STABILITY, CYCLICITY AND BIFURCATION DIAGRAMS OF POLYCYCLES IN NON-SMOOTH PLANAR VECTOR FIELDS

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Abstract. In this paper we extend the main results about polycycles (also known as graphs) of planar smooth vector field to planar non-smooth vector fields (also known as piecewise vector fields, or Filippov systems). The polycycles considered here may contain hyperbolic saddles, semi-hyperbolic saddles, saddle-nodes and tangential singularities of any degree. We determine when the polycycle is stable or unstable. We prove the bifurcation of at most one limit cycle in some conditions and at least one limit cycle for each singularity in other conditions. We also give the bifurcation diagram of the polycycles composed of a hyperbolic saddle and a quadratic-regular tangential singularity.

1. Introduction and description of the results

The field of Dynamic Systems had developed and now have many branches, being one of them the field of non-smooth vector fields (also known as piecewise vector fields, or Filippov systems), a common frontier between mathematics, physics and engineering. See [3, 10] for the pioneering works in this area. A polycycle is a simple closed curve composed of a collection of singularities and regular orbits, inducing a first return map. There are many works in the literature about polycycles in smooth vector fields, take for example some works about its stability [5, 7, 11, 21], the number of limit cycles which bifurcates from it [8, 9, 13, 15, 22], the displacement maps [6, 12, 14, 19] and some bifurcation diagrams [8, 16]. There are also some literature about polycycles in non-smooth vector fields, dealing with bifurcation diagrams [1, 17, 18] and the Dulac’s problem [2].

The goal of this paper is to extend to the realm of non-smooth vector fields some of the well established results about polycycles in smooth vector fields. To do this, we lay, as in the smooth case, mainly in the idea of obtaining global properties of a polycycle from local properties of its singularities. Based on the works of Mourtada [15], Dumortier et al [8] and Andrade et al [1], we established some results about the stability of a polycycle and the number of limit cycles which can bifurcate from it. Take for example one of the two polycycles \( \Gamma \) given by Figure 1. Let \( 2k \) be the contact of the non-smooth vector field at \( p_2 \). Let also \( \nu < 0 < \lambda \) be the two eigenvalues of the hyperbolic saddle \( p_1 \) and \( r_1 = \frac{\nu}{\lambda} \) the hyperbolicity ratio of \( p_1 \). We know that, locally, \( p_1 \) contracts the flow if \( r_1 > 1 \) and repels it if \( r_1 < 1 \) (see more details in Section 3.1). Therefore, it was possible to prove that \( \Gamma \) attracts the flow in the bounded region if \( 2kr_1 > 1 \) and repels it if \( 2kr_1 < 1 \). See Theorem 1. Moreover, besides \( 2kr_1 > 1 \), if we also have \( r_1 > 0 \), then it follows from Theorem 2 that at most one limit cycle can bifurcate from \( \Gamma \) and, if it does, then the limit cycle is hyperbolic and stable. Moreover, if \( 2kr_1 > 1 \)
and $r_1 < 1$, then it follows from Theorem 3 that there exists an arbitrarily small perturbation of $\Gamma$ such that at least two limit cycles bifurcates from it. In a more general way, if we have a polycycle $\Gamma$ with $n$ hyperbolic singularities and $m$ tangential singularities, then, under some hypothesis, there is an arbitrarily small perturbation such that at least $n + m$ limit cycles bifurcates from $\Gamma$. On the other hand, if every singularity contracts (resp. repels) the flow, then at most one limit cycle can bifurcate from $\Gamma$ and if it does, then such limit cycle is hyperbolic and stable (resp. unstable). Finally, as an application of Theorems 1, 2 and 3, we give the unfolding of both polycycles given by Figure 1, with $k = 1$ and $r_1 \in \mathbb{R}^+ \setminus \{\frac{1}{2}, 1\}$. See Theorem 4 and Figures 2, 3 and 4.
The paper is organized as follows. In Section 2 we establish the main theorems. In Section 3 we have some preliminaries about the transitions maps near a hyperbolic saddle, a semi-hyperbolic singularity with a hyperbolic sector, and a tangential singularity. Theorems 1 and 2 are proved in Section 4. In Sections 5 and 6 we study some tools to approach Theorem 3, which is proved in Section 7. Finally, Theorem 4 is proved in Section 8.

2. Main Results

Let $Z = (X_1,\ldots,X_M)$, $M \geq 2$, be a planar non-smooth vector field depending on a parameter $\mu \in \mathbb{R}^r$ with $N \geq 1$ discontinuities $\Sigma_1,\ldots,\Sigma_N$ given by,

$$h_i(x) = 0, \ldots, h_N(x) = 0, \quad x \in \mathbb{R}^2,$$

with each $X_i = X_i(x,\mu)$ of class $C^\infty$ in $(x,\mu) \in \mathbb{R}^2 \times \mathbb{R}^r$, each $h_j$ of class $C^\infty$ in $x \in \mathbb{R}^2$ with 0 as a regular value. From now on let,

$$\Sigma = \bigcup_{i=1}^N \Sigma_i,$$

denote the set of discontinuities of $Z$. Let also $A_1,\ldots,A_M$ be the open connected components of $\mathbb{R}^2 \setminus \Sigma$ with each $X_i$ defined over $\overline{A_i}$ (where $\overline{A}$ denotes the topological closure of the set $A$). Let $p \in \Sigma_1 \subset \Sigma$, with $p \notin \Sigma_j \cap \Sigma_k$ for any $j \neq k$, and let $X$ be one of the two components of $Z$ defined at $p$. The Lie derivative of $h_i$ in the direction of the vector field $X$ at $p$ is defined as,

$$Xh_i(p) = \langle X(p), \nabla h_i(p) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^2$.

**Definition 1.** We say that a point $p \in \Sigma_1 \subset \Sigma$ is a tangential singularity if $p \notin \Sigma_j \cap \Sigma_k$ for any $j \neq k$, and if,

$$X_a h_i(p)X_b h_i(p) = 0, \quad X_a(p) \neq 0, \quad X_b(p) \neq 0,$$

where $X_a$ and $X_b$ are the two components of $Z$ defined at $p$.

We also suppose that at $\mu = \mu_0$ we have a polycycle $\Gamma^n$ composed of $n$ singularities $p_1,\ldots,p_n \in \mathbb{R}^2$ such that each $p_i$ is either a hyperbolic saddle or a tangential singularity. Let $L_1,\ldots,L_n$ denote the heteroclinic connections of $\Gamma^n$ such that $\omega(L_i) = p_i$ and $\alpha(L_i) = p_{i+1}$, with $\alpha(L_n) = p_1$. Let us also assume that if $L_i$ intersects $\Sigma\setminus\{p_1,\ldots,p_n\}$, then it does at most in a finite number of points $\{x_{i,0},x_{i,1},\ldots,x_{i,n(i)}\}$ and $x_{i,j} \notin \Sigma_r \cap \Sigma_s$, for any $r \neq s$, $j \in \{0,\ldots,n(i)\}$. In this case, let $\gamma_i(t)$ be a parametrization of $L_i$ such that,

$$\gamma_i(t_{i,j}) = x_{i,j}, \quad t_{i,n(i)} < \cdots < t_{i,1} < t_{i,0} = 0.$$

Let us also assume that around each $x_{i,j}$ there is a neighborhood $N_{i,j}$ of $x_{i,j}$ such that $\Sigma \cap N_{i,j}$ is a crossing region of $\Sigma$, i.e. if $x_{i,j} \in \Sigma_r$ and $x_{b}, X_a$ are the two components of $Z$ defined at $x_{i,j}$, then,

$$X_a h_r(x_{i,j})X_b h_i(x_{i,j}) > 0.$$ 

See Figure 5. On the other hand, if $L_i$ does not intersect $\Sigma$, then take any point $x_{i,0} \in L_i$ and a parametrization $\gamma_i(t)$ of $L_i$ such that $\gamma_i(0) = x_{i,0}$. Let us also assume that $\Gamma^n$ is a connected component of the boundary of some open ring $A \subset \mathbb{R}^2$. See Figure 5. Furthermore we say that the cyclicity of $\Gamma^n$ is $k$ if at most $k$ limit cycles can bifurcate from an arbitrarily small perturbation of $\Gamma^n$.

**Definition 2.** Let $p \in \Sigma_1 \subset \Sigma$ be a tangential singularity, $X$ one of the components of $Z$ defined at $p$ and let $X^k h_i(p) = \langle X(p), \nabla X^{k-1} h_i(p) \rangle$, $k \geq 2$. We say that $X$ has $m$-order contact with $\Sigma$ at $p$, $m \geq 1$, if $m$ is the first positive integer such that $X^m h_i(p) \neq 0$.

**Definition 3.** Let $p \in \Sigma_1 \subset \Sigma$ be a tangential singularity of $\Gamma^n$, $L_a$ and $L_b$ the heteroclinic connections of $\Gamma^n$ such that $\omega(L_a) = p$ and $\alpha(L_a) = p$. Let $X_a$ and $X_b$ be the two components of $Z$ defined at $p$ and let $A_a$, $A_b$ be the respective connected components of $\mathbb{R}^2 \setminus \Sigma$ such that $X_a$ and $X_b$ is defined over $\overline{A_a}$ and $\overline{A_b}$. Given two parametrizations $\gamma_a(t)$ and $\gamma_b(t)$ of $L_a$ and $L_b$ such that $\gamma_a(0) = \gamma_b(0) = p$, let $A_s, A_u \in \{A_a,A_b\}$ be such that $A_a \cap \gamma_s([-\infty,0]) \neq \emptyset$ and $A_u \cap \gamma_u([0,\infty]) \neq \emptyset$, for any $\varepsilon > 0$ small. Let also $X_s, X_u \in \{X_a,X_b\}$ denote the components of $Z$ defined at $A_s$ and $A_u$. Observe that we may have $A_s = A_u$ and thus $X_s = X_u$. See Figure 6. We define the stable and unstable contact order of $p$
as the contact order \( n_s \) and \( n_u \) of \( X_s \) and \( X_u \) with \( \Sigma \) at \( p \), respectively. Furthermore we also say that \( X_s \) and \( X_u \) are the stable and unstable components of \( Z \) defined at \( p \).

**Figure 5.** An example of \( \Gamma^3 \) with the open ring \( A \) contained in the bounded region.

\[ \Sigma \]

\( \sum \)

**Figure 6.** Examples of a tangential singularity \( p \) such that (a) \( A_s \neq A_u \) and (b) \( A_s = A_u \).

About polycycles in smooth vector fields, Cherkas [5] proved that if \( \Gamma \) is a polycycle composed of \( n \) hyperbolic saddles \( p_1, \ldots, p_n \), with hyperbolicity ratios \( r_1, \ldots, r_n \), then \( \Gamma \) is stable if,

\[ r := r_1 \ldots r_n > 1, \]

and unstable if \( r < 1 \). Therefore, our first main theorem is an extension, to the realm of non-smooth vector fields, of such classic result.

**Theorem 1.** Let \( Z \) be and \( \Gamma^n \) be as in Section 2. Let \( n_{i,s} \) and \( n_{i,u} \) be the stable and unstable contact orders of the tangential singularities \( p_i \) of \( \Gamma^n \), \( \nu_i < 0 < \lambda_i \) be the eigenvalues of the hyperbolic saddles \( p_i \) of \( \Gamma^n \) and in either case define \( r_i = \frac{n_{i,u}}{n_{i,s}} \) or \( r_i = \frac{\nu_i}{\lambda_i} \), respectively. Let also,

\[ r(\Gamma^n) = \prod_{i=1}^{n} r_i. \]

If \( r(\Gamma^n) > 1 \) (resp. \( r(\Gamma^n) < 1 \)), then there is a neighborhood \( N_0 \) of \( \Gamma^n \) such that the orbit of \( Z \) through any point \( p \in N_0 \cap A \) has \( \Gamma^n \) as \( \omega \)-limit (resp. \( \alpha \)-limit).

In Theorem 1, it was possible to use \( r_i \) to quantify the attractive or repulsive force of the singularities. In what follows we will relax the hypothesis and let \( \Gamma^n \) have some non-hyperbolic singularities (e.g. a saddle-node or a semi-hyperbolic saddle). In this case, we will see that such singularities overwhelm all the previous one and thus will dictate the stability of the polycycle.

More precisely, let \( \Gamma^{n,l} \) denote a polycycle composed of \( n \) singularities \( p_1, \ldots, p_n \in \mathbb{R}^2 \) such that each \( p_i \) is either a hyperbolic saddle or a tangential singularity, and by \( l \) semi-hyperbolic singularities
Let $q_1, \ldots, q_l \in \mathbb{R}^2 \setminus \Sigma$ such that each $q_i$ has at least one hyperbolic sector. Let us also suppose that the heteroclinic connections $L_{s,i}$ and $L_{u,i}$ of $\Gamma^{n,l}$ such that $\omega(L_{s,i}) = q_i$, $\alpha(L_{u,i}) = q_i$ are the boundaries of a hyperbolic sector of $q_i$, contained in the open ring $A$, in such a way that a first-return map can be defined. With little adaptations in the proof of Theorem 1, one can prove the following result.

**Corollary 1.** Let $Z$ and $\Gamma^{n,l}$, $l \geq 1$, be as above. Let $\lambda_i \in \mathbb{R} \setminus \{0\}$ denote the unique non-zero eigenvalue of the semi-hyperbolic singularity $q_i$. If every $\lambda_i < 0$ (resp. every $\lambda_i > 0$), then there is a neighborhood $N_0$ of $\Gamma^{n,l}$ such that the orbit of $Z$ through any point $p \in N_0 \cap A$ has $\Gamma^{n,l}$ as $\omega$-limit (resp. $\alpha$-limit).

**Definition 4.** Given a hyperbolic saddle $p_i$ of a polycycle $\Gamma^{n,l}$, let $r_i \in \mathbb{R}$ be as in Theorem 1. We say that $p_i$ is a attracting (resp. repelling) singular point if $r_i > 1$ (resp. $r_i < 1$). Given a tangential singularity $p_i$ of $\Gamma^{n,l}$ we say that $p_i$ is attracting (resp. repelling) singular point if $n_s = 1$ (resp. $n_s = 1$). Furthermore if $q_i$ is a semi-hyperbolic singularity of $\Gamma^{n,l}$ with its unique non-zero eigenvalue denoted by $\lambda_i$, then $q_i$ is a attracting (resp. repelling) singular point if $\lambda_i < 0$ (resp. $\lambda_i > 0$).

In Definition 4, unlike the case of the hyperbolic saddle, if $p_i$ is a tangential singularity of $\Gamma^{n,l}$, the hypothesis of $r_i > 1$ is not enough to say that $p_i$ is stable. The reason for this, is that even if $r_i > 1$, given a small enough perturbation $p_i^*$ we cannot ensure that $r_i^* > 1$. See Figure 7. On the other hand, if we had $n_s = 1$, then any small enough perturbation would be, at most, a crossing singularity and therefore play no role on the stability of the polycycle. As we shall see at Section 5, this lack of stability of the tangential singularities will be of great role.

In the realm of smooth vector fields, Dumortier et al [8] proved that if $\Gamma$ is a polycycle of a smooth vector field composed only by stable (resp. unstable) singular points, then $\Gamma$ has cyclicity one. Furthermore if any small perturbation of $\Gamma$ has a limit cycle, then it is hyperbolic and stable (resp. unstable). Therefore, our second main theorem is an extension of such result to the realm of non-smooth vector fields.

**Theorem 2.** Let $Z$ and $\Gamma^{n,l}$, $n + l \geq 1$, be as in Corollary 1. Suppose that each singularity $p_i$ and $q_j$ is a stable (resp. unstable) singular point of $\Gamma^{n,l}$. If a small perturbation of $\Gamma^{n,l}$ has a limit cycle, then it is unique, hyperbolic and stable (resp. unstable). Furthermore the cyclicity of $\Gamma^{n,l}$ is one.

Again in the realm of smooth vector fields, let $\Gamma$ be a polycycle composed of $n$ hyperbolic saddles $p_1, \ldots, p_n$, with hyperbolicity ratios $r_1, \ldots, r_n$. Let also, $R_i = r_1 \cdots r_i$. Han et al [13] proved that if $(R_i - 1)(R_{i+1} - 1) < 0$, $i \in \{1, \ldots, n - 1\}$, then there exists an arbitrarily small perturbation of $\Gamma$ with at least $n$ limit cycles. Therefore, our third main theorem is an extension of such result to the realm of non-smooth vector fields.

**Theorem 3.** Let $Z_0$ and $\Gamma^n$ be as in Theorem 1 and define $R_i = r_1 \cdots r_i$, $i \in \{1, \ldots, n\}$. Suppose $R_n \neq 1$ and, if $n \geq 2$, suppose $(R_i - 1)(R_{i+1} - 1) < 0$ for $i \in \{1, \ldots, n - 1\}$. Then, there exists a $C^\infty$ map $g(x, \mu) = O(||\mu||)$, $\mu \in \mathbb{R}^n$, such that the vector field $Z = Z_0 + g$ has at least $n$ limit cycles in a neighborhood of $\Gamma^n$ for some $\mu$ arbitrarily near the origin. Therefore, the cyclicity of $\Gamma^n$ is at least $n$.

Finally, as an application of Theorems 1-3, we obtain the bifurcations diagrams of the polycycles given by Figure 1 for $r_1 \notin \{1, 1/2\}$.

**Theorem 4.** Let $\Gamma$ be one of the polycycles given by Figure 1. Then, the bifurcations diagrams of $\Gamma$ for $r_1 > 1$, $\frac{1}{2} < r_1 < 1$ and $r_1 < \frac{1}{2}$ are those given by Figures 2, 3 and 4.
3. Preliminaries

3.1. Transition map near a hyperbolic saddle. Let $X_\mu$ be a $C^\infty$ planar vector field depending in a $C^\infty$-way on a parameter $\mu \in \mathbb{R}^r$, $r \geq 1$, defined in a neighborhood of a hyperbolic saddle $p_0$ at $\mu = \mu_0$. Let $\Lambda \subset \mathbb{R}^r$ be a small enough neighborhood of $\mu_0$, $\nu(\mu) < 0 < \lambda(\mu)$ be the eigenvalues of the hyperbolic saddle $p(\mu)$, $\mu \in \Lambda$, and $r(\mu) = \frac{\nu(\mu)}{\lambda(\mu)}$ be the hyperbolicity ratio of $p(\mu)$. Let $B$ be a small enough neighborhood of $p_0$ and $\Phi : B \times \Lambda \to \mathbb{R}^2$ be a $C^\infty$-change of coordinates such that $\Phi$ sends the hyperbolic saddle $p(\mu)$ to the origin and its unstable and stable manifolds $W^u(\mu)$ and $W^s(\mu)$ to the axis $Ox$ and $Oy$, respectively. We can suppose that $\sigma$ and $\tau$ are parametrized by $x \in [0, x_0]$ and $y \in [0, y_0]$, with $x = 0$ and $y = 0$ corresponding to $Oy \cap \sigma$ and $Ox \cap \tau$, respectively. Clearly the flow of $X_\mu$, in this new coordinate system, defines a transition map:

$$D : (0, x_0] \times \Lambda \to (0, y_0],$$

called the Dulac map, since it was study first by Dulac in [7]. See Figure 8. Observe that $D$ is of class $C^\infty$ for $x \neq 0$ and it can be continuously extend by $D(0, \mu) = 0$ for all $\mu \in \Lambda$.

![Figure 8. The Dulac map near a hyperbolic saddle.](image)

**Definition 5.** Let $I_k$, $k \geq 0$, denote the set of functions $f : [0, x_0] \times \Lambda_k \to \mathbb{R}$ with the following properties:

(a) $f$ is $C^\infty$ on $(0, x_0] \times \Lambda_k$;

(b) For each $j \in \{0, \ldots, k\}$ we have that $\varphi_j = x^j \frac{\partial f}{\partial x}(x, \mu)$ is continuous on $(0, x_0] \times \Lambda_k$ with $\varphi_j(x, \mu) \to 0$ for $x \to 0$, uniformly in $\mu$.

A function $f : [0, x_0] \times \Lambda \to \mathbb{R}$ is said to be of class $I$ if $f$ is $C^\infty$ on $(0, x_0] \times \Lambda$ and for every $k \geq 0$ there exists a neighborhood $\Lambda_k \subset \Lambda$ of $\mu_0$ such that $f$ is of class $I^k$ on $(0, x_0] \times \Lambda_k$.

**Theorem 5** (Mourtada). Let $X_\mu$, $\sigma$, $\tau$, and $D$ be as above. Then

$$D(x, \mu) = x^{r(\mu)} (A(\mu) + \phi(x, \mu)),$$

with $\phi \in I$ and $A$ a positive $C^\infty$ function.

**Proof.** See [15]. For a sketch of the proof in English, see [8].

Following Dumortier et al [8], we call Mourtada’s form the expression (2) of the Dulac map and denote by $\mathcal{D}$ the class of maps given by (2).

**Proposition 1.** Given $D(x, \mu) = x^{r(\mu)} (A(\mu) + \phi(x, \mu)) \in \mathcal{D}$, the following statements hold.

(a) $D^{-1}$ is well defined and $D^{-1}(x, \mu) = x^{1/r(\mu)} (B(\mu) + \psi(x, \mu)) \in \mathcal{D}$;

(b) $\frac{\partial D}{\partial x}$ is well defined and

$$\frac{\partial D}{\partial x}(x, \mu) = r(\mu)x^{r(\mu)-1}(A(\mu) + \xi(x, \mu)),$$

with $\xi \in I$. 


Proof. See [15]. For a sketch of the proof in English, see [8]. □

3.2. Transition map near a semi-hyperbolic singularity. Let $X_\mu$ be a $C^\infty$ planar vector field depending in a $C^\infty$-way on a parameter $\mu \in \mathbb{R}^r$, $r \geq 1$, defined, at $\mu = \mu_0$, in a neighborhood of a singularity $p_0$ such that $p_0$ has an unique non-zero eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ and one of its sectors is hyperbolic, e.g. a saddle-node or a degenerated saddle. Reversing the time if necessary, we can assume $\lambda < 0$.

**Theorem 6.** Let $X_\mu$ and $p_0 \in \mathbb{R}^2$ be as above. Then, for each $k \geq 1$, there exists a $C^k$-family of diffeomorphisms of $\mathbb{R}^2$ such that at this new coordinate system, $X_\mu$ is given by,

$$\dot{x} = g(x, \mu), \quad \dot{y} = -y,$$

except by the multiplication of a $C^k$-positive function. Furthermore

$$g(0, \mu_0) = \frac{\partial g}{\partial x}(0, \mu_0) = 0,$$

and $g(x, \mu_0) > 0$ for $0 < x < \varepsilon$ small enough.

Proof. See [8] and the references therein. □

In this new coordinate system given by Theorem 6, let $\sigma(\mu)$ and $\tau$ be two small cross sections of the axis $Oy$ and $Ox$ (which are, respectively, the stable and the central manifolds of the origin at $\mu = \mu_0$). Let us parametrize $\sigma(\mu)$ and $\tau$ by $x \in [x^*(\mu), x_0]$ and $y \in [0, y_0]$, where $x^*(\mu)$ is the largest zero of $g(x, \mu) = 0$ (observe that $x^*(\mu_0) = 0$). As in Section 3.1, in this new coordinate system the flow of $X_\mu$ defines a transition map,

$$F: (x^*(\mu), x_0) \times \Lambda \to (0, y_0).$$

**Theorem 7.** Let $X_\mu$ and $F$ be as above. Then,

$$F(x, \mu) = C e^{-T(x, \mu)},$$

where $C > 0$ and $T: [x^*(\mu), x_0] \times \Lambda \to \mathbb{R}^+$ is the time function from $\sigma(\mu)$ to $\tau$.

Proof. See [8]. □

3.3. Transition map near a tangential singularity. Let $p_0$ be a tangential singularity of $\Gamma^\circ$ and $X_s, X_u$ be the stable and unstable components of $Z$ defined at $p_0$ with $\mu = \mu_0$. Let $B$ be a small enough neighborhood of $p_0$ and $\Phi: B \times \Lambda \to \mathbb{R}^2$ be a $C^\infty$ change of coordinates such that $\Phi(p_0, \mu_0) = (0, 0)$ and $\Phi(B \cap \Sigma) = Ox$. Let $l_s = \Phi(B \cap L_s)$, $l_u = \Phi(B \cap L_u)$ and $\tau_s, \tau_u$ two small enough cross sections of $l_s$ and $l_u$, respectively. Let also,

$$\sigma = [0, \varepsilon] \times \{0\}, \quad \sigma = (-\varepsilon, 0] \times \{0\}, \text{ or } \sigma = \{0\} \times [0, \varepsilon),$$

depending on $\Gamma^\circ$. It follows from [1] that $\Phi$ can be choose such that the transition maps $T^{s, u}: \sigma \times \Lambda \to \tau_s, \tau_u$, given by the flow of $X_{s,u}$ in this new coordinate system, are well defined and given by,

$$T^s(h_\mu(x), \mu) = k_u(\mu)x^{n_u} + O(x^{n_u+1}) + \sum_{i=0}^{n_u-2} \lambda_i^u(\mu)x_i,$$

$$T^s(h_\mu(x), \mu) = k_s(\mu)x^{n_s} + O(x^{n_s+1}) + \sum_{i=0}^{n_s-2} \lambda_i^s(\mu)x_i,$$

with $\lambda_i^{s,u}(\mu_0) = 0$, $k_{s,u}(\mu_0) \neq 0$, $h_\mu: \mathbb{R} \to \mathbb{R}$ a diffeomorphism and with $h_\mu$ and $\lambda_i^{s,u}$ depending continuously on $\mu$. For examples of such maps, see Figure 9.
Let $X$ be a smooth vector field with a hyperbolic saddle at the origin, with eigenvalues $\nu < 0 < \lambda$, and let $D: (0, x_0) \rightarrow (0, y_0)$ be its Dulac map, as in Section 3.1. In the first part of his paper, Cherkas [5] proved that given $\varepsilon > 0$ we can choose $x_0 > 0$ small enough such that,

$$x_0^{r(1-\varepsilon)} < D(x) < x_0^{r(1+\varepsilon)},$$

where $r = -\frac{\nu}{\lambda}$ is the hyperbolicity ratio of the origin. However, due to Mourtada’s form (2), we now have a more general way to describe the Dulac map, even if its depends on a parameter. Hence, once equipped with Mourtada’s form, Cherkas proof can be not only simplified, but extended to smooth vector fields depending on a parameter. Therefore, with the advent of the formulas given at Andrade et al [1], it was natural to use such formulas to extend Cherkas result to non-smooth vector fields.

**Proof of Theorem 1.** For simplicity let us assume that $\Sigma = h^{-1}(0)$ with 0 a regular value of $h$, and $Z = (X_1, X_2)$. Let also $T = \Gamma^3$ be composed of two tangential singularities $p_1$, $p_2$ and by a hyperbolic saddle $p_3$. See Figure 10. Let $B_i$ be a small enough neighborhood of $p_i$ and let $\Phi_i: B_i \times \{\mu_0\} \rightarrow \mathbb{R}^2$ be chosen as in Section 3.3, $i \in \{1, 2\}$. Let also $B_3$ be a neighborhood of $p_3$ and $\Phi_3: B_3 \times \{\mu_0\} \rightarrow \mathbb{R}^2$ be chosen as in Section 3.1. Knowing that $T_{i}^{\nu,u}: \sigma_i \times \{\mu_0\} \rightarrow \tau_{i}^{\nu,u}$ and $D: \sigma \times \{\mu_0\} \rightarrow \tau$, let,

$$J_1 = \Phi_1^{-1}(\sigma), \quad J_2 = \Phi_2^{-1}(\sigma), \quad J_3 = \Phi_3^{-1}(\sigma),$$

with $i \in \{1, 2\}$. Let also,

$$\omega_1: \tau_1^{\nu,u} \rightarrow \tau_2^{\nu,u}, \quad \omega_2: \tau_2^{\nu,u} \rightarrow \tau_3^{\nu,u}.$$ 

be defined by the flow of $X_1$ and $X_2$. See Figure 10. Finally let,

$$\rho_1 = \Phi_2 \circ \omega_1 \circ \Phi_1^{-1}, \quad \rho_2 = \Phi_3 \circ \omega_2 \circ \Phi_2^{-1}, \quad \rho_3 = \Phi_1 \circ \omega_3 \circ \Phi_3^{-1},$$

and,

$$T_i^{\nu} = \Phi_i^{-1} \circ T_i^{\nu,u} \circ \Phi_i, \quad T_i^{\nu,u} = \Phi_i^{-1} \circ T_i^{\nu} \circ \Phi_i, \quad T_i = \Phi_i^{-1} \circ D \circ \Phi_i,$$

with $i \in \{1, 2\}$. See Figure 10. Let $\nu < 0 < \lambda$ be the eigenvalues of $p_3$ and denote $r = \frac{|\nu|}{\lambda}$. Let also $n_{i,s}$ and $n_{i,u}$ denote the stable and unstable order of $q_i$, $i \in \{1, 2\}$. Therefore, it follows from Sections 3.1 and 3.3 that,

$$T_i^{\nu}(x) = k_{i,s}x^{n_{i,s}} + O(x^{n_{i,u}+1}), \quad T_i^{\nu,(x)} = k_{i,u}x^{n_{i,s}} + O(x^{n_{i,u}+1}),$$

$$D(x) = ax^r + O(x^{r+1}), \quad \rho_j(x) = a_jx + O(x^2),$$

with $k_{i,s}, k_{i,u}, a_j, a \neq 0, i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Therefore, if we define,

$$\pi = \rho_2 \circ T_2^{\nu} \circ (T_2^{\nu})^{-1} \circ \rho_1 \circ T_1^{\nu} \circ (T_1^{\nu})^{-1} \circ \rho_3 \circ D,$$

then one can conclude that,

$$\pi(x) = Kx^{n_0} + O(x^{n_0+1}),$$

with $K \neq 0$ and $n_0 = r(\Gamma^n)$, as in (1). Hence, if $x$ is small enough we conclude that $\pi(x) < x$ if $n_0 > 1$ and $\pi(x) > x$ if $n_0 < 1$. The result now follows from the fact that the Poincaré map,

$$P = \omega_2 \circ T_2^{\nu} \circ (T_2^{\nu})^{-1} \circ \omega_1 \circ T_1^{\nu} \circ (T_1^{\nu})^{-1} \circ \omega_3 \circ \omega_2 \circ \omega_1,$$
satisfies \( P = \Phi^{-1}_3 \circ \pi \circ \Phi_3 \).

It is clear from the demonstration of Theorem 1 that if there are no perturbations, then in the case of a tangential singularity the ratio \( r = \frac{n_u}{n_s} \) has the same role as the hyperbolicity ratio in the case of a hyperbolic saddle, i.e. both the singularities contracts the flow if \( r > 1 \), and repels it if \( r < 1 \). Unfortunately, unlike a hyperbolic saddle, a tangential singularity is not stable and thus the inequality \( r > 1 \) (\( r < 1 \)) may change under any perturbation, unless \( n_s = 1 \) (\( n_u = 1 \)). See Figure 7. As we shall see in the proof of Theorem 3, in order to avoid this lack of stability, we will make use of bump-functions to control the bifurcations.

**Remark 1.** With clear adaptations in the proof of Theorem 1, we observe that one may use Section 3.2 to prove Corollary 1.

With the advent of Mourtada’s form (5), Dumortier et al [8] defined the notion of stable (resp. unstable) singular points and thus proved that if a polycycle in a smooth vector field has stable (resp. unstable) singular points only, then its cyclicity is one. Therefore, as in the proof of Theorem 1, it is natural to use the new formulas of [1] to extend the definition of stable and unstable singular points (see Definition 4 and Figure 7) and thus extend the result itself.

**Proof of Theorem 2.** Let us suppose that every singular point of \( \Gamma^{n,l} \) is attracting. Following [8] and the proof of Theorem 1, we observe that the Poincaré map, when well defined, can be written as the composition

\[
P'_\mu(x_1) = G'_k(y_k)F'_k(x_k) \cdots G'_1(y_1)F'_1(x_1),
\]

where each \( F_i \) is the transition map near a hyperbolic saddle (given by (2)), a semi-hyperbolic singularity (given by (3)), or a tangential singularity (given by (4)), and each \( G_i \) is the composition of a finite number of regular transitions given by the flow of \( Z \), i.e. a \( C^\infty \) diffeomorphism in \( x \). We call \( y_1 = F_1(x_1) \), \( x_2 = G_1(y_1) \), \( \ldots \), \( y_k = F_k(x_k) \), \( x_{k+1} = G_k(y_k) \). Thus,

\[
\left| P'_\mu(x_1) \right| < 1 \quad \text{for all } \mu \in \Lambda.
\]

Therefore, it follows that for all \( \varepsilon > 0 \) there exists a neighborhood \( \Lambda \) of \( \mu_0 \) and neighborhoods \( W_i \) of \( x_i = 0 \), \( i \in \{1, \ldots, k+1\} \), such that if \( x_1 \in W_1 \), then \( x_i \in W_i \) and \( |F'_i(x_i)| < \varepsilon \), for all \( i \in \{1, \ldots, k+1\} \) and for all \( \mu \in \Lambda \). Also, if \( \Lambda \) and each \( W_i \) are small enough, then each \( G'_i(y_i) \) is bounded, and bounded away from zero. Hence, for \( \varepsilon > 0 \) small enough it follows that \( P'_\mu(x_1) < 1 \) for \( (x_1, \mu) \in W_1 \times \Lambda \) and thus an unique limit cycle exists and it is hyperbolic and attracting. The repelling case follows by time reversing.

Unlike Theorems 1 and 2, to obtain Theorem 3 it will be necessary to work on some technicalities to understand how such a polycycle in a non-smooth vector field may bifurcate. Sections 5 and 6 will deal with that. It is in such technicalities that the bifurcations presented at Figure 7 will play a great role and thus present the differences between the smooth and non-smooth case.
5. The displacement map

Let $Z$ and $\Gamma^n$ be as in Section 2. For simplicity, at $\mu = \mu_0$, let $p_1 \in A_1$ and $p_2 \in A_2$ be two hyperbolic saddles of $\Gamma^n$ with the heteroclinic connection $L_0$ such that $\omega(L_0) = p_1$, $\alpha(L_0) = p_2$ and $L_0 \cap \Sigma = \{x_0\}$, $\Sigma = h^{-1}(0)$. Let $\gamma_0(t)$ be a parametrization of $L_0$ such that $\gamma_0(0) = x_0$ and $u_0$ be an unitarian vector orthogonal to $T_{x_0} \Sigma$ such that $\text{sign}(\langle u_0, \nabla h(x_0) \rangle) = \text{sign}(X_1 h(x_0)) = \text{sign}(X_2 h(x_0))$. See Figure 11. Also, define $\omega_0 \in \{-1, 1\}$ such that $\omega_0 = 1$ if the orientation of $\Gamma^n$ is counterclockwise and $\omega_0 = -1$ if the orientation of $\Gamma^n$ is clockwise. We denote by $DX(p, \mu^*)$ the Jacobian matrix of $X|_{\mu=\mu^*}$ at $p$, i.e. if $X = (P, Q)$, then

$$DX(p, \mu^*) = \begin{pmatrix}
\frac{\partial P}{\partial x_1}(p, \mu^*) & \frac{\partial P}{\partial x_2}(p, \mu^*) \\
\frac{\partial Q}{\partial x_1}(p, \mu^*) & \frac{\partial Q}{\partial x_2}(p, \mu^*)
\end{pmatrix}.$$  

If $\Lambda$ is a small enough neighborhood of $\mu_0$, then, following [19], we know from the Implicit Function Theorem that if $\mu \in \Lambda$, then $p_i(\mu) \in A_i$ is a hyperbolic saddle of $X_i$ and $p_i(\mu) \to p_i$ as $\mu \to \mu_0$, with $p_i(\mu)$ of class $C^\infty$, $i \in \{1, 2\}$. Let $(y_{i,1}, y_{i,2}) = (y_{i,1}(\mu), y_{i,2}(\mu))$ be a coordinate system with its origin at $p_i(\mu)$ and such that the $y_{i,1}$-axis and the $y_{i,2}$-axis are the one-dimensional stable and unstable spaces $E_s^i(\mu)$ and $E_u^i(\mu)$ of the linearization of $X_i(\cdot, \mu)$ at $p_i(\mu)$, $i \in \{1, 2\}$. It follows from the Stable Manifold Theorem (see [20]) that the stable and unstable manifolds $S_i^\mu$ and $U_i^\mu$ of $X_i(\cdot, \mu)$ at $p_i(\mu)$ are given by,

$$S_i^\mu : y_{i,1} = \Psi_{i,1}(y_{i,1}, \mu) \quad \text{and} \quad U_i^\mu : y_{i,1} = \Psi_{i,1}(y_{i,2}, \mu),$$  

where $\Psi_{i,1}$ and $\Psi_{i,2}$ are $C^\infty$ functions, $i \in \{1, 2\}$. Restricting $\Lambda$ if necessary, it follows that there exist $\delta > 0$ such that,

$$y_{i}^\mu(\mu) = (\delta, \Psi_{i,2}(\delta, \mu)) \in S_i^\mu \quad \text{and} \quad U_i^\mu,$$

$i \in \{1, 2\}$. If $C_i(\mu)$ is the diagonalization of $DX_i(p_i(\mu), \mu)$, then at the original coordinate system $(x_1, x_2)$ we obtain,  

$$x_{\mu}^i(\mu) = p_i(\mu) + C_i(\mu)y_{i}^\mu(\mu) \in S_i^\mu \quad \text{and} \quad x_{\mu}^\mu(\mu) = p_i(\mu) + C_i(\mu)u_{i}^\mu(\mu) \in U_i^\mu,$$

$i \in \{1, 2\}$. Furthermore $x_{\mu}^i(\mu)$ and $x_{\mu}^\mu(\mu)$ are also $C^\infty$ at $\Lambda$. Let $\phi_i(t, \xi, \mu)$ be the flow of $X_i(\cdot, \mu)$ such that $\phi_i(0, \xi, \mu) = \xi$ and $L_0^i = L_0^i(\mu)$, $L_u^i = L_u^i(\mu)$ be the perturbations of $L_0$ such that $\omega(L_0^i(\mu)) = p_i(\mu)$ and $\alpha(L_0^i(\mu)) = p_2(\mu)$. Then it follows that,

$$x_{\mu}^i(t, \mu) = \phi_i(t, x_{\mu}^i(\mu), \mu) \quad \text{and} \quad x_{\mu}^\mu(t, \mu) = \phi_2(t, x_{\mu}^\mu(\mu), \mu)$$

are parametrizations of $L_0^i(\mu)$ and $L_u^i(\mu)$, respectively. Since $L_0$ intersects $\Sigma$, it follows that there are $t_0^s < 0$ and $t_0^u > 0$ such that $x_{\mu}^i(t_0^s, \mu_0) = x_0 = x_{\mu}^u(t_0^u, \mu_0)$ and thus by the uniqueness of solutions we have,

$$x_{\mu}^i(t + t_0^s, \mu_0) = \gamma_0(t) = x_{\mu}^u(t + t_0^u, \mu_0),$$

for $t \in [0, +\infty)$ and $t \in (-\infty, 0]$, respectively.
Definition 6. It follows from Lemma 1 that the displacement function,

where \( \tau \) the existence of \( \tau \big|_{\mathcal{L}} \) is well defined for all \( \mu \in \Lambda \). See Figure 12.

![Figure 12. Illustration of \( x^s_0(\mu) \) and \( x^u_0(\mu) \).](image)

Proof. Let \( X_1 \) denote a \( C^\infty \) extension of \( X \) to a neighborhood of \( \overline{A}_1 \) and observe that now \( x^s(t, \mu_0) \) is well defined for \( |t - t_0^s| \) small enough. Knowing that \( \Sigma = h^{-1}(0) \), define \( S(t, \mu) = h(x(t, \mu)) \) and observe that \( S(t_0^s, \mu_0) = h(x_0) = 0 \) and,

\[
\frac{\partial S}{\partial t}(t_0^s, \mu_0) = \langle \nabla h(x_0), X_1(x_0) \rangle \neq 0.
\]

It then follows from the Implicit Function Theorem that there exist a \( C^\infty \) function \( \tau^s(\mu) \) such that \( \tau^s(\mu_0) = t_0^s \) and \( S(\tau^s(\mu), \mu) = 0 \) and thus \( x^s_0(\mu) = x^s(\tau^s(\mu), \mu) \in \Sigma \). In the same way one can prove the existence of \( \tau^u \).

\( \square \)

Definition 6. It follows from Lemma 1 that the displacement function,

\[
d(\mu) = \omega_0[x^s_0(\mu) - x^u_0(\mu)] \wedge u_0,
\]

where \( (x_1, x_2) \wedge (y_1, y_2) = x_1y_2 - y_1x_2 \), is well defined near \( \mu_0 \) and it is of class \( C^\infty \). See Figure 13.

![Figure 13. Illustration of \( d(\mu) > 0 \) and \( d(\mu) < 0 \).](image)

Remark 2. We observe that \( L_0 \) can intersect \( \Sigma \) multiple times. In this case, following Section 2 we would have \( L_0 \cap \Sigma = \{x_0, x_1, \ldots, x_n\} \) and \( \gamma_0(t) \) a parametrization of \( L_0 \) such that \( \gamma_0(t_i) = x_i \), with \( t_n < \cdots < t_1 < t_0 = 0 \). Therefore, applying Lemma 1 one shall obtain \( x^u_n(\mu) \) and then applying the Implicit Function Theorem multiple times one shall obtain \( x^u_i(\mu) \) as a function of \( x^u_{i+1}(\mu) \), \( i \in \{0, \ldots, n-1\} \), and thus the displacement function would still be defined at \( x_0 \).

Let us define,

\[
x^s_\mu(t) = \phi_1(t, x^s_0(\mu), \mu) \text{ for } t \geq 0 \quad \text{and} \quad x^u_\mu(t) = \phi_2(t, x^u_0(\mu), \mu) \text{ for } t \leq 0,
\]
new parametrizations of $L_0^s(\mu)$ and $L_0^u(\mu)$, respectively. In the following Lemma we will denote by $X_i$ some $C^\infty$ extension of $X_i$ at some neighborhood of $\overline{A}_i$, $i \in \{1, 2\}$, and thus $x_\mu^s(t)$ and $x_\mu^u(t)$ are well defined for $|t|$ small enough.

**Lemma 2.** For any $\mu^* \in \Lambda$ and any $i \in \{1, \ldots, n\}$ the maps,
\[
\frac{\partial x_\mu^s}{\partial \mu_i}(t) \text{ and } \frac{\partial x_\mu^u}{\partial \mu_i}(t),
\]
are bounded as $t \to +\infty$ and $t \to -\infty$, respectively.

**Proof.** Let us consider a small perturbation of the parameter in the form,
\[
\mu = \mu^* + \varepsilon e_i,
\]
where $e_i$ is the $i$th vector of the canonical base of $\mathbb{R}^r$. The corresponding perturbation of the singularity $p_2(\mu^*)$ takes the form,
\[
p_2(\mu) = p_2(\mu^*) + \varepsilon y_0 + o(\varepsilon).
\]
Knowing that
\[
X_2(p_2(\mu), \mu) = 0 \text{ for any } \varepsilon \text{ it follows that,}
\]
\[
0 = \frac{\partial X_2}{\partial \varepsilon}(p_2(\mu), \mu) = DX_2(p_2(\mu), \mu)[y_0 + o(\varepsilon)] + \frac{\partial X_2}{\partial \mu}(p_2(\mu), \mu)e_i,
\]
and thus applying $\varepsilon \to 0$ we obtain,
\[
y_0 = -F_0^{-1}G_0 e_i,
\]
where $F_0 = DX_2(p_2(\mu^*), \mu^*)$ and $G_0 = \frac{\partial X_2}{\partial \mu}(p_2(\mu^*), \mu^*)$. Hence,
\[
\frac{\partial p_2}{\partial \mu}(\mu^*) = -F_0^{-1}G_0 e_i.
\]
Therefore, it follows from the $C^\infty$-differentiability of the flow near $p_2(\mu)$ that,
\[
\lim_{t \to -\infty} \frac{\partial x_\mu^u}{\partial \mu_i}(t) = \frac{\partial p_2}{\partial \mu_i}(\mu^*) = -F_0^{-1}G_0 e_i
\]
and thus we have the proof for $x_\mu^u$. The proof for $x_\mu^s$, is similar. □

Let $\theta_i \in (-\pi, \pi)$ be the angle between $X_i(x_0)$ and $u_0$, $i \in \{1, 2\}$. See Figure 14. For $i \in \{1, 2\}$ we denote by $M_i$ the rotation matrix of angle $\theta_i$, i.e.
\[
M_i = \begin{pmatrix}
\cos \theta_i & -\sin \theta_i \\
\sin \theta_i & \cos \theta_i
\end{pmatrix}.
\]
Following Perko [19], we define,
\[
n^u(t, \mu) = \omega_0[x_\mu^u(t) - x_0] \wedge u_0 \text{ and } n^s(t, \mu) = \omega_0[x_\mu^s(t) - x_0] \wedge u_0.
\]
It then follows from Definition 6 that,
\[
d(\mu) = n^u(0, \mu) - n^s(0, \mu),
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig14.png}
\caption{Illustration of $\theta_1 > 0$ and $\theta_2 < 0$.}
\end{figure}
and thus,

\[
\frac{\partial d}{\partial \mu_j}(\mu_0) = \frac{\partial n^u}{\partial \mu_j}(0, \mu_0) \frac{\partial n^s}{\partial \mu_j}(0, \mu_0).
\]

Therefore, to understand the displacement function \(d(\mu)\), it is enough to understand \(n^u\) and \(n^s\). Let \(X_i = (P_i, Q_i), \; i \in \{1, 2\}\). Knowing that \(\gamma_0\) is a parametrization of \(L_0\) such that \(\gamma_0(0) = x_0\), let \(L_0^+ = \{\gamma_0(t) : t > 0\} \subset A_1\) and,

\[
I_j^+ = \int_{L_0^+} e^{D_i(t)} \left[(M_1 X_1) \wedge \frac{\partial X_1}{\partial \mu_j}(\gamma_0(t), \mu_0) - \sin \theta_1 R_{1,j}(\gamma_0(t), \mu_0)\right] dt,
\]

where,

\[
D_i(t) = -\int_0^t \text{Div} X_1(\gamma_0(s), \mu_0) ds,
\]

and,

\[
R_{i,j} = \frac{\partial P_i}{\partial \mu_j} \left[\left(\frac{\partial Q_1}{\partial x_1} + \frac{\partial P_1}{\partial x_2}\right) Q_i + \left(\frac{\partial P_1}{\partial x_1} - \frac{\partial Q_1}{\partial x_2}\right) P_i \right] + \frac{\partial Q_i}{\partial \mu_j} \left[\left(\frac{\partial P_1}{\partial x_1} + \frac{\partial Q_1}{\partial x_2}\right) P_i + \left(\frac{\partial Q_1}{\partial x_2} - \frac{\partial P_1}{\partial x_1}\right) Q_i \right],
\]

\(i \in \{1, 2\}\); \(j \in \{1, \ldots, r\}\).

**Proposition 2.** For any \(j \in \{1, \ldots, r\}\) it follows that,

\[
\frac{\partial n^s}{\partial \mu_j}(0, \mu_0) = \frac{\omega_0}{||X_1(x_0, \mu_0)||} I_j^+.
\]

**Proof.** From now on in this proof we will denote \(X_1\) some \(C^\infty\) extension of \(X_1\) at some neighborhood of \(A_1\) and thus \(x^s_\mu(t)\) is well define for \(|t|\) small enough. Let \(j \in \{1, \ldots, r\}\). Defining,

\[
\xi(t, \mu) = \frac{\partial x^s_\mu}{\partial \mu_j}(t),
\]

it then follows that,

\[
\dot{\xi}(t, \mu) = \frac{\partial x^s_\mu}{\partial \mu_j}(t) = \frac{\partial}{\partial \mu_j} \left(X_1(x^s_\mu(t), \mu)\right) = D X_1(x^s_\mu(t), \mu)\xi(t, \mu) + \frac{\partial X_1}{\partial \mu_j}(x^s_\mu(t), \mu).
\]

Let \((s, n) = (s(t, \mu), n(t, \mu))\) be the coordinate system with origin at \(x^s_\mu(t)\) such that the angle between \(X_1(x^s_\mu(t), \mu)\) and \(s\) equals \(\theta_1\) and \(n\) is orthogonal to \(s\), pointing outwards in relation to \(G\). See Figure 15. It follows from the definition of \(n^s\) that the component of \(\dot{\xi}\) in the direction of \(n\) is precisely equals to

\[
\frac{\partial x^s_\mu}{\partial \mu_j}(t, \mu). \]

Therefore, we conclude,

\[
\frac{\partial n^s}{\partial \mu_j}(t, \mu) = \omega_0 \xi \wedge M_1 X_1(x^s_\mu(t), \mu).
\]

Denoting \(M_1 X_1 = (P_0, Q_0), \xi = (\xi_1, \xi_2)\) and,

\[
\rho(t, \mu) = \xi \wedge M_1 X_1(x^s_\mu(t), \mu),
\]

\[
\text{Figure 15. Illustration of } (s, n) \text{ along } x^s_\mu(t).\]
we conclude that,

\[ \rho = \xi_1 Q_0 - P_0 \xi_2, \]

where,

\[ P_0 = P_1 \cos \theta_1 - Q_1 \sin \theta_1, \quad Q_0 = Q_1 \cos \theta_1 + P_1 \sin \theta_1. \]

Hence,

\[ \dot{\rho} = \dot{\xi}_1 Q_0 - \dot{\xi}_2 P_0 + \xi_1 \dot{Q}_0 - \xi_2 \dot{P}_0. \]

Knowing that,

\[ \dot{P}_1 = \dot{x}_1 = \frac{\partial P_1}{\partial x_1} P_1 + \frac{\partial P_1}{\partial x_2} Q_1, \quad \dot{Q}_1 = \dot{x}_2 = \frac{\partial Q_1}{\partial x_1} P_1 + \frac{\partial Q_1}{\partial x_2} Q_1, \]

we conclude,

\[ \dot{P}_0 = \frac{\partial P_1}{\partial x_1} P_1 \cos \theta_1 + \frac{\partial P_1}{\partial x_2} Q_1 \cos \theta_1 - \frac{\partial Q_1}{\partial x_1} P_1 \sin \theta_1 - \frac{\partial Q_1}{\partial x_2} Q_1 \sin \theta_1, \]

\[ \dot{Q}_0 = \frac{\partial Q_1}{\partial x_1} P_1 \cos \theta_1 + \frac{\partial Q_1}{\partial x_2} Q_1 \cos \theta_1 + \frac{\partial P_1}{\partial x_1} P_1 \sin \theta_1 + \frac{\partial P_1}{\partial x_2} Q_1 \sin \theta_1. \]

Replacing (10) in (9) one can conclude,

\[ \dot{\rho} = \text{Div} X_1 \rho - M_1 X_1 \wedge \frac{\partial X_1}{\partial \mu_j} + \sin \theta_1 R_{1,j}. \]

Solving (11) we obtain,

\[ \rho(t, \mu)e^{D_1(t)} \bigg|_{t_0}^{t_1} = \int_{t_0}^{t_1} e^{D_1(t)} \left[ \sin \theta_1 R_{1,j}(x^*_{\mu}(t, \mu)) - M_1 X_1 \wedge \frac{\partial X_1}{\partial \mu_j}(x^*_{\mu}(t, \mu)) \right] dt. \]

Observe that \( X_1(x^*_{\mu}(t, \mu), \mu) \to 0 \) as \( t \to +\infty \). Therefore, it follows from (7) and (8) that \( \rho(t, \mu) \to 0 \) as \( t \to +\infty \) (since from Lemma 2 we know that \( \frac{\partial n^*}{\partial \mu} \) is bounded). Thus, if we take \( t_0 = 0 \) and let \( t_1 \to +\infty \) in (12), then it follows that,

\[ \rho(0, \mu) = \int_0^{+\infty} e^{D_1(t)} \left[ M_1 X_1 \wedge \frac{\partial X_1}{\partial \mu_j}(x^*_{\mu}(t, \mu)) - \sin \theta_1 R_{1,j}(t, \mu) \right] dt, \]

and thus it follows from (7) and (8) we have that,

\[ \frac{\partial n^*}{\partial \mu_j}(0, \mu_0) = \frac{\omega_0}{||X_1(x_0, \mu_0)||} I^+_j. \]

\[ \square \]

**Remark 3.** Observe that even if \( L_0 \) intersects \( \Sigma \) in multiple points, \( L_0^+ \) was defined in such a way that there is no discontinuities on it. Therefore, if \( L_0 \cap \Sigma = \{ x_0 \} \), then \( L_0^- = \{ \gamma_0(t) : t < 0 \} \) also has no discontinuities and thus, as in Proposition 2, one can prove that,

\[ \frac{\partial n^*}{\partial \mu_j}(t, \mu) = \frac{\omega_0}{||X_2(x^*_{\mu}(t, \mu)||} \bar{\rho}(t, \mu), \]

with \( \bar{\rho} \) satisfying (11), but with \( x^*_{\mu} \) instead of \( x^* \) and \( X_2 \) instead of \( X_1 \). Furthermore we have \( \bar{\rho}(t, \mu) \to 0 \) as \( t \to -\infty \) and thus by setting \( t_1 = 0 \) and letting \( t_0 \to -\infty \) we obtain,

\[ \bar{\rho}(0, \mu) = -\int_0^{+\infty} e^{D_2(t)} \left[ M_2 X_2 \wedge \frac{\partial X_2}{\partial \mu_j}(x^*_{\mu}(t, \mu)) - \sin \theta_2 R_{2,j}(t, \mu) \right] dt. \]

Hence, in the simple case where \( L_0 \) intersects \( \Sigma \) in an unique point \( x_0 \), it follows from (6) and from Proposition 2 that,

\[ \frac{\partial d}{\partial \mu_j}(\mu_0) = -\omega_0 \left( \frac{1}{||X_2(x_0, \mu_0)||} I^+_j + \frac{1}{||X_1(x_0, \mu_0)||} I^-_j \right), \]

with,

\[ I^+_j = \int_{L_0^-} e^{D_1(t)} \left[ (M_1 X_1) \wedge \frac{\partial X_1}{\partial \mu_j}(\gamma_0(t), \mu_0) - \sin \theta_1 R_{1,j}(\gamma_0(t), \mu_0) \right] dt, \]

\[ I^-_j = \int_{L_0^+} e^{D_2(t)} \left[ (M_2 X_2) \wedge \frac{\partial X_2}{\partial \mu_j}(\gamma_0(t), \mu_0) - \sin \theta_2 R_{2,j}(\gamma_0(t), \mu_0) \right] dt. \]
Remark 4. If instead of a non-smooth vector field we suppose that $Z = X$ is smooth, then we can assume $X_1 = X_2$ and take $u_0 = \frac{X(x_0, \mu_0)}{\|X(x_0, \mu_0)\|}$. In this case we would have $\theta_1 = \theta_2 = 0$ and therefore conclude that,
\[
\frac{\partial d}{\partial \mu_j}(\mu_0) = -\frac{\omega_0}{\|X(x_0, \mu_0)\|} \int_{-\infty}^{+\infty} e^{-\int_0^t \text{Div}X(\gamma_0(s), \mu_0)ds} \left[ X \wedge \frac{\partial X}{\partial \mu_j}(\gamma_0(t), \mu_0) \right] dt,
\]
as in the works of Perko, Holmes and Guckenheimer [12, 14, 19].

Remark 5. Within this section, the hypothesis of a polycycle is not necessary. In fact if suppose only a heteroclinic connection between saddles, then can define the displacement function as,
\[
d(\mu) = [x_0^u(\mu) - x_0^s(\mu)] \wedge u_0,
\]
and therefore obtain,
\[
\frac{\partial d}{\partial \mu_j}(\mu_0) = -\left( \frac{1}{\|X_2(x_0, \mu_0)\|} I_j^+ + \frac{1}{\|X_1(x_0, \mu_0)\|} J_j^+ \right),
\]
It is only necessary to pay attention at which direction we have $d(\mu) > 0$ or $d(\mu) < 0$.

Let us now study the case where at least one of the endpoints of the heteroclinic connection is a tangential singularity. In the case of the hyperbolic saddle, we use the Stable Manifold Theorem to take a point $x_1^s(\mu)$ within the stable manifold of the hyperbolic saddle. Then, we define,
\[
x^s(t, \mu) = \phi_1(t, x_1^s(\mu), \mu),
\]
where $\phi_1$ is the flow of $X_1$, the component of $Z$ which contains the hyperbolic saddle. Then, we use the Implicit Function Theorem at Lemma 1 to obtain a smooth function $\tau^s(\mu)$ such that,
\[
x_0^s(\mu) = x^s(\tau^s(\mu), \mu) \in \Sigma,
\]
for every $\mu \in \Lambda$, where $\Lambda$ is a small enough neighborhood of $\mu_0$. Therefore, in the case of the tangential singularity, we define,
\[
x_1^s(\mu) = T^s(h_\mu(0), \mu),
\]
where $T^s$ and $h_\mu$ are given by (4). At first glance, the problem with such definition is that the hyperbolic saddle is stable, while a tangential singularity is not. See Figure 7. Moreover, the parameters at (4), related with the transitions maps near tangential singularities, depend continuously on the parameter $\mu$. Hence, to avoid such problems, we need to suppose either that the tangential singularity is quadratic-regular (i.e. $n_s = 2$ and $n_u = 1$, or vice-versa), or that it does not depend on the parameter $\mu$. In either case, it is clear that we can obtain Lemmas 1 and 2.

Remark 6. From now on, given a tangential singularity $p$, we suppose either that $p$ is quadratic-regular or that $p$ does not depend on the parameter $\mu$.

To avoid such restrictions and prove Theorem 3, we will make use of bump-functions to construct perturbations that does not affect any tangential singularity.

Proposition 3. Let $p_1$ be a tangential singularity satisfying Remark 6. Then for any $j \in \{1, \ldots, r\}$ it follows that,
\[
\frac{\partial n^s}{\partial \mu_j}(0, \mu_0) = \frac{\omega_0}{\|X_1(x_0, \mu_0)\|} (I_j^+ + H_j^+),
\]
where,
\[
H_j^+ = \epsilon^{D_1(t_1)} \frac{\partial \gamma_0}{\partial \mu_j}(t_1) \wedge M_1 X_1(p_1, \mu_0),
\]
and $\gamma_0$ is a parametrization of $L_0^+$ such that $\gamma_0(0) = x_0$ and $\gamma_0(t_1) = p_1$.

Proof. It follows from Proposition 2 that,
\[
\frac{\partial n^s}{\partial \mu_j}(t, \mu) = \omega_0 \frac{\rho(t, \mu)}{\|X_1(x^s_\mu(t), \mu)\|},
\]
with,
\[
\rho(t, \mu) = \xi \wedge M_1 X_1(x^s_\mu(t), \mu),
\]
Proposition 4. Let \( Z \) and \( \Gamma^n \) be as in Section 2, with the tangential singularities satisfying Remark 6, and \( d_i : \Lambda \to \mathbb{R} \), \( i \in \{1, \ldots, n\} \), be the displacement maps defined at the heteroclinic connections of \( \Gamma^n \). Let \( \sigma_0 \in \{-1, 1\} \) be a constant such that \( \sigma_0 = 1 \) (resp. \( \sigma_0 = -1 \)) if the Poincaré map is defined in the bounded (resp. unbounded) region of \( \Gamma^n \). Then following statements holds.

(a) If \( r(\Gamma^n) > 1 \) and \( \mu \in \Lambda \) is such that \( \sigma_0 d_1(\mu) \leq 0, \ldots, \sigma_0 d_n(\mu) \leq 0 \) with \( \sigma_0 d_1(\mu) < 0 \) for some \( i \in \{1, \ldots, n\} \), then at least one stable limit cycle \( \Gamma \) bifurcates from \( \Gamma^n \).

(b) If \( r(\Gamma^n) < 1 \) and \( \mu \in \Lambda \) is such that \( \sigma_0 d_1(\mu) \geq 0, \ldots, \sigma_0 d_n(\mu) \geq 0 \) with \( \sigma_0 d_1(\mu) > 0 \) for some \( i \in \{1, \ldots, n\} \), then at least one unstable limit cycle \( \Gamma \) bifurcates from \( \Gamma^n \).

Proof. For the simplicity we will use the same polycycle \( \Gamma \) used in the proof of Theorem 1. Let \( x_{i, 0} \in L_i \) be as in Section 2 and \( l_i \) be transversal sections of \( L_i \) through \( x_{i, 0}, i \in \{1, 2, 3\} \). Let \( R_i : l_i \times \Lambda \to l_{i-1} \) be functions given by the compositions of the functions used in the proof of Theorem 1, \( i \in \{1, 2, 3\} \).

Figure 16. Observe that if \( d_i(\mu) > 0 \), then the composition may not be well defined.

as \( P(x, \mu) = R_3(R_2(R_1(x, \mu), \mu), \mu) \). We observe that \( P \) is \( C^\infty \) in \( x \), continuous in \( \mu \), and it follows from the proof of Theorem 1 that \( P(\cdot, \mu_0) \) is non-flat. It follows from Theorem 1 that there is an open ring \( A \) in the bounded region delimited by \( \Gamma \), such that the orbit \( \Gamma \) through any point \( q \in A_0 \) spiral towards \( \Gamma^3 \) as \( t \to +\infty \). Let \( p \) be the interception of \( \Gamma \) and \( l_1 \), \( q_0 \in A_0 \cap l_1 \), \( \xi \) a coordinate system along \( l_1 \) such that \( \xi = 0 \) at \( p \) and \( \xi > 0 \) at \( q_0 \) and let we identify this coordinate system with \( \mathbb{R}^+ \). Observe that \( P(q_0, \mu_0) < q_0 \) and thus by continuity \( P(q_0, \mu) < q_0 \) for any \( \mu \in \Lambda \). See Figure 17. Therefore, it follows from the Poincaré-Bendixson theory and from the non-flatness of \( P \) that at least one stable limit cycle \( \Gamma_0 \) bifurcates from \( \Gamma^3 \). Statement (b) can be prove by time reversing. \( \square \)

6. THE FURTHER DISPLACEMENT MAP

Let \( Z \) and \( \Gamma^n \) be as in Section 2, with the tangential singularities satisfying Remark 6. Let \( L_i^n(\mu) \) and \( L_i^s(\mu) \) be the perturbations of \( L_i \) such that \( \alpha(L_i^n) = p_{i+1} \) and \( \omega(L_i^s) = p_i \), \( i \in \{1, \ldots, n\} \), with each index being modulo \( n \). Following the work of Han et al [13], let \( C_i = x_{i, 0} \). If \( C_i \notin \Sigma \), then let \( v_i \) be the unique unitarian vector orthogonal to \( Z(C_i, \mu_0) \) and pointing outwards in relation to \( \Gamma^n \). See Figure 18. On the other hand, if \( C_i \in \Sigma \), then let \( v_i \) be the unique unitarian vector tangent to \( T_{C_i} \Sigma \).
and pointing outwards in relation to $\Gamma^n$. In both cases, let $l_i$ be the transversal section normal to $L_i$ at $C_i$. It is clear that any point $B \in l_i$ can be written as $B = C_i + \lambda v_i$, with $\lambda \in \mathbb{R}$. Moreover, let $N_i$ be a small enough neighborhood of $C_i$ and $J_i = N_i \cap \Sigma$. It then follows that any point $B \in J_i$ can be orthogonally projected on the line $L_i : C_i + \lambda v_i$, $\lambda \in \mathbb{R}$, and thus it can be uniquely, and smoothly, identified with $C_i + \lambda B v_i$, for some $\lambda B \in \mathbb{R}$. In either case $C_i \in \Sigma$ or $C_i \notin \Sigma$, observe that if $\lambda > 0$, then $B$ is outside $\Gamma^n$ and if $\lambda < 0$, then $B$ is inside $\Gamma^n$. For each $i \in \{1, \ldots, n\}$ we define,

\[
B^u_i = L^u_i \cap l_i = C_i + b^u_i(\mu) v_i, \quad B^s_i = L^s_i \cap l_i = C_i + b^s_i(\mu) v_i.
\]

Therefore, it follows from Section 5 that,

\[
d_i(\mu) = b^u_i(\mu) - b^s_i(\mu),
\]

$i \in \{1, \ldots, n\}$. Let $r_i = \frac{\mu_i(\mu_0)}{\lambda_i(\mu_0)}$ if $p_i$ is a hyperbolic saddle or $r_i = \frac{v_i \cdot n_i}{m_i}$ if $p_i$ is a tangential singularity, $i \in \{1, \ldots, n\}$. If $r_i > 1$ and $d_i(\mu) < 0$, then following [13], we observe that,

\[
B^*_i = L^*_i \cap l_i = C_i + b^*_i(\mu) v_i,
\]

is well defined and thus we define the further displacement map as,

\[
d^*_i(\mu) = b^*_i(\mu) - b^s_i(\mu).
\]

See Figure 19. On the other hand, if $r_i < 1$ and $d_i(\mu) > 0$, then,

\[
B^*_i = L^*_i \cap l_i = C_i + b^*_i(\mu) v_i,
\]

is well defined and thus we define the further displacement map as,

\[
d^*_i(\mu) = b^*_i(\mu) - b^s_i(\mu).
\]
Remark 7. We observe that in both cases $r_i < 1$ and $r_i > 1$ we have that $d_{i-1}(\mu) > 0$ and $d_i(\mu) < 0$ are necessary conditions for $d_{i-1}^*(\mu) = 0$. Furthermore observe that the signal of $d_{i-1}^*$ has the same topological meaning whether $r_i < 1$ or $r_i > 1$.

Proposition 5. For $i \in \{1, \ldots, n\}$ and $\Lambda \subset \mathbb{R}^\tau$ small enough we have,

$$d_{i-1}^*(\mu) = \begin{cases} d_{i-1}(\mu) + O(||\mu - \mu_0||^{\tau_i}) & \text{if } r_i > 1, \\ d_i(\mu) + O(||\mu - \mu_0||^{\tau_i}) & \text{if } r_i < 1. \end{cases}$$

Proof. For simplicity let us assume $i = n$ and $r_n > 1$. Following Han et al [13], it follows from the definition of $d_n^*$ and $d_{n-1}$ that,

$$d_n^* = (b^*_n - b^*_n) + d_{n-1}.$$  

Let $B = B_n^* + \lambda v_n \in l_n$, $\lambda < 0$, with $|\lambda|$ small enough and observe that the orbit through $B$ will intersect $l_{n-1}$ in a point $C$ which can be written as,

$$C = B_{n-1}^* + F(\lambda, \mu)v_{n-1}.$$  

Therefore, we have a function $F : l_n \to l_{n-1}$ with $F(\lambda, \mu) < 0$ for $\lambda < 0$, $|\lambda|$ small enough, such that $F(\lambda, \mu) \to 0$ as $\lambda \to 0$. From (14) we have,

$$B_n^* = (C_n^* + b_n^* v_n) + (b_n - b_n^*)v_n = B_n^* + d_n v_n$$  

and

$$B_{n-1}^* = (C_{n-1}^* + b_{n-1}^* v_{n-1}) + (b_n^* - b_{n-1}^*)v_{n-1} = B_{n-1}^* + (b_n^* - b_{n-1}^*)v_{n-1}.$$  

Since $B_{n-1}^*$ is the intersection of the positive orbit through $B_n^*$ with $l_{n-1}$ it follows from (16) that,

$$b_n^* - b_{n-1}^* = F(d_n, \mu).$$  

Therefore, it follows from (15) that,

$$d_{n-1}^* = F(d_n, \mu) + d_{n-1}.$$  

If $p_i$ is a hyperbolic saddle, then $F$ is, up to the composition of some diffeomorphisms given by the flow of the components of $Z$, the Dulac map $D_i$ defined at Section 3.1. If $p_i$ is a tangential singularity (we remember that we are under the hypothesis of Remark 6), then $F$ is, up to the composition of some diffeomorphisms given by the flow of the components of $Z$, the composition $T_i^* \circ (T_i^*)^{-1}$ defined at Section 3.3. In either case, it follows from Section 3 that,

$$-F(\lambda, \mu) = |\lambda|^\tau_n(A(\mu) + O(1)),$$

with $A(\mu_0) \neq 0$. Since $d_n = O(||\mu||)$, it follows from (18) that,

$$F(d_n, \mu) = O(||\mu||^\tau_n),$$

and thus from (17) we have the result. The case $r_n < 1$ follows similarly from the fact that the inverse $F^{-1}$ has order $r_n^{-1}$ in $u$.  

Corollary 2. For each $i \in \{1, \ldots, n\}$ the further displacement map $d_i^*$ is continuous differentiable with the $j$-partial derivative given either by the $j$-partial derivative of $d_i$ or $d_{i-1}$. Furthermore a connection between $p_i$ and $p_{i-2}$ exists if and only if $d_i^*(\mu) = 0$ and $d_{i-1}(\mu) \neq 0$.  

\[ \text{Figure 19. Illustration of } d_1^* < 0 \text{ for both } r_1 > 1 \text{ and } r_1 < 1. \]
7. Proof of Theorem 3

Proof of Theorem 3. Let \( Z = (X_1, \ldots, X_M) \) and denote \( X_i = (P_i, Q_i), i \in \{1, \ldots, M\} \). Let \( \{p_1, \ldots, p_n\} \) be the singularities of \( \Gamma^n \) and \( L_i \) the heteroclinic connections between them such that \( \omega(L_i) = p_i \) and \( \alpha(L_i) = p_{i+1} \). If \( L_i \cap \Sigma = \emptyset \), then take \( x_{i,0} \in L_i \) and \( \gamma_i(t) \) a parametrization of \( L_i \) such that \( \gamma_i(0) = x_{i,0} \). If \( L_i \cap \Sigma \neq \emptyset \), then let \( L_i \cap \Sigma = \{x_{i,0}, \ldots, x_{i,n(i)}\} \) and take \( \gamma_i(t) \) a parametrization of \( L_i \) such that \( \gamma_i(t_{i,j}) = x_{i,j} \) with \( 0 = t_{i,0} > t_{i,1} > \cdots > t_{i,n(i)} \). In either case denote \( L_i^+ = \{\gamma_i(t) : t > 0\} \) if \( p_i \) is a hyperbolic saddle or \( L_i^- = \{\gamma_i(t) : t < 0\} \), where \( t_i \) is such that \( \gamma_i(t_i) = p_i \), if \( p_i \) is a \( \Sigma \)-singularity. Following [13], for each \( i \in \{1, \ldots, n\} \) let \( G_{i,j}, j \in \{1, 2\} \), be two compact disks small enough such that,

1) \( \Gamma^n \cap G_{i,j} = L_i^+ \cap G_{i,j} \neq \emptyset, j \in \{1, 2\} \);
2) \( G_{i,1} \subset \text{Int} G_{i,2} \);
3) \( G_{i,2} \cap G_{s,2} = \emptyset \) for any \( i \neq s \);
4) \( G_{i,j} \cap \Sigma = \emptyset \).

Let \( k_i : \mathbb{R}^2 \to [0, 1] \) be a \( C^\infty \)-bump function such that,

\[
k_i(x) = \begin{cases} 0, & x \notin G_{i,2}, \\ 1, & x \in G_{i,1}. \end{cases}
\]

See Figure 20. Let \( \mu \in \mathbb{R}^n \) and \( g_i : \mathbb{R}^2 \to \mathbb{R}^2, i \in \{1, \ldots, n\} \), be maps that we yet have to define. Let

\[ g(x, \mu) = \sum_{i=1}^{n} \mu_i k_i(x) g_i(x), \]

and for now one let us denote \( X_i = X_i + g \). Let \( \Lambda \) be a small enough neighborhood of the origin of \( \mathbb{R}^n \). It follows from Section 5 that each displacement map \( d_i : \Lambda \to \mathbb{R} \) controls the bifurcations of \( L_i \) near \( x_{i,0} \). It follows from Definition 6 that,

\[ d_i(\mu) = \omega_0[\omega_i(\mu) - x_{i,0}^s(\mu)] \wedge \eta_i, \]

where \( \eta_i \) is the analogous of \( u_0 \) in Figure 11. But from the definition of \( g \) we have that each \( x_{i,0}^u(\mu) \) does not depend on \( \mu \) and thus \( x_{i,0}^u \equiv x_{i,0} \). Furthermore it follows from the definition of \( k_i \) that each singularity \( p_i \) of \( \Gamma^n \) also does not depend on \( \mu \) and thus \( \frac{\partial}{\partial \mu_j}(t_i) = 0 \) for every tangential singularity \( p_i \). Therefore, it follows from Propositions 2 and 3 that,

\[
\frac{\partial d_i}{\partial \mu_j}(0) = -\frac{\omega_0}{||X_i(x_0, \mu_0)||} \int_{L_i^+} e^{D_i(t)} \left[ (M_i X_i) \wedge \frac{\partial X_i}{\partial \mu_j}(\gamma_i(t), 0) - \sin \theta_i R_{i,j}(\gamma_i(t), 0) \right] dt,
\]
with,

$$D_i(t) = - \int_0^t \text{Div} X_i(\gamma_0(s), \mu_0)ds,$$

and,

$$R_{i,j} = \frac{\partial P_i}{\partial D_j}\left[\left(\frac{\partial Q_i}{\partial x_1} + \frac{\partial P_i}{\partial x_2}\right) Q_i + \left(\frac{\partial P_i}{\partial x_1} - \frac{\partial Q_i}{\partial x_2}\right) P_i\right] + \frac{\partial Q_i}{\partial D_j}\left[\left(\frac{\partial Q_i}{\partial x_1} + \frac{\partial Q_i}{\partial x_2}\right) P_i + \left(\frac{\partial Q_i}{\partial x_1} - \frac{\partial P_i}{\partial x_2}\right) Q_i\right],$$

$$i, j \in \{1, \ldots, n\}.$$ We observe that if $$L_i \cap \Sigma = \emptyset,$$ then $$\theta_i = 0.$$ It follows from the definition of the sets $$G_{i,j}$$ that $$\frac{\partial d}{\partial D_j}(0) = 0$$ if $$i \neq j.$$ Let $$M_{i,X} = (\overline{P_i}, \overline{Q_i})$$ and $$R_{i,i} = \frac{\partial P_i}{\partial D_i}F_{i,1} + \frac{\partial Q_i}{\partial D_i}F_{i,2},$$ where,

$$F_{i,1} = \left(\frac{\partial Q_i}{\partial x_1} + \frac{\partial P_i}{\partial x_2}\right) Q_i + \left(\frac{\partial P_i}{\partial x_1} - \frac{\partial Q_i}{\partial x_2}\right) P_i,$$

$$F_{i,2} = \left(\frac{\partial P_i}{\partial x_1} + \frac{\partial Q_i}{\partial x_2}\right) P_i + \left(\frac{\partial Q_i}{\partial x_1} - \frac{\partial P_i}{\partial x_2}\right) Q_i.$$ Let $$g_i = (g_{i,1}, g_{i,2})$$ and observe that,

$$(M_{i,X}) \wedge \frac{\partial X_i}{\partial D_i} - \sin \theta_i R_{i,i} = k_i[g_{i,2}(\overline{P_i} - \sin \theta_i F_{i,2}) - g_{i,1}(\overline{Q_i} + \sin \theta_i F_{i,1})].$$

Therefore, if we take $$g_i = -\omega_0(\overline{Q_i} - \sin \theta_i F_{i,1}, \overline{P_i} - \sin \theta_i F_{i,2}),$$ then we can conclude that,

$$d_i(\mu) = a_i \mu_i + O(||\mu||^2),$$

with $$a_i = \frac{\partial d}{\partial \mu_i}(0) > 0,$$ $$i \in \{1, \ldots, n\}.$$ If $$n = 1,$$ then it follows from Proposition 4 that any $$\mu \in \mathbb{R}$$ arbitrarily small such that $$(R_{1,1})_{\sigma_0 \mu} < 0$$ result in the bifurcation of at least one limit cycle. Suppose $$n \geq 2$$ and that the result had been proved in the case $$n - 1.$$ We will now prove by induction in $$n.$$ For definiteness we can assume $$R_n > 1$$ and therefore $$R_{n-1} < 1$$ and thus $$r_n > 1.$$ Moreover, it follows from Theorem 1 that $$\Gamma^0$$ is stable. Define,

$$D = (d_1, \ldots, d_{n-2}, d_{n-1}).$$

It follows from Proposition 5 and from (19) that we can apply the Implicit Function Theorem on $$D$$ and thus obtain unique $$C^\infty$$ functions $$\mu_i = \mu_i(\mu_n), \mu_i(0) = 0, i \in \{1, \ldots, n - 1\},$$ such that,

$$D(\mu_1(\mu_n), \ldots, \mu_{n-1}(\mu_n), \mu_n) = 0,$$

for $$|\mu_n|$$ small enough. It also follows from (19) that $$d_n \neq 0$$ if $$\mu_n \neq 0,$$ with $$|\mu_n|$$ small enough. Therefore, if $$\mu_i = \mu_i(\mu_n)$$ and $$\mu_n \neq 0,$$ then it follows from the definition of $$D = 0$$ that there exist a $$\Gamma^{n-1} = \Gamma^{n-1}(\mu_n)$$ polycycle formed by $$n-1$$ singularities and $$n-1$$ heteroclinic connections $$L_i^* = L_i^*(\mu_n)$$ such that,

1) $$\Gamma^{n-1} \to \Gamma^n,$$
2) $$L_{n-1}^{*} \to L_n \cup L_{n-1}$$ and,
3) $$L_i^* \to L_i, \quad i \in \{1, \ldots, n-2\},$$

as $$\mu_n \to 0.$$ See Figure 21. Let,

$$R^*_j = r_1 \cdots r_j \big|_{\mu_i=\mu_i(\mu_n), i \in \{1,\ldots, n-1\}}.$$
Then it follows from the hypothesis,

$$(R_i - 1)(R_{i+1} - 1) < 0,$$

for $i \in \{1, \ldots, n - 1\}$ and from the hypothesis $R_{n-1} < 1$, that,

$$(R_i^* - 1)(R_{i+1}^* - 1) < 0,$$

for $i \in \{1, \ldots, n - 2\}$ and $R_n^* < 1$ for $\mu_n \neq 0$ small enough. Thus, it follows from Theorem 1 that $\Gamma_{n-1}$ is unstable while $\Gamma_n$ is stable. It then follows from the Poincaré-Bendixson theory (see [4]) and from the non-flats of the Poincaré map that at least one stable limit cycle $\tau_n(\mu_n)$ exists near $\Gamma_{n-1}$. In fact both the limit cycle and $\Gamma_{n-1}$ bifurcates from $\Gamma_n$. Now fix $\mu_n \neq 0$, $|\mu_n|$ arbitrarily small, and define the non-smooth system,

$$Z_i^* = Z^*(x) + g^*(x, \overline{\mu}),$$

where $Z^*(x) = Z(x) + g(x, \mu_1(\mu_n), \ldots, \mu_{n-1}(\mu_n), \mu_n)$ and,

$$g^*(x, \overline{\mu}) = \sum_{i=1}^{n-1} \overline{\mu}_i k_i(x) g_i(x),$$

with $\overline{\mu}_i = \mu_i - \mu_i(\mu_n)$. It then follows by the definitions of $G_{i,j}$ and $L_i^*$ that,

$$\Gamma_{n-1} \cap G_{i,j} = (L_i^*)^+ \cap G_{i,j} \neq \emptyset,$$

$i \in \{1, \ldots, n - 1\}$ and $j \in \{1, 2\}$. In this new parameter coordinate system the bump functions $k_i$ still ensures that $\frac{\partial k_i}{\partial \mu_j}(\mu) = 0$ if $i \neq j$. Since $a_i > 0$, it also follows that at the origin of this new coordinate system we still have $\frac{\partial k_i}{\partial \mu_j}(0) > 0$. Therefore, it follows by induction that at least $n - 1$ crossing limit cycles $\tau_j(\overline{\mu})$, $j \in \{1, \ldots, n - 1\}$, bifurcates near $\Gamma_{n-1}$ for arbitrarily small $|\overline{\mu}|$. We observe that $\tau_n(\mu_n)$ persists for $\overline{\mu}$ small enough. □

**Remark 8.** It follows from the proof of Theorem 3 that there exist an open set $V$ in the space of parameters $\mu \in \mathbb{R}^n$, with the origin in its closure, such that at least $n$ limit cycles bifurcates from $\Gamma^n$ for any $\mu \in V$.

### 8. Proof of Theorem 4

Let $Z_0 = (X_1, X_2)$ be a planar non-smooth vector field with a discontinuity $\Sigma = h^{-1}(0)$ and $\Gamma = \Gamma^2$ be a polycycle composed of a hyperbolic saddle $p_1 \in A_1$ and a regular-quadratic tangential singularity $p_2 \in \Sigma$ such that $n_u = 2$ and $n_s = 1$. Let also $L_1$ and $L_2$ be the heteroclinic connections such that $\omega(L_i) = p_i$, $i \in \{1, 2\}$, and without loss of generality let us suppose $L_1 \cap \Sigma = \{x_{1,0}\}$ and $L_2 \cap \Sigma = \emptyset$. See Figure 22. Following Section 5, we take any point $x_{2,0} \in L_2$ and any small enough neighborhood $U$ of $Z_0$ and define the displacement maps $d_i : U \to \mathbb{R}$ near the point $x_{i,0}$, $i \in \{1, 2\}$. Since $p_1$ and $p_2$ are both structurally stable, it follows from the previous sections that to describe any small enough bifurcation of $\Gamma$, it is enough to look at the two small parameters $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ given by $\beta_i = d_i(Z)$, $i \in \{1, 2\}$. See Figure 23. Let $\nu < 0 < \lambda$ be the eigenvalues of $p_1 = p_1(Z_0)$ and $r_1 = \frac{|\nu|}{\lambda}$ the hyperbolicity ratio of $p_1$. 

![Figure 22. Illustration of $\Gamma$.](image-url)
Lemma 3. If \((\beta_1, \beta_2) \in \mathbb{R}^2\) is close enough to the origin, then following statements holds.

(a) If \(r_1 > 1\), then the cyclicity of \(\Gamma\) is one, and if a limit cycle bifurcate from \(\Gamma\), it is hyperbolic and stable;

(b) If \(\frac{1}{2} < r_1 < 1\), then the following statements holds:
   (i) The cyclicity of \(\Gamma\) is two and when two limit cycles bifurcates from \(\Gamma\), then both are hyperbolic and the inner one is stable while the outer one is unstable;
   (ii) Let \(\beta^* = (\beta_1^*, \beta_2^*)\), \(\beta_1^* > 0\), be such that two limit cycles exists. Then \(\beta_2^* < 0\) and there exists at least one \(\beta_2 \in (\beta_2^*, 0)\) such that a saddle-node bifurcation happens between these two limit cycles at \(\beta = (\beta_1^*, \beta_2)\);

(c) If \(r_1 < \frac{1}{2}\), then the following statement holds:
   (i) The cyclicity of \(\Gamma\) is two and when two limit cycles bifurcates from \(\Gamma\), then both are hyperbolic and the inner one is unstable while the outer one is stable;
   (ii) Let \(\beta^* = (\beta_1^*, \beta_2^*)\), \(\beta_1^* < 0\), be such that two limit cycles exists. Then \(\beta_2^* > 0\) and there exists at least one \(\beta_2 \in (0, \beta_2^*)\) such that a saddle-node bifurcation happens between these two limit cycles at \(\beta = (\beta_1^*, \beta_2)\).

**Proof.** First observe that statement \((a)\) follows directly from Theorem 2. Let \(L_i^s = L_i^s(\beta_i)\) and \(L_i^u = L_i^u(\beta_i)\) be the perturbations of \(L_i\) such that \(\omega(L_i^s) = \rho_i\), \(i \in \{1, 2\}\), \(\alpha(L_i^s) = p_2\) and \(\alpha(L_i^u) = p_1\). Let \(\tau_i\) be a cross section through \(L_i\) at \(x_i, 0\), \(\sigma_1\) be a cross section through \(L_1\) close enough to \(p_1\) and \(\sigma_2 = \Sigma \cap \overline{A}\). See Figure 24. Let \(G_i : \tau_i \to \sigma_i\) and \(F_i : \sigma_i \to \tau_{i-1}\) be given by the flow of \(Z\). It follows from Section 3 that we can assume,

\[ G_i(y_i) = y_i - \beta_i, \quad F_i^{-1}(y_2) = x^{s1}(A(\beta) + \varphi(y_2, \beta)), \quad F_2(x_2) = kx_2^2 + O(x_2^3), \]

Figure 23. Illustration of a small perturbation of \(\Gamma\) with \(\beta_1 > 0\) and \(\beta_2 < 0\).

Figure 24. Illustration of the maps \(F_i\) and \(G_i, i \in \{1, 2\}\), with \(\beta_1 > 0\) and \(\beta_2 < 0\). Observe that the point \(p_2\) is equivalent to \(x_2 = 0\).
with \( k > 0, A_1(\beta) > 0 \) and \( s_1 = \frac{1}{r_1} \). See Figure 24. Following [8] we define,

\[
H(x_2) = G_1 \circ F_2(x_2) - F_1^{-1} \circ G_2^{-1}(x_2)
\]

and observe that,

\[
H'(x_2) = 2kx_2 + O(x_2^3) - s_1(x_2 + \beta_2)^{s_1}(A(\beta) + \varphi(x_2 + \beta_2, \beta)),
\]

\[
H''(x_2) = 2k + O(x_2) - s_1(s_1 - 1)(x_2 + \beta_2)^{s_1-2}(A(\beta) + \xi(x_2 + \beta_2, \beta)).
\]

Therefore, if \( \frac{1}{r} < r_1 < 1 \), then \( s_1 - 2 < 0 \) and thus it follows from (21), together with the boundedness of \( A \) and \( \xi \), that,

\[
\lim_{(x_2, \beta_2) \to 0} H''(x_2) = -\infty,
\]

and hence at most two limit cycles can bifurcate from \( \Gamma \), i.e. the cyclicity of \( \Gamma \) is at most two. From Theorem 3 we have that there is some bifurcation which two limit cycles bifurcates and thus the cyclicity of \( \Gamma \) is two. Moreover, since \( H''(x_2) < 0 \) for \( ||(x_2, \beta)|| \) small enough, it follows that when we have two limit cycles, then the inner one is stable and the outer one is unstable and thus we have statement (b)(i). Statement (b)(ii) follows from an analysis of the degrees of \( G_1 \circ F_2 \) and \( F_1^{-1} \circ G_2^{-1} \). Let \( \beta^* = (\beta^*_1, \beta^*_2) \), \( \beta^*_1 > 0 \), be such that we have two limit cycles near \( \Gamma \). Let \( 0 < x_{2,1}(\beta) < x_{2,2}(\beta) \) be the respective zeros of \( H \) which gives those limit cycles and observe that \( H'(x_{2,1}) > 0 \) and \( H'(x_{2,2}) < 0 \). Observe also that \( x_2 + \beta_2 \geq 0 \) is a necessary condition for the well definition of \( H \). Since \( H'(x_{2,1}) > 0 \) and \( 0 < s_1 - 1 < 1 \), it follows that \( \beta^*_2 < 0 \). Moreover, since \( 1 < s_1 < 2 \), it follows from (20) and (21) that \( H(x_2) < 0 \) and \( H'(x_2) < 0 \) if \( |\beta| \) is small enough. Then, if we fix \( \beta_1 = \beta^*_1 \) and make \( \beta_2 \to 0 \), we will have the collapse of \( x_{2,1} \) and \( x_{2,2} \) and thus the birth of a semi-stable limit cycle at some \( \beta_2 \in (\beta^*_2, 0) \). About statement (c), observe that \( r_1 < \frac{1}{r} \) implies \( s_1 > 2 \) and thus,

\[
\lim_{(x_2, \beta_2) \to 0} H''(x_2) = 2k > 0.
\]

The demonstration is now similar. \( \square \)

**Lemma 4.** The contacts of the curves \( d_1^* = 0, d_2^* = 0 \) and \( \gamma \) at \( \beta = 0 \) are those given by Figures 2-4.

**Proof.** Since the tangential singularity \( p_2 \) is structural stable, it follows from Section 6 and Proposition 5 that \( d_1^*(\beta) = \beta_1 + f(\beta) \), with \( f(0) = \frac{\partial f}{\partial x}(0) = 0, i \in \{1, 2\} \), and,

\[
d_2^*(\beta) = \begin{cases} 
\beta_2 + g(\beta), & \text{if } r_1 > 1, \\
\beta_1 + g(\beta), & \text{if } r_1 < 1,
\end{cases}
\]

with \( g(0) = \frac{\partial g}{\partial x}(0) = 0, i \in \{1, 2\} \). Therefore, \( \nabla d_1^*(0) = (1, 0) \) and,

\[
\nabla d_2^*(0) = \begin{cases} 
(0, 1), & \text{if } r_1 > 1, \\
(1, 0), & \text{if } r_1 < 1.
\end{cases}
\]

Hence, the contacts of the curves \( d_1^* = 0 \) and \( d_2^* = 0 \) at \( \beta = 0 \) are those given by Figures 2-4. Let \( F = (F_1, F_2) \) be given by,

\[
F_1(x, \beta_1, \beta_2) = H(x, \beta_1, \beta_2), \quad F_2(x, \beta_1, \beta_2) = \frac{\partial H}{\partial x}(x, \beta_1, \beta_2).
\]

It follows from (20), (21) and from Section 3.1 that,

\[
\lim_{(x, \beta) \to 0} \frac{\partial F_1}{\partial \beta_1} = -1, \quad \lim_{(x, \beta) \to 0} \frac{\partial F_1}{\partial \beta_2} = 0, \quad \lim_{(x, \beta) \to 0} \frac{\partial F_2}{\partial \beta_2} = -\infty,
\]

and that \( \frac{\partial F_2}{\partial \beta_1} \) is bounded for \((x, \beta)\) near the origin. Therefore,

\[
\frac{\partial F}{\partial \beta_1 \beta_2} = \begin{pmatrix}
\frac{\partial F_1}{\partial \beta_1} & \frac{\partial F_1}{\partial \beta_2} \\
\frac{\partial F_2}{\partial \beta_1} & \frac{\partial F_2}{\partial \beta_2}
\end{pmatrix},
\]

is invertible for all \((x, \beta)\) near the origin. Let \((x^*, \beta^*_1, \beta^*_2)\) small enough be such that \( F(x^*, \beta^*_1, \beta^*_2) = 0 \). In fact, it follows from Theorem 3 that we can take \( \beta^* \) arbitrarily near to the origin. It follows from
the Implicit Function Theorem that there exist two unique functions \( \beta_1 = \beta_1(x) \) and \( \beta_2 = \beta_2(x) \), at least of class \( C^1 \), such that \( F(x, \beta_1(x), \beta_2(x)) \equiv 0 \), with \( \beta_i(x^*) = \beta_i^* \), \( i \in \{1, 2\} \). It also follows from the Implicit Function Theorem that,

\[
\begin{pmatrix}
\beta_1'(x^*) \\
\beta_2'(x^*)
\end{pmatrix} = -\left[ \frac{\partial F}{\partial \beta_1 \partial \beta_2} \right]^{-1} \cdot \left( \frac{\partial F_1}{\partial x} \right),
\]

Since \( \frac{\partial F_1}{\partial x} = F_2 \), with the latter vanishing at \( (x, \beta_1(x), \beta_2(x)) \), it follows from (22) that,

\[
\frac{d\beta_1}{d\beta_2} = -\frac{\partial F_1}{\partial \beta_2} \cdot \left( \frac{\partial F_1}{\partial \beta_1} \right)^{-1}.
\]

Hence, it follows from (20) that \( \beta_i^* \to 0 \) implies \( (x^*, \beta^*) \to 0 \) and thus from (23) we have that,

\[
\lim_{\beta_i^* \to 0} \frac{d\beta_1}{d\beta_2} = 0.
\]

Hence, the contact of the curve \( \gamma \) is the one given at Figures 2-4.

**Remark 9.** It follows from the uniqueness of the functions \( \beta_i(x) \) of Lemma 4 that \( \beta_i \), from statements (b)(ii) and (c)(ii) of Lemma 3, are also unique. Furthermore as we shall see, the restrictions \( \beta_i > 0 \) and \( \beta_1 < 0 \) of statements (b) and (c) of Lemma 3 are in fact no restrictions at all since these conditions are necessary for the existence of two limit cycles.

Before we prove Theorem 4, let us draw again the bifurcations diagrams, this time with a more precise identification of the phase portraits. See Figure 25.

**Figure 25.** Bifurcation diagrams with a identification of the phase portraits.

**Proof of Theorem 4.** First let us suppose \( r_1 > 1 \). Then, from Theorem 1 we know that \( \Gamma \) is stable if \( \beta = 0 \) and thus we have phase portrait 1. It follows from Theorem 2 that if a limit cycle bifurcates from \( \Gamma \), then it is unique, hyperbolic and stable. Therefore, if \( \beta_1 \leq 0 \), \( \beta_2 \leq 0 \) and \( \beta_1^2 + \beta_2^2 > 0 \), then it follows from the Poincaré-Bendixson theory for non-smooth vector fields (see [4]) that an unique stable and hyperbolic limit cycle bifurcates from \( \Gamma \) and thus we have phase portraits 8, 9 and 10. It also follows from the Poincaré-Bendixson theory for non-smooth vector fields that if \( d_1^* < 0 \) or \( d_2^* < 0 \), then again an unique stable and hyperbolic limit cycle bifurcates from \( \Gamma \) and thus we have phase portraits 7 and 11. By definition of \( d_i^* = 0 \), it follows that if \( d_i^* = 0 \) or \( d_2^* = 0 \), then a new polycycle \( \Gamma^* \) composed either by \( p_1 \) or \( p_2 \), respectively. Since \( \Gamma^* \) is stable in both cases, we can ensure from Theorem 2 that no limit cycle bifurcates from it and hence we have phase portraits 6 and 12. Clearly if \( \beta_2 > 0 \) and \( d_2^* \geq 0 \), then a sliding polycycle \( \Gamma^* \) composed of \( p_1 \) and \( p_2 \) bifurcates from \( \Gamma \), giving us phase portraits 2 and 13. We remember from the proof of Lemma 3 that,

\[
H(x_2) = kx_2^2 + O(x_2^3) - \beta_1 - (x_2 + \beta_2)^{s_1}(A(\beta) + \varphi(x_2 + \beta_2, \beta)),
\]

where \( s_1 = \frac{1}{s_2} < 1 \). Thus, if \( \beta_1 > 0 \), then it follows from (24) that \( H(x_2) < 0 \) for \( x_2 > 0 \) and \( x_2 + \beta_2 > 0 \) small. Hence, we have phase portraits 3, 4 and 5.
Let us now suppose $\frac{1}{2} < r_1 < 1$. The phase portraits of the first, second and third quadrant follows in a similar way to the previous case. Hence, we will focus on the fourth quadrant. It follows from statement (b) of Lemma 3 that at most two limit cycles can bifurcate from $\Gamma$ and if it does, then the outer one is necessary unstable. This, together with the Poincaré-Bendixson Theory for non-smooth vector fields, give us phase portrait 5. If $d_1^* = 0$, then the polycycle $\Gamma^*$ composed of $p_1$ is unstable and thus we have phase portrait 4. Again from the Poincaré-Bendixson Theory for non-smooth vector fields, an unique hyperbolic unstable limit cycle bifurcate from $\Gamma^*$ if $d_1^* > 0$ and thus we have phase portrait 3. Phases portrait 1 and 2 follows from statement (b)(ii) of Lemma 3. The case $r_1 < \frac{1}{2}$ is similar to the previous two cases.

Finally, the proof about the other polycycle is identical. One need only to consider the maps given by Figure 26 and consider once again the map $H = G_1 \circ F_2 - F_1^{-1} \circ G_2^{-1}$.

![Figure 26. Illustration with $\beta_1 > 0$ and $\beta_2 < 0$.](image)

9. Conclusion and Further Directions

The goal of this work was to extend some well established results about polycycles in smooth vector fields, to the realm of non-smooth vector fields. To do that, first we saw that given a polycycle, the ratio between the contact orders of a tangential singularity dictates whether such singularity will locally contract or repel the flow. Therefore, such ratio plays the same role as the hyperbolicity ratio in the case of the hyperbolic saddles. Then, we saw when such a tangential singularity has some degree of stability, allowing us to perturb the polycycle without changing the local stability of the singularities and thus controlling the cyclicity of the polycycle and the hyperbolicity of the limit cycles. After that, we make use of the bump-functions to avoid the lack of stability of the tangential singularities and to prove that, under some conditions, any small perturbation of a polycycle with $n$ hyperbolic saddles and $m$ tangential singularities may have at least $n + m$ limit cycles. Finally, we applied such tools to obtain the bifurcation diagram of two polycycles composed of a hyperbolic saddle and a quadratic-regular tangential singularity.

Regarding further directions, since the quadratical-regular singularities are the only stable tangential singularity, it may be fruitful to study the cyclicity of a polycycle composed of $n$ hyperbolic saddles and $m$ quadratical-regular tangential singularities. From the work presented here, we already know that it is at least $n + m$.

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References

[1] K. ANDRADE, O. GOMIDE AND D. NOVAES, Qualitative Analysis of Polycycles in Filippov Systems, arXiv:1905.11950v2 (2019).
[2] K. Andrade, M. Jeffrey, R. Martins and M. Teixeira, On the Dulac’s problem for piecewise analytic vector fields, Journal of Differential Equations, 4 (2019), p. 2259-2273.
[3] A. Andronov, A. Vitt and S. Khaikin, Theory of Oscillators, Translated from the Russian by F. Immirzi; translation edited and abridged by W. Fishwick. Pergamon Press, Oxford-New York-Toronto, Ont., 1966.
[4] C. Buzzi, T. Carvalho and R. Euzébio, On Poincaré-Bendixson theorem and non-trivial minimal sets in planar non-smooth vector fields, Publicacions Matemàtiques 62 (2018), p. 113-131.
[5] L. Cherkas, The stability of singular cycles, Differentsial’nye Uravneniya, 4 (1968), p. 1012-1017.
[6] G. Duff, Limit-Cycles and Rotated Vector Fields, Annals of Mathematics, Second Series, 57 (1953), p. 15-31.
[7] H. Dulac, Sur les cycles limites, Bulletin de la Société Mathématique de France, 62 (1923), p. 45-188.
[8] F. Dumortier, R. Roussarie and C. Rousseau, Elementary graphics of cyclicity 1 and 2, Nonlinearity, 7 (1994), p. 1001-1043.
[9] F. Dumortier, M. Morsalani and C. Rousseau, Hilbert’s 16th problem for quadratic systems and cyclicity of elementary graphics, Nonlinearity, 9 (1996), p. 1209–1261.
[10] A. Filippov, Differential Equations with Discontinuous Right-hand Sides, Translated from the Russian, Mathematics and its Applications (Soviet Series) 18, Kluwer Academic Publishers Group, Dordrecht (1988).
[11] A. Gasull, V. Mañosa and F. Mañosas, Stability of certain planar unbounded polycycles, J. Math. Anal. Appl. 269 (2002), p. 332–351.
[12] J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Springer-Verlag, New York, 42 (1983), p. XVI+453.
[13] M. Han, Y. Wu, P. Bi, Bifurcation of limit cycles near polycycles with $n$ vertices, Chaos, Solitons and Fractals, 22 (2004) p. 383-394.
[14] P. Holmes, Averaging and chaotic motions in forced oscillations, SIAM Journal on Applied Mathematics, 38 (1980), p. 65-80.
[15] A. Mourtada, Cyclicité finie des polycycles hyperboliques de champs de vecteurs du plan mise sous forme normale, Springer Berlin Heidelberg (1990), p. 272-314.
[16] A. Mourtada, Degenerate and Non-trivial Hyperbolic Polycycles with Two Vertices, Journal of Differential Equations, 113 (1994) p. 68-83.
[17] D. Novaes and G. Rondon, On limit cycles in regularized Filippov systems bifurcating from homoclinic-like connections to regular-tangential singularities, Physica D: Nonlinear Phenomena, 442 (2022) p. 135326.
[18] D. Novaes, M. Teixeira and I. Zeli, The generic unfolding of a codimension-two connection to a two-fold singularity of planar Filippov systems, Nonlinearity, 31 (2018) p. 2083-2104.
[19] L. Perko, Homoclinic loop and multiple limit cycle bifurcation surfaces, Trans. Amer. Math. Soc, 344 (1994), p. 101-130.
[20] L. Perko, Differential equations and dynamical systems, vol. 7 of Texts in Applied Mathematics, Springer-Verlag, New York, third ed, 2001.
[21] J. Sotomayor, Curvas Definidas por Equações Diferenciais no Plano, Rio de Janeiro: Instituto de Matemática Pura e Aplicada, 1981.
[22] Y. Ye, S. Cai and C. Lo, Theory of limit cycles, Providence, R.I. : American Mathematical Society, 1986.

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