PERIODIC ORBITS FOR PERIODIC ECO-EPIEDEMILOGICAL
SYSTEMS WITH INFECTED PREY

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Abstract. We address the existence of periodic orbits for periodic eco-epidemiological system with disease in the prey. To do it, we consider three main steps. Firstly we study a one parameter family of systems and obtain uniform bounds for the components of any periodic solution of these systems. Next, we make a suitable change of variables in our family of systems to establish the setting where we are able to apply Mawhin’s continuation Theorem. Finally, we use Mawhin’s continuation Theorem to obtain our result. Later on, we present two examples that include previous results in the literature and some numerical simulations to illustrate our results.

1. Introduction

Eco-epidemiological models are ecological models that include infected compartments. In many situations, these models describe more accurately the real ecological system than models where the disease is not taken into account.

There is already a large number of works concerning eco-epidemiological models. To mention just a few recent works, we refer [4] where a mathematical study on disease persistence and extinction is carried out; [5] where the authors study the global stability of a delayed eco-epidemiological model with holling type III functional response, and [2] where an eco-epidemiological model with harvesting is considered.

One of the main concerns when studying eco-epidemiological models is to determine conditions under which one can predict if the disease persists or dies out. In mathematical epidemiology, these conditions are usually given in terms of the so called basic reproduction ratio \( R_0 \), defined in [8] for autonomous systems as the spectral radius of the next generation matrix.

In [7], \( R_0 \) was introduced for the periodic models, and later on, in [10], the definition of \( R_0 \) was adapted to the study of periodic patchy models. In the recent article [6] the theory in [10] was used in the study of persistence of the predator in a general periodic predator-prey models.

When persistence is guaranteed, the obtention of conditions that assure the existence of periodic orbits for periodic eco-epidemiological models is an important issue in the deepening of the description of these models since these orbits correspond to situations where possibly there is some equilibrium in the described ecological system, reflected in the fact that the behaviour of the theoretical model is the same over the years. In [3] it was proved that there is an endemic periodic
orbit for the periodic version of the model considered in [11] when the infected prey is permanent and some additional conditions are fulfilled, partially giving a positive answer to a conjecture in this last paper.

The models in [11] and [3] assume that there is no predation on uninfected preys. In spite of that, this assumption is not suitable for the description of many eco-epidemiological models. The main purpose of this paper is to present some results on the existence of an endemic periodic orbit for periodic eco-epidemiological systems with disease in the prey that generalize the systems in [11] and [3] by including in the model a general function corresponding to the predation of uninfected preys. The proof of our result relies on Mawhin’s continuation theorem. Following the approach in [3], we begin by locating the components of possible periodic orbits for the one parameter family of systems that arise in Mawhin’s result, allowing us to check that the conditions of that theorem are fulfilled. Although the main steps in our proof correspond to the ones in [3], several additional nontrivial arguments are needed in our case. Additionally, there is also a substantial difference between our approach and the one in [11, 3]. In fact, we take as a departure point some prescribed behaviour of the uninfected subsystem, corresponding to the dynamics of preys and predators in the absence of disease: we will assume in this work that we have global asymptotic stability of solutions of some special perturbations of the bidimensional predator-prey system (the system obtained by letting $I = 0$ in the first and third equations in (1)). Thus, when applying our results to particular situations, one must verify that the underlying uninfected subsystem satisfies our assumptions. On the other hand, our approach allows us to construct an eco-epidemiological model from a previously studied predator-prey model (the uninfected subsystem) that satisfies our assumptions. This approach has the advantage of highlighting the link between the dynamics of the eco-epidemiological model and the dynamics of the predator-prey model used in its construction.

2. A GENERAL ECO-EPIDEMIOLOGICAL MODEL WITH DISEASE IN PREY

As a generalization of the model considered in [3], a periodic version of the general non-autonomous model introduced in [11], we consider the following periodic eco-epidemiological model:

\[
\begin{align*}
S' &= \Lambda(t) - \mu(t)S - a(t)f(S, P)P - \beta(t)SI \\
I' &= \beta(t)SI - \eta(t)PI - c(t)I \\
P' &= (r(t) - b(t)P)P + \gamma(t)a(t)f(S, P)P + \theta(t)\eta(t)PI
\end{align*}
\]

(1)

where $S$, $I$ and $P$ correspond, respectively, to the susceptible prey, infected prey and predator, $\Lambda(t)$ is the recruited rate of the prey population, $\mu(t)$ is the natural death rate of the prey population, $a(t)$ predation rate of susceptible prey, $\beta(t)$ is the incidence rate, $\eta(t)$ is the predation rate of infected prey, $c(t)$ is the death rate in the infective class ($c(t) \geq \mu(t)$), $\gamma(t)$ is the rate converting susceptible prey into predator (biomass transfer), $\theta(t)$ is the rate of converting infected prey into predator, $r(t)$ and $b(t)$ are parameters related the vital dynamics of the predator population that is assumed to follow a logistic law and includes the intra-specific competition between predators. It is assumed that only susceptible preys $S$ are capable of reproducing, i.e., the infected prey is removed by death (including natural and disease-related death) or by predation before having the possibility of reproducing.
Given a $\omega$-periodic function $f$ we will use throughout the paper the notations $f^c = \inf_{t \in [0, \omega]} f(t)$, $f^u = \sup_{t \in [0, \omega]} f(t)$ and \( \bar{f} = \frac{1}{\omega} \int_0^\omega f(s) \, ds \). We will assume the following structural hypothesis concerning the parameter functions and the function $f$ appearing in our model:

\begin{itemize}
  \item [S1)] The real valued functions $\Lambda$, $\mu$, $\beta$, $\eta$, $c$, $\gamma$, $r$, $\theta$ and $b$ are periodic with period $\omega$, nonnegative and continuous;
  \item [S2)] Function $f$ is nonnegative and continuous;
  \item [S3)] Function $x \mapsto f(x, z)$ is nondecreasing;
  \item [S4)] Function $z \mapsto f(x, z)$ is nonincreasing;
  \item [S5)] For all $(x, z)$ we have $\beta \frac{\partial f}{\partial x}(x, z) + \eta \frac{\partial f}{\partial z}(x, z) \geq 0$;
  \item [S6)] For any $C_1, C_2 > 0$, function $x \mapsto f(C_1 x + C_2, x)$ is nondecreasing;
  \item [S7)] $\bar{\Lambda} > 0$, $\bar{\mu} > 0$, $\bar{r} > 0$ and $\bar{b} > 0$.
\end{itemize}

To formulate our next assumptions we need to consider two auxiliary equations and one auxiliary system. First, for each $\lambda \in (0, 1]$, we need to consider the following equations:

\begin{equation}
  x' = \lambda (\Lambda(t) - \mu(t)x)
\end{equation}

and

\begin{equation}
  z' = \lambda (r(t) - b(t)z)z.
\end{equation}

Note that, if we identify $x$ with the susceptible prey population, equation \( 2 \) gives the behavior of the susceptible preys in the absence of infected preys and predator and identifying $z$ with the predator population, equation \( 3 \) gives the behavior of the predator in the absence of preys.

Equations \( 2 \) and \( 3 \) have a well known behavior, given in the following lemmas:

**Lemma 1** (Lemma 1 in \cite{11}). For each $\lambda \in (0, 1]$ there is a unique $\omega$-periodic solution of equation \( 2 \), $x^*_\lambda(t)$, that is globally asymptotically stable in $\mathbb{R}^+$.  

**Lemma 2** (Lemma 2 in \cite{11}). For each $\lambda \in (0, 1]$ there is a unique $\omega$-periodic solution of equation \( 3 \), $z^*_\lambda(t)$, that is globally asymptotically stable in $\mathbb{R}^+$.  

For each $\lambda \in (0, 1]$, we also need to consider the next family of systems, which correspond to behavior of the preys and predators in the absence of infected preys (system \( 4 \) with $I = 0$, $S = x$ and $P = z$):

\begin{align*}
  x' &= \lambda (\Lambda(t) - \mu(t)x - a(t)f(x, z)z - \varepsilon_1 x) \\
  z' &= \lambda (\gamma(t)a(t)f(x, z) + r(t) - b(t)z + \varepsilon_2)z
\end{align*}

\begin{equation}
  \tag{4}
\end{equation}

We now make our last structural assumption on system \( 4 \):

\begin{itemize}
  \item [S9)] For each $\lambda \in (0, 1]$ and each $\varepsilon_1, \varepsilon_2 \geq 0$ sufficiently small, system \( 4 \) has a unique $\omega$-periodic solution, $(x^*_\lambda, x^*_\varepsilon_1, x^*_\varepsilon_2)(t)$, with $x^*_\lambda, x^*_\varepsilon_1, x^*_\varepsilon_2(t) > 0$ and $z^*_\lambda, z^*_\varepsilon_1, z^*_\varepsilon_2(t) > 0$, that is globally asymptotically stable in the set $\{(x, z) \in (\mathbb{R}^+)_0^2 : x \geq 0 \land z > 0\}$. We assume that $(\varepsilon_1, \varepsilon_2) \mapsto (x^*_\lambda, x^*_\varepsilon_1, x^*_\varepsilon_2(t), z^*_\lambda, z^*_\varepsilon_1, z^*_\varepsilon_2(t))$ is continuous.

Denoting $x^*_\lambda = x^*_\lambda, 0, 0$ and $z^*_\lambda = z^*_\lambda, 0, 0$, we introduce the numbers

\begin{equation}
  \mathcal{R}_0 = \frac{\beta \Lambda / \mu}{\varepsilon + \eta r / \bar{b}}, \quad \mathcal{R}_\lambda = \frac{\beta x^*_\lambda}{\varepsilon + \eta z^*_\lambda} \quad \text{and} \quad \bar{R}_0 = \inf_{\lambda \in (0, 1]} \mathcal{R}_\lambda
\end{equation}

\begin{equation}
  \tag{5}
\end{equation}
Theorem 1. If \( \bar{R}_0 > 1 \), \( \gamma a \beta - \beta a \leq 0 \) and
\[
\bar{R}_0 > 1 + \frac{\beta}{\mu} \left( \frac{\eta}{\eta + \sigma} - \frac{\gamma a}{\beta} \right) f \left( \frac{\eta \eta/\beta + \tau/\gamma}{\beta} \right)
\] (6)
then system (11) possesses an endemic periodic orbit of period \( \omega \).

Our proof relies on an application of Mawhin’s continuation theorem. We will proceed in several steps. Firstly, in subsection 3.1 we consider a one parameter family of systems and obtain uniform bounds for the components of any periodic solution of these systems. Next, in subsection 3.2 we make a suitable change of variables in our family of systems to establish the setting where we will apply Mawhin’s continuation Theorem. Finally, in subsection 3.3 we use Mawhin’s continuation Theorem to obtain our result.

3.1. Uniform Persistence for the periodic orbits of a one parameter family of systems. In this section, to obtain uniform bounds for the components of any periodic solution of the family of systems that we can obtain multiplying the right hand side of (11) by \( \lambda \in (0,1] \), we need to consider the auxiliary systems:

\[
\begin{align*}
S_\lambda' &= \lambda(\Lambda(t) - \mu(t)S_\lambda - a(t)f(S_\lambda, P_\lambda)P_\lambda - \beta(t)S_\lambda I_\lambda) \\
I_\lambda' &= \lambda(\beta(t)S_\lambda I_\lambda - \eta(t)P_\lambda I_\lambda - c(t)I_\lambda) \\
P_\lambda' &= \lambda(\gamma(t)a(t)f(S_\lambda, P_\lambda)P_\lambda + \theta(t)\eta(t)P_\lambda I_\lambda + r(t)P_\lambda - b(t)P_\lambda^2)
\end{align*}
\] (7)

We will consider separately each of the several components of any periodic orbit.

Lemma 3. Let \( x_\lambda^*(t) \) be the unique solution of (2). There is \( L_1 > 0 \) such that, for any \( \lambda \in (0,1] \) and any periodic solution \( (S_\lambda(t), I_\lambda(t), P_\lambda(t)) \) of (7) with initial conditions \( S_\lambda(t_0) = S_0 > 0, I_\lambda(t_0) = I_0 > 0 \) and \( P_\lambda(t_0) = P_0 > 0 \), we have \( S_\lambda(t) + I_\lambda(t) \leq x_\lambda^*(t) \leq \Lambda^* / \mu \) and \( S_\lambda \geq L_1 \), for all \( t \in \mathbb{R} \).

Proof. Let \( (S_\lambda(t), I_\lambda(t), P_\lambda(t)) \) be some periodic solution of (7) with initial conditions \( S_\lambda(t_0) = S_0 > 0, I_\lambda(t_0) = I_0 > 0 \) and \( P_\lambda(t_0) = P_0 > 0 \). Since \( c(t) \geq \mu(t) \), we have, by the first and second equations of (7),

\[ (S_\lambda + I_\lambda)' \leq \lambda(\Lambda(t) - \mu(t)S_\lambda - \nu(t)I_\lambda) \leq \lambda\Lambda(t) - \lambda\mu(t)(S_\lambda + I_\lambda). \]

Since, by Lemma 1, equation (2) has a unique periodic orbit, \( x_\lambda^*(t) \), that is globally asymptotically stable, we conclude that \( S_\lambda(t) + I_\lambda(t) \leq x_\lambda^*(t) \) for all \( t \in \mathbb{R} \). Comparing equation (2) with equation \( x' = \lambda \Lambda x - \lambda \mu x \), we conclude that \( x_\lambda^*(t) \leq \Lambda^* / \mu \).

Using conditions (3) and (4), by the third equation of (7), we have

\[ P_\lambda' \leq \lambda(r(t) + \gamma(t)a(t)f(x_\lambda^*(t), 0) + \theta(t)\eta(t)x_\lambda^*(t)) - b(t)P_\lambda P_\lambda \leq (\Theta^* - b^2P_\lambda)P_\lambda, \]

where function \( \Theta \) is given by \( \Theta(t) = r(t) + \gamma(t)a(t)f(x_\lambda^*(t), 0) + \theta(t)\eta(t)x_\lambda^*(t) \). Thus, comparing with equation (3) and using Lemma 2 we get \( P_\lambda(t) \leq P_\lambda^*(t) \leq \Theta^* / b^2 \).

Using the bound obtained above, since \( -\beta(t)S_\lambda(t) \geq -\beta(t)x_\lambda^*(t) \), we have, by
Proof. Let\( S_λ(t) = S_0 > 0, I_λ(t) = I_0 > 0 \) and \( P_λ(t) = P_0 > 0 \), we have \( r^f/b^u ≤ z_λ^*(t) ≤ P_λ(t) ≤ L_2 \), for all \( t ∈ R \).

Comparing the previous inequality with equation \( S_9 \) and using Lemma 2, we get \( P_λ(t) ≥ z_λ^*(t) \). Moreover, comparing equation \( S_9 \) with equation \( \lambda R \), we conclude that \( z_λ^*(t) ≥ r^f/b^u \).

Using the computations in proof of the previous lemma, we have \( P_λ(t) ≤ L_1 \) and we take \( L_2 = L_1 \).

Lemma 5. Let \( \bar{R}_0 > 1 \). There are \( L_3, L_4 > 0 \) such that, for any \( λ ∈ (0, 1] \) and any periodic solution \( (S_λ(t), I_λ(t), P_λ(t)) \) of \( 7 \) with initial conditions \( S_λ(t_0) = S_0 > 0, I_λ(t_0) = I_0 > 0 \) and \( P_λ(t_0) = P_0 > 0 \), we have \( L_3 ≤ I_λ(t) ≤ L_4 \), for all \( t ∈ R \).

Proof. We will first prove that there is \( \varepsilon_1 > 0 \) such that, for any \( λ ∈ (0, 1] \), we have

\[
\limsup_{t \to +\infty} I_λ(t) ≥ \varepsilon_1.
\]

By contradiction, assume that \( 8 \) does not hold. Then, for any \( ε > 0 \), there must be \( λ > 0 \) such that \( I_λ(t) < ε \) for all \( t ∈ R \). We have

\[
\begin{align*}
S_λ' &≤ λA(t) - λμ(t)S_λ - λa(t)f(S_λ, P_λ)P_λ \\
P_λ' &≤ λ(γ(t)a(t)f(S_λ, P_λ) + r(t) - b(t)P_λ + λεθ^uη^u)P_λ
\end{align*}
\]

and

\[
\begin{align*}
S_λ' &≥ λA(t) - λμ(t)S_λ - λa(t)f(S_λ, P_λ)P_λ - ελβ^uS_λ \\
P_λ' &≥ λ(γ(t)a(t)f(S_λ, P_λ) + r(t) - b(t)P_λ)P_λ
\end{align*}
\]

By condition \( S_9 \), we conclude that

\[
x_{λ,ελβ^u,0}^*(t) ≤ S_λ(t) ≤ x_{λ,0,ελθ^uη^u}^*(t)
\]

and

\[
z_{λ,ελβ^u,0}^*(t) ≤ P_λ(t) ≤ z_{λ,0,ελθ^uη^u}^*(t).
\]
Thus, using condition (9), we have 
\[
I'_\lambda = \lambda(\beta(t)S_\lambda - \gamma(t)P_\lambda - c(t))I_\lambda \\
\geq (\lambda \beta(t)x_{\lambda,\beta,\gamma}(t) - \lambda \gamma(t)z_{\lambda,0,\lambda,\mu}(t) - \lambda c(t))I_\lambda
\]
and thus is independent of $\lambda$. Integrating in $[0, \omega]$ and using (9), we get
\[
0 = \frac{1}{\omega} (\ln I_\lambda(\omega) - \ln I_\lambda(0)) = \frac{1}{\omega} \int_0^\omega I'_\lambda(s)/I_\lambda(s) \, ds \\
\geq \lambda (\beta_x - \bar{c} - \gamma\bar{z}) + \phi(\varepsilon) = \lambda(\bar{c} + \gamma\bar{z}) (R_0^\lambda - 1) + \phi(\varepsilon)
\]
and since 
\[
R_0^\lambda \geq \inf_{\varepsilon \in (0, 1]} R_0^\lambda = \bar{R}_0 > 1,
\]
we have a contradiction. We conclude that (8) holds. Next we will prove that there is $\varepsilon_2 > 0$ such that, for any $\lambda \in (0, 1]$, we have
\[
\liminf_{t \to +\infty} I_\lambda(t) \geq \varepsilon_2. \quad (10)
\]
Assuming by contradiction that (10) does not hold, we conclude that there is a sequence $(\lambda_n, I_{\lambda_n}(s_n), I_{\lambda_n}(t_n)) \subset (0, 1] \times R_0^+ \times R_0^+$ such that $s_n < t_n$, $t_n - s_n \leq \omega$, 
$I_{\lambda_n}(s_n) = 1/n$, $I_{\lambda_n}(t_n) = \varepsilon_2/2$ and $I_{\lambda_n}(t) \in (1/n, \varepsilon_2/2)$, for all $t \in (s_n, t_n)$.
Since $\lambda_n \leq 1$, by Lemma 8 we have
\[
I'_{\lambda_n} = (\lambda_n \beta(t)S_{\lambda_n} - \lambda_n \gamma(t)P_{\lambda_n} - \lambda_n c(t))I_{\lambda_n} \leq \beta^u \Lambda^u I_{\lambda_n}/\mu^\ell
\]
and thus
\[
\ln(\varepsilon_2 n/2) = \ln(I_{\lambda_n}(t_n)/I_{\lambda_n}(s_n)) = \int_{s_n}^{t_n} I'_{\lambda_n}(s)/I_{\lambda_n}(s) \, ds \leq \beta^u \Lambda^u \omega/\mu^\ell,
\]
which is a contradiction since the sequence $(\ln(\varepsilon_2 n/2))_{n \in N}$ goes to $+\infty$ as $n \to +\infty$, and thus is not bounded.

We conclude that there is $\varepsilon_2 > 0$ such that (10) holds. Letting $L_3 = \varepsilon_2$, we obtain $I_\lambda(t) \geq L_3$ for all $\lambda \in (0, 1]$. Since $I_{\lambda}(t) \leq S_{\lambda}(t) + I_\lambda(t)$, by Lemma 3 we can take $L_4 = L_2$ and the result is established. \(\square\)

### 3.2. Setting where Mawhin’s continuation theorem will be applied
To apply Mawhin’s continuation theorem to our model we make the change of variables: $S(t) = e^{u_1(t)}$, $I(t) = e^{u_2(t)}$ and $P(t) = e^{u_3(t)}$. With this change of variables, system (11) becomes
\[
\begin{align*}
    u'_1 &= \Lambda(t)e^{-u_1} - a(t)f(e^{u_1}, e^{u_2})e^{u_3} - \beta(t)e^{u_2} - \mu(t) \\
    u'_2 &= \beta(t)e^{u_1} - \eta(t)e^{u_2} - c(t) \\
    u'_3 &= \gamma(t)a(t)f(e^{u_1}, e^{u_2}) + \theta(t)\eta(t)e^{u_2} - b(t)e^{u_3} + r(t)
\end{align*}
\]

Note that, if $(u'_1(t), u'_2(t), u'_3(t))$ is an $\omega$-periodic solution of the system (11) then $(e^{u_1(t)}, e^{u_2(t)}, e^{u_3(t)})$ is an $\omega$-periodic solution of system (1).
To define the operators in Mawhin’s theorem (see appendix A), we need to consider the Banach spaces \((X, \| \cdot \|)\) and \((Z, \| \cdot \|)\) where \(X\) and \(Z\) are the space of \(\omega\)-periodic continuous functions \(u : \mathbb{R} \to \mathbb{R}^3:\)
\[
X = Z = \{ u = (u_1, u_2, u_3) \in C(\mathbb{R}, \mathbb{R}^3) : u(t) = u(t + \omega) \}
\]
and
\[
\|u\| = \max_{t \in [0, \omega]} |u_1(t)| + \max_{t \in [0, \omega]} |u_2(t)| + \max_{t \in [0, \omega]} |u_3(t)|.
\]

Next, we consider the linear map \(L : \mathbb{R}^3 \to Z\) given by
\[
L(u(t)) = \frac{du(t)}{dt} \quad (12)
\]
and the map \(N : X \to Z\) defined by
\[
N(u(t)) = \begin{bmatrix}
\Lambda(t)e^{-u_1(t)} - a(t)f(e^{u_1(t)}, e^{u_3(t)})e^{u_3(t)} - u_1(t) - \beta(t)e^{u_2(t)} - \mu(t) \\
\beta(t)e^{u_1(t)} - \eta(t)e^{u_3(t)} - c(t) \\
\gamma(t)a(t)f(e^{u_1(t)}, e^{u_3(t)}) + \theta(t)\eta(t)e^{u_2(t)} - b(t)e^{u_3(t)} + r(t)
\end{bmatrix} \quad (13)
\]

In the following lemma we show that the linear map in (12) is a Fredholm mapping of index zero

**Lemma 6.** The linear map \(L\) in (12) is a Fredholm mapping of index zero.

**Proof.** We have
\[
\ker L = \{(u_1, u_2, u_3) \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : \frac{du_i(t)}{dt} = 0, \; i = 1, 2, 3\}
= \{(u_1, u_2, u_3) \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : u_i \text{ is constant, } i = 1, 2, 3\}
\]
and thus \(\ker L\) can be identified with \(\mathbb{R}^3\). Therefore \(\dim \ker L = 3\). On the other hand
\[
\text{Im } L = \{(z_1, z_2, z_3) \in Z : \exists u \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : \frac{du_i(t)}{dt} = z_i(t), \; i = 1, 2, 3\}
= \{(z_1, z_2, z_3) \in Z : \int_0^\omega z_i(s) \, ds = 0, \; i = 1, 2, 3\}.
\]
and any \(z \in Z\) can be written as \(z = \tilde{z} + \alpha\), where \(\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3\) and \(\tilde{z} \in \text{Im } L\). Thus the complementary space of \(\text{Im } L\) consists of the constant functions. Thus, the complementary space has dimension 3 and therefore \(\text{codim } \text{Im } L = 3\).

Given any sequence \((z_n)\) in \(\text{Im } L\) such that
\[
z_n = ((z_1)_n, (z_2)_n, (z_3)_n) \to z = (z_1, z_2, z_3),
\]
we have, for \(i = 1, 2, 3\) (note that \(z \in Z\) since \(Z\) is a Banach space and thus it is integrable in \([0, \omega]\) since it is continuous in that interval),
\[
\int_0^\omega z_i(s) \, ds = \int_0^\omega \lim_{n \to +\infty} (z_i)_n(s) \, ds = \lim_{n \to +\infty} \int_0^\omega (z_i)_n(s) \, ds = 0.
\]
Thus, \(z \in \text{Im } L\) and we conclude that \(\text{Im } L\) is closed in \(Z\). Thus \(L\) is a Fredholm mapping of index zero. \(\square\)
Consider the projectors $P : X \to X$ and $Q : Z \to Z$ given by

$$Pu(t) = \frac{1}{\omega} \int_0^\omega u(s) ds \quad \text{and} \quad Qz(t) = \frac{1}{\omega} \int_0^\omega z(s) ds.$$ 

Note that $\text{Im } P = \ker \mathcal{L}$ and that $\ker Q = \text{Im}(I - Q) = \text{Im} \mathcal{L}$.

Consider the generalized inverse of $\mathcal{L}$, $\mathcal{K} : \text{Im} \mathcal{L} \to D \cap \ker P$, given by

$$\mathcal{K}z(t) = \int_0^t z(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s) ds dr$$

the operator $Q\mathcal{N} : X \to Z$ given by

$$Q\mathcal{N}u(t) = \begin{bmatrix}
\frac{1}{\omega} \int_0^\omega \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} - \beta(s)e^{u_2(s)} ds - \pi \\
\frac{1}{\omega} \int_0^\omega \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} ds - \pi \\
\frac{1}{\omega} \int_0^\omega \gamma(s)a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} ds + \pi
\end{bmatrix}$$

and the mapping $\mathcal{K}(I - Q)\mathcal{N} : X \to D \cap \ker P$ given by

$$\mathcal{K}(I - Q)\mathcal{N}u(t) = B_1(t) - B_2(t) - B_3(t),$$

where

$$B_1(t) = \begin{bmatrix}
\int_0^t \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s) ds \\
\int_0^t \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s) ds \\
\int_0^t \gamma(s)a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} dt + r(s) ds
\end{bmatrix},$$

$$B_2(t) = \begin{bmatrix}
\frac{1}{\omega} \int_0^\omega \int_0^t \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s) ds dr \\
\frac{1}{\omega} \int_0^\omega \int_0^t \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s) ds dr \\
\frac{1}{\omega} \int_0^\omega \int_0^t \gamma(s)a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} + r(s) ds dr
\end{bmatrix},$$

and

$$B_3(t) = \left( \frac{t}{\omega} - \frac{1}{2} \right) \begin{bmatrix}
\int_0^\omega \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s) ds \\
\int_0^\omega \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s) ds \\
\int_0^\omega \gamma(s)a(s)f(e^{u_1(s)}, e^{u_3(s)})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} + r(s) ds
\end{bmatrix}.$$ 

The next lemma shows that $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any open bounded subset of its domain.

**Lemma 7.** The map $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any open bounded set $U \subseteq X$. 
Proof. Let $U \subseteq X$ be an open bounded set and $\overline{U}$ its closure in $X$. Then, there is $M > 0$ such that, for any $u = (u_1, u_2, u_3) \in U$, we have that $|u_i| \leq M$, $i = 1, 2, 3$. Letting $\mathcal{N}u = ((\mathcal{N})_1u, (\mathcal{N})_2u, (\mathcal{N})_3u)$, we have

$$|(\mathcal{N})_1u(t)| \leq e^M \left( \Lambda + a f(e^M, 0) + \beta \right) + \mu,$$

$$|(\mathcal{N})_2u(t)| \leq e^M (\beta + \eta) + \tau,$$

$$|(\mathcal{N})_3u(t)| \leq e^M \left( -\alpha f(e^M, 0) + \bar{b} t + \bar{b} \right) + \tau$$

and we conclude that $\mathcal{N}u(\overline{U})$ is bounded.

Let now

$$\mathcal{N}u = (\mathcal{N}_1u, \mathcal{N}_2u, \mathcal{N}_3u) : u \in B.$$

Let $B \subseteq X$ be a bounded set. Note that the boundedness of $B$ implies that there is $M$ such that $|u_i| < M$, for all $i = 1, 2, 3$ and all $u = (u_1, u_2, u_3) \in B$. It is immediate that $\{\mathcal{N}u : u \in B\}$ is pointwise bounded. Given $u = (u_1, u_2, u_3) \in B$ we have

$$(\mathcal{N}_1u)(t) - (\mathcal{N}_1u)(v)$$

and similarly

$$(\mathcal{N}_3u)(t) - (\mathcal{N}_3u)(v)$$

and similarly

$$(\mathcal{N}_2u)(t) - (\mathcal{N}_2u)(v)$$

and

$$(\mathcal{N}_3u)(t) - (\mathcal{N}_3u)(v)$$

By (14), (15) and (16), we conclude that $\{\mathcal{N}u : u \in B\}$ is equicontinuous. Therefore, by Ascoli-Arzela’s theorem, $\mathcal{N}B$ is relatively compact. Thus the operator $\mathcal{N} : B \to \mathcal{N}$ is compact.

We conclude that $\mathcal{N}$ is $\mathcal{C}$-compact in the closure of any bounded set contained in $X$. \qed

3.3. Application of Mawhin’s continuation theorem. In this section we will construct the set where, applying Mawhin’s continuation theorem, we will find the periodic orbit in the statement of our result.

Consider the system of algebraic equations:

$$\begin{cases} \overline{\mathcal{N}} e^{-u_1} - \overline{\mathcal{N}} f(e^{u_1}, e^{u_2}) e^{u_3} - u_1 - \beta e^{u_2} - \overline{\mathcal{N}} e^{u_3} - \sigma = 0 \\ \overline{\mathcal{N}} e^{u_1} - \overline{\mathcal{N}} e^{u_2} - \overline{\mathcal{N}} = 0 \\ \overline{\mathcal{N}} f(e^{u_1}, e^{u_2}) + \overline{\mathcal{N}} e^{u_2} - \overline{\mathcal{N}} e^{u_3} + \sigma = 0 \end{cases}$$

(17)

By the second and third equations we get

$$e^{u_1} = \frac{\overline{\mathcal{N}} e^{u_2} + \sigma}{\beta} \quad \text{and} \quad e^{u_2} = \frac{\overline{\mathcal{N}}}{\beta} f\left( \frac{\overline{\mathcal{N}} e^{u_2} + \sigma}{\beta}, e^{u_3} \right) + \frac{\overline{\mathcal{N}}}{\beta} e^{u_3} - \frac{\sigma}{\beta}.$$

Therefore, using the first equation, we get

$$G_1(e^{u_3}) - G_2(e^{u_3}) f\left( \frac{\overline{\mathcal{N}} e^{u_2} + \sigma}{\beta}, e^{u_3} \right) - G_3(e^{u_3}) = 0,$$
where

\[ G_1(x) = \frac{\Lambda \beta x}{\eta x + c}, \quad G_2(x) = \frac{\pi \beta x}{\eta x + c} - \frac{\beta m}{\eta}, \quad G_3(x) = \frac{\beta b}{\beta} x - \frac{\beta r}{\beta} + \mu \]

Consider the function \( G : [\frac{r}{b}, +\infty) \rightarrow \mathbb{R} \) (notice that, by the third equation in (17), we have \( e^{u_3} \geq \frac{r}{b} \)), given by

\[ G(x) = G_1(x) - G_2(x)f \left( \frac{\eta x + c}{\beta}, x \right) - G_3(x) \]

and observe that function \( G_1 \) is decreasing and functions \( G_2 \) and \( G_3 \) are increasing. Thus, since the function \( [\frac{r}{b}, +\infty) \ni x \mapsto f \left( \frac{\eta x + c}{\beta}, x \right) \) is nondecreasing, we conclude that

\[ [\frac{r}{b}, +\infty) \ni x \mapsto -G_2(x)f \left( \frac{\eta x + c}{\beta}, x \right) \]

is a decreasing function. It is immediate the function \( [\frac{r}{b}, +\infty) \ni x \mapsto -G_3(x) \) is decreasing. Consequently \( G \) is a decreasing function and equation (17) has, at most, one solution.

It is easy to verify that

\[ \lim_{x \to +\infty} G(x) = -\infty \]

and, by the hypothesis in our theorem

\[ G(\frac{r}{b}) = \frac{\Lambda \beta}{\eta \frac{r}{b} + c} - \left( \frac{\pi \beta}{\eta \frac{r}{b} + c} - \frac{\beta m}{\eta} \right) f \left( \frac{\eta \frac{r}{b} + c}{\beta}, \frac{r}{b} \right) - \mu \]

\[ = \mu \left( R_0 - 1 - \frac{\beta}{\mu} \left( \frac{\pi}{\eta} + \frac{c}{\beta} \right) f \left( \frac{\eta \frac{r}{b} + c}{\beta}, \frac{r}{b} \right) \right) > 0. \]

Thus we conclude that there is a unique solution of equation (17). Denote this solution by \( p^* = (p_1^*, p_2^*, p_3^*) \).

By Lemmas 3-5 there is a constant \( M_0 > 0 \) such that \( \|u_3(t)\| < M_0 \), for any \( t \in [0, \omega] \) and any periodic solution \( u_3(t) \) of (7). Let

\[ U = \{(u_1, u_2, u_3) \in X : \|(u_1, u_2, u_3)\| < M_0 + \|p^*\|\}. \quad (18) \]

Conditions M1. and M2. in Mawhin’s continuation theorem (see appendix A) are fulfilled in the set \( U \) defined in (15).

Using the notation \( v = (e^{p_1^*}, e^{p_2^*}) \), the Jacobian matrix of the vector field corresponding to (17) computed in \( (p_1^*, p_2^*, p_3^*) \) is

\[ J = \begin{bmatrix}
-\frac{\pi f}{\beta g} (v) e^{p_1^*} - \frac{\beta}{\beta} e^{p_2^*} - \mu & -\frac{\pi g}{\beta} (v) e^{p_2^* - p_1^*} - \eta f(v) e^{p_2^* - p_1^*} & 0 \\
\frac{\beta}{\beta} e^{p_1^*} & 0 & -\frac{\eta g}{\beta} (v) e^{p_2^* - p_1^*}
\end{bmatrix}. \]
Corollary 1. If $A$ model with no predation on susceptible preys.

4.1. Letting $a(t) \equiv 0$ or $f \equiv 0$ in system (1), and still assuming that the real valued functions $\Lambda$, $\mu$, $\eta$, $c$, $r$, $\theta$ and $b$ are periodic with period $\omega$, nonnegative, continuous and also that $\bar{\Lambda} > 0$, $\bar{\mu} > 0$, $\bar{r} > 0$ and $\bar{b} > 0$, we obtain the periodic model considered in [3]:

$$ \begin{align*}
S' &= \Lambda(t)S - \mu(t)S - \beta(t)SI \\
I' &= \beta(t)SI - \eta(t)PI - c(t)I \\
P' &= (r(t) - b(t)P)P + \theta(t)\eta(t)PI
\end{align*} $$

(20)

Note that conditions S1 and S7 are assumed and conditions S2 to S6 and S8 are trivially satisfied since $f \equiv 0$. Also note that system (4) becomes in this context

$$ \begin{align*}
x' &= \lambda(\Lambda(t) - \mu(t)x - \epsilon_1x) \\
z' &= \lambda(r(t) - b(t)z + \epsilon_2z)
\end{align*} $$

(21)

and, by Lemmas 1 to 4 in [11] we conclude that condition S3 holds in this setting. Note also that condition (13) becomes $\bar{\Lambda}_0 > 1$ and condition $\bar{\mu} - \bar{\eta} \bar{r} \bar{b} \leq 0$ is trivially satisfied since we can take $\gamma = 0$. We obtain the following corollary that recovers the result in [3]:

Corollary 1. If $\bar{\Lambda}_0 > 1$ and $\bar{\eta}_0 > 1$ hold, then system (20) possesses an endemic periodic orbit of period $\omega$.

4.2. A model with Holling-type I functional response. Letting $f(S, P) = S$ (Holling-type I functional response) in system (1), and assuming that the real valued functions $\beta$, $\eta$ and $c$ are periodic with period $\omega$ and that the real valued functions $\Lambda$, $\mu$, $\gamma$, $r$, $\theta$, $b$ and $a$ are constant and positive, we obtain the periodic model:

$$ \begin{align*}
S' &= \Lambda - \mu S - aSP + \beta(t)SI \\
I' &= \beta(t)SI - \eta(t)PI - c(t)I \\
P' &= (r - bP)P + \gamma aSP + \theta(t)PI
\end{align*} $$

(22)
Since \( f(S, P) = S \), conditions (22) to (26) are trivially satisfied. Conditions (14) and (17) are assumed and (13) is satisfied with \( K = 1 \). Notice additionally that system (11) becomes in our context
\[
\begin{aligned}
x' &= \lambda(\mu x - axz - \varepsilon_1 x) \\
z' &= \lambda(r - bz + \gamma ax + \varepsilon_2)z
\end{aligned}
\]  
(23)

System (23) has two equilibriums: \( E_1 = (\Lambda/(\mu + \varepsilon_1), 0) \) and
\[
E_2 = \left( \frac{\sqrt{V^2 + 4\Lambda r^2/a^2} - V}{2\gamma a^2/b}, \frac{\sqrt{V^2 + 4\Lambda r^2/a^2} - V}{2\gamma a^2/b} + r + \varepsilon_2 \right),
\]
where \( V = \mu + \varepsilon_1 + a(r + \varepsilon_2)/b \). It is easy to check that \( E_2 \) is locally attractive and that \( E_1 \) is a saddle point whose stable manifold coincides with the x-axis. If \( 0 < \alpha < (r + \varepsilon_2)/b \) then, in the line \( z = \alpha \) the flow points upward. Additionally, if \( \Lambda < \mu(\mu + \varepsilon_1)/a \), in the line \( x = \mu/a \) the flow points to the left and the \( x \)-coordinate of \( E_1 \) is less than \( \mu/a \). Thus the region \( R = \{(x, z) \in \mathbb{R}^2 : 0 \leq x \leq \mu/a \wedge z \geq \alpha \} \) is positively invariant. Since the divergence of the vector field is given by
\[ -z \mu/a \]
then it is null on the line \( z = \frac{-\mu - \varepsilon_1 + \varepsilon_2}{a + 2b} + \frac{\gamma a}{a + 2b} x \). Thus the divergence of the vector field doesn’t change sign on the region \( R \) and this forbids the existence of a periodic orbit on \( R \). There is also no periodic orbit on \((\mathbb{R}^2)^c \setminus R\) since there is no additional equilibrium in \((\mathbb{R}^2)^c \). Since \( E_2 \) is locally asymptotically stable, there is no homoclinic orbit connecting \( E_2 \) to itself. Therefore, the \( \omega \)-limit of any orbit in \((\mathbb{R}^2)^c \) must be the equilibrium point \( E_2 \) and the global asymptotic stability of (23) for sufficiently small \( \varepsilon_1, \varepsilon_2 > 0 \) follows. We conclude that condition (30) holds.

Notice that condition (3) becomes \( \overline{R}_{0} > 1 + \frac{a(r \bar{\theta} \bar{r} - \bar{\gamma} \bar{\gamma} - \bar{\theta} \bar{\gamma})}{\mu \bar{\theta} \bar{b}} \) and condition \( \gamma \beta - \theta \bar{\gamma} \leq 0 \) is trivially satisfied. We obtain the corollary that generalizes the result in [3]:

**Corollary 2.** If \( \Lambda < \mu^2/a, \gamma \beta - \theta \bar{\gamma} \leq 0, \overline{R}_{0} > 1 \) and
\[
\overline{R}_{0} > 1 + \frac{a(r \bar{\theta} \bar{r} - \bar{\gamma} \bar{\gamma} - \bar{\theta} \bar{\gamma})}{\mu \bar{\theta} \bar{b}}
\]
then system (22) possesses an endemic periodic orbit of period \( \omega \).

To do some simulation, we consider the following particular set of parameters: \( \Lambda = 0.1, \mu = 0.6, \beta(t) = 20(1 + 0.9 \cos(2\pi t)), \eta(t) = 0.7(1 + 0.7 \cos(\pi + 2\pi t)), c(t) = 0.1, r = 0.2, b = 0.3, \theta = 10, \gamma(t) = 0.1 \) and \( a = 3 \). We obtain the model:
\[
\begin{aligned}
S' &= 0.1 - 0.6S - 20(1 + 0.9 \cos(2\pi t))SI - 3SP \\
I' &= 20(1 + 0.9 \cos(2\pi t))SI - 0.7(1 + 0.7 \cos(\pi + 2\pi t))PI - 0.1I \\
P' &= (0.2 - 0.3P)P + 7(1 + 0.7 \cos(\pi + 2\pi t))PI + 0.3SP
\end{aligned}
\]  
(24)

Notice that, for our model, \( \Lambda = 0.1 > 0.012 = \mu^2/a, \gamma \beta - \theta \bar{\gamma} = -5 > 0, \overline{R}_{0} \approx 5.88 > 1 + 3.3 \) and \( \overline{R}_{0} \approx 24.8 > 1 \), and thus the conditions in Corollary 2 are fulfilled. Considering the initial condition \((S_0, I_0, P_0) = (0.03567, 0.02047, 0.88021)\) we obtain the periodic orbit in figure 1. Although our theoretical result doesn’t imply the attractivity of the periodic solution, the simulations carried out suggest that this is the case.

**Appendix A. Mawhin’s continuation theorem**

In this appendix we state Mawhin’s continuation theorem [13, Part IV]. Let \( X \) and \( Z \) be Banach spaces.

**Definition 1.** A linear map \( L : D \subseteq X \rightarrow Z \) is called a Fredholm mapping of index zero if
1. \( \dim \ker L = \codim \text{Im} L \leq \infty \);
2. \( \text{Im} L \) is closed in \( Z \).
Given a Fredholm mapping of index zero $L : D \subseteq X \to Z$ it is well known that there are continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that:

1. $\text{Im } P = \ker L$;
2. $\ker Q = \text{Im } L = \text{Im } (I - Q)$;
3. $X = \ker L \oplus \ker P$;
4. $Z = \text{Im } L \oplus \text{Im } Q$.

It follows that $L|_{D \cap \ker P} : (I - P)X \to \text{Im } L$ is invertible. We denote the inverse of that map by $K$.

**Definition 2.** A continuous mapping $N : X \to Z$ is called $L$-compact on $U \subset X$, where $U$ is an open bounded set, if

1. $QN(U)$ is bounded;
2. $K(I - Q)N : U \to X$ is compact.

Note that, since $\text{Im } Q$ is isomorphic to $\ker L$, there is an isomorphism $I : \text{Im } Q \to \ker L$.

We are now prepared to state the Mawhin’s continuation theorem.

**Theorem 2** (Mawhin’s continuation theorem). Let $X$ and $Z$ be Banach spaces and let $U \subset X$ be an open set. Assume that $L : D \subseteq X \to Z$ is a Fredholm mapping of index zero and let $N : X \to Z$ be $L$-compact on $\overline{U}$. Additionally, assume that

M1) for each $\lambda \in (0, 1)$ and $x \in \partial U \cap D$ we have $Lx \neq \lambda Nx$;
M2) for each $x \in \partial U \cap \ker L$ we have $QNx \neq 0$;
M3) $\text{deg}(IQN, U \cap \ker L, 0) \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $D \cap U$.

**References**

[1] C. Rebelo, A. Margheri, N. Bacaër, Persistence in seasonally forced epidemiological models, J. Math. Biol. 64 (6) (2012) 933–949.
[2] A. S. Purnomo, I. Darti, A. Suryanto, Dynamics of eco-epidemiological model with harvesting, AIP Conference Proceeding 1913, 020018 (2017).
[3] C. M. Silva, Existence of Periodic Solutions for Eco-Epidemic Model with Disease in the Prey, accepted for publications in J. Math. Anal. Appl.
[4] Chakraborty, K., Das K., Haldar, S., Kar,T.K, A mathematical study of an eco-epidemiological system on disease persistence and extinction perspective, Applied Mathematics and Computation 254, 99-112 (2015)
[5] H. Bai and R. Xo, Global stability of a delayed eco-epidemiological model with holling type III functional response, Springer Proceedings in mathematics ans Statistics 225, 119–130 (2018)
[6] M. Garrione and C. Rebelo, Persistence in seasonally varying predator-prey systems via the basic reproduction, Nonlinear Anal. Real World Appl. 30, 73–98 (2016)
[7] N. Bacaër, S. Guernaoui. The epidemic Threshold of vector-borne diseases with seasonality, J. Math. Biol. 53 (2006), 421–436.
[8] O. Diekmann, J.A.P Heesterbeek, J.A.J Metz, On the definition and the computation of the basic reproduction ratio $R_0$ in models for infectious diseases in heterogeneous population, J. Math. Biol 28 (1990) 365.
[9] P. Van den Driessche, J. Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math Biosci. 180 (2002)29-48.
[10] W. Wang, X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, J. Dynam. Differential Equations 20 (3), 699-717 (2008)
[11] Xingge Niu, Tailei Zhang, Zhidong Teng, The asymptotic behavior of a nonautonomous eco-epidemic model with disease in the prey, Applied Mathematical Modelling 35, 457-470 (2011)
[12] Xiao-Qiang Zhao, Dynamical Systems in Population Biology, Springer, 2003
[13] R. E. Gaines, J. L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Mathematics 568, Springer-Verlag Berlin Heidelberg, 1977

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