DEFORMATION OF DELONE DYNAMICAL SYSTEMS 
AND PURE POINT DIFFRACTION

MICHAEL BAAKE AND DANIEL Lenz

Abstract. This paper deals with certain dynamical systems built from point sets and, more 
generally, measures on locally compact Abelian groups. These systems arise in the study 
of quasicrystals and aperiodic order, and important subclasses of them exhibit pure point 
diffraction spectra. We discuss the relevant framework and recall fundamental results and 
examples. In particular, we show that pure point diffraction is stable under “equivariant” 
local perturbations and discuss various examples, including deformed model sets. A key step 
in the proof of stability consists in transforming the problem into a question on factors of 
dynamical systems.

1. Introduction

Aperiodic order has become a topic of intense research over the last two decades [34, 38, 7, 
47, 49]. While the term is not rigorously defined (yet), it roughly refers to forms of order at the 
very verge between periodic and non-periodic structures. As such, it has attracted attention in 
various branches of mathematics including geometry, combinatorics, ergodic theory, operator 
theory and harmonic analysis.

An important trigger in these developments has been the actual discovery of physical 
substances with strong aperiodic order [42], which are now called quasicrystals. They owe 
their discovery to their remarkable diffraction patterns: These patterns imply a high degree 
of order as they are pure point spectra (or Bragg spectra), while, at the same time, they 
exclude periodicity by their non-crystallographic symmetries. Accordingly, the study of pure 
diffraction has been an important topic in this context ever since.

This paper is concerned with pure point diffraction. More precisely, we study the stability 
of pure point diffraction under certain deformations. This is a very natural one, both 
from the physical and the mathematical point of view. In order to study stability under 
deformations, we need to review the undeformed case first. To make the paper essentially 
self-contained, this discussion is carried out at some length, including relevant concepts and 
examples. Moreover, we hope that the paper can serve as an introductory survey on the 
treatment of diffraction via dynamical systems for the reader unfamiliar with the field.

Delone sets provide an important model class for the description of aperiodic order. In 
picular, they can be viewed as a mathematical abstraction of the set of atomic positions 
of a physical quasicrystal (at zero temperature, or at a given instant of time). Many of 
the rather intriguing spectral properties of quasicrystals can be formulated, in a simplified 
manner, on the basis of Delone sets. This is also a rather common class of structures in 
the mathematical theory of aperiodic order [29]. It is attractive because it admits a direct 
geometric interpretation with two Delone sets being close to one another if large patches 
(around some fixed point of reference, say) coincide, possibly after a tiny local rearrangement 
of the individual points.
However, from a more physical point of view, other scenarios are also very important. In particular, the description of an aperiodic distribution of matter by means of (continuous) quasi- or almost periodic density functions has been emphasized right from the very beginning of quasicrystal theory [9]. Here, closeness of two structures is more adequately described by means of the supremum norm, as in the theory of almost periodic functions.

As is apparent, these two pictures are not compatible — unless they are embedded into a larger class of structures that admit both the Delone (or tiling) picture and the continuous density description as special cases. One possibility is the use of translation bounded (complex) measures, equipped with the vague topology. Here, two structures (i.e., measures) are close if their evaluations with continuous functions supported on a large compact set \( K \) are close. This entails both situations mentioned above, one being described by pure point measures, the other by absolutely continuous measures with continuous Radon-Nikodym densities.

In view of the fact that the original distinction between the discrete and the continuous approach led to rather hefty disputes on the justification and appropriateness of the two approaches, we believe that the systematic development of a unified frame is overdue. We take this as our main motivation for a dynamical systems approach based on measures, though we shall also spell out the details for the more conventional (and perhaps more intuitive) approach via Delone sets.

As mentioned already, one important issue in this context is that of the stability of certain features, e.g., stability under slight modifications or deformations. The question of stability of pure point diffractivity is addressed in this paper.

Our main abstract result shows that pure point diffraction is stable under local “equivariant” perturbations. The proof relies on two steps: We use a recent result of ours [3] (see [30, 22] for related material) which establishes when pure point dynamical spectrum is equivalent to pure point diffraction spectrum. This effectively transforms the stability problem into a question on dynamical systems. This question is then solved by studying certain factors of the original dynamical system.

To give the reader a flavour of this procedure, we include the following rather informal statement of our main result, when restricted to Delone sets.

**Result.** The hull of an admissibly deformed Delone set is a topological factor of the hull of the original Delone set. In particular, if a Delone set has pure point diffraction spectrum, its deformation has pure point diffraction spectrum as well.

A precise version of this result is given in Theorem 3. As mentioned already, our setting is general enough to treat not only the case of Delone sets but rather the case of arbitrary measure dynamical systems. This is made precise in Theorem 4.

The abstract result is applied to various examples. In particular, we study perturbations of model sets in the context of cut and project schemes. This generalizes the corresponding considerations of Hof [24] and Bernou and Duneau [11]. It also shows that related results of Clark and Sadun [12] fall well within our framework.

Our results should be compared to complementary results of Hof [26]. They show that random perturbations do not leave a pure point spectrum unchanged, but rather introduce an absolutely continuous component, see also [2, 15] for further examples.

We are well aware of the fact that considerable parts of the following investigation dealing with topological dynamical systems can be generalized to measurable dynamical systems.
However, by its very nature, the subject of aperiodic order seems to be a topological one. For this reason, we stick to the topological category.

The paper is organized as follows. In Section 2, we introduce some basic notation concerning topological dynamical systems. In Section 3, we recall and establish various facts on factors. These considerations are the abstract core behind our deformation procedure. Section 4 is devoted to a discussion of diffraction in the context of dynamical systems of Delone sets and measures. The abstract deformation procedure and the stability of pure point diffraction under this type of deformation is discussed in Section 5. Applications to model sets are studied in Section 6, which also contains a brief summary of their general definition. The various concepts and results will then be illustrated with a concrete example, the silver mean chain, in Section 7. Further aspects of the deformation procedure, in particular concerning topological conjugacy, are discussed in Section 8.

2. Generalities on dynamical systems

Our considerations are set in the framework of topological dynamical systems. We are dealing with \( \sigma \)-compact locally compact topological groups and compact spaces. Thus, we start with some basic notation and facts concerning locally compact topological spaces used throughout the paper.

Whenever \( X \) is a \( \sigma \)-compact locally compact space (by which we mean to include the Hausdorff property), we denote the space of continuous functions on \( X \) by \( C(X) \) and the subspace of continuous functions with compact support by \( C_c(X) \). This space is equipped with the locally convex limit topology induced by the canonical embeddings \( C_K(X) \hookrightarrow C_c(X) \), where \( C_K(X) \) is the space of complex continuous functions with support in a given compact set \( K \subset X \). Here, each \( C_K(X) \) is equipped with the topology induced by the standard supremum norm.

As \( X \) is a topological space, it carries a natural \( \sigma \)-algebra, namely the Borel \( \sigma \)-algebra generated by all closed subsets of \( X \). The set \( M(X) \) of all complex regular Borel measures on \( G \) can then be identified with the space \( C_c(X)^* \) of complex valued, continuous linear functionals on \( C_c(G) \). This is justified by the Riesz-Markov representation theorem, compare [39, Ch. 6.5] for details. In particular, we can write \( \int_X f \, d\mu = \mu(f) \) for \( f \in C_c(X) \) and simplify the notation this way. The space \( M(X) \) carries the vague topology, i.e., the weakest topology that makes all functionals \( \mu \mapsto \mu(\varphi) \), \( \varphi \in C_c(X) \), continuous. The total variation of a measure \( \mu \in M(X) \) is denoted by \( |\mu| \).

We now fix a \( \sigma \)-compact locally compact Abelian (LCA) group \( G \) for the remainder of the paper. The dual group of \( G \) is denoted by \( \hat{G} \), and the pairing between a character \( \hat{s} \in \hat{G} \) and \( t \in G \) is written as \( (\hat{s}, t) \). Whenever \( G \) acts on the compact space \( \Omega \) (which is then also Hausdorff by our convention) by a continuous action

\[
\alpha: G \times \Omega \rightarrow \Omega, \quad (t, \omega) \mapsto \alpha_t(\omega),
\]

where \( G \times \Omega \) carries the product topology, the pair \( (\Omega, \alpha) \) is called a topological dynamical system over \( G \). We shall often write \( \alpha_t \omega \) for \( \alpha_t(\omega) \). If \( \omega \in \Omega \) satisfies \( \alpha_t \omega = \omega \), \( t \) is called a period of \( \omega \). If all \( t \in G \) are periods, \( \omega \) is called \( G \)-invariant, or \( \alpha \)-invariant to refer to the action involved.

The set of all Borel probability measures on \( \Omega \) is denoted by \( \mathcal{P}(\Omega) \), and the subset of \( \alpha \)-invariant probability measures by \( \mathcal{P}_G(\Omega) \). As \( \Omega \) is compact, \( C_c(\Omega) \) equipped with the supremum norm is a Banach space. The vague topology on \( M(\Omega) \) is then just the weak-\( * \)
topology. By Alaoglu’s theorem on weak-* compactness of the unit sphere (compare [39 Thm. 2.5.2]), we easily conclude that \( P(\Omega) \) is compact. As \( P_G(\Omega) \) is obviously closed in \( P(\Omega) \), it is then compact as well. Apparently, \( P_G(\Omega) \) is convex. More importantly, it is always non-empty. For \( G = \mathbb{Z} \), this is standard, compare Section 6.2 in [50]. This proof only uses the existence of a van Hove sequence and the compactness of \( P(\Omega) \). Thus, it can be carried over to our setting (for the existence of van Hove sequences, we refer the reader to [44 p. 145] and [48 Appendix, Sec. 3.3]).

An \( \alpha \)-invariant probability measure is called \textit{ergodic} if every (measurable) invariant subset of \( \Omega \) has either measure zero or measure one. The ergodic measures are exactly the extremal points of the convex set \( P_G(\Omega) \). The dynamical system \((\Omega, \alpha)\) is called \textit{uniquely ergodic} if \( P_G(\Omega) \) is a singleton set, i.e., if it consists of exactly one element. As usual, \((\Omega, \alpha)\) is called \textit{minimal} if, for all \( \omega \in \Omega \), the \( G \)-orbit \( \{\alpha_t \omega : t \in G\} \) is dense in \( \Omega \). If \((\Omega, \alpha)\) is both uniquely ergodic and minimal, it is called \textit{strictly ergodic}.

Given an \( m \in P_G(\Omega) \), we can form the Hilbert space \( L^2(\Omega, m) \) of square integrable measurable functions on \( \Omega \). This space is equipped with the inner product

\[
\langle f, g \rangle = \langle f, g \rangle_\Omega := \int_\Omega \overline{f(\omega)} g(\omega) \, dm(\omega).
\]

The action \( \alpha \) gives rise to a unitary representation \( T = T^\Omega := T^{(\Omega, \alpha, m)} \) of \( G \) on \( L^2(\Omega, m) \) by

\[
T_t: L^2(\Omega, m) \rightarrow L^2(\Omega, m), \quad (T_t f)(\omega) := f(\alpha_{-t} \omega),
\]

for every \( f \in L^2(\Omega, m) \) and arbitrary \( t \in G \).

An \( f \in L^2(\Omega, m) \) is called an \textit{eigenfunction of} \( T \) with \textit{eigenvalue} \( \hat{s} \in \hat{G} \) if \( T_t f = (\hat{s}, t) f \) for every \( t \in G \). An eigenfunction \((\hat{s}, \text{say})\) is called \textit{continuous} if it has a continuous representative \( f \) with \( f(\alpha_{-t} \omega) = (\hat{s}, t) f(\omega) \), for all \( \omega \in \Omega \) and \( t \in G \). The representation \( T \) is said to have \textit{pure point spectrum} if the set of its eigenfunctions is total in \( L^2(\Omega, m) \). One then also says that the dynamical system \((\Omega, \alpha)\) has \textit{pure point dynamical spectrum}.

By Stone’s theorem, compare [32 Sec. 36D], there exists a projection valued measure

\[
E_T: \text{Borel sets of } \hat{G} \rightarrow \text{projections on } L^2(\Omega, m)
\]

with

\[
\langle f, T_t f \rangle = \int_{\hat{G}} \langle \hat{s}, t \rangle \, d\langle f, E_T(\cdot) f \rangle(\hat{s}) := \int_{\hat{G}} \langle \hat{s}, t \rangle \, d\rho_f(\hat{s}),
\]

where \( \rho_f = \rho_f^\Omega := \rho_f^{(\Omega, \alpha, m)} \) is the measure on \( \hat{G} \) defined by \( \rho_f(B) := \langle f, E_T(B) f \rangle \). In fact, by Bochner’s theorem [41], \( \rho_f \) is the unique measure on \( \hat{G} \) with \( \langle f, T_t f \rangle = \int_{\hat{G}} \langle \hat{s}, t \rangle \, d\rho_f(\hat{s}) \) for every \( t \in \hat{G} \).

3. Factors

Factors of dynamical systems and the corresponding subrepresentations will be an important tool in our study of deformations. In this section, we recall their basic theory, most of which is well known. Since details are somewhat scattered in the literature, we sketch some of the proofs for the sake of completeness, or give precise references. Readers who are familiar with it, or are more interested to first learn about diffraction, may skip this section at first reading.
Let \((\Omega, \alpha)\) and \((\Theta, \beta)\) be two topological dynamical systems under the action of \(G\), with a mapping \(\Phi: \Omega \rightarrow \Theta\) that gives rise to the following diagram:

\[
\begin{array}{ccc}
\Omega & \xrightarrow{\alpha} & \Omega \\
\downarrow{\phi} & & \downarrow{\phi} \\
\Theta & \xrightarrow{\beta} & \Theta
\end{array}
\]

(1)

**Definition 1.** Let two topological dynamical systems \((\Omega, \alpha)\) and \((\Theta, \beta)\) under the action of \(G\) and a mapping \(\Phi: \Omega \rightarrow \Theta\) be given. Then, \((\Theta, \beta)\) is called a factor of \((\Omega, \alpha)\), with factor map \(\Phi\), if \(\Phi\) is a continuous surjection that makes the diagram \(\begin{array}{ccc} \Omega & \xrightarrow{\alpha} & \Omega \\
\downarrow{\phi} & & \downarrow{\phi} \\
\Theta & \xrightarrow{\beta} & \Theta \end{array}\) commutative, i.e., \(\Phi(\alpha_t(\omega)) = \beta_t(\Phi(\omega))\) for all \(\omega \in \Omega\) and \(t \in G\).

Factors inherit many features from the underlying dynamical system. Due to the commutativity of diagram \(\begin{array}{ccc} \Omega & \xrightarrow{\alpha} & \Omega \\
\downarrow{\phi} & & \downarrow{\phi} \\
\Theta & \xrightarrow{\beta} & \Theta \end{array}\), a period \(t \in G\) of \(\omega\) is also a period of \(\Phi(\omega)\). Clearly, the converse need not be true, as we shall see in an example later on. Let us next recall three other properties of dynamical systems which are inherited by factors.

**Fact 1.** Let \((\Theta, \beta)\) be a factor of \((\Omega, \alpha)\), with factor map \(\Phi: \Omega \rightarrow \Theta\). Then, \(U \subset \Theta\) is open if and only if \(\Phi^{-1}(U)\) is open in \(\Omega\).

**Proof.** As \(\Phi\) is continuous, the only if part is clear. So, assume that \(U \subset \Theta\) is given with \(\Phi^{-1}(U)\) open. Then, \(\Phi^{-1}(\Theta \setminus U) = \Omega \setminus \Phi^{-1}(U)\) is closed and thus compact, as \(\Omega\) is compact. Thus, by continuity and surjectivity of \(\Phi\), the set \(\Theta \setminus U = \Phi(\Phi^{-1}(\Theta \setminus U))\) is compact and, in particular, closed. Thus, \(U\) is open.

Clearly, \(\Phi\) induces a mapping \(\Phi_*: \mathcal{M}(\Omega) \rightarrow \mathcal{M}(\Theta), \mu \mapsto \Phi_*\mu\), via \((\Phi_*\mu)(g) := \mu(g \circ \Phi)\) for all \(g \in C(\Theta)\). If \(\mu\) is a probability measure on \(\Omega\), its image, \(\Phi_*\mu\), is a probability measure on \(\Theta\). Moreover, if \(\Phi\) is a factor map, invariance under the group action is preserved. So, in this case, we obtain the mapping

\[
\Phi_*: \mathcal{P}_G(\Omega) \rightarrow \mathcal{P}_G(\Theta), \quad \mu \mapsto \Phi_*\mu,
\]

where we stick to the same symbol, \(\Phi_*\), for simplicity.

**Fact 2.** Let \((\Theta, \beta)\) be a factor of \((\Omega, \alpha)\), with factor map \(\Phi: \Omega \rightarrow \Theta\). Then, \(\Phi_*\) is continuous, and by \([15]\) Prop. 3.11, it is onto. Direct calculations show \(\Phi_*\left( \sum c_i \mu_i \right) = \sum c_i \Phi_*\mu_i\) for every finite convex combination \(\sum c_i \mu_i\) of measures in \(\mathcal{P}_G(\Omega)\).

Let \(\mu \in \mathcal{P}_G(\Omega)\) be ergodic, i.e., any \(\alpha\)-invariant measurable subset \(A\) of \(\Omega\) satisfies either \(\mu(A) = 0\) or \(\mu(A) = 1\). Consider \(\nu := \Phi_*\mu \in \mathcal{P}_G(\Theta)\), and let \(B\) be a \(\beta\)-invariant measurable subset of \(\Theta\), i.e., \(\beta_t(B) = B\) for all \(t \in G\). Clearly, one has \(\nu(B) = \mu(\Phi^{-1}(B))\), where \(A := \Phi^{-1}(B) = \{\omega \in \Omega: \Phi(\omega) \in B\}\) is \(\alpha\)-invariant, as a consequence of \(\begin{array}{ccc} \Omega & \xrightarrow{\alpha} & \Omega \\
\downarrow{\phi} & & \downarrow{\phi} \\
\Theta & \xrightarrow{\beta} & \Theta \end{array}\). So, \(\nu(B) = \mu(A)\) is either 0 or 1, and \(\nu\) is also ergodic. The final claim about the extremal points is then standard, compare \([15]\) Prop. 5.6.

**Fact 3.** Let \((\Theta, \beta)\) be a factor of \((\Omega, \alpha)\), with factor map \(\Phi: \Omega \rightarrow \Theta\). If \((\Omega, \alpha)\) is uniquely ergodic, minimal or strictly ergodic, the analogous property holds for \((\Theta, \beta)\) as well.
Proof. If $(\Omega, \alpha)$ is uniquely ergodic, $\mathcal{P}_G(\Omega)$ is a singleton set, and $\mathcal{P}_G(\Theta) = \Phi_*(\mathcal{P}_G(\Omega))$ must then also be a singleton set, by Fact 2. So, also $(\Theta, \beta)$ is uniquely ergodic. Apparently, every $G$-orbit in $\Theta$ is the image of a $G$-orbit in $\Omega$, under the factor map $\Phi$. Continuity of $\Phi$ implies $\Phi(C) \subset \Phi(\overline{C}) \subset \overline{\Phi(C)}$ for arbitrary $C \subset \Omega$. If $C$ is dense, $\overline{C} = \Omega$, and $\overline{\Phi(C)} = \Phi(\overline{C}) = \Theta$ because $\Phi$ is onto. This shows that minimality is properly inherited, and the last claim on strict ergodicity is then obvious. \hfill \Box

Now, let $(\Theta, \beta)$ be a factor of $(\Omega, \alpha)$ with factor map $\Phi : \Omega \rightarrow \Theta$ and let $m \in \mathcal{P}_G(\Omega)$ be fixed. For the remainder of this section, we denote the induced measure on $\Theta$ by $n = \Phi_*(m)$. Consider the mapping

$$i^\Phi : L^2(\Theta, n) \rightarrow L^2(\Omega, m), \quad f \mapsto f \circ \Phi,$$

and let $p_\Phi : L^2(\Omega, m) \rightarrow L^2(\Theta, n)$ be the adjoint of $i^\Phi$. The maps $i^\Phi$ and $p_\Phi$ are partial isometries. More precisely, $i^\Phi$ is even an isometric embedding because

$$\langle i^\Phi(g), i^\Phi(f) \rangle_\Theta = \int_{\Omega} (g \circ \Phi)(f \circ \Phi) \, dm = \langle \Phi_*(m)(\overline{f}), \overline{g} \rangle = \langle g, f \rangle_\Theta$$

for arbitrary $f, g \in L^2(\Theta, n)$. As $i^\Phi$ is an isometry from $L^2(\Theta, n)$ with range $i^\Phi(L^2(\Theta, n))$, standard theory of partial isometries (compare [51, Thm. 4.34]) implies

$$p_\Phi \circ i^\Phi = \text{id}_{L^2(\Theta, n)} \quad \text{and} \quad i^\Phi \circ p_\Phi = P_{i^\Phi}(L^2(\Theta, n)),$$

where $\text{id}_{L^2(\Theta, n)}$ is the identity on $L^2(\Theta, n)$ and $P_{i^\Phi}(L^2(\Theta, n))$ is the orthogonal projection of $L^2(\Omega, m)$ onto $\mathcal{V} := i^\Phi(L^2(\Theta, n))$.

Given these maps, we can discuss the relation between the spectral theory of $T^\Theta$ and $T^\Omega$.

**Theorem 1.** Let $L^2(\Omega, m)$ and $L^2(\Theta, n)$ be the canonical Hilbert spaces attached to the dynamical systems $(\Omega, \alpha)$ and $(\Theta, \beta)$, with factor map $\Phi$ and $n = \Phi_*(m)$. Then, the partial isometries $i^\Phi$ and $p_\Phi$ are compatible with the unitary representations $T^\Theta$ and $T^\Omega$ of $G$ on $L^2(\Omega, m)$ and $L^2(\Theta, n)$, i.e.,

$$i^\Phi \circ T^\Theta_t = T^\Omega_t \circ i^\Phi \quad \text{and} \quad T^\Theta_t \circ p_\Phi = p_\Phi \circ T^\Omega_t,$$

for all $t \in G$. Similarly, the spectral families $E_{T^\Theta}$ and $E_{T^\Omega}$ satisfy

$$i^\Phi \circ E_{T^\Theta}(\cdot) = E_{T^\Omega}(\cdot) \circ i^\Phi \quad \text{and} \quad E_{T^\Theta}(\cdot) \circ p_\Phi = p_\Phi \circ E_{T^\Omega}(\cdot).$$

The corresponding measures satisfy $\rho^\Theta_g = \rho^\Omega_{i^\Phi(g)}$ for every $g \in L^2(\Theta, n)$.

**Proof.** Let $g \in L^2(\Theta, n)$ be given. As $\Phi$ is a factor map, a short calculation gives

$$(T^\Theta_t(i^\Phi(g))(\omega) = g(\Phi(\alpha^{-t} \omega)) = g(\beta^{-t} \Phi(\omega)) = (i^\Phi(T^\Omega_t)(g))(\omega)$$

and the first of the equations stated above follows. The second follows by taking adjoints.

Choose $g \in L^2(\Theta, n)$. As discussed above, $\rho^\Theta_g$ is the unique measure on $\hat{G}$ with

$$\langle g, T^\Theta_t \rangle_\Theta = \int_{\hat{G}} \langle \hat{s}, t \rangle \, d\rho^\Theta_g(\hat{s}), \quad \text{for all } t \in G.$$

Similarly, $\rho^\Omega_{i^\Phi(g)}$ is the unique measure on $\hat{G}$ with

$$\langle i^\Phi(g), T^\Omega_t \rangle_\Omega = \int_{\hat{G}} \langle \hat{s}, t \rangle \, d\rho^\Omega_{i^\Phi(g)}(\hat{s}), \quad \text{for all } t \in G.$$
Moreover, as \( i^\Phi \) is an isometry, we obtain from the statements proved so far that
\[
\langle g, T^\Theta_i g \rangle_\Theta = \langle i^\Phi(g), i^\Phi(T^\Theta_i g) \rangle_\Omega = \langle i^\Phi(g), T^\Omega_i i^\Phi(g) \rangle_\Omega.
\]
Putting the last three equations together, we obtain
\[
\int_G (\hat{s}^\Theta, t) \, d\rho^\Theta_g(\hat{s}) = \int_G (\hat{s}^\Theta, t) \, d\rho^\Omega_g(\hat{s})
\]
for every \( t \in G \). By the mentioned uniqueness of the involved measures, this gives
\[
\rho^\Theta_g = \rho^\Omega_g.
\]
This, in turn, implies
\[
\langle g, E_T^{\Theta}(B)g \rangle_\Theta = \rho^\Theta_g(B) = \rho^\Omega_g(B) = \langle i^\Phi(g), E_T^{\alpha}(B)i^\Phi(g) \rangle_\Omega = \langle g, p_\Phi E_T^{\alpha}(B)i^\Phi(g) \rangle_\Omega
\]
for all Borel measurable \( B \subset \hat{G} \) and every \( g \in L^2(\Theta, n) \). As \( g \in L^2(\Theta, n) \) is arbitrary, we infer \( E_T^{\theta}(\cdot) = p_\Phi E_T^{\alpha}(\cdot)i^\Phi \). \( \square \)

One succinct way to summarize the core of Theorem 1 is to say that the following diagram is commutative, with the map \( i^\Phi \) (resp. \( p_\Phi \)) being injective (resp. surjective).

\[
\begin{array}{ccc}
L^2(\Theta, n) & \xrightarrow{i^\Phi} & L^2(\Omega, m) \\
& \xrightarrow{T^\Theta} & \\
L^2(\Theta, n) & \xrightarrow{i^\Phi} & L^2(\Omega, m) \\
\end{array}
\]

**Corollary 1.** Assume the situation of Theorem 1 and define \( V = i^\Phi(L^2(\Theta, n)) \). Then, \( U : L^2(\Theta, n) \rightarrow V, f \mapsto i^\Phi(f) \), is a unitary map, the subspace \( V \) of \( L^2(\Omega, m) \) is invariant under \( T^\Omega \), and the restriction \( T^\Omega |_V \) of \( T^\Omega \) to \( V \) is unitarily equivalent to \( T^\Theta \) via \( U \).

**Proof.** As \( i^\Phi \) is an isometric embedding, the map \( U : L^2(\Theta, n) \longrightarrow i^\Phi(L^2(\Theta, n)) \) is unitary. By Theorem 1 we have \( i^\Phi \circ T^\Theta = T^\Omega \circ i^\Phi \). Consequently, the space \( V = i^\Phi(L^2(\Theta, n)) \) is invariant under \( T^\Omega \), with \( T^\Omega |_V U g = UT^\Theta g \) for every \( g \in L^2(\Theta, n) \). \( \square \)

The foregoing results describe the relationship between \( T^\Theta \) and \( T^\Omega \) in the general case. In the special case of pure point spectrum, we can be more explicit as follows.

**Proposition 1.** Let \( (\Theta, \beta) \) be a factor of the dynamical system \( (\Omega, \alpha) \), with factor map \( \Phi : \Omega \longrightarrow \Theta \). Let \( m \in \mathcal{P}_G(\Omega) \) be given, \( n = \Phi_*(m) \), and let \( L^2(\Theta, n) \) and \( L^2(\Omega, m) \) be the corresponding Hilbert spaces. Then, the following assertions hold.

(a) If \( g \) is an eigenfunction of \( T^\Theta \) to the eigenvalue \( \hat{s} \), \( i^\Phi(g) = g \circ \Phi \) is an eigenfunction of \( T^\Omega \) to the eigenvalue \( \hat{s} \).

(b) If \( T^\Omega \) has pure point dynamical spectrum, the same is true of \( T^\Theta \).

**Proof.** (a): Let \( g \) be an eigenfunction of \( T^\Theta \). Then, \( i^\Phi(g) = g \circ \Phi \) is an eigenfunction of \( T^\Omega \), as \( i^\Phi \circ T^\Theta_i = T^\Omega_i \circ i^\Phi \) by Theorem 1.

(b): If \( T^\Omega \) has pure point dynamical spectrum, there exists an orthonormal basis of \( L^2(\Omega, m) \) which entirely consists of eigenfunctions of \( T^\Omega \). Now, by Theorem 1 we have \( T^\Theta \circ p_\Phi = p_\Phi T^\Omega \). Therefore, \( p_\Phi f \) is an eigenfunction of \( T^\Theta \) (or zero) if \( f \) is an eigenfunctions of \( T^\Omega \). As \( p_\Phi \) is onto, the statement follows. \( \square \)
Let us now discuss the continuity of eigenfunctions. Recall that a sequence \((B_n)_{n \in \mathbb{N}}\) of compact sets in \(G\) with non-empty interior is called van Hove, if it exhausts \(G\) and if
\[
\lim_{n \to \infty} \frac{|\partial^K B_n|}{|B_n|} = 0
\]
for every compact \(K\) in \(G\), where \(\partial^K B := ((B + K) \setminus B) \cup ((G \setminus B - K) \cap B)\).

For \(G = \mathbb{R}^d\) and \(G = \mathbb{Z}^d\), the following lemma (and much more) was shown by Robinson in [40]. His proof carries over easily to our situation. For the convenience of the reader, we include a brief discussion.

**Lemma 1.** Let \((\Omega, \alpha)\) be a uniquely ergodic dynamical system. Denote the unique invariant probability measure on \(\Omega\) by \(m\). Let \(\hat{s}\) be an eigenvalue of \(T = T^{(\Omega, \alpha, m)}\). Then, the following assertions are equivalent.

(i) There exists a continuous eigenfunction \(f\) to \(\hat{s}\) (i.e., \(f\) is continuous with \(f(\alpha^{-t}(\omega)) = (\hat{s}, t)f(\omega)\) for all \(t \in G\) and \(\omega \in \Omega\)).

(ii) The sequence \(A_{B_n}(h)\) of continuous functions on \(\Omega\), defined by
\[
A_{B_n}(h)(\omega) := \frac{1}{|B_n|} \int_{B_n}(\hat{s}, t)h(\alpha^{-t}(\omega))\,dt,
\]
converges uniformly, for every van Hove sequence \((B_n)\) and every \(h \in C(\Omega)\).

**Proof.** (i) \(\Rightarrow\) (ii) (cf. [40]). If \(f\) is the continuous eigenfunction, \(|f|\) is invariant and continuous. As \((\Omega, \alpha)\) is uniquely ergodic, we may assume, without loss of generality, that \(|f(\omega)| = 1\) for every \(\omega \in \Omega\). Let \(h \in C(\Omega)\) be given. Apparently, the function \(g = h\overline{f}\) is continuous. Therefore, by unique ergodicity, the functions
\[
\frac{1}{|B_n|} \int_{B_n} g(\alpha^{-t}(\omega))\,dt = \frac{1}{|B_n|} \int_{B_n} h(\alpha^{-t}(\omega))\overline{f(\alpha^{-t}(\omega))}\,dt = \frac{\overline{f(\omega)}}{|B_n|} \int_{B_n} (\hat{s}, t)h(\alpha^{-t}(\omega))\,dt
\]
converge uniformly in \(\omega \in \Omega\). Multiplying by \(f\) and using \(\int f = 1\), we infer (ii).

(ii) \(\Rightarrow\) (i). As \(\hat{s}\) is an eigenvalue of \(T\), the projection \(E(\{\hat{s}\})\) onto the eigenspace of \(\hat{s}\) is not zero. Since \(C(\Omega)\) is dense in \(L^2(\Omega, m)\), there exists an \(h \in C(\Omega)\) with \(E(\{\hat{s}\})h \neq 0\). Now, by the von Neumann ergodic theorem, see [28, Thm. 6.4.1] for a formulation that allows its derivation in the generality we need it here, the sequence \(A_{B_n}(h)\) converges in \(L^2(\Omega, m)\) to \(E(\{\hat{s}\})h\). By assumption (ii), this sequence converges uniformly to a function \(g\). Thus, \(g = E(\{\hat{s}\})h \in L^2(\Omega, m)\). Moreover, by uniform convergence, \(g\) is continuous and satisfies \(g(\alpha^{-t}(\omega)) = (\hat{s}, t)g(\omega)\) for every \(\omega \in \Omega\) and \(t \in G\). This gives (i).

Lemma 1 has the following interesting consequence.

**Proposition 2.** Let \((\Omega, \alpha)\) be a uniquely ergodic dynamical system, all eigenfunctions of which are continuous. If \((\Theta, \beta)\) is a factor of \((\Omega, \alpha)\), with factor map \(\Phi\), it is a uniquely ergodic dynamical system, all eigenfunctions of which are continuous as well.

**Proof.** Fact 3 gives that \((\Theta, \beta)\) is uniquely ergodic. Let \(\hat{s}\) be an eigenvalue of \(T^\Theta\). Then, \(\hat{s}\) is an eigenvalue of \(T^\Omega\) by Proposition 1. We now apply Lemma 1 to infer continuity. To that end, choose an arbitrary \(g \in C(\Theta)\), wherefore \(h = g \circ \Phi\) belongs to \(C(\Omega)\). By (i) \(\Rightarrow\) (ii) of Lemma 1, the sequence \((A_{B_n}(h))\) converges uniformly for every van Hove sequence \((B_n)\). A
short calculation then gives
\[ A_{B_n}(h)(\omega) = \frac{1}{|B_n|} \int_{B_n} (g \circ \Phi)((\alpha_t(\omega))(\tilde{s}, t)) \, dt = \frac{1}{|B_n|} \int_{B_n} g(\beta_{-t}(\Phi(\omega)))(\tilde{s}, t) \, dt. \]
As \( \Phi \) is onto, this shows uniform convergence of \( \theta \mapsto \frac{1}{|B_n|} \int_{B_n} g(\beta_{-t}(\theta))(\tilde{s}, t) \, dt \). As \( g \in C(\Theta) \) was arbitrary, this gives the desired continuity statement, by (ii) \( \implies \) (i) of Lemma 1. \( \square \)

Although we have not made use of it so far, it is possible to express \( p_\Phi \) via a disintegration. Since it is instructive and also useful in applications, we finish this section by giving the details for the case when \( \Omega \) and \( \Theta \) are metrizable. We are in the somewhat simpler situation that a continuous map \( \Phi: \Omega \to \Theta \) exists. By standard theory, compare [19, Thm. 5.8] and [37, Thm. 4.5], there exists a measurable map
\[ k: \Theta \to \mathcal{M}(\Omega), \quad \vartheta \mapsto k^\vartheta \]
that satisfies the following three properties.

1. For \( n \)-almost every \( \vartheta \in \Theta \), \( k^\vartheta \) is a probability measure on \( \Omega \) supported in \( \Phi^{-1}(\vartheta) \).
2. For all \( f \in L^1(\Omega, m) \), the function \( f^{k^\vartheta}: \Theta \to \mathbb{C} \), \( \vartheta \mapsto k^\vartheta(f) \), is integrable with respect to \( n = \Phi_* (m) \).
3. For all \( f \in L^1(\Omega, m) \), one has \( n(f^{k^\vartheta}) = m(f) \).

In terms of integrals, the last property reads
\[ \int_\Omega f^{k^\vartheta} \, dn = \int_\Theta \int_{\Phi^{-1}(\vartheta)} f(\omega) \, dk^\vartheta(\omega) \, dn(\vartheta) = \int_\Omega f \, dm. \]

**Remarks.** (1) Note that [37, Thm. 4.5] only deals with bounded functions \( f \). However, using standard monotone class arguments, it is not hard to extend the statements given there to functions \( f \in L^1(\Omega, m) \). This yields (2) and (3).
(2) The function \( f^{k^\vartheta} \) can also be considered as a conditional expectation of \( f \) (see part (i) of [19, Thm. 5.8] or part (b) of [37, Thm. 4.5]).

Given this disintegration, one can now describe the action of \( p_\Phi \) on \( f \in L^2(\Omega, m) \) explicitly, namely in terms of partial averages over the fibres \( \Phi^{-1}(\vartheta) \).

**Proposition 3.** Assume that \( \Omega \) and \( \Theta \) are compact metric spaces, and let \( n = \Phi_* (m) \) as before. Then, the equation \( (p_\Phi(f))(\vartheta) = k^\vartheta(f) \) holds for all \( f \in L^2(\Omega, m) \) and \( n \)-almost every \( \vartheta \in \Theta \).

**Proof.** Fix \( f \in L^2(\Omega, m) \), and let \( g \in L^2(\Theta, n) \) be arbitrary. Then, \( f \) belongs to \( L^1(\Omega, m) \), since \( \Omega \) is compact and \( g \circ \Phi \cdot f \) belongs to \( L^1(\Omega, m) \), as it is the product of two \( L^2 \) functions. Using the properties of \( k \), we can then calculate

\[ \langle g, f^{k^\vartheta} \rangle_{\Theta} = \int_\Theta \bar{g} f^{k^\vartheta} \, dn = \int_\Theta \bar{g}(\vartheta) \int_{\Phi^{-1}(\vartheta)} f(\omega) \, dk^\vartheta(\omega) \, dn(\vartheta) \]
\[ = \int_\Theta \int_{\Phi^{-1}(\vartheta)} \frac{g(\Phi(\omega))}{|\Phi(\omega)|} f(\omega) \, dk^\vartheta(\omega) \, dn(\vartheta) = n \left( \langle \Phi(g), f^{k^\vartheta} \rangle_{\Theta} \right) \]
\[ = m(\Phi(g) f) = \int_{\Omega} \frac{g(\Phi(\omega))}{|\Phi(\omega)|} f(\omega) \, dm(\omega) = \langle \Phi(g), f \rangle_{\Omega} \]
\[ = \langle g, p_\Phi(f) \rangle_{\Theta}. \]
As \( g \in L^2(\Theta, n) \) is arbitrary, this gives \( f^{k^\vartheta} = p_\Phi(f) \) in \( L^2(\Theta, n) \), and our claim follows. \( \square \)
4. Diffraction theory of measure and Delone dynamical systems

In this section, we specify the dynamical systems we are dealing with and discuss the necessary background from diffraction theory. The material is taken from [3], where the proofs and further details can be found. For related material dealing with point dynamical systems, we refer the reader to [16, 23, 30, 44, 45, 46].

As discussed in the introduction, our main focus is on measure dynamical systems which includes the case of point dynamical systems. For the convenience of the reader, however, we start this section with a short discussion of point dynamical systems and discuss the general case of measures only afterwards.

Let $V$ be an open neighbourhood of 0 in $G$. A subset $A$ of $G$ is called $V$-discrete if every translate of $V$ contains at most one point of $A$. Such sets are necessarily closed. A set is uniformly discrete if it is $V$-discrete for some open neighbourhood $V$ of 0. The set of $V$-discrete point sets in $G$ is abbreviated as $\mathcal{D}_V(G)$, while the set of all uniformly discrete subsets of $G$ is denoted by $\mathcal{UD}(G)$. The set $\mathcal{UD}(G)$ (and actually even the set $\mathcal{C}(G)$ of all closed subsets of $G$) can be topologized by a uniformity as follows. For $K \subset G$ compact and $V$ an open neighbourhood of 0 in $G$, we set

$$U_{K,V} := \{ (P_1, P_2) \in \mathcal{UD}(G) \times \mathcal{UD}(G) : P_1 \cap K \subset P_2 + V \text{ and } P_2 \cap K \subset P_1 + V \}. $$

It is not hard to check that $\{ U_{K,V} : K \text{ compact, } V \text{ open with } 0 \in V \}$ generates a uniformity (see [27] Ch. 6) for basics about uniformities), and hence, via the neighbourhoods

$$U_{K,V}(P) := \{ Q : (Q, P) \in U_{K,V} \}, \quad P \in \mathcal{UD}(G),$$

a topology on $\mathcal{UD}(G)$. This topology is called the local rubber topology (LRT). For each open neighbourhood $V$ of 0 in $G$, the set $\mathcal{D}_V(G)$ is compact in LRT. Apparently, $G$ acts on $\mathcal{UD}(G)$ by translation. By slight abuse of notation, this action is again called $\alpha$, i.e., we define

$$\alpha_t(L) := \{ t + x : x \in L \} = t + L.$$

To distinguish (compact) sets of measures $\omega$ from sets of point sets $\Lambda$, we shall use the suggestive notation $\Omega$ versus $\Omega_p$ from now on.

**Definition 2.** The pair $(\Omega_p, \alpha)$ is called a point dynamical system if $\Omega_p$ is a closed $\alpha$-invariant subset of $\mathcal{D}_V(G)$ for a suitable neighbourhood $V$ of 0 in $G$.

Apparently, every $A \in \mathcal{UD}(G)$ gives rise to a point dynamical system $(\Omega(A), \alpha)$, where $\Omega(A)$ is the closure of $\{ \alpha_t(A) : t \in G \}$ in LRT and $\alpha$ is the action induced from the natural action of $G$ on $\mathcal{UD}(G)$.

After this short look at point dynamical systems, we now introduce our main object of interest: measure dynamical systems. As mentioned already, they generalize point dynamical systems (see below for details).

Let $C > 0$ and a relatively compact open set $V$ in $G$ be given. A measure $\mu \in \mathcal{M}(G)$ is called $(C, V)$-translation bounded if $|\mu|(t+V) \leq C$ for all $t \in G$. It is called translation bounded if there exists such a pair $C, V$ so that $\mu$ is $(C, V)$-translation bounded. The set of all $(C, V)$-translation bounded measures is denoted by $\mathcal{M}_{C,V}(G)$, the set of all translation bounded measures by $\mathcal{M}^\infty(G)$. In the vague topology, the set $\mathcal{M}_{C,V}(G)$ is a compact Hausdorff space. There is an obvious action of $G$ on $\mathcal{M}^\infty(G)$, again denoted by $\alpha$, given by

$$\alpha : G \times \mathcal{M}^\infty(G) \longrightarrow \mathcal{M}^\infty(G), \quad (t, \mu) \mapsto \alpha_t \mu \quad \text{with} \quad (\alpha_t \mu) := \delta_t * \mu.$$

Restricted to $\mathcal{M}_{C,V}(G)$, this action is continuous.
Here, the convolution of two convolvable measures $\mu, \nu$ is defined by

$$(\mu * \nu)(\varphi) = \int_G \varphi(r + s) \, d\mu(r) \, d\nu(s).$$

**Definition 3.** $(\Omega, \alpha)$ is called a dynamical system on the translation bounded measures on $G$ (TMDS for short) if there exist a constant $C > 0$ and a relatively compact open set $V \subset G$ such that $\Omega$ is a closed $\alpha$-invariant subset of $\mathcal{M}_{C,V}(G)$.

It is possible to consider a point dynamical system as a TDMS. Namely, define

$$\delta : UD(G) \rightarrow \mathcal{M}(G), \quad \delta(\Lambda) := \sum_{x \in \Lambda} \delta_x,$$

where $\delta_x$ is the unit point (or Dirac) measure at $x$. The mapping $\delta$ is continuous and injective.

**Lemma 2.** If $(\Omega_p, \alpha)$ is a point dynamical system, the mapping $\delta : \Omega_p \rightarrow \Omega := \delta(\Omega_p)$ establishes a topological conjugacy between the point dynamical system $(\Omega_p, \alpha)$ and its image, the TMDS $(\Omega, \alpha)$.

**Proof.** By [3, Lemma 2], $\delta : \Omega_p \rightarrow \delta(\Omega_p)$ is a homeomorphism that is compatible with the $G$-action $\alpha$, i.e., $\delta(\alpha_t(\Lambda)) = \alpha_t(\delta(A))$ for all $A \in \Omega_p$ and all $t \in G$. So, $\delta$ provides a topological conjugacy as claimed. □

Having introduced our models, we can now discuss some key issues of diffraction theory. Let $(\Omega, \alpha)$ be a TMDS, equipped with an $\alpha$-invariant measure $m \in \mathcal{P}_G(\Omega)$. We shall need the mapping

$$f : C_c(G) \rightarrow C(\Omega), \quad f_\varphi(\omega) := \int_G \varphi(-s) \, d\omega(s).$$

Then, there exists a unique measure $\gamma = \gamma_m$ on $G$, called the **autocorrelation** (often called Patterson function in crystallography [14], though it is a measure in our setting) with

$$\gamma(\varphi * \psi) = \langle f_\varphi, f_\psi \rangle$$

for all $\varphi, \psi \in C_c(G)$, where $\psi(s) := \psi(-s)$. The convolution $\varphi * \psi$ is defined by $(\varphi * \psi)(t) = \int \varphi(t - s) \psi(s) \, ds$. For a more explicit formulation in terms of a weighted average, see [3, Prop. 6].

The measure $\gamma$ is positive definite. Therefore, its Fourier transform is a positive measure $\hat{\gamma}$; it is called the **diffraction measure**. This measure describes the outcome of a diffraction experiment, see [14] for background material.

**Remark.** This concept of an autocorrelation is defined via the entire dynamical system, which implicitly involves a local averaging procedure. The conventional approach uses a limit of a sequence of finite measures along a van Hove averaging sequence in $G$. If the dynamical system is (uniquely) ergodic, the two notions coincide [3]. In general, the definition we use here has the advantage of removing the dependence of the averaging sequence and automatically deals with the typical autocorrelation, at least with reference to the measure $m$.

In view of the fact that, in reality, one always faces finite structures, one can give a justification along the following lines. Among all elements of the full system that are compatible with a given finite part, “typical” ones are those to be considered, if no other piece of information is available. This means to take into account all structures which, after a small translation and/or up to some tiny local deformation, coincide with a fixed finite patch. One way to do so is to take an average over all these possibilities (on the level of their autocorrelations), which is essentially what our $\gamma$ does. In the situation of unique ergodicity, compare [3], the precise method for forming the average is irrelevant – the result is independent of it.
Theorem 2 (Theorem 7 in [3]). Let \((\Omega, \alpha)\) be a TMDS with invariant measure \(m\). Then, the following assertions are equivalent.

(i) The measure \(\hat{\gamma}\) is a pure point measure.
(ii) \(T^\Omega\) has pure point dynamical spectrum. 

\[\square\]

Theorem 2 links pure point diffraction spectrum to pure point dynamical spectrum. This is of particular relevance for our considerations. It will allow us to set up a perturbation and stability theory for pure point diffraction spectrum by studying (perturbations of) dynamical systems. This is the abstract core of our investigation, to be analyzed next.

5. Deforming measure and Delone dynamical systems: Abstract setting

In this section, we introduce a deformation procedure for dynamical systems that, under certain conditions, is isospectral, i.e., the deformation does not change the dynamical spectrum. In particular, we shall later consider deformations of regular model sets and show that a relevant class of deformations preserves their pure point diffraction property. As discussed in the introduction, these considerations are motivated by questions from the mathematical theory of quasicrystals. They generalize the corresponding results in [24, 11].

For pedagogic reasons, we start with a short discussion of deformations of Delone dynamical systems. This results in Theorem 3. The general case of measure dynamical systems is treated afterwards.

Let \((\Omega_p, \alpha)\) be a Delone dynamical system, with \(\Omega_p\) contained in \(D_V(G)\), and consider a continuous mapping \(q : \Omega_p \rightarrow G\) whose image then is a compact set. In fact, let us assume that \(q(\Omega_p) - q(\Omega_p) \subset V\) for some neighbourhood \(V\) of \(0\) in \(G\). Note that there exists an open neighbourhood \(V'\) of \(0\) in \(G\) with \(V' + q(\Omega_p) - q(\Omega_p) \subset V\).

In particular, for arbitrary \(\Lambda \in \Omega_p\) and \(y, z \in \Lambda\) with \(y \neq z\), we have \(y + q(\Lambda - y) \neq z + q(\Lambda - z)\) as well as

\[\Lambda_q := \{x + q(\Lambda - x) : x \in \Lambda\} \subset D_{V'}(G)\]

\(\Lambda_q\) can be viewed as a “deformed” version of \(\Lambda\), which explains the terminology. Moreover, \(\Omega_{p}^q := \{\Lambda_q : \Lambda \in \Omega_p\}\) can rather directly be seen to be \(\alpha\)-invariant and closed in \(D_{V'}(G)\). Thus, \((\Omega_{p}^q, \alpha)\) is a point dynamical system, and we have a mapping \(\Phi^q : \Omega_p \rightarrow \Omega_p^q\) given by \(\Phi^q(\Lambda) = \Lambda_q\). This map can easily be seen to be a factor map.

In fact, it turns out that we do not need \(q\) to be defined on the whole of \(\Omega_p\) to obtain a factor map. It suffices to have it defined on a “transversal”. To be more precise here, we introduce the following subset of \(\Omega_p\),

\[\Xi := \{\Lambda \in \Omega_p : 0 \in \Lambda\}\]

Since the elements of \(\Omega_p\) are non-empty point sets of \(G\), it is clear that each \(G\)-orbit in \(\Omega_p\) contains at least one element of \(\Xi\). Moreover, the following holds.

Lemma 3. If \(\Omega_p\) is a point dynamical system under the action of the LCA group \(G\), the subset \(\Xi\) of \(\Omega_p\) is compact.

Proof. By definition, \(\Omega_p\) is a closed subset of \(D_V(G)\) for a suitable neighbourhood \(V\) of \(0\) in \(G\). As \(D_V(G)\) is compact in LRT, \(\Omega_p\) is compact in LRT as well. So, we need to show that \(\Xi \subset \Omega_p\) is a closed set.
Let \((\Gamma_i)\) be a net in \(\Xi\) (so, \(0 \in \Gamma_i\) for all \(i\)) which converges to some \(\Lambda\), where the latter must then lie in \(\Omega_p\). Assume that \(0 \notin \Lambda\). Since \(\Lambda\) is itself a closed subset of \(G\), we know that \(G \setminus \Lambda\) is an open set. By assumption, this open set would contain \(0\), and hence also an entire open neighbourhood of \(0\). This, however, contradicts the convergence \(\Gamma_i \rightarrow \Lambda\) in the LRT.

As \(\Xi\) is compact, every continuous function \(q\) on \(\Xi\) can be extended to a continuous function \(\tilde{q}\) on \(\Omega_p\) (the latter being compact and hence normal) by Tietze’s extension theorem, compare [39, Prop. 1.5.8]. The very definition of \(\Omega_p^{\tilde{q}}\), compare (5), shows that it only depends on \(q\) (and not on the extension chosen). In this situation, we can thus consistently define

\[
\Omega_p^q := \Omega_p^{\tilde{q}}.
\]

Now, we can state our result on Delone dynamical systems.

**Theorem 3.** Let \((\Omega_p, \alpha)\) be a point dynamical system under the action of the LCA group \(G\) with \(\Omega_p \subset D_V(G)\) for a suitable neighbourhood \(V\) of \(0\) in \(G\). Let \(q: \Xi \rightarrow G\) be continuous with \(q(\Omega_p) - q(\Omega_p) \subset V\). Then, the following assertions hold.

(a) If \((\Omega_p, \alpha)\) has pure point diffraction spectrum (w.r.t. an invariant probability measure \(m\)), so does \((\Omega_p^q, \alpha)\) (w.r.t. the measure \(\Phi^q(m)\)).

(b) If \((\Omega_p, \alpha)\) is minimal or uniquely ergodic, then so is \((\Omega_p^q, \alpha)\).

(c) If \((\Omega_p, \alpha)\) is uniquely ergodic with pure point diffraction spectrum and all of its eigenfunctions are continuous, the same holds for \((\Omega_p^q, \alpha)\).

**Proof.** Let \(\tilde{q}\) be a continuous extension of \(q\) from \(\Xi\) to \(\Omega_p\). As discussed above in (7), we then have a factor map \(\Phi^q: \Omega_p \rightarrow \Omega_p^q\). Now, we can prove the assertions.

(a): If \((\Omega_p, \alpha)\) has pure point diffraction spectrum, it has pure point dynamical spectrum, by Theorem 2. As \((\Omega_p^q, \alpha)\) is a factor of \((\Omega, \alpha)\), it has pure point dynamical spectrum as well, by Proposition 1. Now, another application of Theorem 2 shows that \((\Omega_p^q, \alpha)\) has pure point diffraction spectrum.

(b): This follows from Fact 3.

(c): The statement about continuity of the eigenfunctions is immediate from Proposition 2. The other statements follow by (a) and (b). □

Having discussed the special case of point dynamical systems, we now treat the general case. Let \((\Omega, \alpha)\) be a TMDS. We shall deform \((\Omega, \alpha)\) by means of a measure-valued mapping

\[
\lambda: \Omega \rightarrow \mathcal{M}(G), \quad \omega \mapsto \lambda^\omega,
\]

which satisfies the following two properties.

(D1) The mapping \(\Omega \times C_c(G) \rightarrow C, (\omega, \varphi) \mapsto \lambda^\omega(\varphi)\), is continuous.

(D2) There exists a compact \(K \subset G\) such that \(\text{supp}(\lambda^\omega) \subset K\) for all \(\omega \in \Omega\).

Such a deformation map \(\lambda\) will be called *admissible*. This definition entails the case that \(\lambda^\omega \equiv \delta_0\), which we shall call the trivial deformation map.

**Proposition 4.** Let \((\Omega, \alpha)\) be a TMDS and let \(\lambda: \Omega \rightarrow \mathcal{M}(G)\) be an admissible deformation map. Then, \(\omega \mapsto |\lambda^\omega|(1)\) is bounded.
Proof. Let $K$ be given according to (D2), and let $V \subset G$ be open and relatively compact. Since $K + V$ is compact, one has
\[ |\lambda^\omega|(1) = |\lambda^\omega|(K + V) = \sup\{|\lambda^\omega(\varphi)| : \text{supp}(\varphi) \subset K + V, \|\varphi\|_\infty \leq 1\}, \]
where we used [3] Prop. 1 in the last step. Due to compactness of $\Omega$, the statement now follows from (D1) and the uniform boundedness principle (see [39] Thm. 2.2.9). \(\square\)

For $\omega \in \Omega$ and $\varphi \in C_c(G)$, we define the actual deformation of $\omega$ into $\Phi^\lambda(\omega)$ via
\[ (\Phi^\lambda(\omega))(\varphi) := \int_G \int_G \varphi(r + s) \, d\lambda^{\alpha - r}(\omega)(s) \, d\omega(r), \]
where the double integral exists by (D1) and (D2). The constant deformation map $\lambda^\omega \equiv \delta_t$, with $t \in G$, results in a translation, i.e., $\Phi^\lambda(\omega) = \delta_t * \omega$ in this case, for all $\omega \in \Omega$. The trivial deformation map thus induces the identity. In general, the following is true.

**Proposition 5.** Let a TMDS $(\Omega, \alpha)$ be given and let $\lambda$ be an admissible deformation map. Then, the following assertions hold.

(a) For every $\omega \in \Omega$, the map $\Phi^\lambda(\omega) : C_c(G) \to \mathbb{C}$, $\varphi \mapsto (\Phi^\lambda(\omega))(\varphi)$, is continuous, i.e., $\Phi^\lambda(\omega)$ belongs to $\mathcal{M}(G)$. Moreover, the map $\Phi^\lambda : \Omega \to \mathcal{M}(G)$, $\omega \mapsto \Phi^\lambda(\omega)$, is continuous as well.

(b) There exists a constant $C > 0$ and an open neighbourhood $V$ of 0 in $G$ such that $\Phi^\lambda(\omega)$ belongs to $\mathcal{M}_{G,V}(G)$, for all $\omega \in \Omega$.

(c) For all $t \in G$ and $\omega \in \Omega$, one has $\Phi^\lambda(\alpha_t(\omega)) = \alpha_t(\Phi^\lambda(\omega))$.

**Proof.** (a): Let $K$ be compact according to (D2). For fixed $\omega \in \Omega$ and $\varphi \in C_c(G)$ with support in the compact set $L$, the function
\[ r \mapsto \int_G \varphi(r + s) \, d\lambda^{\alpha - r}(\omega)(s) \]
has support contained in $L - K$. Moreover, this function is continuous, since it can easily be expressed as a composition of continuous functions. In fact, extending this type of reasoning, one can show that
\[ F : \Omega \times C_c(G) \to C_c(G), \quad F(\omega, \varphi)(r) := \int_G \varphi(r + s) \, d\lambda^{\alpha - r}(\omega)(s), \]
is continuous. In particular, $C_c(G) \to \mathbb{C}$, $\varphi \mapsto \omega(F(\omega, \varphi))$, is continuous for fixed $\omega \in \Omega$ and $\omega : \Omega \to \mathbb{C}$, $\omega \mapsto \omega(F(\omega, \varphi))$, is continuous for $\varphi \in C_c(G)$. As $\Phi^\lambda(\omega)(\varphi) = \omega(F(\omega, \varphi))$, we infer (a).

(b): Let $L$ be an arbitrary non-empty open set with compact closure. Let $1_{\overline{L} - K}$ be the characteristic function of $\overline{L} - K$, where $K$ is taken from (D1). Then, for every $\varphi \in C_L(G)$, we have
\[ |\Phi^\lambda(\omega)(\varphi)| \leq \int_G |\lambda|^{\alpha - r}(1) \|\varphi\|_\infty \, 1_{\overline{L} - K}(r) \, d|\omega|(r) \leq C(\lambda)\|\varphi\|_\infty |\omega|(\overline{L} - K), \]
where $C(\lambda)$ is the bound on $\omega \mapsto |\lambda^\omega|(1)$ obtained in Proposition. Thus,
\[ |\Phi^\lambda(\omega)|(L + t) = \sup\{|\Phi^\lambda(\omega)(\varphi)| : \varphi \in C_{L+t}(G), \|\varphi\|_\infty \leq 1\} \leq C(\lambda)\|\varphi\|_\infty |\omega|(t + \overline{L} - K) \]
is uniformly bounded in $t \in G$, as $\omega$ is translation bounded, and (b) follows.

(c): This is immediate from

$$
(\Phi^\lambda(\alpha_t \omega))(\varphi) = \int_G \int_G \varphi(r + s) \, d\lambda^{\alpha - r + t}(\omega)(s) \, d(\alpha_t \omega)(r)
= \int_G \int_G \varphi(r + s + t) \, d\lambda^{\alpha - r}(\omega)(s) \, d\omega(r)
= (\alpha_t(\Phi^\lambda(\omega)))(\varphi),
$$

which is valid for every $\varphi \in C_c(G)$.

Define the set of periods of a measure $\omega$ as

$$(8) \quad \text{Per}(\omega) := \{ t \in G : \alpha_t \omega = \omega \}.$$

We then have the following consequence.

**Corollary 2.** Let $(\Omega, \alpha)$ be given and let $\lambda$ be an admissible deformation map. For any $\omega \in \Omega$, with resulting deformation $\Phi^\lambda(\omega)$, one has

$$\text{Per}(\omega) \subset \text{Per}(\Phi^\lambda(\omega)).$$

Moreover, if any $\omega \in \Omega$ exists where $\text{Per}(\Phi^\lambda(\omega))$ is a true superset of $\text{Per}(\omega)$, the mapping $\Phi^\lambda : \Omega \to M(G)$ fails to be injective.

**Proof.** The first claim follows at once from part (c) of Proposition 5. For the second claim, let $t$ be a period of $\Phi^\lambda(\omega)$ that is not a period of $\omega$. Then, $\omega \neq \alpha_t \omega$, but their images under $\Phi^\lambda$ are equal. \qed

Part (a) of Proposition 5 implies that, for a given TMDS $(\Omega, \alpha)$, the set

$$\Omega^\lambda := \{ \Phi^\lambda(\omega) : \omega \in \Omega \}$$

is compact, as it is the image of a compact set under a continuous map. Furthermore, by part (c) of the same proposition, $\Omega^\lambda$ is invariant under $\alpha$. In fact, by part (b) of Proposition 5, $\Omega^\lambda$ is a subset of $M_{C,V}(G)$ for suitable $C,V$. Putting this together, we have proved the following result.

**Lemma 4.** Let $(\Omega, \alpha)$ be a TMDS and let $\lambda : \Omega \to M(G)$ be an admissible deformation map. Then, $(\Omega^\lambda, \alpha)$ is a TDMS. Moreover, $(\Omega^\lambda, \alpha)$ is a factor of $(\Omega, \alpha)$, with factor map $\Phi^\lambda : \Omega \to \Omega^\lambda$. \qed

If the situation of Lemma 4 applies, we call $(\Omega^\lambda, \alpha)$ an admissible deformation of $(\Omega, \alpha)$, with deformation map $\lambda$. The main abstract result of this paper now reads as follows.

**Theorem 4.** Let $(\Omega, \alpha)$ be a TMDS and let $\lambda : \Omega \to M(G)$ be an admissible deformation map. Then, the following assertions hold.

(a) If $(\Omega, \alpha)$ has pure point diffraction spectrum (w.r.t. some invariant probability measure $m$), so does $(\Omega^\lambda, \alpha)$ (w.r.t. the corresponding induced measure).

(b) If $(\Omega, \alpha)$ is minimal or uniquely ergodic, then so is $(\Omega^\lambda, \alpha)$.

(c) If $(\Omega, \alpha)$ is uniquely ergodic with pure point diffraction spectrum and all of its eigenfunctions are continuous, the same holds for $(\Omega^\lambda, \alpha)$. 
Proof. The proof is essentially the same as the proof of Theorem 3.
(a): If \((\Omega, \alpha)\) has pure point diffraction spectrum, it has pure point dynamical spectrum, by Theorem 2. As \((\Omega^\lambda, \alpha)\) is a factor of \((\Omega, \alpha)\) by Lemma 4, it has pure point dynamical spectrum as well, by Proposition 1. Now, another application of Theorem 2 shows that \((\Omega^\lambda, \alpha)\) has pure point diffraction spectrum.
(b): This follows from Fact 3.
(c): The statement about continuity of the eigenfunctions is immediate from Proposition 2. The other statements follow by (a) and (b).

\[ \square \]

Remarks. (1) Of course, the previous discussion of TMDS includes the case of Delone dynamical systems treated at the beginning of the section. To see this, one has to apply the mapping \(\delta: \Omega_p \rightarrow \Omega\) introduced in the previous section.
(2) The discussion of point dynamical systems given above requires a non-overlapping condition under deformation, here written as \(q(\Omega_p) - q(\Omega_p) \subset V\) for a suitable open set \(V\). In the TMDS setting, such a restriction is not necessary, which shows once more the greater flexibility of the approach via measures.

6. Model sets and their deformation

Model sets probably form the most important class of examples of aperiodic order. In their case, one starts with a periodic structure in a high-dimensional space and considers a partial “image” in a lower dimensional space. This image will not be periodic any more but still preserve many regularity features due to the periodicity of the underlying high dimensional structure. For a survey and further references, we refer the reader to [33, 35].

Let us start with a brief recapitulation of the setting of a cut and project scheme and the definition of a model set. We need two locally compact Abelian groups, \(G\) and \(H\), where \(G\) is also assumed to be \(\sigma\)-compact, see [44] for the reasons why this is needed. As usual, neutral elements will be denoted by \(0\) (or by \(0^G, 0^H\), if necessary). A cut and project scheme emerges out of the following collection of groups and mappings:

\[ G \xleftarrow{\pi} G \times H \xrightarrow{\pi_{\text{int}}} H \]
\[ \cup \quad \cup \quad \cup \text{ dense} \]
\[ L \xleftarrow{1-1} \tilde{L} \xrightarrow{\ast} L^\ast \]
\[ \| \quad \| \]

Here, \(\tilde{L}\) is a lattice in \(G \times H\), i.e., a cocompact discrete subgroup. The canonical projection \(\pi\) is one-to-one between \(\tilde{L}\) and \(L\) (in other words, \(\tilde{L} \cap \{0^G\} \times H = \{0\}\)), and the image \(L^\ast = \pi_{\text{int}}(\tilde{L})\) is dense in \(H\), which is often called the internal space. In view of these properties of the projections \(\pi\) and \(\pi_{\text{int}}\), one usually defines the *-map as \((. \ast): L \rightarrow H\) via \(x^\ast := (\pi_{\text{int}} \circ (\pi_{\tilde{L}})^{-1})(x)\), where \((\pi_{\tilde{L}})^{-1}(x) = \pi^{-1}(x) \cap \tilde{L}\), for all \(x \in L\).

A model set is now any translate of a set of the form

\[ \mathcal{A}(W) := \{ x \in L : x^\ast \in W \} \]

where the window \(W\) is a relatively compact subset of \(H\) with non-empty interior. Without loss of generality, we may assume that the stabilizer of the window,

\[ H_W := \{ c \in H : c + W = W \}, \]
is the trivial subgroup of $H$, i.e., $H_W = \{0\}$. If this were not the case (which could happen in compact groups $H$ for instance), one could factor by $H_W$ and reduce the cut and project scheme accordingly [14]. Furthermore, we may assume that $\langle W - W \rangle$, the subgroup of $H$ that is algebraically generated by the subset $W - W$, is the entire group, i.e., $\langle W - W \rangle = H$, again by reducing the cut and project scheme to this situation, compare [13] for details.

There are variations on the precise requirement to $W$ which depend on the fine properties of the model sets one is interested in, compare [35, 44]. In particular, a model set is called regular if $\partial W$ has Haar measure 0 in $H$, and generic if, in addition, $\partial W \cap L^* = \emptyset$.

As discussed immediately after Definition [2], every model set $\Lambda$ gives rise to the dynamical system $(\Omega(\Lambda), \alpha)$. It is one of the central results of this area, compare [35, 44] and references given there, that (regular) model sets provide a very natural generalization of the concept of a lattice.

**Theorem 5.** [44] Regular model sets are pure point diffraction. In fact, $(\Omega(\Lambda), \alpha)$ is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions. □

For our purposes, it is sufficient to restrict our attention to regular model sets where $W$ is a compact subset of $H$ with $W^0 = W$ (in particular, $W$ then has non-empty interior and, due to regularity, a boundary of Haar measure 0). This is motivated by the fact that diffraction cannot distinguish two model sets $\Lambda(W)$ and $\Lambda(W')$ if the symmetric difference $W \triangle W'$ of the windows has Haar measure 0 in $H$.

A regular model set with compact window $W$ can be deformed as follows [23, 11]. Let $\vartheta: H \rightarrow G$ be a continuous function with compact support, which, in view of the discussion around (7), we may assume to include $W$ if necessary. If $\Lambda = \Lambda(W)$, one defines

$$\Lambda_{\vartheta} := \{ x + \vartheta(x^*) : x \in \Lambda \} = \{ x + \vartheta(x^*) : x \in L \text{ and } x^* \in W \}.$$  

To make sure that $\Lambda_{\vartheta}$ is still a Delone set, one usually requires that the compact set $K := \vartheta(H) - \vartheta(H)$ satisfies $K \subset V$ where $V$ is an open neighbourhood of $0 \in G$ so that $\Lambda \in D_V(G)$.

Note that $\Lambda_{\vartheta}$ (if it is Delone) has a well defined density, and one obtains

$$\text{dens}(\Lambda_{\vartheta}) = \text{dens}(\Lambda).$$

In other words, an admissible deformation does not change the density.

Our aim is now to show that the continuous mapping $\vartheta$ induces a deformation map $q$ on $\Xi$. To do so, we shall need the following lemma. It essentially says that the $\star$-map on $\Lambda$ can be extended to a unique continuous map on $\Xi$.

**Lemma 5.** Let $\Lambda = \Lambda(W)$, with $W = W^0$ compact, be a regular model set and assume that $H_W = \{0\}$. Then, the set $\{ \Lambda - x : x \in \Lambda \}$ is dense in the compact set $\Xi$ and there is precisely one continuous mapping $\sigma: \Xi \rightarrow W$ with $\sigma(\Lambda - x) = x^*$ for every $x \in \Lambda$.

**Proof.** First, let us show that $\{ \Lambda - x : x \in \Lambda \}$ is dense in $\Xi$, the latter being compact by Lemma 3.

To this end, let $\Gamma \in \Xi$ be given and consider an arbitrary neighbourhood $U_{K,V}(\Gamma)$ of $\Gamma$, where $K \subset G$ is compact and $V$ is an open neighbourhood of 0 in $G$. Replacing $K$ by $K \cup \{0\}$ if necessary, we can assume $0 \in K$ without loss of generality. We have to provide an element of the form $\Lambda - p$ with $p \in \Lambda$ which belongs to $U_{K,V}(\Gamma)$.

To do so, choose a compact neighbourhood $V'$ of 0 in $G$ with $V' + V' \subset V$ and $V' = -V'$.
As $Ξ$ is a subset of $Ω_p(Λ)$, which is the orbit closure of $\{t + Λ : t \in G\}$, there exists a $t \in G$ with

$$t + Λ \in U_{K + V', V'}(Γ).$$

As $0$ belongs to both $Γ$ and $K$, we infer that

$$0 \in Γ \cap K \subset Γ \cap (K + V') \subset (t + Λ) + V'.$$

Therefore, $0 = t + p + v'$ with $p \in Λ$ and $v' \in V'$, or, put differently, $p = -t - v' \in Λ$. This gives

$$Λ - p = Λ + t + v' \in U_{K + V', V'}(Γ) + v' \subset U_{K, V}(Γ),$$

where the last inclusion follows by our choice of $V'$. As discussed above, this proves the density statement.

It remains to show the existence and uniqueness of a continuous map $σ : Ξ \rightarrow H$ with $σ(Λ - x) = x^*$ for every $x \in Λ$, where the uniqueness will be an immediate consequence of the continuity of $σ$ and the already established denseness of $\{Λ - x : x \in Λ\}$ in $Ξ$.

Existence: By [44, Lemma 4.1], for every $Γ \in Ξ$, the set

$$(14) \quad σ(Γ) = \bigcap_{y \in Γ}(W - y^*)$$

is a singleton set in $H$ (note that the sign change in our formulation does not affect this statement). In the sequel, we shall tacitly identify the singleton set $σ(Γ)$ with its unique element. Then, $σ$ can be considered as a map on $Ξ$ with values in $H$.

By [44], $Γ \subset W - σ(Γ)$. As $0 \in Γ$, we infer $0 = w - σ(Γ)$ for some $w \in W$, and hence $σ(Γ) \in W$. If $Γ = Λ - x$ for some $x \in Λ$, then we claim that $x^* \in σ(Λ - x) = \bigcap_{y \in Λ - x}(W - y^*)$. This is so because $y \in Λ - x$ implies $y = ℓ - x$ for some $ℓ \in Λ$, hence $W - y^* = W - (ℓ^* - x^*) = (W - ℓ^*) + x^*$. Clearly, $ℓ^* \in W$, so $0 \in W - ℓ^*$, and this gives $x^* \in W - y^*$. With $y \in Λ - x$ arbitrary, we obtain $σ(Λ - x) = \{x^*\}$, as $σ(Γ)$ is a singleton set.

Next, following [44, Prop. 4.3], we can show continuity of the mapping $σ$. Let $Γ \in Ξ$, and let $V = V(σ(Γ))$ be an open neighbourhood of $σ(Γ)$ in $H$. Since $σ(Γ) = \bigcap_{y \in Γ}(W - y^*)$ is a singleton set, one has

$$\bigcap_{y \in Γ}(W - y^*) \setminus V = \emptyset.$$ 

As $V$ is open, each $(W - y^*) \setminus V$ is closed, hence also compact. So, there must be a finite set $F \subset Γ$ such that we already have $\bigcap_{y \in F}(W - y^*) \setminus V = \emptyset$. This implies that a compact set $K$ exists such that $\bigcap_{y \in Γ \cap K}(W - y^*) \setminus V = \emptyset$, so

$$\bigcap_{y \in Γ \cap K}(W - y^*) \subset V.$$ 

This inclusion means that $Γ' \cap K = Γ \cap K$, for any $Γ' \in Ξ$, implies $σ(Γ') \subset V$. By a standard argument, this can now be turned into the claimed continuity of $σ$.

We can now show how $σ$ induces a deformation $q$.

**Proposition 6.** Let $Λ = \lambda(W)$, with $W = W^c$ compact, be a regular model set and assume that $H_W = \{0\}$. Let $θ : W \rightarrow G$ be continuous. Then, there is precisely one continuous mapping $q : Ξ \rightarrow G$ with $q(Λ - x) = θ(Λ - x)$ for all $x \in Λ$.

**Proof.** This follows directly from Lemma 5. Uniqueness follows because $\{Λ - x : x \in Λ\}$ is dense in $Ξ$. Existence follows as we can simply define $q := θ \circ σ$ with the $σ$ of Lemma 5. □
Remark. Let us point out that continuity of \( \vartheta \) is not necessary to obtain continuity of \( \vartheta \circ \sigma \). In fact, it is easy to construct examples where \( \vartheta \) may even have countably many points of discontinuity (at points of \( L^* \), in fact).

Based on Proposition 6, we can now prove our result on deformed model sets.

**Theorem 6.** Let \( \Lambda \) be a regular model set and \( \vartheta : H \rightarrow G \) a continuous map. Let \( \Lambda_\vartheta \) be defined according to (12), with the restriction that it is still a Delone set. Then, \( \Lambda_\vartheta \) is pure point diffractive. In fact, the dynamical system \((\Omega(\Lambda_\vartheta), \alpha)\) is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions.

**Proof.** Consider the map \( q : \Xi \rightarrow G \) constructed in Proposition 6. Plugging in the definitions, we easily find \( \Lambda_q = \Lambda_\vartheta \). This, in turn, gives \( (\Omega(\Lambda))^q = \Omega(\Lambda_q) = \Omega(\Lambda_\vartheta) \).

Thus, it suffices to show that \((\Omega(\Lambda))^q, \alpha)\) is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions. This, however, is immediate from Theorem 3.

Remark. Let us mention that the abstract result of Theorem 6 has a very concrete extension in that it is possible to calculate the diffraction of \( \Lambda_\vartheta \) explicitly. For the Euclidean setting, this is explained in [24, 11], and we illustrate it below in a concrete example.

**7. Example: The silver mean chain**

Let us explain the various notions with a simple example in one dimension, compare [6, Sec. 8.1]. To this end, consider the two letter substitution rule

\[
\sigma : \begin{align*}
a & \mapsto aba \\
b & \mapsto a
\end{align*}
\]

which allows the construction of a bi-infinite (and reflection symmetric) fixed point as follows. Starting from the (admissible) seed \( w_1 = a|a \), where \( | \) denotes the reference point, and defining \( w_{n+1} = \sigma(w_n) \), one obtains the iteration sequence

\[
a|a \xrightarrow{\sigma} aba|aba \xrightarrow{\sigma} abaaaba|abaaaba \xrightarrow{\sigma} \ldots \xrightarrow{n \to \infty} w = \sigma(w)
\]

where \( w \) is a bi-infinite word in the alphabet \( \{a, b\} \) and convergence is in the obvious product topology as generated from the alphabet together with the discrete topology.

The corresponding substitution matrix reads

\[
M_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}
\]

where \( M_{k\ell} \) is the number of symbols of type \( \ell \) in the word \( \sigma(k) \), for \( k, \ell \in \{a, b\} \). This matrix is primitive, with Perron-Frobenius eigenvalue \( s = 1 + \sqrt{2} \), which happens to be a Pisot-Vijayaraghavan number. It is often called the silver mean, due to its continued fraction expansion \( s = [2; 2, 2, 2, \ldots] \), in contrast to \([1; 1, 1, 1, \ldots] = (1 + \sqrt{5})/2 \) for the golden mean. The corresponding eigenvectors (left and right) code the frequencies of the letters \( a \) and \( b \) in \( w \), and also the information for a proper geometric representation of \( w \) as a point set in \( \mathbb{R} \), such that the substitution turns into a geometric inflation rule. One convenient choice here is to represent \( a \) by an interval of length \( 1 + \sqrt{2} \), and \( b \) by one of length \( 1 \). Their frequencies are \( \frac{1}{2} \sqrt{2} \) and \( \frac{1}{2} (2 - \sqrt{2}) \), respectively.

This is an example of a so-called Pisot substitution with two symbols, and the derived point set is known to be a regular model set (with the projection scheme yet to be derived).
Also, the fixed point is a non-singular (or generic) member of the LI-class defined by it. At the same time, it is a Sturmian sequence, and we could have started with a concrete cut and project scheme (then with the compatibility with the inflation to be established). We prefer the former possibility here, as there is a rather elegant number theoretic formulation which we shall now use.

Let \( \Lambda_a \) and \( \Lambda_b \) denote the left endpoints of the intervals of type \( a \) and \( b \), with our reference point (formerly marked by \( | \) ) being mapped to 0 in this process. Both point sets are subsets of the \( \mathbb{Z} \)-module
\[
\mathbb{Z}[\sqrt{2}] := \{ m + n\sqrt{2} : m, n \in \mathbb{Z} \}
\]
which happens to be the ring of integers in the quadratic field \( \mathbb{Q}(\sqrt{2}) \). There is one non-trivial algebraic conjugation in this field, defined by \( \star : \sqrt{2} \mapsto -\sqrt{2} \), which maps \( \mathbb{Z}[\sqrt{2}] \) onto itself. This will take the role of the \( \star \)-map in the cut and project scheme, which looks as follows.

\[
\begin{align*}
\mathbb{R} & \xrightarrow{\pi} \mathbb{R} \times \mathbb{R} \xrightarrow{\pi_{\text{int}}} \mathbb{R} \\
\text{dense} & \cup \mathbb{Z}[\sqrt{2}] & \xrightarrow{1-1} & \tilde{L} = \mathbb{Z}[\sqrt{2}]
\end{align*}
\]

where \( \tilde{L} = \{(x,x^*) : x \in \mathbb{Z}[\sqrt{2}]\} \) is a (rectangular) lattice in \( \mathbb{R}^2 \). In comparison to the standard situation of model sets, compare [35], this cut and project scheme is self-dual, see also [33, p. 418]. In particular, the \( \star \)-map is then one-to-one on \( \mathbb{Z}[\sqrt{2}] \).

An explicit geometric realization of \( \tilde{L} \) with basis vectors is
\[
(16) \quad \tilde{L} = \left\langle \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right), \left( 1, 1 \right) \right\rangle_\mathbb{Z}
\]
which has the nice property that we can directly work with the standard Euclidean scalar product for our further analysis (rather than with the quadratic form defined by the lattice).

In particular, we shall later also need the dual lattice
\[
(17) \quad \tilde{L}^* = \{ y \in \mathbb{R}^2 : xy \in \mathbb{Z} \text{ for all } x \in \tilde{L} \} = \left\langle \frac{1}{4} \left( \frac{1}{2} \sqrt{2} \right), \frac{1}{2} \left( 1, 1 \right) \right\rangle_\mathbb{Z}
\]
(note the different star symbol), which has the projections
\[
L^o = \pi(\tilde{L}^*) = \left\{ \frac{1}{2} \left( m + \frac{n}{\sqrt{2}} \right) : m, n \in \mathbb{Z} \right\} = \pi_{\text{int}}(\tilde{L}^*) = (L^o)^{\star}
\]
Note that the \( \star \)-map is well defined on the rational span of \( L \) which includes \( L^o \).

Let us continue with the construction of our model set. By standard theory for the fixed point of a primitive substitution, the sets \( A_a \) and \( A_b \) satisfy the equations
\[
A_a = sA_a \cup (sA_a + (1 + s)) \cup sA_b \\
A_b = sA_a + s
\]
with \( s = 1 + \sqrt{2} \) from above, and \( \cup \) denoting the disjoint union of sets. Under the \( \star \)-map followed by taking the closure, one obtains a new set of equations for the windows \( W_a = A_a^\star \) and \( W_b = A_b^\star \),
\[
W_a = sW_a \cup (sW_a + (1 + s)) \cup sW_b \\
W_b = sW_a + s
\]
where \( s^\star = 1 - \sqrt{2} \) is less than 1 in absolute value. This new set of equations constitutes a coupled iterated functions system that is a contraction. By standard Hutchinson theory, there
is a unique pair of compact sets \( W_a \) and \( W_b \) that solves this system, compare [6, Thm. 1.1 and Sec. 4] for details. It is easy to check that this solution is given by

\[
W_a = \left[ \frac{\sqrt{2} - 2}{2}, \frac{\sqrt{2}}{2} \right], \quad W_b = \left[ -\frac{\sqrt{2}}{2}, \frac{\sqrt{2} - 2}{2} \right].
\]

From here, one can also see that \( W = W_a \cup W_b = \left[ -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right] \) is the window for the full set \( \Lambda = \Lambda_a \cup \Lambda_b \), with \( W = W^\circ \). Moreover, since \( \pm 1/\sqrt{2} \) are not elements of \( \mathbb{Z}\sqrt{2} \), we see that \( \Lambda = \Lambda(W) = \Lambda(W^\circ) \), so that \( \Lambda \) (and also \( \Lambda_a \) and \( \Lambda_b \)) are regular, generic (or non-singular) model sets. The density of \( \Lambda \) is \( \text{dens}(\Lambda) = 1/2 \).

The deformation is now achieved by a suitable function \( \vartheta: \mathbb{R} \to \mathbb{R} \) which is continuous on \( W \) and vanishes on its complement. This is consistent with (17) because the deformation rule (12) does not require the knowledge of \( \vartheta \) for any value outside of \( W \). A simple but interesting candidate is

\[
\vartheta(y) = \begin{cases} 
\alpha y + \beta, & y \in W \\
0, & y \notin W
\end{cases}
\]

with some constants \( \alpha, \beta \in \mathbb{R} \). For admissible values of \( \alpha \), the affine nature of \( \vartheta \) on \( W \) has the effect of changing the relative length ratio of the \( a \) and \( b \) intervals, with \( \beta \) being a global translation. It is easy to check that the admissible values of \( \alpha \) include

\[-1 < \alpha < 3 + \sqrt{2}\]

which results in the ratio

\[
\varrho = \frac{\text{length}(a_{\vartheta})}{\text{length}(b_{\vartheta})} = 1 + \frac{1 - \alpha}{1 + \alpha} \sqrt{2}.
\]

Here, we use \( a_{\vartheta} \) and \( b_{\vartheta} \) for the intervals that result from the deformation (19). For a given ratio, the parameter \( \alpha \) is given by

\[
\alpha = (\sqrt{2} + 1 - \varrho)/((\sqrt{2} - 1 + \varrho).
\]

We shall come back to this discussion in the next section.

Of particular interest is the fact that one does not only get the theoretical result of pure point diffraction, but also an explicit formula for the diffraction measure. A detailed account for its calculation can be found in [11], which can also be derived explicitly via Weyl’s lemma on uniform distribution, compare [43, 36] for a formulation of the latter in the context of model sets. The result is

\[
\widehat{\gamma}_{\Lambda_{\vartheta}} = \sum_{k \in L^\circ} |A_{\vartheta}(k)|^2 \delta_k
\]

where the so-called Fourier-Bohr coefficients (or diffraction amplitudes) are given by

\[
A_{\vartheta}(k) = \frac{1}{2\sqrt{2}} \int_W e^{2\pi i(k^*y - k\vartheta(y))} \, dy
\]

for all \( k \in L^\circ \), and \( A_{\vartheta}(k) = 0 \) otherwise. Note that \( A_{\vartheta}(0) \equiv 1/2 = \text{dens}(\Lambda) \) in agreement with a previous remark.

To arrive at (21) and (22), one first shows that \( A_{\vartheta}(k) \) must vanish for all \( k \notin L^\circ \), which is part of [11, Thm. 2.6]. Then, let \( k \in L^\circ \), and consider the points of \( \Lambda \) in a (large) finite patch, e.g., in the interval \( B_r(0) \) of radius \( r \) around 0. We denote such a patch by \( \Lambda^{(r)} \) and set

\[
A_{\vartheta}^{(r)} = \{ x + \vartheta(x^*) : x \in \Lambda^{(r)} \}.
\]
If we place unit point measures at the points of $\Lambda^{(r)}$, we obtain a finite measure whose Fourier transform exists and reads
\[
\sum_{x' \in A_0^{(r)}} e^{-2\pi i k x'} = \sum_{x \in A^{(r)}} e^{-2\pi i (k x + k \vartheta(x^*))} = \sum_{x \in A^{(r)}} e^{2\pi i (k^* x^* - k \vartheta(x^*))}
\]
where the last step used the fact that $e^{-2\pi i (k x + k^* x^*)} = 1$ for $k \in L^0$ and $x \in L$. Now, after dividing by the length of $B_r(0)$, one obtains the coefficient $A_\vartheta(k)$ by taking the limit as $r \to \infty$, which exists and gives \[22\] by Weyl’s lemma.

Let us also mention that, if we use the formulation via measures, the diffraction formula \[21\] remains valid for all (continuous) functions $\vartheta$, not just for those which preserve the Delone property.

For our special choice \[19\], one obtains
\[
A_{\alpha,\beta}(k) = e^{-2\pi i \beta k \sin(z)} \left| \frac{z}{2} \right|_{z = \pi(\alpha k^* - k^* \vartheta) \sqrt{2}}
\]
for all $k \in L^0$.

### 8. Topological conjugacy and further aspects

In this section, we briefly comment on the question whether $(\Omega^{\lambda}, \alpha)$ is topologically conjugate to $(\Omega, \alpha)$. A deformed model set need not be topologically conjugate to the undeformed system. In our silver mean example, with the deformation function $\vartheta$ of \[19\], we can find values of the scaling parameter $\alpha$ where the factor becomes periodic, while $\Lambda$ itself (which corresponds to $\alpha = \beta = 0$) is aperiodic. In such a case, in view of Corollary 2, we cannot have topological conjugacy. Note that, in contrast to \[12\], we do not keep track of the type of the intervals here. If we did that (e.g., by giving different weights to the points of $a$ and $b$ intervals), topological conjugacy would always be preserved under the deformation.

In particular, $\alpha = 1$ (which gives $\vartheta = 1$) results in $A_\vartheta = 2Z + \beta$. Eq. \[21\] then reduces to $\gamma A_\vartheta = \frac{1}{4}\delta_{Z/2}$, as it has to. This is a concrete example of the phenomenon of an extinction rule, which can often be used to detect situations where topological conjugacy fails. Here, by analyzing \[23\] in detail, one finds that the Fourier-Bohr spectrum
\[
\Sigma_{\alpha,\beta} := \{ k \in \mathbb{R} : A_{\alpha,\beta}(k) \neq 0 \}
\]
is independent of $\beta$, but depends on $\alpha$. Concretely, one has
\[
\langle \Sigma_{\alpha,\beta} \rangle \mathbb{Z} = \begin{cases} 
\frac{1}{2}\mathbb{Z}, & \alpha = 1 \\
L^0, & \text{otherwise.}
\end{cases}
\]
Here, the $\mathbb{Z}$-span is needed because one can have systematic extinctions also for $\alpha \neq 1$. This happens for $\alpha \in \mathbb{Q}$ and for $\alpha = 1 + r\sqrt{2}$ with $r \in \mathbb{Q}$, through solutions of $\sin(z) = 0$ in \[23\]. Such an extinction phenomenon is usually linked to the existence of symmetries. In our case, for these special values of $\alpha$, the point set $A_\vartheta$ admits an inflation symmetry, and the extinctions can be understood from that \[17\], see \[18\] for a general discussion.

Whenever $\alpha \neq 1$, the deformed model $A_\vartheta$ set is actually topologically conjugate to the original model set $\Lambda$, though in general not via a local derivation rule, compare \[13\] for a recent clarification of the relation between these concepts.

Another interesting phenomenon is the appearance of periodic diffraction, even if the underlying structure is non-periodic. For simplicity, let us concentrate on the case $\beta = 0$. 

Whenever $\varrho$ of (20) is a rational number, $\varrho = p/q$ say with $p,q$ coprime, the set of positions of $A_\varrho$ is a subset of a lattice in $\mathbb{R}$ (of period $\lambda = \text{length}(a_\varrho)/p = \text{length}(b_\varrho)/q$). Consequently, by [1, Thm. 1], the diffraction measure of the corresponding Dirac comb is periodic, with period $1/\lambda$. As the diffraction is also pure point, by our Theorem [6] it is of the form $\mu * \delta_{\mathbb{Z}/\lambda}$, where $\mu$ is a finite positive pure point measure on $[0,1/\lambda)$. Unless $\alpha = 1$, the Fourier-Bohr spectrum is dense in $\mathbb{R}$, and the underlying Dirac comb based on $A_\varrho$ is not periodic. So, in our example, failure of topological conjugacy coincides with the existence of periods for $A_\varrho$.

In the example, and also in our general discussion, we started from a model set and constructed a deformation scheme. In general, a deformation will not result in another model set, though its Fourier-Bohr spectrum remains unchanged. The latter is of central importance for the actual structure determination in crystallography, e.g., from a diffraction experiment. It is often implicitly assumed that the underlying structure is a model set, but our above analysis shows that this need not be the case. An important open question is thus how to effectively characterize model sets versus deformed model sets by means of intrinsic properties, preferably by easily accessible ones. Some first results can be inferred from [4], but more has to be done in this direction.

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
E-mail address: mbaake@math.uni-bielefeld.de

Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany
E-mail address: dlenz@mathematik.tu-chemnitz.de