HARTOGS’ PHENOMENON AND THE PROBLEM OF
COMPLEX SPHERES

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In memory of Pierre Dolbeault (1924-2015)

Abstract
In this paper, we solve in the negative the following open problem.
Problem of complex spheres. Is there any integrable almost complex structure on the sphere $S_6$?

Contents

1. Statement of results.
   1.1. Statement of the theorem of complex spheres
   1.2. The main theorem: statement and strategy of proof.
   1.3. Hypothetic $A$–submanifolds of $S_{2n}$
2. Topology of the $A$–submanifolds of $S_{2n}$
   2.1. Some special homeomorphisms of $\mathbb{C}^n$
   2.2. Applications to the topology of the $A$–submanifolds of $S_{2n}$
   2.3. Some open sets homeomorphic to balls
3. The $\bar{\partial}$–cohomology of $A$–submanifolds of $S_{2n}$
   3.1. The hypothetic complex manifold $(\mathbb{C}^n, A)$
   3.2. The almost everywhere psh function $f_A$
   3.3. A vanishing theorem
4. Proof of the main theorem
   4.1. $\{Hath\}_n \Rightarrow \{Necs\}_n$
   4.2. $\{Necs\}_n \Rightarrow \{Hath\}_n$
5. Appendix
   5.1. Pseudoconvex manifolds
   5.2. Čech cohomology
References

Introduction

The story of this long standing problem began more than sixty years ago when C. Ehresmann has introduced in differential geometry the notion of almost complex structure on a differentiable manifold of even dimension (see [5], [6]). The first result in the direction of this problem was given by Borel and Serre [1] and we know according to this result that the only spheres of even dimension which admit an almost complex structure are $S_2$ and $S_6$; therefore no spheres of even dimension
except perhaps $S_2$ and $S_6$ admit a complex structure. Since it is well known that $S_2$ is a Riemann surface, that is already a complex manifold of $\dim_{\mathbb{C}}S_2 = 1$, it remained then to study only the case $S_6$. It was Ehresmann himself (see [3], [16]) who showed for the first time that the sphere $S_6$ admitted almost complex structures, and it was Kirchoff (see [13], [16]) who gave the first example of an almost complex structure on $S_6$. This example is the standard almost complex structure $J : TS_6 \to TS_6$ defined by the cross product $\times$ on imaginary parts $u, v \in \mathbb{R}^7$ of the Cayley’s Octonions

$$T_\omega(v) = u \times v \quad \text{for all} \quad v \in T_\omega S_6 = u^\perp \subset \mathbb{R}^7.$$  

But it turned out that this famous almost complex structure is not integrable. The question was then: How to determine whether there exists or not an integrable almost complex structure on the sphere $S_6$? One great difficulty in solving this question is that the theorem of integrability of almost complex structures of Newlander Nirenberg [17], can not be easily implemented. We mention however two kinds of fundamental partial results in the direction of the problem of the "complex" sphere $S_6$. The first result shows that, roughly speaking, although the sphere $S_6$ is "simple" as real analytic manifold, an hypothetic complex structure makes it more "complicated" as complex analytic manifold. Indeed, we know that the Betti numbers $b_j$ of $S_6$ with $1 \leq j \leq 5$ are null, however a surprising result due to A. Gray [8] (see also Ughart [19]), shows that the Hodge number $h^{0,1}$ of an hypothetic complex structure on $S_6$, is $\neq 0$, which means that not any $\overline{\partial}$–closed differential form of bidegree $(0, 1)$ in $S_6$ is $\overline{\partial}$–exact; or in other words, the Cauchy-Riemann equation $\overline{\partial}u = f$ can not be always resoluble in the whole $S_6$! The second kind of results shows that some almost complex structures on $S_6$ which satisfy some "additional conditions" can also not be integrable, see [2], [3], [15]. For instance, C. LeBrun has shown in [15], that there exist no integrable almost complex structures $J$ on $S_6$ orthogonal with respect to the Riemannian metric of $S_6$. Such was the strategy adopted by almost all geometers interested in this problem!, and all attempts to solve the problem of the existence (or not) of a complex structure on the sphere $S_6$, have been partially unsuccessful. The question remains then still open! In this paper, we will not be occupied by the integrability of any a priori almost complex structure on $S_6$, since we will not have to use the result of Borel and Serre [11]. Indeed, we propose an approach which shows that for $n \geq 2$, the obstruction for the spheres $S_{2n}$ to admit a complex structure, is not due only to the fact that they do not admit almost complex structures, as is the case for the spheres other than $S_6$, but that this phenomenon is due to Hartogs’ theorem which is valid for all $n \geq 2$; and then, even if the the spheres $S_{2n}$ with $n \geq 2$ other than $S_6$ admitted almost complex structures, they do in any way, admit a complex structure. Although we adopt a cohomological approach, we don’t search to apply the $\overline{\partial}$–cohomology of the whole sphere $S_6$ as in [8] and [19], because this turned out to be not conclusive. We will rather focus our attention on the study of the Dolbeault cohomology of a family of hypothetic open complex submanifolds $X_{\alpha, \beta}$ of $S_{2n}$, called $A$–submanifolds of $S_{2n}$ (see definition [10], [19]), and then we look for the problem of complex spheres from the point of view of Hartogs phenomenon. Indeed, and surprisingly enough, it turns out that the problem of complex spheres on $S_{2n}$, viewed under $\overline{C}$hech cohomology, is a geometric phenomenon, deeply linked to the problem of analytic continuation of holomorphic functions defined in a connected domain around a compact subset of $\mathbb{C}^n$ with $n \geq 2$, that is, to the so-called Hartogs phenomenon! [10], [11], [18]. We prove in this paper two fundamental
results concerning the so called $\mathcal{A}$--submanifolds of $S_{2n}$ mentioned above: The first result shows that from the topological point of view, any $\mathcal{A}$--submanifolds of $S_{2n}$ is homeomorphic to $\mathbb{C}^n$, and the second one shows that any $\mathcal{A}$--submanifolds of $S_{2n}$ is strictly pseudoconvex, and then by the well known Cartan’s theorem B, the Dolbeault group of cohomology of bidegree $(0,1)$ of such manifold vanishes, that is
\[ H^{0,1}(X_{\alpha,\beta}, \mathbb{C}^n) = \{0\}. \]

Then equipped with this vanishing result, we show using Čech cohomology, that the assumption that $S_{2n}$ with $n \geq 2$, admits a complex structure is in contradiction with Hartogs’ theorem, which gives a negative solution to the problem of complex spheres for all spheres $S_{2n}$ with $n \geq 2$ and then for $S_6$ as a particular case \footnote{The approach proposed in this paper does not suppose the result of Borel Serre, and then we consider the problem of complex spheres for all spheres $S_{2n}$ with $n \geq 2$ and not only for $S_6$.}. The problem of complex spheres becomes then a geometric phenomenon induced by the analytic Hartogs’ phenomenon, and we prove even that both phenomenons are in fact equivalent. It is rather curious that for more than a century, we have always viewed Hartog’s phenomenon only from its analytic aspect! We will see in this paper, that Hartogs’ phenomenon and the problem of complex spheres represent two faces of the same phenomenon, which answers in the negative the not yet solved problem of complex spheres. The proof of such a link is the main motivation for this work.

1. Statement of results.

1.0.1. Notations. It is useful before beginning our exposition to specify once for all some frequent notations.

1. $B(a,r)$ denotes always an open ball of $\mathbb{C}^n$ centered at $a$, and of radius $r > 0$, and when we note simply $B_r$ without specifying the center, this always means that the ball is centered at the origin. The corresponding closed balls, will be noted respectively by $\overline{B}(a,r)$, and $\overline{B}_r$.

2. $S_k(a,r)$ denotes a sphere of dimension $k < 2n$ embedded in $\mathbb{C}^n$, of radius $r > 0$ and centered at the point $a \in \mathbb{C}^n$, and when we note $S_k$, this always means that the sphere embedded in $\mathbb{C}^n$, is viewed as a unit sphere centered at the origin.

3. For two subsets $X$ and $Y$ of $\mathbb{C}^n$, we note
\[ Y - X = \left\{ z \in Y, \quad z \notin X \cap Y \right\}. \]

4. If $X$ and $Y$ are topological spaces, the notation $X \approx Y$ means always that $X$ is homeomorphic to $Y$.

5. We use the abbreviate notation ”\textit{psh}” for plurisubharmonic functions.

6. If $X$ is a complex manifold and $f : X \to \mathbb{C}$ is any piecewise $C^k$–function, we note then by $\tilde{f}$ the expression of $f$ in a chart $(V, \psi)$, that is, in the
complex coordinates $ζ = ψ(z) ∈ C^n$

$$
\begin{array}{c}
V \\ ψ
\end{array} \xrightarrow{f} \begin{array}{c}
K \\ \psi(V)
\end{array} \xrightarrow{f} \begin{array}{c}
\tilde{f}
\end{array}
$$

1.1. Statement of the theorem of complex spheres.
We begin by stating the following theorem.

**Theorem 1.1. (Theorem of complex spheres)**
Among all spheres of even dimension $S_{2n}$, only the sphere $S_2$ admits a complex structure.

One of the fundamental problems in several complex variables solved in the last century (period 1942-1954) was the so called Levi’s problem which characterizes domains of holomorphy in $C^n$. For the commodity of the reader, we recall here that a domain of holomorphy is roughly speaking a maximum domain $D ⊆ C^n$ of definition of a holomorphic function $f : D → C$. (By "maximum" we mean that $f$ cannot be analytically extended across any part of the boundary $∂D$).

Two fundamental results in several complex analysis marked the beginning of the 20th century, both proved in 1906. One of them (2) shows that not any domain in several complex variables is a domain of holomorphy. This result is the now classical Hartogs’ theorem, See [4], [7], [10], [12], [15]. We will prove that the theorem of complex spheres stated above is a corollary of the main theorem of this paper (theorem 1.6) combined with Hartogs’ theorem, whose statement is as follows:

**Theorem 1.2. (Hartogs’ theorem)**
Let $n ≥ 2$, and let $D ⊂ C^n$ be an open set, and let $K$ be a compact subset of $D$, such that $D − K$ is connected. Then for every holomorphic mapping $f : D − K − → C^k$, there exists a holomorphic mapping $F : D − → C^k$ such that $F = f$ in $D − K$.

Before stating our main theorem (theorem 1.6), let us first consider the following two propositions:

**Proposition 1.3.** $\{Hath\}_n$
Let $n ∈ N^*$, and let $B ⊂ C^n$ be an open ball and let $K$ be a compact subset of $B$, such that $B − K$ is connected. Then for every holomorphic mapping $f : B − K − → C^k$, there exists a holomorphic mapping $F : B − → C^k$ such that $F = f$ in $B − K$.

and

**Proposition 1.4.** $\{Necs\}_n$
Let $n ∈ N^*$, then the sphere $S_{2n}$ does not admit a complex structure.

**Remark 1.5.**

1. In this paper, we will refer to propositions 1.3 and 1.4 respectively, by the following abbreviated notations:
   (a) $\{Hath\}_n$ which is short for Hartog’s theorem in $C^n$,
   (b) $\{Necs\}_n$ which is short for Non existence of complex spheres $S_{2n}$.

2The second result is the theorem proved by H. Poincaré which says that in $C^n$ with $n ≥ 2$, a ball is never biholomorphic to a polydisc.
(2) It is clear that for \( n \geq 2 \), proposition \( \{Hath\}_n \) is the statement for balls of Hartogs theorem, and for our purpose, it will be enough sufficient to restrict the statement of Hartogs’ theorem to balls.

1.2. The main theorem: statement and strategy of proof.

1.2.1. Statement of the main theorem.

The ultimatum goal of this paper is then to prove the following main theorem.

**Theorem 1.6.** For all \( n \geq 1 \), the following equivalence holds:

\[
\{Hath\}_n \iff \{Necs\}_n.
\]

**Remark 1.7.**

(1) Once theorem 1.6 proved, proposition \( \{Necs\}_n \) for \( n \geq 2 \), becomes then a geometric phenomenon equivalent to the analytic phenomenon \( \{Hath\}_n \).

(2) It is obvious that for \( n = 1 \), the equivalence \( \{Hath\}_1 \iff \{Necs\}_1 \) holds. Indeed, both propositions \( \{Hath\}_1 \) and \( \{Necs\}_1 \) are false, because for \( D = \mathbb{B}_1 \) and \( K = \{0\} \), the holomorphic function \( f(z) = \frac{1}{z} \) defined on \( \mathbb{B}_1 - \{0\} \), does not admit an analytic continuation to \( \mathbb{B}_1 \), which means that \( \{Hath\}_1 \) is false, and in the same time, the sphere \( S_2 \) is a Riemann surface, which means that \( S_2 \) admits a complex structure, and then \( \{Necs\}_1 \) is also false.

(3) Observe that for \( n \geq 2 \), \( \{Hath\}_n \) is already true, it is as mentioned above the statement for balls of the well known theorem on analytic continuation of holomorphic functions in \( \mathbb{C}^n \), proved by Hartogs in 1906 (see [10]). However, until proving the main theorem (theorem 1.6), we will not consider \( \{Hath\}_n \) for \( n \geq 2 \), as already a theorem, but only as a logical proposition, without looking whether it is true or false. But once we prove theorem 1.6 we will finally take into account the fact that for \( n \geq 2 \), \( \{Hath\}_n \) is already true, and then we will deduce immediately the exactness of \( \{Necs\}_n \) for \( n \geq 2 \), that, there exist no complex structures on the sphere \( S_{2n} \) for \( n \geq 2 \). Hence, the problem of complex spheres will finally be solved. In the same time, we obtain back a geometric interpretation of Hartogs’ theorem in terms of non existence of complex structures on the spheres \( S_{2n} \) for \( n \geq 2 \). As far as I know, this is the first time that a geometric interpretation of Hartogs’ theorem in terms of complex analytic geometry is given, since it is proved in 1906.

1.2.2. A version of \( \{Hath\}_n \) for \( n \geq 2 \), in a complex chart.

Since it will be question to apply Hartogs’ theorem on a sphere, it becomes then interesting to formulate a version of \( \{Hath\}_n \) in a chart of a complex analytic manifold \( X \) of \( \dim_{\mathbb{C}} X \geq 2 \). This is given in lemma 1.8 below. But let’s recall that we shall continue temporarily (that is, before proving the main theorem) to consider proposition \( \{Hath\}_n \) not as already a theorem, but only as a logical proposition.

**Lemma 1.8.** Let \((U, \varphi)\) be a complex chart of a complex analytic manifold \( X \) of \( \dim_{\mathbb{C}} X \geq 2 \), and let \( D \) be an open subset of \( U \) and let \( K \) be a compact subset of \( D \) such that:

(1) \( D - K \) is connected,

(2) \( \varphi(D) \) is a ball of \( \mathbb{C}^n \).
Assume that proposition \( \{\text{Hath}\}_n \) is true, then for every holomorphic mapping \( f : D - K \rightarrow \mathbb{C}^k \), there exists a holomorphic mapping \( F : D \rightarrow \mathbb{C}^k \) such that \( F = f \) on \( D - K \).

**Proof.** Let \( f : D - K \rightarrow \mathbb{C}^k \) be a holomorphic mapping. This means that the mapping \( \tilde{f} = f \circ \varphi^{-1} : \varphi(D) - \varphi(K) \rightarrow \mathbb{C}^k \) (which is the expression of \( f \) in the chart \((U, \varphi)\)) is holomorphic. Since \( \varphi \) is a homeomorphism, then \( \varphi(K) \) is a compact subset of \( \varphi(D) \subset \mathbb{C}^n \) and \( \varphi(D) - \varphi(K) \) is connected in \( \mathbb{C}^n \). \( \varphi(D) \) being a ball of \( \mathbb{C}^n \), then according to \( \{\text{Hath}\}_n \), there exists a holomorphic mapping \( \tilde{F} : \varphi(D) \rightarrow \mathbb{C}^k \) such that \( \tilde{F} = f \) in \( \varphi(D) - \varphi(K) \). But \( \tilde{F} \) is exactly the expression in the chart \((U, \varphi)\) of the mapping \( F = \tilde{F} \circ \varphi : D \rightarrow \mathbb{C}^k \) which is holomorphic in \( D \) and satisfies \( F = f \) in \( D - K \). Lemma 1.8 is then proved. \( \square \)

1.3. **Hypothetic \( \mathcal{A} \)-submanifolds of \( S_{2n} \).**

As mentioned in the introduction, the main geometric object to be studied in this paper is the notion of \( \mathcal{A} \)-submanifold of \( S_{2n} \). This notion is introduced in the definition 1.10 below, and it will be a powerful tool to prove the main theorem 1.6. But let’s insist on the fact that the notion of \( \mathcal{A} \)-submanifold of \( S_{2n} \) is purely hypothetic, since its existence depends only on the assumption that proposition \( \{\text{Necs}\}_n \) is false, that is, on the assumption that \( S_{2n} \) admits a complex structure \( \mathcal{A} \), and then it will be used only as a technical tool to prove by contradiction theorem 1.6.

1.3.1. **Notations and definition.**

Assume that \( \{\text{Necs}\}_n \) is false, that is, \( S_{2n} \) admits a complex structure defined by an hypothetic complex atlas

\[
\mathcal{A} = \left\{ (U_j, \varphi_j), \quad j \in \mathcal{J} \right\}
\]

where \( \{U_j, \quad j \in \mathcal{J}\} \) is an open covering of \( S_{2n} \), and \( \varphi_j : U_j \subset S_{2n} \rightarrow \varphi_j(U_j) \subset \mathbb{C}^n \) are homeomorphisms such that for all \((i, j) \in \mathcal{J}^2 \) with \( U_i \cap U_j \neq \emptyset \), the following change of charts

\[
\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)
\]

is biholomorphic. By refining if necessarily the atlas \( \mathcal{A} \), we can always assume taking into account the compactness of \( S_{2n} \), that \( \mathcal{A} \) has a finite number of charts

\[
\text{card}(\mathcal{A}) = N + 1
\]

and that \( \mathcal{A} \) satisfies furthermore the following conditions:

\[
(1.1) \quad \text{For all } j \in \mathcal{J}, \quad \varphi_j(U_j) \text{ is a ball of } \mathbb{C}^n.
\]

\[
(1.2) \quad \text{For all } (i, j) \in \mathcal{J}^2, \quad U_i \cap U_j \text{ is simply connected.}
\]

**Remark 1.9.** In all that follows, the hypothetic complex atlas \( \mathcal{A} \) will be fixed once for all, and whenever we refer to an hypothetic complex structure on \( S_{2n} \), this will always be the complex structure \( \mathcal{A} \) considered above.
Definition 1.10.
Assume that proposition \{Necs\}_n is false, and let then
\[
\mathcal{A} = \left\{ (U, \varphi) \mid \gamma \in J \right\}
\]
be the hypothetic complex structure on \(S_{2n}\) considered above. An hypothetic complex submanifold \(X\) of \(S_{2n}\) is said to be an \(\mathcal{A}\)-submanifold of \(S_{2n}\), if there exist a chart \((U_d, \varphi_d) \in \mathcal{A}\), and a compact subset \(K\) of \(S_{2n}\) such that the following conditions are satisfied:

(a) \(X = S_{2n} - K\) (\(X\) is then open).
(b) \(K \subset U_d\).
(c) \(\varphi_d(K) = S_{2n-2}(b, r)\) (that is, \(\varphi_d(K)\) is a sphere of \(\mathbb{C}^n\) of dim = \(2n-2\)).

\((U_d, \varphi_d)\) is said to be a distinguished chart of \(\mathcal{A}\) for the \(\mathcal{A}\)-submanifold \(X\).

Remark 1.11.

(a) By observing that the translations of \(\mathbb{C}^n\) are biholomorphic mappings, then without changing the hypothetic complex structure of \(S_{2n}\), we can always assume that \(\varphi_d(K)\) is a sphere centered at \(b = 0\), that is

\[
\varphi_d(K) = S_{2n-2}(0, r_2).
\]

(b) The distinguished chart \((U_d, \varphi_d) \in \mathcal{A}\) is not necessarily unique, but the following proposition shows that for a given \(\mathcal{A}\)-submanifold \(X\), we can always choose an hypothetic complex atlas \(\tilde{\mathcal{A}}\) of the sphere \(S_{2n}\) finer than \(\mathcal{A}\) such that \(X\) admits one and only one distinguished chart.

Proposition 1.12. Let \(X\) be an \(\mathcal{A}\)-submanifold of \(S_{2n}\). Then there exists on \(S_{2n}\) an atlas \(\tilde{\mathcal{A}}\) equal or finer than \(\mathcal{A}\), such that \(X\) viewed as \(\tilde{\mathcal{A}}\)-submanifold of \(S_{2n}\) admits one and only one distinguished chart.

Proof.
Let \((U_d, \varphi_d)\) and \((U_{d'}, \varphi_{d'})\) be two distinguished charts for the \(\mathcal{A}\)-submanifold \(X\). \(X\) can then be written as follows

\[
X = S_{2n} - K
\]
where \(K\) is a compact subset of \(S_{2n}\) and
\[
K \subset U_d \cap U_{d'}.
\]
Let \(T\) be an open neighborhood of \(K\) such that
\[
K \subset T \subset U_d \cap U_{d'},
\]
that is, such that
\[
(U_{d'} - T) \cap K = \emptyset.
\]
By recovering in \(\varphi_{d'}(U_{d'}) \subset \mathbb{C}^n\), the following open set
\[
\varphi_{d'}(U_{d'}) - \varphi_{d'}(T)
\]
by small open balls \(B(a, r)\) such that
\[
B(a, r) \cap \varphi_{d'}(T) = \emptyset,
\]
we can then define new complex charts \((U_d^{a,r}, \varphi_d^{a,r})\) by setting
\[
\begin{align*}
U_d^{a,r} &= \varphi_d^{-1}(B(a,r)) \\
\varphi_d^{a,r} &= \varphi_d / U_d^{a,r}
\end{align*}
\]
(that is, by restriction).

Since \(S_{2n}\) is compact, then by replacing in \(A\), the distinguished chart \((U_d, \varphi_d)\) by a union of a finite number of new charts as defined above, we obtain a new atlas denoted \(\tilde{A}\), finer than \(A\), in which, the chart \((U_d, \varphi_d)\) is the only distinguished chart for the \(\tilde{A}\)–submanifold \(X\).

\[\square\]

Remark 1.13.
1) If there exist more than two distinguished charts for an \(A\)–submanifold \(X\), we prove proposition 1.12 by induction.
2) By substituting if necessarily the atlas \(A\) by \(\tilde{A}\), we can according to proposition 1.12, always assume that a given \(A\)–submanifold \(X\) admits one and only one distinguished chart \((U_d, \varphi_d)\).

1.3.2. Complex structure of an \(A\)–submanifold of \(S_{2n}\).

Let \(X = S_{2n} - K\) be an \(A\)–submanifold of \(S_{2n}\). We want to describe the complex structure of \(X\) inherited from \(A\). For this, let the distinguished chart \((U_d, \varphi_d) \in A\) for \(X\), and let by condition (1.1) the ball \(B(a,r_1) = \varphi_d(U_d) \subset \mathbb{C}^n\) and let by (1.3) the sphere \(S_{2n-2}(b,r_2) = \varphi_d(K) \subset B(a,r_1)\).

Since \(\dim S_{2n-2}(0,r_2) = 2n - 2\), there exists then a real vector space \(F \subset \mathbb{C}^n\)
\[
F = \left\{ \zeta \in \mathbb{C}^n, \quad \langle w, \zeta \rangle = 0 \right\}
\]
of \(\dim F = 2n - 1\), with \(w \neq 0\), such that
\[
S_{2n-2}(0,r_2) \subset F \subset \mathbb{C}^n.
\]

Set now
\[
(1.4) \quad N(a,r_1,r_2) := B(a,r_1) - S_{2n-2}(0,r_2) \subset \mathbb{C}^n.
\]

With these notations, the complex structure of the \(A\)–submanifold \(X\) is then given by the restricted atlas
\[
A/X := \left\{ U_j, \varphi_j \right\}, \quad j \in \mathcal{J}
\]
where the charts \((U_j, \varphi_j)\) of \(A/X\) are defined as follows:

1. For \(j = d\), the chart \((\tilde{U}_d, \tilde{\varphi}_d)\) called distinguished chart for \(X\) is given by
   (a) \(\tilde{U}_d := U_d - K \subset X\).
   (b) \(\tilde{\varphi}_d := \varphi_d / (U_d - K)\) (\(\tilde{\varphi}_d\) is the restriction of \(\varphi_d\) to \(U_d - K\)).
The restricted mapping
\[ \varphi_d : U_d - K \longrightarrow N(a,r_1,r_2). \]
is then a homeomorphism.

(2) For all \( j \in J - \{d\}, \quad (\tilde{U}_j, \tilde{\varphi}_j) = (U_j, \varphi_j) \)
which implies that for all \( j \neq d, \quad \tilde{\varphi}_j (\tilde{U}_j) = \varphi_j (U_j) \) is a ball of \( \mathbb{C}^n \).

Let \( W \) be the relatively compact open set of \( X \) defined by
\[ W = S_{2n} - \overline{U_d}. \]
\( W \) is then covered by all domains of charts \( U_j \) with \( j \neq d \), that is
\[ W \subset \bigcup_{j \neq d} U_j. \]
Since by (1.5) \( \varphi_d (U_d) \) is an open ball of \( \mathbb{C}^n \), we obtain
\[ \partial W = \partial U_d = S_{2n-1} \]
and since by stereographic projection, \( \varphi_d (W) \) is relatively compact in \( \mathbb{C}^n \), we deduce that
\[ W \approx B(0, r). \]

**Remark 1.14.**
The geometry of an \( A- \)submanifold \( X \) of \( S_{2n} \) is characterized by two objects:

1. The distinguished complex chart \( \psi_d : U_d - K \longrightarrow N(a,r_1,r_2). \)
2. The relatively compact open subset \( W = S_{2n} - \overline{U_d}, \) which is covered by the other charts \( (U_j, \varphi_j) \) and which is homeomorphic to an open ball of \( \mathbb{C}^n \).

In this paper, we will prove for \( A- \)submanifolds of \( S_{2n}, \) two fundamental properties:

1. Every \( A- \)submanifold of \( S_{2n} \) is homeomorphic to \( \mathbb{C}^n \).
2. Every \( A- \)submanifold \( X \) of \( S_{2n} \) is strictly pseudoconvex, and then by Cartan’s theorem B, the group of cohomology of bidegree \((0,1)\) of \( X \) is trivial
\[ H^{0,1} (X, \mathbb{C}^n) = \{0\}. \]

**1.3.3. The strategy of proof.**
It is now legitimate to ask how one can apply Hartogs’ theorem to prove the theorem of complex spheres? Or in other words, how one can prove the implication \( \{Hath\}_n \Longrightarrow \{Necs\} \) for \( n \geq 2? \)

Before presenting our approach of proof, let’s start with some remarks:

**Remark 1.15.**
1. It is obvious for reason of compactness, that \( S_{2n} \) can never admit a complex structure in one chart.
2. Hartogs’ theorem 1.2 implies immediately that \( S_{2n} \) with \( n \geq 2, \) can neither admit a complex structure in two charts.

Indeed, If \( (U_1, \varphi_1) \) and \( (U_2, \varphi_2) \) are two complex charts on \( S_{2n} \) with
\[ S_{2n} = U_1 \bigcup U_2 \]
then the holomorphic mapping
\[ \varphi_2 : U_1 \cap U_2 \subset S_{2n} \longrightarrow \varphi_2 (U_1 \cap U_2) \subset \mathbb{C}^n \]
\((U_1 \cap U_2 \) connected) admits by Hartogs’ theorem an analytic continuation to \( U_1. \) Therefore, \( \varphi_2 \) admits an analytic continuation to the whole sphere
\[ S_{2n}, \text{ and then becomes constant. For the same reason, we obtain the same result for } \varphi_1. \] But this contradicts the fact that \( \varphi_2 \circ \varphi_1^{-1} \) is bi-holomorphic.

The proof that \( S_{2n} \) with \( n \geq 2 \), could never admit a complex structure in more than two charts, seems somewhat more complicated, but it is precisely this hypothetic situation which we will show in this paper that it is impossible.

**What is the main idea of proof?**

This idea consists roughly speaking, in proving by contradiction that if for \( n \geq 2 \), \( \{Necs\} \) is false, then for an appropriate atlas \( \mathcal{F} = \{ (W_\gamma, \Phi_\gamma) , \ \gamma \in \mathcal{L} \} \) finer than \( \mathcal{A} \), one can construct for all pairing of different charts \( (W_\alpha, \Phi_\alpha), (W_\beta, \Phi_\beta) \in \mathcal{F} \) satisfying the condition \( W_\alpha \cap W_\beta \neq \emptyset \), an \( \mathcal{A} \)-submanifold \( X_{\alpha,\beta} \) of \( S_{2n} \) in the sense of definition [1,10] and a harmonic mapping

\[ H : X_{\alpha,\beta} \rightarrow \mathbb{C}^n \]

with \( H = H_1 + \overline{H_2}, \) \( (H_1 \text{ and } H_2 \text{ are holomorphic}) \) such that for all \( \alpha \in \mathcal{L} \)

\[ \partial H/W_\alpha = \partial \Phi_\alpha \]

which implies by Hartogs’ theorem that \( \Phi_\alpha \) is constant. The proof of the existence of such harmonic mapping \( H \) needs some preparation, and it will be done later in section 4. Leaving then the details to section 4), we sketch below just the procedure of proof.

**The procedure of proof.**

(1) We begin by proving that \( X_{\alpha,\beta} \) is homeomorphic to \( \mathbb{C}^n \). This result is given in section 2).

(2) Then we prove that \( H^0,1 (X_{\alpha,\beta}, \mathbb{C}^n) = \{0\} \), which implies by Dolbeault’s resolution of the sheaf \( \mathcal{O}^n \) that the first Čech cohomology group is trivial

\[ H^1 (X_{\alpha,\beta}, \mathcal{O}^n) = \{0\}. \]

The proof of the triviality of the group \( H^1 (X_{\alpha,\beta}, \mathbb{C}^n) \) will be the main object of section 3).

(3) We deduce from 2) that the 1-cycle \( g_{\mu,\nu} \) with coefficients in the sheaf \( \mathcal{O}^n \) defined on \( X_{\alpha,\beta} \) by

\[ f_{\mu,\nu} = \begin{cases} \Phi_\beta - \Phi_\alpha = & \text{if } (\mu, \nu) = (\alpha, \beta) \\ 0 = & \text{if } (\mu, \nu) \neq (\alpha, \beta). \end{cases} \]

is a 1-coboundary, that is

\[ f_{\mu,\nu} = F_\nu - F_\mu. \]

(4) The restrictions \( F_\alpha/(X_{\alpha,\beta} - \overline{W_\alpha}) \) and \( F_\beta/(X_{\alpha,\beta} - \overline{W_\beta}) \) being holomorphic, admit both by Hartogs’ theorem [3] an analytic continuation to the sphere \( S_{2n} \), and therefore become constant on \( S_{2n} \), and then

\[ \partial \Phi_\alpha = \partial \Phi_\beta \quad \text{on} \quad W_\alpha \cap W_\beta. \]

Gluing the different pieces \( \partial \Phi_\gamma \) together, there exists then a \( \partial \)-closed form \( \omega \) with values in \( \mathbb{C}^n \), of bidegree \((1,0)\), such that \( \omega/W_\gamma = \partial \Phi_\gamma \)

---

[^3]: It is at this stage of the proof that we will use the fact that \( n \geq 2 \).
Once again, using the triviality of the group $H^{1,0}(X_{\alpha,\beta},\mathbb{C}^n)$, we prove that there exists a harmonic mapping $H = H_1 + H_2 : X_{\alpha,\beta} \to \mathbb{C}^n$ (with $H_1$ and $H_2$ holomorphic), such that $\partial H = \omega$.

$H_1$ and $H_2$ admit by Hartogs’ theorem analytic continuations to the sphere $S_{2n}$, and then become constant. Hence, by (3) and (4) we obtain $\partial \Phi_\gamma = 0$. But this contradicts the fact that $(W_\gamma, \Phi_\gamma)$ is a complex analytic chart.

**Remark 1.16.**

1) It appears clearly that the main piece which feeds into the procedure of proof sketched above, is the fact that the group of cohomology of Dolbeault of bidgree $(0,1)$ of any $A-$submanifold of $S_{2n}$ is trivial, and then the reader who would admit temporarily this result could read directly the proof of the main theorem 1.6 in section 4).

2) As mentioned in the introduction, although the problem of complex spheres stands only for $S_6$, our approach is however valid for all spheres $S_{2n}$ with $n \geq 2$, and then, we don’t have to use the result of Borel-Serre [1].

### 2. Topology of the $A-$submanifolds of $S_{2n}$

This section is devoted to prove two topological results: The first one is that every $A-$submanifold of $S_{2n}$ is homeomorphic to $\mathbb{C}^n$. The second one (see proposition 2.9) shows that under some supplementary conditions, if $A$ and $B$ are open sets of $\mathbb{C}^n$, both homeomorphic to balls of $\mathbb{C}^n$, then $B - A$ is homeomorphic to a ball of $\mathbb{C}^n$. As a consequence of this result, we deduce for the relatively compact open set defined by $W = S_{2n} - \overline{U_d}$ where $(U_d, \varphi_d)$ is the distinguished chart of an $A-$submanifold of $S_{2n}$, that there exist open sets

$$W_1, W_2, ..., W_N$$

all homeomorphic to balls of $\mathbb{C}^n$ such that

$$W_1 \subset W_2 \subset ... \subset W_N = W.$$

Such a result will be very useful in the next section to prove that any $A-$submanifolds of $S_{2n}$ is strictly pseudoconvex. To prove these results, we need first to check some lemmas concerning some special homeomorphisms of $\mathbb{C}^n$.

#### 2.1. Some special homeomorphisms of $\mathbb{C}^n$.

**Lemma 2.1.**

Let the ball $B_r \subset \mathbb{C}^n$, and let $K$ be a compact subset of $B_r$. Then for every $\varepsilon > 0$, there exists a homeomorphism $f : \mathbb{C}^n \to \mathbb{C}^n$ satisfying the following conditions:

$$\begin{cases} 
  f(z) = z & \text{if } z \notin \overline{B_r} \\
  \text{diam}(f(K)) < \varepsilon.
\end{cases}$$

**Proof.** Since the ball $B_r$ is open and contains the compact $K$, then there exists an open ball $B_{r'}$ with $(0 < r' < r)$, such that

$$K \subset B_{r'} \subset B_r.$$
Let the piecewise linear homeomorphism \( \chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) given by:

\[
\chi(t) := \begin{cases} 
\varepsilon \frac{t}{r'} & \text{if } 0 \leq t \leq r' \\
\frac{(r - \frac{\varepsilon}{2})(t - r')}{r - r'} + \frac{\varepsilon}{2} & \text{if } r' \leq t \leq r \\
t & \text{if } t \geq r.
\end{cases}
\] (2.1)

and define the mapping \( f : \mathbb{C}^n \rightarrow \mathbb{C}^n \) by the following expression:

\[
f(z) := \begin{cases} 
z \frac{\chi(\|z\|)}{\|z\|} & \text{if } z \neq 0 \\
0 & \text{if } z = 0.
\end{cases}
\] (2.2)

By writing \( z \in \mathbb{C}^n \) in spherical coordinates, we observe that the homeomorphism \( \chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) acts only on the norm \( \|z\| \), and not on the angles of \( z \). The mapping \( f \) is then a homeomorphism of \( \mathbb{C}^n \). Since for \( \|z\| \geq 1 \), we have \( \chi(\|z\|) = \|z\| \), we deduce then that \( f(z) = z \) in \( \mathbb{C}^n - B_r \). Now let \( z, \xi \in K \), that is, \( \|z\| \leq r' \) and \( \|\xi\| \leq r' \).

1) If \( \xi = 0 \), then \( \|f(z) - f(0)\| = \|f(z)\| = \frac{\varepsilon\|z\|}{r'} \leq \frac{\varepsilon}{2} \leq \varepsilon \).
2) If both \( z \) and \( \xi \) are \( \neq 0 \), then an elementary calculation gives

\[
\|f(z) - f(\xi)\| \leq \|f(z)\| + \|f(\xi)\| \\
\leq \frac{\|z\|}{2\varepsilon r'} + \frac{\|\xi\|}{2\varepsilon r'} \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\leq \varepsilon
\] (2.3)

which means that \( \text{diam}(f(K)) \leq \varepsilon \). The proof of lemma 2.1 is then complete. □

**Lemma 2.2. (Flattening homeomorphism)**

Let \( B(a, r) \) be a closed ball of \( \mathbb{C}^n \), and let \( K \) be a closed subset of \( \overline{B}(a, r) \) such that the center \( a \notin K \). Then for all \( \varepsilon \in (0, r] \), there exists a homeomorphism \( f : \mathbb{C}^n \rightarrow \mathbb{C}^n \) satisfying the conditions:

\[
\begin{cases} 
f(z) = z & \text{if } z \notin \overline{B}(a, r) \\
f(K) \subset \overline{B}(a, r) - B(a, r - \varepsilon).
\end{cases}
\]

**Remark 2.3.**

We don’t know if the term "flattening homeomorphism" exists already in mathematics, but we adopt it here in order to give an exact idea on how the homeomorphism considered above acts. Indeed, \( f \) acts really by "flattening" the compact \( K \) into the annulus \( \overline{B}(a, r) - B(a, r - \varepsilon) \).

**Proof.** Let \( \varepsilon \in (0, r] \). Since the center of the ball \( a \notin K \), then

\[
b := \inf_{z \in K} \|z - a\| > 0.
\]
Consider the piecewise linear homeomorphism \( \chi : \mathbb{R}^+ \to \mathbb{R}^+ \) given by
\[
(2.4) \quad \chi(t) := \begin{cases} 
\frac{t(r - \varepsilon)}{b} & \text{if } 0 \leq t \leq b \\
\varepsilon \frac{(t - b)}{(r - b)} + r - \varepsilon & \text{if } b \leq t \leq r \\
t & \text{if } r \leq t.
\end{cases}
\]
and define the mapping \( f : \mathbb{C}^n \to \mathbb{C}^n \) by
\[
(2.5) \quad f(z) := \begin{cases} 
\frac{a + \chi(\|z - a\|)}{\|z - a\|}(z - a) & \text{if } z \neq a \\
a & \text{if } z = a.
\end{cases}
\]
As in the proof of the previous lemma \( \text{2.1} \), we observe by writing \( z \in \mathbb{C}^n \) in spherical coordinates, that the homeomorphism \( \chi : \mathbb{R}^+ \to \mathbb{R}^+ \) defined by (2.4), acts only on the norm \( \|z\| \), and not on the angles of \( z \). The mapping \( f \) is then a homeomorphism of \( \mathbb{C}^n \). It is clear from the expression (2.5) that \( f \) satisfies the required conditions, that is
\[
\begin{cases} 
f(z) = z & \text{if } z \notin \mathcal{B}(a, r) \\
f(K) \subset \mathcal{B}(a, r) - \mathcal{B}(a, r - \varepsilon).
\end{cases}
\]

**Lemma 2.4.**
Let the open ball \( \mathcal{B}_r \) of \( \mathbb{C}^n \), and let \( \mathcal{M}_0 \) be an open subset of \( \mathbb{C}^n \) satisfying the following conditions:
1. \( \mathcal{M}_0 \cap \mathcal{B}_r \neq \emptyset \).
2. \( \mathcal{B}_r \cap \partial \mathcal{M}_0 \) is simply connected.
3. \( 0 \notin \mathcal{M}_0 \).
4. For all \( z \in \mathcal{B}_r \cap \partial \mathcal{M}_0 \), we have
   \[
   [0, z] \cap \mathcal{B}_r \cap \partial \mathcal{M}_0 = \{z\}
   \]
   where \( [0, z] \) denotes the half-line starting from the center \( 0 \) of \( \mathcal{B}_r \) and containing \( z \).

For every point \( z \in \mathcal{B}_r \cap \partial \mathcal{M}_0 \), define the point \( s(z) \) by
\[
s(z) = [0, z] \cap \mathcal{S}_{2n-1}(0, r)
\]
and let the closed cone
\[
\widehat{\mathcal{M}} := \bigcup_{z \in \mathcal{B}_r \cap \partial \mathcal{M}_0} [0, z].
\]
Then the mapping \( \mathcal{H}_0 : (\mathcal{B}_r - \mathcal{M}_0) \to \mathcal{B}_r \) defined by:
\[
(2.6) \quad \begin{cases} 
\mathcal{H}_0(z) = z & \text{if } z \notin \widehat{\mathcal{M}} \\
\mathcal{H}_0(z) = r \frac{z}{\|z\|} & \text{if } z \in \widehat{\mathcal{M}}
\end{cases}
\]
is a homeomorphism.

**Proof.** We leave to the reader to check lemma 2.4. \( \square \)
2.2. Applications to the topology of the $\mathcal{A}$–submanifolds of $S_{2n}$.

2.2.1. Quotient topology.
Let $E$ be a metric space, and let $K$ be a compact subset of $E$. We define then on $E$ a relation of equivalence $\sim$ by
\begin{equation}
z \sim z' \iff \begin{cases} 
z = z' \\
or\ z, z' \in K
\end{cases}
\end{equation}
and we note by $\pi : E \to E/K$ the canonical projection. We want to prove that under some conditions on $K$, the quotient space $E/K$ is homeomorphic to $E$. This will be the object of proposition 2.5 below, and then we deduce that every $\mathcal{A}$–submanifold of $S_{2n}$ is homeomorphic to $\mathbb{C}^n$.

Proposition 2.5.
Let $(E,d)$ be a metric space, and let $K$ be a compact subset of $(E,d)$, and consider on $E$ the relation of equivalence $\sim$ defined by (2.7). Assume that the compact $K$ admits in $(E,d)$ a fundamental system of open neighborhoods $\{U_k\}_{k \in \mathbb{N}}$, such that each $U_k$ is relatively compact and is homeomorphic to $\mathbb{C}^n$. Then for every open neighborhood $U$ of $K$ which is relatively compact, there exists a homeomorphism $h : E/K \to E$ such that $h \circ \pi$ coincides with $I_E$ in $E - U$.

Proof.
Let us first begin by the following remark.

Remark 2.6. If the compact $K$ is a singleton set, that is $K = \{z_0\}$, then the equivalence relation $\sim$ coincides with the equality relation $=$, and then $E/K = E$. In this case, we have nothing to prove, because the identity mapping $h = I_E$ answers the proposition. Therefore, we can assume that $K$ is not a singleton set.

Since by hypothesis, the open sets $U_k$ are relatively compact, that is, each $\overline{U_{k+1}}$ is compact in $U_k$, and since they constitute a fundamental system of neighborhoods of $K$, then
\[ K = \bigcap_{k=1}^{\infty} U_k. \]
Without loss of generality, we can suppose $U = U_1$. We claim that, there exists a sequence of homeomorphisms $g_k : E \to E$, such that:

1. $g_1 = I_E$.

2. For all $k > 1$, $g_k$ satisfies the conditions:
\begin{equation}
\begin{cases}
g_k(z) = g_{k-1}(z) & \text{for all } z \in E - U_k \\
diam g_k(U_{k+1}) < \frac{1}{k}.
\end{cases}
\end{equation}
Indeed, suppose by induction, that we have already constructed the homeomorphism $g_{k-1}$ satisfying condition (2.8). Since $g_{k-1}$ is continuous (and then uniformly continuous on every compact subset of $E$), then for every fixed compact $C \subset E$ and for every subset $Y \subset C$, we have
\begin{equation}
\lim_{\text{diam}(Y) \to 0} \text{diam} (g_{k-1}(Y)) = 0.
\end{equation}
That is, for any subset $Y \subset C$, and for all $k \in \mathbb{N}^*$, there exists a number $\delta_k > 0$ such that,

$$diam(Y) < \delta_k \implies diam\left(g_{k-1}(Y)\right) < \frac{1}{k}.$$  

(2.10)

To construct the homeomorphism $g_k : E \rightarrow E$ satisfying (2.8), consider the relatively compact neighborhoods $U_k$ and $U_{k+1}$ of $K$, and recall that there exists by hypothesis, a homeomorphism $\varphi_k : U_k \rightarrow \mathbb{C}^n$.

Since $U_{k+1}$ is compact in $U_k$, then $\varphi_k\left(U_{k+1}\right)$ is compact in $\mathbb{C}^n$. Now let $B_{r_k}$ be an open ball in $\mathbb{C}^n$ centered at 0 and containing the compact $\varphi_k\left(U_{k+1}\right)$, then by lemma 2.1 there exists a homeomorphism $f_k : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that

$$\begin{cases} 
    f_k(\varphi_k(z)) = \varphi_k(z) & \text{if } \varphi_k(z) \notin \overline{B_{r_k}} \\
    diam(f_k(\varphi_k(U_{k+1}))) < \delta_k.
\end{cases}$$

According to the commutative diagram

$$
\begin{array}{ccc}
U_k & \xrightarrow{\tilde{g}_k} & U_k \\
\varphi_k \downarrow & & \downarrow \varphi_k \\
\mathbb{C}^n & \xrightarrow{f_k} & \mathbb{C}^n
\end{array}
$$

there exists a homeomorphism $\tilde{g}_k : U_k \rightarrow U_k$ satisfying the conditions:

$$\begin{cases} 
    \tilde{g}_k(z) = z & \text{if } z \in E - U_k \\
    diam(\tilde{g}_k(U_{k+1})) < \delta_k.
\end{cases}$$

Taking into account that we have already $\tilde{g}_k(z) = z$ on $U_k - \varphi_k^{-1}\left(\overline{B_{r_k}}\right)$, we can then extend the local homeomorphism $\tilde{g}_k : U_k \rightarrow U_k$ to a global homeomorphism $\tilde{g}_k : E \rightarrow E$, by setting:

$$\tilde{g}_k(z) = z \quad \text{if } z \notin E - U_k.$$

Hence, the global homeomorphism $\tilde{g}_k : E \rightarrow E$ satisfies the conditions:

$$\begin{cases} 
    \tilde{g}_k(z) = z & \text{if } z \in E - U_k \\
    diam(\tilde{g}_k(U_{k+1})) < \delta_k.
\end{cases}$$

Define now the required mapping $g_k : E \rightarrow E$ by setting:

$$g_k = g_{k-1} \circ \tilde{g}_k.$$

We claim that:

1. $g_k : E \rightarrow E$ is a homeomorphism,

2. $g_k$ satisfies condition (2.8).

Indeed, $g_k$ is a homeomorphism because both $g_{k-1}$ and $\tilde{g}_k$ are homeomorphisms. Since for all $z \in E - U_k$, we have $\tilde{g}_k(z) = z$, we obtain then

$$g_k(z) = g_{k-1}(\tilde{g}_k(z)) = g_{k-1}(z) \quad \text{on } E - U_k.$$  

(2.11)

Furthermore, since $diam(\tilde{g}_k(U_{k+1})) < \delta_k$, it follows by (2.10) that

$$diam\left(g_k(U_{k+1})\right) = diam\left(g_{k-1}(\tilde{g}_k(U_{k+1}))\right) < \frac{1}{k}.$$  

(2.12)
Hence condition (2.8) is satisfied. Consider now the sequence of homeomorphisms \( \{g_k\}_{k \geq 1} \) satisfying condition (2.8). We claim that \( \{g_k\}_{k \geq 1} \) admits a subsequence \( \{g_{k_j}\}_{j \geq 1} \) which converges point wise to a continuous mapping \( g : E \rightarrow E \).

Indeed, fix a point \( z_0 \in \mathcal{K} \), and consider the sequence of points \( \{g_k(z_0)\}_{k \geq 1} \) of the metric space \( E \). Since the compact \( \mathcal{K} \) is assumed to be non reduced to a singleton set, then by condition (2.8), we obtain

\[
\{g_k(z_0)\}_{k \geq 1} \subset \mathcal{K} = \bigcap_{k=1}^{\infty} U_k.
\]

By the well known Bolzano-Weirstrass theorem, the sequence of points \( \{g_k(z_0)\}_{k \geq 1} \) admits then a subsequence \( \{g_{k_j}(z_0)\}_{j \geq 1} \) which converges to a point \( \ell \in \mathcal{K} = \bigcap_{k=1}^{\infty} U_k \).

Consider now the mapping \( g : E \rightarrow E \) defined as follows:

\[
g(z) = g_{k_j}(z) \quad \text{if} \quad z \in E - U_{k_j},
\]

\[
g(z) = \ell \quad \text{if} \quad z \in \mathcal{K}.
\]

By condition (2.8), the mapping \( g \) is well defined. It is clear that \( g \) is continuous in \( E - \mathcal{K} \), because it coincides with one of the homeomorphisms \( g_{k_j} \). We claim that \( g \) is also continuous at every point \( z_0 \in \mathcal{K} \).

Indeed, Let the number \( \varepsilon > 0 \). Since \( \lim_{z \to z_0} g_{k_j}(z_0) = \ell \), there exists then \( N_0 \in \mathbb{N} \) such that for all \( k_j \geq N_0 \), we have \( d\left(g_{k_j}(z_0), \ell\right) \leq \frac{\varepsilon}{2} \).

In another hand, there exists a neighborhood \( U_{N_1} \) of \( \mathcal{K} \), such that

\[
z \in U_{N_1} \implies d\left(g_{k_j}(z), g_{k_j}(z_0)\right) \leq \frac{\varepsilon}{2}.
\]

Now, let \( N = \max\{N_0, N_1\} \), and let \( z \in U_N \subset U_{N_1} \).

1) If \( z \in \mathcal{K} \), we have then \( d(g(z), g(z_0)) = 0 \).

2) If \( z \in (U_N - \mathcal{K}) \), there exists \( k_j \geq N \), such that \( g(z) = g_{k_j}(z) \). In this case, we obtain:

\[
d\left(g(z), g(z_0)\right) = d\left(g(z), \ell\right)
= d\left(g_{k_j}(z), \ell\right)
\leq d\left(g_{k_j}(z), g_{k_j}(z_0)\right) + d\left(g_{k_j}(z_0), \ell\right)
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
= \varepsilon.
\]

Then \( g \) is a continuous mapping on \( E \), but since \( \mathcal{K} \) is not a singleton set, \( g \) is not a homeomorphism because \( \text{diam}(g(\mathcal{K})) = 0 \).

Now let \( \pi : X \rightarrow E/\mathcal{K} \) be the canonical projection, and let the relatively compact neighborhood \( U = U_1 \) of \( \mathcal{K} \). Since the mapping \( g : E \rightarrow E \) is constant in the compact \( \mathcal{K} \), and \( g_{k_j} \) are homeomorphisms, then \( g \) factorizes as follows

\[
g = h \circ \pi
\]
where $h : E/\mathcal{K} \rightarrow E$ is a homeomorphism such that the following diagram commutes

$$
\begin{array}{ccc}
E & \xrightarrow{g} & E \\
\downarrow_{\pi} & & \downarrow_{\pi} \\
E/\mathcal{K} & \xrightarrow{h} & E
\end{array}
$$

This means that the homeomorphism $h : E/\mathcal{K} \rightarrow E$ is such that $h \circ \pi = g = I_E$ in $E - U$. The proof of proposition 2.6 is then complete. □

**Corollary 2.7.**
If $\mathcal{K}$ is a compact subset of $S_{2n}$, then $S_{2n} - \mathcal{K}$ is homeomorphic to $\mathbb{C}^n$.

**Proof.** Indeed, since by stereographic projection, the compact $\mathcal{K}$ admits a fundamental system of relatively compact neighborhoods each of them homeomorphic to $\mathbb{C}^n$, we deduce then according to proposition 2.5 that for every neighborhood $U$ of $\mathcal{K}$, there exists a homeomorphism

$$
h : S_{2n}/\mathcal{K} \rightarrow S_{2n}
$$

such that the following diagram commutes

$$
\begin{array}{ccc}
S_{2n} & \xrightarrow{g} & S_{2n} \\
\downarrow_{\pi} & & \downarrow_{\pi} \\
S_{n}/\mathcal{K} & \xrightarrow{h} & S_{2n}
\end{array}
$$

and that $g = h \circ \pi$ coincides with $\mathbb{1}_{S_{2n}}$ in $U$.

Since for all $z \in (S_{2n} - \mathcal{K})$, we have $\pi(z) = z$, then

$$
S_{n}/\mathcal{K} = (S_{2n} - \mathcal{K}) \bigcup \pi(\mathcal{K}).
$$

By removing from the sphere $S_{2n}$ the point $N := h(\pi(\mathcal{K})) \in S_{2n}$, we obtain then a homeomorphism

$$
h : S_{2n} - \mathcal{K} \rightarrow S_{2n} - \{N\}.
$$

But we know that by stereographic projection with respect to the pole $N$, $S_{2n} - \{N\}$ is already homeomorphic to $\mathbb{C}^n$. Then $S_{2n} - \mathcal{K}$ is homeomorphic to $\mathbb{C}^n$. □

**Corollary 2.8.**
Every $\mathcal{A}$–submanifold of $S_{2n}$ is homeomorphic to $\mathbb{C}^n$.

**Proof.** Indeed, this is a particular case of the previous corollary 2.7 □

2.3. Some open sets homeomorphic to balls.

**Proposition 2.9.**
Let $B_r$ be an open ball of $\mathbb{C}^n$ and let $\mathcal{M} \subset \mathbb{C}^n$ be an open set homeomorphic to a ball of $\mathbb{C}^n$ such that:

(a) The boundary $\partial \mathcal{M}$ is analytic.

(b) $B_r \bigcap \mathcal{M} \neq \emptyset$.

(c) $B_r \bigcap \partial \mathcal{M}$ is simply connected.

Then $B_r - \overline{\mathcal{M}}$ and $B_r \bigcap \mathcal{M}$ are homeomorphic to $B_r$. 
Proof. Since the boundary $\partial M$ is assumed to be analytic, then $B_r \cap \partial M$ admits at more a finite number $K_1, K_2, ..., K_m$ of $(2n-2)$-dimensional singular surfaces. By applying to $B_r \cap \partial M$ if necessarily a finite number $f_1, f_2, ..., f_s$ of "flattering" homeomorphisms as defined in lemma 2.2, we obtain then a subset

$$B_r \cap M_0 = (f_1 \circ f_2 \circ ... \circ f_s) (B_r - M)$$

where $M_0$ is an open set satisfying conditions of lemma 2.4. Hence by lemma 2.4, $B_r - M$ is homeomorphic to $B_r$. We prove in the same way that $B_r \cap M$ is homeomorphic to $B_r$. The proof of proposition 2.9 is then complete. □

3. The $\overline{\partial}$-cohomology of $A$–submanifolds of $S_{2n}$

This section is devoted to prove that the Dolbeault cohomology group of bidegree $(0, 1)$ of any $A$–submanifold of $S_{2n}$ is trivial.

3.1. The hypothetic complex manifold $(\mathbb{C}^n, \mathfrak{A})$.

3.1.1. Notations and definition.

Definition 3.1.
With the notations of 1.3.2 let $X$ be an $A$–submanifold of $S_{2n}$, and let

$$A/X := \left\{ \left( \overline{U}_j, \overline{\varphi}_j \right), \quad j \in \mathcal{J} \right\}.$$ 

be the hypothetic complex structure on $X$. By corollary 2.7 there exists a homeomorphism

$$h : X \longrightarrow \mathbb{C}^n.$$ 

The numerical space $\mathbb{C}^n$ is then endowed with a new atlas $\mathfrak{A} = h(A/X)$

$$\mathfrak{A} := \left\{ (V_j, \psi_j), \quad j \in \mathcal{J} \right\}$$

obtained by carrying the atlas $A/X$ to $\mathbb{C}^n$. The complex charts $(V_j, \psi_j)$ of $\mathfrak{A}$ are then defined for all $j \in \mathcal{J}$, by

$$\begin{align*}
V_j &= h (\overline{U}_j) \\
\psi_j &= \overline{\varphi}_j \circ h^{-1}.
\end{align*}$$

Taking into account the properties of $A/X$ given in subsubsection 1.3.2, the atlas $\mathfrak{A}$ inherits then from $A/X$ the following properties:

1. For all $j \in \mathcal{J} - \{d\}$, the domain of chart $V_j$ is bounded in $\mathbb{C}^n$ and it follows from condition (1.1), that

$$\forall \ j \neq d, \quad \psi_j (V_j) \quad \text{is a ball of } \mathbb{C}^n.$$  

2. It follows also from condition (1.2), that

$$\forall \ i, j \neq d, \quad V_i \cap \partial V_j \quad \text{is simply connected.}$$

4 The hypothetic complex structure $\mathfrak{A}$ on $\mathbb{C}^n$ supposed here is of course different from the standard complex structure defined usually on $\mathbb{C}^n$. 


(3) For \( j = d \), the mapping
\[
\psi_d : V_d \rightarrow \mathcal{N}(a, r_1, r_2)
\]
is a homeomorphism.
The chart \((V_d, \psi_d)\) is said to be the distinguished chart of \( Y = (\mathbb{C}^n, \mathfrak{A}) \).
Observe that since for all \( j \in J - \{d\} \), the domains \( V_j \) are bounded, then
the domain \( V_d \) is the only one which is non bounded in \( Y = (\mathbb{C}^n, \mathfrak{A}) \) and
its boundary \( \partial V_d \) satisfies
\[
\partial V_d \approx S_{2n-1}.
\]
(4) The subset \( \mathfrak{W} := h(\mathcal{W}) = \mathbb{C}^n - \overline{V_d} \) is a relatively compact open set covered
by all domains \( V_j \) with \( j \neq d \)
\[
\mathfrak{W} \subset \bigcup_{j \neq d} V_j
\]
and \( \mathfrak{W} \) is homeomorphic to a ball of \( \mathbb{C}^n \), and its boundary satisfies
\[
\partial \mathfrak{W} = \partial V_d \approx S_{2n-1}.
\]

**Remark 3.2.**
In order to avoid any confusion with the usual complex structure of the numerical
space \( \mathbb{C}^n \), we note by \( Y = (\mathbb{C}^n, \mathfrak{A}) \) the complex analytic structure of \( \mathbb{C}^n \) defined by
the hypothetic complex atlas \( \mathfrak{A} \) given above.

3.1.2. **The ordered atlas** \( (\mathfrak{A}, \preceq) \).

To prove that the hypothetic \( \mathfrak{A} \)-submanifold \( X \) of \( S_{2n} \) is strictly pseudoconvex,
we are lead by the biholomorphic mapping \( h : X \rightarrow Y \) to prove that the complex
manifold \( Y = (\mathbb{C}^n, \mathfrak{A}) \) is strictly pseudoconvex, which will be done by constructing
an exhaustive smooth strictly \textbf{psh} function \( \Psi : Y \rightarrow [0, +\infty[ \). But to do this, we
need to introduce in the hypothetic complex atlas \( \mathfrak{A} \) an appropriate order: This
order helps much first to construct an exhaustive almost everywhere \( C^\infty \) strictly
\textbf{psh} function \( f_a : Y \rightarrow [0, +\infty[ \), then by applying an appropriate regularizing
process to \( f_a \), we construct the required strictly \textbf{psh} function \( \Psi \).
Let then the hypothetic complex atlas
\[
\mathfrak{A} = \left\{ (V_j, \psi_j), \quad j \in J \right\}
\]
with
\[
\text{card}(J) = N + 1
\]
and let the relatively compact open set
\[
\mathfrak{W} = \mathbb{C}^n - \overline{V_d}
\]
as given in definition 3.1, where \( V_d \) is the domain of the distinguished chart \((V_d, \psi_d)\).

**Proposition 3.3.** With these notations, there exists a bijective mapping
\[
\sigma : \{1, 2, \ldots, N + 1\} \rightarrow J
\]
such that, for all \( 1 \leq k \leq N + 1 \), the open set
\[
\mathfrak{W} \cap \left( \bigcup_{j=1}^{k} V_{\sigma(j)} \right)
\]
is homeomorphic to a ball of \( \mathbb{C}^n \).
Proof.

1st step. Let’s organize the domains of charts of the subatlas
\[ \mathcal{A}^* := \mathcal{A} - \{ (V_d, \psi_d) \} \]
in a "spherical" arrangement.

Indeed, by observing that \( \mathcal{M} = \mathbb{C}^n - V_d \) is covered by the domains of the subatlas \( \mathcal{A}^* \), and that
\[ \partial \mathcal{M} \approx S_{2n-1} \]
we define \( \mathcal{A}_1 \) to be the set of all domains \( V_j \in \mathcal{A}^* \) which intersect \( \partial \mathcal{M} \), that is
\[ \mathcal{A}_1 := \left\{ V_j \in \mathcal{A}^*, \quad V_j \cap \partial \mathcal{M} \neq \emptyset \right\} \]
and we define \( Y_1 \) to be the open subset of \( Y = (\mathbb{C}^n, \mathcal{A}) \) covered by \( \mathcal{A}_1 \), that is
\[ Y_1 := \bigcup_{V_j \in \mathcal{A}_1} V_j. \]

Since by property (3.1), all domains \( V_j \in \mathcal{A}_1 \) are homeomorphic to balls of \( \mathbb{C}^n \) and satisfy condition (3.2), it follows then by proposition 2.9 that
\[ \partial Y_1 = \Gamma_1 \bigcup \Gamma_2 \]
where
\[ \begin{cases} 
\Gamma_1 \approx \partial \mathcal{M} \approx S_{2n-1} \\
\Gamma_2 \approx S_{2n-1} \quad \text{or eventually} \quad \Gamma_2 = \emptyset.
\end{cases} \]

If \( \Gamma_2 \neq \emptyset \), that is, if \( \Gamma_2 \approx S_{2n-1} \), we continue the construction, and we define
\[ \mathcal{A}_2 := \left\{ V_j \in \mathcal{A}, \quad V_j \cap \Gamma_2 \neq \emptyset \right\} \]
and
\[ Y_2 := \bigcup_{V_j \in \mathcal{A}_2} V_j. \]

We observe as before that
\[ \partial Y_2 = \Gamma_2 \bigcup \Gamma_3 \]
where
\[ \begin{cases} 
\Gamma_2 \approx S_{2n-1} \\
\Gamma_3 \approx S_{2n-1} \quad \text{or eventually} \quad \Gamma_3 = \emptyset.
\end{cases} \]

By induction, we obtain a finite family of "spherical" coverings \( \{ \mathcal{A}_r \}_{1 \leq r \leq m} \subset \mathcal{A}^* \) defined by
\[ \mathcal{A}_r = \left\{ V_j \in \mathcal{A}^*, \quad V \cap \Gamma_r \neq \emptyset \right\} \]
and a finite family of open sets \( \{ Y_r \}_{1 \leq r \leq m} \) of \( (\mathbb{C}^n, \mathcal{A}) \), where \( Y_r \) is covered by \( \mathcal{A}_r \), that is
\[ Y_r := \bigcup_{V_j \in \mathcal{A}_r} V_j. \]
and where the boundary of $Y_r$ is given according to proposition 2.9 by

$$\begin{cases}
\partial Y_r = \Gamma_r \bigcup \Gamma_{r+1} & \text{if } 1 \leq r \leq m - 1 \\
\partial Y_m = \Gamma_m \approx S_{2n-1}
\end{cases}$$

and for all $1 \leq r \leq m$

$$\Gamma_r \approx S_{2n-1}.$$ 

Hence the subatlas $\mathfrak{A}^*$ can be written as follows

$$\mathfrak{A}^* = \bigcup_{r=1}^m \mathfrak{A}_r.$$ 

2nd step. Now write

$$\mathfrak{W} = \bigcup_{j \in J} \left( \mathfrak{W} \cap V_j \right),$$

Since we know that $\mathfrak{W}$ is homeomorphic to an open ball of $\mathbb{C}^n$, we can then by proposition 2.9 define a bijective mapping

$$\sigma : \{1, 2, \ldots, N + 1\} \to J$$

satisfying condition of proposition 3.3.

Indeed, let us proceed as follows: First, set

$$\sigma(N + 1) := d \in J$$

that is

$$V_{\sigma(N+1)} := V_d$$

then define by induction the respective values

$$\sigma(N), \sigma(N-1), \sigma(N-2), \sigma(N-3), \ldots, \sigma(3), \sigma(2), \sigma(1)$$

by removing from $\mathfrak{W}$ the sets $\mathfrak{W} \cap V_j$ one by one, while respecting the following rule: whenever we remove a domain $V_{\sigma(j)}$, the next domain $V_{\sigma(j-1)}$ chosen to be removed must intersect the maximum number of the previous ones, that is

$$V_{\sigma(j-1)} \cap V_{\sigma(j)} \cap V_{\sigma(j+1)} \cap \ldots \cap V_{\sigma(j+k)} \neq \emptyset.$$ 

We begin then by removing the domains of $\mathfrak{A}_m$ one by one until exhausting $\mathfrak{A}_m$, then we remove the domains of $\mathfrak{A}_{m-1}$ one by one until exhausting $\mathfrak{A}_{m-1}$, and so on, until exhausting $\mathfrak{A}_1$.

Recall that the domains $V_j$ with $j \neq d$ satisfy condition (3.2), and then we are in a position to apply proposition 2.9.

Since $\mathfrak{W}$ is homeomorphic to an open ball of $\mathbb{C}^n$, the next open set obtained by removing a chosen domain $\mathfrak{W} \cap V_{\sigma(N)}$ from $\mathfrak{W}$, that is $\mathfrak{W} - \mathfrak{W} \cap V_{\sigma(N)}$ is according to proposition 2.9 also homeomorphic to an open ball of $\mathbb{C}^n$.

Taking into account that the domains $V_j$ with $j \neq d$ satisfy condition (3.2), we prove by induction that whenever

$$\mathfrak{W} - \bigcup_{j=k+1}^{N} \left( \mathfrak{W} \cap V_{\sigma(j)} \right)$$
is homeomorphic to an open ball of $\mathbb{C}^n$, then by proposition 2.9, the next open set obtained by removing a chosen domain $\mathcal{W} \cap V_{\sigma(k+1)}$ from $\mathcal{W}$, that is

$$\mathcal{W} - \bigcup_{j=k}^{N} (\mathcal{W} \cap V_{\sigma(j)})$$

is also homeomorphic to an open ball of $\mathbb{C}^n$.

Hence, we construct a mapping

$$\sigma : \{1, 2, ..., N + 1\} \longrightarrow J$$

such that, for all $1 \leq k \leq N$, the open set

$$\bigcup_{j=1}^{k} (\mathcal{W} \cap V_{\sigma(j)})$$

is homeomorphic to an open ball of $\mathbb{C}^n$. □

**Remark 3.4.** The bijective mapping $\sigma$ is not unique.

**Corollary 3.5.** The relation $\preceq$ defined in $\mathfrak{A}$ by

$$(V_{\sigma(j)}, \psi_{\sigma(j)}) \preceq (V_{\sigma(k)}, \psi_{\sigma(k)}) \iff j \leq k$$

is a total order. The atlas $\mathfrak{A}$ endowed with this relation is denoted by $(\mathfrak{A}, \preceq)$.

**Notation.** Once the relation of order $\preceq$ in $\mathfrak{A}$ is acquired, we can note the charts of $\mathfrak{A}$ simply by $(V_j, \psi_j)$ instead of $(V_{\sigma(j)}, \psi_{\sigma(j)})$, and then the atlas $\mathfrak{A}$ can be written

$$\mathfrak{A} = \left\{ (V_j, \psi_j), \quad 1 \leq j \leq N + 1 \right\}.$$ 

With this notation, the distinguished chart is then $(V_{N+1}, \psi_{N+1})$.

**Corollary 3.6.**

Let the relatively compact set

$$\mathfrak{W} = \mathbb{C}^n - \overline{V_d}.$$ 

With the notations above, there exists a finite increasing sequence of open sets $\mathfrak{W}_k$ all homeomorphic to balls of $\mathbb{C}^n$:

$$\mathfrak{W}_1 \subset \mathfrak{W}_2 \subset \mathfrak{W}_3 \subset ... \subset \mathfrak{W}_{N-1} \subset \mathfrak{W}_N$$

such that $\mathfrak{W} = \mathfrak{W}_N$.

**Proof.** Indeed, it suffices according to proposition 3.3 to take for all $1 \leq k \leq N$

$$(3.4) \quad \mathfrak{W}_k = \bigcup_{j=1}^{k} (\mathfrak{W} \cap V_j).$$

□
3.1.3. Points of first and points of second kind with respect to \((\mathfrak{A}, \preceq)\).

It emerges from the relation of order \(\preceq\) on the atlas \(\mathfrak{A}\) a natural distinction between points of \(Y = (\mathbb{C}^n, \mathfrak{A})\). This distinction is made precise in the following definition.

**Definition 3.7.** We say that a point \(z \in Y = (\mathbb{C}^n, \mathfrak{A})\) is of first kind with respect to the ordered atlas \((\mathfrak{A}, \preceq)\), if \(z\) satisfies the following condition

\[(3.5) \text{ For all } j, k \in \{1, \ldots, N + 1\} \text{ with } j \leq k, \quad z \not\in (\partial V_j) \bigcap V_k.\]

A point \(z \in Y = (\mathbb{C}^n, \mathfrak{A})\) is said to be of second kind with respect to \((\mathfrak{A}, \preceq)\), if

\[(3.6) \text{ There exist } j, k \in \{1, \ldots, N + 1\} \text{ with } j \leq k, \quad z \in (\partial V_j) \bigcap V_k.\]

The condition for a point \(z\) to be of second kind with respect to \((\mathfrak{A}, \preceq)\), means that \(z\) belongs to a domain of chart \(V_k\), and in the same time to the boundary of a previous one.

**Notation.** We note by \(K_1\) respectively \(K_2\) the sets of points of first kind respectively of second kind with respect to \((\mathfrak{A}, \preceq)\).

**Remark 3.8.**

1. Not any point of the boundary of a domain of chart is of second kind with respect to \((\mathfrak{A}, \preceq)\), because \(\partial V_{N+1} \not\subseteq K_2\).
2. The set \(K_2\) is neglected with respect to the Lebeague measure, because

\[K_2 \subset \bigcup_{j=1}^{N} \partial V_j\]

3. \(K_1\) is an open subset of \(Y = (\mathbb{C}^n, \mathfrak{A})\).

3.2. The almost everywhere psh function \(f_\mathfrak{A}\).

We show below how we can associate to the ordered atlas \((\mathfrak{A}, \preceq)\) an exhaustive function \(f_\mathfrak{A} : Y = (\mathbb{C}^n, \mathfrak{A}) \rightarrow [0, +\infty[\) almost everywhere \(\mathcal{C}^\infty\) and strictly psh.

**3.2.1. Notations and definitions.**

With the notations of definition 3.1, let the hypothetic ordered atlas

\[\mathfrak{A} = \left\{(V_j, \psi_j), \quad 1 \leq j \leq N + 1\right\}\]

and let \(\mathfrak{W}\) be the relatively compact open subset of \(Y = (\mathbb{C}^n, \mathfrak{A})\) defined by

\[\mathfrak{W} = \mathbb{C}^n - V_{N+1},\]

recall that \(\mathfrak{W}\) is covered by the domains of charts \(V_j\), with \(1 \leq j \leq N\)

\[\mathfrak{W} \subset \bigcup_{j=1}^{N} V_j.\]

Now, let \(a_1 \in V_1\), such that \(\psi_1(a_1)\) is the center of the ball \(\psi_1(V_1)\), and chose in the open set \(\mathfrak{W}\), \(N - 1\) points of second kind

\[\left\{a_2, a_3, \ldots, a_N\right\} \subset \mathfrak{W} \bigcap K_2\]

such that, for \(2 \leq k \leq N\)

\[a_k \in (\partial V_{k-1}) \bigcap V_k.\]
and define by induction the real valued functions

\[ f_1, f_2, f_3, \ldots, f_N, f_{N+1} \]

as follows:

The function \( f_1 \) is defined on \( \text{Dom} (f_1) = V_1 \cap \mathfrak{M} \) by

\[ f_1(z) = \| \psi_1(z) - \psi_1(a_1) \|^2 \tag{3.7} \]

and for \( 2 \leq k \leq N \), the function \( f_k \) is defined on \( \text{Dom} (f_k) = (V_k - V_{k-1}) \cap \mathfrak{M} \) by

\[ f_k(z) = \| \psi_k(z) - \psi_k(a_k) \|^2 + b_{k-1} \tag{3.8} \]

and for \( k = N + 1 \), the function \( f_{N+1} \) is defined on \( \text{Dom} (f_{N+1}) = V_{N+1} = \mathbb{C}^n - \mathfrak{M} \) by

\[ f_{N+1}(z) = -\log \left( \max \left\{ \left( \frac{\| \psi_{N+1}(z) \|^2 - r_2^2}{r_1^2 - r_2^2} \right)^2, \left( \frac{\text{Re} (\psi_{N+1}(z), w)}{r_1 \| w \|} \right)^2 \right\} \right) + b_N. \tag{3.9} \]

where for all \( 1 \leq k \leq N \), the numbers \( b_k \) are given by induction by

\[ b_k = \sup_{z \in \text{Dom}(f_k)} f_k(z). \tag{3.10} \]

By gluing the functions \( f_1, f_2, \ldots, f_{N+1} \) defined by (3.7), (3.8), (3.9) together, we define then the following global function

\[ f_a : (\mathbb{C}^n, \mathfrak{M}) \rightarrow [0, +\infty[ \]

by

\[ f_a(z) := \begin{cases} f_1(z) & \text{if } z \in V_1 \cap \mathfrak{M} \\ f_k(z) & \text{if } z \in (V_k - V_{k-1}) \cap \mathfrak{M}, \text{ with } 2 \leq k \leq N \\ f_{N+1}(z) & \text{if } z \in V_{N+1}. \end{cases} \tag{3.11} \]

Remark 3.9. Observe that the function \( f_a \) defined by (3.11) is not continuous, and that the points of discontinuity of \( f_a \) belong to the set of second kind \( K_2 \).

Lemma 3.10.

For \( r_1 > r_2 > 0 \), let

\[ \overline{N}(a, r_1, r_2) := \overline{B}(a, r_1) - S_{2n-2}(0, r_2) \]

and let \( \tilde{f}_{N+1} = f_{N+1} \circ \psi_{N+1}^{-1} \) be the representation in the local coordinates \( \zeta = \psi_{N+1}(z) \) of the function \( f_{N+1} \) defined in (3.9), that is, \( \tilde{f}_{N+1} : \overline{N}(a, r_1, r_2) \rightarrow \mathbb{R} \) defined by

\[ \tilde{f}_{N+1}(\zeta) = -\log \left( \max \left\{ \left( \frac{\| \zeta \|^2 - r_2^2}{r_1^2 - r_2^2} \right)^2, \left( \frac{\text{Re} (\zeta, w)}{r_1 \| w \|} \right)^2 \right\} \right) + b_N. \]

Then the function \( \tilde{f}_{N+1} \) satisfies the following properties:

1. \( \tilde{f}_{N+1} \) is almost everywhere a \( C^\infty \) strictly psh.
2. The following equivalence holds

\[ \tilde{f}_{N+1}(\zeta) = 0 \iff \zeta \in S_{2n-2}(0, r_2), \tag{3.12} \]
(3) For all $c > b_N$, there exists $0 < \rho < r_1$ such that the sublevel set

$$L_c \left( \tilde{f}_{N+1} \right) := \left\{ \zeta \in \mathcal{N}(a, r_1, r_2), \quad \tilde{f}_{N+1}(\zeta) \leq c \right\}$$

is homeomorphic to the closed annulus $\overline{B}(a, r_1) - B(a, \rho)$

Proof. Consider the functions

$$g_1(\zeta) = \left( \frac{\|\zeta\|^2 - r_2^2}{r_1^2 - r_2^2} \right)^2$$

and

$$g_2(\zeta) = \left( \frac{\text{Re} \langle \zeta, w \rangle}{r_1 \cdot \|w\|} \right)^2.$$  

1) The function $g_1$ is clearly $C^\infty$ and is strictly psh because it is composed of the strictly psh function $\frac{\|\zeta\|^2 - r_2^2}{r_1^2 - r_2^2}$ with the strictly convex function $x \mapsto -x^2$.

The function $g_2$ is clearly of class $C^\infty$ and is also strictly psh because we have

$$g_2(\zeta) = \left( \frac{\text{Re} \langle \zeta, w \rangle}{r_1 \cdot \|w\|} \right)^2 = \left( \frac{\langle \zeta, w \rangle}{r_1 \cdot \|w\|} \right)^2 + 2 \left( \frac{\|\zeta, w\|}{r_1 \cdot \|w\|} \right)^2$$

and then the Levi form of $g_2$ coincides with the Levi form of the function

$$\zeta \mapsto \frac{\|\zeta, w\|^2}{2 \left( r_1 \cdot \|w\| \right)^2}$$

which is clearly strictly psh. Since the set

$$\left\{ z \in \mathcal{N}(a, r_1, r_2), \quad |g_1(\zeta)| = |g_2(\zeta)| \right\}$$

is neglected with respect to Lebesgue measure, it follows then that $\tilde{f}_{N+1}$ is almost everywhere a $C^\infty$ strictly psh function.

2) It is clear that the set of solutions of the equation $\tilde{f}_{N+1}(\zeta) = 0$ is

$$S_{2n-2}(0, r_2) = \left\{ \zeta \in \mathbb{C}^n, \quad \tilde{f}_{N+1}(\zeta) = 0 \right\}.$$  

3) Let the sets

$$F_1 = \left\{ \zeta \in \mathcal{N}(a, r_1, r_2), \quad |g_1(\zeta)| < e^{-c - b_N} \right\}$$

and

$$F_2 = \left\{ \zeta \in \mathcal{N}(a, r_1, r_2), \quad |g_2(\zeta)| < e^{-c - b_N} \right\}.$$  

We check easily that $F_1 \cap F_2$ is a bounded open convex set, and then $F_1 \cap F_2$ is homeomorphic to a ball $B(a, \rho)$ of $\mathbb{C}^n$. Since $L_c \left( \tilde{f}_{N+1} \right)$ is the complement of $F_1 \cap F_2$ with respect to the closed ball $\overline{B}(a, r_1)$, then $L_c \left( \tilde{f}_{N+1} \right)$ is homeomorphic to a closed annulus of the form $\overline{B}(a, r_1) - B(a, \rho)$. The proof of the lemma [3.10] is then complete. □
Proposition 3.11. (Properties of the function $f_a$)

Let the function $f_a : Y = (\mathbb{C}^n, \mathfrak{A}) \rightarrow [0, +\infty[$ defined by (3.11), and for $c > 0$, let the sublevel set

$$L_c(f_a) = \left\{ z \in Y, \quad f_a(z) \leq c \right\}.$$

Then

1. On the set of first kind $\mathbb{K}_1$, $f_a$ is of class $C^\infty$.
2. On the set of first kind $\mathbb{K}_1$, $f_a$ is strictly psh.
3. For all $c > 0$, the set $L_c(f_a)$ is homeomorphic to a closed ball of $\mathbb{C}^n$.

Proof.

1. By recalling the definition of the open set of the points of first kind $\mathbb{K}_1$ and by examining the expressions of the different functions $f_k$, we observe clearly that the function $f_a$ is of class $C^\infty$ in $\mathbb{K}_1$.

2. Since the property of positivity of the Levi form of a function, is independent of the choice of local coordinates, it suffices then for a point $z \in V_k$ to check this positivity only in the chart $(V_k, \psi_k)$. Let then $z \in \mathbb{K}_1$, and for all $1 \leq k \leq N + 1$, let $\tilde{f}_k = f \circ \psi_k^{-1}$ be the expression of $f_k$ in the local coordinates $\zeta = \psi_k(z)$. By a careful examination of the different expressions of the function $f_k$, we obtain:

   a. If $z \in V_1 \cap \mathfrak{M}$, then

   $$\tilde{f}_1(\zeta) = ||\zeta - a_1||^2$$

   which means that the function $f_1$ is strictly psh on $V_1 \cap \mathfrak{M}$.

   b. For $2 \leq k \leq N$, and $z \in (V_k - \overline{V}_{k-1}) \cap \mathfrak{M}$, we have

   $$\tilde{f}_k(\zeta) = ||\zeta - \psi_k(a_k)||^2 + b_{k-1}.$$

   The function $f_k$ in also strictly psh on $(V_k - \overline{V}_{k-1}) \cap \mathfrak{M}$.

   c. If $z \in \overline{V}_{N+1}$, we know by lemma (3.10) that the function

   $$\tilde{f}_{N+1}(\zeta) = -\log \left( \max \left\{ \left( \frac{||\zeta||^2 - r_1^2}{r_1^2 - r_2^2} \right)^2, \left( \frac{\text{Re}(\zeta, w)}{r_1||w||} \right)^2 \right\} \right) + b_N.$$

   is almost everywhere strictly psh on $\psi_{N+1}(V_{N+1})$.

We conclude then by this discussion that on the set of points of first kind $\mathbb{K}_1$, the function $f_a$ is strictly psh, and therefore its Levi form given in the local coordinates $\zeta = \psi_k(z)$ by

$$\mathcal{L}_\zeta(\tilde{f}_k)[t, t] = \sum_{j,l} \frac{\partial^2 \tilde{f}_k(\zeta)}{\partial \zeta_j \partial \zeta_l} t_j t_l$$

where $t = (t_1, ..., t_n) \in \mathbb{C}^n$.

3. Let $c > 0$, and let the sublevel set

$$L_c(f_a) := \left\{ z \in Y, \quad f_a(z) \leq c \right\}.$$

1st case. $c = b_k$.

By the definition of the number $b_k$ given by equation (3.10), we have

$$L_{b_k}(f_a) = \overline{\mathfrak{M}}.$$
and then \( L_{b_k}(f_{a}) \) is homeomorphic to a closed ball of \( \mathbb{C}^n \).

**2nd case.** \( b_k < c < b_{k+1} \).

Let the closed ball \( \overline{B}(a_{k+1}, c - b_k) \), since \( \psi_{k+1}(V_{k+1}) \) is a ball, then the set

\[
D_{c,k}(f_{a}) := \psi_{k+1}^{-1}(\overline{B}(a_{k+1}, c - b_k) \cap \psi_{k+1}(V_{k+1}))
\]

is homeomorphic to a closed ball of \( \mathbb{C}^n \). For \( b_k < c < b_{k+1} \), we can write \( L_c(f_{a}) \) as follows:

\[
L_c(f_{a}) = \overline{W}_{k+1} - \left( \left( V_{k+1} - \overline{W}_k \right) - \left( D_{c,k}(f_{a}) - \overline{W}_k \right) \right).
\]

By proposition \( \ref{prop:3.9} \) we deduce that \( L_c(f_{a}) \) is homeomorphic to a closed ball of \( \mathbb{C}^n \).

**3rd case.** \( b_N < c \).

In this case we have with the notations of lemma \( \ref{lem:3.10} \)

\[
(3.15) \quad L_c(f_{a}) = L_{b_N}(f_{a}) \cup \psi_{N+1}^{-1} \left( L_c(\tilde{f}_{N+1}) \right).
\]

and

\[
(3.16) \quad \partial L_c(f_{a}) \cap \partial \left( \psi_{N+1}^{-1} \left( L_c(\tilde{f}_{N+1}) \right) \right) = \partial V_{N+1} \approx S_{2n-1}(a, r_1).
\]

Since \( L_{b_N}(f_{a}) \) is homeomorphic to the ball and from lemma \( \ref{lem:3.10} \) \( \psi_{N+1}^{-1} \left( L_c(\tilde{f}_{N+1}) \right) \) is homeomorphic to an annulus, we deduce from \( \ref{prop:3.11} \) and \( \ref{lem:3.10} \) that \( L_c(f_{a}) \) is homeomorphic to a closed ball of \( \mathbb{C}^n \). The proof of proposition \( \ref{prop:3.11} \) is then complete. \( \square \)

### 3.3. A vanishing theorem.

#### 3.3.1. A regularization process.

In this subsection, we show how the properties of the atlas \( \mathfrak{A} \) enable us to introduce on the hypothenic complex manifold \( Y = (\mathbb{C}^n, \mathfrak{A}) \) an appropriate regularization process.

Indeed, let \( 1 \leq j \leq N + 1 \) and let \( \varepsilon > 0 \) small enough, and consider the open set

\[
V_j^\varepsilon := \left\{ z \in V_j, \quad d\left( \psi_j(z), \partial(\psi_j(V_j)) \right) > \varepsilon \right\}.
\]

Since the domain \( V_j \) is assumed to be a ball of \( \mathbb{C}^n \), then \( V_j^\varepsilon \) is also a ball of \( \mathbb{C}^n \). Fix \( \varepsilon > 0 \) so small that the union of open sets \( V_j^\varepsilon \) cover \( \mathbb{C}^n \).

We want now to associate to the atlas \( \mathfrak{A} \) a matrix regularizing operator. For this, let \( B_1 \) be the unit ball of \( \mathbb{C}^n \), and let \( \chi \in C_0^\infty(B_1) \) be a cutoff function with \( \chi(\zeta) \geq 0 \), and \( \int_{B_1} \chi(\zeta)d\zeta = 1 \). We note \( \chi_\varepsilon(z) = \chi(z\varepsilon) \).

Since the open sets \( \psi_j(V_j) \) are balls of \( \mathbb{C}^n \), then by translating them if necessarily and independently of one another, we can always assume that for all \( j, k \in \{1, \ldots, N + 1\} \), and for all \( (\zeta_j, \zeta_k) \in \psi_j(V_j) \times \psi_k(V_k) \), the following condition is fulfilled

\[
\zeta_j - \zeta_k \in B_\varepsilon \iff j = k \quad (B_\varepsilon \text{ is the ball of radius } \varepsilon).
\]

That is, in other words, one can translate in \( \mathbb{C}^n \) the domains of coordinates \( \psi_j(V_j) \) so far away, that for \( j \neq k \), the points of \( \psi_j(V_j) \) become fairly distant from the points of \( \psi_k(V_k) \) that \( \zeta_j - \zeta_k \notin B_\varepsilon \).
Let $L^1_{\text{loc}}(Y)$ be the space of locally integrable complex valued functions with respect to Lebesgue measure
\[ dz = \frac{1}{(2\pi)^n} \prod_{j=1}^n (d\xi_j \wedge dz_j), \]
and let the mapping $\hat{\psi}_j : Y \to \mathbb{C}^n$ given by:
\[ \hat{\psi}_j(z) := \begin{cases} \psi_j(z) & \text{if } z \in V_j \\ 0 & \text{if } z \notin V_j. \end{cases} \]
With these notations, we define then a linear $(N+1) \times (N+1)$ matrix operator
\[ \mathcal{R}_\mathbb{A} : L^1_{\text{loc}}(Y) \to (L^1_{\text{loc}}(Y))^{(N+1)^2} \]
by
\[ f \mapsto \mathcal{R}_\mathbb{A}[f] = \left( R_{j,k}[f] \right)_{1 \leq j,k \leq N+1} \]
by
\[ R_{j,k}[f](z) = \varepsilon^{2n} \int_{\eta \in V_{j,\varepsilon}} f(\xi) \cdot \chi_{V_j} \left( \hat{\psi}_k(z) - \hat{\psi}_j(\xi) \right) d\eta. \]

**Proposition 3.12.**
For all $f \in L^1_{\text{loc}}(Y)$, we have the following properties of $\mathcal{R}_\mathbb{A}[f]$:

1. The linear operator $\mathcal{R}_\mathbb{A}$ is diagonal, that is
   \[ R_{j,k}[f] = 0 \iff j \neq k \]
2. The linear operator $\mathcal{R}_\mathbb{A}$ is a regularizing operator, that is
   \[ R_{j,k}[f] \in C^\infty(Y). \]
3. For all $z \notin V_k^\varepsilon$, we have
   \[ R_{k,k}[f](z) = 0 \]
   which means that $\text{supp}(R_{k,k}[f]) \subseteq V_k^\varepsilon$.
4. By the following change of variables $\eta = \hat{\psi}_k(z) - \hat{\psi}_k(\xi)$, we obtain
   \[ R_{k,k}[f](z) = \varepsilon^{2n} \int_{\eta \in B_\varepsilon} \left( f \circ \psi_k^{-1} \right) \left( \hat{\psi}_k(z) - \eta \right) \cdot \chi_{V_k}(\eta) d\eta. \]
5. Let $\tilde{f}_k = f \circ \psi_k^{-1}$ denote the expression of the function $f$ in the local coordinates $\zeta = \psi_k(z)$, and let $\mathcal{R}_{k,k}[f] = \mathcal{R}_{k,k}[f \circ \psi_k^{-1}]$ denote the expression of the function $\mathcal{R}_{k,k}[f]$ in the same coordinates $\zeta = \psi_k(z)$,

\[ V_k \xrightarrow{f} \mathbb{C} \quad \xrightarrow{\psi_k} \hat{\psi}_k(V_k) \quad \psi_k(V_k) \xrightarrow{\psi_j} \mathbb{C} \quad \xrightarrow{\mathcal{R}_{k,k}[f]} \psi_j(V_j) \]

Then
\[ \mathcal{R}_{k,k}[f](\zeta) = \varepsilon^{2n} \int_{\eta \in B_\varepsilon} \tilde{f}_k(\zeta - \eta) \cdot \chi_{V_k}(\eta) d\eta. \]
Using (3.19), we can write $R_{k,k}$ in terms of convolution operation as follows

$$R_{k,k}[f] = \left( (f \circ \psi_k^{-1}) * \chi_{\varepsilon} \right) \circ \psi_k.$$  

Proof.
Property (1) follows from the fact that $\chi_{\varepsilon}$ is of class $C^{\infty}$.
Property (2) is a consequence of condition (3.17) and the fact that $\text{supp} \chi_{\varepsilon} \subset B_{\varepsilon}$.
Property (3) holds because to compute $R_{k,k}[f]$ we integrate on $\xi \in V_{\varepsilon}^k$ and if $z \not\in V_{\varepsilon}^k$, then $\chi_{\varepsilon} \left( \hat{\psi}_k(z) - \hat{\psi}_k(\xi) \right) = 0$ and therefore $R_{k,k}[f] = 0$.
Properties (4) and (5) follow from the definition of $R_{k,k}$ given in (3.18). □

Corollary 3.13. Fix a point $z_0 \in \partial V_{\varepsilon}^k$, and let $z \in V_{\varepsilon}^k$. Then for all multi-indexes $\alpha, \beta \in \mathbb{N}^n$ we obtain

$$\lim_{z \to z_0} \partial_\alpha \partial_\beta R_{k,k}[f](z) = 0.$$  

Proof. This is a consequence of properties (1) and (3) of the operator $R_{\mathfrak{A}}$. □

We can now state the main result of this section.

**Theorem 3.14.**

*The hypothetic complex analytic manifold $Y = (\mathbb{C}^n, \mathfrak{A})$ is strictly pseudoconvex.*

Before proving theorem 3.14 let us deduce the following corollary.

**Corollary 3.15. (Vanishing theorem.)**

Let $X$ be an $\mathfrak{A}$–submanifold of $\mathcal{S}_{2n}$, and let $H^{0,1}(X, \mathbb{C}^n)$ be its Dolbeault group of cohomology of type $(0,1)$ with coefficients in $\mathbb{C}^n$. Then

$$H^{0,1}(X, \mathbb{C}^n) = \{0\}.$$  

Proof. (of corollary)

The hypothetic complex manifold $Y = (\mathbb{C}^n, \mathfrak{A})$ is by construction biholomorphic to the $\mathfrak{A}$–submanifold $X$. The corollary follows from Cartan’s theorem B; see [12]. □

Proof. (of theorem 3.14)

Let the function $f_{\mathfrak{A}}$ defined by (3.11). Since $f_{\mathfrak{A}}$ is almost everywhere of class $C^{\infty}$ and then locally integrable, we can then apply to $f_{\mathfrak{A}}$ the regularizing operator $R_{\mathfrak{A}}$ defined by equation (3.18). For $\varepsilon > 0$ small enough, let then the function

$$\Psi : Y = (\mathbb{C}^n, \mathfrak{A}) \to [0, +\infty[$$  

defined by

$$\Psi(z) = \text{trace} \left( R_{\mathfrak{A}}[f] \right)(z).$$

Let $z \in V_{\varepsilon}^k$ and consider the following subset of $\{1, 2, ..., N\}$

$$\text{Ind}(z) := \left\{ k, \quad \text{with} \quad 1 \leq k \leq N, \quad z \in V_{\varepsilon}^k \right\}.$$
We can then write (3.21) as follows

\[
\Psi(z) = \text{trace}\left( R_{\mathfrak{A}} [f] \right)(z)
\]

\[
= \sum_{k=1}^{N+1} R_{k,k} [f](z)
\]

\[
= \sum_{k=1}^{N+1} \varepsilon^{2n} \int_{\xi \in V_{k}^{2e}} f(\xi) \cdot \chi_{e} \left( \hat{\psi}_{k}(z) - \hat{\psi}_{k}(\xi) \right) d\hat{\psi}_{k}(\xi)
\]

\[
= \varepsilon^{2n} \sum_{k \in \text{Ind}(z)} \int_{\xi \in V_{k}^{2e}} f(\xi) \cdot \chi_{e} \left( \hat{\psi}_{k}(z) - \hat{\psi}_{k}(\xi) \right) d\hat{\psi}_{k}.
\]

We claim that

1) \( \Psi \) is a \( C^{\infty} \) function on \( Y = (\mathbb{C}^{n}, \mathfrak{A}) \).

2) \( \Psi \) is strictly \( psh \) on \( Y = (\mathbb{C}^{n}, \mathfrak{A}) \).

3) \( \Psi \) is an exhaustion of \( Y = (\mathbb{C}^{n}, \mathfrak{A}) \), that is, for all \( c > 0 \), the sublevel set

\[
L_{c}(\Psi) := \{ z \in Y, \quad \Psi(z) < c \}
\]

is relatively compact.

Indeed.

1) Fix \( z_{0} \in Y = (\mathbb{C}^{n}, \mathfrak{A}) \). To check that \( \Psi \) is a \( C^{\infty} \) function at the point \( z_{0} \), we consider two situations:

(a) For all \( 1 \leq k \leq N+1 \), \( z_{0} \notin \partial V_{k}^{e} \), then there exists a open neighborhood \( W_{z_{0}} \subset \bigcap_{k \in \text{Ind}(z_{0})} V_{k} \) of \( z_{0} \) such that, for all \( z \in W_{z_{0}} \):

\[
\Psi(z) = \sum_{k \in \text{Ind}(z_{0})} R_{k,k} [f](z).
\]

In this case, \( \Psi \) is \( C^{\infty} \) on \( W_{z_{0}} \) by property (1) of \( R_{\mathfrak{A}} \), (see proposition 3.12).

(b) There exists \( 1 \leq j \leq N+1 \) such that \( z_{0} \in \partial V_{j}^{e} \). In this case, there exists a neighborhood \( W_{z_{0}} \) such that for all \( z \in W_{z_{0}} \)

\[
\Psi(z) = \begin{cases} 
\sum_{k \in \text{Ind}(z_{0})} R_{k,k} [f](z) & \text{if } z \in W_{z_{0}} - V_{j} \\
R_{j,j} [f](z) + \sum_{k \in \text{Ind}(z_{0})} R_{k,k} [f](z) & \text{if } z \in W_{z_{0}} \cap V_{j}.
\end{cases}
\]

Since we have by corollary 3.13 \( \lim_{z \to z_{0}} \partial_{z}^{\mathfrak{A}} \partial_{\bar{z}}^{\mathfrak{A}} R_{j,j} [f](z) = 0 \), and since

\[
\sum_{k \in \text{Ind}(z_{0})} R_{k,k} [f](z) \text{ is } C^{\infty} \text{ on } W_{z_{0}}, \text{ then } \Psi \text{ is } C^{\infty} \text{ on } W_{z_{0}}. \text{ By letting } z_{0}
\]

running over \( Y = (\mathbb{C}^{n}, \mathfrak{A}) \), we obtain then from (a) and (b), that \( \Psi \) is \( C^{\infty} \)

on \( Y = (\mathbb{C}^{n}, \mathfrak{A}) \).
2) Let \( z \in Y = (\mathbb{C}^n, \mathfrak{A}) \), and let
\[
\Psi(z) = \text{trace} \left( \mathcal{R}_A \left[ f \right] \right)(z) = \varepsilon^{2n} \sum_{k \in \text{Ind}(z)} \int_{\xi \in V_k^{2\varepsilon}} f(\xi) \cdot \chi_\varepsilon \left( \hat{\psi}_k(z) - \hat{\psi}_k(\xi) \right) d\hat{\psi}_k.
\]
Since the property of plurisubharmonicity is independent of the choice of local coordinates, it suffices then to check the plurisubharmonicity of each function
\[
\Psi_k(z) := \mathcal{R}_{k,k}^\varepsilon \left[ f \right](z) = \int_{\xi \in V_k^{2\varepsilon}} f(\xi) \cdot \chi_\varepsilon \left( \hat{\psi}_k(z) - \hat{\psi}_k(\xi) \right) d\hat{\psi}_k
\]
only in the chart \((V_k, \psi_k)\). Let then \( \tilde{f}_k \) and \( \tilde{\Psi}_k \) be the expressions of \( f \) and \( \Psi_k \) respectively in the local coordinates \( \zeta = \psi_k(z) \). We have by formula (3.19)
\[
\tilde{\Psi}_k(\zeta) = \varepsilon^{2n} \int_{\eta \in B_s} \tilde{f}_k(\zeta - \eta) \cdot \chi_\varepsilon \left( \eta \right) d\eta.
\]
By differentiating under the sign sum, we obtain
\[
\frac{\partial^2 \tilde{\Psi}_k(\zeta)}{\partial \zeta_j \partial \zeta_\ell}(\zeta) = \varepsilon^{2n} \int_{\eta \in B_s} \frac{\partial^2 \tilde{f}_k(\zeta - \eta)}{\partial \zeta_j \partial \zeta_\ell} \cdot \chi_\varepsilon \left( \eta \right) d\eta,
\]
which means that for all \( t = (t_1, ..., t_n) \in \mathbb{C}^n \)
\[
\mathcal{L}_t(\tilde{\Psi}_k)[t,t] = \varepsilon^{2n} \int_{\eta \in B_s} \mathcal{L}_{(\zeta - \eta)}(\tilde{f}_k)[t,t] \cdot \chi_\varepsilon \left( \eta \right) d\eta.
\]
Since by (5.11), the Levi form \( \mathcal{L}_{(\zeta - \eta)}(\tilde{f}_k)[t,t] \) is positive defined on \( \mathbb{K}_1 \), that is, almost everywhere positive defined, then by (3.22), the Levi form of \( \Psi_k \) is positive defined in \( V_k^{2\varepsilon} \). Hence the Levi form of \( \Psi \) is positive defined in the open neighborhood \( W_z \) of the point \( z \)
\[
W_z = \bigcap_{k \in \text{Ind}(z)} V_k^{2\varepsilon}.
\]
Therefore the function \( \Psi \) which is \( C^\infty \) is strictly \textbf{psh} on \( W_z \). By letting \( z \) running over \( Y = (\mathbb{C}^n, \mathfrak{A}) \), we obtain that the function \( \Psi \) is strictly \textbf{psh} on \( Y = (\mathbb{C}^n, \mathfrak{A}) \).

3) Let \( c > 0 \), and assume by contradiction that the sublevel set
\[
L_c(\Psi) := \left\{ z \in Y, \quad \Psi(z) < c \right\}
\]
is not relatively compact. This implies that \( L_c(\Psi) \) is not bounded. Since the domain \( V_{N+1} \) of the distinguished chart \( (V_{N+1}, \psi_{N+1}) \) is the only one which is non bounded, and since \( f_n \) is already an exhaustion of \( Y = (\mathbb{C}^n, \mathfrak{A}) \), and is continuous on \( V_{N+1} \), then there exists a sequence of points \( z_m \in V_{N+1} \) such that
\[
\lim_{m \to +\infty} \| z_m \| = +\infty
\]
and
\[
\lim_{m \to +\infty} \left\{ \min_{\xi \in B(z_m,c)} f_n(\xi) \right\} = +\infty
\]
and

\[ (3.24) \quad \text{For all } m \in \mathbb{N}, \quad \Psi(z_m) < c. \]

Taking into account the definition of \( \Psi \), we obtain from (3.24) by an elementary calculus

\[
c \geq \Psi(z_m) \\
\geq R_{N+1,N+1} [f_a](z_m) \\
\geq \varepsilon^{2n} \int_{\eta \in B_\varepsilon} \left( f_a \circ \psi_{N+1}^{-1} \right) \left( \psi_{N+1}(z_m) - \eta \right) \cdot \chi_{\varepsilon}(\eta) \, d\eta \\
\geq \min_{\xi \in B(z_m, \varepsilon)} f_a(\xi) \cdot \left( \varepsilon^{2n} \int_{\eta \in B_\varepsilon} \chi_{\varepsilon}(\eta) \, d\eta \right) \\
\geq \min_{\xi \in B(z_m, \varepsilon)} f_a(\xi),
\]

which means that for all \( m \in \mathbb{N} \)

\[
c \geq \min_{\xi \in B(z_m, \varepsilon)} f_a(\xi).
\]

But this contradicts (3.23). The hypothesis that the sublevel set \( L_c(\Psi) \) is not relatively compact, is then false.

The previous properties of \( \Psi \) mean that \( \Psi \) is an exhaustive \( C^\infty \) strictly \text{psh} function on \((\mathbb{C}^n, \mathfrak{A})\). The hypothetic complex manifold \( Y = (\mathbb{C}^n, \mathfrak{A}) \) is then strictly pseudoconvex. The proof of theorem 3.14 is then complete. \( \square \)

4. Proof of the main theorem

We are now in a position to prove the main theorem 1.6. We have already observed in remark 1.7 that \( \{Hath\}_1 \iff \{Necs\}_1 \) holds, and then we have to prove theorem 1.6 only for \( n \geq 2 \). We begin first by proving the implication

\[
\{Hath\}_n \implies \{Necs\}_n
\]

then we prove the converse

\[
\{Necs\}_n \implies \{Hath\}_n.
\]

4.1. \( \{Hath\}_n \implies \{Necs\}_n \)

\[ \text{Proof.} \]

For \( n \geq 2 \), we prove \( \{Hath\}_n \implies \{Necs\}_n \) by contradiction, that is, by showing that if \( \{Necs\}_n \) is false, which means that the sphere \( S_{2n} \) admits a complex analytic structure, then \( \{Hath\}_n \) implies that the homeomorphisms defining the charts of this structure are constant, which contradicts the fact that the change of charts is biholomorphic. The proof will be done in three steps.

1st step. Assume that proposition \( \{Necs\}_n \) is false, and let the hypothetic complex atlas

\[
\mathcal{A} = \left\{ (U_j, \varphi_j), \quad j \in \mathcal{J} \right\}
\]
of \( S_{2n} \) as considered in subsubsection 1.3.1. In this first step, we want to construct (without changing the hypothetic complex structure of \( S_{2n} \)) an appropriate complex atlas \( \mathcal{F} \) of \( S_{2n} \) finer than \( \mathcal{A} \)

\[
\mathcal{F} = \left\{ (W_\gamma, \Phi_\gamma), \quad \gamma \in \mathcal{L} \right\}
\]

and to attach to each pairing of different charts \( \left( (W_\alpha, \Phi_\alpha), (W_\beta, \Phi_\beta) \right) \in \mathcal{F}^2 \) satisfying the condition

\[
W_\alpha \cap W_\beta \neq \emptyset
\]

an \( \mathcal{A} \)-submanifold \( X_{\alpha, \beta} \) of \( S_{2n} \). The new atlas \( \mathcal{F} \) will be constructed as follows: First, recover each ball \( \varphi_j(U_j) \subset \mathbb{C}^n \) by all smaller balls \( B_{(a,r)}^j \) which are centered at points \( a \in \varphi_j(U_j) \) and which have radii \( r > 0 \), such that \( B_{(a,r)}^j \subset \subset \varphi_j(U_j) \).

Every ball \( \varphi_j(U_j) \) becomes then union of smaller balls \( B_{(a,r)}^j \)

\[
\varphi_j(U_j) = \bigcup_{(a,r)} B_{(a,r)}^j.
\]

Observe that the new family of balls \( B_{(a,r)}^j \) is indexed by the following set

\[
\mathcal{L} := \left\{ (j,a,r), \quad \text{such that } j \in \mathcal{J}, \quad a \in \varphi_j(U_j), \quad r > 0, \quad \text{with } B_{(a,r)}^j \subset \subset \varphi_j(U_j) \right\}
\]

and that by the condition \( B_{(a,r)}^j \subset \subset \varphi_j(U_j) \), the set \( \mathcal{L} \) is not a cartesian product.

We define then the new complex charts \( \left( W_{(j,a,r)}, \Phi_{(j,a,r)} \right) \) of the sphere \( S_{2n} \), by setting for every ball \( B_{(a,r)}^j \subset \subset \varphi_j(U_j) \):

\[
\begin{align*}
W_{(j,a,r)} &= \varphi_j^{-1}(B_{(a,r)}^j) \\
\Phi_{(j,a,r)} &= \varphi_j|W_{(j,a,r)}.
\end{align*}
\]

Hence, we obtain a new complex atlas \( \mathcal{F} \) of \( S_{2n} \) finer than \( \mathcal{A} \)

\[
\mathcal{F} := \left\{ (W_{(j,a,r)}, \Phi_{(j,a,r)}), \quad (j,a,r) \in \mathcal{L} \right\}.
\]

Observe that by construction of the charts \( (W_{(j,a,r)}, \Phi_{(j,a,r)}) \), we have \( U_j = \bigcup_{(a,r)} W_{(j,a,r)} \).

**Notations.** To simplify the notations, we note every triplet \( (j,a,r) \in \mathcal{L} \) by one Greek character, which enables us to write the new atlas \( \mathcal{F} \) simply as:

\[
\mathcal{F} = \left\{ (W_\gamma, \Phi_\gamma), \quad \gamma \in \mathcal{L} \right\}.
\]

Now fix a pairing of charts \( \left( (W_\alpha, \Phi_\alpha), (W_\beta, \Phi_\beta) \right) \) satisfying condition (4.1), that is

\[
W_\alpha \cap W_\beta \neq \emptyset
\]

and collect all other charts \( (W_\gamma, \Phi_\gamma) \in \mathcal{F} \) whose domain \( W_\gamma \) does not intersect both \( W_\alpha \) and \( W_\beta \), that is, such that

\[
W_\gamma \cap W_\alpha \cap W_\beta = \emptyset.
\]
We obtain then a sub-atlas $F_{\alpha,\beta} \subset F$ defined by

$$F_{\alpha,\beta} := \left\{(W_{\alpha}, \Phi_{\alpha}), (W_{\beta}, \Phi_{\beta}) \right\} \bigcup_{\gamma \neq \alpha, \beta} \left\{(W_{\gamma}, \Phi_{\gamma}) \in F, \text{ with } W_{\gamma} \text{ satisfying (1.2)} \right\}.$$ 

Let

$$X_{\alpha,\beta} := \bigcup_{(W_{\gamma}, \Phi_{\gamma}) \in F_{\alpha,\beta}} W_{\gamma}.$$ 

The open set $X_{\alpha,\beta}$ is then an open complex submanifold of $S_{2n}$ whose complex structure is defined by the subatlas $F_{\alpha,\beta}$, and $X_{\alpha,\beta}$ is covered by the following family of open sets

$$\Psi_{\alpha,\beta} := \left\{W_{\gamma}, \text{ such that } (W_{\gamma}, \Phi_{\gamma}) \in F_{\alpha,\beta} \right\}.$$ 

By condition (1.2), the complementary of $X_{\alpha,\beta}$ with respect to $S_{2n}$ is

$$S_{2n} - X_{\alpha,\beta} = \partial W_{\alpha} \cap \partial W_{\beta}$$

and then

$$X_{\alpha,\beta} = S_{2n} - \mathcal{K}$$

where the compact $\mathcal{K}$ is given by

$$\mathcal{K} = \partial W_{\alpha} \cap \partial W_{\beta} \approx S_{2n-2}.$$ 

By writing $\alpha = (j, a, r) \in L$ and $\beta = (k, b, s) \in L$, we observe that

$$\mathcal{K} \subset U_j \cap U_k \subset U_j$$

and that

$$\varphi_j(\mathcal{K}) = S_{2n-2}(b, r).$$

Equations (4.5), (4.3), (4.7) imply that the open complex manifold $X_{\alpha,\beta}$ is an $A$–submanifold of $S_{2n}$ in the sense of definition 1.10.

2nd–step. Our goal in this step, is to construct two holomorphic mappings

$$F : X_{\alpha,\beta} - W_{\alpha} \longrightarrow \mathbb{C}^n$$

and

$$G : X_{\alpha,\beta} - W_{\beta} \longrightarrow \mathbb{C}^n$$

such that

$$G / (W_{\alpha} \cap W_{\beta}) - F / (W_{\alpha} \cap W_{\beta}) = \Phi_{\beta} - \Phi_{\alpha} \quad \text{on } W_{\alpha} \cap W_{\beta}.$$ 

Indeed, let $\mathcal{O}^n$ be the sheaf of holomorphic mappings on $S_{2n}$ with values in $\mathbb{C}^n$, and let $\mathcal{O}^n_{\alpha,\beta}$ denote the restriction of the sheaf $\mathcal{O}^n$ to the $A$–submanifold $X_{\alpha,\beta}$, that is

$$\mathcal{O}^n_{\alpha,\beta} := \mathcal{O}^n / X_{\alpha,\beta}.$$ 

Consider with respect to the open covering $\Psi_{\alpha,\beta}$ of $X_{\alpha,\beta}$, the following groups of cochains with values in the sheaf $\mathcal{O}^n_{\alpha,\beta}$:

$$C^0 \left( \Psi_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) = \prod_{\gamma \in \mathcal{L}} \mathcal{O}^n_{\alpha,\beta} \left( W_{\gamma} \right)$$

and

$$C^1 \left( \Psi_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) = \prod_{(\gamma, \mu) \in \mathcal{L}^2} \mathcal{O}^n_{\alpha,\beta} \left( W_{\gamma} \cap W_{\mu} \right).$$
HARTOGS’ PHENOMENON AND THE PROBLEM OF COMPLEX SPHERES 35

\[ C^2\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) = \prod_{(\gamma, \mu, \nu) \in \mathcal{L}^3} \mathcal{O}^n_{\alpha,\beta}(W_\gamma \cap W_\mu \cap W_\nu) \]

and let the coboundary operators:

\[ \delta: C^0\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) \rightarrow C^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) \]

and

\[ \delta: C^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) \rightarrow C^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) \]

defined by:

1) For the 0-cochains \((f_\gamma)_{(\gamma) \in \mathcal{L}} \in C^0\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right)\)

\[ \delta\left( (f_\gamma)_{(\gamma) \in \mathcal{L}} \right) = (g_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2} \]

where

\[ g_{\gamma,\mu} = f_\mu - f_\gamma \quad \text{on } W_\gamma \cap W_\mu. \]

2) For the 1-cochains \((f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2} \in C^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right)\)

\[ \delta\left( (f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2} \right) = (g_{\gamma,\mu,\nu})_{(\gamma,\mu,\nu) \in \mathcal{L}^3} \]

where

\[ g_{\gamma,\mu,\nu} = f_{\mu,\nu} - f_{\gamma,\nu} + f_{\gamma,\mu} \quad \text{on } W_\gamma \cap W_\mu \cap W_\nu. \]

Since the boundary operators are trivially group homomorphisms, we obtain then the group of 1-cycles

\[ Z^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) := \ker C^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) \]

and the group of 1-coboundaries

\[ B^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) := \text{Im } C^0\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) \]

and the cohomology group with coefficients in the sheaf \(\mathcal{O}^n_{\alpha,\beta}\), with respect to the covering \(\mathcal{V}_{\alpha,\beta}\) is

\[ H^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) := Z^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right) / B^1\left( \mathcal{V}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta} \right). \]

Consider now on the \(A\)–submanifold \(X_{\alpha,\beta}\) the following 1–cochain \((f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2}\) with \(f_{\gamma,\mu} \in \mathcal{O}^n_{\alpha,\beta}(W_\gamma \cap W_\mu)\) defined by:

\[ \begin{cases} 
  f_{\alpha,\beta} = \Phi_\beta - \Phi_\alpha \\
  f_{\gamma,\mu} = 0 
\end{cases} \quad \text{for all } (\gamma, \mu) \neq (\alpha, \beta). \]  

\text{(4.8)}

Since the homeomorphism \(\Phi_\alpha\) and \(\Phi_\beta\) are holomorphic by the hypothetic complex structure of the sphere \(\mathcal{S}_{2n}\), then the 1–cochain \((f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2}\) takes its values in the sheaf \(\mathcal{O}^n_{\alpha,\beta}\), and is therefore well defined.

We claim that the 1-chain \((f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2}\) is a 1-coboundary. To prove this, let’s first check that \((f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2}\) is a 1-cocycle. Indeed, since by condition \text{(4.2)}, the
covering $\mathfrak{M}_{\alpha,\beta}$ does not contain any domain $W_\gamma$ which intersects both $W_\alpha$ and $W_\beta$, then for every triplet of domains $W_\gamma, W_\mu, W_\nu$ of $X_{\alpha,\beta}$, satisfying
\[ W_\gamma \cap W_\mu \cap W_\nu \neq \emptyset \]
we have trivially
\[ \delta \left( f_{\gamma,\mu} \right)_{(\gamma,\mu) \in \mathcal{L}^2} = 0 \]
which means that the 1-chain $(f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2}$ is a 1-cocycle.

Furthermore, we know by the vanishing theorem 3.15 that:
\[ H^{0,1}(X_{\alpha,\beta}, \mathbb{C}^n) = \{0\}, \]
which implies by Dolbeault’s resolution of the sheaf $\mathcal{O}^n_{\alpha,\beta}$, that
\[ H^1\left(X_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta}\right) = \{0\}, \]
and then according to Leray’s theorem 5.3 (see index), the cohomology group with respect to the covering $\mathfrak{M}_{\alpha,\beta}$ is trivial, that is
\[ H^1\left(\mathfrak{M}_{\alpha,\beta}, \mathcal{O}^n_{\alpha,\beta}\right) = \{0\}. \]
Thus the 1-cochain $(f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2}$ is a 1-boundary.

This means that there exists over $X_{\alpha,\beta}$ a 0-cochain $(g_\gamma)_{\gamma \in \mathcal{L}}$ with values in the sheaf $\mathcal{O}^n_{\alpha,\beta}$, such that
\[ \delta \left( (g_\gamma)_{\gamma \in \mathcal{L}} \right) = (f_{\gamma,\mu})_{(\gamma,\mu) \in \mathcal{L}^2} \]
that is
\[ g_\mu - g_\nu = f_{\mu,\nu} \quad \text{on} \quad W_\mu \cap W_\nu. \]

Observe from the definition of the 1-chain $(f_{\mu,\nu})_{(\mu,\nu) \in \mathcal{L}^2}$, that the restrictions of the 0-chain $(g_\gamma)_{\gamma \in \mathcal{L}}$ to the open sets $X_{\alpha,\beta} - \overline{W_\alpha}$ and $X_{\alpha,\beta} - \overline{W_\beta}$, satisfy the following compatibility conditions:

1. On the open set $X_{\alpha,\beta} - \overline{W_\alpha}$, we have
\[ g_\mu - g_\nu = 0 \quad \text{on} \quad W_\mu \cap W_\nu. \]
2. On the open set $X_{\alpha,\beta} - \overline{W_\beta}$, we have
\[ g_\mu - g_\nu = 0 \quad \text{on} \quad W_\mu \cap W_\nu. \]

Then there exist two holomorphic mappings with values in $\mathbb{C}^n$:
\[ F : X_{\alpha,\beta} - \overline{W_\alpha} \rightarrow \mathbb{C}^n \]
and
\[ G : X_{\alpha,\beta} - \overline{W_\beta} \rightarrow \mathbb{C}^n \]
such that
\[ F/W_\mu = g_\mu \quad \text{on} \quad W_\mu \]
and
\[ G/W_\nu = g_\nu \quad \text{on} \quad W_\nu. \]
that is, for $\mu = \alpha$ and $\nu = \beta$
\[
F/ (W_\alpha \cap W_\beta) - G/ (W_\alpha \cap W_\beta) = g_\alpha/ (W_\alpha \cap W_\beta) - g_\beta/ (W_\alpha \cap W_\beta)
\]
(4.9)
\[
= f_{\alpha,\beta}/ (W_\alpha \cap W_\beta)
\]
\[
= \Phi_{\alpha} (W_\alpha \cap W_\beta) - \Phi_{\beta} (W_\alpha \cap W_\beta).
\]

3rd step. In this third step, we are going to deduce that for $n \geq 2$, \{Hath\}_n implies that the homeomorphisms $\Phi_{\alpha}$ defining the charts of the finer atlas $F$ are constant. Indeed, assume that for $n \geq 2$, proposition \{Hath\}_n is true, and let the complex chart $(U_{ij}, \Phi_{ij})$ of the hypothetic atlas $A$, (recall that $U_{ij}$ is assumed to be homeomorphic to a ball of $\mathbb{C}^n$). Observe that the open set $(X_{\alpha,\beta} - W_{\alpha}) \cap U_i$ is connected. Since the restricted mapping
\[
F : (X_{\alpha,\beta} - W_{\alpha}) \cap U_j \rightarrow \mathbb{C}^n
\]
is holomorphic, it follows then by lemma 1.8, that $F/ (X_{\alpha,\beta} - W_{\alpha}) \cap U_j$ admits an analytic continuation to the whole domain $U_j$. Hence the holomorphic mapping
\[
F : X_{\alpha,\beta} - W_{\alpha} \rightarrow \mathbb{C}^n
\]
 admits an analytic continuation to the whole sphere $\mathcal{S}_{2n}$ (still denoted $F$)
\[
F : \mathcal{S}_{2n} \rightarrow \mathbb{C}^n.
\]
But, since $\mathcal{S}_{2n}$ is compact, then $F$ must be constant on the sphere $\mathcal{S}_{2n}$. By a similar argument, we show that the holomorphic mapping
\[
G : X_{\alpha,\beta} - W_{\beta} \rightarrow \mathbb{C}^n
\]
 admits an analytic continuation to the whole sphere $\mathcal{S}_{2n}$, and then becomes also constant on $\mathcal{S}_{2n}$.
Taking into account the fact that $F$ and $G$ are both constant mappings on $\mathcal{S}_{2n}$, we obtain then by differentiating (4.9) on $(W_\alpha \cap W_\beta)$
(4.10)
\[
\partial \Phi_{\alpha} = \partial \Phi_{\beta} \quad \text{on} \quad W_\alpha \cap W_\beta.
\]
Recall that until now, the indexes $\alpha$ and $\beta \in L$ are assumed to be fixed, but when we let $\alpha$ and $\beta$ running over the index set $L$, the identity (4.10) becomes then valid for all $\alpha,\beta \in L$, that is
(4.11)
\[
\partial \Phi_{\alpha} = \partial \Phi_{\beta} \quad \text{n} \quad W_\alpha \cap W_\beta.
\]
Hence, there exists by the compatibility conditions (4.11), a global 1-differential vectorial form $\omega$ of type $(1,0)$ on the whole sphere $\mathcal{S}_{2n}$, with holomorphic coefficients, such that
(4.12)
\[
\omega/ W_{\gamma} = \partial \Phi_{\gamma} \quad \text{for all} \quad \gamma \in L.
\]
By differentiating (4.12) we obtain trivially
\[
\partial \omega = 0 \quad \text{on} \quad \mathcal{S}_{2n}.
\]
Since by the vanishing theorem 3.15, the cohomology group $\mathcal{H}^{1,0} (X_{\alpha,\beta}, \mathbb{C}^n)$ is also trivial
\[
\mathcal{H}^{1,0} (X_{\alpha,\beta}, \mathbb{C}^n) = \{0\}
\]
there exists then a $\mathcal{C}^\infty$ mapping $H : X_{\alpha,\beta} \rightarrow \mathbb{C}^n$ such that
(4.13)
\[
\partial H = \omega.
Observing that $\omega$ has holomorphic coefficients, and that $\partial H = \omega$, we deduce that $H : X_{\alpha, \beta} \rightarrow \mathbb{C}^n$ is a harmonic mapping and that $H$ can be written in the form

$$H = H_1 + \overline{H_2}$$

where both mappings $H_1, H_2 : X_{\alpha, \beta} \rightarrow \mathbb{C}^n$ are both holomorphic on $X_{\alpha, \beta}$, and where $\overline{H_2}$ denotes the conjugate of $H_2$.

Using once again the same arguments as for the holomorphic mappings $F$ and $G$ above, one can prove that both holomorphic mappings $H_1$ and $H_2$ which are already holomorphic on $X_{\alpha, \beta}$, admit analytic continuations to the whole sphere $S_{2n}$. Both mappings $H_1$ and $H_2$ become then constant on $S_{2n}$. According to equation (4.13) we obtain $\omega = 0$, and thanks to equation (4.12), we can finally conclude that:

for all $\gamma \in \mathcal{L}$, $\Phi_\gamma$ is constant on $W_\gamma$.

But this last conclusion contradicts the fact that $(W_\gamma, \Phi_\gamma)$ is a chart of a complex structure on the sphere $S_{2n}$. Then the assumption that $\{\text{Necs}\}_n$ is false for $n \geq 2$, has lead to a contradiction. The implication $\{\text{Hath}\}_n \Rightarrow \{\text{Necs}\}_n$ is then true for all $n \geq 2$. □

4.2. $\{\text{Necs}\}_n \Rightarrow \{\text{Hath}\}_n$.

to prove the implication $\{\text{Necs}\}_n \Rightarrow \{\text{Hath}\}_n$, we need first to prove the following lemma.

**Lemma 4.1.**

Let the unit ball $B_1$ of $\mathbb{C}^n$, and let $K$ be a compact subset of $B_1$ such that $B_1 - K$ is connected. Assume that $\{\text{Hath}\}_n$ is false, that is, there exists a holomorphic function $f : B_1 - K \rightarrow \mathbb{C}$ which does not admit an analytic continuation to the ball $B_1$. Then there exist two balls $B_{r_1}$ and $B_{r_2}$ with

$$K \subset B_{r_1} \subset B_{r_2} \subset B_1$$

and there exist two open convex sets $C_1$ and $C_2$ with

$$K \subset C_1 \subset C_2 \subset B_1$$

and there exists a bi-holomorphic mapping

$$F : C_2 - C_1 \rightarrow B_{r_2} - B_{r_1}$$

which does not admit an analytic continuation to the open convex set $C_2$.

Proof.

For $0 < r < r' < 1$, Let $B_r$ and $B_{r'}$ be two balls of $\mathbb{C}^n$ such that

$$K \subset B_r \subset B_{r'} \subset B_1$$

and observe that the annulus $\mathcal{K}_{r, r'} := B_{r'} - B_r$ is a compact subset of $B_1 - K$.

Let the holomorphic function $f : B_1 - K \rightarrow \mathbb{C}$ satisfying the hypothesis of the lemma, and for a positive number $\lambda > 0$, let the mapping

$$F_\lambda : B_{r'} - B_r \rightarrow F_\lambda(B_{r'} - B_r) \subset \mathbb{C}^n$$

defined for $z = (z_1, \ldots, z_n) \in B_{r'} - B_r$ by:

$$F_\lambda(z) = \left( z_1 + \frac{f(z_1, z_2, \ldots, z_n)}{\lambda}, z_2, \ldots, z_n \right).$$
It is clear that $F_\lambda$ is holomorphic in $B_{r'} - B_r$ and doesn’t admit an analytic continuation to $B_{r'}$, and that $f$ and its derivatives $\frac{\partial f}{\partial z_j}$ and $\frac{\partial^2 f}{\partial z_j \partial z_k}$ are bounded on the compact annulus $K_{r,r'}$. Let:

\[ \theta = \sup_{z \in K_{r,r'}} |f(z)| \]

\[ \theta_j = \sup_{z \in K_{r,r'}} \left| \frac{\partial f}{\partial z_j}(z) \right| \]

\[ \theta_{j,k} = \sup_{z \in K_{r,r'}} \left| \frac{\partial^2 f}{\partial z_j \partial z_k}(z) \right| \]

and

\[ \bar{\theta} = \max_{1 \leq j,k \leq n} \{ \theta, \theta_j, \theta_{j,k} \} . \]

The $n \times n$-matrix

\[
A(z) = \begin{pmatrix}
\frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2} & \cdots & \frac{\partial f}{\partial z_n} \\
0 & 0 & \ddots & 0 \\
0 & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

is then bounded on the compact annulus $K_{r,r'}$

\[ \|A(z)\| \leq \bar{\theta} . \]

Now let the jacobian matrix of $F_\lambda(z)$

\[ \text{Jac}(F_\lambda)(z) = I_n + A(z) . \]

If we choose $\lambda > n \bar{\theta}$, the jacobian $\text{Jac}(F_\lambda)(z)$ is then invertible, and therefore the mapping

\[ F_\lambda : B_{r'} - B_r \rightarrow F_\lambda (B_{r'} - B_r) \subset \mathbb{C}^n \]

is bi-holomorphic.

From the definition (4.14) of $F_\lambda$, we have for all $z \in B_{r'} - B_r$

\[ \lim_{\lambda \rightarrow +\infty} F_\lambda(z) = z . \]

This means that for all $r_1, r_2 > 0$, with $r < r_1 < r_2 < r'$, there exists a positive number $\lambda_0 > 0$, such that

(4.15) \[ \lambda > \lambda_0 \Rightarrow B_{r_2} - B_{r_1} \subset F_\lambda (B_1 - K) . \]

It follows from (4.15), that for $\lambda > \lambda_0$, there exist two open subsets $C_1$ and $C_2$, of $B_1$ with

\[ K \subset C_1 \subset C_2 \subset B_1 \]

such that:

\[ \partial C_1 := \left\{ z \in \mathbb{C}^n, \; \|F_\lambda(z)\|^2 = r_1 \right\} \]

\[ \partial C_2 := \left\{ z \in \mathbb{C}^n, \; \|F_\lambda(z)\|^2 = r_2 \right\} \]

and

\[ F_\lambda (C_2 - C_1)) = B_{r_2} - B_{r_1} . \]
We claim that both open sets $C_1$ and $C_2$ are convex. Indeed, consider the function $\rho_\lambda : B_1 - K \rightarrow \mathbb{R}_+$ defined by

$$\rho_\lambda(z) := \|F_\lambda(z)\|^2.$$  

By differentiating of (4.16), we obtain at a point $z$ near $\partial C_j$, (with $j = 1$ or $j = 2$)

$$(4.17) \quad Hess(\rho_\lambda(z)) = I_{2n} + R(z) \quad z \in \partial C_j$$

where $R(z)$ is a $2n \times 2n$ real matrix involving the first and second real derivatives of the holomorphic function $f$. Since the derivatives of $f$ are bounded on the compact annulus $B_r' - B_r$ by $\tilde{\theta}$, then by choosing $\lambda > \max\{\lambda_0, 2n\tilde{\theta}\}$, we obtain $\|R(z)\| < 1$.

From (4.17), the hessian matrix $Hess(\rho_\lambda(z))$ is then positive defined, which means that the open sets $C_1$ and $C_2$ are convex. The proof of lemma 4.1 is then complete. \qed

We are now ready to prove the implication $\{Necs\}_n \Rightarrow \{Hath\}_n$.

Proof.

Let

$$\{e_1 + ie_2, ..., e_{2n-1} + ie_{2n}, e_{2n+1}\}$$

be an orthonormal basis of $\mathbb{C}^n \times \mathbb{R}$. Without loss of generality, we can assume that $\mathcal{S}_{2n}$ is the unit sphere embedded in $\mathbb{C}^n \times \mathbb{R}$, $\mathcal{S}_{2n} \hookrightarrow \mathbb{C}^n \times \mathbb{R}$ with equation $|z_1|^2 + ... + |z_n|^2 + s^2 = 1$.

Let $N = (0, ..., 0, 1)$ and $S = (0, ..., 0, -1)$ be respectively the north and the south poles of $\mathcal{S}_{2n}$, and let the half spheres

$$W_+ := \{(z_1, ..., z_n, s) \in \mathcal{S}_{2n}, \quad 0 < s \leq 1\}$$

and

$$W_- := \{(z_1, ..., z_n, s) \in \mathcal{S}_{2n}, \quad -1 \leq s < 0\}$$

and for $0 < \sigma < 1$, let the following open set of $\mathcal{S}_{2n}$

$$W_\sigma := \{(z_1, ..., z_n, s) \in \mathcal{S}_{2n}, \quad -\sigma < s < 0\}.$$

Recall that the stereographic projection of pole $N$, is the homeomorphism

$$pr_N : \mathcal{S}_{2n} - \{N\} \rightarrow \mathbb{C}^n$$

which sends every point $P \in \mathcal{S}_{2n} - \{N\}$ to the point $pr_N(P) \in \mathbb{C}^n$, which is the intersection of the line $(N, P)$ with the hyperplane $s = 0$. The stereographic projection of a point $P = (z_1, ..., z_n, s) \in \mathcal{S}_{2n} - \{N\}$ is given by

$$pr_N(P) = \frac{1}{1 - s}\left\{P - s. e_{2n+1}\right\}.$$  

Now assume by contradiction that proposition $\{Hath\}_n$ is false. We claim that the sphere $\mathcal{S}_{2n}$ admits a complex atlas $\mathcal{A}$ formed of exactly two complex charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$. Indeed, Since

$$pr_N(W_-) = B_1$$
we can define the first chart \((U_1, \varphi_1) \in A\), by setting
\[
U_1 := W_- = \left\{ (z_1, ..., z_n, \zeta) \in S_{2n}, \quad -1 \leq \zeta < 0 \right\}
\]
and by choosing the homeomorphism \(\varphi_1\) to be the stereographic projection of pole \(N\), restricted to \(W_-\), that is
\[
\varphi_1 := pr_N : W_- \rightarrow B_1.
\]
Observe that to define the first chart, we haven’t need to use the assumption that \(\{Hath\}_n\) is false, but for the construction of the second chart \((U_2, \varphi_2)\), this assumption will play a crucial role. Indeed, by the assumption that \(\{Hath\}_n\) is false, there exist according to lemma 4.1 a compact subset \(K \subset B_1\) such that \(B_1 - K\) is connected, and there exist two convex sets \(C_1\) and \(C_2\) and two balls \(B_{r_1}\) and \(B_{r_2}\) satisfying the conditions:
\[
K \subset C_1 \subseteq C_2 \subset B_1
\]
and
\[
K \subset B_{r_1} \subseteq B_{r_2} \subset B_1
\]
and there exists a bi-holomorphic mapping
\[
F : C_2 - C_1 \longrightarrow B_{r_2} - B_{r_1} \subset \mathbb{C}^n
\]
which does not admit an analytic continuation to \(C_2\).

**Remark 4.2.** by replacing if necessarily, the mapping \(F\) by \(\frac{1}{r_2} F\), we can assume that \(r_2 = 1\), which means that the bi-homomorphic mapping \(F\) can be defined as follows
\[
F : C_2 - C_1 \longrightarrow B_1 - B_{r_1}.
\]
Now choose the domain of the second chart to be \(U_2 := W_+ \bigcup W_\sigma\), and observe that
\[
U_1 \bigcap U_2 = W_\sigma.
\]
The homeomorphism of the second chart will be defined by a mapping
\[
\varphi_2 : U_2 = W_+ \bigcup W_\sigma \longrightarrow C_2 - C_1
\]
where the restrictions
\[
(4.18) \quad \varphi_2/W_\sigma : W_\sigma \longrightarrow C_2 - \overline{C_1}
\]
and
\[
(4.19) \quad \varphi_2/W_+ : W_+ \longrightarrow \overline{C_1}
\]
are homeomorphisms.
To define the restriction \(\varphi_2/W_\sigma\), that is \(4.18\), observe first that if we choose
\[
\sigma = \frac{1 - r_1}{r_1}
\]
then an elementary calculation, shows that
\[
\varphi_1(W_\sigma) = pr_N (W_\sigma) = B_1 - B_{r_1}.
\]
Thus, for \(P \in W_\sigma\), we set:
\[
(4.20) \quad \varphi_2(P) := F^{-1} (\varphi_1(P)) \in C_2 - \overline{C_1}.
\]
To define the second restriction \( \varphi_2 / W_+ \), that is (4.19), we need to use the fact that \( C_1 \) is convex, and that the half sphere \( W_+ \) is geodesically convex.

Indeed,

(1) Since \( W_+ \) is geodesically convex, there exists a mapping
\[
\Gamma : \left[ 0, \frac{\pi}{2} \right] \times \partial W_+ \rightarrow W_+
\]
\[
(t, \xi) \mapsto \Gamma(t, \xi)
\]
where \( \Gamma(t, \xi) \) is the geodesic curve starting from the north pole \( N \) at \( t = 0 \) and arriving to the point \( \xi \in \partial W_+ \) at \( t = \frac{\pi}{2} \); that is \( \Gamma(0, \xi) = N \), and \( \Gamma(\frac{\pi}{2}, \xi) = \xi \in \partial W_+ \), and then for every point \( P \in U_+ \), there exists a unique pairing
\[
(t, \xi) \in \left[ 0, \frac{\pi}{2} \right] \times \partial W_+
\]
such that the point \( P \) can be written
\[
P = \Gamma(t, \xi).
\]
The parameter \( t \in \left[ 0, \frac{\pi}{2} \right) \) represents the geodesic distance from \( P \) to \( N \).

(2) Fix a point \( M \in C_1 \). By convexity of \( C_1 \), every point \( \eta \in \partial C_1 \) can be joined to the fixed point \( M \) by a linear segment \( [M, \eta] \), and then we can define the mapping
\[
\Lambda : \left[ 0, \frac{\pi}{2} \right] \times \partial C_1 \rightarrow C_1
\]
\[
(t, \eta) \mapsto \Lambda(t, \eta) := \frac{2}{\pi} \left( \frac{\pi}{2} - t \right) M + t \eta
\]
\( \Lambda(t, \eta) \) is the linear curve starting from the fixed point \( M \) at \( t = 0 \), and arriving to the point \( \eta \in \partial C_1 \) at \( t = \frac{\pi}{2} \). For every point \( Q \in C_1 \), there exists then a unique pairing
\[
(t, \eta) \in \left[ 0, \frac{\pi}{2} \right] \times \partial C_1
\]
such that the point \( Q \) can be written
\[
Q = \Lambda(t, \xi).
\]
The parameter \( t \in \left[ 0, \frac{\pi}{2} \right) \) represents the Euclidean distance from \( Q \) to the fixed point \( M \).

The observations (1) and (2) enable us to define the second restriction \( \varphi_2 / W_+ \) for every point \( P = \Gamma(t, \xi) \in U_+ \), by:

\[
(\varphi_2 / W_+) (P) = (\varphi_2 / W_+) (\Gamma(t, \xi)) = \Lambda(t, F^{-1}(\varphi_1(\xi))) \in \overline{C_1}.
\]
In other words, \( \varphi_2 / W_+ \) sends the extremity \( \xi \in \partial W_+ \) to the extremity \( \eta = F^{-1}(\varphi_1(\xi)) \in \partial C_1 \) and sends the point \( P = \Gamma(t, \xi) \) of the geodesic arc joining the north pole \( N \) to \( \xi \in \partial U_+ \) which is at the geodesic distance \( t \) from \( N \) to the point \( Q = \Lambda(t, F^{-1}(\varphi_1(\xi))) \), of the segment \( [M, F^{-1}(\varphi_1(\xi))] \) which is at the Euclidean distance \( t \) from the fixed point \( M \in C_1 \).

Taking into account the restrictions \( \varphi_2 / W_\sigma \) and \( \varphi_2 / W_+ \), we can now define the global homeomorphism \( \varphi_2 : U_2 = W_+ \cup W_\sigma \rightarrow \overline{C_1} \cup (C_2 - \overline{C_1}) \) by the following expression

\[
\varphi_2(P) := \begin{cases} 
F^{-1}(\varphi_1(P)) & \text{if } P \in W_\sigma \\
\Lambda(t, F^{-1}(\varphi_1(\xi))) & \text{if } P = \Gamma(t, \xi) \in W_+.
\end{cases}
\]
Remark 4.3.

(1) Since by construction, the restrictions $\varphi_2/W$ and $\varphi_2/W_+$ are homeomorphisms, and for all $P \in \partial W_+ \cap \partial W$, we have
\[
(\varphi_2/W_\sigma)(P) = (\varphi_2/W_+)(P)
\]
then $\varphi_2$ is a homeomorphism.
(2) It follows from the definition of $\varphi_1$ and $\varphi_2$, that the change of charts in the domain $U_1 \cap U_2 = W_\sigma$, is the bi-holomorphic mapping
\[
\varphi_2 \circ \varphi_1^{-1} = F^{-1} : \mathcal{B}_1 - \mathcal{B}_{r_1} \rightarrow \mathcal{C}_2 - \mathcal{C}_1.
\]
Then $A := \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a complex analytic atlas on the sphere $S_{2n}$.

Hence the sphere $S_{2n}$ admits a complex structure, which means that $\{Necs\}_n$ is false. The implication $\{Necs\}_n \Rightarrow \{Hath\}_n$ is then true and the proof of theorem 1.6 is then complete. □

Corollary 4.4. (Theorem of complex spheres) Among all spheres $S_{2n}$, only the sphere $S_2$ admits a complex structure.

Proof. This is a corollary of the main theorem 1.6 and Hartogs’ theorem. □

5. Appendix

We summarize below the main tools of pseudoconvexity, and Čech cohomology we need to prove the main theorem 1.6. For pseudoconvexity, see Hörmander [12], and for Čech cohomology, see O. Forster [7].

5.1. Pseudoconvex manifolds.

5.1.1. Plurisubharmonic functions.

Recall that a function $f : \Omega \subset \mathbb{C}^n \rightarrow [-\infty, +\infty]$ is said to be plurisubharmonic in the open set $\Omega$ (we note $f \in \text{Psh}(\Omega)$), if $f$ is upper semicontinuous and satisfies the following mean value inequality
\[
f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + e^{i\theta} \xi) \, d\theta
\]
for every $\xi \in \mathbb{C}^n$ such that $z_0 + \xi \mathcal{D} \in \Omega$, where $\mathcal{D}$ is the unit disc of $\mathbb{C}$.

The following properties will be very useful.

1) If $f$ is convex on $\Omega$, then $f \in \text{Psh}(\Omega)$. Indeed, convexity implies continuity, and the inequality of convexity
\[
f(z_0) \leq \frac{1}{2} \left\{ \{f(z_0 + \xi e^{i\theta}) + f(z_0 - \xi e^{i\theta}) \right\}
\]
implies inequality of plurisubharmonicity (5.1).

2) Let $F : \Omega_1 \subset \mathbb{C}^n \rightarrow \Omega_2 \subset \mathbb{C}^n$ be a bi-holomorphic mapping, then
\[
g \in \text{Psh}(\Omega_2) \iff g \circ F \in \text{Psh}(\Omega_1).
\]

3) Let $X$ be a complex manifold, a function $f : X \rightarrow [-\infty, +\infty]$ is said to be plurisubharmonic on $X$ if for every complex chart $(U, \varphi)$, the function
\[
f \circ \varphi^{-1} : \varphi(U) \rightarrow [-\infty, +\infty]
\]
is plurisubharmonic.
Remark 5.1. By property 2) above, the notion of plurisubharmonicity on a complex manifold is independent of the choice of the chart \((U, \varphi)\).

5.1.2. Pseudoconvex manifolds. Let \(X\) be a complex manifold, and let \(f: X \rightarrow \mathbb{R}\) be a smooth function. \(f\) is said to be an exhaustion on \(X\) if for every \(c \in \mathbb{R}\), the sublevel set \(X_c = f^{-1}(c)\) is relatively compact in \(X\).

The complex manifold \(X\) is said to be strongly pseudoconvex if there exists a smooth strictly psh function \(f\) exhaustion on \(X\). We write in this case \(X = \bigcup_{c \in \mathbb{R}} f^{-1}(c)\).

Theorem 5.2. (Theorem B of Cartan.) If \(X\) is a strongly pseudoconvex manifold, then for every \(q \geq 1\), \(H^{0,q}(X, \mathbb{C}^n) = \{0\}\).

5.2. Čech cohomology. We recall below the necessary tools from Čech cohomology, we have also to use in the proof of the main theorem 1.6.

Let \(X\) be a topological space, and let \(\mathcal{V} = \{V_j\}_{j \in J}\) be a covering of \(X\), and let \(\mathcal{G}\) be a sheaf of abelian groups on \(X\). We define the \(q\)th group of cochains of \(\mathcal{G}\) with respect to the covering \(\mathcal{V}\), by:

\[
C^q(\mathcal{V}, \mathcal{G}) := \prod_{(j_0, \ldots, j_q) \in J^{q+1}} \mathcal{G}(V_{j_0} \cap \ldots \cap V_{j_q}).
\]

A \(q\)-cochain is then a family \((f_{j_0, \ldots, j_q})_{(j_0, \ldots, j_q) \in J^{q+1}}\) such that \(f_{j_0, \ldots, j_q} \in \mathcal{G}(V_{j_0} \cap \ldots \cap V_{j_q})\) for all \((j_0, \ldots, j_q) \in J^{q+1}.

We define the coboundary operators

\[
\delta: C^0(\mathcal{V}, \mathcal{G}) \rightarrow C^1(\mathcal{V}, \mathcal{G})
\]

and

\[
\delta: C^1(\mathcal{V}, \mathcal{G}) \rightarrow C^2(\mathcal{V}, \mathcal{G})
\]

as follows:

1) For \((f_i)_{i \in J} \in C^0(\mathcal{V}, \mathcal{G})\), we define \(\delta((f_i)_{i \in J}) = (g_{i,j})_{(i,j) \in J^2} \in C^1(\mathcal{V}, \mathcal{G})\) by

\[
g_{i,j} = f_j - f_i \in \mathcal{G}(V_i \cap V_j).
\]

2) For \((f_{i,j})_{(i,j) \in J^2} \in C^1(\mathcal{V}, \mathcal{G})\), we define \(\delta((f_{i,j})_{(i,j) \in J^2}) = (g_{i,j,k})_{(i,j,k) \in J^3} \in C^2(\mathcal{V}, \mathcal{G})\) by

\[
g_{i,j,k} = f_{j,k} - f_{i,k} + f_{i,j} \in \mathcal{G}(V_i \cap V_j \cap V_k).
\]

Since the boundary operators are group homomorphisms, we are lead then to consider the group of 1-cycles

\[
Z^1(\mathcal{V}, \mathcal{G}) := \ker \delta: C^1(\mathcal{V}, \mathcal{G}) \rightarrow C^2(\mathcal{V}, \mathcal{G})
\]

and the group of 1-coboundaries

\[
B^1(\mathcal{V}, \mathcal{G}) := \text{Im} \delta: C^0(\mathcal{V}, \mathcal{G}) \rightarrow C^1(\mathcal{V}, \mathcal{G})
\]
and to define with respect to the covering $\mathcal{U}$, the cohomology group with coefficients in the sheaf $\mathcal{G}$, that is the following quotient group

$$H^1(\mathcal{U}, \mathcal{G}) := Z^1(\mathcal{U}, \mathcal{G}) / B^1(\mathcal{U}, \mathcal{G}).$$

We need about the cohomology group $H^1(\mathcal{U}, \mathcal{G})$ with coefficients in the sheaf $\mathcal{G}$, to recall the following two fundamental facts.

1) The cohomology group $H^1(\mathcal{U}, \mathcal{G})$ depends obviously of the covering $\mathcal{U}$, but we can define the cohomology group $H^1(X, \mathcal{G})$ of the topological space $X$ as the inductive limit of the groups $H^1(\mathcal{U}, \mathcal{G})$ over all finer coverings $\mathcal{U}$ of $X$, that is

$$H^1(X, \mathcal{G}) := \lim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{G}).$$

2) To compute the cohomology group $H^1(X, \mathcal{G})$ from the group $H^1(\mathcal{U}, \mathcal{G})$, we use the following powerful theorem.

**Theorem 5.3. (Leray)** Let $X$ be a topological space, and let $\mathcal{G}$ be a sheaf over $X$, and let $\mathcal{U} = \{U_j\}_{j \in J}$ be a covering of $X$ such that for all $j \in J$, $H^q(U_j, \mathcal{G}) = \{0\}$. Then

$$H^q(X, \mathcal{G}) = H^q(\mathcal{U}, \mathcal{G}).$$

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