Singularities of Codimension Two Mean Curvature Flow of Symplectic Surfaces

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1 Introduction

Geometers have been interested in constructing minimal surfaces for long time. One possible way to produce such surfaces is to deform a given surface by its mean curvature vector. More precisely, the surface evolves in the gradient flow of the area functional, and such a flow is the so-called mean curvature flow. A mean curvature flow, however, develops singularity after finite time in general [H2]. It is therefore desirable to understand behavior of the flow near the singular points (cf. [B], [CL], [E1-E2], [H1-H3], [HS1-HS2], [I1-I2], [Wa1], [Wh1-Wh2] and so on). We consider this problem, in this paper, for compact symplectic surfaces moving by mean curvature flow in a Kähler-Einstein surface. It was observed in [CT2] that symplectic surface remains symplectic along the flow (also see [CL], [Wa1]), and the flow has long time existence in the graphic case as discussed in [CLT] and [Wa2]. One of our motivations of considering symplectic surfaces is inspired by the symplectic isotopic problem for symplectic surfaces in Del Pezzo surfaces, i.e., those complex surfaces with positive first Chern class. It was conjectured in [T] that every embedded orientable closed symplectic surface in a compact Kähler-Einstein surface is isotopic to a symplectic minimal surface in a suitable sense. When Kähler-Einstein surfaces are of positive scalar curvature, this was proved for lower degrees by using pseudo-holomorphic curves (cf. [ST], [Sh]). It would be interesting to have a proof of this result, even for lower degrees, by using the mean curvature flow. In the negative

*Chen is supported partially by a Sloan fellowship and a NSERC grant.
†Li is partially supported by the National Science Foundation of China and by the Partner Group of MPI for Mathematics.
scalar curvature case, Arezzo [Ar] pointed out that a symplectic minimal surface may not be holomorphic even if it represents a \((1, 1)\)-type class, by constructing examples of symplectic minimal surfaces which are not holomorphic. Moving Lagrange surfaces by mean curvature flow was studied by Smoczyk [Sm1-Sm3].

Let \(M\) be a compact Kähler-Einstein surface. On \(M\), let \(\omega\) be the Kähler form, \(\langle \cdot, \cdot \rangle\) the Kähler metric and \(J\) the complex structure which is compatible with \(\omega\) and \(\langle \cdot, \cdot \rangle\), i.e., for any tangent vectors \(X, Y\) to \(M\) at an arbitrary point in \(M\),

\[
\langle X, Y \rangle = \omega(X, JY), \\
\omega(X, Y) = \omega(JX, JY).
\]

For a compact oriented real surface \(\Sigma\) without boundary which is smoothly immersed in \(M\), one defines, following [CW], the Kähler angle \(\alpha\) of \(\Sigma\) in \(M\) by

\[
\omega|_{\Sigma} = \cos \alpha \, d\mu_{\Sigma},
\]

where \(d\mu_{\Sigma}\) is the area element of \(\Sigma\) in the induced metric from \(\langle \cdot, \cdot \rangle\). As a function on \(\Sigma\), \(\alpha\) is continuous everywhere and is smooth possibly except at the complex or anti-complex points of \(\Sigma\), i.e., where \(\alpha = 0\) or \(\pi\). We say that \(\Sigma\) is a holomorphic curve if \(\cos \alpha \equiv 1\), \(\Sigma\) is a Lagrangian surface if \(\cos \alpha \equiv 0\) and \(\Sigma\) is a symplectic surface if \(\cos \alpha > 0\).

Given an immersion \(F_0 : \Sigma \to M\), we consider a one-parameter family of immersions \(F_t = F(\cdot, t) : \Sigma \to M\), and denote the image surfaces by \(\Sigma_t = F_t(\Sigma)\). The immersed surfaces \(\Sigma_t\) satisfy a mean curvature flow if

\[
\begin{cases}
\frac{d}{dt} F(x, t) = H(x, t) \\
F(x, 0) = F_0(x),
\end{cases}
\]

where \(H(x, t)\) is the mean curvature vector of \(\Sigma_t\) at \(F(x, t)\) in \(M\).

The standard parabolic theory implies that the mean curvature flow \((\text{1})\) has a smooth solution for short time. More precisely, there exists \(T > 0\) such that \((\text{1})\) has a smooth solution in the time interval \([0, T]\). If the second fundamental form \(|A|^2\) on \(\Sigma_t\) is bounded uniformly in \(t\) near \(T\), then the solution can be extended smoothly to \([0, T + \epsilon]\) for some \(\epsilon > 0\). However, in general \(\max_{\Sigma_t} |A|^2\) becomes unbounded as \(t \to T\). In this case we say that the mean curvature flow blows up at \(T\); moreover, to classify the singularities of mean curvature flows, Huisken, according to the blowing up rate of \(|A|\), introduced

**Definition 1.1** (Huisken [H1]) On a mean curvature flow \(\Sigma_t\), suppose that

\[
\lim_{t \to T^-} \max_{\Sigma_t} |A|^2 = \infty.
\]

If there exists a positive constant \(C\) such that

\[
\limsup_{t \to T^-} \left( (T - t) \max_{\Sigma_t} |A|^2 \right) \leq C,
\]
the mean curvature flow $F$ has a *Type I singularity at $T$*; otherwise it has a *Type II singularity at $T$*. 

In the codimension one case, singularities of mean curvature flows have been studied in depth (cf. [E1], [H1-H3], [HS1-HS2], [I1-I2], [Wh2]). For higher codimensions, if the initial compact surface is symplectic in a compact Kähler-Einstein surface, the motion of the mean curvature flow preserves symplecticity of $\Sigma_t$ as long as the smooth solution exists; and furthermore the flow does not develop any Type I singularities (cf. [CT2], [CL], [Wa1]).

In this paper, we shall study the Type II singularities of the mean curvature flow of a compact symplectic surface in a compact Kähler-Einstein surface. Especially, we shall focus on the structure of tangent cones of the mean curvature flow where a singularity occurs at the first singular time $T < \infty$.

To describe the tangent cones, suppose $(X_0, T)$ is a singular point of the flow (1), i.e. $|A(x, t)|$ becomes unbounded when $(x, t) \to (X_0, T)$. For an arbitrary sequence of numbers $\lambda \to \infty$ and any $t < 0$, if $T + \lambda^{-2}t > 0$ we set

$$F_{\lambda}(x, t) = \lambda(F(x, T + \lambda^{-2}t) - X_0).$$

We denote the scaled surface by $(\Sigma_{\lambda}^t, d\mu_{\lambda}^t)$. If the initial surface is symplectic, it is proved in Lemma 2.3 that there is a subsequence $\lambda_i \to \infty$ such that for any $t < 0$, $(\Sigma_{\lambda_i}^t, d\mu_{\lambda_i}^t)$ converges to $(\Sigma^\infty, d\mu^\infty)$ in the sense of measures; the limit $\Sigma^\infty$ is called a *tangent cone arising from the rescaling $\lambda_i$*, or simply a *$\lambda$ tangent cone at $(X_0, T)$*. This tangent cone is independent of $t$ as shown in Lemma 2.3. There is also a time dependent scaling which we would like to consider

$$\bar{F}(\cdot, s) = \frac{1}{\sqrt{2(T-t)}} F(\cdot, t),$$

where $s = -\frac{1}{2} \log(T - t)$, $c_0 \leq s < \infty$. Here we have chosen the coordinates so that $X_0 = 0$. Rescaling of this type arises naturally in classifying Type I and Type II singularities for mean curvature flows [H2]. Denote $\bar{\Sigma}_s$ the rescaled surface by $\bar{F}(\cdot, s)$. If a subsequence of $\bar{\Sigma}_s$ converges in measures to a limit $\bar{\Sigma}_\infty$, then the limit is called a *tangent cone arising from the time dependent scaling at $(X_0, T)$*, or simply a *$t$ tangent cone*. In this paper, a *tangent cone* of the mean curvature flow at $(X_0, T)$ means either a *$\lambda$ tangent cone* or a *$t$ tangent cone* at $(X_0, T)$.

The main result of this paper is

**Theorem 1.2** Let $M$ be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and $T > 0$ is the first blow-up time of the mean curvature flow (1) and $(X_0, T)$ is a singular point, then the tangent cone of the mean curvature flow at $(X_0, T)$ consists of a finite union of more than one 2-planes in $\mathbb{R}^4$ which are complex in a complex structure on $\mathbb{R}^4$. 

3
It is interesting to compare our result with the one obtained by Morgan in [M] (also see [MW]): the tangent cone to an oriented area minimizing surface in $\mathbb{R}^4$ consists of 2-planes which are complex in $\mathbb{R}^4$ (see Remark 6.8).

An important technical device in the present paper is a monotonicity formula, which is established in [CL] and also see Proposition 2.1, for
\[
\int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, t) \phi d\mu_t,
\]
where $F : \Sigma_0 \times [0, T) \to M$ is the mean curvature flow, $\phi$ is a cut-off function supported in a small geodesic ball in $M$ which is contained in a single coordinate chart, and $\rho$ is defined by the backward heat kernel on $\mathbb{R}^4$. The weight function $1/\cos \alpha$ captures some key geometric information of the symplectic surfaces in the flow, and this quantity is sensitive to orientation.

The organization of the paper is follows. From section 2 to section 5 of this paper, we consider the $\lambda$ scaling near a singular point $(X_0, T)$. In section 2, first we derive four integral estimates in Proposition 2.2, by using the weighted monotonicity formula (10). These estimates play a crucial role in our analysis of the tangent cones. Then we prove in Lemma 2.3 that a subsequence of the rescaled surfaces $\Sigma^\lambda_t$ converges in measure to a limit $\Sigma^\infty$, which is independent of $t$. In section 3, we prove that the tangent cone $\Sigma^\infty$ is rectifiable. This is done by first showing that the density function of the tangent cone exists and then applying a local regularity theorem of White for classical mean curvature flows (cf. [Wh1], [E1]) to obtain a positive lower bound for the density function of $\Sigma^\infty$, and then using Priess’s theorem in [P] to conclude the rectifiability of $\Sigma^\infty$. The positive lower bound can also be obtained by using the isoperimetric inequality (cf. the proof of Lemma 6.4). In section 4, using Proposition 2.2 and Allard’s compactness theorem in [A] we first show, in Proposition 4.2, that $\Sigma^\infty$ is stationary. Then we prove that the restriction of the Kähler form on the scaled surface converges to the restriction of a constant 2-form $\omega_0$ in $\mathbb{R}^4$ along $\Sigma^\infty$, and further we prove that $\omega_0|_{\Sigma^\infty} = \theta_0 d\mu^\infty$ for an $\mathcal{H}^2$ a.e. constant function $\theta_0$ on $\Sigma^\infty$. A theorem of Harvey-Shiffman in [HS] then implies that $\Sigma^\infty$ is a holomorphic subvariety. In section 5, we further use Proposition 2.2 and Allard’s compactness theorem to conclude $F^\perp$ equals zero and then show $\Sigma^\infty$ is flat away from its singular locus, which is a collection of finitely many points. Section 6 concerns with the time dependent scaling. The arguments essentially proceed in a similar fashion as those in the previous sections.

2 Monotonicity formula and integral estimates

Let $\Sigma_t = F(\Sigma, t)$ be the family of immersed surfaces, which are determined by the mean curvature flow $F$, in the 4-dimensional manifold $M$. Denote the Riemannian metric on $M$ by $\langle \cdot, \cdot \rangle$. In a normal coordinate chart around a point in $\Sigma_t$, the induced
metric on $\Sigma_t$ from $\langle \cdot , \cdot \rangle$ is given by
$$g_{ij} = \langle \partial_i F, \partial_j F \rangle,$$
where $\partial_i$ ($i = 1, 2$) are the partial derivatives with respect to the local coordinates. In the sequel, we denote by $\Delta$ and $\nabla$ the Laplace operator and covariant derivative for the induced metric on $\Sigma_t$ respectively. We choose a local field of orthonormal frames $e_1, e_2, v_1, v_2$ of $M$ along $\Sigma_t$ such that $e_1, e_2$ are tangent vectors of $\Sigma_t$ and $v_1, v_2$ are in the normal bundle over $\Sigma_t$. The second fundamental form $A$ and the mean curvature vector $H$ of $\Sigma_t$ can be expressed, in the local frame, as
$$A = A^\alpha v_\alpha, \quad H = -H^\alpha v_\alpha,$$
where and throughout this paper all repeated indices are summed over suitable range. For each $\alpha$, the coefficient $A^\alpha$ is a $2 \times 2$ matrix ($h^\alpha_{ij}$). By Weingarten’s equation (cf. [Sp]), we have
$$h^\alpha_{ij} = \langle \partial_i v_\alpha, \partial_j F \rangle = \langle \partial_j v_\alpha, \partial_i F \rangle = h^\alpha_{ji}.$$
The trace and the norm of the second fundamental form of $\Sigma_t$ in $M$ are:
$$H^\alpha = g^{ij} h^\alpha_{ij} = h^\alpha_{ii}$$
$$|A|^2 = \sum_{\alpha} |A^\alpha|^2 = g^{ij} g^{kl} h^\alpha_{ik} h^\alpha_{jl} = h^\alpha_{ik} h^\alpha_{ik}. $$
The area element of the induced metric $g_{ij}$ on $\Sigma_t$ is $\sqrt{\det(g_{ij})}dxdy$. Along the mean curvature flow, it is well known that
$$\frac{d}{dt} \sqrt{\det(g_{ij})} = -|H|^2 \sqrt{\det(g_{ij})}.$$ 
Logarithmic integration implies that $F$ remains immersed as long as the smooth solution of (1) exists.

Let $J_{\Sigma_t}$ be an almost complex structure in a tubular neighborhood of $\Sigma_t$ on $M$ with
$$\begin{cases}
J_{\Sigma_t} e_1 = e_2 \\
J_{\Sigma_t} e_2 = -e_1 \\
J_{\Sigma_t} v_1 = v_2 \\
J_{\Sigma_t} v_2 = -v_1.
\end{cases} \quad (3)$$
It is not difficult to verify (cf. Lemma 3.1 in [CL]), with $\nabla$ being the covariant derivative of the metric $\langle \cdot , \cdot \rangle$ on $M$, that
$$|\nabla J_{\Sigma_t}|^2 = |h^2_{11} + h^1_{12}|^2 + |h^2_{21} + h^1_{22}|^2 + |h^2_{12} - h^1_{11}|^2 + |h^2_{22} - h^1_{21}|^2$$
$$= \frac{1}{2} |H|^2 + \frac{1}{2} \left( ((h^2_{11} + h^1_{12}) + 2(h^2_{12} - h^1_{11}))^2 + (h^2_{11} + h^2_{22} + 2h^1_{21} - 2h^1_{11})^2 \right)$$
$$\geq \frac{1}{2} |H|^2. \quad (4)$$
Let \( H(X, X_0, t) \) be the backward heat kernel on \( \mathbb{R}^4 \). Define

\[
\rho(X, X_0, t, t_0) = 4\pi(t_0 - t)H(X, X_0, t) = \frac{1}{4\pi(t_0 - t)} \exp \left( -\frac{|X - X_0|^2}{4(t_0 - t)} \right)
\]

for \( t < t_0 \). Let \( i_M \) be the injective radius of \( M^4 \). We choose a cut off function \( \phi \in C^\infty_0(B_{2r}(X_0)) \) with \( \phi \equiv 1 \) in \( B_r(X_0) \), where \( X_0 \in M, 0 < 2r < i_M \). Choose a normal coordinates in \( B_{2r}(X_0) \) and express \( F \), by the coordinates \((F^1, F^2, F^3, F^4)\), as a surface in \( \mathbb{R}^4 \). We define

\[
\Phi(X_0, t_0, t) = \int_{\Sigma_t} \phi(F) \rho(F, X_0, t, t_0) d\mu_t.
\]

(Huisken derived the following useful monotonicity formula in \([H1]\): there are positive constants \( c_1 \) and \( c_2 \) depending only on \( M^4, F_0 \) and \( r \) where \( r \) is the constant in the definition of \( \phi \), such that

\[
\frac{\partial}{\partial t} \left( e^{c_1\sqrt{t_0 - t}} \Phi(X_0, t_0, t) \right) \leq -e^{c_1\sqrt{t_0 - t}} \int_{\Sigma_t} \phi \rho(F, X_0, t, t_0) \left( |H + \frac{(F - X_0)^\perp}{2(t_0 - t)}|^2 + c_2(t_0 - t) \right) d\mu_t.
\]

(6)

Note that \( c_1 \) and \( c_2 \) equal 0 when \( M \) is a Euclidean space.

When \( M \) is a Kähler-Einstein surface with scalar curvature \( R \) and \( \Sigma_t \) evolves under the mean curvature flow, the Kähler angle \( \alpha \) of \( \Sigma_t \) in \( M \) satisfies the parabolic equation (cf. \([CL\], [CT]\)):

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \cos \alpha = |\nabla J_{\Sigma_t}|^2 \cos \alpha + R \sin^2 \alpha \cos \alpha.
\]

(7)

Suppose that the initial surface is symplectic, i.e., \( \cos \alpha(\cdot, 0) \) has a positive lower bound. Then by applying the parabolic maximum principle to the evolution equation (7), one concludes that \( \cos \alpha \) remains positive as long as the mean curvature flow has a smooth solution, no matter \( R \) is positive, 0 or negative (cf. \([CT2\], [CL\], [Wa1]\)).

Let \( R_0 = \max\{0, -R\} \) and set

\[
v(x, t) = e^{R_0 t} \cos \alpha(x, t).
\]

By (7), we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \frac{1}{v} \leq -|\nabla J_{\Sigma_t}|^2 \frac{1}{v} - \frac{2}{v^3} |\nabla v|^2.
\]

(8)

Along the flow, we introduce a function

\[
\Psi(X_0, t_0, t) = \int_{\Sigma_t} \frac{1}{v} \phi \rho(F, X_0, t, t_0) d\mu_t.
\]

(9)

The following weighted monotonicity formula in \([CL]\) will play a crucial role in this paper.
Proposition 2.1 (Weighted Monotonicity Formula) If the initial compact surface \( \Sigma_0 \) is symplectic in a Kähler-Einstein surface \( M \) and \( \Sigma_t \) evolves under the mean curvature flow (1), then

\[
\frac{\partial}{\partial t} \left( e^{c_1 \sqrt{t_0}} \Psi(X_0, t_0, t) \right) \leq -e^{c_1 \sqrt{t_0}} \left( \int_{\Sigma_t} \frac{1}{v} \phi \rho(F, X_0, t, t_0) \left| H + \frac{(F - X_0)^2}{2(t_0 - t)} \right|^2 \, d\mu_t \right.
\]
\[
+ \int_{\Sigma_t} \frac{1}{v} \phi \rho(F, X_0, t, t_0) \left| \nabla J_{\Sigma_t} \right|^2 \, d\mu_t
\]
\[
+ \int_{\Sigma_t} \frac{2}{v^3} \left| \nabla v \right|^2 \phi \rho(F, X_0, t, t_0) d\mu_t \bigg) + c_2(t_0 - t).
\] (10)

Here the positive constants \( c_1 \) and \( c_2 \) depend on \( M, F_0 \) and \( r \) where \( r \) is the constant in the definition of \( \phi \).

Suppose that \((X_0, T)\) is a singular point of the mean curvature flow (1). We now describe the rescaling process around \((X_0, T)\). For any \( t < t_0 \), we set

\[
F_\lambda(x, t) = \lambda(F(x, T + \lambda^{-2}t) - X_0),
\]

where \( \lambda \) are positive constants which go to infinity. The scaled surface is denoted by \( \Sigma^\lambda_t = F_\lambda(\Sigma, t) \) on which \( d\mu^\lambda_t \) is the area element obtained from \( d\mu_t \). If \( g^\lambda \) is the metric on \( \Sigma^\lambda_t \), it is clear that

\[
g^\lambda_{ij} = \lambda^2 g_{ij}, \quad (g^\lambda)^{ij} = \lambda^{-2} g^{ij}.
\]

We therefore have

\[
\frac{\partial F^\lambda}{\partial t} = \lambda^{-1} \frac{\partial F}{\partial t} \quad \text{(11)}
\]
\[
H^\lambda = \lambda^{-1} H \quad \text{(12)}
\]
\[
|A^\lambda|^2 = \lambda^{-2} |A|^2. \quad \text{(13)}
\]

It follows that the scaled surface also evolves by a mean curvature flow

\[
\frac{\partial F^\lambda}{\partial t} = H^\lambda. \quad \text{(14)}
\]

Proposition 2.2 Let \( M \) be a Kähler-Einstein surface. If the initial compact surface is symplectic, then for any \( R > 0 \) and any \(-\infty < s_1 < s_2 < 0\), we have

\[
\int_{s_1}^{s_2} \int_{\Sigma^\lambda_t \cap B_R(0)} \left| \nabla J_{\Sigma^\lambda_t} \right|^2 d\mu^\lambda_t \, dt \to 0 \quad \text{as} \quad \lambda \to \infty, \quad \text{(15)}
\]
\[
\int_{s_1}^{s_2} \int_{\Sigma^\lambda_t \cap B_R(0)} \left| \nabla \cos \alpha^\lambda \right|^2 d\mu^\lambda_t \, dt \to 0 \quad \text{as} \quad \lambda \to \infty, \quad \text{(16)}
\]
\[ \int_{s_1}^{s_2} \int_{\Sigma_t \cap B_R(0)} |H_\lambda|^2 d\mu_\lambda^t dt \to 0 \quad \text{as} \quad \lambda \to \infty, \quad (17) \]

and
\[ \int_{s_1}^{s_2} \int_{\Sigma_t \cap B_R(0)} |F_\lambda|^2 d\mu_\lambda^t dt \to 0 \quad \text{as} \quad \lambda \to \infty. \quad (18) \]

**Proof:** For any \( R > 0 \), we choose a cut-off function \( \phi_R \in C_0^\infty(B_{2R}(0)) \) with \( \phi_R \equiv 1 \) in \( B_R(0) \), where \( B_r(0) \) is the metric ball centered at \( 0 \) with radius \( r \) in \( \mathbb{R}^4 \). For any fixed \( t < T \), the mean curvature flow (1) has a smooth solution near \( T + \lambda^{-2}t < T \) for sufficiently large \( \lambda \), since \( T > 0 \) is the first blow-up time of the flow. It is clear
\[
\int_{\Sigma_t^\lambda} \frac{1}{v_\lambda} \frac{1}{0-t} \phi_R(F_\lambda) \exp \left( -\frac{|F_\lambda|^2}{4(0-t)} \right) d\mu_\lambda^t
\]
\[
= \int_{\Sigma_t^{\lambda+\lambda^{-2}s}} \frac{1}{v_\lambda} \phi(F_\lambda) \frac{1}{T - (T + \lambda^{-2}t)} \exp \left( -\frac{|F(x, T + \lambda^{-2}t) - X_0|^2}{4(T - (T + \lambda^{-2}t))} \right) d\mu_t,
\]
where \( \phi \) is the function defined in the definition of \( \Phi \). Note that \( T + \lambda^{-2}t \to T \) for any fixed \( t \) as \( \lambda \to \infty \). By (19),
\[
\frac{\partial}{\partial t} \left( e^{c_1 \sqrt{t_0-t}} \Psi \right) \leq c_2(t_0-t),
\]
and it then follows that \( \lim_{t \to t_0} e^{c_1 \sqrt{t_0-t}} \Psi \) exists. This implies, by taking \( t_0 = T \) and \( t = T + \lambda^{-2}s \), that for any fixed \( s_1 \) and \( s_2 \) with \( -\infty < s_1 < s_2 < 0 \),
\[
e^{-c_1 \sqrt{T-(T+\lambda^{-2}s_2)}} \int_{\Sigma_{s_2}^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{0-s_2} \exp \left( -\frac{|F_\lambda|^2}{4(0-s_2)} \right) d\mu_{s_2}^\lambda
\]
\[
- e^{-c_1 \sqrt{T-(T+\lambda^{-2}s_1)}} \int_{\Sigma_{s_1}^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{0-s_1} \exp \left( -\frac{|F_\lambda|^2}{4(0-s_1)} \right) d\mu_{s_1}^\lambda
\]
\[
\to 0 \quad \text{as} \quad \lambda \to \infty. \quad (19)
\]

Integrating (19) from \( s_1 \) to \( s_2 \) yields
\[
e^{-c_1 \sqrt{-\lambda^{-2}s_2}} \int_{\Sigma_{s_2}^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{0-s_2} \exp \left( -\frac{|F_\lambda|^2}{4(0-s_2)} \right) d\mu_{s_2}^\lambda
\]
\[
+ e^{c_1 \sqrt{-\lambda^{-2}s_1}} \int_{\Sigma_{s_1}^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{0-s_1} \exp \left( -\frac{|F_\lambda|^2}{4(0-s_1)} \right) d\mu_{s_1}^\lambda
\]
\[
\geq \int_{s_1}^{s_2} e^{c_1 \sqrt{-\lambda^{-2}t}} \int_{\Sigma_t^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{0-\lambda^{-2}t} \exp \left( -\frac{|F_\lambda|^2}{4(0-\lambda^{-2}t)} \right) d\mu_t^\lambda
\]
\[
+ \int_{s_1}^{s_2} e^{c_1 \sqrt{-\lambda^{-2}t}} \int_{\Sigma_t^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{0-\lambda^{-2}t} \exp \left( -\frac{|F_\lambda|^2}{4(0-\lambda^{-2}t)} \right) d\mu_t^\lambda
\]
\[
- e_2 \lambda^{-2}(s_2^2 - s_1^2). \quad (20)
\]
Putting (19) and (20) together, we have
\[ \lim_{\lambda \to \infty} \int_{s_1}^{s_2} \sum_{t} \left| \sum_{\lambda} J_{\Sigma_{\lambda}^t} \right|^2 d\mu^\lambda_t = 0, \]
and
\[ \lim_{\lambda \to \infty} \int_{s_1}^{s_2} \sum_{t} \left| \nabla v_{\lambda} \right|^2 \rho(F_{\lambda}, t) d\mu^\lambda_t = 0, \]
which yield (15) and (16) respectively, and
\[ \lim_{\lambda \to \infty} \int_{s_1}^{s_2} \sum_{t} \left| \nabla v_{\lambda} \right|^2 \rho(F_{\lambda}, t) d\mu^\lambda_t = 0. \]
Finally, (4) and (15) imply (17), and (17) and (21) imply (18). Q.E.D.

**Lemma 2.3** For any \( \lambda, R > 0 \) and any \( t < 0 \),
\[ \mu^\lambda_t(\Sigma_{\lambda}^t \cap B_R(0)) \leq CR^2, \]  
where \( B_R(0) \) is a metric ball in \( R^4 \) and \( C > 0 \) is independent of \( \lambda \). For any sequence \( \lambda_i \to \infty \), there is a subsequence \( \lambda_k \to \infty \) such that \( (\Sigma_{\lambda_k}^t, \mu_{\lambda_k}^t) \to (\Sigma^\infty, \mu^\infty) \) in the sense of measure, for any fixed \( t < 0 \), where \( (\Sigma^\infty, \mu^\infty) \) is independent of \( t \). The multiplicity of \( \Sigma^\infty \) is finite.

**Proof:** We shall first prove the inequality (22). We shall use \( C \) below for uniform positive constants which are independent of \( R \) and \( \lambda \). Straightforward computation shows
\[
\mu^\lambda_t(\Sigma_{\lambda}^t \cap B_R(0)) = \lambda^2 \int_{\sum_{\lambda}^{t+\lambda^{-2}R} \cap B_R} d\mu_t \\
= R^2(\lambda^{-1}R)^{-2} \int_{\sum_{\lambda}^{t+\lambda^{-2}R} \cap B_R} d\mu_t \\
\leq CR^2 \int_{\sum_{\lambda}^{t+\lambda^{-2}R} \cap B_R} \frac{1}{4\pi(\lambda^{-1}R)^2} e^{-\frac{|X-X_0|^2}{4(\lambda^{-1}R)^2}} d\mu_t \\
= CR^2 \Phi(X_0, T + (\lambda^{-1}R)^2 + \lambda^{-2}t, T + \lambda^{-2}t).
\]
By the monotonicity inequality (1), we have
\[
\mu^\lambda_t(\Sigma_{\lambda}^t \cap B_R(0)) \leq CR^2 \left( \Phi(X_0, T + (\lambda^{-1}R)^2 + \lambda^{-2}t, T/2) + C \right) \\
\leq CR^2 \left( \mu_{T/2}(\Sigma_{T/2}) + C \right).
\]
Since
\[ \frac{\partial}{\partial t} \mu_t(\Sigma_t) = -\int_{\Sigma_t} |H|^2 d\mu_t, \]
we can now conclude (22)
\[ \mu_t^\lambda(\Sigma_t^\lambda \cap B_R(0)) \leq CR^2. \]

By (22), the compactness theorem of the measures (c.f. [Si1], 4.4) and a diagonal subsequence argument, we conclude that there is a subsequence \( \lambda_k \to \infty \) such that 
\( (\Sigma_{t_0}^{\lambda_k}, \mu_{t_0}^{\lambda_k}) \to (\Sigma_{t_0}^\infty, \mu_{t_0}^\infty) \) in the sense of measures for a fixed \( t_0 < 0 \).

We now show that, for any \( t < 0 \), the subsequence \( \lambda_k \) which we have chosen above satisfies 
\( (\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}) \to (\Sigma_t^\infty, \mu_t^\infty) \) in the sense of measure. And consequently the limiting surface \( (\Sigma_t^\infty, \mu_t^\infty) \) is independent of \( t_0 \). Recall that the following standard formula for mean curvature flows
\[
\frac{d}{dt} \int_{\Sigma_t^\lambda} \phi \, d\mu_t^\lambda = -\int_{\Sigma_t^\lambda} \left( \phi |H_\lambda|^2 + \nabla \phi \cdot H_\lambda \right) \, d\mu_t^\lambda
\]
is valid for any test function \( \phi \in C_0^\infty(M) \) (cf. (1) in Section 6 in [12] and [B] in the varifold setting).

Then for any given \( t < 0 \) integrating (23) yields
\[
\int_{\Sigma_t^{\lambda_k}} \phi \, d\mu_t^{\lambda_k} - \int_{\Sigma_{t_0}^{\lambda_k}} \phi \, d\mu_{t_0}^{\lambda_k} = \int_t^{t_0} \int_{\Sigma_t^{\lambda_k}} \left( \phi |H_\lambda|^2 + \nabla \phi \cdot H_\lambda \right) \, d\mu_t^{\lambda_k} \, dt \\
\to 0 \text{ as } k \to \infty \text{ by (14).}
\]

So, for any fixed \( t < 0 \), \( (\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}) \to (\Sigma_t^\infty, \mu_t^\infty) \) in the sense of measures as \( k \to \infty \). We denote \( (\Sigma_t^\infty, \mu_t^\infty) \) by \( (\Sigma^\infty, \mu^\infty) \), which is independent of \( t_0 \).

The inequality (22) yields a uniform upper bound on \( R^{-2} \mu_t^{\lambda_k}(\Sigma_t^\lambda \cap B_R(0)) \), which yields finiteness of the multiplicity of \( \Sigma^\infty \). Q.E.D.

**Definition 2.4** Let \((X_0, T)\) be a singular point of the mean curvature flow of a closed symplectic surface \( \Sigma_0 \) in a compact Kähler-Einstein surface \( M \). We call \((\Sigma^\infty, d\mu^\infty)\) obtained in Lemma 2.3 a \( \lambda \) tangent cone of the mean curvature flow \( \Sigma_t \) at \((X_0, T)\).

### 3 Rectifiability of the \( \lambda \) tangent cones

In this section we shall show that the \( \lambda \) tangent cone \( \Sigma^\infty \) is \( \mathcal{H}^2 \)-rectifiable, where \( \mathcal{H}^2 \) is the 2-dimensional Hausdorff measure.

**Proposition 3.1** Let \( M \) be a compact Kähler-Einstein surface. If the initial compact surface \( \Sigma_0 \) is symplectic, then the \( \lambda \) tangent cone \((\Sigma^\infty, d\mu^\infty)\) of the mean curvature flow at \((X_0, T)\) is \( \mathcal{H}^2 \)-rectifiable.

**Proof:** We set
\[
A_R = \left\{ t \in (-\infty, 0) \mid \lim_{k \to \infty} \int_{\Sigma_t^\lambda \cap B_R(0)} |H_k|^2 \, d\mu_t^k \neq 0 \right\},
\]

10
and

\[ A = \bigcup_{R > 0} A_R. \]

Denote the measures of \( A_R \) and \( A \) by \( |A_R| \) and \( |A| \) respectively. It is clear from (17) that \( |A_R| = 0 \) for any \( R > 0 \). So \( |A| = 0 \).

For any \( \xi \in \Sigma^\infty \), choose \( \xi_k \in \Sigma^k_t \) with \( \xi_k \to \xi \) as \( k \to \infty \). By the monotonicity identity (17.4) in [Si1], we have

\[
\sigma^2 \mu^k_t(B_\sigma(\xi_k)) = \rho^2 \mu^k_t(B_\rho(\xi_k)) - \int_{B_\rho(\xi_k) \setminus B_\sigma(\xi_k)} \frac{|D_t r|^2}{r^2} d\mu^k_t
- \frac{1}{2} \int_{B_\rho(\xi_k)} (x - \xi_k) \cdot \mathbf{H}_k \left( \frac{1}{r^2 \sigma^2} - \frac{1}{\rho^2} \right) d\mu^k_t,
\]

for all \( 0 < \sigma \leq \rho \), where \( \mu^k_t(B_\sigma(\xi_k)) \) is the area of \( \Sigma^k_t \cap B_\sigma(\xi_k) \), \( r = r(x) \) is the distance from \( \xi_k \) to \( x \), \( r_\sigma = \max\{r, \sigma\} \), and \( D_t r \) denotes the orthogonal projection of \( Dr \) (which is a vector of length 1) onto \( \langle T_{\xi_k} \Sigma^k_t \rangle^{\perp} \). Choosing \( t \not\in A \), we have

\[
\lim_{k \to \infty} \int_{B_\rho(\xi_k)} |\mathbf{H}_k|^2 d\mu^k_t = 0.
\]

Hölder’s inequality and (22) then lead to

\[
\lim_{k \to \infty} \left| \int_{B_\rho(\xi_k)} (x - \xi_k) \cdot \mathbf{H}_k \left( \frac{1}{r^2 \sigma^2} - \frac{1}{\rho^2} \right) d\mu^k_t \right|
\leq \ C \rho \left( \frac{1}{\sigma^2} - \frac{1}{\rho^2} \right) \lim_{k \to \infty} \left( \sqrt{\mu^k_t(B_\rho(\xi_k))} \sqrt{\int_{B_\rho(\xi_k)} |\mathbf{H}_k|^2 d\mu^k_t} \right)
\leq \ C \rho^2 \left( \frac{1}{\sigma^2} - \frac{1}{\rho^2} \right) \lim_{k \to \infty} \int_{B_\rho(\xi_k)} |\mathbf{H}_k|^2 d\mu^k_t
= \ 0.
\]

Letting \( k \to \infty \) in (24) and using (25), we obtain

\[
\lim_{\rho \to 0} \rho^{-2} \mu^\infty(B_\rho(\xi)) \leq \sigma^{-2} \mu^\infty(B_\sigma(\xi)),
\]

for all \( 0 < \sigma \leq \rho \). By (22) we know that

\[
\lim_{\rho \to 0} \rho^{-2} \mu^\infty(B_\rho(\xi)) < C < \infty.
\]

Therefore, \( \lim_{\rho \to 0} \rho^{-2} \mu^\infty(B_\rho(\xi)) \) exists.

We shall show that there exists a positive number \( r_0 \) such that for any \( 0 < r < r_0 \) the following density estimate holds

\[
\lim_{\rho \to 0} \rho^{-2} \mu^\infty(B_\rho(\xi)) \geq \frac{1}{16}.
\]
Assume (24) fails to hold. Then there is \( \rho_0 > 0 \) such that

\[
(2\rho_0)^{-2} \mu^\infty(B_{2\rho_0}(\xi)) < \frac{1}{16}.
\]

By the monotonicity formula (24) and that \( \mu^\infty \) converges to \( \mu^\infty \) as measures, there exists \( k_0 > 0 \) such that, for all \( 0 < \rho < 2\rho_0 \) and \( k > k_0 \), we have

\[
\rho^{-2} \mu^k_t(B_{\rho}(\xi)) < \frac{1}{8}. \tag{27}
\]

Take a cut-off function \( \phi_\rho \in C_0^\infty(B_{\rho}(\xi)) \) on the 4-dimensional ball \( B_\rho(\xi_k) \) so that

\[
\phi_\rho \equiv 1 \quad \text{in} \quad B_{\frac{\rho}{2}}(\xi)
\]

\[
0 \leq \phi_\rho \leq 1, \quad \text{and} \quad |\nabla \phi_\rho| \leq \frac{C}{\rho}, \quad \text{in} \quad B_\rho(\xi).
\]

From (23), we have

\[
\rho^{-2} \int_{B_\rho(\xi)} \phi_\rho \, d\mu^k_{t-r^2} - \rho^{-2} \int_{B_\rho(\xi)} \phi_\rho \, d\mu^k_t
\]

\[
\leq 2 \rho^{-2} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \, ds + C \rho^{-3} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \, ds
\]

\[
\leq C \rho^{-2} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \, ds + C \rho^{-3} \int_{t-r^2}^t \left( \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \right)^{1/2} \mu^k_s(B_{\rho}(\xi))^{1/2} \, ds
\]

\[
\leq C \rho^{-2} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \, ds + C \rho^{-2} \int_{t-r^2}^t \left( \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \right)^{1/2} \, ds \quad \text{by (24)}
\]

\[
\rightarrow 0, \quad \text{as} \quad k \rightarrow \infty \quad \text{by (17)}.
\]

Here we have used \( C \) for uniform positive constants which are independent of \( k \) and \( \rho \). Therefore, there are constants \( \delta_1 > 0 \) and \( \kappa_1 > 0 \) such that for all \( \rho \) and \( k \) with \( 0 < \rho < \delta_1 \), \( 0 < r < 1 \), and \( k > \kappa_1 \) the estimate

\[
\rho^{-2} \mu^k_{t-r^2}(B_\rho(\xi)) < \frac{1}{4} < 1 \tag{28}
\]

holds. Let \( d\sigma^k_{t-r^2} \) be the arc-length element of \( \partial B_\rho(\xi) \cap \Sigma^k_{t-r^2} \). By the co-area formula, for \( 0 < r << 1 \),

\[
\Phi_k(\xi, t, t-r^2) = \frac{1}{4\pi r^2} \int_{\Sigma^k_{t-r^2}} e^{-\frac{|\xi_k - \xi|^2}{4r^2}} \, d\mu^k_{t-r^2}
\]

\[
\leq \frac{1}{4\pi r^2} \int_0^{\delta_1} \int_{\partial B_\rho(\xi) \cap \Sigma^k_{t-r^2}} e^{-\frac{2}{4\pi r^2}} \, d\sigma^k_{t-r^2} \, d\rho
\]

\[
+ \frac{1}{4\pi r^2} \int_{\delta_2}^\infty \int_{\partial B_\rho(\xi) \cap \Sigma^k_{t-r^2}} e^{-\frac{2}{4\pi r^2}} \, d\sigma^k_{t-r^2} \, d\rho.
\]
\begin{align*}
&\leq \frac{1}{4\pi r^2} \int_{0}^{\delta_1} e^{-\frac{\rho^2}{4r^2}} \frac{d}{d\rho} \text{Vol}(B_\rho(\xi) \cap \Sigma_{t-r^2}^k) \, d\rho \\
&\quad + \frac{1}{4\pi r^2} \int_{\delta_1}^{\infty} e^{-\frac{\rho^2}{4r^2}} \frac{d}{d\rho} \text{Vol}(B_\rho(\xi) \cap \Sigma_{t-r^2}^k) \, d\rho \\
&\leq \frac{1}{4\pi r^2} \int_{0}^{\delta_1} e^{-\frac{\rho^2}{4r^2}} \rho \frac{d}{d\rho} \text{Vol}(B_\rho(\xi) \cap \Sigma_{t-r^2}^k) \, d\rho \\
&\quad + \frac{1}{4\pi r^2} \int_{\delta_1}^{\infty} e^{-\frac{\rho^2}{4r^2}} \frac{d}{d\rho} \text{Vol}(B_\rho(\xi) \cap \Sigma_{t-r^2}^k) \, d\rho \\
&\leq \frac{1}{4\pi r^2} \int_{0}^{\delta_1} e^{-\frac{\rho^2}{4r^2}} \rho^3 \, d\rho + o(r) \quad \text{by (28) and (22)} \\
&\leq 1 + o(r). \tag{29}
\end{align*}

For any classical mean curvature flow \( \Gamma_t \) in a compact Riemannian manifold which is isometrically embedded in \( \mathbb{R}^N \), White proves a local regularity theorem (Theorem 3.1 and Theorem 4.1 in \cite{Wh1}). When \( \dim \Gamma_t = 2 \), White’s theorem asserts that there is a constant \( \epsilon > 0 \) such that if the Gaussian density satisfies

\[ \lim_{r \to 0} \int_{\Gamma_{t-r^2}} \frac{1}{4\pi r^2} \exp \left( -\frac{|y-x|^2}{4r^2} \right) \, d\mu(y) < 1 + \epsilon, \]

then the mean curvature flow is smooth in a neighborhood of \( x \). Combining this regularity result with (29), we are led to choose \( r > 0 \) sufficiently small and then conclude that

\[ \sup_{B_r(\xi) \cap \Sigma_{t-r^2}^k} |A_k| \leq C \]

and consequently \( \Sigma_k \) converges strongly in \( B_r(\xi) \cap \Sigma_{t-r^2}^k \) to \( \Sigma_{t-r^2}^{\infty} \cap B_r(\xi) \), as \( k \to \infty \). So \( \Sigma_{t-r^2}^{\infty} \cap B_r(\xi) \) is smooth. Smoothness of \( \Sigma_{t-r^2}^{\infty} \cap B_r(\xi) \) immediately implies

\[ \lim_{\rho \to 0} \rho^{-2} \mu^{\infty}(B_\rho(\xi)) = 1. \]

This contradicts (27). Hence we have established (26).

In summary, we have shown that \( \lim_{\rho \to 0} \rho^{-2} \mu^{\infty}(B_\rho(\xi)) \) exists and for \( \mathcal{H}^2 \) almost all \( \xi \in \Sigma^{\infty} \),

\[ \frac{1}{16} \leq \lim_{\rho \to 0} \rho^{-2} \mu^{\infty}(B_\rho(\xi)) < \infty. \tag{30} \]

Finally, we recall a fundamental theorem of Priess in \cite{P}: if \( 0 \leq m \leq n \) are integers and \( \Omega \) is a Borel measure on \( \mathbb{R}^n \) such that

\[ 0 < \lim_{r \to 0} \frac{\Omega(B_r(x))}{r^m} < \infty, \]

for almost all \( x \in \Omega \), then \( \Omega \) is \( m \)-rectifiable. Now we conclude from (30) that \( (\Sigma^{\infty}, \mu^{\infty}) \) is \( \mathcal{H}^2 \)-rectifiable. Q.E.D.
4 Holomorphicity of the $\lambda$ tangent cones

In this section, we shall first show that the $\lambda$ tangent cone $\Sigma^\infty$ is stationary and then prove that $\Sigma^\infty$ is a complex subvariety in $\mathbb{R}^4$. This result allows us to assert that the set of singular points of $\Sigma^\infty$ consists of discrete points.

A $k$-varifold is a Radon measure on $G^k(M)$, where $G^k(M)$ is the Grassmann bundle of all $k$-planes tangent to $M$. Allard’s compactness theorem for rectifiable varifolds (6.4 in [A], also see 1.9 in [I2] and Theorem 42.7 in [Si1]) can be stated as follows.

**Theorem 4.1** (Allard’s compactness theorem) Let $(V_i, \mu_i)$ be a sequence of rectifiable $k$-varifolds in $M$ with

$$\sup_{i \geq 1}(\mu_i(U) + |\delta V_i|(U)) < \infty$$

for each $U \subset M$. Then there is a varifold $(V, \mu)$ of locally bounded first variation and a subsequence, which we also denote by $(V_i, \mu_i)$, such that

1. Convergence of measures: $\mu_i \rightarrow \mu$ as Radon measures on $M$,
2. Convergence of tangent planes: $V_i \rightarrow V$ as Radon measures on $G^k(M)$,
3. Convergence of first variations: $\delta V_i \rightarrow \delta V$ as $TM$-valued Radon measures,
4. Lower semicontinuity of total first variations: $|\delta V| \leq \liminf_{i \to \infty} |\delta V_i|$ as Radon measures.

We first show that the $\lambda$ tangent cone is stationary.

**Proposition 4.2** Let $M$ be a compact Kähler-Einstein surface. If the initial compact surface is symplectic, then the $\lambda$ tangent cone $\Sigma^\infty$ is stationary.

**Proof:** Let $V_t^k$ be the varifold defined by $\Sigma_t^k$. By the definition of varifolds, we have

$$V_t^k(\psi) = \int_{\Sigma_t^k} \psi(x, T\Sigma_t^k) d\mu_t^k$$

for any $\psi \in C_0^0(G^2(\mathbb{R}^4), R)$, where $G^2(\mathbb{R}^4)$ is the Grassmanian bundle of all 2-planes tangent to $\Sigma_t^\infty$ in $\mathbb{R}^4$. For each smooth surface $\Sigma_t^k$, the first variation $\delta V_t^k$ of $V_t^k$ (cf. [A], (39.4) in [Si1] and (1.7) in [I2]) is

$$\delta V_t^k = -\mu_t^k |H_k|.$$

By Proposition 2.2, we have that $\delta V_t^k \rightarrow 0$ at $t$ as $k \to \infty$.

By (iii) in Theorem 4.1, we have that

$$-\mu^\infty |H_\infty| = \delta V^\infty = \lim_{k \to \infty} \delta V_t^k = 0.$$

Therefore $\Sigma^\infty$ is stationary. Q.E.D.
**Theorem 4.3** Let $M$ be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and $T > 0$ is the first blow-up time of the mean curvature flow, then the tangent cone $\Sigma^\infty$ of the mean curvature flow at $(X_0, T)$ is a holomorphic subvariety of complex dimension one in some complex structure on $\mathbb{R}^4$, with multiplicity more than one in $\mathbb{R}^4$.

**Proof:** For a smoothly immersed real surface $\Sigma$ in a Kähler manifold $(M, \omega)$ of complex dimension two, we may choose a local orthogonal frame $\{e_1, e_2, e_3, e_4\}$ on $M$ along $\Sigma$ with $e_1, e_2$ tangent to $\Sigma$, so that along $\Sigma$ the Kähler form $\omega$ has the expression (cf. [CT1, CW]):

$$\omega = \cos \alpha (e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) + \sin \alpha (e_1^* \wedge e_3^* - e_2^* \wedge e_4^*),$$

where $e_j^*$ is the dual of $e_j$ for $j = 1, 2, 3, 4$.

Along each surface $\Sigma_t$, we have

$$\omega = \cos \alpha_t (e_1^*(\Sigma_t^k) \wedge e_2^*(\Sigma_t^k) + e_3^*(\Sigma_t^k) \wedge e_4^*(\Sigma_t^k)) + \sin \alpha_t (e_1^*(\Sigma_t^k) \wedge e_3^*(\Sigma_t^k) - e_2^*(\Sigma_t^k) \wedge e_4^*(\Sigma_t^k)),$$

and

$$\omega|_{\Sigma_t^k} = \cos \alpha_t d\mu_t^k = \cos \alpha_k e_1^*(\Sigma_t^k) \wedge e_2^*(\Sigma_t^k).$$

Since $\cos \alpha_k > c > 0$ along the flow, the compactness theorem for Radon measures (cf. Theorem 4.4 in [Si]) implies that the bounded positive measures $\cos \alpha_k d\mu_t^k$ and $d\mu_t^k$ converge to nonnegative measures $\theta_0 d\mu^\infty$ and $d\mu^\infty$ respectively on $\Sigma^\infty$ in the sense of measures, for some measurable function $\theta_0$ on $\Sigma^\infty$ with $0 < \theta_0 \leq 1$. Here we take a subsequence if necessary.

In the scaling process on a small neighborhood of the singular point $X_0$ in $M$, the Riemannian metric $g^k$ tends to the flat metric on $\mathbb{R}^4$ as $k \to \infty$, equivalently, the Kähler form $\omega^k$ converges to a self-dual, positive definite, constant 2-form $\omega_0$ on $\mathbb{R}^4$ with $\omega_0(0) = \omega(X_0)$, where 0 is the origin of $\mathbb{R}^4$.

By (ii) in Theorem [4.3], along $\Sigma_t^k$,

$$\omega^k|_{\Sigma_t^k} \to \omega_0|_{\Sigma^\infty} = \theta_0 e_1^*(\Sigma^\infty) \wedge e_2^*(\Sigma^\infty),$$

as measures. Note that Allard’s compactness theorem only provides convergence of tangent planes to $\Sigma_t^k$ so the other components in $\omega^k$ may not converge to those in $\omega_0$. Nevertheless, we have

$$\theta_0 d\mu^\infty = \theta_0 e_1^*(\Sigma^\infty) \wedge e_2^*(\Sigma^\infty) = \omega_0|_{\Sigma^\infty}.$$

Next, we shall show that $\theta_0$ is constant $\mathcal{H}^2$ a.e. on $\Sigma^\infty$. To do so, we claim that for any $r > 0$, $\xi_1, \xi_2 \in \Sigma_t^k \cap B_{R/2}(0)$ the following holds

$$\left| \frac{1}{\text{Vol}(B_r(\xi_1) \cap \Sigma_t^k)} \int_{B_r(\xi_1) \cap \Sigma_t^k} \cos \alpha_k d\mu_t^k - \frac{1}{\text{Vol}(B_r(\xi_2) \cap \Sigma_t^k)} \int_{B_r(\xi_2) \cap \Sigma_t^k} \cos \alpha_k d\mu_t^k \right|$$
\[
\leq \frac{C_1(r)}{\text{Vol}(B_r(\xi_1) \cap \Sigma_t^k)} \cdot \frac{C_2(r)}{\text{Vol}(B_r(\xi_2) \cap \Sigma_t^k)} \int_{B_R(0) \cap \Sigma_t^k} |\nabla \cos \alpha_k| \, d\mu_t^k, \quad (31)
\]

where \( B_r(\xi_i), i = 1, 2, \) are the 4-dimensional balls in \( M \). To prove (31), let us first recall the isoperimetric inequality on \( \Sigma_t^k \) (c.f. [HSp] and [MS]): let \( B^k_\rho(p) \) be the geodesic ball in \( \Sigma_t^k \), with radius \( \rho \) and center \( p \), then

\[
\text{Vol}(B^k_\rho(p)) \leq C \left( \text{length}(\partial B^k_\rho(p)) + \int_{B^k_\rho(p)} |H_k|^2 \, d\mu_t^k \right)^2,
\]

for any \( p \in \Sigma_t^k \), and any \( \rho > 0 \), where \( C \) does not depend on \( k, \rho, \) and \( p \). By Proposition 2.2, we have

\[
\int_{B^k_\rho(p)} |H_k|^2 \, d\mu_t^k \to 0 \quad \text{as} \quad k \to \infty.
\]

So, for \( k \) sufficiently large, we obtain:

\[
\text{Vol}(B^k_\rho(p)) \leq C \left( \text{length}(\partial B^k_\rho(p)) \right)^2.
\]

In particular, for \( k \) sufficiently large, the isoperimetric inequality implies

\[
\text{Vol}(B^k_\rho(p)) \geq C \rho^2,
\]

where \( C \) is a positive constant independent of \( k, \rho \) and \( p \).

Suppose that the diameter of \( B_r(\xi) \cap \Sigma_t^k \) is \( d_k(\xi) \). Then

\[
C \rho^2 \geq \int_{B_r(\xi) \cap \Sigma_t^k} d\mu_t^k \quad \text{by (22)}
\]

\[
= \int_0^{d_k(\xi)/2} \int_{\partial B^k_\rho(p)} d\sigma d\rho \quad \text{for some} \quad p \in \Sigma_t^k
\]

\[
\geq c \int_0^{d_k(\xi)/2} \text{Vol}^{1/2}(B^k_\rho(p)) d\rho + o(1), \quad o(1) \to 0 \quad \text{as} \quad k \to \infty
\]

\[
\geq c \int_0^{d_k(\xi)/2} C \rho d\rho + o(1) \quad \text{by (32)}
\]

\[
\geq c d_k(\xi)^2 + o(1).
\]

We therefore have, for any \( \xi \),

\[
d_k(\xi) \leq C r + o(1) \quad (33)
\]

where the constant \( C \) is independent of \( \xi \) and \( k \).
For any fixed \( \eta \in B_r(\xi_2) \cap \Sigma^k_t \) and any \( \xi \in B_r(\xi_1) \cap \Sigma^k_t \), we choose a geodesic \( l_{\eta\xi} \) connecting \( \eta \) and \( \xi \), call it a ray from \( \eta \) to \( \xi \). Take an open tubular neighborhood \( U(l_{\eta\xi}) \) of \( l_{\eta\xi} \) in \( \Sigma^k_t \). Within this neighborhood \( U(l_{\eta\xi}) \), we call the line in the normal direction of the ray \( l_{\eta\xi} \) the normal line which we denote by \( n(l_{\eta\xi}) \). It is clear that

\[
\cos \alpha_k(\xi) - \cos \alpha_k(\eta) = \int_{l_{\eta\xi}} \partial_l \cos \alpha_k dl
\]

where \( dl \) is the arc-length element of \( l_{\eta\xi} \). Choose \( r \) small enough so that \( B_r(\xi_1) \cap \Sigma^k_t \) is contained in \( U(l_{\eta\xi_1}) \). Keeping \( \eta \) fixed and integrating \((34)\) with respect to the variable \( \xi \), first along the normal direction \( n(l_{\eta\xi_1}) \) and then on the ray direction \( l_{\eta\xi_1} \), we have

\[
\leq \frac{1}{\text{Vol}(B_r(\xi_1) \cap \Sigma^k_t)} \int_0^{d_k(\xi_1)} \int_{n(l_{\eta\xi_1})} |\nabla \cos \alpha_k| \, d\mu^k_t \, d\rho
\]

\[
\leq \frac{1}{\text{Vol}(B_r(\xi_1) \cap \Sigma^k_t)} \int_0^{d_k(\xi_1)} \int_{B(0)} |\nabla \cos \alpha_k| \, d\mu^k_t \, d\rho
\]

\[
\leq \frac{Cr}{\text{Vol}(B_r(\xi_1) \cap \Sigma^k_t)} \int_{B(0)} |\nabla \cos \alpha_k| \, d\mu^k_t,
\]

where \( \theta \) is fixed. From \((34)\), integrating with respect to \( \eta \) in \( B_r(\xi_2) \cap \Sigma^k_t \) and dividing by \( \text{Vol}(B_r(\xi_2) \cap \Sigma^k_t) \), we get the desired inequality \((35)\).

For \( i = 1, 2 \) Hölder’s inequality and \((22)\) lead to

\[
\int_{B_r(\xi_1) \cap \Sigma^k_t} |\nabla \cos \alpha_k| \, d\mu^k_t \leq Cr \left( \int_{B_r(\xi_1) \cap \Sigma^k_t} |\nabla \cos \alpha_k|^2 \, d\mu^k_t \right)^{1/2}.
\]

The triangle inequality implies \( B^k_i(\xi_1) \subseteq B_r(\xi_1) \cap \Sigma^k_t \) for \( i = 1, 2 \); therefore by \((22)\)

\[
\text{Vol}(B_r(\xi_1) \cap \Sigma^k_t) \geq \text{Vol}(B^k_i(\xi_1)) \geq Cr^2.
\]

Now first letting \( k \to \infty \) in \((31)\) and using that the right hand side of \((31)\) tends to 0 by Proposition 2.2, and then letting \( r \to 0 \), we conclude that \( \theta \) is constant \( H^2 \) a.e. on \( \Sigma^\infty \).

We choose a new complex structure on \( \mathbb{R}^4 \) so that with respect to the corresponding Kähler form \( \omega_1 \) we have

\[
\omega_1|_{\Sigma^\infty} = \theta_1 d\mu^\infty \quad \text{and} \quad \theta_1 = 1 \quad \text{at one point} \ x \in \Sigma^\infty.
\]

It is clear that, in the new complex structure on \( \mathbb{R}^4 \), Proposition 2.2 remains true. Therefore, by the same argument as above for \( \theta \), we conclude that \( \theta_1 \equiv 1 \) and accordingly

\[
\omega_1|_{\Sigma^\infty} \equiv d\mu^\infty.
\]
In other words, we have shown that \( \Sigma^\infty \) is calibrated by the closed 2-form \( \omega_1 \). Applying Theorem 2.1 in [HS], we see that \( \Sigma^\infty \) is a holomorphic subvariety, in the chosen complex structure, of complex dimension one. More precisely, Harvey-Shiffman’s theorem says that, if \( T \) is a locally rectifiable \((k, k)\) type current and the \((2k + 1)\)-dim measure of \( \text{supp}(T) = 0 \), then \( T \) is a holomorphic \( k \)-chain, i.e., away from the support of its boundary \( T = \sum_j n_j[V_j] \), where \( n_j \) are positive integers and \( V_j \) are pure 1-dimensional complex subvarieties (cf. [HS], [HL]). We proved that \( \Sigma^\infty \) is \( H^2 \)-rectifiable (see Proposition 3.1) and it is stationary (see Lemma 4.2), which implies that the 3-dimensional measure of \( \text{supp}(\Sigma^\infty) = 0 \) (cf. Remark. after theorem 2.1 in [HS]). By (36), we can see that \( \Sigma^\infty \) is a bidegree \((1, 1)\) current, hence a holomorphic curve according to Harvey-Shiffman’s theorem.

We are left to show that the holomorphic curve has multiplicity more than 1. Otherwise, we would have

\[
\lim_{\rho \to 0} \frac{1}{\pi \rho^2} \mu^\infty(B_\rho(0)) = 1.
\]

It then follows from (24) that for any \( \epsilon > 0 \), there are \( \delta > 0 \) and \( k > k_0 \),

\[
\rho^{-2} \mu_{-r^2}(B_\rho(\xi)) < \pi(1 + \epsilon)
\]

for any fixed \( r > 0 \). The choice of \( r \) will be based on the following observation

\[
\Phi(F_k, 0, 0, -r^2) \leq \frac{1}{4\pi r^2} \int_0^\delta \int_{\partial B_\rho(0) \cap \Sigma^k_{0, -r^2}} e^{-\frac{\rho^2}{4r^2}} d\mu_{0, -r^2}^k + \frac{1}{4\pi r^2} \int_\delta^\infty \int_{\partial B_\rho(0) \cap \Sigma^k_{0, -r^2}} e^{-\frac{\rho^2}{4r^2}} d\mu_{0, -r^2}^k \\
\leq \frac{1}{4\pi r^2} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \int_{\partial B_\rho(0) \cap \Sigma^k_{0, -r^2}} d\mu_{0, -r^2}^k + \frac{1}{4\pi r^2} \int_\delta^\infty e^{-\frac{\rho^2}{4r^2}} \int_{\partial B_\rho(0) \cap \Sigma^k_{0, -r^2}} d\mu_{0, -r^2}^k \\
\leq \frac{1}{4\pi r^2} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \frac{\rho^2}{2r^2} \text{Vol}(B_\rho(0) \cap \Sigma^k_{0, -r^2}) d\rho + \frac{1}{4\pi r^2} \int_\delta^\infty e^{-\frac{\rho^2}{4r^2}} \frac{\rho^2}{2r^2} \text{Vol}(B_\rho(0) \cap \Sigma^k_{0, -r^2}) d\rho \\
\leq \frac{1}{4\pi r^2} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \frac{\rho^3}{2r^2} d\rho + \epsilon + o(r) \text{ by (37) and (22)} \\
\leq 1 + o(r).
\]

Choosing \( r > 0 \) sufficiently small, we therefore have

\[
\Phi(F, X_0, T, T - \lambda_k^{-2} r^2) = \Phi(F_k, 0, 0, -r^2) \leq 1 + \epsilon.
\]

18
Now by White’s local regularity theorem ([Wh1] Theorem 3.1 and Theorem 4.1, also see [E2]), \((X_0, T)\) could not be a singular point of the mean curvature flow. This is a contradiction. Now the proof of Theorem 4.3 is complete. Q.E.D.

5 Flatness of the \(\lambda\) tangent cones

In this section, we prove that the \(\lambda\) tangent cones are flat.

**Theorem 5.1** Let \(M\) be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and \(T > 0\) is the first blow-up time of the mean curvature flow, then the \(\lambda\) tangent cone \(\Sigma^\infty\) of the mean curvature flow at \((X_0, T)\) consists of a finite union of more than one 2-planes which are complex in a complex structure on \(\mathbb{R}^4\).

**Proof:** Without loss of any generality, we may assume \(0 \in \Sigma^\infty\) where 0 is the origin of \(\mathbb{R}^4\). In fact, if not, \(\Sigma^\infty\) would move to infinity, then we would have

\[
\Phi(F, X_0, T, -\lambda_k^{-2}r^2) = \Phi(F_k, 0, 0, 0 - r^2) \to 0 \text{ as } k \to \infty.
\]

But White’s regularity theorem then implies that \((X_0, T)\) is a regular point. This is impossible.

There is a sequence of points \(X_k \in \Sigma^k_t\) satisfying \(X_k \to 0\) as \(k \to \infty\). By Proposition 2.2, for any \(s_1\) and \(s_2\) with \(-\infty < s_1 < s_2 < 0\) and any \(R > 0\), we have

\[
\int_{s_1}^{s_2} \int_{\Sigma^k_t \cap B_R(0)} |\nabla_k|^2 d\mu_k dt \to 0 \text{ as } k \to \infty.
\]

Thus, by (22)

\[
\lim_{k \to \infty} \int_{s_1}^{s_2} \int_{\Sigma^k_t \cap B_R(0)} |(F_k - X_k)^\perp|^2 d\mu_k dt
\]

\[
\leq 2 \lim_{k \to \infty} \int_{s_1}^{s_2} \int_{\Sigma^k_t \cap B_R(0)} |F_k|^2 d\mu_k dt + C(s_2 - s_1)R^2 \lim_{k \to \infty} |X_k|^2 = 0.
\]

Let us denote the tangent spaces of \(\Sigma^k_t\) at the point \(F_k(x, t)\) and of \(\Sigma^\infty\) at the point \(F^\infty(x, t)\) by \(T\Sigma^k_t\) and \(T\Sigma^\infty\) respectively. It is clear that

\((F_k - X_k)^\perp = \text{dist } (X_k, T\Sigma^k_t),\)

and

\((F^\infty)^\perp = \text{dist } (0, T\Sigma^\infty).\)

By Allard’s compactness theorem, i.e. Theorem 4.1 (ii), we have

\[
\int_{s_1}^{s_2} \int_{\Sigma^\infty \cap B_R(0)} |(F^\infty)^\perp|^2 d\mu^\infty dt = \int_{s_1}^{s_2} \int_{\Sigma^\infty \cap B_R(0)} |\text{dist } (0, T\Sigma^\infty)|^2 d\mu^\infty dt
\]
By Theorem 4.3, we know that each sheet of \( \Sigma^\infty \) is smooth outside a discrete set of points \( S \). So outside \( S \), we have

\[
\langle F_\infty, v_\alpha \rangle = 0.
\]

Note that the above inner product is taken in \( \mathbb{R}^4 \), and differentiating in \( \mathbb{R}^4 \) then yields

\[
0 = \langle \partial_t F_\infty, v_\alpha \rangle + \langle F_\infty, \partial_t v_\alpha \rangle = \langle F_\infty, \partial_t v_\alpha \rangle.
\]

Because \( \partial_t F_\infty \) is tangential to \( \Sigma^\infty \), by Weingarten’s equation we observe

\[
(h_\infty)^\alpha_{ij} \langle F_\infty, e_j \rangle = 0 \quad \text{for all} \quad \alpha, \ i = 1, 2.
\]

So for \( \alpha = 1, 2 \), we have

\[
\det((h_\infty)^\alpha_{ij}) = 0.
\]

Since \( H = 0 \), for \( \alpha = 1, 2 \) we also have

\[
\text{tr}((h_\infty)^\alpha_{ij}) = 0.
\]

It then follows immediately that the symmetric matrix \( ((h_\infty)^\alpha_{ij}) \) is in fact the zero matrix, for all \( i, j, \alpha = 1, 2 \), which obviously yields \( |A_\infty| \equiv 0 \).

Since the second fundamental form \( A_\infty \) of \( \Sigma^\infty \) is identically zero on the smooth locus of \( \Sigma^\infty \) whose multiplicity is finite but bigger than 1, \( \Sigma^\infty \) is a finite union of more than one 2-planes. Moreover, if any two of these 2-planes meet at two distinct points they would intersect along a line containing these two points; but this contradicts to that \( \Sigma^\infty \) is a holomorphic subvariety.

This completes the proof of Theorem 5.1. Q.E.D.

6 Tangent cones from time dependent scaling

In this section, we consider the tangent cones which arise from the rescaled surface \( \tilde{\Sigma}_s \) defined by

\[
\tilde{F}(\cdot, s) = \frac{1}{\sqrt{2(T-t)}} F(\cdot, t), \quad (38)
\]

where \( s = -\frac{1}{2} \log(T-t) \), \( c_0 \leq s < \infty \). Here we choose the coordinates so that \( X_0 = 0 \). Rescaling of this type was used by Huisken [H2] to distinguish Type I and
Type II singularities for mean curvature flows. Denote the rescaled surface by $\tilde{\Sigma}_s$. From the evolution equation of $F$ we derive the flow equation for $\tilde{F}$

$$\frac{\partial}{\partial s} \tilde{F}(x, s) = \tilde{H}(x, s) + \tilde{F}(x, s).$$

(39)

It is clear that

$$\cos \tilde{\alpha}(x, s) = \cos \alpha(x, s),$$

$$|\tilde{A}|^2(x, s) = 2(T - t)|A|^2(x, t).$$

Recall that we set, in section 2, $v(x, t) = e^{R_0(t-\tilde{t})} \cos \alpha(x, t)$ where $R_0 = \max\{0, -R\}$ and $R$ is the scalar curvature of $M$. The function $v(x, t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) v(x, t) \geq |\nabla J_{\Sigma_t}|^2 v(x, t).$$

The corresponding version of this evolution inequality for the scaled flow is in the following lemma.

**Lemma 6.1** Assume that $M$ is a Kähler-Einstein surface with scalar curvature $R$ and $\Sigma_t$ evolves by a mean curvature flow in $M$ with the initial surface being compact and symplectic. Let $\tilde{v}(x, s) = e^{R_0(T-\tilde{t})} \cos \tilde{\alpha}(x, s)$ where $R_0 = \max\{0, -R\}$. Then

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \tilde{v}(x, s) \geq |\nabla J_{\tilde{\Sigma}_s}|^2 \tilde{v}(x, s).$$

(40)

**Proof:** One can check directly that

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \cos \tilde{\alpha}(x, s) = 2(T - t) \left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha(x, t).$$

It follows that

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \tilde{v}(x, s) = 2(T - t) \left(\frac{\partial}{\partial t} - \Delta\right) v(x, t)$$

$$\geq 2(T - t) |\nabla J_{\Sigma_t}|^2 v(x, t)$$

$$= |\nabla J_{\tilde{\Sigma}_s}|^2 \tilde{v}(x, s).$$

This proves the lemma. Q.E.D.

Next, we shall derive the corresponding weighted monotonicity formula for the scaled flow. By (40), we have

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \frac{1}{v} \leq -|\nabla J_{\tilde{\Sigma}_s}|^2 \frac{1}{v} - \frac{2}{\tilde{v}^3} |\nabla \tilde{v}|^2.$$
Let
\[ \tilde{\rho}(X) = \exp \left( -\frac{1}{2} |X|^2 \right), \]
\[ \Psi(s) = \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(\tilde{F}) d\tilde{\mu}_s. \]

**Lemma 6.2** There are positive constants \( c_1 \) and \( c_2 \) which depend on \( M \), \( F_0 \) and \( r \) which is the constant in the definition of \( \phi \), so that the following monotonicity formula holds
\[
\frac{\partial}{\partial s} \exp(c_1 e^{-s}) \Psi(s) \leq -\exp(c_1 e^{-s}) \left( \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(\tilde{F}) \left| \tilde{H} + \tilde{F}^\perp \right|^2 d\tilde{\mu}_s + \int_{\Sigma_s} \frac{2}{v^3} \left| \tilde{\nabla} \right|^2 \phi \tilde{\rho}(\tilde{F}) d\tilde{\mu}_s \right) + c_2 e^{-2s}. \tag{41}
\]

**Proof:** Note that
\[
\tilde{F}(x, s) = \frac{F(x, t)}{\sqrt{2(T-t)}},
\]
\[
\tilde{H}(x, s) = \sqrt{2(T-t)} H(x, t),
\]
\[
\left| \tilde{\nabla} J_{\Sigma_s} \right|^2(x, s) = 2(T-t) \left| \nabla J_{\Sigma_t} \right|^2(x, t),
\]
\[
\left| \tilde{\nabla} \nu \right|^2(x, s) = 2(T-t) \left| \nabla \nu \right|^2(x, t).
\]

By the chain rule
\[
\frac{\partial}{\partial s} = 2e^{-2s} \frac{\partial}{\partial t}
\]
and the monotonicity inequality (10) for the unscaled surface, we obtain the desired inequality. \( \text{Q.E.D.} \)

**Lemma 6.3** Let \( M \) be a compact Kähler-Einstein surface. If the initial compact surface is symplectic, then there is a sequence \( s_k \to \infty \) such that, for any \( R > 0 \),
\[
\int_{\Sigma_{s_k} \cap B_R(0)} \left| \nabla J_{\Sigma} \right|^2 d\tilde{\mu}_{s_k} \to 0 \quad \text{as} \quad k \to \infty, \tag{42}
\]
\[
\int_{\Sigma_{s_k} \cap B_R(0)} \left| \nabla \cos \tilde{\alpha} \right|^2 d\tilde{\mu}_{s_k} \to 0 \quad \text{as} \quad k \to \infty, \tag{43}
\]
\[
\int_{\Sigma_{s_k} \cap B_R(0)} \left| \tilde{H} \right|^2 d\tilde{\mu}_{s_k} \to 0 \quad \text{as} \quad k \to \infty, \tag{44}
\]

and
\[
\int_{\Sigma_{s_k} \cap B_R(0)} \left| \tilde{F}^\perp \right|^2 d\tilde{\mu}_{s_k} \to 0 \quad \text{as} \quad k \to \infty. \tag{45}
\]
Proof: By (41), we have

$$\int_{s_0}^{\infty} \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(\tilde{F}) \left| \nabla J_{\Sigma_s} \right|^2 d\tilde{\mu}_s ds$$

$$+ \int_{s_0}^{\infty} \left( \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(\tilde{F}) \left| \nabla J_{\Sigma_s} \right|^2 d\tilde{\mu}_s + \int_{\tilde{\mu}_s} \frac{2}{v^3} \nabla v \phi \tilde{\rho}(\tilde{F}) d\tilde{\mu}_s \right) ds.$$

Hence there is a sequence $s_k \to \infty$, such that as $k \to \infty$

$$\int_{\Sigma_{s_k}} \frac{1}{v} \phi \tilde{\rho}(\tilde{F}) \left| \nabla J_{\Sigma_{s_k}} \right|^2 d\tilde{\mu}_{s_k} \to 0$$

and

$$\int_{\Sigma_{s_k}} \frac{2}{v^3} \nabla v \phi \tilde{\rho}(\tilde{F}) d\tilde{\mu}_{s_k} \to 0,$$

which yields (42) and (43), and

$$\int_{\Sigma_{s_k}} \frac{1}{v} \phi \tilde{\rho}(\tilde{F}) \left| \nabla J_{\Sigma_{s_k}} \right|^2 d\tilde{\mu}_{s_k} \to 0.$$

(46)

By (4) and (42) we see that (44) holds. By (44) and (46) we get (45). This proves the proposition. Q.E.D.

The proof of the following lemma is essentially the same as the one for Proposition 3.1, except there are two parameters $\lambda, t$ for the $\lambda$ tangent cones but only one parameter $t$ for the time dependent tangent cones. For the sake of completeness, we shall provide a proof.

**Lemma 6.4** There is a subsequence of $s_k$, which we also denote by $s_k$, such that $(\Sigma_{s_k}, d\tilde{\mu}_{s_k}) \to (\Sigma_{\infty}, d\tilde{\mu}_{\infty})$ in the sense of measures. And $(\Sigma_{\infty}, d\tilde{\mu}_{\infty})$ is $\mathcal{H}^2$-rectifiable.

**Proof:** To show the subconvergence, it suffices to show that, for any $R > 0$,

$$\tilde{\mu}_{s_k} (\Sigma_{s_k} \cap B_R(0)) \leq CR^2,$$

where $B_R(0)$ is a metric ball in $\mathbb{R}^4$, $C > 0$ is independent of $k$. Direct calculation leads to

$$\tilde{\mu}_{s_k} (\Sigma_{s_k} \cap B_R(0)) = (2(T-t))^{-1} \int_{\Sigma_{T-\epsilon^{2s_k} \cap B(\sqrt{2(T-t)}R(0))}^1} d\mu_t$$

$$= R^2 \left( \sqrt{2} e^{-s} R \right)^{-2} \int_{\Sigma_{T-\epsilon^{2s_k} \cap B(\sqrt{2(T-t)}R(0))}^1} d\mu_t$$

$$\leq CR^2 \Phi \left( 0, T + (\sqrt{2} e^{-s} R)^2 - e^{2s_k}, T - e^{2s_k} \right)$$

By the monotonicity inequality (6), we have

$$\tilde{\mu}_{s_k} (\Sigma_{s_k} \cap B_R(0)) \leq CR^2 \left( \Phi(0, T + (\sqrt{2} e^{-s} R)^2 - e^{2s_k}, T/2) + C \right)$$

$$\leq C \frac{R^2}{T} \left( \mu_{T/2}(\Sigma_{T/2}) + C \right).$$
Since 
\[
\frac{\partial}{\partial t} \mu_t(\Sigma_t) = - \int_{\Sigma_t} |H|^2 d\mu_t, 
\]
we have 
\[
\tilde{\mu}_{s_k}(\tilde{\Sigma}_{s_k} \cap B_R(0)) \leq CR^2.
\]

We now prove that \((\Sigma_\infty, d_{\tilde{\mu}_\infty})\) is \(H^2\)-rectifiable. For any \(\xi \in \tilde{\Sigma}_\infty\), choose \(\xi_k \in \tilde{\Sigma}_{s_k}\) with \(\xi_k \to \xi\) as \(k \to \infty\). By the monotonicity identity (17.4) in [Si1], we have 
\[
\sigma^{-2} \tilde{\mu}_{s_k}(B_{\sigma}(\xi_k)) = \rho^{-2} \tilde{\mu}_{s_k}(B_{\rho}(\xi_k)) - \int_{B_{\rho}(\xi_k) \setminus B_{\rho}(\xi_k)} \frac{|D^\perp r|^2}{r^2} d\tilde{\mu}_{s_k} - \frac{1}{2} \int_{B_{\rho}(\xi_k)} (x - \xi_k) \cdot \overline{H}_k \left( \frac{1}{r^2} - \frac{1}{\rho^2} \right) d\tilde{\mu}_{s_k},
\]
for all \(0 < \sigma \leq \rho\), where \(\tilde{\mu}_{s_k}(B_{\sigma}(\xi_k))\) is the area of \(\tilde{\Sigma}_{s_k} \cap B_{\sigma}(\xi_k)\), \(r_\sigma = \max\{r, \sigma\}\) and \(D^\perp r\) denotes the orthogonal projection of \(Dr\) (which is a vector of length 1) onto \((T_{\xi_k} \tilde{\Sigma}_{s_k})^\perp\). Letting \(k \to \infty\), by Lemma 6.3, we have 
\[
\sigma^{-2} \tilde{\mu}_\infty(B_{\sigma}(\xi)) \leq \rho^{-2} \tilde{\mu}_\infty(B_{\rho}(\xi)),
\]
for all \(0 < \sigma \leq \rho\). Therefore, \(\lim_{\rho \to 0} \rho^{-2} \tilde{\mu}_\infty(B_{\rho}(\xi))\) exists. By (47) we know that \(\lim_{\rho \to 0} \rho^{-2} \tilde{\mu}_\infty(B_{\rho}(\xi))\) is finite; and we now show that it has a positive lower bound.

By the isoperimetric inequality on \(\tilde{\Sigma}_{s_k}\) (c.f. [HSp] and [MS]), we have 
\[
\text{Vol}(B_{\rho}(\xi_k)) \leq C \left( \text{length}(\partial(B_{\rho}(\xi_k))) + \int_{B_{\rho}(\xi_k)} |\overline{H}| d\tilde{\mu}_{s_k} \right)^2 
\leq C \left( \text{length}(\partial(B_{\rho}(\xi_k))) + \left( \int_{B_{\rho}(\xi_k)} |\overline{H}|^2 d\tilde{\mu}_{s_k} \right)^{1/2} \text{Vol}^{1/2}(B_{\rho}(\xi_k)) \right)^2,
\]
for any \(\rho > 0\), where \(B_{\rho}(\xi_k)\) is the geodesic ball in \(\tilde{\Sigma}_{s_k}\), with radius \(\rho\) and center \(\xi_k\), \(C\) does not depend on \(k, \rho\). By Lemma 6.3, we have 
\[
\int_{B_{\rho}(\xi_k)} |\overline{H}|^2 d\tilde{\mu}_{s_k} \to 0 \text{ as } k \to \infty.
\]
Hence, for \(k\) sufficiently large, we have 
\[
\text{Vol}(B_{\rho}(\xi_k)) \geq C \rho^2,
\]
where \(C\) is a positive constant independent of \(k, \rho\).

The triangle inequality implies \(B_{r}(\xi_k) \subset B_{r}(\xi_k) \cap \tilde{\Sigma}_{s_k}\) for \(k = 1, 2, \ldots\), therefore by (49) 
\[
\text{Vol}(B_{r}(\xi_k) \cap \tilde{\Sigma}_{s_k}) \geq \text{Vol}(B_{r}(\xi_k)) \geq C r^2.
\]
It concludes that \( \lim_{\rho \to 0} \rho^{-2} \mu_\infty(B_\rho(\xi)) \) exists and for \( \mathcal{H}^2 \) almost all \( \xi \in \tilde{\Sigma}_\infty \),

\[
0 < C \leq \lim_{\rho \to 0} \rho^{-2} \mu_\infty(B_\rho(\xi)) < \infty. \tag{50}
\]

By Priess’s theorem in [P] we can see from (50) that, \((\tilde{\Sigma}_\infty, d\tilde{\mu}_\infty)\) is \(\mathcal{H}^2\)-rectifiable. This proves the lemma. Q.E.D.

**Definition 6.5** Let \((X_0, T)\) be a singular point of the mean curvature flow of a closed symplectic surface in a compact Kähler-Einstein surface \(M\). We call \((\tilde{\Sigma}_\infty, d\tilde{\mu}_\infty)\) obtained in the last lemma a tangent cone of the mean curvature flow \(\Sigma_t\) at \((X_0, T)\) in the time dependent scaling.

With the lemmas established in this section, we can derive, by using arguments completely similar to those for the \(\lambda\) tangent cones in section 4 and section 5, holomorphicity and flatness of the tangent cones coming from time dependent scaling.

**Theorem 6.6** Let \(M\) be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and \(T > 0\) is the first blow-up time of the mean curvature flow, then the tangent cone \(\tilde{\Sigma}_\infty\) of the mean curvature flow at \((X_0, T)\) coming from time dependent scaling consists of a finite union of more than one 2-planes in \(\mathbb{R}^4\) which are complex in a complex structure on \(\mathbb{R}^4\).

Finally, we give two remarks.

**Remark 6.7** Let \(M\) be a compact Kähler-Einstein surface. Assume that the initial surface is symplectic. If \((X_0, T)\) is the Type I singularity for the mean curvature flow, then

\[
0 < c \leq (T - t)|A|^2 \leq C < \infty,
\]

and consequently, \(\tilde{\Sigma}_{s_k}\) converges strongly to \(\tilde{\Sigma}_\infty\) with

\[
0 < c \leq |\tilde{A}_\infty|^2 \leq C < \infty.
\]

However, by Theorem 6.6, we have \(\tilde{A}_\infty \equiv 0\). This contradiction shows that there is no Type I singularity for mean curvature flow of symplectic surface in K-E surfaces. This result was proved in [CL] and [W1].

**Remark 6.8** When one considers the singularity of holomorphic curves, using the monotonicity identities (24) and (1.2) in [Si2], by an argument similar to the one used in the present paper, one can show that the bubbles are two-dimensional planes which are complex under some complex structure on \(\mathbb{R}^4\) (also see [M]). In fact, in [M], Morgan obtained more general results, in particular, he proved that any tangent cone to a two-dimensional oriented area minimizing surface in \(\mathbb{R}^4\) consists of such planes.
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