A subgeometric convergence formula for finite-level M/G/1-type Markov chains via the Poisson equation of the deviation matrix

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Abstract

This paper considers finite-level M/G/1-type Markov chains. We introduce the fundamental deviation matrix of the infinite-level M/G/1-type Markov chain, which is a solution of the Poisson equation that the deviation matrix satisfies. With the fundamental deviation matrix, we describe a difference formula for the respective stationary distributions of the finite-level chain and its infinite-level limit. From the difference formula, we derive a subgeometric convergence formula for the stationary distribution of the finite-level chain as its maximum level goes to infinity. Using the obtained formula, we show an asymptotic formula for the loss probability in the MAP/G/1/N + 1 queue.

Keywords: Finite-level M/G/1-type Markov chain; Subgeometric convergence; Poisson equation; Deviation matrix; MAP/G/1/N + 1 queue; Loss probability

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1 Introduction

This paper considers finite-level M/G/1-type Markov chains, which belong to a special class of upper block-Hessenberg Markov chains. Finite-level M/G/1-type Markov chains appear in the analysis of finite semi-Markovian queues (see, e.g., [4, 5, 8, 12]). Except for a few special cases [1, 15], the stationary distribution of the finite-level M/G/1-type Markov chain does not have any simple and analytical expression. Hence, several researchers have derived approximate and/or asymptotic formulae for the stationary distribution of the finite-level M/G/1-type Markov chain and related ones, such as finite-level GI/M/1-type Markov chains and finite-level quasi-birth-and-death processes (QBDs).

Miyazawa et al. [28] present an asymptotic formula for the stationary probability of the finite-level QBD being in the maximum level. Using the asymptotic formula, they also investigate an

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asymptotic behavior of the loss probability of a MAP/MSP/c/1 + c queue. J. Kim and B. Kim [16] extend the asymptotic formula in [28] to the finite-level GI/M/1-type Markov chain. Ishizaki and Takine [13] consider a special finite-level M/G/1-type Markov chain with level-decreasing jumps governed by a block matrix of rank one, and show a direct relation of such a chain to its infinite-level version. Baiocchi [4] derives a geometric asymptotic formula for the loss probability in a MAP/G/1/K queue, through the analysis of a finite-level M/G/1-type Markov chain with light-tailed level increments. Liu and Zhao [20] present power-law asymptotic formulas for the loss probability in an M/G/1/N queue with vacations, where the embedded queue length process is a special finite-level M/G/1-type Markov chain with a single background state.

The main contribution of this paper is to present a subgeometric convergence formula for the stationary distribution of the finite-level M/G/1-type Markov chain in the *infinite-level limit*; that is, in the limit as the maximum level goes to infinity. Note that the infinite-level limit of a finite-level M/G/1-type Markov chain is the infinite-level (and thus ordinary) M/G/1-type Markov chain. For simplicity, we may refer to finite- and infinite-level M/G/1-type Markov chains as *finite- and infinite-level chains*, respectively.

To discuss the convergence of a finite-level chain to the infinite-level limit, we introduce the fundamental deviation matrix $H$. The matrix $H$ satisfies the Poisson equation of the deviation matrix $D$ (see, i.e., [7]). With the fundamental deviation matrix $H$, we show a difference formula for the respective stationary distributions of the finite-level chain and its infinite-level limit. We also provide the block-decomposition results of $H$ and $D$. Moreover, combining the difference formula with the block-decomposition result of $H$, we derive a subgeometric convergence formula for the stationary distribution of the finite-level chain in the infinite-level limit, where the equilibrium distribution of level increments is assumed to be subexponential. Finally, using the subgeometric convergence formula, we show an asymptotic formula for the loss probability in the MAP/G/1/1 + 1 queue with the subexponential equilibrium service-time distribution.

The rest of this paper consists of five sections. Section 2 describes finite- and infinite-level M/G/1-type Markov chains. Section 3 provides basic results under the second-order moment conditions for the level increments of the infinite-level M/G/1-type Markov chain. Section 4 discusses the difference between the respective stationary distributions of the finite-level chain and its infinite-level limit, through the fundamental deviation matrix $H$. Section 5 presents the main results of this paper, which are concerned with the convergence of the stationary distribution of the finite-level chain as its maximum level goes to infinity. Section 6 considers the application of the main results to the asymptotic analysis of the loss probability in the MAP/G/1/N + 1 queue.

## 2 Model description and basic results

We introduce mathematical symbols and notation. Let

$$Z = \{0, \pm 1, \pm 2, \ldots\}, \quad Z_+ = \{0, 1, 2, \ldots\}, \quad N = \{1, 2, 3, \ldots\},$$

and

$$Z_{\geq k} = \{\ell \in Z : \ell \geq k\}, \quad k \in Z,$$

$$Z_{[k,\ell]} = \{k, k + 1, \ldots, \ell\}, \quad k, \ell \in Z, \ k \leq \ell.$$
We then define
\[ M_0 = \mathbb{Z}_{[1,M_0]} = \{1, 2, \ldots, M_0\} \subset \mathbb{N}, \quad M_1 = \mathbb{Z}_{[1,M_1]} = \{1, 2, \ldots, M_1\} \subset \mathbb{N}. \]

We also define \( x \wedge y = \min(x, y) \) for \( x, y \in (-\infty, \infty) \). In addition, for matrices (including vectors), the absolute value operator \(| \cdot |\) works on them elementwise. Let \( O \) and \( I \) denote the zero matrix and identity matrix, respectively, with appropriate sizes (i.e., with appropriate numbers of rows and columns). Finally, let \( \mathbb{I}(\cdot) \) denote an indicator function that takes value of one if the statement in the parentheses is true; otherwise takes the value of zero.

### 2.1 Infinite-level M/G/1-type Markov chains

Let \( \{ (X_\nu, J_\nu); \nu \in \mathbb{Z}_+ \} \) denote a discrete-time Markov chain on state space \( \mathbb{F} := \bigcup_{k=0}^{\infty} \mathbb{L}_k \), where \( \mathbb{L}_k = \{k\} \times \mathbb{M}_{k+1} \) for \( k \in \mathbb{Z}_+ \). Furthermore, let \( P \) denote the transition probability matrix of the Markov chain \( \{ (X_\nu, J_\nu) \} \), which is in the form:

\[
P = \begin{pmatrix}
\mathbb{L}_0 & \mathbb{L}_1 & \mathbb{L}_2 & \mathbb{L}_3 & \cdots \\
\mathbb{L}_0 & B(0) & B(1) & B(2) & \cdots \\
\mathbb{L}_1 & B(-1) & A(0) & A(1) & \cdots \\
\mathbb{L}_2 & O & A(-1) & A(0) & \cdots \\
\mathbb{L}_3 & O & O & A(-1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(2.1)

By definition,
\[
\sum_{k=-1}^{\infty} A(k)e = e, \quad (2.2)
\]
\[
\sum_{k=0}^{\infty} B(k)e = e, \quad (2.3)
\]
\[
B(-1)e = A(-1)e, \quad (2.4)
\]

where \( e \) denotes the column vector of ones with an appropriate dimension. The Markov chain \( \{ (X_\nu, J_\nu) \} \) is referred to as an infinite-level M/G/1-type Markov chain or M/G/1-type Markov chain, for short (see [29]). The subset \( \mathbb{L}_k \) of state space \( \mathbb{F} \) is referred to as level \( k \). For later use, we define \( \mathbb{L}_{\geq k} \) and \( \mathbb{L}_{\leq k} \), \( k \in \mathbb{Z}_+ \), as

\[
\mathbb{L}_{\geq k} = \bigcup_{\ell=k}^{\infty} \mathbb{L}_\ell, \quad \mathbb{L}_{\leq k} = \bigcup_{\ell=0}^{k} \mathbb{L}_\ell, \quad k \in \mathbb{Z}_+.
\]

We also define \( \overline{A}(k) \) and \( \overline{B}(k) \) as

\[
\overline{A}(k) = \sum_{\ell=k+1}^{\infty} A(\ell), \quad k \in \mathbb{Z}_{\geq -2}, \quad (2.5)
\]
\[
\overline{B}(k) = \sum_{\ell=k+1}^{\infty} B(\ell), \quad k \in \mathbb{Z}_+, \quad (2.6)
\]
It thus follows from (2.2) and (2.3) that
\[
\lim_{k \to \infty} \overline{A}(k) = O, \quad (2.7)
\]
\[
\lim_{k \to \infty} \overline{B}(k) = O. \quad (2.8)
\]

It should be noted that \( P \) does not always have a stationary distribution. Assumption 2.1 below ensures (see, e.g., [3, Chapter XI, Proposition 3.1]) that \( \{(X_\nu, J_\nu)\} \) is ergodic (i.e., irreducible, aperiodic and positive recurrent) and thus has an unique stationary distribution, denoted by \( \pi = (\pi(k, i))_{(k, i) \in \mathbb{F}}. \)

Assumption 2.1 (i) The transition probability matrix \( P \) is irreducible and aperiodic; (ii) \( A \) is irreducible; (iii) \( \sum_{k=1}^{\infty} k B(k) e < \infty \); and (iv) \( \sigma := \varpi \sum_{k=-1}^{\infty} k A(k) e < 0 \), where \( \varpi \) denotes a unique stationary distribution of \( A \).

To describe the stationary distribution \( \pi \), we introduce the \( G \) - and \( R \)-matrices of the infinite-level \( M/G/1 \)-type Markov chain. Let \( G := (G_{i,j})_{i,j \in M_1} \) denote an \( M_1 \times M_1 \) matrix such that
\[
G_{i,j} = P(J_{T_n} = j \mid (X_0, J_0) = (n + 1, i) \in \mathbb{L}_{\geq 2}),
\]
where \( T_n = \inf\{\nu \in \mathbb{N} : X_\nu = n\} \) for \( n \in \mathbb{Z}_+. \) Assumption 2.1 (ii) and (iv) ensures that \( G \) is a stochastic matrix that is the minimal nonnegative solution of the matrix equation (see [29, Eq. (2.3.3) and Theorem 2.3.1]):
\[
G = \sum_{m=-1}^{\infty} A(m)G^{m+1}.
\]
Furthermore, \( G \) has a unique closed communicating class [17, Proposition 2.1] and thus a unique stationary distribution, denoted by \( g \).

For \( k \in \mathbb{N} \), let \( R_0(k) := (R_{0, i,j}(k))_{(i,j) \in \mathbb{M}_0 \times \mathbb{M}_1} \) and \( R(k) := (R_{i,j}(k))_{(i,j) \in \mathbb{M}_1 \times \mathbb{M}_1} \) denote
\[
R_0(k) = \sum_{m=k}^{\infty} B(m)G^{m-k}(I - \Phi(0))^{-1}, \quad k \in \mathbb{N}, \quad (2.9)
\]
\[
R(k) = \sum_{m=k}^{\infty} A(m)G^{m-k}(I - \Phi(0))^{-1}, \quad k \in \mathbb{N}, \quad (2.10)
\]
respectively, where
\[
\Phi(0) = \sum_{m=0}^{\infty} A(m)G^m. \quad (2.11)
\]
Note here that \( R_0(k) \) and \( R(k) \) have the following probabilistic interpretations:
\[
R_{0, i,j}(k) = \mathbb{E}_{(0,i)} \left[ \sum_{\nu=1}^{T_{\leq k-1}} \mathbb{1}( (X_\nu, J_\nu) = (k, j) ) \right],
\]
\[
R_{i,j}(k) = \mathbb{E}_{(n,i)} \left[ \sum_{\nu=1}^{T_{\leq n+k-1}} \mathbb{1}( (X_\nu, J_\nu) = (n+k, j) ) \right] \quad \text{with } (n, i) \in \mathbb{L}_{\geq 1},
\]
where \( T_{\leq k} = \inf \{ \nu \in \mathbb{N} : X_\nu \leq k \} \) for \( k \in \mathbb{Z}_+ \), and where
\[
E_{(k,i)}[\cdot] = E[\cdot | (X_0, J_0) = (k,i)], \quad (k,i) \in \mathbb{F}.
\]

We now define \( \pi(k), k \in \mathbb{Z}_+ \), as
\[
\pi(k) = (\pi(k,1), \pi(k,2), \ldots, \pi(k, M_{k\wedge 1})), \quad k \in \mathbb{Z}_+.
\]
We then have
\[
\pi(k) = \pi(0)R_0(k) + \sum_{\ell=1}^{k-1} \pi(\ell)R(k-\ell), \quad k \in \mathbb{N},
\]
which is referred to as Ramaswami’s recursion \([30]\).

2.2 Finite-level M/G/1-type Markov chains

For \( N \in \mathbb{N} \), let \( P^{(N)} \) denote a stochastic matrix such that
\[
P^{(N)} = \begin{pmatrix}
B(0) & B(1) & B(2) & \cdots & B(N-2) & B(N-1) & B(N-1) \\
B(-1) & A(0) & A(1) & \cdots & A(N-3) & A(N-2) & A(N-2) \\
O & A(-1) & A(0) & \cdots & A(N-4) & A(N-3) & A(N-3) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
O & O & O & \cdots & A(0) & A(1) & A(1) \\
O & O & O & \cdots & A(-1) & A(0) & A(0) \\
O & O & O & \cdots & O & A(-1) & A(-1)
\end{pmatrix},
\]
where \( \tilde{A}(k) \) and \( \tilde{B}(k) \) are \( M_1 \times M_1 \) and \( M_0 \times M_1 \) substochastic matrices such that
\[
\tilde{A}(k)e = \overline{A}(k)e, \quad k \in \mathbb{Z}_{\geq -1},
\]
\[
\tilde{B}(k)e = \overline{B}(k)e, \quad k \in \mathbb{Z}_+.
\]

It follows from (2.7), (2.8), (2.14) and (2.15) that
\[
\lim_{k \to \infty} \tilde{A}(k) = O, \quad \lim_{k \to \infty} \tilde{B}(k) = O,
\]
and thus (see (2.1) and (2.13))
\[
\lim_{N \to \infty} P^{(N)} = P.
\]

Clearly, the stochastic matrix \( P^{(N)} \) specifies a Markov chain. We refer to this Markov chain as a finite-level M/G/1-type Markov chain. The finite-level M/G/1-type Markov chain always has at least one stationary distribution. We define \( \pi^{(N)} := (x^{(N)}(k, i))_{(k,i) \in \mathbb{F}^{(N)}} \) as an arbitrary stationary distribution of \( P^{(N)} \), i.e.,
\[
\pi^{(N)} P^{(N)} = \pi^{(N)}, \quad \pi^{(N)} e = e.
\]
Remark 2.1 If
\[ \tilde{A}(k) = \overline{A}(k), \quad k \in \mathbb{Z}_{\geq -1}, \]
\[ \tilde{B}(k) = \overline{B}(k), \quad k \in \mathbb{Z}_+, \]
then \( P(N)'s, N \in \mathbb{N} \), are the last-column-block-augmented truncations of \( P \) (see [23, 24, 26, 27]).

As mentioned in the introduction, this paper discusses the difference \( \pi^{(N)} - \pi \). For this purpose, we append zeros to \( \pi^{(N)} \) and \( P^{(N)} \) (keeping their original elements in the original positions) so that they have the same sizes as those of \( \pi \) and \( P \), respectively. Therefore, the differences \( \pi^{(N)} - \pi \) and \( P^{(N)} - P \) are well-defined.

3 The second-order moment condition for level increments

In this section, we present some preliminary results under the second-order moment condition for level increments.

Assumption 3.1 (Second-order moment condition)
\[ \sum_{k=1}^{\infty} k^2 A(k) < \infty, \quad \sum_{k=1}^{\infty} k^2 B(k) < \infty. \]

Under Assumptions 2.1 and 3.1, we establish a Foster-Lyapunov-type drift condition. With the drift condition, we prove that the stationary distributions \( \pi^{(N)} \) and \( \pi \) have finite means; that is, \( \sum_{k=1}^{\infty} k \pi^{(N)}(k)e < \infty \) and \( \sum_{k=1}^{\infty} k \pi(k)e < \infty. \) We also discuss the mean first passage time to level zero. The contents of this section are related to the existence of the deviation matrix and also to the convergence of \( \{\pi^{(N)}; N \in \mathbb{N}\} \) to \( \pi \), which are discussed in the subsequent sections.

3.1 Drift condition

Let \( \alpha \) denote
\[ \alpha = (I - A + e \varpi)^{-1} (-\sigma e + \beta_A) + ce, \]
where \( c \) is an arbitrary real number, and where
\[ \beta_A = \sum_{k=-1}^{\infty} k A(k)e. \]

The vector \( \alpha \) satisfies the following Poisson equation:
\[ (I - A)x = -\sigma e + \beta_A, \]
where \( A = \sum_{\ell=-1}^{\infty} A(\ell) \) and \( \sigma = \varpi \beta_A < 0 \).

To proceed further, we assume that \( c > 0 \) is sufficiently large so that \( \alpha > 0 \). We then define \( v := (v(k, i))_{i \in M_{k, \Lambda + 1}} \) as a column vector such that
\[ v(k) := (v(k, i))_{i \in M_{k, \Lambda + 1}} = k^2 e + 2k \alpha + e, \quad k \in \mathbb{Z}_+. \]
We also define \( f := (f(k, i))_{i \in \mathbb{M}_{k+1}} \) as a column vector such that
\[
f(k) := (f(k, i))_{i \in \mathbb{M}_{k+1}} = \begin{cases} 
|\sigma|e = -\sigma e, & k = 0, \\
k|\sigma|e = -k\sigma e, & k \in \mathbb{N}.
\end{cases}
\] (3.5)

In addition, for \( C \subseteq F \), let \( 1_C := (1_C(k, i))_{(k, i) \in F} \) denote a column vector such that
\[
1_C(k, i) = \begin{cases} 
1, & (k, i) \in C, \\
0, & (k, i) \notin C.
\end{cases}
\]

**Lemma 3.1** If Assumptions 2.7 and 3.1 hold, there exists some \( b \in (0, \infty) \) such that, for all \( N \in \mathbb{N} \),
\[
\mathbf{P}^{(N)} \mathbf{v} \leq \mathbf{P} \mathbf{v} \leq \mathbf{v} - f + b1_{1 \leq K},
\] (3.6)
where
\[
K + 1 = \inf \left\{ k \in \mathbb{Z}_+ : -|\sigma|k e + \sum_{\ell=-1}^{\infty} \ell^2 \mathbf{A}(\ell)e + 2 \sum_{\ell=-1}^{\infty} \ell \mathbf{A}(\ell)\alpha \leq 0 \right\}.
\] (3.7)

**Proof.** From (3.4), we have
\[
0 \leq \mathbf{v}(0) \leq \mathbf{v}(1) \leq \mathbf{v}(2) \leq \cdots.
\]
Using this, (2.1) and (2.13), we obtain
\[
\mathbf{P}^{(N)} \mathbf{v} \leq \mathbf{P} \mathbf{v}, \quad N \in \mathbb{N}.
\]
Thus, it suffices to show that there exists some \( b \in (0, \infty) \) such that
\[
\mathbf{P} \mathbf{v} \leq \mathbf{v} - f + b1_{1 \leq K}.
\] (3.8)

Let \( \mathbf{P}(k; \ell) = P(k, i; \ell, j)_{(i, j) \in \mathbb{M}_{k+1} \times \mathbb{M}_{\ell+1}} \) for \( k, \ell \in \mathbb{Z}_+ \). It then follows from (2.1), (3.2) and (3.4) that, for all \( k \in \mathbb{Z}_{\geq 2} \),
\[
\sum_{\ell=0}^{\infty} \mathbf{P}(k; \ell) \mathbf{v}(\ell) = \sum_{\ell=-1}^{\infty} \mathbf{A}(\ell)\mathbf{v}(k + \ell)
\]
\[
= \sum_{\ell=-1}^{\infty} (k + \ell)^2 \mathbf{A}(\ell)e + 2 \sum_{\ell=-1}^{\infty} (k + \ell)\mathbf{A}(\ell)\alpha + e
\]
\[
= k^2 e + e + 2k (\beta \alpha + \mathbf{A}\alpha) + \sum_{\ell=-1}^{\infty} \ell^2 \mathbf{A}(\ell)e + 2 \sum_{\ell=-1}^{\infty} \ell \mathbf{A}(\ell)\alpha.
\] (3.9)

Recall here that \( \alpha \) is a solution of Poisson equation (3.3); that is, \( \alpha \) satisfies
\[
\mathbf{A}\alpha + \beta = \alpha + \sigma e.
\]
Substituting this into (3.9), and using (3.4) and (3.5), we obtain

\[
\sum_{\ell=0}^{\infty} P(k; \ell) v(\ell) = k^2 e + e + 2k(\alpha + \sigma e) + \sum_{\ell=-1}^{\infty} \ell^2 A(\ell)e + 2 \sum_{\ell=-1}^{\infty} \ell A(\ell)\alpha
\]

\[
= v(k) + 2k\sigma e + \sum_{\ell=-1}^{\infty} \ell^2 A(\ell)e + 2 \sum_{\ell=-1}^{\infty} \ell A(\ell)\alpha
\]

\[
= v(k) - f(k) + \left( -|\sigma|k e + \sum_{\ell=-1}^{\infty} \ell^2 A(\ell)e + 2 \sum_{\ell=-1}^{\infty} \ell A(\ell)\alpha \right), \quad k \in \mathbb{Z}_{\geq 2}. \quad (3.10)
\]

It follows from (3.7) and (3.10) that

\[
\sum_{\ell=0}^{\infty} P(k; \ell) v(\ell) \leq v(k) - f(k), \quad k \in \mathbb{Z}_{\geq K+1}.
\]

It also follows from (3.4) and Assumption 3.1 that \(\sum_{\ell=0}^{\infty} P(k; \ell) v(\ell)\) is finite for all \(k \in \mathbb{Z}_{[0,K]}\). As a result, (3.8) holds for some \(b \in (0, \infty)\).

**Theorem 3.1** If Assumption 2.1 is satisfied, then the following hold:

(i) Assumption 3.1 holds if and only if

\[
\sum_{k=1}^{\infty} k\pi(k)e < \infty. \quad (3.11)
\]

(ii) If (3.11) holds, then

\[
\sup_{N \in \mathbb{N}} \sum_{k=1}^{N} k\pi^{(N)}(k)e < \infty. \quad (3.12)
\]

**Proof.** To prove this theorem, we show that Assumption 3.1 implies (3.11) and (3.12), and also show that (3.11) implies Assumption 3.1.

We first assume that Assumption 3.1 holds. It then follows from Lemma 3.1 that (3.6) holds for some \(b \in (0, \infty)\). Premultiplying (3.6) by \(\pi^{(N)}\), and using \(\pi^{(N)} P^{(N)} = \pi^{(N)}\), we obtain

\[
\pi^{(N)} f < b \quad \text{for all } N \in \mathbb{N}.
\]

This inequality together with (3.5) implies that

\[
\sup_{N \in \mathbb{N}} \sum_{k=1}^{\infty} k\pi^{(N)}(k)e < b.
\]

Therefore, (3.12) holds. Similarly, we can prove that (3.11) hold.
Next, we assume that (3.11) holds. It then follows from (2.12) that
\[
\sum_{k=1}^{\infty} k\pi(k)e = \pi(0) \sum_{k=1}^{\infty} kR_0(k)e + \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \ell\pi(\ell) \cdot R(k-\ell)e
\]
\[+ \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \pi(\ell) \cdot (k-\ell)R(k-\ell)e\]
\[= \pi(0) \sum_{k=1}^{\infty} kR_0(k)e + \sum_{\ell=1}^{\infty} \ell\pi(\ell) \sum_{k=1}^{\infty} R(k)e\]
\[+ \sum_{\ell=1}^{\infty} \pi(\ell) \sum_{k=1}^{\infty} kR(k)e,
\]
which is finite. Thus, \(\sum_{k=1}^{\infty} kR_0(k)e\) and \(\sum_{k=1}^{\infty} kR(k)e\) are finite. It also follows from (2.9) and \((I - \Phi(0))^{-1}e \geq e\) that
\[
\sum_{k=1}^{\infty} kR_0(k)e = \sum_{k=1}^{\infty} k \sum_{m=k}^{\infty} B(m)G^{m-k}(I - \Phi(0))^{-1}e
\]
\[\geq \sum_{k=1}^{\infty} k \sum_{m=k}^{\infty} B(m)G^{m-k}e = \sum_{k=1}^{\infty} k \sum_{m=k}^{\infty} B(m)e
\]
\[= \frac{1}{2} \sum_{m=1}^{\infty} m(m+1)B(m)e.
\]
Therefore, \(\sum_{m=1}^{\infty} m^2B(m)e\) is finite. Similarly, using (2.10), we can prove that \(\sum_{m=1}^{\infty} m^2A(m)e\) is finite. As a result, Assumption 3.1 holds.

3.2 The first passage time to level zero

Let \(u(k) := (u(k, i))_{i \in \mathbb{M}_{k-1}}, k \in \mathbb{Z}_+,\) denote a column vector such that
\[u(k, i) = E_{(k, i)}[T_0] \geq 1, \quad (k, i) \in \mathbb{F}.
\] (3.13)

**Lemma 3.2** If Assumption 2.7 holds, then
\[u(0) = e + \sum_{m=1}^{\infty} B(m)(I - G^m + meg)(I - A - \beta A g)^{-1}e,\] (3.14)
\[u(k) = (I - G^k + keg)(I - A - \beta A g)^{-1}e, \quad k \in \mathbb{N}.
\] (3.15)

**Proof.** Let \(\hat{G}(z) := (\hat{G}_{i,j}(z))_{(i,j) \in \mathbb{M}_1 \times \mathbb{M}_1}, \quad \hat{G}_0(z) := (\hat{G}_{0,i,j}(z))_{(i,j) \in \mathbb{M}_1 \times \mathbb{M}_0}\) and \(\hat{K}(z) := (\hat{K}_{i,j}(z))_{(i,j) \in \mathbb{M}_0 \times \mathbb{M}_0}\) denote \(M_1 \times M_1, M_1 \times M_0\) and \(M_0 \times M_0\) matrices, respectively, such that
\[
\hat{G}_{i,j}(z) = E_{(n+1,i)}[z^{T_n} \mathbb{I}(X_{T_n} = j)], \quad (i, j) \in \mathbb{M}_1 \times \mathbb{M}_1, n \in \mathbb{N},
\]
\[
\hat{G}_{0,i,j}(z) = E_{(i, i)}[z^{T_0} \mathbb{I}(X_{T_0} = j)], \quad (i, j) \in \mathbb{M}_1 \times \mathbb{M}_0,
\]
\[
\hat{K}_{i,j}(z) = E_{(0, i)}[z^{T_0} \mathbb{I}(X_{T_0} = j)], \quad (i, j) \in \mathbb{M}_0 \times \mathbb{M}_0.
\]
By definition,
\[
\begin{align*}
\mathbf{u}(0) &= \frac{d}{dz} \tilde{K}(z) \bigg|_{z=1} \mathbf{e}, \\
\mathbf{u}(k) &= \frac{d}{dz} [\tilde{G}(z)]^{k-1} \tilde{G}_0(z) \bigg|_{z=1} \mathbf{e}, \quad k \in \mathbb{N}.
\end{align*}
\] (3.16) (3.17)

It follows (see [29, Eqs. 2.2.9, 2.4.3 and 2.4.8]) that
\[
\tilde{G}(z) = \left[ I - z \sum_{m=0}^{\infty} A(m) \{ \tilde{G}(z) \}^m \right]^{-1} zA(-1),
\] (3.18)
\[
\tilde{G}_0(z) = \left[ I - z \sum_{m=0}^{\infty} A(m) \{ \tilde{G}(z) \}^m \right]^{-1} zB(-1),
\] (3.19)
\[
\tilde{K}(z) = zB(0) + z \sum_{m=1}^{\infty} B(m) \{ \tilde{G}(z) \}^{m-1} \tilde{G}_0(z).
\] (3.20)

It also follows from (2.4), (3.18) and (3.19) that
\[
\tilde{G}_0(z) \mathbf{e} = \tilde{G}(z) \mathbf{e}.
\] (3.21)

Applying (3.21) to (3.17) and to (3.20) post-multiplied by \( \mathbf{e} \), we have
\[
\begin{align*}
\mathbf{u}(k) &= \frac{d}{dz} \tilde{G}(z) \bigg|_{z=1}^k \mathbf{e}, \quad k \in \mathbb{N}, \\
\tilde{K}(z) \mathbf{e} &= zB(0) \mathbf{e} + z \sum_{m=1}^{\infty} B(m) \{ \tilde{G}(z) \}^m \mathbf{e}.
\end{align*}
\] (3.22) (3.23)

Substituting (3.23) into (3.16), and using \( \tilde{G}(1) \mathbf{e} = \mathbf{G} \mathbf{e} = \mathbf{e} \) and \( \tilde{G}_0(1) \mathbf{e} = \mathbf{e} \), we obtain
\[
\begin{align*}
\mathbf{u}(0) &= B(0) \mathbf{e} + \sum_{m=1}^{\infty} B(m) \mathbf{e} + \sum_{m=1}^{\infty} B(m) \frac{d}{dz} \tilde{G}(z) \bigg|_{z=1}^m \mathbf{e} \\
&= \mathbf{e} + \sum_{m=1}^{\infty} B(m) \mathbf{u}(m),
\end{align*}
\] (3.24)

where the second equality holds due to (3.22) and \( \sum_{m=0}^{\infty} B(m) \mathbf{e} = \mathbf{e} \).

We note that (3.15) together with (3.24) yields (3.14). Thus, to complete the proof, it suffices to show that (3.15) holds. From (3.22) and \( \tilde{G}(1) \mathbf{e} = \mathbf{G} \mathbf{e} = \mathbf{e} \), we have
\[
\mathbf{u}(k) = \sum_{n=0}^{k-1} \mathbf{G}^n \frac{d}{dz} \tilde{G}(z) \bigg|_{z=1} \mathbf{e}, \quad k \in \mathbb{N}.
\]

We also have (see [29, Eqs. 3.1.3 and 3.1.12])
\[
\frac{d}{dz} \tilde{G}(z) \bigg|_{z=1} \mathbf{e} = (\mathbf{I} - \mathbf{G} + \mathbf{e} \mathbf{g})(\mathbf{I} - \mathbf{A} - \beta \mathbf{g})^{-1} \mathbf{e}.
\]
Combining these two equations yields

\[ u(k) = \sum_{n=0}^{k-1} G^n (I - G + eg)(I - A - \beta_A g)^{-1} e \]

\[ = (I - G^k + keg)(I - A - \beta_A g)^{-1} e, \quad k \in \mathbb{N}, \]

which shows that (3.15) holds.

**Theorem 3.2** Suppose that Assumption 2.1 is satisfied. Assumption 3.1 holds if and only if

\[ \sum_{k=0}^{\infty} \pi(k) u(k) < \infty. \] (3.25)

**Proof.** From (3.13), we have \( u(k) \geq e \) for \( k \in \mathbb{Z}_+ \). From (3.15), we also have

\[ \lim_{k \to \infty} \frac{u(k)}{k} = eg(I - A - \beta_A g)^{-1} e \geq e, \]

which shows that (3.25) is equivalent to (3.11). Consequently, Theorem 3.1 ensures that Theorem 3.2 holds.

### 4 A difference formula and the fundamental deviation matrix

This section presents a difference formula for \( \pi^{(N)} \) and \( \pi \) through the fundamental deviation matrix \( H \), introduced in this section. The fundamental deviation matrix is a solution of the Poisson equation satisfied by the deviation matrix \( D \) (see, e.g., [7]). Using the M/G/1-type structure of \( P \), we obtain a block-decomposition result of \( H \) and, as its by-product, that of \( D \). The result of \( H \), as well as the difference formula for \( \pi^{(N)} \) and \( \pi \), is used to derive an asymptotic formula for \( \pi^{(N)} - \pi \) in the next section.

#### 4.1 A difference formula via the fundamental deviation matrix

We begin with the following proposition.

**Proposition 4.1 (Basic difference formula)** Let \( \mathcal{H} \) denote an arbitrary solution \( \mathcal{H} \) of the Poisson equation (see, e.g., [22]):

\[ (I - P) \mathcal{H} = I - e\pi. \] (4.1)

We then have

\[ \pi^{(N)} - \pi = \pi^{(N)} (P^{(N)} - P) \mathcal{H}. \] (4.2)

**Proof.** It is implied in [18] and [19] that this proposition holds. However, for completeness, we provide the proof: Using (4.1) and \( \pi^{(N)} = P^{(N)} \pi^{(N)} \), we have

\[ \pi^{(N)} (P^{(N)} - P) \mathcal{H} = \pi^{(N)} (I - P) \mathcal{H} = \pi^{(N)} (I - e\pi) = \pi^{(N)} - \pi, \]
which shows that (4.2) holds.

Fix \((k_*, i_*) \in \mathbb{F}\) arbitrarily, and let \(H := (H(k, i; \ell, j))_{(k, i, \ell, j) \in \mathbb{F}^2}\) denote a matrix such that

\[
H(k, i; \ell, j) = E_{(k,i)} \left[ T(k_*, i_*)^{-1} \sum_{\nu=0}^{T(k_*, i_*)-1} \mathbb{1}( (X_\nu, J_\nu) = (\ell, j) ) \right] - \pi(\ell, j) E_{(k,i)} \left[ T(k_*, i_*) \right],
\]  

where \(T(\ell, j) = \inf\{ \nu \in \mathbb{N} : (X_\nu, J_\nu) = (\ell, j) \}\). The matrix \(H\) is a solution of Poisson equation (4.1) (see, e.g., [9, Lemma 2.1]). Proposition 4.1 thus yields a concrete difference formula:

\[
\pi^{(N)} - \pi = \pi^{(N)} (P^{(N)} - P) H.
\]  

We now define \(D\) as the deviation matrix of the infinite-level M/G/1-type Markov chain \(\{(X_\nu, J_\nu)\}\); that is,

\[
D = \sum_{n=0}^{\infty} (P^n - e \pi).
\]

The deviation matrix \(D\) is a unique solution of Poisson equation (4.1) with the constraints (see, e.g., [9, Lemma 2.7]):

\[
\pi_{\mathcal{H}} = 0, \quad \pi|_{\mathcal{H}} \text{ is finite.}
\]

Thus, we have another difference formula:

\[
\pi^{(N)} - \pi = \pi^{(N)} (P^{(N)} - P) D.
\]  

**Proposition 4.2** Suppose that Assumption 2.1 is satisfied. Assumption 3.1 holds if and only if \(D\) exists.

**Proof.** Recall that Assumption 2.1 ensures that the Markov chain \(\{(X_\nu, J_\nu)\}\) is ergodic. It then follows from [7, Theorem 4.1] that the deviation matrix \(D\) exists if and only if

\[
\sum_{(k,i) \in \mathbb{F}} \pi(k, i) E_{(k,i)} [T(0,j)] < \infty.
\]  

Moreover, for all \(j \in \mathbb{M}_0\) and \((k, i) \in \mathbb{F},

\[
E_{(k,i)}[T_0] \leq E_{(k,i)}[T(0,j)] = E_{(k,i)}[T_0 \mathbb{1}( J_0 = j )] + E_{(k,i)}[ \mathbb{1}(J_0 \neq j) \{ T_0 + E[T(0,j) - T_0 \mid J_0 \neq j]\} ] 
\leq 2E_{(k,i)}[T_0] + E[T(0,j) \mid X_0 = 0, J_0 \neq j],
\]

where \(E[T(0,j) \mid X_0 = 0, J_0 \neq j] < \infty\) due to the ergodicity of \(\{(X_\nu, J_\nu)\}\). Thus, (4.8) holds if and only if

\[
\sum_{(k,i) \in \mathbb{F}} \pi(k, i) E_{(k,i)} [T_0] = \sum_{k=0}^{\infty} \pi(k) u(k) < \infty,
\]
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where the equality is due to (3.13). As a result, \( \sum_{k=0}^{\infty} \pi(k)u(k) < \infty \) and thus Assumption 3.1 holds (see Theorem 3.2) if and only if the deviation matrix \( D \) exists.

With either of \( H \) and \( D \), we can express the difference \( \pi^{(N)} - \pi \), as shown in (4.4) and (4.7). Proposition 4.2 however shows that the existence condition of \( D \) is stronger than that of \( H \). More specifically, \( D \) requires Assumption 3.1 whereas \( H \) does not necessarily it. Furthermore, \( D \) can be expressed with \( H \) (see Remark 4.1 below). These facts imply that \( H \) is more fundamental than the deviation matrix \( D \). Thus, for convenience, we refer to \( H \) as the fundamental deviation matrix.

### 4.2 Block decomposition of the fundamental deviation matrix

We first partition \( P \) in (2.1) as follows:

\[
P = \begin{pmatrix}
\mathbb{L}_0 & \mathbb{L}_{\geq 1} \\
\mathbb{L}_0 & P_{0+} & P_+
\end{pmatrix},
\]

(4.9)

where

\[
P_{0+} = \begin{pmatrix}
B(1) & B(2) & B(3) & \cdots \\
B(-1) & O & O & \cdots \\
B(-1) & O & O & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\[
P_+ = \begin{pmatrix}
A(0) & A(1) & A(2) & A(3) & \cdots \\
A(-1) & A(0) & A(1) & A(2) & \cdots \\
0 & A(-1) & A(0) & A(1) & \cdots \\
0 & 0 & A(-1) & A(0) & \cdots \\
\dots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

(4.10)

We then define \( F_+ := (F_+(k, i; \ell, j))(k,i,\ell,j)\in(\mathbb{L}_{\geq 1})^2 \) as the fundamental matrix of \( P_+ \), i.e.,

\[
F_+ = \sum_{m=0}^{\infty} P_+^m = (I - P_+)^{-1}.
\]

(4.11)

Note that

\[
F_+(k, i; \ell, j) = \mathbb{E}_{(k,i)} \left[ \sum_{\nu=0}^{T_0-1} \mathbb{1}((X_\nu, J_\nu) = (\ell, j)) \right], \quad (k, i; \ell, j) \in (\mathbb{L}_{\geq 1})^2,
\]

(4.12)

where \( \{(X_\nu, J_\nu)\} \) is the infinite-level chain. From (4.9), (4.11) and \( \pi P = \pi \), we have

\[
(\pi(1), \pi(2), \ldots) = \pi(0)P_{0+}F_+.
\]

This equation together with (4.10) yields

\[
\pi(\ell) = \pi(0)\sum_{m=1}^{\infty} B(m)F_+(m; \ell), \quad \ell \in \mathbb{N},
\]

(4.13)

where \( F_+(k; \ell) = (F_+(k, i; \ell, j))(i,j)\in(\mathbb{M}_1)^2 \) for \( k, \ell \in \mathbb{N} \). The block matrices \( F_+(k; \ell) \)'s of \( F_+ \) satisfy the following recursive formula.
Proposition 4.3 ([31, Theorem 9]) If Assumption 2.1 holds, then, for \( k, \ell \in \mathbb{N} \),

\[
F_+(k; \ell) = \begin{cases} 
GF_+(k-1; \ell), & 1 \leq \ell \leq k - 1, \\
F_+(1; 1) + GF_+(k-1; k), & \ell = k, \\
\sum_{n=1}^{\ell-1} F_+(k; n) R(\ell - n), & \ell \geq k + 1,
\end{cases}
\]  

(4.14)

where \( F_+(0; m) = O \) for \( m \in \mathbb{N} \).

We note that \( \Phi(0) \) in (2.11) is a substochastic matrix that contains the transition probabilities of an absorbing Markov chain obtained by observing the infinite-level Markov chain \( \{(X_\nu, J_\nu)\} \) only when it is in \( \mathbb{L}_1 \) and until it reaches \( \mathbb{L}_0 \). It thus follows from (4.12) that

\[
F_+(1; 1) = (I - \Phi(0))^{-1}.
\]

From this equation and (4.14), we have

\[
F_+(k; 1) = G^{k-1}(I - \Phi(0))^{-1}, \quad k \in \mathbb{N}.
\]  

(4.15)

To describe the block-decomposition result of \( H \), we introduce a stochastic matrix associated with \( F_+ \). Let \( \tilde{P}_0 \) denote

\[
\tilde{P}_0 = B(0) + P_0F_+P_0.
\]

The matrix \( \tilde{P}_0 \) can be interpreted as the transition probability matrix of a censored Markov chain obtained by observing the Markov chain \( \{(X_\nu, J_\nu)\} \) only when it is in \( \mathbb{L}_0 \) (see [32, Theorem 2]). Therefore, the probability vector

\[
\tilde{\pi}_0 := \frac{\pi(0)}{\pi(0)e}
\]  

(4.16)

is a unique stationary distribution of \( \tilde{P}_0 \), and

\[
\tilde{\pi}_0\tilde{P}_0 = \tilde{\pi}_0.
\]

Using Proposition 4.3 we obtain the following theorem.

**Theorem 4.1** Suppose that Assumption 2.1 holds. Let \( H(k; \ell), k, \ell \in \mathbb{Z}_+ \), denote the block matrix \( (H(k, i; \ell, j))_{(i, j) \in M_{k \land 1} \times M_{\ell \land 1}} \) of the fundamental deviation matrix \( H \). We then have

\[
(I - \tilde{P}_0)H(0; 0) = I - u(0)\pi(0),
\]  

(4.17)

\[
(I - \tilde{P}_0)H(0; \ell) = \sum_{m=1}^{\infty} B(m)F_+(m; \ell) - u(0)\pi(\ell), \quad \ell \in \mathbb{N},
\]  

(4.18)

and, for \( k \in \mathbb{N} \),

\[
H(k; \ell) = (1 - \delta_{k,\ell})F_+(k; \ell) \\
+ G^{k-1}(I - \Phi(0))^{-1}B(-1)H(0; \ell) - u(k)\pi(\ell), \quad \ell \in \mathbb{Z}_+,
\]  

(4.19)

where \( \delta_{k,\ell} \) denotes the Kronecker delta, and where \( u(m) \)'s, \( m \in \mathbb{Z}_+ \), are given in (3.14) and (3.15).
Proof. Let \( \tilde{H}(k; \ell) := (\tilde{H}(k, i; \ell, j))_{(i,j) \in \mathbb{M}_{k+1} \times \mathbb{M}_{\ell+1}}, k, \ell \in \mathbb{Z}_+, \) denote a matrix such that
\[
\tilde{H}(k, i; \ell, j) = \mathbb{E}(k,i) \left[ \sum_{\nu=0}^{T_{i,-1}} \mathbb{1}((X, J) = (\ell, j)) \right].
\] (4.20)

It follows from (4.12) and (4.20) that
\[
\tilde{H}(k; \ell) = \begin{cases} 
\delta_{k,0} I, & k \in \mathbb{Z}_+, \ell = 0, \\
\sum_{m=1}^{\infty} B(m) F_+(m; \ell), & k = 0, \ell \in \mathbb{N}, \\
F_+(k; \ell), & k \in \mathbb{N}, \ell \in \mathbb{N}.
\end{cases}
\] (4.21)

It also follows from [9, Theorem 2.5] that
\[
H(0; \ell) = \tilde{H}(0; \ell) - u(0) \pi(\ell) + \tilde{P}_0 H(0; \ell), \quad \ell \in \mathbb{Z}_+, \quad (4.22)
\]
\[
H(k; \ell) = \tilde{H}(k; \ell) - u(k) \pi(\ell) + F_+(k; 1) B(-1) H(0; \ell), \quad k \in \mathbb{N}, \ell \in \mathbb{Z}_+. \quad (4.23)
\]

Combining (4.21) and (4.22) leads to (4.17) and (4.18). Furthermore, substituting (4.15) and (4.21) into (4.23) yields (4.19).

From Theorem 4.1, we have two corollaries. The first corollary shows that \( H \) satisfies the same inequality constraint (4.6) as \( D \).

**Corollary 4.1** Suppose that Assumption 2.7 is satisfied. Assumption 3.1 holds if and only if
\[
\sum_{(k,i) \in \mathcal{F}} \pi(k,i) |H(k,i; \ell,j)| < \infty, \quad (\ell,j) \in \mathcal{F}. \quad (4.24)
\]

**Proof.** From (4.14), we have
\[
F_+(k; \ell) = G^{k-\ell} F_+(\ell; \ell), \quad \ell \in \mathbb{N}, k \in \mathbb{Z}_{\geq \ell}. \quad (4.25)
\]

It follows from (3.15), (4.19) and (4.25) that, for \( \ell \in \mathbb{Z}_+ \),
\[
H(k; \ell) = (1 - \delta_{0,\ell}) G^{k-\ell} F_+(\ell; \ell) + G^{k-1}(I - \Phi(0))^{-1} B(-1) H(0; \ell)
- (I - G^k + k e g)(I - A - \beta A g)^{-1} e \pi(\ell), \quad k \in \mathbb{Z}_{\geq \ell} \cap \mathbb{N},
\] (4.26)
which implies that
\[
\lim_{k \to \infty, k \geq \ell} \frac{H(k; \ell)}{k} = -e g (I - A - \beta A g)^{-1} e \pi(\ell), \quad \ell \in \mathbb{Z}_+. \quad (4.27)
\]

It also follows from (3.13) and (3.15) that
\[
gu(1) = g(I - G + e g)(I - A - \beta A g)^{-1} e
= g(I - A - \beta A g)^{-1} e \geq 1. \quad (4.28)
\]

Combining (4.27) and (4.28) leads to
\[ \lim_{k \to \infty} \frac{|H(k; \ell)|}{k} = eg(I - A - \beta_A g)^{-1}e\pi(\ell) \geq e\pi(\ell) > 0, \quad \ell \in \mathbb{Z}_+. \]

Thus, (4.24) holds if and only if \( \sum_{k=1}^{\infty} k\pi(k)e \) is finite. This equivalence and Theorem 3.1 imply Corollary 4.1.

The second one presents the block-decomposition result of \( D \).

**Corollary 4.2** Suppose Assumptions 2.1 and 3.1 hold. Let \( D(k; \ell) \), \( k, \ell \in \mathbb{Z}_+ \), denote the block matrix \( (D(k; i, j))_{(i,j) \in M_k \times M_\ell} \) of the deviation matrix \( D \). We then have
\[ D(k; \ell) = \sum_{m=0}^{\infty} (\delta_{k,m} I - \pi(m)) Z(m; \ell), \quad k, \ell, \in \mathbb{Z}_+, \tag{4.29} \]
where
\[ Z(0; 0) = (I - \tilde{P}_0 + e\tilde{\pi}_0)^{-1}(I - u(0)\pi(0)), \tag{4.30} \]
\[ Z(0; \ell) = (I - \tilde{P}_0 + e\tilde{\pi}_0)^{-1}\left( \sum_{m=1}^{\infty} B(m) F_+ (m; \ell) - u(0)\pi(\ell) \right), \quad \ell \in \mathbb{N}, \tag{4.31} \]
and, for \( k \in \mathbb{N} \),
\[ Z(k; \ell) = (1 - \delta_{0,\ell}) F_+ (k; \ell) + G^{k-1}(I - \Phi(0))^{-1}B(-1) Z(0; \ell) - u(k)\pi(\ell), \quad \ell \in \mathbb{Z}_+. \tag{4.32} \]

**Remark 4.1** Clearly, (4.29) is equivalent to
\[ D = (I - e\pi)Z, \tag{4.33} \]
where \( Z = (Z(k; \ell))_{k,\ell \in \mathbb{Z}_+} \); that is, \( Z(k; \ell) \) is the \((k; \ell)\)-th block matrix of \( Z \). Note here that the constraint (4.6) leads to
\[ \pi D = 0. \tag{4.34} \]
From this equation and (4.41) below, we have \( \eta = -\pi H \) and thus
\[ D = (I - e\pi)H, \tag{4.35} \]
which connects the deviation matrix \( D \) with the fundamental deviation matrix \( H \).

**Proof.** [Proof of Corollary 4.2] We note (see, e.g., [29, Theorem 3.2.1]) that
\[ \pi(0) = \frac{\tilde{\pi}_0}{\pi_0 u(0)}, \tag{4.36} \]
which leads to
\[ \tilde{\pi}_0(I - u(0)\pi(0)) = 0. \]
From this equation and (4.30), we have
\[(I - \tilde{P}_0)Z(0; 0) = I - u(0)\pi(0) - e\tilde{\pi}_0(I - u(0)\pi(0)) = I - u(0)\pi(0).\]

Thus, \(Z(0; 0)\) is a solution of the Poisson equation:
\[(I - \tilde{P}_0)\mathcal{H}_0 = I - u(0)\pi(0).\]

According to (4.17), \(H(0; 0)\) is also a solution of this Poisson equation. In addition, from (4.13), (4.16) and (4.36), we have
\[\tilde{\pi}_0 \left( \sum_{m=1}^{\infty} B(m) F_+(m; \ell) - u(0)\pi(\ell) \right) = \frac{\pi(\ell)}{\pi(0)e} (1 - \pi(0)u(0)) = 0.\]

Therefore, (4.18) and (4.31) imply that \(H(0; \ell)\) and \(Z(0; \ell), \ell \in \mathbb{N},\) are solutions of the Poisson equation:
\[(I - \tilde{P}_0)\mathcal{H}_\ell = \sum_{m=1}^{\infty} B(m) F_+(m; \ell) - u(0)\pi(\ell).\]

The above argument together with [11, Proposition 1.1] shows that for each \(\ell \in \mathbb{Z}_+\) there exists some \(1 \times M_{\ell,1}\) vector \(\zeta(\ell)\) such that
\[H(0; \ell) = Z(0; \ell) + e\zeta(\ell), \quad \ell \in \mathbb{Z}_+.\] (4.37)

Substituting (4.37) into (4.19) yields
\[H(k; \ell) = F_+(k; \ell) + G^{k-1}(I - \Phi(0))^{-1} B(-1) Z(0; \ell) - u(k)\pi(\ell)\]
\[\quad + G^{k-1}(I - \Phi(0))^{-1} B(-1) e\zeta(\ell), \quad k \in \mathbb{N}, \ell \in \mathbb{Z}_+.\] (4.38)

Note that \(G^{k-1}(I - \Phi(0))^{-1} B(-1) e = e\) for \(k \in \mathbb{N}.\) Thus, from (4.38) and (4.32), we have
\[H(k; \ell) = Z(k; \ell) + e\zeta(\ell), \quad k \in \mathbb{N}, \ell \in \mathbb{Z}_+.\] (4.39)

Combining (4.37) and (4.39) yields
\[H = Z + e\zeta,\] (4.40)
where \(\zeta = (\zeta(0), \zeta(1), \zeta(2), \ldots).\) Furthermore, it follows from (4.24) and [11, Proposition 1.1] that there exists some row vector \(\eta := (\eta(k, i))_{(k, i) \in \mathcal{F}}\) such that
\[D = H + e\eta = Z + e(\zeta + \eta),\] (4.41)

where the second equality holds due to (4.40). From (4.34) and (4.41), we have \(\zeta + \eta = -\pi Z.\) Substituting this result into (4.41) yields (4.33).
5 Subgeometric convergence of finite-level chains

This section discusses the infinite-level limit of the finite-level chain. We first prove \( \lim_{N \to \infty} \pi^{(N)} = \pi \) under the condition \( \sum_{k=1}^{\infty} k \pi(k) e < \infty \); that is, Assumption 3.1 (see Theorem 3.1). We then derive a subgeometric convergence formula for \( \pi^{(N)} - \pi \) under Assumption 5.1 (introduced below) together with Assumption 3.1.

5.1 Basic results on convergence

Let \( v' := (v'(k, i))_{(k, i) \in F} \) denote a column vector such that
\[
v'(k) := (v'(k, i))_{i \in M_{k \wedge 1}} = (k + 1)e + \alpha, \quad k \in \mathbb{Z}_+,
\]
where \( \alpha \) is given in (5.1).

Lemma 5.1 If Assumption 2.1 holds, then there exist some \( b' \in (0, \infty) \) and \( K' \in \mathbb{N} \) such that
\[
|P^{(N)}v' - P v'| \leq |\sigma|e + b'1_{\ell \leq K'} \quad \text{for all } N \in \mathbb{N}.
\]

Proof. Following the derivation of (3.10), we obtain, for \( k \in \mathbb{Z}_2 \),
\[
\sum_{\ell=0}^{\infty} P(k; \ell)v'(\ell) = e + \sum_{\ell=1}^{\infty} (\ell + k)A(\ell)e + \sum_{\ell=1}^{\infty} A(\ell)\alpha
= (k + 1)e + (A\alpha + \beta_A)
= (k + 1)e + \sigma e + (A\alpha + \beta_A - \sigma e)
= (k + 1)e + \alpha - |\sigma|e
= v'(k) - |\sigma|e,
\]
which implies that, for some \( b' \in (0, \infty) \) and \( K' \in \mathbb{N} \),
\[
Pv' \leq v' - |\sigma|e + b'1_{\ell \leq K'}.
\]
Furthermore, \( P^{(N)}v' \leq P v' \) for \( N \in \mathbb{N} \). As a result, (5.2) holds.

Using Lemma 5.1, we prove the following theorem.

Theorem 5.1 If Assumptions 2.1 and 3.1 hold, then there exists some function \( E : \mathbb{N} \to \mathbb{R}_+ := [0, \infty) \) such that \( \lim_{N \to \infty} E(N) = 0 \) and
\[
\sup_{0 \leq w \leq e \atop \pi w > 0} \frac{|\pi^{(N)} - \pi| w}{\pi w} \leq E(N),
\]
where \( w := (w(k, i))_{(k, i) \in F} \) denotes a nonnegative column vector.
Proof. Let $Q = P - I$ and $Q^{(N)} = P^{(N)} - I$ for $N \in \mathbb{N}$. Clearly, $Q$ and $Q^{(N)}$ are $Q$-matrices (see [2, Section 2.1, page 64]) with the invariant (stationary) probability vectors $\pi$ and $\pi^{(N)}$, respectively. It thus follows from Lemma 5.1 and [19, Theorem 2.1] that, for any bounded $w \geq 0$,

$$|\pi^{(N)} - \pi| w \leq \left(1 + \frac{\pi w}{|\sigma|}\right)\pi^{(N)} |P^{(N)} - P| (v' + Ce), \quad N \in \mathbb{N},$$

(5.3)

where $v'$ is given in (5.1), and where $C > 0$ is some constant independent of $w$. Note (see [27, Eq. (2.26)]) that

$$\sup_{0 \leq w \leq e, 0 < \varepsilon \leq 1} \frac{\pi^{(N)} - \pi| w}{\pi w} = \sup_{0 \leq w \leq e, 0 < \varepsilon \leq 1} \frac{\pi^{(N)} - \pi| (w/\varepsilon)}{\pi (w/\varepsilon)} = \frac{\sup_{0 \leq w \leq e/\varepsilon, \pi w \geq 1, 0 < \varepsilon \leq 1} |\pi^{(N)} - \pi| w}{\pi w}, \quad N \in \mathbb{N}. \quad (5.4)$$

Using this and (5.3), we obtain

$$\sup_{0 \leq w \leq e} \frac{|\pi^{(N)} - \pi| w}{\pi w} \leq \sup_{0 \leq w \leq e/\varepsilon, \pi w \geq 1, 0 < \varepsilon \leq 1} \left(\frac{1}{\pi w} + \frac{1}{|\sigma|}\right)\pi^{(N)} |P^{(N)} - P| (v' + Ce)$$

$$= \left(1 + \frac{1}{|\sigma|}\right)\pi^{(N)} |P^{(N)} - P| (v' + Ce), \quad N \in \mathbb{N}. \quad (5.4)$$

We now fix

$$E(N) = \pi^{(N)} |P^{(N)} - P| (v' + Ce).$$

It then follows from (5.4) that

$$\sup_{0 \leq w \leq e} \frac{|\pi^{(N)} - \pi| w}{\pi w} \leq \left(1 + \frac{1}{|\sigma|}\right) E(N).$$

Therefore, to prove the present theorem, it suffices to show that

$$\lim_{N \to \infty} \pi^{(N)} |P^{(N)} - P| v' = 0. \quad (5.5)$$

Recall that (3.12) holds under Assumption 3.1 (see Theorem 3.1). It also follows from (5.2) that

$$|P^{(N)} - P| v' \leq v' + b'e, \quad N \in \mathbb{N}.$$ 

From this inequality, (3.12) and (5.1), we obtain

$$\sup_{N \in \mathbb{N}} \pi^{(N)} |P^{(N)} - P| v' \leq \sup_{N \in \mathbb{N}} \pi^{(N)} (v' + b'e)$$

$$= \sup_{N \in \mathbb{N}} \sum_{k=1}^{\infty} k\pi^{(N)}(k)e + \sup_{N \in \mathbb{N}} \sum_{k=0}^{\infty} \pi^{(N)}(k)\alpha + b' + 1 < \infty.$$ 

Therefore, using $\lim_{N \to \infty} |P^{(N)} - P| = O$ (due to (2.16)) and the dominated convergence theorem, we have (5.5).
Corollary 5.1 If Assumptions 2.1 and 3.1 hold, then there exists some function $E : \mathbb{N} \rightarrow \mathbb{R}_+$ such that $\lim_{N \rightarrow \infty} E(N) = 0$ and

$$\sum_{(k,i) \in A} |\pi^{(N)}(k,i) - \pi(k,i)| \leq E(N)$$

for all $A \subseteq \mathbb{F}$.

Proof. Fix $w = (w(k,i))_{(k,i) \in \mathbb{F}}$ such that

$$w(k,i) = \begin{cases} 1, & (k,i) \in A, \\ 0, & (k,i) \not\in A. \end{cases}$$

It then follows from Theorem 5.1 that the present corollary holds.

5.2 Subgeometric convergence formula

We introduce two classes of discrete long-tailed distributions.

Definition 5.1 A discrete distribution $F$ on $\mathbb{Z}_+$ is said to be long-tailed if $1 - F(k) > 0$ for all $k \in \mathbb{Z}_+$ and

$$\lim_{k \rightarrow \infty} \frac{F(k + \ell)}{F(k)} = 1$$

for any fixed $\ell \in \mathbb{N}$, where $F(k) = \sum_{\ell=0}^{\infty} F(\ell)$. The set of long-tailed distributions is denoted by $\mathcal{L}$.

Definition 5.2 A discrete distribution $F$ on $\mathbb{Z}_+$ is said to be subexponential if $1 - F(k) > 0$ for all $k \in \mathbb{Z}_+$ and

$$\lim_{k \rightarrow \infty} \frac{1 - F^{*2}(k)}{1 - F(k)} = 2,$$

where $F^{*2}$ is the two-fold convolution of $F$; that is,

$$F^{*2}(k) = \sum_{\ell=0}^{k} F(k - \ell) F(\ell), \quad k \in \mathbb{Z}_+.$$

The set of subexponential distributions is denoted by $\mathcal{S}$.

Remark 5.1 Definitions 5.1 and 5.2 are the discrete versions of the definitions of long-tailed and subexponential distributions in $\mathbb{R}_+$ (see, e.g., [10, Definitions 2.21 and 3.1]). Indeed, if the distribution of a random variable $Y$ in $\mathbb{R}_+$ is long tailed (resp. subexponential), then those of $\lceil Y \rceil$ and $\lfloor Y \rfloor$ are in $\mathcal{L}$ specified by Definition 5.1 (resp. $\mathcal{S}$ specified by Definition 5.2), because the long-tailed and subexponential classes are closed with respect to tail equivalence (see, e.g., [10, Lemma 2.23 and Corollary 3.13]).

To proceed further, we assume the following.
Assumption 5.1 There exists some $F \in \mathcal{S}$ such that

$$
\lim_{k \to \infty} \frac{A(k)e}{F(k)} = c_A, \quad (5.6)
$$

$$
\lim_{k \to \infty} \frac{B(k)e}{F(k)} = c_B, \quad (5.7)
$$

where either $c_A \neq 0$ or $c_B \neq 0$; and

$$
\overline{A}(n) = \sum_{m=n+1}^{\infty} A(m), \quad \overline{B}(n) = \sum_{m=n+1}^{\infty} B(m), \quad n \in \mathbb{Z}_{\geq -1}. \quad (5.8)
$$

Under Assumption 5.1 we have a subexponential asymptotic formula for the stationary distribution $\pi$ of the infinite-level M/G/1-type Markov chain.

Proposition 5.1 ([25, Theorem 3.1]) If Assumptions 2.1 and 5.1 hold, then

$$
\lim_{k \to \infty} \frac{\overline{\pi}(k)}{F(k)} = \frac{\pi(0)c_B + \overline{\pi}(0)c_A}{-\sigma},
$$

where $\overline{\pi}(k) = \sum_{\ell=k+1}^{\infty} \pi(\ell)$ for all $k \in \mathbb{Z}_+.$

Using Proposition 5.1 and some technical lemmas (presented in Appendix A), we prove the following theorem.

Theorem 5.2 If Assumptions 2.1, 3.1 and 5.1 hold, then

$$
\lim_{N \to \infty} \frac{\pi^{(N)} - \pi}{\overline{\pi}(N)e} = \pi. \quad (5.9)
$$

Proof. See Appendix A

Remark 5.2 The asymptotic formula (5.9) does not necessarily require that $P$ is aperiodic. Indeed, when $P$ is periodic, we consider $Q := (I + P)/2$ instead of $P$. Clearly, $Q$ is an ergodic (thus aperiodic) M/G/1-type stochastic matrix that has the same stationary distribution $\pi$ as $P$. The arguments leading to the formula (5.9) are basically valid for $Q$ and its finite-level version (i.e., $Q^{(N)} := (I + P^{(N)})/2$) though minor and appropriate modifications are made. In addition, Proposition 5.1 holds without the aperiodicity of the M/G/1-type Markov chain (see [25, Theorem 3.1]). As a result, (5.9) holds if all the conditions of Theorem 5.2 are satisfied except for the aperiodicity of $P$.

6 Application to the loss probability in the MAP/G/1 queue

We first describe the Markovian arrival process (MAP) [21]. Let $\{J(t); t \geq 0\}$ denote the background Markov chain of the MAP, which is defined on the state space $M := \{1, 2, \ldots, M\}$. Let $\{U(t); t \geq 0\}$ denote the counting process of the MAP; that is, $U(t)$ is the total number
of arrivals in time interval \((0, t]\), where \(U(0) = 0\) is assumed. The bivariate stochastic process \(\{(U(t), J(t)) : t \geq 0\}\) is a continuous-time Markov chain on state space \(\mathbb{Z}_+ \times M\) with the infinitesimal generator given by

\[
\begin{pmatrix}
\Lambda_0 & \Lambda_1 & 0 & 0 & \cdots \\
0 & \Lambda_0 & \Lambda_1 & 0 & \cdots \\
0 & 0 & \Lambda_0 & \Lambda_1 & \cdots \\
0 & 0 & 0 & \Lambda_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(6.1)

where \(\Lambda_1\) is an \(M \times M\) nonnegative matrix, and where \(\Lambda_0\) is an \(M \times M\) matrix with negative diagonal elements and nonnegative off-diagonal ones. We denote by MAP \((\Lambda_0, \Lambda_1)\) the MAP characterized in (6.1). By definition, \(\Lambda := \Lambda_0 + \Lambda_1\) is the infinitesimal generator of the background Markov chain \(\{j(t)\}\). For analytical convenience, we assume that \(\Lambda\) is irreducible, and then define \(\varpi > 0\) as the unique stationary probability vector of \(\Lambda\). We also define \(\lambda\) as the mean arrival rate, i.e.,

\[\lambda = \varpi \Lambda_1 e.\]

To exclude trivial cases, we assume \(\lambda > 0\) and thus \(\Lambda_1 \neq O\).

Next we describe the MAP/G/1/N + 1 queue. The system has a single server and a buffer of capacity \(N\), and thus the system capacity is equal to \(N + 1\). Customers arrive at the system according to MAP \((\Lambda_0, \Lambda_1)\). Arriving customers are allowed to join the system until the queue length reaches the system capacity \(N + 1\). Accepted customers are served on a first-come-first-served basis, and their service times are independent of MAP \((\Lambda_0, \Lambda_1)\), and are independent and identically distributed (i.i.d.) according to a general distribution function \(\beta\) on \(\mathbb{R}_+\) with mean \(\beta_1 \in (0, \infty)\). Thus, \(\rho := \lambda \beta_1\), which is the traffic intensity.

In what follows, we discuss the loss probability, denoted by \(P_{\text{loss}}^{(N)}\), in the MAP/G/1/N + 1 queue, described above. Let \(X^{(N)}(t), t \in \mathbb{R}_+\), denote the queue length at time \(t\). Let \(p^{(N)}(0)\) denote

\[p^{(N)}(0) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{1}(X^{(N)}(u) = 0)\, du.\]

It then follows that

\[P_{\text{loss}}^{(N)} = 1 - \frac{1 - p^{(N)}(0)}{\rho}.\]

(6.2)

To express \(P_{\text{loss}}^{(N)}\) more specifically, we introduce the embedded queue length process in the MAP/G/1/N + 1 queue. Let \(X^{(N)}_{\nu} = X^{(N)}(t_{\nu}+)\) and \(J_{\nu} = J(t_{\nu}+)\) for \(\nu \in \mathbb{Z}_+\), where \(t_{\nu}\) is the \(\nu\)-th service completion time. It is known that \(\{(X^{(N)}_{\nu}, J_{\nu}) : \nu \in \mathbb{Z}_+\}\) is a Markov chain with the
following transition probability matrix:

\[
P^{(N)} = \begin{pmatrix}
B(0) & B(1) & B(2) & \cdots & B(N-2) & B(N-1) & \bar{B}(N-1) \\
A(-1) & A(0) & A(1) & \cdots & A(N-3) & A(N-2) & \bar{A}(N-2) \\
O & A(-1) & A(0) & \cdots & A(N-4) & A(N-3) & \bar{A}(N-3) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
O & O & O & \cdots & A(0) & A(1) & \bar{A}(1) \\
O & O & O & \cdots & A(-1) & A(0) & \bar{A}(0) \\
O & O & O & \cdots & O & A(-1) & \bar{A}(-1) \\
\end{pmatrix},
\]

where

\[
B(k) = (-\Lambda_0)^{-1} A(k + 1), \quad k \in \mathbb{Z}_+,
\]

and where \(A(k), k \in \mathbb{Z}_{\geq -1}\), denotes an \(M \times M\) matrix such that

\[
\hat{A}(z) := \sum_{k=-1}^{\infty} z^k A(k) = z^{-1} \int_0^\infty \exp\{(\Lambda_0 + z\Lambda_1)x\}d\beta(x).
\]

Since \(\Lambda = \Lambda_0 + \Lambda_1\) is irreducible and

\[
\bar{A}(1) = \sum_{k=2}^{\infty} A(k)
\geq \int_0^\infty d\beta(x) \int_0^x dx_2 \int_0^{x_2} dx_1 e^{\Lambda x_1} \Lambda_1 e^{A(x_2-x_1)} \Lambda_1 e^{A(x-x_2)} > O.
\]

Furthermore, all the diagonal elements of \(A(-1) = \int_0^\infty d\beta(x) \exp\{A_0 x\}\) are positive. As a result, the stochastic matrix \(P^{(N)}\) in (6.3) is irreducible and thus has a unique stationary distribution, denoted by \(\pi^{(N)} := (\pi^{(N)}(k, i))_{(k,i) \in \mathbb{Z}_{\{0,N\}} \times M}\). It is known (see [4, Page 873]) that

\[
p^{(N)}(0) = \frac{\pi^{(N)}(0)(-\Lambda_0)^{-1} e}{\pi^{(N)}(0)(-\Lambda_0)^{-1} e + \beta_1},
\]

where \(\pi^{(N)}(k) = (\pi^{(N)}(k, i))_{i \in M}\) for \(k \in \mathbb{Z}_{\{0,N\}}\). Substituting (6.7) into (6.2) yields

\[
P^{(N)}_{\text{loss}} = 1 - \frac{1}{\rho} \frac{\beta_1}{\pi^{(N)}(0)(-\Lambda_0)^{-1} e + \beta_1}.
\]

We now assume the following.

**Condition 1**

\[
\sigma := \rho - 1 = \infty \sum_{k=-1}^{\infty} k A(k) e < 0.
\]

It follows from (6.4) and (6.9) that \(\sum_{k=1}^{\infty} k B(k) e\) is finite. Therefore, Condition 1 ensures that the stochastic matrix

\[
P := \lim_{N \to \infty} P^{(N)} = \begin{pmatrix}
B(0) & B(1) & B(2) & B(3) & \cdots \\
A(-1) & A(0) & A(1) & A(2) & \cdots \\
O & A(-1) & A(0) & A(1) & \cdots \\
O & O & A(-1) & A(0) & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

(6.10)
satisfies Assumption 2.1 except aperiodicity, and \( P \) in (6.10) thus has a unique stationary distribution (see, e.g., [3, Chapter XI, Proposition 3.1]), denoted by \( \pi := (\pi(k,i))_{(k,i) \in \mathbb{Z}_+ \times \mathcal{M}} \).

Let \( \pi(k) = (\pi(k,i))_{i \in \mathcal{M}} \) and \( \bar{\pi}(k) = \sum_{\ell=k+1}^{\infty} \pi(\ell) \) for \( k \in \mathbb{Z}_+ \). It then follows from [25, Theorem 4.2] that
\[
\lim_{k \to \infty} \frac{\bar{\pi}(k)}{P(S_e > k/\lambda)} = \frac{\rho}{1 - \rho}, \quad (6.11)
\]
under Condition 2 below together with Condition 1.

**Condition 2** Let \( S \) denote a generic random variable for i.i.d. service times according to distribution \( \beta \). Let \( S_e \) denote the equilibrium random variable of \( S \), which is distributed as follows:
\[
P(S_e \leq x) = \frac{1}{\beta_1} \int_0^x (1 - \beta(y)) dy, \quad x \in \mathbb{R}_+.
\]
In addition, (i) \( S_e \) is subexponential; and (ii) \( \sqrt{S_e} \) is long-tailed.

From (6.11) and Theorem 5.2, we obtain an asymptotic formula for \( P^{(N)}_{\text{loss}} \).

**Theorem 6.1** Consider the MAP/G/1/N + 1 queue, and suppose that Conditions 1 and 2 hold. If \( \mathbb{E}[S^2] < \infty \) or equivalently, \( \mathbb{E}[S_e] < \infty \), then
\[
P^{(N)}_{\text{loss}} \sim \frac{\rho P(S_e > N/\lambda)}{1 + \rho P(S_e > N/\lambda)}, \quad (6.12)
\]
where \( \xi_1(x) \sim \xi_2(x) \) presents \( \lim_{x \to \infty} \xi_1(x)/\xi_2(x) = 1 \).

**Proof.** It follows from (6.11) and \( \mathbb{E}[S_e] < \infty \) that
\[
\sum_{k=0}^{\infty} \bar{\pi}(k)e = \sum_{k=1}^{\infty} k\pi(k)e < \infty.
\]
Theorem 3.1 thus implies that Assumption 3.1 is satisfied. Note here that \( P \) in (6.10) may be periodic (the other conditions of Assumption 2.1 are satisfied). Nevertheless, according to Remark 5.2, Theorem 5.2 is applicable to the present setting. Applying Theorem 5.2 to (6.8) yields
\[
P^{(N)}_{\text{loss}} \sim 1 - \frac{1}{\rho \{1 + \pi(N)e\}} \pi(0)(-\mathcal{A}_0)^{-1}e + \beta_1. \quad (6.13)
\]
Furthermore, applying the Markov renewal theory (see, e.g., [6, Chapter 10, Theorem 4.3], and see also [21]) to the queue length process in the MAP/G/1 queue (with infinite capacity), we have
\[
\frac{1}{\lambda} = \pi(0)(-\mathcal{A}_0)^{-1}e + \beta_1 + \sum_{k=1}^{\infty} \pi(k)e \cdot \beta_1.
\]
\[
= \pi(0)(-\mathcal{A}_0)^{-1}e + \beta_1. \quad (6.14)
\]
Combining (6.13) and (6.14), and using $\rho = \lambda \beta_1$ and (6.11), we obtain

$$
P^{(N)}_{\text{loss}} \sim 1 - \frac{1}{\rho \{1 + \pi(N) e\} (\lambda - \beta_1) + \beta_1} \frac{\beta_1}{\rho}$$

$$= 1 - \frac{1}{\{1 + \pi(N) e\} (1 - \rho) + \rho} \frac{1}{1 + (1 - \rho) \pi(N) e}$$

$$\sim \frac{\rho P(S_e > N/\lambda)}{1 + \rho P(S_e > N/\lambda)},$$

which shows that (6.12) holds.

### A Proof of Theorem 5.2

We begin with the following lemma.

**Lemma A.1** If Assumption 2.1 holds, then, for $k \in \mathbb{Z}_{[0,N]}$,

$$
\pi^{(N)}(k) - \pi(k) = \pi^{(N)}(0) \left[ \left( \overline{B}(N - 1) - \overline{B}(N - 1) \right) H(N; k) + \sum_{n=N+1}^{\infty} B(n) S^{(N)}(n; k) + \frac{1}{-\sigma} \overline{B}(N - 1) e \pi(k) \right] \\
+ \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \left[ \left( \overline{A}(N - \ell - 1) - \overline{A}(N - \ell - 1) \right) H(N; k) + \sum_{n=N+1}^{\infty} A(n - \ell) S^{(N)}(n; k) + \frac{1}{-\sigma} \overline{A}(N - \ell - 1) e \pi(k) \right],
$$

(A.1)

where

$$
S^{(N)}(n; k) = (G^{N-k} - G_{n-k}) F_+(k; k) \\
+ (G^{N-1} - G^{n-1})(I - \Phi(0))^{-1} B(-1) H(0; k) \\
+ (G^{N} - G^{n}) (I - A - \beta A g)^{-1} e \pi(k), \quad 0 \leq k \leq N < n.
$$

(A.2)
Proof. It follows from (2.1) and (2.13) that
\[
P^{(N)} - P = \begin{pmatrix}
\mathbb{I}_{N-1} & \mathbb{I}_N & \mathbb{I}_{N+1} & \mathbb{I}_{N+2} & \ldots \\
\mathbb{L}_0 & O & \tilde{B}(N-1) - B(N) & -B(N+1) & -B(N+2) & \ldots \\
\mathbb{L}_1 & O & \tilde{A}(N-2) - A(N-1) & -A(N) & -A(N+1) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbb{L}_{N-1} & O & \tilde{A}(0) - A(1) & -A(2) & -A(3) & \ldots \\
\mathbb{L}_N & O & \tilde{A}(-1) - A(0) & -A(1) & -A(2) & \ldots \\
\mathbb{L}_{N+1} & O & -A(-1) & -A(0) & -A(1) & \ldots \\
\mathbb{L}_{N+2} & O & O & -A(-1) & -A(0) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}.
\]

Using this equation and (4.4), we have
\[
\pi^{(N)}(k) - \pi(k) = \pi^{(N)}(0) \left( \tilde{B}(N-1) - B(N) \right) H(N; k)
+ \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \left( \tilde{A}(N - \ell - 1) - A(N - \ell) \right) H(N; k)
- \sum_{n=N+1}^{\infty} \left\{ \pi^{(N)}(0) B(n) + \sum_{\ell=1}^{N} \pi^{(N)}(\ell) A(n - \ell) \right\} H(n; k), \quad k \in \mathbb{Z}_{[0,N]}.
\] (A.3)

From (2.5) and (2.6), we also have
\[
A(N - \ell) = \tilde{A}(N - \ell - 1) - \sum_{n=N+1}^{\infty} A(n - \ell), \quad \ell \in \mathbb{Z}_{[1,N]},
\]
\[
B(N) = \tilde{B}(N-1) - \sum_{n=N+1}^{\infty} B(n).
\]

Substituting these equations into (A.3) yields, for \(k \in \mathbb{Z}_{[0,N]}\),
\[
\pi^{(N)}(k) - \pi(k) = \pi^{(N)}(0) \left( \tilde{B}(N-1) - \tilde{B}(N-1) \right) H(N; k)
+ \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \left( \tilde{A}(N - \ell - 1) - \tilde{A}(N - \ell - 1) \right) H(N; k)
+ \pi^{(N)}(0) \sum_{n=N+1}^{\infty} B(n) \left\{ H(N; k) - H(n; k) \right\}
+ \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \sum_{n=N+1}^{\infty} A(n - \ell) \left\{ H(N; k) - H(n; k) \right\}.
\] (A.4)

Next, we discuss the term \(H(N; k) - H(n; k)\) in (A.4). Combining (4.19) and (4.25), we
obtain, for $n \in \mathbb{Z}_{\geq N+1}$ and $k \in \mathbb{Z}_{[0,N]}$,

\[
H(N; k) - H(n; k) = (1 - \delta_{0,k})(G^{N-k} - G^{n-k})F_+(k; k) \\
+ (G^{N-1} - G^{n-1})(I - \Phi(0))^{-1}B(-1)H(0; k) \\
+ \{u(n) - u(N)\} \pi(k).
\] (A.5)

Using (3.15), we rewrite the third term of (A.5) as

\[
u(n) - u(N) = (G^N - G^n)(I - A - \beta_A g)^{-1}e \\
+ (n - N)e g(I - A - \beta_A g)^{-1}e, \quad n \in \mathbb{Z}_{\geq N+1}.
\] (A.6)

Moreover, from (3.15) and [29, Theorem 3.1.1], we have

\[
\frac{1}{-\sigma} = g u(1) = g(I - G - eg)(I - A - \beta_A g)^{-1}e \\
= g(I - A - \beta_A g)^{-1}e.
\]

Thus, (A.6) leads to

\[
u(n) - u(N) = (G^N - G^n)(I - A - \beta_A g)^{-1}e + \frac{1}{-\sigma}(n - N)e, \quad n \in \mathbb{Z}_{\geq N+1}.
\] (A.7)

Substituting (A.7) into (A.5), and using (A.2), we obtain, for $n \in \mathbb{Z}_{\geq N+1}$ and $k \in \mathbb{Z}_{[0,N]}$,

\[
H(N; k) - H(n; k) = (G^{N-k} - G^{n-k})F_+(k; k) \\
+ (G^{N-1} - G^{n-1})(I - \Phi(0))^{-1}B(-1)H(0; k) \\
+ (G^N - G^n)(I - A - \beta_A g)^{-1}e \pi(k) \\
+ \frac{1}{-\sigma}(n - N)e \pi(k) \\
= S^{(N)}(n; k) + \frac{1}{-\sigma}(n - N) e \pi(k).
\] (A.8)

Furthermore, incorporating (A.8) into (A.4) results in the following: For $k \in \mathbb{Z}_{[0,N]}$,

\[
\pi^{(N)}(k) - \pi(k) \\
= \pi^{(N)}(0) \left( \tilde{B}(N - 1) - \overline{B}(N - 1) \right) H(N; k) \\
+ \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \left( \tilde{A}(N - \ell - 1) - \overline{A}(N - \ell - 1) \right) H(N; k) \\
+ \pi^{(N)}(0) \sum_{n=N+1}^{\infty} B(n) \left\{ S^{(N)}(n; k) + \frac{1}{-\sigma}(n - N)e \pi(k) \right\} \\
+ \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \sum_{n=N+1}^{\infty} A(n - \ell) \left\{ S^{(N)}(n; k) + \frac{1}{-\sigma}(n - N)e \pi(k) \right\}.
\] (A.9)
To obtain (A.1), we arrange third and fourth terms of (A.9). From (2.5) and (5.8), we have, for \( \ell \in \mathbb{Z}_{[1,N]} \),

\[
\sum_{n=N+1}^{\infty} nA(n-\ell) = \sum_{n=N+1}^{\infty} \sum_{m=1}^{n} A(n-\ell)
= \sum_{m=1}^{N+1} \sum_{n=N+1}^{\infty} A(n-\ell) + \sum_{m=N+2}^{\infty} \sum_{n=m}^{\infty} A(n-\ell)
= (N+1)\overline{A}(N-\ell) + \sum_{m=N+2}^{\infty} \overline{A}(m-\ell-1)
= N\overline{A}(N-\ell) + \sum_{m=N+1}^{\infty} \overline{A}(m-\ell-1)
= N\overline{A}(N-\ell) + \overline{A}(N-\ell-1),
\]

which leads to

\[
\sum_{n=N+1}^{\infty} (n-N)A(n-\ell) = \sum_{n=N+1}^{\infty} nA(n-\ell) - N\overline{A}(N-\ell)
= \overline{A}(N-\ell-1). \tag{A.10}
\]

Similarly, we have

\[
\sum_{n=N+1}^{\infty} (n-N)B(n) = \overline{B}(N-1). \tag{A.11}
\]

As a result, substituting (A.10) and (A.11) into (A.9) yields (A.1).

In what follows, we present the asymptotic results of the terms on the right hand side of (A.1).

**Lemma A.2** If Assumptions 2.1, 3.1 and 5.1 hold, then

\[
\lim_{N \to \infty} \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\overline{A}(N-\ell-1)e}{\overline{F}(N)} = \pi(0)c_A. \tag{A.12}
\]

**Proof.** It follows from Corollary 5.1 that for any \( \varepsilon \in (0,1) \) there exists some \( N_0 \in \mathbb{N} \) such that, for all \( N \geq N_0 \),

\[
(1 - \varepsilon)\pi(k) \leq \pi^{(N)}(k) \leq (1 + \varepsilon)\pi(k), \quad k \in \mathbb{Z}_{[0,N]}.
\]

Thus, for all \( \varepsilon \in (0,1) \) and \( N \geq N_0 \),

\[
\sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\overline{A}(N-\ell-1)e}{\overline{F}(N)} \leq (1 + \varepsilon) \sum_{\ell=1}^{N} \pi(\ell) \frac{\overline{A}(N-\ell-1)e}{\overline{F}(N)}, \tag{A.13}
\]

\[
\sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\overline{A}(N-\ell-1)e}{\overline{F}(N)} \geq (1 - \varepsilon) \sum_{\ell=1}^{N} \pi(\ell) \frac{\overline{A}(N-\ell-1)e}{\overline{F}(N)}. \tag{A.14}
\]
It follows from (5.8) that
\[
\sum_{k=N}^{\infty} \sum_{\ell=0}^{k} \pi(\ell) \overline{A}(k - \ell) = \sum_{\ell=0}^{N} \pi(\ell) \sum_{k=N}^{\infty} \overline{A}(k - \ell) + \sum_{\ell=N+1}^{\infty} \pi(\ell) \sum_{k=\ell}^{\infty} \overline{A}(k - \ell)
\]
\[
= \sum_{\ell=0}^{N} \pi(\ell) \overline{A}(N - \ell - 1) + \overline{\pi}(N) \overline{A}(-1),
\]
which leads to
\[
\sum_{\ell=1}^{N} \pi(\ell) \overline{A}(N - \ell - 1) = \sum_{k=N}^{\infty} \sum_{\ell=0}^{k} \pi(\ell) \overline{A}(k - \ell) - \pi(0) \overline{A}(N - 1) - \overline{\pi}(N) \overline{A}(-1).
\]
(A.15)

It also follows from (A.15), together with Lemma 6 in [14], Assumption 5.1 and Proposition 5.1, that
\[
\lim_{N \to \infty} \sum_{\ell=1}^{N} \pi(\ell) \overline{A}(N - \ell - 1) = \frac{\pi(0) c_B + \overline{\pi}(0) c_A}{-\sigma} \overline{A}(-1) + \sum_{k=0}^{\infty} \pi(k) c_A
\]
\[
- \pi(0) c_A - \frac{\pi(0) c_B + \overline{\pi}(0) c_A}{-\sigma} \overline{A}(-1)
\]
\[
= \sum_{k=1}^{\infty} \pi(k) c_A = \overline{\pi}(0) c_A.
\]
(A.16)

Combining (A.13), (A.14) and (A.16), and letting \( \varepsilon \to 0 \), we obtain (A.12).

**Lemma A.3** If Assumptions 2.1, 3.1 and 5.1 hold, then

\[
\lim_{N \to \infty} \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \sum_{n=N+1}^{\infty} \frac{A(n - \ell) S^{(N)}(n; k)}{F(N)} = O,
\]
(A.17)

\[
\lim_{N \to \infty} \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \left( \overline{A}(N - \ell - 1) - \overline{A}(N - \ell - 1) \right) H(N; k) \overline{F}(N) = O.
\]
(A.18)

**Proof.** We first prove (A.17). The matrix \( G \) is stochastic. Thus, (A.2) implies that for each \( k \in \mathbb{Z}_{[0, N]} \) there exists some \( C_S(k) > 0 \) such that
\[
|S^{(N)}(n, k)| \leq C_S(k) c e^T \quad \text{for all } N \in \mathbb{N} \text{ and } n \in \mathbb{Z}_{\geq N+1}.
\]
(A.19)
The inequality (A.19) yields
\[ \left| \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \sum_{n=N+1}^{\infty} \frac{A(n-\ell)S^{(N)}(n;k)}{F(N)} \right| \]

\[ = C_S(k) \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \sum_{n=N+1}^{\infty} \frac{A(n-\ell)ee^{T}}{F(N)} \]

\[ = C_S(k) \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{A(N-\ell)ee^{T}}{F(N)} \]

\[ \leq C_S(k) \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{A(N-\ell-1)ee^{T}}{F(N)}. \]  \hspace{1cm} (A.20)

It also follows from (5.6) and (5.8) that
\[ \lim_{k \to \infty} \frac{\overline{A}(k)e}{F(k)} = \lim_{k \to \infty} \frac{\overline{A}(k-1)e}{F(k-1)} - \frac{\overline{A}(k)e}{F(k)} = c_A - c_A = 0. \]  \hspace{1cm} (A.21)

Using (A.21), and proceeding as in the proof of Lemma A.2, we can readily show that
\[ \lim_{N \to \infty} \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\overline{A}(N-\ell-1)e}{F(N)} = 0. \]  \hspace{1cm} (A.22)

Combining (A.20) and (A.22) leads to (A.17).

Next we prove (A.18). It follows from (4.26) that \( H(N;k), N \in \mathbb{N}, k \in \mathbb{Z}_{[0,N]} \), can be decomposed as follows:
\[ H(N;k) = \tilde{H}(N;k) - \text{Neg}(I - A - \beta_A g)^{-1} e\pi(k), \]  \hspace{1cm} (A.23)

where
\[ \tilde{H}(N;k) = G^{N-k}F_+(k;k) + G^{N-1}(I - \Phi(0))^{-1}B(-1)H(0;k) \]
\[ - (I - G^{N})(I - A - \beta_A g)^{-1}e\pi(k). \]  \hspace{1cm} (A.24)

The decomposition (A.23) together with (2.14) yields
\[ \left| \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\overline{A}(N-\ell-1) - \overline{A}(N-\ell-1)\tilde{H}(N;k)}{F(N)} \right| \]
\[ = \left| \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\overline{A}(N-\ell-1) - \overline{A}(N-\ell-1)\tilde{H}(N;k)}{F(N)} \right| \]
\[ \leq \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\overline{A}(N-\ell-1) - \overline{A}(N-\ell-1)}{F(N)} |\tilde{H}(N;k)|. \]  \hspace{1cm} (A.25)
Furthermore, \((A.24)\) implies that for any \(k \in \mathbb{Z}_+\) there exists some constant \(\tilde{C}_H(k) > 0\) such that
\[
|\hat{H}(N; k)| \leq \tilde{C}_H(k)ee^T \quad \text{for all } N \in \mathbb{N}.
\] (A.26)

Applying \((A.26)\) to \((A.25)\), and using \((2.14)\) and \((A.22)\), we obtain
\[
\limsup_{N \to \infty} \left| \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \left( \frac{\tilde{A}(N - \ell - 1) - \tilde{A}(N - \ell - 1)}{F(N)} \right) H(N; k) \right|
\leq 2\tilde{C}_H(k) \limsup_{N \to \infty} \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\tilde{A}(N - \ell - 1)ee^T}{F(N)} = O,
\]
which shows that \((A.18)\) holds.

**Lemma A.4** If Assumptions 2.7, 3.1 and 5.1 hold, then
\[
\lim_{N \to \infty} \sum_{n=N+1}^{\infty} \frac{B(n)S^{(N)}(n; k)}{F(N)} = O,
\] (A.27)
\[
\lim_{N \to \infty} \frac{\left( \tilde{B}(N - 1) - \tilde{B}(N - 1) \right) H(N; k)}{F(N)} = O.
\] (A.28)

**Proof.** We first prove \((A.27)\). From \((A.19)\), we have
\[
\left| \sum_{n=N+1}^{\infty} \frac{B(n)S^{(N)}(n; k)}{F(N)} \right| \leq C_S(k) \sum_{n=N+1}^{\infty} \frac{B(n)ee^T}{F(N)} = C_S(k) \frac{\bar{B}(N)ee^T}{F(N)}
\]
\[
= C_S(k) \frac{\bar{B}(N - 1)ee^T - \bar{B}(N)ee^T}{F(N)}.
\]
Therefore, it follows from \((5.7)\) and \((5.8)\) that
\[
\limsup_{N \to \infty} \left| \sum_{n=N+1}^{\infty} \frac{B(n)S^{(N)}(n; k)}{F(N)} \right| \leq C_S(k) \limsup_{N \to \infty} \frac{\bar{B}(N - 1)ee^T - \bar{B}(N)ee^T}{F(N)} = O,
\]
which shows that \((A.27)\) holds.

Next we prove \((A.28)\). Using \((2.15)\), \((A.23)\) and \((A.26)\), we obtain
\[
\limsup_{N \to \infty} \left| \frac{\left( \tilde{B}(N - 1) - \tilde{B}(N - 1) \right) H(N; k)}{F(N)} \right|
\leq \limsup_{N \to \infty} \frac{|\tilde{B}(N - 1) - \tilde{B}(N - 1)| |\hat{H}(N; k)|}{F(N)}
\leq \bar{C}_H(k) \limsup_{N \to \infty} \frac{|\tilde{B}(N - 1) - \tilde{B}(N - 1)| ee^T}{F(N)}
\leq 2\bar{C}_H(k) \limsup_{N \to \infty} \frac{\bar{B}(N - 1)ee^T}{F(N)}
\leq 2\bar{C}_H(k) \limsup_{N \to \infty} \frac{\bar{B}(N - 2)ee^T - \bar{B}(N - 1)ee^T}{F(N)},
\] (A.29)
where the last equality holds due to (5.8). Applying to (5.7) to (A.29), we have
\[
\limsup_{N \to \infty} \left| \left( \tilde{B}(N - 1) - \bar{B}(N - 1) \right) H(N; k) \right| = O,
\]
which shows that (A.28) holds.

We are now ready to prove Theorem 5.2. Combining Lemmas A.1, A.3 and A.4, we obtain, for \( k \in \mathbb{Z}_+ \),
\[
\lim_{N \to \infty} \frac{\pi^{(N)}(k) - \pi(k)}{F(N)} = \frac{1}{-\sigma} \lim_{N \to \infty} \pi^{(N)}(0) \frac{\bar{B}(N - 1)e}{F(N)} \cdot \pi(k) 
+ \frac{1}{-\sigma} \lim_{N \to \infty} \sum_{\ell=1}^{N} \pi^{(N)}(\ell) \frac{\bar{A}(N - \ell - 1)e}{F(N)} \cdot \pi(k).
\]
(A.30)

Applying (5.7), Lemma A.2 and Corollary 5.1 to (A.30) yields
\[
\lim_{N \to \infty} \frac{\pi^{(N)}(k) - \pi(k)}{F(N)} = \frac{\pi(0)c_B + \pi(0)c_A}{-\sigma} \pi(k), \quad k \in \mathbb{Z}_+.
\]
Combining this result and Proposition 5.1 leads to
\[
\lim_{N \to \infty} \frac{\pi^{(N)}(k) - \pi(k)}{\bar{\pi}(N)e} = \lim_{N \to \infty} \frac{\pi^{(N)}(k) - \pi(k)}{F(N)} \frac{\bar{F}(N)}{\bar{\pi}(N)e} = \pi(k), \quad k \in \mathbb{Z}_+, \]
which shows that (5.9) holds.

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