( \Gamma_L(0) = L_k \). By Lemma 5.7 of [15], there exists \( \epsilon > 0 \) such that \( \Gamma_L \) is well-defined in \( [0, 2\epsilon] \) and \( L_\epsilon = \Gamma_L(\epsilon) \in \text{Pos}_n \). We have \( L_\theta \leq L_k \ll L_\epsilon \) and therefore \( L_\theta \ll L_\epsilon \) (Lemma 5.2 and Equation (15) of [15]). By Lemma 5.3 of [15],
there exists a locally convex curve \( \Gamma : [0, \varepsilon] \to \text{Lo}^{1}_{n+1} \), \( \Gamma(0) = L_\theta \) and \( \Gamma(\varepsilon) = L_\varepsilon \). Define
\[
\Gamma_1(t) = \begin{cases} Q(\Gamma_\varepsilon(t)), & t \in [0, \varepsilon], \\ \Gamma(t), & t \in [\varepsilon, 1]. \end{cases}
\]
Notice that, for \( t \in (0, \varepsilon) \), we have \( \Gamma_1(t) \in \text{Pos}_\eta \) and therefore \( \Gamma_1(t) \in \text{Bru}_\eta \). The curve \( \Gamma_1 : [0, 1] \to \text{Spin}_{n+1} \) is locally convex and satisfies \( \Gamma_1(0) = z_{k-1} \alpha_i \), \( \Gamma(1) = z_x \) and \( \Gamma(t) \in \text{Bru}_\eta \) for all \( t \in (0, 1) \). By definition, \( z_{k-1} \alpha_i(\theta) \ll z_x \).

In general, there is an upper triangular matrix \( U \in \text{Up}_{n+1}^1 \) such that the corresponding projective transformation takes \( z_k \) to \( z_k^U = Q(U^{-1}z_k) \in Q(\text{Pos}_{\sigma_z}) \), reducing the situation to the previous case. \( \square \)

We now present an algebraic definition. Consider \( z_x \in \text{Bru}^1_\theta \). Consider \( \rho_0 \in S_{n+1} \) such that \( z_x \in \text{Bru}_\hat{\eta}(\rho_0)^{-1} \), \( y_0 = z_x^{-1}\hat{\eta} \in \text{Bru}_{\rho_0} \). We first define sets \( \text{Ac}_{(i_1, \ldots, i_k)}(z_x) \subseteq \text{Bru}_\sigma \) where \( \sigma = a_{i_1} \cdots a_{i_k} \) is a reduced word. When \( z_x \) is fixed (and thus so are \( y_0 \) and \( \rho_0 \)) we write for simplicity \( \text{Ac}_{(i_1, \ldots, i_k)} = \text{Ac}_{(i_1, \ldots, i_k)}(z_x) \).

For each \( j \in [k] \), set \( \sigma_j = a_{i_1} \cdots a_{i_j} \), so that \( \sigma_{j-1} \triangleleft \sigma_j = \sigma_{j-1} a_{i_j} \); also, define recursively
\[
\rho_j = \begin{cases} \rho_{j-1} a_{i_j}, & \text{if } \rho_{j-1} \triangleleft \rho_{j-1} a_{i_j}, \\ \rho_{j-1}, & \text{otherwise}, \end{cases}
\]
so that either \( \rho_{j-1} = \rho_j \) or \( \rho_{j-1} \triangleleft \rho_j = \rho_{j-1} a_{i_j} \). For those \( j \) such that \( \rho_{j-1} = \rho_j \), define auxiliary functions \( \Theta_{i_j} : \text{Bru}_{\rho_j} \to (0, \pi) \) as follows: \( \Theta_{i_j}(z) = \theta \) if and only \( z a_{i_j}(-\theta) \in \text{Bru}_{\rho_{j-1} a_{i_j}} \). It is a consequence of the proof of Theorem 1 of [15] that these functions are well-defined and smooth (see Remark 6.6 of [15]).

Set \( \text{Ac}_{(i)} = \text{Bru}_1 = \{1\} \). We assume \( \text{Ac}_{(i_1, \ldots, i_{j-1})} \) defined and proceed to construct \( \text{Ac}_{(i_1, \ldots, i_j)} \):

\[
\text{Ac}_{(i_1, \ldots, i_j)} = \{ z_{j-1} \alpha_{i_j}(\theta_j) \mid z_{j-1} \in \text{Ac}_{(i_1, \ldots, i_{j-1})}, \theta_j \in (0, \Theta_{i_j}(z_{j-1})) \};
\]

\[
\vartheta_{i_j} : \text{Ac}_{(i_1, \ldots, i_{j-1})} \to (0, \pi), \quad \vartheta_{i_j}(z_{j-1}) = \begin{cases} \pi, & \rho_{j-1} \triangleleft \rho_j, \\ \pi - \Theta_{i_j}(y_0 z_{j-1}), & \rho_{j-1} = \rho_j. \end{cases}
\]

**Lemma 5.4.** The sets \( \text{Ac}_{(i_1, \ldots, i_j)}, 1 \leq j \leq k \), defined above satisfy

\[
\text{Ac}_{(i_1, \ldots, i_j)} = \{ \alpha_{i_1}(\theta_1) \cdots a_{i_j}(\theta_j) \mid (\theta_1, \ldots, \theta_j) \in X_j \} \subseteq \text{Bru}_{\rho_j} \cap (y_0^{-1} \text{Bru}_{\rho_j})
\]

where \( (X_j)_{1 \leq j \leq k} \) is a quasiproduct; \( \text{Ac}_{(i_1, \ldots, i_j)} \) is diffeomorphic to \( \mathbb{R}^j \).

Notice that the inclusion in the statement is necessary to make sense of the definition of \( \vartheta_{i_j} \). The reader should compare this result with Lemma 4.2.
**Proof.** The proof is by induction on \( k \); the case \( k = 0 \) is trivial. Take \( z_k = z_{k-1} \alpha_{i_k}(\theta_k) \in A c_{(i_1,\ldots,i_k)} \), \( z'_{k-1} \in A c_{(i_1,\ldots,i_{k-1})} \), \( \theta_k \in (0, \theta_{i_k}^{-1}(z_{k-1})) \). We assume by induction hypothesis that \( A c_{(i_1,\ldots,i_{k-1})} \subseteq B r u_{\delta_{k-1}} \). We therefore have \( z_k \in B r u_{\delta_{k-1}} \) \( B r u_{\delta_{i_k}} = B r u_{\delta_k} \) (by Corollary 6.2 of [15]). We also assume by induction hypothesis that \( y_k = y_0 z_{k-1} \in B r u_{\rho_{k-1}} \). If \( \rho_{k-1} < \rho_k \), Corollary 6.2 of [15] implies that \( y_k = y_0 z_k = y_k \alpha_{i_k}(\theta_k) \in B r u_{\rho_{k-1}} B r u_{\delta_{i_k}} = B r u_{\rho_k} \). If \( \rho_{k-1} = \rho_k \), take \( \tilde{\theta} = \Theta_{i_k}(y_{k-1}) \) and \( \tilde{y} \in B r u_{\rho_k\alpha_{i_k}} \) such that \( y_k = \tilde{y} \alpha_{i_k}(\tilde{\theta}) \). By our recursive definition, \( \tilde{\theta} + \theta_k < \pi \); by Theorem 1 of [15], we have \( y_k = \tilde{y} \alpha_{i_k}(\tilde{\theta} + \theta_k) \in B r u_{\rho_k} \). \( \square \)

We now prove that the two definitions are equivalent.

**Lemma 5.5.** Consider \( z_{\chi} \in B r u_{\eta}^1 \) and \( \sigma_k = a_{i_1} \cdots a_{i_k} \) a reduced word in \( S_{n+1} \). Then \( A c_{\sigma_k}(z_{\chi}) = A c_{(i_1,\ldots,i_k)}(z_{\chi}) \).

**Proof.** The proof is by induction on \( k \); the case \( k = 0 \) is trivial. Assume therefore \( A c_{\sigma_{k-1}}(z_{\chi}) = A c_{(i_1,\ldots,i_{k-1})}(z_{\chi}) \) for \( \sigma_{k-1} = a_{i_1} \cdots a_{i_{k-1}} \).

Consider \( z_k = z_{k-1} \alpha_{i_k}(\theta_k), z_{k-1} \in B r u_{\sigma_{k-1}}, z_k \in B r u_{\delta_k}, \theta_k \in (0, \pi) \). It follows from Lemma 5.3 that \( z_k \in A c_{\sigma_k} \) implies \( z_{k-1} \in A c_{\sigma_{k-1}} \) and therefore

\[
A c_{\sigma_k} \subseteq A c_{\sigma_{k-1}} B r u_{\delta_k}, \quad A c_{(i_1,\ldots,i_k)} \subseteq A c_{\sigma_{k-1}} B r u_{\delta_k};
\]

we have to prove that these two sets are equal.

Given \( z_{k-1} \in A c_{\sigma_{k-1}} \), let \( J_{z_{k-1}} \subseteq (0, \pi) \) be the set such that, for all \( \theta_k \in (0, \pi), \theta_k \in J_{z_{k-1}} \) if and only if \( z_{k-1} \alpha_{i_k}(\theta_k) \in A c_{\sigma_k} \). It follows from Lemma 5.3 that \( J_{z_{k-1}} \) is either empty or an interval.

We claim that \( J_{z_{k-1}} \) is not empty. By applying a projective transformation, we may assume \( z_{k-1} = Q(L_{k-1}), L_{k-1} \in P o s_{\sigma_{k-1}} \). Take \( \Gamma \in L_{\eta}^\circ(z_{k-1}; z_{\chi}) \). Define \( \Gamma_L = L \circ \Gamma \), with maximal connected domain containing \( t = 0 \). Consider \( \Gamma_{\bullet} > 0 \) in this domain and \( L_{\bullet} = \Gamma_L(t_{\bullet}), L_{\bullet} \in P o s_{\eta}, L_{k-1} \ll L_{\bullet} \). Take \( t_k > 0 \) such that \( L_{k-1} \lambda_{i_k}(t_k) \ll L_{\bullet} \); define \( \theta_k > 0 \) by \( Q(L_{k-1} \lambda_{i_k}(t_k)) = z_{k-1} \alpha_{i_k}(\theta_k) \). Take convex \( \Gamma_{L,1} : [0, t_{\bullet}] \to L_{\eta}^1_{n+1} \) such that \( \Gamma_{L,1}(0) = L_{k-1} \lambda_{i_k}(t_k) \) and \( \Gamma_{L,1}(t_{\bullet}) = L_{\bullet} \). Finally, take \( \Gamma_1 : [0, 1] \to S p i n_{n+1}, \)

\[
\Gamma_1(t) =\begin{cases} Q(\Gamma_{L,1}(t)), & t \in [0, t_{\bullet}], \\ \Gamma(t), & t \in [t_{\bullet}, 1]. \end{cases}
\]

We have \( \Gamma_1 \in L_{\eta}^\circ(z_{k-1} \alpha_{i_k}(\theta_k); z_{\chi}) \) and therefore \( \theta_k \in J_{z_{k-1}} \), as claimed.

We claim that \( J_{z_{k-1}} \) is open. Assume by contradiction \( \theta_k^* = \max(J_{z_{k-1}}) \), \( z_k^* = z_{k-1} \alpha_{i_k}(\theta_k^*) \in A c_{\sigma_k} \). By applying a projective transformation, we may assume that \( z_k^* \in Q[P o s_{\sigma_k}] \). As in the previous paragraph, take a locally convex curve \( \Gamma \in L_{\eta}^\circ(z_k^*; z_{\chi}) \), use \( L \) to take its initial segment to \( L_{\eta}^1_{n+1} \) and slightly
perturb it to obtain \( \theta_k \in J_{z_{k-1}} \), \( \theta_k > \theta_k^* \). The argument is so similar that we feel that a repetition is pointless.

At this point we know that there exists a function \( \tilde{\vartheta}_k : \mathcal{A}_{\sigma_{k-1}} \to (0, \pi) \) such that \( J_{z_{k-1}} = (0, \tilde{\vartheta}_k(z_{k-1})) \). We are left with proving that \( \vartheta_k = \tilde{\vartheta}_k \).

We first prove that \( \tilde{\vartheta}_k(z_{k-1}) \leq \vartheta_k(z_{k-1}) \) for all \( z_{k-1} \). If \( \rho_{k-1} \leq \rho_k \) then \( \vartheta_k(z_{k-1}) = \pi \) and we are done. If \( \rho_{k-1} = \rho_k \), take \( \theta_k^* = \vartheta_k(z_{k-1}) \), \( z_k = z_{k-1} \alpha_{i_k}(\theta_k^*) \) and \( y_k^* = y_0z_k^* \). Recall that in this case there exists \( \rho_0 \in S_{n+1}, \rho_0 < \rho_{k-1} = \rho_k = \rho_0 a_{i_k} \). By definition of \( \vartheta_k, y_k^* \in \text{Bru}_y^{\bullet} \) so that \( \text{adv}(y_k^*) = q^* \hat{\eta} \) for \( q^* \in \text{Quat}_{n+1}, q^* \neq 1 \). By Theorem 3 of [15], any locally convex curve starting at \( y_k^* \) immediately enters \( \text{Bru}_y^{\bullet} \). Thus, \( \mathcal{L}_{z_k}^{\bullet}(y_k^*; \hat{\eta}) = \emptyset \) and therefore \( \mathcal{L}_{z_k}^{\bullet}(z_k; z_k) = \emptyset \). It follows that \( z_k \not\in \mathcal{A}_{\sigma_k}(z_x) \) and therefore \( \theta_k^* \geq \vartheta_k(z_{k-1}) \), proving our claim.

We finally prove that \( \tilde{\vartheta}_k(z_{k-1}) \geq \vartheta_k(z_{k-1}) \). Consider \( \theta_k < \vartheta_k(z_{k-1}) \), \( z_k = z_{k-1} \alpha_{i_k}(\theta_k) \) and \( y_k = y_0z_k \in \text{Bru}_y^{\bullet} \). Notice that \( z_{k-1} \alpha_{i_k}(\theta) \in \text{Bru}_y^{\bullet} \) and \( y_{k-1} \alpha_{i_k}(\theta) \in \text{Bru}_y^{\bullet} \) for all \( \theta \in [0, \theta_k] \). By compactness and Theorem 3 of [15], there exists \( c > 0 \) such that, for all \( \theta \in [0, \theta_k] \) and for all \( t \in (0, c] \), we have both \( z_{k-1} \alpha_{i_k}(\theta) \exp(th) \in \text{Bru}_y^{\bullet} \) and \( y_{k-1} \alpha_{i_k}(\theta) \exp(th) \in \text{Bru}_y^{\bullet} \). Apply Lemma 6.1 of [15] to obtain a continuous family \( H : [0, \theta_k] \times [\frac{1}{2}, 1] \to \text{Spin}_{n+1} \) of convex curves \( H(\theta) : [\frac{1}{2}, 1] \to \text{Spin}_{n+1} \) going from \( y_{k-1} \alpha_{i_k}(\theta) \exp(ch) \) to \( \hat{\eta} \). Extend this to \( H : [0, \theta_k] \times [0, 1] \to \text{Spin}_{n+1} \) by defining \( H(\theta)(t) = y_{k-1} \alpha_{i_k}(\theta) \exp(2cth) \) for \( t \in [0, \frac{1}{2}] \). This extension is still continuous. For each \( \theta \in [0, \theta_k] \), the arc \( H(\theta) : [0, 1] \to \text{Spin}_{n+1} \) is in \( \mathcal{L}_{z_k}^{\bullet}(y_{k-1} \alpha_{i_k}(\theta); \hat{\eta}) \), since we have \( H(\theta)(t) \in \text{Bru}_y^{\bullet} \) for all \( t \in (0, 1) \). Multiply by \( y_0^{-1} \) to obtain a family \( y_0^{-1} H(\theta) : [0, 1] \to \text{Spin}_{n+1} \) going from \( z_{k-1} \alpha_{i_k}(\theta) \) to \( z_x \). We prove that, for all \( \theta \), we have \( \Gamma_\theta \in \mathcal{L}_{z_k}^{\bullet}(z_{k-1} \alpha_{i_k}(\theta); z_x) \), i.e., that \( \Gamma_\theta(t) \in \text{Bru}_y^{\bullet} \) for all \( t \in (0, 1) \). We know that \( \Gamma_0 \) is convex and that \( z_k \in \mathcal{A}_{\sigma_{k-1}}(z_x) \) and therefore, from Lemma 5.1, \( \Gamma_0 \in \mathcal{L}_{z_k}^{\bullet}(z_{k-1}; z_x) \). We know by construction that \( \Gamma_\theta(t) \in \text{Bru}_y^{\bullet} \) for all \( t \in (0, 1) \). Apply again Lemma 6.1 of [15] to construct a continuous family of convex arcs \( \hat{\Gamma}_\theta : [0, \frac{1}{2}] \to \text{Spin}_{n+1} \) from \( \hat{\Gamma}_\theta(0) = 0 \) to \( \hat{\Gamma}_\theta(\frac{1}{2}) = \Gamma_\theta(\frac{1}{2}) \). Extend \( \hat{\Gamma}_\theta \) to \( [0, 1] \) by \( \hat{\Gamma}_\theta(t) = \Gamma_\theta(t) \) for \( t \in [\frac{1}{2}, 1] \). The corresponding family of extended locally convex curves is again continuous. We have \( \text{sing}(\hat{\Gamma}_0) = \emptyset \). Also, from Theorem 1, \( \text{sing}(\hat{\Gamma}_\theta) \) is a continuous function of \( \theta \). Since \( \emptyset \in \mathcal{H}([0, 1]) \) is an isolated point, we have \( \text{sing}(\hat{\Gamma}_\theta) = \emptyset \) for all \( \theta \), as desired. This implies that \( z_k = z_{k-1} \alpha_{i_k}(\theta_k) \ll z_x \) and therefore \( \theta_k < \vartheta_k(z_{k-1}) \). Since this holds for any \( \theta_k < \vartheta_k(z_{k-1}) \) we have \( \tilde{\vartheta}_k(z_{k-1}) \geq \vartheta_k(z_{k-1}) \), completing our proof.

\[ \square \]

**Remark 5.6.** We saw in Lemmas 3.3 and 5.1 that, given \( z_0 \in \text{Bru}_y^{\bullet} \) and \( z_1 \in \text{Bru}_y^{\bullet} \), the set \( \mathcal{L}_{z_0}^{\bullet}(z_0; z_1) \) is either empty or equal to \( \mathcal{L}_{n, \text{convex}}(z_0; z_1) \) and contractible. In Lemma 3.3 we saw an explicit contraction if \( z_0^{-1} z_1 \in \text{Bru}_y^{\bullet} \) but otherwise used Proposition 6.4 of [31]. We now present a more explicit contraction in general. For any \( \Gamma \in \mathcal{L}_{z_0}^{\bullet}(z_0; z_1) \), we have \( (\Gamma(0))^{-1} \Gamma(\frac{1}{2}) \in \text{Bru}_y^{\bullet} \) and \( (\Gamma(\frac{1}{2}))^{-1} \Gamma(1) \in \text{Bru}_y^{\bullet} \). Apply the contraction in the proof of Lemma 3.3.
Let \( \Gamma(\frac{1}{2}) \) be the corresponding itinerary with \( \ell = \ell(w) = \text{card}(\text{sing}(\Gamma)) \in \mathbb{N} \). Recall from Lemma 5.1 that \( \Gamma \in \mathcal{L}_{n}(\hat{\eta}) \) is convex if and only if its itinerary iti(\( \Gamma \)) is the empty word of length 0.

Given the path \((z_1, \ldots, z_\ell)\) of some \( \Gamma \in \mathcal{L}_n \), it is trivial to determine the corresponding itinerary \( w = (\sigma_1, \ldots, \sigma_\ell) \in \mathcal{W}_n \). Conversely, given an itinerary \( w = (\sigma_1, \ldots, \sigma_\ell) \in \mathcal{W}_n \), define \( B(w, j) \in \tilde{B}_{n+1}^+ \) for \( j \in \mathbb{Z} \), \( 0 \leq j \leq \ell + 1 \), and \( B(w, j + \frac{1}{2}) \in \tilde{B}_{n+1}^+ \) for \( j \in \mathbb{Z} \), \( 0 \leq j \leq \ell \), by

\[
\begin{align*}
B(w, 0) &= 1, \\
B\left(w, \frac{1}{2}\right) &= \hat{\eta}, \\
B(w, j) &= B\left(w, j - \frac{1}{2}\right) \hat{\sigma}_j, \\
B\left(w, j + \frac{1}{2}\right) &= B\left(w, j - \frac{1}{2}\right) \hat{\sigma}_j, \\
B(w, \ell + 1) &= B\left(w, \ell + \frac{1}{2}\right) \hat{\eta}.
\end{align*}
\]

In particular, we have \( B(w, \ell + 1) = \hat{\eta} \hat{\sigma}_1 \cdots \hat{\sigma}_\ell \in \text{Quat}_{n+1} \), where we define the hat of a word by \( \hat{w} = \hat{\sigma}_1 \cdots \hat{\sigma}_\ell \in \text{Quat}_{n+1} \). We adopt here the conventions \( t_0 = 0 \), \( t_{\ell+1} = 1 \), \( z_0 = 1 \), \( z_{\ell+1} = B(w, \ell + 1) \), \( \sigma_0 = \sigma_{\ell+1} = \eta \).

It follows from Theorem 3 of [15] that if \( \Gamma \in \mathcal{L}_n[w] \) and \( \text{sing}(\Gamma) = \{t_1 < \cdots < t_\ell\} \) then

\( \Gamma \in \mathcal{L}_n(\hat{\eta} \hat{\sigma}_1 \cdots \hat{\sigma}_\ell), \quad \Gamma(t_j) \in \text{Bru}_{B(w, j)}, \quad \forall t \in (t_j, t_{j+1}), \quad \Gamma(t) \in \text{Bru}_{B(w, j + \frac{1}{2})}. \)

Thus, if \( \Gamma \in \mathcal{L}_n[w] \) then \( \text{path}(\Gamma) \in \text{Bru}_{B(w, 1)} \times \cdots \times \text{Bru}_{B(w, \ell)}. \)

Given \( w = (\sigma_1, \ldots, \sigma_\ell) \in \mathcal{W}_n \) and \( j \in \mathbb{Z} \), \( 0 \leq j \leq \ell \), define \( q_j \in \text{Quat}_{n+1} \) by

\[
B(w, j) = q_j \text{acute}(\eta \sigma_j) \in q_j \text{Bru}_0^{\hat{\eta}}, \\
B\left(w, j + \frac{1}{2}\right) = q_j \hat{\eta}, \\
B(w, j + 1) = q_j \text{grave}(\eta \sigma_{j+1}) \in q_j \text{Bru}_1^{\hat{\eta}}.
\]

A sequence \((z_1, \ldots, z_\ell)\) is an accessible path for \( w \) if

\[
\forall j \in [\ell] \quad (q_j^{-1} z_j \in \text{Ac}_{\hat{\eta} \sigma_j}(q_j^{-1} z_{j+1})).
\]

Let \( \text{Path}(w) \subseteq \text{Bru}_{B(w, 1)} \times \cdots \times \text{Bru}_{B(w, \ell)} \) be the set of accessible paths for \( w \).
Lemma 6.1. Consider \( w \in W_n \). For any \( \Gamma \in L_n[w] \), \( \text{path}(\Gamma) \) is accessible, i.e., belongs to \( \text{Path}(w) \).

Proof. Consider \( t_j < t_{j+1} \) and the arc \( q_j^{-1}\Gamma|_{[t_j,t_{j+1}]} \). Except for the modified \( q_j^{-1}\Gamma|_{[t_j,t_{j+1}]} \) domain, this arc belongs to \( L_n(q_j^{-1}\Gamma(t_j); q_j^{-1}\Gamma(t_{j+1})) \) and therefore \( q_j^{-1}\Gamma(t_j) \in Ac_{\eta^j}(q_j^{-1}\Gamma(t_{j+1})) \), as desired. \( \square \)

Lemma 6.2. Consider \( w \in W_n \). For any accessible path \( (z_1, \ldots, z_\ell) \in \text{Path}(w) \), the set \( \{ \Gamma \in L_n[H^1][w] \mid \text{path}(\Gamma) = (z_1, \ldots, z_\ell) \} \) is nonempty and contractible.

Proof. The collection of sets \( \{ t_1 < \cdots < t_\ell \} \in H([0,1]) \) is a contractible subset. At this point, the values of \( q_j \in \text{Quat}_{n+1} \), of \( t_j < t_{j+1} \), of \( z_j \in q_j \text{Bru}_{\text{acute}(\eta^j)} \) and of \( z_{j+1} \in q_j \text{Bru}^1_{q} \) with \( q_j^{-1}z_j \in Ac_{\eta^j}(q_j^{-1}z_{j+1}) \) are all given. The set of locally convex arcs \( \Gamma : [t_j, t_{j+1}] \to \text{Spin}_{n+1} \) with \( \Gamma(t_j) = z_j \), \( \Gamma(t_{j+1}) = z_{j+1} \) and \( \Gamma(t) \in q_j \text{Bru}_{\eta^j} \) for all \( t \in (t_j, t_{j+1}) \) is homeomorphic to \( L_n(q_j^{-1}z_j; q_j^{-1}z_{j+1}) \); by Lemma 6.3 this set is contractible (with an explicit contraction given by Remark 5.6). Concatenate the above arcs to construct \( \Gamma \); this yields the desired result. \( \square \)

Lemma 6.3. Consider \( w \in W_n \). The set \( \text{Path}(w) \subseteq \text{Bru}_{B(w,1)} \times \cdots \times \text{Bru}_{B(w,\ell)} \) is diffeomorphic to \( \mathbb{R}^d \), \( d = \text{inv}(\eta^1) + \cdots + \text{inv}(\eta^\ell) \). In particular, \( \text{Path}(w) \) is contractible (and nonempty).

Proof. Start constructing the set from the \( \ell \)-th coordinate \( \text{Bru}_{B(w,\ell)} \) and proceed backwards. Use Lemma 5.4 for the inductive step. The set \( \text{Path}(w) \) is parametrized by a quasiprocess. \( \square \)

Lemma 6.4. For any \( w \in W_n \), the subset \( L_n[H^1][w] \subseteq L_n[H^1](\eta^1 \ldots \eta^\ell) \) is contractible.

Proof. We omit the superscript \( [H^1] \) throughout the proof. Let \( \text{Path}_1(w) \subseteq L_n[w] \) be the set of locally convex curves \( \Gamma \) such that the arcs \( \Gamma|_{[t_i-1,t_i]} \) are assigned base points to the contractible sets \( L_n,\text{convex}(\Gamma(t_i); \Gamma(t_i)) \) (up to a reparameterization). Here we assume that \( \text{sing}(\Gamma) = \{ t_1 < \cdots < t_\ell \} \); we may use the construction in Remark 5.6 to select a basepoint.

Lemma 6.2 yields a deformation retract from \( L_n[w] \) to \( \text{Path}_1(w) \), a homotopy \( H_0 : [0,1] \times L_n[w] \to L_n[w] \) which starts with an arbitrary curve \( \Gamma_0 \in L_n[w] \) and deforms it through \( \Gamma_s = H_0(s, \Gamma_0) \) for \( s \in [0,1] \). The homotopy satisfies \( \text{sing}(\Gamma_s) = \text{sing}(\Gamma_0) = \{ t_1 < \cdots < t_\ell \} \) and \( \text{path}(\Gamma_s) = \text{path}(\Gamma_0) \) for all \( s \in [0,1] \). We have \( \Gamma_1 \in \text{Path}_1(w) \), i.e., the arcs \( \Gamma_1|_{[t_{i-1},t_i]} \) are the base points of the contractible sets \( L_n,\text{convex}(\Gamma_0(t_{i-1}); \Gamma_0(t_i)) \). Also, if \( \Gamma_0 \in \text{Path}_1(w) \) then \( \Gamma_s = \Gamma_0 \) for all \( s \in [0,1] \).

Let \( \text{Path}_2(w) \subseteq \text{Path}_1(w) \) be the set of paths \( \Gamma \in \text{Path}_1(w) \) such that \( \text{sing}(\Gamma) = \{ \frac{1}{\ell+1} < \cdots < \frac{\ell}{\ell+1} \} \). There is an easy deformation retract \( H_1 : \)
\[ [1, 2] \times \text{Path}_1(w) \rightarrow \text{Path}_1(w) \] from \( \text{Path}_1(w) \) to \( \text{Path}_2(w) \): affinely reparameterize each interval \([t_{i-1}, t_i]\).

Lemma 6.1 shows that \( \text{Path}_2(w) \) is homeomorphic to \( \text{Path}(w) \): the homeomorphism takes \( \Gamma \) to \( \text{path}(\Gamma) \). Lemma 6.3 shows us how to construct a homotopy \( \tilde{H}_2 : [2, 3] \times \text{Path}(w) \rightarrow \text{Path}(w) \) with \( \tilde{H}_2(2, z) = z \) and \( \tilde{H}_2(3, z) = z_0 \) where \( z_0 \in \text{Path}(w) \) is a base point. Compose with the homeomorphism above to define a deformation retract \( H_2 \) from \( \text{Path}_2(w) \) to a point. Concatenate \( H_0, H_1, H_2 \) to construct the desired contraction.

**Remark 6.5.** The proof of Lemma 6.4 above obtains a rather explicit contraction. A slightly shorter proof is possible using the metrizable topological manifold structure provided in the proof of Theorem 2 below: use Theorem 15 of [28] and the long exact sequence of homotopy groups for the fibration \( \text{path} : L_n[w] \rightarrow \text{Path}(w) \), via Lemmas 6.1, 6.2 and 6.3. The longer proof above is more self-contained.

Up to this point in this section, all arguments relied solely on the fact that \( \mathcal{L}_n^{[H^r]}(z_0; z_1) \) is a metrizable manifold including piecewise \( C^1 \) curves. Herein, by piecewise \( C^1 \) we mean that there exists a finite family of compact intervals \([0, t_1], [t_1, t_2], \ldots, [t_k, 1]\) covering \([0, 1]\) such that \( \Gamma \) is of class \( C^1 \) in each interval \([t_i, t_{i+1}]\). In certain situations though, we prefer to work in a space of curves whose derivatives of certain orders are well-defined. The Hilbert manifolds \( \mathcal{L}_n^{[H^r]}(z_0; z_1) \) and \( \mathcal{L}_n^{[H^r]} \), for \( r > 2 \), were introduced in Subsection 2.2 to fulfill this role. Therein, we prove that the inclusions \( \mathcal{L}_n^{[H^r]}(z_0; z_1) \hookrightarrow \mathcal{L}_n^{[H^1]}(z_0; z_1) \) are homotopy equivalences homotopic to diffeomorphisms. We now verify that there exist similar stratifications for \( r > 2 \):

\[
\mathcal{L}_n^{[H^r]} = \bigcup_{w \in \mathcal{W}_n} \mathcal{L}_n^{[H^r]}[w], \quad \mathcal{L}_n^{[H^r]}[w] = \mathcal{L}_n^{[H^1]}[w] \cap \mathcal{L}_n^{[H^r]}.
\]

It turns out that the adjacency relations between strata are different in the two cases (compare Theorem 3 and Equation (6)).

**Lemma 6.6.** Consider \( w \in \mathcal{W}_n, r \in \mathbb{N}, r > 2 \). The set \( \mathcal{L}_n^{[H^r]}[w] \subset \mathcal{L}_n^{[H^r]}(\hat{\eta}_w\hat{\eta}) \) is nonempty and contractible.

Fact 2.2 is used in Section 2.2 to prove that the inclusions \( \mathcal{L}_n^{[H^r]}(z_0; z_1) \hookrightarrow \mathcal{L}_n^{[H^1]}(z_0; z_1) \) are weak homotopy equivalences. It is tempting to want to use the same fact to prove that \( \mathcal{L}_n^{[H^r]}[w] \subset \mathcal{L}_n^{[H^1]}[w] \) is also a weak homotopy equivalence. This proof is not valid at this point, however, since we do not know that these sets are manifolds. We shall prove below that they are indeed topological manifolds, but this is not sufficient to apply Fact 2.2.
The idea is to imitate the proof of Lemma 6.4. Notice that some of the building blocks, i.e., Lemmas 6.1 and 6.3, apply just the same to the case \( r > 2 \). We need to state and prove results leading to an alternate version of Lemma 6.2. In order to do so, we introduce the concepts of \( r \)-jet and enhanced path. Given \( r \in \mathbb{N}^* \), \( J \subset \mathbb{R} \) an interval, \( t \in J \) and \( \Gamma : J \to \text{Spin}_{n+1} \) locally convex of class \( C^r \), we define the \( r \)-jet of \( \Gamma \) at \( t \), \( \text{jet}^r(\Gamma; t) \in \text{Spin}_{n+1} \times \mathbb{R}^{nr} \), by

\[
\text{jet}^r(\Gamma; t) = (\Gamma(t), \kappa_1(t), \ldots, \kappa_1^{(r-1)}(t), \ldots, \kappa_n(t), \ldots, \kappa_n^{(r-1)}(t)),
\]

where the real functions \( \kappa_i(t) = \kappa_i(\Gamma; t) \) are described in Equation (11) in the Introduction. Notice that, given \( \Gamma_0 : [t_0, t_1] \to \text{Spin}_{n+1} \) and \( \Gamma_1 : [t_1, t_2] \to \text{Spin}_{n+1} \), \( \Gamma_0 \in \mathcal{L}_n^{[H^r]}(z_0; z_1) \), \( \Gamma_1 \in \mathcal{L}_n^{[H^r]}(z_1; z_2) \) (up to reparameterizations), the concatenation \( \Gamma_0 \star \Gamma_1 : [t_0, t_2] \to \text{Spin}_{n+1} \) belongs to \( \mathcal{L}_n^{[H^r]}(z_0; z_2) \) (up to a reparameterization) if and only if \( \text{jet}^{r-1}(\Gamma_0; t_1) = \text{jet}^{r-1}(\Gamma_1; t_1) \). For sing(\( \Gamma \)) = \{\( t_1 < \cdots < t_\ell \)\} \( \subset (0,1) \), define the enhanced path of \( \Gamma \) as

\[
\text{path}^{[H^r]}(\Gamma) = (\text{jet}^{r-1}(t_1), \ldots, \text{jet}^{r-1}(t_\ell)).
\]

Notice that \( \text{path}(\Gamma) \) is obtained from \( \text{path}^{[H^r]}(\Gamma) \) by coordinate-wise application \( \Pi : \text{Spin}_{n+1} \times \mathbb{R}^{n(r-1)} \to \text{Spin}_{n+1} \).

Given \( (z_1, \ldots, z_\ell) \in \text{Path}(w) \), we consider in the proof of Lemma 6.2 the contractible set of curves \( \mathcal{L}^c_n(q_j^{-1}z_j; q_j^{-1}z_{j+1}) \approx \mathcal{L}_n^{[H^r]}(z_j; z_{j+1}) \) (the bijection is obtained by multiplication by \( q_j \in \text{Quat}_{n+1} \)). Given jets \( j_j \in \text{Spin}_{n+1} \times \mathbb{R}^{n(r-1)} \) with \( \Pi(j_j) = z_j \), we are now interested in the subsets

\[
\mathcal{L}_n^{[H^r]}(j_j; j_{j+1}) \subset \mathcal{L}_n^{[H^r]}(z_j; j_{j+1}), \mathcal{L}_n^{[H^r]}(j_j; z_{j+1}) \subset \mathcal{L}_n^{[H^r]}(z_j; z_{j+1}),
\]

where, for instance, \( \Gamma : [t_j, t_{j+1}] \to \text{Spin}_{n+1} \), \( \Gamma \in \mathcal{L}_n^{[H^r]}(z_j; z_{j+1}) \) (up to a reparameterization) belongs to \( \mathcal{L}_n^{[H^r]}(z_j; j_{j+1}) \) if and only if \( \text{jet}^{r-1}(\Gamma; t_{j+1}) = j_{j+1} \). In each case we consider the corresponding subset of convex curves: thus, for instance, \( \mathcal{L}_n^{[H^r]}(z_j; j_{j+1}) \) is the subset of convex curves in \( \mathcal{L}_n^{[H^r]}(z_j; j_{j+1}) \).

**Lemma 6.7.** The subsets below are contractible connected components:

\[
\mathcal{L}_n^{[H^r]}(j_j; j_{j+1}) \subset \mathcal{L}_n^{[H^r]}(j_j; j_{j+1}), \quad \mathcal{L}_n^{[H^r]}(z_j; j_{j+1}) \subset \mathcal{L}_n^{[H^r]}(z_j; j_{j+1}),
\]

\[
\mathcal{L}_n^{[H^r]}(j_j; z_{j+1}) \subset \mathcal{L}_n^{[H^r]}(j_j; z_{j+1}), \quad \mathcal{L}_n^{[H^r]}(z_j; z_{j+1}) \subset \mathcal{L}_n^{[H^r]}(z_j; z_{j+1}).
\]

**Proof.** In this proof we use the following notation: \( H^{r,n} = (H^{r-1}(t_j, t_{j+1}; \mathbb{R}))^n \). We also consider \( \mathcal{L}_n^{[H^r]}(z_j; z_{j+1}) \) as a submanifold of \( H^{r,n} \), given by the functions \( \xi_1, \ldots, \xi_n \) of Equation (11). For \( B_1 = H^{r,n} \) and \( B_2 = H^{1,n} \), Fact 2.2 implies that the inclusion

\[
\mathcal{L}_n^{[H^r]}(z_j; z_{j+1}) \subset \mathcal{L}_n^{[H^r]}(z_j; z_{j+1})
\]

is a homeomorphism.
is a homotopy equivalence between Hilbert manifolds. Since $L_{n,\text{convex}}^H(z_j; z_{j+1})$ is a contractible connected component, so is $L_{n,\text{convex}}^H(z_j; z_{j+1})$, as we already pointed out in the proof of Lemma 3.3.

Now, we deal with the inclusion $L_{n,\text{convex}}^H(j_j; z_{j+1}) \subset L_{n,\text{convex}}^H(j_j; z_{j+1})$. We want to use Fact 2.2. We have however not a linear subspace, but an affine subspace. This requires a minor adaptation. Take

$$B_1 = \left\{ (\xi_1, \ldots, \xi_n) \in H^{r,n} \mid \forall i, j \in [n] \left( \xi_i(t_j) = \xi_i(t_j) = \cdots = \xi_i^{(r-2)}(t_j) = 0 \right) \right\}$$

and $B_2 = H^{1,n}$. Take $\tilde{\kappa} \in C^\infty([t_j, t_{j+1}], \mathbb{R}^n)$ with $j_i = (z_j, \tilde{\kappa}_1(t_j), \ldots, \tilde{\kappa}_n^{(r-2)}(t_j))$ and the corresponding $\tilde{\xi} = (\xi_1, \ldots, \xi_n)$. Consider the translated submanifold $M_2 = L_{n}^H(z_j; z_{j+1}) - \tilde{\xi} \subset B_2$. Apply Fact 2.2 in order to obtain the desired conclusion. The other cases are similar. \hfill \qed

**Proof of Lemma 6.5.** Let $\text{Path}^H(w) = \text{Path}(w) \times \mathbb{R}^{\ell(w)}$ be the contractible set of accessible enhanced paths, defined in the obvious manner (here, $\ell = \ell(w)$). Lemma 6.7 shows that the set of $H^r$ locally convex curves with a prescribed enhanced path is contractible. Thus, the map from $L_{n,\text{convex}}^{H^r}[w]$ to $\text{Path}^{H^r}(w)$ taking a curve $\Gamma$ to its enhanced path is a fibration with a fiber homeomorphic to the separable Hilbert space and base space homeomorphic to an Euclidean space, proving our claim. \hfill \qed

We now present a smooth example of tubular neighborhood, used in the proof of Theorem 2. We use the notation of Remark 1.1.

**Remark 6.8.** For all $z_0 \in \overline{B}_{n+1}$, the open set $U_{z_0}$ is a smooth tubular neighborhood in $\text{Spin}_{n+1}$ of the signed Bruhat cell $\text{Bru}_{z_0}$ (with $B = \mathbb{R}^k$). We denote its smooth projection map by $\Pi_{z_0} : U_{z_0} \to \text{Bru}_{z_0} \subseteq U_{z_0}$. Write $z_0 = q\sigma$ for $q \in \text{Quat}_{n+1}$ and $\sigma \in S_{n+1}$. If $\sigma \neq \eta$, we have that $\text{Bru}_{z_0}$ is a signed Bruhat cell of $\text{Spin}_{n+1}$ with positive codimension $k = \text{inv}(\eta) - \text{inv}(\sigma)$. In this case, there is a smooth submersion $f_{z_0} = (f_{z_0,1}, \ldots, f_{z_0,k}) : U_{z_0} \to \mathbb{R}^k$ satisfying the following conditions:

1. $\text{Bru}_{z_0} = f_{z_0}^{-1}(0) = \{ z \in U_{z_0} \mid f_{z_0}(z) = 0 \}$;
2. $(\Pi_{z_0}, f_{z_0}) : U_{z_0} \to \text{Bru}_{z_0} \times \mathbb{R}^k$ is a smooth diffeomorphism;
3. Given a locally convex curve $\Gamma : (-\epsilon, \epsilon) \to U_{z_0}$, if $\Gamma$ is differentiable in $t$, then $(f_{z_0,k} \circ \Gamma)'(t) > 0$.

The pair of maps $(\Pi_{z_0}, f_{z_0})$ is explicitly constructed in Theorem 2 of [15] using a triangular system of coordinates (see also Remark 6.7 therein). \hfill \diamond
For the proof of Theorem 2 below, we also need the following technical result. Notice that the proof uses the concept of positivity (see Subsection 2.1).

**Lemma 6.9.** Consider \( \sigma \in S_{n+1}, \sigma \neq \eta \) and \( z_0 = q\hat{\sigma} \in \mathbb{B}^+_{n+1}, q \in \text{Quat}_{n+1} \). If \( \Gamma : [-\epsilon, \epsilon] \to \mathcal{U}_{z_0} \) is locally convex and there exists \( t_1 \in [-\epsilon, \epsilon] \) with \( \Gamma(t_1) \in \text{Bru}_{z_0} \) then \( \text{sing}(\Gamma) = \{ t_1 \} \).

**Proof.** Consider a projective transformation \( \phi \) for which \( \phi(z_0) \in q\text{Quat}^{\text{Pos}}_\sigma \subset \mathcal{U}_q \). By continuity, there exists an open set \( A \subset \mathcal{U}_{z_0}, z_0 \in A \), such that \( \phi[A] \subset \mathcal{U}_q \). We may furthermore assume that \( z \in \phi[A] \cap \text{Bru}_\sigma \) implies \( z \in q\text{Quat}^{\text{Pos}}_\sigma \).

Consider \( \Gamma \) as in the statement. Apply triangular coordinates to \( \mathcal{U}_{z_0} \) to define the convex curve \( \Gamma_L : [-\epsilon, \epsilon] \to \text{Lo}^1_{n+1}, \Gamma(t) = z_0Q[\Gamma_L(t)] \). For \( \lambda \in [1, +\infty) \), consider the projective transform

\[
\Gamma^\lambda_L(t) = \text{diag}(1, \lambda^{-1}, \ldots, \lambda^{-n})\Gamma_L(t) \text{diag}(1, \lambda, \ldots, \lambda^n)
\]

and the locally convex curve \( \Gamma^\lambda(t) = z_0Q[\Gamma^\lambda_L(t)] \). Notice that \( \Gamma^\lambda : [-\epsilon, \epsilon] \to \mathcal{U}_{z_0} \) satisfies \( \text{sing}(\Gamma^\lambda) = \text{sing}(\Gamma) \) and \( \text{iti}(\Gamma^\lambda) = \text{iti}(\Gamma) \). Given \( t_0 \in [-\epsilon, \epsilon] \), we have \( \lim_{\lambda \to +\infty} \Gamma^\lambda(t_0) = z_0 \); by compactness, there exists \( \lambda_0 \) such that \( \Gamma^{\lambda_0}([-\epsilon, \epsilon]) \subset A \). The curve \( \tilde{\Gamma} = \phi \circ \Gamma^{\lambda_0} \) therefore admits triangular coordinates \( \tilde{\Gamma}_L : [-\epsilon, \epsilon] \to \text{Lo}^1_{n+1}, \tilde{\Gamma}(t) = q\text{Q}[\tilde{\Gamma}_L(t)] \). We have \( \tilde{\Gamma}(t_1) \in \phi[A] \cap \text{Bru}_\sigma \) and therefore \( \Gamma_L(t_1) \in \text{Pos}_\sigma \). From Lemma 5.7 of [13], \( t > t_1 \) implies \( \Gamma_L(t) \in \text{Pos}_\sigma \), i.e., \( \text{sing}(\tilde{\Gamma}) \cap (t_1, \epsilon) = \emptyset \). Thus \( t_1 \) is the last element of \( \text{sing}(\Gamma) \). A similar argument using the sets \( \text{Neg}_\sigma \) instead of \( \text{Pos}_\sigma \) proves that \( t_1 \) is also the first element of \( \text{sing}(\Gamma) \).

**Proof of Theorem 2** The nonemptiness and contractibility of \( \mathcal{L}_{n}^{[H^r]}[w] \) is already established by previous lemmas in this section. It remains to be shown that, for \( r \neq 2 \), \( \mathcal{L}_{n}^{[H^r]}[w] \) is a globally collared topological submanifold of \( \mathcal{L}_{n}^{[H^r]}(q_w) \), \( q_w = \eta\hat{\eta}[\eta] \in \text{Quat}_{n+1} \), with codimension \( \text{dim}(w) \); also that, if \( r > 2 \), then \( \mathcal{L}_{n}^{[H^r]}[w] \) is in fact an embedded submanifold of differentiability class \( C^{r-1} \).

For \( w = \sigma_1 \cdots \sigma_\ell = (\sigma_1, \ldots, \sigma_\ell) \) and \( 2j \in \mathbb{Z} \cap [0, 2\ell + 2] \), set \( B(w, j) \in \mathbb{B}^+_{n+1} \) as in Equation (12) above; in particular, \( B(w, \ell + 1) = q_w = \eta\hat{\eta}[\eta] \in \text{Quat}_{n+1} \). We first define an open subset \( \mathcal{A}^j_w \subset \mathcal{L}_{n}^{[H^r]}(q_w) \times (0, 1) \). A pair \((\Gamma, \bar{\epsilon})\) belongs to \( \mathcal{A}^j_w \) if there exist \( 0 = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_\ell < \tilde{t}_{\ell+1} = 1 \) such that:

(i) For each \( i, \ell_{i+1} - \ell_i > 2\bar{\epsilon} \).

(ii) Each arc \( \Gamma|_{[\ell_i, 2\ell_i, \ell_{i+2}]} \) is convex, with image in \( \mathcal{U}_{B(w, i)} \). In particular, for \( \bar{z}_i = \Gamma(\tilde{t}_i) \), we have \( \bar{z}_i \in \mathcal{U}_{B(w, i)} \).

(iii) Each arc \( \Gamma|_{[\ell_{i+2}, \ell_{i+2+\bar{\epsilon}}]} \) is convex, with image in \( \mathcal{U}_{B(w, i+1/2)} = \text{Bru}_{B(w, i+1/2)} \).

(iv) For \( f_{i,k_i} : \mathcal{U}_{B(w, i)} \to \mathbb{R} \) and \( k_i \) as in Remark 6.8 we have \( f_{i,k_i}(\bar{z}_i) = 0 \).
(v) Let $\Pi_{B(w,i)} : U_{B(w,i)} \to \text{Br}u_{B(w,i)} \subset U_{B(w,i)}$ be the smooth projection of Remark 6.8. Set $\tilde{z}_i = \Pi_{B(w,i)}(\check{z}_i)$. There exist convex arcs in $U_{B(w,i)}$ from $\Gamma(\check{t}_i - \frac{3}{2})$ to $\check{z}_i$ and from $\check{z}_i$ to $\Gamma(\check{t}_i + \frac{3}{2})$.

For $\Gamma \in L_n^{[H^r]}(q_w)$, set $J_{\Gamma} = \{ \tilde{\epsilon} \in (0,1) \mid (\Gamma, \tilde{\epsilon}) \in A_w^i \}$; clearly, $J_{\Gamma}$ is either an open interval or empty; for $\Gamma \in L_n^{[H^r]}[w]$, $J_{\Gamma}$ is an interval of the form $J_{\Gamma} = (0, \epsilon)$ for some $\epsilon > 0$. Set

$$A_w = \{ \Gamma \in L_n^{[H^r]}(q_w) \mid J_{\Gamma} \neq \emptyset \} \subseteq L_n^{[H^r]}(q_w),$$

an open subset. For $\Gamma \in A_w$, the times $\check{t}_i$ are well-defined and, from Condition 3 of Remark 6.8, the functions $\Gamma \mapsto \check{t}_i$ are of class $C^{r-1}$. For $\Gamma \in A_w$, select $\epsilon = \epsilon_\Gamma \in J_{\Gamma}$; from the $C^{r-1}$ regularity of $\check{t}_i$ and several uses of Lemma 5.5 of [15], the function $\Gamma \mapsto \epsilon$ can be taken to be of class $C^{r-1}$. We have $\check{z}_i = \Gamma(\check{t}_i) \in U_{B(w,i)}$; we define $\check{z}_i^- = \Gamma(\check{t}_i - \frac{3}{2})$, $\check{z}_i^+ = \Gamma(\check{t}_i + \frac{3}{2})$, and $\check{z}_i = \Pi_{B(w,i)}(\check{z}_i)$. Also, the map $\Gamma \mapsto (\check{z}_i, \check{z}_i^-, \check{z}_i^+, \check{z}_i)$ is of class $C^{r-1}$. For $\Gamma \in A_w$, $\epsilon = \epsilon_\Gamma$, $\check{t}_i$, $\check{z}_i$ and $\check{z}_i$ as above we therefore have the following properties:

(a) For each $i$, $\check{t}_{i+1} - \check{t}_i \geq 8\epsilon$.

(b) Each arc $\Gamma|_{[\check{t}_i - \epsilon, \check{t}_i + \epsilon]}$ is convex, with image in $U_{B(w,i)}$; also, $\check{z}_i \in U_{B(w,i)}$.

(c) Each arc $\Gamma|_{[\check{t}_i + \epsilon, \check{t}_{i+1} - \epsilon]}$ is convex, with image in $U_{B(w,i+\frac{1}{2})}$.

(d) For $f_{i,k} : U_{B(w,i)} \to \mathbb{R}$ as in Remark 6.8 we have $f_{i,k}(\check{z}_i) = 0$.

(e) There exist convex arcs in $U_{B(w,i)}$ from $\check{z}_i^-$ to $\check{z}_i$ and from $\check{z}_i$ to $\check{z}_i^+$.

Set $d = \dim(w) = d_1 + \cdots + d_k$, where $d_i = k_i - 1$. Define $F : A_w \to \mathbb{R}^d$, $F(\Gamma) = (F_1(\Gamma), \ldots, F_r(\Gamma))$, where $F_i : A_w \to \mathbb{R}^{d_i}$, $F_i(\Gamma) = (f_{i,1}(\check{z}_i), \ldots, f_{i,d_i}(\check{z}_i))$. The coordinate functions $f_{i,j}$ above are the first $d_i$ coordinate functions of the smooth submersion $f_{B(w,i)} = (f_{i,1}, \ldots, f_{i,k_i}) : U_{B(w,i)} \to \mathbb{R}^{k_i}$ of Remark 6.8.

We claim that, for $\Gamma \in A_w$, $F(\Gamma) = 0$ if and only if $\Gamma \in L_n^{[H^r]}[w]$.

Indeed, if $\Gamma \in A_w$ and $F(\Gamma) = 0$ we have $\check{z}_i = \Gamma(\check{t}_i) \in \text{Br}u_{pr}$. We already know that $\{\check{t}_1 < \cdots < \check{t}_k\} \subseteq \text{sing}(\Gamma) \subseteq \bigcup_{t_i}(\check{t}_i - \epsilon, \check{t}_i + \epsilon)$. By Lemma 6.9 we have $\text{sing}(\Gamma) = \{\check{t}_1 < \cdots < \check{t}_k\}$ and therefore $\text{it}(\Gamma) = w$.

For $r > 2$, the Regular Value Theorem applied to the submersion $F$ shows that $L_n^{[H^r]}[w]$ is an embedded submanifold of $L_n^{[H^r]}(q_w)$ of class $C^{r-1}$ and codimension $d = \dim(w)$, as claimed. The set $A_w$ is a promising candidate for a tubular neighborhood: all we would have to do is to construct a projection. We prefer, however, to use the normal bundle. Indeed, there is a well-defined tubular neighborhood, i.e., a $C^{r-1}$ embedding of the normal bundle $N L_n^{[H^r]}[w] \hookrightarrow A_w$. 

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that extends the inclusion of $\mathcal{L}_n^{[H^r]}[w]$ (regarded as the zero section of its normal bundle) in $\mathcal{A}_w$. This tubular neighborhood is foliated by normal sections.

We now deal with the case $r = 1$. We construct a projection $\Pi : \mathcal{A}_w \to \mathcal{L}_n^{[H^1]}[w] \subset \mathcal{A}_w$. Given $\Gamma \in \mathcal{A}_w$, the curve $\hat{\Gamma} = \Pi(\Gamma)$, $\hat{\Gamma} : [0, 1] \to \text{Spin}_{n+1}$, will coincide with $\Gamma$ except in the intervals $[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]$ and will satisfy $\hat{\Gamma}(\frac{t_i}{2}) = z_i$. The restrictions $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ and $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ will be convex arcs contained in $\mathcal{U}_{B(w, i)}$ joining $z_i^-$ to $\hat{z}_i$ and $\hat{z}_i$ to $z_i^+$, respectively. These two convex arcs are obtained from the convex arcs $\Gamma|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ and $\Gamma|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ by projective transformations as follows. We have $\hat{z}_i, \hat{z}_i \in z_i^- \text{Bru}_i$; take $U \in \mathcal{U}_{B(w, i)}[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]$ such that $((\hat{z}_i^-)^{-1}\hat{z}_i)^U = (z_i^-)^{-1}z_i$ and set $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]} = \hat{z}_i^-((\hat{z}_i^-)^{-1}\Gamma)^U|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}. The convex arc $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ is obtained likewise.

The function $\Gamma \to \epsilon$ can be constructed so as to satisfy $\epsilon_{\hat{\Gamma}} = \epsilon_{\Gamma}$. If $\Gamma \in \mathcal{L}_n[w]$ we have $\hat{z}_i = z_i$ and therefore $\hat{\Gamma} = \Gamma$.

We now have a continuous map $(\Pi, F) : \mathcal{A}_w \to \mathcal{L}_n^{[H^1]}[w] \times \mathbb{R}^d$; let $\mathcal{B}_w \subseteq \mathcal{L}_n^{[H^1]}[w] \times \mathbb{R}^d$ be its image. We construct the inverse map $\Phi : \mathcal{B}_w \to \mathcal{A}_w$ in the process we see that the set $\mathcal{B}_w$ is an open neighborhood of $\mathcal{L}_n^{[H^1]}[w] \times \{0\}$. Indeed, given $\hat{\Gamma} \in \mathcal{L}_n^{[H^1]}[w]$ construct $\epsilon, \hat{t}_i, \hat{z}_i = \hat{\Gamma}(\hat{t}_i)$ and $z_i^\pm = \hat{\Gamma}(\hat{t}_i \pm \frac{\epsilon}{2})$ as above. Given $x = (x_1, \ldots, x_{d}) \in \mathbb{R}^{d_1 + \cdots + d_i}$, there exist unique $\hat{z}_i \in \mathcal{U}_{B(w, i)}$ with $\Pi_{B(w, i)}(\hat{z}_i) = \hat{z}_i$ and $f_{B(w, i)}(\hat{z}_i) = x_i$. If there exist convex arcs contained in $\mathcal{U}_{B(w, i)}$ from $\hat{z}_i^-$ to $z_i$ and from $\hat{z}_i$ to $\hat{z}_i^+$ then $(\hat{\Gamma}, x_i) \in \mathcal{B}_w$ and the curve $\hat{\Gamma} = \Phi(\hat{\Gamma}, x_i)$ is constructed as before. More precisely, $\hat{\Gamma}$ coincides with $\hat{\Gamma}$ except in the intervals $[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]$. The convex arcs $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ and $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ are obtained from the arcs $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ and $\hat{\Gamma}|_{[\frac{t_i}{2}, \frac{t_i}{2} + \frac{\epsilon}{2}]}$ by projective transformations.

In the proof of Theorem 2 of [15], it was shown that there exists a natural diffeomorphism between $\mathcal{U}_{B(w, i)}$ and the cartesian product $U_{\eta, \sigma} \times \text{Lo}_{\sigma^{-1}}$ of certain affine spaces of triangular matrices defined in Equation (4) of [15]. Endow the affine spaces $U_{\eta, \sigma}$ and $\text{Lo}_{\sigma^{-1}}$ with the natural Euclidean metrics coming from the sets of free coordinates (i.e., entries not obligatorily equal to 0 or 1). Use the above diffeomorphism to endow $\mathcal{U}_{B(w, i)}$ with a flat Euclidean metric. Similarly, endow the cartesian product $U_{B(w, i)}^3 = U_{B(w, i)} \times U_{B(w, i)} \times U_{B(w, i)}$ with a flat Euclidean metric. Let $\mathcal{W}_i \subseteq U_{B(w, i)}^3$ be the open set of triples $(z_i^-, z_i, z_i^+)$ such that there exist convex arcs contained in $U_{B(w, i)}$ from $z_i^-$ to $z_i$ and from $z_i$ to $z_i^+$. Let $\delta_i : \mathcal{W}_i \to (0, +\infty)$ be the continuous function taking a triple $(z_i^-, z_i, z_i^+) \in \mathcal{W}_i$ to one half of the distance (in the flat Euclidean metric constructed above) from the complement $U_{B(w, i)} \setminus \mathcal{W}_i$, i.e., $\delta_i((z_i^-, z_i, z_i^+)) = \frac{1}{2}d((z_i^-, z_i, z_i^+), U_{B(w, i)}^3 \setminus \mathcal{W}_i)$. Given $\hat{\Gamma} \in \mathcal{L}_n^{[H^1]}[w]$, define $\delta(\hat{\Gamma}) = \min\delta_i((\hat{z}_i^-, \hat{z}_i, \hat{z}_i^+))$ where, as above, $\hat{z}_i^\pm = \Gamma(\hat{t}_i \pm \frac{\epsilon}{2})$. Notice that $\delta : \mathcal{L}_n^{[H^1]}[w] \to (0, +\infty)$ is continuous and that if $|x| \leq \delta(\Gamma)$ then $(\Gamma, x) \in \mathcal{B}_w$ (by construction).
Let \( \mathbb{B}^d \subset \mathbb{D}^d \subset \mathbb{R}^d \) be the open and closed balls of radius 1, respectively. Define \( \hat{\Phi} : \mathcal{L}^{[H^1]}[w] \times \mathbb{D}^d \to \mathcal{A}_w \) by \( \hat{\Phi}(\Gamma, \mathbf{x}) = \Phi(\Gamma, \delta(\Gamma)\mathbf{x}) \). Let \( \hat{\mathcal{A}}_w = \hat{\Phi} \left[ \mathcal{L}^{[H^1]}[w] \times \mathbb{B}^d \right] \subset \mathcal{A}_w \) and define \( \hat{\mathcal{F}} : \hat{\mathcal{A}}_w \to \mathbb{B}^d \) so that \( (\Pi, \hat{\mathcal{F}}) = \hat{\Pi}^{-1} : \hat{\mathcal{A}}_w \to \mathcal{L}^{[H^1]}[w] \times \mathbb{B}^d \). This completes the construction of the tubular neighborhood of \( \mathcal{L}^{[H^1]}[w] \).

\[ \ast \]

7 Transversal sections

The proof of Theorem 2 implicitly gives us (topologically) transversal sections. We now construct explicit transversal sections to \( \mathcal{L}^{[H^1]}[w] \) in \( \mathcal{L}^{[H^1]} \). We omit the superscript \([H^1]\) throughout this section. The construction roughly corresponds to going back to Theorem 2, then to Theorem 2 and Remark 6.7 of [15], then to Lemma 5.5 and Remark 5.6 of [15], and following the steps. A key difference is that strictly following the proof of Theorem 2 given in Section 6 yields curves which fail to be smooth precisely at the times \( t_i \) (defined in the mentioned proof); the curves produced by our construction in this section are smooth (indeed algebraic) in a neighborhood of \( t_i \) (though they are not globally smooth). A standard procedure can be applied to smoothen out the transversal section \( \phi : \mathbb{D}^d \to \mathcal{L}^{[H^1]} \) to \( \mathcal{L}^{[H^1]}[w] \) just constructed. The result is, for each \( r \geq 3 \), a smooth map \( \tilde{\phi} : \mathbb{D}^d \to \mathcal{L}^{[H^1]} \) such that \( \tilde{\Phi} : \mathbb{D}^d \times [0, 1] \to \text{Spin}_{n+1}, \tilde{\Phi}(\mathbf{x}, t) = \tilde{\phi}(\mathbf{x})(t) \), coincides with \( \Phi(\mathbf{x}, t) = \phi(\mathbf{x})(t) \) (and hence is algebraic) in \( \mathbb{D}^d \times (\cup_i(\tilde{t}_i - \delta, \tilde{t}_i + \delta)) \) for some \( \delta > 0 \). Of course, \( \tilde{\phi} \) is transversal to \( \mathcal{L}^{[H^1]}[w] \). We first present the construction as an algorithm, then provide examples.

Consider \( \sigma \in S_{n+1}, \sigma \neq e, \rho = \eta \sigma \) and \( d = \dim(\sigma) = \text{inv}(\sigma) - 1 \). Consider \( z_0 = q\eta \tilde{\sigma} \in \tilde{B}_{n+1}^+, q \in \text{Quat}_{n+1}, \) so that \( \text{chop}(z_0) = q\eta \) and \( \text{adv}(z_0) = q\eta \tilde{\sigma} \). Let \( Q_0 = \Pi(z_0) \in \mathbb{B}_{n+1}^+ \subset \text{SO}_{n+1} \). We first construct an explicit transversal section \( \psi : \mathbb{R}^{d+1} \to \text{SO}_{n+1} \) to the Bruhat cell \( \text{Bru}_{Q_0} = \Pi[Bru_{z_0}] \subset \text{SO}_{n+1} \) passing through \( Q_0 = \psi(0) \) (compare with Remarks 5.6 and 6.7 of [15]). First we define a matrix \( \check{M} \in (\mathbb{R}[x_1, \ldots, x_{d+1}])^{(n+1) \times (n+1)} \) with polynomial entries in the variables \( x_1, 1 \leq l \leq d + 1 \). For \( i \in \llbracket n+1 \rrbracket \), set \( (\check{M})_{i,i} = (Q_0)_{i,i} = \pm 1 \). There are \( d + 1 \) zero entries in \( Q_0 \) which are simultaneously below a nonzero entry and to the left of a nonzero entry: these are the pairs \( (i, j) \) for which \( j < i \) and \( j^{d+1} < i \). Assign to each such position \( (i, j) \) an integer \( l \) from 1 to \( d + 1 \) in the same order you would read or write them on a page (top to bottom and left to right). For each such position \( (i, j) \), set \( (\check{M})_{i,j} = (Q_0)_{i,j} x_1 \). The other entries of \( \check{M} \) are set to 0: this defines the desired matrix \( \check{M} \in (\mathbb{R}[x_1, \ldots, x_{d+1}])^{(n+1) \times (n+1)} \) or, equivalently, a smooth map \( \psi_L : \mathbb{R}^{d+1} \to \text{GL}_{n+1}^+ \) where \( \psi_L(\mathbf{x}) \) is obtained by evaluating \( \check{M} \) at \( \mathbf{x} \in \mathbb{R}^{d+1} \). As an example, the matrices below correspond respectively to \( n = 2, \)
For each nilpotent matrix whose only nonzero entries are \( \psi \) in the variable  

\[
\sigma_0 = [321] = aba \ (d = 2), \text{ and } n = 3, \ \sigma_1 = [3142] = acb \ (d = 2):
\]

\[
\tilde{M}_0 = \begin{pmatrix}
1 & 0 & 0 \\
x_1 & 1 & 0 \\
x_2 & x_3 & 1
\end{pmatrix}; \quad \tilde{M}_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & -x_1 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
x_2 & x_3 & 1 & 0
\end{pmatrix}
\]

(we take \( q = 1 \) in both examples). Notice that the map \( \psi_L \) is a smooth diffeomorphism from \( \mathbb{R}^{d+1} \) to \( Q_0 \text{Lo}_{\sigma^{-1}} \subset GL^+_{n+1} \) (see Equation (4) of [15] for the definition of \( \text{Lo}_{\sigma^{-1}} \)). Recall that we denote by \( Q : GL^+_{n+1} \to SO_{n+1} \) the map that takes \( M \) to the orthogonal part \( Q(M) \) in the \( QR \) decomposition \( M = Q(M)R, \ R \in Up^+_{n+1} \). The smooth algebraic map \( \psi_A = Q \circ \psi_L : \mathbb{R}^{d+1} \to SO_{n+1} \) is the desired transversal section to the Bruhat cell \( \text{Bru}_{Q} \). In order to define \( \psi : \mathbb{R}^{d+1} \to \text{Spin}_{n+1} \), \( \psi_A = \Pi \circ \psi \), lift the map \( \psi_A \) starting at \( \psi(0) = z_0 \).

Consider \( \mathbb{R}^d \subset \mathbb{R}^{d+1} \) defined by \( x_{d+1} = 0 \). Let \( \mathbf{n} = \sum_i a_i \mathbf{i}_i \) be the lower triangular nilpotent matrix whose only nonzero entries are \( \mathbf{n}_{i+1,j} = 1 \) (see Equation (9)). For each \( \mathbf{x} \in \mathbb{R}^d \), define a curve \( \phi_L(x; \cdot) : \mathbb{R} \to Q_0 \text{Lo}_{n+1} \subset GL^+_{n+1} \) by the IVP

\[
\frac{\partial}{\partial t} \phi_L(x; t) = \phi_L(x; t) \mathbf{n}, \quad \phi_L(x; 0) = \psi_L(x),
\]

so that \( \phi_L(x; t) = \psi_L(x) \exp(t \mathbf{n}) \). Since entries of \( \phi_L(x; t) \) are polynomials in \( \mathbf{x} \) and \( t \), we may equivalently consider the matrix \( M \in (\mathbb{R}[x; t])^{(n+1) \times (n+1)} \), \( M(x, t) = \phi_L(x; t) \), whose entries are polynomials in \( \mathbf{x} \) and \( t \), of degree at most \( n \) in the variable \( t \) and satisfying

\[
(M)_{i,j+1} = \frac{\partial}{\partial t} (M)_{i,j}.
\]

As an example, the two matrices below again correspond to \( n = 2, \sigma_0 = [321] = aba \) and \( n = 3, \sigma_1 = [3142] = acb \) (and \( q = 1 \) in both cases):

\[
M_0 = \begin{pmatrix}
1 & 0 & 0 \\
t + x_1 & 1 & 0 \\
t^2 + x_2 & x_3 & 1
\end{pmatrix}; \quad M_1 = \begin{pmatrix}
-t & -1 & 0 & 0 \\
-t^3 - x_1 t & -t^2 \frac{x_1}{2} & x_1 & -t & 1 \\
-1 & 0 & 0 & 0 \\
\frac{x^2}{2} + x_2 & t & 1 & 0
\end{pmatrix}.
\]

Notice that, given \( \mathbf{x} \in \mathbb{R}^d \), the map \( Q_0^{-1} \phi_L(\mathbf{x}; \cdot) : \mathbb{R} \to \text{Lo}_{n+1}^1 \) is a smooth (indeed algebraic) convex curve. Let \( \Gamma_\mathbf{x} : \mathbb{R} \to \text{Spin}_{n+1} \) be the locally convex curve defined by \( \Gamma_\mathbf{x}(t) = Q(\phi_L(\mathbf{x}, t)) \), \( \Gamma_\mathbf{x}(0) = \psi(\mathbf{x}) \). Clearly, \( \Gamma_0(0) = z_0, \Gamma_0(t) \in \text{Bru}_{\text{chop}(z_0)} \) for \( t < 0 \) and \( \Gamma_0(t) \in \text{Bru}_{\text{adv}(z_0)} \) for \( t > 0 \).

We now construct the desired transversal surface \( \phi : \mathbb{D}^d \to \mathcal{L}_n \). Choose \( z_0 \) above such that \( \text{chop}(z_0) = \hat{\mathbf{n}} \) and \( \text{adv}(z_0) = \mathbf{n} \hat{\mathbf{n}} \); let \( q_1 = \mathbf{n} \hat{\mathbf{n}} \hat{\mathbf{n}} \in \text{Quat}_{n+1} \) for sufficiently small \( r \in (0, \frac{\pi}{4}) \), there exists a convex arc contained in \( \text{Bru}_{\text{chop}(z_0)} \).
proof of Theorem 2). Define $\phi$ is a positive multiple of the identity. is constructed in the proof of Theorem 2 in Section 6. The map $\hat{\phi}$ has a unique intersection at $x = 0$. By continuity, there exists a convex arc contained in $\text{Bru}_{\text{adv}(\sigma)}$ going from $\Gamma_0(r)$ to $q_1 \exp(-r h)$. Fix such a small $r = 0$. By continuity, there exists a small $\bar{s} > 0$ such that, if $|x| \leq \bar{s}$ then there exists a convex arc contained in $\text{Bru}_{\text{adv}(\sigma)}$ going from $\exp(rh)$ to $\Gamma_x(-r)$. Similarly, for sufficiently small $s > 0$, if $|x| \leq s$ then there exists a convex arc contained in $\text{Bru}_{\text{adv}(\sigma)}$ going from $\Gamma_x(r)$ to $q_1 \exp(-r h)$. Fix such a small $s > 0$. We thus define, for each $x \in \mathbb{D}^d$, convex arcs $\tilde{\phi}(x)|_{[\bar{s}, \bar{s}]}$ going from $\exp(rh)$ to $\Gamma_x(-r)$ and $\tilde{\phi}(x)|_{[\bar{s}, \bar{s}]}$ going from $\Gamma_{\text{adv}}(r)$ to $q_1 \exp(-r h)$. For $t \in [0, \frac{1}{8}]$, set $\tilde{\phi}(x)(t) = \exp(8 \gamma in(t \eta))$; for $t \in [\frac{7}{8}, 1]$, set $\tilde{\phi}(x)(t) = \Gamma_x(8 \gamma (t - \frac{7}{8})). Consider now $s \in (0, \bar{s})$ sufficiently small so that, for all $x \in \mathbb{D}^d$ with $|x| \leq \frac{3}{8}$, we have $\tilde{\phi}(x) \in \hat{\mathcal{A}}$ (where $\hat{\mathcal{A}}$ is the open neighborhood of $\mathcal{L}_n[\sigma]$ constructed in the proof of Theorem 2). Define $\phi : \mathbb{D}^d \to \mathcal{A}_n \subset \mathcal{L}_n$ by $\phi(x) = \phi(\frac{1}{2} x).$

**Lemma 7.1.** Consider $\sigma \in S_{n+1}$, $\sigma \neq e$, $\dim(\sigma) = d$ and construct the map $\phi : \mathbb{D}^d \to \mathcal{L}_n$ as above. This map is topologically transversal to $\mathcal{L}_n[\sigma]$, with a unique intersection at $x = 0 \in \mathbb{D}^d$.

**Proof.** Uniqueness of intersection follows from Theorem 2 of [15]. Topological transversality follows from taking the composition $\hat{F} \circ \phi$, where $\hat{F} : \mathcal{A}_n \to \mathbb{D}^d \subset \mathbb{R}^d$ is constructed in the proof of Theorem 2 in Section 6. The map $\hat{F} \circ \phi : \mathbb{D}^d \to \mathbb{D}^d$ is a positive multiple of the identity. \qed

Notice that the maps $\hat{F} : \mathcal{A}_n \to \mathbb{R}^d$ and $\phi : \mathbb{D}^d \to \mathcal{A}_n \subset \mathcal{L}_n$ consistently provide us with a transversal orientation to $\mathcal{L}_n[\sigma]$.

This completes the construction of a transversal section to $\mathcal{L}_n[\sigma_1]$ at the path $(z_1) \in \text{Path}((\sigma_1))$, $z_1 = q \sigma_1(q), q \in \text{Quat}_{n+1}$. By applying affine transformations in the interval and projective transformations in the group $\text{Spin}_{n+1}$, this defines a map $\phi$ taking each $x \in \mathbb{D}^d$ ($d_1 = \dim(\sigma_1)$) to a convex arc $\Gamma_{\hat{x}} : [t_1 - \ell, t_1 + \ell] \to \text{Spin}_{n+1}$ with $\text{sing}(\Gamma_{\hat{x}}) \neq \emptyset$, $\text{sing}(\Gamma_{\hat{x}}) \subset (t_1 - \frac{1}{2}, t_1 + \frac{1}{2})$ and satisfying $\text{iti}(\Gamma_{\hat{x}}) = (\sigma_1)$ if and only if $x = 0$. We may furthermore assume that $\Gamma_{\hat{x}}(t_1 \pm \ell) = z_1 \exp(\pm \eta h)$ for all $x \in \mathbb{D}^d$ and that $\Gamma_{\hat{x}}(t) = z_1 \exp((t - t_1)\eta)$ for $x = 0 \in \mathbb{D}^d$.

More generally, for any $w = \sigma_1 \cdots \sigma_\ell = (\sigma_1, \ldots, \sigma_\ell) \in \mathcal{W}_n$, for any path $(z_1, \ldots, z_\ell) \in \text{Path}(w)$ and for any set $\{t_1 < \cdots < t_\ell\} \subset (0, 1)$, we show how to construct a smooth map $\phi : \mathbb{D}^d \to \mathcal{L}_n$, $d = \dim(w)$, transversal to $\mathcal{L}_n[w]$ at $\phi(0) \in \mathcal{L}_n[w]$, $\text{path}(\phi(0)) = (z_1, \ldots, z_\ell)$, $\text{sing}(\phi(0)) = \{t_1 < \cdots < t_\ell\}$. Make the convention $t_0 = 0$, $z_0 = 1$, $t_{\ell+1} = 1$ and $z_{\ell+1} = \eta \sigma_1 \cdots \sigma_\ell \eta$. For each $i \in [\ell + 1]$, define $q_i \in \text{Quat}_{n+1}$ such that $q_i \eta = \text{adv}(z_i - 1) = \text{chop}(z_i)$. First, choose $\epsilon > 0$ such that, for all $i \in [\ell + 1]$, $t_{i-1} + \epsilon < t_i - \epsilon$ and $z_i \exp(-\epsilon h) \in \text{Bru}_{q_i \eta}$ and $z_i \exp(-\epsilon h) \in \text{Bru}_{q_i \eta}$. Define $L_i-, L_i+ \in \text{Lo}^i_{n+1}$ by $z_i \exp(\epsilon h) = q_i \eta Q(L_i-)$ and $z_i \exp(-\epsilon h) = q_i \eta Q(L_i+)$. by taking $\epsilon$ sufficiently small we may assume
that $L_{i-} \ll L_{i+}$. Choose fixed convex arcs $\Gamma_{i-\frac{1}{2}} : [t_{i-1} + \epsilon, t_i - \epsilon] \to \text{Bru}_{q_i \theta}$ satisfying $\Gamma_{i-\frac{1}{2}}(t_{i-1} + \epsilon) = z_{i-1} \exp(\epsilon h)$, $\Gamma_{i-\frac{1}{2}}(t_i - \epsilon) = z_i \exp(-\epsilon h)$. In each interval $[t_i - \epsilon, t_i + \epsilon]$, define as above a map $\phi_i$ associating to each $x_i \in \mathbb{D}^d_i$ a convex arc $\Gamma_{i,x_i} : [t_i - \epsilon, t_i + \epsilon] \to \text{Spin}_{n+1}$ with $\Gamma_{i,x_i}(t_i - \epsilon) = z_i \exp(-\epsilon h)$, $\Gamma_{i,x_i}(t_i + \epsilon) = z_i \exp(\epsilon h)$. Set $\Gamma_0 : [0, \epsilon] \to \text{Spin}_{n+1}$, $\Gamma_0(t) = \exp(\epsilon h)$ and $\Gamma_{\ell+1} : [1 - \epsilon, 1] \to \text{Spin}_{n+1}$, $\Gamma_{\ell+1}(t) = z_{\ell+1} \exp((t - 1)\epsilon h)$. Finally, for $x = (x_1, \ldots, x_\ell)$, concatenate these arcs to define $\phi(x) = \Gamma_x \in \mathcal{L}_n$: the map $\phi$ is the desired transversal section.

These explicit transversal sections allow us to explore the vicinity of a given stratum $\mathcal{L}_n[w]$. In the examples below, we follow the above algorithm: we consider a family of convex arcs $\phi(x) = \Gamma_x \in \mathcal{L}_n$, $\phi(0) = \Gamma_0 \in \mathcal{L}_n[w]$, obtained from a matrix with polynomial entries $M = M(x, t) = \phi_L(x; t)$.

Let $m_j(x, t)$ be the southwest $j \times j$ minor of $M$, so that $m_j$ is an explicit element of $\mathbb{R}[x_1, \ldots, x_d, t]$. By construction, $m_j(x, t)$ is a positive multiple of the southwest $j \times j$ minor of $\Pi(\Gamma_x(t))$. From Theorem 4 of [14], given $t_s \in \mathbb{R}$ and $\sigma \in S_{n+1}$, we have $\Gamma_x(t_s) \in \text{Bru}_{w_\sigma}$ if and only if, for each $j \in [n]$, $t = t_s$ is a zero of $m_j(t)$ of multiplicity $\text{mult}_j(\sigma)$. Here, $\text{mult}_j(\sigma) = (1^\sigma - 1) + \cdots + (j^\sigma - j)$, as in Equation (1). The permutation $\sigma \in S_{n+1}$ can be readily recovered from the list of its multiplicities $\text{mult}(\sigma) = (\text{mult}_1(\sigma), \ldots, \text{mult}_n(\sigma)) \in \mathbb{N}^n$: we have $j^{\sigma} = \text{mult}_j(\sigma) - \text{mult}_{j-1}(\sigma) + j$ (with the convention $\text{mult}_0 = \text{mult}_{n+1} = 0$).

Adjacent strata $\mathcal{L}_n[w']$ of codimension $\dim(w') = 0$ are such that $w' = (a_{i_1}, \ldots, a_{i_d})$ is a string of Coxeter generators. In this case, $\Gamma_x \in \mathcal{L}_n[w']$ if and only if the real roots of $m_1(t), \ldots, m_n(t)$ are all simple and distinct. Multiple or common real roots correspond to more profound strata. More explicitly, if, for some value of $x$ and some $t_i \in \text{sing}(\Gamma_x)$, there exists a subset $\{j_1, \ldots, j_k\} \subseteq [n]$ such that $m_{j_1}(t_i) = \cdots = m_{j_k}(t_i) = 0$, then the corresponding letter $\sigma_i$ in the itinerary $\text{iti}(\Gamma_x) = (\sigma_1, \ldots, \sigma_\ell)$ has reduced words involving all the generators $a_{j_1}, \ldots, a_{j_k}$. The set of $x = (x_1, \ldots, x_d)$ for which a given profound letter occurs is a subset of the zero locus of discriminants and resultants of the polynomials $m_j$. Let

$$d_j(x) = \text{disc}(m_j(x, t)) \in \mathbb{R}[x], \quad j \in [n];$$

$$r_{i,j}(x) = \text{res}_i(m_i(x, t), m_j(x, t)) \in \mathbb{R}[x], \quad i, j \in [n], \quad i < j.$$

Thus, for instance, if a letter $[ab]$ occurs in the itinerary of $\Gamma_x$, then $d_1(x) = r_{12}(x) = 0$; we shall see other examples below.

Example 7.2. In our first example, $n = 2$, $w = (\sigma)$, $\sigma = [321] = aba$ (see matrix $M_0$ in Equation (14)), we have

$$m_1 = \frac{t^2}{2} + x_2, \quad m_2 = \frac{t^2}{2} + x_1t - x_2, \quad d_1 = -2x_2, \quad d_2 = x_1^2 + 2x_2, \quad r_{1,2} = -\frac{d_1d_2}{4}.$$
Thus, \( m_1(t) \) has two simple real roots \( t = \pm \sqrt{-2x_2} \) if \( x_2 < 0 \) and \( m_2(t) \) has two simple real roots \( t = -x_1 \pm \sqrt{x_1^2 + 2x_2} \) if \( x_2 > -\frac{x_1^2}{2} \). Thus, if \( x_2 > 0 \) the itinerary of \( \Gamma_x \) is \( bb \) and if \( x_2 < -\frac{x_1^2}{2} \) the itinerary is \( aa \). If \( x_1 < 0 \) (resp. \( x_1 > 0 \)) and \( -\frac{x_1^2}{2} < x_2 < 0 \) the itinerary is \( abab \) (resp. \( babab \)). These itineraries correspond to adjacent strata of codimension zero; more profound strata occur for \( x_2 = 0 \) or \( x_1^2 + 2x_2 = 0 \). If \( x_1 < 0 \) (resp. \( x_1 > 0 \)) and \( x_2 = 0 \), the itinerary is \( [ab]b \) (resp. \( b[ab] \)). If \( x_1 < 0 \) (resp. \( x_1 > 0 \)) and \( x_1^2 + 2x_2 = 0 \), the itinerary is \( a[ba] \) (resp. \( [ba]a \)). For instance, let \( x = (x_1, x_2) \) with \( x_1 > 0 \) and \( x_1^2 + 2x_2 = 0 \). Then, \( m_1(t) \) has two simple roots at \( t = \pm x_1 \) and \( d_2 = 0 \), so that \( m_2(t) \) has a double root at \( t = -x_1 \). Therefore, \( \text{sing}(\Gamma_x) = \{-x_1, x_1\} \), \( w' = \text{iti}(\Gamma_x) = (\sigma_1, \sigma_2) \) and, by Theorem 4 of [15], \( \text{mult}(\sigma_1) = (1, 2) = \text{mult}([ba]) \) and \( \text{mult}(\sigma_2) = (1, 0) = \text{mult}(a) \). Thus, \( w' = [ba]a \). The other cases are similar. The reader should compare these results, summarized in Figure 3, with Example 1.2 and Figure 1 in the introduction. Notice that, the more profound the stratum, the less generic are the curves therein.

Figure 3: Left: transversal section \( \phi : \mathbb{D}^2 \to \mathcal{L}_2 \to \mathcal{L}_2[[aba]] \) (see Example 7.2). Right: for \( x_1 = 1/3 \) and \( x_2 = -x_1^2/2 = -1/18 \) we have \( \text{sing}(\Gamma_x) = \{\pm 1/3\} \) and \( \text{iti}(\Gamma_x) = [ba]a \). Compare with Figure 1 from Example 1.2.

**Example 7.3.** In our second example, \( n = 3, w = (\sigma), \sigma = [3142] = \text{acb} \) (see matrix \( M_1 \) in Equation (14)), we have

\[
m_1 = \frac{t^2}{2} + x_2, \quad m_2 = -t, \quad m_3 = \frac{t^2}{2} - x_1, \quad m_4 = \frac{t^2}{2}.
\]

\[
d_1 = -2r_{1,2} = -2x_2, \quad d_2 = 1, \quad d_3 = -2r_{2,3} = 2x_1, \quad r_{1,3} = \frac{(x_1 + x_2)^2}{4}.
\]

Thus, \( m_2(t) \) has a simple root at \( t = 0 \) for all values of \( x_1, x_2 \). If \( x_2 > 0 \), \( m_1(t) \) has no real roots; if \( x_2 < 0 \), \( m_1(t) \) has roots \( t = \pm \sqrt{-2x_2} \). Similarly, for \( x_1 < 0 \), \( m_3(t) \) has no real roots and for \( x_1 > 0 \), \( m_3(t) \) has roots \( t = \pm \sqrt{2x_1} \). It is now easy to verify the itineraries of \( \Gamma_x \) in Figure 4 using resultants and
multiplicities. For instance, for \( \mathbf{x} = (x_1, x_2) \) with \( x_1 > 0 \) and \( x_2 = -x_1 \), the simple roots of \( m_1(t) \) and \( m_3(t) \) coincide pairwise at \( t = \pm \sqrt{2x_1} \). We therefore have \( \text{sing}(\Gamma_x) = \{-\sqrt{2x_1}, 0, +\sqrt{2x_1}\} \) and \( \text{iti}(\Gamma_x) = (\sigma_1, \sigma_2, \sigma_3) = [ac]b[ac] \), since, by Theorem 4 of [15], \( \text{mult}(\sigma_1) = \text{mult}(\sigma_3) = (1, 0, 1) = \text{mult}([ac]) \) and \( \text{mult}(\sigma_2) = (0, 1, 0) = \text{mult}(b) \). Notice that if \( x_1 < 0 \) and \( x_2 = -x_1 \), then \( \text{sing}(\Gamma_x) = \{0\} \) and \( \text{iti}(\Gamma_x) = (b) \), even if we are in the zero locus of \( r_{1,3} \). Also notice that the stratum \( \mathcal{L}_3[[ac]b[ac]] \) is as profound as \( \mathcal{L}_3[[acb]] \), since \( \dim([ac]b[ac]) = \dim([acb]) = 2 \).

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & -x_1 & 0 & -1 \\
1 & 0 & 0 & 0 \\
x_2 & 0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
g_{1,1}(t) & -1 & 0 & 0 \\
g_{2,1}(t) & g_{2,2}(t) & g_{2,3}(t) & -1 \\
-1 & 0 & 0 & 0 \\
g_{4,1}(t) & g_{4,2}(t) & 1 & 0
\end{pmatrix}.
\]

This can be explicitly integrated, yielding polynomial coefficients \( g_{i,j} \). As before, consider the \( j \times j \) southwest minors \( m_j \) of \( \Gamma_{x,u}(t) \) as polynomials in the

\[m = m_1(t) = m_3(t)\]

\[m = m_2(t)\]

Figure 4: Left: transversal section \( \phi : \mathbb{D}^2 \to \mathcal{L}_3 \) to \( \mathcal{L}_3[[acb]] \) (see Example 7.3). Right: for \( x_1 = -x_2 = 1/18 \) we have \( \text{sing}(\Gamma_x) = \{0, \pm 1/3\} \) and \( \text{iti}(\Gamma_x) = [ac]b[ac] \).

**Example 7.4.** Consider the 1-parameter family of perturbations \( \phi_u : \mathbb{D}^2 \to \mathcal{L}_3 \), \( u \in (-\epsilon, \epsilon) \) (for some fixed \( \epsilon \in (0, 1) \)), of the transversal section \( \phi = \phi_0 \) of Example 7.3 given by \( \phi_u(x) = Q_{\eta \sigma} \circ \Gamma_{x,u} \), where \( \Gamma_{x,u} : [-1, 1] \to \eta \sigma \mathbb{L}_0^{1} \) is the solution to the ODE

\[
\Gamma'_{x,u}(t) = \Gamma_{x,u}(t) \left( \beta_1(t)I_1 + \beta_2(t)I_2 + \beta_3(t)I_3 \right),
\]

\[
\beta_1(t) = 1 + ut > 0, \quad \beta_2(t) = 1, \quad \beta_3(t) = 1 - ut > 0,
\]

with the initial condition below:

\[
\Gamma_{x,u}(0) = \left\|egin{array}{cccc}
0 & -1 & 0 & 0 \\
0 & -x_1 & 0 & -1 \\
1 & 0 & 0 & 0 \\
x_2 & 0 & 1 & 0
\end{array}\right\|, \quad
\Gamma_{x,u}(t) = \left\|egin{array}{cccc}
g_{1,1}(t) & -1 & 0 & 0 \\
g_{2,1}(t) & g_{2,2}(t) & g_{2,3}(t) & -1 \\
-1 & 0 & 0 & 0 \\
g_{4,1}(t) & g_{4,2}(t) & 1 & 0
\end{array}\right\|.
\]
Figure 5: Above: transversal sections $\phi_u : \mathbb{D}^2 \to \mathcal{L}_3$ to $\mathcal{L}_3[[ac]]$ for $u = \pm 2/5$ (see Example 7.4). Below: for $x_1 = 1/3$, and $x_2 > -1/3$ such that $r_{1,3}(x_1, x_2, u) = 0$, $\text{iti}(\phi_u(x_1, x_2)) = [ac]bac$ for $u = 2/5$ while $\text{iti}(\phi_u(x_1, x_2)) = cab[ac]$ for $u = -2/5$. The slopes of the graphs of $m_1(t)$ and $m_3(t)$ in this example are suggestive of the fact that no perturbation of $\phi_u(0, 0)$ will produce the itinerary $cabca$ in the case $u > 0$, and similarly, no perturbation of $\phi_u(0, 0)$ will produce the itinerary $acbca$ in the case $u < 0$. We shall go back to this issue in Section 9. The reader may want to compare this figure with Figure 7 therein.
indeterminates \( x_1, x_2, u, t \) and compute their discriminants and resultants:

\[
m_1 = \frac{ut^3}{3} + \frac{t^2}{2} + x_2, \quad m_2 = -t, \quad m_3 = -\frac{ut^3}{3} + \frac{t^2}{2} - x_1, \\
d_1 = -\frac{x_2(6u^2x_2 + 1)}{2}, \quad d_2 = 1, \quad d_3 = -\frac{x_1(6u^2x_1 - 1)}{2}, \\
r_{1,2} = x_2, \quad r_{1,3} = \frac{u}{108} (4u^2(x_2 - x_1)^3 + 9(x_1 + x_2)^2), \quad r_{2,3} = x_1.
\]

Figure 5 shows the itineraries of curves in the section \( \phi_u \) for fixed values of \( u > 0 \) and \( u < 0 \). The zero loci of the discriminants \( d_j \) and resultants \( r_{i,j} \) contain the coordinate axes \( x_1 \) and \( x_2 \) as before, and a singular cusp \( r_{1,3} \). The zero loci of \( d_1 \) and \( d_3 \) include lines far from the origin, which do not concern us. Notice the intersection of the zero loci of \( d_1 \), \( d_3 \) and \( r_{i,j} \) at the origin. The two diagrams differ combinatorially: for \( u > 0 \), the itinerary \( acbac \) appears and \( cabca \) does not; for \( u < 0 \), it is the other way around; this will be discussed in Section 9.

\[ \diamond \]

**Example 7.5.** Alternatively, consider another 1-parameter family of perturbations \( \psi_u : \mathbb{D}^2 \to \mathcal{L}_3, u \in (-\epsilon, \epsilon) \), of the transversal section \( \phi = \psi_0 \) of Example 7.3 given by taking

\[
\tilde{M}_{1,u} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -x_1 & 0 & -1 \\ -1 & -u & 0 & 0 \\ x_2 & x_3 & 1 & 0 \end{pmatrix}, \quad M_{1,u} = \begin{pmatrix} -t & -1 & 0 & 0 \\ -t^3 - x_1 t & -t^2 - x_1 & -t & -1 \\ -ut - 1 & -u & 0 & 0 \\ t^2 + x_2 & t & 1 & 0 \end{pmatrix}
\]

instead of \( \tilde{M}_1 \) and \( M_1 \) of Equation (14). As in Example 7.4, the functions \( m_j \) are explicit polynomials. Figure 6 shows the zero loci of the new resultants near the origin (there are complications far away which do not concern us). Notice the similarity between Figures 3 and 6.

\[ \diamond \]

### 8 Proof of Theorem 3

Let \( \sigma \in S_{n+1}, \sigma \neq e \); let \( \sigma_1 = \eta \sigma \). Let \( z_1 = \hat{\eta} \sigma = q \bar{\sigma}_1 \in \mathbb{B}^+_{n+1} \), \( q = \hat{\eta} \sigma \bar{\eta} \in \text{Quat}^+_{n+1} \). Recall from Example 4.2 of [15] that \( L(\exp(\theta \bar{h})) = \exp(\tan(\theta) \bar{h}_L) \) for \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). Let \( \theta_0 \in (0, \frac{\pi}{2}) \). The smooth curve \( \Gamma_{z_1,\bar{h}} : [-\theta_0, +\theta_0] \to \text{Spin}_{n+1}, \Gamma_{z_1,\bar{h}}(\theta) = z_1 \exp(\theta \bar{h}), \) is locally convex and has image contained in \( \mathcal{U}_{z_1} \). It is therefore strictly convex and can be expressed in triangular coordinates (see Subsections 2.1 and 2.3): \( \Gamma_{z_1,\bar{h}}(\theta) = z_1 Q(\Gamma_L(\theta)), \) where \( \Gamma_L(\theta) : [-\theta_0, \theta_0] \to \text{Lo}_{n+1}^1, \Gamma_L(\theta) = \exp(\tan(\theta) \bar{h}_L) \). We have \( \text{sing}(\Gamma_{z_1,\bar{h}}) = \{0\} \) and \( \text{iti}(\Gamma_{z_1,\bar{h}}) = (\sigma) \).

For \( w \in W_n \), we define \( w \vdash \sigma \) if there exists a convex curve \( \Gamma_1 \) with \( \text{iti}(\Gamma_1) = w \) in the set \( \mathcal{L}_{n,\text{convex}}(z_1 \exp(-\theta_0 \bar{h}); z_1 \exp(\theta_0 \bar{h})) \). Lemma 3.2 implies that we have \( (\) \neq \sigma \) (here, \( (\) \in W_n \) is the empty word). Our next result shows that the condition above does not depend on the particular choice of \( \theta_0 \in (0, \frac{\pi}{2}) \).

\[ 41 \]
Lemma 8.1. Consider \( z_1 \in \tilde{B}_{n+1}^+, \theta_1, \theta_2 \in (0, \frac{\pi}{2}) \) and \( w \in \mathbf{W}_n \). Then there exists \( \Gamma_1 \in \mathcal{L}_{n,\text{convex}}^{|H^1|}(z_1 \exp(-\theta_1 h); z_1 \exp(\theta_1 h)) \) with \( \text{iti}(\Gamma_1) = w \) if and only if there exists \( \Gamma_2 \in \mathcal{L}_{n,\text{convex}}^{|H^1|}(z_1 \exp(-\theta_2 h); z_1 \exp(\theta_2 h)) \) with \( \text{iti}(\Gamma_2) = w \). Furthermore, if there exists \( \Gamma_1 \in \mathcal{L}_{n,\text{convex}}^{|H^1|}(z_1 \exp(-\theta_1 h); z_1 \exp(\theta_1 h)) \) such that \( \text{iti}(\Gamma_1) = w \) then there exists a homotopy \( H : [0, 1] \to \mathcal{L}_{n,\text{convex}}^{|H^1|}(z_1 \exp(-\theta_1 h); z_1 \exp(\theta_1 h)) \),

\[
H(0) = \Gamma_0 = \Gamma_{z_1, h}, \quad H(1) = \Gamma_1, \quad H(s)|_{\theta_1 = \theta_1 - s \theta_1, \theta_2 = \theta_2 + s \theta_1, \theta_3 = \theta_3} = \Gamma_0|_{\theta_1 = \theta_1 - s \theta_1, \theta_2 = \theta_2 + s \theta_1, \theta_3 = \theta_3}
\]
such that \( \text{iti}(H(s)) = w \) for all \( s \in (0, 1) \).

Proof. We start with the first claim. Assume without loss of generality that \( \theta_1 < \theta_2 \). Given \( \Gamma_1 \) as above, \( \Gamma_2 \) can be constructed by attaching arcs: set

\[
\Gamma_2(\theta) = \begin{cases} 
\Gamma_1(\theta), & \theta \in [-\theta_1, \theta_1], \\
\left[ z_1 \exp(\theta h) \right], & \theta \in [-\theta_2, -\theta_1] \cup [\theta_1, \theta_2].
\end{cases}
\]

Conversely, given \( \Gamma_2 \) we apply a projective transformation to obtain \( \Gamma_1 \). More precisely, set \( \Gamma_{2, L} : [-\theta_2, \theta_2] \to \text{Lo}_{n+1}^1 \), \( \Gamma_{2, L}(\theta) = \text{L}(z_1^{-1} \Gamma_2(\theta)) \). Notice that a diagonal projective transformation takes \( \exp(\pm \tan(\theta_2) h L) \) to \( \exp(\pm \tan(\theta_1) h L) \):

\[
\exp(\pm \tan(\theta_1) h L) = \text{diag}(1, \lambda, \ldots, \lambda^n) \exp(\pm \tan(\theta_2) h L) \text{diag}(1, \lambda^{-1}, \ldots, \lambda^{-n})
\]
for \( \lambda = \tan(\theta_1)/\tan(\theta_2) \); apply this projective transformation and reparametrize the domain to obtain \( \Gamma_{1,L} \) and therefore \( \Gamma_1 \in \mathcal{L}^{[H]}_{n,\text{convex}}(z_1 \exp(-\theta_1 h); z_1 \exp(\theta_1 h)) \) with \( \text{itti}(\Gamma_1) = \text{itti}(\Gamma_2) \). For the second claim, given \( \Gamma_1 \), apply a projective transformation as above to define \( H(s) \) satisfying the conditions in the statement (compare with the construction of the homotopy in the proof of Lemma 8.3).

\[
\text{Lemma 8.2.} \quad \text{Consider } w \in \mathcal{W}_n, \sigma \in S_{n+1}, \sigma \neq e, \text{ and } \tilde{\Gamma} \in \mathcal{L}^{[H]}_n[(\sigma)]. \text{ There exists a sequence } (\Gamma_k)_{k \in \mathbb{N}} \text{ of curves } \Gamma_k \in \mathcal{L}^{[H]}_n[w] \text{ with } \lim_{k \to \infty} \Gamma_k = \tilde{\Gamma} \text{ in } \mathcal{L}^{[H]}_n \text{ if and only if } w \vdash \sigma.
\]

Notice that one implication is already known for the special case \( \tilde{\Gamma} = \Gamma_{z_1,h} \), \( \Gamma_1 = \tilde{\Gamma} \) for \( w \vdash \sigma \), we constructed in Lemma 8.1 a path \( H \) of curves of itinerary \( w \) tending to \( \Gamma_{z_1,h} \).

\[
\text{Proof.} \quad \text{Assume first that a sequence } (\Gamma_k) \text{ as above exists: we prove that } w \vdash \sigma. \quad \text{Let } z_1 = \eta \hat{\sigma}. \quad \text{For } w \vdash \sigma, \text{ we constructed in Lemma 8.1 a path } H \text{ of curves of itinerary } w \text{ tending to } \Gamma_{z_1,h}.
\]

\[
\text{Take open neighborhoods } A_{0\vdash}, A_{2\vdash}, A_{2\vdash}^+ \text{ and } A_{0\vdash}^+ \subset \mathcal{L}^{1}_{n+1} \text{ of } \tilde{\Gamma}_L(1/2 - \epsilon_0), \tilde{\Gamma}_L(1/2 - \epsilon_2), \tilde{\Gamma}_L(1/2 + \epsilon_2) \text{ and } \tilde{\Gamma}_L(1/2 + \epsilon_0), \text{ respectively, such that for all } L_{i;\pm} \subset A_{i;\pm}, \text{ } i \in \{0,2\}, \text{ we have } L_{0\vdash} \ll L_{1\vdash} \ll L_{2\vdash} \ll I \ll L_{2\vdash}^+ \ll L_{1\vdash}^+ \ll L_{0\vdash}^+. \quad \text{Let } B_{i;\pm} = z_1 \mathcal{Q}[A_{i;\pm}] \subset \mathcal{U}_1, \text{ } i \in \{0,2\}; \text{ notice that } \tilde{\Gamma}(1/2 \pm \epsilon_i) \in B_{i;\pm}, \text{ } i \in \{0,2\}.
\]

For sufficiently large \( k \), we have \( \Gamma_k(1/2 \pm \epsilon_i) \in B_{i;\pm}, \text{ } i \in \{0,2\} \). By Theorem 1 for sufficiently large \( k \), we also have \( \text{sing}(\Gamma_k) \subset (1/2 - \epsilon_2, 1/2 + \epsilon_2) \). For such large \( k \), define a locally convex curve \( \tilde{\Gamma}_k \) which coincides with \( \Gamma_k \) except in the intervals \( [1/2 - \epsilon_0, 1/2 - \epsilon_2] \text{ and } [1/2 + \epsilon_2, 1/2 + \epsilon_0] \). In these arcs, \( \tilde{\Gamma}_k \) is defined so that \( \tilde{\Gamma}_k(1/2 - \epsilon_1) = z_1 \mathcal{Q}(L_{1\vdash}) = \Gamma(1/2 - \epsilon_1) \) and \( \tilde{\Gamma}_k(1/2 + \epsilon_1) = z_1 \mathcal{Q}(L_{1\vdash}^+) = \Gamma(1/2 + \epsilon_1) \). the above conditions guarantee that this is possible. The restriction of any such curve \( \tilde{\Gamma}_k \) to the interval \( [1/2 - \epsilon_1, 1/2 + \epsilon_1] \) yields, by definition, \( w \vdash \sigma \).

Now, assume \( w \vdash \sigma \) and take \( \tilde{\Gamma} \in \mathcal{L}^{[H]}_n[(\sigma)] \text{ with } \text{sing}(\tilde{\Gamma}) = \{1/2 \} \). As before, take \( \Gamma(t) = \Gamma_{z_1,h}((t - 1/2)\pi) \). Define \( \tilde{\Gamma}_L(t) = \mathcal{L}(z_1^{-1}\tilde{\Gamma}(t)) \) and \( \Gamma_L(t) = \mathcal{L}(z_1^{-1}\Gamma(t)) \) for \( t \) in some interval \( [1/2 - \epsilon_0, 1/2 + \epsilon_0] \). Given \( k \in \mathbb{N}^* \), take \( \epsilon_1 \in (0, \epsilon_2) \). For sufficiently small \( \epsilon_2 \in (0, \epsilon_1) \), we have \( \Gamma_L(1/2 - \epsilon_1) \ll \Gamma_L(1/2 - \epsilon_2) \) and \( \Gamma_L(1/2 + \epsilon_2) \ll \Gamma_L(1/2 + \epsilon_1) \).

By Lemma 5.3 of \([15]\), there exist convex arcs \( \Gamma_{k,L,-} : [1/2 - \epsilon_1, 1/2 - \epsilon_2] \to \mathcal{L}^{1}_{n+1} \) and
\( \Gamma_{k,L,*} : [\frac{1}{2} + \epsilon_2, \frac{1}{2} + \epsilon_1] \rightarrow \mathcal{L}^{1}_{n+1} \) with \( \Gamma_{k,L,*}(\frac{1}{2} - \epsilon) = \tilde{\Gamma}_L(\frac{1}{2} - \epsilon_1), \Gamma_{k,L,*}(\frac{1}{2} - \epsilon_2) = \Gamma_L(\frac{1}{2} - \epsilon_2), \Gamma_{k,L,*}(\frac{1}{2} + \epsilon_2) = \Gamma_L(\frac{1}{2} + \epsilon_2), \Gamma_{k,L,*}(\frac{1}{2} + \epsilon_1) = \Gamma_L(\frac{1}{2} + \epsilon_1) \). Since \( w \perp \sigma \), there exists a convex arc \( \Gamma_{k,L,0} : [\frac{1}{2} - \epsilon_2, \frac{1}{2} + \epsilon_2] \rightarrow \mathcal{L}^{1}_{n+1} \) with itinerary \( w \) such that \( \Gamma_{k,L,0}(\frac{1}{2} \pm \epsilon_2) = \Gamma_L(\frac{1}{2} \pm \epsilon_2) \). For each \( k \in \mathbb{N} \), define

\[
\Gamma_k(t) = \begin{cases} 
\tilde{\Gamma}(t), & t \in [0, \frac{1}{2} - \epsilon_1], \\
z_1Q(\Gamma_{k,L,-}(t)), & t \in [\frac{1}{2} - \epsilon_1, \frac{1}{2} - \epsilon_2], \\
z_1Q(\Gamma_{k,L,0}(t)), & t \in [\frac{1}{2} - \epsilon_2, \frac{1}{2} + \epsilon_2], \\
z_1Q(\Gamma_{k,L,+}(t)), & t \in [\frac{1}{2} + \epsilon_2, \frac{1}{2} + \epsilon_1], \\
\hat{\Gamma}(t), & t \in [\frac{1}{2} + \epsilon_1, 1]. 
\end{cases}
\]

Of course we have \( \lim_{k \to \infty} \Gamma_k = \hat{\Gamma} \) in \( \mathcal{L}^{H\!}\!_1 \), as desired. \( \Box \)

**Remark 8.3.** In the statement of Lemma 8.2 the curves \( \hat{\Gamma} \) and \( \Gamma_k \) start at \( \hat{\Gamma}(0) = \Gamma_k(0) = 1 \) and end at \( \hat{\Gamma}(1) = \Gamma_k(1) = \hat{\eta}\hat{\sigma}\hat{\eta} \). The reader will notice, however, that only small convex arcs containing the singular sets are relevant to the proof. We may therefore apply Lemma 8.2 whenever both \( \hat{\Gamma} \) and \( \Gamma_k \) are convex arcs in the open subset \( \mathcal{U}_z \subset \text{Spin}_{n+1} \). Such arcs have free endpoints in the appropriate connected components of \( \mathcal{U}_z \cap \text{Br}_{\hat{\eta}} \) and \( \mathcal{U}_z \cap \text{Br}_{\hat{\eta}} \). This is an equivalent statement since we can always append initial and final arcs obtained by projective transformations. \( \Diamond \)

**Lemma 8.4.** For \( \sigma \in S_{n+1} \setminus \{e\} \) and \( w \in \mathcal{W}_n \), the following conditions are equivalent:

(i) \( w \perp \sigma \);

(ii) \( w \preceq (\sigma) \);

(iii) \( \mathcal{L}^{H\!\!}_n[(\sigma)] \cap \mathcal{L}^{H\!\!}_n[w] \neq \emptyset \);

(iv) \( \mathcal{L}^{H\!\!}_n[(\sigma)] \subseteq \mathcal{L}^{H\!\!}_n[w] \);

(v) given \( \tilde{\Gamma} \in \mathcal{L}^{H\!\!}_n[(\sigma)] \), \( \epsilon > 0 \), \( \text{sing}(\tilde{\Gamma}) = \{t_*\} \) and an open neighborhood \( U \subset \mathcal{L}^{H\!\!}_n \) of \( \tilde{\Gamma} \) there exists \( \Gamma \in U \cap \mathcal{L}^{H\!\!}_n[w] \) with \( \Gamma \) and \( \tilde{\Gamma} \) coinciding outside \( (t_* - \epsilon, t_* + \epsilon) \).

**Proof.** Conditions (ii) and (iv) are equivalent by definition. Condition (iv) clearly implies (iii); Lemma 8.2 shows that (iii) implies (i) and that (i) implies (iv). The proof of Lemma 8.2 shows that (i) implies (v). Finally, (v) clearly implies (iii). \( \Box \)

**Remark 8.5.** The known fact that convex curves form a connected component of \( \mathcal{L}^{H\!\!}_n \) (as in Lemma 8.3) gives us a second proof of the fact that \( (\cdot) \not\preceq (\sigma) \). \( \Diamond \)
Lemma 8.4 is a local version of Theorem 3, which we are now ready to prove.

**Proof of Theorem 3** Condition (iii) implies (i); Condition (i) implies (ii). We now show that Condition (ii) implies (iv). Indeed, take $\tilde{\Gamma} \in \mathcal{L}^{[H^r]}_n[\omega_1] \cap \mathcal{L}^{[H^r]}_n[\omega_0]$ and a sequence $(\Gamma_k)_{k \in \mathbb{N}}$ of curves in $\mathcal{L}^{[H^r]}_n[\omega_0]$ tending to $\tilde{\Gamma}$. Set $\text{sing}(\Gamma) = \{ t_1 < \cdots < t_k \}$ and $\epsilon > 0$ such that $3\epsilon < \min\{ t_{i+1} - t_i ; i \in \mathbb{N} \}$ (where $t_0 = 0$ and $t_{k+1} = 1$, as usual). Notice that the intervals $J_i = [t_i - \epsilon, t_i + \epsilon]$ are disjoint.

By Theorem 1, for sufficiently large $k$, we have $\text{sing}(\Gamma_k) \subset \cup_i J_i$. The restrictions $\Gamma_k|_{J_i}$ tend to $\tilde{\Gamma}|_{J_i}$ and therefore, for large $k$, $\text{iti}(\Gamma_k|_{J_i}) = \tilde{w}_i \leq (\sigma_i)$, by Lemmas 8.2 and 8.4 (see also Remark 8.3). We have $\tilde{w}_i \neq (\cdot)$ by Remark 8.5.

Now we prove that Condition (iv) implies (iii). The idea is to slightly perturb $\tilde{\Gamma}$ about each singular point $t_i$ while leaving the curve unchanged outside the supports of these perturbations. Implication (ii) to (v) in Lemma 8.4 ensures that the resulting curve $\Gamma$ can be made to have the desired itinerary $w_0$.

**9 Proof of Theorem 4 and the example [acb]**

The proof of Theorem 4 could have been given immediately after the proof of Theorem 1 but we prefer to discuss in this section questions related to the $H^r$ metric for large $r$.

**Proof of Theorem 4** We fix $w_1 \in \mathbb{W}_n$ and $\Gamma_1 \in \mathcal{L}^{[H^r]}_n[\omega_1]$, where we take

$$r > r_\bullet(n) = \left(\frac{n + 1}{2}\right)^2 = \max\{ \text{mult}_j(\sigma) ; \sigma \in S_{n+1}, j \in \mathbb{N} \}.$$  

We construct an open neighbourhood $U$ of $\Gamma_1$ in $\mathcal{L}^{[H^r]}_n$ such that $\Gamma \in U$, $\text{iti}(\Gamma) = w_0$, implies $\text{mult}(w_0) \leq \text{mult}(w_1)$. Write $\text{sing}(\Gamma_1) = \{ t_1 < \cdots < t_k \}$. As before, for each $j \in \mathbb{N}$, consider the function $m_{\Gamma_1;j} : [0,1] \rightarrow \mathbb{R}$ given by the southwest $j \times j$ minor of the matrix $\Pi(\Gamma_1(t))$. Set $\mu_{i,j} = \text{mult}_j(\Gamma_1; t_i)$, the multiplicity of $t = t_i$ as a zero of $m_{\Gamma_1;j}(t)$, so that $\sum_i \mu_{i,j} = \text{mult}_j(w_1)$, by Theorem 4 of [15]. Notice that $m_{\Gamma_1;j}(t_i) \neq 0$ for all $i, j$. The value of $r_\bullet(n)$ above was chosen so that these derivatives are all known to be continuous. Take $\epsilon > 0$ and disjoint open intervals $J_i \ni t_i$ such that, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$, we have $|m_{\Gamma_1;j}(t_i)| > \epsilon$ and, for all $t \in J_i$, $|m_{\Gamma_1;j}(t)| > \epsilon/2$. Using Theorem 1, we take an open set $U \in \mathcal{L}^{[H^r]}_n$ containing $\Gamma_1$ such that $\Gamma \in U$ implies $\text{sing}(\Gamma) \subset \cup_i J_i$ and $t \in J_i$ implies $|m_{\Gamma_1;j}(t)| > \epsilon/4$. The fact that the derivative of order $\mu_{i,j}$ of $m_{\Gamma_1;j}$ has constant sign in $J_i$ implies that the number of zeroes of $m_{\Gamma_1;j}$ in $J_i$ (counted with multiplicity) is at most $\mu_{i,j}$. Now, Theorem 4 of [15] implies the desired result.  

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Given $\sigma \in S_{n+1} \setminus \{e\}$, $w \in W_n$, we write $w \vdash \sigma [H^r]$ if and only if there exists a strictly convex curve $\Gamma_0 : [-1, 1] \to \Spin_{n+1}$ of class $H^r$ satisfying:

1. $\Gamma_0(-1), \Gamma_0(1) \in \Bru_n$, sing($\Gamma_0$) = $\{0\}$ and iti($\Gamma_0$) = ($\sigma$);

2. for each $\epsilon > 0$, there is a locally convex curve $\Gamma_\epsilon : [-1, 1] \to \Spin_{n+1}$ of class $H^r$ satisfying iti($\Gamma_\epsilon$) = $w$ and $d^{[H^r]}(\Gamma_0, \Gamma_\epsilon) < \epsilon$.

Of course, for all $\sigma \in S_{n+1} \setminus \{e\}$, there is a strictly convex curve $\Gamma_0$ satisfying Condition 1 if just take $\Gamma_0(t) = \sigma \exp \left( \frac{\pi}{2} ith \right)$. We stress though that, in the definition above, the curve $\Gamma_0$ need not have this special form. Under both Conditions 1 and 2 above, Theorem 1 implies that, for all $\epsilon > 0$, $\epsilon < \epsilon_\text{convex}$ implies $\Gamma_\epsilon$ being strictly convex.

**Lemma 9.1.** Given $w_0, w_1 \in W_n$, we have $\mathcal{L}^{[H^r]}_n[w_1] \cap \mathcal{L}^{[H^r]}_n[w_0] \neq \emptyset$ if and only if there are $\sigma_1, \ldots, \sigma_\ell \in S_{n+1} \setminus \{e\}$ and $w_{0,1}, \ldots, w_{0,\ell} \in W_n$ such that $w_0 = w_{0,1} \cdots w_{0,\ell}$, $w_1 = (\sigma_1, \ldots, \sigma_\ell)$ and, for all $i \in [\ell]$, $w_{0,i} \vdash \sigma_i [H^r]$.

In the proof of Lemma 9.1 we are going to use the following alternate characterization of the relation $\vdash [H^r]$.

**Lemma 9.2.** Given $\sigma \in S_{n+1} \setminus \{e\}$ and $w \in W_n$, we have $w \vdash \sigma [H^r]$ if and only if there exists a strictly convex curve $\Gamma_0 : [-1, 1] \to \Spin_{n+1}$ of class $H^r$ for which the following conditions hold:

1'. $\Gamma_0(-1), \Gamma_0(1) \in \Bru_n$, sing($\Gamma_0$) = $\{0\}$ and iti($\Gamma_0$) = ($\sigma$);

2'. given $\epsilon > 0$, $\tau \in (0, 1)$, there is a locally convex curve $\tilde{\Gamma} : [-1, 1] \to \Spin_{n+1}$ of class $H^r$ satisfying iti($\tilde{\Gamma}$) = $w$, $d^{[H^r]}(\Gamma_0, \tilde{\Gamma}) < \epsilon$ and $\tilde{\Gamma}|_{[-1,-\tau] \cup [\tau,1]} = \Gamma_0|_{[-1,-\tau] \cup [\tau,1]}$.

**Proof.** The only nontrivial claim is: if $\Gamma_0$ satisfies Conditions 1 and 2, then it also satisfies Condition 2. Given $\epsilon > 0$ and $\tau \in (0, 1)$, take $\Gamma_{\epsilon'}$ satisfying Condition 2 for $\epsilon' \in (0, \epsilon)$ such that sing($\Gamma_{\epsilon'}$) $\subset (-\frac{\pi}{2}, \frac{\pi}{2})$. The idea is now to obtain locally convex arcs $\Gamma_- : [-\tau, -\frac{\tau}{2}] \to \Spin_{n+1}$ and $\Gamma_+ : [\frac{\tau}{2}, \tau] \to \Spin_{n+1}$ such that $\tilde{\Gamma} : [-1, 1] \to \Spin_{n+1}$, given by

$$
\tilde{\Gamma}(t) = \begin{cases}
\Gamma_0(t), & t \in [-1, -\tau] \cup [\tau, 1], \\
\Gamma_-(t), & t \in [-\tau, -\frac{\tau}{2}], \\
\Gamma_+(t), & t \in [\frac{\tau}{2}, \tau], \\
\Gamma_{\epsilon'}(t), & t \in [-\frac{\tau}{2}, \frac{\tau}{2}]
\end{cases}
$$

has the desired properties. One may do so by producing standard convex arcs between $\Gamma_0(\pm \tau)$ and $\Gamma_{\epsilon'}(\pm \frac{\tau}{2})$ by means of projective transformations and then applying a smoothening procedure. We omit the details (but see [16]).  

\[\square\]
Proof of Lemma 9.2. One direction is easy: write \( w_1 = (\sigma_1, \ldots, \sigma_\ell) \), \( \sigma_i \in S_{n+1} \setminus \{e\} \). Given \( \Gamma_0 \in L_n^{[H^r]}[w_1] \), let \( \text{sing}(\Gamma_0) = \{t_1 < \cdots < t_\ell\} \) and take \( \tau > 0 \) such that \( \Gamma_{0,i} = \Gamma_0|_{(t_i - \tau, t_i + \tau)} \) is strictly convex and \( \text{iti}(\Gamma_{0,i}) = (\sigma_i) \) for all \( i \). Now, assume there is a sequence \( (\Gamma_k) \), \( k \in \mathbb{N}^* \), in \( L_n^{[H^r]}[w_0] \) such that \( \lim_{k \to \infty} d^{[H^r]}(\Gamma_k, \Gamma_0) = 0 \) and consider the restrictions \( \Gamma_{k,i} = \Gamma_k|_{(t_i - \tau, t_i + \tau)} \). By Theorem 1 for sufficiently large \( k \), we can assume \( \text{sing}(\Gamma_{k,i}) \subset (t_i - \tau, t_i + \tau) \) for all \( i \). For each \( k \), write \( \text{iti}(\Gamma_{k,i}) = w_0^{k,i} \) so that \( w_0 = w_0^{k,0} \cdots w_0^{k,\ell} \). Since there are finitely many decompositions of \( w_0 \) in subwords, we take a subsequence and assume \( (\Gamma_k) \) is such that there are fixed subwords \( w_{0,1}, \ldots, w_{0,\ell} \in W_n \) with \( \text{iti}(\Gamma_{k,i}) = w_0^{k,i} \) for all \( k \). It follows that \( w_0, 1^{\sigma_1 [H^r]} \) for all \( i \). Now for the reciprocal, let \( \sigma_1, \ldots, \sigma_\ell \in S_{n+1} \setminus \{e\} \) and \( w_{0,i}, \ldots, w_{0,\ell} \in W_n \) be such that \( w_{0,i} \rightarrow \sigma_i [H^r] \) for all \( i \in [\ell] \). Also, fix once and for all a single \( \tau \in (0, 1) \) and, for each \( i \) and each \( \epsilon > 0 \), let \( \Gamma_{0,i} \), \( \text{iti}(\Gamma_{0,i}) = (\sigma_i) \), and \( \Gamma_{\epsilon,i} \), \( \text{iti}(\Gamma_{\epsilon,i}) = w_0^{\epsilon,i} \), be as in Conditions 1 and 2 of Lemma 9.2. We shall produce from these ingredients a locally convex curve \( \tilde{\Gamma}_0 \in L_n^{[H^r]}[w_1] \), \( w_1 = (\sigma_1, \ldots, \sigma_\ell) \), and a sequence of locally convex curves \( \tilde{\Gamma}_k \in L_n^{[H^r]}[w_0] \), \( k \in \mathbb{N}^* \), \( w_0 = w_{0,1} \cdots w_{0,\ell} \), such that \( \lim_{k \to \infty} d^{[H^r]}(\tilde{\Gamma}_k, \tilde{\Gamma}_0) = 0 \). We begin by setting \( \tilde{\Gamma}_{k,i} = \tilde{\Gamma}_{k,i}^{+} \) (i.e., take \( \epsilon = \frac{1}{k} \)) for all \( i \in [\ell] \), and all \( k \in \mathbb{N}^* \). Let \( q_0 = 0, q_1 = \hat{\sigma}_1, q_2 = \hat{\sigma}_1 \hat{\sigma}_2, \ldots, q_\ell = \hat{\sigma}_1 \cdots \hat{\sigma}_\ell = \hat{w}_1 \in \text{Quat}_{n+1} \). We have, for all \( i \) and all \( k \), \( \tilde{\Gamma}_{k,i}(1) = \Gamma_{0,i}(1) \in \text{Bru}_{q_{i-1}} = U_{q_{i-1}} \). Choose recursively a sequence of matrices \( U_1, \ldots, U_\ell \in U_{n+1} \) such that \( 1 \leq \Gamma_{0,1}^{U_i}(1) \leq \Gamma_{0,1}(1)q_1^{-1} \leq \Gamma_{0,2}(1)q_1^{-1} \cdots \leq \cdot \). We now fix, for all \( i \in [\ell - 1] \) and all \( k \), smooth strictly convex arcs \( \Gamma_{0,i+\frac{1}{2}} : [-1, 1] \to \text{Bru}_{q_{i}} \) such that \( \Gamma_{0,i+\frac{1}{2}}(1) = \Gamma_{0,i}^{U_i}(1) \) and \( \Gamma_{0,i+\frac{1}{2}}(1) = \Gamma_{0,i+1}^{U_i+1}(1) \). Of course, there are smooth strictly convex arcs \( \Gamma_{0,\frac{1}{2}} : [-1, 1] \to \text{Bru}_q \) and \( \Gamma_{0,\frac{1}{2}} : [-1, 1] \to \text{Bru}_{q_1} \) such that \( \Gamma_{0,\frac{1}{2}}(1) = 1, \Gamma_{0,\frac{1}{2}}(1) = \Gamma_{0,1}^{U_1}(1), \Gamma_{0,\frac{1}{2}}(1) = \Gamma_{0,1}^{U_1}(1) \). Furthermore, \( \tilde{\Gamma}_k = \Gamma_{0,\frac{1}{2}} \hat{w}_1 \hat{w}_2 \cdots \hat{w}_\ell \), \( k \in \mathbb{N}^* \); the same piecewise affine reparameterization on \( [0, 1] \) is used in all these concatenations. These locally convex curves are of class \( H^r \) except at the finitely many welding points \( 0 < \tau_1 < \tau_2 < \cdots < \tau_\ell < 1 \), all of them far away from the singular sets \( \text{sing}(\Gamma_0), \text{sing}(\tilde{\Gamma}_0) \subset (0, 1) \). Also, notice that, by Condition 2 of Lemma 9.2, satisfied by the sequences \( \tilde{\Gamma}_{k,i} \) (recall we have fixed \( \tau > 0 \) right from the start), there is \( \delta > 0 \) such that, for all \( j \in [\ell] \), \( \text{we have } \Gamma_0|_{[\tau_j - \delta, \tau_j + \delta]} = \Gamma_k|_{[\tau_j - \delta, \tau_j + \delta]} \) for all \( k \in \mathbb{N}^* \). Apply a standard smoothing procedure to each one of these \( \ell \) arcs (of class \( H^r \) except at \( t = \tau_j \)), obtaining the corresponding locally convex arcs \( \tilde{\Gamma}_{k,j} : [\tau_j - \delta, \tau_j + \delta] \to \text{Spin}_{n+1} \) of class \( H^r \) (coinciding with the original ones on an initial and on a final segment). It is now easily seen that the maps
\[ \tilde{\Gamma}_k : [0, 1] \to \text{Spin}_{n+1}, \quad k \in \mathbb{N}, \]

\[ \tilde{\Gamma}_k(t) = \begin{cases} \tilde{\Gamma}_{0,j}(t), & t \in [\tau_j - \delta, \tau_j + \delta], \\ \Gamma_k(t), & t \in [0, 1] \setminus (\cup_j [\tau_j - \delta, \tau_j + \delta]) \end{cases} \]

satisfy all the desired properties. \( \square \)

**Corollary 9.3** (of Theorem 4 and Lemma 9.1). Given \( w_0, w_1 \in W_n \), if there are \( \sigma_1, \ldots, \sigma_{\ell} \in S_{n+1} \setminus \{ e \} \) and \( w_{0,1}, \ldots, w_{0,\ell} \in W_n \) such that \( w_0 = w_{0,1} \cdots w_{0,\ell} \), \( w_1 = (\sigma_1, \ldots, \sigma_{\ell}) \) and for all \( i \in [\ell] \), \( w_{0,i} \vdash \sigma_i [H^r] \), then, \( \text{mult}(w_0) \leq \text{mult}(w_1) \).

The following statement is already known to be true for \( n = 2 \).

**Conjecture 9.4.** For \( \dim(\sigma) = \text{inv}(\sigma) - 1 < n \) and all \( w \in W_n \), we have \( w \vdash \sigma [H^1] \) (i.e., \( w \leq (\sigma) \)) if and only if \( w \vdash \sigma [H^r] \) for all \( r > 2 \).

We now discuss in greater detail the example \([acb]\), with emphasis on the \( H^r \) metric, \( r \geq 3 \). This example has already been mentioned in Equation (6) in the Introduction and in Examples 7.3 and 7.5. We already know from Example 7.5 (via Lemma 9.1) that \( L_3[H^r][acb] \cap L_3[H^r][cabca] \neq \emptyset \), for all \( r \). By Theorem 3 (or Lemma 8.4), we have the inclusion \( L_3[H^1][acb] \subset L_3[H^1][cabca] \).

**Proposition 9.5.** Take \( n = 3 \) and \( r \geq 3 \). There exists a continuous function \( u : L_3[H^r][acb] \to \mathbb{R} \) with the following properties:

1. The set \( u^{-1}([0]) \subset L_3[H^r][acb] \) is a non empty closed subset and a topological submanifold of codimension 1.

2. The function \( u \) is topologically transversal to \( u^{-1}([0]) \subset L_3[H^r][acb] \).

3. If \( \Gamma_0 = \phi_a(0) \) for \( \phi_a \) as in Example 7.4 then \( u(\Gamma_0) = u \).

4. If \( \Gamma_0 \in L_3[H^r][acb] \) and \( u(\Gamma_0) > 0 \) then the left hand side of Figure 5 provides a local topological normal form for the itinerary of locally convex curves near the curve \( \Gamma \).

5. Conversely, if \( u(\Gamma_0) < 0 \) then the right hand side of Figure 5 provides a local topological normal form.

The concept of a local topological normal form needs clarification. Let \( H \) be an infinite dimensional separable Hilbert space. We claim, for instance, that if \( u(\Gamma_0) > 0 \) then there exists a neighborhood \( W \subset L_3[H^r] \) of \( \Gamma_0 \) and a homeomorphism \( \tilde{\psi} : \mathbb{R}^2 \times H \to W \) such that the itinerary of \( \tilde{\psi}(x_1, x_2, *) \) is given by \( (x_1, x_2) \) in the left hand side of Figure 5.
Notice, in particular, that if a curve $\Gamma \in \mathcal{L}^{[Hr]}_3$, $r \geq 3$, has a letter $[acb]$ in its itinerary and $u > 0$, then there exist perturbations $\tilde{\Gamma}$ of $\Gamma$ where the letter $[acb]$ splits into the string $acbac$ but there are no perturbations $\tilde{\Gamma}$ of $\Gamma$ where $[acb]$ becomes $cabca$. Similarly, for $u < 0$, the letter $[acb]$ can become $cabca$, but not the itinerary $acbac$.

**Remark 9.6.** Figure 7 shows the itineraries near a specific curve $\Gamma_0 \in \mathcal{L}^{[Hr]}_3[[acb]]$ with $u(\Gamma_0) = 0$. In fact, Figure 7 provides a local topological normal form near all such curves. The proof of this last fact shall not be given, but is similar (but more laborious) than that of Proposition 9.5.

**Proof of Proposition 9.5.** Consider a locally convex curve $\Gamma \in \mathcal{L}^{[Hr]}_3(\cdot ; \cdot ; \cdot)$ of class
\( H^r \) with iti(\( \Gamma \)) = [\( acb \)] and local triangular presentation \( \Gamma_L : [-\epsilon, \epsilon] \to \eta \sigma L_1^1 \),

\[
\Gamma_L(t) = \begin{pmatrix}
-g_{1,1}(t) & -1 & 0 & 0 \\
-g_{2,1}(t) & -g_{2,2}(t) & -g_{2,3}(t) & -1 \\
-1 & 0 & 0 & 0 \\
g_{4,1}(t) & g_{4,2}(t) & 1 & 0
\end{pmatrix}
\]

with logarithmic derivative \( (\Gamma_L(t))^{-1} \Gamma_L'(t) = \beta_1(t)I_1 + \beta_2(t)I_2 + \beta_3(t)I_3 \), i.e.,

\[
g_{1,1}' = \beta_1, \quad g_{2,1}' = g_{2,2}\beta_1, \quad g_{2,2}' = g_{2,3}\beta_2, \quad g_{2,3}' = \beta_3,
\]

\[
g_{4,1}' = g_{4,2}\beta_1, \quad g_{4,2}' = \beta_2.
\]

Let as before \( m_j = m_j(t) \) be the \( j \times j \) southwest minor of \( \Gamma_L(t) \). We have

\[
m_1 = g_{4,1}, \quad m_1' = g_{4,2}\beta_1, \quad m_2 = -g_{4,2}, \quad m_2' = -\beta_2,
\]

\[
m_3 = g_{4,2}g_{2,3} - g_{2,2}, \quad m_3' = g_{4,2}\beta_3.
\]

The so called local triangular presentation is obtained as follows. Fix a compact subinterval \( J \subset (0,1) \) containing the singular set sing(\( \Gamma \)) = \{ \( t_\Gamma \) \} in its interior and such that \( \Gamma[J] \subset U_{\eta\sigma} \) and consider an orientation-preserving diffeomorphism \( \theta : [-\epsilon, \epsilon] \to J \) satisfying \( \theta(0) = t_\Gamma \). We set \( \Gamma_L(t) = \eta \sigma L_{\eta\sigma}(\Gamma(\theta(t))) \).

The positive functions \( \beta_1, \beta_2, \beta_3 : [-\epsilon, \epsilon] \to (0, +\infty) \) thus obtained are of class \( H^{r-1} \). To simplify the computations, we assume without loss of generality that the reparameterization \( \theta \) is chosen so as to produce \( \beta_3(t) = 1 \) constant (at least in a small neighborhood of \( t = 0 \)). We already know from Theorem 4 of [15] that iti(\( \Gamma \)) = [\( acb \)] if and only if \( m_2(t) \) has a simple zero at \( t = 0 \) and both \( m_1(t) \) and \( m_3(t) \) have a double zero at \( t = 0 \). From [16], we therefore have

\[
g_{4,1}(0) = g_{4,2}(0) = g_{2,2}(0) = 0.
\]

Notice that the parameters \( g_{1,1}(0), g_{2,1}(0), g_{2,3}(0) \) parameterize the intersection point \( \Gamma(t_\Gamma) \in \text{Bru}_{\eta\sigma} \), but do not change either the singular set or the itinerary of \( \Gamma \). All these simplifications taken into account, Equation (16) boils down to

\[
m_1(0) = 0, \quad m_1'(t) = t\beta_1(t), \quad m_2(t) = -t,
\]

\[
m_3(0) = 0, \quad m_3'(t) = t\beta_3(t).
\]

We are ready to define the desired function \( u \):

\[
b_1 = \beta_1(0) > 0, \quad b_3 = \beta_3(0) > 0, \quad u = u(\Gamma) = \frac{b_3\beta_1'(0) - b_1\beta_3'(0)}{2b_1b_3}.
\]

The first three items are clear.

We now study the ways the common double zero of \( m_1(t) \) and \( m_3(t) \) at \( t = 0 \) can possibly split as we slightly perturb \( \Gamma \) to obtain a curve \( \tilde{\Gamma} \). Consider the
hyperplane
\[ S = \left\{ M(x, y) = \begin{pmatrix}
-y_1 & -1 & 0 & 0 \\
-y_2 & -x_1 & -y_3 & -1 \\
-1 & 0 & 0 & 0 \\
x_2 & 0 & 1 & 0
\end{pmatrix} \in \mathcal{L} \sigma \mathcal{L}^1_1; \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad y = (y_1, y_2, y_3) \in \mathbb{R}^3 \right\}. \]

Notice that the locally convex curve \( \Gamma \) intersects the smooth codimension one submanifold \( Q_{\mathcal{L} \sigma}[S] \subset \mathcal{U}_{\mathcal{L} \sigma} \subset \text{Spin}_{n+1} \) transversally and at a single value of the parameter \( t \in (0, 1) \): transversality comes from \( g_{s, t} = \beta \) Transversality can also be assumed for \( \Gamma \). For each \( \Gamma \), we also consider the corresponding local triangular presentation \( \tilde{\Gamma}_L \), with logarithmic derivative \( \tilde{\beta}_1(t)I_1 + I_2 + \tilde{\beta}_3(t)I_3 \) (as before, we assume the reparameterization \( \tilde{\theta} \) is such that \( \tilde{\beta}_2(t) = 1 \), constant). After a reparametrization of \( t \) by a translation, we may assume that \( \tilde{\Gamma}_L \) intersects \( S \) at \( t = 0 \). The corresponding version of Equation (17) is
\[
\tilde{m}_1(t) = x_2, \quad \tilde{m}_1'(t) = t\tilde{\beta}_1(t), \quad \tilde{m}_2(t) = -t, \\
\tilde{m}_3(t) = -x_1, \quad \tilde{m}_3'(t) = t\tilde{\beta}_3(t).
\] (18)

Notice that the graphs of the functions \( \tilde{m}_1 = \tilde{m}_1(t) \) and \( \tilde{m}_3 = \tilde{m}_3(t) \) are convex in a neighborhood of their local minima \( (0, x_2) \) and \( (0, -x_1) \), respectively.

Assume \( u > 0 \): we prove the topological normal form. If \( x_2 > 0 \) or \( x_1 < 0 \) the itinerary is as in Figure 5. We are left with studying the quadrant \( x_1 > 0 \), \( x_2 < 0 \). Let \( \tilde{b}_1 = \tilde{\beta}_1(0) \approx b_1, \tilde{b}_3 = \tilde{\beta}_3(0) \approx b_3 \). For fixed \( \tilde{b}_1 \) and \( \tilde{b}_3 \), we consider the line segment \( (x_1, x_2) = (2s\tilde{b}_3c, -2(1-s)\tilde{b}_1c) \) where \( c > 0 \) is fixed and \( s \in [0, 1] \). When \( s \) moves from 0 to 1, the roots of \( \tilde{m}_1 \) move monotonically towards 0. Similarly, the roots of \( \tilde{m}_3 \) move monotonically away from 0.

We study the point \( s = \frac{1}{2} \). We have \( \tilde{m}_1(0) = x_2 = -\tilde{b}_1c, \tilde{m}_3(0) = -x_1 = -\tilde{b}_3c \) and therefore \( \tilde{b}_3\tilde{m}_1(0) = \tilde{b}_1\tilde{m}_3(0) \). For small \( |t| \) and \( \tilde{\beta}_i \) near \( \tilde{\beta}_i \), we have \( \tilde{b}_3\tilde{\beta}_1'(t) > \tilde{b}_1\tilde{\beta}_3'(t) \). Thus, for small \( \tilde{\beta}_i \), \( t \neq 0 \), we have \( \tilde{b}_3t\tilde{\beta}_1(t) > \tilde{b}_1t\tilde{\beta}_3(t) \) and therefore \( \tilde{b}_3\tilde{m}_1'(t) > \tilde{b}_1\tilde{m}_3'(t) \). Thus, \( t > 0 \) implies \( \tilde{b}_3\tilde{m}_1(t) > \tilde{b}_1\tilde{m}_3(t) \) and \( t < 0 \) implies \( \tilde{b}_3\tilde{m}_1(t) < \tilde{b}_1\tilde{m}_3(t) \). Thus, if \( t > 0 \) and \( \tilde{m}_1(t) = 0 \) we have \( \tilde{m}_3(t) < 0 \); if \( t < 0 \) and \( \tilde{m}_1(t) = 0 \) we have \( \tilde{m}_3(t) > 0 \). At this point the itinerary is therefore \( acbac \).

Monotonicity implies that as \( s \) goes from 0 to 1 the itinerary changes from \( a[cb]a \) (at \( s = 0 \)) to \( acbca \) to \( acb[ac] \) (for a unique \( s_\in \in (0, \frac{1}{2}) \)) to \( acbac \) (at an open neighborhood of \( s = \frac{1}{2} \)) to \( [ac]bac \) (for a unique \( s_\in \in (\frac{1}{2}, 1) \)) to \( cabac \) to \( c[ab]c \) (at \( s = 1 \)). A piecewise linear reparamatization leads to Figure 5. The case \( u < 0 \) is of course similar. \( \square \)
For $q \in \text{Quat}_{n+1}$, let $L_n(q)$ be the space of locally convex curves $\Gamma$ in Spin$_{n+1}$ with $\Gamma(0) = 1$ and $\Gamma(1) = q$. The present paper constructs a stratification of the space $L_n(q)$. We proved the necessary results for the construction of a homotopy equivalent CW complex. We detail this construction and prove some consequences in [14]. In this final section we discuss some of the methods involved, state a few consequences proved in [14, 1] and also mention a conjectural result.

One important construction in [36] and [30] is the add-loop procedure, which, in certain cases, is used to loosen up compact families of nondegenerate curves through a homotopy in $L_n(q)$. The resulting families of curly curves are then maleable: if a homotopy exists in the space of immersions, another homotopy exists in the space of locally convex curves. In [30], for instance, open dense subsets $\mathcal{Y}_\pm \subset L_2(\pm 1)$ are shown to be homotopy equivalent to the space of loops $\Omega S^3$. This approach is reminiscent of classical constructions such as Thurston’s eversion of the sphere by corrugations [25] and the proof of Hirsch-Smale Theorem [19, 40]. It can be considered as an elementary instance of the h-principle of Eliashberg and Gromov [11, 17]. Theorem 3, Corollary 1.1 and Lemma 10.3 in [14] are based on this method and apply to higher dimensions. We restate here that Corollary 1.1:

**Corollary 10.1.** If $q \in \text{Quat}_{n+1} \setminus Z(\text{Quat}_{n+1})$ then the inclusion $i_q : L_n(q) \to \Omega \text{Spin}_{n+1}$ is a weak homotopy equivalence.

We now restate the main result from [1], which gives the homotopy type of spaces of locally convex curves in the sphere $S^3$. A similar result for $S^2$ is the main result in [30]. Recall that Spin$_4 = S^3 \times S^3$; the subgroup Quat$_4 \subset$ Spin$_4$ is generated by $(1, -1)$, $(i, i)$ and $(j, j)$. The center of Quat$_4$ is $Z(\text{Quat}_4) = \{(\pm 1, \pm 1)\}$.

**Theorem 5.** We have the following weak homotopy equivalences:

- $L_3((+1, +1)) \approx \Omega(S^3 \times S^3) \lor S^4 \lor S^8 \lor S^{12} \lor S^{12} \lor \cdots$,
- $L_3((-1, -1)) \approx \Omega(S^3 \times S^3) \lor S^2 \lor S^6 \lor S^{10} \lor S^{10} \lor \cdots$,
- $L_3((+1, -1)) \approx \Omega(S^3 \times S^3) \lor S^0 \lor S^4 \lor S^4 \lor S^8 \lor S^8 \lor \cdots$,
- $L_3((-1, +1)) \approx \Omega(S^3 \times S^3) \lor S^2 \lor S^6 \lor S^{10} \lor S^{10} \lor \cdots$.

The above bouquets include one copy of $S^k$, two copies of $S^{(k+4)}$, \ldots, $j + 1$ copies of $S^{(k+4j)}$, \ldots, and so on.

The presence of $S^0$ in the bouquet indicates the presence of the contractible connected component of convex curves.
Our methods allow us to study the corresponding problem for locally convex curves in $\mathbb{S}^n$, $n > 3$. We do not have a conjectural homotopy type in general, but we hope to be able to prove the following result.

**Conjecture 10.2.** If $q \in Z(\text{Quat}_{n+1})$ then the space $L_n(q)$ is not homotopy equivalent to $\Omega \text{Spin}_{n+1}$.

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