ON ALL REAL ZEROS FOR A NEW CLASS OF THE EVEN ENTIRE FUNCTION

XIAO-JUN YANG

Abstract. In this article we propose a new class of the even entire function connected with the product and series with the real coefficients. We address a sufficient condition for all real zeros for it. As a typical example, we give an answer to the problem of Lagarias and Montague. We suggest the open problems for the class of the even entire function.

Contents
1. Introduction 1
2. A special class of the even entire functions 3
   2.1. Change the product of $\aleph(\tau)$ 3
   2.2. Structure the function $\psi(\vartheta)$ 3
3. The proof of Theorem 1 5
   3.1. Set up a class of $\psi(\vartheta)$ 5
   3.2. Find a class of $\varphi(\vartheta)$ 5
   3.3. Propose a class of $\psi(t)$ 6
   3.4. Consider the convergence of $\psi(\vartheta)$ 7
   3.5. Present the identity $\Re(\vartheta_k) = 0$ 7
   3.6. Prove that all zeros of $\aleph(\tau)$ are real 9
   3.7. Equivalently sufficient conditions 10
4. A typical application 10
   4.1. Prove the conjecture of Lagarias and Montague 10
   4.2. Two open problems 13
5. Conclusion 14
References 15

1. Introduction

The theory of the entire functions has played an important role in the study of the behaviors of the zeros of the functions of the real and complex variables (see for instance

2020 Mathematics Subject Classification. Primary: 30D10; Secondary: 30D15, 30D99.

Key words and phrases. entire function, real zeros, conjecture of Lagarias and Montague, even entire function.
[1, 2] and the references therein). The entire functions have the popular applications of the polynomials of the entire functions with the distributions of their zeros in the fields of mathematics, physics, and engineering (see [3, 4, 5, 6]). Let $\mathbb{C}$ and $\mathbb{R}$ be the sets of the complex and real numbers, respectively, and let $i = \sqrt{-1}$. In 2015 Csordas the entire function $\Xi(\tau)$ has a nice connection with the Fourier transform [7]

\[
\Omega(\tau) = \int_{-\infty}^{\infty} h(\ell) e^{i\tau \ell} d\ell = \int_{0}^{\infty} h(\ell) \cos(\tau \ell) d\ell,
\]

where $h(\ell)$ is a positive-value kernel in the domain $\ell \in \mathbb{R}$ and $t \in \mathbb{C}$. In fact, the integral equation (1) was proposed by Jensen in 1913 [8] and developed by Pólya in 1927 [9, 10] due to the connection with the Riemann zeta function [11]. In Bruijn’s 1950 paper, he considered (1) to set up the zeroes of the Ramanujan zeta function [12]. Jensen [8] proposed that Riemann xi function $\Xi(t)$ has the Fourier cosine integral representation as follows:

\[
\Xi(\tau) = 2 \int_{0}^{\infty} g(\ell) \cos(\tau \ell) d\ell,
\]

where

\[
g(\ell) = \sum_{n=1}^{\infty} \left( 4\pi^2 n^4 e^{\frac{2\ell}{\pi}} - 6\pi n^2 e^{\frac{2\ell}{\pi}} \right) e^{-\pi n^2 e^{2\ell}}.
\]

In 2019 Griffin, Ono, Rolen and Zagier [13] made a big contribution for the real zeros of (2). In 1976 Newman [14] introduced the a family of the Fourier cosine integral

\[
H(\tau) = 2 \int_{0}^{\infty} e^{-\nu \ell^2} g(\ell) \cos(\tau \ell) d\ell,
\]

where $\nu \in \mathbb{R}$ is the de Bruijn-Newman constant. In 2020 Rodgers and Tao [15] reported that $\nu$ is non-negative can be considered as the non-negative value. Ki, Kim and Lee [16] suggested that the de Bruijn-Newman constant $\nu < 1/2$. In 2011 Lagarias and Montague [17] suggested the Fourier sine integral

\[
\Phi(\tau) = 2 \int_{0}^{\infty} \ell^{-1} g(\ell) \sin(\tau \ell) d\ell.
\]

The problem of Lagarias and Montague is a conjecture given by Lagarias and Montague [17], which states all zeros of the function $\Phi(\tau)$ are real. As a generalization of the work in [18] to solve the problem of Jensen [8], the main target of the present paper is to consider a family of the even entire function as follows:

**Definition 1.** An even entire function $\mathcal{N}(\tau)$ of order $\gamma = 1$ for $\tau \in \mathbb{C}$, represented by the series

\[
\mathcal{N}(\tau) = \sum_{m=0}^{\infty} \alpha_m \tau^{2m},
\]

is said to be in the class $\varphi$, written as $\mathcal{N} \in \varphi$, if $\mathcal{N}(\tau)$ can be expressed as

\[
\mathcal{N}(\tau) = \mathcal{N}(0) \prod_{\vartheta_k} \left( 1 - \frac{\vartheta_k}{\vartheta_k} \right),
\]
where $\theta_k \neq 0$ run the zeros of $\mathcal{R}(\tau)$, $\alpha_m = (-1)^m B_m$ are the coefficients for $\mathcal{R}(\tau)$ with $B_m > 0$, the series

\[ p = \sum_{k=1}^{\infty} \frac{1}{|\theta_k|^2}, \]

is convergent, and

\[ \mathcal{R}(0) \neq 0. \]

It is obvious that the even entire function $\mathcal{R}(\tau)$ has the infinite zeros because it is not constant and finite polynomials. The outline of this paper is given as follows. In Section 2 we introduce the equivalent idea for the class of the even entire function. In Section 3 we prove:

**Theorem 1.** Let $\mathcal{R} \in \wp$. Then all of the zeros of $\mathcal{R}(\tau)$ are real.

In Section 4 we give a detailed answer for the problem of Lagarias and Montague.

2. A special class of the even entire functions

2.1. Change the product of $\mathcal{R}(\tau)$. We start with the following result:

**Lemma 1.** If $\mathcal{R} \in \wp$, then there is

\[ \mathcal{R}(\tau) = \mathcal{R}(0) \prod_{\Im(\theta_k) > 0} \left( 1 + \frac{\tau^2}{\theta_k^2} \right). \]

*Proof.* Because $\mathcal{R}(\tau)$ is an even function, we present

\[ \mathcal{R}(\theta_k) = \mathcal{R}(-\theta_k). \]

By (11), the function (7) can be written as

\[ \mathcal{R}(\tau) = \mathcal{R}(0) \prod_{\theta_k} \left( 1 - \frac{\tau i}{\theta_k} \right) = \mathcal{R}(0) \prod_{\Im(\theta_k) > 0} \left( 1 - \frac{\tau i}{\theta_k} \right) \left( 1 - \frac{\tau i}{-\theta_k} \right) = \mathcal{R}(0) \prod_{\Im(\theta_k) > 0} \left( 1 + \frac{\tau^2}{\theta_k^2} \right). \]

We thus finish the proof. □

2.2. Structure the function $\psi(\vartheta)$. Now, let us give the special class of the even entire functions.

We first set

\[ x = -i\vartheta, \]

where $\vartheta \in \mathbb{C}$.

Substituting (13) back into (7), we obtain

\[ \mathcal{R}(-i\vartheta) = \sum_{m=0}^{\infty} B_m x^{2m} \]

and

\[ \mathcal{R}(-i\vartheta) = \mathcal{R}(0) \prod_{\vartheta_k} \left( 1 - \frac{\vartheta}{\vartheta_k} \right). \]
Let us define the function $\psi(\vartheta)$ by
\begin{equation}
\psi(\vartheta) := \aleph(-i\vartheta).
\end{equation}
Then (14) and (15) can be rewritten as
\begin{equation}
\psi(\vartheta) = \sum_{m=0}^{\infty} B_m \vartheta^{2m}
\end{equation}
and
\begin{equation}
\psi(\vartheta) = \psi(0) \prod_{\vartheta_k}(1 - \frac{\vartheta}{\vartheta_k}),
\end{equation}
respectively.

**Lemma 2.** There exists the functional equation
\begin{equation}
\psi(\vartheta) = \psi(-\vartheta).
\end{equation}
**Proof.** From (17) we have
\begin{equation}
\psi(-\vartheta) = \sum_{m=0}^{\infty} B_m (-\vartheta)^{2m} = \sum_{m=0}^{\infty} B_m \vartheta^{2m}.
\end{equation}
Thus, this is the required result. \qed

**Lemma 3.** There exists the identity
\begin{equation}
\psi(0) \prod_{\vartheta_k}(1 - \frac{\vartheta}{\vartheta_k}) = \psi(0) \prod_{\Im(\vartheta_k) > 0} (1 - \frac{\vartheta^2}{\vartheta_k^2}).
\end{equation}
**Proof.** By using Lemma 2, we get
\begin{equation}
\psi(\vartheta) = \psi(0) \prod_{\vartheta_k}(1 - \frac{\vartheta}{\vartheta_k}) = \psi(0) \prod_{\Im(\vartheta_k) > 0} (1 - \frac{\vartheta}{\vartheta_k}) (1 + \frac{\vartheta}{\vartheta_k}) = \psi(0) \prod_{\Im(\vartheta_k) > 0} (1 - \frac{\vartheta^2}{\vartheta_k^2}).
\end{equation}
We thus complete the proof. \qed

Combining (17), (18) and Lemma 3, we obtain
\begin{equation}
\psi(\vartheta) = \psi(0) \prod_{\vartheta_k}(1 - \frac{\vartheta}{\vartheta_k}) = \psi(0) \prod_{\vartheta_k}(1 - \frac{\vartheta}{\vartheta_k}) \prod_{\Im(\vartheta_k) > 0} (1 - \frac{\vartheta^2}{\vartheta_k^2}).
\end{equation}
Considering the given condition that $\aleph(\tau)$ is an even function of order $\gamma = 1$, the argument in Levin’s book (see Theorem 13 in [1], p.24) said that there exists $\varepsilon > 0$ such that
\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{|\vartheta_k|^{1+\varepsilon}} = \sum_{k=1}^{\infty} \frac{1}{|\vartheta_k|^{1+\varepsilon}}
\end{equation}
is convergent. Taking $\varepsilon = 1$ in (24), we know that
\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{|\vartheta_k|^2}
\end{equation}
is convergent.
3. The proof of Theorem 1

We now begin with the proof of the Theorem 1. We need to divide it into six steps.

3.1. Set up a class of $\psi(\vartheta)$. By (6) and (7), we structure

$$\sum_{m=0}^{\infty} B_m \vartheta^{2m} = \psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\vartheta}{\vartheta_k}\right).$$

Owing to Lemma 3, we easily demonstrate that

$$\psi(\vartheta) = \psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\vartheta}{\vartheta_k}\right).$$

Combining (26) into (27), we show that

$$\psi(\vartheta) = \sum_{m=0}^{\infty} B_m \vartheta^{2m} = \psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).$$

3.2. Find a class of $\bar{\psi}(\bar{\vartheta})$. Let $\psi(\vartheta)$, $\bar{\vartheta}$ and $\vartheta_k$ be the complex conjugates of $\psi(\vartheta)$, $\vartheta$ and $\vartheta_k$, respectively.

By (28), the function $\bar{\psi}(\bar{\vartheta})$ can be expressed as

$$\bar{\psi}(\bar{\vartheta}) = \left[\sum_{m=0}^{\infty} B_m \vartheta^{2m}\right] = \sum_{m=0}^{\infty} B_m \bar{\vartheta}^{2m} \Rightarrow \bar{\psi}(\bar{\vartheta}) = \psi(\vartheta).$$

Making use of (28), we have

$$\psi(\vartheta) = \psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right),$$

which implies from (30) that

$$\bar{\psi}(\bar{\vartheta}) = \psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\bar{\vartheta}^2}{\bar{\vartheta}_k^2}\right).$$

Similarly, by (28), we have

$$\psi(\vartheta) = \psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right),$$

such that

$$\bar{\psi}(\bar{\vartheta}) = \left[\psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right)\right] = \psi(0) \prod_{\Im(\vartheta_k)>0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right),$$

since

$$\psi(0) = B_0 > 0,$$
proved that one takes $\vartheta = 0$ into (28).

Combining (31) and (32), we suggest

\[(34) \quad \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).\]

In view of (30) and (34), we demonstrate the identity

\[(35) \quad \overline{\psi(\vartheta)} = \psi(\overline{\vartheta}) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).\]

3.3. **Propose a class of $\psi(t)$**. Putting

\[(36) \quad \vartheta = t \in \mathbb{R}\]

into (29), we find that

\[(37) \quad \overline{\psi(t)} = \psi(\overline{t}) = \psi(t).\]

With (31) and (38), we arrive at

\[(39) \quad \overline{\psi(t)} = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).\]

With the aid of (39), we may get

\[(40) \quad \psi(t) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).\]

In a similar way, by using (32) and (38), we show

\[(41) \quad \overline{\psi(t)} = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).\]

With (38) and (41), we obtain

\[(42) \quad \psi(t) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).\]

Combining (40) and (42), we suggest

\[(43) \quad \psi(t) = \sum_{m=0}^{\infty} B_m t^{2m} = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{\vartheta^2}{\vartheta_k^2}\right).\]
3.4. Consider the convergence of \( \psi(\vartheta) \). Considering the fact \( B_m > 0 \) and using (43), we have

\[
\psi(1) = \sum_{m=0}^{\infty} B_m
\]

such that

\[
\psi(1) > 0. \tag{45}
\]

With use of (45) and (43), we find that

\[
\psi(1) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{1}{\vartheta_k^2}\right) > 0, \tag{46}
\]

\[
\psi(1) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{1}{\vartheta_k^2}\right) > 0, \tag{47}
\]

and

\[
\psi(1) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{1}{\vartheta_k^2}\right) = \psi(0) \prod_{\Im(\vartheta_k) > 0} \left(1 - \frac{1}{\vartheta_k^2}\right). \tag{48}
\]

It follows from (24) that the products of (48) are absolutely convergent. This implies that the products of (48) are convergent.

Titchmarsh (see [19], p.14-15) argued that (48) is convergent if and only if

\[
\sum_{k=1}^{\infty} \frac{1}{\vartheta_k^2}
\]

are convergent and there always exists

\[
\sum_{k=1}^{\infty} \frac{1}{\vartheta_k}
\]

Applying (49) and (50), we see that (28) and (43) are convergent.

3.5. **Present the identity** \( \Re(\vartheta_k) = 0 \). Since (49) and (50) are valid, (51) can be written as

\[
\sum_{k=1}^{\infty} \frac{1}{\vartheta_k} - \sum_{k=1}^{\infty} \frac{1}{\vartheta_k^2} = 0. \tag{52}
\]

From (52) we obtain

\[
\overline{\vartheta_k^2} - \vartheta_k^2 = 0 \tag{53}
\]
or, alternatively,

\[ (v_k - \overline{v}_k) (v_k + \overline{v}_k) = 0. \]  

Since \( \Im (v_k) > 0 \) and \( v_k - \overline{v}_k = 2i \Im (v_k) \), we have from (54) that

\[ v_k + \overline{v}_k = 2 \Re (v_k) = 0, \]

or, alternatively,

\[ \Re (v_k) = 0. \]

Taking

\[ |\Im (v_k)| = \sigma_k > 0 \]

and substituting (56) back into (43), we obtain

\[ \psi (t) = \psi (0) \prod_{\Im (v_k) > 0} \left( 1 - \frac{\varphi^2}{\sigma_k^2} \right) = \psi (0) \prod_{\sigma_k} \left( 1 + \frac{\varphi^2}{\sigma_k^2} \right) = \psi (0) \prod_{k=1}^{\infty} \left( 1 + \frac{\varphi^2}{\sigma_k^2} \right). \]

Adopting (53), (56) and (57), we have

\[ \varphi^2_k = \overline{\varphi^2}_k = -\sigma_k^2. \]

It follows from (51) that

\[ p = \sum_{k=1}^{\infty} \frac{1}{v_k} = \sum_{k=1}^{\infty} \frac{1}{\varphi_k} = -\sum_{k=1}^{\infty} \frac{1}{\sigma_k}. \]

From (25) we have

\[ \sum_{k=1}^{\infty} \frac{1}{|\varphi_k|^2} = \sum_{k=1}^{\infty} \frac{1}{\sigma_k^2} = -p. \]

Consequently, (56) is true.

Combining (23) and (56), we present

\[ \psi (\vartheta) = \sum_{m=0}^{\infty} B_m \vartheta^{2m} = \psi (0) \prod_{\varphi_k} \left( 1 - \frac{\vartheta}{\varphi_k} \right) \]

\[ = \psi (0) \prod_{\Im (\varphi_k) > 0} \left( 1 - \frac{\vartheta^2}{\varphi_k^2} \right) \]

\[ = \psi (0) \prod_{\Im (\varphi_k) > 0} \left( 1 + \frac{\vartheta^2}{\sigma_k^2} \right) \]

\[ = \psi (0) \prod_{k=1}^{\infty} \left( 1 + \frac{\vartheta^2}{\sigma_k^2} \right). \]
3.6. **Prove that all zeros of $\mathbb{N}(\tau)$ are real.** Taking $\vartheta = i\tau$ in (62) implies that

\[ (63) \quad \psi (i\tau) = \sum_{m=0}^{\infty} B_m (i\tau)^{2m} = \psi (0) \prod_{k=1}^{\infty} \left[ 1 + \frac{(i\tau)^2}{\sigma_k^2} \right]. \]

To simplify (63), we obtain

\[ (64) \quad \psi (i\tau) = \sum_{m=0}^{\infty} (-1)^m B_m \tau^{2m} = \psi (0) \prod_{k=1}^{\infty} \left( 1 - \frac{\tau^2}{\sigma_k^2} \right). \]

Taking

\[ (65) \quad \mathbb{N}(\tau) = \psi (i\tau) \]

in (64), we have

\[ (66) \quad \mathbb{N}(\tau) = \sum_{m=0}^{\infty} (-1)^m B_m \tau^{2m} = \psi (0) \prod_{k=1}^{\infty} \left( 1 - \frac{\tau^2}{\sigma_k^2} \right) = \mathbb{N}(0) \prod_{k=1}^{\infty} \left( 1 - \frac{\tau^2}{\sigma_k^2} \right), \]

which leads to

\[ (67) \quad \mathbb{N}(0) = B_0 = \psi (0) \]

when one substitutes $\tau = 0$ into (66).

Similarly, combining (62), (65) and (67), we have

\[ (68) \quad \psi (i\tau) = \psi (0) \prod_{\vartheta_k} \left( 1 - \frac{\vartheta^2}{\vartheta_k^2} \right) \]

such that

\[ (69) \quad \mathbb{N}(\tau) = \psi (0) \prod_{\vartheta_k} \left( 1 - \frac{\vartheta^2}{\vartheta_k^2} \right) = \mathbb{N}(0) \prod_{\vartheta_k} \left( 1 - \frac{\vartheta^2}{\vartheta_k^2} \right). \]

Since (66) is equivalent to (69), we have the identity

\[ (70) \quad \mathbb{N}(\tau) = \sum_{m=0}^{\infty} (-1)^m B_m \tau^{2m} = \mathbb{N}(0) \prod_{\vartheta_k} \left( 1 - \frac{\vartheta^2}{\vartheta_k^2} \right) = \mathbb{N}(0) \prod_{k=1}^{\infty} \left( 1 - \frac{\tau^2}{\sigma_k^2} \right). \]

From (70) it is observed that all zeros of $\mathbb{N}(\tau)$ are real.

Thus, this is required result.

**Remark.** Replacing $\vartheta \in \mathbb{C}$ by $\vartheta \in \mathbb{C}$ in (35), we also deduce that

\[ \psi (\vartheta) = \psi (0) \prod_{\Im (\vartheta_k) > 0} \left( 1 - \frac{\vartheta^2}{\vartheta_k^2} \right) = \psi (0) \prod_{\Im (\vartheta_k) > 0} \left( 1 - \frac{\vartheta^2}{\vartheta_k^2} \right), \]

which leads to (43) and (48) when $\vartheta = t \in \mathbb{R}.$
3.7. **Equivalently sufficient conditions.** With use of Lemma 1 and (70), we obtain

\[
\mathfrak{N} (\tau) = \sum_{m=0}^{\infty} \alpha_{m} \tau^{2m} = \mathfrak{N} (0) \prod_{\Im (\vartheta_{k}) > 0} \left( 1 + \frac{\tau^{2}}{\vartheta_{k}^{2}} \right) = \mathfrak{N} (0) \prod_{k=1}^{\infty} \left( 1 - \frac{\tau^{2}}{\sigma_{k}^{2}} \right).
\]

(71)

From (71) we know that ±σ_k (σ_k > 0) are all real zeros of \( \mathfrak{N} (\tau) \).

As a direct result of (71), we have the following:

**Corollary 1.** If \( \mathfrak{N} \in \wp \), then there exist the following equivalent representations:

(A) All of the zeros of \( \mathfrak{N} (\tau) \) are real.

(B) There exists the identity

\[
\sum_{m=0}^{\infty} \alpha_{m} \tau^{2m} = \mathfrak{N} (0) \prod_{\Im (\vartheta_{k}) > 0} \left( 1 + \frac{\tau^{2}}{\vartheta_{k}^{2}} \right).
\]

(72)

(C) There exists the identity

\[
\sum_{m=0}^{\infty} \alpha_{m} \tau^{2m} = \Lambda (0) \prod_{\vartheta_{k}} \left( 1 - \frac{\tau^{2}}{\vartheta_{k}} \right).
\]

(73)

(D) There exists the identity

\[
\sum_{m=0}^{\infty} \alpha_{m} \tau^{2m} = \Lambda (0) \prod_{k=1}^{\infty} \left( 1 - \frac{\tau^{2}}{\sigma_{k}^{2}} \right).
\]

(74)

By Corollary 1, we see that there are some sufficient conditions that all of the zeros of \( \mathfrak{N} (\tau) \) are real.

4. **A TYPICAL APPLICATION**

4.1. **Prove the conjecture of Lagarias and Montague.** Let us start with its proof.

By using (5), the Lagarias-Montague function \( \Phi (\tau) \) can be written as

\[
\Phi (\tau) = 2 \int_{0}^{\infty} g (\ell) \ell^{-1} \sin (\tau \ell) d\ell = 2 \int_{0}^{\infty} g (\ell) \ell^{-1} \left[ \sum_{m=0}^{\infty} \frac{(-1)^{n} (\tau \ell)^{2m+1}}{(2m+1)!} \right] d\ell.
\]

(75)

To simplify (75), we obtain

\[
\Phi (\tau) = \sum_{m=0}^{\infty} (-1)^{n} \lambda_{2m+1} \tau^{2m+1},
\]

(76)

where

\[
\lambda_{2m+1} = 2 \int_{0}^{\infty} g (\ell) \frac{\ell^{2m}}{(2m+1)} d\ell > 0.
\]

(77)

Because of

\[
\Xi (\tau) = \Phi^{(1)} (\tau),
\]

(78)
In fact, Boas (see Theorem 2.4.1 in [2], p.13) argued that \( \Phi (\tau) \) and \( \Xi (\tau) \) are of the same order and type. In view of the work of Dimitrov and Lucas [20], \( \Phi (\tau) \) and \( \Xi (\tau) \) are the functions order \( \rho_1 = 1 \).

Let
\[
\hat{\Phi}_1 (\tau) = \frac{\Phi (i\tau)}{i\tau} = \sum_{m=0}^{\infty} \lambda_{2m+1} \tau^{2m}
\]
and
\[
\hat{\Phi}_2 (\tau) = \frac{\Phi (\tau)}{\tau} = \sum_{m=0}^{\infty} (-1)^m \lambda_{2m+1} \tau^{2m}
\]
such that
\[
F (\tau) = i\tau \hat{\Phi}_1 (\tau)
\]
and
\[
\Phi (\tau) = \tau \hat{\Phi}_2 (\tau).
\]
Since (79) and (80) are of the same order and type due to the fact
\[
\lambda_{2m+1} = |(-1)^m \lambda_{2m+1}|,
\]
(81) and (82) are also of the same order and type. This implies that (79), (80), (81) and (82) are of order \( \rho_1 = 1 \). Moreover, (79) and (80) are the even entire functions.

Since (79) is an even entire function of order \( \rho_1 = 1 \) with the positive real coefficients \( \lambda_{2m+1} > 0 \), Theorem 3 in Levin’s book (see [1], p.8) said that the product presentation of \( \Phi (it) \) reads
\[
F (\tau) = it \hat{\Phi}_1 (\tau) = i\tau \hat{\Phi}_1 (0) \prod_{u_k} \left( 1 - \frac{\tau}{u_k} \right)
\]
with
\[
\hat{\Phi}_1 (\tau) = \hat{\Phi}_1 (0) \prod_{u_k} \left( 1 - \frac{\tau}{u_k} \right),
\]
where \( u_k \) run the zeros of \( \hat{\Phi}_1 (\tau) \).

If \( \Im (u_k) > 0 \) and (79) is an even function with the complex zeros, then there is the functional equation
\[
\hat{\Phi}_1 (\tau) = \hat{\Phi}_1 (-\tau)
\]
such that
\[
\hat{\Phi}_1 (\tau) = \hat{\Phi}_1 (0) \prod_{u_k} \left( 1 - \frac{\tau}{u_k} \right)
\]
\[
= \hat{\Phi}_1 (0) \prod_{\Im (u_k) > 0} \left( 1 - \frac{\tau}{u_k} \right) \left( 1 + \frac{\tau}{u_k} \right)
\]
\[
= \hat{\Phi}_1 (0) \prod_{\Im (u_k) > 0} \left( 1 - \frac{\tau^2}{u_k^2} \right).
\]
From (79) and (80) we have

\[ \hat{\Phi}_1 (it) = \hat{\Phi}_2 (t) \]  

and

\[ \hat{\Phi}_1 (0) = \hat{\Phi}_2 (0) = 2 \int_0^{\infty} g (\ell) \frac{\ell^{2m}}{(2n+1)!} d\ell > 0. \]

With the aid of (86), (87) and (88), we may get

\[ \hat{\Phi}_2 (\tau) = \hat{\Phi}_1 (0) \prod_{\Im (u_k) > 0} \left( 1 + \frac{\tau^2}{u_k^2} \right) = \hat{\Phi}_2 (0) \prod_{\Im (u_k) > 0} \left( 1 + \frac{\tau^2}{u_k^2} \right). \]

By combination of (80) and (89), we present

\[ \hat{\Phi}_2 (\tau) = \sum_{m=0}^{\infty} (-1)^m \lambda_{2m+1} \tau^{2m} = \hat{\Phi}_2 (0) \prod_{\Im (u_k) > 0} \left( 1 + \frac{\tau^2}{u_k^2} \right). \]

Because of the fact \( \hat{\Phi}_2 (\tau) \) is of order \( \rho_1 = 1 \), Theorem 13 ([1], p.24) has reported that there exists any \( \hat{\varepsilon} > 0 \) such that the series

\[ \sum_{k=1}^{\infty} \frac{1}{|u_k|^{1+\hat{\varepsilon}}} \]

is convergent.

Taking \( \hat{\varepsilon} = 1 \) in (91), the series

\[ \sum_{k=1}^{\infty} \frac{1}{|u_k|^2} \]

is convergent and (89) is also convergent.

In sum, we have the followings three conditions:

(A1) \( \hat{\Phi}_2 (\tau) \) is of order \( \rho_1 = 1 \).

(A2) The identity (90) holds for \( \tau \in \mathbb{C} \).

(A3) The series (92) is convergent.

Then, we obtain

\[ \hat{\Phi}_2 \in \wp. \]

By using Theorem 1 and (B) in Corollary 1, we have from (90) that all zeros \( u_k \) of the even entire function \( \hat{\Phi}_2 (\tau) \) are real if \( \hat{\Phi}_2 \in \wp \).

This implies that

\[ u_k = \pm i \beta_k, \]

where \( \beta_k > 0 \).

By using (94), the identity (90) can be written as

\[ \hat{\Phi}_2 (\tau) = \sum_{m=0}^{\infty} (-1)^m \lambda_{2m+1} \tau^{2m} = \hat{\Phi}_2 (0) \prod_{\Im (u_k) > 0} \left( 1 - \frac{\tau^2}{\beta_k^2} \right). \]
With (76), (80) and (95), we clearly see that

\[
\Phi(\tau) = \sum_{m=0}^{\infty} (-1)^n \lambda_{2m+1} \tau^{2m+1}
\]

\[
= \tau \Phi_2(0) \prod_{\beta_k > 0} \left( 1 - \frac{\tau^2}{\beta_k^2} \right)
\]

(96)

\[
= \tau \Phi_2(0) \prod_{u_k} \left( 1 - \frac{\tau}{i\beta_k} \right)
\]

\[
= \tau \Phi_2(0) \prod_{k=1}^{\infty} \left( 1 - \frac{\tau^2}{\beta_k^2} \right).
\]

It follows from (96) that all zeros of \( \Phi(\tau) \) are real because \( \beta_k > 0 \).

Thus, we prove the conjecture of Lagarias and Montague.

**Remark.** By substitution of (94) into (86), we have

\[
\hat{\Phi}_1(\tau) = \sum_{m=0}^{\infty} \lambda_{2m+1} \tau^{2m}
\]

\[
= \hat{\Phi}_1(0) \prod_{u_k} \left( 1 - \frac{\tau}{i\beta_k} \right)
\]

(97)

\[
= \hat{\Phi}_1(0) \prod_{\beta_k > 0} \left( 1 - \frac{\tau}{i\beta_k} \right) \left( 1 + \frac{\tau}{i\beta_k} \right)
\]

\[
= \hat{\Phi}_1(0) \prod_{k=1}^{\infty} \left( 1 + \frac{\tau^2}{\beta_k^2} \right),
\]

which implies that

\[
F(\tau) = i\tau \sum_{m=0}^{\infty} \lambda_{2m+1} \tau^{2m} = i\tau \hat{\Phi}_1(0) \prod_{u_k} \left( 1 - \frac{\tau}{u_k} \right) = i\tau \hat{\Phi}_1(0) \prod_{k=1}^{\infty} \left( 1 + \frac{\tau^2}{\beta_k^2} \right).
\]

(98)

4.2. **Two open problems.** From (97) and (98) it is easily seen that all zeros of \( \hat{\Phi}_1(\tau) \) are \( \tau = \pm i\beta_k \), where \( \beta_k > 0 \), and that the function \( F(\tau) \) has the purely imaginary number zeros \( \tau = \pm i\beta_k \), where \( \beta_k > 0 \), and real zero \( t = 0 \). Similarly, by (95) and (96), it is also observed that all real zeros of \( \hat{\Phi}_2(\tau) \) are \( \tau = \pm \beta_k \), where \( \beta_k > 0 \), and that the function \( \Phi(\tau) \) has the real zeros \( \tau = \pm \beta_k \), where \( \beta_k > 0 \) and real zero \( \tau = 0 \). Here, we call \( \hat{\Phi}_1(\tau) \) as the hungry pair of \( \hat{\Phi}_2(\tau) \) if there exist all zeros \( \tau = \pm \beta_k \) of \( \hat{\Phi}_2(\tau) \) and all zeros \( \tau = \pm i\beta_k \) of \( \hat{\Phi}_1(\tau) \). Computing real zeros of (5) is still an open problem in the theory of the Lagarias-Montague function.
By using the observation that all zeros of
\[ \Phi(t) = \int_0^t \Xi(t) \, dt \]
and \( \Xi(t) = \Phi^{(1)}(t) \) are real, we have the followings:

**Problem 1.** Let \( \aleph \in \wp \). Then all zeros of \( G(\tau) \) are real if \( G(t) = \aleph^{(1)}(t) \).

As an analogous problem 1, the real zeros of the derivative of the entire function in the Laguerre-Pólya class was proposed by Pólya in 1913 [19] and proved by Hellerstein and Williamson [20, 21].

**Problem 2.** Let \( \aleph \in \wp \). Then all zeros of \( M(\tau) \) are real if there exists the integral

(99) \[ M(t) = \int_0^t \aleph(t) \, dt. \]

Here, we easily find that all zeros of the function \( M(\tau) = \cos(\tau) \) are real if
\[ \cos(t) = \int_0^t \sin(t) \, dt, \]
and that all zeros of the function \( G(\tau) = -\sin(\tau) \) are real if \( \cos^{(1)}(t) = -\sin(t) \). Also, it is easy to see that \( \cosh(\tau) \) is considered as the hungry pair of \( \cos(\tau) \). As a direct result of Corollary 1, we have the following:

**Corollary 2.** \( \cos(\tau) \) belongs to the class $\wp$.

**Proof.** Adopting the product representations of \( \cos(\tau) \) (see [22], p.114), we structure the class of the series and product representations of \( \cos(\tau) \), given as

(100) \[ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \tau^{2m} = \prod_{k=1}^{\infty} \left[ 1 - \frac{\tau^2}{(k - \frac{1}{2})^2 \pi^2} \right]. \]

By Corollary 1 and (100), we directly obtain the required result because \( \cos(\tau) \) is an even function of order \( \gamma = 1 \) (see [22], p.255) and the Euler’s product of (100) is convergent. □

5. Conclusion

In the present article we have proposed a sufficient condition for all real zeros of a class of the even entire function. With the aid of the obtained result, we have proved that the conjecture of Lagarias and Montague is true. By comparison between the real zeros of the Riemann Xi and the Lagarias-Montague functions, we have suggested two open problems for the even entire functions. The result may be proposed as a new mathematical approach to open a new door for handling the de Bruijn-Newman constant.
REFERENCES

[1] B. Y. Levin, Distribution of zeros of entire functions, Vol. 150, American Mathematical Society, 1980.
[2] R. P. Boas, Entire functions, Academic press, 2011.
[3] A. A. Requicha, The zeros of entire functions: Theory and engineering applications, Proceedings of the IEEE, 68 (1980) (3), 308-328.
[4] B. Q. Li, Two elementary properties of entire functions and their applications, The American Mathematical Monthly, 122 (2015) (2), 169-172.
[5] G. T. Deng and T. Qian, An application of entire function theory to analytic signals, Journal of Mathematical Analysis and Applications, 389 (2012) (1), 54-57.
[6] N. Anghel, Entire functions of finite order as solutions to certain complex linear differential equations, Proceedings of the American Mathematical Society, 140 (2012) (7), 2319-2332.
[7] G. Csordas, Fourier transforms of positive definite kernels and the Riemann \( \xi \)-Function. Computational Methods and Function Theory, 15 (2015) (3), 373-391.
[8] J. L. W. V. Jensen, Recherches sur la théorie des equations, Acta Mathematica, 36 (1913) (1) 181-195.
[9] G. Pólya, Über trigonometrische Integrale mit nur reellen Nullstellen, Journal für die reine und angewandte Mathematik, 158 (1927), 6-18.
[10] G. Pólya, Über die algebraisch-funktionentheoretischen Untersuchungen von J L W V Jensen, Mathematisk-fysiske Meddelelse VIII, 17 (1927), 1-33.
[11] G. Csordas and R. S. Varga, Integral transforms and the laguerre-pólya class, Complex Variables and Elliptic Equations, 12 (1989) (1-4), 211-230.
[12] D. N. G. Bruijn, The roots of trigonometric integrals, Duke Mathematical Journal, 17 (1950) (3), 197-226.
[13] M. Griffin, K. Ono, L. Rolen and D. Zagier, Jensen polynomials for the Riemann zeta function and other sequences, Proceedings of the National Academy of Sciences, 116 (24) (2019), 11103-11110.
[14] C. M. Newman, Fourier transforms with only real zeros, Proceedings of the American Mathematical Society, 61 (1976) (2), 245-251.
[15] B. Rodgers and T. Tao, The de Bruijn-Newman constant is non-negative, Forum of Mathematics, Pi, Vol. 8, Cambridge University Press, 2020.
[16] H. Ki, Y. O. Kim and J. Lee, On the de Bruijn-Newman constant, Advances in Mathematics, 222 (2009) (1), 281-306.
[17] J. C. Lagarias and D. Montague, The integral of the Riemann xi-function, Rikkyo Daigaku sugaku zasshi, 60 (2011) (1-2), 143-169.
[18] X. J. Yang, All nontrivial zeros for the Riemann zeta function are on the critical line \( \Re(s) = 1/2 \), arXiv: 1811.02418v15.
[19] G. Pólya, Über Annäherung durch Polynome mit lauter reellen Wurzeln, Rendiconti del Circolo Matematico di Palermo, 36 (1913) (1), 279-295.
[20] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question of Pólya, Bulletin of the American Mathematical Society, 81 (1975) (2), 453-455.
[21] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question of Pólya. II, Transactions of the American Mathematical Society, 234 (1977) (2), 497-503.
[22] E. C. Titchmarsh, The theory of functions, Oxford University Press, 1939.

Email address: dyangxiaojun@163.com; xjyang@cumt.edu.cn

1 School of Mathematics, and State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China

2 Department of Mathematics, Faculty of Science, King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia