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What Can Possibly Go Wrong?

Abstract A lot.

1 Motivation

There is no principle built into the laws of Nature that says that theoretical physicists have to be happy [1]. In this hard, unbiased and objective look at some past and continuing blunders in following Weinberg’s suggestions to arrive at a comprehensive description of Nuclear Physics using Effective Field Theories, some names and citations are withheld to protect the innocent.

Scientific history is often told as inevitable and steady progress towards a more perfect theory, with amusing asides about a few endearing follies of key protagonists. This volume may provide a good excuse to replace the pressure of result-oriented rigour in novelty-research articles by a few qualitative remarks. None are original, and all are most likely standard lore. They were triggered in part by presentations and discussions at the workshops The Tower of Effective (Field) Theories and the Emergence of Nuclear Phenomena (EFT and Philosophy of Science) at CEA/SPhN Saclay in 2017 [2], Lattice Nuclei, Nuclear Physics and QCD—Bridging the Gap and New Ideas in Constraining Nuclear Forces at the ECT* in 2015 and 2018, respectively, and by the “Folk Theorem” of Effective Field Theories (EFTs), originally formulated by Weinberg in 1979 [3] and here quoted in the 1997 version [4]:

When you use quantum field theory to study low-energy phenomena, then according to the folk theorem you’re not really making any assumption that could be wrong, unless of course Lorentz invariance or quantum mechanics or cluster decomposition is wrong, […] As long as you let it be the most general possible Lagrangian consistent with the symmetries of the theory, you’re simply writing down the most general theory you could possibly write down. This point of view has been used in the last fifteen years or so to justify the use of effective field theories, […]

At Chiral Dynamics 2009 in Bern, he replied to the question how to prove that: I know of no proof, but I am sure it’s true. That’s why it’s called a folk theorem. It constitutes a lemma to what has since time immemorial been known as “Totalitarian Principle”1 or “Swiss Basic Law” [6]: Everything not forbidden is compulsory. That, in turn, is a corollary to the fundamental theorem What-ever can happen will happen.2

If the “Folk Theorem” were all there is to it, an EFT would indeed be little less than symmetries plus parametrisation of ignorance3 – a Hail Mary to throw the kitchen sink at any question. But an EFT involves

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1 Often attributed to Gell-Mann, the origin is lost in words spoken long before 1956 [5].
2 In compliance with the Zeroth Theorem of the History of Science, the phrase’s likely first appearance in print is a 1866 article by De Morgan [7], well before the famed Murphy was even born.
3 Originally a put-down, this dictum was embraced by EFT advocates early on; cf. e.g. [8,9].
much more: for an efficient/effective description at the scales of interest, one needs to identify the relevant symmetries; the appropriate degrees of freedom; a workable separation of scales; and a consistent scheme to bring order to the infinite of possible contributions.

So an EFT actually offers ample opportunity to verify that anything that can go wrong will go wrong. As discussed in Sect. 2, a series of choices is based on assumptions, some obvious, some carefully stated, some hidden, and at times even most dangerously hidden in plain sight. Moreover, the abstract theorem may hold, but since Physics is done by humans, Sociology enters as discussed in Sect. 3. We theorists are often wrong, for example out of convenience (because doing the right thing is hard); prejudice (because we know this to be wrong but it is not); stubbornness (because we have always done it this way); lack of foresight (thirty years later); or even sheer bad luck (why didn’t anybody think of this earlier?). What follows are a few instances which touched me; there are more and better examples.

To illustrate the point that Physics is done by humans, Fig. 1 reproduces Manoel Roberto Robilotta’s cartoon capturing the spirit of the 1999 Workshop on Nuclear Forces. Notice how familiar the topics sound even today. This was, I think, the first and most epic clash between iconoclasts (commonly referred to as “cockroaches” there) and traditionalists (then called “dinosaurs”), recalled also in Ubirajara van Kolck’s contribution to this issue [10]. It had been eight years since Weinberg’s Nuclear Trifecta to whose central part this issue is dedicated [11–13], but the nuclear community had taken little notice, despite a seminal PhD thesis by an upstart about everything subsequent EFT generations re-discovered [14].

2 Input: Trust But Verify

Let us first turn to some assumptions in Nuclear EFT – from known knowns to unknown unknowns [16].

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4 Throughout, “we” serves not as pluralis majestatis, but as shorthand for the part of the nuclear community which is everybody but You, dear Reader.

5 Reflecting another seismic change, [13] was Weinberg’s first submission to the arXiv, which, incidentally, turns 30, too.
2.1 Size Matters, Or: Issues with the Expansion Parameter

Confronted with an infinite number of possible interactions, one must devise a power counting scheme to order contributions and observables by their relative importance in some small, dimensionless quantity

\[ Q := \frac{\text{typical low momenta } p_{\text{typ}}}{\text{breakdown scale } \Lambda_{\text{EFT}}} < 1. \]  

(1)

The numerator summarily depends on intrinsic low scales \( p_{\text{typ}} \) at which the EFT is supposed to be applicable, including the relative momentum \( k \) between scattering particles, the masses of light particles (\( m_\pi \) for the pion), and the scales associated with binding within the EFT’s reach. Around the breakdown scale \( \Lambda_{\text{EFT}} \), new dynamical degrees of freedom enter which are not explicitly accounted for by the EFT but whose effects at short distances \( \lesssim 1/\Lambda_{\text{EFT}} \) are simplified into Low-Energy Coefficients (LECs). A breakdown scale is not a number, but a gradual corridor of values over which an EFT becomes increasingly unreliable until it eventually makes no sense at all.

The range is also limited from below. A Nuclear EFT assumes \( p_{\text{typ}} \gg 1 \text{eV} \) and dispatches atomic effects because that is not the relevant Physics at the requested scale. This implies one determines parameters most efficiently around \( p_{\text{typ}} \), not at much smaller scales; see the discussion of the fit corridor in Sect. 3.7.

Take an observable \( \text{Obs} \) whose first nonzero piece starts at order \( Q^{n_0} \). Let us label the \( i \)-th order contribution relative to leading order (LO) as \( Q^i \text{Obs}_i \), making the order of \( Q \) explicit. Discarding logarithmic corrections for brevity’s sake, it can hence be expanded as

\[ \text{Obs} = Q^{n_0} \left[ \sum_{i=0}^{n} Q^i \text{Obs}_i + \mathcal{O}(Q^{n+1}) \right]. \]  

(2)

One estimates the theory uncertainty from truncating the series at order \( \mathcal{O}(Q^{n+\theta n}) \), or (Next-to-)Leading Order (N\(^n\)LO) as being one order higher, namely \( Q^{n+1} \) relative to LO and hence with an associated truncation error of \( Q^{n+1} \).

Consider the expansion of electro-dynamic effects. It proceeds in powers of \( \alpha = \frac{1}{137} \) \cite{17}, so its higher-order corrections contribute typically just \( \lesssim 1\% \). Compton scattering on a charge, for example, starts at \( \mathcal{O}(\alpha^2) \), i.e. \( n_0 = 2 \), and the first correction is at order \( \alpha^3 \), providing a correction to the LO result of \( \mathcal{O}(\alpha^4) \approx 1\% \).

But in Nuclear Physics, we are not blessed with an exceptionally small \( Q \) even at typical low scales: \( Q \approx 0.4 \) in \( \chi \text{EFT} \) with a dynamical Delta resonance degree of freedom, at best \( \frac{1}{2} \) without Deltas, and \( \frac{1}{4} \) in EFT(\( \pi \)) are common numbers; cf. Sect. 2.4. That makes estimating theory uncertainties even the more imperative. On top of that, there are quite a few options how to expand; cf. Sect. 2.6. Fortunately, one can check if expectation and outcome match by carefully checking convergence patterns; see Sect. 3.7. When one invests that effort to dot the i’s, includes quite a few orders and consequently faces quite a number of non-trivial interactions whose parameters are usually determined well from the cornucopia of nuclear data, one is rewarded by results with theory truncation errors which are both credible and competitive with experimental errors – and even overlap.

2.2 It Ain’t Natural, Or: Naïve Dimensional Analysis and Error Bars

EFTs carry the seed of their own destruction \cite{8}: At \( Q \gtrsim 1 \) (\( p_{\text{typ}} \gtrsim \Lambda_{\text{EFT}} \)), the Lagrangean may still perfectly reasonably reflect the symmetries of the problem, but there is no power counting. Additional arguments must then justify why some terms are kept while an infinity of others is dropped. Practicality is a good one, as is the hope to “model” one’s way to a more comprehensive understanding which eventually may be cast into an EFT. Maybe one can rearrange the deck chairs to find a converging result...

As \( p_{\text{typ}} \gtrsim \Lambda_{\text{EFT}} \) (\( Q \gtrsim 1 \)), the demise of the power counting is of course not sudden but gradual. This decrease of expected accuracy must be reflected in larger theory errors, for example at higher energies, and accounted for both in data fits and in comparing different EFTs; see e.g.\cite{18–21}.

The expansion of \( \text{Obs} \) in eq. (2) is based on a key assumption not only of EFTs but of Physics in general: “Weak Naturalness” requires that higher orders (namely so-called details) do generally not spoil the perturbative series, i.e. \( |\text{Obs}_i| > Q|\text{Obs}_{i+1}| \), with only “a few” exceptions; see also \cite{22–25}. When \( Q \approx 10^{-20} \) as in nuclear corrections from Quantum Gravity at the Planck scale, ratios of \( |\text{Obs}_{i+1}|/|\text{Obs}_i| \approx 10^{15} \) may appear prohibitively large, but the contribution of the \((i+1)\text{st} \) term is still suppressed by \( 10^{-20+15} = 10^{-3} \) against
the $i$th term and hence provides a negligible correction for all practical purposes. If, however, $Q \approx \frac{1}{4}$ as in $\chi$ EFT, then ratios of $|\text{Obs}_{i+1}|/|\text{Obs}_i| \approx 3$ or so are already precarious. Thus, one often considers for example contributions from isovector nucleonic magnetic moments, $\kappa_v \approx 4.7 \sim 1/Q$ one order sooner, avoiding relatively large but well-understood higher-order corrections; see e.g.[26]. Naturalness flows into another fundamental assumption: Higher-order terms can reliably be estimated by Naïve Dimensional Analysis [27–31]. Without these variants of Occam’s Razor [32], one cannot rule out alternative explanations via extraordinarily large higher-order corrections. Since the dawn of the quantitative Scientific Method, researchers have implicitly assumed that Nature is not malevolent$^6$. That makes the difference between Theory and Conspiracy Theory.

A comprehensive and quantitative theory of Weak Naturalness and Naïve Dimensional Analysis has been emerging this past decade, based on checking assumptions against outcomes using Bayesian statistics with reasonable expectations clearly formulated as priors; see e.g.[18–21,25] and references therein. A cornerstone of any EFT is to actually provide quantitative estimates of theory errors, rather than “educated guesses” based on “years of experience”, and we should have paid attention sooner; cf. Sect. 3.7.

2.3 No Freedom in the Degrees of Freedom?, Or: The Relevant Particle Content

In Nuclear Theory, we may have largely found the “right” degrees of freedom for efficient versions of the most general Lagrangean: “pion-less EFT” (EFT($\chi$)) employs contact interactions between only nucleons (and external probes) at very low momenta $p_{\text{typ}} \ll \Lambda_\chi \approx m_\pi$; and Chiral EFT (EFT) adds pions and the Delta resonance in nuclear processes at more generic nuclear scales $p_{\text{typ}} \sim m_\pi$; cf. Sect. 2.4. The strange few-hadron sector has also been explored; see [34] in this issue and references therein.

The correlated two-pion state $f_0(500)$ could possibly be added as its own degree of freedom. It has the quantum numbers of the QCD vacuum and a mass somewhere around $(400 \ldots 550) - (200 \ldots 350)\text{ MeV}$ just around or below $\Lambda_\chi$, according to the PDG’s 2020 edition [35]. Some ideas about its significance for the NN potential are emerging, especially related to its rôle in the two-pion exchange; cf.[36,37]. It would be interesting to further explore its impact on few-nucleon systems, and it would definitely be amusing if the pre-EFT controversy$^7$ about the meson formerly known as $\sigma$ would be revived. However, its enormous width and large mass suggests that it is just as well captured by only very mildly energy/momentum-dependent LECs already in $\chi$ EFT.

We have not yet found a path to formulate a clear separation of scales through the jungle of GeV-scale meson and nucleon resonances. In such uncertain territory, models and less-than-rigorous Ansätze provide crucial insight into what degrees of freedom and symmetries may be appropriate and relevant – if one optimistically assumes that we just have not yet found the right EFT there.

As one moves to heavier nuclei, it is however no surprise that interactions between “free” nucleons and pions become less efficient ways to describe the relevant Physics. Since the advent of the liquid-drop model, we know that nucleons in heavy nuclei are not free but subject to some collective motion. The bridge to descriptions which utilise more collective degrees of freedoms, like shell-and-core or quasi-particles, is one we have now started to explore with more confidence [38–41]; cf. discussion in Sect. 3.9.

2.4 The Delta Variant, Or: An Often-Overlooked Degree of Freedom

However, the $\Delta(1232)$ resonance still plays the rôle of the understudy: used if unavoidable. Its resonance peak energy of about $300 \text{ MeV} \approx 2m_\pi$ above the nucleon mass, its width of $\Gamma_\Delta \approx 70 \text{ MeV} \approx \frac{m_\pi}{2}$, and its rather sizeable coupling to pions and photons, means its effects are manifest even at energies $E \approx \Delta_M - \frac{\Gamma_\Delta}{2} \lesssim 200 \text{ MeV}$. The Delta channel opens immediately with the pion threshold and has a dramatic energy dependence, as well known from the textbook plots of cross sections in the first dozen MeV of pion-photoproduction and pion-nucleon scattering. But even below that, at $100 \text{ MeV}$ or so, its impact is obvious in processes like few-nucleon Compton scattering [42] where energy and momentum of the probe are actually identical. In that case, $E \approx p_{\text{typ}} \approx \Delta_M$, so that the breakdown of $\chi$ EFT without a dynamical Delta is at best set by the Delta-nucleon mass splitting as $\Lambda_\chi(\Delta) \lesssim \Delta_M \approx 300 \text{ MeV}$ – but that neglects that the large width makes it contribute even

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$^6$ Raffiniert ist der Herrgott, aber boshafft ist er nicht. [33].

$^7$ Come on, we can do without that sigma crap. [9].
well before. Even in that case, the expansion parameter \( Q = \frac{p_{\text{typ}}}{\Lambda_\chi} \) would become uncomfortably large in processes like Compton scattering where energy and momentum scales of external probes are identical. Whether results actually converge, needs therefore close examination. Delta effects are at times somewhat suppressed in isoscalar nuclei, but that is the exception to the rule.

On the other hand, the breakdown scale \( \Lambda_\chi \approx [700 \ldots 1000] \text{MeV} \) of \( \chi \text{EFT} \) with dynamical Delta is consistent with the masses of the \( \omega \) and \( \rho \) as the next-lightest exchange mesons, and with the chiral symmetry breaking scale; see [43] for an oft-employed variant. At \( p_{\text{typ}} \approx m_\pi \), the expansion parameter is then about \( \frac{1}{6} \) for pion physics and about 0.4 for the perturbative Delta. At energies \( \sim \Delta_M \), the Delta resonance dominates and constitutes LO, \( \pi N \Delta \) interactions must be resummed, and the expansion parameter is now about 0.4 for pions as well. In either régime, \( Q \) is not very small, but convergence appears quite reasonable, reaching \( \lesssim \pm 3\% \) around \( m_\pi \) at \( N^4 \text{LO} \). That is confirmed by a quantitative Bayesian analysis of uncertainties which also bears in mind that both the power counting and \( Q \) itself changes with \( p_{\text{typ}} \) [20].

Even at low energies, the Delta variant helps with Naturalness; cf. Sect. 2.2. Without it, some LECs like the \( \pi \pi N \) coefficients \( c_{2,3} \) are unnaturally large. Most of that strength is resolved by resonance saturation as coming from the dynamical Delta [44]. What remains of \( c_{2,3} \) becomes natural-sized. Even at low \( p_{\text{typ}} \sim m_\pi \), that reduces the risk of large corrections which are formally of higher order. So, the information from additional degrees of freedom can actually improve predictions. So an improved resolution, namely a “more fundamental” theory, can lead to an increase of information, with fewer unknowns and fewer mysteries.

At \( p_{\text{typ}} \sim m_\pi \), high-accuracy \( \chi \text{EFT} \) interactions with a perturbative Delta dramatically impact nuclear structure and nuclear matter [38–41]. But for energies \( \sim \Delta_M \), a typical scale if not in finite nuclei, then in neutron stars, the Delta dominates over pion-exchange and cannot be treated perturbatively. A theory without the Delta is there computationally certainly more convenient than a coupled-channel problem with it. Conceptually, \( \chi \text{EFT} \)’s standard should in the long run be to include the Delta as a matter of course.

2.5 These are not the Symmetries You are Looking for, Or: The Importance of Conservation Laws

Chiral symmetry, gauge and Lorentz invariance (often as perturbation in powers of velocities; cf. Sect. 3.8), and other symmetries are certainly not over-constraining Nuclear EFTs. The cornucopia of non-trivial high-accuracy agreements between theory and data makes it unlikely that we have imposed an exact or approximate symmetry that is not there. Requiring parity in weak interaction would be a counter-example.

More tricky is the question whether we are missing symmetries. These would show up as correlations between observables (and between parameters in the Lagrangean) which appear accidental or fine-tuned.

Theorists do not like fine-tuning. They are much happier with an underlying symmetry, even if approximate, to protect combinations of parameters from deviating a lot after renormalisation. For example, chiral symmetry explains why chiral-symmetry-breaking interactions are small and disappear as \( Q \to \infty \).

The most famed fine-tuning in Nuclear Physics is related to the anomalously large S-wave scattering lengths \( a \) and corresponding anomalously small binding energies of few-N systems: the deuteron, triton and both Helium isotopes. These need an intricate balance between attractive long-range and repulsive short-range effects. Chiral symmetry alone does not explain this fine-tuning.

Therefore, a few groups have proposed an expansion about a point where several protective symmetries coincide [45–51]. In the unitarity limit \( a \to \infty \), Nuclear Physics becomes scale-invariant at low scales. The NN cross section saturates and no dimension-ful scale is left. This may suggest that all nuclear binding energies must thus be infinite or zero. But in \( \text{EFT}(\pi) \), renormalisation via the Efimov effect breaks continuous scale invariance down to a discrete one. That introduces one dimension-ful 3N scale, set for example by the binding energy of the triton. When unitarity is imposed for both NN S-waves, Wigner’s \( SU(4) \) symmetry of arbitrary combined spin and isospin rotations is manifest as well. So the unitarity limit is a point of increased symmetry, and Nature appears to break it only by a small amount, \( \frac{1}{m_\pi} \lesssim 0.3 \). This expansion reproduces well the ground state of \(^4\text{He} \), and that its first excitation is very close to breakup. There is even evidence that the Coester correlation between binding energy per nucleon and nuclear matter density can be explained, as well as the symmetry energy of nuclear matter, its slope and compressibility.

Surprising here is not so much that the detailed value of the scattering length becomes irrelevant in nuclei and nuclear matter – after all, the momentum scales are \( p_{\text{typ}} \gg \frac{1}{m_\pi} \). Surprising is that one might get away with a theory of nuclear matter that does not know about pions if or because \( p_{\text{typ}} < m_\pi \). The typical binding
momentum in heavy nuclei is $\gamma_A \sim \sqrt{2M_B A/\Lambda} \rightarrow 120\text{MeV} \lesssim m_\pi$ for $B_A/A \approx 8\text{ MeV}$, which is still less than the pion mass. Only more, and more non-trivial, applications can shed light on this idea.

The goal here is not a detailed reproduction of Nuclear Physics, but a conceptual understanding of its gross structure from its most important symmetries. The proposal constitutes a paradigm shift away from emphasising details of QCD and NN scattering, towards the importance of renormalised scales in the 3N system and of approximate symmetries. It even speculates that patterns like a Nuclear Chart are not unique to QCD but analogues emerge in any many-body system with anomalously large scattering lengths, like clusters of Rb atoms; cf.[52]. It also begs the obvious question how this protective symmetry emerges for the EFT which includes pionic degrees of freedom, and on the quark level. Finally, it sheds light on the fundamental question how complex phenomena emerge from seemingly simple foundations, and how simple patterns emerge in turn.

2.6 The Original Sin\(^8\), Or: Power-Counting a Non-Perturbative LO

Different choices how to count powers in the small dimension-less expansion parameter $Q$ lead to vastly different physical situations, but they all need to be consistent, not some \textit{ad hoc} prescription to be thrown overboard once one encounters problems. Only self-consistent theories can be falsified in observations. Inconsistent ones are wrong from the start\(^9\).

When all interactions are perturbative, as in the mesonic and single-baryon sectors, this amounts to little more than counting powers of $k$ and $p_\text{typ} \propto \Lambda$ but not quite. One must still classify them as large or small relative to another scale: $m_\pi \gg m_e$ but $m_\pi \ll M_\text{physicist}$. Usually, that is understood, but dimension-ful quantities quickly trick one into paying less attention to relevant high-momentum scales.

However, if there are shallow real or virtual bound states at scales $\frac{1}{a} \lesssim p_\text{typ} \ll \Lambda_{\text{EFT}}$ in the EFT’s range of validity, some interactions must be treated non-perturbatively at leading order. In other words, an infinite number of terms needs to be summed because bound particles are never free – they never do not interact with each other. Weinberg ingeniously proposed to pragmatically power-count the potential, truncate it at a given order, and then iterate that to create bound states [12]; see also his more qualitative discussion in [11]. That appears to conflict with the fundamental EFT tenet that the power counting only applies to physical (renormalised and/or observable) quantities, but it was a convenient and clever way to enrol for quick results the by-then well-matured technologies to solve Schrödinger and Lippmann-Schwinger equations: plug a ready-made potential into an accepted formalism.

Since then, who employed this idea appears to have at times become more important than \textit{whether} to employ it\(^10\). Especially in the first decade of the third millennium, an exegetic, and at times even hermeneutic, reading of the Sacred Texts [11–13] attempted to extricate more than a plain, constitutionalist interpretation allows. While research is often inspired by particular phrases we encounter, scientists can fortunately claim to be agnostic about an author’s intentions or reputation. What eventually counts (or should count) is logical deduction and reproducible observation.

Consider a general argument much older than ref. [25]. Denote the NN scattering amplitude $T_{\text{NN}}$ by an ellipse and the interaction kernel $K_{2\text{N}}$ by a rectangle. For two nucleons, the kernel is the two-nucleon potential $V_{\text{NN}}$ of strong interactions. The semi-graphical representation of the well-known LO Lippmann-Schwinger integral equation is:

\[
\begin{align*}
\begin{array}{c}
\text{k} \\
\text{K_{2N}} \\
\text{k}
\end{array}
\end{align*}
\]

\[
(T_{\text{NN}} \sim Q^m) = (K_{2\text{N}} \sim Q^m) + (T_{\text{NN}} G_{\text{NN}} K_{2\text{N}} \sim Q^{2m} Q^{m_e})
\]

where $q$ is the relative momentum of the nucleons in the intermediate state, $k$ is the scattering momentum, and nucleons are close to their non-relativistic mass-shell, $E \sim \frac{k^2}{2M} \sim Q^2$ (potential régime; see e.g. [56,57]). The

\(^8\) Note Added in Proof: D. R. Phillips uses the same phrase to explore a slightly different perspective of the same issue [53].

\(^9\) \text{Das ist nicht nur nicht richtig; es ist nicht einmal falsch!} [54].

\(^{10}\) \text{When the President does it, that means that it is not illegal.} [55].
intermediate NN state of free two-nucleon propagation is described by a propagator (free Green’s function of the NN system) and an integration. Let us say this operator scales with some power of $Q$,

$$G_{NN} := \int \frac{d^3q}{(2\pi)^3} \frac{1}{k^2 - q^2} \sim Q^{mG}. \quad (4)$$

Only a nonperturbative solution, namely summing at least some subset of interactions an infinite number of times, keeps the particles always close and hence creates a bound state. Therefore, all terms, including the interaction, must be of the same order. Without that, one term could be treated as perturbation of the others. For example, if the driving term (in the middle) were of higher order, we would see bound states but no scattering states. If the homogeneity (last term) were of higher order, one would find the Born approximation.

That resummation is not just a good idea when shallow bound states exist, but is compulsory, imposes consistency conditions on both interaction and amplitude: they both must count the same, $T_{NN} \sim K_{2N} \sim Q^m$, and the last term in eq. (3) imposes they must count inverse to $G_{NN}$, i.e. $m = -mG$. The integral in $G_{NN}$ is dominated by the parts with large integrand, namely when typical scales of loop momenta are the external, low momenta: $k \approx q \sim p_{\text{typ}} \sim Q$. That is also a fundamental tenet of Naïve Dimensional Analysis justified in the threshold expansion formalism [56,57]. Therefore, the scaling is fixed as $G_{NN} \sim Q^{3/2} = Q^1 \sim Q^{-m}$ and the mere existence of a shallow real or virtual bound state mandates $T_{NN} \sim K_{2N} \sim Q^{-1}$.

Remember that $G_{NN}$ describes the propagation of two non-interacting nucleons between two interactions. It is agnostic about the kernel $K_{2N}$. Ultimately, binding must be explained by the interaction as the origin of an intimate correlation of nucleons, and not by the free-nucleon propagator.

This reasoning has several intriguing aspects. It is simple. It only relies on the qualitative feature of the existence of an anomalously shallow bound state, not on any particular value of $a$. It does not reveal which terms constitute the LO kernel or how the shallow scale emerges from it; only how those terms must be power-counted. It thus equally well applies to any systems with shallow bound states, including halo-EFT, EFT($\vec{q}$) and Non-Relativistic QED/QCD. It is consistent at LO and permits corrections to be treated in strict perturbation theory; see Sect. 3.5 below. It imposes a power-counting from an observable, namely the scattering amplitude $T_{NN}$, via a free propagator $G_{NN}$ onto the non-observable $K_{2N}$, not vice versa.

It also leads to a surprising take on the one-pion exchange, as it appears to scale like $\frac{(\sigma \cdot q)(\sigma \cdot q)}{q^2 + m^2} \sim Q^0$ if one counts only explicit low-momentum scales, but must be of order $Q^{-1}$ if its iteration is required.

Consequently, a choice of $\chi$EFTs exist with the same symmetries and degrees of freedom but different power countings, corresponding to different worlds. In the “KSW” version, the system is at LO ($Q^{-1}$) bound by contact interactions only, like in EFT($\vec{q}$), and the one-pion exchange scales indeed as $Q^0$ to enter at NLO. Its analytic results in the NN system pass every test of self-consistency [58–60]; cf. Sect. 3.6.

In the most popular version of $\chi$EFT, the one-pion exchange is taken to enter at leading order [11–13] and therefore must scale as $\frac{(\sigma \cdot q)(\sigma \cdot q)}{q^2 + m^2} \sim Q^{-1}$. One cannot just count momenta.

There are many other versions, all consistent, but all describing different worlds. One has no shallow scales ($\frac{1}{a} \sim \Lambda_{\text{EFT}}$), one-pion exchange $\frac{(\sigma \cdot q)(\sigma \cdot q)}{q^2 + m^2} \sim Q^0$ is perturbative (Born approximation) and only detailed knowledge of QED explains nuclear states. We quickly throw that one away because Nature has bound states in low partial waves [11]. But its power-counting is appropriate for higher ones [61].

Most likely, the real world is reproduced at LO by mix-and-match: contact interactions plus one-pion exchange in the $3\text{SD}_1$ and some other low partial waves, KSW in $^1\text{S}_0$ and others, and perturbative in higher partial waves $l \geq 3$ or so [62–65]; cf. Sect. 3.1.

So what makes one decide whether resumming one-pion exchange (OPE) at LO is mandatory or discretionary? Its scale appears in $\chi$EFT to be set by $\Lambda_{NN} = \frac{16\pi f_\pi^2}{g_A M} \approx 300 \text{ MeV}$ [58,59,63]. That lies right between the typical low scale $m_\sigma$ and the expected breakdown scale $\Lambda_\chi$. This scale is dynamic, dictated by interactions, and thus most naturally accommodated in the kernel/potential $K_{2N}$. Below it, pions are higher-order effects and hence perturbative (KSW); above, they are LO and hence nonperturbative. However, that does not explain why shallow bound states exist. Chiral extrapolations show that the QCD parameters are fine tuned to produce large scattering lengths at the physical pion mass; see [66] and references therein. Even small variations in $m_\pi$ bring one quickly to a world with $a \lesssim \frac{1}{m_\sigma}$. But $\Lambda_{NN}$ is largely constant in $m_\sigma$ since it only involves $g_A$, $f_\pi$ and $M$, none of which change dramatically with $m_\sigma$. For $p_{\text{typ}} \gtrsim \Lambda_{NN}$, one may thus be forced to resum the

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11 Note Added In Proof: See footnote 2 in D. R. Phillips’ contribution to this issue [53].
Weinberg’s pragmatic proposal is widely interpreted as counting powers of $p_{\text{typ}} \sim q$ only, with $K_{2N} \sim q^0 \sim Q_{m=0}^{m=0}$ as LO. Since there is a shallow bound state, eq. (3) mandates then $G_{NN} \sim Q^0$, and therefore $q \sim Q^0$. That leads to the contradiction that all powers of $q$ actually count the same, $q^n \sim Q^0$ – unless one resorts to fine-tuning in $G_{NN}$. One could try to count powers of $q$ and of $(k^2 - q^2)$ differently in eq. (4). But that contradicts that typical scales of loop momenta are the external, low momenta $p_{\text{typ}}$. One could also try to fine-tune otherwise independent contributions. For example, re-interpret $G_{NN}$ as doing more than just propagating two free nucleons. Since eq. (3) is form-invariant under

$$G_{NN} \rightarrow Q^{-1} G_{NN} \sim Q^0, \quad T_{NN} \rightarrow Q T_{NN} \sim Q^0, \quad K_{2N} \rightarrow Q K_{2N} \sim Q^0,$$

(5)

this shifts the fine-tuning burden from the kernel $K_{2N}$ to the two-nucleon propagator $G_{NN}$ and turns the free propagator into a quasi-correlated one – except that there are no interactions in it. I fear this leads to a contradiction: The propagator is both propagating without interactions and knows it is inside a nucleus, subject to binding forces and correlations with other nucleons – it is born free but everywhere is in chains [67]. On top of that, none of this explains how to proceed with the power counting at higher orders. Often, Weinberg’s pragmatic proposal is advocated as advantageous because one only needs to count powers of $p_{\text{typ}}$ in the kernel $K$, but what about $G_{NN}$? Does its fine-tuning persist at higher orders, or does it also contain contributions which scale naturally? As to be discussed in Sect. 2.8, the idea already fails at NLO. Without the alleged fine tuning, on the other hand, it counts reproducibly as $Q^1$.

Finally, one can extend this construction to $n$-nucleon systems with a kernel $K_{nN}$ by induction from the scaling of the amplitude $T_{(n-1)N}$ of the anomalous $(n-1)$-particle subsystem with kernel $K_{(n-1)N}$:

$$T_{nN} \sim K_{nN} \sim Q^{1-n}.$$

(6)

Remember that $K_{nN}$ is not necessarily what is often called a $n$-nucleon interaction, but rather a kernel which involves interactions of possibly up to $n$ nucleons. It could also only contain NN interactions. The construction is agnostic about which interactions are needed – it only addresses how those which are needed, are to be power-counted. In $\chi$EFT, for example, the leading-order kernel $K_{3N}$ between 3 nucleons appears to contain only NN interactions, and no 3N interactions [70]. In EFT(\&), however, it is well known that a 3N interaction is needed at LO to stabilise the system against collapse, in addition to an iterated NN interaction. So there is no contradiction between the 3N interactions of these two theories because their information content is different. What exactly constitutes the LO kernel depends on the particles, symmetries, resolution scales etc. of the theory.

### 2.7 It Works Until It Doesn’t, Or: Power Counting for Uncorrelated Nucleons

The preceding argument also defines a correlated few-nucleon propagator: Even the $m$th iteration of the $n$-nucleon interaction counts the same, namely $K_{nN} \sim K_{nN}(G_{nN}K_{nN})^m \sim T_{nN} \sim Q^{1-n}$. It is this rescattering series which turns the free Green’s operator into the correlated propagator of nucleons inside a nucleus, 

$$G_{nN} \rightarrow G_{nN}^{\text{correlated}} = G_{nN} \sum_{i=0}^{\infty} (K_{nN}G_{nN})^i = G_{nN} [1 - K_{nN}G_{nN}]^{-1}.$$

(7)

One can now understand the transition from correlated to quasi-free/uncorrelated propagation of nucleons inside nuclei, as formalised in Threshold Expansion [56, 57]. The question becomes: At which scales is the resummation of the geometric series in the equation above mandatory, and at which is it merely optional? A key assumption above was that the scattering particles are not far off their mass shell, $k^2 \sim q^2 \sim ME$. Since this implies that momenta are much bigger than kinetic energies, $k \sim q \gg E$, energy flow in interactions can be treated perturbatively, which leads to a kernel $K_{2N}$ which describes an instantaneous, energy-independent potential $V_{NN}$ between two nucleons at LO – like in one-pion exchange. What happens when an external probe, say a photon, pumps more energy $\omega$ into the system? The relative importance of terms, and hence the power counting, changes with energy because the Physics that is relevant changes qualitatively. First, adding or subtracting external energy modifies the propagator in $G_{NN}$ to

$$\frac{1}{k^2 - q^2} \rightarrow \frac{1}{k^2 \pm M\omega - q^2}.$$

(8)
For $\omega \sim E \sim k^2 q^2 M_\pi^2 \ll k, q$, that little more of energy does not knock nucleons significantly off their mass shell. The scale of the $q$-integration in eq. (4) is still set by $k, q$, i.e. $G_{nN} \sim Q^1$ and $T \sim Q^{-1}$ as before. Nucleons interact for a long time $\sim \frac{1}{\omega} \gg \frac{1}{M}$, until the uncertainty principle makes them radiate the excess energy, e.g. as another photon. In that time between energy absorption and emission, such rescattering maintains the the correlated nuclear state $G_{nN}^{\text{correlated}}$. This is precisely what is needed in Compton scattering at $\omega \ll m_\pi$ to maintain the Thomson limit as low-energy theorem on the nucleus as a whole. One can show that it is fulfilled exactly at LO, with zero higher-order corrections; see [26] and references therein.

On the other hand, nuclear coherence breaks at higher energies $\omega \sim k \gg E \sim k^2 M_\pi$ (but of course still $\omega \ll \bar{K}_{\text{EFT}}$). Such scales are relevant in Compton scattering [26], and in particular in pion photo-production [68] and pion-nucleus scattering [13], where the pion’s absorption by nucleus adds an energy of at least $\omega \geq m_\pi$. Intuitively, the intermediate-nucleon state is in that case far off-shell and the time-scale $\frac{1}{\omega} \sim \frac{1}{E^2 m_\pi}$ between absorbing and emitting the large excess energy is too short for much rescattering. Instead, nucleons propagate largely un-correlated, namely quasi-free via $G_{nN}$. Recattering is demoted to a higher-order effect. Indeed, expand eq. (8) as

$$\frac{1}{k^2 \pm M \omega - q^2} \to \frac{1}{\pm M \omega - q^2} \left[ 1 + \mathcal{O}\left( \frac{k^2}{M \omega} \right) \right]$$

reveals that the information about binding (momentum $k$) is lost at LO and the momentum scale in the integration is set by the external probe as $q^2 \sim M \omega$. The binding scale $k \sim \frac{1}{a}$ has disappeared, and there is no need to iterate an instantaneous kernel $K_{nN}$ for $T_{nN}$. Instead, the energy $\sim \omega$ and momentum $q$ of the intermediate state are of the same size, which means retardation becomes important. The interaction between nucleons is not any more instantaneous, and $K_{nN} \sim Q^0$.

Weinberg’s pion-deuteron scattering used that the pion transfers $\omega \geq m_\pi$ on the nucleus and allows for at most one instantaneous charged-pion exchange between nucleons before the pion is radiated off again [13]. He intuitively identified a two-nucleon irreducible part which was made coherent by iteration to produce the nucleus, and then sandwiched irreducible chiral interactions with external probes once between the resulting wave functions, at energies so high that intermediate states had no coherence.

### 2.8 Child’s Play, Or: Why Getting Counting Right Beyond LO is Important

Toy models are helpful idealisations because lessons in simpler settings inform the search for solutions of complicated issues. While they can validate hypotheses only for the model studied, and not in the general case, they can invalidate general hypotheses by counter-example. In Nuclear Physics, we have a toy model which even provides useful Physics insight: EFT($q$); cf. the ideas about the unitarity limit in Sect. 2.5.

EFT($q$) also illustrates that power-counting is not just Mathematics, no Physics [9]: One cannot cure deficits by just going to sufficiently high orders until the actually-lower-order LECs are all included. An illustration might be useful. In Weinberg’s pragmatic proposal, the order at which each term in S-wave EFT($q$) contributes should just follow the number of momenta, namely $p^{2n} C_{2n} \sim Q^{2n}$ enters at $N^{2n}$LO. At LO or alleged $\mathcal{O}(Q^4)$, the scattering lengths suffice as sole input. No correction enters at what is believed to be $\mathcal{O}(Q^1)$, and $p^2 C_2$ enters only at putative $\mathcal{O}(Q^2)$ or $N^2$LO, determined by one new datum. Assuming for concreteness $Q \approx \frac{1}{3}$, $N^2$LO would only provide a correction of $Q^2 \approx \pm 10\%$; the next term, allegedly $N^3$LO, just $Q^4 \approx 1\%$.

Actually, one can show analytically that the contact interactions of EFT($q$) scale as $p^{2n} C_{2n} \sim Q^{n-1}$. LO is $Q^{-1}$ as expected, and corrections enter at $N^0$LO, namely much earlier than anticipated in the pragmatic proposal. For example, $p^2 C_2$ is NLO and contributes $Q^1 \approx \pm 30\%$. That re-ordering ricochets across orders. In a little twist, the renormalisation group flow of the $n$th term allows for a new parameter to be determined from data only at $N^{2n-1}$LO [60]. So, every subsequent fit parameter $C_{2n}$ enters one order earlier than conjectured, and its effect is $\frac{1}{Q} \approx 3$ times stronger than when simplistically counting momenta.

In conclusion, an incorrect power counting scheme to classify interactions makes one work not to an accuracy one had hoped for, but bears the very real danger of doing worse. The optimist is then frustrated that Bayesian uncertainty quantification [18–21] reveals the corrections from one alleged order to another to be quite frequently larger than anticipated, until one is finally forced to acknowledge that something must be wrong. One may then ask if fighting the obvious at all cost is worth years of trouble.
3 Development: Humanity Amongst Physicists

Let us therefore talk about pride and prejudice.

3.1 The Whole is Greater than the Sum of Its Parts, Or: Testing Consistency in Non-Perturbative LO

So developing a consistent power counting is a trifle more involved than just counting powers of momenta and asserting that that is all there is to it. Instead, consistency needs to be checked at each order. It is no wonder that once Particle Physicists had figured out the basics of QCD and a possible path to Confinement, some moved on to perturbative Physics at higher energies and thus intellectually less taxing issues than nonperturbative power counting, emergence and complexity12.

After a series of critiques of Weinberg’s approach based on analysing diagrams or classes of diagrams which need to be resummed starting with [69], Beane et al.[62] and Nogga et al.[70] employed fully non-perturbative arguments to demonstrate that Weinberg’s pragmatic proposal is fundamentally and irredeemably flawed not just because the argument why LO should be resummed is not consistent.

Beane et al.[62] showed that in Weinberg’s pragmatic proposal, the \( m_\chi \)-dependence of contact interactions between two nucleons does not match that of the cutoff-dependencies they are to cure, for example in the \( ^1S_0 \) wave. But extrapolations to physical pion masses used in lattice-QCD rely on the correct \( m_\chi \)-dependence of LECs. To those uninterested in relating \( \chi \) EFT to QCD, that appears irrelevant in the real world where \( m_\pi \approx 140 \text{ MeV} \) is fixed, until one realises that chiral minimal substitution generates interactions from such dependencies. Chiral symmetry dictates that a \( m_\pi^2 \)-dependent NN interaction creates at the next chiral order an interaction between two pions and two nucleons which has the same strength: \( D_2 m_\pi^2 \text{NN} \xrightarrow{\chi \text{sym}} D_2 m_\pi^2 \pi \pi \text{NN etc.} \) This child inherits a power counting and strength from its parent.

Just as concerning is the finding by Nogga et al.[70] that the pragmatic proposal cannot cure strong cutoff-dependencies in some of the important attractive NN partial waves. The argument is actually quite intuitive [63–65]. The tensor part of one-pion exchange contributes at LO. When it is attractive \( \propto r^{-3} \) at small distances, the wave function collapses into the origin, meaning one is sensitive to details of short-distance Physics. To avoid that, one must add a repulsive LEC with one parameter determined by data. That is the familiar scenario in the \( ^3S_1 \) wave. Whether that happens in other waves as well depends on the height of the repulsive centrifugal barrier \( \propto l(l+1)r^{-2} \) of orbital angular momentum \( l \), relative to the scattering energy and the strength of the tensor interaction, at distances \( r \gtrsim \Lambda_\chi \). Peripheral waves remain unaffected because the centrifugal barrier is too strong [61], but low partial waves, most notably \( ^3P_0 \) and \( ^3P_2 \), need LECs even at LO. Since these are momentum-dependent like \( k^{2,4,6,8} \), they would only enter at \( N^{2,4,6,8} \) LO in Weinberg’s pragmatic proposal, but are actually needed at LO to prevent collapse. The effect of these momentum-dependent, stabilising interactions is large, and again ricochets across orders like in EFT(\( \pi \)); cf. Sect. 2.8. On top of that, minimal substitution turns them again into additional interactions with external probes at leading and subsequent orders, e.g. in photo-nuclear reactions via LECs for \( \gamma \text{NN}, \gamma \pi \text{NN etc.} \) Likewise, gauge and chirally invariant few-nucleon LECs are re-ordered [71,72].

As sociological footnote, it took EFT practitioners more than a decade, and several since the discovery of the tensor force, to numerically test if it is properly renormalised by momentum-independent LECs. What Nogga et al.[70] did is so endearing because it is straightforward, almost trivial – in hindsight – and thus begs the question why it had not been done before.

The mis-classification in Weinberg’s pragmatic proposal translates thus into mis-estimates of coupling strengths, and thus of the accuracy to which single-nucleon observables like the \( \pi N \) scattering lengths can be extracted from few-nucleon data before few-nucleon LECs like a \( \pi \pi \text{NN term} \) enter. Under-estimating their importance leads to a false sense of accuracy. Weinberg’s pragmatic proposal may predict that one can extract some one-nucleon observables from few-nucleon data with an accuracy of \( \pm 5\% \), when it is actually only \( \pm 20\% \) – unless data from different nuclei allow one to extract the new few-nucleon LEC or these can be determined in lattice-QCD computations like in [73].

J. de Vries and collaborators recently demonstrated that the consequences are beyond ivory tower theatrials; see e.g.[74,75]. Building on [71], they showed that in neutrinoless double-\( \beta \) decay, a short-range, lepton-number-violating interaction \( \text{nn} \rightarrow \text{pee} \) between two neutrons already enters at LO, and not at \( N^2 \) LO as Weinberg’s pragmatic proposal would have it. Therefore, one cannot make even LO predictions of \( 0\nu\beta\beta \)

\[ ^{12} \text{I write as someone raised as Particle Physicist, inoculated early with a contempt for the messiness of Nuclear Physics.} \]
matrix elements in nuclei without at least estimating the strength of that interaction. Fortunately, they were also able to calculate the corresponding LEC. That is of course indispensable for any interpretation. Computations of the pertinent matrix elements are now being redone by several groups. Likewise, few-nucleon interactions with unknown coefficients enter at NLO in the direct detection of Dark Matter via nuclei and in the search for Electric Dipole Moments. Notice that these are all processes in which planning and analysis of multi-million-dollar experimental efforts to look for beyond-the-Standard-Model Physics rely on theory predicting, not post-dicting, effects.

3.2 There is Always a Well-Known Solution to Every Human Problem – Neat, Plausible, and Wrong [76]

So, ordering interactions in $\chi$EFT is not as simple a prescription as adding and subtracting powers of momenta. It is a set of operational instructions: Include at each order only those interactions needed to renormalise the problem or with coefficients which Naïve Dimensional Analysis predicts at that order.

Fortunately, a small band of brave theorists has taken on the ungrateful and gruelling task to tirelessly turn such abstract rules into tables for the rest of us what term needs to be added at what order [64,65,77–81]; see [82] for an even-handed review. The dispute which one is correct is not yet completely settled and in affectionate circles known as Power Counting Wars\textsuperscript{13}. A lack of universally accepted analytic solutions obfuscates the relation between cutoff-independence, convergence pattern and numerics, so Bayesian analysis and renormalisation-group consistency checks are reasonable tools to convince the community [18–21,25].

Whether we will listen to and implement the outcome, will determine the fate of $\chi$EFT as either another set of models which have at least (largely) the correct symmetries but to which parameters are added as needed to match data – or as a comprehensive and consistent theory of nuclear phenomena; cf. Sect. 3.9.

That does not mean we on the sidelines need to wait with bated breath until the appropriate power counting is established and a consistent set of interactions between pions, nucleons, Deltas and external probes is available, with tested and widely-accepted prescriptions to estimate residual theory errors, to assess residual renormalisation-group (cutoff) dependence, and to establish World Peace. Instead, work progresses now in parallel to update few- and many-body codes with new chiral interactions, all of which will eventually contribute at some order, and with routines to assess uncertainties.

It is indisputable, however, that Weinberg’s pragmatic proposal is not the way forward because ultimately, and leaving all arguments of the preceding sub-section about its self-consistency aside, it pays too much attention to terms that do not matter that much, and not enough attention to terms that matter more than one might have thought. It makes us work both more and less than needed, and it lulls us into a false sense of accuracy. A brilliant idea got us started on the right track, and it turned out to be pioneering but wrong after we learned from it how to think for ourselves.

While the final verdict on what interactions to add at which order is still pending, a number of crucial features are already decided, including that there are more LECs in the attractive triplet-P waves. Today’s potentials, flawed as they are, already show interesting trends. Important lessons are already learned from consistent power counting in not-so-light nuclei; see \textit{e.g.} most recently [83], and [21] in this issue.

3.3 Fit to Shrink, Or: How Much Can We Learn From NN Data?

With well over 6,000 NN scattering data, the temptation is big to turn into “chisquare afficiados” [9], trying to reproduce relatively narrow but cornucopious information extremely well. But if we really need a $\chi^2$ close to 1 in NN phase shifts and binding energies, plus maybe in 3N and 4N, to achieve even just several percent of accuracy for ground and excited energies in nuclei – arguably the observables least sensitive to how well one’s wave function actually captures reality – then that may indicate another fine-tuning: very many orders in $\chi$EFT conspiring, overlapping, cancelling and enhancing each other. The idea is not far fetched that the power-counting/relative importance changes in not-so-few and many-nucleon systems to de-emphasise one-pion exchange. After all, the precise values of the scattering lengths play certainly no substantial rôle in nuclear matter; \textit{cf.} Sect. 2.5. Indeed, combinatoric arguments have recently been employed to advocate for greater relevance of three- and more-nucleon interactions [84].

In a way, the strange sector of $\chi$EFT is more fortunate: Data exists but is rare and not of the kind of alleged extraordinary quality which distracts from the core mission to explain, rather than to fit. It also encourages

\textsuperscript{13} I am sure there is a dissertation about pop culture references in the oral scientific discourse. Please let me know.
one to be much more ingenious to determine parameters; see, for example, [34] in this issue and references therein. Sometimes, less is more, and too much can be a curse.

3.4 The World is Not All that is NN, Or: The Value of External Probes

Of course, Nuclear Physics is more than NN and few-N bound and scattering states. One can learn just as much, and often complementing, information from external probes, breakup, fusion, etc. For example, pion scattering [13], pion photoproduction [68] and Compton scattering [26] on light nuclei all test the charged pion-exchange contribution to nuclear binding, and thus chiral symmetry.

3.5 Do Not Listen to Your Elders, Or: Calculating Higher Orders Made Simpler

Following Weinberg’s example [12, 13], observables beyond LO have traditionally been found by “partial resummation”: Power-count the strong interactions in few-nucleon systems (usually the wrong way; cf. discussions above); truncate at a desired order; and then iterate by inserting it into eq. (3). Likewise, power-count interactions with perturbative external probes, and then sandwich between the wave functions derived from the partially resummed few-nucleon interactions. Long ago, we actually followed Weinberg’s “hybrid approach” [13] to use any high-precision wave function, chiral or not. However, this leads to a mismatch of unphysical high-momentum components and exacerbates theory errors; see also Sect. 3.8. Fortunately, the advent of high-quality chiral potentials allowed us to move on.

For the EFT power counting to make sense, higher-order corrections must be ever-smaller. So including them in “strict perturbation” must be allowed. The geometric series provides a nice example. For $|x| \ll 1$, the resummed and expanded-but-truncated versions must agree within truncation errors:

$$\frac{1}{1-x} - (1 + x + x^2) = \mathcal{O}(x^3) \ll 1 \text{ for } |x| \ll 1.$$  (10)

In general, if the resummed and strictly perturbative results differ significantly, then corrections are obviously not actually small and one is faced with fine-tuning. Whether to resum at higher order or not should therefore not be a matter of principle, but of choice and convenience; cf. kinematics example in Sect. 3.7. But strict perturbation also avoids a number of problems.

First and pragmatically, including complicated interactions in strict perturbation often avoids solving differential or integral equations and leads to simpler, more stable numerics. It is thus not an uncommon trick when 3N and 4N interactions are added in nuclei and nuclear matter; see e.g. [85] in this issue.

Second, iterations usually generate spurious deeply bound states. While by definition outside the EFT’s range of applicability, these are often precariously close and infect observables even at $p_{\text{typ}} < \Lambda_{\text{EFT}}$. With higher order or higher cutoff $\Lambda$, their number proliferates and they become more problematic [86, 87]. Take the resummed Effective Range Expansion:

$$\frac{4\pi}{M} \frac{1}{\frac{1}{a} + i \frac{1}{k}} \left[ 1 + \frac{\rho}{2} \frac{k^2}{a + i k} + \ldots \right] \Rightarrow \mathcal{A}(k) = -\frac{4\pi}{M} \frac{1}{\frac{1}{a} - \frac{\rho}{2} k^2 + \ldots + i k}$$  (11)

Its LO is found for effective range $r_0 = 0$. The strictly-perturbative result on the left provides a small correction as long as $a \sim \frac{1}{k} \gg r_0 \sim \frac{1}{m_\pi}$. Its LO pole is at $k_0 = \frac{1}{a}$, and shifted at NLO to $k_0 \approx \frac{1}{a} (1 + \frac{\rho}{2 a})$.

These same poles are also found in the resummed version on the right hand side at LO ($r_0 = 0$) and NLO. But there is another NLO pole at $k_1 \approx \frac{2}{r_0} (1 - \frac{\rho}{2 a}) \approx \frac{2}{r_0} \gtrsim m_\pi$ with equal but opposite, and hence unphysical, residue. Since $\frac{\rho}{a} \ll 1$, it is never far from the breakdown scale. In the $^1S_0$ channel, for example, $k_1 \approx 150i$ MeV $\approx m_\pi$. At $^3S_1$LO, this pole moves to 130i MeV and one encounters two more poles at $[-60i \pm 350] $ MeV. Two more unphysical poles appear at every odd order. Similarly, the attractive $\frac{1}{r}$ part of the tensor one-pion exchange leads to additional deeply bound states.

Such spurious states not only lead to numerical issues which need nontrivial solutions, like projecting them out. Partial resummation also softens the (unphysical) ultraviolet behaviour of the amplitude: The resummed

\[14\] Yes, poles can be shifted in perturbation theory; see for example [59] and also formalistic details and references in [88].
NLO version of eq. (11), $A_{NLO}(k \to \infty) \sim k^{-2}$ converges more quickly than the LO form, $A_{LO}(k \to \infty) \sim k^{-1}$, while the strictly perturbative NLO version is $\sim k^0$. Therefore, if these amplitudes are inserted into 3N processes, fewer LECs appear to be necessary to cure residual cutoff dependence at higher orders. In a striking example, Gabbiani demonstrated that a careless resummation of effective-range contributions appears to eliminate the need for the very 3N interaction which is so central to the Efimov effect [89], and that it also happens to lead to phase shifts which are not supported by data. The problem might be mitigated by defining a much smaller applicable cutoff window $\Lambda_{EFT} \lesssim \Lambda \ll k_{spur}$. But that makes it much harder to analyse consistency and cutoff-independence of the EFT power counting. All power counting developers therefore use strict perturbation theory around a non-perturbative LO result.

In it, both observables and interactions are expanded in powers of $Q$, and only matching powers are kept. For example, the NLO correction $T_0$ (blue hatched) to eq. (3) is determined by terms which involve the NLO interaction $V_0$ once and once only, and the LO $T_{-1}$ (red shaded) only in half-off-shell kinematics:

\[
\begin{align*}
\begin{array}{c}
\raisebox{-0.5em}{\includegraphics[width=0.2\textwidth]{fig12}}
\end{array}
\end{align*}
\]

The total amplitude is then the sum of LO and NLO, $T = T_{-1} + T_0$. Starting at N^2LO, terms like

\[
\begin{align*}
\begin{array}{c}
\raisebox{-0.5em}{\includegraphics[width=0.2\textwidth]{fig13}}
\end{array}
\end{align*}
\]

appear to require LO amplitudes with both incident and outgoing momenta off-shell, sandwiched between NLO corrections $V_0$. Many, like me, were happy with strict perturbation at NLO but thought N^2LO and beyond was just too much work, turned to partial resummation, and discouraged others to push further [90].

Fortunately, Jared Vanasse [87] did not listen and in 2013 re-discovered what, embarrassingly, has for over a century been known in mathematical Perturbation Theory. I even taught it regularly in my Mathematical Methods class without making the connection. One never needs a full-off-shell amplitude! The lower-order correction to the half-off-shell amplitude $T_{<n}$ and to the interactions $V_{<n}$ (both red) fully determine the $n$th correction in an integral equation for the half-off-shell $T_n$ from term $V_n$ (both blue):

\[
\begin{align*}
\begin{array}{c}
\raisebox{-0.5em}{\includegraphics[width=0.2\textwidth]{fig14}}
\end{array}
\end{align*}
\]

This is simply the Distorted-Wave Born Approximation for the Lippmann-Schwinger equation and implemented quite easily into existing codes; notes on a number of other approximation methods exist [88].

3.6 Listen to Your Elders, Or: Bridging between EFT($\pi$) and $\chi$EFT

The KSW variant of perturbative pions with a non-perturbative contact interaction at LO [58, 59] is a consistent $\chi$EFT, with analytic results in two-nucleon processes. Unfortunately, this beautiful theory was slain by the ugly fact that the momentum-dependence of the corresponding counter term at N^2LO in the $^3$SD$_1$ wave is not versatile enough to limit corrections to remain smaller than the leading pieces beyond momenta $\lesssim 200$ MeV or so. It quickly had no resemblance with data [60].

Indeed since the inception, pre-EFT people had warned us that in their experience, the attractive tensor part of one-pion exchange was so strong that it needed to be iterated to get anywhere near observed phase shifts even at relatively low energies. I flat-out denied the relevance of that, but You don’t know what you are talking about [9]. So we buried perturbative pions for not living up to their promise.

And yet, there is life in the old dog. Not only are perturbative pions the consistent $\chi$EFT both of the $^1$S$_0$ wave [62] and of the $^3$SD$_1$ wave at least up to $m_\pi$ [60]. It is my firm prejudice, untainted by evidence, that they are also our best hope to capture the transition from pionless EFT to non-perturbative pions.
3.7 The Thin Blue Line, Or: Reporting Results Needs Theory Uncertainties

EFTs make a specific promise to facilitate a core tenet of the Scientific Method: To provide pre- and postdictions that can be falsified by data. For that, not only experiments must provide error bars; theorists, too, must clearly and reproducibly assess their uncertainties, preferably before a closer look at the data to be explained [91]. That means theorists must provide a probability distribution function encapsulating the likelihood of their answer, so that its overlap with data can be quantified. It is insufficient to compare numbers; one also must judge their reliability. Only then can one enter into an informed scientific discussion weighing one interpretation against another. A priceless advantage of EFTs is that its assumptions can be tested ex-post: Are higher orders indeed small? Is the expansion parameter what it is supposed to be?

Reasonable people can reasonably disagree about to which degree reasonable assumptions are actually reasonable, but no reasonable dialogue is possible without disclosing those assumptions in full. Error bars have error bars [94]; that is why in modern statistics language, they are called “confidence intervals”15.

In retrospect, it is astounding how we EFT advocates in the heat on the top floor of the ECT*’s Rustico in 1999 could endlessly speak of model-independence, consistency and convergence, while at the same time showing precious few quantified estimates of EFT truncation errors [9]. In what surely is a sign of progress, referees now routinely request a discussion of theory uncertainties. Simply stating that this is difficult [9] is no sufficient excuse any more [91]. While such discussions were few and far between before, this past decade saw a barrage of articles on sophisticated tools to quantitatively test the EFT assumptions. Many of these techniques are accompanied by software which makes them easy to employ by the average user; see e.g. [92,93]. Bayesian statistical analysis, starting from reasonable expectations clearly formulated as priors, sets the standard; see e.g. [18–21,25] and references therein. One can use it to quantify to which extent the fundamental EFT assumptions actually bear out: order-by-order convergence; the putative effect of higher-order terms; the values of the momentum-dependent expansion parameter \( Q \) and of the breakdown scale \( \Lambda_{\text{EFT}} \); and whether fitted LECs are indeed of natural size, as Naturalness requires. To be taken seriously, authors have to demonstrate, at the very least, that what is classified as higher-order terms does indeed decrease in importance from one order to the next. Bygone the days of plots with lines of infinitesimal width – corridors of uncertainties are the New Normal. Real theorists have error bars. [9]

While a Bayesian interpretation is only as reasonable as its assumptions, those can, in turn, fortunately be tested for consistency within the formalism itself, but also outside it. Responsible Scientists are schizophrenic at heart: both convinced of a result and constantly questioning it at the same time. The more of our own questions it survives, the more confident we become – and sometimes arrogant. So, it is advantageous to query the prior’s posterior by also assessing how stable results are under reasonable variations not fully captured by Bayesian analysis.

Each of the following methods uses the “democratic principle” that different, reasonable choices must agree up to higher-order corrections. Like in a democracy, fringe choices will lead to extremist results which should however be discarded in a healthy discourse. In particular, one can check if the impact of different choices on observables decrease order-by-order. This way, one yet again maps out a corridor of theory uncertainties which usually complement the corridors of Bayesian analyses because they test assumptions which are at least in part different.

Top of the list is using different numerical cutoffs, and different ways to regulate, like hard, Gaußian and Pauli-Villars cutoff functions. A dimensionful cutoff \( \Lambda \) has no physical significance; it is merely a tool to cut off integrals at high momenta or small distances to test to which degree answers depend on those high-end loop momenta \( q \gtrsim \Lambda_{\text{EFT}} \) at which the EFT does not capture the correct Physics. That means \( \Lambda \gtrsim \Lambda_{\text{EFT}} \) or a bit smaller. While one often talks of “divergent” and “non-renormalised” answers, this is just short-speech for “answers which, at any given order, depend more than they should on what happens at scales at which the theory does not make sense”. But since this is a mouthful, we use trigger words.

It is tempting to call the result for \( \Lambda \to \infty \) “the” answer, but any cutoff is equally legitimate and valid, and none is preferred, as long as \( \Lambda \gtrsim \Lambda_{\text{EFT}} \). This “democratic principle” is a neat tool to turn lines into corridors of uncertainty, even when one is not a purist who explores a wide cutoff range to develop the correct power counting as in Sect. 3.2. Fortunately, modern \( \chi \) EFT interactions come with strengths determined over a reasonable range of cutoffs – but a wider range would of course be better. Unfortunately, these also routinely appear to under-estimate higher-order effects (unless one also samples different cutoff functions). Therefore, a wide corridor indicates that the question whether or not data and \( \chi \) EFT agree remains (at best) undecided.

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15 The aim is to estimate the uncertainty, not to state the exact amount of the error or provide a rigorous bound, [91].

16 Some only vary the cutoff in a window around the breakdown scale, \( \Lambda \approx \Lambda_{\text{EFT}} \), for philosophical or numerical reasons.
But a narrow corridor does not make a result highly reliable without additional corroborating evidence, like a Bayesian uncertainty quantification.

One can also vary the renormalisation point, namely at which energy and from which observable one determines parameters. For example, the Effective Range Expansion of eq. (11) can be about $k = 0$, giving the correct scattering length etc. Likewise, one can expand about the pole position $k_0 = \frac{1}{a}(1 + \frac{m^2}{\not{Q}^2})$, so that one starts out with the correct binding energy even at LO. Indeed, here is another “democratic principle”: fitting at any point or corridor $k \ll \Lambda_{\text{EFT}}$ is equally legitimate if all data are of the same quality. Of course, one must avoid $k \not\sim \Lambda_{\text{EFT}}$ where the expansion parameter becomes unreliably large. Overall, fits must be weighted with $Q$: more constraining at small $Q$, less so as $Q \not\sim 1$. The “Goldilocks corridor” $k \sim p_{\text{typ}}$ is much preferred since it captures the Physics at the scales the EFT is designed for. User-friendly Bayesian methods are readily available [18–21].

$\chi$EFT is designed for $k \sim m_\pi$, so fits for $k \ll m_\pi$ are not as efficient or meaningful. At these lower scales, fine tuning and universal correlations take over. For example, the deuteron mean-square radius is intricately correlated to its binding energy. In the Effective Range Expansion around $k = 0$, eq. (11), it moves from $r_d = a$ at LO to $r_d = a(1 - \frac{m^2}{\not{Q}^2}) = \frac{1}{a} \frac{m^2}{\not{Q}^2}$ at NLO, which also happens to be the inverse of where the pole of the amplitude is at each of these orders; cf. discussion below eq. (11). So, one can avoid the trivial correlation between binding energy $B$ and system size if one starts from the right binding energy both at LO and NLO:

$$r_d = \frac{1}{|k_0|=\sqrt{M_B}} \pm 4\%.$$ Consequently, a fit to these two data is highly correlated. That is but one example of a kinematic point which is, in a particular context, more important than others.

Likewise, the pion-production threshold is in the cm system at $m_\pi$ in LO $\chi$EFT, with nucleon-recoil corrections at higher orders restoring the kinematically correct position. To avoid trivial correlations, one agrees with data on the threshold position by resumming some kinematic contributions; see e.g. [26].

One can also include a sub-set of higher-order corrections which by itself both obeys all symmetries and is not needed to cure other cutoff-dependencies at the given order (i.e. is renormalisation group invariant by itself). That does not include the intrinsic accuracy of the EFT result without additional $a$-priori justification why to include that particular term, and not others (see threshold and pole positions above). One just adds some terms one did not have to add. In return, one can estimate the corridor mapped out by a democracy of higher-order corrections. In the end, however, the accuracy is only improved to that of the next order if one includes all contributions at that next order, not just a few which are easy to compute.

All this makes for more work: more computational resources, more thought, more discussions. But it is time well spent and prevents a quick adrenaline rush of prematurely claiming victory.

It goes without saying that the corridors of theoretical uncertainties should honestly be assessed before comparing to data. The more diverse the methods, the more confident one becomes. After all, we want to determine if the EFT, with its symmetries, degrees of freedom and power counting, is in contradiction to Nature or not, namely if data and theory corridors do consistently and significantly overlap or not. We cannot prove that we are right, but we can be proven wrong, or become – with accumulating evidence – increasingly confident about being right. As frustrating as inconclusive results may be, they are just as worth reporting as agreements or disagreements, so that others who are smarter can build on them.

3.8 Variatio Delectat, Or: What EFT Should I Choose?

EFT( ), KSW and $\chi$EFT both without and with a dynamical Delta are all perfectly consistent theories of Nuclear Physics. They share symmetries but contain quite different particles. Each converges order-by-order at some scale, but the scales where they start to disagree with data is quite different: $p_{\text{typ}} \ll m_\pi \approx 200$ MeV, $\Delta M$, or 700 MeV, respectively. There are also different proposals to order interactions; cf. Sect. 3.2. Likewise, “non-relativistic” versions add relativistic corrections as perturbations and exist alongside kinematically covariant ones. Finally, people determine parameters from different data, fit regions and cutoffs.

\[\text{17 That is the curse of statistics, that it can never prove things, only disprove them! At best, you can substantiate a hypothesis by ruling out, statistically, a whole long list of competing hypotheses, every one that has ever been proposed. After a while your adversaries and competitors will give up trying to think of alternative hypotheses, or else they will grow old and die, and then your hypothesis will become accepted. Sounds crazy, we know, but that's how science works! [96, Sect. 14.0]; cf. Sect. 4.}\]

\[\text{18 As none contain nucleon-anti-nucleon loops, they are not covariant Quantum Field Theories – merely covariant Quantum Mechanics, which is well known to be conceptually inconsistent [95].}\]
This proliferation of variants is confusing – is $\chi$EFT now simply a collection of different models (sometimes merely of different geographical origin) which vie for supremacy, often with less-than-convincing logical arguments? Are we back to my model is better than yours\textsuperscript{9}?

Yes. No. $\chi$EFT is not a fixed set of numbers and interactions but a set of operational instructions. Any two variants which are known to be consistent and applicable in some kinematic overlap must agree to within their respective levels of accuracy inside that region. However, some are technically simpler\textsuperscript{19}, converge more quickly (order-by-order, not to data), avoid fine tuning, or even need fewer parameters at the same level of accuracy. Where a bouquet of variants overlaps, it estimates yet again residual uncertainties.

The variant to choose depends thus also on the problem’s scale and required accuracy – and, most importantly, on what question to answer. Interactions fitted to NN and few-N data robustly explain the gross structures of heavy nuclei. But to get more details right, one better uses parameters from, say, light nuclei. EFTs build bridges between simpler and more complex systems, and explain patterns in the latter.

However, it is not “anything goes”. As much as one might be tempted, one can\textsuperscript{20}, for example, not sandwich pionless interactions whose strengths are determined in $\chi$EFT between wave functions of $\chi$EFT to compute nuclear matrix elements. These two theories have an entirely different particle content (pions or not), in part even different symmetries, and definitely different short-distance behaviour, so the result is inherently unstable against variations of the cutoff. Such mix-and-match can only be compared to wearing a pair of red trousers with a polka-dot shirt.

3.9 Endgame, Or: What Do We Want to Achieve\textsuperscript{21}

What is the goal of an EFT of Nuclear Physics? EFTs are bridges between the microscopic and the macroscopic – $\chi$EFT is one from the quark-gluon version of QCD both to EFT($\pi$) and to Nuclear Structure. They aim to explain relations and structures and are set to perform reasonably in the overwhelming majority of tests. $\chi$EFT will not precisely predict the intricate energy spacing and ordering in Linoleum-314\textsuperscript{9}; if necessary, we have another EFT for that. Indeed, EFTs do not attempt to describe all aspects of the real world at a given scale completely. They cannot be beaten by “nuclear engineering”, namely models fine-tuned to particular details of particular systems, at the cost of failing in almost all other situations\textsuperscript{22}.

A central EFT promise is that it encodes the unresolved short-distance information at a given accuracy into not just some, but the smallest-possible number of independent LECs constrained by a set of symmetries. If a set we thought should be minimal shows actually correlations, then the theory is not yet reduced to its minimal information content, for example because we are missing pivotal symmetries; cf. Sect. 2.5.

That maximally-compressed information is what survives as important to the low-resolution version of the high-resolution theory. And that is why counting powers and finding the smallest-possible number of parameters is so imperative. If two EFTs are renormalised and describe the same data with the same accuracy, the one with the least number of parameters wins because it needs the least information.

But trust in EFT methods and methodologies must be earned by demonstrating that results agree with Nature at least for a few “signal observables”: non-trivial data which have ideally both eluded explanations in the past and are of great importance, for example for key astrophysical processes which help us interpret our place in the Universe. The general public that so gracefully and patiently finances our passion can expect the community to formulate overarching goals and to coordinate the effort towards them. While one may be reluctant to declare success, whatever that means, in “explaining” Nuclear Physics, it is important to do so for at least some subsections we care about.

3.10 Gripes of Wrath

One could write about many more mistakes and things we are doing better now than before Weinberg’s contributions. For decades, we have used few-N potentials which largely neglect retardation effects in pion

\textsuperscript{19} In QCD, the $\overline{\text{MS}}$ scheme is popular because it is simple, but its convergence pattern is actually not that stellar.

\textsuperscript{20} One “can” (is able to), but one “should” not (is not allowed to).

\textsuperscript{21} For the thoughts in this subsection, I am particularly indebted to M. Savage’s moderation of a discussion at the 2015 ECT* workshop NEW IDEAS IN CONSTRAINTING NUCLEAR FORCES with uncomfortable and therefore thought-provoking questions.

\textsuperscript{22} In fact the trouble in the recent past has been a surfeit of different models [of the nucleus], each of them successful in explaining the behavior of nuclei in some situations, and each in apparent contradiction with other successful models or with our ideas about nuclear forces.\textsuperscript{97}.
exchange (some recent work includes it perturbatively). For decades, we fit to NN data above the pion-production threshold with potentials that do not allow for pion production (some recent work stops the fitting just below the pion-production threshold).

Despite two decades of complaints, we also still do not have a common standard to share code or results, be it potentials, interactions, wave functions or matrix elements, so that our peers can use them as input. There are a few but notable exceptions, including the github repositories of the Buqeye [92] and Band [93] collaborations, the self-consistent Green’s function code by Barbieri [98], Stroberg’s in-medium Similarity Renormalisation Group code [99] and shell model codes like NuShellX [100], KSHELL [101], BigStick [102] and ANTOINE [103]. That lattice-QCD is mandated to make configurations and codes fully available after an embargo period, has made the groundbreaking research by upstarts like NPLQCD possible. We know that coding needs dedicated experts, but we have no agreed mechanism to credit them and reward their dedication such that they feel safe to invest the work to document their codes and make them public, without fear of the question what “Actual Physics” they accomplished. On both these issues, see also a recent memorandum on an open-source toolchain for ab initio Nuclear Physics [104].

We continue what we have done so far because it is less dangerous and produces more publications per year than changing course23. We set publications aside which introduce conceptual ideas but do not immediately find a killer application. We are risk-averse because articles about what did not work are difficult to get published. We spin even defeats into victories. We are doing a million things wrong, and only a few middling-right, amongst them that we listen and learn – sometimes.

### 4 Oh Now That’s a Blueprint for an Impossibly Rosy Future [105]

Max Planck had a very dark take on scientific progress, often condensed into the dictum *Science advances one funeral at a time*24. But despite all our shortcomings, Nuclear Physics has revolutionised and re-invented itself in the thirty years since Weinberg’s foundational contributions [11–13]. Jim Friar’s prediction in 1999 became true, he was just too optimistic by a factor of ten: *In 1-to-2 years, we will all be using χ PT-designed products.* [9] In this issue, Ruprecht Machleidt eloquently describes how, after initial scepticism, the Nuclear community not just adapted and adopted, but embraced χ EFT [107].

On the way, we doled out foolish advice and ridiculed sage one. In the next thirty years and beyond, we will forget lessons learned, indulge in new mistakes, explore new cul-de-sacs and commit new follies. To crawl or even more-than-crawl forward on the road of progress will continue to need blood, toil, tears and sweat [108]. We do Nuclear Physics not because it is easy, but because it is hard – but hopefully, new generations will make light of present and past struggles as trivial [109].

*Wrong theories are not an impediment to the progress of science. They are a central part of the struggle.* [110]

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23 One of the referees reminds me, however, that we sometimes continue what we have done so far because we actually believe in it.

24 *Eine neue wissenschaftliche Wahrheit pflegt sich nicht in der Weise durchzusetzen, daß ihre Gegner überzeugt werden und sich als belehrt erklären, sondern vielmehr dadurch, daß ihre Gegner allmählich aussterben und daß die heranwachsende Generation von vornherein mit der Wahrheit vertraut gemacht ist.* [106].
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