CONVERGENCE OF THE FULL COMPRESSIBLE NAVIER-STOKES-MAXWELL SYSTEM TO THE INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS IN A BOUNDED DOMAIN

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Abstract. In this paper we establish the uniform estimates of strong solutions with respect to the Mach number and the dielectric constant to the full compressible Navier-Stokes-Maxwell system in a bounded domain. Based on these uniform estimates, we obtain the convergence of the full compressible Navier-Stokes-Maxwell system to the incompressible magnetohydrodynamic equations for well-prepared data.

1. Introduction. In this paper we consider the singular limit of the following full compressible Navier-Stokes-Maxwell system in a bounded domain $\Omega \subset \mathbb{R}^3$ ([6]):

\begin{equation}
\partial_t \rho + \text{div} (\rho u) = 0,
\end{equation}

\begin{equation}
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \frac{1}{\epsilon^2} \nabla p - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u = (E + u \times b) \times b,
\end{equation}

\begin{equation}
\partial_t (\rho e) + \text{div} (\rho ue) + p \text{div} u - \text{div} (\kappa \nabla T) = \epsilon^2 \left( 2\mu |D(u)|^2 + \lambda (\text{div} u)^2 + (E + u \times b)^2 \right),
\end{equation}

\begin{equation}
\epsilon_2 \partial_t E - \text{rot} b + E + u \times b = 0,
\end{equation}

\begin{equation}
\partial_t b + \text{rot} E = 0, \quad \text{div} b = 0,
\end{equation}

where the unknowns $\rho, u, p, e, T, E,$ and $b$ stand for the density, velocity, pressure, internal energy, temperature, electric field, and magnetic field, respectively. The
physical constants $\mu$ and $\lambda$ are the shear viscosity and bulk viscosity coefficients of the flow and satisfy $\mu > 0$ and $\lambda + \frac{4}{3}\mu \geq 0$. $\kappa > 0$ is the heat conductivity. $\epsilon_1 > 0$ is the (scaled) Mach number, and $\epsilon_2 > 0$ is the (scaled) dielectric constant. $D(u) := \frac{1}{2}(\nabla u + \nabla u^T)$, where $\nabla u^T$ denotes the transpose of the matrix $\nabla u$.

In [9, 10], Kawashima and Shizuta established the global existence of smooth solutions for small data and studied its zero dielectric constant limit $\epsilon_2 \to 0$ to the system (1)-(5) and obtained the convergence of the system (1)-(5) to the incompressible Navier-Stokes-Maxwell system in the torus $\mathbb{T}^3$, see also [8] on the similar results to the inviscid case of (1)-(5). In [11], Li and Mu studied the low Mach number limit $\epsilon_1 \to 0$ to the system (1)-(5) and obtained the convergence of the system (1)-(5) to the incompressible Navier-Stokes-Maxwell system in the torus $\mathbb{T}^3$.

It should be pointed out that no boundary effect is considered in the references mentioned above. The purpose of this paper is to investigate the singular limits $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ to the system (1)-(5) in a bounded domain. For simplicity, we denote $\epsilon := (\epsilon_1, \epsilon_2)$ and consider the case that the fluid is a polytropic ideal gas, that is

$$e := C_V T, \quad p := \mathcal{R} \rho T$$

with $C_V > 0$ and $\mathcal{R}$ being the specific heat at constant volume and the generic gas constant, respectively.

To state the main result of this paper, we denote the density and temperature variations by $\sigma^\epsilon$ and $\theta^\epsilon$:

$$\rho^\epsilon := 1 + \epsilon_1 \sigma^\epsilon, \quad T^\epsilon := 1 + \epsilon_1 \theta^\epsilon.$$ (7)

Then we can rewrite the system (1)-(5) as follows:

$$\partial_t \sigma^\epsilon + \text{div} (\sigma^\epsilon u^\epsilon) + \frac{1}{\epsilon_1} \text{div} u^\epsilon = 0,$$ (8)

$$\rho^\epsilon(\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon) + \frac{\mathcal{R}}{\epsilon_1}(\nabla \sigma^\epsilon + \nabla \theta^\epsilon) + \mathcal{R} \nabla (\sigma^\epsilon \theta^\epsilon) - \mu \Delta u^\epsilon - (\lambda + \mu) \nabla \text{div} u^\epsilon = (E^\epsilon + u^\epsilon \times b^\epsilon) \times b^\epsilon,$$ (9)

$$C_V \rho^\epsilon(\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon) + \mathcal{R}(\rho^\epsilon \theta^\epsilon + \sigma^\epsilon) \text{div} u^\epsilon + \frac{\mathcal{R}}{\epsilon_1} \text{div} u^\epsilon = \kappa \Delta \theta^\epsilon + \epsilon_1 [2\mu |D(u^\epsilon)|^2 + \lambda (\text{div} u^\epsilon)^2 + (E^\epsilon + u^\epsilon \times b^\epsilon)^2],$$ (10)

$$\epsilon_2 \partial_t E^\epsilon - \text{rot} b^\epsilon + E^\epsilon + u^\epsilon \times b^\epsilon = 0,$$ (11)

$$\partial_t b^\epsilon + \text{rot} E^\epsilon = 0, \quad \text{div} b^\epsilon = 0.$$ (12)

Here we have added the superscript $\epsilon$ on the unknowns $(\sigma, u, \theta, E, b)$ to emphasise the dependence of $\epsilon$. The system (8)-(12) are supplemented with the following initial and boundary conditions:

$$(\sigma^\epsilon, u^\epsilon, \theta^\epsilon, E^\epsilon, b^\epsilon)(\cdot, 0) = (\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)(\cdot) \quad \text{in} \quad \Omega,$$ (13)

$$u^\epsilon \cdot n = 0, \quad \text{rot} u^\epsilon \times n = 0, \quad \frac{\partial \theta^\epsilon}{\partial n} = 0, \quad E^\epsilon \times n = 0, \quad b^\epsilon \cdot n = 0 \quad \text{on} \quad \partial \Omega,$$ (14)

where $n$ is the unit outer normal vector to the smooth boundary $\partial \Omega$.

Formally, if we let $\epsilon_1 \to 0$ in (8) and (9), then we obtain that $\text{div} u^\epsilon \to 0, \nabla \theta^\epsilon \to 0$, and $\nabla \sigma^\epsilon \to 0$. Letting $\epsilon_2 = 0$ in (11) gives $E^\epsilon = \text{rot} b^\epsilon - u^\epsilon \times b^\epsilon$. Pulling it into (12)
and taking the limit $\epsilon_2 \to 0$ we obtain the following limit system (suppose that the limits $(u^\epsilon, b^\epsilon) \to (u, B)$ as $\epsilon_1, \epsilon_2 \to 0$ exist):

\begin{align}
\begin{cases}
    v_t + v \cdot \nabla v + \nabla \pi - \mu \Delta v = \text{rot } B \times B, \\
    B_t + \text{rot } (B \times v) - \Delta B = 0, \\
    \text{div } v = 0, \\
    \text{div } B = 0.
\end{cases}
\end{align}

We shall give a rigorous proof of the above formal analysis below. Denote

\[ M^\epsilon(t) := \sup_{0 \leq s \leq t} \left\{ \| (\sigma^\epsilon, u^\epsilon, \theta^\epsilon, \sqrt{\epsilon^2 E^\epsilon}, b^\epsilon)(\cdot, s) \|_{H^2} + \| \partial_t (\sigma^\epsilon, u^\epsilon, \theta^\epsilon, \sqrt{\epsilon^2 E^\epsilon}, b^\epsilon)(\cdot, s) \|_{H^1} \\
    + \epsilon_1 \| \partial_t^2 (\sigma^\epsilon, u^\epsilon, \theta^\epsilon)(\cdot, s) \|_{L^2} + \left\| \frac{1}{1 + \epsilon_1 \sigma^\epsilon(\cdot, s)} \right\|_{L^\infty} \right\} \\
    + \left\{ \int_0^t \left( \| (u^\epsilon, \theta^\epsilon)(\cdot, s) \|_{H^1}^2 + \| \partial_t (u^\epsilon, \theta^\epsilon)(\cdot, s) \|_{H^1}^2 + \| \epsilon_1 \partial_t^2 (\sigma^\epsilon, u^\epsilon, \theta^\epsilon)(\cdot, s) \|_{H^1}^2 \\
    + \| E^\epsilon(\cdot, s) \|_{H^1}^2 + \| \partial_t (E^\epsilon, b^\epsilon)(\cdot, s) \|_{H^1}^2 \right) ds \right\}^{\frac{1}{2}}. \]

First, we have

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be a simply connected, bounded domain with smooth boundary $\partial \Omega$ and $0 < \epsilon < 1$. Suppose that the initial data $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)$ satisfy the following regularity conditions:

\[ 0 \leq \theta_0^\epsilon, \quad 0 < \frac{1}{K_0} \leq 1 + \epsilon_1 \sigma_0^\epsilon \leq K_0, \]

\[ \| (\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon) \|_{H^2} + \| \partial_t (\sigma^\epsilon, u^\epsilon, \theta^\epsilon, E^\epsilon, b^\epsilon)(\cdot, 0) \|_{H^1} \]

\[ + \epsilon_1 \| \partial_t^2 (\sigma^\epsilon, u^\epsilon, \theta^\epsilon)(\cdot, 0) \|_{L^2} \leq K_1 \]

for some positive constants $K_0 > 1$ and $K_1$ independent of $\epsilon > 0$. Then there exist a small time $\bar{T} > 0$ independent of $\epsilon > 0$ and a unique strong solution $(\sigma, u, \theta, E, b)$ to the initial boundary value problem (8)-(14) such that

\[ M^\epsilon(\bar{T}) \leq K \]

for some positive constant $K$ independent of $\epsilon > 0$.

**Remark 1.1.** In the assumption (18), $\sigma_0^\epsilon(\cdot, 0)$ is indeed defined by $-\text{div} (\sigma_0^\epsilon u_0^\epsilon) + \frac{1}{\epsilon} \text{div } u_0^\epsilon$ through the density equation and the other quantities are defined by an analogous way.

Based on the uniform estimates of the solutions, we can prove the following convergence result by applying the Arzelà-Ascoli theorem in a standard way.

**Theorem 1.2.** Let $(\sigma^\epsilon, u^\epsilon, \theta^\epsilon, E^\epsilon, b^\epsilon)$ be the solution of the problem (8)-(14) with the initial data $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)$ satisfying the conditions in Theorem 1.1. Assume further that the initial data $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)$ satisfy that

\[ (e \sigma_0^\epsilon, u_0^\epsilon, e \theta_0^\epsilon, b_0^\epsilon) \to (0, v_0, 0, B_0) \text{ strongly in } H^s \text{ for any } 0 \leq s < 2 \text{ as } \epsilon \to 0, \]

\[ E_0^\epsilon \to \text{rot } B_0 - v_0 \times B_0 \text{ strongly in } H^s \text{ for any } 0 \leq s < 1 \text{ as } \epsilon \to 0. \]

Then $(e \sigma^\epsilon, u^\epsilon, e \theta^\epsilon, b^\epsilon) \to (0, v, 0, B)$ strongly in $L^\infty(0, \bar{T}; H^1)$ and $E^\epsilon \to B - v \times B$ strongly in $L^\infty(0, \bar{T}; L^2)$ as $\epsilon \to 0$, where $(v, B)$ satisfies (15) with the following...
initial and boundary conditions:
\[
\begin{cases}
v \cdot n = B \cdot n = 0, \quad \text{rot } v \times n = \text{rot } B \times n = 0 \quad \text{on } \partial \Omega \times (0, \tilde{T} ], \\
(v, B)(\cdot, 0) = (v_0, B_0)(\cdot) \quad \text{in } \Omega \subseteq \mathbb{R}^3.
\end{cases}
\] (20)

The remainder of this paper is devoted to the proof of Theorem 1.1 which will be given in the next section.

2. Proof of Theorem 1.1. In this section we shall prove Theorem 1.1 by combining the ideas developed in [1, 12, 4, 5]. First, by taking the similar arguments to that [1, 12], we know that in order to prove (19), it suffices to show the following inequality
\[
M^t(t) \leq C_0(M^t(0)) \exp[t^{1/2}C(M^t(t))], \quad \forall t \in [0, \tilde{T}],[21]
\]
where \(C_0(\cdot)\) and \(C(\cdot)\) are some given positive nondecreasing continuous functions.

Below we shall omit the spatial domain \(\Omega\) in the integrals and drop the superscript \(\epsilon\) of \(\epsilon\) for the sake of simplicity; moreover, we write \(M^t(t)\) and \(M^t(0)\) as \(M\) and \(M_0\), respectively. Since the physical constants \(\kappa, C_V, \) and \(R\) do not bring any essential difficulties in our arguments, we shall take \(\kappa = C_V = R = 1\).

We will also use the following two inequalities:
\[
\|u\|_{H^s(\Omega)} \leq C(\|\text{div } u\|_{H^{s-1}(\Omega)} + \|\text{rot } u\|_{H^{s-1}(\Omega)}
+ \|u\|_{H^{s-1}(\Omega)} + \|u \cdot n\|_{H^{s-1/2}(\partial \Omega)}),
\] (22)
\[
\|\text{div } u\|_{H^{s-1}(\Omega)} + \|\text{rot } u\|_{H^{s-1}(\Omega)}
+ \|u\|_{H^{s-1}(\Omega)} + \|u \cdot n\|_{H^{s-1/2}(\partial \Omega)}
\] (23)
for any \(u \in H^s(\Omega)\) with \(s \geq 1\), which were obtained in [2] and [13] respectively.

Because the local existence for the problem (8)-(14) with fixed \(\epsilon > 0\) is essential similar to that in [14], we only need to prove (21). We will use the methods developed in [4, 5].

First, by the same calculations as that in [3], we get
\[
\left\| \frac{1}{\rho}(\cdot, t) \right\|_{L^\infty} + \|\rho(\cdot, t)\|_{H^2} \leq C_0(M_0) \exp(\sqrt{t}C(M)),
\] (24)
\[
\|\rho(\cdot, t)\|_{H^1} \leq C_0(M_0) \exp(\sqrt{t}C(M)).
\] (25)

Now we use the same method as that in [5] to prove some a priori estimates on \((E, b)\).

Testing (11) and (12) by \(E\) and \(b\), respectively, and summing up the results, we see that
\[
\frac{1}{2} \frac{d}{dt} \int (\epsilon_2 E^2 + b^2)dx + \int E^2 dx = \int (b \times u)E dx 
\leq \|b\|_{L^2} \|E\|_{L^\infty} \|u\|_{L^\infty} \leq \frac{1}{2} \int E^2 dx + C(M).
\]

Integrating the above inequality over \((0, t)\), we find that
\[
\int (\epsilon_2 E^2 + b^2)dx + \int_0^t \int E^2 dx ds \leq C_0(M_0) + tC(M).
\] (26)

Using (14) and the formula
\[-(u \times b) \times n = (b \cdot n)u - (n \cdot u)b = 0 \quad \text{on } \partial \Omega,\] (27)
we infer that
\[
\text{rot } b \times n = 0 \quad \text{on } \partial \Omega.
\] (28)
Taking rot to (11) and (12), testing the results by rot $E$ and rot $b$, respectively, summing up the results, and using (28) and integration by parts, we have

$$
\frac{1}{2} \int \left(\epsilon_2 |\text{rot } E|^2 + |\text{rot } b|^2 \right) dx + \int |\text{rot } E|^2 dx
= - \int \text{rot} (u \times b) \cdot \text{rot } E dx
\leq \frac{1}{2} \int |\text{rot } E|^2 dx + C\|u\|_{H^2}^2 \|\text{rot } b\|_{L^2}^2
\leq \frac{1}{2} \int |\text{rot } E|^2 dx + C(M).
$$

Integrating the above inequality over $(0,t)$, we have

$$
\int (\epsilon_2 |\text{rot } E|^2 + |\text{rot } b|^2) dx + \int_0^t \int |\text{rot } E|^2 dx ds \leq C_0(M_0) + tC(M). \tag{29}
$$

Taking div to (11) and testing the result by div $E$, we infer that

$$
\frac{\epsilon_2}{2} \frac{d}{dt} \int (\text{div } E)^2 dx + \int (\text{div } E)^2 dx = \int \text{div} (b \times u) \text{div } E dx
\leq \frac{1}{2} \int (\text{div } E)^2 dx + C(M).
$$

Integrating the above inequality over $(0,t)$, we deduce that

$$
\epsilon_2 \int (\text{div } E)^2 dx + \int_0^t \int (\text{div } E)^2 dx ds \leq C_0(M_0) + tC(M). \tag{30}
$$

Taking $\partial_t$ to (11) and (12), testing the results by $E_t$ and $b_t$, respectively, summing up the results, we get

$$
\frac{1}{2} \frac{d}{dt} \int (\epsilon_2 |E_t|^2 + |b_t|^2) dx + \int |E_t|^2 dx = \int \partial_t (b \times u) \partial_t E dx
\leq (\|b_t\|_{L^\infty} \|u\|_{L^\infty} + \|b\|_{L^\infty} \|u_t\|_{L^2}) \|E_t\|_{L^2}
\leq \frac{1}{2} \int |E_t|^2 dx + C(M).
$$

Integrating the above inequality over $(0,t)$, we get

$$
\int (\epsilon_2 |E_t|^2 + |b_t|^2) dx + \int_0^t \int |E_t|^2 dx ds \leq C_0(M_0) + tC(M). \tag{31}
$$

(12) and (28) give the boundary condition

$$
\text{rot }^2 E \times n = 0 \quad \text{on} \quad \partial \Omega. \tag{32}
$$

Taking rot$^2$ to (11) and (12), testing the results by rot$^2 E$ and rot$^2 b$, respectively, summing up the results, we derive that

$$
\frac{1}{2} \frac{d}{dt} \int (\epsilon_2 |\text{rot }^2 E|^2 + |\text{rot }^2 b|^2) dx + \int |\text{rot }^2 E|^2 dx
= \int \text{rot}^2 (b \times u) \text{rot }^2 E dx
\leq \frac{1}{2} \int |\text{rot }^2 E|^2 dx + C\|b\|_{H^2}^2 \|u\|_{H^2}^2
\leq \frac{1}{2} \int |\text{rot }^2 E|^2 dx + C(M).
$$
Integrating the above inequality over $(0, t)$, we have
\[
\int (\epsilon_2 |\text{rot}^2 E|^2 + |\text{rot}^2 b|^2)dx + \int_0^t \int |\text{rot}^2 E|^2 dx ds \leq C_0(M_0) + tC(M).
\] (33)

Taking $\nabla \text{div}$ to (11), testing the result by $\nabla \text{div} E$, we have
\[
\frac{\epsilon_2}{2} \frac{d}{dt} \int |\nabla \text{div} E|^2 dx + \int |\nabla \text{div} E|^2 dx = \int \nabla \text{div}(b \times u) \cdot \nabla \text{div} Edx
\leq \frac{1}{2} \int |\nabla \text{div} E|^2 dx + C\|b\|_{H^2}^2 \|u\|_{H^2}^2
\leq \frac{1}{2} \int |\nabla \text{div} E|^2 dx + C(M).
\]

Integrating the above inequality over $(0, t)$, we obtain
\[
\epsilon_2 \int |\nabla \text{div} E|^2 dx + \int_0^t \int |\nabla \text{div} E|^2 dx ds \leq C_0(M_0) + tC(M).
\] (34)

Taking $\partial_t \text{rot}$ to (11) and (12), testing the results by $\partial_t \text{rot}^2 E$ and $\partial_t \text{rot} b$, respectively, summing up the results, and using (28), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int (\epsilon_2 |\text{rot} E^2|^2 + |\text{rot} b|^2)dx + \int |\text{rot} E|^2 dx = \int \text{rot}(b_t \times u + b \times u_t) \cdot \text{rot} E_t dx
\leq C(\|b_t\|_{H^1} \|u\|_{H^2} + \|b\|_{H^2} \|u_t\|_{H^1}) \|\text{rot} E_t\|_{L^2}
\leq \frac{1}{2} \int |\text{rot} E|^2 dx + C(M).
\]

Integrating the above inequality over $(0, t)$, we obtain
\[
\int (\epsilon_2 |\text{rot} E|^2 + |\text{rot} b|^2)dx + \int_0^t \int |\text{rot} E|^2 dx ds \leq C_0(M_0) + tC(M).
\] (35)

Applying $\partial_t \text{div}$ to (11), testing the result by $\text{div} E_t$, we have
\[
\frac{\epsilon_2}{2} \frac{d}{dt} \int (\text{div} E_t)^2 dx + \int (\text{div} E_t)^2 dx = \int \text{div}(b_t \times u + b \times u_t) \cdot \text{div} E_t dx
\leq C(\|b_t\|_{H^1} \|u\|_{H^2} + \|u_t\|_{H^1} \|b\|_{H^2}) \|\text{div} E_t\|_{L^2}
\leq \frac{1}{2} \int (\text{div} E_t)^2 dx + C(M).
\]

Integrating the above inequality over $(0, t)$, we have
\[
\epsilon_2 \int (\text{div} E_t)^2 dx + \int_0^t \int (\text{div} E_t)^2 dx ds \leq C_0(M_0) + tC(M).
\] (36)

Now we use the method in [4] to prove some a priori estimates on $(\sigma, u, \theta)$. Testing (8), (9) and (10) by $\sigma, u$ and $\theta$, respectively, summing up the results, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int (\sigma^2 + \rho u^2 + \rho \theta^2) dx + \int (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 + |\nabla \theta|^2) dx
\]
Integrating the above inequality over \((0, t)\), we obtain
\[
\int (\sigma^2 + \rho \theta^2)dx + \int_0^t (|\nabla u|^2 + |\nabla \theta|^2)dxds \leq C_0(M_0) + \sqrt{t}C(M). \tag{37}
\]
Applying \(\partial_t\) to (8), (9) and (10), we see that
\[
\frac{1}{\epsilon_1} \partial_{tt} + \text{div} u_t = -\text{div} (\sigma u)_t, \tag{38}
\]
and
\[
\rho (u_{tt} + u \cdot \nabla u_t) = \frac{1}{\epsilon_1} (\nabla \sigma_t + \nabla \theta_t) - \mu \Delta u_t - (\lambda + \mu) \text{div} u_t
\]
\[
= -\rho u_{tt} - (\rho u)_t \nabla u - \nabla (\sigma u)_t + [(E + u \times b) \times b]_t. \tag{39}
\]
\[
\rho (\theta_{tt} + u \cdot \nabla \theta_t) = \frac{1}{\epsilon_1} \text{div} u_t - \Delta \theta_t = -\rho \theta_{tt} - (\rho u)_t \cdot \nabla \theta - ((\rho \theta + \sigma) \text{div} u)_t
\]
\[
+ \epsilon_1 (2\mu |D(u)|^2 + \lambda (\text{div} u)^2 + (E + u \times b)^2)_t. \tag{40}
\]
Testing (38), (39) and (40) by \(\sigma_t, u_t\) and \(\theta_t\), respectively, summing up the results, we reach
\[
\int_0^t \int (\sigma_t^2 + \rho u_t^2 + \rho \theta_t^2)dx + \int \|\nabla u_t\|^2 + (\lambda + \mu) (\text{div} u_t)^2 + |\nabla \theta_t|^2 dx
\]
\[
= \int (\sigma u)_t \nabla \sigma_t dx - \int [\rho u_{tt} + (\rho u)_t \nabla u + \nabla (\sigma u)_t] u_t dx
\]
\[
+ \int [(E + u \times b) \times b]_t u_t dx - \int [\rho u_{tt} + (\rho u)_t \nabla \theta + ((\rho \theta + \sigma) \text{div} u)_t] \theta_t dx
\]
\[
+ \epsilon_1 \int (2\mu |D(u)|^2 + \lambda (\text{div} u)^2 + (E + u \times b)^2)_t \theta_t dx
\]
\[
\leq C(M) + \int ((E + u \times b) \times b) u_t dx + \epsilon_1 \int ((E + u \times b)^2)_t \theta_t dx
\]
\[
\leq C(M) + \|E_t\|_{L^2}^2 C(M). \tag{41}
\]
Integrating the above inequality over \((0, t)\), we have
\[
\int (\sigma_t^2 + \rho u_t^2 + \rho \theta_t^2)dx + \int_0^t \int (|\nabla u_t|^2 + |\nabla \theta_t|^2)dxds \leq C_0(M_0) + \sqrt{t}C(M). \tag{41}
\]
Testing (39) by \(-\nabla \text{div} u\) in \(L^2(\Omega \times (0, t))\), we find that
\[
\frac{\mu + \lambda}{2} \|\nabla \text{div} u(\cdot, t)\|^2_{L^2} - \frac{1}{\epsilon_1} \int_0^t \int (\nabla \sigma_t + \nabla \theta_t) \cdot \nabla \text{div} u dx ds
\]
\[
= \frac{\mu + \lambda}{2} \|\nabla \text{div} u_0\|^2_{L^2} + \int_0^t \int \rho (u_{tt} + u \cdot \nabla u_t) \nabla \text{div} u dx ds
\]
\[
+ \int_0^t \int (\rho u_{tt} + (\rho u)_t \nabla u + \nabla (\sigma u)_t) \cdot \nabla \text{div} u dx ds
\]
\[
- \int_0^t \int ((E + u \times b) \times b)_t \nabla \text{div} u dx ds
\]
\[ =: \frac{\mu + \lambda}{2} \|
abla \text{div } u_0\|_{L^2}^2 + I_1 + I_2 + I_3. \quad (42) \]

We bound \(I_1, I_2\) and \(I_3\) as follows.

\[
I_1 \leq C(M) \int_0^t \|u_{tt}\|_{L^2} ds + tC(M) \leq \sqrt{t}C(M),
\]
\[
I_2 \leq tC(M),
\]
\[
I_3 \leq C(M) \int_0^t \|E_t\|_{H^1} ds + tC(M) \leq \sqrt{t}C(M).
\]

Applying \(\nabla\) to (8) and (10), testing the results by \(\nabla \sigma_i\) and \(\nabla \theta_i\) in \(L^2(\Omega \times (0, t))\), respectively, we derive

\[
\int_0^t \int |
\nabla \sigma_i|^2 dx ds + \frac{1}{\epsilon_1} \int_0^t \int \nabla \sigma_i \nabla \text{div } u dx ds
\]
\[
= - \int_0^t \int \nabla \text{div } (\sigma u) \nabla \sigma_i dx ds \leq tC(M), \quad (43)
\]

and

\[
\frac{1}{2} \int |\Delta \sigma|^2 dx + \int_0^t \int |\nabla \theta|^2 dx ds + \frac{1}{\epsilon_1} \int_0^t \int \nabla \theta \nabla \text{div } u dx ds
\]
\[
= \frac{1}{2} \int |\Delta \sigma_0|^2 dx + \int_0^t \int \epsilon_1 \nabla [2\mu |D(u)|^2 + \lambda (\text{div } u)^2 + (E + u \times b)]^2 \nabla \theta dx ds
\]
\[
- \int_0^t \int \nabla (\rho u + \sigma) \nabla \theta_i dx ds - \int_0^t \int \nabla \rho \theta_t + \nabla \rho u \cdot \nabla \theta \nabla \theta_i dx ds
\]
\[
\leq C_0(M_0) + \sqrt{t}C(M). \quad (44)
\]

Summing up (42), (43) and (44), we arrive at

\[
\int (|\nabla \text{div } u|^2 + |\Delta \theta|^2) dx + \int_0^t \int (|\nabla \sigma_i|^2 + |\nabla \theta|^2) dx ds \leq C_0(M_0) \exp(\sqrt{t}C(M)). \quad (45)
\]

Testing (38), (39) and (40) by \(-\Delta \sigma_i, -\nabla \text{div } u_i,\) and \(-\Delta \theta_i\), respectively, we derive

\[
\frac{1}{2} \int |\nabla \sigma_i|^2 dx + \frac{1}{2} \int_0^t \int \nabla \text{div } u_i \nabla \sigma_i dx ds
\]
\[
= \frac{1}{2} \int |\nabla \sigma_i(0)|^2 dx + \int_0^t \int \text{div } (\sigma_i u + \sigma u_i) \cdot \Delta \sigma_i dx ds
\]
\[
=: \frac{1}{2} \int |\nabla \sigma_i(0)|^2 dx + I_4. \quad (46)
\]

We bound \(I_4\) as follows.

\[
I_4 = \int_0^t \int \nabla \sigma_i \Delta \sigma dx - \int_0^t \int \nabla (\sigma_i \text{div } u + u_i \nabla \sigma + \sigma \text{div } u_i) \nabla \sigma_i dx ds
\]
\[
= - \sum_i \int_0^t \int \partial_i u \nabla \sigma_i \partial_i \sigma_i dx ds - \frac{1}{2} \int \text{div } u |\nabla \sigma_i|^2 dx ds
\]
\[
- \int_0^t \int \nabla (\sigma_i \text{div } u + u_i \nabla \sigma + \sigma \text{div } u_i) \nabla \sigma_i dx ds
\]
\[
\leq C(M) \int_0^t \|u\|_{L^\infty} ds + C(M) \int_0^t \|u\|_{H^3} ds + C(M) \int_0^t \|u_i\|_{H^2} ds
\]
\[
\leq \sqrt{t}C(M).
\]
Similarly, we have
\[
\frac{1}{2} \int \rho (\nabla u_t)^2 dx + (\lambda + 2\mu) \int_0^t \int |\nabla \nabla u_t|^2 dx ds \\
- \frac{1}{\epsilon_1} \int_0^t \int \nabla \nabla u_t (\nabla \sigma_t + \nabla \theta_t) dx ds
\]
\[
= \frac{1}{2} \int \rho_0 (\nabla u_t(0))^2 dx + \int_0^t \int [\rho_t u_t + (\rho u)_t \nabla u + \nabla (\sigma \theta)] \nabla \nabla u_t dx ds \\
+ \int_0^t \int (E \times b)_t \nabla \nabla u_t dx ds - \int_0^t \int [(u \times b) \times b_t] \nabla \nabla u_t dx ds \\
+ \int_0^t \int (\epsilon_1 \nabla \theta_t + u \cdot \nabla u_t) \nabla \nabla u_t dx ds - \int_0^t \int \sum \nabla u_i \partial_i u_t \nabla \nabla u_t dx ds
\]
\[
\leq C_0(M_0) + \sqrt{t}C(M),
\]
(47)
and
\[
\frac{1}{2} \int \rho |\nabla \theta_t|^2 dx + \int_0^t \int |\Delta \theta_t|^2 dx ds + \frac{1}{\epsilon_1} \int_0^t \int \nabla \nabla u_t \nabla \theta_t dx ds
\]
\[
= \frac{1}{2} \int \rho_0 |\nabla \theta_t(0)|^2 dx - \int_0^t \int \epsilon_1 \nabla \sigma_t \nabla \theta_t dx ds - \int_0^t \int \sum (\rho u)_t \partial_i \theta_t \nabla \theta_t dx ds
\]
\[
+ \int_0^t \int \Delta \theta_t [\rho_t \theta_t + (\rho u)_t \nabla \theta + (\rho \sigma + \div u)_t] dx ds \\
- \epsilon_1 \int_0^t \int \Delta \theta_t (2\mu |D(u)|^2 + \lambda (\div u)^2 + (E + u \times b)^2) dx ds
\]
\[
\leq C_0(M_0) + \sqrt{t}C(M).
\]
(48)
Summing up (46), (47) and (48), we arrive at
\[
\int \int (|\nabla \sigma_t|^2 + (\div u_t)^2 + |\nabla \theta_t|^2) dx ds + \int_0^t \int (|\nabla \nabla u_t|^2 + (\Delta \theta_t)^2) dx ds
\]
\[
\leq C_0(M_0) \exp(\sqrt{t}C(M)).
\]
(49)
Now, testing \( \partial_t \nabla (8) \) by \( \partial_t \nabla \sigma + \partial_t \nabla \theta \) and carrying out the same calculations as those in [4], we have
\[
\frac{1}{2} \int \int |\partial_t \nabla \sigma|^2 dx ds + \int \partial_t \nabla \sigma \cdot \partial_t \nabla \theta dx ds + \frac{1}{\epsilon_1} \int_0^t \int \partial_t \nabla \theta \div (\partial_t \nabla \sigma + \partial_t \nabla \theta) dx ds
\]
\[
\leq C_0(M_0) + \sqrt{t}C(M).
\]
(50)
Testing \( \partial_t (9) \) by \( \partial_t \div u \) in \( L^2(\Omega \times (0, t)) \) and carrying out the same calculations as those in [4], we obtain
\[
\frac{1}{2} \int_0^t \int |\partial_t \div u|^2 dx ds - \frac{1}{\epsilon_1} \int_0^t \int \partial_t \div u (\partial_t \nabla \sigma + \partial_t \nabla \theta) dx ds \leq tC(M).
\]
(51)
(50), (51) and (45) give
\[
\int |\nabla^2 \sigma|^2 dx + \int_0^t \int |\nabla^2 \div u|^2 dx ds \leq C_0(M_0) \exp(\sqrt{t}C(M)).
\]
(52)
Applying rot to (9) and denoting the vorticity \( \omega := \text{rot} u \), we see that
\[
\rho \omega_t + \rho u \cdot \nabla \omega - \mu \Delta \omega = (\partial_j \rho u_i - \partial_i \rho u_j) + (\partial_j (\rho u_k) \partial_k u_i - \partial_i (\rho u_k) \partial_k u_j)
+ \text{rot} [(E + u \times b) \times b].
\]
We test (53) by \( \Delta \omega \) in \( L^2(\Omega \times (0, t)) \) to get
\[
\int \rho |\text{rot} \omega|^2 \, dx + \mu \int_0^t \int |\Delta \omega|^2 \, dx \, ds \leq C_0(M_0) + \sqrt{t}C(M).
\]
Similarly, we apply \( \partial_t \) to (53) and test the resulting equations by \( \omega_t \) in \( L^2(\Omega \times (0, t)) \) to deduce that
\[
\int \rho |\omega_t|^2 \, dx + \mu \int_0^t \int |\text{rot} \omega_t|^2 \, dx \, ds \leq C_0(M_0) + \sqrt{t}C(M).
\]
By carrying out the same calculations as that in [4], we have
\[
\|\Delta \theta\|_{L^2(0, t; H^1)} \leq C_0(M_0) \exp(t \frac{1}{4} C(M)).
\]
It follows from (11), (12), (26), (29), (30), (31), (33), (34), (35) and (36) that
\[
\epsilon_2 \|E_{tt}\|_{L^2(0, t; L^2)} \leq \|\text{rot} b_t - E_t - (u \times b)_t\|_{L^2(0, t; L^2)} \leq C_0(M_0) + tC(M).
\]
\[
\sqrt{\epsilon_2} \|b_{tt}(t)\|_{L^2} \leq \sqrt{\epsilon_2} \|\text{rot} E_{t}(\cdot, t)\|_{L^2} \leq C_0(M_0) + tC(M).
\]
Finally, we need to estimate \( \epsilon_1 \sigma_{tt} \), \( \epsilon_1 u_{tt} \) and \( \epsilon_1 \theta_{tt} \) to close the energy estimates. Testing \( \partial_2^2(8) \), \( \partial_2^2(9) \) and \( \partial_2^2(10) \) by \( \epsilon_1 \sigma_{tt} \), \( \epsilon_1^2 u_{tt} \) and \( \epsilon_1^2 \theta_{tt} \), respectively, then by carrying out the same calculations as that in [4], we conclude that
\[
\epsilon_1 \|(\sigma_{tt}, u_{tt}, \theta_{tt})(t)\|_{L^2} + \epsilon_1 \|(u_{tt}, \theta_{tt})(t)\|_{L^2(0, t; H^1)} \leq C_0(M_0) \exp(t \frac{1}{4} C(M))
\]
and thus (19) hold true.

This completes the proof of Theorem 1.1. \( \square \)

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