Charge dynamics and optical conductivity of the $t - J$ model

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Abstract

The dynamic charge susceptibility and the optical conductivity are calculated in the planar $t - J$ model within the memory function method, working directly in terms of Hubbard operators. The density fluctuation spectrum consists of a damped sound-like mode for small wave vectors and a broad high energy peak ($\sim t$) for large momenta. The study of the optical conductivity shows that electron scattering from spin fluctuations leads to the Drude-frequency dependent relaxation rate which exhibits a crossover from $\omega^{3/2}$ behavior at low frequencies ($\omega < 2|\mu|$), to a linear $\omega$-dependence for frequencies larger than $2|\mu|$. Due to the spin-polaron nature of charge carriers, extra absorptions arise starting at a frequency $\omega \gtrsim J$. The obtained results are in a good agreement with exact diagonalization studies.

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I. INTRODUCTION

Among other unconventional normal state properties of high-$T_c$ superconductors, an anomalous charge dynamics has also been detected in the optical measurements of the underdoped samples.\(^1\) Namely, a non-Drude fall-off of the low-frequency absorption indicating a linear $\omega$–dependence of the relaxation rate and an anomalous mid-infrared (MIR) band with a typical energy $\sim 0.1$ eV have been observed.\(^2\)\(^3\)

It is widely believed that unusual properties of the superconducting cuprates are due to the strong electron correlations.\(^1\) The minimal model to describe correlation effects in the cuprates is the $t – J$ model. While a number of analytic works have been carried out to investigate spin dynamics within the $t – J$ model, only few of authors have studied charge dynamics.\(^4\)\(^5\)\(^6\) In Refs. 4,5 charge fluctuations have been studied by the slave boson and Hubbard operator (HO) formalism within the leading order of $1/N$ expansion, respectively. It was found that the density fluctuations at large momenta show a sharp high-energy peak corresponding to the collective mode which reduces to the sound mode in the long-wavelength limit.\(^4\)\(^5\) Later, the authors of Ref. 6 showed that next-order corrections in the $1/N$ expansion lead to the broadening of the high-energy peak due to incoherent motion of bare holes. Similar features of the density response have been previously observed in exact diagonalization studies of small clusters.\(^7\) Recently, the charge correlations in the AFM phase of the $t – J$ model has been investigated by commulant version of projection technique.\(^8\)

In the present paper, we investigate the charge fluctuation spectrum of the $t – J$ model in the paramagnetic state with short-range antiferromagnetic (AFM) correlations. We develop a self-consistent theory for the dynamic charge susceptibility (DCS) by applying the memory function method in terms of HO’s. The employment of the HO technique has a twofold advantage; By using the equations of motion for the HO’s we automatically take into account scattering of electrons on spin and charge fluctuations originated from the strong correlations, as it has first been pointed out by Hubbard.\(^9\) Moreover, the HO formalism allows us to preserve rigorously the local constraint of no double occupancy.
We calculate the memory function within the mode coupling approximation (MCA) in terms of the dressed particle-hole (p-h) and spin fluctuations. Similarly to the nearly antiferromagnetic Fermi liquid approach, we treat fermionic and localized spin excitations as independent degrees of freedom. We show that the memory function involves two contributions. The first one stems from the hopping term and describes a particle-hole contribution from the itinerant hole subsystem. The second one involves scattering processes of electrons on charge and spin fluctuations and comes both from kinematics and exchange interactions.

Further, we perform an analytic analysis of different limiting behavior of DCS to show that the essential features observed in the exact diagonalization studies can be reproduced within the present formalism. We find out that for small $q$ the DCS is mainly governed by the sound mode. Although the unrenormalized sound velocity is larger than the Fermi velocity, unlike the Fermi liquid theory, the “self-energy” corrections lead to softening of the sound. The renormalized sound falls down into the p-h continuum getting a finite damping due to the decay into pair excitations. In the short-wavelength limit, density fluctuation spectrum mainly consists of a broad high-energy peak. At large enough wave vectors the peak is dispersed out of the coherent p-h continuum and broadens due to high energy ($\sim t$) transitions involving the incoherent band of one-particle excitations.

We also discuss the optical conductivity $\sigma(\omega)$. For low frequencies we analyze $\sigma(\omega)$ in terms of the generalized Drude law. We show that there is a mass enhancement of order $m^*/m \simeq 6$, due to the electron scattering on spin fluctuations. These scattering processes also lead to a frequency-dependent relaxation rate which exhibits a crossover from $\omega^{3/2}$ behavior at low frequencies, $\omega < 2|\mu|$, to a linear $\omega$-dependence for $\omega > 2|\mu|$. A possible origin of the MIR band is also discussed.

The paper is organized as follows. In the next section, we give the basic definitions and sketch the memory function formalism. In Sec. III, we employ MCA to calculate a memory function. The dynamic charge susceptibility and optical conductivity are discussed in Secs. IV and V, respectively. The last section summarizes our main results.
II. MODEL AND MEMORY FUNCTION FORMALISM

The $t - J$ model expressed in terms of HO’s, $X_{i}^{\alpha \beta} = |i, \alpha \rangle \langle i, \beta|$, reads as

$$
H = H_t + H_J = - \sum_{i,j} t_{ij} X_i^{\sigma 0} X_j^{0 \sigma} + \frac{1}{4} \sum_{i,j,\sigma} J_{ij} \{ X_i^{\sigma \delta} X_j^{\delta \sigma} - X_i^{\sigma \sigma} X_j^{\delta \delta} \},
$$

(1)

where the indices 0 and $\sigma = \pm 1$ correspond to a hole and an electron with spin $\sigma/2$, respectively, $t_{ij} = t$ and $J_{ij} = J$ for the nearest-neighbor (n.n.) sites on a planar lattice. The HO’s can be either Bose-like or Fermi-like and obey the following on-site multiplication rules $X_{i}^{\alpha \beta} X_{i}^{\gamma \delta} = \delta_{\beta \gamma} X_{i}^{\alpha \delta}$ and the commutation relations

$$
[X_{i}^{\alpha \beta} X_{j}^{\gamma \delta}]_{\pm} = \delta_{ij} \left( \delta_{\beta \gamma} X_{i}^{\alpha \delta} \pm \delta_{\delta \alpha} X_{i}^{\beta \gamma} \right),
$$

(2)

where the upper sign stands for the case when both HO’s are Fermi-like, otherwise the lower sign should be adopted. In the $t - J$ model only singly occupied sites are retained and the completeness relation for the HO’s reads as

$$
X_{i}^{0 0} + \sum_{\sigma} X_{i}^{\sigma \sigma} = 1.
$$

(3)

The spin and density operators are expressed by HO’s as

$$
S_{i}^{\sigma} = X_{i}^{\sigma \sigma}, \quad S_{i}^{z} = \frac{1}{2} \sum_{\sigma} \sigma X_{i}^{\sigma \sigma}, \quad n_{i} = \sum_{\sigma} X_{i}^{\sigma \sigma}.
$$

(4)

The dynamic charge susceptibility $N_{q}(\omega)$ is given by a Fourier transformed two-time retarded Green function (GF)[1]

$$
N_{q}(\omega) = -\langle \langle n_{q} | n_{-q} \rangle \rangle_{\omega} = i \int_{0}^{\infty} dt e^{i\omega t} \langle [n_{q}(t), n_{-q}] \rangle.
$$

(5)

To calculate $N_{q}(\omega)$, we employ the memory formalism as discussed in Refs. [12] [14]. First we introduce density-density relaxation function

$$
\Phi_{q}(\omega) = \langle \langle n_{q} | n_{-q} \rangle \rangle_{\omega} = -i \int_{0}^{\infty} dt e^{i\omega t} \langle n_{q}(t) | n_{-q} \rangle,
$$

(6)
where the Kubo-Mori scalar product is defined as

$$(A(t), B) = \int_0^\beta d\lambda \langle A(t - i\lambda) B \rangle,$$  \hspace{1cm} (7)$$

with $\beta = 1/T$. The DCS $N_q(\omega)$ is coupled to the relaxation function $\Phi_q(\omega)$ by the equation

$$N_q(\omega) = N_q - \omega \Phi_q(\omega),$$  \hspace{1cm} (8)$$

where $N_q = N_q(0)$ is the static susceptibility.

We introduce the memory function $M^0_q(\omega)$ for the relaxation function $\Phi_q(\omega)$ as

$$\Phi_q(\omega) = \frac{N_q}{\omega - M^0_q(\omega)/N_q}.$$  \hspace{1cm} (9)$$

By adopting the equation of motion method for the relaxation function $\Phi_q(\omega)$ one finds that the memory function $M^0_q(\omega)$ is given by the irreducible part of the relaxation function for "currents"\cite{15,16}

$$M^0_q(\omega) = \langle (j_q|j_{-q}) \rangle^{irr}_\omega.$$  \hspace{1cm} (10)$$

The "current" operator $j_q$ in the site representation reads as

$$j_i = \dot{n}_i = -i[n_i, H] = -i \sum_{j,\sigma} t_{ij} (X^{\sigma 0}_j X^{0\sigma}_i - \text{H.c.}).$$  \hspace{1cm} (11)$$

The Heisenberg part of the Hamiltonian $H$ conserves the local particle number and thus gives no contribution to the "current" operator $\langle j_q|j_{-q} \rangle$. To treat properly a contribution from the $H_J$ term, we go one step further and similarly to Eq.(9) we introduce the memory function $M_q(\omega)$ for the relaxation function for "currents" $M^0_q(\omega)$ \cite{11}, by the following equation

$$M^0_q(\omega) = \frac{m_q}{\omega - M_q(\omega)/m_q},$$  \hspace{1cm} (12)$$

where

$$m_q = -\langle (j_q|j_{-q}) \rangle_{\omega=0} = i\langle [j_q, n_{-q}] \rangle.$$  \hspace{1cm} (13)$$
is the first moment of DCS and the memory function $M_q(\omega)$ is given by the irreducible part of the relaxation function for “forces”:

$$M_q(\omega) = ((F_q|F_{-q}))^{\text{irr}}_{\omega}$$  \hspace{1cm} (14)

with

$$F_q = \dot{j}_q = -i[j_q,H].$$  \hspace{1cm} (15)

Further, to close the system of equations [see Eqs. (8), (9), and (12)], we employ a mode coupling approximation for the memory function $M_q(\omega)$.

**III. MODE COUPLING APPROXIMATION**

First we express the memory function in terms of the irreducible part of time-dependent correlation function for “forces” by means of the fluctuation-dissipation theorem

$$M_q(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi \omega'} \frac{e^{i\omega'} - 1}{\omega - \omega' + i\eta} \times \int_{-\infty}^{\infty} dt e^{-i\omega't} \langle F_{-q}(t)|F_q \rangle^{\text{irr}}.$$  \hspace{1cm} (16)

The “force” operator is given by

$$F_i = \sum_{m,j,\sigma\sigma'} \left[ \Pi_{i,m,j}^{\sigma\sigma'} - \Pi_{m,i,j}^{\sigma\sigma'} + \text{H.c.} \right],$$  \hspace{1cm} (17)

$$\Pi_{m,i,j}^{\sigma\sigma'} = t_{im} \left[ t_{mj}(X_i^{\sigma0} X_j^{0\sigma'} \delta_{\sigma\sigma'} - X_i^{\sigma0} X_j^{0\sigma'} B_{m}^{\sigma\sigma'}) + J_{mj} X_i^{\sigma0} X_{m}^{0\sigma'} B_{j}^{\sigma\sigma'} \right],$$  \hspace{1cm} (18)

where the Bose-like operator

$$B_i^{\sigma\sigma'} = X_i^{\sigma\bar{\sigma}} \delta_{\sigma\sigma'} - X_i^{\bar{\sigma}\sigma} \delta_{\sigma\sigma'}$$

$$= \left[ \frac{1}{2} n_i - \sigma S_i^z \right] \delta_{\sigma\sigma'} - S_i^z \delta_{\sigma\sigma'}$$  \hspace{1cm} (19)

describes electron scattering on spin and charge fluctuations.
The sum in Eq. (17) contains the products of HO’s from the same site. As follows, such products give no contribution to the memory function, while being decoupled they do contribute. That is a result of the complexity of HO’s algebra (2). To show this, let us consider the term given by Eq. (18); since \((i, m)\) and \((m, j)\) are n.n. pairs \(i \neq m\) and \(m \neq j\), however \(i\) can be equal to \(j\). For the latter case, \(i = j\), the first term in Eq. (18) is linear in the density operator \(X_{0i}^{0\sigma} X_{i\sigma}^{0}\) and thus gives no contribution to the irreducible part of the correlation function for “forces”.16 As for the second term (18), one can easily verify that in the case \(i = j\) it is canceled out by its counter part from the sum (17). Finally, the last term in Eq. (18) vanishes for \(i = j\) since \(X_{i}^{0\sigma} B_{i}^{\sigma'\sigma} = 0\) due to the constraint. Therefore we have to subtract these terms from the “force” operator. As a result, we come to the following expression in the momentum space

\[
F_{q} = -\frac{1}{\sqrt{N}} \sum_{k,\sigma} G_{k,q} X_{k+q}^{0\sigma} X_{k}^{0\sigma} - \frac{1}{N} \sum_{k,p,\sigma,\sigma'} M_{k,q,p} X_{k+q-p}^{0\sigma} X_{k}^{0\sigma'} B_{p}^{\sigma'\sigma},
\]

(20)

the vertices \(G_{k,q}\) and \(M_{k,q,p}\) are given by

\[
G_{k,q} = g_{k,q} - \overline{g}_{k,q} \tag{21}
\]

\[
M_{k,q,p} = \sum_{i} [m^{(i)}_{k,q,p} - \overline{m}^{(i)}_{k,q,p}] \tag{22}
\]

where

\[
g_{k,q} = (zt)^{2} \gamma_{k,q}^{2}, \quad m^{(1)}_{k,q,p} = \frac{zt}{2} J \gamma_{p}[\gamma_{k-p,q} - \gamma_{k,q}]
\]

\[
m^{(2)}_{k,q,p} = (zt)^{2}[\gamma_{k,q}\gamma_{k+q-p} - \gamma_{k-p,q}\gamma_{k}]. \tag{23}
\]

In Eqs. (21) and (22) \(\overline{g}(\overline{m})\) denote \(g(m)\) averaged over the Brillouin zone and are given by

\[
\overline{g}_{k,q} = 2zt^{2}(1 - \gamma_{q}), \quad \overline{m}^{(1)}_{k,q,p} = \frac{zt}{2} J \gamma_{k,q} - \gamma_{k-p,q} \gamma_{k,q}
\]

\[
\overline{m}^{(2)}_{k,q,p} = 2zt^{2}\gamma_{p-q,q}, \tag{24}
\]

where \(\gamma_{k,q} = \gamma_{k+q} - \gamma_{k}, \gamma_{q} = 1/2[\cos(q_{x}) + \cos(q_{y})]\) and \(z = 4\) for a 2-dimensional square lattice. This form of the renormalized vertices (21)-(22) insures that all the operators in the
products of Eq. (20) are from different sites. Therefore HOs can be simply permuted within the decoupling procedure.

To calculate the irreducible part of the time-dependent correlation function in the right-hand side of Eq. (16), we apply the mode-coupling approximation in terms of an independent propagation of dressed p-h pairs and charge-spin fluctuations. The proposed approximation is defined by the following decoupling of the time-dependent correlation functions

$$
\langle X_{k-q}^{\sigma_0}(t)X_{k}^{0\sigma}(t)|X_{k'}^{0\sigma'}(t)X_{k'+q}^{\sigma'}\rangle \simeq
\delta_{\sigma,\sigma'}\delta_{k-q,k'}\langle X_{k-q}^{\sigma_0}(t)X_{k'}^{0\sigma}(t)\rangle\langle X_{k}^{0\sigma}(t)X_{k'+q}^{\sigma'}\rangle,
$$

(25)

$$
\langle X_{k-q-p}^{\sigma_0}(t)X_{k}^{0\sigma}(t)B_{p}^{\sigma'}(t)|X_{k'-q-p'}^{0\sigma'}X_{k'+q-p'}^{\sigma'}B_{p'}^{\sigma'}\rangle
\simeq
\delta_{\sigma,\sigma'}\delta_{\sigma',\sigma''}\delta_{k-q-p,k'}\delta_{p,-p'}\delta_{k'-q-p',k''}.
$$

(26)

By using the decoupling scheme and the spectral representation for the two-time retarded GF's we obtain for the memory function

$$
M_{q}(\omega) = \frac{1}{\omega}[\Pi(q, \omega) - \Pi(q, 0)],
$$

$$
\Pi(q, \omega) = \Pi_{1}(q, \omega) + \Pi_{2}(q, \omega),
$$

(27)

where $$\Pi_{1}(q, \omega)$$ and $$\Pi_{2}(q, \omega)$$ stem from the first and the second term of Eq. (20), respectively. Their imaginary parts are given by

$$
\Pi_{1}''(q, \omega) = \frac{-2\pi}{N}\sum_{k}G_{k,q}^{2}\int_{-\infty}^{\infty} d\omega_{1} n_{\omega_{1},\omega} \times A_{k}(\omega_{1})A_{k+q}(\omega_{1} + \omega),
$$

(28)

$$
\Pi_{2}''(q, \omega) = \frac{-2\pi}{N^{2}}\sum_{k,p}M_{k,q,p}^{2}\int_{-\infty}^{\infty} d\omega_{1} d\omega_{2} N_{\omega_{1},\omega_{2}} \times A_{k+q-p}(\omega - \omega_{1} + \omega_{2})A_{k}(\omega_{2})\chi''_{cs}(p, \omega_{1}),
$$

(29)

where $$n_{\omega_{1},\omega} = n(\omega_{1}) - n(\omega_{1} + \omega)$$, $$N_{\omega_{1},\omega_{2}} = [1 + N(\omega_{1}) + N(\omega - \omega_{1})]n_{\omega_{2},\omega - \omega_{1}}$$ with $$n(\omega)$$ and $$N(\omega)$$ being Fermi and Bose distribution functions, respectively, and
A_k(\omega) = -\frac{1}{\pi} \text{Im} \langle X^{0\sigma}_q \mid X^{\sigma 0}_q \rangle_\omega, \tag{30}

is a single-particle spectral function which does not depend on spin \(\sigma\) in the paramagnetic state. We have also introduced the unified spin-charge fluctuation spectrum

$$\chi''_{cs}(q, \omega) = \frac{1}{4\pi} \sum_p \chi''_p(\omega) + \chi''_q(\omega), \tag{31}$$

with

$$\chi_q(\omega) = -\frac{1}{\pi} \langle \langle S_q \mid S_{-q} \rangle \rangle_\omega \tag{32}$$

being the dynamic spin susceptibility. In obtaining Eq.(29) we have also used the identity \(\langle \langle S^{\sigma}_q \mid S^{\sigma}_{-q} \rangle \rangle_\omega = 2\langle \langle S^z_q \mid S^z_{-q} \rangle \rangle_\omega\) which holds in the paramagnetic state.

As it is clear from Eq.(27) the memory function involves two contributions; the first one (28) stems from the \(H_t\) term (1) and describes the p-h contribution, while the second one comes both from \(H_t\) and \(H_J\) parts and involves electron scattering on spin and charge fluctuations. Further we assume that the charge carriers are mainly relaxed by the scattering on spin fluctuations and retain only the second term \(\chi''_q(\omega)\) in Eq.(31).

For further discussion it is more convenient to integrate out the fermionic degrees of freedom \((k, \omega_2)\) in \(\Pi''_2(q, \omega)\) (29), which results

$$\Pi''_2(q, \omega) = \frac{1}{N} \sum_p \int_{-\infty}^{\infty} d\omega_1 \left[1 + N_{\omega_1} + N_{\omega - \omega_1}\right] \times \tilde{\Pi}''_{q-p}(\omega - \omega_1) \chi''_p(\omega_1), \tag{33}$$

where we have introduced an effective spectral function

$$\tilde{\Pi}''_{q-p}(\omega) = \frac{-2\pi}{N} \sum_k M^2_{k, q-p} \int_{-\infty}^{\infty} d\omega_1 n_{\omega_1, \omega} \times A_{k+q-p}(\omega + \omega_1) A_k(\omega), \tag{34}$$

for particle-hole excitations coupled to a particular \((p, \omega)\) state of spin fluctuations.

To conclude the section, we calculate the first moment of DCS (13). By performing the commutation between the density and “current” (11) operators we readily get
\[ m_q = 4ztN_1(1 - \gamma_q), \tag{35} \]

where

\[ N_m = \frac{1}{N} \sum_q \gamma_q^m \langle X_q^{\sigma_0} X_q^{\sigma_0} \rangle \tag{36} \]

is the p-h correlation function.

\section*{IV. DYNAMIC CHARGE SUSCEPTIBILITY}

Equations (8), (9), and (12) result in the following form of DCS

\[ N_q(\omega) = -\frac{m_q}{\omega^2 - [\Pi(q, \omega) - \Pi(q, 0)]} / m_q - \Omega_q^2, \tag{37} \]

where \( m_q \) is the first moment of DCS given by Eq. (35), and \( \Omega_q^2 = m_q / N_q \) is a mean field (MF) spectrum for the density fluctuations. The memory function formalism does not provide itself the static susceptibility \( N_q \). The latter is calculated within the same approximation scheme as for the static spin susceptibility,\(^{17}\) which results in the following form of the MF spectrum

\[ \Omega_q^2 = 2z^2t^2C_q(1 - \gamma_q) \tag{38} \]

where

\[ C_q = \frac{1}{z}(1 - \frac{n}{2}) + N_2 - \frac{J}{2zt}N_1(1 + z\gamma_q) \tag{39} \]

and the parameters \( N_m \) are defined by Eq. (36). Here we note that the MF spectrum \( \Omega_q \) resembles the dispersion of an undamped collective mode in the charge channel found in the leading order of \( 1/N \) expansion.\(^3\)

In a proper analysis, Eqs. (27)-(29) and (37) should be treated self-consistently with the equations for the single-particle spectral function \( A_k(\omega) \)\(^3\) and the spin susceptibility \( \chi_q(\omega) \).\(^3\) This problem, to our knowledge, can be solved only numerically. Here we show, that the main features of charge fluctuation spectrum observed in the exact diagonalization
studies can be, at least qualitatively, reproduced in an analytic way based on the physically justified anzats for the single–particle spectral function and the dynamic spin susceptibility. First, we discuss the one-particle spectral function $A_k(\omega)$.

Actually, the spectral characteristics of the $t–J$ model have been investigated by various analytic and numerical approaches. Those results led to the consensus that the single–particle spectrum involves a narrow quasiparticle (QP) band of coherent states and a broad continuum of incoherent states. The corresponding spectral function can be represented as

$$A_k(\omega) = A_k^{\text{coh}}(\omega) + A_k^{\text{inc}}(\omega),$$

(40)

where the QP part is given by

$$A_k^{\text{coh}}(\omega) = Z_k\delta(\omega - \epsilon_k),$$

(41)

with $Z_k$ and $\epsilon_k = \varepsilon_k - \mu$ being the QP weight and dispersion referred to the chemical potential $\mu$, respectively. While the incoherent part $A_k^{\text{inc}}(\omega)$ is little affected by the doping, the coherent band structure strongly depends on the magnetic background. Namely, in the low doping regime (ordered phase) the QP dispersion is determined by the hopping within a given AFM sublattice and the Fermi surface (FS) consists of small hole–pockets centered around $(\pm \pi/2, \pm \pi/2)$. While for a moderate doping (paramagnetic state) the dispersion reflects the dominance of n.n. hopping, there exists large electronic FS which encloses a fraction of the Brillouin zone equal to the electron concentration $n$. For the latter doping regime, the exact diagonalization results are well fitted by a simple tight-binding dispersion with some effective hopping amplitude $t_{\text{eff}}$ which scales with $J$ (for $\delta = 0.1$ and $J = 0.4t$ $t_{\text{eff}} = 0.24t$). Hence, in the paramagnetic phase we can put $\varepsilon_k = -zt_{\text{eff}}\gamma_k$.

A nearly structureless incoherent part is predominantly distributed below the QP band (in the electronic picture) and we approximate $A_k^{\text{inc}}(\omega)$ as follows

$$A_k^{\text{inc}}(\omega') = \frac{1}{\Gamma}\theta(-\omega')\theta(W_{\text{inc}} + \omega'),$$

(42)

where $W_{\text{inc}} \approx 5t$ is the incoherent bandwidth and $\omega'$ is measured from the bottom of the QP.
band. The spectral weights of the incoherent band $\Gamma$ and QP $Z_k$ are provided by the sum rules

$$\frac{1}{N} \sum_{k} \int_{-\infty}^{\infty} d\omega A_k(\omega) = 1 - \frac{n}{2},$$

$$\frac{1}{N} \sum_{k,\sigma} \int_{-\infty}^{0} d\omega A_k(\omega) = n,$$

(43)

and are given by

$$Z = \frac{2\delta}{1+\delta}, \quad \Gamma = \frac{2(1+\delta)}{(1-\delta)^2} W_{\text{inc}},$$

(44)

where $Z = \langle Z_k \rangle$ is the averaged QP weight and it coincides with that obtained within the Gutzwiller approximation.\[21\]

With the above form of the spectral function (40)-(42), the equal-time correlation functions $N_{1,2}$ (39) are estimated to be

$$N_1 \approx \frac{2}{\pi^2} nZ, \quad N_2 \approx \frac{n}{2z}$$

(45)

Finally, we assume that the spin susceptibility $\chi_q(\omega)$ (32) is peaked at the AFM wave vector $Q = (\pi, \pi)$.\[22\] Then, at $T = 0$, we approximate $\Pi''_2(q, \omega)$ (33) as follows

$$\Pi''_2(q, \omega) \approx \int_0^\omega d\omega_1 \Pi''_{q-Q}(\omega - \omega_1) \chi''(\omega_1),$$

(46)

where

$$\chi(\omega) = \frac{1}{N} \sum_p \chi''_p(\omega)$$

(47)

is the local spin susceptibility. Since the detailed form of $\chi(\omega)$ is not essential for our study, we use the following relaxation-type susceptibility

$$\chi(\omega) = \frac{\chi}{1 - i\omega/\omega_0}$$

(48)

which seems to be in agreement with the exact diagonalization data.\[23\] In Eq.(48) $\omega_0 \propto J$ is the energy scale of spin fluctuations and $\chi$ is the static susceptibility provided by the sum rule
\[ \int_0^\omega d\omega \chi(\omega) = \frac{3}{4} n, \]  

(49)

with the high-energy cutoff \( \omega_c = 2J \). Equations (48) and (49) result in

\[ \chi = \frac{3n}{2\omega_0 \ln[1 + (\omega_c/\omega_0)^2]} . \]  

(50)

A. Long–wavelength limit

First, we discuss a small \( q, \omega \) limit of DCS. In the long wave-length limit, \( q \to 0 \), the MF spectrum (38) reduces to the sound mode \( \Omega_q = v_s q \) with the sound velocity \( v_s = zt\sqrt{C_0/2} \) larger than the Fermi velocity \( v_F(\mu) = zt_{\text{eff}}\sqrt{(1 - \tilde{\mu}^2)/2} \) ( \( \tilde{\mu} = |\mu|/zt_{\text{eff}} \approx \pi\delta/4 \) for small \( \delta \)).

At small \( q \) the vertex functions (21)-(24) are given by

\[ G_{k,q} = 2t^2[2z(\hat{q}\nabla_k \gamma_k)^2 - 1]q^2, \]  

(51)

\[ M_{k,Q,p} = z(z - 1)tJ(\hat{q}\nabla_k \gamma_k)q. \]  

(52)

with \( \hat{q} = q/q \).

First we consider the real part of “self–energy” \( \Pi(q, \omega) \) (27)–(29). Since for small \( q \), the vertex functions \( G_{k,q} \sim q^2 \) and \( M_{k,Q,p} \sim q \), we approximate \( \Pi'(q, \omega) \simeq \Pi'_2(q, \omega) \) to keep the contributions leading in small \( q \). Moreover, for small \( \omega \) we expand \( \Pi'_2(q, \omega) \) as

\[ \Pi'_2(q, \omega) \simeq \Pi'_2(q, 0) - \alpha_q \omega^2, \]  

(53)

where \( \alpha_q > 0 \) and is given by

\[ \alpha_q = -\frac{1}{2} \left. \frac{d^2\Pi'_2(q, \omega)}{d\omega^2} \right|_{\omega=0} = -\frac{1}{\pi} \left. \int_{-\infty}^{\omega} \frac{\Pi'_2(q, \omega)}{\omega^3} d\omega, \right. \]  

(54)

Since \( \Pi''_2(q, \omega) \) (40) is an odd function of \( \omega \), there is no term linear in \( \omega \) in the expansion (53). Equations (37) and (53) result in the following form of DCS for small \( q, \omega \)

\[ N_q(\omega) \simeq \frac{-m_q/(1 + \lambda)}{\omega^2 - v_s^2 q^2 + 2i\omega\Gamma_q}, \]  

(55)

where
\[
\lambda = \lim_{q \to 0} \frac{c_q}{m_q}, \quad \tilde{v}_s = \frac{v_s}{\sqrt{1 + \lambda}}, \quad \Gamma_q = \frac{-\Pi''(q, \omega)}{2m_q(1 + \lambda)\omega},
\]

(56)

\(\tilde{v}_s\) and \(\Gamma_q\) are the renormalized sound velocity and sound damping, respectively.

First, we consider renormalization factor \(\lambda\). The estimation of \(\lambda\) is interesting by itself, since it represents the electron mass enhancement factor [see Sec. V] and can be related to experiment.\(^2\) Approximation (40) for the one-particle spectral function \(A_k(\omega)\) leads to two different contributions to the effective spectral function for p-h excitations \(\tilde{\Pi}''_{q-Q}(\omega)\) (51). The first one \(\tilde{\Pi}''_{q-Q}(\omega)^{c\_c}\) is due to the transitions within the QP band and the remaining part \(\tilde{\Pi}''_{q-Q}(\omega)^{i\_c}\) is provided by the incoherent-coherent transitions.

First, considering \(\tilde{\Pi}''_{q-p}(\omega)^{c\_c}\) (54), (51) we come to the following expression for small \(q\)

\[
\tilde{\Pi}''_{q-Q}(\omega)^{c\_c} = \frac{-2\pi \Lambda Z^2}{N} q^2 \sum_k (q\nabla_k \gamma_k)^2 \times [n(\epsilon_k) - n(\epsilon_k + \omega)]\delta(\omega - \epsilon_{k+q-Q} + \epsilon_k),
\]

(57)

with \(\Lambda = [z(z - 1)tJ]^2\). As it has been previously discussed by several authors,\(^2\) the p-h spectral function for tight-binding electrons exhibits a crossover at frequency \(\omega = 2|\mu|\). Namely, for \(\omega < 2|\mu|\) it is peaked at the incommensurate wave vectors \(Q_\ast = (\pi \pm \delta^\ast, \pi), (\pi, \pi \pm \delta^\ast)\) where the displacement \(\delta^\ast\) for small \(\mu\) is given by \(\delta^\ast \simeq \mu/\text{t}_{\text{eff}}\), whereas for \(\omega > 2|\mu|\) the p-h spectral function gets its maximum value at AFM wave vector \(Q\) and follows the nested Fermi liquid scaling.\(^2\) Hence we consider the cases \(\omega < 2|\mu|\) and \(\omega > 2|\mu|\) separately.

In the case \(\omega < 2|\mu|\) we put \(\epsilon_{k+q-Q} = \epsilon_{k-Q_\ast}\) in Eq.(57) and for \(\delta^\ast \ll 1\) expand \(\epsilon_{k+q_\ast} \simeq -\epsilon_k - \delta^\ast \partial\epsilon_k/\partial k_x\). Moreover, since the dominant contribution to the integral in Eq.(54) comes from \(\omega \sim 0\), at \(T = 0\) we approximate \(n(\epsilon_k) - n(\epsilon_{k+\omega}) \simeq \omega \delta(\epsilon_k)\) and, as a result, we get

\[
\tilde{\Pi}''_{q-Q}(\omega)^{c\_c} = \frac{-2\pi \Lambda t Z^2}{W_{\text{coh}}^2|\mu|} q^2 \omega I_1(\omega) \theta(2|\mu| - \omega),
\]

(58)

where \(W_{\text{coh}} = 2zt_{\text{eff}}\) is the coherent bandwidth, and

\[
I_1(\omega) = \frac{2\sin^2 k_x^\ast}{\pi^2 |\cos k_y^\ast \sin k_x^\ast|}
\]

with \(\sin k_x^\ast = 1 - \omega/(2|\mu|)\) and \(\cos k_y^\ast = 2\tilde{\mu} - \cos k_x^\ast\). For \(\omega \ll 2|\mu|\) we have \(I_1(\omega) \simeq 2\sqrt{\omega/|\mu|}/\pi^2\) and thus
\( \tilde{\Pi}''_{\mathbf{q}-\mathbf{Q}}(\omega)^{c-c} \approx \frac{-4\Lambda t Z^2}{\pi W_{\text{coh}}^2} \sqrt{\frac{\omega}{|\mu|}} q^2. \) \hspace{1cm} (59)

Then from Eqs. (46) and (59) for the velocity renormalization factor \( \lambda \) (56) we get

\[ \lambda_{1}^{c-c} \approx \frac{\sqrt{2} \Lambda Z^2 \chi}{15 \pi^2 z t_{\text{eff}}^2 N_1 \omega_0} \]

(60)

where the QP weight \( Z \) and n.n p-h correlator \( N_1 \) are given by Eqs. (44) and (45). For the actual values of the parameters \( J = 0.4t \), \( t_{\text{eff}} = 0.24t \), and \( \delta = 0.1 \) we obtain \( \lambda_{1}^{c-c} \approx 4 \).

Next we consider the case \( \omega > 2|\mu| \). In this case the p-h spectral function is peaked at the AFM wave vector \( Q \) and from Eq.(57) we come to

\[ \tilde{\Pi}''_{\mathbf{q}-\mathbf{Q}}(\omega)^{c-c} = \frac{-\pi \Lambda Z^2}{W_{\text{coh}}} q^2 I(\tilde{\omega}) \theta(\tilde{\omega} - 2\tilde{\mu}) \theta(2 - \tilde{\omega}), \]

(61)

where \( \tilde{\omega} = 2\omega/W_{\text{coh}} \) and

\[ I(\tilde{\omega}) = \frac{1}{\pi^2} [\tilde{\omega}_+ E(\tilde{\omega}_+/\tilde{\omega}_+) - 2\tilde{\omega} K(\tilde{\omega}_+/\tilde{\omega}_+)] \]

(62)

with \( \tilde{\omega}_\pm = 2 \pm \tilde{\omega} \), \( K(x) \) and \( E(x) \) are the complete elliptic integrals of the first and second kind, respectively. The function \( I(x) \) (62) is normalized to 1/4 in the interval \( 0 < x < 2 \) and well approximated by the following linear dependence

\[ I(x) = (W_{\text{coh}} - \omega) \theta(\omega - 2|\mu|) \theta(W_{\text{coh}} - \omega). \]

(63)

That results in the following contribution to \( \lambda \)

\[ \lambda_{2}^{c-c} \approx \frac{2(z - 1)^2 t J Z^2}{N_1 W_{\text{coh}}^2} \tilde{I}, \]

(64)

with

\[ \tilde{I} = \int_0^\infty \int_0^x dxdy \frac{\tilde{f}(x-y)}{x^3} \tilde{\chi}(y) \theta(2 - y) \]

(65)

where the dimensionless functions \( \tilde{f}(x) \) and \( \tilde{\chi}(y) \) stand for \( f(\omega) \) (63) and spin susceptibility \( \chi(\omega) \) (18) measured in units of \( J \). For \( J = 0.4t \) and \( \delta = 0.1 \) we calculated integral (65) numerically and found \( \tilde{I} = 0.5 \). Then from Eq.(64) we estimate \( \lambda_{2}^{c-c} \approx 0.8 \).
As for the remaining contribution $\Pi''_{q-Q}(\omega)^{i-c}$, with the help of Eqs. (34), (42), and (52) at $T = 0$ we obtain

$$\Pi''_{q-Q}(\omega)^{i-c} \simeq \frac{-\pi \Lambda Z}{4z \Gamma} q^2 \times \theta(2\omega - W_{coh} + 2|\mu|) \theta(W_{tot} - \omega),$$  

(66)

where $W_{tot} = W_{coh} + W_{inc}$ is the total bandwidth. From Eq. (56) and by using the sum rule (49), the upper value of the incoherent-coherent contributions to $\Pi''_{2}(q, \omega)^{i-c}$ is estimated as

$$\Pi''_{2}(q, \omega)^{i-c} \simeq \frac{3}{4} n \Pi''_{q-Q}(\omega)^{i-c}.$$  

(67)

That results in

$$\lambda^{i-c} \simeq \frac{3n(z-1)^2 Z t J^2}{16 \Gamma N_1 W_{coh}^2} \left[ 4 - \frac{W_{coh}^2}{W_{tot}^2} \right] \simeq 0.1.$$  

(68)

A small value of $\lambda^{i-c} \simeq 0.1$ in comparison with $\lambda^{i-c} \simeq 4$ is due to a large threshold energy $(W_{coh}/2 - |\mu| \simeq t)$ (63) for incoherent-coherent transitions which are important only in describing high-energy density fluctuations.

Finally, by summing all three contributions (12), (66), and (70) for the velocity renormalization factor we get $\lambda \simeq 5$. Here we notice, that in the Fermi liquid theory, for $v_s > v_F$ "self-energy" corrections stiffen the sound. Contrary to this, in the present case the sound softens. That is due to the scattering on spin fluctuations given by the $\Pi''_{2}(q, \omega)$ term (46). One can easily verify that the renormalized sound velocity (50) gets smaller than the Fermi one $\bar{v}_s = v_s/\sqrt{1 + \lambda} < v_F$. It falls down into a particle-hole continuum getting finite damping due to the decay into particle-hole pairs. This process is described by $\Pi''_{1}(q, \omega)$ (28). The latter in the small $q, \omega$ limit reads as [see Eqs. (28) and (51)]

$$\Pi''_{1}(q, \omega) = -\frac{8\pi t^4 Z^2}{N} \omega q^3 \sum_k [2z(\hat{q} \nabla_k \gamma_k)^2 - 1]^2 \times \delta(\epsilon_k) \delta(\omega/q - \hat{q} v_k),$$  

(69)

where $v_k = \nabla_k \epsilon_k$ is the QP velocity. The integration over $k$ in Eq. (69) results in

$$\Pi''_{1}(q, \omega) = \frac{-\sqrt{2} t^4 Z^2}{\pi t^2 e_0^2 \sin \theta} \left[ 1 - \left( \frac{2 \omega}{q v_F} \right)^2 \right] \omega q^3,$$  

(70)
where \( \cos \theta = \left[ \mu^2 + 2 \omega^2/q^2 \right]/v_F^2 - 1 \). This gives the following form of the sound damping

\[
\Gamma_q \simeq \beta \tilde{v}_s q, \quad \beta = \frac{Z^2 t^3}{\pi z t_{\text{eff}} N_1 v_s^3},
\]

(71)

for actual values of the parameters \( \beta < 1 \), and thus, in accordance with Ref. 6, one obtains that the sound damping is only numerically smaller than its energy.

### B. Short–wavelength limit

At large momenta the main spectral weight of density fluctuations is located at high energies, \( (\sim t) \), near the MF spectrum \( \Omega_q \) (38). For instance, at \( q = Q \) we have \( \Omega_Q \simeq zt \) while in the exact diagonalization studies the peak is observed at \( \omega \simeq 6t \). However, considering “self-energy” corrections and noting that \( \Pi'_q(\omega) \) falls off as \( 1/\omega^2 \) at large frequencies, from Eq. (37) we obtain the renormalized spectrum as \( \tilde{\Omega}_q \simeq \sqrt{\Omega_q^2 - \Pi'_q(0)/m_q} \). One can easily show that \( \Pi'_q(0) < 0 \) and hence the spectrum is shifted to higher energies. At large momenta the peak is dispersed out of the coherent p-h continuum and its broadening is only due to high-energy transitions involving the incoherent band. Since the latter has been neglected in Ref. 5, the authors observed an infinitely sharp peak. However, as it follows, the damping of the high energy mode is comparable to its energy. Near the pole \( \tilde{\Omega}_q \) we estimate the damping as

\[
\tilde{\Gamma}_q = \frac{-\Pi''(q, \tilde{\Omega}_q)}{2 m_q \tilde{\Omega}_q},
\]

(72)

where \( \Pi''(q, \omega) = \Pi''_1(q, \omega)^{\text{i-c}} + \Pi''_2(q, \omega)^{\text{i-c}} \) describes incoherent-coherent transitions. From Eqs. (28) and (42) for \( q = Q \) we obtain

\[
\Pi''_1(Q, \omega)^{\text{i-c}} \simeq -\frac{5\pi (zt)^4 Z}{4 \Gamma}.
\]

(73)

The second contribution \( \Pi''_2(q, \omega)^{\text{i-c}} \) from Eqs. (14), (12), and (37) is estimated as

\[
\Pi''_2(Q, \omega)^{\text{i-c}} \simeq -\frac{6n \pi (zt)^4 Z}{5 \Gamma}.
\]

(74)

Equations (72), (73), and (74) result in
\[ \bar{\Gamma}_q \simeq \frac{2(25 + 24n)\pi t^3 Z}{5N_1 \Gamma \Omega_q} \sim 3t. \] (75)

Thus, the peak gets rather broad in accordance with the exact diagonalization results.

For large momenta but low energies, charge excitation spectrum should show some low energy structure related to the contribution from the p-h continuum to \( N''_q(\omega) \). Since \( \Omega_q \) is larger in \((\xi, \xi)\) direction than in \((\xi, 0)\), the low energy structure should be less pronounced in the latter case. The same anisotropy has been observed in the exact diagonalization studies.

V. OPTICAL CONDUCTIVITY

In this section, we discuss the optical conductivity \( \sigma(\omega) \). In the linear response theory of Kubo\(^{29}\), the frequency-dependent conductivity is given by the relaxation function for currents

\[ \sigma_{xx}(\omega) = \frac{ie^2}{V} (|J_x| J_x) \omega. \] (76)

By using the continuity equation and equation of motion for the GF’s, one can easily relate the longitudinal conductivity to the dynamic charge susceptibility (5)

\[ \sigma_{xx}(\omega) = -\frac{ie^2}{V} \lim_{q \to 0} \frac{\omega N_q(\omega)}{q^2}, \] (77)

where \( q = q_x \). From Eqs. (37) and (77) we express conductivity in terms of the memory function

\[ \sigma_{xx}(\omega) = \frac{ie^2}{V} \frac{D}{\omega - M(\omega)} \] (78)

where \( D = \lim_{q \to 0} m_q / q^2 = ztN_1 \) is the Drude weight which is given by one-half the averaged kinetic energy \( D = -\langle H_\ell \rangle / 2 \), \( m_q \) and \( N_1 \) are defined by Eqs. (35) and (36), respectively.

The memory function \( M(\omega) \) reads as

\[ M(\omega) = \frac{\Pi(\omega) - \Pi(0)}{\omega}, \] (79)
where

\[ \Pi(\omega) = \lim_{q \to 0} \frac{\Pi(q, \omega)}{D q^2}, \quad (80) \]

with \( \Pi(q, \omega) \) defined by Eq. (27). Since for small \( q \), \( \Pi_1(q, \omega) \sim q^4 \) and \( \Pi_2(q, \omega) \sim q^2 \), only the second one contributes to \( \Pi(\omega) \). The latter is given by Eqs. (33) and (34) at \( q = 0 \) with \( M_{k,q,p} \) replaced by the transport vertex given by

\[
M_{k,p} = \lim_{q \to 0} \frac{M_{k,q,p}}{q^2} = \left[ t_{k-p} v_k - t_k v_{k-p} - 2t v_p \right] \\
\quad + \frac{1}{2} [J + J_p] [v_{k-p} - v_k], \quad (81)\]

where \( t_k = z t \gamma_k \), \( J_k = z J \gamma_k \), and \( v_k = \partial t_k / \partial k_x \).

We rewrite conductivity (78) in the form of the generalized Drude law as follows

\[ \sigma_{xx}(\omega) = \frac{e^2}{V} \frac{\tilde{D}(\omega)}{1/\tilde{\tau}(\omega) - i\omega}, \quad (82) \]

where an effective Drude weight and the relaxation time are given by

\[
\tilde{D}(\omega) = \frac{D}{1 + \lambda(\omega)} \quad \frac{1}{\tilde{\tau}(\omega)} = \frac{1}{\tau(\omega)(1 + \lambda(\omega))}, \quad (83)\]

with

\[
\lambda(\omega) = -\frac{M'(\omega)}{\omega} \quad \frac{1}{\tau(\omega)} = -M''(\omega), \quad (84)\]

and \( 1 + \lambda(\omega) \) is the interaction-induced optical mass enhancement factor. The latter in the static limit is calculated in the preceding section and is estimated to be of order 6. That is in a good agreement with the optical measurement data.\cite{24} Optical conductivity of the \( t-J \) model within the present formalism was studied by one of us (N.P.) in Ref. \cite{14}, where temperature and frequency dependence of \( \sigma(\omega) \) where discussed. Here we mainly focus on the analysis of the low-frequency behavior of the relaxation rate

\[ \Gamma(\omega) = \frac{1}{\tilde{\tau}(\omega)} = -\frac{\Pi''(\omega)}{\omega}. \quad (85) \]

Following Sec. IV we approximate \( \Pi''(\omega) \) as
\[
\Pi''(\omega) \simeq \int_{0}^{\mu} d\omega_1 \chi''(\omega_1) \times \left\{ \begin{array}{ll}
\tilde{\Pi}''_{Q,\omega}(\omega - \omega_1) & \omega < 2|\mu| \\
\tilde{\Pi}''_{Q}(\omega - \omega_1) & \omega > 2|\mu| 
\end{array} \right.
\]

(86)

In the case \( \omega < 2|\mu| \), the effective spectral function of p-h excitations \( \tilde{\Pi}''_{Q,\omega} \) (59) is given by

\[
\tilde{\Pi}''_{Q,\omega}(\omega) \simeq -\frac{4\Lambda t Z^2}{\pi W_{coh}^2 D \sqrt{|\mu|}} \omega 
\]

(87)

We remark the square-root behavior of the p-h spectral function \( \Pi''_{Q,\omega} \sim \sqrt{\omega} \) instead of the conventional linear \( \omega \)-dependence. This behavior results in the square-root singularity of the structure factor that is known as \( 2k_F \) anomaly familiar for the electron system in low dimensions. It also leads to the deviation from the conventional square law resulting in the following form of the relaxation rate

\[
\Gamma(\omega) \simeq \frac{16\Lambda Z^2 t \chi}{15\pi D W_{coh}^2 \sqrt{|\mu|}} \omega^{3/2}.
\]

(88)

Here we note that the \( \omega^{3/2} \)-law of inverse life time for electron states near the saddle points has been obtained in Refs. [30,31]. In the present case, the Van Hove singularity plays no role. The obtained \( \omega^{3/2} \)-dependence of the relaxation time is rather due to the coexistence of the peak in the spin fluctuation spectrum and the \( 2k_F \) “anomaly” in the p-h spectral function at \( q \sim Q \). We notice that the former one, \( 2k_F \) “anomaly”, is not related to the FS topology and is inherent in a low-dimensional electron system.

Now we consider the region \( \omega > 2|\mu| \). In this case the p-h spectral function is peaked at the AFM wave vector and is almost \( \omega \)-independent for low frequencies \( \omega \ll W_{coh} \) (63), which results in

\[
\Gamma(\omega) \simeq \frac{\pi \Lambda Z^2 \chi}{8 W_{coh} D \omega_0} \omega.
\]

(89)

Unlike the previous case, now the electron band structure is mainly responsible for obtained behavior. Of course, the AFM character of spin fluctuations favors the scatter process with momentum transfer \( Q \) and, thus, enhances its contribution to the relaxation rate.
To summarize the low-energy behavior of optical conductivity, we have shown that the relaxation rate due to the electron scattering on spin fluctuations exhibits the crossover from the \( \Gamma(\omega) \sim \omega^{3/2} \) behavior at low frequencies \( \omega \ll 2|\mu| \) to a linear \( \omega \)-dependence at \( \omega > 2|\mu| \).

Now we discuss the conductivity at intermediate frequencies. The exact diagonalization studies of the latter quantity have suggested a possible explanation of the MIR absorption within the one-band model.\(^{19,32}\) For instance, as it has been observed in Ref. 32, the finite frequency part of \( \sigma(\omega) \) is dominated by a single excitation which scales with \( J \) in the underdoped regime.\(^{32}\) The origin of this excitation was ascribed to transitions in which internal degrees of freedom of the spin-bag QP are excited. The presence of extra absorption ranging from MIR frequency to \( \sim 1 \) eV was also observed in the \( 1/N \) expansion study of the \( t-J \) model.\(^{33}\) The authors of Ref. 33 interpreted this feature as being due to the incoherent motion of charge carriers. Since the existence of broad incoherent band in the density of states of charge carriers is due to internal degrees of freedom of the spin-bag QP it follows that the underlying physics of both these points are the same. Below we also support this explanation of the MIR band.

Actually, with increasing energy an extra channel of optical transitions opens. These are the transitions which involve an incoherent band of the single-particle spectral function. As we have already discussed, the incoherent–coherent transitions are characterized by the energy scale \( \Delta = W_{\text{coh}} - 2|\mu| \) being a threshold energy for creating “particle-hole” pairs with a “hole” in the incoherent band. Due to this extra channel at \( \omega > \Delta \), the real part of \( \sigma(\omega) \) starts to increase. Since \( \sigma(\omega) \) vanishes in the limit \( \omega \to \infty \), there should be a peak in conductivity at energies of order \( \Delta \). Since the coherent bandwidth \( W_{\text{coh}} \) (and hence \( \Delta \)) scales with \( J \) it follows that the typical energy of the peak is also \( J \) that coincides with the energy scale of fine structure in high-energy absorption found in Ref. 32.

To conclude this section, we discuss the doping dependence of the Drude and regular parts of the optical conductivity. The doping dependence of the renormalized Drude weight \( \tilde{D} \)\(^{33}\) is due to the \( \delta \)-dependence of both kinetic energy and mass enhancement factor. For small \( \delta \) the former one scales as \( |< H_t >| \sim Z N_1 \sim \delta \) and increases with doping.
While the mass enhancement factor, $1 + \lambda$, that is due to the electron scattering on spin fluctuations gets smaller upon doping since the system moves away from the AFM phase boundary. Combining both points we conclude that the Drude weight increases faster than the number of doped holes. Since the sum rule for the optical conductivity has to be fulfilled we expect transfer of the spectral weight from the regular to the Drude part of the conductivity. The loosing of the spectral weight by the mid infrared band is provided by the fact that the normalized rate of the incoherent-coherent transitions $\Pi''_1(\omega)^{ic}/D \sim Z/D\Gamma \sim (1 - \delta)$ decreases upon doping. This qualitative picture of doping dependence of $\sigma(\omega)$ is in general agreement with finite clusters calculations. To give some quantitative estimates on evaluation of $\sigma(\omega)$ with doping one needs a more detailed information concerning $\delta$-dependence of one-particle spectral characteristics as well as spin fluctuation spectrum used as the inputs in our theory. However, the presently available numerical data on the $\delta$-dependence of these quantities is far from convincing.

VI. CONCLUSION

To summarize, we have developed a self-consistent theory both for the dynamic charge susceptibility and the optical conductivity within the memory function formalism in terms of the Hubbard operators. In this framework the charge susceptibility has been expressed via the memory function $M_q(\omega)$, that is given by fluctuating force correlation function. The essential approximation of the preceding theory is the mode-coupling approximation. Within the MCA the memory function has been factorized in terms of single-particle GF and spin and charge susceptibilities. As a result a type of the “golden-rule formula” has been obtained: the density fluctuation created at some place and time will decay either into p-h pair excitations and/or p-h pairs and spin (charge) density fluctuations if the system evolves into the future. Though, at present time the precise range of validity of MCA can not be given the following general arguments can be used to justify it. First, being a self-consistent scheme of calculations MCA does not use any unphysical for strongly correlated
systems the zero-order GF and can be viewed as an analog of the non-crossing or the self-consistent Born approximation (SCBA) for the single-particle GF. However in the SCBA vertex corrections are neglected and only the skeleton loop diagrams are taken into account. While MCA correctly reproduces the back-scattering term in the impurity problem and thus partially accounts for vertex corrections [see Eqs. (21) and (22) for the present case].

Next, this approach allows to deal with constrained electron operators avoiding an auxiliary field representation that rises the problem of local constraint. Moreover, within our decoupling scheme the operators from the same site is never decoupled and thus the strong local correlations are retained. Finally, it is expected that there is no divergence in renormalized vertexes signaling the existence of a critical low energy mode and the vertex renormalizations that is not accounted by our theory can change only our numerical estimates but not the qualitative behavior of the susceptibility.

Our main findings are as follows. We have shown that in the long-wavelength limit the charge fluctuation spectrum is mainly governed by the sound mode \(\text{55}\). Although unrenormalized sound velocity is larger than the Fermi velocity, the “self-energy” corrections soften the sound. Sound falls down into the the particle-hole continuum and thus acquires a finite damping due to the decay into pair excitations. The sound damping \(\text{71}\) is only numerically smaller than its energy and hence there is no well-defined sound mode.

At large momenta the density fluctuation spectrum mainly consists of a broad high-energy peak which nearly follows the MF dispersion \(\text{38}\). At wave vectors large enough the peak is dispersed out of the coherent particle-hole continuum and its broadening \(\text{72}\) is due to the high energy \(\sim t\) transitions involving the incoherent band of the single–particle excitations.

We have also discussed the optical conductivity. At low frequencies we have analyzed \(\sigma(\omega)\) in terms of the generalized Drude law. We have shown that there is a large mass enhancement of order \(m^*/m \simeq 6\), due to the electron scattering on spin fluctuations. This scattering process also leads to the non-Drude fall-off of the low energy part of \(\sigma'(\omega)\). Namely, the relaxation rate shows a power law \(\omega\)–dependence with the exponent \(3/2\) at low frequencies.
\( \omega < 2|\mu| \) and it is linear in \( \omega \) at frequencies \( \omega > 2|\mu| \). As for the intermediate frequency conductivity, we have pointed out the existence of a characteristic energy \( \Delta \) (of order \( J \)) above which an extra channel of the optical transitions opens. These are the transitions in which particle-hole pairs with a “hole” in the incoherent band are excited and they might be responsible for the experimentally observed MIR absorption.

The obtained results are in good agreement with the exact diagonalization studies of small clusters.\(^7\)

In this paper we have focused on the moderate doping regime, when the Fermi surface is large and there exist a strong short-range AFM correlations in the system. Our assumption about the large electronic FS gives a lower boundary \( \delta_c \) of the range of validity of the present study with respect to the hole doping (\( \delta_c \) being the hole concentration when the topology of the FS changes, i.e. when a transition from large electronic to the hole pocket-like FS takes place when doping becomes smaller than \( \delta_c \))\(^3\). To discuss the low doping regime \( \delta < \delta_c \) one has to take into account the changes in the QP spectra. Here, we only point out, that for \( \delta < \delta_c \) the new low energy scale \( v_F \sim J\sqrt{\delta} \) appears in the charge dynamics. Due to this and the different topology of FS one can expect the different behavior of the low energy charge fluctuations that are driven by FS related particle-hole excitations. The existence of the upper boundary is due to the assumption on the presence of strong short range AFM correlations. When these correlations are already destroyed the assumption that the scattering on spin-fluctuations gives the main input to the charge relaxation is not valid anymore and in order to relate the theory with experiment one has to include also the scattering on other bosonic degrees of freedom. However, this doping regime is beyond the scope of the present consideration.

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