Divisibility Relations for the Dimensions and Hilbert series of Nichols Algebras of Non-Abelian Group Type

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Abstract

We present a divisibility relation for the dimensions and Hilbert series of certain classes of Nichols algebras of non-abelian group type, which generalizes Nichols algebras over Coxeter groups with constant cocycle $-1$. For this we introduce three groups of isomorphisms acting on Nichols algebras, which generalizes the exchange operator introduced by Milinski and Schneider in [17] for Coxeter groups.

1 Introduction

In [11], Theorem 4.14, Graña, Heckenberger and Vendramin gave a full classification of finite-dimensional Nichols algebras of non-abelian group type with absolutely irreducible Yetter-Drinfeld modules under the assumption that their Hilbert series factorize as

$$H_B(t) = r \prod_{j=1}^{r} (\alpha_j)_t \quad \text{with} \quad (\alpha_j)_t := 1 + t + t^2 + \cdots + t^{\alpha_j-1}$$

for some $r, \alpha_j \in \mathbb{N}$. From the examples in Subsection 2.3 they covered all Nichols algebras except $\mathcal{B}(Q_{3,1}, E_3)^{(2)}$ and $\mathcal{B}(Q_{4,1}, \chi_4)$, whose Hilbert series do not factorize as above. In a subsequent paper ([15]), Heckenberger, Vendramin and the author classified all finite-dimensional Nichols algebras of non-abelian group type with absolutely irreducible Yetter-Drinfeld modules under the more general assumption

$$H_B(t) = \prod_{j=1}^{r} (\alpha_j)_t \cdot \prod_{j=1}^{s} (\beta_j)_t^2, \quad \alpha_j, \beta_j \in \mathbb{N}, \quad (1)$$

but restricting the calculations to the special class of braided racks. With this approach, the Nichols algebras $\mathcal{B}(Q_{3,1}, E_3)^{(2)}$ and $\mathcal{B}(Q_{4,1}, \chi_4)$ were found to be finite-dimensional. Though the calculations are intricate already in the case of braided racks, the classification of finite-dimensional Nichols algebras along the approach proposed by Graña, Heckenberger and Vendramin seems to be feasible; if it is possible to show that the Hilbert series always factorizes in a way similar to Equation (1) this paper wants to contribute to the last question.

Shortly after publication of [15], Heckenberger brought to our attention that the dimensions of the Nichols algebras identified so far are always divisible by the order of the inner group of their underlying racks,
a fact which has been proven for Nichols algebras over Coxeter groups by Milinski and Schneider in 1999 ([17]). For their proof, they constructed a family of isomorphisms between the homogeneous components, based on interchanging the coefficients of a decomposition of each vector. This decomposition is possible due to the Nichols-Zoeller-Theorem ([20], see also Theorem 7.2.9 in [7]), which essentially states that a finite-dimensional Hopf algebra \( H \) is a free left \( B \)-module for all Hopf subalgebras \( B \) of \( H \).

Similar freeness theorems are abundant in Hopf theory and can be found e.g. in [5] (Theorem 1 and Lemma 3.2), [22] (Theorem 6.1), and [21]. We will use a version by Graña:

**Theorem 1 (Graña)**

(Cf. [9], Theorem 3.8.1) Let \( V \) be a finite-dimensional Yetter-Drinfeld module over a group \( G \) and assume \( V = V' \oplus U \) with

- \( V' \subseteq V \) a \( G' \)-stable \( KG \)-subcomodule, where \( G' \) is the smallest subgroup of \( G \) with \( \delta(V') \subseteq KG' \otimes V' \), and
- \( U \subseteq V \) a \( KG \)-subcomodule and \( KG' \)-submodule.

Let \( \{ e_i : i = 1, \ldots, d \} \) be a basis of \( V' \) and \( \partial_i \) the corresponding braided (skew) derivatives on the Nichols algebra \( \mathfrak{B}(V) \) over \( V \). Then

\[
\mathfrak{B}(V) \cong \left( \bigcap_{i=1}^{d} \ker \partial_i \right) \otimes \mathfrak{B}(V')
\]

as right \( \mathfrak{B}(V') \)-modules and left \( \left( \bigcap_{i=1}^{d} \ker \partial_i \right) \)-modules.

In particular, \( \dim \mathfrak{B}(V') \) divides \( \dim \mathfrak{B}(V) \).

Applying this together with specialized maps similar to those used by Milinski and Schneider, our main results are as follows:

**Theorem 2**

Let \( \mathfrak{B} \) be a finite-dimensional Nichols algebra over the indecomposable quandle \( X \) with a 2-cocycle \( \chi \). Assume that the degree of \( X \) divides the order of the diagonal elements of \( \chi \). Then each \( \text{Inn} X \)-homogeneous component of \( \mathfrak{B} \) has the same dimension (this is wrong for \( \deg X \nmid \text{ord} \chi \) in general) and thus \( \# \text{Inn} X \) divides \( \dim \mathfrak{B} \).

Moreover, let \( X' \) be a non-empty proper sub-rack of \( X \) and \( \mathfrak{B}' \) the Nichols sub-algebra generated by \( X' \). Assume that \( X \setminus X' \) still generates \( \text{Inn} X \). Then \( \# \text{Inn} X \cdot \dim \mathfrak{B}' \) divides \( \dim \mathfrak{B} \).

If we drop the assumption that the degree of \( X \) divides the order of the diagonal elements of \( \chi \), we can still prove the following:

**Theorem 3**

Let \( \mathfrak{B} \) be a finite-dimensional Nichols algebra over an indecomposable rack \( X \) and a 2-cocycle with diagonal elements of order \( m \). Let \( X' \) be a non-empty proper subrack of \( X \) and \( \mathfrak{B}' \) its corresponding Nichols sub-algebra of \( \mathfrak{B} \). Then the Hilbert series \( \mathcal{H}_\mathfrak{B}(t) \) is divisible by \( (m)_1 \cdot \mathcal{H}_{\mathfrak{B}'}(t) \).

Our paper is organized as follows. In Section 2, we define the core notions to understand the main results (racks, quandles, Nichols algebras, (opposite) braided derivations, Hilbert series) as well as a list of known Nichols algebras of non-abelian group type with absolutely irreducible Yetter-Drinfeld modules and the corresponding quandles in Subsection

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In Section 3 we introduce maps, so-called shifts, which are direct generalisations of the maps defined by Milinski and Schneider in [17]. An even more generalized form is currently considered by Angiono, Vay, and Vendramin. In Subsection 3.1 we apply them to show the first half of Theorem 2, i.e. \( \# \text{Inn} X \mid \dim \mathcal{B} \) for a slightly larger class of Nichols algebras than those examined in [17]. In Section 4 we define two improved variations of shifts and apply them to Graña’s Freeness Theorem to show the second half of Theorem 2 for the same class of Nichols algebras as before. We will then use our methods to analyze arbitrary Nichols algebras over non-trivial, indecomposable racks and proof Theorem 3. Finally, we will show in Subsection 5.2 that the direct approach to proof \( \# \text{Inn} X \mid \dim \mathcal{B} \) along the lines of [17] and Subsection 3.1 is not feasible in the general case \( \deg X \nmid \text{ord} \chi \).

2 Preliminaries

2.1 Notations

Denote \( \mathbb{N} = \mathbb{N}_0 \setminus \{0\} \). With \( C_k \) we denote the cyclic group of order \( k \), and with \( [j]_k \) the equivalence class of \( j \) in \( C_k \), for \( k \in \mathbb{N} \) and \( j \in \mathbb{Z} \). We use \( S_n \) to refer to the symmetric group on \( n \) symbols. Furthermore, we use the notation “\( Q_{x,y} \)” to refer to the \( y \)-th indecomposable quandle of size \( x \) in Graña’s and Vendramin’s list of small indecomposable quandles ([24], implemented in Rig, see [12]). \( \mathbb{K} \) shall always be a field of arbitrary characteristic, if not said otherwise.

2.2 Nichols Algebras from Racks

For a detailed account on racks in the context of Nichols Algebras, see [1].

**Definition 4**

A rack \( X \) is a set with a binary operation \( \triangleright \), which fulfills:

- **Left-self-distributivity:** For all \( t_1, t_2, t_3 \in X \) holds \( t_1 \triangleright (t_2 \triangleright t_3) = (t_1 \triangleright t_2) \triangleright (t_1 \triangleright t_3) \).

- **The operations** \( g_t : X \to X, s \mapsto t \triangleright s \) **are bijections.**

An idempotent rack is called quandle.

Due to left-self-distributivity, each \( g_t \) as defined above is an automorphism of \( (X, \triangleright) \). The permutation subgroup generated by the \( g_t \) is called the inner group \( \text{Inn} X \) of \( X \). It is a quotient of the enveloping (or structure) group

\[
\text{Env} X := \{ t \in X \mid s \cdot t = (s \triangleright t) \cdot s \ \forall \ s, t \in X \}_{\text{group}}.
\]

A rack \( X \) is called indecomposable, if \( \text{Inn} X \) acts transitively on \( X \). If the map \( g : X \to \text{Inn} X, t \mapsto g_t \) is injective, \( X \) is called faithful.

If \( X \) is a faithful quandle, then \( X \) is realized as a conjugation-closed generating subset of a group. On the other hand, each such subset is a faithful quandle.

Throughout this article, let \( X \) denote a finite indecomposable faithful quandle and \( \mathbb{K} \) our ground-field.
Example 5  ................................................  5
(Cf. [11], subsection 1.3) Let $A$ be an abelian group and $\alpha : A \to A$ some automorphism of $A$. Then $\triangleright : A \times A \to A$, $(t, s) \mapsto \alpha(t - s) + s$ defines a quandle structure on $A$. Quandles of this kind are called affine quandles. Many of the quandles used to construct Nichols algebras are affine (see the tables in subsection [23]), though not all. Assume $A$ is an affine quandle with two commuting elements $t, s \in A$ (i.e. $t \triangleright s = s$ and vice versa), then inserting into the definition gives $t = s$. This excludes many quandles, e.g. the quandles given by transpositions in the symmetric group $S_n$ for $n \geq 4$. The smallest non-affine quandles without commuting elements are $Q_{15,5}$ and $Q_{15,6}$ ([12]).

On the other hand, there are many non-faithful affine quandles.

Definition 6  ................................................  6
(Cf. [12]; for terms of Hopf algebra theory, see e.g. [2]) Let $V_0$ be a finite-dimensional vector space and set $V = KX \otimes V_0$. Denote $V_t := \{t\} \otimes V_0$. Assume the inner group $\text{Inn}X$ acts on $V$ with $g_t(V_s) \subseteq V_{t \triangleright s}$ for all $t, s \in X$. Then $V$ is an $\text{Inn}X$-Yetter-Drinfeld module, and a braiding

$$c(v \otimes w) = g_t(w) \otimes v$$

on the tensor product $V \otimes V$ is induced, where $v \in V_t$ and $w \in V$ are arbitrary. This in turn induces a co-algebra structure on the tensor algebra $(TV, \mu)$, which is uniquely determined by the two requirements

1. $\Delta \mu = (\mu \otimes \mu) (\text{id}_V \otimes c \otimes \text{id}_V)(\Delta \otimes \Delta)$
2. and each $v \in V \subseteq TV$ is primitive.

The co-unit is given by $\varepsilon(1) := 1$ and $\varepsilon(v) := 0$ for all $v \in V \subseteq TV$. Furthermore, an antipode $S : TV \to TV$ can be defined which endows $TV$ with the structure of an $N_0$-graded Hopf algebra (actually $F_X$-graded, where $F_X$ is the free group over the set $X$).

There is a unique maximal homogeneous ideal and coideal $I$ of $TV$ such that

$$\Delta(I) \subseteq I \otimes TV + TV \otimes I$$

and such that all homogeneous elements of $I$ are of degree $\geq 2$. The quotient $\mathfrak{B} := TV/I$ is called the Nichols algebra of $V$. It is an $\text{Env}X$-graded braided Hopf algebra, whose primitive elements are exactly those of degree 1 and generate $\mathfrak{B}$.

The Nichols algebra $\mathfrak{B}$ can be completely described in terms of the rack $X$ and a 2-cocycle $\chi : X \times X \to \text{End}(V_0)$ satisfying

$$\chi(t, s \triangleright r) \chi(s, r) = \chi(t \triangleright s, t \triangleright r) \chi(t, r),$$

which induces the action of $\text{Inn}X$ on $V$ ([11], [3]).

If $V$ is finite-dimensional and absolutely irreducible as a Yetter-Drinfeld module, the Lemma of Schur shows that $\chi(t, t)$ actually is a scalar multiple $q_t$ of the identity for each $t \in X$ ([11]; Theorem 2.7 in [10]). If $X$ is indecomposable, the transitive action of $\text{Inn}X$ ensures that $q_t$ does not depend on $t$; we drop the index in this case. However, Graña showed in [10], Lemma 3.1, that the cases $\dim V_0 \geq 2$ impose severe restrictions on $q$ and $\chi$ if $\mathfrak{B}$ is to be finite-dimensional. Therefore, for the most part of this paper, we will restrict to the case $\dim V_0 = 1$, without losing too many cases.
Definition 7. Let $X$ be an indecomposable rack. Then the order of $g$, does not depend on $t \in X$ and we define the degree $\deg X$ to be this number.

The nilpotency order $\text{nord} v$ of an element $v \in \mathcal{B}$ is the minimal $m \in \mathbb{N}$ with $v^m = 0$.

The order $\text{ord} \chi$ of a 2-cocycle $\chi$ is the minimal $m \in \mathbb{N} \cup \{\infty\}$ with $\chi(t, t)^m = 1$ for all $t \in X$. (If $\mathcal{B}$ is finite dimensional, one easily shows that $q_t := \chi(t, t)$ has to be a root of unity, such that $\text{ord} \chi$ is finite.)

For the rest of this paper, set $n := \deg X$ and $m := \text{ord} \chi$.

Denote with $\{e_t\}_{t \in X}$ the standard base in $KX$. If $V_0$ is one-dimensional, we use it as standard base for $V$ as well; else, we denote $e_t \otimes v$ with $e_t v$ (see [3]). If $X$ is indecomposable, $\text{nord}(e_t v)$ does not depend on $t \in X$ nor $v \in V_0$. Using Equation (9) in Proposition 11 one easily calculates $\text{nord}(e_t v) = m$ in this case for $v \in V_0 \setminus \{0\}$. There is however no obvious relation between $m$ and $n$, as can be seen in the examples of subsection 2.3.

Definition 8. Given a $G$-graded vector space $U$, denote with $U(g)$ the $g$-homogeneous subspace of $U$. We say the grading is balanced, if $\dim U(g)$ is finite and constant for all $g \in G$. A vector space $U$ is $G$-balanced, if $U$ is $G$-graded and the grading is balanced.

Let $U$ be a $G$-graded vector space and $H$ a quotient of $G$. Then $U$ is $H$-graded as well. Any Nichols algebra $\mathcal{B}$ is $\text{Env}X$-graded. As there are canonical surjective homomorphisms $\text{Env}X \to \mathbb{Z}$ and $\text{Env}X \to \text{Inn}X$, we will use the notation $\mathcal{B}(x)$ for any $x \in \text{Env}X$, $\mathbb{Z}$ or $\text{Inn}X$ without further notice, as the latter two gradings are induced by the first one. If the $G$-grading is balanced, then the induced $H$-grading is balanced as well.

Definition 9. If $U$ is a $\mathbb{Z}$-graded vector space and each $U(g)$ is finite-dimensional, define the (formal) Hilbert series $H_U$ by

$$H_U(t) := \sum_{j \in \mathbb{Z}} \dim U(j) t^j.$$ 

For any $k \in \mathbb{N} \cup \{\infty\}$ define $(k)_t$ to be the series $\sum_{j=1}^{k-1} t^j$.

Finite dimensional Nichols algebras of abelian group type (i.e. over trivial quandles) have been completely classified by I. Heckenberger in [13] and [14]. The classification of finite dimensional Nichols algebras of non-abelian group type, particularly over indecomposable quandles, is advanced by several strategies. A very interesting ansatz is to identify the set of appropriate racks, so by identifying racks of type D. This led to the exclusion of conjugacy class racks of whole classes of groups, notably the alternating groups $A_m$ for $m \geq 6$ ([2]) and many sporadic groups ([4]). An alternative is to derive inequalities on the maximal dimensions of the lower homogeneous degrees, as has been done in [11] and [13], and connect these to a certain factorization of $H_{\mathcal{B}}$ in terms of $(k)_t$ and $(k)_{2t}$.
2.3 Examples for Nichols Algebras

There are nine indecomposable and faithful quandles known to provide examples of finite-dimensional Nichols algebras, with the following properties.

| $X$     | $\deg X$ | $\text{Inn} X$ | (Size) | $g_t \in \text{Inn} X$ for generating $t \in X$ (as perm. of $X$ in cycle notation) |
|---------|----------|-----------------|--------|-------------------------------------------------------------------|
| $Q_{3,1}$ | 2        | $S_3$           | (6)    | $g_1 = (2,3)$, $g_2 = (1,3)$                                      |
| $Q_{4,1}$ | 3        | $A_4$           | (12)   | $g_1 = (2,3,4)$, $g_2 = (1,4,3)$                                 |
| $Q_{5,2}$ | 4        | $C_5 \rtimes C_4$ | (20)   | $g_1 = (2,4,5,3)$, $g_2 = (1,4,3,5)$                             |
| $Q_{5,3}$ | 4        | $C_5 \rtimes C_4$ | (20)   | $g_1 = (2,3,5,4)$, $g_2 = (1,5,3,4)$                             |
| $Q_{6,1}$ | 2        | $S_4$           | (24)   | $g_1 = (3,5)(4,6)$, $g_2 = (3,6)(4,5)$, $g_3 = (1,5)(2,6)$    |
| $Q_{6,2}$ | 4        | $S_4$           | (24)   | $g_1 = (3,5,4,6)$, $g_3 = (1,6,2,5)$                             |
| $Q_{7,4}$ | 6        | $(C_7 \rtimes C_3) \rtimes C_2$ | (42)   | $g_1 = (2,6,5,7,3,4)$, $g_2 = (1,4,5,3,7,6)$                    |
| $Q_{7,5}$ | 6        | $(C_7 \rtimes C_3) \rtimes C_2$ | (42)   | $g_1 = (2,4,3,7,5,6)$, $g_2 = (1,6,7,3,5,4)$                    |
| $Q_{10,1}$ | 2       | $S_5$           | (120)  | $g_1 = (2,7)(3,5)(4,6)$, $g_2 = (1,7)(3,8)(4,10)$, $g_3 = (1,5)(2,8)(4,9)$,
|           |          |                 |        | $g_4 = (1,6)(2,10)(3,9)$                                        |

The quandles $Q_{3,1}$, $Q_{5,2}$, $Q_{5,3}$, $Q_{7,4}$, and $Q_{7,5}$ are affine quandles over the cyclic abelian groups of order $\#X$, with $\alpha$ the multiplication with 2, 3, 2, 5, and 3, respectively. $Q_{4,1}$ also is an affine quandle over $C_2 \times C_2$ with $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The quandles $Q_{3,1}$, $Q_{6,1}$ and $Q_{10,1}$ can also be defined as the conjugacy classes of transpositions in the symmetric group $S_n$, for $n = 3, 4, 5$, respectively.

The following fourteen finite-dimensional Nichols algebras over $\mathbb{K}$ of non-abelian group type are our basic examples, sorted by quandle and dimension.

| $\mathcal{B}(X, \tau)$ | $\text{char}(\mathbb{K})$ | $n$ | $m$ | dimension | $\mathcal{H}_\mathbb{F}(t)$ |
|-------------------------|---------------------------|-----|-----|------------|-----------------------------|
| $\mathcal{B}(Q_{3,1}, -1)$ | $*$                        | 2   | 2   | 12         | $(2)^1_1(3)^1_1$            |
| $\mathcal{B}(Q_{4,1}, E_3)^{(2)}$ | 2                         | 2   | 3   | 432        | $(3)^2_2(4)^1_1(6)^1_2$     |
| $\mathcal{B}(Q_{4,1}, -1)^{(2)}$ | 2                         | 3   | 2   | 36         | $(2)^1_1(3)^1_1$            |
| $\mathcal{B}(Q_{4,1}, -1)^{(2), \neq 2}$ | 3                         | 3   | 2   | 72         | $(2)^1_1(3)^1_1(6)^1_2$     |
| $\mathcal{B}(Q_{4,1}, \chi_4)$ | $*$                        | 4   | 2   | 1,280      | $(4)^1_1(5)^1_1$            |
| $\mathcal{B}(Q_{5,2}, -1)$ | $*$                        | 4   | 2   | 1,280      | $(4)^1_1(5)^1_1$            |
| $\mathcal{B}(Q_{5,3}, -1)$ | $*$                        | 4   | 2   | 576        | $(2)^2_2(3)^1_1(4)^1_1$     |
| $\mathcal{B}(Q_{6,1}, \chi_6)$ | $*$                        | 2   | 2   | 576        | $(2)^2_2(3)^1_1(4)^1_1$     |
| $\mathcal{B}(Q_{6,2}, -1)$ | $*$                        | 4   | 2   | 576        | $(2)^2_2(3)^1_1(4)^1_1$     |
| $\mathcal{B}(Q_{7,4}, -1)$ | $*$                        | 6   | 2   | 326,592    | $(6)^1_1(7)^1_1$            |
| $\mathcal{B}(Q_{7,5}, -1)$ | $*$                        | 6   | 2   | 326,592    | $(6)^1_1(7)^1_1$            |
| $\mathcal{B}(Q_{10,1}, -1)$ | $*$                        | 2   | 2   | 8,294,400  | $(4)^1_1(5)^1_1(6)^1_4$     |

In all cases we have $\dim V_0 = 1$. $E_N$ denotes an $N$-th root of unity and the superscripts $^{(2)}$ and $^{(\neq 2)}$ refer to the field’s characteristic. The
non-constant cocycles \( \chi_4, \chi_6 \) and \( \chi_{10} \) are defined as follows:

\[
\chi_4 := \begin{pmatrix}
E_3 & -E_3 & -E_3 & E_3 \\
-E_3 & E_3 & -E_3 & E_3 \\
-E_3 & -E_3 & E_3 & E_3 \\
E_3 & E_3 & E_3 & E_3
\end{pmatrix}
\]

\[
\chi_6 := \begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\[
\chi_{10} := \begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

A more concise description of \( \chi_6 \) and \( \chi_{10} \) in terms of transpositions in \( S_4 \) and \( S_5 \) is given e.g. in [17] (Example 5.3) and in [23].

Note that \( \mathcal{B}(Q_{5,2}, -1) \) and \( \mathcal{B}(Q_{5,3}, -1) \) as well as \( \mathcal{B}(Q_{7,4}, -1) \) and \( \mathcal{B}(Q_{7,5}, -1) \) are dual algebras (see Example 2.1 in [3]), \( \mathcal{B}(Q_{6,1}, -1) \) and \( \mathcal{B}(Q_{6,1}, \chi_6) \) as well as \( \mathcal{B}(Q_{10,1}, -1) \) and \( \mathcal{B}(Q_{10,1}, \chi_{10}) \) are twist-equivalent to each other ([23]).

Also note that the factorization of \( \mathcal{H}_3(t) \) in terms of \( (k)_t \) and \( (k)_{12} \) is not unique.

### 2.4 Braided Derivations and Braided Commutator

For the rest of the paper, we assume \( \dim V_0 = 1 \) for simplicity.

**Definition 10**

*Given the comultiplication \( \Delta : \mathcal{B} \to \mathcal{B} \), we can uniquely define linear maps \( \partial_t \) and \( \partial_t^{op} : \mathcal{B} \to \mathcal{B} \) for arbitrary \( t \in X \) via*

\[
\Delta(v) = v \otimes 1 + \sum_{t \in X} \partial_t(v) \otimes e_t + \text{some element of } \mathcal{B} \otimes \bigoplus_{j=2}^{\infty} \mathcal{B}(j)
\]

\[
= 1 \otimes v + \sum_{t \in X} e_t \otimes \partial_t^{op}(v) + \text{some element of } \bigoplus_{j=2}^{\infty} \mathcal{B}(j) \otimes \mathcal{B}.
\]

*We call these maps braided derivations and opposite braided derivations, respectively.*

The braided derivations \( \partial \) and \( \partial^{op} \), have been introduced by Nichols in subsection 3.3 in [15] under the name “quantum differential operators”; for an account on them, we refer to [5].
The maps $\partial_t$ and $\partial_{op}^t$ satisfy the following properties for all $t, s \in X$ and $v, w \in \mathcal{B}$:

\[
\begin{align*}
\partial_t(1) &= 0 \quad (2) \\
\partial_t(e_s) &= \delta_{t,s} \quad (3) \\
\partial_t(vw) &= v\partial_t(w) + \partial_t(v)g_t(w) \quad (4) \\
\partial_{op}^t(1) &= 0 \quad (5) \\
\partial_{op}^t(e_s) &= \delta_{t,s} \quad (6) \\
\partial_{op}^t(vw) &= \partial_{op}^t(v)w + v\partial_{op}^t\left(\frac{g_{s^{-1}(t)}(w)}{\chi(v,t)}\right) \quad (if \ v \ is \ homog.) \quad (7) \\
\partial_s^op \partial_t &= \partial_t \partial_s^op \\
\bigcap_{t \in X} \ker \partial_t &= \bigcap_{s \in X} \ker \partial_s^op = \mathcal{B}(0) \quad (9)
\end{align*}
\]

(where $\delta_{t,s}$ is the Kronecker symbol.)

**Proof** (2), (5): Obvious.

(3), (6): Follows from primitivity of $\mathcal{B}(1)$.

(8): Follows from co-associativity.

(9): Follows from [18] and [9].

(4): Let $v, w \in \mathcal{B}$ be arbitrary. Then holds:

\[
\begin{align*}
\Delta(vw) &= \Delta(v) \cdot \Delta(w) \\
&= \left(v \otimes 1 + \sum_{t \in X} \partial_t(v) \otimes e_t + \text{higher terms}\right) \\
&\quad \cdot \left(w \otimes 1 + \sum_{t \in X} \partial_t(w) \otimes e_t + \text{higher terms}\right) \\
&= (vw) \otimes 1 + \sum_{t \in X} \left(\partial_t(v) \otimes e_t\right) \cdot \left(w \otimes 1\right) \\
&\quad + \sum_{t \in X} \left(v \otimes 1\right) \cdot \left(\partial_t(w) \otimes e_t\right) + \text{higher terms} \\
&= (vw) \otimes 1 + \sum_{t \in X} \left(\partial_t(v) g_t(w) + v\partial_t(w)\right) \otimes e_t + \text{h.t.}
\end{align*}
\]

(7): Similar to (4) we have:

\[
\begin{align*}
\Delta(vw) &= \Delta(v) \cdot \Delta(w) \\
&= \left(1 \otimes v + \sum_{t \in X} e_t \otimes \partial_{op}^t(v) + \text{higher terms}\right) \\
&\quad \cdot \left(1 \otimes w + \sum_{s \in X} e_s \otimes \partial_{s}^{op}(w) + \text{higher terms}\right) \\
&= 1 \otimes (vw) + \sum_{t \in X} \left(e_t \otimes \partial_{op}^t(v)\right) \cdot \left(1 \otimes w\right) \\
&\quad + \sum_{s \in X} \left(1 \otimes v\right) \cdot \left(e_s \otimes \partial_{s}^{op}(w)\right) + \text{higher terms} \\
&= 1 \otimes (vw) + \sum_{t \in X} \left(e_t \otimes \partial_{op}^t(v)w\right) + \sum_{s \in X} \left(g_v(e_s) \otimes v\partial_{s}^{op}(w)\right) + \text{h.t.}
\end{align*}
\]
where $g_v$ is defined such that $v \in \mathcal{B}(g_t)$. By definition we have $g_v(c) = \chi(v, s) e_{vd}^t$ (where $v \triangleright s$ is short-hand for $g_v(s)$ and the 2-cocycle $\chi$ is extended in the obvious way). Choosing $t$ such that $e_{vd}^t = e_t$, we conclude

$$\Delta(vw) = 1 \otimes (vw) + \sum_{t \in X} e_t \otimes (\partial_t^{op}(v) w + v \partial_t^{op}(v) w / \chi(v, t)) + \text{h.t.}$$

where $v \triangleright^{-1} t := g_v^{-1}(t)$.

$\partial_t$ is a right $\sigma$-skew-derivation, as one sees from Equation 13 with the endomorphism $\sigma = g_t$. $\partial_t^{op}$ is not a right skew-derivation; so we chose the word “opposite braided derivation”, to emphasize its kinship with $\partial_t$. Also note that $\text{ker } \partial_t^{op} \neq \text{ker } \partial_t$ in general.

Definition 12 (6)  ........................................ 12

Let $\mathcal{B}$ be a Nichols algebra with braiding $c$. Define the braided commutator

$$[x, y]_c := \mu \circ (\text{id} - c)(x \otimes y)$$

for all $x, y \in \mathcal{B}$.

Proposition 13  ........................................ 13

For all $t \in X$ holds: $\partial_t g_t = q_t \cdot g_t \partial_t$.

Proof By induction over the $\mathbb{N}$-degree $d$ of $v \in \mathcal{B}$. For $d \in \{0, 1\}$, this is clear. For each $v, w \in \mathcal{B}$ we have

$$\partial_t g_t (v w) = (g_t v) \cdot (\partial_t g_t w) + (\partial_t g_t v) \cdot (g_t^2 w)$$

and $g_t \partial_t (v w) = (g_t v) \cdot (g_t \partial_t w) + (g_t \partial_t v) \cdot (g_t^2 w)$. $\square$

Proposition 14  ........................................ 14

Let $t \in X$ and $v \in \text{ker } \partial_t$ be arbitrary. Then $[e_t, v]_c \in \text{ker } \partial_t$.

Proof With Proposition 13 one finds

$$\partial_t ([e_t, v]_c) = \partial_t (e_t v) - \partial_t ((g_t v)e_t) = g_t v - g_t v = 0.$$  $\square$

3 The Shift Group of a Nichols Algebra

Milinski and Schneider showed in [17], Theorem 5.8, that the grading of a Nichols algebra is balanced, if $\text{Im } X$ is a Coxeter group and a certain type of cocycle is given. In general, the grading of a Nichols algebra need not be balanced, as we will see in the case of the 72-dimensional Nichols algebra in subsection 5.2.

Let $\mathcal{B}$ be a finite-dimensional Nichols-Algebra over the rack $X$ and cocycle $\chi$ with $\text{dim } V_0 = 1$ (see Definition 6). Recall that $\mathcal{B}$ is generated as an algebra by the elements $e_t \in \mathcal{B}(1), t \in X$.

It is a well-known fact that for each finite-dimensional Nichols algebra $\mathcal{B}$ over the quandle $X$ and the field $K$ and for each $t \in X$ holds:
Proposition 17. For each \( t, s \) pose would be a non-trivial linear dependency. Due to uniqueness, all summands by (4) hold:

\[
\mathcal{B} \cong (\ker \partial_t) \otimes (\Bbbk[e_t]/e_t^m) \quad \text{and} \quad \mathcal{B} \cong (\Bbbk[e_t]/e_t^m) \otimes (\ker \partial_t^{op}).
\]

Proof. 1) This directly follows from Graña’s Freeness Theorem. We reproduce a short proof to compare it to part (2).

Let \( s \in X \setminus \{ t \} \) be arbitrary. Then by definition of the braided commutator holds:

\[
e_t e_s = [e_t, e_s]_c + (g_t e_s) e_t.
\]

For \([e_t, e_s]_c\) we use Proposition 14 to see that \([e_t, e_s]_c \in \ker \partial_t\). On the other hand, we have \(\partial_t(g_t e_s) = 0\), because \( t \uplus s \neq t \) for \( s \neq t \) and any quandle \( X \). By induction we find that for each \( v \in \mathcal{B} \) there are \( v_j \in \ker \partial_t\) with

\[
v = \sum_{j=0}^{m-1} v_j e_t^j.
\]

We now show that these \( v_j \) are uniquely determined (this is analog to Lemma 2.5 in [14]): Assume \( \sum_{j=0}^{m-1} v_j e_t^j = 0 \). Apply \( \partial_t \) \((m-1)\)-times to find \( v_{m-1} = 0 \). Then apply \( \partial_t \) \((m-2)\)-times to see \( v_{m-2} = 0 \), induction.

2) Let \( v \in \mathcal{B} \) be homogeneous with \( \partial_t^{op}(v) \neq 0 \). By induction over the length, we can restrict to \( v = u e_s \) with \( \partial_t^{op}(u) \neq 0 \) but \( \partial_t^{op}(u e_s) \neq 0 \). Set \( w := \chi(u, t)^{-1} u \). Then \( w \in \ker \partial_t^{op} \) and

\[
\partial_t^{op}(u e_s - e_t w) = \chi(u, t)^{-1} u \partial_t^{op}(u e_s) - w - q_t^{-1} e_t \partial_t^{op} w = 0,
\]

hence \( v = e_t w - (v - e_t w) \in (\Bbbk[e_t]/e_t^m) \otimes (\ker \partial_t^{op}) \). Like in (1), linear independence is shown by applying \( \partial_t^{op} \).

One would expect that for each element \( v \in \mathcal{B}(1) \), there is a decomposition \( \mathcal{B} = U \otimes \Bbbk[v]/v^{\text{nord} v} \) similar to the one of Lemma 15. This, however, is wrong: Take \( v = e_1 + e_2 \in \mathcal{B}(Q_{3,1}, -1) \). If \( K \) is of characteristic \( \neq 2 \), \( v \) has nilpotency order 4. If \( \mathcal{B}(Q_{3,1}, -1) \) would decompose into a tensor product with factor \( \Bbbk[v]/v^4 \), its Hilbert series would be divisible by \((4)_t\), which is not the case.

Proposition 16. Let \( v \in \mathcal{B}(g) \) for some \( g \in \text{Env} X \) and \( t \in X \) be arbitrary. If we decompose \( v \) into the sum \( \sum_{j=0}^{m-1} v_j e_t^j \) with Lemma 15, we have \( v_j \in \mathcal{B}(g t^{-j}) \) for each \( j \). If decomposed into \( v = \sum_{j=0}^{m-1} e_t^j v_j \), each \( v_j \in \mathcal{B}(t^{-j} g) \).

Proof. Each summand \( v_j e_t^j \) is itself \( \text{Env} X \)-homogeneous (otherwise there would be a non-trivial linear dependency). Due to uniqueness, all \( v_j e_t^j \) are linearly independent, hence each \( v_j e_t^j \) must be element of \( \mathcal{B}(g) \ni v \). Then \( v_j \in \mathcal{B}(h) \) with \( h = g t^{-j} \). The second statement follows the same way.

Proposition 17. Let \( t, s \in X \) be arbitrary, \( t \neq s \). The shift

\[
\phi_t : \mathcal{B} \to \mathcal{B}, \quad v = \sum_{j=0}^{m-1} v_j e_t^j \mapsto v_{m-1} + \sum_{j=1}^{m-1} v_{j-1} e_t^j
\]

\[\forall j : v_j \in \ker \partial_t\]
is a well-defined linear isomorphism. We call the group generated by the shifts $\phi_t$ the shift group $\Phi(B)$ (or just $\Phi$). By definition, its operation on $B$ is free.

**Proof** $\phi_t$ is bijective because $\phi_t^m = \text{id}_B$ and $\phi_t$ obviously is linear. □

**Lemma 18** The $\Phi$-orbit of 1 linearly spans $B$.

**Proof** By induction over the $N_0$-degree $d$ of $v \in B$. For $d \in \{0, 1\}$, this is clear. Assume $\Phi(1)$ spans the whole of $B(d)$. Let $w \in B(d)$ and $t \in X$ be arbitrary. By Lemma 15, $w$ decomposes into

$$w = \sum_{j=0}^{m-1} w_j e_t^j$$

with $w_j \in \ker \partial_t$, $j = 0 \ldots (m - 1)$. Due to the grading, each $w_j$ can be chosen to be of length $d - j$. Now we see

$$w e_t = \sum_{j=0}^{m-2} w_j e_t^{j+1} = \phi_t \left( \sum_{j=0}^{m-2} w_j e_t^j \right).$$

$\sum_{j=0}^{m-2} w_j e_t^j$ is of length $d$ and can be spanned by $\Phi(1)$. Hence, $w e_t$ can be spanned by $\Phi(1)$ as well. □

As an (unused) corollary, we see that if $\Phi$ is finite, $B$ must be finite-dimensional. However, the converse is not true: $\Phi(B(Q_4, 1, -1)^{(\#2)})$ contains the infinite group $C_2 \times C_2$ as a subgroup (see subsection 5.2).

The dimension of the matrix algebra $\text{Alg} \Phi$ spanned by the maps $\phi_t$ is bounded from above by $(\text{dim } B)^2$; and if $B$ is infinite, it must be infinite-dimensional as well, due to Lemma 18. In the case of the Nichols algebra $B(Q_3, -1)$, we find that the dimension of this shift algebra is 12, which equals the dimension of $B$. But we have $\phi_t^2 = \text{id}$ in this algebra, so it cannot be $N_0$-graded. Hence, the shift algebra and $B$ are not isomorphic.

In the case of $B(Q_4, 1, -1)^{(\#2)}$, even the dimensions differ: $B$ has dimension 36, but the shift algebra of $B$ has dimension 648 = $18 \cdot 36$. For the 72-dimensional algebra $B(Q_4, 1, -1)^{(\#2)}$ one finds that the shift algebra has dimension $2592 = 36 \cdot 72$.

### 3.1 Case $n$ divides $m$

In the following, let $B$ be a Nichols algebra with $n | m$, $\Phi$ its shift group and $K\Phi$ the group algebra of $\Phi$. (Instead of $K\Phi$ we might just as well take the shift algebra of $B$.)

The evaluation at $1 \in B$ yields a linear map $ev_1 : K\Phi \to B$, which by Lemma 18 is surjective. In particular, we may define subspaces $K\Phi_g := ev_1^{-1}(B(g))$.

**Proposition 19** If $\phi_g$ is restricted to $B(g)$ for any $g \in \text{Inn } X$, its image restricts to $B(gt)$ and hence yields a linear isomorphism $B(g) \cong B(gt)$. 

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Proof We use the notations from Proposition \[17\] Let \( v \in \mathcal{B}(g) \). Then \( v_j \in \mathcal{B}(gt^{-j}) \) and hence \( v_{j-1}e_i^j \in \mathcal{B}(gt) \) for each \( j = 1 \ldots (m-1) \). Because of \( n \mid m \) we have \( t^m = 1 \) in \( \text{Inn} X \) and \( v_{m-1} \in \mathcal{B}(h) \) with \( h = gt^{-(m-1)} = gt \), so the sum \( \phi_t(v) \) is in \( \mathcal{B}(gt) \). Apply \( \phi_t \) \( m \)-times to see that \( \phi_t|_{\mathcal{B}(g)} : \mathcal{B}(g) \to \mathcal{B}(gt) \) is surjective and hence an isomorphism. \( \square \)

Corollary 20 \[ \] The grading of \( \mathcal{B} \) is balanced. The order of \( \text{Inn} X \) divides \( \dim \mathcal{B} \).

Proof In particular generates \( \text{Inn} X \). \( \square \)

Lemma 21 \[ \] \( \text{Inn} X \) is a quotient of \( \Phi \).

Proof By Proposition \[19\] the map \( \gamma := \deg \circ \text{ev}_1 : \Phi \to G \) is a surjective homomorphism. \( \square \)

Given a subset \( X' \) of \( X \), let \( \mathcal{B}' \) be the subalgebra-with-one of \( \mathcal{B} \) generated by \( \mathcal{K}X' \subseteq \mathcal{B}(1) \). If \( X' \) is a subrack of \( X \), then \( \mathcal{B}' \) is its corresponding Nichols subalgebra.

Proposition 22 \[ \] Let \( X' \) be a subrack of \( X \) and \( v \in \mathcal{B}' \).

1) If \( \partial_t v = 0 \) for all \( t \in X' \), then \( v \in \mathcal{B}(0) \).

2) If \( \partial^{\text{op}}_t v = 0 \) for all \( t \in X' \), then \( v \in \mathcal{B}(0) \).

Proof From \( v \in \mathcal{B}' \) we know \( \partial_s v = 0 = \partial^{\text{op}}_s v \) for all \( s \in X \setminus X' \), so \( v \) must be a multiple of \( 1 \). Note that (1) actually holds for arbitrary subsets \( X' \) of \( X \) when \( \mathcal{B}' \) is defined as the subalgebra generated by all \( e_t \) with \( t \in X' \); this is not true for (2). \( \square \)

Lemma 23 \[ \] Let \( X' \) be a non-empty proper subrack of \( X \) and \( G' \) the subgroup of \( \text{Inn} X \) spanned by the operations \( g_t : X \to X \) for all \( t \in X' \). Let \( V' \) be the linear span of the elements \( e_t \) with \( t \in X' \) and \( U \) the linear span of all \( e_s \) with \( s \in X \setminus X' \). Then holds:

1. \( G' \) is the smallest subgroup of \( G = \text{Inn} X \) with \( \delta(V') \subseteq \mathcal{K}G' \otimes V' \),
2. \( V' \) is \( G' \)-stable,
3. \( V' \) and \( U \) are \( \mathcal{K}G \)-subcomodules, and
4. \( U \) is a \( \mathcal{K}G' \)-submodule.

In particular, Graña’s Freeness Theorem (Theorem \[7\]) applies and we find

\[
\mathcal{B} \cong \left( \bigcap_{t \in X'} \ker \partial_t \right) \otimes \mathcal{B}',
\]

where \( \mathcal{B}' \) is the Nichols sub-algebra generated by \( X' \).
Proof (1) holds by definition of $G'$. (2) holds because $X'$ is a subrack. (3) is due to the diagonal comodule structure $\delta(e_t) = g_t \otimes e_t$ for all $t \in X$. To show (4), let $t \in X'$ and $s \in X \setminus X'$ be arbitrary. Then $g_t(e_s)$ is a multiple of $e_{ts}$. The element $t \triangleright s$ cannot be in $X'$ (otherwise $s$ would be in $X'$), so $g_t(e_s) \in U$. 

Freeness theorems as Grana’s allow for recursion, such that $\mathcal{B}$ can be written as tensor product of terms of the form $\bigcap_{t \in X'} \ker \partial_t$ for ever decreasing subracks $X'$. Such a factorization induces a factorization of the Hilbert series as well. In the case of the related Fomin-Kirillov algebras, the analogous factorization has been conjectured in \[16\], Conjecture 8.6, and has been proven by Fomin and Procesi in \[8\]. Their factors are subalgebras generated by transpositions $(i, n)$ for fixed $n$ and $1 \leq i < n$. This corresponds to the intersection $\bigcap_{t \in X'} \ker \partial_t$ if $X$ is the rack generated by the transpositions in $S_n$ and $X'$ the subrack generated by the transpositions of $S_{n-1} < S_n$. The subalgebra generated by $X \setminus X'$ is a subspace of $\bigcap_{t \in X'} \ker \partial_t$, but not necessarily all of it.

Proposition 24 Let $X'$ be a non-empty subset of $X$. Let $G = \text{Env } X$ or any quotient of $\text{Env } X$. Then $\bigcap_{t \in X'} \ker \partial_t$ is a $G$-homogeneous basis.

Proof Each $\partial_t$ is a $G$-graded map, hence $\ker \partial_t$ is a $G$-graded sub-$\mathcal{B}$-module of $\mathcal{B}$; same for their intersection $\bigcap_{t \in X'} \ker \partial_t$. Each graded submodule has a homogeneous basis, see e.g. section 2.1 in \[19\]. \qed

4 The Modified Shift Groups

Each shift $\phi_t$ can be written in the form

$$\phi_t(v) = \frac{1}{(1 + q_t)(1 + q_t + q_t^2) \cdots (1 + q_t + \cdots + q_t^{m-2})} \cdot \partial_t^{m-1}v + v e_t$$

by inserting the decomposition of $v$ into the right-hand side. We modify this definition by removing the leading factor and subtracting the braided commutator for one variant; and by issuing the opposite braided derivative for another.

Definition 25 Let $\mathcal{B}$ be a Nichols algebra with rack $X$ and $t \in X$ arbitrary. Define the modified shifts

$$\psi_t(v) := \partial_t^{m-1}(v) + e_t g_t^{-1}(v)$$

and

$$\xi_t(v) := (\partial_t^{\op})^{m-1}(v) + e_t v$$

and the corresponding modified shift groups $\Psi$ and $\Xi$, generated by all $\psi_t$ (respectively $\xi_t$) with $t \in X$. If $X'$ is a subset of $X$, define $\Psi|_{X'}$ and $\Xi|_{X'}$ to be the groups generated by all $\psi_t$ (respectively $\xi_t$) with $t \in X'$.

Proposition 26 Let $s, t \in X$ and $g \in G$ be arbitrary, where $G$ is any quotient of $\text{Env } X$. Then holds:

1. $\psi_t$ is a linear isomorphism.
2. If $t^m = e$ in $G$, then $\psi_t$ maps $B(g)$ to $B(gt)$.

3. If $t \neq s$, then $\psi_t(\ker \partial_s^{op}) = \ker \partial_t^{op}$.

4. The $\Psi$-orbit of 1 linearly spans $B$.

**Proof** 1. Linearity is obvious. Now assume $v \in \ker \psi_1 \setminus \{0\}$. Then $\partial_t^{m-1}(v) = -e_t g_t^{-1}(v)$. Use Lemma [15] to decompose $v = \sum_{j=0}^{m-1} v_j e_j$ of $v \in \ker \partial_t$. Inserting this yields $v_{m-1} = \sum \lambda_j e_t g_t^{-1}(v_j) e_j$ for some $\lambda_j \in \mathbb{K} \setminus \{0\}$. Using the braided commutator, we find

$$e_t g_t^{-1}(v_j) e_j = v_j e_j^{+1} + [e_t, g_t^{-1}(v_j)] e_j \in \ker \partial_t$$

by Propositions [13] and [14]. We know $v_{m-1} \in \ker \partial_t$, and by comparing the coefficients of $e_j$, we find:

$$v_{m-1} = \lambda_0 [e_t, g_t^{-1}(v_0)] e$$
$$v_{j-1} = - [e_t, g_t^{-1}(v_j)] e$$ \quad if 1 \leq j \leq m-1

In particular, the minimal length of the $\mathbb{N}_0$-homogeneous components of $v_{j-1}$ is at least one plus the minimal length of $v_j$, and the minimal length of $v_{m-1}$ is at least one plus the minimal length of $v_0$. This cannot be, hence all $v_j$ are zero.

2. Let $v \in B(g) \setminus \{0\}$ be arbitrary. Then the degree of $\partial_t^{m-1}(v) = gt^{1-m} = gt$ and the degree of $e_t g_t^{-1}(v)$ also is $tt^{-1} gt = gt$. Thereby, $\psi_t(v)$ is homogeneous of degree $gt$.

3. Let $v \in \ker \partial_s^{op}$ be arbitrary. From Proposition [11] we conclude that $\partial_t^{m-1}(v) \in \ker \partial_s^{op}$. For the other summand, we find

$$\partial_s^{op}(e_t g_t^{-1}(v)) = q e_t \partial_s^{op} g_t^{-1}(v).$$

$v$ is in $\ker \partial_s^{op}$, so $g_t^{-1}(v)$ is in $\ker \partial_s^{op} g_t^{-1}$. s.

4. We may use the same induction as in the proof of Lemma [18]. We write

$$e_t v = \psi_t(g_t v) - \partial_t^{m-1}(g_t v)$$

and note that both summands on the right hand side are in the linear span of $\Phi(1)$.

Property (3) above is something not found in the shifts introduced in Proposition [17]. In the Nichols algebra $B(Q_{3,1}, -1)$ choose $v = e_3$. Then $v \in \ker \partial_3^{op}$, but $\phi_2(v) = e_3 e_2 \notin \ker \partial_3^{op}$, while $\psi_2(v) = -e_2 e_1 \in \ker \partial_3^{op}$. This justifies the introduction of the modified shifts.

**Proposition 27** .................................................. 27
Let $s, t \in X$ and $g \in G$ be arbitrary, where $G$ is any quotient of $\text{Env } X$. Then holds:

1. $\xi_s$ is a linear isomorphism.

2. If $s^m = e$ in $G$, then $\xi_s$ maps $B(g)$ to $B(sg)$.

3. If $t \neq s$, then $\xi_s(\ker \partial_t) = \ker \partial_t$.

4. The $\Xi$-orbit of 1 linearly spans $B$. 

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Proof The proof is very similar to the proof of Proposition 26 but differs in details.

1. Again, linearity is obvious. Let \( v \in \ker \xi_s \setminus \{0\} \) be arbitrary and use the right-hand decomposition in Lemma 15, 
\[
v = \sum_{j=0}^{m-1} e_j^s v_j
\] 
with \( v_j \in \ker \partial_{op} s \). Inserting this into \( \xi_s(v) = 0 \) yields 
\[
v_{m-1} = \sum_{j=0}^{m-2} \mu_j e_j^{s+1} v_j
\] 
for some \( \mu_j \in K \setminus \{0\} \). Due to the linear independence in the decomposition in Lemma 15, we conclude 
\( v_j = 0 \) for all \( j \) and thus \( v = 0 \).

2. Straightforward, see Proposition 26.

3. Let \( v \in \ker \partial_t \) be arbitrary. Due to Proposition 11 \( (\partial_{op})^{m-1} \) and \( \partial_t \) commute, so \( (\partial_{op})^{m-1}(v) \in \ker \partial_t \). The other summand vanishes due to \( \partial_t(e_s v) = e_s \partial_t(v) = 0 \).

4. Analog to Proposition 26 \( \square \)

We will mainly use the shifts \( \xi_t \) in the following, due to the special form of Grañas Freeness Theorem we are using.

Lemma 28 
Let \( X' \) be a non-empty subset of \( X \). Let \( G \) be a quotient of \( \text{Env} X \) with \( s^{m} = e \) for all \( s \in X \setminus X' \), such that \( X \setminus X' \) still generates \( G \). Then 
\[
\bigcap_{t \in X'} \ker \partial_t
\] 
is \( G \)-balanced.

Proof By Proposition 24, \( K := \bigcap_{t \in X'} \ker \partial_t \) has a \( G \)-homogeneous basis. For any \( s \in X \setminus X' \) and \( g \in G \), \( \xi_s \) will map \( K \cap \mathfrak{B}(g) \) to \( K \cap \mathfrak{B}(sg) \) by Proposition 27, so \( \dim(K \cap \mathfrak{B}(g)) = \dim(K \cap \mathfrak{B}(sg)) \). The Lemma now follows from the assumption that \( G \) is generated by \( X \setminus X' \). \( \square \)

Theorem 29 
Assume \( n \mid m \). Let \( X' \) be a non-empty proper subrack of \( X \), and assume that \( X \setminus X' \) still generates \( \text{Inn} X \). Set \( \mathfrak{B}' \) to be the sub-Nichols algebra generated by \( X' \). Then holds 
\[
\dim \mathfrak{B} = \# \text{Inn} X \cdot \dim \mathfrak{B}' \cdot \dim \left( \mathfrak{B}(e) \cap \bigcap_{t \in X'} \ker \partial_t \right). \quad (10)
\]

Proof The proof follows directly from Grañas Freeness Theorem by applying Lemma 28 to Lemma 29 (note that \( g^m_n = e \) holds for the inner group if and only if \( n \mid m \)). \( \square \)

Assume \( X \) is indecomposable and consists of at least three elements (there is no indecomposable quandle of two elements). Choose \( X' = \{t\} \subseteq X \), thus \( \mathfrak{B}' \cong K[t]/t^n \). Due to irreducibility, there must exist \( r, s \in X \setminus X' \) with \( r \triangleright s = t \), so \( g_t = g_r g_s g_r^{-1} \) and \( \text{Inn} X \) is generated by \( X \setminus X' \). Applying Theorem 29 we find \( \# \text{Inn} X \cdot n \mid \dim \mathfrak{B} \).

Example 30
From the examples of subsection 2.3, four Nichols algebras fulfill \( n \mid m \):

| \( \mathfrak{B} \) | \( m \) | \( \dim \mathfrak{B} \) | \( \#G \) | \( \mathfrak{B}' \) | \( \#G \cdot \dim \mathfrak{B}' \) | \( \dim \mathfrak{B} \cdot \#G / \dim \mathfrak{B}' \) |
|---|---|---|---|---|---|---|
| \( \mathfrak{B}(q_{3,1}, -1) \) | 2 | 12 | 6 | \( \mathbb{K}[t]/t^2 \) | 12 | 1 |
| \( \mathfrak{B}(q_{4,1}, q_{4}) \) | 3 | 5,184 | 12 | \( \mathbb{K}[t]/t^3 \) | 36 | 144 |
| \( \mathfrak{B}(q_{6,1}, -1) \) | 2 | 576 | 24 | \( \mathfrak{B}(q_{3,1}, -1) \) | 288 | 2 |
| \( \mathfrak{B}(q_{10,1}, -1) \) | 2 | 8,294,400 | 120 | \( \mathfrak{B}(q_{6,1}, -1) \) | 69,120 | 120 |

where we use \( G = \text{Inn} X \).

It is still unclear, whether \( \mathfrak{B}(q_{15,7}, -1) \) (the Nichols algebra of the transpositions in the symmetric group \( S_6 \) with constant cocycle \(-1\)) is finite-dimensional or not. If it is, its dimension must be divisible by

\[
\#S_6 \cdot \dim \mathfrak{B}(q_{10,1}, -1) = 720 \cdot 8,294,400 = 5,971,968,000.
\]

Taking a look at the quotients \( \frac{\dim \mathfrak{B}}{\#G \cdot \dim \mathfrak{B}'} \) in the above table, one might guess that \( \dim \mathfrak{B}(q_{15,7}, -1) \) will probably be at least another factor 720 larger, and thus divisible by 4,299,816,960,000.

5 General Case

We remember from Lemma 21 that \( \text{Inn} X \) is a quotient of the shift group \( \Phi \) if \( n \mid m \). Moreover, this induces a \( G \)-grading on \( \mathbb{K}\Phi \) (which is balanced if \( \mathbb{K}\Phi \) is finite-dimensional), which in turn induces a balanced \( G \)-grading on \( \mathfrak{B} \). One might ask, how to generalize this idea to the case \( n \nmid m \).

Let \( ev_1 : \mathbb{K}\Phi \to \mathfrak{B} \) be the evaluation at 1 \( \in \mathfrak{B} \). From Lemma 18 we know that \( ev_1 \) is a surjective linear map. \( ev_1 \) is neither an algebra homomorphism, nor is its kernel an ideal of \( \mathbb{K}\Phi \); still, there is an identification of \( \mathbb{K}\Phi / \ker ev_1 \) and \( \mathfrak{B} \) as linear spaces. Assume there is a surjective homomorphism \( \pi : \Phi \to G \) to some finite quotient \( G \) of \( \text{Env} X \). Define \( \Phi_g := \pi^{-1}(g) \) and \( U_g := \mathbb{K}\Phi_g \subseteq \mathbb{K}\Phi \), such that \( \mathbb{K}\Phi = \bigoplus_{g \in G} U_g \). Choose a system of representatives \( \phi_g \in U_g \setminus \{0\} \) and define the translations \( \tau_g : \mathbb{K}\Phi \to \mathbb{K}\Phi, \phi \mapsto \phi_g \phi \). Each \( \tau_g \) is a linear isomorphism of \( \mathbb{K}\Phi \) and \( \tau_g(U_h) = U_{gh} \) for each \( g, h \in G \). Now assume \( \phi \in \ker ev_1 \). Then \( \tau_g(\phi)(1) = \phi_g(\phi(1)) = 0 \), hence \( \tau_g(\ker ev_1) = \ker ev_1 \). We may therefore define linear maps \( \tau_g : \mathbb{K}\Phi / \ker ev_1 \to \mathbb{K}\Phi / \ker ev_1 \) with \( \tau_g(U_h / \ker ev_1) = U_{gh} / \ker ev_1 \). Obviously, we have \( \mathbb{K}\Phi / \ker ev_1 = \sum_{g \in G} U_g / \ker ev_1 \). Assume this sum is direct. Then the isomorphisms \( \tau_g \) show that this grading is balanced, and hence \( \#G \) divides \( \dim \mathfrak{B} \). So there currently are two open questions to transfer the results of Subsection 3.1 to the general case:

1. Is \( G \) a quotient of \( \Phi \)?
2. Is the sum \( \sum_{g \in G} U_g / \ker ev_1 \) direct?

We will now concentrate on \( G = C_k \), which is a quotient of \( \text{Env} X \) by \( t \mapsto [1]_k \in C_k \) for all \( t \in X \).

5.1 Factors in the Hilbert Series

Each Nichols Algebra \( \mathfrak{B} \) is \( \mathbb{Z} \)-graded. Taking quotients, we find \( C_k \)-gradings of \( \mathfrak{B} \) for each \( k > 1 \).
Lemma 31. \( \mathcal{B} \) is \( C_m \)-balanced. In particular, we have for each \([k]_m \in C_m\)

\[
\sum_{j \in \mathbb{N}_0} \dim \mathcal{B}(j) = \frac{1}{m} \dim \mathcal{B}.
\]

\( j \equiv k \pmod{m} \).

Proof. Let \( t \in X \) be arbitrary. We may define \( \phi_t \) as in Proposition 17 (or, equivalently, any of the shifts of Definition 25) and find that \( \phi_t \) maps \( \mathcal{B}(j) \) to \( \mathcal{B}(j+1) \oplus \mathcal{B}(j-m+1) \). Hence, \( \phi_t \) is a linear isomorphism between \( \mathcal{B}([j]_m) \) and \( \mathcal{B}([j+1]_m) \).

Clearly, if \( j \mid k \) and \( \mathcal{B} \) is \( C_k \)-balanced, then \( \mathcal{B} \) is \( C_j \)-balanced as well. The following table shows for some Nichols algebras \( \mathcal{B} \) those \( k > 1 \) such that \( \mathcal{B} \) is \( C_k \)-balanced.

| \( \mathcal{B} \)       | \( n \) | \( m \) | \( \dim \mathcal{B} \) | \( C_k \)-balanced for ... |
|------------------------|--------|--------|------------------------|-------------------------|
| \( \mathbb{K}[t]/t^m \) | 1      | \( m \) | \( k \mid m \)         |                         |
| \( \mathcal{B}(Q_{3,1},-1) \) | 2      | 2      | \( 12 \) | \( k = 2, 3 \)         |
| \( \mathcal{B}(Q_{4,1},\mathbb{E}_3)(2) \) | 2      | 3      | 432       | \( k = 2, 3, 4, 6 \)    |
| \( \mathcal{B}(Q_{4,1,-1})(2) \) | 3      | 2      | \( 36 \) | \( k = 2, 3 \)         |
| \( \mathcal{B}(Q_{4,1,-1})(\neq 2) \) | 3      | 2      | \( 72 \) | \( k = 2, 3, 6 \)      |
| \( \mathcal{B}(Q_{4,1},\chi_4) \) | 3      | 3      | 5,184     | \( k = 2, 3, 4, 6 \)    |
| \( \mathcal{B}(Q_{5,\ast},-1) \) | 4      | 2      | \( 1,280 \) | \( k = 2, 4, 5 \)      |
| \( \mathcal{B}(Q_{6,\ast}) \) | 2/2/4 | 2      | \( 576 \) | \( k = 2, 3, 4 \)      |
| \( \mathcal{B}(Q_{7,\ast},-1) \) | 6      | 2      | 326,592   | \( k = 2, 3, 6, 7 \)    |
| \( \mathcal{B}(Q_{10,1,\ast}) \) | 2      | 2      | \( 8,294,400 \) | \( k = 2, 3, 4, 5, 6 \) |

Lemma 32. \( \mathcal{B} \) is \( C_k \)-balanced (as a quotient of its \( \mathbb{Z} \)-grading) if and only if \( (k)_t := \sum_{j=0}^{k-1} t^j \) is a divisor of the Hilbert series \( \mathcal{H}_\mathcal{B}(t) \) of \( \mathcal{B} \), such that the quotient polynomial has integer coefficients only.

Proof. To simplify notation, if \( p \) is a polynomial, set \( p_j \) to be the coefficient of \( t^j \) in \( p(t) \) (or zero if \( j < 0 \)) and \( b_j := (\mathcal{H}_\mathcal{B})_j = \dim \mathcal{B}(j) \) for any \( j \in \mathbb{N}_0 \).

We will show that \( \mathcal{B} \) is \( C_k \)-balanced by assuming that \( \sum_{j=0}^{k-1} b_j \cdot p_{j-k} \) is a divisor of \( \mathcal{H}_\mathcal{B}(t) \) for each \( 0 \leq l \leq k-1 \). A telescoping sum is needed to show that this implies all \( \mathcal{B} \) is \( C_k \)-balanced.

\[
\sum_{0 \leq j \leq d} b_{j+k+l} - \sum_{0 \leq j \leq d} b_{j+k+l-1} = \sum_{0 \leq j \leq d} (p_{j+k+l} - p_{j+k+l-1}) = -p_{l-k} + p_{d+k+l}.
\]

\( \mathcal{B} \) is \( C_k \)-balanced by assumption, so the two sums on the left hand side must sum to the same value (namely \( \frac{1}{k} \dim \mathcal{B} \)). \( p_{l-k} \) is zero by definition \( (l - k < 0) \), hence \( p_{d+k+l} \) is zero as well. This shows that \( p \) actually is a polynomial, and by definition its coefficients are integers. From \( b_j = \sum_{i=0}^{k-1} p_{j-i} \), we also see \( \mathcal{H}_\mathcal{B}(t) = (k)_t \cdot p(t) \).
“⇐”: Let \( \mathcal{H}_\mathfrak{B}(t) = (k)_1 \cdot p(t) \) for some polynomial \( p \) with integer coefficients. We have \( b_j = \sum_{i=0}^{k-1} p_{j-i} \) and therefore for each \( \lfloor l \rfloor_k \in C_k \)

\[
\dim \mathfrak{B}(\lfloor l \rfloor_k) = \sum_{j \in \mathbb{N}_0, j \equiv l \pmod{k}} \dim \mathfrak{B}(j) = \sum_{j \in \mathbb{N}_0, j \equiv l \pmod{k}} \sum_{i=0}^{k-1} p_{j-i} = \sum_{j \in \mathbb{N}_0} p_j,
\]

which does not depend on \( \lfloor l \rfloor_k \), so \( \mathfrak{B} \) is \( C_k \)-balanced.

From Lemmas \( 31 \) and \( 32 \) follows that \( (m)_l \) is a divisor of the Hilbert series of \( \mathfrak{B} \). This result is well-known and can be seen directly from any of the Freeness Theorems applied to a trivial subrack.

**Theorem 33**

Let \( \mathfrak{B} \) be a finite-dimensional Nichols algebra over a rack \( X \) and a 2-cocycle of order \( m \). Let \( X' \) be a non-empty proper subrack of \( X \) and \( \mathfrak{B}' \) its corresponding Nichols sub-algebra of \( \mathfrak{B} \). Then the Hilbert series \( \mathcal{H}_\mathfrak{B}(t) \) is divisible by \( (m)_l \cdot \mathcal{H}_{\mathfrak{B}'}(t) \).

**Proof** We use the notation of Lemmas \( 31 \) and \( 32 \). Let \( t \in X \) be arbitrary. We have seen in the proof of Lemma \( 31 \) that \( \phi_t \) is a linear isomorphism between \( \mathfrak{B}(\langle j \rangle_m) \) and \( \mathfrak{B}(\langle j+1 \rangle_m) \) for all \( j \); so is \( \xi_t \) from Proposition \( 27 \).

Applying the same techniques of the proof of Theorem \( 29 \) to the quotient \( C_m \)-grading yields the proposition.

**Corollary 34**

Let \( \mathfrak{B} \) be a finite-dimensional Nichols algebra over an indecomposable rack \( X \) with \( \#X \geq 3 \) and a 2-cocycle of order \( m \). Then its Hilbert series \( \mathcal{H}_\mathfrak{B}(t) \) is divisible by \( (m)_l \).

**Example 35**

We know that there is a sequence of embeddings of quandles

\[
\{t\} \hookrightarrow Q_{3,1} \hookrightarrow Q_{6,1} \hookrightarrow Q_{10,1}
\]

associated to the Nichols-algebra-embeddings

\[
\mathbb{K}[t]/t^2 \hookrightarrow \mathfrak{B}(Q_{3,1},-1) \hookrightarrow \mathfrak{B}(Q_{6,1},-1) \hookrightarrow \mathfrak{B}(Q_{10,1},-1).
\]

In addition to this, one easily sees that \( Q_{6,1} \setminus Q_{3,1} \) still generates \( \text{Inn} \ Q_{6,1} \) and \( Q_{10,1} \setminus Q_{6,1} \) still generates \( \text{Inn} \ Q_{10,1} \). Applying Theorem \( 29 \) three times now shows that \( (2)_l^3 \) is a factor of \( \mathcal{H}_\mathfrak{B}(Q_{10,1},-1)(t) \). Following Example \( 37 \) we conclude that \( (2)_l^5 \) must be a factor of \( \mathcal{H}_\mathfrak{B}(Q_{15,7},-1)(t) \), if this Nichols algebra is finite dimensional.

### 5.2 The 72-dimensional Nichols Algebra

We now concentrate on one example with \( n \nmid m \), the 72-dimensional Nichols algebra \( \mathfrak{B}(Q_{4,1},-1)^{(\#2)} \) first introduced in \( 10 \).

Let \( X = (\{1,2,3,4\}, \triangleright) \) be the quandle with operation

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 3 | 4 | 2 |
| 2 | 4 | 2 | 1 | 3 |
| 3 | 2 | 4 | 3 | 1 |
| 4 | 3 | 1 | 2 | 4 |
Choose $\dim V_0 = 1$ and as cocycle choose the constant cocycle $\chi = -1$ over any field $K$ of characteristic $\neq 2$. The resulting Nichols Algebra $\mathcal{B}$ has dimension 72 \([10]\), a possible basis is given by the following products, written in syntax notation:

$$
[\ e_1\ ]\  \ [\ e_2\ [\ e_1\ ]\ ]\  \ [\ e_3\ e_2\ e_1\ ]\  \ [\ e_3\ [\ e_2\ ]\ ]\  \ [\ e_4\ ]
$$

(each argument in square brackets is optional). Its relations are generated by the relations

\[
\begin{align*}
0 &= e_t^2 \quad \forall t \in X \\
0 &= e_r e_s + e_s e_t + e_t e_r \\
0 &= (e_3 e_2 e_1)^2 + (e_2 e_1 e_3)^2 + (e_1 e_3 e_2)^2
\end{align*}
\]

The inner group of $X$ is isomorphic to the alternating group $A_4$. With respect to this grading, $\mathcal{B}$ is not balanced:

| $g \in A_4$ | $\dim \mathcal{B}(g)$ |
|-------------|-----------------|
| $()$        | 12              |
| $(1,3)(2,4)$| 4               |
| $(1,4)(2,3)$| 4               |
| $(1,2)(3,4)$| 4               |

| $g \in A_4$ | $\dim \mathcal{B}(g)$ |
|-------------|-----------------|
| $(1,2,3)$   | 6               |
| $(1,3,4)$   | 6               |
| $(1,4,2)$   | 6               |
| $(2,3,4)$   | 6               |

| $g \in A_4$ | $\dim \mathcal{B}(g)$ |
|-------------|-----------------|
| $(1,3,2)$   | 6               |
| $(1,4,3)$   | 6               |
| $(1,2,4)$   | 6               |
| $(2,4,3)$   | 6               |

(Elements in cycle notation; calculations have been performed with Rig, see \([12]\). As one sees, the dimension is preserved by conjugation; this is due to the operation of $\text{Env} \ X$ on $\mathcal{B}$, which conjugates the grading. The grading of $\mathcal{B}$ with respect to the enveloping group $\text{Env} \ X$ of $X$ must be unbalanced, because $\text{Env} \ X$ is infinite. Indeed, the two elements $g_1g_2g_1$ and $g_2g_3g_2g_1g_3g_4 \in \text{Env} \ X$ fulfill $\dim \mathcal{B}(g) = 5$, eight elements have $\dim \mathcal{B}(g) = 3$, another eight elements, 22 elements have $\dim \mathcal{B}(g) = 1$ (including the identity element) and the remaining elements 0. One would therefore ask, whether there is a quotient $G$ of $\text{Env} \ X$, such that $\mathcal{B}$ is $G$-balanced and $G$ is large enough to have $\text{Inn} \ X$ as a quotient itself. However:

**Proposition 36**

There is no quotient $G$ of $\text{Env} \ X$, such that $\text{Inn} \ X$ is a quotient of $G$ and $\mathcal{B}$ is $G$-balanced.

**Proof** Taking a quotient cannot lower the dimensions of the grade. For $g = g_1g_2g_1 \in \text{Env} \ X$ we therefore find, that the dimension of the grade of the image of $g$ under the canonical projection $\text{Env} \ X \to G$ (and hence for each element in $G$) must be at least 5. By hypothesis, $\text{Inn} \ X$ is a quotient of $G$ and therefore fulfills $\dim \mathcal{B}(g) \geq 5$ for each $g \in \text{Inn} \ X$: Contradiction. $\Box$

**Proposition 37**

The shift group $\Phi$ of $\mathcal{B}(Q_{4,1}, -1)^{(\neq 2)}$ is infinite in characteristic 0.

**Proof** The endomorphism $\phi_1 \phi_2$ has a Jacobi normal form with eight blocks of each of the three types

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad
\begin{pmatrix}
\rho & 1 & 0 \\
0 & \rho & 1 \\
0 & 0 & \rho \\
\end{pmatrix}, \quad
\begin{pmatrix}
\bar{\rho} & 1 & 0 \\
0 & \bar{\rho} & 1 \\
0 & 0 & \bar{\rho} \\
\end{pmatrix}
$$

19
where \( \rho \) and \( \bar{\rho} \) are different third roots of unity. From this decomposition one sees that \( \phi_1 \phi_2 \) has infinite order if \( K \) is of characteristic 0, and thus \( \langle \phi_1, \phi_2 \rangle \cong C_2 \ast C_2 \).

\( \square \)

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**References**

[1] N. Andruskiewitsch and M. Graña. From Racks to Pointed Hopf Algebras. In *Advances in Mathematics*, volume 178 (2003), no. 2, pages 177–243.

[2] N. Andruskiewitsch, F. Fantino, M. Graña, L. Vendramin. Finite-dimensional pointed Hopf algebras with alternating groups are trivial. In *Annali di Matematica Pura ed Applicata (4)*, volume 190 (2011), no. 2, pages 225–245. Available at [math.QA/0812.4628](http://arxiv.org/abs/math.QA/0812.4628).

[3] N. Andruskiewitsch, F. Fantino, G. A. García, L. Vendramin. On Nichols algebras associated to simple racks. In *Contemporary Mathematics*, volume 537 (2011), pages 31–56. Available at [http://arxiv.org/abs/1006.5727](http://arxiv.org/abs/1006.5727).

[4] N. Andruskiewitsch, F. Fantino, M. Graña, L. Vendramin. Pointed Hopf algebras over the sporadic simple groups. In *Journal of Algebra*, volume 325 (2011), no. 1, pages 305–320. Available at [http://arxiv.org/abs/1001.1108](http://arxiv.org/abs/1001.1108).

[5] N. Andruskiewitsch, I. Heckenberger, H.-J. Schneider. The Nichols algebra of a semisimple Yetter-Drinfeld module. In *American Journal of Mathematics*, volume 132 (2010), no. 6, pages 1493–1547.

[6] N. Andruskiewitsch and H.-J. Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 1–68. Cambridge Univ. Press, Cambridge, 2002.

[7] S. Dăscălescu, C. Năstăsescu, Ş. Raianu Hopf algebras: an introduction. *Pure and Applied Mathematics*, volume 235, Marcel Dekker, Inc., 2001.

[8] S. Fomin, C. Procesi. Fibered Quadratic Hopf Algebras Related to Schubert Calculus. In *Journal of Algebra*, volume 230 (2000), pages 174–183.

[9] M. Graña. A freeness theorem for Nichols algebras. In *Journal of Algebra*, volume 231 (2000), no. 1, pages 235–257. Available at [http://mate.dm.uba.ar/~matiasg/](http://mate.dm.uba.ar/~matiasg/).

[10] M. Graña. On Nichols algebras of low dimension. In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 111–136. Amer. Math. Soc., Providence, RI, 2000. Available at [http://mate.dm.uba.ar/~matiasg/](http://mate.dm.uba.ar/~matiasg/).

[11] M. Graña, I. Heckenberger, L. Vendramin. Nichols algebras of group type with many quadratic relations. In *Advances in Mathematics*, volume 227 (2011), no. 5, pages 1956–1989.

[12] M. Graña and L. Vendramin. Rig, A GAP package for racks and Nichols Algebras. Available at [http://code.google.com/p/rig/](http://code.google.com/p/rig/).
[13] I. Heckenberger. Nichols algebras of diagonal type and arithmetic root systems. Habilitation thesis 2005, available at http://www.mathematik.uni-marburg.de/~heckenberger.

[14] I. Heckenberger. Classification of arithmetic root systems. In Advances in Mathematics, volume 220 (2009), pages 59–124, available at http://www.mathematik.uni-marburg.de/~heckenberger.

[15] I. Heckenberger, A. Lochmann, L. Vendramin. Braided racks, Hurwitz actions and Nichols algebras with many cubic relations. In Transformation Groups, volume 17 (2012), no. 1, 157–194.

[16] A. N. Kirillov. On some quadratic algebras. In D. Faddeev’s Seminar on Mathematical Physics, Amer. Math. Soc. Transl. Ser, volume 2, 1997, available at q-alg/9705003.

[17] A. Milinski and H.-J. Schneider. Pointed indecomposable Hopf algebras over Coxeter groups. In New trends in Hopf algebra theory (La Falda, 1999), volume 267 of Contemp. Math., pages 215–236. Amer. Math. Soc., Providence, RI, 2000.

[18] W. Nichols. Bialgebras of type one. In Communications in Algebra, volume 6, no. 15, pages 1521–1552.

[19] C. Năstăsescu, F. Van Oystaeyen Methods of Graded Rings. Lecture Notes in Mathematics, volume 1836, Springer, 2004.

[20] W. D. Nichols, M. B. Zoeller. Finite-Dimensional Hopf Algebras are Free over Grouplike Subalgebras. In Journal of Pure and Applied Algebra, volume 56, pages 51–57, North-Holland, 1989.

[21] D. E. Radford. Pointed Hopf Algebras Are Free over Hopf Subalgebras. In Journal of Algebra, volume 45, pages 266–273, Academic Press, Inc., 1977.

[22] S. Skryabin. Projectivity and freeness over comodule algebras. In Transactions of the American Society, volume 359 (2007), pages 2597–2623.

[23] L. Vendramin. Nichols algebras associated to the transpositions of the symmetric group are twist-equivalent. Accepted for publication in the Proceedings of the American Mathematical Society.

[24] L. Vendramin. On the classification of quandles of low order. In Journal of Knot Theory Ramifications, volume 21 (2012), no. 9, 1250088.