STOCHASTIC MAPPINGS AND RANDOM DISTRIBUTION FIELDS. A CORRELATION APPROACH

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ABSTRACT. This paper contains a study of multivariate second order stochastic mappings indexed by an abstract set \( \Lambda \) in close connection to their operator covariance functions. The characterizations of the normal Hilbert module or of Hilbert spaces associated to such a m.s.o. stochastic mapping in terms of reproducing kernel structures are given, aiming not only to gather into a unified way some concepts from the field, but also indicate an instrument to extend the very well elaborated theory of multivariate second order stochastic processes (or random fields) to the case of multivariate random distribution fields, treated in the second half of the paper, where also a first step in prediction is given.

1. INTRODUCTION

The study of stochastic or random processes is nowadays among the important topics in mathematical research. This is the area where our present attempt fits. Let us briefly describe how our research can be placed in the scientific literature. The necessity of simultaneous study of more stochastic processes led to the concept of multivariate (finite or infinite variate) stochastic process. Throughout this paper multivariate random variables on a probability space \((\Omega, \mathcal{A}, \varnothing)\) means that they are \(H\)-valued, with \(H\) an infinite dimensional separable complex Hilbert space, for which \(\mathcal{B}(H)\) denotes the algebra of bounded linear operators on \(H\), while \(\mathcal{C}_1(H)\) is the ideal of trace class operators from \(\mathcal{B}(H)\). The set of multivariate random variables will be denoted by \(L^0(\varnothing, H)\) (where \(H\) will be omitted if \(H = \mathbb{C}\)) and will be organized with natural operations and convergence in measure as a complete topological left \(\mathcal{B}(H)\)-module. Observing that, via the conjugate \(\mathcal{B}(H)\)-linear embedding

\[
L^0(\varnothing, H) \ni f \mapsto W_f \in \mathcal{B} \left( H, L^0(\varnothing) \right),
\]

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where \( W_f x := \langle x, f(\cdot) \rangle_H \), \( x \in H \), \( L^0(\vartheta, H) \) can be embedded into the complete topological right \( \mathcal{B}(H) \)-module \( \mathcal{B}(H, L^0(\vartheta)) \) of all continuous linear operators from \( H \) on \( L^0(\vartheta) \). So it is natural to use for the elements of \( \mathcal{B}(H, L^0(\vartheta)) \) the terminology generalized multivariate random variables.

Generally speaking a (generalized) multivariate stochastic process is a family \( \{ \Phi_\lambda \}_{\lambda \in \Lambda} \) of (generalized) multivariate random variables, where if \( d \) is a non zero natural number and the index set \( \Lambda \) is the cartesian product \( \mathbb{Z}^d \) of \( d \) copies of the set of integers \( \mathbb{Z} \), we have a discrete \( d \)-time parameter (generalized) multivariate stochastic process or, if \( \Lambda \) is the \( d \)-dimensional real euclidian space \( \mathbb{R}^d \), we have a continuous \( d \)-time parameter (generalized) multivariate stochastic process.

It is often required for \( \Phi_\lambda \), \( \lambda \in \Lambda \), to be strongly or weakly square integrable and of zero mean, i.e. \( \Phi_\lambda \in L^2_{s,0}(\vartheta, H) \) or \( \Phi_\lambda \in L^2_{w,0}(\vartheta, H) \), respectively. Let’s note that the mapping

\[
(1.2) \quad L^2_{s,0}(\vartheta, H) \ni f \mapsto V_f \in C_2(L^2_0(\vartheta), H),
\]

where

\[
(1.3) \quad V_f \chi = \int_\Omega \chi(\omega)f(\omega)d\vartheta(\omega), \quad \chi \in L^2_0(\vartheta),
\]

is a module isomorphism of \( L^2_{s,0}[\vartheta, H] \) (under this notation regarded as a normal Hilbert \( \mathcal{B}(H) \)-module, [10, Section1.3]) onto the class of Hilbert-Schmidt operators \( C_2(L^2_0(\vartheta), H) \), while its extension to \( L^2_{w,0}(\vartheta, H) \) embeds it in a natural way into the Hilbert left \( \mathcal{B}(H) \)-module \( \mathcal{B}(L^2_0(\vartheta), H) \). So the elements of \( \mathcal{H} \), which as in [10] represents a common notation for \( L^2_{s,0}[\vartheta, H] \) and \( C_2(L^2_0(\vartheta), H) \), respectively the elements of \( \mathcal{B}(L^2_0(\vartheta), H) \) will be called strong second order multivariate random variables of zero mean, respectively generalized multivariate second order random variables of zero mean. This last frame is the most adequate structure to treat the weakly square integrable (i.e. weak second order) multivariate random variables. In what follows instead of strong second order we use simply the term second order. So, in these cases we speak about \( \{ \Phi_\lambda \}_{\lambda \in \Lambda} \) as being a (generalized if \( \mathcal{B}(L^2_0(\vartheta), H) \)-valued) multivariate second order stochastic process. For an embedding as before, see also [21] or [3], where Banach space valued random variables are considered and the term of generalized second order stochastic process in this sense first appears.

When \( d > 1 \), or even when \( \Lambda \) is a locally compact abelian group, the term (generalized) random field is preferred instead of (generalized) stochastic process, while when \( \Lambda \) is an arbitrary index set, the term (generalized) stochastic mapping is the most appropriate. When \( \Lambda \) is a separable metric space, then we find ourself with multivariate stochastic mappings in the particular situation of the infinite dimensional random mappings from the relatively recent paper [18]. Also covered by this concept are the random integrals and the random operators.
For example, when $\Lambda = H$ and $\Phi$ is a continuous linear operator from $H$ to $L^0(\varphi, H)$, then it is known as a random operator in the sense of Skorohod, whereas if $\Lambda$ is a dense subspace in $H$, then a closed linear operator from $\Lambda$ to $L^0(\varphi, H)$ is a random operator in the sense of Hackenbroch (see [8]). The concept of univariate second order stochastic mapping mentioned as such by H. Niemi in the introduction of the paper [12], while the concept of the generalized multivariate second order stochastic mapping was introduced by P. Masani in [11] under the name hilbertian variety. In the framework of this last concept we may place also the works of W. Hackenbroch [7] on Hilbert space operator valued processes, S. A. Chobanyan and A. Weron [3] on prediction theory in Banach spaces and of I. Suciu and I. Valuşescu [16, 19, 20] about the study of stochastic processes in the context of complete correlated actions.

An important tool in the development of the theory in all mentioned areas is the reproducing kernel technique for Hilbert spaces and Hilbert $C^*$-modules which are well presented in [2], respectively in [17].

In what follows we restrain ourselves to the study of multivariate second order (m.s.o.) stochastic mappings, which cover not only some particular concepts used in the very well developed theory of m.s.o. random fields (see [10]), but also an extension which we have in view. Namely, if we intent to consider the m.s.o. stochastic processes not only as $\mathcal{H}$-valued or $\mathcal{B}(L^2_0(\varphi), H)$-valued functions on $\mathbb{R}^d$, but more generally as $\mathcal{H}$-valued respectively $\mathcal{B}(L^2_0(\varphi), H)$-valued distributions on $\mathbb{R}^d$, then it is necessary to have $\Lambda = D(\mathbb{R}^d) = \mathcal{D}_d$, the space of test functions in the theory of distributions. For such a (generalized) m.s.o. stochastic mapping we shall use the term (generalized) m.s.o. random distribution field (m.s.o.r.d.f.).

Mentioning that, for the univariate one time parameter case, such an extension of stochastic processes was considered for the first time by K. Itô ([9]) and I.M. Gelfand ([1]) in 1953 and 1955 respectively and then for the finite variate $d$ time parameter case in 1957 by A. M. Yaglom [25] (see also [5], [24], [26], [23]), we emphasize that the starting point of our research was the extension of the theory of m.s.o. random fields on $\mathbb{R}^d$ (treated in [10]) to the m.s.o. random distribution fields.

Some basic results regarding the correlation theory formulated in the general setting of m.s.o. stochastic mappings are given in Section 2, while Section 3 is devoted to its application to the class of m.s.o. random distribution fields on $\mathbb{R}^d$ and to some of its remarkable subclasses consisting of m.s.o. stochastic not necessarily bounded measures of finite operator semivariation, or of m.s.o. not necessarily summable random Radon measures. Some elements of prediction for m.s.o.r.d.f. are also given, including a general Wold-type decomposition into deterministic and purely nondeterministic parts.
2. Multivariate second order stochastic mappings

In this section we consider (continuous) m.s.o. stochastic mappings, indexed over an abstract set (topological space) Λ, which in what follows will be denoted (in both cases) by \( M(\Lambda, \mathcal{H}) \). In analogy to [10, Section 4.1] we associate to such stochastic mappings the vector domain, the measurements space and the operator and scalar (cross) covariance functions. The corresponding modular and vector domains are then characterized as reproducing kernel structures (a Hilbert module and a Hilbert space) considered in [10, Section 2.4], reproduced by a positive definite \( C_1(H) \)-valued kernel on Λ, which is just the operator covariance function of the stochastic mapping. Analogously the measurements space will be identified as a Hilbert space reproduced by the scalar covariance function of the stochastic mapping. Also for a general positive definite \( C_1(H) \)-valued kernel, as considered in [10, Section 2.4], a Kolmogorov type factorization in terms of a m.s.o. stochastic mapping is obtained. Thus, this Section can be regarded as being complementary to Section 2.4 from the book [10].

For a given m.s.o.s.m. \( \Phi \in M(\Lambda, \mathcal{H}) \) we denote by \( H(\Phi) \) the closed linear subspace of \( L^2_{\mathcal{H}_0}(\mathcal{H}) \) generated by the values \( \{ \Phi(\mu), \mu \in \Lambda \} \) and call it the vector domain of \( \Phi \), while the modular domain of \( \Phi \) is the closure \( H_\Phi \) in \( H \) of the submodule

\[
H_\Phi^0 := \left\{ \sum_{i \in \mathbb{N}_m} a_i \Phi(\lambda_i), a_i \in B(H), \lambda_i \in \Lambda, i \in \mathbb{N}_m, m \in \mathbb{N} \right\},
\]

where \( \mathbb{N}_m = \{1, 2, \ldots, m\} \). Obviously, \( H(\Phi) \subset H_\Phi \).

Using the operator model of the normal Hilbert \( B(H) \)-module \( H_\Phi \) (see Corr. 7 pp. 30 of [10]) and denoting \( G_\Phi := G_{H_\Phi} \), which will be called the measurements space of \( \Phi \), we have the module isomorphisms

\[
H_\Phi \cong C_2(G_\Phi, H) \cong H \hat{\otimes} G_\Phi,
\]

where \( H \hat{\otimes} G \) is the hilbertian tensor product of the Hilbert spaces \( H \) and \( G \) as in [10] pp.20.

In such terms we can introduce in \( M(\Lambda, \mathcal{H}) \) the relation of subordination. Namely, we shall say that \( \Phi \) is subordinate (operator subordinate) to \( \Psi \) if \( H(\Phi) \subset H(\Psi) \) (\( H_\Phi \subset H_\Psi \), respectively).

Also for two elements \( \Phi, \Psi \in M(\Lambda, \mathcal{H}) \) the operator cross covariance function \( \Gamma_{\Phi, \Psi} \) will be defined as

\[
\Gamma_{\Phi, \Psi}(\lambda, \mu) := [\Phi(\lambda), \Psi(\mu)]_{\mathcal{H}}; \quad \lambda, \mu \in \Lambda,
\]

while the scalar cross covariance function \( \gamma_{\Phi, \Psi} \) is defined by

\[
\gamma_{\Phi, \Psi}(\lambda, \mu) := \text{tr}\Gamma_{\Phi, \Psi}(\lambda, \mu); \quad \lambda, \mu \in \Lambda.
\]

When \( \Phi = \Psi \), then we denote simply \( \gamma_{\Phi, \Phi} = : \gamma_\Phi \) and \( \Gamma_{\Phi, \Phi} = : \Gamma_\Phi \), which will be called the scalar covariance function, respectively the operator covariance function of \( \Phi \).
Now, the operator covariance function $\Gamma_\Phi$ (the scalar one $\gamma_\Phi$, respectively) of $\Phi$ is a $C^1_1(H)$-valued ($C$-valued) positive definite kernel on $\Lambda$, in the sense of the positivity from $B(H)$, i.e. it holds

\[(2.5) \quad \sum_{i,j \in \mathbb{N}_m} a_i \Gamma_\Phi(\lambda_i, \lambda_j) a_j^* \geq 0,\]

for any $m \in \mathbb{N}$ and any finite systems $a_1, \ldots, a_m \in B(H)$, $\lambda_1, \ldots, \lambda_m \in \Lambda$, (respectively in the sense of the usual positivity in $C$)

\[(2.6) \quad \sum_{i,j \in \mathbb{N}_m} \alpha_i \alpha_j \gamma_\Phi(\lambda_i, \lambda_j) \geq 0\]

for any $m \in \mathbb{N}$ and any finite systems $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, $\lambda_1, \ldots, \lambda_m \in \Lambda$.

Indeed the relation (2.5) results by applying (2.3) and computing:

\[
\sum_{i,j \in \mathbb{N}_m} a_i \Gamma_\Phi(\lambda_i, \lambda_j) a_j^* = \sum_{i,j \in \mathbb{N}_m} a_i [\Phi(\lambda_i), \Phi(\lambda_j)]_\mathcal{F} a_j^* =
\]

\[
\sum_{i,j \in \mathbb{N}_m} [a_i \Phi(\lambda_i), a_j \Phi(\lambda_j)]_\mathcal{F} = \left[ \sum_{i \in \mathbb{N}_m} a_i \Phi(\lambda_i), \sum_{j \in \mathbb{N}_m} a_j \Phi(\lambda_j) \right]_\mathcal{F} \geq 0,
\]

while (2.6) appears also in [12].

Note also that the positive definite $C^1_1(H)$-valued kernel $\Gamma = \Gamma_\Phi$ reproduces a normal Hilbert $B(H)$-module $\mathcal{H}_{\Gamma_\Phi}$ (see [10] Section 2.4, Thm. 13, pp.37), as well as a Hilbert space $G_{\Gamma_\Phi}$ of $H$-valued functions on $\Lambda$ (see [10], Prop. 23, Section 2.4, pp.44), while $\gamma = \gamma_\Phi$ as complex valued positive definite kernel reproduces a Hilbert space $K_{\gamma_\Phi}$, (see for example [4]).

It is not hard to infer now, extending the correspondence $\Phi(\lambda) \mapsto \Gamma_\Phi(\lambda, \cdot)$ to a $B(H)$-linear mapping between the generating submodules of $\mathcal{H}_\Phi$ and $\mathcal{H}_{\Gamma_\Phi}$, that these are isomorph (i.e. $\mathcal{H}_\Phi \cong \mathcal{H}_{\Gamma_\Phi}$) as normal Hilbert $B(H)$-modules and also that the measurements space $G_{\Phi}$ associated to $\Phi$ is isomorph to the Hilbert space $G_{\Gamma_\Phi}$ ($G_\Phi \cong G_{\Gamma_\Phi}$) having $\Gamma_\Phi$ as operator reproducing kernel.

Moreover extending the correspondence $\Phi(\lambda) \mapsto \gamma_\Phi(\lambda, \cdot)$ to a linear mapping between generating subspaces of $\mathcal{H}(\Phi)$ and of $K_{\gamma_\Phi}$, respectively, we have that these are also isomorph as Hilbert spaces.

On the other hand, similarly to (2.2) the isomorphisms

\[(2.7) \quad \mathcal{H}_{\Gamma_\Phi} \cong \mathcal{C}_2(G_{\Gamma_\Phi}, H) \cong H \hat{\otimes} G_{\Gamma_\Phi}\]

also hold.

Noticing that $G_{\Gamma_\Phi}$ is generated from elements of the form $\Gamma_\Phi(\cdot, \lambda)x$ with $\lambda \in \Lambda, x \in H$, as well as the fact that the normal Hilbert $B(H)$-module isomorphism $\mathcal{H}_\Phi \cong \mathcal{H}_{\Gamma_\Phi}$ was constructed by the $B(H)$-linear extension of the correspondence $\Phi(\lambda) \mapsto \Gamma_\Phi(\lambda, \cdot)$, the measurements
space $G_\Phi$ associated to the m.s.o. stochastic mapping $\Phi$ coincides with the closed subspace in $L^2_0(\mathcal{F})$ generated by

\[(2.8) \quad \left\{ V^*_{\Phi(\lambda)} x, \, \lambda \in \Lambda, \, x \in H \right\}, \]

where $V$ is the module isomorphism \[(1.2)\].

The above results can be gathered in

**Theorem 2.1.** Given an arbitrary set (a topological space) $\Lambda$ and a (continuous) m.s.o. stochastic mapping $\Phi : \Lambda \rightarrow \mathcal{H}$ the following assertions hold:

(i) The scalar covariance function $\gamma_\Phi$, the operator covariance function $\Gamma_\Phi$ respectively, is a complex, a $C_1(H)$-valued respectively, positive definite (and continuous) kernel on $\Lambda$.

(ii) The space $\mathcal{H}(\Phi)$ (the module $\mathcal{H}_\Phi$ respectively), associated to $\Phi$ is isomorph as Hilbert space, (as normal Hilbert $\mathcal{B}(H)$-module respectively) to the reproducing kernel Hilbert space $K_{\gamma_\Phi}$, (the reproducing kernel normal Hilbert $\mathcal{B}(H)$-module $\mathcal{H}_{\Gamma_\Phi}$ respectively).

(iii) The measurements space $G_\Phi$, associated to $\Phi$ is isomorph to the Hilbert space $G_{\Gamma_\Phi}$, having $\Gamma_\Phi$ as operatorial reproducing kernel. In this way the two Hilbert spaces ensure respectively the description of the $\mathcal{B}(H)$-modules $\mathcal{H}_\Phi, \mathcal{H}_{\Gamma_\Phi}$ by means of the operatorial models mentioned in \[(2.4)\], respectively \[(2.7)\]. Moreover, $G_\Phi$ can be described directly in terms of the m.s.o. stochastic mapping $\Phi$ by \[(2.8)\].

The proof being contained in the considerations preceding the theorem we shall only mention that, in the case where $\Lambda$ is a topological space, the continuity of $\Phi$, easily implies the continuity of $\Gamma_\Phi$, which in fact is equivalent to the continuity on the diagonal of $\Lambda \times \Lambda$.

The reproducing kernel technique for normal $\mathcal{B}(H)$-modules will be also used to characterize the subordination of two m.s.o.s.m. in terms of their operator (cross) covariance functions. Namely, in analogy to the characterization given in \[10\] we have

**Theorem 2.2.** If $\Phi$ and $\Psi$ are two m.s.o.s.m. having the operator covariance functions $\Gamma_\Phi$ and $\Gamma_\Psi$ respectively, and the operator cross covariance function $\Gamma_{\Phi,\Psi}$, then $\Phi$ is subordinate to $\Psi$, iff the function $K_\lambda(\cdot) = \Gamma_{\Phi,\Psi}(\lambda, \cdot)$ is in the $\mathcal{B}(H)$-module $\mathcal{H}_{\Gamma_\Psi}$, having $\Gamma_\Psi$ as reproducing kernel and

\[(2.9) \quad [K_\lambda, K_\mu]_{\mathcal{H}_{\Gamma_\Psi}} = \Gamma_\Phi(\lambda, \mu); \quad \lambda, \mu \in \Lambda.\]

The proof is based on the fact that the operator cross covariance function $\Gamma_{\Phi,\Psi}$ of $\Phi, \Psi \in \mathcal{M}(\Lambda, \mathcal{H})$ appears in the expression of the operator covariance function of the product $\Phi \times \Psi \in \mathcal{M}(\Lambda, \mathcal{H} \times \mathcal{H})$ \[(\mathcal{H} \times \mathcal{H}\)}
being the “product” $\mathcal{B}(H)$-module as in [10, Section 4.9, pp. 192]), namely

$$\Gamma_{\Phi \times \Psi}(\lambda, \mu) = \left( \begin{array}{cc} \Gamma_{\Phi}(\lambda, \mu) & \Gamma_{\phi, \Psi}(\lambda, \mu) \\ \Gamma_{\psi, \Phi}(\lambda, \mu) & \Gamma_{\Psi}(\lambda, \mu) \end{array} \right), \quad \lambda, \mu \in \Lambda.$$  

Having in view the preceding results, the rest of the proof runs similarly as in Theorem 9.3, pp. 193 from [10, Chap.4]. □

We shall now prove two important properties of the operator correlation mapping

\begin{equation} \tag{2.10} M(\Lambda, \mathcal{H}) \ni \Phi \mapsto \Gamma_{\Phi} \in \Gamma(\Lambda, \mathcal{C}_1(H)), \end{equation}

namely its surjectivity and the description of the pre-image of each $\Gamma \in \Gamma(\Lambda, \mathcal{C}_1(H))$.

**Theorem 2.3.** (i) If $\Lambda$ is an arbitrary set (a topological space), then for any positive definite $\mathcal{C}_1(H)$-valued kernel $\Gamma$ on $\Lambda$ (continuous on $\Lambda \times \Lambda$), there exists a (continuous) multivariate second order stochastic mapping $\Phi : \Lambda \to \mathcal{H}$, which assures for $\Gamma$ a Kolmogorov factorization:

\begin{equation} \tag{2.11} \Gamma(\lambda, \mu) = [\Phi(\lambda), \Phi(\mu)]_{\mathcal{H}} ; \quad \lambda, \mu \in \Lambda, \end{equation}

i.e. $\Gamma$ coincides with the operator covariance function $\Gamma_{\Phi}$ of $\Phi$.

(ii) If $\Phi_1$ and $\Phi_2$ are two (continuous) m.s.o. stochastic mappings on $\Lambda$ for which $\Gamma_{\Phi_1} = \Gamma_{\Phi_2}$, then $\Phi_1$ and $\Phi_2$ are gramian unitary equivalent, i.e. there exists a gramian unitary operator $W : \mathcal{H}_{\Phi_2} \to \mathcal{H}_{\Phi_1}$ such that

\begin{equation} \tag{2.12} \Phi_1(\lambda) = W\Phi_2(\lambda), \quad \lambda \in \Lambda. \end{equation}

**Proof.** (i) Let $\Gamma$ be a positive definite $\mathcal{C}_1(H)$-valued kernel on $\Lambda$ and $\mathcal{H}_{\Gamma}$ ($G_{\Gamma}$ respectively) the normal Hilbert $\mathcal{B}(H)$-module (Hilbert space respectively) reproduced by $\Gamma$, generated as in [10] Sec.2.4, Thm.13, pp.37 or Prop.23, pp.44, with the gramian (respectively the scalar product) defined there. Since, by Thm. 2.4., pp. 45 of [10], $\mathcal{C}_2(G_{\Gamma}, H)$ is an operatorial model for $\mathcal{H}_{\Gamma}$, embedding $G_{\Gamma}$ into a space of the form $L^0_2(\wp)$ (see [13]) $\mathcal{H}_{\Gamma}$ embeds into a normal Hilbert $\mathcal{B}(H)$-module of the form $\mathcal{C}_2(L^0_2(\wp), H)$.

In this way the $\mathcal{H}_{\Gamma}$-valued function defined by $\Phi(\lambda) := \Gamma(\lambda, \cdot)$, $(\lambda \in \Lambda)$ gives us exactly the m.s.o. stochastic mapping we seek, since we get for any $\lambda, \mu \in \Lambda$

$$\Gamma_{\Phi}(\lambda, \mu) = [\Phi(\lambda), \Phi(\mu)]_{\mathcal{H}_{\Gamma}} = [\Gamma(\lambda, \cdot), \Gamma(\mu, \cdot)]_{\mathcal{H}_{\Gamma}} = \Gamma(\lambda, \mu).$$

(ii) From the hypothesis on $\Phi_1$ and $\Phi_2$ we have

$$[\Phi_1(\lambda), \Phi_1(\mu)]_{\mathcal{H}_{\Gamma}} = [\Phi_2(\lambda), \Phi_2(\mu)]_{\mathcal{H}_{\Gamma}}, \quad \lambda, \mu \in \Lambda,$n

which means that the mapping $W$ defined on $\{\Phi_2(\mu), \mu \in \Lambda\}$ by
preserves the gramian, is easily extended by $\mathcal{B}(H)$-linearity, still conserving the gramian and is clearly surjective between the $\mathcal{B}(H)$-modules $\mathcal{H}_2^0$ and $\mathcal{H}_1^0$ (see (2.1)). This allows the extension of $W$ by continuity to a gramian unitary operator from $\mathcal{H}_2^0$ to $\mathcal{H}_1^0$, satisfying (2.12). □

Remark 2.1. Analogous results, formulated in a Hilbert space setting, hold for the scalar correlation mapping

\begin{equation}
M(\Lambda, \mathcal{H}) \ni \Phi \mapsto \gamma_\Phi \in \gamma(\Lambda),
\end{equation}

where $\gamma(\Lambda)$ is the set of complex valued positive definite kernels on $\Lambda$.

3. Multivariate second order random distribution fields and their covariance distributions

In this Section we apply the results from the previous Section to the concrete case of multivariate second order (m.s.o.) random distribution fields.

For completeness we shall work with the $m$-order ($0 \leq m \leq \infty$) m.s.o. random distribution field $U$, which is a continuous linear m.s.o stochastic mapping having the index set $\Lambda = \mathcal{D}_d^m$, the space of complex valued compactly supported continuously derivable functions up to the order $m$ ($m = 0, 1, 2, \ldots, \infty$). In other words $U \in (\mathcal{D}_d^m)'(\mathcal{H}) := \mathcal{B}(\mathcal{D}_d^m, \mathcal{H})$, i.e. it is a $\mathcal{H}$-valued $m$-order distribution on $\mathbb{R}^d$. For $m = 0$, $U$ will be a ($\mathcal{H}$-valued) m.s.o. random Radon measure and for $m = \infty$, the index $m$ and the "$m$-order" will be omitted, otherwise $U$ will be called of finite $m$-order.

Now for $U, V \in (\mathcal{D}_d^m)'(\mathcal{H})$ the scalar cross covariance function $\gamma_{UV}$ and the operator cross covariance function $\Gamma_{UV}$ are defined by (2.4) and (2.3). We shall comment only the properties of $\Gamma_{UV}$, which – from the linearity and continuity of $U$ and $V$ – is sesquilinear and continuous on $\mathcal{D}_d^m \times \mathcal{D}_d^m$. Moreover $\Gamma_U$ is obviously positive definite, i.e. it satisfies a relation of the form (2.5).

It is possible as in Theorems 2.1 and 2.3 to describe and to characterize the m.s.o. random distribution fields, their modular and vector domains and measurements space in terms of such kernels. We prefer but to do that by using everywhere a distributional framework, where instead of $\mathcal{C}_1(H)$-valued sesquilinear continuous kernels on $\mathcal{D}_d^m \times \mathcal{D}_d^m$, we use the so-called distribution kernels on $\mathbb{R}^d$ in the sense of L. Schwartz (see [15], I, pp. 138]), which are $m$-order distributions on $\mathbb{R}^{2d}$. Therefore we need the following Lemma, where we identify $\mathcal{D}_d^m$ with the inductive tensor product $\mathcal{D}_d^m \otimes \mathcal{D}_d^m$ (see [6], pp. 84]).

Lemma 3.1. Let $\Gamma$ be a $\mathcal{C}_1(H)$-valued sesquilinear kernel on $\mathcal{D}_d^m$, which is continuous on the diagonal of $\mathcal{D}_d^m \times \mathcal{D}_d^m$. Then there is an $m$-order
distribution kernel \( C_{\Gamma} \) on \( \mathbb{R}^d \) such that

\[
C_{\Gamma}(\varphi \otimes \psi) = \Gamma(\varphi, \bar{\psi}), \quad \varphi, \psi \in \mathcal{D}_d^m.
\]

Moreover, when \( \Gamma \) is positive on the diagonal, then \( C_{\Gamma} \) is a positive definite \( m \)-order distribution kernel.

Proof. First, from the hypothesis, it is easily seen that the kernel \( \Gamma \) is continuous on \( \mathcal{D}_d^m \times \mathcal{D}_d^m \) and when \( \Gamma(\varphi, \varphi) \geq 0, \quad \varphi \in \mathcal{D}_d^m \), then \( \Gamma \) is also positive definite. We shall attach an \( m \)-order \( \mathcal{G}_1(H) \)-valued distribution \( C_{\Gamma} \) on \( \mathbb{R}^{2d} \) as follows. Define first \( C_{\Gamma} \) on the elementary tensors \( \varphi \otimes \psi \) from \( \mathcal{D}_d^m \times \mathcal{D}_d^m \) by \( C_{\Gamma}(\varphi \otimes \psi) := \Gamma(\varphi, \bar{\psi}) \), extend that by linearity, and then by continuity to the whole \( \mathcal{D}_d^m \otimes \mathcal{D}_d^m = \mathcal{D}_d^{2m} \) preserving the notation \( C_{\Gamma} \).

So \( C_{\Gamma} \in (\mathcal{D}_d^{2m})'(\mathcal{G}_1(H)) \), which, when \( \Gamma \) is positive on the diagonal, will be a positive definite distribution kernel on \( \mathbb{R}^d \). We shall refer to this as \( C_{\Gamma} \in pd(\mathcal{D}_d^{2m})'(\mathcal{G}_1(H)) \).

Since, for each \( U, V \in (\mathcal{D}_d^{2m})'(\mathcal{H}) \) the operator cross covariance functions \( \Gamma_{U,V}, \Gamma_U \) satisfy the hypotheses from Lemma 3.1 the existence of the distribution kernels \( C_{\Gamma_{U,V}} \) and \( C_{\Gamma_U} \) is assured, the last one being even positive definite. These will be called the operator cross covariance distribution of \( U \) and \( V \), respectively the operator covariance distribution of the m.s.o. random distribution field \( U \) and denoted by \( C_{U,V} \), respectively \( C_U \). Correspondingly \( c_{U,V} \) defined by \( c_{U,V}(\chi) := TrC_{U,V}(\chi), \chi \in \mathcal{D}_d^{2m} \), will be called the scalar cross covariance distribution of \( U \) and \( V \), respectively \( c_U = c_{U,U} \) the scalar covariance distribution of \( U \).

By applying Theorem 2.1 the following description of the domains associated to a m.s.o.r.d.f. in terms of its covariance distribution holds.

**Corollary 3.2.** For a given m.s.o. random distribution field \( U \in (\mathcal{D}_d^{2m})'(\mathcal{H}) \), the modular domain \( \mathcal{H}_U \), respectively the measurements space \( G_U \) will be isomorph to the \( \mathcal{B}(H) \)-module \( \mathcal{H}_{C_{\Gamma_U}} \), respectively the Hilbert space \( G_{C_{\Gamma_U}} \) both reproduced by the \( \mathcal{G}_1(H) \)-valued kernel \( \Gamma^{C_{\Gamma_U}} \), while the vector domain \( \mathcal{H}_{(U)} \) of \( U \) is isomorphic to the Hilbert space reproduced by the complex valued kernel \( \gamma^{C_{\Gamma_U}} \).

The operator cross covariance distribution together with the operator covariance distributions of two m.s.o.r.d.f. will be used to obtain from Theorems 2.2 the following characterization of their subordination.

**Corollary 3.3.** Given two m.s.o.r.d.f. \( U \) and \( V \) then \( U \) is subordinate to \( V \) iff for each fixed \( \varphi \in \mathcal{D}_d \), the function \( K_\varphi(\cdot) \) given by

\[
\mathcal{D}_d \ni \psi \mapsto C_{U,V}(\varphi \otimes \bar{\psi}) \in \mathcal{G}_1(H)
\]

lies in the \( \mathcal{B}(H) \)-module \( \mathcal{H}_{K_{\Gamma_{CV}}} \) and \( [K_\varphi, K_\psi]_{\mathcal{H}_{K_{\Gamma_{CV}}}} = C_U(\varphi \otimes \bar{\psi}) \).
The operator correlation mapping (2.10) becomes
\begin{equation}
(\mathcal{D}^m_d)'(\mathcal{H}) \ni U \mapsto C_U \in \text{pd}(\mathcal{D}^m_d)'(\mathcal{C}_1(H)) ,
\end{equation}
which we shall call the operator covariance distribution mapping. Its properties are contained in the following Corollaries.

**Corollary 3.4.** For an m-order distribution kernel on \(\mathbb{R}^d\), \(C \in (\mathcal{D}^m_d)'(\mathcal{C}_1(H))\) the following statements are equivalent:

(i) \(C \in \text{pd}(\mathcal{D}^m_d)'(\mathcal{C}_1(H))\); i.e. the operatorial kernel \(\Gamma^C\) on \(\mathcal{D}^m_d\) defined by \(\Gamma^C(\varphi,\psi) := C(\varphi \otimes \overline{\psi}) (\varphi,\psi \in \mathcal{D}^m_d)\) is positive definite.

(ii) The operatorial kernel \(\Gamma^C\) is sesquilinear and positive (in the sense of positivity from \(\mathcal{B}(H)\)) on the diagonal of \(\mathcal{D}^m_d \times \mathcal{D}^m_d\).

(iii) There is a m.s.o. random distribution field \(U \in (\mathcal{D}^m_d)'(\mathcal{H})\) such that \(C = C_U\) (i.e. \(C\) has a Kolmogorov type factorization).

If, for a given \(C \in \text{pd}(\mathcal{D}^m_d)'(\mathcal{C}_1(H))\), there exists a m.s.o. random distribution field \(U\), such that \(C_U = C\), its uniqueness (up to a gramian unitary equivalence) results now from Theorem 2.3(ii). More precisely it holds

**Corollary 3.5.** If the m.s.o. random distribution fields \(U^1, U^2 \in (\mathcal{D}^m_d)'(\mathcal{H})\) have the same operator covariance distribution, i.e. \(C_{U^1} = C_{U^2}\), then there exists a gramian unitary operator \(W : \mathcal{H}_{U^1} \to \mathcal{H}_{U^2}\) such that
\begin{equation}
W U^1_\varphi = U^2_\varphi, \quad \varphi \in \mathcal{D}^m_d.
\end{equation}

It would be of course interesting to know if each \(\mathcal{C}_1(H)\)-valued distribution kernel on \(\mathbb{R}^d\) is the operator cross covariance distribution of some pair \(U,V \in (\mathcal{D}^m_d)'(\mathcal{H})\), but this won’t be our goal here.

Now we shall introduce in our general frame the concepts of determinism and nondeterminism and we give a general decomposition of Wold type of a m.s.o.r.d.f. into deterministic and purely nondeterministic parts.

In [14] (see also [1]), the observable space up to the moment \(t_0\) for a random distribution for the case \(d = 1\) and \(H = \mathbb{C}\) was defined. By using in \(\mathbb{R}^d\) the ordering relation \(s = (s_1,\ldots,s_d) \leq t = (t_1,\ldots,t_d)\) if \(s_j \leq t_j, \quad j = 1,\ldots,d\), we shall define by analogy these observable structures for m.s.o.r.d.f. up to the moment \(t_0\).

To this purpose we introduce first the subspace \(\mathcal{D}^m_d(t_0)\) in \(\mathcal{D}_d\) by
\begin{equation}
\varphi \in \mathcal{D}^m_d(t_0) \iff \text{supp}\varphi \subset \{t \in \mathbb{R}^d : t \leq t_0\}.
\end{equation}

Thus, for a m.s.o.r.d.f. \(U = \{U_\varphi\}_{\varphi \in \mathcal{D}^m_d}\) we call the observable module \(\mathcal{H}^m_U(t_0)\) (the observable space \(\mathcal{H}^m_U(t_0)\) up to the moment \(t_0\), the closed \(\mathcal{B}(H)\)-module (space) in \(\mathcal{H}\), generated by the set \(\{U_\varphi, \varphi \in \mathcal{D}^m_d(t_0)\}\).
We notice that the Hilbert $\mathcal{B}(H)$-module (Hilbert space) generated by $\bigcup_{t \in \mathbb{R}^d} \mathcal{H}^t_U$ (respectively) is just the modular domain $\mathcal{H}^\infty_U$ (vector domain respectively) of the m.s.o.r.d.f. $U$, while $\bigcap_{t \in \mathbb{R}^d} \mathcal{H}^t_U$ (respectively) will be called the remote past module (space) of $U$ and denoted by $\mathcal{H}^{-\infty}_U$ (respectively). We shall say that $\varphi \mapsto U_\varphi$ is operator deterministic (or, simply deterministic), if $\mathcal{H}^\infty_U = \mathcal{H}^{-\infty}_U$ (respectively). If $\mathcal{H}^{-\infty}_U \subseteq \mathcal{H}_U$ ($\mathcal{H}_U \subseteq \mathcal{H}^{-\infty}_U$ respectively) we say that $U$ is operator nondeterministic (nondeterministic). For extreme situations, where $\mathcal{H}^{-\infty}_U = \{0\}$ ($\mathcal{H}_U = \{0\}$ respectively), the term non-deterministic becomes purely non-deterministic (see also [22], [10, Sec. 5.1] for the first, respectively the classical result). In this context we have the following general Wold decomposition for m.s.o.r.d.f.

**Theorem 3.6.** For a m.s.o.r.d.f. $\{U_\varphi\}_{\varphi \in \mathcal{D}_d}$ taking values in $\mathcal{H}$, there exists a unique decomposition

$$U_\varphi = U^{\text{det}}_\varphi + U^p_\varphi, \quad \varphi \in \mathcal{D}_d$$

of $U$ such that

(i) $\{U^{\text{det}}_\varphi\}_{\varphi \in \mathcal{D}_d}$ is operator deterministic and $\{U^p_\varphi\}_{\varphi \in \mathcal{D}_d}$ is operator purely nondeterministic,

(ii) $U^{\text{det}}$ and $U^p$ are operator subordinate to $U$,

(iii) $U^{\text{det}}$ and $U^p$ are operator uncorrelated, i.e. the operator cross covariance distribution $C_{U^{\text{det}} U^p}$ vanishes on $\mathcal{D}_d$.

Moreover the gramian orthogonal decomposition

$$\mathcal{H}_U = \mathcal{H}_{U^{\text{det}}} \oplus \mathcal{H}_{U^p}$$

holds.

**Proof.** Since in normal Hilbert $\mathcal{B}(H)$-modules, any closed submodule has a gramian projection (Lemma 2, Section 2.2, pp.22 [10]), let $P$ be the gramian projection associated to the submodule $\mathcal{H}^{-\infty}_U$ and let

$$U^{\text{det}}_\varphi = PU_\varphi, \quad U^p_\varphi = (I - P)U_\varphi, \quad \varphi \in \mathcal{D}_d.$$

One can verify by using standard arguments (as in [10, Section 5.1]) that (3.6) gives us the very m.s.o.r.d.f. we seek, the uniqueness of decomposition (3.5) being obtained by direct reasoning. □

### 4. Multivariate second order stochastic measures and their associated bimeasures

In this Section we shall see how from the theory of m.s.o.r.d.f. we can indeed infer the classical theory of m.s.o. random fields, as well as how to fit in this general theory the special class of m.s.o.r.d.f., namely that of (not necessarily bounded) measures, which is very useful for the
spectral representations of random (distribution) fields.

Each very well known continuous m.s.o. random field $F$ on $\mathbb{R}^d$ (i.e. $F \in \mathcal{E}_0^d(\mathcal{H})$) will be identified with the m.s.o. random Radon measure $U^F$, defined by

$$U^F_\varphi = \int_{\mathbb{R}^d} \varphi(t) F(t) dt, \quad \varphi \in \mathcal{D}_d.$$  \hfill (4.1)

Moreover, since $F$ is bounded on each bounded Borel subset $A$ of $\mathbb{R}^d$ (i.e. $A \in \tilde{\text{Bor}}(\mathbb{R}^d)$), it can be even regarded as a not necessarily bounded regular measure $\xi^F$ through

$$\xi^F(A) = \int_A F(t) dt, \quad A \in \tilde{\text{Bor}}(\mathbb{R}^d).$$  \hfill (4.2)

In such a way the locally convex $\mathcal{B}(H)$-module $\mathcal{E}_0^d(\mathcal{H})$ is embedded in the class of m.s.o. random distribution fields of zero order consisting of the multivariate second order stochastic (not necessarily bounded) regular measures, for which the index set $\Lambda$ is the $\delta$-ring $\tilde{\text{Bor}}(\mathbb{R}^d)$. Playing an important role in the definitions of various kinds of harmonizability of m.s.o. random distribution fields, they are sometimes supposed to have finite operator semivariation i.e.

$$\|\xi\|_o(A) < \infty, \quad A \in \tilde{\text{Bor}}(\mathbb{R}^d),$$  \hfill (4.3)

where $\|\cdot\|_o$ is defined as in [10, pp. 56 Def. 4 (1)], or finite semivariation i.e.

$$\|\xi\|(A) < \infty, \quad A \in \tilde{\text{Bor}}(\mathbb{R}^d),$$  \hfill (4.4)

where $\|\cdot\|$ is defined as in [10, pp. 54 Def. 1], the corresponding classes of m.s.o. stochastic regular measures being denoted by $\mathcal{FsvrM}_d(\mathcal{H})$, or by $\mathcal{FsVrM}_d(\mathcal{H})$, respectively. They are locally convex $\mathcal{B}(H)$-modules with the seminorms given by (4.3) and (4.4) respectively. However, since $\|\xi\|_o(A) \leq \|\xi\|(A), \quad A \in \tilde{\text{Bor}}(\mathbb{R}^d)$, it holds $\mathcal{FsvrM}_d(\mathcal{H}) \subset \mathcal{FsVrM}_d(\mathcal{H})$ with continuous embedding.

Now an element $\xi \in \mathcal{FsVrM}_d(\mathcal{H})$ is to be regarded as a $\mathcal{H}$-valued distribution $U^\xi$, which will be a $\mathcal{H}$-valued Radon measure, by putting

$$U^\xi_\varphi = \int_{\mathbb{R}^d} \varphi(t) d\xi(t), \quad \varphi \in \mathcal{D}_d^0.$$  \hfill (4.5)

Thus, we have obtained
Proposition 4.1. For the above mentioned classes of m.s.o. random distribution fields the following inclusions hold with continuous embedding

\[(4.6) \quad \mathcal{E}_d^{0}(\mathcal{H}) \subset f\text{osur }\mathcal{M}_d(\mathcal{H}) \subset f\text{osur }\mathcal{M}_d(\mathcal{H}) \subset \mathcal{D}_d' \subset (\mathcal{D}_d')' \subset (\mathcal{D}_d''')' \subset \mathcal{D}_d'(\mathcal{H}).\]

Let’s observe that a m.s.o. stochastic mapping from one of the mentioned classes can have more than one index set. For example, looking to the above identifications:

\[\xi = \{\xi(A)\}_{A \in \widetilde{\text{Bor}}(\mathbb{R}^d)} \quad \text{with} \quad U^\xi = \{U^\xi_{\varphi}\}_{\varphi \in \mathcal{D}_d^0},\]

respectively

\[F = \{F(t)\}_{t \in \mathbb{R}^d} \quad \text{with} \quad \xi^F = \{\xi^F(A)\}_{A \in \widetilde{\text{Bor}}(\mathbb{R}^d)} \quad \text{and} \quad U^F = U^{\xi^F},\]

it is not difficult to see that

\[\mathcal{H}_\xi = \mathcal{H}_{U^\xi}, \quad \text{respectively} \quad \mathcal{H}_F = \mathcal{H}_{\xi^F} = \mathcal{H}_{U^F},\]

analogous relations being true for the corresponding vector domains and measurements spaces. It is of interest how those classes of positive definite kernels which are co-domains of the “restrictions” of the operator covariance distribution mapping \((3.2)\) to the submodules from \((1.6)\) can be suitable described. Let’s mention that for the operator covariance distribution of an element \(\xi \in f\text{osur }\mathcal{M}_d(\mathcal{H})\) regarded as a m.s.o. random distribution field we naturally use the notation \(\Gamma_{U^\xi}\), while if \(\xi\) is regarded as a m.s.o. stochastic mapping on \(\widetilde{\text{Bor}}(\mathbb{R}^d)\), then its operator covariance function \(\Gamma_{\xi}\) represents a positive definite regular bimeasure on \(\text{Bor}(\mathbb{R}^d) \times \text{Bor}(\mathbb{R}^d)\), for which we shall use the notation \(\tau_{\xi}\). It will be also called the operator covariance bimeasure associated to the m.s.o stochastic measure \(\xi\).

For the bimeasures on \(\widetilde{\text{Bor}}(\mathbb{R}^d) \times \widetilde{\text{Bor}}(\mathbb{R}^d)\) and their semivariation or operator semivariation we adopt analogue definitions as in \([10]\) (Definition 9 (1) and (3) pp.62 and Definition 16 pp.65). For the spaces of such \((\mathcal{C}_1(\mathbb{H})\text{-valued})\) regular bimeasures with finite (operator) semivariation we shall use the notation \(f\text{osur }\mathcal{M}_{2d}(\mathcal{C}_1(\mathbb{H}))\) (respectively \(f\text{osur }\mathcal{M}_{2d}'(\mathcal{C}_1(\mathbb{H}))\)). However, since in our case, the measure and the bimeasure are defined on a \(\delta\)-ring, some properties from \([10]\) do not automatically hold. The corresponding classes of positive definite bimeasures will be denoted by \(f\text{osur }\mathcal{M}_{2d}^{pd}(\mathcal{C}_1(\mathbb{H}))\) and \(f\text{osur }\mathcal{M}_{2d}^{pd}(\mathcal{C}_1(\mathbb{H}))\). So in the above notation \(\tau_{\xi} \in f\text{osur }\mathcal{M}_{2d}^{pd}(\mathcal{C}_1(\mathbb{H}))\).

Let’s also mention that it is not hard to see that a Morse-Transue strict integral as in \([10]\) Section 1.2 pp.5] can be defined for bimeasures \(\tau\) on \(\text{Bor}(\mathbb{R}^d) \times \text{Bor}(\mathbb{R}^d)\). With such a strict MT-integral to each \(\tau \in f\text{osur }\mathcal{M}_{2d}(\mathcal{C}_1(\mathbb{H}))\) we can attach a distribution \(C^\tau\) on \(\mathbb{R}^{2d}\) (a distribution kernel on \(\mathbb{R}^d\), in the sense of L. Schwartz), first defined on elementary tensors trough...
\[
(C^\tau(\varphi \otimes \psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(s)\psi(t)\,d\tau(s,t), \quad \varphi, \psi \in D''_m,
\]
and then, by the usual extension, to the whole \(D''_m\).

On the other hand, each \(K \in E^0_{2d}(\mathcal{E}_1(H))\) can be regarded as a regular bimeasure with finite operator semivariation \(\tau^K\) by putting
\[
(\tau^K(A,B) := \int_A \int_B K(s,t)\,dsdt, \quad A,B \in \widetilde{\text{Bor}}(\mathbb{R}^d).
\]

This infers

**Proposition 4.2.** The valued domains of the restrictions of the operator covariance distribution mapping (3.2) to the spaces from (4.6) satisfy respectively the following inclusions
\[
(4.9) \quad pdE^0_{2d}(\mathcal{E}_1(H)) \subset fosvr M^pd_{2d}(\mathcal{E}_1(H)) \subset fsvr M^pd_{2d}(\mathcal{E}_1(H)) \subset pd(D^0_{2d})'(\mathcal{E}_1(H)) \subset pd(D''_{2d})'(\mathcal{E}_1(H)).
\]

For a more complete image we discuss in detail the restrictions of the operator covariance distribution mapping (3.2) to the space \(fsvr M_d(\mathcal{H})\), respectively to \(fsvr M_d(\mathcal{H})\) and more particular to \(E^0_0(\mathcal{H})\). Since
\[
\tau_\xi(A,B) = (\xi \otimes \xi)(A,B) = [\xi(A),\xi(B)]_\mathcal{H}, \quad A,B \in \widetilde{\text{Bor}}(\mathbb{R}^d)
\]
we can apply Lemma 19(1) and (3) of [10, pp. 66], from where we deduce that \(\tau_\xi \in fsvr M^pd_{2d}(\mathcal{E}_1(H))\), respectively \(\tau_\xi \in fsvr M^pd_{2d}(\mathcal{E}_1(H))\).

It is not difficult to see that it results

**Proposition 4.3.** Statements analogous as for the operator covariance distribution mapping (3.2) in the corollaries above, hold for the corresponding mappings
\[
(4.10) \quad fosvr M_d(\mathcal{H}) \ni \xi \mapsto \tau_\xi \in fsvr M^pd_{2d}(\mathcal{E}_1(H))
\]

Moreover, these operator covariance bimeasure mappings are natural extensions of the covariance function mapping
\[
(4.11) \quad E^0_0(\mathcal{H}) \ni F \mapsto \Gamma_F \in pdE^0_{2d}(\mathcal{E}_1(H)),
\]
associated to classical m.s.o. random fields, as was defined in [10, Section 4.1, pp. 148].

Now it is naturally to ask how the mappings (4.11),(4.10) can be regarded as restrictions of the covariance distribution mapping (3.2) to the first three subspaces from (4.6), i.e our generalization is coherent to the classical case in [10].

We shall show that the operator covariance distributions of (4.10) and
of (4.11) will be regarded as $C_1(H)$-valued distribution $C^\tau$ on $\mathbb{R}^{2d}$ corresponding to (generated by) the bimeasure $\tau$, respectively $C^{\Gamma_F}$ corresponding to (generated by) the correlation function $\Gamma_F$. More precisely it holds

**Proposition 4.4.** The positive definite operator valued bimeasures from (4.10) and the operator covariance function of (4.11) satisfy

\[ (4.12) \quad C_{U\xi} = C^\tau \quad \text{and} \quad C_{U\xi F} = C^{\Gamma_F}, \]

respectively.

**Proof.** Let $\xi \in fsvr \mathcal{M}_d(\mathcal{H})$, and $U^\xi \in (\mathcal{D}^m_d(\mathcal{H}))' \mathcal{H}$ given by (4.5). Then we successively obtain for each $\varphi, \psi \in \mathcal{D}^m_d(\mathcal{H})$

\[
C_{U\xi}(\varphi \otimes \overline{\psi}) = \left[ U^\xi \varphi, U^\xi \psi \right]_{\mathcal{H}} = \left[ \int_{\mathbb{R}^d} \varphi(s) d\xi(s), \int_{\mathbb{R}^d} \psi(t) d\xi(t) \right]_{\mathcal{H}} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(s) \psi(t) d\xi(s,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(s) \psi(t) d\tau(s,t),
\]

which by (4.7) gives the first relation in (4.12).

In particular for $\xi = \xi^F$ with $\xi^F$ given by (4.2), we have first $U^\xi = U^F$, given by (4.5) and secondly, for each $\varphi, \psi \in \mathcal{D}^m_d(\mathcal{H})$

\[
C'^{\tau^F}(\varphi \otimes \overline{\psi}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(s) \psi(t) \Gamma_F(s,t) ds dt = C^{\Gamma_F}(\varphi \otimes \overline{\psi}),
\]

which finally means the second relation from (4.12). \[\square\]

For a m.s.o.r.d.f. $F$ the observable structure $\mathcal{H}_F^{t_0}$ (respectively $\mathcal{H}_F^{\{t\}}$) is the closed $\mathcal{B}(H)$-module (space) in $\mathcal{H}$ generated by the set $\{F(t), t \leq t_0\}$. Moreover $\mathcal{H}_F^{\xi}$, $(\mathcal{H}_F^{\xi})$ is the closed $\mathcal{B}(H)$-module (space) in $\mathcal{H}$ generated by the set $\{\xi(A), A \in \overset{\sim}{\text{Or}}(\mathbb{R}^d) : t \in A \Rightarrow t \leq t_0\}$.

**Remark 4.1.** These definitions of the observable structures are also coherent to the classical ones, i.e. $\mathcal{H}_F^{t_0} = \mathcal{H}_{U\xi^{t_0}}$, $\mathcal{H}_{(F)} = \mathcal{H}_{U\xi^{t_0}}$. Moreover, for $\xi \in fsvr \mathcal{M}_d(\mathcal{H})$ it holds

\[
\mathcal{H}_\xi^{t_0} = \mathcal{H}_{U\xi^{t_0}}, \quad \mathcal{H}_{(\xi)}^{t_0} = \mathcal{H}_{(U\xi)^{t_0}}.
\]

Consequently we can formulate
Remark 4.2. The above m.s.o.r.d.f. $F$, $\xi^F$, $U^F$ as well as $\xi$, $U^\xi$ are simultaneously (operator) deterministic, (operator) nondeterministic or purely nondeterministic. Also the summands in the Wold decomposition are the same, i.e.

$$(U^F)^{\text{det}} = U^{F^{\text{det}}}, \quad (U^F)^p = U^{F^p},$$

as well as

$$(U^\xi)^{\text{det}} = U^{\xi^{\text{det}}}, \quad (U^\xi)^p = U^{\xi^p}.$$  

Finally let’s mention that the invariance to the gramian unitary equivalences is an important property not only for a pair of m.s.o. random distribution fields with the same operator covariance distribution (Corollary 3.5), but for a larger class of m.s.o. random distribution fields, namely those having the property of operator stationarity. The study of this class, as well as of some other classes of m.s.o. random distribution fields, which are invariant to similarities (bounded linearly operator stationary m.s.o. random distribution fields), or which are invariant to the actions of bounded $\mathcal{B}(H)$-linear operators (especially harmonizable m.s.o. random distribution fields), where the concept of propagator, see [11] or [17], as well as some “intertwining” properties of the Fourier transform with the covariance distribution mapping (3.2), or with the covariance distribution bimeasure mappings (4.10), (4.11) play important roles, will be given in some forthcoming papers.



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