RYSER TYPE CONDITIONS FOR EXTENDING COLORINGS OF TRIPLE SYSTEMS

AMIN BAHMANIAN

Abstract. In 1951, Ryser showed that an \( n \times n \) array \( L \) whose top left \( r \times s \) subarray is filled with \( n \) different symbols, each occurring at most once in each row and at most once in each column, can be completed to a latin square of order \( n \) if and only if the number of occurrences of each symbol in \( L \) is at least \( r + s - n \). We prove a Ryser type result on extending partial coloring of 3-uniform hypergraphs. Let \( X,Y \) be finite sets with \( X \subseteq Y \) and \( |Y| \equiv 0 \pmod{3} \). When can we extend a (proper) coloring of \( \lambda(X) \) (all triples on a ground set \( X \), each one being repeated \( \lambda \) times) to a coloring of \( \lambda(Y) \) using the fewest possible number of colors? It is necessary that the number of triples of each color in \( \lambda(X) \) is at least \( |X| - 2|Y|/3 \). Using hypergraph detachments (Combin. Probab. Comput. 21 (2012), 483–495), we establish a necessary and sufficient condition in terms of list coloring complete multigraphs. Using Häggkvist-Janssen’s bound (Combin. Probab. Comput. 6 (1997), 295–313), we show that the number of triples of each color being at least \( |X|/2 - |Y|/6 \) is sufficient. Finally we prove an Evans type result by showing that if \( |Y| \geq 3|X| \), then any \( q \)-coloring of any subset of \( \lambda(X)^3 \) can be embedded into a \( \lambda(3)^2 \)-coloring of \( \lambda(Y)^3 \) as long as \( q \leq \lambda(3)^2 - \lambda(3)^3/|X|/3 \).

1. Introduction

Ryser showed that any \( r \times s \) latin rectangle \( L \) can be embedded into an \( n \times n \) latin square if and only if the number of occurrences of each symbol in \( L \) is at least \( r + s - n \) [40]. In other words, an \( n \)-coloring of the complete bipartite graph \( K_{r,s} \) can be extended into an \( n \)-coloring of \( K_{n,n} \) if and only if the number of edges of each color in \( K_{r,s} \) is at least \( r + s - n \). The main motivation of this note is to find a three-dimensional analogue of Ryser’s theorem for symmetric latin cubes for which there has been very little success over the last seventy years. To our best knowledge, our main result is the first result of this kind. The study of latin cubes was initiated in the 1940’s by Fisher [17] and Kishen [27, 28]. These objects are also closely related to orthogonal arrays [23, 26] which themselves have applications in statistics, coding theory, and cryptography [10].

Let \( L \) be an \( n \times n \times n \) array. A layer in \( L \) is obtained by fixing one coordinate. More precisely, for each \( i \in \{1, \ldots, n\} \), \( L_{j**} = \{L_{ijk} \mid 1 \leq j,k \leq n\} \), \( L_{s**} = \{L_{ijk} \mid 1 \leq j,k \leq n\} \), and \( L_{**k} = \{L_{jki} \mid 1 \leq j,k \leq n\} \). Then, \( L \) is a layer-equitable latin cube if it is filled with \( n \) different symbols such that each layer contains each symbol \( n \) times. A layer-equitable latin cube in which each layer is a latin square, is called a layer-latin latin cube. If \( L \) is filled with \( n^2 \) symbols such that in each layer every symbol occurs exactly once, then \( L \) is a layer-rainbow latin cube. There is no consensus on which of the above three notions should be called a latin cube, but most statisticians refer to layer-rainbow latin cubes as latin cubes, and most combinatorialists refer to layer-latin latin cubes as permutation cubes or latin cubes. For the sake of convenience, throughout the rest of this note we shall refer to layer-latin latin cubes as latin cubes.

Ryser’s theorem implies that every \( r \times n \) latin rectangle can be extended to an \( n \times n \) latin square [22]. This result is not true in the 3-dimensional case, as there are many examples of \( r \times n \times n \) partial...
latin cubes that cannot be completed to $n \times n \times n$ latin cubes [9, 31, 36]. There are only a few results on embedding partial latin cubes, and in fact, most of the current literature is focused on the cases where a partial latin cube cannot be completed. Cruse showed that a partial latin cube of order $n$ can be embedded into a latin cube of order $16n^4$ [12]; this result was extended to idempotent latin cubes by Lindner [35] and Csima [13]. Potapov improved Cruse’s bound to $n^3$ [39]. Denley and Öhman [14] found sufficient conditions for when certain $k \times l \times m$ partial Latin cubes, namely latin boxes (or latin parallelepipeds), can be extended to $k \times n \times m$ latin boxes, and subsequently, to $k \times n \times n$ latin boxes, though they were not able to extend in the third dimension to obtain latin cubes. For related embedding results in higher dimensions and connections with maximum distance separable codes, see Krotov and Sotnikova [33], and Potapov [37, 39]. Various results on latin cubes. For related embedding results in higher dimensions and connections with maximum

For further results on latin cubes, see [5, 15, 25, 38] and references therein. latin boxes that cannot be completed to latin cubes can be found in [7, 8, 18, 24, 30, 29, 31, 32, 34]. For further results on latin cubes, see [5, 15, 25, 38] and references therein.

Cruse settled conditions that ensure an $r \times r$ symmetric latin rectangle can be extended to a symmetric latin square [11] (see Theorem 1.1). We nearly extend this result to symmetric layer-rainbow latin cubes (see Theorem 1.4). A layer-rainbow latin cube $L$ of order $n$ is symmetric if it meets the following conditions.

1. $L_{ij\ell} = L_{i\ell j} = L_{\ell ij}$ for distinct $i, j, \ell \in \{1, \ldots, n\}$,
2. $L_{ij} = L_{jji}, L_{ij\ell} = L_{ijj}, L_{ijj} = L_{ji\ell}$ for $i, j \in \{1, \ldots, n\}$.

Here is an example of a symmetric layer-rainbow latin cube of order 5.

$$
\begin{array}{cccccc}
A & F & K & P & U & G \\
B & G & L & Q & V & F \\
C & H & M & R & W & L \\
D & I & N & S & X & T \\
E & J & O & T & Y & V \\
\end{array}
\begin{array}{cccccc}
S & Q & R & D & T & Y \\
I & A & Y & O & C & J \\
N & E & F & V & G & O \\
P & W & J & U & H & T \\
X & K & B & L & M & U \\
\end{array}
\begin{array}{cccccc}
M & L & C & N & O & E \\
H & U & T & E & D & A \\
K & X & P & J & A & B \\
R & Y & V & F & B & C \\
W & S & Q & G & I & F \\
\end{array}
$$

For $k \in \mathbb{N}$, and a finite set $X$, let $[k] = \{1, \ldots, k\}$, and let $\binom{X}{k}$ be the collection of all subsets of size $k$ of $X$. Let $G \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of $X$. We call $G$ a simple hypergraph and each element of $G$ is an edge. In a general hypergraph, we allow edges to be repeated and in addition we allow the vertices to be repeated within each edge. We use $\text{mult}_G(e)$ to denote the number of occurrences of $e$ in $G$. The degree of $x \in X$ in $G$, written $\text{deg}_G(x)$, is the number of occurrences of $x$ in $G$. For $\lambda \in \mathbb{N}$, $\lambda G$ is obtained by replacing each $e \in G$ by $\lambda$ copies of $e$. A $k$-coloring of $G$ is a partition of $G$ into color classes $G(1), \ldots, G(k)$ such that in each color class each pair of edges are disjoint. If each color class is a partition of $X$, then the coloring is a one-factorization.

We restate Cruse’s theorem as follows.

**Theorem 1.1.** [11, Theorem 1] A $|Y|$-coloring of $F := \binom{X}{2}$ can be extended to a one-factorization of $\binom{X}{2}$ if and only if $|Y|$ is even and

$$
|F(i)| \geq |X| - \frac{|Y|}{2} \quad \text{for } i = 1, \ldots, |Y|.
$$

Symmetric latin squares of order $|X|$ can be seen as one-factorizations of $\binom{X}{1} \cup \binom{X}{2}$, so to account for $\binom{X}{1}$, a further condition is needed (see [11]). Symmetric layer-rainbow latin cubes can be seen as one-factorizations of $\binom{X}{1} \cup 3\binom{X}{2} \cup 2\binom{X}{3}$ [2]. For example, consider the following one-factorization
F of \( \binom{X}{1} \cup 3(\binom{X}{3}) \cup 2(\binom{X}{5}) \) for \( X = \{1, 2, 3, 4, 5\} \) (We abbreviate sets such as \( \{x, y, z\} \) to \( xyz \)).

\[
\begin{align*}
\{1, 24, 35\}, \{12, 345\}, \{13, 245\}, \{14, 235\}, \{15, 234\}, \{5, 12, 34\}, \{12, 345\}, \\
\{12, 345\}, \{34, 125\}, \{14, 235\}, \{45, 123\}, \{2, 13, 45\}, \{15, 134\}, \{14, 235\}, \{3, 14, 25\}, \\
\{35, 124\}, \{25, 134\}, \{14, 235\}, \{23, 145\}, \{4, 15, 23\}, \{34, 125\}, \{24, 135\}, \\
\{23, 145\}, \{15, 234\}. 
\end{align*}
\]

Let us assign symbols \( A, B, \ldots, Y \) to the 26 color classes, respectively. Then we fill \( L_{ij} \) with the color of \( i \) in \( F \) for \( i \in \{1, \ldots, 5\} \). For distinct \( i, j \in \{1, \ldots, 5\} \), suppose the color of the three copies of \( \{ij\} \) in \( F \) are \( c \), \( c' \) and \( c'' \). We fill the cells \( L_{ii} \) and \( L_{ij} \) with \( c' \), and \( L_{ji} \) and \( L_{jj} \) with \( c'' \). For distinct \( i, j, \ell \in \{1, \ldots, 5\} \), suppose the color of the two copies of \( \{ij\ell\} \) are \( c \) and \( c' \). We fill the cells \( L_{ii} \), \( L_{ji} \) and \( L_{ij} \) with \( c, \) and \( L_{jj} \) and \( L_{jj} \) with \( c' \). The result is the symmetric layer-rainbow Latin cube of order 5 we mentioned earlier.

To investigate Ryser’s theorem for symmetric layer-rainbow Latin cubes we establish conditions that ensure a coloring of \( \lambda(\binom{X}{3}) \) can be extended to a one-factorization of \( \lambda(\binom{X}{3}) \) (see Theorem 1.4). First, let us recall a necessary condition.

**Lemma 1.2.** [1, Lemma 7.2] \textit{If a} \( \lambda(\binom{Y}{2}) \)-coloring of \( F := \lambda(\binom{X}{3}) \) \textit{is extended to a one-factorization of} \( \lambda(\binom{X}{3}) \), then

\[
|F(i)| \geq |X| - \frac{2|Y|}{3} \quad \text{for } i = 1, \ldots, \lambda\left(\frac{|Y| - 1}{2}\right). 
\]

If \(|X| = |Y| = 0 \pmod{3}\) and \(|Y| \leq 2|X| - 1\), then a \( \lambda(\binom{X}{2}) \)-coloring of \( \lambda(\binom{X}{3}) \) cannot be embedded into a \( \lambda(\binom{Y}{2}) \)-coloring of \( \lambda(\binom{X}{3}) \) [1, 20], hence condition (1) is not sufficient. We show that a slightly weaker condition is sufficient.

**Theorem 1.3.** \( \lambda(\binom{Y}{2}) \)-coloring of \( F := \lambda(\binom{X}{3}) \) satisfying

\[
|F(i)| \geq \frac{|X|}{2} - \frac{|Y|}{6} \quad \text{for } i = 1, \ldots, \lambda\left(\frac{|Y| - 1}{2}\right),
\]

can be extended to a one-factorization of \( \lambda(\binom{X}{3}) \) if and only if \(|Y| = 0 \pmod{3}\).

The proof relies on Theorem 1.4 together with a list-chromatic index bound for complete graphs due to Häggkvist and Janssen [21]. For a function \( \gamma : G \rightarrow P([k]) \), a \( \gamma \)-coloring of \( G \) is a coloring of \( G \) in which the color of each edge \( e \in G \) is chosen from the set \( \gamma(e) \). Here is our main result.

**Theorem 1.4.** Let \( X, Y \) be finite sets with \( X \subseteq Y \), and let \( F = \lambda(\binom{X}{3}), G = \lambda(\binom{Y}{3}), H = \lambda(|Y| - |X|)(\binom{X}{2}), k = \lambda(|Y| - 1) \). A \( k \)-coloring of \( F \) can be embedded into a one-factorization of \( G \) if and only if \(|Y| = 0 \pmod{3}\), and there exists a \( \gamma \)-coloring of \( H \) such that

\[
|H(i)| \geq |X| - \frac{|Y|}{3} - 2|F(i)| \quad \text{for } i \in [k],
\]

where

\[
\gamma(uv) = \{i \in [k] \mid \deg_{F(i)}(u) = \deg_{F(i)}(v) = 0\} \quad \text{for } u, v \in X.
\]

We remark that Baranyai [6] showed that \( \binom{X}{h} \) is one-factorable if and only if \(|X| = 0 \pmod{h}\). Last but not least, Evans, and independently, S. K. Stein [16] used Ryser’s theorem to show that a partial latin square of order \( m \) on \( m \) symbols can be extended to a latin square of order \( n \) for \( n \geq 2m \). Cruse extended this idea to symmetric latin squares which we restate as follows.

**Theorem 1.5.** [11] If \(|Y| \geq 2|X|\), then an \((|X| - 1)\)-coloring of any \( F \subseteq \binom{X}{2} \) can be embedded into a one-factorization of \( \binom{Y}{2} \) if and only if \(|Y| = 0 \pmod{2}\).
Here is our next result.

**Theorem 1.6.** If \(|Y| \geq 3|X|\) and

\[ q \leq \lambda \left( \frac{|Y| - 1}{2} \right) - \frac{\lambda \left( \frac{|X|}{3} \right)}{\left[ \frac{|X|}{3} \right]}, \]

then a \(q\)-coloring of any \(F \subseteq \lambda \left( \frac{Y}{3} \right)\) can be embedded into a one-factorization of \(\lambda \left( \frac{Y}{3} \right)\) if and only if \(|Y| \equiv 0 \pmod{3}\).

To avoid trivial cases, we will always assume that \(|X| \geq 3\) and \(|Y| > |X|\). We abbreviate sets like \(\{u, v\}\) to \(uv\), \(\{u, v, w\}\) to \(uvw\), \(\{u, u, v\}\) to \(u^2v\), \(\{u, u, u\}\) to \(u^3\), etc.

2. **Proof of Theorem 1.4**

To prove the necessity, suppose that a \(k\)-coloring of \(F\) is extended to a one-factorization of \(G\). The existence of a one-factor in \(G\) implies that \(|Y| \equiv 0 \pmod{3}\). Let

\[ H' = \{e \cap X \mid e \in G, |e \cap X| = 2\}. \]

The \(k\)-coloring of \(G\) induces a \(\gamma\)-coloring of \(H' \cong H\). For \(i \in [k]\), let \(c_i\) and \(d_i\) be the number of edges of \(G\) colored \(i\) with exactly 1 and 0 vertices, respectively, in \(X\). We have

\[
\begin{align*}
|F(i)| + |H(i)| + c_i + d_i &= \frac{|Y|}{3} \quad \text{for } i \in [k], \\
3|F(i)| + 2|H(i)| + c_i &= |X| \quad \text{for } i \in [k].
\end{align*}
\]

Therefore,

\[ d_i = 2|F(i)| + |H(i)| - |X| + \frac{|Y|}{3} \quad \text{for } i \in [k]. \]

Since \(d_i \geq 0\) for \(i \in [k]\), we have

\[ |H(i)| \geq |X| - \frac{|Y|}{3} - 2|F(i)| \quad \text{for } i \in [k]. \]

To prove the sufficiency, suppose that a \(k\)-coloring of \(F\) is given and that there exists a \(\gamma\)-coloring of \(H\) such that (3) holds. Let \(G_1\) be obtained by adding a new vertex \(\alpha\) and the following multi-set

\[ \lambda \left( |Y| - |X| \right) \{ \alpha uv \mid u, v \in X, u \neq v \} \]

of edges to \(F\). In other words, for each pair of distinct vertices \(u, v \in X\), we add \(\lambda \left( |Y| - |X| \right)\) copies of an edge of the form \(\alpha uv\) to \(G_1\). We use the \(\gamma\)-coloring of each edge \(uv\) of \(H\) to color the corresponding edge \(\alpha uv\) of \(G_1\). Observe that for \(i \in [k]\),

\[ \sum_{u \in X} \deg_{G_1(i)}(u) = 3|F(i)| + 2|H(i)|. \]

Let \(G_2\) be obtained by adding the following multi-set

\[ \left( \lambda \left( \frac{|Y| - |X|}{2} \right) \{ \alpha^2 u \mid u \in X \} \right) \cup \left( \lambda \left( \frac{|Y| - |X|}{3} \right) \{ \alpha^3 \} \right) \]

of edges to \(G_1\). In other words, we add \(\lambda \left( \frac{|Y| - |X|}{3} \right)\) copies of an edge of the form \(\alpha^3\), and for each \(u \in X\), we add \(\lambda \left( \frac{|Y| - |X|}{2} \right)\) copies of an edge of the form \(\alpha^2 u\) to \(G_2\). Recall that by (3),
such that the edges incident with in \( G \) for some \( i \)

By [3, Theorem 4.1], there exists a hypergraph \( G \) in the following way.

\[
\begin{align*}
\text{mult}_{G_2(i)}(\alpha^2 u) &= 1 - \deg_{G_1(i)}(u) & \text{for } i \in [k], \\
\text{mult}_{G_2(i)}(\alpha^3) &= 2|F(i)| + |H(i)| - |X| + \frac{|Y|}{3} & \text{for } i \in [k].
\end{align*}
\]

This is possible, for

\[
\sum_{i \in [k]} \left( 1 - \deg_{G_1(i)}(u) \right) = k - \deg_{G_1}(u)
\]

\[
= \lambda\left( \frac{|Y| - 1}{2} \right) - \lambda\left( \frac{|X| - 1}{2} \right) - \lambda(|X| - 1)(|Y| - |X|)
\]

\[
= \lambda\left( \frac{|Y| - |X|}{2} \right)
\]

and

\[
\frac{1}{\lambda} \sum_{i \in [k]} \left( 2|F(i)| + |H(i)| - |X| + \frac{|Y|}{3} \right)
\]

\[
= 2\left( \frac{|X|}{3} \right) + (|Y| - |X|) \left( \frac{|X|}{2} \right) + \left( \frac{|Y|}{3} - |X| \right) \left( \frac{|Y| - 1}{2} \right)
\]

\[
= \left( \frac{|Y| - |X|}{3} \right).
\]

For \( i \in [k] \), we have the following.

\[
\deg_{G_2(i)}(\alpha) = 3 \text{mult}_{G_2(i)}(\alpha^3) + 2 \sum_{u \in X} \text{mult}_{G_2(i)}(\alpha^2 u) + |H(i)|
\]

\[
= (6|F(i)| + 3|H(i)| - 3|X| + |Y|) + 2 \sum_{u \in X} \left( 1 - \deg_{G_1(i)}(u) \right) + |H(i)|
\]

\[
= 6|F(i)| + 4|H(i)| - 3|X| + |Y| + (2|X| - 6|F(i)| - 4|H(i)|)
\]

\[
= |Y| - |X|.
\]

By [3, Theorem 4.1], there exists a hypergraph \( G \), obtained by replacing the vertex \( \alpha \) of \( G_2 \) by \( |Y| - |X| \) new vertices \( \alpha_1, \ldots, \alpha_{|Y| - |X|} \) in \( G \), replacing each \( \alpha uv \)-edge in \( G_2 \) by an \( \alpha_i uv \)-edge in \( G \) (for some \( i \in [|Y| - |X|] \)), replacing each \( \alpha^2 u \)-edge in \( G_2 \) by an \( \alpha_i \alpha_j uv \)-edge in \( G \) (for distinct \( i, j \in [|Y| - |X|] \)), replacing each \( \alpha^3 \)-edge in \( G_2 \) by an \( \alpha_i \alpha_j \alpha_k \)-edge in \( G \) (for distinct \( i, j, \ell \in [|Y| - |X|] \)), such that the edges incident with \( i \) in \( G_2 \) are shared as evenly as possible among \( \alpha_1, \ldots, \alpha_{|Y| - |X|} \) in \( G \) in the following way.

(a) For \( i \in [|Y| - |X|] \), and \( j \in [k] \)

\[
\deg_{G(j)}(\alpha_i) = \frac{\deg_{G_2(j)}(\alpha)}{|Y| - |X|} = 1;
\]

(b) For \( i \in [|Y| - |X|] \) and distinct \( u, v \in X \),

\[
\text{mult}_G(\alpha_i uv) = \frac{\text{mult}_{G_2}(\alpha uv)}{|Y| - |X|} = \lambda;
\]

(c) For distinct \( i, j \in [|Y| - |X|] \) and \( u \in X \),

\[
\text{mult}_G(\alpha_i \alpha_j u) = \frac{\text{mult}_{G_2}(\alpha^2 u)}{|Y| - |X|} = \lambda;\]
Corollary 3.1. For distinct \( i, j, \ell \in [|Y| - |X|] \),

\[
\text{mult}_G(\alpha_i \alpha_j \alpha_\ell) = \frac{\text{mult}_{G_3}(\alpha^3)}{(|Y| - |X|)} = \lambda.
\]

By (b)–(d), \( G \cong \lambda X \), and by (a), each color class of \( G \) is a perfect matching. This completes the proof. \( \square \)

3. Proof of Theorem 1.3

Häggkvist and Janssen \cite{21} showed that for any \( \gamma : G \to \mathcal{P}([k]) \) with \( |\gamma(e)| \geq |X| \) for \( e \in \binom{X}{2} \), \( \binom{X}{2} \) has a \( \gamma \)-coloring. Consequently,

\[
\lambda \binom{X}{2} \text{ has a } \gamma \text{-coloring } \quad \forall \gamma : \lambda \binom{X}{2} \to \mathcal{P}([k]) \text{ with } |\gamma(e)| \geq \lambda |X| \text{ for } e \in \lambda \binom{X}{2}.
\]

Since (2) holds, (3) is trivial, and so by Theorem 1.4, it is enough to show that \( H := \lambda (|Y| - |X|) \binom{X}{2} \) has a \( \gamma \)-coloring where \( \gamma \) satisfies (4). Thus, by (5), if we show that \( |\gamma(e)| \geq \lambda |X| (|Y| - |X|) \) for \( e \in H \), we are done. Let

\[
C(u) = \{ i \in [k] \mid \text{deg}_{F(i)}(u) = 1 \} \quad \text{for } u \in X.
\]

Observe that \( |C(u)| = \text{deg}_F(u) = \lambda \binom{|X| - 1}{2} \) for \( u \in X \). Moreover, for distinct \( u, v \in X \), there are exactly \( \lambda(|X| - 2) \) edges in \( F \) containing both \( u \) and \( v \), and so \( |C(u) \cap C(v)| \geq \lambda(|X| - 2) \). For \( e = uv \in H \), we have the following which completes the proof.

\[
|\gamma(uv)| \geq \lambda |X| (|Y| - |X|) = |\{ i \in [k] \mid \text{deg}_{F(i)}(u) = \text{deg}_{F(i)}(v) = 0 \}| - \lambda |X| (|Y| - |X|)
\]

\[
\geq |\{ i \in [k] \mid i \in C(u) \cap C(v) \}| - \lambda |X| (|Y| - |X|)
\]

\[
= k - |C(u) \cup C(v)| - \lambda |X| (|Y| - |X|)
\]

\[
\geq \lambda \binom{|Y| - 1}{2} - 2\lambda \binom{|X| - 1}{2} + \lambda |X| (|Y| - 2) - \lambda |X| (|Y| - |X|)
\]

\[
\geq \frac{\lambda}{2} (|Y|^2 - 2|X||Y| - 3|Y| + 8|X| - 6)
\]

\[
\geq \frac{\lambda}{2} (|Y|^2 - 2|X||Y| - 3|Y| - 2|X|^2 + 6|X| + 2)
\]

\[
= \frac{6\lambda}{|Y| - |X|} \left( \frac{|X|}{3} \right) - \frac{\lambda}{|Y| - |X|} \left( \frac{|Y| - 1}{2} \right) (3|X| - |Y|)
\]

\[
= \frac{6}{|Y| - |X|} \sum_{i \in [k]} |F(i)| - \frac{6\lambda}{|Y| - |X|} \left( \frac{|Y| - 1}{2} \right) \left( \frac{|X|}{2} - \frac{|Y|}{6} \right) \geq 0.
\]

\( \square \)

An immediate consequence of Theorem 1.3 is the following which improves the best previous bound of \( |Y| \geq (2 + \sqrt{2})|X| \) \cite{4}.

Corollary 3.1. If \(|Y| \geq 3|X|\), then a \( \lambda \binom{|Y| - 1}{2} \)-coloring of \( \lambda \binom{X}{3} \) can be extended to a one-factorization of \( \lambda \binom{X}{3} \) if and only if \(|Y| \equiv 0 \pmod{3}\).
4. Proof of Theorem 1.6

Let \( k = \lambda^{|Y| - 1}/2 \). Suppose that \(|Y| \geq 3|X|, |Y| \equiv 0 \pmod{3} \), and \( k - q \geq \lambda(|X|)/|X|/3 \). We show that we can extend the given \( q \)-coloring of \( F \subseteq \lambda^{(X)}_3 \) to a \( k \)-coloring of \( \lambda^{(Y)}_3 \). First, let us color \( G := \lambda^{(X)}_3\{\alpha^3\} \) with colors \( \{q + 1, \ldots, k\} \) such that

\[
mult_{G(i)}(\alpha^3) \leq \left\lfloor \frac{|X|}{3} \right\rfloor \quad \text{for } i = q + 1, \ldots, k.
\]

This is possible, for \( \lambda^{(X)}_3 \leq (k - q)/|X|/3 \). We have

\[
\deg_{G(i)}(\alpha) = 3 \mult_{i}(\alpha^3) \leq |X| \quad \text{for } i = q + 1, \ldots, k.
\]

By [3, Theorem 4.1], there exists a hypergraph \( G' \), obtained by replacing the vertex \( \alpha \) of \( G \) by \( |X| \) new vertices \( \alpha_1, \ldots, \alpha_{|X|} \) in \( G' \), replacing each \( \alpha^3 \)-edge in \( G \) by an \( \alpha_i\alpha_j\alpha_\ell \)-edge in \( G' \) such that the following conditions hold (Here, \( a \approx b \) means \( a \in \{[b], [b]\} \)).

\[
\deg_{G'}(\alpha_i) \approx \frac{\deg_{G(j)}(\alpha)}{|X|} \leq 1 \quad \text{for } i = 1, \ldots, |X|, j = q + 1, \ldots, k;
\]

\[
\mult_{G'}(\alpha_i \alpha_j \alpha_\ell) = \frac{\mult_{G}(\alpha^3)}{|X|/3} = \lambda \quad \text{for distinct } i, j, \ell \in \{1, \ldots, |X|\}.
\]

Thus, we obtain a \( (k - q) \)-coloring of \( G' \). Since there is a one-to-one correspondence between \( G' \) and \( \lambda^{(X)}_3 \), we can color \( \lambda^{(X)}_3 \setminus F \) using the \( (k - q) \)-coloring of \( G' \). This together with the given \( q \)-coloring of \( F \) leads to a \( k \)-coloring of \( \lambda^{(Y)}_3 \). Applying Corollary 3.1 completes the proof. \( \square \)

5. The case \(|Y| \equiv 0 \pmod{3} \)

One may consider our main problem without assuming that \(|Y| \equiv 0 \pmod{3} \). Analogous problems are difficult even for the cases of graphs [19].

Let \( \chi'(\lambda^{(Y)}_3) \) be the smallest number of colors needed to color \( \lambda^{(Y)}_3 \).

**Theorem 5.1.**

\[
\chi'(\lambda^{(Y)}_3) = \begin{cases} 
\lambda\left(\frac{|Y| - 1}{2}\right) & \text{if } |Y| \equiv 0 \pmod{3}, \\
\lambda\left(\frac{|Y|}{2}\right) & \text{if } |Y| \equiv 2 \pmod{3}, \\
\frac{\lambda|Y|(|Y| - 2)}{2} & \text{if } |Y| \equiv 4 \pmod{6}, \text{ or if } |Y| \equiv 1 \pmod{3}, \lambda \equiv 0 \pmod{2}, \\
\frac{\lambda|Y|^2 - 2\lambda|Y| + 1}{2} & \text{if } |Y| \equiv 1 \pmod{6}, \lambda \equiv 1 \pmod{2}.
\end{cases}
\]

**Proof.** We show that

\[
(6) \quad \chi'(\lambda^{(X)}_3) = \left\lfloor \frac{\lambda\left(\frac{|X|}{3}\right)}{|Y|/3} \right\rfloor.
\]

We remark that the case \( \lambda = 1 \) of (6) was previously settled in [6]. Here, for convenience we provide a proof using hypergraph detachment. Our argument also works if we replace 3 by any \( h < |Y| \). It is clear that in any coloring of \( \lambda^{(Y)}_3 \), each color class has at most \( |Y|/3 \) edges. Therefore,
\( \chi'(\lambda(Y)_3) \geq \frac{\lambda(Y)_3}{3} \) \( \geq k \). To complete the proof, we find a \( k \)-coloring of \( \lambda(Y)_3 \). We color \( G := \lambda(Y)_3 \{\alpha^3\} \) with colors \( \{1, \ldots, k\} \) such that
\[
\text{mult}_{G(i)}(\alpha^3) \leq \left\lfloor \frac{|Y|}{3} \right\rfloor \quad \text{for } i = 1, \ldots, k.
\]
This is possible, for \( \lambda(Y)_3 \) \( \leq k \left\lfloor |Y|/3 \right\rfloor \). By [3, Theorem 4.1], there exists a hypergraph \( G' \), obtained by replacing the vertex \( \alpha \) of \( G \) by \( \alpha_1, \ldots, \alpha_{|Y|} \) in \( G' \), replacing each \( \alpha^3 \)-edge in \( G \) by an \( \alpha_i \alpha_j \alpha_\ell \)-edge in \( G' \) such that the following conditions hold.
\[
\deg_{G'(j)}(\alpha_i) \approx \deg_{G(j)}(\alpha_i) \times \frac{\text{mult}_{G(j)}(\alpha^3)}{|Y|} \leq 1 \quad \text{for } i = 1, \ldots, |Y|, j = 1, \ldots, k;
\]
\[
\text{mult}_{G'}(\alpha_i \alpha_j \alpha_\ell) = \frac{\text{mult}_{G}(\alpha^3)}{\binom{|Y|}{3}} = \lambda \quad \text{for distinct } i, j, \ell \in \{1, \ldots, |Y|\}.
\]
This leads to a \( k \)-coloring of \( G' \cong \lambda(Y)_3 \).

If \( |Y| \equiv 0 \pmod{3} \), then \( |Y|/3 \in \mathbb{Z} \) and \( k = 3\lambda(Y)_3/|Y| = \lambda \left( |Y|_2 - 1 \right) \). If \( |Y| \equiv 2 \pmod{3} \), then \( |Y|/3 = (|Y| - 2)/3 \), and \( k = 3\lambda(Y)_3/(|Y| - 2) = \lambda(Y)_2 \). Finally, if \( |Y| \equiv 1 \pmod{3} \), then \( |Y|/3 = (|Y| - 1)/3 \), and we have
\[
k = \left\lfloor 3\lambda \left( \frac{|Y|}{3} \right) \middle/ \left( |Y| - 1 \right) \right\rfloor = \left\lfloor \frac{|Y|\lambda(|Y| - 2)}{2} \right\rfloor = \begin{cases} \frac{|Y|\lambda(|Y| - 2)}{2} & \text{if } \lambda |Y| \equiv 0 \pmod{2}, \\ \frac{|Y|\lambda(|Y| - 2) + 1}{2} & \text{if } \lambda = |Y| \equiv 1 \pmod{2}. \end{cases}
\]

An immediate consequence of the previous theorem is the following.

**Corollary 5.2.** In any \( \chi'(\lambda(Y)_3) \)-coloring of \( \lambda(Y)_3 \), all color classes are isomorphic if and only if \( |Y| \equiv 1 \pmod{6} \) or \( \lambda \equiv 0 \pmod{2} \).

Corollary 5.2 reveals the very challenging case when \( |Y| \equiv 1 \pmod{6}, \lambda \equiv 1 \pmod{2} \), for in this case not all color classes are isomorphic. Even in the other cases where all color classes are isomorphic but \( |Y| \equiv 0 \pmod{3} \), we encounter another issue. In order to extend the main result of this note, Theorem 1.4, to say, the case \( |Y| \equiv 1 \pmod{3}, \lambda \equiv 0 \pmod{2} \), one has to decide which color classes have an isolated vertex in \( X \) or in \( |Y| \setminus X \).

6. CONCLUDING REMARKS

Beside improving the bounds in Theorems 1.3 and 1.6, one may consider our main problem without assuming that \( |Y| \equiv 0 \pmod{3} \).

**Problem 1.** Find conditions that ensure a coloring of \( \lambda(X)_3 \) can be extended to a coloring of \( \lambda(Y)_3 \) using fewest possible number of colors?

Recall that for \( \gamma : G \to \mathcal{P}([k]) \), a \( \gamma \)-coloring of \( G \) is a coloring of \( G \) in which the color of each \( e \in G \) is chosen from \( \gamma(e) \). Let \( \chi'_\gamma(G) \) denote the smallest number \( t \) such that \( G \) has a \( \gamma \)-coloring whenever \( |\gamma(e)| \geq t \) for each \( e \in G \) (Here, \( k \) is sufficiently larger than \( \chi'_\gamma(G) \)). Let \( f : [k] \to \mathbb{N} \cup \{0\} \) with \( g(i) \leq \min\{f(i), |\{e \in G \mid i \in \gamma(e)\}| \} \) for \( i \in [k] \). The following list coloring problem with restrictions deserves investigating specially in connection with Theorem 1.4 for the case where \( G = \binom{X}{2} \).

**Problem 2.** Let \( \gamma : G \to \mathcal{P}([k]) \) with \( |\gamma(e)| \geq \chi'_\gamma(G) \). Find conditions under which \( G \) has a \( \gamma \)-coloring such that the following condition holds.
\[
g(i) \leq |G(i)| \leq f(i) \text{ for } i \in [k].
\]
References

[1] Amin Bahmanian. Connected Fair Detachments of Hypergraphs. arXiv e-prints, page arXiv:2009.09674.
[2] Amin Bahmanian. Symmetric Layer-Rainbow Colorations of Cubes. arXiv e-prints, page arXiv:2205.02210.
[3] Amin Bahmanian. Detachments of hypergraphs I. The Berge-Johnson problem. Combin. Probab. Comput., 21(4):483–495, 2012.
[4] Amin Bahmanian and Chris Rodger. Embedding factorizations for 3-uniform hypergraphs. J. Graph Theory, 73(2):216–224, 2013.
[5] R. A. Bailey, Peter Cameron, Cheryl Praeger, and Csaba Schneider. The geometry of diagonal groups. Transactions of the AMS, to appear.
[6] Zs. Baranyai. On the factorization of the complete uniform hypergraph. In Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I, pages 91–108. 1975.
[7] Thomas Britz, Nicholas J. Cavenagh, and Henrik Kragh Sørensen. Maximal partial Latin cubes. Electron. J. Combin., 22(1):Paper 1.81, 17, 2015.
[8] Darryn Bryant, Nicholas J. Cavenagh, Barbara Maenhaut, Kyle Pula, and Ian M. Wanless. Nonextendible Latin cuboids. SIAM J. Discrete Math., 26(1):239–249, 2012.
[9] Nam-Po Chiang and Hung-Lin Fu. A note on embedding of a latin parallelepiped into a latin cube. Tatung J., XII:311–313, 1992.
[10] Charles J. Colbourn and Jeffrey H. Dinitz, editors. Handbook of combinatorial designs. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, second edition, 2007.
[11] Allan B. Cruse. On embedding incomplete symmetric Latin squares. J. Combinatorial Theory Ser. A, 16:18–22, 1974.
[12] Allan B. Cruse. On the finite completion of partial Latin cubes. J. Combinatorial Theory Ser. A, 130:26–41, 2015.
[13] C. Csima. Embedding partial idempotent d-ary quasigroups. Pacific J. Math., 80(2):351–357, 1979.
[14] Tristan Denley and Lars-Daniel Öhman. Extending partial Latin cubes. Ars Combin., 113:405–414, 2014.
[15] John T. Ethier and Gary L. Mullen. Sets of mutually orthogonal Sudoku frequency squares. Des. Codes Cryptogr., 87(1):57–65, 2019.
[16] Trevor Evans. Embedding incomplete latin squares. Amer. Math. Monthly, 67:958–961, 1960.
[17] R. A. Fisher. A system of confounding for factors with more than two alternatives, giving completely orthogonal cubes and higher powers. Ann. Eugenics, 12:283–290, 1945.
[18] Hung-Lin Fu. On Latin \((n \times n \times (n-2))\)-parallelepipeds. Tamkang J. Math., 17(1):107–111, 1986.
[19] J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman, and Sibel Özkan. Hall’s theorem and extending partial Latinized rectangles. J. Combin. Theory Ser. A, 130:26–41, 2015.
[20] R. Håggkvist and T. Hellgren. Extensions of edge-colourings in hypergraphs. I. In Combinatorics, Paul Erdős is eighty, Vol. I, Bolyai Soc. Math. Stud., pages 215–238. János Bolyai Math. Soc., Budapest, 1993.
[21] Roland Håggkvist and Jeannette Janssen. New bounds on the list-chromatic index of the complete graph and other simple graphs. Combin. Probab. Comput., 6(3):295–313, 1997.
[22] Marshall Hall. An existence theorem for Latin squares. Bull. Amer. Math. Soc., 51:387–388, 1945.
[23] A. S. Hedayat, N. J. A. Sloane, and John Stufken. Orthogonal arrays. Springer Series in Statistics. Springer-Verlag, New York, 1999. Theory and applications, With a foreword by C. R. Rao.
[24] Peter Horák. Latin parallelepipeds and cubes. J. Combin. Theory Ser. A, 33(2):213–214, 1982.
[25] M. Huggan, G. L. Mullen, B. Stevens, and D. Thomson. Sudoku-like arrays, codes and orthogonality. Des. Codes Cryptogr., 82(3):675–693, 2017.
[26] A. Donald Keedwell and József Dénes. Latin squares and their applications. Elsevier/North-Holland, Amsterdam, second edition, 2015. With a foreword to the previous edition by Paul Erdős.
[27] K. Kishen. On latin and hyper-graeco-latin cubes and hyper-cubes. Current Sci., 11:98–99, 1942.
[28] K. Kishen. On the construction of latin and hyper-graeco-latin cubes and hypercubes. J. Indian Soc. Agric. Statist., 2:20–48, 1949.
[29] Martin Kochol. Latin \((n \times n \times (n-2))\)-parallelepipeds not completing to a Latin cube. Math. Slovaca, 39(2):121–125, 1989.
[30] Martin Kochol. Latin parallelepipeds not completing to a cube. Math. Slovaca, 41(1):3–9, 1991.
[31] Martin Kochol. Relatively narrow Latin parallelepipeds that cannot be extended to a Latin cube. Ars Combin., 40:247–260, 1995.
[32] Martin Kochol. Non-extendible Latin parallelepipeds. Inform. Process. Lett., 112(24):942–943, 2012.
[33] Denis S. Krotov and Ev V. Sotnikova. Embedding in q-ary 1-perfect codes and partitions. Discrete Math., 338(11):1856–1859, 2015.
[34] Jaromy Kuhl and Tristan Denley. Some partial Latin cubes and their completions. European J. Combin., 32(8):1345–1352, 2011.
[35] Charles C. Lindner. A finite partial idempotent Latin cube can be embedded in a finite idempotent latin cube. J. Combinatorial Theory Ser. A, 21(1):104–109, 1976.

[36] Brendan D. McKay and Ian M. Wanless. A census of small Latin hypercubes. SIAM J. Discrete Math., 22(2):719–736, 2008.

[37] V. N. Potapov. On the complementability of partial $n$-quasigroups of order 4. Mat. Tr., 14(2):147–172, 2011.

[38] Vladimir N. Potapov. Constructions of pairs of orthogonal Latin cubes. J. Combin. Des., 28(8):604–613, 2020.

[39] Vladimir N. Potapov. Embedding in MDS codes and Latin cubes. J. Combin. Des., 30(9):626–633, 2022.

[40] H. J. Ryser. A combinatorial theorem with an application to latin rectangles. Proc. Amer. Math. Soc., 2:550–552, 1951.

Department of Mathematics, Illinois State University, Normal, IL USA 61790-4520