A Set of Sequences of Complexity $2n + 1$

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Abstract. We prove the existence of a ternary sequence of factor complexity $2n + 1$ for any given vector of rationally independent letter frequencies. Such sequences are constructed from an infinite product of two substitutions according to a particular Multidimensional Continued Fraction algorithm. We show that this algorithm is conjugate to a well-known one, the Selmer algorithm. Experimentations (Baldwin, 1992) suggest that their second Lyapunov exponent is negative which presages finite balance properties.

Keywords: Substitutions, factor complexity, Selmer, continued fraction, bispecial.

1 Introduction

Words of complexity $2n + 1$ were considered in [2] with the condition that there is exactly one left and one right special factor of each length. These words are called Arnoux-Rauzy sequences and are a generalization of Sturmian sequences on a ternary alphabet. It is known that the frequencies of any Arnoux-Rauzy word are well defined and belong to the Rauzy Gasket [3], a fractal set of Lebesgue measure zero. Thus the above condition on the number of special factors is very restrictive for the possible letter frequencies.

Sequences of complexity $p(n) \leq 2n + 1$ include Arnoux-Rauzy words, codings of interval exchange transformations and more [12]. For any given letter frequencies one can construct sequences of factor complexity $2n + 1$ by the coding of a 3-interval exchange transformation. It is known that these sequences are unbalanced [14]. Thus the question of finding balanced ternary sequences of factor complexity $2n + 1$ for all letter frequencies remains. This article intends to give a positive answer to this question for almost all vectors of letter frequencies (with respect to Lebesgue measure).

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In recent years, multidimensional continued fraction algorithms were used to obtain ternary balanced sequences with low factor complexity for any given letter frequency vector. Indeed the Brun algorithm leads to balanced sequences [10] and it was shown that the Arnoux-Rauzy-Poincaré algorithm leads to sequences of factor complexity \( p(n) \leq \frac{5}{2}n + 1 \) [5].

In 2015, the first author introduced a new Multidimensional Continued Fraction algorithm [9] based on the study of Rauzy graphs. In this work, we formalize the algorithm, its matrices, substitutions and associated cocycles and \( S \)-adic words. We show that \( S \)-adic words obtained from these substitutions have complexity \( 2n + 1 \). We also show that the algorithm is conjugate to the Selmer algorithm, a well-known Multidimensional Continued Fraction algorithm. We believe that almost all sequences generated by the algorithm are balanced.

2 A Bidimensional Continued Fraction Algorithm

On \( \Lambda = \mathbb{R}^3_{\geq 0} \), the bidimensional continued fraction algorithm introduced by the first author [9] is

\[
F_C(x_1, x_2, x_3) = \begin{cases} 
(x_1 - x_3, x_3, x_2), & \text{if } x_1 \geq x_3; \\
(x_2, x_1, x_3 - x_1), & \text{if } x_1 < x_3.
\end{cases}
\]

More information on Multidimensional Continued Fraction Algorithms can be found in [8,13].

2.1 The Matrices

Alternatively, the map \( F_C \) can be defined by associating nonnegative matrices to each part of a partition of \( \Lambda \) into \( \Lambda_1 \cup \Lambda_2 \) where

\[
\Lambda_1 = \{ (x_1, x_2, x_3) \in \Lambda \mid x_1 \geq x_3 \}, \\
\Lambda_2 = \{ (x_1, x_2, x_3) \in \Lambda \mid x_1 < x_3 \}.
\]

The matrices are given by the rule \( M(x) = C_i \) if and only if \( x \in \Lambda_i \) where

\[
C_1 = \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}.
\]

The map \( F_C \) on \( \Lambda \) and the projective map \( f_C \) on \( \Delta = \{ x \in \Lambda \mid \|x\|_1 = 1 \} \) are then defined as:

\[
F_C(x) = M(x)^{-1}x \quad \text{and} \quad f_C(x) = \frac{F_C(x)}{\|F_C(x)\|_1}.
\]

Many of its properties can be found in [11] and the density function of the invariant measure of \( f_C \) was computed in [1].
2.2 The Cocycle

The algorithm \( F_C \) defines a cocycle \( M_n : A \rightarrow SL(3, \mathbb{Z}) \) by

\[
M_0(x) = I \quad \text{and} \quad M_n(x) = M(x)M(F_C x)M(F_C^2 x) \cdots M(F_C^{n-1} x)
\]

satisfying the cocycle property \( M_{n+m}(x) = M_n(x) \cdot M_m(F_C x) \).

For example starting with \( x = (1, e, \pi)^T \), the first iterates (approximate to the nearest hundredth) under \( F_C \) are

\[
\begin{pmatrix}
1.00 \\
2.72 \\
3.14
\end{pmatrix} \xrightarrow{F_C} \begin{pmatrix}
2.72 \\
1.00 \\
2.14
\end{pmatrix} \xrightarrow{F_C} \begin{pmatrix}
0.58 \\
2.14 \\
0.82
\end{pmatrix} \xrightarrow{F_C} \begin{pmatrix}
2.14 \\
0.58 \\
0.42
\end{pmatrix} \xrightarrow{F_C} \begin{pmatrix}
1.72 \\
0.42 \\
0.58
\end{pmatrix} \xrightarrow{F_C} \begin{pmatrix}
1.14 \\
0.42 \\
0.58
\end{pmatrix}
\]

The associated cocycle at \( x = (1, e, \pi)^T \) when \( n = 5 \) is

\[
M_5(x) = M(x)M(F_C x)M(F_C^2 x)M(F_C^3 x)M(F_C^4 x)
\]

\[
= C_2 C_1 C_2 C_1 C_1
\]

\[
= \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 1 \\
1 & 2 & 1 \\
1 & 2 & 2
\end{pmatrix}.
\]

2.3 The Substitutions

Let \( A = \{1, 2, 3\} \). The substitutions on \( A^* \) are given by the rule \( \sigma(x) = c_i \) if and only if \( x \in A_i \) for \( i = 1, 2 \) where

\[
c_1 = \begin{cases}
1 &\mapsto 1 \\
2 &\mapsto 13 \\
3 &\mapsto 2
\end{cases}
\quad \text{and} \quad
\]

\[
c_2 = \begin{cases}
1 &\mapsto 2 \\
2 &\mapsto 13 \\
3 &\mapsto 3
\end{cases}
\]

One may check that \( C_i \) is the incidence matrix of \( c_i \) for \( i = 1, 2 \). For any word \( w \in A^* \), we denote \( \bar{w} = |w|_1, |w|_2, |w|_3 \in \mathbb{N}^3 \) where \( |w|_i \) means the number of occurrences of the letter \( i \) in \( w \). Therefore, for all \( x \in A \), \( \sigma(x) : A^* \rightarrow A^* \) is a monoid morphism such that its incidence matrix is \( M(x) \), i.e., \( \sigma(x)(w) = M(x) \cdot \bar{w} \).

2.4 S-adic Words

Let \( S \) be a set of morphisms. A word \( w \) is said to be \( S \)-adic if there is a sequence \( s = (\tau_n : A_{n+1} \rightarrow A_n)_{n \in \mathbb{N}} \in S^\mathbb{N} \) and a sequence \( a = (a_n) \in \prod_{n \in \mathbb{N}} A_n \) such that \( w = \lim_{n \rightarrow +\infty} \tau_0 \tau_1 \cdots \tau_{n-1}(a_n) \). The pair \((s, a)\) is called an \( S \)-adic representation of \( w \) and the sequence \( s \) a direct sequence of \( w \). The \( S \)-adic representation is said to be primitive whenever the direct sequence \( s \) is primitive, i.e., for all \( r \geq 0 \), there exists \( r' > r \) such that all letters of \( A_r \) occur in all images \( \tau_r \tau_{r+1} \cdots \tau_{r'-1}(a) \), \( a \in A_{r'} \). Observe that if \( w \) has a primitive \( S \)-adic representation, then \( w \) is uniformly recurrent. For all \( n \), we set \( w^{(n)} = \lim_{m \rightarrow +\infty} \tau_n \tau_{n+1} \cdots \tau_{m-1}(a) \).
2.5 S-adic Words Associated with the Algorithm $F_\mathcal{C}$

The algorithm $F_\mathcal{C}$ defines the function $\sigma_n : A \to \text{End}(A^*)$, $\sigma_n(x) = \sigma(F_\mathcal{C}^n x)$

When the sequence $(\sigma_n(x))_{n \in \mathbb{N}}$ contains infinitely many occurrences of $c_1$ and $c_2$, this defines a $\mathcal{C}$-adic word, $\mathcal{C} = \{c_1, c_2\}$,

$$W(x) = \lim_{n \to \infty} \sigma_0(x)\sigma_1(x) \cdots \sigma_n(x)(1).$$

Indeed, let $w_n = \sigma_0(x) \cdots \sigma_n(x)(1)$. As $c_1$ and $c_2$ occur infinitely often, there exist infinitely many indices $m$ such that $\sigma_{m+1}(x) = c_1$ and $\sigma_{m+2}(x) = c_2$. For all $n \geq m+2$, let $z = \sigma_{m+3}(x) \cdots \sigma_n(x)(1)$. Since $\{1, 2\}A^*$ is stable under both $c_1$ and $c_2$, we have $z \in \{1, 2\}A^*$, so that $c_1c_2(z) \in \{13, 12\}A^*$. Then 1 is a proper prefix of $c_1c_2(z) = \sigma_{m+1}(x) \cdots \sigma_n(x)(1)$, and therefore $w_m$ is a proper prefix of $w_n$. It follows that the limit of $(w_n)$ exists.

For example, using vector $x = (1, e, \pi)^T$, we have

$$\sigma(x)\sigma(F_\mathcal{C}x)\sigma(F_\mathcal{C}^2x)\sigma(F_\mathcal{C}^3x)\sigma(F_\mathcal{C}^4x) = c_2c_1c_2c_1c_1 = \begin{cases} 1 &\mapsto 23 \\ 2 &\mapsto 23213 \\ 3 &\mapsto 2313 \end{cases},$$

whose incidence matrix is $M_5(x)$. The associated infinite $\mathcal{C}$-adic word is

$$W(x) = 23232132323231323223213233223132323232323232\cdots.$$

**Lemma 1.** Let $x \in \Delta$. The following conditions are equivalent.

(i) the entries of $x$ are rationally independent,

(ii) the directive sequence of $W(x)$ is primitive,

(iii) the directive sequence of $W(x)$ does not belong to $\mathcal{C}^*\{c_1^2, c_2^2\}^\omega$.

Furthermore, the vector of letter frequencies of 1, 2 and 3 in $W(x)$ is $x$.

**Proof.** Let us first prove that (ii) and (iii) are equivalent. Assume that $s = (\tau_n) \in \mathcal{C}^*\{c_1^2, c_2^2\}^\omega$. Then there exists $r \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $\tau_{r+2i} = \tau_{r+2i+1}$, and $\tau_{r+2i} \tau_{r+2i+1}$ is either $c_1^2$ or $c_2^2$. Observe that $c_1^2(1) = 1$, $c_2^2(3) = 3$, and $c_2^2(3) = c_2^2(13) = 13$. Let $r' > r$. If $r' - r$ is even, then $\tau_{r} \tau_{r+1} \cdots \tau_{r'-1}(1)$ does not contain the letter 2. If $r' - r$ is odd, then $\tau_{r} \tau_{r+1} \cdots \tau_{r'-1}(2)$ does not contain the letter 2. Therefore the directive sequence $s$ is not primitive.

Conversely, if $s \not\in \mathcal{C}^*\{c_1^2, c_2^2\}^\omega$, then $s$ contains infinitely many occurrences of words in $\{c_1c_2^{2i+1}c_1c_2^{2j+1}c_2c_1^{2k+1}c_2c_1^{2l+1}c_2c_1^{2m+1} : i, j, k, l, m \in \mathbb{N} \setminus \{0\}\}$. It can be checked that all the matrices of these substitutions have positive entries, so that $s$ is primitive.

Let us now assume that (iii) does not hold. Then, as above, there exists $r \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $\tau_{r+2i} = \tau_{r+2i+1}$. Note that, if $y = (y_1, y_2, y_3)$, then $C^{-2}_1y = (y_1 - y_2 - y_3, y_2, y_3)$ and $C^{-2}_2y = (y_1, y_2, y_3 - y_1 - y_2)$. In both cases, the middle entry is unchanged, and the sum of the two other entries decreases by at least $y_2$. Let $F_\mathcal{C}(x) = (y_1, y_2, y_3)$ and $F_\mathcal{C}^{r+2i}(x) = (z_1, z_2, z_3)$. Then $z_2 = y_2$
and \(z_1 + z_2 \leq y_1 + y_2 - iy_2\). This is possible for all \(i\) only if \(y_2 = 0\), and then \(\ell' F_C(x) = 0\), where \(\ell'\) is the row vector \(\ell' = (0, 1, 0)\). Then \(\ell x = 0\) where \(\ell = \ell' M(F_C^{-1}(x))^{-1} \ldots M(x)^{-1}\) is a nonzero integer row vector, showing that the entries of \(x\) are rationally dependent.

Finally, let us assume that (iii) holds and (i) does not hold. Observe first that, if \(F_C^{r'}(x)\) has a zero entry for some \(r\), then either \(F_C(x)\) or \(F_C^{r'}(x)\) has a zero middle entry, and from this point on the directive sequence can be factored over \(\{c_1^2, c_2^2\}\), contradicting (iii). From now on we assume that all entries of \(F_C(x)\) are positive for all \(n\).

Let \(\ell_0\) be a nonzero integer row vector such that \(\ell_0 x = 0\). The directive sequence can be factored over \(\{c_1c_2^2, c_1c_2^2c_2: k \in \mathbb{N}\}\). Let us consider the sequence \((\tau_m)\) such that \(\tau_0 = 0\) and \(\tau_m \ldots \tau_{m+1}-1\) is in this set for all \(m \in \mathbb{N}\). Let \(\ell_m = \ell_0 M(x) \ldots M(F_C^{-1}(x))\). Then \(\ell_m\) is a nonzero integer row vector such that \(\ell_m F_C^{m+1}(x) = 0\), and \(\ell_{m+1}\) is either \(\ell_m C_1 C_2^2 C_1\) or \(\ell_m C_2 C_2^2 C_2\) for some \(k\).

Assume that \(\ell_m = (a, b, c)\). Then \(\ell_{m+1}\) is one of

\[
\ell_{m+1} C_1 C_2^{2k} C_1 = (a', a' + b, a' + c) \quad \text{with} \quad a' = a + k b,
\]

\[
\ell_{m+1} C_1 C_2^{2k+1} C_1 = (a' - b, a', a' - c) \quad \text{with} \quad a' = a + (k + 1) b + c,
\]

\[
\ell_{m+1} C_2 C_2^{2k} C_2 = (c' + a, c' + b, c') \quad \text{with} \quad c' = c + k b,
\]

\[
\ell_{m+1} C_2 C_2^{2k+1} C_2 = (c' - a, c', c' - b) \quad \text{with} \quad c' = c + (k + 1) b + a.
\]

Define \(D_m\) as the difference between the maximum and the minimum entry of \(\ell_m\). Note that, as \(F_C(x)\) has positive entries, the maximum entry of \(\ell_m\) is positive and the minimum entry is negative. Then \(D_m = \max(|b-a|, |c-b|, |c-a|)\). In the first two cases \(D_{m+1} = \max(|b|, |c|, |c-b|)\). If \(a\) is (inclusively) between \(b\) and \(c\), which must then have opposite signs, then \(D_{m+1} = D_m = |c-b|\). Otherwise \(D_{m+1} < D_m\). Similarly, in the other two cases \(D_{m+1} = \max(|a|, |b|, |b-a|)\), and \(D_{m+1} = D_m\) if \(c\) is inclusively between \(a\) and \(b\), while \(D_{m+1} < D_m\) otherwise.

The sequence of positive integers \((D_m)\) is non-increasing. To reach a contradiction, we need to show that it decreases infinitely often.

If for large enough \(m\) all transitions between \(\ell_m\) and \(\ell_{m+1}\) are of the first type, then (iii) is not satisfied. Similarly, if for large enough \(m\) all transitions are of the third type, then (iii) is not satisfied. So we must either have infinitely often transitions of the second or fourth type, or infinitely often a transition of the first type followed by a transition of the third type.

Assume first that the transition between \(\ell_m\) and \(\ell_{m+1}\) is of the second type. Then \(\ell_{m+1} = (a' - b, a', a' - c)\) and \(\ell_{m+2}\) is one of

\[
\ell_{m+1} C_1 C_2^{2k} C_1 = (a'', a'' + a', a'' + a' - c) \quad \text{with} \quad a'' = a' - b + k' a',
\]

\[
\ell_{m+1} C_1 C_2^{2k+1} C_1 = (a'' - a', a'', a'' - a' + c) \quad \text{with} \quad a'' = a' - b + (k' + 1) a' + a' - c,
\]

\[
\ell_{m+1} C_2 C_1^{2k} C_2 = (c'' + a' - b, c'', a'' + c, a'' - c) \quad \text{with} \quad c'' = a' - c + k' a',
\]

\[
\ell_{m+1} C_2 C_1^{2k+1} C_2 = (c'' - a' + c, c'', c'' - a') \quad \text{with} \quad c'' = a' - c + (k' + 1) a' + a' - b.
\]

If \(D_{m+1} = D_m\), then \(a\) is between \(b\) and \(c\) which must have opposite signs. Then \(a'\) is strictly between \(a' - b\) and \(a' - c\), which implies in all four cases that \(D_{m+2} < D_{m+1}\). So we always have \(D_{m+2} < D_m\).
The case where the transition between $\ell_m$ and $\ell_{m+1}$ is of the fourth type is similar. Assume now that this transition is of the first type, and the transition between $\ell_{m+1}$ and $\ell_{m+2}$ is of the third type. Then $\ell_{m+1} = (a', a' + b, a' + c)$ and $\ell_{m+2} = (c' + a', c'' + a' + b, c'')$ with $c'' = a' + c + k'(a' + b)$. If $D_{m+1} = D_m$, then $a$ is between $b$ and $c$ which must have opposite signs, so that $a'$ is strictly between $a' + b$ and $a' + c$, which implies that $D_{m+2} < D_{m+1}$. So again we always have $D_{m+2} < D_m$, and this concludes the proof.

3 Factor Complexity of Primitive $C$-adic Words

Let $w$ be a (infinite) word over some alphabet $A$. We let $\text{Fac}(w)$ denote the set of factors of $w$, i.e., $\text{Fac}(w) = \{ u \in A^* \mid \exists i \in \mathbb{N} : w_i \cdots w_{i+|u|-1} = u \}$. The extension set of $u \in \text{Fac}(w)$ is the set $E(u, w) = \{ (a, b) \in A \times A \mid au \in \text{Fac}(w) \}$.

We represent it by an array of the form

$$E(u, w) = \begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
1 \rightarrow 1 & 1 \rightarrow 12 & 1 \rightarrow 13 & 1 \rightarrow 13 \\
2 \rightarrow 12 & 2 \rightarrow 132 & 2 \rightarrow 132 & 2 \rightarrow 132 \\
3 \rightarrow 13 & 3 \rightarrow 2 & 3 \rightarrow 12 & 3 \rightarrow 12 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}$$

where a symbol $\times$ in position $(i, j)$ means that $(i, j)$ belongs to $E(u, w)$. When the context is clear we omit the information on $w$ and simply write $E(u)$. We also represent it as an undirected bipartite graph, called the extension graph, whose set of vertices is the disjoint union of $\pi_1(E(u, w))$ and $\pi_2(E(u, w))$ ($\pi_1$ and $\pi_2$ respectively being the projection on the first and on the second component) and its edges are the pairs $(a, b) \in E(u, w)$. A factor $u$ of $w$ is said to be bispecial whenever $\#\pi_1(E(u, w)) > 1$ and $\#\pi_2(E(u, w)) > 1$. A bispecial factor $u \in \text{Fac}(w)$ is said to be ordinary if there exists $(a, b) \in E(u, w)$ such that $E(u, w) \subset \{(a) \times A \} \cup \{ A \times \{b\} \}$.

To simplify proofs, we consider $C' = \{ c_{11}, c_{22}, c_{122}, c_{211}, c_{121}, c_{212} \}$, where

$$c_{11} = c_1^2 : \begin{cases} 
1 \rightarrow 1 \\
2 \rightarrow 12 \\
3 \rightarrow 13 
\end{cases}, \quad c_{122} = c_1 c_2^2 : \begin{cases} 
1 \rightarrow 12 \\
2 \rightarrow 132 \\
3 \rightarrow 2 
\end{cases}, \quad c_{211} = c_2 c_1^2 : \begin{cases} 
1 \rightarrow 2 \\
2 \rightarrow 213 \\
3 \rightarrow 23 
\end{cases}, \quad c_{212} = c_2 c_1 c_2 : \begin{cases} 
1 \rightarrow 23 \\
2 \rightarrow 213 \\
3 \rightarrow 13 
\end{cases}$$

Any (primitive) $C$-adic word is a (primitive) $C'$-adic word and conversely. We let $\varepsilon$ denote the empty word. We have the following result, where uniqueness follows from the fact that $\tau(A)$ is a code.

Lemma 2 (Synchronization). Let $w$ be a $C'$-adic word with directive sequence $(\tau_n)_{n \in \mathbb{N}} \in C'^\mathbb{N}$. If $u \in \text{Fac}(w)$ is a non-empty bispecial factor, then
1. If $\tau_0 = c_{11}$, there is a unique word $v \in \text{Fac}(w^{(1)})$ such that $u = \tau_0(v)1$.
2. If $\tau_0 = c_{22}$, there is a unique word $v \in \text{Fac}(w^{(1)})$ such that $u = 3\tau_0(v)$.
3. If $\tau_0 = c_{122}$, there is a unique word $v \in \text{Fac}(w^{(1)})$ such that $u \in 2\tau_0(v)[1, \varepsilon]$.
4. If $\tau_0 = c_{211}$, there is a unique word $v \in \text{Fac}(w^{(1)})$ such that $u \in \{3, \varepsilon\}\tau_0(v)2$.
5. If $\tau_0 = c_{121}$, there is a unique word $v \in \text{Fac}(w^{(1)})$ such that $u \in \{2, \varepsilon\}\tau_0(v)[1, 13]$.
6. If $\tau_0 = c_{212}$, there is a unique word $v \in \text{Fac}(w^{(1)})$ such that $u \in \{3, 13\}\tau_0(v)[2, \varepsilon]$.

Furthermore, $v$ is a bispecial factor of $w^{(1)}$ and is shorter than $u$.

Let $w, u$ and $v$ be as in Lemma 2. The word $v$ is called the bispecial antecedent of $u$ under $\tau_0$. Similarly, $u$ is called a bispecial extended image of $v$ under $\tau_0$. Since the bispecial antecedent of a non-empty bispecial word is always shorter, for any bispecial factor $u$ of $w$, there is a unique sequence $(u_i)_{0 \leq i \leq n}$ such that

- $u_0 = u$, $u_n = \varepsilon$ and $u_i \neq \varepsilon$ for all $i < n$;
- for all $i < n$, $u_{i+1} \in \text{Fac}(w^{(i+1)})$ is the bispecial antecedent of $u_i$.

All bispecial factors of the sequence $(u_i)_{0 \leq i < n}$ are called the bispecial descendants of $\varepsilon$ in $w^{(n)}$.

As any bispecial factor of a primitive $C$-adic word is a descendant of the empty word, to understand the extension sets of any bispecial word in $w$, we need to know the possible extension sets of $\varepsilon$ in $w^{(n)}$ and to understand how the extension set of a bispecial factor governs the extension sets of its bispecial extended images.

**Lemma 3.** If $w$ is a primitive $C$-adic word with directive sequence $(\tau_n)_{n \in \mathbb{N}} \in C^{\mathbb{N}}$, then the extension set of $\varepsilon$ is one of the following, depending on $\tau_0$.

| $\tau_0 = c_{11}$ | $\tau_0 = c_{122}$ | $\tau_0 = c_{211}$ |
|-------------------|---------------------|---------------------|
| 1 \times \times \times | 1 \times \times | 1 \times \times |
| 2 \times | 2 \times \times | 2 \times |
| 3 \times | 3 \times \times | 3 \times |

| $\tau_0 = c_{212}$ | $\tau_0 = c_{211}$ | $\tau_0 = c_{212}$ |
|-------------------|---------------------|---------------------|
| 1 \times | 1 \times \times | 1 \times |
| 2 \times | 2 \times \times | 2 \times \times |
| 3 \times \times | 3 \times | 3 \times \times |

**Proof.** The directive sequence being primitive, all letters of $A$ occur in $w^{(1)}$. The result then follows from the fact that all morphisms $\tau$ in $C'$ are either left proper $(\tau(A) \subset aA^*$ for some letter $a$) or right proper $(\tau(A) \subset A^*a$ for some letter $a$).

The next lemma describes how the extension set of a bispecial word determines the extension set of any of its bispecial extended images.

**Lemma 4.** Let $w$ be a $C'$-adic word with directive sequence $(\tau_n)_{n \in \mathbb{N}} \in C^{\mathbb{N}}$. If $u \in \text{Fac}(w)$ is the bispecial extended image of $v \in \text{Fac}(w^{(1)})$ and if $x, y \in A^*$ are such that $u = x\tau_0(v)yz$, then
1. if \( \tau_0(\mathcal{A}) \subseteq \mathcal{A}^i \) for some letter \( i \in \mathcal{A} \), we have

\[
E(u, w) = \{(a, b) \mid \exists (a', b') \in E(v, w^{(1)}) : \tau_0(a') \in \mathcal{A}^*ax \land \tau_0(b')i \in yb\mathcal{A}^* \};
\]

2. if \( \tau_0(\mathcal{A}) \subseteq \mathcal{A}^i \) for some letter \( i \in \mathcal{A} \), we have

\[
E(u, w) = \{(a, b) \mid \exists (a', b') \in E(v, w^{(1)}) : i\tau_0(a') \in \mathcal{A}^*ax \land \tau_0(b') \in yb\mathcal{A}^* \}.
\]

Proof. Let us prove the first equality, the second one being symmetric.

For the inclusion \( \supseteq \), consider \((a', b') \in E(v) \) such that \( \tau_0(a') \in \mathcal{A}^*ax \) and \( \tau_0(b')i \in yb\mathcal{A}^* \). Let \( c \in \mathcal{A} \) be such that \( a'vb'c \) is a factor of \( w^{(1)} \). Then \( \tau_0(a'vb'c) \in \tau_0(\mathcal{A}) \subseteq \mathcal{A}^*ax\tau_0(v)yb\mathcal{A}^* \) is a factor of \( w \) and we have \((a, b) \in E(u) \).

For the inclusion \( \subseteq \), consider \((a, b) \in E(u) \). Using Lemma 2, the word \( ax \) (resp., \( yb \)) is the suffix (resp., prefix) of a word \( \tau_0(x') \), \( x' \in \mathcal{A}^+ \) (resp., \( \tau_0(y') \), \( y' \in \mathcal{A}^+ \)) such that \( x'y' \in \text{Fac}(w^{(1)}) \). Furthermore, still using Lemma 2, \( x \) is a strict suffix of \( \tau_0(a') \), where \( x' \in \mathcal{A}^*a' \) and \( y \) is a prefix of \( \tau_0(b') \), where \( y' \in b'\mathcal{A}^* \). If \( y \) is a strict prefix of \( \tau_0(b') \), then \((a', b') \) is an extension of \( v \) such that \( \tau_0(a') \in \mathcal{A}^*ax \) and \( \tau_0(b') \in yb\mathcal{A}^* \). Otherwise, if \( \tau_0(b') = y \), we have \( b = i \) since \( \tau_0(\mathcal{A}) \subseteq \mathcal{A}^i \) and \((a', b') \) is an extension of \( v \) such that \( \tau_0(a') \in \mathcal{A}^*ax \) and \( \tau_0(b')i = yi \), which concludes the proof.

Lemma 4 can be more easily understood using the tabular representation of the extension sets. Indeed, for the first case (\( \tau_0(\mathcal{A}) \subseteq \mathcal{A}^i \)), the extensions of \( v = x\tau(v)y \) can be obtained as follows: 1) replace any left extensions \( a \) by \( \tau(a) \) and any right extension \( b \) by \( \tau(b)i \); 2) remove the suffix \( x \) from the left extensions whenever it is possible (otherwise, delete the row) and remove the prefix \( y \) from the right extensions whenever it is possible (otherwise, delete the column); 3) keep only the last letter of the left extensions and the first letter of the right extensions; 4) permute and merge the rows and columns with the same label. The second case (\( \tau_0(\mathcal{A}) \subseteq \mathcal{A}^i \)) is similar.

Let us make this more clear on an example and consider the extension set \( E(v) = \{(1, 3), (2, 1), (2, 2), (2, 3), (3, 2) \} \). This extension set corresponds to the extension set of the empty word whenever the last applied substitution is \( c_{211} \) (see Lemma 3). Using Lemma 4 the extension sets of \( 2c_{122}(v) \) and \( 2c_{121}(v) \) are obtained as follows (arrow labels indicate above step number):

\[
\begin{array}{c|ccc} \hline E(v) & 1 & 2 & 3 \\
\hline 1 & \times & 1 \to 212 & 2 \\
2 & \times & \times & 2132 \times \times \\
3 & \times & 22 & \times \\
\hline \end{array}
\]

\[
\begin{array}{c|ccc} \hline E(v) & 1 & 2 & 3 \\
\hline 1 & \times & 1 \to 13 & 131 \times 3 \\
2 & \times & \times & 132 \times \times \\
3 & \times & 12 & \times \\
\hline \end{array}
\]

\[
\begin{array}{c|ccc} \hline E(c_{122}(v)) & 1 \to 1 & 2 \to 212 & 2 \\
\hline 1 & \times & 3 \times & 2 \times \\
2 & \times & \times & 3 \times \times \\
3 & \times & \times & \times \times \\
\hline \end{array}
\]

\[
\begin{array}{c|ccc} \hline E(c_{121}(v)) & 1 \to 1 & 2 \to 13 & 131 \times 3 \\
\hline 1 & \times & 3 \times & 2 \times \\
2 & \times & \times & 3 \times \times \\
3 & \times & \times & \times \times \\
\hline \end{array}
\]

\[
\begin{array}{c|ccc} \hline E(2c_{122}(v)) & 1 \to 1 & 2 \to 1 \to 212 & 2 \\
\hline 1 & \times & 3 \times & 2 \times \\
2 & \times & \times & 3 \times \times \\
3 & \times & \times & \times \times \\
\hline \end{array}
\]

\[
\begin{array}{c|ccc} \hline E(2c_{122}(v)) & 1 \to 1 & 2 \to 1 \to 13 & 131 \times 3 \\
\hline 1 & \times & 3 \times & 2 \times \\
2 & \times & \times & 3 \times \times \\
3 & \times & \times & \times \times \\
\hline \end{array}
\]

\[
\begin{array}{c|ccc} \hline E(2c_{121}(v)) & 1 \to 1 & 2 \to 1 \to 13 & 131 \times 3 \\
\hline 1 & \times & 3 \times & 2 \times \\
2 & \times & \times & 3 \times \times \\
3 & \times & \times & \times \times \\
\hline \end{array}
\]

\[
\begin{array}{c|ccc} \hline E(2c_{121}(v)) & 1 \to 1 & 2 \to 1 \to 13 & 131 \times 3 \\
\hline 1 & \times & 3 \times & 2 \times \\
2 & \times & \times & 3 \times \times \\
3 & \times & \times & \times \times \\
\hline \end{array}
\]
The proof of Proposition 6 will essentially consist in describing how ordinary bispecial words occur. The next lemma allows to understand when bispecial words have ordinary bispecial extended images.

Lemma 5. Let \( w \) be a \( C' \)-adic word with directive sequence \( (\tau_n)_{n \in \mathbb{N}} \in C'^{\mathbb{N}} \). Let \( u \in \text{Fac}(w) \) be a non-empty bispecial factor and \( v \) be its bispecial antecedent. We have the following.

1. If \( \tau_0 \in \{c_{11}, c_{22}\} \), then \( E(u) = E(v) \);
2. if \( v = \varepsilon \) and \( \tau_0 \in \{c_{121}, c_{212}\} \), then \( u \) is ordinary;
3. if \( \tau_0 \in \{c_{122}, c_{121}, c_{212}\} \), if \( E(v) \subseteq (A \times \{1, 2\}) \cup \{(a, 3)\} \) for some letter \( a \in A \) with \( E(v) \cap \{(a, 1), (a, 2)\} \neq \emptyset \) and if \( E(v) \setminus \{(a, 3)\} \) is the extension set of an ordinary bispecial word, then \( u \) is ordinary;
4. if \( \tau_0 \in \{c_{211}, c_{121}, c_{212}\} \), if \( E(v) \subseteq \{(2, 3) \times A\} \cup \{(1, a)\} \) for some letter \( a \in A \) with \( E(v) \cap \{(2, a), (3, a)\} \neq \emptyset \) and if \( E(v) \setminus \{(1, a)\} \) is the extension set of an ordinary bispecial word, then \( u \) is ordinary;
5. if \( v \) is ordinary, then \( u \) is ordinary

Proof. Items 1 and 2 directly follow from Lemma 4. Item 3 can be checked by hand using Lemma 3 and Lemma 4. Let us prove Item 3. Item 4 being symmetric.

Let us say that two extension sets \( E \) and \( E' \) are equivalent whenever there exist two permutations \( p_1 \) and \( p_2 \) of \( A \) such that \( E = \{(p_1(a), p_2(b)) \mid (a, b) \in E'\} \). If \( \tau_0 = c_{122} \), then \( u \in \{2\sigma(v), 2\sigma(v)1\} \) by Lemma 2. We make use of Lemma 4. If \( u = 2\sigma(v) \), then the extension set of \( u \) is equivalent to the one obtained from \( E(v) \) by merging the columns with labels 1 and 2. If \( u = 2\sigma(v)1 \), then the extension set of \( u \) is equivalent to the one obtained from \( E(v) \) by deleting the column with label 3. In both cases, \( u \) is ordinary.

The same reasoning applies when \( \tau_0 \in \{c_{121}, c_{212}\} \): depending on the word \( x \) such that \( u \in A'\sigma(v)x \), either we delete the column with label 3, or we merge the columns with labels 1 and 2.

Recall that an infinite word is a tree word if the extension graph of any of its bispecial factors is a tree. Obviously, if \( u \) is an ordinary bispecial word, its extension graph is a tree. If \( w \in A^{\mathbb{N}} \) is a tree word in which all letters of \( A \) occur, then \( w \) has factor complexity \( p(n) = (\text{Card}(A) - 1)n + 1 \) for all \( n \).

Proposition 6. Any primitive \( C' \)-adic word is a uniformly recurrent tree word. In particular, any primitive \( C' \)-adic word has factor complexity \( p(n) = 2n + 1 \).

Proof. Any primitive \( C' \)-adic word has a primitive \( C' \)-adic representation, hence is uniformly recurrent.

To show that the extension graphs of all bispecial factors are trees, we make use of Lemma 5. If \( u \) is a bispecial factor of \( w \), it is a descendant of \( \varepsilon \in \text{Fac}(w^{(n)}) \) for some \( n \). If \( \tau_n \in \{c_{11}, c_{22}\} \), then from Lemma 3 and Lemma 5 all descendants of \( \varepsilon \) are ordinary. The extension graph of \( u \) is thus a tree.

For \( \tau_n \in \{c_{122}, c_{211}, c_{121}, c_{212}\} \), we represent the extension sets of the descendants of \( \varepsilon \) in the graphs represented in Figure 1 and Figure 2. Observe that the situation is symmetric for \( c_{122} \) and \( c_{211} \), and for \( c_{121} \) and \( c_{212} \) so we only represent
The matrices are given by the rule $\sigma_{M}$.

Fig. 1. Non-ordinary bispecial descendants of $\varepsilon \in \text{Fac}(w^n)$ whenever $\tau_n = c_{122}$.

Fig. 2. Non-ordinary bispecial descendants of $\varepsilon \in \text{Fac}(w^n)$ whenever $\tau_n = c_{121}$.

the graphs for $c_{122}$ and $c_{121}$. Furthermore, in these graphs, we do not represent the extension sets of ordinary bispecial factors as the property of being ordinary is preserved by taking bispecial extended images (Lemma 5). Given an extension set of some bispecial word $v$, if $u$ is a bispecial extended image of $v$ such that $u = x \tau(v)y$, we label the edge from $E(v)$ to $E(u)$ by $x \cdot \tau \cdot y$. Finally, for all $v$, we have $E(c_{11}(v)) = E(v)$ and $E(3c_{22}(v)) = E(v)$, but for the sake of clarity, we do not draw the loops labeled by $c_{11} \cdot 1$ and by $3 \cdot c_{22}$. We conclude the proof by observing that the extension graphs of all descendants are trees.

4 Selmer Algorithm

Selmer algorithm [13] (also called the GMA algorithm in [11]) is an algorithm which subtracts the smallest entry to the largest. Here we introduce a semi-sorted version of it which keeps the largest entry at index 1. On $\Gamma = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3_{\geq 0} \mid \max(x_2, x_3) \leq x_1 \leq x_2 + x_3\}$, it is defined as

$$F_S(x_1, x_2, x_3) = \begin{cases} (x_2, x_1 - x_3, x_3) & \text{if } x_2 \geq x_3, \\ (x_3, x_2, x_1 - x_2) & \text{if } x_2 < x_3. \end{cases}$$

The partition of $\Gamma$ into $\Gamma_1 \cup \Gamma_2$ is

$$\begin{align*} \Gamma_1 &= \{(x_1, x_2, x_3) \in \Gamma \mid x_2 \geq x_3\}, \\
\Gamma_2 &= \{(x_1, x_2, x_3) \in \Gamma \mid x_2 < x_3\}. \end{align*}$$

For semi-sorted Selmer algorithm, the matrices and associated substitutions are

$$S_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $s_1 = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \end{cases}$, $s_2 = \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 12 \end{cases}$.

The matrices are given by the rule $M(x) = S_i$ if and only if $x \in \Gamma_i$. The map $F_S$ on $\Gamma$ is then defined as: $F_S(x) = M(x)^{-1}x$. The substitutions on $A^*$ are given by the rule $\sigma(x) = s_i$ if and only if $x \in \Gamma_i$ for $i = 1, 2$. 
A Set of Sequences of Complexity 2

5 Conjugacy of $F_C$ and $F_S$

The numerical computation of Lyapunov exponents made in [11] indicate that exponents for the unsorted Selmer algorithm and $F_C$ have statistically equal values. The next proposition gives an explanation for this observation.

Proposition 7. Algorithms $F_C : \Lambda \rightarrow \Lambda$ and $F_S : \Gamma \rightarrow \Gamma$ are topologically conjugate.

Proof. Let $z : \Lambda \rightarrow \Gamma$ be the homeomorphism defined by $x \mapsto Zx$ with

$$Z = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$  

We verify that $C_i$ is conjugate to $S_i$ through matrix $Z$ for $i = 1, 2$:

$$S_1Z = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = ZC_1 \quad \text{and} \quad S_2Z = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = ZC_2.$$  

Thus we have $z \circ F_C = F_S \circ z$.

An infinite word $u \in A^\infty$ is said to be finitely balanced if there exists a constant $C > 0$ such that for any pair $v, w$ of factors of the same length of $u$, and for any letter $i \in A$, $|v_i| - |w_i| \leq C$.

Based on [7, Theorem 6.4], and considering that computer experiments suggest that the second Lyapunov exponent of Selmer algorithm is negative ($\theta_1 \approx \log(1.200) \approx 0.182$ and $\theta_2 \approx \log(0.9318) \approx -0.0706$ in [4, p. 1522], $\theta_1 \approx 0.18269$ and $\theta_2 \approx -0.07072$ in [11]), we believe that the following conjecture holds.

Conjecture 8. For almost every $x \in \Delta$, the word $W(x)$ is finitely balanced.

5.1 Substitutive Conjugacy

Let $z_l$ and $z_r$ be the following two substitutions:

$$z_l : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 123 \\ 3 \mapsto 13 \end{cases} \quad \text{and} \quad z_r : \begin{cases} 1 \mapsto 21 \\ 2 \mapsto 231 \\ 3 \mapsto 31 \end{cases}.$$  

The substitution $z_l$ is left proper while $z_r$ is right proper. Moreover they are conjugate through the equation

$$z_l(w) \cdot 1 = 1 \cdot z_r(w) \quad \text{for every} \ w \in A^*.$$  

Notice that $Z$ is the incidence matrix of both $z_l$ and $z_r$.

The substitutions $c_i$ are not conjugate to $s_i$ but are related through substitutions $z_l$ and $z_r$ for $i = 1, 2$:

$$s_1 \circ z_l = z_r \circ c_1 = (1 \mapsto 21, 2 \mapsto 2131, 3 \mapsto 231),$$

$$s_2 \circ z_r = z_l \circ c_2 = (1 \mapsto 123, 2 \mapsto 1213, 3 \mapsto 13).$$  

We deduce that
Proposition 9. $S$-adic sequences when $S = \{s_1, s_2\}$ restricted to the application of the semi-sorted Selmer algorithm $F_S$ on totally irrational vectors $x \in \Gamma$ have factor complexity $2n + 1$.

The problem of finding an analogue of $F_C$ in dimension $d \geq 4$ (i.e. projective dimension $d - 1$), generating $S$-adic sequences with complexity $(d-1)n + 1$ for almost every vector of letter frequencies is still open.

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