A Constructive Approach to Topological Invariants for One-dimensional
Strictly Local Operators

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Abstract

In this paper we shall focus on one-dimensional strictly local operators, the notion of which naturally arises in the context of discrete-time quantum walks on the one-dimensional integer lattice $\mathbb{Z}$. In particular, we give an elementary constructive approach to the following two topological invariants associated with such operators: Fredholm index and essential spectrum. As a direct application, we shall explicitly compute and fully classify these topological invariants for a well-known quantum walk model.

Keywords: Strictly local operators, Fredholm index, Essential spectrum, Toeplitz operators, Quantum walks

1. Introduction

The underlying Hilbert space of this manuscript is $\ell^2(\mathbb{Z}, \mathbb{C}^n)$ of square-summable $\mathbb{C}^n$-valued sequences indexed by the set $\mathbb{Z}$ of all integers. With the obvious orthogonal decomposition $\ell^2(\mathbb{Z}, \mathbb{C}^n) = \bigoplus_{j=1}^{n} \ell^2(\mathbb{Z}, \mathbb{C})$ in mind, we shall consider the following finite sum of operators;

$$A = \sum_{y=-k}^{k} \begin{pmatrix} a_{11}(y, \cdot) L^y & \cdots & a_{1n}(y, \cdot) L^y \\ \vdots & \ddots & \vdots \\ a_{n1}(y, \cdot) L^y & \cdots & a_{nn}(y, \cdot) L^y \end{pmatrix} = \begin{pmatrix} \sum_{y=-k}^{k} a_{11}(y, \cdot) L^y & \cdots & \sum_{y=-k}^{k} a_{1n}(y, \cdot) L^y \\ \vdots & \ddots & \vdots \\ \sum_{y=-k}^{k} a_{n1}(y, \cdot) L^y & \cdots & \sum_{y=-k}^{k} a_{nn}(y, \cdot) L^y \end{pmatrix},$$

(1)

where $L$ denotes the bilateral left-shift operator on $\ell^2(\mathbb{Z}, \mathbb{C})$ (see Equation (2) for definition) and where each $a_{ij}(y, \cdot) = (a_{ij}(y, x))_{x \in \mathbb{Z}}$ is an arbitrary bounded $\mathbb{C}$-valued sequence viewed as a multiplication operator on $\ell^2(\mathbb{Z}, \mathbb{C}) = \bigoplus_{x \in \mathbb{Z}} \mathbb{C}$. An operator of the form (1) is known as a (one-dimensional) strictly local operator [CGS+18, §1.2]. Such an operator naturally arises, for example, in the context of $n$-state quantum walks defined on the integer lattice $\mathbb{Z}$, where we regard $\ell^2(\mathbb{Z}, \mathbb{C}^n)$ as the state Hilbert space of the walker. The purpose of this paper is to prove the following general theorem:

**Theorem A.** Let $A$ be a strictly local operator of the form (1) with the property that the following two-sided limits exist:

$$a_{ij}(y, \pm \infty) := \lim_{x \to \pm \infty} a_{ij}(y, x) \in \mathbb{C}, \quad i, j = 1, \ldots, n, \quad -k \leq y \leq k.$$ (A1)
Let

\[
A(\pm \infty) := \begin{pmatrix}
\sum_{y=-k}^{k} a_{11}(y, \pm \infty)L^y & \cdots & \sum_{y=-k}^{k} a_{1n}(y, \pm \infty)L^y \\
\vdots & \ddots & \vdots \\
\sum_{y=-k}^{k} a_{n1}(y, \pm \infty)L^y & \cdots & \sum_{y=-k}^{k} a_{nn}(y, \pm \infty)L^y
\end{pmatrix}, \quad (A2)
\]

\[
\hat{A}(z, \pm \infty) := \begin{pmatrix}
\sum_{y=-k}^{k} a_{11}(y, \pm \infty)z^y & \cdots & \sum_{y=-k}^{k} a_{1n}(y, \pm \infty)z^y \\
\vdots & \ddots & \vdots \\
\sum_{y=-k}^{k} a_{n1}(y, \pm \infty)z^y & \cdots & \sum_{y=-k}^{k} a_{nn}(y, \pm \infty)z^y
\end{pmatrix}, \quad z \in \mathbb{T}. \quad (A3)
\]

Then the following assertions hold true:

(i) We have that $A$ is Fredholm if and only if $\mathbb{T} \ni z \mapsto \det \hat{A}(z, \ast) \in \mathbb{C}$ is nowhere vanishing on $\mathbb{T}$ for each $\ast = \pm \infty$. In this case, the Fredholm index of $A$ is given by

\[
\text{ind} (A) = \text{wn} \left( \det \hat{A}(\cdot, +\infty) \right) - \text{wn} \left( \det \hat{A}(\cdot, -\infty) \right), \quad (A4)
\]

where $\text{wn} \left( \det \hat{A}(\cdot, \ast) \right)$ denotes the winding number of the function $\mathbb{T} \ni z \mapsto \det \hat{A}(z, \ast) \in \mathbb{C}$ with respect to the origin.

(ii) The essential spectrum of $A$ is given by

\[
\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A(+\infty)) \cup \sigma_{\text{ess}}(A(-\infty)), \quad (A5)
\]

\[
\sigma_{\text{ess}}(A(\ast)) = \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{A}(z, \ast) \right), \quad \ast = \pm \infty. \quad (A6)
\]

Note that the Fredholm index and essential spectrum can be viewed as topological invariants in the sense that these are stable under compact perturbations as is well-known. Theorem A(i) allows us to explicitly compute these topological invariants for strictly local operators, both of which depend only on the asymptotic values $\{A(1, \pm \infty)\}$. As we shall see in this article, Theorem A(ii) can be viewed as an abstract version of the one-dimensional bulk-boundary correspondence (see, for example, Corollary 4.3). In the setting of discrete-time quantum walks, a standard approach to Theorem A(ii) involves the use of the discrete Fourier transform and Weyl’s criterion for the essential spectrum; see, for example, Lemma 3.3, where the essential spectrum of the self-adjoint discriminant operator is determined via a certain spectral mapping theorem for quantum walks. Weyl’s criterion is capable of dealing with a wide range of perturbations that are not necessarily compact (see, for example, [SS17]), but its usage is obviously restricted to normal operators, unlike Theorem A(ii).

This paper is organised as follows. Proof of Theorem A is given in §2. In particular, our proof of the index formula $\{A(1, \pm \infty)\}$ is entirely motivated by $\text{Mat20}$, where the special case $n = 1$ of the index formula is established by making use of the notion of Toeplitz operators with $\mathbb{C}$-valued symbols. We show that Theorem A(ii) can be proved without relying on Weyl’s criterion, if we allow Toeplitz operators to have $\mathbb{C}^n \times \mathbb{C}^n$-valued symbols. The paper concludes with §3 where we explicitly compute and fully classify the Witten index of a certain quantum walk model on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ with the aid of Theorem A(i). This is the one-dimensional split-step quantum walk model considered in $\text{FFS17}$, $\text{FFS18}$, $\text{FFS19}$, $\text{ST19}$. $\text{Mat20}$, $\text{ST19}$, $\text{Mat20}$ with a modification that all parameters of the model depend freely on $\mathbb{Z}$. It is shown that this seemingly minor modification leads to the new index formula taking values from $\{-2, -1, 0, +1, +2\}$, where the indices $\pm 2$ do not appear in the existing literature $\text{ST19}$, $\text{Mat20}$. This result turns out to be significant improvement, since the Witten index gives a lower bound for the number of topologically protected bound states in the sense of §3.4.2. As a direct application of Theorem A(ii), we shall also compute the essential spectrum of the associated time-evolution operator.
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2. Analysis of Strictly Local Operators

2.1. Notation and terminology

By operators we shall always mean everywhere-defined bounded linear operators between Banach spaces throughout this paper. An operator \( A \) on a Hilbert space \( \mathcal{H} \) is said to be Fredholm, if \( \ker A, \ker A^* \) are finite-dimensional and if \( A \) has a closed range. Given such \( A \), we define the Fredholm index of \( A \) by

\[
\text{ind}(A) := \dim \ker A - \dim \ker A^*.
\]

It is well-known that the Fredholm index is invariant under compact perturbations. That is, given an operator \( A \) on \( \mathcal{H} \) and a compact operator \( K \) on \( \mathcal{H} \), we have that \( A \) is Fredholm if and only if so is \( A + K \), and in this case \( \text{ind}(A) = \text{ind}(A + K) \). The (Fredholm) essential spectrum of an operator \( A \) on \( \mathcal{H} \) is defined as the set \( \sigma_{\text{ess}}(A) \) of all \( \lambda \in \mathbb{C} \), such that \( A - \lambda I \) fails to be Fredholm. Note that \( \sigma_{\text{ess}}(A) \) is also stable under compact perturbations.

The Hilbert space of all square-summable \( \mathbb{C} \)-valued sequences \( \Psi = (\Psi(x))_{x \in \mathbb{Z}} \) is denoted by the shorthand \( \ell^2(\mathbb{Z}, \mathbb{C}) \). We have a natural orthogonal decomposition \( \ell^2(\mathbb{Z}) = \bigoplus_{i=1}^{\infty} \ell^2(\mathbb{Z}) \), where

\[
\ell^2_{\mathbb{L}}(\mathbb{Z}) := \{ \Psi \in \ell^2(\mathbb{Z}) \mid \Psi(x) = 0 \ \forall x \geq 0 \}, \quad \ell^2_{\mathbb{R}}(\mathbb{Z}) := \{ \Psi \in \ell^2(\mathbb{Z}) \mid \Psi(x) = 0 \ \forall x < 0 \}.
\]

The orthogonal projections of \( \ell^2(\mathbb{Z}) \) onto the above subspaces shall be denoted by \( P_{\mathbb{L}} \) and \( P_{\mathbb{R}} = 1 - P_{\mathbb{L}} \) respectively. For each \( \sharp \in \{ \mathbb{L}, \mathbb{R} \} \), the orthogonal projection \( P_{\sharp} \) can be written as \( P_{\sharp} = \iota_{\sharp}\iota_{\sharp}^* \), where \( \iota_{\sharp} : \ell^2_{\sharp}(\mathbb{Z}) \hookrightarrow \ell^2(\mathbb{Z}) \) is the inclusion mapping. The left-shift operator \( L \) on \( \ell^2(\mathbb{Z}) \) is defined by

\[
L\Psi := \Psi(-1), \quad \Psi \in \ell^2(\mathbb{Z}).
\]

Let \( n \in \mathbb{N} \) be fixed throughout the current section. Any operator \( A \) on \( \ell^2(\mathbb{Z}, \mathbb{C}^n) := \bigoplus_{j=1}^{n} \ell^2(\mathbb{Z}) \) admits the following unique block-operator matrix representation:

\[
A = \begin{pmatrix}
A_{11} & \ldots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \ldots & A_{nn}
\end{pmatrix}
\bigoplus_{j=1}^{n} \ell^2(\mathbb{Z}),
\]

where each \( A_{ij} \) is an operator on \( \ell^2(\mathbb{Z}) \). We shall agree to use the shorthand \( A = (A_{ij}) \) to mean that \( \Psi \in \ell^2(\mathbb{Z}, \mathbb{C}^n) \).

2.2. Strictly local operators

**Lemma 2.1.** Let \( (\delta_x)_{x \in \mathbb{Z}} \) be the standard complete orthonormal basis for \( \ell^2(\mathbb{Z}) \), and let \( A \) be an operator on \( \ell^2(\mathbb{Z}, \mathbb{C}^n) \) with the block-operator matrix representation \( A \). Then the following are equivalent:

\[
A_{\sharp} := \begin{pmatrix}
\iota_{\sharp}^* A_{11}\iota_{\sharp} & \ldots & \iota_{\sharp}^* A_{1n}\iota_{\sharp} \\
\vdots & \ddots & \vdots \\
\iota_{\sharp}^* A_{n1}\iota_{\sharp} & \ldots & \iota_{\sharp}^* A_{nn}\iota_{\sharp}
\end{pmatrix}
\bigoplus_{j=1}^{n} \ell^2_{\sharp}(\mathbb{Z})
\]
For each $i, j \in \{1, \ldots, n\}$, the operator $A_{ij}$ is a finite sum of the form $A_{ij} = \sum_{y=-k}^{k} a_{ij}(y, \cdot)L^y$ for some $\mathbb{C}$-valued sequences $a_{ij}(y, \cdot) = (a_{ij}(y, x))_{x \in \mathbb{Z}}$, where $-k \leq y \leq k$, viewed as multiplication operators on $\ell^2(\mathbb{Z}) = \bigoplus_{x \in \mathbb{Z}} \mathbb{C}$.

There exists a large enough positive integer $k$, such that for each $x \in \mathbb{Z}$ and for each $i, j \in \{1, \ldots, n\}$, the vector $A_{ij}\delta_y \in \ell^2(\mathbb{Z})$ belongs to the linear span of the finite set $\{\delta_{x-y} \mid -k \leq y \leq k\}$.

Following [CGS+18, §1.2], any operator $A$ satisfying the above equivalent conditions shall be referred to as a strictly local operator from here on. It follows from (i) that such $A$ admits a block-matrix representation of the form (1).

Proof. It is obvious that (i) implies (ii), since $L^y\delta_y = \delta_{x-y}$ for each $x, y \in \mathbb{Z}$. This equality shall be repeatedly used throughout the current section. To prove the converse, let (ii) hold true, and let $i, j$ be both fixed. For each $x \in \mathbb{Z}$, there exist finitely many scalars $a'_{ij}(y, x) \in \mathbb{C}$, where $-k \leq y \leq k$, such that

$$A_{ij}\delta_x = \sum_{y=-k}^{k} a'_{ij}(y, x)\delta_{x-y}. \quad (5)$$

Note that $a'_{ij}(y, \cdot) = (a'_{ij}(y, x))_{x \in \mathbb{Z}}$ is a bounded sequence for $-k \leq y \leq k$;

$$|a'_{ij}(y, x)| = |(\delta_{x-y}, A_{ij}\delta_y)_{\ell^2(\mathbb{Z})}| \leq \|A_{ij}\|\|\delta_{x-y}\|_{\ell^2(\mathbb{Z})}\|\delta_x\|_{\ell^2(\mathbb{Z})} \leq \|A_{ij}\|, \quad x \in \mathbb{Z}.$$  

Let $a_{ij}(y, x) := a'_{ij}(y, x + y)$ for each $x, y$. Then we obtain the following equality for each $x \in \mathbb{Z}$;

$$\sum_{y=-k}^{k} a_{ij}(y, \cdot)L^y\delta_x = \sum_{y=-k}^{k} a_{ij}(y, \cdot)\delta_{x-y} = \sum_{y=-k}^{k} a_{ij}(y, x - y)\delta_{x-y} = \sum_{y=-k}^{k} a'_{ij}(y, x)\delta_{x-y} = A_{ij}\delta_x,$$

where the last equality follows from (5). That is, (i) holds true.  

Corollary 2.2. If $A$ is a strictly local operator on $\ell^2(\mathbb{Z}, \mathbb{C}^n)$, then the difference $A - A_L \oplus A_R$ is finite-rank. Moreover, the following assertions hold true:

(i) The operator $A$ is Fredholm if and only if $A_L, A_R$ are both Fredholm. In this case, we have

$$\text{ind}(A) = \text{ind}(A_L) + \text{ind}(A_R). \quad (6)$$

(ii) The essential spectrum of $A$ is given by

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_L) \cup \sigma_{\text{ess}}(A_R). \quad (7)$$

Proof. Note that $P := \bigoplus_{j=1}^{n} P_R$ is the orthogonal projection onto $\ell^2_R(\mathbb{Z}, \mathbb{C}^n) = \bigoplus_{j=1}^{n} \ell^2_R(\mathbb{Z})$. We have

$$A - A_L \oplus A_R = PA(1 - P) + (1 - P)AP = PA - PAP + AP - PAP = P[P, A] + [A, P]P,$$

where $[X, Y] := XY - YX$ denotes the commutator of two operators $X, Y$. It remains to show that $[A, P]$ is finite-rank, where we may assume without loss of generality that $A$ is of the form (1). Since $P = \bigoplus_{j=1}^{n} P_R$ is a diagonal block-operator matrix, we obtain

$$[A, P] = \begin{pmatrix}
\sum_{y=-k}^{k} a_{11}(y, \cdot)L^y, P_R & \ldots & \sum_{y=-k}^{k} a_{1n}(y, \cdot)L^y, P_R \\
\vdots & \ddots & \vdots \\
\sum_{y=-k}^{k} a_{n1}(y, \cdot)L^y, P_R & \ldots & \sum_{y=-k}^{k} a_{nn}(y, \cdot)L^y, P_R
\end{pmatrix}.$$  

Since $[\cdot, P_R]$ is linear with respect to the first variable, each $(i, j)$-entry of the above block-operator matrix is given by $\sum_{y=-k}^{k} a_{ij}(y, \cdot)[L^y, P_R]$, where the commutator $[L^y, P_R]$ is finite-rank for $-k \leq y \leq k$. It follows that $A - A_L \oplus A_R$ is finite-rank, and so the remaining assertions immediately follow.
Note that a strictly local operator of the form (i) has the simplest formula, if each sequence \( a_{ij}(y, \cdot) \) is constant. Such an operator admits the following characterisation:

**Lemma 2.3.** Let \( A \) be an operator on \( \ell^2(\mathbb{Z}) \) with the \( \ell^2 \)-block-operator matrix representation (ii). Then the following are equivalent:

(i) For each \( i, j \in \{1, \ldots, n\} \), the operator \( A_{ij} \) is a finite sum of the form \( A_{ij} = \sum_{y=-k}^{k} a_{ij}(y)L^y \) for some complex numbers \( a_{ij}(y) \), where \( -k \leq y \leq k \).

(ii) The operator \( A \) is strictly local and \( [A_{ij}, L^x] = 0 \) for each \( x \in \mathbb{Z} \) and each \( i, j \in \{1, \ldots, n\} \).

The operator \( A \) is said to be uniform, if it satisfies the above equivalent conditions. It follows from (i) that such \( A \) admits a block-matrix representation of the following form:

\[
A = \sum_{y=-k}^{k} \begin{pmatrix}
    a_{11}(y)L^y & \cdots & a_{1n}(y)L^y \\
    \vdots & \ddots & \vdots \\
    a_{n1}(y)L^y & \cdots & a_{nn}(y)L^y
\end{pmatrix} = \begin{pmatrix}
    \sum_{y=-k}^{k} a_{11}(y)L^y & \cdots & \sum_{y=-k}^{k} a_{1n}(y)L^y \\
    \vdots & \ddots & \vdots \\
    \sum_{y=-k}^{k} a_{n1}(y)L^y & \cdots & \sum_{y=-k}^{k} a_{nn}(y)L^y
\end{pmatrix}.
\]  

(8)

**Proof.** It is obvious that (i) implies (ii). To prove the converse, let \( A = (A_{ij}) \) be strictly local, and let \([A_{ij}, L^x] = 0 \) for each \( x \in \mathbb{Z} \) and for each \( i, j \in \{1, \ldots, n\} \). It follows from Lemma 2.3(ii) that for each \( i, j \in \{1, \ldots, n\} \) we have \( A_{ij} = \sum_{y=-k}^{k} a_{ij}(y)L^y \). It remains to show that the sequence \( a_{ij}(y, \cdot) = (a_{ij}(y, x))_{x \in \mathbb{Z}} \) is constant for a fixed pair \( i, j \in \{1, \ldots, n\} \). Since \( A_{ij}\delta_x = \sum_{y=-k}^{k} a_{ij}(y, x-y)\delta_{x-y} \) for each \( x \in \mathbb{Z} \), we get

\[
A_{ij}\delta_x = A_{ij}L^{-x}\delta_0 = L^{-x}A_{ij}\delta_0 = L^{-x}\left( \sum_{y=-k}^{k} a_{ij}(y, -y)\delta_{-y} \right) = \sum_{y=-k}^{k} a_{ij}(y, -y)\delta_{x-y}.
\]

It follows that \( a_{ij}(y, x-y) = a_{ij}(y, -y) \) for each \( x \in \mathbb{Z} \) and for each \( y \in \{-k, \ldots, k\} \). The claim follows.

2.3. Uniform operators

The following result is one of the main theorems of the current section:

**Theorem 2.4.** Let \( A \) be a uniform operator on \( \mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^n) \) of the form (ii), and let

\[
\hat{A}(z) := \begin{pmatrix}
    \sum_{y=-k}^{k} a_{11}(y)z^y & \cdots & \sum_{y=-k}^{k} a_{1n}(y)z^y \\
    \vdots & \ddots & \vdots \\
    \sum_{y=-k}^{k} a_{n1}(y)z^y & \cdots & \sum_{y=-k}^{k} a_{nn}(y)z^y
\end{pmatrix}, \quad z \in \mathbb{T}.
\]  

(9)

Then the following assertions hold true:

(i) The operator \( A \) is Fredholm if and only if \( A_L, A_R \) are both Fredholm if and only if \( \mathbb{T} \ni z \mapsto \det \hat{A}(z) \in \mathbb{C} \) is nowhere vanishing on \( \mathbb{T} \). In this case, we have \( \text{ind } A = \text{ind } A_R + \text{ind } A_L = 0 \), and

\[
\text{ind } A_R = \text{wn } (\det \hat{A}).
\]

(ii) The essential spectrum of \( A \) is given by

\[
\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_L) \cup \sigma_{\text{ess}}(A_R) = \bigcup_{z \in \mathbb{T}} \sigma(\hat{A}(z)).
\]  

(10)

A proof of Theorem 2.4 shall be given at the end of the current subsection. Let us first recall the notion of Toeplitz operators. Let \( L^2(\mathbb{T}) \) be the Hilbert space of square-summable functions on the unit-circle \( \mathbb{T} \), where \( \mathbb{T} \) is endowed with the normalised arc-length measure. It is well-known that \( L^2(\mathbb{T}) \) admits the standard complete orthonormal basis \( (e_x)_{x \in \mathbb{Z}} \), where each \( e_x \) is defined by \( \mathbb{T} \ni z \mapsto z^x \in \mathbb{C} \). The Hardy-Hilbert
**space** \( H^2 \) is the closure of the linear span of the set \( \{ e_x \mid x \geq 0 \} \). Let \( \iota : H^2 \hookrightarrow L^2(T) \) be the inclusion mapping, and let \( f \in C(T) \). Then the **Toeplitz operator** \( T_f \) with symbol \( f \) is defined by

\[
T_f := \iota^* M_f \iota,
\]

where \( M_f : L^2(T) \to L^2(T) \) is the multiplication operator by \( f \). More generally, let us consider the Banach space \( C(T, \mathbb{C}^{n \times n}) \) of continuous matrix-valued functions on \( T \). Given a function \( F \in C(T, \mathbb{C}^{n \times n}) \) of the form \( F(z) = (f_{ij}(z)) \) for each \( z \in T \), the Toeplitz operator with symbol \( F \) is defined by

\[
T_F := \left( T_{f_{11}}, \ldots, T_{f_{nn}} \right)
\]

where \( \left( T_{f_{ij}} \right) \forall i, j \in \mathbb{Z} \).

The following result is standard;

**Theorem 2.5.** Let \( F \in C(T, \mathbb{C}^{n \times n}) \) be a matrix-valued function of the form \( F(\cdot) = (f_{ij}(\cdot)) \), and let \( T_F \) be the corresponding Toeplitz operator given by (12). Then the following assertions hold true:

(i) The Toeplitz operator \( T_F \) is Fredholm if and only if \( T \ni z \mapsto \det F(z) \in \mathbb{C} \) is nowhere vanishing on \( T \). In this case,

\[
\text{ind } T_F = - \text{wn}(\det F).
\]

(ii) The **essential spectrum** of \( T_F \) is given by

\[
\sigma_{\text{ess}}(T_F) = \bigcup_{z \in T} \sigma(F(z)).
\]

**Proof.** Note that (i) is the celebrated theorem of Gohberg-Krein (see, for example, [Mur06, Theorem 3.3]). It remains to prove (ii). Let \( B_n(H^2) := \mathcal{B} \left( \bigoplus_{j=1}^{n} H^2 \right) \) be the \( C^* \)-algebra of operators on \( \bigoplus_{j=1}^{n} H^2 \), and let \( \mathcal{K}_n(H^2) := \mathcal{K} \left( \bigoplus_{j=1}^{n} H^2 \right) \) be the ideal of compact operators on \( \bigoplus_{j=1}^{n} H^2 \). Let \( \mathcal{A}_n(H^2) \) be the closed \( * \)-subalgebra of \( B_n(H^2) \) generated by \( \{ T_F \mid F \in C(T, \mathbb{C}^{n \times n}) \} \). It is a well-known result that the following mapping is \( * \)-isomorphic (see, for example, [Dou73, §1]):

\[
C(T, \mathbb{C}^{n \times n}) \ni F \mapsto [T_F] \in \mathcal{A}_n(H^2)/\mathcal{K}_n(H^2).
\]

That is, for each \( F \in C(T, \mathbb{C}^{n \times n}) \) we have that \( F \) is invertible in \( C(T, \mathbb{C}^{n \times n}) \) if and only if \( [T_F] \) is invertible in the Calkyn algebra \( \mathcal{B}_n(H^2)/\mathcal{K}_n(H^2) \). The equality (13) follows.

Let us consider two unitary operators \( \mathcal{F}_L : H^2 \to \ell_2^L(\mathbb{Z}) \) and \( \mathcal{F}_R : H^2 \to \ell_2^R(\mathbb{Z}) \) defined respectively by

\[
\mathcal{F}_L e_x := \delta_{-x-1}, \quad \mathcal{F}_R e_x := \delta_x, \quad x \geq 0,
\]

where \( (\delta_x)_{x \in \mathbb{Z}}, (e_x)_{x \in \mathbb{Z}} \) are the standard bases of \( \ell^2(\mathbb{Z}) \), \( H^2 \) respectively.

**Lemma 2.6.** Let \( A \) be a uniform operator on \( \mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^n) \) of the form (8), and let \( \hat{A} \) be given by (9). Then

\[
\left( \bigoplus_{j=1}^{n} \mathcal{F}_j^* \right) A \left( \bigoplus_{j=1}^{n} \mathcal{F}_j \right) = \begin{pmatrix} T_{\hat{A}} & z = L, \ 
T_{\hat{A}(-)} & z = R \end{pmatrix},
\]

where \( \hat{A}(-) \) denote the matrix-valued function \( T \ni z \mapsto \hat{A}(z^*) \in \mathbb{C}^{n \times n} \).
Proof. Note first that the inverses of $F_L, F_R$ are given respectively by $F_L^{-1}\delta_x = F_L^1\delta_x = e_{-x-1}$ for each $x < 0$ and $F_R^{-1}\delta_x = F_R^1\delta_x = e_x$ for each $x \geq 0$. Let us first prove the following non-trivial equalities:

$$T_{ey} = F_L^e L^y t_L F_L = F_R^e L^y t_R F_R, \quad y \in \mathbb{Z}.$$  \hspace{1cm} (16)

Note that for each $y \geq 0$ each $x \geq 0$ we have

$$\begin{align*}
T_{ey} e_x &= e^* M_{ey} e_x = e^* M_{ey+y} e_x = e_{x+y}, \\
F_L^e L^y t_L F_L e_x &= F_L^{e^*} L^y t_L \delta_{x-1} = F_L^{e^*} \delta_{x-1-y} = e_{x+y}, \\
F_R^e L^y t_R F_R e_x &= F_R^{e^*} L^y t_R \delta_x = F_R^{e^*} \delta_{x+y} = e_{x+y}.
\end{align*}$$

That is, we have shown that (16) holds true for any $y \geq 0$. On the other hand, if $y < 0$, then $-y > 0$, and so

$$T_{ey} = (T_{-ey})^* = (F_L^e L^y t_L F_L)^* = (F_R^e L^y t_R F_R)^*, \quad y < 0.$$ \hspace{1cm} (16)

That is, (16) holds true for any $y \in \mathbb{Z}$. Let

$$f_{ij}(z) := \sum_{y=-k}^{k} a_{ij}(y) z^y = \sum_{y=-k}^{k} a_{ij}(y) e_y(z), \quad z \in \mathbb{T}.$$ Then the block-operator matrix representation of $A$ is given by

$$A = \begin{pmatrix}
\sum_{y=-k}^{k} a_{11}(y) L^y & \ldots & \sum_{y=-k}^{k} a_{1n}(y) L^y \\
\vdots & \ddots & \vdots \\
\sum_{y=-k}^{k} a_{n1}(y) L^y & \ldots & \sum_{y=-k}^{k} a_{nn}(y) L^y
\end{pmatrix} = \begin{pmatrix}
f_{11}(L) & \ldots & f_{1n}(L) \\
\vdots & \ddots & \vdots \\
f_{n1}(L) & \ldots & f_{nn}(L)
\end{pmatrix}.$$ With the representation (14) in mind, we obtain

$$\left( \bigoplus_{j=1}^{n} F_j^* \right) \left( \bigoplus_{j=1}^{n} F_j \right) = \begin{pmatrix}
F_1^* L_{ij} F_1^* L_{ij} F_1 & \ldots & F_1^* L_{ij} F_1 F_2 \\
\vdots & \ddots & \vdots \\
F_1^* L_{ij} F_2 & \ldots & F_1^* L_{ij} F_2 F_2
\end{pmatrix}, \quad \# = L, R,$$

where (16) gives the following equalities for each $i, j \in \{1, \ldots, n\}$:

$$F_1^* L_{ij} L_F F_2 = \sum_{y=-k}^{k} a_{ij}(y) (F_1^* L_{ij} L_F F_2) = \begin{pmatrix}
\sum_{y=-k}^{k} a_{ij}(y) T_{ey} = T_{f_{ij}}, \\
\sum_{y=-k}^{k} a_{ij}(y) T_{ey} = T_{f_{ij}}, \\
\sum_{y=-k}^{k} a_{ij}(y) T_{ey} = T_{f_{ij}}.
\end{pmatrix} \leq L, \quad \# = R.$$ It follows that (16) holds true, since the Toeplitz operator $T_A$ is given by (14) with $F := \hat{A}$. \qed

Proof of Theorem 2.4. Let $A$ be a uniform operator on $H = L^2(\mathbb{Z}, \mathbb{C}^n)$ of the form (8), and let $\hat{A}$ be given by (9). It follows from (15) that $A_L \cong T_{\hat{A}}$ and $A_R \cong T_{\hat{A}(-)}$, where $\cong$ denotes unitary equivalence. The Fredholmness and essential spectra are invariant under unitary transforms.

(i) It follows from Corollary 2.2(i) that the operator $A$ is Fredholm if and only if $A_L \cong T_{\hat{A}}$, $A_R \cong T_{\hat{A}(-)}$ are both Fredholm, and in this case we have $\text{ind} A = \text{ind} T_{\hat{A}} + \text{ind} T_{\hat{A}(-)}$. On the other hand, it follows from Theorem 2.5(i) that $A_L \cong T_{\hat{A}}$ is Fredholm if and only if $A_R \cong T_{\hat{A}(-)}$ is Fredholm if and only if $\text{det} \hat{A}$ is nowhere vanishing. In this case, we have $\text{ind} A_L = -\text{wn} \left( \text{det} \hat{A} \right)$ and $\text{ind} A_R = -\text{wn} \left( \text{det} \hat{A} \right)$.

Therefore, $\text{ind} A = \text{ind} A_L + \text{ind} A_R = -\text{wn} \left( \text{det} \hat{A} \right) - \text{wn} \left( \text{det} \hat{A} \right) = -\text{wn} \left( \text{det} \hat{A} \right) + \text{wn} \left( \text{det} \hat{A} \right) = 0$.

(ii) It follows from Corollary 2.2(ii) that $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_L) \cup \sigma_{\text{ess}}(A_R)$, where $A_L \cong T_{\hat{A}}$, $A_R \cong T_{\hat{A}(-)}$. It follows from Theorem 2.5(ii) that

$$\sigma_{\text{ess}}(A_L) = \sigma_{\text{ess}}(T_{\hat{A}}) = \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{A}(z) \right) = \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{A}(z) \right) = \sigma_{\text{ess}} \left( T_{\hat{A}(-)} \right) = \sigma_{\text{ess}}(A_R),$$

where the third equality follows from the fact that the ranges of $\hat{A}, \hat{A}(-)$ are identical. \qed
2.4. Proof of Theorem A

We are now in a position to prove Theorem A with the aid of the following lemma;

**Lemma 2.7. Under the assumption of Theorem A, the difference \( A_L \ominus A_R = A(-\infty)_L \oplus A(+\infty)_R \) is compact.**

It follows that \( A_L - A(-\infty)_L, A_R - A(+\infty)_R \) are both compact.

**Proof.** Let \( P_+ = \bigoplus_{j=1}^n P_j \), and let \( P_- := \bigoplus_{j=1}^n P_j \). We have

\[
A_L \ominus A_R = A_L \ominus 0 + 0 \ominus A_R - (A(-\infty)_L \ominus 0 + 0 \ominus A(+\infty)_R) = P_- A \ominus P_- A + P_- A \ominus A(-\infty)_R + P_+ A(+\infty)_R
\]

It remains to show that \( B(\ast) := P_+(A - A(\ast)) P_- \) is compact for each \( \ast = \pm \infty \). We have

\[
B(\ast) = P_+(a_{ij}(y, \ast) - a_{ij}(y, \ast))L^y \quad \text{for each } \ast = \pm \infty
\]

If \( B(\ast) = (B_{ij}(\ast)) \) is the block-operator matrix representation of \( B \), then

\[
B_{ij}(\ast) = \begin{cases}
\sum_{y=-k}^k P_\ell(a_{ij}(y, \ast) - a_{ij}(y, \ast))L^y, & \ast = +\infty, \\
\sum_{y=-k}^k P_\ell(a_{ij}(y, \ast) - a_{ij}(y, \ast))L^y, & \ast = -\infty.
\end{cases}
\]

Note that the projection \( P_\ell \) on \( \ell^2(\mathbb{Z}) = \bigoplus_{x \in \mathbb{Z}} \mathbb{C} \) is a multiplication operator of the form \( P_\ell = \bigoplus_{x \in \mathbb{Z}} \delta_\ell(x) \), where \( \delta_\ell(x) := 1 \) for each \( x \geq 0 \) and \( \delta_\ell(x) := 0 \) for each \( x < 0 \). That is, for each \( y \in \{-k, \ldots, k\} \),

\[
P_\ell(a_{ij}(y, \ast) - a_{ij}(y, \ast)) = \bigoplus_{x \in \mathbb{Z}} \delta_\ell(x)(a_{ij}(y, x) - a_{ij}(y, +\infty)),
\]

\[
\lim_{x \to \pm \infty} \delta_\ell(x)(a_{ij}(y, x) - a_{ij}(y, +\infty)) = 0.
\]

It follows that \( P_\ell(a_{ij}(y, \ast) - a_{ij}(y, +\infty)) \) is compact for each \( i, j, y \), and so \( B_{ij}(\pm \infty) \) given by (17) is compact. Hence, \( B(\pm \infty) = (B_{ij}(\pm \infty)) \) is compact. An analogous argument can be used to show that \( B(\ast) = (B_{ij}(\ast)) \) is compact. The claim follows.

**Proof of Theorem A.** Under the assumption of Theorem A it follows from Corollary 2.2 and Lemma 2.7 that \( A - A(-\infty)_L \ominus A(+\infty)_R \) is compact, where \( A(-\infty)_L \ominus A(+\infty)_R \) are uniform operators.

(i) We shall make use of Theorem 2.4 (i). We have that \( A \) is Fredholm if and only if \( A(-\infty)_L \ominus A(+\infty)_R \) are Fredholm if and only if \( \hat{A}(\ast, \ast) \) is nowhere vanishing on \( \mathbb{T} \) for each \( \ast = \pm \infty \). In this case, we have

\[
\text{ind } A = \text{ind } A(-\infty)_L \ominus \text{ind } A(+\infty)_R = -\text{wn } \left( \hat{A}(\ast, -\infty) \right) + \text{wn } \left( \hat{A}(\ast, +\infty) \right).
\]

(ii) We shall make use of Theorem 2.4 (ii). We have

\[
\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A(-\infty)_L) \cup \sigma_{\text{ess}}(A(+\infty)_R) = \sigma_{\text{ess}}(A(-\infty)) \cup \sigma_{\text{ess}}(A(+\infty)),
\]

where each \( \sigma_{\text{ess}}(A(\pm \infty)) \) is given by the formula (A6).
3. Applications of Theorem A

Quantum walk theory is a quantum-mechanical counterpart of the classical random walk theory, and it constitutes a vast multidisciplinary branch of modern Science. Certain primitive forms of this ubiquitous notion can be found, for example, in [Gud88, AD293, Mey96, ABN+01]. Mathematically rigorous studies of discrete-time quantum walks include: scattering-theoretic analysis [Seg11, CGML12, Suz16, FFS18, FFS19], localisation and weak-limit theorems [Kon02, IKK04, Seg11, CGML12, Suz16, FFS18, FFS19], classification theorems [Ohn16, Ohn17, CGG+16, CGS+18], discrete analogues of the time-operator [ST19a, FMS18, FMS19], and index theorems [CGS+18, ST19a, ST20].

In this section we shall give a new index theorem in align with the setting of [CGG+18, ST19a, Mat20, as a direct application of Theorem A]. More precisely, we consider a concrete quantum walk model defined on the underlying Hilbert space \( \ell^2(\mathbb{Z}, \mathbb{C}^2) \), which unifies all of the following models: the one-dimensional quantum walk considered in [ABN+01, Kon02, Suz16, Kit], Kitagawa’s split-step quantum walk [KRB10, KKB12, Kit12], another split-step quantum walk [FFS17, FFS18, FFS19, ST19b, Mat20]. We compute the following two associated topological invariants: (i) a certain well-defined Fredholm index known as the Witten index, and (ii) the essential spectrum of the evolution operator. The complete classification of these invariants can be found in Theorem B(i), (ii) respectively. The precise statement of Theorem B including the definition of the model, appears in §3.1. Proof of Theorem B(i), (ii) will be given in §3.3. This paper concludes with several remarks in §3.3.

3.1. Statement of the main theorem (Theorem B)

What follows is a brief overview of [Suz19, §2] and [ST19a, §2.1]. An operator \( \Gamma \) defined on an abstract Hilbert space \( \mathcal{H} \) is called an involution, if \( \Gamma^2 = 1 \), where 1 denotes the identity operator on \( \mathcal{H} \). Note first that any unitary involution is self-adjoint (in fact, if an operator possess any two of the properties “self-adjoint”, “unitary”, “involutory”, then it automatically has the third). The two operators \(-1, 1\) are referred to as trivial unitary involutions. We have \( \mathcal{H} = \ker(\Gamma - 1) \oplus \ker(\Gamma + 1) \) for any unitary involution \( \Gamma \). The model we shall consider in this section is based on the following simple finite-dimensional example;

**Example 3.1.** If \( X \) denotes a \( 2 \times 2 \) matrix viewed as an operator on \( \mathbb{C}^2 \), then \( X \) is a non-trivial unitary involution if and only if there exist \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{C} \) satisfying the following equalities:

\[
X = \begin{pmatrix} \alpha & \beta \\ \beta^* & -\alpha \end{pmatrix}, \quad \alpha^2 + |\beta|^2 = 1. \tag{18}
\]

Note that the spectrum of such a matrix \( X \) is \( \{1, -1\} \), and so the following diagonalisation is possible:

\[
\eta^* \begin{pmatrix} \alpha & \beta \\ \beta^* & -\alpha \end{pmatrix} \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sqrt{\Theta} \\ 0 & e^{-i\Theta} \end{pmatrix} \begin{pmatrix} \alpha_+ & -\alpha_- \\ -\alpha_+ & \alpha_- \end{pmatrix}, \tag{19}
\]

where \( \Theta \) is any real number satisfying \( \beta = e^{i\Theta} |\beta| \) and \( \alpha_\pm := \sqrt{1 \pm \alpha} \). Note that \( \eta \) is unitary.

A chiral pair on \( \mathcal{H} \) is any pair \( (\Gamma, \Gamma') \) of two unitary involutions \( \Gamma, \Gamma' \) on \( \mathcal{H} \). Given such a pair \( (\Gamma, \Gamma') \), the operator \( U := \Gamma \Gamma' \) is called the evolution operator of \( (\Gamma, \Gamma') \). Let \( R, Q \) be the real and imaginary parts of \( U \) respectively:

\[
R := \text{Re} \, U = \frac{U + U^*}{2} = \begin{pmatrix} \Gamma & \Gamma' \end{pmatrix}, \quad Q := \text{Im} \, U = \frac{U - U^*}{2i} = \begin{pmatrix} \Gamma & \Gamma' \end{pmatrix},
\]

where \( \{X,Y\} := XY + YX \) and \( \{X,Y\} := XY - YX \). Note that the evolution operator \( U \) satisfies the following chiral symmetry conditions with respect to \( \Gamma, \Gamma' \) respectively:

\[
U^* = \Gamma' \Gamma = \Gamma^2 \Gamma' \Gamma = \Gamma'(\Gamma \Gamma') \Gamma = \Gamma \Gamma', \quad U' = \Gamma' \Gamma = \Gamma \Gamma'(\Gamma')^2 = \Gamma' (\Gamma \Gamma') \Gamma' = \Gamma' \Gamma \Gamma'. \tag{20}
\]

The above equality immediately implies the commutation relations \( [\Gamma, R] = [\Gamma', R] = 0 \) and anti-commutation relations \( \{\Gamma, Q\} = \{\Gamma', Q\} = 0 \), and so \( R, Q, U \) admit the following block-operator matrix representations.
with respect to \( \mathcal{H} = \ker(\Gamma - 1) \oplus \ker(\Gamma + 1) = \ker(\Gamma' - 1) \oplus \ker(\Gamma' + 1) \):

\[
R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)} = \begin{pmatrix} R'_1 & 0 \\ 0 & R'_2 \end{pmatrix}_{\ker(\Gamma' - 1) \oplus \ker(\Gamma' + 1)},
\]

\[
Q = \begin{pmatrix} 0 & Q_0' \\ Q_0 & 0 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)} = \begin{pmatrix} 0 & (Q_0')^* \\ Q_0' & 0 \end{pmatrix}_{\ker(\Gamma' - 1) \oplus \ker(\Gamma' + 1)},
\]

\[
U = \begin{pmatrix} R_1 & iQ_0' \\ iQ_0 & R_2 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)} = \begin{pmatrix} R'_1 & i(Q_0')^* \\ iQ_0' & R'_2 \end{pmatrix}_{\ker(\Gamma' - 1) \oplus \ker(\Gamma' + 1)},
\]

where \( R_j, R'_j \) are self-adjoint for each \( j = 1, 2 \). The chiral pair \((\Gamma, \Gamma')\) is said to be Fredholm, if the operator \( Q \) is Fredholm (or, equivalently, \( Q_0, Q_0' \) are Fredholm). In this case, we define the Witten indices of the pairs \((\Gamma, \Gamma'), (\Gamma', \Gamma)\) by the following formulas respectively:

\[
\text{ind} (\Gamma, \Gamma') := \text{ind} Q_0, \quad \text{ind} (\Gamma', \Gamma) := \text{ind} Q_0',
\]

where \( \text{ind} Q_0, \text{ind} Q_0' \) denote the Fredholm indices of \( Q_0, Q_0' \) respectively. The Witten indices introduced above are unitarily invariant in the following precise sense (see [ST19b, Theorem 3] for details); for any unitary operator \( \epsilon \) on \( \mathcal{H} \), we have that \((\Gamma, \Gamma')\) is Fredholm if and only if so is \((\epsilon^* \Gamma \epsilon, \epsilon^* \Gamma' \epsilon)\), and that in this case

\[
\text{ind} (\Gamma, \Gamma') = \text{ind} (\epsilon^* \Gamma \epsilon, \epsilon^* \Gamma' \epsilon), \quad \text{ind} (\Gamma', \Gamma) = \text{ind} (\epsilon^* \Gamma' \epsilon, \epsilon^* \Gamma \epsilon).
\]

With Example 5.1 in mind, we are now in a position to state the main theorem of the current section;

**Theorem B.** Let \( \Gamma, \Gamma' \) be two unitary involutions defined respectively as the following block-operator matrices with respect to \( \ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \):

\[
\Gamma := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & q \\ q^* & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} p & qL \\ L^*q^* & -p(-1) \end{pmatrix}, \quad (B1)
\]

\[
\Gamma' := \begin{pmatrix} a & b^* \\ b & -a \end{pmatrix}, \quad (B2)
\]

where we assume that two convergent \( \mathbb{R} \)-valued sequences \( p = (p(x))_{x \in \mathbb{Z}}, a = (a(x))_{x \in \mathbb{Z}} \) and two convergent \( \mathbb{C} \)-valued sequences \( q = (q(x))_{x \in \mathbb{Z}}, b = (b(x))_{x \in \mathbb{Z}} \) satisfy the following:

\[
p(x)^2 + |q(x)|^2 = 1, \quad x \in \mathbb{Z}, \quad (B3)
\]

\[
a(x)^2 + |b(x)|^2 = 1, \quad x \in \mathbb{Z}, \quad (B4)
\]

\[
p(\pm \infty) := \lim_{x \to \pm \infty} p(x) \in \mathbb{R}, \quad a(\pm \infty) := \lim_{x \to \pm \infty} a(x) \in \mathbb{R}, \quad (B5)
\]

\[
q(\pm \infty) := \lim_{x \to \pm \infty} q(x) \in \mathbb{C}, \quad b(\pm \infty) := \lim_{x \to \pm \infty} b(x) \in \mathbb{C}, \quad (B6)
\]

\[
\theta(\pm \infty) := \begin{cases} \arg g(\pm \infty), & q(\pm \infty) \neq 0, \\ 0, & q(\pm \infty) = 0, \end{cases} \quad \phi(\pm \infty) := \begin{cases} \arg b(\pm \infty), & b(\pm \infty) \neq 0, \\ 0, & b(\pm \infty) = 0, \end{cases} \quad (B7)
\]

where \( \arg \) of a non-zero complex number \( w \) is uniquely defined by \( w = e^{i \arg w} \) and \( \arg w \in [0, 2\pi) \). Then the following two assertions hold true:
(i) The chiral pair \((\Gamma, \Gamma')\) is Fredholm if and only if \(|p(\ast)| \neq |a(\ast)|\) for each \(\ast = \pm \infty\). In this case, we have

\[
\text{ind} (\Gamma, \Gamma') = \begin{cases} 
0, & |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
+\text{sgn} p(+\infty), & |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|, \\
-\text{sgn} p(-\infty), & |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
-\text{sgn} p(+\infty) - \text{sgn} p(-\infty), & |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|,
\end{cases}
\]  

\[
\text{ind} (\Gamma', \Gamma) = \begin{cases} 
0, & |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|, \\
-\text{sgn} a(-\infty) + \text{sgn} a(+\infty), & |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
+\text{sgn} a(-\infty), & |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|, \\
-\text{sgn} a(+\infty), & |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|,
\end{cases}
\]  

where the sign function \(\text{sgn} : \mathbb{R} \to \{-1, 1\}\) is defined by

\[
\text{sgn} x := \begin{cases} 
\frac{x}{|x|}, & x \neq 0, \\
1, & x = 0.
\end{cases}
\]  

(ii) The essential spectrum of the evolution operator \(U := \Gamma \Gamma'\) is given by

\[
\sigma_{\text{ess}}(U) = \bigcup_{\ast = \pm \infty} \{ z \in \mathbb{T} \mid \text{sgn} (p(\ast)a(\ast)) \cdot \text{Re} z \in I(\ast) \},
\]  

\[
I(\ast) := |p(\ast)a(\ast)| - |q(\ast)b(\ast)|, |p(\ast)a(\ast)| + |q(\ast)b(\ast)|, \quad \ast = \pm \infty.
\]  

Moreover, \(\sigma_{\text{ess}}(U)\) does not contain both \(-1, +1\) if and only if \(|p(\ast)| \neq |a(\ast)|\) for each \(\ast = \pm \infty\).

The chiral pair \((\Gamma, \Gamma')\) in Theorem \([3]\) is a one-dimensional split-step quantum walk model considered in \([17, 13, 19, 11, 20]\) with a modification that the two parameters \(p, q\) depend freely on \(Z\). Note that this seemingly minor modification leads to the new Witten index formula \([8]\) taking values from \(-2, -1, 0, +1, +2\), where the indices \(\pm 2\) never appear in the existing formula (see \([19, 2]\) or \([20, 11, \text{Theorem 1.1}]\) for details), since \(p, q\) are kept constant in the five papers mentioned above. The index formula \([19]\) is new to the best of the author’s knowledge. As we shall see in Section 3.3, we can now naturally provide a lower bound for \(\dim \ker (U \mp 1)\).

3.2. Proof of Theorem B (i)  

What follows is a generalisation of \([19, \text{§3}]\). With the notation introduced in Theorem \([3]\), on one hand, the imaginary part \(Q\) admits two off-diagonal matrix representations as in \([22]\), where \(\text{ind} (\Gamma, \Gamma') = \text{ind} Q_0\) and \(\text{ind} (\Gamma', \Gamma) = \text{ind} Q_0^\ast\) by definition. On the other hand, an easy computation shows that the same operator \(Q\) does not admit an off-diagonal representation with respect to the orthogonal decomposition \(\mathbb{L}^2 (\mathbb{Z}, \mathbb{C}^2) = \mathbb{L}^2 (\mathbb{Z}) \oplus \mathbb{L}^2 (\mathbb{Z})\). The unitary invariance property \([24]\) motivates us to construct explicit unitary operators \(\epsilon, \gamma : \mathbb{L}^2 (\mathbb{Z}) \to \mathbb{L}^2 (\mathbb{Z})\), such that \(\epsilon^* Q_0 \epsilon, \gamma^* Q_0 \gamma^*\) become off-diagonal with respect to this decomposition.

**Lemma 3.2.** Let \((\Gamma, \Gamma')\) be the chiral pair in Theorem \([3]\) and let \(U := \Gamma \Gamma'\) be the associated evolution operator. Let \(R, Q\) be the real and imaginary parts of \(U\) respectively. For each \(x \in \mathbb{Z}\), let \(\theta(x), \phi(x)\) be any real numbers satisfying \(q(x) = |q(x)| e^{i\phi(x)}\) and \(b(x) = |b(x)| e^{i\phi(x)}\). Let \(p_{\pm} := \sqrt{1 \pm \ast p}\), and let \(a_{\pm} := \sqrt{1 \pm \ast a}\). Let

\[
\epsilon := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & L^* e^{-i\theta} \end{pmatrix}, \quad \gamma := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} a_+ & -a_- \\ -a_+ & a_- \end{pmatrix}.
\]  

Then the unitary operators \(\epsilon, \gamma\) give the following decompositions with respect to \(\mathbb{L}^2 (\mathbb{Z}, \mathbb{C}^2) = \mathbb{L}^2 (\mathbb{Z}) \oplus \mathbb{L}^2 (\mathbb{Z})\):

\[
\epsilon^* \Gamma \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon^* \Gamma' \epsilon = \begin{pmatrix} R_{12} & 0 \\ i Q_{10} & R_{22} \end{pmatrix}, \quad \epsilon^* R \epsilon = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix}, \quad \epsilon^* Q \epsilon = \begin{pmatrix} Q_{10} & 0 \\ 0 & Q_{20} \end{pmatrix}, \quad \gamma^* \Gamma \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^* \Gamma' \gamma = \begin{pmatrix} R_{10} & 0 \\ i Q_{12} & R_{20} \end{pmatrix}, \quad \gamma^* R \gamma = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix}, \quad \gamma^* Q \gamma = \begin{pmatrix} Q_{10} & 0 \\ 0 & Q_{20} \end{pmatrix}.
\]
where the six operators $Q_{c_0}, Q_{\gamma_0}, R_{c_1}, R_{c_2}, R_{c_3}, R_{c_4}$ are defined respectively by the following formulas:

\begin{align}
-2iQ_{c_0} &:= p_+ e^{i\theta} L' p_+ + p_- b^* L^* e^{-i\theta} p_- - |q|(a + a(\cdot + 1)), \\
2iQ_{c_0} &:= e^{i\theta} L' Q_{c_0} a_+ - a Q_{c_0} L e^{i\theta} a_- - |b|(p + p(\cdot - 1)), \\
2R_{c_1} &:= p_- e^{i\theta} L' p_+ + p_+ b^* L^* e^{-i\theta} p_- + p_2 a - p_- a(\cdot + 1), \\
2R_{c_1} &:= a_- e^{i\theta} L' Q_{c_0} a_+ + a_+ Q_{c_0} L e^{i\theta} a_- + a_+ p - a^- p(\cdot - 1), \\
2R_{c_2} &:= p_+ e^{i\theta} L' p_+ + p_- b^* L^* e^{-i\theta} p_- - p_2 a + p_+ a(\cdot + 1), \\
2R_{c_2} &:= a_+ e^{i\theta} L' Q_{c_0} a_+ + a_- Q_{c_0} L e^{i\theta} a_- - a_+^2 p + a^2 p(\cdot - 1) \\
\end{align}

Moreover, the chiral pair $(\Gamma, \Gamma')$ is Fredholm if and only if $Q_{c_0}, Q_{\gamma_0}$ are Fredholm. In this case,

\begin{align}
\text{ind}(\Gamma, \Gamma') = \text{ind} Q_{c_0}, \quad \text{ind}(\Gamma', \Gamma) = \text{ind} Q_{\gamma_0}.
\end{align}

**Proof.** It follows from (19) that we have the following diagonalisation:

\begin{align}
\epsilon_0^* \left( \begin{array}{c}
q \\
\eta
\end{array} \right) q_0 = \left( \begin{array}{c}
1 \\
0
\end{array} \right), \quad \epsilon_0 := \frac{1}{\sqrt{2}} \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
p_+ \\
p_-
\end{array} \right), \\
\gamma_0^* \left( \begin{array}{c}
a \\
\eta
\end{array} \right) \eta_0 = \left( \begin{array}{c}
1 \\
-1
\end{array} \right), \\
\gamma_0 := \frac{1}{\sqrt{2}} \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
a_+ \\
a_-
\end{array} \right).
\end{align}

The operator $\epsilon$ given by the first equality in (25) can be written as the product $\epsilon = (1 \oplus L^*)\epsilon_0$ of two unitary operators $1 \oplus L^*$ and $\epsilon_0$. With the first equality in (11), we obtain

\begin{align}
\epsilon \Gamma \epsilon = \epsilon_0 \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
0 & L^*
\end{array} \right) \left( \begin{array}{c}
p_+ & q
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
1 + L^* & 0
\end{array} \right) \epsilon_0 = \left( \begin{array}{c}
1 \\
-1
\end{array} \right).
\end{align}

Given an operator $X$ on $\ell^2(Z, \mathbb{C}^2)$, we introduce the shorthand $X_\epsilon := \epsilon^* X \epsilon$ and $X_\eta := \eta^* X \eta$. With this convention in mind, we have the commutation relations $[\Gamma_\epsilon, R_\epsilon] = [\Gamma'_\epsilon, R_\epsilon] = 0$ and anti-commutation relations $\{\Gamma_\epsilon, Q_\epsilon\} = \{\Gamma'_\epsilon, Q_\epsilon\} = 0$, where $\Gamma_\epsilon = \Gamma'_\epsilon = 1 \oplus -1$ with respect to $\ell^2(Z, \mathbb{C}^2) = \ell^2(Z) \oplus \ell^2(Z)$. It follows that we have the following representations:

\begin{align}
R_\epsilon &= \left( \begin{array}{cc}
R_{c_1} & 0 \\
0 & R_{c_2}
\end{array} \right), \\
Q_\epsilon &= \left( \begin{array}{c}
0 \\
Q_{c_0}
\end{array} \right), \\
U_\epsilon &= R_\epsilon + iQ_\epsilon = \left( \begin{array}{c}
R_{c_1} & i(Q_{c_0})^* \\
iQ_{c_0} & R_{c_2}
\end{array} \right), \\
R_\gamma &= \left( \begin{array}{cc}
R_{\gamma_1} & 0 \\
0 & R_{\gamma_2}
\end{array} \right), \\
Q_\gamma &= \left( \begin{array}{c}
0 \\
Q_{\gamma_0}
\end{array} \right), \\
U_\gamma &= R_\gamma + iQ_\gamma = \left( \begin{array}{c}
R_{\gamma_1} & i(Q_{\gamma_0})^* \\
iQ_{\gamma_0} & R_{\gamma_2}
\end{array} \right).
\end{align}

It remains to show that the six operators introduced above coincide with the ones defined by the formulas (28) to (33). Note that

\begin{align}
2\Gamma_\epsilon = 2\Gamma U_\epsilon = 2 \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
R_{c_1} & i(Q_{c_0})^* \\
iQ_{c_0} & R_{c_2}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) = 2 \left( \begin{array}{c}
R_{c_1} & 0 \\
0 & R_{c_2}
\end{array} \right), \\
2\Gamma_\epsilon = 2U_\epsilon \Gamma_\epsilon = 2 \left( \begin{array}{c}
R_{c_1} & i(Q_{c_0})^* \\
iQ_{c_0} & R_{c_2}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) = 2 \left( \begin{array}{c}
R_{c_1} & 0 \\
0 & R_{c_2}
\end{array} \right).
\end{align}

It remains to compute $2\Gamma'_\epsilon, 2\Gamma'_\gamma$;

\begin{align}
2\epsilon^* \left( \begin{array}{c}
b \\
0
\end{array} \right) \epsilon &= \left( \begin{array}{c}
p_+ e^{i\theta} Lb_+ + p_- b^* L^* e^{-i\theta} p_- - p_- e^{i\theta} Lb_- + p_+ b^* L^* e^{-i\theta} p_+
\end{array} \right), \\
2\epsilon^* \left( \begin{array}{c}
a \\
-\eta
\end{array} \right) \epsilon &= \left( \begin{array}{c}
|q|(a + a(\cdot + 1)) \\
p_2 a - p_2 a(\cdot + 1)
\end{array} \right), \\
2\gamma^* \left( \begin{array}{c}
q \\
L^* q
\end{array} \right) \gamma &= \left( \begin{array}{c}
a_- e^{-i\theta} L q^* a_+ + a_+ q Le^{i\theta} a_- - a_- e^{-i\theta} L^* q^* a_+ + a_+ q Le^{i\theta} a_- \\
a_+ e^{-i\theta} L^* q^* a_+ - a_- q Le^{i\theta} a_- - a_- e^{-i\theta} L^* q^* a_+ - a_+ q Le^{i\theta} a_-
\end{array} \right), \\
2\gamma^* \left( \begin{array}{c}
p \\
-p(\cdot - 1)
\end{array} \right) \gamma &= \left( \begin{array}{c}
a^2 p - a^2 p(\cdot - 1) - |b|(p + p(\cdot - 1)) \\
-a_2 p - a_2 p(\cdot - 1)
\end{array} \right).
\end{align}
It follows from the above equalities that

\[
2\Gamma' = 2e^\ast \begin{pmatrix} 0 & b^\ast \\ b & 0 \end{pmatrix} \Gamma + 2e^\ast \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = \begin{pmatrix} 2R_{\ell_1} & 2iQ_{\ell_0}^\ast \\ -2iQ_{\ell_0} & -2R_{\ell_2} \end{pmatrix}, 
\]

(41)

\[
2\Gamma = 2\gamma^\ast \begin{pmatrix} L^* & qL \\ 0 & 0 \end{pmatrix} \gamma + 2\gamma^\ast \begin{pmatrix} p & 0 \\ 0 & p(-1) \end{pmatrix} = \begin{pmatrix} 2R_{\ell_1} & 2iQ_{\ell_0}^\ast \\ 2iQ_{\ell_0} & -2R_{\ell_2} \end{pmatrix}. 
\]

(42)

By comparing (39) to (40) with (41) to (42), we see that (26) to (27) hold true.

Note that \( \ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \) can be identified with the orthogonal sum \( \ell^2(\mathbb{Z}) \oplus \{0\} \oplus \ell^2(\mathbb{Z}) \) through the following unitary transform:

\[
ell^2(\mathbb{Z}, \mathbb{C}^2) \ni (\Psi_1, \Psi_2) \mapsto (\Psi_1, 0, \Psi_2) \in \ell^2(\mathbb{Z}) \oplus \{0\} \oplus \ell^2(\mathbb{Z}).
\]

It is then easy to see that the operator \( Q_\ell \) admits the following block-operator matrix representations:

\[
Q_\ell = \begin{pmatrix} 0 & Q_{\ell_0}^\ast \\ Q_{\ell_0} & 0 \end{pmatrix} \in \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}),
\]

(43)

where \( 0 \) denotes the zero operator of the form \( 0 : \{0\} \to \{0\} \), and where \( \ell^2(\mathbb{Z}) \oplus \{0\} = \ker(\Gamma_\ell - 1) \) and \( \{0\} \oplus \ell^2(\mathbb{Z}) = \ker(\Gamma_\ell + 1) \). On the other hand, the same operator \( Q_\ell \) is the imaginary part of \( U_\ell = \Gamma_\ell \Gamma_\ell' \), and so it admits the following off-diagonal block-operator matrix representation according to (22):

\[
Q_\ell = \begin{pmatrix} 0 & (Q_{\ell_0}^\ast) \rangle \\ \langle Q_{\ell_0} & 0 \end{pmatrix}_{\ker(\Gamma_\ell - 1) \oplus \ker(\Gamma_\ell + 1)} = \begin{pmatrix} 0 & (Q_{\ell_0}^\ast) \rangle \\ \langle Q_{\ell_0} & 0 \end{pmatrix}_{(\ell^2(\mathbb{Z}) \oplus \{0\}) \oplus ((\{0\} \oplus \ell^2(\mathbb{Z}))}
\]

(44)

It follows from (43) to (44) that \( Q_{\ell_0}^\ast \) is an off-diagonal block-operator matrix of the form:

\[
Q_{\ell_0}^\ast = \begin{pmatrix} 0 & 0 \\ Q_{\ell_0} & 0 \end{pmatrix}. 
\]

Since \( 0 : \{0\} \to \{0\} \) is a Fredholm operator of zero index, we have that \( Q_{\ell_0}^\ast \) is Fredholm if and only if \( Q_{\ell_0} \) is Fredholm. In this case, we have \( \text{ind} Q_{\ell_0}^\ast = \text{ind} Q_{\ell_0} + \text{ind} (0) = -\text{ind} Q_{\ell_0} + 0 = \text{ind} Q_{\ell_0} \). The first equality in (43) follows from the unitary invariance of the Witten index [24]. An analogous argument can be used to show that the second equality in (43) also holds true.

**Remark 3.3.** The above derivation of (43) only requires the sequences \( p, q, a, b \) to be bounded, and so the existence of the two-sided limits (43) to (44) turns out to be redundant. Note, however, that from here on we shall impose (43) to (44) to prove the index formulas (38) to (39).

With Lemma 2.2 in mind, it remains to compute the Fredholm indices of the following operators:

\[
-2iQ_{\ell_0} = p_+p_+(\cdot + 1)b(\cdot + 1)e^{\theta L} - p_-p_-(\cdot - 1)b^\ast e^{-i\theta(\cdot - 1)}L^* - |q|(a + a(\cdot + 1)), 
\]

(45)

\[
2iQ_{\ell_0} = a_+a_+(\cdot - 1)q(\cdot - 1)e^{-i\phi}L^* + a_-a_-(\cdot + 1)qe^{i\phi(\cdot + 1)}L - |b|(p + p(-1)),
\]

(46)

where \( \theta, \phi \) can be any \( \mathbb{R} \)-valued sequences satisfying \( q(x) = |q(x)|e^{i\theta(x)} \) and \( b(x) = |b(x)|e^{i\phi(x)} \) for each \( x \in \mathbb{Z} \). Note that Theorem A(i) is not immediately applicable to the above strictly local operators, since it is not necessarily true that \( \theta \) and \( \phi \) are convergent. More precisely, for each \( \ast = \pm \), if \( q(\ast) \neq 0 \), then we can explicitly construct \( \theta \) in such a way that \( \theta(\ast) = \lim_{x \to \ast} \theta(x) \) holds true. On the other hand, if \( q(\ast) = 0 \), then the same conclusion cannot be drawn in general, because there are some pathological examples. Note that the same remark applies to \( \phi \). The purpose of the current section is to overcome this hindrance, which does not appear under the setting of [ST19b, Mat20], where \( p, q \) are held constant.
Lemma 3.4. The following assertions hold true:

(i) There exist two \( \mathbb{R} \)-valued sequences \( \theta_+ = (\theta_+(x))_{x \in \mathbb{Z}}, \theta_- = (\theta_-(x))_{x \in \mathbb{Z}}, \) such that

\[
e^{-i\theta} (2iQ_{na}) e^{i\theta} = p_+ p_+ (\cdot + 1) b (\cdot + 1) e^{i(\theta - \theta_+ + \theta_- (x + 1))} L^* - p_- p_- (\cdot - 1) b^* e^{-i(\theta_+ - \theta_- (x - 1) + \theta_+)} L^* - |q| (a + a (\cdot + 1)) e^{i(\theta - \theta_+)} ,
\]

where the three coefficients of the above strictly local operator have the following limits for each \( \star = \pm \infty : \)

\[
\lim_{x \to \star} \left( p_+(x) p_+(x + 1) b(x + 1) e^{i(\theta(x) - \theta_+(x) + \theta_-(x + 1))} \right) = (p(\star) + 1) b(\star) e^{i\theta(\star)},
\]

\[
\lim_{x \to \star} \left( - p_-(x) p_-(x - 1) b(x)^* e^{-i(\theta(x-1) - \theta_-(x - 1) + \theta_+)} \right) = (p(\star) - 1) b(\star)^* e^{-i\theta(\star)},
\]

\[
\lim_{x \to \star} \left( - |q|(a(x) + a(x + 1)) e^{i(\theta_-(x) - \theta_+)} \right) = -2 |q(\star)| a(\star).
\]

(ii) There exist two \( \mathbb{R} \)-valued sequences \( \phi_+ = (\phi_+(x))_{x \in \mathbb{Z}}, \phi_- = (\phi_-(x))_{x \in \mathbb{Z}}, \) such that

\[
e^{i\phi} (2iQ_{na}) e^{-i\phi} = a_+ a_+ (\cdot - 1) q (\cdot - 1)^* e^{-i(\phi - \phi_+ + \phi_- (x - 1))} L^* - a_- a_- (\cdot + 1) q e^{i(\phi_+ (x) - \phi_- (x + 1) + \phi_+)} L^* - |b| (p + p(\cdot - 1)) e^{i(\phi_+ - \phi_-)},
\]

where the three coefficients of the above strictly local operator have the following limits for each \( \star = \pm \infty : \)

\[
\lim_{x \to \star} \left( a_+(x) a_+(x - 1) q(x - 1)^* e^{-i(\phi(x) - \phi_+(x) + \phi_- (x - 1))} \right) = (a(\star) + 1) q(\star)^* e^{-i\phi(\star)},
\]

\[
\lim_{x \to \star} \left( - a_-(x) a_- (x + 1) q(x) e^{i(\phi(x + 1) - \phi_-(x + 1) + \phi_+ (x))} \right) = (a(\star) - 1) q(\star) e^{i\phi(\star)},
\]

\[
\lim_{x \to \star} \left( - |b(x)| (p(x) + p(x - 1)) e^{i(\phi_-(x) - \phi_+(x))} \right) = -2 |b(\star)| p(\star).
\]

Proof. For each \( x \in \mathbb{Z} \) we let

\[
\star(x) := \begin{cases} +\infty, & x \geq 0, \\ -\infty, & x < 0, \end{cases} \quad \theta_\pm(x) := \begin{cases} \theta(x), & p(\star(x)) = \pm 1, \\ 0, & p(\star(x)) \neq \pm 1, \end{cases} \quad \phi_\pm(x) := \begin{cases} \phi(x), & a(\star(x)) = \pm 1, \\ 0, & a(\star(x)) \neq \pm 1. \end{cases}
\]

Note that [47] follows from [45], and [51] follows from [46]. For each \( x \in \mathbb{Z} \) we let

\[
\Lambda_1(x) := \theta(x) - \theta_+(x) + \theta_-(x + 1), \quad \Lambda_2(x) := \theta(x) - 1 - \theta_-(x - 1) + \theta_+(x), \quad \Lambda_3(x) := \theta_-(x) - \theta_+(x).
\]

(i) It suffices to prove the following equalities:

\[
\lim_{x \to \star} \left( p_+(x) p_+(x + 1) e^{i\Lambda_1(x)} \right) = (p(\star) + 1) e^{i\theta(\star)},
\]

\[
\lim_{x \to \star} \left( - p_-(x) p_-(x - 1) e^{-i\Lambda_2(x)} \right) = -(p(\star) - 1) e^{-i\theta(\star)},
\]

\[
\lim_{x \to \star} \left( |q(x)| e^{i\Lambda_3(x)} \right) = |q(\star)|.
\]

Let \( \star = \pm \infty \) and \( |x| > 1 \) be fixed. If \( |p(\star)| < 1, \) then \( \theta_+(x) = \theta_-(x) = 0. \) In this case, [52] to [54] follow from the fact that as \( x \to \star \) we have \( \Lambda_j(x) \to \theta(\star) \) for each \( j = 1, 2, \) and \( \Lambda_3(x) \to 0. \) On the other hand, if \( |p(\star)| = 1, \) then \( q(\star) = 0, \) and so [57] becomes trivial. We need to check the following two cases separately: \( p(\star) = -1 \) and \( p(\star) = +1. \) If \( p(\star) = -1, \) then [55] holds trivially, and [56] follows from \( \theta_-(x - 1) = \theta(x - 1) \)
and \( \theta_+(x) = 0 = \theta(*) \), where the last equality follows from \([B7]\). Similarly, if \( p(*) = +1 \), then \((55)\) holds trivially, and \((55)\) follows from \( \theta_+(x) = \theta(x) \) and \( \theta_-(x + 1) = 0 = \theta(*) \).

(ii) The claim follows from the following analogous equalities:

\[
\begin{align*}
\lim_{x \to x^+} \left( a_+ (x) a_+ (x - 1) e^{-i(\phi(x) - \phi_+ (x) + \phi_- (x - 1))} \right) &= (a(*) + 1) e^{-i\phi(*)}, \\
\lim_{x \to x^+} \left( a_- (x) a_- (x + 1) e^{i(\phi(x+1) - \phi_- (x+1) + \phi_+ (x))} \right) &= -(a(*) - 1) e^{i\phi(*)}, \\
\lim_{x \to x^-} \left( b(x) e^{i(\phi_+ (x) - \phi_-(x))} \right) &= |b(*)|.
\end{align*}
\]

We omit the proof. \(\square\)

Since the Fredholm index is invariant under multiplication by invertible operators, we have

\[
\text{ind} \left( e^{-i\theta + Q_{\alpha_0} e^{i\theta -}} \right) = \text{ind} \left( I, I' \right), \quad \text{ind} \left( e^{i\phi + Q_{\gamma_0} e^{-i\phi -}} \right) = \text{ind} \left( I', I \right).
\]

We are now in a position to apply Theorem \([A1]\) to the strictly local operators \( A_\epsilon := e^{-i\theta + Q_{\alpha_0} e^{i\theta -}} \) and \( A_\gamma := e^{i\phi + Q_{\gamma_0} e^{-i\phi -}} \). Since the two-sided limits of the coefficients of \(-2iA_\epsilon \) and \( 2iA_\gamma \) are given respectively by \((48)\) to \((50)\) and \((52)\) to \((54)\), we introduce the following two functions for each \( * = \pm \infty \) according to \([A3]\):

\[
\begin{align*}
-2i f_\epsilon (z, \cdot, \cdot) &:= (p(*) + 1) b(*) e^{i\theta(*)} z + (p(*) - 1) b(*) e^{-i\theta(*)} z^* - 2|q(*)|a(*) \quad z \in \mathbb{T}, \quad (58) \\
2i f_\gamma (z, \cdot, \cdot) &:= (a(*) + 1) q(*) e^{-i\phi(*)} z^* + (a(*) - 1) q(*) e^{i\phi(*)} z - 2|b(*)|p(*) \quad z \in \mathbb{T}. \quad (59)
\end{align*}
\]

It follows from Theorem \([A1]\) that \( A_\epsilon \) is Fredholm (resp. \( A_\gamma \) is Fredholm) if and only if for each \( * = \pm \infty \) the function \( f_\epsilon (\cdot, *, \cdot) \) is nowhere vanishing (resp. \( f_\gamma (\cdot, *, \cdot) \) is nowhere vanishing). Moreover, in this case,

\[
\begin{align*}
\text{ind} \left( I, I' \right) &= \text{wn}(f_\epsilon (\cdot, +\infty)) - \text{wn}(f_\epsilon (\cdot, -\infty)), \quad (60) \\
\text{ind} \left( I', I \right) &= \text{wn}(f_\gamma (\cdot, +\infty)) - \text{wn}(f_\gamma (\cdot, -\infty)). \quad (61)
\end{align*}
\]

It remains to compute the winding numbers of \( f_\epsilon (\cdot, *, \cdot) \), \( f_\gamma (\cdot, *, \cdot) \) by making use of the following elementary fact;

**Lemma 3.5.** For each \( j = 1, 2 \), let \( (\alpha_j, \beta_j, \Theta_j) \in \mathbb{R} \times \mathbb{C} \times [0, 2\pi) \) be a fixed triple satisfying \( \alpha_j^2 = 1 \) and \( \beta_j = |\beta_j| e^{i\Theta_j} \). Let \( f : \mathbb{T} \to \mathbb{C} \) be the trigonometric polynomial defined by

\[
2f(z) := (\alpha_1 + 1) \beta_2 e^{i\Theta_1} z + (\alpha_1 - 1) \beta_2^* e^{-i\Theta_1} z^* - 2|\beta_1| a_2, \quad z \in \mathbb{T}. \quad (62)
\]

Then the function \( \Gamma \ni z \mapsto f(z) \in \mathbb{C} \) is nowhere vanishing if and only if \( |\alpha_1| \neq |\alpha_2| \). In this case, we have

\[
\text{wn}(f) = \begin{cases} 
\text{sgn} \alpha_1, & |\alpha_1| > |\alpha_2|, \\
0, & |\alpha_1| < |\alpha_2|,
\end{cases} \quad (63)
\]

where the sign function \( \text{sgn} \) is defined by \([B10]\).

As we shall see below, if \( \alpha_1 \beta_2 \neq 0 \), then the image of the function \( f \) turns out to be an ellipse. In this case, the curve \([0, 2\pi] \ni t \mapsto f(e^{it}) \in \mathbb{C} \) makes precise one revolution around the fixed point \(-|\beta_1| a_2 \) on the real axis, and so we have \( \text{wn}(f) \in \{-1, 0, +1\} \).

**Proof.** Let us first prove that \( f \) is nowhere vanishing if and only if \( |\alpha_1 \beta_2| \neq |\beta_1 a_2| \). In this case,

\[
\text{wn}(f) = \begin{cases} 
\text{sgn} \alpha_1, & |\alpha_1 \beta_2| > |\beta_1 a_2|, \\
0, & |\alpha_1 \beta_2| < |\beta_1 a_2|.
\end{cases} \quad (64)
\]
Let us consider the following function on $\mathbb{R}$:

$$2F(s) := (|\alpha_1\beta_2| + |\beta_2|)e^{is} + (|\alpha_1\beta_2| - |\beta_2|)e^{-is} = 2|\alpha_1\beta_2| \cos s + i2|\beta_2| \sin s, \quad s \in \mathbb{R}.$$ 

On one hand, if $\alpha_1\beta_2 = 0$, then the image of $F$ is a vertical line segment passing through the origin $0 + i0$. On the other hand, if $\alpha_1\beta_2 \neq 0$, then the image of $F$ is an ellipse centred at the origin. For each $t \in [0, 2\pi]$

$$2f(e^{it}) + 2|\beta_1|\alpha_2 = (\alpha_1 + 1)\beta_2e^{i\Theta}e^{it} + (\alpha_1 - 1)\beta_2^*e^{-i\Theta}e^{-it}$$

$$= (\text{sgn } \alpha_1|\alpha_1| + 1)|\beta_2|e^{i(\Theta_1 + \Theta_2 + t)} + (\text{sgn } \alpha_1|\alpha_1| - 1)|\beta_2|e^{-i(\Theta_1 + \Theta_2 + t)}$$

$$= \text{sgn } \alpha_1 \cdot 2F(\text{sgn } \alpha_1(\Theta_1 + \Theta_2 + t)).$$

If $\alpha_1\beta_2 = 0$, then the image of the function $[0, 2\pi] \ni t \mapsto f(e^{it}) \in \mathbb{C}$ coincides with that of the vertical line segment $[-1, 1] \ni t \mapsto -|\beta_1|\alpha_2 + it|\beta_2| \in \mathbb{C}$ passing through $-|\beta_1|\alpha_2$. That is, $f$ does not go through the origin if and only if $\beta_1\alpha_2 \neq 0 = \alpha_1\beta_2$, and in this case $\text{wn}(f) = 0$. This is a special case of (64). If $\alpha_1\beta_2 \neq 0$, then the image of the curve $[0, 2\pi] \ni t \mapsto f(e^{it}) \in \mathbb{C}$ is the following ellipse with $\text{sgn } \alpha_1$ being its winding number with respect to the center $-|\beta_1|\alpha_2$ on the real axis; 

If $|\alpha_1\beta_2| > |\beta_1\alpha_2|$, then the origin is inside the interior of the ellipse, and so $\text{wn}(f) = \text{sgn } \alpha_1$. If $|\alpha_1\beta_2| < |\beta_1\alpha_2|$, then the origin is inside the exterior of the ellipse, and so $\text{wn}(f) = 0$. Clearly, the ellipse $f$ goes through the origin if and only if $|\alpha_1\beta_2| = |\beta_1\alpha_2|$.

It remains to check that (63) coincides with (64). If the notation $\equiv$ simultaneously denotes $>$, $=\equiv$, then $|\alpha_1\beta_2| \leq |\beta_1\alpha_2|$ if and only if $|\alpha_1|^2|\beta_2|^2 \leq |\beta_1|^2|\alpha_2|^2$ if and only if $|\alpha_1|^2 \leq |\alpha_2|^2$ if and only if $|\alpha_1| \leq |\alpha_2|$. The claim follows.

**Proof of Theorem 17(i).** (1) Let $\alpha_1 := p(\ast)$, $\beta_1 := q(\ast)$, $\alpha_2 := a(\ast)$, $\beta_2 := b(\ast)$, $\Theta := \Theta(\ast)$. Then (62) becomes

$$2f(z) = (p(\ast) + 1)b(\ast)e^{i\Theta(\ast)}z + (p(\ast) - 1)b(\ast)^*e^{-i\Theta(\ast)}z^* - 2[q(\ast)]a(\ast) = -2if_z(\ast), \quad z \in \mathbb{T}.$$ 

That is, $f = -if_z(\ast)$, where the constant $-i$ does not play a significant role in this proof. It follows from Lemma 5.3 that $f_z(\cdot, \ast)$ is nowhere vanishing if and only if $|p(\ast)| \neq |a(\ast)|$. In this case, we have

$$\text{wn}(f_z(\cdot, \ast)) = \text{wn}(f) = \begin{cases} \text{sgn } p(\ast), & |p(\ast)| > |a(\ast)|, \\ 0, & |p(\ast)| < |a(\ast)|. \end{cases} \quad (65)$$

The index formula (63) is now an immediate consequence of (62) and (65).

(2) Let $\alpha_1 := a(\ast)$, $\beta_1 := b(\ast)$, $\alpha_2 := p(\ast)$, $\beta_2 := q(\ast)$, $\Theta := \Theta(\ast)$. Then (62) becomes

$$2f(z) = (a(\ast) + 1)q(\ast)e^{i\Theta(\ast)}z + (a(\ast) - 1)q(\ast)^*e^{-i\Theta(\ast)}z^* - 2|b(\ast)||p(\ast)| = (2if_z(\ast)\ast), \quad z \in \mathbb{T}.$$
That is, \( f^* = -if_\gamma(\cdot, \ast) \). It follows from Lemma 3.5 that \( f_\gamma(\cdot, \ast) \) is nowhere vanishing if and only if \(|a(\ast)| \neq |p(\ast)|\). In this case, we have

\[
\operatorname{wn}(f_\gamma(\cdot, \ast)) = \operatorname{wn}(f^*) = -\operatorname{wn}(f) = \begin{cases} -\text{sgn} a(\ast), & |a(\ast)| > |p(\ast)|, \\ 0, & |a(\ast)| < |p(\ast)|, \end{cases}
\]

where the last equality follows from \( \operatorname{wn}(f^*) = -\operatorname{wn}(f) \). The index formula (63) is now an immediate consequence of (61) and (65).

Theorem 3(i) can also be proved by a purely analytic method without relying on Lemma 3.5 (see, for example, [Mat20, §4]).

3.3. Proof of Theorem B (ii)

Proof of Theorem B (ii). It follows from a direct computation that \( U = \Gamma \Gamma^* \) is a strictly local operator of the following form:

\[
U = \begin{pmatrix} p & qL \\ L^{-1}q^* & -p(-1) \end{pmatrix} \begin{pmatrix} a & b^* \\ b & -a \end{pmatrix} = \begin{pmatrix} qLb + pa \\ L^{-1}q*a - p(-1)b \\ L^{-1}q*b + p(-1)a \end{pmatrix}.
\]

For each \( \ast = \pm \infty \) and each \( z \in \mathcal{T} \), we introduce the following matrix according to (A3):

\[
\hat{U}(z, \ast) := \begin{pmatrix} q(\ast)b(\ast)z + p(\ast)a(\ast) \\ (q(\ast)a(\ast)z - p(\ast)b(\ast)) \\ (q(\ast)^2b(\ast)^*z - p(\ast)a(\ast)) \end{pmatrix} = \begin{pmatrix} X(z, \ast) & -Y(z, \ast)^* \\ Y(z, \ast) & X(z, \ast) \end{pmatrix}.
\]

It follows from Theorem (A3(ii)) that the essential spectrum of the evolution operator \( U \) is given by

\[
\sigma_{\text{ess}}(U) = \sigma(+\infty) \cup \sigma(-\infty), \quad \sigma(\pm \infty) := \bigcup_{z \in \Gamma} \sigma(\hat{U}(z, \pm \infty)) = \bigcup_{t \in [0, 2\pi]} \sigma(\hat{U}(e^{it}, \pm \infty)).
\]

(67)

It remains to compute \( \sigma(\ast) \) for each \( \ast = \pm \infty \). Recall that we have \( q(\ast) = |q(\ast)|e^{i\theta(\ast)} \) and \( b(\ast) = |b(\ast)|e^{i\phi(\ast)} \) by (77). For each \( t \in [0, 2\pi] \) we have

\[
X(e^{it}, \ast) = q(\ast)b(\ast)e^{it} + p(\ast)a(\ast) = |q(\ast)|b(\ast)|e^{it+\theta(\ast)+\phi(\ast)}| + p(\ast)a(\ast),
\]

\[
Y(e^{it}, \ast) = q(\ast)a(\ast)e^{it} - p(\ast)b(\ast)^* = |q(\ast)|a(\ast)|e^{it+\theta(\ast)}| - p(\ast)|b(\ast)|e^{-i\phi(\ast)}.
\]

We get the following characteristic equation for each \( t \in [0, 2\pi] \):

\[
\det \left( \hat{U}(e^{it}, \ast) - \lambda \right) = \lambda^2 - (X(e^{it}, \ast) + X(e^{it}, \ast)^*) \lambda + X(e^{it}, \ast)X(e^{it}, \ast)^* + Y(e^{it}, \ast)Y(e^{it}, \ast)^* = 0.
\]

where \( X(e^{it}, \ast)X(e^{it}, \ast)^* + Y(e^{it}, \ast)Y(e^{it}, \ast)^* = 1 \) for each \( t \in [0, 2\pi] \) by a direct computation. The above characteristic equation becomes the following quadratic equation:

\[
\lambda^2 - 2(p(\ast)a(\ast) + |q(\ast)b(\ast)|) \cos(\theta(\ast) + \phi(\ast)) + 1 = 0, \quad t \in [0, 2\pi].
\]

Thus, if we let \( \tau(t, \ast) := p(\ast)a(\ast) + |q(\ast)b(\ast)| \cos(\theta(\ast) + \phi(\ast)) \) for each \( t \in [0, 2\pi] \), then the above equation has the following two solutions:

\[
\lambda_\pm(t, \ast) := \tau(t, \ast) \pm \sqrt{\tau(t, \ast)^2 - 1},
\]

where \( \{\tau(t, \ast)\}_{t \in [0, 2\pi]} = [p(\ast)a(\ast) - |q(\ast)b(\ast)|, p(\ast)a(\ast) + |q(\ast)b(\ast)|] \subseteq [-1, 1] \), since

\[
\sup_{t \in [0, 2\pi]} |\tau(t, \ast)| \leq |p(\ast)a(\ast)| + |q(\ast)b(\ast)| \leq \frac{|p(\ast)|^2 + |a(\ast)|^2}{2} + \frac{|q(\ast)|^2 + |b(\ast)|^2}{2} = 1.
\]
It follows that
\[
\sigma(*) = \bigcup_{t \in [0, 2\pi]} \sigma(\tilde{U}(e^{it}, *)) = \left\{ \lambda_\pm(t, *) \right\}_{t \in [0, 2\pi]} = \left\{ x \pm i \sqrt{1 - x^2} \right\}_{x \in [p(*) a(*) - |q(*) b(*)|, p(*) a(*) + |q(*) b(*)]|}.
\]

Note that ±1 ∈ σ(·) if and only if p(*) a(*) ± |q(*) b(*)| = ±1. It can be shown that the last equations are equivalent to (p(*) ± a(*))^2 = 0, and so it follows from (67) that σ_{ess}(U) does not contain both −1, +1 if and only if |p(*)| ≠ |a(*)| for each * ∈ ±∞. Finally, the formula (61) follows from the fact that x ∈ [p(*) a(*) − |q(*) b(*)|, p(*) a(*) + |q(*) b(*)]| if and only if sgn (p(*) a*) x ∈ I(*) for each x ∈ [−1, 1].

3.4. Several concluding remarks

3.4.1. The essential spectrum of the imaginary part

**Lemma 3.6.** Let H be an abstract Hilbert space, and let (Γ, Γ′) be a chiral pair on H. Let U := ΓΓ′ be the associated evolution operator, and let R, Q be the real and imaginary parts of U respectively. Then

\[
\sigma_{ess}(R) = \left\{ \frac{z + z^*}{2} \mid z \in \sigma_{ess}(U) \right\}
\]

(68)

\[
\sigma_{ess}(Q) = \left\{ \frac{z - z^*}{2i} \mid z \in \sigma_{ess}(U) \right\}
\]

(69)

**Proof.** Let B(H) ⊃ A → [A] ∈ B(H)/K(H) be the natural surjection onto the Calkin algebra B(H)/K(H).

If \( p: \mathbb{T} \to \mathbb{C} \) is a trigonometric polynomial of the form \( p(z) = \sum_{y=-k}^k a(y)z^y \) for each \( z \in \mathbb{T} \), then

\[
\sigma_{ess}(p(U)) = \sigma\left( \left[ \sum_{y=-k}^k a(y)U^y \right] \right) = \sigma\left( \sum_{y=-k}^k a(y)U^y \right) = \sigma(p([U])) = p(\sigma([U])) = p(\sigma_{ess}(U)).
\]

(70)

where the second last equality follows from the spectral mapping theorem. If we let \( p(z) := (z + z^*)/2 \) (resp. \( p(z) := (z - z^*)/(2i) \)) for each \( z \in \mathbb{T} \), then \( p(U) = R \) (resp. \( p(U) = Q \)). The claim follows.

It follows from (69) that the following characterisations holds true:

The chiral pair (Γ, Γ′) is Fredholm if and only if 0 ∉ σ_{ess}(Q) if and only if −1, +1 ∉ σ_{ess}(U). (71)

Note that (71) explains why we have the same characterisation |p(*)| ≠ |a(*)| for each * ∈ ±∞ in Theorem B(i) and Theorem B(ii).

**Theorem 3.7.** If (Γ, Γ′) is the chiral pair in Theorem B and if Q is the imaginary part of the evolution operator \( U := \Gamma\Gamma' \), then the following formula holds true;

\[
\sigma_{ess}(Q) = \bigcup_{* = \pm \infty} \{ \pm |f_\gamma(z, *)| \}_{z \in \mathbb{T}},
\]

(72)

where \( f_\epsilon(z, *) \) and \( f_\gamma(z, *) \) are given respectively by (58) and (59).

**Proof.** It follows from Lemma 3.2 that the block-operator matrix representation of \( Q_\epsilon := \epsilon^*Q_\epsilon \) is given explicitly by the last equality in (20). Let \( \theta_+, \theta_- \) be as in Lemma 3.4 and let us consider the following unitary transformation of \( Q_\epsilon \):

\[
A := \begin{pmatrix}
0 & e^{i\theta_-} & 0 & 0 \\
e^{i\theta_+} & 0 & e^{i\theta_+} & 0 \\
0 & Q_{\epsilon_{\theta_+}} & 0 & 0 \\
e^{-i\theta_-} & 0 & e^{-i\theta_-} & 0
\end{pmatrix} = \begin{pmatrix}
0 & e^{i\theta_-} & 0 & 0 \\
e^{i\theta_+} & 0 & e^{i\theta_+} & 0 \\
0 & Q_{\epsilon_{\theta_+}} & 0 & 0 \\
e^{-i\theta_-} & 0 & e^{-i\theta_-} & 0
\end{pmatrix} = \begin{pmatrix}
0 & e^{i\theta_-} & 0 & 0 \\
e^{i\theta_+} & 0 & e^{i\theta_+} & 0 \\
0 & Q_{\epsilon_{\theta_+}} & 0 & 0 \\
e^{-i\theta_-} & 0 & e^{-i\theta_-} & 0
\end{pmatrix}.
\]

We are in a position to apply Theorem A(i) to the above strictly local operator. We introduce the following matrix-valued function according to A(ii):

\[
\hat{A}(\cdot, \cdot) := \begin{pmatrix}
f_\epsilon(\cdot, \cdot) & 0 \\
0 & f_\epsilon(\cdot, \cdot)^*
\end{pmatrix}, \quad * = \pm \infty,
\]

(73)
where \( \sigma \left( \hat{A}(z, \star) \right) = \{ \pm |f_{\star}(z, \star)| \} \) for each \( z \in \mathbb{T} \) and each \( \star = \pm \infty \). It follows from Theorem \( \text{A ii} \) that

\[
\sigma_{\text{ess}}(A) = \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{A}(\cdot, -\infty) \right) \cup \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{A}(\cdot, +\infty) \right) = \bigcup_{z \in \mathbb{T}} \{ \pm |f_{\star}(z, -\infty)| \} \cup \bigcup_{z \in \mathbb{T}} \{ \pm |f_{\star}(z, +\infty)| \}.
\]

Since the essential spectrum is invariant under unitary transforms, we have \( \sigma_{\text{ess}}(Q) = \sigma_{\text{ess}}(Q_1) = \sigma_{\text{ess}}(A) \). It follows that the first equality in \( (72) \) holds true. The second equality follows similarly. \( \square \)

### 3.4.2. Topologically protected bound states

What follows is the subject of another paper in preparation. Let \( (\Gamma', \Gamma'') \) be a chiral pair defined on an abstract Hilbert space \( \mathcal{H} \), and let \( R, Q \) be the real and imaginary parts of the evolution operator \( U := \Gamma' \Gamma'' \) respectively. With the block-operator matrix representations \( (21) \) to \( (23) \) in mind, we define the following two indices:

\[
\text{ind}_{\pm}(\Gamma', \Gamma'') := \dim \ker \left( R_1 \mp 1 \right) - \dim \ker \left( R_2 \pm 1 \right).
\]

The operator \( U = R + iQ \) is unitary, and so \( [R, Q] = 0 \) and \( R^2 + Q^2 = 1 \). It immediately follows from the second equality that the two indices \( \text{ind}_{\pm}(\Gamma', \Gamma'') \) given by \( (73) \) are well-defined, if the chiral pair \( (\Gamma', \Gamma'') \) is Fredholm. In this case, it is not difficult to prove:

\[
\text{ind}_{\pm}(\Gamma', \Gamma'') = \frac{\text{ind}(\Gamma', \Gamma') \pm \text{ind}(\Gamma', \Gamma')}{2},
\]

\[
|\text{ind}_{\pm}(\Gamma', \Gamma'')| \leq \dim \ker \left( U \mp 1 \right),
\]

where non-zero vectors in \( \ker(U \mp 1) \) may be referred to as **topologically protected bound states** as in Physics literature (see, for example, \( \text{KRBD10, KBF+12} \)). The three indices \( \text{ind}(\Gamma', \Gamma'), \text{ind}_{\pm}(\Gamma', \Gamma'') \) coincide with the **symmetry indices** \( \text{si}(U), \text{si}_{\pm}(U) \) discussed in \( \text{CGG+18, CGS+18} \).

If \( (\Gamma', \Gamma'') \) is the chiral pair in Theorem \( \text{E} \) then it follows from Theorem \( \text{E i} \) and \( (73) \) that \( \text{ind}_{\pm}(\Gamma', \Gamma'') \) are given explicitly by the following formulas:

\[
2 \cdot \text{ind}_{\pm}(\Gamma', \Gamma'') = \begin{cases} 
\mp \text{sgn} \ a(\pm \infty) \pm \text{sgn} \ a(-\infty), & |p(-\infty)| < |a(-\infty)| \text{ and } |p(\pm \infty)| < |a(\pm \infty)|, \\
\pm \text{sgn} \ p(\pm \infty) \pm \text{sgn} \ a(-\infty), & |p(-\infty)| < |a(-\infty)| \text{ and } |p(\pm \infty)| > |a(\pm \infty)|, \\
-\text{sgn} \ p(-\infty) \mp \text{sgn} \ a(\pm \infty), & |p(-\infty)| > |a(-\infty)| \text{ and } |p(\pm \infty)| < |a(\pm \infty)|, \\
+\text{sgn} \ p(\pm \infty) - \text{sgn} \ p(-\infty), & |p(-\infty)| > |a(-\infty)| \text{ and } |p(\pm \infty)| > |a(\pm \infty)|, 
\end{cases}
\]

where we assume \( |p(\star)| \neq |a(\star)| \) for each \( \star = \pm \infty \). With \( (75) \) in mind, the formula \( (76) \) can be used to give a lower bound for the number of topologically protected bound states associated with this explicit one-dimensional quantum walk model. This estimate is robust in the sense that the lower bounds \( |\text{ind}_{\pm}(\Gamma', \Gamma'')| \) depend only on the four asymptotic values \( p(\pm \infty), a(\pm \infty) \).

Note that an estimate of this kind can also be obtained via the **spectral mapping for discrete-time quantum walks** discussed in \( \text{SS16, SS19} \). Indeed, \( \text{Suz19} \), Theorem 3.1(iii)] states that if \( U := \Gamma' \Gamma'' \) denotes the evolution operator of an abstract chiral pair \( (\Gamma', \Gamma'') \), where \( \Gamma' \) is expressed as \( \Gamma' = 2d^*d - 1 \) for some fixed co-isometry (i.e. \( dd^* = 1 \)), then the following equalities hold true:

\[
\dim \ker \left( U \mp 1 \right) = m_{\pm} + M_{\pm},
\]

where \( m_{\pm} := \dim \ker(d\Gamma' d^* \mp 1) \) and \( M_{\pm} := \dim \ker(U \mp 1) \cap \ker d \). For example, as in \( \text{FFS18} \) Remark 6.2, if \( U = \Gamma' \Gamma'' \) is the evolution operator associated with the chiral pair \((\Gamma', \Gamma'')\), defined in Theorem \text{E} with \( p, q \) being held constant, then one can solve some first-order difference equations to show that there exists a well-defined constant \( k > 0 \), such that the condition \( |q| < k \) ensures \( \dim \ker(U \mp 1) = 1 \).
References

[ABN+01] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous. One-dimensional quantum walks. In Proceedings of 33rd ACM Symposium of the Theory of Computing, pages 37–49. ACM Press, 2001.

[ADD20] Y. Aharonov, L. Davidovich, and N. Zagury. Quantum random walks. Phys. Rev. A, 48:1687–1690, Aug 1993.

[CGG+18] C. Cedzich, T. Geib, F. A. Grünbaum, C. Stahl, L. Velázquez, A. H. Werner, and R. F. Werner. The Topological Classification of One-Dimensional Symmetric Quantum Walks. Ann. Henri Poincaré, 19(2):325–383, Feb 2018.

[CGL12] M. J. Cantero, F. A. Grünbaum, L. Moral, and L.Velázquez. One-dimensional quantum walks with one defect. Rev. Math. Phys., 24(02):1250092, Mar 2012.

[CGS+18] C. Cedzich, T. Geib, C. Stahl, L. Velázquez, A. H. Werner, and R. F. Werner. Complete homotopy invariants for translation invariant symmetric quantum walks on a chain. Quantum, 2:95, September 2018.

[Dou73] R. G. Douglas. Banach Algebra Techniques in the Theory of Toeplitz Operators. Regional Conference Series in Mathematics 15. AMS, 1973.

[FKS17] T. Fuda, D. Funakawa, and A. Suzuki. Localization of a multi-dimensional quantum walk with one defect. Quantum Inf. Process., 16(8), Jul 2017.

[FKS18] T. Fuda, D. Funakawa, and A. Suzuki. Localization for a one-dimensional split-step quantum walk with bound states robust against perturbations. J. Math. Phys., 59(8):082201, Aug 2018.

[FKS19] T. Fuda, D. Funakawa, and A. Suzuki. Weak limit theorem for a one-dimensional split-step quantum walk. Rev. Roumaine Math. Pures Appl., 64:157–165, 2019.

[FMS+20] D. Funakawa, Y. Matsuzawa, I. Sasaki, A. Suzuki, and N. Taranishi. Time operators for quantum walks. Lett. Math. Phys., 110(9):2471–2490, Jun 2020.

[Gud88] S. Gudder. Quantum Probability. Probability and Mathematical Statistics : a series of monographs and textbooks. Elsevier Science, 1988.

[IKK04] N. Inui, Y. Konishi, and N. Konno. Localization of two-dimensional quantum walks. Phys. Rev. A, 69(5):052323, May 2004. Publisher: American Physical Society.

[KBF+12] T. Kitagawa, M. A. Broome, A. Fedrizzi, M. S. Rudner, E. Berg, I. Kassal, A. Aspuru-Guzik, E. Demler, and A. G. White. Observation of topologically protected bound states in photonic quantum walks. Nat. Commun., 3(1), Jan 2012.

[Kit12] T. Kitagawa. Topological phenomena in quantum walks: elementary introduction to the physics of topological phases. Quantum Inf. Process., 11(5):1107–1148, Aug 2012.

[Kon02] N. Konno. Quantum random walks in one dimension. Quant. Inf. Process., 1(5):345–354, 2002.

[KRB10] T. Kitagawa, M. S. Rudner, E. Berg, and E. Demler. Exploring topological phases with quantum walks. Phys. Rev. A, 82:033429, Sep 2010.

[Mey96] D. A. Meyer. From quantum cellular automata to quantum lattice gases. J. Statist. Phys., 85(5-6):551–574, Dec 1996.

[Mor19] H. Morikiti. Generalized eigenfunctions and scattering matrices for position-dependent quantum walks. Rev. Math. Phys., 31(07):1950019, Jan 2019.

[MSS+18a] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K. Suzuki. Weak limit theorem for a nonlinear quantum walk. Quantum Inf. Process., 17(9), Jul 2018.

[MSS+18b] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K. Suzuki. Scattering and inverse scattering for nonlinear quantum walks. Discrete Contin. Dyn. Syst., 38(7):3687–3703, 2018.

[MSS+19] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K. Suzuki. Dynamics of solitons for nonlinear quantum walks. J. Phys. Commun., 3(7):075002, Jul 2019.

[Mur06] G. J. Murphy. Topological and analytical indices in C*-algebras. J. Funct. Anal., 234(2):261–276, 2006.

[Ohn16] H. Ohno. Unitary equivalent classes of one-dimensional quantum walks. Quantum Inf. Process., 15(9):3599–3617, Jun 2016.

[Ohn17] H. Ohno. Unitary equivalence classes of one-dimensional quantum walks ii. Quantum Inf. Process., 16(12), Oct 2017.

[RST17] S. Richard, A. Suzuki, and R. Tiedra de Aldecoa. Quantum walks with an anisotropic coin i: spectral theory. Lett. Math. Phys., 108(2):331–357, Sep 2017.

[RST18] S. Richard, A. Suzuki, and R. Tiedra de Aldecoa. Quantum walks with an anisotropic coin ii: scattering theory. Lett. Math. Phys., 109(1):61–88, May 2019.

[Seg11] E. Segawa. Localization of quantum walks induced by recurrence properties of random walks. J. Comput. Theor. Nanos., 10, 12 2011.

[SS16] E. Segawa and A. Suzuki. Generator of an abstract quantum walk. Quantum Stud.: Math. Found., 3(1):11–30, Jan 2016.

[SS17] E. Segawa and A. Suzuki. Essential spectrum of the discrete laplacian on a perturbed periodic graph. J. Math. Anal. Appl., 446(2):1863–1881, 2017.

[SS19] E. Segawa and A. Suzuki. Spectral mapping theorem of an abstract quantum walk. Quantum Inf. Process., 18(11), Sep 2019.

[ST19a] D. Sambou and R. Tiedra de Aldecoa. Quantum time delay for unitary operators: General theory. Rev. Math. Phys., 31(06):1950018, Jun 2019.

[ST19b] A. Suzuki and Y. Tanaka. The witten index for 1d supersymmetric quantum walks with anisotropic coins. Quantum Inf. Process., 18(12), Nov 2019.
[Suz16] A. Suzuki. Asymptotic velocity of a position-dependent quantum walk. *Quantum Inf. Process.*, 15(1):103–119, Nov 2016.

[Suz19] A. Suzuki. Supersymmetry for chiral symmetric quantum walks. *Quantum Inf. Process.*, 18(12), Oct 2019.

[Wad20] K. Wada. A weak limit theorem for a class of long-range-type quantum walks in 1d. *Quantum Inf. Process.*, 19(1), 1 2020.