Chaotic inflation in modified gravitational theories

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We study chaotic inflation in the context of modified gravitational theories. Our analysis covers models based on (i) a field coupling \( \omega(\phi) \) with the kinetic energy \( X = -(1/2)g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi \) and a nonminimal coupling \( \xi \phi^2 R/2 \) with a Ricci scalar \( R \), (ii) Brans-Dicke (BD) theories, (iii) Gauss-Bonnet (GB) gravity, and (iv) gravity with a Galileon correction. Dilatonic coupling with the kinetic energy and/or negative nonminimal coupling are shown to lead to compatibility with observations of the Cosmic Microwave Background (CMB) temperature anisotropies for the self-coupling inflaton potential \( V(\phi) = \lambda \phi^4/4 \). BD theory with a quadratic inflaton potential, which covers Starobinsky’s \( f(R) = R + R^2/(6M^2) \) with the BD parameter \( \omega_{BD} = 0 \), gives rise to a smaller tensor-to-scalar ratio for decreasing \( \omega_{BD} \). In the presence of a GB term coupled to the field \( \phi \), we express the scalar/tensor spectral indices \( n_s \) and \( n_t \) as well as the tensor-to-scalar ratio \( r \) in terms of two slow-roll parameters and place bounds on the strength of the GB coupling from the joint data analysis of WMAP 7yr combined with other observations. We also study the Galileon-like self-interaction \( \Phi(\phi)X \Box \phi \) with exponential coupling \( \Phi(\phi) \propto e^{\mu \phi} \). Using a CMB likelihood analysis we put bounds on the strength of the Galileon coupling and show that the self-coupling potential can in fact be made compatible with observations in the presence of the exponential coupling with \( \mu > 0 \).

I. INTRODUCTION

Inflation, which was originally proposed by a number of authors independently in the early 1980s, is at present the main theoretical framework employed in describing the early universe evolution and accounting for the observational data, specially the observed spectrum of primordial perturbations. The sustained success of this framework over the last 3 decades has been impressive, particularly given the enormous improvement in the accuracy and resolution of the cosmological data over this period. The simplest and most common models of inflation considered so far have employed a single scalar field, minimally coupled to the curvature and possessing a canonical kinetic term (see [9] for reviews). As a result, until recently, much effort has gone into the study of such models, an important example of which has been the chaotic inflationary model [10].

Despite its successes in accounting for important features of observations, however, there is no unique mechanism which underpins inflation. Indeed almost since its inception it has been known that an accelerated phase of cosmic evolution could be produced by a wide range of mechanisms such as \( f(R) \) theories (see the reviews [11, 12] and references therein). Thus, an important task in cosmology has been to narrow down the range of possible alternatives and ultimately to situate inflationary models within fundamental theories of physical interactions. There have been two approaches to this problem. The first aims to construct individual models that are directly suggested by supersymmetric theories. For example, chaotic inflationary models have been constructed in the framework of supergravity [13] or superstring theory [14]. The second, on the other hand, considers classes of generalized models of inflation which possess ingredients motivated by field theories such as string theory [15, 24].

In the absence of a unique, fully successful and non-fine tuned model of the first kind so far, a great deal of effort has recently gone into the study of the models of the second kind. In general such models are expected to possess a number of ingredients motivated by fundamental theories, including (a) nonminimal couplings of the field to the Ricci scalar \( R \), (b) non-canonical kinetic terms, and (c) higher derivative quantum gravity corrections in their actions, such as the Gauss-Bonnet term

\[
\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta},
\]

where \( R \) is the Ricci scalar, \( R_{\alpha\beta} \) is the Ricci tensor and \( R_{\alpha\beta\gamma\delta} \) is the Riemann tensor, or a nonlinear field-interaction, for example in the form

\[
G(\phi, X) \Box \phi,
\]

where \( G \) is in general a differential function of the field \( \phi \) and \( X = -(1/2)g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi \). In addition, such models would also be expected to possess multiple scalar fields, but in order to separate the effects of different ingredients we shall confine ourselves to models possessing a single scalar field.
In standard chaotic inflation the self coupling $\lambda$ for the inflaton potential $V(\phi) = \lambda \phi^4/4$ is constrained to be small ($\lambda \approx 10^{-13}$) from the WMAP normalization $[5]$. Moreover, this model is in tension with the observations of CMB temperature anisotropies because it predicts a large tensor-to-scalar ratio ($r \approx 0.3$) $[6]$. If we take into account the nonminimal coupling $\zeta \phi^2 R/2$, it is possible to realise larger $\lambda$ compatible with the natural values appearing in particle physics ($\lambda = 0.01-0.1$). In the limit where the negative nonminimal coupling $\zeta$ satisfies the condition $|\zeta| \gg 1$, the tensor-to-scalar ratio $r$ reduces to the order of $10^{-3}$ with the scalar spectral index $n_s \approx 0.96$ $[25]$. This agrees well with the CMB observations $[24, 28]$. Recently, there has been renewed interest in nonminimally coupled inflation models by identifying the inflaton as a Higgs boson appearing in standard model of particle physics $[17, 29]$.

It is of interest to see whether the chaotic inflationary models that are in tension with observations can be rescued by taking into account the various field couplings mentioned above. In the presence of the nonlinear kinetic interaction $\phi R$, taking into account the various field couplings mentioned above. In the presence of the nonlinear kinetic interaction $\phi R$, it is possible to realise larger $\lambda$ compatible with the natural values appearing in particle physics ($\lambda = 0.01-0.1$). Of course this outcome would be expected to be different if we choose more general functions of $G(\phi, X)$ that depend on both $\phi$ and $X$.

In this paper we wish to make a detailed and unified study of inflation in the context of modified gravitational theories. To this end, and for concreteness and comparison with previous works, we shall study chaotic inflation, sourced by potentials of the form

$$V(\phi) = V_0(\phi/M_{pl})^p, \quad (p > 0),$$

where $V_0$ and $p$ are real constants and $M_{pl} = 1/\sqrt{8 \pi G_N} = 2.44 \times 10^{18}$ GeV is the reduced Planck mass ($G_N$ is the gravitational constant). We shall consider a number of field couplings such as (i) the non-canonical kinetic term $\omega(\phi) X$ as well as the nonminimal coupling $\zeta \phi^2 R/2$, (ii) Brans-Dicke (BD) theories having explicit couplings $\phi R$ and $(\omega_{BD}/\phi) X$ (including $f(R)$ gravity), (iii) Gauss-Bonnet (GB) coupling of the form $\xi_0 \phi^{(n-4)/2}$, and (iv) the generalized Galileon coupling of the form $(\phi^{(n-4)/2}) X^n \Box \phi$ (which reproduces the pure Galileon term $[22, 31]$ in the limit that $n \to 1$). The terms of the forms (i), (iii), and (iv) appear as a next order correction to the tree-level action in low energy effective string theory $[32]$.

In each model we evaluate the three inflationary observables: (a) the scalar spectral index $n_s$, (b) the tensor spectral index $n_t$, and (c) the tensor-to-scalar ratio $r$. We place observational constraints on the model parameters by carrying out a CMB likelihood analysis. We find that in most cases it is possible for the chaotic inflationary potentials with $p = 2$ and $p = 4$ to be consistent with the current observations. We also show that the equilateral nonlinear parameter $f_{NL}^{equi}$ describing the scalar non-Gaussianity is smaller than the order of unity, apart from in the generalized Galileon model with $n > 1$.

The structure of the paper is as follows. In Sec. II we derive the background equations of motion for the general action $[4]$ and introduce a number of slow-roll parameters. In Sec. III we present the power spectra of scalar and tensor perturbations derived under the framework of linear cosmological perturbation theory. The formula of the equilateral non-Gaussianity parameter $f_{NL}^{equi}$ is also given there. In Sec. IV we show the three inflationary observables $n_s$, $n_t$, and $r$ in the Einstein frame for the theories without the GB or Galileon terms, which are convenient for the analysis in Secs. V and VI. Sec. V is devoted to the study of the nonminimal coupling $\zeta \phi^2 R/2$ as well as the coupling $e^{\mu \phi/\sqrt{M_{pl}^2}} X$. In Sec. VI we consider BD theories with the two potentials $V = V_0(\phi/M_{pl})^p$ and $V(\phi) = V_0(\phi - M_{pl})^p$ in the Jordan frame for arbitrary BD parameters $\omega_{BD}$. For $p = 2$ and $\omega_{BD} = 0$ the latter potential covers the Starobinsky’s $f(R)$ model $f(R) = R + R^2/(6M^2)$. In Sec. VII we study chaotic inflation in the presence of the exponential GB coupling and place observational constraints on the strength of the GB coupling. In Sec. VIII we show that the exponential Galileon coupling can lead to the consistency of self-coupling chaotic inflation with the observational data. Sec. IX is devoted to our conclusions.

II. MODELS AND BACKGROUND EQUATIONS

We start with the generalized action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{pl}^2}{2} F(\phi) R + \omega(\phi) X - V(\phi) - \xi(\phi) G - G(\phi, X) \Box \phi \right],$$

where $g$ is a determinant of the space-time metric $g_{\mu\nu}$, and $\phi$ is a scalar field with a kinetic term $X$. The functions $F(\phi), \omega(\phi)$, and $\xi(\phi)$ are differentiable functions of $\phi$, whereas $G(\phi, X)$ depend on both $\phi$ and $X$. The field $\phi$ couples to both the Ricci scalar $R$ as well as the Gauss-Bonnet term $G$. 
We consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with a scale factor $a(t)$, where $t$ is cosmic time. The background equations are then given by

$$
E_1 \equiv 3M_{pl}^2 H^2 + 3M_{pl}^2 H \dot{F} - \omega X - V - 2H \dot{\phi} X G, X + 2XG, \phi = 0, \tag{5}
$$

$$
E_2 \equiv 3M_{pl}^2 H^2 + 2M_{pl}^2 H H + 2M_{pl}^2 \dot{F} + \omega X - V - 16H \dot{\phi} \dot{X} - 8H^2 \dot{\phi} X - G, X \dot{\phi} \dot{X} - G, \phi \phi^2 = 0, \tag{6}
$$

$$
E_3 \equiv (\omega + 6H \dot{\phi} G, X + 6H \dot{\phi} X G, X - 2XG, \phi X - 2G, \phi) \phi 
+ (3\omega H + \phi \omega, \phi + 9H^2 \phi G, X + 3H \dot{\phi} G, X + 3H \dot{\phi}^2 G, X - 6HG, \phi - G, \phi \phi) \phi 
- \omega X + V, \phi - 6M_{pl}^2 H^2 F, \phi - 3M_{pl}^2 H F, \phi + 24H^4 \dot{\xi} \phi + 24H^2 \dot{H} \dot{\xi} \phi = 0, \tag{7}
$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, a dot represents a derivative with respect to $t$, and a comma represents a partial derivative in terms of $\phi$ or $X$. Only two of the above equations are independent due to the Bianchi identities, $\phi E_1 + \dot{E}_1 + 3H (E_1 - E_2) = 0$. The combined equation, $(E_2 - E_1)/(M_{pl}^2 H^2 F) = 0$, gives

$$\epsilon \equiv - \frac{\dot{H}}{H^2} \equiv \frac{\dot{F}}{2HF} + \frac{\dot{F}}{2H^2F} + \frac{\omega X}{M_{pl}^2 H^2 F} + \frac{4H \dot{\xi}}{M_{pl}^2 H^2 F} - \frac{8H \dot{\xi}}{M_{pl}^2 H^2 F} - \frac{4\dot{\phi} G, X}{M_{pl}^2 F} - 3\phi X G, X - \frac{\dot{\phi} X G, X}{M_{pl}^2 H F} - \frac{2XG, \phi}{M_{pl}^2 H^2 F} \equiv \frac{1}{2} \frac{\dot{F}}{H^2 F} \equiv \frac{\dot{F}}{2HF} = \delta_F(\delta_F + \eta_F - \epsilon), \frac{\dot{\xi}}{M_{pl}^2 F} = \delta_\xi(\delta_F + \eta_\xi + \epsilon). \tag{8}
$$

Since $\epsilon \ll 1$ during inflation, the modulus of each term on the r.h.s. of Eq. 8 is much smaller than unity (unless some cancellation occurs between those terms). We introduce the following slow-roll parameters

$$
\delta_F \equiv \frac{\dot{F}}{H^2 F}, \delta_X \equiv \frac{\omega X}{M_{pl}^2 H^2 F}, \delta_\xi \equiv \frac{H \dot{\xi}}{M_{pl}^2 F}, \delta_{GX} \equiv \frac{\dot{\phi} G, X}{M_{pl}^2 H F}, \delta_\phi \equiv \frac{\phi}{H \phi}, \delta_{G\phi} \equiv \frac{XG, \phi}{M_{pl}^2 H^2 F}, \eta_F \equiv \frac{\eta_F}{H \delta_F}, \eta_\xi \equiv \frac{\eta_\xi}{H \delta_\xi}. \tag{9}
$$

by which we have

$$\frac{\dot{F}}{2HF} = \delta_F(\delta_F + \eta_F - \epsilon), \frac{\dot{\xi}}{M_{pl}^2 F} = \delta_\xi(\delta_F + \eta_\xi + \epsilon). \tag{10}
$$

From Eq. 8 we obtain

$$
\epsilon = \frac{2\delta_X - \delta_F + 8\delta_\xi + 6\delta_{GX} - 4\delta_{G\phi} + \delta_F(\delta_F \eta_F) - 8\delta_\xi(\delta_F + \eta_\xi) - 2\delta_\phi \delta_{G\phi}}{2 + \delta_F - 8\delta_\xi} \tag{11}
$$

$$
= \delta_X - \delta_F/2 + 4\delta_\xi + 3\delta_{GX} - 2\delta_{G\phi} + O(\epsilon^2), \tag{12}
$$

where in the latter step we have taken the leading-order contribution.

### III. COSMOLOGICAL PERTURBATIONS

Let us consider cosmological perturbations about the flat FLRW background. We take into account, up to a gauge choice, both the perturbations in the scalar field $\phi$ and in the scalar and tensor modes of the metric. For the calculations of observables, including primordial non-Gaussianities, it is convenient to employ the 4-dimensional ADM perturbed metric [33] of the form

$$
ds^2 = - [(1 + \alpha)^2 - a^{-2}(t) e^{-2R}(\partial \psi)^2] dt^2 + 2\partial_\psi dt dx^i + a^2(t) e^{2R} \delta_{ij}(x^i + h_{ij}) dx^i dx^j, \tag{13}
$$

where $R$ is the curvature perturbation, $\alpha$ and $\psi$ are related with the lapse $(1 + \alpha)$ and the shift vector $\partial_i \psi$, and $h_{ij}$ are tensor perturbations. In the metric [13] we have gauged away a field $E$ appearing as $E_{ij}$ inside $h_{ij}$, to fix the spatial components of a gauge-transformation vector $\xi^\mu$. We choose the uniform-field gauge where $\delta \phi = 0$, in order to fix the time-component of $\xi^\mu$ [34].

Expanding the action [4] up to second order for the metric [13], performing integration by parts and using the Hamiltonian and momentum constraints to eliminate the contribution coming from $\alpha$ and $\psi$, we obtain the following second-order action [35]

$$S_2 = \int dt d^3x a^3Q \left[ \dot{R}^2 - \frac{\dot{\psi}^2}{a^2} (\partial R)^2 \right], \tag{14}
$$
In order to avoid the appearance of ghosts and Laplacian instabilities we require that
\[ Q > 0, \quad c_2^2 > 0, \] (21)
respectively. One can express \( w_i \) \((i = 1, \cdots, 4)\) in terms of the slow-roll parameters. For example one has
\[ w_3 = -9M_p^2FH^2 \left( 1 + \frac{\delta_F}{3} - \frac{1}{3} \delta_X - 16\delta_\xi - 4\delta_{GX} + \frac{2}{3} \delta_{G\phi} - 2\delta_{GX} \lambda_{GX} + \frac{2}{3} \delta_{G\phi} \lambda_{G\phi} \right), \] (22)
where
\[ \lambda_{GX} \equiv \frac{X_{GX}}{G}, \quad \lambda_{G\phi} \equiv \frac{X_{G\phi}}{G}. \] (23)
The quantities \( \lambda_{GX} \) and \( \lambda_{G\phi} \) are not necessarily small.

The expansion in terms of the slow-roll parameters gives
\[
c_2^2 \simeq \frac{\delta_X + 4\delta_{GX} - 2\delta_{G\phi} + 2\delta_{G\phi} \lambda_{G\phi}}{\delta_X + 6\delta_{GX} - 2\delta_{G\phi} + 6\delta_{GX} \lambda_{GX} - 2\delta_{G\phi} \lambda_{G\phi}},
\] (24)
\[
\epsilon_s \equiv \frac{Qc_2^2}{M_p^2F} = \delta_X + 4\delta_{GX} - 2\delta_{G\phi} + 2\delta_{G\phi} \lambda_{G\phi} - 2\delta_{G\phi} \delta_F \lambda_{G\phi} + 16\delta_{G\phi} \delta_\xi \lambda_{G\phi} + 2\delta_{G\phi} \delta_{GX} \lambda_{GX} + 4\delta_{G\phi} \delta_{GX} \lambda_{G\phi} + 3\delta_F^2/4 - 12\delta_\xi \delta_F + 2\delta_{GX} \lambda_{G\phi} - \delta_F \delta_X - 5\delta_F \delta_{GX} + 2\delta_F \delta_{G\phi} + 16\delta_X - 16\delta_\xi \delta_{G\phi} - 4\delta_{GX} \delta_{G\phi} + 2\delta_{GX} \delta_X + 12\delta_\xi + 7\delta_{GX} + O(\epsilon^3),
\] (25)
where in the expression for \( c_2^2 \) we have picked up the leading-order contributions. In standard slow-roll inflation with \( F = 1, \omega = 1, \xi = 0, \) and \( G = 0 \) we obtain the exact expressions \( c_2^2 = 1 \) and \( Q/M_p^2 = \delta_X = \epsilon. \) Equation (24) shows that the nonminimal coupling \( F(\phi)R \) and the Gauss-Bonnet term \( \xi(\phi)G \) do not give rise to contributions to \( c_2^2 \) at linear order. The effects of those terms on \( c_2^2 \) appear at the next order.

The power spectrum of the curvature perturbation is given by
\[
\mathcal{P}_s = \frac{H^2}{8\pi^2 Qc_2^2} = \frac{H^2}{8\pi^2 M_p^2 F\epsilon_s c_s},
\] (26)
which gives the scalar spectral index
\[
n_s - 1 = \frac{d \ln \mathcal{P}_s}{d \ln k} \bigg|_{c_s, k = aH} = -2\epsilon - \delta_Q - 3s \] (27)
\[
= -2\epsilon - \delta_F - \eta_s - s,
\] (28)
where
\[
\delta_Q \equiv \frac{\dot{Q}}{HQ}, \quad s \equiv \frac{\dot{c}_s}{Hc_s}, \quad \eta_s \equiv \frac{\dot{\epsilon}_s}{H\epsilon_s}.
\] (29)
We have assumed that both $H$ and $c_s$ vary slowly, such that $d \ln k$ at $c_s k = aH$ may be approximated by $d \ln k = d \ln a = H dt$.

The tensor power spectrum is given by

$$ P_t = \frac{H^2}{2^n^2 Q_t c_t^4},$$

(30)

where $Q_t = w_1/4 = M_{pl}^2 F(1 - 8\delta_\xi)/4$ and $c_t^2 = w_1/w_1 = 1 + 8\delta_\xi + O(\epsilon^2)$. Taking the leading-order contribution in $P_t$, it follows that $P_t \simeq 2H^2/(\pi^2 M_{pl}^2 F)$. The tensor spectral index is

$$ n_t = \frac{d \ln P_t}{d \ln k} \bigg|_{c_s k = aH} = -2\epsilon - 2\delta_F,$$

(31)

which is valid at first order in slow-roll. At times before the end of inflation ($\epsilon \ll 1$) when both $P_s$ and $P_t$ remain approximately constants, we can estimate the tensor-to-scalar ratio, as

$$ r = \frac{P_t}{P_s} \simeq \frac{16Qc_t^2}{M_{pl}^2 F} = 16c_s \epsilon_s.$$

(32)

The non-Gaussianities of scalar perturbations for the action (4) have been evaluated in Ref. 35 (see also Refs. 34, 36, 37 for related works). Under the slow-roll approximation the nonlinear parameter $f_{NL}^{\text{equil}}$ in the equilateral configuration is

$$ f_{NL}^{\text{equil}} \simeq \frac{85}{324} \left(1 - \frac{1}{c_s^2}\right) \left(1 - \frac{\Lambda}{81 \Sigma} + \frac{55}{36} \epsilon_s + \frac{5}{12} \eta_s - \frac{85}{54} \delta^2 F \right) + \frac{5}{162} \delta_F \left(1 - \frac{1}{c_s^2}\right) \left(1 - \frac{\Lambda}{81 \Sigma} + \frac{29}{12} \epsilon_s \right) + \delta_G X \left(\frac{20(1 + \lambda_{GX})}{81 \epsilon_s} + \frac{65}{162 c_s^2 \epsilon_s}\right),$$

(33)

where

$$ \Lambda \equiv F^2 \left[\phi H(XG_{,X} + 5X^2G_{,XX} + 2X^3G_{,XXX}) - 2(2X^2G_{,\phi,X} + X^3G_{,\phi,XX,X})/3\right],$$

(34)

$$ \Sigma \equiv \frac{w_1(4w_1w_3 + 9w_2^2)}{12 M_{pl}^4} \simeq M_{pl}^2 F^3 H^2 (\delta_X + 6\delta_{G \phi} - 2\delta_{G \phi,X} - 6\delta_{G \phi} + 6\delta_{G X} + 2\delta_{G \phi} \lambda_{G \phi}) \lambda_{G \phi}.$$

(35)

In the absence of the Galileon term ($\delta_{G X} = 0 = \delta_{G \phi}$) one has $c_s^2 \simeq 1$ and $\epsilon_s \simeq \delta_X$ from Eqs. (24) and (25) at linear order in slow-roll. In this case, the expansion of $c_s^2$ up to second order gives

$$ c_s^2 \simeq 1 - \frac{2\delta_\xi(\delta_F - 8\delta_{\xi}) (3\delta_F - 24\delta_\xi - 4\delta_X)}{\delta_X},$$

(36)

which shows that the GB contribution can only lead to small changes to the value $c_s^2 = 1$. Then the nonlinear parameter in Eq. (35) is approximately given by

$$ f_{NL}^{\text{equil}} \simeq \frac{55}{36} \epsilon_s + \frac{5}{12} \eta_s + \frac{10}{3} \delta_\xi,$$

(37)

which means that the non-Gaussianity is small for the theories with $G = 0$. However, the presence of the Galileon term can potentially give rise to large non-Gaussianities.

IV. THE ACTION IN THE EINSTEIN FRAME

We start by considering non-minimally coupled theories in the absence of the GB and Galileon terms ($\xi = 0 = G$), i.e. with actions of the form

$$ S = \int d^4 x \sqrt{-g} \left[\frac{M_{pl}^2}{2} F(\phi) R + \omega(\phi) X - V(\phi) \right].$$

(38)
Since $c_s^2 = 1$ and $s = 0$ in these theories, it follows that

\[ n_s - 1 = -2\epsilon - \delta_Q = -2\epsilon - \delta_F - \eta_s \approx -2\epsilon_s - \eta_s, \]  
\[ n_t = -2\epsilon - \delta_F \approx -2\epsilon_s, \]  
\[ r = 16 \frac{Q}{M_{pl}^2} = 16\epsilon_s \approx -8n_t, \]  
where

\[ Q = \frac{F(2F\omega^2 + 3M_{pl}^2\dot{F}^2)}{(2HF + F)^2}. \]  

In the last approximate equalities of Eqs. (39), (40), and (41) we have used the relation $\epsilon_s \approx \epsilon + \delta_F/2$ valid at linear order in slow roll. This follows from Eqs. (12) and (25), i.e. $\epsilon \approx \delta_X - \delta_F/2$ and $\epsilon_s \approx \delta_X$, respectively.

It is convenient to transform the action (38), expressed in the so called Jordan frame, into the one having a scalar field minimally coupled to gravity (the Einstein frame), via the conformal transformation

\[ \hat{g}_{\mu\nu} = F(\phi)g_{\mu\nu}. \]  

The transformed action is given by

\[ S_E = \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{2}M_{pl}^2\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - U(\chi) \right], \]  

where a hat represents the quantities in the Einstein frame, and

\[ U = \frac{V}{F^2}, \quad \chi = \int B(\phi)\,d\phi, \quad B(\phi) \equiv \sqrt{\frac{3}{2}\left(\frac{M_{pl}F^2}{F}\right)^2 + \omega F}. \]  

The following relations hold between the variables in the two frames:

\[ d\hat{t} = \sqrt{F}\,dt, \quad \hat{a} = \sqrt{F}a, \quad \hat{H} = \frac{1}{\sqrt{F}}\left(H + \frac{\dot{F}}{2F}\right). \]  

Defining the variables

\[ \hat{\epsilon} \equiv -\frac{1}{\hat{H}^2}\frac{d\hat{H}}{d\hat{t}}, \quad \hat{Q} \equiv \frac{1}{2\hat{H}^2}\left(\frac{d\chi}{d\hat{t}}\right)^2, \quad \hat{\delta}_Q \equiv \frac{1}{\hat{HQ}}\frac{d\hat{Q}}{d\hat{t}}, \]  

we obtain

\[ \hat{\epsilon} = \frac{\epsilon + \delta_F/2}{1 + \delta_F/2} - \frac{\delta_F}{2\hat{H}(1 + \delta_F/2)^2}, \quad \hat{Q} = \frac{Q}{F}, \quad \hat{\delta}_Q = \frac{\delta_Q - \delta_F}{1 + \delta_F/2}. \]  

Since $\hat{\epsilon} \approx \epsilon + \delta_F/2$ and $\hat{\delta}_Q \approx \delta_Q - \delta_F$ at linear order in slow roll, we find that Eqs. (39), (40), and (41) reduce to

\[ n_s - 1 \approx -2\hat{\epsilon} - \hat{\delta}_Q, \]  
\[ n_t \approx -2\hat{\epsilon}, \]  
\[ r \approx 16 \frac{\hat{Q}}{M_{pl}^2} = 16\hat{\epsilon}. \]  

In the last equality of Eq. (51) we have used the relation $\dot{\epsilon} = \hat{Q}/M_{pl}^2$, which follows from the background equation $d\hat{H}/d\hat{t} = -(d\chi/d\hat{t})^2/(2M_{pl}^2)$. The results (49)-(51) coincide with those derived in the Einstein frame [26, 27]. This equivalence is a consequence of the fact that both the scalar and tensor spectra are unchanged under the conformal transformation ($\mathcal{P}_s = \mathcal{P}_s$ and $\mathcal{P}_t = \mathcal{P}_t$) [26].
Under the slow-roll conditions \( |d^2 \chi / dt^2| \ll |3 \ddot{H} d \chi / d t| \) and \( (d \chi / dt)^2 / 2 \ll U \) the background equations are approximately given by

\[
3M_{\text{pl}}^2 \ddot{H}^2 \simeq U, \quad 3 \ddot{H} \frac{d \chi}{dt} \simeq -U \chi. \tag{52}
\]

We then have

\[
\dot{\iota} = \frac{\dot{Q}}{M_{\text{pl}}^2} \simeq \frac{M_{\text{pl}}^2}{2} \left( \frac{U_{,XX}}{U} \right)^2, \quad \dot{\delta} \dot{Q} \simeq 2M_{\text{pl}}^2 \left[ \frac{U_{,XX}}{U} - \frac{U_{,XXX}}{U} \right]. \tag{53}
\]

The observables (49)–(51) can be explicitly written as

\[
n_s - 1 \simeq -3M_{\text{pl}}^2 \left( \frac{U_{,XX}}{U} \right)^2 + 2M_{\text{pl}}^2 \frac{U_{,XXX}}{U} \simeq \frac{M_{\text{pl}}^2}{B^2} \left[ \frac{2V_{,\phi\phi}}{V} - \frac{3V_{,\phi}^2}{V^2} - 4\frac{F_{,\phi\phi}}{F} + 4\frac{V_{,\phi} F_{,\phi}}{V F} - 2\frac{B_{,\phi}}{B} \left( \frac{V_{,\phi}}{V} - \frac{2F_{,\phi}}{F} \right) \right], \tag{54}
\]

\[
r \simeq -8n_t \simeq 8M_{\text{pl}}^2 \left( \frac{U_{,XX}}{U} \right)^2 \simeq 8 \frac{M_{\text{pl}}^2}{B^2} \left( \frac{V_{,\phi}^2}{V} - 2\frac{F_{,\phi}}{F} \right)^2. \tag{55}
\]

In the Jordan frame the number of e-foldings from the time \( t \) (with the field value \( \phi \)) to the time \( t_f \) at the end of inflation (with the field value \( \phi_f \)) is given by

\[
N = \int_{t}^{t_f} \frac{H dt}{\dot{H}} = \int_{t}^{t_f} \frac{F}{\dot{F}} d t + \frac{1}{2} \ln \frac{F}{F_f}, \tag{56}
\]

where \( F_f \equiv F(\phi_f) \). Note that in the last equality we have used Eq. (16). The scales relevant to the CMB temperature anisotropies correspond to \( N = 50-60 \) [39]. The number of e-foldings in the Einstein frame should be equivalent to that in the Jordan frame by properly choosing some reference length scale [40]. Using the slow-roll approximation in the Einstein frame, the frame-independent quantity \( \dot{\delta} \dot{Q} \) can be written as

\[
N \simeq \int_{x_f}^{x} \frac{U}{M_{\text{pl}}^2 U_{,XX}} d x + \frac{1}{2} \ln \frac{F}{F_f}, \tag{57}
\]

which we will use in the following sections.

V. INFLATION WITH NONMINIMAL COUPLING AND FIELD COUPLING WITH THE KINETIC TERM

In this section we study, in turn, models with the nonminimal coupling \( \zeta \phi^2 R / 2 \) and the non-canonical kinetic term \( \omega(\phi)X \). These models are described by the action

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} R - \frac{1}{2} \zeta \phi^2 R + \omega(\phi)X - V(\phi) \right]. \tag{58}
\]

In this case the function \( F \) is given by

\[
F = 1 - \zeta x^2, \quad x \equiv \phi / M_{\text{pl}}. \tag{59}
\]

Note that in our notation the conformal coupling corresponds to \( \zeta = 1/6 \). For the canonical field with \( \omega(\phi) = 1 \), the observational constraints were studied for the chaotic potential of the type [3] by using the WMAP 1yr data combined with the large-scale structure data [27]. Recently the observational compatibility of this type of potential as well as \( V(\phi) = \lambda(\phi^2 - v^2)^2 / 4 \) was examined in Ref. [28] by using the WMAP 7yr data. The latter potential appears in the context of Higgs inflation with the electroweak scale \( v \sim 10^3 \text{ GeV} \) [17]. If the nonminimal coupling is negative with \( |\zeta| \gg 1 \), it is possible to use the Higgs field as an inflaton because the self coupling \( \lambda \) can be of the order of 0.01-0.1 from the WMAP normalization. Since the field \( \phi \) is much larger than the electroweak scale during inflation, the observational prediction of the potential \( V(\phi) = \lambda(\phi^2 - v^2)^2 / 4 \) is very similar to that of the potential [9] with \( p = 4 \).

In this work we shall take into account the non-canonical kinetic term \( \omega(\phi)X \) in addition to the nonminimal coupling \( \zeta \phi^2 R / 2 \). We provide general formulae for \( n_s \), \( r \), and \( n_t \) in terms of the function of \( x = \phi / M_{\text{pl}} \) and then apply them
to the cases where $\omega(\phi) = \text{constant}$ and where the exponential coupling $\omega(\phi) = e^{\mu\phi/M_{pl}}$ is present. In the Einstein frame this potential takes the form

$$U = \frac{V_0 x^p}{(1 - \zeta x^2)^2}. \quad (60)$$

For $p < 4$ this has a local maximum at $x = \sqrt{p/(4 - p)|\zeta|}$ and hence the nonminimal coupling makes it more difficult to realise inflation. If $p = 4$ the potential (60) is asymptotically flat in the region $\phi \gg M_{pl}$. If $p > 4$ the potential does not possess a local maximum, but for $p > 5 + \sqrt{13}$ inflation does not occur.

From Eqs. (54) and (55) it follows that

$$n_s - 1 \simeq -\frac{1}{[\omega + (6\zeta - \omega)\zeta x^2]^2} \left\{ (p - 4)^2(6\zeta - \omega)(\zeta x^2)^3 + (24\omega - 14p\omega + 3p^2\omega + 24p\zeta - 12p^2\zeta)(\zeta x^2)^2 
+ (-8\omega + 4p\omega - 3p^2\omega + 24p\zeta + 6p^2\zeta)\zeta x^2 + p\omega(p + 2) - \mu\omega x(1 - \zeta x^2)^2[(p - 4)\zeta x^2 - p] \right\}, \quad (61)$$

$$r \simeq -8n_t \simeq \frac{8[p + (4 - p)\zeta x^2]^2}{x^2[\omega + (6\zeta - \omega)\zeta x^2]}, \quad (62)$$

where

$$\mu \equiv \frac{M_{pl}\omega \phi}{\omega}. \quad (63)$$

For the dilatonic coupling $\omega(\phi) = e^{\mu\phi/M_{pl}}$ the parameter $\mu$ is constant. Using the approximate equations (52), the scalar power spectrum is given by

$$P_s \simeq \frac{U^3}{12\pi^2 M_{pl}^6 U^2_{\chi}} = \frac{V_0}{12\pi^2 M_{pl}^6} \frac{x^{p+2}[6\zeta^2 x^2 + \omega(1 - \zeta x^2)]}{(1 - \zeta x^2)^2[p + (4 - p)\zeta x^2]^2}. \quad (64)$$

The WMAP normalization corresponds to $P_s \simeq 2.4 \times 10^{-9}$ at the scale $k = 0.002\text{Mpc}^{-1}$. In the following we shall first consider the nonminimally coupled theories with $\mu = 0$ and then proceed to the case in which the dilatonic kinetic term $e^{\mu\phi/M_{pl}}X$ is present.

### A. Effect of the nonminimal coupling $\zeta \phi^2 R/2$ with constant $\omega$

We first discuss the effect of the nonminimal coupling for the theories with

$$\mu = 0, \quad (65)$$

in which case $\omega$ is constant. Introducing a new field $\varphi = \sqrt{\omega}\phi$ the kinetic term $\omega X$ reduces to the canonical form $-g^{\mu\nu}\partial_\mu \varphi \partial_\nu \varphi/2$. Then the nonminimal coupling $\zeta \phi^2 R/2$ can be written as $\zeta \varphi^2 R/2$, where $\zeta = \zeta/\omega$. The potential $V(\phi) = V_0(\phi/M_{pl})^p$ for the scalar field $\varphi$, takes the power-law form $V = V_0(\varphi/M_{pl})^p$, where $V_0 = V_0/\omega^{p/2}$. This means that these theories reduce to nonminimally coupled theories with $\omega = 1$ in terms of the field $\phi$. The ratio $\zeta = \zeta/\omega$ characterizes the effect of the nonminimal coupling on the inflationary observables $n_s$, $n_t$, and $r$, and $V_0 = V_0/\omega^{p/2}$ sets the scale for the scalar power spectrum.

From Eq. (57) the number of e-foldings is given by

$$N \simeq -\frac{1}{4\zeta} \ln \left| \frac{(p - 4)\zeta x_f^2 - p}{(p - 4)\zeta x^2 - p} \right|^{\frac{3n_s - 2n_t}{p - 4}} - \frac{1}{4} \ln \left| \frac{1 - \zeta x_f^2}{1 - \zeta x^2} \right| \quad (p \neq 4), \quad (66)$$

$$N \simeq \frac{\omega - 6\zeta}{8} (x^2 - x_f^2) - \frac{1}{4} \ln \left| \frac{1 - \zeta x^2}{1 - \zeta x_f^2} \right| \quad (p = 4), \quad (67)$$

where $x_f \equiv \phi_f/M_{pl}$. The result (67) can be also reproduced by taking the limit $p \to 4$ in Eq. (66). We identify the end of inflation by the condition $\dot{\epsilon} = 1$, which gives

$$x_f^2 = \frac{\omega - \zeta p(4 - p) - \sqrt{(\omega - 2p\zeta)(\omega - 6p\zeta)}}{\zeta(4 - p)^2 + 2(\omega - 6\zeta)}.$$  

(68)
Let us consider the limits where $|\zeta/\omega| \ll 1$. We implicitly assume that $\omega$ is not different from the order of 1. We expand the right hand side of Eqs. (66) and (67) up to first order in $\zeta$ and then solve them for $x$ by using Eq. (68). This gives

$$x^2 \approx \frac{p(p+4N)}{2\omega} \left[ 1 - \frac{8(p-4)N^2 + 4p(p-6)N + p^2(p-8)}{2(p+4N)} \frac{\zeta}{\omega} \right],$$  

(69)

which is valid for both $p \neq 4$ and $p = 4$. The spectral index (61) and the tensor-to-scalar ratio (62) are approximately given by

$$n_s - 1 \approx -\frac{2(p+2)}{p+4N} \left[ 1 - \frac{4(p-2)(p-12)N^2 + 2p(p^2 - 12p + 28)N + p^2(12-p)}{(p+4N)(p+2)} \frac{\zeta}{\omega} \right],$$  

(70)

$$r \approx \frac{16p}{p+4N} \left[ 1 - \frac{2N(2(p-12)N + p(p-10))}{p+4N} \frac{\zeta}{\omega} \right],$$  

(71)

This shows that the effect of the nonminimal coupling appears in terms of the ratio $\zeta/\omega$.

Substituting Eq. (69) into Eq. (61) and expanding it up to first order in $\zeta$, it follows that

$$P_s \approx \frac{\hat{V}_0}{M_{\text{pl}}^4} \frac{p(p+4N)}{24\pi^2p} \left[ 1 - \frac{p^4 + (4N-12)p^3 + (8N^2 - 64N)p^2 + (80N - 112N^2)p + 192N^2}{4(p+4N)} \frac{\zeta}{\omega} \right]^{p/2},$$

(72)

around $N = 55$. In the absence of the nonminimal coupling ($\zeta = 0$) the WMAP normalization for the canonical scalar field $\varphi$ gives $m \simeq 6.8 \times 10^{-6} M_{\text{pl}}$ for $p = 2$ (where $\hat{V}_0 = m^2 M_{\text{pl}}^2/2$) and $\lambda \simeq 2.0 \times 10^{-13}$ for $p = 4$ (where $\hat{V}_0 = \lambda M_{\text{pl}}^4/4$). If $\zeta \neq 0$, then the inside of the last parenthesis in Eq. (72) is approximately given by $1 + 4\zeta/\omega$ for $p = 2$ and $1 + 460\zeta/\omega$ for $p = 4$. As long as $|\zeta/\omega| \ll 1$, the order of $\hat{V}_0$ is not subject to change by the presence of the nonminimal coupling.

In the following we derive the numerical values of $n_s$ and $r$ for $p = 2$ and $p = 4$ separately to compare the models with observations.

1. $p = 2$

In order to obtain the theoretical values of $n_s$ and $r$ for $p = 2$, we numerically solve the background equations of motion in the Jordan frame by identifying the end of inflation under the condition (68). We derive the numerical values of $x$ corresponding to the number of e-foldings $N = 55$ and then evaluate $n_s$ and $r$ by using the formulas (61) and (62).

In Fig. 1 we show the 1σ and 2σ observational contours constrained by the joint data analysis of WMAP 7yr [8], Baryon Acoustic Oscillations (BAO) [41], and the Hubble constant measurement (HST) [42]. This is derived by varying the two parameters $n_s$ and $r$ with the consistency relation $r = -8n_s$ [see Eq. (11)]. Since the runnings of scalar and tensor spectral indices are suppressed to be of the order of $\epsilon^2$, they are set to be 0 in the likelihood analysis. These results are valid for the theories with $\xi = 0 = G$.

In the limit $|\zeta| \ll 1$, Eqs. (61) and (62) give

$$n_s - 1 \approx -\frac{4}{2N+1} \left[ 1 - \frac{4N + 5 \zeta}{2N+1} \frac{\zeta}{\omega} + \frac{2(104N^4 + 160N^3 + 84N^2 - 30N - 9)}{3(2N+1)^2} \frac{\zeta^2}{\omega^2} \right],$$

(73)

$$r \approx \frac{16}{2N+1} \left[ 1 + \frac{4N(5N + 4) \zeta}{2N+1} \frac{\zeta}{\omega} \right].$$

(74)

For the scalar index we have included the second-order correction in $\zeta/\omega$ because the dominant contribution to the first-order term in $\zeta/\omega$ in Eq. (70) vanishes for $p = 2$. In the absence of the nonminimal coupling ($\zeta = 0$) one has $n_s = 0.964$ and $r = 0.144$ for $N = 55$, which is inside the 2σ observational bound (see Fig. 1). A positive nonminimal coupling leads to an increase of $r$ relative to the case $\zeta = 0$. Since $r$ is bounded from above observationally, this puts an upper bound on the positive value of $\zeta$. The negative nonminimal coupling gives rise to the deviation from the scale-invariant spectrum ($n_s = 1$) and the decrease of $r$.

From the observational constraints on $n_s$ we can place the bound on the negative nonminimal coupling. We find that the ratio $\zeta/\omega$ is constrained to be

$$-7.0 \times 10^{-3} < \zeta/\omega < 7.0 \times 10^{-4} \quad (95\% \text{ CL}),$$

(75)
Figure 1: 1σ and 2σ observational contours in the \((n_s, r)\) plane constrained by the joint data analysis of WMAP 7yr, BAO, and HST with the pivot scale \(k_0 = 0.002\, \text{Mpc}^{-1}\). Shown also are the theoretical predictions for the potential \(V(\phi) = m^2\phi^2/2\) for \(N = 55\) in three cases: (a) constant \(\omega\) (i.e. \(\mu = 0\)) in the presence of the nonminimal coupling \(\zeta/\omega = 0.001, 0, -0.001, -0.005, -0.01\) (from top to bottom), (b) the exponential coupling \(e^{\mu \phi/M_{\text{pl}}} X\) with \(\mu = -0.05, 0, 0.1, 1, 10\) (from top to bottom) in the absence of the nonminimal coupling, and (c) the exponential coupling \(e^{\phi/M_{\text{pl}}} X\) (i.e. \(\mu = 1\)) in the presence of the nonminimal coupling with \(\zeta = 0.03, 0.01, -0.05, -0.1\) (from top to bottom).

which agrees with that derived in Ref. \[28\] for \(\omega = 1\). The lower bound in Eq. (75) is slightly tighter than the constraint \(\zeta > -1.1 \times 10^{-2}\) (with \(\omega = 1\)) \[27\] obtained by using the WMAP 1yr data combined with the large-scale structure data.

2. \(p = 4\)

We proceed to the case of the self-coupling inflaton potential \(V(\phi) = \lambda \phi^4/4\). In the regime \(|\zeta/\omega| \ll 1\) Eqs. (70) and (71) give

\[
\begin{align*}
n_s - 1 &\simeq -\frac{3}{N+1} \left[ 1 + \frac{4(2N^2 + N - 4) \zeta}{3(N+1) \omega} \right], \\
r &\simeq \frac{16}{N+1} \left[ 1 + \frac{4N(2N+3) \zeta}{N+1 \omega} \right].
\end{align*}
\]

In the absence of the nonminimal coupling one has \(n_s = 0.946\) and \(r = 0.286\) for \(N = 55\), which is outside the 2σ observational bound (see Fig. 2). The presence of the negative nonminimal coupling leads to the increase of \(n_s\), whereas \(r\) gets smaller. Hence it is possible for the self-coupling inflaton potential to be consistent with observations. From the joint data analysis of WMAP 7yr, BAO, and HST the nonminimal coupling is constrained to be

\[
\zeta/\omega < -2.0 \times 10^{-3} \quad (95\% \, \text{CL}),
\]

which is tighter than the bound \(\zeta < -3.0 \times 10^{-4}\) (with \(\omega = 1\)) derived in Ref. \[27\].

In another limit where \(|\zeta/\omega| \to \infty\), inflation is realised by the flat potential \(U\) in the Einstein frame in the regime \(x \gg 1\). In this case one has \(x_f^2 \simeq -2\sqrt{3}/(3\zeta)\) and \(N \simeq -3\zeta x_f^2/4\) from Eqs. (65) and (77), respectively. From Eqs. (61)
Figure 2: The same observational constraints as shown in Fig. 1 with the theoretical prediction of the potential $V(\phi) = \lambda \phi^4 / 4$ for $N = 55$. Each curve corresponds to (a) constant $\omega$ (i.e. $\mu = 0$) in the presence of the nonminimal coupling $\zeta \phi^2 R / 2$ with $\zeta / \omega = 0, -0.001, -0.005, -0.01, -0.03$ (from top to bottom), (b) the exponential coupling $e^{\mu \phi / M_{pl}} X$ with $\mu = 0.1, 1, 10$ (from top to bottom) in the absence of the nonminimal coupling, and (c) the exponential coupling $e^{\phi / M_{pl}} X$ (i.e. $\mu = 1$) in the presence of the nonminimal coupling with $\zeta = 0.03, 0.02, -0.03$ (from top to bottom). The label “FU” corresponds to the Fakir-Unruh scenario [16] with $\zeta \to -\infty$.

and (62) the leading contributions to $n_s$ and $r$ in the regime $N \gg 1$ are

$$n_s - 1 \simeq -2 / N,$$

$$r \simeq 12 / N^2.$$  \hspace{1cm} (79)

As long as $|\zeta|$ is sufficiently large relative to $\omega$, the effect of the term $\omega$ appears only as the next order corrections to (79) and (80) with the order of $\omega / (\zeta N^2)$. For $N = 55$ one has $n_s = 0.964$ and $r = 0.004$ from (79) and (80), which are well inside the $1\sigma$ observational bound.

In the regime $|\zeta / \omega| \gg 1$ the power spectrum (64) reduces to $P_s \simeq \lambda N^2 / (72 \pi^2 \zeta^2)$, so that the WMAP normalization $P_s \simeq 2.4 \times 10^{-9}$ at $N = 55$ gives

$$\lambda / \zeta^2 \simeq 5.6 \times 10^{-10}.$$  \hspace{1cm} (81)

For large negative nonminimal couplings, such as $\zeta \sim -10^4$, the self coupling $\lambda$ can be of the order of $10^{-2}$. This property was used in the context of Higgs inflation.

B. Effect of the non-canonical kinetic term $e^{\mu \phi / M_{pl}} X$ with $\zeta = 0$

Let us consider the case in which the field-dependent coupling $\omega(\phi) = e^{\mu \phi / M_{pl}}$ with the kinetic energy $X$ is present, without taking into account the nonminimal coupling ($\zeta = 0$). After the field settles down to the potential minimum, $\phi = 0$, the coupling $\omega(\phi) \to 1$ and one recovers the standard kinetic energy $X$.

The number of e-foldings (57) is given by

$$N = \frac{1}{p \mu^2} \left[ (\mu x - 1) e^{\mu x} - (\mu x f - 1) e^{\mu x f} \right].$$  \hspace{1cm} (82)
Since \( \dot{\epsilon} = p^2/(2x^2\omega) \), we can estimate \( x_f \) by setting \( \dot{\epsilon} = 1 \):

\[
x_f^2 e^{\mu x_f} = \frac{p^2}{2}, \quad \text{or} \quad x_f = \frac{2W(\sqrt{2}|\mu|p/4)}{\mu},
\]

where \( W \) is the Lambert’s \( W \) function \(^{13}\). The scalar spectral index \(^{11}\) and the tensor-to-scalar ratio \(^{12}\) are

\[
n_s - 1 = -\frac{p}{2x^2e^{\mu x}} (p + 2 + \mu x),
\]

\[
r = \frac{8p^2}{x^2e^{\mu x}}.
\]

In the limit \(|\mu| \ll 1\), using Eq. (83), one can rewrite Eq. (82) in the form

\[
N \simeq \frac{2x^2 - p^2}{4p} + \frac{(8x^3 + \sqrt{2}p^3)}{24p} \mu,
\]

which can be solved for \( x \), as

\[
x^2 \simeq \frac{p^2}{2} + 2pN - \frac{\mu^2}{12} \left[ \sqrt{2}p^3 + (2p^2 + 8pN)^{3/2} \right].
\]

By replacing this relation into Eqs. (84) and (85), we obtain

\[
n_s - 1 \simeq -\frac{p + 2}{2N} \left[ 1 - \frac{(p - 1)\mu\sqrt{2pN}}{3(p + 2)} \right],
\]

\[
r \simeq \frac{4p}{N} \left[ 1 - \frac{\mu\sqrt{2pN}}{3} \right].
\]

which are valid up to the first order in \( \mu \). The presence of the positive \( \mu \) leads to the approach to the scale-invariant spectrum, whereas \( r \) gets smaller. In Figs. 1 and 2 we plot the theoretical values of \( n_s \) and \( r \) in the \((n_s, r)\) plane for \( p = 2 \) and \( p = 4 \), respectively, with several different values of \( \mu \). These are derived numerically by integrating the background equations without using the approximation given above (because the approximation loses its validity for \( \mu \gtrsim 1 \)). Interestingly the models with large positive values of \( \mu \) can be favoured observationally. On the other hand, the models with negative \( \mu \) lead to the deviation from the observationally allowed region. The joint observational constraints from WMAP 7yr, BAO, and HST give the following bounds on \( \mu \):

\[
\mu > -0.04 \quad (95\% \text{ CL}) \quad \text{for} \quad p = 2,
\]

\[
\mu > 0.2 \quad (95\% \text{ CL}) \quad \text{for} \quad p = 4.
\]

Let us now consider the limit where \( \mu \gg 1 \). In this regime the condition \( \mu x \gg 1 \) is satisfied, so that \( N \simeq x e^{\mu x}/(p\mu) \) from Eq. (82). Then the scalar index \(^{13}\) and the tensor-to-scalar ratio \(^{13}\) reduce to

\[
n_s - 1 \simeq -\frac{1}{N},
\]

\[
r \simeq \frac{8p}{N} \frac{1}{\mu x}.
\]

For a given \( N \), \( \mu x \) increases for larger \( \mu \). This means that, in the limit \( \mu \gg 1 \), \( n_s \) and \( r \) approach \( n_s \to 1 - 1/N \simeq 0.982 \) (for \( N = 55 \)) and \( r \to 0 \), respectively, which is inside the 1\( \sigma \) observational bound. We have also confirmed numerically that inflation is followed by a reheating phase with oscillations of \( \phi \).

C. Combined effects of the nonminimal coupling \( \zeta \phi^2 R/2 \) and the non-canonical kinetic term \( e^{\mu \phi/M_{pl}} X \)

Finally we consider the case in which both the nonminimal coupling \( \zeta \phi^2 R/2 \) and the non-canonical kinetic term \( e^{\mu \phi/M_{pl}} X \) are taken into account. Since it is difficult to derive an analytic form for the number of e-foldings \( N \), we solve the background equations numerically to identify the values \( x \) corresponding to \( N = 55 \) before \( x = x_f \) which happens when \( \dot{\epsilon} = 1 \). We then use the formulae \(^{54}\) and \(^{55}\) to evaluate \( n_s \) and \( r \) for given values of \( p \), \( \mu \), and \( \zeta \).
In Fig. 1 we plot the numerical values of $n_s$ and $r$ in the two-dimensional plane for $p = 2$ and $\mu = 1$ with $\zeta = 0.03, 0.01, -0.05, -0.1$. The presence of the term $e^{\mu\phi/M_{pl}}X$ with $\mu > 0$ leads to the compatibility of the nonminimally coupled models with larger values of $|\zeta|$ than those for $\mu = 0$ and $\omega = 1$. When $\mu = 1$ we find that the nonminimal coupling is constrained to be

$$-0.12 < \zeta < 0.035 \quad (95\% \text{ CL}),$$

which is wider than the range (75). For values of $|\zeta|$ larger than the bounds given by (94) the effect of the nonminimal coupling is more important than that of the non-canonical kinetic term.

For $p = 4$ and $\mu = 0$ a positive nonminimal coupling is not allowed observationally because both $|n_s - 1|$ and $r$ tend to be larger than those for $\zeta = 0$. However, the non-canonical kinetic term with $\mu > 0$ allows the compatibility of the positive nonminimally coupled models with observations (see Fig. 2). If $\mu = 1$, $\zeta$ is constrained to be

$$\zeta < 0.025 \quad (95\% \text{ CL}).$$

VI. INFLATION IN THE CONTEXT OF BRANS-DICKE THEORIES

Let us proceed to Brans-Dicke (BD) theory [44] with the action

$$S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} M_{pl} \phi R + \frac{M_{pl}}{\phi} \omega_{BD} X - V(\phi) \right],$$

where $\omega_{BD}$ is the BD parameter. Here we have introduced the reduced Planck mass $M_{pl}$ in the first two terms, so that the field $\phi$ has a dimension of mass. Under the conformal transformation (43) we obtain the action (44) in the Einstein frame with

$$F = \frac{\phi}{M_{pl}} = e^{\mu\chi/M_{pl}}, \quad U = e^{-2\mu\chi/M_{pl}} V,$$

where

$$\mu \equiv 1/\sqrt{3/2 + \omega_{BD}}.$$ (98)

The integration constant for the field $\chi$ is chosen such that $\chi = 0$ corresponds to $\phi = M_{pl}$.

A. Case of the power-law potential

Let us consider the power-law potential [43] in the Jordan frame. In the Einstein frame the potential is given by

$$U(\chi) = V_0 e^{\lambda \chi/M_{pl}}, \quad \lambda \equiv \frac{p - 2}{\sqrt{3/2 + \omega_{BD}}}. $$

For the BD parameter of the order of 1 inflation does not occur unless $p$ is close to 2. However, for $\omega_{BD} \gg 1$, it is possible to realise $|\lambda| \ll 1$ even if $p$ is away from 2.

On using Eqs. (54) and (55) for the potential (99), it follows that

$$n_s - 1 = -\lambda^2, \quad r = -8n_t = 8\lambda^2.$$ (101)

The CMB likelihood analysis using the data of WMAP 7yr [6] combined with BAO [41] and HST [42] gives the following bound on $\lambda$ [45]:

$$0.09 < \lambda < 0.23 \quad (95\% \text{ CL}).$$ (102)
This translates into the constraint on $\omega_{\text{BD}}$:

$$19(p-2)^2 - 3/2 < \omega_{\text{BD}} < 123(p-2)^2 - 3/2. \quad (103)$$

The reason why the $p = 2$ case (i.e. $\lambda = 0$) is disfavoured is that the Harrison-Zel’dovich spectrum ($n_s = 1$ and $r = 0$) is in tension with observations [6]. If $p = 4$, Eq. (103) gives the bound $75 < \omega_{\text{BD}} < 491$.

The exponential potential in the Einstein frame does not lead to the end of inflation, so the above scenario has to be modified in a way that the potential has a minimum to lead to a successful reheating. In the following we shall consider the modification of the power-law potential in the Jordan frame, such that inflation ends as in the Starobinsky’s $f(R)$ model [1].

B. Models including the Starobinsky’s $f(R)$ scenario

The $f(R)$ theory with the action

$$S = \int d^4x \sqrt{-g} M_{\text{pl}}^2 f(R), \quad (104)$$

is equivalent to BD theory with $\omega_{\text{BD}} = 0$ [46]. In fact the action (104) can be written as

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} F(\phi) R - V(\phi) \right], \quad (105)$$

where

$$F = \frac{\phi}{M_{\text{pl}}} = \frac{\partial f}{\partial R}, \quad V(\phi) = \frac{M_{\text{pl}}^2}{2} \left( R \frac{\partial f}{\partial R} - f \right). \quad (106)$$

In Starobinsky’s model $f(R) = R + R^2/(6M^2)$, we have $R = 3M^2(\phi/M_{\text{pl}} - 1)$ and

$$V(\phi) = \frac{3M^2}{4}(\phi - M_{\text{pl}})^2. \quad (107)$$

We consider the following more general potential

$$V(\phi) = V_0(\phi - M_{\text{pl}})^p, \quad (108)$$

with arbitrary values of $\omega_{\text{BD}}$, so that the model $f(R) = R + R^2/(6M^2)$ is covered as a special case with $p = 2$ and $\omega_{\text{BD}} = 0$. The potential in the Einstein frame reads

$$U = V_0 M_{\text{pl}}^p e^{(p-2)\mu/\chi/M_{\text{pl}}} \left( 1 - e^{-\mu/\chi/M_{\text{pl}}} \right)^p, \quad (109)$$

where $\mu$ is defined in Eq. (98). For $|\omega_{\text{BD}}| \sim O(1)$, i.e. $\mu \sim O(1)$, inflation occurs in the regime $\chi \gg M_{\text{pl}}$ (including the Starobinsky’s $f(R)$ model). The behaviour of the potential (109) depends on the values of $p$:

- If $p = 2$ the potential (109) becomes constant for $\chi \gg M_{\text{pl}}$. We note that it is not necessary for $p$ to exactly equal 2 for this behaviour to occur. Since $U$ is approximated as $U \propto \chi^2$ in the regime $\chi \ll M_{\text{pl}}$, inflation is followed by a successful reheating.

- For $p > 2$ the field rolls down the potential towards $\chi = 0$.

- When $p < 2$ the field rolls down the potential towards $\chi = +\infty$ or towards $\chi = 0$. In the latter case the potential does not have a minimum at $\phi = 0$. As a result, reheating is problematic for $p < 2$.

From Eqs. (154) and (155) it follows that

$$n_s - 1 = -\frac{\mu^2 (4 + 2(3p - 4)F + (p-2)^2 F^2)}{(F-1)^2}, \quad (110)$$

$$r = \frac{8\mu^2[2 + (p-2)F]^2}{(F-1)^2}. \quad (111)$$
The number of e-foldings \([57]\) reads
\[
N = \frac{\frac{1}{2\mu^2}(F - F_f) + \frac{1}{2} \left( 1 - \frac{1}{\mu^2} \right) \ln \left( \frac{F}{F_f} \right)}{(p = 2),}
\]
\[
N = \frac{p}{2\mu^2(p-2)} \ln \left( \frac{2 + (p-2)F}{2 + (p-2)F_f} \right) + \frac{1}{2} \left( 1 - \frac{1}{\mu^2} \right) \ln \left( \frac{F}{F_f} \right) \quad (p \neq 2),
\]
where \(F_f\) is the value of \(F\) at the end of inflation. Using the criterion \(\dot{\epsilon} = 1\) for the end of inflation, we have
\[
F_f = \frac{1 + \sqrt{2}\mu}{1 - (p - 2)\mu/\sqrt{2}}.
\]

1. Case: \(p = 2\)

Let us consider the case \(p = 2\). For the theories with \(|\omega_{BD}| \sim O(1)\) (i.e. \(\mu \sim O(1)\)) one has \(F_f = 1 + \sqrt{2}\mu = O(1)\) and \(N \simeq F/(2\mu^2)\), which means that \(F \gg 1\) for \(N \gg 1\). From Eqs. \([110]\) and \([111]\) it follows that
\[
n_s - 1 \simeq -\frac{4\mu^2}{F} \simeq -\frac{2}{N}, \quad r \simeq \frac{32\mu^2}{F^2} \simeq \frac{8}{\mu^2N^2} = \frac{4(3 + 2\omega_{BD})}{N^2},
\]
which are valid for \(-3/2 < \omega_{BD} < O(1)\). The metric \(f(R)\) gravity corresponds to \(\omega_{BD} = 0\), which gives \(r \simeq 12/N^2\). This result matches with the one derived in other papers \([47]\). In the limit that \(\omega_{BD} \rightarrow -3/2\) the tensor-to-scale ratio vanishes. The BD parameter \(\omega_{BD} = -3/2\) corresponds to Palatini \(f(R)\) gravity \([11, 12]\), in which case a separate analysis is required as in Ref. \([48]\).

If \(\omega_{BD} \gg 1\), then one has \(\mu \ll 1\) and hence \(F\) is close to 1 even during inflation. The end of inflation is characterized by the condition \(\dot{\epsilon} = 1\), which gives \(F_f = 1 + \sqrt{2}\mu \simeq 1\). Then the number of e-foldings \([112]\) is approximately given by \(N \simeq (\chi/M_{pl})^2/4\). The scalar spectral index and the tensor-to-scalar ratio are
\[
n_s - 1 \simeq -\frac{M_{pl}^2}{\chi^2} \simeq -\frac{2}{N}, \quad r \simeq \frac{32M_{pl}^2}{\chi^2} \simeq \frac{8}{N},
\]
which match with those for the chaotic inflation model with the potential \(U(\phi) = m^2\phi^2/2\) \([49]\).

The tensor-to-scalar ratio depends on the BD parameter \(\omega_{BD}\), while the scalar index is practically independent of \(\omega_{BD}\). In Fig. \(3\) we plot the theoretical predictions of \(n_s\) and \(r\) for several different values of \(\omega_{BD}\) by fixing \(N = 55\). Shown also are the \(1\sigma\) and \(2\sigma\) observational contours constrained by the joint data analysis of WMAP 7yr \([8]\), BAO \([11]\), and HST \([42]\). The \(f(R)\) model \(f(R) = R + R^2/(6M^2)\), which corresponds to \(\omega_{BD} = 0\), is well within the \(1\sigma\) observational contour. While the present observations allow the large BD parameter with \(\omega_{BD} \gg 1\), it will be of interest to see how the PLANCK satellite \([50]\) can provide an upper bound on \(\omega_{BD}\).

Using the approximate relation \(F \simeq 2\mu^2N + F_f\) following from Eq. \([112]\), the WMAP normalization for the scalar power spectrum \(P_s = U^3/(12\pi^2M_{pl}^6U^2\chi)\) is given by
\[
P_s \simeq \frac{V_0}{2\pi^3 M_{pl}^4 (2\mu^2N + 1 + \sqrt{2}\mu)^2} = 2.4 \times 10^{-9},
\]
around \(N = 55\). Since \(\omega_{BD} = 0\) and \(\mu = 1/\sqrt{3/2}\) for the \(f(R)\) model \(f(R) = R + R^2/(6M^2)\), the mass scale \(M\) is constrained to be \(M \simeq 3 \times 10^{13}\) GeV. The energy scale \(V_0\) is different depending on the BD parameter.

2. Case: \(p \neq 2\)

We proceed to the case \(p \neq 2\). For the BD parameter \(\omega_{BD}\) of the order of unity, the number of e-foldings \([113]\) cannot be much greater than 1 unless \(F\) is enormously larger than \(F_f\) (\(~O(1)\)). If \(F \gg 1\), then Eqs. \([110]\) and \([111]\)
Figure 3: 1σ and 2σ observational contours in the \((n_s, r)\) plane constrained by the joint data analysis of WMAP 7yr, BAO, and HST with the pivot scale \(k_0 = 0.002\) Mpc\(^{-1}\) (logarithmic scale for the vertical line). The dotted points show the theoretical predictions for the BD theories with the potential \(V(\phi) = V_0(\phi - M_{pl})^2\). The number of e-foldings is chosen to be \(N = 55\). From bottom to top the points correspond to \(\omega_{BD} = -1.4, -1, 0, 10, 10^2, 10^3, 10^4\) and \(\omega_{BD} \to \infty\), where \(\omega_{BD} = 0\) represents the model \(f(R) = R + R^2/(6M^2)\). For larger \(\omega_{BD}\) the two observables \(n_s\) and \(r\) approach those for the chaotic inflation with the quadratic potential \(m^2\phi^2/2\).

\[
\begin{align*}
  n_s - 1 &\approx -\mu^2(p-2)^2, \\
  r &\approx 8\mu^2(p-2)^2.
\end{align*}
\]

(120) (121)

Since \(\mu \sim \mathcal{O}(1)\) the results (120) and (121) mean that for small \(\omega_{BD}\) both the scalar index and the tensor-to-scalar ratio are incompatible with observations apart from the case where \(p\) is close to 2. On reflection this is unsurprising since only for \(p \approx 2\) is there a flat region of the potential that will give slow-roll along with its signatures of near scale invariance and suppressed tensor modes.

When \(\omega_{BD} \gg 1\) one has \(\mu \ll 1\) and hence \(F = e^{\mu \chi/M_{pl}}\) is close to 1. Then Eqs. (110) and (111) give

\[
\begin{align*}
  n_s - 1 &\approx -p(p+2)\frac{M^2_{pl}}{\chi^2} \approx \frac{p+2}{2N}, \\
  r &\approx 8p^2\frac{M^2_{pl}}{\chi^2} \approx 4p \frac{2N}{N}.
\end{align*}
\]

(122) (123)

where we have used the approximate relation \(N \approx \chi^2/(2pM^2_{pl})\). These results match with those of chaotic inflation with the potential (3). The case \(p = 4\) is excluded observationally both in the regimes \(\omega_{BD} \gg 1\) and \(\omega_{BD} = \mathcal{O}(1)\). Even for other values of \(\omega_{BD}\) it is difficult to satisfy observational constraints unless \(p\) is close to 2.
VII. INFLATION IN THE PRESENCE OF A GAUSS-BONNET TERM

In this section we study the effects of the Gauss-Bonnet (GB) term on the chaotic inflationary scenario, described by the action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M^2_{\text{pl}}}{2} R + X - V(\phi) - \xi(\phi) \mathcal{G} \right]. \]

In order to confront the model with observations, it is convenient to rewrite inflationary observables in terms of the following slow-roll parameters:

\[ \epsilon_v \equiv \frac{M^2_{\text{pl}}}{2} \frac{V_{,\phi}}{V}, \quad \eta_v \equiv \frac{M^2_{\text{pl}} V_{,\phi\phi}}{V}. \]

The background equations are

\[ 3M^2_{\text{pl}} H^2 = \frac{\dot{\phi}^2}{2} + V + 24H^2 \dot{\xi}, \]

\[ \ddot{\phi} + 3H \dot{\phi} + V_{,\phi} + 24H^2 \xi_{,\phi}(H^2 + \dot{H}) = 0. \]

At linear order Eqs. (25) and (12) give

\[ \epsilon_s = \delta X, \quad \epsilon = \epsilon_s + 4\delta \xi. \]

From Eqs. (126) and (127) the potential \( V \) and its derivative \( V_{,\phi} \) can be expressed as

\[ V = 3M^2_{\text{pl}} H^2 \left( 1 - \frac{1}{3} \epsilon_s - 8\delta \xi \right), \]

\[ V_{,\phi} = -H \dot{\phi} \left[ 3 - \epsilon + 1 - \frac{1}{2} \eta_s + 12 \frac{\delta \xi}{\epsilon_s} (1 - \epsilon) \right]. \]

Taking the leading-order contribution in Eq. (130), it follows that

\[ V_{,\phi} \approx -3H \dot{\phi} \left( 1 + \frac{4\delta \xi}{\epsilon_s} \right), \]

\[ V_{,\phi\phi} \approx -3H^2 \left[ \frac{1}{2} \eta_s - 2\epsilon_s - 16\delta \xi - \frac{4\delta \xi}{\epsilon_s} \left( 8\delta \xi + \frac{1}{2} \eta_s - \eta \xi \right) \right], \]

which lead to

\[ \epsilon_v \approx \epsilon_s \left( 1 + \frac{4\delta \xi}{\epsilon_s} \right)^2, \]

\[ \eta_v \approx -\frac{1}{2} \eta_s \left( 1 - \frac{4\delta \xi}{\epsilon_s} \right) + 2\epsilon_s + 4\delta \xi \left[ 4 + \frac{1}{\epsilon_s} (8\delta \xi - \eta) \right]. \]

From this we obtain the inversion formulas

\[ \epsilon_s \approx \frac{1}{2} \left[ \epsilon_v - 8\delta \xi + \sqrt{\epsilon_v^2 - 16\epsilon_v \delta \xi} \right], \]

\[ \eta_s \approx -\frac{2 \left( \eta_v - 2\epsilon_s - 4\delta \xi [4 + (8\delta \xi - \eta)/\epsilon_s] \right)}{1 - 4\delta \xi/\epsilon_s}, \]

where we have taken the positive sign in Eq. (135) to reproduce \( \epsilon_s \rightarrow \epsilon_v \) for \( \delta \xi \rightarrow 0 \).

From Eqs. (28), (51), and (52) the inflationary observables are given by

\[ n_s - 1 = -2\epsilon_s - \eta_s - 8\delta \xi, \]

\[ n_t = -2\epsilon_s - 8\delta \xi, \]

\[ r = 16\epsilon_s, \]
which are written in terms of the four variables: $\epsilon_V$, $\eta_V$, $\delta_{\xi}$, and $\eta_\xi$. By specifying the functional forms of $V(\phi)$ and $\xi(\phi)$, we can reduce the number of those variables. For the chaotic inflation potential [3] one has $\epsilon_V = (p^2/2)(M_{pl}/\phi)^2$ and $\eta_V = p(p - 1)(M_{pl}/\phi)^2$, so that they are related with each other via the relation

$$\eta_V = \frac{2(p - 1)}{p} \epsilon_V.$$  

(140)

For the Gauss-Bonnet coupling, we take

$$\xi(\phi) = \xi_0 e^{\mu \phi/M_{pl}},$$  

(141)

where $\xi_0$ and $\mu$ are constants. It then follows that

$$\eta_\xi = -2\epsilon_s + \eta_s/2 - 8\delta_{\xi} \pm \sqrt{\frac{2\epsilon_s}{\epsilon_s}},$$  

(142)

where the plus and minus signs correspond to $\phi > 0$ and $\phi < 0$, respectively. Combining Eq. (142) with Eq. (136), we obtain

$$\eta_s \simeq -2 \left[ \eta_V - 2\epsilon_s + 4\delta_{\xi} \left( \frac{\pm\mu\sqrt{2\epsilon_s} - 16\delta_{\xi}}{\epsilon_s} - 6 \right) \right].$$  

(143)

Substituting Eq. (143) into Eq. (137) and choosing the negative sign of $\dot{\phi}$, the scalar spectral index can be written as

$$n_s - 1 \simeq -6\epsilon_s + 2\eta_V - 8\delta_{\xi} \left( \frac{7 + \mu\sqrt{2\epsilon_s} + 16\delta_{\xi}}{\epsilon_s} \right),$$  

(144)

where

$$\eta_V = \frac{2(p - 1)}{p} \epsilon_V \simeq \frac{2(p - 1)}{p} \epsilon_s \left( 1 + \frac{4\delta_{\xi}}{\epsilon_s} \right)^2.$$  

(145)

For fixed values of $p$ and $\mu$ one can carry out the CMB likelihood analysis in terms of $n_s$, $r$, and $n_l$ by varying the two parameters $\epsilon_s$ and $\delta_{\xi}$.

In Fig. 4 the observational constraints on the parameters $\epsilon_s$ and $r_\xi \equiv \delta_{\xi}/\epsilon_s$ are plotted for $p = 2$ and $\mu = 1$. We run the Cosmological Monte Carlo (CosmoMC) code [51] with the data of WMAP 7yr [6] combined with large-scale structure [8] (including BAO [41]), HST [42], Supernovae type Ia (SN Ia) [52], and Big Bang Nucleosynthesis (BBN) [53], by assuming a $\Lambda$CDM universe. The ratio $r_\xi$ is constrained to be $|r_\xi| < 0.1$ (95% CL), which means that the effect of the GB term needs to be suppressed. Hence the energy scale $V_0$ is similar to that in the standard chaotic inflation.

From Fig. 4 we find that the slow-roll parameter $\epsilon_s$ is bounded to be $\epsilon_s < 0.025$ (95% CL). In the presence of the GB term the small values of $\epsilon_s$ can give rise to the scalar index close to $n_s = 0.96$. For example, when $\epsilon_s = 0.002$, $r_\xi = 0.05$, $\mu = 1$, and $p = 2$, one has $n_s = 0.962$ from Eq. (144). This is different from the standard chaotic inflation in which the small values of $\epsilon_s$ lead to the spectrum close to the Harrison-Zel’dovich one (which is not favored observationally). Hence the allowed range of $\epsilon_s$ tends to be wider in the presence of the GB coupling.

Let us estimate the two observables $n_l$ and $r$ in terms of the number of e-foldings $N$ under the condition $|r_\xi| \ll 1$. Since $\epsilon_s \simeq \epsilon_V - 8\delta_{\xi}$ from Eq. (136), Eqs. (144) and (137) reduce to

$$n_s - 1 \simeq - \left( 2 + \frac{4}{p} \right) \epsilon_V - 8\delta_{\xi} \left( 1 + \mu \sqrt{\frac{2}{\epsilon_V}} \right),$$  

(146)

$$r \simeq 16\epsilon_V \left( 1 - \frac{8\delta_{\xi}}{\epsilon_V} \right).$$  

(147)

From Eqs. (129) and (131) one has $H/\dot{\phi} \simeq -(1 + 4\delta_{\xi}/\epsilon_s)V/(M_{pl}^2 V_{\phi})$. Using the relation $\epsilon_s = \delta_X$ and the definition of $\delta_{\xi}$, it follows that $H/\dot{\phi} = -\phi/(M_{pl}^2 + 8H^2\xi_0\phi)$. Since we are considering the case where $H^2\xi_0\phi/M_{pl}^2 \ll 1$, the number of e-foldings for the potential [3] is

$$N = \int_{\phi}^{\phi_f} \frac{H}{\phi} d\phi \simeq \frac{x^2 - x_f^2}{2p} + N_p,$$  

where

$$N_p \equiv \frac{8\xi_0\mu}{3p^2} \frac{V_0}{M_{pl}^4} \int_{x_f}^{x} e^{\mu x} x^{p+2} dx.$$  

(148)
Here $x_f$ is the value of $x = \phi/M_{pl}$ at the end of inflation. We identify the end of inflation by the condition $\epsilon_V = 1$, i.e. $x_f = p/\sqrt{2}$. When $p = 2$ and $p = 4$, Eq. (148) is integrated to give

$$\bar{N}_2 = \frac{-m^2}{3M_{pl}^2 \mu^2} \frac{\xi_0}{\mu^4} \left\{ e^{\mu x}[\mu^4 x^4 + 4(-\mu^3 x^3 + 3\mu^2 x^2 - 6\mu x + 6)] - 4e^{\sqrt{2}\mu} [\mu^4 - 2\sqrt{2}\mu^3 + 6(\mu^2 - \sqrt{2}\mu + 1)] \right\} ,$$

(149)

$$\bar{N}_4 = -\frac{\lambda \xi_0}{2^3 \mu^6} \left\{ e^{\mu x}[(\mu^6 x^6 - 6\mu^5 x^5 + 30\mu^4 x^4 - 120\mu^3 x^3 + 360\mu^2 x^2 - 720\mu x + 720) - 16e^{2\sqrt{2}\mu} \left[ 32\mu^6 - 48\sqrt{2}\mu^5 + 120\mu^3(\mu - \sqrt{2}) + 90(2\mu - \sqrt{2}) + 45 \right] \right\} ,$$

(150)

where we have set $V_0 = m^2 M_{pl}^2/2$ for $p = 2$ and $V_0 = \lambda M_{pl}^4/4$ for $p = 4$.

For positive $\mu$ of the order of unity, the dominant contributions to $\bar{N}_p$ come from the first terms in Eqs. (149) and (150), i.e. $\bar{N}_2 \approx -\frac{m^2}{3M_{pl}^2 \mu^2} \xi_0 x^4 / (3M_{pl}^2)$ and $\bar{N}_4 \approx -\lambda \xi_0 x^6 e^{\mu x} / 24$, for the scales relevant to CMB ($\mu x \gg 1$). In this case the number of e-foldings (148) is approximately given by

$$N \approx \frac{1}{4} x^2 \left[ 1 - \frac{4}{3} \left( \frac{m}{M_{pl}} \right)^2 \xi x^2 \right] - \frac{1}{2} \quad (p = 2) ,$$

(151)

$$N \approx \frac{1}{8} x^2 \left[ 1 - \frac{1}{3} \lambda \xi x^4 \right] - 1 \quad (p = 4) .$$

(152)

Since $\delta_{c} \approx -\mu V_0 / (3M_{pl}^4) x^{p-1} \xi$ and $\epsilon_V = p^2 / (2x^2)$, one can express the scalar index (146) and the tensor-to-scalar ratio (147) in terms of $x$. By treating the $\xi$-dependent terms in Eqs. (151) and (152) as small corrections, $n_s$ and $r$ can be written in terms of $N$:

$$n_s - 1 \approx -\frac{2}{N} \left[ 1 - \frac{16}{3} N^2 \left( \frac{m}{M_{pl}} \right)^2 \mu^2 \xi \right] , \quad r \approx \frac{8}{N} \left[ 1 + \frac{32}{3} N^{3/2} \left( \frac{m}{M_{pl}} \right)^2 \mu \xi \right] , \quad (p = 2) ,$$

(153)

$$n_s - 1 \approx -\frac{3}{N} \left[ 1 - \frac{256}{9} N^3 \lambda \mu^2 \xi \right] , \quad r \approx \frac{16}{N} \left[ 1 + \frac{128\sqrt{2}}{3} N^{5/2} \lambda \mu \xi \right] , \quad (p = 4) .$$

(154)
which are valid for positive $\mu$ of the order of unity. If $\xi > 0$ (i.e. $\delta \xi < 0$ for $\dot{\phi} < 0$), then the effect of the GB coupling leads to the approach to the scale-invariant spectrum, while $r$ gets larger. The negative values of $\xi$ lead to the decrease of $r$, but $n_s$ deviates from 1. Since $\epsilon_s$ is approximately given by $\epsilon_s \approx \epsilon_{\psi} \approx p^2/(8N)$, the scales relevant to the CMB anisotropies ($N = 50$-60) correspond to $0.008 < \epsilon_s < 0.01$ for $p = 2$. Figure 3 shows that the ratio $r_{\xi}$ is constrained to be $-0.04 < r_{\xi} < 0.03$ (95% CL) for this range of $\epsilon_s$. The self-coupling potential $V(\phi) = \lambda \phi^4/4$ is not saved by taking into account the GB term with positive $\mu$, because the GB coupling does not lead to the increase of $n_s$ and the decrease of $r$ simultaneously.

For negative $\mu$ with $|\mu| = O(1)$ the exponential term $e^{\mu x}$ in Eqs. (149) and (150) is much smaller than 1 for the scales relevant to CMB ($x \gg 1$). In this case we have

$$\dot{N}_2 \simeq \frac{4m^2 \xi_0 e^{\sqrt{\mu}}}{3M_{pl}^2 \mu^4} \left[ \mu^4 - 2\sqrt{2}\mu^3 + 6(\mu^2 - \sqrt{2}\mu + 1) \right],$$

(155)

$$\dot{N}_4 \simeq \frac{2\xi_0 e^{2\sqrt{\mu}}}{3\mu^6} \left[ 32\mu^6 - 48\sqrt{2}\mu^5 + 120\mu^3(\mu - \sqrt{2}) + 90\mu(2\mu - \sqrt{2}) + 45 \right].$$

(156)

The scalar index and the tensor-to-scalar ratio are approximately given by

$$n_s - 1 \simeq -\frac{2}{N} \left( 1 + \frac{N_2 - 1/2}{N} \right), \quad r \simeq \frac{8}{N} \left( 1 + \frac{N_2 - 1/2}{N} \right), \quad (p = 2),$$

(157)

$$n_s - 1 \simeq -\frac{3}{N} \left( 1 + \frac{N_4 - 1}{N} \right), \quad r \simeq \frac{16}{N} \left( 1 + \frac{N_4 - 1}{N} \right), \quad (p = 4),$$

(158)

where we have assumed $N \gg N_p$, and ignored the exponential term $e^{\mu x}$. When $\mu < 0$, one can show that $\dot{N}_2$ and $\dot{N}_4$ in Eqs. (155) and (156) are positive for $\xi_0 > 0$ and negative for $\xi_0 < 0$. In the latter case the presence of the GB term leads to the approach to the Harrison Zel’dovich spectrum. In fact, such a scenario was discussed in Ref. [21]. Since $m/M_{pl}$ and $\lambda$ are much smaller than 1 by the WMAP normalization ($m/M_{pl} \simeq 6.8 \times 10^{-6}$ and $\lambda \simeq 2.0 \times 10^{-13}$), one has $|\dot{N}_p| \ll 1$ for $|\xi_0|$ smaller than the order of 1. For $\mu < 0$ the effect of the GB term on the inflationary observables appears for very large values of $\xi_0$ such as $|\xi_0| \sim 10^{10}$ [21].

VIII. G-INFLATION WITH A FIELD POTENTIAL

Finally we study chaotic inflation in the presence of the Galileon-like self-interaction $G(\phi, X)\square \phi$ (called “G-inflation”). We specify the functional form of $G(\phi, X)$, as

$$G(\phi, X) = \Phi(\phi)X^n, \quad \Phi(\phi) = \frac{\theta}{M^{4n-1}}e^{\mu \phi/M_{pl}},$$

(159)

where $n$ and $\mu$ are constants, and $\theta = \pm 1$. The constant $M$ has a dimension of mass with $M > 0$. Here we have introduced the exponential form for $\Phi$ motivated by the dilaton coupling in the low-energy effective bosonic string theory. We also consider the power-law function $X^n$ by generalizing previous studies [30]. Equations 59 and 71 can be written as

$$V = 3M_{pl}^2H^2 \left( 1 - \frac{1}{3}\delta_X - 2\delta_{GX} + \frac{2}{3}\delta_{G\phi} \right),$$

(160)

$$V_{,\phi} = -3H\phi \left( 1 + (3 - \epsilon)\frac{\delta_{GX}}{\delta_X} - \frac{\mu^2}{3n}\delta_{GX} + 2(n - 1)\frac{\delta_{G\phi}}{\delta_X} + \frac{\delta_{\psi}}{3} \right) \left[ 1 + 6n\frac{\delta_{GX}}{\delta_X} - 2(n + 1)\frac{\delta_{G\phi}}{\delta_X} \right].$$

(161)

To compare with observations, we seek an expression for $n_s - 1 = \dot{N}_s/(H\dot{P}_s)$ in terms of a minimal set of independent slow-roll parameters. Since $\dot{P}_s = H^2/(8\pi^2Qc_s^2)$, it is important to find the expressions for $Q$ and $c_s$. From Eqs. (15) and (16) it follows that

$$Q_{,\phi} = \frac{\delta_X + 6n\delta_{GX} - 2(n + 1)\delta_{G\phi} + 3\delta_{G\phi}}{(1 - \delta_{GX})^2},$$

(162)

$$c_s^2 = \frac{\delta_X + 2(2 + n\delta_{\psi})\delta_{GX} + 2(n - 1)\delta_{G\phi} - \delta_{G\phi}}{\delta_X + 6n\delta_{GX} - 2(n + 1)\delta_{G\phi} + 3\delta_{G\phi}^2},$$

(163)
The relation (140) between $\lambda_{G_X} = n - 1$, and $\lambda_{G_\phi} = n$. Hence, we can derive an exact expression for $n_s - 1$ in terms of the slow-roll parameters entering Eqs. (162) and (163) and their first derivatives, which introduce other slow-roll parameters, however these are not all independent. We shall now discuss the relations which reduce the number of independent slow-roll parameters.

For the choice of the function $\Phi$ in Eq. (159) we have

$$\delta_{G_\phi} = \pm \frac{\mu}{\sqrt{2n}} \delta_{G_X} \sqrt{\delta_X}, \quad (164)$$

where the $\pm$ signs in this expression are compatible with those in the expression $\dot{\phi} = \pm \sqrt{2M_{pl}H} \sqrt{\delta_X}$. Equation (164) shows that $\delta_{G_\phi}$ is in general suppressed relative to $\delta_{G_X}$. This relation also implies that

$$\eta_{G_\phi} = \frac{\dot{\delta}_{G_\phi}}{H \delta_{G_\phi}} = \frac{\eta_X}{2} + \eta_{G_X}, \quad (165)$$

where

$$\eta_X = \frac{\dot{\delta}_X}{H \delta_X}, \quad \text{and} \quad \eta_{G_X} = \frac{\dot{\delta}_{G_X}}{H \delta_{G_X}}. \quad (166)$$

From the definition of $\delta_X$ and $\delta_{G_X}$ we obtain

$$\eta_X = 2(1 - \eta_{G_X}) \delta_\phi + 2 \delta_X + 6 \delta_{G_X} - 4 \delta_{G_\phi},$$

$$\eta_{G_X} = (2n + 1 - \eta_{G_X}) \delta_\phi \pm \mu \sqrt{2 \delta_X + \delta_X - 2 \delta_{G_\phi} + 3 \delta_{G_X}}, \quad (167)$$

where we have used the relation

$$\epsilon = \delta_X + 3 \delta_{G_X} - 2 \delta_{G_\phi} - \delta_\phi \delta_{G_X}. \quad (169)$$

It should be noted that the four relations (164)-(169) are all exact. From Eqs. (165), (167), (168) with Eq. (164) we find that $\eta_{G_\phi}, \eta_X,$ and $\eta_{G_X}$ can be expressed in terms of the three slow-roll parameters $\delta_\phi, \delta_X, \text{and} \delta_{G_X}$.

We finally use a last constraint coming from the fact that we have chosen a power-law form (3) for the potential. The relation (140) between $\epsilon \nu$ and $\eta_X$ leads to

$$\frac{V_{,\phi}}{H \nu_{,\phi}} = \frac{p - 1}{p} \frac{V_{,\phi}}{H \nu_{,\phi}} \dot{\phi}. \quad (170)$$

This equation can be used to set the last constraint on the slow-roll variables. At lowest order we have

$$\delta_\phi = \frac{(\delta_X + 3 \delta_{G_X}) (2 - p) \delta_X + 6 \delta_{G_X})}{p(\delta_X + 6 n \delta_{G_X})} \sqrt{\delta_X} \delta_{G_X} - \frac{2(2n + 1 - \eta_{G_X})}{n(\delta_X + 6 n \delta_{G_X})^2} \mu^2 + O(\epsilon^{3/2}). \quad (171)$$

Using this relation we can express $\eta_{G_\phi}, \eta_X,$ and $\eta_{G_X}$ in terms of two slow-roll parameters $\delta_X$ and $\delta_{G_X}$.

We are now ready to explicitly calculate the scalar index $n_s - 1 = -2 \epsilon - \eta_{Q} - 3 s$, where $\eta_{Q} = \dot{Q}/(HQ)$ and $s = \dot{s}/(HC_s)$ are evaluated by taking the time derivatives of Eqs. (162) and (163). This gives

$$n_s - 1 \approx - \frac{2(\delta_X + 3 \delta_{G_X})}{p(\delta_X + 4 \delta_{G_X}) (\delta_X + 6 n \delta_{G_X})^2} \sqrt{\delta_X} (p + 2) + \delta_X^3 \delta_{G_X} [(3p - 6)n^2 + (12p + 27)n + 4p + 8]$$

$$+ \delta_X \delta_{G_X}^2 [(57p + 30)n^2 + (54p + 105)n + 6] + 72 n \delta_{G_X}^3 (3np + 2n + 1)$$

$$\pm \frac{3\sqrt{2} \delta_X \delta_{G_X}^2 [7(n + 2) \delta_X \delta_{G_X} + n \delta_X^2 + 24n \delta_{G_X}]}{\delta_X + 6 n \delta_{G_X})^2 (\delta_X + 4 \delta_{G_X})^2} \mu$$

$$- \frac{2 \delta_{G_X}}{n(\delta_X + 6 n \delta_{G_X})^3 (4 \delta_{G_X} + \delta_X)^2} \left[ \delta_X^4 + (9 n + 8 + 6 n) \delta_{G_X} + 4 \delta_{G_X} \right] \mu^2$$

$$+ (4 - 99 n^2 + 54 n - 42 n^2) \delta_X^2 \delta_{G_X} + (32 n^2 - 282 n^3 + 24 - 122 n) \delta_X \delta_{G_X}^3 - 72 n (n^2 - 3n + 6) \delta_{G_X}^4 \mu^2, \quad (172)$$
where, in order to derive this result, we have also included the terms of order $O(x^{3/2})$ not shown in Eq. (171). Again the $\pm$ signs in the term involving $\mu$ in Eq. (172) are compatible with those in the expression for $\phi$ in terms of $\delta_X$. The tensor-to-scalar ratio (32) and the tensor index (31) are approximately given by

$$r \simeq 16 \frac{(\delta_X + 4\delta_{GX})^{3/2}}{(\delta_X + 6n\delta_{GX})^{1/2}},$$

$$n_t \simeq -2(\delta_X + 3\delta_{GX}),$$

and the scalar propagation speed squared is

$$c_s^2 \simeq \frac{\delta_X + 4\delta_{GX}}{\delta_X + 6n\delta_{GX}}.$$ (175)

If $|\delta_{GX}| \ll \delta_X$, then these observables reduce to

$$n_s - 1 \simeq -\frac{2(p + 2)}{p} \delta_X \pm 3\sqrt{2n\mu\delta_{GX}} \sqrt{\delta_X},$$

$$r \simeq 16\delta_X \simeq -8n_t.$$ (176)

On the other hand, in the limit where $\delta_{GX} \gg \delta_X$, one has

$$n_s - 1 \simeq -\frac{3(3np + 2n + 1)}{pn} \delta_{GX} \pm \frac{\mu}{\sqrt{2n}} \sqrt{\delta_X},$$

$$r \simeq \frac{64}{3} \frac{\sqrt{6}}{n} \delta_{GX} \simeq -\frac{32}{9} \frac{\sqrt{6}}{n} n_t \simeq -\frac{8.7}{\sqrt{n}} n_t,$$ (179)

which agree with the results in Ref. [30] derived for $n = 1$ and $\mu = 0$.

To be concrete, in the following discussion we focus on the theories with $n = 1$, $\mu \neq 0$, and $\theta = -1$. Then $\delta_{GX} > 0$ for $\phi < 0$, so that the conditions for the avoidance of ghosts and Laplacian instabilities ($Q > 0$ and $c_s^2 > 0$) are always satisfied. In this case we need to take the minus sign for the term $\mu$ in Eqs. (172), (176), and (178). Since $V_\phi \simeq -3H\phi(1 + 3H\dot{\phi})$ from Eq. (181), the field velocity corresponding to $\dot{\phi} < 0$ is

$$\dot{\phi} \simeq \frac{\sqrt{1 - 4V_\phi - 1}}{6H},$$ (180)

where we used $\Phi < 0$. Employing the approximate relation $V \simeq 3H^2M_{pl}^2$, the two slow-roll parameters $\delta_X$ and $\delta_{GX}$ can be expressed in terms of $\delta_X$ as

$$\delta_X \simeq \frac{M_{pl}^2(\sqrt{1 - 4V_\phi - 1})^2}{8V^2\phi^2}, \quad \delta_{GX} \simeq \frac{\delta_X}{6(\sqrt{1 - 4V_\phi - 1})}.$$ (181)

The number of e-foldings is given by

$$N = \int_{\phi_f}^{\phi_i} \frac{H}{\phi} d\phi \simeq \frac{2}{M_{pl}^2} \int_{\phi_f}^{\phi_i} \frac{\Phi V}{\sqrt{1 - 4V_\phi - 1}} d\phi = 2B^4 \int_{x_f}^{x_i} \frac{x^p e^{\mu x}}{\sqrt{1 + 4B^4p x^p e^{\mu x} - 1}} dx,$$ (182)

where

$$B \equiv \left(\frac{V_0}{M^4 M_{pl}}\right)^{1/4}, \quad x \equiv \frac{\phi}{M_{pl}}, \quad x_f \equiv \frac{\phi_f}{M_{pl}}.$$ (183)

We determine the value of $x_f$ at the end of inflation using the condition $\epsilon \simeq \delta_X + 3\delta_{GX} = 1$.

In the limit $B \rightarrow 0$ (i.e. $\delta_{GX} \rightarrow 0$) we have $\epsilon \simeq \delta_X \simeq p^2/(2x^2)$ and $N \simeq x^2/(2p) - p/4$, so that Eq. (177) gives

$$n_s \simeq 1 - \frac{2(p + 2)}{4N + p}, \quad r \simeq \frac{16p}{4N + p} \simeq -8n_t.$$ (184)

In Eq. (184) we have not taken into account the contributions coming from the term $\mu$, because we do not have analytic an expression for general $p$. Numerical calculations show that in the regime $B \ll 1$ both $n_s$ and $r$ become smaller for $\mu > 0$. If $\mu < 0$, then $n_s$ get smaller, whereas $r$ increases.
Figure 5: Theoretical values of \( n_s \) and \( r \) for the potential \( V(\phi) = m^2 \phi^2/2 \) in the presence of the Galileon-type coupling \( G = -(1/M^3)e^{\mu \phi/M^p}X \) with \( \mu = 1 \) (solid line). The points correspond to the cases with \( B = 0, 10^{-5/2}, 10^{-9/4}, 10^{-2}, 10^{-7/4}, 10^{-3/2}, 10^{-5/4}, 0.1, 10^{-1/2}, 10, 10^{3/2}, 10^2 \) with \( N = 55 \). In the limit \( B \to \infty \) one has \( n_s = 0.9675 \) and \( r = 0.1258 \). The dotted curve corresponds to the case where \( \mu = 0 \). We also show the 1\( \sigma \) and 2\( \sigma \) observational contours derived by the joint data analysis of WMAP 7 yr, BAO, and HST with the consistency relation \( r = -8n_t \).

Figure 6: Similar to Fig. 5 but for the potential \( V(\phi) = \lambda \phi^4/4 \) with \( \mu = 1 \) (solid line). The points correspond to the cases with \( B = 0, 10^{-9/2}, 10^{-17/4}, 10^{-3}, 10^{-15/4}, 10^{-7/4}, 10^{-13/4}, 10^{-3/2}, 10^{-11/4}, 10^{-5/2}, 10^{-2}, 10^{-3/4}, 10^{-1/2}, 10^{1/2}, 10^{3/4}, 10^{7/2}, 10^3 \) with \( N = 55 \). In the limit where \( B \to \infty \) one has \( n_s = 0.9614 \) and \( r = 0.1791 \). The dotted curve corresponds to the case where \( \mu = 0 \). The same observational contours as those in Fig. 5 are also plotted.
In the opposite limit where $B \gg 1$ it follows that

$$\epsilon \simeq 3\delta_{GX} \simeq \frac{p^{3/2}}{2B^2}x^{-(p+3)/2}e^{-\mu x/2}, \qquad N \simeq \frac{B^2}{\sqrt{p}} \int_{x_1}^{x} x^{(p+1)/2}e^{\mu x/2}dx,$$

(185)

and $\delta_X \simeq px^{-(p+1)}e^{-\mu x}/(2B^4)$. In order to have $N \approx 55$ for $B \gg 1$ the integral inside the expression of $N$ needs to be much smaller than 1, so that $x < 1$ for $|\mu| = O(1)$. Using the approximation $|\mu x| \ll 1$, we have $x_f^{(p+3)/2} \simeq p^{3/2}/(2B^2)$ and

$$N \simeq \frac{B^2}{\sqrt{p}} \frac{2}{p+3} x^{(p+3)/2} \left[1 + \frac{p + 3}{2(p+5)} \mu x\right] - \frac{p}{p+3},$$

(186)

From Eqs. (178) and (179) it follows that

$$n_s \simeq 1 - \frac{3(p+1)}{(p+3)N + p} \left[1 - \frac{2(p-1)}{3(p+1)(p+5)} \mu x\right],$$

(187)

$$r \simeq \frac{64\sqrt{6}}{9} \frac{p}{(p+3)N + p} \left(1 - \frac{\mu x}{p+5}\right).$$

(188)

For $N = 55$, in the limit where $\mu \to 0$, one has $n_s = 0.9675$, $r = 0.1258$ for $p = 2$ and $n_s = 0.9614$, $r = 0.1791$ for $p = 4$. In the regime $B \gg 1$ the current observations can be consistent with both models. In the presence of the exponential coupling with positive $\mu$ the scalar spectral index gets larger for $p > 1$, while the tensor-to-scalar ratio is smaller.

In the intermediate regime between $B < 1$ and $B \gg 1$ we evaluate $n_s$ and $r$ as follows. For given values of $p$, $\mu$, and $B$ we identify the field value $x = \phi/M_\mu$ corresponding to $N = 55$ by integrating Eq. (182) numerically. We derive $\delta_X$ and $\delta_{GX}$ from Eq. (181) which allows us to obtain $n_s$ and $r$ by using the formulas (172) and (173). We have also solved the background equations numerically to the end of inflation and confirmed that the above method provides accurate estimation for $n_s$ and $r$.

The theoretical values of $n_s$ and $r$ for $\mu = 1$ are plotted in Figs. 5 and 6 (corresponding to $p = 2$ and $p = 4$, respectively) with several different values of $B$. If we choose larger $B$ starting from $B = 0$, $n_s$ decreases up to some value of $B$, starts to increase, and finally decreases towards the point given by Eq. (187). Meanwhile $r$ decreases up to some value of $B$ with a minimum smaller than 0.1, before starting to increase towards the asymptotic value $\delta_{int}$. The above peculiar curved trajectories in the $(n_s, r)$ plane occur because of the presence of the exponential Galileon coupling with $\mu > 0$. For $\mu = 0$ the theoretical curve can be approximated by a line that connects the two asymptotic points corresponding to $B \to 0$ and $B \to \infty$, see Figs. 5 and 6.

If $\mu < 0$ and $B$ is increasing, then $r$ increases up to some value of $B$, whereas $n_s$ decreases. The maximum values of $r$ for $\mu = -1$ are about 0.35 and 0.68 for $p = 2$ and $p = 4$, respectively. If $B$ is increased further, $r$ starts to decrease towards the point given by Eq. (188) (with $n_s$ starting to increase at some value of $B$). Compared to the case $\mu > 0$ this behaviour is not desirable to satisfy the observational bounds, especially for $p = 4$. In the following discussion we shall therefore focus on the case of the positive $\mu$.

From Eqs. (173) and (174) one has

$$\frac{r}{n_t} = -8 \frac{(1 + 4R_G)^{3/2}}{(1 + 6R_G)^{1/2}(1 + 3R_G)},$$

(189)

where $R_G \equiv \delta_{GX}/\delta_X$. For $0 \leq R_G < \infty$ the ratio $r/n_t$ is constrained to be in the narrow range $-8.71 < r/n_t \leq -8$. We carry out the CMB likelihood analysis in terms of $n_s$ and $r$ by using the two consistency relations $r = -8n_t$ and $r = -8.71n_t$. We find that the observational constraints on $n_s$ and $r$ are similar in both cases. Hence the constraints using the standard consistency relation $r = -8n_s$ should be trustable even in the intermediate regime. Figure 5 shows that the quadratic inflaton potential is consistent with observations even in the presence of the exponential Galileon coupling with $\mu = 1$. From Fig. 6 we find that the self coupling inflaton potential can also be saved by taking into account the exponential Galileon coupling.

For the theoretical points shown in Figs. 5 and 6 we can evaluate the values of $\delta_X$ and $\delta_{GX}$ corresponding to $N = 55$. It is then possible to derive fitting functions that relate $\delta_{GX}$ with $\delta_X$. The fitting function for $p = 4$ and $\mu = 1$ is given in Eq. (A1) in the Appendix. This allows us to run the CosmoMC code in terms of one inflationary parameter $\delta_X$. In Fig. 7 we show the 1-dimensional marginalized probability distribution for $p = 4$ and $\mu = 1$ constrained by the joint data analysis of WMAP 7 yr, LSS, HST, SN Ia, and BBN. In the absence of the Galileon coupling ($\delta_{GX} = 0$) one has $\delta_X = p/(4N + p) \simeq 0.018$ for $N = 55$, which is observationally excluded. In the opposite limit of large
Galileon coupling such that $\delta_{G,X} \gg \delta_X$, it follows that $\delta_{G,X} \approx p/\{3[(p + 3)N + p]\} = 3.4 \times 10^{-3}$ for $N = 55$. Since this case is marginally inside the $2\sigma$ observational contour in Fig. 6, we find a suppressed probability distribution for smaller $\delta_X$ in Fig. 7. The intermediate regime such as $10^{-4} \lesssim \delta_X \lesssim 10^{-3}$ is most favored observationally, because the corresponding theoretical points can be deep inside the $2\sigma$ bound in Fig. 6. In Fig. 8 the theoretical point for $B = 10^{-3/2}$ gives $\delta_X = 3.5 \times 10^{-4}$, which actually corresponds to the highest probability in Fig. 7. Hence the effect of the exponential Galileon coupling works to save the self-coupling inflaton potential.

The scalar spectrum $P_s$ at the scale $k = 0.002$ Mpc$^{-1}$ (for $n = 1$) is subject to the WMAP normalization:

$$P_s = \frac{\sqrt{3}}{\pi^2} \left( \frac{M_{pl}}{M} \right)^6 \left( \frac{V_0}{M_{pl}^4} \right)^3 \frac{y^{1/2} x^p e^{2\mu x}}{(y - 1)^2 (2y + 1)^{3/2}} \approx 2.4 \times 10^{-9},$$

where $y \equiv (1 + 4pB^4x^{p-1}e^{\mu x})^{1/2}$. In the limit that $B \gg 1$ we obtain $m \approx 10^{16}(10^{12} \text{GeV}/M)$ GeV for $p = 2$ (with $V_0 = m^2 M_{pl}^2/2$) and $\lambda \approx (10^{12} \text{GeV}/M)^4$ for $p = 4$ (with $V_0 = \lambda M_{pl}^4/4$), which agree with those obtained in Ref. [30] for $\mu = 0$. In order to have $B \gg 1$ for $p = 4$, we require that $\lambda \gg (M/M_{pl})^3$. Combining this with the WMAP normalization, the mass scale $M$ is constrained to be $M \ll 10^{-5} M_{pl}$. If we demand that the coupling $\lambda$ is smaller than 1, this gives another constraint $M > 4 \times 10^{-7} M_{pl}$. In the intermediate regime between $B \gg 1$ and $B \ll 1$ we need to solve Eq. (190) to relate $M$ and $V_0$ after identifying the values of $x$ at $N = 55$ numerically. In the regime $B \ll 1$ we recover the standard mass scales of inflaton: $m/M_{pl} \approx 6.8 \times 10^{-6}$ for $p = 2$ and $\lambda \approx 2.0 \times 10^{-13}$ for $p = 4$.

Since we regard $M$ to be a cutoff scale for the function $G(\phi, X)$, the effective theory can be trusted as long as $H \lesssim M$. This relation yields the constraint $B^4 x^p \lesssim M_{pl}/M$. For the case $B \gg 1$ and $p = 2$, we find that the effective theory can be trusted for $x \lesssim M/m \approx (M/10^{14} \text{GeV})^2$. For $B \gg 1$ and $p = 4$, this constraint reduces to $x \lesssim \lambda^{-1/4} (M/M_{pl})^{1/2} \approx (M/10^{14} \text{GeV})^{3/2}$.

Finally we note that the scalar propagation speed squared (175), in the regime $\delta_{G,X} \gg \delta_X$, reduces to $c_s^2 \approx 2/(3n)$. Also, although the non-Gaussianity parameter $f_{NL}^{\text{equil}}$ is constrained to be small for $n = 1$, it is possible to have $|f_{NL}^{\text{equil}}| \gg 1$ for $n \gg 1$. It would be of interest to see whether or not such models can be compatible with observations.

IX. CONCLUSIONS

In this paper we have studied the observational signatures of chaotic inflationary models with the potential $V(\phi) = V_0(\phi/M_{pl})^p$ in the context of modified gravitational theories. In Einstein gravity the self-coupling potential $V(\phi) = \lambda \phi^4/4$ is excluded by CMB temperature anisotropy data, while the quadratic potential $V(\phi) = m^2 \phi^2/2$ is within the
2σ observational contour. Our main aim here has been to clarify how various field couplings present in low-energy effective string theory modify the scalar/tensor power spectra generated during inflation.

We have found a number of new results summarized below.

- (i) The inclusion of a non-canonical kinetic term $\omega(\phi)X$ with the exponential coupling $\omega(\phi) = e^{\mu \phi/M_{\text{pl}}} (\mu > 0)$ allows the chaotic inflation models that are in tension with observations to be made compatible with them. We have studied the effects of the non-canonical kinetic coupling as well as the nonminimal coupling on the inflationary observables and have placed bounds on the strength of the couplings by using recent data from WMAP 7yr, BAO, and HST.

- (ii) In Brans-Dicke theory we have found that the field potential of the form $V(\phi) = V_0(\phi - M_{\text{pl}})^p$, where $p$ is close to 2, can be viable for inflation followed by a successful reheating. We have evaluated the scalar index $n_s$ and the tensor-to-scalar ratio $r$ for the potential $V(\phi) = V_0(\phi - M_{\text{pl}})^2$ and have shown that $r$ decreases for smaller values of the BD parameter $\omega_{\text{BD}}$. The models where $\omega_{\text{BD}}$ is around the order of unity, which includes the $f(R) = R + R^2/(6M^2)$ model, is well within the 1σ observational bound constrained by WMAP 7yr, BAO, and HST.

- (iii) In the presence of the Gauss-Bonnet coupling of the form $\xi(\phi)\mathcal{G}$, where $\xi(\phi) = \xi_0 e^{\mu \phi/M_{\text{pl}}}$, we have found that the GB coupling with positive $\mu$ does not save the self-coupling potential $V(\phi) = \lambda \phi^4/4$. For the quadratic potential $V(\phi) = m^2 \phi^2/2$ we have shown that the GB coupling needs to be suppressed ($|\delta_{GB}/\epsilon_s| < 0.1$) from the CMB likelihood analysis. If $\mu$ is negative then it is possible to lead to the decrease of both $|n_s - 1|$ and $r$ for negative $\xi_0$, but we require a large coupling constant, such as $|\xi_0| \sim 10^{10}$, in order to produce a sizable effect on the inflationary observables.

- (iv) In the presence of the Galileon-like self-interaction $G(\phi, X) \Box \phi$, where $G(\phi, X) = \Phi(\phi) X^n$ and $\Phi \propto e^{\mu \phi/M_{\text{pl}}}$, we have expressed the three inflationary observables $n_s$, $r$, and $n_t$ in terms of two slow-roll parameters $\delta_X$ and $\delta_{GX}$. In the regime where the Galileon term dominates over the standard kinetic term ($\delta_{GX} \gg \delta_X$) we have derived analytic formulas for $n_s$ and $r$ in terms of the number of e-foldings $N$, which recover the results obtained for $\mu = 0$. We have shown that, for $\mu > 0$, the Galileon term can lead to the compatibility of the chaotic inflationary potentials with current observations. We have confirmed this property for the potential $V(\phi) = \lambda \phi^4/4$ by carrying out the CMB likelihood analysis.

In summary, we have undertaken a unified study of the effects of a number of generalisations to the standard inflationary picture as motivated by low-energy effective string theory. We have found that a number of chaotic inflationary models which are in tension with observations can be made compatible with them through the addition of such terms. The stronger constraints on $n_s$ and $r$ expected from the PLANCK satellite will provide an opportunity to further test the viability of such scenarios.

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Appendix A: Fitting function for G-inflation with an exponential coupling

In this Appendix we present the fitting function for the quartic potential $V(\phi) = \lambda \phi^4/4$ in the presence of the Galileon-type coupling $G = -(1/M^3) e^{\mu \phi/M_{\text{pl}}} X$ with $\mu = 1$. Numerically we find the field value $\phi$ giving $N = 55$ before the end of inflation and evaluate $\delta_X$ and $\delta_{GX}$ for several different values of $B$ ($B = 10^{i/8}$ with $i = -32, \ldots, 32$). These slow-roll parameters can be approximated by the following fitting function (found by using the method of least
Figure 8: Numerical data points corresponding to the values of $\delta_{GX}$ and $\delta_X$ satisfying the constraint $N = 55$. Each data point corresponds to a particular value $B = 10^{i/8}$ with $i = -32, \ldots, 32$. The fitting function used for the CMB likelihood analysis is also plotted.

\[
\delta_{GX} = -5.25192634579698 + 540.21080897015 \delta_X^{1/2} - 350.55978587371 \delta_X^{1/3} + 15290.159752272 \delta_X^{1/4} \\
-38509.952672544 \delta_X^{1/5} + 53949.4042466374 \delta_X^{1/6} - 38908.1718682253 \delta_X^{1/7} \\
+11232.4296410833 \delta_X^{1/8} - 224.28358682764 \delta_X + 6155.30047533836 \delta_X^5 \\
-519243.629001884 \delta_X + 38227861.764318 \delta_X^4 - 1897688289.07932 \delta_X^3 + 54200448383.7942 \delta_X^2 \\
-665839723646.196 \delta_X^{1/4}. \tag{A1}
\]

We have used this expression in the regime $10^{-8} < \delta_X < 0.018$ for the CMB likelihood analysis in Fig. 7. Finally, in Fig. 8 we show both the numerical data and the fitting function $\delta_{GX} = \delta_{GX}(\delta_X)$. Since its inverse function, on the whole interval, is multivalued, we have used $\delta_X$ as the independent slow-roll parameter.

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