Synthesis of Output-Feedback Controllers for Mixed Traffic Systems in Presence of Disturbances and Uncertainties

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Abstract—In this paper, we study mixed traffic systems that move along a single-lane ring-road or open-road. The traffic flow forms a platoon, which includes a number of heterogeneous human-driven vehicles (HDVs) together with only one connected and automated vehicle (CAV) that receives information from several neighbors. The dynamics of HDVs are assumed to follow the optimal velocity model (OVM), and the acceleration of the single CAV is directly controlled by a dynamical output-feedback controller. The ultimate goal of this work is to present a robust control strategy that can smoothen the traffic flow in the presence of undesired disturbances (e.g., abrupt deceleration) and parametric uncertainties. A prerequisite for synthesizing a dynamical output controller is the stabilizability and detectability of the underlying system. Accordingly, a theoretical analysis is presented first to prove the stabilizability and detectability of the mixed traffic flow system. Then, two $H_\infty$ control strategies, with and without considering uncertainties in the system dynamics, are designed. The efficiency of the two control methods is subsequently illustrated through numerical simulations, and various experimental results are presented to demonstrate the effectiveness of the proposed controller to mitigate disturbance amplification and achieve platoon stability.

I. INTRODUCTION

In recent decades, thanks to developments in automation, such as the emergence of automated vehicles or automated infrastructures, a tremendous revolution has occurred in transportation systems. The transition phase from using only human-driven vehicles (HDVs) to fully connected and automated vehicles (CAVs) results in new challenges and creates a strong motivation to study the mixed traffic systems, that include both HDVs and CAVs (e.g., see [1], [2] and the references therein). In this direction, new opportunities arise to utilize the potential abilities of CAVs to control a transportation network, manage congestion, and promote the efficiency and safety of traffic systems.

More traditional methodologies for controlling traffic flow employ controllers and actuators at fixed locations, among which variable speed limits (VSLs) and ramp metering (RMs) can be mentioned [3]. However, the installation of these actuators is not cost-effective and reduces the flexibility of the control system. On the other hand, the advent of CAVs as mobile actuators—so-called Lagrangian actuators—paves the way for applying traffic flow control in a more effective and flexible manner. For instance, if all the involved vehicles are CAVs, efficient control strategies, such as adaptive cruise control (ACC) and cooperative adaptive cruise control (CACC), can be employed and lead to a desirable system performance [4], [5], [6], [7]. Nevertheless, in a mixed traffic system, where the penetration rate of CAVs is less than one, new challenges are introduced that require further theoretical and experimental analyses.

In this direction, we consider a mixed traffic system, including one CAV and numerous HDVs, for the two common scenarios of a ring-road and an open-road and reveal how the single CAV is capable of controlling the entire network. In the following, we first present a review of some relevant works in the literature, and subsequently, we discuss the main contributions of this work.

A. Literature Review

There are a few experimental studies that verify the emergence of stop-and-go waves in traffic flow systems. For instance, in the study in [8], the outcome of a practical experiment on a single-lane ring-road demonstrated that a platoon consisting only of HDVs has the potential to initiate stop-and-go waves. These waves that travel upstream along the road make a uniform flow unstable and create a so-called phantom traffic jam. This phenomenon of instability has been studied in the literature from macroscopic [9], cellular automaton [10], and microscopic [11] point of view. The emerging nonlinear waves can be amplified by some effects such as stochastic behaviour of human drivers, lane changing, road characteristics, and ramps, to name a few. In [12], a field experiment was conducted to show that utilizing a single CAV in a platoon on a circular roadway can dissipate the undesired waves. Moreover, in [13], through some theoretical analysis, the capability of a single CAV to control the traffic flow on a ring-road was investigated.

In fact, since the platoon is connected, and neighboring vehicles can interact, a sparse number of CAVs—that act as mobile actuators—can influence the whole network and stabilize the traffic system. The notions of string and ring stability have been employed here for the stability analysis of interconnected vehicles on a string and on a ring roadway, respectively [14], [15], [16]. In the same direction, and for mixed traffic systems, the string stability of a mixed platoon of infinite length has been analyzed in [17]. Furthermore, in [18], a linear stability condition has been stated in terms of the penetration rate and spatial distribution of CAVs.

In order to dissipate stop-and-go waves in a mixed traffic system, an appropriate control strategy can be provided that is applied to the CAVs as the controllers. To establish a theoretical analysis for these systems, we should first derive a mathematical model that represents the dynamical behaviour of HDVs. In this direction, there are different car-following models, among which the optimal-velocity-follow-the-leader

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(OV-FTL) model [11] and the intelligent driver model (IDM) [19] can be indicated. These models are nonlinear in principle, but most of studies in the field of mixed traffic systems utilize a linearized version of the nonlinear dynamics around the equilibrium flow (see e.g., [20], [21], [16], [22], [23]).

A fundamental network property that should be checked before designing a controller is its controllability or stabilizability [24], [25], [26]. If a linear system is controllable, then a control signal can be designed to steer the system from any initial state to any final state within finite time. A weaker condition that is necessary for the existence of a controller is the stabilizability of a system. A linear system is stabilizable if with a suitable choice of control signals, all the states remain bounded or converge to constant values [27]. Recently, some works in the literature have focused on providing a rigorous controllability and stabilizability analysis for ring-road mixed traffic systems with one single CAV.

In [13], [22], it is assumed that all HDVs in the platoon are homogeneous, which is a strong assumption for practical scenarios. In fact, the problem where heterogeneity of HDVs is considered is closer to reality, but theoretically more challenging. In [28], [29], a controllability analysis for mixed traffic systems, including one CAV and a number of heterogeneous HDVs, is provided. It is stated that under a restrictive condition on parameters of the dynamic model, all nonzero eigenvalues of the system are controllable; while there exists only one uncontrollable eigenvalue at origin. More recently, in [29], by considering the similar condition on the parameters of the system, the controllability of a so-called “1+n” mixed platoon, forming a string at a signalized intersection, is provided. Moreover, in [30], by defining a new notion of leading cruise control, the controllability of a platoon along a string is analyzed. None of these works provide the controllability analysis of a mixed platoon in the most general case and without assuming any constraint on the system parameters.

There are numerous works in the literature, proposing various control strategies to stabilize a mixed platoon on an open-road or a ring-road [31], [32], [13], [33], [12], [34], [35]. However, in most of these works, the communication topology of the network which represents the capability of the CAV to receive information from its neighboring vehicles has been neglected [31], [36], [20], [37], [35], and it has been assumed that it can be connected to any vehicle in the platoon. More recently, in [28], the issue of limited communication has been considered, and a structured optimal control has been proposed, which is in general computationally intractable [38] and results in a sub-optimal solution. In addition, the HDVs usually do not follow deterministic dynamic models, and there exists uncertainties in the model parameters of HDVs, that can affect the efficiency of the control strategies. Accordingly, it is needed to provide robust control methods that dissipate the perturbations of the traffic flow in the presence of system uncertainties (see e.g., [39], [40], [41]).

B. Contributions

In this paper, we consider a mixed traffic system with one CAV and numerous heterogeneous HDVs. The mixed platoon is studied in two cases of a ring-road and an open-road. The traffic system is modeled at a microscopic scale, and the dynamics of HDVs are represented by the optimal velocity model. Furthermore, the acceleration of the CAV is directly controlled. For this system, we first establish a stabilizability analysis for the two cases of a ring-road and an open-road. Moreover, since the goal is to synthesize an output-feedback controller, the detectability analysis of the system is also necessary, which is presented subsequently. As a real-world scenario, we also consider the limited communication capability of the CAV in this work. In order to deal with the topological communication constraints, we propose an output-feedback controller that employs the information of a sparse set of vehicles in the control signal design. Our proposed method offers a robust $H_{\infty}$ controller, that not only increases the efficiency of the CAV, but it also dampens the disturbances appearing as nonlinear waves and improves the performance in the behavior of the entire traffic network.

In summary, the main contributions of this work are listed as follows:

- We analyze the controllability and observability of a mixed traffic system with one CAV and numerous heterogeneous HDVs for both common scenarios of a ring-road and an open-road in the most general case. In fact, unlike the existing works in the literature, such as [23], that investigates the stabilizability under restrictive parameter constraints, we prove that, for any value of system parameters, the mixed platoon is stabilizable. This analysis verifies the ability of the single CAV to make the states of all HDVs converge to desired values. Further, since we aim to synthesize a dynamic output controller that can utilize the states of only a subset of HDVs, we prove in this work that the mixed traffic system is also detectable.

- In order to dampen the undesired perturbations occurring in a mixed traffic flow where there is no uncertainties in the system parameters, we propose a solution based on synthesizing an $H_{\infty}$ output dynamic feedback controller. This controller also tackles the issue of the CAV’s communication constraints. To the best of our knowledge, this is the first time in the literature that an output dynamic controller is utilized for the control of a mixed platoon. Unlike some existing control strategies, e.g., [7], [42], that aim to increase the local efficiency around the CAVs, our method offers a controller that improves the performance in the behavior of the entire traffic network. More importantly, as we consider a higher degree of freedom in designing the control strategy, our proposed control method leads to a global optimal solution, while the structural control method in [23] results is a sub-optimal one.

- As the next and the more practical scenario, we consider the model mismatch and parametric uncertainties in the dynamics of HDVs and provide a robust control strategy that can smoothen the traffic flow in the presence of disturbances and uncertainties.
C. Outline

The rest of this paper is organized as follow. In Section II, the model of a mixed traffic system is presented, and the main problems of this work are formulated. Section III discusses the stabilizability and detectability of a mixed platoon with a single CAV for both ring-road and open-road setups. In section IV, we propose an output dynamic controller for a mixed traffic flow that has no uncertainty in the system parameters. Section V establishes a control strategy that smoothen the traffic flow in the presence of uncertainties. In Section VI, numerical validations of the results as well as a numerical comparison with some of existing works in the literature are provided. Finally, Section VII concludes the paper.

II. Preliminaries

In this section, we first present a dynamic model for a mixed traffic system on a single-lane ring-road, and then we formulate the problem.

We denote the set of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. For $a \in \mathbb{C}$, $\text{Re}(a)$ represents its real part. We denote the transpose of matrix $M$ by $M^T$. Also, $\det(M)$ represents its determinant. For a vector space $\mathcal{V}$, $\dim(\mathcal{V})$ indicates its dimension. The identity matrix is denoted by $I$, and its $j$-th column is designated by $e_j$. Also, $0_{n \times m}$ represents an $n \times m$ zero matrix. We also show by $0$ a zero matrix of an appropriate dimension. For $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $A = \text{diag}(\alpha_1, \ldots, \alpha_n)$ is an $n \times n$ diagonal matrix whose diagonal elements are $\alpha_1, \ldots, \alpha_n$. For a matrix $M \in \mathbb{R}^{n \times n}$, where $M = M^T$, $M \succ 0$ (resp., $M \succeq 0$) implies that $M$ is a positive definite (resp., positive semi-definite) matrix. $||.||$ denotes the Euclidean norm of vectors and the induced norm of matrices. Also, $||.||_F$ denotes Frobenius norm of matrices.

Fact 1: [43] For any matrix $A \in \mathbb{R}^{n \times n}$, we have $||A|| < ||A||_F$, where $||A||_F = (\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2)^{1/2}$.

Fact 2: [43] For any matrix $A \in \mathbb{R}^{n \times n}$, $||A||_F \leq 1$ if and only if $A^T A \preceq I$.

A. Modeling a Mixed Traffic System

We study a mixed traffic system that is a network of $n$ vehicles, including one CAV and $n - 1$ HDVs. We consider two cases of a single-lane ring-road and a single-lane open-road with length $D$. In Fig. 1(a), a schematic diagram of this network is illustrated, where the red car denotes the CAV, indexed with 1, and all others are HDVs. The position and velocity of vehicle $i$ are denoted by $p_i$ and $v_i$, respectively. We define as $s_i = p_{i-1} - p_i$ the back-to-back distance of the $i$-th vehicle from the $i-1$-th vehicle.

There are different models in the literature to represent the car-following dynamics of human-driven vehicles (see e.g., [19, 21, 11]). For instance, the optimal velocity model (OVM) [11] can be described as:

$$\dot{s}_i(t) = v_{i-1}(t) - v_i(t)$$
$$\dot{v}_i(t) = H_i(v_i(t), s_i(t), \dot{s}_i(t)),$$

where $H_i(\cdot)$ is the acceleration of vehicle $i$, that is a nonlinear function of its velocity $v_i$, the spacing $s_i$, and the relative velocity $\dot{s}_i$. Note that unlike most of the works in literature (e.g., [13, 31, 2]), in this paper, HDVs are assumed to be heterogeneous, and thus, the dynamics of each vehicle $i$ is described by a distinct nonlinear function $H_i(\cdot)$. One can see that at the equilibrium point of dynamics [11], all vehicles have the same velocity $v^*$. Moreover, since we have $\dot{v}^* = 0$, the spacing $s_i^*$ is computed by $0 = H_i(v^*, s_i^*, 0)$. Now, let us define the state error $x_i^T = [\bar{s}_i, \bar{v}_i] = [s_i - s_i^*, v_i - v^*]$. Then, by linearization of [11] around the equilibrium point $[s_i^*, v^*]^T$, for $i = 2, \ldots, n$, one can derive a linear time-invariant (LTI) model for the $i$-th HDV as:

$$\dot{\bar{s}}_i(t) = \bar{v}_{i-1}(t) - \bar{v}_i(t)$$
$$\dot{\bar{v}}_i(t) = \beta_{i1}\bar{s}_i(t) - \beta_{i2}\bar{v}_i(t) + \beta_{i3}\bar{v}_{i-1}(t),$$

where

$$\beta_{i1} = \frac{\partial H_i}{\partial s_i}, \quad \beta_{i2} = \frac{\partial H_i}{\partial v_i}, \quad \beta_{i3} = \frac{\partial H_i}{\partial \dot{s}_i},$$

computed at the equilibrium point. Due to some physical constraints imposed by the behavior of HDVs in practice [13], one should consider

$$\beta_{i1} > 0, \quad \beta_{i2} > 0, \quad \beta_{i3} > 0.$$

Now, for the dynamics of the single CAV, we consider two cases of a ring-road and an open-road: 1) Corresponding to the case of a ring-road, the dynamics of the single CAV whose acceleration can be directly controlled are given by:

$$\dot{\bar{s}}_1(t) = \bar{v}_n(t) - \bar{v}_1(t)$$
$$\dot{\bar{v}}_1(t) = u(t),$$

where $u(t) \in \mathbb{R}$ is the control signal. 2) In the case of an open road, we have:

$$\dot{\bar{s}}_1(t) = u_1(t) - \bar{v}_1(t)$$
$$\dot{\bar{v}}_1(t) = u_2(t),$$

where $u_1(t), u_2(t) \in \mathbb{R}$ are two external inputs, and $u(t) = [u_1 \ u_2]^T \in \mathbb{R}^2$. In fact, in the case of an open-road, both

![Fig. 1](https://via.placeholder.com/150)
acceleration and velocity of the CAV are directed by external inputs.

Now, by defining the aggregated vector of states of all vehicles as $x = [x_1^T, x_2^T, \ldots, x_n^T]^T \in \mathbb{R}^{2n}$, one can derive the following LTI dynamics for the overall system:

$$
\dot{x}(t) = Ax(t) + Bu(t),
$$

where $A = \begin{bmatrix} J_1 & 0 & \cdots & 0 & J_2 \\ A_{21} & A_{22} & \cdots & 0 & B_1 \\ 0 & A_{31} & A_{32} & \cdots & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & A_{n1} & A_{n2} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$,

with $J_i = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, and for $i = 2, \ldots, n$, we have:

$$
A_{i1} = \begin{bmatrix} 0 & 1 \\ 0 & \beta_i \end{bmatrix}, \quad A_{i2} = \begin{bmatrix} 0 & -1 \\ \beta_i & \beta \end{bmatrix}.
$$

Now, for the case of a ring-road, we have $B \in \mathbb{R}^{n \times 1}$ and define:

$$
J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

Moreover, for an open-road, we have $B \in \mathbb{R}^{n \times 2}$, and $J_2, B_1, B_2$ are defined as:

$$
J_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
$$

Finally, we note that the CAV can receive the state information associated with only a number of HDVs, due to its communications constraints. For example, in Fig. 1 the green links show HDVs whose information is available to the CAV. In this direction, an output vector $y(t) \in \mathbb{R}^{2n}$ is defined, that includes the states of the CAV together with the states of all HDVs whose information is accessible by the CAV. For $k = 1, \ldots, m$, let $j_k$ be the index of the vehicle whose state information, i.e. $\tilde{s}_{j_k}$ and $\tilde{v}_{j_k}$, can be directly measured and observed. Consequently, we have:

$$
y(t) = Cx(t),
$$

where $CT = \begin{bmatrix} e_{(2j_1-1)} & e_{2j_1} & \cdots & e_{(2j_m-1)} & e_{2j_m} \end{bmatrix}$. Note that we can have $j_1 = 1$, since the CAV has access to its own information.

In this paper, we design a dynamical output-feedback controller that has the following dynamics:

$$
\begin{align*}
\dot{x}_k(t) &= A_kx_k(t) + B_ky(t), \\
u(t) &= C_kx_k(t),
\end{align*}
$$

where $A_k \in \mathbb{R}^{2n \times 2n}$, $B_k \in \mathbb{R}^{2n \times 2\alpha}$, and $C_k \in \mathbb{R}^{1 \times 2\alpha}$, and $x_k \in \mathbb{R}^{2n}$ is state of the controller.

**Problem 1:** In order to ensure the existence of an output-feedback controller, based on the separation principle, the first goal of this paper is to prove the stabilizability of pair $(A, B)$ and detectability of pair $(A, C)$ in (9) and (10).

Pair $(A, B)$ is stabilizable if the uncontrollable modes are all stable. Similarly, pair $(A, C)$ is detectable if the unobservable modes are all stable. We can use the Popov-Belevitch-Hautus (PBH) test for checking the controllability and observability of a specific eigenvalue.

**Property 1:** An eigenvalue $\lambda$ of a pair $(A, B)$ (resp., $(A, C)$) is controllable (resp., observable) if and only if for all nonzero $\rho$ for which $\rho^T A = \lambda \rho^T$ (resp., $A \rho = \lambda \rho$), $\rho^T B \neq 0$ (resp., $C \rho \neq 0$).

**Problem 2:** The next problem of this work is dedicated to computing matrices $A_k, B_k, C_k,$ and $D_k$ in (11), for the mixed traffic system described in (5) and (10), such that the undesired perturbations in the traffic flow are dissipated. We first solve this problem without considering any uncertainty in the parameters of heterogeneous vehicles. Subsequently, we assume that there might be uncertainties in the dynamical model of HDVs, and we compute a robust output-feedback controller in the presence of disturbance and parametric uncertainty.

Accordingly, in the next section, we analyze the stabilizability and detectability of the mixed traffic system (5) and (10), and subsequently, we design a dynamic controller.

**III. STABILIZABILITY AND DETECTABILITY**

In this section, the stabilizability and detectability of dynamical system (5) and (10) are discussed.

**A. Stabilizability Analysis**

In order to prove the stabilizability of the mixed traffic system, we first consider a ring-road and prove that there is only one uncontrollable eigenvalue at the origin.

**Property 2:** In the case of a ring-road, the pair $(A, B)$ in (5), where $B$ is defined in (9), has only one uncontrollable eigenvalue at the origin.

**Proof:** Let $\rho = [\rho_1 \ldots \rho_n]^T \in \mathbb{R}^{2n}$, where $\rho_i = [\rho_{i1} \rho_{i2}]^T$. Considering the expression of $A$ in (6), equation $\rho^T A = 0$ leads to:

$$
\begin{align*}
\rho_{i1} A_{i2} + \rho_{i+1} A_{(i+1)1} &= 0, \quad i = 2, \ldots, n-1 \\
\rho_{i1} J_1 + \rho_{i2} A_{21} &= 0 \\
\rho_{i2} J_2 + \rho_{i+1} A_{n2} &= 0
\end{align*}
$$

Now, from (12), one can derive

$$
\begin{align*}
\rho_{i2} &= 0, \quad i = 2, \ldots, n \\
\rho_{i1} &= \rho_{(i+1)1}, \quad i = 1, \ldots, n-1
\end{align*}
$$

Thus, a left eigenvector $\rho$ of $A$ associated with $\lambda = 0$ can be written as $\rho = [\alpha, \beta, \alpha, 0, \ldots, \alpha, 0]^T$. Now, assume that $\lambda = 0$ is an uncontrollable eigenvalue. Thus, from Proposition 1, one can see that $\rho^T B = 0$, which essentially means $\rho_{i2} = 0$. Now, define $V_u = \{ \rho \in \mathbb{R}^{2n} : \rho^T A = 0, \rho^T B = 0 \}$. Then, one can see that $V_u = \{ \rho \in \mathbb{R}^{2n} : \rho = [\alpha, \beta, \alpha, \ldots, \alpha, 0]^T, \text{for some } \alpha \neq 0 \}$. Thus, dim$(V_u) = 1$, which implies that there is only one uncontrollable eigenvalue of $A$ at the origin.

Now, consider an open-road. Next, we show that in this case, there is no uncontrollable eigenvalue at the origin.

**Property 3:** In the case of an open-road, the pair $(A, B)$ in (5), where $B$ is defined in (9), has no uncontrollable eigenvalue at the origin.
Proof: In this case, from (12), we have:

$$\rho_{12} = 0, \quad i = 2, \ldots, n$$

$$\rho_{1i} = \rho_{(i+1)1}, \quad i = 1, \ldots, n-1$$

$$\rho_{n1} = 0.$$  

Therefore, a nonzero left eigenvector $\rho$ of $A$ associated with $\lambda = 0$ has a form as $\rho = [0 \beta_0 0 \ldots 0]^T$. Assume that we have an uncontrollable eigenvalue at the origin. Thus, from Proposition 1, $\rho^T B = 0$ leads to $\rho_{11} = \rho_{12} = 0$, implying that $\rho = 0$, that is a contradiction.

Next, it suffices to show that all unstable eigenvalues of $A$ in (5) are negative. Moreover, if the roots are complex and $S$ is nonsingular. Accordingly, for $i = 1, \ldots, n$, one can conclude that $\lambda_{1i} = 0$, implying that for $i = 2, \ldots, n,$ $(\lambda I - A_{i2})$ is nonsingular.

Theorem 1: For both cases of a ring-road and an open-road, the pair $(A, B)$ associated with the mixed traffic system described in (5) is stabilizable.

Proof: Let $\lambda$ be an eigenvalue of $A$ with Re($\lambda$) > 0, and let also $\rho = [\rho_1^T \ldots \rho_n^T]^T \in \mathbb{R}^{2n}$ be its nonzero left eigenvector, where $\rho_i^T = [\rho_{1i} \rho_{2i}]$. Define $A_{11} = J_2$ and $A_{12} = J_1$. Then, for $i = 1, \ldots, n$, equation $\rho_i^T A = \lambda \rho_i^T$ implies that

$$\rho_i^T (\lambda I - A_{i2}) = \rho_{i+1}^T A_{(i+1)1}. \quad (13)$$

Now, one can see that det($\lambda I - A_{i2}$) = $\lambda^2$, which is nonzero, since $\lambda \neq 0$. Moreover, since Re($\lambda$) > 0, Lemma 1 implies that $(\lambda I - A_{i2})$ is nonsingular. Accordingly, for $i = 1, \ldots, n$, one can rewrite (13) as

$$\rho_i^T = \rho_{i+1}^T A_{(i+1)1} (\lambda I - A_{i2})^{-1}. \quad (14)$$

Let $L_i = (\lambda I - A_{i2})^{-1}$. Now, by recursively employing equation (14) for $i = 1, \ldots, n$, we can derive

$$\rho_1^T = \rho_2^T A_{21} L_1$$

$$= \rho_3^T A_{31} L_2 A_{21} L_1$$

$$= \ldots$$

$$= \rho_n^T A_{n1} L_{n-1} A_{(n-1)1} \ldots L_2 A_{21} L_1$$

$$= \rho_1^T A_{11} (L_{n1} A_{n1}) \ldots (L_2 A_{21} L_1). \quad (15)$$

In the case an open-road, since $A_{11} = 0$, one can conclude from (15) that $\rho_{11} = 0$. Now, we want to prove that $\rho = 0$ holds for a ring-road as well. Note that we have

$$L_i A_{11} = \left[ \begin{array}{cc} 0 & s_1^T \\ s_2^T & 0 \end{array} \right], \quad i = 2, \ldots, n, \quad (16)$$

where $s_1^T = \beta_2 A_{11} + \beta_1$, $s_2^T = \lambda + \beta_2 - \beta_3$, and $s_3^T = \beta_3 A_{11} + \beta_1$. As shown before, $s_1^T \neq 0$. Now, we substitute (16) into (15), and for the case of a ring-road, we obtain

$$[\rho_{11} \rho_{12}] = \frac{\prod_{i=2}^{n} s_1^T}{\lambda_1^2 - s_1^T} [\rho_{11} \rho_{12}] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right]. \quad (17)$$

Equation (17) leads to $\rho_{11} = 0$ and $\rho_{12} = \prod_{i=2}^{n} s_1^T \rho_{11} = 0$. Thus, $\rho_1 = 0$. Now, for both cases of a ring-road and an open-road, by recursively applying (14) for $i = n, n-1, \ldots, 2$, one can conclude that $\rho_i = 0$. Therefore, we have $\rho = 0$ that contradicts the assumption. Thus, $A$ has no eigenvalue which lies on the right half-plane. In addition, from Proposition 2, for the case of a ring-road, there is only one uncontrollable eigenvalue at origin. Furthermore, Proposition 3 implies that in the case of an open-road, there is no uncontrollable eigenvalue at origin. Hence, in both cases, the system is stabilizable.

Remark 1: In [23], it has been stated that the mixed traffic system described in (5) is stabilizable if for all $i, k \in \{1, 2, \ldots, n\}$, we have $\beta_{k1}^2 - \beta_{2i} \beta_{k1} \beta_{k3} + \beta_{1i} \beta_{k2}^2 \neq 0$. Moreover, through a different approach, it has been shown that there is only one uncontrollable eigenvalue at origin. However, in Theorem 1, without assuming any restrictive constraint, we have demonstrated the stabilizability of a mixed traffic system with heterogeneous HDVs for both cases of a ring-road and an open-road and proved that the system is stabilizable even if for some $i, k \in \{1, 2, \ldots, n\}$, we have $\beta_{k1}^2 - \beta_{2i} \beta_{k1} \beta_{k3} + \beta_{1i} \beta_{k2}^2 = 0$.

B. Detectability Analysis

Here, the detectability of the mixed traffic system is studied, and we show that by observing only the states of the single CAV, the detectability of the whole system can be ensured.

Property 4: The zero eigenvalue of the mixed traffic system described in (5) and (10), in both cases of a ring-road and an open-road, is observable even if the CAV has access to only its own states.

Proof: Let $C = [e_1 e_2]$, and assume that $\lambda = 0$ is not observable. Then, from Proposition 1, $A$ has a nonzero right eigenvector $\rho$, where we have both $A \rho = 0$ and $C \rho = 0$. Let $\rho = [\rho_1^T \ldots \rho_n^T]^T \in \mathbb{R}^{2n}$, where $\rho_i^T = [\rho_{1i} \rho_{2i}]$. From equation $A \rho = 0$, one can write:

$$\rho_{12} - \rho_{(i+1)2} = 0, \quad i = 2, \ldots, n$$

$$\beta_{3i} \rho_{i+1} - \beta_{1i} \rho_{i1} - \beta_{2i} \rho_{i2} = 0, \quad i = 2, \ldots, n. \quad (18)$$

Moreover, from $C \rho = 0$, one can conclude that $\rho_{11} = \rho_{12} = 0$. Thus, (18) leads to $\rho = 0$ in both cases of a ring-road and an open-road, which is a contradiction. Therefore, the zero eigenvalue is observable.

Theorem 2: In both cases of a ring-road and an open-road, the mixed traffic system described in (5) and (10) is detectable even if only the states of the single CAV are directly observed.

Proof: Let $C = [e_1 e_2]$. Assume that the system is not detectable. Then, $A$ has an eigenvalue $\lambda$ on the right half-plane with a nonzero right eigenvector $\rho$ such that $C \rho = 0$. Denote $\rho = [\rho_1^T \ldots \rho_n^T]^T \in \mathbb{R}^{2n}$, where $\rho_i^T = [\rho_{1i} \rho_{2i}]$. Then, we should have $\rho_{11} = \rho_{12} = 0$. Define $A_{11} = J_2$ and
where for the case of a ring-road, one can define the controlled output (performance output)

\[ (\lambda I - A_{12}) \rho_i = A_{i1} \rho_{i-1}. \]

Since \( \text{Re}(\lambda) > 0 \), from Lemma 1, one can see that, for \( i = 2, \ldots, n \), \( (\lambda I - A_{12}) \) is invertible. Moreover, \( \det(\lambda I - A_{12}) \neq 0 \). Let \( L_i = (\lambda I - A_{12})^{-1} \). Then, for \( i = 1, \ldots, n \), we can write:

\[ \rho_i = L_i A_{i1} \rho_{i-1}. \]  

(19)

Now, since \( \rho_1 = 0 \), recursively using equation (19) for \( i = 2, \ldots, n \) leads to \( \rho = 0 \), which contradicts the assumption. In addition, based on Proposition 4, the zero eigenvalue is observable. Thus, in summary, one can conclude that the system is detectable.

IV. CONTROLLER SYNTHESIS: WITHOUT UNCERTAINTIES

In this section, we aim to design a dynamic output-feedback controller for the mixed traffic system (5) to dissipate undesired perturbations. In the first step, we assume there is no uncertainty in the model of the traffic system and design an output-feedback controller.

A. Disturbances and Performance Outputs

In fact, the perturbations may appear due to lane changes or merges or the stochastic behavior of HDVs in ring-roads with no bottlenecks [8, 23].

Perturbations are modeled as disturbance signals added to the acceleration of each vehicle. Thus, by defining \( d \) or merges or the stochastic behavior of HDVs in ring-roads A. Disturbances and Performance Outputs.

Consider an output-feedback controller with the dynamics described in (11). We aim to design a dynamic controller that stabilizes the system (20) and minimizes the influence of the disturbance \( d \) on the performance output \( z \).

By applying the controller (11) to (20), the closed-loop system is expressed as

\[ \dot{x}(t) = \tilde{A} \bar{x}(t) + \tilde{B} d(t), \]

\[ z(t) = \tilde{C} \bar{x}(t), \]  

(22)

where \( \bar{x} = [x(t) \ x_k(t)] \), \( \tilde{A} = [A \ B C_k \ A_k] \), \( \tilde{B} = [B_d \ 0] \) and \( \tilde{C} = [C_z \ D_z C_k] \).

C. \( H_{\infty} \) Control Problem

Let \( T_{zd} \) be the closed-loop transfer function from the disturbance \( d \) to the performance output \( z \).

**Problem:** Find the matrices \( A_k, B_k, \) and \( C_k \) for the controller (11) such that the closed-loop system (22) satisfies the inequality

\[ ||T_{zd}||_{\infty} = \max_{d(t) \neq 0} ||z(t)||_2 < \gamma, \]  

(23)

and \( \gamma > 0 \) is minimized. Note that \( ||T_{zd}||_{\infty} \) denotes the \( H_{\infty} \) norm of \( T_{zd} \), which measures the largest input-output gain for energy or power input signals [45].

**Problem solution:** Based on the bounded real lemma (BRL) [46], \( ||T_{zd}||_{\infty} \) is smaller than \( \gamma \) if and only if there exists a positive definite matrix \( P > 0 \), and matrices \( A_k, B_k, \) and \( C_k \) satisfying

\[ \begin{bmatrix} \tilde{A}^T P + \frac{P \tilde{A}}{2} & \frac{P \tilde{B}}{2} & \tilde{C}^T \\ \frac{\tilde{B}^T P}{2} & -\gamma^2 I & 0 \\ \tilde{C} & 0 & -I \end{bmatrix} \prec 0. \]  

(24)

Now, note that the inequality (24) is not a linear matrix inequality (LMI) with respect to the variables \( P, A_k, B_k, \) and \( C_k \), because it is not linear with respect to these variables. In order to extract an LMI for computing the controller parameters \( A_k, B_k, \) and \( C_k \), we apply a method that has been presented in [45] in details, and we provide a summarized description of this method in the following.

Let us partition the matrices \( P \) and \( P^{-1} \) as the following form:

\[ P = \begin{bmatrix} Y & N \\ MT & * \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} X & M \\ T & * \end{bmatrix}, \]  

where \( X, Y \in \mathbb{R}^{2n \times 2n} \). Note that since \( P > 0 \), we have \( X, Y > 0 \). From the equation \( PP^{-1} = I \), one can deduce that \( NMT + XY = I \). Further, one can find that \( P = \lambda_2 \Lambda_1^{-1} \), where

\[ \Lambda_1 = \begin{bmatrix} X & I \\ MT & 0 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}. \]

Therefore, we have

\[ \Lambda_1^T PA_1 = \Lambda_1^T A_2 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \]
Substituting $P = \Lambda_2 \Lambda_1^{-1}$ in (24), we obtain
\[
\Omega_1 = \begin{bmatrix}
\bar{A}^T \Lambda_2 \Lambda_1^{-1} + \Lambda_1^{-T} \Lambda_2^T \bar{A} & \Lambda_1^{-T} \Lambda_2^T \bar{B} \\
\bar{B}^T \Lambda_2 \Lambda_1^{-1} & -\gamma^2 I
\end{bmatrix} \prec 0.
\] (25)

Now, let us define a matrix $\Omega_2$ by the following congruent transformation of $\Omega_1$
\[
\Omega_2 = \begin{bmatrix}
\Lambda_1^T & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
\Omega_1 & 0 & 0 \\
0 & \Omega_1 & 0 \\
0 & 0 & \Omega_1
\end{bmatrix}
\begin{bmatrix}
\Lambda_1^T & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}^{-1}
\begin{bmatrix}
\Lambda_2^T \bar{A} + \Lambda_2^T \hat{A}_1 \\
\bar{B}^T \Lambda_2 & -\gamma^2 I & 0 \\
\bar{C} \Lambda_1 & 0 & -I
\end{bmatrix} \prec 0.
\] (26)

Now, substituting the values of $\Lambda_1$, $\Lambda_2$, $\hat{A}$, $\hat{B}$ and $\bar{C}$, we obtain the following matrix inequality:
\[
\Omega_2 = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44}
\end{bmatrix} \prec 0,
\] (27)

where
\[
\begin{align*}
\Omega_{11} &= AX + XA^T + BC_k M^T + MC_k^T B^T, \\
\Omega_{12} &= \Omega_{21} = MA_k^T N^T + XC_k^T B_k^T N^T + MC_k^T B^T Y + XA^T Y + A, \\
\Omega_{13} &= \Omega_{31} = B_d, \quad \Omega_{14} = \Omega_{41} = XC_k^T \bar{z} + MC_k^T B_d^T, \\
\Omega_{23} &= \Omega_{32} = Y B_d, \quad \Omega_{24} = \Omega_{42} = C_k^T \bar{z}, \quad \Omega_{33} = -\gamma^2 I, \\
\Omega_{34} &= \Omega_{43} = 0, \quad \Omega_{44} = -I.
\end{align*}
\]

Finally, by defining a new set of variables as
\[
\begin{align*}
\hat{A} &= NA_k M^T + NB_k CX + YBC_k M^T + YAX, \\
\hat{B} &= NB_k, \\
\hat{C} &= C_k M^T, \\
\gamma &= \sqrt{2},
\end{align*}
\] (28)

we obtain the LMIs described in (29) with respect to variables $\gamma, X, Y, \hat{A}, \hat{B}, \hat{C}$. Therefore, if the optimization problem is feasible, then we can find $\gamma, X, Y, \hat{A}, \hat{B},$ and $\hat{C}$ and solve the matrix equation $NM^T = I - YX$ for the non-singular matrices $M$ and $N$. Moreover, matrices $A_k$, $B_k$, and $C_k$ in the state-space realization of the output-feedback controller can be derived based on (28) as follows:
\[
\begin{align*}
A_k &= N^{-1}(\hat{A} - NB_k CX - YBC_k M^T - YAX)M^{-T}, \\
B_k &= N^{-1} \hat{B}, \\
C_k &= \hat{C} M^{-T}.
\end{align*}
\] (30)

V. CONTROLLER SYNTHESIS: WITH UNCERTAINTIES

In this section, we consider parametric uncertainties in the dynamic model of the HDVs. In fact, we assume that $\beta_1$, $\beta_2$, and $\beta_3$, appearing in (7), are unknown parameters for the mixed traffic system. Then, we synthesize an output-feedback controller with dynamics (11) that stabilizes the entire closed-loop system in the presence of disturbances and parametric uncertainties.

After the linearization of the overall dynamics of the system, the uncertainty is assumed to appear in the system matrix $A$ in (6) as
\[
A = A_N + \Delta A.
\] (31)

The matrix $A_N$ is the mean-valued matrix that is constant and known. One can calculate $A_N$ as $A_N = [a_{ij} + \frac{1}{2}(a_{ij,\text{max}} - a_{ij,\text{min}})]$, where $a_{ij,\text{min}}$ (resp., $a_{ij,\text{max}}$) is the minimum (resp., maximum) value that an entry of $A$ in its $i$th row and $j$th column can have. On the other hand, $\Delta A$ represents parametric uncertainties, that is assumed to be structurally bounded. One can write $\Delta A$ as
\[
\Delta A = LFR,
\]

where the matrices $L \in \mathbb{R}^{2n \times 2n}$ and $R \in \mathbb{R}^{2n \times 2n}$ are known constant matrices. Moreover, $F \in \mathbb{R}^{2n \times 2n}$ is an unknown matrix, which satisfies the following condition:
\[
FTF \preceq I.
\] (32)

Remark 2: (Computing $L$ and $R$) We can choose $L$ and $R$ as $L = gl$ and $R = gI$. Therefore, $\Delta A = LFR = g\rho F$, and based on Fact 1, we get $||F|| = (\rho g)^{-1}||\Delta A|| \leq (\rho g)^{-1}||\bar{A}||F$. On the other hand, $\Delta A = A - A_N = [\tilde{a}_{ij}]$, where $|\tilde{a}_{ij}| \leq \frac{1}{2}(a_{ij,\text{max}} - a_{ij,\text{min}})$. Therefore, $||F|| \leq (2\rho g)^{-1}(\sum_{i=1}^{2n} \sum_{j=1}^{2n} (a_{ij,\text{max}} - a_{ij,\text{min}})^2)^{\frac{1}{2}}$. Now, if one sets $\rho g = \frac{1}{2}(\sum_{i=1}^{2n} \sum_{j=1}^{2n} (a_{ij,\text{max}} - a_{ij,\text{min}})^2)^{\frac{1}{2}}$, then $||F|| \leq 1$, and the condition (32) is satisfied.

In the following, we show how we can ensure the existence of an output-feedback controller that attenuates the effect of the disturbance on the performance output, while stabilizing the closed-loop system.

Theorem 3: Consider a system with dynamics (20), which has uncertainties that are modelled as (31). Then, there exists an output-feedback controller that stabilizes the closed-loop system (22) and minimizes $\gamma$ in the inequality (23) if the optimization problem (33) is feasible.

Proof of Theorem 3: See Appendix.

Remark 3: By solving the optimization problem (33), $\eta$, $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, $\bar{A}$, $\bar{B}$, $\bar{C}$, $X$, and $Y$ can be computed. We note that $\eta = \gamma^2$, where $\gamma$ is an upper bound of the $||T_{zd}||_\infty$. In addition, we can solve the matrix equation $NM^T = I - YX$ for the non-singular matrices $M$ and $N$. Then, the matrices $A_k$, $B_k$, and $C_k$ in the state-space realization of the output-feedback controller can be obtained as
\[
\begin{align*}
A_k &= N^{-1}(\hat{A} - NB_k CX - YBC_k M^T - YAX)M^{-T}, \\
B_k &= N^{-1} \hat{B}, \\
C_k &= \hat{C} M^{-T}.
\end{align*}
\] (30)

VI. SIMULATION RESULTS

In this section, the efficiency of the proposed control strategies are validated through numerical simulations. The minimization problems (29) and (33) are solved in YALMIP interface for MATLAB with Sedumi solver. For a better comparison of the results, we simulate an experimental setup
Moreover, we define
\[ V_\theta = \min_{\eta, X, Y, \theta} \eta \]
subject to:
\[
\begin{bmatrix}
X & I \\
I & Y
\end{bmatrix} \succ 0 \\
AX + XA^T + B\hat{C} + \hat{C}^TB^T + \hat{A}^T + A \\
\ast \\
\ast \\
\ast
\end{bmatrix}
\begin{bmatrix}
A^TY + YA + B\hat{C} + C^TB^T \\
\ast \\
\ast \\
\ast
\end{bmatrix}
\begin{bmatrix}
B_d \\
\ast \\
\ast \\
\ast
\end{bmatrix}
X_{\theta}^T + \hat{C}^TD_{\theta}^T < 0
\] (29)

that is similar to the ones considered in [28], [23], where a ring-road with a circumference \( D = 400 \text{ m} \) and 20 vehicles has been studied (since the results for the case of an open-road is analogous to the case of a ring-road, we illustrate here the simulation results only for a ring-road). We assume that the 1st vehicle can be a CAV that has access to the state information of the five HDVs ahead and the five HDVs behind.

\[ \min_{\eta, \epsilon_1, \epsilon_2, \epsilon_3, X, Y, \theta} \eta \]
subject to:
\[
\begin{bmatrix}
X & I \\
I & Y
\end{bmatrix} \succ 0 \\
\Gamma_{11} \Gamma_{12} \prec 0 \\
\begin{bmatrix}
A_NX + XA_N^T + B\hat{C} + \hat{C}^TB^T + L(\epsilon_1 + \epsilon_2)T \\
\ast \\
\ast \\
\ast
\end{bmatrix}
\begin{bmatrix}
A_N^TY + YA_N + B\hat{C} + C^TB^T + \epsilon_3 R^TR \\
\ast \\
\ast \\
\ast
\end{bmatrix}
\begin{bmatrix}
B_d \\
\ast \\
\ast \\
\ast
\end{bmatrix}
X_{\theta}^T + \hat{C}^TD_{\theta}^T
\] (33)

\[ \Gamma_{11} = \\
\Gamma_{12} = \\
\Gamma_{22} = -\text{diag}(\epsilon_1 I, \epsilon_2 I, \epsilon_3 I, I, I)
\]

where we set \( s_{i,\text{st}} = 5, s_{i,\text{go}} = 35 + U[-5, 5], v_{i,\text{max}} = 30, \) and \( h_{i,v}(s_i) \) is a nonlinear function chosen as [31]
\[
h_{i,v}(s_i) = \frac{v_{i,\text{max}}}{2} (1 - \cos(\frac{s_i - s_{i,\text{st}}}{s_{i,\text{go}} - s_{i,\text{st}}}))
\]

In order to ensure the safety and prevent collisions, every vehicle is also assumed to be equipped with an automatic braking system, described as
\[
\dot{v}(t) = a_{\text{min}} \quad \text{if} \quad \frac{v(t)^2 - v(t-\tau)^2}{2s_i(t)} \geq |a_{\text{min}}|
\]

where \( a_{\text{min}} = -5 \text{ m/s}^2 \). Moreover, we set the maximum acceleration of any vehicle as \( a_{\text{max}} = 2 \text{ m/s}^2 \).

A. Simulation setup

Through using an optimal velocity model (OVM), any nonlinear function \( H_i(\cdot) \) in (1), \( i = 2, \ldots, 20 \), that describes the acceleration function of the \( i \)th HDV is written as \( H_i(\cdot) = \alpha_i(V_i(s_i(t)) - v_i(t)) + \theta_i s_i(t) \), where \( \alpha_i \) and \( \theta_i \) are sensitivity coefficient in an OVM. Moreover, \( V_i(s_i) \) is the desired speed of HDV \( i \) that is a function of spacing \( s_i \). Due to heterogeneity of HDVs, we set \( \alpha_i = 0.6 + U[-0.1, 0.1] \) and \( \theta_i = 0.9 + U[-0.1, 0.1] \), where \( U[a, b] \) represents a uniform distribution function that takes values from the interval \([a, b]\). Moreover, we define \( V_i(s_i) \) as a piecewise function
\[
V_i(s_i) = \begin{cases} 
0, & s_i \leq s_{i,\text{st}}, \\
h_{i,v}(s_i), & s_{i,\text{st}} < s_i < s_{i,\text{go}}, \\
v_{i,\text{max}}, & s_i \geq s_{i,\text{go}},
\end{cases}
\]

B. Stabilizability Verification

As the first scenario, assume that all vehicles are randomly distributed along the ring-road and start their movement with initial velocity \( v_i(0) \) from the distribution \( 15 + U[-4, 4] \text{ m/s} \). Notice that we consider no uncertainties in the dynamic model of the system in this case. First, assume that all vehicles of the mixed traffic system are HDVs. In Fig. (a), it can be seen that multiple perturbations occur in this system, which are amplified over time and generate an unstable nonlinear wave moving upstream the traffic flow.

Next, assume that the 1st vehicle is a CAV that is controlled by an output-feedback controller described in Section IV.C
First, we set the equilibrium velocity as $v^* = 15 \text{ m/s}$. Then, the equilibrium spacing of vehicle $i$ is obtained by solving the equation $0 = H_i(15, s^*_i, 0)$. Note that in a circular path, the sum of spacing of all vehicles should equal to the circumference $D$. In other words, we should have $\sum_{i=1}^2 s_i(t) = \sum s^*_i = 400$. Now, one can see in Fig. 2(b) that the perturbations can be attenuated within a short time.

As the next experiments, we change the equilibrium velocity to $16 \text{ m/s}$ and $14 \text{ m/s}$, respectively. Notice that in this case, the parameters of the linearized system (5) that are computed around the equilibrium point will be changed. In these two cases, one can observe in Figs. 2(c,d) that the single CAV is still capable of stabilizing traffic flow and steering it towards the new equilibrium points, which verifies the stabilizability of the mixed traffic system with a single CAV.

C. Robustness Against Disturbances

In this part, we aim to illustrate the performance of the output-feedback controller, proposed in Section 4C, to dampen the undesired disturbances.

We note that the parameters $\gamma_s$, $\gamma_v$, and $\gamma_u$ in the performance output $z(t)$ in (21), should be selected such that the optimization problem (29) is feasible. Moreover, we should prevent rapid oscillations in the output of the system. If we choose smaller values for the parameters $\gamma_s$ and $\gamma_v$, the system oscillations increase, while the system response converges more slowly that is not desirable. As a result, there is a trade-off between the convergence rate of the system response and its quality in terms of the amplitude of the oscillations. Accordingly, adjusting the values of the parameters $\gamma_s$ and $\gamma_v$ is an important part of the controller design. In order to have an appropriate output behaviour of the traffic flow system, we choose $\gamma_s = 0.03$, $\gamma_v = 0.15$, and $\gamma_u = 1$.

As the next experiment, we assume that, at $t = 20 \text{ s}$, the 7th vehicle is decelerated at $-3 \text{ m/s}^2$ for $3 \text{ s}$ (this perturbation can be due to road bottlenecks). In Figs. 3(a,b), the trajectory and the velocity profile of all vehicles when there is no CAV in the system are illustrated. One can see in these figures that the perturbation does not vanish, and a nonlinear wave appears that propagates against the traffic flow. On the other hand, when one CAV is added to the traffic system and is controlled by the proposed strategy in Section 4C, one can see in Figs. 4(a,b) that the stop-and-go-wave can be quickly dissipated, and the traffic flow is stabilized to the equilibrium point.

D. Comparison with Existing Results

In this part, we compare the efficiency of the output-feedback controller proposed in Section 4C with some of the existing control strategies in the literature. In particular, we
are considerably smaller compared to the other strategies. The whole process, and its fluctuations around the desired spacing remains close to the equilibrium spacing over the optimal control strategy in [23]. In fact, with our controller, the absolute value of the spacing error is also [15 m].

Fig. 5. Comparison of the results for four different control strategies; (a) \( \max_i |s_i(t) - s_0^i| \) w.r.t. the index of the perturbed HDV; (b) Energy of the performance output, i.e., \( ||z(t)||^2 = \int_{t=0}^{\infty} x^T(t)Tx(t) + u^T(t)Qu(t) \), w.r.t. the index of the perturbed HDV.

E. Robustness against Disturbances and Uncertainties

As the last experiment, we assume that some parameters appearing in the dynamic model of HDVs are uncertain. However, in our case study, the nominal values of the parameters are the same for all HDVs. In fact, instead of considering heterogeneous HDVs, we assume that there are a number of vehicles with homogeneous nominal dynamic models that may also include uncertainties. In order to dampen the perturbations of the mixed traffic system in the presence of parametric uncertainties, we design an output-feedback controller, using the procedure proposed in Section V.

In this experiment, we assume that the parameters \( \alpha_i, \theta_i, \) and \( s_{i, go} \) (see Section VI-A) are unknown. The uncertain parameters are distributed around the nominal values \( \alpha_N = 0.6, \theta_N = 0.9, \) and \( s_{N, go} = 35 \). Then, one can define \( \alpha_i = \alpha_N + \Delta \alpha, \theta_i = \theta_N + \Delta \theta, s_{i, go} = s_{N, go} + \Delta s_{go}, \) where \(-0.1 \leq \Delta \alpha \leq 0.1, -0.1 \leq \Delta \theta \leq 0.1, \) and \(-5 \leq \Delta s_{go} \leq 5 \).

We first assume in this experiment that the initial velocity of the vehicles is randomly chosen from the distribution \( 15 + U[-4, 4] \) m/s. We compute the controller parameters associated with three different equilibrium velocities, that is, 14, 15, and 16 m/s. As observed in Figs. 6 (a–c), in these three cases, by using the control strategy proposed in Section V the perturbations occurring in the traffic flow system are dampened, and the velocities of all vehicles converge to the equilibrium points within a short time.

Next, we assume that, at \( t = 20s \), the vehicle no. 7 brakes at \(-3 m/s^2\) for \( 3s \), and the single CAV is under the proposed control in Section V. In Fig. 6 (a,b), the velocity profile of all vehicles from a 3-dimensional (3D) and 2-dimensional (2D) perspective are illustrated. One can see the with the proposed output-feedback controller, the perturbation is attenuated very quickly in this traffic system. Moreover, as observed in Fig. 8 the spacing of HDVs and the CAV from the preceding vehicles converges to the equilibrium value in a few seconds.

VII. Conclusion

In this work, the stabilizability and detectability of a mixed traffic system along a ring-road and an open-road has been studied and analyzed. It has been shown that in both cases,
The effectiveness of the proposed methods to achieve traffic flow stability has been verified and demonstrated through numerical simulation experiments. Finally, the numerical results are compared to another approach from the literature. Future work is going to deal with the generalization of these results to more complex traffic patterns.

**APPENDIX**

We first present two results that will be used in the proof of Theorem 3 and then discuss the proof.

Lemma 2: [47] (Young inequality) Given matrices $D$, $F$, and $S$ of appropriate dimensions, the inequality $DFS + (DFS)^T \preceq \epsilon^{-1}DD^T + \epsilon S^T S$ holds for some scalar $\epsilon > 0$ if we have $FTF \preceq I$.

Lemma 3: [48] (Schur complement) Consider the matrices $W_1$, $W_2$, and $W_3$ with the appropriate dimensions, where $W_1 = W_1^T > 0$. Then, the matrix inequality $W_1 + W_3^{-1}W_2 < 0$ is equivalent to

\[
\begin{bmatrix}
W_1 & W_3^T \\
W_3 & -W_2
\end{bmatrix} < 0.
\]

**Proof of Theorem 3** As shown in section (IV-C), based on the bounded real lemma (BRL), if the matrix inequality (27) is satisfied, then $||T_{zd}||_\infty < \gamma$. Incorporating $A$ from (31) to (27), we get

\[
\Omega_4 = \Omega_3 + \Delta \Omega < 0,
\]

where

\[
\Omega_3 = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44}
\end{bmatrix},
\]

with

\[
\begin{align*}
\Omega_{11} &= A_N^T X + X A_N^T + BC_k M^T + MC_k^T B^T, \\
\Omega_{12} &= \Omega_{21}^T = MA_k^T N_T + X C^T B_k^T N_T^T + MC_k^T B^T Y, \\
\Omega_{13} &= \Omega_{41} = B_d, \\
\Omega_{14} &= \Omega_{43} = X C_{z_1}^T + MC_k^T D_z^T, \\
\Omega_{22} &= A_k^T Y + Y A_N + N B_k C + C^T B_k^T N_T, \\
\Omega_{23} &= \Omega_{42} = Y B_d, \\
\Omega_{24} &= \Omega_{43} = C_z^T, \\
\Omega_{33} &= \Omega_{34} = 0, \\
\Omega_{34} &= \Omega_{44} = -I,
\end{align*}
\]

and

\[
\Delta \Omega = \begin{bmatrix}
\Delta A X + X \Delta A^T & \Delta A + X \Delta A^T Y & 0 & 0 \\
\Delta A^T + Y \Delta A & \Delta A^T Y + Y \Delta A & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

By substituting $\Delta A = LFR$, and applying condition (32) and Lemma 2, $\Omega_4$ can be bounded as

\[
\Omega_4 \preceq \Omega_3 + \sum_{i=1}^{41} \Upsilon_i < 0,
\]
Now, by defining \( \dot{A} = N A_k M^T + N B_k C X + Y B C_k M^T + Y A_N X, \dot{B} = N B_k \dot{C} = C_k M^T, \eta = \gamma^2 \) and selecting \( \delta = 1 \), we obtain
\[
\Omega_5 + \Gamma_{12} \Gamma_{22}^{-1} \Gamma_{12}^T \prec 0
\]
where
\[
\Omega_5 = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{12}^T & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
\Omega_{13} & \Omega_{23}^T & \Omega_{33} & \Omega_{34} \\
\Omega_{14} & \Omega_{24}^T & \Omega_{34} & \Omega_{44}
\end{bmatrix}
\]
with
\[
\begin{align*}
\Omega_{11} &= AX + XA_N^T + B \dot{C} + \dot{C}^T B^T + (\epsilon_1 + \epsilon_2) LL^T, \\
\Omega_{12} &= \Omega_{21}^T = A^T + A_N, \\
\Omega_{22} &= A_N^T Y + Y A_N + \hat{B} C + C^T \hat{B}^T + \epsilon_3 R R^T, \\
\Omega_{13} &= \Omega_{31}^T = B_d, \\
\Omega_{14} &= \Omega_{41}^T = X C_{\tau}^T + \dot{C}^T D_{\tau}, \\
\Omega_{23} &= \Omega_{32}^T = Y B_d, \quad \Omega_{24} = \Omega_{42}^T = C_{\tau}^T, \quad \Omega_{33} = -\eta I, \\
\Omega_{34} &= \Omega_{43} = 0, \quad \Omega_{44} = -I.
\end{align*}
\]
Moreover, we have
\[
\Gamma_{12} = \begin{bmatrix}
XR^T & 0 & 0 & YL \\
0 & R^T & YL & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & XR^T
\end{bmatrix},
\]
\[
\Gamma_{22} = -\text{diag}(\epsilon_1 I, \epsilon_2 I, \epsilon_3 I, I, I).
\]
Finally, by applying Lemma 3, the LMIs in (33) are obtained.

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