JACOB’S LADDERS, HETEROGENEOUS QUADRATURE FORMULAE, BIG ASYMMETRY AND RELATED FORMULAE FOR THE RIEMANN ZETA-FUNCTION

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Abstract. In this paper we obtain as our main result new class of formulae expressing correlation integrals of the third-order in $Z$ on disconnected sets $G_1(x), G_2(y)$ by means of an autocorrelative sum of the second order in $Z$. Moreover, the distance of the sets $G_1(x), G_2(y)$ from the set of arguments of autocorrelative sum is extremely big, namely $\sim A\pi(T), T \to \infty$, where $\pi(T)$ is the prime-counting function.

1. Introduction

1.1. Let us remind the following formulae

\begin{equation}
\int_T^{2T} Z^4(t)dt \sim \frac{2\pi}{\ln T} \sum_{T \leq t_v \leq T+2T} Z^4(t_v), \quad T \to \infty,
\end{equation}

\begin{equation}
\int_T^{T+U} Z^2(t)dt \sim \frac{2\pi}{\ln T} \sum_{T \leq t_v \leq T+U} Z^2(t_v), \quad U = \sqrt{T} \ln T, \quad T \to \infty
\end{equation}

(1.1)

(see [4], (4.4), (4.7)), where

\begin{equation}
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right),
\end{equation}

\begin{equation}
\vartheta(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + \frac{it}{2} \right)
\end{equation}

(1.2)

(see [4], pp. 79, 329), and $\{t_v\}$ is the Gram sequence (comp. [4], p. 99). The formulae (1.1) were proved by us in connection with the Kotelnikov-Whittaker-Nyquist theorem. Namely, by these formulae we have expressed the biquadratic and quadratic effects for the continuous signals $Z(t)$, $t \in [T, 2T]$; $t \in [T, T+U]$ from the point of view of information theory.

1.2. The formulae (1.1) are:

(a) asymptotic quadrature formulae (from the left to the right),
(b) asymptotic summation formulae (from the right to the left).

Next, for the second formula in (1.1) (for example) we have

\begin{equation}
t \in [T, T+U] \to t_v \in [T, T+U],
\end{equation}

\begin{equation}
Z^2(t) \to Z^2(t_v),
\end{equation}

(1.3)

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i.e. the segment \([T, T + U]\) and the exponent 2 are conserved.

**Remark 1.** By (1.3) it is natural to call the formulae of the kind (1.1) **homogeneous formulae**.

On the contrary, we obtain in this paper some heterogeneous asymptotic quadrature formulae – such formulae that the properties (1.3) are not fulfilled.

2. **HETEROGENEOUS QUADRATURE FORMULAE**

2.1. Let (see [3], (3))

\[
G_1(x) = G_1(x; T, H) = \bigcup_{T \leq t_\nu \leq T + H} \{ t : t_2\nu(-x) < t < t_2\nu(x) \}, \quad 0 < x \leq \frac{\pi}{2},
\]

(2.1) \[
G_2(y) = G_2(y; T, H) = \bigcup_{T \leq t_{2\nu+1} \leq T + H} \{ t : t_{2\nu+1}(-y) < t < t_{2\nu+1}(y) \}, \quad 0 < y \leq \frac{\pi}{2},
\]

\[H = T^{1/6 + 2\epsilon},\]

where the collection of sequences

\[\{ t_\nu(\tau) \}, \quad \tau \in [-\pi, \pi], \quad \nu = 1, 2, \ldots\]

is defined by the equation (see [3], (1))

\[\varphi[t_\nu(\tau)] = \pi \nu + \tau; \quad t_\nu(0) = t_\nu,\]

and (see [3], (8))

(2.2) \[m\{G_1(x)\} = \frac{\pi}{\pi} H + O\left(\frac{x}{\ln T}\right), \quad m\{G_2(y)\} = \frac{y}{\pi} H + O\left(\frac{y}{\ln T}\right),\]

where \(m\{G_1(x)\}, \ldots\) is the measure of the set \(G_1, \ldots\).

Let \(\varphi_1\{G_1(x)\} = G_1(x), \quad \varphi_1\{G_2(y)\} = G_2(y).\)

The following Theorem holds true.

**Theorem 1.**

\[
\int_{G_1(x)} \omega(t)Z[\varphi_1(t)]Z^2(t)dt = \]

\[= \frac{m\{G_1(x)\}}{Q_1 \ln P_0} \sum_{T \leq t_\nu \leq T + U_1} Z(t_\nu)Z\left(t_\nu + \frac{x}{\ln P_0}\right) + O\left(\frac{H}{\ln T}\right),\]

(2.3)

\[
\int_{G_2(y)} \omega(t)Z[\varphi_1(t)]Z^2(t)dt = \]

\[= -\frac{m\{G_2(y)\}}{Q_1 \ln P_0} \sum_{T \leq t_\nu \leq T + U_1} Z(t_\nu)Z\left(t_\nu + \frac{y}{\ln P_0}\right) + O\left(\frac{H}{\ln T}\right),\]

\[x, y \in \left(0, \frac{\pi}{2}\right), \quad T \to \infty,\]

where

\[\omega(t) = \frac{1}{\ln t} \left\{1 + O\left(\frac{\ln \ln t}{\ln t}\right)\right\},\]

\[
= \frac{1}{\ln t} \left\{1 + O\left(\frac{\ln \ln t}{\ln t}\right)\right\},\]

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and (see [1], (38); \( H \to U_1 \))

\[
Q_1 = \sum_{T \leq t \leq T + U_1} 1 = \frac{1}{\pi} U_1 \ln P_0 + O \left( \frac{U_1^2}{T} \right),
\]

(2.5)

\[
U_1 = \sqrt{T} \ln P_0, \quad P_0 = \sqrt{\frac{T}{2\pi}}.
\]

Furthermore we have

(2.6)

\[
m\{\hat{G}_1(x)\}, \quad m\{\hat{G}_2(y)\} < T^{1/3+\epsilon}.
\]

2.2.

Remark 2. Every of the correlation integrals in (2.3) contains the product

(2.7)

\[
Z [\varphi_1(t)] Z^2(t)
\]

and, as usually, we have (see [6], (6.2); [7], (8.4))

(2.8)

\[
t - \varphi_1(t) \sim (1 - c) \frac{t}{\ln t} \sim (1 - c) \pi(t), \quad t \to \infty,
\]

i. e. we have big difference of arguments in (2.7), where \( c \) is the Euler constant and \( \pi(t) \) is the prime-counting function.

Next, in the case

\[
t = \hat{T}; \quad \varphi_1(\hat{T}) = T \Rightarrow T \to \infty \Leftrightarrow \hat{T} \to \infty
\]

we obtain from (2.8)

\[
\hat{T} - T \sim (1 - c) \frac{\hat{T}}{\ln \hat{T}} \Rightarrow 1 - \frac{T}{\hat{T}} \sim (1 - c) \frac{1}{\ln T} \Rightarrow
\]

\[
\Rightarrow \hat{T} \sim T \Rightarrow \ln \hat{T} \sim \ln T,
\]

i. e.

\[
\hat{T} - T \sim (1 - c) \frac{T}{\ln T},
\]

and (see (2.5))

\[
\hat{T} - (T + U_1) \sim (1 - c) \frac{T}{\ln T} - U_1 \sim (1 - c) \frac{T}{\ln T}.
\]

Consequently we have

(2.9)

\[
\rho ([\hat{T}, \hat{T} + H]; [T, T + U_1]) \sim (1 - c) \pi(T);
\]

\[
[T, T + U_1] \prec [\hat{T}, \hat{T} + H],
\]

where \( \rho \) stands for the distance of the corresponding segments. We may, of course, put

(2.10)

\[
G_1(x) \cap [T, T + H] = \hat{G}_1(x) \to G_1(x), \ldots
\]

if necessary.

Remark 3. We have the following properties
(a) \[
\hat{G}_1(x), \hat{G}_2(y) \in [\hat{T}, \hat{T} + \hat{H}] \rightarrow [T, T + U],
\]
(comp. (1.3), (2.10)), where \(\hat{G}_1(x), \hat{G}_2(y)\) are disconnected sets,
(b) if \(\hat{G} = \hat{G}_1(x), \hat{G}_2(y)\)
then extremely big distance occurs, namely (comp. (2.9), (2.10))
\[
\rho(\hat{G}; [T, T + U]) \sim (1 - c)\pi(T), \ T \rightarrow \infty,
\]
(c) for the corresponding orders of \(Z\) (comp. (1.3))
\[
1 + 2 \rightarrow 1 + 1
\]
then the formulae (2.3) are strongly heterogeneous (comp. Remark 1).

**Remark 4.** Moreover we explicitly notice the following:
(a) the formulae (2.3) are not accessible by the current methods in the theory of the Riemann zeta-function,
(b) small improvements of the exponents
\[
\frac{1}{6}, \frac{1}{2}, \ldots
\]
are irrelevant for main direction of this paper (comp. our paper [2], Appendix A: On I.M. Vinogradov’s scepticism on possibilities of the method of trigonometric sums).

3. **Big asymmetry and related formulae**

3.1. Let (see [1], p. 29)
\[
G_3(x) = G_3(x; T, U_2) = \bigcup_{T \leq g_2 \leq T + U_2} \{t : g_2(-x) < t < g_2(x)\}, \quad 0 < x \leq \frac{\pi}{2},
\]
\[
G_4(y) = G_4(y; T, U_2) = \bigcup_{T \leq g_2+1 \leq T + U_2} \{t : g_2+1(-y) < t < g_2+1(y)\}, \quad 0 < y \leq \frac{\pi}{2},
\]
where the collection of sequences
\[
\{g_\nu(\tau)\}, \ \tau \in [-\pi, \pi], \ \nu = 1, 2, \ldots
\]
is defined by the equation (see [1], (6))
\[
\vartheta_1[g_\nu(\tau)] = \frac{\pi}{2} \nu + \frac{\tau}{2}; \ g_\nu(0) = g_\nu,
\]
where (comp. (1.22))
\[
\vartheta(t) = \vartheta_1(t) + O \left(\frac{1}{t}\right), \ \vartheta_1(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8},
\]
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and (see [4], (13))

\[
m\{G_3\} = \frac{x}{\pi} U_2 + O\left(\frac{x}{\ln T}\right),
\]

\[
m\{G_4\} = \frac{y}{\pi} U_2 + O\left(\frac{y}{\ln T}\right).
\]

Let

\[
\varphi_1\{\hat{G}_3(x)\} = G_3(x), \quad \varphi_1\{\hat{G}_4(y)\} = G_4(y).
\]

The following Theorem holds true.

**Theorem 2.**

\[
\int_{\hat{G}_3(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt \sim \frac{4}{\pi} U_2 \sin x + \int_{\hat{G}_4(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt,
\]

\[
x \in \left(0, \frac{\pi}{2}\right], \quad T \to \infty.
\]

**Remark 5.** By the asymptotic formula (3.3) is expressed the property of big asymmetry in the distribution of the values of \(Z\) on disconnected sets \(\hat{G}_3(x), \hat{G}_4(x)\). Namely, the correlation integral (comp. (2.8)) on the disconnected set \(\hat{G}_3(x)\) essentially exceeds that on the set \(\hat{G}_4(x)\). For example

\[
\int_{\hat{G}_3(\pi/2)} - \int_{\hat{G}_4(\pi/2)} \sim \frac{4}{\pi} U_2, \quad T \to \infty,
\]

where

\[
m\{\hat{G}_3(\pi/2)\} + m\{\hat{G}_4(\pi/2)\} = U_2
\]

if (comp. (2.10))

\[
G_3(x) \cap [T, T + U_2] \to \hat{G}_3(x), \ldots
\]

3.2. Finally, the following Theorem holds true.

**Theorem 3.**

\[
\int_{\hat{G}_3(x)} \omega(t) Z[\varphi_1(t)] Z^2(t) dt =
\]

\[
= \frac{m\{G_1(x)\}}{2m\{G_3(x)\}} \int_{\hat{G}_3(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt - \frac{m\{G_1(x)\}}{2m\{G_4(x)\}} \int_{\hat{G}_4(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt + O(HT^{-\epsilon}),
\]

\[
\int_{\hat{G}_2(y)} \omega(t) Z[\varphi_1(t)] Z^2(t) dt =
\]

\[
= \frac{m\{G_2(y)\}}{2m\{G_4(y)\}} \int_{\hat{G}_4(y)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt - \frac{m\{G_2(y)\}}{2m\{G_3(y)\}} \int_{\hat{G}_3(y)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt + O(HT^{-\epsilon}),
\]

\[
x, y \in \left(0, \frac{\pi}{2}\right], \quad T \to \infty.
\]
Remark 6. For the disconnected sets
\[ G_1(x), G_2(y), \hat{G}_3(x), \hat{G}_4(x), \hat{G}_3(y), \hat{G}_4(y); \hat{G}_1 \cap \hat{G}_2 = \emptyset, \hat{G}_3 \cap \hat{G}_4 = \emptyset \]
we have the following property: the correlation integrals of the order 1 + 2 on \( G_1(x), G_2(y) \) are expressed as the linear combinations of the correlation integrals of the order 2 + 2 on \( \hat{G}_3(x), \hat{G}_4(x); \hat{G}_3(y), \hat{G}_4(y) \) correspondingly.

4. PROOF OF THEOREM 1

4.1. In the paper [1] (see (10)) we have proved the following autocorrelative formula
\[
\sum_{T \leq \nu \leq T + U_1} Z(t_\nu)Z(t_\nu + \beta) = 2 \pi \sin(\beta \ln P_0) + O(\sqrt{T \ln^2 T}),
\]
where
\[
\beta = \mathcal{O}\left(\frac{1}{\ln T}\right), \quad U_1 = \sqrt{T \ln P_0}.
\]
In the case
\[
\beta \ln P_0 = x \Rightarrow \beta = \frac{x}{\ln P_0}, \quad x \in \left(0, \frac{\pi}{2}\right]
\]
we obtain from (4.1), (4.2)
\[
\sum_{T \leq \nu \leq T + U_1} Z(t_\nu)Z\left(t_\nu + \frac{x}{\ln P_0}\right) = 2 \pi \frac{\sin x}{x} U_1 \ln^2 P_0 + O(\sqrt{T \ln^2 T}).
\]
Consequently we have (see (2.3), (4.2))
\[
\frac{1}{Q_1 \ln P_0} \sum_{T \leq \nu \leq T + U_1} Z(t_\nu)Z\left(t_\nu + \frac{x}{\ln P_0}\right) = 2 \sin x + O\left(\frac{1}{\ln T}\right); \quad x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin x \in \left[\frac{2}{\pi}, 1\right].
\]

4.2. Next, in the paper [3], (5), (9) we have obtained the following mean-value formula on the disconnected sets \( G_1(x), G_2(y) \) (see (2.1))
\[
\int_{G_1(x)} Z(t)dt = \frac{2}{\pi} H \sin x + \mathcal{O}(T^{1/6+\epsilon}),
\]
\[
\int_{G_2(y)} Z(t)dt = -\frac{2}{\pi} H \sin y + \mathcal{O}(T^{1/6+\epsilon}),
\]
\[
H = T^{1/6+2\epsilon}; \quad T^{1/6}\psi^2 \ln^5 T \rightarrow T^{1/6+\epsilon}.
\]
Hence, from (4.4) by (2.3) we obtain
\[
\frac{1}{m\{G_1(x)\}} \int_{G_1(x)} Z(t)dt = 2 \frac{\sin x}{x} + \mathcal{O}(T^{-\epsilon}),
\]
\[
\frac{1}{m\{G_2(y)\}} \int_{G_2(y)} Z(t)dt = -2 \frac{\sin y}{y} + \mathcal{O}(T^{-\epsilon}).
\]
4.3. In the paper [7], (9.2), (9.5) we have proved the following Lemma: if

$$\varphi_1([\overline{T}, \overline{T+U}]) = [T, T+U],$$

then for every Lebesgue-integrable function

$$f(x), \ x \in [T, T+U]$$

we have

$$\int_{\overline{T}}^{T+U} f[\varphi_1(t)] \tilde{Z}^2(t) dt = \int_{T}^{T+U} f(x) dx,$$

$$T \geq T_0[\varphi_1], \ U \in \left(0, \frac{T}{\ln T}\right),$$

where

$$\tilde{Z}^2(t) = \frac{Z^2(t)}{\left\{1 + \mathcal{O}\left(\frac{\ln t}{\ln \ln t}\right)\right\} \ln t} = \omega(t)Z^2(t);$$

$$\omega(t) = \frac{1}{\left\{1 + \mathcal{O}\left(\frac{\ln t}{\ln \ln t}\right)\right\} \ln t} = \frac{1}{\ln t} \left\{1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right\}. $$

Consequently, we have (see (4.7), (4.8))

$$\int_{\overline{T}}^{T+U} \omega(t)f[\varphi_1(t)]Z^2(t) dt = \int_{T}^{T+U} f(x) dx,$$

$$U \in \left(0, \frac{T}{\ln T}\right).$$

4.4. Hence, by (1.6), (1.9) we obtain following formulae

$$\frac{1}{m\{G_1(x)\}} \int_{\tilde{G}_1(x)} \omega(t)Z[\varphi_1(t)]Z^2(t) dt = \frac{1}{m\{G_2(y)\}} \int_{\tilde{G}_2(y)} \omega(t)Z[\varphi_1(t)]Z^2(t) dt =$$

$$= 2 \sin \frac{x}{x} + \mathcal{O}(T^{-\epsilon}),$$

$$= -2 \sin \frac{y}{y} + \mathcal{O}(T^{-\epsilon}).$$

Finally, simple elimination of the values

$$2 \frac{\sin x}{x}, \ 2 \frac{\sin y}{y}$$

from (4.3), (4.9) gives (2.3).
4.5. In the case $f(x) = 1$

in (4.7) we obtain (see (2.1), (2.10), (4.8))

$$\int_{G_1(x)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt < \int_{T}^{T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim$$

$$\sim \ln T \int_{T}^{T+H} 1 \cdot dt,$$

i. e.

(4.11) $$\int_{T}^{T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt < AH \ln T, \ T \to \infty.$$

Let

(4.12) $$\hat{T} + H - \hat{T} \geq T^{1/3+\epsilon},$$

(comp. [6], (2.5); where $\frac{1}{3}$ is the Balasubramanian exponent). Then

(4.13) $$\int_{T}^{T+H} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim (\hat{T} + H - \hat{T}) \ln T > B(\hat{T} + H - \hat{T}) \ln T,$$

and (see (4.11), (4.13))

(4.14) $$\hat{T} + H - \hat{T} < CH = CT^{1/6+\epsilon}.$$

Now, (4.14) contradicts (4.12). Consequently we have that

$$m \{ \hat{G}_1(x) \} < \hat{T} + H - \hat{T} < T^{1/3+\epsilon}$$

and we obtain the second inequality in (2.6) by the similar way.

5. PROOFS OF THEOREM 2 AND THEOREM 3

5.1. In the paper [4], (14), (15) we have proved the following mean-value formulae

$$\int_{G_3(x)} Z^2(t) dt =$$

$$= \frac{x}{\pi} U_2 \ln \frac{T}{2\pi} + \frac{2}{\pi} (cx + \sin x) U_2 + O(T^{5/12} \ln^2 T),$$

(5.1)

$$\int_{G_4(y)} Z^2(t) dt =$$

$$= \frac{y}{\pi} U_2 \ln \frac{T}{2\pi} + \frac{2}{\pi} (cy - \sin y) U_2 + O(T^{5/12} \ln^2 T)$$
on the disconnected sets $G_3(x), G_4(y)$, (comp. (3.1)). From (5.1) we have in the case $x = y$

$$
\int_{G_3(x)} Z^2(t) dt - \int_{G_4(x)} Z^2(t) dt = \frac{4}{\pi} U_2 \sin x + O(x T^{5/12} \ln^2 T).
$$

Consequently, we obtain from (5.1) by (4.7) – (4.9) with $f(x) = Z^2(x)$ the formula (3.3).

5.2. Next, from the formula (see [4], (16))

$$
\frac{1}{m\{G_3(x)\}} \int_{G_3(x)} Z^2(t) dt - \frac{1}{m\{G_4(x)\}} \int_{G_4(x)} Z^2(t) dt \sim
$$

$$
\sim \frac{4 \sin x}{x}, \quad T \to \infty
$$

we obtain by (4.7) – (4.9) ($f(x) = Z^2(x)$) the following formula

$$
\int_{G_3(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt - \int_{G_4(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt \sim \frac{4 \sin x}{x}, \quad T \to \infty
$$

(5.2)

and, by the similar way, we obtain the formula for $y \in (0, \pi/2]$. Hence, the elimination of

$$
2 \frac{\sin x}{x}, \quad 2 \frac{\sin y}{y}
$$

from (4.10), (5.2) implies the formula (3.4).

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REFERENCES

[1] J. Moser, ‘On autocorrelative sum in the theory of the Riemann zeta-function’, Acta Math. Univ. Comen., 37 (1980), 121-134, (in Russian).
[2] J. Moser, ‘On the roots of the equation $Z\prime(t) = 0$', Acta Arith. 40 (1981), 97-107, (in Russian), arXiv: 1303.0967.
[3] J. Moser, ‘New consequences of the Riemann-Siegel formula’, Acta Arith., 42 (1982), 1-10, (in Russian), arXiv: 1312.4767.
[4] J. Moser, ‘New mean-value theorems for the function $|\zeta(1/2 + it)|^2\$, Acta Math. Univ. Comen., 46-47 (1985), 21-40.
[5] J. Moser, ‘On the order of Titchmarsh sum in the theory of the Riemann’s zeta-function and on the biquadratic effect in the information theory’, Czechoslovak Math. J., 41 (116), (1991), 663-684, (in Russian), arXiv: 1112.9548.
[6] J. Moser, ‘Jacob’s ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral’, Math. Notes 88, (2010) 414-422, arXiv: 0901.3937.
[7] J. Moser, ‘Jacob’s ladders, the structure of the Hardy-Littlewood integral and some new class of nonlinear integral equations’, Proc. Stek. Inst. 276, (2011), 208-221, arXiv: 1103.0359.
[8] E.C. Titchmarsh, ‘On van der Corput ’s method and the zeta-function of Riemann (IV)’, Quart. J. Math., (Oxford), 5, (1934), 98-105.
[9] E.C. Titchmarsh, ‘The theory of the Riemann zeta-function’ Clarendon Press, Oxford, 1951.