Dynamic pricing in retail with diffusion process demand

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Abstract

When randomness in demand affects the sales of a product, retailers use dynamic pricing strategies to maximize their profits. In this article, we formulate the pricing problem as a continuous-time stochastic optimal control problem, and find the optimal policy by solving the associated Hamilton-Jacobi-Bellman (HJB) equation. We propose a new approach to modelling the randomness in the dynamics of sales based on diffusion processes. The model assumes a continuum approximation to the stock levels of the retailer which should scale much better to large-inventory problems than the existing Poisson process models in the revenue management literature. The diffusion process approach also enables modelling of the demand volatility, whereas Poisson process models do not.

We present closed-form solutions to the HJB equation when there is no randomness in the system. It turns out that the deterministic pricing policy is near-optimal for systems with demand uncertainty. Numerical errors in calculating the optimal pricing policy may, in fact, result in a lower profit on average than with the heuristic pricing policy.

Keywords: Diffusion processes; dynamic pricing; Hamilton-Jacobi-Bellman; stochastic optimal control.

1 Introduction

Consider a monopolist retailer who wants to design a dynamic pricing policy for a product over a given period, in order to maximize their total revenue and minimize the cost associated with handling unsold items at a given terminal time. Two important components in the decision process are the ability to take into account the uncertainty associated with future cost and demand and to optimally adjust for new knowledge as it arrives. We will illustrate how to address both components in this article, focusing on large-inventory limits and multiplicative demand uncertainty. Assume the retailer sells large volumes of its product at high frequency compared to the total pricing period. In such a setting, it is appropriate to model the sales process in a continuum limit, both for product volume and for time, similar to Kalish (1983).

In the revenue management literature, most efforts to model demand uncertainty in continuous time have focused on Poisson processes. See, for example, the overviews by Bitran & Caldentey (2003) or Aviv & Vulcano (2012). We stress that in other communities, such as financial markets, both diffusion processes and jump-diffusions are common for modelling demand and spot-prices (Benth et al. 2014). The survey by Carmona & Coulon (2014) gives an indication of how flexible more advanced models used for commodity markets can be. With this article, we wish to inspire the revenue management community to take advantage of this research when modelling uncertainty in their own domain. For the remainder of this section, however, we mainly focus on modelling in the revenue management literature.

A pure Poisson process assumption is not compatible with taking a time-continuum limit for sales volume with demand uncertainty and may be better suited for demand modelling in industries with lower product sales volumes, such as the airline and hotel industries. Maglaras & Meissner (2006), Schlosser (2015), and Schlosser (2015b) propose pricing heuristics similar to this article, by considering a deterministic model based on the asymptotic continuum limit of Poisson processes. To the author’s knowledge, however, few attempts have been made to unify demand uncertainty with the continuum limit. For example, Raman & Chatterjee (1995) and Wu & Wu (2016) model demand uncertainty as increments of a Brownian motion. As we will show in this article, their approaches lead to demand processes that admit

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negative sales, with a probability approaching 1/2 over infinitesimally small time periods. We believe this is an important factor for why so little research has been done in this area. This article proposes a different approach to modelling demand uncertainty in order to remedy this. In our approach, the parameters of the system are described as diffusion processes that are solutions to stochastic differential equations (SDEs). This enables modelling of the demand volatility, which Poisson processes do not. Our proposed approach can be combined with parameter estimation methods already used by retailers and also extends naturally to multiple products. Modelling the demand over time as a diffusion process has previously been done by Chambers (1992) at macroeconomic scale with UK national data. His focus was on the data assimilation aspect and was not applied to a setting of optimal control.

The retailer’s dynamic pricing policies are stochastic processes that control the SDE which describes the depletion of stock. In this article, we seek an optimal pricing policy that maximizes the expected value of the profit over a given pricing period. One way to find an optimal pricing policy is to solve an associated nonlinear partial differential equation (PDE), known as the Hamilton-Jacobi-Bellman (HJB) equation (Pham 2009). We provide closed-form solutions for the HJB equation in the deterministic case, for both linear and exponential demand functions. The solution identifies two pricing regimes, one where the retailer maximizes their profits without depleting the inventory and another where the retailer aims to maximize the price and still deplete its inventory. Xu & Hopp (2006) have considered a continuous-time pricing problem where the uncertainty is modelled by a geometric Brownian motion. Their expected demand function is unbounded, with the result that the optimal pricing strategy ensures that all stock is sold by the terminal time. The demand functions we consider are bounded, and by introducing a penalty on unsold stock we capture the pricing regime change that does not arise in Xu & Hopp (2006).

In financial markets traders face a similar problem to that presented in this article, known as the optimal execution, or liquidation, problem. There, a trader tries to sell, or purchase, a particular amount of an asset by a predetermined time. See, for example, Cartea et al. (2015) for an overview of this problem. Instead of controlling the price, the focus of the retailer pricing problem, the trader directly controls how much of the product to sell at a particular time. If the expected demand model is invertible, the retailer’s pricing problem can be reformulated to control the expected amount of stock to sell at each time. This reformulation is sometimes chosen in the revenue management community as well, see, for example, Bitran & Caldentey (2003). In this article we focus on the formulation that writes the expected demand model in terms of the price.

By investigating the terms in the HJB equation, we identify the cases where the deterministic-case solution is appropriate. Potentially significant changes to the pricing policy for the stochastic system are at the interface between the two pricing regimes: far away from this interface the deterministic pricing policy is near-optimal. For example, the expected price path is decreasing when one takes into account uncertainty, while it is not for the deterministic heuristic. For a risk-neutral decision maker, however, the differences in profit are insignificant for most cases that may be relevant in industry.

The article is structured as follows: In Section 2, we describe the modelling of the system and compare the new parameter uncertainty approach to the existing Brownian increments approach. Then, a formulation of the pricing problem and the associated HJB equation is given in Section 3. We also propose a method to estimate the multiplicative factor in our model, in order to implement the pricing policy in practice. The optimal pricing policy in the deterministic limit is covered in Section 4, and the comparison to the stochastic system is shown in Section 5. Extensions to the problem, such as other models for uncertainty, and risk aversion, are discussed in Section 6. Finally, we conclude and suggest avenues for further research in Section 7.

2 Modelling demand and uncertainty

For a given, positive amount of initial stock of a product, we are interested in modelling the product sales over some finite time period. Assume that the initial quantity of stock is large, and that there is a substantial volume sold over time periods that are small compared to the total period of interest. These assumptions can apply to many products sold by large retailers. For example, in the monopoly setting, this leads to a continuum model similar to that of Kalish (1983). For a given product, denote the amount of stock left at time \( t \) by \( S(t) \), and let \( q(a) \) represent product demand at price \( a \), per unit time. For simplicity, we assume \( q \) does not explicitly depend on time. In the continuum limit, the change in stock
at time $t$ is thus

$$
dS(t) = \begin{cases} 
-\dot{q}(\hat{a}) \, dt & \text{if } \dot{S}(t) > 0, \\
0 & \text{if } \dot{S}(t) \leq 0.
\end{cases}
$$

(1)

For a pricing policy $\alpha(t)$, the remaining stock at time $t$ is then

$$
\hat{S}(t) = \hat{S}(0) - \int_0^t \hat{q}(\hat{a}(\hat{u})) \, du.
$$

(2)

In the remainder of the article, we emphasise the dependence of remaining stock on a particular pricing policy $\alpha(t)$ using the superscript $S^\alpha$.

At the start of the prediction period, it is not known exactly what the demand will be at future times. We will now discuss how to represent this uncertainty in the model. First, we note that the uncertainty in future demand is due to the changes in $W(t)$, $\sigma(t,s,a)$ denote a Brownian motion, and let $\sigma(t,s,a)$ be the volatility in demand as a function of time, stock, and price. Then one may say that the uncertainty in increments of a Brownian motion. Instead, the uncertainty in demand could be modelled more realistically $\sim N(0,1)$, and thus

$$
\mathbb{P}(S(\Delta t) > S(0)) = \mathbb{P}(Z \leq -\sqrt{\Delta t} \, \bar{q}/\bar{\sigma}) \to 0.5^-, \quad \text{as } \Delta t \to 0^+.
$$

(4)

We therefore argue that it is generally inappropriate to model the uncertainty in future demand with increments of a Brownian motion. Instead, the uncertainty in demand could be modelled more realistically.
by describing the evolution of parameters in the demand function. This article will focus on multiplicative
demand uncertainty, which may arise from estimates of seasonality or long-term trends in demand.
As such, we introduce a multiplicative parameter \( g \geq 0 \), and consider the demand function given by
\((a, g) \mapsto q(a)g\). Assume that the parameter is not directly affected by price, but only represents exogenous
information outside the retailer’s control. For our purposes, we will model the multiplicative parameter
as a geometric Brownian motion (GBM),
\[ G(t) = \exp \left(-\frac{\sigma^2}{2} t + \sigma W(t) \right), \]
with volatility coefficient \( \sigma \geq 0 \).
The GBM approach is also considered by Xu & Hopp (2006) for a different dynamic pricing problem
from that presented in this article. We restrict ourselves to a GBM with no drift to focus on the impact
of the uncertainty, rather than modelling time-dependent behaviour such as seasonality. For any \( \Delta t \geq 0 \),
some relevant properties of \( G(t) \) are as follows.

\[
\begin{align*}
G(0) &= 1, & \text{martingale property,} \\
\mathbb{E}[G(t + \Delta t) \mid G(t)] &= G(t), & \text{increasing variance.}
\end{align*}
\]

With this model, we expect future demand \( q(a)G(t) \) at a given price \( a \geq 0 \) to be the current experienced
demand, but with decreasing certainty the further ahead we forecast. The SDE governing the system,
started at \( S(0) > 0 \), \( G(0) = 1 \), is
\[
\begin{align*}
dG(t) &= \sigma G(t) dW(t), \\
dS^\alpha(t) &= -q(\alpha(t))G(t) dt, & \text{stopped at zero.}
\end{align*}
\]
The sample paths of \( S^\alpha(t) \) following the GBM model, as shown in Figure 2, are more regular than of the
Brownian noise model.

![Figure 2](image-url)

Figure 2: Sample paths of \( S^\alpha(t) \) following the GBM model (8), with \( q(\alpha(t)) = 1 \), and GBM volatility
\( \sigma = 0.1 \). The paths are more regular than that of the Brownian noise model shown in Figure 1.

### 2.1 Non-dimensionalized system

In order to capture similarities between different pricing decisions, irrespective of units such as a particular
currency, it is helpful to work in a dimensionless system. We thus non-dimensionalize the model, which
also helps to reduce the number of parameters in the decision problem. The units at play in our system
are the time and the product price, for example measured in weeks and £. If \( s, a, \) and \( t \) denotes unscaled
quantities of stock, price, and time respectively, we rescale them with dimensionless hat-quantities
\[
\begin{align*}
\hat{s} &= \frac{s}{S(0)}, & \hat{t} &= \frac{t}{T}, & \hat{a} &= \frac{a}{\bar{a}}.
\end{align*}
\]
Here, \( S(0) \gg 1 \) is the initial quantity of stock, \( T \) is the time-horizon for the pricing problem, and \( \hat{a} \) is some reference price chosen to make typical prices \( \hat{a} \) continuous and of order one. In order to write down a dimensionless formulation of the system’s SDE, we need to work with the expected demand function and volatility parameter defined as

\[
\hat{q}(\hat{a}) = \frac{T}{S(0)} q(\hat{a}\hat{\sigma}), \quad \hat{\sigma} = \sqrt{T\sigma}.
\]

Thus, for a given pricing policy \( \hat{\alpha}(\hat{t}) \), the system starts at \( \hat{S}(0), \hat{G}(0) = 1 \), and evolves according to

\[
\begin{align*}
\hat{d}\hat{G}(\hat{t}) &= \hat{\sigma}\hat{G}(\hat{t}) \, dW(\hat{t}), \\
\hat{d}\hat{S}^{\hat{\alpha}}(t) &= -\hat{q}(\hat{a}(\hat{t}))\hat{G}(\hat{t}) \, d\hat{t}, \quad \text{stopped at zero.}
\end{align*}
\]

This scaling means that we work solely with stock and time on the unit interval, \( \hat{S}^{\hat{\alpha}}(\hat{t}), \hat{\alpha} \in [0,1] \). In the remainder of the article, we will omit the hats, and assume all quantities are dimensionless.

### 3 Formulation of the decision problem

We restate the retailer’s objective which guides the choice of pricing policy: Over some decision horizon, continuously adjust the price of a given product in order to maximize the total profit, generated from sales revenue minus the cost of handling unsold items at the terminal time. We formulate this mathematically as a stochastic optimal control problem, for which the optimal pricing policy can be found by solving the associated HJB equation. In Section 3.1 we address the real-world restrictions of the continuous-time assumption.

Let \( C > 0 \) denote the handling cost per unit of stock at the terminal time. Define \( T_h = \min\{1, T_0\} \), where \( T_0 = \inf\{t \geq 0 \mid S^\alpha(t) = 0\} \). The total profit accrued from a pricing policy \( \alpha(t) \) is then

\[
P(\alpha) = \int_0^{T_h} \alpha(u)q(\alpha(u))G(u) \, du - CS^\alpha(1).
\]

Note that we focus on time horizons where we believe discounting future cash is negligible. This profit is a random variable that depends on the event \( \omega \) or, equivalently, the path of the Brownian motion \( W(t) \).

In the context of this article, we assume a retailer that wants to maximize the expected value of profit, \( \mathbb{E}[P(\alpha)] \).

Further, we restrict the product price to be in some closed interval \( A \subset \mathbb{R}_{\geq 0} \). In addition, we will only look for Markovian pricing policies in an admissible set \( A \). We say that \( \alpha(t) \) is Markovian if there is a function of the form \( a(t, s, g) \) such that for each event \( \omega \),

\[
\alpha(t)(\omega) = a(t, S^\alpha(t)(\omega), G(t)(\omega)) \in A.
\]

Thus, we seek pricing policies that set the price at time \( t \), based on the knowledge of the state at that time. We also assume that \( A \) only contains pricing policies such that the integral in (12) exists and \( P(\alpha) \) is integrable. We can now state the mathematical problem we seek to solve in the remainder of the article.

**Definition 3.1** (Pricing problem). In order to maximize the retailer’s expected profit, find a solution to the stochastic optimal control problem

\[
\max_{\alpha \in A} \mathbb{E}[P(\alpha)].
\]

Our chosen strategy for finding the corresponding pricing function \( a^*(t, s, g) \) of a maximizer \( \alpha^* \in A \) is to solve the associated HJB equation for the pricing problem. For a detailed explanation of the theory behind stochastic optimal control and HJB equations, see, for example, Pham (2009). The HJB equation is a nonlinear PDE, where the solution describes the value of being in a particular state. We will not worry about uniqueness of solutions to the HJB equation in this article. From the value function defined below, one can calculate the optimal pricing function \( a^*(t, s, g) \).

If at time \( t \), we know the value of \( S(t) \) and \( G(t) \), then the value function represents the expected value of applying the optimal pricing policy for the remainder of the pricing period. We define the value
function \( v(t, s, g) \) for \( t, s \in [0, 1] \) and \( g \geq 0 \) by

\[
v(t, s, g) = \max_{a \in A} \mathbb{E}_t \left[ \int_t^{T_h} \alpha(u)q(\alpha(u))G(u) \, du - CS^a(1) \right],
\]

\[(15)\]

\[
v(t, 0, g) = 0,
\]

\[(16)\]

\[
v(t, s, 0) = -Cs.
\]

\[(17)\]

The subscript on the expectation denotes that we condition on \( G \), which gives the derivation and properties of this estimator.

In the numerical examples in this article, we estimate \( G \) for example in pricing of options using the Black-Scholes equations (Black & Scholes 1973). In order to use a pricing function for the problem. Such a continuous-time approximation to inherently discrete systems is common, only observed at these time points. When computing an optimal pricing strategy we will still consider updated frequently at fixed time points \( t \), \( G \) in a real retail application we cannot update the price in continuous time, and it is not possible to infer solutions to HJB equations.

Therefore consider the solutions in the viscosity sense. See Pham (2009) for a description of viscosity solutions to HJB equations.

Remark 1. The value function is not necessarily sufficiently smooth to satisfy the HJB equation in the classical sense. This is indeed the case for some examples in this article when \( \sigma = 0 \), and we must therefore consider the solutions in the viscosity sense. See Pham (2009) for a description of viscosity solutions to HJB equations.

3.1 Parameter estimation

In a real retail application we cannot update the price in continuous time, and it is not possible to infer \( G(t) \) exactly. To represent real world conditions, we assume that the price is piecewise constant and updated frequently at fixed time points \( t_0 < t_1 < \cdots < T_h \). Further, we assume that stock levels are only observed at these time points. When computing an optimal pricing strategy we will still consider the continuous-time function \( v(t, g) \) that can be computed from the HJB equation as the optimal pricing function for the problem. Such a continuous-time approximation to inherently discrete systems is common, for example in pricing of options using the Black-Scholes equations (Black & Scholes 1973). In order to use a pricing function \( a(t_k, g) \) we must estimate \( G(t_k) \). Due to the Markov properties of \( G(t) \) and \( S^a(t) \), information about the process for \( t < t_k-1 \) is not needed, and we can estimate \( G(t_k) \) based on \( a(t_{k-1}), \alpha(t_{k-1}), S^a(t_{k-1}) \), and \( S^a(t_k) \). Let \( k = 1 \) and leave out the superscript \( a \) of \( S^a(t) \) for the remainder of this section. By assumption, the price has been constant, \( \alpha(u) = a_0 \), for the time period \( u \in [t_0, t_1] \). Say \( S^a(t_1) > 0 \), and that we wish to update the price at time \( t_1 \). From (11), the SDE describing our system, we have

\[
S(t_1) = S(t_0) - q(a_0) \int_{t_0}^{t_1} G(u) \, du.
\]

In the numerical examples in this article, we estimate \( G(t_1) \) with \( \tilde{G}(t_1) = \frac{S(t_0) - S(t_1)}{q(a_0)(t_1 - t_0)} \). We now discuss the derivation and properties of this estimator.

Define \( B(t) = \exp(-\sigma^2t/2 + \sigma\sqrt{t}Z_0) \), where \( Z_0 \sim \mathcal{N}(0, 1) \). The evolution of \( G(t) \) is known in closed form, which gives \( G(t_1) = G(t_0)B(\Delta t) \), with \( \Delta t = t_1 - t_0 \). From (20) it follows that

\[
G(t_1) = \frac{S(t_0) - S(t_1)}{q(a_0)} \int_0^{\Delta t} B(u) \, du.
\]

So long as \( \Delta t = t_1 - t_0 \), and the variance of \( B(u) \), are sufficiently small, we may use the approximation \( \int_0^{\Delta t} B(u) \, du \approx \frac{\Delta t}{2}(B(0) + B(\Delta t)) = \frac{\Delta t}{2}(1 + B(\Delta t)) \). Hence, the conditional distribution of \( G(t_1) \) is
approximated as
\[
G(t_1) \mid (a_0, S(t_0), S(t_1)) \approx \frac{S(t_0) - S(t_1)}{q(a_0)} \frac{B(\Delta t)}{\Delta t 1 + B(\Delta t)} \frac{2 B(\Delta t)}{B(\Delta t)} \frac{S(t_0)}{q(a_0)} \Delta t.
\]

The second approximate equality comes from the Taylor expansion \( x_1^+ x \approx 1 + x \) about \( x = 1 \). This gives us the following expressions for the first two moments of \( G(t_1) \):

\[
E[G(t_1) \mid a_0, S(t_0), S(t_1)] \approx S(t_0) - S(t_1) q(a_0) \Delta t,
\]

\[
\text{Var}[G(t_1) \mid a_0, S(t_0), S(t_1)] \approx \frac{1}{4} \left( \frac{S(t_0) - S(t_1)}{q(a_0) \Delta t} \right)^2 \left( e^{\sigma^2 \Delta t} - 1 \right).
\]

The conditional expectation in (24) is equal to our estimator \( \hat{G}(t) \). Figure 3 shows the distribution of the relative difference \( 1 - \hat{G}(t)/G(t) \) between the estimate and the true \( G(t) \), for the parameters used in Section 5. The relative estimator error when \( \sigma = 0.1 \) and \( \Delta t = 0.01 \) is typically within 1%.

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\]

\[
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\]

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Figure 3: Distribution of the relative error in estimating \( G(t) \) with \( \sigma = 0.1 \) and \( \Delta t = 0.01 \), based on 100,000 samples.

### 4 Solution to the deterministic system

In this section, we provide solutions to the pricing problem in the deterministic case, \( \sigma = 0 \), for families of linear and exponential demand functions \( q_l(a) \) and \( q_e(a) \) respectively.

\[
q_l(a) = q_1 - q_2 a, \quad \text{for } q_1, q_2 > 0,
\]

\[
q_e(a) = q_1 e^{-q_2 a}, \quad \text{for } q_1, q_2 > 0.
\]

These demand functions are often used in the literature. For a discussion about their properties and usage in modelling demand, see (Talluri & van Ryzin 2006, Ch. 7). With both demand functions, the optimal pricing function policy is to hold the price constant over the pricing horizon. They translate to the following intuition: For small amounts of stock, relative to demand, sell at the highest price so that all stock is depleted by the terminal time. As the amount of stock increases, the retailer should decrease the price, until it arrives at the price that maximizes the profits balancing revenue and cost of unsold stock at the terminal time. The ansatz pricing policies referred to below come from the discrete-time pricing problem considered by Riseth et al. (2017).
4.1 Linear demand function

When \( q(a) = q_1 - q_2a \) for \( q_1, q_2 > 0 \), the maximization in the HJB equation can be solved in closed form. To reduce the number of parameters, we can rescale the price per product with \( q_2 \) by setting \( q_2a \) to \( a \). Then \( q(a) = q_1 - a \), and we set the pricing interval to \( A = [0, q_1] \) so that the demand function is non-negative. Now, use the ansatz that \( a^D(t, s, g) \) sets the price so that \( \alpha^D(t) \) is constant for the remainder of the pricing period. From the expression of the value function in (15), the ansatz pricing policy must also be given by

\[
a^D(t, s, g) = \arg \max_{a \in A} \left\{ \int_t^1 aq(a)g \, du - C \left( s - \int_t^1 q(a)g \, du \right) \right\} \quad (28)
\]

\[
= \arg \max_{a \in A} \{(1 - t)(a + C)q(a)g - Cs \mid s \geq (1 - t)q(a)g\} . \quad (29)
\]

The maximizer therefore satisfies the equality constraints that \( s \) equals \( (1 - t)gg(a) \), is zero, or is in the interior of the feasible set, given by \( A \). For \( q(a) = q_1 - a \), we can verify that this implies

\[
a^D(t, s, g) = \begin{cases} 
q_1 - \frac{D}{(1-t)g}, & \text{if } 0 \leq s \leq (1-t)g \min\{q_1, \frac{1}{2}(q_1 + C)\}, \\
\max(0, q_1 - C)/2, & \text{otherwise}.
\end{cases} \quad (30)
\]

It follows that \( T_h = 1 \) when following the optimal pricing strategy. This is an intuitive result, because if \( T_h < 1 \), one can increase the price until \( T_h = 1 \) and \( S^\alpha(1) = 0 \), which earns extra revenue at no extra cost. The pricing policy suggested by the deterministic assumption provides the following, obvious, heuristic: First, find the price that maximizes profits, ignoring inventory constraints. Second, if sales forecasts suggest you will deplete stock before the end of the time horizon at this price, increase it accordingly. This is consistent with the heuristic proposed by Schlosser (2015b), Schlosser (2015a), which he finds by considering a deterministic continuum approximation to a Poisson process demand model.

**Example 4.1.** To demonstrate what form the pricing function may take, Figure 4 shows a plot of \( a^D(t, s, 1.0) \) for a given set of parameters. We have chosen the parameters so that the pricing problem starts at the most interesting point: at the kink separating the regimes where all the stock is sold out and where it is not. This corresponds to a combination of parameters such that \( \min(q_1, \frac{1}{2}(q_1 + C)) = 1 \).

![Figure 4: Example optimal, deterministic, pricing function \( a^D(t, s, 1.0) \) from (30), as a function of time and stock. Notice how sensitive \( a^D(t, s, 1.0) \) is to changes in \( s \), for large \( t \). The parameters used are \( q_1 = 3/2 \), and \( C = 1/2 \). In Figure 5 we compare this function to the optimal price when \( \sigma = 0.1 \).](image-url)
To verify that \( a^D (t, s, g) \) given by (30) indeed is the solution to the deterministic pricing problem, we show that it satisfies the HJB equation. On the interior of the \((t, s, g)\)-domain, \( a^D (t, s, g) \) must solve

\[
\max_{a \in A} \{ (a - v_s(t, s, g))(q_1 - a) \}. \tag{31}
\]

Let \( P_A \) denote the projection of the real line to \( A \). The objective is concave, and hence the maximizer is

\[
a^D (t, s, g) = P_A \left[ \frac{q_1 + v_s}{2} \right]. \tag{32}
\]

Define \( \Gamma \) to be the boundary of the terminal time problem, that is when \( t = 1, s = 0, \) or \( g = 0 \). Let us assume that \( C < q_1 \), then the deterministic-case HJB equation for \( v \) is

\[
v_t(t, s, g) + \frac{q_1}{4} (q_1 - v_s(t, s, g))^2 = 0, \quad (t, s, g) \in (0, 1)^2 \times \mathbb{R}_{>0}, \tag{33}
\]

\[
v(t, s, g) = -Cs, \quad (t, s, g) \in \Gamma. \tag{34}
\]

If \( C \geq q_1 \), then there are regions where \( a^D (t, s, g) = 0 \). In particular, this means that for \( t, s, g \) such that \( s > (1-t)gq_1 \), \( v \) must satisfy

\[
v_t(t, s, g) - gq_1 v_s(t, s, g) = 0. \tag{35}
\]

From the two expressions for \( a^D (t, s, g) \) in (30) and (32), the ansatz implies that the value function must satisfy

\[
v^D (t, s, g) = \begin{cases} 
q_1 s - \frac{s^2}{(1-t)g}, & \text{if } 0 \leq s \leq (1-t)g \min\{q_1, \frac{1}{2}(q_1 + C)\}, \\
-Cs + V(C)(1-t)g, & \text{otherwise.}
\end{cases} \tag{36}
\]

\[
V(C) = \begin{cases} 
\frac{1}{2}(q_1 + C)^2, & C < q_1, \\
q_1 C, & C \geq q_1.
\end{cases} \tag{37}
\]

This \( v \) does indeed satisfy the deterministic HJB equation given by (33)-(35), and hence we can conclude that \( a^D (t, s, g) \) is an optimal pricing function. Note that \( v \) is not smooth for all parameter combinations, and is therefore considered a solution in the viscosity sense as noted in Remark 1.

### 4.2 Exponential demand function

With the same ansatz that was used for the linear demand function, we can also find the optimal, deterministic, pricing function for exponential demand \( q(a) = q_1 e^{-a} \). Indeed, this is true for any demand function for which a closed form solution exists for \( \max_{a \in A} \{ (a + C)q(a) \} \) and \( (1-t)qq(a) = a \).

As with the linear demand, we can eliminate the parameter \( q_2 \) so that \( q(a) = q_1 e^{-a} \), given by replacing \( q_2 a \) by \( a \). In the exponential demand case, with \( A = [0, \infty) \), the ansatz gives us the optimal pricing function

\[
a^D (t, s, g) = \begin{cases} 
\log \frac{q_1 e^a}{s}, & \text{if } 0 \leq \frac{s}{q_1 e^{q_1(1-t)}} \leq e^{C-1}, \\
\max(0, 1 - C), & \text{otherwise.}
\end{cases} \tag{38}
\]

For completeness, we state the HJB equation for the exponential demand case when \( C < 1 \), and provide the solution so that the optimality of (38) can be verified. The maximizer of \( \max_{a \in A} \{ (a - v_s)q(a) \} \) in the HJB equation is \( a^D = 1 + v_s \). Thus, the value function must satisfy the HJB equation

\[
v_t(t, s, g) + gq_1 e^{-v_s(t, s, g)} = 0, \quad (t, s, g) \in (0, 1)^2 \times \mathbb{R}_{>0}, \tag{39}
\]

\[
v(t, s, g) = -Cs, \quad (t, s, g) \in \Gamma. \tag{40}
\]

The viscosity solution of the HJB equation, acquired from the ansatz \( a^D (t, s, g) \) in (38), is

\[
v^D (t, s, g) = \begin{cases} 
s \log \frac{q_1 e^a}{s}, & \text{if } 0 \leq \frac{s}{q_1 e^{q_1(1-t)}} \leq e^{C-1}, \\
-Cs + gV(C)(1-t), & \text{otherwise.}
\end{cases} \tag{41}
\]

\[
V(C) = q_1 e^{C-1}. \tag{42}
\]
5 Impact of uncertainty

We now discuss to what degree multiplicative uncertainty changes our policy. In this section, we solve the HJB equation numerically with the linear demand function \( q(a) = q_1 - a \) defined on \( A = [0, q_1] \), and compare the resulting pricing policy to the deterministic-system policy from the previous section. With the diffusion term in the HJB equation, one can expect the kink in the deterministic pricing function to smooth out. It turns out that the difference between an optimal policy and a heuristic policy based on the solution to the deterministic system is at most \( \mathcal{O}(\sigma \sqrt{T-t}) \). Further, numerical tests indicate that the closed-form pricing functions found in the previous section perform sufficiently well in most situations.

We assume in the following that \( (q_1 + v_s) \in [0, 2] \) for \( t \in [0, 1], s, g > 0 \), so that the pricing function satisfies \( a(t, s, g) = (q_1 + v_s(t, s, g))/2 \) as given by [32]. Using (18) and (33), it then follows that \( v \) should satisfy

\[
v_t(t, s, g) + \frac{\sigma^2}{2} v_{yy}(t, s, g) + \frac{g}{4} (q_1 - v_s)^2 = 0, \quad (t, s, g) \in (0, 1)^2 \times \mathbb{R}_{>0},
\]

\[
v(t, s, g) = -Cs, \quad (t, s, g) \in \Gamma.
\]

The numerical solution to the HJB equation is solved with the following procedure: (i) Reformulate the PDE with the similarity transformation \( \xi = s/g \) and \( v = g \phi \). (ii) Truncate the boundary for \( \xi \rightarrow \infty \) and set an asymptotic Dirichlet boundary condition based on the deterministic-system solution. (iii) Approximate the PDE for \( \phi(t, \xi) \) with central finite differences and the Tsit5 time stepping procedure in DifferentialEquations.jl [Rackauckas & Nie 2017], implemented in the Julia programming language [Bezanson et al. 2017]. We denote the computed pricing function and pricing policy by \( a^D(t, s, g) \) and \( a^D(t) \) respectively.

**Example 5.1.** Let us consider the particular example system used in Example 4.1. That is, a linear demand function \( q(a) = q_1 - a \), with \( q_1 = 3/2 \) and \( C = 1/2 \). We set the volatility level of \( G(t) \) to \( \sigma = 0.1 \), which corresponds to a true demand near the terminal time within 20% of the expected demand \( q(a) \), with probability 0.95. Figure 5 shows the optimal pricing function for \( g = 1 \), and a plot of the difference \( a^B(t, s, g) - a^D(t, s, g) \). The only visible difference is along the kink line, \( s = g(1-t) \), where \( a^B \) smooths out the transition between the two regions, and hence sells the product at a slightly higher price.

![Figure 5: The optimal pricing function \( a^B(t, s, g) \), left) smooths out the kink as compared to the deterministic heuristic \( a^D(t, s, g) \), Figure 4.](image)

The right plot shows the impact of uncertainty on the optimal pricing function: (i) When we do not expect to sell out of the product, the prices are the same. (ii) When we expect to sell out of the product, the deterministic heuristic takes a slightly larger price. (iii) In the transition between the two regions, uncertainty increases the optimal price.
The difference between \( a^B \) and \( a^D \) is of order \( \sigma \sqrt{1 - t} \) at the kink, and \( -\sigma^2 \xi / 2 \) as we move to the left in the plots. This example uses \( q_1 = 3/2, C = 1/2, \) and \( \sigma = 0.2. \)

Figure 6 indicates that one can do an asymptotic analysis of the impact of \( 0 < \sigma \ll 1 \) on the pricing function. The appendix provides details of the analysis. We summarise the results here, and refer to Figure 6 for a visualisation of the differences between \( a^B \) and \( a^D \).

**Result 1.** Define \( \beta = \min(q_1, (q_1 + C)/2) \). There is an inner layer around the surface \( s/g = (1 - t)\beta \) which smooths out the kink of the solution \( a^D(t, s, g) \) that arises when \( \sigma = 0. \) At the kink, the first-order correction in the pricing function is of order \( \mathcal{O}(\sigma \sqrt{1 - t}) \). The width of the layer is of the order \( \mathcal{O}(\sigma(1 - t)^{3/2}) \), which connects the two pricing regimes identified in the deterministic case. As we move from the inner layer to larger values of \( \xi := s/g > (1 - t)\beta \), the solution tends to the value \( \delta = \max(0, q_1 - C)/2 \), which coincides with \( a^D \). As we move from the inner layer to smaller values of \( \xi \), the leading order solution of the inner layer coincides with \( a^D \). Further, there is a second order correction in the region \( \xi < (1 - t)\beta \) equal to \( -\sigma^2 \xi / 2 \).

The first and second order corrections reflect the insights one can arrive at when taking into account the deviations of actual future demand from expected demand. At the interface where we expect to sell all the inventory at the optimal lower-bound, “infinite-inventory” price \( \delta \), taking into account the possibility of higher future demand means that we can increase the price. When the inventory is so low that we expect to sell it all at some price higher than \( \delta \), taking into account the possibility of lower future demand means that we should price the product lower than in the deterministic-case in order to reduce the probability of having excess inventory at the terminal time.

The asymptotic results together with Figure 6 indicate that one can expect a price decrease over time when following the optimal pricing policy \( a^B(t) \). The numerical investigation in the next example verifies this, but highlights that there are negligible gains in total profit from pricing according to \( a^B(t) \) rather than the deterministic-case pricing policy \( a^D(t) \).

**Example 5.2.** Let us simulate the system from Example 5.1 with the numerical approximation to \( a^B(t) \) and compare it to \( a^D(t) \). We draw 100,000 sample paths from the underlying Brownian motion \( W(t) \), and set the price to be constant on intervals of size \( \Delta t = 0.01 \) using a policy \( \alpha(t) \). We use the estimator described in Section 3.1 as an approximation to \( G(t) \). The simulations are run with both \( a^B(t) \) and \( a^D(t) \), and statistics of their paths are shown in Figure 6. The prices \( a^B(t) \) start slightly higher than \( a^D(t) \), but will then over time decrease, on average, towards the lower bound \( \delta = 1/2 \). We also see that the deterministic heuristic is less anticipative, and will begin increasing the prices compared to the optimal policy after \( t = 0.2 \).

The measure of interest, however, is how much profit the different policies make. Recall that the
Evolution of quantile levels 0.05, 0.5, 0.95

From top to bottom, the optimal policy, the deterministic heuristic, and their difference. The optimal policy starts slightly higher, and decreases over time. Note that the 0.05 and 0.5 quantiles lie on top of each other for $\alpha^D(t)$, so prices are likely to stay constant at $1/2$.

The profit of following a policy $\alpha(t)$ is the random variable

$$P(\alpha) = \int_0^{T_h} q(\alpha(u))\alpha(u)G(u) \, du - CS^\alpha(1).$$

From simulations we estimate the distribution of $P(\alpha)$ for the optimal and deterministic policies, and compare their performance. The improvement is negligible, as we see from the relative statistics

$$\mathbb{E} \left[ 1 - \frac{P(\alpha^D)}{P(\alpha^B)} \right] \approx 2 \times 10^{-4}, \quad \text{std} \left[ 1 - \frac{P(\alpha^D)}{P(\alpha^B)} \right] \approx 6 \times 10^{-4},$$

$$\text{Median} \left[ 1 - \frac{P(\alpha^D)}{P(\alpha^B)} \right] \approx -1 \times 10^{-4}.$$

The calculated optimal pricing policy results in 0.02% higher profits than the heuristic, on average. It even results in lower profits than the heuristic more than 50% of the realizations. Figure 8 shows a histogram that approximates the distribution of the relative loss from using $\alpha^D(t)$ over the optimal pricing policy. The differences between the two are small, but the distribution is non-symmetric: The heuristic $\alpha^D(t)$ results in slightly larger profit for more than half of the realizations, at the expense of performing worse for the remaining realizations.

In Example 5.2 and Figure 8, it appears that the relative improvement in profits by pricing according to $\alpha^B(t)$, rather than the heuristic policy $\alpha^D(t)$, is negligible. Further numerical experiments for other values of $\sigma$ strengthen these results. See Table 1 for summary statistics of the relative difference in profits $1 - P(\alpha^D)/P(\alpha^B)$. The relative improvement of following the strategy $\alpha^B(t)$ increases with $\sigma$, however the standard deviations are all on the order of 0.01% to 0.1%.

6 Extensions of the pricing problem

For our one-product system with multiplicative parameter dynamics with uncertainty, two natural extensions to the pricing problem are: (i) To incorporate other forms of uncertainty, and (ii) formulate
Figure 8: The histogram shows the distribution of the relative profit by using the deterministic pricing heuristic $\alpha^D(t)$ to using the optimal policy $\alpha^B(t)$, as described in Example 5.2. Positive values correspond to realizations of $W(t)$ where $\alpha^B(t)$ is better. The shape of the distribution is similar to the discrete-time pricing problem studied by Riseth et al. (2017).

Table 1: Summary statistics of the relative profit difference $1 - P(\alpha^D)/P(\alpha^B)$ from the model in Example 5.2 with different levels of uncertainty $\sigma$. The headings $Q_z$ denote the $z$-quantile of the distribution.

| $\sigma$ | mean | std  | $Q_{0.05}$ | $Q_{0.5}$ | $Q_{0.95}$ |
|----------|------|------|------------|----------|------------|
| 0.05     | 0.0  | 0.4  | -0.5       | -0.2     | 0.9        |
| 0.1      | 0.2  | 0.6  | -0.4       | -0.1     | 1.4        |
| 0.2      | 0.3  | 0.7  | -0.3       | 0.0      | 1.8        |
| 0.4      | 0.4  | 1.2  | -0.3       | 0.1      | 1.8        |

6.1 Other forms of uncertainty

In the prior sections, the source of randomness in the system has come from the multiplicative term $G(t)$, modelled as a geometric Brownian motion martingale. A different non-negative stochastic process may be more appropriate, and the choice of dynamics can be guided by existing sales data. More generally, the demand function $q(a)$ depends on multiple parameters that exhibit different levels of uncertainty and dynamics in time. Another parameter in the pricing problem is the unit cost $C$, where its value at the terminal time may depend on factors unknown at times $t < 1$. Let $\theta(t) \in \mathbb{R}^n$ denote the vector of parameters that are relevant to the problem, and say the demand function and the unit cost depend explicitly on $\theta$. We write $q(a; \theta)$ and $C(\theta)$ for this dependence. Within the diffusion based stochastic framework, we can model the dynamics of $\theta(t)$ with functions $b(t, \theta) \in \mathbb{R}^n$, $\sigma(t, \theta) \in \mathbb{R}^{n \times p}$, and a vector-valued, uncorrelated, Brownian motion $W(t) \in \mathbb{R}^p$. We assume that $\theta(t)$ does not depend on the pricing policy $\alpha(t)$ or the remaining stock $S^\alpha(t)$. Thus, the system for the pricing problem is described
by the SDE,
\[
\begin{align*}
    d\theta(t) &= b(t, \theta(t)) \, dt + \sigma(t, \theta(t)) \, dW(t), \\
    dS^\alpha(t) &= -q(\alpha(t); \theta(t)).
\end{align*}
\]

Let \( \Theta \subset \mathbb{R}^n \) denote the state space for \( \theta(t) \). Let \( D_{\theta}v \) and \( D^2_{\theta}v \) denote the gradient and Hessian of \( v(t, s, \theta) \) with respect to \( \theta \). We denote the transpose operator with a superscript asterisk, and introduce the volatility matrix \( \Sigma(t, \theta) = \sigma(t, \theta)\sigma(t, \theta)^T \). The HJB approach for the pricing problem is then to find \( v : [0,1]^2 \times \Theta \rightarrow \mathbb{R} \) which satisfies
\[
\begin{align*}
    v_t + b(t, \theta)^* D_{\theta}v + \frac{1}{2} \text{tr} \left( \Sigma(t, \theta) D^2_{\theta}v \right) + \max_a \{ q(a; \theta)(a - v_s) \} &= 0, \\
    v(1, s, \theta) &= -C(\theta)s.
\end{align*}
\]

Additional boundary conditions may be necessary, depending on \( \theta(t) \). For example, the boundary condition for \( q = 0 \) in our multiplicative model from the previous sections. The value function and the pricing function now depend explicitly on each element in \( \theta \), thus increasing the dimension of the corresponding HJB equation. Efficient algorithms for solving high-dimensional PDEs of this form exist. See, for example, the multigrid preconditioning approach by [Reisinger & Arto 2017], or the splitting into a sequence of lower-dimensional PDEs by [Reisinger & Wissmann 2018]. In higher dimensions, it is of even greater importance to critically balance computational cost with the suboptimality of approximations. As Section 5 shows, approximate policies can perform well. This underscores the importance of assessing whether new sources of randomness in the model have significant impact on the objective and the optimal policy.

### 6.2 Expected utility risk aversion

One may argue that for a retailer as a whole, an assumption of a risk-neutral decision maker is valid. For individual product managers, whose performance is evaluated over shorter time horizons, a degree of risk aversion can be preferable from their point of view. Formulations and investigations of the impact of risk aversion on pricing policies is also noted as an interesting area of research by [Bitran & Caldentey 2003]. For investigations of the expected utility problem with Poisson process demand, see, for example, [Lim & Shanthikumar 2007], [Feng & Xiao 2008].

In this section, we assume that the decision maker is evaluated based on the performance of the total profit from selling a product over the time interval \([0, 1]\). Then, the pricing decision at time \( t \) may also depend on how much revenue has been accrued at that time. So we introduce a state variable \( R^a(t) \) representing the accrued revenue at time \( t \), with dynamics \( dR^a(t) = \alpha(t)q(\alpha(t); \theta(t)) \, dt \). We consider risk aversion based on an expected utility-maximizing decision maker. Given a utility function \( U(x) \), the pricing problem is then to find a pricing policy \( \alpha \) that maximizes the expected utility \( \mathbb{E}[U(R^a(1) - C(\theta(1))S^\alpha(1))] \).

For this utility function, the value function is defined as
\[
v(t, s, \theta, r) = \max_{\alpha \in \mathcal{A}} \mathbb{E}_t \left[ U \left( R^\alpha(1) - C(\theta(1))S^\alpha(1) \right) \right].
\]

The corresponding HJB problem is then to find \( v : [0,1]^2 \times \Theta \times [0, \infty) \rightarrow \mathbb{R} \) that solves
\[
\begin{align*}
    v_t + b(t, \theta)^* D_{\theta}v + \frac{1}{2} \text{tr} \left( \Sigma(t, \theta) D^2_{\theta}v \right) + \max_a \{ q(a; \theta)(av_r - v_s) \} &= 0, \\
    v(1, s, \theta, r) &= U(r - C(\theta)s),
\end{align*}
\]

plus additional boundary conditions. The impact of the risk aversion on pricing decisions appears in the maximization term in the HJB equation, \( \max_a \{ q(a; \theta)(av_r - v_s) \} \). The \( v_r \) term represents the relative importance of accruing more revenue to the utility of selling more stock, as represented by the \( v_s \) term.

### 7 Conclusion

This article focuses on continuum approximations for dynamic pricing problems under uncertainty. Most of the existing literature on continuous time dynamic pricing for revenue management is based on Poisson
processes. This is suitable for many applications, however for large retailers, approximating the number of sales and stock as a continuum can simplify calculation of pricing rules. We present an approach for modelling the sales as a continuous time dynamical system, where the uncertainty in demand arises from stochastic processes. An advantage over the Poisson process model is that this approach allows us to directly model the demand volatility. Under this model, we consider a pricing problem where the retailer aims to deplete inventory of a product at maximum profit. By formulating the problem as a stochastic optimal control problem, we can express the optimal pricing policy in terms of the solution to a nonlinear PDE. For linear and exponential demand functions, we find closed-form expressions for the pricing policy when the system is deterministic. It turns out that, for a risk-neutral decision maker, the deterministic pricing policy is a near-optimal heuristic for systems with demand uncertainty. Numerical errors in calculating the optimal pricing policy may in fact result in lower profits on average than with the heuristic pricing policy.

There are two topics of particular interest for future study. The first is to understand why demand uncertainty has such a small effect on the optimal pricing policy for risk-neutral decision makers, and whether constraints such as requiring monotone-in-time pricing policies may increase this effect. Second, a case study of the continuum model framework for multiple products and time-dependent demand is needed, in order to understand how well this approach can scale to revenue management implementations for retailers.

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Appendix: Asymptotic analysis

We carry out the asymptotic analysis of the pricing function that arises from the linear demand HJB equation (43) under the assumptions of Section 5. Further, we assume that the value and pricing functions are sufficiently differentiable to carry out the operations below. The smoothing effect of the diffusion in the PDEs when $\sigma > 0$ justifies this assumption. Result 1 in Section 5 discusses the interpretation of the following results.

Let us first reduce the dimension of the problem with a similarity transform working in reverse time, and then present a PDE satisfied by the pricing function. Consider the transformations $\xi(s,g) = s/g$, $\tau(t) = 1 - t$, and $v(t,s,g) = g\phi(t,\xi(s,g))$, which satisfies

$$\phi_\tau = \frac{1}{2}\sigma^2 \xi^2 \phi_{\xi\xi} + \frac{1}{4}(q_1 - \phi_\xi)^2$$

(55)

$$\phi(0,\xi) = -C\xi$$

(56)

$$\phi(\tau,0) = 0,$$

(57)

$$\phi_\xi(\tau,\infty) = -C.$$  

(58)

In the new coordinates, define the pricing function in terms of the function $\psi(\tau,\xi)$ so that $a^B(t,s,g) = \psi(\tau(t),\xi(s,g))$. Then $\psi = (q_1 + \phi_\xi)/2$ satisfies the following PDE that arises from differentiating (55)
with respect to $\xi$,

$$
\psi = \frac{1}{2} \sigma^2 \xi^2 \psi_{\xi\xi} + (\sigma^2 \xi - q_1 + \psi) \psi, \quad (59)
$$

$$
\psi(\tau, 0) = q_1, \quad (60)
$$

$$
\psi(\tau, \infty) = \delta := \max(0, (q_1 - C)/2). \quad (61)
$$

When $\sigma = 0$, we know from Section 4 that the viscosity solution $\psi^{(0)}$ is

$$
\psi^{(0)}(\tau, \xi) = \begin{cases} 
q_1 - \xi/\tau, & \text{if } 0 \leq \xi \leq \beta \tau, \\
\delta, & \text{if } \xi > \beta \tau,
\end{cases} \quad \text{where } \beta := \min(q_1, (q_1 + C)/2). \quad (62)
$$

This leads us to consider an inner layer asymptotic analysis near the kink $\xi = \beta \tau$. Zoom in near the kink using the coordinates $x(\tau, \xi) = (\xi - \beta \tau)/\sigma$ and write $\psi(\tau, \xi) = \sigma u(\tau, x(\tau, \xi)) + \delta$. In the new coordinates (59) becomes

$$
\tau \frac{\partial u}{\partial \tau} = \frac{1}{2}(\beta \tau)^2 u_{xx} + \sigma(\sigma x + \beta \tau) u_x + uu_x, \quad \tau > 0, \ x > -\beta \tau/\sigma, \quad (63)
$$

with matching conditions $\lim_{x \to -\infty} u_x(\tau, x) = -1/\tau$ and $\lim_{x \to \infty} u(\tau, x) = 0$. The leading order equation for $u = u^{(0)} + \sigma u^{(1)} + \sigma^2 u^{(2)} + \cdots$ is therefore

$$
\tau \frac{\partial u^{(0)}}{\partial \tau} = \frac{1}{2}(\beta \tau)^2 u_{xx}^{(0)} + u^{(0)} u_x^{(0)}. \quad (64)
$$

This equation has a similarity solution using the transformations $\eta(\tau, x) = x/\tau^{3/2}$ and $u^{(0)}(\tau, x) = \tau^{1/2} f(\eta(\tau, x))$. The function $f(\eta)$ must satisfy the boundary-value ODE

$$
\beta^2 f'' + (3\eta + 2)f' - f = 0, \quad (65)
$$

$$
f \to 0 \quad \text{as } \eta \to \infty, \quad (66)
$$

$$
f' \to -1 \quad \text{as } \eta \to -\infty. \quad (67)
$$

Figure 9 shows numerically computed solutions to the ODE for different values of $\beta$.

![Figure 9: Numerically computed solutions of (65). Note the similarity to the optimal pricing function curves zoomed in near the kink in Figure 6.](image)

Figure 9: Numerically computed solutions of (65). Note the similarity to the optimal pricing function curves zoomed in near the kink in Figure 6.

We briefly note that an asymptotic expansion of $\phi = \phi^{(0)} + \sigma \phi^{(1)} + \sigma^2 \phi^{(2)} + \cdots$ in the outer region $\xi < \beta t$ reveals that $\phi^{(1)} = 0$ and $\phi^{(2)}(t, \xi) = -\xi^2/2$. It follows that $\psi(t, \xi) \approx \psi^{(0)}(t, \xi) - \sigma^2 \xi^2/2$ in this region.

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