Path tracking and backstepping control for a wheeled mobile robot (WMR) in a slipping environment

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Abstract. This work studies the motion of a two-wheeled differential drive mobile robot by presenting the kinematic and dynamic model of a nonholonomic wheeled mobile robot (WMR) with and without slip effect. The traction forces are derived and a control law designed using the backstepping method to drive the WMR to the desired trajectory. The parameters of the kinematic and dynamic controllers are tuned and modified to reach the desired performance. In addition, the input torque is controlled to avoid slip occurring. Simulation results show the effectiveness of the proposed control algorithm that is demonstrated by applying these controllers at different case studies (circular trajectory, elliptical trajectory, sinusoidal trajectory, infinity trajectory, and line trajectory), these results show good matching between desired trajectory and simulation, while the errors converge to zero rapidly. It shows good performance in minimizing the overshoot and reducing energy and the transient response time.

Table 1: Nomenclature

Table 1.a

| Abbreviation | Definition                  |
|--------------|-----------------------------|
| WMR          | Wheeled mobile robot        |
| PID          | Proportional integral derivative |
| FOPID        | Fractional order proportional integral derivative |
| GA           | Genetic algorithms          |

Table 1.b

| Subscripts | Definition                  |
|------------|-----------------------------|
| g          | Global frame                |
| r (up)     | Robot frame                 |
| R          | Pertaining to the right wheel |
| L          | Pertaining to the left wheel |
| o          | Pertaining to the center point between the driving wheels |
| c(down)    | Pertaining to the center of mass |
| w(down)    | Wheel                        |
| long       | Longitudinal                |
| lat        | Lateral                      |
| s          | Static                       |
| k          | Kinetic                      |
| d          | Desired                      |
Table 1.c

| Symbol | Description | units |
|--------|-------------|-------|
| OXY Z  | Main or global axis                    |       |
| oxy z  | Robot or local axis                    |       |
| \( \mathbf{v} \) | Liner velocity                        | m/s   |
| \( \omega \) | Angular velocity                      | rad/s |
| \( \beta \) | Angular position of the robot         | rad   |
| \( \dot{\gamma} \) | Angular speed of the wheels           | rad/s^2 |
| 2L     | Distance between wheels                | m     |
| b      | Distance from wheel’s axis to center of mass | m  |
| R      | Wheels radius                          | m     |
| M(q)   | Inertia matrix                         | Kg    |
| B(q)   | The input transformation matrix        |       |
| V(q)   | The centripetal and coriolis matrix    | m/s^2 |
| \( \mathbf{\tau} \) | The input torque vector                | N m   |
| \( \mathbf{A}^T(q) \) | The nonholonomic constraint matrix     |       |
| \( \lambda \) | The Lagrange multiplier vector        |       |
| \( \mathbf{E} \) | The Lagrange function which represents the kinetic energy of the system in this case | |
| \( q_i \) | The generalized coordinates            |       |
| \( \mathbf{F} \) | Force                                  | N     |
| \( \mathbf{a} \) | Acceleration                           | m/s^2 |
| \( \mathbf{k} \) | Gain                                   |       |
| \( \mathbf{m} \) | The mass of the platform               | Kg    |
| \( \mathbf{I} \) | The inertia of the platform about its rotational axis | Kg m^2 |
| \( \mathbf{m}_w \) | The mass of each driving wheel (with actuator) | kg |
| \( \mathbf{I}_c \) | The moment of inertia of the DDMR about the vertical axis through the center of mass | kg m^2 |
| \( \mathbf{I}_w \) | The moment of inertia of each driving wheel with a motor about the wheel axis | kg m^2 |
| \( \mathbf{I}_m \) | The moment of inertia of each driving wheel with a motor about the wheel diameter | kg m^2 |
| \( \mathbf{Z} \) | The longitudinal slip variable (total linear displacement of the wheel) | m     |
| \( \delta \) | The displacement loss due to slip      | m     |
| \( \rho \) | The lateral slip variable              | m     |
| \( \mu_k \) | Coefficient of kinetic friction       |       |
| \( \mu_s \) | Coefficient of static friction         |       |
1. Introduction

The wheeled mobile robot (WMR) has interested many researchers because it has wide application in unlimited fields, where it could be used with or without human action. Such fields include electrical, aerospace, mining transportation, under-water, inspection, medicine, military, and for use in place of people in harmful environments, to transport goods or find explosive materials.

Pure rolling has often been assumed in the literature as a condition of developing the kinematic and dynamic model of the differential drive WMR, because nonholonomic WMR are distinguished by no-slip constraints. But when the WMR moves on a slippery surface or at high speed, then slip is inescapable. Controllers should be used to drive the WMR to a desired trajectory and to ensure efficient performance.

There have been many studies of the WMR with and without slip, and studies of the trajectory tracking control such as Falsafi M.H., (2018) who introduced the kinematic model of a differential drive WMR with two active wheels and one castor wheel in the front. The trajectory tracking had been analyzed using fuzzy, fuzzy logic, and model predictive control methods. The controllers were designed to minimize tracking error. Results showed that the fuzzy controller generated the best performance [1].

Saleh A.L. et al, (2018) simulated the kinematic and dynamic models of WMR for a desired trajectory tracking, and presented an optimal controller [fractional order PID (FOPID)]. Results revealed minimization in the distance and the deviation angle [2]. Alouache A. et al, (2018) investigated the trajectory tracking of nonholonomic WMR. A PID controller was used to control the velocity and minimizing the tracking error. The genetic algorithm (GA) was used to improve the PID controller performance. Results showed that the GA.PID controller yielded better performance than the PID controller [3].

Tinh N.V. et al (2016) built a kinematic and dynamic model for a nonholonomic WMR with slip effect. The WMR had two active wheels and two passive wheels. A control law was designed by the input-output feedback linearization method, for the trajectory tracking. Results showed the effectiveness of the control law by converging the errors to zero [8]. Tian Y. et al (2013) modeled the WMR dynamics with wheel slip, and a discontinuous control was proposed for regulation control and a sliding mod control for sharp turning control. The results of simulation validated the theoretical results [13]. Sidek N. el, 2008 developed theoretical lateral slip dynamics of WMR, then a nonlinear feedback controller was designed for navigation efficiency. Results revealed the worth of the model and the control techniques [14].

The contribution of this work includes:

❖ Investigation of the kinematic and dynamic model of WMR with and without slip (longitudinal and lateral slip) and determination of the traction forces, taking into account the caster wheel effect.
❖ Design of a control law to track the desired trajectory using the backstepping method and creation of a control strategy to control the torque and prevent slip occurring.
❖ Building of a MATLAB and Simulink model (using MATLAB R2018a) to simulate the results.

2- Coordinate Systems

The position and orientation of WMR may be described by two coordinate systems (frames) which are the initial (global) coordinate system and the robot (local) coordinate system. The global frame is fixed in the plane of the robot movement and denoted as \( \{x^g, y^g, \beta^g \} \) or \( \{X, Y, Z\} \).
Y, β}, while the robot frame is attached to the robot and moves with it, this coordinate denoted as \( \{x^r, y^r, \beta\} \) or \( \{x, y, \beta\} \) as shown in Figure 1.

![Figure 1: Two-wheeled DDMR](image)

3- Modeling of DDWMR under pure rolling

3.1 Kinematic of the Differential-Drive Robot

To derive the kinematics of the WMR, there are two assumptions to obtain the non-holonomic condition should be known, which are as follows:

1- There is no slippage in wheels rolling which means that the robot can’t move sideward (only in a forward and backward curved motion); this means that the velocity at the mid-point o of the robot lateral axis is zero.

\[
\dot{y}_o^r = 0
\]  

(1)

2- Pure rolling constraint which means that in the robot frame there is no longitudinal slipping in the \( (x^r) \) axis, and no lateral skidding in the \( (y^r) \) axis.

Now according to the assumption of rigid body the linear velocity of the right wheel \( v_R \) and the left wheel \( v_L \) are as follows:

\[
v_R = v_O + L \dot{\beta}, \quad v_L = v_O - L \dot{\beta}
\]  

[15] (2.a,b)

Adding and subtracting equation (2.a) and (2.b) to get

\[
x_O^r = v = \frac{v_R + v_L}{2} = R \frac{\dot{y}_R + \dot{y}_L}{2}
\]  

[10] (3)

\[
\dot{\beta} = \omega = \frac{v_R - v_L}{2L} = R \frac{\dot{y}_R - \dot{y}_L}{2L}
\]  

[10] (4)

From Equations (3), and (4) the DDMR velocities in terms of the wheels velocities in the local frame can be written in matrix form as follows:

\[
\begin{bmatrix}
\dot{v} \\
\dot{\omega}
\end{bmatrix} = \begin{bmatrix}
\frac{R}{2} & \frac{R}{2} \\
\frac{R}{2L} & -\frac{R}{2L}
\end{bmatrix} \begin{bmatrix}
\dot{y}_R \\
\dot{y}_L
\end{bmatrix}
\]  

[5] (5)

Or an alternative form of the kinematic model can be written in terms of the linear and angular velocities in the global coordinates as follows:
\[
\dot{q} = \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\beta}
\end{bmatrix} = \begin{bmatrix}
\cos \beta & 0 & v \\
\sin \beta & 0 & \omega \\
0 & 1 & 0
\end{bmatrix}
\]

This leads to the following relation: \( \dot{x} \sin \beta = \dot{y} \cos \beta \)

\[ (7) \]

Now from Equations (5) and (6) the constraint equations are as follows

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\beta}
\end{bmatrix} = \begin{bmatrix}
\cos \beta & \sin \beta & 0 \\
-\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\beta}
\end{bmatrix}
\]

Then the pure constraint equations in term of the center of rotation \(o\) as follows:

\[
R \dot{y}_R = \dot{X}_o \cos \beta + \dot{Y}_0 \sin \beta + L \dot{\beta}
\]

\[
R \dot{y}_L = \dot{X}_o \cos \beta + \dot{Y}_0 \sin \beta - L \dot{\beta}
\]

\[ 0 = -\dot{X}_o \sin \beta + \dot{X}_0 \cos \beta \]

\[ (9) \]

### 3.2- Dynamic Modeling of the DDMR

The dynamic equation of motion for non-holonomic system could be investigated by assuming there is no friction and no unknown disturbance, on the other hand due to the robot’s movement on the horizontal plane the gravitational vector and the potential energy will be equal to zero, so the equation of motion becomes:

\[ M(q) \ddot{q} = B(q) \tau + A^T(q) \lambda \]

\[ (10) \]

Now the Lagrange dynamic approach can be used to derive the dynamic equation of motion which can be described in the following form:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \frac{\partial \mathcal{L}}{\partial q} = B(q) \tau + A^T(q) \lambda
\]

\[ (11) \]

\[ \mathcal{L} = \frac{1}{2} m \dot{v}_o^2 + \frac{1}{2} I \dot{\beta}^2 \]

\[ (12) \]

Where the general velocity equation in the global frame \( v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 \)

\[ (13) \]

\[
M(q) = \begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
\dot{x}_o \\
\dot{y}_o \\
\dot{\beta}
\end{bmatrix} = \frac{1}{R} \begin{bmatrix}
\cos \beta & \cos \beta \\
\sin \beta & \sin \beta \\
L & -L
\end{bmatrix} \begin{bmatrix}
\tau_R \\
\tau_L
\end{bmatrix} + \begin{bmatrix}
\sin \beta \\
- \cos \beta \\
0
\end{bmatrix} \lambda \]

\[ (14) \]

Then, after eliminating the Lagrange multiplier the dynamic equations referred to the center of rotation \(o\) of the robot are as follows:

\[ \dot{v} = \frac{\tau_1}{m}, \quad \dot{\omega} = \frac{\tau_2}{m} \]

\[ (15,a,b) \]

\[ \text{Where: } \tau_1 = \frac{\tau_R + \tau_L}{R}, \quad \tau_2 = \frac{2L(\tau_R - \tau_L)}{R} \]

\[ (16,a,b) \]

Note that these dynamic equations are the equations used to design the motor controller depending on the torque.
Now by including the wheels of robot in the dynamic model and considering the center of mass as the position point, then the dynamic equation of motion can be written in the following form:

\[ M(q)\ddot{q} + V(q, \dot{q}) \dot{q} = B(q)\tau + A^T(q)\lambda. \]  

\[
M(q) = \begin{bmatrix}
m_t & 0 & -mb \sin \beta & 0 & 0 \\
0 & m_t & mb \cos \beta & 0 & 0 \\
-mb \sin \beta & mb \cos \beta & I_t & 0 & 0 \\
0 & 0 & 0 & I_w & 0 \\
0 & 0 & 0 & 0 & I_w \\
\end{bmatrix},
\]

\[
V(q, \dot{q}) = \begin{bmatrix}
0 & 0 & -mb \dot{\beta} \cos \beta & 0 & 0 \\
0 & 0 & -mb \dot{\beta} \sin \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad B(q) = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}.
\]

\[
A^T(q) \lambda = \begin{bmatrix}
-\sin \beta & \cos \beta & \cos \beta \\
\cos \beta & -\sin \beta & \sin \beta \\
0 & L & -L \\
0 & -R & 0 \\
0 & 0 & -R \\
\end{bmatrix} \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5 \\
\end{bmatrix}
\]

4- DDWMR with slip effect

4.1 DDWMR Kinematics with slip effect

The constraint equations in the presence of slip and in term of the center of mass (C) are as follows:

\[
R \dot{R}_R = \dot{X}_c \cos \beta + \dot{Y}_c \sin \beta + L \dot{\beta}
\]

\[
R \dot{R}_L = \dot{X}_c \cos \beta + \dot{Y}_c \sin \beta - L \dot{\beta}
\]

\[
0 = -\dot{X}_c \sin \beta + \dot{X}_c \cos \beta - b \dot{\beta}
\]

Now by assuming linear displacement of the wheels as \(Z=R \gamma - \delta\) where \(R \gamma\) represent the transverse arc length and \(\delta\) represents the displacement loss due to slip and the steering angles equal to zero we get

\[
\dot{Z}_R = \dot{X}_c \cos \beta + \dot{Y}_c \sin \beta + L \dot{\beta}
\]

\[
\dot{Z}_L = \dot{X}_c \cos \beta + \dot{Y}_c \sin \beta - L \dot{\beta}
\]

\[
\dot{\rho} = -\dot{X}_c \sin \beta + \dot{X}_c \cos \beta - b \dot{\beta}
\]

4.2 DDWMR dynamics with slip effect

By substituting \(\dot{Z}_R, \dot{Z}_L,\) and \(\dot{\rho}\) into Lagrange equation and evaluate the final form as follows:
\[ E = \frac{1}{2} [m(\dot{X}_c^2 + \dot{Y}_c^2)] + \frac{1}{2} I_{rz} \dot{\beta}^2 + \frac{1}{2} [m_w(\dot{Z}_R^2 + \dot{\rho}^2)] + \frac{1}{2} [m_w(\dot{Z}_L^2 + \dot{\rho}^2)] + \frac{1}{2} [I_{wy} \dot{\gamma}_R^2 + I_{wz} \dot{\beta}^2] + \frac{1}{2} [I_{wy} \dot{\gamma}_L^2 + I_{wz} \dot{\beta}^2] \]

(20)

Where:

\[ q = [x_c \ y_c \ \beta \ \rho \ z_R \ z_L \ \gamma_R \ \gamma_L]^T \]

\[ m_t \ddot{X}_c + 2m_w b (\dot{\beta} \sin \beta + \dot{\beta}^2 \cos \beta) = (F_{longR} + F_{longL}) \cos \beta - (F_{latR} + F_{latL}) \sin \beta \]

(21)

\[ m_t \ddot{Y}_c - 2m_w b (\dot{\beta} \cos \beta + \dot{\beta}^2 \sin \beta) = (F_{longR} + F_{longL}) \sin \beta + (F_{latR} + F_{latL}) \cos \beta \]

(22)

\[ I_t \ddot{\beta} + 2m_w b (\dot{X}_c \sin \beta + \dot{Y}_c \cos \beta) = (F_{longR} - F_{longL}) L - (F_{latR} + F_{latL}) b \]

(23)

\[ I_w \ddot{\gamma}_R = \tau_R - R F_{longR} \]

(24)

\[ I_w \ddot{\gamma}_L = \tau_L - R F_{longL} \]

(25)

\[ \ddot{\rho} = -\ddot{X}_c \sin \beta + \ddot{Y}_c \cos \beta - \dot{\beta} (\dot{X}_c \cos \beta - \dot{Y}_c \sin \beta) - b \ddot{\beta} \]

(26)

\[ \ddot{Z}_R = \dot{X}_c \cos \beta + \dot{Y}_c \sin \beta - \dot{\beta} (\dot{X}_c \sin \beta - \dot{Y}_c \cos \beta) + L \ddot{\beta} \]

(27)

\[ \ddot{Z}_L = \dot{X}_c \cos \beta + \dot{Y}_c \sin \beta - \dot{\beta} (\dot{X}_c \sin \beta - \dot{Y}_c \cos \beta) - L \ddot{\beta} \]

(28)

Now we can write equations (21)-(28) in matrix form as follows:

\[ M(q) \ddot{q} + V(q, \dot{q}) \dot{q} = B(q) \tau + F(q) \]

(29)

Where:

\[
M(q) = \begin{bmatrix}
    m_t & 0 & 2m_w b \sin \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & m_t & -2m_w b \cos \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
    2m_w b \sin \beta & -2m_w b \cos \beta & I_t & 0 & 0 & 0 & 0 & 0 & 0 \\
    -\sin \beta & \cos \beta & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
    \cos \beta & \sin \beta & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
    \cos \beta & \sin \beta & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & I_{wy} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{wy} & 0 
\end{bmatrix}
\]

\[
V(q, \dot{q}) = \begin{bmatrix}
    2m_w b \dot{\beta} \cos \beta & \frac{2m_w b \dot{\beta} \sin \beta}{2} & -\dot{X}_c \cos \beta - \dot{Y}_c \sin \beta & -\dot{X}_c \sin \beta - \dot{Y}_c \cos \beta & -\dot{X}_c \cos \beta - \dot{Y}_c \sin \beta \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B(q) = \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_R \\ \tau_L \end{bmatrix}.
\]
\[
F(q) = \begin{bmatrix}
(F_{longR} + F_{longL}) \cos \beta - (F_{latR} + F_{latL}) \sin \beta \\
(F_{longR} + F_{longL}) \sin \beta + (F_{latR} + F_{latL}) \cos \beta \\
(F_{longR} - F_{longL}) L - (F_{latR} + F_{latL}) b \\
0 \\
0 \\
0 \\
-R F_{longR} \\
-R F_{longL}
\end{bmatrix}
\]

5- Wheels’ traction forces

5.1 Wheels’ traction forces in pure rolling

By assuming rigid ground with rigid wheels as shown in Figure 3 where \(F_{long}\) is the longitudinal force, the wheel equations will be as follows:

\[m_w a_w = F_{long}\]  \(\text{(30)}\)

Where the linear acceleration in pure rolling is \(a_w = R \ddot{y}\)  \(\text{(31)}\)

Now for each wheel we have \(I_{wy} \ddot{y} = \tau - R F_{long}\)  \(\text{(32)}\)

From the above equation we get \(\ddot{y} = \frac{\tau - R F_{long}}{I_{wy}}\)  \(\text{(33)}\)

Substitute Equation (33) in (31) to obtain \(a_w = R \frac{\tau - R F_{long}}{I_{wy}}\)  \(\text{(34)}\)

Substitute Equation (34) in (30) to obtain \(m_w R \frac{\tau - R F_{long}}{I_{wy}} = F_{long}\)  \(\text{(35)}\)

Rearrange the above equation (35) to get \(F_{long} = \frac{\tau m_w R}{m_w R^2 + I_{wy}}\)  \(\text{(36)}\)

5.2 Wheels’ traction forces with slip

Due to the nonuniformity in the contact region between the wheels and the ground, we will consider Coulomb’s model of friction. This nonuniformity acts as moors which should be broken by the maximum force to make relative motion between the wheels and the ground, so we use the static coefficient of friction (\(\mu_s\)) with the normal reaction force on the wheel to evaluate the value of force that should be applied to the wheel before the commencement of wheel slip \((F_{long}^s)\) as follows:

\[F_{long}^s = \mu_s N\]  \(\text{(37)}\)

Now if \(F_{long} < F_{long}^s\) then there is no slip because all the value of \(F_{long}\) transmitted to the ground.

If \(F_{long} > F_{long}^s\) slip occurs because of the difference between these two forces’ values that makes \(F_{long}\) exceed the value of \(F_{long}^s\). Now if slip occurs the kinetic coefficient of friction \(\mu_k\) is used to determine the force results in the linear \((F_{long}^k)\) motion as follows:

\[F_{long}^k = \mu_k N\]  \(\text{(38)}\)
Now by decomposing the total force that was exerted due to the applied torque $\mu_kN$ into longitudinal and lateral forces as shown in Figure 4, we get the relation between the applied force and the slip values.

\[
F_{\text{long}R} = \mu_k N_R \left( \frac{Z_R}{\sqrt{Z_R^2 + \rho^2}} \right) 
\]

\[
F_{\text{lat}R} = \mu_k N_R \left( \frac{b}{\sqrt{Z_R^2 + \rho^2}} \right) 
\]

\[
F_{\text{long}L} = \mu_k N_L \left( \frac{Z_L}{\sqrt{Z_L^2 + \rho^2}} \right) 
\]

\[
F_{\text{lat}L} = \mu_k N_L \left( \frac{b}{\sqrt{Z_L^2 + \rho^2}} \right) 
\]

6- DDWMR control

This part will include three sections:

1- kinematic control
2- dynamic control
3- torque control to prevent slip

The linear and angular velocities resulting from the kinematic controller will be the input velocities for the dynamic controller that generates the torques of the wheels. The procedure of this control is classified as a backstepping control.

6.1 Kinematic controller

The desired trajectory should satisfy the nonholonomic and the kinematic equations, so we get the following:

\[
\dot{q}_d = \begin{bmatrix}
\dot{x}_d \\
\dot{y}_d \\
\dot{\beta}_d
\end{bmatrix} = \begin{bmatrix}
\cos \beta_d & 0 \\
\sin \beta_d & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
v_d \\
\omega_d
\end{bmatrix} 
\]

\[
\dot{x}_d \sin \beta_d = \dot{y}_d \cos \beta_d 
\]
position and orientation errors of the robot in the global frame representing by the difference between these two coordinates as follows:

\[ \dot{q} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\beta} \end{bmatrix}, \quad \text{where:} \quad \ddot{x} = x_d - x, \quad \ddot{y} = y_d - y, \quad \ddot{\beta} = \beta_d - \beta \]

(45)

![Figure 5: Robot trajectory model](image)

Then we use the transformation matrix to get the errors in the local frame as follows:

\[ \begin{bmatrix} \ddot{x}_r \\ \ddot{y}_r \\ \ddot{\beta}_r \end{bmatrix} = \begin{bmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\beta} \end{bmatrix} \]

(46)

Now by differentiating Equation (46) and considering Equations (7) and (44) we get the error dynamics \( \ddot{q}_r = [\ddot{x}_r \quad \ddot{y}_r \quad \ddot{\beta}_r]^T \) as follows:

\[ \ddot{x}_r = v_d \cos \ddot{\beta}_r - v + \ddot{y}_r \]

(7) (47)

\[ \ddot{y}_r = v_d \sin \ddot{\beta}_r - \ddot{x}_r \omega \]

(7) (48)

\[ \ddot{\beta}_r = \omega_d - \omega \]

(7) (49)

The aim of this control procedure is to track a desired trajectory with error convergence to zero, so the structure of the controller will be determined using Lyapunov’s second method, starting by selecting the following candidate function

\[ V (\ddot{q}_r) = \frac{1}{2} (\dddot{x}_r^2 + \dddot{y}_r^2) + (1- \cos \dddot{\beta}_r) \]

(50)

This candidate function should satisfy the following Lyapunov function properties [12]:

1- \( V (\ddot{q}_r) \) this function and its derivative is continuous
2- \( V (0) = 0 \)
3- \( V (\ddot{q}_r) > 0 \) for all \( \ddot{q}_r \neq 0 \)
4- \( \frac{dV (\ddot{q}_r)}{dt} < 0 \) for \( \ddot{q}_r \neq 0 \)

Since Equation (140) satisfies the first three properties we should check the fourth one by taking the time derivative of (V) to get the following:

\[ \dot{V} (\ddot{q}_r) = (v_d \cos \dddot{\beta}_r - v) \dddot{x}_r + (\omega_d - \omega + v_d \dddot{y}_r) \]

(51)

The error dynamics will be asymptotically stable by using the following control law [6][9]:
\[ v_c = k_x \ddot{x}_r + v_d \cos \dot{\beta}_r \]
\[ \omega_c = k_\beta \sin \dot{\beta}_r + v_d \ddot{y}_r + \omega_d \]  
(52.a,b)

Where \( k_x \) and \( k_\beta \) are positive gains, it is clear that this velocity control inputs makes \( \dot{V}(\ddot{q}_r) \) < 0 as follows:

\[ \dot{V}(\ddot{q}_r) = -(k_x \dddot{x}_r^2 + k_\beta \sin^2 \dot{\beta}_r) \]  
(53)

### 6.2 Dynamic controller

By defining the velocity errors as
\[ \ddot{v}_c = v_c - v, \quad \ddot{\omega}_c = \omega_c - \omega \]  
(54.a,b)

The velocity error dynamics will be
\[ \ddot{v}_c = \dot{v}_c - v = \ddot{v}_c - \frac{\tau_1}{m}, \quad \ddot{\omega}_c = \dot{\omega}_c - \omega = \ddot{\omega}_c - \frac{\tau_2}{I} \]  
(55.a,b)

Now by selecting the input torques as
\[ \tau_1 = m \ddot{v}_c + k_a \dddot{v}_c, \quad \tau_2 = I \ddot{\omega}_c + k_b \dddot{\omega}_c \]  
(56.a,b)

The velocity error equation will be
\[ m \dddot{v}_c + k_a \dddot{v}_c = 0, \quad I \dddot{\omega}_c + k_b \dddot{\omega}_c = 0 \]  
(57.a,b)

Where the gains \( k_a \) and \( k_b \) are positive which means that the velocity errors \( (\dddot{v}_c, \dddot{\omega}_c) \) are stable and converge to zero

### 6.3 Torque control to prevent slip

The aim of this control is to limit the applied torque so its value does not exceed the maximum value, but first we should limit the longitudinal force to be less than the static frictional force \( \mu_s N \).

For a single wheel the longitudinal force is
\[ F_{long} = \frac{\tau R}{\frac{m_w}{2} R^2 + I_{wy}} \]  
(58)

Rearrange the above equation to get
\[ F_{long} = \frac{\tau m_w R}{m_w R^2 + 2 I_{wy}} \]  
(59)

Now by limiting the longitudinal force to the static frictional force as follows:
\[ \mu_s N = \frac{\tau m_w R}{m_w R^2 + 2 I_{wy}} \]  
(60)

We get the limit for the maximum applied torque in general, torque in the right wheel, and torque in the left wheel respectively in the following equations:

\[ \tau_{max} = \frac{\mu_s N(m_w R^2 + 2 I_{wy})}{m_w R} \]  
(61)

\[ \tau_{maxR} = \frac{\mu_s N_R(m_w R^2 + 2 I_{wy})}{m_w R} \]  
(62)

\[ \tau_{maxL} = \frac{\mu_s N_L(m_w R^2 + 2 I_{wy})}{m_w R} \]  
(63)

From equation (16.a,b) we obtain
\[ \tau_R = (\tau_1 - \frac{1}{2b} \tau_2) \frac{R}{2}, \quad \tau_L = (\tau_1 + \frac{1}{2b} \tau_2) \frac{R}{2} \]  
(64)

To avoid slip, we compare the torque value of equations (64) with the maximum allowable torque of the right and left wheel from equations (62) and (63) as follows:

If \( \tau_R, \tau_L < \tau_{maxR}, \tau_{maxL} \), then the controller allows applying the torque
If \( \tau_R, \tau_L > \tau_{maxR}, \tau_{maxL} \) then \( \tau_R, \tau_L \) limited to the value of the maximum allowable torques.

Now we can represent the entire trajectory tracking as shown in the block diagram Figure 6, and by using MATLAB/Simulink software with the simulation variables listed in Table 2 the system could be simulated.

![Figure 6: The entire robot trajectory tracking](image)

**Table 2: Simulation Variables**

| symbol | value  | unit     |
|--------|--------|----------|
| M      | 2      | kg       |
| I      | 0.009735 | kg.m^2  |
| L      | 0.1    | m        |
| R      | 0.03   | m        |

### 7- Path tracking results:

1. Circular trajectory: it is required that the robot follow the desired trajectory described by \( x_d = 1 + \sin \omega_d t \) and \( y_d = 1 - \cos \omega_d t \) as shown in Figure 7.a. Figure 7.b shows the simulation result of factual circular trajectory. Figure 7.c shows the desired and factual values of \( x \). Figure 7.d shows the desired and factual values of \( y \). Figure 7.e shows the position and orientation errors. Figure 7.f shows the right and left input torques. Figure 7.g shows the controller output velocities. Figure 7.h shows the velocity error. Figure 7.i shows the energy.

![Images](image)
2. Infinity trajectory: it is required that the robot follow the desired trajectory described by 
\[ x_d = 3\sin\frac{2\pi}{105}t \]
and 
\[ y_d = \sin\frac{4\pi}{105}t \]
as shown in Figure 8.a. Figure 8.b shows the simulation result of factual infinity trajectory. Figure 8.c shows the desired and factual values of \(x\). Figure 8.d shows the desired and factual values of \(y\). Figure 8.e shows the position and orientation errors. Figure 8.f shows the right and left input torques. Figure 8.g shows the controller output velocities. Figure 8.h shows the velocity error. Figure 8.i shows the energy.

Figure 7: Circular trajectory.
3. Sinusoidal trajectory: it is required that the robot follow the desired trajectory described by 
\[ x_d = \sin \left( \frac{\pi}{2} \omega_d t \right) \] 
and 
\[ y_d = \sin \left( \frac{\pi}{2} \omega_d t + \frac{\pi}{2} \right) \] 
as shown in Figure 9.a. Figure 9.b shows the simulation result of factual Sinusoidal trajectory. Figure 9.c shows the desired and factual values of \( x \). Figure 9.d shows the desired and factual values of \( y \). Figure 9.e shows the position and orientation errors. Figure 9.f shows the right and left input torques. Figure 9.g shows the controller output velocities. Figure 9.h shows the velocity error. Figure 9.i shows the energy.

Line trajectory: it is required that the robot follow the desired trajectory described by 
\[ x_d = t + 3 \] 
and 
\[ y_d = 3 \] 
as shown in figure 10.a. Figure 10.b shows the simulation result of factual line
trajectory. Figure 10.c shows the desired and factual values of \(x\). Figure 10.d shows the desired and factual values of \(y\). Figure 10.e shows the position and orientation errors. Figure 10.f shows the right and left input torques. Figure 10.g shows the controller output velocities. Figure 10.h shows the velocity error. Figure 10.i shows the energy.

Elliptical trajectory: it is required that the robot follow the desired trajectory described by

\[ x_d = 1 + \sin \omega_d t \]
\[ y_d = 1 - \cos \frac{\pi}{4} \omega_d t \]

as shown in figure 11.a. Figure 11.b shows the simulation.
result of factual elliptical trajectory. Figure 11.c shows the desired and factual values of $(x)$. Figure 11.d shows the desired and factual values of $(y)$. Figure 11.e shows the position and orientation errors. Figure 11.f shows the right and left input torques. Figure 11.g shows the controller output velocities. Figure 11.h shows the velocity error. Figure 11.i shows the energy.

![Graphs](image)

**Figure 11: Elliptical trajectory.**

8- **Discussion**

It’s clear that a very good trajectory tracking performance is achieved by the proposed controllers. Among the five trajectory shapes the maximum position error appeared in the line trajectory where the mean error reaches 0.8379 in x direction and 0.4367 in y direction, while the minimum position error appeared in the infinity trajectory where the mean position error in x and y direction was approximately zero. In contrast, the maximum orientation error appeared in the infinity trajectory due to the presence of tipping points. On the other hand, we note that the torque didn’t exceed 10 N.m in all trajectories because of torque limiting control.

9- **Conclusion**

In this work, the kinematic and dynamic model of DDWMR with and without slip and the traction forces are derived. Backstepping-based controllers have been designed and the input torque limited to avoid slip. MATLAB and Simulink were used to simulate the WMR trajectory tracking. The parameters of the kinematic and dynamic controllers were tuned and modified to reach the desired performance. Simulation results showed good matching between desired and actual trajectory and ensured that the errors converged to zero. A good performance was shown in terms of minimizing the overshoot and reducing the energy and the transient response time.
References

[1] Falsafi M, Alipour K, and Tarvirdizadeh B 2018. Tracking-error fuzzy-based control for nonholonomic wheeled robots. *J. Arabian Journal for Science and Engineering*.

[2] Saleh A, Hussain M, and Klim S 2018 Iraq. Optimal trajectory tracking control for a wheeled mobile robot using fractional order PID controller. *J. Journal of University of Babylon*. V26.P292-306.

[3] Alouache A, and Wu Q 2018 China. Genetic algorithms for trajectory tracking of mobile robot based on PID controller.

[4] Hirpo B, and Zhongmin W 2017 China. Design and control for differential drive mobile robot. *J. International Journal of Engineering Research & Technology*. V6. PP327-334.

[5] Shih C, and Lin L 2017 Taiwan. Trajectory planning and tracking control of a differential drive mobile robot in a picture drawing appication. P1-15.

[6] García-Sánchez J, et al 2017 Mexico. Tracking control for mobile robots considering the dynamics of all their subsystems: experimental implementation. P1-18.

[7] Fareh R, Saad M, Khadraoui S, and Rabie T 2016. Lyapunov-based tracking control for nonholonomic wheeled mobile robot. *J. International Journal of Electrical and Computer Engineering*. V10. PP1042-1047.

[8] Tinh N, Linh N, Cat P, Tuan P, Anh M, and Anh N 2016 USA. Modeling and feedback linearization control of a nonholonomic wheeled mobile robot with longitudinal, lateral slips. *International Conference on Automation Science and Engineering*. P996-1001.

[9] Obaid M, Husain A, Al-Kubati A 2016. Robust backstepping tracking control of mobile Robot Based on Nonlinear Disturbance Observer. *J. International Journal of Electrical and computer engineering (IJECE)*. V6. PP901-908.

[10] Petrović E, et al 2016 Serbia. Kinematic model and control of mobile robot for trajectory tracking. *J. ANNALS of Faculty Engineering Hunedoara – International Journal of Engineering*. P161-164.

[11] Tawfik M, Abdulwahb E, and Swadi S 2014 Iraq. Trajectory tracking control for a wheeled mobile robot using fractional order PID controller. *J. Al-Khwarizmi Engineering Journal*. V10. PP39-52.

[12] Tzafestas S 2014 Greece. B. Introduction to mobile robot control.

[13] Tian Y, and Sarkar N 2013 USA. Control of a mobile robot subject to wheel slip. *J. Intell Robot Syst*.

[14] Sidek N, and Sarkar N 2008 UK. Dynamic modeling and control of nonholonomic mobile robot with lateral slip. *International conference on signal processing, robotics and automation*. p66-74.

[15] Hadi N 2005 Iraq. Fuzzy control of mobile robot in slippery environment. *J. Al-Khwarizmi Engineering Journal*. V1.pp41-51.