Probabilistic Models over Ordered Partitions with Application in Learning to Rank

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Abstract

This paper addresses the general problem of modelling and learning rank data with ties. We propose a probabilistic generative model, that models the process as permutations over partitions. This results in super-exponential combinatorial state space with unknown numbers of partitions and unknown ordering among them. We approach the problem from the discrete choice theory, where subsets are chosen in a stagewise manner, reducing the state space per each stage significantly. Further, we show that with suitable parameterisation, we can still learn the models in linear time. We evaluate the proposed models on the problem of learning to rank with the data from the recently held Yahoo! challenge, and demonstrate that the models are competitive against well-known rivals.

1 Introduction

Ranking appears to be natural to humans as we often express preference over things. Consequently, rank data has been widely studied in statistical sciences (e.g. see [20] for a comprehensive survey). More recently, the intersection between machine learning and information retrieval has resulted in a fruitful sub-area called learning to rank (e.g. see [17] for a recent review), where the goal is to learn rank functions that can accurately order objects from retrieval systems. Broadly speaking, a rank is a type of permutation, where the ordering of objects has some meaningful interpretation - e.g. the rank of student performance in a class. Although we would like to obtain a complete ordering over a set of objects, often this is possible only in small sets. In larger sets, it is more natural to rate an object from a rating scale, and the result is that many objects may have the same rating. Such phenomena is common in large sets such as movies, books or web-pages wherein many objects may have tied ratings.

This paper focuses on the modelling and learning rank data with ties. Previous work often involves paired comparisons (e.g. see [7][11][24]), ignoring simultaneous interactions among objects. Such interactions can be strong - in the case of learning to rank, objects are often returned from a query, and thus clearly related to the query and to each other. We take an alternative approach by modelling objects with the same tie as a partition, translating the problem into ranking
or ordering these partitions. This problem transformation results in a combinatorial problem - set partitioning with unknown numbers of subsets with unknown order amongst them. For a given number of partitions, the order amongst them is a permutation of the partitions being considered, wherein each partition has objects of the same rank. A generative view of the problem can then be as follows: Choose the first partition with elements of rank 1, then choose the next partition from the remaining objects with elements ranked 2 and so on. The number of partitions then does not have to be specified in advance, and can be treated as a random variable. The joint distribution for each ordered partition can then be composed using a variant of the Plackett-Luce model \[18\] [23], substituting object potentials by the partition potential. We propose two choices for these potential functions: First, we consider the potential of each partition to be the normalised sum of individual object potentials in that partition, leading to a simple normalisation factor in the estimation of the joint distribution. Second, we propose a MCMC based parameter estimation for the general choice of potential functions. We specify this model as the Probabilistic Model over Ordered Partitions. Demonstrating its application to the learning to rank problem, we use the dataset from the recently held Yahoo! challenge [28]. Besides the regular first-order features, we study second-order features constructed as the Cartesian product over the feature set. We show that our results both in terms of predictive performance and training time are competitive with other well-known methods such as RankNet [3], Ranking SVM [15] and ListMLE [27]. With the choice of our proposed simple potential function, we get the added advantage of lower computational cost as it is linear in the query size compared to quadratic complexity for the pairwise methods.

Our main contributions are the construction of a probabilistic model over ordered partitions and associated inference and learning techniques. The complexity of this problem is super-exponential with respect to number of objects \(N\) because both the number of partitions and their order are unknown - it grows exponentially as \(N!/(2(\ln 2)^{N+1})\) [21, pp. 396–397]. Our contribution is to overcome this computational complexity through the choice of suitable potential functions, yielding learning algorithms with linear complexity, thus making the algorithm deployable in real settings. The novelty lies in the rigorous examination of probabilistic models over ordered partitions, extending earlier work in discrete choice theory [9] [18] [23]. The significance of the model is its potential for use in many applications. One example is the learning to rank with ties problem and is studies in this paper. Further, the model opens new potential applications for example, novel types of clustering, in which the clusters are automatically ordered.

2 Background

In this section, we review some background in rank modelling and learning to rank which are related to our work.

Rank models. Probabilistic models of permutation in general and of rank in particular have been widely analysed in statistical sciences (e.g. [20] for a comprehensive survey). Since the number of all possible permutations over \(N\) objects is \(N!\), multinomial models are only computationally feasible for small \(N\) (e.g. \(N \leq 10\)). One approach to avoid this state space explosion is to deal directly with the data space, i.e. based on the distance between two ranks. The assumption is that there exists a modal ranking over all objects, and what we observe are ranks randomly distributed around the mode. The most well-know model is perhaps the Mallows [19], where the probability of a rank decreases exponentially with the distance from the mode. Depending on
the distance measures, the model may differ; and the popular distance measures include those by Kendall and Spearman. The problem with this approach is that it is hard to handle the cases of multiple modes, with ties and incomplete ranking.

Another line of reasoning is largely associated with the discrete choice theory (e.g. see [18]), which assumes that each object has an intrinsic worth which is the basis for the ordering between them. For example, Bradley and Terry [1] assumed that the probability of object preference is proportional to its worth, resulting in the logistic style distribution for pairwise comparison. Subsequently, Luce [18] and Plackett [23] extended this model to multiple objects. More precisely, for a set of \( N \) objects denoted by \( \{x_1, x_2, \ldots, x_N\} \) the probability of ordering \( x_1 \succ x_2 \succ \ldots \succ x_N \) is defined as

\[
P(x_1 \succ x_2 \succ \ldots \succ x_N) = \prod_{i=1}^{N} \frac{\phi(x_i)}{\sum_{j=1}^{N} \phi(x_j)}
\]

where \( x_i \succ x_j \) denotes the preference of object \( x_i \) over \( x_j \), and \( \phi(x_i) \in \mathbb{R} \) is the worth of the object \( x_i \). The idea is that, we proceed in selecting objects in a stagewise manner: Choose the first object among \( N \) objects with probability of \( \frac{\phi(x_1)}{\sum_{j=1}^{N} \phi(x_j)} \), then choose the second object among the remaining \( N-1 \) objects with probability of \( \frac{\phi(x_2)}{\sum_{j=2}^{N} \phi(x_j)} \) and so on until all objects are chosen. It can be verified that the distribution is proper, that is \( P(x_1 \succ x_2 \succ \ldots \succ x_N) > 0 \) and the probabilities of all possible orderings will sum to one. This paper will follow this approach as it is easily interpretable and flexible to incorporate ties and incomplete ranks.

Finally, for completeness, we mention in passing the third approach, which treats a permutation as a symmetric group and applying spectral decomposition techniques [8] [13].

Learning to rank. Learning-to-rank is an active topic in the intersection between machine learning and information retrieval (e.g. see [17] for a recent survey). The basic idea is that we can learn ranking functions that can capture the relevance of an object (e.g. document or image) with respect to a query. Although it appears to be an application of rank theory, the setting and goal are inherently different from traditional rank data in statistical sciences. Often, the pool of all possible objects in a typical retrieval system is very large, and often changes over time. Thus, it is not possible to enumerate objects in the rank models. Instead, each object-query pair is associated with a feature vector, which often describes how relevant the object is with respect to the query. As a result, the distribution over objects is query-specific, and these distributions share the same parameter set. As discussed in [17], machine learning methods extended to ranking can be divided into:

- **Pointwise approach** which includes methods such as ordinal regression [5] [6]. Each query-document pair is assigned a ordinal label, e.g. from the set \{0, 1, 2, ..., M\}. This simplifies the problem as we do not need to worry about the exponential number of permutations. The complexity is therefore linear in the number of query-document pairs. The drawback is that the ordering relation between documents is not explicitly modelled.

- **Pairwise approach** which spans preference to binary classification [3] [10] [15] methods, where the goal is to learn a classifier that can separate two documents (per query). This casts the ranking problem into a standard classification framework, wherein many algorithms are readily available, for example, SVM [15], neural network and logistic regression [3], and boosting [10]. The complexity is quadratic in number of documents per query and linear in number of queries. Again, this approach ignores the simultaneous interaction about objects within the same query.
Listwise approach which models the distribution of permutations \[4\][26][27]. The ultimate goal is to model a full distribution of all permutations, and the prediction phase outputs the most probable permutation. This approach appears to be most natural for the ranking problem. In fact, the methods suggested in \[4\][27] are applications of the Plackett-Luce model.

3 Modelling Sets with Ordered Partitions

3.1 Problem Description

Let \(X = \{x_1, x_2, \ldots, x_N\}\) be a collection of \(N\) objects. In a complete ranking setting, each object \(x_i\) is further assigned with a ranking index \(\pi_i\), resulting in the ranked list of \(\{x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_N}\}\) where \(\pi = (\pi_1, \ldots, \pi_N)\) is a permutation over \(\{1, 2, \ldots, N\}\). For example, \(X\) might be a set of documents returned by a search engine in response to a query, and \(\pi_1\) is the index to the first document, \(\pi_2\) is the index to second document and so on. Ideally \(\pi\) should contain ordering information for all returned documents; however, this task is not always possible for any non-trivial size \(N\) due to the labor cost involved\(\footnote{We are aware that clickthrough data can help to obtain a complete ordering, but the data may be noisy.}\). Instead, in many situations, during training a document is rated\(\footnote{We caution the confusion between ‘rating’ and ‘ranking’ here. Ranking is the process of sorting a set of objects in an increasing or decreasing order, whereas in ‘rating’ each object is given with a value indicating its preference.}\) to indicate its degree of relevance for the query. This creates a scenario where more than one document will be assigned to the same rating – a situation known as ‘ties’ in learning-to-rank. When we enumerate over each object \(x_i\) and putting those with the same rating together, the set of \(N\) objects \(X\) can now be viewed as being divided into \(K\) partitions with each partition is assigned with a number to indicate the its unique rank \(k \in \{1, 2, \ldots, K\}\). The ranks are obtained by sorting ratings associated with each partition in the decreasing order. Our essential contribution in this section is a probabilistic model over this set of partitions, learning its parameter from data, and performing inference.

Consider a more generic setting in which we know that objects will be rated against an ordinal value from 1 to \(K\) but do not know individual ratings. This means that we have to consider all possible ways to split the set \(X\) into exactly \(K\) partitions, and then rank those partitions from 1 to \(K\) wherein the \(k\)th partition contains all objects rated with the same value \(k\). This is the first rough description of state space for our model. Formally, for a given \(K\) and the order among the partitions \(\sigma\), we write the set \(X = \{x_1, \ldots, x_N\}\) as a union of \(K\) partitions

\[X = \bigcup_{j=1}^{K} X_{\sigma_j}\]  

\(\Box\)
where \( \sigma = (\sigma_1, \ldots, \sigma_K) \) is a permutation over \( \{1, 2, \ldots, K\} \) and each partition \( X_k \) is a non-empty subset of objects with the same rating \( k \). These partitions are pairwise disjoint and having cardinality range from 1 to \( N \). It is easy to see that when \( K = N \), each \( X_k \) is a singleton, \( \sigma \) is now a complete permutation over \( \{1, \ldots, N\} \) and the problem reduces exactly to the complete ranking setting mentioned earlier. To get an idea of the state space, it is not hard to see that there are \( \binom{N}{K} K! \) ways to partition and order \( X \) where \( \binom{N}{K} \) is the number of possible ways to divide a set of \( N \) objects into \( K \) partitions, otherwise known as Stirling numbers of second kind \[25\] p. 105. If we consider all the possible values of \( K \), the size of our state space is

\[
\sum_{k=1}^{N} \frac{N!}{k!} = \sum_{j=1}^{\infty} \frac{j^N}{2j+1} \tag{2}
\]

which is also known in combinatorics as the Fubini’s number \[21\] pp. 396–397. This is a super-exponential growth number. For instance, Fubini(1) = 1, Fubini(3) = 13, Fubini(5) = 541 and Fubini(10) = 102,247,563. Its asymptotic behaviour can also be shown \[21\] pp. 396–397 to approach \( N!/(2(\ln 2)^{N+1}) \) as \( N \to \infty \) where we note that \( \ln(2) < 1 \), and thus it grows much faster than \( N! \). Clearly, for unknown \( K \) this presents a very challenging problem. In this paper, we shall present an efficient and a generic approach to tackle this state-space explosion.

### 3.2 Probabilistic Model over Ordered Partitions

Return to our problem, our task now to model a distribution over the ordered partitioning of set \( X \) into \( K \) partitions and the ordering \( \sigma = (\sigma_1, \ldots, \sigma_K) \) among \( K \) partitions given in Eq (1):

\[
p(X) = p(X_{\sigma_1}, \ldots, X_{\sigma_K}) \tag{3}
\]

A two-stage view has been given thus far: first \( X \) is partitioned in any arbitrary way so long as it creates \( K \) partitions and then these partitions are ranked, result in a ranking index vector \( \sigma \). This description is generic and one can proceed in different ways to further characterise Eq (3). We present here a generative, multistage view to this same problem so that it lends naturally to the specification of the distribution in Eq (17). First, we construct a subset \( X_1 \) from \( X \) by collecting all objects which (supposedly) have the largest ratings. If there are more elements in the the remainder set \( \{X \setminus X_1\} \) to be selected, we construct a subset \( X_2 \) from \( \{X \setminus X_1\} \) whose elements have the second largest ratings. This process continues until there is no more object to be selected\[3\]. An advantage of this view is that the resulting total number of partitions \( K_\sigma \) is automatically generated, no need to be specified in advance and can be treated as a random variable. If our data truly contains \( K \) partitions then \( K_\sigma \) should be equal to \( K \). Using the chain rule, we write the joint distribution over \( K_\sigma \) ranked partitions as

\[
p(X_1, \ldots, X_{K_\sigma}) = p(X_1) \prod_{k=2}^{K_\sigma} p(X_k | X_1, \ldots, X_{k-1}) = p_1(X_1) \prod_{k=2}^{K_\sigma} p_k(X_k | X_{1:k-1}) \tag{4}
\]

where we have used \( X_{1:k-1} = \{X_1, \ldots, X_{k-1}\} \) for brevity.

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\[3\]This process resembles the generative process of Plackett-Luce discrete choice model \[18\] \[23\], except we apply on partitions rather than single element. It clear from here that Plackett-Luce model is a special case of ours wherein each partition \( X_k \) reduces to a singleton.
3.3 Parameterisation, Learning and Inference

It remains to specify the local distribution \( P(X_k | X_{1:k-1}) \). Let us first consider what choices do we have after the first \((k-1)\) partitions have been selected. It is clear that we can select any objects from the remainder set \( \{X \setminus X_{1:k-1}\} \) for our next partition \( k \). If we denote this remainder set by \( R_k = \{X \setminus X_{1:k-1}\} \) and \( N_k = |R_k| \) is the number of remaining objects, then our next partition \( X_k \) is a subset of \( R_k \); furthermore, there is precisely \((2^{N_k} - 1)\) such non-empty subsets. Using the notation \( 2^{R_k} \) to denote the power set of the set \( R_k \), i.e., \( 2^{R_k} \) contains all possible non-empty subsets of \( R \), we are ready to specify each local conditional distribution in Eq (17) as:

\[
p_k (X_k | X_{1:k-1}) = \frac{\Phi_k (X_k)}{\sum_{S \subseteq 2^{R_k}} \Phi_k (S)}
\]

where \( \Phi_k (S) > 0 \) is an order-invariant set function defined over a set or partition \( S \), and the summation in the denominator clearly makes the definition in Eq (5) a proper distribution. The set function \( \Phi_k (\cdot) \) can also be interpreted as the potential function in standard probabilistic graphical models literature.

Although the state space \( 2^{R_k} \) for this local conditional distribution is significantly smaller than the space of all possible ordered partitions of \( N \) objects, it is still exponential as we have shown earlier to be \( 2^{N_k} - 1 \). In general, directly computing the normalising term is still not possible, let alone learning the model parameters. In what follows, we will study an efficient special case which has (sub)-quadratic complexity in learning, and a general case with MCMC approximation. We further term our Probabilistic Model over Ordered Partition as PMOP.

3.3.1 Full-Decomposition PMOP

Under a full-decomposition setting, we assume the following local additive decomposition at each \( k \)th step:

\[
\Phi_k (X_k) = \frac{1}{|X_k|} \sum_{x \in X_k} \phi_k (x)
\]

The normalising term \( |X_k| \) is to ensure that the probability is not monotonically increasing with number of objects in the partition. Given this form, the local normalisation factor represented in the denominator of Eq (5) can now efficiently represented as the sum of all weighted sums in the denominator of Eq (5) can now efficiently represented as the sum of all weighted sums of objects. Since each object \( x \) in the remainder set \( R_k \) participates in the same additive manner towards the construction of the denominator in Eq (5), it must admit the following form:

\[
\sum_{S \subseteq 2^{R_k}} \Phi_k (S) = \sum_{S \subseteq 2^{R_k}} \frac{1}{|S|} \sum_{x \in S} \phi_k (x) = C \times \sum_{x \in R_k} \phi_k (x)
\]

where \( C \) is some constant and its exact value is not essential under a maximum likelihood parameter learning treatment (readers are referred to Appendix A for the computation of \( C \)). To see this, substitute Eq (6) and (7) into Eq (5).

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4 The usual understanding would also contain the empty set, but we exclude it in this paper.

5 i.e., the function value does not depend on the order of elements within the partition.

6 To illustrate this intuition, suppose the remainder set is \( R_k = \{a, b\} \), hence its power set, excluding \( \emptyset \), contains 3 subsets \( \{a\}, \{b\}, \{a, b\} \). Under the full-decomposition assumption, the denominator in Eq (5) becomes \( \phi (r_a) + \phi (r_b) + \frac{1}{2} \phi (r_a) + \phi (r_b) \} = (1 + \frac{1}{2}) \sum_{x \in \{a, b\}} \phi (r_x) \). The constant term is \( C = \frac{3}{2} \) in this case.
\[
\log p(X_k \mid X_{1:k-1}) = \log \frac{\Phi_k(X_k)}{\sum_{S \in 2^k} \Phi_k(S)} = \log \frac{1}{C(x)} \sum_{x \in X_k} \phi_k(x) = \log \frac{\sum_{x \in X_k} \phi_k(x)}{\sum_{x \in R_k} \phi_k(x)} - \log C(X_k)
\]  

(8)

Since \( \log C(X_k) \) is a constant w.r.t the parameters used to parameterise the potential functions \( \phi_k(\cdot) \), it does not affect the gradient of the log-likelihood. It is also clear that maximising the likelihood given in Eq (17) is equivalent to maximising each local log-likelihood function given in Eq (8) for each \( k \). Discarding the constant term in Eq (8), we re-write it in this simpler form:

\[
\log p(X_k \mid X_{1:k-1}) = \log \sum_{x \in X_k} g_k(x \mid X_{1:k-1}) \quad \text{where} \quad g_k(x \mid X_{1:k-1}) = \frac{\phi_k(x)}{\sum_{x \in R_k} \phi_k(x)},
\]

(9)

Depend on the specific form chosen for \( \phi_k(x) \), maximising log-likelihood in the form of Eq (9) can be carried on in most cases. Gradient-based learning this type of model is generally takes \( N^2 \) time complexity. However, using dynamic programming technique, we show that if the function \( \phi_k(x) \) does not depend on its position \( k \), then the gradient-based learning complexity can be reduced to \( \log N \).

To see how, dropping the explicit dependency of the subscript \( k \) in the definition of \( \phi_k(\cdot) \), we maintain an auxiliary array \( a_k = \sum_{x \in X_k} \phi(x) \) where \( a_{K_\sigma} = \sum_{x \in X_{K_\sigma}} \phi(x) \) and \( a_k = a_{k+1} + \sum_{x \in X_k} \phi(x) \) for \( k < K_\sigma \). Clearly \( a_{1:K_\sigma} \) can be computed in \( N \) time in a backward fashion. Thus, \( g_k(\cdot) \) in Eq (9) can also be computed linearly via the relation \( g_k(x) = \phi(x) / a_k \). This also implies that the total log-likelihood can also computed linearly in \( N \).

Furthermore, the gradient of log-likelihood function can also be computed linearly in \( N \). Given the likelihood function in Eq (17), using Eq (9), the log-likelihood function and its gradient, without explicit mention of the parameters, can be shown to be:

\[
\mathcal{L} = \log p(X_1, \ldots, X_{K_\sigma}) = \sum_{k=1}^{K} \log \sum_{x \in X_k} g_k(x \mid X_{1:k-1}) = \sum_{k=1}^{N} \log \frac{\phi(x)}{a_k}
\]

(10)

\[
\partial \mathcal{L} = \sum_k \partial \log \sum_{x \in X_k} \phi(x) - \sum_k \partial \log a_k = \sum_k \sum_{x \in X_k} \partial \phi(x) / \phi(x) - \sum_k a_k \sum_{x \in R_k} \partial \phi(x)
\]

(11)

It is clear that the first summation over \( k \) in the RHS of the last equation takes exactly \( N \) time since \( \sum_{k=1}^{N} X_k = N \). For the second summation over \( k \), it is more involved because both \( k \) and \( R_k \) can possibly range from 1 to \( N \), so direct computation will cost at most \( N(N-1)/2 \) time. Similar to the case of \( a_k \), we now maintain an 2-D auxiliary array \( b_k = \sum_{x \in R_k} \partial \phi(x) \), where \( b_{K_\sigma} = \sum_{x \in X_{K_\sigma}} \partial \phi(x) \) and \( b_k = b_{k+1} + \sum_{x \in X_k} \partial \phi(x) \) for \( k < K_\sigma \). Thus, \( b_{1:K_\sigma} \), and therefore the gradient \( \partial \mathcal{L} \), can be computed in \( NF \) time in a backward fashion, where \( F \) is the number of parameters.

### 3.3.2 General State PMOP and MCMC Inference

In the general case without any assumption on the form of the potential function \( \Phi_k(\cdot) \) using only Eq (5) and (17), the log-likelihood function and its gradient, again without explicit mention of the

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Footnotes:

1. To be more precise, for \( k = 1 \) we define \( X_{1:0} \) to be \( \theta \).
2. This is 2-D because we also need to index the parameters as well as the subsets.
model parameter, are:

\[
\mathcal{L} = \log p(X_1) + \sum_{k=2}^{K_n} \log p_k(X_k | X_{1:k-1})
\]

and:

\[
\partial \mathcal{L} = \sum_{k=1}^{K_n} \partial \log \Phi_k(X_k) - \sum_{k=1}^{K_n} \left\{ \sum_{S \in 2^{R_k}} p_k(S | X_{1:k-1}) \partial \log \Phi_k(S) \right\}
\]

Clearly, both the distribution \( p_k(X_k | X_{1:k-1}) \) and the expectation \( \sum_{S \in 2^{R_k}} p_k(S | X_{1:k-1}) \partial \log \Phi_k(S) \) are generally intractable to evaluate. In this paper, we make use of MCMC methods to approximate \( p_k(X_k | X_{1:k-1}) \). There are two natural choices: the Gibbs sampling and Metropolis-Hastings sampling. For Gibbs sampling we note that this problem can be viewed as sampling from a random field with binary variables. Each object is attached with binary variable whose states are either 'selected' or 'not selected' at \( k \)th stage. Thus, there will be \( 2^{N_k} - 1 \) joint states in the random field, where we recall that \( N_k \) is the total number of remaining objects after \( (k-1) \)-th stage. The pseudo code for Gibbs and Metropolis-Hastings routines performed at \( k \)th stage is illustrated in Alg. 1.

**Algorithm 1** MCMC sampling approaches for PMOP in general case.

**Gibbs sampling**

1. Randomly choose an initial subset \( X_k \)
2. Repeat until stopping criteria met
   - For each remaining object \( x \) at stage \( k \), randomly select the object with the probability
     \[
     \frac{\Phi_k(X^+_k)}{\Phi_k(X^+_k) + \Phi_k(X^-_k)}
     \]
     where \( \Phi_k(X^+_k) \) is the potential of the currently selected subset \( X_k \) if \( x \) is included and \( \Phi_k(X^-_k) \) is when \( x \) is not.

**Metropolis-Hastings sampling**

1. Randomly choose an initial subset \( X_k \)
2. Repeat until stopping criteria met
   - Randomly choose number of objects \( m \), subject to \( 1 \leq m \leq N_k \).
   - Randomly choose \( m \) distinct objects from remaining set \( R_k = \{X \setminus X_{1:k-1}\} \) to construct a new partition denoted by \( S \)
   - Set \( X_k \leftarrow S \) with the probability of
     \[
     \min \left\{ 1, \frac{\Phi_k(S)}{\Phi_k(X_k)} \right\}
     \]

Finally, we note that in practical implementation of learning, we follow the proposal in [12] wherein for each local distribution at \( k \)th round we run the MCMC for only a few steps starting from the observed subset \( X_k \). This technique is known to produce a biased estimate, but empirical evidences have so far indicated that the bias is small and the estimate is effective. Importantly, it is very fast compared to full sampling.

### 3.4 Learning-to-Rank with PMOP

To conclude the presentation of our proposed model for probabilistic modelling over ordered partitions (PMOP), we present a specific application of PMOP for the problem of leaning-to-rank. The ultimate goal after training is that, for each query the system needs to return a list of
related objects and their ranking. Slightly different from the standard rank setting in statistics, the objects in learning-to-rank problem are often not indexed (e.g. the identity of the object is not captured in any parameter). Instead, we will assume that for each query-object pair \((q, x)\) we can extract a feature vector \(x^q\). Model distribution specified in this way is thus query-specific. As a result, we are not interested in finding the single mode for the rank distribution over all queries, but in finding the rank mode for each query.

At the ranking phase, suppose for a unseen query \(q\) a list of \(X^q = \{x_1^q, \ldots, x_{N_q}^q\}\) objects related to \(q\) is returned. The task is then to rank these objects in decreasing order of relevance w.r.t \(q\). Enumerating over all possible ranking take an order of \(N_q!\) time. Instead we would like to establish a scoring function \(f(x^q, w) \in \mathbb{R}\) for the query \(q\) and each object \(x\) returned where \(w\) is now introduced as the parameter. Sorting can then be carried out much more efficiently in the complexity order of \(N_q \log N_q\) instead of \(N_q!\). The function specification can be a simple a linear combination of features \(f(x^q, w) = w^\top x^q\) or more complicated form, such as a multilayer neural network, can be used.

In the practice of learning-to-rank, the dimensionality of feature vector \(x^q\) is often remains the same across all queries, and since it is observed, we use PMOP described before to specify conditional model specific to \(q\) over the set of returned objects \(X^q\) as follows.

\[
p(X^q | w) = p(X_1^q, X_2^q, \ldots, X_{K_q}^q | w) = P(X_1^q | w) \prod_{k=2}^{K_q} p(X_k^q | X_{1:k-1}^q, w) \tag{14}
\]

We can see that Eq (14) has exactly the same form of Eq (17) specified for PMOP, but applied instead on the query-specific set of objects \(X^q\) and additional parameter \(w\). During training, each query-object pair is labelled by a relevance score, which is typically an integer from the set \(\{0, \ldots, M\}\) where 0 means the object is irrelevant w.r.t. the query \(q\), and \(M\) means the object is highly relevant. The value of \(M\) is typically much smaller than \(N_q\), thus, the issue of ties, described at the beginning of this section, occur frequently. In a nutshell, for each training query \(q\) and its rated associated list of objects a PMOP is created. The important parameterisation to note here is that the parameter \(w\) is shared across all queries; and thus, enabling ranking for unseen query in the future.

Using the scoring function \(f(x, w)\) we specify the individual potential function \(\phi(\cdot)\) in the exponential form:

\[
\phi_k(x, w) = \exp \{ f(x, w) \}
\]

The local potential function defined over for partition \(\Phi_k(X_k^q)\) can now be explicitly constructed under full-decomposition (Subsection 3.3.1) and general case (Subsection 3.3.2) as respectively follows.

\[
\text{Full-decomposition: } \Phi_k(X_k^q) = \frac{1}{|X_k^q|} \sum_{x \in X_k^q} \exp \{ f(x, w) \} \tag{15}
\]

\footnote{We note a confusion that may arise here is that, although during training each training query \(q\) is supplied with a list of related objects and their ratings, during the ranking phase the system still needs to return a ranking over the list of related objects for an unseen query.}

\footnote{This would lead to something like the static rank over all possible objects in the database - like those in Google’s PageRank.}

\footnote{Note that generally \(K \neq M + 1\) because there may be gaps in rating scales for a specific query.}
General case: \( \Phi_k (X^q_k) = \exp \left\{ \frac{1}{|X^q_k|} \sum_{x \in X^q_k} f(x, w) \right\} \) (16)

The gradient of the log-likelihood function can also be computed efficiently. For full-decomposition, it can be shown to be:

\[
\frac{\partial \log p (X^q_k | X^q_{1:k-1})}{\partial w} = \sum_{x \in X^q_k} \phi_k(x, w) x - \sum_{x \in R^q_k} \phi_k(x, w) x
\]

For the general case, the gradient of the log-likelihood function can be shown to be:

\[
\frac{\partial \log p (X^q_k | X^q_{1:k-1})}{\partial w} = X^q_k - \sum_{S_k \in 2^q_k} P(S_k | X^q_{1:k-1}) \bar{s}_k
\]

where

\[
X^q_k = \frac{1}{|X^q_k|} \sum_{x \in X^q_k} x
\]

The quantity \( p (X^q_k | X^q_{1:k-1}) \) can be interpreted as the probability that the subset \( X^q_k \) is chosen out of all possible subsets at stage \( k \), and \( \bar{s}_k \) is the centre of the chosen subset.

The expectation \( \sum_{S_k} P(S_k | X^q_{1:k-1}) \bar{s}_k \) is expensive to evaluate, since there are \( 2^{N_k} - 1 \) possible subsets. Thus, we resort to MCMC techniques. We follow the suggestion in [12] to start the Markov chain from the observed subset \( X_k \) and run for a few iterations. The parameter update is stochastic

\[
w \leftarrow w + \eta \sum_k \left( X^q_k - \frac{1}{n} \sum_{l=1}^n \bar{s}^{(l)}_k \right)
\]

where \( \bar{s}^{(l)}_k \) is the centre of the subset sampled at iteration \( l \), and \( \eta > 0 \) is the learning rate, and \( n \) is number of samples. Typically we choose \( n \) to be small, e.g. \( n = 1, 2, 3 \).

### 4 Discussion

In our specific choice of the local distribution in Eq (5), we share the same idea with that of Plackett-Luce, in which the probability of choosing the subset is proportional to the subset’s worth, which is realised by the subset potential. In fact, when we limit the subset size to 1, i.e. there are no ties, the proposed model reduces to the well-known Plackett-Luce models.

It is worth mentioning that the factorisation in Eq (17) and the choice of local distribution in Eq (5) are not unique. In fact, the chain-rule can be applied to any sequence of choices. For example, we can factorise in a backward manner

\[
p (X_1, \ldots, X_{K\sigma}) = p_1 (X_{K\sigma}) \prod_{k=1}^{K\sigma-1} p_k (X_k | X_{k+1:K\sigma})
\] (17)

where \( X_{k+1:K\sigma} \) is a shorthand for \( \{X_{k+1}, X_{k+2}, \ldots, X_{K\sigma}\} \). Interestingly, we can interpret this reverse process as subset elimination: First we choose to eliminate the worst subset, then the second
worst, and so on. This line of reasoning has been discussed in [9] but it is limited to 1-element subsets. However, if we are free to choose the parameterisation of $p_k(X_k | X_{k+1:k+1})$ as we have done for $p_k(X_k | X_{1:k-1})$ in Eq (5), there are not guarantee that the forward and backward factorisations admit the same distribution.

Our model can be placed into the framework of probabilistic graphical models (e.g. see [16][22]). Recall that in standard probabilistic graphical models, we have a set of variables, each of which receives values from a fixed set of states. Generally, variables and states are orthogonal concepts, and the state space of a variable do not explicitly depends on the states of other variables. In our setting, the objects play the role of the variables, and their memberships in the subsets are their states. However, since there are exponentially many subsets, enumerating the state spaces as in standard graphical models is not possible. Instead, we can consider the ranks of the subsets in the list as the states, since the ranks only range from 1 to $N$. Different from the standard graphical models, the variables and the states are not always independent, e.g. when the subset sizes are limited to 1, then the state assignments of variables are mutually exclusive, since for each position, there is only one object. Probabilistic graphical models are generally directed (such as Bayesian networks) or undirected (such as Markov random fields), and our PMOP can be thought as a directed model. The undirected setting is also of great interest, but it is beyond the scope of this paper.

With respect to tie handling, most previous work focuses on pairwise models. The basic idea is to assign some probability mass for the event of ties [7][11][24]. For instance, denote by $x_i \succ x_j$ the preference of $x_i$ over $x_j$, and by $x_i \approx x_j$ the tie between the two objects, Rao and Kupper [24] proposed the following models

$$P(x_i \succ x_j) = \frac{\phi(x_i)}{\phi(x_i) + \theta \phi(x_j)}$$
$$P(x_i \approx x_j) = \frac{(\theta^2 - 1) \phi(x_i) \phi(x_j)}{[\phi(x_i) + \theta \phi(x_j)] [\theta \phi(x_i) + \phi(x_j)]}$$

(18)

where $\theta \geq 1$ is the parameter to control the contribution of ties. When $\theta = 1$, the model reduces to the standard Bradley-Terry model [1]. This method of ties handling is further studied in [29] in the context of learning to rank. Another method is introduced in [7], where the probability masses are defined as

$$P(x_i \succ x_j) = \frac{\phi(x_i)}{\phi(x_i) + \phi(x_j) + \nu \sqrt{\phi(x_i) \phi(x_j)}}$$
$$P(x_i \approx x_j) = \frac{\nu \sqrt{\phi(x_i) \phi(x_j)}}{\phi(x_i) + \phi(x_j) + \nu \sqrt{\phi(x_i) \phi(x_j)}}$$

(19)

where $\nu \geq 0$. The applications of these two tie-handling models to learning to rank are detailed in Appendix[C].

For ties of multiple objects, we can create a group of objects, and work directly on groups. For example, let $X_i$ and $X_j$ be two sport teams, the pairwise team ordering can be defined using the Bradley-Terry model as

$$P(X_i \succ X_j) = \frac{\sum_{x \in X_i} \phi(x)}{\sum_{x \in X_i} \phi(x) + \sum_{x \in X_j} \phi(x)}$$

12Note that, this is different from saying the states of variables are independent.
The extension of the Plackett-Luce model to multiple groups has been discussed in [14]. However, we should emphasize that this setting is not the same as ours, because the partitioning is known in advance, and the groups behave just like standard super-objects. Our setting, on the other hand, assumes no fixed partitioning, and the membership of the objects in a group is arbitrary.

5 Evaluation

5.1 Setting

The data is from Yahoo! learning to rank challenge [28]. This is currently the largest dataset available for research. At the time of this writing, the data contains the groundtruth labels of 473,134 documents returned from 19,944 queries. The label is the relevance judgment from 0 (irrelevant) to 4 (perfectly relevant). Features for each document-query pairs are also supplied by Yahoo!, and there are 519 unique features.

We split the data into two sets: the training set contains roughly 90% queries, and the test set is the remaining 10%. Two performance metrics are reported: the Normalised Discounted Cumulative Gain at position $T$ (NDCG@$T$), and the Expected Reciprocal Rank (ERR). NDCG@$T$ metric is defined as

$$
\text{NDCG}@T = \frac{1}{\kappa(T)} \sum_{i=1}^{T} \frac{2^r_i - 1}{\log_2(1+i)}
$$

where $r_i$ is the relevance judgment of the document at position $i$, $\kappa(T)$ is a normalisation constant to make sure that the gain is 1 if the rank is correct. The ERR is defined as

$$
\text{ERR} = \sum_i \frac{1}{i} V(r_i) \prod_{j=1}^{i-1} (1 - V(r_j)) \text{ where } V(r) = \frac{2^r - 1}{16}
$$

which puts even more emphasis on the top-ranked documents.

For comparison, we implement several well-known methods, including RankNet [3], Ranking SVM [15] and ListMLE [27]. The RankNet and Ranking SVM are pairwise methods, and they differ on the choice of loss functions, i.e. logistic loss for the RankNet and hinge loss for the Ranking SVM [13]. Similarly, choosing quadratic loss gives us a rank regression method, which we will call Rank Regress. From rank modelling point of view, the RankNet is essentially the Bradley-Terry model [1] applied to learning to rank. Likewise, the ListMLE is essentially the Plackett-Luce model. We also implement two variants of the Bradley-Terry model with ties handling, one by Rao-Kupper [24] (denoted by PairTies-RK; this also appears to be implemented in [29] under the functional gradient setting) and another by Davidson [7] (denoted by PairTies-D; and this is the first time the Davidson method is applied to learning to rank). See Appendix C for implementation details.

There are three methods resulted from our framework (see description in Section 3.4). The first is the PMOP with full-decomposition (denoted by PMOP-FD), the second is with Gibbs sampling (denoted by PMOP-Gibbs), and the third is with Metropolis-Hastings sampling (denoted by PMOP-MH).

12Strictly speaking, RankNet makes use of neural networks as the scoring function, but the overall loss is still logistic, and for simplicity, we use simple perceptron.
For those pairwise methods without ties handling, we simply ignore the tied document pairs. For the ListMLE, we simply sort the documents within a query by relevance scores, and those with ties are ordered according to the sorting algorithm. All methods, except for PMOP-Gibbs/MH, are trained using the Limited Memory Newton Method known as L-BFGS. The L-BFGS is stopped if the relative improvement over the loss is less than $10^{-5}$ or after 100 iterations. As the PMOP-Gibbs/MH are stochastic, we run the MCMC for a few steps per query, then update the parameter using the Stochastic Gradient Ascent. The learning rate is fixed to 0.1, and the learning is stopped after 1,000 iterations.

As for feature representation, we first normalised the features across the whole training set to roughly have mean 0 and standard deviation 1. We then employ both the first-order features and second-order features (by taking the Cartesian product of first-order features). The rationale for the second-order features is that since the first-order features are selected manually based on Yahoo! experience, features are highly correlated. Thus second-order features may capture aspects not previously thought by feature designers. Since the number of second-order features is large, we perform a correlation-based selection. First, we compute the Pearson’s correlation between each second-order feature with the label, then choose those features whose absolute correlation is beyond a threshold. For this particular data, we found the threshold of 0.15 is useful, although we did not perform an extensive search. The number of selected second-order features is 14,188.

### 5.2 Results

The results are reported in Table 1. The following conclusions can be drawn. First, the use of second order features improves the performance for nearly all the baseline methods. In our algorithms, the second order features yield better performance for PMOP-FD (incorporating the full decomposition).

Second, using either first or second order features, all our algorithms outperform the baseline methods. For example, the PMOP-MH wins over the best performing baseline, ListMLE, by 1.82%, using first-order features. In our view, this is a significant improvement given the scope of
Table 2: Learning complexity of models, where \( F \) is the number of unique features. For pairwise models, see Appendix B for the details.

| Pairwise models | PMOP/ListMLE |
|-----------------|--------------|
| \( \max\{O(N^2), O(NF)\} \) | \( O(NF) \) |

The dataset. We note that the difference in the top 20 in the leaderboard of the Yahoo! challenge is just 1.56%.

As for training time, the PMOP-FD is numerically the fastest method. Theoretically, it has the linear complexity similar to ListMLE. All other pairwise methods are quadratic in query size, and thus numerically slower. The PMOP-Gibbs/MH is also linear in the query size, by a constant factor that is determined by the number of iterations. See Table 2 for a summary.

6 Conclusions

Addressing the general problem of ranking with ties, we have proposed a generative probabilistic model, with suitable parameterisation to address the problem complexity. We present efficient algorithms for learning and inference. We evaluate the proposed models on the problem of learning to rank with the data from the currently held Yahoo! challenge, demonstrating that the models are competitive against well-known rivals designed specifically for the problem, both in predictive performance and training time.

References

[1] R.A. Bradley and M.E. Terry. Rank analysis of incomplete block designs. *Biometrika*, 39:324–345, 1952.

[2] S. Brin and L. Page. The anatomy of a large-scale hypertextual Web search engine. *Computer networks and ISDN systems*, 30(1-7):107–117, 1998.

[3] C. Burges, T. Shaked, E. Renshaw, A. Lazier, M. Deeds, N. Hamilton, and G. Hullender. Learning to rank using gradient descent. In *Proc. of ICML*, page 96, 2005.

[4] Z. Cao, T. Qin, T.Y. Liu, M.F. Tsai, and H. Li. Learning to rank: from pairwise approach to listwise approach. In *Proceedings of the 24th international conference on Machine learning*, page 136. ACM, 2007.

[5] W. Chu and Z. Ghahramani. Gaussian processes for ordinal regression. *Journal of Machine Learning Research*, 6(1):1019, 2006.

[6] D. Cossock and T. Zhang. Statistical analysis of bayes optimal subset ranking. *IEEE Transactions on Information Theory*, 54(11):5140–5154, 2008.

[7] R.R. Davidson. On extending the Bradley-Terry model to accommodate ties in paired comparison experiments. *Journal of the American Statistical Association*, 65(329):317–328, 1970.
[8] P. Diaconis. A generalization of spectral analysis with application to ranked data. *The Annals of Statistics*, pages 949–979, 1989.

[9] M.A. Fligner and J.S. Verducci. Multistage ranking models. *Journal of the American Statistical Association*, 83(403):892–901, 1988.

[10] Y. Freund, R. Iyer, R.E. Schapire, and Y. Singer. An efficient boosting algorithm for combining preferences. *Journal of Machine Learning Research*, 4(6):933–969, 2004.

[11] WA Glenn and HA David. Ties in paired-comparison experiments using a modified Thurstone–Mosteller model. *Biometrics*, 16(1):86–109, 1960.

[12] G.E. Hinton. Training products of experts by minimizing contrastive divergence. *Neural Computation*, 14:1771–1800, 2002.

[13] J. Huang, C. Guestrin, and L. Guibas. Fourier theoretic probabilistic inference over permutations. *The Journal of Machine Learning Research*, 10:997–1070, 2009.

[14] T.K. Huang, R.C. Weng, and C.J. Lin. Generalized Bradley-Terry models and multi-class probability estimates. *The Journal of Machine Learning Research*, 7:115, 2006.

[15] T. Joachims. Optimizing search engines using clickthrough data. In *Proc. of SIGKDD*, pages 133–142. ACM New York, NY, USA, 2002.

[16] S.L. Lauritzen. *Graphical Models*. Oxford Science Publications, 1996.

[17] T.Y. Liu. Learning to rank for information retrieval. *Foundations and Trends in Information Retrieval*, 3(3):225–331, 2009.

[18] R.D. Luce. *Individual choice behavior*. Wiley New York, 1959.

[19] C.L. Mallows. Non-null ranking models. I. *Biometrika*, 44(1):114–130, 1957.

[20] J.I. Marden. *Analyzing and modeling rank data*. Chapman & Hall/CRC, 1995.

[21] M. Mureşan. *A concrete approach to classical analysis*. Springer Verlag, 2008.

[22] J. Pearl. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann, San Francisco, CA, 1988.

[23] R.L. Plackett. The analysis of permutations. *Applied Statistics*, pages 193–202, 1975.

[24] P.V. Rao and L.L. Kupper. Ties in paired-comparison experiments: A generalization of the Bradley-Terry model. *Journal of the American Statistical Association*, pages 194–204, 1967.

[25] J.H. van Lint and R.M. Wilson. *A course in combinatorics*. Cambridge Univ Pr, 1992.

[26] M.N. Volkovs and R.S. Zemel. BoltzRank: learning to maximize expected ranking gain. In *Proceedings of the 26th Annual International Conference on Machine Learning*. ACM New York, NY, USA, 2009.

[27] F. Xia, T.Y. Liu, J. Wang, W. Zhang, and H. Li. Listwise approach to learning to rank: theory and algorithm. In *Proc. of ICML*, pages 1192–1199, 2008.
A Computing $C$

Let us calculate the constant $C$ in Eq (7). Let use rewrite the equation for ease of comprehension

$$\sum_{S \in 2^R_k} \frac{1}{|S|} \sum_{x \in S} \phi_k(x) = C \times \sum_{x \in R_k} \phi_k(x)$$

where $2^R_i$ is the power set with respect to the set $R_k$, or the set of all non-empty subsets of $R_k$. Equivalently

$$C = \sum_{S \in 2^R_k} \frac{1}{|S|} \sum_{x \in S} \sum_{x \in R_k} \phi_k(x)$$

If all objects are the same, then this can be simplified to

$$C = \sum_{S \in 2^R_k} \frac{1}{|S|} \sum_{x \in S} 1 = \frac{1}{N_k} \sum_{S \in 2^R_k} 1 = \frac{2^{N_k} - 1}{N_k}$$

where $N_k = |R_k|$. In the last equation, we have made use of the fact that $\sum_{S \in 2^R_k} 1$ is the number of all possible non-empty subsets, or equivalently, the size of the power set, which is known to be $2^{N_k} - 1$. One way to derive this result is the imagine a collection of $N_k$ variables, each has two states: ‘selected’ and ‘not selected’, where ‘selected’ means the object belongs to a subset. Since there are $2^{N_k}$ such configurations over all states, the number of non-empty subsets must be $2^{N_k} - 1$.

For arbitrary objects, let us examine the the probability that the object $x$ belong to a subset of size $m$, which is $\frac{N_k}{N_k}$. Recall from standard combinatorics that the number of $m$-element subsets is the binomial coefficient $\binom{N_k}{m}$, where $1 \leq m \leq N_k$, and . Thus the number of times an object appears in any $m$-subset is $\frac{N_k}{N_k}$. Taking into account that this number is weighted down by $m$ (i.e. $|S|$ in Eq (7)), the the contribution towards $C$ is then $\binom{N_k}{m} \frac{1}{N_k}$. Finally, we can compute the constant $C$, which is the weighted number of times an object belongs to any subset of any size, as follows

$$C = \sum_{m=1}^{N_k} \frac{N_k}{m} \frac{1}{N_k} = \frac{1}{N_k} \sum_{m=1}^{N_k} \binom{N_k}{m}$$

$$= \frac{2^{N_k} - 1}{N_k}$$

We have made use of the known identity $\sum_{m=1}^{N_k} \binom{N_k}{m} = 2^{N_k} - 1$.\[16\]
B Pairwise Losses

Let \( \delta_{ij}(w) = \phi(x_i, w) - \phi(x_j, w) \), the pairwise losses are

\[
\text{loss}(x_i \succ x_j; w) = \begin{cases} 
\log(1 + \exp(-\delta_{ij}(w))) & \text{for logistic loss in RankNet} \\
\max\{0, 1 - \delta_{ij}(w)\} & \text{for hinge loss in Ranking SVM} \\
(1 - \delta_{ij}(w))^2 & \text{for quadratic loss in Pair Regress}
\end{cases}
\]

The overall loss is then

\[ \text{Loss} = \sum_{i < j} \text{loss}(x_i \succ x_j; w) \]

Taking derivative with respect to \( w \) yields

\[
\frac{\partial \text{Loss}}{\partial w} = \sum_i \sum_{j < i} \frac{\partial \text{loss}(x_i \succ x_j; w)}{\partial \delta_{ij}(w)} \left( \frac{\partial \phi(x_i, w)}{\partial w} - \frac{\partial \phi(x_j, w)}{\partial w} \right) = \sum_i \left( \sum_{j < i} \frac{\partial \text{loss}(x_i \succ x_j; w)}{\partial \delta_{ij}(w)} \right) \frac{\partial \phi(x_i, w)}{\partial w} - \sum_j \left( \sum_{i > j} \frac{\partial \text{loss}(x_i \succ x_j; w)}{\partial \delta_{ij}(w)} \right) \frac{\partial \phi(x_j, w)}{\partial w}
\]

As it takes \( N^2 \) time to compute all the partial derivatives \( \frac{\partial \text{loss}(x_i \succ x_j; w)}{\partial \delta_{ij}(w)} \) for all \( i, j \) where \( j < i \), the overall gradient requires \( N^2 + NF \) time. Thus the complexity of the pairwise methods is \( O(\max\{N^2, NF\}) \).

C Learning the Paired Ties Models

This section describes the details of learning the paired ties models discussed in Section 4.

Rao-Kupper method. Recall that the Rao-Kupper model defines the following probability masses

\[
\begin{align*}
P(x_i \succ x_j | w) &= \frac{\phi(x_i, w)}{\phi(x_i, w) + \theta \phi(x_j, w)} \\
P(x_i \prec x_j | w) &= \frac{\phi(x_j, w)}{\theta \phi(x_i, w) + \phi(x_j, w)} \\
P(x_i \approx x_j | w) &= \frac{(\theta^2 - 1)\phi(x_i, w)\phi(x_j, w)}{[\phi(x_i, w) + \theta \phi(x_j, w)][\theta \phi(x_i, w) + \phi(x_j, w)]}
\end{align*}
\]
where $\theta \geq 1$ is the ties factor and $w$ is the model parameter. For ease of unconstrained optimisation, let $\theta = 1 + e^\alpha$ for $\alpha \in \mathbb{R}$. In learning, we want to estimate both $\alpha$ and $w$. Let

$$
P_i = \frac{\phi(x_i, w)}{\phi(x_i, w) + (1 + e^\alpha)\phi(x_j, w)}$$

$$
P_j^* = \frac{\phi(x_j, w)}{\phi(x_i, w) + (1 + e^\alpha)\phi(x_j, w)}$$

$$
P_i^* = \frac{\phi(x_i, w)}{(1 + e^\alpha)\phi(x_i, w) + \phi(x_j, w)}$$

$$
P_j = \frac{\phi(x_j, w)}{(1 + e^\alpha)\phi(x_i, w) + \phi(x_j, w)}$$

Taking partial derivatives of the log-likelihood gives

$$
\frac{\partial \log P(x_i > x_j | w)}{\partial w} = (1 - P_i) \frac{\partial \log \phi(x_i, w)}{\partial w} - (1 + e^\alpha)P_j \frac{\partial \log \phi(x_j, w)}{\partial w}
$$

$$
\frac{\partial \log P(x_i > x_j | w)}{\partial \alpha} = -P_i e^\alpha
$$

$$
\frac{\partial \log P(x_i \approx x_j | w)}{\partial w} = (1 - P_i - (1 + e^\alpha)P_j^*) \frac{\partial \log \phi(x_i, w)}{\partial w} + (1 - P_j - (1 + e^\alpha)P_j^*) \frac{\partial \log \phi(x_j, w)}{\partial w}
$$

$$
\frac{\partial \log P(x_i \approx x_j | w)}{\partial \alpha} = \frac{2(1 + e^\alpha)}{(1 + e^\alpha)^2 - 1} - P_i^* - P_j^* e^\alpha
$$

**Davidson method.** Recall that in the Davidson method the probability masses are defined as

$$
P(x_i > x_j | w) = \frac{\phi(x_i, w)}{\phi(x_i, w) + \phi(x_j, w) + \sqrt{\phi(x_i, w)\phi(x_j, w)}}$$

$$
P(x_i < x_j | w) = \frac{\phi(x_j, w)}{\phi(x_i, w) + \phi(x_j, w) + \sqrt{\phi(x_i, w)\phi(x_j, w)}}$$

$$
P(x_i \approx x_j | w) = \frac{\sqrt{\phi(x_i, w)\phi(x_j, w)}}{\phi(x_i, w) + \phi(x_j, w) + \sqrt{\phi(x_i, w)\phi(x_j, w)}}$$

where $\nu \geq 0$. Again, for simplicity of unconstrained optimisation, let $\nu = e^\beta$ for $\beta \in \mathbb{R}$. Let

$$
P_i = \frac{\phi(x_i, w)}{\phi(x_i, w) + \phi(x_j, w) + e^\beta \sqrt{\phi(x_i, w)\phi(x_j, w)}}$$

$$
P_j = \frac{\phi(x_j, w)}{\phi(x_i, w) + \phi(x_j, w) + e^\beta \sqrt{\phi(x_i, w)\phi(x_j, w)}}$$

$$
P_{ij} = \frac{e^\beta \sqrt{\phi(x_i, w)\phi(x_j, w)}}{\phi(x_i, w) + \phi(x_j, w) + e^\beta \sqrt{\phi(x_i, w)\phi(x_j, w)}}$$
Taking derivatives of the log-likelihood gives

\[
\frac{\partial \log P(x_i > x_j|w)}{\partial w} = (1 - P_i - 0.5P_{ij}) \frac{\partial \log \phi(x_i, w)}{\partial w} - (P_i + 0.5P_{ij}) \frac{\partial \log \phi(x_j, w)}{\partial w}
\]

\[
\frac{\partial \log P(x_i > x_j|w)}{\partial \beta} = -P_{ij}
\]

\[
\frac{\partial \log P(x_i \approx x_j|w)}{\partial w} = (0.5 - P_i - 0.5P_{ij}) \frac{\partial \log \phi(x_i, w)}{\partial w} + (0.5 - P_j - 0.5P_{ij}) \frac{\partial \log \phi(x_j, w)}{\partial w}
\]

\[
\frac{\partial \log P(x_i \approx x_j|w)}{\partial \beta} = 1 - P_{ij}
\]