An Extended Galerkin Analysis for Elliptic Problems

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Abstract

A general analysis framework is presented in this paper for many different types of finite element methods (including various discontinuous Galerkin methods). For second order elliptic equation, this framework employs 4 different discretization variables, \( u_h, p_h, \bar{u}_h, \bar{p}_h \), where \( u_h \) and \( p_h \) are for approximation of \( u \) and \( p = -\alpha \nabla u \) inside each element, and \( \bar{u}_h \) and \( \bar{p}_h \) are for approximation of residual of \( u \) and \( p \cdot n \) on the boundary of each element. The resulting 4-field discretization is proved to satisfy inf-sup conditions that are uniform with respect to all discretization and penalization parameters. As a result, most existing finite element and discontinuous Galerkin methods can be analyzed using this general framework by making appropriate choices of discretization spaces and penalization parameters.

1 Introduction

In this paper, we propose an extended Galerkin analysis framework for most of the existing finite element methods (FEMs). We will illustrate the main idea by using the following elliptic boundary value problem

\[
\begin{aligned}
- \text{div}(\alpha \nabla u) &= f \quad \text{in } \Omega, \\
u &= g_D \quad \text{on } \Gamma_D, \\
-(\alpha \nabla u) \cdot n &= g_N \quad \text{on } \Gamma_N,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^d \) (\( d \geq 1 \)) is a bounded domain and its boundary, \( \partial \Omega \), is split into Dirichlet and Neumann parts, namely \( \partial \Omega = \Gamma_D \cup \Gamma_N \). For simplicity, we assume that the \((d-1)\)-dimensional measure of \( \Gamma_D \) is nonzero. Here \( n \) is the outward unit normal direction of \( \Gamma_N \), and \( \alpha : \mathbb{R}^d \to \mathbb{R}^d \) is a bounded and symmetric positive definite matrix, with its inverse denoted by \( c = \alpha^{-1} \). Setting \( p = -\alpha \nabla u \), the above problem can be written as

\[
\begin{aligned}
cp + \nabla u &= 0 \quad \text{in } \Omega, \\
-\text{div}p &= -f \quad \text{in } \Omega,
\end{aligned}
\]

with the boundary condition \( u = g_D \) on \( \Gamma_D \) and \( p \cdot n = g_N \) on \( \Gamma_N \).

There are two major variational formulations for (1.1). The first is to find \( u \in H_1^1(\Omega) := \{ v \in H^1(\Omega) : v|_{\Gamma_D} = g_D \} \) such that for any \( v \in H_1^1(\Omega) := \{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \} \),

\[
\int_{\Omega} (\alpha \nabla u) \cdot \nabla v \, dx = \int_{\Omega} fv \, dx - \int_{\Gamma_N} g_N v \, ds.
\]
The second one is to find $p \in H_N(\text{div}; \Omega) := \{ q \in H(\text{div}) : q \cdot n = g_N \}$, $u \in L^2(\Omega)$ such that for any $q \in H_{N0}(\text{div}; \Omega) := \{ q \in H(\text{div}) : q \cdot n = 0 \}$ and $v \in L^2(\Omega)$,

\[
\begin{aligned}
\int_{\Omega} p \cdot q \, dx - \int_{\Omega} u \, \text{div} q \, dx &= - \int_{\Gamma_D} g_D q \cdot n \, ds, \\
- \int_{\Omega} v \, \text{div} p \, dx &= - \int_{\Omega} f v \, dx.
\end{aligned}
\]  

(1.4)

In correspondence to the two variational formulations, two different conforming finite element methods have been developed. The first one, which approximates $u \in H^1_0(\Omega)$, can be traced back to the 1940s [1] and the Courant element [2]. After a decade, many works, such as [3, 4, 5, 6, 7, 8, 9, 10], proposed more conforming elements and presented serious mathematical proofs concerning error analysis and, hence, established the basic theory of FEMs. These \textit{primal FEMs} contain one unknown, namely $u$, to solve. The second one, which approximates $p \in H_N(\text{div}; \Omega)$ and $u \in L^2(\Omega)$ based on a mixed variational principal, is called the \textit{mixed FEMs} [11, 12, 13, 14, 15, 16]. These mixed methods solve two variables, namely flux variable $p$ and $u$, and the condition for the well-posedness of mixed formulations is known as inf-sup or the Ladyzhenskaya-Babuska-Brezzi (LBB) condition [11].

Contrary to the continuous Galerkin methods, the discontinuous Galerkin (DG) methods, which can be traced back to the late 1960s [17, 18], aim to relax the conforming constraint on $u$ or $p \cdot n$. To maintain consistency of the DG discretization, additional finite element spaces need to be introduced on the element boundaries. In essence, the numerical fluxes on the element boundaries were introduced explicitly and therefore eliminated. In most existing DG methods, only one such boundary space is introduced as, for example, Lagrangian multiplier space, either for $u$ as the primal DG methods [19, 20, 21] or for $p \cdot n$ as the mixed DG methods [22]. Primal DG methods have been applied to purely elliptic problems; examples include the interior penalty methods studied in [23, 24, 25] and the local DG method for elliptic problem in [20]. Primal DG methods for diffusion and elliptic problems were considered in [26]. A review of the development of DG methods up to 1999 can be found in [27].

Given a triangulation of $\Omega$, let $u_h \in V_h$ and $p_h \in Q_h$ be discontinuous piecewise polynomial approximations of $u$ and $p$, respectively. In [28], Arnold, Brezzi, Cockburn, and Marini unified the analysis of DG methods for elliptic problems (1.1) with $c = 1$ and $\Gamma_D = \partial \Omega$, which hinges on the unified formulation [28, Eqn. (3.11)] (Here, we change the notation $\tilde{u}_h \mapsto \bar{u}_h$, $\sigma_h \mapsto -p_h$ and $\bar{\sigma}_h \mapsto -\bar{p}_h$, see also (2.1) for the DG notation):

\[
(\nabla_h u_h, \nabla_h v_h) + \langle [\bar{u}_h - u_h], [\nabla_h v_h] \rangle_{\tilde{E}_h} + \langle \{ \bar{p}_h \}, \{ v_h \} \rangle_{\tilde{E}_h} \\
+ \langle \{ \bar{u}_h - u_h \}, [\nabla_h v_h] \rangle_{\tilde{E}_h} + \langle \{ p_h \}, \{ v_h \} \rangle_{\tilde{E}_h} = (f, v_h) \quad \forall v_h \in V_h, 
\]  

(1.5)

where the numerical traces $\bar{u}_h$ and $\bar{p}_h$ (i.e., $-\bar{\sigma}_h$ in [28]) are explicitly given in [28, Table 3.1].

As a key step in our extended Galerkin analysis, we introduce two additional residual corrections to the numerical traces $\tilde{u}_h$ and $\tilde{p}_h$ in (3.4), which gain the flexibility of boundary finite element spaces for both $u$ and $p \cdot n$. More specifically, in addition to the $\tilde{u}_h$ and $\tilde{p}_h$ given explicitly, our extended Galerkin analysis is presented in terms of four discretization variables, namely

\[
p_h, \quad \bar{p}_h, \quad u_h, \quad \bar{u}_h.
\]

The variables $\tilde{u}_h$ and $\tilde{p}_h$ are introduced for the following approximation on element boundary

\[
u \approx \tilde{u}_h + \bar{u}_h, \quad p \cdot n_e \approx \bar{p}_h \cdot n_e + \bar{p}_h, \quad e = K^+ \cap K^-,
\]

which gives the following formulation by adopting the DG notation (2.1),

\[
\begin{aligned}
(\nabla_h v_h, q_h) - (u_h, \text{div}_h q_h) + [\bar{u}_h + \bar{u}_h, [q_h]]_{\tilde{E}_h} &= - (g_D, q_h \cdot n)_{\Gamma_D} \\
\langle \bar{p}_h \cdot n_e + \bar{p}_h, [v_h]_e \rangle_{\tilde{E}_h} &= - (f, v_h) + (g_N, v_h)_{\Gamma_N} \\
\forall q_h &\in Q_h, \\
\forall v_h &\in V_h.
\end{aligned}
\]  

(1.6)
As a direct consequence, the formulation (1.6) is equivalent to the formulation [28, Equ. (3.4)-(3.5)] if we simply choose \( \bar{u}_h = \bar{p}_h = 0 \) and \( e = 1 \), which leads to (1.5) by eliminating \( p_h \) (i.e., \(-\sigma_h \) in [28]). As in most DG methods, the Nitsche’s trick (see (3.6) below) for \( \bar{u}_h \) and \( \bar{p}_h \) will be used. In this paper, we develop a concise formulation (see (3.10) below) in terms of four variables \( p_h, \bar{p}_h, u_h, \bar{u}_h \), which contain all the possible variables in most of the existing FEMs. Therefore, it has the flexibility to unify the analysis of most of the existing FEMs:

1. Under proper choices of the discrete spaces, formulation (3.10) recovers the analysis of \( H^1 \) conforming finite element if we eliminate all the discretization variables except \( u_h \). By eliminating \( \bar{p}_h \), formulation (3.10) recovers some special cases of the hybrid methods [29] in which \( \bar{u}_h \) serves as a Lagrange multiplier to force the continuity of \( p \cdot n \) across the element boundary. If we further eliminate the Lagrange multiplier, the resulting system needs to solve two variables \( p_h \) and \( u_h \), which recovers the \( H(\text{div}) \) conforming mixed finite element method.

2. The relationship between the formulation (3.10) and DG methods is twofold. First, by simply taking the trivial spaces for \( \bar{u}_h \) and \( \bar{p}_h \), formulation (3.10) recovers most of DG methods shown in [28]. Second, if we confine to a special choice \( \bar{u}_h = \{ u_h \} \) and \( \bar{p}_h = \{ p_h \} \), by virtue of the characterization of hybridization and DG method [29], formulation (3.10) can be related to most of DG methods if we eliminate both \( \bar{p}_h \) and \( \bar{u}_h \) (see Section 6.3).

3. In Section 6.1, formulation (3.10) can be compared with most hybridized discontinuous Galerkin (HDG) methods if we eliminate \( \bar{p}_h \). In 2009, a unified formulation of the hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems was presented in [30]. The resulting system needs to solve three variables, one approximating \( u \), one approximating \( p \), and the third one approximating the trace of \( u \) on the element boundary. A projection-based error analysis of HDG methods was presented in [31], in which a projection operator was tailored to obtain the \( L^2 \) error estimates for both potential and flux. More references to the recent developments of HDG methods can be found in [32].

4. In Section 6.2, formulation (3.10) can be compared with most weak Galerkin (WG) methods if we eliminate \( \bar{u}_h \). With the introduction of weak gradient and weak divergence, a WG method for a second-order elliptic equation formulated as a system of two first-order linear equations was proposed and analyzed in [33, 34]. In fact, the weak Galerkin methods in [34] also solve three variables, one approximating \( u \), one approximating \( p \), and the third one approximating the flux \( p \cdot n \) on the element boundary. A summary of the idea and applications of WG methods for various problems can be found in [35].

In addition, we study two types of uniform inf-sup conditions for the proposed formulation in Section 4, by which the well-posedness of the formulation (3.10) follows naturally. With these uniform inf-sup conditions, we obtain some limiting of formulation (3.10) in Section 5:

1. If the parameters in the Nitsche’s trick are set to be \( \tau = (\rho \varepsilon)^{-1} \), \( \eta \cong \tau^{-1} \), formulation (3.10) is shown to converge to \( H^1 \) conforming method as \( \rho \to 0 \) under certain conditions pertaining to the discrete spaces.

2. If the parameters in the Nitsche’s trick are set to be \( \eta = (\rho \varepsilon)^{-1} \), \( \tau \cong \eta^{-1} \), formulation (3.10) is shown to converge to \( H(\text{div}) \) conforming method as \( \rho \to 0 \) under certain conditions pertaining to the discrete spaces.

Throughout this paper, we shall use letter \( C \), which is independent of mesh-size and stabilization parameters, to denote a generic positive constant which may stand for different values at different occurrences. The notations \( x \lesssim y \) and \( x \gtrsim y \) mean \( x \leq C y \) and \( x \geq C y \), respectively.
2 Preliminaries

Given $\Omega \subset \mathbb{R}^d$, for any $D \subseteq \Omega$, and any positive integer $m$, let $H^m(D)$ be the Sobolev space with the corresponding usual norm and semi-norm, denoted by $\| \cdot \|_{m,D}$ and $| \cdot |_{m,D}$, respectively. The $L^2$-inner product on $D$ and $\partial D$ are denoted by $(\cdot, \cdot)_D$ and $(\cdot, \cdot)_{\partial D}$, respectively. $\| \cdot \|_0,D$ and $| \cdot |_{0,\partial D}$ are the norms of Lebesgue spaces $L^2(D)$ and $L^2(\partial D)$, respectively. We abbreviate $\| \cdot \|_{m,D}$ and $| \cdot |_{m,D}$ by $\| \cdot \|_m$ and $| \cdot |_m$, respectively, when $D = \Omega$, and $\| \cdot \|_0 = \| \cdot \|_{0,\Omega}$.

2.1 DG notation

We denote by $\{T_h\}_h$ a family of shape-regular triangulations of $\overline{\Omega}$. Let $h_K = \text{diam}(K)$ and $h = \max\{h_K : K \in T_h\}$. For any $K \in T_h$, denote $n_K$ as the outward unit normal of $K$. Denote by $\mathcal{E}_h$ the union of the boundaries of the elements $K$ of $T_h$.

Let $\mathcal{E}_i^h = \mathcal{E}_h \setminus \partial \Omega$ be the set of interior edges and $\mathcal{E}_i^h = \mathcal{E}_h \setminus \mathcal{E}_i^h$ be the set of boundary edges. Further, for any $e \in \mathcal{E}_h$, let $h_e = \text{diam}(e)$. For $e \in \mathcal{E}_i^h$, we select a fixed normal unit direction, denoted by $n_e$. For $e \in \mathcal{E}_i^h$, we specify the unit outward normal of $\Omega$ as $n_e$. Let $e$ be the common edge of two elements $K^+$ and $K^-$, and let $n^e = n_{|_{\partial K}}$ be the unit outward normal vector on $\partial K_i$ with $i = +, -$. For any scalar-valued function $v$ and vector-valued function $q$, let $v^\pm = v|_{\partial K^\pm}$, $q^\pm = q|_{\partial K^\pm}$. Then, we define averages $\{\cdot\}$, $\{\cdot\}_0$ and jumps $\llbracket\cdot\rrbracket$, $\llbracket\cdot\rrbracket_e$, $\llbracket\cdot\rrbracket_e$, $\llbracket\cdot\rrbracket$ as follows:

\[
\begin{align*}
\{v\} &= \frac{1}{2}(v^+ + v^-), \quad \{q\} = \frac{1}{2}(q^+ + q^-), \quad \{q\}_e = \frac{1}{2}(q^+ + q^-) \cdot n_e & \text{on } e \in \mathcal{E}_i^h, \\
\llbracket v \rrbracket &= v^+ n^+ + v^- n^-, \quad [v]_e = [v] \cdot n_e, \quad [q] = q^+ n^+ + q^- n^- & \text{on } e \in \mathcal{E}_i^h, \quad (2.1) \\
\llbracket q \rrbracket &= q, \quad \{q\}_e = q \cdot n, \quad [q] = q \cdot n & \text{on } e \in \Gamma_D, \\
\llbracket v \rrbracket_0 &= 0, \quad {\{v\}}_e = 0, \quad \{v\} = v, \quad \{q\} = q, \quad \{q\}_e = q \cdot n, \quad [q] = q \cdot n & \text{on } e \in \Gamma_N.
\end{align*}
\]

The notation follows the rules: (i) $\llbracket \cdot \rrbracket$ and $\llbracket \cdot \rrbracket$ are vector-valued operators; (ii) $\{\cdot\}$, $\{\cdot\}$, $\{\cdot\}_e$ and $\llbracket \cdot \rrbracket_e$ are scalar-valued operators; (iii) $\{\cdot\}_e$ and $\llbracket \cdot \rrbracket_e$ are orientation-dependent operators. Clearly, $\llbracket q \rrbracket \cdot [v] = \{q\}_e [v]_e$.

For simplicity of exposition, we use the following convention:

\[
(\cdot, \cdot) := \sum_{K \in T_h} (\cdot, \cdot)_{|K}, \quad \langle \cdot, \cdot \rangle := \sum_{e \in \mathcal{E}_i^h} \langle \cdot, \cdot \rangle_e, \quad \langle \cdot, \cdot \rangle_{\partial K} := \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{\partial K}.
\]

We now give more details about the last inner product. For any scalar-valued function $v$ and vector-valued function $q$, we denote

\[
\langle v, q \cdot n \mathcal{T}_h \rangle := \sum_{K \in \mathcal{T}_h} \langle v, q \cdot n_K \rangle_{\partial K}.
\]

Here, we specify the outward unit normal direction $n$ corresponding to the element $K$, namely $n_K$. In addition, let $\nabla_h$ and $\text{div}_h$ be defined as

\[
\nabla_h v|_K := \nabla v|_K, \quad \text{div}_h q|_K := \text{div} q|_K \quad \forall K \in T_h.
\]

Lemma 2.1 With the averages and jumps defined in (2.1), we have the following identities [28]:

\[
(v, \text{div}_h q) + (\nabla_h v, q) = \langle v, q \cdot n \mathcal{T}_h \rangle = \llbracket q \rrbracket \cdot [v] + \{q\} \cdot [v] = \llbracket q \rrbracket_0 \cdot [v] + \{q\}_0 [v], (2.3)
\]

\[
\langle u_h, v_h \rangle_{\partial K} = 2\langle u_h, v_h \rangle + \frac{1}{2} \llbracket u_h \rrbracket \cdot [v_h] + \frac{1}{2} \llbracket [u_h]_e \rrbracket_0 [v_h]_e. (2.4)
\]

Proof. On each $e = \partial K^+ \cap \partial K^-$, the following identity can be verified by a direct calculation:

\[
q^+ \cdot n^+ v^+ + q^- \cdot n^- v^- = \llbracket q \rrbracket \cdot [v] + \{q\} \cdot [v]. (2.5)
\]
Consequently, by the averages and jumps defined on $\Gamma_D$ and $\Gamma_N$ in (2.1), we have

$$\langle v, q \cdot n \rangle_{\partial \Omega} = \{[v], \{q\}\} + \{[v], [q]\}. \quad (2.6)$$

By integrating by parts and (2.6), we have identity (2.3). Identity (2.4) can be obtained by a direct calculation.

**DG finite element spaces.** Before discussing various Galerkin methods, we need to introduce the finite element spaces associated with the triangulation $\mathcal{T}_h$. First, $V_h$ and $Q_h$ are the piecewise scalar and vector-valued discrete spaces on the triangulation $\mathcal{T}_h$, respectively and for $k \geq 0$, we define the spaces as follows

$$V^k_h := \{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

$$Q^k_h := \{p_h \in L^2(\Omega) : p_h|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

$$Q^{k,RT}_h := \{p_h \in L^2(\Omega) : p_h|_K \in \mathcal{P}_k(K) + x\mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

(2.7)

where $\mathcal{P}_k(K)$ is the space of polynomial functions of degree at most $k$ on $K$, and $\mathcal{P}_k(K) := |\mathcal{P}_k(K)|^d$.

Second, $\hat{V}_h$ and $\hat{Q}_h$ are the piecewise scalar-valued discrete spaces on $\mathcal{E}_h$, respectively and for $k \geq 0$, we define the spaces as follows

$$\hat{Q}^k_h := \{\hat{p}_h \in L^2(\mathcal{E}_h) : \hat{p}_h|_e \in \mathcal{P}_k(e), \forall e \in \mathcal{E}_h, \hat{p}_h|_{\Gamma_D} = 0\},$$

$$\hat{V}^k_h := \{\hat{v}_h \in L^2(\mathcal{E}_h) : \hat{v}_h|_e \in \mathcal{P}_k(e), \forall e \in \mathcal{E}_h, \hat{v}_h|_{\Gamma_D} = 0\},$$

(2.8)

where $\mathcal{P}_k(e)$ is the space of polynomial functions of degree at most $k$ on $e$. Further, let $Q(e), V(e)$ denote some local spaces on $e$ which will be specified at their occurrences.

### 3 A Unified Four Field Formulation

We start with equation (1.2), namely

$$\begin{cases}
  c p + \nabla u = 0 & \text{in } \Omega, \\
  -\text{div} p = -f & \text{in } \Omega.
\end{cases} \quad (3.1)$$

Multiplying the first and second equations by $q_h \in Q_h$ and $v_h \in V_h$, and summing on all $K \in \mathcal{T}_h$, we get

$$\begin{cases}
  (cp, q_h) + (\nabla u, q_h) = 0 & \forall q_h \in Q_h, \\
  -(\text{div} p, v_h) = -(f, v_h) & \forall v_h \in V_h.
\end{cases}$$

Using the identity (2.3), we have

$$\begin{cases}
  (cp, q_h) - (u, \text{div} q_h) + \{[u], [q_h]\} + \{[u]|_e, [q_h]|_e\} = 0 & \forall q_h \in Q_h, \\
  (p, \nabla v_h) - (p \cdot n_e, [v_h]|_e) - \{[p], [v_h]\} = -(f, v_h) & \forall v_h \in V_h.
\end{cases}$$

(3.2)

Noting that $u \in H^1(\Omega), p \in H(\text{div}, \Omega), u = g_D$ on $\Gamma_D$ and $p \cdot n = g_N$ on $\Gamma_N$, we obtain

$$\begin{cases}
  (cp, q_h) - (u, \text{div} q_h) + \{[u], [q_h]\} = -(g_D, q_h \cdot n)_{\Gamma_D} & \forall q_h \in Q_h, \\
  (p, \nabla v_h) - (p \cdot n_e, [v_h]|_e) = -(f, v_h) + (g_N, v_h)_{\Gamma_N} & \forall v_h \in V_h.
\end{cases}$$

(3.2)
The unified formulation. It is natural to approximate \( u, p \) on the interior of the elements of \( T_h \) by

\[
    u \approx u_h, \quad p \approx p_h, \tag{3.3}
\]

for \( u_h \in V_h \) and \( p_h \in Q_h \). Our key observation is that most DG methods can be obtained by approximating \( u \) and \( p \cdot n_e \) on \( E_h \) by

\[
    u \approx \bar{u}_h(u_h) + \tilde{u}_h, \quad p \cdot n_e \approx \tilde{p}_h(u_h, p_h) + \bar{p}_h, \tag{3.4}
\]

where \( \bar{u}_h(u_h) \), \( \tilde{p}_h(u_h, p_h) \) are given in terms of \( u_h, p_h \) as shown in [28, Table 3.1] (by changing the notation \( \bar{\tau}_h \cdot n_e \mapsto -\tilde{\tau}_h \) and \( \bar{u}_h \in \tilde{V}_h, \bar{p}_h \in \tilde{Q}_h \) are some residual corrections to \( \bar{u}_h(u_h), \bar{p}_h(u_h, p_h) \), respectively. As a result, we obtain

\[
\begin{align*}
    (c p_h, q_h) - (u_h, \text{div}_h q_h) + \langle \bar{u}_h(u_h) + \tilde{u}_h, [q_h] \rangle &= -\langle g_D, q_h \rangle_{\Gamma_D} \quad \forall q_h \in Q_h, \\
    (p_h, \nabla_h u_h) - \langle \bar{p}_h(u_h, p_h) + \tilde{p}_h, [v_h]_e \rangle &= -(f, v_h) + \langle g_N, v_h \rangle_{\Gamma_N} \quad \forall v_h \in V_h.
\end{align*}
\tag{3.5}
\]

Besides (3.5), two additional equations are required to determine \( \bar{p}_h \) and \( \tilde{u}_h \). On the interior edges, we adopt

\[
    \bar{p}_h \approx \tau [u_h]_e, \quad \tilde{u}_h \approx \eta [p_h].
\]

More specifically,

\[
\begin{align*}
    \langle [u_h]_e - \tau^{-1} \bar{p}_h, \tilde{q}_h \rangle_{\gamma_h} &= 0 \quad \forall \tilde{q}_h \in \tilde{Q}_h, \tag{3.6a} \\
    \langle [p_h] - \eta^{-1} \tilde{u}_h, \bar{v}_h \rangle_{\gamma_h} &= 0 \quad \forall \bar{v}_h \in \bar{V}_h. \tag{3.6b}
\end{align*}
\]

On the boundary edges, we naturally adopt

\[
\hat{p}_h \approx \begin{cases}
    \tau (u_h - g_D) & \text{on } \Gamma_D, \\
    0 & \text{on } \Gamma_N,
\end{cases}
\quad \hat{u}_h \approx \begin{cases}
    0 & \text{on } \Gamma_D, \\
    \eta (p_h \cdot n - g_N) & \text{on } \Gamma_N,
\end{cases}
\]

namely

\[
\begin{align*}
    -\langle u_h - \tau^{-1} \bar{p}_h, \tilde{q}_h \rangle_{\gamma_h} &= -\langle g_D, \tilde{q}_h \rangle_{\Gamma_D} \quad \forall \tilde{q}_h \in \tilde{Q}_h, \tag{3.7a} \\
    \langle p_h \cdot n - \eta^{-1} \tilde{u}_h, \bar{v}_h \rangle_{\gamma_h} &= \langle g_N, \bar{v}_h \rangle_{\Gamma_N} \quad \forall \bar{v}_h \in \bar{V}_h. \tag{3.7b}
\end{align*}
\]

Collectively, we obtain a concise formulation of (3.6)-(3.7) as follows

\[
\begin{align*}
    -\langle [u_h]_e - \tau^{-1} \bar{p}_h, \tilde{q}_h \rangle_{\gamma_h} &= -\langle g_D, \tilde{q}_h \rangle_{\Gamma_D} \quad \forall \tilde{q}_h \in \tilde{Q}_h, \tag{3.8a} \\
    \langle [p_h] - \eta^{-1} \tilde{u}_h, \bar{v}_h \rangle_{\gamma_h} &= \langle g_N, \bar{v}_h \rangle_{\Gamma_N} \quad \forall \bar{v}_h \in \bar{V}_h. \tag{3.8b}
\end{align*}
\]

The combination of (3.5) and (3.8) obtains formulation: Find \( (p_h, \bar{p}_h, u_h, \tilde{u}_h) \in Q_h \times \tilde{Q}_h \times V_h \times \tilde{V}_h \) such that for any \( (q_h, \tilde{q}_h, v_h, \bar{v}_h) \in Q_h \times \tilde{Q}_h \times V_h \times \tilde{V}_h \)

\[
\begin{align*}
    (c p_h, q_h) - (u_h, \text{div}_h q_h) + \langle \bar{u}_h(u_h) + \tilde{u}_h, [q_h] \rangle &= -\langle g_D, q_h \rangle_{\Gamma_D} \quad \forall q_h \in Q_h, \\
    (p_h, \nabla_h u_h) - \langle \bar{p}_h(u_h, p_h) + \tilde{p}_h, [v_h]_e \rangle &= -(f, v_h) + \langle g_N, v_h \rangle_{\Gamma_N} \quad \forall v_h \in V_h, \\
    -\langle [u_h]_e - \tau^{-1} \bar{p}_h, \tilde{q}_h \rangle_{\gamma_h} &= -\langle g_D, \tilde{q}_h \rangle_{\Gamma_D} \quad \forall \tilde{q}_h \in \tilde{Q}_h, \\
    \langle [p_h] - \eta^{-1} \tilde{u}_h, \bar{v}_h \rangle_{\gamma_h} &= \langle g_N, \bar{v}_h \rangle_{\Gamma_N} \quad \forall \bar{v}_h \in \bar{V}_h.
\end{align*}
\tag{3.9}
\]

We point out here that if \( \tilde{Q}_h = \{0\}, \tilde{V}_h = \{0\} \), then the above method (3.9) induce to the consistent methods listed in [28, Table 3.1].
Compact form for a special case. In what follows, in this paper, we consider a special case: \( \bar{u}_h(u_h) = \{u_h\} \) and \( \bar{p}_h(u_h, p_h) = \{p_h\} \). In this case, the formulation (3.9) can be recast into the following compact form: Find \( (p_h, \bar{p}_h, u_h, \bar{u}_h) \in Q_h \times \bar{Q}_h \times V_h \times \bar{V}_h \) such that for any \( (q_h, \bar{q}_h, v_h, \bar{v}_h) \in Q_h \times \bar{Q}_h \times V_h \times \bar{V}_h \)

\[
\begin{align*}
\begin{cases}
  a(\bar{p}_h, \bar{q}_h) + b(\bar{p}_h, \bar{u}_h) &= - (g_D, q_h \cdot n + q_h)_{\Gamma_D}, \\
  b(\bar{p}_h, \bar{v}_h) - c(\bar{u}_h, \bar{v}_h) &= - (f, v_h) + (g_N, v_h + \bar{v}_h)_{\Gamma_N}
\end{cases}
\end{align*}
\]

\( \forall \bar{q}_h \in \bar{Q}_h := Q_h \times \bar{Q}_h, \forall \bar{v}_h \in \bar{V}_h := V_h \times \bar{V}_h, \quad (3.10) \)

where \( \bar{p}_h := (p_h, \bar{p}_h), \bar{u}_h := (u_h, \bar{u}_h) \) and

\[
\begin{align*}
  a(\bar{p}_h, \bar{q}_h) := (cp_h, q_h) + \langle \tau^{-1} \bar{p}_h, \bar{q}_h \rangle, \\
  c(\bar{u}_h, \bar{v}_h) := \langle \eta^{-1} \bar{u}_h, \bar{v}_h \rangle, \\
  b(\bar{q}_h, \bar{u}_h) := (\nabla_h u_h, q_h) - (u_h | e, q_h) + \langle \bar{u}_h, [q_h] \rangle - (u_h | e, \bar{q}_h), \\
  b(\bar{p}_h, \bar{v}_h) := -(u_h, \text{div}_h q_h) + \langle \{ u_h \}, [q_h] \rangle + \langle \bar{u}_h, q_h \rangle - (u_h | e, \bar{q}_h),
\end{align*}
\]

where (2.3) is used to rewrite the bilinear form \( b(\bar{q}_h, \bar{u}_h) \).

**Remark 3.1** We note that if \( (p, u) \) is the solution of (1.2), then \( (p, 0; u, 0) \) satisfies the equations (3.10). Namely, the formulation (3.10) is consistent.

Let

\[
\hat{a}(\bar{p}_h, (\bar{q}_h, \bar{u}_h)) := a(\bar{p}_h, \bar{q}_h) + b(\bar{q}_h, \bar{u}_h) + b(\bar{p}_h, \bar{v}_h) - c(\bar{u}_h, \bar{v}_h).
\]

Motivated by the two formulations of \( b(\bar{q}_h, \bar{u}_h) \) in (3.11c) and (3.11d), we have two types of inf-sup conditions for the formulation (3.10), which will be discussed in next section.

### 4 Unified Analysis of the Four Filed Formulation

In this section, we shall present two types of the inf-sup condition for the formulation (3.10).

#### 4.1 Gradient-based uniform inf-sup condition

Let us consider the well-posedness of formulation (3.10) in the gradient-based case. For any \( p_h \in Q_h, \bar{p}_h \in \bar{Q}_h, u_h \in V_h, \bar{u}_h \in \bar{V}_h \), define

\[
\begin{align*}
\| \bar{p}_h \|_{\rho, p_h}^2 := &\ (c p_h, p_h) + \langle \rho \epsilon \bar{p}_h, \bar{p}_h \rangle, \\
\| \bar{u}_h \|_{\rho, u_h}^2 := &\ (\nabla_h u_h, \nabla_h u_h) + \langle \rho^{-1} \epsilon^{-1} \bar{Q}_h^\rho [u_h] | e, \bar{Q}_h^\rho [u_h] | e \rangle + \langle \rho^{-1} \epsilon^{-1} \bar{u}_h, \bar{u}_h \rangle \quad (4.1)
\end{align*}
\]

where \( \bar{Q}_h^\rho : L^2(\mathcal{E}_h) \to \bar{Q}_h \) and \( \bar{Q}_h^\rho : L^2(\mathcal{E}_h) \to \bar{V}_h \) are the \( L^2 \) projections. Here, we abbreviate the dependence of both \( \rho \) and mesh size \( h \) in the norms to \( \rho_h \).

We are now ready to state the first main result.

**Theorem 4.1** If we choose \( \tau = (\rho \epsilon)^{-1}, \eta \equiv \tau^{-1} = \rho \epsilon \) in formulation (3.10) and the spaces \( Q_h, \bar{Q}_h, V_h \) satisfy the conditions:

(a) \( \bar{Q}_h \) contains piecewise constant function space;
(b) $\nabla_h V_h \subset Q_h$;

(c) $\{\nabla_h V_h\}_e \subset \hat{Q}_h$.

Then we have:

1. There exists $\rho_0 > 0$ such that $\tilde{a}((\cdot, \cdot), (\cdot, \cdot))$ in (3.12) is uniformly well-posed with respect to the norms $\| \cdot \|_{0, \rho_h}$, $\| \cdot \|_{1, \rho_h}$ when $\rho \in (0, \rho_0]$ and the following estimates holds:

$$\| p_h \|_{0,c} + \| \tilde{p}_h \|_{0,\rho_h} + \| u_h \|_{1,\rho_h} + \| \tilde{u}_h \|_{0,1,\rho_h} \lesssim \| f \|_{-1,\rho_h} + \| g_D \|_{1,\rho_h, \Gamma_D} + \| g_N \|_{-1,\rho_h, \Gamma_N},$$

where

$$\| f \|_{-1,\rho_h} := \sup_{v_h \in V_h \setminus \{0\}} \frac{(f, v_h)}{\| v_h \|_{1,\rho_h}};$$

$$\| g_D \|_{1,\rho_h, \Gamma_D} := \sup_{q_h \in Q_h \setminus \{0\}} \frac{(g_D, q_h \cdot n)_{\Gamma_D}}{\| q_h \|_{0,c}}$$

$$\| g_N \|_{-1,\rho_h, \Gamma_N} := \sup_{v_h \in V_h \setminus \{0\}} \frac{(g_N, v_h)_{\Gamma_N}}{\| v_h \|_{1,\rho_h}}.$$ 

2. Let $(p, u) \in L^2(\Omega) \times H^1(\Omega)$ be the solution of (1.2) and $(\tilde{p}_h, \tilde{u}_h) \in Q_h \times \tilde{V}_h$ be the solution of (3.10), we have the quasi-optimal approximation as follows:

$$\| p - p_h \|_{0,c} + \| \tilde{p}_h \|_{0,\rho_h} + \| u - u_h \|_{1,\rho_h} + \| \tilde{u}_h \|_{0,1,\rho_h} \lesssim \inf_{q_h, v_h \in Q_h \times V_h \setminus \{0\}} (\| p - q_h \|_{0,c} + \| u - v_h \|_{1,\rho_h}).$$

3. If $p \in H^{k+1}(\Omega), u \in H^{k+2}(\Omega)$ ($k \geq 0$) and we choose the spaces $Q_h \times \hat{Q}_h \times V_h \times \tilde{V}_h = Q_h^k \times Q_h^k \times V_h^{k+1} \times \tilde{V}_h$ for any $\tilde{V}_h$, then we have the error estimate

$$\| p - p_h \|_{0,c} + \| \tilde{p}_h \|_{0,\rho_h} + \| u - u_h \|_{1,\rho_h} + \| \tilde{u}_h \|_{0,1,\rho_h} \lesssim h^{k+1}(\| p \|_{k+1} + \| u \|_{k+2}).$$

**Proof.** First, we consider the boundedness of formulation (3.12). The boundedness of $a(\cdot, \cdot)$ and $c(\cdot, \cdot)$ follows directly from the definition of parameter-dependent norms (4.1). For the boundedness of $b(\cdot, \cdot)$ given in (3.11c), we use the Cauchy-Schwarz inequality, trace inequality, and the following inequality

$$h^{-1}_c \| u_h \|_{0,c}^2 \lesssim \| \nabla_h u_h \|_{0,\omega_c}^2 + h^{-1}_c \| \bar{Q}_h^c \|_{0,c}^2,$$

provided that $\hat{Q}_h$ contains piecewise constant function space. Here, $\omega_c = \bigcup_{c \in \partial K}$.

Next we consider the inf-sup condition for the bilinear form $\tilde{a}((\cdot, \cdot), (\cdot, \cdot))$ defined in (3.12). The proof follows from the technique shown in [36]. For any given $(\tilde{p}_h, \tilde{u}_h)$, since $\nabla_h V_h \subset Q_h$, we choose

$$\tilde{q}_h = \gamma \tilde{p}_h + \tilde{s}_h := \gamma \tilde{p}_h + \left( -\rho^{-1} \nabla_h \tilde{u}_h \right), \tilde{v}_h = -\gamma \tilde{u}_h,$$

where $\gamma$ is a constant that will be determined later. The boundedness of $\tilde{q}_h$ and $\tilde{v}_h$ under the parameter-dependent norms (4.1) is straightforward. Next, we have

$$\tilde{a}((\tilde{p}_h, \tilde{u}_h), (\tilde{q}_h, \tilde{v}_h)) = a(\tilde{p}_h, \gamma \tilde{p}_h + \tilde{s}_h) + b(\gamma \tilde{p}_h + \tilde{s}_h, \tilde{u}_h) + b(\tilde{p}_h, -\gamma \tilde{u}_h) + c(\tilde{u}_h, \tilde{u}_h)
\begin{align*}
&= \gamma a(\tilde{p}_h, \tilde{p}_h) + a(\tilde{p}_h, \tilde{s}_h) + b(\tilde{s}_h, \tilde{u}_h) + b(\tilde{p}_h, -\gamma \tilde{u}_h) + c(\tilde{u}_h, \tilde{u}_h) \\
&= \gamma \| \tilde{p}_h \|_{1,\rho_h}^2 + \gamma (\eta^{-1} \tilde{u}_h, \tilde{u}_h) + a(\tilde{p}_h, \tilde{s}_h) + b(\tilde{s}_h, \tilde{u}_h)
\end{align*}
\begin{align*}
&\geq \gamma \| \tilde{p}_h \|_{1,\rho_h}^2 + C_0 \gamma \| \tilde{u}_h \|_{1,\rho_h}^2 + a(\tilde{p}_h, \tilde{s}_h) + b(\tilde{s}_h, \tilde{u}_h).
\end{align*}
Clearly, from the definitions of \(a(\cdot, \cdot)\) in (3.11a) and \(b(\cdot, \cdot)\) in (3.11c), we have

\[
a(\tilde{p}_h, \tilde{u}_h) \geq -\epsilon_1 \|\tilde{u}_h\|_{0, \rho_h}^2 - \epsilon_1^{-1}\|\tilde{p}_h\|_{0, \rho_h}^2
\]

\[
= -\epsilon_1(\nabla_h u_h, \nabla_h u_h) - \epsilon_1\langle \rho^{-1}h^{-1}\tilde{Q}_h[u_h], \tilde{Q}_h[u_h]\rangle - \epsilon_1^{-1}\|\tilde{p}_h\|_{0, \rho_h}^2,
\]

\[
b(\tilde{u}_h, \tilde{u}_h) = \|\nabla_h u_h\|_0^2 + \langle \rho^{-1}h^{-1}\tilde{Q}_h[u_h], \tilde{Q}_h[u_h]\rangle + \langle \tilde{u}_h, [\nabla_h u_h]\rangle - \langle [u_h], [\nabla_h u_h]\rangle .
\]

The standard Cauchy-Schwarz inequality, trace inequality, inverse inequality and the third condition \{\nabla_h V_h\}_e \subset \tilde{Q}_h\) imply that

\[
\langle \tilde{u}_h, [\nabla_h u_h]\rangle \geq -\epsilon_2 \sum_{e \in \mathcal{E}_h} \|h^{-2}_e \tilde{u}_h\|_{0,e}^2 - \epsilon_2 \sum_{e \in \mathcal{E}_h} \|h^2_e [\nabla_h u_h]\|_{0,e}^2
\]

\[
\geq -\epsilon_2 \{h^{-2}_e \tilde{u}_h, \tilde{u}_h\} - C_1 \epsilon_2 \sum_{e \in \mathcal{E}_h} \|\nabla_h u_h\|_{0,\omega_e}^2
\]

\[
\geq -\rho \epsilon_2 \|\nabla_h u_h\|_{0,\rho_h}^2 - C_2 \epsilon_2 \|\nabla_h u_h\|_0^2,
\]

\[
-\langle [u_h], [\nabla_h u_h]\rangle \geq -\epsilon_3 \|\nabla_h u_h\|_0^2 - C_3 \epsilon_3^{-1}\{h^{-2}_e \tilde{Q}_h[u_h], \tilde{Q}_h[u_h]\}.
\]

Therefore, from the above inequalities, we deduce that when \(\rho \in (0, \rho_0)\),

\[
\tilde{a}((\tilde{p}_h, \tilde{u}_h), (\tilde{q}_h, \tilde{v}_h)) \geq (\gamma - \epsilon_1^{-1})\|\tilde{p}_h\|_{0, \rho_h}^2 + (C_0 \gamma - \rho \epsilon_2^{-1})\|\tilde{u}_h\|_{0, \rho_h^{-1}}^2
\]

\[
+ (1 - \epsilon_1 \epsilon_2 \gamma - C_2 \epsilon_2 - \epsilon_3)\|\nabla_h u_h\|_0^2 + (1 - \epsilon_1 - C_3 \rho \epsilon_3^{-1})\{h^{-2}_e \tilde{Q}_h[u_h], \tilde{Q}_h[u_h]\}
\]

\[
\geq \frac{1}{4} \left(\|\tilde{p}_h\|_{0, \rho_h}^2 + \|\tilde{u}_h\|_{0, \rho_h}^2\right),
\]

by choosing \(\epsilon_1, \epsilon_2, \epsilon_3, \gamma\) and \(\rho_0\) as

\[
\epsilon_1 = \frac{1}{4} \max\{\|e\|_{\infty}, 1\}, \quad \epsilon_2 = \frac{1}{4C_2}, \quad \epsilon_3 = \frac{1}{4}, \quad \gamma = \frac{1}{4} + \frac{1}{2C_0} + 4 \max\{\|e\|_{\infty}, 1\}, \quad \rho_0 = \min\{\frac{1}{16C_2}, \frac{1}{8C_3}\}.
\]

Hence, we have the inf-sup condition for \(\tilde{a}(\cdot, \cdot, \cdot)\) under the parameter-dependent norms (4.1). The stability result (4.2), quasi-optimal error estimates (4.3) and (4.4) then follow directly from the Babuška theory and interpolation theory.

### 4.2 Divergence-based uniform inf-sup condition

In light of the formulation of \(b(\cdot, \cdot)\) in (3.11d), we then establish the divergence-based uniform inf-sup condition. For any \(p_h \in Q_h, \tilde{p}_h \in \tilde{Q}_h, u_h \in V_h, \tilde{u}_h \in \tilde{V}_h\), the norms are defined by

\[
\|\tilde{p}_h\|_{0, \rho_h}^2 := \frac{(c p_h, p_h) + (\text{div}_h p_h, \text{div}_h p_h) + \langle \rho^{-1}h^{-1}\tilde{Q}_h[p_h], \tilde{Q}_h[p_h]\rangle + \langle \rho^{-1}h^{-1}\tilde{p}_h, \tilde{p}_h\rangle}{\|p_h\|_{0, \rho_h}^2},
\]

\[
\frac{\|\tilde{u}_h\|_{0, \rho_h}^2 := \langle u_h, u_h \rangle + \langle \rho \epsilon_2 \tilde{u}_h, \tilde{u}_h \rangle}{\|u_h\|_{0, \rho_h}^2},
\]

We are now in the position to state the second main result.

**Theorem 4.2** If we choose \(\eta = (\rho \epsilon_2)^{-1}, \tau \equiv \eta^{-1} = \rho \epsilon_2\) in the formulation (3.10) and the spaces \(Q_h, V_h, \tilde{V}_h\) satisfy the conditions

(a) Let \(R_h := Q_h \cap H(\text{div}, \Omega)\) and \(R_h \times V_h\) is a stable pair for mixed method;

(b) \(\text{div}_h Q_h = V_h\);
(c) \( \{ \text{div}_h Q_h \} \subset \tilde{V}_h \).

Then we have

1. There exists \( \rho_0 > 0 \), \( \tilde{a}(\cdot,\cdot,\cdot) \) in (3.12) is uniformly well-posed with respect to the norms \( \| \cdot \|_{\text{div},\rho_0} \), \( \| \cdot \|_{0,\rho_0} \) when \( \rho \in (0,\rho_0) \) and the following estimate holds:

\[
\| p_h \|_{\text{div},\rho_0} + \| \tilde{p}_h \|_{0,\rho_0}^{-1} + \| u_h \|_0 + \| \tilde{u}_h \|_{0,\rho_0} \lesssim \| f \|_0 + \| g_D \|_{-\frac{1}{2},\rho_0,\Gamma_D} + \| g_N \|_{\frac{1}{2},\rho_0,\Gamma_N}. \tag{4.7}
\]

where

\[
\| g_D \|_{-\frac{1}{2},\rho_0,\Gamma_D} := \sup_{q_h \in Q_h \setminus \{0\}} \frac{(g_D, q_h \cdot n)_{\Gamma_D}}{\| q_h \|_{\text{div},\rho_0}} + \sup_{\tilde{q}_h \in Q_h \setminus \{0\}} \frac{(g_D, \tilde{q}_h)_{\Gamma_D}}{\| \tilde{q}_h \|_{1,\rho_0}}.
\]

\[
\| g_N \|_{\frac{1}{2},\rho_0,\Gamma_N} := \sup_{v_h \in V_h \setminus \{0\}} \frac{(g_N, v_h)_{\Gamma_N}}{\| v_h \|_0} + \sup_{\tilde{v}_h \in V_h \setminus \{0\}} \frac{(g_N, \tilde{v}_h)_{\Gamma_N}}{\| \tilde{v}_h \|_{0,\rho_0}}.
\]

2. Let \( (p, u) \in H(\text{div},\Omega) \times L^2(\Omega) \) be the solution of (1.2) and \( (\tilde{p}_h, \tilde{u}_h) \in \tilde{Q}_h \times \tilde{V}_h \) be the solution of (3.10), we have the following quasi-optimal approximation:

\[
\| p - p_h \|_{\text{div},\rho_0} + \| \tilde{p}_h \|_{0,\rho_0}^{-1} + \| u - u_h \|_0 + \| \tilde{u}_h \|_{0,\rho_0} \lesssim \inf_{q_h \in Q_h, v_h \in V_h} (\| p - q_h \|_{\text{div},\rho_0} + \| u - v_h \|_0). \tag{4.8}
\]

3. If \( p \in H^{k+2}(\Omega), u \in H^{k+1}(\Omega) \) (\( k \geq 0 \)), and we choose the spaces \( Q_h \times \tilde{Q}_h \times V_h \times \tilde{V}_h = Q_h^{\text{RT}} \) (or \( Q_h^{k+1} \times Q_h \times V_h \times \tilde{V}_h \) for any \( \tilde{Q}_h \), then the following estimate holds:

\[
\| p - p_h \|_{\text{div},\rho_0} + \| p_h \|_{0,\rho_0}^{-1} + \| u - u_h \|_0 + \| \tilde{u}_h \|_{0,\rho_0} \lesssim H^{k+1}(\| p \|_{k+2} + \| u \|_{k+1}). \tag{4.9}
\]

**Proof.** Since \( \{ V_h \} = \{ \text{div}_h Q_h \} \subset \tilde{V}_h \), the boundedness of \( \tilde{a}(\cdot,\cdot,\cdot) \) under the divergence-based norms (4.6) is standard (by the Piola’s transformation) and is therefore omitted.

Next we consider the inf-sup condition for the bilinear form \( \tilde{a}(\cdot,\cdot,\cdot) \) defined in (3.12). The proof follows from the technique shown in [36]. Since \( R_h \times V_h \) is a stable pair for mixed method, for any given \( (\tilde{p}_h, \tilde{u}_h) \), there exists \( r_h \in R_h \) such that

\[
-\text{div} r_h = u_h \quad \text{and} \quad \| r_h \|_0 + \| \text{div} r_h \|_0 \leq C_{\text{stab}} \| u_h \|_0. \tag{4.10}
\]

Now, we choose

\[
\tilde{q}_h = \gamma \tilde{p}_h + \tilde{s}_h := \gamma \tilde{p}_h + \begin{pmatrix} r_h \\ 0 \end{pmatrix}, \quad \tilde{v}_h = -\gamma \tilde{u}_h - \tilde{w}_h := -\gamma \tilde{u}_h - \begin{pmatrix} \text{div}_h p_h \\ -\rho^{-1} \rho^{-1} \tilde{Q}_h^u \{ p_h \} \end{pmatrix}, \tag{4.11}
\]

where \( \gamma \) is a constant which will be determined later. First, we have the boundedness of \( \tilde{q}_h \) and \( \tilde{v}_h \) by using (4.10),

\[
\| \tilde{q}_h \|_{\text{div},\rho_0} \leq \| \tilde{p}_h \|_{\text{div},\rho_0} + \| \tilde{s}_h \|_{\text{div},\rho_0} \lesssim \| \tilde{p}_h \|_{\text{div},\rho_0} + \| u_h \|_0, \quad \| \tilde{v}_h \|_{0,\rho_0} \leq \| \tilde{u}_h \|_{0,\rho_0} + \| \tilde{w}_h \|_{0,\rho_0} \lesssim \| \tilde{u}_h \|_{0,\rho_0} + \| \text{div}_h p_h \|_0 + (\rho^{-1} \tilde{Q}_h^u \{ p_h \}, \tilde{Q}_h^u \{ p_h \})^{1/2}.
\]

Next, we have

\[
\tilde{a}(\tilde{p}_h, \tilde{u}_h, (\tilde{q}_h, \tilde{v}_h)) = a(\tilde{p}_h, \gamma \tilde{p}_h + \tilde{s}_h) + b(\gamma \tilde{p}_h + \tilde{s}_h, \tilde{u}_h) + b(\tilde{p}_h, -\gamma \tilde{u}_h - \tilde{w}_h) + c(\tilde{u}_h, \gamma \tilde{u}_h + \tilde{w}_h)
\]

\[
= \gamma a(\tilde{p}_h, \tilde{p}_h) + c(\tilde{p}_h, \tilde{p}_h) + (\gamma^{-1} \tilde{p}_h, \tilde{p}_h) + \gamma a(\tilde{p}_h, \tilde{p}_h) + b(\tilde{s}_h, \tilde{u}_h) - b(\tilde{p}_h, \tilde{w}_h) + c(\tilde{u}_h, \tilde{u}_h + \gamma \tilde{u}_h + \tilde{w}_h)
\]

\[
= \gamma a(\tilde{p}_h, \tilde{p}_h) + (\gamma^{-1} \tilde{p}_h, \tilde{p}_h) + \gamma a(\tilde{p}_h, \tilde{p}_h) + (\tilde{p}_h, \tilde{p}_h) + \gamma a(\tilde{p}_h, \tilde{p}_h) + b(\tilde{s}_h, \tilde{u}_h) - b(\tilde{p}_h, \tilde{w}_h) + c(\tilde{u}_h, \tilde{u}_h + \gamma \tilde{u}_h + \tilde{w}_h)
\]

\[
\geq \gamma a(\tilde{p}_h, \tilde{p}_h) + C_{a} \| \tilde{p}_h \|_{\text{div},\rho_0}^2 + \| \tilde{u}_h \|_{0,\rho_0}^2 + \| \tilde{v}_h \|_{0,\rho_0} + \| \tilde{w}_h \|_{0,\rho_0} + \| \text{div}_h p_h \|_0 + (\rho^{-1} \tilde{Q}_h^u \{ p_h \}, \tilde{Q}_h^u \{ p_h \})^{1/2}.
\]
Clearly, from the definitions of \(a(\cdot, \cdot)\) in (3.11a) and \(c(\cdot, \cdot)\) in (3.11b), we have

\[
a(\tilde{p}_h, \tilde{u}_h) \geq -\epsilon_1 r_h \frac{1}{\eta} - C_1 \epsilon_1^{-1} c(p_h, p_h) \geq -C_{stab}^2 \epsilon_1 \|u_h\|^2 - C_1 \epsilon_1^{-1} (c(p_h, p_h), c(u_h, w_h)) \geq -C_2 \epsilon_2^{-1} (\tilde{u}_h, \tilde{w}_h).
\]

Further, from (4.10) and the formulation of \(b(\cdot, \cdot)\) in (3.11d), we have

\[
b(\tilde{u}_h, \tilde{w}_h) = \|\nabla u_h\|^2 - C_3 \epsilon_3^{-1} \|\tilde{u}_h\|^2 + (1 - C_2 \epsilon_2^{-1}) \|\tilde{u}_h\|^2 - C_4 \epsilon_4^{-1} \|\tilde{w}_h\|^2 + (1 - C_2 \epsilon_2^{-1}) \|\tilde{w}_h\|^2,
\]

by choosing \(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \gamma\) and \(\rho_0\) as

\[
\epsilon_1 = \frac{1}{2C_{stab}}, \quad \epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{1}{4}, \quad \gamma = \frac{1}{2} + \frac{1}{2C_0} + \max\{2C_{stab} C_1, 4C_2, \frac{C_3}{4C_0 C_4}\}, \quad \rho_0 = \frac{1}{16C_4}.
\]

Hence, we have the inf-sup condition for \(\tilde{a}(\cdot, \cdot), (\cdot, \cdot)\) under the parameter-dependent norms (4.6). The stability result (4.7), quasi-optimal error estimates (4.8) and (4.9) then follow directly from the Babuška theory and interpolation theory.

5 Some limiting case of four filed formulation

With the uniform inf-sup conditions, we revisit some limiting of formulation (3.10) in case of \(\rho \to 0\) [22].

First, having the gradient-based inf-sup condition, we discuss the limiting of formulation (3.10) with \(g_D = 0\) in case of \(\tau = (\rho \alpha)^{-1}\), \(\eta \simeq \tau^{-1} = \rho \alpha\) as \(\rho \to 0\). Denote \(H_{0, \Gamma_D}^1(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}\). Consider the \(H^1\) conforming subspace \(V_{\rho, \alpha}^\epsilon = V_h \cap H_{0, \Gamma_D}^1(\Omega) \subset V_h\), then the primal method when applying to the Poisson equation (1.1) can be written as: Find \((u_h^\epsilon, p_h^\epsilon) \in V_{\rho, \alpha}^\epsilon \times Q_h\) such that

\[
\begin{cases}
\langle \nabla u_h^\epsilon, \phi_h \rangle + \langle \nabla \phi_h, u_h^\epsilon \rangle = G_p(\phi_h), & \forall \phi_h \in Q_h, \\
\langle \nabla v_h^\epsilon, p_h \rangle = F_p(v_h), & \forall v_h^\epsilon \in V_{\rho, \alpha}^\epsilon.
\end{cases}
\]

where \(G_p(\phi_h) = 0, F_p(v_h^\epsilon) = -(f, v_h^\epsilon) + (g_N, v_h^\epsilon)|_{\Gamma_N}\). Then, by \(\nabla V_{\rho, \alpha}^\epsilon \subset \nabla V_h \subset Q_h\), the well-posedness of the primal method (cf. [37]) implies that

\[
\|p_h^\epsilon\|_{0, c} + \|u_h^\epsilon\| \leq C_p \left( \sup_{\phi_h \in Q_h \setminus \{0\}} \frac{G_p(\phi_h)}{\|\phi_h\|_{0, c}} + \sup_{v_h^\epsilon \in V_{\rho, \alpha}^\epsilon \setminus \{0\}} \frac{F_p(v_h^\epsilon)}{\|v_h^\epsilon\|} \right).
\]

We have the following theorem.
Theorem 5.1 Assume that the spaces $Q_h, V_h$ and $Q^c_h$ satisfy

(a) $\nabla_h V_h \subset Q_h$;
(b) $\{Q_h\}_c \subset Q^c_h$;
(c) $V_h = V^k_h$ ($k \geq 1$).

Then formulation (3.10) with $g_D = 0$ and $\tau = (\rho h_c)^{-1}, \eta \equiv \tau^{-1} = \rho h_c$ converges to primal method (5.1) as $\rho \to 0$. Further, let $(p^e_h, p^0_h, u^c_h, \tilde{u}^e_h)$ be the solution of (3.10) and $(p^e_h, u^e_h)$ be the solution of (5.1), we have

$$\|p^e_h - p^0_h\|_{0,c} + (\|\nabla_h (u^e_h - u^0_h)\| + \sum_{c \in E_h} h_c^{-1} \|u^e_h - u^0_h\|_{0,c}^2)^{1/2} \leq \rho^{3/2} R_p,$$

where $R_p := \|f\|_{-1, \rho h} + \|g_N\|_{-1/2, \rho h, \Gamma_N}$.

Proof. Taking $v_h = v^c_h$ in the second equation in (3.5), we see that

$$\begin{cases}
(c p^e_h, q_h) + (\nabla_h u^e_h, q_h) - \langle [u^e_h], [q_h] \rangle_{c} - \langle \tilde{u}^e_h, [q_h] \rangle = -(g_D, q_h \cdot n)_{\Gamma_D} & \forall q_h \in Q_h, \\
(p^e_h, \nabla v^c_h) = -(f, v^c_h) + (g_N, v^c_h)_{\Gamma_N} & \forall v^c_h \in V^c_h.
\end{cases}$$

Let

$$\delta^e_h = p^e_h - p^0_h, \quad \delta^0_h = u^e_h - u^0_h.$$

Subtracting (5.1) from the equation (5.4), we have

$$\begin{cases}
(c \delta^e_h, q_h) + (\nabla_h \delta^e_h, q_h) = \langle [u^e_h], [q_h] \rangle_{c} - \langle \tilde{u}^e_h, [q_h] \rangle - \langle g_D, q_h \cdot n \rangle_{\Gamma_D} & \forall q_h \in Q_h, \\
(\delta^e_h, \nabla v^c_h) = 0 & \forall v^c_h \in V^c_h.\end{cases}$$

By the assumption $\{Q_h\}_c \subset Q^c_h$ and noting that $u^e_h$ satisfies (3.8a), we have

$$\begin{cases}
(c \delta^e_h, q_h) + (\nabla_h \delta^e_h, q_h) = \langle \rho h_c \tilde{p}^0_h, [q_h] \rangle_{c} - \langle \tilde{u}^e_h, [q_h] \rangle & \forall q_h \in Q_h, \\
(\delta^e_h, \nabla v^c_h) = 0 & \forall v^c_h \in V^c_h.\end{cases}$$

Further, for any $u^0_h \in V^c_{h,0}$, we have

$$\begin{cases}
(c \delta^e_h, q_h) + (\nabla u^0_h - \nabla u^e_h, q_h) = \langle \rho h_c \tilde{p}^0_h, [q_h] \rangle_{c} - \langle \tilde{u}^e_h, [q_h] \rangle + (\nabla u^0_h - \nabla u^e_h, q_h) & \forall q_h \in Q_h, \\
(\delta^e_h, \nabla v^0_h) = 0 & \forall v^0_h \in V^c_{h,0}.\end{cases}$$

By the assumption $\nabla_h V_h \subset Q_h$, using (5.2), trace inequality, inverse inequality and Cauchy-Schwarz inequality, we obtain

$$\|p^e_h\|_{0,c} + \|u^e_h - u^0_h\|_1 \leq C_P \sup_{q_h \in Q_h \setminus \{0\}} \langle \rho h_c \tilde{p}^0_h, [q_h] \rangle_{c} - \langle \tilde{u}^e_h, [q_h] \rangle + (\nabla u^0_h - \nabla u^e_h, q_h) \|q_h\|_{0,c}$$

$$\lesssim \|\nabla u^0_h - \nabla u^e_h\|_0 + \rho^{3/2} (\|\tilde{p}^0_h\|_{0,\rho h} + \|\tilde{u}^e_h\|_{0,\rho h}^{-1}).$$
Therefore, noting that $V_h = V_h^k$ ($k \geq 1$), (5.6) and (4.2) imply that
\[ \|\delta_p^h\|_{0,c} + (\|\nabla h\delta_p^h\|_0^2 + \sum_{e \in E_h} h_e^{-1} \|\delta_p^h\|_{0,e}^2)^{\frac{1}{2}} \]
\[ \leq \inf_{u_h^c \in V_{h,c}} \left( (\|\delta_p^h\|_{0,c} + (\|\nabla h((u_h^c - u_h^d))\|_0 + (\|\nabla h(u_h^c - u_h^d))\|_0^2 + \sum_{e \in E_h} h_e^{-1} \|u_h^c - u_h^d\|_{0,e}^2)^{\frac{1}{2}} \right) \]
\[ \lesssim \rho^{\frac{1}{2}}(\|\delta_p^h\|_{0,c} + (\|\nabla h((u_h^c - u_h^d))\|_0 + (\|\nabla h(u_h^c - u_h^d))\|_0^2 + \sum_{e \in E_h} h_e^{-1} \|u_h^c - u_h^d\|_{0,e}^2)^{\frac{1}{2}} \]
\[ \lesssim \rho^{\frac{1}{2}}(\|\delta_p^h\|_{0,c} + (\|\nabla h((u_h^c - u_h^d))\|_0 + (\|\nabla h(u_h^c - u_h^d))\|_0^2 + \sum_{e \in E_h} h_e^{-1} \|u_h^c - u_h^d\|_{0,e}^2)^{\frac{1}{2}} \]
\[ \lesssim \rho^{\frac{1}{2}}(\|\delta_p^h\|_{0,c} + (\|\nabla h((u_h^c - u_h^d))\|_0 + (\|\nabla h(u_h^c - u_h^d))\|_0^2 + \sum_{e \in E_h} h_e^{-1} \|u_h^c - u_h^d\|_{0,e}^2)^{\frac{1}{2}} \]
\[ \lesssim \rho^{\frac{1}{2}}(\|\delta_p^h\|_{0,c} + (\|\nabla h((u_h^c - u_h^d))\|_0 + (\|\nabla h(u_h^c - u_h^d))\|_0^2 + \sum_{e \in E_h} h_e^{-1} \|u_h^c - u_h^d\|_{0,e}^2)^{\frac{1}{2}} \]
\[ \lesssim \rho^{\frac{1}{2}}(\|\delta_p^h\|_{0,c} + (\|\nabla h((u_h^c - u_h^d))\|_0 + (\|\nabla h(u_h^c - u_h^d))\|_0^2 + \sum_{e \in E_h} h_e^{-1} \|u_h^c - u_h^d\|_{0,e}^2)^{\frac{1}{2}} \]

This completes the proof. ■

Next, having the divergence-based inf-sup condition, we discuss the limiting of formulation (3.10) with $g_N = 0$ in case of $\eta = (\rho h_e)^{-1}$, $\tau \equiv \eta^{-1} = \rho h_e$ as $\rho \to 0$. Denote $H_{0,\Gamma_D} = \{p \in H(D, \Omega) : p \cdot n|_{\Gamma_D} = 0\}$. Consider the $H(D)$ conforming subspace $Q_{h,g} := Q_h \cap H_{0,\Gamma_D} \subset Q_h$, the mixed method when applying to the Poisson equation (1.1) can be written as: Find $(p_h^c, u_h^c) \in Q_{h,g} \times V_h$ such that
\[
\begin{cases}
(cp_{h,c}^c, q_{h,c}^c) - (u_{h,c}^c, \text{div} q_{h,c}^c) = G_m(q_{h,c}^c) & \forall q_{h,c}^c \in Q_{h,0}, \\
-(\text{div} p_{h,c}^c, v_h) = F_m(v_h) & \forall v_h \in V_h,
\end{cases}
\]
where $G_m(q_{h,c}) = -(g_D, q_{h,c} \cdot n)|_{\Gamma_D}, F_m(v_h) = -(f, v_h)$. Then, by the fact that $\text{div} Q_{h,c} = \text{div} Q_h = V_h$, the well-posedness of the mixed method (cf. [15, 16]) implies that
\[
\|
p_{h,c}^c\|_{H(D)} + \|
q_{h,c}^c\|_0 \lesssim \sup_{q_{h,c}^c \in Q_{h,0} \setminus \{0\}} \frac{G_m(q_{h,c}^c)}{\|q_{h,c}^c\|_{H(D)}} + \sup_{v_h \in V_h \setminus \{0\}} \frac{F_m(v_h)}{\|v_h\|_0}.
\]
We have the following theorem.

**Theorem 5.2** Assume that the spaces $Q_h, \hat{V}_h$ and $V_h$ satisfy
\[
(a) \ \text{div}_h Q_h = V_h; \\
(b) \ \{V_h\} \subset \hat{V}_h; \\
(c) \ Q_h = Q_h^{k,RT} \text{ or } Q_h^{k+1}, k \geq 0.
\]
Then formulation (3.10) with $g_N = 0$ and $\eta = (\rho h_e)^{-1}$, $\tau \equiv \eta^{-1} = \rho h_e$ converges to mixed method (5.7) as $\rho \to 0$. Further, let $(p_h^c, p_h^c, u_h^c, u_h^c)$ be the solution of (3.10) and $(p_h^c, u_h^c)$ be the solution of (5.7), we have
\[
\|
p_{h,c}^c - p_{h,c}^c\|_{0,c} + \|
\text{div}_h (p_{h,c}^c - p_{h,c}^c)\|_0 + \|
u_{h,c}^c - u_{h,c}^c\|_0 \lesssim \rho^{\frac{1}{2}} R_m,
\]
where $R_m := \|f\|_0 + \|g_D\|_0^{-\frac{1}{2}} \rho_{h,D}\Gamma_D$.

**Proof.** Taking $q_{h,c}^c = q_{h,c}^c$ in the first equation in (3.5), we see that
\[
\begin{cases}
(cp_{h,c}^c, q_{h,c}^c) - (u_{h,c}^c, \text{div} q_{h,c}^c) = -(g_D, q_{h,c} \cdot n)|_{\Gamma_D} & \forall q_{h,c}^c \in Q_{h,0}, \\
-(\text{div} p_{h,c}^c, v_h) + \langle [p_{h,c}^c], [v_h]\rangle_c = -(f, v_h) + \langle g_N, v_h\rangle_{\Gamma_N} & \forall v_h \in V_h.
\end{cases}
\]
Let
\[ \delta_h^n = p_h^n - p_h^c, \quad \delta_h^u = u_h^n - u_h^c. \]
Subtracting (5.7) from (5.10), we have
\[
\begin{cases}
(c \delta_h^n, q_h^n) - (\delta_h^n, \text{div} q_h^n) = 0 & \forall q_h^n \in Q_{h,0}^c, \\
(\text{div} h^n, v_h) = -\langle (p_h^n, \{v_h\}) + (\bar{p}_h^n, [v_h]_c) + (g_N, v_h) \rangle_{\Gamma_N} & \forall v_h \in V_h.
\end{cases}
\]
By the assumption \( \{V_h\} \subset \widetilde{V}_h \) and noting that \( p_h^n \) satisfies (3.8b), we have
\[
\begin{cases}
(c \delta_h^n, q_h^n) - (\delta_h^n, \text{div} q_h^n) = 0 & \forall q_h^n \in Q_{h,0}^c, \\
(\text{div} h^n, v_h) = -\langle \rho h c \bar{u}_h^n, \{v_h\} \rangle + (\bar{p}_h^n, [v_h]_c) & \forall v_h \in V_h.
\end{cases}
\]
Further, for any \( p_h^l \in Q_{h,0}^c \),
\[
\begin{cases}
(c(p_h^l - p_h^c), q_h^n) - (\delta_h^n, \text{div} q_h^n) = (c(p_h^l - p_h^c), q_h^n) & \forall q_h^n \in Q_{h,0}^c, \\
(\text{div}(p_h^l - p_h^c), v_h) = -\langle \rho h c \bar{u}_h^n, \{v_h\} \rangle + (\bar{p}_h^n, [v_h]_c) + (\text{div}(p_h^l - p_h^c), v_h) & \forall v_h \in V_h.
\end{cases}
\]
By the well-posedness of the mixed methods (5.8), trace inequality, inverse inequality and Cauchy-Schwarz inequality, we have
\[
\begin{align*}
&\|p_h^l - p_h^c\|_{\mathcal{H}(\text{div})} + \|\delta_h^n\|_0 \\
\leq &\ C_m \left( \sup_{q_h^n \in Q_{h,0}^c} \frac{(c(p_h^l - p_h^c), q_h^n)}{\|q_h^n\|_{\mathcal{H}(\text{div})}} + \sup_{v_h \in V_h \setminus \{0\}} \frac{-\langle \rho h c \bar{u}_h^n, \{v_h\} \rangle + (\bar{p}_h^n, [v_h]_c) + (\text{div}(p_h^l - p_h^c), v_h)}{\|v_h\|_0} \right) \\
\lesssim &\ \|p_h^l - p_h^c\|_{0,c} + \|\text{div}(p_h^l - p_h^c)\|_0 + \rho h (\|\bar{u}_h^n\|_{0,\rho_h} + \|\bar{p}_h^n\|_{0,\rho_h^{-1}}).
\end{align*}
\]
Hence, by (5.12) and (4.7), we have
\[
\begin{align*}
&\|\delta_h^n\|_{0,c} + \|\text{div} h^n, \delta_h^n\|_0 + \|\delta_h^n\|_0 \\
\lesssim &\ \rho h (\|\bar{u}_h^n\|_{0,\rho_h} + \|\bar{p}_h^n\|_{0,\rho_h^{-1}}) + \inf_{p_h^l \in Q_{h,0}^c} \left( \|p_h^l - p_h^c\|_{0,c} + \|\text{div}(p_h^l - p_h^c)\|_0 \right) \\
\lesssim &\ \rho h (\|\bar{u}_h^n\|_{0,\rho_h} + \|\bar{p}_h^n\|_{0,\rho_h^{-1}} + \left( \sum_{c \in \mathcal{E}_h} h_c^{-1} \|p_h^l\|_{0,\rho_c} \right)^{\frac{1}{2}}) \\
\lesssim &\ \rho h (\|\bar{u}_h^n\|_{0,\rho_h} + \|\bar{p}_h^n\|_{0,\rho_h^{-1}} + \|p_h^l\|_{\text{div},\rho_h}) \lesssim \rho h R_m.
\end{align*}
\]
This completes the proof. \( \blacksquare \)

## 6 Unified Extended Galerkin Analysis of Existing Methods

In this section, we exploit the relationship between the formulation (3.10) and several existing numerical methods, which leads to the well-posedness and error estimates of the existing numerical methods. We consider three different variants of the 4-field system (3.10) by eliminating either \( \bar{p}_h \) or \( \bar{u}_h \), or both.
6.1 Eliminating $\bar{p}_h$

By (3.8a), we have the explicit expression of $\bar{p}_h$ as

$$
\bar{p}_h = \begin{cases} \tau \hat{Q}^p_h[u_h]_e & \text{on } E_h^i, \\
\tau \hat{Q}^p_h(u_h - g_D) & \text{on } \Gamma_D, \\
0 & \text{on } \Gamma_N.
\end{cases}
$$

(6.1)

Then formulation (3.10) is reduced to

$$
\begin{cases}
\begin{aligned}
a_H(p_h, q_h) + b_H(q_h, \hat{u}_h) &= -\langle g_D, q_h \cdot n \rangle_{\Gamma_D} \\
b_H(p_h, \hat{v}_h) - c_H(u_h, \hat{v}_h) &= -(f, v_h) + \langle g_N, v_h + \hat{v}_h \rangle_{\Gamma_N} - \langle \tau \hat{Q}^p_h g_D, v_h \rangle_{\Gamma_D} & \forall q_h \in Q_h,
\end{aligned}
\end{cases}
$$

(6.2)

where

$$
a_H(p_h, q_h) = (c p_h, q_h),
b_H(q_h, u_h) = -(u_h, \text{div} q_h) + \langle u_h + \{ u_h \}, [q_h] \rangle,
$$

$$
c_H(u_h, \hat{v}_h) = \langle \eta^{-1} \hat{u}_h, \hat{v}_h \rangle + \langle \tau \hat{Q}^p_h[u_h]_e, \hat{Q}^p_h[v_h]_e \rangle.
$$

Now let us transform $\hat{u}_h := \hat{Q}^n_h\{u_h\} + \hat{u}_h$, then we can rewrite the above formulation as: Find $(p_h, u_h, \hat{u}_h) \in Q_h \times V_h \times \hat{V}_h$ such that

$$
\begin{cases}
\begin{aligned}
a_H(p_h, q_h) + b_H(q_h, u_h, \hat{u}_h) &= -\langle g_D, q_h \cdot n \rangle_{\Gamma_D} & \forall q_h \in Q_h, \\
b_H(p_h, v_h, \hat{v}_h) - c_H(u_h, \hat{v}_h; v_h, \hat{v}_h) &= -(f, v_h) + \langle g_N, v_h + \hat{v}_h - \hat{Q}^n_h\{v_h\} \rangle_{\Gamma_N} - \langle \tau \hat{Q}^p_h g_D, v_h \rangle_{\Gamma_D} & \forall (v_h, \hat{v}_h) \in \hat{V}_h,
\end{aligned}
\end{cases}
$$

(6.3)

where

$$
a_H(p_h, q_h) = (c p_h, q_h),
b_H(q_h, u_h, \hat{u}_h) = -(u_h, \text{div} q_h) + \langle u_h - \hat{Q}^n_h\{u_h\} + \{ u_h \}, [q_h] \rangle,
$$

$$
c_H(u_h, \hat{v}_h; v_h, \hat{v}_h) = \langle \eta^{-1} (\hat{u}_h - \hat{Q}^n_h\{u_h\}), \hat{v}_h - \hat{Q}^n_h\{v_h\} \rangle + \langle \tau \hat{Q}^p_h[u_h]_e, \hat{Q}^p_h[v_h]_e \rangle.
$$

The resulting three-field formulation (6.2) is a generalization of the stabilized hybrid mixed method [22], or some special cases of the HDG method [29, 30, 31, 38, 39].

Some special cases: More precisely, under the conditions that $\eta = \frac{1}{\tau} - 1$ and $\hat{V}_h = \hat{Q}_h$, (6.2) is shown to be the standard HDG method [29, 30, 31], if

$$
Q_h \cap n|_{E_h} \subset \hat{V}_h \quad \text{and} \quad V_h|_{E_h} \subset \hat{V}_h.
$$

(6.4)

Under the condition (6.4) and $\eta = \frac{1}{\tau} - 1$, using the identity (2.4), a hybridizable formulation of (6.3) is obtained: Find $(p_h, u_h, \hat{u}_h) \in Q_h \times V_h \times \hat{V}_h$ such that for any $(q_h, v_h, \hat{v}_h) \in Q_h \times V_h \times \hat{V}_h$

$$
\begin{cases}
\langle c p_h, q_h \rangle - (u_h, \text{div} q_h) + (\hat{u}_h, q_h \cdot n)_{\partial \Omega_h} = -\langle g_D, q_h \cdot n \rangle_{\Gamma_D}, \\
-(\text{div} p_h, v_h) + \langle 2(\hat{u}_h - \hat{Q}^n_h u_h), \hat{Q}^n_h v_h \rangle_{\partial \Omega_h} = -(f, v_h) + \langle g_N, v_h - \hat{Q}^n_h\{v_h\} \rangle_{\Gamma_N} - \langle \tau \hat{Q}^p_h g_D, v_h \rangle_{\Gamma_D}, \\
\langle p_h \cdot n, \hat{v}_h \rangle_{\partial \Omega_h} - \langle 2(\hat{u}_h - \hat{Q}^n_h u_h), \hat{v}_h \rangle_{\partial \Omega_h} = \langle g_N, \hat{v}_h \rangle_{\Gamma_N}.
\end{cases}
$$

(6.5)

The above formulation shows that $p_h$ and $u_h$ can be represented by $\hat{u}_h$ locally from the first and the second equations. As a result, a globally coupled equation solely for $\hat{u}_h$ on $E_h$ can be obtained.

Moreover, (6.3) reduces to the HDG with reduced stabilization method [38, 39] if

$$
Q_h \cap n|_{E_h} \subset \hat{V}_h.
$$

(6.6)

Specific choices of the discrete space and the corresponding numerical methods are summarized in Table 6.1. We refer to [22] for discussion from the HDG to the hybrid mixed methods [14, 15, 29] and the mixed methods [12, 13, 40, 41, 15, 16].
Remark 6.1 We should note that the uniform inf-sup condition for the HDG method when $\eta = \frac{1}{4} \tau^{-1} = O(1)$, $Q_h = Q_h^{k}$, $V_h = V_h^{k}$, $V_h^{k}$ is not proved in Section 4.

Minimal stabilized divergence-based method. In light of Theorem 4.2, the divergence-based inf-sup condition holds for any $Q_h$. Hence, when choosing $Q_h = \{0\}$, the formulation (6.2) reduces to a stabilized divergence-based method with minimal stabilization, which reads: Find $(p_h, u_h, \bar{u}_h) \in Q_h \times V_h \times \bar{V}_h$, such that for any $(q_h, v_h, \bar{v}_h) \in Q_h \times V_h \times \bar{V}_h$

\[
\begin{align*}
(c p_h, q_h) - (u_h, \text{div} q_h) + \langle \bar{u}_h + \{u_h\}, \{q_h\} \rangle = & -\langle D, q_h \cdot n \rangle_{\Gamma_D}, \\
- (\text{div} p_h, v_h) + \langle \{p_h\}, \bar{v}_h + \{v_h\} \rangle - \langle \eta^{-1} \bar{u}_h, \bar{v}_h \rangle = & -\langle f, v_h \rangle + \langle g_N, \bar{v}_h + v_h \rangle_{\Gamma_N}. \\
\end{align*}
\]

Consequently, the scheme (6.7) is stable provided that $Q_h$, $V_h$, and $\bar{V}_h$ satisfy the conditions in Theorem 4.2.

Further, by assuming $Q_h \cdot n|_{\mathcal{E}_h} \subset \bar{V}_h$ and eliminating $\bar{u}_h$ (see (6.9) below), we obtain the mixed DG method [22]: Find $(p_h, u_h) \in Q_h \times V_h$ such that for any $(q_h, v_h) \in Q_h \times V_h$

\[
\begin{align*}
(c p_h, q_h) + \langle \eta [p_h], [q_h] \rangle + (\nabla u_h, \theta_h) - \langle \{u_h\}, \{q_h\} \rangle = & -\langle D, q_h \cdot n \rangle_{\Gamma_D} + \langle \eta N, q_h \cdot n \rangle_{\Gamma_N}, \\
(p_h, \nabla v_h) - \langle \{p_h\}, [v_h] \rangle = & -\langle f, v_h \rangle + \langle g_N, v_h \rangle_{\Gamma_N}. \\
\end{align*}
\]

This implies that the mixed DG method proposed in [22] can be interpreted as the minimal stabilized divergence-based method.

Mixed method. Finally, we remark that, if we take $\tau \to 0$ and choose $Q_h \times V_h \times \bar{V}_h = Q_h^{k+1} \times V_h \times \bar{V}_h$ or $Q_h \times V_h \times \bar{V}_h = Q_h^{k,RRT} \times V_h \times \bar{V}_h$, the (6.2) implies the mixed method by eliminating $\bar{u}_h$.

6.2 Eliminating $\bar{u}_h$

By (3.8b), we have the explicit expression of $\bar{u}_h$ as

\[
\bar{u}_h = \begin{cases} 
\eta \bar{Q}_h^n[p_h] & \text{on } \mathcal{E}_h^i, \\
0 & \text{on } \Gamma_D, \\
\eta \bar{Q}_h^n[p_h] - n \cdot g_N & \text{on } \Gamma_N.
\end{cases}
\]

Then formulation (3.10) can be recast as

\[
\begin{align*}
a_w(p_h, q_h) + b_w(q_h, v_h) = & -\langle D, q_h \cdot n + \bar{q}_h \rangle_{\Gamma_D} + \langle \eta \bar{Q}_h^n g_N, q_h \cdot n \rangle_{\Gamma_N} \quad \forall q_h \in \bar{Q}_h, \\
b_w(p_h, v_h) = & -\langle f, v_h \rangle + \langle g_N, v_h \rangle_{\Gamma_N} \quad \forall v_h \in V_h,
\end{align*}
\]

where

\[
a_w(p_h, \bar{q}_h) = (c p_h, q_h) + \langle \eta \bar{Q}_h^n[p_h], \bar{Q}_h^n[q_h] \rangle + \langle \tau^{-1} \bar{p}_h, \bar{q}_h \rangle, \\
b_w(q_h, u_h) = (\nabla u_h, q_h) - \langle \{u_h\}_c, \bar{q}_h + \{q_h\}_c \rangle.
\]

Now let us transform $\hat{p}_h := \bar{Q}_h^n[p_h]_c + \bar{p}_h$, then we can rewrite the above formulation as: Find $(p_h, \hat{p}_h, u_h) \in Q_h \times \bar{Q}_h \times \bar{V}_h$ such that

\[
\begin{align*}
a_w(p_h, \hat{p}_h; q_h, \hat{q}_h) + b_w(q_h, \hat{p}_h; u_h) = & -\langle D, q_h \cdot n + \hat{q}_h - \bar{Q}_h^n \{q_h\}_c \rangle_{\Gamma_D} + \langle \eta \bar{Q}_h^n g_N, q_h \cdot n \rangle_{\Gamma_N} \quad \forall (q_h, \hat{q}_h) \in \bar{Q}_h, \\
b_w(p_h, \hat{p}_h; v_h) = & -\langle f, v_h \rangle + \langle g_N, v_h \rangle_{\Gamma_N} \quad \forall v_h \in V_h,
\end{align*}
\]

where

\[
a_w(p_h, \hat{p}_h; q_h, \hat{q}_h) = (c p_h, q_h) + \langle \tau^{-1} (\hat{p}_h - \bar{Q}_h^n \{p_h\}_c), \hat{q}_h - \bar{Q}_h^n \{q_h\}_c \rangle + \langle \eta \bar{Q}_h^n[p_h], \bar{Q}_h^n[q_h] \rangle, \\
b_w(q_h, \hat{q}_h; u_h) = (\nabla u_h, q_h) - \langle \{u_h\}_c, \hat{q}_h - \bar{Q}_h^n \{q_h\}_c + \{q_h\}_c \rangle.
\]
The resulting three-field formulation (6.10) is a generalization of the stabilized hybrid primal method [22], or some special cases of the WG-MFEM method [34].

Some special cases: Again, under the conditions that $\tau = \frac{4}{\epsilon} \eta^{-1}$ and $\hat{Q}_h = \hat{V}_h$, (6.11) is the WG-MFEM method [34], if

$$Q_h \cdot n_{\varepsilon} |_{\partial_{\varepsilon} h} \in \hat{Q}_h.$$  \hspace{1cm} (6.12)

That is, we have the following formulation: Find $(p_h, \hat{p}_h, u_h) \in Q_h \times \hat{Q}_h \times V_h$ such that for any $(q_h, \hat{q}_h, v_h) \in Q_h \times \hat{Q}_h \times V_h$

$$\begin{cases}
(c \mathbf{p}_h, q_h) - (2\eta \mathbf{p}_h \cdot \mathbf{n}, q_h)_{\partial_{\varepsilon} h} + (\nabla_h u_h, q_h) - (u_h, \mathbf{n})_{\partial_{\varepsilon} h} = -(g_D, q_h \cdot \mathbf{n} + \hat{q}_h \mathbf{n})_{\Gamma_D}, \\
(p_h, \nabla_h v_h) - (\hat{p}_h, v_h)_{\partial_{\varepsilon} h} = -(f, v_h) + (g_N, v_h)_{\Gamma_N}, \\
\langle 2\eta \mathbf{p}_h - \mathbf{p}_h \cdot \mathbf{n}, \hat{q}_h \rangle_{\partial_{\varepsilon} h} = -(g_D, \hat{q}_h)_{\Gamma_D}.
\end{cases}$$

Several possible discrete spaces for (6.13) are

$$V_h = V_h^{k+1}, Q_h = Q_h^k, \hat{Q}_h = \hat{Q}_h^k, \text{ or } V_h = V_h^k, Q_h = Q_h^{k,RT}, \hat{Q}_h = \hat{Q}_h^k.$$  \hspace{1cm} (6.13)

We refer to [22] for discussion from the WG to the hybrid primal methods [42, 43, 44] and the primal methods [45, 46, 47, 48, 49, 4, 50].

Minimal stabilized gradient-based method. In light of Theorem 4.1, the gradient-based inf-sup condition holds for any $\hat{V}_h$. Hence, we relax the condition in WG by choosing $\hat{V}_h = \{0\}$ in (6.10) to obtain a stabilized gradient-based method with minimal stabilization, which reads: Find $(p_h, \hat{p}_h, u_h) \in Q_h \times \hat{Q}_h \times V_h$, such that for any $(q_h, \hat{q}_h, v_h) \in Q_h \times \hat{Q}_h \times V_h$

$$\begin{cases}
(c \mathbf{p}_h, q_h) + (\tau^{-1} \mathbf{p}_h, \hat{q}_h) + (\nabla_h u_h, q_h) - (\mathbf{u}_h, \{q_h\})_{\varepsilon} = -(g_D, q_h \cdot \mathbf{n} + \hat{q}_h \mathbf{n})_{\Gamma_D}, \\
(p_h, \nabla_h v_h) - (\hat{p}_h, \{p_h\})_{\varepsilon} = -(f, v_h) + (g_N, v_h)_{\Gamma_N}.
\end{cases}$$

Consequently, the scheme (6.14) is also stable provided that $Q_h$, $\hat{Q}_h$ and $V_h$ satisfy the conditions in Theorem 4.1. Further, by the elimination of $\hat{p}_h$ using (6.1), we obtain an LDG method [20] in mixed form: Find $(p_h, u_h) \in Q_h \times V_h$ such that for any $(q_h, v_h) \in Q_h \times V_h$

$$\begin{cases}
(c \mathbf{p}_h, q_h) + (\tau \mathbf{Q}_h^{[p]} u_h, q_h)_{\varepsilon} = -(g_D, q_h \cdot \mathbf{n})_{\Gamma_D}, \\
(p_h, \nabla_h v_h) - (\mathbf{Q}_h^{[p]} u_h, \{p_h\})_{\varepsilon} = -(f, v_h) + (g_N, v_h)_{\Gamma_N} + (\tau \mathbf{Q}_h^{[p]} g_D, v_h)_{\Gamma_D}.
\end{cases}$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$Q_h$ & $Q_h$ & $V_h$ & $V_h$ & reference & inf-sup condition \\
\hline
$Q_h^{k+1}$ & $Q_h^{k+1}$ & $V_h^{k+1}$ & $V_h^{k+1}$ & HDG in [38] & gradient-based \\
\hline
$Q_h^{k,RT}$ & $Q_h^k$ & $V_h^k$ & $V_h^k$ & HDG in [30] & divergence-based \\
\hline
$Q_h^k$ & $Q_h^k$ & $V_h^{k+1}$ & $V_h^{k+1}$ & HDG with reduced stabilization in [38, 39] & gradient-based \\
\hline
$Q_h^k$ & $Q_h^k$ & $V_h^k$ & $V_h^k$ & HDG in [31] & not proved \\
\hline
$Q_h^{k+1}$ & $\{0\}$ & $V_h^k$ & $V_h^k$ & Mixed DG in [22] & divergence-based \\
\hline
$Q_h^{k,RT}$ & $Q_h^k$ & $V_h^k$ & $V_h^{k+1}$ & WG in [33] & divergence-based \\
\hline
$Q_h^k$ & $Q_h^k$ & $V_h^{k+1}$ & $V_h^{k+1}$ & WG-MFEM in [34] & gradient-based \\
\hline
$Q_h^k$ & $Q_h^k$ & $\{0\}$ & $\{0\}$ & LDG in [20] & gradient-based \\
\hline
\end{tabular}
\caption{From (3.10) to existing methods}
\end{table}
**Primal method.** We remark that, if we take \( \eta \to 0 \) and choose \( \mathbf{Q}_h \times \hat{\mathbf{Q}}_h \times \mathbf{V}_h = \mathbf{Q}^0_h \times \hat{\mathbf{Q}}^0_h \times \mathbf{V}^1_h \), the WG method (6.11) is equivalent to the nonconforming finite element method discretized by Crouzeix-Raviart element. However, when choosing \( \mathbf{Q}_h \times \hat{\mathbf{Q}}_h \times \mathbf{V}_h = \mathbf{Q}^1_h \times \hat{\mathbf{Q}}^1_h \times \mathbf{V}^2_h \) and taking \( \eta \to 0 \), the WG method (6.11) is getting unstable. In this case, the stabilization is needed for the hybrid primal method which induces to the WG method.

### 6.3 Eliminating both \( \hat{\mathbf{p}}_h \) and \( \hat{\mathbf{u}}_h \)

Plugging in (6.1) and (6.9) into (3.8a) and (3.8b), respectively, we obtain a DG method: Find \( (\mathbf{p}_h, \mathbf{u}_h) \in \mathbf{Q}_h \times \mathbf{V}_h \) such that for any \( (\mathbf{q}_h, \mathbf{v}_h) \in \mathbf{Q}_h \times \mathbf{V}_h \)

\[
\begin{align*}
    (\mathbf{p}_h, \nabla \mathbf{u}_h) + (\mathbf{q}_h, \mathbf{q}_h) &+ \langle \eta \mathbf{Q}^0_h [\mathbf{p}_h], \hat{\mathbf{Q}}^0_h [\mathbf{q}_h] \rangle - \langle \mathbf{u}_h, \mathbf{q}_h \rangle = -(g_D, \mathbf{q}_h \cdot \mathbf{n})_{\Gamma_D} + \langle \eta \mathbf{Q}^1_h \mathbf{g}_N, \mathbf{q}_h \cdot \mathbf{n} \rangle_{\Gamma_N}, \\
    (\mathbf{p}_h, \nabla \mathbf{v}_h) - \langle \mathbf{p}_h, \mathbf{v}_h \rangle - \langle \tau \mathbf{Q}^0_h [\mathbf{u}_h]|_{\Gamma_D}, \hat{\mathbf{Q}}^0_h [\mathbf{v}_h]|_{\Gamma_D} \rangle = -(f, \mathbf{v}_h) + \langle \mathbf{g}_N, \mathbf{v}_h \rangle_{\Gamma_N} + \langle \tau \mathbf{Q}^1_h \mathbf{g}_D, \mathbf{v}_h \rangle_{\Gamma_D}.
\end{align*}
\]

(6.16)

We note that (6.16) is equivalent to the formulation (3.10). Firstly, the solution \( \mathbf{p}_h, \mathbf{u}_h \) obtained from (3.10) coincides the solution of (6.16). On the other hand, having the solution \( \mathbf{p}_h, \mathbf{u}_h \) of (6.16), by using (6.1) and (6.9), we can construct \( \hat{\mathbf{p}}_h \) and \( \hat{\mathbf{u}}_h \). It is straightforward to show that \( (\mathbf{p}_h, \mathbf{u}_h, \hat{\mathbf{p}}_h, \hat{\mathbf{u}}_h) \) is the solution of (3.10).

If the choice of the spaces \( \mathbf{Q}_h, \mathbf{V}_h, \hat{\mathbf{Q}}_h \) satisfying \( \mathbf{Q}_h \subset \mathbf{V}_h \subset \hat{\mathbf{Q}}_h \) then the projections \( \mathbf{Q}^0_h \) and \( \hat{\mathbf{Q}}^0_h \) reduce to identities. Then in this case, (6.16) reduces to the LDG method proposed in [52].

**Remark 6.2** There are four filed: \( \mathbf{u}_h, \mathbf{p}_h, \hat{\mathbf{u}}_h, \hat{\mathbf{p}}_h \). Theoretically by eliminating any \( m \)-fields for \( m \leq 3 \), we obtain:

\[
C_4^1 + C_4^2 + C_4^3 = 4 + 6 + 4 = 14
\]

namely 14 methods. Some of the methods should be hybridized algorithms. These algorithms have special interesting case under special assumption, e.g. primal method and mixed method.

### 7 Conclusion

The unified formulation, presented in this paper, is a 4-field formulation that deduces most existing finite element methods and DG method as special cases. In particular, we deduce HDG method and WG method from the formulation and show that they can both be recast into a DG method derived from the unified formulation. In addition, we prove two types of uniform inf-sup conditions for the formulation, which naturally lead to uniform inf-sup conditions of HDG, WG and the DG method.

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