Banach space valued $H^p$ spaces with $A_p$ weight

Sakin Demir
Agri Ibrahim Cecen University
Faculty of Education
Department of Basic Education
04100 Ağrı, Turkey
e-mail: sakin.demir@gmail.com
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Abstract

In this research we introduce the Banach space valued $H^p$ spaces with $A_p$ weight, and prove the following results:

Let $\mathbb{A}$ and $\mathbb{B}$ Banach spaces, and $T$ be a convolution operator mapping $\mathbb{A}$-valued functions into $\mathbb{B}$-valued functions, i.e.,

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) \cdot f(y) \, dy,$$

where $K$ is a strongly measurable function defined on $\mathbb{R}^n$ such that $\|K(x)\|_{\mathbb{B}}$ is locally integrable away from the origin. Suppose that $w$ is a positive weight function defined on $\mathbb{R}^n$, and that

(i) For some $q \in [1, \infty]$, there exists a positive constant $C_1$ such that

$$\int_{\mathbb{R}^n} \|Tf(x)\|_{\mathbb{B}}^q w(x) \, dx \leq C_1 \int_{\mathbb{R}^n} \|f(x)\|_{\mathbb{A}}^q w(x) \, dx$$

for all $f \in L^q_{\mathbb{A}}(\mathbb{R}^n)$.

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There exists a positive constant $C_2$ independent of $y \in \mathbb{R}^n$ such that
\[ \int_{|x| > 2|y|} \|K(x-y) - K(x)\|_{B} \, dx < C_2. \]
Then there exists a positive constant $C_3$ such that
\[ \|Tf\|_{L^1_B(w)} \leq C_3 \|f\|_{H^1_A(w)} \]
for all $f \in H^1_A(w)$.

Let $w \in A_1$. Assume that $K \in L_{loc}(\mathbb{R}^n \setminus \{0\})$ satisfies
\[ \|K \ast f\|_{L^2_B(w)} \leq C_1 \|f\|_{L^2_A(w)} \]
and
\[ \int_{|x| \geq C_2|y|} \|K(x-y)-K(x)\|_{B} w(x+h) \, dx \leq C_3 w(y+h) \quad (\forall y \neq 0, \forall h \in \mathbb{R}^n) \]
for certain absolute constants $C_1$, $C_2$, and $C_3$. Then there exists a positive constant $C$ independent of $f$ such that
\[ \|K \ast f\|_{L^1_B(w)} \leq C \|f\|_{H^1_A(w)} \]
for all $f \in H^1_A(w)$.

1 Introduction

Extending $L^p$ spaces to Banach space valued $L^p$ spaces first started with the work of A. Benedek et al [1]. J. Bourgain [2] extended some part of their results to a lattice with UMD-property. Later, the results of A. Benedek et al [1] have been reconstructed with a little more modern notations by J. L. Rubio de Francia et al [10].

Obviously, Banach space valued setting is more general than the usual structure because we have a Banach space norm instead of absolute value. When a theorem can be extended from $L^p$ spaces to Banach space valued $L^p$ spaces it becomes a much more powerful theorem than its initial version.

Let us first recall some basic definitions and theorems from Banach space valued $L^p$ theory:
Let $B$ be a Banach space, and $p < \infty$. By $L^p_B = L^p_B(\mathbb{R}^n)$ we denote the
Bochner-Lebesgue space consisting of all $\mathcal{B}$-valued (strongly) measurable functions $f$ in $\mathbb{R}^n$ such that

$$\|f\|_{L^p_B} = \left(\int_{\mathbb{R}^n} \|f(x)\|_B^p \, dx\right)^{1/p} < \infty.$$ 

For $p = \infty$, norm of an element of $L^{\infty}(\mathcal{B}) = L^{\infty}_B(\mathbb{R}^n)$ is

$$\|f\|_{L^{\infty}_B} = \text{ess sup} \|f(x)\|_B < \infty$$

and $L^{\infty}_B(\mathcal{B})$ denotes the space of all compactly supported members of $L^{\infty}(\mathcal{B})$. Let $f$ be a locally integrable $\mathcal{B}$-valued function, and $1 \leq r \leq \infty$. We define the maximal functions

$$M_r f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q \|f(y)\|_B^r \, dy\right)^{1/r}$$

and

$$f^r(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q \|f(y) - f_Q\|_B \, dy,$$

where $Q$ denotes an arbitrary cube in $\mathbb{R}^n$ and $f_Q$ is the average of $f$ over $Q$, an element of $\mathcal{B}$.

Given a weight $w$ on $\mathbb{R}^n$, we denote by $L^p_B(w)$ the space of all functions satisfying

$$\|f\|_{L^p_B(w)} = \int_{\mathbb{R}^n} \|f(x)\|_B^p w(x) \, dx < \infty.$$ 

When $p = \infty$, $L^{\infty}_B(w)$ will be taken to mean $L^{\infty}_B(\mathbb{R}^n)$ and

$$\|f\|_{L^{\infty}_B(w)} = \|f\|_{L^{\infty}_B}.$$ 

## 2 Banach Space Valued $H^p$ Spaces with $A_p$ Weight

Analogous to the classical weighted Hardy spaces we can also define the weighted Hardy spaces $H^p_B(w)$ of $\mathcal{B}$-valued functions for $p > 0$. Let $\phi$ be a function in $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$. Define

$$\phi_t(x) = t^{-n} \phi(x/t), \quad t > 0, \quad x \in \mathbb{R}^n,$$
and the maximal function $f^*$ by

$$f^*(x) = \sup_{t>0} \| f * \phi_t(x) \|_B.$$  

Then $H^p_B(w)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $f^* \in L^p_B(w)$ with $\|f\|_{H^p_B(w)} = \|f^*\|_{L^p_B(w)}$.

As in the classical case these spaces can also be characterized in terms of atoms in the following way.

**Definition 1.** Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index $q_w$. Set $\lfloor \cdot \rfloor$ the integer function. For $s \in \mathbb{Z}$ satisfying $s \geq \lfloor n(q_w/p - 1) \rfloor$, a $B$-valued function $a$ defined on $\mathbb{R}^n$ is called a $(p, q, s)$-atom with respect to $w$ if

(i) $a \in L^q_B(w)$ and is supported on a cube $Q$,

(ii) $\|a\|_{L^q_B(w)} \leq w(Q)^{1/q - 1/p},$

(iii) $\int_{\mathbb{R}^n} a(x) x^\alpha \, dx = 0$ for every multi-index $\alpha$ with $|\alpha| \leq s$.

The $B$-valued atom defined above is called $(p, q, s)$-atom centered at $x_0$ with respect to $w$ (or $w - (p, q, s)$-atom centered at $x_0$), where $x_0$ is the center of the cube $Q$.

**Lemma 1.** Let $a$ be any $B$-valued $w - (p, q, s)$-atom supported in a cube $Q$. Then we have

$$\int_Q \|a(x)\|^p_B w(x) \, dx \leq 1.$$  

**Proof.** Let $a$ be any $B$-valued $w - (p, q, s)$-atom. It is clear that $a \in L^p_B(w)$ and $\|a\|_{L^p_B(w)} \leq 1$, since by Hölder’s inequality

$$\int_Q \|a(x)\|^p_B w(x) \, dx \leq \|a^p\|_{L^r_B(w)} \left( \int_Q w(x) \, dx \right)^{1/r'}$$

$$= \|a\|_{L^p_B(w)}^p \cdot w(Q)^{1-p/q} \leq 1,$$

where $r = q/p$ and $1/r' = 1 - 1/r = 1 - p/q$. 

□
Analog to the classical case $H^p_B(w)$ can be characterized by $B$-valued $w$–$(p, q, s)$-atoms.

We state the following few theorems without proof since their proofs are similar to the scalar case, i.e., one only needs to replace the absolute value with the $B$-norm in the proofs for classical cases.

**Theorem 1.** Let $w \in A_\infty$ and $0 < p \leq 1$. For each $f \in H^p_B(w)$, there exists a sequence of $B$-valued $(p, \infty, N)$-atoms with respect to $w$ and a sequence $\{\lambda_i\}$ of real numbers with $\sum_j |\lambda_i|^p \leq C \|f\|_{H^p_B(w)}$ such that

$$f(x) = \sum_j \lambda_j a_j(x); \quad (\lambda_j \in \ell^p)$$

both in the sense of distribution and in the $H^p_B(w)$ norm.

Let $H^{p,q,s}_B(w)$ denote the space consisting of tempered distributions admitting a decomposition

$$f(x) = \sum_j \lambda_j a_j(x); \quad (\lambda_j \in \ell^p),$$

where $a_i$’s are $B$-valued $w$–$(p, q, s)$-atoms and $\sum_i |\lambda_i|^p < \infty$. For fixed functions $w$ and $f \in H^p_B(w)$, we also set

$$\mathcal{N}_{p,q,s}(f) = \inf_{\{\lambda_i\}} \left\{ \left( \sum_i |\lambda_i|^p \right)^{1/p} : f = \sum_i \lambda_i a_i \text{ is an atomic decomposition} \right\}. $$

**Theorem 2.** If both triples $(p, q, N)$ and $(p, q_2, N)$ satisfy the conditions in definition of $B$-valued $w$-atom, then

$$H^{p,q,N}_B(w) = H^{p,q_2,N}_B(w)$$

and, for all $q$, the gauges $\mathcal{N}_{p,q,N}(f)$ are equivalent.

**Theorem 3.** For $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$, every $B$-valued $(p, q, N)$-atom with respect to $w$ is in $H^p_B(w)$, and its $H^p_B(w)$-norm is bounded by a constant independent of the atom.

**Theorem 4.** All spaces $H^{p,q,s}_B(w)$ coincide with $H^p_B(w)$ and $\mathcal{N}_{p,q,s}(f) \approx \|f\|_{H^p_B(w)}$ provided that the triple $(p, q, s)$ satisfies the conditions in the definition of $B$-valued $w$-atom.
Definition 2. Let \( B \) be a Banach space. For \( 0 < p \leq 1 \leq q \leq \infty \) and \( p \neq q \), let \( w \in A_q \) with critical index \( q_w \) and critical index \( r_w \) for the reverse Hölder condition. Set \( s \geq N, \epsilon > \max \{ sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, 1/p - 1 \} \), \( a = 1 - 1/p + \epsilon \), and \( b = 1 - 1/p + \epsilon \). A \( B \)-valued \( (p, q, s, \epsilon) \)-molecule centered at \( x_0 \) with respect to \( w \) (or \( w - (p, q, s, \epsilon) \)-molecule centered at \( x_0 \)) is a function \( M \in L^q_B(w) \) satisfying

(i) \( M(x) \cdot w(I^{x_0}_{|x-x_0|})^b \in L^q_B(w) \),

(ii) \( \| M \|^{a/b}_{L^q_B(w)} \cdot \| M(x) \cdot w(I^{x_0}_{|x-x_0|})^b \|^{1-a/b}_{L^q_B(w)} \equiv R_w(M) < \infty \),

(iii) \( \int_{\mathbb{R}^n} x^\alpha \, dx = 0 \) for every multi-index \( \alpha \) with \( |\alpha| \leq s \).

In the above definition \( R_w(M) \) is called the molecular norm of \( M \) with respect to \( w \) (or \( w \)-molecular norm of \( M \)). If \( w(x) \equiv \text{constant} \), then \( q_w = 1 \) and \( r_w = \infty \).

3 The Results

Let \( A \) and \( B \) be Banach spaces, and \( T \) be a convolution operator mapping \( A \)-valued functions into \( B \)-valued functions, i.e.,

\[
Tf(x) = \int_{\mathbb{R}^n} K(x - y) \cdot f(y) \, dy,
\]

where \( K \) is a strongly measurable function defined on \( \mathbb{R}^n \) such that \( \| K(x) \|_B \) is locally integrable away from the origin.

The following theorem is our first result:

Theorem 5. Let \( A \) and \( B \) be Banach spaces, and \( T \) be a convolution operator mapping \( A \)-valued functions into \( B \)-valued functions. Suppose that \( w \) is a positive weight function defined on \( \mathbb{R}^n \), and that

(i) For some \( q \in [1, \infty] \), there exists a positive constant \( C_1 \) such that

\[
\int_{\mathbb{R}^n} \| T f(x) \|_B^q w(x) \, dx \leq C_1 \int_{\mathbb{R}^n} \| f(x) \|_A^q w(x) \, dx
\]

for all \( f \in L^q_A(\mathbb{R}^n) \).
(ii) There exists a positive constant $C_2$ independent of $y \in \mathbb{R}^n$ such that

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\|_B \, dx \leq C_2.$$ 

Then there exists a positive constant $C_3$ such that

$$\|Tf\|_{L^1_A(w)} \leq C_3 \|f\|_{H^1_A(w)}$$

for all $f \in H^1_A(w)$.

**Proof.** Given a ball $U = U(x_0; R)$ in $\mathbb{R}^n$ with center $x_0$ and radius $R$, and denoting by $\tilde{U}$ the double ball, $\tilde{U} = U(x_0; 2R)$, we first claim that

$$\int_{\mathbb{R}^n \setminus \tilde{U}} \|Tf(x)\|_B \, w(x) \, dx \leq C \|f\|_{L^1_A(w)}$$

for every $f \in L^1_A(w)$ supported in $U$ such that $\int f(x) \, dx = 0$. But, for such a function $f$,

$$Tf(x) = \int_U \{K(x-y) - K(x-x_0)\} \cdot f(y) \, dy \quad (x \in \tilde{U})$$

and therefore

$$\int_{\mathbb{R}^n \setminus \tilde{U}} \|Tf(x)\|_B \, w(x) \, dx \leq \int \int_{|x-x_0| \geq 2R \geq |y-x_0|} \|\{K(x-y) - K(x-x_0)\} \cdot f(y)\|_B \, dy \, w(x) \, dx \leq C \int_{|y-x_0| < R} \|f(y)\|_A \, w(y) \, dy,$$

which proves our claim.

Let now $a$ be an $A$-valued atom with supporting cube $Q$, and let $U$ be the smallest ball containing $Q$, and $\tilde{Q}$ as before. Then there exits a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^n \setminus \tilde{U}} \|Ta(x)\|_B \, w(x) \, dx \leq C_1.$$ 

On the other hand, since

$$\int_{\mathbb{R}^n} \|Ta(x)\|_B^q \, w(x) \, dx \leq C_2 \int_{\mathbb{R}^n} \|a(x)\|_A^q \, w(x) \, dx,$$
we have
\[
\int_{\mathbb{U}} \|Ta(x)\|_{Bw(x)} \, dx \leq C_3 \|a(x)\|_{L^2_A(w)} (C_n w(Q))^{1/q'} \\
\leq \text{Constant}.
\]

Our second result is the following:

**Theorem 6.** Let \( w \in A_1 \). Assume that \( K \in L_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) satisfies
\[
\|K * f\|_{L^2_A(w)} \leq C_1 \|f\|_{L^2_A(w)}
\]
and
\[
\int_{|x| \geq C_2 |y|} \|K(x - y) - K(x)\|_{Bw(x + h)} \, dx \leq C_3 w(y + h) \quad (\forall y \neq 0, \forall h \in \mathbb{R}^n)
\]
for certain absolute constants \( C_1, C_2, \) and \( C_3 \). Then there exists a constant \( C \) independent of \( f \) such that
\[
\|K * f\|_{L^2_A(w)} \leq C \|f\|_{H^1_A(w)}
\]
for all \( f \in H^1_A(w) \).

**Proof.** Because of the atomic decomposition of a function in \( H^1_A(w) \), it suffices to show that
\[
\|K * a\|_{L^1_A(w)} \leq C
\]
for any \( B \)-valued \( w - (1, 2, 0) \)-atom \( a \) with constant \( C \) independent of the choice of \( a \). Let us first consider a weighted 1-atom \( a \) centered at 0 with support \( \text{supp}(f) \subset I_R \), we have
\[
\|a\|_{L^2_B(w)} \leq w(I_R)^{-1/2}
\]
and
\[
\int_{I_R} a(x) \, dx = 0.
\]
Thus, we have
\[
\int_{|x| \geq C_2 \sqrt{nR}} \| K \ast a(x) \|_{B} w(x) \, dx \\
= \int_{|x| \geq C_2 \sqrt{nR}} \left\| \int_{I_R} \{ K(x - y) - K(x) \} a(y) \, dy \right\|_{B} w(x) \, dx \\
\leq \int_{I_R} \| a(y) \|_B \, dy \int_{|x| \geq C_2 |y|} \| K(x - y) - K(x) \|_{B} w(x) \, dx \\
\leq C_3 \int_{I_R} \| a(y) \|_B w(y) \, dy \\
\leq C_3.
\]
Furthermore, we have by Schwarz’s inequality and the doubling condition,
\[
\int_{|x| < C_2 \sqrt{nR}} \| K \ast a(x) \|_{B} w(x) \, dx \leq \| K \ast a \|_{L^2_B(w)} \left( \int_{|x| < C_2 \sqrt{nR}} w(x) \, dx \right)^{1/2} \\
\leq C_1 \| a \|_{L^2_B(w)} w(C_2 \sqrt{n} I_R)^{1/2} \leq C.
\]
So in both cases for any $B$-valued $w - (1, 2, 0)$-atom $a$ centered at the origin we have obtained
\[
\| K \ast a \|_{L^1_B(w)} \leq C.
\]
Let now $a$ be a $B$-valued $w - (1, 2, 0)$-atom centered $x_0 \in \mathbb{R}^n$. Then $b(x) = a(x - x_0)$ is a $B$-valued $w_1 - (1, 2, 0)$-atom centered at 0, where $w_1(x) = w(x - x_0) \in A_1$. Furthermore, $K$ satisfies
\[
\| K \ast b \|_{L^2_B(w_1)} \leq C_1 \| b \|_{L^2_B(w_1)}
\]
and
\[
\int_{|x| \geq C_2 |y|} \| K(x - y) - K(x) \|_{B} w_1(x) \leq C_3 w_1(y) \quad (\forall y \neq 0).
\]
Thus, we have as above
\[
\| K \ast b \|_{L^1_B(w_1)} \leq C.
\]
Hence, we obtain
\[
\| K \ast a \|_{L^1_B(w)} = \| K \ast b \|_{L^1_B(w_1)} \leq C
\]
as desired. \qed
4 An Application

Let $f$ be a measurable functions defined on $\mathbb{R}$, and for each $n \in \mathbb{Z}$ define the averaging operator

$$A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y) \, dy.$$ 

Consider the variation operator

$$V f(x) = \left( \sum_{-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^s \right)^{1/s}$$

for $2 \leq s < \infty$.

Given a locally integrable function $f$ we define the sequence-valued operator $T$ as follows:

$$T f(x) = \{A_n f(x) - A_{n-1} f(x)\}_n$$

$$= \left\{ \int_{\mathbb{R}} \left( \frac{1}{2^n} \chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x-y) \right) f(y) \, dy \right\}_n$$

$$= \int_{\mathbb{R}} K(x-y) \cdot f(y) \, dy,$$

where $K$ is the sequence-valued function

$$K(x) = \left\{ \frac{1}{2^n} \chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x) \right\}_{n \in \mathbb{Z}} = \{K_n(x)\}_{n \in \mathbb{Z}}.$$

It is clear that

$$\|T f(x)\|_{\ell^s(\mathbb{Z})} = V f(x).$$

It is proven in Lemma 1 of S. Demir [4] that $K$ satisfies the $D_r$ condition for $r \geq 1$. For $r = 1$ this condition is equivalent to Theorem 5 (ii) known as Hörmander condition with $\mathbb{B} = \ell^s(\mathbb{Z})$ for $s \geq 2$.

Also, Theorem 2 of S. Demir [4] shows that Theorem 5 (i) is satisfied for $1 \leq q < \infty$ with the absolute value as $\mathbb{A}$. This shows that Theorem 5 can be applied to $T f$, and thus for $s \geq 2$ there exists a positive constant $C$ such that

$$\|T f\|_{L_{\ell^s(\mathbb{Z})}(w)} = \|V f\|_{L^1(w)} \leq C \|f\|_{H^1(w)}$$

for all $f \in H^1(w)$.


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