CHARACTERIZATIONS OF EQUIVARIANT LITTLE DISCS AND LINEAR ISOMETRIES OPERADS

JONATHAN RUBIN

Abstract. We study the indexing systems that correspond to equivariant linear isometries operads and infinite little discs operads. When $G$ is a finite abelian group, we prove that a $G$-indexing system is realized by a little discs operad if and only if it is generated by cyclic $G$-orbits. When $G = C_n$ is a finite cyclic group, and $n$ is either a prime power or $n = pq$ for primes $3 < p < q$, we prove that a $G$-indexing system is realized by a linear isometries operad if and only if it satisfies Blumberg and Hill’s horn-filling condition.

We also develop equivariant algebra, at times necessary for, and at times inspired by the work above. We introduce transfer systems, a finite reformulation of the data in an indexing system, and we construct image and inverse image adjunctions for transfer systems that are analogous to equivariant induction, restriction, and coinduction. We construct derived induction, restriction, and coinduction functors for $N_\infty$ operads, and we prove that they correspond to their algebraic counterparts for injective maps.

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1. Introduction

Recent work in equivariant algebraic topology has unearthed a multitude of interesting, new $G$-operads, by which we mean symmetric operads in the category of $G$-spaces for a finite group $G$. These operads $\mathcal{O}$, called $N_\infty G$-operads, are characterized by three conditions:

(1) for any integer $n \geq 0$, the $n$th space $\mathcal{O}(n)$ is $\Sigma_n$-free,
(2) for any subgroup $\Gamma \subset G \times \Sigma_n$, the fixed point subspace $\mathcal{O}(n)^\Gamma$ is either empty or contractible, and
(3) for any integer $n \geq 0$, the coaction $\epsilon: \mathcal{O}(n) \to \Sigma_n \times \mathcal{O}(n)$ is a $G$-map.

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Informally, such operads parametrize homotopy-coherent associative, commutative, and unital operations, together with partial systems of transfer maps. One often thinks of $N_\infty$ operads as encoding multiplicative structure, and hence the transfer maps are often thought of as norms.

There are several canonical examples of $N_\infty$ operads. Let $U$ be a $G$-universe, i.e. a countably infinite-dimensional real $G$-inner product space that contains each of its finite-dimensional subrepresentations infinitely often, and which contains trivial summands. Then we have an infinite little discs operad $\mathcal{D}(U)$, a Steiner operad $\mathcal{K}(U)$, and a linear isometries operad $\mathcal{L}(U)$ structured by $U$. The operads $\mathcal{D}(U)$ and $\mathcal{K}(U)$ are equivalent, but the operads $\mathcal{D}(U)$ and $\mathcal{L}(U)$ need not be [2, Theorem 4.22]. From a classical standpoint, the operads $\mathcal{D}(U)$ and $\mathcal{L}(U)$ parametrize additive and multiplicative structures on spectra over $U$, and they were a major motivation behind the general theory of $N_\infty$ operads. Accordingly, one might hope that every $N_\infty$ $G$-operad is equivalent to $\mathcal{D}(U)$ or $\mathcal{L}(U)$ for some $G$-universe $U$. This is false (cf. Theorem 3.5). General $N_\infty$ operads are essentially algebraic in nature, in the sense that every $N_\infty$ operad is determined by a combinatorial invariant called an indexing system [2, Theorem 3.24], and every indexing system is realized by an $N_\infty$ operad that is unique up to homotopy [4], [6], [7]. Notably, the algebra of equivariant function composition is all that is relevant in this classification. On the other hand, the operads $\mathcal{D}(U)$ and $\mathcal{L}(U)$ have representation-theoretic properties that are lost at this level of abstraction.

In this paper, we study the features peculiar to equivariant little discs and linear isometries operads. In the first half, we describe properties that distinguish infinite little discs and linear isometries operads from general $N_\infty$ operads. In the second half, we develop algebra that is sometimes necessary for, and sometimes inspired by, the considerations in the first half. We review some relevant terminology below, and then we state the results.

Suppose that $G$ is a finite group and that $\mathcal{O}$ is a $G$-operad, i.e. a symmetric operad valued in $G$-spaces. For any subgroup $H \subset G$ and finite $H$-set $\Gamma$, let $\Gamma^T = \{(h, \sigma(h)) \mid h \in H\} \subset G \times \Sigma_{|\Gamma|}$, where $\sigma : H \to \Sigma_{|\Gamma|}$ is a chosen permutation representation of $T$. We say that $T$ is admissible for $\mathcal{O}$ if $\mathcal{O}((T))^T \neq \emptyset$. This condition is independent of the choice of permutation representation, and if $H/K$ is admissible for $\mathcal{O}$, then $\mathcal{O}$ parametrizes a transfer map $X^K \to X^H$ on every $\mathcal{O}$-algebra $G$-space $X$. The class of all admissible sets of $\mathcal{O}$ is graded over the set of subgroups of $G$, and if $\mathcal{O}$ is an $N_\infty$ operad, then the class of admissible sets of $\mathcal{O}$ contains all trivial actions, and is closed under: isomorphism, conjugation, restriction, subobjects, finite coproducts, and self-induction, i.e. if $T$ is an admissible $K$-set and $H/K$ is also admissible, then $\text{ind}^H_K T = H \times_K T$ is admissible. A class of $G$-subgroup actions that satisfies these conditions is called an indexing system. Every class of finite $G$-subgroup actions generates an indexing system, namely the intersection of all indexing systems that contain it, and every indexing system is generated by the set of orbits it contains.

Many of our results in the first half boil down to calculations of the indexing systems associated to $\mathcal{D}(U)$ and $\mathcal{L}(U)$, in terms of the universe $U$. We give modest elaborations on Blumberg and Hill’s calculations (cf. Propositions 2.3 and 5.4), and then we deduce the following consequences. Say that an orbit $G/H$ is cyclic if $H \subset G$ is normal and the quotient group $G/H$ is cyclic.
Theorem 2.7. Let $G$ be a finite abelian group and let $\mathcal{I}$ be a $G$-indexing system. Then $\mathcal{I}$ is the class of admissible sets of a $G$-little discs operad if and only if $\mathcal{I}$ is generated by cyclic $G$-orbits.

This provides a great deal of control over little discs operads in the finite abelian case, and we develop techniques for computing the indexing system generated from a prescribed set of orbits in section 6 and appendix A. We use these methods to construct minimal universes $U$ such that $D(U)$ admits $H/K$ (cf. Theorem 3.3). We also use them to produce indexing systems that are not realized by any little discs or linear isometries operad.

Theorem 3.5. Suppose that $G$ is a non-cyclic finite abelian group. Then the $G$-indexing system generated by $G/\{e\}$ alone is not realized by a $G$-little discs operad or a $G$-linear isometries operad.

On the other hand, the classification of $N_\infty$ operads guarantees that there is some algebraically defined $N_\infty$ operad $\mathcal{O}$ that realizes the indexing system generated by $G/\{e\}$. Therefore $\mathcal{O} \not\simeq D(U)$ and $\mathcal{O} \not\simeq L(U)$ for all $G$-universes $U$.

We now turn to the situation for linear isometries operads. In [2, p. 17], Blumberg and Hill identify an extra closure condition on the admissible sets of linear isometries operads, namely: if $H/K$ is admissible for $L(U)$ and $K \subset L \subset H$, then $H/L$ is also admissible for $L(U)$. This is a kind of horn-filling condition, so we refer to indexing systems with this additional property as $\Lambda$-indexing systems. In general, not every $\Lambda$-indexing system is realized by a linear isometries operad (cf. Example 4.11), but there are special cases where they all are.

Theorem 5.7. Let $G = C_{p^n}$ for a prime $p$ and natural number $n$, and let $\mathcal{J}$ be a $C_{p^n}$-indexing system. Then $\mathcal{J}$ is realized by a linear isometries operad if and only if $\mathcal{J}$ is a $\Lambda$-indexing system.

That being said, the situation is already more delicate when the cyclic group is not of prime power order.

Theorem 5.11. Suppose that $p < q$ are primes and consider the cyclic group $C_{pq}$. If $p, q > 3$, then a $C_{pq}$-indexing system is realized by a linear isometries operad if and only if it is a $\Lambda$-indexing system. If $p = 2$ or $p = 3$, then there are $C_{pq}$-$\Lambda$-indexing systems that are not realized by any linear isometries operad.

The difficulty with $C_{2q}$ and $C_{3q}$ is that there are not enough irreducible representations. We believe that the analogue to Theorem 5.7 should be true for cyclic groups $C_n$ when $n$ is suitably large. We do not pursue the matter any further here, but we hope to follow up in future work.

Theorems 2.7, 5.7, and 5.11 are the most specific characterizations of little discs and linear isometries operads that we obtain, but they are not the end of the story. The remainder of this work focuses on more general structural features of the problem. Fix a finite group $G$, let $\text{Uni}$ be the set of all isomorphism classes of $G$-universes, and let $\text{Ind}$ denote the set of all $G$-indexing systems. Both $\text{Uni}$ and $\text{Ind}$ are lattices equipped with a right action by $\text{Aut}(G)$. In studying the properties of little discs and linear isometries operads, there are two fundamental maps $\text{Uni} \rightrightarrows \text{Ind}$ to consider. One sends the isoclass of $U$ to the class of admissible sets of $D(U)$, and the other does the same for $L(U)$. One would hope that these functions preserve all structure in sight, but that is just too optimistic. The next two results are not difficult, but they are revealing.
Theorem 3.7. Let $\Psi_D : \text{Uni} \to \text{Ind}$ be the function that sends an isoclass $[U]$ to the class of admissible sets of $D(U)$. Then $\Psi_D$ is $\text{Aut}(G)$-equivariant, and it preserves the order, the maximum element, the minimum element, and joins. It is not always order-reflecting, meet-preserving, or injective.

The map for linear isometries operads is somewhat more unsettling.

Theorem 4.4. Let $\Psi_L : \text{Uni} \to \text{Ind}$ be the function that sends an isoclass $[U]$ to the class of admissible sets of $L(U)$. Then $\Psi_L$ is $\text{Aut}(G)$-equivariant, and it preserves the maximum element and the minimum element. It is not always order-preserving, order-reflecting, join-preserving, meet-preserving, or injective.

The failure of $\Psi_D$ to preserve structure reflects the fact that there are nonisomorphic irreducible $G$-representations whose points have the same stabilizers. This “overabundance” of $G$-representations is also related to why $\Psi_L$ fails to be injective and to reflect order relations. The order preservation properties of $\Psi_D$ and $\Psi_L$ are more subtle. While the admissible sets of $D(U)$ can be computed one subrepresentation of $U$ at a time (cf. Proposition 2.3), the same cannot be done for the admissibles of $L(U)$ (cf. [2, Theorem 4.18] and Theorem 4.1). As a consequence, the map $\Psi_D$ preserves joins and order relations, while the map $\Psi_L$ does not. Fortunately, we do have $\text{Aut}(G)$-equivariance in both cases, which helps us identify universes $U$ such that $D(U)$ and $L(U)$ admit a set $T$ (cf. Corollary 3.8).

The $\text{Aut}(G)$-action on $\text{Ind}$ is also quite intriguing, in and of itself. The equivalence between indexing systems and $N_\infty$ operads suggests that the $\text{Aut}(G)$-action on $\text{Ind}$ should correspond to restriction on $N_\infty$ operads, and that it should be part of a larger functoriality of $\text{Ind} = \text{Ind}(G)$ in the group $G$. We show that this is, indeed, the case. Given a homomorphism $f : G_1 \to G_2$ between finite groups, we construct a pair of image and inverse image adjunctions $f_L \dashv f^{-1}_R$ and $f^{-1}_L \dashv f_R$ between the lattices of $G_1$-indexing systems and $G_2$-indexing systems. These maps are analogous to induction, restriction, and coinduction (cf. Definition 7.5), but the definitions are complicated by the fact that the obvious constructions do not preserve indexing systems. Moreover, it turns out the maps $f^{-1}_R$ and $f^{-1}_L$ are only equal when $f$ is injective (cf. Theorem 7.12). All told, we obtain four natural, functorial extensions of the $\text{Aut}(G)$-action on $\text{Ind}$.

Theorem 7.8. Let $\text{FinGrp}$ and $\text{FinPos}$ denote the categories of finite groups and finite posets. Then there are functors

$(-)_L, (-)_R : \text{FinGrp} \cong \text{FinPos}$ and $(-)^{-1}_L, (-)^{-1}_R : \text{FinGrp}^{op} \cong \text{FinPos}$

such that for any homomorphism $f : G \to G'$ in $\text{FinGrp}$, we have order adjunctions $f_L \dashv f^{-1}_R$ and $f^{-1}_L \dashv f_R$. For any finite group $G$, these functors extend the natural action of $\text{Aut}(G)$ on $\text{Ind}(G)$ (cf. Corollary 7.14).

The proof does not seem to be entirely formal. Our argument relies on some of the combinatorics developed in appendix A.

As one would hope, the adjunctions $f_L \dashv f^{-1}_R$ and $f^{-1}_L \dashv f_R$ often correspond to their natural operadic counterparts.

Theorem 8.9. For any homomorphism $f : G_1 \to G_2$ between finite groups, the adjunction $f_L^{-1} \dashv f_R$ corresponds to the derived adjunction $\text{Res}_{f} \dashv \text{Coind}_{f}$ on $N_\infty$ operads. If $f$ is injective, the adjunction $f_L \dashv f^{-1}_R$ also corresponds to the derived adjunction $\text{Ind}_{f} \dashv \text{Res}_{f}$.
It is unclear what the operadic lift of $f_L \dashv f_R^{-1}$ should be when $f$ is noninjective, because we do not know how to derive the adjunction $\text{ind}_f \dashv \text{res}_f$ in this case (cf. Theorem 8.6). Note also that the natural space-level functor $\text{ind}_f$ does not preserve $N_\infty$ operads, so some care must be taken to construct a viable $N_\infty$ induction functor. We work in a certain model category of operads where the desired homotopical properties are automatic (cf. §8).

We now come to transfer systems. We have delayed discussing these objects for the sake of clarity, but in reality, they make most of our combinatorially intensive work possible. Here is the basic idea. In §6, we reformulate the structure contained in an indexing system. Every indexing system is generated by the orbits it contains, and with a bit of thought, one can also recast the axioms for an indexing system purely in terms of orbits. The result is what we call a transfer system (cf. Definition 6.4). We give explicit equivalences between indexing systems, transfer systems, and the indexing categories that appear in Blumberg and Hill’s theory of incomplete Tambara functors [3] (cf. Theorem 6.6 and Corollary 6.9). In contrast to indexing systems and indexing categories, transfer systems are finite.

Switching over to transfer systems was quite helpful in several parts of this paper. For small enough groups $G$, they are good notational devices. We refer the reader to Figure 1 for a picture of the 19-element lattice of all $K_4$-transfer systems (p. 12). Figure 2 displays the 10-element lattice of all $C_{pq}$-transfer systems, where $p$ and $q$ are distinct primes (p. 20). Furthermore, all of our computations of indexing systems in appendix A are done on the level of transfer systems, and we only pass to indexing systems at the end. Finally, the adjunctions $f_L \dashv f_R^{-1}$ and $f_L^{-1} \dashv f_R$ in Theorem 7.8 are also constructed on the level of transfer systems. Here, we were compelled to identify explicitly the smallest indexing system containing a given set of orbits, and dually, the largest indexing system contained in a suitable class of group actions. We found the size of indexing systems and indexing categories made them too unwieldy for these purposes. We are hopeful that transfer systems will have further applications beyond this work.

The remainder of this paper is organized as follows. In §2, we continue Blumberg and Hill’s analysis of little discs operads, and we find conditions for when an indexing system corresponds to an operad $\mathcal{D}(U)$. We turn the problem around in §3, where we show how to find a universe $U$ such that $\mathcal{D}(U)$ has a prescribed admissible orbit $H/K$, and then we analyze the map that sends a universe $U$ to the admissible sets of $\mathcal{D}(U)$. In §4, we study the map that sends a universe $U$ to the admissible sets of $\mathcal{L}(U)$, and we continue Blumberg and Hill’s analysis of linear isometries operads. We describe necessary conditions for an indexing system to correspond to a linear isometries operad. We specialize to $C_n$-linear isometries operads in §5. We prove that Blumberg and Hill’s horn-filling condition characterizes the indexing systems for $C_n$-linear isometries operads when $n$ is a prime power, or when $n = pq$ for primes $3 < p < q$. In §6, we introduce transfer systems, and we prove that they are equivalent to indexing systems and indexing categories. We continue their analysis in §7, where we construct transfer system analogues to induction, restriction, and coinduction. In §8, we construct derived induction, restriction, and coinduction functors for $N_\infty$ operads, and we examine when they correspond to their algebraic counterparts. Appendix A explains how to compute the indexing system generated from a prescribed set of orbits, and then works out a few cases that are relevant to the earlier sections.
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2. INDEXING SYSTEMS FROM LITTLE DISCS OPERADS

In this section, we continue Blumberg and Hill’s analysis of the admissible sets of $G$-little discs operads $\mathcal{D}(U)$ (Proposition 2.3), and then we characterize the corresponding indexing systems when $G$ is a finite abelian group (Theorem 2.7). Our techniques also apply when $G$ is nonabelian, but we do not obtain as clean a description in this case (Corollary 2.4).

The strategy is to break a $G$-universe $U$ apart into irreducible representations $V$, and then to express the admissible sets of the operad $\mathcal{D}(U)$ in terms of those for the $\mathcal{D}(V)$. To start, note that every $G$-indexing system $I$ is generated by the orbits $H/K$ that it contains, where $H$ ranges over all subgroups of $G$. However, if $I$ is the class of admissible sets of a little discs operad $\mathcal{D}(U)$, then the following lemma shows that we can restrict attention to $H = G$. Recall that a finite $H$-set $T$ is admissible for $\mathcal{D}(U)$ if and only if there is an $H$-embedding $T \hookrightarrow V^G$ (cf. [2, Theorem 4.19]).

**Lemma 2.1.** Suppose that $U$ is a $G$-universe, and let $I$ be the class of admissible sets of $\mathcal{D}(U)$. Then $I$ is an indexing system, and it is generated by the $G$-orbits $G/K$ that it contains.

**Proof.** The class $I$ is an indexing system because $\mathcal{D}(U)$ is an $N_\infty$ operad. Let $J$ be the smallest indexing system that contains all $G$-orbits $G/K \in I$. Then $J \subset I$.

On the other hand, if $H \subset G$ is a subgroup and $H/K \in I$, then there is an $H$-equivariant embedding $\varphi : H/K \hookrightarrow U$. Let $x = \varphi(eK) \in U$. Then $K = H_x = G_x \cap H$, and hence $\varphi$ factors as a composite $H/K \hookrightarrow G/G_x \hookrightarrow U$ of embeddings, the first of which is $H$-equivariant, and the second of which is $G$-equivariant. The embedding $G/G_x \hookrightarrow U$ implies that $G/G_x \in I$, and hence $G/G_x \in J$ as well. The embedding $H/K \hookrightarrow V^G$ then implies that $H/K \in J$.

We see that $I$ and $J$ contain the same orbits $H/K$, and therefore $I = J$. □

Next, we express the admissible sets of the operad $\mathcal{D}(U)$ in terms of the irreducible representations contained in $U$.

**Definition 2.2.** For any $G$-representation $V$, let $O_V$ denote the set of all $G$-orbits $G/K$ that $G$-embed in $V$.

**Proposition 2.3.** Let $U$ be a $G$-universe, and suppose that $U \cong \bigoplus_{i \in I} V_i$ for some $G$-representations $V_i$, indexed over a possibly infinite set $I$. Then the class of admissible sets of $\mathcal{D}(U)$ is the indexing system generated by $\bigcup_{i \in I} O_{V_i}$.

**Proof.** Let $I$ be the class of admissible sets of $\mathcal{D}(U)$, and let $J$ indexing system generated by $\bigcup_{i \in I} O_{V_i}$. For each of the generators $G/K \in \bigcup_{i \in I} O_{V_i}$, there is a composite $G$-embedding $G/K \hookrightarrow V_i \hookrightarrow U$, and therefore $G/K$ is admissible for $\mathcal{D}(U)$. Therefore $\bigcup_{i \in I} O_{V_i} \subset I$, and the inclusion $J \subset I$ follows.

Conversely, suppose that $G/K$ is admissible for $\mathcal{D}(U)$, and choose an embedding $\varphi : G/K \hookrightarrow \bigoplus_{i} V_i$. Since $G$ is finite, the map $\varphi$ factors through a finite sum $V_{i_1} \oplus \cdots \oplus V_{i_n} \hookrightarrow \bigoplus_{i} V_i$. Let $(x_1, \ldots, x_n) = \varphi(eK) \in V_{i_1} \oplus \cdots \oplus V_{i_n}$. Then we
obtain embeddings $G/G_{x_k} \hookrightarrow V_k$ for $k = 1, \ldots, n$, and the original embedding $\varphi$ factors as

$$G/K \hookrightarrow G/G_{x_1} \times \cdots \times G/G_{x_n} \hookrightarrow V_1 \oplus \cdots \oplus V_n \hookrightarrow \bigoplus_{i \in I} V_i.$$  

The embeddings $G/G_{x_k} \hookrightarrow V_k$ imply that $G/G_{x_k} \in \bigcup_{i \in I} O_{V_i} \subset \mathcal{J}$ for $k = 1, \ldots, n$. It follows that $G/G_{x_1} \times \cdots \times G/G_{x_n} \in \mathcal{J}$ as well. From here, the embedding $G/K \hookrightarrow G/G_{x_1} \times \cdots \times G/G_{x_n}$ implies that $G/K \in \mathcal{J}$. We have shown that every admissible $G$-orbit of $\mathcal{D}(U)$ is contained in the indexing system $\mathcal{J}$, and the inclusion $\mathcal{I} \subset \mathcal{J}$ follows from Lemma 2.1. \hfill \Box

**Corollary 2.4.** Fix a $G$-indexing system $\mathcal{I}$. The following are equivalent:

1. The indexing system $\mathcal{I}$ is the class of admissible sets of a little discs operad $\mathcal{D}(U)$ for some $G$-universe $U$.
2. The indexing system $\mathcal{I}$ is generated by the set $O_{V_1} \cup O_{V_2} \cup \cdots \cup O_{V_n}$ for some finite-dimensional, irreducible real $G$-representations $V_1, V_2, \ldots, V_n$.

Thus, we can determine all $G$-indexing systems that correspond to little discs operads by computing the irreducible real representations of $G$, and then identifying the stabilizers of all points contained therein. Here is how Proposition 2.3 plays out when $G$ is finite abelian (cf. Theorem 2.7).

**Definition 2.5.** Suppose that $G$ is a group and that $N \subset G$ is a normal subgroup of $G$. We say that the $G$-orbit $G/N$ is **cyclic** if the quotient group $G/N$ is cyclic.

**Example 2.6.** Suppose that $G$ is the underlying additive abelian group of $\mathbb{F}^p_2$, where $\mathbb{F}_p$ denotes the field with $p$ elements for some prime $p$. Let $W \subset \mathbb{F}^n_p$ be a subgroup. Then $\mathbb{F}^n_p/W$ is a nontrivial cyclic orbit if and only if $W$ is a codimension 1 subspace of $\mathbb{F}^n_p$.

For more general abelian groups $G$, a subgroup $H \subset G$ for which $G/H$ is a cyclic orbit decomposes $G$ as a (non-direct) sum of $H$ and a cyclic subgroup of $G$.

**Theorem 2.7.** Let $G$ be a finite abelian group and let $\mathcal{I}$ be a $G$-indexing system. Then $\mathcal{I}$ is the class of admissible sets of a $G$-little discs operad if and only if $\mathcal{I}$ is generated by cyclic $G$-orbits.

**Proof.** There are two kinds of irreducible representations of $G$ over $\mathbb{R}$. We have one-dimensional representations, where each $g \in G$ acts as multiplication by $+1$ or $-1$, and we have two-dimensional representations, where each $g \in G$ acts by a rotation of angle $\theta(g) \in [0, 2\pi)$, and at least one angle $\theta(g)$ is not equal to $0$ or $\pi$. In the former case, we obtain a map $V : G \to O(1) \cong C_2$, and in the latter case we obtain a map $V : G \to C_{|G|} \to SO(2)$, where $C_{|G|}$ embeds in $SO(2)$ as the rotations by multiples of $2\pi/|G|$. Therefore $G/\ker V$ is a cyclic group for every irreducible real $G$-representation $V$.

Now consider the stabilizers of the points in an irreducible, real $G$-representation. The actions of $C_2$ on $\mathbb{R}$ and $C_{|G|}$ on $\mathbb{R}^2$ are free away from the origin. Pulling back to $G$, we see that $G_0 = G$ and $G_x = \ker V$ for every $x \neq 0$. Therefore $O_V = \{G/G, G/\ker V\}$. Thus, if $\mathcal{D}(U)$ is a $G$-little discs operad, and the irreducible representations appearing in $U$ are $V_1, \ldots, V_n$, then the class of admissible sets of $\mathcal{D}(U)$ is generated by the set of cyclic orbits $\{G/G, G/\ker V_1, \ldots, G/\ker V_n\}$.

Conversely, suppose that $\{G/H_1, \ldots, G/H_n\}$ is a set of cyclic $G$-orbits, and let $\mathcal{I}$ be the indexing system they generate. We shall construct a little discs operad...
that realizes \( I \). For \( i = 1, \ldots, n \), choose an embedding \( G/H_i \rightharpoonup O(2) \) of \( G/H_i \) as the rotations of \( \mathbb{R}^2 \) by multiples of \( 2\pi/G : H_i \), and let the representation \( \lambda_i : G \rightarrow G/H_i \rightharpoonup O(2) \) be the pullback to \( G \). Then \( O_{\lambda_i} = \{ G/G, G/H_i \} \), and by Proposition 2.3, the class of admissible sets of \( D(\mathbb{R}^\infty \oplus \lambda_1^\infty \oplus \cdots \oplus \lambda_n^\infty) \) is \( I \). □

**Corollary 2.8.** Fix \( n > 0 \). A \( C_n \)-indexing system corresponds to a little discs operad if and only if it is generated by \( C_n \)-orbits.

**Example 2.9.** Suppose that \( G = (\mathbb{F}_p^n, +) \cong C_p^\times n \) for a prime \( p \) and integer \( n > 0 \). By Example 2.6, a \( \mathbb{F}_p^n \)-indexing system corresponds to a \( \mathbb{F}_p^n \)-little discs operad if and only if it generated by a set of orbits \( \{ \mathbb{F}_p^n/W_1, \ldots, \mathbb{F}_p^n/W_m \} \), where \( W_1, \ldots, W_m \subset \mathbb{F}_p^n \) are codimension 1 subspaces.

3. **Parametrizing norms with little discs operads**

Now that we know how to compute the admissible sets of \( D(U) \) in terms of the universe \( U \), we can try to turn the problem around. Given a finite \( H \)-set \( T \), one might ask which universes \( U \) make \( T \) admissible for \( D(U) \), and which \( U \) are minimal with this property. Theorem 2.7, combined with our calculations in appendix A (cf. Proposition A.8) provide some leverage over this problem when \( G \) is finite abelian. We give a recipe for producing minimal universes \( U \) such that \( D(U) \) admits \( H/K \) (Theorem 3.3), but the solution is not generally unique. In Theorem 3.7 and the subsequent discussion, we examine some structural features of the problem, and explain how some of the non-uniqueness arises from \( \text{Aut}(G) \)-actions on \( \text{Set} \) and on the poset of all indexing systems (Example 3.9).

For clarity, we shall focus on the case that \( T = H/K \). More general results can be obtained by working one orbit at a time. First, a bit of notation.

**Definition 3.1.** Let \( \text{Uni} \) denote the set of all isomorphism classes \( [U] \) of \( G \)-universes \( U \).

We declare \( [U] \leq [U'] \) if there is a \( G \)-embedding \( U \rightharpoonup U' \) for some representatives \( U \) and \( U' \). The trivial universe and the complete universe are the minimum and maximum elements of \( \text{Uni} \), respectively. The join of \( [U] \) and \( [U'] \) is represented by \( U \vee U' \). The meet \( [U] \wedge [U'] \) is the universe that contains infinitely many copies of each irreducible \( V \) that embeds in both \( U \) and \( U' \). Thus \( \text{Uni} \) is a lattice.

Now, for any nontrivial cyclic \( G \)-orbit \( G/H \), let \( \lambda_H \) be a two-dimensional real \( G \)-representation \( G \rightarrow G/H \cong C_n \rightharpoonup SO(2) \) obtained by choosing an isomorphism \( G/H \cong C_n \), and then embedding \( C_n \) as the \( n \)th roots of unity in \( S^1 \cong SO(2) \). These representations appeared in the proof of Theorem 2.7, and we shall need them again below.

**Lemma 3.2.** Suppose that \( V \subset \lambda_H \) is an irreducible \( G \)-representation. Then for every nonzero \( x \in V \), we have \( G_x = H \).

**Proof.** The action of \( G/H \cong C_n \) on \( \mathbb{R}^2 \) as the \( n \)th roots of unity is free away from the origin. Hence every nonzero \( x \in \lambda_H \) has \( G_x = H \). This proves the lemma when \( \lambda_H \) is irreducible. If \( \lambda_H \) is reducible, then \( G/H \cong C_2 \), and hence we have a splitting \( \lambda_H \cong \sigma_H \oplus \sigma_H \) where \( \sigma_H \) is the one-dimensional representation \( G \rightarrow G/H \cong C_2 \rightharpoonup O(1) \). In this case, \( V \cong \sigma_H \) and we argue as before. □

**Theorem 3.3.** Suppose that \( G \) is a finite abelian group and that \( K \subset H \subset G \) are subgroups. Let \( H_1, \ldots, H_m \subset G \) be distinct, proper subgroups such that
(1) \( G/H_i \) is a cyclic orbit for every \( i \), and
(2) \( H \cap H_1 \cap \cdots \cap H_m = K \),
and write \( \lambda_i = \lambda_{H_i} \) and \( U = (\mathbb{R} \oplus \lambda_1 \oplus \cdots \oplus \lambda_m)^\infty \). Then \( H/K \) is admissible for \( \mathcal{D}(U) \), and \( [U] \) is minimal among the \([U] \in \text{Uni} \) such that \( \mathcal{D}(U) \) admits \( H/K \) if and only if \( H \cap H_1 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_m \supseteq K \) for every \( i = 1, \ldots, m \).

**Proof.** By Lemma 3.2 and Proposition 2.3, the class of admissible sets of \( \mathcal{D}(U) \) is the indexing system generated by \( G/H_1, \ldots, G/H_m \). This contains \( H/K \) by (2) and Proposition A.8.

Next, observe that \( \lambda_i \) does not embed in \( U_i := \mathbb{R}^\infty \oplus \bigoplus_{j \neq i} \lambda_j^\infty \). For if there were an embedding, then an irreducible subrepresentation \( V \subset \lambda_i \) would embed in \( \mathbb{R} \) or \( \lambda_j \) for some \( j \neq i \), but Lemma 3.2 implies this is impossible because the subgroups \( G, H_1, \ldots, H_m \) are all distinct. Therefore \( U_i \) is a proper subuniverse of \( U \) for every \( i = 1, \ldots, m \). In fact, each \( U_i \) is a maximal proper subuniverse, because each \( \lambda_i \) is either irreducible, or splits as \( \lambda_i \cong \sigma_1 \oplus \sigma_2 \).

We now consider the minimality of \( U \). If \( H \cap H_1 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_m = K \) for some \( i \), then \( H/K \) is admissible for \( \mathcal{D}(U_i) \) by Proposition A.8. Thus \( U \) is not minimal among the universes \( U \) making \( H/K \) admissible for \( \mathcal{D}(U) \).

Now suppose that each \( H \cap H_1 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_m \) strictly contains \( K \). By Proposition A.8, the orbit \( H/K \) is not a member of any of the indexing systems \( \langle G/H_1, \ldots, G/H_{i-1}, G/H_{i+1}, \ldots, G/H_m \rangle \), because all nontrivial \( H \)-orbits in these indexing systems are of the form \( H/L \) for a subgroup \( L \supseteq K \). Therefore \( H/K \) is not admissible for any of the little discs operads \( \mathcal{D}(U_i) \). It follows that \( U \) is minimal among the universes \( U \) making \( H/K \) admissible for \( \mathcal{D}(U) \), because any proper subuniverse \( U' \hookrightarrow U \) of \( U \) \( G \)-embeds into one of the universes \( U_i \).

**Example 3.4.** Let \( G = (\mathbb{F}_p^n, +) \) and suppose that \( V \subset \mathbb{F}_p^n \) is a proper subspace. Choose one-dimensional subspaces \( \ell_1, \ldots, \ell_m \) such that \( \mathbb{F}_p^n = V \oplus \ell_1 \oplus \cdots \oplus \ell_m \), and let \( \lambda_i \) be the pullback of the representation \( \lambda : C_p \to SO(2) \) along the quotient map \( \pi_i : \mathbb{F}_p^n \to \mathbb{F}_p^n/W_i \cong C_p \). Then \( U = (\mathbb{R} \oplus \lambda_1 \oplus \cdots \oplus \lambda_m)^\infty \) is a minimal universe such that the orbit \( \mathbb{F}_p^n/V \) is admissible for \( \mathcal{D}(U) \).

Thus, if the free \( G \)-orbit \( \langle G/\{e\} \rangle \) is admissible for \( \mathcal{D}(U) \) and \( G \) is finite abelian, then we should generally expect \( U \) to be large. A similar phenomenon occurs for linear isometries operads. Pushing this observation a bit further yields the next proposition. Compare to the proof of [2, Theorem 4.22].

**Theorem 3.5.** Suppose that \( G \) is a non-cyclic finite abelian group. Then the \( G \)-indexing system generated by \( \langle G/\{e\} \rangle \) alone is not realized by a \( G \)-little discs operad or a \( G \)-linear isometries operad.

**Proof.** The only nontrivial \( G \)-orbit in \( \langle G/\{e\} \rangle \) is \( G/\{e\} \), by Proposition A.6.

If the orbit \( G/\{e\} \) is admissible for \( \mathcal{D}(U) \), then some other cyclic orbit \( G/H \) must also be admissible for \( \mathcal{D}(U) \), because the admissible sets of \( \mathcal{D}(U) \) form a nontrivial indexing system generated by cyclic orbits (Theorem 2.7). Therefore the class of admissible sets of \( \mathcal{D}(U) \) cannot be exactly \( \langle G/\{e\} \rangle \).

If the orbit \( G/\{e\} \) is admissible for \( \mathcal{L}(U) \), then every \( G \)-orbit \( G/H \) is admissible for \( \mathcal{L}(U) \) by [2, p. 17]. Once again, the class of admissible sets of \( \mathcal{L}(U) \) cannot be precisely \( \langle G/\{e\} \rangle \). \( \Box \)

---

1. Reproduced in this paper as Proposition 4.6.
Suppose once more that $H/K$ is an $H$-orbit, and that we wish to produce a $G$-universe $U$ such that $D(U)$ admits $H/K$. For simplicity, suppose that $G$ is finite abelian. There are many choices going into the construction in Theorem 3.3, and therefore $U$ is not generally unique. Actions by $\text{Aut}(G)$ track some of the nonuniqueness that arises from different presentations of $G$ (cf. Corollary 3.8).

Let $G$ be any finite, not necessarily abelian, group. Consider the following right actions of $\text{Aut}(G)$:

(i) On the group $G$, where $g \cdot \sigma := \sigma^{-1}(g)$ for all group elements $g \in G$.

(ii) On the lattice $\text{Sub}(G)$ of all subgroups of $G$, where $H \cdot \sigma := \sigma^{-1}H$ for all subgroups $H \subseteq G$.

(iii) On the indexing system $\text{Set}$, where for any finite $H$-set $T$, we define $T \cdot \sigma$ to be the $\sigma^{-1}H$-set $\sigma^{-1}H \sigma \to H \to \text{Perm}(T)$.

(iv) On the lattice $\text{Ind}$ of all $G$-indexing systems, where $\mathcal{I} \cdot \sigma = \{T \cdot \sigma | T \in \mathcal{I}\}$.

(v) On the lattice $\text{Uni}$ of all isomorphism classes of $G$-universes $[U]$, where $[U] \cdot \sigma$ is represented by the pulled back $G$-universe $G \overset{\sigma}{\to} G \to \text{Aut}(U)$.

Note that the $\text{Aut}(G)$-action on $\text{Ind}$ is a special case of the functoriality of $\text{Ind} = \text{Ind}(G)$ in the group $G$. We refer the reader to section 7 for further discussion.

**Lemma 3.6.** The formulas in (i) – (v) define structure-preserving right actions.

**Proof.** Claims (i), (ii), and (iii) are immediate.

Claim (iv) follows from the functoriality of $\text{Ind}(G)$ in $G$ (cf. Theorem 7.8), but it is easy enough to check directly. The main point is that $\mathcal{I} \cdot \sigma$ is an indexing system whenever $\mathcal{I}$ is. We use the formulation in Definition 6.1. Conditions (1), (2), (5), and (6) hold because $(-) \cdot \sigma$ preserves trivial actions, isomorphisms, inclusions, and coproducts. Conditions (3), (4), and (7) follow from the fact that $(-) \cdot \sigma$ commutes with conjugation, restriction, and induction.

For (v), observe that pulling back along any $\sigma \in \text{Aut}(G)$ yields an automorphism $(-) \cdot \sigma$ of the category $\text{Rep}(G)$ of real $G$-representations. Therefore $(-) \cdot \sigma$ preserves direct sums and irreducible representations. Applying $(-) \cdot \sigma$ also preserves trivial representations. Therefore $U \cdot \sigma$ is universe whenever $U$ is, and $[U] \cdot \sigma := [U \cdot \sigma]$ is well-defined. We obtain a right action of $G$ on $\text{Uni}$ through order-isomorphisms because $(-) \cdot \sigma$ preserves embeddings.

The next theorem describes how little discs operads relate $\text{Uni}$ and $\text{Ind}$.

**Theorem 3.7.** Let $\mathcal{A}_D : \text{Uni} \to \text{Ind}$ be the function that sends an isoclass $[U]$ to the class of admissible sets of $D(U)$. Then $\mathcal{A}_D$ is $\text{Aut}(G)$-equivariant, and it preserves the order, the maximum element, the minimum element, and joins. It is not always order-reflecting, meet-preserving, or injective.

**Proof.** A finite $H$-set $T$ is admissible for $D(U)$ if and only if there is an $H$-embedding of $T$ into $\text{res}_H^GU$. By composing with an embedding from the inequality $[U] \subseteq [U']$, we deduce that $\mathcal{A}_D$ is order-preserving. Next, by applying $(-) \cdot \sigma$ and $(-) \cdot \sigma^{-1}$, we see that there is a $H$-embedding of $T$ into $\text{res}_H^GU$ if and only if there is a $\sigma^{-1}H$-embedding of $T \cdot \sigma$ into $U \cdot \sigma$. It follows that $\mathcal{A}_D([U]) \cdot \sigma = \mathcal{A}_D([U \cdot \sigma])$ for every $[U] \in \text{Uni}$ and $\sigma \in \text{Aut}(G)$.

We have $\mathcal{A}_D([\mathbb{R}^\infty]) = \text{triv}$ because the only orbits that embed in $\mathbb{R}^\infty$ are trivial, and $\mathcal{A}_D([\mathbb{R}G[\infty]]) = \text{Set}$ because every orbit embeds in $\mathbb{R}[G]$. Now suppose that $[U_1], [U_2] \in \text{Uni}$. Then we have $\mathcal{A}_D([U_1] \cup [U_2]) = \mathcal{A}_D([U_1] \cup [U_2]) = O_{U_1} \cup O_{U_2} = O_{U_1} \cup (O_{U_2} = \mathcal{A}_D([U_1]) \cup \mathcal{A}_D([U_2])$ by Proposition 2.3, so $\mathcal{A}_D$ preserves joins.
For the negative results, we give an example. Let $G = C_p = \langle g \mid g^p = 1 \rangle$ for a prime $p > 3$, and consider the $C_p$-representations

\[
\begin{align*}
\lambda(1) : C_p &\to O(2), \quad g \mapsto \text{rotation by } 2\pi/p \text{ counterclockwise} \\
\lambda(2) : C_p &\to O(2), \quad g \mapsto \text{rotation by } 4\pi/p \text{ counterclockwise}.
\end{align*}
\]

Take the universes $U_1 = (\mathbb{R} \oplus \lambda(1))\infty$ and $U_2 = (\mathbb{R} \oplus \lambda(2))\infty$. Then $[U_1]$ and $[U_2]$ are distinct and incomparable in $\text{Uni}$ because $\lambda(1)$ and $\lambda(2)$ are nonisomorphic. Moreover, $\lambda(1)$ and $\lambda(2)$ specify $C_p$ actions that are free away from the origin, so that $C_p/\{1\}$ embeds in both $U_1$ and $U_2$. It follows that $\mathcal{A}_D([U_1]) = \mathcal{A}_D([U_2]) = \text{Set}$, which means that $\mathcal{A}_D$ is neither injective nor order-reflecting. To make matters worse, we have $[U_1] \wedge [U_2] = [\mathbb{R}^\infty]$, and therefore $\mathcal{A}_D([U_1] \wedge [U_2]) = \text{triv}$, while $\mathcal{A}_D([U_1]) \wedge \mathcal{A}_D([U_2]) = \text{Set}$. Therefore $\mathcal{A}_D$ does not preserve all meets, either. \hfill \Box

The $\text{Aut}(G)$-equivariance of $\mathcal{A}_D : \text{Uni} \to \text{Ind}$ has the following consequence.

**Corollary 3.8.** If $T$ is an admissible $H$-set of $\mathcal{D}(U)$, and $\sigma \in \text{Aut}(G)$ is such that $T \cdot \sigma \equiv T$, then $T$ is also an admissible $H$-set of $\mathcal{D}(U \cdot \sigma)$. If $U$ is a minimal universe such that $T$ is admissible for $\mathcal{D}(U)$, then so is $U \cdot \sigma$.

**Proof.** Suppose that $T$ is a finite $H$-set and that $\sigma \in \text{Aut}(G)$ is such that $T\sigma \equiv T$. If $T \in \mathcal{A}_D([U])$, then $T \equiv T\sigma \in \mathcal{A}_D([U])\sigma = \mathcal{A}_D([U\sigma])$. The preservation of minimality follows because $\text{Aut}(G)$ acts on $\text{Uni}$ through monotone maps. \hfill \Box

We shall soon specialize to the Klein four-group $K_4$ to illustrate how this all looks in practice, but first a bit of notation. Recall that an indexing system is determined by the orbits that it contains. Thus, we can specify a $G$-indexing system $\mathcal{T}$ by giving the set of all pairs $(K,H)$ for which $H/K \in \mathcal{T}$. These data correspond to a binary relation on the set $\text{Sub}(G)$ of all subgroups of $G$, and since indexing systems contain all trivial actions and are closed under self-induction, this relation is actually a partial order that refines inclusion. Thus, there is a Hasse diagram that encodes the data in any given indexing system. Strictly speaking, we are describing the *transfer system* associated to the indexing system $\mathcal{T}$, and we refer the reader to section 6 for details.

Now let $K_4 = \{1, a, b, c\}$, where 1 is the identity, and the product of any two of $a$, $b$, or $c$ is the third. The subgroup lattice of $K_4$ is

\[
\begin{array}{c}
\text{triv} \\
\text{Set} = \Phi \\
\langle\{1, a\}/1\rangle \\
\langle K_4/\{1, a\}\rangle \quad \vdots
\end{array}
\]

\[
\begin{array}{c}
\{1, a\} \\
\{1, b\} \\
\{1, c\} \\
1
\end{array}
\]

and we shall write

\[
\begin{align*}
\langle\{1, a\}/1\rangle &= \mathcal{S} \quad \text{\textbullet} \\
\langle K_4/\{1, a\}\rangle &= \mathcal{X}
\end{align*}
\]

Using Theorem A.2, it is relatively straightforward to enumerate all $K_4$-indexing systems (e.g. by considering the number of $K_4$-orbits contained therein). The lattice of all $K_4$-indexing systems is given in Figure 1 (p. 12).

The next example identifies the $K_4$-indexing systems that correspond to little discs operads, and it also highlights a few cases in which a given orbit is admissible. Example 4.11 does something similar for linear isometries operads.
Example 3.9. Keep notation as above. There are four irreducible, real representations of $K_4$, namely: the trivial representation $\mathbb{R} : K_4 \to O(1)$, the sign representation $\sigma_a : K_4 \to K_4/\{1,a\} \xrightarrow{\sigma} O(1)$, and similarly for $\sigma_b$ and $\sigma_c$. For any set $I \subset \{a,b,c\}$, let $U(I) = \mathbb{R}^\infty \oplus \bigoplus_{i \in I} \sigma_i^\infty$. Then the lattice $\text{Uni}$ of all $K_4$-universes, and the indexing systems for the corresponding little discs operads are:

Every permutation of the elements of $\{a,b,c\}$ corresponds to an automorphism of $K_4$. Therefore $\text{Aut}(K_4) \cong S_3$, and the $S_3$ action on $\text{Uni}$ permutes the universes
in each horizontal layer. Similarly for the action on \( \text{Ind} \). We see that the map 
\[ \kappa_D : \text{Uni} \rightarrow \text{Ind} \] 
is a \( S_3 \)-equivariant embedding.

Consider the orbit \( K_4/1 \). It is stabilized, up to isomorphism, by all of \( \text{Aut}(K_4) \).
On the other hand, it is admissible for \( D(U) \) whenever \( U \) contains at least two of \( \sigma_a, \sigma_b, \) or \( \sigma_c \). These universes correspond to the upper half of the cube, which is closed under the entire action of \( \text{Aut}(K_4) \). As for the orbit \( K_4/\{1,a\} \), the only elements of \( \text{Aut}(K_4) \) that stabilize it are the identity and the transposition \( \tau_{bc} \) that switches \( b \) and \( c \). The orbit \( K_4/\{1,a\} \) is admissible for \( D(U) \) whenever \( \sigma_a \subset U \), and these universes correspond to the face containing \( U(a) \) and \( U(a,b,c) \). This face is closed under the transposition \( \tau_{bc} \). Finally, consider the orbit \( \{1,a\}/1 \). Its \( \text{Aut}(K_4) \)-stabilizer is also generated by \( \tau_{bc} \), and it is admissible for \( D(U) \) whenever one of \( \sigma_b \) or \( \sigma_c \) is in \( U \). These universes correspond to the two upper faces opposite to \( U(a) \), and these faces are also closed \( \tau_{bc} \).

4. Indexing systems from linear isometries operads

This section highlights a few general features of the indexing systems that correspond to linear isometries operads. As we shall explain, the structure theory in this case is more subtle than the corresponding theory for little discs operads. The basic problem is encapsulated in [2, Theorem 4.18], which we restate below as Theorem 4.1. While one can study the operad \( D(U) \) one irreducible subrepresentation of \( U \) at a time, we have no such luck for the linear isometries operad \( \mathcal{L}(U) \).

One can salvage the situation to some extent. Proposition 4.6 reviews a necessary condition from [2] for an indexing system to correspond to a linear isometries operad, and Corollary 4.2 and Proposition 4.10 give two further necessary conditions. Unfortunately, none of these conditions are generally sufficient. What’s more, the map \( \text{Uni} \rightarrow \text{Ind} \) that sends \( \{U\} \) to the class of admissible sets of \( \mathcal{L}(U) \) is no longer order-preserving, but it is still \( \text{Aut}(G) \)-equivariant (cf. Theorem 4.4).

We begin with the crux of the matter. In [2], it is proven that a finite \( H \)-set \( T \) is admissible for \( \mathcal{L}(U) \) if and only if there is an \( H \)-embedding \( (\text{res}_H^G T) \hookrightarrow \text{res}_H^G U \). We quickly rephrase their result in terms of irreducible \( G \)-representations.

**Theorem 4.1** ([2, Theorem 4.18]). Suppose that \( U \) is a \( G \)-universe, and let \( \mathcal{L}(U) \) be the linear isometries operad over \( U \). Fix subgroups \( K \subset H \subset G \). Then \( H/K \) is admissible for \( \mathcal{L}(U) \) if and only if for every irreducible \( H \)-representation \( V \subset \text{res}_H^G U \), every irreducible \( H \)-representation \( W \subset \text{ind}_K^H \text{res}_K^H V \) also appears in \( \text{res}_H^G U \).

**Proof.** The orbit \( H/K \) is admissible for \( \mathcal{L}(U) \) if and only if there is a \( H \)-embedding
\[ \text{ind}_K^H \text{res}_K^H \text{res}_H^G U \cong (\text{res}_H^G U)^{\oplus H/K} \hookrightarrow \text{res}_H^G U, \]
and both restriction and induction commute with direct sums.

Suppose we are given the embedding above. If \( V \subset \text{res}_H^G U \) and \( W \subset \text{ind}_K^H \text{res}_K^H V \) are irreducible, then we obtain a \( H \)-embedding
\[ W \subset \text{ind}_K^H \text{res}_K^H V \hookrightarrow \text{ind}_K^H \text{res}_K^H \text{res}_H^G U \hookrightarrow \text{res}_H^G U. \]

Conversely, if we have such embeddings for all \( V \) and \( W \), then we can construct a \( H \)-embedding \( (\text{res}_H^G U)^{\oplus H/K} \hookrightarrow \text{res}_H^G U \) one summand at a time. \( \square \)

**Corollary 4.2.** Keep notation as in Theorem 4.1. If the \( H \)-orbit \( H/K \) is admissible for \( \mathcal{L}(U) \), then every irreducible \( H \)-representation \( W \) satisfying \( \dim W^K > 0 \) must appear in \( \text{res}_H^G U \).
Proof. Suppose that $H/K$ is admissible for $L(U)$. The $H$-universe $\text{res}^H_U$ contains the trivial $H$-representation over $\mathbb{R}$, and therefore it contains all irreducible sub-$H$-representations of $\text{ind}^H_K \text{res}^H_K \mathbb{R} \cong \mathbb{R}[H/K]$ by Theorem 4.1. If $W$ is an irreducible $H$-representation with $\dim W^K > 0$, then there is a surjection $\mathbb{R}[H/K] \twoheadrightarrow W$, and hence $W$ appears in $\mathbb{R}[H/K]$. Thus $W$ also appears in $\text{res}^H_U$. 

Specializing to the case that $H/K = G/\{e\}$, we recover the well-known fact that a linear isometries operad $L(U)$ admits $G/\{e\}$ if and only if $U$ is a complete $G$-universe. However, the following counterexample illustrates that the condition in Corollary 4.2 is not always sufficient.

**Example 4.3.** Let $G = K_4$ be the Klein four-group, and keep notation as in Example 3.9. Consider the universe $U(a, b)$. The representations $\mathbb{R}$ and $\sigma_a$ are the only irreducible $K_4$-representations with nontrivial $\{1, a\}$-fixed points, and both appear in $U(a, b)$. However, the orbit $K_4/\{1, a\}$ is not admissible for $L(U)$ because $\text{ind}^{K_4}_{\{1, a\}} \text{res}^{K_4}_{\{1, a\}} \sigma_b \cong \text{ind}^{K_4}_{\{1, a\}} \sigma_c \cong \sigma_b \oplus \sigma_c$, and $\sigma_c$ does not appear in $U(a, b)$.

We can feel the influence of Theorem 4.1 when we consider the analogue to Theorem 3.7 for linear isometries operads.

**Theorem 4.4.** The map $\Lambda_L : \text{Uni} \rightarrow \text{Ind}$ that sends an isoclass $[U]$ to the class of admissible sets of $L(U)$ is $\text{Aut}(G)$-equivariant, and it preserves maximum and minimum elements. It is not always order-preserving, order-reflecting, join-preserving, meet-preserving, or injective.

Proof. We begin with the $\text{Aut}(G)$-equivariance. Right multiplication $(-)\sigma$ commutes with restriction and induction for actions of subgroups of $G$ on sets, and for linear representations of $G$ over $\mathbb{R}$. Then, since $(-)\sigma$ also preserves embeddings, we conclude that $\text{ind}^H_K \text{res}^H_U$ embeds into $\text{res}^H_U$ if and only if $\text{ind}^G_{\sigma^{-1} H} \text{res}^G_{\sigma^{-1} H}(U \sigma)$ embeds into $\text{res}^G_{\sigma^{-1} H}(U \sigma)$. Therefore $H/K \in \Lambda_L([U])$ if and only if $(H/K)\sigma \in \Lambda_L([U] \sigma)$, and the analogous statement for finite $H$-sets $T$ follows by passage to coproducts. Thus $\Lambda_L([U]) \sigma = \Lambda_L([U] \sigma)$.

The map $\Lambda_L$ preserves minimum elements because no nontrivial universe embeds in a trivial one, and every restriction of $\mathbb{R}^\infty$ is trivial. Dually for complete universes.

The counterexample from the proof of Theorem 3.7 also works here to show that $\Lambda_L$ need not be injective or order-reflecting. Keep the same notation. For a prime $p > 3$, we know that the $C_p$-universes $U_1 = (\mathbb{R} \oplus \lambda(1))^\infty$ and $U_2 = (\mathbb{R} \oplus \lambda(2))^\infty$ represent distinct, incomparable classes in $\text{Uni}$. However, Corollary 4.2 implies that $C_p/\{1\}$ is not admissible for either $L(U_1)$ or $L(U_2)$, because $U_1$ and $U_2$ are incomplete. Therefore $\Lambda_L([U_1]) = \Lambda_L([U_2]) = \text{triv}$.

In Examples 4.11 and 5.8, we show that $\Lambda_L$ is not generally order-preserving, even when $G$ is a finite cyclic group. Here is the basic idea: suppose that $H/K$ is admissible for $L(U)$. If we enlarge $U$ to $U'$, then $H/K$ will not be admissible for $L(U')$ unless $\text{ind}^H_K \text{res}^H_V \cong \text{res}^H_{U'}$ for every new irreducible representation $V \in L(U') - U$. Thus, to disprove order-preservation along $\Lambda_L$, take $U'$ a bit larger than $U$, but not too much.

That being said, if $\Lambda_L$ is not order-preserving, then it also does not preserve all meets and joins, because $a \leq b$ if and only if $a \cap b = a$ if and only if $a \lor b = b$. 

We deduce the following corollary as before.
Corollary 4.5. If $T$ is an admissible $H$-set of $\mathcal{L}(U)$, and $\sigma \in \text{Aut}(G)$ is such that $T \cdot \sigma \cong T$, then $T$ is also an admissible $H$-set of $\mathcal{L}(U \cdot \sigma)$. If $U$ is a minimal universe such that $T$ is admissible for $\mathcal{L}(U)$, then so is $U \cdot \sigma$.

We return to the problem of finding necessary conditions for an indexing system to correspond to a linear isometries operad. The following was first observed by Blumberg and Hill.

**Proposition 4.6** ([2, p. 17]). Let $\mathcal{I}$ be the class of admissible sets of $\mathcal{L}(U)$. Then $\mathcal{I}$ must satisfy the following additional property:

\[
(\Lambda) \quad \text{For any subgroups } K \subset L \subset H \subset G, \text{ if } H/K \in \mathcal{I} \text{ then } H/L \in \mathcal{I}.
\]

We give an alternate proof, just for variety.

**Proof.** For any subgroups $K \subset L \subset G$, the unit $\eta$ of the adjunction

\[
\text{res}^L_K : \text{Rep}(L) \xrightarrow{\eta} \text{Rep}(K) : \text{coind}^L_K \cong \text{ind}^L_K
\]

is a monomorphism, and the right adjoint $\text{ind}^L_K$ preserves monomorphisms. Thus

\[
\text{ind}^L_K \eta_{\text{res}^L_K L} : \text{ind}^H_K \text{res}^L_K L \hookrightarrow \text{ind}^H_K \text{res}^L_K \text{res}^L_K U \cong \text{ind}^H_K \text{res}^L_K U
\]

is a $H$-equivariant embedding. We conclude that $\text{ind}^H_K \text{res}^L_K U -$embeds in $\text{res}^H_K U$ whenever $\text{ind}^H_K \text{res}^L_K U$ does. \qed

**Remark 4.7.** Condition ($\Lambda$) can be thought of as a horn-filling property. Suppose that we have a chain of subgroups

\[
H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n,
\]

regarded as (the spine of) an $n$-simplex in the subgroup lattice of $G$. If the orbit $H_n/H_0$ is admissible for the indexing system $\mathcal{I}$, and $\mathcal{I}$ satisfies condition ($\Lambda$), then every suborbit $H_i/H_j$ for $i \geq j$ must also be admissible for $\mathcal{I}$.

**Definition 4.8.** A $\Lambda$-indexing system is an indexing system that satisfies ($\Lambda$).

Every indexing system $\mathcal{I}$ is generated by the orbits it contains. If $\mathcal{I}$ is a $\Lambda$-indexing system, then we can be more specific.

**Definition 4.9.** For any subgroups $K \subset H \subset G$, we shall say that the orbit $H/K$ is simple if $K$ is a maximal proper subgroup of $H$.

**Proposition 4.10.** Let $G$ be a finite group and let $\mathcal{I}$ be a $G$-$\Lambda$-indexing system. Then $\mathcal{I}$ is generated by its admissible simple orbits.

**Proof.** Suppose that $\mathcal{I}$ satisfies ($\Lambda$), and let $\mathcal{J}$ be the indexing system generated by the simple orbits contained in $\mathcal{I}$. We must prove that $\mathcal{I} = \mathcal{J}$. The inclusion $\mathcal{J} \subset \mathcal{I}$ is immediate. For the reverse inclusion, let $H/K \in \mathcal{I}$ be nontrivial. Since $G$ is finite, we can choose a (nonunique) chain of subgroups

\[
K = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = H
\]

such that the orbit $K_i/K_{i-1}$ is simple for $i = 1, \ldots, n$. Since $\mathcal{I}$ is an indexing system and $K_i/K \subset \text{res}^H_{K_i} H/K$, each orbit $K_i/K$ is in $\mathcal{I}$. Then since $\mathcal{I}$ satisfies ($\Lambda$), each simple orbit $K_i/K_{i-1}$ is also in $\mathcal{I}$. It follows that $K_i/K_{i-1} \in \mathcal{J}$ for $i = 1, \ldots, n$, and therefore

\[
\text{ind}^K_{K_{i-1}} \cdots \text{ind}^K_{K_1} K_1/K_0 \cong K_n/K_0 = H/K
\]

must also be in $\mathcal{J}$. Since the indexing system $\mathcal{J}$ contains all nontrivial orbits of $\mathcal{I}$, the inclusion $\mathcal{I} \subset \mathcal{J}$ follows. \qed
At one point, it was hoped that condition (Λ) would completely characterize the indexing systems that correspond to linear isometries operads. Unfortunately, this is false in general. We illustrate by example.

**Example 4.11.** Let $G = K_4$ be the Klein four-group, and keep notation as in Example 3.9. The lattice of all $K_4$-universes, and the indexing systems for the corresponding linear isometries operads are:

$\begin{array}{c}
U(\emptyset) \\
U(a) \\
U(b) \\
U(c) \\
U(a, b) \\
U(a, c) \\
U(b, c) \\
U(a, b, c)
\end{array}$

In particular, we miss the following six $K_4$-Λ-indexing systems

$\begin{array}{c}
\searrow \\
\nearrow \\
\searrow \\
\nearrow \\
\searrow
\end{array}$

and none of the order relations between the second and third horizontal layers of the cube are preserved. Combined with Example 3.9, we see that little discs and linear isometries operads account for only nine of the nineteen $K_4$-indexing systems.

We shall encounter similar issues when $G = C_6$, and when $G = C_{2q}$ or $G = C_{3q}$ for a prime $q > 3$ (cf. Theorem 5.11). Both the representation theory and the subgroup lattice of $G$ can prevent (Λ) from being a sufficient condition for realizing an indexing system by a linear isometries operad. That said, it turns out condition (Λ) actually is sufficient in a some special cases. We shall consider these matters further in the next section.

5. Linear isometries operads for finite cyclic groups

In this section, we specialize [2, Theorem 4.18] to the case $G = C_n$ (Proposition 5.4), and then we specialize the number $n$ further. We show that Blumberg and Hill’s condition (Λ) characterizes the $C_n$-indexing systems that correspond to linear isometries operads if $n$ is a prime power, or if $n = pq$ for primes $3 < p < q$ (Theorems 5.7 and 5.11). If $n = 2q$ or $3q$ for a prime $q > 3$, the set irreducible $C_n$-representations is overcrowded, and not every Λ-indexing system is realized by a $C_n$-linear isometries operad.

Fix a natural number $n$. We shall use the following notation for real $C_n$-representations.

**Notation 5.1.** For any finite cyclic group $C_n = \langle g \mid g^n = 1 \rangle$, define

$\lambda_n(m) : C_n \to S^1 \cong SO(2)$ by $g \mapsto e^{2\pi im/n}$. 

The character of $\lambda_n(m)$ is

$g^j \mapsto 2\cos(2\pi m j / n) = e^{2\pi imj/n} + e^{-2\pi imj/n}$. 

If \( d \mid n \), then we shall write \( \text{res}_d^n \) and \( \text{ind}_d^n \) for restriction and induction along the map \( C_d \hookrightarrow C_n \), which sends the chosen generator of \( C_d \) to the \( \frac{d}{2} \)th power of the chosen generator of \( C_n \).

The representations \( \lambda_n(m) \) have the following properties.

**Lemma 5.2.**

1. If \( m \equiv m' \mod n \), then \( \lambda_n(m) = \lambda_n(m') \).
2. \( \lambda_n(m) \cong \lambda_n(-m) \).
3. If \( d \mid n \), then \( \text{res}_d^n \lambda_n(m) = \lambda_d(m) \).
4. If \( d \mid n \), then \( \text{ind}_d^n \lambda_d(m) \equiv \bigoplus_{a=0}^{n/d-1} \lambda_n(m + da) \).

**Proof.** The first three statements are clear. For the fourth statement, we compute characters. The character of \( \text{ind}_d^n \lambda_d(m) \) is

\[
g^j \mapsto \begin{cases} \frac{n}{d}(e^{2\pi imj/n} + e^{-2\pi imj/n}) & \text{if } \frac{n}{d} \mid j \\ 0 & \text{otherwise} \end{cases},
\]

and the character of \( \bigoplus_{a=0}^{n/d-1} \lambda_n(m + da) \) is

\[
g^j \mapsto \left[ e^{2\pi imj/n} \cdot \sum_{a=0}^{n/d-1} (e^{2\pi idj/n})^a \right] + \left[ e^{-2\pi imj/n} \cdot \sum_{a=0}^{n/d-1} (e^{-2\pi idj/n})^a \right].
\]

These two functions are equal. \( \square \)

Next, observe that the representation \( \lambda_n(m) \) is irreducible, unless

1. \( m \equiv 0 \mod n \), in which case \( \lambda_n(m) \cong \mathbb{R} \oplus \mathbb{R} \), or
2. \( m \equiv n/2 \mod n \), in which case \( \lambda_n(m) \cong \sigma \oplus \sigma \).

Then, since \( \lambda_n(m) = \lambda_n(m') \) whenever \( m \equiv m' \mod n \), and \( \lambda_n(m) \cong \lambda_n(-m) \) for all \( m \), it follows that the irreducible, real \( C_n \) representations are

| \( n \) odd | \( n \) even |
|---|---|
| \( \mathbb{R} \) | \( \mathbb{R} \) |
| \( \lambda_n(1) \cong \lambda_n(n - 1) \) | \( \lambda_n(1) \cong \lambda_n(n - 1) \) |
| \( \lambda_n(2) \cong \lambda_n(n - 2) \) | \( \lambda_n(2) \cong \lambda_n(n - 2) \) |
| \( \vdots \) | \( \vdots \) |
| \( \lambda_n\left(\frac{n-1}{2}\right) \cong \lambda_n\left(\frac{n+1}{2}\right) \) | \( \lambda_n\left(\frac{n}{2} - 1\right) \cong \lambda_n\left(\frac{n}{2} + 1\right) \) |

Since every \( C_n \) universe \( U \) must contain \( \mathbb{R} \), along with infinitely many copies of its irreducible subrepresentations, we arrive at the following conclusion.

**Lemma 5.3.** Every \( C_n \)-universe \( U \) is of the form

\[
U \cong \bigoplus_{i \in I} \lambda_n(i)^\infty,
\]

where \( I \subset \mathbb{Z}/n \cong \{0, 1, \ldots, n - 1\} \) is a set that contains 0, and which is closed under additive inversion.

**Proof.** The representation \( \lambda_n(i) \) is well-defined for every \( [i] \in \mathbb{Z}/n \), by Lemma 5.2. Given an arbitrary \( C_n \)-universe \( U \), rewrite the \( \mathbb{R}^\infty \)-summand of \( U \) as \( \lambda_n(0)^\infty \), and rewrite each \( \lambda_n(i)^\infty \)-summand as \( \lambda_n(i)^\infty \oplus \lambda_n(n - i)^\infty \). If \( n \) is even, then rewrite any \( \sigma^\infty \)-summand as \( \lambda_n\left(\frac{n}{2}\right)^\infty \). \( \square \)
In what follows, we shall describe everything in terms of the representations \( \lambda_n(i) \), and sets \( I \subset \mathbb{Z}/n \) that contain 0 and are closed under additive inversion. These conventions allow us to express the admissible sets of \( \mathcal{L}(\bigoplus_{i \in I} \lambda_n(i)^\infty) \) in terms of the translation invariance of \( I \subset \mathbb{Z}/n \) and its reductions.

**Proposition 5.4.** Let \( U = \bigoplus_{i \in I} \lambda_n(i)^\infty \), where \( I \subset \mathbb{Z}/n \) contains 0 and is closed under additive inversion. Then the set of admissible orbits of \( \mathcal{L}(U) \) is

\[
\mathcal{O}_I = \left\{ C_e/C_d \mid \text{d} \mid \text{e} \mid \text{n} \text{ and } (\text{I mod e}) + \text{d} = (\text{I mod e}) \right\},
\]

and the class of admissible sets of \( \mathcal{L}(U) \) is the indexing system generated by \( \mathcal{O}_I \).

**Proof.** By [2, Theorem 4.18] and Lemma 5.2, the orbit \( C_e/C_d \) is admissible for the operad \( \mathcal{L}(U) \) if and only if there is a \( C_e \)-equivariant embedding

\[
\bigoplus_{i \in I} \bigoplus_{a=0}^{e/d-1} \lambda_e(i + da) \cong \text{ind}_d^e \text{res}_d^n U \hookrightarrow \text{res}_e^n U \cong \bigoplus_{i \in I} \lambda_e(i)^\infty.
\]

We analyze this condition in stages. First, note that we have a \( C_e \)-equivariant embedding as above if and only if we have \( C_e \)-embedding \( \lambda_e(i + da) \hookrightarrow \bigoplus_{i \in I} \lambda_e(i)^\infty \) for every \( i \in I \) and \( a = 0, \ldots, e/d - 1 \). In turn, we have such embeddings if and only if for every such \( i \) and \( e \), there is some \( j \in I \) such that \( \lambda_e(i + da) \cong \lambda_e(j) \).

Now \( \lambda_e(a) \cong \lambda_e(b) \) if and only if \( a \equiv \pm b \text{ mod } e \). Since \( I \) is closed under additive inversion, we deduce that \( C_e/C_d \) is admissible for \( \mathcal{L}(U) \) if and only if for every \( i \in I \) and \( a = 0, \ldots, e/d - 1 \), there is some \( j \in I \) such that \( i + da \equiv j \text{ mod } e \). By induction, this is equivalent to saying that for every \( i \in I \), there is \( j \in I \) such that \( i + d \equiv j \text{ mod } e \), i.e. \( (I \text{ mod } e) + d \subset (I \text{ mod } e) \). The equality \((I \text{ mod } e) + d = (I \text{ mod } e)\) follows because \( I \) is finite.

**Corollary 5.5.** Let \( G = C_n \) be a finite cyclic group and let \( \mathcal{J} \) be a \( C_n \)-indexing system. Then \( \mathcal{J} \) is realized by a \( C_n \)-linear isometries operad if and only if \( \mathcal{J} \) is generated by a set of orbits \( \mathcal{O}_I \) as above.

**Example 5.6.** Let \( G = C_{10} \) and \( U = [\lambda_{10}(0) \oplus \lambda_{10}(1) \oplus \lambda_{10}(9)]^\infty \), so that \( I = \{0, 1, 9\} \). Then the only nontrivial admissible orbit for \( \mathcal{L}(U) \) is \( C_2/C_1 \), because \((I \text{ mod } 10) = \{0, 1, 9\} \) and \((I \text{ mod } 5) = \{0, 1, 4\} \) have no nontrivial translation invariance, while \((I \text{ mod } 2) = \{0, 1\}\) is invariant under \((-) + 1 : \mathbb{Z}/2 \rightarrow \mathbb{Z}/2\). Thus, the class of admissible sets of \( \mathcal{L}(U) \) is \( \langle C_2/C_1 \rangle \).

The remainder of this section is studies two cases where condition (\( \Lambda \)) characterizes the \( C_n \)-indexing systems that correspond to linear isometries operads. We begin with the case where \( G \) is cyclic of prime power order.

Write \( G_k = C_{p^k} \) for \( k = 0, \ldots, n \), so that the subgroup lattice of \( C_{p^n} \) is

\[
\{1\} = G_0 \hookrightarrow G_1 \hookrightarrow \cdots \hookrightarrow G_{n-1} \hookrightarrow G_n = C_{p^n}.
\]

Each inclusion \( G_j \hookrightarrow G_{j+1} \) above sends the generator of \( G_j \) to the \( p \)-th power of the generator of \( G_{j+1} \).

**Theorem 5.7.** Let \( G = C_{p^n} \) for a prime \( p \) and natural number \( n \), and let \( \mathcal{J} \) be a \( C_{p^n} \)-indexing system. Then \( \mathcal{J} \) is realized by a linear isometries operad if and only if \( \mathcal{J} \) is a \( \Lambda \)-indexing system.
Proof: The “only if” direction was established in [2, p. 17]. For the “if” direction, suppose that \(J\) is a \(C_{p^n}\)-\(\Lambda\)-indexing system. By Proposition 4.10, \(J\) is generated by a set of orbits of the form \(O = \{G_{k_1+1}/G_{k_1}, \ldots, G_{k_m+1}/G_{k_m}\}\), where \(0 \leq k_1 < \cdots < k_m < n\). Let \(I \subset \mathbb{Z}/p^n\) be the set

\[
I = \left\{ \pm (a_1 p^{k_1} + \cdots + a_m p^{k_m}) \mid 0 \leq a_1, \ldots, a_m < p \right\},
\]

and let \(U = \bigoplus_{i \in I} \lambda_p(i)\). We claim that \(L(U)\) realizes \(J\).

Let \(J'\) be the class of admissible sets of \(L(U)\). Then \(J'\) is also a \(C_{p^n}\)-\(\Lambda\)-indexing system, and therefore it is generated by its admissible simple orbits. We shall use Proposition 5.4 repeatedly to show that \(G_{j+1}/G_j = C_{p^{j+1}}/C_{p^j}\) is admissible for \(L(U)\) if and only if \(j = k_1, \ldots, k_m\).

Consider the orbit \(G_{k_1+1}/G_{k_1}\). The set \((I \mod p^{j+1})\) consists of all residues of the form \(\pm (a_1 p^{k_1} + \cdots + a_j p^{k_j})\) with \(0 \leq a_1, \ldots, a_j < p\), and this subset of \(\mathbb{Z}/p^{j+1}\) is closed under \((-) + p^{j+1}\). Therefore \(G_{k_1+1}/G_{k_1}\) is admissible for \(L(U)\).

Now consider the orbit \(G_{j+1}/G_j\) with \(j \neq k_1, \ldots, k_m\). We shall show \(G_{j+1}/G_j\) is not admissible for \(L(U)\) by studying the cases \(j < k_1, k_i < j < k_{i+1}\), and \(k_m < j\) separately. In each case, we must check that \((I \mod p^{j+1})\) is not closed under \((-) + p^{j+1}\), so it will be enough to show \(p^j \not\equiv (I \mod p^{j+1})\). If \(j < k_1\), then \((I \mod p^{j+1}) = \{0\} \subset \mathbb{Z}/p^{j+1}\), which does not contain \(p^j\). If \(k_i < j < k_{i+1}\), then \((I \mod p^{j+1}) = \{\pm (a_1 p^{k_1} + \cdots + a_i p^{k_i})\}\) as above, and \(0 \leq a_1 p^{k_1} + \cdots + a_i p^{k_i} < p\) for all \(0 \leq a_1, \ldots, a_i < p\). Therefore \(0 < p^j \not\equiv (a_1 p^{k_1} + \cdots + a_i p^{k_i}) < p^{j+1}\), and it follows that \(p^j \not\equiv (I \mod p^{j+1})\). The case where \(k_m < j\) is similar.

We have shown that the admissible simple orbits of \(J'\) are precisely the generating simple orbits \(G_{k_1+1}/G_{k_1}\) of \(J\). By Proposition 4.10, the \(\Lambda\)-indexing systems \(J\) and \(J'\) are equal, because they are generated by the same set.

Now suppose that \(p < q\) are distinct primes. As in section 3, we shall use transfer system notation, i.e. we shall depict a \(C_{pq}\)-transfer system by drawing an edge into the subgroup lattice of \(C_{pq}\) for each admissible orbit, while omitting the redundant ones. Thus, we have

\[
\begin{align*}
\text{triv} & \quad \Rightarrow \\
\text{Set} & = \bigodot \\
\langle C_p/C_1 \rangle & \quad \vdash \\
\langle C_{pq}/C_p \rangle & \quad \vdash
\end{align*}
\]

More formally, we are drawing Hasse diagrams of \(C_{pq}\)-transfer systems (cf. section 6). As in section 3, we can use Theorem A.2 to enumerate all \(C_{pq}\)-indexing systems, and one finds that they form the lattice in Figure 2 (p. 20). Corollary 2.8 identifies the \(C_{pq}\)-indexing systems that correspond to little discs operads. We shall now determine which indexing systems correspond to linear isometries operads.

**Example 5.8.** Let \(G = C_6\). For any subset \(I \subset \mathbb{Z}/6\) that contains 0 and is closed under additive inversion, let \(U(I) = \bigoplus_{i \in I} \lambda_6(i)\). The lattice of \(C_6\)-universes, and
Figure 2. The lattice of all $C_{pq}$-indexing systems.

the Λ-indexing systems for the corresponding linear isometries operads are:

Note that the Λ-indexing systems $\Lambda$ and $\Lambda$ do not appear on the right, and that not all order relations are preserved (cf. Theorem 4.4).

We miss two $C_6$-Λ-indexing systems in Example 5.8 because there are too few $C_6$-representations. We now explain what happens as $p$ and $q$ get larger.

Lemma 5.9. Suppose that $p < q$ are prime, and that $q > 3$. Then the six $C_{pq}$-indexing systems below are realized by the linear isometries operads $\mathcal{L}(U(I))$ for the
indicated sets $I \subset \mathbb{Z}/pq$. 

| $\Lambda$-indexing system | set $I$                          |
|---------------------------|---------------------------------|
| :                        | \{0\}                           |
| ≺                        | \{0, ±1, ±2, . . . , ±(p/2)]    |
| ≺                        | \{0, ±1, ±2, . . . , ±(q/2)]    |
| ≺                        | \{0, p, 2p, . . . , p(q − 1)\}  |
| ≺                        | \{0, 2q, . . . , (p−1)q\}       |
| ≺                        | \{0, 1, 2, . . . , pq − 1\}     |

Proof. We apply Proposition 5.4 repeatedly. The claims for $I = \{0\}$ and $I = \{0, 1, . . . , pq − 1\}$ are immediate, because these index sets correspond to a trivial universe and a complete universe.

For $\cdot$, note that if $I = \{0, 1, . . . , [p/2], pq − [p/2], . . . , pq − 1\}$, then $\lfloor p/2 \rfloor + 1$ implies $(I \bmod pq)$ has no translation invariance. Similarly, $\lfloor q/2 \rfloor + 1$ implies $(I \bmod q)$ has no translation invariance.

For $\cdot$, we have $I = \{0, 1, . . . , [q/2], pq − [q/2], . . . , pq − 1\}$. The inequalities $\lfloor q/2 \rfloor < \lfloor q/2 \rfloor + 1 + \lfloor q/2 \rfloor$ imply that $I$ is invariant under $(-) + 1$. Thus, the only nontrivial admissible orbits of $\mathcal{L}(U(I))$ are $C_{pq}/C_1$.

For $\cdot$, we have $I = \{0, p, 2p, . . . , p(q − 1)\}$. Since $0 < 1 < p$ and $q \notin I$, $I$ is only invariant under $(-) + p$ and $(-) + pq$. Next, $(I \bmod p) = \{0\}$, so it has no translation invariance. Finally, $(I \bmod q) = \{0, 1, . . . , q − 1\}$, which is invariant under $(-) + 1$. Thus, the only nontrivial admissible orbits of $\mathcal{L}(U(I))$ are $C_{pq}/C_p$ and $C_q/C_1$. A similar argument applies for $\cdot$.  

Lemma 5.10. Suppose that $p < q$ are prime. If $p = 2$ or 3, then $\cdot$ is not realized by any $C_{pq}$-linear isometries operad. If $p > 3$, then it is realized by the $C_{pq}$-linear isometries operad over $U(\pm 1, 0, p, 2p, . . . , p(q − 1))$.

Proof. Suppose first that $p > 3$, and let $I = \{0, 1, p, 2p, . . . , p(q − 1), pq − 1\}$. The inequality $p + 1 < 2p$ and the fact that $q \notin I$ imply that $I$ has no translation invariance. Next, $(I \bmod p) = \{0, 1, p − 1\}$, and $1 < 2 < p − 1$ implies that $(I \bmod p)$ also has no translation invariance. Finally, $(I \bmod q) = \{0, 1, . . . , q − 1\}$, which is invariant under $(-) + 1$. Therefore $C_q/C_1$ is the only nontrivial admissible orbit of the operad $\mathcal{L}(U(I))$.

Now suppose that $p = 2$ or 3. We shall prove that $\cdot$ cannot be realized by a linear isometries operad. Suppose that $C_q/C_1$ is admissible for $\mathcal{L}(U(I))$, but that $C_p/C_1$ is not. Then $I \subset p\mathbb{Z}/pq$, because if $(I \bmod p) \neq \{0\}$, then $\text{res}_{pq}^\pi U(I)$ is complete, and hence $C_p/C_1$ would be admissible for $\mathcal{L}(U(I))$. The reduction map $\pi : \mathbb{Z}/pq \to \mathbb{Z}/q$ induces a bijection $\pi : p\mathbb{Z}/pq \to \mathbb{Z}/q$, and since $C_q/C_1$ is
admissible for $L(U(I))$, we must have $\pi(I) = \mathbb{Z}/q$. It follows that $I = p(\mathbb{Z}/pq)$, but then $C_{pq}/C_p$ must also be admissible for $L(U(I))$. Therefore $L(U(I))$ cannot realize $\langle C_q/C_1 \rangle$ alone.

In summary, we obtain the following result.

**Theorem 5.11.** Suppose that $p < q$ are primes and consider the cyclic group $C_{pq}$.

1. If $p = 2$ and $q = 3$, then all $C_6$-$\Lambda$-indexing systems except $\cdot \cdot \cdot$ and $\cdot \cdot \cdot$ are realized by linear isometries operads.
2. If $p = 2$ or $3$ and $q > 3$, then every $C_{pq}$-$\Lambda$-indexing system except $\cdot \cdot \cdot$ is realized by a linear isometries operad.
3. If $p, q > 3$, then a $C_{pq}$-indexing system is realized by a linear isometries operad if and only if it is a $\Lambda$-indexing system.

6. Transfer systems

In this section, we introduce transfer systems (Definition 6.4) and we explain their relationship to indexing systems, and to the indexing categories studied in [3, §3]. This discussion gives formal meaning to our pictures in §§3–5, it aids our work in §§7–8 on induction, restriction, and coinduction, and it streamlines the calculations in appendix A.

Roughly speaking, a transfer system is a graphical representation of the orbits in an indexing system (cf. Theorem 6.6). Alternatively, a transfer system is the intersection of the orbit category $O_G$ with an indexing category, i.e. a wide, pull-back stable, finite coproduct complete subcategory of the category $\text{Set}^{G}_{\text{fin}}$ of finite $G$-sets (cf. Corollary 6.9). Indexing systems naturally encode the structure on equivariant operads, while indexing categories naturally parametrize the transfers on incomplete Mackey functors and the norms on incomplete Tambara functors. The main point of working with transfer systems is that they are finite.

We begin with a review of indexing systems. These objects are defined in [2, Definition 3.22] as (full) truncation sub-symmetric monoidal coefficient systems of $\text{Set}$, which contain all trivial actions, and which are closed under self-induction and cartesian products. As explained in [7], the following definition is equivalent to the original formulation. Indeed, indexing systems in the sense below are precisely the object classes of indexing systems in the sense of [2].

**Definition 6.1.** Let $G$ be a finite group and let $\text{Sub}(G)$ denote the set of all subgroups of $G$. A *class of finite $G$-subgroup actions* is a class $\mathcal{X}$, equipped with a function $\mathcal{X} \rightarrow \text{Sub}(G)$, such that the fiber over $H$ is a class of finite $H$-sets for every $H \in \text{Sub}(G)$. Write $\mathcal{X}(H)$ for the fiber over $H$.

A *$G$-indexing system* $\mathcal{I}$ is a class of finite $G$-subgroup actions which satisfies the following closure conditions:

1. (trivial sets) For any subgroup $H \subseteq G$, $\emptyset \in \mathcal{I}(H)$ and $H/H \sqcup H/H \in \mathcal{I}(H)$.
2. (isomorphism) For any subgroup $H \subseteq G$ and finite $H$-sets $S$ and $T$, if $S \in \mathcal{I}(H)$ and $S \cong T$, then $T \in \mathcal{I}(H)$.
3. (restriction) For any subgroups $K \subseteq H \subseteq G$ and finite $H$-set $T$, if $T \in \mathcal{I}(H)$, then $\text{res}_H^K T \in \mathcal{I}(K)$.
4. (conjugation) For any subgroup $H \subseteq G$, group element $g \in G$, and finite $H$-set $T$, if $T \in \mathcal{I}(H)$, then $c_g T \in \mathcal{I}(ghg^{-1})$. 


(5) (subobjects) For any subgroup $H \subset G$ and finite $H$-sets $S$ and $T$, if $T \in \mathcal{I}(H)$ and $S \subset T$, then $S \in \mathcal{I}(H)$.

(6) (coproducts) For any subgroup $H \subset G$ and finite $H$-sets $S$ and $T$, if $S \in \mathcal{I}(H)$ and $T \in \mathcal{I}(H)$, then $S \sqcup T \in \mathcal{I}(H)$.

(7) (self-induction) For any subgroups $K \subset H \subset G$ and finite $K$-set $T$, if $H/K \in \mathcal{I}(H)$ and $T \in \mathcal{I}(K)$, then $\text{ind}_K^H T \in \mathcal{I}(H)$.

Recall that a $\Lambda$-indexing system (cf. Definition 4.8) also satisfies:

(A) For any subgroups $K \subset L \subset H \subset G$, if $H/K \in \mathcal{I}(H)$ then $H/L \in \mathcal{I}(H)$.

We call the elements of $\mathcal{I}(H)$ the admissible $H$-sets of $\mathcal{I}$. Let $\text{Ind}$ denote the poset of all indexing systems, ordered under inclusion.

For any group $G$, there is a maximum indexing system $\text{Set}$, whose $H$-fiber is the class of all finite $H$-sets, and there is a minimum indexing system $\text{triv}$, whose $H$-fiber is the class of all finite, trivial $H$-sets.

Every $G$-indexing system is determined by the orbits that it contains, because it is closed under subobjects and coproducts. We can use this to give an alternate description of $G$-indexing systems in terms of the subgroup lattice of $G$.

**Definition 6.2.** For any $G$-indexing system $\mathcal{I}$, we define the graph of $\mathcal{I}$ to be the set $\text{Sub}(G)$, equipped with the binary relation $\rightarrow_{\mathcal{I}}$:

$$K \rightarrow_{\mathcal{I}} H \quad \text{if and only if} \quad H/K \in \mathcal{I}.$$ 

We think of subgroups $H \subset G$ as vertices, and relations $K \rightarrow_{\mathcal{I}} H$ as directed edges. We can characterize the binary relations $(\text{Sub}(G), \rightarrow_{\mathcal{I}})$ that arise from indexing systems and from $\Lambda$-indexing systems.

**Proposition 6.3.** Suppose that $\mathcal{I}$ is a $G$-indexing system. Then $\rightarrow = \rightarrow_{\mathcal{I}}$ is:

(a) a refinement of the subset relation: if $K \rightarrow H$, then $K \subset H$,

(b) a partial order,

(c) closed under conjugation: if $K \rightarrow H$, then $(gKg^{-1}) \rightarrow (gHg^{-1})$ for every group element $g \in G$, and

(d) closed under restriction: if $K \rightarrow H$ and $L \subset H$, then $(K \cap L) \rightarrow L$.

If $\mathcal{I}$ is a $\Lambda$-indexing system, then $\rightarrow$ also is:

(e) saturated: if $K \rightarrow H$ and $K \subset L \subset H$, then $K \rightarrow L$ and $L \rightarrow H$.

**Proof.** For (a), we have $K \subset H$ for every orbit $H/K$. For (b), reflexivity holds because $H/H \in \mathcal{I}$ for every $H \subset G$, antisymmetry follows from (a), and transitivity holds because if $K \rightarrow L$ and $L \rightarrow H$, then $L/K \in \mathcal{I}$ and $H/L \in \mathcal{I}$, and hence $H/K \cong \text{ind}_L^H L/K \in \mathcal{I}$. Condition (c) holds because if $K \rightarrow H$, then $H/K \in \mathcal{I}$, and hence $gHg^{-1}/gKg^{-1} \cong c_gH/K \in \mathcal{I}$. Condition (d) holds because we have an embedding $L/(L \cap K) \hookrightarrow \text{res}_L^K H/K$, and $\mathcal{I}$ is closed under restriction and subobjects. Condition (e) is just a restatement of condition (A). \qed

Proposition 6.3 motivates the following definition.

**Definition 6.4.** Let $G$ be a finite group. A $G$-transfer system is a partial order on $\text{Sub}(G)$, which

(1) refines the subset relation, and

(2) is closed under conjugation and restriction.
We use arrows → to denote transfer systems. We say → is saturated if it satisfies condition (e) above. Let \( \text{Tr} = \text{Tr}(G) \) denote the poset of all \( G \)-transfer systems →, ordered under refinement, i.e. declare →1 ≤ →2 if and only if \( K \to_1 H \) implies \( K \to_2 H \) for all \( K, H \subset G \).

We can reverse the construction of →\( _T \) from \( I \) by regarding a given transfer system → as a set of orbits, and then closing up under finite coproducts.

**Proposition 6.5.** If → is a \( G \)-transfer system, then there is a unique \( G \)-indexing system \( I = I \to \) such that →\( _T = \to \). More specifically, \( I \to(H) \) is the class of all finite coproducts of \( H \)-orbits \( H/K \) such that \( K \to H \). The transfer system → is saturated if and only if \( I \to \) is a \( \Lambda \)-indexing system.

**Proof.** Fix a transfer system →. If \( I \) is an indexing system such that →\( _T = \to \), then the orbits of \( I \) must be those \( H/K \) such that \( K \to H \), and \( I \) must be the class of all finite coproducts of such orbits. Therefore \( I \) is unique if it exists.

We check that the recipe above works. Define a class \( I \) of finite \( G \)-subgroup actions by the

\[
\mathcal{I}(H) := \left\{ \text{finite sets } T \mid \text{there exist } n \geq 0 \text{ and } K_1, \ldots, K_n \subset H \text{ such that } T \cong \bigsqcup_{i=1}^n H/K_i \text{ and } K_i \to H \text{ for } i = 1, \ldots, n \right\},
\]

where empty coproducts are understood to be \( \emptyset \). We must check that \( I = I \to \) is a \( G \)-indexing system, and that →\( _T = \to \).

We verify the axioms in Definition 6.1. Condition (1) holds because → is reflexive. Condition (2) holds because coproducts are only defined up to isomorphism. Condition (3) holds because if \( T \cong \bigsqcup_{i=1}^n H/K_i \) with \( K_i \to H \), then for any \( L \subset H \),

\[
\text{res}_L^HT \cong \bigsqcup_{i=1}^n \text{res}_L^HH/K_i \cong \bigsqcup_{i=1}^n \bigsqcup_{a \in L \setminus H/K_i} L/(L \cap aK_i a^{-1}).
\]

The right hand side is a finite coproduct, and if \( K_i \to H \), then for any \( a \in L \setminus H/K_i \), we have \((aK_i a^{-1}) \to (aHa^{-1}) = H \) and also \((L \cap aK_i a^{-1}) \to L \), because → is closed under conjugation and restriction. Condition (4) holds because if \( T \cong \bigsqcup_{i=1}^n H/K_i \), then \( c_gT \cong \bigsqcup_{i=1}^n gHg^{-1}/gK_ig^{-1}, \) and → is closed under conjugation. Condition (5) holds because every subobject of \( T \cong \bigsqcup_{i=1}^n H/K_i \in I \), is still just a finite coproduct of orbits \( H/K \) with \( K \to H \). Similarly for condition (6).

Suppose that \( H/K \in I \). Then \( H/K \cong H/K' \) for some \( K' \to H \). Therefore \( K = hK' h^{-1} \) for some \( h \in H \), and thus \( K = hK' h^{-1} \to hHHh^{-1} = H \). From here, we deduce (7), because if \( T \cong \bigsqcup_{i=1}^n K/L_i \in I \) for \( L_i \to K \) and \( H/K \in I \), then \( K \to H \), and therefore \( L_i \to H \) by transitivity. Thus, \( \text{ind}_{K}^HT \cong \bigsqcup_{i=1}^n H/L_i \in I \). This proves that \( I \) is an indexing system.

If the transfer system → is saturated, and we have \( H/K \in I \) and \( K \subset L \subset H \), then \( K \to H \) by the previous paragraph, and thus \( K \to L \to H \). Therefore \( H/L \in I \), which proves that \( I \) is a \( \Lambda \)-indexing system. The converse is similar.

Finally, we consider the graph →\( _T \). If \( K \to \Lambda H \), then \( H/K \in I \) by the definition of →\( _T \), and therefore \( K \to H \) as above. Conversely, if \( K \to H \), then \( H/K \in I \) by the definition of \( I = I \to \), and therefore \( K \to \Lambda H \). Therefore →\( _T \) and → are the same partial order.

\( \square \)

Thus, indexing systems and transfer systems are in a one-to-one correspondence. We obtain a lattice isomorphism with minor elaboration.
Theorem 6.6. The maps $\rightarrow_{\bullet} : \text{Ind} \xrightarrow{\cong} \text{Tr} : \mathcal{I}_{\bullet}$ are inverse order isomorphisms, and they restrict to an isomorphism between the subposet of $\Lambda$-indexing systems and the subposet of saturated transfer systems.

Proof. Propositions 6.3 and 6.5 prove that $\rightarrow_{\bullet}$ and $\mathcal{I}_{\bullet}$ are inverse set maps, and that they restrict to a pair of inverses between the set of all $\Lambda$-indexing systems and the set of all saturated transfer systems. It remains to check that for any indexing systems $\mathcal{I}$ and $\mathcal{J}$, we have $\mathcal{I} \subset \mathcal{J}$ if and only if $\rightarrow_{\mathcal{I}}$ refines $\rightarrow_{\mathcal{J}}$.

Suppose that $\mathcal{I} \subset \mathcal{J}$. If $K \rightarrow_{\mathcal{I}} H$, then $H/K \in \mathcal{I} \subset \mathcal{J}$, and therefore $K \rightarrow_{\mathcal{J}} H$. Thus $\rightarrow_{\mathcal{I}}$ refines $\rightarrow_{\mathcal{J}}$. Conversely, if $\rightarrow_{\mathcal{I}}$ refines $\rightarrow_{\mathcal{J}}$, then every orbit in $\mathcal{I}$ is also contained in $\mathcal{J}$. Therefore $\mathcal{I} \subset \mathcal{J}$, because $\mathcal{I}$ is generated by its orbits. □

There is another perspective on indexing systems, developed in [3]. The following terminology was suggested by Mike Hill.

Definition 6.7. Let $\text{Set}_{fin}^G$ denote the category of finite $G$-sets. An indexing category is a wide, pullback stable, finite coproduct complete subcategory $\mathcal{P} \subset \text{Set}_{fin}^G$. We write $\text{IndCat}$ for the poset of all indexing categories.

Blumberg and Hill give an explicit isomorphism $\text{Ind} \cong \text{IndCat}$, and we shall end this section by describing the composite isomorphism $\text{Tr} \cong \text{Ind} \cong \text{IndCat}$. This will make precise the intuition that transfer systems are the intersections of indexing categories with the orbit category $\mathcal{O}_G \subset \text{Set}_{fin}^G$.

We begin with a review of the isomorphism $\text{Ind} \cong \text{IndCat}$. For any indexing system $\mathcal{I}$, let $\text{Set}_{\mathcal{I}}^G$ to be the indexing category consisting of those morphisms $f : S \rightarrow T$ such that for any $s \in S$, we have $G_{f(s)}/G_s \in \mathcal{I}$. Conversely, given any indexing category $\mathcal{P} \subset \text{Set}_{fin}^G$, let $\mathcal{I}_\mathcal{P}$ be the indexing system whose admissible $H$-sets are those $T$ such that $T = p^{-1}(eH)$ for some $p : S \rightarrow G/H$ in $\mathcal{P}$.

Theorem 6.8 ([3, Theorem 3.17]). The maps $\text{Set}_{\bullet}^G : \text{Ind} \xrightarrow{\cong} \text{IndCat} : \mathcal{I}_{\bullet}$ are inverse lattice isomorphisms.

We now consider the composite $\text{Tr} \cong \text{Ind} \cong \text{IndCat}$. For any transfer system $\rightarrow_{\bullet} \in \text{Tr}$, let $\text{Set}_{\mathcal{I}_{\bullet}}^G \in \text{IndCat}$ consist of those morphisms $f : S \rightarrow T$ such that for any $s \in S$, we have $G_s \rightarrow G_{f(s)}$. Conversely, for any $\mathcal{P} \in \text{IndCat}$, let $\rightarrow_{\mathcal{P}} \in \text{Tr}$ be the transfer system defined by

$K \rightarrow_{\mathcal{P}} H$ if and only if $(\pi : G/K \rightarrow G/H) \in \mathcal{P},$

where $\pi$ is the projection map $\pi(gK) = gH$. Unwinding the definitions and simplifying proves the following.

Corollary 6.9. The lattice maps $\text{Set}_{\bullet}^G : \text{Tr} \xrightarrow{\cong} \text{IndCat} : \rightarrow_{\bullet}$ are inverse.

Thus, there is a chain of equivalences $\text{Ho}(N_\infty\text{-Op}) \simeq \text{Ind} \cong \text{IndCat} \cong \text{Tr}$, where $\text{Ho}(N_\infty\text{-Op})$ denotes the homotopy category of $N_\infty$ operads in $G$-spaces. From the perspective of transfer systems, we conclude that the following group-theoretic data determines everything about these structures.

Corollary 6.10. The lattices $\text{Ind}, \text{IndCat}$, and $\text{Tr}$, and the category $\text{Ho}(N_\infty\text{-Op})$ are determined by the lattice $\text{Sub}(G)$, and the orbit space of $\text{Sub}(G) \times \text{Sub}(G)$ under the diagonal conjugation $G$-action.

\footnote{The indexing system $\mathcal{I}_{\mathcal{P}}$ is obtained from the construction in [3, Lemma 3.18] by composing with the equivalence $\text{Set}_{\mathcal{P}}^{G/H} \simeq \text{Set}^H$, and then taking object classes.}
If we restrict to finite abelian groups $G$, or to finite groups for which all subgroups are normal, then the lattice $\text{Sub}(G)$ determines $\text{Ind}$, $\text{IndCat}$, $\text{Tr}$, and $\text{Ho}(\text{N}_{\text{cts}} \text{-Op})$. This is why some of our analysis of $C_{pq}$-indexing systems and $C_{pq}$-indexing systems in §5 could be carried out uniformly in $p$ and $q$.

7. Images and inverse images of transfer systems

Suppose that $f : G \to G'$ is a homomorphism between finite groups. In this section, we explain how to induce, restrict, and coinduce transfer systems along the map $f$ (Definition 7.5), and we prove that these constructions are suitably functorial (Theorem 7.8). Restricting to the automorphisms of a group $G$ recovers the action on $\text{Ind}(G) \cong \text{Tr}(G)$ considered in §§3–4 (Corollary 7.14).

We outline the constructions now, before diving into the details. What follows is loosely inspired by the image-inverse image adjunctions associated to a set map. Suppose that $f : X \to Y$ is a function between sets, and let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ denote the power sets of $X$ and $Y$, ordered under inclusion. Then there are order adjunctions $f : \mathcal{P}(X) \to \mathcal{P}(Y) : f^{-1}$ and $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X) : f_*$, where $f$ and $f^{-1}$ denote the usual image and inverse image along $f$, and for any $A \subseteq X$, $f_* A = \{ y \in Y | f^{-1}(y) \subseteq A \}$.

Now, given a $G$-transfer system $\rightarrow$ and a group homomorphism $f : G \to G'$, one might guess that the image $f(\rightarrow)$ is obtained by applying $f$ componentwise to the pairs $(K, H)$ in $\rightarrow$. Unfortunately, the result is not usually a transfer system, and so we close it up. Dually, the natural inverse image along $f^2$ does not always yield a transfer system, but we can cut it down to one. The result is an adjunction analogous to $f \dashv f^{-1}$, which we denote $f_! : \text{Tr}(G) \vdash \text{Tr}(G') : f^{-1}_R$.

Now for the second adjunction $f^{-1} \dashv f_*$. There is another way one might try to construct an inverse image along $f : G \to G'$: instead of pulling back along $f^2$, one could also just apply $f^{-1}$ componentwise to the pairs in a $G'$-transfer system. As before, the result is not usually a transfer system, but we can close it up. Dually, inverse images of $G$-transfer systems along $(f^{-1})^2$ can be cut down to $G'$-transfer systems, and we obtain an adjunction $f_*^L : \text{Tr}(G') \vdash \text{Tr}(G) : f^R$.

With a bit of work, one can show that $f_!$ and $f_*$ are covariantly functorial in $f$, while $f^L_!$ and $f^R_*$ are contravariantly functorial. In contrast to the adjunction $f \dashv f^{-1} \dashv f_*$, the two inverse images $f^R_!$ and $f^L_!$ are not always equal. In general, $f^L_!$ refines $f^R_!$ (Proposition 7.10), and equality holds if and only if $f$ is injective (Theorem 7.12). When $f$ is injective, we have adjunctions $f_! \dashv f^{-1}_R = f^L_! \dashv f_*$, analogous to induction, restriction, and coinduction.

Now for the details. Let $G$ be a finite group, and suppose that $R$ is a binary relation on $\text{Sub}(G)$. As usual, we identify $R$ with the set $\{(K, H) \in \text{Sub}(G)^2 | K \mathcal{R} H \}$. If $R$ refines inclusion, then it is contained the maximum $G$-transfer system

$$\text{Sub}(G) \subseteq := \{(K, H) \in \text{Sub}(G)^2 | K \subseteq H \}.$$ 

Thus, we can make the following definition.

**Definition 7.1.** For any binary relation $R$ on $\text{Sub}(G)$ that refines inclusion, let $\langle R \rangle$ be the intersection of all $G$-transfer systems that contain $R$.

The relation $\langle R \rangle$ is the smallest $G$-transfer system that contains $R$, because the intersection of a nonempty family of transfer systems is still a transfer system. We shall give a more explicit description of $\langle R \rangle$ in Appendix A.
Dually, suppose that $R$ is a reflexive binary relation on $\text{Sub}(G)$. Then $R$ contains the minimum $G$-transfer system $\Delta \text{Sub}(G) = \{(H, H) \mid H \subset G\}$. It follows that there are maximal $G$-transfer systems contained in $R$, but there need not be a maximum, because the set-theoretic union of a family of transfer systems is not necessarily their join. However, if $R$ is a partial order that refines inclusion, then there is a maximum.

**Proposition 7.2.** Suppose that $\leq$ is a partial order on $\text{Sub}(G)$ that refines inclusion. Then

$$\langle \rangle \leq \langle \rangle := \left\{ (K, H) \mid K \subset H, \text{ and } gKg^{-1} \cap L \leq L \text{ for all } g \in G \text{ and } L \subset gHg^{-1} \right\}$$

is the largest $G$-transfer system contained in $\leq$.

**Proof.** We begin by showing $\langle \rangle \leq \langle \rangle$ is a transfer system. The reflexivity of $\langle \rangle \leq \langle \rangle$ follows from that of $\leq$. By definition, the relation $\langle \rangle \leq \langle \rangle$ refines $\subset$, and therefore it is also antisymmetric. For transitivity, suppose that $(K, J), (J, H) \in \langle \rangle$. Given $g \in G$ and $L \subset gHg^{-1}$, let $M = gJg^{-1} \cap L$. Then $M \subset gHg^{-1}$, and we have

$$gKg^{-1} \cap L = gKg^{-1} \cap M \leq M = gJg^{-1} \cap L \leq L,$$

so that $gKg^{-1} \cap L \leq L$ by transitivity of $\leq$. It is clear that $\langle \rangle \leq \langle \rangle$ is closed under conjugation. For restriction, suppose $(K, H) \in \langle \rangle \leq \langle \rangle$ and $L \subset H$. Then $(K \cap L, L) \in \langle \rangle$ because if $g \in G$ and $M \subset gHg^{-1}$, then $M \subset gHg^{-1}$ and hence

$$g(K \cap L)g^{-1} \cap M = gKg^{-1} \cap M \leq M.$$

Thus $\langle \rangle \leq \langle \rangle$ is a transfer system.

Next, suppose that $(K, H) \in \langle \rangle \leq \langle \rangle$. Taking $g = e \in G$ and $L = H \subset eHe^{-1}$, we find $K = eKe^{-1} \cap H \leq H$, because $K \subset H$. Therefore $\langle \rangle \leq \langle \rangle$ refines $\leq$.

Finally, suppose $\rightarrow$ is a transfer system that refines $\leq$, and suppose $K \rightarrow H$. Then $K \subset H$ because $\rightarrow$ refines inclusion. Then, for any $g \in G$ and $L \subset gHg^{-1}$, we have $gKg^{-1} \rightarrow gHg^{-1}$ and $gKg^{-1} \cap L \rightarrow L$, so that $gKg^{-1} \cap L \leq L$. Therefore $\rightarrow$ refines $\langle \rangle \leq \langle \rangle$. \qed

**Remark 7.3.** The formula for $\langle \rangle \leq \langle \rangle$ should be compared to Blumberg and Hill’s calculation of the admissible sets of a coinduced $N_\infty$ operad [2, Proposition 6.16].

We shall use $\langle \bullet \rangle$ and $\langle \bullet \rangle \bullet \langle \rangle$ to set up image and inverse image adjunctions, but we work a little more generally for now to treat both cases simultaneously. Suppose that $F : \text{Sub}(G) \to \text{Sub}(G')$ is an order-preserving map, let $F_\subset = F^2 : \text{Sub}(G)_\subset \to \text{Sub}(G')_\subset$, and consider the image and inverse image maps $F_\subset : \mathcal{P}(\text{Sub}(G)_\subset) \to \mathcal{P}(\text{Sub}(G')_\subset) = (F_\subset)^{-1}$. The elements of $\mathcal{P}(\text{Sub}(G)_\subset)$ are binary relations on $\text{Sub}(G)$ that refine inclusion, and similarly for $G'$.

**Definition 7.4.** Keep notation as above. For any $G$-transfer system $\rightarrow$ and $G'$-transfer system $\rightarrow$, define

$$F_L(\rightarrow) := \langle F_\subset(\rightarrow) \rangle \quad \text{and} \quad F_R^{-1}(\rightarrow) := \langle (F_\subset)^{-1}(\rightarrow) \rangle.$$

The definition of $F_R^{-1}$ makes sense because $(F_\subset)^{-1}$ preserves partial orders that refine inclusion. Specializing to the case where $F = f$ or $F = f^{-1}$ for a group homomorphism $f$ gives the desired image and inverse image maps.
Definition 7.5. Let $f : G \to G'$ be a homomorphism between finite groups. We use the image map $f : \text{Sub}(G) \to \text{Sub}(G')$ to define the maps

$$f_L : \text{Tr}(G) \subseteq \text{Tr}(G') : f_R^{-1}$$

and we use the inverse image map $f^{-1} : \text{Sub}(G') \to \text{Sub}(G)$ to define the maps

$$f_L^{-1} := (f^{-1})_L : \text{Tr}(G') \subseteq \text{Tr}(G) : (f^{-1})_R^{-1} =: f_R.$$

We now analyze the adjointness and functoriality properties of these maps.

Lemma 7.6.

(1) For any inclusion-preserving map $F : \text{Sub}(G) \to \text{Sub}(G')$, there is an adjunction $F_L : \text{Tr}(G) \rightleftarrows \text{Tr}(G') : F_R^{-1}$. If $F_\subset$ preserves transfer systems, then $F_L = F_{\subset}$. If $(F_\subset)^{-1}$ preserves transfer systems, then $F_R^{-1} = (F_\subset)^{-1}$.

(2) For any pair of inclusion-preserving maps $E : \text{Sub}(G) \to \text{Sub}(G')$ and $F : \text{Sub}(G') \to \text{Sub}(G''')$, we have refinements

$$(FE)_L(\rightarrow) \subseteq F_LE_L(\rightarrow) \quad \text{and} \quad E_R^{-1}F_R^{-1}(\rightarrow) \subseteq (FE)_R^{-1}(\rightarrow).$$

Moreover, if either of the equalities $(FE)_L = F_LE_L$ or $(FE)_R^{-1} = E_R^{-1}F_R^{-1}$ hold, then both of them hold.

(3) If either $E_\subset$ or $(F_\subset)^{-1}$ preserve transfer systems, then $(FE)_L = F_LE_L$ and $(FE)_R^{-1} = E_R^{-1}F_R^{-1}$.

Proof. For (1), the adjunction $F_L \vdash F_R^{-1}$ follows from the adjunction $F_\subset \vdash (F_\subset)^{-1}$ and the adjointness properties of $\bullet$ and $\bullet'$. If $F_\subset$ preserves transfer systems, then applying $\bullet$ does nothing to $F_\subset(\rightarrow)$, and similarly for $(F_\subset)^{-1}$.

Now for (2). Suppose $E : \text{Sub}(G) \to \text{Sub}(G')$ and $F : \text{Sub}(G') \to \text{Sub}(G''')$ are order-preserving. For any $G$-transfer system $\rightarrow$, we have $E_\subset(\rightarrow) \subseteq E_L(\rightarrow)$, and hence $(FE)_\subset(\rightarrow) \subseteq F_\subset E_L(\rightarrow) \subseteq F_L E_L(\rightarrow)$. It follows $(FE)_L(\rightarrow) \subseteq F_L E_L(\rightarrow)$. Dually, $E_R^{-1}F_R^{-1}(\rightarrow) \subseteq (FE)_R^{-1}(\rightarrow)$ for every $G''$-transfer system $\rightarrow$.

Suppose further that $(FE)_L = F_LE_L$. Then by the uniqueness of adjoints, the functors $E_R^{-1}F_R^{-1}$ and $(FE)_R$ are naturally isomorphic maps $\text{Tr}(G'') \rightleftarrows \text{Tr}(G)$, but the codomain is a poset. Therefore $E_R^{-1}F_R^{-1} = (FE)_R^{-1}$. The argument when $(FE)_R^{-1} = E_R^{-1}F_R^{-1}$ is dual.

For (3), suppose that $E_\subset$ preserves transfer systems. Then

$$F_LE_L(\rightarrow) = (F_\subset(E_\subset(\rightarrow))) = (F_\subset E_\subset(\rightarrow)) = (FE)_L(\rightarrow)$$

for every $G$-transfer system $\rightarrow$. The equality $(FE)_R^{-1} = E_R^{-1}F_R^{-1}$ follows from (2). Dually if $(F_\subset)^{-1}$ preserves transfer systems. \qed

Suppose that $h : G \to G'$ and $k : G' \to G''$ are composable homomorphisms between finite groups, and that $\rightarrow$ and $\rightarrow'$ are $G$ and $G''$-transfer systems, respectively. It follows from Lemma 7.6 that there are inclusions

$$kh_L(\rightarrow) \subseteq k_L h_L(\rightarrow), \quad (kh)_R^{-1}(\rightarrow) \supseteq h_L^{-1}k_R^{-1}(\rightarrow)$$

$$kh_L^{-1}(\rightarrow) \subseteq h_L^{-1}k_L^{-1}(\rightarrow), \quad (kh)_R(\rightarrow) \supseteq k_R h_R(\rightarrow).$$

We establish the reverse inclusions using the description of $\bullet$ in Construction A.1.

Lemma 7.7. Suppose that $h : G \to G'$ and $k : G' \to G''$ are homomorphisms between finite groups.

(1) For any $G$-transfer system $\rightarrow$, we have $k_R h_R(\rightarrow) \supseteq (kh)_R^{-1}(\rightarrow)$.

(2) For any $G''$ transfer system $\rightarrow'$, we have $h_L^{-1}k_L^{-1}(\rightarrow') \supseteq (kh)_L^{-1}(\rightarrow')$. 

Proof. For (1), note that the relation \( h \subset (\to) \) is closed under restriction. Indeed, if \((hK, hH) \in h \subset (\to) \) for some \( K \to H \), and \( L' \subset hH \), then for \( L = h^{-1}L' \cap H \) we have \( K \cap L \to L \), and therefore \((hK \cap L', L') = (h(K \cap L), hL) \in h \subset (\to) \). It follows that \( h_L(\to) = (h \subset (\to)) \) is the the reflexive and transitive closure of the relation \( \{(g(hK)g^{-1}, g(hH)g^{-1}) \mid g \in G' \text{ and } K \to H\} \), and by applying \( h \) to chains of these relations, we deduce that \( k_\subset h_L(\to) \subset ((kh) \subset (\to)) = (kh)_L(\to) \). Therefore \( k_L h_L(\to) \subset (kh)_L(\to) \).

For (2), note that the relation \( (k^{-1}) \subset (\to) \) is closed under conjugation. Indeed, if \((k^{-1}K'', k^{-1}H'') \in (k^{-1}) \subset (\to) \) for some \( K'' \to H'' \), and \( \alpha \in G' \), then we have \( k(\alpha)K''k(\alpha)^{-1} \to \alpha k(\alpha)H''k(\alpha)^{-1} \), and hence \( \alpha(k^{-1}K'')\alpha^{-1}, \alpha(k^{-1}H'')\alpha^{-1} = (k^{-1}(k(\alpha)K''k(\alpha)^{-1}), k^{-1}(k(\alpha)H''k(\alpha)^{-1})) \in (k^{-1}) \subset (\to) \). Therefore \( k_L^{-1}(\to) \) is the reflexive and transitive closure of the relation
\[
\{(k^{-1}K'' \cap L', L') \mid K'' \to H'' \text{ and } L' \subset k^{-1}H''\},
\]
and applying \( h^{-1} \) to chains of such relations implies \((h^{-1}) \subset k_L^{-1}(\to) \subset (kh)_{L'}^{-1}(\to) \). Therefore \( h_L^{-1} k_L^{-1}(\to) = ((h^{-1}) \subset k_L^{-1}(\to)) \subset (kh)_{L'}^{-1}(\to) \). \( \square \)

Thus \( (kh)_L = k_L h_L \) and \( (kh)^{-1}_L = h_{L'}^{-1} k_L^{-1} \), and the analogous equations for \((\to)_R \) and \((\to)^{-1}_R \) follow from part (2) of Lemma 7.6. In summary:

**Theorem 7.8.** Let \( \text{FinGrp} \) and \( \text{FinPos} \) denote the categories of finite groups and finite posets. Then the constructions in Definition 7.5 determine functors
\[
(\cdot)_L, (\cdot)_R : \text{FinGrp} \Rightarrow \text{FinPos} \quad \text{and} \quad (\cdot)^{-1}_L, (\cdot)^{-1}_R : \text{FinGrp}^{op} \Rightarrow \text{FinPos},
\]
such that for any homomorphism \( f : G \to G' \) in \( \text{FinGrp} \), we have order adjunctions \( f_L \dashv f_R^{-1} \) and \( f_L^{-1} \dashv f_R \).

As mentioned earlier, the maps \( f_L^{-1} \) and \( f_R^{-1} \) are not always equal. We illustrate by example.

**Example 7.9.** Let \( ! : G \to 1 \) be the unique morphism. There is only one transfer system \( \to \in \text{Tr}(1) \), and it is both initial and terminal. Applying \( !^{-1}_L \) yields the initial \( G \)-transfer system, because \( !^{-1}_L \) is a left adjoint, and applying \( !^{-1}_R \) yields the terminal transfer system.

More generally, we have the following inequality.

**Proposition 7.10.** Suppose that \( f : G \to G' \) is a homomorphism between finite groups. Then for any \( \to \in \text{Tr}(G') \), we have \( f_L^{-1}(\to) \subset f_R^{-1}(\to) \).

Proof. For any \( \to \in \text{Tr}(G') \), we claim that \( (f^{-1}) \subset (\to) \subset (fL)^{-1}(\to) \). For suppose \( (K, H) \in (f^{-1}) \subset (\to) \). Then \( (K, H) = (f^{-1}K', f^{-1}H') \) for some \( K' \to H' \). Given \( g \in G \) and \( L \subset gHg^{-1} \), we must check that \( (gKg^{-1} \cap L, L) \in (f^{-1}) \subset (\to) \). We have \( f(gKg^{-1} \cap L) = f(g)K'f(g)^{-1} \cap fL \), where \( f(g) \in G' \) and \( fL \subset fL \to fL \). Since \( K' \to H' \) and \( \to \) is a transfer system, we also have \( f(g)K'f(g)^{-1} \cap fL \to fL \). Therefore \( (f^{-1}) \subset (\to) \subset (fL)^{-1}(\to) \), and \( f_L^{-1}(\to) \subset f_R^{-1}(\to) \) follows. \( \square \)

Fortunately, we can completely characterize when \( f_L^{-1} = f_R^{-1} \). First, a lemma.

**Lemma 7.11.** Suppose that \( m : G \to G' \) is an injective homomorphism between finite groups. Then for every \( \to \in \text{Tr}(G') \), the relation \( (m^{-1})^{-1}(\to) \) is a \( G \)-transfer system, and there is an equality \( (m^{-1})^{-1}(\to) = (m^{-1}) \subset (\to) \).
Proof. As observed earlier, $(m_{\subset})^{-1}(\sim\subset)$ is a partial order on $\text{Sub}(G)$ that refines inclusion. It is closed under conjugation because applying $m$ preserves conjugation, and it is closed under restriction because if $(K, H) \in (m_{\subset})^{-1}(\sim\subset)$ and $L \subset H$, then the injectivity of $m$ ensures that $(m(K \cap L), m(L)) = (m(K) \cap m(L), m(L)) \in \sim\subset$. Thus $(m_{\subset})^{-1}(\sim\subset)$ is a $G$-transfer system.

The inclusion $(m_{\subset})^{-1}(\sim\subset) \subset (m^{-1})_{\subset}(\sim\subset)$ also follows from the injectivity of $m$, because if $(mK, mH) \in \sim\subset$, then $(K, H) = (m^{-1}mK, m^{-1}mH) \in (m^{-1})_{\subset}(\sim\subset)$. The other inclusion $(m_{\subset})(\sim\subset) \supset (m_{\subset})^{-1}(\sim\subset)$ holds because if $(K', H') \in \sim\subset$, then $(mm^{-1}K', mm^{-1}H') = (m(G) \cap K', m(G) \cap H') \in \sim\subset$ because $\sim\subset$ is closed under restriction along $m(G) \cap H' \subset H'$. \hfill $\square$

**Theorem 7.12.** Suppose that $f : G \to G'$ is a homomorphism between finite groups. Then the following are equivalent:

1. $f$ is injective.
2. $f^{-1}_{L} = f^{-1}_{R}$.
3. $f^{-1}_{L}$ has a left adjoint.

Proof. (1) $\Rightarrow$ (2) follows from Lemma 7.11 and (2) $\Rightarrow$ (3) is immediate from the adjunction $f_{L} \dashv f^{-1}_{R}$. Now for (3) $\Rightarrow$ (1). Assume that $f$ is not injective. We shall show that $f^{-1}_{L}$ does not preserve all limits. For any $\sim\subset \in \text{Tr}(G')$ and $(K, H) \in (f^{-1})_{\subset}(\sim\subset)$, we have $\ker(f) \subset K$. By part (2) of Proposition A.3, it follows that $H \not\subset \ker(f)$ for every nontrivial relation $(K, H) \in f^{-1}_{L}(\sim\subset)$. On the other hand, if $K \subset H \subset \ker(f)$, then $(K, H) \in (f^{-1}_{L})(\sim\subset)$. It follows that $(1, \ker(f)) \in f^{-1}_{R}(\sim\subset) \setminus f^{-1}_{L}(\sim\subset)$, so the inclusion $f^{-1}_{L}(\sim\subset) \subset f^{-1}_{R}(\sim\subset)$ of Proposition 7.10 is strict. Therefore $f^{-1}_{L}$ does not preserve the terminal transfer system. \hfill $\square$

Similar pathologies for non-injective homomorphisms $f : G \to G'$ also appear on the operad level (cf. part (2) of Theorem 8.6), but all is well when we restrict attention to injective homomorphisms.

**Corollary 7.13.** If $m : G \to G'$ is an injective homomorphism between finite groups, then there is a chain of order adjunctions $m_{L} : m_{L}^{-1} \dashv m_{R}^{-1} \dashv m_{R}$.

We conclude by relating the functors above to the $\text{Aut}(G)$-action on $\text{Ind}(G) \cong \text{Tr}(G)$ considered in §§3–4.

**Corollary 7.14.** For any finite group $G$, the right action

$$\text{Aut}(G)^{\text{op}} \hookrightarrow \text{FinGrp}^{\text{op}} \xrightarrow{(-)^{L}_{-1}} \text{FinPos}$$

on $\text{Tr}(G)$ is isomorphic to the right $\text{Aut}(G)$-action on $\text{Ind}(G)$ described on p. 10.

Proof. Consider the right $\text{Aut}(G)$-action on $\text{Tr}(G)$. For any $\rightarrow \in \text{Tr}(G)$ and $\sigma \in \text{Aut}(G)$, we have $\rightarrow \cdot \sigma = \{(\sigma^{-1}K, \sigma^{-1}H) | K \to H\}$ by Lemma 7.11. On the other hand, the $\text{Aut}(G)$-action on $\text{Ind}(G)$ is given on orbits by $H/K \cdot \sigma \cong \sigma^{-1}H/\sigma^{-1}K$. Therefore the map $\rightarrow : \text{Ind}(G) \to \text{Tr}(G)$ in Theorem 6.6 is an $\text{Aut}(G)$-isomorphism. \hfill $\square$

**Remark 7.15.** We could also have used $(-)^{L}_{-1}$, $((-)^{L}_{-1})$, and $((-)^{R}_{-1})$ to define right $\text{Aut}(G)$ actions on $\text{Tr}(G)$. The adjunctions $\sigma_{L} : \sigma_{L}^{-1} \dashv \sigma_{R}^{-1} \dashv \sigma_{R}$ imply that we get the same result. Therefore $(-)^{L}_{-1}$ and $(-)^{R}_{-1}$ determine functorial extensions of the right $\text{Aut}(G)$-action on $\text{Ind}(G)$, while $(-)^{L}$ and $(-)^{R}$ determine functorial extensions of the corresponding left action.
8. $N_\infty$ Induction, Restriction, and Coinduction

The purpose of this section is to relate the algebraic adjunctions $f_L \dashv f_R^1$ and $f_L^{-1} \dashv f_R$ constructed last section to derived induction-restriction-coinduction adjunctions for $N_\infty$ operads (Theorem 8.9). We prove that for any homomorphism $f : G_1 \to G_2$ between finite groups, the adjunction $\text{res}_f \dashv \text{coind}_f$ derives (Theorem 8.6), and that $\mathbb{L}\text{res}_f \dashv \mathbb{R}\text{coind}_f$ corresponds to $f_L^{-1} \dashv f_R$ via the equivalence $\text{Ho}(N_\infty - \text{Op}) \simeq \text{Ind} \cong \text{Tr}$. We also prove that $\text{ind}_f \dashv \text{res}_f$ derives whenever $f$ is injective, and that $\text{Lind}_f \dashv \mathbb{R}\text{res}_f$ corresponds to $f_L \dashv f_R^{-1}$ in this case.

Our first obstacle is making sense of induction for $N_\infty$ operads. The category $\text{Op}_{\top}$ of operads in topological spaces is complete and cocomplete, and therefore the usual left and right Kan extensions determine induction and coinduction functors. In particular, if $\Theta$ is an $H$-operad for some subgroup $H \subset G$, then the induced $G$-operad $\text{ind}_H^G \Theta$ is a $G/H$-fold coproduct of copies of $\Theta$. Unfortunately, the functor $\text{ind}_H^G : \text{Op}_G^H \to \text{Op}_G^G$ does not preserve $N_\infty$ operads, because $\text{ind}_H^G \Theta$ does not have $G$-fixed points, and its fixed-point subspaces need not be contractible. One could imagine attaching a $G$-fixed operad and then killing all higher homotopy groups, but we take a different route. In [7], we show how to model $N_\infty$ operads in spaces using operads whose components are contractible 1-groupoids. With a bit more elaboration (cf. [8, §§3.5 – 3.6]), one can organize these categorical operads into a combinatorial category $\text{Op}(G)$, similar to those in [4] and [6]. The issues with the functor $\text{ind}_H^G : \text{Op}_G^H \to \text{Op}_G^G$ vanish once we replace $\text{Op}_G^H$ with a suitable slice category of $\text{Op}_G^G$.  

We begin with a review of some material in [8]. Let $\text{Set}$ and $\text{Cat}$ denote the categories of small sets and small categories. The object functor $\text{Ob} : \text{Cat} \to \text{Set}$ has a right adjoint $(-) : \text{Set} \to \text{Cat}$, which sends a set $X$ to the category $\mathcal{X}$ whose object set is $X$, and which has exactly one morphism $(x, y) : x \to y$ for every $x, y \in X$. The functor $(-)$ is full and faithful, and its essential image consists of those categories that are either empty or equivalent to the terminal category. Let $\mathcal{Set}$ denote the essential image. Then $\text{Ob}$ and $(-)$ induce an equivalence $\text{Set} \simeq \mathcal{Set}$. This observation allows us to work with homotopy-coherent structures as if they were discrete. For example, the category $\mathcal{Set}$ is complete and cocomplete, with limits and colimits computed by forgetting down to sets, computing the limit or colimit there, and then applying $(-)$. These are not homotopy limits and colimits in the standard sense, but they are useful.

Now suppose that $G$ is a finite group, and let $\text{Op}(G)$ be the category of operads in $\mathcal{Set}^G$. Since $\mathcal{Set} \subset \text{Cat}$ is naturally a 2-category, so is $\text{Op}(G)$, and its 2-morphisms are tuples of natural transformations that are compatible with the operad structure. Taking nerves gives a simplicial enrichment. Limits and colimits in $\text{Op}(G)$ are computed by forgetting down to operads in $G$-sets, computing limits and colimits there, and then applying $(-)$. Now let $\mathcal{SM}$ denote the free operad in $\text{Op}(G)$ generated by the symmetric sequence $G \times \Sigma_0/G \sqcup G \times \Sigma_2/G$. We define $\text{Op}(G)$ to be the slice category $\mathcal{SM}/\text{Op}(G)$. By adjunction, an object of $\text{Op}_*(G)$ is an operad $\Theta \in \text{Op}(G)$, equipped with a marked constant $u \in \Theta(0)^G$ and product $p \in \Theta(2)^G$, and all 1-morphisms and 2-morphisms in $\text{Op}_*(G)$ preserve markings.3

3The category $\text{Op}_*(G)$ is denoted $\text{Op}_{h,m}$ in [8].
Next, we make $\tilde{\text{Op}}_\ast(G)$ into a homotopical category. Declare a morphism $f : \mathcal{O} \to \mathcal{O}'$ between operads in $G$-spaces to be a weak equivalence if and only if $f : \mathcal{O}(n)^F \to \mathcal{O}'(n)^F$ is a weak homotopy equivalence for every $n \geq 0$ and subgroup $\Gamma \subset G \times \Sigma_n$ that intersects $\Sigma_n$ trivially. We call such $\Gamma$ graph subgroups. Declare a morphism $f : \mathcal{O} \to \mathcal{O}'$ in $\tilde{\text{Op}}_\ast(G)$ to be a weak equivalence if and only if the map $Bf : B\mathcal{O} \to B\mathcal{O}'$ is a weak equivalence. Equivalently, a map $f : \mathcal{O} \to \mathcal{O}'$ in $\tilde{\text{Op}}_\ast(G)$ is a weak equivalence if and only if the operads $\mathcal{O}$ and $\mathcal{O}'$ have the same admissible sets.

Now let $\text{Sym}(G)$ denote the category of symmetric sequences in $G$-sets. There is a free-forgetful adjunction $F : \text{Sym}(G) \rightleftarrows \tilde{\text{Op}}_\ast(G) : U$, where $U\mathcal{O}$ is the underlying symmetric sequence of $\text{Ob}\mathcal{O}$. The operad $FS$ is constructed from $S$ by adjoining $G \times \Sigma_0/G \sqcup G \times \Sigma_2/G$, generating a discrete free operad in $G$-sets, and then applying $(-)$ levelwise. Define sets $I$ and $J$ of generating cofibrations and generating acyclic cofibrations by

$$I := \left\{ F(\varnothing \to G \times \Sigma_n/\Gamma) \right\},$$

$$J := \left\{ F\left(G \times \Sigma_n/\Gamma \cong G \times \Sigma_n/\Gamma \sqcup G \times \Sigma_n/\Gamma \right) \right\},$$

where $n$ ranges over all nonnegative integers, $\Gamma$ ranges over all graph subgroups of $G \times \Sigma_n$, and $\iota_1$ is the inclusion into the first summand. The morphisms in $J$ are analogous to inclusions $D^n \hookrightarrow D^n \times [0,1]$, because $B(\ast \sqcup \ast) \cong S^n$.

**Theorem 8.1** ([8, Theorems 3.40 and 3.41]). Consider the category $\tilde{\text{Op}}_\ast(G)$, equipped with the preceding weak equivalences, generating cofibrations $I$, and generating acyclic cofibrations $J$. Then:

1. These data make $\tilde{\text{Op}}_\ast(G)$ is a locally finitely presentable, cofibrantly generated, right proper, simplicial model category.
2. Every object of $\tilde{\text{Op}}_\ast(G)$ is fibrant, every cofibrant object of $\tilde{\text{Op}}_\ast(G)$ is an $N_\infty$ operad (but not conversely), and every simplicial mapping space in $\tilde{\text{Op}}_\ast(G)$ is either empty or contractible.
3. Let $LB : \tilde{\text{Op}}_\ast(G) \to N_\infty\text{-Op}(\text{Top}^G)$ denote the composite of cofibrant replacement in $\tilde{\text{Op}}_\ast(G)$ with the classifying space functor. Then $LB$ induces a Dwyer-Kan equivalence on simplicial localizations.

In general, an operad $\mathcal{O} \in \tilde{\text{Op}}_\ast(G)$ is $N_\infty$ if and only if it is $\Sigma$-cofibrant in the sense of Berger and Moerdijk [1]. It follows from part (3) that the classification of $N_\infty$ operads applies equally well to the objects of $\tilde{\text{Op}}_\ast(G)$.

**Definition 8.2.** For any operad $\mathcal{O}$ in $G$-spaces or $G$-categories, let $\to_\mathcal{O}$ be the relation on $\text{Sub}(G)$ defined by

$$K \to_\mathcal{O} H \quad \text{if and only if} \quad \mathcal{O}(|H : K|)^{F_{W/K}} \neq \varnothing.$$

**Corollary 8.3.** For any $\mathcal{O} \in \tilde{\text{Op}}_\ast(G)$, the relation $\to_\mathcal{O}$ is a $G$-transfer system, and the functor $\to_\mathcal{O} : \text{Ho}(\tilde{\text{Op}}_\ast(G)) \to \text{Tr}(G)$ is an equivalence of categories.

**Proof.** Suppose that $\mathcal{O} \in \tilde{\text{Op}}_\ast(G)$ and let $Q\mathcal{O}$ be a cofibrant replacement of $\mathcal{O}$. Then the operads $LQ\mathcal{O} = BQ\mathcal{O}$, $Q\mathcal{O}$, and $\mathcal{O}$ have the same admissible sets. Therefore $\to_\mathcal{O} = \to_{LQ\mathcal{O}}$, and the latter is a transfer system because $LQ\mathcal{O}$ is an
\(N_\infty\) operad. It follows that the functor \(\to\) factors as a chain \(\text{Ho}(\\tilde{\text{Op}}_*(G)) \simeq \text{Ho}(N_\infty\text{-Op}) \simeq \text{Ind}(G) \simeq \text{Tr}(G)\).

We now consider point-set level induction, restriction, and coinduction adjunctions. Write \(\\tilde{\text{Op}}_* = \text{Op}_*(1)\) and \(\text{Sym} = \text{Sym}(1)\), where 1 denotes the trivial group. Then \(\text{Sym} = \text{Set}^\Sigma\), where \(\Sigma\) is the category \(\bigsqcup_{n \geq 0} B\Sigma_n\). The canonical isomorphism \(\text{Sym}(G) = (\text{Set}^G)^\Sigma \simeq (\text{Set}^\Sigma)^G = \text{Sym}^G\) lifts to an isomorphism \(\\tilde{\text{Op}}_*(G) \simeq \text{Op}_*\), and the category \(\text{Op}_*\) is bicomplete. Therefore we may construct induction, restriction, and coinduction via Kan extension and pullback.

**Definition 8.4.** Suppose that \(f : G \to G'\) is a homomorphism between finite groups and write \(Bf : BG \to BG'\) for the corresponding functor on one-object categories. Define adjoints \(\text{ind}_f \dashv \text{res}_f \dashv \text{coind}_f\) between \(\text{Sym}^G\) and \(\text{Sym}^{G'}\) and between \(\text{Op}^G\) and \(\text{Op}^{G'}\) by \(\text{ind}_f := \text{Lan}_{Bf}\), \(\text{res}_f := (Bf)^*\), and \(\text{coind}_f := \text{Ran}_{Bf}\).

More explicitly, the end and coend formulas imply that \(\text{coind}_f\) and \(\text{ind}_f\) are given by the usual equalizers and coequalizers

\[
\text{coind}_f X \cong \text{eq} \left( \prod_{G'} X \Rightarrow \prod_{G} \prod_{G'} X \right) \quad \text{and} \quad \text{ind}_f X \cong \text{coeq} \left( \prod_{G'} \prod_{G} X \Rightarrow \prod_{G'} X \right),
\]

where \(X\) is either an object of \(\text{Sym}\) or \(\tilde{\text{Op}}_*\), and all products and coproducts are taken in the corresponding category.

We are primarily concerned with deriving the adjunctions \(\text{ind}_f \dashv \text{res}_f \dashv \text{coind}_f\), and thus we shall analyze how they interact with the generating cofibrations of \(\tilde{\text{Op}}_*(G)\). To start, we consider how these functors interact with the free-forgetful adjunction \(F : \text{Sym}(G) \rightleftarrows \tilde{\text{Op}}_*(G) : U\). For every finite group \(G\), the adjunction \(F : \text{Sym} \rightleftarrows \tilde{\text{Op}}_* : U\) induces an adjunction \(F_* : \text{Sym}^G \rightleftarrows \tilde{\text{Op}}_*(G) : U_*\) by post-composition. The isomorphisms \(\text{Sym}(G) \cong \text{Sym}^G\) and \(\tilde{\text{Op}}_*(G) \cong \tilde{\text{Op}}_*(G)\) identify \(F_* \dashv U_*\) with the adjunction \(F : \text{Sym}(G) \rightleftarrows \tilde{\text{Op}}_*(G) : U\) used to construct the model structure on \(\tilde{\text{Op}}_*(G)\), and this implies the following commutation relations.

**Lemma 8.5.** Suppose that \(f : G \to G'\) is a homomorphism between finite groups, and write \(F : \text{Sym}(-) \rightleftarrows \tilde{\text{Op}}_*(-) : U\) for the free-forgetful adjunctions. Then there are natural isomorphisms

\[
\text{ind}_f \circ F \cong F \circ \text{ind}_f \quad , \quad \text{res}_f \circ F \cong F \circ \text{res}_f \\
\text{coind}_f \circ U \cong U \circ \text{coind}_f \quad , \quad \text{res}_f \circ U \cong U \circ \text{res}_f.
\]

**Proof.** The functor \(\text{res}_f\) commutes with \(F\) and \(U\) because pre-composition commutes with post-composition. The commutation relations for \(\text{ind}_f\) and \(\text{coind}_f\) follow from the uniqueness of adjoints.

Thus, we are reduced to studying \(\text{ind}_f\) and \(\text{res}_f\) on symmetric sequences. Thinking of the components of a \(G\)-symmetric sequence as \(G \times \Sigma_n\)-sets, we have

\[
(\text{res}_f S)_n \cong \text{res}_{f \times \text{id}} S'_n \quad \text{and} \quad (\text{ind}_f S)_n \cong \text{ind}_{f \times \text{id}} S_n
\]

for every homomorphism \(f : G \to G'\), \(S \in \text{Sym}(G)\), and \(S' \in \text{Sym}(G')\). We arrive the following result.

**Theorem 8.6.** Suppose that \(f : G \to G'\) is an arbitrary homomorphism between finite groups. Then:
(1) The pair $\text{res}_f : \overline{\text{Op}}_\ast(G') \rightleftarrows \overline{\text{Op}}_\ast(G) : \text{coind}_f$ is a Quillen adjunction.

(2) The pair $\text{ind}_f : \overline{\text{Op}}_\ast(G) \rightleftarrows \overline{\text{Op}}_\ast(G') : \text{res}_f$ is a Quillen adjunction if and only if the map $f$ is injective.

Proof. We consider the $\text{res}_f \rightleftarrows \text{coind}_f$ adjunction first. Suppose that the morphism $i = F(\varnothing \to G' \times \Sigma_n / \Gamma')$ is a generating cofibration of $\overline{\text{Op}}_\ast(G')$. By Lemma 8.5, we have $\text{res}_f(i) \cong F(\varnothing \to \text{res}_f \times \text{id}(G' \times \Sigma_n / \Gamma'))$. The $G \times \Sigma_n$-set $\text{res}_f \times \text{id}(G' \times \Sigma_n / \Gamma')$ is $\Sigma_n$-free because $\text{res}_f \times \text{id}$ does not change the $\Sigma_n$-action, and therefore it splits as

$$\text{res}_f \times \text{id}(G' \times \Sigma_n / \Gamma') \cong \coprod_{k=1}^m G \times \Sigma_n / \Gamma_k$$

for some graph subgroups $\Gamma_1, \ldots, \Gamma_m \subset G \times \Sigma_n$. Since $F$ commutes with coproducts, we deduce that $\text{res}_f(i) \cong \coprod_{k=1}^m F(\varnothing \to G \times \Sigma_n / \Gamma_k)$, which is a cofibration of $\overline{\text{Op}}_\ast(G)$. Inducting up cell complexes and passing to retracts proves that $\text{res}_f$ preserves all cofibrations. An analogous argument shows that $\text{res}_f$ also preserves acyclic cofibrations, and therefore $\text{res}_f \rightleftarrows \text{coind}_f$ is a Quillen adjunction.

Now consider the adjunction $\text{ind}_f \rightleftarrows \text{res}_f$. If $f$ is injective, then we may assume $f : G \hookrightarrow G'$ is the inclusion of a subgroup, and that $\text{ind}_f \times \text{id} = \text{ind}_{G' \times \Sigma_n}$ is induction in the usual sense. Arguing as above proves that $\text{ind}_f$ is left Quillen.

Now suppose that $f$ is not injective. We shall prove that the functor $\text{ind}_f : \overline{\text{Op}}_\ast(G) \to \overline{\text{Op}}_\ast(G')$ does not preserve all cofibrant objects. By Theorem 8.1, it is enough to find a cofibrant operad $\mathcal{O} \in \overline{\text{Op}}_\ast(G)$ such that $\text{ind}_f \mathcal{O}$ is not $\Sigma$-free.

Let $\Gamma_G$ be the graph of a permutation representation $\sigma : G \to \Sigma[G]$ for $G/e$, and let $\mathcal{O} = F(G \times \Sigma[G] / \Gamma_G)$. Then $\text{ind}_f \mathcal{O} \cong \text{Find}_f \mathcal{O} = F_{G \times \Sigma[G]}(G \times \Sigma[G] / \Gamma_G)$, and $S = \text{ind}_f \mathcal{O}$ is not $\Sigma$-free. Hence $\text{ind}_f \mathcal{O}$ also is not $\Sigma$-free because it receives a unit map from $S$.

Remark 8.7. Theorem 8.1 states that every object of $\overline{\text{Op}}_\ast(G)$ is fibrant. Therefore $\text{coind}_f$ preserves all weak equivalences, and hence $\mathbb{R}\text{coind}_f \cong \text{Ho}(\text{coind}_f)$ and $\mathbb{L}\text{res}_f \cong \text{Ho}(\text{coind}_f)$. If $f$ is injective, then also $\text{res}_f$ preserves all weak equivalences, and therefore $\mathbb{L}\text{res}_f \cong \text{Ho}(\text{res}_f) \cong \mathbb{R}\text{res}_f$ and $\text{Ind}_f \cong \text{Ho}(\text{res}_f) \cong \text{Ho}(\text{coind}_f)$.

Now that we know how to derive the operadic adjunctions $\text{ind}_f \rightleftarrows \text{res}_f \rightleftarrows \text{coind}_f$, we can relate them to the adjunctions $f_L \dashv f_R^{-1}$ and $f_L^{-1} \dashv f_R$ in 57. First, a calculation.

Lemma 8.8. Suppose that $f : G_1 \to G_2$ is a homomorphism between finite groups, let $H \subset G_2$ be a subgroup, and let $T$ be a $H$-set of finite cardinality $n$. Write $\Gamma(T)$ for the graph of a permutation representation of $T$. Then

$$\text{res}_f \times \text{id}(G_2 \times \Sigma_n / \Gamma(T)) \cong \coprod_{r \in \text{im}(f) \times \Sigma_n \backslash G_2 \times \Sigma_n / \Gamma(T)} G_1 \times \Sigma_n / \Gamma(f^* \text{res}_r \text{res}_f \text{H}_r^{-1} \cap \text{im}(f) c_r T),$$

where the double-coset representatives $r$ are taken in the subgroup $G_2 \times \{\text{id}\}, c_r T$ is the conjugate $r H r^{-1}$-action to $T$, and $f^* \text{res}_r \text{H}_r^{-1} \cap \text{im}(f) c_r T$ is the $f^{-1}(r H r^{-1})$-action obtained by pulling back the $r H r^{-1} \cap \text{im}(f)$-action on $\text{res}_r \text{H}_r^{-1} \cap \text{im}(f) c_r T$.

Proof. Compute $\text{res}_f \times \text{id}$ by first restricting to $\text{im}(f) \times \Sigma_n$ and applying the double-coset formula, and then pull back along $f \times \text{id} : G_1 \times \Sigma_n \to \text{im}(f) \times \Sigma_n$. \qed
Theorem 8.9. Suppose that \( f : G_1 \to G_2 \) is an arbitrary homomorphism between finite groups. Then the squares
\[
\begin{array}{ccc}
\text{Ho}(\overline{\text{Op}}_*(G_1)) & \xrightarrow{\text{Lres}_f} & \text{Ho}(\overline{\text{Op}}_*(G_2)) \\
\Downarrow \text{Tr}(G_1) & \xrightarrow{f_L^{-1}} & \Downarrow \text{Tr}(G_2)
\end{array}
\quad \begin{array}{ccc}
\text{Ho}(\overline{\text{Op}}_*(G_1)) & \xrightarrow{\text{Ho}(\text{coind}_f)} & \text{Ho}(\overline{\text{Op}}_*(G_2)) \\
\Downarrow \text{Tr}(G_1) & \xrightarrow{f_R} & \Downarrow \text{Tr}(G_2)
\end{array}
\]

commute. Suppose additionally that the map \( f \) is injective. Then the squares
\[
\begin{array}{ccc}
\text{Ho}(\overline{\text{Op}}_*(G_1)) & \xrightarrow{\text{Lind}_f} & \text{Ho}(\overline{\text{Op}}_*(G_2)) \\
\Downarrow \text{Tr}(G_1) & \xrightarrow{f_L} & \Downarrow \text{Tr}(G_2)
\end{array}
\quad \begin{array}{ccc}
\text{Ho}(\overline{\text{Op}}_*(G_1)) & \xrightarrow{\text{Ho}(\text{res}_f)} & \text{Ho}(\overline{\text{Op}}_*(G_2)) \\
\Downarrow \text{Tr}(G_1) & \xleftarrow{f_R^{-1} = f_L^{-1}} & \Downarrow \text{Tr}(G_2)
\end{array}
\]
also commute.

Proof. We check the equation \( \rightarrow \circ \text{Lind}_f = f_L^{-1} \circ \rightarrow \) directly, and then deduce the others from the uniqueness of adjoints. Suppose that \( \Theta \in \overline{\text{Op}}_*(G_2) \), and let
\[ Q^\Theta = F \left( \prod_{K \to \Theta H} G_2 \times \Sigma_{[H:K]} / \Gamma(H/K) \right). \]

Then \( Q^\Theta \) is a cofibrant replacement for \( \Theta \), because the class of admissible sets of \( Q^\Theta \) is generated by the admissible orbits of \( \Theta \) (cf. [7, Theorem 2.16]). By Lemma 8.5 and Lemma 8.8, the operad \( \text{res}_f(Q^\Theta) \) is isomorphic to
\[
F \left( \prod_{r \in \text{im}(f) \times \Sigma_{[H:K]} / G_2 \times \Sigma_{[H:K]} / \Gamma(H/K)} G_1 \times \Sigma_{[H:K]} / \Gamma(f^* \text{res}^* r H_{H^{-1}} \cap \text{im}(f) r H/K) \right).
\]

We now compute \( \rightarrow_{\text{Lres}_f \Theta} \) by identifying the admissibles of \( \text{Lres}_f \Theta \simeq \text{res}_f(Q^\Theta) \), and then passing to transfer systems. Applying [7, Theorem 2.16] once more, we see that the class of admissible sets of \( \text{res}_f Q^\Theta \) is the indexing system generated as
\[
\left\{ f^* \text{res}^* r H_{H^{-1}} \cap \text{im}(f) G_1 H/K \mid r \in \text{im}(f) \times \Sigma_{[H:K]} / G_2 \times \Sigma_{[H:K]} / \Gamma(H/K), K \to \Theta H \text{ and } H \subset \text{im}(f) \right\}.
\]

Then since indexing systems are closed under conjugation, restriction, and subobjects, and \( f^* \) commutes with coproducts, this expression simplifies to
\[
\langle f^* H/K \mid K \to \Theta H \text{ and } H \subset \text{im}(f) \rangle = \langle f^{-1} H/f^{-1} K \mid K \to \Theta H \rangle.
\]

By Proposition A.4, the associated transfer system is \( \langle (f^{-1} K, f^{-1} H) \mid K \to \Theta H \rangle \), which is precisely \( f_L^{-1}(\Theta) \). This proves that \( \rightarrow_{\text{Lres}_f \Theta} = f_L^{-1}(\Theta) \) for every \( \Theta \in \overline{\text{Op}}_*(G_2) \), and the equality \( \rightarrow \circ \text{Lres}_f = f_L^{-1} \circ \rightarrow \) of functors follows because parallel morphisms in the poset \( \text{Tr}(G_1) \) are equal.

Now let \( \rightarrow^{-1} \) be a pseudoinverse to \( \rightarrow \). The equation \( \rightarrow \circ \text{Lres}_f = f_L^{-1} \circ \rightarrow \) implies an isomorphism \( \text{Ho}(\text{coind}_f) \circ \rightarrow^{-1} \cong \rightarrow^{-1} \circ f_R \) of right adjoints, and hence...
→* Ho(coind_f) ≅ f_R ◦ →* as well. Since Tr(G_2) is a poset, we actually have an equality →* Ho(coind_f) = f_R ◦ →*.

Suppose further that the morphism f : G_1 → G_2 is injective. Then f_L^{-1} = f_R^{-1} by Theorem 7.12, and Lres_f ≅ Ho(res_f) ≅ Res_f by Remark 8.7. Therefore →* Ho(res_f) = f_R ◦ →*, and the equality →* Lind_f = f_L ◦ →* for left adjoints follows as above. □

**Corollary 8.10.** Keep notation as above. If θ ∈ Op_{∞}(G_2) is an N_{∞} operad, then Lres_f θ ≅ res_f θ and →_res_f θ = f_L^{-1} (→ θ).

**Proof.** Let q : Qθ → θ be a cofibrant replacement. Since Qθ and θ are Σ-free, the map q induces an equivalence q : Qθ(n)≡ θ(n)≡ for every n ≥ 0 and every subgroup Ξ ⊂ G_2 × Σ_n. Therefore q : Qθ(n)(f × id)Γ → θ(n)(f × id)Γ is an equivalence for every graph subgroup Γ ⊂ G_1 × Σ_n, so that res_f(q) : Lres_f θ ≅ res_f Qθ → res_f θ is an equivalence. Therefore →_res_f θ = →_Lres_f θ = f_L^{-1} (→ θ).

Following is a quick application of Theorem 8.9.

**Example 8.11.** Suppose that f : G → G’ is a homomorphism between finite groups, and that θ ∈ Op_{∞}(G) is genuine G-E_{∞}, i.e. →_θ is the maximum transfer system. Then coinind_f θ is genuine G’-E_{∞}, because →_coinind_f θ = f_R (→ θ), and the right adjoint f_R preserves terminal transfer systems.

Now specialize to the case that f is the unique map ! : 1 → G, and θ is the categorical Barratt-Eccles operad P(n) = Σ_n ∈ Op_{∞}(1), marked with the operations id_0 ∈ P(0) and id_2 ∈ P(2). Then (coinind(P))(n) ≅ Prod_G Σ_n ⊕ Fun(G^{disc}, Σ_n) ≅ Fun(1, Σ_n). The G-category Fun(G, Σ_n) is the nth component of the G-Barratt-Eccles operad P_G studied in [5] and in subsequent work of Guillou-May-Merling-Osorno. Since the operad P is genuine 1-E_{∞}, we recover the well-known fact that P_G is genuine G-E_{∞}.

We end with some space-level consequences of the above. Part (2) of the next result specializes to [2, Proposition 6.16] when f is the inclusion of a subgroup.

**Proposition 8.12.** Suppose that f : G_1 → G_2 is a homomorphism between finite groups. Then:

1. If θ is an N_{∞}-operad in G_2-spaces, then →_res_f θ = f_L^{-1} (→ θ).
2. If θ is an N_{∞}-operad in G_1-spaces, then →_coinind_f θ = f_R (→ θ).

**Proof.** We start with (1). Suppose that θ is an N_{∞} operad in G_2-spaces, and choose a categorical N_{∞} operad N ∈ Op_{∞}(G_2) with the same admissible sets. Then the operads B_N and θ × B_N are also N_{∞}, and both projections in the product diagram θ ← θ × B_N → B_N are weak equivalences. As in the proof of Corollary 8.10, applying res_f gives a zig-zag res_f θ ← res_f (θ × B_N) → res_f B_N ≅ Bres_f N of weak equivalences. Therefore →_res_f θ = →_Bres_f N = →_res_f N, because B preserves admissible sets, and the chain →_res_f N = f_L^{-1} (→ θ) follows from Corollary 8.10 and the choice of N.

The proof of (2) goes the same way, once we know that coinind_f preserves weak equivalences and commutes with B. The functor coinind_f preserves all weak equivalences, because given any graph subgroup Γ ⊂ G_2 × Σ_n, if we write

res_{f × id}(G_2 × Σ_n/Γ) ≅ \prod_{k=1}^{m} G_1 × Σ_n/Γ_k

for graph subgroups $\Gamma_1, \ldots, \Gamma_m \subset G_1 \times \Sigma_n$, as in the proof of Theorem 8.6, then $(\text{coind}_f \theta)(n)^k$ is naturally isomorphic to $\prod_{k=1}^m \theta(n)^k$. Coinduction commutes with $B$ because the functor $B$ preserves finite limits. Now we argue as before. \hfill \Box

These arguments do not yield an analogous result for $\text{ind}_f$, because the classifying space functor does not commute with operadic induction.

**Appendix A. Generating indexing systems**

The purpose of this appendix is to describe how to generate an indexing system from a prescribed set of orbits. Construction A.1 gives an explicit presentation on earlier sections of the paper. and the remainder of this appendix specializes to cases that were of interest in

**Construction A.1.** Suppose that $G$ is a finite group, and that $R$ is a binary relation on $\text{Sub}(G)$ that refines inclusion, i.e. if $KRH$, then $K \subset H$. Define

\[
R_0 := R,
R_1 := \{(gKg^{-1}, gHg^{-1}) \mid KR_0H \text{ and } g \in G\},
R_2 := \{(L \cap K, L) \mid \text{there are subgroups } K \subset H \supset L \text{ such that } KR_1H\},
R_3 := \{(K, H) \mid \text{there is } n \geq 0 \text{ and subgroups } H_0, H_1, \ldots, H_n \text{ such that } K = H_0R_2H_1R_2 \cdots R_2H_n = H\} = \text{rt}(R_2)
\]

Thus, we close $R$ under conjugation to get $R_1$, we close $R_1$ under restriction to get $R_2$, and we take the reflexive and transitive closure of $R_2$ to get $R_3$.

**Theorem A.2.** Suppose that $R$ is a binary relation on $\text{Sub}(G)$ that refines inclusion. Then $\langle R \rangle := R_3$ is the transfer system generated by $R$, i.e. $R_3$ is the least transfer system that is refined by $R$.

**Proof.** Let $R$ be a binary relation on $\text{Sub}(G)$ that refines inclusion. Then $R = R_0 \subset R_1 \subset R_2 \subset R_3$, and if $S$ is any $G$-transfer system that contains $R$, then its closure properties imply that it must also contain $R_3$. Thus, the argument will be complete once we prove that $R_3$ is a transfer system.

To start, observe that $R_2$ is closed under conjugation and restriction, and that it refines inclusion. Now consider $R_3$. It is a preorder by construction, and it refines inclusion because $R_2$ does. Therefore $R_3$ is also antisymmetric. Conjugating $R_2$-chains proves that $R_3$ is closed under conjugation. To see that $R_3$ is closed under restriction, suppose that $H_0R_2H_1R_2 \cdots R_2H_n$ and let $L \subset H_n$. Define $L_i = L \cap H_i$. Restricting the relation $H_iR_2H_{i+1}$ to $L_{i+1}$ yields $L_i = (L_{i+1} \cap H_i)R_2L_{i+1}$ for $0 \leq i < n$. Therefore we obtain the chain $(L \cap H_0) = L_0R_2L_1R_2 \cdots R_2L_n = L$. \hfill \Box

Theorem A.2 implies the following rough bounds on the transfer system $\langle R \rangle$.

**Proposition A.3.** Let $R$ be a binary relation on $\text{Sub}(G)$ that refines inclusion, and let $N \subset G$ be a normal subgroup.

1. Suppose that for every relation $KRH$, we have $H \subset N$. Then $H \subset N$ for every nontrivial relation $(K, H) \in \langle R \rangle$.
2. Suppose that for every relation $KRH$, we have $N \subset K$. Then $H \not\subset N$ for every nontrivial relation $(K, H) \in \langle R \rangle$. 
Proof. We start with (1). Assume that $KR_0H$ implies $H \subset N$. Then $KR_1H$ implies $H \subset N$, because $N$ is normal, and $KR_2H$ implies $H \subset N$ from the transitivity of $\subset$. Finally, if $(K, H) \in R_3$ is nontrivial, then there is a chain $K = H_0R_2H_2R_3 \cdots R_2H_n = H$ with $n > 0$, and $H_{n-1}R_2H_n$ implies $H = H_n \subset N$.

Now consider (2). Assume that $KR_0H$ implies $N \subset K$. Then $KR_1H$ implies $N \subset K$ because $N$ is normal. Now suppose that $KR_2H$. We shall prove that if $H \subset N$, then $K = H$. Indeed, there are subgroups $K' \subset H' \supset L'$ such that $K'R_1H'$ and $(K, H) = (L' \cap K', L')$. If $H \subset N$, then $L' = H \subset N \subset K'$ and therefore $K = L' \cap K' = L' = H$. Finally, we prove that for every $(K, H) \in R_3$, if $H \subset N$, then $K = H$. For suppose $K = H_0R_2H_1R_2 \cdots R_2H_n = H \subset N$ for $n > 0$. If $n = 0$, there is nothing to check. If $n > 0$, then since $R_2$ refines inclusion, we have $H_{i+1} \subset N$ and $H_iR_2H_{i+1}$ for every $0 \leq i < n$. It follows from the above that $K = H_0 = H_1 = \cdots = H_n = H$. □

Now suppose that $O$ is a set of orbits $H/K$, for subgroups $K \subset H \subset G$. Combining Theorems A.2 and 6.6 to gives an explicit description of the indexing system $\langle O \rangle$ generated by $O$. Define the graph $\rightarrow_o$ of $O$ exactly as in Definition 6.4:

$$K \rightarrow_o H$$

if and only if $H/K \in O$.

Thus $\rightarrow_o$ is a binary relation on $\text{Sub}(G)$ that refines inclusion.

**Proposition A.4.** Suppose that $O$ is a set of orbits, $\rightarrow$ is a transfer system, and write $I_{\rightarrow}$ for the indexing system associated to $\rightarrow$ (cf. Proposition 6.5). Then $I_{\rightarrow_o}$ is the indexing system generated by $O$. Equivalently, $\langle O \rangle = \langle \rightarrow_o \rangle$.

**Proof.** For any indexing system $I$, we have:

$$O \subset I \iff \rightarrow_o \text{ refines } I \iff \langle O \rangle \text{ refines } I \iff I_{\rightarrow_o} \subset I.$$

Taking $I = I_{\rightarrow_o}$ proves that $O$ is contained in the indexing system $I_{\rightarrow_o}$, and the equivalence of (i) and (iv) proves that $I_{\rightarrow_o}$ is the least such indexing system. □

**Corollary A.5.** Suppose that $O$ is a set of orbits, and let $\langle O \rangle$ be the indexing system that it generates. Then $H/K \in \langle O \rangle$ if and only if $K \rightarrow H$ in $\langle \rightarrow_o \rangle$.

Thus, we can compute the orbits in $\langle O \rangle$ by running through the description of $\langle \rightarrow_o \rangle$ in Construction A.1. The next results do this in some cases of interest. We start by considering the indexing system generated by a single, well-behaved orbit.

**Proposition A.6.** Suppose that $G$ is a finite group and that $H, K \subset G$ are normal subgroups of $G$ such that $K \subset H$. Let $\mathcal{I} = \langle H/K \rangle$ be the $G$-indexing system generated by the single orbit $H/K$. Then the orbits of $\langle H/K \rangle$ are precisely the trivial orbits $L/L$ for $L \subset G$, and the orbits of the form $L/L \cap K$ for $L \subset H$.

**Proof.** We compute the transfer system generated by the single relation $K \rightarrow H$. Let $R = \{(K, H)\}$ and keep notation as in Construction A.1. Then $R_1 = R$ because $K$ and $H$ are normal in $G$, and $R_2 = \{(L \cap K, L) | L \subset H\}$. Finally, $R_3 = \{(L, L) | L \subset G\} \cup R_2$. Indeed, the inclusion $\supset$ holds because $R_3$ is a reflexive relation containing $R_2$. On the other hand, given any nontrivial $R_2$-chain $L_0R_2L_1R_2 \cdots R_2L_n$ with $n > 0$, we must have $L_i = L_{i+1} \cap K$ for every $i < n$. It follows that $L_0 = L_n \cap K$, so that $(L_0, L_n) = (L_n \cap K, L_n) \in R_2$. We conclude by noting that $R_3 = \langle \rightarrow_{\langle H/K \rangle} \rangle$, and applying Corollary A.5. □
We usually apply Proposition A.6 when \( G \) is abelian, or when \( H/K = G/\{e\} \). The situation is less straightforward when \( H \) and \( K \) are not normal in \( G \). Recall that the normal core of a subgroup \( K \subset G \) is the intersection of all conjugates of \( K \) in \( G \), while the normal closure of \( K \) is their join.

**Proposition A.7.** Suppose that \( K \subseteq H \subset G \) are subgroups of a finite group \( G \), and let \( N \) be the normal core of \( K \) in \( H \). Then \( H/N \in \langle H/K \rangle \).

**Proof.** Consider the transfer system \( \rightarrow = \langle (K, H) \rangle \), and let \( K = K_1, \ldots, K_n \) be the conjugates of \( K \) in \( H \). Then \( K \to H \), and hence \( K_i \to H \) for every \( 1 \leq i \leq n \). Restricting \( K_m \to H \) along \( \bigcap_{i=1}^{m-1} K_i \subset H \) shows that \( \bigcap_{i=1}^{m} K_i \to \bigcap_{i=1}^{m-1} K_i \). Therefore \( N = \bigcap_{i=1}^{n} K_i \to \bigcap_{i=1}^{n-1} K_i \to \cdots \to K_1 \to H \), so that \( N \to H \) and \( H/N \in \langle H/K \rangle \). \( \square \)

Proposition A.3 gives additional bounds on \( \langle H/K \rangle \) in terms of the normal closure of \( H \) and the normal core of \( K \) in \( G \).

As explained in §2, every indexing system that comes from a little discs operads is generated by \( G \)-orbits. If \( G \) is finite abelian, then such indexing systems have a quick description. We prove a slightly more general result.

**Proposition A.8.** Suppose that \( G \) is finite abelian, that \( H \subset G \) is a subgroup, and that \( \mathcal{O} = \{ H/K_1, \ldots, H/K_n \} \) is a set of \( H \)-orbits. Then the orbits of \( \langle \mathcal{O} \rangle \) are precisely the trivial orbits \( L/L \) for subgroups \( L \subset G \), and the orbits of the form \( L/(L \cap K_{i_1} \cap K_{i_2} \cdots \cap K_{i_m}) \), where \( 1 \leq i_1, i_2, \ldots, i_m \leq n \) and \( L \subset H \).

**Proof.** We compute the transfer system generated by \( R = \{(K_1, H), \ldots, (K_n, H)\} \), and then apply Corollary A.5. We have \( R_1 = R \) because \( G \) is abelian, and \( R_2 = \{(L \cap K_i, L) \mid L \subset H \text{ and } i = 1, \ldots, n\} \). It remains to identify \( \langle \to \mathcal{O} \rangle = R_3 \).

We claim that
\[
R_3 = \{(L, L) \mid L \subset G\} \cup \{(L \cap K_{i_1} \cap \cdots \cap K_{i_m}, L) \mid 1 \leq i_1, \ldots, i_m \leq n \text{ and } L \subset H\}.
\]
Indeed, we have \((L, L) \in R_3\) because \( R_3 \) is reflexive. Next, if \( 1 \leq i_1, \ldots, i_m \leq n \) and \( L \subset H \), then setting \( L_0 = L \) and \( L_j = L \cap K_{i_1} \cap \cdots \cap K_{i_j} \) for \( j = 1, \ldots, m \) shows \( L_{j+1} = (L_j \cap K_{i_{j+1}})R_3L_j \) for \( j = 0, \ldots, m-1 \), because \( L_j \subset H \). Therefore \( L_mR_3L_{m-1}R_3L_0 \), and therefore \( (L \cap K_{i_1} \cap \cdots \cap K_{i_m}) = L_mR_3L_0 = L \). This proves the inclusion \( \supseteq \).

Conversely, suppose that \( MR_3M' \). Either \( M = M' \), or there is a nontrivial chain \( M = M_0R_2M_1R_2 \cdots R_2M_m = M' \) where \( m > 0 \). In the latter case, we find that for every \( j = 1, \ldots, m \): \( M_j \subset H \) and \( M_{j-1} = M_j \cap K_{i_j} \) for some \( 1 \leq i_j \leq n \). Therefore \( M = M_0 = M_1 \cap K_{i_1} = \cdots = M_m \cap K_{i_1} \cap \cdots \cap K_{i_m} = M' \cap K_{i_1} \cap \cdots \cap K_{i_m} \), and \( M' \subset H \). This proves the inclusion \( \subseteq \). \( \square \)

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University of California Los Angeles, Los Angeles, CA 90095

E-mail address: jrubin@math.ucla.edu