Impact of Uncertainties in the Cosmological Parameters on the Measurement of Primordial non-Gaussianity

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We study the impact of cosmological parameters' uncertainties on estimates of the primordial NG parameter $f_{\text{NL}}$ in local and equilateral models of non-Gaussianity. We show that propagating these errors increases the $f_{\text{NL}}$ $1\sigma$ uncertainty by a term $\delta f_{\text{NL}}^{\text{local}}/f_{\text{NL}}^{\text{local}} \approx 16\%$ for WMAP and $\delta f_{\text{NL}}^{\text{local}}/f_{\text{NL}}^{\text{local}} \approx 5\%$ for Planck in the local case, whereas for equilateral configurations the correction term are $\delta f_{\text{NL}}^{\text{equil}}/f_{\text{NL}}^{\text{equil}} \approx 14\%$ and $\delta f_{\text{NL}}^{\text{equil}}/f_{\text{NL}}^{\text{equil}} \approx 4\%$, respectively. If we assume for $f_{\text{NL}}$ a central value $(f_{\text{NL}}^{\text{local}})^{\text{central}} \approx 60$, according to recent WMAP 5-years estimates, we obtain for Planck a final correction $\Delta f_{\text{NL}}^{\text{local}} \approx 3$. Although not dramatic, this correction is at the level of the expected estimator uncertainty for Planck, and should then be taken into account when quoting the significance of an eventual future detection. In current estimates of $f_{\text{NL}}$, the cosmological parameters are held fixed at their best-fit values. We finally note that the impact of uncertainties in the cosmological parameters on the final $f_{\text{NL}}$ error bar would become totally negligible if the parameters were allowed to vary in the analysis and then marginalized over.

I. INTRODUCTION

Research of primordial non-Gaussianity in the Cosmic Microwave Background (CMB) is an important field of cosmology today. In a recent work, Yadav and Wandelt \cite{1} claim a $3\sigma$ detection of a large NG signal in the WMAP 3-years data. The following WMAP 5-years analysis produced similar results but showed a reduction of statistical significance from about 3 to $2\sigma$ \cite{2}. Yadav and Wandelt estimate, using WMAP 3-years data, is

$$27 \leq f_{\text{NL}}^{\text{local}} \leq 147 \ (95\% \text{ c.l.}) .$$

The WMAP 5-years estimate is:

$$-9 \leq f_{\text{NL}}^{\text{local}} \leq 111 \ (95\% \text{ c.l.}) ,$$

where $f_{\text{NL}}$ is a dimensionless parameter defining the strength of primordial non-Gaussianity (NG) \cite{3}, and the superscript "local" indicates that we are considering a primordial NG curvature perturbation of the form

$$\Phi(x) = \Phi_L(x) + f_{\text{NL}}^{\text{local}} (\Phi_L^2(x) - \langle \Phi_L^2(x) \rangle) ,$$

where $\Phi(x)$ is the total gravitational potential, $\Phi_L(x)$ is the gravitational potential computed at the linear level and $\langle \cdots \rangle$ stands for the ensemble average. The reason for calling this a local NG model relies in the fact that the NG part of the primordial curvature perturbation is a local functional of the Gaussian part. The local shape of NG arises from standard single-field slow-roll inflation \cite{4} as well as from alternative inflationary scenarios for the generation of primordial perturbations, like the curvaton \cite{5} or inhomogenous (pre)reheating models \cite{6}, or even from alternatives to inflation, such as ekpyrotic and cyclic models \cite{7}. Other models, such as DBI inflation \cite{8} and ghost inflation \cite{9}, predict a different kind of primordial NG, called "equilateral", because the three point function for this kind of NG is peaked on equilateral configurations, in which the lengths of the three wavevectors forming a triangle in Fourier space are equal \cite{10}. This second form of NG is characterised by the parameter $f_{\text{NL}}^{\text{equil}}$, which defines the amplitude of the equilateral triangles. In the following we will sometimes write the amplitude of NG simply as $f_{\text{NL}}$, without any superscript. When we do so, we mean that our conclusions apply to both the local and the equilateral case, with no need for distinction.

Standard single field inflation predicts $f_{\text{NL}}^{\text{local}} \sim 10^{-2}$ at the end of inflation \cite{4} (and therefore a final value $f_{\text{NL}}^{\text{local}} \sim$ unity after general relativistic second-order perturbation effects are taken into account \cite{11}). It is thus clear that large central values of $f_{\text{NL}}^{\text{local}}$, like those obtained in the above mentioned analyses, are going to rule out the simplest scenarios of inflation as viable models of the Early Universe. On the other hand, the low statistical significance of the final WMAP 5-years result makes any conclusion premature at this stage. With its high angular resolution and sensitivity Planck will allow to significantly improve the statistical estimate of $f_{\text{NL}}^{\text{local}}$, reducing the final error bars...
from the present $\Delta f_{NL}^{\text{local}} \simeq 30$ to a final value of $\Delta f_{NL}^{\text{local}} \simeq 5$ \[12\] thus allowing a many $\sigma$ detection of non-Gaussianity if the present large central values of $f_{NL}^{\text{local}}$ were to be confirmed. An eventual detection of a large $f_{NL}$ by Planck would not however automatically imply that the observed non-Gaussianity is primordial in origin. A number of effects can produce a spurious NG signal that can bias the final estimate. The most relevant examples of NG contaminants are probably given by diffuse foregrounds emission, unresolved point sources contamination, NG noise. Both the analyses by Yadav and Wandelt and by the WMAP team consider all these effects and conclude that effects such as the cross-correlations SZ-lensing and ISW-lensing produce a bias in the estimate of $f_{NL}^{\text{local}}$ which is at the level of the expected estimator variance at Planck angular resolution. Analogous conclusions have been reached by Babich and Pierpaoli \[14\] for the cross correlations of density and lensing magnification of radio and SZ point sources with the ISW effect. Note that all these effects are unimportant for present analyses both because they involve higher multipoles than those reached at the WMAP angular resolution, and because they produce a bias $\Delta f_{NL}^{\text{local}} \sim 1$, thus much smaller than the WMAP sensitivity to $f_{NL}^{\text{local}}$, which is $\Delta f_{NL}^{\text{local}} \sim 30$. However, the much higher angular resolution achieved by Planck and an expected predicted sensitivity on $f_{NL}^{\text{local}}$ given by $\Delta f_{NL}^{\text{local}} \sim 5$ (and $\Delta f_{NL}^{\text{local}} \sim 3$ if polarisation data are included in the analysis) will make the above mentioned sources of NG contamination no longer negligible in the future. In other words, the same nice properties that make Planck more sensitive to the detection of a primordial NG signal (i.e. high angular resolution and sensitivity) make it in fact also much more sensitive to the observation of NG contaminants and require a very careful investigation of all the potential sources of bias in the estimate of $f_{NL}$.

In this paper we will consider another potential source of uncertainty in the detection of NG, namely the propagation of the uncertainties in the cosmological parameters on the measured value of $f_{NL}$. This effect can be summarised as follows: the estimator usually employed to measure $f_{NL}$ assumes a given underlying cosmological model obtained by fixing the cosmological parameters at their best-fit values (obtained from the two-point CMB likelihood analysis of the experiment under exam). However the cosmological parameter estimates are characterised by uncertainties that should be propagated into the $f_{NL}$ estimate in order to accurately quote the final error bars. The uncertainties in the cosmological parameters can be safely neglected as long as they are much smaller than the variance of the NG estimator. While this works well for WMAP, it is a priori unclear whether it is still a good approximation for Planck.

The paper is structured as follow: in section \textbf{II} we will describe the $f_{NL}$ estimator commonly employed in the analysis and study in detail the effect of cosmological parameters uncertainties on this estimator. After deriving the error propagation formulae \[13\] and \[14\] we will apply them to obtain analytical and numerical estimates of the expected $f_{NL}$ uncertainty for WMAP and Planck, both in the local and equilateral case. In section \textbf{III} we will then consider the possibility of applying a more complex analysis in which $f_{NL}$ is estimated by firstly allowing the cosmological parameters to vary and then by marginalising over them, rather than by fixing the cosmological model. In this case we adopt a Fisher matrix approach to propagate the parameter uncertainties on the final predicted $\Delta f_{NL}$. We will finally discuss our results and draw our conclusion in section \textbf{IV}.

\section{II. ERRORS PROPAGATION}

The estimate of $f_{NL}$ from CMB data are usually obtained from measurements of the three point function in harmonic space, called the angular bispectrum and defined as:

$$B_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} \equiv \langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3} \rangle .$$ \hspace{1cm} (4)

Due to rotational invariance of the CMB sky the angular bispectrum can be written as:

$$B_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} = G_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} b_{\ell_1 \ell_2 \ell_3} ,$$ \hspace{1cm} (5)

where $G_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3}$ is the Gaunt integral:

$$G_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} = \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} ,$$ \hspace{1cm} (6)

while $b_{\ell_1 \ell_2 \ell_3}$ is called the reduced bispectrum and contains all the relevant physical information. Analytic formulae for both the local and equilateral cases have been computed \[12\], \[16\]. The local reduced bispectrum can be written as:

$$b_{\ell_1 \ell_2 \ell_3}^{\text{local}} = 2 f_{NL}^{\text{local}} \int drr^2 \alpha_1 (r) \beta_2 (r) \beta_3 (r) + (2 \text{ perm.}) ,$$ \hspace{1cm} (7)
whereas the equilateral bispectrum is

\[ b_{\ell_1\ell_2\ell_3}^{\text{NL}} = 6 f_{N\text{L}} \int drr^2 \alpha_\ell (r) \beta_\ell (r) \beta_\ell (r) + (2 \text{ perm.}) + \delta_\ell (r) \delta_\ell (r) \delta_\ell (r) + \beta_\ell (r) \gamma_\ell (r) \delta_\ell (r) + (5 \text{ perm.}) . \]  

(8)

The functions \( \alpha_\ell (r) \), \( \beta_\ell (r) \), \( \gamma_\ell (r) \), \( \delta_\ell (r) \) appearing in the previous formulae are defined as:

\[ \alpha_\ell (r) = \frac{2}{\pi} \int dk k^2 \Delta_\ell (k) j_\ell (kr) , \]

(9)

\[ \beta_\ell (r) = \frac{2}{\pi} \int dk k^2 P_\Phi (k) \Delta_\ell (k) j_\ell (kr) , \]

\[ \gamma_\ell (r) = \frac{2}{\pi} \int dk k^2 P_\Phi^{1/3} (k) \Delta_\ell (k) j_\ell (kr) , \]

\[ \delta_\ell (r) = \frac{2}{\pi} \int dk k^2 P_\Phi^{2/3} (k) \Delta_\ell (k) j_\ell (kr) . \]

In the previous set of formulae \( \Delta_\ell (k) \) indicates the CMB radiation transfer function and \( P_\Phi (k) \) is the power spectrum of primordial curvature perturbation. It is thus clear that the reduced bispectrum will be dependent on the cosmological parameters.

The NG estimator which is generally employed to analyse CMB data can be written as [17, 18):

\[ \hat{f}_{\text{NL}} = \frac{1}{N} \sum_{\ell_1 \ell_2 \ell_3} \sum_{m_1 m_2 m_3} G_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} \frac{b_{\ell_1 \ell_2 \ell_3}}{C_{\ell_1} C_{\ell_2} C_{\ell_3}} a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} , \]

(10)

where \( b_{\ell_1 \ell_2 \ell_3} \) is the analytical form of the primordial reduced bispectrum for the model we are considering (i.e. either local or equilateral), whereas \( N \) is a normalisation factor designed to produce unitary response when \( f_{N\text{L}} = 1 \):

\[ N = \sum_{\ell_1 < \ell_2 < \ell_3} \frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{array} \right)^2 \frac{b_{\ell_1 \ell_2 \ell_3}^{N\text{L}=1}}{C_{\ell_1} C_{\ell_2} C_{\ell_3}} . \]

(11)

An additional linear term is added to the estimator when dealing with anisotropic noise in the data. We will not include such term here as it is not dependent on the cosmological parameters and thus does not affect our estimates. The estimated value of \( f_{N\text{L}} \) is then obtained by correlating the observed bispectrum with the theoretically expected one for a given primordial shape (local or equilateral) and given cosmological model, and dividing by a suitable normalisation factor. Also the normalisation will be dependent on the bispectrum shape and on the cosmological parameters, as both \( b_{\ell_1 \ell_2 \ell_3} \) and \( C_\ell \) are. The \( C_\ell \) and \( b_{\ell_1 \ell_2 \ell_3} \) that enter the estimator are calculated by assuming the best-fit cosmological model for the experiment under consideration. However, the best-fit cosmological parameters are characterised by uncertainties that propagate to \( \hat{f}_{\text{NL}} \). The aim of this work is to analyse this effect in detail and assess its significance for WMAP and Planck. As a start, following [16] let us assume that the cosmological model assumed in the calculation of \( b_{\ell_1 \ell_2 \ell_3} \) and \( C_\ell \) is not the “real” one. Let us call the reduced bispectrum obtained from the real cosmological parameters \( \tilde{b}_{\ell_1 \ell_2 \ell_3} \). It is then easy to see that the average value of the estimator will be [16]:

\[ \langle \hat{f}_{\text{NL}} \rangle = \frac{1}{N} \sum_{\ell_1 < \ell_2 < \ell_3} \frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{array} \right)^2 \frac{b_{\ell_1 \ell_2 \ell_3}^{N\text{L}=1} a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3}}{C_{\ell_1} C_{\ell_2} C_{\ell_3}} , \]

(12)

throughout the paper we will use a superscript \( ^* \) on \( f_{N\text{L}} \) whenever we want to indicate a statistical estimate. If we now want to estimate the bias \( \delta \hat{f}_{\text{NL}} \) due to the mismatch between the assumed cosmological model and the real one, we can write:

\[ \delta \hat{f}_{\text{NL}} = \langle \hat{f}_{\text{NL}} \rangle - f_{\text{NL}} = \frac{1}{N} \sum_{\ell_1 < \ell_2 < \ell_3} \frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{array} \right)^2 \frac{b_{\ell_1 \ell_2 \ell_3}^{N\text{L}=1} \delta b_{\ell_1 \ell_2 \ell_3}}{C_{\ell_1} C_{\ell_2} C_{\ell_3}} , \]

(13)
In this last formula, \( \delta b_{1,\ell_2\ell_3} \) is the difference between the theoretical bispectra computed for the “real” and “assumed” cosmological model; all the quantities with a superscript \( \sim \) are computed in the “real” cosmological model. If we then have a cosmological parameter characterised by an uncertainty \( \delta p \) we simply propagate this uncertainty on the \( f_{NL} \) estimate as:

\[
\delta \hat{f}_{NL} = \frac{\partial \hat{f}_{NL}}{\partial p} \delta p \simeq \frac{f_{NL}}{N} \sum_{\ell_1,\ell_2<\ell_3} \frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi} \left( \ell_1 \ell_2 \ell_3 \right) C_{\ell_1} C_{\ell_2} C_{\ell_3} \left[ \frac{\delta p \delta p}{\delta f_{NL}} \right] \delta p .
\]

(14)

The term in square brackets expresses the derivative of \( \hat{f}_{NL} \) with respect to the parameter \( p \) as a function of the derivative of the bispectrum with respect to \( p \). As an estimate of the uncertainty on the parameter \( p \) we can use the standard deviation and thus substitute \( \delta p \), indicating a general small variation of the parameter in the previous formula, with \( \sigma_p \), expressing its 1\( \sigma \) uncertainty. The formula above then simply reads:

\[
\sigma_{f_{NL}} = \frac{\partial \hat{f}_{NL}}{\partial p} \sigma_p ;
\]

(15)

this is the standard formula of error propagation. Note that here and in the following equations, we sometimes indicate uncertainties in a parameter using the letter \( \delta \) (e.g. \( \delta p \)), sometimes we use \( \sigma \), e.g. \( \sigma_{f_{NL}} \). With \( \delta \) we just generically indicate a small variation in the parameter (possibly dependent on the variation in one or more other parameters), whereas with \( \sigma \) we specifically refer to the standard deviation.

In general we have a cosmological model defined by a set of parameters \( \{p_i\} \). All these parameters are allowed to vary and can be correlated. The standard error propagation formula in this case becomes:

\[
\delta \hat{f}_{NL} = \sqrt{\sum_{ij} \frac{\partial f_{NL}}{\partial p_i} \frac{\partial f_{NL}}{\partial p_j} \text{Cov}(p_i, p_j) \frac{\partial f_{NL}}{\partial p_j} \delta p_j} \delta p_i .
\]

(16)

where the average values \( \bar{p}_i \) of the parameters and their covariance matrix \( \text{Cov}(p_i, p_j) \) are determined from a standard CMB-likelihood analysis, in which the anisotropies are assumed Gaussian. In our analysis we considered a model characterised by the six parameters \( A_s, n_s, \tau, \omega_b = \Omega_b h^2, \omega_c = \Omega_c h^2, \Omega_L \), respectively defining the amplitude of curvature perturbation at \( k_0 = 0.002/\text{Mpc} \), the scalar spectral index, the optical depth to reionization, the physical density of baryons, the physical density of matter and the dark energy density. We fixed these parameters at their maximum likelihood values from the WMAP 5-year analysis and as an estimate of the covariance matrix we computed the Fisher matrix of the parameters. In the computation of the Fisher matrix we consider two cases: in the first case we use the 41, 61 and 94 Ghz frequency channels of WMAP (Q+V+W bands) and in the second we consider the combination of the 143 and 217 Ghz frequency channels of Planck. For the numerical details of the computation (e.g. choice of the step size for the derivatives w.r.t. cosmological parameters) we closely followed the methodology presented in [20]. We then computed the CMB local and equilateral bispectra numerically using formulae [7,8,9] and we took two sided numerical derivatives to evaluate \( \partial B / \partial p \). For the steps of the derivatives we again followed the prescriptions of [20]. Our calculation shows that the \( f_{NL} \) error bars get relative corrections \( \delta f_{NL}^{\text{local}} / f_{NL} \simeq 16.5\% \) and \( \delta f_{NL}^{\text{equi}} / f_{NL} \simeq 14.5\% \) for WMAP in the local and equilateral case, while for Planck we have \( \delta f_{NL}^{\text{local}} / f_{NL} \simeq 5\% \) and \( \delta f_{NL}^{\text{equi}} / f_{NL} \simeq 4.5\% \). Before discussing the significance of this correction let us try to understand this result in a more intuitive way by employing some analytical approximations. From figures 1 and 2 we see that most of the contribution to \( \delta f_{NL} \) from error propagation comes from only 3 of the 6 considered parameters: \( A_s, n_s \) and \( \tau \) (this is in agreement with the results of [1]). We will then restrict the following simplified analysis to these three parameters.

The first step of this analysis is to get analytical expressions for the derivatives of the bispectrum with respect to \( A_s, n_s, \tau \). From formula (9) it can be easily seen that \( \partial B / \partial A_s = 2B / A_s \). The fractional variation in \( \hat{f}_{NL} \) corresponding to a variation \( \delta A_s \) is then:

\[
\frac{\delta \hat{f}_{NL}}{\delta A_s} \delta A_s = 2 \hat{f}_{NL} \frac{\delta A_s}{A_s} .
\]

(17)

The parameter \( \tau \) defines the optical depth to reionization and the effect of changing it can be described by introducing a multiplicative factor \( e^{-\tau} \) in front of the radiation transfer function at high \( \ell \)s. Note that the radiation transfer functions appear in the definition of the bispectrum through the functions \( \alpha, \beta, \gamma, \delta \) defined in formula (9). The bispectrum modes that give the largest contributions in the local case to the final signal-to-noise ratio are the so called squeezed configuration, i.e. configuration where one of the three \( \ell \)s is much smaller than the other two, and
FIG. 1: Contribution of the different cosmological parameter uncertainties to the final error in the estimate of \( f_{NL}^{\text{local}} \) (lower panel) and \( f_{NL}^{\text{equil}} \) (upper panel) as a function of \( \ell_{\text{max}} \). The quantity \( \langle \partial f_{NL}/\partial p_i \rangle \sigma_{p_i} \) is plotted for each of the six parameters in the model. In this figure we considered an experiment with the characteristics of WMAP.

Equilateral triangles in the other case (see e.g. [16]). For this reason, in the local case one of the three \( \ell \)'s will be super-horizon and the corresponding transfer function will not show a multiplicative factor \( e^{-\tau} \) in front. In the equilateral case all modes are sub-horizon in the important configurations. We can then write \( \tilde{b}_{\text{local}}^{\ell_1 \ell_2 \ell_3} = \exp(-2\tau)b_{\text{local}}^{\ell_1 \ell_2 \ell_3} \) and \( \tilde{b}_{\text{equil}}^{\ell_1 \ell_2 \ell_3} = \exp(-3\tau)b_{\text{equil}}^{\ell_1 \ell_2 \ell_3} \). Substituting into equation (14) we find \( \delta f_{\text{local}}^{\ell_1 \ell_2 \ell_3} \approx -2f_{\text{local}}^{\ell_1 \ell_2 \ell_3} \delta \tau \) and \( \delta f_{\text{equil}}^{\ell_1 \ell_2 \ell_3} \approx -3f_{\text{equil}}^{\ell_1 \ell_2 \ell_3} \delta \tau \).

Note though that the parameter \( \tau \) obtained from the \( C_\ell \) likelihood analysis is degenerate with the amplitude of the spectrum of primordial curvature perturbations. In order to include this degeneration in our simplified description, for a given variation in \( \tau \) we will also introduce a variation in the power spectrum amplitude that leaves the final \( C_\ell \) unchanged. This is obtained by multiplying the amplitude by a factor \( \exp(a\delta \tau) \), where \( a = 2, 3 \) in the local and equilateral case respectively. A small shift \( \delta \tau \) in the ionisation optical depth then implies a shift in the amplitude equal to \( \delta A_S \approx 2A_S \delta \tau \). The total bispectrum variation \( \delta B \) is then given by (we omit the subscript \( \ell_1 \ell_2 \ell_3 \) for simplicity of notation):

\[
\frac{\delta B}{\delta \tau} = \frac{\partial B}{\partial \tau} \delta \tau + \frac{\partial B}{\partial A_S} \frac{\delta A_S}{\delta \tau} \delta \tau
\]

\[
= -aB e^{-a \delta \tau} \delta \tau + 4Be^{a \delta \tau} \delta \tau
\]

\[
\approx (4-a)B \delta \tau ,
\]

(18)
where in the last line we neglected second order terms in \( \delta \tau \). The total variation in \( \hat{f}_{\text{NL}} \) for a given \( \delta \tau \) is then:

\[
\delta \hat{f}_{\text{NL}} = (4 - a) \hat{f}_{\text{NL}} \delta \tau
\]  

(19)

The remaining parameter to take into account is the scalar spectral index \( n \). Our next step is then the evaluation of \( \delta B/\delta n \). First of all we note that when changing \( n \) we have to change the power spectrum normalisation accordingly because the normalisation is defined at a given pivot scale. To compute \( \delta B \) arising from a small change in the spectral index we then have to evaluate again:

\[
\frac{\delta B}{\delta n} = \frac{\partial B}{\partial n} \delta n + \frac{\partial B}{\partial A} \frac{\partial A}{\partial n} \delta n ,
\]  

(20)

where the partial derivative with respect to \( n \) is taken by assuming \( A \) fixed. The authors of [22] use WMAP 3-years data to find that the normalisation is well fit by the following expression:

\[
A_S^{\text{WMAP}} = \tilde{A}_S \exp(-1.24 + 1.04r)(1 - n) \sqrt{1 + 0.53r} ,
\]  

(21)
where \( r \) is the tensor-to-scalar ratio. We will use this ansatz, with the additional assumption \( r = 0 \). In this way we obtain, for a given variation \( \delta n \) of the scalar spectral index: \( \delta A = 1.24A\delta n \) and, correspondingly, \( \delta B \approx 2.5\delta n \). Finally, to approximately evaluate \( \partial B/\partial n \) we work in the pure SW regime. Estimates of the signal-to-noise ratio have in this case been obtained by Komatsu and Spergel [12] for the equilateral configurations and by Babich and Zaldarriaga [15] for the squeezed ones for \( n = 1 \). Extending their results, we obtain that in both cases:

\[
\frac{\partial \hat{f}_{NL}}{\partial n} \approx \frac{f_{NL}}{2} \left[ \log (\ell_{\text{max}}) - \frac{1}{(1 - n)} \right];
\]

this allows to write the variation in \( \hat{f}_{NL} \) for a given \( \delta n \)

\[
\frac{\delta \hat{f}_{NL}}{\hat{f}_{NL}} \approx \left[ 2.5 - \frac{1}{2(1 - n)} + \frac{1}{2} \log (\ell_{\text{max}}) \right] \delta n.
\]

Having an expression for the derivatives of the bispectrum with respect to each of the three parameters \( A_S, n \) and \( \tau \) we can now propagate the error using Eq. (16), that for this particular case reads

\[
\sigma_{f_{NL}} = \sqrt{\left( \frac{\delta \hat{f}_{NL}}{\delta \hat{A}_S} \right)^2 \sigma_{A_S}^2 + \left( \frac{\delta \hat{f}_{NL}}{\delta \tau} \right)^2 \sigma_{\tau}^2 + \left( \frac{\delta \hat{f}_{NL}}{\delta n} \right)^2 \sigma_n^2}.
\]

Note that, as mentioned above, the correlation of \( A_S \) with \( \tau \) and \( n \) is accounted for by the ansatz we have made for \( A_S \):

\[
A_S = \hat{A}_S \exp[-a\tau - 1.24(1-n)],
\]

whereas the correlation between \( \tau \) and \( n \) has been neglected in the previous formula. Using the expressions just derived above for \( \delta f_{NL}/\delta \hat{A}_S, \delta f_{NL}/\delta \tau \) and \( \delta f_{NL}/\delta n \) we finally get:

\[
\frac{\sigma_{f_{NL}}}{f_{NL}} = \sqrt{\left( \frac{2\sigma_{\hat{A}_S}}{\hat{A}_S} \right)^2 + [(4 - a)\sigma_{\tau}]^2 + \left[ 2.5 - \frac{1}{2(1 - n)} + \frac{1}{2} \log (\ell_{\text{max}}) \right]^2 \sigma_n^2}.
\]

The present WMAP 5-years analysis yields a fractional uncertainty on the amplitude of the curvature power spectrum of order 3\%, while \( \sigma_{\tau} \approx 0.016 \) and \( \sigma_n = 0.015 \). Substituting these numbers in the last formula yields a total fractional correction of order 14\% on \( \hat{f}_{NL} \), in very good agreement with the numerical results. If we now consider an experiment with the characteristics of Planck, our Fisher matrix analysis give a fractional uncertainty on \( A_S \) of order 1.5\%, \( \sigma_n = 0.004, \sigma_\tau = 0.005 \). This produces a final fractional correction of order \( \approx 5\% \) on \( f_{NL} \), again in very good agreement with the numerical estimate.

To understand whether these corrections are negligible or not we have now to compare it with the 1\sigma uncertainty \( \Delta f_{NL} \) of the estimator, obtained with fixed cosmological parameters. The WMAP analysis finds \( \Delta f_{NL}^{\text{local}} \approx 30, \Delta f_{NL}^{\text{equil}} \approx 100 \), and central values \( f_{NL}^{\text{local}} \approx 60 \) and \( f_{NL}^{\text{equil}} \approx 70 \). Using the fractional uncertainties above we get a contribution to the final error bar from cosmological parameters uncertainties that amounts to \( \Delta f_{NL}^{\text{local}} \approx 10 \) and \( \Delta f_{NL}^{\text{equil}} \approx 9 \). The effect of propagating cosmological parameters uncertainties can then be neglected for the equilateral shape, where the error bars are larger, whereas it is more important for the local shape (about 30\% of the presently quoted error bar). If we now consider Planck, Fisher matrix based forecasts predict \( \Delta f_{NL}^{\text{local}} \approx 5 \) and \( \Delta f_{NL}^{\text{equil}} \approx 60 \). If we assume the \( f_{NL} \) central values found by the WMAP analysis we obtain a small effect for the equilateral case, whereas \( \delta f_{NL}^{\text{local}} \approx 3, \) i.e. of the same order of magnitude as \( \Delta f_{NL}^{\text{local}} \). This analysis then suggests that the effect of propagating uncertainties in the cosmological parameters on the final \( f_{NL}^{\text{local}} \) error bar should be taken into account if large central values of \( f_{NL}^{\text{local}} \) are found with Planck. Note that a value of \( f_{NL}^{\text{local}} \) of order 60 would mean a many \( \sigma \) detection with Planck. Correcting the error bar in order to account for error propagation effects would not change this result but it would on the other hand modify the level of significance of such a detection.

Before concluding this section, we would like to stress again that the estimator of \( f_{NL} \) currently employed in the analyses fixes the cosmological parameters at their best-fit values. A way to reduce the impact of the uncertainties on the parameters would be to perform a joint likelihood analysis in which the cosmological parameters are allowed to vary and then marginalise over their uncertainties. Obtaining a forecast of the final \( f_{NL} \) error if this approach is taken is the purpose of the next section.
III. FISHER MATRIX

As we were mentioning in the previous section, the optimal approach to the $f_{NL}$ measurement would be to treat the cosmological parameters as nuisance parameters and to marginalise over their distributions in order to get the final $f_{NL}$ estimate. The error on $f_{NL}$ can in this case be estimated by a Fisher matrix analysis. If we consider a set $p = \{p_i\}$ of cosmological parameters we can express the Fisher matrix as $F_{ij}$:

$$F_{ij} = \sum_{2<\ell_1<\ell_2<\ell_3} \frac{\partial B_{\ell_1\ell_2\ell_3}}{\partial p_i} \frac{\partial B_{\ell_1\ell_2\ell_3}}{\partial p_j} \frac{1}{\sigma^2},$$

(27)

where $\sigma^2$ is the bispectrum variance. In the limit of small non-Gaussianity we can take $\sigma^2 = C_\ell C_{\ell_2} C_{\ell_3} \Delta C_{\ell_1\ell_2\ell_3}$, where $\Delta$ takes the values 1, 2, 6 when two $\ell$'s are different, two of them are the same and all are the same respectively. Following the results of the previous section, we know that the relevant set of parameters to consider is $p = \{f_{NL}, A, n, \tau\}$. Two account for cosmological parameter uncertainties we add a Gaussian prior on the $ith parameter with variance $\sigma^2_i$, where $\sigma_i$ is the standard deviation obtained from the two-point function likelihood analysis. This approach is feasible as long as we deal with weak non-Gaussianity and the two and three point function can then be treated as uncorrelated. A Gaussian prior on the $ith parameter with variance $\sigma^2_i$ is imposed by simply adding a $\frac{1}{\sigma^2_j}$ term to the $ii$ entry of the Fisher matrix (see e.g. [24]). Once the Fisher matrix has been computed, the error on the $i$-th parameter after marginalising over the others can be estimated in the standard way as:

$$\sigma_{p_i} = \sqrt{F_{ii}^{-1}}. \quad (28)$$

Before moving to the numerical evaluation of formula (27) for the full set of parameters let us start with a simplified case in which only $f_{NL}$ and $\tau$ are considered in the analysis and let us for simplicity restrict ourselves to the local case. In this case, having made the approximation (explained in the previous section) $\partial B/\partial \tau = \exp(-2\tau)B$, a simple analytical calculation gives the following Fisher matrix:

$$F = \begin{pmatrix}
\sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} & -2f_{NL} \sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} \\
-2f_{NL} \sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} & 4f_{NL}^2 \sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} + \frac{1}{\sigma^2}\n\end{pmatrix}, \quad (29)$$

where $\sigma_B^2$ is the bispectrum variance defined above. This matrix is singular for $f_{NL} \neq 0$, meaning that $f_{NL}$ and $\tau$ are degenerate parameters. Adding a Gaussian prior on $\tau$ with variance $\sigma^2_\tau$ breaks the degeneracy:

$$F = \begin{pmatrix}
\sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} & -2f_{NL} \sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} \\
-2f_{NL} \sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} & 4f_{NL}^2 \sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} + \frac{1}{\sigma^2} + \frac{1}{\sigma^2_\tau}\n\end{pmatrix}. \quad (30)$$

Inverting the Fisher matrix and taking the square roots yields the final error on $f_{NL}$:

$$\sigma_{f_{NL}} = \sqrt{\frac{1}{\sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2} \left(1 + 4\sigma^2 f_{NL}^2 \sum \frac{B^2_{\ell_1\ell_2\ell_3}}{\sigma_B^2}\right)}}. \quad (31)$$

If we call $\Delta f_{NL}$ the estimated Fisher matrix error when we do not marginalise over $\tau$ (i.e. the error usually quoted in the literature) then we see from the previous formula that:

$$\sigma_{f_{NL}} = \sqrt{(\Delta f_{NL})^2 + 4\sigma^2 f_{NL}^2}; \quad (32)$$

note the difference with respect to the previous approach in which cosmological parameters were fixed. In that case the cosmological parameter errors biased the estimator and the uncertainties propagated linearly (see also [16]):

$$\sigma_{f_{NL}} = \Delta f_{NL} + 2f_{NL} \sigma_\tau. \quad (33)$$

As we saw in the previous section, the error propagation scheme arising from the standard approach of fixing cosmological parameters produces a relative correction of a few percent for WMAP and Planck. We concluded that this correction is small but not always negligible for Planck. On the other hand the marginalisation approach used...
FIG. 3: Correction to $\Delta f_{NL}$ after marginalisation over $\tau$ is performed in a toy model where only uncertainties over $\tau$ are considered. We plot the correction as a function of $\Delta f_{NL}$. The correction becomes significant only when $\Delta f_{NL}$ is small enough to produce a many $\sigma$ detection for a given $f_{NL}$.

As long as only the parameter $\tau$ is considered we can then conclude that both for WMAP ($\sigma_\tau = 0.016$, $\Delta f_{NL}^{\text{local}} \simeq 30$) and Planck ($\sigma_\tau = 0.016$, $\Delta f_{NL}^{\text{local}} \simeq 5$) the correction to the $f_{NL}$ error bars is totally negligible: $\delta f_{NL}^{\text{local}} < 0.2\% f_{NL}$, assuming a central value $f_{NL} \simeq 60$. From the formula above we basically see that the effect of marginalising over $\tau$ is to suppress the correction mentioned in the previous section by a further factor $\Delta f_{NL}$. Moreover we recover the correction mentioned in the previous section in the limit $\Delta f_{NL} \to 0$. All this makes sense: the correction from cosmological parameters uncertainties is significant only if the error bar on $f_{NL}$ before marginalisation is comparable to the error bars on the other parameters; moreover a full likelihood estimation optimises the final error bar on $f_{NL}$ with respect to an analysis in which the cosmological parameters are held fixed. The same results arise when we account not only for $\tau$, but we consider the full set $\{A, n, \tau, f_{NL}\}$. In this case we evaluated $\delta f_{NL}$ numerically from formula \eqref{27} and obtained that the correction on the $f_{NL}$ error bar after marginalisation is always less than 0.5%. The conclusion is that if a full likelihood analysis including the two and three point functions is applied in order to estimate $f_{NL}$, then the impact of cosmological parameters uncertainties is totally negligible.
IV. CONCLUSIONS

In this paper we considered the effect of propagating cosmological parameters uncertainties on the estimate of the primordial NG parameter \( f_{\text{NL}} \). We firstly show that, accounting for the large central value of \( f_{\text{NL}} \) presently measured [1,2], the final correction from parameters uncertainties is of order 30\% of the quoted \( f_{\text{NL}} \) error bars for WMAP and at about the same level of the predicted \( f_{\text{NL}} \) error bars for Planck. If a large \( f_{\text{NL}} \) will be observed by Planck, the effect of these uncertainties will then be not big enough to change the conclusion that a large level of primordial non-Gaussianity is present in the data. However the effect is important enough to change the significance of the detection and should be taken into account when quoting the error bars. We finally show that the effect of cosmological parameters uncertainties becomes totally negligible if we do not fix the cosmological parameters in the analysis, but we treat them as nuisance parameters and marginalise over their distribution in order to obtain the final \( f_{\text{NL}} \) estimate.

Even if optimal, this last approach is nevertheless probably still inconvenient. A joint-likelihood evaluation would require a large amount of time and the final gain in the error bar would be significant only for large values of \( f_{\text{NL}} \), but those would produce a significant detection even in the sub-optimal approach.

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