HOMOGENIZATION AND SINGULAR PERTURBATION IN POROUS MEDIA

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ABSTRACT. We study a Dirichlet problem in periodic porous medium depending on two small parameters, the hydraulic permeability of the porous inclusions δ and the period ε. We study the situation as δ → 0, ε → 0 and ε → 0, δ → 0 and prove that the two limits do not commute.

1. Introduction. Modeling a flow in porous media is an important subject due to its numerous applications. A standard and very efficient approach is to use a periodic model of porous medium and the method of homogenization.

Frequently, the porous medium is modeled as a periodic array of solid structures with fluid flowing around them. Such geometry allows to derive a model formally and to prove rigorously the convergence of the homogenization process, as the period of the medium tends to zero.

The formal computation using two-scale asymptotic expansions as well as the proof of the homogenization procedure can be found, for instance, in Sanchez-Palencia [23] or Bakhvalov and Panasenko [5].

The rigorous proof of convergence for the homogenization procedure can be done using the Tartar approach of oscillating test-functions (also called the energy method) presented in the appendix of [23]. Another approach involves the two-scale convergence, introduced by Nguetseng [22] and popularized by Allaire [1]. One can also use an efficient method called the unfolding, based on the notion of the local unfolding operator (see e.g. [8, 3, 25]).

Here, we consider a double-porosity situation (see e.g. [8, 3, 25]). We have a periodic porous medium, with period ε, consisting of periodic cells, each of them having a fluid part and containing a porous inclusion with very small permeability δ ≪ 1. The fluid flows in both, larger pores and the embedded porous block. The flow in the block is supposed to be governed by the Brinkman equation (see [21]) with small permeability δ.

Thus, our model has two small parameters, the period ε and the permeability of the porous inclusions δ. Obviously, letting the permeability δ → 0 gives the standard

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model with periodically distributed solid inclusions. And then the homogenization procedure is applied to take the limit as \( \varepsilon \to 0 \) leading to the Darcy law.

In situations when the boundary value problems depend on two small parameters, and we study the limits as both parameters tend to zero, it is always interesting to see whether the two limits commute. So, in the second part of the paper, we can first take the limit as the period \( \varepsilon \to 0 \), using the homogenization, and then let the permeability in the inclusion \( \delta \to 0 \) in the homogenized problem. It turns out that the obtained results are not the same.

To minimize the technicalities and present the main feature in a more evident way, we consider the mono-directional flow governed by the scalar equations.

Boundary value problems depending on two parameters and combining homogenization and singular perturbation are not new in the literature, and the question of commutation of two limits naturally arises. Most notable contribution are related to homogenization of processes in thin domains where the singular perturbation part is related to the lower-dimensional approximation, as for instance in \([4, 9, 10, 12, 14, 15, 18, 17]\). See also \([7]\) (periodic porous medium with thin fracture), \([11]\) (singular perturbation of the boundary condition), \([24]\) (p-Laplace equation with singularly perturbed right-hand side), \([2]\) and \([16]\) (periodic porous medium with vanishing inclusions) or \([6]\) and the references therein. The list is by no means exhaustive.

2. The problem. In this section we introduce precisely the mathematical problem that we want to study, as well as the conditions on the geometry and given data needed for our study.

2.1. The geometry. We start by a bounded smooth domain \( \Omega \subset \mathbb{R}^2 \). Then we construct a periodic cell

\[
Y^* = Y \setminus \overline{A}.
\]

Here, \( Y = [0, 1]^2 \) is a unit square and \( A \subset \subset Y \) is its smooth (\( \partial A \) is of class \( C^1 \)) subset.

![Diagram of a periodic cell](image)

We take some small parameter \( \varepsilon \ll 1 \) (the average pore size) and define

\[
\Omega_\varepsilon = \Omega \cap \left( \sum_{k \in \mathbb{Z}^2} \varepsilon (k + Y^*) \right).
\]

![Diagram of \( \Omega_\varepsilon \)](image)

We denote by \( \partial \Omega \) the exterior boundary of the domain.
2.2. The equations. By $u_\delta^\varepsilon$ we denote the unique solution to the Dirichlet problem for the scalar Brinkman equation

$$\begin{cases}
-\Delta u_\delta^\varepsilon + \frac{1}{\delta} \chi^\varepsilon u_\delta^\varepsilon = f & \text{in } \Omega \\
u_\delta^\varepsilon = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\chi^\varepsilon(x) = \chi\left(\frac{x}{\varepsilon}\right)$ is the characteristic function of the set $A_\varepsilon = \Omega \setminus \Omega_\varepsilon$ and $\chi(y) = \begin{cases} 
1, & \text{in } A \\
0, & \text{in } Y \setminus A
\end{cases}$
is the characteristic function of the set $A$, extended by periodicity.

The source term $f$ is assumed to be an $L^2(\Omega)$ function. Additional assumptions on $f$ will be added later, when needed.

The physical interpretation is the unidirectional flow of the fluid through porous medium with double porosity. We have the free flow in $\Omega_\varepsilon$ described by the Laplace equation, and in the porous blocks $A_\varepsilon$ the flow is described by the Brinkman equation with the small permeability $\delta$.

We want to study the limits as both parameters $\varepsilon$ and $\delta$ tend to zero and to see whether the two limits commute.

3. The case $\lim_{\varepsilon \to 0} \lim_{\delta \to 0} u_\delta^\varepsilon$. We first consider the limit as $\delta \to 0$. That is a classical singular perturbation of the boundary value problem. We start with a priori estimates:

**Lemma 3.1.** Let $u_\delta^\varepsilon$ be the solution to the Dirichlet problem (2.2). Then there exists a constant $C > 0$ such that, for all $\varepsilon > 0$ and $\delta > 0$ the following estimates hold

$$|u_\delta^\varepsilon|_{H^1(\Omega)} \leq C \quad \text{and} \quad |u_\delta^\varepsilon|_{L^2(A_\varepsilon)} \leq C\sqrt{\delta}.$$  \hspace{1cm} (3.1)

The proof is straightforward.

Consequently, we have:

**Lemma 3.2.** Let $u_\delta^\varepsilon$ be the solution to the problem (2.2). Then there exists a function $u_\varepsilon \in H^1(\Omega)$ such that

$$u_\delta^\varepsilon \rightharpoonup u_\varepsilon \quad \text{weakly in } H^1(\Omega)$$
as $\delta \to 0$. Furthermore the limit function is the unique solution to the Dirichlet problem

$$\begin{cases}
-\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\
u_\varepsilon = 0 & \text{on } \partial \Omega 
\end{cases}$$

and

$$u_\varepsilon = 0 \quad \text{on } A_\varepsilon.$$  \hspace{1cm} (3.2)

Now (3.2) is a well known Dirichlet problem for the Laplace equation in periodic perforated domain and (see e.g. [23] or [5]) and the homogenized limit is the Darcy law

$$\varepsilon^{-2} u_\varepsilon \rightharpoonup u = K f \quad \text{weakly in } L^2(\Omega),$$

where $K = \int_Y w(y)dy$ and $Y^* = Y \setminus \overline{A}$ and $w$ is the solution to the auxiliary problem

$$-\Delta w = 1 \quad \text{in } Y^*.$$
and \( w \) is \( Y \)-periodic.

4. **Asymptotic expansion in case** \( \delta \ll \varepsilon \). To get better approximation, we can derive an asymptotic expansion for the problem in powers of both parameters \( \varepsilon \) and \( \delta \). To do so, we need more regularity of \( f \). We assume that \( f \in C^\infty_c(\Omega) \).

We start with an expansion in powers of \( \delta \):

\[
u^0_\delta(x) = u^0_\delta(x) + \delta u^1_\delta(x) + \delta^2 u^2_\delta(x) + \cdots.
\]

Inserting into equation (2.2) and collecting equal powers of \( \delta \) leads to the recursive system of equations (\( y = x/\varepsilon \))

\[
\delta^{-1} : \chi(y)u^0_\delta = 0 \quad \Rightarrow \quad u^0_\delta = 0 \quad \text{in} \quad A
\]

\[
\delta^0 : -\Delta u^0_\delta + \chi(y)u^1_\delta = f(x)
\]

\[
\Rightarrow \quad -\Delta u^0_\delta = f(x) \quad \text{in} \quad \Omega, \quad u^1_\delta = f \quad \text{in} \quad A
\]

\[
\delta^1 : -\Delta u^1_\delta + \chi(y)u^2_\delta = 0
\]

\[
\Rightarrow \quad -\Delta u^1_\delta = 0 \quad \text{in} \quad \Omega, \quad u^2_\delta = \Delta f \quad \text{in} \quad A
\]

\[
\delta^2 : -\Delta u^2_\delta + \chi(y)u^3_\delta = 0
\]

\[
\Rightarrow \quad -\Delta u^2_\delta = 0 \quad \text{in} \quad \Omega, \quad u^3_\delta = \Delta^2 f \quad \text{in} \quad A \ldots.
\]

Now we look for an expansion for \( u^0_\delta \) in powers of \( \varepsilon \). It is well known (see e.g.\([5]\)) that it has an expansion of the form

\[
u^0_\varepsilon(x) = \varepsilon^2 u^2_\varepsilon(x,y) + \varepsilon^3 u^3_\varepsilon(x,y) + \cdots, \quad y = x/\varepsilon.
\]

Inserting into equation (3.2) and collecting equal powers of \( \varepsilon \) and denoting

\[
\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}
\]

\[
\Delta_y = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}
\]

\[
\Delta_{xy} = \frac{\partial^2}{\partial y_1 \partial x_1} + \frac{\partial^2}{\partial y_2 \partial x_2},
\]

leads to the recursive system of equations

\[
\varepsilon^0 : \begin{cases}
-\Delta_y u^0_\varepsilon = f \\
u^0_\varepsilon = 0 \quad \text{on} \quad \partial \Omega
\end{cases} \quad \Rightarrow \quad u^0_\varepsilon(x,y) = w(y)f(x)
\]

the boundary conditions are satisfied since \( f(x) = 0 \) on \( \partial \Omega \) and \( w(y) = 0 \) on \( \partial A \)

\[
\varepsilon : \begin{cases}
-\Delta_y u^0_\varepsilon(x,y) - 2\Delta_{xy} u^0_\varepsilon(x,y) = 0 \\
u^0_\varepsilon = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

\[
\varepsilon^2 : \begin{cases}
-\Delta_y u^0_\varepsilon(x,y) - 2\Delta_{xy} u^0_\varepsilon(x,y) - \Delta_x u^0_\varepsilon(x,y) = 0 \\
u^0_\varepsilon = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

\[
\varepsilon^2 : \begin{cases}
-\Delta_y u^0_\varepsilon(x,y) - 2\Delta_{xy} u^0_\varepsilon(x,y) - \Delta_x u^0_\varepsilon(x,y) = 0 \\
u^0_\varepsilon = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

\[
\varepsilon^2 : \begin{cases}
-\Delta_y u^0_\varepsilon(x,y) - 2\Delta_{xy} u^0_\varepsilon(x,y) - \Delta_x u^0_\varepsilon(x,y) = 0 \\
u^0_\varepsilon = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

Since

\[
\Delta_{xy} u^0_\varepsilon(x,y) = \nabla_y w(y) \cdot \nabla_x f(x) = \sum_{k=1}^2 \frac{\partial w}{\partial y_k}(y) \frac{\partial f}{\partial x_k}(x)
\]
for the second term we have an ansatz in the form

\[ u_0^0(x, y) = \sum_{k=1}^{2} z_k(y) \frac{\partial f}{\partial x_k}(x) \]

with

\[-\Delta z_k = 2 \frac{\partial w}{\partial y_k} \quad \text{in} \ Y^* \]
\[ z_k = 0 \quad \text{on} \ \partial A \] (4.8)

and \( z_k \) is \( Y \)-periodic.

The boundary conditions are satisfied like in the case of \( u_0^2 \). Again, since we have chosen \( f \) to be compactly supported \( \frac{\partial f}{\partial x_k}(x) = 0 \) on \( \partial \Omega \) and \( z_k \) equals zero on the boundaries of the perforations \( \partial A_\epsilon \).

Thus

\[ u_0^0(x) = \varepsilon^2 w(y) f(x) + \varepsilon^3 \sum_{k=1}^{2} z_k(y) \frac{\partial f}{\partial x_k}(x) + O(\varepsilon^4) \] . (4.9)

Next we pass to \( u_1^\varepsilon \). The problem reads

\[-\Delta u_1^\varepsilon = 0 \quad \text{in} \ \Omega_\varepsilon \ , \ u_1^\varepsilon = f \quad \text{on} \ \partial A_\varepsilon \ . \] (4.10)

We look for \( u_1^\varepsilon \) in the form

\[ u_1^\varepsilon(x) = f(x) + \varepsilon^2 w(y) \Delta f(x) - \varepsilon^3 \sum_{k=1}^{2} z_k(y) \frac{\partial \Delta f}{\partial x_k}(x) + O(\varepsilon^4) \]

where \( w \) is the solution to the auxiliary problem (3.3) and \( z_k \) of (4.8).

Finally we compute \( u_2^\varepsilon \). The problem reads

\[-\Delta u_2^\varepsilon = 0 \quad \text{in} \ \Omega_\varepsilon \ , \ u_2^\varepsilon = \Delta f \quad \text{on} \ \partial A_\varepsilon \ . \] (4.11)

We look for \( u_2^\varepsilon \) in the form

\[ u_2^\varepsilon(x) = \Delta f(x) - \varepsilon^2 w(y) \Delta^2 f(x) - \varepsilon^3 \sum_{k=1}^{2} z_k(y) \frac{\partial \Delta^2 f}{\partial x_k}(x) + O(\varepsilon^4) \]

Again \( w \) and \( z_k \) are the solution to the auxiliary problems (3.3) and (4.8), respectively.

Finally we get an approximation

\[ u(\delta, \varepsilon)(x) = \delta^2 f(x) + \delta^2 \Delta_x f(x) + \varepsilon^2 w(y) f(x) + \varepsilon^2 \delta w(y) \Delta_x f(x) \] (4.12)

and we hope that

\[ u_0^\delta = u(\delta, \varepsilon) + O(\varepsilon^3 + \varepsilon^2 \delta^2 + \delta^3) \]

However this formal expansion would be difficult to justify since the approximation is not an \( H^2(\Omega) \) function. Indeed on the boundary of perforations the approximation is continuous but it’s normal derivative is not. There is a jump of the normal derivative on \( \partial A_\varepsilon \) of the form

\[ \left[ \frac{\partial u(\delta, \varepsilon)}{\partial n} \right] = \varepsilon \left[ \frac{\partial w}{\partial n} \right] f + O(\varepsilon^2) \] .

Thus, the best we can get is an \( H^1(\Omega) \) estimate of order \( \varepsilon \) that hardly justifies our formal approximation. Computing further order terms in the expansion, obviously would not help since the problem is in the nature of our formal approximation and the micro boundary layer that appears on the boundary of each perforation. If
the perforations were balls, those boundary layers could be corrected by using the matching procedure, but that is out of the scope of this paper.

5. **The case** \( \lim_{\delta \to 0} \lim_{\varepsilon \to 0} u_{\varepsilon}^\delta \). The second case is more complicated. Indeed, the estimate (3.1) still holds, so that we can extract a subsequence and pass to the limit. It is an easy exercise to prove that

\[
\begin{aligned}
u_{\varepsilon}^\delta &\to u^\delta \quad \text{weakly in } H^1(\Omega) \\
a \rightarrow 0,
\end{aligned}
\]

as \( \varepsilon \to 0 \), where

\[
\begin{aligned}
-\Delta u + \frac{1}{\delta} |A| \ u^\delta &= f \quad \text{in } \Omega \\
\ u^\delta &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

And now, clearly

\[
\begin{aligned}
u_{\delta}^\varepsilon &\to 0 \quad \text{in } L^2(\Omega),
\end{aligned}
\]

as \( \delta \to 0 \). That is not the desired result. However, it is hardly surprising since we expect to get the non-trivial limit of \( \varepsilon^{-2} u_{\varepsilon}^\delta \) and not \( u_{\delta}^\varepsilon \). Therefore, in the first step, we need to derive an asymptotic expansion in powers of \( \varepsilon \) and not just the limit as \( \varepsilon \to 0 \).

6. **Asymptotic expansion in case** \( \varepsilon \ll \delta \). In this section we need \( f \) to be smooth function. For simplicity, to avoid the boundary layer on \( \partial \Omega \), we also choose \( f \) as a compactly supported function from \( C^\infty_c(\Omega) \). We hope that the first two terms in such expansion will have the trivial limits as \( \delta \to 0 \) and that the term with power \( \varepsilon^2 \) will have some more interesting limit as \( \delta \to 0 \). Hopefully the same limit as in the previous section, i.e. \( u(x) = K \ f(x) \). Thus we look for an approximation of the form

\[
u_{\delta}^\varepsilon(x) = u_{0}^\delta(x,y) + \varepsilon u_{1}^\delta(x,y) + \varepsilon^2 u_{2}^\delta(x,y) + \cdots, \quad y = x/\varepsilon.
\]

Inserting into equation (2.2) and collecting equal powers of \( \varepsilon \) leads to the recursive system of equations

\[
\begin{aligned}
\frac{1}{\varepsilon^2} : &- \Delta_y u_{0}^\delta = 0 \quad \Rightarrow \quad u_{0}^\delta(x,y) = u_{0}^\delta(x) \\
\frac{1}{\varepsilon} : &- \Delta_y u_{1}^\delta = 0 \quad \Rightarrow \quad u_{1}^\delta(x,y) = u_{1}^\delta(x) \\
\varepsilon^0 : &- \Delta_y u_{2}^\delta(x,y) - \Delta_x u_{0}^\delta(x) + \frac{1}{\delta} \chi(y) \ u_{0}^\delta(x) = f(x) \\
\varepsilon : &- \Delta_y u_{3}^\delta(x,y) - \Delta_x u_{1}^\delta(x) - 2 \Delta_{xy} u_{2}^\delta(x,y) + \frac{1}{\delta} \chi(y) \ u_{1}^\delta(x) = 0 \\
\varepsilon^2 : &- \Delta_y u_{4}^\delta(x,y) - \Delta_x u_{2}^\delta(x,y) - 2 \Delta_{xy} u_{3}^\delta(x,y) + \frac{\chi(y)}{\delta} \ u_{2}^\delta(x,y) + \frac{1}{\delta} \chi(y) \ u_{1}^\delta(x) = 0 \\
\varepsilon^m : &- \Delta_y u_{m+2}^\delta - \Delta_x u_{m}^\delta - 2 \Delta_{xy} u_{m+1}^\delta + \frac{1}{\delta} \chi(y) u_{m}^\delta = 0.
\end{aligned}
\]

Integrating (6.1) with respect to \( y \) over \( Y \), and using periodicity, gives

\[
- \Delta_x u_{0}^\delta(x) + \frac{1}{\delta} \ |A| \ u_{0}^\delta(x) = f(x)
\]

and then

\[
- \Delta_y u_{2}^\delta(x,y) + \frac{1}{\delta} \ (\chi(y) \ - |A|) \ u_{0}^\delta(x) = 0
\]
Now, the equation (6.4) gives
\[ u_0^\delta(x) = \delta |A|^{-1} f(x) + \delta^2 |A|^{-2} \Delta_x f(x) + \delta^3 |A|^{-3} \Delta_x^2 f(x) + \cdots = \delta |A|^{-1} V_0^\delta(x) \]
with
\[ V_0^\delta(x) = \sum_{k=0}^{\infty} (\delta |A|^{-1} \Delta_x)^k f(x). \]

Plugging it in (6.5) gives
\[ -\Delta_y u_2^\delta(x, y) + (\chi(y) - |A|) |A|^{-1} V_0^\delta(x) = 0. \]

Thus
\[ u_2^\delta(x, y) = w_A(y) V_0^\delta(x) + v_2(x), \quad (6.6) \]
with
\[ -\Delta_y w_A(y) = 1 - |A|^{-1} \chi(y) \quad \text{in} \quad Y, \quad w_A \quad \text{is} \quad Y\text{-periodic}. \quad (6.7) \]

The function \( v_2(x) \) is to be determined later in a way to keep (6.3) solvable. We integrate (6.3) with respect to \( y \) and get
\[ -\Delta x v_2^\delta(x) + \frac{|A|}{\delta} v_2^\delta(x) = \langle w_A \rangle \Delta_x V_0^\delta(x) - \frac{1}{\delta} \langle \chi w_A \rangle V_0^\delta(x) \]
where \( \langle \cdot \rangle = \int_Y \cdot \, dy \) denotes the mean value with respect to \( y \). We look for \( v_2^\delta \) in the form of an asymptotic expansion
\[ v_2^\delta(x) = \sum_{k=0}^{\infty} \delta^k v_2^k(x) \]
so that
\[ v_2^0(x) = -|A|^{-1} \langle \chi w_A \rangle f(x) \quad (6.8) \]
\[ v_2^1(x) = |A|^{-1} \{ \langle w_A \rangle - 2 |A|^{-1} \langle \chi w_A \rangle \} \Delta_x f(x) \quad (6.9) \]
\[ \ldots \]
\[ v_2^k(x) = |A|^{-k} \{ k \langle w_A \rangle - (k+1) |A|^{-1} \langle \chi w_A \rangle \} \Delta_x^k f(x) \quad (6.10) \]
and
\[ u_2^\delta(x, y) = \{ w_A(y) - |A|^{-1} \langle \chi w_A \rangle \} f(x) + \]
\[ + \delta |A|^{-1} \{ w_A(y) + \langle w_A \rangle - 2 |A|^{-1} \langle \chi w_A \rangle \} \Delta_x f(x) + \]
\[ + \cdots + \delta^k |A|^{-k} \{ w_A(y) + k \langle w_A \rangle - (k+1) |A|^{-1} \langle \chi w_A \rangle \} \Delta_x^k f(x) + \cdots. \quad (6.12) \]

Multiplying (6.7) with \( w_A \) and integrating over \( Y \), we get
\[ \langle w_A \rangle - |A|^{-1} \langle \chi w_A \rangle = \int_Y |\nabla w_A|^2 \equiv \kappa_A. \]
Thus
\[ u_2^\delta(x) = \left\{ j \lambda_A - |A|^{-1} \langle \chi w_A \rangle \right\} \Delta_x^j f(x). \]
Finally by (6.6)
\[ u_2^\delta(x, y) = \sum_{j=0}^{\infty} \delta^j |A|^{-j} \left\{ w_A(y) - j \lambda_A - |A|^{-1} \langle \chi w_A \rangle \right\} \Delta_x^j f(x). \]
Integrating (6.2) with respect to $y$ gives
$$-\Delta_x u_1^\delta(x) + \frac{|A|}{\delta} u_1^\delta(x) = 0,$$
implying that $u_1^\delta = 0$.

For the next term $u_3^\delta$ we obviously (see 6.3) have
$$u_3^\delta(x, y) = \sum_{k=1}^{2} z_A^k(y) \frac{\partial V_0^\delta}{\partial x_k}(x) + v_3^\delta(x),$$
(6.13)
where
$$-\Delta_y z_A^k + 2 \frac{\partial w_A}{\partial y_k} = 0, \quad w_A \text{ is } Y\text{-periodic.}$$
(6.14)

For this auxiliary problem we can easily prove that

**Lemma 6.1.** Let $z_A^k$ be the solution to the auxiliary problem (6.14). Then
$$|A| \langle z_A^k \rangle = \langle \chi z_A^k \rangle.$$
(6.15)

**Proof.** We multiply (6.14) by $w_A$ and (6.7) by $z_A^k$ and integrate over $Y$. We get
$$\int_Y \nabla z_A^k \nabla w_A = -2 \int_Y \frac{\partial w_A}{\partial y_k} w_A = - \int_Y \frac{\partial w_A^2}{\partial y_k} = 0,$$
$$\int_Y \nabla w_A \nabla z_A^k = \int_Y (1 - |A|^{-1} \chi) z_A^k dy = (z_A^k) - |A|^{-1} \langle \chi z_A^k \rangle.$$

To compute $v_3^\delta$ we take the equation for $u_3^\delta$
$$-\Delta_y u_3^\delta - \Delta_x u_3^\delta - 2\Delta_{xy} u_3^\delta + \frac{1}{\delta} \chi(y) u_3^\delta = 0$$
and integrate with respect to $y$ over $Y$. It gives
$$-\Delta_x \langle u_3^\delta \rangle + \frac{1}{\delta} \langle \chi(y) u_3^\delta \rangle = 0.$$

Due to (6.13), we have
$$\langle u_3^\delta \rangle = \sum_{k=1}^{2} \langle z_A^k \rangle \frac{\partial V_0^\delta}{\partial x_k}(x) + v_3^\delta(x),$$
so that
$$-\Delta_x v_3^\delta + \frac{|A|}{\delta} v_3^\delta = \sum_{k=1}^{2} \left( \langle z_A^k \rangle \frac{\partial \Delta_x V_0^\delta}{\partial x_k} - \frac{1}{\delta} \langle \chi z_A^k \rangle \frac{\partial V_0^\delta}{\partial x_k} \right).$$
The definition of $V_1^\delta$ and lemma 6.15 imply
$$\langle z_A^k \rangle \frac{\partial \Delta_x V_0^\delta}{\partial x_k} - \frac{1}{\delta} \langle \chi z_A^k \rangle \frac{\partial V_0^\delta}{\partial x_k} = - |A| \frac{\delta}{\delta} \langle z_A^k \rangle \frac{\partial f}{\partial x_k},$$
so that
$$-\Delta_x v_3^\delta + \frac{|A|}{\delta} v_3^\delta = - |A| \frac{\delta}{\delta} \sum_{k=1}^{2} \langle z_A^k \rangle \frac{\partial f}{\partial x_k}.$$
(6.16)

Comparing with (6.4) we easily conclude that
$$v_3^\delta = - \frac{|A|}{\delta} \sum_{k=1}^{n} \langle z_A^k \rangle \frac{\partial u_0^\delta}{\partial x_k} = \sum_{k=1}^{2} \langle z_A^k \rangle \frac{\partial V_0^\delta}{\partial x_k} = - \sum_{j=0}^{\infty} (|A|^{-1} \Delta_x)^j \left( \sum_{k=1}^{2} \langle z_A^k \rangle \frac{\partial f}{\partial x_k} \right).$$
We could continue this procedure up to any order of accuracy with respect to $\varepsilon$ and compute $u_m^{\varepsilon}$ for any $m \in \mathbb{N}$.

6.1. Error estimate. For the asymptotic expansion derived in the previous section we have the following error estimate:

**Theorem 6.2.** For any $m \in \mathbb{N}$, $m > 2$, there exists a constant $C > 0$, independent on $\varepsilon$ and $\delta$, such that for any $k \in \mathbb{N}_0$ and

\[
U_m(\varepsilon, \delta) = u_0^\varepsilon + \varepsilon u_1^\varepsilon + \varepsilon^2 u_2^\varepsilon + \cdots + \varepsilon^m u_m^\varepsilon
\]

\[
|u_j^\varepsilon - U_m(\varepsilon, \delta)|_{H^1(\Omega)} \leq C(\varepsilon^{m+k} \delta^{-1/2} + \varepsilon^m) \tag{6.17}
\]

\[
|u_j^\varepsilon - U_m(\varepsilon, \delta)|_{L^2(\Omega)} \leq C(\varepsilon^{m+k} \delta^{-1/2} + \varepsilon^{m+1}) \tag{6.18}
\]

\[
|u_j^\varepsilon - U_m(\varepsilon, \delta)|_{L^2(A_\varepsilon)} \leq C\varepsilon^{m+1} \tag{6.19}
\]

**Proof.** By direct computation, for $m \in \mathbb{N}$, the function $R = u_j^\varepsilon - U_{m+j}(\varepsilon, \delta)$ satisfies

\[
-\Delta R + \frac{1}{\varepsilon} \chi(y) R = -\varepsilon^{m+j-1}(-\Delta x u_{m+j-1}^\varepsilon - 2\Delta x y u_{m+j}^\varepsilon + \frac{1}{\varepsilon} \chi(y) u_{m+j-1}^\varepsilon) - \varepsilon^{m+j}(-\Delta x u_{m+j}^\varepsilon + \frac{1}{\varepsilon} \chi(y) u_{m+j}^\varepsilon) = -\varepsilon^{m+j-1} \delta^{-1} \chi(y)(u_{m+j-1}^\varepsilon + \varepsilon u_{m+j}^\varepsilon) + \varepsilon^{m+j-1}(\Delta x u_{m+j}^\varepsilon + \varepsilon \Delta x u_{m+j}^\varepsilon) = \varepsilon^{m+j-1}(I + \delta^{-1} \chi(y) J)
\]

where $y = x/\varepsilon$ and $|I|_{L^\infty(\Omega)} \leq C$, $|J|_{L^\infty(A_\varepsilon)} \leq C$. Using $R$ as the test function in the above equation and the Poincaré inequality, we arrive at

\[
\int_\Omega |\nabla R|^2 + \frac{1}{\varepsilon} \int_{A_\varepsilon} R^2 = \varepsilon^{m+j-1} \left( \int_\Omega I R + \frac{1}{\varepsilon} \int_{A_\varepsilon} J R \right) \leq C \frac{\varepsilon^{2(m+j-1)}}{\delta} + \frac{1}{2} \int_\Omega |\nabla R|^2 + \frac{1}{2\delta} \int_{A_\varepsilon} R^2
\]

giving the estimate

\[
|R|_{H^1(\Omega)} \leq C \varepsilon^{m+j-1} \delta^{-1/2}, \quad |R|_{L^2(A_\varepsilon)} \leq C \varepsilon^{m+j-1}.
\]

For

\[
E \equiv U_{m+j}(\varepsilon, \delta) - U_m(\varepsilon, \delta) = \varepsilon^{m+1} u_{m+1}^\varepsilon + \cdots + \varepsilon^{m+j} u_{m+j}^\varepsilon
\]

we obviously have

\[
|E|_{H^1(\Omega)} \leq C \varepsilon^m, \quad |E|_{L^2(A_\varepsilon)} \leq C \varepsilon^{m+1},
\]

proving the theorem (with $k = j - 1$).

Although $k$ is arbitrary, that estimate does not allow to pass to the limit as $\delta \to 0$, before we pass to the limit as $\varepsilon \to 0$.

7. Comparing the limits. We have concluded before that

\[
\lim_{\varepsilon \to 0} \left( \lim_{\delta \to 0} \frac{u_j^\varepsilon}{\varepsilon^2} \right) = K f.
\]

On the other hand, since we have derived an approximation of the form

\[
u_j^\varepsilon(x) = \delta |A|^{-1} f(x) + \delta^2 |A|^{-2} \Delta_x f(x) + \delta^3 |A|^{-3} \Delta_x^2 f(x) + \cdots + \varepsilon^2 \left\{ w_A(y) - |A|^{-1} (\chi w_A) \right\} f(x) +
\]

\[
+ \delta^2 \left\{ \left[ w_A(y) - |A|^{-1} (\chi w_A) \right] f(x) + \right\}
\]

...
$$+ \delta |A|^{-1} \{ w_A(y) + \kappa_A + |A|^{-1} \langle \chi w_A \rangle \} \Delta_x f(x) + \cdots \} + \cdots .$$

we have

$$\lim_{\delta \to 0} \left( \lim_{\varepsilon \to 0} \frac{u_0^\delta - u_0^\varepsilon}{\varepsilon^2} \right) = \kappa_A f,$$

and

$$\lim_{\varepsilon \to 0} u_0^\varepsilon = 0 .$$

So, in both cases the limit (in some sense) leads to the Darcy law, but in the first case the permeability is given by

$$K = \int_Y |\nabla_y w|^2$$

with \(w\) given by the auxiliary problem (3.3)

$$-\Delta_y w = 1 \quad \text{in} \quad Y^* , \quad w = 0 \quad \text{on} \quad \partial A \quad \text{and} \quad w \quad \text{is} \quad Y - \text{periodic} .$$

extended by zero to \(A\). In the second case the permeability \(\kappa_A\) is also given by

$$\kappa_A = \int_Y |\nabla_y w_A|^2$$

but with different auxiliary function, denoted \(w_A\) and defined by (6.7)

$$-\Delta_y w_A(y) = 1 - |A|^{-1} \chi(y) \quad \text{in} \quad Y , \quad w_A \quad \text{is} \quad Y - \text{periodic}.$$ 

Although both auxiliary functions satisfy the same equation in \(Y^* :\)

$$-\Delta_y w = 1 \quad \text{in} \quad Y^* \quad \text{and} \quad -\Delta_y w_A = 1 \quad \text{in} \quad Y^* ,$$

they do not satisfy the same equation in \(A:\)

$$-\Delta_y w = 0 \quad \text{in} \quad A \quad \text{and} \quad -\Delta_y w_A = 1 - |A|^{-1} \quad \text{in} \quad A.$$ 

Also the normal derivative of \(w_A\) is continuous across \(\partial A\), while that of \(w\) is not. That proves \(w \neq w_A\), but it does not prove that \(K \neq \kappa_A\). Testing (6.7) with \(w\) gives

$$- \int_Y \Delta_y w_A w = [w = 0 \quad \text{on} \quad A] = - \int_{Y^*} \Delta_y w_A w = \int_{Y^*} \nabla_y w_A \nabla_y w
= \int_Y (1 - |A|^{-1} \chi) w = \int_{Y^*} w = K$$

implying

$$K = \int_Y |\nabla_y w|^2 = \int_Y \nabla_y w \cdot \nabla y w_A .$$

On the other hand, testing (3.3) with \(w_A\) gives \(w\) denotes the interior, relative to \(A\), unit normal on \(\partial A\))

$$- \int_{Y^*} \Delta_y w w_A = \int_{Y^*} \nabla_y w_A \nabla_y w - \int_{\partial A} \frac{\partial w}{\partial n} w_A = \int_{Y^*} w_A = \int_Y (1 - \chi) w_A = \int_Y w
= \int_Y (1 - |A|^{-1} \chi) w_A + \left( \frac{1}{|A|} - 1 \right) \int_Y \chi w_A = \kappa_A + \left( \frac{1}{|A|} - 1 \right) \int_Y \chi w_A$$

so that

$$\int_Y \nabla w \cdot \nabla w_A = \kappa_A + \int_{\partial A} \frac{\partial w}{\partial n} w_A + \left( \frac{1}{|A|} - 1 \right) \int_Y \chi w_A .$$
On the other hand
\[
\left(1 - \frac{1}{|A|}\right) \int_A w_A = - \int_A \Delta_y w_A w_A = \int_A |\nabla w_A|^2 + \int_{\partial A} \frac{\partial w_A}{\partial n} w_A
\]
so that
\[
K - \kappa_A = \int_{\partial A} \frac{\partial w}{\partial n} w_A + \left(\frac{1}{|A|} - 1\right) \int_Y \chi w_A = \int_{\partial A} \left(\frac{\partial w}{\partial n} - \frac{\partial w_A}{\partial n}\right) w_A - \int_A |\nabla w_A|^2.
\]
Still, it does not prove that \(\kappa_A\) and \(K\) are different as the expression on the left-hand side could be zero. To prove that the two permeabilities are, in general, different, we address one case when they can be computed using the asymptotic analysis. We consider the case when the inclusion \(A\) is a rectangle and the part of the periodicity cell where the fluid flows freely, \(Y^*\), consists of two narrow channels of width \(\sigma \ll 1\). Due to the periodicity, we can translate the cell \(Y\) and get
\[
Y_\sigma^* = \left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right]\right) \cup \left(\left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right)
\]
It can be proved (see [19] or [20]) that \(w\), the solution of the problem (3.3) satisfies
\[
w(y_1, y_2) = \begin{cases} 
\sigma^2 \frac{2}{8} - \frac{y_2^2}{2} & \text{in } Y_1 \\
\sigma^2 \frac{2}{8} - \frac{y_2^2}{2} & \text{in } Y_2 \setminus Y_1 
\end{cases} + O(\sigma^3)
\]
where
\[
Y_1 = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right], \quad Y_2 = \left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] 
\]
So that
\[
K = \int_{Y_\sigma^*} w(y)dy = \frac{\sigma^3}{6} + O(\sigma^4) .
\]
On the other hand the equation (6.7) for \(w_A\) reads
\[-\Delta_y w_A(y) = 1 - |A|^{-1} \chi(y) .
\]
It is easy to see that
\[
|A| = (1 - \sigma)^2.
\]
Thus, the right hand side \(F_\sigma = 1 - |A|^{-1} \chi(y)\) satisfies (as \(\sigma \to 0\))
\[
\sigma^{-1} \int_Y F_\sigma(y) \psi(y)dy = \frac{\sigma - 2}{(1 - \sigma)^2} \int_Y \psi(y)dy + \frac{1}{\sigma(1 - \sigma)^2} \int_{Y \setminus A_\sigma} \psi(y)dy
\]
\[ \rightarrow -2 \int_Y \psi(y) dy + \int_{-1/2}^{1/2} \psi(0, y_2) dy_2 + \int_{-1/2}^{1/2} \psi(y_1, 0) dy_1, \]

for any \( Y \)-periodic function from \( H^1(Y) \). Therefore

\[ \sigma^{-1} w_A \rightharpoonup w_0 \quad \text{weakly in} \quad H^1(Y), \]

where \( w_0 \) is the solution to the problem

\[ -\Delta_y w_0 (y) = -2 + \delta_{\{y_1=0\}} + \delta_{\{y_2=0\}}, \]

with periodic boundary condition, where

\[ \langle \delta_{\{y_1=0\}} | \psi \rangle = \int_{-1/2}^{1/2} \psi(0, y_2) dy_2, \quad \langle \delta_{\{y_2=0\}} | \psi \rangle = \int_{-1/2}^{1/2} \psi(y_1, 0) dy_1, \]

are the Dirac measures concentrated on \( \{y_1=0\} \) and \( \{y_2=0\} \), respectively. Since

\[ \kappa_A = \int_Y w_A (y) dy - |A|^{-1} \int_A w_A (y) dy = (1 - |A\sigma|^{-1}) \int_{A\sigma} w_A (y) dy + \int_{Y \setminus A\sigma} w_A (y) dy \]

we have

\[ \sigma^{-2} \kappa_A \rightarrow -2 \int_Y w_0 (y) dy + \int_{-1/2}^{1/2} w_0 (0, y_2) dy_2 + \int_{-1/2}^{1/2} w_0 (y_1, 0) dy_1 = \int_Y |\nabla w_0 (y)|^2 dy > 0, \]

so that

\[ \kappa_A = \sigma^2 \int_Y |\nabla w_0 (y)|^2 dy + o(\sigma^2). \]

So, for this particular geometry, we have proved that

\[ K \neq \kappa_A, \]

meaning that, in general the two permeabilities are different.

8. Simultaneous limit \( \delta = \delta(\varepsilon) \). The asymptotic expansion derived in Section 6 applies here and it is justified by error estimates (6.17)-(6.19), since \( k \) was arbitrary. The limit depends on the function \( \delta \). We can distinguish two cases:

- If

\[ \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\delta(\varepsilon)} = 0, \]

then

\[ \lim_{\varepsilon \to 0} \frac{u_{\varepsilon}^\delta}{\varepsilon^2} = |A|^{-1} f, \]

- The most interesting case for us is when

\[ \lim_{\varepsilon \to 0} \frac{\delta(\varepsilon)}{\varepsilon^2} = \alpha. \]

In that case we obviously have

\[ \lim_{\varepsilon \to 0} \frac{u_{\varepsilon}^\delta}{\varepsilon^2} = M f, \]

where

\[ M = \kappa_A + \alpha |A|^{-1}. \]

Of course, \( \alpha \) can be equal to zero, which gives the same effective law, with permeability \( \kappa_A \), as in section 6.
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