CONSTRUCTION AND SET THEORY

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Abstract. This paper argues that mathematical objects are constructions and that constructions introduce a flexibility in the ways that mathematical objects are represented (as sets of binary sequences for example) and presented (in a particular order for example). The construction approach is then applied to searching for a mathematical object in a set, and a logarithm-time search algorithm outlined which applies to a set \( X \) of all binary sequences of length ordinal \( \beta \) with a binary label appended to each sequence to indicate that sequence is a member of \( X \) or not. It follows that deciding membership of a set of binary sequences of length ordinal \( \beta \) takes \( \beta + 1 \) bits, which is shown to be equivalent to the Generalised Continuum Hypothesis.

1. Philosophical Introduction

This is a short paper about set theory as a foundation for mathematics. It is not my intention to repeat what many authors have already written on the subject of set theory, so there is no discussion of the iterative conception of sets, forcing or limitation of size arguments, and only a mention of large cardinal axioms as a complexity measure. Rather the aim of this paper is to convince the reader about a certain way of looking at mathematics, which has some implications for set theory. That way of looking at mathematics owes something to information theory and computer science, and a great deal to P. Lorenzen’s notion of construction (see [8] and [9]).

The basic idea is that all of the objects and activities of mathematics are constructed by functions, and that the existence of the functions enables objects (including sets) to be defined. To give a simple example, the function of successor defines the set of natural numbers (subject to the condition that there is an initial number, 0, and the successor function does not output 0) given that the construction defines the smallest such set because an agent with unbounded but finite resource would construct exactly the set of the natural numbers. Moreover constructions can also be carried out on much larger sets than the set of natural numbers, in much the

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1See [6] for an encyclopedic overview of set theory up to the millennium and [11, 10] for very readable introductions to the iterative conception of set, which remains the standard motivation for set theory in terms of motivating the axioms of first-order Zermelo Fraenkel set theory. [11] gives an excellent background in the development of the concept of set, while [15] gives a structuralist interpretation of set theory that is still unsurpassed in clarity. Large cardinal axioms (axioms asserting the existence of infinite cardinal numbers with certain defining properties that are not theorems of first-order Zermelo Fraenkel set theory) have a vast literature, but [17] is a good introduction.

2Strictly, in terms of an ontology each mathematical “object” is really a function (or type) over a set of concrete individuals, because there is an issue of non-unique types, such as in the statement “1, 2 and 3 are 3 numbers”.

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same way as intuitionists admit for natural numbers and real numbers, namely by
free choice. The axiom of choice in the form of the well-order-ability of any well-
founded set is a key principle of infinite construction, and is constructive because
an agent with sufficient (i.e. infinite) resource could choose elements successively
and at infinite limits form the sequence of all elements chosen so far. If one accepts
infinite constructions, then the truth or falsehood of any proposition of first-order
set theory follows. For example, the truth of a quantified proposition has a clear
inductive construction in terms of a sequence of truth values of its subformulas that
follows the constant true sequence or constant false sequence of truth values or that
does not follow those sequences. While constructions determine how objects come
to exist, that does not mean that relationships between the objects cannot exist
that were not intended as part of the construction. Mathematics does not need to
be predicative (i.e. defining sets in stages only in terms of sets that are already
defined) provided the rule or process of construction is clear (which in my view
includes the process of choosing members of a set). As truth is well defined, the
logic of mathematics does not need to be constructive or intuitionistic. However,
according to this view the objects of mathematics are no more than constructions,
and we should not imagine that they exist independently of the process of their
construction. The objects of mathematics are possibilities of construction, in the
modal-structural sense of [5], and it is the clarity of their rules of construction that
grants them existence.

One problem with this approach is the status of these agents with infinite resources
(actually bounded by some infinite ordinal). I do not claim that such agents exist
in our physical world, but I do claim that their existence is possible if a rule of
construction that an agent uses is clear. In the same way that Euclid’s proof of the
infinite of primes gives a bound on finding the next prime in the sequence of prime
natural numbers, and thereby shows that the number of prime natural numbers
is infinite even though there are only finitely many atoms in the universe, rules
of construction that require infinite resources can have interesting properties that
help frame our theories of the physical world.

This may be all very well as a philosophical position (or not of course), but what
practical value does it have? Put briefly, the value of this position is the recognition
that mathematicians have freedom to represent a set of objects as they wish subject
to the constraints of the construction, including the presentation of the set in terms
of ordering. That is to say, if a mathematical object does not come equipped with
its own intrinsic ordering, an ordering can be added without affecting the intrinsic
properties of the mathematical object. It turns out that freedom to present and
represent mathematics does have practical consequences.

3See for example [11].
4For example, $(\forall x) P(x)$ is true in a model $M$ if $\{a : a \in M\}$ can be well ordered as $\{a_\alpha : \alpha < \aleph\}$
using the axiom of choice and the truth values of $< P(a_\alpha) : \alpha < \aleph>$ form a constant sequence
of value “true” of length $\aleph$. The constant sequence of value “false” corresponds to $(\forall x) \neg P(x)$ and
not following constant sequence of value “false” corresponds to $(\exists x) P(x)$.
5This is a deviation from the view of Lorenzen and the school that includes H. Poincaré, H.
Weyl and S. Feferman, see [3] for example.
2. Search for a Member of a Set

As an example of the constructive nature of mathematics, consider the question of what it means to search for a member of a set. In theory, if we represent the members of a set as binary sequences (or bitstrings for short), then you could read the bitstring and then append a label (say 1) to the bitstring if the bitstring were a member of the set and another label (say 0) if it were not a member of the set. In general, we would have to rely on an oracle to decide whether a set defined in this way were (equivalent to) the same set as a defined by a property of the members, but this lack of decidability is a problem with properties rather than with sets. We can say that if a set of bitstrings have length of least upper bound cardinal number \( \aleph_0 \), then the amount of information in searching for a member of the set is, adding 1 to the length of the sequence for the binary label, \( \text{Ord}(\aleph_0) + 1 \), where \( \text{Ord} \) is a function which returns the ordinal number corresponding to a cardinal number. In practice, for any reasonably large set we will be faced with a lot of bitstrings, and have no way to search for a particular bitstring \( x \) other than to enumerate the set of bitstrings somehow. Let us suppose (using our freedom of construction) that we can linearly order lexicographically (written \( \leq \)) the members of the set such that there is a least upper bound and greatest lower bound (in terms of bitstrings of length \( \aleph_0 \)) for the set as a whole and we can assign a distance between any two members of the set. It is reasonable to suppose that a set can be presented already linearly ordered, not when we are faced with a list to sort, but when we can choose how to present a set in the first place.

To justify our assumptions, we can define an interval \( X \) of binary sequences of length ordinal \( \beta \) as a set of all such binary \( \beta \)-sequences (binary sequences of length \( \beta \)) with the properties that every path through the tree of sequences from root to leaves is a branch of the tree, i.e., \( (\forall f : \beta \rightarrow \{0, 1\})((\forall x)(x \in f \rightarrow x \in X) \rightarrow (f \in X)) \), where \( x \in y \) is defined as \( (\exists z)(x \in z \land z \in y) \). Intervals defined in this way are not uniquely determined by ordinal \( \aleph_0 \) as the tree could have gaps between the sequences, but it is possible to make them unique by stipulating that for interval \( X \), \( (\forall f : \beta \rightarrow \{0, 1\})(f \in X) \). We can also stipulate that the root represents 0., so that in a sense the interval represents the maximal interval from 0 to 1 comprising binary \( \beta \)-sequences. Intervals of this type are written \( ([0, 1])^{(\beta)} \).

To justify that any two members \( x, y \) of \( ([0, 1])^{(\beta)} \) can be assigned a distance \( d(x, y) \) to be constant 0 from 0 onwards. Then \( d \) can be seen to be a Generalised \( \beta \)-ultrametric (i.e. \( \max(d(x, y), d(y, z)) \geq d(x, z) \)).

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6This is possible by fixing an enumeration of a set \( X \), \( (x_\alpha : \alpha < \aleph) \) (by the Axiom of Choice), and for any subset \( Y \subseteq X \) forming the binary \( \aleph \)-sequence \( (b_\alpha : (x_\alpha \in Y \rightarrow b_\alpha = 1) \lor (x_\alpha \notin Y \rightarrow b_\alpha = 0)) \), where the ordinal index of any member \( y \in Y \) is taken from the enumeration of \( X \) (which includes all members of \( Y \)). Thus a subset of \( X \) can be identified with a binary \( \aleph \)-sequence, and a set of subsets of \( X \) can be identified with a set of binary \( \aleph \)-sequences.

7\( z \leq y \) if \( (\exists \alpha < \aleph)(z_\alpha < y_\alpha) \land (\forall \beta < \alpha)(z_\beta = y_\beta) \) or \( (\forall \beta < \aleph)(z_\beta = y_\beta) \)

8\( d \) is a Generalised ultrametric because distances are not real numbers but binary \( \beta \)-sequences.

9To see this, fix labels \( x, y, z \) arbitrarily. Then if \( x \) splits from \( y \) before \( x \) splits from \( z \), then \( d(x, y) \geq d(x, z) \) and \( d(y, z) = d(x, y) \), so \( \max(d(x, y), d(y, z)) = d(x, y) \geq d(x, z) \). If \( x \) splits from \( y \) after \( x \) splits from \( z \), then \( d(x, z) \geq d(x, y) \) and \( d(y, z) = d(x, z) \), so \( \max(d(x, y), d(y, z)) = d(x, z) \geq d(x, z) \). Finally, if \( x \) splits from \( y \) at the same position that \( x \) splits from \( z \), then \( d(x, z) =
It is possible to losslessly compress any binary $\beta$-sequence to a binary $\mathbb{D}$-sequence where $\mathbb{D}$ is a cardinal $\mathbb{D} \leq \beta < \mathbb{D} + 1$. We can thus represent any set of $\beta$-sequences $X$ as $\subseteq (\mathbb{0}, \mathbb{1})L(\mathbb{D})$. But we actually want for the construction below to use sets such that each $\mathbb{D}$-sequence is labelled with a 1 (if $x \in X$) and 0 (if $x \notin X$). These labelled sets of binary $\mathbb{D}$-sequences are then $\subseteq (\mathbb{0}, \mathbb{1})\mathbb{O}(\mathbb{D})$ such that the set of binary $\mathbb{D}$-sequences without the labels $= (\mathbb{0}, \mathbb{1})L(\mathbb{D})$. We will write the labelled set of binary $\mathbb{D}$-sequences corresponding to set $X$ as $L(X)$.

Any set $X$ of size $\leq 2^\aleph$ can be searched for the bitstring $x$ of length $\aleph$ in $\aleph$ steps by representing the set $X$ by the labelled set $L(X)$ and then dividing $\mathbb{R}$ into two equal intervals (which is possible whether the midpoint is $\in X$ or not), choosing the interval that contains $x$ based on the value of the next bit of $x$ (because $a \leq x \leq b$ for $a, b$ the lower and upper limits of the interval) and iterating $\aleph$ times (taking the intersection of intervals at any limit ordinal stages), and checking the label of $x$ in $L(X)$ at the $\mathbb{O}(\aleph) + 1$-th step.

This enumeration (well-ordering) of intervals can also be regarded as an enumeration of members of the intervals. Members of the intervals may be members of $X$ but they do not have to be. For definiteness and balance we alternate the $\mathbb{R}$-sequences in $X$ and $(\mathbb{0}, \mathbb{1})(\aleph) - X$ as successive elements of the enumeration as far as possible (ending when an interval has all members $\in X$ or $\notin X$), and we see that there are $\leq (\mathbb{O}(\aleph) + 1) \times \mathbb{O}(\aleph) + 1$ steps to decide $x \in X$. Thus for any given binary $\aleph$-sequence $x$ we have an enumeration of $X$ and $(\mathbb{0}, \mathbb{1})(\aleph) - X$ that takes $< \aleph + 1$ steps to decide $x \in X$. We call this last statement (*).

In the simplest case of the real numbers, we can see that the search method amounts to binary search for a binary $\omega$-sequence in a labelled set that extends the closed interval $[0, 1]$. That us to say, every binary $\omega$-sequence is represented (starting with 0, in the case of $[0, 1]$) and every $\omega$-sequence has an extension at position $\omega + 1$ which states whether $x \in X$, where $X$ is coded as a set of binary $\omega$-sequences. It is clear that $x \in X$, for $X$ a set of real numbers, can be decided in $\leq \omega + 1$ steps. But does that mean that the set of real numbers is the closed interval $[0, 1]$? No, but it does mean that the set of real numbers are represented by $[0, 1]$ insofar as purely set theoretic properties, such as cardinality, are concerned.

**Theorem 1.** $GCH$ is equivalent to (*).

**Proof.** Assume $GCH$, and fix a binary $\aleph$-sequence $x$. Then if $x \in X$ then by $GCH$ $x$ will be enumerated in $< |X| \leq 2^{\aleph} = \aleph + 1$ steps. While if $x \notin X$ then $x$ will be enumerated in $< |(\mathbb{0}, \mathbb{1})(\aleph) - X| = 2^{\aleph} = \aleph + 1$ steps. In either case then $x \in X$ can be decided by enumeration in $< \aleph + 1$ steps, i.e. decided in $\leq \aleph$ steps since $\aleph$ is a cardinal. But if $x \notin X$ can be decided in $\leq \aleph$ steps, then it can be decided in $\leq \mathbb{O}(\aleph) + 1$ steps.

Now assume (*) and that $GCH$ is false, i.e. $X$ has cardinality $\aleph < c < 2^\aleph$, and fix a binary $\aleph$-sequence $x$. Then if $x \in X$, we would always find $x$ in $< c$ steps by enumeration since there are $|(\mathbb{0}, \mathbb{1})(\aleph) - X| = 2^{\aleph}$ members of $(\mathbb{0}, \mathbb{1})(\aleph) - X$.

$d(x, y)$ and $d(y, z) \leq d(x, y)$, so $\max(d(x, y), d(y, z)) = d(x, y) \geq d(x, z)$. These inequalities are not strict and allow for the cases of $x = y$, $y = z$ or $z = x$. 

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to be enumerated otherwise (and $c < 2^\aleph_0$). We can now check that $c = \aleph_1$ is consistent with (*), but $c > \aleph_1$ leads to $x$ being enumerated in $\geq \aleph_1 + 1$ steps (contradiction) and $\aleph_1 + 1 > c$ leads to $\aleph_1 + 1 > c > \aleph_0$ (contradiction). If $x \notin X$, then we could either enumerate all $c$ members of $X$ or $< 2^\aleph_0$ members of $([0,1])(\aleph_1) \setminus X$. But enumerating all of $c$ members of $X$ contradicts (*) because $c = \aleph_1 + 1$ leads to $x$ being enumerated in $\aleph_1 + 1$ steps (contradiction), $c > \aleph_1 + 1$ leads to $x$ being enumerated almost always in $\geq \aleph_1 + 1$ steps (contradiction) and $\aleph_1 + 1 > c$ leads to $\aleph_1 + 1 > c > \aleph_0$ (contradiction). The remaining possibility if $x \notin X$ is that $x$ is enumerated in $< 2^\aleph_0$ steps in $([0,1])((\aleph_1) \setminus X)$. Then $\aleph_1 + 1 = 2^\aleph_0$ is consistent with (*), and $\aleph_1 + 1 < 2^\aleph_0$ leads to $x$ being enumerated almost always in $\geq \aleph_1 + 1$ steps (contradiction) and $\aleph_1 + 1 > 2^\aleph_0$ contradicts Cantor’s theorem that $\aleph_1 + 1 \leq 2^\aleph_0$ (contradiction). Since $X$ is not empty and $\neq ([0,1])((\aleph_1) \setminus X)$ because $X$ has cardinality $c$, then both $c = \aleph_1 + 1$ and $\aleph_1 + 1 = 2^\aleph_0$ are witnessed as $x$ and the associated enumerations vary; hence $c = 2^\aleph_0$ (contradiction). Hence GCH is true. \hfill \Box

It is worth noting that GCH seems to have a uniformity that (*) lacks, in that the enumeration of $x$ when GCH is true does not depend on $x$, but only on the enumeration of $X$. However, the proof in Theorem 1 does not require that uniformity, since in both directions of the proof $x$ can be given in advance. In the direction where (*) is assumed, only those enumerations of $X$ which are most efficient in deciding whether $x \in X$ are compared\[10\], namely those decide $x \in X$ in $< \aleph_1 + 1$ steps and in $< c$ steps if $x \in X$ and in $< 2^\aleph_0$ steps if $x \notin X$.

The reason why a statement like GCH that is independent of first-order Zermelo-Fraenkel set theory turns out to be true for almost all sets if (*) is true is that (*) requires a very rich theory to be true. If we were to measure the complexity of a decision problem by the size of any set (i.e. possibly “a large cardinal”\[11\]) that is needed to solve the decision problem by deduction from the axioms of a first-order theory of sets\[12\], then (*) indicates that for infinite cardinal $\aleph_0, \aleph_1$ rather than a large cardinal would be the measure of complexity\[13\]. We can conclude that (*) is not compatible with decidability by a first-order deductive theory (that uses set cardinality as a complexity measure of decidability), but is compatible with truth in an initial segment of the Von Neumann hierarchy of pure sets, $V$. $V$ itself is a class model of first-order set theory.

Labelled sets are a good way to see the power of the decision criterion (*). Labelled sets clearly represent a standard binary coding of any set, but with the advantage that it is easy to tell which binary sequences are members of the set or not. There are for uncountable sets more labelled sets than there can be sets defined in any countable formal language of set theory, because each $\subseteq ([0,1])(\aleph_1)$,

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\[10\] That is $2^{\aleph_0}$ of $2^{2^\aleph_0}$ possible enumerations of $X$.

\[11\] If the axiom of choice is assumed, then the size a set is its only distinguishing feature, since all sets are well-orderable and are isomorphic to ordinals; and ordinals can be losslessly compressed to cardinals.

\[12\] See \[2\] 417 for an example from the work of H. Friedman of statements that can be encoded in first-order arithmetic that require a large cardinal axiom.

\[13\] The assumption is that the large cardinal would be needed for the proof, so should be enumerated as a measure of run time complexity (with an finite proof).
has labels for all its members and non-members (which is not true for membership defined by means of formulas through the axiom schemas of separation or replacement). In fact we can see that all sets $\subseteq ([0, 1])(\alpha)$ can be labelled for all cardinals $\alpha$. It is worth noting that labelled sets do not satisfy the axioms of first-order Zermelo-Fraenkel set theory, because functions cannot be applied to labels in the same way as to the data they label, but the axioms could be easily modified by stripping out the labels (i.e. the $\text{Ord}(\alpha) + 1$-th nodes), applying the function to binary sequences of length $\alpha$ and adding back the labels. That is, if $L(X)$ is a labelled set of binary $\alpha$-sequences then we can form the labelled set $\{\langle y, 0 \rangle : y \in ([0, 1])(\alpha)\} \cup \{\langle y, 1 \rangle : \langle x, 1 \rangle \in L(X) \Rightarrow y = f(x)\}$, where $\langle \rangle$ is a $\text{Ord}(\alpha) + 1$-sequence and $\cup$ is a union operator with the property that $\langle y, 0 \rangle \cup \langle y, 1 \rangle = \langle y, 1 \rangle$. It is clear though that labelled sets preserve what sets can be formed in initial segments of $V$.

3. Conclusions

I think the example of search for a member of a set shows, at least in principle, that taking mathematical objects as constructions (for example, labelled sets) which can be represented and ordered in different ways has mathematical consequences. The alternative to the labelled set approach discussed above is to suppose that there are sets which in principle we cannot define (by means of finite formulas) and of which we are not even permitted to see their shadows.

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