Reliability-based analysis of machine structures using second-order reliability method

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Abstract
In design and manufacture of machine structures and elements, the characteristics and sizes are randomly changing factors because of the change of loads, the mechanical properties of materials, and the quality of manufacturing process (size tolerances). Thus, the reliability of structures is extremely important to be considered carefully. This paper presents the application of Second-Order Reliability Method (SORM) to analyze and design of CNC milling machine structure. Many examples are utilized to validate the use of the proposed SORM approach. Then, the analytical findings are verified by comparing with First-Order Reliability Method (FORM) and Monte Carlo Method (MCS). The obtained findings highlighted that the SORM method is more reliable than the FORM method. Besides, the result deviations between the SORM method and the MCS method are negligible. Therefore, SORM method is a potentially effective approach in calculation and design with the aims of improving the reliability, increasing safety, and working efficiency of machine structures.

Keywords: First-order reliability, Monte Carlo simulation, Second-order reliability

1. Introduction

In the traditional design method, known as deterministic design, machine structures and elements are commonly designed based on the criteria of ability to operate and their safety, which is presented as a safety factor. However, the design qualities are random variables because of the three main factors which are the change of parameters (the manufacturing errors, randomly changing external loads, mechanical properties of materials, etc.), errors of the modeling of calculation, and errors of the computational method. In the design of complex systems, small changes in the input parameters are the cause of loss of quality or impairment of the reliability and safety of products. The probabilistic design methods have been applied to determine the probability distribution of design variables rather than using the calculated value. The probabilistic design approach is one of many methods that ensure reliability, safety, quality, and economical products in the field of mechanical engineering (Nguyen, 2015) (Zhang, Jiang, Wang, & Han, 2015).

In reliability analysis, the popularly applied method is sensitivity-based approximation access, which comprises the matching moment method and the worst-case analysis (Du & Chen, 2001) (Ganji & Jowkarshorijeh, 2012). For the worst-case analysis method, all the changes are supposed to happen concurrently in the worst probable combinations. Based on this assumption, the first order Taylor expansion or optimization method are used to determine the worst value of the limit-state function. The standard deviation and mean value of a limit-state function are gathered with the moment matching methods. However, the moment matching method is commonly inadequately precise, specifically, in the case of a large uncertainty value. Generally, the sensitivity-based approach is subject to the accuracy issue, especially when the random variables are not presented in the normal distribution and have large variations. Analysis
and design based on the reliability, which is one of the Most Probable Points (MPP)-based approaches, is very effective to determine the failure probability (Haldar & Mahadevan, 2000) (Tu, Choi, & Park, 2001) (Baran, Tutum, & Hattel, 2013). However, the limit-state function is approximated by the technique of first-order Taylor series; thus, the accuracy of the method is not high and is sacrificed for highly nonlinear or high-dimensional limit-state function. Based on data sampling, Monte Carlo Simulation (MCS) is more comprehensive in the generation of the probability density function (PDF) and the cumulative distribution function (CDF) in the output of a system (Radoń, 2015) (Valdebenito, Pradlwarter, & Schueller, 2010).

Nevertheless, the shortcomings of MCS include high computational cost and the fact that great computational effort needs in some common cases. Therefore, this method is only suitable for computer calculations (Padmanabhan, Agarwal, Renaud, & Batill, 2006) (Wang & Ma, 2017). Response surface method (RSM) uses polynomial functions instead of limit-state functions and based on these functions to determine the reliability according to the approximation method (Qu, 2004) (Cheng, Zhao, Zhao, Sun, & Gu, 2015) (Gao, Yan, Xie, & Wu, 2013). RSM method overcomes the drawbacks of computing time and interference when compared with MCS methods. However, RSM necessitates identifying alternative forms of equations (first-order, second-order), planning matrices and determining the unknown coefficients of the replacement model. Replacement function should be revised correctly when compared with the original function and needs the least experimental effort (Nguyen, 2011) (Montgomery, 2012).

The above literature review indicates that Monte Carlo Simulation is the very high accuracy method, but it needs a large number of sample points, so it takes a lot of time for calculation and has a high computational cost. Therefore, it is necessary to use the alternative suitable method which reduces the calculation time but with the acceptable accuracy in some large and complicated structures which require a long calculation time. SORM and FORM are ones of these alternative methods. To compare FORM and SORM, the relative error of FORM and SORM compared to MCS are determined. The relative error value of any method is smaller, the method has higher accuracy. Reliability assessment based on Monte Carlo simulation (MCS) is used as a reference.

In this paper, the utilization of second-order reliability method (SORM) in analysis and design of machine elements according to reliability to enhance the efficiency and accuracy of calculations were presented. The analytical results were compared with the existing methods of first-order reliability (FORM) based on Monte Carlo methods (MCS). Reliability assessment based on Monte Carlo simulation (MCS) is used as a reference. To ensure that the MCS approach produces an accurate estimation, 5.10^8 sample points are used. The observations indicated that SORM method is more reliable than the FORM method and deviations between the MCS method and SORM method are insignificant.

2. Theoretical background

Reliability is determined as the probability of limit-state function \( g(\mathbf{X}) > 0 \), corresponding to the probability parameter of the random variable of \( \mathbf{X} \) which is located in the safety region, determined by the formula:

\[
R = P(g(\mathbf{X}) > 0) = \int_{g(\mathbf{X}) > 0} f(\mathbf{X}) d\mathbf{X}
\]  

(1)

The common method used in the reliability analysis is the current matching method, which includes the second-order approximate and first-order approximate method. The basis of this method is a simple calculation process by simplifying the formula under the integral sign \( f(\mathbf{X}) \) and using the approximate value of limit-state function \( g(\mathbf{X}) \).

There is similar to the method of FORM, the procedure of SORM method consists of two steps: first transforming original random variables into standard space, followed by the second-order approximation.

The original space transforms randomly into standard space, the first standardized mandibular integral \( f(\mathbf{X}) \) simplifies by transforming the random variables. Space of original random variables of \( \mathbf{X} = (x_1, x_2, \ldots, x_n) \) is called the space \( \mathbf{X} \) (Fig. 1a). All random design variables of the standard space \( \mathbf{U} = (u_1, u_2, \ldots, u_n) \) are transferred from space \( \mathbf{X} \). The standard deviation equals one, and the mean value of this variable is zero (Fig. 1b).

\[
f(x_i) = f(u_i) \quad \text{Where} \quad x_i = m_{x_i} + u_i S_{x_i}
\]  

(2)

The limit-state function after the transformation is: \( Y = g(\mathbf{U}) \).

Equation (1) can be rewritten as:
\[ R = P(g(U) > 0) = \int_{g(U)>0} \phi_U(U)du \]  

(3)

Since \( u_i \) is the independent variable, the function \( \phi_U(U) \) is determined:

\[ \phi_U(U) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{u_i^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\sum u_i^2}{2}} \]  

(4)

\[ g(\mathbf{U}) \approx g(\mathbf{u}^*) + \sum_{i=1}^{n} \frac{\partial g(\mathbf{U})}{\partial u_i} \bigg|_{\mathbf{u}^*} (u_i - u_i^*) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 g(\mathbf{U})}{\partial u_i \partial u_j} \bigg|_{\mathbf{u}^*} (u_i - u_i^*) (u_j - u_j^*) \]  

(5)

Equation (5) is rewritten into a vector form:

\[ g(\mathbf{U}) \approx g(\mathbf{u}^*) + \nabla g(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*)^T + \frac{1}{2} (\mathbf{U} - \mathbf{u}^*)^T \mathbf{H}(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*)^T \]  

(6)

Where \( \mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_n^*) \) is the expansion point, \( \nabla g(\mathbf{u}^*) \) is the gradient of the function \( g(\mathbf{U}) \) at \( \mathbf{u}^* \).

\( \mathbf{H}(\mathbf{u}^*) \) is a quadratic Hessian matrix of the function \( g(\mathbf{U}) \) at MPP given by:

\[ \mathbf{H}(\mathbf{u}^*) = \nabla^2 g(\mathbf{u}^*)_{ij} = \frac{\partial^2 g(\mathbf{u}^*)}{\partial u_i \partial u_j} \]  

(7)
An analytical solution for failure probability cannot be taken directly from Eq. (6) due to the quadratic Hessian matrix. Orthogonal and linear transformations are necessary to simplify Eq. (5) or Eq. (6) further. After the transformation is completed, \( g(U) \) will be a limit-state function of independent and standard normal variables. Then failure probability is estimated by (Du & Chen, 2001):

\[
F = P \left( g(X) < 0 \right) \Phi(-\beta) \prod_{i=1}^{n-1} \left( 1 + \beta k_i \right)^{-0.5}
\]

where \( \beta \) is the reliability index using FORM and \( k_i \) is principal curvatures of the limit-state function \( g(U) \) at the MPP.

**Orthogonal transform:**

For the convenience of computation of the failure probability, the transformations are coordinate rotations from \( U \) standard space to \( Y \) standard space so that the axis \( y_n \) coincides with the vector \( \beta \) (Fig. 2) through orthogonal matrix \( R_1 \):

\[
Y = R_1 U
\]

Orthogonal matrix \( R_1 \) is determined in two following steps:

- **Step 1:** Firstly, \( R_0 \) matrix can be established as below:

\[
R_0 = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \ldots & \ldots & \alpha_n
\end{bmatrix}
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the direction cosines of the unit gradient vector at the MPP

- **Step 2:** Calculate the orthogonal matrix \( R_1 \) with Gram–Schmidt orthogonalization

![Fig. 2 Second-order approximation in space Y.](image)

Based on the Gram-Schmidt orthogonalization formula, orthogonal matrix \( R_1 \) can be determined from \( R_0 \):

\[
\begin{aligned}
r_n &= r_{0n} \\
r'_l &= r_{0l} - \sum_{j=l+1}^{n} \frac{r_j r'_l}{r_j r_j} r_j, \quad r_i = \frac{r'_i}{\|r'_i\|}, \quad l = n - 1, n - 2, \ldots, 1
\end{aligned}
\]

**Calculate the principal curvatures \( k_i \):**

Excluding the last column and the last row of the matrix \( A_{ij} \), the principal curvatures \( k_i \) can be calculated as the eigenvalues of the matrix \( A_{i-1,j-1} \).

Matrix \( A_{ij} \) is given by:

\[
A_{ij} = \frac{R_i H(u^*) R_j^T}{\|\nabla g(u^*)\|}
\]

Solving the eigenvalue problem:

\[
A_{(i-1,j-1)} Z = k_i Z
\]

Then the equation is solved as follows:

\[
\text{Det}(A_{(i-1,j-1)} - k_i) = 0
\]
The eigenvalues are acquired when Det designates the determinant of a matrix.

\[ k = (k_1, k_2, \ldots, k_{n-1}) \]  

(15)

The general process of the SORM method is presented in Fig. 3:

![Fig. 3 The general procedure of the SORM.](image)

3. Numerical example

SORM, FORM, and MCS methods are applied to determine the reliability of the body of the CNC milling machine as in Fig. 4. In Fig. 4, hypothesize that the body is composed of four elements linked together such as the body head (1), the body beam (2), the body column (3) and the body base (4). To simplify the calculations and ensure that there is no loss of generality, it is realized that the body head, the body beam, the body column is a cantilever beam corresponding to the length \( L_1, L_2, L_3 \), and the body base is the length of the beam, \( L_3 \) is supported on two bearings.

![Fig. 4 Model of a CNC milling machine.](image)

Thus, the problem of reliability analysis for CNC milling machine body is equivalent to the problem of reliability analysis for the system which consists of 4 elements (1), (2), (3), and (4). The reliability of each element in the system is first analyzed, and then they are combined to consider the reliability of the system. This Section presents only the limited results for the reliability analysis of element (1).

The cantilever beam element (1) with hollow square cross-section is subjected to forces \( F_X, F_Y, F_Z \), and moment \( M_Z \) (Fig. 5). The forces \( F_X, F_Y, F_Z \), yield strength and length of the beam are the random variables with values given in
Table 1.

Table 1 The value of the random variables

| Parameters                  | Mean value | Standard deviation |
|-----------------------------|------------|--------------------|
| Force $F_x$, N              | 7500       | 750                |
| Force $F_y$, N              | 7500       | 750                |
| Force $F_z$, N              | 5500       | 550                |
| Yield strength $\sigma_y$, MPa | 400       | 45                 |
| Length of beam $L$, mm      | 350        | 10                 |
| Elastic modulus $E$, MPa    | $2.10^5$   | $2.10^5$           |
| Moment $M_z$, Nmm           | $5.10^5$   | 0                  |

Firstly, the Most Probable Point (MPP) - based approaches was used to design (with reliability $R = 0.9999$):
- According to working target as durability: $a = 5\text{mm}$ and $b = 50\text{mm}$
- According to working target as stiffness: $a = 16\text{mm}$ and $b = 120\text{mm}$

In the case of a machine structure, the stiffness index is more important than durability. According to working target as durability, the limit - state function has been written as below:

$$
 g(X) = \sigma_y - \sigma_{\max} = \sigma_{\text{ch}} - \sqrt{\left(\frac{F_z}{4a(b-a)} + \frac{6LbF_x}{b^4 - (b-2a)^4}\right)^2 + 4\left(\frac{M_z}{2a(b-a)^2}\right)^2}
$$

with $a = 5\text{mm}$ and $b = 50\text{mm}$, the results with different methods are presented:

**FORM method:**
After four loops, the convergence problem. Findings are illustrated in Table 2.
**MCS method:**
The simulation results with N = 5000000 are represented in Fig. 6 and Table 3.

![Fig. 6 Histograms for stress, strength, and performance function with Monte Carlo simulation.](image)

**Table 3 Result of Monte Carlo simulation**

| Simulation samples N | Mean value m_p | Standard deviation S_p | Reliability R |
|----------------------|----------------|------------------------|--------------|
| 5000000              | 174.91943265   | 49.93970127            | 0.9997531    |

**SORM method:**
Probability of failure:  F = 0.000241
Reliability:          R = 0.999759
The result of the reliability analysis by MCS, FORM, and SORM is shown in Table 4.

![Fig. 6 Histograms for stress, strength, and performance function with Monte Carlo simulation.](image)

**Table 4 Result of reliability analysis according to working target as durability**

| Method    | MCS     | FORM    | SORM    |
|-----------|---------|---------|---------|
| Reliability R | 0.9997531 | 0.9997619 | 0.999759 |

Relative errors of the SORM and FORM methods compared with MCS are as follows:

\[ \epsilon_{\text{FORM}} = 3.56\% \]
\[ \epsilon_{\text{SORM}} = 2.39\% \]

\[ \epsilon_{\text{SORM}} < \epsilon_{\text{FORM}}: \text{For this problem, the accuracy level in the SORM method is higher than the FORM method.} \]

According to the worked target as stiffness, the limit-state function has been written as below:

\[ g(X) = f_{\text{lim}} - f_{\max} = f_{\text{lim}} - \sqrt{\left(\frac{F_x L^3}{3EI_y}\right)^2 + \left(\frac{F_y L^3}{3EI_x}\right)^2} \]

with \( f_{\text{lim}} \) = limited stiffness; a = 16mm and b = 120mm. The results with different methods are presented in Table 5.

![Fig. 6 Histograms for stress, strength, and performance function with Monte Carlo simulation.](image)

**Table 5 Result of reliability analysis according to working target as stiffness**

| Method    | MCS     | FORM    | SORM    |
|-----------|---------|---------|---------|
| Reliability R | 0.9997439 | 0.9997528 | 0.9997407 |

Relative errors of the FORM and SORM methods are compared with MCS:

\[ \epsilon_{\text{FORM}} = 3.48\% \]
\[ \epsilon_{\text{SORM}} = 1.25\% \]

\( \epsilon_{\text{SORM}} < \epsilon_{\text{FORM}} \): For this problem, the accuracy level in the SORM method is higher than the FORM method.

This study only focuses on comparing the accuracy of FORM and SORM methods based on the MCS method because MCS is the method with the highest accuracy. To compare FORM and SORM, the relative error of FORM and SORM compared to MCS are determined. The relative error value of any method is smaller; the method has higher accuracy. Reliability assessment based on Monte Carlo simulation (MCS) is used as a reference. The results also demonstrated that the deviations between two methods of SORM and MCS are smaller the deviations between FORM and MCS. Moreover, the calculation time of FORM and SORM are nearly the same but equal half of the calculation time of MCS. The result of the reliability analysis of machine structure under two indexes suggests that the CNC machine body is firstly calculated by stiffness standard, and then re-checked by the reliability standard.

**Sensitivity analysis:**

The sensitivity analysis of the random variables should be conducted to determine the degree of influence of the random variables to the failure probability. The analytical results are illustrated in Table 6.

| Random variable | Sensitivity          |
|-----------------|----------------------|
| Mean value \( F_x \) | 5.0268x10^{-07} |
| Standard deviation \( S_{F_x} \) | 7.4246x10^{-07} |
| Mean value \( F_z \) | 1.9353x10^{-08} |
| Standard deviation \( S_{F_z} \) | 8.0702x10^{-10} |
| Mean value \( L \) | 1.2196x10^{-05} |
| Standard deviation \( S_L \) | 5.8273x10^{-06} |
| Mean value \( \sigma_y \) | 1.7753x10^{-05} |
| Standard deviation \( S_{\sigma_y} \) | 5.5561x10^{-05} |

Table 6 showed that the yield strength’s standard deviation is the most significant impact on failure probability. A survey of the influence of the yield strength’s standard deviation on the failure probability of a cantilever beam element using three methods of MCS, FORM, and SORM is conducted to make the reader clearer.

![Graphs](image1)

a. According to the working target as durability

b. According to the working target as stiffness

Fig. 7 Variation of failure probability concerning the standard deviation of durability and stiffness indexes.

The relationship between the failure probability of a cantilever beam and the standard deviation of durability and stiffness is shown in Fig. 7a and b, respectively. It is found from Fig. 7 that the failure probability obtained by using the FORM, the SORM, and the MCS increase with the increase of the standard deviation and the accuracy of the SORM method is higher than the FORM method. Reliability assessment based on Monte Carlo simulation (MCS) is used as a reference to access the accuracy of the FORM and SORM for this problem.
4. Conclusions

This paper presented the Second-Order Reliability Method (SORM) to analyze and design the structure of a CNC milling machine under working targets of durability and stiffness. The accuracy of the SORM results is investigated by comparing with the outcomes of both FORM and MCS. The following conclusions can be drawn from the obtained results:

- Monte Carlo Simulation is the very high accuracy method, but it needs a large number of sample points, so it takes a lot of time for calculation and has a high computational cost. In this paper, $5 \times 10^6$ sample points were employed to make sure the MCS approach generates an accurate estimation.

- Reliability assessment based on Monte Carlo simulation (MCS) is used as a reference for FORM and SORM methods. The relative error of FORM and SORM compared to MCS were determined. The relative error value of any method is smaller; the method has higher accuracy.

- The deviations between SORM and MCS are lower than the deviations between FORM and MCS in both durabilities ($\varepsilon_{\text{SORM}} = 2.39\%$ compared with $\varepsilon_{\text{FORM}} = 3.56\%$) and stiffness ($\varepsilon_{\text{SORM}} = 1.25\%$ compared with $\varepsilon_{\text{FORM}} = 3.48\%$) indexes. In other words, the accuracy level of the SORM method is higher than the FORM method. Moreover, the calculation time of FORM and SORM are nearly the same but equal half of the calculation time of MCS.

- The calculated results indicate that the SORM is more accurate than FORM in the stiffness index. So, it is recommended that SORM should be used to calculate the stiffness of the machine structure.

- The sensitivity factors indicated the important role of each random variable in the probability of failure. The reliability index $\beta$ or the sensitivity of the failure probability demonstrated insignificant differences in the random variables to be explored, which related to much useful information of the statistical variation. Moreover, the sensitivity analysis is also conducted for random variables; the results showed that the standard deviation of the yield strength $S_{\text{y}}$ and the limited stiffness $S_{\text{lim}}$ are the most significant impact on the reliability of the beam element. Thus, in some cases, adjusting the reliability is the most appropriate plan to change the value of this random variable.

- Additionally, this paper has also surveyed the influence of the yield strength’s and limited stiffness standard deviations to the reliability of the cantilever beam element using three methods of MCS, FORM, and SORM. The results have also highlighted that the SORM method is more reliable than the FORM method. Based on the achieved analytical results, SORM method is considerable potential in calculation and design to improve the safety, working efficiency, and reliability of machine structures.

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