GENERALIZED COVERING MAP THEORY FOR GRAPHS

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ABSTRACT. In this paper, we introduce the notions of $r$-covering maps and $r$-fundamental groups for a positive integer $r$ and investigate their basic properties. There exists a natural relation between $r$-covering maps and the subgroups of $r$-fundamental groups. After establishing the basic properties about $r$-coverings and $r$-fundamental groups, we obtain some conditions a graph has a map to odd cycles from $r$-fundamental groups.

1. Introduction

The covering space theory is a basic theory in the algebraic topology, which states that subgroups of the fundamental group naturally correspond the covering spaces over the space, and is applied in many branches of the topology. A reference for classical covering space theory is [3] for example.

The author introduced the notions of 2-covering maps and 2-fundamental groups for graphs in [5], and gives a relation between the chromatic number. In this paper, we generalize this theory to $r$-covering maps and $r$-fundamental groups for any positive integer $r$. $r$-covering maps and $r$-fundamental groups are closely related, as is the case of the covering space theory in topology. For example, the following theorem hold. This is the main theorem in this paper.

Theorem 5.11. Let $(G, v)$ be a based graph and $r$ a positive integer. We write $\mathcal{X}_r(G, v)$ for the category of connected based $r$-covering over $(G, v)$ whose morphism is a graph map over $(G, v)$, and $\mathcal{Y}_r(G, v)$ for the small category of all subgroups of $\pi_1^r(G, v)$ whose morphisms are inclusion maps. Then the functor $F : \mathcal{X}_r(G, v) \to \mathcal{Y}_r(G, v), p \mapsto \text{Im} p_*$ is a categorical equivalence.

The theory of $r$-covering maps gives some obstructions of the existence of the graph map $G$ to an odd cycle as follows.

Corollary 6.5. Let $G$ be a connected graph with $\chi(G) \neq 2$ and $m$ an odd integer greater than 1. If there is a graph map $G \to C_m$, the abelianization of $\pi_1^r(G, v)$ has $\mathbb{Z}$ as a direct summand for any $r < m$.

This paper is organized as follows. In Section 2, we collect the basic terminologies and definitions about graphs. Especially, we need the notion of the $\times$-homotopy defined in [2]. In Section 3, we give the definition of $r$-covering maps and investigate their basic properties. In Section 4, we give the definition of $r$-fundamental groups and investigate their basic properties. In Section 5, we investigate the relation between based $r$-covering maps and $r$-fundamental groups and prove the first main theorem. In Section 6, we compute all $r$-coverings over cycles and prove Corollary 6.5.
2. Definition

A graph is a pair \((V, E)\), where \(V\) is a set and \(E\) is a subset of \(V \times V\) such that \((x, y) \in E\) implies \((y, x) \in E\). So our graphs are nondirected, have no parallel edges and may have loops. For a graph \(G = (V, E)\), \(V\) is called the vertex set of \(G\), denoted by \(V(G)\), and \(E\) is called the edge set of \(G\) and is denoted by \(E(G)\). For vertices \(v, w \in V(G)\), we often write \(v \simeq w\) if \((v, w)\). A graph map or a graph homomorphism from a graph \(G\) to a graph \(H\) is a map \(f : V(G) \to V(H)\) with \((f \times f)(E(G)) \subset E(H)\). For a vertex \(v \in V(G)\), we write \(N(v)\) for the set \(\{w \in V(G) \mid (v, w) \in E(G)\}\). More generally we write \(N(A) = \bigcup_{v \in A} N(v)\) for any subset \(A \subset V(G)\).

For a graph \(G\) and \(H\), we define the product graph \(G \times H\) by \(V(G \times H) = V(G) \times V(H)\) and \(E(G \times H) = \{( (x_0, y_0), (x_1, y_1) ) \mid (x_0, x_1) \in E(G)\) and \((y_0, y_1) \in E(G)\} \). A based graph is a pair \((G, v)\) where \(G\) is a graph and \(v\) is a vertex of \(G\). A based graph map is a graph map preserving basepoints.

For a nonnegative integer \(n \geq 0\), the graph \(K_n\) is defined by \(V(K_n) = \{0, 1, \cdots , n - 1\}\) and \(E(K_n) = \{(i, j) \mid i \neq j\}\). For a graph \(G\), we define the chromatic number of \(G\) by the number
\[
\chi(G) = \inf \{n \geq 0 \mid \text{There is a graph map } G \to K_n.\}
\]

Let \(G\) and \(H\) be graphs. Then a map \(\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}\) is called a multihomomorphism from \(G\) to \(H\) if \(\eta(v) \times \eta(w) \subset E(H)\) for any \((v, w) \in E(G)\). Let \(\eta_0\) and \(\eta_1\) be multihomomorphisms from \(G\) to \(H\). Then we write \(\eta_0 \leq \eta_1\) if \(\eta_0(v) \subset \eta_1(v)\) for any \(v \in V(G)\). We write \(\text{Hom}(G, H)\) for the poset of all multihomomorphisms from \(G\) to \(H\) and call the \(\text{Hom}\) complex from \(G\) to \(H\). For further reference about \(\text{Hom}\) complex, see [1], [4] for example.

\(\times\)-homotopy is defined by Dochtermann in [2] as follows. For nonnegative integer \(n\), a graph \(I_n\) is defined by \(V(I_n) = \{0, 1, \cdots , n\}\) and \(E(I_n) = \{(x, y) \mid |x - y| \leq 1\}\). Let \(f\) and \(g\) be graph maps from a graph \(G\) to a graph \(H\). A \(\times\)-homotopy from \(f\) to \(g\) is a graph map \(F : G \times I_n \to H\) such that \(F(x,0) = f(x)\) and \(F(x,n) = g(x)\) for any \(x \in V(G)\). If there is a \(\times\)-homotopy from \(f\) to \(g\), we say that \(F\) is \(\times\)-homotopic to \(g\), and written by \(f \simeq_{\times} g\) or simply \(f \simeq g\). It is easy to see that \(f\) is \(\times\)-homotopic to \(g\) if and only if \(f\) and \(g\) are in the same component of \(\text{Hom}(G, H)\).

A based \(\times\)-homotopy is defined similarly considered in [5]. For based graph maps \(f, g : (G, v) \to (H, w)\), a based \(\times\)-homotopy from \(f\) to \(g\) is a \(\times\)-homotopy \(F : G \times I_n \to H\) from \(f\) to \(g\) such that \(F(v, i) = w\) for \(i \in \{0, 1, \cdots , n\}\). If there is a based \(\times\)-homotopy from \(f\) to \(g\), we say that \(f\) is \(\times\)-homotopic to \(g\) in the based sense. A based multihomomorphism from \((G, v)\) to \((H, w)\) is a multihomomorphism \(\eta\) from \(G\) to \(H\) such that \(\eta(v) = \{w\}\). We write \(\text{Hom}((G, v), (H, w))\) for the poset of all based multihomomorphism from \((G, v)\) to \((H, w)\). One can easily show that for based graph maps \(f\) and \(g\) from \((G, v)\) to \((H, w)\), \(f\) and \(g\) are \(\times\)-homotopic if and only if \(f\) and \(g\) are in the same component in \(\text{Hom}((G, v), (H, w))\).

3. \(r\)-Covering Maps

Let \(G\) be a graph and \(v \in V(G)\). We write \(N(v)\) for the set \(\{w \in V(G) \mid (v, w) \in E(G)\}\). We define inductively \(N_r(v)\) as follows:
\[
N_0(v) = \{v\}, N_r(v) = \bigcup_{w \in N_{r-1}(v)} N(w), (r \geq 1).
\]
Definition 3.1. Let $r$ be a positive integer. A graph map $p : G \to H$ is called an $r$-covering map if $p|_{N_i(v)} : N_i(v) \to N_i(p(v))$ is bijective for any $0 \leq i \leq r$.

1-covering map is called a covering map in the usual sense.

We do not assume that $r$-covering maps are surjective on vertex sets. Hence $\emptyset \to G$ is an $r$-covering map.

Example 3.2. The followings are the fairly simple examples of $r$-covering maps.

1. An identity map is an $r$-covering map.
2. For a graph $G$, the second projection $K_2 \times G \to G$ is a $r$-covering map. More generally if a graph $T$ such that $\sharp N(x) = 1$ for any vertex $x \in V(T)$, the second projection $T \times G \to G$ is an $r$-covering map.

Lemma 3.3. Let $r$ be a positive integer and $p : G \to H$ a graph map. Then $p$ is an $r$-covering map if and only if $p|_{N(v)} : N(v) \to N(p(v))$ is surjective and $p|_{N_i(v)} : N_i(v) \to N_i(p(v))$ is injective for any $v \in V(G)$.

Proof. The “only if” part is obvious. Hence we only prove “if” part.

Suppose $p|_{N(v)}$ is surjective and $p|_{N_i(v)}$ is injective for any $v \in V(G)$. First we show that $p|_{N_i(v)}$ is surjective for any $i \geq 1$ by the induction on $i$. Suppose $p|_{N_i(v)} : N_i(v) \to N_i(p(v))$ is surjective for any $v \in V(G)$. Let $v \in V(G)$ and let $x \in N_{i+1}(p(v))$. Since $p|_{N_i(v)}$ is surjective, there exists $w \in N_i(v)$ with $x \in N(p(w))$. Since $p|_{N(v)}$ is surjective, there is $u \in N(w)$ with $p(u) = x$. Since $u \in N_{i+1}(v)$, we have that $p|_{N_{i+1}(v)}$ is surjective.

Next we show that $p|_{N_i(v)}$ is injective for any $1 \leq i \leq r$ and $v \in V(G)$. Suppose $i < r$ and $p|_{N_{i+1}(v)}$ is injective for any $v \in V(G)$. Let $v$ be a vertex of $G$. Let $w_0, w_1 \in N_i(v)$ with $p(w_0) = p(w_1)$. Since $w_0, w_1 \in N_i(v)$, we have $N_i(v) \neq \emptyset$, and hence we have $N(v) \neq \emptyset$. Let $u \in N(v)$. Then $w_0, w_1 \in N_{i+1}(u)$. Since $p|_{N_{i+1}(w)}$ is injective, we have that $w_0 = w_1$. Hence $p|_{N_i(v)}$ is injective.

Lemma 3.4. Let $f : G \to H, g : H \to K$ be graph homomorphisms and $r$ a positive integer. Then the followings hold.

1. If $g$ and $f$ are $r$-covering maps, then $gf$ is an $r$-covering map.
2. If $g$ and $gf$ are $r$-covering, then $g$ is an $r$-covering map.
3. If $f$ is surjective on vertex sets, and $gf$ is an $r$-covering map, then $g$ is an $r$-covering map.

Proof. Let $v \in V(G)$. We have the following commutative diagrams

$$
\begin{array}{ccc}
N_i(v) & \xrightarrow{f|_{N_i(v)}} & N_i(f(v)) \\
\downarrow g|_{N_i(v)} & & \downarrow g|_{N_i(f(v))} \\
N_i(gf(v)) & \xrightarrow{N_i(gf(v))} & N_i(gf(v))
\end{array}
$$

If one of (1), (2), (3) holds, two of three arrows in the diagram are bijective. Hence so is third.

Definition 3.5. Let $\Gamma$ be a group and $G$ a graph and $\alpha$ a right $\Gamma$-action on $G$. $\alpha$ is called an $r$-covering action if for every $v \in V(G)$ and $\gamma \in \Gamma \setminus \{e\}$, we have $N_r(v) \cap N_r(v\gamma) = \emptyset$. We define similarly that a left $\Gamma$-action on a graph is a $r$-covering action.
If an action \( G \rtimes \Gamma \) is an \( r \)-covering action, we have \( N_i(v) \cap N_i(v\gamma) = \emptyset \) for \( v \in V(G) \), \( \gamma \in \Gamma \setminus \{e_{\Gamma}\} \), and for \( 1 \leq i \leq r \). Hence for \( 1 \leq s \leq r \), an \( r \)-covering action is an \( s \)-covering action.

For a graph \( G \) we write \( \hat{G} \) for a graph with the vertex set is the set of nonisolated vertices of \( G \) and \( E(\hat{G}) = E(G) \). We remark that a \( \Gamma \)-action on a graph \( G \) induces the \( \Gamma \)-action on \( \hat{G} \) and \( (\hat{G}/\Gamma) = \hat{G}/\Gamma \).

The \( \Gamma \)-action on \( G \) is an \( r \)-covering action if and only if the induced action on \( \hat{G} \) is an \( r \)-covering action. A graph map \( p : G \to H \) is an \( r \)-covering map if and only if \( \hat{p} : \hat{G} \to \hat{H} \) is an \( r \)-covering map.

**Proposition 3.6.** Let \( G \) be a graph, \( \Gamma \) a group, \( \alpha \) a right \( \Gamma \)-action on \( G \), and \( r \) a positive integer greater than 1. Consider the following three conditions.

1. \( \alpha \) is an \( r \)-covering action.
2. \( \alpha \) is free on \( V(\hat{G}) \) and the quotient map \( p : G \to G/\Gamma \) is an \( r \)-covering map.
3. \( \alpha \) is effective on \( V(\hat{G}) \) and the quotient map \( p : G \to G/\Gamma \) is an \( r \)-covering map.

In any case, (1) and (2) are equivalent. If \( \hat{G} \) is connected, then the above conditions are equivalent.

**Proof.** Replacing \( G \) to \( \hat{G} \), we can assume that \( G \) has no isolated vertices.

(1) \( \Rightarrow \) (2) : Suppose \( \alpha \) is an \( r \)-covering action. Since we assume that \( G \) has no isolated vertices, we have \( v \in N_2(v) \) for every vertex \( v \) of \( G \). Since \( N_2(v) \cap N_2(v\gamma) = \emptyset \) for \( \gamma \in \Gamma \setminus \{e_{\Gamma}\} \).

Let \( v \in V(G) \). We want to show that \( p|_{N(v)} \) is surjective and \( p|_{N_r(v)} \) is injective. Let \( a \in N(p(v)) \) and \( w \in a \). Then there exists \( \gamma_1, \gamma_2 \in \Gamma \) with \( (v\gamma_1, w\gamma_2) \in E(G) \). Then \( w\gamma_2\gamma_1^{-1} \in N(v) \) and \( p(w\gamma_2\gamma_1^{-1}) = a \). Hence \( p|_{N(v)} \) is surjective. Let \( w_1, w_2 \in N_r(v) \) with \( p(w_1) = p(w_2) \). Then there is \( \gamma \in \Gamma \) with \( w_1\gamma = w_2 \). Since \( v \in N_r(w_1) \cap N_r(w_2) \), we have \( \gamma = e_{\Gamma} \) from the definition of \( r \)-covering actions and hence \( w_1 = w_2 \). Thus \( p|_{N_r(v)} \) is injective.

(1) \( \Leftarrow \) (2) : Let \( v \in V(G) \) and \( \gamma \in \Gamma \). Suppose \( N_r(v) \cap N_r(v\gamma) \neq \emptyset \) and let \( w \in N_r(v) \cap N_r(v\gamma) \). Then we have \( v, v\gamma \in N_r(w) \) and \( p(v) = p(v\gamma) \). Hence we have \( v = v\gamma \) since \( G \to G/\Gamma \) is an \( r \)-covering map. Since the \( \Gamma \)-action \( \alpha \) on \( G \) is free, we have \( \gamma \) is the identity of \( \Gamma \). Therefore \( \alpha \) is an \( r \)-covering action.

(2) \( \Rightarrow \) (3) is obvious. We suppose that \( G \) is connected and prove (2) \( \Leftarrow \) (3). Let \( v \in V(G) \) and \( \gamma \in \Gamma \) and suppose \( v = v\gamma \). Then the map \( f_\gamma : G \to G \), \( x \mapsto x\gamma \) has a fixed point \( v \). Then for any vertex \( w \in N(v) \) is fixed point of \( \gamma \) since \( w, w\gamma \in N(v) = N(v\gamma) \) and \( p(w) = p(w\gamma) \). By induction, we have that \( f_\gamma(w) = w \) for any \( w \in V(G) \). Hence we have \( f_\gamma = \text{id}_G \). Since the action \( \alpha \) is effective, we have \( \gamma \) is the identity of \( \Gamma \). Hence we have \( \alpha \) is free. \( \square \)

**Example 3.7.** Let \( G \) be a connected graph with \( \chi(G) = 2 \). Then there exist two independent subsets \( A_0, A_1 \) of \( G \) such that \( A_0 \cup A_1 = V(G) \) and \( A_0 \cap A_1 = \emptyset \). Furthermore, this unordered pair \( \{A_0, A_1\} \) is uniquely determined. An involution \( \tau : G \to G \) is said to be odd if \( \tau(A_0) = A_1 \). Given an odd involution \( \tau \) on \( G \), the associated \( \mathbb{Z}/2\mathbb{Z} \)-action on \( G \) is \( r \)-covering action for any positive integer \( r \). Indeed, for any \( v \in A_0 \), \( N_r(v) \subset A_{(r \mod 2)} \) and \( N_r(\tau(v)) \subset A_{(r+1 \mod 2)} \).

### 4. \( r \)-FUNDAMENTAL GROUPS

Let \( n \) be a nonnegative integer. The graph \( L_n \) is defined by \( V(L_n) = \{0,1,\cdots,n\} \) and \( E(L_n) = \{(x,y) \mid |x-y| = 1\} \). A graph map \( f : L_n \to G \) is called a path of \( G \) with length \( n \). For \( v, w \in V(G) \), we denote the set of all paths from \( v \) to \( w \) by \( P(G;v,w) \). For a path \( \varphi \) of \( G \), we write
Lemma 4.1. \(l(\varphi)\) for \(\varphi\). Let \(r\) be a positive integer. We consider the following two conditions for \((\varphi, \psi) \in P(G; v, w) \times P(G; v, w)\).

1. \(l(\varphi) + 2 = l(\psi)\) and there exists \(x \in \{0, 1, \cdots, l(\varphi)\}\) with \(\varphi(i) = \psi(i)\) for \(i \leq x\) and \(\varphi(i) = \psi(i+2)\) for \(i \geq x\).

2. \(l(\varphi) = l(\psi)\) and there exists \(x \in \{1, 2, \cdots, l(\varphi)-1\}\) such that \(\varphi(i) = \psi(i)\) if \(i \notin \{x, x+1, \cdots x + r-2\}\).

The condition (ii) implies \(\varphi = \psi\). We write \(\sim_r\) for the equivalence relation on \(P(G; v, w)\) generated by the above two conditions, and write \(\pi^r_1(G; v, w)\) for the quotient set \(P(G; v, w)/\sim_r\). For paths \(\varphi\) and \(\psi\) from \(v\) to \(w\), we say that \(\varphi\) is \(r\)-homotopic if \(\varphi \sim_r \psi\). We write \([\varphi]_r\) for the equivalence class of \(\sim_r\) represented by \(\varphi\).

Let \(\varphi : L_n \to G\) and \(\psi : L_m \to G\) be paths with \(\varphi(n) = \psi(0)\). We define the path \(\psi \circ \varphi : L_{n+m} \to G\) by \((\psi \circ \varphi)(i) = \varphi(i)\) for \(i \leq n\) and \((\psi \circ \varphi)(i) = \psi(i-n)\) for \(i \geq n\).

**Lemma 4.1.** Let \(r\) be a positive integer and \(G\) a graph and \(u, v, w \in V(G)\) and \(\varphi, \varphi' \in P(G; v, w)\) and \(\psi, \psi' \in P(G; u, v)\). If \(\varphi \sim_r \varphi'\) and \(\psi \sim_r \psi'\) then we have \(\varphi \circ \psi \sim_r \varphi' \circ \psi'\).

**Proof.** We can assume \(\varphi = \varphi'\) or \(\psi = \psi'\). Suppose \(\varphi = \varphi'\). It is sufficient to show that if \(\psi\) and \(\psi'\) satisfy the condition (i) or (ii), then \(\varphi \circ \psi \sim_r \varphi' \circ \psi'\). But this is obvious. The case \(\psi = \psi'\) is similar. \(\Box\)

From Lemma 4.1, the composition map \(P(G; v, w) \times P(G; u, v) \to P(G; u, w)\) induces a map \(\pi^r_1(G; v, w) \times \pi^r_1(G; u, v) \to \pi^r_1(G; u, w)\).

**Definition 4.2.** For a based graph \((G, v)\), \(\pi^r_1(G; v, v)\) is denoted by \(\pi^r_1(G, v)\). \(\pi^r_1(G, v)\) is a group with the composition and is called the \(r\)-fundamental group of \((G, v)\).

Let \(f : (G, v) \to (H, w)\) be a based graph map. Then it is easy to see that \(f\) induces a group homomorphism \(\pi^r_1(G, v) \to \pi^r_1(H, w)\), \([\varphi]_r \mapsto [f \circ \varphi]_r\). We denote this group map by \(\pi^r_1(f)\) or \(f_*\). Let \(r\) and \(s\) be positive integers with \(s \leq r\). From the definition of \(r\)-homotopy of paths, \(\varphi \sim_s \psi\) implies \(\varphi \sim_r \psi\) for any loops \(\varphi\) and \(\psi\) of a based graph \((G, v)\). Hence there is a surjective group homomorphism \(\pi^r_1(G, v) \to \pi^s_1(G, v)\). This group maps are natural with respect to based graph maps.

We remark that for each loops \(\varphi\) and \(\psi\) of \((G, v)\), \(\varphi \sim_r \psi\) implies \(l(\varphi) = l(\psi)\) modulo 2. Therefore we have a well-defined group homomorphism

\[\pi^r_1(G, v) \to \mathbb{Z}/2\mathbb{Z}, [\varphi] \mapsto (l(\varphi) \mod 2).\]

We write \(\pi^r_1(G, v)_{\text{ev}}\) for the kernel of the above group homomorphism and call the even part of the \(r\)-fundamental group of \((G, v)\). For any positive integer \(r\), \(\pi^r_1(G, v)_{\text{ev}} = \pi^r_1(G, v)\) if and only if the chromatic number of the component containing \(v\) is equal to 2.

**Relation with \(\times\)-homotopy.**

**Lemma 4.3.** Let \(G\) be a graph, and \(v, w \in V(G)\). We write \(\sim_2\) for the equivalence relation on \(P(G; v, w)\) generated by the condition (i) in the definition of \(r\)-homotopy of paths and the condition (ii) for paths \(\varphi, \psi \in P(G; v, w)\) defined as follows :

(ii) \(l(\varphi) = l(\psi)\) and \((\varphi \times \psi)(E_{L(\varphi)}) \subset E(G)\).
Then \(\simeq_2\) coincides \(\simeq'_2\).

**Proof.** Let \(\varphi\) and \(\psi\) be paths from \(v\) to \(w\). If \(\varphi\) and \(\psi\) satisfy the condition \((ii)_2\), then \(\varphi\) and \(\psi\) satisfy the condition \((ii)'_2\). Hence \(\approx_2\) implies \(\approx'_2\). Suppose \(\varphi\) and \(\psi\) satisfy the condition \((ii)'_2\). Write \(n\) for the length of \(\varphi\). We write \(\varphi_k : L_n \to G\) \((k = 1, \ldots, n - 1)\) for the path defined by \(\varphi_k(i) = \psi(i)\) for \(i < k\) and \(\varphi_k(i) = \psi(i)\) for \(i \geq k\). Then \(\varphi_k\) and \(\varphi_{k-1}\) satisfy the condition \((ii)'_2\). Hence \(\approx'_2\) implies \(\approx_2\). \(\square\)

**Lemma 4.4.** Suppose \(r\) is an integer greater than 1. Let \(f, g : (G, v) \to (H, w)\) be graph maps with \(f \simeq_\times g\) in the based sense. Then we have that \(\pi^r_1(f) = \pi^r_1(g) : \pi^r_1(G, v) \to \pi^r_1(H, w)\).

**Proof.** We can assume that \((f \times g)(E(G)) \subset E(H)\). Let \(\varphi : L_n \to G\) be a loop of \((G, v)\). Then we have that \((f \circ \varphi) \times (g \circ \varphi)(E(L_n)) \subset (f \times g) \circ (\varphi \times \varphi)(E(L_n)) \subset E(G)\). Hence \(f \circ \varphi\) is 2-homotopic to \(g \circ \varphi\). \(\square\)

Next we consider nonbased \(\times\)-homotopy. To formulate this, we require the adjoint map of paths.

Let \(G\) be a graph, \(v, w \in V(G)\), and \(\alpha \in \pi^r_1(G; v, w)\). Then \(\text{Ad}(\alpha) : \pi^r_1(G, v) \to \pi^r_1(G, w)\) be a map defined by \(\beta \mapsto \alpha \cdot \beta \cdot \overline{\pi}\). For a path \(\gamma\), we often abbreviate \(\text{Ad}(\gamma)\) for \(\text{Ad}(\gamma)\). Then we have that

- \(\text{Ad}(\beta \cdot \alpha) = \text{Ad}(\beta) \circ \text{Ad}(\alpha)\) if composable.
- \(\text{Ad}([*]_v) = \text{id}_{\pi^r_1(G, v)}\)

Hence we have that \(\text{Ad}([\pi]) \circ \text{Ad}(\alpha) = \text{id}\) and \(\text{Ad}(\alpha) \circ \text{Ad}([\pi]) = \text{id}\). Therefore \(\text{Ad}(\alpha)\) is an isomorphism of groups whose inverse is \(\text{Ad}([\overline{\pi}])\). In particular, if \(G\) is connected, the isomorphism class of the \(r\)-fundamental group of \(G\) is independent of the choice of the basepoint of \(G\). So we often abbreviate \(\pi^r_1(G, v)\) to \(\pi^r_1(G)\) if \(G\) is connected.

**Proposition 4.5.** Let \(r\) be an integer greater than 1, \(f\) and \(g\) graph maps from a graph \(G\) to a graph \(H\) with \(f \simeq_\times g\). Let \(v\) be a nonisolated point of \(G\). Then there is a path \(\gamma\) from \(f(v)\) to \(g(v)\) such that \(\text{Ad}(\gamma) \circ \pi^r_1(f) = \pi^r_1(g) : \pi^r_1(G, v) \to \pi^r_1(H, g(v))\).

**Proof.** Let \(F : G \times I_n \to H\) be a \(\times\)-homotopy from \(f\) to \(g\). Let \(w \in N(v)\). We write \(\gamma' : L_{2n} \to G \times I_n\) for the path of \(G \times I_n\) defined by \(\gamma'(2i) = (v, i)\) for \(0 \leq i \leq n\) and \(\gamma'(2i - 1) = (w, i)\) for \(1 \leq i \leq n\). We write \(\gamma\) for a path \(F \circ \gamma'\). We want to prove that \(\gamma\) is the desired one.

Let \(\varphi\) be a loop of \((G, v)\). Put \(\varphi_0 = i_0 \circ \varphi\) and \(\varphi_n = i_n \circ \varphi\), where \(i_k : G \to G \times I_n\) is a graph map defined by \(x \mapsto (x, k)\) for \(k \in \{0, 1, \ldots, n\}\). We claim that \(\varphi_0 \simeq_r \overline{\gamma} \cdot \varphi_n \cdot \gamma'\). Let \(q : G \times I_n \to G \times I_n\) be the graph map defined by \(q(x, t) = (x, 0)\). Then \(q\) is \(\times\)-homotopic to the identity in the based sense, where we consider the basepoint of \(G \times I_n\) is \((v, 0)\). Hence we have

\[ \overline{\gamma} \cdot \varphi_n \cdot \gamma' \simeq_r q(\overline{\gamma} \cdot \varphi_n \cdot \gamma') \simeq_r \varphi_0 \]

from the previous lemma. Therefore we have

\[ f_*([\varphi]_r) = F_*([\varphi_0]_r) = F_*([\overline{\gamma} \cdot \varphi_n \cdot \gamma']_r) \overline{\gamma} \cdot g_*([\varphi]_r) \cdot \gamma = \text{Ad}(\gamma)(g_*([\varphi]_r)). \]

Hence we have \(g_* = \text{Ad}(\gamma) \circ f_*\). \(\square\)

**Corollary 4.6.** Let \(r\) be an integer greater than 1 and \(f : G \to H\) be a \(\times\)-homotopy equivalence. Then for any nonisolated vertex \(v\) of \(G\), the map \(\pi^r_1(f) : \pi^r_1(G, v) \to \pi^r_1(H, f(v))\) is an isomorphism.
Proposition 5.4. Let \( g \) be a \( \times \)-homotopy inverse of \( f \). Then from the previous proposition, \( \pi^1_1(g \circ f) \) is an adjoint of paths and, in particular, is an isomorphism. Hence \( \pi^1_1(f) \) is injective. Similarly, we can prove that \( \pi^1_1(f \circ g) \) is an isomorphism. Hence \( \pi^1_1(f) \) is surjective. 
\[ \square \]

5. Relation to \( r \)-covering maps and \( r \)-fundamental groups

In this section, we assume that \( r \) is a fixed positive integer. A based graph map \( p : (G, v) \to (H, w) \) is called a based \( r \)-covering map if \( p : G \to H \) is an \( r \)-covering map.

Lemma 5.1. Let \( p : (G, v) \to (H, w) \) be a based \( r \)-covering map for \( r \geq 1 \).

(1) For a path \( \varphi : (L_n, 0) \to (H, w) \), there exists a unique \( \tilde{\varphi} : (L_n, 0) \to (G, v) \) with \( \varphi = p \circ \tilde{\varphi} \).

(2) Let \( u \in V(H) \) and paths \( \varphi \) and \( \psi \) of \( H \) from \( w \) to \( u \). We write \( \tilde{\varphi} \) and \( \tilde{\psi} \) for the lifts of \( \varphi \) and \( \psi \) whose initial points are \( v \). If \( \varphi \simeq_r \psi \), then the terminal points of \( \tilde{\varphi} \) and \( \tilde{\psi} \) coincide and \( \tilde{\varphi} \simeq_r \tilde{\psi} \).

Proof. The proof of (1) is easy and is left to the reader.

We can assume that \( \varphi \) and \( \psi \) satisfies the condition (i) or (ii) in the definition of \( r \)-homotopy of paths. Suppose \( (\varphi, \psi) \) satisfies (i). We write \( l(\varphi) = n \) and \( x \in \{0, 1, \ldots, n\} \) with \( \varphi(i) = \psi(i) \) for \( i \leq x \) and \( \varphi(i) = \psi(i + 2) \) for \( i \geq x \). Then the uniqueness of the lift of \( \varphi|_{L_n} \) implies \( \tilde{\varphi}(i) = \tilde{\psi}(i) \) for \( i \leq x \) and the uniqueness of the lift of \( L_{n-x} \to G, i \mapsto \varphi(i + x) \) implies \( \tilde{\varphi}(i) = \tilde{\psi}(i + 2) \) for \( i \geq x \). Hence the terminal points of \( \tilde{\varphi} \) and \( \tilde{\psi} \) coincide and \( (\tilde{\varphi}, \tilde{\psi}) \) satisfies the condition (i).

Secondly we assume that \( (\varphi, \psi) \) satisfy the condition (ii). Then \( n := l(\varphi) = l(\psi) \) and there is \( x \in \{1, \ldots, n-r+1\} \) with \( \varphi(i) = \psi(i) \) for \( i \notin \{x, x+1, \ldots, x+r-2\} \). From the uniqueness of the lift, we have \( \tilde{\varphi}(x-1) = \tilde{\psi}(x-1) \). Since \( \varphi|_{L_{n-x}} \to G, i \mapsto \varphi(i + x) \) implies \( \tilde{\varphi}(i) = \tilde{\psi}(i + x) \) and \( \varphi(x + r - 1) = \psi(x + r - 1) \), we have \( \tilde{\varphi}(x + r - 1) = \tilde{\psi}(x + r - 1) \) and hence we have \( \tilde{\varphi}(n) = \tilde{\psi}(n) \). Hence \( (\tilde{\varphi}, \tilde{\psi}) \) satisfies the condition (ii).

\[ \square \]

Corollary 5.2. Let \( p : (G, v) \to (H, w) \) be a based \( r \)-covering map. Then \( p_* : \pi^1_1(G, v) \to \pi^1_1(H, w) \) is injective.

Proof. Let \( \varphi \) be a loop in \( (G, v) \) with \( p \circ \varphi \simeq_* \). From the previous lemma, we habe that the \( r \)-homotopy class of \( \varphi \) is equal to the \( r \)-homotopy class of the lift of \( *_w \). Hence we have that \( \varphi \simeq *_v \).

\[ \square \]

Proposition 5.3. Let \( p : (G, v) \to (H, w) \) be a based \( r \)-covering map. Then \( p_* : \pi^1_1(G, v) \to \pi^1_1(H, w) \) is injective. Let \( \varphi \) be a loop of \( (H, w) \). Then \( [\varphi] \in p_*(\pi^1_1(G, v)) \) if and only if the lift of \( \varphi \) whose initial point is \( v \) is a loop of \( (G, v) \).

Proof. Let \( \varphi \) be a loop of \( (H, w) \) and \( \tilde{\varphi} \) be the lift of \( \varphi \) whose initial point is \( v \). If \( \tilde{\varphi} \) is a loop of \( (G, v) \), we have \( [\varphi] = p_*[\tilde{\varphi}] \in p_*([\pi^1_1(G, v)]) \). On the other hand, suppose \( [\varphi] \in p_*([\pi^1_1(G, v)]) \). Then there exists a loop \( \tilde{\psi} \) of \( (G, v) \) with \( p \circ \tilde{\psi} \simeq_r \varphi \). From Lemma 5.1, the terminal point of \( \tilde{\varphi} \) is equal to the one of \( \psi \), and hence \( \tilde{\varphi} \) is a loop of \( (G, v) \).

\[ \square \]

Proposition 5.4. Let \( p : (G, v) \to (H, w) \) be a based \( r \)-covering map and \( (T, x) \) a connected based graph, and \( f : (T, x) \to (H, w) \) be a based map. Then there exists a graph map \( \tilde{f} : (T, x) \to (G, v) \) such that \( p \circ \tilde{f} = f \) if and only if \( f_*([\pi^1_1(T, x)]) \subset p_*([\pi^1_1(G, v)]) \).

\[ \square \]
Proof. Suppose there exists \( \hat{f} : (T, x) \to (G, v) \) such that \( p \circ \hat{f} = f \). Then we have \( f_*(\pi_1^r(T, x)) = p_* \circ \hat{f}_*(\pi_1^r(T, x)) \subset p_*(\pi_1^r(G, v)) \).

Suppose \( f_*(\pi_1^r(T, x)) \subset p_*(\pi_1^r(G, v)) \). Let \( y \in V(T) \) and \( \varphi \) a path from \( x \) to \( y \). We write \( \tilde{\varphi} \) for the lift of \( f \circ \varphi \). Hence we want to define that \( \hat{f}(y) \) is the terminal point of \( \tilde{\varphi} \).

We prove that \( \hat{f} \) is well-defined. Let \( \psi \) be another path from \( x \) to \( y \) and \( \tilde{\psi} \) be the lift of \( f \circ \psi \) whose initial point is \( v \). We write \( \gamma \) for the lift of \( f \circ (\psi \cdot \varphi) \) whose initial point is \( v \). Then \( \gamma \) is a loop since \( f_*(\pi_1^r(T, x)) \subset p_*(\pi_1^r(G, v)) \). Since the lifts of two paths which are \( r \)-homotopic have the same terminal point and the lift of \( f \circ (\psi \cdot \varphi) \) is \( \tilde{\psi} \cdot \gamma \), we have that \( \tilde{\psi} \) and \( \tilde{\varphi} \) have the same terminal point. Hence \( \hat{f} \) is well-defined.

Next we prove that \( \hat{f} \) is a graph map. Let \( (y_0, y_1) \in E(T) \). Let \( \varphi_0 \) be a path from \( x \) to \( y_0 \), and \( \varphi_1 : L_\varphi \to T \) be a path defined by \( \varphi_1|_{L_\varphi} = \varphi_0 \) and \( \varphi_1(i) = y_1 \). We write \( \tilde{\varphi}_i \) for the lift of \( f \circ \varphi \) whose initial point is equal to \( v \) for each \( i = 1, 2 \). From the uniqueness of the lift of paths, we have that \( \tilde{\varphi}_1|_{L_\varphi} = \tilde{\varphi}_0 \). Hence \( \hat{f}(y_1) = \tilde{\varphi}_1(i(y_0) + 1) \sim \tilde{\varphi}_0(i(y_0)) = \hat{f}(y_0) \). Hence we have that \( \hat{f} \) is a graph map. \( \square \)

Remark 5.5. The lift \( \hat{f} \) is uniquely determined. More generally, the followings hold.

Let \( p : G \to H \) be a 1-covering map, \( T \) a connected graph and two graph maps \( f, g : T \to G \) with \( p \circ f = p \circ g \). If there exists \( x \in V(G) \) with \( f(x) = g(x) \), then \( f = g \).

Proposition 5.6. Let \( p : (G, v) \to (H, w) \) be a based \( r \)-covering map where \( G \) is connected. Then there is a bijection

\[
\Phi : \pi_1^r(H, w)/p_*(\pi_1^r(G, v)) \to p^{-1}(w).
\]

This bijection is constructed as follows. Let \( [\varphi] \in \pi_1^r(H, w)/p_*(\pi_1^r(G, v)) \). Then \( \Phi([\varphi]) \) is the terminal point of the lift of \( \varphi \) whose initial point is \( v \).

Proof. First we prove that \( \Phi \) is well-defined. Let \( \varphi \) and \( \psi \) be loops in \( (H, w) \) with \( [\varphi] = [\psi] \) in \( \pi_1^r(H, w)/p_*(\pi_1^r(G, v)) \). We write \( \tilde{\varphi} \) for the lift of \( \varphi \) with respect to \( p \) whose initial point is \( v \). There exists a loop \( \gamma \) in \( (G, v) \) such that \( \varphi \cdot (p \circ \gamma) \simeq \psi \). Then the terminal points of lifts of \( \varphi \cdot (p \circ \gamma) \) and \( \psi \) are coincide. Since the lift of \( \varphi \cdot (p \circ \gamma) \) is equal to \( \tilde{\varphi} \cdot \gamma \), we have that the terminal points of \( \tilde{\varphi} \) and \( \tilde{\psi} \) are coincide. Hence \( \Phi \) is well-defined.

The surjectivity of \( \Phi \) is easily deduced from the connectivity of \( G \). We prove that \( \Phi \) is injective. Let \( \varphi \) and \( \psi \) be loops in \( (H, w) \) with \( \Phi([\varphi]) = \Phi([\psi]) \). Let \( \gamma = \tilde{\varphi} \cdot \tilde{\psi} \). Then \( \tilde{\varphi} \cdot \gamma \simeq \tilde{\psi} \). Hence we have \( \varphi \cdot (p \circ \gamma) \simeq \psi \). This means that \( [\varphi] = [\psi] \) in \( \pi_1^r(H, w)/p_*(\pi_1^r(G, v)) \). Hence \( \Phi \) is injective. \( \square \)

A based \( r \)-covering map \( \hat{p} : (\hat{G}, \hat{v}) \to (G, v) \) is said to be universal if \( \hat{G} \) is connected and \( \pi_1^r(G, v) \) is trivial. We can easily prove that a universal \( r \)-covering over \( (G, v) \) is unique up to isomorphism over \( (G, v) \) from Proposition 5.4 and Remark 5.5.

Corollary 5.7. Let \( p : (\hat{G}, \hat{v}) \to (G, v) \) be a universal \( r \)-covering map. For each \( x \in p^{-1}(v) \), let \( \varphi_x \) be a path from \( \hat{v} \) to \( x \). Then \( [p \circ \varphi_x] \neq [p \circ \varphi_y]_r \) for \( x, y \in p^{-1}(v) \) with \( x \neq y \), and \( \pi_1^r(G, v) = \{[p \circ \varphi_x]_r \mid x \in p^{-1}(v)\} \).
Proposition 5.8. Let \((G,v)\) be a graph. Then there exists a universal covering over \((G,v)\).

Proof. We define \(V(\tilde{G})\) by the set \(\coprod_{\alpha \in \Gamma} \pi_1^\alpha(G,v,w)\) and \(E(\tilde{G}) = \{(\alpha, \beta) \mid \text{There is } \varphi \in \beta \text{ with } \varphi|_{L_{\tilde{G}}} = \varphi\} \subset \{w\} \text{ and } \tilde{v} = \{v\} \).

First we show that \(\tilde{G} = (V(\tilde{G}), E(\tilde{G}))\) is a graph. We claim that the following three conditions for \((\alpha, \beta) \in V(\tilde{G}) \times V(\tilde{G})\) are equivalent.

1. \((\alpha, \beta) \in E(\tilde{G})\).
2. For each \(\varphi \in \alpha\), the map \(\varphi' : V(L_{\tilde{G}}) \to V(G)\) defined by \(\varphi'|_{V(L_{\tilde{G}})} = \varphi\) and \(\varphi'(l(\varphi) + 1) = p(\beta)\) is a graph map and an element of \(\beta\).
3. There is \(\varphi \in \alpha\) such that the map \(\varphi' : V(L_{\tilde{G}}) \to V(G)\) defined by \(\varphi'|_{V(L_{\tilde{G}})}\) and \(\varphi'(l(\varphi) + 1) = p(\beta)\) is a graph map and an element of \(\beta\).

In fact (2) \(\Rightarrow\) (3) \(\Leftrightarrow\) (1) is obvious, and (3) \(\Rightarrow\) (2) is deduced from Lemma 4.1.

Let \((\alpha, \beta) \in E(\tilde{G})\) and \(\varphi \in \alpha\) and \(\varphi' \in \beta\) with \(\varphi'|_{L_{\tilde{G}}} = \varphi\). We define \(\varphi'' : L_{\tilde{G}} \to G\) by \(\varphi''(i) = \varphi'(i)\) for \(i \leq l(\varphi')\) and \(\varphi''(l(\varphi') + 1) = p(\alpha) = \varphi(l(\varphi)).\) Then since \(\varphi'' \simeq \varphi\) and hence \([\varphi''] \in \beta\). Therefore we have \((\beta, \alpha) \in E(\tilde{G})\).

Next we prove that \(p\) is an \(r\)-covering map. It is obvious that \(p\) is a graph map. Let \(\alpha \in V(\tilde{G})\). It is easy to show that \(p|_{N(\alpha)} : N(\alpha) \to N(p(\alpha))\) is surjective. So we only prove that \(p|_{N_r(\alpha)} : N_r(\alpha) \to N_r(p(\alpha))\) is injective. Let \(\beta_1, \beta_2 \in N_r(\alpha)\) with \(p(\beta_1) = p(\beta_2)\). Then there are sequences \(\gamma_0, \ldots, \gamma_r\) and \(\gamma'_0, \ldots, \gamma'_r\) of \(V(\tilde{G})\) with

\[
\begin{align*}
\alpha &= \gamma_0 \sim \gamma_1 \sim \cdots \sim \gamma_r = \beta_1, \\
\alpha &= \gamma'_0 \sim \gamma'_1 \sim \cdots \sim \gamma'_r = \beta_2.
\end{align*}
\]

Let a representative \(\varphi : L_n \to G\) of \(\alpha\). We define \(\varphi_1 : L_{n+r} \to G\) by \(\varphi_1|_{L_n} = \varphi\) and \(\varphi_1(n + i) = p(\gamma_i)\) for \(0 \leq i \leq r\) and \(\varphi_2 : L_{n+r} \to G\) by \(\varphi_2|_{L_n} = \varphi\) and \(\varphi_2(n + i) = p(\gamma'_i)\) for \(0 \leq i \leq r\). Then \(\varphi_1 \in \beta_1\) and \(\varphi_2 \in \beta_2\) and \(\varphi_1 \simeq \varphi_2\). Therefore \(p|_{N_r(\alpha)}\) is injective.

Next we prove that \(\tilde{G}\) is connected. Let \(\alpha \in V(\tilde{G})\) and \(\varphi : L_n \to G\) be a representative of \(\alpha\). Then

\[
\tilde{v} = [\varphi|_{L_0}] \sim [\varphi|_{L_1}] \sim \cdots \sim [\varphi|_{L_n}] = \alpha
\]

Hence \(\tilde{G}\) is connected.

Finally we prove that \(\pi_1^\alpha(G,v)\) is trivial. Let \([\varphi]\in p_*(\pi_1^\alpha(\tilde{G}, \tilde{v}))\) and put \(n = l(\varphi)\). Then the lift \(\tilde{\varphi}\) of \(\varphi\) whose initial point is \(\tilde{v}\) is described as \(\tilde{\varphi}(i) = [\varphi|_{L_i}]\). Since the terminal point of \(\tilde{\varphi}\) is equal to \(\tilde{v}\), hence we have \([*v]\sim [\varphi]\). Hence \(p_*\) is trivial. Since \(p_*\) is injective, \(\pi_1^\alpha(G,v)\) is trivial. \(\Box\)

Lemma 5.9. Let \((G,v)\) be a graph and \((\tilde{G}, \tilde{v})\) the universal \(r\)-covering over \((G,v)\) constructed in the proof of Proposition 4.8. Then the action

\[
V(\tilde{G}) \times \pi_1^\alpha(G,v) \to V(\tilde{G}), (\beta, \alpha) \mapsto \beta \cdot \alpha
\]

is an \(r\)-covering action.
Proof. Let $\beta \in V(\hat{G})$ and $\alpha \in \pi_1^r(G, v)$. Suppose $N_r(\beta) \cap N_r(\beta \cdot \alpha) \neq \emptyset$ and let $\gamma \in N_r(\beta) \cap N_r(\beta \cdot \alpha)$. Then $\beta, \beta \cdot \alpha \in N_r(\gamma)$. Since $p : \hat{G} \to G$ is an $r$-covering map, we have that $\beta = \beta \cdot \alpha$. Since this action is obviously free, we have that $\alpha$ is equal to the identity of $\Gamma$. Hence this action on $\hat{G}$ is an $r$-covering action.

Proposition 5.10. Let $r$ be a positive integer. Let $(G, v)$ be a based graph and $\Gamma$ a subgraph of $\pi_1^r(G, v)$. Then there is a connected based $r$-covering $p : (G_\Gamma, v_\Gamma) \to (G, v)$ with $p_{r*}(\pi_1^r(G_\Gamma, v_\Gamma)) = \Gamma$.

Proof. Let $p : (\hat{G}, \hat{v}) \to (G, v)$ be the universal $r$-covering over $(G, v)$ constructed in Proposition 5.7. Put $G_\Gamma = \hat{G}/\Gamma$. Since the action on $\hat{G}$ is an $r$-covering action, the quotient map $q : \hat{G} \to G_\Gamma$ is an $r$-covering map. From the universality of the quotient, there is $p_\Gamma : G_\Gamma \to H$ such that $p_\Gamma \circ q = p$. Since $p$ and $q$ are $r$-covering maps and $q$ is surjective on vertex sets, $p_\Gamma$ is an $r$-covering map.

We want to show that $p_{r*}(\pi_1^r(G_\Gamma, v_\Gamma)) = \Gamma$. Let $\varphi$ be a loop of $(G, v)$. We write $\varphi_\Gamma$ for the lift of $\varphi$ with respect to $p_\Gamma$ whose initial point is $v_\Gamma$, and $\check{\varphi}$ for the lift of $\varphi$ with respect to $p$ whose initial point is $\hat{v}$. Then $q \circ \check{\varphi} = \varphi_\Gamma$. Then we have that $\"[\varphi] \in p_{r*}(\pi_1^r(G_\Gamma, v_\Gamma))\" \iff \"\varphi_\Gamma \text{ is a loop}\" \iff \"The terminal point of } \check{\varphi} \text{ is included in } \Gamma.\" \iff \"[\varphi] \in \Gamma.\" \text{ The last } \"\iff\" \text{ is obtained from the fact that } \check{\varphi}(k) = [\varphi]|_{L_k}.\"

Theorem 5.11. Let $(G, v)$ be a based graph and $r$ a positive integer. We write $X_r(G, v)$ for the category of connected based $r$-covering over $(G, v)$ whose morphism is a graph map over $(G, v)$, and $Y_r(G, v)$ for the small category of all subgroups of $\pi_1^r(G, v)$ whose morphisms are inclusion maps. Then the functor $F : X_r(G, v) \to Y_r(G, v), p \mapsto \text{Imp}_r$ is a categorical equivalence.

Proof. We have that $F$ is essentially surjective from Proposition 4.10, and $F$ is fully and faithful from Proposition 5.4 and Remark 5.5.

Proposition 5.12. Let $(G, v)$ be a connected based graph with $\chi(G) \neq 2$. Then the subgroup of $\pi_1^r(G, v)$ associated to $(K_2 \times G, (0, v)) \to (G, v)$ is $\pi_1^r(G, v)_{ev}$.

Proof. Let $\varphi : L_n \to G$ be the loop of $(G, v)$. Then the lift $\check{\varphi}$ of $\varphi$ whose initial point is $(0, v)$ is $x \mapsto (x \mod 2, \varphi(x))$. Hence $\check{\varphi}$ is a loop if and only if $l(\varphi)$ is even.

6. $r$-COVERING MAPS OVER CYCLES

For a positive integer $m$, a cycle $C_m$ is the graph defined by $V(C_m) = \mathbb{Z}/m\mathbb{Z}$ and $E(C_m) = \{(x, x + 1), (x + 1, x) \mid x \in \mathbb{Z}/m\mathbb{Z}\}$. Hence $C_1$ is the one looped vertex, $C_2 = K_2$ and $C_3 = K_3$. In this section, we compute all covering maps over $C_m$ for $m \geq 1$. $r$ and $m$ are assumed to be positive integers. Next we consider the condition a graph has a graph map to $C_m$ with odd $m$.

Let $L$ be a graph defined by $V(L) = \mathbb{Z}$ and $E(L) = \{(x, y) \mid |x - y| = 1\}$. First we remark that $\pi_1^r(L, 0)$ is trivial for any $r \geq 1$. Indeed, to prove this fact, it is sufficient to prove that $\pi_1^1(L, 0)$ is trivial. But this is well-known.

Lemma 6.1. Let $r$ and $m$ be positive integers and $p_m : L \to C_m$ a graph map defined by $x \mapsto (x \mod m)$. Then the followings hold.

1. Suppose $m$ is even. Then $p_m$ is an $r$-covering map if and only if $2r < m$. 

(2) Suppose $m$ is odd. Then $p_m$ is an $r$-covering map if and only if $r < m$.

Proof. Let $x \in V(L) = \mathbb{Z}$. Then $p_m|_{N(x)} : N(x) \to N(x \mod m)$ is surjective. So we want to determine the condition $p_m|_{N(x)}$ is injective for all $x \in V(L)$. Since $N_r(x) = \{x - r, x - r + 2, \cdots, x + r\}$, $p_m|_{N_r(x)}$ is injective for all $x \in V(L)$ if and only if for each $x \in V(C_m) = \mathbb{Z}/m\mathbb{Z}$, $x, x + 2, \cdots, x + 2r$ are distinct $(r + 1)$-elements of $\mathbb{Z}/m\mathbb{Z}$. This completes the proof. \qed

Lemma 6.2. Let $r$ and $m$ be positive integers. Then the followings hold.

1. Suppose $m$ is even and $2r < m$. Then $\pi_1^r(C_m) \cong \mathbb{Z}$.
2. Suppose $m$ is odd and $r < m$. Then $\pi_1^r(C_m) \cong \mathbb{Z}$.

Moreover, the generator of $\pi_1^r(C_m)$ is represented by $L_m \to C_m$, $x \mapsto (x \mod m)$ in these cases.

Proof. Let $m$ be a positive integer satisfying one of the following two conditions.

- $m$ is even and $2r < m$.
- $m$ is odd and $r < m$.

In these cases, the map $p_m : L \to C_m$ is the universal $r$-covering map. For a nonnegative integer $n$, we define $\varphi_n : L_{nm} \to L$ by $\varphi_n(x) = x$. For a negative integer $n$, we define $\varphi_n : L_{-nm} \to L$ by $\varphi_n(x) = -x$. Put $\alpha_n = [p \circ \varphi_n]_r \in \pi_1^r(C_m)$. Then we have

- $\alpha_n \neq \alpha_k$ if $n \neq k$.
- $\pi_1^r(C_m) = \{\alpha_n \mid n \in \mathbb{Z}\}$.
- $\alpha_n \cdot \alpha_k = \alpha_{n+k}$.

Hence the map $\mathbb{Z} \to \pi_1^r(C_m)$, $k \mapsto \alpha_k$ is a group isomorphism. \qed

This lemma implies that if $m$ is even and $2r < m$ or $m$ is odd and $r < m$, the $r$-covering map over $C_m$ is represented as a natural projection $C_{nm} \to C_m$ for any positive integer $n$.

Lemma 6.3. Let $r$ and $m$ be positive integers. Then the followings hold.

1. Suppose $m$ is even. Then $\pi_1^r(C_m)$ is trivial if $2r \geq m$.
2. Suppose $m$ is odd. Then $\pi_1^r(C_m)_{ev}$ is trivial if $r \geq m$.

Proof. First we prove (1). The case $m = 2$ is obvious. We assume $m \geq 4$. In this case, $\pi_1^r(C_m) \cong \mathbb{Z}$ from the previous lemma, and the generator is represented by $\varphi : L_m \to C_m$. So we want to prove that $[\varphi]_r = 0$ in the case $2r \geq m$. In this case, we have that $\varphi$ is homotopic to the map $\psi : L_m \to C_m$ such that $\psi(i) = \varphi(i)$ for $i \leq \frac{m}{2}$ and $\psi(i) = \varphi(m - i)$ for $i \geq \frac{m}{2}$, and $\psi$ is obviously $r$-homotopic to the trivial one. This completes the proof of (1).

The (2) is deduced from (1), since $K_2 \times C_m \cong C_{2m}$ if $m$ is odd. \qed

Summarizing these two lemmas, we have the following.

Theorem 6.4. Let $r$ and $m$ be positive integers. Then the followings hold.

1. Suppose $m$ is even. Then $\pi_1^r(C_m)$ is trivial if $2r \geq m$ and $\pi_1^r(C_m) \cong \mathbb{Z}$ if $2r < m$.
2. Suppose $m$ is odd. Then $\pi_1^r(C_m) \cong \mathbb{Z}/2\mathbb{Z}$ if $r \geq m$ and $\pi_1^r(C_m) \cong \mathbb{Z}$ if $r < m$.

Corollary 6.5. Let $G$ be a connected graph with $\chi(G) \neq 2$ and $m$ an odd integer greater than 1. If there is a graph map $G \to C_m$, the abelianization of $\pi_1^r(G, v)$ has $\mathbb{Z}$ as a direct summand for any $r < m$. 

Proof. Let $r$ be a positive integer with $r < m$. Suppose that the abelianization of $\pi_1^r(G)$ does not have $\mathbb{Z}$ as a direct summand and there exists a map $G \to C_m$. Then we have that $\pi_1^r(G) \to \pi_1^r(C_m) \cong \mathbb{Z}$ is trivial. Hence there is a map $G \to L$. But this contradicts the assumption $\chi(G) \neq 2$. \hfill $\square$

Remark 6.6. If the abelianization of $\pi_1^{m-1}(G, v)$ has $\mathbb{Z}$ as a direct summand, then the abelianization of $\pi_1^r(G, v)$ has $\mathbb{Z}$ as a direct summand for any $r \leq m - 1$ since there is a surjective group homomorphism $\pi_1^r(G, v) \to \pi_1^{m-1}(G, v)$.

Example 6.7. Let $G$ be a graph obtained from the graph of Figure 1 by identifying each vertex on the boundary with its antipodal vertex. Then it is easy to see that $\chi(G) = 3$. But there is no graph map $G \to C_5$, since $\pi_1^3(G) \cong \mathbb{Z}/2\mathbb{Z}$. We remark that the odd girth of $G$ is equal to 9.

![Figure 1.](image-url)

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