A NOTE ON LOGARITHMIC SPACE STREAM ALGORITHMS FOR MATCHINGS IN LOW ARBORICITY GRAPHS

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1. Introduction

We present a data stream algorithm for estimating the size of the maximum matching of a low arboricity graph. Recall that a graph has arboricity $\alpha$ if its edges can be partitioned into at most $\alpha$ forests and that a planar graph has arboricity $\alpha = 3$. Estimating the size of the maximum matching in such graphs has been a focus of recent data stream research [1–5, 7]. See also [6] for a survey of the general area of graph algorithms in the stream model.

A surprising result on this problem was recently proved by Cormode et al. [3]. They designed an ingenious algorithm that returned a $(22.5\alpha + 6)(1 + \epsilon)$ approximation using a single pass over the edges of the graph (ordered arbitrarily) and $O(\epsilon^{-2}\alpha \cdot \log n \cdot \log_2 n)$ space. We improve the approximation factor to $(\alpha + 2)(1 + \epsilon)$ via a tighter analysis and show that, with a modification of their algorithm, the space required can be reduced to $O(\epsilon^{-2} \log n)$.

2. Results

Let $\operatorname{match}(G)$ be the maximum size of a matching in a graph $G$ and let $E_\alpha$ be the set of edges $uv$ where the number of edges incident to $u$ or $v$ that appear in the stream after $uv$ are both at most $\alpha$.

2.1. A Better Approximation Factor. We first show a bound for $\operatorname{match}(G)$ in terms of $|E_\alpha|$.

Cormode et al. proved a similar but looser bound.

**Theorem 1.** $\operatorname{match}(G) \leq |E_\alpha| \leq (\alpha + 2) \operatorname{match}(G)$.

**Proof.** We first prove the left inequality. To do this define $y_e = 1/(\alpha + 1)$ if $e$ is in $E_\alpha$ and 0 otherwise. Note that $y_e$ is a fractional matching with maximum weight $1/(\alpha + 1)$ and hence

$$\frac{|E_\alpha|}{\alpha + 1} = \sum_e y_e \leq \left(1 + \frac{1}{\alpha + 1}\right) \operatorname{match}(G) = \frac{\alpha + 2}{\alpha + 1} \operatorname{match}(G).$$

It remains to prove the right inequality. Define $H$ to be the set of vertices with degree $\alpha + 1$ or greater. We refer to these as the heavy vertices. For $u \in H$, let $B_u$ be the set of the last $\alpha + 1$ edges incident to $u$ that arrive in the stream.

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2 Here, and throughout, space is specified in words and we assume that an edge or a counter (between 0 and $\alpha$) can be stored in one word of space.

3 It can be shown as a corollary of Edmonds Matching Polytope Theorem [3] that any fractional matching in which all edge weights are bounded by $\epsilon$ is at most a factor $1 + \epsilon$ larger than the maximum integral matching. See [3] Theorem 5 for details.
Corollary 2. Let $\epsilon$-A (Slightly) Better Algorithm.

2.2. **Better Algorithm.** See Figure 1 for an algorithm that approximates $E^*$ to a $(1 + \epsilon)$-factor in the insert-only graph stream model. The algorithm is a modification of the algorithm for estimating $|E_\alpha|$ designed by Cormode et al. [3]. The basic idea is to independently sample edges from $E_\alpha^t$ with probability that is high enough to obtain an accurate approximation of $|E_\alpha^t|$ and yet low enough to use a small amount of space. For every sampled edge $e = uv$, the algorithm stores the edge itself and two counters $c_v^t$ and $c_u^t$ for degrees of its endpoints in the rest of the stream. If we detect that a sampled edge is not in $E_\alpha^t$, i.e., either of the associated counters exceed $\alpha$, it is deleted.

Cormode et al. ran multiple instances of this basic algorithm corresponding to sampling probabilities $1, (1 + \epsilon)^{-1}, (1 + \epsilon)^{-2}, \ldots$ in parallel; terminated any instance that used too much space; and returned an estimate based on one of the remaining instantiations. Instead, we start sampling with probability 1 and put a cap on the number of edges stored by the algorithm. Whenever the capacity is reached, the algorithm halves the sampling probability and deletes every edge currently stored with probability 1/2. This modification saves a factor of $O(\epsilon^{-1} \log n)$ in the space use and update time of the algorithm. We save a further $O(\alpha)$ factor in the analysis by using the algorithm to estimate $E^*$ rather than $|E_\alpha|$. The proof of correctness is similar to that for the original algorithm.

**Theorem 3.** With high probability, Algorithm 1 outputs a $(1 + \epsilon)$ approximation of $E^*$.

**Proof.** Let $k$ be such that $2^{k-1} \leq E^* < 2^k \tau$ where $\tau = 20 \epsilon^{-2} \log n$. First suppose we toss $O(\log n)$ coins for each edge in $E_\alpha^t$ and say that an edge $e$ is sampled at level $i$ if at least the first $i - 1$ coin tosses at heads. Hence, the probability that an edge is sampled at level $i$ is $p_i = 1/2^i$ and that the probability an edge is sampled at level $i$ conditioned on being sampled at level $i - 1$ is 1/2. Let $s_i^t$ be the number of edges sampled. It follows from the Chernoff bound that for $i \leq k$, $s_i^t - p_t E_i^t \leq \exp \left( -\frac{\epsilon^2 E_i^t p_t}{4} \right)$ $\leq \exp \left( -\frac{\epsilon^2 E^* p_t}{4} \right) \leq \exp \left( -\frac{\epsilon^2 \tau}{8} \right) = 1/\text{poly}(n)$. By the union bound, with high probability, $s_i^t/p_t = |E_\alpha^t| \pm \epsilon E^*$ for all $0 \leq i \leq k$, $1 \leq t \leq n$. 

Say an edge $uv$ is good if $uv \in B_u \cap B_v$ and wasted if $uv \in B_u \oplus B_v$, i.e., the symmetric difference. Then $|E_\alpha|$ is exactly the number of good edges. Define

\begin{align*}
    w &= \text{number of good edges with exactly no end points in } H, \\
    x &= \text{number of good edges with exactly one end point in } H, \\
    y &= \text{number of good edges with two end points in } H, \\
    z &= \text{number of wasted edges with two end points in } H,
\end{align*}

and note that $|E_\alpha| = w + x + y$.

We know $x + 2y + z = (\alpha + 1)|H|$ because $B_u$ contains exactly $\alpha + 1$ edges if $u \in H$. Furthermore, $z + y \leq \alpha|H|$ because the graph has arboricity $\alpha$. Therefore

$$x + y \geq (\alpha + 1)|H| - \alpha|H| = |H|.$$
Algorithm 1: APPROXIMATING $E^*$

1. Initialize $S \leftarrow \emptyset$, $p = 1$, max $= 0$
2. For each edge $e = uv$ in the stream:
   (a) With probability $p$ add $e$ to $S$ and initialize counters $c^w_e \leftarrow 0$ and $c^v_e \leftarrow 0$
   (b) For each edge $e' \in S$, if $e'$ shares endpoint $w$ with $e$:  
      • Increment $c^w_{e'}$  
      • If $c^w_{e'} > \alpha$, remove $e'$ from $S$ and corresponding counters
   (c) If $|S| > 30\epsilon^{-2}\log n$:
      • $p \leftarrow p/2$
      • Remove each edge in $S$ and corresponding counters with probability $1/2$
   (d) max $\leftarrow \max(\max, |S|/p)$
3. Return max

Figure 1. APPROXIMATING $E^*$ Algorithm.

The algorithm initially maintains the edges in $E^*_\alpha$ sampled at level $i = 0$. If the number of these edges exceeds the threshold, we subsample these to construct the set of edges sampled at level $i = 1$. If this set of edges also exceeds the threshold, we again subsample these to construct the set of edges at level $i = 2$ and so on. If $i$ never exceeds $k$, then the above calculation implies that the output is $(1 \pm \epsilon)E^*$. But if $s^*_t$ is bounded above by $(1 + \epsilon)E^*/2^k < (1 + \epsilon)\tau$ for all $t$ with high probability, then $i$ never exceeds $k$. □

It is immediate that the algorithm uses $O(\epsilon^{-2}\log n)$ space since this is the maximum number of edges stored at any one time. By Corollary 2, $E^*$ is an $(\alpha + 2)$ approximation of $\text{match}(G)$ and hence we have proved the following theorem.

Theorem 4. The size of the maximum matching of a graph with arboricity $\alpha$ can be $(\alpha + 2)(1 + \epsilon)$-approximated with high probability using a single pass over the edges of $G$ given $O(\epsilon^{-2}\log n)$ space.

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