AN ELEMENTARY PROOF FOR THE DIMENSION OF THE GRAPH OF THE CLASSICAL WEIERSTRASS FUNCTION

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Abstract. Let $W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n g(b^n x)$ where $b \geq 2$ is an integer and $g(u) = \cos(2\pi u)$ (classical Weierstrass function) or $b = 2$ and $g(u) = \text{dist}(u, \mathbb{Z})$. Building on work by Baránsky, Bárány and Romanowska [1] and Tsujii [15], we provide elementary proofs that the Hausdorff dimension of $W_{\lambda,b}$ equals $2 + \frac{\log \lambda}{\log b}$ for all $\lambda \in (\lambda_b, 1)$ with a suitable $\lambda_b < 1$. This reproduces results by Ledrappier [7] and Baránsky, Bárány and Romanowska [1] without using the dimension theory for hyperbolic measures of Ledrappier and Young [8, 9], which is replaced by a simple telescoping argument together with a recursive multi-scale estimate.

1. Introduction

The classical Weierstrass function $W_{\lambda,b} : \mathbb{I} := [0,1) \to \mathbb{R}$ with parameters $b \in \mathbb{N}$, $\lambda \in (0,1)$ and $b\lambda > 1$ is defined by

$$W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x).$$

The box dimension of its graph is equal to

$$D = 2 + \frac{\log \lambda}{\log b},$$

as proved by Kaplan, Mallet-Paret and Yorke in [5]. In 1977, Mandelbrot conjectured in his monograph [10] that $D$ is also the Hausdorff dimension of this graph. Despite many efforts, this conjecture is not yet proved in full generality. Among others it is known to be true for sufficiently large integers [2, 3]. The history of the problem and the present state of knowledge are summarized in the introduction to a recent paper by Baránsky, Bárány and Romanowska [1], in which the authors prove that for each integer $b \geq 2$ there exist $\tilde{\lambda}_b < \lambda_b < 1$ such that $D = 2 + \frac{\log \lambda}{\log b} \leq 2$ equals the Hausdorff dimension of the graph of $W_{\lambda,b}$ for every $\lambda \in (\lambda_b, 1)$ and for Lebesgue-a.e. $\lambda \in (\tilde{\lambda}_b, 1)$. They determine $\lambda_b$ and $\tilde{\lambda}_b$ as unique zeroes of certain functions and provide a number of numerical and asymptotic values for them, among others

$$\lambda_2 = 0.9531, \quad \lambda_3 = 0.7269, \quad \lambda_4 = 0.6083, \quad \text{and} \quad \lim_{b \to \infty} \lambda_b = 1/\pi = 0.3183.$$

For their proof they interpret the graph of $W_{\lambda,b}$ as the unique invariant repellor of the dynamical system

$$\Phi_{\lambda,b} : \mathbb{I} \times \mathbb{R} \to \mathbb{I} \times \mathbb{R}, \quad \Phi_{\lambda,b}(u,v) = \left( bu \mod 1, \frac{v - g(u)}{\lambda} \right)$$

with $g(u) = \cos(2\pi u)$, and observe that it suffices to show that $D$ is the Hausdorff dimension of the lift of the Lebesgue measure on $\mathbb{I}$ to the graph of $W_{\lambda,b}$, denoted by $\mu_{\lambda,b}$. Then they extend the transformation $u \mapsto bu \mod 1$ of the first coordinate to an invertible 'b-baker' map. The resulting 3-dimensional system is hyperbolic, and the extension of $\mu_{\lambda,b}$ is a hyperbolic invariant measure for $\Phi_{\lambda,b}$. 

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1More precisely, they consider the extension of $u \mapsto bu \mod 1$ by a full one-sided $b$-shift.
it. This sets the stage to combine dimension results for hyperbolic measures by Ledrappier and Young [8] and [9], an observation by Ledrappier [7] and a transversality estimate by Tsujii [15] to determine $\lambda_0$. Finally, additional effort is needed to determine $\tilde{\lambda}_b$ based on the transversality approach of Peres and Solomyak [13].

In this note I propose a much more elementary approach to reduce the calculation of the dimension of the graph of the Weierstrass function to the basic estimates provided by Tsujii [15] combined with the numerical estimates by Barany, Bárány and Romanowska [1], or, in the case of the piecewise linear function $g(u) = \text{dist}(u, Z)$ and $b = 2$, to the problem of whether an infinite Bernoulli convolution is absolutely continuous. An additional benefit of this approach is that it avoids reference to a result of [7] the proof of which is only briefly sketched along the line of some arguments in [9].

Most proofs in the present note are, nevertheless, the result of my efforts to understand the basic lines of arguments in the papers mentioned above. The only exception is Proposition 3.3 that seems to introduce a new point of view.

2. The main results

Throughout this note we use the notation

$$\gamma := \frac{1}{b\lambda} < 1.$$ 

Our main results are the new proofs for the following two theorems - not the theorems themselves. The first one is due to Ledrappier [7]:

**Theorem 2.1.** Let $g(u) = \text{dist}(u, Z)$, $b = 2$, and let $\lambda \in (0, 1)$ be such that the infinite Bernoulli convolution with parameter $\gamma$ has a square-integrable density w.r.t. Lebesgue measure. Then the graph of $W_{\lambda, 2}$ has Hausdorff dimension $D = 2 + \frac{\log \lambda}{\log 2}$.

**Remark 2.2.** The infinite Bernoulli convolution with parameter $\gamma$ is the distribution of the random variable $\Theta = \sum_{n=1}^{\infty} \gamma^n Z_n$, where the $Z_n$ are independent random variables with $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2}$. The investigation of $\Theta$ has a long history, see e.g. [11, 12] and, for more recent results, also [14]. In particular, the set of parameters $\lambda \in (1/2, 1)$ for which the corresponding Bernoulli convolution with parameter $\gamma$ has a square integrable density, has full Lebesgue measure in this interval.

The second theorem is due to Barański, Bárány and Romanowska [1], building upon work of Tsujii [15]:

**Theorem 2.3.** Let $g(u) = \cos(2\pi u)$. For each integer $b \geq 2$ there exists $\lambda_b < 1$ such that the graph of $W_{\lambda, b}$ described by (1.1) has Hausdorff dimension $D = 2 + \frac{\log \lambda}{\log b}$ for every $\lambda \in (\lambda_b, 1)$.

**Remark 2.4.** $\lambda_b$ is the unique zero of the function

$$h_b(\lambda) = \begin{cases} \frac{1}{4\lambda^2 (2\lambda-1)^2} + \frac{1}{16\lambda^4 (4\lambda-1)^2} - \frac{5}{6\lambda^4} + \frac{\sqrt{2}}{2\lambda} & \text{for } b = 2 \\ \frac{1}{(b\lambda-1)^2} + \frac{1}{(b^2\lambda-1)^2} - \sin^2 \left( \frac{\pi}{b} \right) & \text{for } b \geq 3 \end{cases}$$

on the interval $(1/b, 1)$, see [1] Theorem B.

3. Proofs

In Sections 3.1 and 3.2 we recall some observations from [7] and [1], and in Section 3.3 we provide a fresh look at the strong stable manifolds from those references. Section 3.4 contains the telescoping argument already used in a similar situation in [6], and the proof is finished in Sections 3.5 and 3.6 by combining some of the more elementary arguments from [7], [15] and [1].
3.1. The Weierstrass graph as an attractor. Recall from (1.1) that
\[ \Phi_{\lambda,b}(u,v) = \left( bu \mod 1, \frac{u - g(u)}{\lambda} \right). \]

We are mostly interested in the classical case \( g(u) = \cos(2\pi u) \) and in \( g(u) = \text{dist}(u,\mathbb{Z}) \) where \( g'(\xi) = (-1)^{\lfloor 2\xi \rfloor} \). For notational convenience we denote the map \( u \mapsto bu \mod 1 \) by \( \tau \) so that \( \Phi_{\lambda,b}(u,v) = (\tau(u), \frac{u - g(u)}{\lambda}) \). Then the Weierstrass function \( W = W_{\lambda,b} \) satisfies
\[ \Phi(u,W(u)) = (\tau(u),W(\tau(u))) \]

In particular,
\[ \lambda W(\tau(u)) = W(u) - g(u). \quad (3.1) \]

Denote by \( (\xi, x) \mapsto B(\xi, x) \) the \( b \)-baker map on \( \mathbb{I}^2 \) for the integer \( b \geq 2 \), i.e.
\[ B(\xi, x) = \left( \tau(\xi), \frac{x + k(\xi)}{b} \right) \quad \text{with} \quad k(\xi) = j \in \{0, \ldots, b - 1 \} \text{ if } \xi \in [j/b, (j + 1)/b), \]
and define \( F : \mathbb{I}^2 \times \mathbb{R} \to \mathbb{I}^2 \times \mathbb{R} \) as
\[ F(\xi, x, y) = (B(\xi, x), \lambda y + f(\xi, x)) \quad \text{with} \quad f(\xi, x) := g\left( \frac{x + k(\xi)}{b} \right). \]

Then the graph of the Weierstrass function \( W = W_{\lambda,b} \) is an invariant attractor for \( F \) in the following sense:
\[ F(\xi, x, W(x)) = \left( B(\xi, x), \lambda W(x) + g\left( \frac{x + k(\xi)}{b} \right) \right) = \left( B(\xi, x), W\left( \frac{x + k(\xi)}{b} \right) \right) \]
\[ = (B(\xi, x), W(B\xi(\xi, x))) \]
where the second identity follow from (3.1) with \( u = \frac{x + k(\xi)}{b} \). As \( F \) has skew-product structure over the base \( B \) and as \( \left| \frac{\partial F}{\partial y} \right| = \lambda < 1 \), the graph of \( W \) (interpreted as a function of \( \xi \) and \( x \)) is an attractor for \( F \).

3.1.1. Notation for orbits. Given a point \( (\xi, x) \in \mathbb{I}^2 \) we denote by \( (\xi_n, x_n) \) the point \( B^n(\xi, x) \) \((n \in \mathbb{Z})\). Note that \( \xi_n = \tau^n(\xi) \) \((n \geq 0)\) and \( x_n = \tau^{-n} x \) \((n \leq 0)\) and that
\[ k(\xi_i) = k(x_{i+1}) \quad \text{for all } i \in \mathbb{Z}. \]

We also write
\[ k_n(\xi) = \sum_{i=0}^{n-1} b^i k(\tau^i \xi). \]

For later use we note that
\[ k_n(\xi) = \sum_{i=0}^{n-1} b^i k(\xi_i) = b^n \sum_{i=0}^{n-1} b^{j-i} k(x_{i+1}) = b^n \sum_{j=0}^{n-1} b^{-j-1} k(x_{n-1}) = b^n \sum_{j=0}^{n-1} b^{-j-1} k(\tau^j x_n) \]
\[ = b^n x_n - b^n \sum_{j=n}^{\infty} b^{-j-1} k(x_{n-j}) = b^n x_n - b^n x_n = b^n x_n - \sum_{i=0}^{\infty} b^{-i-1} k(\tau^i x) = b^n x_n - x. \]

In particular,
\[ k_n(\xi_i) = b^n \tau^{j-i}(x) - \tau^i(x) \quad \text{for } n \leq i, \]
\[ k_n(\xi_{i+1}) = b^n x_{n-1} - \tau^{j-i}(x) \quad \text{for } n \geq i, \]
and
\[ \frac{x_n + k(\xi_n)}{b} = \frac{x + k_n(\xi) + b^n k(\xi_n)}{b^{n+1}} = \frac{x + k_{n+1}(\xi)}{b^{n+1}} = x_{n+1}. \]

For comparison with the notation of (1.1) note also that
\[ x_n = \frac{x + k_n(\xi)}{b^n} = \frac{x + k(x_1)}{b^n} + \frac{k(x_2)}{b^n} + \frac{k(x_3)}{b^n} + \cdots + \frac{k(x_n)}{b^n} = \frac{x + k(\xi_0)}{b^n} + \cdots + \frac{k(\xi_{n-1})}{b^n}. \quad (3.2) \]
As the foliation into strong stable fibres is invariant, we have

3.3. Distances between strong stable fibres.

Following [1] and also the earlier paper [7], we describe the stable and unstable manifolds of $F$. The derivative $DF$ is well defined except when $\xi \in S := \{ j/b : j = 0, \ldots, b - 1 \}$, namely

$$DF(\xi, x, y) = \begin{pmatrix} b & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{\partial \gamma}{\partial x}(\xi, x) \end{pmatrix}.$$  

The Lyapunov exponents of the corresponding cocycle are $\log b$, $-\log b$ and $\log \lambda$. Indeed, they correspond to the invariant vector fields

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X(\xi, x, y) = \begin{pmatrix} 0 \\ 1 \gamma^{-1} \sum_{n=0}^{\infty} \gamma^n \frac{\partial f}{\partial x}(B^n(\xi, x)) \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\gamma = (b\lambda)^{-1}$. Observe that none of these fields depends on the variable $y$, so we write $X(\xi, x)$ henceforth.

**Remark 3.1.** As $\frac{\partial f}{\partial x}(B^n(\xi, x)) = b^{-1}g'(\frac{x_n + k_n(\xi)}{b}) = b^{-1}g'(x_{n+1})$, the third component of $X$ can be written as

$$X_3(\xi, x) = -\sum_{n=1}^{\infty} \gamma^n g'(x_n) = -\sum_{n=1}^{\infty} \gamma^n g'\left(\frac{x + k_n(\xi)}{b^n}\right)$$

which is precisely the second component of the field $J_{x,1}$ of [1], also denoted $Y_{x,\gamma}(i)$ in that paper.

For each fixed $\xi$, the field $X$ defines the strong stable foliation in the $(x, y)$-plane $H_\xi$ over $\xi$. The fibres are parallel graphs over $x$ with uniformly bounded slopes. That means, for all $(\xi, x, y) \in (1 \setminus S) \times \mathbb{R}$ there is an open interval $J$ containing $x$ such that the fibre through $(\xi, x, y)$ is the graph of a function $\ell_{ss}(\xi, x, y) : J \to \mathbb{R}$ defined by

$$\frac{\partial}{\partial v} \ell_{ss}(\xi, x, y)(v) = X_3(\xi, v) \quad \text{and} \quad \ell_{ss}(\xi, x, y)(x) = y.$$  

Denote by $\gamma \ell_{ss}(\xi, x, y)$ the graph of the function $\ell_{ss}(\xi, x, y)$ in the hyperplane $H_\xi$, i.e.

$$\gamma \ell_{ss}(\xi, x, y) = \left\{ (\xi, u, \ell_{ss}(\xi, x, y)(u)) : u \in I \right\}.$$  

As the foliation into strong stable fibres is invariant, we have

$$F(\gamma \ell_{ss}(\xi, x, y)) \subseteq \gamma \ell_{ss}(\xi, x, y).$$

3.3. Distances between strong stable fibres. Given two points $(\xi, x), (\xi, x') \in \mathbb{R}^2$ we denote by $|\Delta_\xi(x, x')|$ the vertical distance of the strong stable fibres through the points $(\xi, x, W(x))$ and $(\xi, x', W(x'))$, respectively. More precisely,

$$\Delta_\xi(x, x') = \ell_{ss}(\xi, x', W(x')) - \ell_{ss}(\xi, x, W(x)) = \ell_{ss}(\xi, x', W(x'))(x') - \ell_{ss}(\xi, x, W(x))(x')$$

$$= W(x') - W(x) - \left( \ell_{ss}(\xi, x', W(x'))(x') - \ell_{ss}(\xi, x, W(x))(x') \right)$$

$$= W(x') - W(x) - \int_x^{x'} X_3(\xi, t) dt$$

$$= W(x') - W(x) + \sum_{n=1}^{\infty} \gamma^n \int_x^{x'} g'\left(\frac{t + k_n(\xi)}{b^n}\right) dt$$
3.3.1. The piecewise linear case. In the piecewise linear case \( g'(u) = (-1)^{2u} \) and \( b = 2 \) we have
\[
g'(\frac{t + k_n(\xi)}{b^n}) = g'\left(\frac{t}{b^n} + \frac{k(\xi)}{b^n} + \cdots + \frac{k(\xi_{n-1})}{b}\right) = g'(\frac{k(\xi_{n-1})}{2}) = (-1)^k(\xi_{n-1})
\]
so that
\[
\Delta_\xi(x, x') = W(x') - W(x) + (x' - x) \cdot \sum_{n=1}^{\infty} \gamma^n (-1)^k(\xi_{n-1})
\]
\[
= W(x') - W(x) + (x' - x) \cdot \Theta(\xi)
\]
where \( \Theta(\xi) := \sum_{n=1}^{\infty} \gamma^n (-1)^k(\xi_{n-1}) \) is an infinite Bernoulli convolution. It is known \[14\] that for Lebesgue-almost \( \gamma \in (1/2, 1) \) the distribution of the random variable \( \Theta \) has a square-integrable density with respect to Lebesgue measure.

3.3.2. The case \( g(u) = \cos(2\pi u) \). If \( g(u) = \cos(2\pi u) \), then
\[
\int_x^{x+1} g'(\frac{t + k_n(\xi)}{b^n}) \, dt = b^n \left( \cos\left(2\pi \frac{x'}{b^n} + k_n(\xi)\right) - \cos\left(2\pi \frac{x}{b^n} + k_n(\xi)\right) \right)
\]
\[
= -2b^n \sin\left(2\pi \frac{x'}{b^n} - x\right) \sin\left(2\pi \frac{x'}{2b^n} + k_n(\xi)\right) / b^n
\]
\[
= -2b^n \sin\left(2\pi \frac{x'}{b^n} - x\right) \sin\left(2\pi \frac{x + x' - x}{2b^n} + k_n(\xi)\right) / b^n
\]
so that with \( s(t) := (t/2)^{-1} \sin(2\pi t/2) \),
\[
\Delta_\xi(x, x') = W(x') - W(x) - (x' - x) \sum_{n=1}^{\infty} \gamma^n s\left(\frac{x'}{b^n}\right) \sin\left(2\pi \left(x + \frac{x' - x}{2} + k_n(\xi)\right) / b^n\right).
\]
With
\[
\Theta_z(\xi, x) := \sum_{n=1}^{\infty} \gamma^n s\left(\frac{z}{b^n}\right) \sin\left(2\pi \left(x + \frac{z}{2b^n}\right)\right)
\]
this can be written as
\[
\Delta_\xi(x, x') = W(x') - W(x) - (x' - x) \cdot \Theta_{x' - x}(\xi, x),
\]
(3.4)
because \( \frac{x + k_n(\xi)}{b^n} = x_n \), see (3.2).

The function \( \Theta_0(\xi, x) := -2\pi \sum_{n=1}^{\infty} \gamma^n \sin(2\pi x_n) \) is, up to some constant factor and different notation, just the function \( S(x, 1) \) from \[1\] and Proposition 4.2 of this paper (which is proved via some explicit estimates) together with the more elementary part of Tsujii’s paper \[13\] Sections 3, 4] yields the following fact: Define
\[
\Psi_z : l^2 \to \mathbb{R}, \quad (\xi, x) \mapsto (x, \Theta_z(\xi, x))
\]
and let \( \mu_{x, z} := (m \times \delta_x) \circ \Psi_z^{-1} \) for \( x \in \mathbb{I} \). (Denote \( \Pi_2 : l \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto y \). Then \( \mu_{x, z} \circ \Pi_2^{-1} \) is the conditional distribution of \( \Theta_z(\xi, x) \) given \( x \).

Proposition 3.2. Let \( \lambda \in (\lambda_b, 1) \), i.e. \( b \gamma \in (1, \lambda_b^{-1}) \). For m-a.e. \( x \in \mathbb{I} \), the measure \( \mu_{x, 0} \circ \Pi_2^{-1} \) is absolutely continuous with respect to \( \mu_{x, 0} \) and m. Its density \( h_{x, 0} \) satisfies \( H := \int_{\mathbb{R}} ||h_{x, 0}||_2^2 \, dx < \infty \).

A major technical problem is that this estimate is needed also for \( z \neq 0 \). One approach could be to imitate Tsujii’s recursion from \[15\], and indeed, one obtains densities \( h_{x, z} \) with
\[
\sup_{|z| \leq 1} \int_{\mathbb{R}} ||h_{x, z}||_2^2 \, dx < \infty.
\]
But this approach does not provide any local information on the \( h_{x, z} \) uniformly in \( z \): the set of \( (\xi, x) \) where \( h_{x, z}(\xi) \) is exceptionally big, depends in a complicated way on \( x \). Therefore we follow a different approach here. Naively, one can start with comparing \( \Theta_z \) to \( \Theta_0 \): it is easily seen that there is a constant \( C > 0 \) such that \( ||\Theta_z - \Theta_0||_\infty \leq C |z|^2 \). As in later steps of the proof we have to approximate \( \Theta_z \) by \( \Theta_0 \) up to an error of order \( r \) for small \( r > 0 \), this would cover only \( |z| < \sqrt{T} \). \[3\] However, if one treats a finite part of the sum defining \( \Theta_z \) and

\[\text{The same problem occurs also in Ledrappier’s sketch of a related proof \[4\]. He solves it by using formulas relating dimensions and exponents of various conditional and projected measures as in \[3\].}\]
the remaining tail separately, one sees that a tail starting at \( n = n_0 \) varies with \( z \) only of the order \( \left( \frac{1}{2} \right)^{n_0} |z|^2 \). Using this observation recursively we will prove the following result in Section 3.4.

**Proposition 3.3.** Let \( \lambda \in (A_0, 1) \). For each \( \eta > 0 \) there are \( \delta \in (0, \eta) \) and \( C > 0 \) such that for each \( r > 0 \) there is a measurable set \( E_r \subset \mathbb{I}^2 \) with \( m^2(E_r) \leq C r^\delta \) and the following property: For each measurable family \((\mathcal{I}_x)_{x \in \mathbb{I}}\) of intervals of length \( r \) and for each \( z \in [-1, 1] \),

\[
m^2 \{ (\xi, x) \in \mathbb{I}^2 \setminus E_r : \Theta_x(\xi, x) \in \mathcal{I}_x \} \leq C r^{1-2\eta}.
\]

A crucial ingredient of the proof is the following observation:

**Remark 3.4.** Recall from 3.2 that \( x_n = \frac{x}{b^n} + \frac{k(\xi_0)}{b^n} + \cdots + \frac{k(\xi_{n-1})}{b} \). Hence the conditional distribution, given \((x, k(\xi_0), \ldots, k(\xi_{N-1}))\), of

\[
\Theta_0(B^N(\xi, x)) = \Theta_0(\xi_N, x_N) = -2\pi \sum_{n=1}^{\infty} \gamma^n \sin(2\pi x_{N+n})
\]

\[
= -2\pi \sum_{n=1}^{\infty} \gamma^n \sin \left( 2\pi \left( \frac{x_N}{b^n} + \frac{k(\xi_N)}{b^n} + \frac{k(\xi_{n+1})}{b^{n-1}} + \cdots + \frac{k(\xi_{n+N-1})}{b} \right) \right)
\]

is \( \mu_{x_n,0} \), the distribution of \( \Theta_0(x_N, \ldots) \), because the \( k(\xi_n) \) are independent and uniformly distributed on \( \{0, \ldots, b-1\} \).

3.4. **Telescopings - a replacement for the Ledrappier-Young argument.**

3.4.1. **Neighbourhoods bounded by strong stable fibres.** We define a kind of \( \epsilon \)-neighbourhoods of points \((\xi, x, W(x))\) in \((x, y)\)-direction. To that end fix a constant \( K > 0 \) (to be determined later) and, for any \( \xi \in \mathbb{I} \) and a \( b \)-adic \( \epsilon \)-neighbourhood \( I_N(x) \) of \( x \in \mathbb{I} \) with \( \epsilon = b^{-N} \), let

\[
V_N(\xi, x) = \left\{ (v, w) \in \mathbb{I} \times \mathbb{R} : v \in I_N(x), |w - \ell^{_{\xi,x,W(x)}}(v)| \leq Kb^{-N} \right\}.
\]

The sets \( \{\xi\} \times V_N(\xi, x) \) are quadrilaterals in \( H_{\xi} \), which are bounded in \( x \)-direction by two vertical lines of distance \( b^{-N} \) and in \( y \)-direction by the strong stable fibres through \((\xi, x, W(x) \pm Kb^{-N})\) (which are parallel!). Denote by \( G := \{(x, W(x)) : x \in \mathbb{I}\} \) the graph of \( W, \) let

\[
A_N(\xi, x) = V_N(\xi, x) \cap G
\]

and let \( \mu \) be the Lebesgue measure \( m \) on \( \mathbb{I} \) lifted to \( G \). We will evaluate the local dimension (in \( H_{\xi} \)) of \( \mu \) at \((x, W(x)) \in G \) along \( b \)-adic neighbourhoods \( V_N(\xi, x) \), i.e. we are going to determine the limit

\[
\lim_{N \to \infty} \frac{\log \mu(V_N(\xi, x))}{\log(b^{-N})}.
\]

(3.5)

Observe that this limit, if it exists, does not depend on \( \xi \), as the next remark shows among others.

**Remark 3.5.** As \( X_3 \) is uniformly bounded by some constant \( K_1 \), all \( \ell^{_{\xi,x,W(x)}} \) have \( K_1 \) as a common Lipschitz constant. Fixing the constant \( K \) as \( K_1 + 1 \) and choosing \( n_1 \in \mathbb{N} \) such that \( b^{n_1} > 2K_1 + 1 \), elementary geometric arguments show that

\[
V_{N+n_1}(\xi, x) \subseteq \{(v, w) \in \mathbb{I} \times \mathbb{R} : v \in I_N(x), |w - W(x)| \leq b^{-N} \} \subseteq V_N(\xi, x).
\]

This proves not only that the limit in (3.5) does not depend on \( \xi \), but also that the \( V_N(\xi, x) \) can be replaced by rectangles of height \( 2 \cdot 2^{-N} \) over the base \( I_N(x) \).

Furthermore, for \( m \text{-a.e.} \ x \), one can replace the dyadic intervals \( I_N(x) \) by symmetric intervals \( I_N'(x) := [x-2^{-N}, x+2^{-N}] \) and hence \( V_N(\xi, x) \) by \( V_N'(\xi, x) := I_N'(x) \times I_N(W(x)) \). Indeed, it is immediate that \( V_{N+n_1}(\xi, x) \subseteq V_N'(\xi, x) \) and, by Borel-Cantelli, for \( m \text{-a.e.} \ x \) there is \( N(\xi, x) \in \mathbb{N} \) such that \( V_{N+\lfloor 2\log_2 N \rfloor}(\xi, x) \subseteq V_N(\xi, x) \) for all \( N \geq N(\xi, x) \).
3.4.2. The telescoping step. $F^{-N}((\xi) \times V_N(\xi, x))$ is the image of the quadrilateral $\{\xi \} \times V_N(\xi, x)$ in $H_{\xi-N}$ under a map with derivative $\text{diag}(b^N, \lambda^{-N})$ which maps strong stable fibres to strong stable fibres. Hence

$$F^{-N}((\xi) \times V_N(\xi, x)) = \{\xi-N\} \times \Sigma_N(\xi-N, x-N)$$

where

$$\Sigma_N(\xi, u) := \{(v, w) \in I \times \mathbb{R} : |w - \ell_{(\xi, x, W(u))}^s(v)| \leq K(b\lambda)^{-N}\}$$

is a strip in $H_{\xi}$ of width 1 and height $2K(b\lambda)^{-N} = 2K\gamma^{-N}$. Therefore,

$$\frac{\mu(V_N(\xi, x))}{\mu(I_N(x))} = \frac{m \left( \left\{ v \in I_N(x) : |W(v) - \ell_{(\xi, x, W(x))}^s(v)| \leq K(b\lambda)^{-N} \right\} \right)}{m(I_N(x))}$$

$$= \frac{m \left( \left\{ x' \in \mathbb{I} : |W(x') - \ell_{(\xi, x, W(x))}^s(x')| \leq K\gamma^{-N} \right\} \right)}{m(\mathbb{I})}$$

$$= \frac{m \left( \left\{ x' \in \mathbb{I} : |\Delta_{\xi-N}(x-N, x')| \leq K\gamma^{-N} \right\} \right)}{m(\mathbb{I})}$$

so that

$$\lim_{N \to \infty} \frac{\log \mu(V_N(\xi, x))}{\log(b^{-N})} = 1 + \lim_{N \to \infty} \frac{\log m \left( \left\{ x' \in \mathbb{I} : |\Delta_{\xi-N}(x-N, x')| \leq K\gamma^{-N} \right\} \right)}{\log(b^{-N})}$$

(3.7)

provided the limits exist. This corresponds to identity (2.2) in [1], which states that $\dim \mu = 1 + \frac{\log \Omega}{\log b}$, $\dim \nu_{N, 1}$. Indeed, the remaining task in that paper, namely to show that $\dim \nu_{N, 1} \geq 1$, corresponds in our approach to showing that

$$\lim_{N \to \infty} m \left( \left\{ x' \in \mathbb{I} : |\Delta_{\xi-N}(x-N, x')| \leq K\gamma^{-N} \right\} \right) \geq 1$$

for $m^2$-a.e. $(\xi, x) \in \mathbb{I}^2$. (3.8)

We prove this in Section 5.5. It can be interpreted in the following way: for "typical" $(\xi, x)$ the distribution of the random variable $\Delta_{\xi-N}(x-N, \cdot)$ has local dimension (at least) 1 at $\Delta = 0$. Indeed, these distributions are closely related to the $\nu_{N, 1}$ of [1].

Remark 3.6. Instead of projecting along stable fibres $\ell_{(\xi, x, W(x))}^s$ that depend on the additional variable $\xi$, one could as well choose a new coordinate system for each $\xi$, describe the Weierstrass function $W$ in this new coordinate system (resulting in a transformed version of $W_\xi$) and project the measure $\mu_\xi$, denoting Lebesgue measure lifted to the graph of $W_\xi$, horizontally to the real axis. These projected measures would typically be different one from each other (they depend on $\xi$), but the arguments above show that they all have the same dimension. In this sense our approach is equivalent to determining the dimension of the graph of $W_\xi$ for almost all realisations of this random collection of graphs. For Weierstrass graphs with random phase shifts this was done by Hunt [4]. The difference to our situation is that Hunt introduced additional external randomness to the problem, while in our case the randomness is generated by the unstable coordinate of the underlying baker map.

3.5. A Marstrand projection estimate. For our further discussion we use the assumption covering both theorems that the parameter $\gamma$ is such that the random variables $\Theta_\xi$ on $(\mathbb{T}, m^2)$ have distributions of dimension 1 in the sense that they obey the conclusion of Proposition 3.3 which we prove for the classical Weierstrass function at the end of this note. In the piecewise linear case where $\Theta_\xi(\xi, x) = \Theta(x)$ is an infinite Bernoulli convolution, this is an additional assumption satisfied for Lebesgue-almost $\gamma \in (1/2, 1)$. Indeed, for such $\gamma$ the distribution of $\Theta$ has a square-integrable density w.r.t. Lebesgue measure, see [11, 14], and this implies rather immediately that the conclusion of Proposition 3.3 is satisfied.

The following argument is inspired by [7]. Let $\eta > 0$ and let $\delta \in (0, \eta)$, $C > 0$ and the sets $E_\eta \subseteq \mathbb{T}$ be as in Proposition 3.3. Let $K = \{(\xi, x, z) \in \mathbb{I}^2 \times [-1, 1] : 0 \leq x + z \leq 1\}$ and
By Borel-Cantelli we thus conclude with \( r = 3.6 \).

\[
\int_1^r \left( \frac{2}{|z|} \right)^{1-2\eta} \, dz \quad \text{(by Proposition 3.3)}
\]

\[\leq C \int_{-1}^1 \left( \frac{2}{|z|} \right)^{1-2\eta} \, dz \]

Here and in the sequel, \( C \) denotes a generic constant whose value may change from occurrence to occurrence. Therefore, writing again (\( \xi, x, x \)) for \( B^{-N}(\xi, x) \) and using the \( B \)-invariance of \( m^2 \),

\[
m^2 \{ (\xi, x) \in \mathbb{I}^2 : m \{ x' \in \mathbb{I} : |\Delta_{\xi, x} (x_N, x') | \leq r \} \geq r^{1-3\eta} \} \leq C r^{1-3\eta}.
\]

\[\leq C r^{1-3\eta} \int_{\mathbb{I}^2 \setminus E_r} m \{ x' \in \mathbb{I} : |\Delta_{\xi, x} (x') | \leq r \} \, dm^2(\xi, x)
\]

\[\leq C r^{1-3\eta} + r^{1-3\eta} \int_{E_r} \Delta_{\xi, x} (x') \, dm^2(\xi, x)
\]

\[\leq C r^{1-3\eta} + r^{1-3\eta} m^2 \{ (\xi, x) \in \mathbb{I}^3 : (\xi, x) \notin E_r, |\Delta_{\xi, x} (x') | \leq r \} \leq C r^{1-3\eta} \]

By Borel-Cantelli we thus conclude with \( r = K \gamma^N \):

\[\limsup_{N \to \infty} \gamma^{-1-3\eta} \leq K \gamma^N \leq K^{1-3\eta} \]

for \( m^2 \)-a.e. \( (\xi, x) \in \mathbb{T}^2 \). On a logarithmic scale this implies

\[\liminf_{N \to \infty} \frac{\log m \{ x' \in \mathbb{T}^1 : |\Delta_{\xi, x} (x_N, x') | \leq K \gamma^N \}}{\log b^{(N)}} \geq (1 - 3\eta) \frac{\log \gamma}{\log b^{(1)}},
\]

and as this holds for all \( \eta > 0 \), it proves 3.3 and thus finishes the proofs of Theorems 2.1 and 2.3.

**3.6. Proof of Proposition 3.3**

Let \( \eta > 0 \) and fix \( \alpha \in \left( \frac{\log \gamma}{\log b^{(1)}} \right) \). Choose \( \ell \in \mathbb{N} \) such that \( \alpha^\ell < \eta \) and let \( \delta = \frac{\alpha}{\eta} \). Given \( r \in (0, 1) \), let \( n_0 := \left[ \frac{\log r}{\log \gamma} \right] \) and \( n_k := [\alpha n_{k-1}] (k = 1, \ldots, \ell) \).

Observe that \( n_{\ell} \leq n_{\ell-1} \leq \cdots \leq n_0 \). On the other hand, for \( k = 1, \ldots, \ell \),

\[n_k \geq \alpha n_{k-1} \geq \cdots \geq \alpha^k n_0 \geq \alpha^k \log r \log b \] (for \( k = 0, \ldots, \ell \))

\[
\gamma^r = r^{\log b^{(1)}} \leq e^{\alpha^k \} \leq \left( \frac{\alpha}{\eta} \right)^{n_k} \leq \gamma^r \text{ for } k = 1, \ldots, \ell.
\]

(For the last inequality we used that \( \alpha > \frac{\log r}{\log \gamma} \)). We will also use that \( n_k \leq n_{k-1} + 1 \), which yields by induction

\[
n_k \leq \alpha^k n_0 + \frac{1 - \alpha^k}{1 - \alpha} \text{ for } k = 0, \ldots, \ell.
\]

For \( k = 1, \ldots, \ell \) define truncated versions of \( \Theta_z \),

\[
\Theta_{z,k}(\xi, x) := \sum_{n=1}^{n_k} \gamma^n s \left( \frac{z}{b^n} \right) \sin \left( 2\pi \left( x_n + \frac{z}{2b^n} \right) \right),
\]

and remainders

\[
R_{z,k}(\xi, x) := \Theta_z(\xi, x) - \Theta_{z,k}(\xi, x).
\]
Hence, observe $m$ and, for $G$ where we used (3.9) and the definition of $L$

\[
\gamma^m \Theta_{z/b^n} (B^{n_0} (\xi, x)) \]

so that

\[
\|\Theta_z - \Theta_0\|_\infty = \|R_{z,0}\|_\infty \leq C \sum_{n=n_0+1}^{\infty} \gamma^n = C\gamma^{n_0} \leq C r \tag{3.11}
\]

by definition of $n_0$, and

\[
\|\Theta_{z,k-1} - (\Theta_{z,k} + \gamma^{n_k} \cdot (\Theta_0 \circ B^{n_k}))\|_\infty \leq \|\Theta_{z,k-1} - \Theta_{z,k} - R_{z,k}\|_\infty + \gamma^{n_k} \|\Theta_{z/b^{n_k}} - \Theta_0\|_\infty \leq \|R_{z,k-1}\|_\infty + \gamma^{n_k} \|\Theta_{z/b^{n_k}} - \Theta_0\|_\infty \leq C \cdot (\gamma^{n_k-1} + \left(\frac{\gamma}{b}\right)^{n_k}) \leq C \cdot r^{\alpha k-1}, \tag{3.12}
\]

where we used \[\text{3.5}\] for the last inequality. As before, $C$ denotes a generic constant whose value may change from occurrence to occurrence. It will finally depend on $\ell$, but that poses no problem. All we have to make sure is that it does not depend on $r$.

Now we define $E_r := \ell^2 \setminus G_r$, where

\[
G_r := \bigcap_{k=1}^\ell B^{-n_k} \{(\xi, x) \in \ell^2 : h_{x,0}(\Theta_0(\xi, x)) \leq r^{-\delta}\}. \tag{3.11}
\]

Let $(L_r)_{r \in \mathbb{R}}$ be a measurable family of intervals of length $r$, and denote by $L_x(t)$ the $t$-neighbourhood of $L_x$. Then, by (3.11),

\[
m^2 \{(\xi, x) \in \ell^2 \setminus E_r : \Theta_{z,0}(\xi, x) \in L_x(Cr)\} \leq m^2 \{(\xi, x) \in G_r : \Theta_{z,0}(\xi, x) \in L_x(\ell^2)\} \leq C \sum_{n=n_0+1}^{\infty} \gamma^n = C\gamma^{n_0} \leq C r \tag{3.12}
\]

and, for $k = 1, \ldots, \ell$, estimate (3.12) implies

\[
m^2 \{(\xi, x) \in G_r : \Theta_{z,k-1}(\xi, x) \in L_x(Cr^{\alpha k-1})\} \leq C \sum_{n=n_0+1}^{\infty} \gamma^n \cdot \Theta_0(B^{n_k}(\xi, x)) \in L_x(Cr^{\alpha k-1}) \}
\]

and (3.11) implies

\[
m^2 \{(\xi, x) \in G_r : \Theta_{z,k}(\xi, x) \in L_x(Cr^{\alpha k-1}) + \gamma^{n_k} \Theta_0(B^{n_k}(\xi, x)) \} \}
\]

where we used \[\text{3.5}\] and the definition of $G_r$ for the last inequality. To continue this estimate we observe

- $\Theta_{z,k}(\xi, x)$ depends on $(\xi, x)$ only through $x_1, \ldots, x_k$, i.e. through $x_1, \ldots, k(\xi_{n_k-1})$.
- the conditional distribution of $\Theta_{z,b^n}(\xi, x)$ given $(x, k(\xi_0), \ldots, k(\xi_{n_k-1}))$ is $\mu_{x_{n_k,0} \circ \Pi_2}$ and has density $h_{x_{n_k,0}}$ with respect to Lebesgue measure $m$ (Proposition 3.2).

Hence,

\[
m^2 \{(\xi, x) \in G_r : \Theta_{z,k-1}(\xi, x) \in L_x(Cr^{\alpha k-1})\} \]

\[
\leq \int_{\{(\xi, x) \in G_r : \Theta_{z,k}(\xi, x) \in L_x(Cr^{\alpha k})\}} m^2 \left\{(\Theta_0(B^{n_k}(\xi, x)) \in \gamma^{-n_k} \left(L_x(Cr^{\alpha k-1}) - \Theta_{z,k}(\xi, x)\right) \}
\]

\[
\cap \left\{h_{x_{n_k,0}}(\Theta_0(B^{n_k}(\xi, x))) \leq r^{-\delta}\right\} \right| x, k(\xi_0), \ldots, k(\xi_{n_k-1}) \right\} dm^2(\xi, x) \]

\[
= \int_{\{(\xi, x) \in G_r : \Theta_{z,k}(\xi, x) \in L_x(Cr^{\alpha k})\}} \left(\int_{\Theta_0(B^{n_k}(\xi, x)) \in \gamma^{-n_k} \left(L_x(Cr^{\alpha k-1}) - \Theta_{z,k}(\xi, x)\right)} h_{x_{n_k,0}}(\Theta_0(B^{n_k}(\xi, x))) \right) d\theta \right\} dm^2(\xi, x) \]

\[
\leq C\gamma^{-n_k} r^{\alpha k-1} r^{-\delta} \cdot m^2 \{(\xi, x) \in G_r : \Theta_{z,k}(\xi, x) \in L_x(Cr^{\alpha k})\}. \]
Because of (3.10),

\[ \gamma^{-n_k} \leq C \gamma^{-\alpha k_0} \leq C r^{-\alpha k}, \]

so that

\[ m^2 \{ (\xi, x) \in G_r : \Theta_{z,k-1}(\xi, x) \in \mathbb{L}_x(Cr^{-\alpha k-1}) \} \]

\[ \leq C r^{\alpha k - 1 - \alpha k - \delta} \cdot m^2 \{ (\xi, x) \in G_r : \Theta_{z,k}(\xi, x) \in \mathbb{L}_x(Cr^{-\alpha k}) \}. \]

Applying this estimate inductively and observing (3.13), we obtain

\[ m^2 \{ (\xi, x) \in I^2 \setminus E_r : \Theta_z(\xi, x) \in \mathbb{L}_x \} \]

\[ \leq m^2 \{ (\xi, x) \in I^2 : \Theta_0(\xi, x) \in \mathbb{L}_x(Cr^{-\alpha}) \} \]

\[ \leq C r^{1 - 2\eta} . \]

This finishes the proof of Proposition 3.3.

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