A note on the \((h, q)\)-zeta-type function with weight \(\alpha\)

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Abstract

The objective of this paper is to derive the symmetric property of an \((h, q)\)-zeta function with weight \(\alpha\). By using this property, we give some interesting identities for \((h, q)\)-Genocchi polynomials with weight \(\alpha\). As a result, our applications possess a number of interesting properties which we state in this paper.

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1 Introduction

Recently, Kim has developed a new method by using the \(q\)-Volkenborn integral (or \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\)) and has added weight to \(q\)-Bernoulli numbers and polynomials and investigated their interesting properties (see [1]). He also showed that these polynomials are closely related to weighted \(q\)-Bernstein polynomials and derived novel properties of \(q\)-Bernoulli numbers with weight \(\alpha\) by using the symmetric property of weighted \(q\)-Bernstein polynomials with the help of the \(q\)-Volkenborn integral (for more details, see [2]). Afterward, Araci et al. have introduced weighted \((h, q)\)-Genocchi polynomials and defined \((h, q)\)-zeta-type function with weight \(\alpha\) by applying the Mellin transformation to the generating function of the \((h, q)\)-Genocchi polynomials with weight \(\alpha\) which interpolates for \((h, q)\)-Genocchi polynomials with weight \(\alpha\) at negative integers (for details, see [3]). In this paper, we also consider a \((h, q)\)-zeta-type function with weight \(\alpha\) and derive some interesting properties.

We firstly list some notations as follows.

Imagine that \(p\) is a fixed odd prime. Throughout this work, \(\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p\) and \(\mathbb{C}_p\) will denote by the ring of integers, the field of \(p\)-adic rational numbers and the completion of the algebraic closure of \(\mathbb{Q}_p\), respectively. Also, we denote \(\mathbb{N}^* = \mathbb{N} \cup \{0\}\) and \(\exp(x) = e^x\). Let \(v_p : \mathbb{C}_p \rightarrow \mathbb{Q} \cup \{\infty\}\) (\(\mathbb{Q}\) is the field of rational numbers) denote the \(p\)-adic valuation of \(\mathbb{C}_p\) normalized so that \(v_p(p) = 1\). The absolute value on \(\mathbb{C}_p\) will be denoted as \(|\cdot|\), and \(|x|_p = p^{-v_p(x)}\) for \(x \in \mathbb{C}_p\). When one speaks of \(q\)-extensions, \(q\) is considered in many ways, e.g., as an indeterminate, a complex number \(q \in \mathbb{C}\), or a \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}\), we assume that \(|q| < 1\). If \(q \in \mathbb{C}_p\), we assume \(|1 - q|_p < p^{-\frac{1}{p-1}}\) so that \(q^x = \exp(x \log q)\) for \(|x|_p \leq 1\). We use the following notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.
\] (1.1)
We want to note that \( \lim_{q \to 1} [x]_q = x; \) cf. [1–23].

For a fixed positive integer \( d \), set

\[
X = X_d = \lim_{n \to \infty} \mathbb{Z}/dp^n \mathbb{Z},
\]

\[
X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp^n \mathbb{Z}_p
\]

and

\[
a + dp^n \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^n} \},
\]

where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < dp^n \) (see [1–23]).

The following \( p \)-adic \( q \)-Haar distribution was defined by Kim:

\[
\mu_q(x + p^n \mathbb{Z}_p) = q^x \frac{[p^n]_q}{[p^n]}_q
\]

for any positive \( n \) (see [12, 13]).

Let \( UDF(\mathbb{Z}_p) \) be the set of uniformly differentiable functions on \( \mathbb{Z}_p \). We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) if the difference quotient

\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
\]

has a limit \( f'(a) \) as \( (x, y) \to (a, a) \) and denote this by \( f \in UDF(\mathbb{Z}_p) \). In [12] and [13], the \( p \)-adic \( q \)-integral of the function \( f \in UDF(\mathbb{Z}_p) \) is defined by Kim as follows:

\[
I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q(\xi) = \lim_{n \to \infty} \sum_{\xi = 0}^{p^n - 1} f(\xi) \mu_q(\xi + p^n \mathbb{Z}_p). \tag{1.2}
\]

The bosonic integral is considered as the bosonic limit \( q \to 1, I_1(f) = \lim_{q \to 1} I_q(f) \). Similarly, the \( p \)-adic fermionic integration on \( \mathbb{Z}_p \) is defined by Kim [8] as follows:

\[
I_{-q}(f) = \lim_{q \to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x). \tag{1.3}
\]

By using a fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \), \( (h, q) \)-Genocchi polynomials are defined by [3]

\[
\frac{G_{n+1}^{(h, q)}(x)}{n + 1} = \int_{\mathbb{Z}_p} q^{(h-1)\xi} [x + \xi]_q^n d\mu_{-q}(\xi)
\]

\[
= \lim_{n \to \infty} \frac{1}{[p^n]_{-q}} \sum_{\xi = 0}^{p^n - 1} (-1)^{\xi} [x + \xi]_q^n q^{\xi}. \tag{1.4}
\]
For \( x = 0 \) in (1.4), we have \( \widetilde{G}_{n,q}(0) : = \widetilde{G}_{n,q} \) are called \((h,q)\)-Genocchi numbers with weight \( \alpha \) which is defined by

\[
\widetilde{G}_{n,q}(\alpha,h) = 0, \quad \text{and} \quad q^h \frac{\widetilde{G}_{m+1}(\alpha,h)}{m+1} + \frac{\widetilde{G}_{m+1}(\alpha)}{m+1} = \begin{cases} 
[2]_q & \text{if } m = 0, \\
0 & \text{if } m \neq 0.
\end{cases}
\]

By (1.4), we have a distribution formula for \((h,q)\)-Genocchi polynomials, which is shown by [3]

\[
\widetilde{G}_{n+1,q}(\alpha,h)(x) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[m+x]_q^a}.
\]

By applying some elementary methods, we will give symmetric properties of weighted \((h,q)\)-Genocchi polynomials and a weighted \((h,q)\)-zeta-type function. Consequently, our applications seem to be interesting and worthwhile for further works of many mathematicians in analytic number theory.

\section*{2 On the \((h,q)\)-zeta-type function}

In this part, we firstly recall the \((h,q)\)-zeta-type function with weight \( \alpha \) which is derived in [3] as follows:

\[
\widetilde{\zeta}_{q}(\alpha,h)(s,x) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[m+x]_q^a},
\]

where \( q \in \mathbb{C}, h \in \mathbb{N} \) and \( \Re(s) > 1 \). It is clear that the special case \( h = 0 \) and \( q \to 1 \) in (2.1) reduces to the ordinary Hurwitz-Euler zeta function. Now, we consider (2.1) in the following form:

\[
\widetilde{\zeta}_{q}(\alpha,h)(s,bx + \frac{bj}{a}) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[m+bx+\frac{bj}{a}]_q^a}.
\]

By applying some basic operations to the above identity, that is, for any positive integers \( m \) and \( b \), there exist unique non-negative integers \( k \) and \( i \) such that \( m = bk + i \) with \( 0 \leq i \leq b - 1 \). For \( a \equiv 1(\text{mod} \ 2) \) and \( b \equiv 1(\text{mod} \ 2) \). Thus, we can compute as follows:

\[
\widetilde{\zeta}_{q}(\alpha,h)(s,bx + \frac{bj}{a}) = [a]_q^s [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[ma + abx + b]_q^a} = [a]_q^s [2]_q \sum_{m=0}^{\infty} \sum_{i=0}^{b-1} \frac{(-1)^{i+mb} q^{i+mb}ah}{[(i+mb)a + abx + bj]_q^a} = [a]_q^s [2]_q \sum_{i=0}^{b-1} \frac{(-1)^i q^{iab} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[ab(m+X) + a + bj]_q^a}}.
\]

\[(2.2)\]
From this, we can easily discover the following:

\[
\sum_{j=0}^{a-1} (-1)^j q^{jb} \zeta_q^{(a,h)}(s, bx + bj/a) = [a]_{q^a} [2]_{q^2} \sum_{j=0}^{a-1} (-1)^j q^{jb} \sum_{i=0}^{b-1} (-1)^i q^{ih} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbh}}{[ab(m + x) + ai + bj]_{q^3}}.
\]  

(2.3)

Replacing \(a\) by \(b\) and \(j\) by \(i\) in (2.2), we have the following:

\[
\tilde{\zeta}^{(a,h)}(s, ax + ai/b) = [b]_{q^b} [2]_{q^2} \sum_{j=0}^{a-1} (-1)^j q^{jb} \sum_{i=0}^{b-1} (-1)^i q^{ih} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbh}}{[ab(m + x) + ai + bj]_{q^3}}.
\]

By considering the above identity in (2.3), we can easily state the following theorem.

**Theorem 1** The following identity is true:

\[
\frac{[2]_{q^2}}{[a]_{q^a} [2]_{q^2}} \sum_{i=0}^{a-1} (-1)^i q^{ib} \tilde{\zeta}^{(a,h)}(s, bx + bi/a) = \frac{[2]_{q^2}}{[b]_{q^b} [2]_{q^2}} \sum_{i=0}^{b-1} (-1)^i q^{ih} \tilde{\zeta}^{(a,h)}(s, ax + ai/b).
\]

Now, setting \(b = 1\) in Theorem 1, we have the following distribution formula:

\[
\tilde{\zeta}^{(a,h)}(s, ax) = \frac{[2]_{q^2}}{[2]_{q^2} [a]_{q^a}} \sum_{i=0}^{a-1} (-1)^i q^{ib} \tilde{\zeta}^{(a,h)}(s, x + i/a).
\]  

(2.4)

Putting \(a = 2\) in (2.4) leads to the following corollary.

**Corollary 1** The following identity holds true:

\[
\tilde{\zeta}^{(a,h)}(s, 2x) = \frac{[2]_{q^2}}{[2]_{q^2} [2]_{q^2}} \left( \tilde{\zeta}^{(a,h)}(s, x) - q^x \tilde{\zeta}^{(a,h)}(s, x + 1/2) \right).
\]

Taking \(s = -m\) into Theorem 1, we have the symmetric property of \((h, q)\)-Genocchi polynomials by the following theorem.

**Theorem 2** The following identity is true:

\[
[2]_{q^2} [a]_{q^a} \sum_{j=0}^{m-1} (-1)^j q^{jb} \tilde{G}^{(a,h)}_{m,q^a}(bx + bi/a) = [2]_{q^2} [b]_{q^b} \sum_{i=0}^{b-1} (-1)^i q^{ih} \tilde{G}^{(a,h)}_{m,q^b}(ax + ai/b).
\]

Now also, setting \(b = 1\) and replacing \(x\) by \(x/a\) in the above theorem, we can rewrite the following \((h, q)\)-Genocchi polynomials with weight \(a\):

\[
\tilde{G}^{(a,h)}_{n,q^a}(x) = \frac{[2]_{q^2}}{[2]_{q^2} [a]_{q^a}} \sum_{i=0}^{a-1} (-1)^i q^{ib} \tilde{G}^{(a,h)}_{a,q^a}(x + i/a).
\]  

(2 a).
Due to Araci et al. [3], we develop as follows:

\[
\sum_{n=0}^{\infty} \binom{n}{m} (x+y)^m t_m = [2] q^t \sum_{m=0}^{\infty} (-1)^m q^{\binom{m}{2}} e^{x-y+mt}.
\]

\[
= [2] q^t \sum_{m=0}^{\infty} (-1)^m q^{\binom{m}{2}} e^{\binom{x+y}{m}} t_m = \left( \sum_{n=0}^{\infty} \binom{n}{m} t^n / n! \right) \left( \sum_{n=0}^{\infty} q^{\binom{n}{m}} \sum_{k=0}^{n} \binom{n}{k} t^k / k! \right).
\]

By using the Cauchy product, we see that

\[
\sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} q^{j(j-1)} \binom{x+y}{j} \right) t^n / n!.
\]

Thus, by comparing the coefficients of \( \frac{t^n}{n!} \), we state the following corollary.

**Corollary 2**  The following equality holds true:

\[
Q_{a,b}(x+y) = \sum_{j=0}^{n} \binom{n}{j} q^{j(j-1)} \binom{x+y}{j} \frac{t^n}{n!}.
\] (2.5)

By using Theorem 2 and (2.5), we readily derive the following symmetric relation after some applications.

**Theorem 3**  The following equality holds true:

\[
[2] q^t \sum_{m=0}^{\infty} \binom{m}{i} \left( \frac{a}{b} \right)^{m-i} \binom{x+y}{m-i} \binom{m}{i} \left( \frac{b}{a} \right)^{m-i} \binom{x+y}{m-i} \sum_{j=0}^{n} \binom{n}{j} q^{j(j-1)} \binom{x+y}{j} \frac{t^n}{n!}.
\]

where \( \binom{x+y}{j} \) = \( \sum_{i=0}^{n} (-1)^i q^{i(i-1)} / i! \).

When \( q \rightarrow 1 \) into Theorem 3, it leads to the following corollary.

**Corollary 3**  The following identity holds true:

\[
\sum_{i=0}^{m} \binom{m}{i} a^{m-i} b^i \frac{G_i(bx)S_{m-i}(a)}{S_{m-i}(bx)G_i(a)}.
\]
where $S_m(a) = \sum_{n=0}^{a-1} (-1)^n f^n$ and $G_n(x)$ are called the ordinary Genocchi polynomials which are defined via the following generating function:

$$
\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1}. 
$$

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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