Can we (fore)see a microscopic texture beyond the experimental window?

Corneliu Sochichiu
GIST College, Gwangju Institute of Science and Technology,
123 Cheomdan-gwagiro(Oryong-dong), Buk-gu
Gwangju 500-712 Republic of Korea

We conjecture the existence of microscopic structures in the intermediate region between the observable high energy particle physics and the Planck scale that can be described as graphs or networks of fermionic states and examine a possible role played by such structures in the emergence of the four-dimensional space-time, particle spectrum as well as interactions.

INTRODUCTION

Standard Model is the most fundamental particle theory based on observations (see e.g. [1]). Like other quantum field theory models it relies on three principles: Lorentz invariance, continuous space-time, and locality. Although firmly established empirically, this combination of principles was in the past and still is a source of mathematical and conceptual difficulties. Most of mathematical issues, e.g. UV divergencies were successfully solved within the the quantum field theory framework. However, when it comes to gravity in which the local Lorentz symmetry extended to the diffeomorphism invariance assumes the role of gauge symmetry, there is no successful model combining all three principles. Moreover, we have an argument [2], according to which a quantum gravity theory combining these concepts is in principle impossible. Hence, a successful theory of gravity should possess these properties only in an approximate way which is precise only within the experimental window available to us today.

There is also string theory, which is outside of our experimental window. It successfully quantises gravity, albeit with no locality. In fact, a non-perturbative approach to strings opens a Pandora’s box of models, all of which are unified under the paradigm of M-theory, which itself is not yet a fully understood model. In particular, there are string theory inspired matrix models in which the space-time emerges as an arena for large scale dynamics (see Ref. [8], for some recent results).

Although mathematically elegant and feature rich, string theory fails to fit naturally with Standard Model. The problem is that it predicts too many structures, while those compatible with observations are by no means special [8]. Also, there is a wide scale gap between the regime claimed by string theory and our experimental knowledge described within Standard Model.

On the other hand, in condensed matter theory we have situations that are well understood from both the bottom and the top. A condensed matter system is characterised by a hierarchy of structures where at the ‘microscopic’ level we have atomic structures resulting from ‘more fundamental’ electronic exchange forces organised at lower energies into ‘less fundamental’ collective excitation modes naturally described in a field theory framework. In the case of Dirac materials, these models are also Lorentz invariant with most notorious example being graphene [9].

Under deeper mathematical scrutiny the emergence of Lorentz invariant Dirac, Weyl or Majorana fermions in such models does not seem accidental. In fact, when the Fermi surface shrinks to a point, low energy fluctuations organise themselves into relativistic spinor objects [10]. This conclusion is based on ABS construction [11].

If we assume that a quantum field theory emerges from a similar structure, evolved from either string theory or some unknown model, then what can be said about the basic properties of the above structure?

In this letter we attempt a study of this hypothetical structure, which we call microscopic texture. We do so by relying on minimal assumptions, such as the structure being organised as a graph, or a network of fermionic states. The graph is parameterised by a Hermitian adjacency matrix $T$. The statistical weight of such a graph is given by the partition function,

$$Z(T) = \int [d\psi^\dagger] [d\psi] e^{iS[\psi^\dagger, \psi; T]},$$

where $[d\psi] = \prod_{k \in \text{Graph nodes}} d\psi_k$ is the fermionic Grassmann measure and, correspondingly, $[d\psi^\dagger]$ is its conjugate. The ‘action’ is given in terms of $T$,

$$S[\psi^\dagger, \psi; T] = \psi^\dagger \cdot T \cdot \psi.$$  

The matrix $T$ is the main focus of our study. We assume that the graph structure is in broad equilibrium (as dictated by the microscopic theory). Therefore, the main contribution to the graph’s dynamics, when we include it into consideration, should come from the fermionic back reaction. We consider the situation in which exactly one-half of states are excited. This is an analogue of half filling in condensed matter systems.

Furthermore, we assume that statistically relevant contributions come from graph configurations leading to smooth local geometries and renormalisable interactions in the saddle point approximation. This belief is based on the role of RG flow [8], according to which the deformations moving away from local and continuous quantum
field theory are given by irrelevant operators. Throughout this work we follow this ‘quantum Darwinistic principle’. Therefore, for the beginning we consider graphs of a special structure, which are (a) invariant under a $D$-dimensional translation group, i.e. split in blocks (unit cells) forming a $D$-dimensional lattice, and (b) have non-trivial adjacency only between nearest neighbour cells. Thus, an adjacency matrix takes the form,

$$T = \alpha + \sum_{I} (\alpha^I T^I + \alpha^I T^I)^{\dagger}, \quad (3)$$

where $T_I$, $I = 1, \ldots, D$, are the generators of translations while $\alpha$, $\alpha^I$, and $\alpha^{I\dagger}$ are, respectively, the Hermitian unit cell internal adjacency and the adjacency to the neighbour cell in the direction $I$.

Here we aim to consider the smallest non-trivial unit cell. The case of one site unit cell is degenerate for any $D > 1$. Therefore, the simplest non-trivial case is that of unit cell consisting of two sites. This implies that $\alpha$ and $\alpha^I$ are $2 \times 2$ matrices.

In what follows, we consider the saddle point limit of such graphs and show that within some range of parameters in the large scale limit these graphs generically lead to a free relativistic fermionic model. Then we show that a deformation of such translational invariant graph by an arbitrary local operator having smooth limit leads to the coupling of the Dirac fermion to background geometry and gauge field. The technical details behind the results presented here will be reported in a separate manuscript [10].

**PSEUDO-FERMI POINTS AND DIRAC FERMION**

Although not containing an evolution parameter like time, the graph partition function [11] is designed to be a sum of complex phase factors rather than that of well-behaving exponentially suppressed factors. We do this in a way where the partition function is not to be defined directly but requires an analytical continuation for complete definition.

Since the action is quadratic in the fermionic variable, the main contribution in the saddle point limit comes from configurations close to zero modes of the matrix $T$. In the Fourier basis given by functions,

$$\psi(k) = \sum_{n} e^{ik_{n}} n\psi_{n}, \quad (4)$$

where $-\pi \leq k_{I} < \pi$ is the first Brillouin zone, the adjacency matrix becomes a $2 \times 2$ matrix function. It can be expanded in terms of the extended set of Pauli matrices $\sigma^A$: $T(k) = T_A(k)\sigma^A$, $(A = 0, 1, 2, 3)$, where

$$T_A(k) = \alpha_A + \sum_{I} (\alpha^I_A e^{-ik_I} + \bar{\alpha}^I_A e^{ik_I}). \quad (5)$$

In this basis the zero mode condition takes the form,

$$\det T(k) \equiv T_0(k) - T_1^2(k) - T_2^2(k) - T_3^2(k) = 0. \quad (6)$$

Eq. (6) points to a possible way for the analytic continuation of the integral [11]. Thus, exponentially suppressing factors emerge if we replace,

$$\sigma_0 = \mathbb{I} \mapsto i\sigma_0. \quad (7)$$

This is equivalent to analytic continuation of the trace part of $T$ to imaginary values. Such a modification is similar to Wick rotation in field theory.

With such an analytic continuation in mind, the statistically dominant configuration in the saddle point limit is given by an analogue of Fermi sea, which we will call pseudo-Fermi sea. The analogue of the Fermi surface is given by a variety of momentum space points satisfying $T_A(k) = 0$. The ABS construction [2] implies that when it is a set of linearly non-degenerate disjoint points the fluctuations in vicinity of such points are organised into Weyl fermions (see also Refs. [6, 11]).

Concerning the occurrence of such points that will be called pseudo-Fermi points (pFp), engineering an adjacency matrix with a pFp at a desired location $K$ can be accomplished by picking some coefficients $\alpha^I_A$, then the coefficients $\alpha_A$ are unambiguously determined by the condition $T_A(K) = 0$,

$$\alpha_A = -\sum_{I} (\alpha^I_A e^{-ik_I} + \bar{\alpha}^I_A e^{ik_I}). \quad (8)$$

The only restriction imposed on the choice of $\alpha^I_A$ and $K_I$, following from the linear non-degeneracy, is for the matrix,

$$\frac{\partial T_A}{\partial k_I} = -i (\alpha^I_A e^{-ik_I} - \bar{\alpha}^I_A e^{ik_I}), \quad (9)$$

to have the maximal rank at $k = K$. When $D = 4$, this implies non-vanishing of its determinant. Moreover, in this case any small change to parameters $\alpha_A$ and $\alpha^I_A$ results at most in a motion of the pFp without destroying it. This is a manifestation of stability, resulting from the appropriate choice of the translational symmetry dictated by the ABS construction [2]. Hence, we have to assume $D = 4$. Only for this choice, pFps exist for a continuous range of graph parameters.

It might be asked whether $K$ is a unique pFp or if it can be made such by a clever choice of parameters. The answer to both questions is ‘No’, and for that there is an argument similar to one used in the Nielsen-Ninomiya theorem [12]. For a pFp $K_\sigma$ consider the topological charge [8],

$$n_\sigma = \frac{1}{24\pi^2} \int_{S^2_3} \text{tr}(dTT^{-1} \wedge dTT^{-1} \wedge dTT^{-1}), \quad (10)$$

where the integration is performed over a small momentum space sphere $S^2_{3}$ surrounding the point $K_\sigma$. As we
show below, the fluctuations around the pFp are described by a Weyl fermion with chirality \( n_\sigma = \pm 1 \). The charge \( n_\sigma \) is restricted to \( \pm 1 \) due to the linear nature of pFp. In a more general context, \( n_\sigma \in \mathbb{Z} \), and one ends up with a multiplet of Weyl fermions.

The compactness of the Brillouin zone implies that the sum over all points of the zone vanishes,

\[
\sum_\sigma n_\sigma = 0. \tag{11}
\]

The sign of the topological charge is determined by that of \( \det(\partial T/\partial k) \) at \( K_\sigma \). Therefore, in addition to the original pFp, there must be at least one more separate pFp with the opposite sign of \( \det(\partial T/\partial k) \).

Alternatively, a pFp can be regarded as a common point of intersection between four three-dimensional surfaces \( T_A(k) = 0 \) in \( \mathbb{R}^4 \). We are unaware of any theory describing how we can control such intersections. However, by using Mathematica we were able to model a number of such intersections. Thus, the simplest case, apart from no intersection, is the two-point intersection with \( n_1 = -n_2 \). In particular, within the parameter range where the surfaces are small enough, their shapes approach that of spheres, and four three-spheres intersect by either zero or two points, one-point intersection being the limiting degenerate case.

Now we shall show that the saddle point effective action is given by a free Dirac fermion model. Since the main contribution to the partition function in this approximation comes from modes near pFps, we can replace the adjacency operator \( T(k) \) by its linear expansion near these points. Therefore, the action takes the form,

\[
S = \sum_\sigma \int \frac{d^4k}{(2\pi)^D} \bar{\psi}_I \left( \frac{\partial T_A}{\partial k_I} \right)_{k = K_\sigma} k_I \sigma^A \psi_\sigma + \ldots, \tag{12}
\]

where we introduced the notation: \( \psi_\sigma = \psi(K_\sigma + k) \), and dots denote higher orders in \( k \).

At every pFp, we could relate the local momentum \( k_I \) to the macroscopic Cartesian momentum \( q_A \) by a linear transformation with matrix \( (\partial T/\partial k)_{K_\sigma} \). When the determinant of this matrix is positive, this is a legitimate transformation. However, when it is negative, the orientation is not preserved, but we can correct it by a modification of the sign of an odd number of columns or rows. Thus, the Cartesian momentum \( q_A \) can be related to \( k_I \) as follows,

\[
q_A = h^{\sigma I}_A k_I, \tag{13}
\]

where the positive determinant matrix \( h^\sigma \) is defined by,

\[
h^\sigma_0 = \left( \frac{\partial T_0}{\partial k_I} \right)_{K_\sigma}, \quad h^\sigma_a = n_\sigma \left( \frac{\partial T_a}{\partial k_I} \right)_{K_\sigma} \tag{14}
\]

and \( a = 1, 2, 3 \). A different choice would related to this one by a unitary transformation.

In terms of new variables the action (12) becomes,

\[
S = \int \frac{d^4q}{(2\pi)^D} \bar{\Psi} (I \otimes \sigma_0 + \sigma^a \otimes \sigma^3 q_a) \Psi, \tag{15}
\]

where we introduced the four component Dirac spinor \( \Psi \) given by,

\[
\Psi_\sigma = (\det h_\sigma)^{-1/2} \psi_\sigma. \tag{16}
\]

By identifying the Dirac matrix basis as,

\[
\gamma^0 = I \otimes \sigma^1, \quad \gamma^a = \gamma^0 \cdot (\sigma^a \otimes \sigma^3) = -i(\sigma^a \otimes \sigma^2), \tag{17}
\]

and by doing a continuous inverse Fourier transform, we can cast the effective action in the standard form of the free Dirac particle action,

\[
S_{\text{eff}} = -i \int d^4x \bar{\Psi} \gamma^4 \partial_A \Psi, \tag{18}
\]

where the bar denotes the Dirac conjugate: \( \bar{\Psi} = \Psi^\dagger \gamma^0 \).

**GRAPH DEFORMATIONS AND GAUGE/ GRAVITY BACKGROUND**

We have just shown that a translational invariant adjacency matrix under certain conditions brings a free Dirac fermion. A small variation of parameters that does not break the translational invariance does not change the spectrum of the theory. Large modification of parameters, however, can bring transitions to a phase with a different fermionic field content. But now, let us discuss what happens if we allow deformations which do not respect translational invariance. For example, let us consider an operator \( D \), whose coefficients are cell dependent,

\[
D_{n,m} = \alpha_n \delta_{n,m} + \sum_I (\alpha^I_n \delta_{n+I,m} + \alpha^I_n \delta_{n-I,m}). \tag{19}
\]

The quantum Darwinistic principle implies that the coefficients \( \alpha_n \) and \( \alpha^I_n \) are such that the continuum limit exists (see Ref. [11]). Interestingly, in the case of two pFps, there are two distinct situations depending on whether the matrices \( h^{1,2} \) are equal or not. Thus, the kernel of the continuum operator corresponding to (19), takes the form,

\[
D_{\sigma\sigma'}(x,y) = \varphi_{\sigma\sigma'}(x) \delta(h^\sigma \cdot x - h^{\sigma'} \cdot y) + i \xi_{\sigma\sigma'}(x) \frac{\partial}{\partial y}(\delta(h^\sigma \cdot x - h^{\sigma'} \cdot y) - i \chi_{\sigma\sigma'}(y) \frac{\partial}{\partial x}(\delta(h^\sigma \cdot x - h^{\sigma'} \cdot y). \tag{20}
\]

The contribution to components \( \xi_{\sigma\sigma'} \) and \( \chi_{\sigma\sigma'} \) comes from the Fourier modes of \( \alpha_n + \sum_I (\alpha_n^I e^{-ik_{n+I}} + \alpha_n^I e^{ik_{n-I}}) \), and \( \alpha_n^I e^{-ik_{n}} \), near \( K_\sigma - K_{\sigma'} \). Therefore, the off-diagonal
part is non-local as long as matrices $h^\sigma$ are not equal. In this case it should be dropped, and the resulting continuum action takes the form,

$$S_{\text{eff}} =\int d^4x (-i\bar{\Psi}^I\gamma^A\partial_I\Psi + \bar{\Psi}\chi_A\gamma^5\gamma^A\Psi + \bar{\Psi}\nu_A\gamma^A\Psi),$$  \hspace{1cm} (21)

where the fields $\xi, \chi,$ and $\nu$ are independent for $h_\sigma \neq h_{\sigma'}$ \hspace{1cm} ($\sigma \neq \sigma'$).

Although the action \hspace{1cm} (21) does not appear explicitly diffeomorphism invariant, it can be made such by an appropriate change of (field) variables \hspace{1cm} [13],

$$S_{\text{eff}} = -i\int d^4x \gamma^A\nabla_A\Psi,$$  \hspace{1cm} (22)

where $\gamma^I$ are the co-moving Dirac matrices in the vierbein $\xi^I,$

$$\gamma^I = \xi^I_A\gamma^A, \quad \{\gamma^I, \gamma^J\} = 2G^{IJ} \equiv 2\hbar\gamma^{AB}\xi^A_B,$$  \hspace{1cm} (23)

and $\nabla_A$ is the spinor covariant derivative in this background,

$$\nabla_I = \partial_I - \frac{1}{4}\omega_{ABI}\gamma^A\gamma^B + i\gamma^5\omega_{ABI} + V_I, $$  \hspace{1cm} (24)

$\omega_{ABI}$ being the corresponding spin connection. The fields $A_I$ and $V_I$ are, respectively, the external axial and vector Abelian gauge fields. By not receiving off-diagonal contribution in \hspace{1cm} (23), in the case of pFp asymmetry, the action can not get a mass term or an interaction which can produce it, as it happens in \hspace{1cm} (2 + 1)-dimensional case \hspace{1cm} [14].

By contrast, the off-diagonal terms in Eq. \hspace{1cm} (20) in the symmetric case are local and, therefore, produce a Yukawa coupling term,

$$\Delta S_{\text{symm}} = \int d^4x \phi \bar{\Psi}\Psi,$$  \hspace{1cm} (25)

where $\phi$ is a scalar field born by modes near momenta $K_\sigma - K_{\sigma'},$ of \hspace{1cm} (20). In symmetric case, however, there is no contribution leading to axial coupling, $A_I \equiv 0$. It is also worth noting that the symmetry of pFps is not stable: Unless it is enhanced dynamically, a generic graph deformation drives the system away from the symmetric point.

Turning now to the the gauge/gravity dynamics, it is determined by two main factors. First, there are microscopic forces which are completely unknown to us. There is no \textit{a priori} reason to believe that they share the same symmetry with the low energy theory, but as we know, they do not contribute in the large scale limit. The second factor is the fermionic back reaction. Apart from anomalies, it produces gauge/diffeomorphism invariant dynamical parts through what is known as Sakharov's mechanism \hspace{1cm} [15]. Unlike the fermionic part, which did not depend on the detailed parameters of the graph except the number and types of pFps, the couplings of induced gauge and gravity interactions are expected to depend on the locations of pFps as well as on the slopes of adjacency at these points.

**DISCUSSION**

In this letter, we raised the problem regarding a possible fine texture below the scale of applicability for quantum field theory approach. Some questions have been answered in this or will be answered in the accompanying paper \hspace{1cm} [10].

We have shown that the simplest configurations of fermionic graphs lead to Lorentz invariant gauge models in four-dimensional space-time as low energy theories. We restricted the adjacency to the nearest cell neighbour to make the analysis transparent. In fact, it is possible to add various non-local operators without changing the large scale behaviour, provided they vanish quickly enough with cell separation distance.

Also, including deformations beyond the nearest neighbour range increases the diversity of interaction terms, giving more control over the large scale theory. It makes the adjacency matrix an almost generic function, subject only to topological restrictions. Thus, graphs leading to Standard Model field content are easily obtainable once we know the singularity structure of the adjacency in the momentum space studied in Ref. \hspace{1cm} [8].

The analysis of non-local graphs as well as the analysis of the dependence of the gauge couplings on the details of the graph structure is a possible direction for a future research. It would be interesting to establish the range of the parameter space corresponding to Standard Model and to look for a matrix model in which the configurations from this range dominate in some regime.

**Acknowledgements.**

During two years of research on this project I benefited from many discussions in various places. I want to thank Ellis Lee, for the hard effort to make the language of this manuscript seem English.

---

\textsuperscript{*} E-mail: corneliu@gist.ac.kr

[1] S. Weinberg, *The Quantum Theory of Fields* 1st ed. (Cambridge University Press, 1995).

[2] R. Penrose, in *Magic Without Magic*, edited by J. Klauder (Freeman, San Francisco, 1972) pp. 333–354.

[3] S.-W. Kim, J. Nishimura, and A. Tsuchiya, Phys.Rev.Lett. \textbf{108}, 011601 (2012), arXiv:1108.1540 [hep-th].
[4] M. R. Douglas, JHEP 0305, 046 (2003), arXiv:hep-th/0303194 [hep-th].

[5] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Reviews of Modern Physics 81, 109 (2009), arXiv:0709.1163.

[6] P. Hořava, Phys. Rev. Lett. 95, 016405 (2005); G. Volovik, The Universe in a Helium Droplet The International Series of Monographs on Physics (Oxford University Press, 2009).

[7] M. Atiyah, R. Bott, and A. Shapiro, Topology 3, Supplement 1, 3 (1964).

[8] K. G. Wilson, Phys.Rev. B4, 3174 (1971).

[9] Unless otherwise stated, we will assume summation over repeated index of this type.

[10] C. Sochichi, In preparation.

[11] P. Hořava, JHEP 0305, 046 (2003), arXiv:hep-th/0303194 [hep-th].

[12] H. B. Nielsen and M. Ninomiya, Nucl.Phys. B185, 20 (1981).

[13] T. Kleinert, Multivalued Fields in Condensed Matter, Electromagnetism (World Scientific, 2008).

[14] C. Sochichi, Int.J.Mod.Phys. B26, 1250055 (2012), arXiv:1012.5354 [cond-mat.stat-mech].

[15] A. D. Sakharov, Sov. Phys. Dokl. 12, 1040 (1968).