Abstract

We extend previously proposed generalized gauge theory formulation of Chern-Simons type and topological Yang-Mills type actions into Yang-Mills type actions. We formulate gauge fields and Dirac-Kähler matter fermions by all degrees of differential forms. The simplest version of the model which includes only zero and one form gauge fields accommodated with the graded Lie algebra of $SU(2|1)$ super-group leads Weinberg-Salam model. Thus the Weinberg-Salam model formulated by noncommutative geometry is a particular example of the present formulation.
1 Introduction

It is obviously the most challenging question how we can formulate the standard model together with quantum gravity in a unified way. The superstring related topics are motivated to challenge on this question. It is, however, not obvious that this is the unique way to find a clue to this question.

Empirically to say lattice theories are successful to describe the quantum and non-perturbative aspects of gauge field theories. The lattice QCD is one example and two-dimensional quantum gravity is another successful example. In the standard model there are several parameters; quark and lepton masses, weak mixing angles, \cdots, which, we hope, would be numerically evaluated by a quantitative description of the unified theory. We believe that the lattice theory might again play an important role in the quantitative formulation of the unified theory.

In formulating a gauge theory on a simplicial lattice manifold, we naively expect that we should formulate the gauge theory by differential forms since general coordinate invariance can be easily accommodated by differential forms. Furthermore the form degree corresponds to the dimensions of simplex on the simplicial lattice manifold and thus \( n \)-form field variable may be assigned on the \( n \)-simplex of the simplicial lattice manifold.

One of the present authors (N. K.) and Watabiki proposed the generalized gauge theory formulation for Chern-Simons type actions and topological Yang-Mills actions which include all the degrees of differential forms yet has the same algebraic structure as the ordinary gauge theory \([1-3]\). The field variables are quaternion valued which classifies bosonic even, bosonic odd, fermionic even, and fermionic odd forms. \( Z_2 \) grading structure for the field variables is a natural consequence of the quaternion structure and thus the graded Lie algebra naturally comes in as a gauge algebra. There are similar formulations related with the generalized gauge theory \([1, 3]\).

We can then expect that this type of the generalized gauge theory formulation may provide a gauge theory formulation with gravity on the simplicial lattice. In fact three-dimensional lattice gravity and four-dimensional \( BF \) lattice gravity have been formulated by using the leading terms of the generalized Chern-Simons action which are the standard Chern-Simons action in three dimensions and \( BF \) action in four dimensions \([5-7]\). These experiences of the formulation of lattice gravity including 0-, 1- and 2-form field variables provide us a feeling that the form variables may play an important role in the formulation of lattice gravity.
It has been clarified that the quantization of the generalized Chern-Simons action is highly nontrivial, in fact infinitely reducible [8–10]. The quantized minimal action has the same generalized Chern-Simons type structure as the classical one. This would mean that we have quantized the topological gravity in two dimensions and the topological conformal gravity in four dimensions which were classically formulated by the even-dimensional version of the generalized gauge theories [11, 12].

Connes pointed out that the Weinberg-Salam model can be formulated as a particular case of the noncommutative geometry formulation of a gauge theory [13–15]. Consider a manifold which is composed of a direct product of discrete two points and four-dimensional flat space, $Z_2 \times M_4$, and define a connection and differential operator on this manifold. Due to the discrete nature of the two points the differential operator can be represented by two by two matrix. Thus the connection or equivalently the gauge field is now represented by two by two matrix as well and thus possesses diagonal and off-diagonal components. Then the weak and electromagnetic 1-form gauge fields are assigned to the diagonal component while the 0-form Higgs fields are assigned to the off-diagonal components. Then the spontaneously broken Weinberg-Salam model comes out naturally from the pure Yang-Mills action on this manifold by taking the group $SU(2) \times U(1)$ [13–15]. This type of noncommutative geometry formulation of Weinberg-Salam model have been intensively investigated. Here we give partial lists of those investigations and the further references are therein [16–29].

We show that the two by two matrix representation of the gauge fields are easily accommodated by the quaternions of the generalized gauge theory since the quaternion algebra can be represented by two by two matrices. We can, however, point out that our generalized gauge theory is more general formulation as noncommutative geometry since it includes not only 0- and 1-form of gauge fields but also all the possible bosonic and fermionic form degrees of gauge fields and gauge parameters and it can accommodate a graded Lie algebra naturally. In fact we show in this paper that the graded Lie algebra of $SU(2|1)$ supergroup leads naturally to the Weinberg-Salam model. The importance of the $SU(2|1)$ graded Lie algebra in connection with the standard model was first pointed out by Ne’eman [30]. Later the noncommutative geometry formulation of $SU(2|1)$ graded Lie algebra were given by Coquereaux et al. [17–19]. We believe that our formulation of the Weinberg-Salam model will provide new insights into the formulation of the standard model.
Witten pointed out that four-dimensional $N = 2$ super Yang-Mills action comes out from the topological Yang-Mills action with instanton gauge fixing via twisting mechanism \[31\]. The quantization of the generalized topological Yang-Mills action in two dimensions with instanton gauge fixing was investigated. It was clarified that $N = 2$ super Yang-Mills action comes out naturally and the matter fermions appears from ghosts of quantization via twisting mechanism \[32\]. The twisting mechanism and the Dirac-Kähler fermion formulation \[33–35\] are essentially related through $N = 2$ supersymmetry.

In the lattice gauge theory there is the well known chiral fermion problem. The staggered fermion formulation \[36\] made it clear that the Kogut-Susskind fermion formulation \[37, 38\] can avoid the problem in such a way that there appear 4 copies of flavor suffices. The curved space version of the Susskind fermion formulation or equivalently the staggered fermion formulation on the flat spacetime is the Dirac-Kähler fermion formulation which is formulated by differential forms \[33–35, 39–44\]. In formulating matter fermions in the Weinberg-Salam model, we employ the Dirac-Kähler fermion formulation so that all the gauge fields and matter fermions are formulated by the differential forms. This would provide us a hope that the standard model with matter fermion coupled to the gravity could be formulated purely by differential forms and thus leads to a unified model with gravity on the simplicial lattice. This kind of non-standard overview on unifying the standard model with gravity on the lattice was reviewed in \[45\].

This paper is organized as follows: We summarize the generalized gauge theory formulation of Chern-Simons type and topological Yang-Mills type actions in section 2. We then extend the generalized gauge theory formulation into Yang-Mills type action in section 3. In section 4 we provide a generalized theory version of the Dirac-Kähler fermion formulation. We then formulate the Weinberg-Salam model by the generalized Yang-Mills action and Dirac-Kähler fermion formulation with $SU(2|1)$ graded Lie algebra in section 5. Summary and discussions will be given in the last section.

2 Generalized gauge theory in arbitrary dimensions

The generalized Chern-Simons actions, which were proposed by one of the present authors and Watabiki about ten years ago, is a generalization of the ordinary three-dimensional Chern-Simons theory into arbitrary dimensions \[1, 2\]. The essential point of the generalization is to extend a 1-form gauge field and 0-form gauge parameter to a quaternion valued generalized gauge field and gauge parameter which contain all the possible degrees
of differential forms. Correspondingly the standard gauge symmetry is extended to much higher topological symmetry. These generalizations are formulated in such a way that the generalized actions have the same algebraic structure as the ordinary three-dimensional Chern-Simons action.

Since this generalized Chern-Simons action can be formulated completely parallel to the ordinary gauge theory, the generalization can be extended further to the topological Yang-Mills actions. The generalized Chern-Simons actions and topological Yang-Mills actions have topological nature from the construction of the actions themselves. This could naively be understood from the fact that in the generalized gauge theory formulation the generalized gauge fields and gauge parameters contain exactly the same number of field variables and thus all the generalized gauge fields could be gauged away. If we, however, try to construct generalized Yang-Mills actions, the construction does not proceed completely parallel to the generalized Chern-Simons actions and topological Yang-Mills actions since the topological nature is lost in the construction of the generalized Yang-Mills action. It is one of the main subjects of this paper to formulate the generalized Yang-Mills actions together with matter fermions in terms of differential forms. Before getting into the details of formulating the generalized Yang-Mills action we summarize the results of generalized gauge theory for Chern-Simons type and topological Yang-Mills type actions.

In the most general form, a generalized gauge field $A$ and a gauge parameter $V$ are defined by the following component form:

$$A = 1 \psi + i \hat{\psi} + jA + k \hat{A}, \quad (2.1)$$

$$V = 1 \hat{a} + i a + j \hat{\alpha} + k \alpha, \quad (2.2)$$

where $(\psi, \alpha), (\hat{\psi}, \hat{\alpha}), (A, a)$ and $(\hat{A}, \hat{a})$ are direct sums of fermionic odd forms, fermionic even forms, bosonic odd forms and bosonic even forms, respectively, and they take values on a gauge algebra. The bold face symbols $1, i, j$ and $k$ satisfy the algebra

$$1^2 = 1, \quad i^2 = \epsilon_1 1, \quad j^2 = \epsilon_2 1, \quad k^2 = -\epsilon_1 \epsilon_2 1,$$

$$ij = -ji = k, \quad jk = -kj = -\epsilon_2 i, \quad ki = -ik = -\epsilon_1 j, \quad (2.3)$$

where $(\epsilon_1, \epsilon_2)$ takes the value $(-1, -1), (-1, +1), (+1, -1)$ or $(+1, +1)$. We may call this algebra as “quaternion algebra”.

The components of the gauge field $A$ and parameter $V$ are assigned to the elements of
the gauge algebra in a specific way:

\[ A = T_a A^a, \quad \dot{\psi} = T_a \dot{\psi}^a, \quad \psi = \Sigma_\alpha \psi^\alpha, \quad \dot{A} = \Sigma_\alpha \dot{A}^\alpha, \]

\[ \dot{\alpha} = T_a \dot{\alpha}^a, \quad \alpha = T_a \alpha^a, \quad \dot{\alpha} = \Sigma_\alpha \dot{\alpha}^\alpha, \quad a = \Sigma_\alpha a^\alpha. \]  

The following graded Lie algebra can be adopted as a gauge algebra:

\[ [T_a, T_b] = f^c_{ab} T_c, \quad [T_a, \Sigma_\beta] = g^\gamma_{a\beta} \Sigma_\gamma, \quad \{\Sigma_\alpha, \Sigma_\beta\} = h^c_{a\beta} T_c, \]  

(2.5)

where \([ , ]\) and \(\{ , \}\) are commutator and anticommutator respectively. All the structure constants are subject to consistency conditions which follow from the graded Jacobi identities. If we choose \(\Sigma_\alpha = T_a\) especially, this algebra reduces to \(T_a T_b = k^c_{ab} T_c\) which is closed under multiplication. A specific example of such algebra is realized by Clifford algebra [11].

An element having the same type of component expansion as \(A\) or \(V\) belongs to \(\Lambda_-\) or \(\Lambda_+\) class, respectively,

\[ \lambda_- = 1 \text{(fermionic odd)} + i \text{(fermionic even)} \]

\[ + j \text{(bosonic odd)} + k \text{(bosonic even)} \quad \in \Lambda_- , \]  

(2.6)

\[ \lambda_+ = 1 \text{(bosonic even)} + i \text{(bosonic odd)} \]

\[ + j \text{(fermionic even)} + k \text{(fermionic odd)} \quad \in \Lambda_+ . \]  

(2.7)

These elements fulfill the following \(Z_2\) grading structure:

\[ [\lambda_+, \lambda_+] \in \Lambda_+, \quad [\lambda_+, \lambda_-] \in \Lambda_- , \quad \{\lambda_-, \lambda_-\} \in \Lambda_+. \]

The elements of \(\Lambda_-\) and \(\Lambda_+\) can be regarded as generalizations of odd forms and even forms, respectively. The generalized exterior derivative which belongs to \(\Lambda_-\) is given by

\[ Q = j d \quad \in \Lambda_- , \]  

(2.8)

and the following graded Leibniz rule similar to the ordinary differential algebra holds:

\[ \{Q, \lambda_-\} = Q\lambda_- , \quad [Q, \lambda_+] = Q\lambda_+ , \quad Q^2 = 0, \]

(2.9)

where \(\lambda_+ \in \Lambda_+\) and \(\lambda_- \in \Lambda_-\). The arrow of the differential operator denotes \(\vec{Q}\lambda_- = Q\lambda_- - \lambda_- Q\). In the ordinary differential algebra of exterior derivative with \(Q\) replaced by the exterior derivative \(d\), \(\Lambda_-\) and \(\Lambda_+\) are odd and even form variables, respectively.
To construct the generalized Chern-Simons and topological Yang-Mills actions, we need to introduce two kinds of traces satisfying

\[
\text{Tr}[T_a, \cdots] = 0, \quad \text{Str}[T_a, \cdots] = 0,
\]

where \((\cdots)\) in the commutators or the anticommutators denotes a product of generators. In particular \((\cdots)\) should include an odd number of \(\Sigma_\alpha\)'s in the last equation of (2.10). \(\text{Tr}\) is the usual trace while \(\text{Str}\) is the super trace satisfying the above relations. We can then derive the following relations:

\[
\begin{align*}
\text{Tr}_k [\lambda_+, \lambda_-] &= \text{Tr}_k [\lambda_+, \lambda'_+] = 0, \\
\text{Str}_j [\lambda_+, \lambda_-] &= \text{Str}_j [\lambda_+, \lambda'_+] = 0, \\
\text{Str}_1 [\lambda_+, \lambda'_+] &= \text{Str}_1 \{\lambda_-, \lambda'_-\} = 0,
\end{align*}
\]

where \(\text{Tr}_q(\cdots)\) and \(\text{Str}_q(\cdots)\) \((q = 1, j, k)\) are defined so as to pick up only the coefficients of \(q\) from \((\cdots)\) and take the traces defined by eq.(2.10). These definitions of the traces will be crucial to show that the generalized gauge theory action can be invariant under the generalized gauge transformation below.

As we have seen in the above the generalized gauge field \(A\) and parameter \(V\), and the generalized differential operator \(Q\) play the same role as the 1-form gauge field, 0-form gauge parameter and differential operator of the usual gauge theory, respectively. We can then construct generalized actions in terms of these generalized quantities. We first define a generalized curvature

\[
\mathcal{F} \equiv \frac{1}{2} \{Q + A, Q + A\} = QA + A^2.
\]

We can construct the generalized actions of Chern-Simons type which have the standard form with respect to the generalized quantities

\[
\begin{align*}
S_{GCS}^e &= \int_M \text{Tr}_k \left( \frac{1}{2} QA + \frac{1}{3} A^3 \right), \\
S_{GCS}^o &= \int_M \text{Str}_j \left( \frac{1}{2} QA + \frac{1}{3} A^3 \right),
\end{align*}
\]

where \(S_{GCS}^e\) is even-dimensional generalized Chern-Simons action while \(S_{GCS}^o\) is odd-dimensional generalized Chern-Simons action. When we consider \(D\)-dimensional manifold \(M\), we need to pick up \(D\)-form terms in the action.
Similarly we can obtain the generalized topological Yang Mills action by taking the 1-th component which is even-dimensional counterpart of the bosonic component,

\[ S_{GTYM} = \int_M \text{Str}_1 (\mathcal{F} \mathcal{F}). \] (2.14)

In these generalized actions we have given the examples of taking the \( k, j, 1 \)-th component to pick up the even- and odd-dimensional counterpart of bosonic generalized Chern-Simons actions and even-dimensional counterpart of bosonic generalized topological Yang-Mills action, respectively. We can, however, construct all the different types; bosonic even, bosonic odd, fermionic even, fermionic odd, Chern-Simons type and topological Yang-Mills type actions by taking \( q = 1, i, j, k \)-th components of \( \text{Tr}_q(\cdots) \) and \( \text{Str}_q(\cdots) \).

Those generalized Chern-Simons actions (2.13) and the topological Yang-Mills actions (2.14) are invariant under the following generalized gauge transformations:

\[ \delta \mathcal{A} = [Q + \mathcal{A}, \mathcal{V}], \] (2.15)

where \( \mathcal{V} \) is the generalized gauge parameter defined by eq.(2.2). Correspondingly the generalized gauge transformation of the generalized curvature (2.12) is given by

\[ \delta \mathcal{F} = [\mathcal{F}, \mathcal{V}]. \] (2.16)

It should be noted that this symmetry is much larger than the usual gauge symmetry, in fact topological symmetry, since the gauge parameter \( \mathcal{V} \) contains as many gauge parameters as gauge fields of various forms in \( \mathcal{A} \).

We now show the explicit form of these generalized actions. Firstly we define the explicit form of the generalized gauge fields and gauge parameters by

\[ \mathcal{A} = jA + k\hat{A} \]

\( \equiv j(\omega^a + \Omega^a + \cdots)T_a + k(\phi^\alpha + B^\alpha + H^\alpha + \cdots)\Sigma_\alpha, \)

\[ \mathcal{V} = 1\hat{a} + i\hat{a} \]

\( \equiv 1(v^a + b^a + V^a + \cdots)T_a + i(u^\alpha + U^\alpha + \cdots)\Sigma_\alpha, \)

where \( \phi, \omega, B, \Omega, H \) are bosonic 0-, 1-, 2-, 3-, 4-form gauge fields and \( v, u, b, U, V \) are bosonic 0-, 1-, 2-, 3-, 4-form gauge parameters, respectively. We have omitted the fermionic degrees for simplicity. The generators \( T_a \) and \( \Sigma_\alpha \) satisfy the graded Lie algebra (2.5).
We show concrete expressions of the generalized Chern-Simons actions in two, three and four dimensions where we choose particular quaternion algebra \((\epsilon_1, \epsilon_2) = (-1, -1)\) in this section for simplicity,

\[
S_2 = - \int \text{Tr}\{\phi(d\omega + \omega^2) + \phi^2 B\},
\]
\[
S_3 = - \int \text{Str}\left\{\frac{1}{2}\omega d\omega + \frac{1}{3}\omega^3 - \phi(dB + [\omega, B]) + \phi^2 \Omega\right\},
\]
\[
S_4 = - \int \text{Tr}\{B(d\omega + \omega^2) + \phi(d\Omega + \{\omega, \Omega\}) + \phi B^2 + \phi^2 H\},
\]

which are invariant under the following gauge transformations:

\[
\delta \phi = [\phi, v],
\]
\[
\delta \omega = dv + [\omega, v] - \{\phi, u\},
\]
\[
\delta B = du + \{\omega, u\} + [B, v] + [\phi, b],
\]
\[
\delta \Omega = db + [\omega, b] + [\Omega, v] - \{B, u\} + \{\phi, U\},
\]
\[
\delta H = -dU - \{\omega, U\} + \{\Omega, u\} + [H, v] + [B, b] + [\phi, V].
\]

In ordinary gauge theory the integral for the trace of the \(n\)-th power of curvature is called \(n\)-th Chern character and has topological nature. In the generalized gauge theory it is possible to define generalized Chern character which is expected to have topological nature related to “generalized index theorem"

\[
\text{Str}_1(F^n) = \text{Str}_1(Q\Omega_{2n-1}), \quad (2.17)
\]

where \(\Omega_{2n-1}\) is the “generalized” Chern-Simons forms. Especially, for \(n = 2\) case in (2.17), we obtain the topological Yang-Mills type action of (2.14) on an even-dimensional manifold \(M\) related to the generalized Chern-Simons action with one dimension lower,

\[
\int_M \text{Str}_1\left(\frac{1}{2} F^2\right) = \int_M \text{Str}_1\left(Q\left(\frac{1}{2} AQA + \frac{1}{3} A^3\right)\right), \quad (2.18)
\]

which has the same form of the standard relation. The generalized gauge theory version of this topological relation (2.18) in four dimensions can be explicitly given by

\[
\int_{M_4} \text{Str}\left[\frac{1}{2}(d\omega + \omega^2)^2 + (d\omega + \omega^2)\{\phi, B\} + \phi^2(d\Omega + \{\omega, \Omega\})
\right.
\]
\[
- (d\phi + [\omega, \phi])(dB + [\omega, B] + [\Omega, \phi]) + \frac{1}{2}B^4 + \frac{1}{2}\{\phi, B\}^2 + \phi^2 B^2
\)
\]
\[
= \int_{M_4} \text{Str}\left[\frac{1}{2}\omega d\omega + \frac{1}{3}\omega^3 - \phi(dB + [\omega, B]) + \phi^2 \Omega\right],
\]

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where the first term in the left and right hand side includes the ordinary 2-form curvature square term and Chern-Simons term. This relation, however, includes all the degrees of differential forms up to 4-form, and yet fulfills highly non-trivial topological relation which is expected to have mathematical background.

### 3 Generalized Yang-Mills actions

In this section we formulate Yang-Mills action in terms of differential forms. In particular we introduce higher degrees of differential forms as gauge fields in contrast with the standard Yang-Mills action where only 1-form gauge field is introduced. We have defined generalized gauge field in (2.1) where all degrees of differential forms with bosonic and fermionic antisymmetric tensor fields are introduced. For simplicity we omit to introduce fermionic gauge fields in this section. We here again show the explicit form of the generalized gauge field

\[
A = jA + k\hat{A}
\equiv j(\omega^a + \Omega^a + \cdots)T_a + k(\phi^a + B^a + H^a + \cdots)\Sigma_a.
\] (3.1)

It is important to realize here that the graded Lie algebra naturally comes into the generalized gauge theory formulation from the beginning as we saw in the previous section.

Here we generalize the differential operator (2.8) in the following form:

\[
\mathcal{D} = jd + km
= jd + km^\alpha\Sigma_\alpha,
\] (3.2)

where \(d\) is the standard exterior derivative and \(m\) is a constant matrix. It should be noted that this newly defined generalized differential operator \(\mathcal{D}\) stays to be an element of \(\Lambda_-\) class and satisfies the graded Leibniz rule similar to (2.9)

\[
\{\mathcal{D}, \lambda_-\} = \mathcal{D}\lambda_-,
[\mathcal{D}, \lambda_+] = \mathcal{D}\lambda_+,
\] (3.3)

where \(\lambda_+ \in \Lambda_+\) and \(\lambda_- \in \Lambda_-\). It should, however, be noted that the nilpotency of \(\mathcal{D}\) is not satisfied in general since \(\mathcal{D}^2 = k^2m^2\).

The products of the form-valued elements \(\lambda_-\), \(\lambda_+\) are meant to be the wedge product and we do not write the wedge \(\wedge\) explicitly in the case of wedge product unless it is necessary to stress the difference from the cup product \(\vee\) which will be defined shortly.
Here we define the following generalized curvature which is the naive generalization of the generalized curvature (2.12):

\[ F \equiv \frac{1}{2} \{ D + A, D + A \} = D^2 + \{ D, A \} + A^2 \]
\[ \equiv \epsilon_2 \left[ 1(F^{(0)} + F^{(2)} + F^{(4)} + \cdots) + i(F^{(1)} + F^{(3)} + \cdots) \right], \quad (3.4) \]

where

\[ F^{(0)} = -\epsilon_1 (\phi + m)^2, \]
\[ F^{(1)} = F^{(1)}_{\mu} dx^{\mu} = -d\phi - [\omega, \phi + m], \]
\[ F^{(2)} = \frac{1}{2} F^{(2)}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = d\omega + \omega^2 - \epsilon_1 \{ \phi + m, B \}, \]
\[ F^{(3)} = \frac{1}{3!} F^{(3)}_{\mu\nu\rho} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} = -dB - [\omega, B] - [\Omega, \phi + m], \]
\[ F^{(4)} = \frac{1}{4!} F^{(4)}_{\mu\nu\rho\sigma} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} = d\Omega + \{ \omega, \Omega \} - \epsilon_1 (B^2 + \{ \phi + m, H \}), \]
\[ \vdots \]

Here in this definition it can be recognized that the addition of the constant matrix to the differential operator simply shift the value of constant term in the 0-form gauge field \( \phi \).

In order to define generalized Yang-Mills action we introduce the notion of Clifford product or simply cup product \( \vee \) which was introduced by Kähler \[34\]. This cup product \( \vee \) should be differentiated from the wedge product \( \wedge \). We consider an element \( u \) which is the direct sum of forms

\[ u = \sum_{p=0}^{m} \frac{1}{p!} u_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \quad (3.6) \]

where \( m \leq D \) with \( D \) as spacetime dimension.

We then define the linear operator \( e_\mu \),

\[ e_\mu u = \sum_{p=0}^{m-1} \frac{1}{p!} u_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \quad (3.7) \]

which is understood as a differentiation of the polynomial of differential form with respect to \( dx^\mu \). In particular it plays the role of contracting operator as

\[ e_\mu dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge dx^{\alpha_3} \wedge dx^{\alpha_4} \wedge \cdots \]
\[ = g^{\alpha_1}_{\mu} dx^{\alpha_2} \wedge dx^{\alpha_3} \wedge dx^{\alpha_4} \wedge \cdots - g^{\alpha_2}_{\mu} dx^{\alpha_1} \wedge dx^{\alpha_3} \wedge dx^{\alpha_4} \wedge \cdots \]
\[ + g^{\alpha_3}_{\mu} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge dx^{\alpha_4} \wedge \cdots - g^{\alpha_4}_{\mu} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge dx^{\alpha_3} \wedge \cdots \]
\[ + \cdots. \]
We can now define the cup product of \( u \) and \( v \), which have the same expansion form as (3.6), by
\[
 u \vee v = \sum_{p=0}^{m} \frac{1}{p!} \zeta_p \{ \eta^p(e_{\mu_1} \cdots e_{\mu_p} u) \} \wedge e^{\mu_1} \cdots e^{\mu_p} v,
\]
(3.8)
where we need to introduce the sign operators \( \eta \) and \( \zeta \) which generate the following sign factors:
\[
 \zeta u_p = \zeta_p u_p \equiv (-1)^{\frac{p(p-1)}{2}} u_p, \\
 \eta u_p = (-1)^p u_p,
\]
(3.9)
where \( u_p \) is a \( p \)-form variable. These sign operators satisfy the following properties:
\[
 \eta(u_p \wedge w_q) = (\eta u_p) \wedge (\eta w_q), \quad \eta(u_p \vee w_q) = (\eta u_p) \vee (\eta w_q), \\
 \zeta(u_p \wedge w_q) = (\zeta w_q) \wedge (\zeta u_p), \quad \zeta(u_p \vee w_q) = (\zeta w_q) \vee (\zeta u_p),
\]
(3.10)
where \( u_p \) and \( w_q \) are arbitrary \( p \)- and \( q \)-form variables, respectively. The sign factors \( \zeta_p \) and the operator \( \eta^p \) are necessary to ensure the associativity of the cup product
\[
 (u \vee v) \vee w = u \vee (v \vee w).
\]
(3.11)

We now define the generalized Yang-Mills action in \( D \) dimensions with \( e \) as the coupling constant
\[
 S_G = \frac{1}{e^2} \int \text{Tr} [\mathcal{F} \vee \mathcal{F}] \mathbf{1} \ast 1 \\
 = \frac{1}{e^2} \int d^D x \sqrt{g} \text{Tr} \left[ \sum_{p=0}^{D} \frac{1}{p!} \mathcal{F}_{\mu_1 \cdots \mu_p}^{(p)} \mathcal{F}_{\mu_{1} \cdots \mu_{p}}^{(p)} \right],
\]
(3.12)
where we have taken the particular choice \( \epsilon_1 = 1 \) which will be shown as a natural choice later. The Hodge star \( \ast \) acting on \( \mathbf{1} \) denotes the invariant volume element. For example in four dimensions \( \ast 1 = \sqrt{g} \frac{1}{4!} \epsilon_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge dx^{\alpha_3} \wedge dx^{\alpha_4} = \sqrt{g} d^4 x \) with \( g \) as the metric determinant. We need to pick up 0-form components in the trace due to the presence of \( \ast 1 \). The symbol \( \mathbf{1} \) denotes to pick up the coefficient of \( \mathbf{1} \) for the quaternion expansion in the trace. The reason of this particular choice of the component is due to the fact that the coefficient of \( \mathbf{1} \) is even form and thus includes 0-form terms.

The generalized gauge parameter \( V \) includes all degrees of differential forms as in (2.2) where we do not consider the fermionic gauge parameters here. The generalized gauge field \( A \) with the new definition of the generalized differential operator (3.2) transforms similarly as (2.15)
\[
 \delta A = [\mathcal{D} + A, V].
\]
(3.13)
The gauge transformation of newly defined generalized curvature \((3.4)\) leads
\[
\delta \mathcal{F} = \{ \mathcal{D} + \mathcal{A}, \delta \mathcal{A} \} = [\mathcal{F}, \mathcal{V}] .
\tag{3.14}
\]

If we now consider the gauge invariance of the generalized Yang-Mills action \((3.12)\) under the generalized gauge transformation \((3.13)\), we notice that the gauge invariance is lost in general due to the loss of the associativity since the wedge product and the cup product are mixed up in the generalized gauge transformation of the action. There is, however, an exception that the associativity is recovered when the gauge parameter includes only a 0-form. This is the standard situation where the gauge parameter normally includes 0-form only for the gauge transformation of the Yang-Mills action. We thus define the generalized gauge parameter
\[
\mathcal{V}_0 = \mathbf{1} v^a T_a ,
\tag{3.15}
\]
which includes only 0-form gauge parameter \(v\). This should be compared with the generalized gauge parameter of generalized Chern-Simons formulation where all the degrees of differential forms were introduced. The generalized gauge transformation now leads to
\[
\delta \mathcal{A} = [\mathcal{D} + \mathcal{A}, \mathcal{V}_0] = [\mathcal{D} + \mathcal{A}, \mathcal{V}_0]_{\mathcal{V}} ,
\tag{3.16}
\]
where the first commutator is the standard commutator with wedge product while the second commutator is defined as:
\[
[\mathcal{A}, \mathcal{B}]_{\mathcal{V}} = \mathcal{A} \wedge \mathcal{B} - \mathcal{B} \wedge \mathcal{A} .
\]
The last equality of \((3.16)\) is satisfied since \(\mathcal{V}_0\) includes only 0-form. Then the gauge transformation of the generalized curvature is given by
\[
\delta \mathcal{F} = [\mathcal{F}, \mathcal{V}_0] = [\mathcal{F}, \mathcal{V}_0]_{\mathcal{V}} ,
\tag{3.17}
\]
or equivalently
\[
\delta \mathcal{F}^{(p)} = [\mathcal{F}^{(p)}, v]_{\mathcal{V}} .
\tag{3.18}
\]
We can then show that the generalized Yang-Mills action \(S_G\) is gauge invariant under the generalized gauge transformation \((3.16)\) since the gauge transformation of the action can be written in the following form:
\[
\delta S_G = \frac{1}{e^2} \int \text{Tr} \left[ \{ \delta \mathcal{F}, \mathcal{F} \}_{\mathcal{V}} \right] \mathbf{1} \ast 1
\]
\[
= \frac{1}{e^2} \int \sum_{p=0}^{D} \text{Tr} \left[ \{ \delta \mathcal{F}^{(p)}, \mathcal{F}^{(p)} \}_{\mathcal{V}} \right] \ast 1
\]
\[
= 0 .
\]
In the proof of the gauge invariance the associativity of the cup product (3.11) and the following relation should be used:

\[ \text{Tr}[\lambda_+ \lor \lambda_+'] \ast 1 = \text{Tr}[\lambda_+ \lor \lambda_+'] \ast 1, \]

where \( \lambda_+, \lambda_+' \in \Lambda_+ \). It should be noted that the above relation is satisfied only when we use the following relation: \( \text{Tr}(\Sigma_\alpha \Sigma_\beta) = \text{Tr}(\Sigma_\beta \Sigma_\alpha) \). This is in contrast with the generalized gauge theory of the previous section where we needed to introduce supertrace:

\[ \text{Str}(\Sigma_\alpha \Sigma_\beta) = -\text{Str}(\Sigma_\beta \Sigma_\alpha) \]

to define odd-dimensional version of generalized Chern-Simons actions and even-dimensional topological Yang-Mills actions [2].

It is important to recognize here that the gauge invariance is assured only with the 0-form gauge parameter as is defined in (3.15). As we mentioned already in the previous section, the gauge invariance of the action \( S_G \) is lost if we introduce higher form gauge parameters due to the breakdown of associativity by the mixture of wedge and cup products. This is in contrast with the generalized Chern-Simons actions and the generalized topological Yang-Mills actions where the gauge invariance with full degrees of differential form for the gauge parameters is assured. It is important to realize at this stage that the associativity of the generalized gauge invariance with full degrees of differential form for the gauge parameters will be recovered if we introduce the cup product from the beginning for the definition of the curvature and the generalized gauge transformation. In this case we are forced to omit exterior derivative in the definition of the generalized curvature and the generalized gauge transformation since it includes prohibited terms otherwise. This type of the generalized Yang-Mills actions without derivative terms, which we shall write down in the last section, may be related to the “reduced model” and need further investigations.

Here we explicitly show the four-dimensional generalized Yang-Mills action.

\[
S_4 = \frac{1}{e^2} \int d^D x \sqrt{g} \text{Tr} \left[ \left( \mathcal{F}^{(0)} \right)^2 + \mathcal{F}_\mu^{(1)} \mathcal{F}^{(1)\mu} - \frac{1}{2!} \mathcal{F}^{(2)}_{\mu\nu} \mathcal{F}^{(2)\mu\nu} - \frac{1}{3!} \mathcal{F}^{(3)}_{\mu\nu\rho} \mathcal{F}^{(3)\mu\nu\rho} + \frac{1}{4!} \mathcal{F}^{(4)}_{\mu\nu\rho\sigma} \mathcal{F}^{(4)\mu\nu\rho\sigma} \right],
\]

(3.19)

where the explicit form of \( \mathcal{F}^{(p)}_{\mu_1^p \cdots \mu_p} \) \( (p = 0 \sim 4) \) in terms of the generalized gauge fields including all the degrees of differential forms \( \phi, \omega, B, \Omega, H \) are given in (3.5). The explicit form of two- and three-dimensional generalized Yang-Mills actions can be obtained similarly.

These generalized Yang-Mills actions are gauge invariant under the generalized gauge
transformation (3.16) or equivalently,
\[ \delta \phi = [\phi + m, v], \quad \delta \omega_\mu = \partial_\mu v + [\omega_\mu, v], \]
\[ \delta B_\mu = [B_\mu, v], \quad \delta \Omega_\mu = [\Omega_\mu, v], \quad \delta H_\mu = [H_\mu, v]. \]
where we have used the notation for the p-form gauge field: \( A^{(p)} = \frac{1}{p!} A^{(p)}_{\mu_1 \cdots \mu_p} dx^{\mu_1} \cdots dx^{\mu_p} \).

Here we explicitly give the simplest version of the four-dimensional generalized Yang-Mills action including 0- and 1-form gauge fields only. In this particular case \( \mathcal{F}^{(3)} = \mathcal{F}^{(4)} = 0 \) and then the action leads much simpler form than (3.19),
\[
S_4 = \frac{1}{e^2} \int d^D x \sqrt{g} \text{Tr} \left[ -\frac{1}{2} F_{\mu \nu} F^{\mu \nu} + (\partial_\mu \phi + [\omega_\mu, \phi + m]) (\partial^\mu \phi + [\omega^\mu, \phi + m]) + (\phi + m)^4 \right], \tag{3.20}
\]
where \( F_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \). It is interesting to note that this action is closely related with the Yang-Mills action à la noncommutative geometry formulation of Connes which includes only 0- and 1-form gauge fields \([13, 15]\).

4 Dirac-Kähler matter fermions

The generalized Chern-Simons actions formulated in section 2 have topological nature because all the gauge fields and parameters are expressed by differential forms whose metric independence are trivial from the definitions. If we, however, try to generalize the formulation to Yang-Mills action, the topological nature is lost because we need to introduce Hodge star operation to define the dual of curvature, and thus need to choose a particular metric.

As we have shown in the previous section, the generalized Yang-Mills actions are formulated by the differential forms. In this section we formulate matter fermions by the antisymmetric tensor of differential forms, Dirac-Kähler fermion formulation. The basic idea is as follows \([33, 35, 39, 44]\). We first note the following well known relations on the flat spacetime:
\[
(d + \delta)^2 = \partial ^\mu \partial_\mu = (\gamma ^\mu \partial_\mu)^2, \tag{4.1}
\]
where \( \delta \) is the adjoint of the exterior derivative \( d \) and can be expressed as \( \delta = (-1)^{Dp+D+1} * d * \) for p-form in D-dimensional Minkowski spacetime. In even dimensions it has the following form:
\[
\delta = - * d * \equiv e^\mu \partial_\mu, \tag{4.2}
\]
where we have introduced the operator \( e^\mu \) which coincides with the one defined in (3.7). The above relation suggests the following correspondences:

\[
\begin{align*}
d + \delta &= (dx^\mu \wedge +e^\mu)\partial_\mu \sim \gamma^\mu \partial_\mu, \\
dx^\mu \wedge +e^\mu &\sim \gamma^\mu.
\end{align*}
\]

We now reintroduce the simplest version of the cup product by

\[
(dx^\mu \wedge +e^\mu)\Phi \equiv dx^\mu \vee \Phi. \tag{4.3}
\]

This is the particular example of the general definition (3.8) with \( u = dx_\mu \). We can then show that the cup product satisfies the Clifford algebra:

\[
\{dx^\mu, dx^\nu\}_\vee = 2g^{\mu\nu}, \tag{4.4}
\]

where the anticommutator with \( \vee \) is defined as

\[
\{u, v\}_\vee = u \vee v + v \vee u,
\]

for arbitrary differential forms \( u \) and \( v \).

We now define the following \( 2^D \times 2^D \) matrix \( Z_{ij} \)

\[
Z_{ij} = \sum_{p=0}^{D} \frac{1}{p!} (\gamma^T_{\mu_1} \cdots \gamma^T_{\mu_p})_{ij} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{4.5}
\]

which satisfies the following orthogonality:

\[
\int Z_{ij} \vee Z_{kl} * 1 = 2^D \delta_{il} \delta_{jk} \int *1. \tag{4.6}
\]

We can then show the following crucial relation:

\[
dx^\mu \vee Z_{ij} = (\gamma^{\mu^T} Z)_{ij} = \gamma^\mu_{ki} Z_{kj}. \tag{4.7}
\]

We now consider an arbitrary fermionic field \( \Psi \) which is a direct sum of fermionic antisymmetric tensor fields contracted with differential forms. Then this form valued generalized fermionic field \( \Psi \) can be expanded by \( Z_{ij} \),

\[
\Psi(x) = \sum_{i,j} \psi_{ij}(x)Z_{ij}, \tag{4.8}
\]

where

\[
\psi_{ij}(x) = 2^{-\frac{D}{2}} \{\Psi(x) \vee Z_{ji}\}_0, \tag{4.9}
\]

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where \([ \ ]_0\) denotes to pick up 0-form term. By employing the relation (4.7), we obtain
\[
d \vee \Psi = (\gamma^\mu)_{ki} \partial_\mu \psi_{ij} Z_{kj}.
\] (4.10)

Considering the completeness of \(Z_{ij}\) to be understood from the orthogonality relation (4.6), we obtain the following Dirac-Kähler equations which have \(2^D\) “flavor” component suffices \(i\):
\[
d \vee \Psi = 0 \longleftrightarrow (\gamma^\mu)_{\alpha\beta} \partial_\mu \psi_{\beta(i)} = 0,
\] (4.11)
where the first suffix of \(\psi_{ij}\) is identified as spinor suffix. We then obtain the free Dirac fermion Lagrangian from the Dirac-Kähler fermion formulation,
\[
\int \overline{\Psi} \vee d \vee \Psi = 2^D \sum_j \int d^D x \overline{\psi}_{(j)} \partial \psi_{(j)},
\] (4.12)
where we obtain \(2^D\) copy of flavor suffices \((j)\).

We now introduce generalized version of our fermionic matter fields by Dirac-Kähler fermion formulation. We first define a direct sum of the quaternion valued fermionic antisymmetric tensor fields contracted with differential forms:
\[
\Psi = 1 \psi + i \hat{\psi} \in \Lambda_-, \\
\overline{\Psi} = j \hat{\psi} + k \psi \in \Lambda_+, \tag{4.13}
\]
where \(\psi\) and \(\overline{\psi}\) are fermionic odd forms while \(\hat{\psi}\) and \(\hat{\overline{\psi}}\) are fermionic even forms. The conjugate fermionic field \(\overline{\Psi}\) is defined to be related with \(\Psi\) as:
\[
\overline{\Psi} = \zeta (\Psi^\dagger \vee j dx^0) \in \Lambda_+ \text{ (Minkowski)}, \tag{4.14}
\]
\[
\overline{\Psi} = \zeta (\Psi^\dagger k) \in \Lambda_+ \text{ (Euclid)}, \tag{4.15}
\]
where we define \(\Psi^\dagger = (1 \psi + i \hat{\psi})^\dagger = 1 \psi^\dagger + i \hat{\psi}^\dagger\). The field \(\overline{\Psi}\) has different \(\Psi^\dagger\) dependence from the Minkowskian case to the Euclidean case analogous to the usual expressions. They, however, belong to the same \(\Lambda_+\) class. We then obtain the following relations:
\[
\hat{\psi} = \zeta (\psi^\dagger \vee dx^0), \quad \overline{\psi} = \zeta (\hat{\psi}^\dagger \vee dx^0) \text{ (Minkowski)}, \tag{4.16}
\]
\[
\hat{\overline{\psi}} = \epsilon_1 \zeta \hat{\psi}^\dagger, \quad \overline{\overline{\psi}} = \zeta \psi^\dagger \text{ (Euclid)}. \tag{4.17}
\]

With all these definitions given we introduce generalized matter action coupled to the generalized gauge fields
\[
S_M = \int [\overline{\Psi} \vee (D + A) \vee \Psi] \star 1, \tag{4.18}
\]
where the generalized differential operator $\mathcal{D}$ and the generalized gauge field $\mathcal{A}$ are given in (3.2) and (3.1), respectively. The reason of taking the coefficient of $1$ for the quaternion expansion in the trace is the same as the generalized Yang-Mills action. In other words we need to pick up 0-form components in the trace and thus take the even form part of $1$-th component due to the presence of $*1$. In this paper we consider that fermions belong to the fundamental representation of the gauge algebra.

Substituting the generalized fermionic fields (4.13) into the generalized matter action $S_M$ and collecting the $1$-th components, we obtain

$$S_M = \epsilon_2 \int \left[ (\epsilon_1 \overline{\psi} + \hat{\psi}) \lor (d + A) \lor (\psi + \hat{\psi}) \ight. \\
- \epsilon_1 (\overline{\psi} + \hat{\psi}) \lor (\hat{A} + m) \lor (\psi + \hat{\psi}) \left. \right] \ast 1,$$

where $\epsilon_1$ and $\epsilon_2$ are the sign factors of quaternion algebra (2.3). It should be noted that only the 0-form terms are allowed in the square bracket of the generalized matter action $S_M$ due to the presence of $*1$. There are thus several trivial terms included in the matter action such as $\overline{\psi} \lor (d + A) \lor \psi \ast 1 = 0$. In order that $\overline{\psi} = \overline{\psi} + \hat{\psi}$ be the common conjugate fermionic field in the generalized matter action (1.13), we need to take the particular choice of the sign factor $\epsilon_1 = +1$ for the quaternion algebra (2.3) while the overall sign factor $\epsilon_2$ could be arbitrary and thus $\epsilon_2 = \pm 1$.

There are two possible quaternion algebra which are consistent with the conjugate definitions of the generalized fermionic fields (4.14) and (4.15). The first “quaternion algebra” corresponds to the choice $(\epsilon_1, \epsilon_2) = (+1, +1)$:

(1) \( i^2 = j^2 = 1, \quad k^2 = -1, \quad ij = k, \quad jk = -i, \quad ki = -j, \quad (4.20) \)

while the second “quaternion algebra” corresponds to the choice $(\epsilon_1, \epsilon_2) = (+1, -1)$:

(2) \( i^2 = k^2 = 1, \quad j^2 = -1, \quad ij = k, \quad jk = i, \quad ki = -j. \quad (4.21) \)

These algebra are common to Minkowski and Euclidean cases.

Hereafter we take the choice of the first “quaternion algebra” (4.20) and then the generalized matter action leads

$$S_M = i2^{-\frac{D}{2}} \int \left[ \overline{\Psi} \lor (\mathcal{D} + \mathcal{A}) \lor \Psi \right]_1 \ast 1$$
$$= i2^{-\frac{D}{2}} \int \left[ (\overline{\psi} + \hat{\psi}) \lor (d + A - \hat{A} - m) \lor (\psi + \hat{\psi}) \right] \ast 1,$$

where we have introduced the normalization factor.
Let us now define
\[
\psi + \hat{\psi} = \psi_{ij} Z_{ij},
\]
\[
\bar{\psi} + \bar{\hat{\psi}} = \bar{\psi}_{ij} Z_{ij},
\]
\[
A - \hat{A} = - \sum_{p=0}^{D} \frac{(-1)^{p}}{p!} A_{\mu_1 \cdots \mu_p}^{(p)} d\mu_1 \wedge \cdots \wedge d\mu_p.
\] (4.23)

Using the relation \(dx^\mu \wedge dx^\nu = dx^\mu \vee dx^\nu - g^\mu\nu\) successively and the relation (4.11), we can prove the following relation:
\[
A_{\mu_1 \cdots \mu_p}^{(p)} (dx^\mu_1 \wedge \cdots \wedge dx^\mu_p) \vee \psi_{ij} Z_{ij}
\]
\[
= A_{\mu_1 \cdots \mu_p}^{(p)} (dx^\mu_1 \vee \cdots \vee dx^\mu_p) \vee \psi_{ij} Z_{ij}
\]
\[
= A_{\mu_1 \cdots \mu_p}^{(p)} (\gamma^\mu_1 \cdots \gamma^\mu_p)_{ki} \psi_{kj} Z_{ij}.
\]

We can then obtain the following concrete expression of the generalized matter action coupled to the generalized gauge fields:
\[
S_M = \int d^D x \left[ \bar{\psi}_{ik} i(\gamma^\mu \partial_\mu - \sum_{p=0}^{D} \frac{(-1)^{p}}{p!} A_{\mu_1 \cdots \mu_p}^{(p)} (\gamma^\mu_1 \cdots \gamma^\mu_p) - m)_{kl} \psi_{li} \right].
\] (4.24)

We next consider the gauge invariance of the generalized matter action \(S_M\). It is easy to show that \(S_M\) is gauge invariant if the generalized gauge fields and matter fields transform as:
\[
\delta A = [\mathcal{D} + A, \mathcal{V}]_\vee,
\]
\[
\delta \Psi = -\mathcal{V} \vee \Psi, \quad \delta \bar{\Psi} = \bar{\Psi} \vee \mathcal{V},
\] (4.25)

where the gauge transformation of the generalized gauge field \(A\) has the similar transformation form as the generalized gauge transformation (3.13) except that the commutator with cup product is adapted. It is important to realize here again that the generalized gauge transformation of the gauge field (4.23) includes prohibited terms unless we omit exterior derivative. This situation is similar to the generalized Yang-Mills action as we have mentioned in the previous section. If we, however, introduce only 0-form gauge parameter as in (3.15), the gauge invariance with differential operator is recovered even with full degrees of differential forms for the generalized gauge fields and matter fermions.

Since \(\bar{\Psi}\) is related to \(\Psi\) by the relations (4.14) for Minkowski case and (4.15) for Euclidean case, it is not obvious if the generalized gauge transformation (4.26) is consistent.
As far as the generalized gauge parameter includes only 0-form, we can consistently impose the gauge transformation (4.26) by taking anti-Hermite 0-form gauge parameter.

Here we investigate the consistency condition between the generalized gauge transformation (1.26) and the definition of the conjugate fermionic field (4.14) in the Minkowski case. We study the general case for the generalized gauge parameter including all the degrees of differential forms

\[ V = \hat{1} \hat{a} + i a, \]  

(4.27)

where \( \hat{a} \) and \( a \) are the direct sum of bosonic even and odd forms, respectively. For simplicity we have omitted the fermionic gauge parameters here, to be compared with the general expression (2.2).

Due to the relation between the conjugate fermionic field \( \overline{\Psi} \) and the original fermionic field \( \Psi \) given in (4.14), the gauge transformation of the conjugate fermionic field should obey

\[ \delta \overline{\Psi} = \zeta (\delta \Psi^\dagger \vee j dx^0) \]
\[ = j (\overline{\psi} \vee \zeta \hat{a}^\dagger - \overline{\psi} \vee \zeta a^\dagger) + k (\overline{\psi} \vee \zeta \hat{a}^\dagger - \overline{\psi} \vee \zeta a^\dagger), \]

(4.28)

while \( \delta \overline{\Psi} \) needs to satisfy the gauge transformation (1.26).

\[ \delta \overline{\Psi} = \overline{\Psi} \vee V \]
\[ = j (\overline{\psi} \vee \hat{a} - \overline{\psi} \vee a) + k (\overline{\psi} \vee \hat{a} - \overline{\psi} \vee a). \]

(4.29)

To be consistent, these two expressions should coincide and thus we obtain the following consistency constraints:

\[ \zeta \hat{a}^\dagger = -\hat{a}, \quad \zeta a^\dagger = a. \]  

(4.30)

It turns out that these constraints for the Euclidean case exactly coincide with those of the Minkowski case.

In these derivations we have used the sign operator property (3.10) and adapted the following Hermite conjugate:

\[ \zeta (\hat{a} \vee \psi)^\dagger = \zeta (\hat{a}^{bx} \vee \psi^\dagger) T_b^\dagger = (\zeta \psi^\dagger) \vee (\zeta \hat{a}^{bx}) T_b^\dagger. \]

5 Weinberg-Salam model from generalized gauge theory

In formulating the generalized Yang-Mills actions with Dirac-Kähler fermions, we have not specified the gauge algebra. In formulating the generalized Chern-Simons actions and
topological Yang-Mills actions, the graded Lie algebra as a gauge algebra of the generalized
gauge theory was the natural consequence of the formulation. This characteristic of the
natural introduction of the graded Lie algebra in the generalized gauge theory transfers to
the formulation of the generalized Yang-Mills actions. In this section we take a particular
graded Lie algebra of supergroup \( SU(2|1) \), as the generalized gauge algebra and show
that the Weinberg-Salam model with spontaneously broken symmetry can be formulated
naturally from the generalized Yang-Mills action. Though the noncommutative geometry
formulation of the Weinberg-Salam model based on the \( SU(2|1) \) graded Lie algebra was
intensively studied by Coquereaux et al. \[19\], we believe that our formulation based on
the generalized gauge theory will give new insights into the formulation.

In order to accommodate the Dirac-Kähler fermion formulation into our generalized
gauge theory formulation we needed to specify the quaternion algebra in such a way that
the generalized gauge transformation of the fermion and the conjugate definitions of the
generalized fermionic field are consistent. Here in this section we choose the following
"quaternion algebra" defined in (4.20)

\[
(1) \quad i^2 = j^2 = 1, \quad k^2 = -1, \quad ij = k, \quad jk = -i, \quad ki = -j. \tag{5.1}
\]

We now introduce the algebra of supergroup \( SU(2|1) \) as the graded Lie algebra. \( SU(2|1) \) generators can be represented by \( 3 \times 3 \) matrices \[19, 47–48\]

\[
T_i = \begin{pmatrix} \sigma_i & 0 \\ \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},
\]

\[
\Sigma_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Sigma'_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma'_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

where \( \sigma_i \)'s are Pauli matrices. They satisfy the following graded Lie algebra:

\[
[T_i, T_j] = i\epsilon_{ijk}T_k, \quad [Y, T_i] = 0,
\]

\[
[T_\pm, \Sigma_\pm] = 0, \quad [T_\pm, \Sigma_\mp] = \frac{1}{\sqrt{2}}\Sigma_\pm,
\]

\[
[T_\pm, \Sigma'_\pm] = 0, \quad [T_\pm, \Sigma'_\mp] = -\frac{1}{\sqrt{2}}\Sigma'_\pm,
\]

\[
[T_3, \Sigma_\pm] = \pm\frac{1}{2}\Sigma_\pm, \quad [T_3, \Sigma'_\pm] = \pm\frac{1}{2}\Sigma'_\pm,
\]

\[
[Y, \Sigma_\pm] = -\Sigma_\pm, \quad [Y, \Sigma'_\pm] = \Sigma'_\pm,
\]
\{\Sigma_\pm, \Sigma_\mp\} = \{\Sigma_\pm, \Sigma_\mp\} = 0, \quad \{\Sigma'_\pm, \Sigma'_\mp\} = \{\Sigma'_\pm, \Sigma'_\mp\} = 0,
\{\Sigma_\pm, \Sigma'_\pm\} = \sqrt{2}T_\pm, \quad \{\Sigma_\pm, \Sigma'_\mp\} = \pm T_3 + \frac{1}{2} Y,
with \( T_\pm = \frac{1}{\sqrt{2}}(T_1 \pm iT_2) \). \( T_3, Y \) correspond to a generator of a weak isospin and a weak hypercharge, respectively. This algebra contains \( SU(2) \times U(1)_Y \) in the even graded parts \( T_i, Y \), and \( SU(2) \) doublets in the odd graded parts \( \Sigma_\pm, \Sigma'_\pm \) whose subscripts \( \pm \) correspond to the generators with eigenvalues \( \pm \frac{1}{2} \) of the generator \( T_3 \). Indeed Higgs doublet which we have introduced as Lie algebra valued gauge fields corresponds to these odd parts. It is interesting to note that the supersymmetric algebra of \( SU(2|1) \) can be accommodated in the generalized gauge theory even without fermionic fields.

In the formulation of generalized Yang-Mills action of the previous section we have introduced all the degrees of differential forms but only 0-form for the generalized gauge parameter. The reason why we have introduced only 0-form gauge parameter is that the action is not gauge invariant under the higher form gauge parameters. Hereafter we introduce only 0-form and 1-form gauge fields in accordance with the standard gauge theory.

The gauge field is now expanded by corresponding fields of the generators
\[
\mathcal{A} = jie(A^i T_i + \frac{1}{2\sqrt{3}} BY) + kie^{\frac{1}{\sqrt{2}}}(\phi_0 \Sigma_+ + \phi_- \Sigma_- + \phi^*_- \Sigma'_+ + \phi^*_+ \Sigma'_-),
\]
where \( A^i, B \) and \( \phi_0, \phi_+ \) are real \( SU(2) \times U(1)_Y \) 1-form gauge fields and 0-form complex Higgs scalar fields respectively. The normalization factors and the pure imaginary constant \( i \) of each fields are adjusted to give the standard kinetic terms in the final Weinberg-Salam action. The generalized gauge field is rescaled by the coupling constant \( e: \mathcal{A} \rightarrow e \mathcal{A} \).

We can choose a particular form of the constant matrix \( m \) of the generalized differential operator (3.2) parametrized by a complex number \( v \) as
\[
\mathcal{D} = jd + km^a \Sigma_a
= jd + k \frac{i}{\sqrt{2}} (v \Sigma_+ + v^* \Sigma'_-),
\]
which leads
\[
\mathcal{D}^2 = -1m^2 = \frac{1}{2} |v|^2 (T_3 + \frac{1}{2} Y).
\]
Then \( \mathcal{D}^2 \) can be taken to be proportional to the generator of the electromagnetic charge of the Weinberg-Salam model.
The generalized curvature is now given by

\[ F \equiv D^2 + \{D, A\} + A^2 \]
\[ = 1 \left( e^2 F^{(0)} - \frac{i}{2} e F^{(2)} d\mu \wedge d\nu \right) + i e F^{(1)} d\mu. \]  

(5.5)

The kinetic terms of \( SU(2) \times U(1)_Y \) gauge fields are

\[ F^{(2)}_{\mu \nu} = F^k_{\mu \nu} T_k + \frac{1}{2\sqrt{3}} G_{\mu \nu} Y, \]  

(5.6)

where

\[ F^k_{\mu \nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu - e \epsilon^{ij} A^i_\mu A^j_\nu, \]
\[ G_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \]

The kinetic terms of Higgs fields and the gauge-Higgs interaction terms are

\[ F^{(1)}_\mu = -\frac{1}{\sqrt{2}} \left\{ \left( \partial_\mu \phi_0 + \frac{i}{\sqrt{2}} e W_\mu \phi_0 + \frac{i}{\sqrt{3}} e Z_\mu \left( \phi_0 + \frac{v}{e} \right) \right) \Sigma_+ \right. \]
\[ + \left( \partial_\mu \phi_+ + \frac{i}{\sqrt{2}} \epsilon W_\mu^* (\phi_0 + \frac{v}{e}) - i \frac{\sqrt{3}}{6} e Z_\mu \phi_+ - \frac{i}{2} e A_\mu \phi_+ \right) \Sigma_- \]
\[ + \text{h.c.} \left\}, \]  

(5.7)

where

\[ W_\mu = \frac{1}{\sqrt{2}} \left( A^1_\mu - i A^2_\mu \right), \]

and

\[ Z_\mu = \frac{\sqrt{3}}{2} A^3_\mu - \frac{1}{2} B_\mu, \]
\[ A_\mu = \frac{1}{2} A^3_\mu + \frac{\sqrt{3}}{2} B_\mu. \]  

(5.8)

These identifications (5.8) fix the Weinberg angle to be \( \theta_W = \frac{\pi}{6} \) which is an arbitrary parameter in the Weinberg-Salam model. Thus the direction of spontaneous breaking is particularly chosen by the model itself. The Higgs potential term is given by

\[ F^{(0)} = \frac{1}{2} \left\{ \left| \phi_0 + \frac{v}{e} \right|^2 \left( T_3 + \frac{1}{2} Y \right) + \left( \phi_0 + \frac{v}{e} \right) \phi_+^* \sqrt{2} T_+ \right. \]
\[ \left. + \phi_+ \left( \phi_0^* + \frac{v^*}{e} \right) \sqrt{2} T_- + \left| \phi_+ \right|^2 \left( - T_3 + \frac{1}{2} Y \right) \right\}. \]  

(5.9)

As we have seen before, the generalized gauge transformation of the generalized curvature is given by (3.17): \( \delta F = [F, V_0]_\gamma. \) The 0-form generalized gauge parameter
\[ V_0 = v^a T_a \] depends only on the even part of generators of the graded SU(2\|1) algebra and thus commutes with the generator \( Y \). Therefore the gauge transformation of the generalized curvature is form invariant even if we add the term proportional to the generator \( Y \) to the curvature. In other words there is a particular arbitrariness of the constant term in the definition of the generalized curvature. We thus define a new curvature which includes the term,

\[ \mathcal{F}' = \mathcal{F} + y|v|^2 Y, \]

where we introduce a new parameter \( y \) with previously introduced dimensionful parameter \( v \) in (5.3).

Using the new definition of the generalized curvature, we obtain the full expression of the generalized Yang-Mills action with SU(2\|1) graded Lie algebra in flat Minkowski spacetime

\[
S_G = -\frac{1}{e^2} \int \text{Tr} \left[ \mathcal{F}' \vee \mathcal{F}' \right] 1 \ast 1 \\
= -\int d^4x \text{Tr} \left[ \frac{1}{2} F_{\mu
u}^{(2)} F_{\mu\nu}^{(2)} - F_{\mu}^{(1)} F_{\mu}^{(1)} + e^2 (F^{(0)})^2 \right] \\
= \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z_{\mu\nu} \\
- \frac{1}{2} (D^{\mu} W^{\mu\dagger} - D^{\mu\dagger} W^{\mu}) (D_{\mu} W_{\nu} - D_{\mu\dagger} W_{\nu}) \\
- i e \left( \frac{\sqrt{3}}{2} Z_{\mu\nu} + \frac{1}{2} F_{\mu\nu} \right) W^{\mu} W^{\nu\dagger} \\
+ \frac{e^2}{2} \left( |W_{\mu} W^{\mu}|^2 - (W_{\mu} W^{\mu\dagger})^2 \right) \\
+ \partial_{\mu} \phi_0 + \frac{i}{\sqrt{2}} e W_{\mu} \phi_+ + \frac{i}{\sqrt{3}} e Z_{\mu} \left( \phi_0 + \frac{|e|}{v}\right) |^2 \\
+ |\partial_{\mu} \phi_+ + \frac{i}{\sqrt{2}} e W_{\mu} \phi_0 + \frac{v}{e} - \frac{i}{\sqrt{3}} e Z_{\mu} \phi_+ - \frac{i}{2} e A_{\mu} \phi_+|^2 \\
- \frac{e^2}{2} \left( |\phi_0 + \frac{v}{e}|^2 + |\phi_+|^2 \right) \\
- 3y e^2 \left| \frac{v}{e} \right|^2 \left( |\phi_0 + \frac{v}{e}|^2 + |\phi_+|^2 \right) \\
- 6y^2 e^2 \left| \frac{v}{e} \right|^2 \right\}, \tag{5.11}
\]

where

\[
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \\
Z_{\mu\nu} = \partial_{\mu} Z_{\nu} - \partial_{\nu} Z_{\mu}, \\
D_{\mu} = \partial_{\mu} + ie \left( \frac{\sqrt{3}}{2} Z_{\mu} + \frac{1}{2} A_{\mu} \right).
\]
This is the Weinberg-Salam model with the Weinberg angle $\theta_W = \frac{\pi}{6}$. If we look at the Higgs potential term, it has the local minimum at

1. $|\phi_0 + \frac{y}{e}| = 0$, $|\phi_+| = 0$ for $y \geq 0$,
2. $|\phi_0 + \frac{y}{e}| = \sqrt{3|y|v}$, $|\phi_+| = 0$ for $y < 0$.

Then the masses of the weak bosons $W^\pm$, $Z$ and Higgs are, respectively, given by

1. $M_W = 0$, $M_Z = 0$, $M_\phi = \sqrt{3|y|v}$,
2. $M_W = \sqrt{\frac{3|y|}{2}}v$, $M_Z = \sqrt{2|y|v}$, $M_\phi = \sqrt{6|y|v}$, and thus $\frac{M_\phi}{M_W} = 2$.

The Higgs-weak boson mass ratio coincides with the result of noncommutative geometry formulation à la Connes [13–15] [19]. It is important to recognize here that the spontaneously broken phase can be realized only when $y < 0$.

Phenomenologically the Weinberg angle $\sin^2 \theta_W = 0.25$ is close to the experimental value $\approx 0.23$, while the Higgs-weak boson mass ratio might be away from the recent informal result of LEP which is very close to $\sqrt{2}$ [19]. It is, however, interesting to note that 0-, 1- and 2-form curvature terms, $\text{Tr}[(\mathcal{F}(0))^2]$, $\text{Tr}[\mathcal{F}(1)\mathcal{F}(1)]$ and $\text{Tr}[\mathcal{F}(2)\mathcal{F}(2)]$ in the action (5.11) are independently gauge invariant as we pointed out in (3.18). As far as the gauge invariance of the even generators of the graded Lie algebra $SU(2|1)$ is concerned, there thus come in two free parameters in the action except for the overall normalization factor. If we introduce these parameters, the Weinberg angle is unchanged but the Higgs-weak boson mass ratio will get the parameter dependence. Thus the above predicted mass ratio, $\frac{M_\phi}{M_W}$, may be changed if we introduce these free parameters.

We now consider the generalized matter action (4.22) with the generalized gauge field (5.2) including only 0- and 1-form gauge fields,

$$S_M = \frac{i}{4} \int \left[ \overline{\Psi} \left( D + A \right) \Psi \right] \bar{1} * 1$$

$$= \frac{i}{4} \int \left[ \left( \overline{\psi} + \overline{\hat{\psi}} \right) \left( d + A - \hat{A} - m \right) \left( \psi + \hat{\psi} \right) \right] * 1$$

$$= \int d^4x \overline{\psi}^{(j)} \left\{ \gamma^\mu (i \partial_\mu - e(A_i^\mu T_i + \frac{1}{2\sqrt{3}}B_\mu Y)) \right.$$  
$$\left. + \frac{e}{\sqrt{2}} \left( (\phi_0 + \frac{v}{e}) \Sigma_+ + \phi_+ \Sigma_- + (\phi_0^* + \frac{v^*}{e}) \Sigma_-' + \phi_+^* \Sigma_- ' \right) \right\} \psi^{(j)}, \quad (5.12)$$

where the following relations should be understood:

$$A^{ab} = ie(A_i^\mu T_i + \frac{1}{2\sqrt{3}}B_\mu Y)^{ab},$$
$$\hat{A}^{ab} = \frac{ie}{\sqrt{2}}(\phi_0 \Sigma_+ + \phi_+ \Sigma_- + \phi_0^* \Sigma_- ' + \phi_+^* \Sigma_-'^{ab}),$$
$$m^{ab} = \frac{i}{\sqrt{2}}(v \Sigma_+ + v^* \Sigma_- ')^{ab},$$

$$24$$
\[ \psi^{(j)a}_\alpha = \frac{1}{4}[(\psi + \hat{\psi})^a \vee Z_j]_0, \quad \bar{\psi}^{(j)a}_\alpha = \frac{1}{4}[(\bar{\psi} + \hat{\bar{\psi}})^a \vee Z_\alpha]_0. \]

Here we have explicitly shown the suffices of gauge algebra to stress the difference of the gauge and flavor suffices. \([\,]_0\) denotes the same notation as \([1.9]\).

In order to introduce the realistic leptons and quarks, we need to identify those states with the eigenstates of the isospin and hypercharge corresponding to the graded Lie algebra of the supergroup \(SU(2|1)\). We first consider the lepton multiplet of the electron sector by assuming that the electron neutrino possesses a small mass according to the recent neutrino experiments [50]. Correspondingly we consider quartet states \(\nu_L, e_L, e_R, \nu_R\) for the electron sector which are classified by the quantum number of the hypercharge \(y\), the magnitude and the third component of isospin \(t\) and \(t_3\). Denoting the eigenstate as \(|y, t, t_3\rangle\), we identify

\[ |\nu_L\rangle = | -1, \frac{1}{2}, \frac{1}{2} \rangle, \quad |e_L\rangle = | -1, \frac{1}{2}, -\frac{1}{2} \rangle, \quad |e_R\rangle = | -2, 0, 0 \rangle, \quad |\nu_R\rangle = |0, 0, 0\rangle, \]

which corresponds to the representation

\[ T_i = \left( \begin{array}{ccc} \frac{\sigma_i}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad Y = \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{array} \right). \]

The above identification of the electron sector satisfies the Nishijima-Gellmann relation: electric charge \(Q = T_3 + \frac{Y}{2}\).

The matrix elements corresponding to the odd counterpart of \(SU(2|1)\) generators which can be read from the relations in the appendix are given by

\[ \Sigma_+ = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \Sigma_- = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \Sigma'_+ = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \Sigma'_- = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \]

where four free parameters \(\alpha, \beta, \gamma, \epsilon\) are introduced. To be consistent with the generator relations:

\[ \{\Sigma_+, \Sigma'_+\} = \sqrt{2}T_\pm, \quad \{\Sigma_+, \Sigma'_\pm\} = \pm T_3 + \frac{1}{2} Y, \]

25
these parameters must satisfy
\[ \alpha \gamma + \beta \epsilon = 1, \quad \alpha \gamma = 0, \quad \beta \epsilon = 1. \] (5.16)

We can then take the following choice:
\[ \epsilon = \frac{1}{\beta}, \quad \alpha = 0, \] (5.17)
for the odd generators in (5.13).

Identifying \( \psi^{(j)\alpha} = (\nu_L, e_L, e_R, \nu_R)^t \) and introducing the above generators in the matter action \( S_M \), we obtain the Higgs-fermion coupling action of the Weinberg-Salam model. The quartet difference of the electron sector can be denoted by the suffix \( a \) while the flavor suffix \( (j) \) is neglected here. We simply consider that there are 4 copies of electron sector and thus we neglect the physical significance of the result of the Dirac-Kähler formulation.

In order to obtain the mass term of the fermion sector we need some care since the odd generators are usually not Hermitian. We can, however, use an automorphism \( c \) which is special to the Lie superalgebra \( SU(2|1) \) and corresponds to the charge conjugation. In other words there is an equivalent representation of the algebra which induces charge conjugate states \([19, 47, 48]\),

\[
\begin{align*}
(T^\pm_3)^c &= T^\mp_3, \quad (Y)^c = -Y, \\
(\Sigma^\pm_3)^c &= \pm \Sigma^\mp_3.
\end{align*}
\] (5.18)

We then identify the charge conjugate states of the electron sector as
\[
\begin{align*}
|e^c_R\rangle &= |1, \frac{1}{2}, \frac{1}{2}\rangle, \quad -|\nu^c_R\rangle = |1, \frac{1}{2}, -\frac{1}{2}\rangle, \\
|e^c_L\rangle &= |2, 0, 0\rangle, \quad |\nu^c_L\rangle = |0, 0, 0\rangle,
\end{align*}
\] (5.19)

which are consistent with the Nishijima-Gellmann relation: \( Q^c = T^c_3 + \frac{Y^c}{2} \) with the following even generators:

\[
T^c_i = \begin{pmatrix}
\sigma_i & 0 & 0 \\
\frac{2}{0} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad Y^c = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\] (5.20)

We can now identify the charge conjugate electron sector \( \psi^{(j)\alpha} = (e^c_R, -\nu^c_R, e^c_L, \nu^c_L)^t \), where the flavor suffix \( (j) \) is again neglected here.

This choice of the even generators \( T^c_i \) does not directly satisfy the relations (5.18) with respect to the original generators \( T_i \) of (5.14). In this representation of charge conjugation
matrix, the first and second suffices of row and column are interchanged with respect to the naive charge conjugation matrix. Correspondingly the odd generators of charge conjugation matrix $\Sigma^c$ can be obtained by a similar change

\[
\Sigma^c_+ = \begin{pmatrix}
0 & 0 & -\frac{1}{\beta} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \Sigma^c_- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\beta} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(5.21)

\[
\Sigma'^c_+ = \begin{pmatrix}
0 & 0 & 0 & -\gamma \\
0 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \Sigma'^c_- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma \\
-\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The mass term of the electron sector can be obtained by adding the original fermionic mass term and the charge conjugate fermionic mass term with Hermitian conjugate,

\[
S_{\text{mass}} = \int \left[ -\frac{i}{4}(\bar{\psi} + \bar{\psi}^c) \vee (m + \hat{A}) \vee (\psi + \hat{\psi}) \\
-\frac{i}{4}(\bar{\psi} + \bar{\psi}^c)^c \vee (m^c + \hat{A}^c) \vee (\psi + \hat{\psi}) \vee + h.c. \right] \ast 1
\]

\[
\int d^4x \left[ \sqrt{6}|y|v(-\frac{1}{\beta}\bar{e}e + \gamma \bar{\nu} \nu + \cdots) \right], \quad (5.22)
\]

where the terms $[\cdots]$ include Higgs-lepton Yukawa coupling terms. We have chosen $v$ to be real and $y < 0$ for the spontaneously broken phase. As we can see from (5.22), the electron mass and neutrino mass are given by $-\frac{1}{\beta}\sqrt{6}|y|v$ and $\gamma \sqrt{6}|y|v$, respectively. These masses are then controlled by the free parameters $\beta$ and $\gamma$.

The quark sector can be treated in a similar way as the lepton sector. In the case of the quark sector we assign the quartet of $u, d$ quark eigenstates as

\[
|u_L\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad |d_L\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -2 \end{pmatrix},
\]

(5.23)

\[
|u_R\rangle = \frac{4}{3} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |d_R\rangle = \frac{2}{3} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

where the Nishijima-Gellmann relation leads

\[
Q = T_3 + \frac{Y}{2} = \begin{pmatrix}
\frac{2}{3} & 0 & 0 & 0 \\
0 & -\frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & -\frac{1}{3}
\end{pmatrix}, \quad (5.24)
\]
for the hypercharge generator

\[ Y = \begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & -\frac{2}{3}
\end{pmatrix}. \] (5.25)

It is important to note here that the lepton hypercharge generator and the quark hypercharge generator both satisfy \( \text{Str} Y = 0 \). This is one of the most elegant points of the \( SU(2|1) \) graded Lie algebra that the natural choice of the hypercharge generator leads to satisfy the Nishijima-Gellmann relation. This point was stressed by Ne’eman [30] and Coquereaux et al. [19].

6 Summary and discussions

We have extended the formulation of generalized gauge theory of Chern-Simons type and topological Yang-Mills type actions to Yang-Mills type actions. In extending the generalized gauge theory formulation to Yang-Mills type actions we needed to introduce the cup product of differential forms, which breaks the topological nature of the generalized gauge theory since the cup product includes Hodge star operation and thus needs a particular choice of the metric.

As an application of the generalized Yang-Mills action we have introduced only 0-form and 1-form gauge fields which correspond to Higgs and gauge fields, respectively. Since the graded Lie algebra is the natural gauge algebra of the generalized gauge theory, we have chosen the graded Lie algebra of \( SU(2|1) \) supergroup. We can then derive the Weinberg-Salam model from the generalized gauge theory formulation where the quaternion plays the role of the two by two matrix representing the discrete two points of noncommutative geometry formulation à la Connes. Matter fermions are formulated by Dirac-Kähler fermion formulation by employing fermionic differential forms. In this way the Weinberg-Salam model has been formulated purely by differential forms.

In the formulation of the generalized Yang-Mills action, only the 0-form gauge parameter appears as an effective gauge parameter. The generalized gauge invariance is lost for the higher form gauge parameters due to the break down of the associativity by the mixture of the product between the wedge product and the cup product. If we, however, formulate the generalized Yang-Mills action introducing only the cup product from the beginning, we can construct an interesting Yang-Mills action which lacks differential
operator but possesses the gauge invariance of all the degrees of differential forms of the
generalized gauge parameters. If we define the generalized curvature by anticommutator
with the cup product instead of the standard wedge product as
\[ \mathcal{F} = \frac{1}{2}\{\mathcal{D} + \mathcal{A}, \mathcal{D} + \mathcal{A}\}_\vee \]
there appear unwanted derivative terms. We then define a new curvature neglecting the
derivative terms in this definition. We can now define the generalized Yang-Mills action
with matter fermions without derivative terms
\[ S = \int \left[ \text{Tr}[\mathcal{A} \vee \mathcal{A} \vee \mathcal{A} \vee \mathcal{A}] + \overline{\Psi} \vee \mathcal{A} \vee \Psi \right] \ast 1, \]
which is invariant under the following generalized gauge transformation since the associativity is recovered now,
\[ \delta \mathcal{A} = [\mathcal{A}, \mathcal{V}]_\vee, \]
\[ \delta \Psi = -\mathcal{V} \vee \Psi, \quad \delta \overline{\Psi} = \overline{\Psi} \vee \mathcal{V}. \]
The gauge parameters include all the degrees of differential forms as in (4.27) and should
satisfy the consistency constraints (4.30). We need to choose one of the quaternion algebra
(4.20) or (4.21) in this formulation. As we can see that this action has similar structure
as the reduced model [51] yet this is not the reduced model in the standard sense since
the gauge fields and parameters have spacetime dependence. In the present formulation
fermions are in the fundamental representation, we can, however, introduce adjoint rep-
resentations as well. We expect that this model may have essential connection with the
formulation of the generalized gauge theory formulation on the simplicial lattice.
In the formulation of matter fermion of the Weinberg-Salam model by the Dirac-Kähler
fermion formulation, the flavor or, it is better to say, family suffix naturally appears but
this family suffix has been neglected in the formulation. In our real nature the number
of family is three instead of four which is the natural consequence of the Dirac-Kähler
fermion formulation. We expect that the family suffix and the suffix of the gauge algebra
would be interrelated in the real unified theory of the formulation which we are hoping to
derive.
In fact there is a toy model of this type that the gauge algebra and the Dirac-Kähler
matter fermion and supersymmetry are naturally interrelated. This is the formulation of
two-dimensional version of the generalized topological Yang-Mills action with the instanton
gauge fixing \[32\]. In this formulation \( N = 2 \) super Yang-Mills action comes out naturally and the matter fermions appears from ghosts of quantization via twisting mechanism just like in four-dimensional topological field theory \[31\]. The twisting mechanism and the Dirac-Kähler fermion formulation are essentially related through supersymmetry.

We hope that the generalized gauge theory presented in this paper may play an essential role in unifying the standard model and quantum gravity on the lattice.

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**Appendix**

The matrix elements of \( SU(2|1) \) generators \[19, 46–48\] are given by

\[
Y|y, t, t_3\rangle = y|y, t, t_3\rangle, \\
T_3|y, t, t_3\rangle = t_3|y, t, t_3\rangle, \\
T_{\pm}|y, t, t_3\rangle = \sqrt{(t \mp t_3)(t \pm t_3 + 1)}|y, t, t_3 \pm 1\rangle, \\
\Sigma_{\pm}|y, t, t_3\rangle = \pm \beta \sqrt{t \mp t_3}|y - 1, t - \frac{1}{2}, t_3 \pm \frac{1}{2}\rangle, \\
\Sigma'_{\pm}|y, t, t_3\rangle = \alpha \sqrt{t + t_3}|y + 1, t - \frac{1}{2}, t_3 \pm \frac{1}{2}\rangle, \\
\Sigma_{\pm}|y - 1, t - \frac{1}{2}, t_3\rangle = 0, \\
\Sigma'_{\pm}|y - 1, t - \frac{1}{2}, t_3\rangle = \pm \epsilon \sqrt{t \mp t_3 + \frac{1}{2}}|y, t, t_3 \pm \frac{1}{2}\rangle \\
+ \zeta \sqrt{t \mp t_3 - \frac{1}{2}}|y - 1, t_3 \pm \frac{1}{2}\rangle, \\
\Sigma_{\pm}|y + 1, t - \frac{1}{2}, t_3\rangle = \gamma \sqrt{t \pm t_3 + \frac{1}{2}}|y, t_3 \pm \frac{1}{2}\rangle \\
\pm \delta \sqrt{t \mp t_3 - \frac{1}{2}}|y - 1, t_3 \pm \frac{1}{2}\rangle, \\
\Sigma'_{\pm}|y + 1, t - \frac{1}{2}, t_3\rangle = 0,
\]
\[
\Sigma_\pm |y, t-1, t_3\rangle = \omega \sqrt{t + t_3} |y - 1, t - \frac{1}{2}, t_3 \pm \frac{1}{2}\rangle,
\]

\[
\Sigma'_\pm |y, t-1, t_3\rangle = \pm \tau \sqrt{t + t_3} |y + 1, t - \frac{1}{2}, t_3 \pm \frac{1}{2}\rangle.
\]

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