Categoricity for Patterns of Order 2

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In this paper we show how a Categoricity Theorem for patterns of resemblance of order 2, in analogy to Theorem 9.1 of [1] for $\mathcal{R}_1$, follows from [2]. This is the result alluded to in the last paragraph of the introduction to [2] where it is stated

... a method of generating the core is established which shows that the order in which patterns of embeddings of this level occur is the same for reasonable hierarchies.

As a consequence, if a reasonable hierarchy $\mathcal{B}$ (see the Categoricity Theorem below) has arbitrary long finite chains in the interpretation of $\preceq_2$ then a finite structure is a pattern of resemblance of order two iff it is isomorphic to a finite substructure of $\mathcal{B}$ (see Corollary 0.8). These results apply to the version of $\mathcal{R}_2$ defined in the introduction to [2] as initial segments are reasonable hierarchies.

Our basic reference is [2].

We will work in the theory $\text{KP}\omega$ i.e. Kripke-Platek Set Theory plus the Axiom of Infinity.

Fix a language $\mathcal{L}$ including the binary relation symbol $\preceq$. Let $\mathcal{L}_2$ be the expansion of $\mathcal{L}$ by binary relation symbols $\preceq_1$ and $\preceq_2$. We also write $\preceq_0$ for $\preceq$. We use structure to refer to what is more commonly called a partial structure where the interpretations of the function symbols are allowed to be partial. We will write $|\mathcal{P}|$ for the universe of a structure $\mathcal{P}$.

For the remainder of the paper, let $\mathcal{R}$ be an EM structure (see Section 3 of [2]) for $\mathcal{L}$ on the class of ordinals with $\preceq^\mathcal{R}$ the usual ordering. We assume the restriction of $\mathcal{R}$ to any ordinal is a set, there is a restriction with $\omega$ indecomposables and the indecomposables are cofinal in the ordinals.
Since \( \mathcal{R} \) is an EM structure, it can be recovered by its restriction to the \( \omega \)-th indecomposable which implies the set of indecomposables is \( \Delta \)-definable and the function which maps an indecomposable \( \lambda \) to \( \mathcal{R} \upharpoonright \lambda \) is \( \Sigma \)-definable.

We also assume \( \mathcal{B} \) is a structure for \( \mathcal{L}_2 \) whose arithmetic part (i.e. restriction to \( \mathcal{L} \)) is an arithmetic structure with respect to \( \mathcal{R} \) (Definition 4.1 of [2]) in which the interpretation of each function symbol is total. We do not require that \( \mathcal{B} \) be well-ordered with respect to the interpretation of \( \preceq \) though our main focus will be on those \( \mathcal{B} \) which are. Recall that \( \preceq^\mathcal{B}_k \) respects \( \preceq^\mathcal{B}_{k-1} \) if
\[
\alpha \preceq^\mathcal{B}_{k-1} \beta \preceq^\mathcal{B}_{k-1} \gamma \quad \text{and} \quad \alpha \preceq^\mathcal{B}_k \gamma \quad \implies \quad \alpha \preceq^\mathcal{B}_k \beta
\]
for all \( \alpha, \beta, \gamma \).

**Categoricity Theorem for \( \mathcal{R}_2 \).** If

(a) For \( k = 1, 2 \), \( \mathcal{B} \upharpoonright \alpha \preceq^\mathcal{B}_k \mathcal{B} \upharpoonright \beta \) whenever \( \alpha \preceq^\mathcal{B}_k \beta \).

(b) \( \preceq^\mathcal{B}_1 \) and \( \preceq^\mathcal{B}_2 \) are partial orderings of the universe of \( \mathcal{B} \) with \( \preceq^\mathcal{B}_2 \subseteq \preceq^\mathcal{B}_1 \subseteq \preceq^\mathcal{B}_0 \).

(c) \( \preceq^\mathcal{B}_k \) respects \( \preceq^\mathcal{B}_{k-1} \) for \( k = 1, 2 \).

(d) The arithmetic part of \( \mathcal{B} \) is \( \mathcal{R} \upharpoonright \lambda \) for some \( \lambda \) which is indecomposable in \( \mathcal{R} \).

then the core of \( \mathcal{B} \) is isomorphic to an initial segment of the core of \( \mathcal{R}_2 \upharpoonright \lambda \).

\( \mathcal{R}_2 \) is defined in Definition 5.4 of [2].

For the rest of the paper, assume \( \mathcal{B} \) satisfies (a)-(c) of the theorem. We do not assume that \( \mathcal{B} \) is necessarily well-ordered by \( \preceq^\mathcal{B} \).

**Definition 0.1** A pattern \( \mathbf{P} \) is \( \mathcal{B} \)-covered if there is a covering of \( \mathbf{P} \) in \( \mathcal{B} \).

See Definition 5.3 of [2] for the definition of pattern (short for pattern of resemblance of order two). See Definition 5.2 of [2] for the definition of covering. That definition is slightly different from that used in [1] in that the range of a covering is required to be closed (Definition 2.3 of [2]).

**Definition 0.2** Assume \( \mathbf{P} \) is a pattern, \( h \) is a function from the universe of \( \mathbf{P} \) into the universe of \( \mathcal{B} \) and \( \varphi \) is a regressive function on the nonminimal indecomposable elements in the range of \( h \) (i.e. \( h(\alpha) < \alpha \) for any nonminimal
element in the range of $h$ which is indecomposable in $R\)$. Suppose also that $P$ is a closed substructure of the pattern $P^+$. A function $h^+$ of the universe of $P^+$ into the universe of $B$ extends $h$ above $\varphi$ if $h^+$ extends $h$ and $\varphi(h(a)) < h^+(b)$

for any indecomposable $b$ in $P^+$ and any indecomposable $a$ in $P$ such that $(-\infty, a)^P \prec b \prec a$.

**Definition 0.3** Assume $P$ and $P^+$ are patterns and $P$ is a closed substructure of $P^+$. The rule $P|P^+$ is cofinally valid in $B$ if for every covering $h$ of $P$ in $B$ and every regressive function $\varphi$ on the nonminimal indecomposable elements in the range of $h$ there is a covering $h^+$ of $P^+$ into $B$ which extends $h$ above $\varphi$.

**Lemma 0.4** Every generating rule is cofinally valid in $B$.

**Proof.** The only properties of $R_2$ used in the proof of part 2 of Lemma 13.11 of [2] and the supporting lemmas are the preliminary properties we have assumed of $B$. Therefore, the proof carries over with $R_2$ replaced by $B$.

The proof is by induction on the generation of the generating rules (Definition 13.10 of [2]).

Suppose $P^+$ is 1-correct arithmetic extension of $P$. The proof that $P|P^+$ is cofinally valid in $B$ is analogous to the proof of Lemma 8.4 of [2]. Assume $h$ is a covering of $P$ in $B$ and $\varphi$ is a regressive function on the nonminimal indecomposables in the range of $h$. Notice that any covering of $P^+$ in $B$ which extends $h$ vacuously extends $h$ above $\varphi$ since there are no new indecomposable elements (by Lemma 4.9 of [2]). By Lemma 4.5 of [2], there is an embedding $h^+$ of the arithmetic part of $P^+$ in $R$ which extends $h$. Clearly, the range of $h^+$ is contained in $\lambda$. A straightforward argument using the fact that $P^+$ is a 1-correct arithmetic extension of $P$ (Definitions 4.8, 7.1 and 8.1 of [2]) shows $h^+$ is a covering of $P^+$ in $B$.

Suppose $P^+$ is obtained from $P$ by 1-reflecting $X$ downward from $b$ to $a$. The proof that $P|P^+$ is cofinally valid in $B$ is analogous to the proof of Lemma 9.3 of [2]. Assume $h$ is a covering of $P$ in $B$ and $\varphi$ is a regressive function on the nonminimal indecomposables in the range of $h$. Since $h$ is a covering and $a \prec b$, $h(a) \prec b$ implying $h(a) \prec^\infty h(b)$ in $B$. Therefore, there is $\bar{X}$ such that $h[(-\infty, a)^P] \cup \varphi(h(a)) \prec \bar{X} \prec h(b)$ and $h[(-\infty, a)^P] \cup \bar{X}$ is both closed and a covering of $h[(-\infty, a)^P] \cup h[\bar{X}]$. Let $h^+$ be the order
isomorphism of $P^+$ and $h[|P|] \cup \tilde{X}$. A straightforward argument using the fact that $P^+$ is obtained from $P$ by 1-reflecting $X$ downward from $a$ to $b$ (Definition 9.1 of [2]) shows that $h^+$ is a covering of $P^+$ which extends $h$ above $\varphi$.

Suppose $P^+$ is obtained from $P$ by 2-reflecting $X$ downward from $b$ to $a$. The proof that $P|P^+$ is cofinally valid in $B$ is analogous to the proof of Lemma 9.6 of [2] and similar to the proof in the previous paragraph (using Definition 9.4 of [2] instead of Definition 9.1).

Assume $P|P^+$ is a generating rule which is cofinally valid in $B$ and $P|P^*$ is obtained by 2-reflecting $P|P^+$ upward from $a$ to $b$. The proof that $P|P^*$ is cofinally valid in $B$ is analogous to the proof of Lemma 10.3 of [2]. Let $X = |P^+| \setminus |P|$. By Definition 10.1 of [2], $P^+$ is a continuous extension of $P$ at $a$ (see Definitions 7.1 and 7.4 of [2]) and $a \leq_2 b$. Assume $h$ is a covering of $P$ in $B$ and $\varphi$ is a regressive function on the nonminimal indecomposables in the range of $h$. Since $a \leq_2 h(b)$, $h(a) \leq_2 h(b)$ implying $h(a) \leq_2^2 h(b)$ in $B$. Since $P|P^+$ is cofinally valid in $B$, there are cofinally many $\tilde{X}$ below $h(a)$ such that $h[(\omega, a)^P] \cup \tilde{X}$ is closed and a covering of $(\omega, a)^P \cup X$ (as a substructure of $P^+$). Since $h(a) \leq_2 h(b)$ in $B$, there are cofinally many $\tilde{X}$ below $h(b)$ such that $h[(\omega, a)^P] \cup \tilde{X}$ is closed and a covering of $(\omega, a)^P \cup X$. Choose such $\tilde{X}$ such that $\varphi(h(b)) < \tilde{X}$. A straightforward argument using the fact that $P|P^*$ is obtained by 2-reflecting $P|P^+$ upward from $a$ to $b$ (Definition 10.1 of [2]) shows that $h^+$ is a covering of $P^*$ which extends $h$ above $\varphi$.

Assume $P_i|P_{i+1}$ is a generating rule which is cofinally valid in $B$ for $i < n$ and $P^+$ is a closed substructure of $P_n$ which extends $P_0$. An easy argument by induction shows $P_0|P_i$ is cofinally valid in $B$ for $i \leq n$. The fact that $P_0|P_n$ is cofinally valid in $B$ clearly implies that $P_0|P^+$ is also.

Assume $P|P^+$ is a generating rule which is cofinally valid in $B$ and $h$ is a continuous embedding of $P$ in $Q$. Let $Q^+$ be a minimal lifting (Definitions 12.1 and 12.4 of [2]) of $P|P^+$ to $Q$ with respect to $h$ and let $h^+$ be the lifting map. The proof that $Q|Q^+$ is cofinally valid in $B$ is analogous to the proof of Lemma 13.8 of [2]. By identifying $P$ and $P^+$ with their images under $h^+$, we may assume $h^+$ is the identity on $|P^+|$. Assume $f$ is a covering of $Q$ in $B$ and assume $\varphi$ is a regressive function on the nonminimal indecomposables in the range of $f$. By increasing the values of $\varphi$ if necessary, we may assume that $f[(\omega, a)^Q] \leq \varphi(h(a))$ whenever $a \in |P|$ and $h(a)$ is indecomposable. Since $P|P^+$ is cofinally valid in $B$, there is a covering $g$ of $P^+$ in $B$ which extends the restriction of $f$ to $|P|$ above the restriction of $\varphi$ to the indecomposables in $f[|P|]$. The restriction of $f \cup g$ to the indecomposables of $Q^+$ is order
preserving. By Lemma 4.5 of [2], this map extends to a unique arithmetic
embedding of the arithmetic part of \( Q^+ \) in \( B \) which must extend both \( f \) and
\( g \). Therefore, \( f \cup g \) is an arithmetic embedding of the arithmetic part of \( Q^+ \)
in \( B \). Let \( Q^\ast \) be the pattern with the same arithmetic part as \( Q^+ \) which
is induced by \( B \) through \( f \cup g \) i.e. so that \( f \cup g \) is an embedding of \( Q^\ast \) in
\( B \). Consider the structure \( Q' \) which has the same arithmetic part as \( Q^\ast \) so
that the interpretation of \( \preceq_k \) is the intersection of the interpretations of \( \preceq_k \)
in \( Q^+ \) and \( Q^\ast \). A straightforward argument shows \( Q' \) is a lifting of \( P|P^+ \) to
\( Q^+ \) (actually, equal to \( Q^+ \)) implying \( Q^\ast \) is a cover of \( Q^+ \). Therefore, \( f \cup g \) is a
covering of \( Q^+ \) in \( B \). Clearly, \( f \cup g \) extends \( f \) above \( \varphi \). QED

Statement 0.5
Assume \( P \) and \( Q \) are patterns and \( P \) generates \( Q \). Any covering of \( P \) in \( B \) extends to a covering of \( Q \) in \( B \).

Proof. Straightforward from the previous lemma (see Definition 14.2 of [2]). QED

The following two lemmas will be used only to show that if the arithmetic
part of \( B \) is the restriction of \( R \) to an indecomposable of \( R \) then every \( B \)-
covered pattern is covered i.e. \( R_2 \)-covered. Hence, if one is willing to accept
the assumption that every pattern is covered (which increases the proof-
theoretic strength of the metatheory to just beyond KP\(_\ell\)) then these lemmas
can be omitted.

The next lemma is an observation that the proofs of parts 3, 4, 6 and 8 of
Lemma 14.8 in [2] actually prove stronger assertions. Notice that in our base
theory KP\(_\omega\), saying that a linear ordering is order isomorphic to an ordinal
is stronger than saying it is a well-ordering.

Statement 0.6
Assume \( P_n \) (\( n \in \omega \)) is an increasing sequence of patterns such
that \( P_n|P_{n+1} \) is a generating rule for each \( n \in \omega \). Let \( P_\infty \) be the union of
the \( P_n \) (\( n \in \omega \)).

3*. Every covering of \( P_0 \) in \( B \) extends to a covering of \( P_\infty \) in \( B \).

4*. Assume \(|B|, \preceq^B \) is order isomorphic to an ordinal. If \( P_0 \) is \( B \)-covered
then \(|P_\infty|, \preceq^{P_\infty} \) is order isomorphic to an ordinal

6* If \( P_\infty \) is a well-ordered structure (i.e. \( \preceq^{P_\infty} \) is a well-ordering of \( P_\infty \))
and \( Q \) is a closed substructure of \( P_\infty \) which is a covering of \( P_n \) then
\(|P_n|, \preceq^{P_\infty}_{pw} |Q| \).
8*. Assume $P_n \ (n \in \omega)$ is fair and $P_\infty$ is a well-ordered structure.

(a) For $k = 1, 2$ and $a, b \in |R|$

$$a \preccurlyeq_k b \implies a \preccurlyeq_{P_\infty} b$$

(b) If $(|P_\infty|, \preccurlyeq_{P_\infty})$ is order isomorphic to an ordinal then $P_\infty$ is isomorphic to $R_2|\delta$ for some $\delta$ which is indecomposable in $R$.

**Proof.** Part 3* follows from Lemma 0.5.

Part 4* follows from part 3*.

For part 6*, notice that parts 1, 5 and 7 of Lemma 14.8 of [2] implies that $P_\infty$ satisfies our preliminary assumptions on $\mathcal{B}$ i.e. the arithmetic part of $P_\infty$ is an arithmetic structure with respect to $R$ and parts (a)-(c) of the Categoricity Theorem hold. Taking $\mathcal{B}$ to be $P_\infty$ in part 3* we see there is a covering of $P_\infty$ into itself which extends the covering of $P_n$ onto $Q$. Since $P_\infty$ is well-ordered, we must have $|P_n| \preccurlyeq_{P_\infty} |Q|$.

The proof of part 8 of Lemma 14.8 of [2] actually shows part 8*(a) if we replace applications of part 6 of Lemma 14.8 by applications of part 6* above.

For part 8*(b), we may assume the arithmetic part of $\mathcal{B}$ is $R \upharpoonright \delta$ for some ordinal $\delta$ which is indecomposable in $R$ by parts 1 and 5 of Lemma 14.8 and Lemmas 4.4 and 4.5 of [2]. A simple induction using part 7 of Lemma 14.8 of [2] and part 8*(a) shows that for $\alpha \leq \delta$, the restriction of $\preccurlyeq_{\mathcal{B}}$ to $\alpha$ is the same as the restriction of $\preccurlyeq_{R_2}$ to $\alpha$ for $k = 1, 2$.

QED

**Lemma 0.7** If the arithmetic part of $\mathcal{B}$ is isomorphic to an initial segment of $R$ then any $\mathcal{B}$-covered pattern is covered.

**Proof.** Assume $h$ is a covering of the pattern $P$ in $\mathcal{B}$. Let $P_n \ (n \in \omega)$ be a fair sequence of patterns with $P_0 = P$.

By part 3* of the previous lemma, there is a covering $h^+$ of $P_\infty$ in $\mathcal{B}$ which extends $h$. By part 8*(b) of the previous lemma, $P_\infty$ is isomorphic to an initial segment of $R_2$. The restriction of that isomorphism to $|P|$ is a covering of $P$ in $R_2$.

QED

**Proof of the Categoricity Theorem.** Our proof will follow the general lines of the proof of Theorem 9.1 of [1].

6
Claim 1. Assume $P$ is $B$-covered and $P'$ is a minimal element with respect to $\preceq_{pw}$ (the pointwise partial ordering of finite subsets of $B$) among the closed substructures of $B$ which are coverings of $P$.

(i) If $Q$ is a substructure of $B$ which is a cover of $P$ then $|P'| \leq_{pw} |Q|$.

(ii) $P \simeq P'$.

For (i), suppose $Q$ is a substructure of $B$ which is a cover of $P$. By Theorem 14.10 of [2], there are finite closed substructures $R$ and $P^*$ of $R_2$ which are isominimal in $R_2$ and isomorphic to $P' \cup Q$ (with a slight abuse of notation) and $P$ respectively. Let $\overline{P'}$ and $\overline{Q}$ be the images of $P'$ and $Q$ respectively under the isomorphism of $P' \cup Q$ and $R$. By part 2 of Theorem 14.10 of [2], $|P*| \leq_{pw} |\overline{P'}|, |\overline{Q}|$. By part 5 of Theorem 14.10 of [2], $\overline{P'} \cup \overline{Q}$ generates $P^* \cup \overline{P'} \cup \overline{Q}$. By Lemma 0.3 there is a covering $h$ of $P^* \cup \overline{P'} \cup \overline{Q}$ in $B$ which extends the isomorphism of $\overline{P'} \cup \overline{Q}$ with $P' \cup Q$. Let $P''$ be the image of $P^*$ under $h$. We have $|P''| \leq_{pw} |P'|, |Q|$. By the minimality of $P'$, $P'' = P'$. Therefore, $|P'| \leq_{pw} |Q|$.

For part (ii), follow the argument for part (i) (one may take $Q = P'$) to conclude from $P'' = P'$ that $P^* = \overline{P'}$. Since $P^* \simeq P$ and $\overline{P'} \simeq P'$, $P \simeq P'$.

For any covered pattern $P$, let $P^*$ be the isominimal substructure of $R_2$ which is isomorphic to $P$. For $P$ an isominimal substructure of $B$, define $f_P$ to be the isomorphism of $P$ and $P^*$. Let $f$ be the union of the $f_P$. A straightforward argument shows $f$ is an embedding of the core of $B$ into the core of $R_2$.

To show the range of $f$ is an initial segment of $R_2$, assume $\alpha < \beta$ where $\beta$ is in the range of $f$. There is an isominimal substructure $P$ of $B$ such that $\beta$ is in the range of $f_P$. Let $P_n$ $(n \in \omega)$ be a fair sequence with $P_0 = P$ and let $P_\infty$ be the union of the $P_n$. By Lemma 14.9 of [2], there is an isomorphism $g$ of $P_\infty$ with $R_2 \upharpoonright \delta$ for some $\delta$ which is indecomposable in $R$ and the image of $P_n$ under $g$ is $P_n'$ for each $n \in \omega$. Fix $n$ such that $\alpha$ is in $P_n'$. By Lemma 0.6 and Claim 1, there is an isominimal substructure $Q$ of $B$ which is isomorphic to $P_n$. Since $\alpha$ is in $P^*$ which the range of $f_Q$, $\alpha$ is in the range of $f$. QED

Corollary 0.8 Assume $B$ satisfies (a)-(d) of the Categoricity Theorem for $R_2$. If there are arbitrarily long finite chains in $\preceq_2$ then the core of $B$ is isomorphic to the core of $R_2$ and a finite structure is isomorphic to a finite closed substructure of $B$ iff it is a pattern of resemblance of order 2.
Proof. Assume there are arbitrarily long finite chains in $\leq_2^B$. By the Categoricity Theorem, the core of $B$ is isomorphic to an initial segment of the core of $R_2$. Since this initial segment contains arbitrarily long finite chains in $\leq_2$, it must be the entire core of $R_2$ by part 2 of Theorem 14.10 of [2]. Hence, every pattern of resemblance of order two is isomorphic to a substructure of $B$. The converse is straightforward after noticing that condition (a) of the Categoricity Theorem implies that $\alpha$ is indecomposable whenever $\alpha \leq_1^B \beta$ and both $\alpha$ and $\beta$ are indecomposable whenever $\alpha \leq_2^B \beta$.

Corollary 0.9 Assume $R'_2$ is the alternate definition of $R_2$ from the introduction to [2] using $\Sigma_1$ and $\Sigma_2$ elementarity. $R'_2 \upharpoonright \delta$ satisfies the (a)-(d) of the Categoricity Theorem for each indecomposable $\delta$ and, hence, the conclusions of the Categoricity Theorem and the previous corollary hold for $R_2$.

Proof. Straightforward after noting that in $R'_2$, if $\alpha < \beta$, $\beta$ is a limit ordinal and $\alpha \leq_1 \xi$ for all $\xi$ with $\alpha \leq \xi < \beta$ then $\alpha \leq_1 \beta$.

One can prove that $\leq_2$ in $R'_2$ has arbitrarily long finite chains well within ZF.

References

1. Elementary patterns of resemblance, Annals of Pure and Applied Logic 108 (2001), pp. 19-77.
2. Patterns of resemblance of order 2, Annals of Pure and Applied Logic 158 (2009), pp. 90-124.