Auxetic two-dimensional lattices with Poisson’s ratio arbitrarily close to $-1$

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In this paper, we propose a class of lattice structures with macroscopic Poisson’s ratio arbitrarily close to the stability limit $-1$. We tested experimentally the effective Poisson’s ratio of the microstructured medium; the uniaxial test was performed on a thermoplastic lattice produced with a three-dimensional printing technology. A theoretical analysis of the effective properties was performed, and the expression of the macroscopic constitutive properties is given in full analytical form as a function of the constitutive properties of the elements of the lattice and on the geometry of the microstructure. The analysis was performed on three microgeometries leading to an isotropic behaviour for the cases of three- and sixfold symmetries and to a cubic behaviour for the case of fourfold symmetry.

1. Introduction

Auxetic materials are important in practical applications for civil and aeronautical engineering, defence equipments, smart sensors, filter cleaning and biomechanics; in recent years, the number of patent applications and publications has increased exponentially. There is also a strong interest in the theoretical modelling and in the reformulation of several aspects of mechanics, where the interval of negative Poisson’s ratio, although admissible in terms of thermodynamic stability, has often been omitted in the past. This interest must be accompanied by the design of new man-made microstructured media that can lead to a negative Poisson’s...
ratio. The term ‘auxetic’ comes from the Greek word ‘αυξησίζ’ (auxesis: increase, grow) and was first used by Evans [1] (see also [2]) to indicate materials with a negative Poisson’s ratio, expanding in the direction perpendicular to the applied tensile stress, and contracting for perpendicular compressive stress.

Poisson’s ratio $\nu$ is an indication of the mechanical properties of a medium to deform mainly deviatorically or isotropically, as described by the ratio $K/\mu$ between the bulk and the shear modulus, ranging from a so-called rubbery behaviour at the upper limit of $\nu$ to a ‘dilatational’ behaviour at the lower limit of $\nu$ [3], where for a three-dimensional isotropic medium $-1 \leq \nu \leq \frac{1}{2}$. Rubber, most liquids and granular solids are almost incompressible ($K/\mu \gg 1$, $\nu \to \frac{1}{2}$), while examples of extremely compressible materials ($K/\mu \ll 1$, $\nu < 0$) are re-entrant foams [4,5] and several molecular structures [6–8].

Auxetic systems perform better than classical materials in a number of applications, owing to their superior properties. They have been shown to provide better indentation resistance [9–11]; the material flows in the vicinity of an impact as a result of lateral contraction accompanying the longitudinal compression owing to the impact loading. Hence, the auxetic material densifies in both the longitudinal and transverse directions, leading to increased indentation resistance. This behaviour has also been correlated to the atomic packing [12], which has been found to be proportional to Poisson’s ratio, and to the densification mechanism under high contact pressure [13]. In an isotropic material, indentation resistance is roughly proportional to the ratio $E/(1-\nu^2)$, where $E$ is Young’s modulus and $\nu$ is Poisson’s ratio, meaning that the resistance can be strongly increased, even with respect to an incompressible material, when Poisson’s ratio is below $-1/2$. Resistance to damage is also associated with the capacity of negative Poisson’s ratio materials to distribute internal energy over a larger region as opposed to common materials which, in the presence of stress concentrators as point forces or geometrical singularities, accumulate internal energy in a neighbourhood of the concentrator leading to possible damage of the material and consequent failure. In this sense, auxetic materials can be applied to improve protective materials or energy-absorbing materials [14]. Furthermore, they can also be applied as an efficient membrane filter with variable permeability [15,16], fasteners [17,18], shape memory materials [19] and acoustic dampeners [20,21]. They have the ability to form dome-shaped structures when bent [4,22], undergoing double (synclastic) curvature, as opposed to the saddle shape (anticlastic curvature) that non-auxetic materials adopt when subjected to out-of-plane bending moments. In addition, they have better acoustic and vibration properties over their conventional counterparts [23–26].

There are several natural materials that have been discovered to possess a negative Poisson’s ratio: iron pyrites [27], arsenic and bismuth [28], cadmium [29], several cubic and face-centred cubic rare gas solids along a specific crystallographic direction [7] and also biomaterials such as cow teat skin [30] and load-bearing cancellous bones [31].

Artificial auxetic structures are two- and three-dimensional re-entrant structures, chiral structures, rotating rigid/semi-rigid units, hard molecules, liquid crystalline polymers and microporous polymers. Extended reviews can be found in [32–34], and here we pay particular attention to the most interesting models in terms of mechanics. Almost all these models are based on a simple mechanism that is treated as a unit cell leading to a global stiffening effect. Conceptually, the auxetic structure has been known since 1944 [27], but the first artificial experimental samples concerning a re-entrant structure, first proposed by Almgren [35] and Kolpakov [36], were presented in 1987 by Lakes [4]. Since then, different models have been proposed and analysed [37–47]. In the chiral structure proposed by Prall & Lakes [48], the basic unit is formed by connecting straight ligaments to central nodes and the auxetic effect is achieved through wrapping or unwrapping of the ligaments around the nodes in response to an applied force. Its static and dynamic behaviour has been studied in the context of the generalized micropolar theory of elasticity [49–51]. Other models, see for example [52], derive their auxetic behaviour from the rotation of the shapes when loaded; this kind of structure was developed to produce auxetic behaviour in micro- and nano-structure networks by joining the rigid or semi-rigid shapes [53–55]; three-dimensional models in the linear [56,57] and nonlinear [58] regimes...
were also proposed. Different structures were obtained using topology optimization \[59,60\] and periodic tessellation \[61\].

There are fewer examples of auxetic materials with continuous microstructure: the design of a family of two-dimensional, two-phase composites with Poisson’s ratios arbitrarily close to \(-1\) is given in \[3\]. Successively, in the seminal paper by Milton & Cherkaev \[62\], it is shown that every combination of positive-definite effective constitutive tensor can be obtained from a two-phase composite and particular attention was given to multi-rank laminates. In that paper, the important concept of \(n\)-mode materials, indicating the number of easy modes of deformation, was also introduced; such a concept has strict analogies with the number of degrees of freedom in a mechanism of partially constrained structure, which is common in structural mechanics.

In this paper, we propose a novel lattice model with three different realizations that leads to a Poisson’s ratio arbitrarily close to the thermodynamic limit corresponding to \(\nu = -1\). The effect is achieved by the superposition of clockwise and anticlockwise internal rotations leading to a macroscopic non-chiral effect. In §2, we present experimental evidence of the negative Poisson’s ratio \(\nu = -0.9931 \pm 0.0025\). In §3, we detail the kinematics of the mechanical system for three types of periodic lattices and, in §4, we determine analytically the macroscopic constitutive properties of these structures. The dependence of the effective properties on the constitutive and geometrical parameters of the microstructure is shown in §5, where a comparative analysis with hexagonal, triangular and square honeycombs is also performed. Final remarks in §6 conclude the paper.

2. The lattice with Poisson’s ratio close to \(-1\)

Here, we show experimental evidence of the microstructured media with Poisson’s ratio approaching \(-1\). The elements of the structure have been produced with a three-dimensional printer (Dimension SST 1200es) in thermoplastic polymer ABS with two different colours, blue and white. In figure 1, some snapshots of the experiment are shown, the elastic structure is subjected to a uniaxial tension in the horizontal direction. The images were taken at a distance of approximately 1.2 m from the sample, which was considered sufficient to minimize image distortions.

The displacements of the junction points at the central hinge of each couple of cross-shaped elements of different colours are equal to the macroscopic displacement of the perfectly periodic structure. The displacements of these junction points in the \(4 \times 4\) central unit cells are tracked from the movie of the experiment (see the electronic supplementary material). To this purpose, an algorithm for image processing in Matlab (release 2011b) was implemented and, in figure 1d, the progressive position of these points is highlighted with white dots. The corresponding deformation is plotted in figure 1e; the resulting Poisson’s ratio is \(\nu = -0.9931 \pm 0.0025\)!

3. Model of periodic lattice with auxetic macroscopic behaviour

The microstructured media fall within the class of unimode materials as shown in \[62; 63, \text{ch. 30}\] and \[64,65\]. In our plane linear elastic system, a single eigenvalue of the effective elasticity matrix is very small (approaching zero), and the other two are very large. As is common to all isotropic and cubic materials with Poisson’s ratio approaching \(-1\) the only easy mode of deformation is dilatation (plane dilatation in a two-dimensional system). Here, we focus on three affine materials, two isotropic and one cubic, presented in §2. The kinematic analysis of a single radially foldable structure is used to determine Poisson’s ratio of the perfect lattice and its class. Here, we use the term perfect to indicate that the lattice is composed of rigid elements.

(a) Kinematics of radially foldable structures

We consider the angulated element \(\overline{ABC}\), shown in grey in figure 2. This element represents two arms of a single cross-shaped structure that will be assembled with a second one to create the unit cell. The rigid element \(\overline{ABC}\) is supposed to roto-translate with a single degree of freedom,
Figure 1. Deformation of the auxetic lattice subjected to a horizontal tensile traction. (a–c) Three configurations of the structured media at increasing magnitude of deformation. (d) The white dots indicate the progressive position of the central points in the $4 \times 4$ central unit cells of the structured media. (e) Deformations $\bar{\varepsilon}_{11}$ and $\bar{\varepsilon}_{22}$ as a result of the applied uniaxial stress $\bar{\sigma}_{11}$. The grey lines correspond to the points highlighted in (d). The thick black line indicates the average deformation. Poisson’s ratio is equal to $-0.9931 \pm 0.0025$. (Online version in colour.)

Figure 2. Pair of linkages movable with a single degree of freedom $\gamma$. The two rigid linkages $\overline{ABC}$ and $\overline{EBD}$ are shown in grey and black, respectively. They are constrained at the ‘coupler’ point $B$ to have the same displacement. Points $A$ and $D$ and $E$ and $C$ can only move along straight lines.
where \( A \) moves along the \( Ox_1 \)-axis and \( C \) moves along the axis inclined by the angle \( \alpha \) with respect to the \( Ox_1 \)-axis. In analysing the trajectory of the central point \( B \), we also follow the more general formulation given in [66,67]. In figure 2, \( p \) is the length of each arm, \( \theta \) is the internal angle between them and \( \alpha \) is the angle between the two straight lines along which the points \( A \) and \( C \) are constrained to move. \( B \) is the ‘coupler’ point of the linkage. The equation for the one-parameter trajectory followed by the point \( B \) is obtained by fixing the values of the geometrical variables \( p \), \( \theta \), \( \alpha \); then, the position of \( B \) is determined by the angle \( \gamma \).

The coordinates of points \( A \), \( B \) and \( C \) are (figure 2)

\[
A \equiv (x_1 - p \cos \gamma, x_2 - p \sin \gamma), \quad B \equiv (x_1, x_2) \quad \text{and} \quad C \equiv (x_1 + p \cos(\pi - \theta + \gamma), x_2 + p \sin(\pi - \theta + \gamma)).
\]

Then, the ‘coupler’ point \( B \) is constrained to move within the rotated ellipse

\[
C_1 x_1^2 + C_2 x_2^2 + C_{12} x_1 x_2 + C = 0,
\]

where

\[
C_1 = p^2 \tan^2 \alpha, \quad C_2 = p^2 (2 + \tan^2 \alpha - 2 \cos \theta + 2 \tan \alpha \sin \theta),
\]

\[
C_{12} = -2p^2 (\tan^2 \alpha \sin \theta - \tan \alpha \cos \theta + \tan \alpha)
\]

and

\[
C = -p^4 (\tan \alpha \cos \theta + \sin \theta)^2.
\]

Note that, for a single linkage, angles \( \alpha \) and \( \theta \) are independent.

When we couple the movement of the linkage \( \overline{ABC} \) with the linkage \( \overline{EBD} \), depicted in black in figure 2, we obtain a relation between angles \( \alpha \) and \( \theta \). The two linkages share the same coupler curve (3.2) at their common point \( B \) at which they are connected by means of a hinge. This implies the condition

\[
\alpha = \pi - \theta.
\]

Consequently, in the trajectory equation (3.2)

\[
C_1 = p^2 \tan^2 \theta, \quad C_2 = p^2 \frac{(1 - \cos \theta)^2}{\cos^2 \theta}, \quad C_{12} = 2p^2 \left(1 - \frac{1}{\cos \theta}\right) \tan \theta, \quad C = 0,
\]

yielding

\[
x_2 = \frac{\sin \theta}{1 - \cos \theta} x_1.
\]

The common coupler curve for the two linkages is aligned with the radial line \( \overline{OB} \) and to avoid crossover with other pairs in a polar arrangement of the fully radially foldable linkage the angle \( \gamma \) has to satisfy the bounds

\[
\alpha - \eta \leq \gamma \leq \pi - \eta,
\]

where \( \eta = B \hat{A}C = B \hat{D}E = (\pi - \theta)/2 \).

We consider three geometries: hexagonal (\( \alpha = \pi/3 \)), figure 3a; square (\( \alpha = \pi/2 \)), figure 3b; (see also [59]) and triangular (\( \alpha = \pi/6 \)), figure 3c. The two-arm linkages are assembled in order to create radially foldable structures. Different configurations are shown in figure 3d-f, for the hexagonal, square and triangular structures, respectively; the ‘coupler’ point for each pair of linkages moves radially, and the corresponding Poisson’s ratios are equal to \(-1\).

The relative position of the point \( B \) with respect to the centre \( O \) of the structure, shown in the right column of figure 3d-f as a function of the angle parameter \( \gamma \), shows that the maximum volumetric expansion increases when we move from the triangular to the square and, then, to the hexagonal case.
Figure 3. Radially foldable structures. Poisson’s ratio is equal to \(-1\). (a) \(\alpha = \pi / 3\), hexagonal structure composed of six couples of rigid two-arm linkages. (b) \(\alpha = \pi / 2\), square structure. (c) \(\alpha = 2\pi / 3\), triangular structure. (d–f) Configurations of the single degree of freedom unit cells at different values of the geometrical parameter \(\gamma\) for the hexagonal, square and triangular structures, respectively. The radial distance \(OB\) \((p = 1)\) as a function of \(\gamma\) is also given.

(b) Construction of periodic lattice

The two-arm linkages presented in §3a are assembled in order to create the kinematically compatible periodic structures shown in figure 4. The microstructure is composed of cross-shaped elements with arms of the same length. The number of arms in each cross-shaped element is 3, 4 and 6 for the hexagonal, square and triangular geometries, respectively. A couple of cross-shaped elements are built where the two crosses are disposed in two different planes; in figure 2, they are indicated in black and grey. Each couple is mutually constrained to have the same displacement at the central point where a hinge is introduced. Different couples are then constrained by internal hinges at the external end of each arm. In §4, we also introduce some springs to provide stability of the constitutive behaviour.
Figure 4. Periodic microstructures. (a) Hexagonal, (b) square and (c) triangular geometries. Three different configurations, for different values of $\gamma$, are shown for each geometry. The grey dashed regions are the unit cells of the Bravais lattice where $t_1$ and $t_2$ are the primitive vectors.

The periodic structures have a Bravais periodic lattice [68] consisting of points

$$R = n_1 t_1 + n_2 t_2,$$

(3.8)

where $n_{1,2}$ are integers and $t_{1,2}$ are the primitive vectors spanning the lattice. The three different geometries described in figure 4 correspond to the fundamental centred rectangular (rhombic), square and hexagonal Bravais lattices, respectively. Following the systematic analysis for finite deformation as in [64,65], we show here that the lattice is a unimode material. Let

$$T = \begin{bmatrix} t_1 & t_2 \end{bmatrix}$$

(3.9)

be the ‘lattice matrix’. During the deformation, the primitive vectors undergo an affine transformation, and the matrix $T$ describes a motion starting at $t = t_0$, with $\gamma(t_0) = \gamma_0$. At time $t$, the deformation gradient is described by

$$F(t, t_0) = [T(t)][T(t_0)]^{-1}$$

(3.10)

and the associated Cauchy–Green tensor is a path

$$C(t, t_0) = [F(t, t_0)]^T [F(t, t_0)] = [T(t_0)]^{-T} [T(t)]^T [T(t)] [T(t_0)]^{-1}$$

(3.11)

beginning at $C(t_0, t_0) = I$. Note that any other possible Bravais lattice is associated with the same path (3.11).
In particular, we have the following cases.

— Hexagonal lattice (rhombic Bravais lattice as in figure 4a):

\[
T(t) = p \sin \gamma \begin{pmatrix} 0 & -\sqrt{3} \\ 6 & 3 \end{pmatrix}, \quad T(t_0) = p \sin \gamma_0 \begin{pmatrix} 0 & -\sqrt{3} \\ 6 & 3 \end{pmatrix}.
\]  
(3.12)

— Square lattice (square Bravais lattice as in figure 4b):

\[
T(t) = 2p \sin \gamma \mathbf{I}, \quad T(t_0) = 2p \sin \gamma_0 \mathbf{I}.
\]  
(3.13)

— Triangular lattice (hexagonal Bravais lattice as in figure 4c):

\[
T(t) = 2p \sin \gamma \begin{pmatrix} 0 & -\sqrt{3} \\ 2 & 3 \end{pmatrix}, \quad T(t_0) = 2p \sin \gamma_0 \begin{pmatrix} 0 & -\sqrt{3} \\ 2 & 3 \end{pmatrix}.
\]  
(3.14)

For all three cases, the path is

\[
C(t, t_0) = \left( \frac{\sin \gamma}{\sin \gamma_0} \right)^2 \mathbf{I}.
\]  
(3.15)

Then, we conclude that the three materials are unimode, because the possible paths \(C(t, t_0)\), for any Bravais lattice, lie on the same one-dimensional curve.

4. Effective properties of the periodic auxetic lattices

Here, we derive the effective constitutive parameters of the lattices. A perfect lattice would clearly have zero in-plane bulk modulus and unbounded shear moduli. In order to estimate the macroscopic behaviour for a real lattice, we consider the elastic behaviour of the elements of the lattice, and we compute the effective constitutive behaviour as a function of the constitutive behaviour of the single constituents and of the microstructure. To ensure stability, we introduce extensional or rotational elastic springs that can also mimic the effect of friction in a loading branch or model elastic hinges (see figure 5 for the hexagonal lattice).

For the sake of simplicity, we restrict our attention to macroscopic linear elasticity. The linearized behaviour depends nonlinearly on the reference configuration determined by \(\gamma_0\), which will be indicated by \(\gamma\) in the following for ease of notation.
Figure 6. Application of the principle of virtual work. Hexagonal lattice reinforced with longitudinal springs. (a) Simplified structure analysed for the computation of the effective properties. The applied forces $F_N$ and $F_T_1$ and $F_T_2$, with $F_T = (F_T_1 + F_T_2)/2$, correspond to macroscopic stress components as in (4.7). (b) Disconnected statically determined structure introduced for the determination of the internal actions $(M, N, S_L)$. (c) Statically determined structure adopted for the computation of the horizontal displacement $u_1$ of the point $B$.

The hexagonal and triangular lattices have threefold symmetries, and basic considerations on the symmetry group of the material lead to the conclusion that the constitutive behaviour is isotropic (in the plane of deformation). Similar considerations, based on the tetragonal symmetry of the square lattice, lead to the conclusion that the square lattice is cubic. Therefore, it will be necessary to compute two effective elastic constants for the isotropic structures and three for the cubic one. Stability constrains the in-plane Poisson’s ratio to range between $-1$ and $+1$. Effective properties are denoted as $K^*$ (bulk modulus), $E^*$ (Young’s modulus), $\mu^*$ (shear modulus) and $\nu^*$ (Poisson’s ratio) and macroscopic stress and strain as $\bar{\sigma}$ and $\bar{\varepsilon}$, respectively.

(a) Analysis of hexagonal lattice

The analytical derivation of the macroscopic properties for the periodic hexagonal lattice is given. The structure is composed of slender crosses, and classical structural theories can be conveniently applied to analyse the response of the elastic system. In particular, each arm of a single cross-shaped element is modelled as a Euler beam undergoing flexural and extensional deformations. Each beam has Young’s modulus $E$, cross-sectional area $A$ and moment of inertia $J$. Additional springs have longitudinal stiffness equal to $k_L$ or rotational stiffness equal to $k_R$ (figure 5). We also introduce the non-dimensional stiffness ratio parameters $\alpha_1 = k_L p/(E A)$, $\alpha_2 = k_L p^3/(E J)$, $\alpha_3 = k_R p/(E A)$ and $\alpha_4 = k_R p^3/(E J)$. Macroscopic stresses are computed averaging the resultant forces on the boundary of the unit cell. Periodic boundary conditions have been applied on the boundary of the unit cell, so that displacements are periodic and forces are anti-periodic. Additional constraints are introduced to prevent rigid body motions. To solve the structure, we apply the principle of virtual work (PVW) ([69], ch. 15, §326). In the following, we apply the PVW in two steps: in the first, we find the internal actions (bending moments $M$, shear forces $V$, axial forces $N$ and spring forces $S_L$ and moments $M_R$) of the structure searching for the kinematic admissible configuration in the set of statically admissible ones (flexibility method), and in the second step we compute the macroscopic displacements. This procedure has the advantage of maintaining a sufficiently simple analytical treatment. We point out that all the results have also been verified numerically implementing a finite-element code in Comsol Multiphysics.

We consider the elastic structure shown in figure 6a, subjected to known normal and tangential external forces, corresponding to a macroscopic stress with components $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ different from zero. We define an equivalent statically determined (or isostatic) system disconnecting two springs and introducing the dual static parameter as unknown $X$, equal for the two springs, as
shown in figure 6b. Then, the general field of tension $\Sigma$ ($\Sigma = M, V, N, S^L, M^R$) in equilibrium with the external loads is

$$\Sigma = \Sigma_0 + X\Sigma_1, \quad (4.1)$$

where $\Sigma_0$ is the solution of the static scheme in equilibrium with the external loads and $X$, whereas the field $\Sigma_1$ is the solution of the static scheme in equilibrium with zero external loads (autosolution of the problem) and $X = 1$.

The kinematic constraints, suppressed in the isostatic structure, are restored imposing the kinematic compatibility equation\(^1\) that determines the values of the unknown $X$ and uniquely defines the elastic solution of the problem, i.e.

$$X = -\frac{\sum_{\text{beam}} \int_0^p (M_0(M_1/EJ) + N_0(N_1/EA)) d\xi + \sum_{\text{spring}} S_{L}^1 S_{L}^1/(k_L/2)}{\sum_{\text{beam}} \int_0^p (M_1(M_1/EJ) + N_1(N_1/EA)) d\xi + \sum_{\text{spring}} S_{L}^1 S_{L}^1/(k_L/2)} = \left( F_N - \frac{\sqrt{3}(1 + \alpha_1)}{3 + 3\alpha_1 \cos^2 \gamma + \alpha_2 \sin^2 \gamma} \right) \cot \gamma, \quad (4.2)$$

where $F_T = (F_{T_1} + F_{T_2})/2$. We note that, for sufficiently slender beam structures, the contribution owing to the shear deformation is negligible compared with that owing to flexural and axial deformations and, therefore, it has been neglected.

We reconstruct the distribution of internal actions by a linear combination of partial diagrams of $N$ and $M$ and of the spring forces $S^L$, as functions of external forces $F_N$ and $F_T = (F_{T_1} + F_{T_2})/2$:

\[
\begin{align*}
N &= N_0 + XN_1, \\
M &= M_0 + XM_1 \\
S^L &= S_{L}^0 + XS_{L}^1.
\end{align*}
\]

(4.3)

Applying the PVW for the second time, we calculate the displacement of point $B$ at the centre of the spring as shown in figure 6c. To this purpose, we consider the real structure as kinematically admissible, and an isostatic structure subjected to horizontal and vertical forces of magnitude equal $1/4$ as statically admissible, so that the virtual external works coincide exactly with the horizontal and vertical displacements of the point $B$, $u_1$ and $u_2$, respectively. In particular, the PVW equations have the form

$$u_i = \sum_{\text{beam}} \int_0^p \left( M_0^* M/EJ + N_0^* N/EA \right) d\xi + \sum_{\text{spring}} S_{L}^* S_{L}/k_L/2, \quad (i = 1, 2), \quad (4.4)$$

where $(M_0^*, N_0^*, S_{L}^*) (i = 1, 2)$ are the internal actions of the statically admissible structure subjected to forces applied in direction $x_i (i = 1, 2)$. The corresponding displacements are

\[
\begin{align*}
u_1 &= A_1 F_N + B_1 F_T \\
u_2 &= A_2 F_N + B_2 F_T.
\end{align*}
\]

(4.5)

\(^1\)The equation describes the virtual work done by the statically admissible actions $\Sigma$ owing to the kinematically admissible displacements and deformations associated with $\Sigma_1$. 

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\[\sum_{\text{beam}} \int_0^p (M_0(M_1/EJ) + N_0(N_1/EA)) d\xi + \sum_{\text{spring}} S_{L}^1 S_{L}^1/(k_L/2)\]
where

\[ A_1 = \frac{\sqrt{3}(\alpha_1 + \cos^2 \gamma)}{2 \sin^3 \gamma k_L}, \]

\[ B_1 = \frac{9 \cos^2 \gamma(\cos^2 \gamma \sin^2 \gamma - 1)\alpha_1 - 9 \cos^2 \gamma(\cos^2 \gamma + 1)\alpha_1 + 6 \cos^2 \gamma \sin^4 \gamma \alpha_1 \alpha_2}{6 \sin^2 \gamma(3 + 3 \alpha_1 \cos^2 \gamma + \alpha_2 \sin^2 \gamma) k_L} + \frac{\sin^6 \gamma \alpha_2^2 + 3 \sin^4 \gamma \alpha_2 - 9 \cos^4 \gamma}{6 \sin^2 \gamma(3 + 3 \alpha_1 \cos^2 \gamma + \alpha_2 \sin^2 \gamma) k_L}, \]

\[ A_2 = 3 \frac{\sqrt{3} \cos^2 \gamma \alpha_1^2 + \sin^2 \gamma \alpha_1 \alpha_2 + (1 + \cos^4 \gamma) \alpha_1 + \cos^2 \gamma \sin^2 \gamma \alpha_2 + 3 \cos^2 \gamma}{2 \sin^2 \gamma(3 + 3 \alpha_1 \cos^2 \gamma + \alpha_2 \sin^2 \gamma) k_L} \]

and

\[ B_2 = -\sqrt{3} \frac{9 \cos^2 \gamma(\cos^2 \gamma \sin^2 \gamma + 1)\alpha_1^2 + 6 \sin^2 \gamma(\cos^2 \gamma \sin^2 \gamma + 1)\alpha_1 \alpha_2}{6 \sin^2 \gamma(3 + 3 \alpha_1 \cos^2 \gamma + \alpha_2 \sin^2 \gamma) k_L} - \frac{9(2 - \cos^2 \gamma \sin^2 \gamma)\alpha_1 + \sin^6 \gamma \alpha_2^2 + 3(1 - \cos^4 \gamma) \alpha_2 + 9 \cos^2 \gamma}{6 \sin^2 \gamma(3 + 3 \alpha_1 \cos^2 \gamma + \alpha_2 \sin^2 \gamma) k_L}. \]

Equations (4.5) are explicit linear relations between the forces \( F_N \) and \( F_T \) associated with macroscopic stresses

\[ \bar{\sigma}_{11} = \frac{\sqrt{3} F_N + F_T}{3 p \sin \gamma}, \quad \bar{\sigma}_{22} = \frac{\sqrt{3} F_N - 3 F_T}{3 p \sin \gamma} \]  

and displacement of the point B associated with macroscopic strains

\[ \bar{e}_{11} = \frac{2 u_1}{\sqrt{3} p \sin \gamma} = 2 A_1 F_N + B_1 F_T, \quad \bar{e}_{22} = \frac{2 u_2}{3 p \sin \gamma} = 2 A_2 F_N + B_2 F_T. \]

Solving relations (4.8) in terms of \( F_N \) and \( F_T \) and substituting the result into equation (4.7) leads to the macroscopic constitutive relation between the macroscopic stress \( \bar{\sigma} \) and macroscopic strain \( \bar{e} \). Clearly, appropriate choices of the forces \( F_N \) and \( F_T \) can be considered in order to set to zero some components of the stress.

Poisson’s ratio \( v^* \) of the hexagonal lattice is

\[ v^*_{\text{HL}} = \frac{\bar{\sigma}_{22} \bar{e}_{11} - \bar{\sigma}_{11} \bar{e}_{22}}{\bar{\sigma}_{11} \bar{e}_{11} - \bar{\sigma}_{22} \bar{e}_{22}} = \frac{c_1 \alpha_1^2 + c_2 \alpha_2^2 + c_3 \alpha_1 \alpha_2 + c_4 \alpha_1 + c_5 \alpha_2 + c_6}{c_7 \alpha_1^2 + c_8 \alpha_2^2 + c_9 \alpha_1 \alpha_2 + c_9 \alpha_1 + c_{10} \alpha_2 - c_6}, \]

where

\[ \begin{align*}
    c_1 &= 9 \cos^2 \gamma(\cos^4 \gamma - \cos^2 \gamma + 2), \\
    c_2 &= -\sin^6 \gamma, \\
    c_3 &= 3 \sin^2 \gamma(2 \cos^4 \gamma - 2 \cos^2 \gamma + 1), \\
    c_4 &= 9(2 \cos^4 \gamma + \cos^2 \gamma + 1), \\
    c_5 &= -3(2 \cos^4 \gamma - 3 \cos^2 \gamma + 1), \\
    c_6 &= 18 \cos^2 \gamma, \\
    c_7 &= 9 \cos^2 \gamma(\cos^4 \gamma - \cos^2 \gamma - 2), \\
    c_8 &= 3 \sin^2 \gamma(2 \cos^4 \gamma - 2 \cos^2 \gamma - 3), \\
    c_9 &= -9(2 \cos^4 \gamma - \cos^2 \gamma + 3), \\
    c_{10} &= -3 \sin^2 \gamma(2 \cos^2 \gamma + 1).
\end{align*} \]  

The effective in-plane bulk modulus is

\[ K^*_{\text{HL}} = \frac{1}{2} \frac{\bar{\sigma}_{11}}{\bar{e}_{11}} - \frac{\bar{\sigma}_{22}}{\bar{e}_{22}} = \frac{\sqrt{3} \sin^2 \gamma k_L}{6(\cos^2 \gamma + \alpha_1)}. \]  

Consequently, Young’s modulus of the hexagonal lattice with extensional springs is

\[ E^*_{\text{HL}} = \frac{\bar{\sigma}_{11} - \bar{\sigma}_{22}}{\bar{e}_{11} - \bar{e}_{22}} = 2 K^*_{\text{HL}}(1 - v^*_{\text{HL}}) = \frac{4 \sqrt{3} k_L \sin^2 \gamma (3 + 3 \cos^2 \gamma \alpha_1 + \sin^2 \gamma \alpha_2)}{-c_7 \alpha_1^2 - c_2 \alpha_2^2 - c_8 \alpha_1 \alpha_2 - c_9 \alpha_1 + c_{10} \alpha_2 + c_6}, \]
where the constants $c_2, c_6$ are given in equation (4.10), and the shear modulus is given by

$$\mu^*_H = \frac{3(1 - 2\cos^2 \gamma)\alpha_3 - (1 - 2\cos^2 \gamma)\alpha_4 + 18\cos^2 \gamma}{3(-3 + 2\cos^2 \gamma)\alpha_3 - (1 + 2\cos^2 \gamma)\alpha_4 - 18\cos^2 \gamma},$$

$$\nu^*_H = \frac{\sqrt{3}(k_R/p^2)}{2(3\sin^2 \gamma\alpha_3 + \cos^2 \gamma\alpha_4 + 9\cos^2 \gamma)},$$

$$K^*_H = \frac{8\sqrt{3}(k_R/p^2)}{3(3 - 2\cos^2 \gamma)\alpha_3 + (1 + 2\cos^2 \gamma)\alpha_4 + 18\cos^2 \gamma},$$

and

$$\mu^*_H = \frac{2\sqrt{3}(k_R/p^2)}{\alpha_4 + 3\alpha_3},$$

where we remember that $\alpha_3 = k_R/(EAp)$ and $\alpha_4 = (k_Rp)/(EF)$.

**Figure 7.** Triangular lattice reinforced with elastic springs. (a) Longitudinal springs of stiffness $k_L$. (b) Rotational spring of stiffness $k_R$. The dashed area represents a typical unit cell of the periodic elastic system. (c) Simplified structure analysed for the computation of the effective properties.

(b) **Analysis of the triangular lattice**

The isotropic triangular lattice structures with longitudinal and rotational springs are given in figure 7a,b. In figure 7c, the simplified structure implemented for the computation of the macroscopic constitutive properties is also shown. The effective properties are given below.

(i) **Triangular lattice with longitudinal springs**

— Poisson’s ratio

$$v^*_L = \frac{d_1\alpha_1^3 - 3\alpha_1^2\alpha_2 + d_2\alpha_1\alpha_2^2 + d_3\alpha_1^2 + d_4\alpha_2^2 + d_5\alpha_1\alpha_2 + d_6\alpha_1 + d_7\alpha_2}{-d_1\alpha_1^2 + d_8\alpha_1^2\alpha_2 + 3d_2\alpha_1\alpha_2^2 + d_9\alpha_1^2 + 3d_4\alpha_2^2 + d_{10}\alpha_1\alpha_2 - d_6\alpha_1 - d_7\alpha_2},$$
(ii) Triangular lattice with rotational springs

When rotational springs are considered, the triangular lattice structure is statically determined, and the effective constants are as follows.

— Poisson’s ratio

\[
v_{Tr}^* = \frac{\varepsilon_1 \alpha_4^2 + \varepsilon_2 \alpha_5^2 + \varepsilon_3 \alpha_3 \alpha_4 + \varepsilon_4 \alpha_3 + \varepsilon_5 \alpha_4}{\varepsilon_6 \alpha_3^2 + \varepsilon_7 \alpha_4^2 + \varepsilon_8 \alpha_3 \alpha_4 - \varepsilon_4 \alpha_3 - \varepsilon_5 \alpha_4},
\]

where

\[
\begin{align*}
\varepsilon_1 &= 9(2 \cos^4 \gamma - 3 \cos^2 \gamma + 1), & \varepsilon_2 &= (2 \cos^4 \gamma - \cos^2 \gamma), \\
\varepsilon_3 &= 3(4 \cos^2 \gamma \sin^2 \gamma - 1), & \varepsilon_4 &= 27 \cos^2 \gamma, \\
\varepsilon_5 &= 9 \cos^2 \gamma, & \varepsilon_6 &= 9(2 \cos^4 \gamma - \cos^2 \gamma - 1) \\
\varepsilon_7 &= 2 \cos^4 \gamma - 3 \cos^2 \gamma, & \varepsilon_8 &= 12 \cos^2 \gamma \sin^2 \gamma - 9.
\end{align*}
\]

— Bulk modulus

\[
K_{Tr}^* = \frac{3 \sqrt{3} (k_R / p^2)}{2 (3 \sin^2 \gamma \alpha_3 + \cos^2 \gamma \alpha_4 + 9 \cos^2 \gamma)}.
\]

— Young’s modulus

\[
E_{Tr}^* = -\frac{6 \sqrt{3} (3 \alpha_3 + \alpha_4)(k_R / p^2)}{e_6 \alpha_3^2 + e_7 \alpha_4^2 + e_8 \alpha_3 \alpha_4 - e_4 \alpha_3 - e_5 \alpha_4},
\]

where the coefficients \(e_4 - 8\) are given in equation (4.21).

— Shear modulus

\[
\mu_{Tr}^* = \frac{3 \sqrt{3} (3 \alpha_3 + \alpha_4)(k_R / p^2)}{9 \sin^2 \gamma \alpha_3^2 + \sin^2 \gamma \alpha_4^2 + 6(2 - \sin^2 \gamma) \alpha_3 \alpha_4}.
\]
Figure 8. Square lattice reinforced with longitudinal elastic springs. (a) The lattice structure. The dashed area represents a typical unit cell of the periodic elastic system. (b) Simplified structure used for the computation of the effective in-plane Poisson’s ratio $\nu_{\text{SL}}^e$, bulk modulus $K_{\text{SL}}^e$ and Young’s modulus $E_{\text{SL}}^e$. The forces $F_{H,V}$ are associated with macroscopic stress components $\tilde{\sigma}_{11,22} = F_{H,V}/(2p \sin \gamma)$. (c) Simplified structure used for the computation of the in-plane shear modulus $\mu_{\text{SL}}^e$. The force $F_V$ is associated with the macroscopic stress $\tilde{\sigma}_2 = F_V/(2p \sin \gamma)$.

(c) Analysis of the square lattice

The square lattice has cubic symmetry, and it is necessary to compute three independent elastic constants to determine the effective behaviour of the structure. The lattice with longitudinal springs is given in figure 8a, and the simplified structure used to compute the in-plane Poisson’s ratio $\nu_{\text{SL}}^e$, bulk modulus $K_{\text{SL}}^e$ or Young’s modulus $E_{\text{SL}}^e$ is shown in figure 8b. Statically, the structure is twice overdetermined, and it is therefore necessary to introduce two disconnections and two dual static variables to find the internal actions within the elastic system. The simplified structure introduced to compute the in-plane shear modulus $\mu_{\text{SL}}^e$ is given in figure 8c. In this case, the springs are not activated and they can be neglected, so that the structure can be considered as statically determined. The same structural models of figure 8 have been considered with rotational springs instead of longitudinal ones. The effective properties are reported in the following.

(i) Square lattice with longitudinal springs

--- Poisson’s ratio

$$\nu_{\text{SL}}^e = \frac{-3(\alpha_1 + 1)^2 \cos^2 \gamma}{3 \cos^2 \gamma \alpha_1^2 + 2 \sin^2 \gamma \alpha_1 \alpha_2 + 6(1 - \sin^2 \gamma \cos^2 \gamma)\alpha_1 + 2 \sin^2 \gamma \cos^2 \gamma \alpha_2 + 3 \cos^2 \gamma}.$$  \hspace{1cm} (4.25)

--- Bulk modulus

$$K_{\text{SL}}^e = \frac{k_L \sin^2 \gamma}{2(\alpha_1 + \cos^2 \gamma)}. \hspace{1cm} (4.26)$$

--- Young’s modulus

$$E_{\text{SL}}^e = \frac{2k_L \sin^2 \gamma(3 \cos^2 \gamma \alpha_1 + \sin^2 \gamma \alpha_2 + 3)}{3 \cos^2 \gamma \alpha_1^2 + 2 \sin^2 \gamma \alpha_1 \alpha_2 + 6(1 - \sin^2 \gamma \cos^2 \gamma)\alpha_1 + 2 \sin^2 \gamma \cos^2 \gamma \alpha_2 + 3 \cos^2 \gamma}.$$  \hspace{1cm} (4.27)

--- Shear modulus

$$\mu_{\text{SL}}^e = \frac{3k_L}{2(3 \cos^2 \gamma \alpha_1 + \sin^2 \gamma \alpha_2)}.$$  \hspace{1cm} (4.28)

(ii) Square lattice with rotational springs

The structure is statically undetermined with degree 1. The effective constitutive parameters are as follows.
5. Analysis of effective properties

Here, the effective properties of the microstructured media are analysed in detail. We consider at first the case of vanishing stiffness of the springs \( k_L, k_R \rightarrow 0 \). For every lattice

\[
v^* \simeq -1 + v^*_{1L,R} + \mathcal{O}(k^2_{L,R}), \quad K^* \simeq 0 + K^*_{1L,R} + \mathcal{O}(k^2_{L,R}) \tag{5.1}
\]

and

\[
E^* \simeq 0 + E^*_{1L,R} + \mathcal{O}(k^2_{L,R}), \quad \mu^* \simeq \mu^*_{0L,R} + \mu^*_{1L,R} + \mathcal{O}(k^2_{L,R}),
\]

where \( v^*_{1L,R}, K^*_{1L,R}, \mu^*_{1L,R} > 0 \) and their explicit expressions are given in table 1.

It is shown in (5.1) that also for deformable structures the Poisson’s ratio remains \(-1\) when the spring stiffnesses are zero, whereas the effect of the springs is to increase the value of \( v^* \). In such a limit, the bulk and Young’s moduli vanish, whereas the shear modulus remains finite. This is clearly associated with the deformation mechanism of the lattice, which involves deformation of the cross-shaped elements when a macroscopic shear stress or shear deformation is applied, whereas macroscopic volumetric deformations can be sustained by rigid internal rotations of the elements of the microstructure. The limiting behaviour described in (5.1) can also be understood in terms of relative stiffness between the spring elements and the elements of the lattice as described by the coefficients \( \alpha_{1 \ldots 4} \). In this respect, when \( \alpha_{1 \ldots 4} \rightarrow 0 \), the same outcomes of equations (5.1) are obtained. The dependence of Poisson’s ratio on the stiffnesses \( k_L \) and \( (k_R/p^2) \) is shown in figure 9a and b, respectively, for the three microstructures. Results confirm that Poisson’s ratio approaches \(-1\) when the spring constants are zero. They also show that \( v^*_{1L,R} < v^*_{2L,R} < v^*_{H,L,R} \).

It is worthwhile to note the maximum theoretical values that can be reached by Poisson’s ratios at the limit \( k_L, k_R/p^2 \rightarrow \infty \); the limiting expressions are

\[
\begin{align*}
v^*_{H,L} & \simeq 1 - \frac{(s/p)^2}{\sin^2 \gamma} + \mathcal{O} \left( \frac{s}{p} \right)^4, \\
v^*_{H,R} & \simeq 1 - \frac{2 \cos^2 \gamma}{1 + 2 \cos^2 \gamma} \left[ 1 - \frac{(s/p)^2}{1 + 2 \cos^2 \gamma} \right] + \mathcal{O} \left( \frac{s}{p} \right)^4, \\
v^*_{I,L} & \simeq \frac{1}{3} - \frac{1 + \sin^2 \gamma \cos^2 \gamma}{\sin^2 \gamma} \left( \frac{s}{p} \right)^2 + \mathcal{O} \left( \frac{s}{p} \right)^4, \\
v^*_{I,R} & \simeq \frac{1}{3} - \frac{2 \cos^2 \gamma}{3 - 2 \cos^2 \gamma} \left[ 1 - \frac{(s/p)^2}{3 - 2 \cos^2 \gamma} \right] + \mathcal{O} \left( \frac{s}{p} \right)^4.
\end{align*}
\]

and

\[
v^*_{1L,R} \simeq 0 - \frac{\cot^2 \gamma}{8} \left( \frac{s}{p} \right)^2 + \mathcal{O} \left( \frac{s}{p} \right)^4, \quad v^*_{H,L,R} \rightarrow 0,
\]
Table 1. Explicit expression of the coefficients in the asymptotic formulae in equation (5.1). Here, \( \eta_1 = 3 \sin^2 \gamma / (EJ) \) and \( \eta_2 = 3 \cos^2 \gamma / (AE) + p^2 \sin^2 \gamma / (EI) \).

| lattice   | longitudinal springs | rotational springs |
|-----------|----------------------|---------------------|
| hexagonal | \( \nu_1^* = \frac{p}{3} (\eta_1 + \eta_2) \) | \( \nu_1^* = \frac{\eta_1 + \eta_2}{9p \cos^2 \gamma} \) |
|           | \( K_1^* = \frac{\sqrt{3}}{6} \tan^2 \gamma \) | \( K_1^* = \frac{\sqrt{3}}{18p^2 \cos^2 \gamma} \) |
|           | \( E_1^* = \frac{2}{3} \tan^2 \gamma \) | \( E_1^* = \frac{2 \sqrt{3}}{9p^2 \cos^2 \gamma} \) |
|           | \( \mu_0^* = \frac{\sqrt{3}}{p(\eta_1 + \eta_2)} \) | \( \mu_0^* = \frac{\sqrt{3}}{p(\eta_1 + \eta_2)} \) |
| triangular | \( \nu_1^* = \frac{4p}{3} \eta_1 \eta_2 \tan^2 \gamma \) | \( \nu_1^* = \frac{4}{9p \cos^2 \gamma (\eta_1 + \eta_2)} \eta_1 \eta_2 \) |
|           | \( K_1^* = \frac{\sqrt{3}}{2} \tan^2 \gamma \) | \( K_1^* = \frac{\sqrt{3}}{6p^2 \cos^2 \gamma} \) |
|           | \( E_1^* = 2 \sqrt{3} \tan^2 \gamma \) | \( E_1^* = \frac{2 \sqrt{3}}{3p^2 \cos^2 \gamma} \) |
|           | \( \mu_0^* = \frac{3 \sqrt{3} \eta_1 + \eta_2}{4p \eta_1 \eta_2} \) | \( \mu_0^* = \frac{3 \sqrt{3} \eta_1 + \eta_2}{4p \eta_1 \eta_2} \) |
| square    | \( \nu_1^* = \frac{2}{3} \frac{p}{\eta_1} \tan^2 \gamma \) | \( \nu_1^* = \frac{1}{3 \cos^2 \gamma} \eta_1 \) |
|           | \( K_1^* = \frac{1}{2} \tan^2 \gamma \) | \( K_1^* = \frac{1}{4p^2 \cos^2 \gamma} \) |
|           | \( E_1^* = 2 \tan^2 \gamma \) | \( E_1^* = \frac{1}{p^2 \cos^2 \gamma} \) |
|           | \( \mu_0^* = \frac{3}{2p \eta_1} \) | \( \mu_0^* = \frac{3}{2p \eta_1} \) |

where in (5.2) and in the following, we consider, for the sake of simplicity, rectangular cross sections of the arms of the cross-shaped elements, so that \( A = ts \) and \( J = ts^3 / 12 \), where \( s \) and \( t \) are the in-plane and out-of-plane thicknesses, respectively. Therefore, \( s/p \ll 1 \). We note that Poisson’s ratio remains always negative for the square lattice approaching zero in the limit. Interestingly, the hexagonal lattice with longitudinal springs has a completely different behaviour approaching the upper limit for Poisson’s ratio corresponding to an incompressible material. For the structures with rotational springs the limit depends on the actual configuration described by the angle \( \gamma \) and ranges between \(-0.2\) and 1 for the hexagonal lattice and between 0.2 and 1/3 for the triangular one. For general values of \( k_R \) and \( k_L \), we note that the shear modulus \( \mu^* \) is independent of the rotational stiffness \( k_R \). In addition, for the square lattice, the shear modulus does not depend on the longitudinal stiffness \( k_L \); in fact, \( \mu^*_{\text{pl}} = \mu^*_{\text{sk}} \).

The polar diagrams of Poisson’s ratio and Young’s modulus are given in figure 10. In a reference system rotated by an angle \( \beta \) with respect to the system of reference \( O_{x_1, x_2} \), Poisson’s ratio and Young’s modulus are given by

\[
\nu^*(\beta) = -\frac{b_i b_j S^*_{ijkl} n_k n_l}{n_i n_j S^*_{ijkl} n_k n_l}, \quad E^*(\beta) = (n_i n_j S^*_{ijkl} n_k n_l)^{-1}, \quad (i, j, k, l = 1, 2), \quad (5.3)
\]
and therefore, \( \alpha \) also noted in [49]. On physical grounds, it is reasonable to consider the microstructures. It is also inversely proportional to the density of the effective medium, as \( p \). The ratio \( K \), \( \alpha \), is proportional to the slenderness of the arms of the cross-shaped elements of the microstructure, whereas the other effective constitutive parameters are governed by the flexural behaviour of the elements of the microstructures associated with the parameters \( \alpha \).

\[ K_1 \propto \frac{k_L}{E} \]  
\[ \alpha_1 \propto \frac{k_L}{E} \]  
\[ \alpha_2 \propto \frac{k_R}{p^2} \]  
\[ \alpha_3 \propto \frac{k_R}{p^2} \]  
\[ \alpha_4 \propto \frac{k_R}{p^2} \]

The ratio \( p/s \gg 1 \) is proportional to the slenderness of the arms of the cross-shaped elements of the microstructures. It is also inversely proportional to the density of the effective medium, as also noted in [49]. On physical grounds, it is reasonable to consider \( \alpha_1 < 10 \) (\( \alpha_2/p^2 < 10 \)) and, therefore, \( \alpha_1 \ll \alpha_2^{1/3} \). It follows that, for sufficiently low values of \( k_L \) and \( k_R \), only \( K_{1H}^* \) and \( K_{2L}^* \) are governed by the axial stiffness of the arms of the cross-shaped elements of the microstructure, whereas the other effective constitutive parameters are governed by the flexural behaviour of the elements of the microstructures associated with the parameters \( \alpha_2 \).
Young’s modulus among others. For the purpose of comparison, we consider the behaviour in terms of the effective in [48].

... and the same geometrical parameters of the elements, the stored-energy density of the triangular lattice is exactly three times the stored-energy density of the hexagonal one when dilatational deformations are applied. The nearly linear dependence of $K_{H1}^s$, $K_{L1}^s$, and $K_{S1}^s$ on the ‘slenderness’ $p/s$ highlights the dependence of the bulk moduli on the axial stiffness of the element of the microstructure, whereas for $K_{Hr}^s$, $K_{Lr}^s$, and $K_{Sr}^s$, it is the bending stiffness of the elements of the microstructure that governs the effective behaviour. This is evident from the comparison between the dashed black lines, corresponding to formulae in equations (4.14), (4.22) and (4.30), and the grey continuous lines where the effect of the axial stiffness of the element has been neglected ($\alpha_3 = 0$).

We conclude our analysis with a comparison of the proposed model with classical cellular solids. Square, triangular and hexagonal lattices are common topologies encountered in physical models and engineering applications and their static behaviour is discussed in [42,49,70,71], among others. For the purpose of comparison, we consider the behaviour in terms of the effective Young’s modulus $E^s$, and we introduce the relative density parameter $\tilde{\rho}$, defined as the ratio between the volume occupied by the thin elements of the microstructure and the volumes of the unit cell. In particular, for the proposed hexagonal, square and triangular auxetic lattices, we have

$$\tilde{\rho}_H^s = \frac{\sqrt{3}}{3 \sin^2 \gamma} \frac{s}{p}, \quad \tilde{\rho}_S^s = \frac{1}{\sin^2 \gamma} \frac{s}{p}, \quad \tilde{\rho}_T^s = \frac{\sqrt{3}}{\sin^2 \gamma} \frac{s}{p},$$

respectively, where, for the purpose of comparison, the physical parameters are reported for unit out-of-plane thickness.

The dominant deformation mechanism in classical honeycomb structures may be of extensional or bending nature. The effective behaviour of honeycombs, with triangular and square microstructures, is dominated by the axial deformations of their internal components [70], whereas the corresponding hexagonal lattice is dominated by cell-wall bending [42]. The same bending-dominated behaviour has been observed experimentally for the chiral lattice proposed in [48].

Figure 10. Polar diagrams of (a) Poisson’s ratio and (b) Young’s modulus as a function of the angle $\beta$. Results are given for the three microgeometries with longitudinal springs corresponding to different values of the non-dimensional stiffness ratio $\alpha_1 = (s/p)^2 \alpha_2/12$. 

and $\alpha_4$. In figure 11, the effective bulk modulus $K^s$ is shown as a function of the geometrical parameter $p/s$; lattices with longitudinal and rotational springs are shown in figure 11a and b, respectively. The triangular lattice is the stiffest, and its bulk modulus is exactly three times the bulk modulus for the hexagonal microstructure; in fact it is easy to check that, for the same $\gamma$ and the same geometrical parameters of the elements, the stored-energy density of the triangular lattice is exactly three times the stored-energy density of the hexagonal one when dilatational deformations are applied. The nearly linear dependence of $K_{H1}^s$, $K_{L1}^s$, and $K_{S1}^s$ on the ‘slenderness’ $p/s$ highlights the dependence of the bulk moduli on the axial stiffness of the element of the microstructure, whereas for $K_{Hr}^s$, $K_{Lr}^s$, and $K_{Sr}^s$, it is the bending stiffness of the elements of the microstructure that governs the effective behaviour. This is evident from the comparison between the dashed black lines, corresponding to formulae in equations (4.14), (4.22) and (4.30), and the grey continuous lines where the effect of the axial stiffness of the element has been neglected ($\alpha_3 = 0$).

We conclude our analysis with a comparison of the proposed model with classical cellular solids. Square, triangular and hexagonal lattices are common topologies encountered in physical models and engineering applications and their static behaviour is discussed in [42,49,70,71], among others. For the purpose of comparison, we consider the behaviour in terms of the effective Young’s modulus $E^s$, and we introduce the relative density parameter $\tilde{\rho}$, defined as the ratio between the volume occupied by the thin elements of the microstructure and the volumes of the unit cell. In particular, for the proposed hexagonal, square and triangular auxetic lattices, we have

$$\tilde{\rho}_H^s = \frac{\sqrt{3}}{3 \sin^2 \gamma} \frac{s}{p}, \quad \tilde{\rho}_S^s = \frac{1}{\sin^2 \gamma} \frac{s}{p}, \quad \tilde{\rho}_T^s = \frac{\sqrt{3}}{\sin^2 \gamma} \frac{s}{p},$$

respectively, where, for the purpose of comparison, the physical parameters are reported for unit out-of-plane thickness.

The dominant deformation mechanism in classical honeycomb structures may be of extensional or bending nature. The effective behaviour of honeycombs, with triangular and square microstructures, is dominated by the axial deformations of their internal components [70], whereas the corresponding hexagonal lattice is dominated by cell-wall bending [42]. The same bending-dominated behaviour has been observed experimentally for the chiral lattice proposed in [48].
**Figure 11.** Effective bulk modulus $K^*$ (MPa) as a function of the ‘slenderness’ $p/s$. (a) $K^*_L$, $K^*_T$, and $K^*_H$. (b) $K^*_R$, $K^*_T$, and $K^*_H$ where black dashed lines correspond to formulae in equations (4.14), (4.22) and (4.30) while continuous grey lines correspond to the same formulae with $\alpha_3 = 0$. Results are given for: Young’s modulus $E = 3000$ MPa, $t = 5$ mm, $k_L = k_R/p^2 = 1$ N mm$^{-1}$.  

**Figure 12.** Comparison between auxetic and honeycomb solids. Effective Young’s modulus $E^*$ (MPa) is given as a function of the relative density $\bar{\rho}^*$. The results correspond to the parameters: Young’s modulus $E = 3000$ MPa (thermoplastic polymer ABS), longitudinal spring stiffness $k_L = E s/(2p \cos \gamma)$ and $\gamma = 4\pi/9$. The Hashin–Shtrikman upper bound $E^* = E\bar{\rho}^*/(3 - 2\bar{\rho}^*)$ for isotropic media is also shown.

In figure 12, we compare the effective Young’s modulus $E^*$ of the proposed auxetic lattices with those of the honeycomb microstructures. The curves are shown for $p/s \geq 5$, in the range of validity of the beam theory, which was used for the computation of the effective behaviour. For the purpose of comparison, we also consider the stiffness of the spring equal to the longitudinal stiffness of an element with the same cross section of the arms of the cross-shaped elements. We note that the auxetic lattices can reach values of $E^*$ comparable to the corresponding honeycomb structures and that the square topology gives the stiffer behaviour, when the results are given in terms of relative density $\bar{\rho}$. 
6. Conclusion

A new family of auxetic lattices with Poisson’s ratio arbitrarily close to −1 has been proposed, and the extreme properties of the microstructured medium have been proved experimentally. A complete analysis of the static behaviour has been performed, and the effective properties have been given in closed analytical form. They depend on the constitutive properties of the single constituents and on the topology of the microstructure. Comparisons with classical honeycomb microstructured media show that the parameter of the geometry can be set in order to have effective properties of the same order of magnitude. For this type of structure, we envisage quite direct applications in structural aeronautics and civil engineering, where the design of the internal hinges is a problem that has already been solved technologically, and different joint structures have already been produced. In addition, with the advent of three-dimensional printing technology the capability to create the proposed microstructure at different scales opens new and exciting prospects in terms of possible technological applications.

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