Abstract

In this work, we use martingale theory to derive formulas for the expected decision time, error rates, and first passage times associated with a multistage drift diffusion model, defined as a Wiener diffusion model with piecewise constant time-varying drift rates and decision boundaries. The model we study is a generalization of that considered in Ratcliff (1980). The derivation relies on using the optional stopping theorem for properly chosen martingales, thus providing formulae which may be used to compute performance metrics for a particular stage of the stochastic decision process. We also explicitly solve the case of a two stage diffusion model, and provide numerical demonstrations of the computations suggested by our analysis. We discuss the applications of these formulae for experiments involving time pressure and/or changes in attention over the course of the decision process. We further show how these formulae can be used to semi-analytically calculate reward rate in the service of optimizing the speed-accuracy trade-off. Finally we present calculations that allow our techniques to approximate time-varying Ornstein-Uhlenbeck processes. By presenting these explicit formulae, we aim to foster the development of refined numerical methods and analytical techniques for studying diffusion models with time-varying parameters.

1. Introduction

Continuous time stochastic processes based on drift and diffusion between two absorbing boundaries have been used in a wide variety of applications including statistical physics (Farkas and Fulop, 2001), finance (Lin, 1998), and health science (Horrocks and Thompson, 2004). In this article, we focus on applications to decision making tasks, where such models have successfully accounted for behavior and neural activity in a wide array of two alternative forced choice tasks, including phenomena such as the speed-accuracy tradeoff and the dynamics of neural activity during decision making in such tasks (Ratcliff and Rouder, 1998; Bogacz et al., 2006; Ratcliff and McKoon, 2008; Simen et al., 2009; Gold and Shadlen, 2001, 2007; Brunton et al., 2013; Feng et al., 2009; Shadlen and Newsome, 2001). In particular, we will discuss extensions of a specific class of diffusion model referred to as the pure drift diffusion model (DDM; Eq. 1 below), which can be shown to be statistically optimal (Wald, 1945; Wald and Wolfowitz, 1948). Varieties of diffusion models have been applied within and outside of psychology and neuroscience to elucidate mechanisms for perception, decision-making, memory, attention, and cognitive control (see reviews in Ratcliff and Smith (2004); Bogacz et al. (2006)).

In the pure DDM (and related models), the state variable $x(t)$ is thought to represent the amount of accumulated noisy evidence at time $t$ for decisions represented by the two absorbing boundaries, which we refer to as the upper (+) and lower (-) boundaries. The evidence $x(t)$ evolves in time according to a biased random walk with Gaussian increments, which may be written as $dx(t) \sim \text{Normal}(A dt, \sigma^2 dt)$, and a decision is made at time $\tau$, the smallest time $t$ for which $x(t)$ hits either the top boundary ($x(\tau) = +z$) or bottom
boundary \((x(\tau) = -z)\). Hitting either of the two boundaries corresponds to making one of two possible decisions at time \(\tau\). The resulting decision dynamics are thus described by the first passage times of the underlying drift diffusion process. In studying these processes one is often interested in relating performance metrics such as the mean decision time and error rate (i.e. the probability of hitting the boundary opposite to the direction of drift) to empirical data. For example, one may be interested in studying how actions and cognitive processes might seek to maximize reward rate, which is a simple function of error rate and mean decision time [Bogacz et al., 2006].

However, not all decisions can be well described by a pure DDM with time-invariant decision parameters. Many decisions may require time-varying drift rates, diffusion rates, and/or thresholds in order to model "bottom-up" signals, e.g. sensory processing, or to model "top-down" effects, for example, when there are changes in attentional focus or cognitive control [Krajbich et al., 2010, White et al., 2011]. In this article, we study performance metrics for such extensions of the DDM, in which model parameters are time-varying. In doing so, we build on recent work that is focused on similar time-varying random walk models [Hubner et al., 2010, Diederich and Oswald, 2014]. We offer an alternative approach to deriving performance metrics in this setting, including first passage time distributions as well as expected error rates and decision times, and describe how our approach can be applied to the more general class of Ornstein-Uhlenbeck (O-U) processes. An in-depth discussion of how the present article interfaces with other studies of time-varying DDMs is given in Section 7.

2. Drift diffusion models

In this section we recall the pure DDM and introduce the multistage drift diffusion model (MSDDM).

2.1. The (single stage) drift diffusion model

The single stage pure DDM [Ratcliff and Rouder, 1998, Bogacz et al., 2006] models human decision making in two alternative forced choice tasks. The DDM models the evolution of the evidence for decision making using the following stochastic differential equation (SDE):

\[
dx(t) = Adt + \sigma dW(t), \quad x(t_0) = x_0,
\]

where parameters \(A\) and \(\sigma\) are constants referred to as the drift and diffusion rate, respectively; \(x_0\) is the initial condition (the starting point of the decision process), and \(\sigma dW(t)\) are independent Wiener increments with variance \(\sigma^2 dt\). The pure DDM (1) models a decision process in which an agent is integrating noisy evidence until sufficient evidence is gathered in favor of one of the two alternatives.

A decision is made when the evidence \(x(t)\) crosses one of the two symmetric decision thresholds \(\pm z\) for the first time (also referred to as its first passage time). In other words, the decision time is the first passage time of the drift diffusion process (1) with respect to the set of points within boundaries \(\pm z\).

We will use the terms correct and incorrect to refer to responses that cross the threshold of equivalent or opposite sign of the drift rate for the decision process. For instance, if \(A > 0\), a correct decision is one for which threshold \(+z\) is crossed first; conversely, an incorrect response refers to the case in which threshold \(-z\) was crossed first. Accordingly, we refer to the probability of crossing the negative threshold in this case as the error rate and the time at which the decision variable crosses one of the thresholds as the decision time.

The "pure" DDM contrasts with Ratcliff’s (1978) “extended” DDM, in which the drift rates and the initial conditions across trials in an experimental session are assumed to be random variables drawn from stationary distributions. This paper will focus entirely on analyses of the former model, (as well as its applications to Ornstein-Uhlenbeck processes). Note also that the parameterization we use for the pure DDM is different from those in some other cases, e.g., Smith (2000), Ratcliff and Smith (2004), Navarro and Fuss (2009), although the underlying model is equivalent.
2.2. The multistage drift diffusion model

The DDM has been extremely successful in explaining behavioral and neural data from two alternative forced choice tasks. However, its basic assumption — that model parameters such as drift rate and threshold remain constant throughout the decision process — is unlikely to hold in a number of cases. For instance, in several experimental (and real-world) contexts the quality of evidence is not stationary (i.e., the drift rate and possibly even the diffusion rate are not a constant function of time) or decision urgency leads thresholds to decay with time.

For purposes of modeling situations described above, we introduce an extension of the DDM, the MSDDM, which is a generalization of the two stage process considered in Ratcliff (1980). In an MSDDM, the drift rate, the diffusion rate, and the thresholds are piecewise constant functions of time.

To implement an MSDDM, we partition the set of non-negative real numbers (time axis) into $n$ sets $[t_{i-1}, t_i]$, $i \in \{1, \ldots, n\}$ with $t_0 = 0$ and $t_n = +\infty$. We then assume that the drift rate, the diffusion rate, and the decision thresholds are constant within each interval, but that their values may be different for different intervals. In particular, the evidence integration in the MSDDM is modeled by

$$dx(t) = a(t) dt + \sigma(t)dW(t), \quad x(t_0) = x_0,$$

where

$$a(t) = a_i, \quad \text{for } t_{i-1} \leq t < t_i,$$

$$\sigma(t) = \sigma_i, \quad \text{for } t_{i-1} \leq t < t_i,$$

for each $i \in \{1, \ldots, n\}$. Note that for $n = 1$, the MSDDM reduces to the pure DDM. For simplicity of presentation, we first consider the case with constant decision thresholds at $\pm z$. In section 3 we describe how to incorporate piecewise constant time-varying thresholds into the MSDDM.

We will frequently refer to the $i$-th stage DDM of the MSDDM. For $t > t_{i-1}$, it is written as

$$dx(t) = a_i dt + \sigma_i dW(t), \quad x(t_{i-1}) = X_{i-1}, \quad \text{decision thresholds } \pm z.$$

The initial condition is now a random variable $X_{i-1}$ defined as $x(t_{i-1})$ conditioned on no decision until time $t_{i-1}$, i.e., the density of $X_{i-1}$ is the conditional distribution of $x(t_{i-1})$ given that $\tau > t_{i-1}$. Thus, the random variable $X_{i-1}$ corresponds to realizations of the MSDDM that remain within the thresholds $\pm z$ until time $t_{i-1}$. Note that $X_0$ is a point mass distribution centered at $x_0$.

We introduce the following notations throughout this paper:

- $\tau = \inf\{t > 0 | x(t) \notin (-z, z)\}$, the first passage time for the entire multistage process
- $\tau_i = \tau | \tau > t_{i-1}$, the first passage time for the $i$-th stage DDM.
- $\theta_i = (a_i, \sigma_i, z)$, and $\theta_{i, \ell} = (a_1, \ldots, a_\ell, \sigma_1, \ldots, \sigma_\ell, z)$.
- $s_i = a_i / \sigma_i^2$, the $i$-th stage signal to noise square ratio.

3. First passage time metrics for the MSDDM

In this section we derive first passage time (FPT) properties of the MSDDM. At a higher level, our approach can be summarized as follows:

The MSDDM can be viewed as a cascade of $n$ modified DDMs in which for each stage of the cascade, the initial condition is a random variable and only the decisions made before a deadline are considered. For the $i$-th stage DDM with a known distribution of initial condition $X_{i-1}$, we derive (i) properties of the FPT conditioned on a decision before the deadline $t_i$, and (ii) distribution of $X_i$, i.e., the distribution of $x(t_i)$ conditioned on the FPT for the $i$-th stage DDM greater than $t_i$. We then use these properties sequentially for $i \in \{1, \ldots, n\}$ to determine FPT properties during each stage. Finally, we aggregate FPT properties at
each stage to compute FPT properties for the MSDDM as a whole, that is, for the properties of the FPT that are not conditioned on the stage.

In the following, we first recall FPT properties for the pure DDM. We then use these properties, together with tools from martingale theory, to derive FPT metrics for the modified $i$-th stage DDM described above. Finally, we use total probability formulae to merge these metrics and obtain FPT metrics for the MSDDM. Throughout this section, we assume the decision thresholds are fixed at $\pm z$, the number of stages is $n$, and $t_0 < t_1 < \ldots < t_n = \infty$.

3.1. FPT properties of the pure DDM

We first review some well known results for the pure DDM \cite{1}, which is equivalent to a 1-stage MSDDM. Without loss of generality, we assume that the drift rate $a_1 \geq 0$. Recall that $\theta_1 = (a_1, \sigma_1, z)$ and $s_1 = a_1/\sigma_1^2$. For the 1-state MSDDM:

(i) The error rate $ER(x_0, \theta_1)$ is

$$ER(x_0, \theta_1) = P(x(\tau_1) = -z) = \left\{ \begin{array}{ll}
\frac{e^{-2s_1 x_0} - e^{-2s_1 z}}{z^2 - 2z}, & \text{if } a_1 > 0, \\
\frac{e^{-2s_1 z}}{z^2}, & \text{if } a_1 = 0.
\end{array} \right. \quad \text{(4)}$$

Throughout the remainder of the section, for brevity, we let $ER = ER(x_0, \theta_1)$.

(ii) The mean decision time $mDT(x_0, \theta_1)$ is given by

$$mDT(x_0, \theta_1) = \mathbb{E}[\tau_1] = \mathbb{E}[\tau] = \left\{ \begin{array}{ll}
\frac{(1-2ER)z - x_0}{2s_1}, & \text{if } a_1 > 0, \\
\frac{z^2 - 2s_1 x_0}{2s_1}, & \text{if } a_1 = 0.
\end{array} \right. \quad \text{(5)}$$

(iii) The first passage time density $f(t; x_0, \theta_1)$, i.e., the probability density function of the decision time is

$$f(t; x_0, \theta_1) = \frac{d}{dt} P(\tau_1 \leq t) = e^{-\frac{a_1^2 t^2}{2\sigma_1^2}} \left( e^{-s_1(z+x_0)} B(t; \frac{z - x_0}{\sigma_1}, \frac{2z}{\sigma_1}) + e^{s_1(z-x_0)} B(t; \frac{z + x_0}{\sigma_1}, \frac{2z}{\sigma_1}) \right), \quad \text{(6)}$$

where

$$B(t; u, v) = \sum_{k=-\infty}^{+\infty} \frac{v - u + 2kv}{\sqrt{2\pi t}^{3/2}} e^{-(v-u+2kv)^2/2t}, \quad u < v.$$

(iv) The expected decision time conditioned on the correct response $mDT_+$ and the expected decision time conditioned on the incorrect response $mDT_-$ are

$$mDT_+ = \frac{\mathbb{E}[\tau_1 1(x(\tau_1) = +z)]}{1 - ER} = \left\{ \begin{array}{ll}
\frac{2z}{a_1} \coth(2s_1 z) - \frac{z + x_0}{a_1} \coth(s_1(z + x_0)), & \text{if } a_1 > 0, \\
\frac{2z}{a_1} \coth(s_1(z + x_0)), & \text{if } a_1 = 0;
\end{array} \right. \quad \text{(7)}$$

$$mDT_- = \frac{\mathbb{E}[\tau_1 1(x(\tau_1) = -z)]}{ER} = \left\{ \begin{array}{ll}
\frac{2z}{a_1} \coth(2s_1 z) - \frac{z - x_0}{a_1} \coth(s_1(z - x_0)), & \text{if } a_1 > 0, \\
\frac{2z}{a_1} \coth(s_1(z - x_0)), & \text{if } a_1 = 0,
\end{array} \right. \quad \text{(8)}$$

where $\mathbb{E}[\tau_1 1(x(\tau_1) = \pm z)]$ and $1(\cdot)$ is the indicator function.

Note that $mDT_+$ corresponds to the average decision time computed by dividing the sum of decision times associated with the correct decision by the instances of such decisions; while $mDT_+$ corresponds to the average decision time computed by dividing the sum of decision times associated with the correct decision by the instances of all (correct and incorrect) decisions. $mDT_-$ and $mDT_-$ are computed analogously.
(v) The first passage time density conditioned on a particular decision is given by
\[
\frac{d}{dt} P(\tau_1 \leq t | x(\tau_1) = z) = f^+(t; x_0, \theta_1) = \frac{e^{-\frac{x_1^2}{2\sigma_1^2}} + s_1(z-x_0)}{1-ER} \beta(t; \frac{z+x_0}{\sigma_1}, \frac{2z}{\sigma_1}) \tag{9}
\]
\[
\frac{d}{dt} P(\tau_1 \leq t | x(\tau_1) = -z) = f^-(t; x_0, \theta_1) = \frac{e^{-\frac{x_1^2}{2\sigma_1^2}} - s_1(z+x_0)}{ER} \beta(t; \frac{z-x_0}{\sigma_1}, \frac{2z}{\sigma_1}), \tag{10}
\]
where \( f^\pm(t; x_0, \theta_1) = \frac{d}{dt} P(\tau \leq t \& x(\tau) = \pm z) \), i.e., \( f^\pm(t; x_0, \theta_1) dt \) is the probability of the event \( \tau \in [t, t+dt] \) and \( x(\tau) = \pm z \). Note that \( f \) defined in (9) is the sum of \( f^+ \) and \( f^- \).

(vi) The joint density \( g_{\text{ddin}}(x, t; x_0, \theta_1) \) of the evidence \( x(t) \) and the event \( \tau \geq t \) is
\[
g_{\text{ddin}}(x, t; x_0, \theta_1) = \frac{d}{dx} P(x(t) \leq x \& \tau \geq t) = \frac{d}{dx} P(x(t) \leq x \& \tau_1 \geq t) = \mathbf{1}(x \in (-z, z)) e^{-\frac{x^2 + 2a(x-x_0)}{2\sigma_1^2}} \sum_{n=-\infty}^{\infty} \left( e^{-\frac{(x-x_0+4\pi n)^2}{2\sigma_1^2}} - e^{-\frac{(2z-x-x_0+4\pi n)^2}{2\sigma_1^2}} \right). \tag{11}
\]
where \( \mathbf{1}(\cdot) \) is the indicator function.

The joint density \( g_{\text{ddin}} \) can also be used to determine the FPT distribution by integrating it over the range of \( x \). More importantly, dividing \( g_{\text{ddin}} \) by \( P(\tau \geq t) \) yields the conditional density on the evidence \( x(t) \) conditioned on no decision until time \( t \). This is critical for the analysis of the MSDDM. In particular, we use a slight modification of \( g_{\text{ddin}} \) at the \( i \)-th stage to determine the density of the initial condition \( X_i \) for the \((i+1)\)-th stage.

Various derivations for the error rate, decision time, and FPT densities may be found in the decision making literature \cite{Ratcliff2004, Bogacz2006, NavarroFuss2009}. In the probability literature, the expressions for the error rate and the expected decision time may be derived using a differential equation approach \cite{Gardiner2009} or a martingale-based approach \cite{Durrett2010}. Under the latter, the FPT densities are found by first determining the Laplace transform by constructing an appropriate martingale \cite{Durrett2010, BorodinSalminen2002, Lin1998}. Taking the inverse Laplace transforms yields the conditional FPT densities. The negative of the derivative of the Laplace transform with respect to the frequency yields the conditional mean decision times. The final statement is derived by repeatedly applying the reflection principle, followed by the Cameron-Martin formula \cite{Douady1999, Durrett2010}. Formulas for the Laplace transform are given in Appendix A.

It is worth noting that the infinite series solutions for the FPT given in (6) are equivalent to the small-time representations for the FPT analyzed in \cite{NavarroFuss2009, Blurton2012}. For completeness we also state the large-time representation that can be obtained by solving the Fokker-Planck equation \cite{Feller1968}:
\[
f(t; x_0, \theta_1) = \frac{\pi \sigma_1^2}{4z^2} e^{-\frac{x_1^2}{2\sigma_1^2}} \sum_{n=1}^{+\infty} (-1)^{n-1} ne^{-\frac{n^2+x_0^2}{\sigma_1^2}} \left( e^{\frac{2n(z-x_0)}{\sigma_1^2}} \sin \left( \frac{n\pi(z+x_0)}{2z} \right) + e^{\frac{-2n(z-x_0)}{\sigma_1^2}} \sin \left( \frac{n\pi(z-x_0)}{2z} \right) \right).
\]
The small-time and the large-time representations mean that the associated series has nice convergence properties (e.g., monotonicity) for small and large values of decision times, respectively.

3.2. FPT properties of the \( i \)-th stage DDM

In this section we analyze the \( i \)-th stage DDM. Recall from above that \( X_{i-1} \) is the random variable \( x(t_{i-1}) \) conditioned on the event \( \tau > t_{i-1} \), and that \( \tau_i \) is the FPT conditioned on the event \( \tau > t_{i-1} \).
For the i-th DDM, the initial condition is $X_{i-1}$ and only decisions made before the deadline $t_i$ are relevant. For the analysis of such a system, the two key ingredients are the density of $X_{i-1}$ and the density of the FPT $\tau_i$.

Conditioned on a realization of $X_{i-1}$, the density of $X_i$ can be computed using (11). If the density of $X_{i-1}$ is known, then the unconditional density of $X_i$ can be obtained by computing the expected value of the conditional density of $X_i$ with respect to $X_{i-1}$. Since the density of $X_0$ is known, this procedure can be recursively applied to obtain densities of $X_{i-1}$, for each $i \in \{1, \ldots, n\}$. Formally, the joint density $g_i^{\text{ddln}}(x; x_0, \theta_{1:i})$ of the evidence $x(t_i)$ and the event $\tau \geq t_i$ is

$$g_i^{\text{ddln}}(x; x_0, \theta_{1:i}) = \frac{d}{dx} \mathbb{P}(x(t_i) \leq x \& \tau > t_i) = \frac{d}{dx} \mathbb{P}(X \leq x) \mathbb{P}(\tau > t_i) = \mathbb{E}_{X_{i-1}}[g_i^{\text{ddln}}(x, t_i - t_i - 1; X_{i-1}, \theta_i)],$$

(12)

where $\mathbb{E}_{X_{i-1}}[\cdot]$ denotes the expected value with respect to $X_{i-1}$. The density of $X_i$, i.e., $x(t_i)$ conditioned on $\tau > t_i$ is determined by dividing $g_i^{\text{ddln}}$ by $\mathbb{P}(\tau > t_i)$ which can be computed by integrating $g_i^{\text{ddln}}$ over the range of $x(t_i)$. Note that the parameters in $g_i^{\text{ddln}}$ are $x_0$ and $\theta_{1:i}$; this highlights the fact that the distribution of $X_i$ depends on all previous stages.

Similarly, the FPT density for the i-th stage DDM conditioned on a realization of $X_{i-1}$ can be computed using (10), and the unconditional density can be obtained by computing the expected value of the conditional density with respect to $X_{i-1}$. Formally, the FPT density for i-th stage DDM $f_i(t; x_0, \theta_{1:i})$ is

$$f_i(t; x_0, \theta_{1:i}) = \frac{d}{dt} \mathbb{P}(\tau_i \leq t) = \mathbb{E}_{X_{i-1}}[f(t - t_i - 1; X_{i-1}, \theta_i)],$$

(13)

where $t > t_i$. The cumulative distribution function $F_i(t; x_0, \theta_{1:i}) = \mathbb{P}(\tau_i \leq t)$ is obtained by integrating $f_i(t; x_0, \theta_{1:i})$. Note that every trajectory crossing the decision threshold before $t \leq t_i$ does so irrespective of the deadline at $t_i$. Thus, the expression for density $f_i$ does not depend on $t_i$.

We are now ready to establish performance metrics for the i-th DDM. The detailed derivations of the expressions presented in this section are contained in Appendix B.

(i) The error rate during the i-th stage $ER_i(x_0, \theta_{1:i})$, i.e., the probability of an incorrect decision given that a response is made during the i-th stage, $\mathbb{P}(x(\tau) = -z | t_{i-1} < \tau \leq t_i)$, is

$$ER_i(x_0, \theta_{1:i}) = \begin{cases} \mathbb{E}_{X_{i-1}}[e^{-2sX_{i-1}} - \mathbb{E}_{X_i}[e^{-2sX_i}]\mathbb{P}(\tau_i > t_i) - e^{-2sX_i}\mathbb{P}(\tau_i \leq t_i)] \mathbb{P}(\tau_i \leq t_i), & \text{if } a_i > 0, \\ 1 - \frac{(e^{2sX_i} - e^{-2sX_i})\mathbb{P}(\tau_i \leq t_i)}{2s\mathbb{P}(\tau_i \leq t_i)}, & \text{if } a_i = 0. \end{cases}$$

(14)

In the following we drop the arguments of $ER_i(x_0, \theta_{1:i})$ and refer it by $ER_i$.

(ii) The joint FPT density for the i-th stage DDM and a given response $f_i^\pm(t; x_0, \theta_{1:i}) = \frac{d}{dt} \mathbb{P}(\tau_i \leq t \& x(\tau_i) = \pm z)$ is

$$f_i^+(t; x_0, \theta_{1:i}) = \frac{d}{dt} \mathbb{P}(\tau_i \leq t \& x(\tau_i) = z) = \mathbb{E}_{X_{i-1}}[f_i^+(t - t_i - 1; X_{i-1}, \theta_i)],$$

(15)

$$f_i^-(t; x_0, \theta_{1:i}) = \frac{d}{dt} \mathbb{P}(\tau_i \leq t \& x(\tau_i) = -z) = \mathbb{E}_{X_{i-1}}[f_i^-(t - t_i - 1; X_{i-1}, \theta_i)],$$

(16)

where the functions $f_i^\pm(t; x_0, \theta_{1:i})$ are taken from [9] and [10].

(iii) The mean decision time given that a response is made during the i-th stage $\text{mDT}_i(x_0, \theta_{1:i})$ is

$$\text{mDT}_i(x_0, \theta_{1:i}) = \mathbb{E}[\tau_i | \tau_i \leq t_i]$$

$$= \begin{cases} t_{i-1} + \frac{1 - 2a_i \sigma_x^2 \mathbb{P}(\tau_i \leq t_i) - \mathbb{E}_{X_{i-1}}[X_i] \mathbb{P}(\tau_i > t_i) - a_i \mathbb{P}(t_i - t_{i-1}) \mathbb{P}(\tau_i > t_i)}{a_i \mathbb{P}(\tau_i \leq t_i)} \sigma_x^2 \mathbb{P}(\tau_i \leq t_i), & \text{if } a_i > 0, \\ t_{i-1} + \frac{\mathbb{E}_{X_{i-1}}[X_i] \mathbb{P}(\tau_i > t_i) - a_i \mathbb{P}(t_i - t_{i-1}) \mathbb{P}(\tau_i > t_i)}{a_i \mathbb{P}(\tau_i \leq t_i)}, & \text{if } a_i = 0. \end{cases}$$

(17)
In particular, we view the MSDDM as a cascade of modified DDMs in which the initial condition is a random contained in Appendix C.

Then, we compute the properties of the FPT associated with the

### 3.3. FPT properties of the MSDDM

We now describe the FPT properties of the MSDDM (2). The derivations of these expressions are

$$\hat{E}[\tau_x(x_1, \theta_1, i)] = \frac{\hat{m}_{DT}^+(x_0, \theta_1, i)}{(1 - ER_i)\mathbb{P}(\tau_x \leq t_i)}$$

$$= \frac{1}{(1 - ER_i)\mathbb{P}(\tau_x \leq t_i)} \left( t_{i-1}\mathbb{P}(x(x_i) = z) + \mathbb{E}_{X_{i-1}}[\hat{m}_{DT}^+(X_{i-1}, \theta_i)] \right)$$

$$- \mathbb{E}_{X_i}[\hat{m}_{DT}^+(X_i, \theta_i)]\mathbb{P}(\tau_x > t_i) + t_i\mathbb{P}(x(x_i) = z \& \tau_x > t_i) \right) \quad (18)$$

$$\mathbb{E}[\tau_x(x_1, \theta_1, i)] = \frac{\hat{m}_{DT}^-(x_0, \theta_1, i)}{ER_{i}\mathbb{P}(\tau_x \leq t_i)}$$

$$= \frac{1}{ER_i\mathbb{P}(\tau_x \leq t_i)} \left( t_{i-1}\mathbb{P}(x(x_i) = -z) + \mathbb{E}_{X_{i-1}}[\hat{m}_{DT}^-(X_{i-1}, \theta_i)] \right)$$

$$- \mathbb{E}_{X_i}[\hat{m}_{DT}^-(X_i, \theta_i)]\mathbb{P}(\tau_x > t_i) + t_i\mathbb{P}(x(x_i) = -z \& \tau_x > t_i) \right) \quad (19)$$

where $\hat{m}_{DT}^+(x_0, \theta_1, i) = \mathbb{E}[\tau_x(x_1(x_i) = \pm z \& \tau_x \leq t_i)]$, $\hat{m}_{DT}^+(X_{i-1}, \theta_i)$ is calculated using (7) and (8), $\mathbb{P}(x(x_i) = \pm z)$ is calculated using (4), and $\mathbb{P}(x(x_i) = \pm z \& \tau_x > t_i)$ is calculated using (18) and (19).

We now describe the FPT properties of the MSDDM (2). The derivations of these expressions are contained in Appendix C.

(i) For $t \in (t_{k-1}, t_k)$, for some $k \in \{1, \ldots, n\}$, the FPT distribution for the MSDDM is

$$\mathbb{P}(\tau_x \leq t) = 1 - \prod_{i=1}^{k-1} \mathbb{P}(\tau_x > t_i) + \mathbb{P}(\tau_x \leq t) \prod_{i=1}^{k-1} \mathbb{P}(\tau_x > t_i) \quad (20)$$

Note that $\prod_{i=1}^{k-1} \mathbb{P}(\tau_x > t_i) = \mathbb{P}(\tau > t_{k-1})$ and $\mathbb{P}(\tau_x \leq t) = \mathbb{P}(\tau \leq t)\mathbb{P}(\tau > t_{k-1})$.

(ii) The expected decision time for the MSDDM is

$$\mathbb{E}[\tau] = \sum_{i=1}^{n} \left( \mathbb{E}[	au_x | \tau_i \leq t_i] \mathbb{P}(\tau_i \leq t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_i) \right)$$

$$= \sum_{i=1}^{n} \left( m_{DT}(x_0, \theta_1, i) \mathbb{P}(\tau_i \leq t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_i) \right). \quad (21)$$

Note that the expected decision time is the weighted sum of the expected decision times for the DDMs at each stage and the weight is the probability of the decision in that stage.

(iii) The error rate for the MSDDM is

$$ER = \sum_{i=1}^{n} \left( ER_i \mathbb{P}(\tau_x < t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_i) \right). \quad (22)$$

Similar to the case of the expected decision time, the error rate is the weighted sum of the error rates during each stage.
(iv) The expected decision time conditioned on a particular decision is

\[
E[\tau|x(\tau) = z] = \frac{1}{1 - ER} \sum_{i=1}^{n} \left( E[\tau_i|x(\tau_i) = z & \tau_i \leq t_i](1 - ER_i) \mathbb{P}(\tau_i \leq t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_i) \right) \tag{23}
\]

\[
E[\tau|x(\tau) = -z] = \frac{1}{ER} \sum_{i=1}^{n} \left( E[\tau_i|x(\tau_i) = z & \tau_i \leq t_i]ER_i \mathbb{P}(\tau_i \leq t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_i) \right). \tag{24}
\]

Note that \( E[\tau_i|x(\tau_i) = z & \tau_i \leq t] \mathbb{P}(\tau_i \leq t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_i) = \mathbb{P}(t_i - 1 < \tau \leq t_i) \).

(v) For \( t \in (t_{k-1}, t_k] \), for some \( k \in \{1, \ldots, n + 1\} \), the FPT distribution conditioned on a particular decision is

\[
P(\tau \leq t|x(\tau) = z)(1 - ER) = \mathbb{P}(\tau_k \leq t & x(\tau_k) = z) \prod_{j=1}^{k-1} \mathbb{P}(\tau_j > t_j)
\]

\[+ \sum_{i=1}^{k-1} \mathbb{P}(\tau_i \leq t_i & x(\tau_i) = z) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_j) \tag{25}\]

\[
P(\tau \leq t|x(\tau) = -z)ER = \mathbb{P}(\tau_k \leq t & x(\tau_k) = -z) \prod_{j=1}^{k-1} \mathbb{P}(\tau_j > t_j)
\]

\[+ \sum_{i=1}^{k-1} \mathbb{P}(\tau_i \leq t_i & x(\tau_i) = -z) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_j). \tag{26}\]

Note that \( \prod_{i=1}^{i-1} \mathbb{P}(\tau_j > t_j) = \mathbb{P}(\tau > t_{i-1}) \) and \( \mathbb{P}(\tau_i \leq t_i & x(\tau_i) = z)\mathbb{P}(\tau > t_{i-1}) = \mathbb{P}(t_{i-1} < \tau \leq t_i \& x(\tau) = z) \).

4. Time-varying thresholds

For clarity of exposition, the argument leading to results in \[3\] assumes the same thresholds for each stage. Now suppose that the thresholds for the \( i \)-th stage DDM are \( \pm z_i \). The case of the same thresholds in \[3\] is then the specialization \( z_i = z > 0 \) for each \( i \in \{1, \ldots, n\} \).

Suppose that \( z_{i+1} \) is smaller than \( z_i \). Then the probability of instantaneous absorption into the top boundary at \( t_i \) is found by integrating \[12\] from \( z_{i+1} \) to \( z_i \). Likewise, the probability of instantaneous absorption into the bottom boundary at \( t_i \) is determined by integrating \[12\] from \(-z_i\) to \(-z_{i+1}\). The density of \( X_i \) is then found by truncating the support of the density in \[12\] to \((-z_{i+1}, z_{i+1})\) and normalizing the truncated density. In the cases where \( z_{i+1} \geq z_i \), there is no instantaneous absorption, and the density of \( X_i \) is found by extending of the density in \[12\], assigning zero density to the previously undefined support. In all cases, the new, updated \( X_i \) may be used for computations dealing with the \((i + 1)\)-th stage of the MSDDM.

5. Time-varying Ornstein-Uhlenbeck processes

In this section we study FPT properties of an O-U process with piecewise constant parameters. Similarly to the MSDDM, the \( n \)-stage O-U process is defined by

\[
dx(t) = a(t)dt - \lambda(t)x(t)dt + \sigma(t)dW(t), \quad x(t_0) = x_0, \tag{27}\]
where
\[ a(t) = a_i, \quad \text{for } t_{i-1} \leq t < t_i, \]
\[ \sigma(t) = \sigma_i, \quad \text{for } t_{i-1} \leq t < t_i, \]
\[ \lambda(t) = \lambda_i, \quad \text{for } t_{i-1} \leq t < t_i, \]
for each \( i \in \{1, \ldots, n\} \). The multistage O-U process is obtained from the multistage drift-diffusion process by adding a term \(-\lambda(t)x(t)dt\), i.e., the O-U process is a leaky integrator, while the drift-diffusion process is a perfect integrator. In this section, we compute an approximation of the distribution for the FPT for the O-U process (27) defined by \( \tau = \inf\{t \in \mathbb{R}_0 \mid x(t) \in \{-z, z\}\} \), where \( \pm z \) are predefined thresholds.

First, we show that a 1-stage O-U process can be written as a transformation of the Wiener process, and then use this transformation to write the FPT problem for the O-U process as a FPT problem for the Wiener process with time-varying thresholds. Finally, we approximate the time-varying thresholds with piecewise constant time-varying thresholds to compute the approximate FPT distribution of the O-U process and use it to recursively compute the FPT distribution of the multistage O-U process.

### 5.1. The O-U process as a transformation of the Wiener process

Consider the 1-stage O-U process
\[ dx(t) = a_1 dt - \lambda_1 x(t) dt + \sigma_1 dW(t), \; x(t_0) = x_0. \]  
\( (28) \)

It is known (Cox and Miller 1965) that the solution to (28) is
\[ x(t) = \frac{a_1}{\lambda_1}(1 - e^{-\lambda_1 t}) + e^{-\lambda_1 t}x_0 + e^{-\lambda_1 t}W\left(\frac{\sigma_1^2(e^{2\lambda_1 t} - 1)}{2\lambda_1}\right). \]

Define the transformed time \( u_1(t) := \frac{\sigma_1^2(e^{2\lambda_1 t} - 1)}{2\lambda_1} \). Equivalently, \( t = \frac{1}{2\lambda_1} \log(1 + \frac{2\lambda_1 u_1}{\sigma_1^2}) \). Thus,
\[ x_0 + W(u_1(t)) = \left(x(t) - \frac{a_1}{\lambda_1}\right)e^{\lambda_1 t} + \frac{a_1}{\lambda_1} = \left(x(t) - \frac{a_1}{\lambda_1}\right)\sqrt{1 + \frac{2\lambda_1 u_1(t)}{\sigma_1^2}} + \frac{a_1}{\lambda_1}. \]

Therefore, if \( x(t) = \pm z \), we have \( x_0 + W(u_1) = (\pm z - \frac{a_1}{\lambda_1})\sqrt{1 + \frac{2\lambda_1 u_1}{\sigma_1^2}} + \frac{a_1}{\lambda_1} = z^2_{\pm}(u_1) \). Consequently, the FPT for \( x(t) \) with respect to thresholds \( \pm z \) is homeomorphic to the FPT of a Wiener process starting at \( x_0 \) and evolving on a transformed time \( u_1 \) with respect to time-varying thresholds \( z^2_{\pm}(u_1) \). Since \( u_1 \) is a monotonically increasing function, the FPT distribution \( \tau \) is readily obtained from \( u_1(\tau) \), the FPT distribution of the Wiener process. Also, the error rates for the two processes are also the same.

The computation of FPT distribution for the Wiener process with time-varying thresholds is, in general, not analytically tractable. However, the time-varying thresholds can be approximated by piecewise constant time-varying thresholds and approximate FPT distributions can be computed using the MSDDM. While the thresholds \( z^2_{\pm} \) are asymmetric in this setting, unlike in the case described for the MSDDM, such a case can be easily handled by replacing the expression in (6) and (12) with corresponding expressions for asymmetric thresholds (see Douady 1999, Borodin and Salminen 2002).

### 5.2. Approximate computation of the FPT distribution of the multistage O-U process

The solution to the multistage O-U process (27) for \( t \in [t_{i-1}, t_i) \) is
\[ x(t) = \frac{a_i}{\lambda_i}(1 - e^{-\lambda_i(t-t_{i-1})}) + e^{-\lambda_i(t-t_{i-1})}x(t_{i-1}) + e^{-\lambda_i(t-t_{i-1})}W\left(\frac{\sigma_i^2(e^{2\lambda_i(t-t_{i-1})} - 1)}{2\lambda_i}\right). \]  
\( (29) \)
Let \( u_i(t) = \frac{\sigma_i^2 e^{2\lambda_i(t-t_{i-1})}}{2\lambda_i} \). Also, let \( \tau_i \) and \( X_{i-1}, \) \( i \in \{1, \ldots, n\} \) be defined similarly to the MSDDM. Then, conditioned on a realization of \( X_{i-1} \), the FPT problem of the \( i \)-th stage O-U process can be equivalently written as the FPT problem of the Wiener process \( X_{i-1} + W(u_i(t)) \) with respect to thresholds

\[
\tau^{\pm}_i(u_i(t)) = \left( \pm z - \frac{a_i}{\lambda_i} \right) \sqrt{1 + \frac{2\lambda_i u_i(t)}{\sigma_i^2} + \frac{a_i}{\lambda_i}}
\]

If the distribution of \( X_{i-1} \) is known, then, analogous to the case of MSDDM, the distribution of \( \tau_i | \tau_i \leq t_i \) can be computed. However, in this case the distribution will be approximately computed using an MSDDM for the \( i \)-th stage O-U process. Furthermore, analogous to the MSDDM, the distribution of \( X_{i-1} + W(u_i(t_i)) \) conditioned on \( \tau_i > t_i \) can also be computed, which in conjunction with (29) yields the distribution of \( z \). Then, a recursive procedure akin to the MSDDM yields the approximate FPT distribution for the multistage O-U process.

6. Illustrative examples

In this section we discuss the application of the expressions derived in §3 for experiments involving time pressure or changes in attention over the course of decision process, and for optimizing the speed-accuracy trade-off using the reward rate. We compare the theoretical predictions obtained from the analysis in this paper with the numerical values obtained through Monte-Carlo simulations. The simulation results in this section were obtained using 1000 Monte Carlo runs. The diffusion models were simulated using the Euler-Maruyama method with step size \( 10^{-3} \). Codes used to produce all figures may be found at https://github.com/PrincetonUniversity/msddm

6.1. Two stage DDM

Consider the MSDDM (2) with two stages, i.e., \( n = 2 \). For such a two stage DDM, the following expressions for the error rate and the expected decision time can be computed using the theory in §3

\[
ER = ER_1 \mathbb{P}(\tau_1 \leq t_1) + ER_2 \mathbb{P}(\tau_1 > t_1),
\]

where

\[
ER_1 = \frac{e^{-2s_1 x_0} - \mathbb{E}[e^{-2s_1 X_1}] \mathbb{P}(\tau_1 > t_1) - e^{-2s_1 z} \mathbb{P}(\tau_1 \leq t_1)}{(e^{2s_1 z} - e^{-2s_1 z}) \mathbb{P}(\tau_1 \leq t_1)},
\]

\[
ER_2 = \frac{\mathbb{E}[e^{-2s_2 X_2}] - e^{-2s_2 z}}{e^{2s_2 z} - e^{-2s_2 z}},
\]

and

\[
\mathbb{E}[\tau] = \frac{(1 - 2ER_1) e^{2s_1 z} \mathbb{P}(\tau_1 \leq t_1) - x_0 + \mathbb{E}[X_1] \mathbb{P}(\tau_1 > t_1)}{a_1} + \frac{(1 - 2ER_2) z - \mathbb{E}[X(t_1)]}{a_2} \mathbb{P}(\tau_1 > t_1).
\]

6.2. First passage times of a four stage process

To illustrate the expressions derived in §3 we consider a four stage MSDDM. The drift rates and diffusion rates in each stage are \( (a_1, a_2, a_3, a_4) = (0.1, 0.2, 0.05, 0.3) \) and \( (\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (1, 1.5, 1.25, 2) \), respectively. Let the stage initiation times be \( (t_0, t_1, t_2, t_3) = (0, 1, 2, 3) \) and the initial condition be \( x_0 = -0.2 \). The FPT of the unconditional and conditional decision time for \( z = 2 \) obtained using analytic expressions and Monte-Carlo simulations is shown in Figure 1(a). Similarly, the error rate, expected overall decision times, and expected decision times conditional on which threshold was crossed obtained using analytic expressions and Monte-Carlo simulations are shown in Figure 1(b) as a function of threshold \( z \).
Figure 1: Performance metrics for a four stage DDM with drift rates \((a_1, a_2, a_3, a_4) = (0.1, 0.2, 0.05, 0.3)\), diffusion rates \((\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (1, 1.5, 1.25, 2)\), stage initiation times \((t_0, t_1, t_2, t_3) = (0, 1, 2, 3)\), and initial condition \(x_0 = -0.2\). The FPT distribution is computed for threshold \(z = 2\).
Figure 2: Unconditional and conditional FPT distributions for a 30-stage DDM with alternating drift rate. The drift rates are \((a_1, a_2, a_3, \ldots) = (1, -0.75, 1, \ldots)\), diffusion rate at each stage is unity, the threshold \(z = 2\), and stage initiation times are chosen randomly in the interval \((0, 10)\).

### 6.3. First Passage Time Distribution for DDM with Alternating Drift

One potential application of these analyses is to describe situations in which evidence accumulation changes dynamically with the decision-maker’s changing focus of attention. For instance, Krajbich and colleagues (2010) have shown that the process of weighing two value-based options (e.g., foods) can be modeled with a DDM in which drift rates vary during the decision process based on the option being attended to at any given moment. We consider such a case using a 30-stage DDM in which the drift rates 1 and \(-0.75\) alternate (i.e., \(a_1 = 1\), \(a_2 = -0.75\), \(a_3 = 1\), \(\ldots\)) to capture a situation in which the decision maker’s attention alternates between two options, one of which has greater perceived value (higher drift rate) than the other. Let \(t_1 = 0\) and the remaining 29 stage initiation times be randomly and uniformly selected between \((0, 10)\). Assume \(x_0 = 0\), \(z = 2\), and let the diffusion rate be stationary and equal to unity. The unconditional and conditional FPT distributions in this scenario obtained using the analytic expressions and using Monte-Carlo simulations are shown in Figure 2.

### 6.4. First Passage Time Distribution for DDM with Gradually Time-varying Drift

These analyses can also be applied to situations in which changes in evidence accumulation occur gradually over time. For instance, White and colleagues (2011) proposed a “shrinking spotlight” model of the Eriksen-Flanker Task, a task in which participants responding to the direction of a central arrow are influenced by the direction of arrows in the periphery (see also Servan-Schreiber et al. [1990] Liu et al. [2009] [Servant et al.] [2015]). According to these models, evidence accumulation is initially influenced by all of the arrows (central as well as flanksers, which may drive an incorrect response) but as the attentional spotlight narrows the drift rate is gradually more influenced by the central arrow alone. We now consider such a situation, using a 20-stage DDM as an approximation to the DDM with time-varying drift rate. Assume that the diffusion rate is constant and equal to unity. Let \(x_0 = 0\), \(z = 2\), and stage initiation times be uniformly selected in the interval \([0, 5]\). Let the drift rate in the \(i\)-th stage be \(-0.2 + 0.0263(i - 1)\). The unconditional and conditional FPT distributions for such a 20-stage DDM obtained using the analytic expressions and using Monte-Carlo simulations are shown in Figure 3.
6.5. First Passage Time Distribution for DDM with Collapsing Thresholds

Analogously to the previous section, one might also be interested in modeling a decision process in which thresholds are dynamic rather than uniform across stages. This can be used to describe discrete changes in choice strategy, or a continuous change in thresholds over time, e.g., collapsing boundaries); the latter approach has been successful at describing behavior under conditions that either involve an explicit response deadline (e.g., Milosavljevic et al., 2010; Frazier and Yu, 2008) or where there is an implicit opportunity cost for longer time spent accumulating evidence (Drugowitsch et al., 2012). We model such a situation, using a 20-stage DDM as an approximation to the DDM with collapsing threshold. Assume that the drift rate and the diffusion rate are constant and equal to 0.15 and 1, respectively. Let $x_0 = 0$, and stage initiation times be uniformly selected in the interval [0, 5]. Let the threshold in the $i$-th stage be $z_i = 3 - 0.1579(i - 1)$. The unconditional and conditional FPT distributions for such a 20-stage DDM obtained using the analytic expressions and using Monte-Carlo simulations are shown in Figure 4.

6.6. Optimizing Speed-Accuracy Trade-off in the Pure versus Multistage DDM

So far, we have assumed that the thresholds for decision-making are known and we have characterized the associated error rate and first passage time properties. The choice of threshold dictates the speed-accuracy trade-off, i.e., the trade-off between a fast decision and an accurate decision. This joint function of speed and accuracy is believed to be critical to how animals choose to set and adjust their threshold. In particular, it has been proposed (Bogacz et al., 2006) that human subjects choose a threshold that maximizes the reward rate defined by

$$RR = \frac{1 - ER}{E[\tau] + T_{\text{nd}}},$$

where $T_{\text{nd}}$ is the sensory and motor processing time (non-decision time) and ER and $E[\tau]$ are computed using the expressions derived in §3.

We now investigate the reward rate as a function of the threshold for a two-stage DDM. We assume that the threshold is the same for the two stages. The reward rate for the pure DDM as shown in Figure 5(a).
is a unimodal function. In contrast, the reward rate for the two-stage DDM can be a bimodal function, as shown in Figure 5(b).

The bimodality of the reward rate leads to peculiar behavior of the optimal threshold as $a_1$ and $t_1$ are varied, as shown in Figure 6. In Figure 6 we fix $a_2 = 0.5$, $\sigma_1 = \sigma_2 = 0.1$, $x_0 = 0$, and $T_{nd} = 0.3$ and study the effect of the drift rate $a_1$ and the switching time $t_1$ on the optimal threshold that maximizes the reward rate. For a given $a_1$, the optimal threshold first increases as $t_1$ is increased and then jumps down at a critical $t_1$. The jump is attributed to the fact that one of the peaks in the bimodal function increases with $t_1$, while the other decreases. At the critical $t_1$, the global maximum switches from one peak to the other.

The reward rate for the multistage DDM is a univariate function of the threshold and the globally optimal threshold can be efficiently determined using algorithms in Hansen et al. (1992). However, compared to the algorithms for the maximization of unimodal functions, these algorithms require additional information. In particular, an upper bound on the slope of the reward rate over the domain of interest is needed to implement these algorithms.

7. Closing discussion

We have analyzed in detail the FPT properties of the multistage drift diffusion model, which is a Wiener diffusion model with piecewise time-varying drift rate, noise parameter, and decision thresholds. Ratcliff (1980) studied the two stage version of the MSDDM with constant thresholds and described a procedure for how to compute the FPT density. Here, we have extended this result to $n$-stages and time-varying thresholds, which required relaxing the assumption that the initial condition $x_0$ is a point mass. Indeed, the initial condition of the $i$-th DDM is not a deterministic quantity but is a random variable. Furthermore, rather than requiring integration over FPT density to obtain conditional and unconditional expected decision times and error rates, our martingale-based approach allows for direct computation of these quantities. Another major contribution of the paper is to show how various other performance metrics, such as the error rate during each stage, evolve as the underlying dynamics change. Using these, one may compute a variety of behavioral performance metrics, without resorting to first computing the FPT densities. We also independently derived the FPT density for the MSDDM.
Figure 5: Reward rate as a function of the threshold for the pure DDM and the two stage DDM. For pure DDM $a_1 = 0.5, \sigma_1 = 0.1$ and $x_0 = 0$, while for two stage DDM $a_1 = 0.5, a_2 = 0.1, \sigma_1 = \sigma_2 = 0.1, x_0 = 0$ and $t_1 = 0.15$. The reward rate for the pure DDM is a unimodal function and achieves a unique local maximum, while the reward rate for the two stage DDM has two local maxima.

Figure 6: Optimal threshold for 2-stage DDM obtained by maximizing reward rate. The left panel shows the variation of the optimal threshold as a function of $t_1$ and $a_1$. The other parameters are $a_2 = 0.5, \sigma_1 = \sigma_2 = 0.1$, and $x_0 = 0$. The right panel shows the associated contour plot. The regions of the contour plot associated with $t_1 = 0$ and $a_1 = 0.5$ correspond to the pure DDM.

The calculations in §3 are relatively straightforward to implement, and code for doing so is available online along with code that reproduces the figures in this article. It is important to note, however, the other highly optimized software packages for computing FPT statistics for time-varying diffusion models. One such package in this domain is that of Smith (2000), which solves an integral equation (code for this is available online Drugowitsch (2014)). Diederich and Busemeyer (2003) introduced a matrix based approach, similar to Markov chain Monte-Carlo methods, to efficiently implement and analyze performance metrics for a variety of extensions of the DDM. The matrix approach has been used to analyze multistage processes associated with multiattribute choice (Diederich and Oswald, 2014). Also relevant is the paper of Voss and Voss (2008), which develops an efficient numerical algorithm for estimating parameters of the time-varying

\[1\text{https://github.com/PrincetonUniversity/msddm}\]
diffusion model from reaction time data (i.e., first passage times). More recently, very fast codes for a broad class of diffusion models have been developed by Verdonck et al. (2015), with implementations on both CPUs and GPUs for considerable performance increases. Compared to these previous efforts, our work is not immediately focused on developing a rapid numerical tool for simulation, but rather introducing martingale theory as a useful approach for understanding and analyzing DDMs with multiple stages. Thus, the codes released with this report are not intended to compete with the efficiency of the aforementioned codes, which have been highly optimized and tuned for throughput, but instead demonstrate the simplicity and effectiveness of our analysis.

This said, our results do suggest promising avenues for future numerical work. Particularly relevant is work by Navarro and Fuss (2009); Blurton et al. (2012); Gondan et al. (2014) who develop efficient numerical schemes for evaluating the relevant infinite sums involved in FPT calculations. Similar methods could be applied to results in §3.2 and §3.3 to develop efficient and accurate MSDDM codes, which could in turn contribute to the growing collection of numerical tools available for practitioners using diffusion models to study decision making. Our results may also serve as a starting point for further analysis of more complicated stochastic decision models. As an important step in this direction, we have also shown how the equations for the MSDDM can be applied to Ornstein-Uhlenbeck processes, which can approximate leaky integration over the course of evidence accumulation, e.g., the Leaky Competing Accumulator model [LCA] (Usher and McClelland, 2001). Given that the LCA itself can in certain cases approximate a reduced form of more complex and biologically plausible models of interactions across neuronal populations (e.g., Wang, 2002; Wong and Wang, 2006; Bogacz, 2007), our analyses offer an important step toward better understanding time-varying dynamics within and across neural networks, and how these might explain complex cognitive phenomena. More broadly, we believe our analysis furthers the theory surrounding diffusion models with time-varying drift rates, and that the tools and formulae introduced will contribute to the ongoing effort to develop and understand psychologically and neurally plausible models of decision making.

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Appendix A. Laplace transforms for the pure DDM

The Laplace transform, or moment generating function, of the FPT density conditioned on response is given for $\alpha \in \mathbb{R}$ by

$$
\mathbb{E}[e^{-\alpha \tau_1} | \tau_1 = z] = \frac{e^{-a_1(z-x_0)} - \alpha \tau_1}{1 - \text{ER}_1} \frac{\sinh(\frac{(z-x_0)\sqrt{2\alpha \sigma_1^2 + \sigma^2}}{\sigma_1})}{\sinh(\frac{2z\sqrt{2\alpha \sigma_1^2 + \sigma^2}}{\sigma_1})},
$$

$$
\mathbb{E}[e^{-\alpha \tau_1} | \tau_1 = -z] = \frac{e^{-a_1(z+x_0)} - \alpha \tau_1}{\text{ER}_1} \frac{\sinh(\frac{(z-x_0)\sqrt{2\alpha \sigma_1^2 + \sigma^2}}{\sigma_1})}{\sinh(\frac{2z\sqrt{2\alpha \sigma_1^2 + \sigma^2}}{\sigma_1})}.
$$

References for these expressions may be found in the main text.

Appendix B. Derivation of expressions in §3.2

We first establish (14). First consider the case $a_i > 0$. Let $\{\mathcal{F}_t^i\}_{t \geq t_i-1}$ be the filtration defined by the evolution of the MSDDM (2) until time $t$ conditioned on $\tau > t_i-1$. For some $s \in (t_i-1, t)$, it can be shown that $\mathbb{E}[e^{-2s_i\tau(t)} | \mathcal{F}_s^i] = e^{-2s_i\tau(s)}$. Thus, $\{e^{-2s_i\tau(t)}\}_{t \geq t_i-1}$ is a martingale. Furthermore, $\hat{t}_i := \min\{\tau_i, t_i\}$ is a stopping time. Therefore, it follows from optional stopping theorem that

$$
\mathbb{E}[e^{-2s_iX_{\hat{t}_i} - 1}] = \mathbb{E}[e^{-2s_i\tau(\hat{t}_i)}] = \mathbb{E}[e^{-2s_i\tau(t_i)}] \mathbb{P}(\tau_i \leq t_i) + \mathbb{E}[e^{-2s_iX_i}] \mathbb{P}(\tau_i > t_i) = (e^{-2s_i^2(1 - \text{ER}_i) + \sigma^2(1 - \text{ER}_i)} \mathbb{P}(\tau_i \leq t_i) + \mathbb{E}[e^{-2s_iX_i}] \mathbb{P}(\tau_i > t_i)].
$$
Solving the above equation for \( \text{ER}_i \), we obtain the desired expression.

For \( a_i = 0 \), we note that \( \{ x(t) \}_{t \geq t_i} \) is a martingale. Therefore, applying the optional stopping theorem, we obtain

\[
\mathbb{E}[X_{t_i - 1}] = \mathbb{E}[x(\tau_i)] \\
= \mathbb{E}[x(\tau_i) | \tau_i \leq t_i] \mathbb{P}(\tau_i \leq t_i) + \mathbb{E}[X_i | \mathbb{P}(\tau_i > t_i)] \\
= (1 - 2\text{ER}_i)z \mathbb{P}(\tau_i \leq t_i) + \mathbb{E}[X_i | \mathbb{P}(\tau_i > t_i)].
\]

Solving the above equation for \( \text{ER}_i \), we obtain the desired expression.

The formulas (15) and (16) immediately follow from applying expectation to (9) and (10), respectively.

To establish (17) for \( a_i > 0 \), we note that for the DDM (1), \( \{ x(t) - a_i t \}_{t \geq t_i} \) is a martingale. Therefore, applying the optional stopping theorem, we obtain

\[
\mathbb{E}[X_{t_i - 1} - a_i t_{t_i - 1}] = \mathbb{E}[x(\tau_i) - a_i \tau_i] \\
= \mathbb{E}[x(\tau_i) - a_i \tau_i | \tau_i \leq t_i] \mathbb{P}(\tau_i \leq t_i) + \mathbb{E}[(X_i - a_i t_i) | \mathbb{P}(\tau_i > t_i)] \\
= (z(1 - \text{ER}_i) - z \mathbb{E}[\tau_i | \tau_i \leq t_i]) \mathbb{P}(\tau_i \leq t_i) + (\mathbb{E}[X_i] - a_i t_i) \mathbb{P}(\tau_i > t_i).
\]

Solving the above equation for \( \mathbb{P}(\tau_i | \tau_i \leq t_i) \) yields the desired expression. For \( a_i = 0 \), we note that \( \{ x(t)^2 - \sigma^2_i t \}_{t \geq t_i} \) is a martingale. Therefore, applying the optional stopping theorem, we obtain

\[
\mathbb{E}[X_{t_i - 1}^2] - \sigma^2_i t_{t_i - 1} = \mathbb{E}[x(\tau_i)^2 - \sigma^2_i \tau_i] \\
= \mathbb{E}[x(\tau_i)^2 - \sigma^2_i \tau_i | \tau_i \leq t_i] \mathbb{P}(\tau_i \leq t_i) + \mathbb{E}[X_i^2 - \sigma^2_i t_i | \mathbb{P}(\tau_i > t_i)] \\
= (z^2 - \sigma^2_i \mathbb{E}[\tau_i | \tau_i \leq t_i]) \mathbb{P}(\tau_i \leq t_i) + (\mathbb{E}[X_i^2] - \sigma^2_i t_i) \mathbb{P}(\tau_i > t_i).
\]

Solving the above equation for \( \mathbb{P}(\tau_i | \tau_i \leq t_i) \) yields the desired expression.

Next, we need to establish that the Laplace transform of the density for the FPT for a particular decision made before \( t_i \) is

\[
\mathbb{E}[e^{-\lambda t} \mathbb{1}(x(\tau_i) = z | \tau_i \leq t_i)] = \frac{e^{-\lambda \mathbb{E}[x(\tau_i)]} - e^{-\lambda t} \mathbb{E}[\mathbb{1}(x(\tau_i) = z | \tau_i \leq t_i)]}{\mathbb{P}(\tau_i \leq t_i)}, \\
\mathbb{E}[e^{-\lambda t} \mathbb{1}(x(\tau_i) = -z | \tau_i \leq t_i)] = \frac{e^{-\lambda \mathbb{E}[x(\tau_i)]} - e^{-\lambda t} \mathbb{E}[\mathbb{1}(x(\tau_i) = -z | \tau_i \leq t_i)]}{\mathbb{P}(\tau_i \leq t_i)}.
\]

To establish this, we consider the stochastic process \( e^{\lambda x(t)-\lambda a_i t-\lambda^2 \sigma^2_i / 2} \). Since

\[
\mathbb{E}[e^{\lambda x(t)-\lambda a_i t-\lambda^2 \sigma^2_i / 2} \mathbb{1}(x(\tau_i) = z | \tau_i \leq t_i)] = e^{\lambda x(s)-\lambda a_i s-\lambda^2 \sigma^2_i / 2},
\]

it follows that \( \{ e^{\lambda x(t)-\lambda a_i t-\lambda^2 \sigma^2_i / 2} \}_{t \geq t_i} \) is a martingale for each \( \lambda \in \mathbb{R} \). We choose two particular values of \( \lambda \):

\[
\lambda_1 = \frac{-a_i - \sqrt{a_i^2 + 2a_i \sigma^2}}{\sigma^2}, \quad \text{and} \quad \lambda_2 = \frac{-a_i + \sqrt{a_i^2 + 2a_i \sigma^2}}{\sigma^2}.
\]

Note that for \( \lambda \in \{ \lambda_1, \lambda_2 \} \), \( \lambda a_i t + \lambda^2 \sigma^2_i / 2 = \alpha \). Therefore, stochastic processes \( \{ e^{\lambda_1 x(t)-\alpha t} \}_{t \geq 0} \) and \( \{ e^{\lambda_2 x(t)-\alpha t} \}_{t \geq t_i} \) are martingales. Now applying the optional stopping theorem, we obtain

\[
\mathbb{E}[e^{\lambda_1 X_{t_i - 1} - \alpha t_{t_i - 1}}] = \mathbb{E}[e^{\lambda_1 x(\tau) - \alpha \tau}] = e^{\lambda_1 \mathbb{E}[e^{-\alpha t} \mathbb{1}(x(\tau_i) = z \& \tau_i \leq t_i)]} + e^{-\lambda_1 \mathbb{E}[e^{-\alpha t} \mathbb{1}(x(\tau_i) = -z \& \tau_i \leq t_i)] + e^{-\alpha t} \mathbb{E}[e^{\lambda_1 X_i} \mathbb{P}(\tau_i > t_i)]} \quad \text{(B.1)}
\]

Similarly,

\[
\mathbb{E}[e^{\lambda_2 X_{t_i - 1} - \alpha t_{t_i - 1}}] = e^{\lambda_2 \mathbb{E}[e^{-\alpha t} \mathbb{1}(x(\tau) = z \& \tau_i \leq t_i)]} + e^{-\lambda_2 \mathbb{E}[e^{-\alpha t} \mathbb{1}(x(\tau) = -z \& \tau_i \leq t_i)] + e^{-\alpha t} \mathbb{E}[e^{\lambda_2 X_i} \mathbb{P}(\tau_i > t_i)]} \quad \text{(B.2)}
\]
Equations (B.1) and (B.2) are two simultaneous equations in two unknowns \( \mathbb{E}[e^{-\alpha t_i} 1(x(\tau_i) = z \& \tau_i \leq t_i)] \) and \( \mathbb{E}[e^{-\alpha t_i} 1(x(\tau_i) = -z \& \tau_i \leq t_i)] \). Solving for these unknowns, we obtain

\[
\mathbb{E}[e^{-\alpha t_i} 1(x(\tau_i) = z \& \tau_i \leq t_i)] = \frac{e^{-\alpha t_i} \mathbb{E}[e^{\lambda_1(X_{i-1}+z)} - e^{\lambda_2(X_{i-1}+z)}] - e^{-\alpha t_i} \mathbb{E}[e^{\lambda_1(X_i) - e^{\lambda_2(X_i)}]} \mathbb{P}(\tau_i > t_i)}{e^{2\lambda_1 z} - e^{2\lambda_2 z}},
\]

and

\[
\mathbb{E}[e^{-\alpha t_i} 1(x(\tau_i) = -z \& \tau_i \leq t_i)] = \frac{e^{-\alpha t_i} \mathbb{E}[e^{-\lambda_1(z-X_{i-1})} - e^{-\lambda_2(z-X_{i-1})}] - e^{-\alpha t_i} \mathbb{E}[e^{-\lambda_1(z-X_i)} - e^{-\lambda_2(z-X_i)}]}{e^{-2\lambda_1 z} - e^{-2\lambda_2 z}} \mathbb{P}(\tau_i > t_i).
\]

Simplifying these expressions, we obtain the desired expression.

Finally, (18) and (19) follow from differentiating the Laplace transform with respect to \( \alpha \), flipping sign, and then evaluating at \( \alpha = 0 \).

Appendix C. Performance metrics for the overall process

We start by establishing (20). Since \( t \in (t_{k-1}, t_k) \),

\[
P(\tau \leq t) = \mathbb{P}(\tau \leq t \& \tau \leq t_{k-1}) + \mathbb{P}(\tau \leq t \& \tau > t_{k-1})
= \mathbb{P}(\tau \leq t_{k-1}) + \mathbb{P}(\tau \leq t | \tau > t_{k-1}) \mathbb{P}(\tau > t_{k-1})
= 1 - \prod_{i=1}^{k-1} \mathbb{P}(\tau > t_i | \tau > t_{i-1}) + \mathbb{P}(\tau_k \leq t) \prod_{i=1}^{k-1} \mathbb{P}(\tau > t_i | \tau > t_{i-1})
= 1 - \prod_{i=1}^{k-1} \mathbb{P}(\tau_i > t_i) + \mathbb{P}(\tau_k \leq t) \prod_{i=1}^{k-1} \mathbb{P}(\tau_i > t_i).
\]

We now establish (21). We note that

\[
\mathbb{E}[\tau] = \sum_{i=1}^{n} \mathbb{E}[\tau_t | t_{i-1} < \tau \leq t_i]
= \sum_{i=1}^{n} \mathbb{E}[\tau | t_i \leq \tau | t_i \leq t_i] \mathbb{P}(\tau > t_i - 1)
= \sum_{i=1}^{n} \left( \mathbb{E}[\tau | \tau_i \leq t_i] \mathbb{P}(\tau_i \leq t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_j) \right).
\]

To establish (22), we note that

\[
\mathbb{E}R = \sum_{i=1}^{n+1} \mathbb{P}(x(\tau) = -z \text{ and } t_{i-1} \leq \tau \leq t_i)
= \sum_{i=1}^{n+1} \mathbb{P}(x(\tau) = -z \text{ and } \tau < t_i | \tau > t_{i-1}) \mathbb{P}(\tau > t_{i-1})
= \sum_{i=1}^{n} \left( \mathbb{E}R \mathbb{P}(\tau_i < t_i) \prod_{j=1}^{i-1} \mathbb{P}(\tau_j > t_j) \right).
\]

Equations (23) and (24) follow similarly to (21), and Equations (25) and (26) follow similarly to (20).