Aug. 21, 2003.

To: JIIP editor, Professor M.M.Lavrentiev, Applicable Analysis editor, Professor R. P. Gilbert

A brief information to the reader: I have received from the editor of the Journal of Inverse and Ill-Posed Problems (JIIP), Prof. M.M.Lavrentiev, in March, 2003, a letter of P.Sabatier to the editor of JIIP. In this letter Sabatier wrote that my statement "The Newton-Sabatier (NS) method for inverting the fixed-energy phase shifts for a potential is fundamentally wrong" is erroneous. This statement appeared in [3]. I have proposed to the editor of JIIP to publish Sabatier's letter and my reply. This reply the reader can find below. The editor of JIIP informed me that Sabatier's letter and my reply will not be published in JIIP.

However, I have found out on Aug.17, 2003, that Sabatier's letter was published in Applicable Analysis, 82, N3, (2003), 399-400. The editor of Applicable Analysis, Prof. R. P. Gilbert, did not send me Sabatier's letter and I was not given a chance to reply to this letter. Below the reader can find my comments on the published letter of P.Sabatier.

Comments on the letter of P.Sabatier, An erroneous statement, Applicable Analysis, 82, N3, (2003), 399-400.

The published letter of Professor Sabatier is an edited version of his letter, sent to me in March, 2003, by the JIIP editor, and my reply is appended below.

The editor of Applicable Analysis did not send me Sabatier's letter, so I was not able to reply to this letter right away.

In his letter Sabatier writes that my statement "The Newton-Sabatier (NS) method for inverting the fixed-energy phase shifts for a potential is fundamentally wrong" is erroneous.

In [2] my statement is proved. It is explained in [2] that the foundations of the NS method are wrong, and, in particular, that the basic integral equation of this method, in general is not uniquely solvable for some $r > 0$, contrary to the implicit assumption of the NS method. If this equation is not uniquely solvable for some $r > 0$, then the NS method breaks down: it produces a potential which is not locally integrable.

Thus, my conclusion was: the NS theory as an inverse scattering theory is fundamentally wrong in the sense that its foundations are wrong. This conclusion is fully justified in [2].

Sabatier does not discuss this main issue. In fact he does not discuss any of my concrete statements, given in [2].

In [2] I have discussed the following questions:

a) Is the NS theory a valid inverse scattering theory?
b) Are the foundations of the NS theory correct?

I gave the answer NO to both questions, and justified my answers.

Sabatier does not argue with my justification of these answers. The validity of the NS theory as an inverse scattering theory is not even mentioned in his letter.

Instead, he tries to create an impression that the NS method can serve, under some conditions, as a parameter-fitting method, which produces some potential, given the set of ”non-exceptional” phase shifts. The question of whether this potential generates the original phase shifts is not discussed in Sabatier’s letter. In [1, p.205], there is a discussion of this basic question, but the proof of the ”consistency” of the NS method is not correct.

I have demonstrated in [2] that the NS method cannot serve as a parameter-fitting method, because, in general, the basic integral equation is not uniquely solvable for some $r > 0$, and in this case the NS method breaks down. I do not want to go into further discussion of the errors in the presentation of the NS method in [1]. However, one wrong statement, which is given as a conclusion of the ”proof of the consistency” of the NS method in [1, p.205], may be of interest to the reader, and its discussion is short. Sabatier claims in [1, p.205], that the NS method gives ”one (only one) potential which decreases faster than $r^{-\frac{3}{2}}$. ” This statement is incorrect: take any compactly supported, integrable, real-valued potential $q$, construct its phase shifts at a fixed positive energy, and apply the NS method for the reconstruction. If the NS method produces a potential $q_{NS}$, decaying faster than $r^{-\frac{3}{2}}$, and if this potential generates the original phase shifts, then one has two potentials, $q$ and $q_{NS}$, which both decay faster than $r^{-\frac{3}{2}}$, and which are different, because the NS method cannot produce a compactly supported potential ([2]). Thus, Sabatier’s statement in [1, p.205] is incorrect. Since Sabatier claims in his letter that the NS method should work for ”almost all” phase shifts (the meaning of ”almost all” is not specified), one may hope that the phase shifts, generated by some compactly supported potential, are not ”exceptional”, because the set of compactly supported integrable potentials is dense in many sets of integrable potentials.

The other statements in Sabatier’s letter are discussed in detail in the appended reply to his original letter of March, 2003.

To summarize: my statement that ”the NS inversion theory is fundamentally wrong, in the sense that its foundations are wrong” is correct.

Alexander Ramm

=============== March 14, 2003.

To: JIIP editor, Professor M.M.Lavrent’ev, Applicable Analysis editor, Professor R. P. Gilbert
Some of the conclusions proved in [2] are:

1. In the NS theory R. Newton and P. Sabatier assume *without proof* that equation (12.1.2) in [1], with \( f \) defined in (12.2.1) in [1], is uniquely solvable for all \( r > 0 \). This assumption is fundamentally wrong. In general, this equation is not uniquely solvable for some \( r > 0 \) and, in this case, the NS method breaks down: it produces a potential which is not locally integrable.

2. In the NS theory R. Newton and P. Sabatier assume *without proof* that the transformation kernel can be represented by formula (12.2.3) from [1]. This assumption is fundamentally wrong, because, in general, the transformation kernel does not have the form (12.2.3) from [1]. In [1], the existence and uniqueness of the transformation kernel, used there, is not proved, and its properties are not studied. A proof of the existence and uniqueness of it is given in [6], where the properties of the transformation kernel, used in [1], are studied.

The above assumptions form the basis of the NS inversion theory. *Therefore the NS theory is fundamentally wrong; that is, its basic assumptions are wrong.*

This theory is *not* an inversion method for solving the inverse scattering problem. An inversion method is a method that allows one to recover a potential \( q \), given the scattering data (fixed-energy phase shifts in our case) generated by this \( q \). For the NS procedure, (even if it can be carried through), it is not proved that the potential it produces generates the original scattering data (the proof given in [1], p.205, is not convincing [2]).

The above argument does not depend on the class of potentials in which the solution to the inverse scattering problem is sought. Therefore, Professor Sabatier’s remark about ”favorite classes of potentials” is irrelevant and shows that he has missed the main point.

Professor Sabatier writes that the inverse scattering problem is to find \( c_\ell \) given the phase shifts \( \delta_\ell \). This is incorrect: the problem is to find a potential \( q \) from the corresponding phase shifts. In [2] the questions related to finding \( c_\ell \) from \( \delta_\ell \) are not discussed.

Professor Sabatier writes that he ”does not object against Ramm republishing 40 years old results”. Apparently Professor Sabatier has missed the basic point: I have proved that *the 40 year-old NS theory is fundamentally wrong*, and presented a detailed justification of this conclusion.

Professor Sabatier mentions that the uniqueness theorem for the inverse scattering problem in the class of compactly supported potentials belongs to Loeffel, and, he continues: ” Ramm claims that he proved the theorem (1987)” . This statement of Professor Sabatier is erroneous: whereas Loeffel dealt with 1D case, namely, with the spherically-symmetric potentials, my result, mentioned in [2], is the uniqueness theorem for the 3D inverse scattering problem with fixed-energy data. My uniqueness theorem implies the uniqueness theorem for the case of the spherically-symmetric compactly supported potentials.
Professor Sabatier writes "there are many potentials fitting the same set of phase shifts". It is not clear what he means: if "fitting" means that two compactly supported potentials, $q_1$ and $q_2$, produce identical fixed-energy phase shifts for all $\ell$, then Professor Sabatier's statement is incorrect, and $q_1 = q_2$. If "fitting" means approximately equal (in some sense) then Professor Sabatier's statement is true if it is taken out of the context; however, in the context it shows that he does not understand the difference between the notion of inverse scattering theory and the notion of ill-posedness of the inverse scattering problem with fixed-energy data. An inverse scattering theory is the theory that reconstructs a potential $q$, which belongs to some class of potentials, from the scattering data, generated by this potential. This problem may be ill-posed in the sense that $q$ does not depend continuously on the data. A problem may have a unique solution and be ill-posed. The inverse scattering theory may give results of the following nature: given the scattering data, generated by a potential $q$ from some class, one can uniquely reconstruct this $q$. Also, it may give an algorithm for finding $q$ from the scattering data. However, there can be many different potentials (even in the same class, but, also in other classes as well), which produce the scattering data close in some sense to the original data. If $q$ does not depend continuously (in some sense) on the data, then the inverse scattering problem is ill-posed. Theoretical investigation of the uniqueness of the solution to the inverse scattering problem and finding a reconstruction formula or algorithm for finding $q$ from the exact data are questions that are separate from the numerical reconstruction given noisy data and from the related notion of ill-posedness. All this is well known; I am stating this, because Professor Sabatier confuses the issues in his letter.

Professor Sabatier writes that the NS inversion procedure works in "most cases". Exactly the opposite is true: the NS procedure does not work in most cases, because in most cases equation (12.1.2) in [1] does not have a unique solution for some $r > 0$. This was explained in [2]. In [4] there was an attempt to prove that a modified version of equation (12.1.2) in [1] has a unique solution for all $r > 0$, but the claim in [4] is erroneous: a counterexample to this claim is constructed in [5].

Professor Sabatier writes that Fredholm's integral equation with spectral parameter is "not generically solvable in this sense". This is incorrect: such an equation has a discrete set of eigenvalues and therefore, by the Fredholm alternative, it is uniquely solvable "generically"-that is, for all values of the spectral parameter, except for a countable set (of Lebesgue's measure zero).

Professor Sabatier writes that a student can work out "one or two parameters examples...". This is true: these examples of solving linear Fredholm integral equations with degenerate kernels are trivial. However, these examples have nothing to do with the inverse scattering problem. They are not examples of finding a potential from the given phase shifts.

It follows from the above, that the claim in [3] stating that the NS inversion theory is fundamentally wrong, is correct and justified completely and in detail in [2].
I do not find the last paragraph of Professor Sabatier’s letter worthy of discussion.

One may argue that the NS theory, being fundamentally wrong as an inverse scattering theory, can be used as a parameter-fitting procedure. However, it is proved in [2] that the set of potentials, which can possibly be obtained by the NS procedure, is not dense in the standard class $L_{1,1}$ of potentials $q$ in the norm $||q|| := \int_0^\infty x|q(x)|dx$, for example. Therefore, if one wishes to solve the inverse scattering problem with the given fixed-energy phase shifts, corresponding to a potential in $L_{1,1}$, one cannot do this using the NS procedure as a parameter-fitting procedure, in general.

In the literature there are papers with examples showing that although the original NS procedure does not work numerically, its modification, proposed by Professor Scheid and his colleagues, can be used as a parameter-fitting procedure in some cases. This, however, is the point which was not discussed in [2], where I have analyzed the foundations of the NS theory.

References: [1] Chadan K., Sabatier P., Inverse Problems in Quantum Scattering Theory, Springer, New York, 1989.

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[3] Gutman, S., Ramm A.G., Scheid W., Inverse scattering by the stability index method, Jour. of Inverse and Ill-Posed Probl., 10, N5, (2002), 487-502.

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[5] Ramm A.G., A counterexample to a uniqueness result, Applic. Analysis, 81, N4, (2002), 833-836.

[6] Ramm A.G., Inverse scattering problem with part of the fixed-energy phase shifts, Comm. Math. Phys. 207, N1, (1999), 231-247.

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