KISSING NUMBER IN HYPERBOLIC SPACE

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Abstract. This paper provides upper and lower bounds on the kissing number of congruent radius $r > 0$ spheres in $\mathbb{H}^n$, for $n \geq 2$. For that purpose, the kissing number is replaced by the kissing function $\kappa(n, r)$ which depends on the radius $r$. After we obtain some theoretical lower and upper bounds for $\kappa(n, r)$, we study their asymptotic behaviour and show, in particular, that $\lim_{r \to \infty} \frac{\log \kappa(n, r)}{r} = n - 1$. Finally, we compare them with the numeric upper bounds obtained by solving a suitable semidefinite program.

Key words: hyperbolic geometry, kissing number, semidefinite programming.

2010 AMS Classification: Primary: 05B40. Secondary: 52C17, 51M09.

1. Introduction

The kissing number $\kappa(n)$ is the maximal number of unit spheres that can simultaneously touch a central unit sphere in $n$-dimensional Euclidean space $\mathbb{R}^n$ without pairwise overlapping. The research on the kissing number leads back to 1694, when Isaac Newton and David Gregory had a discussion whether $\kappa(3)$ is equal to 12 or 13 [3].

The exact value of $\kappa(n)$ is only known for $n = 1, 2, 3, 4, 8, 24$, whereas for $n = 1, 2$ the problem is trivial. In 1953, Schütte and van der Waerden proved that $\kappa(3) = 12$ [18]. Furthermore, Delsarte, Goethals, and Seidel [4] developed a linear programming (LP) bound, which was used by Odlyzko and Sloan [15] and independently by Levenshtein [11] to prove $\kappa(8) = 240$ and $\kappa(24) = 196560$. Later, Musin [14] showed that $\kappa(4) = 24$ by using a stronger version of the LP bound. Bachoc and Vallentin [1] strengthened the LP bound further by using a semidefinite program (SDP), which is a generalisation of a linear program. In [13], Mittelmann and Vallentin give a table with the best upper bounds for the kissing number for $n \leq 24$ by using SDP bounds. Moreover, Machado and Oliveira [12] improved some of these results, by exploiting polynomial symmetry in the SDP.

The study of the Euclidean kissing number is still an active area in geometry and optimisation. In this paper, we consider an analogous problem in $n$-dimensional hyperbolic space $\mathbb{H}^n$. In a kissing configuration every sphere in $\mathbb{H}^n$ has the same radius $r$. Unlike in Euclidean spaces, the kissing number in $\mathbb{H}^n$ depends on the radius $r$, and we denote it by $\kappa(n, r)$. By using a purely Euclidean picture of the arrangement of $k$ spheres of radius $r$ in $\mathbb{H}^n$ seen in the Poincaré ball model, the kissing number in $\mathbb{H}^n$ coincides with the maximal number of spheres of radius $\frac{1}{2} \left( \tanh \frac{r}{2} - \tanh \frac{3}{2} \right)$ that can simultaneously touch a central sphere of radius $\tanh \frac{r}{2}$ in $\mathbb{R}^n$ without pairwise intersecting.

In Section 2 we determine upper and lower bounds for $\kappa(n, r)$ in order to evaluate its asymptotic behaviour for large values of the respective parameters. In Section 3 we adapt the SDP of Bachoc and Vallentin to obtain upper bounds for $\kappa(n, r)$. Moreover, for $r = 0.5, 0.9, 1.1$ concrete kissing configurations in $\mathbb{H}^3$ are given in Section 4. For certain dimensions and certain radii, we compute lower and upper bounds by using the results of Section 2 and compare those results with upper bounds given by the SDP and lower bounds given by concrete kissing configurations.
Acknowledgements

A. K. was supported by the Swiss National Science Foundation, project no. PP00P2-170560. The idea of the project was conceived during the workshop “Discrete geometry and automorphic forms” at the American Institute of Mathematics (AIM) in September, 2018. The authors are grateful to the workshop organisers, Henry Cohn and Maryna Viazovska, all its participants and the AIM administration for creating a unique and stimulating research environment. They would also like to thank Matthew de Courcy–Ireland (EPF Lausanne, Switzerland) for inspiring discussions.

2. Estimating the kissing number

In this section we shall produce some upper and lower bounds for the kissing function $\kappa(n, r)$, so that we can analyse its asymptotic behaviour in the dimension $n$, and in the radius $r$, for large values of the respective parameters. We refer the reader to [16, §2.1] and [16, §4.5] for all the necessary elementary facts about spherical and hyperbolic geometry, and their models.

2.1. Upper bound. First we prove the following upper bound for $\kappa(n, r)$.

**Theorem 2.1.** For any integer $n \geq 2$ and a non-negative number $r \geq 0$, we have that

$$\kappa(n, r) \leq \frac{2B\left(\frac{n-1}{2}, \frac{1}{3}\right)}{B\left(\frac{\text{sech}^2 r}{4}; \frac{n-1}{2}, \frac{1}{2}\right)},$$

where $B(x; y, z) = \int_0^x t^{y-1}(1-t)^{z-1}dt$ is the incomplete beta-function, and $B(y, z) = B(1; y, z)$, for all $x \in [0, 1]$, and $y, z > 0$.

**Proof.** In the proof we shall use a purely Euclidean picture of the arrangement of $k = \kappa(n, r)$ spheres in the hyperbolic space $\mathbb{H}^n$ seen in the Poincaré ball model, c.f. [16, §4.5]. A part of the kissing configuration looks (through the Euclidean eyes) as depicted in Fig. 1. Let $S_0$ be the “large” sphere centred at $O$ of Euclidean radius $r_1 = \tanh \frac{r}{2}$ that is in the kissing configuration with $k$ “small” spheres $S_1, \ldots, S_k$ of equal Euclidean radii $r_2 = \frac{1}{2} \left(\tanh \frac{3r}{2} - \tanh \frac{r}{2}\right)$.

**Figure 1.** A configuration of kissing hyperbolic spheres viewed as Euclidean ones.
Indeed, the Euclidean distance from the centre $O$ of the ball model to any point at hyperbolic distance $r$ from $O$ equals $\tanh \frac{r}{2}$. If $O_i$ is the Euclidean centre of any of the “small” spheres $S_i$, $i = 1, \ldots, k$, then the hyperbolic distance between $O$ and the point of tangency $L_i$ between $S_0$ and $S_i$ is $r$, and it corresponds to the Euclidean radius of $S_0$, which equals $r_1$. The hyperbolic distance between $O$ and the point $R_i$ on the diameter of $S_i$ opposite to $L_i$ is $r + 2r = 3r$, while the corresponding Euclidean distance is $r_1 + 2r_2$. Thus, we obtain $r_1 = \tanh \frac{r}{2}$, and $r_1 + 2r_2 = \tanh \frac{3r}{2}$, from which the above values of $r_1$ and $r_2$ can be found.

Choose one of the small spheres $S_i$, $i = 1, \ldots, k$, with Euclidean centre $O_i$, and project it onto $S_0$ along the rays emanating from $O$: such a projection will be a spherical cap $C_i$ on $S_0$ of angular radius $\theta$ between a tangent line $OT_i$ to $S_i$ (with $T_i$ the point of tangency to $S_i$) and the line $OO_i$ connecting the centres of $S_0$ and $S_i$ (so that $OO_iT_i$ is a right triangle, with $\angle OT_iO_i = \frac{\pi}{2}$). The volume of such a cap is given by the formula

$$\text{Area } C_i = \frac{1}{2} \cdot \text{Area } S_0 \cdot \frac{B \left( \sin^2 \theta; \frac{n-1}{2}, \frac{1}{2} \right)}{B \left( \frac{n-1}{2}, \frac{1}{2} \right)}.$$

If we consider $S_0$ with the standard angular metric, then $\theta$ equals the radius of $C_i$, which can be interpreted as a spherical circle, i.e. the set of all points at angular distance $\theta$ from the centre $O_i = OO_i \cap S_0$.

Since the points $O$, $O_i$ and $T_i$ are situated in a plane, a standard trigonometric computation yields (see Fig. 2)

$$\sin \theta = \frac{r_2}{r_1 + r_2},$$

where $r_1 = \tanh \frac{r}{2}$ and $r_2 = \frac{1}{2} \left( \tanh \frac{3r}{2} - \tanh \frac{r}{2} \right)$, as given above. By substituting all of these quantities and simplifying the resulting formula, we obtain

$$\sin \theta = \frac{\text{sech } r}{2}.$$ 

Since the caps $C_i$ are mutually congruent and they pack the sphere $S_0$, we have that

$$\kappa(n, r) \cdot \text{Area } C_i = \sum_{i=1}^{k} \text{Area } C_i \leq \text{Area } S_0,$$

and the theorem follows. \hfill \square

2.2. Lower bound. Before providing a lower bound for $\kappa(n, r)$, let us first formulate two simple but crucial observations. The first one concerns a relation between packing and covering of the $n$-dimensional unit sphere $S^n$ by closed metric balls. A packing of $S^n$ by closed metric balls of angular radius $r > 0$ with non-intersecting interiors is called maximal if it cannot be enlarged by adding more such balls without overlapping their interiors.

**Lemma 2.2.** Let $S^n$ be packed by closed metric balls $B_i$, $i = 1, 2, \ldots$, of equal (angular) radius $r$, and let such packing be maximal. Then $S^n$ is covered by closed metric balls $B_i'$, $i = 1, 2, \ldots$, concentric to $B_i$, of radius $2r$.

**Proof.** Let $x_i$ be the centre of $B_i$, $i = 1, 2, \ldots$. For $x \in X$ let $\rho(x)$ be the shortest angular distance from $x$ to the set $\cup_i B_i$. Since $B_i$’s form a maximal packing, we have that $\rho(x) < r$. \hfill 1

The hyperbolic centre of $S_i$ appears displaced towards the boundary of the ball model, as compared to $O_i$, and this is the reason for performing our computation of $r_1$ and $r_2$ below in a slightly peculiar way.
Figure 2. The section by the \( OO_iT_i \) plane of a three spheres configuration.

Thus, \( x \) is at distance \( \rho(x) \) from some \( B_j \), which implies that \( x \) is at distance \( \rho(x) + r < 2r \) from its centre \( x_j \). The latter means that \( x \in B'_j \). □

Lemma 2.3. The packing of \( S_0 \) by the spherical caps \( C_i, i = 1, 2, \ldots, k \), from Theorem 2.1 is maximal, if \( k = \kappa(n,r) \).

Proof. Let us consider two small spheres \( S_i \) and \( S_j \) tangent to \( S_0 \), and let \( C_i \) and \( C_j \) be the corresponding caps on \( S_0 \). Let \( P \) be a two-dimensional plane through \( O, O_i, \) and \( O_j \). Let \( s_0 = S_0 \cap P \) be a circle of radius \( r_1 \) in the plane \( P \), while \( s_i = S_i \cap P \) and \( s_j = S_j \cap P \) be the two outer circles of radius \( r_2 \) tangent to \( s_0 \). Let also \( c_i = C_i \cap P \) and \( c_j = C_j \cap P \). Observe that \( s_i \) and \( s_j \) intersect if and only if \( c_i \) and \( c_j \) intersect, for all \( 1 \leq i, j \leq k \). This is equivalent to \( S_i \) intersecting \( S_j \) if and only if \( C_i \) intersects \( C_j \).

After placing an additional cap \( C_{k+1} \) (congruent to any of the already existing \( C_i \)'s) on \( S_0 \), we can create a sphere \( S_{k+1} \) (congruent to any of \( S_i \)'s) producing \( C_i \) as its central projection onto \( S_0 \). By assumption, \( k = \kappa(n,r) \), and thus there exists \( S_i \) such that \( S_{k+1} \) and \( S_i \) intersect. Then, \( C_{k+1} \) and \( C_i \) also intersect, and thus the packing of \( S_0 \) by \( C_i \)'s is indeed maximal. □

Now we can formulate and prove a lower bound for \( \kappa(n,r) \).

Theorem 2.4. For any integer \( n \geq 2 \) and a non-negative number \( r \geq 0 \), we have that

\[
\kappa(n,r) \geq \frac{2 B \left( \frac{n-1}{2}, \frac{1}{2} \right)}{B \left( \text{sech}^2 r - \frac{\text{sech}^4 r}{2}; \frac{n-1}{2}, \frac{1}{2} \right)},
\]

where \( B(x; y, z) = \int_0^x t^{y-1}(1-t)^{z-1} dt \) is the incomplete beta-function, and \( B(y, z) = B(1; y, z) \), for all \( x \in [0, 1] \), and \( y, z > 0 \).

Proof. By Lemma 2.3, the packing of \( S_0 \) by \( k = \kappa(n,r) \) spherical caps \( C_i \) produced in the proof of Theorem 2.1 is maximal. Let \( C'_i \) be a spherical cap concentric to \( C_i \) of angular radius \( 2\theta \). Since \( X = S_0 \) with angular metric on it can be thought of as a rescaled unit sphere \( S^{n-1} \), in
which each $C_i$ is a closed metric ball of radius $\theta$, and each $C'_i$ is a closed metric ball of radius $2\theta$, we obtain that $C'_i$’s cover $S_0$ by Lemma 2.2.

Then,

$$\kappa(n, r) \cdot \text{Area } C'_i = \sum_{i=1}^{k} \text{Area } C'_i \geq \text{Area } S_0,$$

where

$$\text{Area } C'_i = \frac{1}{2} \cdot \text{Area } S_0 \cdot \frac{B \left( \sin^2(2\theta); \frac{n-1}{2}, \frac{1}{2} \right)}{B \left( \frac{n-1}{2}, \frac{1}{2} \right)},$$

with $\theta$ such that

$$\sin \theta = \frac{\text{sech } r}{2}.$$

By using the formula $\sin(2\theta) = 2\sin \theta \cos \theta$, one can express $\sin^2(2\theta)$ through $\sin^2 \theta$, and the theorem follows.

□

2.3. Euclidean kissing numbers. The estimates of Theorems 2.1 and 2.4 can be easily adapted to the case of the Euclidean kissing number, i.e. the kissing number of spheres of unit radii in $\mathbb{R}^n$, for $n \geq 2$. To this end, one needs just to set $\theta = \frac{\pi}{3}$ and $r = 0$ in the preceding argument.

**Theorem 2.5.** For the kissing number $\kappa(n)$ of unit radius spheres in $\mathbb{R}^n$, we have that

$$\frac{2 B \left( \frac{n-1}{2}, \frac{1}{2} \right)}{B \left( \frac{3}{4}, \frac{n-1}{2}, \frac{1}{2} \right)} \leq \kappa(n) \leq \frac{2 B \left( \frac{n-1}{2}, \frac{1}{2} \right)}{B \left( \frac{1}{4}, \frac{n-1}{2}, \frac{1}{2} \right)}.$$

The upper bound in the above theorem is indeed identical to the one obtained by Glazyrin [8, Theorem 6] albeit in a different context. Namely, the work [8] studies the average degree of contact graphs for kissing spheres having arbitrary radii. It is worth mentioning that in this case we shall have exactly the same estimates for hyperbolic and Euclidean spaces, since by allowing varying radii we effectively avoid the ambient metric influencing the combinatorics of kissing configurations.

2.4. Asymptotic behaviour. First of all, we establish the following fact, which can be expressed simply by saying that the kissing number $\kappa(n, r)$ in $\mathbb{H}^n$, for any fixed dimension $n \geq 2$, grows exponentially fast with the radius $r$.

**Theorem 2.6.** Let $n \geq 2$ be a fixed natural number, then

$$\frac{n-1}{2^{n-1}} B \left( \frac{n-1}{2}, \frac{1}{2} \right) e^{(n-1)r} \lesssim \kappa(n, r) \lesssim (n-1) B \left( \frac{n-1}{2}, \frac{1}{2} \right) e^{(n-1)r},$$

for $r \to \infty$.

**Proof.** By using the series expansion of the integrand in the definition of $B(x; \frac{n-1}{2}, \frac{1}{2})$ around $x = 0$ and integrating term-wise, then inverting the series, we obtain:

$$\frac{1}{B(x; \frac{n-1}{2}, \frac{1}{2})} = \frac{n-1}{2} \cdot x^{-\frac{1}{2}} \cdot \left( 1 + O(x) \right).$$

Then, using the fact that $\text{sech } r \sim 2e^{-r}$, as $r \to \infty$, we use the above expansion together with Theorems 2.1 and 2.4 in order to obtain the desired asymptotic inequalities.

□

2This is true since spheres in $\mathbb{H}^n$ (given by Poincaré’s ball model) can be also described as spheres in $\mathbb{R}^n$ (i.e. by quadratic equations), and vice versa.
An easy corollary of Theorem 2.6 is the following.

**Corollary 2.7.** For the kissing number $\kappa(n,r)$ of radius $r$ spheres in $\mathbb{H}^n$, $n \geq 2$, we have that
$$\lim_{r \to \infty} \frac{\log \kappa(n,r)}{r} = n - 1.$$

Note, that an exponential lower bound for the kissing number $\kappa(2,r)$, as $r \to \infty$, follows readily from the work by Bowen [2]. Also, some asymptotic behaviour for the Euclidean kissing number $\kappa(n)$, as $n \to \infty$, can be deduced from Theorem 2.5.

**Corollary 2.8.** As $n \to \infty$, we have that
$$\sqrt{\pi(n-1)} \left(2\sqrt{3}\right)^{-n^{-2}} \lesssim \kappa(n) \lesssim \sqrt{6\pi(n-1)} 2^{n-2}.$$

**Proof.** From [3, Equation (3.16)], we obtain that
$$B \left( \frac{n-1}{2}, \frac{1}{2} \right) \sim \sqrt{\frac{2\pi}{n-1}},$$
while
$$B \left( \frac{1}{4}; \frac{n-1}{2}, \frac{1}{2} \right) \sim \frac{1}{\sqrt{3}} \cdot (n-1)^{-1} \cdot 2^{-(n-3)},$$
and
$$B \left( \frac{3}{4}; \frac{n-1}{2}, \frac{1}{2} \right) \sim 4 \cdot (n-1)^{-1} \cdot \left(\frac{2}{\sqrt{3}}\right)^{-(n-1)}.$$

The proof follows by substituting the above asymptotic expressions into the inequalities of Theorem 2.5 and easy algebraic simplifications. □

However, the upper asymptotic bound of Corollary 2.8 is much poorer than $2^{0.401444(n+o(1))}$ by Kabatiansky and Levenshtein [10]. The lower bound is basically identical to the one by Wyner [19, §5.1], and is thus poorer than the recent one by Jenssen, Joos, and Perkins [9].

### 3. The semidefinite programming bound

For $x, y \in \mathbb{R}^n$, we denote by $x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n$ their Euclidean inner product, and let $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ be the $(n-1)$-dimensional unit sphere. Furthermore, let the **angular distance** between $x, y \in S^{n-1}$ be $d(x,y) = \arccos(x \cdot y)$, c.f. [16, §2.1].

In order to determine upper bounds for the kissing number, we first consider the more general problem of finding the maximal number of minimal points on the unit sphere with minimal angular distance $\theta$. This problem is defined by

$$A(n, \theta) = \max \{|C| : C \in S^{n-1} \text{ and } d(x,y) \leq \cos \theta \text{ for all distinct } x, y \in C\}.$$

Note that the Euclidean kissing number equals $\kappa(n) = A(n, \pi/3)$.

A set of points $C \in S^{n-1}$ with $d(x,y) \leq \cos \theta$ for all distinct $x, y \in C$ is called a **spherical code with minimal angular distance $\theta$**. Then the kissing number $\kappa(n,r)$ of radius $r$ spheres in $\mathbb{H}^n$ is equal to the cardinality of a maximal spherical code $C \in S^{n-1}$ with $d(x,y) \leq 1 - \frac{1}{1+\cosh(2r)}$ for $x, y \in C$.

**Lemma 3.1.**

$$\kappa(n,r) = \max \{|C| : C \in S^{n-1} \text{ and } d(x,y) \leq 1 - \frac{1}{1+\cosh(2r)} \text{ for all distinct } x, y \in C\}.$$
Proof. Consider an equilateral hyperbolic triangle with side length $2r$, and let $\theta$ be one of its inner angles. By the hyperbolic law of cosines, we obtain
\[
\cos \theta = \frac{\cosh(2r) - \cosh(2r)}{\sinh^2(2r)} = \frac{\cosh(2r) - (\cosh(2r) - 1)}{\cosh^2(2r) - 1} = \frac{\cosh(2r)}{1 + \cosh(2r)} = 1 - \frac{1}{1 + \cosh(2r)}.
\]

\[\square\]

Delsarte, Goethals, and Seidel [4] developed a linear programming bound for $A(n, \theta)$, which was used by Odlyzko and Sloan [13] and independently by Levenshtein [11] to prove $\kappa(8) = 240$ and $\kappa(24) = 196560$. Furthermore, Bachoc and Vallentin [1] strengthened the LP bound further via semidefinite programming (SDP). Since we can also use the SDP method to obtain upper bounds for the kissing number in hyperbolic space, we consider the SDP bound given by Bachoc and Vallentin.

For $n \geq 3$, let $P^n_k(u)$ denote the Jacobi polynomial of degree $k$ and parameters $((n-3)/2, (n-3)/2)$, normalized by $P^n_k(1) = 1$. If $n = 2$, then $P^n_k(u)$ denote the Chebyshev polynomial of the first kind of degree $k$. For a fixed integer $d > 0$, we define $Y^n_k$ to be a $(d-k+1) \times (d-k+1)$ matrix whose entries are polynomials on the variables $u, v, t$ defined by
\[
(Y^n_k)_{i,j}(u, v, t) = P^{n+2k}_i(u)P^{n+2k}_j(v)Q^{n-1}_k(u, v, t),
\]
for $0 \leq i, j \leq d - k$, where
\[
Q^{n-1}_k(u, v, t) = ((1 - u^2)(1 - v^2))^{k/2}P^{n-1}_k \left( \frac{t - uv}{\sqrt{(1 - u^2)(1 - v^2)}} \right).
\]

The symmetric group on three elements $S_3$ acts on a triple $(u, v, t)$ by permuting its components. This induces the action
\[
\sigma p(u, v, t) = p(\sigma^{-1}(u, v, t))
\]
on $\mathbb{R}[u, v, t]$, where $\sigma \in S_3$. By taking the group average of $Y^n_k$, we obtain the matrix
\[
S^n_k(u, v, t) = \frac{1}{6} \sum_{\sigma \in S_3} \sigma Y^n_k(u, v, t),
\]
whose entries are invariant under the action of $S_3$.

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called positive semidefinite if all its eigenvalues are non-negative. We write this as $A \succeq 0$. Furthermore, if all its eigenvalues are strictly positive, then $A$ is called positive definite and we write $A \succ 0$. A semidefinite program (SDP) is an optimisation problem for a linear function over a set of positive semidefinite matrices restricted by linear matrix equalities. For $A, B \in \mathbb{R}^{n \times n}$, let $\langle A, B \rangle = \text{tr}(B^TA)$ be the trace inner product. Also, we define
\[
\Delta = \{(u, v, t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq \cos \theta \text{ and } 1 + 2uv - u^2 - v^2 - t^2 \geq 0\}
\]
and
\[
\Delta_0 = \{(u, u, 1) : -1 \leq u \leq \cos \theta\}.
\]
The triples $(u, v, t) \in \Delta$ are possible inner products between three points in a spherical code in $S^{n-1}$ with minimal angular distance $\theta$. Hence,
\[
(u, v, t) \in \Delta \text{ if and only if } \forall x, y, z \in S^{n-1} : x \cdot y \leq \cos \theta, x \cdot z \leq \cos \theta, y \cdot z \leq \cos \theta, x \cdot y = u, x \cdot z = v, y \cdot z = t.
\]

In [1], Bachoc and Vallentin proved the following theorem, where $J$ denotes the “all 1’s” matrix.
Theorem 3.2. Any feasible solution of the following optimisation program gives an upper bound on $A(n, \theta)$:

\[
\begin{align*}
\min & \quad 1 + \sum_{k=1}^{d} a_k + b_{11} + \langle J, F_0 \rangle, \\
\text{subject to} & \quad a_k \geq 0 \text{ for } k = 1, \ldots, d, \\
& \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \succeq 0, \\
& \quad F_k \in \mathbb{R}^{(d-k+1) \times (d-k+1)} \text{ and } F_k \succeq 0 \text{ for } k = 0, \ldots, d, \\
& \quad (i) \sum_{k=1}^{d} a_k P^u_k(u) + 2b_{12} + b_{22} + 3 \sum_{k=0}^{d} \langle S^n_k(u, u, 1), F_k \rangle \leq -1 \text{ for } (u, u, 1) \in \triangle_0, \\
& \quad (ii) b_{22} + \sum_{k=0}^{d} \langle S^n_k(u, v, t), F_k \rangle \leq 0 \text{ for } (u, v, t) \in \triangle.
\end{align*}
\]

The conditions (i) and (ii) of the previous program are polynomial constraints where we have to check that certain polynomials are non-negative in a given domain. Constraints of this kind can be written as sum-of-squares conditions. A polynomial $p$ is said to be a \textit{sum-of-squares polynomial} if and only if there exists polynomials $q_1, \ldots, q_m$ such that

\[ p = q_1^2 + \cdots + q_m^2. \]

To be a sum-of-squares polynomial is a sufficient condition for non-negativity. Analogously, one can also use sum-of-squares to check non-negativity for a certain domain: if there exists sum-of-squares polynomials $q_1, q_2$ such that

\[ p(x) = q_1(x) + (b - x)(x - a)q_2(x) \]

then $p(x) \geq 0$ for $a \leq x \leq b$. Using sum-of-squares relaxations we can formulate the program in Theorem 3.2 as an SDP.

In order to obtain a finite-dimensional SDP which we can solve in practice, we have to fix the degree of the polynomials we consider for the sum-of-squares conditions. A polynomial $p(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $2d$ can be written as a sum-of-squares if and only if there exists a positive semidefinite matrix $X$ such that

\[ p(x_1, \ldots, x_n) = \langle X, v^d(x_1, \ldots, x_n)v^d(x_1, \ldots, x_n)^T \rangle, \]

where $v^d(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]^{(n+d) \choose d}$ is a vector which contains a basis of the space of real polynomials up to degree $d$. We denote $V^d(x) = v^d(x)v^d(x)^T$.

Hence, we obtain for the above conditions (i) and (ii) the following sum-of-squares relaxations

\[
(i) \sum_{k=1}^{d} a_k P^u_k(u) + 2b_{12} + b_{22} + 3 \sum_{k=0}^{d} \langle S^n_k(u, u, 1), F_k \rangle + 1 + \langle Q_0, V^d(u) \rangle + (u + 1)(\cos \theta - u)\langle Q_1, V^{d-1}(u) \rangle = 0
\]
\[(ii) \quad b_{22} + \sum_{k=0}^{d} \langle S^n_k(u, v, t), F_k \rangle + \langle R, V^d(u, v, t) \rangle + (u + 1)(\cos \theta - u) \langle R_0, V^{d-1}(u, v, t) \rangle \\
+ (v + 1)(\cos \theta - v) \langle R_1, V^{d-1}(u, v, t) \rangle + (t + 1)(\cos \theta - t) \langle R_2, V^{d-1}(u, v, t) \rangle \\
+ (1 + 2uvt - u^2 - v^2 - t^2) \langle R_3, V^{d-1}(u, v, t) \rangle = 0,\]

where $Q_0, Q_1, R, R_0, \ldots, R_3$ are positive semidefinite matrices.

By rewriting condition (ii) as sum-of-squares and using $d = 15$ the largest matrix has dimension 816. Hence, in order to be able to apply a high-precision solver to the considered SDP, we can not take large values of $d$, since otherwise the matrices become too large. Bachoc and Vallentin computed upper bounds for the kissing number by using $d = 10$.

The polynomials in the entries of the $S^n_k$ matrices are invariant under the action of the symmetry group $S_3$. Machado and Oliveira \cite{12} used this property to block-diagonalise the matrices which represents the sum-of-squares polynomials in condition (ii). This leads to smaller matrices and more stable programs. Furthermore, as the matrices become smaller one can solve the SDP even for $d = 16$ with high precision.

Bachoc and Vallentin proved that any feasible solution of the SDP in Theorem 3.2 gives an upper bound on

\[
\max \{ |C| : C \in S^{n-1} \text{ and } d(x, y) \leq \cos \theta \text{ for all distinct } x, y \in C \}. \]

Hence, by applying Lemma 3.1 we can replace $\cos \theta$ by $1 - \frac{1}{1 + \cosh(2r)}$.

**Corollary 3.3.** Any feasible solution of the optimisation program in Theorem 3.2 with

\[
\Delta = \left\{ (u, v, t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq 1 - \frac{1}{1 + \cosh(2r)} \text{ and } 1 + 2uvt - u^2 - v^2 - t^2 \geq 0 \right\}
\]

and

\[
\Delta_0 = \left\{ (u, u, 1) : -1 \leq u \leq 1 - \frac{1}{1 + \cosh(2r)} \right\},
\]

gives an upper bound on $\kappa(n, r)$.

The polynomials in $S^n_k$ are still invariant under the action of $S_3$, since the only modification is changing the value of $\cos \theta$ in the domains $\Delta_0$ and $\Delta$ of the corresponding SDP. Hence, we can exploit the symmetries same way as Machado and Oliveira, which allows us to solve the SDP for sufficiently large $d$ with high precision.

In order to obtain rigorous bounds one has to check whether the matrices which we get from the SDP solver are indeed positive semidefinite and satisfy the matrix inequalities. Since, except of the domains $\Delta$ and $\Delta_0$, our program coincides with the program by Machado and Oliveira, we can use their verification script in order to prove our results. The idea of the verification process is similar to the one used by Dostert, Guzmán, Oliveira, and Vallentin \cite{5}. First one has to find an optimal solution of the SDP, where the minimal eigenvalue of any matrix is large compared to the maximal violation of any constraint. If this is accomplished, it is then possible to turn the solution into a feasible one, without changing the objective value. One can compute rigorous bounds on the minimum eigenvalue of each matrix and also on the violation of each constraint by using high-precision interval arithmetic. The verification process is explained in detail in \cite[Chapter 6]{12}. For our SDP, we adapt the verification script by Machado and Oliveira. The verification script runs in Sage 6.6 \cite{17} and can be found in the \texttt{ancillary files}.\[\]
4. Kissing configurations in dimension 3

A feasible kissing configuration in dimension $n = 3$ for radius $r$ is given by a spherical code $C \in S^2$ where $d(x, y) \leq 1 - \frac{1}{1 + \cosh(2r)}$ for all distinct $x, y \in C$. Any feasible spherical code $C$ gives a lower bound on the kissing number, hence $\kappa(n, r) \geq |C|$. In this section, we construct spherical codes in dimension 3 for certain values of $r$, c.f. Figure 3.

Since $1 - \frac{1}{1 + \cosh(2r)}$ increases with $r$ increasing, any kissing configuration for radius $r$ is also a kissing configuration for $\kappa(n, r')$ where $r' \geq r$.

First of all, a classical kissing configuration in the Euclidean 3-space is given by the vertices of a regular icosahedron, that gives a spherical code of cardinality 12. Thus, $\kappa(3, 0) \geq 12$.

Furthermore, the points $\frac{1}{\sqrt{3}} ((\pm 1, \pm 1, \pm 1), (0, \pm 1, 0), (0, 0, \pm 1)$ define a spherical code with cardinality 14. The maximal angular distance between two distinct points in that code is less than 0.5774. Since $\frac{1}{1 + \cosh(1.0)} \approx 0.6068$ this code is a kissing configuration for radius $r = 0.9$. Therefore, $\kappa(3, 0.9) \geq 20$.

The truncated octahedron whose vertices are all permutations of $\frac{1}{\sqrt{5}} ((0, \pm 1, \pm 2)$ provides a spherical code of cardinality 24 and maximal angular distance 0.8. Furthermore, adding the points $\frac{1}{\sqrt{3}} ((\pm 1, \pm 1, \pm 1)$ to this code does not change the maximal angular distance between any two distinct points. Since $1 - \frac{1}{1 + \cosh(1.6)} > 0.8203$, we obtain a feasible kissing configuration in dimension 3 for $r = 1.1$. Therefore, $\kappa(3, 1.1) \geq 32$. 

Figure 3. Kissing configurations for hyperbolic spheres: the dodecahedron for $r = 0.9$ (left) and the truncated octahedron with additional vertices for $r = 1.1$ (right).
5. Lower and upper bounds

In this section, we provide concrete upper bounds for the kissing function in dimensions \( n = 2, 3, 4, 8 \) for certain radii \( r \) by using the SDP of Section 3. In order to solve the SDP with high-precision arithmetic we use the SDPA-GMP solver [7]. Afterwards, we apply the verification script from Section 3 to the numerical solutions and obtain rigorous upper bounds. For solving the SDP we used \( d = 10 \) or \( d = 12 \). Furthermore, we compare the obtained results with the theoretical lower and upper bounds that we get from Theorems 2.1 – 2.4 as well as from the feasible configurations in Section 4.

### Dimension 2

| \( r \) | theoretical lower bound | SDP upper bound | theoretical upper bound |
|---|---|---|---|
| 0 | 3 | 6.000003 | 6 |
| 0.5 | 3.41924 | 6.692287 | 6.83848 |
| 0.7 | 3.83383 | 7.588743 | 7.66765 |
| 0.8 | 4.09962 | 8.141859 | 8.19924 |
| 0.9 | 4.40747 | 8.746839 | 8.81494 |
| 1.0 | 4.76023 | 9.464054 | 9.52046 |
| 1.1 | 5.16126 | 10.295854 | 10.3225 |
| 1.2 | 5.61442 | 11.212263 | 11.2288 |

### Dimension 3

| \( r \) | theoretical lower bound | SDP upper bound | theoretical upper bound |
|---|---|---|---|
| 0 | 4 | 12.439856 | 14.9282 |
| 0.5 | 5.08616 | 16.383058 | 19.2900 |
| 0.7 | 6.30180 | 20.831491 | 24.1640 |
| 0.8 | 7.15493 | 24.052767 | 27.5821 |
| 0.9 | 8.21495 | 27.631899 | 31.8273 |
| 1.0 | 9.52439 | 32.490317 | 37.0698 |
| 1.1 | 11.1358 | 38.621850 | 43.5197 |
| 1.2 | 13.1139 | 46.144945 | 51.4358 |

### Dimension 4

| \( r \) | theoretical lower bound | SDP upper bound | theoretical upper bound |
|---|---|---|---|
| 0 | 5.11506 | 24.140051 | 34.6807 |
| 0.5 | 7.19743 | 36.824236 | 50.7015 |
| 0.6 | 8.30930 | 43.734168 | 59.2993 |
| 0.7 | 9.79914 | 52.769975 | 70.8552 |
| 0.8 | 11.7805 | 65.091141 | 86.2726 |
| 0.9 | 14.4076 | 81.901605 | 106.780 |
It follows from the proof of Theorem 2.1 that by rounding the theoretical upper bound we shall obtain the number of circles in the kissing configuration in dimension $n = 2$. This is also evidenced by the numerical values in the respective table above. Another observation is that the theoretical upper bounds in dimension $n = 3$ keep relatively close to the SDP upper bounds, while in higher dimensions they diverge quite quickly.

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