Symmetric representations of distributions over $\mathbb{R}^2$ by distributions with not more than three-point supports

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January 13, 2013

Abstract. We construct symmetric representations of distributions over $\mathbb{R}^2$ with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values. These representations are two-dimensional analogs of the following easy verified formula for distributions $\mathbf{p}$ over $\mathbb{R}^1$ with a mean value $u$:

$$
\mathbf{p} = \int_{x=u}^{\infty} \mathbf{p}(dx) \int_{y=-\infty}^{u+} \frac{x-y}{\int_{t=u}^{\infty} (t-u) \cdot \mathbf{p}(dt)} \cdot \mathbf{p}_{x,y}^u \cdot \mathbf{p}(dy),
$$

where, for $y < u < x$, distributions $\mathbf{p}_{x,y}^u = ((x-u) \cdot \delta^y + (u-y) \cdot \delta^x)/(x-y)$, $\delta^x$ is the degenerate distribution with the single-point support $x$, and $\mathbf{p}_{x,u}^u = \mathbf{p}_{u,y}^u = \delta^u/2$.

Key words: probability distributions over the plane, mean values, extreme points of convex sets, convex combinations of distributions.

1. Introduction. Setting of problem.

We consider the set $\mathbf{P}(\mathbb{R}^2)$ of probability distributions $\mathbf{p}$ over the plane $\mathbb{R}^2 = \{z = (x, y)\}$ with finite first absolute moments

$$
\int_{\mathbb{R}^2} |x| \cdot \mathbf{p}(dz) < \infty, \quad \int_{\mathbb{R}^2} |y| \cdot \mathbf{p}(dz) < \infty.
$$

We denote by $\mathbf{E}_\mathbf{p}[x]$ and $\mathbf{E}_\mathbf{p}[y]$ the mean values of distribution $\mathbf{p}$:

$$
\mathbf{E}_\mathbf{p}[x] = \int_{\mathbb{R}^2} x \cdot \mathbf{p}(dz) < \infty, \quad \mathbf{E}_\mathbf{p}[y] = \int_{\mathbb{R}^2} y \cdot \mathbf{p}(dz) < \infty.
$$
We construct symmetric representations of the convex set of distributions with given mean values

\[ \Theta(u, v) = \{ p \in P(\mathbb{R}^2) : E_p[x] = u, E_p[y] = v \}, \]

as a convex hull of its extreme points.

This is sufficient to give the representation for the set \( \Theta(0, 0) \). The extreme points of the set \( \Theta(0, 0) \) are the degenerate distribution \( \delta_0 \) with the single-point support \( 0 = (0, 0) \), distributions \( p_{0z_1, z_2}^0 \in \Theta(0, 0) \) with two-point supports \( (z_1, z_2) \), and distributions \( p_{z_1, z_2, z_3}^0 \in \Theta(0, 0) \) with three-point supports \( (z_1, z_2, z_3) \).

This problem arose from investigating multistage bidding models where two types of risky assets are traded [1]. As the example for imitation we take the symmetric representation of one-dimensional probability distributions over the integer lattice that was exploited in [2] for analysis of bidding models with single-type asset. Let \( p \) be a probability distribution over the set of integers \( \mathbb{Z}^1 \) with zero mean value. Then

\[ p = p(0) \cdot \delta^0 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k + l}{\sum_{t=1}^{\infty} t \cdot p(t)} p(-l)p(k) \cdot p_{k, -l}^0, \tag{1} \]

where \( p_{k, -l}^0 \) is the probability distribution with the support \( \{-l, k\} \) and with zero mean value. Formula (1) can be written as

\[ p = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{k + l}{\sum_{t=1}^{\infty} t \cdot p(t)} p(-l)p(k) \cdot p_{k, -l}^0, \]

if we put \( p_{k,0}^0 = p_{0, -l}^0 = \delta^0/2 \).

Observe that the coefficients \( P_p(p_{k, -l}^0) \) of decomposition (1), that may be treated as probabilities of corresponding distributions \( p_{k, -l}^0 \) in the two-step lottery realizing distribution \( p \), have the form

\[ P_p(p_{k, -l}^0) = \alpha(k, -l) \beta(p)p(k)p(-l), \]

where \( \alpha(k, l) = k + l \) and \( \beta(p) = 1/\sum_{t=1}^{\infty} t \cdot p(t) = 1/\sum_{t=1}^{\infty} t \cdot p(-t) \), the last equality playing the crucial role. We mean just this form of coefficients saying
that the representation (1) is symmetric. We aim for constructing the rep-
presentation of two-dimensional probability distributions with the analogous
characteristics.

Formula (1) can be easily generalized for probability distributions over the
set of real numbers $\mathbb{R}^1$ with zero mean value. Namely

$$
P = \int_{x=0^-}^{\infty} p(dx) \int_{y=-\infty}^{x-\infty} \frac{x-y}{t} \cdot p_x(-y) \cdot p(dy),$$

where, for $y < 0 < x$, the distributions $p_{x,-y} = (x \cdot \delta(y) - y \cdot \delta(x))/(x-y)$, and $p_{x,0} = \delta(0)/2$.

Consider the set of three-point sets that form triangles containing the
point $(0,0)$:

$$
\Delta^0 = \{(z_1, z_2, z_3), z_i \neq (0,0) : (0,0) \in \Delta(z_1, z_2, z_3)\}.
$$

The set $\Delta^0$ is a manifold with boundary. Its interior $\text{Int}\Delta^0$ is the set of
three-point sets $(z_1, z_2, z_3) \in \Delta^0$ such that $(0,0)$ belongs to the interior of the
$\Delta(z_1, z_2, z_3)$. Its boundary $\partial\Delta^0$ is the set of three-point sets $(z_1, z_2, z_3) \in \Delta^0$
such that $(0,0)$ belongs to the boundary of the $\Delta(z_1, z_2, z_3)$.

The distribution $p_{z_1, z_2, z_3}^0 \in \Theta(0,0)$ with the support $\{z_1, z_2, z_3\} \in \Delta^0$ is
given by

$$
p_{z_1, z_2, z_3}^0 = \frac{\sum_{j=1}^3 \text{det}[z_{i+1}, z_{i+2}] \cdot \delta(z_i)}{\sum_{j=1}^3 \text{det}[z_j, z_{j+1}]},
$$

(2)

where $\text{det}[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$. All arithmetical operations with
subscripts are fulfilled modulo 3. If the points $(z_1, z_2, z_3) \in \Delta^0$ are indexed
counterclockwise, then $\text{det}[z_i, z_{i+1}] \geq 0$.

If $(z_1, z_2, z_3) \in \partial\Delta^0$, then there is an index $i$ such that $\text{det}[z_i, z_{i+1}] = 0$. In
this case $\arg z_{i+1} = \arg z_i + \pi(\text{mod} \ 2\pi)$, the point $(0,0) \in [z_i, z_{i+1}]$ and the
distribution $p_{z_1, z_2, z_3}^0$ degenerates into the distribution $p_{z_i, z_{i+1}}^0$ with the support
$\{z_i, z_{i+1}\}$.

2. Key invariants for distributions $p \in \Theta(0,0)$.

For $\psi \in [0,2\pi)$, let $R_\psi$ be the half-line

$$
R_\psi = \{z : \arg z = \psi(\text{mod} \ 2\pi)\}.
$$
With each $\psi \in [0, 2\pi)$ we associate the set of two-point sets
\[
\Delta^0(\psi) = \{(z_1, z_2), z_i \neq (0,0) : \forall z \in R_\psi, (0,0) \in \Delta(z_1, z_2, z)\}.
\]
Denote by $\text{Int}\Delta^0(\psi)$ and $\partial\Delta^0(\psi)$ the sets of two-point sets $(z_1, z_2)$ such that, for $z \in R_\psi$, the set $(z_1, z_2, z)$ belongs to $\text{Int}\Delta^0$ and to $\partial\Delta^0$ respectively. We take, that the points $(z_1, z_2)$ are indexed counterclockwise.

Consider the quantity
\[
\Phi(p, \psi) = \int_{\text{Int}\Delta^0(\psi)} \det[z_1, z_2]p(dz_1)p(dz_2) + 1/2 \int_{\partial\Delta^0(\psi)} \det[z_1, z_2]p(dz_1)p(dz_2). \tag{3}
\]
Using polar coordinates $z_1 = (r_1, \varphi_1)$, $z_2 = (r_2, \varphi_2)$ we get
\[
\Phi(p, \psi) = \int_{\varphi_1=\psi}^{\pi+\psi+} \int_{r_1=0+}^{\infty} p(dr_1d\varphi_1) \int_{\varphi_2=\pi+\psi}^{\pi+\varphi_1+} \int_{r_2=0+}^{\infty} r_1r_2\sin(\varphi_2-\varphi_1)p(dr_2d\varphi_2).
\]

Remark 1. The quantity
\[
\partial\Phi(p, \psi) = 1/2 \int_{\partial\Delta^0(\psi)} \det[z_1, z_2]p(dz_1)p(dz_2)
\]
differs from zero only if the measure $p(R_{\psi+\pi})$ is more than zero. In this case
\[
\partial\Phi(p, \psi) = \int_{R_{\psi+\pi}} r_2p(dr_2) \cdot \int_{H_{p\psi}} \det[e_\psi, z_1]p(dz_1)
\]
\[
= \int_{R_{\psi+\pi}} r_1p(dr_1) \cdot \int_{H_{p\psi+\pi}} \det[z_2, e_\psi]p(dz_2), \tag{4}
\]
where $e_\psi = (1, \psi)$ and $H_{p\varphi}$ is the half-plane
\[
H_{p\varphi} = \{ z : \arg z \in (\varphi, \varphi + \pi)(\mod 2\pi) \}.
\]

The next fact produces the base for constructing symmetric representations of distributions over $\mathbb{R}^2$ with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values.
**Theorem 1.** For any distribution \( p \in \Theta(0, 0) \) the quantity \( \Phi(p, \psi) \) does not depend on \( \psi \), i.e. this is an invariant \( \Phi(p) \) of distribution \( p \in \Theta(0, 0) \).

**Proof.** We begin with proving Theorem 1 for distributions \( p \in \Theta^f(0, 0) \) with finite supports. Let \( \psi_1, \psi_2 \in [0, 2\pi), \psi_1 < \psi_2 \), be such two values of argument that the support of the distribution \( p \in \Theta^f(0, 0) \) does not contain points \( z \) with \( \psi_1 < \arg z < \psi_2 \).

Set
\[
U(\psi_1 + \pi, \psi_2 + \pi) = \{ z \in \mathbb{R}^2 : \psi_1 + \pi < \arg z \leq \psi_2 + \pi \}.
\]

We have
\[
\Phi(p, \psi_1) - \Phi(p, \psi_2) = \sum_{z_3 \in U(\psi_1 + \pi, \psi_2 + \pi)} \sum_{z_2 \in \mathbb{R}^2} p(z_2)p(z_3) \det[z_2, z_3].
\]

Since, for distributions \( p \in \Theta(0, 0) \),
\[
\sum_{z_2 \in \mathbb{R}^2} p(z_2) \det[z_2, z_3] = 0,
\]
we obtain
\[
\Phi(p, \psi_1) - \Phi(p, \psi_2) = 0.
\]

Iterating this argument the relevant number of times we obtain the statement of Theorem 1 for any distribution \( p \in \Theta^f(0, 0) \).

As the set \( \Theta^f(0, 0) \) is weakly* everywhere dense in \( \Theta(0, 0) \) we obtain the statement of Theorem 1 for arbitrary distributions \( p \in \Theta(0, 0) \).

\( \square \)

**Remark 2.** This theorem is a two-dimensional analog of the fact that, for \( p \in \Theta(0) \subset P(\mathbb{R}^1) \), the equality
\[
\int_{t=0}^{\infty} t \cdot p(dt) = \int_{t=0}^{\infty} t \cdot p(-dt)
\]
holds.

**Example 1.** Consider the distribution \( p_{z_1, z_2, z_3}^0 \) with \( (z_1, z_2, z_3) \in \text{Int}\Delta^0 \). For this distribution, if \( \arg(z_i) = \varphi_i \) and
\[
\varphi_i + \pi < \psi < \varphi_{i+1} + \pi (\text{mod } 2\pi),
\]

then the support of the measure induced by \( p_{z_1,z_2,z_3}^0 \) over the set \( \Delta^0(\psi) \) is the set \( \{(z_i, z_{i+1}) \} \subset \text{Int}\Delta^0(\psi) \). Thus

\[
\Phi(p_{z_1,z_2,z_3}^0, \psi) = \det[z_i, z_{i+1}] \cdot p_{z_1,z_2,z_3}^0(z_i) p_{z_1,z_2,z_3}^0(z_{i+1}) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}.
\]

If \( \varphi_i + \pi = \psi \pmod{2\pi} \), then the support of the induced measure is the set \( \{(z_{i-1}, z_i), (z_i, z_{i+1}) \} \subset \partial\Delta^0(\psi) \). Thus

\[
\Phi(p_{z_1,z_2,z_3}^0, \psi) = 1/2 \cdot (\det[z_{i-1}, z_i] \cdot p_{z_1,z_2,z_3}^0(z_{i-1}) p_{z_1,z_2,z_3}^0(z_i) + \det[z_i, z_{i+1}] \cdot p_{z_1,z_2,z_3}^0(z_i) p_{z_1,z_2,z_3}^0(z_{i+1})) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}.
\]

Thus, in accordance with Theorem 1, \( \Phi(p_{z_1,z_2,z_3}^0, \psi) \) has the same value \( \Phi(p_{z_1,z_2,z_3}^0) \) for all values of \( \psi \).

**3. Decomposition theorem for distributions \( p \in \Theta(0,0) \).**

The invariance of the quantity \( \Phi(p) \) proved in the previous section allows us to formulate the following preliminary variant of decomposition theorem for two-dimensional distributions. This variant demonstrate a perfect analogy with the decomposition of one-dimensional distributions.

**Proposition 2.** Any distribution \( p \in \Theta(0,0) \) has the following symmetric decomposition into a convex combination of distributions with not more than three-point supports:

\[
p = p(0,0) \cdot \delta^0 + \int_{\text{Int}\Delta^0} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(p)} p_{z_1,z_2,z_3}^0(z_1) p(z_2) p(z_3) dz_1 dz_2 dz_3
\]

\[
+ 1/2 \int_{\partial\Delta^0} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(p)} p_{z_1,z_2,z_3}^0(z_1) p(z_2) p(z_3), \quad (5)
\]

where \( \Phi(p) \) is given by (3).

**Proof.** We begin with proving Proposition 2 for distributions \( p \in \Theta^f(0,0) \) with finite supports. Take a point \( z_1 = (r_1, \varphi_1) \in \text{supp} \ p \). This point occurs
in three point set \((z_1, z_2, z_3)\) if \((z_2, z_3) \in \Delta^0(\psi)\). The probability \(p'(z_1)\) calculated according to formula (5) is

\[
p'(z_1) = \sum_{(z_2, z_3) \in \text{Int} \Delta^0(\psi)} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}] p^0_{z_1, z_2, z_3}(z_1)p(z_1)p(z_2)p(z_3)}{\Phi(p)} + \frac{1}{2} \sum_{(z_2, z_3) \in \partial \Delta^0(\psi)} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}] p^0_{z_1, z_2, z_3}(z_1)p(z_1)p(z_2)p(z_3)}{\Phi(p)}.
\]

Substituting the values \(p^0_{z_1, z_2, z_3}(z_1)\) given by (2) we get

\[
p'(z_1) = \frac{p(z_1)}{\Phi(p)} \left( \sum_{\text{Int} \Delta^0(\psi)} + \frac{1}{2} \sum_{\partial \Delta^0(\psi)} \right) \det[z_2, z_3] \cdot p(z_2)p(z_3) = p(z_1).
\]

This proves Proposition 2 for any distribution \(p \in \Theta_f(0,0)\).

As the set \(\Theta_f(0,0)\) is weakly\(^*\) everywhere dense in \(\Theta(0,0)\) we obtain the statement of Proposition 2 for arbitrary distributions \(p \in \Theta(0,0)\).

\[\square\]

The term

\[
\partial p = \frac{1}{2} \int_{\partial \Delta^0} \sum_{j=1}^{3} \det[z_j, z_{j+1}] p^0_{z_1, z_2, z_3} p(dz_1)p(dz_2)p(dz_3)
\]

of decomposition (5) contains all distributions \(p^0_{z_1, z_{i+1}}\) with two-point supports \((z_i, z_{i+1})\), where \(z_i \in R_{\psi}\) and \(z_{i+1} \in R_{\psi+\pi}\). In order that such combination of points could appear with nonzero probability, it is necessary that the measure \(p(R_\psi)\) and the measure \(p(R_{\psi+\pi})\) are more than zero. This is possible for a not more than countable set \(\Psi(p)\) of values \(\psi\).

These considerations make possible the final formulation of the principal Theorem:

**Theorem 3.** Any probability distribution \(p \in \Theta(0,0)\) has the following symmetric representation as a convex combination of distributions with one-, two-, and three-point supports:

\[
p = p(0,0) \cdot \delta^0 + \int_{\text{Int} \Delta^0} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}] p^0_{z_1, z_2, z_3} p(dz_1)p(dz_2)p(dz_3)}{\Phi(p)}
\]
\[ + \sum_{\Psi(p)} \frac{\partial \Phi(p, \psi)}{\Phi(p)} \int_{R_\psi} \int_{R_{\psi+\pi}} \frac{r_1 + r_2}{t(p \Phi)} \int_{R_{\psi+\pi}} p(r_1, \psi), (r_2, \psi + \pi) p(dr_2) p(dr_1). \quad (6) \]

**Proof.** For a pair of points \( z_1 = (r_1, \psi), z_2 = (r_2, \psi + \pi) \), their combination with any point \( z \) from \( H_p \psi \) or from \( H_p \psi + \pi \) reduces to the distribution \( p_{z_1, z_2}^0 \). Since

\[
\int_{H_p \psi} \det[e_\psi, z] p(dz) = \int_{H_p \psi + \pi} \det[z, e_\psi] p(dz),
\]

where \( e_\psi = (1, \psi) \), we get

\[
\partial p = \sum_{\Psi(p)} \int_{H_p \psi} \det[e_\psi, z] p(dz) \int_{R_\psi} \int_{R_{\psi+\pi}} \frac{r_1 + r_2}{\Phi(p)} p(r_1, \psi), (r_2, \psi + \pi) p(dr_2) p(dr_1).
\]

It follows from (4) that

\[
\int_{H_p \psi} \det[e_\psi, z_1] p(dz_1) = \frac{\partial \Phi(p, \psi)}{\int_{R_{\psi+\pi}} r_2 p(dr_2)}.
\]

Substituting this expression in place of this integral we obtain

\[
\partial p = \sum_{\Psi(p)} \frac{\partial \Phi(p, \psi)}{\Phi(p)} \int_{R_\psi} \int_{R_{\psi+\pi}} \frac{r_1 + r_2}{t(p \Phi)} \int_{R_{\psi+\pi}} p(r_1, \psi), (r_2, \psi + \pi) p(dr_2) p(dr_1).
\]

Substituting this into formula (5) we obtain (6). This proves Theorem 3.

\( \square \)

**Remark 3.** For distributions \( p \in \Theta(0, 0) \) with discrete supports this theorem indicates probabilities \( P_p(p_{z_1, z_2, z_3}^0) \) and \( P_p(p_{z_1, z_2}^0) \) of appearance of distributions with two- and three-point supports in their symmetric representations:

\[
P_p(p_{z_1, z_2, z_3}^0) = \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(p)} p(z_1)p(z_2)p(z_3);
\]

\[
P_p(p_{(r_1, \psi), (r_2, \psi+\pi)}^0) = \frac{\partial \Phi(p, \psi)}{\Phi(p)} \frac{r_1 + r_2}{\sum_{R_{\psi+\pi}} t p(t)} p(r_1, \psi)p(r_2, \psi + \pi).
\]
4. Examples.

Here we give several elementary examples concerning calculation of invariants \( \Phi(p) \) and constructing symmetric representations as a convex combinations of distributions with one-, two-, and three-point supports, for simple distributions with finite supports.

**Example 1’.** We return to the distribution \( p_{z_1, z_2, z_3}^0 \) with \((z_1, z_2, z_3) \in \text{Int}\Delta^0\). For this distribution, as it is shown in Example 1,

\[
\Phi(p_{z_1, z_2, z_3}^0) = \det[z_i, z_{i+1}] \cdot p_{z_1, z_2, z_3}^0(z_i)p_{z_1, z_2, z_3}^0(z_{i+1}) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}.
\]

As the distribution \( p_{z_1, z_2, z_3}^0 \) is an extreme point of the set \( \Theta(0, 0) \) its symmetric representation is trivial. To check it formally put

\[
P_{p_{z_1, z_2, z_3}^0}(p_{z_1, z_2, z_3}^0) = \sum_{i=1}^3 \frac{\det[z_i, z_{i+1}]}{\Phi(p_{z_1, z_2, z_3}^0)}p_{z_1, z_2, z_3}^0(z_i)p_{z_1, z_2, z_3}^0(z_{i+1}) = 1.
\]

**Example 2.** For \( z = (z_1, z_2, z_3) \in \text{Int}\Delta^0, \alpha = (\alpha_1, \alpha_2, \alpha_3) > 0, \sum_{i=1}^3 \alpha_i = 1 \), consider the distribution

\[
p_{z, \alpha} = \sum_{j=1}^3 \alpha_i p_{z_i, -z_i}^0 \in \Theta(0, 0).
\]

For this distribution,

\[
\text{supp } p_{z, \alpha} = \{z_1, z_2, z_3, -z_1, -z_2, -z_3\}, \quad p_{z, \alpha}(z_i) = p_{z, \alpha}(-z_i) = \alpha_i/2.
\]

Let \( \arg(z_i) = \varphi_i \). If

\[
\varphi_i < \psi < \varphi_{i-1} + \pi \text{ (mod } 2\pi),
\]

then the support of the measure induced by \( p_{z, \alpha} \) over the set \( \Delta^0(\psi) \) is the set

\[
\{(-z_i, z_{i-1}), (-z_i, -z_{i+1}), (z_{i+1}, z_{i-1})\} \subset \text{Int}\Delta^0(\psi).
\]

Since \( \det[-z_i, z_{i-1}] = \det[z_{i-1}, z_i], \det[-z_i, -z_{i+1}] = \det[z_i, z_{i+1}] \), we get

\[
\Phi(p_{z, \alpha}) = \sum_{i=1}^3 \det[z_i, z_{i+1}] \cdot p_{z, \alpha}(z_i)p_{z, \alpha}(z_{i+1}) = 1/4 \sum_{i=1}^3 \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}.
\]
The symmetric representation of the distribution $p_{z,\alpha}$ includes five extreme distributions: two three-point distributions $p_{z_1,z_2,z_3}^0$ and $p_{-z_1,-z_2,-z_3}^0$, and three two-point distributions $p_{z_1,-z_1}^0$, $p_{z_2,-z_2}^0$, and $p_{z_3,-z_3}^0$. These distributions occur with probabilities

$$P_{p_{z,\alpha}}(p_{z_1,z_2,z_3}^0) = P_{p_{z,\alpha}}(p_{-z_1,-z_2,-z_3}^0) = \frac{\sum_{i=1}^{3} \det[z_i, z_{i+1}]}{\Phi(p_{z,\alpha})} p_{z,\alpha}(z_1)p_{z,\alpha}(z_2)p_{z,\alpha}(z_3)$$

$$= \frac{1}{2} \frac{\sum_{i=1}^{3} \det[z_i, z_{i+1}]}{\sum_{i=1}^{3} \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}} \alpha_1 \alpha_2 \alpha_3;$$

$$P_{p_{z,\alpha}}(p_{z_i,-z_i}^0) = 2 \frac{\det[z_i, z_{i+1}]p_{z,\alpha}(z_{i+1}) + 2 \det[z_{i+2}, z_i]p_{z,\alpha}(z_{i+2})}{\Phi(p_{z,\alpha})} p_{z,\alpha}(z_i)^2$$

$$= \frac{\det[z_i, z_{i+1}]\alpha_{i+1} + \det[z_{i+2}, z_i] \alpha_{i+2}}{\sum_{i=1}^{3} \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}} \alpha_{i+2}^2.$$

Observe that

$$P_{p_{z,\alpha}}(p_{z_1,z_2,z_3}^0) + P_{p_{z,\alpha}}(p_{-z_1,-z_2,-z_3}^0) + \sum_{i=1}^{3} P_{p_{z,\alpha}}(p_{z_i,-z_i}^0)$$

$$= \sum_{i=1}^{3} \det[z_i, z_{i+1}] \alpha_1 \alpha_2 \alpha_3 + \sum_{i=1}^{3} (\det[z_i, z_{i+1}] \alpha_i^2 \alpha_{i+1} + \det[z_{i+2}, z_i] \alpha_i^2 \alpha_{i+2})$$

$$\frac{\sum_{i=1}^{3} \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}}{\sum_{i=1}^{3} \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}}$$

$$= \frac{\sum_{i=1}^{3} \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}}{\sum_{i=1}^{3} \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}} = 1$$

**Example 3.** For $z = (z_1, z_2, z_3) \in \text{Int} \Delta^0, \beta \in (0, 1)$, consider the distribution

$$p_{\beta,z} = \beta p_{z_1,z_2,z_3}^0 + (1 - \beta) p_{-z_1,-z_2,-z_3}^0 \in \Theta(0, 0).$$

This distribution has the same support as the distribution $p_{z,\alpha}$ of the previous example:

$$\text{supp } p_{\beta,z} = \{z_1, z_2, z_3, -z_1, -z_2, -z_3\},$$

The probabilities of these points are

$$p_{\beta,z}(z_i) = \beta \frac{\det[z_{i+1}, z_{i+2}]}{\sum_{j=1}^{3} \det[z_j, z_{j+1}]} p_{\beta,z}(-z_i) = (1 - \beta) \frac{\det[z_{i+1}, z_{i+2}]}{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}.$$
For this distribution, if 

$$\varphi_i < \psi < \varphi_{i-1} + \pi (\text{mod } 2\pi),$$

then the support of the measure induced by \(p_{\beta,z}\) over the set \(\Delta^0(\psi)\) is the set 

$$\{(-z_i, z_{i-1}), (-z_i, -z_{i+1}), (z_{i+1}, z_{i-1})\} \subset \text{Int}\Delta^0(\psi).$$

Since

$$\det[-z_i, z_{i-1}]p_{\beta,z}(-z_i)p_{\beta,z}(z_{i-1}) = \frac{\prod_{j=1}^{3} \det[z_j, z_{j+1}]}{(\sum_{j=1}^{3} \det[z_j, z_{j+1}])^2} \beta(1 - \beta),$$

$$\det[-z_i, -z_{i+1}]p_{\beta,z}(-z_i)p_{\beta,z}(-z_{i+1}) = \frac{\prod_{j=1}^{3} \det[z_j, z_{j+1}]}{(\sum_{j=1}^{3} \det[z_j, z_{j+1}])^2} (1 - \beta)^2,$$

we get

$$\Phi(p_{\beta,z}) = \frac{\prod_{j=1}^{3} \det[z_j, z_{j+1}]}{(\sum_{j=1}^{3} \det[z_j, z_{j+1}])^2} (\beta^2 + \beta(1 - \beta) + (1 - \beta)^2).$$

The symmetric representation of the distribution \(p_{\beta,z}\) includes the same five extreme distributions as in the previous example: two three-point distributions \(p_{z_1,z_2,z_3}^0\) and \(p_{-z_1,-z_2,-z_3}^0\), and three two-point distributions \(p_{z_1,-z_1}^0\), \(p_{z_2,-z_2}^0\), and \(p_{z_3,-z_3}^0\). These distributions occur with probabilities

$$P_{p_{\beta,z}}(p_{z_1,z_2,z_3}^0) = \frac{\sum_{i=1}^{3} \det[z_i, z_{i+1}]}{\Phi(p_{\beta,z})} p_{\beta,z}(z_1)p_{\beta,z}(z_2)p_{\beta,z}(z_3) = \frac{\beta^3}{\beta^2 + \beta(1 - \beta) + (1 - \beta)^2};$$

$$P_{p_{\beta,z}}(p_{-z_1,-z_2,-z_3}^0) = \frac{\sum_{i=1}^{3} \det[z_i, z_{i+1}]}{\Phi(p_{\beta,z})} p_{\beta,z}(-z_1)p_{\beta,z}(-z_2)p_{\beta,z}(-z_3) = \frac{(1 - \beta)^3}{\beta^2 + \beta(1 - \beta) + (1 - \beta)^2};$$

$$P_{p_{\beta,z}}(p_{z_1,-z_1}^0) = \frac{2 \det[z_i, z_{i+1}]p_{\beta,z}(z_{i+1}) + 2 \det[z_{i+2}, z_i]p_{\beta,z}(-z_{i+2})}{\Phi(p_{\beta,z})} p_{\beta,z}(z_i)p_{\beta,z}(-z_i).$$
\[
\frac{\det[z_{i+1}, z_{i+2}]}{\sum_{j=1}^{3} \det[z_j, z_{j+1}]} \frac{2\beta(1 - \beta)}{\beta^2 + \beta(1 - \beta) + (1 - \beta)^2}.
\]

Observe that

\[
P_{p, \beta, z}(p^0_{z_1, z_2, z_3}) + P_{p, \beta, z}(p^0_{-z_1, -z_2, -z_3}) + \sum_{i=1}^{3} P_{p, \beta, z}(p^0_{-z_i, -z_i})
\]

\[
= \frac{\beta^3 + (1 - \beta)^3 + 2\beta(1 - \beta)}{\beta^2 + \beta(1 - \beta) + (1 - \beta)^2} = \frac{\beta^2 - \beta(1 - \beta) + (1 - \beta)^2 + 2\beta(1 - \beta)}{\beta^2 + \beta(1 - \beta) + (1 - \beta)^2} = 1.
\]

References

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