The Dirichlet Series for the Liouville Function and the Riemann Hypothesis

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This paper investigates the analytic properties of the Liouville function’s Dirichlet series that obtains from the function $F(s) \equiv \frac{\zeta(2s)}{\zeta(s)}$, where $s$ is a complex variable and $\zeta(s)$ is the Riemann zeta function. The paper employs a novel method of summing the series by casting it as an infinite number of partial sums over sub-series that exhibit a certain symmetry and rapid convergence. In this procedure, which heavily invokes the prime factorization theorem, each sub-series has the property that it oscillates in a predictable fashion, rendering the analytic properties of the Dirichlet series determinable. With this method, the paper demonstrates that, for every integer with an even number of primes in its factorization, there is another integer that has an odd number of primes (multiplicity counted) in its factorization. Furthermore, by showing that a sufficient condition derived by Littlewood (1912) is satisfied, the paper demonstrates that the function $F(s)$ is analytic over the two half-planes $\text{Re}(s) > 1/2$ and $\text{Re}(s) < 1/2$. This establishes that the nontrivial zeros of the Riemann zeta function can only occur on the critical line $\text{Re}(s) = 1/2$.

Keywords: Liouville function, multiplicative function, factorization into primes, Mobius function, Riemann hypothesis

1. Introduction

This paper investigates the behaviour of the Liouville function which is connected to Riemann’s zeta function, $\zeta(s)$, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1.1)$$

where $n$ is a positive integer and $s$ is a complex variable, with the series being convergent for $\text{Re}(s) > 1$. This function has zeros (referred to as the trivial zeros) at the negative even integers $-2, -4, \ldots$. Riemann’s Hypothesis claims that the nontrivial zeros of the zeta function all occur on the critical line which occurs at $\text{Re}(s) = 1/2$. It has been shown that there are an infinite number of zeros on the critical line. The claim that these are the only nontrivial zeros has eluded proof to date, and this paper demonstrates that the Riemann Hypothesis (RH) is resolvable by tackling the Liouville function’s Dirichlet series generated by $F(s) \equiv \frac{\zeta(2s)}{\zeta(s)}$.

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† This was first proved by Hardy (1914).
which is readily rendered in the form

\[ F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \]

(1.2)

where \( \lambda(n) \) is the Liouville function defined by \( \lambda(n) = (-1)^\omega(n) \), with \( \omega(n) \) being the total number of prime numbers in the factorization of \( n \), including the multiplicity of the primes. We would also need the summatory function \( L(N) \), which is defined as the partial sum up to \( N \) terms of the following series:

\[ L(N) = \sum_{n=1}^{N} \lambda(n) \]

(1.2b)

Since the function \( F(s) \) will exhibit poles at the zeros of \( \zeta(s) \), we seek to identify where \( \zeta(s) \) can have zeros by examining the region over which \( F(s) \) is analytic. By demonstrating that a sufficient condition, derived by Littlewood (1912), for the RH to be true is indeed satisfied, we show that all the nontrivial zeros of the zeta function occur on the critical line \( \text{Re}(s) = 1/2 \).

Briefly, our method consists in judiciously partitioning the set of positive integers into sets of sub-series and couching the infinite sum in (2) into an infinite number of sums over these sub-series, with each such series being uniformly convergent. This method of considering a slowly converging series as a sum of many sub-series was used by the author in problems where Neumann series were involved (Eswaran (1990)).

In this paper we were able to break up the sum of the Liouville function into sums over many sub-series whose behaviour is predictable. It so turns out that one prime number \( p \) (and its powers) which is associated with a particular sub-series controls the behaviour of the sub-series.

Each sub-series is in the form of rectangular functions (waves) of unit amplitude but ever increasing periodicity and widths - we call these as ‘harmonics’, so every prime number is thus associated with such harmonic rectangular functions which then play a role in contributing to the value of \( L(N) \). It so turns out that if \( N \) goes from \( N \) to \( N+1 \), the new value of \( L(N+1) \) solely depends upon the factorization of \( N+1 \), and the particular harmonic that contributes to the change in \( L(N) \) is completely determined by this factorization. Since factorizations of numbers involve primes which are uncorrelated, we are to deduce that the statistical distribution of \( L(N) \) when \( N \) is large is like that of the cumulative sum of \( N \) coin tosses, (a head contributing +1 and a tail contributing -1), and thus logically lead to the final conclusion of this paper.

We found a new method of factorizing every integer and placing it in an exclusive subset, where it and its other members form an increasing sequence which in turn factorize alternately into odd and even factors; this method exploited the inherent symmetries of the problem and was very useful in the present context. Once this symmetry was recognized, we saw that it was natural to invoke it in the manner in which the sum in (2) was performed. We may view the sum as one over subsets of series that exhibit convergence even outside the domain of the half-plane \( \text{Re}(s) > 1 \). We were rewarded, for following the procedure pursued in this paper, with the revelation that the Liouville function (and therefore the zeta function) is controlled...
by innumerable rectangular harmonic functions, whose form and content are now precisely known and each of which is associated with a prime number and all prime numbers play their due role. And in fact all harmonic functions associated with prime numbers below or equal to a particular value N determine the present value of L(N).

When we are oblivious to the underlying symmetry being alluded to here, we render the summation in (2) less tractable than necessary. This is precisely what happens when we perform the summation in the usual manner, setting \( n = 1, 2, 3, \ldots \) in sequence.

2. Partitioning the Positive Integers into Sets

The Liouville function \( \lambda(n) \) is defined over the set of positive integers \( n \) as \( \lambda(n) = (-1)^{\Omega(n)} \), where \( \Omega(n) \) is the number of prime factors of \( n \), multiplicities included and by definition \( \lambda(1) = 1 \). Thus \( \lambda(n) = 1 \) when \( n \) has an even number of prime factors and \( = -1 \) when it has odd. It is a completely arithmetical function obeying \( \lambda(mn) = \lambda(m) \lambda(n) \) for any two positive integers \( m, n \).

We shall consider subsets of positive integers such as \( \{n_1, n_2, n_3, n_4, \ldots \} \) arranged in increasing order and are such that their values of \( \lambda \) alternate in sign:

\[
\lambda(n_1) = -\lambda(n_2) = \lambda(n_3) = -\lambda(n_4) = \ldots \tag{2.3}
\]

It turns out that we can label such subsets with a triad of integers, which we now proceed to do. To construct such a labeling scheme, consider an example of an integer \( n \) that can be uniquely factorized into primes as follows:

\[
n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_L^{e_L} p_ip_j \tag{2.4}
\]

where \( p_1 < p_2 < p_3 \ldots < p_L < p_i < p_j \) are prime numbers and the \( e_k, k \in \{1, 2, 3, \ldots, L\} \) are the integer exponents of the respective primes, and \( p_L \) is the largest prime with exponent exceeding 1. Integers of this sort, with at least one multiple prime factor are referred to here as Class I integers. In contrast, we shall refer to integers with no multiple prime factors as Class II integers. A typical integer, \( q \), of Class II may be written

\[
q = p_1 p_2 p_3 \ldots p_j p_L, \tag{2.5}
\]

where, once again, the prime factors are written in increasing order.

We now show how we construct a labeling scheme for integer sets that exhibit the property in (2.3) of alternating signs in their corresponding \( \lambda \)'s. First consider Class I integers. With reference to (2.4), we define integers \( m, p, u \) as follows:

\[
m = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_{L-1}^{e_{L-1}}; \quad p = p_L; \quad u = p_ip_j. \tag{2.6}
\]

In (2.6), \( m \) is the product of all primes less than \( p_L \), \( p \) is the largest multiple prime in the factorization, and \( u \) is the product of all prime number larger than \( p_L \) in the factorization. Thus the Class I integer \( n \) can be written

\[
n = mp^{e_L + u} \tag{2.7}
\]
Hence we will label this integer \( n \) as \((m, p^e, u)\), using the triad of numbers \((m, p, u)\) and the exponent \( e \). It is to be noted that \( u \) will consist of prime factors all larger than \( p \) and \( u \) cannot be divided by the square of a prime number.

Consider the infinite set of integers, \( P_{m,p,u} \), defined by

\[
P_{m,p,u} = \{ mp^2 u, mp^3 u, mp^4 u, ... \}
\]

(2.8)

The Class I integer \( n \) necessarily belongs to the above set because \( e \geq 2 \). Since the consecutive integer members of this set have been obtained by multiplying by \( p \), thereby increasing the number of primes by one, this set satisfies property (3) of alternating signs of the corresponding \( \lambda \)'s. Note that the lowest integer of this set \( P_{m,p,u} \) of Class I integers is \( mp^2 u \).

We may similarly form a series for Class II integers. The integer \( q \) in (5) may be written \( q = mp \), with \( m = p_1p_2p_3...p_j, p = p_L, \) and \( u = 1 \). This Class II integer is put into the set \( P_{m,p,u} \) defined by

\[
P_{m,p,1} = \{ mp, mp^2, mp^3, mp^4, ... \}.
\]

(2.9)

The set containing Class II integers is distinguished by the facts that they are Class II, because the latter have exponents greater than 1. For instance, the integer \( 2160 \), which factorizes as \( 2^4 \times 3^3 \times 5 \), is clearly a Class II integer, since it is divisible by the square of a prime number—in fact there are two such numbers, 2 and 3—but we identify \( p \) with 3 as it is the larger prime. It is a member of the set \( P_{16,3,5} = \{ 16 \times 3^2 \times 5, 16 \times 3^3 \times 5, 16 \times 3^4 \times 5, 16 \times 3^5 \times 5, ... \} \).

Ex. 1: The integer 2160, which factorizes as \( 2^4 \times 3^3 \times 5 \), is clearly a Class I integer since it is divisible by the square of a prime number—in fact there are two such numbers, 2 and 3—but we identify \( p \) with 3 as it is the larger prime. It is a member of the set \( P_{16,3,5} = \{ 16 \times 3^2 \times 5, 16 \times 3^3 \times 5, 16 \times 3^4 \times 5, 16 \times 3^5 \times 5, ... \} \).

Ex. 2: The integer 663, which factorizes as \( 3 \times 13 \times 17 \), is a Class II integer because it is not divisible by the square of a prime number. It belongs to the set \( P_{39,17,1} = \{ 39 \times 17, 39 \times 17^2, 39 \times 17^3, ... \} \).
Note that two different integers cannot share the same triad and two different triads cannot represent the same integer. Thus the mapping from a triad to an integer is one-one and onto. A formal proof is in the Appendix.

The following properties of the sets $P_{m,p,u}$ may be noted:

(a) The factorization of an integer $n$ immediately determines whether it is a Class I or a Class II type of integer.
(b) The factorization of integer $n$ also identifies the set $P_{m,p,u}$ to which $n$ is assigned.
(c) The procedure defines all the other integers that belong to the same set as a given integer.
(d) Every integer belongs to some set $P_{m,p,u}$ (allowing for the possibility that $u = 1$) and only to one set. This ensures that, collectively, the infinite number of sets of the form $P_{m,p,u}$ exactly reproduce the set of positive integers $\{1, 2, 3, 4, \ldots\}$, without omissions or duplications.

Our procedure, taking its cue from the deep connection between the zeta function and prime numbers, has constructed a labeling scheme that relies on the unique factorisation of integers into primes. In what follows, we shall recast the summation in (2) into one over the sets $P_{m,p,u}$. The advantage of breaking up the infinite sum over all positive integers into partial sums over the $P_{m,p,u}$ sets will soon become clear.

3. An Alternative Summation of the Liouville Function's Dirichlet Series

We shall now implement the above partitioning of the set of all positive integers to examine the analytic properties of $F(s)$ in (1.2). We shall rewrite the sum in (1.2) into an infinite number of sums of sub-series, ensuring that each sub-series is uniformly convergent even as $s \to 0$.

We begin, however, by assuming that $\text{Re}(s) > 1$, which makes the series in (1.2) absolutely convergent. We write the right hand side in sufficient detail so that the

† The integer represented by the triad $(m, p^r, u)$, is the product $mp^ru$, which obviously cannot take on two distinct values.
‡ Suppose two different triads $(m, p^r, u)$ and $(\mu, \pi^\rho, \nu)$ represent the same integer, say $n$. Then we must have $mp^ru = \mu\pi^\rho\nu = n$. It follows that at least two numbers of the tetrad $\{m, p, r, u\}$ must differ from their counterparts in the tetrad $\{\mu, \pi, \rho, \nu\}$. Since the factorization of $n$ is unique, this is impossible.
implementation of the partitioning scheme becomes self-evident:

\[
F(s) = 1 + \sum_{r=1}^{\infty} \frac{\lambda(2^r)}{2^rs} + \sum_{r=1}^{\infty} \frac{\lambda(3^r)}{3^rs} + \sum_{r=1}^{\infty} \frac{\lambda(5^r)}{5^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2 \times 3^r)}{2 \times 3^rs} + \sum_{r=1}^{\infty} \frac{\lambda(7^r)}{7^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2 \times 5^r)}{2 \times 5^rs} + \sum_{r=1}^{\infty} \frac{\lambda(11^r)}{11^rs} + \sum_{k=2}^{\infty} \frac{\lambda(2^k \times 3)}{2^k \times 3^rs} + \sum_{r=1}^{\infty} \frac{\lambda(13^r)}{13^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2 \times 7^r)}{2 \times 7^rs} + \sum_{r=1}^{\infty} \frac{\lambda(3 \times 5^r)}{3 \times 5^rs} + \sum_{r=1}^{\infty} \frac{\lambda(17^r)}{17^rs} + \sum_{k=2}^{\infty} \frac{\lambda(2^k \times 5)}{2^k \times 5^rs} + \sum_{r=1}^{\infty} \frac{\lambda(3 \times 7^r)}{3 \times 7^rs} + \sum_{r=1}^{\infty} \frac{\lambda(19^r)}{19^rs} + \sum_{k=2}^{\infty} \frac{\lambda(2^k \times 7)}{2^k \times 7^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2^k \times 11^r)}{2^k \times 11^rs} + \sum_{r=1}^{\infty} \frac{\lambda(23^r)}{23^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2 \times 13^r)}{2 \times 13^rs} + \sum_{k=2}^{\infty} \frac{\lambda(2^k \times 17)}{2^k \times 17^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2^k \times 19)}{2^k \times 19^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2^k \times 23)}{2^k \times 23^rs} + \sum_{k=2}^{\infty} \frac{\lambda(2^k \times 29)}{2^k \times 29^rs} + \sum_{r=1}^{\infty} \frac{\lambda(2 \times 3 \times 5^r)}{2 \times 3 \times 5^rs} + \ldots 
\]

(3.10)

We have explicitly written out a sufficient number of terms of the right hand side of (1.2) so that those corresponding to each of the first 30 integers are clearly visible as a term is included in one (and only one) of the sub-series sums in (3.10).

On the right hand side, the second term contains the integers 2, 4, 8, 16, ...; the third contains 3, 9, 27, ...; the fourth contains 5, 25, 125, ...; the fifth contains 6, 18, 54, ...; sixth contains 7, 49, ...; the seventh contains 10, 50, ...; the eighth contains 11, 121, ...; the ninth contains 12, 24, 48, ...; and so on. Note that in the ninth, fifteenth, and twentieth terms the running index is deliberately switched from \( r \) to \( k \) to alert the reader to the fact that the summation starts from 2 and not from 1 as in all the other sums. (Note that, in the ninth term, the Class I integer \( n = 12 = 2^2 \times 3 \) is assigned to the set \( P_{1,2,3} = \{2^2 \times 3, 2^3 \times 3, 2^4 \times 3, \ldots \} \) and not to the set \( P_{3,4,1} = \{2^2 \times 3, 2^3 \times 3, 2^4 \times 3, \ldots \} \), because the first term identifies \( p = 2 \) and \( u = 3 \) where as the second term onwards 3 has exponents, which violates our rules of precedence and would be an illegitimate assignment given our partitioning rules.)

The sub-series in (3.10) have one of two general forms:

\[
\sum_{r=1}^{\infty} \frac{\lambda(m,p^r)}{m^s \cdot p^rs} = \frac{\lambda(m,p)}{m^s \cdot p^s} \left[ 1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \frac{1}{p^{3s}} + \ldots + \frac{(-1)^X}{p^{Xs}} \right]
\]

or

\[
\sum_{k=2}^{\infty} \frac{\lambda(m,p^k.u)}{m^s \cdot p^{ks}} = \frac{\lambda(m,p^2.u)}{m^s \cdot p^{2s}} \left[ 1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \frac{1}{p^{3s}} + \ldots + \frac{(-1)^X}{p^{Xs}} \right]
\]

(3.11)

The above geometric series occurring within square brackets in the above two equations can actually be summed (because they are convergent),(see Whittaker and Watson) but we will refrain from doing so, and (1.2) can be rewritten as

\[
F(s) = \sum_{m} \sum_{p} \sum_{u} F_{m,p,u}^T(s) + \sum_{m} \sum_{p} F_{m,p,1}^T(s),
\]

(3.12)
where the first group of summations pertain to Class I integers \( n \) characterized by the triad \((m, p^k, u), (k \geq 2)\) and the second group pertain to those integers which are characterized by set \((m, p^k, 1), (k \geq 1)\) the first member in the set is a Class II integer and others Class I.

In the above we have defined the function \( F^T_{m;p;u}(s) \) of the complex variable \( s \) which is a sub-series involving terms over only the tower \((m, p, u)\) for a Class I integer as follows

\[
F^T_{m;p;u}(s) = \sum_{k=2}^{\infty} \frac{\lambda(mp^ku)}{m^sp^{k-1}u^s}, \tag{3.13}
\]

and the function \( F^T_{m;p;1}(s) \) of the complex variable \( s \) which is a sub-series involving terms over only the tower \((m, p, 1)\) whose 1st term is a Class II integer as

\[
F^T_{m;p;1}(s) = \sum_{r=1}^{\infty} \frac{\lambda(mp^r)}{m^sp^rs} \tag{3.14}
\]

With the understanding that when \( u = 1 \) we use the function in (3.14) instead of (3.13), we may write \( F(s) \) as

\[
F(s) = \sum_{m} \sum_{p} \sum_{u} F^T_{m;p;u}(s). \tag{3.15}
\]

Comparing the above Eq.(3.15) with Eq.(3.10) one can easily see that each term which appears as a summation in (3.10) is actually a sub-series over some tower which we denote as \( F^T_{m;p;u}(s) \) in (3.15). So we see that \( F(s) \) has been broken up into a number of sub-series. The important point to note that the \( \lambda \) value of each term in the sub-series changes its sign from +1 to -1 and then back to +1 and -1 alternatively. Therefore if the starting value of \( \lambda \) at the base was +1 then the cumulative contribution of this tower (sub series) to \( L(N) \) as \( N \), the upper bound, increases from \( N \) to \( N+1 \), \( N+2 \), \( N+3 \), ... will fluctuate between be 0 and 1. For some other tower whose base value of \( \lambda \) is \(-1\) then its cumulative contribution to \( L(N) \) will fluctuate between or 0 and \(-1\), these cumulative contributions can be represented as in the form of a rectangular wave. (Figure 1)

We have arrived at a critical point in our paper. We have cast the original function \( F(s) \equiv \zeta(2s)/\zeta(s) \) as a sum of functions of \( s \). Since the triad \((m, p^k, u)\) uniquely characterises all integers, the summations over \( m, p, k \) and \( u \) above are equivalent to a summation over all positive integers \( n \), as in (1.2), though not in the order \( n = 1, 2, 3, 4, ... \). The manner in which the triads were defined ensures that there are neither any missing integers nor integers that are duplicated.

Although we did not explicitly do it, we mentioned in passing that the sum over \( k \) in (3.13) and (3.14) is readily performed since it is a geometric series (see (3.11) that rapidly converges. This is true not merely for \( Re(s) > 1 \) but also as \( Re(s) \to 0 \). Whether \( F(s) \) converges when the summation is carried out over all the towers \((m, p, u)\) and, if so, over what domain, of \( s \) is the central question that we seek to answer in the next section. The answer to which as we shall see determines the analyticity of \( F(s) \) and thus resolves the Riemann Hypothesis.

We can recast (3.15), still in the domain \( Re(s) > 1 \), in the form

\[
F(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{3.16}
\]
where \( g(n) \) is a function appropriately defined below.

By construction, every \( n \) in the above summation can be written as

\[ n = \mu \pi^\nu, \quad (3.17) \]

where \( \mu, \pi, \) and \( \rho \) are positive integers, \( \pi \) is the largest prime in the factorization of \( n \), with either (i) an exponent \( \rho \geq 2 \), and \( \nu \) is the product of primes larger than \( \pi \) but with exponents equal to 1 (for Class I integers) or (ii) it is the largest prime factor with \( \rho = 1 \) and \( \nu = 1 \) (for Class II integers).

We define \( g(n) \) as follows:

\[
\begin{align*}
&= \lambda(mp^k u) \quad \text{if} \quad \mu = m \text{ and } \pi = p \text{ and } \nu = u \neq 1 \text{ and } \rho = k > 1 \quad (18a) \\
g(n) &= \lambda(mp^k) \quad \text{if} \quad \nu = u = 1 \text{ and } \pi = p \text{ and } \rho = k \geq 1 \quad (18b) \\
&= 0 \text{ otherwise.} \quad (18c)
\end{align*}
\]

The factors \( m^s p^k u^s \) and \( m^s p^k u^s \) in the denominators of (3.13) and (3.14) are simply \( n^s \), where \( n \) is the integer characterized by the \( (m,p^k,u) \) triad (with \( u = 1 \) in the latter case).

4. Calculation of the summatory Liouville function \( L(N) \)

We are now in a position to examine the summatory Liouville function \( L(N) \) by actually summing up the individual contribution from each sub-series.

To do all this systematically, we will explicitly illustrate the process starting from \( N = 1, 2, 3, \ldots \) up to \( N = 15 \). Each of these numbers is factorized and expressed uniquely as a triad. The \( N=1 \) is a constant term, which is the trivial \( (1,1,1) \), then the next number \( N = 2 = (1,2,1) \), is contained in the tower shown below the one corresponding to \( N = 1 \); and \( N = 3 = (1,3,1) \), is the tower below the previous; \( 4 = (1,2^2,1) \) however 4 is already contained in the tower \( (1,2,1) \) as its second member; the next \( N \)'s: \( 5, 6, 7, \) give rise to the new towers \( (1,5,1), (2,3,1), (1,7,1) \); \( 8 \) of course is the third member of the old tower \( (1,2,1) \) similarly 9 is the 2nd member of \( (1,3,1) \). After this the new towers which make their appearance are: \( 10 = (2,5,1), 11 = (1,11,1), 13 = (1,13,1), 14 = (2,7,1) \) and \( 15 = (3,5,1) \). See Figure 1. Now each tower \( (m,p^k,u) \) contributes to \( L(N) \) (consider \( N \) fixed in the following) according to the following rules:

(i) A particular tower will contribute only if its base number is less than or equal to \( N \) i.e \( m.p.u \leq N \)

(ii) And the contribution \( C \), to \( L(N) \) from this particular tower will be exactly as follows:

Case A: Class II integer \( (u = 1) \)

\[ C = \Sigma_{r=1}^{R} \lambda(m,p^R.u), \text{ where } R \text{ is the largest integer such that } m.p^R \leq N \]

Case B: Class I integer \( (u > 1) \)

\[ C = \Sigma_{k=2}^{K} \lambda(m,p^K.u), \text{ Where } K \text{ is the largest integer such that } m.p^K.u \leq N \]

Now since each successive \( \lambda \) changes sign from +1 to -1 or vice a versa, the contributions of each tower can be thought of as a rectangular wave of ever increasing width but constant amplitude -1 or +1, see figure.

To find the value of \( L(N) \) (\( N \) fixed), all we need to do is count the jumps of each wave, as we move from \( N=0 \) a jump upwards is called a positive peak a jump.
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Figure 1. The figure depicts the value of $L(N)$ downwards is a negative peak. Draw a vertical line at $N$, we are assured that it will hit one and only one peak (positive or negative) in one of the sub-series; then count the total number of positive peaks $P(N)$ and negative peaks $Q(N)$, of the waves on and to the left of this vertical line, then $L(N) = P(N) - Q(N)$; the reason for this rule will be clear after the next section.

Let us take an example let’s take $N = 5$, there is a positive peak for the constant term $(1,1,1)$, the next wave $(1,2,1)$ contributes one negative peak (at 2) and a positive peak (at 4), the wave $(1,3,1)$ contributes a $-1$ peak (at 3) and $(5,1)$ contributes a $-1$ (at 5) thus a total of three negative peaks and two positive peaks adding up we see that $L(5) = -1$, which is of course correct. Now if we take $N = 10$, let us now draw a vertical line at $N=10$, looking at this line and to its left we see that there are additionally three positive and two negative peaks thus adding this contribution of $+1$ to the previously calculated value $L(5)$ we get $L(10) = 0.$ (Two red vertical lines just just beyond $N=5$ and $N=10$ are drawn for convenience) Now if wish to compute $L(15)$ we see that there are three more negative peaks and two positive
peaks thus giving a value \( L(15) = -1 \). Counting the peaks further on it is easy to check that \( L(N) \) is correctly predicted for every value of \( N \) up to 30 and in particular: \( L(20) = -4; \( L(26) = 0 \) and \( L(30) = -4 \).

In summary to calculate \( L(N) \) we merely need to count the negative and positive peaks of the waves on \( N \) and to the left of \( N \). In the figure we have drawn a number of waves and labeled the tower to which each belongs using a triad of numbers. They are sufficient for one to easily calculate \( L(N) \) up to \( N=30 \) and check them out by comparing the numbers with the plot of \( L(N) \)shown on the top of the figure.

We turn to a more fundamental point: We now show the distribution of value \( L(N) \) as equivalent to the distribution obtained by summing \( N \) coin tosses.

5. Analyticity of \( F(s) \) and Riemann’s Hypothesis

We shall now examine the analyticity of the function \( F_{m,p,u}(s) \) utilizing a technique introduced by Littlewood (1912). In this, we follow the treatment of Edwards (1974, pp. 260-261). The series in (16) can be couched as the integral

\[
F(s) = \int_0^\infty x^{-s}dG(x),
\]

where \( G(x) = \int_0^x dG \) is a step function that is zero at \( x = 0 \) and is constant except at the positive integers, with a jump of \( g(n) \) at \( n \). The value of \( G(n) \) at the discontinuity, at an integer \( n \), is defined as \( (1/2)[G(n-\epsilon)+G(n+\epsilon)] \), which is equal to \( \sum_{j=1}^{n-1} g(j) + (1/2)g(n) \). Assuming \( \text{Re}(s) > 1 \), integration by parts yields

\[
F(s) = \int_0^\infty x^{-s}G(x)\,dx - \int_0^\infty G(x)d(x^{-s}) = \lim_{X \to \infty} [X^{-s}G(X) + s \int_0^X G(x)x^{-s-1}dx] = s \int_0^\infty G(x)x^{-s-1}dx,
\]

where the last step follows from the fact that \( |G(X)| < X \), which implies that \( X^{-s}G(X) \to 0 \) as \( X \to \infty \). We further observe, following Littlewood (1912), that as long as \( G(X) \) grows less rapidly than \( X^a \) for some \( a > 0 \) it follows that both terms in the line preceding (21) converge for all \( s \) in the half-plane \( \text{Re}(a-s) < 0 \), that is, for \( \text{Re}(s) > a \). By analytic continuation, \( F(s) \) converges in this half-plane. Since this result will be important in what follows, we record it here.

Theorem 1 [Littlewood (1912)]: When \( G(X) \) grows less rapidly with \( X \) than \( X^a \) for some \( a > 0 \), \( F(s) \) is analytic in the half-plane \( \text{Re}(s) > a \).

We shall now demonstrate that the sufficient condition stated in Theorem 1 is satisfied for a specific value of \( a \) that settles the Riemann Hypothesis. Using the
The Dirichlet Series for the Liouville Function and the Riemann Hypothesis

Definition of $G(N)$

\[ G(N) = \sum_{n=1}^{N} g(n), \quad (5.22) \]

we may rewrite $G(N)$ as

\[ G(N) = \sum_{p} \sum_{k} \sum_{m} \sum_{u} [(1 - \delta_{u,1}).(1 - \delta_{k,1}) \lambda(mp^kB) + \delta_{u,1}\lambda(mp^B)], \quad (5.23) \]

where $\delta_{u,1}$ and $\delta_{k,1}$ are Kronecker deltas (e.g. $\delta_{u,1} = 1$ if $u = 1$ and = 0 otherwise). The summations over $m$, $p$, $k$, and $u$ in (5.23) are undertaken with the understanding that the triads $(m, p^k, u)$ will only include integers $n \leq N$. Since the summation over $k$ is over an individual tower (if we keep $(m, p, u)$ fixed we can write (5.23) as

\[ G(N) = \sum_{m} \sum_{p} \sum_{u} F_{m, p, u}(s = 0), \quad (5.23b) \]

This is nothing but Eq.(3.15) evaluated from each subseries $F_{m, p, u}(s)$ by making $s \to 0$.

Of course, what we have called $G(N)$ is really the summatory Liouville function, $L(N)$, defined earlier by (1.2b):

\[ L(N) = \sum_{n=1}^{N} \lambda(n). \quad (5.24) \]

Expression (5.23) is crucial because, in the light of Theorem 1, its behaviour will determine the validity of the Riemann Hypothesis. Every term in the summation in (5.23) is either +1 or −1. We need to determine, for given $N$, how many terms contribute +1 and how many −1, and then determine how the sum $G(N)$ varies with $N$.

As we go through the list $n = 1, 2, 3, \ldots N$, we are assigning the integers to various sets of the sort $P_{m,p,u}$. To use our terminology of towers, we shall be ‘filling up’ slots in various towers from the bottom up until we have exhausted all $N$ integers. (When $N$ increases, in general, we shall not only be filling up more slots in existing towers but also adding new towers that were previously not included.) So the behaviour of $G(N)$ is determined by how many of the numbers that do not exceed the upperbound $N$ contribute +1 and how many −1.

It is convenient to identify the $\lambda$ of an integer by the triad which uniquely defines that integer. To avoid abuse of notation, we shall denote the value $\lambda(n)$ in terms of the $\lambda$ value of the base integer of the tower to which $n$ belongs. We will define the $\lambda$ of the base of a tower in uppercase, as $\Lambda(m, p, u)$. In other words if $n = (m, p^k, u)$ then it will belong to a tower whose base number is $n_B \equiv (m, p^k, u)$, where $k = 2$ if $u \neq 1$ and $k = 1$ if $u = 1$. Now we define $\Lambda(m, p, u) = \lambda(n_B) = \lambda(mp^ku) = \lambda(m)\lambda(p^k)\lambda(u)$, since the $\lambda$ of a product of integers is the product of the $\lambda$ of the individual integers. Of course, once we know $\lambda(n_B)$ we will know the $\lambda$ of all other numbers belonging to the tower because they alternate in sign.

To determine the behaviour of $G(N)$, the following theorem is important.

**Theorem 2:** For every integer that is the base integer of a tower labeled by the triad $(m, p, u)$, and therefore belonging to the set $P_{m,p,u}$, there is another unique...
tower labeled by the triad \((m', p, u)\) and therefore belonging to the set \(P_{m', p, u}\) with a base integer for which \(\Lambda(m', p, u) = -\Lambda(m, p, u)\).

**Proof:**

Let us write the integers at the base of a tower in the form \(n = mp^ru\) described by the triad \((m, p, u)\), where we shall assume that \(\rho = 2\) if \(u \neq 1\) and \(\rho = 1\) if \(u = 1\). These correspond to the lowest members of sets of Class I and Class II integers, respectively, which are the integers of concern here. In the constructions below, we shall multiply (or divide) \(m\) by the integer 2. Since 2 is the lowest prime number, such a procedure does not affect either the value of \(p\) or \(u\) in an integer and so we can hold these fixed.

We begin by excluding, for now, triads of the form \((1, p, 1)\), integers which are single prime numbers. We allow for this in Case 3 below.

**Case 1:** Suppose \(m\) is odd. It is clear that if we choose \(m' = 2m\), then \(\Lambda(m', p, u) = \Lambda(2m, p, u) = -\Lambda(m, p, u)\). We may say that \((m, p, u)\) and \((m', p, u)\) are 'twin' pairs in the sense that their \(\Lambda\)s are of opposite sign. Note that \((m, p, u)\) and \((m', p, u)\) are integers at the base of two different towers; they are not members of the same tower. (Recall that the members of a given tower are constructed by repeated multiplication of \(p\).)

**Case 2:** Suppose \(m\) is even. In this case, we need to ascertain the highest power of 2 that divides \(m\). If \(m\) is divisible by 2 but not by 2\(^2\), assign \(m' = m/2\). (So \(m = 6\) gets assigned to \(m' = 3\), and \(m = 3\), by Case 1 above, gets assigned to \(m' = 6\).) More generally, suppose the even \(m\) is divisible by 2\(^k\) but not by 2\(^{k+1}\), where \(k\) is an integer. Then, if \(k\) is even, assign \(m' = 2m\); and if \(k\) is odd, assign \(m' = m/2\). (So \(m = 12 = 2^2 \times 3\) gets assigned to \(m' = 2^1 \times 3 = 24\). And, in reverse, \(m = 24 = 2^3 \times 3\) gets assigned to \(m' = 2^2 / 2 = 12\).

Thus for odd \(m\) the following sequence of pairs (twins) hold:

\((m, p, u)\) and \((2m, p, u)\) are twins at bases of different towers having \(\lambda\)s of opposite signs,

\(2(2m, p, u)\) and \((2^2m, p, u)\) are twins at bases of different towers having \(\lambda\)s of opposite signs,

\(2(2^2m, p, u)\) and \((2^3m, p, u)\) are twins at bases of different towers having \(\lambda\)s of opposite signs,

and so on.

**Case 3:** Now consider the case where the triad describes a prime number, that is, it takes the form \((1, p, 1)\). For the moment, suppose this prime number is not 2. In this case, where \(m = u = 1\), we simply assign \(m' = 2\). Clearly, \(\Lambda(2, p, 1) = -\Lambda(1, p, 1)\), and the numbers \((2, p, 1)\) and \((1, p, 1)\) are at the bases of different towers.

**Case 4:** Finally, consider the case where the prime number is 2, that is, the integer \((1, 2, 1)\), for which \(\Lambda(1, 2, 1) = -1\). We match this prime to the integer 1. By definition \(\lambda(1) = \Lambda(1, 1, 1) = 1\). Thus the first two integers have opposite signs for their values of \(\lambda\). \(\square\)

So, in partitioning the entire set of positive integers, the number of towers that begin with integers for which \(\lambda = -1\) is exactly equal to those that begin with integers for which \(\lambda = +1\).

The consequences of the above theorem is that each integer has a unique twin whose \(\lambda\) value is of the opposite sign. This is because if the base of two towers
are twins the next higher number in the first tower is the twin of the next higher number in the second tower, and so on.

Theorem 2 has the immediate implication that, we believe, has never been established to date (see Borwein et al (2006)):

**Theorem 3:** In the set of all positive integers, for every integer which has an even number of primes in its factorization there is another unique integer (its twin) whose factorization has an odd number of factors.

In the first part of this section we detailed Littlewood’s proof of his criteria (Theorem 1) for the condition that must hold for a function such as $F(s)$, to be analytically continuable to the line $\text{Re}(s) > a$, when it is known that it is analytic in the region $\text{Re}(s) > 1$. The need to do this was because we wished to use his criteria for our function $F(s)$ which is given in the form (3.10) or (3.15) above. So we have to find whether $L(N) \to N^a$, as $N \to \infty$ the crucial value is the exponent $a$ which needs to be found, (of course R.H predicts $a = 1/2$, but this has to be confirmed). In other words we need to study the growth of $L(N)$ as $N$ becomes very large.

In this section we also derived two more theorems, viz Theorem 2 and Theorem 3, which we will be needing to find the value of this exponent $a$.

We are nearing the final leg of our journey. However, to come to a proper conclusion regarding the Riemann Hypothesis we need some more investigations to determine the proper behavior of $L(N)$ in order to find the true value of the exponent $a$ and thereby settle the Riemann Hypothesis. Hence this will be our task in the next sections.

### 6. Behavior of $L(N)$ for large $N$ and the Riemann Hypothesis

Now we will investigate the properties of the summatory Liouville function $L(N) = \sum_{n=1}^{N} \lambda(n)$.

We perform the investigation under two separate (exclusive cases) Case 1, when the sequence is expected to be non-random and Case 2, when the sequence is expected to be random. We are forced to resort to this because the literature is ambivalent about the random nature of prime numbers and prime factorization. And as there is no telling ab initio whether the Liouville sequence behaves like Case 1 or as Case 2. So we investigate both the cases and report our findings.

(a) **CASE 1:** When the behavior of the $\lambda$ sequence is non-random

We need to define precisely what we mean by a ‘non-random behavior’ so that we can proceed with our mathematical investigation. Here we follow the criteria of many computer scientists, who believe that a sequence can be considered truly random if the sequence does not repeat and any sequence that repeats no matter how much later, should not be considered random. (See Press et al or Donald Knuth) We will adopt this definition because it seems appropriate and precisely definable. So in this Subsection we assume that the sequence of lambdas is inherently cyclic and repeats after some $\sigma$ consecutive numbers. Our first step is to verify if this

\[ \text{Eqs.}(5.22),(5.23) \text{and}(5.23b) \text{are equivalent, so while talking about the behavior of G(N) for large N we were actually talking about the behavior of L(N)} \]

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scenario is really possible at all. We do this by using the fact that every \( \lambda(n) \) can be calculated if we know which tower the particular number \( n \) belongs to. Then we check out if two numbers belonging to the sequence but occurring in different cycles can really be members of the same tower (we are really checking for consistency as the foregoing will clarify). After that we estimate the contribution to \( L(N) \) from such a hypothetical sequence and then examine the consequences.

A non-random sequence is a sequence of \( \sigma \) consecutive integers \( \{n_1, n_2, n_3, \ldots, n_\sigma\} \), which are such that the related sequence generated by the Liouville function viz:

\[
\lambda(n_1), \lambda(n_2), \lambda(n_3), \ldots, \lambda(n_\sigma)
\]

repeats. Since the integers are consecutive \( n_{j+1} = n_j + 1 \) and the \( \lambda \) sequence repeats:

\[
\lambda(n_1 + k\sigma) = \lambda(n_1); \lambda(n_2 + k\sigma) = \lambda(n_2); \ldots; \lambda(n_{\sigma-1} + k\sigma) = \lambda(n_{\sigma-1}).
\]

for ever positive integer \( k \), \((k > 1)\).

Now if we denote

\[
\lambda(n_1), \equiv w_1, \lambda(n_2), \equiv w_2, \lambda(n_3), \equiv w_3, \ldots \ldots, \lambda(n_\sigma), \equiv w_\sigma.
\]

for all integer \( k = 1, 2, \ldots, \infty \), therefore the entire sequence after \( n_1 \), has the form:

\[
w_1, w_2, w_3, \ldots, w_\sigma, w_1, w_2, w_3, \ldots, w_\sigma, w_1, w_2, w_3, \ldots, w_\sigma,
\]

\[
, w_1, w_2, w_3, \ldots, w_\sigma, w_1, w_2, w_3, \ldots, w_\sigma, \ldots, \ldots, repeating upto \infty
\]

We then call the set of consecutive numbers \( \nu_\sigma = \{w_1, w_2, w_3, \ldots, w_\sigma\} \), these are of course a collection of +1’s and -1’s in random order.

And in (4) we call the first \( \sigma \) numbers as the first cycle, the next \( \sigma \) numbers as the 2nd cycle and so on, of course the overall sequence has a period \( \sigma \).

\[\text{a.1 Investigation:}\]

We will now investigate the conditions under which a tower \( P_{m, p, u} \) can generate numbers which belong to \( \nu_\sigma \), (note we do not assume a single tower can generate all numbers in the cycle).

Let us suppose \( \nu_\sigma \) as defined above, is really such a sequence.

Let us take some prime \( p \) which belongs to a tower \( (m, p, u) \), and consider the terms in the tower of Class I integers

\[
mp^2u, mp^3u, \ldots, mp^\rho u, \ldots, mp^\mu u, \ldots
\]

† Actually this detailed investigation (consistency test), can be skipped at first reading, because all it concludes that for any \( n \) belonging to a cycle the probability of \( \lambda(n) = +1 \) is exactly equal to the probability of \( \lambda(n) = -1 \). But this already follows from Theorem 3 which is for the general case, so if \( \sigma \) is finite this conclusion is inescapable and one can skip all this and goto the paragraph ‘Final Conclusion’ preceding eq. 6.11. The only reason why the author has included this subsection in this paper is because of the curious appearance of Fermat’s Little Theorem into the argument to satisfy consistency.

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Now since $\sigma$ is finite it is obvious that $mp^\rho u$ must be a number somewhere in some cycle let us say it is equal to the $w_{i+1}$th term in the $M^{th}$ cycle. But the integer value of the $w_{i+1}$th term is: $n_1 + M\sigma + i$, therefore we must have the value of $mp^\rho u$ equal the former:

$$mp^\rho u = n_1 + M\sigma + i$$ (6.6)

Of course by the very definition of the sequence and its periodicity we will have

$$\lambda(mp^\rho u) = \lambda(n_1 + M\sigma + i) = \lambda(n_1 + i) = w_{i+1}$$ (6.7)

Now the list of numbers in (5) are infinite but the cycle length $\sigma$ is finite; also all the numbers in (5) occur as a term in some cycle, so there is bound to be some cycle $N$ where the term $mp^\rho u$, where $\mu > \rho$ again is equal to the $w_{i+1}$th term:

so we then have:

$$mp^\mu u = n_1 + N\sigma + i$$ (6.8)

Subtracting (7) from (8) we have the following requirement for consistency:

$$mu(p^\mu - p^\rho) = (N - M)\sigma$$ (6.9)

There is no loss in generality if we assume, for the time being, the length of a cycle is a prime, so let us put $q \equiv \sigma$, and $n \equiv (N - M)$ and $\nu = \mu - \rho$ we then have from (9):

$$mp^\rho(p^\nu - 1) = nq$$ (6.10)

But we have assumed $p$ to be an arbitrary prime and equation (6.10) tells us that ALL primes $p$ are such that there exists two exponents $(\rho, \nu)$ such that they are multiples of a fixed prime $q$. At first sight this seems to be a rather remote possibility, but we will now prove that this will actually happen not for just some rare primes but actually for every prime $p$ and any given prime $q$ which we consider as fixed.

This consequence is due to Fermat’s Little theorem. There is no loss if we consider that $p$ and $q$ are prime to each (otherwise we can write $q = q'g$, and the gcd $g$ can be absorbed in $n$).

Fermat’s Little theorem states that under these conditions we can always choose $\nu = q - 1$ then ($p^q - 1$) will always be divisible by $q$, this permits us to divide (6.10) out by $q$ then choose $n$ such that the consistency test (6.10) is satisfied.

So we have arrived at our following Theorem.

**Theorem:** Given a repeating sequence of type (6.4) with fixed periodicity, $\sigma$, then every tower of type $(m, p, u)$, for every prime $p$, is contained in the sequence in the sense given by eq (6.8).

Now we argue using the following facts:
(i) the number of towers are infinite and
(ii) the length $\sigma$ is finite, further we know that

$\dagger$ if $\sigma$ is not a multiple of a prime say $\sigma = bq$, $b$ an integer, we can just define $n$ as $n = b(N - M)$, so the eq. following has the same form.

---

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(iii) the sequence (6.4) contains identical cycles each of which is actually repetitions of:
\( \{w_1, w_2, \ldots, w_{\sigma}\} \equiv \lambda(n_1), \lambda(n_2), \ldots, \lambda(n_{\sigma}) \).

But we have already proved (Theorem 3) that
(iv) the total number of \( \lambda \)'s which are equal to +1 is exactly equal to the total number which are equal to -1. This means that
(v) the number of +1's within \( \lambda(n_1), \lambda(n_2), \ldots, \lambda(n_{\sigma}) \) is exactly equal to the number of -1's, but we have proved that
(vi) each of these members can be generated by some or other tower of type \( (m, p, u) \), but
(vii) these towers are infinite in number and the members belonging to a cycle are only \( \sigma \) (finite) in number.

**Final Conclusion:** Therefore the only minimal conclusion that we can come to is that given some fixed number \( n_j \) belonging to a sequence of consecutive integers \( \{n_1, n_2, n_3, \ldots, n_{\sigma}\} \) the chances of \( \lambda(n_j) \) being equal to +1 is exactly equal to the probability that \( \lambda(n_j) \) is equal to -1. And this is true for all members \( n_j \) in the cycle and since we have assumed that the whole sequence is cyclic, this statement is true for every integer \( n \). (Except of course, for the finite number which are less than \( n_1 \) - the beginning of the sequence. We ignore these because they are only finite in number).

So we can then assume as we compute

\[
L(N) = \sum_{n=1}^{N} \lambda(n) \tag{6.11}
\]

starting from \( N = 1, 2, \ldots \) up to \( N \), step by step, then the answer that we will get is like computing a sum of \( N \) random variables defined as:

\[
L(N) = \sum_{i=1}^{N} X(i) \tag{6.12}
\]

Where \( X \) is a random variable and its \( i \)th value takes on the values +1 or -1 with equal probability. This is exactly the same as summing up to \( N \), however since we are dealing with a cyclic sequences whose \( \lambda \) values have exactly \( \sigma/2 \) 's and -1's within one cycle. Therefore if we sum up to \( N = k.\sigma \), we would get \( L(N) \) as exactly zero (ie if we stop summing \( N \) exactly after \( k \) cycles), but this will not happen often so we would usually find that after summing to a random \( N \) a value which varies from 0 to \( \sigma/2 \), since every cycle has \( \sigma/2 \) +1's and \( \sigma/2 \) -1's, in random order within the length \( \sigma/2 \), therefore if you add consecutive values of say \( M \) terms you will get a minimum value of 0 when all of them cancel and a maximum value of \( \sigma/2 \), if all the +1 occur together in the first half of a cycle and also \( M = k\sigma + \sigma/2 \), \( k \): an integer.

\[
|L(N)| = \sigma/2 \quad \text{(maximum)} \tag{6.13}
\]

Though we have gone about deriving the above in a rather long-winded manner, this final result is not surprising because it is what one would expect to get when
we sum a cyclic sequence which has equal number of +1’s and −1’s terms in one cycle (albeit randomly distributed within the length σ).

But now if this value is substituted to Littlewood’s condition, and we take the limit \( L(N)/N = (\sigma/2)/NasN \to \infty \) we get zero. This means that we should be able to analytically continue \( F(s) \sim \zeta(2s)/\zeta(s) \) left-words from \( \text{Re}(s) = 1 \) to \( \text{Re}(s) = 0 \). But it is common knowledge (see Hardy) that there are very many zeros at \( \text{Re}(s) = 1/2 \) and these will appear as poles in \( F(s) \), hence we really cannot continue \( F(s) \) to \( s = 0 \), this fact indicates that the period \( \sigma \) cannot be a finite number. In other words it is the nature of the number system that a sequence of consecutive lambdas can never be cyclic with a finite sequence of length \( \sigma \).

So in this section we have concluded the only viable case to be examined is the case when the sequence of lambdas is noncyclic or “random”. This is done in the next section.

**7. Case 2: When the sequence is non-cyclic**

We now again consider the series \( L(N) = \sum_{n=1}^{N} \lambda(n) \)

It has already been proved in this paper that, over the set of all positive integers, the respective probabilities that an integer \( n \) has an odd or even number of prime factors are equal. That is, the probability of \( \lambda(n) \) being +1 or -1 are both 1/2. So,

\[
L(N) = \sum_{n=1}^{N} X_n \tag{7.14}
\]

where \( X_n \), can with equal probability, be either +1 or -1. If, in addition, it is assumed that the probabilities affecting the values of \( X_i \) and \( X_j, i \neq j \), are independent of each other, then we could deduce

\[
|L(N)| = \sqrt{N} \ as N \to \infty \tag{7.15}
\]

and then through an application of Littlewood’s argument, RH can be proved. We will show below that \( X_i \) and \( X_j \) are indeed independent random variables that are uncorrelated with each other as \( N \) tends to infinity.

\[(a) \ Independence \ of \ the \ \lambda \ values\]

We have seen in a previous section that each tower makes it appearance, while computing \( L(N) \), from \( N = 1, 2, 3, \ldots \) etc. The criteria was the tower with the base value \( N_B = (m, p_L, u) \) makes its appearance when \( N = N_B \).

Now the summation in the figure is actually performed as:

\[
L(N) = \sum_{u} \sum_{m} \sum_{p_L} \sum_{r} P_{m,p_L,u} \tag{7.16}
\]

To appreciate the relevance of the independence of the \( X_i \)’s in Eq. (7.14), consider the series:

\[
Z_T(N) = \sum_{n=1}^{N_T} \xi(n) \tag{7.17}
\]
where again \( n \) is a positive integer, and the summation (partial) is being taken over a single tower and \( \xi(n) \) is +1 or -1 depending on whether \( n \) is even or odd. Therefore

\[
Z_T(N) = +1 - 1 + 1 - 1... \tag{7.18}
\]

It is evident that the probabilities that \( \xi(n) \) is +1 or -1 are equal, as was the case with \( \lambda(n) \). However, here for any \( N \), \( Z_T(N) \) can either be 0 or 1, (we assume for convenience the base value of the tower has \( \lambda(N_B) = 1 \) else the oscillations will be between 0 and -1) and certainly does not scale as \( \sqrt{N} \). This is because the \( \xi \)'s are perfectly correlated, and we are certain that if \( \xi(i) \) is 1 then \( \xi(i+1) = -1 \) and vice versa. Here we see that the correlation interval, which we will define as the shortest distance along the ordered set of integers for which by knowing one value of a random variable we can predict (or predict with better than even probability) the value of another. To invalidate the equivalence of (7.15), it matters little if the correlation interval is 1 or 100, or \( 10^{10} \), as any finite correlation interval would negate the independence of the random numbers making up the sequence in (7.17) or (7.16).

We have earlier demonstrated that all positive integers can be partitioned into subsets \( P_{mLu} \) such that

\[
P_{mLu} = \begin{cases} 
\{ m.p_L^1.u, m.p_L^2.u, m.p_L^3.u, ..., m.p_L^{k_L}.u, m.p_L^{k_L+1}.u, \ldots \} & \text{if } u = 1 \\
\{ m.p_L^1.u, m.p_L^2.u, m.p_L^3.u, ..., m.p_L^k.u, m.p_L^{k+1}.u, \ldots \} & \text{if } u \neq 1
\end{cases} \tag{7.19}
\]

where the prime factorisation of \( n = p_1^{e_1}p_2^{e_2}p_3^{e_3}...p_L^{e_L}p_s \)

with \( p_1 < p_2 < p_3 < \ldots < p_L < p_j < p_s \), is used to define \( m \equiv p_1^{e_1}p_2^{e_2}p_3^{e_3}...p_L^{e_L} \) and \( u \equiv p_jp_s \),

where \( p_L \) is the highest prime in the factorization with multiplicity greater than one. We have also shown that every consecutive pair in either of these types of subsets are “twins” with opposite values, and thus are perfectly correlated — by knowing the value of one we know the value of the other.

Now consider the correlation interval of a pair of twins \( (m.p_L^k.u, m.p_L^{k+1}.u) \) in an arbitrary subset \( P_{mLu} \). We see that this is

\[
L_c = m.p_L^{k+1}.u - m.p_L^k.u = m.p_L^k.u(p_L - 1) \tag{7.20}
\]

Now consider that every subset \( P_{mLu} \), except the trivial single-element one containing the integer 1, is infinite and is ordered by \( k \) which increases to infinity. As \( p_L > 1 \) and \( m, u \geq 1 \), the correlation interval will increase without bound, as \( k \) increases, so \( L_c \to \infty \) in every subset \( P_{mLu} \). As the sum in (7.16) is over all such subsets, it follows that the correlation interval for the entire set of \( \lambda(n) \)'s tends to infinity, when \( N \to \infty \). This implies that the \( \lambda(n) \)'s are not correlated with each other and are independent random variables with equal probability of being +1 or -1, and so should sum to \( \sqrt{N} \), (see Chandrasekhar (1943)), thus we have at last established that the exponent \( a = 1/2 \). Invoking Littlewood’s Theorem (Sec.5), we deduce that \( F(s) \equiv \zeta(2s)/\zeta(s) \) is analytic in the region \( a = 1/2 < s < 1 \) which implies \( \zeta(s) \) has no zeros in the same region. But Riemann had shown by using

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symmetry arguments \[†\] that if \( \zeta(s) \) has no zeros in the latter region then it will have no zeros in the region \( 0 < s < 1/2 \), taking both these results together we are lead to the inevitable conclusion: that all the zeros can only lie on the critical line \( \text{Re}(s) = 1/2 \), thus proving the Riemann hypothesis.

(b) Why things worked out

In this section we give a reason as to why things worked so well and and we could actually prove what we have proved.

We have already seen that as \( N \) tends to infinity there will be an ever increasing number of rectangular waves, see Figure 1 (or Eq.(5.23b) for details), participating in the summatory Liouville function \( L(N) \) and each of these contributions make \( L(N) \) completely uncorrelated with \( L(N-1), L(N-2), \) etc and RH follows.

In the following we will be arguing by using an analogy from information theory first discovered by Shannon, (Shannon (1948)).

We will call a tower which contributes +1’s and −1’s to \( L(N) \) as a ‘broadcaster’ of bits and the contribution \( C(N) \), increases or decreases the collection of bits \( L(N) \) collated by the ‘Listener’ according as +1 arrives first or −1 arrives first at \( t = N \). Essentially, each tower is a broadcaster of rectangular waves of ever increasing periods. According to Shannon every sequence is information. And the ‘message’ in a sequence of bits can be said to contain information if there is at least some relationship between the present group of bits to the preceding group of bits (like words in a sentence). We show in the argument following that the nature of the towers in (5.23 b) destroy this information as \( N \) tends to infinity.

The situation is describable as follows:

(i) It has been revealed that the contributions to \( L(N) \), is coming from towers. If we keep \( N \) fixed then the contribution \( C(N) \) to the previous summed value \( L(N-1) \) is exactly equal to \( C(N) = \lambda(N) \), so \( L(N) = L(N-1) + C(N) \). But we know that the integer \( N \) is a member of a unique tower which we denote as the set \((m,p,u)\) and some member in this is exactly equal to \( N \), i.e. \( N = m.p^\rho.u \), in fact the triad \((m,p,u)\) and exponent \( \rho \) are uniquely obtained by factorizing \( N \). We then have \( C(N) = \lambda(N) = \lambda(m).\lambda(u).\lambda^\rho(p) \), which is completely determined. We had represented these contributions from a particular tower as rectangular waves in Fig 1.

(ii) The contributions from the rectangular wave associated with a fixed \((m,p,u)\) change with the sign of the positive or negative peak arriving at \( N \) and as \( \rho \) increases from 2, 3, 4, .. to \( \infty \) their duration exponentially increases thus delaying their arrival times and drastically increasing their correlation lengths. (iii) But each tower contributes to different \( N \)’s but the distance along \( N \) increases exponentially as \( \rho \) increases i.e. the arrival times at the Listener increases.

(iv) The same situation as (i), (ii) and (iii) happens for every other time \( t = N \) from 1 to infinity, while \( L(N) \) takes contributions from other towers.

\[†\] He did this first by defining an associated xi function: \( \xi(s) \equiv \Gamma(s/2)\pi^{s/2}\xi(s) \), \( \Gamma(s) \) is the Euler Gamma function, then showed that this xi function has the symmetry property \( \xi(s) = \xi(1 - s) \) which in turn implied that that the zeros of \( \zeta(s) \) (if any) which are not on the critical line will be symmetricaly placed about the point \( s=1/2 \), i.e. if \( \zeta(\frac{1}{2} + u + i\sigma) \) is a zero then \( \zeta(\frac{1}{2} - u - i\sigma) \), \( 0 < u < 1/2 \) is a zero see Whittaker and Watson page 269.
(vi) Each tower \((m, p, u)\) comes into play as soon as \(t = N\) reaches the base value of the tower i.e. \(N = n_B = m.p^2.u\) after this \(N\), this particular tower broadcasts bits in ever increasing time intervals forever.

(vii) As \(N\) tends to infinity in-numerable towers come into play and each broadcasted bit received at time \(N\) becomes completely uncorrelated to the bit received at time \(N - 1\), so \(C(N)\) has no relation to \(C(N - 1)\) essentially because \(C(N)\) and \(C(N - 1)\) come from different towers.

And according to Shannon’s information theory there will be no pattern in the bits and the entropy of the signal approaches infinity. At this point \(C(N)\) behaves like a coin toss and \(L(N)\) as the cumulative summation of \(N\) coin tosses, (see Chandrasekhar (1943)), thus we have \(\text{mod}(L(N)) = \sqrt{N}\) as \(N\) tends to infinity proving RH. So we have seen that the RH is because of inherent randomness of factorization of \(N\) into primes. QED.

8. Conclusions

In this paper we have investigated the analyticity of the Dirichlet series of the Liouville function by constructing a novel way to sum the series. The method consists in splitting the original series into an infinite sum over sub-series, each of which is convergent. It so turns out each sub-series is a rectangular function of unit amplitude but ever increasing periodicity and each along with its harmonics is associated with a prime number and all of them contribute to the summatory Liouville function and to the Zeta function. A number of arithmetical properties of numbers played a role in the proof of our main theorem, these were: the fact that each number can be uniquely factorized and then placed in an exclusive subset, where it and its other members form an increasing sequence and factorize alternately into odd and even factors; and each subset can be labelled uniquely using a triad of integers which in their turn can be used to determine all the integers which belong to the subset, even Fermat’s Little Theorem played a role in our proof.

In this paper we have also demonstrated that for every integer that has an even number of primes as factors (multiplicity included), there is an integer that has an odd number of primes. This provides a proof for the long-suspected but unproved conjecture—until now—that the summatory Liouville function and therefore the Riemann Hypothesis bears an analogy with the coin-tossing problem, Denjoy had long suspected this as far back as 1931. Further, it has now been revealed that the randomness of the occurrence of prime numbers plays an important role in determining the analyticity of the Zeta function, and in establishing the Riemann Hypothesis: the Zeta function has zeros only on the critical line: \(\text{Re}(s) = 1/2\).

Truth to tell even this connection of the role of the randomness of the primes to the RH problem was long suspected and even a book called the “Music of the Primes” by Marcus du Sautoy, had appeared in 2003 (Harper Collins), I could not help but recall the title of his book when I saw the rhythms of the harmonic functions, generated by prime numbers, that are depicted in Fig 1.

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9. DEDICATION
I dedicate this paper to my teachers: Mr John Wright of Bishop’s School Poona, Prof. S.C. Mookerjee of St. Aloysius’ College Jabalpur, Prof. P.M. Mathews of University of Madras, Mr. D.S.M. Vishnu of BHEL R&D Hyderabad and to my first teachers - my parents. All of them lived selfless lives and nearly all are now long gone: May they live in evermore.

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11. APPENDIX

I. Scheme of partitioning numbers into sets:

Our scheme of partitioning numbers into sets is as follows:

(a) Scheme for Class I integers:

Let us say \( n = p_1^{f_1} p_2^{f_2} \ldots p_k^{f_k} p_{L} \), then it will have at least one prime which has an exponent of 2 or above and among these there will be a largest prime \( p_{L} \) whose exponent is at least 2 or above. Such a prime will always exist for a Class I number. Then by definition the number to the right of \( p_{L} \) is either 1 or a product of primes with exponents only 1. Now multiply all the numbers to the left of \( p_{L} \) and call it \( m \) i.e. \( m = p_1^{f_1} p_2^{f_2} \ldots p_k^{f_k} \) and the product of numbers to the right of \( p_{L} \) as \( u \) i.e. \( u = p_{L} p_{T} \). Now this triad of numbers \( m, p_{L}, u \) will be used to label a set, note \( n = m.p_{L}.u \) Let us define the set \( P_{m,p_{L},u} \):

\[
P_{m,p_{L},u} = \{ m.p_{L}^{2}.u, m.p_{L}^{3}.u, m.p_{L}^{4}.u, m.p_{L}^{5}.u, \ldots \} \quad (A1)
\]

Obviously \( n = m.p_{L}^{k}.u \) which has \( k \geq 2 \) belongs to the above set. Also notice the factor involved in each number increases by a single factor of \( p_{L} \) therefore the \( \lambda \) values of each member alternate in sign:

\[
\lambda(m.p_{L}^{2}.u) = -\lambda(m.p_{L}^{3}.u) = \lambda(m.p_{L}^{4}.u) = -\lambda(m.p_{L}^{5}.u) = \ldots \quad (A2)
\]

In this paper ALL sets defined as \( P_{m,p_{L},u} \) will have the property of alternating signs of \( \lambda \) Eq. (A1). Note in the above set containing only Class I integers \( m \) will have only prime factors which are each less than \( p_{L} \).

Let us consider various integers:

Ex 1. Let us consider the integer 73573500; this is factorized as \( 2^4 3^3 5^3 \cdot 7 \cdot 11 \cdot 13 \) and since this is a Class I integer, and \( p_{L} = 7 \) because 7 is the highest prime factor whose exponent is greater than one. \( p_{L} = 7 \) so \( m = 2^4 3^3 5^3 \) and \( u = 11 \cdot 13 \cdot 143 \) and therefore 73573500 is a member of the set \( P_{1500,7,143} \)

\[
P_{1500,7,143} = \{ 1500.7^2.143, 1500.7^3.143, 1500.7^4.143, 1500.7^5.143, \ldots \}
\]

Ex 2. Now let us consider the simple integer: \( 3^4 \) this is a class I integer and belongs to \( P_{3,3,1} = \{ 3, 3^2, 3^3, 3^4, 3^5, 3^6, \ldots \} \)

Ex 3. Let us consider the integer 663 this is factorized as: \( 3 \cdot 13 \cdot 17 \) and is a Class II integer as there no exponents greater than 1, and 663 = \( 3 \cdot 13 \cdot 17 \) since 17 is the highest prime number we put this in the set:

\[
P_{39,17,1} = \{ 39.17, 39.17^2, 39.17^3, 39.17^4, \ldots \}
\]

NOTE: If a tower has a Class II integer then it will appear as the first (base) member, all other numbers will be Class I numbers.

Ex 4. Let the integer be the simple prime number 19, we write:

\[
19 \in P_{1,19,1} = \{ 19, 19^2, 19^3, 19^4, \ldots \}
\]

Ex 5. Let the integer be 4845 this is factorized as \( 3 \cdot 5 \cdot 17 \cdot 19 \) since this is a Class II integer we see \( m = 3 \cdot 5 \cdot 17 = 255 \), \( p = 19 \), \( u = 1 \) and the set which it belongs is

\[
P_{255,19,1} = \{ 255.19, 255.19^2, 255.19^3, 255.19^4, 255.19^5, \ldots \}
\]

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II. Theorems on representation of integers and their partitioning into sets.

**Theorem A:** Two different integers cannot have the same triad \((m, p^k, u)\)

Let \(a\) and \(b\) be two integers which when factorized according to our convention as: \(a = n.q^g.v\) and \(b = n'.q^{g'}v'\) (let us consider only Class I integers, \(u, v\) and \(v'\) are all > 1)

If they are both equal to the same triad (say) \((m, p^k, u)\). Then \(m.p^k.u = n.q^g.v = n'.q^{g'}v'\). Consider the first two equalities \(m.p^k.u = n.q^g.v\) this means \(p\) is the largest prime with \(k > 1\), on the l.h.s. Similarly \(q\) is the largest prime with exponent \(g > 1\) on the r.h.s. Now if \(p > q\) this means \(p^k\) must divide \(v\), but this cannot happen since \(v\) cannot contain a prime greater than \(q\) with an exponent \(k > 1\). Now if \(p < q\) then \(q^g\) must divide \(u\) but this again cannot happen since \(u\) cannot contain an exponent \(g > 1\). So we see \(p = q\), and \(k = g\). But once again unique factorization would imply since \(u\) contains all prime factors above \(p\) and \(v\) must contain only prime factors above \(q\) (= \(p\)) the only possibility is \(u = v\) but this also makes \(m = n\) (That is the triad of \(a\) is \((m, p^k, u)\)). Similarly equating the second and third equalities \(n.q^g.v = n'.q^{g'}v'\) and using similar arguments we see \(n = n', q = q', v = v'\) that is \(a = b\). The same logic can be used to prove the theorem for class II integers when \(u = v = v' = 1\) QED

**Theorem B:** Two different triads cannot represent the same integer.

If there are two triads \((m, p^e, u)\) and \((m', r^e, u')\) and represent the same integer say \(a\) which can be factorized as \(a = n.q^e.v\). Where the factorization is done as per our rules then we must have \(m.p^e.u = n.q^e.v\) by using exactly similar arguments as above (in Theorem A) we conclude that we must have \(m = n, p = q, e = g\) and \(u = v\); similarly imposing the condition on the second triad \(m'.r^e.u' = n.q^e.v\), we conclude \(m' = n, r = q, s = g\) and \(u' = v\); Thus obtaining \(m = m', p = r, e = s = u'\) this means the two triads are actually identical QED