Source identity and kernel functions for Inozemtsev-type systems

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Abstract

The Inozemtsev Hamiltonian is an elliptic generalization of the differential operator defining the BC\textsubscript{N} trigonometric quantum Calogero-Sutherland model, and its eigenvalue equation is a natural many-variable generalization of the Heun differential equation. We present kernel functions for Inozemtsev Hamiltonians and Chalykh-Feigin-Veselov-Sergeev-type deformations thereof. Our main result is a solution of a heat-type equation for a generalized Inozemtsev Hamiltonian which is the source for all these kernel functions. Applications are given, including a derivation of simple exact eigenfunctions and eigenvalues for the Inozemtsev Hamiltonian.

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1 Introduction

Integrable models in quantum mechanics are closely related to the mathematical theory of special functions. A famous example are Calogero-Moser-Sutherland models which describe an arbitrary number of identical particles moving in one dimension under the influence of particular one- and two-body potentials \[1, 2, 3, 4\]. The Hamiltonians of these models are differential operators that have eigenfunctions providing natural many-body generalizations of the classical orthogonal polynomials. For example, the so-called BC\textsubscript{N} trigonometric Calogero-Sutherland model has energy eigenfunctions given by many-variable Jacobi polynomials; see e.g. \[5, 6\].

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The BC$_N$ trigonometric Calogero-Sutherland model has an elliptic generalization defined by the Hamiltonian

$$H_N = \sum_{j=1}^{N} \left( -\frac{\partial^2}{\partial x_j^2} + \sum_{\nu=0}^{3} g_\nu (g_\nu - 1) \wp(x_j + \omega_\nu) \right)$$

with \(\wp(x)\) the usual Weierstrass elliptic function with periods 2\(\omega_1\) and 2\(\omega_3\) and where we use the notation \(\omega_0 = 0, \quad \omega_2 = -\omega_1 - \omega_3, \)

here and in the following\(^{\ddagger}\). This Hamiltonian depends on the particle number \(N\) and five coupling parameters \(g_0, g_1, g_2, g_3, \lambda\). It defines the natural quantum-analogue of a classical Liouville integrable system first presented by Inozemtsev in [7], and we refer to this quantum-many body system as the Inozemtsev model. The integrability of the Inozemtsev model was partially established by van Diejen [8], and Oshima [9] described its commuting operators (higher-order Hamiltonians) completely. In the following we sometimes write \(H_N(x; \{g_\nu\}_{\nu=0}^{3}, \lambda)\) for the Inozemtsev Hamiltonian in (1), to indicate the argument \(x = (x_1, \ldots, x_N)\) and the coupling parameters. Note that this differential operator allows for a quantum mechanical interpretation only if one assumes that \(\omega_1 > 0, -i\omega_3 > 0,\) and that all coupling parameters are real. However, many of our results hold true with lesser restrictions.

The Inozemtsev model is interesting already in the one-variable case: the eigenvalue equation \(H_1 \psi(x) = E \psi(x)\) of the differential operator

$$H_1 = -\frac{\partial^2}{\partial x^2} + \sum_{\nu=0}^{3} g_\nu (g_\nu - 1) \wp(x + \omega_\nu)$$

is equivalent to the Heun differential equation, which is a standard form of a second-order Fuchsian differential equation with four singularities and a topic of current research in special function theory; see [10, 11, 12]. The Heun differential equation, and the differential equations of its confluent type, appear in several physics contexts, including quantum mechanics, general relativity, models of crystal imperfections [11], and the AdS/CFT correspondence [13]. In this paper we present generalizations of two known functional identities involving the Heun differential operator in (2) and functions that are products of powers of Jacobi theta functions \(\theta_{\nu+1}(x), \quad \nu = 0, 1, 2, 3\) (see Appendix A.1 for precise definitions). The first known identity is as follows: The function

$$\Psi_1(x) = \prod_{\nu=0}^{3} \theta_{\nu+1}(x)^{g_\nu}$$

obeys the equation

$$\left\{ 2(g_0 + g_1 + g_2 + g_3) \frac{\partial}{\partial \beta} + H_1 - C_1 \right\} \Psi_1(x) = 0$$

Our notation for elliptic functions is as by Whittaker and Watson [14], except that \(\omega_2\) is denoted by us as \(\omega_3\); for the convenience of the reader we collect the definitions of the functions we use in Appendix A.1.
with a known constant $C_1$ (see (20)), where $\beta = 2\omega_1 \omega_3/(\pi i)$ ($\omega_1$ is fixed). This non-stationary Heun equation appears in several physics contexts, including the Wess-Zumino-Witten model [15], the eight-vertex model [16] and Liouville field theory [17]. The second known identity provides a kernel function for a pair of Heun differential operators (recall that a function $F(x, y)$ of two variables $x$ and $y$ is called a kernel function of two differential operators $D(x)$ and $\tilde{D}(y)$ if $[D(x) - \tilde{D}(y) - c]F(x, y) = 0$ for some constant $c$): It is known that the function

$$\Psi_{1,1}(x, y) = \prod_{\nu=0}^{3} \frac{\theta_{\nu+1}(x)^{g_\nu} \theta_{\nu+1}(y)^{\tilde{g}_\nu}}{\theta_1(x-y)^{-\lambda} \theta_1(x+y)^{\lambda}}$$

is a kernel function of the Heun differential operators $H(x; \{g_\nu\}_{\nu=0}^3)$ and $H(y; \{\tilde{g}_\nu\}_{\nu=0}^3)$ provided that $\tilde{g}_\nu = \lambda - g_\nu$ and $\lambda = (g_0 + g_1 + g_2 + g_3)/2$ [18, 20]. In fact, we find that, for arbitrary $\lambda$, the function in (5) is a generalized kernel function of these differential operators in the following sense,

$$\left\{ 2(g_0 + g_1 + g_2 + g_3 - 2\lambda) \frac{\partial}{\partial \beta} + H_1(x; \{g_\nu\}_{\nu=0}^3) - H_1(y; \{\tilde{g}_\nu\}_{\nu=0}^3) + C_{1,1} \right\} \Psi_{1,1}(x, y) = 0$$

with a known constant $C_{1,1}$ (see (30)). Moreover, our results include another generalized kernel function for two Heun-type differential operators: the function

$$\tilde{\Psi}_1(x, y) = \theta_1(x-y)\theta_1(x+y) \prod_{\nu=0}^{3} \theta_{\nu+1}(x)^{g_\nu} \theta_{\nu+1}(y)^{\tilde{g}_\nu}$$

obeys

$$\left\{ 2(g_0 + g_1 + g_2 + g_3 + 2\lambda) \frac{\partial}{\partial \beta} + H_1(x; \{g_\nu\}_{\nu=0}^3) + \lambda H_1(y; \{\tilde{g}_\nu\}_{\nu=0}^3) + \tilde{C}_{1,1} \right\} \tilde{\Psi}_{1,1}(x, y) = 0$$

provided that $\tilde{g}_\nu = (2g_\nu + 1 - \lambda)/(2\lambda)$, for some known constant $\tilde{C}_{1,1}$ (see (31)) and arbitrary $\lambda \neq 0$.

In this paper we present and prove many-variable generalizations of the identities in the previous paragraph: we obtain a generalizations of the non-stationary Heun equation in (3)–(4), and of the two kinds of generalized kernel function identities in (5)–(6) and (7)–(8), to Inozemtsev Hamiltonians, for arbitrary particle numbers; see Corollaries 3.1, 3.2 and 3.3 respectively. Note that, in the latter two cases, the two Inozemtsev Hamiltonians can have different particle numbers $N$ and $M$. The most general kernel function identity we obtain is for a pair of differential operators

$$H_{N,\tilde{N}}(x, \tilde{x}; \{g_\nu\}_{\nu=0}^3, \lambda) = H_N(x; \{g_\nu\}_{\nu=0}^3, \lambda) - \lambda H_{\tilde{N}}(\tilde{x}; \{(\lambda + 1 - 2g_\nu)/(2\lambda)\}_{\nu=0}^3, 1/\lambda)$$

$$+ \sum_{j=1}^{N} \sum_{k=1}^{\tilde{N}} 2(1 - \lambda) \{ \varphi(x_j - \tilde{x}_k) + \varphi(x_j + \tilde{x}_k) \}$$

and $H_{M,M}(y, \tilde{y}; (\lambda - g_\nu)_{\nu=0}^3, \lambda)$, with $H_N(x; \{g_\nu\}_{\nu=0}^3, \lambda)$ in (11), for arbitrary particle numbers $N$, $\tilde{N}$, $M$, $\tilde{M}$; see Corollary 3.4. All our results are special cases of a many-variable generalization of the non-stationary Heun equation in (3)–(4) to a generalized Inozemtsev-type differential operator where all particle mass parameters can be different and where the
interaction strengths depend on these "masses" in a particular way; see Theorem 2.1. We refer to this as source identity since it is the source of all our other results: the latter are obtained in a simple way as special cases. Remarkably, direct proofs of these special cases are often more complicated than the proof of the source identity.

It is known that integrable quantum mechanical models of Calogero-Moser-Sutherland type allow for deformations that share many of their beautiful mathematical properties [22], [23], [24], and for the BC_N trigonometric Calogero-Sutherland system this deformation corresponds to the trigonometric limit of the differential operator in (9). It thus is natural to conjecture that the generalization of this deformation to the Inozemtsev model is given by (9).

The differential operator defining the Sutherland model (= A_{N-1} trigonometric Calogero-Sutherland model) has a well-known kernel functions which can be used to construct the eigenfunctions and eigenvalues of the Sutherland model [25], and this approach can be generalized to the elliptic case [26] and to all quantum Calogero-Moser-Sutherland models associated with classical orthogonal polynomials [6]. Moreover, the latter results allow for a natural generalization to the deformed models [6]. Our results in the present paper provide the starting point to generalize these results to the Inozemtsev model and its deformation in (9).

The first example of a source identity was found for the Sutherland model by Sen [27], and that this identities can be used to obtain kernel functions for the Sutherland model and its deformations was pointed out in [28]. Source identities for all quantum Calogero-Moser-Sutherland models where obtained and used to derive kernel functions in [6]. A source identity allowing to derive kernel functions for the elliptic generalizations of the Sutherland model and their deformations was presented in [29]. The present paper generalizes, to the elliptic case, results previously obtained in [6].

We mention four further topics for future research suggested by the results in the present paper. First, kernel functions of Calogero-Sutherland models can be regarded as a natural quantum analogue of Bäcklund transformations found by Wojciechowski [30]; see also [31]. This suggests that our results can provide Bäcklund transformations for the classical version of the Inozemtsev model. Second, kernel functions for the Sutherland model have been used to construct Q-operators that allow to derive integral representations for the eigenfunctions of this model [32]. This suggests that it is possible to extend Sklyanin’s separation-of-variable approach [33] to the Inozemtsev model using our results. Third, as pointed in [34] (see also [35]), some special cases of the Inozemtsev model are quasi-exactly solvable in the sense that the computation of a finite number, m, of eigenfunctions can be reduced to diagonalizing a m × m matrix (note that the Hamiltonian in Eqs. (2)–(4) in [34] is identical with the Inozemtsev Hamiltonian as in (1) for λ = a, g_0 = g_1 = g_3 = b, g_2 = −3b − 2[m + λ(N − 1)]). We find that these cases are special with regard to our kernel functions (since, by applying the result in Corollary 3.3 to the parameters above and (N,M) = (N,m), we obtain A_{N,M} = 0). This suggests that our results might shed new light, and possibly allow to extend, the results in [34]. Fourth, while many kernel functions for Ruijsenaars’ relativistic generalizations of Calogero-Sutherland-type systems [36] are known [37], the results in [6] and the present paper suggest that many more such relativistic kernel functions should
exist. We believe that a good strategy to find all such kernel identities would be to find relativistic generalizations of the source identities obtained in [6, 29] and the present paper.

2 Main result

As discussed, our main result is a heat-type equation for a Schrödinger-type differential operator. Our notations for elliptic functions is defined in Appendix A.

Theorem 2.1. (Source Identity): For $N$ a positive integer, $\lambda$, $d_\nu$ ($\nu = 0, 1, 2, 3$) and $m_J \neq 0$ ($J = 1, 2, \ldots, N$) complex constants, and $X_J$ ($J = 1, 2, \ldots, N$) complex variables, let

$$
\Phi_0(X) = \left( \prod_{J=1}^N \prod_{\nu=0}^3 \theta_{\nu+1}(X_J)^{g_{\nu,J}} \right) \prod_{1 \leq J < K \leq N} \theta_1(X_J - X_K)^{m_J m_K \lambda} \theta_1(X_J + X_K)^{m_J m_K \lambda}
$$

(10)

and

$$
\mathcal{H} = \sum_{J=1}^N \frac{1}{m_J} \left( -\frac{\partial^2}{\partial X_J^2} + \sum_{\nu=0}^3 g_{\nu,J}(g_{\nu,J} - 1)\varphi(X_J + \omega_\nu) \right) + \sum_{1 \leq J < K \leq N} \gamma_{J,K} \{\varphi(X_J - X_K) + \varphi(X_J + X_K)\}
$$

(11)

with

$$
\gamma_{J,K} = \lambda(m_J + m_K)(\lambda m_J m_K - 1)
$$

(12)

$$
g_{\nu,J} = m_J d_\nu + \frac{\lambda}{2} m_J^2
$$

(13)

Then

$$
\left\{ (4\lambda|m| + 2|d|) \frac{\partial}{\partial \beta} + \mathcal{H} - \mathcal{E}_0 \right\} \Phi_0(X) = 0
$$

(14)

with

$$
\mathcal{E}_0 = (2\lambda|m| + |d|) \left\{ N - \lambda(|m|^2 + |m^2|) - |m||d| \right\} \frac{\eta_1}{\omega_1} + |m| \{(d_0d_1 + d_2d_3)e_1 + (d_0d_2 + d_1d_3)e_2 + (d_0d_3 + d_1d_2)e_3\}
$$

(15)

$$
|d| = \sum_{\nu=0}^3 d_\nu, \quad |m| = \sum_{J=1}^N m_J, \quad |m^2| = \sum_{J=1}^N m_J^2
$$

(16)

Proof. Consider the differential operator

$$
\bar{\mathcal{H}} = \sum_j \frac{1}{m_J} Q_j^+ Q_j^{-}
$$

(17)

with

$$
Q_j^+ = \pm \frac{\partial}{\partial X_J} + \nu_J, \quad \nu_J = \frac{1}{\Phi_0(X)} \frac{\partial \Phi_0(X)}{\partial X_J}
$$

(18)
Using identities of elliptic functions collected and proved in Appendix A.2 we find, by straightforward computations (details are given in Appendix B),
\[ \tilde{H} = (4|m|\lambda + 2|d|) \frac{1}{\Phi_0} \frac{\partial}{\partial \beta} \Phi_0 + H - \mathcal{E}_0. \] (19)
By definition, \( Q^J_\nu \Phi_0 = 0 \) for all \( J \). Thus \( \tilde{H} \Phi_0 = 0 \), and (19) implies our result in (14). \( \square \)

Under suitable restrictions on parameters (see below), the differential operator \( H \) in (11) has a natural physical interpretation as Hamiltonian describing \( \mathcal{N} \) distinguishable quantum particles with interactions, and the results above provide the exact groundstate and groundstate energy of this Hamiltonian. Namely, if \( \omega_1 > 0, -i\omega_3 > 0, \lambda > 0, m_j > 0, d_0 > -\lambda m_j / 2 \) and \( d_1 > -\lambda m_j / 2 \) for \( J = 1, 2, \ldots, \mathcal{N} \), \( d_\nu \) real for \( \nu = 2, 3 \), and \( 4|m|\lambda + 2|d| = 0 \), then the Hamiltonian in (11) defines a unique self-adjoint operator on the Hilbert space \( L^2([0, \omega_1]^N) \) which has \( \Phi_0(X) \) as groundstate and \( \mathcal{E}_0 \) as groundstate energy. (This is true because, under these conditions, \( \mathcal{E}_0 \) and the potential terms in the Hamiltonian \( H \) are real, the function \( \Phi_0(X) \) is square-integrable, and the Hilbert space adjoint of \( Q^J_\nu \) is equal to the closure of \( Q^J_\nu \). This, (17), (19), and the vanishing of \( \partial / \partial \beta \)-term in (19) imply that \( \sum_j (Q^J_\nu)^j Q^J_\nu + \mathcal{E}_0 \) defines such a self-adjoint extension \( \mathcal{H} \); see e.g. [38]). In the rest of this paper the self-adjointness of Inozemtsev-type differential operators will play no role.

3 Special cases

To state important special cases of our main result we use the following notation
\[ H_N(x; \{ g_\nu \}_{\nu=0}^3, \lambda) = \sum_{j=1}^N \left( -\frac{\partial^2}{\partial x_j^2} + \sum_{\nu=0}^3 g_\nu (g_\nu - 1) \varphi(x_j + \omega_\nu) \right) \]
\[ + \sum_{1 \leq j < k \leq N} 2\lambda (\lambda - 1) \left\{ \varphi(x_j - x_k) + \varphi(x_j + x_k) \right\}, \]
\[ \Psi_N(x; \{ g_\nu \}_{\nu=0}^3, \lambda) = \left( \prod_{j=1}^N \prod_{\nu=0}^3 \theta_{\nu+1}(x_j)^{g_\nu} \right) \prod_{1 \leq j < k \leq N} \theta_1(x_j - x_k)^{\lambda} \theta_1(x_j + x_k)^{\lambda} \] (21)
for \( x = (x_1, \ldots, x_N) \) and complex variables \( x_j \). We also use the abbreviations
\[ c_0 = \{(g_0g_1 + g_2g_3)e_1 + (g_0g_2 + g_1g_3)e_2 + (g_0g_3 + g_1g_2)e_3\}, \]
\[ |g| = g_0 + g_1 + g_2 + g_3. \] (22) (23)

We first state the many-variable generalization of the non-stationary Heun equation in (3)-(4).

Corollary 3.1. For \( N \) a positive integer, \( g_\nu (\nu = 0, 1, 2, 3) \) and \( \lambda \) complex parameters, the following holds true
\[ \left\{ A_N \frac{\partial}{\partial \beta} + H_N(x; \{ g_\nu \}_{\nu=0}^3, \lambda) - C_N \right\} \Psi_N(x) = 0 \] (24)
with
\[ A_N = 4\lambda(N - 1) + 2|g| \]
\[ C_N = \frac{A_N}{2}N[1 - \lambda(N - 1) - |g|]\eta_1 + Nc_0. \]  
(25)  
(26)

**Proof.** Set \( N = N, d_\nu = g_\nu - \lambda/2 \) (\( \nu = 0, 1, 2, 3 \)), \((m_J, X_J) = (1, x_J)\) for \( J = 1, 2, \ldots, N \) in Theorem 2.1 and rename \( \mathcal{H}, \Phi_0(x), \mathcal{E}_0 \) to \( H_N(x), \Psi_N(x), C_N \), respectively. Recall \( e_1 + e_2 + e_3 = 0 \), which implies \( (d_0d_1 + d_2d_3)e_1 + (d_0d_2 + d_1d_3)e_2 + (d_0d_3 + d_1d_2)e_3 = c_0. \)

The many-variable generalization of the generalized kernel function identity in (5)–(6) is as follows.

**Corollary 3.2.** For \( N, M \) non-negative integers such that \( N + M > 0 \), \( g_\nu \) (\( \nu = 0, 1, 2, 3 \)) and \( \lambda \) complex parameters, let \( \tilde{g}_\nu = \lambda - g_\nu \) and
\[ \Psi_{N,M}(x, y) = \frac{\Psi_N(x; \{ g_\nu \}_{\nu=0}^3, \lambda)\Psi_M(y; \{ \tilde{g}_\nu \}_{\nu=0}^3, \lambda)}{\prod_{j=1}^N \prod_{k=1}^M \theta_1(x_j - y_k)^{\lambda} \theta_1(x_j + y_k)^{\lambda}}. \] (27)

Then
\[ \left\{ A_{N,M} \frac{\partial}{\partial \beta} + H_N(x; \{ g_\nu \}_{\nu=0}^3, \lambda) - H_M(y; \{ \tilde{g}_\nu \}_{\nu=0}^3, \lambda) - C_{N,M} \right\} \Psi_{N,M}(x, y) = 0 \] (28)
with
\[ A_{N,M} = 4\lambda(N - M - 1) + 2|g| \]
\[ C_{N,M} = \frac{A_{N,M}}{2} \{(N + M)(1 - \lambda) - (N - M)[(N - M - 2)\lambda + |g|]\eta_1 + (N - M)c_0. \] (29)  
(30)

**Proof.** Similarly as above, but now set \( N = N + M, d_\nu = g_\nu - \lambda/2 \) (\( \nu = 0, 1, 2, 3 \)), and \((m_J, X_J) = \begin{cases} (1, x_J), & J = 1, \ldots, N \\ (-1, y_{J-N}), & J = N + 1, \ldots, N + M \end{cases} \) in Theorem 2.1

The many-variable generalization of the generalized kernel function identity in (7)–(8) is as follows.

**Corollary 3.3.** For \( N, M \) non-negative integers such that \( N + M > 0 \), \( g_\nu \) (\( \nu = 0, 1, 2, 3 \)) and \( \lambda \neq 0 \) complex parameters, let \( \tilde{g}_\nu' = (2g_\nu + 1 - \lambda)/(2\lambda) \) and
\[ \tilde{\Psi}_{N,M}(x, y) = \left( \prod_{j=1}^N \prod_{k=1}^M \theta_1(x_j - y_k)^{\theta_1(x_j + y_k)} \right)^{\Psi_N(x; \{ g_\nu \}_{\nu=0}^3, \lambda)\Psi_M(y; \{ \tilde{g}_\nu' \}_{\nu=0}^3, 1/\lambda)}. \] (31)

Then
\[ \left\{ \tilde{A}_{N,M} \frac{\partial}{\partial \beta} + H_N(x; \{ g_\nu \}_{\nu=0}^3, \lambda) + \lambda H_M(y; \{ \tilde{g}_\nu' \}_{\nu=0}^3, 1/\lambda) - \tilde{C}_{N,M} \right\} \tilde{\Psi}_{N,M}(x, y) = 0 \] (32)
Proof. Similarly as above, but now set \( N + \tilde{N} + M + \tilde{M} > 0 \), \( d_\nu \ (\nu = 0, 1, 2, 3) \) and \( \lambda \neq 0 \) complex parameters, let \( g_\nu = d_\nu + \lambda/2 \ (\nu = 0, 1, 2, 3) \) and in Theorem 2.1.

We finally state the generalized kernel function identity for deformed Inozemtsev Hamiltonians. Note that all previous results stated in this section are special cases of this.

**Corollary 3.4.** For \( N, \tilde{N}, M, \tilde{M} \) non-negative integers such that \( N + \tilde{N} + M + \tilde{M} > 0 \), \( d_\nu \ (\nu = 0, 1, 2, 3) \) and \( \lambda \neq 0 \) complex parameters, let \( g_\nu = d_\nu + \lambda/2 \ (\nu = 0, 1, 2, 3) \) and

\[
H_{N,\tilde{N}}^{(\pm)}(x, \tilde{x}) = H_N(x; \{\lambda/2 \pm d_\nu\}_{\nu=0}^3, \lambda) - \lambda H_{\tilde{N}}(\tilde{x}; \{(1/2 \mp d_\nu)/\lambda\}_{\nu=0}^3, 1/\lambda) + \sum_j^{\tilde{N}} \sum_k^N 2(1 - \lambda) \left\{\varphi(x_j - \tilde{x}_k) + \varphi(x_j + \tilde{x}_k)\right\},
\]

\[
\Psi_{N,\tilde{N}}^{(\pm)}(x, \tilde{x}) = \frac{\Psi_N(x; \{\lambda/2 \pm d_\nu\}_{\nu=0}^3, \lambda) \Psi_{\tilde{N}}(\tilde{x}; \{(1/2 \mp d_\nu)/\lambda\}_{\nu=0}^3, 1/\lambda)}{\prod_{j=1}^{\tilde{N}} \prod_{k=1}^N \theta_1(x_j - \tilde{x}_k) \theta_1(x_j + \tilde{x}_k)} \times \prod_{r=\pm} \left( \prod_{j=1}^{\tilde{N}} \prod_{k=1}^M \theta_1(x_j - ry_k) \right) \left( \prod_{j=1}^{\tilde{N}} \prod_{k=1}^M \theta_1(\tilde{x}_j - \tilde{y}_k) \right) \right) \right)
\]

Then

\[
\left(A_{N,\tilde{N},M,\tilde{M}} \frac{\partial}{\partial \beta} + H_{N,\tilde{N}}^{(\mp)}(x, \tilde{x}) - H_{M,\tilde{M}}^{(\pm)}(y, \tilde{y}) - C_{N,\tilde{N},M,\tilde{M}}\right) \Psi_{N,\tilde{N},M,\tilde{M}}(x, \tilde{x}, y, \tilde{y}) = 0
\]

with

\[
A_{N,\tilde{N},M,\tilde{M}} = 4\lambda(N - M - 1) - 4(\tilde{N} - \tilde{M}) + 2|g|
\]

\[
C_{N,\tilde{N},M,\tilde{M}} = \frac{A_{N,\tilde{N},M,\tilde{M}}}{2} \left\{N + \tilde{N} + M + \tilde{M} - |m||(|m| - 2)\lambda + |g| - |m|^2|\lambda\right\} \frac{\eta_1}{\omega_1} + |m|c_0,
\]

\[
|m| = N - M - (\tilde{N} - \tilde{M})/\lambda, \quad |m|^2 = N + M + (\tilde{N} + \tilde{M})/\lambda^2.
\]
Proof. Similarly as above, but now set \( \mathcal{N} = N + \tilde{N} + M + \tilde{M} \) and 

\[
(m_J, X_J) = \begin{cases} 
(1, x_J), & J = 1, \ldots, N \\
(-1/\lambda, \tilde{x}_{J-N}), & J = N + 1, \ldots, N + \tilde{N} \\
(-1, y_{J-N}), & J = N + \tilde{N} + 1, \ldots, N + \tilde{N} + M \\
(1/\lambda, \tilde{y}_{J-\tilde{N}-M}), & J = N + \tilde{N} + M + 1, \ldots N + \tilde{N} + M + \tilde{M} 
\end{cases}
\]

in Theorem 2.1.

Note that \( H^{(+)}_{N,\tilde{N}}(x, \tilde{x}) \) in (35) is equal to \( H_{N,\tilde{N}}(x, \tilde{x}; \{g_\nu\}_{\nu=0}^3, \lambda) \) in (9), and \( H^{(-)}_{M,\tilde{M}}(y, \tilde{y}) \) in (35) is equal to \( H_{M,\tilde{M}}(y, \tilde{y}; \{\lambda - g_\nu\}_{\nu=0}^3, \lambda) \). Thus the general kernel identity in Corollary 3.4 is a natural generalization of the one in Corollary 3.2.

We also note that the generalized kernel function identity on Corollary 3.4 is invariant under the following transformations,

\[
(N, \tilde{N}, M, \tilde{M}, \{g_\nu\}_{\nu=0}^3, \lambda) \rightarrow (M, \tilde{M}, N, \tilde{N}, \{\lambda - g_\nu\}_{\nu=0}^3, \lambda) 
\]

(41)

and these symmetries provide non-trivial checks of our computations. To be more specific: under the transformation in (41), the constants in (39) and (40) change as \( A_{N,\tilde{N},M,\tilde{M}} \rightarrow -A_{N,\tilde{N},M,\tilde{M}}, C_{N,\tilde{N},M,\tilde{M}} \rightarrow -C_{N,\tilde{N},M,\tilde{M}} \), and under the transformations in (42) they change as \( A_{N,\tilde{N},M,\tilde{M}} \rightarrow -A_{N,\tilde{N},M,\tilde{M}}/\lambda, C_{N,\tilde{N},M,\tilde{M}} \rightarrow -C_{N,\tilde{N},M,\tilde{M}}/\lambda \), consistent with the transformation properties of the r.h.s. of the generalized kernel function identity in (38). Note that not only (41) but also (42) is a duality transformation (i.e. applying each of these transformations twice gives the identity).

4 Applications

Using the kernel functions obtained in the previous section it is possible to extend methods developed in [6, 26, 19] (e.g.) to construct eigenfunctions and eigenvalues of Inozemtsev-type differential operators. This section describes a general strategy and two simple examples. More systematic studies are left to future work.

4.1 Integral transformations

We explain how the kernel functions obtained in the previous section can be used to construct integral transformations that map a known generalized eigenfunction of a Inozemtsev-type differential operator to a generalized eigenfunction of another such operator.

Let \( \Psi_{N,\tilde{N},M,\tilde{M}} \equiv \Psi_{N,\tilde{N},M,\tilde{M}}(x, \tilde{x}, y, \tilde{y}) \) be a generalized kernel function, \( H^{(+)}_{N,\tilde{N}}(x, \tilde{x}) \) and \( H^{(-)}_{M,\tilde{M}}(y, \tilde{y}) \) Inozemtsev-type differential operators, and \( A_{N,\tilde{N},M,\tilde{M}} \) and \( C_{N,\tilde{N},M,\tilde{M}} \) constants.
as in Corollary 3.4. If $f(y, \tilde{y})$ is a generalized eigenfunctions of the differential operator $H_{M, \tilde{M}}^{(-)}(y, \tilde{y})$ in the following sense,

$$\left(A_{N, \tilde{N}, M, \tilde{M}} \frac{\partial}{\partial \beta} + H_{M, \tilde{M}}^{(-)}(y, \tilde{y}) - E\right) f(y, \tilde{y}) = 0$$

(43)

for some constant $E$, then (38) implies

$$\left\{ A_{N, \tilde{N}, M, \tilde{M}} \frac{\partial}{\partial \beta} + H_{N, \tilde{N}}^{(+)}(x, \tilde{x}) - E - C_{N, \tilde{N}, M, \tilde{M}} \right\} \Psi_{N, \tilde{N}, M, \tilde{M}} f(y, \tilde{y}) = f(y, \tilde{y}) \{ H_{N, \tilde{N}}^{(-)}(y, \tilde{y}) \Psi_{N, \tilde{N}, M, \tilde{M}} \} - \Psi_{N, \tilde{N}, M, \tilde{M}} \{ H_{N, \tilde{N}}^{(-)}(y, \tilde{y}) f(y, \tilde{y}) \}$$

$$= - \sum_{j=1}^{M} \frac{\partial}{\partial y_j} \left( f(y, \tilde{y}) \frac{\partial}{\partial y_j} \Psi_{N, \tilde{N}, M, \tilde{M}} - \Psi_{N, \tilde{N}, M, \tilde{M}} \frac{\partial}{\partial y_j} f(y, \tilde{y}) \right)$$

$$+ \lambda \sum_{k=1}^{\tilde{M}} \frac{\partial}{\partial \tilde{y}_k} \left( f(y, \tilde{y}) \frac{\partial}{\partial \tilde{y}_k} \Psi_{N, \tilde{N}, M, \tilde{M}} - \Psi_{N, \tilde{N}, M, \tilde{M}} \frac{\partial}{\partial \tilde{y}_k} f(y, \tilde{y}) \right).$$

Integrating this with respect to the variables $(y, \tilde{y})$ over a suitable region $C$, one finds that

$$\tilde{f}(x, \tilde{x}) = \int_{C} \Psi_{N, \tilde{N}, M, \tilde{M}}(x, \tilde{x}, y, \tilde{y}) f(y, \tilde{y}) d^M y d^{\tilde{M}} \tilde{y}$$

(45)

is a generalized eigenfunctions of the differential operator $H_{N, \tilde{N}}^{(+)}(x, \tilde{x})$, i.e.,

$$\left(A_{N, \tilde{N}, M, \tilde{M}} \frac{\partial}{\partial \beta} + H_{N, \tilde{N}}^{(+)}(x, \tilde{x}) - E - C_{N, \tilde{N}, M, \tilde{M}} \right) \tilde{f}(x, \tilde{x}) = 0.$$

(46)

Note that a key point in the derivation of this result is that the region $C$ is suitable in the following sense: First, the integral in (45) has to be well-defined, and second, the integral over the total derivative terms in the last two lines of (43) must vanish (in general, Stokes’ theorem implies that the latter is equal to an integral over the boundary of $C$).

4.2 Example 1

To be specific we assume throughout this section that $\omega_1 > 0$, $-i\omega_3 > 0$, and that $x_j$ and $\tilde{x}_j$ are real variables. As discussed, in this case the Inozemtsev Hamiltonian can be interpreted as a quantum mechanical model of a many-particle system.

We consider the result in the previous section in the case $M = 1$, $\tilde{M} = 0$, and $\tilde{g}_\nu \in \{0, 1\}$ ($\nu = 0, 1, 2, 3$). Then $H_{M, \tilde{M}}^{(-)}(y, \tilde{y}) = -\partial^2/\partial y^2$, and it is trivial to find solutions of (43): $f(y) = \exp(-iy\nu)$ and $E = p^2$, with $p$ an arbitrary constant. Moreover,

$$\Psi_{N, \tilde{N}, 1, 0}(x, \tilde{x}, y) = \Psi_{N, \tilde{N}}^{(+)}(x, \tilde{x}) \left( \prod_{\nu=0}^{3} \theta_{1+\nu}(y)^{\tilde{g}_\nu} \right) \frac{\prod_{j=1}^{N} \theta_1(\tilde{x}_j - y) \theta_1(\tilde{x}_j + y)}{\prod_{j=1}^{N} \theta_1(x_j - y)^{\lambda} \theta_1(x_j + y)^{\lambda}}.$$
A suitable integration region in this case is any path $C$ in the complex $y$-plane such that $\Psi_{N,\tilde{N},1,0}(x,\tilde{x},y)f(y)$ is analytic in some neighborhood $C$ and such that

$$
\int_C \frac{\partial}{\partial y} \left( f(y) \frac{\partial}{\partial y} \Psi_{N,\tilde{N},1,0}(x,\tilde{x},y) - \Psi_{N,\tilde{N},1,0}(x,\tilde{x},y) \frac{\partial}{\partial y} f(y) \right) \, dy = 0. \tag{48}
$$

To find such a path we use that the Theta-functions $\theta_1(x)$ can be expressed in terms of meromorphic functions $\tilde{\theta}_1(x)$ of the variable $x = \exp(-2it/R)$ as follows

$$
\theta_1(x) = e^{-\pi \tau_{1/4} x^2/R} \tilde{\theta}_1(x), \quad \tilde{\theta}_1(z) = \sum_{n=0}^{\infty} (-1)^n e^{\pi i n(n+1)} (z^n - z^{n+1}) \tag{49}
$$

(this is a simple consequence of the definition of the Theta-functions in (66)). For $p = (2n + \tilde{g}_0 + \tilde{g}_1)/R$ and $n$ an integer, one thus finds that $\Psi_{N,\tilde{N},1,0}(x,\tilde{x},y)\exp(-ip\gamma y)$ is a holomorphic function in the variable $\xi = \exp(-2i\gamma y/R)$ in the region $1 < |\xi| < q^{-2}$, and that (48) is satisfied for $C$ the straight line from $y = \epsilon e \to \pi R + \epsilon e$, with $\epsilon > 0$ such that $\exp(2\epsilon/R) < q^{-2}$ (since this corresponds to a closed path in the complex $\xi$-plane where the integrand is holomorphic). Taking the limit $\epsilon \to 0^+$ we obtain the following.

**Proposition 4.1.** For $N, \tilde{N}$ non-negative integers such that $N + \tilde{N} > 0$, $\lambda$ a non-zero constant, $\tilde{g}_\nu \in \{0,1\}$ ($\nu = 0, 1, 2, 3$), and $n$ an arbitrary integer, let $d_\nu = \lambda/2 - \tilde{g}_\nu$, $H_{N,\tilde{N}}^{(+)}(x,\tilde{x})$ as in (36), and $\Psi_{N,\tilde{N}}^{(+)}(x,\tilde{x})$ as in (37). Then the function

$$
\tilde{f}_n(x,\tilde{x}) = \Psi_{N,\tilde{N}}^{(+)}(x,\tilde{x}) \lim_{\epsilon \to 0^+} \int_{\epsilon e}^{\pi R + \epsilon e} \left( \prod_{\nu=0}^{3} \theta_{1+\nu}(y) \tilde{g}_\nu \right) \prod_{j=1}^{N} \frac{\theta_1(\tilde{x}_j - y)\theta_1(\tilde{x}_j + y)}{\theta_1(\tilde{x}_j - y)\theta_1(\tilde{x}_j + y)^\lambda} e^{-i(2n + \tilde{g}_0 + \tilde{g}_1)y/R} \, dy \tag{50}
$$

is well-defined and obeys the equation

$$
\left( A_{N,\tilde{N},1,0} \frac{\partial}{\partial \beta} + H_{N,\tilde{N}}^{(+)}(x,\tilde{x}) - (2n + \tilde{g}_0 + \tilde{g}_1)^2/R^2 - C_{N,\tilde{N},1,0} \right) \tilde{f}_n(x,\tilde{x}) = 0 \tag{51}
$$

with

$$
A_{N,\tilde{N},1,0} = 4\lambda N - 4\tilde{N} - 2|\tilde{g}|, \tag{52}
$$

$$
C_{N,\tilde{N},1,0} = \frac{A_{N,\tilde{N},1,0}}{2} \left\{ N + \tilde{N} + 1 - |m||([|m|+2]\lambda - |\tilde{g}|) - |m^2|\lambda \right\} + |m|c_0. \tag{53}
$$

$$
c_0 = \left\{ (\tilde{g}_0\tilde{g}_1 + \tilde{g}_2\tilde{g}_3)e_1 + (\tilde{g}_0\tilde{g}_2 + \tilde{g}_1\tilde{g}_3)e_2 + (\tilde{g}_0\tilde{g}_3 + \tilde{g}_1\tilde{g}_2)e_3 \right\}, \quad |\tilde{g}| = \tilde{g}_0 + \tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3, \quad |m| = N - 1 - \tilde{N}/\lambda, \quad |m^2| = N + 1 + \tilde{N}/\lambda^2.
$$
A noteworthy special case is $N = 1$, $\tilde{N} = 0$, and $\lambda = |\tilde{g}|/2$. Then the function in (50) is

$$
f_n(x) = \left( \prod_{\nu=0}^{3} \theta_{1+\nu}(x)^{\nu} \right) \int_{0}^{\pi R} \frac{(\prod_{\nu=0}^{3} \theta_{1+\nu}(y)\tilde{g}_\nu)}{\theta_{1}(x-y)^{\lambda} \theta_{1}(x+y)^{\lambda}} e^{-i(2n+\tilde{g}_0+\tilde{g}_1)y/R} dy \tag{54}
$$

with $g_\nu = |\tilde{g}|/2 - \tilde{g}_\nu$, and (51) is the Heun differential equation

$$
\left( -\frac{\partial^2}{\partial x^2} + \sum_{\nu=0}^{3} g_\nu (g_\nu - 1) \varphi(x + \omega_\nu) - (2n + \tilde{g}_0 + \tilde{g}_1)^2/R^2 \right) f_n(x) = 0. \tag{55}
$$

It is worth noting that the integral factor on the r.h.s. in (50) is proportional to the function $f_n(z, \tilde{z})$, $z_j = \exp(-2ix_j/R)$ and $\tilde{z}_j = \exp(-2i\tilde{x}_j/R)$, defined by the following generating function,

$$
\left( \prod_{\nu=0}^{3} \theta_{1+\nu}(\xi)^{\nu} \right) \left( \prod_{j=1}^{N} \tilde{\theta}_{1}(\tilde{z}_j/\xi)\tilde{\theta}_{1}(\tilde{z}_j\xi) \right) = \sum_{n=-\infty}^{\infty} f_n(z, \tilde{z})\xi^{-n}, \tag{56}
$$

where the series on the r.h.s. is absolutely convergent in the region $1 < |\xi| < q^{-2}$.

### 4.3 Example 2

We show how kernel functions can be used to transform Bethe ansatz solutions of the Heun equation obtained in [19] to eigenfunctions of Inozemtsev-type differential operators with arbitrary particle numbers $N$, $\tilde{N}$ (this is a generalization of a result obtained in [12] for the case $N = 1$, $\tilde{N} = 0$).

As shown in [19], for arbitrary non-negative integers $n_\nu$ ($\nu = 0, 1, 2, 3$) and arbitrary constants $\tilde{E}$, the Heun differential equation

$$
\left( -\frac{\partial^2}{\partial y^2} + \sum_{\nu=0}^{3} n_\nu (n_\nu + 1) \varphi(y + \omega_\nu) - \tilde{E} \right) f(y) = 0 \tag{57}
$$

has a non-zero solution which can be written as

$$
f(y) = \frac{\exp(\kappa y)}{\theta_{1}(y)^{n_0} \theta_{2}(y)^{n_1} \theta_{3}(y)^{n_2} \theta_{4}(y)^{n_3} \prod_{j=1}^{n_0+n_1+n_2+n_3} \theta_{1}(y + t_j)}, \tag{58}
$$

for some constants $t_j$ ($j = 1, \ldots, n_0 + n_1 + n_2 + n_3$) and $\kappa$ (see [19] for how these constants can be determined).

Choosing $M = 1$, $\tilde{M} = 0$, $\tilde{g}_\nu \in \{n_\nu+1, -n_\nu\}$, $\lambda$ such that $A_{N,\tilde{N},1,0} = 0$, $\tilde{E} = E + C_{N,\tilde{N},1,0}$, we can use the result in Section 4.1 to transform this solution to an eigenfunction of the Inozemtsev-type differential operator $H^{(\pm)}_{N,\tilde{N}}(x, \tilde{x})$ with $g_\nu = \lambda - \tilde{g}_\nu$ ($\nu = 0, 1, 2, 3$).

As a suitable integration region we now choose a closed path $C$ in the complex $y$-plane such
that \( \Psi_{N,N,1,0}(x, \tilde{x}, y) f(y) \) is analytic in some neighborhood of this path and such that \( \{18\} \) is fulfilled. For example, for \( p \in \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}, \ i \in \{1, 2, \ldots, N\} \), we can choose as \( C \) a figure-eight contour in the \( y \)-plane which encloses \( y = x_i \) counterclockwise, \( y = 2p - x_i \) clockwise, and which does not contain branching points of \( \Psi_{N,N,1,0}(x, \tilde{x}, y) \) other than \( y = x_i, 2p - x_i \) inside see Fig. A (the condition \( \{18\} \) is satisfied, because the function \( f \) is an eigenfunction of \( H \)).

Obviously there are many more such closed paths \( C \), which we call suitable. We thus obtain the following.

**Proposition 4.2.** For \( N, \tilde{N} \) non-negative integers such that \( N + \tilde{N} > 0 \), \( n_\nu \) non-negative integers \( (\nu = 0, 1, 2, 3) \), let \( \bar{g}_\nu \in \{n_\nu + 1, -n_\nu\} \), \( \lambda \) such that \( 2\lambda N - 2\tilde{N} = \bar{g}_0 + \bar{g}_1 + \bar{g}_2 + \bar{g}_3 \), \( d_\nu = \lambda/2 - \bar{g}_\nu \), \( H^{(+)}_{N,N}(x, \tilde{x}) \) as in \( \{35\} \), and \( \Psi^{(+)}_{N,N}(x, \tilde{x}) \) as in \( \{36\} \). Then, for any constant \( E \), there exit constants \( t_i (i = 1, 2, \ldots, n_0 + n_1 + n_2 + n_3) \) and \( \kappa \) such that, for any suitable closed path \( C \) in the complex \( y \)-plane (e.g. the figure-eight contour in Fig. A), the function

\[
\Psi^{(+)}_{N,N}(x, \tilde{x}) \int_C \frac{\prod_{j=1}^{N} \theta_1(\bar{x}_j - y)\theta_1(\bar{x}_j + y)}{\prod_{j=1}^{N} \theta_1(x_j - y)^\lambda \theta_1(x_j + y)^\lambda} \exp(\kappa y) \left( \prod_{j=0}^{3} \theta_1(y)^{\bar{g}_j - n_\nu} \right)^{n_0 + n_1 + n_2 + n_3} \prod_{j=1}^{n_0 + n_1 + n_2 + n_3} \theta_1(y + t_j) dy,
\]

is an eigenfunction of \( H^{(+)}_{N,N}(x, \tilde{x}) \) with the eigenvalue \( E \).

Note that the constants \( t_j \) and \( \kappa \) can be specified by the condition that the function \( \{58\} \) satisfies \( \{57\} \) with the eigenvalue \( \tilde{E} = E + C_{N,N,1,0} \). Note also that there is another expression of solutions of the Heun equation \( \{57\} \) by the Hermite-Krichever ansatz \[21\], and we can obtain a similar result to Proposition \[4.2\].

We now describe a classical example of the Bethe ansatz where the constants \( t_j \) and \( \kappa \) can be specified in a simple manner. If \( n_0 = 1, n_1 = n_2 = n_3 = 0 \), \( \{57\} \) reduces to a Lamé equation, and its solution in \( \{58\} \) can be written as

\[
f(y) = \exp(-\zeta(t)y) \frac{\sigma(y + t)}{\sigma(y)} = \exp(\eta_1 t^2/(2\omega_1)) \exp(-\phi_1(t)y) \frac{\theta_1(y + t)}{\theta_1(y)} \]

with \( t \) such that \( \varphi(t) = -\tilde{E} \); see \[21\] and references therein. By specializing to this case and \( \bar{g}_0 = -1, \bar{g}_1 = \bar{g}_2 = \bar{g}_3 = 0 \) we obtain the following (note that \( c_0 = 0 \) if \( \bar{g}_1 = \bar{g}_2 = \bar{g}_3 = 0 \)).

The second author apologizes for a vague description of the integral routes in \[12\] Proposition 6]. The orientations of them should be specified as the figure-eight contour in this paper.
Proposition 4.3. For $N$ a positive integer, $\tilde{N}$ a non-negative integer, let 
\[ \lambda = \frac{2\tilde{N} + 1}{2N} \]
\[ d_0 = \lambda/2 + 1, \quad d_1 = d_2 = d_3 = \lambda/2, \quad H^{(\pm)}_{N,\tilde{N}}(x,\tilde{x}) \text{ as in (32), and } \Psi^{(\pm)}_{N,\tilde{N}}(x,\tilde{x}) \text{ as in (30)}. \]
Then, for any closed suitable path $C$ in the complex $y$-plane as described above (e.g. the figure-eight contour in Fig. A), and for any constant $t$ such that $\wp(t)$ is finite, the function
\[ \Psi^{(\pm)}_{N,\tilde{N}}(x,\tilde{x}) \int_C \prod_{j=1}^{\tilde{N}} \theta_1(\tilde{x}_j - y)\theta_1(\tilde{x}_j + y) \exp(-\phi_1(t)y)\frac{\theta_1(y + t)}{\theta_1(y)^2} dy \]
(61)
is an eigenfunction of $H^{(\pm)}_{N,\tilde{N}}(x,\tilde{x})$ with the eigenvalue $E = -\wp(t)$.

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A Elliptic functions

A.1 Definitions

The Weierstrass $\wp$-function with periods $(2\omega_1, 2\omega_3)$ is defined as follows:
\[ \wp(x) = \frac{1}{x^2} + \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \left\{ \frac{1}{(x - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right\} \]
(62)
with $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$. We also recall the definitions of the corresponding Weierstrass zeta- and sigma-functions,
\[ \zeta(x) = \frac{1}{x} + \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \left\{ \frac{1}{x - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right\} \]
and
\[ \sigma(x) = x \prod_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \left\{ \left(1 - \frac{x}{\Omega_{m,n}}\right) \exp\left(\frac{x}{\Omega_{m,n}} + \frac{x^2}{2\Omega_{m,n}^2}\right) \right\} , \]
respectively.

We use the following symbols,
\[ \omega_0 = 0, \quad \omega_2 = -\omega_1 - \omega_3, \]
\[ e_\nu = \wp(\omega_\nu), \quad \eta_\nu = \zeta(\omega_\nu) \quad (\nu = 1, 2, 3), \]
(63)
(64)
\[ q = e^{\pi i \tau}, \quad \tau = \frac{\omega_3}{\omega_1}, \quad R = \frac{2\omega_1}{\pi}, \quad \beta = \frac{2\omega_1\omega_3}{\pi} \]

where we regard \( \omega_1 \) as fixed. We also need \( \theta_\nu(x) = \vartheta_\nu(x/R, q) \) (\( \nu = 1, 2, 3, 4 \)) with the Theta-functions \( \vartheta_\nu(x, q) \) as usual [14], i.e.,

\[
\begin{align*}
\theta_1(x) &= 2 \sum_{n=0}^{\infty} (-1)^n e^{\pi i (n+1/2)^2} \sin(2n+1)x/R, \\
\theta_2(x) &= 2 \sum_{n=0}^{\infty} e^{\pi i (n+1/2)^2} \cos(2n+1)x/R, \\
\theta_3(x) &= 1 + 2 \sum_{n=1}^{\infty} e^{\pi i n^2} \cos 2nx/R, \\
\theta_4(x) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{\pi i n^2} \cos 2nx/R.
\end{align*}
\]

We also define

\[
\phi_\nu(x) = \frac{\theta'_\nu(x)}{\theta_\nu(x)} \quad (\nu = 1, 2, 3, 4). \]

Here and in the following we use the following shorthand notation,

\[
\begin{align*}
\theta'_\nu(x) &\equiv \frac{\partial}{\partial x} \theta_\nu(x), \\
\dot{\theta}_\nu(x) &\equiv \frac{\partial}{\partial \beta} \theta_\nu(x) = \frac{2\pi i}{\omega_1^2} \frac{\partial}{\partial \tau} \theta_\nu(x).
\end{align*}
\]

### A.2 Properties

We recall some well-known properties of the Weierstrass elliptic functions that we need (see e.g. [14], Chapter XX):

\[
\begin{align*}
\varphi(x) &= -\zeta'(x), \quad \zeta(x) = \frac{\sigma'(x)}{\sigma(x)}, \\
e_1 + e_2 + e_3 &= \eta_1 + \eta_2 + \eta_3 = 0, \\
\varphi(x + 2\omega_\nu) &= \varphi(x), \quad \zeta(x + 2\omega_\nu) = \zeta(x) + 2\eta_\nu \quad (\nu = 1, 2, 3) \quad (71) \\
\varphi(-x) &= \varphi(x), \quad \xi(-x) = -\xi(x), \quad \sigma(-x) = -\sigma(x), \quad (72)
\end{align*}
\]

and

\[
\left( \zeta(x_1) + \zeta(x_2) + \zeta(x_3) \right)^2 = \varphi(x_1) + \varphi(x_2) + \varphi(x_3) \quad (x_1 + x_2 + x_3 = 0). \quad (73)
\]

As will be seen, the last identity plays an important role in the proof of our main result. Other identities that we need are the heat equation satisfied by the Theta-functions, i.e.,

\[
\theta''_\nu(x) = 2\dot{\theta}_\nu(x) \quad (\nu = 1, 2, 3, 4) \quad (74)
\]

(obvious from the definitions), and two identities obtained from the well-known duplication formula

\[
\theta_1(2x) = C \theta_1(x) \theta_2(x) \theta_3(x) \theta_4(x)
\]
where $C$ is a constant (see e.g. [14], Example 5 at the end of Chapter XXI), i.e.,

$$\frac{\theta_1''(2x)}{\theta_1(2x)} = \frac{1}{2} \left( \frac{\theta_1'(x)}{\theta_1(x)} + \frac{\theta_2'(x)}{\theta_2(x)} + \frac{\theta_3'(x)}{\theta_3(x)} + \frac{\theta_4'(x)}{\theta_4(x)} \right)$$

and

$$\frac{\theta_1''(2x)}{\theta_1(2x)} = \frac{1}{4} \sum_{\nu=1}^{4} \theta_\nu''(x) + \frac{1}{2} \sum_{1 \leq \mu < \nu \leq 4} \theta_\mu'(x) \theta_\nu'(x).$$

Using (67) and the heat equation we can write these identities as

$$\phi_1(2x) = \frac{1}{2} \sum_{\nu=1}^{4} \phi_\nu(x)$$

and

$$\frac{\dot{\theta}_1(2x)}{\theta_1(2x)} = \frac{1}{4} \sum_{\nu=1}^{4} \frac{\dot{\theta}_\nu(x)}{\theta_\nu(x)} + \frac{1}{4} \sum_{1 \leq \mu < \nu \leq 4} \phi_\mu(x) \phi_\nu(x).$$

We also need the relations between the functions defined in (67) and the Weierstrass zeta function:

$$\phi_1(x) = \zeta(x) - \eta_1 \omega_1, \quad \phi_{\nu+1}(x) = \zeta(x + \omega_\nu) - \eta_\nu - \frac{\eta_1 x}{\omega_1} (\nu = 1, 2, 3)$$

(this is a simple consequence of

$$\theta_1(x) = C_0 \sigma(x) \exp\left(-\frac{\eta_1 x^2}{2\omega_1}\right), \quad \theta_{\nu+1}(x) = C_\nu \sigma(x + \omega_\nu) \exp\left(-\eta_\nu x - \frac{\eta_1 x^2}{2\omega_1}\right) (\nu = 1, 2, 3)$$

for constants $C_\nu$ ($\nu = 0, 1, 2, 3$), which can be obtained by comparing the product representation of the Weierstrass sigma function in [14], § 20.421, with the product representations of the Theta-functions in [14], § 21.3).

In the following we collect several identities needed in the proof of our main result.

**Proposition A.1.** The following holds true,

$$\phi'_{\nu+1}(x) = -\varphi(x + \omega_\nu) - \frac{\eta_1}{\omega_1}$$

and

$$\phi_{\nu+1}(x)^2 = 2 \frac{\dot{\theta}_{\nu+1}(x)}{\theta_{\nu+1}(x)} + \varphi(x + \omega_\nu) + \frac{\eta_1}{\omega_1}$$

for $\nu = 0, 1, 2, 3$, and

$$\phi_{\nu+1}(x) \phi_{\mu+1}(x) = \frac{\dot{\theta}_{\nu+1}(x)}{\theta_{\nu+1}(x)} + \frac{\dot{\theta}_{\mu+1}(x)}{\theta_{\mu+1}(x)} + \frac{\eta_1}{\omega_1} - \frac{e_{\nu,\mu}}{2}$$

for $0 \leq \mu < \nu \leq 3$, where

$$e_{\nu,0} = e_\nu \quad (\nu = 1, 2, 3), \quad e_{2,1} = e_3, \quad e_{3,1} = e_2, \quad e_{3,2} = e_1.$$
Moreover,

\[
\phi_1(x - y)\phi_1(x + y) = \frac{1}{2} \sum_{\nu=0}^{3} \sum_{r=\pm} \phi_{\nu+1}(x)\phi_1(x - ry) - \sum_{\nu=0}^{3} \frac{\dot{\phi}_{\nu+1}(x)}{\theta_{\nu+1}} - \sum_{r=\pm} \frac{\dot{\phi}_1(x - ry)}{\theta_1} - \frac{3\eta_1}{\omega_1} \tag{82}
\]

and

\[
\sum_{r=\pm} (\phi_{\nu+1}(x) - r\phi_{\nu+1}(y))\phi(x - ry) = \sum_{r=\pm} \left(\frac{\dot{\phi}_{\nu+1}(x)}{\theta_{\nu+1}} + \frac{\dot{\phi}_{\nu+1}(y)}{\theta_{\nu+1}} + \frac{\dot{\phi}_1(x - ry)}{\theta_1(x - ry)} + \frac{3\eta_1}{2\omega_1}\right) \tag{83}
\]

for \(\nu = 0, 1, 2, 3\), and

\[
\sum_{r, s=\pm} \left(\phi_1(x - ry)\phi_1(x - sz) + \phi_1(y - rx)\phi_1(y - sz) + \phi_1(z - rx)\phi_1(z - sy)\right) = 2 \sum_{r=\pm} \left(\frac{\dot{\phi}_1(x - ry)}{\theta_1} + \frac{\dot{\phi}_1(x - rz)}{\theta_1} + \frac{\dot{\phi}_1(y - rz)}{\theta_1} + \frac{3\eta_1}{2\omega_1}\right). \tag{84}
\]

**Proof.** Differentiate (77) in \(x\) and use (69) to obtain (78).

Differentiate (67) in \(x\) and use (67) again to obtain \(\phi_\nu(x)^2 = \theta_\nu(x)/\theta_\nu(x) - \phi_\nu'(x)\). Insert the heat equation in (74) and use (78) to obtain (79).

We show (80). Substitute \(x_1 = x + \omega_\nu, x_2 = -x - \omega_\mu\) and \(x_3 = \omega_\mu - \omega_\nu\) in (73), and use \(\zeta(\omega_\mu - \omega_\nu) = \eta_\mu - \eta_\nu\) and \(\varphi(\omega_\mu - \omega_\nu) = e_{\nu,\mu}\) to obtain

\[
(\zeta(x + \omega_\nu) - \zeta(x + \omega_\mu) + \eta_\mu - \eta_\nu)^2 = \varphi(x + \omega_\nu) + \varphi(x + \omega_\mu) + e_{\nu,\mu}.
\]

Insert into this

\[
\zeta(x + \omega_\nu) - \zeta(x + \omega_\mu) + \eta_\mu - \eta_\nu = \phi_{\nu+1}(x) - \phi_{\mu+1}(x)
\]

(this follows from (77)), expand the square, insert (79), and obtain an identity equivalent to (80).

We show (82). Substitute \(x_1 = x - y, x_2 = x + y, x_3 = -2x\) in (73), and use (67) and (77) to obtain

\[
(\phi_1(x - y) + \phi_1(x + y) - \phi_1(2x))^2 = \varphi(x - y) + \varphi(x + y) + \varphi(2x).
\]

Expanding the square and using (79) this can be written as

\[
\phi_1(x - y)\phi_1(x + y) = \phi_1(2x) \sum_{r=\pm} \phi_1(x - ry) - \frac{\dot{\phi}_1(2x)}{\theta_1(2x)} - \sum_{r=\pm} \frac{\dot{\phi}_1(x - ry)}{\theta_1(x - ry)} - \frac{3\eta_1}{2\omega_1}.
\]
Insert into this (75) and
\[
\frac{\dot{\theta}_1(2x)}{\theta_1(2x)} = \frac{1}{4} \sum_{\nu=0}^{3} \frac{3}{\theta_{\nu+1}(x)} + \frac{1}{4} \sum_{0 \leq \nu < \mu \leq 3} \left( \frac{\dot{\theta}_{\mu+1}(x)}{\theta_{\mu+1}(x)} + \frac{\dot{\theta}_{\nu+1}(x) + \eta_1}{\theta_{\nu+1}(x)} \right) + \frac{e_{\nu,\mu}}{2} \right)
= \sum_{\nu=0}^{3} \frac{\dot{\theta}_{\nu+1}(x)}{\theta_{\nu+1}(x)} + \frac{3\eta_1}{2\omega_1}
\]
(we used (76), (80) and \(\sum_{\mu<\nu} e_{\nu,\mu} = 2(e_1 + e_2 + e_3) = 0\) to obtain (82).

We show (83). We substitute \(x_1 = x + \omega_\nu, x_2 = -y - \omega_\nu\) and \(x_3 = -x + y\) in (73). Then
\[
(\phi_{\nu+1}(x) - \phi_{\nu+1}(y) - \phi_1(x - y))^2 = (\zeta(x + \omega_\nu) - \zeta(y + \omega_\nu) - \zeta(x - y))^2
= \phi(x + \omega_\nu) + \phi(y + \omega_\nu) + \phi(x - y)
\]
using (77), and it follows from (79) that
\[
\phi_{\nu+1}(x)\phi_{\nu+1}(y) + \phi_{\nu+1}(x)\phi_1(x - y) - \phi_{\nu+1}(y)\phi_1(x - y)
= \frac{\dot{\theta}_{\nu+1}(x)}{\theta_{\nu+1}(x)} + \frac{\dot{\theta}_{\nu+1}(y)}{\theta_{\nu+1}(y)} + \frac{\dot{\theta}_1(x - y)}{\theta_1(x - y)} + \frac{3\eta_1}{2\omega_1}.
\]
Similarly we have
\[
- \phi_{\nu+1}(x)\phi_{\nu+1}(y) + \phi_{\nu+1}(x)\phi_1(x + y) + \phi_{\nu+1}(y)\phi_1(x + y)
= \frac{\dot{\theta}_{\nu+1}(x)}{\theta_{\nu+1}(x)} + \frac{\dot{\theta}_{\nu+1}(y)}{\theta_{\nu+1}(y)} + \frac{\dot{\theta}_1(x + y)}{\theta_1(x + y)} + \frac{3\eta_1}{2\omega_1},
\]
by substituting \(x_1 = x + \omega_\nu, x_2 = y + \omega_\nu\) and \(x_3 = -x - y - 2\omega_\nu\) in (73). Sum up the last two equalities and obtain (83).

We finally show (84). We substitute \(x_1 = x + ry, x_2 = -x - sz\) and \(x_3 = -ry + sz\), for \(r, s = \pm\), in (73). Then
\[
(\phi_1(x + ry) - \phi_1(x + sz) - \phi_1(ry - sz))^2 = \phi(x + ry) + \phi(x + sz) + \phi(y - rsy)
\]
and, by using (79) as above,
\[
\phi_1(x + ry)\phi_1(x + sz) + \phi_1(y + rx)\phi_1(y - rsy) + \phi_1(z + sx)\phi_1(z - rsy)
= \frac{\dot{\theta}_1(x + ry)}{\theta_1(x + ry)} + \frac{\dot{\theta}_1(x + sz)}{\theta_1(x + sz)} + \frac{\dot{\theta}_1(y - rsz)}{\theta_1(y - rsz)} + \frac{3\eta_1}{2\omega_1}.
\]
Sum the last equality over \(r, s = \pm\) to obtain (84).

\section{Proof of Source Identity (details)}

We compute the functions defined in (18),
\[
V_J = \sum_\nu g_{\nu,J} \phi_{\nu+1}(X_J) + \sum_{K \neq J} \sum_{r=\pm} m_J m_K \lambda \phi_1(X_J - rX_K),
\]
and thus
\[ \tilde{H} = -\sum_j \frac{1}{m_j} \partial^2_j + W, \quad W = \sum_j \frac{1}{m_j} \left( \partial_j V_j + V_j^2 \right) \] (86)

with \( \partial_j = \partial/\partial X_j \). Here and in the following, we write \( \sum_\nu \) short for \( \sum_{\nu=0}^3 \), \( \sum_j \) short for \( \sum_{j=1}^N \), etc.

We compute \( W = W_1 + W_2 + W_3 \) with

\[
W_1 = \sum_j \frac{1}{m_j} \left\{ \sum_\nu \left( g_{\nu,J} \phi_{\nu+1}'(X_J) + g_{\nu,J}^2 \phi_{\nu+1}(X_J)^2 \right) + 2 \sum_{\nu<\mu} g_{\nu,J} g_{\mu,J} \phi_{\nu+1}(X_J) \phi_{\mu+1}(X_J) \right\},
\] (87)

the sum of all one-body terms,

\[
W_2 = \sum_j \sum_{K \neq J} \sum_{r=\pm} \left( m_K \lambda \phi_1'(X_J - rX_K) + m_j m_K^2 \lambda^2 \phi_1(X_J - rX_K)^2 \right.
\]
\[+ 2 \sum_\nu g_{\nu,J} m_K \lambda \phi_{\nu+1}(X_J) \phi_1(X_J - rX_K) \]
\[+ 2 m_j m_K^2 \lambda^2 \phi_1(X_J - X_K) \phi_1(X_J + X_K) \},
\] (88)

the sum of all two-body terms, and

\[
W_3 = \sum_j \sum_{K \neq J} \sum_{L \neq J,K} \sum_{r,s=\pm} m_j m_K m_L \lambda^2 \phi_1(X_J - rX_K) \phi_1(X_J - sX_L)
\] (89)

the sum of all three-body terms.

To simplify the one-body terms we use (78)–(80) and obtain

\[
W_1 = \sum_j \frac{1}{m_j} \left\{ \sum_\nu \left( g_{\nu,J}(g_{\nu,J} - 1) \left( \varphi(z + \omega_\nu) + \frac{\eta_1}{\omega_1} \right) + 2 g_{\nu,J}^2 \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)} \right) + 2 \sum_{\nu<\mu} g_{\nu,J} g_{\mu,J} \left( \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)} + \frac{\dot{\theta}_{\mu+1}(X_J)}{\theta_{\mu+1}(X_J)} + \frac{\eta_1}{\omega_1} - \frac{e_{\nu,\mu}}{2} \right) \right\}.
\]

Changing summations in the last sum we obtain, after some computations,

\[
W_1 = \sum_j \sum_\nu \left( \frac{1}{m_j} g_{\nu,J}(g_{\nu,J} - 1) \varphi(z + \omega_\nu) + 2 g_{\nu,J} |d| + 2 m_j \lambda \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)} \right)
\]
\[+ \sum_j \left( (|d| + 2 m_j \lambda)(m_j |d| + 2 m_j^2 \lambda - 1) \frac{\eta_1}{\omega_1} - m_j \sum_{\mu<\nu} d_{\nu} d_{\mu} e_{\nu,\mu} \right),
\] (90)

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where we used
\[ \sum g_{\mu,J} = m_J(|d| + 2m_J\lambda), \quad \sum e_{\nu,\mu} = 0, \quad \sum (d_{\mu} + d_{\nu})e_{\nu,\mu} = 0 \]
following from (13), (70) and (81).

To compute the two-body terms we use (78), (79) and (82) to obtain
\[ \mathcal{W}_2 = \sum_{J<K} \sum_{r=\pm} \left\{ \sum \left( m_K \lambda (m_J m_K \lambda - 1) \left( \varphi(X_J - rX_K) + \frac{\eta_1}{\omega_1} \right) \right) 
\right. \\
+ \sum_{\nu} (2g_{\nu,J} m_K \lambda + m_J m_K^2 \lambda^2) \phi_{\nu+1}(X_J) \phi_1(X_J - rX_K) \right. \\
\left. - 2m_J m_K^2 \lambda^2 \sum_{\nu} \left( \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)} + \frac{3\eta_1}{4\omega_1} \right) \right\} \]

where terms \( \sum_{r=\pm} \dot{\vartheta}_1(X_J - rX_K)/\vartheta_1(X_J - rX_K) \) are cancelled. Symmetrizing the terms in the first two lines using \( (m_K + m_J)\lambda (m_J m_K \lambda - 1) = \gamma_{JK} \),
\[ 2g_{\nu,J} m_K \lambda + m_J m_K^2 \lambda^2 = m_J m_K \lambda (2d_{\nu} + (m_J + m_K)\lambda) \]
and
\[ \phi_1(X_K - rX_J) = -r \phi_1(X_J - rX_K), \]
we can write this as
\[ \mathcal{W}_2 = \sum_{J<K} \sum_{r=\pm} \left\{ \gamma_{JK} \left( \varphi(X_J - rX_K) + \frac{\eta_1}{\omega_1} \right) 
\right. \\
+ \sum_{\nu} m_J m_K \lambda (2d_{\nu} + (m_J + m_K)\lambda) \left( \phi_{\nu+1}(X_J) - r \phi_{\nu+1}(X_K) \right) \phi_1(X_J - rX_K) \right. \\
\left. - \sum_{J} \sum_{K \neq J} \sum_{\nu} 2m_J m_K^2 \lambda^2 \left( \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)} + \frac{3\eta_1}{4\omega_1} \right) \right\} \]

We insert (83) and partly undo the symmetrization,
\[ \mathcal{W}_2 = \sum_{J<K} \sum_{r=\pm} \left\{ \gamma_{JK} \left( \varphi(X_J - rX_K) + \frac{\eta_1}{\omega_1} \right) 
\right. \\
+ \sum_{\nu} m_J m_K \lambda (2d_{\nu} + (m_J + m_K)\lambda) \frac{\dot{\vartheta}_1(X_J - rX_K)}{\vartheta_1(X_J - rX_K)} \right. \\
\left. + \sum_{J} \sum_{K \neq J} \sum_{\nu} \left( 2m_J m_K \lambda (2d_{\nu} + (m_J + m_K)\lambda) - 2m_J m_K^2 \lambda^2 \right) \left( \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)} + \frac{3\eta_1}{4\omega_1} \right) \right\} \]
(a factor 2 in the first term in the last line comes from a $r$-sum) and, after some computations, we obtain

$$\mathcal{W}_2 = \sum_{J<K} \sum_{r=\pm} \left\{ \gamma_{JK} \varphi(X_J - rX_K) + m_J m_K \lambda (2|d| + 4(m_J + m_K)\lambda) \frac{\dot{\theta}_1(X_J - rX_K)}{\theta_1(X_J - rX_K)} \right\}$$

$$+ \sum_{J} \sum_{\nu} 4\lambda(|m| - m_J)g_{\nu,J} \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)}$$

$$+ \left( \sum_{J<K} 2\gamma_{JK} + \sum_{J} 3\lambda(|m| - m_J)m_J(|d| + 2m_J\lambda) \right) \frac{\eta}{\omega_1}$$

using the short-hand notation in (16).

To compute the three-body terms we symmetrize the summations,

$$\mathcal{W}_3 = \sum_{J<K<L} m_J m_K m_L \lambda^2 \sum_{r,s=\pm} \left( \phi_1(X_J - rX_K)\phi_1(X_J - sX_L) \right)$$

$$+ \phi_1(X_K - rX_L)\phi_1(X_K - sX_J) + \phi_1(X_L - rX_J)\phi_1(X_L - sX_K) \right),$$

which allows us to use the identity (84) to obtain

$$\mathcal{W}_3 = \sum_{J<K<L} 2m_J m_K m_L \lambda^2 \sum_{r=\pm} \left( \frac{\dot{\theta}_1(X_J - rX_K)}{\theta_1(X_J - rX_K)} + \frac{\eta}{2\omega_1} \right),$$

Changing summations again we can write this as

$$\mathcal{W}_3 = \sum_{J} \sum_{K \neq J} \sum_{L \neq J, K} 2m_J m_K m_L \lambda^2 \sum_{r=\pm} \left( \frac{\dot{\theta}_1(X_J - rX_K)}{\theta_1(X_J - rX_K)} + \frac{\eta}{2\omega_1} \right),$$

and, by simple computations, we obtain

$$\mathcal{W}_3 = \sum_{J<K} 4m_J m_K (|m| - m_J - m_K) \lambda^2 \sum_{r=\pm} \left( \frac{\dot{\theta}_1(X_J - rX_K)}{\theta_1(X_J - rX_K)} + \frac{\eta}{2\omega_1} \right). \quad (92)$$

Recalling (83) and $\mathcal{H} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3$ we add the results in (90), (91) and (92) and obtain

$$\dot{\mathcal{H}} = \sum_{J} \frac{1}{m_J} \left( -\partial_J^2 + \sum_{\nu} g_{\nu,J} (g_{\nu,J} - 1) \varphi(X_J + \omega_\nu) \right) + \sum_{J<K} \sum_{r=\pm} \gamma_{JK} \varphi(X_J - rX_K)$$

$$+ (|d| + 2|m|\lambda) \left( \sum_{J} \sum_{\nu} 2g_{\nu,J} \frac{\dot{\theta}_{\nu+1}(X_J)}{\theta_{\nu+1}(X_J)} + \sum_{J<K} \sum_{r=\pm} m_J m_K \lambda \frac{\dot{\theta}_1(X_J - rX_K)}{\theta_1(X_J - rX_K)} \right) - \mathcal{E}_0$$

with

$$\mathcal{E}_0 = - \left\{ \sum_{J} \left( (|d| + 2m_J\lambda)(m_J|d| + 2m_J^2\lambda - 1) + 3\lambda(|m| - m_J)m_J(|d| + 2m_J\lambda) \right) \right\} \frac{\eta}{\omega_1} + \sum_{J} m_J \sum_{\mu<\nu} d_{\mu} d_{\nu} \epsilon_{\mu,\nu} \quad (93)$$
Recalling (10) and (11) we can write this as in (19). Some computations show that $E_0$ in (93) can be simplified to the formula given in (15).

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