Bounding the sum of the largest signless Laplacian eigenvalues of a graph

Aida Abiad∗†‡ Leonardo de Lima§ Sina Kalantarzadeh¶ Mona Mohammadi/uni2016 Carla Oliveira∗∗

Abstract

We show several sharp upper and lower bounds for the sum of the largest eigenvalues of the signless Laplacian matrix. These bounds improve and extend previously known bounds.

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1 Introduction

Consider \( G = (V, E) \) to be a simple graph with \( n \) vertices such that \( |V| = n \). Let \( N(v_i) \) be the set of neighbors of a vertex \( v_i \in V \) and \( |N(v_i)| \) its cardinality. The sequence degree of \( G \) is denoted by \( d(G) = (d_1(G), d_2(G), \ldots, d_n(G)) \), such that \( d_i(G) = |N(v_i)| \) is the degree of the vertex \( v_i \in V \) and \( d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G) \). The Laplacian matrix of \( G \) is defined as \( L = D - A \), where \( D \) is the diagonal matrix of the vertex degrees and \( A \) is the adjacency matrix of \( G \). The signless Laplacian matrix (or \( Q \)-matrix), defined as \( Q = A + D \), has received a lot of attention, see, e.g., [6, 7, 8]. The eigenvalues of \( L \) and \( Q \) are denoted as \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0 \) and \( q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \), respectively. For simplicity, the eigenvalues of \( Q \) and \( L \) are called here as \( Q \)-eigenvalues and \( L \)-eigenvalues of \( G \), respectively.

Using Schur’s inequality [17], it is known that

\[
\sum_{i=1}^{m} \lambda_i(G) \geq \sum_{i=1}^{m} d_i(G) \tag{1}
\]

∗Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands
†Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
‡Department of Mathematics and Data Science, Vrije Universiteit Brussel, Belgium (a.abiad.monge@tue.nl)
§Graduate Program in Mathematics, Federal University of Parana, Curitiba, Brazil (leonardo.delima@ufpr.br)
¶Sharif University of Technology, Iran (sinakalantarzadehhh@yahoo.com)
/uni2016 Sharif University of Technology, Iran (mona.mohammadi178@gmail.com)
∗∗Department of Mathematical, National School of Statistical Sciences, Rio de Janeiro, Brazil (carla.oliveira@ibge.gov.br)
and
\[ \sum_{i=1}^{m} q_i(G) \geq \sum_{i=1}^{m} d_i(G) \]  
for \(1 \leq m \leq n\). Note that if \(m = n\), we have equality in (1) and (2), because both terms correspond to the trace of \(L\) and \(Q\), respectively. An improvement of (1) is due to Grone [11], who proved that if \(G\) is connected and \(k < n\) then,
\[ \sum_{i=1}^{m} \lambda_i(G) \geq \sum_{i=1}^{m} d_i(G) + 1. \]  
(3)

The first author, Fiol, Haemers and Perarnau [1] showed a generalization and a variation of (3), as well as an extension of some inequalities by Grone and Merris [12].

There have been some results bounding the sum of the two largest signless Laplacian eigenvalues, see for example [2], [10] and [19]. Cvetković, Rowlinson and Simić [4] proved that \(q_1(G) \geq d_1(G) + 1\). After, Das [9] showed that \(q_2(G) \geq d_2(G) - 1\). An immediate lower bound is \(q_1(G) + q_2(G) \geq d_1(G) + d_2(G)\) (note that Schur’s inequality can also be used to obtain the same result).

In this paper, we use a mix of two types of interlacing (Cauchy and quotient matrix) to obtain several sharp lower and upper bounds on the sum of the largest signless Laplacian eigenvalues. In particular, we show a lower bound for \(q_1(G) + q_2(G)\) and characterize the case of equality. This bound improves previously known bounds. We also show several sharp bounds for the sum of the largest \(Q\)-eigenvalues, providing a \(Q\)-analog of Grone’s inequality (3). The paper is organized such that preliminary results are presented in Section 2, and Sections 3 and 4 are devoted to the main results.

## 2 Preliminaries

Some of our proofs use a classical result in matrix theory, the Cauchy interlacing theorem (see for instance [15, Theorem 4.3.8], [14]).

**Theorem 1 (Interlacing Theorem)** ([14]) Let \(A\) be a real symmetric \(n \times n\) matrix with eigenvalues \(\lambda_1 \geq \cdots \geq \lambda_n\). For some \(m < n\), let \(S\) be a real \(n \times m\) matrix with orthonormal columns, \(S^\top S = I\), and consider the matrix \(B = S^\top A S\), with eigenvalues \(\mu_1 \geq \cdots \geq \mu_m\).

(i) The eigenvalues of \(B\) interlace those of \(A\), that is,
\[ \lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \quad i = 1, \ldots, m, \]  
(4)

(ii) If the interlacing is tight, that is, if exist an integer \(k \in [0, m]\) such that \(\lambda_i = \mu_i\), for \(i = 1, \ldots, k\), and \(\mu_i = \lambda_{n-m+i}\), for \(i = k + 1, \ldots, m\), then \(SB = AS\).

Two interesting types of eigenvalue interlacing appear depending on the choice of \(B\): when \(B\) is a principal submatrix of \(A\) (the so-called Cauchy interlacing), and when \(B\) is the quotient matrix of a certain partition of \(A\). Our proofs in Section 4 will require a novel mix of the two types of eigenvalue interlacing.
The Cauchy interlacing theorem for the signless Laplacian matrix holds in a specific way. In [18, Theorem 2.6], for a vertex \( v \in V \), the authors proved that the \( Q- \)eigenvalues of \( G \) and \( G - v \) interlace, where \( G - v \) is a graph obtained from \( G \) removing the vertex \( v \):

**Theorem 2 ([18])** Let \( G \) be a graph of order \( n \) and \( v \in V \). Then for \( i = 1, \ldots, n - 1 \),

\[
q_{i+1}(G) - 1 \leq q_i(G - v) \leq q_i(G),
\]

where the right inequality holds if and only if \( v \) is an isolated vertex.

Cvetković et al. in [5] presented an edge removal version of the Cauchy interlacing theorem for the \( Q- \)eigenvalues by using line graphs:

**Theorem 3 ([5])** Let \( G \) be a graph on \( n \) vertices and let \( e \) be an edge of \( G \). Let \( q_1 \geq q_2 \geq \cdots \geq q_n \) and \( s_1 \geq s_2 \geq \cdots \geq s_n \) be the \( Q- \)eigenvalues of \( G \) and \( G - e \), respectively. Then

\[
0 \leq s_n \leq q_n \leq \cdots \leq s_2 \leq q_2 \leq s_1 \leq q_1.
\]

It turns out that the Cauchy interlacing theorem also holds for the Laplacian matrix of \( G \) as showed by Godsil and Royle [13, Theorem 13.6.2]:

**Theorem 4 ([13])** Let \( G \) be a graph on \( n \) vertices and let \( e \in E \) be an edge of \( G \). The \( L- \)eigenvalues of \( G \) and \( H = G - e \) interlace, that is,

\[
\lambda_1(G) \geq \lambda_1(H) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(H) = \lambda_n(G) = 0.
\]

Let \( A \) be a symmetric real matrix whose rows and columns are indexed by \( X = \{1, 2, \ldots, n\} \). Let \( \{X_1, X_2, \ldots, X_n\} \) be a partition of \( X \). The characteristic matrix \( S \) is the \( n \times m \) whose \( j-th \) column is the characteristic vector of \( X_j \) \( (j = 1, \ldots, m) \). Define \( n_i = |X_i| \) and the diagonal matrix \( K = \text{diag}(n_1, \ldots, n_m) \). Let \( A \) be partitioned according to \( \{X_1, X_2, \ldots, X_n\} \), that is

\[
A = \begin{bmatrix}
A_{11} & \cdots & A_{1m} \\
\vdots & \ddots & \vdots \\
A_{m1} & \cdots & A_{mm}
\end{bmatrix}
\]

where \( A_{ij} \) denotes the submatrix (block) of \( A \) formed by rows in \( X_i \) and the column in \( X_j \). Let \( b_{ij} \) the average row sum of \( A_{ij} \). Then the matrix \( B = (b_{ij}) \) is called the quotient matrix. We easily have \( KB = S^T AS \) and \( S^T S = K \). If the row sum of each block \( A_{ij} \) is constant the partition is called equitable. If each vertex in \( X_i \) has the same number \( b_{ij} \) of neighbors in part \( X_j \), for any \( j \) (or any \( j \neq i \)), the partition is called almost equitable.

**Lemma 5 ([3])** Let \( A \) be a symmetric matrix of order \( n \), and suppose \( P \) is a partition of \( \{1, \ldots, n\} \) such that the corresponding partition of \( A \) is equitable with quotient matrix \( B \). Then the spectrum of \( B \) is a sub(multi)set of the spectrum of \( A \), and all corresponding eigenvectors of \( A \) are in the column space of the characteristic matrix \( C \) of \( P \) (this means that the entries of the eigenvector are constant on each partition class \( U_i \)). The remaining eigenvectors of \( A \) are orthogonal to the columns of \( C \) and the corresponding eigenvalues remain unchanged if the blocks \( A_{i,j} \) are replaced by \( A_{i,j} + c_{i,j} J \) for certain constants \( c_{i,j} \) (as usual, \( J \) is the all-one matrix).

We will denote by \( K_n \), \( S_n \) and \( K_{n_1, n_2} \) the complete graph, star graph and complete bipartite graph, respectively such that \( n_1 \geq n_2 \) and \( n = n_1 + n_2 \).
3 Bounds on the sum of the two largest eigenvalues

Our main result of this section is a sharp lower bound on the sum of the two largest signless Laplacian (Theorem 12). Some preparation is required. To obtain the main result we first prove some auxiliary results for a subgraph $H$ of $G$ by considering the two vertices of the largest degrees and their neighbors.

Let $u$ and $v$ be the vertices with the two largest degrees of a graph $G$, that is, $|N(u)| = d_1(G)$ and $|N(v)| = d_2(G)$. A subgraph $H(V_H, E_H)$ of $G$ can be obtained by taking the vertex set as $V_H = \{v_i \in V \mid v_i \in N(u) \cup N(v) \cup \{u\} \cup \{v\}\}$ and the edge set as $E_H = \{(v_i, v_j) \in E \mid v_i \in \{u, v\} \text{ and } v_j \in N(u) \cup N(v)\}$.

We improve the lower bound from [4], $q_1(G) \geq d_1(G) + 1$. Roughly speaking, the proofs of our main results in this section (Theorems 12 and 14) follow from the fact that $q_1(G) + q_2(G) \geq q_1(H) + q_2(H)$ (by Theorems 2, 3, 4) and also that $d_1(G) + d_2(G) = d_1(H) + d_2(H)$ since we did not remove any vertex from $N(u)$ and $N(v)$ of $G$ to build the graph $H$. In fact, if we prove that $q_1(H) + q_2(H) \geq d_1(H) + d_2(H) + 1$, we are done. Before proving that, we need to introduce some notation, several preliminary results and two types of graphs obtained by the definition of $H = (V_H, E_H)$.

Let $S_1 = N(u) \setminus (N(v) \cup v)$, $S_2 = N(u) \cap N(v)$ and $S_3 = N(v) \setminus (N(u) \cup u)$, such that $|S_1| = r$, $|S_2| = p$ and $|S_3| = s$. Figure 1 displays the two possible types of graphs isomorphic to $H$. Notice that if $u$ and $v$ are not adjacent, $H$ belongs to $H(p, r, s)$ such that $d_1(G) = d_1(H) = p + r$ and $d_2(G) = d_2(H) = p + s$. If $u$ and $v$ are adjacent, $H$ belongs to $G(p, r, s)$ such that $d_1(G) = d_1(H) = p + r + 1$ and $d_2(G) = d_2(H) = p + s + 1$.

![Figure 1: Families of graphs of the type $H(V_H, E_H)$](image)

Lemmas 6 and 7 establish lower bounds to $q_1(G)$ and $q_2(G)$ in terms of $d_1(G)$ and $d_2(G)$ that will be useful for our purposes here.

**Lemma 6 ([4])** Let $G$ be a connected graph on $n \geq 4$ vertices. Then,

$$q_1(G) \geq d_1(G) + 1$$

with equality if and only if $G$ is the star $S_n$.

**Lemma 7 ([9])** Let $G$ be a graph. Then

$$q_2(G) \geq d_2(G) - 1.$$
Next, we improve the lower bounds of the previous lemmas for all graphs in $\mathcal{H}(p, r, s)$ and $\mathcal{G}(p, r, s)$. This will be crucial to later prove our main result.

**Proposition 8** For $p \geq 1$ and $r \geq s \geq 1$, let $G \in \mathcal{H}(p, r, s)$ be a graph on $n \geq 3$ vertices. Then

$$q_2(G) > d_2(G).$$

**Proof.** For $p \geq 1$, $r \geq s \geq 1$, consider $G \in \mathcal{H}(p, r, s)$. Labeling the vertices in a convenient way, we get

$$Q(G) = \begin{pmatrix} p + r & 0 & 1_{1xp} & 1_{1xr} & 0_{1xs} \\ p + s & 1_{1xp} & 1_{1xr} & 1_{1xs} \\ 1_{p1x} & 1_{p1x} & 2I_{pxp} & 0_{pxr} & 0_{pxs} \\ 1_{r1x} & 0_{r1x} & 0_{rxp} & 1_{rxr} & 0_{rsr} \\ 0_{s1x} & 1_{s1x} & 0_{sxp} & 0_{sxs} & 1_{sxs} \end{pmatrix}.$$

Observe that $x_j = e_3 - e_j$, for $j = 4, \ldots, p + 2$ are eigenvectors associated to the eigenvalue 2 which has multiplicity at least $p - 1$. Also, let us define $y_j = e_{p+3} - e_j$ for each $j = p + 4, \ldots, p + r + 2$ and $z_j = e_{p+r+4} - e_j$ for each $j = p + r + 4, \ldots, p + r + s + 2$. Observe that $y_j$ and $z_j$ are eigenvectors associated to the eigenvalue 1 with multiplicity at least $r + s - 2$. The remaining 5 eigenvalues are the same of the quotient matrix $M$ according to the equitable partition of $Q(G)$:

$$M = \begin{pmatrix} p + r & 0 & p & r & 0 \\ 0 & p + s & p & 0 & s \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of $M$ is given by $f(x, p, r, s) = x^5 - (s + r + p + 6)x^4 + ((r + p + 3)s + (p + 3)r + p^2 + 6p + 5)x^3 + ((-2r - p - 4)s + (-2p - 2)r - 2p^2 - 6p - 2)x^2 + ((s + r)p + p^2 + 2p)x$.

Considering $r = s + k$, where $k \geq 0$ note that $f(d_2(G), p, s + k, s) = (s + p)(ks^2 + s^2 + 2kps + 2p - 2ks - 2s + kps + kp^2 - kp) > 0$. As $f(0, p, r, s) < 0$, if we take $q_2(G) < y < q_1(G)$, then $f(y, p, r, s) < 0$. So, since $f(d_2(G), p, r, s) > 0$ and from Lemma 6, we get $q_2(G) > d_2(G)$. \hfill \Box

**Proposition 9** For $p \geq 1$ and $r \geq 1$, let $G \in \mathcal{H}(p, r, 0)$ be a graph on $n \geq 3$ vertices. Then

$$q_2(G) \geq d_2(G).$$

Equality holds if and only if $G = P_4$.

**Proof.** For $p, r \geq 1$, consider $G \in \mathcal{H}(p, r, 0)$. Labeling the vertices in a convenient way, we get

$$Q(G) = \begin{pmatrix} p + r & 0 & 1_{1xp} & 1_{1xr} \\ 0 & p & 1_{1xp} & 0_{1xr} \\ 1_{p1x} & 1_{p1x} & 2I_{pxp} & 0_{pxr} \\ 1_{r1x} & 0_{r1x} & 0_{rxp} & 1_{rxr} \end{pmatrix}.$$
If \( p = r = 1 \), then \( q_2(G) = d_2(G) = 2 \). If \( p \geq 1 \) and \( r \geq 2 \), observe that \( x_j = e_3 - e_j \), for \( j = 4, \ldots, p+2 \) are eigenvectors associated to the eigenvalue 2 which has multiplicity at least \( p-1 \).

Let us define \( y_j = e_{p+3} - e_j \) for each \( j = p+4, \ldots, p+r+2 \). Observe that \( y_j \) are eigenvectors associated to the eigenvalue 1 with multiplicity at least \( r-1 \). The remaining 4 eigenvalues are the same of the quotient matrix \( M \) according to the equitable partition of \( Q(G) \):

\[
M = \begin{pmatrix} p + r & 0 & p & r \\ 0 & p & p & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
\]

The characteristic polynomial of \( M \) is given by \( f(x, p, r) = x^4 + (-r-2p-3)x^3 + ((p+2)r + p^2 + 4p + 2)x^2 + (-pr - p^2 - 2p)x \). As \( f(-1, p, r) > 0 \), if we take \( q_2(G) < y < q_1(G) \), then \( f(y, p, r) < 0 \).

Note that \( f(d_2(G), p, r) = f(p, p, r) = rp^2 > 0 \). Therefore, from Lemma 6, we have \( q_2(G) > d_2(G) \). So \( q_2(G) \geq d_2(G) \) with equality if and only if \( G = P_4 \).

\[\Box\]

**Proposition 10** For \( r, s \geq 1 \), let \( G \in \mathcal{G}(0, r, s) \). Then,

\[ q_1(G) + q_2(G) > d_1(G) + d_2(G) + 1. \]

**Proof.** For \( r, s \geq 1 \), let \( G \in \mathcal{G}(0, r, s) \). Labeling the vertices of \( G \) conveniently, we get

\[
Q(G) = \begin{pmatrix} r+1 & 1 & 1_{1 \times r} & 0_{1 \times s} \\ 1 & s+1 & 0_{1 \times r} & 1_{1 \times s} \\ 1_{r \times 1} & 0_{r \times 1} & 1_{r \times r} & 0_{r \times s} \\ 0_{s \times 1} & 1_{s \times 1} & 0_{s \times r} & 1_{s \times s} \end{pmatrix}.
\]

Let us define \( y_j = e_3 - e_j \) for each \( j = 4, \ldots, r+2 \), and \( z_j = e_{r+3} - e_j \) for each \( j = r+4, \ldots, r+s+2 \). Observe that \( y_j \) and \( z_j \) are eigenvectors associated to the eigenvalue 1 with multiplicity at least \( r+s-2 \). The remaining 4 eigenvalues are the same of the quotient matrix \( M \) according to the equitable partition of \( Q(G) \):

\[
M = \begin{pmatrix} r+1 & 1 & r & 0 \\ 1 & s+1 & 0 & s \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.
\]

The characteristic polynomial of \( M \) is given by \( f(x, r, s) = x^4 + (-r-s-4)x^3 + ((r+2)s + 2r + 5)x^2 + (-r-s-2)x \). As \( f(0, r, s) > 0 \), if we take \( q_2(G) < y < q_1(G) \), then \( f(y, r, s) < 0 \). Since \( d_2(G) = s+1 \), we get \( f(d_2(G), r, s) = s(s+1)(r-s) \geq 0 \) which implies \( q_2(G) \geq d_2(G) \). From the equality conditions of Lemma 6, \( q_1(G) > d_1(G) + 1 \) and the result follows.

\[\Box\]

Next, in Proposition 11, we present some bounds to \( q_1(G) \) and \( q_2(G) \) when \( G \in \mathcal{G}(p, r, s) \) for \( p \geq 1 \), \( r \geq s \geq 0 \).
Proposition 11 For \( p \geq 1, r \geq s \geq 0 \), let \( G \in \mathcal{G}(p, r, s) \) be a graph on \( n \geq 3 \) vertices. Then

(i) If \( r = p = 1 \) and \( s = 0 \), then \( q_2(G) = d_2(G) \);

(ii) if \( p = 1 \) and \( r = s \), then \( q_1(G) > d_1(G) + \frac{3}{2} \) and \( q_2(G) > d_2(G) - \frac{1}{2} \);

(iii) if \( p \geq 2 \) and \( r = s \), then \( q_1(G) > d_1(G) + 2 \);

(iv) if \( p \geq 1 \) and \( r \geq s + 3 \), then \( q_2(G) > d_2(G) \);

(v) if \( p \geq 1 \) and \( r \in \{s + 1, s + 2\} \), then \( q_1(G) > d_1(G) + 1 + \frac{p}{n} \) and \( q_2(G) > d_2(G) - \frac{p}{n} \).

Proof. For \( p \geq 1, r \geq s \geq 1 \), let \( G \in \mathcal{G}(p, q, r) \). Labeling the vertices in a convenient way, we obtain

\[
Q(G) = \begin{pmatrix}
p + r + 1 & 1 & 1_{1 \times p} & 1_{1 \times r} & 0_{1 \times s} \\
p + s + 1 & 1_{1 \times p} & 0_{1 \times r} & 1_{1 \times s} \\
p_{x1} & 1_{p \times 1} & 2I_{p \times p} & 0_{p \times r} & 0_{p \times s} \\
p_{x1} & 0_{r \times 1} & 0_{r \times p} & 1_{r \times r} & 0_{r \times s} \\
p_{s1} & 0_{s \times 1} & 0_{s \times p} & 0_{s \times r} & 1_{s \times s}
\end{pmatrix}.
\]

Observe that \( x_j = e_3 - e_j \), for \( j = 4, \ldots, p + 2 \) are eigenvectors associated to the eigenvalue 2 which has multiplicity at least \( p - 1 \). Also, let us define \( y_j = e_{p+3} - e_j \) for each \( j = p+4, \ldots, p+r+2 \), and \( z_j = e_{p+r+3} - e_j \) for each \( j = p+r+4, \ldots, p+r+s+2 \). Observe that \( y_j \) and \( z_j \) are eigenvectors associated to the eigenvalue 1 with multiplicity at least \( r + s - 2 \). The others 5 eigenvalues are the same of the reduced matrix

\[
M = \begin{pmatrix}
p + r + 1 & 1 & p & r & 0 \\
p + s + 1 & 1 & p & 0 & s \\
p & 1 & 2 & 0 & 0 \\
p & 0 & 0 & 1 & 0 \\
p & 0 & 1 & 0 & 1
\end{pmatrix}.
\]

The characteristic polynomial of \( M \) is given by \( f(x, p, r, s) = x^5 + (\frac{2r-1}{32})(4r^2 + 12r + 25) \). For \( r = s \geq 1 \), we get \( f(d_1(G) + 3/2, 1, r, s) < 0 \) and \( f(d_2(G) - 1/2, 1, r, s) > 0 \). As from Lemma 6, \( f(d_1(G) + 1, 1, r, s) < 0 \) and we get \( f(d_1(G) + 3/2, 1, r, s) < 0 \), so \( q_1(G) > d_1(G) + 3/2 \). Also, from Lemma 7, \( f(d_2(G) - 1, 1, r, s) > 0 \) and we get \( f(d_2(G) - 1/2, 1, r, s) > 0 \), then \( q_2(G) > d_2(G) - 1/2 \).
(ii) \( p \geq 2 \) and \( r = s \): note that \( f(d_1(G)+2,p,r,r) = -(2r+3p+6)(pr-2r+p^2+p-2) < 0 \) and also from Lemma 6, \( f(d_1(G)+1,p,r,r) < 0 \). So, we can conclude that \( q_1(G) > d_1(G) + 2 \).

(iii) \( p \geq 1 \) and \( r \geq s + 3 \): note that \( f(d_2(G),p,s+k,s) = ks^3 + (3k-2)ps^2 + ((3k-5)p^2 + (k+2)p - k)s + (k-3)p^3 + (k+1)p^2 > 0 \) for \( k \geq 3 \). Also, from Lemma 7, we get \( f(d_2(G) - 1,p,s+k,s) > 0 \). So, \( q_2(G) > d_2(G) \).

(iv) \( p \geq 1 \) and \( r \in \{s+1,s+2\} \): note that \( n = 2s+p+r+2 \). Considering first \( r = s+1 \), we get \( f(d_1(G)+1+p/n,p,r,s) = f(p + s + 3 + p/(p + 2s + 3), p, s + 1, s) < 0 \) and \( f(d_2(G) - p/n,p,r,s) = f(p + s + 1 - p/(p + 2s + 3), p, s + 1, s) > 0 \). Using Lemmas 6 and 7 analogous to the previous cases, we get \( q_1(G) > d_1(G) + 1 + p/n \) and \( q_2(G) > d_2(G) - p/n \).

Now, setting \( r = s+2 \) and using a computational support, we find that \( f(d_1(G)+1+p/(p+2s+4), p, s+2, s) < 0 \) and \( f(d_2(G) - p/(p+2s+4), p, s+2, s) > 0 \). Using Lemmas 6 and 7 analogously to the previous cases, we get \( q_1(G) > d_1(G) + 1 + p/n \) and \( q_2(G) > d_2(G) - p/n \).

For the cases \( p, r \geq 1, s = 0 \), the proof is similar to the previous cases and the result follows. \( \square \)

**Theorem 12** Let \( G \) be a simple connected graph on \( n \geq 3 \) vertices. Then

\[
q_1(G) + q_2(G) \geq d_1(G) + d_2(G) + 1
\]

Equality holds if and only if \( G \) is a complete graph \( K_3 \) or a star \( S_n \).

**Proof.** Let \( G \) be a simple connected graph on \( n \geq 3 \) vertices. Assume that \( u \) and \( v \) are the vertices with largest and second largest degrees of \( G \), i.e., \( d(u) = d_1(G) \) and \( d(v) = d_2(G) \). Take \( H \) as a subgraph of \( G \) containing \( u \) and \( v \) such that \( H \) belongs to either \( \mathcal{H}(p,q,r) \) or \( \mathcal{G}(p,r,s) \). Note that \( d_1(G) + d_2(G) = d_1(H) + d_2(H) \) and from interlacing, Theorems 2 and 3, \( q_1(H) + q_2(H) \geq q_1(G) + q_2(G) \).

Firstly, suppose that \( H \in \mathcal{H}(p,r,s) \). Since \( G \) is connected, the cases \( p = 0 \) with any \( r \) and \( s \) are not possible. If \( p = 1 \) and \( r = s = 0 \), then \( H = \mathcal{H}(1,0,0) = S_3 \) and \( q_1(H) + q_2(H) = 4 = d_1(H) + d_2(H) + 1 \). If \( p = 2 \) and \( r = s = 0 \), then \( H = \mathcal{H}(p,0,0) = K_{2,p} \) and \( q_1(H) + q_2(H) = 2p + 2 > d_1(H) + d_2(H) + 1 = 2p + 1 \). If \( p, r \geq 1 \) and \( s = 0 \), from Proposition 9 and Lemma 6, we get \( q_1(H) + q_2(H) > d_1(H) + d_2(H) + 1 \). Now, if \( p \geq 1 \) and \( r \geq s \geq 1 \), from Proposition 8 and Lemma 6, follows that \( q_1(H) + q_2(H) > d_1(H) + d_2(H) + 1 \).

Now, suppose that \( H \in \mathcal{G}(p,q,r) \). If \( p = s = 0 \) and \( r \geq 1 \), \( H = \mathcal{G}(0,r,0) = S_{r+2} \) and \( q_1(H) + q_2(H) = r + 3 = d_1(H) + d_2(H) + 1 \). If \( p = 0 \) and \( r \geq s \geq 1 \), the result follows from Proposition 10. If \( p = 1 \) and \( r = s = 0 \), then \( H \) is the complete graph \( K_3 \) and \( q_1(H) + q_2(H) = 5 = d_1(H) + d_2(H) + 1 \). If \( p = 2 \) and \( r = s = 0 \), then \( H = \mathcal{G}(p,0,0) = K_2 \cup K_p \), i.e., the complete split graph, and it is well-known that \( q_1(H) = (n + 2 + \sqrt{n^2 + 4n - 12})/2 \) and \( q_2(H) = n - 2 \). It is easy to check that for \( p \geq 2 \), we have \( q_1(H) + q_2(H) > d_1(H) + d_2(H) + 1 \). If \( p \geq 1 \), \( r \geq s \geq 0 \), from Proposition 11 and Lemmas 6 and 7, we get \( q_1(H) + q_2(H) \geq d_1(H) + d_2(H) + 1 \).
From the cases above, the equality conditions are restricted to the graphs $K_3$ and $S_n$, and the result follows. □

Next, we show that a more general bound such as
\[ \sum_{i=1}^{m} q_i(G) \geq 1 + \sum_{i=1}^{m} d_i(G) \]
does not hold for $m \geq 3$. Consider $S_n^+$ to be the graph obtained from a star $S_n$ plus an edge.

**Proposition 13** Let $G$ be isomorphic to $S_n^+$. For $m \geq 3$,
\[ \sum_{i=1}^{m} q_i(G) < 1 + \sum_{i=1}^{m} d_i(G). \]

**Proof.** Let $G$ be isomorphic to $S_n^+$. In this case, $d_1(G) = n - 1, d_2(G) = d_3(G) = 2$ and $d_4(G) = \cdots = d_n(G) = 1$. From [16, Lemma 3.1], we have $q_3(G) = \cdots = q_{n-1}(G) = 1$ and also
\[
\begin{align*}
    &n < q_1(G) < n + \frac{1}{n} \\
    &3 - \frac{2.5}{n} < q_2(G) < 3 - \frac{1}{n}.
\end{align*}
\]
From [9] we know that $q_n(G) < d_n(G)$, and then we obtain
\[ 0 \leq q_n(G) < 1. \]

Since for $m \geq 3$,
\[ 1 + \sum_{i=1}^{m} d_i(G) = n + m + 1, \]
and also
\[ \sum_{i=1}^{m} q_i(G) < n + m + 1, \]
thus the result follows. □

Finally, we consider the inequality $\lambda_1(G) + \lambda_2(G) \geq d_1(G) + d_2(G) + 1$ by Grone [11], and characterize the extremal cases.

**Theorem 14** Let $G$ be a connected graph on $n \geq 3$ vertices. Then
\[ \lambda_1(G) + \lambda_2(G) \geq d_1(G) + d_2(G) + 1 \]
with equality if and only if $G$ is a star $S_n$.  

Theorem 15 (Cauchy and quotient matrix interlacing).  

Let \( G \) be a simple connected graph on \( n \geq 3 \) vertices. The result \( \lambda_1(G) + \lambda_2(G) \geq d_1(G) + d_2(G) + 1 \) follows from Grone in [11]. Now, we need to prove the equality case. Assume that \( u \) and \( v \) are the vertices with largest and second largest degrees of \( G \), i.e., \( d(u) = d_1(G) \) and \( d(v) = d_2(G) \). Let \( tK \) be the graph on \( t \) vertices and no edges. Take \( H \) as a subgraph of \( G \) containing \( u \) and \( v \) and isomorphic to either \( H(p, q, r) \cup (n - p - r - s - 2)K_1 \) or \( G(p, r, s) \cup (n - p - r - s - 2)K_1 \). Note that \( d_1(G) + d_2(G) = d_1(H) + d_2(H) \) and from interlacing Theorem 4, \( \lambda_1(G) + \lambda_2(G) \geq \lambda_1(H) + \lambda_2(H) \).

Firstly, suppose that \( H \in H(p, q, r) \cup (n - p - r - s - 2)K_1 \). In this case, \( H \) is bipartite and \( \lambda_i(H) = q_i(H) \) for \( i = 1, \ldots, n \) (see [4, Proposition 2.5]). The proof is analogous to Theorem 12 and the equality cases are similar. Then equality occurs when \( H = S_3 \).

Now, suppose that \( H \in G(p, r, s) \cup (n - p - r - s - 2)K_1 \). If \( p = 1, r = s = 0 \) then \( H = G(1, 0, 0) \cup (n - 3)K_1 = K_3 \cup (n - 3)K_1 \) and \( 6 = \lambda_1(H) + \lambda_2(H) > d_1(H) + d_2(H) + 1 = 5 \). If \( p \geq 2, r = s = 0 \), then \( 2p + 4 = \lambda_1(H) + \lambda_2(H) > d_1(H) + d_2(H) + 1 + 2p + 1 \). The remaining cases are similar to the ones of the Theorem 12 and equality holds when \( G = S_n \).

\[ \square \]

4 The general case

In this section, using a different approach than in Section 3, we obtain several sharp bounds on the sum of the largest signless Laplacian eigenvalues. We will see that for the case of \( q_1(G) + q_2(G) \), our bounds from Section 3 and 4 are incomparable.

From inequality (2) we have

\[ \sum_{i=1}^{m} q_i(G) \geq \sum_{i=1}^{m} d_i(G). \]  \hspace{1cm} (5)

for \( 1 \leq m \leq n \). If \( m = n \) then we have equality in (5), because both terms correspond to the trace of \( Q \). Similarly we have:

\[ \sum_{i=1}^{m} q_{n-m+i}(G) \leq \sum_{i=1}^{m} d_{n-m+i}(G). \]  \hspace{1cm} (6)

For a vertex set \( U \subset V \) such that \( |U| = m \), write \( \partial(U) \) as the set of vertices in \( U = V \setminus U \) with at least one adjacent vertex in \( U \), and \( \partial(U, \overline{U}) \) as the set of edges connecting vertices in \( U \) with vertices in \( \overline{U} \). The next result shows that the above bounds can be pushed further by using a mix of two types of eigenvalue interlacing (Cauchy and quotient matrix interlacing).

Theorem 15 Let \( G \) be a connected graph on \( n \) vertices. For any given vertex subset \( U = \{u_1, \ldots, u_m\} \) with \( 0 < m < n \), we have

\[ \sum_{i=1}^{m+1} q_{n-i}(G) \leq \sum_{u \in U} d_u + \frac{\sum_{\pi \in \overline{U}} d_{\pi} + 2E[U]}{n-m} \leq \sum_{i=1}^{m+1} q_i(G). \]  \hspace{1cm} (7)
Proof. Let \( U \subset V \) such that \(|U| = m\) where \(0 < m < n\). Consider the partition of the vertex set \( V \) into \( m + 1 \) parts such that \( U_i = \{u_i\} \) for \( u_i \in U, i = 1, \ldots, m \), and \( U_{m+1} = \overline{U} \). Then, the corresponding quotient matrix of this partition is

\[
B' = \begin{bmatrix}
Q_U & b'_{1,m+1} \\
\vdots & \\
b'_{m+1,1} & \cdots & b'_{m+1,m} & b'_{m+1,m+1}
\end{bmatrix},
\]

where \( Q_U \) is the principal submatrix of \( Q \), with rows and columns indexed by the vertices in \( U \), \( b'_{i,m+1} = |\partial(U_i,\overline{U})| \), \( b'_{m+1,i} = \frac{|\partial(U_i,\overline{U})|}{n-m} \), and \( b'_{m+1,m+1} = (\sum_{\pi \in \mathcal{P}} d_{\pi} + 2E[\overline{U}])/(n-m) \). Note \( \mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_{m+1} \) are the eigenvalues of \( B' \), then

\[
\sum_{i=1}^{m+1} \mu'_i = trB' = \sum_{u \in U} d_u + \frac{\sum_{\pi \in \mathcal{P}} d_{\pi} + 2E[\overline{U}]}{n-m}.
\]

From Theorem 1, we have that \( q_i(G) \geq \mu'_i \geq q_{n-m-1+i}(G), \) for \( i = 1, 2, \ldots, m \) and the result follows. \( \square \)

Remark 16 Consider \( G \) as a \( r \)-regular connected graph and let \( m = 1 \) in Theorem 15. Take an arbitrary vertex \( u_1 \in U \). In this case the lower bound provided in Theorem 15 is equal to \( 2r + r(n-2)/(n-1) \) which is better than the lower bound provided in Theorem 12 since \( n \geq 3 \). However, in general, those bounds are incomparable. For instance, for the star \( S_n \), the lower bound provided in Theorem 12 is better than the one in Theorem 15.

Next we investigate the tightness of the bounds from Theorem 15.

Proposition 17 Let \( H \) be the subgraph of \( G \) induced by \( \overline{U} \), and let \( q'_1 \geq \cdots \geq q'_{n-m} \) be the signless Laplacian eigenvalues of \( H \). Define \( b' = \frac{|\partial(U,\overline{U})|}{n-m} \). Then equality holds on the right hand side of (4) if and only if each vertex of \( U \) is adjacent to all or no vertices of \( \overline{U} \), and \( q_{m+1} = q'_1 + b' = d_\pi \), where \( \pi \in \overline{U} \).

Proof. Suppose equality holds on the right hand side of (4). Then

\[
\sum_{i=1}^{m+1} q_i = \sum_{i=1}^{m+1} q'_i \quad \text{and} \quad q_i \geq q'_i
\]

so \( q_i = q'_i \) for \( i = 1, \ldots, m+1 \), therefore the interlacing is tight and hence the partition of \( G \) is almost equitable. Each vertex in \( U \) is adjacent to all or 0 vertices of \( \overline{U} \) since each block should have a constant row and column sum, so each vertex in \( \overline{U} \) has a constant number of neighbors in \( U \) that is \( b' \). By calculation it can be deduced that \( H \) is \( q'_i \)-regular. Now by use of Lemma 5 we have that the eigenvalues of \( Q \) are \( q'_1, \ldots, q'_{m+1} \) together with eigenvalues of \( Q \) with an eigenvector orthogonal to the characteristic matrix \( C \) of the partition. These eigenvalues and eigenvectors remain unchanged if \( Q \) is changed into

\[
\tilde{Q} = \begin{bmatrix}
O & O \\
O & Q_{\pi'} + b'I
\end{bmatrix}.
\]
The considered common eigenvalues of $\tilde{Q}$ and $Q$ are $q'_1 + b' \geq \cdots \geq q'_{n-m-1} + b'$. So $Q$ has eigenvalues $q_1(= q'_1) \geq \cdots \geq q_{m+1}(= q'_{m+1})$, and $q'_1 + b' \geq \cdots \geq q'_{n-m-1} + b'$. Hence, we have $q_{m+1} = q'_1 + b'$. Conversely, if the partition of $G$ is almost equitable, $Q$ has eigenvalues $q'_1 \geq \cdots \geq q_{m+1}$, and $q'_1 + b' \geq \cdots \geq q'_{n-m-1} + b$. Since $q_{m+1} = q'_1 + b'$, it follows that $q'_i = q_i$ for $i = 1, \ldots, m$ (tight interlacing), therefore equality holds on the right-hand side of (4).

**Theorem 18** Let $G$ be a connected graph on $n$ vertices. For any given vertex subset $U = \{u_1, \ldots, u_m\}$ with $0 < m < n$, we have

$$
\sum_{i=1}^m q_{n-i+1} < \sum_{u \in U} d_u + \frac{4E[U]}{n-m} < \sum_{i=1}^m q_i.
$$

(9)

**Proof.** Take matrix $B'$ and its eigenvalues similarly as Theorem 15. By Theorem 15 we have,

$$
\sum_{i=1}^{m+1} \mu'_i = \text{tr}B' = \sum_{u \in U} d_u + \sum_{\pi \in \pi} d_{\pi} + \frac{2E[U]}{n-m}.
$$

(10)

**Claim 19** $\mu'_{m+1} < \frac{|\partial(U,\bar{U})|}{n-m} < \mu'_1$.

**Proof of Claim 19.** Let $\bar{E} = \{e_1, \ldots, e_k\}$ be the set of all edges of $G$ that have at least one endpoint in $U$. Define a $(m+1) \times k$ matrix $D$ as follows:

$$
D_{ij} = \begin{cases} 1, & \text{if } e_j \in \bar{E} \text{ is incident to at least one vertex in } U_i \\ 0, & \text{otherwise} \end{cases}
$$

Also, consider the $(m+1) \times (m+1)$ diagonal matrix $P$ with diagonal entries equal to 1 except the last one equal to $\frac{1}{\sqrt{n-m}}$:

$$
P = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{1}{\sqrt{n-m}} \end{bmatrix}.
$$

Now let $\tilde{B} = DD^T$. It is easy to see that $B' = P^2 \tilde{B}$, and it follows that

$$
P^{-1}B'P = P\tilde{B}P = A,
$$

which means $B'$ and $A$ are similar, so $\mu'_1$ and $\mu'_{m+1}$ are the largest and the smallest eigenvalues of $A$, respectively. Then by the Rayleigh principle we find that:

$$
\mu'_{m+1} \leq \frac{x^T A x}{x^T x} = \frac{x^T P^T DD^T P x}{x^T x} = \frac{||D^T P x||^2}{||x||^2} \leq \mu'_1
$$

Then, taking $x = (0, \ldots, 0, 1)$ the claim follows. Moreover, we show that both inequalities are strict. In Rayleigh principle a necessary condition for equality to be hold in
both inequalities is that $x$ must be an eigenvector of $A$. Assume that $x$ is an eigenvector of $A$. It means that all entries of the last column of $A$ except the last entry must be 0. Similarly, all entries of the last column of $\tilde{B}$ except the last entry must be 0. Since $P$ is a matrix such that by being multiplied from left(right) to $\tilde{B}$ it will only multiply the last row(column) of $\tilde{B}$ by $1/\sqrt{n-m}$. On the other hand, $G$ is connected, thus there exists an edge $e_1 \in \tilde{E}$, which has endpoints in $U_{m+1}$ and $U_1$. Thus the associated entry is always greater than 0 which is a contradiction. Thus both upper and lower bounds are strict.

Now, by multiplying $-1$ to each side of the claim inequality, and adding up $\sum_{i=1}^{m+1} \mu'_i$, we obtain

$$\sum_{i=2}^{m+1} \mu'_i < \sum_{i=1}^{m+1} \mu'_i + \frac{-\partial(U, \tilde{U})}{n-m} = \sum_{u \in U} d_u + \sum_{u \in \tilde{U}} d_i + 2E[\tilde{U}] + \frac{-\partial(U, \tilde{U})}{n-m}$$

$$= \sum_{u \in U} d_u + 4E[\tilde{U}] \frac{n-m}{n-m} < \sum_{i=1}^{m+1} \mu'_i,$$

and applying interlacing, we get $q_i \geq \mu'_i \geq q_{n-m-1+i}$, for $i = 1, \ldots, m+1$. Then (9) follows from observing $\sum_{i=1}^{m+1} \mu'_i \leq \sum_{i=1}^{m+1} q_i$ and $\sum_{i=1}^{m+1} q_{n-i+1} \leq \sum_{i=2}^{m+1} \mu'_i$. □

As a consequence of Theorem 15 we have the following two corollaries.

**Corollary 20** Let $G$ be a connected $k$-regular graph on $n = |V|$ vertices, having adjacency matrix $A$ with eigenvalues $k = \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. For any given partition $(U, \tilde{U})$ such that $|U| = m$, it holds

$$\frac{2E[\tilde{U}]}{n-m} \leq \sum_{i=1}^{m+1} \gamma_i.$$ 

**Proof.** Since $d_v = k$ for any vertex $v \in V(G)$, Theorem 15 implies that

$$(m+1)k + \frac{2E[\tilde{U}]}{n-m} \leq \sum_{i=1}^{m+1} q_i.$$ 

Note that $q_i = \gamma_i + k$ for $1 \leq i \leq n$ and thus the result follows. □

Note that for regular complete multipartite graphs, the bound in Corollary 20 holds with equality if and only if $m = n - 1$.

**Corollary 21** Let $G$ be a connected graph on $n$ vertices. If $I$ is an independent set of cardinality $\alpha > 1$, then

$$\frac{\alpha}{\alpha - 1} \sum_{i=n-\alpha+2}^{n} q_i(G) \leq \sum_{\pi \in I} d_{\pi}.$$ 

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Proof. Let $U = I$. From Theorem 15 and noting that $I$ has no edges we have

$$\sum_{u \in I} d_u + \sum_{\pi \in I} \frac{d_{\pi}}{\alpha} \leq \sum_{i=1}^{n-\alpha+1} q_i(G).$$

Now, note that

$$\sum_{u \in I} d_u = 2E(G) - \sum_{\pi \in I} d_{\pi}$$

and

$$\sum_{i=1}^{n-\alpha+1} q_i(G) = 2E(G) - \sum_{u=n-\alpha+2}^{n} q_i(G).$$

So the result follows. □

The next result is a $Q$-analog version and an extension of a bound by Grone and Merris for the sum of the largest Laplacian eigenvalues [12].

**Theorem 22** Let $G$ be a connected graph of order $n \geq 3$. Given a vertex subset $U \subset V$, with $m = |U| < n$ such that $G[U] = (U, E[U])$ be its induced subgraph. Then

$$\sum_{i=1}^{m+1} q_i \geq \sum_{u \in U} d_u + m - |E[U]|.$$  

Proof. Consider an orientation of $G$ with all edges in $E(U, \overline{U})$ oriented from $U$ to $\overline{U}$, and every vertex in $U \setminus \partial U$ having some outgoing arc (this is always possible as $G$ is connected). Let $Q$ be the corresponding oriented incidence matrix of $G$, and let $(D)_{ij} = |(Q)_{ij}|$. Write $D = [D_1 D_2]$, where $D_1$ corresponds to $E[U] \cup E(U, \overline{U})$, and $D_2$ corresponds to $E[\overline{U}]$. Consider the matrix $M' = D^T D$, with entries $(M)_{ii} = 2$, $(M)_{ij} = 1$ if the arcs $e_i, e_j$ are incident to the same vertex, and $(M)_{ij} = 0$ if the corresponding edges are disjoint, and define $M'_1 = D_1^T D_1$. Then $M'$ has the same nonzero eigenvalues as $Q = DD^T$, the signless Laplacian matrix of $G$, and $M'_1$ is a principal submatrix of $M'$. For every vertex $u \in U$, let $E_u$ be the set of outgoing arcs from $u$. Then $\{E_u|u \in U\}$ is a partition of $E[U] \cup E(U, \overline{U})$. Consider the quotient matrix $B_1 = (b_{ij})$ of $M'_1$ with respect to this partition. Then, $b_{uu} = d_u^+ + 1$ for each $u \in U$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the eigenvalues of $B_1$, then

$$tr B_1 = \sum_{i=1}^{m} \mu_i = \sum_{u \in U} d_u^+ + m = \sum_{u \in U} d_u - |E[U]| + m$$

and the result follows since the eigenvalues of $B_1$ interlace those of $M'_1$, which in turn interlace those of $M'$. □

Note that Theorem 22 can be improved by considering the partition $\mathcal{P} = \{E_u|u \in U\} \cup \{E[U]\}$ of the whole edge set of $G$: 

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Theorem 23 Let $G$ be a connected graph of order $n \geq 3$. Given a vertex subset $U \subset V$, with $m = |U| < n$, let $G[U] = (U, E[U])$ and $G[U]$ be the corresponding induced subgraphs. Let $q'_1$ be the largest signless Laplacian eigenvalue of $G[U]$. Then
\[ \sum_{i=1}^{m+1} q_i \geq \sum_{u \in U} d_u + m - |E[U]| + q'_1. \] (11)

Proof. First observe that the signless Laplacian matrix of $G[U]$ is $D_2D_2^\top$, and therefore $q'_1$ is also the largest eigenvalue of $D_2^\top D_2$. Next we apply interlacing to an $(m+1) \times (m+1)$ quotient matrix $B = S^\top MS$, which is defined slightly different than before. The first $m$ columns of $S$ are the normalized characteristic vectors of $E_u$ (as before), but the last column of $S$ equals $[0 \ v]^\top$, where $v$ is a normalized eigenvector of $D_2^\top D_2$ for the eigenvalue $q'_1$. Then $b_{m+1,m+1} = (D_2v)^\top D_2v = q'_1$, and we find $\text{tr}B = \sum_{u \in U} d_u + m - |E[U]| + q'_1$. \hfill \Box

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