STRICT PARTITIONS OF MAXIMAL PROJECTIVE DEGREE

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Abstract. The projective degrees of strict partitions of \( n \) were computed for all \( n \leq 100 \) and the partitions with maximal projective degree were found for each \( n \). It was observed that maximizing partitions for successive values of \( n \) “lie close to each other” in a certain sense. Conjecturing that this holds for larger values of \( n \), the partitions of maximal degree were computed for all \( n \leq 220 \). The results are consistent with a recent conjecture on the limiting shape of the strict partition of maximal projective degree.

1. Introduction

Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition of \( n \), denoted as usual by \( \lambda \vdash n \). Let \( f^\lambda \) denote the number of standard tableaux of shape \( \lambda \). \( f^\lambda \) is also the number of paths in the Young graph \( Y \) from the root \((1)\) to \( \lambda \), and it is also the degree of the irreducible character \( \chi^\lambda \) of the symmetric group \( S_n \).

A partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \) is said to be strict if \( \lambda_1 > \lambda_2 > \cdots > \lambda_r > 0 \) for some \( r \). In that case we write \( \lambda \models n \). Let \( SY \) be the subgraph of the Young graph \( Y \) formed by the strict partitions. If \( \lambda \models n \), let \( g^\lambda \) denote the number of paths in \( SY \) from the root \((1)\) to \( \lambda \). According to a theorem by Schur, \( g^\lambda \) is the degree of the projective representation of \( S_n \) corresponding to \( \lambda \). It is also the number of standard young tableaux of shifted shape \( \lambda \).

Vershik and Kerov [6, 7] have determined the asymptotic shape of the partition \( \lambda \) that maximizes \( f^\lambda \) as \( |\lambda| = n \to \infty \). They have also shown that the same shape is also the asymptotic expected shape of a random partition with respect to the Plancherel measure. The latter result was reached independently by Logan and Shepp [4] as well. Through the Robinson-Schensted algorithm, the expected shape relates to the expected length of the longest increasing subsequence in a random permutation. For some recent developments related to this problem and the probability distributions involved, see [1, 2, 5].

More precisely, given the Young diagram of a partition \( \lambda \vdash n \) where each box is \( 1 \times 1 \), shrink it along both axes by a factor of \( \sqrt{n} \) to obtain the re-scaled diagram \( \bar{\lambda} \) of total area \( 1 \). For each \( n \), let \( \lambda_{f^\lambda \text{max}}^{(n)} \) be a partition \( \lambda \vdash n \) with maximal \( f^\lambda \), that is, \( f^\lambda_{f^\lambda \text{max}}^{(n)} = \max \{ f^\nu \mid \nu \vdash n \} \). Through slight abuse of notation, where the maximizing partition is not unique for a given \( n \), we shall take \( \lambda_{f^\lambda \text{max}}^{(n)} \) to read “any \( \lambda \vdash n \) of maximal degree”.

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Theorem 1 ([6, 7]). The limit shape as $n \to \infty$ of the re-scaled diagrams $\bar{\lambda}_{f_{\text{max}}}^{(n)}$ exists, and is given by the two axes and by the parametric curve

$$\begin{align*}
    x &= \left(\frac{3}{\pi}\right) (\sin \Theta - \Theta \cos \Theta) + 2 \cos \Theta, \\
    y &= -\left(\frac{2}{\pi}\right) (\sin \Theta - \Theta \cos \Theta),
\end{align*}$$

$0 \leq \Theta \leq \pi$.

Figure 1

The Vershik-Kerov limit shape of Theorem 1 is shown in Figure 1. The problem of determining the asymptotic shape of the partition $\lambda$ which maximizes $g^\lambda$ remains unsolved, and we are unaware of even partial characterizations of the shape. However, recently the following was conjectured.

Conjecture 2 ([3, Conjecture 8.2]). The limit shape $\lambda^*$ of the $\lambda \models n$ maximizing $2^{n-\ell(\lambda)} (g^\lambda)^2$ — and possibly maximizing $g^\lambda$ — is given by the two axes and by the parametric curve

$$\begin{align*}
    x &= 2\sqrt{2} \cos \Theta, \\
    y &= \left(\frac{2\sqrt{2}}{\pi}\right) (\Theta \cos \Theta - \sin \Theta),
\end{align*}$$

$0 \leq \Theta \leq \frac{\pi}{2}$.

Figure 2

The conjectured limit shape of Conjecture 2 is shown in Figure 2. It was obtained from the Vershik-Kerov shape by bisecting it along the line $y = -x$, taking
the upper half, dilating it (so that its area become 1) and applying the shearing
transformation \((x, y) \mapsto (x + y, y)\) (to bring the line \(y = -x\) to the y axis).

In the next section, we give the results of computing the partition maximizing
\(g^\lambda\) over all \(\lambda \models n\) for \(1 \leq n \leq 100\). We observe a property of successive maximizing
partitions in the range \(1 \leq n \leq 100\), and conjecture that it holds for all \(n\). Assuming
the conjecture, we compute the maximizing partitions for \(100 < n \leq 220\). Our
results are consistent with Conjecture 2 for both \(g^\lambda\) and \(2^{-\ell(\lambda)} (g^\lambda)^2\).

2. Results for \(n \leq 100\)

To compute \(g^\lambda\), we used the following formula, due to Schur.

**Theorem 3.** Let \(\lambda = (\lambda_1, \ldots, \lambda_r) \models n\). Then
\[
g^\lambda = \frac{n!}{\lambda_1! \cdots \lambda_r!} \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}
\]

All computations were done with Mathematica.

For \(1 \leq n \leq 100\), \(g^\lambda\) was computed for every \(\lambda \models n\) and the partitions attaining
the maximal value for each \(n\) were identified.

**Observation 4** (Uniqueness of the maximum). For every \(n \in [100] \setminus \{3, 11\}\), there
exists a partition \(\lambda = \lambda_{g_{\text{max}}}^{(n)} \models n\) such that \(g^\lambda > g^\mu\) for all \(\lambda \neq \mu \models n\).

**Observation 5.**
\[
\lambda_{g_{\text{max}}}^{(100)} = (24, 20, 16, 13, 10, 8, 5, 3, 1).
\]

Figure 3 shows the normalized diagram \(\tilde{\lambda}_{g_{\text{max}}}^{(100)}\) overlaid with the conjectured limit
shape.

Recall that by Conjecture 2 the partitions maximizing \(g^\lambda\) are asymptotically
equal to the partitions maximizing \(2^{-\ell(\lambda)} (g^\lambda)^2\). For all strict partitions \(\lambda\) of \(1 \leq n \leq 100\), \(2^{-\ell(\lambda)} (g^\lambda)^2\) was computed and the partition maximizing it was denoted
\(\lambda_{2g_{\text{max}}}^{(n)}\). Comparing the results with the results for \(g^\lambda\), the following was observed.

**Observation 6.** For all
\[
n \in \{1, 2, \ldots, 100\} \setminus \{3, 8, 16, 25, 26, 38, 51, 52, 53, 54, 69, 70, 88, 89, 90, 91\},
\]
\[
\lambda_{g_{\text{max}}}^{(n)} = \lambda_{2g_{\text{max}}}^{(n)}.
\]
Definition 7 \((d\text{-successor})\). Let \(\mu = (\mu_1, \ldots, \mu_r) \models n\) and let \(d \in \mathbb{N}\). The \(d\)-successors of \(\mu\) are the elements of the set

\[
N(\mu, d) := \{ \lambda = (\lambda_1, \ldots, \lambda_s) \models n + 1 \mid |\lambda_i - \mu_i| \leq d \quad 1 \leq i \leq s \}
\]

We have observed the following.

Observation 8. For \(1 \leq n < 100\), \(\lambda_{g \text{max}}^{(n+1)} \in N(\lambda_{g \text{max}}^n, 1)\).

When \(\lambda_{g \text{max}}^{(n)}\) is not unique, read the above to mean “every \(\lambda \models n + 1\) of maximal projective degree is a 1-successor of every \(\lambda \models n\) of maximal degree”.

3. A Conjecture and Results for \(n \leq 220\)

Based on the above observation, we conjecture the following.

Conjecture 9 (Maximizers are successors to a maximizers). For all \(n\), if \(\lambda_{g \text{max}}^{(n+1)} \in N(\lambda_{g \text{max}}^n, 1)\).

Assuming that the conjecture holds, \(\lambda_{g \text{max}}^{(n+1)}\) was computed for \(100 \leq n < 220\) as follows: starting with \(\lambda = \lambda_{g \text{max}}^{(n)}\), for every \(\mu \in N(\lambda, 1)\), the ratio \(\frac{\mu}{g}\) was computed and the \(\mu\) maximizing the ratio was selected.

Observation 10. If Conjecture 9 holds for all \(n < 220\), then

\[\lambda_{g \text{max}}^{(220)} = (37, 32, 28, 24, 21, 18, 16, 13, 11, 8, 6, 4, 2)\].

Figure 4 shows the normalized diagram \(\lambda_{g \text{max}}^{(220)}\) overlaid with the conjectured limit shape.

Figure 5 shows the ratio \(\frac{\lambda_{g \text{max}}^{(n+1)}}{\ell(\lambda_{g \text{max}}^n, 1)}\) for \(1 \leq n \leq 220\), where the values above \(n = 100\) are based on Conjecture 9. According to Conjecture 2, the ratio at the limit is \(\pi\).
REFERENCES

[1] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999.
[2] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the second row of a Young diagram under Plancherel measure. *Geom. Funct. Anal.*, 10(4):702–731, 2000.
[3] D. Bernstein, A. Henke, and A. Regev. Maximal projective degrees for strict partitions. Preprint, 2006.
[4] B. F. Logan and L. A. Shepp. A variational problem for random Young tableaux. *Advances in Math.*, 26(2):206–222, 1977.
[5] R. P. Stanley. Recent progress in algebraic combinatorics. *Bull. Amer. Math. Soc. (N.S.)*, 40(1):55–68 (electronic), 2003. Mathematical challenges of the 21st century (Los Angeles, CA, 2000).
[6] A. M. Vershik and S. V. Kerov. Asymptotics of the Plancherel measure of the symmetric group and the limit form of young tableaux. *Dokl. Akad. Nauk SSSR*, 233:1024–1027, 1977.
[7] A. M. Vershik and S. V. Kerov. Asymptotic behavior of the maximum and generic dimensions of irreducible representations of the symmetric group. *Funktsional. Anal. i Prilozhen.*, 19(1):25–36, 96, 1985.

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