CONSTRANDED TOPOLOGICAL GRAVITY 
FROM TWISTED N=2 LIOUVILLE THEORY

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Abstract

In this paper we show that there exists a new class of topological field theories, whose correlators are intersection numbers of cohomology classes in a constrained moduli space. Our specific example is a formulation of 2D topological gravity. The constrained moduli-space is the Poincaré dual of the top Chern-class of the bundle $E_{hol} \longrightarrow M_g$, whose sections are the holomorphic differentials. Its complex dimension is $2g - 3$, rather then $3g - 3$. We derive our model by performing the A-topological twist of N=2 supergravity, that we identify with N=2 Liouville theory, whose rheonomic construction is also presented. The peculiar field theoretical mechanism, rooted in BRST cohomology, that is responsible for the constraint on moduli space is discussed, the key point being the fact that the graviphoton becomes a Lagrange multiplier after twist. The relation with conformal field theories is also explored. Our formulation of N=2 Liouville theory leads to a representation of the N=2 superconformal algebra with $c = 6$, instead of the value $c = 9$ that is obtained by untwisting the Verlinde and Verlinde formulation of topological gravity. The reduced central charge is the shadow, in conformal field theory, of the constraint on moduli space. Our representation of the N=2 algebra can be split into the direct sum of a minimal model with $c = 3/2$ and a “maximal” model with $c = 9/2$. Considerations on the matter coupling of constrained topological gravity are also presented. Their study requires the analysis of both the A-twist and the B-twist of N=2 matter coupled supergravity, that we postpone to future work.

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1 Introduction

The realm of topological field theories can be divided, according to Witten [1], in two broad classes: the cohomological, or semiclassical theories, whose prototypes are either the topological Yang-Mills theory [2] or the topological $\sigma$-model [3] and the quantum theories, whose prototype is the abelian Chern-Simons theory [4, 5].

In this paper we deal with the cohomological theories and present new properties of these models. In particular, we propose a new formulation of 2D topological gravity [1, 6], leading to correlation functions apparently different from those of Witten’s theory. Nevertheless, our considerations have a more general scope and can apply to a wider number of models.

The new features of cohomological theories that we analyse are the reduction of moduli space to a constrained submanifold of the ordinary moduli space and the field theoretical mechanism that implements such a reduction. Specifically, we derive a theory of 2D topological gravity where the physical correlators are intersection numbers in a proper submanifold $V_{g,s} \subset M_{g,s}$ of the moduli space $M_{g,s}$ of genus $g$ Riemann surfaces $\Sigma_{g,s}$ with $s$ marked points.

$V_{g,s}$ is defined as follows. Consider the $g$-dimensional vector bundle $E_{hol} \to M_{g,s}$, whose sections $s(m)$ are the holomorphic differentials $\omega$ on the Riemann surfaces $\Sigma_{g,s}$, $m$ denoting the point of the base-manifold $M_{g,s}$ (i.e. the polarized Riemann surface). Let $c(E_{hol}) = \det(1 + R)$ be the total Chern class of $E_{hol}$, $R$ being the curvature two-form of a holomorphic connection on $E_{hol}$. For instance, we can choose the canonical connection $\Gamma = h^{-1} \partial h$ associated with the natural fiber metric $h_{jk} = \text{Im } \Omega_{jk}$, $\Omega_{jk}$ being the period matrix of $\Sigma_g$. Then $V_{g,s}$ is the Poincaré dual of the top Chern class $c_g(E_{hol}) = \det R = \det \left( \frac{1}{\Omega_{jk}} \partial \Omega - \frac{1}{\Omega_{jk}} \partial \Omega \right)$, $d = \partial + \bar{\partial}$ being the exterior derivative on the moduli space. $V_{g,s}$ is therefore a submanifold of codimension $g$ described as the locus of those Riemann surfaces $\Sigma_{g,s}(m)$ where some section $s(m)$ of $E_{hol}$ vanishes [7].

Explicitly, the topological correlators of our theory are the intersection numbers of the standard Mumford-Morita cohomology classes $c_1 (L_i)$ on the constrained moduli space, namely

$$< O_1 (x_1) O_2 (x_2) \cdots O_n (x_n) > = \int_{V_{g,s}} [c_1 (L_1)]^{d_1} \wedge \cdots \wedge [c_1 (L_n)]^{d_n} =$$

$$= \int_{M_{g,s}} c_g(E_{hol}) \wedge [c_1 (L_1)]^{d_1} \wedge \cdots \wedge [c_1 (L_n)]^{d_n}. \quad (1)$$

Precisely, $c_1 (L_i)$ are the first Chern-classes of the line bundles $L_i \to M_{g,s}$ defined by the cotangent spaces $T^*_{x_i} \Sigma_g (m)$ at the marked points $x_i$. The above theory will be called 2D constrained topological gravity.

We derive the definition of 2D constrained topological gravity in an algorithmic way, by performing the topological twist of N=2 Liouville theory, that we assume as the correct definition of N=2 supergravity in two-dimensions. The formal set-up for twisting an N=2 locally supersymmetric theory was established in ref. [8].
The origin of a constraint on moduli space is due to the presence of the graviphoton, absent in the existing formulations of 2D topological gravity. The graviphoton is initially a physical gauge-field and after the twist it maintains zero ghost-number. Nevertheless, in the twisted theory, it is no longer a physical field, rather it is a Lagrange multiplier (in the BRST sense). Indeed, it appears in the right-hand side of the BRST-variation of suitable antighosts. Since this Lagrange multiplier possesses global degrees of freedom (the $g$ moduli of the graviphoton), it imposes $g$ constraints on the space $\mathcal{M}_g$, which can be viewed as the space of the global degrees of freedom of the metric tensor. The metric tensor, on the other hand, is the only field that remains physical also after twist. We are lead to conjecture that the inclusion of Lagrange multiplier gauge-fields is a general mechanism producing the appearance of constrained moduli spaces. We recall that in four dimensions, instead, the role of the graviphoton $A$ \cite{8, 9} is that of producing ghosts for ghosts via the self-dual and anti-self-dual components $F^\pm_{ab}$ of the field strength $F_{ab}$.

To develop further the introductory description of our theory, we begin with a brief discussion of cohomological theories from our specific viewpoint.

In Witten’s words \cite{1}, cohomological field theories are concerned with sophisticated counting problems. The fundamental idea is that a generic correlation function of $n$ physical observables $\{O_1, \ldots, O_n\}$ has an interpretation as the intersection number

$$< O_1 O_2 \cdots O_n > = \# (H_1 \cap H_2 \cap \cdots \cap H_n)$$

of $n$ homology cycles $H_i \subset \mathcal{M}$ in the moduli space $\mathcal{M}$ of suitable instanton configurations $\exists [\phi(x)]$ of the basic fields $\phi$ of the theory. For example in the topological $\sigma$-model \cite{3, 10, 11, 12, 13} the basic fields are the maps

$$X : \Sigma_g \longrightarrow \mathcal{N}$$

from a genus $g$ Riemann surface $\Sigma_g$ into a Kählerian manifold $\mathcal{N}$. In this case the instantons $\exists [X(z, \bar{z})]$ are the holomorphic maps $\partial_z X = \partial_{\bar{z}} X = 0$ and the moduli space is, for each homotopy class of holomorphic embeddings of degree $k$, the parameter space $\mathcal{M}_k$ of such class of maps. The degree $k$ is defined by $\int_{\Sigma_g} X^* K = k$, where $X^* K$ is the pull-back of the Kähler two-form $K$ on $\mathcal{N}$. The observables $\mathcal{O}_{A_i}(z_i)$ are in one-to-one correspondence with the de Rahm cohomology classes $A_i \in H^p(\mathcal{N})$ of the target manifold $\mathcal{N}$ and the homology cycles $H_i$ are defined as the subvarieties of $\mathcal{M}_k$ that contain all those instantons such that $X(z_i) \in [A_i]^*$. In this definition we have denoted by $[A_i]^* \subset \mathcal{N}$ the Poincaré dual of the cohomology class $A_i \in H^p(\mathcal{N})$.

It is clear that topological field theories \cite{3} can be defined in completely geometrical terms. However, in every topological model, the right hand side of equation (3) should admit an independent definition as a functional integral in a suitable Lagrangian quantum field theory, in order to be of physical interest. The basic feature of the classical Lagrangian is that of possessing a very large group of gauge symmetries, the topological symmetry, which is the most general continuous deformation of the classical fields. The topological symmetry is treated through the standard techniques of BRST quantization.
and the instanton equations are imposed as a gauge-fixing. In this way, eq. (2), rather than a definition, becomes a map between a physical and a mathematical problem, which evenience is the main source of interest for topological field theories.

From the physical point of view, the basic properties of a topological field theory are encoded in the BRST algebra \( B \) and the anomaly of ghost number.

The moduli space cohomology originates the cohomology of the BRST operator \( s \): the left-hand side of eq. (2) is the vacuum expectation value of the product of \( n \) representatives \( O_i \) of non-trivial BRST cohomology classes

\[
 sO_i = 0, \quad O_i \neq s\{\text{anything}\}. \tag{4}
\]

Correspondingly, the right-hand side of eq. (2) can be expressed as an integral of a product of cocycles over the moduli space.

In full generality, the BRST algebra \( B \) can be decomposed as

\[
 B = B_{\text{gauge-free}} \oplus B_{\text{gauge-fixing}}, \tag{5}
\]

where \( B_{\text{gauge-free}} \subset B \) is the subalgebra that contains only the physical fields and the ghosts (fields of non negative ghost number), while \( B_{\text{gauge-fixing}} \) is the extension of \( B_{\text{gauge-free}} \) by means of antighosts and Lagrange multipliers (or the corresponding gauge-fixing conditions), of non positive ghost number. Usually, \( B_{\text{gauge-fixing}} \) is trivial, but this is not the case we deal with, since the interesting features of our theory come precisely from the nontrivial nature of \( B_{\text{gauge-fixing}} \).

We postpone the discussion of the structure of \( B_{\text{gauge-free}} \) and \( B_{\text{gauge-fixing}} \), in order to consider the mathematical meaning of the other basic aspect of the field theoretical approach, i.e. the anomaly of ghost number. The left-hand side of eq. (2) can be non-zero only if

\[
 \sum_i d_i = \Delta U = \int \partial^\mu J^\text{(ghost)}_\mu d^D x, \tag{6}
\]

where \( J^\text{(ghost)}_\mu \) is the ghost-number current, \( \Delta U \) is its integrated anomaly and \( d_i = gh[O_i] \) is the ghost number of \( O_i \) \([12, 13]\). The divergence of the ghost-current has an interpretation as index-density for some elliptic operator \( \nabla \) that appears in the quantum action through the kinetic term of the ghost(\( C \))-antighost(\( \bar{C} \)) system:

\[
 S_{\text{quantum}} = \int (\cdots + \bar{C}\nabla C + \cdots). \tag{7}
\]

We have

\[
 \text{index}_\nabla = \# \text{ zero modes of ghosts} - \# \text{ zero modes of antighosts}. \tag{8}
\]

On the other hand, the right-hand side of eq. (2) can be non-zero only if the sum of the codimensions of the homology cycles \( H_i \) adds up to the total dimension of the moduli space,

\[
 \sum_i \text{codim } H_i = \dim \mathcal{M}. \tag{9}
\]
In other words, the physical observables must reduce, after functional integration on the irrelevant degrees of freedom, to cocycle forms $\Omega_i$ of degree $d_i$ on the moduli-space $\mathcal{M}$ (the Poincaré duals of the cycles $H_i$) and their wedge product must be a top-form. This means

$$\Delta U = \dim \mathcal{M}. \quad (10)$$

Such an equation is understood in the following way. In the background of an instanton, namely of a gauge-fixed configuration, the zero-modes of the topological ghosts correspond to the residual infinitesimal deformations that preserve the gauge condition. Their number is therefore the dimension of the tangent space to the parameter space of the instanton. The zero-modes of the antighosts correspond, instead, to potential global obstructions to the integration of these infinitesimal deformations \cite{12, 13}. The index $\Delta U$ is therefore named the formal dimension of the moduli space $\mathcal{M}$. The true dimension of the moduli space is larger or equal to its formal dimension,

$$\dim_{\text{true}} \mathcal{M} \geq \dim_{\text{formal}} \mathcal{M} = \Delta U, \quad (11)$$

depending on whether the potential obstructions become real obstructions or not.

In the case of 2D topological gravity, Witten started \cite{1} from the right-hand side of eq. (2), proposing a completely geometrical definition. He assumed that the relevant moduli-space is the standard moduli-space $\mathcal{M}_{g,s}$ of Riemann surfaces of genus $g$ with $s$ marked points, whose dimension is well known to be

$$\dim_{\mathbb{C}} \mathcal{M}_{g,s} = 3g - 3 + s \quad (12)$$

and identified the observables $\mathcal{O}_i$ with the Mumford-Morita cohomology classes, namely the 2$d_i$-forms $[c_1(L_i)]^{d_i}$ on $\mathcal{M}_{g,s}$ introduced in eq. (4). Correspondingly, Witten obtained the selection rule:

$$\sum_{i=1}^{s} d_i = 3g - 3 + s. \quad (13)$$

For a reason that will be clear in a moment, it is convenient, from the field theoretical point of view, to rewrite this condition as

$$\sum_{i=1}^{s} (d_i - 1) = 3g - 3, \quad (14)$$

where now the right hand side is the dimension of the moduli space $\mathcal{M}_g$ without marked points. In this way, Witten assumed that in the field-theoretical formulation of 2D topological gravity, whatever it might be, the integrated anomaly of the ghost-number current should be

$$\Delta U = \int \partial^\alpha J^{(\text{ghost})}_\alpha \, d^2x = 3g - 3. \quad (15)$$

To understand this way of formulating the sum rule, we have to recall the concept of descent equations for the physical observables. Every local observable $\mathcal{O}_i$ of 2D topological
gravity can be written as $\sigma_{d_i}^{(0)}(x_i) = \gamma_0^{d_i}(x_i)$, $\gamma_0(x)$ being a suitable composite field. $\mathcal{O}_i$ is a zero form of ghost-number $2d_i$ and it is related to a one-form $\sigma_{d_i}^{(1)}(x_i)$ of ghost number $2d_i - 1$ and to a two-form $\sigma_{d_i}^{(2)}(x_i)$ of ghost number $2(d_i - 1)$ via the descent equations

$$s\sigma_{d_i}^{(0)} = 0, \quad s\sigma_{d_i}^{(1)} = d\sigma_{d_i}^{(0)}, \quad s\sigma_{d_i}^{(2)} = d\sigma_{d_i}^{(1)}, \quad 0 = d\sigma_{d_i}^{(2)}.$$  

As a consequence, the integrated observables $\int_{\Sigma_g} \sigma_{d_i}^{(2)}$ are BRST-closed,

$$s\int_{\Sigma_g} \sigma_{d_i}^{(2)} = 0,$$  

and can be traded for the local ones, by an equivalence

$$\langle \mathcal{P}(x_1)\sigma_{d_1}^{(0)}(x_1)\cdots\mathcal{P}(x_n)\sigma_{d_n}^{(0)}(x_n) \rangle \approx \langle \int_{\Sigma_g} \sigma_{d_1}^{(2)}(x_1)\cdots\int_{\Sigma_g} \sigma_{d_n}^{(2)}(x_n) \rangle. \quad (18)$$

$\mathcal{P}(x_i)$ denote certain picture changing operators [17], that have ghost number $-2$ and whose mathematical meaning is that of marking the points $x_i$ where the local operators $\sigma^{(0)}(x_i)$ are inserted. Both members of (18) can be calculated as intersection integrals over $\mathcal{M}_g$. The integrations appearing on the right hand side, however, can be also understood as integrations over the positions of “marked points” $x_i$, so that one is allowed to conjecture the correspondence

$$\sigma_{d_i}^{(2)}(x_i) \sim \gamma_0^{d_i}(x_i) \sim [c_1(L_i)]^{d_i}, \quad \gamma_0(x_i) \sim c_1(L_i). \quad (19)$$

Eq. (18) says that the topological amplitude can also be viewed as the correlator of BRST cohomology classes of degree $2(d_i - 1)$ on the space $\mathcal{M}_g$ and Witten’s conjecture (15) on the integrated anomaly of the ghost-current in any field-theoretical formulation of the theory is explained. Indeed, in [17] Verlinde and Verlinde constructed an explicit field theory model where eq. (15) is verified.

On the contrary, the key result of the present paper is the following one. We present a different field theoretical model of topological gravity where eq. (15) is replaced by

$$\Delta U = \int \partial^\alpha J_{\alpha}^{(\text{ghost})} d^2x = 2g - 2 \quad (20)$$

The geometrical interpretation of this fact has already been anticipated. Eq. (20) indicates that we are dealing with a constrained moduli space $\mathcal{V}_g$ whose formal complex dimension is $\dim_{\text{formal}} \mathcal{V}_g = 2g - 2$. Actually the constrained moduli-space $\mathcal{V}_g$ is the Poincaré dual of $c_g(\mathcal{E}_{\text{hot}})$ and its true dimension turns out to be $\dim_{\text{true}} \mathcal{V}_g = 2g - 3$, which is smaller than the formal dimension, another apparently puzzling result. However, if we recall that the effective moduli space emerges from a constraint on a larger moduli-space, then the fact that the formal-dimension is bigger than the actual dimension becomes less mysterious. Indeed, this time, in the sector of the BRST-algebra that implements the constraint, usual rules are inverted. Antighost zero-modes correspond to
local vector fields normal to the constrained surface and ghost zero-modes correspond to possible obstructions to the globalization of such local vector fields. As a consequence, the difference, in the constraint sector of the BRST algebra, of antighost zero-modes minus ghost zero-modes, expresses the minimum number of constraints that are imposed. If the potential obstructions do not occur, then all the antighosts correspond to actual normal directions to the constrained surface and the true dimension of the constraint surface is smaller than its formal dimension.

Notice that the constraint imposed is not a “generic” constraint, but a BRST constraint, i.e. a gauge-fixing of some symmetry. This translates geometrically into the fact, already pointed out, that the constrained moduli space is not a specific hypersurface in $\mathcal{M}_g$, rather it is a “slice choice” of a representative in a homology class of closed submanifolds (the Poincaré dual of $c_g(\mathcal{E}_{hol})$).

Let us now discuss the general structure of our topological field-theory in comparison with that introduced by Verlinde and Verlinde in [17]. The basic idea of [17] is that the moduli-space of Riemann surfaces $\Sigma_g$ can be related to the moduli-space of $SL(2,R)$ flat connections on the same surface. This goes back to the classical Fenchel-Nielsen parametrization of the Teichmüller space. A flat $SL(2,R)$ connection $\{e^{\pm}, e^0\}$ contains the zweibein $e^\pm$ and the spin connection of a constant curvature metric on the imaginary upper half-plane $H$. If that connection is pull-backed to the quotient $H/\Gamma_g$, where $\Gamma_g$ is a Fuchsian group realizing the homotopy group $\pi_1(\Sigma_g)$ of a genus $g$ surface, then the connection $\{e^{\pm}, e^0\}$ realizes a constant curvature metric on that surface. In view of this, the authors of [17] identified the gauge-free BRST algebra $B_{2D \ grav}$ with $B_{gauge-free}$ of 2D topological gravity with $B_{gauge-free}^{SL(2,R)}$, namely the gauge-free topological algebra associated with the Lie-algebra $SL(2,R)$. For a Lie algebra $\mathfrak{g}$ with structure constants $f_{JK}^I$, $I = 1, \ldots, \text{dim } \mathfrak{g}$, $B_{gauge-free}$ is

\begin{align}
 sA^I &= \Psi^I - dC^I - f_{JK}^I A^J C^K, \\
 s\Psi^I &= -d\Gamma^I - f_{JK}^I A^J \Gamma^K - f_{JK}^I C^J \Psi^K, \\
 sC^I &= \Gamma^I - \frac{1}{2} f_{JK}^I C^J C^K, \\
 s\Gamma^I &= f_{JK}^I C^J \Gamma^K, 
\end{align}

(21)

$\Psi$ being the topological ghost, $C$ the ordinary gauge ghost and $\Gamma$ being the ghost for the ghosts. In the case of $SL(2,R)$, the structure constants $f_{JK}^I$ are encoded in the curvature definitions

\begin{align}
 R^\pm &= de^\pm \pm e^0 \wedge e^\pm, \\
 R^0 &= de^0 + a^2 e^+ \wedge e^-, 
\end{align}

(22)

where $a^2 \in \mathbb{R}_+$ expresses the size of the constant negative curvature. The gauge-fixing algebra $B_{gauge-fixing}^{SL(2,R)}$ introduced by Verlinde and Verlinde realizes in an obvious manner the geometrical idea of flat-connections. The primary topological symmetry is broken by introducing antighosts whose BRST-variations are the Lagrange multipliers for the constraints $R^\pm = R^0 = 0$. In addition antighosts and Lagrange multiplier are also
introduced to fix diffeomorphisms, Lorentz invariance and the gauge-symmetry of the topological ghosts, namely superdiffeomorphisms. After gauge-fixing and in the limit $a^2 \to 0$, the model of ref. [17] reduces to the sum of two topological conformal field theories $\text{Liouville} \oplus \text{Ghost}$, that can be untwisted to $N=2$ conformal field-theories of central charges $c_{\text{Liouville}} = 9$ and $c_{\text{Ghost}} = -9$.

As one sees, this construction is in the spirit of the Baulieu-Singer approach to topological field-theories, where the gauge-fixing sector is invented $ad \ hoc$. On the other hand, the topological twisting algorithm produces topological theories where the gauge-fixing part of the BRST-algebra is already encoded in the original untwisted $N=2$ field-theory model. For instance, applying these ideas to the case of $D=4$, $N=2$ $\sigma$-models we discovered the concept of hyperinstantons [9]. In this paper we adopt the twisting strategy also in the case of $2D$ topological gravity. The necessary input is a definition of two-dimensional $N=2$ supergravity. This problem is solved by $N=2$ supersymmetrizing a reasonable definition of two-dimensional gravity. Following [18, 19] we assume as Lagrangian of ordinary $2D$ gravity the following one:

$$L_{2D-\text{grav}} = \Phi(R[g] + a^2)\sqrt{\det g}$$

(23)

where $\Phi$ and the metric $g_{\alpha\beta}$ are treated as independent fields. The variation in $\Phi$ imposes the constant curvature constraint on $g_{\alpha\beta}$. The Lagrangian (23) is equivalent, through the field redefinition $g_{\mu\nu} \rightarrow g_{\mu\nu} e^\Phi$ to the more conventional Liouville Lagrangian

$$L_{\text{Liouville}} = \left[\nabla_\alpha \Phi \nabla^\alpha \Phi + \Phi(R[g] + a^2 e^\Phi)\right] \sqrt{\det g}.$$  

(24)

Both Lagrangians (23) and (24) can be $N=2$ supersymmetrized and the results are related to each other by a field redefinition as in the $N=0$ case. Hence, we work with the $N=2$ analogue of the simpler form (23) and to it we apply the topological twist. The rest follows, although it requires interpretations that are by no means straightforward. As anticipated, the essential new feature is the presence, in the $N=2$ gravitational multiplet, of the graviphoton, a $U(1)$ gauge-connection $A$ that maintains ghost number 0 after the twist. Hence, the geometrical structure we deal with is that of a $U(1)$ bundle on a Riemann surface. The possible deformations of this structure are more than the deformations of a bare Riemann surface. Indeed, the total number of moduli for this bundle is $4g - 3$, $g$ new moduli being contributed by the deformations of $A$. The naive conclusion would be that the gauge-free topological algebra underlying the twisted theory is the direct sum $\mathcal{B}_{gauge-free}^{SL(2,R)} \oplus \mathcal{B}_{gauge-free}^{U(1)}$. This would lead to intersection theory in the $4g - 3$ dimensional moduli-space of the $U(1)$-bundle over $\Sigma_g$, but it is not the case. What actually happens is that the graviphoton belongs to $\mathcal{B}_{gauge-fixing}$, rather than to $\mathcal{B}_{gauge-free}$, satisfying a BRST-algebra of the type

$$s\bar{\psi} = A - d\gamma, \quad sA = -dc, \quad s\gamma = c, \quad sc = 0,$$

(25)

where $\bar{\psi}$ is a one-form of ghost number $-1$, $\gamma$ is a zero-form of ghost number 0 and $c$ is the ordinary gauge ghost (with ghost-number 1). The geometrical meaning of the gauge-fixing algebra (25) has already been anticipated. The deformations of $A$ correspond
to constraints on the allowed deformations of the bundle base-manifold, namely of the Riemann surface. How this mechanism is implemented by the functional integral is what we show in the later technical sections of our paper. We conjecture that BRST-algebras of type \((25)\), associated with a principal \(G\)-bundle \(E \to M\) always correspond to constraints on the moduli-space of the base-manifold \(M\).

Hence, our formulation of 2D topological gravity is based on the same gauge-free algebra \(B_{\text{gauge-free}}^{\text{SL}(2,\mathbb{R})}\) as the model of Verlinde and Verlinde, but it has a much different and more subtle gauge-fixing algebra. At the level of conformal field theories there is also a crucial difference, which keeps trace of the constraint on moduli space. Indeed, after gauge-fixing and in the limit \(a^2 \to 0\), our model also reduces to the sum of two topological conformal field theories \(\text{Liouville} \oplus \text{Ghost}\); the central charges, however, are \(c_{\text{Liouville}} = 6\) and \(c_{\text{Ghost}} = -6\), rather than 9 and \(-9\). We discuss the structure of these conformal field theories in more detail later on.

Let us conclude by summarizing our viewpoint. Equation \((2)\) possesses two deeply different meanings, depending on whether one reads it from the left (= physics) to the right (= mathematics), or from the right to the left.

i) From the right to the left, equation \((2)\) means:

“given a well defined mathematical problem (intersection theory on the moduli space of the instantons of some class of maps), find a quantum field theory that represents the intersection forms as physical amplitudes (averages of products of physical observables)”.

This problem is solved by BRST quantizing the most general continuous deformations of the classical fields (which sort of fields depending on the class of maps that is under consideration) and imposing the instantonic equations as a gauge-fixing.

ii) From the left to the right, \((2)\) means

“given a physically well-defined topological quantum field theory (as it is the topological twist of an N=2 supersymmetric theory), find the mathematical problem (maps, instantons, intersection theory) that it represents.

Of course, there is no general recipe for solving this second problem. We started considering it in four dimensions \([8, 9, 20]\) and this lead to interesting results. The topological twist of the N=2 supersymmetric \(\sigma\)-model permitted to introduce a concept of instantons (\(\text{hyperinstantons}\)) \([3]\) that we later \([20]\) identified with a triholomorphicity condition on the embeddings of four dimensional almost quaternionic manifolds into almost quaternionic manifolds. With this paper, we continue the program begun in ref. \([21]\) of considering the same problem in two dimensions, aiming to uncover the mathematical meanings of the topological field theories obtained by twisting the D=2 N=2 supersymmetric theories. We feel that the amazing secrets of N=2 supersymmetry have not yet been fully uncovered.

Our paper is organized in three main parts. The first part (sections \([2, 4]\)) is devoted to the construction of N=2 Liouville theory and its twist. In section \([4]\) we present D=2 N=2 supergravity in the rheonomic approach, while in section \([3]\) we couple it to N=2 Landau-Ginzburg chiral matter \([22, 23]\). In section \([4]\), we derive the Lagrangian of N=2 Liouville theory, that involves a suitable combination of the gravitational multiplet and a simple
chiral multiplet. We BRST quantize the theory in section 5 and perform the topological twist in section 6. The second part (sections 7 and 8) is devoted to conformal field theory. In section 7, we gauge-fix N=2 Poincaré gravity and show that it corresponds to a conformal field theory with \(c = c_{\text{Liouville}} + c_{\text{ghost}}\), \(c_{\text{Liouville}} = 6\) and \(c_{\text{ghost}} = -6\). We study the N=2 currents, the BRST current and various conformal properties. In section 8 we study the topological twist of the gauge-fixed theory, describing the match between the general twist procedure of section 6 and the procedure that is well known in conformal field theories [24]. We study some properties that emphasize the differences between our model and the Verlinde and Verlinde one. The third part corresponds to section 9, where we suggest the mathematical interpretation of the topological theory and draw the correspondence between quantum field theory and geometry. Finally in section 10 we address some open problems.

2 N=2 D=2 supergravity

Following [18] we assume as the classical Lagrangian of pure gravity the one displayed in eq. (23). Alternatively, in view of the equivalence between (23) and (24), we can also describe the action of pure 2D gravity as the Polyakov action for a Liouville system. We insist on the concept of Polyakov formulation, since the key point in eq. (23) is that both \(\Phi\) and \(g_{\alpha\beta}\) have to be treated as independent fields. This being clarified, we define N=2, D=2 supergravity as the supersymmetrization of eq. (23). To perform such a supersymmetrization, we need the following two ingredients.

i) An off-shell representation of the N=2 algebra, that corresponds to the graviton multiplet containing the metric \(g_{\alpha\beta}\).

ii) An off-shell representation of the N=2 algebra that corresponds to the chiral scalar multiplet containing the field \(\Phi\).

The final Lagrangian is obtained by combining these two multiplets.

For the purpose of this construction, we use the so called rheonomic formalism. This formalism was originally proposed in 1978-79 [25], as an alternative method with respect to the superfield formalism in studying supersymmetric theories. Its main advantage is the reduction of the computational effort for constructing supersymmetric theories to a simple series of geometrically meaningful steps. Rheonomy means “law of the flux” and refers to the fact that the supersymmetrization of a theory can be viewed as a Cauchy problem, in which spacetime, described by the inner (bosonic) components \(x\) of superspace, represents the boundary, while the outer (Grassmann) components \(\theta\) represent the direction of “motion”: the rheonomic principle represents the “equation of motion”. At the end of the rheonomic procedure, the only free choice is the boundary condition, which is the spacetime theory, projection of the superspace theory onto the inner components. The fields (or more generally differential forms) are functions on superspace and the supersymmetry transformations are viewed as odd diffeomorphisms in superspace. We shall give, along the paper, many details on the rheonomic approach in order to furnish enough information for using it.
In a recent paper [21], the rheonomic formalism was applied to the construction of D=2 globally supersymmetric N=2 systems. In particular, the rheonomic formulation of N=2 chiral multiplets coupled to N=2 gauge multiplets was provided. This involved the solution of Bianchi identities and the construction of rheonomic actions in the background of a flat N=2 superspace.

In this section we generalize the construction to curved superspace. To this effect an important preliminary point to be discussed is the following. The formulation of supergravity is performed in the physical Minkowski signature $\eta_{ab} = \text{diag}(+, -)$, yet, after Wick rotation we shall deal with supersymmetry defined on compact Riemann surfaces. These have a positive, null or negative curvature $\mathcal{R}$ depending on their genus $g$, ($\mathcal{R} > 0$ for $g = 0$, $\mathcal{R} = 0$ for $g = 1$ and $\mathcal{R} < 0$ for $g \geq 2$). The sign of the curvature in the Euclidean theory is inherited from the sign of the curvature in the Minkowskian formulation. Since we want to discuss the theory for all genera $g$, we need formulations of supergravity that can accommodate both signs of the curvature. From the group-theoretical point of view, curved superspace with $\mathcal{R} > 0$ is a continuous deformation of the supersymmetric version of the de Sitter space, whose isometry group is $SO(1, D)$, $D$ being the space-time dimension. Hence, for $\mathcal{R} > 0$ the appropriate superalgebra to begin with is, if it exists, the N-extended supersymmetrization of $SO(1, D)$. Similarly, for $\mathcal{R} < 0$, curved superspace is a continuous deformation of the supersymmetric version of the anti de Sitter space, whose isometry group is $SO(2, D - 1)$. Hence, in this second case, the appropriate superalgebra to start from in the construction of supergravity is the N-extended supersymmetrization of $SO(2, D - 1)$. Alternatively, one can start from the Poincaré superalgebra, that corresponds to an Inonü-Wigner contraction of either the de Sitter or the anti de Sitter algebra, and reobtain either one of the decontracted algebras as vacua configurations alternative to the Minkowski one, by giving suitable expectation values to the auxiliary fields appearing in the rheonomic parametrizations of the Poincaré curvatures. Actually, in space-time dimensions different from D=2, the supersymmetric extensions of both the de Sitter and the anti de Sitter algebras are not guaranteed to exist. For instance, in the relevant D=4 case, the real orthosymplectic algebra $Osp(4/N)$ is the N-superextension of the anti de Sitter algebra $SO(2, 3)$ but a superextension of the de Sitter algebra $SO(1, 4)$ does not exist. This is the group-theoretical rationale of some otherwise well known facts. In four-dimensions all supergravity vacua with a positive sign of the cosmological constant (de Sitter vacua) break supersymmetry spontaneously, while the only possible supersymmetric vacua are either in Minkowski or in anti de Sitter space. Indeed, starting from a formulation of D=4 supergravity based on the Poincaré superalgebra, one obtains both de Sitter and anti de Sitter vacuum configurations through suitable expectation values of the auxiliary fields, but it is only in the anti de Sitter case that these constant expectation values are compatible with the Bianchi identities of a superalgebra, namely respect supersymmetry [25]. In the de Sitter case the gravitino develops a mass and supersymmetry is broken. In other words, in D=4, supersymmetry chooses a definite sign for the curvature $\mathcal{R} < 0$.

If this were the case also in $D = 2$, supersymmetric theories could not be constructed.
on all Riemann surfaces, but only either in genus $g \geq 1$ or in genus $g \leq 1$. Fortunately, for D=2 it happens that the de Sitter group $SO(1,2)$ and the anti de Sitter group $SO(2,1)$ are isomorphic. Hence, a supersymmetrization of one is also a supersymmetrization of the other, upon a suitable correspondence. Once we have fixed the conventions for what we call the physical zweibein, spin connection and gravitini, we obtain an off-shell formulation of supergravity where the sign of the curvature is fixed: it is either non-negative or non-positive. Through a field correspondence, we can however make a transition from one case to the other, but a continuous deformation of the auxiliary field vacuum expectation value is not sufficient for this purpose. Furthermore, as we are going to see, the Inonü–Wigner contraction of the N=2 algebra displays also some new features with respect to the $U(1)$ generator associated with the graviphoton.

The most general $D=2$ superalgebra one can write down, through Maurer-Cartan equations, is obtained by setting to zero the following curvatures:

\[
\begin{align*}
T^+ &= de^+ + \omega e^+ - \frac{i}{2} \zeta^+ \zeta^-,
T^- &= de^- - \omega e^- - \varepsilon \frac{i}{2} \tilde{\zeta}^+ \tilde{\zeta}^-,
\rho^+ &= d\zeta^+ + \frac{1}{2} \omega \zeta^+ + \frac{ia_1}{4} A\zeta^+ + a_1 a_2 \tilde{\zeta}^- e^+,
\rho^- &= d\zeta^- + \frac{1}{2} \omega \zeta^- - \frac{ia_1}{4} A\zeta^- - a_1 a_2 \tilde{\zeta}^+ e^+,
\check{\rho}_+ &= d\tilde{\zeta}^+ - \frac{1}{2} \omega \tilde{\zeta}^+ - \frac{ia_1}{4} A\tilde{\zeta}^+ - \varepsilon a_1 a_2 \zeta^- e^-,
\check{\rho}_- &= d\tilde{\zeta}^- - \frac{1}{2} \omega \tilde{\zeta}^- + \frac{ia_1}{4} A\tilde{\zeta}^- - \varepsilon a_1 a_2 \zeta^+ e^-,
R &= d\omega - 2\varepsilon a_1 a_2 e^+ e^- - \frac{i}{2} a_1 a_2 (\zeta^+ \tilde{\zeta}^+ + \zeta^- \tilde{\zeta}^-),
F &= dA - a_2 (\zeta^- \tilde{\zeta}^- - \zeta^+ \tilde{\zeta}^+),
\end{align*}
\]

(26)

where $e^+$ and $e^-$ denote the two components (left and right moving) of the world sheet zweibein one form, while $\zeta^+$, $\tilde{\zeta}^+$ are the two components of the gravitino one form, $\zeta^-$, $\tilde{\zeta}^-$ are the two components of its complex conjugate. $\varepsilon$ can take the values $\pm 1$ and distinguishes the de Sitter ($\varepsilon = 1$) and anti de Sitter ($\varepsilon = -1$) cases. Formally, one can pass from positive to negative curvature by replacing $e^-$ and $T^-$ with $-e^-$ and $-T^-$. 

The algebra (26) contains two free (real) parameters $a_1, a_2$ in its structure constants. Choosing $a_1 = a_2 = \frac{2}{\sqrt{2}} \neq 0$ we have the usual curvature definitions for a de Sitter algebra with cosmological constant $\Lambda = \varepsilon a^2$, namely the superextension of the $SL(2,R)$ Lie algebra. In the limit $a_2 \to 1$ and $a_1 \to 0$ we get the usual $D=2$ analogue of the $N=2$ super Poincaré Lie algebra, where, calling $L$ the $U(1)$ generator dual to the graviphoton, the supercharges $Q^\pm, \tilde{Q}_\pm$ are neutral under $L$:

\[
[L, Q^\pm] = [L, \tilde{Q}_\pm] = 0.
\]

(27)
In this case, the generator $L$ can be interpreted as a “central charge”, since it appears in the supercharges anticommutators:

$$\{Q^-, \tilde{Q}_-\} \sim L, \quad \{Q^+, \tilde{Q}_+\} \sim L.$$  \hfill (28)

Finally, in the limit $a_2 \to 0$, $a_1 \to 1$ we get a new kind of Poincaré superalgebra, named by us “charged Poincaré”, where the supercharges do rotate under the $U(1)$ action:

$$[L, Q^\pm] = \pm Q^\pm, \quad [L, \tilde{Q}_\pm] = \mp \tilde{Q}_\pm,$$

$$\hfill (29)$$

In this case $L$ is not a central charge, since it does not appear in the supercharge anticommutators. Indeed, one has

$$\{Q^+, Q^-\} = P, \quad \{\tilde{Q}_+, \tilde{Q}_-\} = \tilde{P}, \quad \{Q^+, \tilde{Q}_-\} = 0, \quad \{Q^-, \tilde{Q}_+\} = 0,$$

$P$ and $\tilde{P}$ being the left and right translations, dual to $e^+$ and $e^-$ respectively.

In [21] the construction of global $N = 2$ supersymmetric theories was based on the use of the ordinary Poincaré superalgebra. In this case we can always choose the gauge $\omega = A = 0$ and we can altogether forget about these one forms. In the solution of Bianchi identities we simply have to respect global Lorentz and $U(1)$ symmetries. However the flat case is actually unable to distinguish between the ordinary and charged Poincaré algebra. At the level of curved superspace, on the other hand, there is a novelty that distinguishes $D = 2$ from higher dimensions. It turns out that the correct algebra is the charged one.

The field content of the off shell graviton multiplet is easily described. The zweibein describes one bosonic degree of freedom (four components restricted to one by two diffeomorphisms and by the Lorentz symmetry), while each gravitino describes two degrees of freedom (four components restricted by two supersymmetries). Finally, the graviphoton $A$ yields one bosonic degree of freedom (two components restricted by the $U(1)$ gauge symmetry). The mismatch of two bosonic degrees of freedom is filled by a complex scalar auxiliary field $M$ and by its conjugate $\bar{M}$. The problem is therefore that of writing a rheonomic parametrization for the curvatures (26) using as free parameters their space-time components plus an auxiliary complex scalar $M$.

As can be easily read from (26) the curvature two forms satisfy:

$$\nabla T^+ = Re^+ - \frac{i}{2}(\rho^+ \zeta^- - \zeta^+ \rho^-),$$

$$\nabla T^- = -Re^- - \frac{i}{2}\varepsilon(\tilde{\rho}_+ \tilde{\zeta}_- - \tilde{\zeta}_+ \tilde{\rho}_-),$$

$$\nabla \rho^\pm = \frac{1}{2} R\zeta^\pm \pm \frac{ia_1}{4} F\zeta^\pm + a_1 a_2 (\rho^\mp e^+ - \zeta^\pm T^+),$$

$$\nabla \tilde{\rho}_\pm = -\frac{1}{2} R\tilde{\zeta}_\pm \pm \frac{ia_1}{4} F\tilde{\zeta}_\pm - a_1 a_2 \varepsilon (\rho^\mp e^- - \zeta^\mp T^-),$$

$$\nabla R = -2a_1 a_2 \varepsilon (T^+ e^- - e^+ T^-) - \frac{i}{2} a_1 a_2 (\rho^+ \tilde{\zeta}_+ - \zeta^+ \tilde{\rho}_+ + \rho^- \tilde{\zeta}_- - \zeta^- \tilde{\rho}_-),$$

$$\nabla F = -a_2 (\rho^- \tilde{\zeta}_- - \zeta^- \tilde{\rho}_- - \rho^+ \tilde{\zeta}_+ + \zeta^+ \tilde{\rho}_+).$$  \hfill (31)
The general solution for the above Bianchi identities with vanishing torsions is

\[
\begin{align*}
T^+ &= 0, \\
\rho^+ &= \tau^+ e^+ e^- - a_1(M - a_2) \tilde{\zeta} - e^+, \\
\rho^- &= \bar{\tau}^+ e^+ e^- + a_1 \bar{\varepsilon}(M - a_2) \zeta - e^-, \\
R &= (R - 2a_1^2 a_2 \varepsilon) e^+ e^- + \frac{i}{2} \varepsilon e^-(\tau^+ \zeta^- + \tau^- \zeta^+) + \frac{i}{2} e^+(\bar{\tau}_- \bar{\zeta}^- + \bar{\tau}_+ \bar{\zeta}^-) \\
&\quad + \frac{i a_1}{2} \left[ (M - a_2) \zeta^- \bar{\zeta}_- + (M - a_2) \bar{\zeta}^+ \bar{\zeta}^+ \right], \\
F &= \mathcal{F} e^+ e^- + (M - a_2) \zeta^- \bar{\zeta}_- - (M - a_2) \bar{\zeta}^+ \bar{\zeta}^+ - \frac{1}{a_1} \varepsilon (\tau^+ \zeta^- - \tau^- \zeta^+) e^- \\
&\quad + \frac{1}{a_1} (\bar{\tau}_- \bar{\zeta}^- - \bar{\tau}_+ \bar{\zeta}_-) e^+. 
\end{align*}
\]

The formulae for $\rho^-$ and $\bar{\rho}_-$ can be derived from those of $\rho^+$ and $\bar{\rho}_+$ by complex conjugation. In doing this, one has to keep into account that the complex conjugation reverses the order of the fields in a product of fermions.

It is immediate to see in eq.s. (32) that the limit $a_1 \to 0$ is singular, and this reflects the fact that we are not able to find the correct parametrizations for this case. On the contrary the limit $a_2 \to 0$ is perfectly consistent and we call it "charged Poincaré algebra".

From now on, to avoid any confusion in using the formulae for the curvature definition, we will always refer to the symbols $R, F, \rho^+, \bar{\rho}^\pm$ as to the ones defined in (28) with $a_1 = 1, a_2 = 0$.

The general rule for obtaining the solution to the Bianchi identities is rheonomic principle. We briefly describe it in three steps.

i) One expands the curvatures two-forms $\rho^\pm, \bar{\rho}^\pm, R$ and $F$ in a basis of superspace two-forms: the "spacetime" form $e^+ e^-$ and the "superspace" forms, which can be fermionic, like $\zeta^\pm e^\pm$ and $\bar{\zeta}^\pm e^\pm$, or bosonic, like $\zeta^\pm \zeta^\pm, \zeta^\pm \bar{\zeta}^\pm$ and $\bar{\zeta}^\pm \bar{\zeta}^\pm$.

ii) The coefficients $\tau^\pm, \bar{\tau}^\pm, R$ and $\mathcal{F}$ of the spacetime form $e^+ e^-$ are independent ones: the rheonomic parametrizations (28) can be viewed as a definition of them. They are the supercovariantized derivatives of the fields. In particular, $R$ is the supercurvature and $\mathcal{F}$ is the super-field-strength.

iii) The coefficients of the superspace forms, instead, are functions of the fields and of the supercovariantized derivatives $\tau^\pm, \bar{\tau}^\pm, R$ and $\mathcal{F}$. They are determined by solving the Bianchi identities (31) in superspace. Their form is strongly constrained by Lorentz invariance, global $U(1)$ invariance and scale invariance. These restrictions are such that the role of (31) is simply that of fixing some numerical coefficients, while providing also several self-consistency checks. Moreover, imposition of (31) also provides the rheonomic parametrizations of $\nabla \tau^\pm, \nabla \bar{\tau}^\pm, \nabla R$ and $\nabla \mathcal{F}$ and of the covariant derivatives $\nabla M$ and $\nabla \bar{M}$ of the auxiliary fields $M$ and $\bar{M}$, namely

\[
\begin{align*}
\nabla \tau^+ &= \nabla_+ \tau^+ e^+ + \nabla_- \tau^+ e^- + \left( \frac{1}{2} \mathcal{R} + \frac{i}{4} \mathcal{F} - \varepsilon \bar{M} \bar{M} \right) \zeta^+ + \nabla_- M \bar{\zeta}_-, \\
\nabla \bar{\tau}_+ &= \nabla_+ \bar{\tau}_+ e^+ + \nabla_- \bar{\tau}_+ e^- - \left( \frac{1}{2} \mathcal{R} + \frac{i}{4} \mathcal{F} - \varepsilon \bar{M} \bar{M} \right) \bar{\zeta}_+ + \nabla_+ M \zeta-, \\
\nabla M &= \nabla_+ Me^+ + \nabla_- Me^- - \frac{i}{2} \varepsilon (\bar{\tau}_+ \zeta^+ + \tau^+ \bar{\zeta}_+), \\
\end{align*}
\]
\[ \nabla R = \nabla_+ R e^+ + \nabla_- R e^- + \frac{i}{2} (\epsilon \nabla_- \bar{\tau} \bar{\zeta} + \epsilon \nabla_- \bar{\tau} \bar{\zeta} - \nabla_+ \tau \zeta^+ - \nabla_+ \tau \zeta^-) \]

\[ - i \bar{M} (\tau^+ \bar{\zeta} + \bar{\tau} \zeta^+) + M (\tau^- \bar{\zeta} + \bar{\tau} \zeta^-) ] , \]

\[ \nabla F = \nabla_+ F e^+ + \epsilon \nabla_- F e^- - \epsilon \nabla_- \bar{\tau} \bar{\zeta} + \nabla_- \bar{\tau} \bar{\zeta} + \nabla_+ \tau \zeta^+ - \nabla_+ \tau \zeta^- . \] (33)

These equations, in their turn, are the definitions of the supercovariantized derivatives of \( \tau^\pm, \bar{\tau}^\pm, R, F, M \) and \( \bar{M} \). Finally, the \( e^+ e^- \) sector of the Bianchi identities (31) gives the “space-time” counterparts of the Bianchi identities themselves, i.e. the formulæ for \[ [\nabla_+, \nabla_-] \Phi \] of any field \( \Phi \).

The formal correspondence between de Sitter and anti de Sitter theories is summarized by

\[ e^- \rightarrow -e^-, \quad \tau \rightarrow -\tau, \quad R \rightarrow -R, \quad F \rightarrow -F, \quad \nabla_- \rightarrow -\nabla_- . \] (34)

From (33) we can confirm that \( \epsilon = 1 \) corresponds to positive curvature, while \( \epsilon = -1 \) corresponds to negative curvature. Indeed, setting \( R = \text{const} \) and \( F = 0 \), the expressions of \( \nabla R \) and \( \nabla F \) imply either \( M = \bar{M} = 0 \) or \( \tau^\pm = \bar{\tau}^\pm = 0 \). If \( M = \bar{M} = 0 \), then \( \nabla M \) and \( \nabla \bar{M} \) also imply \( \tau^\pm = \bar{\tau}^\pm = 0 \). So, we can conclude that \( \tau^\pm = \bar{\tau}^\pm = 0 \) is in any case true. Finally, \( \nabla \tau \) implies \( R = 2 \epsilon M \bar{M} \) and \( M = \text{const} \). This also shows that one cannot move from the de Sitter to the anti de Sitter case by a continuous deformation of the expectation value of \( M, \bar{M} \).

For simplicity, from now on we set \( \epsilon = +1 \).

### 3 Coupling gravity with chiral matter

In the previous section we have derived the first ingredient we need, namely the off-shell graviton multiplet structure. In the present section we extend the rheonomic construction of chiral multiplets [22] discussed in [21] for flat superspace, to the curved superspace environment. The field content of an off-shell chiral multiplet is \( X^I, \psi^I, \bar{\psi}^I, H^I \) where \( X^I \) is a complex scalar field, \( \psi^I, \bar{\psi}^I \) are complex spin \( \pm \frac{1}{2} \) fields and \( H^I \) is a complex auxiliary scalar field. The complex conjugate fields will be denoted by a star. The index notation is \( I = (0, i), i = 1, \cdots n \). The multiplet corresponding to the value \( I = 0 \) plays a special role in coupling to supergravity, namely it is the multiplet containing the Lagrange multiplier \( \Phi \) introduced in eq.s (23) and (24).

To start our program we need the covariant derivatives for the matter fields \(^2\), which are

\[ \nabla X^I = dX^I, \]

\[ \nabla \psi^I = d\psi^I - \frac{1}{2} \omega \psi^I + \frac{i}{4} A \psi^I , \]

\[ \nabla \bar{\psi}^I = d\bar{\psi}^I + \frac{1}{2} \omega \bar{\psi}^I - \frac{i}{4} A \bar{\psi}^I , \]

\(^2\)Our notation for the covariant derivative is \( \nabla \phi = d\phi - s \omega \phi - \frac{1}{2} q A \), where \( s, q \) are the spin and the \( U(1) \) charge for the field \( \phi \).
\[ \nabla H^I = dH^I. \] (35)

From the Bianchi identities, which are easily read off (33), we find the following rheonomic parametrizations:

\[ \nabla X^I = \nabla_+ X^I e^+ + \nabla_- X^I e^- + \psi^I \zeta^- + \bar{\psi}^I \bar{\zeta}^-, \]
\[ \nabla \psi^I = \nabla_+ \psi^I e^+ + \nabla_- \psi^I e^- - \frac{i}{2} \nabla_+ X^I \zeta^+ + H^I \bar{\zeta}^-, \]
\[ \nabla \bar{\psi}^I = \nabla_+ \bar{\psi}^I e^+ + \nabla_- \bar{\psi}^I e^- - \frac{i}{2} \nabla_- X^I \bar{\zeta}^- - H^I \zeta^-, \]
\[ \nabla H^I = \nabla_+ H^I e^+ + \nabla_- H^I e^- - \frac{i}{2} \nabla_- \psi^I \bar{\zeta}^- + \frac{i}{2} \nabla_+ \bar{\psi}^I \zeta^+. \] (36)

The usual choice for the auxiliary field in the Landau-Ginzburg matter is

\[ H^I = \eta^I J^R \partial_J \bar{W}, \] (37)

\( \eta^{IJ} \) denoting a flat (constant) metric and \( W(X) \) being a (polynomial) chiral potential. If we explicitly make this choice, we also find the fermionic equation of motions from the self-consistency of the parametrization \( \nabla H^I \):

\[ \frac{i}{2} \nabla_- \psi^I - \eta^{IJ} \partial_M \partial_J \bar{W} \psi^{M^*} = 0, \]
\[ \frac{i}{2} \nabla_+ \bar{\psi}^I + \eta^{IJ} \partial_M \partial_J \bar{W} \psi^{M^*} = 0. \] (38)

Finally, from the supersymmetric variations of the fermionic field equation we find the bosonic field equation

\[ [\nabla_- \nabla_+ + \nabla_+ \nabla_-] X^I - 8 \eta^{IJ} \partial_M \partial_J \partial_L \bar{W} \psi^L \psi^{M^*} + 8 \eta^{IJ} \partial_M \partial_J \bar{W} \eta^{M^* L} \partial_L W \]
\[ - 4i \bar{M} \eta^{IJ} \partial_J \bar{W} + \tau^- \psi^I - \bar{\tau}^- \bar{\psi}^I = 0. \] (39)

The coupling of the Landau-Ginzburg matter with N=2 supergravity is described by the following Lagrangian, derived from the field equations (37), (38) and (39)

\[ \mathcal{L}_{Liouville} = \mathcal{L}_{kin} + \mathcal{L}_W, \] (40)

where \( \mathcal{L}_{kin} \) and \( \mathcal{L}_W \) are the kinetic and superpotential terms

\[ \mathcal{L}_{kin} = \eta^{IJ} (\nabla X^I - \psi^I \zeta^- - \bar{\psi}^I \bar{\zeta}^-) (\Pi^I_+ e^+ - \Pi^I_- e^-) + \eta^{IJ} (\nabla X^I - \psi^I \bar{\zeta}^- - \bar{\psi}^I \zeta^-) (\Pi^I_+ e^+ - \Pi^I_- e^-) \]
\[ + \eta^{IJ} (\Pi^I_+ \Pi^R_+ - \Pi^I_- \Pi^R_-) e^+ e^- + 2i \eta^{IJ} (-\psi^I \nabla \psi^J e^+ - \psi^J \nabla \bar{\psi}^I e^- + \bar{\psi}^I \nabla \psi^J e^- + \bar{\psi}^J \nabla \bar{\psi}^I e^-) \]
\[ + \eta^{IJ} (\nabla X^I \psi^J \zeta^- - \nabla X^J \psi^I \zeta^- - \nabla X^I \bar{\psi}^J \bar{\zeta}^- + \nabla X^J \bar{\psi}^I \bar{\zeta}^-) + \eta^{IJ} (\psi^I \bar{\psi}^J \zeta^- \bar{\zeta}^+ + \psi^J \bar{\psi}^I \bar{\zeta}^- \zeta^+) - 8 \eta^{IJ} H^I H^J e^+, \]

\[ \mathcal{L}_W = 4i (\psi^I \partial_J W \bar{\zeta}^+ e^+ + \psi^J \partial_I W \bar{\zeta}^- e^- + \bar{\psi}^I \partial_J W \zeta^- e^- + \bar{\psi}^J \partial_I W \zeta^+ e^- + \bar{\psi}^I \partial_J W \zeta^- e^- + \bar{\psi}^J \partial_I W \zeta^+ e^- + 8 \partial_I \partial_J W \psi^I \bar{\psi}^J \bar{\zeta}^- e^- + 4i (MW - M\bar{W}) e^+ e^- + 2 \bar{W} \bar{\zeta}^- \zeta^- - 2W \bar{\zeta}^- \zeta^- + (8 H^I \partial_I W + 8 H^J \partial_J W) e^+ e^- - 8 \bar{W} \bar{\zeta}^- \zeta^- - 2W \bar{\zeta}^- \zeta^- + (8 H^I \partial_I W + 8 H^J \partial_J W) e^+ e^- . \] (41)
The fields $\Pi_I^\pm$ and $\Pi^{I*}_\pm$ are auxiliary fields for the first order formalism: their equation of motion equates them to the supercovariant derivatives of the $X$-fields,

$$
\Pi_I^\pm = \nabla_\pm X_I^I, \quad \Pi_{I*}^\pm = \nabla_\pm X_{I*}^I. \tag{42}
$$

Substitution of these expressions in $L_{kin}$ gives the usual second order Lagrangian. The rheonomic parametrizations of $\nabla_\pm \Pi_I^\pm$ and $\nabla_\pm \Pi_{I*}^\pm$ are derived from the Bianchi identities and the rheonomic parametrizations (36), in the same way as (33) are derived from the Bianchi identities (31) and the rheonomic parametrizations (32).

### 4 N=2 Liouville gravity

Let us consider the chiral multiplet labelled with the index $I = 0$. We call it “dilaton” multiplet. For convenience, we relabel the dilaton multiplet as

$$(X^0, X^0, \psi^0, \bar\psi^0, H^0, H^0) \rightarrow (X, \bar X, \lambda_-, \lambda_+, \bar\lambda^-, \bar\lambda^+, H, \bar H). \tag{43}$$

The $N = 2$ extension of the Lagrangian $(X + \bar X)R$ is given by

$$
L_1 = (X + \bar X)R - \frac{i}{2}(X - \bar X)F - 2\lambda_-\rho^- + 2\lambda_+\rho^+ + 2\bar\lambda^-\bar\rho_- - 2\bar\lambda^+\bar\rho_+ \\
- 4i\bar MHe^+e^- + 4i\bar MHe^+e^-. \tag{44}
$$

Let us remind the reader how a supersymmetric Lagrangian is constructed in the rheonomic framework. It is sufficient to find an $L$ that satisfies

$$
\nabla L = dL = 0. \tag{45}
$$

In checking this equation one has to use the rheonomic parametrizations (32) and (36) together with the definitions (26) and (35). $L_1$ was determined starting from the first term $(X + \bar X)R$ and guessing the other ones in order to satisfy (44).

One can pass from the second order formalism to the first order one by adding the term

$$
L_T = p_+T^+ + p_-T^-, \tag{46}
$$

where $p_+, p_-$ are (bosonic) Lagrangian multipliers implementing the torsion constraint $T^\pm = 0$. $L_T$ is clearly supersymmetric (the supersymmetry variation of the spin connection is still determined from the variations of zweibein and gravitini: this is the so-called 1.5 order formalism). Moreover, one can add to eq. (44) a “cosmological constant term” compatible with the N=2 local supersymmetry

$$
L_2 = (MX + \bar M\bar X)e^+e^- + \lambda_-\tilde\zeta_+e^+ - \lambda_+\tilde\zeta_-e^+ + \tilde\lambda^-\zeta^+e^- - \tilde\lambda^+\zeta^-e^- \\
+ \frac{i}{2}X\zeta^+\tilde\zeta_+ + \frac{i}{2}\bar X\zeta^-\tilde\zeta_- + 2i(\bar H - H)e^+e^-, \tag{47}
$$

17
so that the total Lagrangian is \( \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \).

The equations for the auxiliary fields are

\[
H = -\frac{i}{4} \bar{X}, \quad \bar{H} = \frac{i}{4} X, \quad M = \bar{M} = -\frac{1}{2}.
\]

Using the equation of motions of \( H, \bar{H} \) and \( X + \bar{X} \), we get precisely a de Sitter supergravity with cosmological constant \( \Lambda = \frac{1}{2} \). The field strength \( F \), on the other hand, is set to zero by the \( X - \bar{X} \) field equation.

Notice that when in the matter Lagrangian (41) the index \( I \) takes the value 0 and \( W = -\frac{i}{4} X^0 \), then \( \mathcal{L}_W \) coincides with the de Sitter term \( \mathcal{L}_2 \).

To conclude this section, let us show that the kinetic term of the dilaton multiplet can be produced from the Poincaré Lagrangian \( \mathcal{L}_1 \) with some field redefinitions. One can perform the substitutions

\[
e^+ \rightarrow e^+ e^{-\frac{i}{4}(X+\bar{X})}, \quad e^- \rightarrow e^- e^{-\frac{i}{4}(X+\bar{X})},
\]

\[
M \rightarrow (M + i\bar{H}) e^{\frac{i}{4}(X+\bar{X})}, \quad \bar{M} \rightarrow (\bar{M} - iH) e^{\frac{i}{4}(X+\bar{X})},
\]

\[
\zeta^+ \rightarrow e^{-\frac{i}{4}X}(\zeta^+ - i\lambda_- e^+), \quad \zeta^- \rightarrow e^{-\frac{i}{4}X}(\zeta^- + i\lambda_+ e^-),
\]

\[
\bar{\zeta}^+ \rightarrow e^{-\frac{i}{4}\bar{X}}(\bar{\zeta}^+ - i\bar{\lambda}_- e^+), \quad \bar{\zeta}^- \rightarrow e^{-\frac{i}{4}\bar{X}}(\bar{\zeta}^- + i\bar{\lambda}_+ e^-),
\]

\[
\lambda_- \rightarrow \lambda_- e^{\frac{i}{4}X}, \quad \lambda^+ \rightarrow \lambda^+ e^{\frac{i}{4}X},
\]

\[
\bar{\lambda}^- \rightarrow \bar{\lambda}^- e^{\frac{i}{4}\bar{X}}, \quad \bar{\lambda}^+ \rightarrow \bar{\lambda}^+ e^{\frac{i}{4}\bar{X}},
\]

and

\[
\Omega \rightarrow \Omega - \frac{1}{2} [\nabla^+ \bar{X} e^+ - \nabla^- \bar{X} e^- - \lambda_+ \zeta^+ + \bar{\lambda}^+ \bar{\zeta}^-],
\]

\[
\bar{\Omega} \rightarrow \bar{\Omega} - \frac{1}{2} [\nabla^+ X e^+ - \nabla^- X e^- + \lambda_- \zeta^- - \bar{\lambda}^- \bar{\zeta}^+],
\]

where \( \Omega = \omega - \frac{i}{2} A, \bar{\Omega} = \omega + \frac{i}{2} A \). Then, the Poincaré Lagrangian \( \mathcal{L}_1 \) (44) goes into

\[
\mathcal{L}_1 + \mathcal{L}_{kin},
\]

\( \mathcal{L}_{kin} \) being the kinetic Lagrangian of the dilaton multiplet, the first formula of (41) with \( I \) restricted to the value 0. As one can easily check, the replacement of \( \omega = \frac{1}{2}(\Omega + \bar{\Omega}) \) implied by (49) is consistent with the preservation of vanishing torsions.

5 Gauge free algebra of the N=2 Liouville theory

The local symmetries of the N=2 Liouville theory are: diffeomorphism, local lorentz rotations, supersymmetries and U(1) gauge symmetry. The procedure for writing down the free BRST algebra [8, 9, 15] is straightforward, once the curvature definitions and
the rheonomic parametrizations are BRST extended to ghost forms, by introducing the ghosts of the local symmetries:

\[
\begin{align*}
\dot{e}^+ &= e^+ + C^+, & \dot{e}^- &= e^- + C^-, \\
\dot{\zeta}^+ &= \zeta^+ + \Gamma^+, & \dot{\zeta}^- &= \zeta^- + \tilde{\Gamma}^-, \\
\dot{\omega} &= \omega + C^0, & \dot{A} &= A + C.
\end{align*}
\] (52)

The exterior derivative is BRST extended to \( \tilde{d} = d + s \). The BRST variations of the fields are easily derived by selecting out the correct ghost number sector in the BRST extensions of formulæ (26), (32) (graviton multiplet) and (35), (36) (dilaton multiplet). We have, in particular, for the graviton multiplet,

\[
\begin{align*}
se^+ &= -\nabla C^+ - C^0e^+ + \frac{i}{2}C^+\Gamma^- + \frac{i}{2}\Gamma^+\zeta^-, \\
se^- &= -\nabla C^- + C^0e^- + \frac{i}{2}C^-\tilde{\Gamma}^- + \frac{i}{2}\tilde{\Gamma}^+\tilde{\zeta}^-, \\
s\zeta^+ &= -\nabla \Gamma^+ - \frac{1}{2}C^0\zeta^+ - \frac{i}{4}C\tau^+(C^+e^- + e^+C^-) - M(\tilde{\Gamma}^+e^- + \tilde{\zeta}^-C^+), \\
s\tilde{\zeta}^+ &= -\nabla \tilde{\Gamma}^+ + \frac{1}{2}C^0\tilde{\zeta}^+ + \frac{i}{4}C\tau^+(C^+e^- + e^+C^-) + M(\Gamma^-e^- + \zeta^-C^-), \\
s\omega &= -dC^0 + \mathcal{R}(C^+e^- + e^+C^-) + \frac{i}{2}C^-(\tau^+\zeta^- + \tau^-\zeta^+) + \frac{i}{2}C^+(\tilde{\tau}^+\tilde{\zeta}^- + \tilde{\tau}^-\tilde{\zeta}^+) \\
&\quad + \frac{i}{2}e^-(\tau^+\Gamma^- + \tau^-\Gamma^+) + \frac{i}{2}e^+(\tilde{\tau}^+\tilde{\Gamma}^- + \tilde{\tau}^-\tilde{\Gamma}^+) \\
&\quad + \frac{i}{2}M(\zeta^+\tilde{\Gamma}^- + \Gamma^+\tilde{\zeta}^- + \tilde{\Gamma}^+\tilde{\zeta}^-), \\
\lambda_+ - \tilde{\lambda}^- &= -dC + \mathcal{F}(C^+e^- + e^+C^-) - (\tau^+\zeta^- - \tau^-\zeta^+)C^- \\
&\quad + (\tilde{\tau}^-\tilde{\zeta}^+ - \tilde{\tau}^+\tilde{\zeta}^-)C^+ - (\tau^+\Gamma^- - \tau^-\Gamma^+)e^- \\
&\quad + (\tilde{\tau}^-\tilde{\Gamma}^+ - \tilde{\tau}^+\tilde{\Gamma}^-)e^+ \\
&\quad + M(\zeta^-\tilde{\Gamma}^- + \Gamma^-\tilde{\zeta}^- + \tilde{\Gamma}^-\tilde{\zeta}^- + \tilde{\Gamma}^+\tilde{\zeta}^- - \tilde{\Gamma}^+\tilde{\zeta}^-). \\
\end{align*}
\] (53)

On the other hand, the BRST transformations of the fields of the dilaton multiplet are

\[
\begin{align*}
sX &= \nabla_+ XC^+ + \nabla_- XC^- + \lambda_+ \Gamma^- + \tilde{\lambda}^- \tilde{\Gamma}^-, \\
s\lambda_+ - \tilde{\lambda}^- &= \nabla_+ \lambda_- C^+ + \nabla_- \lambda_- C^- - \frac{i}{2}\nabla_+ \lambda \Gamma^+ + H\tilde{\Gamma}^-, \\
s\tilde{\lambda}^- &= \nabla_+ \tilde{\lambda}^- C^+ + \nabla_- \tilde{\lambda}^- C^- + \frac{i}{2}\nabla_+ \tilde{\lambda}^+ \tilde{\Gamma}^+ - H\Gamma^-, \\
sH &= \nabla_+ HC^+ + \nabla_- HC^- - \frac{i}{2}\nabla_- \lambda \tilde{\Gamma}^- + \frac{i}{2}\nabla_+ \tilde{\lambda}^+ \Gamma^-.
\end{align*}
\] (54)
6  Topological Twist of the N=2 Liouville Theory

In this section we perform the topological twist of the theory. The formal set-up is analogous to the one in four dimensions \cite{8, 9}. One has to change consistently, the spin, the BRST charge and the ghost number. In particular, the new BRST charge is obtained by the so-called topological shift, which is a simple redefinition of the supersymmetry ghosts that get ghost number zero. Due to the existence of Majorana-Weyl spinors in two dimensions, one has two possibilities, known in the literature as the A and B twists \cite{12, 21}.

The geometrical and physical meaning of the two types of twists was discovered at the level of globally supersymmetric N=2 matter theories. As noticed in eq. (41), the most general interaction of a set of chiral multiplets involves two separately supersymmetric Lagrangian terms, the kinetic term $L_{\text{kin}}$ and the superpotential term $L_{W}$. The choice between the A and B twists decides which term is BRST nontrivial and which one is BRST exact. In the A twist the nontrivial BRST cohomology is carried by $L_{\text{kin}}$, while in the B twist it is carried by $L_{W}$. In the first case, the topologically meaningful coupling parameters are those corresponding to the Kähler class deformations of the target space metric, the correlation functions being instead independent of the deformation parameters of the superpotential. In the B twist the situation is reversed.

The above considerations apply both to globally and locally supersymmetric theories. However, in presence of supergravity and as far as the A twist is concerned, $L_{W}$ cannot be set to zero from the beginning, since it contains the de Sitter term $L_{2}$ (formally obtainable from $L_{W}$ with $W = -\frac{i}{4}X^{0}$) and the parameter $a$ that can be put in front of $L_{2}$ is the cosmological constant, the sign of which has to be compatible with the Euler characteristic $\int R = 2(1 - g)$.

Technically, the A and B twists emerge as follows. We first notice that the Lagrangian (44) of Poincaré gravity, possesses a global $R$-symmetry [which will be denoted by $U(1)'$], under which the fields transform with the charges shown in table 10. $U(1)'$ is not a local symmetry and it is not even a global symmetry for the de Sitter Lagrangian (47). In general R-symmetries and R-dualities play a crucial role [9] in the topological twist of the N=2 theories. Depending on the choice of the twist (A or B), the new Lorentz group is defined as a combination of the old one with the $U(1)'$ or $U(1)$ symmetry; viceversa for the ghost number.

For the A twist the new assignments and the topological shift are

\begin{align*}
\text{spin}' &= \text{spin} + U(1)', \\
\text{ghost}' &= \text{ghost} + 2U(1),
\end{align*}

\begin{align*}
\Gamma^{+} &\to \Gamma^{+} + \alpha, \\
\tilde{\Gamma}_{-} &\to \tilde{\Gamma}_{-} + \beta,
\end{align*}

while for the B twist they are

\begin{align*}
\text{spin}' &= \text{spin} + U(1), \\
\text{ghost}' &= \text{ghost} + 2U(1)',
\end{align*}

\begin{align*}
\Gamma^{+} &\to \Gamma^{+} + \alpha, \\
\tilde{\Gamma}_{+} &\to \tilde{\Gamma}_{+} + \beta.
\end{align*}

$\alpha$ and $\beta$ are the so-called brokers \cite{9}. They are to be treated formally as constant ($d\alpha = d\beta = 0$) and their (purely formal) role is to bring the correct contributions of spin
and ghost number to the fields (see the last column of table 10). Their quantum numbers are given in the table.

In this paper we focus on the A twist. The topological theory that emerges from our analysis is apt to perform the gravitational dressing of the topological theories dealing with Kähler class deformations [21]. The gravitational dressing of the complex structure deformations need the B twist of the N=2 Liouville theory, whose analysis is postponed to future work.

The shift produces a new BRST operator $s'$ which equals $s + \delta_T$, $\delta_T$ being the topological variation (known as $Q_s$ in conformal field theory). On the graviton multiplet, $\delta_T$ acts as

$$
\begin{align*}
\delta_T e^+ &= \frac{i}{2} \alpha \zeta^-, \\
\delta_T \zeta^+ &= -\frac{1}{2} \omega \alpha - \frac{i}{4} A \alpha - M \beta e^+ \equiv B_1 \alpha, \\
\delta_T \zeta^- &= 0, \\
\delta_T M &= -\frac{i}{2} \tilde{\tau}^+ \alpha, \\
\delta_T \omega &= \frac{i}{2} M \zeta^- \beta + \frac{i}{2} \bar{M} \bar{\zeta}^+ + \frac{i}{2} e^- \tau^- \alpha + \frac{i}{2} \bar{e}^+ \tilde{\tau}^+ \beta, \\
\delta_T A &= M \zeta^- \beta - \bar{M} \bar{\zeta}^+ + \tau^- \alpha \bar{e}^+ - \tilde{\tau}^+ \beta \bar{e}^+.
\end{align*}
$$

Taking into account that the BRST algebra closes off-shell, we see that $B_1$ and $B_2$ play the role of Lagrange multipliers, since they are the BRST variations of the antighosts $\zeta^+$ and $\bar{\zeta}^-$. $B_1$ and $B_2$ can be considered as redefinitions of $A$, $M$ and $\bar{M}$. Indeed, since $M$ and $\bar{M}$ have spin 1 and −1 after the twist, $Me^+$ and $\bar{M}e^-$ can be considered as one forms. In particular, we have shown that the graviphoton $A$ belongs to $B_{\text{gauge-fixing}}$. On the other hand, it is clear that the gauge-free topological algebra is that of $SL(2, \mathbb{R})$, since the above formulæ show that the topological symmetry is the most continuous deformation of the zweibein.

On the dilaton multiplet, $\delta_T$ is

$$
\begin{align*}
\delta_T X &= \tilde{\lambda}^- \beta, \\
\delta_T \lambda^- &= -\frac{i}{2} \nabla_+ X \alpha + H \beta \equiv H_1 \alpha, \\
\delta_T \lambda^+ &= 0, \\
\delta_T H &= \frac{i}{2} \nabla_+ \tilde{\lambda}^- \alpha,
\end{align*}
$$

$H_1$ and $H_2$ are also Lagrange multipliers, redefinitions of $H$ and $\bar{H}$.

Finally, the topological variation of the brokers vanishes, but nilpotence of $s'$ and $s$ requires

$$
\begin{align*}
\delta_T \alpha &= -\frac{1}{2} C^0 \alpha - \frac{i}{4} C \alpha = s\alpha, \\
\delta_T \beta &= \frac{1}{2} C^0 \beta - \frac{i}{4} C \beta = s\beta.
\end{align*}
$$

In other words, even if formally, $\alpha$ and $\beta$ have to be considered as sections with definite spin and $U(1)$ charge.

From the above formulæ it is simple to check that $\delta_T$ is nilpotent, $\delta_T^2 = 0$, as expected. Summarizing, we have

$$
\begin{align*}
s' &= \delta_T + s, \\
s'^2 &= s^2 = \delta_T^2 = s\delta_T + \delta_T s = 0.
\end{align*}
$$
Using the above formulae and the notation shown in the last column of table [10], we can write the full Lagrangian $\mathcal{L}$ as the topological variation of a suitable gauge fermion $\Psi$ plus a total derivative term. Precisely,

$$\mathcal{L}_1 = \delta_T \Psi_1 + 2\nabla(XM_-e^- + \bar{X}M_+e^+), \quad \mathcal{L}_2 = \delta_T \Psi_2 + \nabla(\bar{X}e^+ - Xe^-),$$

(59)

where

$$\Psi_1 = -2Xd\bar{\xi} + 2\bar{X}d\xi - 4i\chi_+M_-e^+e^- - 4i\chi_-M_+e^+e^- + 2XB_2\bar{\xi} - 2\bar{X}B_1\xi,$$

$$\Psi_2 = (\bar{X}e^+ - Xe^-)(\bar{\xi} - \xi) - 2i(\chi_+ + \chi_-)e^+e^-.$$

(60)

From the formula of $\Psi_2$, it is apparent that $U(1)'$ and correspondingly spin' are violated by $\mathcal{L}_2$. On the other hand, $\mathcal{L}_2$ is purely a gauge-fixing term, so that this violation does not affect the Lorentz symmetry in the physical correlators. It can be thought as the choice of a noncovariant gauge-fixing.

Let us analyze the gauge-fixing conditions of the twisted theory. We take $\mathcal{L}_{tot} = \mathcal{L}_1 - 2a\mathcal{L}_2$, $\mathcal{L}_1$ and $\mathcal{L}_2$ being given by (47). The Lagrange multipliers $A, M-M$ and $H-H$ impose the following constraints

$$\nabla_+(X - \bar{X}) = \nabla_-(X - \bar{X}) = 0, \quad H = \frac{i}{2}a\bar{X}, \quad \bar{H} = -\frac{i}{2}aX, \quad M = \bar{M} = a.$$

(61)

A check of consistency is that the $\delta_T$ variations of the constraints (61) imposed by the Lagrange multipliers are the field equations of the topological ghosts $\zeta^-, \bar{\zeta}^+, \lambda_+$ and $\bar{\lambda}^-$, obtained from the variations of $\mathcal{L}_{tot}$ with respect to the corresponding antighosts $\lambda_-, \bar{\lambda}^+, \zeta^+$ and $\bar{\zeta}^-$, i.e.

$$\tau^- = 0, \quad \bar{\tau}^+ = 0,$$

$$\nabla_+\lambda_- = 0, \quad \nabla_-\lambda_- = a\bar{\lambda}^+, \quad \nabla_+\bar{\lambda}^- = -a\lambda_+, \quad \nabla_-\bar{\lambda}^- = 0.$$

(62)

To verify this, one has to keep into account that the $\omega$ field equation gives $p_+ = -\nabla_+(X + \bar{X})$, and $p_- = \nabla_-(X + \bar{X})$.

The observables of the topological theory are easily derived, as in the case of the Verlinde and Verlinde model, from the descent equations $d\hat{R}^n = 0$, $\hat{R} = R + \psi_0 + \gamma_0$ being the BRST extension of the curvature $R$. In particular, the local observables are

$$\sigma_n^{(0)}(x) = \gamma_n^0(x),$$

(63)

as anticipated in the introduction. On the other hand, the field strength $F$ does not provide any new observables, due to the fact that $A \in B_{gauge-fixing}$. 

22
7 The conformal field theory associated with N=2 Liouville gravity

In this section, we analyse N=2 Liouville gravity in detail. We gauge-fix it and show that, in the limit $a^2 \to 0$, it reduces to a conformal field theory of vanishing total central charge, summarized by formulæ (88) and (97). The total central charge is the sum of the central charge of the Liouville system, equal to 6, and that of the ghost system, equal to $-6$.

We start from the rheonomic Lagrangian of Poincaré N=2 D=2 supergravity, that we rewrite here for convenience,

$$L_1 = (X + \bar{X}) R - \frac{i}{2} (X - \bar{X}) (F - 2 \lambda_\rho^+ + 2 \lambda^+ \rho - 2 \lambda^- \rho - 2 \lambda^+ \bar{\rho} + 4 i (M \bar{H} - \bar{M} H) E^z E^\bar{z}.$$ (64)

In this section, we use the notation $E^z$ and $E^\bar{z}$ instead of $e^+$ and $e^-$; we do this for the sake of a tensorial notation that is useful in the gauge fixed theory. We shall have tensor indices $t^z \cdots \bar{z} \cdots \bar{\bar{z}}$ that are raised and lowered with the flat metric $\hat{g}^{z \bar{z}} = 1$ and spinor indices $s^\pm$ and $\tilde{s}^\pm$ such that $s^+ - s^- \sim t^z$ and $\tilde{s}^+ - \tilde{s}^- \sim t^\bar{z}$. In this way it is immediate to read the spin assignments of the fields.

We also recall that in order to deal with the torsions, we have to add a term

$$\pi_z T^z + \pi_{\bar{z}} T^\bar{z},$$ (65)

that allows to treat the spin connection $\omega$ as an independent variable.

In the case of locally supersymmetric theories, one has to perform the topological twist on the BRST quantized version of the theory, as discussed in detail in ref. [8]. We have developed the full gauge-free BRST algebra of N=2 Liouville theory in section 5. Now we proceed to gauge-fix deformations, Lorentz rotations, supersymmetries and local $U(1)$ gauge transformations. Then we discuss the gauge-fixed BRST theory.

Diffeomorphisms and Lorentz rotations are fixed by choosing the conformal gauge

$$E^z \land dz = 0, \quad E^z \land d\bar{z} + E^{\bar{z}} \land dz = 0, \quad E^{\bar{z}} \land d\bar{z} = 0.$$ (66)

These conditions permit to express the zweibein as

$$E^z = e^{\varphi(z, \bar{z})} dz, \quad E^{\bar{z}} = e^{\varphi(z, \bar{z})} d\bar{z},$$ (67)

where $\varphi(z, \bar{z})$ is the conformal factor, which is to be identified with the Liouville quantum field.

Supersymmetries are fixed by extending the conformal gauge by means of the conditions

$$\zeta^+ \land E^z = 0, \quad \zeta^- \land E^z = 0, \quad \tilde{\zeta}^+ \land E^{\bar{z}} = 0, \quad \tilde{\zeta}^- \land E^{\bar{z}} = 0,$$ (68)

that, together with (66) make the so-called superconformal gauge. (68) corresponds to the usual condition $\gamma^\mu \zeta^A_{\mu} = 0, A = 1, 2$ and permit to express the gravitini as

$$\zeta^+ = \eta^+ e^\varphi dz, \quad \zeta^- = \eta^- e^\varphi dz, \quad \tilde{\zeta}^+ = \eta^+ e^\varphi d\bar{z}, \quad \tilde{\zeta}^- = \eta^- e^\varphi d\bar{z},$$ (69)
where \( \eta(z, \bar{z}) \) and \( \eta(z, \bar{z}) \) are anticommuting fields of spin \( 1/2 \) and \( -1/2 \) (the super-partners of the Liouville field \( \varphi \)).

The \( U(1) \) gauge transformations have to be treated carefully. The critical N=2 string possesses two local \( U(1) \) gauge-symmetries, that permit to gauge-fix the graviphoton \( A \) to zero and introduce the two \( b\)-\( c \) systems a la Faddeev-Popov, one in the left moving sector and one in the right moving one. In such a way, a complete chiral factorization into two superconformal field theories (left and right moving) is achieved. The theory that we are now dealing with, on the other hand, possesses a single local \( U(1) \) symmetry, the \( U(1)' \) R-symmetry being only global. From a field theoretical point of view, it is not immediate to see how a pair of \( b\)-\( c \) systems can be introduced a la Faddeev-Popov. Indeed, enforcing the usual Lorentz gauge \( \partial \mu A_\mu = 0 \) produces a second order ghost-antighost system. That is why we pay a particular attention to this fact. We arrange things in the correct way by using a trick. Let us introduce an additional trivial BRST system (a “one dimensional topological \( \sigma \)-model”) \( \{ \xi, C' \} \), \( \xi \) being a ghost number zero scalar and \( C' \) being a ghost number one scalar. Their BRST algebra is chosen to be trivial, namely

\[
s\xi = C', \quad sC' = 0.
\]

The meaning of this BRST system is the gauging of the R-symmetry \( U(1)' \). Indeed, \( U(1)' \), which is only a global symmetry of the starting theory, becomes a local symmetry in the gauge-fixed version of the same theory. This will become clear later on, when the complete factorization between left and right moving sectors will be apparent. We fix both the \( U(1) \) gauge symmetry and the trivial symmetry (70) by choosing the following two gauge-fixings

\[
A_z - \partial_z \xi = 0, \quad A_z + \partial_z \xi = 0.
\]

(71)

corresponding to \( A = *d\xi \), where \( A = A_z dz + A_\bar{z} d\bar{z} \).

With obvious notation, the \( \bar{C} \) and \( \bar{\Gamma} \) fields being antighosts, the gauge-fermion \( \Psi \) is

\[
\Psi = \Psi_{\text{diff}} + \Psi_{\text{susy}} + \Psi_{\text{gauge}}
\]

\[
= \bar{C}_{z\bar{z}} E^z \wedge d\bar{z} + \bar{C}_{z\bar{z}} (E^z \wedge d\bar{z} + E^{\bar{z}} \wedge dz) + \bar{C}_{z\bar{z}} E^{\bar{z}} \wedge d\bar{z}
\]

\[
+ \bar{\Gamma}_{+z} \zeta^+ \wedge E^z + \bar{\Gamma}_{-z} \zeta^- \wedge E^z + \bar{\Gamma}_{+\bar{z}} \bar{\zeta}^+ \wedge E^{\bar{z}} + \bar{\Gamma}_{-\bar{z}} \bar{\zeta}^- \wedge E^{\bar{z}}
\]

\[
+ \bar{\zeta} \wedge (A - *d\xi),
\]

(72)

\( \bar{C} \) being a one form, \( \bar{C} = \bar{C}_z dz + \bar{C}_{\bar{z}} d\bar{z} \).

The gauge-fixed Poincaré Lagrangian is thus

\[
\mathcal{L} = \mathcal{L}_1 + s\Psi.
\]

(73)

In writing the gauge-fixed Lagrangian we proceed as follows. The Lagrange multipliers of the algebraic gauge-fixings (i.e. diffeomorphisms, Lorentz rotations and supersymmetries) will be functionally integrated away, thus solving the gauge-fixing conditions (66) and (68). The Lagrange multipliers \( P_z \) and \( P_{\bar{z}} \) of the \( U(1) \) and \( U(1)' \) gauge-fixings (\( s\bar{C}_z = P_z \) and \( s\bar{C}_{\bar{z}} = P_{\bar{z}} \)) will be conveniently retained for now, since the corresponding
gauge-fixings (71) contain derivatives of the fields. The remaining part of the Lagrangian can thus be greatly simplified by using the algebraic gauge-fixing conditions. Moreover, noticing that $\xi^+\xi^- = \xi^+\xi^- = 0$ on the gauge-fixing condition (79), the torsion constraints imposed by the Lagrange multipliers $\pi_\pm$ and $\pi_\mp$ become algebraic “gauge-fixing conditions” on $\omega$ that are solved by
\begin{equation}
\omega = \partial_\nu \varphi \, dz - \partial_\nu \varphi \, d\bar{z}.
\end{equation}

Notice that such an $\omega$ does not depend on the gravitini. The Lorentz ghost $C^0$ appears only algebraically. Consequently, we can eliminate $\bar{C}_{\mp\bar{z}}$, by expressing $C^0$ in terms of the other fields
\begin{equation}
2e^\varphi C^0 d\bar{z} \wedge d\bar{z} = -\nabla C^z \wedge d\bar{z} - \nabla C^\bar{z} \wedge d\bar{z}
\end{equation}
\begin{equation}
+ \frac{i}{2}[(\Gamma^+\xi^- + \xi^+\Gamma^-) \wedge d\bar{z} + (\xi^+_+ \Gamma^- + \Gamma^+_+ \xi^-) \wedge d\bar{z}].
\end{equation}

Due to this, one finds
\begin{equation}
s\Psi_{\text{diff}} + s\Psi_{\text{susy}} = \bar{C}_{\mp\bar{z}} \nabla C^z \wedge d\bar{z} + \bar{C}_{zz} \nabla C^\bar{z} \wedge d\bar{z}
\end{equation}
\begin{equation}
+ \bar{\Gamma}_{zz} \nabla(\Gamma^+ E^z + \xi^+ C^z) + \bar{\Gamma}_\mp \nabla(\Gamma^- E^z + \xi^- C^z)
\end{equation}
\begin{equation}
+ \Gamma^+_\bar{z} \nabla(\Gamma^+ E^\bar{z} + \xi^+ C^\bar{z}) + \Gamma^+_{\bar{z}} \nabla(\Gamma^- E^\bar{z} + \xi^- C^\bar{z}).
\end{equation}

It is convenient to introduce the following substitutions
\begin{align}
\eta^{\pm\bar{z}} &= \eta^{\pm\bar{z}} e^{\frac{i}{2}(\varphi \mp \frac{3}{4} \xi)}, \\
\lambda^{\pm \bar{z}} &= \lambda^{\pm \bar{z}} e^{\frac{i}{2}(\varphi \pm \frac{3}{4} \xi)}, \\
C^{\mp \bar{z}} &= C^{\mp \bar{z}} e^{-\varphi}, \\
\bar{C}_{\mp \bar{z}} &= \bar{C}_{\mp \bar{z}} e^{\varphi}, \\
\Gamma^{\pm \bar{z}} &= \Gamma^{\pm \bar{z}} e^{-\frac{1}{2}(\varphi \pm \frac{3}{4} \xi)} - \eta^{\pm \bar{z}} C^{\mp \bar{z}}, \\
\bar{\Gamma}^{\pm \bar{z}} &= \bar{\Gamma}^{\pm \bar{z}} e^{-\frac{1}{2}(\varphi \pm \frac{3}{4} \xi)} - \eta^{\pm \bar{z}} C^{\mp \bar{z}}, \\
\Gamma^{\pm \bar{z}} &= \Gamma^{\pm \bar{z}} e^{\frac{1}{2}(\varphi \pm \frac{3}{4} \xi)}, \\
\bar{\Gamma}^{\pm \bar{z}} &= \bar{\Gamma}^{\pm \bar{z}} e^{\frac{1}{2}(\varphi \pm \frac{3}{4} \xi)},
\end{align}
which are also allowed in the functional integral, since the Jacobian determinant is one.

The gauge-fixed versions of the gravitini curvatures give
\begin{equation}
\tau^+ + M\eta_{\mp \bar{z}} = -e^{-\frac{3}{4}\varphi \pm \frac{3}{4} \xi} \partial_\nu \eta^{\pm\nu},
\end{equation}
and similar relations, that provide gauge-fixed formula for the supercovariantized derivatives. On the other hand, the field equation of $\omega$ gives expressions for $\pi_\pm$ and $\pi_{\mp}$, that will be useful for computing the BRST charge $Q_{BRST}$. An alternative way of finding $\pi_\pm$ and $\pi_{\mp}$ is that of imposing the independence of $Q_{BRST}$ from $C^0$.

With $\pi = 1/2 (X + \bar{X})$ and $\chi = i/2 (X - \bar{X})$, we have
\begin{equation}
L_1 + s\Psi_{\text{diff}} + s\Psi_{\text{susy}} = -4\pi \partial_\nu \partial_{\bar{\nu}} \varphi + \chi (\partial_\nu A_{\bar{z}} - \partial_{\bar{\nu}} A_z) - 2\lambda^{\nu \pm \bar{z}} \partial_\nu \eta^{\nu \pm \bar{z}} + 2\lambda^{\nu \mp \bar{z}} \partial_\nu \eta^{\nu \mp \bar{z}}
\end{equation}
\begin{equation}
- 2\lambda^{\nu \pm \bar{z}} \partial_\nu \eta^{\nu \pm \bar{z}} + 2\lambda^{\nu \mp \bar{z}} \partial_\nu \eta^{\nu \mp \bar{z}} + C^{\nu \pm \bar{z}} \partial_\nu C^{\nu \mp \bar{z}} - C^{\nu \mp \bar{z}} \partial_\nu C^{\nu \mp \bar{z}}
\end{equation}
Integrating away both superconformal field theory. To show that this is indeed the case, we compute the energy-current $i\delta \bar{\mathcal{C}} = L_{\text{local}} - z\partial z \bar{\mathcal{C}} = -4 + \partial\bar{z} \partial z \bar{\mathcal{C}} + \eta_{\bar{z}}^{(2)} \partial z \bar{\mathcal{C}} + \eta_{\bar{z}}^{(2)} \partial z \bar{\mathcal{C}} = \bar{\mathcal{C}}_{z}[\partial z (C + C') + C''(\partial z A_{\bar{z}} - \partial z A_{\bar{z}}) + \partial z \eta_{\bar{z}}^{(2)} \Gamma^{+} - \partial z \eta_{\bar{z}}^{(2)} \Gamma^{+}]

\begin{align}
\frac{i}{4} (A_{\bar{z}} + \partial z \xi) (2\bar{\lambda}^{*} \eta_{\bar{z}}^{(2)} + 2\bar{\lambda}^{*} \eta_{\bar{z}}^{(2)} - \bar{\mathcal{C}}_{\bar{z}} \Gamma^{+} + \bar{\mathcal{C}}_{\bar{z}} \Gamma^{+})
\end{align}

Now, it remains to deal with the term $s\Psi_{gauge}$. We find

\begin{align}
 s\Psi_{gauge} = P_{\bar{z}} (A_{\bar{z}} - \partial z \xi) - P_{\bar{z}} (A_{\bar{z}} - \partial z \xi)
\end{align}

Integrating away both $P_{\bar{z}}, P_{\bar{z}}$ and $A_{\bar{z}}, A_{\bar{z}}$, we can use the corresponding field equations. Defining

\begin{align}
 c = C + C' - 2C'' \partial z \xi + \eta_{\bar{z}}^{(2)} \Gamma^{+} - \eta_{\bar{z}}^{(2)} \Gamma^{+}, \quad \bar{C}_{\bar{z}} = b_{\bar{z}},
\end{align}

the total gauge-fixed Poincaré Lagrangian takes the form

\begin{align}
 \mathcal{L}_{\text{Poincaré}} = & -4\pi \partial z \partial \bar{z} \varphi + 2\chi \partial z \partial \bar{z} \xi - 2\lambda_{\bar{z}} \partial z \eta_{\bar{z}}^{(2)} + 2\lambda_{\bar{z}} \partial z \eta_{\bar{z}}^{(2)}

\end{align}

Again, the Jacobian determinant corresponding to $[\Sigma]$ is one.

It is natural to conjecture that Poincaré N=2 supergravity corresponds to an N=2 superconformal field theory. To show that this is indeed the case, we compute the energy-momentum tensor $T_{z\bar{z}}$, the supercurrents $G_{z\bar{z}}$ and $G_{z\bar{z}}$, the $U(1)$ current $J_{\bar{z}}$. In order to do this, we first compute the BRST charge $Q^{BRST} = \oint J^{BRST}_{\bar{z}} dz$, $J^{BRST}_{\bar{z}}$ denoting the left moving BRST current. Acting with $Q^{BRST}$ on the various antighost fields it is then simple to derive the “gauge-fixings”, which are, in our case, the N=2 currents. Since the BRST symmetry is a global symmetry and the gauge-fixed action is BRST-invariant, the BRST current $J^{BRST}$ can be found by performing a local BRST transformation $\delta_{BRST} = \kappa(x) s$, where $s$ is the BRST operator and $\kappa(x)$ is a point-dependent ghost number $-1$ scalar parameter. The variation of $\mathcal{L}$ can then be expressed as

\begin{align}
 \delta_{BRST} \mathcal{L} = *d\kappa \land J^{BRST} = (\partial z \kappa J^{BRST}_{\bar{z}} + \partial \bar{z} \kappa J^{BRST}_{\bar{z}}) dz \land d\bar{z}.
\end{align}
As anticipated, expressions for $\pi_z$ and $\bar{\pi}_z$ can be found by requiring the independence of $J^{BRST}$ from $C^0$. In particular, one finds

$$
\pi_z e^\varphi = -2\partial_z \pi + \chi' \eta_z - \chi^2 \eta_z + \bar{G}_{zz} C_{zz}' - \frac{1}{2} b_z (\eta_z^{\Gamma'} - \eta_z^{\Gamma''})
+ \frac{1}{2} (\Gamma_+ \bar{\Gamma}'' + \bar{\Gamma}_+ \Gamma'') - (\bar{\Gamma}_+ \eta_z^{\Gamma'} + \bar{\Gamma}_- \eta_z^{\Gamma''}) C_{zz}'.
$$

We separate the Liouville and ghost sectors by writing

$$
\begin{align*}
T_{zz} &= T_{zz}^{grav} + T_{zz}^{gh}, \\
J_z &= J_z^{grav} + J_z^{gh},
\end{align*}
$$

and similarly for the complex conjugates $T_{zz}$, $J_z$, $G_+^z$ and $G_-^z$. Let us first focus on the Liouville sector. We have, on shell and up to total derivative terms,

$$
\begin{align*}
J_z^{BRST grav} &= -C_{zz} T_{zz}^{grav} - \frac{i}{4} c_z^{grav} + i \frac{1}{2} \Gamma^{zz} C_{zz}^{grav} + \frac{1}{2} \Gamma^{-zz} C_{zz}^{grav},
\end{align*}
$$

where, after a simple redefinition of the fields

$$
\begin{align*}
\varphi &\rightarrow \varphi, \\
\lambda_+ &\rightarrow \frac{1}{4} \lambda_+, \\
\lambda_- &\rightarrow \frac{1}{4} \lambda_-, \\
\xi &\rightarrow -2i \xi, \\
\chi &\rightarrow \frac{i}{4} \chi,
\end{align*}
$$

the N=2 currents are written as

$$
\begin{align*}
T_{zz}^{grav} &= -\partial_z \pi \partial_z \varphi + \frac{1}{2} \partial_z^2 \pi + \partial_z \chi \partial_z \xi + \frac{1}{2} (\partial_z \lambda_- \eta_z^- - \lambda_- \partial_z \eta_z^+) + \frac{1}{2} (\lambda_+ \partial_z \eta_z^- - \partial_z \lambda_+ \eta_z^+), \\
G_+^{grav} &= \partial_z \lambda_+ - \lambda_+ \partial_z (\varphi + \xi) + \eta_z^+ \partial_z (\chi + \pi), \\
G_-^{grav} &= \partial_z \lambda_- - \lambda_- \partial_z (\varphi - \xi) + \eta_z^- \partial_z (\chi - \pi), \\
J_z^{grav} &= \partial_z \chi - \lambda_- \eta_z^- - \lambda_+ \eta_z^+.
\end{align*}
$$

The background charge term $1/2 \partial_z^2 \pi$ has no influence on the central charge, that turns out to be $c^{grav} = 6$. This implies that there is also an N=4 conformal symmetry, according to the analysis of [26]. The fundamental operator product expansions are normalized as follows:

$$
\begin{align*}
\partial_z \pi(z) \partial_w \varphi(w) &= -\frac{1}{(z-w)^2}, \\
\partial_z \chi(z) \partial_w \xi(w) &= \frac{1}{(z-w)^2}, \\
\lambda_+(z) \eta_w^+(w) &= -\frac{1}{z-w}, \\
\lambda_-(z) \eta_w^-(w) &= \frac{1}{z-w}.
\end{align*}
$$

It is easy to check that the N=2 operator product expansions are indeed satisfied by (88).

Before going on, let us dwell for a moment on the above N=2 $c = 6$ superconformal algebra and discuss some of its features. First of all, notice that it can be decomposed into the direct sum of two $N = 2$ superconformal algebras with central charges $c_1 = 3/2$ and $c_2 = 9/2$. We have

$$
\begin{align*}
T_{zz}^{grav} &= T_{zz}^{(1)} + T_{zz}^{(2)}, \\
J_z^{grav} &= J_z^{(1)} + J_z^{(2)}, \\
G_+^{grav} &= G_+^{(1)} + G_+^{(2)}, \\
G_-^{grav} &= G_-^{(1)} + G_-^{(2)},
\end{align*}
$$

(90)
where
\[ T^{(1)}_{zz} = \partial_z \varphi_1 \partial_z \varphi_1' + \frac{1}{4} \partial^2_z (\varphi_1 + \varphi_1') + \frac{1}{2} (\partial_z \lambda^1_+ \lambda^{(1)} + \lambda^{(1)}_+ \partial_z \lambda_{-1}^{(1)}), \]
\[ G^{(1)}_{+z} = \frac{1}{\sqrt{2}} (\partial_z \lambda^{(1)}_+ + 2 \lambda^{(2)}_+ \partial_z \varphi_1), \quad G^{(1)}_{-z} = \frac{1}{\sqrt{2}} (\partial_z \lambda_{-1}^{(1)} + 2 \lambda_{-1}^{(2)} \partial_z \varphi_1), \]
\[ J^{(1)}_z = \frac{1}{2} \partial_z (\varphi_1 - \varphi_1') + \lambda^{(1)}_+ \lambda^{(1)}_-, \quad (91) \]
and
\[ T^{(2)}_{zz} = -\partial_z \varphi_2 \partial_z \varphi_2' - \frac{1}{4} \partial^2_z (\varphi_2 + \varphi_2') - \frac{1}{2} (\partial_z \lambda^{(2)}_+ \lambda^{(2)}_+ - \lambda^{(2)}_+ \partial_z \lambda_{-2}^{(2)}), \]
\[ G^{(2)}_{+z} = \frac{1}{\sqrt{2}} (\partial_z \lambda^{(2)}_+ + 2 \lambda^{(2)}_+ \partial_z \varphi_2), \quad G^{(2)}_{-z} = \frac{1}{\sqrt{2}} (\partial_z \lambda_{-2}^{(2)} + 2 \lambda_{-2}^{(2)} \partial_z \varphi_2), \]
\[ J^{(2)}_z = -\frac{1}{2} \partial_z (\varphi_2 - \varphi_2') - \lambda^{(2)}_+ \lambda^{(2)}_-, \quad (92) \]
The correspondence with the previous fields is
\[ \varphi_1 = -\frac{1}{2} (\varphi + \xi - \chi - \pi), \quad \varphi_1' = -\frac{1}{2} (\varphi - \xi + \chi - \pi), \]
\[ \lambda^{(1)}_- = \frac{1}{\sqrt{2}} (\lambda_- - \eta^+_z), \quad \lambda^{(1)}_+ = \frac{1}{\sqrt{2}} (\lambda_+ + \eta^+_z), \]
\[ \varphi_2 = -\frac{1}{2} (\varphi + \xi + \chi + \pi), \quad \varphi_2' = -\frac{1}{2} (\varphi - \xi - \chi + \pi), \]
\[ \lambda^{(2)}_- = \frac{1}{\sqrt{2}} (\lambda_- + \eta^+_z), \quad \lambda^{(2)}_+ = \frac{1}{\sqrt{2}} (\lambda_+ - \eta^+_z). \quad (93) \]

Let us recall that unitary representations of the \( N=2 \) algebra with \( c < 3 \) are given by the minimal model series, where \( c = \frac{3k}{k+2} \) (\( k \in \mathbb{N} \)), so that our representation \( T^{(1)}_{zz}, G^{(1)}_{+z}, \quad G^{(1)}_{-z}, \quad J^{(1)}_z \) corresponds to a free field realization of the \( k = 2 \) minimal model. Construction of these models \([27]\) as GKO \([28]\) cosets of \( SU(2) \) level \( k \) supersymmetric Kač-Moody algebra are well known. We also recall that unitary representations of the \( N=2 \) algebra with \( c > 3 \) have also been obtained as GKO cosets of the \( SL(2, \mathbb{R}) \) supersymmetric Kač-Moody algebra of level \( k \), yielding a series \( c = \frac{3k}{k+2} \) \([29]\). We nickname these representations “maximal models”. Therefore, our \( T^{(2)}_{zz}, G^{(2)}_{+z}, G^{(2)}_{-z}, J^{(2)}_z \) representation is a free field realization of the \( k = 6 \) “maximal model”.

Moreover, it is also easy to check that the \( N=2 \) \( c = 6 \) superconformal algebra \([88]\) can be decomposed into the direct sum of two \( N=2 \) \( c = 3 \) superconformal algebras. They correspond to the subsets of fields \{\((\varphi + \xi)/\sqrt{2}, (\chi - \pi)/\sqrt{2}, \lambda_+, \lambda^+_z\)\} and \{\((\varphi - \xi)/\sqrt{2}, (\chi + \pi)/\sqrt{2}, \lambda_-, \lambda^-_z\)\}.

Now, let us come to the ghost currents. We proceed as before, namely, we first determine the BRST current \( J^{BRST \, gh}_z \) from a local BRST variation of the action and then act with \( Q^{gh}_{BRST} \equiv \oint J^{BRST \, gh}_z \) on the antighost fields. One finds
\[ J^{BRST \, gh}_z = -\partial_c C^{z} \tilde{C}^{zz}_r C^{z} - \frac{1}{2} \partial_z C^{z} (\tilde{C}^{zz}_r \partial_z \Gamma^+ + \tilde{C}^{zz}_r \Gamma^-) + C^{z} (\tilde{C}^{zz}_r \partial_z \Gamma^+ + \tilde{C}^{zz}_r \partial_z \Gamma^-) \]
\[ - \frac{i}{2} \tilde{C}^{zz}_r \Gamma^+ - \frac{i}{2} C^{z} (\tilde{C}^{zz}_r \partial_z \Gamma^+ + \tilde{C}^{zz}_r \partial_z \Gamma^-) + b_z C^{zz}_r \partial_z c \]
\[ + b_z (\partial_z \Gamma^+ - \Gamma^+ \partial_z \Gamma^-). \quad (94) \]
After the replacements
\[ b_z \rightarrow -1/2 \, i b_z, \quad c \rightarrow 2 i c \] (95)
and the redefinitions
\[ \tilde{C}''_{zz} = -b_{zz}, \quad C'^+ = \bar{c}^z, \]
\[ \Gamma'_{+z} = -i \beta_{+z}, \quad \Gamma'^+ = -i \gamma^+, \]
\[ \Gamma''_{-z} = \beta_{-z}, \quad \Gamma''^{-} = -\gamma^-; \] (96)
one gets
\[ T_{zz}^{gh} = 2 b_{zz} \partial_z c^z + \partial_z b_{zz} c^z + \frac{3}{2} \beta_{+z} \partial_z \gamma^+ + \frac{1}{2} \partial_z \beta_{+z} \gamma^+ + \frac{3}{2} \beta_{-z} \partial_z \gamma^- + \frac{1}{2} \partial_z \beta_{-z} \gamma^- + b_z \partial_z c, \]
\[ G_{+z}^{gh} = 3 \beta_{+z} \partial_z c^z + 2 \partial_z \beta_{+z} c^z - \gamma^- b_{zz} - \gamma^- \partial_z b_z - 2 \partial_z \gamma^- b_z - \beta_{+z} c, \]
\[ G_{-z}^{gh} = 3 \partial_z c^z \beta_{-z} + 2 c^z \partial_z \beta_{-z} - b_{zz} \gamma^+ + \partial_z b_z \gamma^+ + 2 b_z \partial_z \gamma^+ + c \beta_{-z}, \]
\[ J_z^{gh} = \beta_{-z} \gamma^- - \beta_{+z} \gamma^+ - 2 \partial_z (b_z c^z). \] (97)
The fundamental operator product expansions are
\[ b_{zz}(z) c^w (w) = -\frac{1}{1 - w}, \quad b_z (z) c (w) = -\frac{1}{1 - w}, \]
\[ \beta_{+z} (z) \gamma^+ (w) = \frac{1}{1 - w}, \quad \beta_{-z} (z) \gamma^- (w) = \frac{1}{1 - w}. \] (98)
The ghost contribution to the central charge is \( c_{gh} = -6 \), so that \( c_{tot} = c_{grav} + c_{gh} = 0 \), as claimed.

Notice that \( \beta \) and \( \gamma \) commute among themselves, but anticommute with \( b \) and \( c \). This is because they carry an odd ghost number together with an odd fermion number, while \( b \) and \( c \) carry zero fermion number and odd ghost number. In the usual convention, instead, \( \beta \) and \( \gamma \) commute with everything. The above currents satisfy the usual N=2 superconformal algebra with both conventions. This is because we chose an \( \textit{ad hoc} \) ordering between the fields when writing down the supercurrents, namely \( \gamma \) before \( b-c \) in \( G_{+z}^{gh} \) and \( -\gamma \) before \( b-c \) in \( G_{-z}^{gh} \). In all other manipulations the double grading should be taken into account. Only after the topological twist the fermion number grading is absent (things are correctly arranged by the broker \([\S 3,4]\)).

Notice that the ghost sector is made of two \( b\text{-}c\text{-}\beta\text{-}\gamma \) N=1 systems, namely a system \( b_{zz} c^z - \beta_{+z} \gamma^- \) with weight \( \lambda_{\beta_{-z}} = 3/2 \) and a system \( c b_{-z} \gamma^+ - \beta_{+z} \) with weight \( \lambda_{\gamma^+} = -1/2 \). These systems also possess, as it is well-known, an accidental N=2 symmetry \([30]\). However the standard representation of the N=2 \( c = 6 \) superconformal algebra made of these \( b\text{-}c\text{-}\beta\text{-}\gamma \) N=1 systems does not coincide with (97).

It is interesting to notice that in the new notation, one has
\[ J_z^{BRST \ grav} = -c^z \ T_{zz}^{grav} + \frac{1}{2} c J_z^{grav} + \frac{1}{2} \gamma^+ G_{+z}^{grav} - \frac{1}{2} \gamma^- G_{-z}^{grav}, \] (99)
while (94) gives
\[ J_z^{BRST \ gh} = \partial_z c^z b_{zz} c^z + \frac{1}{2} \partial_z c^z (\beta_{+z} \gamma^+ + \beta_{-z} \gamma^-) - c^z (\beta_{+z} \partial_z \gamma^+ + \beta_{-z} \partial_z \gamma^-) - \frac{1}{2} b_{zz} \gamma^+ \gamma^- \]
\[-\frac{1}{2} c (\beta_+ \gamma^+ - \beta_- \gamma^-) + b_z c^z \partial_z c + \frac{1}{2} b_z (\partial_z \gamma^+ \gamma^- - \gamma^+ \partial_z \gamma^-) \]
\[= \frac{1}{2} \left( -c^z T_{zz}^{gh} + \frac{1}{2} c J_z^{gh} + \frac{1}{2} \gamma^+ G_{zz}^{gh} - \frac{1}{2} \gamma^- G_{zz}^{gh} \right), \tag{100} \]

a formula that is analogous to (99), with the usual \( \frac{1}{2} \) overall factor. We have omitted some total derivative terms, that are immaterial as far as \( Q_{BRST} \) is concerned. Thus we can also write
\[ J_z^{BRST} = -c^z T_{zz} + \frac{1}{2} c J_z + \frac{1}{2} \gamma^+ G_{zz} - \frac{1}{2} \gamma^- G_{zz}, \tag{101} \]

where
\[ T_{zz} = T_{zz}^{grav} + \frac{1}{2} T_{zz}^{gh}, \quad J_z = J_z^{grav} + \frac{1}{2} J_z^{gh}, \]
\[ G_{zz} = G_{zz}^{grav} + \frac{1}{2} G_{zz}^{gh}, \quad G_{zz} = G_{zz}^{grav} + \frac{1}{2} G_{zz}^{gh}. \tag{102} \]

The ghost number charge is
\[ Q_{gh} = \oint b_{zz} c^z + \beta_+ \gamma^z + \beta_- \gamma^- + b_z c, \tag{103} \]
so that \( Q_{BRST} = \oint J_z^{BRST} \) has ghost number one:
\[ [Q_{gh}, Q_{BRST}] = Q_{BRST}. \tag{104} \]

In the new notation the Lagrangian (82) is written as
\[ \mathcal{L}_{\text{Poincaré}} = -\pi \partial_z \partial_\xi \varphi + \chi \partial_z \partial_\xi \xi + \lambda_- \partial_z \eta_+ - \lambda_+ \partial_z \eta_- + \lambda_- \partial_z \eta_+ + \lambda_+ \partial_z \eta_- - b_{zz} \partial_z e^z + b_{zz} \partial_z c^z - \beta_+ \partial_z \gamma^z - \beta_- \partial_z \gamma^- + \beta_+ \partial_\xi \gamma^z + \beta_- \partial_\xi \gamma^- - b_z \partial_z c + b_z \partial_z c. \tag{105} \]

Let us make a comment about the addition of the kinetic Lagrangian for the dilaton supermultiplet (see section 4 for an analogous comment before gauge-fixing). Formula (11) with the index \( I \) restricted only to the value 0 and with vanishing superpotential \( W \) gives, after gauge-fixing,
\[ \mathcal{L}_{\text{kin}} = -\frac{1}{2} \partial_z \pi \partial_\xi \pi + \frac{1}{2} \partial_\xi \chi \partial_z \chi + \lambda_- \partial_z \lambda_+ + \lambda_+ \partial_z \lambda_- \tag{106} \]
(a convenient overall numerical factor has been chosen). It is immediate to see that the redefinitions
\[ \varphi \rightarrow \varphi - \frac{1}{2} \pi, \quad \xi \rightarrow \xi - \frac{1}{2} \chi, \]
\[ \eta_\pm \rightarrow \eta_\pm + \frac{1}{2} \lambda_\mp, \quad \eta_{\pm \pm} \rightarrow \eta_{\pm \pm} + \frac{1}{2} \lambda^\pm; \tag{107} \]

turn \( \mathcal{L}_{\text{Poincaré}} \) into \( \mathcal{L}_{\text{Poincaré}} + \mathcal{L}_{\text{kin}} \).

This completes the program of studying the N=2 algebra associated with the gauge-fixed Poincaré Lagrangian. Before making the topological twist, we make some comments on the structure of moduli space, zero modes and amplitudes.
The global degrees of freedom of the metric are the 3(\(g - 1\)) moduli \(m_i\), that are in one-to-one correspondence with the 3\((g - 1)\) zero modes of the spin 2 antighost \(b_{zz}\). There are \(g\) (\(= \dim H^1\)) global degrees of freedom (moduli) \(\nu^j\) of the graviphoton \(A\). These moduli are in one-to-one correspondence with the \(g\) zero modes of the spin 1 antighost \(b\). The total number of moduli is thus \(4g - 3\). The supermoduli \(\hat{m}, \hat{\nu}\) can be counted by computing the number of zero modes of the spin \(3/2\) antighosts \(\beta_{+z}\) and \(\beta_{-z}\), that is \(4g - 4\), one more than the number of moduli. The field \(\xi\) describes the local degrees of freedom of \(A\) that survive the \(U(1)\) gauge-fixing, in the same way as \(\varphi\) describes the local degrees of freedom of the metric that survive the gauge-fixing of diffeomorphisms.

Let us write explicitly the form of the amplitudes of the \(\mathcal{N}=2\) Liouville theory:

\[
< \mathcal{O}_1 \cdots \mathcal{O}_n > = \int d\Phi \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} dm_i d\hat{m}_i \int_{\mathcal{C}^g/\Lambda} \prod_{j=1}^g d\nu_j d\hat{\nu}_j \int d\hat{m} d\hat{\nu} \\
\times \prod_{i=1}^{3g-3} < \mu^i_{zz} b_{zz} > < \mu^i_{zz} \bar{b}_{zz} > \prod_{j=1}^g < \omega^j_{zz} b_z > < \omega^j_{zz} \bar{b}_z > \\
\times \prod_{k=1}^{2g-2} \delta(<\zeta^{-k}_z|\beta_{-z}>) \delta(<\bar{\zeta}^{-k}_z|\beta_{-z}>) \delta(<\zeta^{+k}_z|\beta_{+z}>) \delta(<\bar{\zeta}^{+k}_z|\beta_{+z}>) \\
\times M(\lambda, \eta)c(z_0)\tilde{c}(\bar{z}_0) \prod_i e^{\frac{i}{\hbar} \pi(\zeta_i)} e^{-S(m, \hat{m}, \nu, \hat{\nu})} \mathcal{O}_1 \cdots \mathcal{O}_n. \tag{108}
\]

Let us explain the notation.

i) \(\mu^i_{zz}\) denote the \(3g - 3\) Beltrami differentials, while \(\mu^i_{zz}\) are their complex conjugates. \(<\mu^i_{zz} b_{zz} >= \int_{\Sigma_g} \mu^i_{zz} b_{zz} d^2z\) are the correct insertions that take care of the \(b_{zz}\) zero modes \([31]\) and, at the same time, take into account of the Jacobian determinant (Beltrami differential) coming from the change of variables \(dg_{\mu\nu} \rightarrow \prod_i dm_i d\hat{m}_i\).

ii) \(\zeta^{\pm k}\) and \(\bar{\zeta}^{\pm k}\) are the super Beltrami differentials. The integration \(d\hat{m} d\hat{\nu}\) over supermoduli (that usually produce the supercurrent insertions) is not explicitly performed, since \(S(m, \hat{m}, \nu, \hat{\nu})\) does not depend trivially on them and moreover the observables \(\mathcal{O}\) possibly depend on them.

iii) \(\omega^j_z\) (\(\omega^j_{\bar{z}}\)) are the (anti-)holomorphic differentials parametrizing the global degrees of freedom of the graviphoton \(A\). The \(g\)-dimensional moduli space of \(A\) is the Jacobian variety \([32]\) \(\mathcal{C}_\Lambda^g\), \(\Lambda = \mathbb{Z}^g + \Omega\mathbb{Z}^g\) denoting the lattice \(\nu_j \approx \nu_j + n_j + m_k \Omega_{kj}\), \(n_j\) and \(m_k\) being integer numbers and \(\Omega_{jk}\) being the period matrix. The reason why one has to restrict the integration of the \(U(1)\) moduli to the unit cell is the same as the one that enforces the restriction of the integration on metrics to the proper moduli space \(T/\Gamma\), \(T\) denoting the Teichmüller space and \(\Gamma\) the mapping class group. Indeed, \(\mathcal{C}^g\) is the Teichmüller space parametrizing the deformations of \(A\) that are orthogonal to gauge transformations. There are, however, gauge transformations that are not connectible to the identity and the homotopy classes of these are in one-to-one correspondence with the Jacobian lattice \(\mathbb{Z}^g + \Omega\mathbb{Z}^g\). The \(U(1)\) gauge transformations that are not connectible to the identity correspond to the shifts \(\nu_j \rightarrow \nu_j + n_j + m_k \Omega_{kj}\). For fixed Chern class, two gauge equivalent \(U(1)\) connections \(A_1\) and \(A_2\) are such that \(A_1 - A_2\) can be written as
$U^{-1}dU$ for $U = e^{i\phi}$, $\phi : \Sigma_g \to S^1$ being a map that winds $n_j$ times around the $B_j$ cycles and $m_k$ times around the $A_k$ cycles.

iv) $M(\lambda, \eta)$ generically denotes the insertions that are necessary in order to remove the zero modes of the $\eta$'s and the $\lambda$'s (insertions that can be provided by the observables $O_i$), while $c(z_0)\bar{c}(\bar{z}_0)$ is the insertion for the (constant) zero modes of $c$ and $\bar{c}$. It is understood that the constant zero modes of $\pi$, $\varphi$, $\chi$ and $\xi$ are also reabsorbed.

v) $d\Phi$ denotes the functional integration over the local degrees of freedom of the fields. The action $S$ depends on the local degrees of freedom of the fields, as well as on the moduli $m_i$, $\nu_j$ and supermoduli $\hat{m}$ and $\hat{\nu}$. The other fields (Lagrange multipliers, auxiliary fields and some ghost fields) are those that we have integrated away.

vi) $e^{\frac{\delta}{\gamma} \tilde{\pi}(z_i)}$ are the $\delta$-type insertions that simulate the curvature $R$ such that $\int_{\Sigma_g} R = 2(1 - g)$. $\tilde{\pi}$ is the BRST invariant extension of $\pi$ and the $q_i$ satisfy the condition

$$\sum_i q_i = 2(1 - g).$$

One finds the solution (left moving part)

$$\tilde{\pi} = \pi + \chi - \frac{2}{\gamma} c^z \lambda_-. $$

8 The topological twist on the gauge-fixed theory

In this section, we perform the topological twist on the N=2 gauge-fixed theory. We know that the formal set-up for the topological twist is entirely encoded into the broker, which correctly changes the spin, the ghost number and the BRST charge. However, in the gauge-fixed conformal theory, as we show in a moment, we have more equivalent possibilities. In particular, we can adapt the formalism in order to make more evident contact with the well-known procedure [24] in conformal field theory that consists of redefining the energy momentum tensor by adding to it the derivative of the $U(1)$ current [23]. In particular, we conveniently separate the operation of changing the spin from the rest of the twist procedure (change of ghost number and BRST charge), the rest still being performed by the broker.

In order to produce a twisted energy-momentum tensor equal to $T_{zz} + \frac{1}{2}\partial_z J_z$ we can make a redefinition of the ghost $c$ of the form

$$c' = c - \partial_z c^z.$$ 

Such a replacement, which changes the spin of the fields, is to be viewed as a redefinition of the $U(1)'$ ghost $C'$ rather than the $U(1)$ ghost $C$, since the new spin is defined (see section [3]) by adding the $U(1)'$ charge (not the $U(1)$ charge) to the old spin. $c'$ has a nonvanishing operator product expansion with $b_{zz}$ so that it is also necessary to redefine $b_{zz}$, namely

$$b'_{zz} = b_{zz} - \partial_z b_z. $$ 

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in order to preserve the operator product expansions.

The spin changes can be read from table 10 and justify the following change in notation

\[ \eta_+ \rightarrow \eta, \quad \lambda_+ \rightarrow \lambda, \quad \beta_+ \rightarrow \beta, \quad \gamma^+ \rightarrow \gamma, \]

\[ \eta_\perp \rightarrow \eta, \quad \lambda_\perp \rightarrow \lambda, \quad \beta_\perp \rightarrow \beta, \quad \gamma^- \rightarrow \gamma^\perp. \]  \quad (113)

Similarly, the supercurrents become

\[ G_+ \rightarrow G, \quad G_\perp \rightarrow G. \]  \quad (114)

Redefinitions (111) and (112) produce a new BRST current \( J''_{BRST} \) (equal to the old one apart from a total derivative term) given by

\[ J''_{BRST} = -c^z T''_{zz} + \frac{1}{2} \gamma' G_z - \frac{1}{2} \gamma z G_{zz}, \]  \quad (115)

where \( T''_{zz} = T_{zz} + \frac{1}{2} \partial_z J_z \). As anticipated, \( J''_{BRST} \) generates a new energy-momentum tensor (obtained by acting with the new BRST charge on \( b'_zz \)) equal to

\[ T'''_{zz} = T_{zz} + \frac{1}{2} \partial_z J_z = -\partial_z \pi \partial_z \varphi + \frac{1}{2} \partial_z^2 \pi + \partial_z \chi \partial_z \xi + \frac{1}{2} \partial_z^2 \chi - \lambda_z \partial_z \eta - \partial_z \lambda \eta_z \]  
\[ + 2b'_zz \partial_z c^z + \partial_z b'_zz c^z + 2\beta_z \beta_z \gamma^z + \partial_z \beta \gamma z + \partial_z \gamma \gamma z + \beta \partial_z \gamma + b \partial_z c'. \]  \quad (116)

From this expression, it is immediate to check the new spin assignments. It is interesting to note that the total derivative term in the \( U(1) \) current \( J''_z \) (97) combines with redefinitions (111) and (112) to give the correct energy-momentum tensor for the ghosts \( T''_{zz} \).

The other ingredient of the topological twist is the topological shift \[ \gamma \rightarrow \gamma + \alpha. \]  \quad (117)

Since the spin has been already changed by (111), (117) does not change the spin a second time. Indeed, the new spin of \( \gamma \) is zero and so that of \( \alpha \). Moreover, after twist \( \gamma \) possesses a zero mode (the constant). In this case, \( \alpha \) represents a shift of the zero mode of \( \gamma \).

(111), (112) and (117) permit to move directly from the \( N=2 \) amplitudes to the amplitudes of the topologically twisted theory. This shows that the amplitudes of the topological theory are a subset of the amplitudes of the \( N=2 \) supersymmetric theory. Nevertheless, the twisting procedure cannot be described simply as a change of variables in the functional integral, as (111), (112) and (117) should seem to suggest, since the physical content of the theory could not be changed in this way. This is made apparent by the fact that before twist \( \gamma (\gamma^\perp) \) has spin \( -1/2 \) and possesses no zero mode, so that

\footnote{However, one has to be careful about the zero modes and the global degrees of freedom. Later on we shall come back to this point.}
the choice \( \alpha = \text{const} \) is only consistent after redefining the spin with \([111]\) and \([112]\), while in section \( \S \) it was only a formal position.

For convenience, as far as the gradings are concerned (ghost number and fermion number), we use the same conventions as before twist and leave the broker explicit. Thus \( \alpha \) carries odd fermion number and odd ghost number.

The topological shift \([117]\) produces a total BRST current equal to

\[
J_{BRST}^{tot} = J_{BRST}^z + \frac{1}{2} \alpha G_z, \tag{118}
\]

(again, a total derivative term has been omitted). If we denote, as usual,

\[
Q_{BRST} = \oint J_{BRST}^{tot} \, dz, \quad Q_v = \oint J_{BRST}^v \, dz \quad \text{and} \quad Q_s = \oint G_z \, dz,
\]

we see that the BRST charge is precisely shifted by the supersymmetry charge \( Q_s \), as explained in section \( \S \).

Let us now discuss some properties of the twisted theory. It is convenient to write down the \( Q_s \) transformations of the fields. We denote it by \( \delta_s \).

\[
\begin{align*}
\delta_s (\xi - \varphi) &= 2\eta, & \delta_s \eta &= 0, & \delta_s \lambda_z &= \partial_z (\pi + \chi), & \delta_s (\pi + \chi) &= 0, \\
\delta_s (\pi - \chi) &= 2\lambda, & \delta_s \lambda &= 0, & \delta_s \eta_z &= \partial_z (\xi + \varphi), & \delta_s (\xi + \varphi) &= 0, \\
\delta_s b_{zz}' &= 0, & \delta_s b_{zz} &= -b_{zz}', & \delta_s c^z &= \gamma^z, & \delta_s \gamma^z &= 0, \\
\delta_s b_z &= \beta_z, & \delta_s \beta_z &= 0, & \delta_s c' &= 0, & \delta_s \gamma &= c'.
\end{align*}
\tag{119}
\]

These transformations are the analogue, in the gauge-fixed case, of the \( \delta_T \) transformations \([33]\) and \([50]\). As explained in the previous section, there are two \( b-c-\beta-\gamma \) systems, rearranged in the last two lines of \([119]\). In particular, the last line represents the sector of \( B_{\text{gauge-fixing}} \) that is reminiscent of the constraint on the moduli space. The last but one line represents the usual \( b-c-\beta-\gamma \) ghost for ghost system of topological gravity \([17]\). It is evident that the roles of \( b \) and \( \beta \) and the roles of \( c \) and \( \gamma \) are inverted in the two cases.

The theory is topological, since the energy-momentum tensor \( T'_{zz} \) is a physically trivial left moving operator. Indeed, recalling that \( G_{zz} = -2 \{ Q_v, \beta_{zz} \} \), we have

\[
\alpha T'_{zz} = \{ Q, G_{zz} \}, \quad \{ Q_v, G_{zz} \} = 0. \tag{120}
\]

In ref. \([34]\), it is shown that a “homotopy” operator \( U \) can be defined in the Verlinde and Verlinde model of topological gravity \([17]\), such that \( U Q_{BRST} U^{-1} = Q_s \). This shows that \( Q_{BRST} \) and \( Q_s \) have the same spectra and provides a “matter” representation of the gravitational observables when the theory is coupled to a Landau-Ginzburg model. We now find the operator \( U \) in our case and study some of its properties. This is not only useful for the future program of coupling matter to constrained topological gravity, but also permits to define a third nilpotent operator, called \( S \), that only acts on the ghost

\footnote{In ref. \([33]\), it is claimed that the topological twist of \( N=2 \) supergravity leads to the Verlinde and Verlinde model. To obtain this, a certain reduction mechanism in the ghost sector is advocated, corresponding to setting \( \gamma = 0 \) and \( c' = 0 \). The neglected sector is precisely the sector that is responsible for the constraint on the moduli space, i.e. the last line of \([119]\), which makes the difference between our model and the model of \([17]\).}
sector and further puts into evidence, in some sense, the presence of the constraint on moduli space.

To begin with, it is straightforward to prove that

\[ J_z'_{BRST} = \frac{1}{2} \delta_s (\gamma J_z - c^z G_{zz}). \]  

(121)

Moreover, one also has

\[ \frac{1}{2} (\gamma J_z - c^z G_{zz}) = \delta_v (b_z - c^z \beta_{zz}) + \frac{1}{4} \delta_s (c^z \gamma \beta_{zz} + b_z \gamma \gamma), \]  

(122)

where \( \delta_v \) denotes the action of \( Q_v \). Now, defining

\[ \Theta = \frac{1}{2} \oint (\gamma J_z - c^z G_{zz}) - \frac{1}{4} \oint (c^z \gamma \beta_{zz} + b_z \gamma \gamma), \]  

(123)

one can write

\[ Q_v = [Q_s, \Theta], \quad \{Q_v, \Theta\} = 0. \]  

(124)

Thus, the desired “homotopy” operator is

\[ U = \exp (-2\alpha^{-1}\Theta) \]  

(125)

and we have

\[ UQU^{-1} = \frac{1}{2} \alpha Q_s. \]  

(126)

In this way, it is possible to turn to the “matter picture”, which is also simpler from the computational point of view. The key point, in presence of matter coupling, is that the condition of equivariant cohomology is correspondingly changed \[34\].

Let us now introduce the operator \( S \). We have, from (121) and (122),

\[ J_z'_{BRST} = \delta_v \delta_s (\gamma b_z - c^z \beta_{zz}). \]  

(127)

Due to \( \delta_v J_z = 0 \), we can also write

\[ [Q'_{gh}, Q_v] = Q_v, \]  

(128)

where

\[ Q'_{gh} = Q_{gh} + \oint J_z = \oint b'_z c^z + 2 \beta_{zz} \gamma^z + b_z c' - \lambda_z \eta - \lambda \eta_z \]  

(129)

is the ghost number charge of the twisted theory, equal to the sum of the ghost number charge of the initial N=2 theory plus the \( U(1) \) charge. This corresponds to eq. \(57\).

Define

\[ S = \oint (\gamma b_z - c^z \beta_{zz}). \]  

(130)

Then, eq. (127) implies

\[ Q_v = [Q_v, \{Q_s, S\}], \quad \Theta = [Q_v, S], \]  

(131)
while the ghost charge can be expressed as
\[ Q_{gh} = \oint J_z - \{ Q_s, S \}. \]  
\[ (132) \]

\( S \) is a nilpotent operator, that acts trivially on the Liouville sector, while in the ghost sector it gives
\[ S b_z = 0, \quad S c^z = 0, \quad S \gamma^z = -c^z, \]
\[ S b_z = 0, \quad S \beta_z = b_z, \quad S c = -\gamma, \quad S \gamma = 0. \]  
\[ (133) \]

These rules should be compared with the last two lines of (119). In some sense, the action of \( S \) is dual to the action of \( \delta_s \) in the ghost sector. We have already discussed the last two lines of (113) and the difference between the roles of \( b \) and \( \beta, c \) and \( \gamma \), in the two cases. The action of the operator \( S \), compared to the one of \( \delta_s \), inverts the roles of the two \( b \)-\( c \)-\( \beta \)-\( \gamma \) systems. Clearly, the existence of the operator \( S \) is strictly related to the presence of the graviphoton and thus to the constraint on moduli space.

Let us give some of the transformations corresponding to the change of basis due to \( U \)
\[ U \pi U^{-1} = \pi + \frac{1}{1-\gamma/\alpha} c^z \lambda_z, \]
\[ U \chi U^{-1} = \chi + \frac{1}{1-\gamma/\alpha} c^z \lambda_z, \]
\[ U \lambda_z U^{-1} = \frac{1}{1-\gamma/\alpha} \lambda_z, \]
\[ U \eta_z U^{-1} = \frac{1}{1-\gamma/\alpha} \eta_z, \]
\[ U c U^{-1} = \frac{1}{1-\gamma/\alpha} c^z + f(\gamma, c^z), \]
\[ U c^z U^{-1} = \frac{1}{1-\gamma/\alpha} c^z + f(\gamma, c^z), \]
\[ U \gamma^z U^{-1} = \gamma^z + f(\gamma, c^z, c^z), \]
\[ U \gamma U^{-1} = \frac{1}{1-\gamma/\alpha} \gamma, \]
\[ UT_{zz}U^{-1} = T'_{zz}, \]
\[ UG_{zz}U^{-1} = G_{zz}, \]
\[ (134) \]

where \( \theta_z \) is such that \( \Theta = \oint \theta_z dz \). \( T'_{zz}, \ G_z - \frac{2}{\alpha} \delta_{\theta_z}, \ G_{zz} \) and \( J_z - \frac{1}{2\alpha} \theta_z \) is another representation of the same topological algebra. The functions \( f \) appearing in (134) are complicated expressions of their arguments, which we do not report here. The above information is sufficient to prove that the operator \( U \) defines a changes of variables in the functional integral with unit Jacobian determinant.

Notice that \( \pi + \chi \) is the field that permits the insertion of curvature delta-type singularities. It is \( Q_{s,c} \)-closed and \( \pi = U(\pi + \chi)U^{-1} \) is its \( Q \)-closed generalization. Moreover, since \( UT'_{zz}U^{-1} = T_{zz}' \), it is immediate to prove that \( \pi \) has the same operator product expansion with \( T_{zz}' \) as \( \pi \). In particular, \( e^{\pi} \) is a primary field. The limit \( \alpha \to 0 \) of \( \pi \) is the field (110) that allowed the curvature insertions in the N=2 theory (also called \( \tilde{\pi} \)).

### 9 Geometrical Interpretation

We now discuss the moduli space of the twisted theory and the gauge-fixing sector that implements the constraint defining the submanifold \( \mathcal{V}_g \subseteq \mathcal{M}_g \).
The number of moduli of the twisted theory is $4g - 3$, the same as that of the N=2 theory; $3(g-1)$ moduli $m_i$ corresponding to the metric and $g$ moduli $\nu_j$ corresponding to the $U(1)$ connection $A$. The number of supermoduli, on the other hand, changes by one: it was $4(g - 1)$ for the N=2 theory, it is $4g - 3$ for the topological theory, $3(g-1)$ supermoduli $\hat{m}_i$ corresponding to the zero modes of the spin 2 antighost $\beta_{zz}$ and $g$ supermoduli $\hat{\nu}_j$ corresponding to the zero modes of $\beta_z$. The mismatch of one supermodulus is filled by the presence of one super Killing vector field, corresponding to the (constant) zero mode of $\gamma$.

Thus, comparing the N=2 theory with the twisted one, we can say that the $2(g - 1) + 2(g - 1)$ zero modes of the $\beta_{zz}$ fields rearrange into the $3(g - 1)$ zero modes of $\beta_{zz}$ plus the $g$ zero modes of $\beta_z$ plus one zero mode of $\gamma$. Similarly, the $2(g - 1) + 2(g - 1)$ supermoduli rearrange into $3(g - 1)$ moduli $\hat{m}$ of the topological ghosts plus $g$ moduli $\hat{\nu}$ of the topological antighosts plus one super Killing vector field. The zero modes of the $\lambda$ and $\eta$ fields rearrange among themselves.

In particular, after the twist, the number of bosonic moduli equals the number of fermionic moduli, as expected for a topological theory. However, the two kinds of supermoduli $\hat{m}$ and $\hat{\nu}$ do not carry the same ghost number after the twist. Indeed, $\hat{m}_i$ carry ghost number 1, while $\hat{\nu}_j$ carry ghost number $-1$. Thus, we can interpret $\hat{m}_i$ as the topological variation of $m_i$, but we cannot interpret $\hat{\nu}_j$ as the topological variation of $\nu_j$, rather $\nu_j$ is the topological variation of $\hat{\nu}_j$:

$$s m_i = \hat{m}_i, \quad s \hat{m}_i = 0, \quad s \hat{\nu}_j = \nu_j, \quad s \nu_j = 0. \quad (135)$$

This is in agreement with the interpretation of $A$ as a Lagrange multiplier, so that it is only introduced via the gauge-fixing algebra: $m$ and $\hat{m}$ belong to $B_{\text{gauge-free}}$, while $\nu$ and $\hat{\nu}$ belong to $B_{\text{gauge-fixing}}$.

The amplitudes can be written as

$$< \prod_k \sigma_{n_k} > = \int d\Phi \int_{M_g} \prod_{i=1}^{3g-3} dm_i \int_{C^g/\Lambda} \prod_{j=1}^{g} d\nu_j \int d\hat{m} d\hat{\nu} \prod_i e^{q_i \bar{z}(z_i)} e^{-S(m, \hat{m}, \nu, \hat{\nu})} \prod_k \sigma_{n_k}, \quad (136)$$

where $\sigma_{n_k}$ are the observables. In this expression, the insertions that remove the zero modes of $b_{zz}$, $\beta_{zz}$, $\beta_z$, $b_z$, $\eta$, $\lambda$, $\lambda_z$ and $\eta_z$ are understood, but attention has to be paid to the fact that a super Killing vector field, corresponding to the zero mode of $\gamma$, forbids one fermionic integration. The ghost number of the supermoduli measure adds up to $-2g + 3$. Nevertheless, due to the presence of one super Killing vector field, the selection rule is that the total ghost number of $\prod_k \sigma_{n_k}$ must be equal to $2(g - 1)$ and not to $2g - 3$. This is the mismatch between true dimension and formal dimension addressed in the introduction.

To explain why the graviphoton is responsible for the constraint, let us rewrite the action making the dependence on the $U(1)$-moduli $\nu_j$ and the corresponding supermoduli $\hat{\nu}_j$ explicit.

$$S(m, \hat{m}, \nu, \hat{\nu}) = S(m, \hat{m}, 0, 0) + \nu_j \int_{\Sigma_g} \omega^2 z J_z \omega^2 + \hat{\nu}_j \int_{\Sigma_g} \omega^2 G_z \omega^2 z$$
\[ + \bar{\nu}_{j} \int_{\Sigma_{g}} \omega^{j}_{z} J_{z} d^{2}z + \hat{\nu}_{j} \int_{\Sigma_{g}} \omega^{j}_{z} G_{z} d^{2}z + \nu \hat{\nu} - \text{terms}. \tag{137} \]

The terms that are quadratic in \( \nu \hat{\nu} \) are due to the fact that the gravitini are initially \( U(1) \)-charged. They have not been reported explicitly, since they can be neglected, as we show in a moment. The coefficient of \( \bar{\nu}_{j} \) is the \( U(1) \) current \( J_{z} \) folded with the \( j \)-th (anti)holomorphic differential \( \omega^{j}_{z} \). Similarly, the coefficient of \( \hat{\nu}_{j} \) is the supercurrent \( G_{z} \) folded with the same differential.

We want to perform the \( \nu \hat{\nu} \) integrals explicitly. This is allowed, since the observables should not depend on \( \nu \) and \( \hat{\nu} \). Indeed, \( \nu \) and \( \hat{\nu} \) belong to \( B_{\text{gauge-free}} \) while the observables are constructed entirely from \( B_{\text{gauge-fixing}} \). Anyway, since \( \nu \) and \( \hat{\nu} \) form a closed BRST subsystem, we can consistently project down to the subset \( \nu = \hat{\nu} = 0 \), while retaining the BRST nilpotence. The \( U(1) \) moduli \( \nu \) are not integrated all over \( C_{g} \), which would be nice since the integration would be very easy, rather on the unit cell \( L = \frac{C_{g}}{(Z^{g} + \Omega Z^{g})} \) defined by the Jacobian lattice. To overcome this problem, we take the semiclassical limit, which is exact in a topological field theory. We multiply the action \( S \) by a constant \( \kappa \) that has to be stretched to infinity. \( \kappa \) can be viewed as a gauge-fixing parameter, rescaling the gauge-fermion: no physical amplitude depends on it. Let us define

\[ \nu'_{j} = \kappa \nu_{j}, \quad \hat{\nu}'_{j} = \kappa \hat{\nu}_{j}. \tag{138} \]

We have

\[ \int_{L} \prod_{j=1}^{g} d\nu_{j} d\hat{\nu}_{j} = \int_{\kappa L} \prod_{j=1}^{g} d\nu'_{j} d\hat{\nu}'_{j}, \tag{139} \]

where and \( \kappa L \) is unit cell rescaled. We see that the \( \nu \hat{\nu} \) terms of (137) are suppressed in the \( \kappa \to \infty \) limit, as claimed. We can replace \( \kappa L \) with \( C_{g} \) in this limit. Finally, the integration over the \( U(1) \) moduli and supermoduli produces the insertions

\[ \prod_{j=1}^{g} \int_{\Sigma_{g}} \omega^{j}_{z} G_{z} d^{2}z \cdot \delta \left( \int_{\Sigma_{g}} \omega^{j}_{z} J_{z} d^{2}z \right). \tag{140} \]

The delta-function is the origin of the desired constraint on moduli space. Indeed, the current \( J_{z} \) can be thought as a (field dependent) section of \( \mathcal{E}_{\text{hol}} \). The requirement of its vanishing is equivalent to projecting onto the Poincaré dual of the top Chern class \( c_{g}(\mathcal{E}_{\text{hol}}) \) of \( \mathcal{E}_{\text{hol}} \), due to a theorem that one can find for example in [7]. Changing section only changes the representative in the cohomology class of \( c_{g}(\mathcal{E}_{\text{hol}}) \). Indeed, the Poincaré dual of the top Chern class of a holomorphic vector bundle \( E \to M \) is shown to be the submanifold of the base manifold \( M \) where one holomorphic section \( a \in \Gamma(E, M) \) vanishes identically. In other words, the dual of \( c_{g}(\mathcal{E}_{\text{hol}}) \) is the divisor of some section. For a line bundle \( L \to M \), this is easily seen. Let \( h \) be a fiber metric so that \( ||a||^{2} = a(z) \bar{a}(\bar{z}) h(z, \bar{z}) \) is the norm of the section \( a \). The top Chern class \( c_{1}(L) \) can be written as the cohomology class of the curvature \( R = \partial \bar{\partial} \Gamma \) of the canonical holomorphic connection \( \Gamma = h^{-1} \partial h \), so that

\[ c_{1}(L) = \bar{\partial} \partial \ln ||a(z)||^{2}. \tag{141} \]

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Patchwise, the metric $h$ can be reduced to the identity, but then $c_1(L)$ becomes a de Rahm current, namely a singular $(1,1)$ form with delta-function support on the divisor $\text{Div}[a]$, i.e. the locus of zeroes and poles of $a(z)$. The divisor $\text{Div}[a]$ is the Poincaré dual of $c_1(L)$. For a holomorphic vector bundle $E \to M$ of rank $n$, the same theorem can be understood using the so-called splitting principle. For the purpose of calculating Chern classes, $E$ can always be regarded as the Whitney sum of $n$ line-bundles $L_i$ corresponding, naively, to the eigendirections of the curvature matrix two form $\mathcal{R}^{jk}$,\
\[ E = L_1 \oplus \cdots \oplus L_n. \]

Then we have\
\[ c_n(E) = \prod_{i=1}^{n} c_1(L_i) = \bar{\partial} \partial \ln \|a_1\|^2 \wedge \cdots \wedge \bar{\partial} \partial \ln \|a_n\|^2, \]

where $a_i$ are the components of a section $a$ in a suitable basis. From this formula, we see that $c_n(E)$ has delta-function support on the divisor of $a$. That is why in our derivation of the topological correlators from the functional integral, we do not pay particular attention to the explicit form of $J_z$ and to its dependence on the other fields. What matters is that it is a conserved holomorphic one form, namely a section of $\mathcal{E}_{\text{hol}}$. The functional integral imposes its vanishing, so that the Riemann surfaces that effectively contribute lie in the homology class of the Poincaré dual of $c_g(\mathcal{E}_{\text{hol}})$.

Summarizing, we argue that the topological observables $\sigma_{nk}$ correspond to the Mumford-Morita classes, as in the case of topological gravity $[17]$, but that in constrained topological gravity the correlation functions are intersection forms on the Poincaré dual $\mathcal{V}_g$ of $c_g(\mathcal{E}_{\text{hol}})$ and not on the whole moduli space $\mathcal{M}_g$.

It can be convenient to represent $c_g(\mathcal{E}_{\text{hol}})$ by introducing the natural fiber metric $h_{jk} = \text{Im } \Omega^{jk} = \int_{\Sigma_g} \omega^j \omega^k d^2 z$ on $\mathcal{E}_{\text{hol}}$. The canonical connection associated with this metric is then\
\[ \Gamma = h^{-1} \partial h = \frac{1}{\Omega - \bar{\Omega}} \partial \Omega, \]

which leads to a curvature\
\[ \mathcal{R} = \bar{\partial} \Gamma = \frac{1}{\Omega - \bar{\Omega}} \bar{\partial} \Omega \frac{1}{\Omega - \bar{\Omega}} \partial \Omega. \]

Let $\{\omega^1, \ldots \omega^g\}$ denote a basis of holomorphic differentials. Locally, we can expand $J_z$ in this basis\
\[ J_z = a_j \omega^j. \]

The field dependent coefficients $a_j$ are the components of the section $J_z \in \Gamma(\mathcal{E}_{\text{hol}}, \mathcal{M}_g)$. The constraint then reads\
\[ \text{Im } \Omega^{jk} a_k = 0, \quad \forall j, \]
which, due to the positive definiteness of $\text{Im } \Omega$, is equivalent to\
\[ a_j = 0, \quad \forall j. \]
These are the equations that (locally) identify the submanifold $V_g \subset M_g$. It is also useful to introduce the vectors $v_j = \frac{\partial}{\partial a_j}$ that provide a local basis for the normal bundle $N(V_g)$ to $V_g$. Of course, the vectors $v_j$ commute among themselves:

$$[v_j, v_k] = 0.$$  (149)

In these explicit local coordinates, the top Chern class $c_g(\mathcal{E}_{hol})$ admits the following representation as a de Rham current:

$$c_g(\mathcal{E}_{hol}) = \delta(V_g)\tilde{\Omega}_g,$$  (150)

where

$$\tilde{\Omega}_g = \prod_{j=1}^g da_j, \quad \delta(V_g) = \prod_{j=1}^g \delta(a_j).$$  (151)

This explicit notation is useful to trace back the correspondence between the geometrical and field theoretical definition of the correlators.

To begin with, a convenient representation of the BRST operator (135) on the space $\{m, \hat{m}, \nu, \hat{\nu}\}$ is given by

$$Q_{\text{global}} = \hat{m}_i \frac{\partial}{\partial m_i} + \nu_j \frac{\partial}{\partial \nu_j}.$$  (152)

$Q_{\text{global}}$ is not the total BRST charge, rather it only represents the BRST charge on the sector of the global degrees of freedom. The total BRST charge is the sum of the above operator plus the usual BRST charge $Q = Q_s + Q_v$, that acts only on the local degrees of freedom. Since the total BRST charge acts trivially inside the physical correlation functions, we see that the action of $Q$ inside correlation functions is the opposite of the action of $Q_{\text{global}}$. This means that $Q$ can be identified, apart from an overall immaterial sign, with the operator (152). We know that the geometrical meaning of the supermoduli $\hat{m}_i$ are the differentials $dm_i$ on the moduli space $M_g$ and that the ghost number corresponds to the form degree. In view of this, we argue that the geometrical meaning of the $U(1)$ supermoduli $\hat{\nu}_j$ are contraction operators $\iota_{v_j}$ with respect to the associated vectors $v_j$. Since the $U(1)$ moduli $\nu_j$ are the BRST variations of $\hat{\nu}_j$ and the BRST operation should be identified with the exterior derivative, it is natural to conjecture that $\nu_j$ correspond to the Lie derivatives along the vectors $v_j$.

The correspondence between field theory and geometry is summarized in table 10. We now give arguments in support of this interpretation.

For instance, since $Q \sim d, \text{Im} \Omega^k a_k \sim \int \omega^k_z J_z d^2z$ and $[Q, J_z] = -G_z$, then the insertions $\int \omega^k_z G_z d^2z$ correspond to $d(\text{Im} \Omega^k a_k)$, so that

$$\prod_{j=1}^g \int_{\Sigma_g} \omega^k_z G_z d^2z \cdot \delta \left( \int_{\Sigma_g} \omega^k_z J_z d^2z \right) \sim \tilde{\Omega}_g \delta(V_g) = c_g(\mathcal{E}_{hol}).$$  (153)

If $\alpha_k$ denote the Mumford-Morita classes corresponding to the observables $\mathcal{O}_k$, the amplitudes are

$$< \mathcal{O}_1 \cdots \mathcal{O}_n > = \int_{M_g} \delta(V_g)\tilde{\Omega}_g \wedge \alpha_1 \wedge \cdots \wedge \alpha_n = \int_{V_g} \alpha_1 \wedge \cdots \wedge \alpha_n.$$  (154)
From the geometrical point of view, it is immediate to show that the action of (152) on a correlation function is precisely the exterior derivative, as already advocated. Indeed, we can write the $d$-form $\omega_d$ corresponding to a physical amplitude (not necessarily a top form, if we freeze, for the moment, the integration over the global degrees of freedom) as

$$\omega_d = i_{v_1} \cdots i_{v_g} \Omega_{d+g} = \left( \prod_{j=1}^{g} \hat{\nu}_j \right) \hat{m}_{i_1} \cdots \hat{m}_{i_d} \Omega_{d+g}^{i_1 \cdots i_d}, \quad (155)$$

where $\Omega_{d+g}$ is a suitable $d + g$-form on $\mathcal{M}_g$ (equal to $\tilde{\Omega}_g \wedge \omega_d$). Now, using the representation (152) of the operator $Q$, we find

$$\{Q, \omega_d\} = (-1)^g \left( \prod_{j=1}^{g} \hat{\nu}_j \right) \hat{m}_{i_1} \cdots \hat{m}_{i_d} \frac{\partial \Omega_{d+g}^{i_1 \cdots i_d}}{\partial m_i} + \sum_{k=1}^{g} (-1)^{k+1} \nu_k \left( \prod_{j \neq k} \hat{\nu}_j \right) \hat{m}_{i_1} \cdots \hat{m}_{i_d} \Omega_{d+g}^{i_1 \cdots i_d}. \quad (156)$$

Using the correspondence given in table 10, we have

$$\{Q, \omega_d\} = (-1)^g i_{v_1} \cdots i_{v_g} d\Omega_{d+g} + \sum_{k=1}^{g} (-1)^{k+1} \mathcal{L}_{v_k} i_{v_1} \cdots i_{v_{k-1}} i_{v_{k+1}} \cdots i_{v_g} \Omega_{d+g} = d\omega_d. \quad (157)$$

The second piece of (152) replaces a contraction with the vector $v_j$ with the Lie derivative with respect to the same vector.

Finally, we describe a more intuitive description of the submanifold $\mathcal{V}_g$. $\mathcal{V}_g$ is a representative of a homology cycle, so it can be convenient to make a special choice of this representative, for example, a $\mathcal{V}_g$ lying on the boundary of the moduli space. That means that we are considering degenerate Riemann surfaces. Take $g$ independent and non intersecting homology cycles on the Riemann surface, the A-cycles, and pinch them. You get $g$ nodes. Then, separate the two branches of each node: you get a sphere $S_{2g}$ with $2g$ pairwise identified marked points. We conjecture that $\mathcal{V}_g$ is representable as the space of such spheres. The dimensions turn out to match: indeed, $2g$ complex parameters are the positions of the marked points, but, due to $SL(2, \mathbb{C})$ invariance on the sphere, three points can be fixed to 0, 1 and $\infty$, as usual. Thus, the dimension of $S_{2g}$ equals $2g - 3$, which is the correct result. The $g$ holomorphic differentials of $\Sigma_g$ become differentials of the third kind on $S_{2g}$, with opposite residues on the pairwise identified points.

## 10 Outlook and Open Questions

In the present paper we have shown that there exists a new class of topological field theories whose correlation functions can be interpreted as intersection numbers of cohomology classes in a constrained moduli space. The constrained moduli space is, by itself,
a homology class of cycles in an ordinary, unconstrained, moduli space. The specific example that we have considered is a formulation of 2D topological gravity that is obtained through the A-twist of \(N=2\) Liouville theory.

Usually, in a topological field theory the space of field configurations is projected onto a finite dimensional subspace made of instantons. In two dimensional gravity, however, the space of configurations is the moduli space \(\mathcal{M}_g\) of Riemann surfaces of genus \(g\) and ordinary topological gravity deals with the full \(\mathcal{M}_g\). Constrained topological gravity, instead, deals with a proper submanifold \(\mathcal{V}_g \subset \mathcal{M}_g\), so that also in this case we have a nontrivial concept of instanton configurations. The “instantons” are the solutions to the moduli space constraint. Our formulation of topological gravity bears the same relation with 2D gravity as a generic topological field theory with its non topological version, namely the former is a “proper” projection of the latter.

Our result raises several questions that are so far unanswered.

In Witten’s topological gravity, where the correlators are the intersection numbers of Mumford-Morita classes in the ordinary moduli space \(\mathcal{M}_{g,s}\), the generating function satisfies the integrable KdV hierarchy \([35]\). This result was shown by Kontsevich \([36]\) from algebraic geometry, through a systematic triangulation of moduli space leading to an integral à la matrix-model. It was also justified, in field theoretical terms, by the work of Verlinde and Verlinde \([17]\) and Dijkgraf, Verlinde and Verlinde \([37]\). It is clear that the first question raised by our paper is: which integrable hierarchy is satisfied by the correlators defined in eq.(1)? The answer to this question is left open by our work and it is not yet clear whether it can be more easily obtained from field-theoretical or geometrical considerations: both ways are equally good, since we have established a correspondence between the topological definition (1) and the field-theoretical one \([136]\).

The next open question concerns matter coupling. One should investigate the effects of the moduli space constraint when topological gravity is coupled to topological matter. This involves the study of the topological twist of matter coupled N=2 Liouville theory. In the present paper we have extended to curved superspace the construction of \([21]\) where N=2 matter coupled to gauge theories was analysed. The next step of the program is to write down the most general N=2, D=2 theory that contains the graviton multiplet, the gauge multiplets and the chiral and twisted chiral multiplets \([38]\), interacting through a generalized Kähler metric, a superpotential and a dilaton coupling to the two dimensional curvature. Then, by investigating the A-twist of such a theory, one obtains the matter coupling of the present constrained topological gravity to the topological \(\sigma\)-model.

If we are interested in coupling the topological Landau-Ginzburg model, we should instead perform the B-twist. Indeed, another question raised by our paper that should be addressed in the next future is whether the B-twist of the \(N=2\) Liouville theory produces a similar or different topological gravity. In this paper we have shown that the cohomology of \(\mathcal{Q}_s\) is equivalent to the cohomology of the full BRST operator \(\mathcal{Q}_{BRST}\), the same way as it happens for the Verlinde and Verlinde theory. This property can be exploited fruitfully by coupling the (B-twisted) topological gravity to Landau-Ginzburg matter.

Finally, other open questions are related to conformal field theory. The splitting into
a minimal plus a maximal model of the N=2 superconformal theory associated with the
gauge-fixed Liouville model deserves attention. It might be the way to understand better
the relation with the Polyakov formulation in terms of a level k \( SL(2, R) \) Kač-Moody
algebra [39].

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| Field | spin | ghost | spin' | ghost' | $U(1)$ | $U(1)'$ | New Field |
|-------|------|-------|-------|--------|--------|---------|-----------|
| $e^+$ | $-1$ | $0$   | $-1$  | $0$    | $0$    | $0$     |           |
| $e^-$ | $1$  | $0$   | $1$   | $0$    | $0$    | $0$     |           |
| $C^+$ | $-1$ | $1$   | $-1$  | $1$    | $0$    | $0$     |           |
| $C^-$ | $1$  | $1$   | $1$   | $1$    | $0$    | $0$     |           |
| $\omega$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |           |
| $C^0$ | $0$  | $1$   | $0$   | $1$    | $0$    | $0$     |           |
| $A$   | $0$  | $0$   | $0$   | $0$    | $0$    | $0$     |           |
| $C$   | $0$  | $1$   | $0$   | $1$    | $0$    | $0$     |           |
| $\zeta^+$ | $-1/2$ | $0$ | $0$ | $-1$ | $1/2$ | $1/2$ | $\zeta^+\alpha^{-1} \equiv \xi$ |
| $\zeta^-$ | $-1/2$ | $0$ | $-1$ | $1$ | $1/2$ | $-1/2$ | $\zeta^-\alpha$ |
| $\zeta_+$ | $1/2$ | $0$ | $1$ | $1$ | $1/2$ | $1/2$ | $\zeta_+\beta$ |
| $\tilde{\zeta}_-$ | $1/2$ | $0$ | $0$ | $-1$ | $-1/2$ | $-1/2$ | $\tilde{\zeta}_-\beta^{-1} \equiv \xi$ |
| $\Gamma^+$ | $-1/2$ | $1$ | $0$ | $0$ | $-1/2$ | $1/2$ | $\Gamma^+\alpha^{-1}$ |
| $\Gamma^-$ | $-1/2$ | $1$ | $-1$ | $2$ | $1/2$ | $-1/2$ | $\Gamma^-\alpha$ |
| $\Gamma_+$ | $1/2$ | $1$ | $1$ | $2$ | $1/2$ | $1/2$ | $\Gamma_+\beta$ |
| $\Gamma_-$ | $1/2$ | $1$ | $0$ | $0$ | $-1/2$ | $-1/2$ | $\Gamma_-\beta^{-1}$ |
| $M$ | $0$ | $0$ | $1$ | $0$ | $0$ | $1$ | $M\alpha^{-1}\beta = M_+$ |
| $\bar{M}$ | $0$ | $0$ | $-1$ | $0$ | $0$ | $-1$ | $M\alpha\beta^{-1} = M_-$ |
| $X^I$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |           |
| $\bar{X}^I$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |           |
| $\lambda_{\pm}, \psi^i$ | $1/2$ | $0$ | $1$ | $-1$ | $-1/2$ | $1/2$ | $\lambda_{\pm}\alpha^{-1} \equiv \chi_{\pm}$ |
| $\bar{\lambda}_{\pm}, \psi^i$ | $1/2$ | $0$ | $0$ | $1$ | $1/2$ | $-1/2$ | $\bar{\lambda}_{\pm}\alpha$ |
| $\lambda_{\pm}, \bar{\psi}^i$ | $-1/2$ | $0$ | $0$ | $1$ | $1/2$ | $1/2$ | $\lambda_{\pm}\beta$ |
| $\bar{\lambda}_{\pm}, \bar{\psi}^i$ | $-1/2$ | $0$ | $-1$ | $-1$ | $-1/2$ | $-1/2$ | $\bar{\lambda}_{\pm}\beta^{-1} \equiv \chi_{\pm}$ |
| $H^I$ | $0$ | $0$ | $1$ | $0$ | $0$ | $1$ | $H\alpha^{-1}\beta$ |
| $H^I$ | $0$ | $0$ | $-1$ | $0$ | $0$ | $-1$ | $H\alpha\beta^{-1}$ |
| $\alpha$ | $-1/2$ | $1$ | $0$ | $0$ | $-1/2$ | $1/2$ |           |
| $\beta$ | $1/2$ | $1$ | $0$ | $0$ | $-1/2$ | $-1/2$ |           |
| Field Theory | Geometry |
|-------------|----------|
| $\hat{m}_i$ | $d_m$ |
| $\hat{\nu}_j$ | $i_{v_j}$ |
| $\nu_j$ | $L_{v_j}$ |
| $Q$ | $d$ |

$[Q, m_i] = \hat{m}_i$

$\{Q, \hat{m}_i\} = 0$

$\{Q, \hat{\nu}_j\} = \nu_j$

$[\hat{\nu}_j, \nu_k] = 0$

$[\nu_j, \nu_k] = 0$

$\prod_{j=1}^g \delta \left( \int \omega \, J_z \, d^2 z \right)$

$\prod_{j=1}^g \int \omega \, G_z \, d^2 z$

$\prod_{j=1}^g \int \omega \, G_{zz} \, d^2 z \cdot \delta \left( \int \omega \, J_z \, d^2 z \right)$

$\sigma_{n_j}$

$\sigma_{n_1} \cdots \sigma_{n_k}$

$\int_{V} [c_1 (\mathcal{L}_1)]^{n_1} \wedge \cdots \wedge [c_1 (\mathcal{L}_k)]^{n_k}$