Magnetic relaxation in the Bianchi-I universe

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Abstract

The extended Einstein–Maxwell model and its application to the problem of the evolution of the magnetized Bianchi-I universe are considered. The evolution of medium magnetization is governed by a relaxation type extended constitutive equation. The series of exact solutions to the extended master equations is obtained and discussed. The anisotropic expansion of the Bianchi-I universe is shown to become non-monotonic (accelerated/decelerated) in both principal directions (along the magnetic field and orthogonal to it). A specific type of expansion, the so-called evolution with hidden magnetic field, is shown to appear when the magnetization effectively screens the magnetic field and the latter disappears from the equations for the gravitational field.

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1. Introduction

Observational cosmology and astrophysics indicate that many subsystems of the universe, which have different length-scales: planets, stars, galaxies, clusters and superclusters of galaxies, possess an inherent magnetic field (see, e.g., the reviews [1–3] and references therein). The magnetic field interacts with the material environment and this results in the magnetization of the latter [4]. According to Einstein’s basic ideas, the stress and energy of all three constituents—medium, magnetic field and magnetization—act as sources of gravity field. It seems to be reasonable from a physical point of view that for the majority of astrophysical and cosmological objects the contribution of a material medium, as a source of gravity field, dominates over the contributions of the magnetic field and magnetization; nevertheless, the stress and energy of magnetization itself can be comparable with those of the pure magnetic field. In other words, when we deal with exactly integrable models, for which the magnetic field is considered as a non-negligible source of a gravity field in comparison with the contribution of the material medium, one should also take into account the magnetization as a cross effect.
A cosmological Bianchi-I model seems to be the most convenient for testing this idea due to its three specific features. First, this model is non-stationary; all three constituents, medium, magnetic field and magnetization, can evolve with different rates, thus displaying their dynamic non-equivalence. Second, the Bianchi-I model belongs to the class of exactly integrable ones, thus providing analysis of the singular behaviour of the model. Third, the Bianchi-I model is spatially anisotropic and thus admits a self-consistent description of the uniaxial configuration of the dynamic system containing interacting matter, magnetic field and magnetization. There is a number of anisotropic cosmological models, in which the magnetic field is accompanied by perfect fluid (see, e.g., [5, 6] and references therein), or by non-equilibrium (viscous) cosmic fluid (see, e.g., [7–9]). If in addition to matter and magnetic field the third ‘player’, namely the magnetization, appears in a cosmological dynamics, one can expect that the rate of evolution of the universe modifies. Our expectations are motivated by the analogy with dissipative phenomena in cosmology, described in the framework of causal (extended) thermodynamics [10–24]. In this theory the extended constitutive law for the viscous fluid contains a time parameter, which is known to introduce a specific time scale into the expansion rate. Since the evolution of magnetization usually has the character of relaxation, a new time parameter, say relaxation time, \( \tau \), also must appear and establish a new time scale. The interplay between \( \tau \) and expansion rate parameter(s) can introduce qualitatively new aspects into cosmological dynamics.

In order to formulate a self-consistent Einstein–Maxwell model, taking into account the polarization and magnetization of a non-stationary material medium, two key elements are necessary. The first is an adequate energy–momentum tensor of the electromagnetically active medium, which forms a source term on the right-hand side of the Einstein equations. Here we derive explicitly such a tensor using the Lagrangian formalism for a stationary non-conducting medium with uniaxial symmetry in the case when magneto-electric cross effects are absent. Expressed in terms of induction tensor and Maxwell tensor, this quantity happens to be a symmetrized Minkowski stress–energy tensor (see, e.g., [25–29] for a review, historical details and terminology). When a medium is non-stationary, the required effective stress–energy tensor is assumed to have the same formal structure. Nevertheless, the induction tensor now acquires a new sense: it is the sum of the Maxwell tensor and of a polarization–magnetization tensor; the latter quantity being considered as a new dynamic variable. The second key element of the non-stationary model is the extended constitutive equations, which establish relations between polarization–magnetization of a medium and electromagnetic field strength. We use here the simplest extended constitutive equations of relaxation type, which are formulated phenomenologically, based on the well-known analogues from causal (extended) thermodynamics [10–24]. The main goal of this paper is an application of the formulated extended Einstein–Maxwell (EEM) model to the description of the magnetization dynamics in the universe, considering the corresponding master equations in the context of the Bianchi-I anisotropic cosmological model.

The paper is organized as follows. In section 2 we briefly discuss the principal details of the EEM model. In particular, we introduce the effective stress–energy tensor describing the electromagnetic field and the polarization–magnetization in a medium, as well as introduce the constitutive equations of relaxation type. In section 3 we adopt the EEM model for the symmetry related to the Bianchi-I cosmological model and discuss the reduced Maxwell, Einstein and constitutive equations. In section 4 we consider exact solutions of the dynamic equation for the magnetization with variable relaxation time parameter, discuss general properties of these solutions and some interesting particular cases. In section 5 we obtain exact solutions of the Einstein equations for the case of variable relaxation time parameter. We distinguish two principal submodels in this context. The first submodel describes
a paramagnetic/diamagnetic dust (subsection 5.1) and contains three exactly integrable particular cases. The second submodel describes the so-called longitudinal quasi-vacuum (subsection 5.2) and also contains three exactly integrable particular cases. In section 6 we obtain exact solutions of the EEM model for the case of constant relaxation parameter. In subsection 6.1 we establish a law for the magnetization evolution. In subsections 6.2 we consider the model with hidden induction. In subsections 6.3 three particular cases of exact solutions of the EEM model, describing the submodel with vanishing total longitudinal pressure, are discussed. In subsection 6.4 the example of cosmological dynamics with non-homogeneous nonlinear equation of state of the magnetized matter is studied. Discussions form section 7. The appendix contains a description of the procedure of the variation of the tetrad 4-vectors, which is used in section 2.

2. Extended Einstein–Maxwell model

2.1. Stationary media with uniaxial spatial symmetry

Let us consider a preliminary model with the following action functional:

\[ S = \int d^4x \sqrt{-g} \left\{ R + \frac{2\Lambda}{2\kappa} + L_{\text{matter}} + \frac{1}{4} C_{ikmn} F_{ik} F_{mn} \right\}, \]

as a hint for construction of the extended Einstein–Maxwell model. Here \( R \) is the Ricci tensor, \( g \) is the determinant of \( g_{ik} \), \( \Lambda \) is a cosmological constant, \( \kappa = 8\pi G \), \( G \) is the gravitational constant. The quantity \( F_{mn} \) is the Maxwell tensor, \( F_{mn} = \partial_m A_n - \partial_n A_m, A_m \) is a potential 4-vector of the electromagnetic field. \( L_{\text{matter}} \) is the pure medium contribution to the Lagrangian; we assume that this scalar does not depend on \( F_{mn} \). The quantity \( C_{ikmn} \) is the linear response tensor, which describes the influence of matter on the electromagnetic field. This tensor is assumed to possess the following symmetries:

\[ C_{ikmn} = -C_{kimm} = -C_{iikm} = C_{mnik}, \]

and is a function of the metric \( g_{mn} \), time-like velocity 4-vector of the medium as a whole, \( U^k \), and some space-like vector \( X^k \), pointing out the privileged direction in the medium. These 4-vectors are orthogonal and normalized by unity, i.e.,

\[ g_{pq} U^p X^q = 0, \quad g_{pq} U^p U^q = 1, \quad g_{pq} X^p X^q = -1. \]

When \( C_{ikmn} \) contains the Riemann tensor, the Ricci tensor and the Ricci scalar, we deal with non-minimal Einstein–Maxwell theory \([30, 31]\). When \( C_{ikmn} \) includes the covariant derivative of the velocity 4-vector \( \nabla_i U_k \), the corresponding model describes dynamo-optical effects \([32]\). It is worth stressing that here we restrict ourselves to the case when \( C_{ikmn} = C_{ikmn}[g_{pq}, U^k, X^l] \). The procedure of phenomenological reconstruction of the material tensor \( C_{ikmn} \) is well known \([4, 33]\). In order to obtain \( C_{ikmn} \) as a function of \( g_{mn}, U^k \) and \( X^l \) for the medium with uniaxial symmetry one uses, first, the standard decomposition

\[ C_{ikmn} = \frac{1}{2} U^q U^l \left\{ g_{ik} g_{mn} \xi^{ql} - g_{ik} \xi_{pq} \xi_{mn} (\mu^{-1})^{lp} + \nabla^p \left[ \epsilon_{ikpq} \eta_{mn} + \epsilon_{mnpq} + \epsilon_{ikpn} \eta_{pq} \right] \right\}, \]

where dielectric permittivity tensor, \( \xi^{ql} \), magnetic impermeability tensor, \( (\mu^{-1})^{lp} \), and magneto-electric tensor \( \nabla^p \) are used, defined as

\[ \xi^{ql} = 2C^{ikln} U_k U_l, \quad (\mu^{-1})^{pq} = -\frac{1}{2} \eta^{pik} C_{ikln} \eta_{ln} q, \quad \nabla^p = \eta^{pik} C_{ikln} U_l. \]

The quantity \( \delta_{pq} \) is the Kronecker tensor, \( \epsilon_{ijkl} \) is the Levi-Civita tensor

\[ \delta_{pq} = \delta_{pq}, \quad \epsilon_{ijkl} = \epsilon_{ijkl} U_j, \quad \epsilon_{ikjs} = \frac{E_{ikjs}}{\sqrt{-g}}. \]
$E^{ijk}$ is the completely skew-symmetric Levi-Civita symbol with $E^{0123} = 1$. The Levi-Civita tensor provides the dualization procedure: $F^i_k \equiv \frac{1}{2} \epsilon_{ikmn} F^{mn}$. The second step of reconstruction of the $C^{ikmn}$ tensor for the case of uniaxial symmetry is a phenomenological representation of the tensors $\epsilon^I_p, (\mu^{-1})^I_p$ and $\nu^I_p$. In the simplest case, when the magnetoelectric cross terms are absent ($\nu^I_p = 0$), such relations are

$$e^{lp} = e_\perp \Delta^p + (e_\perp - e_\parallel) X^l X^p, \quad (\mu^{-1})^p = \frac{1}{\mu_\perp} \Delta^p + \left( \frac{1}{\mu_\perp} - \frac{1}{\mu_\parallel} \right) X^l X^p.$$  

(7)

Here the scalar quantities $e_\perp$ and $e_\parallel$ are transversal and longitudinal coefficients of dielectric permittivity, respectively, scalars $\mu_\perp$ and $\mu_\parallel$ represent coefficients of transversal and longitudinal magnetic permeability, respectively, $\Delta^p \equiv g^{lp} - U^l U^p$ is a projector. Finally, (4) can be rewritten as

$$C^{ikmn} = \frac{1}{2\mu_\perp} \left[ (\delta^i_m g^k_n - g^i_n g^k_m) + (e_\perp - 1) \delta^{ik}_p \delta^{mn}_l \right] g^p q^l \frac{1}{\mu_\perp} - \frac{1}{\mu_\parallel} \epsilon^{ik}_{pj} \epsilon^{mn}_{ls} \right].$$  

(8)

When $e_\perp = e_\parallel = e$ and $\mu_\perp = \mu_\parallel = \mu$, the obtained tensor of material coefficients $C^{ikmn}$ covers the well-known isotropic one (see, e.g., [4, 33]).

2.1.1. Maxwell equations. Variation of the action functional (1) with respect to the 4-vector of electromagnetic potential $A_i$ gives the Maxwell equations with vanishing current of free charges

$$\nabla_k (C^{ikmn} F_{mn}) = 0.$$  

(9)

In this case, the induction tensor, $H^{ik}$, is equal to the expression in parentheses.

2.1.2. Equations for gravity field. Variation of the action functional (1) with respect to metric $g_{ik}$ yields

$$R^{ik} - \frac{1}{2} g^{ik} R = \Lambda g^{ik} + \kappa T^{ik}_{(\text{total})}, \quad T^{ik}_{(\text{total})} = T^{ik}_{(\text{matter})} + T^{ik}_{(\text{eff})}.$$  

(10)

The symmetric stress–energy tensor of the material medium $T^{ik}_{(\text{matter})}$, defined as

$$T^{ik}_{(\text{matter})} = \frac{1}{2\sqrt{-g}} \delta_{g_{ik}} \sqrt{-g} L_{(\text{matter})},$$  

(11)

can be written in the standard form

$$T^{ik}_{(\text{matter})} = W U^i U^k + q^i U^k + q^k U^i - P \Delta^{ik} + \Pi^{ik},$$  

(12)

where $W$ is an energy density scalar of the matter, $q^i$ is a heat-flux 4-vector, $P$ is the Pascal pressure and $\Pi^{ik}$ is an anisotropic pressure tensor. The effective stress–energy tensor has the form

$$T^{ik}_{(\text{eff})} = \frac{1}{2\sqrt{-g}} F_{pq} F_{mn} \frac{\delta}{\delta g_{ik}} \sqrt{-g} C^{pqmn},$$  

(13)

it can be calculated directly for the medium with uniaxial symmetry using the decomposition (8) and the following formulae for the variations $\delta U^i$ and $\delta X^i:

$$\delta U^i = \frac{1}{2} \delta g_{ik} (U^i g^k + U^k g^i), \quad \delta X^i = \frac{1}{2} \delta g_{ik} (X^i g^k + X^k g^i).$$  

(14)

The grounds of formulae (14) are presented in the appendix. The variation procedure yields

$$T^{ik}_{(\text{eff})} = \frac{1}{2} g^{ik} C^{pqmn} F_{pq} F_{mn} - \frac{1}{4} \left( C^{mpq} F_{km} + C^{kmpq} F_{im} \right) F_{pq},$$  

(15)

i.e., the effective stress–energy tensor is explicitly symmetric and traceless.
2.1.3. Résumé. When the non-conducting medium with uniaxial symmetry is the stationary one, the Einstein–Maxwell model consists of three ingredients:

(i) Maxwell equations

\[ \nabla_k H^{ik} = 0, \quad \nabla_k F^{*ik} = 0, \]

(ii) constitutive equations

\[ H^{ik} = C^{ikmn} F_{mn}, \]

(iii) gravity field equations

\[ R^{ik} - \frac{1}{2} g^{ik} R = \Lambda g^{ik} + \kappa \left[ W U^i U^k + q^i U^k + q^k U^i - P \Delta^{ik} + \Pi^{ik} \right] + \kappa \left[ \frac{1}{2} g^{ik} F_{mn} F^{mn} - \frac{1}{2} \left( H^m F^k_m + H^k F^m_m \right) \right]. \]

It is the exact result, which follows from the variation procedure. The electromagnetic part of the total stress–energy tensor in the presented form coincides with the symmetrized Minkowski electromagnetic energy–momentum tensor (see, e.g., the review [29]). It is manifestly symmetric, traceless and does not depend explicitly on the choice of \( U^i \).

2.1.4. Extended Einstein–Maxwell model. When the medium is non-stationary, the Einstein–Maxwell model should be modified accordingly; the stationary equations (16)–(18) can be used as a hint. Our ansatz for such extension is the following: we consider the electrodynamic equations in the same form (16), but express the induction tensor in terms of polarization–magnetization tensor \( M^{ik} \); instead of stationary constitutive equations (17) we introduce an extended constitutive equation of relaxation type for the tensor of polarization–magnetization; the gravity field equations have the same form (18), but \( H^{ik} \) is replaced by \( F^{ik} + M^{ik} \), i.e., the basic set of master equations for the extended Einstein–Maxwell model is

\[ \nabla_k (F^{ik} + M^{ik}) = 0, \quad \nabla_k F^{*ik} = 0, \]

\[ \tau DM^{ik} + M^{ik} = \chi^{ikmn} F_{mn}, \]

\[ R^{ik} - \frac{1}{2} g^{ik} R = \Lambda g^{ik} + \kappa \left[ W U^i U^k + q^i U^k + q^k U^i - P \Delta^{ik} + \Pi^{ik} \right] + \kappa \left[ \frac{1}{2} g^{ik} F_{mn} F^{mn} - \frac{1}{2} \left( F^m F^k_m + F^k F^m_m \right) \right] + \kappa \left[ \frac{1}{2} g^{ik} M_{mn} F^{mn} - \frac{1}{2} \left( M^m F^k_m + M^k F^m_m \right) \right]. \]

Here \( \tau \) is relaxation time and \( D \equiv U^k \nabla_k \) is a convective derivative, \( \chi^{ikmn} \) is a linear susceptibility tensor, which can be expressed in terms of \( C^{ikmn} \) as

\[ \chi^{ikmn} = C^{ikmn} - \frac{1}{4} (g^{im} g^{kn} - g^{in} g^{km}). \]

The quantity \( M^{ik} \) is considered in this context as a new variable. Such extension of the constitutive equations can be regarded as phenomenologically motivated, when the well-known analogues from rheology and extended thermodynamics [10–24] are taken into account. Thus, the novelty of the presented model is connected, first, with the extended constitutive equation (20), second, with the modified electromagnetic source in the gravity field equations (see the last line in (21)).

3. Bianchi-I universe

Consider the rotationally isotropic Bianchi-I cosmological model [5, 6] with the line element

\[ ds^2 = dt^2 - a^2(t) \left( (dx^1)^2 + (dx^2)^2 \right) - c^2(t) (dx^3)^2. \]
We suppose that the velocity 4-vector of matter, $U^k$, has the form $U^k = \delta^k_0$ and the space-like vector $X^i$ is $X^i = \delta^i_3 c(t)$. They satisfy the relations $DU^i = 0$ and $DX^i = 0$. The symmetry of the model prescribes the non-diagonal components of the total stress–energy tensor vanish. In this sense, to make the model self-consistent one can consider, for instance, the case when the magnetic field 4-vector and the magnetization 4-vector are parallel to the $x^3$ axis. For such a configuration of the electromagnetic field the total stress–energy tensor, $T_{ik}^{(\text{total})}$, has four non-vanishing components:

$$
T_{00}^{(\text{total})} \equiv W = W + \mathcal{X},
$$

$$
T_{11}^{(\text{total})} = T_{22}^{(\text{total})} \equiv -P_{(\parallel)} = - (P_{(\parallel)} + \mathcal{X}),
$$

$$
T_{33}^{(\text{total})} \equiv -P_{||} = - (P_{||} - \mathcal{X}),
$$

\begin{equation}
\mathcal{X} \equiv \frac{1}{2} \left[ F_{12} F_{12} + M_{12} F_{12} \right].
\end{equation}

(24)

Here $P_{(\parallel)} \equiv P_{(1)} = P_{(2)}$ and $P_{||} \equiv P_{(3)}$ are the transversal and longitudinal pressure components, respectively, coinciding with the corresponding eigenvalues of the material part of the total stress–energy tensor.

3.1. Reduced Maxwell equations

We assume that the solutions inherit the spacetime symmetry and the quantities $F_{ik}$ and $M_{ik}$ are functions of cosmological time only. Thus, it follows from the second subsystem of Maxwell equations (16) that the spatial components of the Maxwell tensor are constant, i.e., $F_{a\beta} = \text{const}$. The only non-vanishing term is $F_{12} = \text{const}$, since the magnetic field points along the $x^3$ axis. Note that the tetrad component of the magnetic field, $B(t)$, is connected with the constant $F_{12}$ by the following relationship:

\begin{equation}
B(t) \equiv F_{ik} U^k X^i = \frac{F_{12}}{a^2(t)} \equiv B_0 \left( \frac{a(t_0)}{a(t)} \right)^2.
\end{equation}

(25)

We assume that the electric field, electric polarization and magneto-electric cross-effect are absent and the medium is locally neutral. These requirements guarantee that the Maxwell equations are satisfied identically. Thus, in the model under consideration the evolution of the magnetization is governed only by the constitutive equation (20).

3.2. Reduced constitutive equations

When $U^i = \delta^i_0$ in the metric (23) the acceleration vector $DU^i$ vanishes, yielding $D\eta^{kl} = D(e^{kl} U_k) = 0$. The magnetization 4-vector has only one component:

$$
M^i \equiv M^{ik} U_k = -X^i M,
$$

\begin{equation}
M^{ik} = -\eta^{ik} M = \eta^{ik} X_i M.
\end{equation}

(26)

Extended constitutive equations (20) reduce to one equation of the relaxation type

\begin{equation}
\tau \dot{M}(t) + M = \left( \frac{1}{\mu ||} - 1 \right) \frac{a^2(t_0)}{a^2(t)} B_0,
\end{equation}

(27)

where the dot denotes the derivative with respect to time.

3.3. Reduced Einstein’s equations

The gravity field equations reduce to the following system:

\begin{equation}
\left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a}}{a} \frac{\dot{c}}{c} = \Lambda + \kappa (W + \mathcal{X}),
\end{equation}

\begin{equation}
\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a} \frac{\dot{c}}{c} = \Lambda - \kappa (P_{(\parallel)} + \mathcal{X}),
\end{equation}

(28)

(29)
2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \kappa (P_\parallel - \mathcal{X}). 
\tag{30}

Differentiation of Einstein’s equations leads to the conservation law

\begin{align*}
W + 2 \left(\frac{\dot{a}}{a}\right) [W + P_{(0)}] + \left(\frac{\dot{c}}{c}\right) [W + P_{(i)}] + \mathcal{X} + 4 \left(\frac{\dot{a}}{a}\right) \mathcal{X} = 0.
\end{align*}
\tag{31}

Note that \( \mathcal{X} \) and its derivative enter the conservation law with the multiplier, which does not depend on \( c(t) \) (on \( a(t) \) only). The function \( \mathcal{X}(t) \) is now

\begin{align*}
\mathcal{X}(t) = \frac{1}{2} B(t) [M(t) + B(t)] = \frac{1}{2} B_0^2 \frac{a^2(t_0)}{a^2(t)} \left[ 1 + \frac{a^2(t)}{a^2(t_0)} \frac{M(t)}{B_0} \right].
\end{align*}
\tag{32}

When the magnetization \( M \) vanishes, the quantity \( \mathcal{X}(t) \) is non-negative. Nevertheless, when \( M \) is non-vanishing, \( \mathcal{X}(t) \) can be negative during some time interval, or be equal to zero. The total energy density \( \mathcal{W} = W + \mathcal{X} \) is assumed to remain positive.

4. Magnetic relaxation with variable time parameter \( \tau(t) \)

The relaxation equation (27) and the quantity \( \mathcal{X}(t) \), the source of the gravitational field (32), contain the function \( a(t) \) and do not contain \( c(t) \). This means that it is reasonable to split the master equations into two subsystems, dealing with the longitudinal \( (P_\parallel, B(t), M(t)) \) and transversal \( (P_{(i)}, a(t)) \) quantities, as well as with functions \( c(t) \) and \( a(t) \), describing the evolution of the universe in the longitudinal and transverse directions, respectively.

Note that the sign of the right-hand side of equation (27) depends on the sign of the expression \( 1/\mu - 1 \). According to the standard definitions, when \( \mu > 1 \) we deal with the so-called paramagnetic medium; in this case the expression in parentheses in (27) is negative.

When \( 0 < \mu \ll 1 \), the medium can be indicated as diamagnetic, this case relates to the positive expression in parentheses in (27). Finally, when \( \mu \gg 1 \), one can say that the medium is in a ferromagnetic phase. We do not consider the cases with negative and vanishing \( \mu \).

In principle, \( \mu \) can be treated as a function of time. This means that the sign of the expression \( 1/\mu - 1 \) may change with time. This problem itself is very interesting, but numerical calculations are needed to solve the corresponding master equations. We focus here on the analytical solutions, and do not consider the case of variable \( \mu \).

Generally, the master equations of the EEM model are self-consistent. In order to find \( M(t) \) from (27) we have to know \( a(t) \). In order to find \( a(t) \) from (30) we should know \( \mathcal{X}(t) \) which includes \( M(t) \). To solve such a problem one should introduce some extra ansatz. We prefer to start from equation (27), and our ansatz concerns the function \( \tau(t) \). There are at least three ways to introduce the relaxation time parameter \( \tau(t) \). The first approach (which is the simplest) assumes that \( \tau \) is constant. We consider the model with \( \tau(t) = \tau_0 \) in section 6.

In the second approach the relaxation time \( \tau(t) \), the bulk viscosity coefficient \( \zeta(t) \), etc, are modelled as power-law functions of the energy density scalar \( W \) (see, e.g., [17–21]). We do not consider such options in this paper. In the third approach \( \tau(t) \) is assumed proportional to the inverse rate of expansion. For instance, in the isotropic Friedmann model \((a(t) = c(t))\), the new dimensionless variable \( \xi = \tau H(t) \) is frequently used, where \( H(t) = \dot{a}/a \) is the Hubble expansion parameter (see, e.g., [34]). In this section, we consider the third version of the representation of \( \tau(t) \).

Since only the function \( a(t) \) enters the relaxation equation and the function \( \mathcal{X}(t) \), we consider \( \tau(t) = \xi \frac{a(t)}{a(t_0)} \). It is convenient to use a new variable \( x = \frac{a(t)}{a(t_0)} \) and a function \( H(a)(x) = \frac{x}{\tau} \), for which

\begin{align*}
\frac{\dot{a}}{a} = H(a)(x), \quad \frac{\ddot{a}}{a} = \frac{1}{2} x \frac{d}{dx} H^2(a)(x) + H^2(a).
\end{align*}
\tag{33}
The three-parameter family of solutions to (27) reads

\[ M(x, \xi, M(t_0), \mu_\parallel) = M_0 x^{-\frac{1}{2}} + \frac{B_0}{(1 - 2\xi^2)} \left( \frac{1}{\mu_\parallel} - 1 \right) \left[ x^{1-\xi} - x^{-\frac{1}{\xi}} \right]. \quad (34) \]

Here \( 2\xi \neq 1 \), the special case \( \xi = \frac{1}{2} \) will be considered in subsection 4.2. The corresponding expression for \( \mathcal{X}(x, \xi, M_0, \mu_\parallel) \) is

\[ \mathcal{X}(x, \xi, M_0, \mu_\parallel) = \frac{1}{2} B_0 \left[ K_1(\xi, \mu_\parallel)x^{-\frac{1}{2}} + K_2(\xi, M_0, \mu_\parallel)x^{-(2+\frac{1}{\xi})} \right], \quad (35) \]

where

\[ K_1(\xi, \mu_\parallel) = \frac{(1 - 2\xi^2\mu_\parallel)}{\mu_\parallel(1 - 2\xi^2)} \quad \text{and} \quad K_2(\xi, M_0, \mu_\parallel) = \frac{M_0}{B_0} + \frac{(\mu_\parallel - 1)}{\mu_\parallel(1 - 2\xi^2)}, \quad (36) \]

are constant. We use for simplicity the definitions \( M_0 \equiv M(t_0) \) and \( B_0 \equiv B(t_0) \), and below we omit the parameters \( \xi, M_0 \) and \( \mu_\parallel \) in the arguments of \( M, \mathcal{X} \) and \( K_1, K_2 \).

4.1. Non-resonant magnetic relaxation: \( \xi > 0, \xi \neq \frac{1}{2} \)

For such \( \xi \) the magnetization decreases when the universe expands, i.e., \( M(t \to \infty) \to 0 \) when \( a(t) \) increases. The relaxation of the initial magnetization \( M_0 \) is characterized by the function \( x^{-\frac{1}{\xi}} \). The relaxation of the induced magnetization \( F_{12}(1/\mu_\parallel - 1) \) is characterized in the limit \( x \to \infty \) by the function \( x^{-\frac{1}{\xi}} \), when \( \xi > \frac{1}{2} \), and by the function \( x^{-2} \), when \( \xi < \frac{1}{2} \).

4.1.1. Extrema and zeros of the function \( \mathcal{X}(t) \). Formula (35) shows that the behaviour of \( \mathcal{X}(x) \) is monotonic when \( K_1, K_2 > 0 \). When \( K_1 K_2 < 0 \), there exists one extremum at \( t = t_* \) such that

\[ \left( \frac{a(t_*)}{a(t_0)} \right)^{2-\xi} = -\frac{4\xi K_1}{K_2(2\xi + 1)}, \quad \mathcal{X}(t_*) = \frac{B^2(t_0)(1 - 2\xi^2\mu_\parallel)}{2\mu_\parallel(2\xi + 1)}. \quad (37) \]

In addition, the inequality \( t_* > t_0 \) is valid for \( 4\xi |K_1| > (2\xi + 1)|K_2| \). It is a requirement to the ratio \( M_0/B_0 \), and we suppose it is valid. The extremum is a minimum with the negative value \( \mathcal{X}(t_*) \), when \( \frac{1}{\xi} < 2\mu_\parallel \) and a maximum with the positive value \( \mathcal{X}(t_*) \), when \( \frac{1}{\xi} > 2\mu_\parallel \), for both paramagnetic and diamagnetic media. Taking into account the definitions of \( K_1, K_2 \), (36), one can find the following possibilities for \( \mathcal{X}(t) \) to have an extremum. When the medium is paramagnetic, i.e., \( \mu_\parallel > 1 \), one obtains three different situations.

(i) \( \frac{1}{\xi} < 2, K_2 < 0 \). In this case there is a minimum. The curve \( \mathcal{X}(t) \) passes its zero-value point, if \( \mathcal{X}(t_0) > 0 \), and tends asymptotically to zero as \( \mathcal{X}(t) \propto a^{-2-\xi} \), when \( a(t) \to \infty \).

(ii) \( \frac{1}{\xi} > 2\mu_\parallel, K_2 < 0 \). This case corresponds to a maximum. The curve \( \mathcal{X}(t) \) passes its zero-value point, if \( \mathcal{X}(t_0) < 0 \), and tends asymptotically to zero as \( \mathcal{X}(t) \propto a^{-4} \), when \( a(t) \to \infty \).

(iii) \( 2 < \frac{1}{\xi} < 2\mu_\parallel, K_2 > 0 \). This case is analogous to the first one.

Likewise, for the diamagnetic medium \( \mu_\parallel < 1 \) one obtains the following situations.

(i) \( \frac{1}{\xi} > 2, K_2 < 0 \) (maximum).

(ii) \( \frac{1}{\xi} < 2\mu_\parallel, K_2 < 0 \) (minimum).

(iii) \( 2 > \frac{1}{\xi} > 2\mu_\parallel, K_2 > 0 \) (maximum).
Besides, there exists a time moment, $t_{\text{on}}$, when the source $\mathcal{X}$ vanishes. According to (35) the condition $\mathcal{X}(t_{\text{on}}) = 0$ can be satisfied when

$$\left(\frac{a(t_{\text{on}})}{a(t_0)}\right)^{2-\frac{1}{\gamma}} = -\frac{K_1}{K_2},$$

but this is possible if the constants $K_1$ and $K_2$ have opposite signs. Thus, the existence of zeros of $\mathcal{X}(t)$ assumes the same requirements $K_1K_2 < 0$, as conditions for the existence of extrema.

4.1.2. Asymptotic case: fast relaxation, $\xi \to 0$. When $\xi \to 0$, formulae (34) and (35) give

$$M(x)|_{\xi \to 0} = B_0x^{-2}\left(\frac{1}{\mu_{||}} - 1\right), \quad \mathcal{X}(x)|_{\xi \to 0} = \frac{B^2(t_0)}{2\mu_{||}}x^{-4}.$$  \hspace{1cm} (39)

In a paramagnetic medium the ratio $M(x)|_{\xi \to 0}/B_0$ is negative, in a diamagnetic medium it is positive. In both cases $\mathcal{X}(x)|_{\xi \to 0}$ is positive.

4.1.3. Asymptotic case: slow relaxation, $\xi \to \infty$. When $\xi \to \infty$, one obtains

$$M(x)|_{\xi \to \infty} = M_0, \quad \mathcal{X}(x)|_{\xi \to \infty} = \frac{1}{2}B_0x^{-4}(B_0 + M_0x^2).$$ \hspace{1cm} (40)

In these formulae $\mu_{||}$ is absent.

4.1.4. Particular case: $2\xi\mu_{||} = 1$. Studying the extrema of the function $\mathcal{X}(t)$ we have found that the condition $2\xi\mu_{||} = 1$ (or, equivalently, $K_1 = 0$) gives some threshold value for the parameter $\xi$. Let us consider this particular case separately. When $2\xi\mu_{||} = 1$

$$M(x)|_{\xi = \frac{1}{2\mu_{||}}} = (M_0 + B_0)x^{-2\mu_{||}} = B_0x^{-2},$$ \hspace{1cm} (41)

and the function $\mathcal{X}(t)$ monotonically tends to zero as $x \to \infty$:

$$\mathcal{X}(x)|_{\xi = \frac{1}{2\mu_{||}}} = \frac{1}{2}B_0\xi(0 + M_0)x^{-2(1+\mu_{||})} \equiv \lambda_0x^{-2(1+\mu_{||})},$$ \hspace{1cm} (42)

remaining positive or negative depending on the sign of its initial value $\lambda_0$. Obviously, the value $\mathcal{X}(x)|_{\xi = \frac{1}{2\mu_{||}}}$ is positive or negative depending on the sign of the sum $M_0 + B_0$, and is equal to zero identically when $M(t_0) = -B(t_0)$. In the last case the information about the magnetic field and magnetization does not enter the Einstein field equations and we deal with a so-called ‘hidden’ magnetic field [31]. Note that in the ferromagnetic phase $\mu_{||} \gg 1$, which corresponds here to the model of fast relaxation $\xi \ll 1$, the source $\mathcal{X}(t)$ disappears very quickly.

4.1.5. Particular case: vanishing initial magnetization, $M_0 = 0$. This particular case can be studied in more detail. The minimum of the function $\mathcal{X}(t)$ exists at the point given by

$$\left(\frac{a(t_{\text{on}})}{a(t_0)}\right)^{2-\frac{1}{\gamma}} = \frac{4\xi(2\xi\mu_{||} - 1)}{(2\xi + 1)(\mu_{||} - 1)}.$$ \hspace{1cm} (43)

Since we put $M_0 = 0$, we should check especially the condition $a(t_{\text{on}}) > a(t_0)$. When the medium is paramagnetic, this condition is satisfied for $\xi > \frac{1}{2}$. This corresponds to the first case of the classification of the extrema: the function $\mathcal{X}(t)$ starts from the positive initial value $\lambda_0$, takes a zero value at $t_{\text{on}}$, reaches its minimum at $t_{\text{on}}$, and tends to zero asymptotically, remaining negative. Here

$$\lambda_0 = \frac{B_0^2}{2}, \quad \left(\frac{a(t_{\text{on}})}{a(t_0)}\right)^{2-\frac{1}{\gamma}} = \frac{(2\xi\mu_{||} - 1)}{(\mu_{||} - 1)} > 1,$$ \hspace{1cm} (44)
solution of the form $W(a)$.\footnote{The vacuum-type equation of state \( c(t) \) picks out (30) from other Einstein field equations.}

In a paramagnetic medium this extremum is a minimum, the condition also depends on \( a \) depends on time via \( X \).

In this case the magnetization decreases when \( 1 + 2|\mu| \to \infty \). The term ‘resonance’ relates to a special case when the relaxation parameter \( \tau(t) \) coincides with half the characteristic expansion time \( 1/H(a) \), where \( H(a) \equiv \frac{\mu}{\rho} \). The function \( \mathcal{X}(x) \) reads

\[
\mathcal{X}(x) = \frac{B_0^2(1 - 2|\mu|)}{2|\mu|(2|\mu| + 1)} \left[ \frac{4|\mu|(2|\mu| - 1)}{(\mu - 1)(2|\mu| + 1)} \right] x^{\frac{1}{2|\mu|}} < 0. \tag{45}
\]

The inequalities for the diamagnetic medium can be obtained analogously.

4.2. Resonance magnetic relaxation: \( \xi = \frac{1}{2} \)

When \( \xi = \frac{1}{2} \), the special solution to (27) exists

\[
M(x)_{|\xi| = \frac{1}{2}} = x^{-2} \left[ M_0 + 2B_0 \left( \frac{1}{\mu} - 1 \right) \log x \right], \tag{46}
\]

which tends to zero as \( t \to \infty \) (\( x \to \infty \)). The term ‘resonance’ relates to a special case when the relaxation parameter \( \tau(t) \) coincides with half the characteristic expansion time \( 1/H(a) \), where \( H(a) \equiv \frac{\mu}{\rho} \). The function \( \mathcal{X}(x)_{|\xi| = \frac{1}{2}} \) reads

\[
\mathcal{X}(x)_{|\xi| = \frac{1}{2}} = \frac{1}{2} B_0 x^{-4} \left[ B_0 + M_0 + 2B_0 \left( \frac{1}{\mu} - 1 \right) \log x \right]. \tag{47}
\]

The function \( \mathcal{X}(x)_{|\xi| = \frac{1}{2}} \) reaches its extremum value

\[
\mathcal{X}(t_*) = \frac{B_0^2(t_*)}{4} \left( \frac{1}{\mu} - 1 \right), \tag{48}
\]

when

\[
\log \left( \frac{a(t_*)}{a(t_0)} \right) = \frac{1}{4} + \frac{\mu}{2(\mu - 1)} \left( 1 + \frac{M_0}{B_0} \right). \tag{49}
\]

In a paramagnetic medium this extremum is a minimum, the condition \( a(t_*) > a(t_0) \) assumes that \( M_0/B_0 > (1 - 3|\mu|)/2|\mu| \). When the medium is diamagnetic, the extremum is a maximum; the condition \( a(t_*) > a(t_0) \) is satisfied if \( M_0/B_0 < (1 - 3|\mu|)/2|\mu| \).

4.3. Magnetic instability: \( \xi < 0 \) and \( M_0 \neq \frac{B_0(1 - \mu_0)}{\mu_0(1 + 2|\mu|)} \)

In this case the magnetization

\[
M(x)_{|\xi| < 0} = M_0 x^{\frac{1}{2|\mu|}} + \frac{B_0}{(1 + 2|\mu|)} \left( \frac{1}{\mu} - 1 \right) \left[ x^{-2} - x^{\frac{1}{2|\mu|}} \right] \tag{50}
\]

increases as \( x^{1/|\xi|} \). In the special case \( M_0 = \frac{B_0(1 - \mu_0)}{\mu_0(1 + 2|\mu|)} \) the term \( x^{\frac{1}{2|\mu|}} \) disappears and the magnetization decreases as \( x^{-2} \). The function

\[
\mathcal{X}(x)_{|\xi| < 0} = \frac{1}{2} B_0^2 \left[ K_1 x^{-4} + K_2 x^{-2|\mu|} \right] \tag{51}
\]

decreases when \( \frac{1}{|\xi|} < 2 \) and increases when \( \frac{1}{|\xi|} > 2 \). When \( \frac{1}{|\xi|} = 2 \), it tends asymptotically to the constant value

\[
\mathcal{X}(\infty) = \frac{1}{2} B_0 \left[ M_0 + B_0 \left( \frac{\mu - 1}{2|\mu|} \right) \right]. \tag{52}
\]

5. Cosmological evolution in the case of variable relaxation time

Below we discuss exactly solvable models of four selected types, describing cosmological evolution of the magnetizable medium. In order to explain our choice let us note that when \( \mathcal{X} \) depends on time via \( a(t) \) only, equation (30) does not contain information about \( c(t) \), if \( P_|| \) also depends on \( a \) only, i.e., \( P_|| = P_||(a(t)) \). The vacuum-type equation of state \( W + P_|| = 0 \) belongs to this class of models, since \( c(t) \) disappears from conservation law (31), and its solution of the form \( W(a) = -P_||(a) \) picks out (30) from other Einstein field equations.
5.1. First example of cosmological evolution: paramagnetic/diamagnetic dust

When the medium behaves as a dust, i.e., $P_{\parallel} = P_{\perp} = 0$, the optimal way to solve the master equations is the following. First, we solve (30), which in terms of variable $x$ takes the form of the equation for $H_{(a)}(x)$:

$$x^{-2} \frac{d}{dx} \left( x^{3} H_{(a)}^{2} \right) = \Lambda + \frac{1}{2} \kappa B_{0}^{2} \left[ K_{1} x^{-4} + K_{2} x^{-(2+\xi)} \right]. \quad (53)$$

Second, we search for the function $a(t)$ using the quadratures

$$t - t_{0} = \pm \int_{1}^{a(t)} \frac{dx}{x H_{(a)}(x)}. \quad (54)$$

When $a(t)$ is found, it is convenient to search for $c(t)$ using the substitution

$$Y(t) = \frac{c(t)}{c(t_{0})} \sqrt{a(t_{0})}. \quad (56)$$

Finally, we search for $W(t)$ using (28) with obtained $a(t)$, $c(t)$, $\dot{a}$ and $\dot{c}$. In the process of integration of equation (53) a new resonance value of the parameter $\xi$, namely $\xi = 1$, appears. Indeed, when $\xi \neq 1$, the solution is

$$H_{(a)}(x) = \pm \sqrt{\Lambda x^{3} + \frac{K_{3} x^{-3} - \kappa B_{0}^{2}}{2} \left[ K_{1} x^{-4} + K_{2} \frac{\xi}{(1-\xi)} x^{-(2+\xi)} \right]}, \quad (57)$$

where

$$K_{3} \equiv H_{(a)}^{2}(t_{0}) - \frac{\Lambda}{3} + \frac{\kappa B_{0}^{2}}{2} \left[ K_{1} + K_{2} \frac{\xi}{(1-\xi)} \right]. \quad (58)$$

When $\xi = 1$, i.e., the relaxation time parameter $\tau(t)$ coincides with $H_{a}^{-1}$, we should replace (57) by

$$H_{(a)}(x) = \pm \sqrt{\Lambda x^{3} + \frac{K_{3} x^{-3} - \kappa B_{0}^{2}}{2} [K_{1} x^{-4} + K_{2} x^{-(2+\xi)} \log x]}, \quad (59)$$

where

$$\tilde{K}_{3} \equiv H_{(a)}^{2}(t_{0}) - \frac{\Lambda}{3} + \frac{\kappa B_{0}^{2}}{2} K_{1}. \quad (60)$$

5.1.1. General properties of solution $H_{(a)}(t)$.

Asymptotics. The quantity $H_{(a)}$ plays the role of the Hubble function, describing the universe evolution in the plane $x^{1} O x^{2}$. When $\Lambda \neq 0$ and $\frac{1}{2} > -2$, the (positive) asymptotic value of this function, $H_{a}(\infty) = \sqrt{\frac{\Lambda}{2}}$, is well known for the Friedmann isotropic model. The corresponding asymptotic behaviour of $a(t)$ is $a(t) \propto \exp \left\{ \sqrt{\frac{\Lambda}{2}} t \right\}$. When $\Lambda = 0$, the asymptotic formula for $H_{(a)}$ is predetermined by the value of the parameter $\xi$:
(a) if \( \frac{1}{\xi} < 1 \), \( H(a) \propto x^{-\left(1+\frac{1}{\xi}\right)} \) and \( a(t) \propto t^{\frac{2}{\xi}} \).
(b) if \( \frac{1}{\xi} > 1 \), \( H(a) \propto x^{-\frac{2}{\xi}} \), and \( a(t) \propto t^{\frac{1}{\xi}} \).

When \( \frac{1}{\xi} = -2 \), the last term in the expression for \( H(a) \) (57) does not depend on time and redefines effectively the cosmological constant

\[
\Lambda \to \Lambda^* = \Lambda + \frac{\kappa B_0^2}{2} \left( \frac{M_0 + \mu_{||}}{2 \mu_{||}} \right).
\]

(61)

When \( \frac{1}{\xi} < -2 \) the last term in the expression for \( H(a) \) (57) is the leading-order term at \( t \to \infty \), and the universe collapses in the cross section \( x^1 \Omega x^2 \). It is an exotic case, and we do not focus on it.

Extrema. Generally, the behaviour of the \( H(a)(x) \) function is non-monotonic, i.e., there are intervals with \( \dot{H}(a)(t) < 0 \) and \( \dot{H}(a)(t) > 0 \). The necessary condition for the existence of extrema marks the points \( x(1), x(2), \ldots, x(s) \), at which

\[
K_{3x(s)} = \frac{\kappa B_0^2}{6} \left[ 4K_1 + K_2 \frac{2\xi + 1}{(1 - \xi)} x^{(2 - \frac{1}{\xi})} \right] = 0.
\]

(62)

Formally speaking, this equation can give \( (m) \) real roots, the function \( H(a) \) can possess \( (m) \) extrema, and, consequently, \( (m) \) points can appear, in which the transition from the accelerated expansion to the decelerated one (and vice versa) takes place. To illustrate these possibilities, consider now several particular cases.

5.1.2. \( K_1 = K_3 = 0 \). Let us choose the initial values \( H(a)(t_0), M_0, F_{12}, a(t_0) \), as well as \( \xi \) and \( \mu_{||} \) parameters in an appropriate manner to provide the relations \( K_1 = 0 \) and \( K_3 = 0 \). It is possible when

\[
\frac{1}{\xi} = 2\mu_{||}, \quad \frac{\Lambda}{3} = H^2(a)(t_0) + \frac{\kappa B_0(B_0 + M_0)}{2(2\mu_{||} - 1)}, \quad K_2 = \frac{B_0 + M_0}{B_0}.
\]

(63)

For this case, \( \lambda(x) \) is the monotonic function

\[
\lambda(t) = \frac{1}{\kappa} (1 - 2\mu_{||}) \left( H^2(a)(t_0) - \frac{\Lambda}{3} \right) x^{-2(1+\mu_{||})},
\]

(64)

and \( H(a)(x) \) simplifies essentially

\[
H(a)(x) = \pm \sqrt{\frac{\Lambda}{3} + \left( H^2(a)(t_0) - \frac{\Lambda}{3} \right) x^{-2(1+\mu_{||})}}.
\]

(65)

Here \( \mu_{||} \neq \frac{1}{\xi} \) to avoid the relation \( \xi = 1 \). For such \( H(a)(x) \) equation (54) can be easily integrated

\[
a(t) = \frac{a(t_0)}{a(t)} = \left\{ \cosh \left[ \sqrt{\frac{\Lambda}{3} (1+\mu_{||})(t - t_0)} \right] + H(a)(t_0) \sqrt{\frac{3}{\Lambda}} \sinh \left[ \sqrt{\frac{\Lambda}{3} (1+\mu_{||})(t - t_0)} \right] \right\}^{\frac{1}{1+\mu_{||}}}.
\]

(66)

\( H(a)(t) \) takes an explicit form

\[
H(a)(t) = \sqrt{\frac{\Lambda}{3}} \left\{ H(a)(t_0) + \sqrt{\frac{2}{3}} \tanh \left[ \sqrt{\frac{2}{3} (1+\mu_{||})(t - t_0)} \right] \right\} \left\{ \sqrt{\frac{2}{3} + H(a)(t_0) \tanh \left[ \sqrt{\frac{2}{3} (1+\mu_{||})(t - t_0)} \right]} \right\}. \]

(67)
The asymptotic behaviour of this solution is given by

\[
a(t \to \infty) \Rightarrow a(t_0) \left[ \frac{1}{2} \left( 1 + H_{(a)}(t_0) \sqrt{\frac{3}{\Lambda}} \right) \right]^{\frac{1}{2|m|}} e^{\sqrt{\frac{\Lambda}{3}}(t - t_0)}, \tag{68}
\]

\[
H_{(a)}(t \to \infty) \to \sqrt{\frac{\Lambda}{3}}. \tag{69}
\]

Returning to the function \(c(t)\), note that equation (55) with \(a(t)\), given by (66), is the Hill equation [35] with imaginary argument. Since \(X(t \to \infty) \to 0\), the appropriate asymptotics of \(Y(t)\) and \(c(t)\) are

\[
Y(t \to \infty) \propto e^{\left\{ \frac{3}{2} \sqrt{\frac{\Lambda}{3}} t \right\}}, \quad c(t \to \infty) \propto e^{\left\{ \frac{\sqrt{\Lambda}}{3} t \right\}}, \tag{70}
\]

thus, the universe isotropizes. To specify \(c(t)\) consider three special examples.

First special example: \(H_{(a)}(t_0) \equiv \sqrt{\frac{\Lambda}{3}}\). When \(H_{(a)}(t_0) \equiv 0\) and the behaviour of \(a(t)\) is governed by the de Sitter expansion law

\[
a(t) = a(t_0) e^{\sqrt{\frac{\Lambda}{3}}(t - t_0)}. \tag{71}
\]

The solution for \(c(t)\) is

\[
c(t) = \left[ c(t_0) + \frac{\kappa W_0}{2\Lambda a^2(t_0)} \right] e^{\sqrt{\frac{\Lambda}{3}}(t - t_0)} - \frac{\kappa W_0}{2\Lambda a^2(t_0)} e^{-2\sqrt{\frac{\Lambda}{3}}(t - t_0)} \tag{72}
\]

The asymptotic behaviour of \(c(t)\) at \(t \to \infty\) is the same as for \(a(t)\). This solution corresponds to the model of hidden induction, mentioned in subsection 4.1.4. The energy density scalar behaves as \(W(t) = W_0 c(t_0) a(t_0)^2 a(t)^2\).

Second special example: \(H_{(a)}(t_0) = 0\). When \(H_{(a)}(t_0) \equiv 0\), one obtains the simplified formulae

\[
H_{(a)}(t) = \sqrt{\frac{\Lambda}{3}} \tanh \left[ \sqrt{\frac{\Lambda}{3}}(1 + \mu_{||})(t - t_0) \right], \tag{73}
\]

\[
a(t) = a(t_0) \left[ \cosh \sqrt{\frac{\Lambda}{3}}(1 + \mu_{||})(t - t_0) \right]^{\frac{1}{2|m|}}, \tag{74}
\]

\[
\kappa \chi(t) = \frac{\Lambda}{3} (2\mu_{||} - 1) \cosh^{-2} \left[ \sqrt{\frac{\Lambda}{3}}(1 + \mu_{||})(t - t_0) \right]. \tag{75}
\]

The equation for \(Y\) can be transformed into the Legendre equation

\[
(1 - z^2)Y''(z) - 2zY'(z) + Y \left[ \nu(v + 1) - \frac{\lambda^2}{1 - z^2} \right] = 0, \tag{76}
\]

where

\[
z = \tanh \left[ \sqrt{\frac{\Lambda}{3}}(1 + \mu_{||})(t - t_0) \right], \quad \nu(v + 1) = \frac{5(2\mu_{||} - 1)}{4(1 + \mu_{||})^2}, \quad \lambda^2 = \frac{9}{4(1 + \mu_{||})^2}. \tag{77}
\]
Thus, one obtains
\[
\frac{c(t)}{c(t_0)} = [1 - z^2(t)]^{\frac{1}{\mu}} \left\{ C_1 P_{\nu}^0(z(t)) + C_2 Q_{\nu}^0(z(t)) \right\},
\]  
where \( P_{\nu}^0(z) \) and \( Q_{\nu}^0(z) \) are the associated Legendre functions of the first and second kinds, respectively (see, \cite{36}, 8.1.1). \( C_1 \) and \( C_2 \) are the constants of integration
\[
C_1 = \frac{1}{\text{Wr}(0)} \left( \left( P_{\nu}^0 \right)'(0) - J Q_{\nu}^0(0) \right), \quad C_2 = \frac{1}{\text{Wr}(0)} \left[ J P_{\nu}^0(0) - \left( P_{\nu}^0 \right)'(0) \right],
\]  
where
\[
\text{Wr}(0) \equiv P_{\nu}^0(0) \left( P_{\nu}^0 \right)'(0) - \left( P_{\nu}^0 \right)'(0) Q_{\nu}^0(0) = \frac{2^{2\nu} \Gamma \left( \frac{1}{2} \nu + \frac{1}{2} \lambda + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \nu + \frac{1}{2} \lambda + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \nu - \frac{1}{2} \lambda + 1 \right) \Gamma \left( \frac{1}{2} \nu - \frac{1}{2} \lambda + \frac{1}{2} \right)}.
\]  
is the Wronski determinant at \( z = 0 \) (\( t = t_0 \)), \( \Gamma(q) \) is the Gamma function,
\[
J = \frac{\dot{c}(t_0)}{c(t_0)} \sqrt{\frac{3}{\Lambda}} (1 + \mu_\parallel)^{-1},
\]
and
\[
\left( P_{\nu}^0 \right)'(0) = 2^{\nu+1} \pi^{-\nu} \frac{\Gamma \left( \frac{1}{2} \nu + \frac{1}{2} \lambda + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \nu - \frac{1}{2} \lambda + 1 \right)} \cos \left[ \frac{\pi (v + \lambda)}{2} \right],
\]
\[
\left( P_{\nu}^0 \right)'(0) = 2^{\nu+1} \pi^{-\nu} \frac{\Gamma \left( \frac{1}{2} \nu + \frac{1}{2} \lambda + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \nu - \frac{1}{2} \lambda + 1 \right)} \sin \left[ \frac{\pi (v + \lambda)}{2} \right].
\]  

Third special example: \( \Lambda = 0 \). The solution of the problem is
\[
\kappa A(t) = (1 - 2\mu_\parallel) H_{(\sigma)}^2(t_0)x^{-2(1+\mu_\parallel)}, \quad H_{(\sigma)}(x) = \pm H_{(\sigma)}(t_0)x^{-\nu(1+\mu_\parallel)}. \]  
\[
a(t) = z^{\frac{1}{\mu}} \frac{a(t_0)}{x(t)}, \quad H_{(\sigma)}(t) = \frac{H_{(\sigma)}(t_0)}{z(t)}, \quad z(t) = 1 + H_{(\sigma)}(t_0)(1 + \mu)(t - t_0). \]  
Equation (55) reduces to the Euler equation
\[
z^2 Y''(z) + Y \frac{5(1 - 2\mu_\parallel)}{4(1 + \mu_\parallel)^2} = 0,
\]  
and \( c(t) \) reads
\[
c(t) = c(t_0)z^{\frac{\mu}{\mu_\parallel}} [z^{-\sigma} + C_3(z^{\sigma} - z^{-\sigma})],
\]  
where
\[
\sigma = \sqrt{\frac{\mu_\parallel^2 + 12\mu_\parallel - 4}{2(1 + \mu_\parallel)}}, \quad C_3 = \frac{1}{2} + \left[ \frac{c(t_0)}{H_{(\sigma)}(t_0)c(t_0)} - \frac{\mu_\parallel}{2} \right] (\mu_\parallel^2 + 12\mu_\parallel - 4)^{-\frac{1}{2}}.
\]  
This solution may not appear in the model without magnetization, since the necessary condition \( 2H_{(\sigma)}^2(t_0)(2\mu_\parallel - 1) + \kappa B_0(M_0 + B_0) = 0 \) is not valid when \( M_0 = 0 \) and \( \mu_\parallel = 1 \). Note that the asymptotics for \( a(t) \) and \( c(t) \),
\[
a(t \to \infty) \propto t^{\frac{1}{\mu_\parallel}}, \quad c(t \to \infty) \propto t^{\frac{\mu_\parallel - \sqrt{\mu_\parallel^2 + 12\mu_\parallel - 4}}{2}},
\]  
coincide only in the critical regime \( \mu_\parallel \to \frac{1}{2} \) or in other words, when \( \xi \to 1 \). Thus, when \( \Lambda = 0 \) the universe does not isotropize for arbitrary \( \mu_\parallel \neq \frac{1}{2} \).
5.2. Second example of cosmological evolution: longitudinal quasi-vacuum

When \( W + P_1 = 0 \), the right-hand sides of (28) and of (30) coincide, and their left-hand sides give
\[
\frac{\dot{c}}{c} = \frac{\dot{a}}{a}.
\]
(89)

One should distinguish two cases: first, when \( \dot{a}(t) \neq 0 \) and \( H_{(a)}(t_0) \neq 0 \), second, when \( \dot{a} = 0 \).

5.2.1. \( a(t) = \text{const} = a(t_0) \). The Einstein equations, supplemented by the equations of state \( P_1 = -W \) and \( P_2 = (\gamma - 1)W \), formally admit the solution \( a(t) = a(t_0) \), if, first, the quantities \( P_1, P_2, W \) are constant and are linked by the relation \( \Lambda + \kappa \left(W + X^2\right) = 0 \), second, the equation for \( c(t) \) has the form
\[
\dot{c}(t) + c(t)Q = 0,
\]
where \( Q \equiv \kappa \left(2 - \lambda\right)X(a(t_0)) - \gamma \Lambda = \text{const}. \)
(90)

When \( Q > 0 \), \( c(t) \) oscillates harmonically with the frequency \( \sqrt{Q} \), and this model is singular. When \( Q < 0 \), \( c(t) \) behaves exponentially. At \( Q = 0 \), \( c(t) \) is a linear function of time and this model can be effectively reduced to the Minkowski spacetime.

5.2.2. \( \dot{a} \neq 0 \), \( H_{(a)}(t_0) \neq 0 \). In this case the consequence (89) gives \( c(t) \) readily in terms of \( a(t) \) and \( H_{(a)}(t) \)
\[
c(t) = c(t_0) \frac{H_{(a)}(t)}{H_{(a)}(t_0)} \frac{a(t)}{a(t_0)}.
\]
(91)

Asymptotic behaviour of \( c(t) \) at \( t \to \infty \) is the same as for \( a(t) \) (i.e., the universe isotropizes), when \( H_{(a)}(t) \to \text{const} \). The optimal strategy to obtain the solution is now the following. First, we find \( W \) from the conservation law (31) transformed into
\[
\frac{d}{dx} \left( x^{2\gamma} W(x) \right) = -x^{\gamma - 4} \frac{d}{dx} \left( x^4 X(x) \right).
\]
(92)

Second, we solve the equation for \( H_{(a)}^2(x) \)
\[
x^{-2} \frac{d}{dx} \left( x^4 H_{(a)}^4 \right) = \Lambda + \kappa \left[ W(x) + X(x) \right],
\]
(93)

which is the direct consequence of equation (30). Third, we consider the solution of (54) for \( a(t) \), and then return to solution (91) for \( c(t) \).

When \( 2(\gamma - 1) \xi \neq 1 \), the solution of (92) is
\[
W(x) = W_0 x^{-2\gamma} + \frac{B_0^2 K_3(2\xi - 1)}{2[2\xi(\gamma - 1) - 1]} x^{-2\gamma - \xi}.
\]
(94)

When \( 2(\gamma - 1) \xi = 1 \), \( W(x) \) behaves according to the formula
\[
W(x) = x^{-2\frac{\xi}{2}} \left[ W_0 - \frac{B_0^2 K_3(2\xi - 1)}{2\xi} \log x \right].
\]
(95)

In the first case, when \( 2(\gamma - 1) \xi \neq 1 \), the solution of (93) takes the form
\[
H_{(a)}^2(x) = \frac{\Lambda}{3} + x^{-3} L_1 + x^{-4} L_2 + x^{-2\gamma} L_3 + x^{-2\frac{\xi}{2}} L_4,
\]
(96)

where
\[
L_1 = H_{(a)}^2(t_0) - \frac{\Lambda}{3} - L_2 - L_3 - L_4,
\]
\[
L_2 = -\frac{1}{2} \kappa B_0^2 K_1,
\]
\[
L_3 = \kappa \frac{B_0^2}{2(3 - 2\gamma)} \left[ \frac{2W_0 + K_3(2\xi - 1)}{B_0^2} \right],
\]
\[
L_4 = \kappa B_0^2 K_2 \left[ \frac{\xi(\gamma - 2)}{(\xi - 1)[2\xi(\gamma - 1) - 1]} \right].
\]
(97)
Analogously to the case of paramagnetic/diamagnetic dust, we draw attention to the fact that generally $H(a)(x)$ is not a monotonic function and has several extrema. This means that in the expansion of the universe there are time intervals characterized by acceleration and deceleration. Asymptotic behaviour of this function is predetermined by the parameters $\Lambda$, $\gamma$ and $\xi$. The novelty in comparison with the previous analysis is that a new parameter $\gamma$ is involved. To obtain $a(t)$ we have to solve equation (54) with $H(a)(x)$ given by (96). To illustrate our conclusions let us consider two particular cases.

5.2.3. Slow relaxation. When $\xi \to \infty$, one obtains a model in which the relaxation time for the magnetization is much larger than the characteristic time of the universe evolution. Consider for simplicity $L_1 = 0$ and $L_3 = 0$. It is possible if $M_0 = 2W_0(1 - \gamma)$, $H(a)(t_0) = \frac{\Lambda}{3} + \kappa W_0(2 - \gamma) - \frac{1}{2}\kappa B_0^2$. (98)

With such a choice of the initial parameters we have

$$H(a)(x) = \sqrt{\frac{\Lambda}{3} + \kappa W_0(2 - \gamma)x^{-2} - \frac{1}{2}\kappa B_0^2x^{-4}},$$

(99)

$$H(a)(x) dH(a)(x) = \kappa B_0^2x^{-5}\left(1 - \frac{x^2}{\chi^2}\right).$$

(100)

Here the definition $\chi^2 = \frac{B_0^2}{W_0(2 - \gamma)}$ is used, and the assumptions $\gamma < 2$ and $B_0^2 > W_0(2 - \gamma)$ are made. At the point $x = \chi$ the function $H(a)(x)$ reaches its maximum with

$$H(a)(\chi) = \sqrt{\frac{\Lambda}{3} + \kappa B_0^2\chi^4},$$

(101)

and tends to the value $\sqrt{\frac{\Lambda}{3}}$ asymptotically at $t \to \infty$. Integration of (54) yields

$$\left(\frac{a(t)}{a(t_0)}\right)^2 = \cosh\left[2\sqrt{\frac{\Lambda}{3}(t-t_0)}\right] + \sqrt{\frac{3}{\Lambda}}H(a)(t_0) \sinh\left[2\sqrt{\frac{\Lambda}{3}(t-t_0)}\right]$$

$$+ \frac{3\kappa W_0(2 - \gamma)}{\Lambda} \sinh^2\left[\sqrt{\frac{\Lambda}{3}(t-t_0)}\right].$$

(102)

Thus, such a model gives $a(t)$, $H(a)(t)$ and $c(t)$ in terms of elementary (hyperbolic) functions and is convenient for qualitative analysis.

5.2.4. Transversal stiff matter, $\gamma = 2$. When $\gamma = 2$, $P_{(\nu)} = W$, i.e., the matter behaves as a stiff one. It follows from (92) that

$$W(x) + X(x) = x^{-4}(W_0 + X_0), \quad X_0 \equiv \frac{1}{2}B_0(M_0 + B_0),$$

(103)

and (30) yields

$$H(a)(x) = \sqrt{\frac{\Lambda}{3} + x^{-3}\left[H^2(a)(t_0) - \frac{\Lambda}{3} + \kappa(W_0 + X_0)\right] - x^{-4}\kappa(W_0 + X_0).}$$

(104)

This means that the parameter $\xi$ becomes hidden if we consider the functions $H(a)(t)$, $a(t)$ and $c(t)$. Nevertheless, it appears when we calculate $W(x)$:

$$W(x) = W_0x^{-4} + \frac{1}{2}B_0^2K_2\left[x^{-4} - x^{-2(1+\gamma)}\right].$$

(105)
To illustrate the problem arising in a particular case \( H_{(a)}(t_0) = 0 \) mentioned in the beginning of subsection 5.2, consider now the solution (104) with a special choice of initial data: \( H_{(a)}(t_0) = 0 \) and \( \kappa (W_0 + \lambda_0) = \frac{1}{\tau} \). Then one obtains

\[
a(t) = a(t_0) \cosh^{\frac{1}{2}} \left[ 2\sqrt{\frac{A}{3}} (t - t_0) \right], \quad H_{(a)}(t) = \frac{A}{\sqrt{3}} \tanh \left[ 2\sqrt{\frac{A}{3}} (t - t_0) \right].
\]

(106)

and

\[
c(t) = \text{const} \cosh^{\frac{1}{2}} \left[ 2\sqrt{\frac{A}{3}} (t - t_0) \right] \sinh \left[ 2\sqrt{\frac{A}{3}} (t - t_0) \right].
\]

(107)

The model is self-consistent, if \( c(t_0) = 0 \). The constant can be chosen from the isotropization condition \( a(t \to \infty) = c(t \to \infty) \), i.e., \( \text{const} = a(t_0) \).

### 6. Constant relaxation parameter

#### 6.1. Evolution of magnetization

When \( \tau \) takes a constant value, \( \tau_0 \), we can readily solve (27) in quadratures to get

\[
M(t) = \exp \left( -\frac{t - t_0}{\tau_0} \right) \left[ M(t_0) + \frac{B_0}{1 - 2H_0\tau_0} \left( 1 - \frac{1}{\mu_{||}} \right) \right] \\
- \exp \left[ -2H_0(t - t_0) \right] \frac{B_0}{1 - 2H_0\tau_0} \left( 1 - \frac{1}{\mu_{||}} \right),
\]

(108)

To obtain some analytical results consider the evolution equations with \( a(t) = a(t_0) e^{H_0(t - t_0)} \), where \( H_0 \) is a constant. Such a two-dimensional de Sitter-type expansion is possible, e.g., when

\[
P_{||} (t) = \chi(t), \quad \Lambda = 3H_0^2.
\]

(109)

In this case the total longitudinal pressure, \( P_{||} \equiv P_{||} (t) - \chi(t) \), vanishes. When \( 2H_0\tau_0 \neq 1 \), formula (108) for the magnetization and formula (32) for \( \chi(t) \) read

\[
M(t) = \exp \left( -\frac{t - t_0}{\tau_0} \right) \left[ M(t_0) + \frac{B_0}{1 - 2H_0\tau_0} \left( 1 - \frac{1}{\mu_{||}} \right) \right] \\
- \exp \left[ -2H_0(t - t_0) \right] \frac{B_0}{1 - 2H_0\tau_0} \left( 1 - \frac{1}{\mu_{||}} \right),
\]

(109)

and

\[
\chi(t) = \frac{B_0}{2} \left[ \left( 1 + \frac{1}{2H_0\tau_0 - 1} \right) \left( 1 - \frac{1}{\mu_{||}} \right) \right] \exp \left[ -4H_0(t - t_0) \right] \\
+ \left[ \frac{M(t_0)}{B_0} - \frac{1}{2H_0\tau_0 - 1} \left( 1 - \frac{1}{\mu_{||}} \right) \right] \exp \left[ - \left( 2H_0 + \frac{1}{\tau_0} \right) (t - t_0) \right],
\]

(110)

respectively. The asymptotic behaviour of \( \chi(t) \) is dominated by the exponential function \( \exp \left[ -4H_0(t - t_0) \right] \) when \( \frac{1}{\tau_0} > 2H_0 \) (i.e., the double relaxation time is less than the characteristic rate of expansion \( 1/H_0 \)), and by another exponent \( \exp \left[ - \left( 2H_0 + \frac{1}{\tau_0} \right) (t - t_0) \right] \), when \( \frac{1}{\tau_0} < 2H_0 \). Note that a special case with \( \chi(t) \equiv 0 \) exists for

\[
M(t_0) = -B_0, \quad 2H_0\mu_{||}\tau_0 = 1.
\]

(111)

The magnetization \( M(t) \) for such a case decreases exponentially

\[
M(t) = -B_0 \exp \left[ -2H_0(t - t_0) \right].
\]

(112)

In the resonance case, when \( \frac{1}{\tau_0} = 2H_0 \), the formulae (108) and (32) give
\[ M(t) = \exp\left( -\frac{t - t_0}{\tau_0} \right) \left[ M(t_0) - \frac{B_0}{\tau_0} \left( 1 - \frac{1}{\mu} \right) (t - t_0) \right], \quad (114) \]

\[ X(t) = \frac{B_0^2}{2} \exp\left[ -\frac{2}{\tau_0} (t - t_0) \right] \left[ 1 + \frac{M(t_0)}{B_0} + \left( \frac{1}{\mu} - 1 \right) \left( \frac{t - t_0}{\tau_0} \right) \right]. \quad (115) \]

Similarly to the case of time-dependent relaxation parameter, the function \( X(t) \) can be monotonic or non-monotonic, depending on the value of the parameters \( M(t_0), H_0, \tau_0 \). To illustrate this fact consider the resonance case \( \frac{1}{\tau_0} = 2H_0 \), when \( \mu > 1 \). If the following inequality takes place:

\[ M(t_0) > \frac{B_0}{2\mu} (1 - 3\mu), \quad (116) \]

the function \( X(t) \) decreases, passes through zero, reaches a minimum at the point

\[ t_* = t_0 + \frac{1}{4H_0} \left( \frac{3\mu - 1)B_0 + 2\mu(M(t_0))}{(\mu - 1)B_0} \right), \quad (117) \]

with

\[ X(t_*) = \frac{B_0}{4\mu} (1 - \mu) \exp\left[ -\frac{3\mu - 1)B_0 + 2\mu(M(t_0))}{(\mu - 1)B_0} \right] < 0, \quad (118) \]

and then increases and tends to zero asymptotically. For the diamagnetic medium we have to change the signs of inequality in (116) and in (118), i.e., the function \( X(t) \) reaches a maximum.

6.2. Third example of cosmological evolution: hidden induction

Consider the special case when \( X(t) = 0 \) despite the magnetic field being non-vanishing (see, also subsection 4.1.4). Assuming that \( P_{\parallel}(t) = 0 \), we guarantee that the first equation in (109) is identically satisfied. Likewise, assume that \( P_{(\nu)} = (\gamma - 1)W \). Then, Einstein’s field equations (28)–(30) effectively reduce to the pair of equations

\[ \dot{c} = H_0 + \frac{\kappa}{2H_0} W, \quad \dot{W} + WH_0(2\gamma + 1) + \frac{\kappa}{2H_0} W^2 = 0. \quad (119) \]

The solution to the second equation is

\[ W(t) = W(t_0) e^{-2\gamma H_0(t - t_0)} \left[ 1 + \frac{\kappa W(t_0)}{2H_0^2 (2\gamma + 1)} \left( e^{-(2\gamma + 1)H_0(t - t_0)} - 1 \right) \right]^{-1}. \quad (120) \]

thus,

\[ c(t) = c(t_0) e^{H_0(t - t_0)} \left[ 1 + \frac{\kappa W(t_0)}{2H_0^2 (2\gamma + 1)} \left( e^{-(2\gamma + 1)H_0(t - t_0)} - 1 \right) \right]. \quad (121) \]

When \( t \to \infty \), one has the asymptotic relationship

\[ \frac{d}{dt} \left[ \log \frac{c(t)}{a(t)} \right] = \left( \frac{\dot{c}}{c} - \frac{\dot{a}}{a} \right) = \frac{\kappa W(t_0)}{2H_0} \left[ e^{(2\gamma + 1)H_0(t - t_0)} - 1 \right]^{-1} \to 0. \quad (122) \]

This means that the universe isotropizes.
6.3. Fourth example of cosmological evolution

Consider now the special case when \( \mathcal{P}(t) = X(t) = 0 \), \( a(t) = a(t_0) \exp[H_0(t - t_0)] \), \( \Lambda = 3H_0^2 \) and \( P(a) = \omega \mathcal{P}(t) \). For such a model the third Einstein equation (30) converts into an identity, the second Einstein equation (29) transforms into the equation for \( c(t) \),

\[
\dot{c} + H_0 \dot{c} + c \left( \kappa (\omega + 1)X(t) - 2H_0^2 \right) = 0,
\]

and the first one, (28), gives \( W(t) \) if \( c(t) \) is known. Note that when \( \omega = 0 \), we have 'transversal material dust', when \( \omega = 1 \), the pressure of matter is isotropic. When \( \omega = -1 \), one obtains that \( \mathcal{P}(a) = \mathcal{P}(t) + X\mathcal{P}(t) = 0 \).

6.3.1. First special case \( \omega = -1 \). This case is the simplest, the solution of (123) is

\[
c(t) = \frac{1}{3} \left[ 2c(t_0) + \frac{c(t_0)}{\mathcal{H}^0} \right] e^{H_0(t - t_0)} + \frac{1}{3} \left[ c(t_0) - \frac{c(t_0)}{\mathcal{H}^0} \right] e^{-2H_0(t - t_0)}.
\]

The energy density can be found from the formula

\[
\kappa (W + X) = -6H_0^2 \left\{ 1 + \left[ \frac{2c(t_0) + \frac{c(t_0)}{\mathcal{H}^0}}{c(t_0) - \frac{c(t_0)}{\mathcal{H}^0}} \right] e^{3H_0(t - t_0)} \right\}^{-1},
\]

where \( X \) is given by (111). Initial value \( \dot{c}(t_0) \) is connected with \( c(t_0) \), \( W_0 \) and \( \mathcal{H}_0 \) by the relation \( \dot{c}(t_0) = H_0 + \frac{\kappa (W_0 + \mathcal{H}_0)}{2\mathcal{H}_0} \), which is a direct consequence of (28). The universe asymptotically isotropizes, i.e., \( a \propto \exp(H_0t) \), \( c \propto \exp(H_0t) \) and the total energy density \( W = W + X \) decreases as \( c \propto \exp(-3H_0t) \).

6.3.2. Second special case \( \omega \neq -1 \). For the special choice of the initial parameter \( M(t_0) \), which yields

\[
M(t_0) = \frac{B_0(\mu_\parallel - 1) - 1}{\mu_\parallel (2H_0\tau_0 - 1)}, \quad X(t) = \frac{B_0^2(2H_0\tau_0\mu_\parallel - 1) - 1}{2\mu_\parallel (2H_0\tau_0 - 1)} e^{-4H_0(t - t_0)},
\]

the substitution

\[
z = A e^{-2H_0(t - t_0)}, \quad c(t) = z^\frac{1}{2} Z(z),
\]

with

\[
A = \sqrt{\frac{\kappa (\omega + 1)B_0^2(2H_0\tau_0\mu_\parallel - 1) - 1}{8H_0^2\mu_\parallel (2H_0\tau_0 - 1)}},
\]

reduces equation (123) to

\[
z^2 \frac{d^2}{dz^2} Z + z \frac{d}{dz} Z + \left( z^2 - \frac{9}{16} \right) Z = 0.
\]

It is the Bessel equation (see, [36], equation (9.1.1)). The solution can be expressed in terms of Bessel functions of the real argument when

\[
\tau_0 > \frac{1}{2H_0} \quad \text{or} \quad \tau_0 < \frac{1}{2H_0\mu_\parallel},
\]

for the paramagnetic medium, and when

\[
\tau_0 < \frac{1}{2H_0} \quad \text{or} \quad \tau_0 > \frac{1}{2H_0\mu_\parallel},
\]
for the diamagnetic medium. This equation can also be reduced to the generalized Bessel equation for the imaginary argument $iz$ (see, [36], equation (9.6.1)), when

$$\frac{1}{2H_0 \mu ||} < \tau_0 < \frac{1}{2H_0},$$

(132) for $\mu || > 1$, and when

$$\frac{1}{2H_0 \mu ||} > \tau_0 > \frac{1}{2H_0},$$

(133) for $\mu || < 1$. In the last case $c(t)$ can be expressed in terms of Bessel functions $I_\nu(z) \equiv i^{-\nu} J_\nu(iz)$. For simplicity we assume that $\omega + 1 > 0$ and $\tau_0$ is positive. Then the solution of (129) is

$$c(t) = e^{-\frac{\kappa}{2} H_0 (t-\tau_0)} \left[ C_1 J_{\frac{3}{4}} (A e^{-2 H_0 (t-\tau_0)}) + C_2 J_{-\frac{3}{4}} (A e^{-2 H_0 (t-\tau_0)}) \right],$$

(134)

where $J_{\frac{3}{4}}(x)$ and $J_{-\frac{3}{4}}(x)$ are the Bessel functions of the indices $\nu = \frac{3}{4}$ and $\nu = -\frac{3}{4}$, respectively. The constants $C_1$ and $C_2$ can be expressed in terms of initial data $c(t_0)$ and $\dot{c}(t_0)$:

$$C_1 = \frac{\pi}{2 \sqrt{2}} \left[ 4c(t_0)A J'_{\frac{3}{4}} (A) + \left( c(t_0) + \frac{2}{H_0} \dot{c}(t_0) \right) J_{-\frac{3}{4}} (A) \right],$$

(135)

$$C_2 = \frac{\pi}{2 \sqrt{2}} \left[ 4c(t_0)A J'_{\frac{3}{4}} (A) + \left( c(t_0) + \frac{2}{H_0} \dot{c}(t_0) \right) J_{-\frac{3}{4}} (A) \right].$$

(136)

When $t \to \infty$ the argument of the Bessel functions in (134) tends to zero, and we have the following asymptotic expression

$$c(t \to \infty) = C_2 \left( \frac{2}{\Gamma \left( \frac{1}{2} \right)} A^\frac{2}{3} \right) e^{H_0 (t-\tau_0)},$$

(137)

where $\Gamma \left( \frac{1}{2} \right)$ is the Gamma function. Thus, the de Sitter regime appears at $t \to \infty$. It is interesting that at $t \to -\infty$, when the argument of the Bessel functions tends to infinity, the corresponding formula

$$c(t) \to \frac{1}{A^\frac{2}{3}} \frac{2}{\sqrt{\pi}} e^{H_0 (t-\tau_0)} \left[ C_1 \cos \left( A e^{-2 H_0 (t-\tau_0)} - \frac{5\pi}{8} \right) + C_2 \sin \left( A e^{-2 H_0 (t-\tau_0)} + \frac{\pi}{8} \right) \right]$$

(138)

demonstrates fast quasi-harmonic oscillations of $c(t)$ with the standard exponential damping. Note that $M(t)$ vanishes when $M(t_0) = 0$ and $\mu || = 1$. This case can also be described by formulae (134), (135) and (136) with

$$A = \left( \frac{\kappa (\omega + 1) B_0^2}{8 H_0^3} \right)^\frac{1}{2},$$

(139)

and $\tau_0 \neq \frac{1}{H_0}$. The energy density $W$ as a solution of (28) inherits the dependence on time via the Bessel functions, we do not reproduce this expression here.

6.3.3. Third special case. When $2H_0 \mu || \tau_0 = 1$, the term $X(t)$ reads

$$X(t) = \frac{B_0^2}{2} \left[ 1 + \frac{M_0}{B_0} \right] e^{-2 H_0 (1 + \mu ||) (t-\tau_0)}.$$

(140)

In this case the solution of equation (123) can also be represented in terms of Bessel functions

$$c(t) = e^{-\frac{\kappa}{2} H_0 (t-\tau_0)} \left[ C_1 J_{\frac{3}{4}} (\tilde{A} e^{-H_0 (1 + \mu ||) (t-\tau_0)}) + C_2 J_{-\frac{3}{4}} (\tilde{A} e^{-H_0 (1 + \mu ||) (t-\tau_0)}) \right].$$

(141)
where
\[ \tilde{\lambda} = \left( \frac{\kappa(\omega + 1)B_0^2}{2H_0^2(1 + \mu^2)} \right)^{1/2} \left( 1 + \frac{M(t_0)}{B_0} \right)^{1/2} \]  
(142)

Note that in the ferromagnetic phase \( \mu_{||} \gg 1 \) the argument of the Bessel function in (141) tends to zero much faster than in case (134), i.e., the isotropization in the ferromagnetic phase takes place faster.

### 6.4. Fifth example of cosmological dynamics: non-homogeneous and nonlinear equation of state

Consider now the special type of equation of state
\[ p(w) = (\gamma - 1)\rho + \lambda \rho^\gamma - \frac{\kappa}{4H_0^2}w^2. \]  
(143)

Numerous non-homogeneous and nonlinear equations of state of such kind are under discussion (see, e.g., [37, 38]). As before, we assume that \( \Lambda = 3H_0^2 \) and \( a(t) = a(t_0) e^{H_0(t-t_0)} \). For the equation of state (143), equation (31) yields
\[ \frac{d}{dt}(W + \chi) + H_0(2\gamma + 1)(W + \chi) = -2H_0\lambda \chi, \]  
whose solution reads
\[ W(t) = -\chi(t) + \tilde{L}_1 e^{-H_0(\gamma+1)(t-t_0)} + \tilde{L}_2 e^{-4H_0(t-t_0)} + \tilde{L}_3 e^{-(2H_0+\frac{1}{\mu})(t-t_0)} \]  
(145)

Here the constants \( \tilde{L}_1, \tilde{L}_2 \) and \( \tilde{L}_3 \) are given by
\[ \tilde{L}_1 \equiv W(t_0) + \frac{B_0^2}{2} - \frac{B_0M(t_0)}{2} \frac{H_0\tau_0(2\gamma + 2\lambda - 1) - 1}{H_0\tau_0(2\gamma - 1) - 1} \]  
\[ + \frac{\lambda B_0^2}{\mu(2\gamma - 3)} \frac{H_0\tau_0(2\gamma - 1) - 1}{H_0\tau_0(2\gamma - 1) - 1} - \frac{\lambda B_0^2}{\mu(2\gamma - 3)} \frac{2H_0\tau_0(\mu - 1)}{2H_0\tau_0 - 1}, \]  
(146)

\[ \tilde{L}_2 \equiv -\frac{\lambda B_0^2}{\mu(2\gamma - 3)} \frac{2H_0\tau_0(\mu - 1)}{2H_0\tau_0 - 1}, \]  
(147)

\[ \tilde{L}_3 \equiv -B_0M(t_0) \frac{H_0\tau_0}{H_0\tau_0(2\gamma - 1) - 1} + \frac{\lambda B_0^2}{\mu(2H_0\tau_0 - 1)} \frac{H_0\tau_0(\mu - 1)}{H_0\tau_0(2\gamma - 1) - 1}. \]  
(148)

Likewise, for \( c(t) \) we obtain
\[ \log \left( \frac{c(t)}{c(t_0)} \right) = H_0(t-t_0) + \frac{\kappa \tilde{L}_1}{2H_0^2(2\gamma + 1)} \left[ 1 - e^{-H_0(2\gamma+1)(t-t_0)} \right] \]  
\[ + \frac{\kappa \tilde{L}_2}{8H_0^2} \left[ 1 - e^{-4H_0(t-t_0)} \right] + \frac{\kappa \tilde{L}_3}{2H_0(2H_0 + \frac{1}{\mu})} \left[ 1 - e^{-(2H_0 + \frac{1}{\mu})(t-t_0)} \right]. \]  
(149)

In the asymptotic regime at \( t \to \infty \) one obtains \( c(t) \to c(\infty) e^{H_0(t-t_0)} \), where
\[ c(\infty) \equiv c(t_0) \exp \left\{ \frac{\kappa}{2H_0} \frac{\tilde{L}_1}{H_0(2\gamma + 1)} + \frac{\tilde{L}_2}{4H_0} + \frac{\tilde{L}_3}{2H_0(2H_0 + \frac{1}{\mu})} \right\}. \]  
(150)

Thus, the isotropization takes place, as it should. For \( t \to -\infty \), \( c(t) \) decreases superexponentially.
7. Discussion

We have considered the simplest model of the one-dimensional relaxation of matter magnetization in a strong magnetic field in the framework of the extended Einstein–Maxwell theory applied to the Bianchi-I cosmological model. We have shown that this model admits a set of exact analytical solutions, depending on the set of guiding parameters. Let us emphasize the main aspects of the obtained results.

1. Analogy with extended (causal) thermodynamics

The key element of the extended Einstein–Maxwell theory in the context of the anisotropic Bianchi-I model is the one-dimensional relaxation equation (27) for the magnetization $M(t)$.

The key element of the extended irreversible thermodynamics in the context of an isotropic Friedmann model is a relaxation equation for the bulk viscous pressure $\sigma(t)$ (see, e.g., [17]).

These two equations

$$\tau \dot{M} + M = \left(\frac{1}{\mu_\parallel} - 1\right) B_0 \left(\frac{a(t_0)}{a(t)}\right)^2$$

and

$$\tau \dot{\sigma} + \sigma = -3\zeta \frac{\dot{a}}{a}$$

look similar. ($\zeta$ is a bulk viscosity coefficient.) In both cases, the rate of evolution of the scale factor $a(t)$ predetermines the relaxation properties of $M(t)$ or $\sigma(t)$. In both cases the function, $M(t)$ or $\sigma(t)$, appears on the right-hand side of the Einstein equations as an element of the source term. In both models new degrees of freedom, activated in matter by the cosmological evolution, change the rate of expansion. In order to check this claim one can simply compare the results obtained for $M(t) = 0$ and $M(t) \neq 0$. As an example, let us compare expressions (94), (96), (97) and (91) with those at $M_0 = 0, \mu_\parallel = 1, \xi = 0$ (i.e., at $K_2 = 0$). The difference is that the magnetization adds a principally new second term in (94), describing the evolution of the energy density scalar, as well as the new last term in (96), describing the rate of expansion in the cross-section $x^1Ox^2$. Taking into account formula (91), one can see that the modifications in $H_{\alpha\alpha}(t)$ lead to the changes in the rate of evolution in the direction $x^3$.

Moreover, the presence of the new terms, $x^{-2-\xi}$, allows us to choose the phenomenological parameter $\xi$ so that this term becomes of the leading order at $x \to \infty$ in comparison with the terms $x^{-3}, x^{-4}$ and $x^{-2\gamma}$ in formula (96). In such a case just the magnetization predetermines the rate of cosmological evolution at $t \to \infty$, and the relaxation time $\tau(t) = \xi H_\alpha(t)$ introduces a new expansion time scale.

In both theories the relaxation time $\tau$ is considered to be a function of cosmological time $t$ and is a subject of modelling. In the extended irreversible thermodynamics the relaxation time is considered as $\tau = \zeta W$, where $\zeta = \alpha W^0$ (in our definition of the energy density scalar) (see, e.g., [17–21]). When the function $W(t)$ is obtained from the cosmological dynamics, the function $\tau(W(t))$ becomes an alternative representation of the function $\tau(H(t))$.

The main difference in the results is connected with the fact that the Bianchi-I model is anisotropic, and $M(t)$ is in fact a projection of the magnetization in the direction pointed by the magnetic field. In the isotropic Friedmann model $\sigma$ is a scalar describing the isotropic bulk viscous pressure. As a consequence, $H(t)$ in the Friedmann model is positive, and the right-hand side of the relaxation equation for $\sigma$ is always negative. The sign of the right-hand side of the relaxation equation for $M(t)$ depends on the sign of $B_0$, as well as on the sign of the difference $(\mu_\parallel - 1)$. Respectively, the obtained magnetization may be positive or negative depending on the (random) initial value $M_0$, relaxation time and magnetic permeability. This option allows a principally new situation to be considered, when magnetic field and magnetization are non-vanishing, nevertheless, the total magnetic source
term $\mathcal{X} = \frac{1}{4} B (B + M)$ is equal to zero and disappears from the Einstein equations. Such solutions are discussed in subsections 4.1.4. and 5.1.2.

2. Monotonic and non-monotonic expansion

Classical models with a pure magnetic field are characterized by the non-negative source term $\mathcal{X} = \frac{1}{4} B^2 \geq 0$, which decreases monotonically as $a^{-3}$. The magnetization changes the situation: $\mathcal{X}$ may be positive, negative or equal to zero. Generally, $\mathcal{X}(t)$ is no longer a monotonic function; it may possess one, two or more extrema. As a consequence of this behaviour, the function $H_{(a)}(t)$ is not monotonic; thus, in the evolution of the universe in the cross-section $x^1 O x^2$ there are periods of (transversal) acceleration and deceleration. The simplest behaviour $H_{(a)}(t)$ is characterized by the presence of one minimum or maximum, and by the asymptotic de Sitter regime $H_n(t) \to \sqrt{\frac{2}{3}}$ (see, e.g., subsection 5.2.3). The behaviour of $c(t)$ is also non-monotonic in this case. A more complicated situation is characterized by the solution for $c(t)$, presented in terms of Bessel functions (see, subsection 6.3.2). The function $c(t)$ behaves quasi-periodically, and one can expect that the number of periods of the longitudinal acceleration and deceleration is infinite.

3. Guiding and resonance parameters

The considered extended Einstein–Maxwell model is characterized by eight guiding parameters: $\xi$ or $\tau_0, \mu_{||} = 1, B_0, M_0, W_0, \gamma, H_{(a)}(t_0)$ and $\Lambda$. There are several underlined values of the parameter $\xi$ (in the model of variable relaxation time) and the parameter $\tau_0$ (in the model of constant relaxation time). The values $\xi = \frac{1}{2}$ and $\frac{1}{\tau_0} = 2 H_0$, respectively, are in fact resonance parameters, which appear in the integration of the differential equation for $M(t)$. In such a resonance case the relaxation time $\tau = \frac{1}{2} H_0^{-1} (t)$ or $\tau_0 = \frac{1}{2} H_0^{-1}$ coincides with the characteristic time of the evolution of the function $B(t) = F_{12} a^{-2}(t)$, which provides the dynamics of magnetization. In the case of resonance the function $x^{-\frac{\xi}{2}}$ as a part of $M(t)$ (34), has to be replaced by $x^{-\frac{3}{2}} \log x$ (see, (46)). It is very interesting to emphasize that in [21] the special value of the parameter $q, q = \frac{1}{2}$, leads to the law $\tau \sim H^{-1}$ for the relaxation of the bulk viscosity pressure. The value $\xi = 1$ is evidently the resonance value of the parameter $\xi$, but it has another origin. When $\xi = 1$, the rate of change of the function $H_{(a)}(t)$ coincides with that of $\mathcal{X}(t)$. As a consequence, the function $x^{-\frac{(2+\xi)}{2}}$ in (57) has to be replaced by $x^{-3} \log x$ (see, (59)). The special values $\frac{1}{\xi} = 2(\gamma - 1), \frac{1}{\tau_0} = H_0(2\gamma - 1)$ (see, (94)–(97) and (146)–(148)) can also be considered as some analogues of the relation $\tau = 3 H_0^{-1} \frac{2 - \epsilon \gamma}{\gamma}$ appeared in [21] in the context of the evolution of the bulk viscous pressure. The special value $\gamma = \frac{1}{3}$, appearing in (97), relates to the vanishing trace of the matter pressure tensor $P_1 + P_2 + P_3 = 2 P_{(a)} + P_{||} = 0$. The special value $\frac{1}{\nu} = 2 H_0 \mu_{||}$ appears in the context of the vanishing $\mathcal{X}(t)$ (see, (112)). Finally, the model characterized by the constant values of $P_{(a)}, P_{||}, W$ and $\mathcal{X}$, discussed in subsection 5.2.1 in the context of the special condition $H_{(a)}(t_0) = 0$, also has an appropriate analogue in [21].

4. Isotropization

The models in which the cosmological constant $\Lambda$ is non-vanishing, isotropize at $t \to \infty$. The exceptional case (see, (88)) corresponds to $\Lambda = 0$. The first novelty of the obtained results is that at $\xi = -\frac{1}{2}$ the magnetized matter can effectively redefine the cosmological constant (see, (61)). This case is exotic, since negative $\xi$ corresponds to magnetic instability and the magnetization increases with time. Nevertheless, when $\Lambda = 0$, this effect can in principle produce a non-vanishing effective cosmological constant. The second novelty is connected with the non-monotonic character of the isotropization.
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Appendix A. Variation of the tetrad vectors

Let \( X^i_{(a)} \) be the set of four tetrad 4-vectors, whose index \((a)\) runs over four values: (0), (1), (2), (3). Let \( X^i_{(0)} \) coincide with \( U^i \), the 4-vector of velocity of the medium as a whole, and let \( X^i_{(3)} \) coincide with \( X^i \), the director 4-vector of the medium with uniaxial symmetry. The tetrad 4-vectors are assumed to satisfy the orthogonality normalization rules

\[
g^{ij} X_{(a)}^i X_{(b)}^j = \eta^{(a)(b)}, \quad \eta^{(a)(b)} X_{(a)}^p X_{(b)}^q = g^{pq},
\]

(A.1)

where \( \eta^{(a)(b)} \) denotes the Minkowski matrix, diagonal \((1, -1, -1, -1)\). Since the tetrad 4-vectors are linked by the relation containing the metric, we have to define the formula for the variation \( \delta X^i_{(a)} / \delta g^{pq} \). Varying the first and second relations (A.1) with respect to the metric, we obtain, respectively,

\[
X_{k(b)} \delta X^k_{(a)} + X_{k(a)} \delta X^k_{(b)} = -X^i_{(a)} X_{k(b)} \delta g^{ik},
\]

(A.2)

\[
\delta g^{pq} = \eta^{(a)(b)} \left[ X^q_{(b)} \delta X^p_{(a)} + X^p_{(a)} \delta X^q_{(b)} \right].
\]

(A.3)

The variation of the arbitrary origin \( \delta X^i_{(a)} \) (not necessarily caused by the metric variation) can be represented as a linear combination of the tetrad 4-vectors:

\[
\delta X^i_{(a)} = X^i_{(c)} Y^{(a)(c)}.
\]

(A.4)

The tetrad tensor \( Y^{(a)(c)} \) is not generally symmetric. Using the convolution of (A.3) with tetrad vectors, we obtain

\[
Y^{(a)(b)} + Y^{(b)(a)} = \delta g^{pq} X^i_{(a)} X^j_{(b)}
\]

(A.5)

where we use standard rules for the indices, e.g., \( X^i_{(f)} = \eta^{(f)(b)} g^{qm} X^m_{(b)} \). Consequently, the symmetric part of the quantity \( Y^{(a)(b)} \), indicated as \( Z^{(a)(b)} \), can be readily found:

\[
Z^{(a)(b)} = \frac{1}{2} \delta g^{pq} X^i_{(a)} X^j_{(b)},
\]

(A.6)

and the law (A.4) reads now

\[
\delta X^i_{(a)} = \frac{1}{4} \delta g^{pq} \left[ X^i_{(p(a)} \delta^q_{j(a)} + X^i_{(q(a)} \delta^p_{j(a)} \right] + X^i_{(c)} Z^{(a)(c)}.
\]

(A.7)

Here \( Z^{(a)(c)} \) is a skew-symmetric part of \( Y^{(a)(c)} \), i.e., \( 2Z^{(a)(c)} \equiv Y^{(a)(c)} - Y^{(c)(a)} \). Therefore, the variation of the metric produces the variation of the tetrad, described by (A.7) with a vanishing skew-symmetric part \( Z^{(a)(c)} \). Thus, one finally has

\[
\frac{\delta X^i_{(a)}}{\delta g^{pq}} = \frac{1}{4} \left[ X^i_{(p(a)} \delta^q_{j(a)} + X^i_{(q(a)} \delta^p_{j(a)} \right],
\]

(A.8)

and we can use this formula for the variation of the 4-velocity vector \( U^i \equiv X^i_{(0)} \) and for the variation of the space-like vector \( X^i \equiv X^i_{(3)} \).
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