A toroidal resolution for the bad reduction of some Shimura varieties

by Alain Genestier

1. — Statement of the main result

Let \( S_K(G, \mu) \) be a Shimura variety. We assume that \( G \) is either the group of unitary similitudes for a hermitian space defined over an imaginary quadratic extension \( E \) of \( \mathbb{Q} \) (unitary case) or a group of symplectic similitudes over \( \mathbb{Q} \) (symplectic case). Let \( p \) be a prime in \( \mathbb{Z} \). We assume that the group \( K_p \) defining the \( p \)-part of the level structures is a parahoric subgroup of \( G(\mathbb{Q}_p) \). In the symplectic case, we assume that \( K_p \) is contained in the Siegel parahoric. In the unitary case, we also assume that the place \( p \) is split in \( E \): \( p = \mathfrak{P}_0 \mathfrak{P}_1 \).

Following Rapoport and Zink ([RZ]), under these conditions the Shimura variety \( S_K(G, \mu) \) has a natural modular model \( S_K(G, \mu)_{\mathbb{Z}(p)} \) over \( \text{Spec} \mathbb{Z}(p) \). In fact in the unitary case this model comes from a scheme over the semi-local ring \( \mathcal{O}_{E,(p)} \) by restriction of scalars.

In some cases the singularities of this modular model have been studied.

For modular curves, the modular model is the one constructed by Deligne and Rapoport ([DR]). The unitary Shimura varieties defined by a hermitian form of signature \((1,n-1)\) in the archimedean place have been studied by M. Rapoport ([R]), who proved that \( S_K(G, \mu)_{\mathcal{O}_{E,(\mathfrak{P}_i)}} \) (\( i = 0, 1 \)) is semi-stable over \( \text{Spec} \mathcal{O}_{E,(\mathfrak{P}_i)} \). The Siegel modular variety with level structures associated to the Siegel parahoric have been studied by Chai and Normann ([CN]). They proved that the singularities are Cohen-Macaulay, and Faltings ([F 1]) even proved that they are rational. Deligne and Pappas ([DP]) also have a result for Hilbert-Blumenthal varieties, when \( p \) divides the discriminant of the defining field (note that these varieties do not satisfy the requirements of the present work).

However, except in the first two cases, the structure morphism is far from being semi-stable and we are interested in constructing resolutions of \( S_K(G, \mu)_{\mathbb{Z}(p)} \) with a “reasonable structure morphism”. By this, we mean a scheme (or eventually an algebraic space) \( \tilde{S}_K(G, \mu)_{\mathbb{Z}(p)} \) over \( S_K(G, \mu)_{\mathbb{Z}(p)} \) such that the morphism \( \tilde{S}_K(G, \mu)_{\mathbb{Z}(p)} \to S_K(G, \mu)_{\mathbb{Z}(p)} \) is proper and becomes an isomorphism over \( \mathbb{Q} \) and the morphism \( \tilde{S}_K(G, \mu)_{\mathbb{Z}(p)} \to \text{Spec} \mathbb{Z}(p) \) has well understood singularities (e.g. is semi-stable, or log-smooth for certain given log-structures).

Results of this type have already been obtained by several authors:
- the Siegel modular variety of genus 2 has been dealt with by A. J. de Jong, who showed ([dJ 1]) that a certain blowing-up of the modular model is semi-stable over \( \mathbb{Z}(p) \)
- using De Concini-Procesi’s compactifications of symmetric spaces (and proving that
they can be performed over $\mathbb{Z}$, Faltings ([F 1]) constructs a semi-stable resolution of the Siegel modular variety with level structures defined by the Siegel parahoric. His methods also work for an analogous parahoric in the unitary case

– in a recent preprint ([F 2]) which was a source of inspiration for the present work, Faltings resolves the unitary Shimura varieties whose defining hermitian form has signature $(2, n-2)$ or $(3, n-3)$ and the Siegel modular variety of genus 3

– a semi-stable resolution of the Siegel modular variety of genus 3 with Iwahori level-structures is also constructed in [G].

In the present work we prove the following one.

**Theorem A.** — **Under the above assumptions for the groups $G$ and $K_p$, $S_K(G,\mu)_{\mathbb{Z}(p)}$ has a canonical log-smooth resolution.**

**Remark:** In fact the resolution $\tilde{S}_K(G,\mu)_{\mathbb{Z}(p)}$ that we obtain satisfies a stronger condition (not appealing to logarithmic geometry: cf. 2.2.6) than log-smoothness over $\mathbb{Z}(p)$. Using this stronger statement and ([KKMS], III. 4), one can see that $S_K(G,\mu)_{\mathbb{Z}(p)}$ has a (non canonical) semi-stable resolution over an extension $\mathbb{Z}(p)[p^{1/\nu}]$ of $\mathbb{Z}(p)$.

To prove theorem A, it will be sufficient to prove an analogous statement for the local model. Following Chaï, Norman, Rapoport and Zink ([CN], [R], [RZ]), $S_K(G,\mu)_{\mathbb{Z}(p)}$ has a local model $\mathcal{M}_p$; this is a projective scheme over $\mathbb{Z}_p$ such that for every point $s$ in $S_K(G,\mu)_{\mathbb{Z}(p)}$, there exists a (non unique) point $m$ in $\mathcal{M}_p$ such that $s$ and $m$ have isomorphic étale neighbourhoods. This scheme is defined in terms of linear algebra and so is easier to deal with than $S(g,\mu)_{\mathbb{Z}_p}$. The couple $(G,\mu)$ enters in its definition only through $G_{\mathbb{Q}_p}$ (i.e. $(\text{GL}_n \times \mathbb{G}_m)_{\mathbb{Q}_p}$ or $\text{GSp}_{2g,\mathbb{Q}_p}$) and (in the unitary case) the signature $(r, n-r)$ of the hermitian form; the parahoric $K_p$ enters in its definition only through the partition of $n$ or $g$ defining it (recall that in the symplectic case we assumed that $K_p$ is contained in the Siegel parahoric).

The local model $\mathcal{M}_p$ is endowed with an action of a certain affine smooth $\mathbb{Z}_p$-group-scheme $\mathcal{K}$ associated to the combinatorial data defining the parahoric $K_p = \mathcal{K}(\mathbb{Z}_p)$ (see [dJ 2], [RZ]), and one has a diagram

$$
\begin{array}{ccc}
\mathcal{T} & \xleftarrow{\sim} & \mathcal{M}_p \\
\downarrow & & \downarrow \\
S_K(G,\mu)_{\mathbb{Z}(p)} & & \mathcal{M}_p \\
\end{array}
$$

where the morphism $\mathcal{T} \rightarrow S_K(G,\mu)_{\mathbb{Z}(p)}$ is a (left) $\mathcal{K}$-principal homogeneous space and the morphism $\mathcal{T} \rightarrow \mathcal{M}_p$ is smooth and $\mathcal{K}$-equivariant. Thus, to obtain a log-smooth resolution of $S_K(G,\mu)_{\mathbb{Z}(p)}$ it will be enough to construct a $\mathcal{K}$-equivariant log-smooth resolution of $\mathcal{M}_p$: if we have such a resolution $\tilde{\mathcal{M}}_p$, the morphism $\mathcal{K} \backslash (\mathcal{T} \times_{\mathcal{M}_p} \tilde{\mathcal{M}}_p) \rightarrow S_K(G,\mu)_{\mathbb{Z}(p)}$ will be a log-smooth resolution. So, the theorem A derives from the following result:

**Theorem B.** — **The local model $\mathcal{M}_p$ has a canonical $\mathcal{K}$-equivariant log-smooth resolution.**
Let us now describe the content of this article.
In section 2, we develop a variant of a reduction principle due to G. Faltings, which reduces the problem of constructing resolutions of the local model to the one of resolving a scheme $\mu$ defined by matrix equations.
In section 3, we use Lafforgue’s work on the compactifications of $\text{PGL}_{r+1}^N/\text{PGL}_r$ ([L 1,2]) to solve this last problem.
In section 4, we give remarks concerning Faltings’s original reduction principle and our variant.

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2. — A reduction

2.1. — In [F 2], G. Faltings introduces the scheme

$$\mu^{r,N} = \{(A_0, \cdots, A_N) \in \text{gl}_r^{N+1} | \begin{align*}
A_0A_1\cdots A_N & = p.\text{Id}_r \\
A_1\cdots A_NA_0 & = p.\text{Id}_r \\
& \vdots \\
A_NA_0\cdots A_{N-1} & = p.\text{Id}_r \},
\end{align*}$$

endowed with a right action of $\text{GL}_{r+1}^N$ defined by

$$(g_0, \cdots , g_N, (A_0, \cdots, A_N) \mapsto (g_N^{-1}A_0g_0, g_0^{-1}A_1g_1, \cdots , g_{N-1}A_Ng_N).$$

and proposes the following strategy to resolve the local models.

1) Establish a reduction principle connecting $K$-equivariant resolutions of $M_p$ to $\text{GL}_{r+1}^N$-equivariant resolutions of $\mu^{r,N}$ (resp. with $r = g$ in the symplectic case).

2) Construct such a $\text{GL}_{r+1}^N$-equivariant resolutions of $\mu^{r,N}$.

In the present work, we will use a variant of Faltings’s original reduction principle: we shall replace $\text{GL}_r$ by a parabolic subgroup $P$ of $\text{GL}_n$ with Levi subgroup $\text{GL}_r \times \text{GL}_{n-r}$ (with $r = n - r = g$ in the symplectic case) and $\mu^{r,N}$ by an analogous scheme $\mu^{P,N}$ also defined by matrix equations (cf. 2.2 for the unitary case and 2.3 for the symplectic one).

2.2. — The unitary case

Let $(d_i)_{1 \leq i \leq N+1}$ be a $(N + 1)$-uple of strictly positive integers with sum $n$. Let $T$ be the matrix

$$\begin{pmatrix}
0 & \text{Id}_{n-1} \\
t & 0
\end{pmatrix} \in \text{gl}_n(\mathbb{Z}[t]),$$

and $V_i$ ($0 \leq i \leq N + 1$) be the lattice $T^{d_1+\cdots+d_i}\mathbb{Z}[t] \subset \mathbb{Z}[t,t^{-1}]^n$. We denote by $\alpha_i$ ($1 \leq i \leq N + 1$) the natural inclusion $V_i \hookrightarrow V_{i-1}$ and by $\alpha_0$ the composite inclusion $V_0 = V_{N+1} \hookrightarrow V_N$. 3
Definition 2.2.1. (cf. [R], [RZ]). — The local model $M_t(\text{GL}_n, r, (d_i)_i)$ consists of $(N + 1)$-uples $(\omega_i \subset V_i)_i$, where $\omega_i$ is locally a direct factor of rank $r$ of $V_i$

and

$\alpha_i(\omega_i) \subset \omega_{i-1}, \forall i \in \mathbb{Z}/(N + 1)\mathbb{Z}$

The local model $M_p(\text{GL}_n, r, (d_i)_i)$ is the restriction of $M_t(\text{GL}_n, r, (d_i)_i)$ along the section $\overline{p} = \{ t = p \}$ of the affine line $\mathbb{A}^1_t = \text{Spec} \mathbb{Z}[t]$.

The functor $M_t(\text{GL}_n, r, (d_i)_i)$ is obviously representable by a closed subscheme in the product of grassmannian schemes $\prod_i \text{Gr}(r, V_i)$; the scheme $M_t(\text{GL}_n, r, (d_i)_i)[t^{-1}]$ is simply $\text{Gr}(r, \mathbb{Z}[t, t^{-1}]^n)$. Let $\mathcal{P}_t$ be the automorphism group of the system $(V_i, \alpha_i)_i$ (i.e. $\mathcal{P}_t$ consists of $(N + 1)$-uples $(g_i)_i \in \prod_i \text{GL}(V_i)$ such that $g_{i-1}\alpha_i = \alpha_i g_i$, $\forall i \in \mathbb{Z}(N + 1)\mathbb{Z}$). The group scheme $\mathcal{P}_t$ acts obviously (on the left) on $M_t(\text{GL}_n, r, (d_i)_i)$. It is affine and smooth over $\mathbb{A}^1_t$ and $\mathcal{P}_t[t^{-1}]$ is naturally identified with $\text{GL}(\mathbb{Z}[t, t^{-1}]^n)$. The group scheme $K$ occuring in the first section is (in the unitary case) the restriction $P_p$ of $\mathcal{P}_t$ along the section $\overline{p}$.

We will now define a certain rigidification of $M_t(\text{GL}_n, r, (d_i)_i)$. Let us denote by $V_i (i \in \mathbb{Z}(N + 1)\mathbb{Z})$ the vector bundle $\mathbb{A}^n$ and by $\Omega_i$ its direct factor $\mathbb{A}^r$.

Definition 2.2.2. — The rigidified local model $M_t^{\text{rig}}(\text{GL}_n, r, (d_i)_i)$ consists of $2(N + 1)$-uples

$((\omega_i \subset V_i)_i, (\phi_i)_i)$,

where

$(\omega_i \subset V_i)_i \in M_t(\text{GL}_n, r, (d_i)_i)$

and

$\phi_i : (\omega_i \subset V_i) \longrightarrow (\Omega_i \subset V_i)\mathbb{Z}[t]$ is an isomorphism of filtered vector bundles, $\forall i \in \mathbb{Z}(N + 1)\mathbb{Z}$.

Let $P$ be the parabolic subgroup

$$
\begin{pmatrix}
\text{GL}_r & * \\
0 & \text{GL}_{n-r}
\end{pmatrix}
$$

of $\text{GL}_n$. The scheme $M_t^{\text{rig}}(\text{GL}_n, r, (d_i)_i)$ is naturally a (right) $P^{N+1}$-principal homogeneous space over $M(\text{GL}_n, r, (d_i)_i)$. It is also endowed with a (left) free action of $\mathcal{P}_t$ and both actions commute.

2.2.3. — Let $\mathfrak{P}$ be the Lie algebra of $P$. We will now re-write the algebraic space $\mathcal{P}_t \setminus M_t^{\text{rig}}(\text{GL}_n, r, (d_i)_i)$ as a locally closed subscheme of $\mathfrak{P}^{N+1} \times \mathbb{A}^1_t$.

Let $\mathfrak{P}_{rk \geq 1}$ be the open subscheme $\mathfrak{P} - \{0\}$ of the affine space $\mathfrak{P}$ and $\mu^{P,N}$ be the closed subscheme of $\mathfrak{P}_{rk \geq 1} \times \mathbb{A}^1_t$ defined by the matrix equations

$$
\begin{align*}
P_0P_1 \cdots P_N &= t.\text{Id}_n \\
P_1 \cdots P_NP_0 &= t.\text{Id}_n \\
&\vdots \\
PNP_0 \cdots P_{N-1} &= t.\text{Id}_n.
\end{align*}
$$

4
The \((N + 1)\)-uple of matrices

\[
\Pi_j := \phi_j^{-1} \alpha_j \psi_j \in \mathfrak{P}_{rk \geq 1}(M^\text{rig}_t (\text{GL}_n, r, (d_i)_i)) \quad (j \in \{0, \cdots, N\})
\]

clearly defines a morphism

\[
M^\text{rig}_t (\text{GL}_n, r, (d_i)_i) \rightarrow \mu_{P,N}
\]
of \(\mathbb{A}^1\)-schemes, with image contained in the biggest open subscheme \(\mu_{(d_i)_i}^{P,N}\) of \(\mu_{P,N}\), on which the minors of size \((n - d_i)\) of the matrix \(\Pi_i \) are invertible, \(\forall i \in \mathbb{Z}(N + 1)\mathbb{Z}\). Moreover, this morphism is \(P^{N+1}\)-equivariant if we endow \(\mu_{P,N}\) with the action of \(P^{N+1}\) defined by

\[
(g_0, \cdots, g_N), (P_0, \cdots, P_N) \mapsto (g_N^{-1} P_0 g_0, g_0^{-1} P_1 g_1, \cdots, g_{N-1} P_N g_N).
\]

**Proposition 2.2.4.** — The \(\mu_{(d_i)_i}^{P,N}\)-scheme \(M^\text{rig}_t (\text{GL}_n, r, (d_i)_i)\) is a \(\mathcal{P}_t\)-principal homogeneous space

**Proof (cf. [RZ], appendix A):** It is clearly sufficient to show that locally for the Zariski topology on \(\mu_{(d_i)_i}^{P,N}\), the morphism

\[
M^\text{rig}_t (\text{GL}_n, r, (d_i)_i) \rightarrow \mu_{(d_i)_i}^{P,N}
\]

has a section. So, let \(R\) be a local ring of \(\mu_{(d_i)_i}^{P,N}\), with maximal ideal \(\mathcal{M}\) and residue field \(k\). The case \(t \in R^\times\) is trivial, and we assume \(t \in \mathcal{M}\). The matrix \(\Pi_i \otimes k\) is of rank \(\geq n - d_i\), hence the \(k\) vectorspace \((V_{i-1} \otimes k)/\Pi_i (V_i \otimes k)\) is of rank \(d'_i \leq d_i\). Let \((e_j[i])_{1 \leq j \leq d'_i}\) be a basis of this \(k\)-vectorspace and let \((\tilde{e}_j[i])_{1 \leq j \leq d'_i}\) be a lifting of this family to a family of elements in \(V_i \otimes R\). The family

\[
(\tilde{e}_j[i])_{1 \leq j \leq d'_i} \cup (\Pi_{i+1} \tilde{e}_j[i + 1])_{1 \leq j \leq d'_{i+1}} \cup \cdots \cup (\Pi_{i+N} \tilde{e}_j[i + N])_{1 \leq j \leq d'_{i+N}}
\]

(where \(i\) is understood as an element of \(\mathbb{Z}(N + 1)\mathbb{Z}\)) generates \(V_i \otimes k\), hence we have \(d'_j = d_j, \forall j\). Let \(U_i\) be the \(R\)-submodule of \(V_i \otimes R\) spanned by the family \((\tilde{e}_j[i])_{1 \leq j \leq d'_i}\). This family obviously defines a morphism \(R^{d_i} \rightarrow U_i\). Let us consider the morphism

\[
\psi_i : R^{d_i} \oplus R^{d_{i+1}} \oplus \cdots \oplus R^{d_{i+N}} \rightarrow V_i \otimes R
\]

induced by \(R^{d_j} \rightarrow U_j\) \((i \leq j \leq i + N)\) and \(\Pi_{i+1} \cdots \Pi_j : U_j \rightarrow V_i \otimes R\) \((i + 1 \leq j \leq i + N)\). It is surjective by Nakayama’s Lemma, and so it must be an isomorphism. The composite

\[
U_i \rightarrow R^{d_{i+N}} \rightarrow V_{i-1} \rightarrow V_{i+N}
\]

of its last factor with \(\Pi_i : V_i \otimes R \rightarrow V_{i-1} \otimes R\) is the composite \((\Pi_i \cdots \Pi_{i+N})|_{U_i} = (t.\text{Id}_n)|_{U_i}\). Hence, the matrix (in the canonical bases) of \(\psi_i^{-1} \Pi_i \psi_i\) is simply \(T^{d_i}\) and the \(2(N + 1)\)-uple \(((\Pi_i)_i, (\psi_i^{-1})_i)\) defines a section of \(M^\text{rig}_t (\text{GL}_n, r, (d_i)_i)_R \rightarrow (\mu_{(d_i)_i}^{P,N})_R\). \(\square\)

2.2.5. — We will now state our variant of G. Faltings’ reduction principle.

Let \(X\) be a \(\mathbb{A}^1\)-scheme. We assume that \(X[t^{-1}]\) is smooth over \(\mathbb{A}^1[t^{-1}] = \text{Spec } \mathbb{Z}[t, t^{-1}]\).
**Definition 2.2.6.** — A toroidal resolution of \( X \) is a proper morphism of \( \mathbb{A}^1 \)-schemes \( r : \widetilde{X} \to X \) such that \( r[t^{-1}] \) is an isomorphism and such that there exists a torus \( T \), a character \( \chi \) of \( T \), a toroidal embedding \( T \hookrightarrow \mathbb{T} \) (cf. [KKMS]) and a diagram

\[
\begin{array}{ccc}
\widetilde{X}^\vee & \to & \widetilde{X} \\
\downarrow & & \downarrow \\
\widetilde{X} & \to & T
\end{array}
\]

of \( \mathbb{A}^1 \)-schemes satisfying the following conditions

- the restriction to \( T \) of the morphism \( \mathbb{T} \to \mathbb{A}^1 \) is the composite \( T \to \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \)
- the left morphism is a principal homogeneous space under a torus \( T' \) endowed with a morphism to the kernel \( T \chi \) of \( \chi \)
- the right morphism is smooth and \( T' \)-equivariant.

We say this toroidal resolution is strongly log-smooth if moreover the group \( T \chi \) is a torus.

**Reduction Principle 2.2.7.** — Assume that we have a \( P^{N+1} \)-equivariant strongly log-smooth toroidal resolution \( \widetilde{\mu}^{P,N} \) of \( \mu^{P,N} \) whose source can be put in a \( P^{N+1} \times T' \)-equivariant diagram (2.2.6), where the action of \( P^{N+1} \times T' \) on \( \mathbb{T} \) is by the second factor. The morphism

\[
\tilde{\mathbb{M}}_t (\text{GL}_n, r, (d_i)_i) := (\tilde{\mathbb{M}}_{t \text{rig}} (\text{GL}_n, r, (d_i)_i) \times_{\mu^{P,N}} \tilde{\mu}^{P,N})/P^{N+1} \to \mathbb{M}_t (\text{GL}_n, r, (d_i)_i)
\]

is then a \( P_t \)-equivariant strongly log-smooth toroidal resolution of \( \mathbb{M}_t (\text{GL}_n, r, (d_i)_i) \).

**Proof:** This morphism is clearly proper, and an isomorphism over \( \mathbb{A}^1_t[t^{-1}] \). As a diagram (2.2.6) for \( \mathbb{M}_t (\text{GL}_n, r, (d_i)_i) \) we can take

\[
((\mathbb{M}_{t \text{rig}} (\text{GL}_n, r, (d_i)_i) \times_{\mu^{P,N}} \tilde{\mu}^{P,N})/P^{N+1} \to \mathbb{T}_t \text{ is smooth. This follows from proposition (2.2.4).} \Box
\]

Taking for granted the existence of a \( P^{N+1} \)-equivariant strongly log-smooth toroidal resolution \( \tilde{\mu}^{P,N} \) of \( \mu^{P,N} \) (cf. section 3), the (unitary case of the) theorem announced in the first section will be a consequence of this reduction principle and of the following proposition (which also explains the terminology 2.2.6)

**Proposition 2.2.8.** —

a) Let \( T \hookrightarrow \mathbb{T} \) be a torus embedding, and \( \chi : \mathbb{T} \to \mathbb{A}^1 \) with restriction to \( T \) given by a character \( \chi \) of \( T \). If we assume that the kernel \( T \chi \) of \( \chi \) is a torus and that \( \mathbb{T} \) and \( \mathbb{A}^1 \) are respectively endowed with the canonical log-structures associated with the torus embeddings \( T \hookrightarrow \mathbb{T} \) and \( \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \), the morphism \( \mathbb{T}_t \) is log-smooth.
b) As a particular case, under the conditions of (a), the restriction $T_p$ of $T_t$ over the section $\overline{p} = \{ t = p \}$ of $\mathbb{A}^1_t$ is log-smooth.

c) Let

$$
\begin{array}{c}
Y \\
\downarrow \\
X \rightarrow Z
\end{array}
$$

be a commutative triangle of morphisms of fine log-schemes. If we assume that the left morphism is smooth and surjective, that the log-structure on $Y$ is the inverse image of the log-structure on $X$, and that $Y$ is log-smooth over $Z$, then $X$ is log-smooth over $Z$.

Proof: a) Let $X^*(T)$ be the character group of the torus $T$. The kernel of $\chi$ is a torus, and so the cokernel of the morphism $Z\chi \rightarrow X^*(T)$ is torsion-free. Hence the morphism $T_t$ is log-smooth.

b) It is a direct consequence of (a).

c) (after I. Vidal) Log-smoothness is a local property for the (classical) étale topology, hence we can assume that $Y$ is an affine space $A^d_X$ over $X$. In this case, we will use Kato’s infinitesimal lifting criterion. Let $S_0 \rightarrow S$ be a strict nilimmersion; every morphism of log-schemes $S_0 \rightarrow X$ can be lifted to a morphism of log-schemes $S_0 \rightarrow A^d_X$ and the log-smoothness of $A^d_X = Y \rightarrow Z$ implies that $S_0 \rightarrow A^d_X$ can be lifted to $S \rightarrow A^d_X$. The projection $S \rightarrow X$ of this last morphism is obviously a lifting of $S_0 \rightarrow X$, and so the proposition is proved. $\square$

2.3. — The symplectic case

Let $(d_i)_{1 \leq i \leq N}$ be a $N$-uple of strictly positive integers with sum $g$ and let $d_{N+1} = g$. The $(N + 1)$-uple $(d_i)_{1 \leq i \leq N+1}$ is a partition of 2$g$ and the local model in the symplectic case, $M_t(Sp_{2g}, (d_i)_i)$, is the following closed subscheme of $M_t(GL_{2g}, g, (d_i)_{1 \leq i \leq N+1})$.

Let $\Delta$ be the matrix associated with the permutation $(g, g-1, \ldots, 1)$ and $J \in GL_{2g}(\mathbb{Z})$ be the matrix

$$
\begin{pmatrix}
0 & -\Delta \\
\Delta & 0
\end{pmatrix}
$$

The matrix $J$ defines on $\mathbb{Z}[t, t^{-1}]$ a nondegenerate symplectic form $\langle , \rangle$. Its restriction $\langle , \rangle_0$ to $\mathcal{V}_0$ is nondegenerate and the restriction $\langle , \rangle_g$ of $t^{-1} \langle , \rangle$ to $\mathcal{V}_g$ has values in $\mathbb{Z}[t]$ and is also nondegenerate.

Definition 2.3.1. (cf. [dJ 2], [RZ]). — The local model $M_t(Sp_{2g}, (d_i)_i)$ consists of $(N + 1)$-uples

$$(\omega_i \subset \mathcal{V}_i)_i \in M_t(GL_{2g}, g, (d_i)_{1 \leq i \leq N+1})$$

where $\omega_0$ and $\omega_N$ are respectively totally isotropic with respect to $\langle , \rangle_0$ and $\langle , \rangle_N$.

The local model $M_p(Sp_{2g}, (d_i)_i)$ is the restriction of $M_t(Sp_{2g}, (d_i)_i)$ along the section $\overline{p} = \{ t = p \}$ of the affine line $\mathbb{A}^1_t = \text{Spec } \mathbb{Z}[t]$.

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The scheme $M_t (\text{Sp}_{2g}, (d_i)_i)[t^{-1}]$ is simply the lagrangian grassmannian of $(\mathbb{Z}[t, t^{-1}]^{2g}, <, >)$.

Let $\mathcal{Q}_t$ be the automorphism group of the system $((\mathcal{V}_i, \alpha_i)_i, <, >, >_0, <, >_N)$ (i.e. $\mathcal{Q}_t$ is the closed subgroup of $\mathcal{P}_t$ defined by the conditions $\gamma_0 \in \text{Sp}(\mathcal{V}_0, <, >_0$ and $\gamma_N \in \text{Sp}(\mathcal{V}_N, <, >_N)$). The group scheme $\mathcal{Q}_t$ acts obviously (on the left) on $M_t (\text{Sp}_{2g}, (d_i)_i)$. It is affine and smooth over $\mathbb{A}^1_\mathbb{Q}$ and $\mathcal{Q}_t[t^{-1}]$ is naturally identified with $\text{Sp}(\mathbb{Z}[t, t^{-1}]^{2g}, <, >)$. The group scheme $\mathcal{J}$ occurring in the first section is (in the symplectic case) the restriction $\mathcal{Q}_p$ of $\mathcal{Q}_t$ along the section $\mathcal{P}$.

We will now define a certain rigidification of $M_t (\text{Sp}_{2g}, (d_i)_i)$.

**Definition 2.3.2.** — The rigidified local model $M_t^{\text{rig}} (\text{Sp}_{2g}, (d_i)_i)$ is the fibre product

$$M_t (\text{Sp}_{2g}, (d_i)_i) \times_{M_t (\text{GL}_{2g}, g, (d_i)_i)} M_t^{\text{rig}} (\text{GL}_{2g}, g, (d_i)_i).$$

The scheme $M_t^{\text{rig}} (\text{Sp}_{2g}, (d_i)_i)$ is naturally a (right) $P_{N+1}$-principal homogeneous space over $M_t (\text{Sp}_{2g}, (d_i)_i)$. It is also endowed with a (left) free action of $\mathcal{Q}_t$ and both actions commute.

2.3.3. — We will now re-write the algebraic space $Q_t \backslash M_t^{\text{rig}} (\text{Sp}_{2g}, (d_i)_i)$.

Let $\Sigma^{g,N}$ be the scheme consisting of triples $((\Pi_i)_{0 \leq i \leq N}, (\, )_0, (\, )_N)$, where

- $(\Pi_i)_{0 \leq i \leq N} \in \mu^{P,N}$
- $(\, )_0$ (resp. $(\, )_N)$ is a non-degenerate symplectic pairing on $V_0$ (resp. $V_N$) such that $\Omega_0$ (resp. $\Omega_N$) is totally isotropic
- $(\Pi_0 \cdots \Pi_{N-1}, x, y)_N = (x, \Pi_N y)_0$, $\forall x \in V_0$, $y \in V_N$

and let $\Sigma^{g,N}_{(d_i)_i}$ be the fibre product $\Sigma^{g,N} \times_{\mu^{P,N}_{(d_i)_i}} \mu^{P,N}_{(d_i)_i}$. Using (2.2.4) and a variant of ([dJ 2], proposition 3.6) (or of [RZ], theorem 3.16) we obtain the following statement.

**Proposition 2.3.4.** — The $\Sigma^{P,N}_{(d_i)_i}$-scheme $M_t^{\text{rig}} (\text{Sp}_{2g}, (d_i)_i)$ is a $Q_t$-principal homogeneous space.

*Proof:* the proof of [ibid.] carries over easily to our context ("$p$ replaced by $t$"). Using (2.2.4), the system $((\mathcal{V}_i)_i, (\Pi_i)_i, (\, )_0, (\, )_N)$ is a system of type II (in the terminology of [dJ 2]) and so the systems $((\mathcal{V}_i)_i, (\Pi_i)_i, (\, )_0, (\, )_N)$ and $((\mathcal{V}_i)_i, (\alpha_i)_i, <, >, >_0, <, >_N)$ are (Zariski-)locally isomorphic on $\Sigma^{P,N}_{(d_i)_i}$. □

2.3.5. — The morphism $\Sigma^{g,N}_{(d_i)_i} \to \mu^{r,N}$ is not smooth. However, as Faltings remarked in his analogous situation, one can turn out this difficulty by using toroidal resolutions with the following additional property.

Let us write

$$\Pi_N = \begin{pmatrix} A_N & B_N \\ 0 & C_N \end{pmatrix}$$

and

$$\Pi_0 \cdots \Pi_{N-1} = \begin{pmatrix} A'_N & B'_N \\ 0 & C'_N \end{pmatrix}$$
and let $\mu_{bl,N}^P$ be the scheme obtained by blowing up $\mu^P$ along the ideal sheaves spanned by the minors of size $i$ of $A_N$, $\forall 1 \leq i \leq g - 1$ and also along the ideal sheaves spanned by the minors of size $i$ of $A'_N$, $\forall 1 \leq i \leq g - 1$.

We shall only consider resolutions of $\mu^P$ factoring through the morphism $\mu_{bl,N}^P \to \mu^P$. The following proposition, essentially due to Faltings ([F 2]), is the crucial step of the reduction principle for these resolutions.

For any $\mathbb{Z}[t]$-scheme $X$ let us denote by $X^+$ the closed subscheme of $X$ defined by the ideal sheaf $\text{Tors}_t(X)$ consisting of sections of $\mathcal{O}_X$ killed by a power of $t$.

**Proposition 2.3.6. —** The morphism

$$(\Sigma_{g,N}^N \times_{\mu^P N} \mu_{bl}^{P,N} \mu_{bl}^{P,N})^+ \to (\mu_{bl}^{P,N})^+$$

is smooth.

**Proof:** Let $(\Sigma_{g,N}^N B)$ (resp. $(\Sigma_{g,N}^N B')$) be the biggest open subscheme of $\Sigma_{g,N}^N$ where the matrix $B_N$ (resp. $B_N'$) is invertible. It is easily seen that the translates of $(\Sigma_{g,N}^N B)$ (resp. $(\Sigma_{g,N}^N B')$) under $P^{N+1}$ cover $\Sigma_{g,N}^N$. Hence their intersection $(\Sigma_{g,N}^N 0)$ is also such that its translates under $P^{N+1}$ cover $\Sigma_{g,N}^N$. The morphism $(\Sigma_{g,N}^N \times_{\mu^P N} \mu_{bl}^{P,N})^+ \to (\mu_{bl}^{P,N})^+$ is $P^{N+1}$-equivariant and so it suffices to prove that the morphism $((\Sigma_{g,N}^N 0) \times_{\mu^P N} \mu_{bl}^{P,N})^+ \to (\mu_{bl}^{P,N})^+$ is smooth.

Changing the bases of $V_0$ and $V_N$ we can assume that $B_N = B_N' = \text{Id}_g$ and (because the ideal sheaves spanned by the minors of size $i$ of $A_N$ (resp. $A'_N$) have been blown up, $\forall 1 \leq i \leq g - 1$) that the matrix $A_N$ (resp. $A'_N$) is the diagonal matrix

$$\text{diag} (a_0, a_0 a_1, \cdots, a_0 \cdots a_{g-1})$$

(resp.

$$\text{diag} (a_1 \cdots a_g, a_2 \cdots a_g, \cdots, a_g),$$

with $a_0 a_1 \cdots a_g = t$). In the new bases we have

$$\Pi_N = \begin{pmatrix} A_N & \text{Id}_g \\ 0 & -A'_N \end{pmatrix} \quad \text{and} \quad \Pi_N' = \begin{pmatrix} A'_N & \text{Id}_g \\ 0 & -A_N \end{pmatrix}$$

(recall that $\Pi_N \Pi_N' = \Pi_N' \Pi_N = t \text{Id}_{2g}$). Let

$$J[0] = \begin{pmatrix} 0 & J_1[0] \\ J_2[0] & J_3[0] \end{pmatrix} \quad \text{and} \quad J[N] = \begin{pmatrix} 0 & J_1[N] \\ J_2[N] & J_3[N] \end{pmatrix}$$

be the matrices defining the symplectic pairings on $V_0$ and $V_N$ in the new bases (the upper left terms vanish because $\Omega_0$ and $\Omega_N$ are totally isotropic). These matrices satisfy the following relations

$$t J_1[0] = -J_2[0], \quad t J_3[0] = -J_3[0], \quad t J_1[N] = -J_2[N], \quad t J_3[N] = -J_3[N],$$

$$A_N J_1[N] + J_1[0] A'_N = 0,$$

$$J_1[N] A_N + A_N J_1[0] = 0,$$

$$J_1[N] - A_N J_3[N] + t J_1[0] + J_3[0] A'_N = 0.$$
Using the first line, we eliminate the coefficients of $J_2[0]$ and of $J_2[N]$. Killing the $t$-torsion, these relations become

$$J_1[0]_j^i + a_i \cdots a_{j-1} J_1[N]_j^i = 0, \forall 1 \leq i \leq j \leq g$$

$$J_1[N]_j^i + a_j \cdots a_{i-1} J_1[0]_j^i = 0, \forall 1 \leq j \leq i \leq g$$

$$J_1[N]_j^i + J_1[0]_i^j - a_0 \cdots a_{i-1} J_3[N]_j^i + a_j \cdots a_g J_3[0]_i^j = 0, \forall 1 \leq i \leq j \leq g$$

Using the two first lines, we eliminate the coefficients $J_1[0] (i \leq j)$ and $J_1[N] (i > j)$. Using the last one, we eliminate $J_1[0] (i > j)$. The other coordinates $J_1[N] (i \leq j)$, $J_3[0] (i < j)$, $J_3[N] (i < j)$ can be fixed freely, and so we see that $((\Sigma g,N)^0 \times _{\mu^{P,N}} \mu_{bl}^{P,N})^+$ is an affine space of dimension $g(3g-1)/2$ over $(\mu_{bl}^{P,N})^+$. This achieves the proof of (2.3.6). $\square$

Using the proposition (2.3.6) we obviously obtain the following variant of Faltings’s reduction principle.

Reduction principle 2.3.7. — Assume that we have a $P^{N+1}$-equivariant toroidal resolution $\tilde{\mu}^{P,N} \longrightarrow \mu_{bl}^{P,N} \longrightarrow \mu^{P,N}$ whose source can be put in a $P^{N+1} \times T'$-equivariant diagram (2.2.6), where the action of $P^{N+1} \times T'$ on $\overline{T}$ is by the second factor. The morphism

$$\overline{M}_t (\operatorname{Sp}_{2g}, (d_i)_i) := (\overline{M}_t^{rig} (\operatorname{Sp}_{2g}, (d_i)_i) \times _{\mu^{P,N}} \tilde{\mu}^{P,N}) / P^{N+1} \longrightarrow \overline{M}_t (\operatorname{Sp}_{2g}, (d_i)_i)$$

is then a $Q_t$-equivariant toroidal resolution of $\overline{M}_t (\operatorname{Sp}_{2g}, (d_i)_i)$. $\square$

Taking for granted the existence of a $P^{N+1}$-equivariant strongly log-smooth toroidal resolution $\tilde{\mu}^{P,N}$ of $\mu^{P,N}$ with this property (cf. section 3), the (symplectic case of the) theorem announced in the first section will be a consequence of this reduction principle and of the proposition (2.2.8).

3. — “Compactification of the multiplication law” for parabolic subgroups of the general linear group

We will now construct a $P^{N+1}$-equivariant strongly log-smooth toroidal resolution of $\mu^{P,N}$. Our construction follows quite closely Lafforgue’s compactification ([L 1,2]) of $\operatorname{PGL}^{N+1}_r / \operatorname{PGL}_r$.

3.1. — Let us first briefly recall Lafforgue’s construction.

Let $P_{ad}$ be the quotient of $P$ by its center $\mathbb{G}_m$. Lafforgue constructs a $P^{N+1}_{ad}$-equivariant open immersion

$$P^{N+1}_{ad} / P_{ad} \hookrightarrow \Omega^{P,N}$$

as a quotient of an open immersion

$$P^{N+1} \times \mathbb{G}_m^{SP,N} / (P \times \mathbb{G}_m^{N+1}) \hookrightarrow \Omega^{P,N}$$

by a torus $\mathbb{G}_m^{SP,N} / \mathbb{G}_m$ acting freely on both sides (see [L 1] for the special case $P = \operatorname{GL}_n$ and [L 2] for arbitrary parabolic subgroups of $\operatorname{GL}_n$). Moreover, $\Omega^{P,N}$ is endowed with
a $\mathbb{G}_m^{S,P,N}/\mathbb{G}_m$-equivariant morphism $\Omega^{P,N} \rightarrow A^{P,N}$ to a toric normal variety $A^{P,N}$ with torus $A^{P,N}_0 = \mathbb{G}_m^{S,P,N}/\mathbb{G}_m$ (we shall recall in the next paragraph the definition of the tori and of the maps between them).

The scheme $\Omega^{P,N}$ is a “toroidal equivariant compactification” of $P^{N+1}_{ad}/P_{ad}$ in the following sense (cf. [L 2], Théorèmes 5 et 6): the scheme $\Omega^{P,N}$ is projective over $\text{Spec} \mathbb{Z}$ and the morphism $\Omega^{P,N} \rightarrow A^{P,N}$ is smooth.

The scheme $\Omega^{P,N}$ is a compactification of the multiplication law for $P_{ad}$ in the following sense ([L 1, L 2]). For every map $\{0, \cdots, M\} \rightarrow \{0, \cdots, N\}$, Lafforgue constructs commutative diagrams

$$P^{N+1} \times \mathbb{G}_m^{S,P,N}/(P \times \mathbb{G}_m^{N+1}) \hookrightarrow \Omega^{P,N} \quad \quad \quad P^{N+1}_{ad}/P_{ad} \hookrightarrow \Omega^{P,N}$$

and

$$P^{M+1} \times \mathbb{G}_m^{S,P,N}/(P \times \mathbb{G}_m^{M+1}) \hookrightarrow \Omega^{P,M} \quad \quad \quad P^{M+1}_{ad}/P_{ad} \hookrightarrow \Omega^{P,M}$$

(the diagram in the right hand side is of course a quotient of the one in the left hand side).

If as a particular case we consider the family of injections

$$\{i, i + 1\} \hookrightarrow \{0, \cdots, N\} \quad (i \in \mathbb{Z}/(N+1)\mathbb{Z} = \{0, \cdots, N\})$$

we obtain a morphism $\Omega^{P,N} \rightarrow (\Omega^{P,N})^{N+1}$ which is a compactification of the closed immersion

$$P^{N+1}/P \hookrightarrow (P^2/P)^{N+1} \quad \quad \quad P^{N+1} \quad \quad \quad (h_0, \cdots, h_N) \quad \quad \quad (h_0 h_1^{-1}, \cdots, h_N h_{N+1}^{-1}, h_N h_0^{-1})$$

with image $\{(g_0, \cdots, g_N)/g_0 \cdots g_N = 1\}$. So, the morphism $\Omega^{P,N} \rightarrow (\Omega^{P,N})^{N+1}$ is a compactification of the morphism giving the product of $N$ elements in $P$.

3.2. — In this paragraph, we shall modify the construction of $\Omega^{P,N}$ to keep track of the center of $P$. More precisely, we shall define a subtorus $\mathcal{T}^{P,N}$ of $\mathbb{G}_m^{S,P,N}/\mathbb{G}_m$ such that the quotient

$$\mathcal{T}^{P,N}\backslash P^{N+1} \times \mathbb{G}_m^{S,P,N}/(P \times \mathbb{G}_m^{N+1})$$

is $\mathbb{G}_m \times P^{N+1}/P$ and so the quotient $\mathcal{T}^{P,N}\backslash \Omega^{P,N}$ will be a “partial compactification” of $\mathbb{G}_m \times P^{N+1}/P$. We will also define a morphism

$$\mathcal{T}^{P,N}\backslash \Omega^{P,N} \rightarrow \mathfrak{P}_{rk \geq 1}^{N+1} \times \mathbb{A}_t^1$$

factoring through the closed immersion $\mu^{P,N} \hookrightarrow \mathfrak{P}_{rk \geq 1}^{N+1} \times \mathbb{A}_t^1$. 11
The (projective) morphism
\[ \tilde{\mu}^{P,N} = T^{P,N} \setminus \Omega^{P,N} \longrightarrow \mu^{P,N} \]
thus obtained will be the desired resolution.

Let us recall the following definitions and facts ([L 2], 1.b).
- The finite set \( S^{P,N} \) appearing in the beginning of the previous paragraph is
  \[
  \{(i_\alpha,1, i_\alpha,2)_{0 \leq \alpha \leq N} \in \mathbb{N}^{2(N+1)} | \sum_\alpha |i_\alpha| = n \text{ and } \sum_\alpha i_{\alpha,1} \geq r\}
  \]
  (where, for simplicity, \( |i_\alpha| \) denotes \( i_{\alpha,1} + i_{\alpha,2} \)). It is endowed with the following partial order
  \[ i \leq j \iff |i_\alpha| = |j_\alpha| \text{ and } i_{\alpha,1} \leq j_{\alpha,1}, \forall \ 0 \leq \alpha \leq N . \]
- The map \( P \times G^{N+1} \hookrightarrow P^{N+1} \times G^{S^{P,N}} \) is
  \[
  (g, \lambda_0, \cdots, \lambda_N) \mapsto (g, \lambda_1 g, \cdots, \lambda_N g, (\lambda_0 \det g^{-1} \lambda_1^{-|i_1|} \cdots \lambda_N^{-|i_N|})_{i \in S^{P,N}}).
  \]
- The map \( P^{N+1} \times G^{S^{P,N}} / (P \times G^{N+1}) \longrightarrow P^2 \times G^{S^{P,2}} / (P \times G^2) \) associated with an injection \( \{j, j+1\} \hookrightarrow \{0, \cdots, N\} \) is simply
  \[
  (g_0, \cdots, g_N, (\lambda_j)_j) \mapsto (g_i, g_{i+1}, (\lambda_{(0,\cdots,0,j_i,j_{i+1},0\cdots,0)}|j_i| + |j_{i+1}| = n)
  \]
- Let \( E = \mathbb{A}^{n(N+1)} \) be the trivial vector bundle of rank \( n(N+1) \), equipped with the obvious gradation
  \[ E = \bigoplus_{0 \leq \alpha \leq N} E_\alpha \quad (E_\alpha \simeq \mathbb{A}^n), \]
  compatible filtration
  \[ (\mathbb{A}^n = F_\alpha = E_{\alpha,1} \subset E_{\alpha,2} = E_\alpha) \]
  and action of \( P^{N+1} \). Let us recall the following notations from [L 2]
  - One has \( \Lambda^n E = \bigoplus_{|i|} \Lambda^{|i|} E \). The sum is over \((N+1)\)-uples
    \[ |i| = (|i|_0, \cdots, |i|_N) \in \mathbb{N}^{N+1} \text{ with } |i|_1 + \cdots |i|_N = n, \]
  and \( \Lambda^{|i|} E \) is \( \Lambda^{|i_0|} E_0 \otimes \cdots \otimes \Lambda^{|i_N|} E_N \).
  - For \( i \in S^{P,N} \), \( \Lambda^i E \) denotes the sub-vector bundle \( \bigotimes_\alpha (\Lambda^{i_{\alpha,1}} F_\alpha \wedge \Lambda^{i_{\alpha,2}} E_\alpha) \) of \( \Lambda^{|i|} E \).
The scheme $\Omega_{P,N}$ inherits from its construction as a Zariski closure in the product

$$\mathbb{G}_m \setminus \prod_{i \in S_{P,N}} [(\Lambda^{|i|}E/ \sum_{j > i} \Lambda^j E) - \{0\}] \times A_{P,N}$$

a morphism

$$\Omega_{P,N} \longrightarrow \mathbb{G}_m \setminus \prod_{i \in S_{P,N}} [(\Lambda^{|i|}E/ \sum_{j > i} \Lambda^j E) - \{0\}]$$

This morphism is generically an immersion. Moreover it is projective ([L 1,2], théorème 5).

The composite of the open immersion $P^{N+1} \times \mathbb{G}^{S_{P,N}}_m / (P \times \mathbb{G}^{N+1}_m) \hookrightarrow \Omega_{P,N}$ with this morphism is

$$(g_0, \cdots, g_N, (\lambda_i)_i) \mapsto (\lambda_i \Lambda^{|i|}_{0}(t g_0) \wedge \cdots \wedge \Lambda^{|i|}_N(t g_N))_i$$

where $\Lambda^{|i|}_{0}(t g_0) \wedge \cdots \wedge \Lambda^{|i|}_N(t g_N)$ is understood as a morphism

$$\Lambda^{|i|}_{0} E_0^\vee \otimes \cdots \otimes \Lambda^{|i|}_N E_N^\vee \longrightarrow \Lambda^{|i|}_{0} E_0^\vee \wedge \cdots \wedge \Lambda^{|i|}_N E_N^\vee = \Lambda^1$$

(the non-invariance of this last identification disappears in the quotient), identified to an element of $\Lambda^{|i|}E$ by biduality.

The composite of the open immersion $P^{N+1} \times \mathbb{G}^{S_{P,N}}_m / (P \times \mathbb{G}^{N+1}_m) \hookrightarrow \Omega_{P,N}$ with the morphism $\Omega_{P,N} \longrightarrow A_{P,N}$ is the obvious morphism $P^{N+1} \times \mathbb{G}^{S_{P,N}}_m / (P \times \mathbb{G}^{N+1}_m) \longrightarrow \mathbb{G}^{S_{P,N}}_m / \mathbb{G}^{N+1}_m \longrightarrow A_{P,N} \hookrightarrow A_{P,N}$.

One has a morphism $\Omega_{P,N} \longrightarrow \Omega^{GL_r,N}$ inducing a morphism $\overline{\Omega}^{P,N} \longrightarrow \overline{\Omega}^{GL_r,N}$ which makes the diagram

$$\begin{align*}
P^{N+1}_{ad} / P_{ad} & \longrightarrow \ PGL_{r}^{N+1} / PGL_r \\
\downarrow & \downarrow \\
\overline{\Omega}^{P,N} & \longrightarrow \overline{\Omega}^{GL_r,N}
\end{align*}$$

commutative. □

Let $\pi_i$ and $\delta_i$ be respectively the $2(N + 1)$-uples

$$((0,0), \cdots, (0,0), (1,0), (n - 1,0), (0,0), \cdots, (0,0))$$

and

$$((0,0), \cdots, (0,0), (0,0), (n,0), (0,0), \cdots, (0,0))$$

(where $n - 1$ and $n$ are in the $(i + 1)$-th position, $i \in \mathbb{Z}/(N + 1)\mathbb{Z}$), and $T^{P,N}$ be the subtorus of $\mathbb{G}^{S_{P,N}}_m$ defined by the $(N + 1)$ equations $\lambda_{\pi_i} = \lambda_{\delta_i}$ ($i \in \mathbb{Z}/(N + 1)\mathbb{Z}$).

The proof of the following proposition is now an easy calculation using the above facts ; it will be left to the reader.
Proposition 3.3. —

a) The morphism

$$P^{N+1} \times \mathbb{G}_m^{SP,N} / (P \times \mathbb{G}_m^{N+1}) \longrightarrow P^{N+1} / P \times \mathbb{G}_m$$

$$= \left\{ (g_0, \cdots, g_N, t) \in P^{N+1} \times \mathbb{G}_m \mid g_0 \cdots g_N = t \right\}$$

defined by

$$(g_0, \cdots g_N, (\lambda_j)_j) \mapsto (\lambda_{\pi_0}/\lambda_{\delta_0} g_0 g_1^{-1}, \cdots, \lambda_{\pi_N}/\lambda_{\delta_N} g_N g_0^{-1}, \prod_i \lambda_{\pi_i}/\lambda_{\delta_i})$$

is an isomorphism

b) The morphism of tori

$$\mathbb{G}_m^{SP,N} \longrightarrow \mathbb{G}_m^{SP,2}$$

associated with an injection $$\{ j, j + 1 \} \hookrightarrow \{ 0, \cdots, N \}$$ sends $$T^{P,N}$$ to $$T^{P,2}$$

c) The composite $$\Pi_i$$ of the obvious morphism

$$\Omega^{P,N} \longrightarrow \mathbb{G}_m \setminus \prod_{i \in SP,N} \left[ (\Lambda^{j|E}/\sum_{j>i} \Lambda^jE) - \{ 0 \} \right] \longrightarrow$$

$$\mathbb{G}_m \setminus \left[ \left( \Lambda^{\delta_i|E}/\sum_{j>\delta_i} \Lambda^jE \right) - \{ 0 \} \right] \times \left[ \left( \Lambda^{\pi_i|E}/\sum_{j>\pi_i} \Lambda^jE \right) - \{ 0 \} \right]$$

with the identifications

$$\mathbb{G}_m \setminus \left[ \left( \Lambda^{\delta_i|E}/\sum_{j>\delta_i} \Lambda^jE \right) - \{ 0 \} \right] \times \left[ \left( \Lambda^{\pi_i|E}/\sum_{j>\pi_i} \Lambda^jE \right) - \{ 0 \} \right]$$

$$= \mathbb{G}_m \setminus \left[ \left( \Lambda^{\delta_i|E} - \{ 0 \} \right) \times \left( \Lambda^{\pi_i|E} - \{ 0 \} \right) \right]$$

and

$$\mathbb{G}_m \setminus \left[ \left( \Lambda^{\delta_i|E} - \{ 0 \} \right) \times \left( \Lambda^{\pi_i|E} - \{ 0 \} \right) \right]$$

$$= E_i \otimes [\Lambda^{n-1} E_{i+1} \otimes (\Lambda^n E_{i+1})^{-1}] - \{ 0 \} = \mathfrak{g}_n - \{ 0 \}$$

is $$T^{P,N}$$-invariant and factors through $$\mathfrak{g}_{rk \geq 1}$$ . Its restriction to $$P^{N+1} \times \mathbb{G}_m^{SP,N} / (P \times \mathbb{G}_m^{N+1})$$

is just

$$(g_0, \cdots g_N, (\lambda_j)_j) \mapsto (\lambda_{\pi_i}/\lambda_{\delta_i}) g_i g_{i+1}^{-1}$$
The ideal sheaves spanned by the minors of size $j$ of the matrix $\Pi_i$ are invertible, $\forall 1 \leq j \leq n - 1$. So are the ideal sheaves spanned by the minors of size $j$ of the matrix $A_i$ deduced from $\Pi_i$ via the obvious projection $\Psi \to \mathfrak{g}_r$, $\forall 1 \leq j \leq r - 1$. In fact, this last matrix is a multiple of the one obtained by composing the morphism $\Omega^{P,N} \to \Omega^{\text{GL}_r,N}$ with the morphism $\Pi_i^{\text{GL}_r}$.

d) The kernel of the morphism $G_{m,N}^{P,N}/G_{m,N}^{r+1} \to G_m$ defined by $(\lambda_j)_j \mapsto \prod_i \lambda_{\pi_i}/\lambda_{\delta_i}$ is a torus. This morphism extends to a $T_{P,N}$-invariant morphism

$$t : A^{P,N} \to \mathbb{A}^1.$$

e) The morphism

$$(\Pi_0, \cdots, \Pi_N, t) : T_{P,N}\setminus\Omega^{P,N} \to \mathcal{G}_{r+1}^{N+1} \times \mathbb{A}^1_1$$

thus defined factors through $\mu^{P,N}$.

f) The morphism $T_{P,N}\setminus\Omega^{P,N} \to \mu^{P,N}$ is projective and birational. It factors through $\mu_{bl}^{P,N}$ (cf. 2.3.6). □

This achieves the construction of $\tilde{\mu}^{P,N} = T_{P,N}\setminus\Omega^{P,N}$.

4. — Complements and remarks

4.1. — Faltings’s original reduction principle.

Using the previous work of L. Lafforgue on the compactification of $\text{PGL}_r^{N+1}/\text{PGL}_r$ ([L 1]), it is also possible to obtain $\text{GL}_r^{N+1}$-equivariant log-smooth resolutions of the schemes $\mu^{r,N}$ (cf. 2.1) introduced by G. Faltings in [F 2].

This construction can be combined with Faltings’s original reduction principle [F 2] and one also obtains log-smooth resolutions

$$\tilde{M}_t' \to M_t$$

of the local models. In fact, the resolution $\tilde{M}_t \to M_t$ constructed in the present work (sections 2-3) factors through $\tilde{M}_t' \to M_t$ ; this reflects the fact that the morphism

$$P_{ad}^{N+1}/P_{ad} \to \text{PGL}_r^{N+1}/\text{PGL}_r$$

extends to a morphism

$$\tilde{\Omega}^{P,N} \to \tilde{\Omega}^{\text{GL}_r,N}$$

of Lafforgue’s compactifications (cf. [L 2]).

The reason why we developed our variant is that it seems to be a little bit more symmetric (cf. remark 4.2.3).
4.1.1. **Remark:** in [F 2], Faltings constructs a proper birational morphism

\[ Y \longrightarrow X = \mu^{r,N} \]

(cf. [F 2], p. 25) and checks that for \( r \leq 3 \) this morphism is a \( GL_n^{N+1} \)-equivariant log-smooth resolution of \( \mu^{r,N} \). Combined with his reduction principle, this result yields resolutions of the local models for \( r \leq 3 \) in the unitary case and \( g \leq 3 \) in the symplectic one. I do not know the relationship between Faltings’s resolutions and the ones we obtain by using Lafforgue’s work. I also do not know the relationship of all these resolutions with the one constructed by de Jong for \( g = 2 \) ([dJ 1]) and with the one obtained in [G] for \( g = 3 \).

4.2. — In some cases the local model \( \mathbb{M}_t \) is in fact endowed with an action of a non-connected group-scheme \( K^{\text{ext}} \) with neutral component \( K \). More precisely, in the unitary case, this occurs when a cyclic permutation of \( \{1, \ldots, N+1\} \) fixes the partition \( (d_i)_{1 \leq i \leq N+1} \) of \( n \) and in the symplectic case, this occurs when the permutation \( (N, N-1, \ldots, 1) \) fixes the partition \( (d_i)_{1 \leq i \leq N} \) of \( g \). We shall see that in these cases the resolution \( \mathbb{M}_t \) is also endowed with an action of \( K^{\text{ext}} \).

4.2.1. — The unitary case. Let \( \sigma : i \mapsto i+s \quad (s \in \mathbb{Z}/(N+1)\mathbb{Z}) \) be a cyclic permutation of \( \{1, \ldots, N+1\} = \mathbb{Z}/(N+1)\mathbb{Z} \) fixing \( (d_i)_i \) and \( S \) be the order of \( \sigma \). The group \( \mathbb{Z}/SZ = <\sigma> \) acts on \( \mathbb{M}_t(G\, L_n, r, (d_i)_i) \) and on \( \mu^{P,N} \) through the permutation of the factors \( V_i \)'s induced by \( \sigma \). This defines an action of \( K^{\text{ext}} := K \rtimes (\mathbb{Z}/SZ) \) on \( \mathbb{M}_t(G\, L_n, r, (d_i)_i) \). It is easily seen that the resolution \( \tilde{\mu}^{P,N} \longrightarrow \mu^{P,N} \) is \( \mathbb{Z}/SZ \)-equivariant: in fact Lafforgue’s compactifications are even endowed with an action of the symmetric group \( \mathfrak{S}_{N+1} \) (cf. [L 1,2], 1.b). Hence the resolution \( \tilde{\mathbb{M}}_t(G\, L_n, r, (d_i)_i) \longrightarrow \mathbb{M}_t(G\, L_n, r, (d_i)_i) \) induced by \( \tilde{\mu}^{P,N} \) is \( K^{\text{ext}} \)-equivariant.

4.2.2. — The symplectic case

Let \( \sigma \) be the permutation \( (N, N-1, \ldots, 1) \) of \( \{1, \ldots, N\} \). We also denote by the same letter the permutation \( (N, N-1, \ldots, 1, N+1) \) of \( \{1, \ldots, N+1\} \). The group \( \mathbb{Z}/2\mathbb{Z} = <\sigma> \) acts on \( \mathbb{K} \), on \( \mathbb{M}_t(Sp_g, (d_i)_i) \) and on \( \mu^{P,N} \), respectively by letting the generator \( \sigma \) act via

\[
(\gamma_i)_i \mapsto (J^{-1} t^{-1} \gamma_{\sigma(i)}^{-1} J)_i,
\]

\[
(\omega_i)_i \mapsto (\omega_{\sigma(i)}^J)_i
\]

and

\[
(\Pi_i)_i \mapsto (J^{-1} t^1 \Pi_{\sigma(i)} J)_i
\]

(where \( J \) is the matrix (2.3), the chain

\[
V_N^\vee \xrightarrow{t^{\alpha_0}} V_0^\vee \xrightarrow{t^{\alpha_3}} V_1^\vee \xrightarrow{t^{\alpha_2}} \cdots \xrightarrow{t^{\alpha_N}} V_N^\vee
\]

is identified to the chain

\[
V_0 \xrightarrow{\alpha_0} V_N \xrightarrow{\alpha_N} V_{N-1} \xrightarrow{\alpha_{N-1}} \cdots \xrightarrow{\alpha_1} V_0
\]
by the perfect pairing on \( \mathbb{Z}[t, t^{-1}]^{2g} \) defined by the matrix

\[
\begin{pmatrix}
0 & \text{Id}_g \\
t \text{Id}_g & 0
\end{pmatrix}
\]

and \( \sigma'^t \) is the permutation \((N, N - 1, \cdots, 0)\) of \(\{0, \cdots, N\}\). This defines an action of \( \mathcal{K}^{\text{ext.}} := \mathcal{K} \rtimes (\mathbb{Z}/2\mathbb{Z}) \) on \( \bar{M}_t(\text{Sp}_{2g}, (d_i)_i) \). One can see that if the resolution \( \bar{\mu}^{P,N} \rightarrow \mu^{P,N} \) is \( \mathbb{Z}/2\mathbb{Z} \)-equivariant (and this will be the case for the resolutions obtained in the third section from Lafforgue’s compactifications: the automorphism \( g \mapsto J^{-1}g^{-1}J \) of \( P \) induces an automorphism of Lafforgue’s compactifications, cf. [L 2], 1.b), the resolution \( \bar{\tilde{M}}_t(\text{Sp}_{2g}, (d_i)_i) \rightarrow \bar{M}_t(\text{Sp}_{2g}, (d_i)_i) \) induced by \( \bar{\tilde{\mu}}^{P,N} \) is \( \mathcal{K}^{\text{ext.}} \)-equivariant.

4.2.3. Remarks:

1) The reason why we have developed our variant and used Lafforgue’s generalization [L 2] of his previous work is that in the symplectic case, the symmetry \( \sigma \) of \( \bar{M}_t(\text{Sp}_{2g}, (d_i)_i) \) does not (or at least not obviously) lift to an automorphism of \( \bar{\tilde{M}}_t(\text{Sp}_{2g}, (d_i)_i) \).

2) The resolution of \( \bar{M}_p(\text{Sp}_4, (1, 1)) \) constructed by de Jong in [dJ 1] is also \( \mathcal{K}_p^{\text{ext.}} \)-equivariant. I do not know wether this is the case for the resolution of \( \bar{M}_p(\text{Sp}_6, (1, 1, 1)) \) constructed in [G].

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