A Basic Structure for Grids in Surfaces

Lowell Abrams∗ and Daniel Slilaty†

January 14, 2019

Abstract

A graph $G$ embedded in a surface $S$ is called an $S$-grid when every facial boundary walk has length four, that is, the topological dual graph of $G$ in $S$ is 4-regular. Aside from the case where $S$ is the torus or Klein bottle, an $S$-grid must have vertices of degrees other than four. Let the sequence of degrees other than four in $G$ be called the curvature sequence of $G$. We give a succinct characterization of $S$-grids with nonempty curvature sequence $L$ in terms of graphs that have degree sequence $L$ and are immersed in a certain way in $S$; furthermore, the immersion associated with the $S$-grid $G$ is unique and so our characterization of $S$-grids also partitions the collection of all $S$-grids.

1 Introduction

The reader is expected to be familiar with the basics of topological graph theory as in Gross and Tucker [7]; all terminology that we do not define is from there.

Given a closed surface $S$, an $S$-grid is an embedding of a graph $G$ in $S$ such that every facial boundary walk has length four, that is, the topological dual graph of $G$ is 4-regular. An $S$-grid might alternatively be termed a quadrangulation of $S$; however, we will use the term “grid” in this paper. This is probably the weakest sort of definition for a “quadrangulation” or “grid”; other studies often place additional constraints on the embedding.

Other than the case in which $S$ is the torus or Klein bottle, any $S$-grid must have vertices of degrees other than four. A very explicit characterization of $S$-grids in the torus and Klein bottle with every vertex of degree four (along with the additional property that the four faces around each vertex along with their boundaries form a $2 \times 2$ square grid) was initially given by Thomassen [13]; a slightly different formulation is given by Márquez, de Mier, Noy, Revuelta [8].

If “most” of the vertices of an $S$-grid are of degree four, then $G$ has “large” areas that are annular or appear as the standard, geometrically-flat, infinite $\{4, 4\}$-planar lattice.

∗Department of Mathematics, The George Washington University, Washington, DC 20052. Email: labrams@gwu.edu
†Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435. Email: daniel.slilaty@wright.edu. Work partially supported by a grant from the Simons Foundation #246380.
In contrast to this, vertices that are not of degree four create the curvature necessary for an $S$-grid to be finite when $S$ is not the torus or Klein bottle. As such, a vertex whose degree is not four is called a curvature vertex. Proposition 1.1 gives a relationship between the quantities and degrees of curvature vertices in an $S$-grid.

**Proposition 1.1.** If $G$ is an $S$-grid with $v_i$ vertices of degree $i$ then,

$$3v_1 + 2v_2 + v_3 = 4\chi(S) + \sum_{i\geq 5}(i-4)v_i.$$  

Furthermore, if $\chi(S) \neq 0$, then there are curvature vertices.

**Proof.** If $G$ has $f$ faces and $e$ edges, then $\sum iv_i = 2e$. Also, $4f = 2e$ and $(\sum iv_i) - e + f = \chi(S)$ which when combined together yield $4(\sum iv_i) = 4\chi(S) + 2e$. Now subtracting we obtain $\sum (4-i)v_i = 4\chi(S)$ which yields our desired results.  

Of course only certain combinations of quantities and degrees of curvature vertices are arithmetically possible. Given an $S$-grid $G$ having some curvature vertices, the degree sequence of $G$ with the 4’s removed is called the curvature sequence of $G$.

Given a graph $H$ and a surface $S$, a transverse immersion of $H$ in $S$ is an immersion of $H$ in $S$ where the only self intersections are transverse crossings of edge segments. Given a transverse immersion of $H$ in $S$, let $H$ be the graph embedding in $S$ obtained by placing a vertex at each transverse crossing of $H$ in $S$. We say that the transverse immersion is quadrangular when $H$ is an $S$-grid.

In Section 2 we will give a characterization of $S$-grids that also yields an equivalence relation on the collection of all $S$-grids with curvature sequence $L$. The equivalence classes will be defined by quadrangular transverse immersions in $S$ of graphs having degree sequence $L$. In Section 3 we discuss a simple arithmetic condition on graphs having transverse immersions in $S$. In Section 4 we motivate the study of $S$-grids by providing an overview of two natural classes of $S$-grids arising from general embeddings of graphs in surfaces.

## 2 Construction

Consider a graph $G$ embedded in a surface $S$ and a vertex $v$ of degree 4 in $G$ with incident edges $e_1,e_2,e_3,e_4$ in rotational order. Say that edges (or a single loop) $e_1$ and $e_{i+2}$ are transverse with respect to $v$. A transverse walk in $G$ is a $uv$-walk in which neither $u$ nor $v$ have degree 4 (possibly with $u = v$), each internal vertex in the walk has degree 4 in $G$, and pairs of successive edges along the walk are transverse. Note that for any choice of vertex $u$ not of degree 4 and incident edge $e$ there is a unique transverse walk starting at $u$ and containing $e$; furthermore, this walk is a trail, that is, no edge is ever used twice in the walk. Also note that no two distinct transverse walks ever share an edge.

Now consider a closed walk $W = v_1,e_1,v_2,e_2,\ldots,v_n,e_n,v_1$ in $G$ in which each $v_i$ has degree 4 in $G$ and pairs of successive edges (including the pair $e_n,e_1$) are transverse. Call such a walk a transverse circuit. Note that if $e$ is an edge in $G$ that is not contained in a transverse walk, then there is a unique transverse circuit $W$ (up to choice of starting vertex and reversal) containing $e$ and $W$ is a trail. Furthermore, if $W_1 \neq W_2$ and $W_i$ is a transverse walk or transverse circuit, then $W_1$ and $W_2$ share no edge in common. These facts yield Proposition 2.1.

**Proposition 2.1.** If $G$ is graph embedded in a surface $S$, then the edges of $G$ partition in exactly one way into transverse walks and transverse circuits.

Now consider a given $S$-grid $G$ with nonempty curvature sequence $L$. Let $G_L$ be the graph whose vertices are the curvature vertices of $G$ with an edge between $u$ and $v$ in $G_L$ if and only if there is a transverse $uv$-walk in $G$. So now the transverse walks in $G$ provide a unique transverse immersion of $G_L$ in $S$ as described in Proposition 2.2.
Proposition 2.2. If $G$ is an $S$-grid with nonempty curvature sequence $L$, then there is a graph $G_L$ on the curvature vertices of $G$ such that: $G_L$ has degree sequence $L$, there is a unique transverse immersion of $G_L$ in $S$ such that $G$ contains a subdivision of $G_L$ as a subgraph, and the subdivided edges of $G_L$ are the transverse walks in $G$.

We call the uniquely obtained embedded graph $\hat{G}_L$ of Proposition 2.2 and Theorem 2.3 the skeleton grid of the $S$-grid $G$.

Theorem 2.3. If $G$ is an $S$-grid with nonempty curvature sequence $L$, then the uniquely obtained embedded graph $\hat{G}_L$ of Proposition 2.2 is an $S$-grid.

Proof. Let $\hat{G}_L$ be the subgraph of $G$ that is a subdivision of $\hat{G}_L$. Let $R$ be the collection of regions into which $\hat{G}_L$ subdivides $S$. It is not a priori true that $R$ is a collection of 2-cells; however, we will see that this is indeed the case. Consider some $R \in \mathbb{R}$ and let $G_R'$ be the subgraph of $G$ that is embedded in $R$ including the boundary which is a closed walk in $\hat{G}_L$. Let $G_R$ be the surface obtained from $G'R$ by cutting along the boundary walk of $R$ in $\hat{G}_L$ so that the resulting boundary is a cycle. Let $v_c$ be the number of times a copy of a vertex of $\hat{G}_L$ (i.e., a branch vertex of $\hat{G}_L$) appears on the boundary walk of $G_R$; $v_s$ be the number of times a copy of a subdividing vertex appears on the boundary walk of $G_R$; $v_I$ be the number of interior vertices of $G_R$; $e$ be the number of edges of $G_R$; $f$ the number of faces of $G_R$ (excluding the outer face); and $l$ be the length of the boundary cycle of $G_R$. So now $4f = 2e - l$ and $2e = 2v_c + 3v_s + 4v_I$. Now calculating the Euler characteristic of $G_R$ we obtain

$$\chi(G_R) = v_c + v_s + v_I - e + f = v_c + v_s + v_I - e + \frac{1}{2}e - \frac{1}{4}l$$
$$= \frac{3}{2}(v_c + v_s) + v_I - \frac{1}{2}v_c - \frac{2}{4}v_s - v_I$$
$$= \frac{1}{2}v_c$$

Of course, $\chi(G_R)$ is an integer and $\chi(G_R) \leq 1$. Also $v_c > 0$ because $G_R$ is defined by a region of $\hat{G}_L$. Thus $0 < \frac{1}{4}v_c = \chi(G_R) \leq 1$ which implies that $v_c = 4$ and that $G_R$ is a disk. Our result follows. \[\Box\]

So now, given an $S$-grid $G$ and its skeleton grid $\hat{G}_L$, again let $\hat{G}_L$ be the subdivision of $\hat{G}_L$ that is a subgraph of $G$. Let $Q$ be a quadrilateral face of $\hat{G}_L$ and let $Q'$ be the corresponding face of $\hat{G}_L$. We claim that the part of $G$ inside of $Q'$ is obtained as follows: subdivide opposite edges on the boundary of $Q$ an equal number of times and then patch with a rectangular grid as shown in Figure 1.

![Figure 1: Patching](image)

Showing that the part of $G$ inside of $Q$ is obtained in this fashion is easily done by the following inductive argument. Let $e_1, e_2, e_3, e_4$ be the boundary walk of $Q$. Consider the edge $e_1$ and say that $e_1$ is subdivided $t$ times in going from $Q$ to $Q'$. Each of these subdividing vertices on $e_1$ has exactly one incident edge in the interior of $Q$. The only way in which quadrilateral faces may now be closed off is with a path of edges from $e_2$ to $e_4$ (see Figure 2). Continuing by induction yields the desired structure.
Figure 2: Induction

The only question remaining is what are the possible choices for the number of subdivisions for each edge. The topological dual graph \((\hat{G}_L)^*\) is 4-regular and so its edges partition into transverse circuits. As stated before, the number of subdivisions for opposing sides of a quadrilateral face \(Q\) must be the same. Hence each of the edges that form a transverse circuit of \((\hat{G}_L)^*\) must be subdivided the same number of times as the others.

We now have the following general construction method for any \(S\)-grid. Furthermore, this method partitions the class of all \(S\)-grids into equivalence classes represented by their skeleton graphs. After choosing the graph \(G_L\) in Step 1, it is not at all clear as to whether or not \(G_L\) has a quadrangular transverse immersion in any closed surface \(S\). Thus most of the detail of \(S\)-grids is contained in Step 2 because Steps 3 and 4 can always be carried out unambiguously after the completion of Step 2.

1. Take a graph \(G_L\) without vertices of degree 4.
2. Take a quadrangular transverse immersion of \(G_L\) in a closed surface \(S\) and its associated skeleton graph \(\hat{G}_L\).
3. Calculate the transverse circuits of \((\hat{G}_L)^*\) and choose a non-negative integer \(n_C\) for each transverse circuit \(C\).
4. Subdivide the edges \(\hat{G}_L\) corresponding to \(C\) \(n_C\) times each and patch the resulting faces.

As an example of this construction consider the Wagner Graph \(V_8\). Two distinct transverse immersions of \(V_8\) in the sphere are shown in Figure 3. The immersion on the right is quadrangular but the one on the left is not. Let \(\hat{V}_8\) be the skeleton graph obtained by the quadrangular transverse immersion.

Figure 3: Two transverse immersions of the Wagner Graph in the sphere. The one on the right is quadrangular.

The edges of the topological dual graph of the spherical grid \(\hat{V}_8\) form a single transverse circuit. Thus each edge of \(\hat{V}_8\) must be subdivided the same number of times and then each face is patched. In Figure 4, each edge is subdivided twice.
3 Arithmetic Conditions

Given a graph $G$ without degree-4 vertices and a surface $S$, Proposition 3.1 provides an arithmetic condition that is necessary for $G$ to have a quadrangular transverse immersion in a given closed surface $S$.

**Proposition 3.1.** If $G$ has a quadrangular transverse immersion in $S$, then $\chi(S) = |V(G)| - \frac{1}{2}|E(G)|$. In particular, a given graph $G$ without vertices of degree 4 can have a quadrangular transverse immersion in surfaces of only one possible Euler characteristic.

**Proof.** Let $\hat{G}$ be the skeleton grid of transverse immersion of $G$ in $S$ with $v_4$ being the number of transverse crossings used. Thus $|V(\hat{G})| = |V(G)| + v_4$, $|E(\hat{G})| = |E(G)| + 2v_4$, and $f$ is the number of faces of the embedding of $\hat{G}$, then $4f = 2(|E(G)| + 2v_4)$. We now have that

$$\chi(S) = |V(G)| + v_4 - (|E(G)| + 2v_4) + \frac{1}{2}(|E(G)| + 2v_4)$$

$$= |V(G)| - \frac{1}{2}|E(G)|$$

If a graph $G$ without degree-4 vertices does have a quadrangular immersion in a closed surface $S$, then even though $G$ satisfies $\chi(S) = |V(G)| - \frac{1}{2}|E(G)|$, two different quadrangular transverse immersions of $G$ may have different numbers of transverse crossings. Figure 5 shows two quadrangular immersions of the alternating 10-wheel with zero and five transverse crossings, respectively. Clearly this example generalizes to the alternating $(4k + 2)$-wheel for any $k \geq 2$. 

Figure 4: A skeleton grid coming from a transverse immersion of the Wagner Graph and a spherical grid obtained by subdividing each edge twice and then patching.
Figure 5: Two different quadrangular transverse immersions of the same graph with different numbers of transverse crossings.

In fact, in general it is not even possible to place an upper bound on the number of transverse crossings (see Figure 6). Interestingly, the graph in Figure 6 is the quotient of the alternating \((4k + 2)\)-wheel under its \((2k + 1)\)-fold rotational symmetry.

Figure 6: Unbounded numbers of transverse crossings for the same graph.

4 Grids coming from arbitrary embeddings

For any graph \(H\) that is cellularly embedded in a closed surface \(S\), there are two \(S\)-grids that are naturally associated with the embedding of \(H\) and its topological dual graph \(H^*\). These two types of \(S\)-grids also form fundamental subclasses within the class of all \(S\)-grids. As such, \(S\)-grids are actually fundamental objects in topological graph theory. In this section we give a short review of these \(S\)-grids.

4.1 Radial Graphs

The well-known radial graph, \(R(H, H^*)\) has vertex set \(V(H) \cup V(H^*)\). To describe the edges of \(R(H, H^*)\) consider a face \(f\) of the embedding of \(H\) in \(S\) and its boundary walk \(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1\). The vertex \(f^* \in V(H^*)\) has edges \(g_1, \ldots, g_m\) connecting respectively to \(v_1, \ldots, v_m\). The radial graph is clearly an \(S\)-grid that is also bipartite with partite sets \(V(H)\) and \(V(H^*)\). The radial graph satisfies \(R(H, H^*) = R(H^*, H)\) and the diagonals of the quadrilateral faces of \(R(H, H^*)\) connecting the vertices of \(V(H)\) form \(E(H)\) and the diagonals connecting the vertices of \(V(H^*)\) form \(E(H^*)\). This latter observation yields Proposition 4.1.

**Proposition 4.1** (Pisanski and Malnič [9]). An \(S\)-grid \(G\) is of the form \(R(H, H^*)\) for some \(H\) embedded in \(S\) if and only if \(G\) is bipartite.
Self-dual embeddings are nicely encoded by the radial graph in that the embeddings of \( H \) and \( H^* \) are map isomorphic if and only if \( \mathcal{R}(H, H^*) \) has a cellular automorphism that switches the partite sets \( V(H) \) and \( V(H^*) \). Self-dual embeddings have been studied from this viewpoint by Archdeacon and Richter [3], Archdeacon and Negami [2], and Abrams and Slilaty [1].

The topological dual graph of the radial graph \( \mathcal{R}(H, H^*) \) is known as the \textit{medial graph} \( \mathcal{M}(H, H^*) \). The medial graph has been used by Archdeacon [4] to give a unified presentation of the concepts of voltage-graph and current-graph covering constructions. In [5], Moffatt and Ellis-Monaghan describe the impressive result that all possible embeddings of the medial graph \( \mathcal{M}(H, H^*) \) in all possible surfaces correspond precisely to the various notions of duality that generalize topological duality, Petrie duality, and their partial versions and associated group actions.

### 4.2 Overlay Graphs

Consider a connected graph \( H \) cellularly embedded in a closed surface \( S \); its topological dual graph \( H^* \) is therefore well defined, connected, and cellularly embedded. Say that all of \( H \) (both vertices and edges) is colored “red” and all of \( H^* \) is colored “blue”. Embed \( H \) and \( H^* \) simultaneously in \( S \) and at each edge/dual-edge crossing point create a new vertex of degree four (which now has alternating red and blue edges in rotation around the vertex) and say that this new vertex is “white”. The graph obtained is called the \textit{overlay graph} \( \mathcal{O}(H, H^*) \). Certainly the overlay graph is an \( S \)-grid that is also bipartite with partite sets \( \text{Red} \cup \text{Blue} \) and White. Since the edge/dual-edge pairs of \( H \) and \( H^* \) are the diagonals of the faces of the radial graph \( \mathcal{R}(H, H^*) \) we also get that \( \mathcal{O}(H, H^*) \) is the radial graph of the radial graph of \( H \) and \( H^* \), that is, \( \mathcal{O}(H, H^*) = \mathcal{R}(\mathcal{R}(H, H^*), \mathcal{M}(H, H^*)) \); recall that \( \mathcal{M}(H, H^*) \) is the topological dual graph of \( \mathcal{R}(H, H^*) \).

The embedding of \( H \) is self dual if and only if \( \mathcal{O}(H, H^*) \) has a cellular automorphism that reverses red and blue colors and preserves white. The overlay graph was used by Servatius and Servatius [10, 11, 12] to classify self-dual embeddings in the sphere along with the pairing of their groups of color-preserving cellular automorphisms of \( \mathcal{O}(H, H^*) \) as an index-2 subgroup of the group of red-blue switching cellular automorphisms of \( \mathcal{O}(H, H^*) \). Graver and Hartung [6] do the same but with more detailed results for the special case of self-dual embeddings of graphs having four trivalent vertices and the remaining vertices all of degree four.

For any closed surface \( S \), \( \mathcal{O}(H, H^*) \) is an \( S \)-grid that is bipartite and with the additional property that all white vertices have degree four. Conversely, however, even if \( G \) is a bipartite \( S \)-grid in which all white vertices have degree four, it is not necessarily true that \( G \) is of the form \( \mathcal{O}(H, H^*) \) for some \( H \). An additional condition that does ensure that \( G \) has the form \( \mathcal{O}(H, H^*) \) is as follows: let \( \mathcal{R}(G) \) be the graph obtained from \( G \) by placing a diagonal edge connecting the black corners of each face and then deleting the white vertices of \( G \).

**Proposition 4.2.** If \( G \) is an \( S \)-grid, then \( G = \mathcal{O}(H, H^*) \) for some \( H \) if and only if \( G \) is bipartite, every white vertex of \( G \) has degree 4, and \( \mathcal{R}(G) \) is bipartite.

**Proof.** The one direction is trivial. For the other direction, the fact that \( \mathcal{R}(G) \) is bipartite allows us to properly 2-color (red and blue) the vertices of \( \mathcal{R}(G) \), which shows \( G \) is of the form \( \mathcal{O}(H, H^*) \), as required.

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