A Consistency Relation for Power Law Inflation in DBI Models

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ABSTRACT: Brane inflation in string theory leads to a new realization of power law inflation which can give rise to significant non-gaussianity. This can happen for any throat geometry if the scalar potential is appropriate. This note presents a consistency relation connecting the running of the nonlinearity parameter characterizing the non-gaussianity and the scalar and tensor indices. The relationship is valid assuming that the throat geometry and scalar potential support power law inflation, regardless of the level of non-gaussianity.

KEYWORDS: String theory; Cosmology

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1. Introduction

Given the importance of inflation in our current view of cosmology it is natural and important to try to understand the details of it in the framework of string theory. From the string theory perspective general relativity is a low energy effective field theory, which receives corrections both at the classical and quantum level. These corrections may be crucial in the very early stages of evolution of our Universe. String theory should also determine the degrees of freedom relevant at the time inflation is expected to occur; specifically, it should provide an inflaton.

One possible scenario is brane inflation [1]–[10], which interprets inflation as the motion of a D3-brane down a throat in a warped Calabi-Yau compactification [11]. Brane inflation has a rather distinct character because the inflaton is identified with a brane position, and the relevant scalar field kinetic energy functional is non-canonical. Its form is determined by T-duality to be a Dirac-Born-Infeld action [12]. When restricted to spatially homogeneous configurations the action of the inflaton
reduces to a DBI scalar field theory studied in a number of papers over the past few years [13]–[20]. The difference between the DBI action and a canonical scalar field action may be interpreted as a classical correction coming from string theory. Brane inflation also introduces a new interpretation for the end of inflation and the thermalization of standard model degrees of freedom: these phenomena come about in consequence of brane annihilation.

It is clearly very important to try to determine observational possibilities which could distinguish this scenario from other options. From the point of view of comparing inflationary models to observation there are a number of properties which significantly restrict the spectrum of possibilities. In this context the most important quantities are inflationary observables like the scalar and tensor spectral indices and the primordial non-gaussianity, as well as the running of these quantities.

One of the interesting features of DBI inflation is the natural appearance of significant levels of primordial non-gaussianity in the spectrum of curvature perturbations. This is a direct consequence of the nonlinear corrections to the scalar kinetic terms. Furthermore, as Chen has emphasized [21], deviations from scale invariance responsible for the running of the scalar and tensor spectral indices also induce running of the non-gaussianity. The current observational limits allow quite significant levels of non-gaussianity and it will be very interesting to see whether it will turn out to be non-vanishing. At the moment one has to regard it as an important dimension of the inflationary parameter space. In terms of the non-gaussianity parameter $f_{NL}$ the current limits [22] give $|f_{NL}| < 300$. Single field inflation models give $|f_{NL}| \approx 1$ at most [23], so if coming experiments observe $|f_{NL}|$ of the order of a few or more it will indicate that models of inflation based on a single canonical field need to be extended either by allowing nontrivial dynamics (as in DBI models) or by having multiple scalars evolving during the inflationary stage.

In an interesting recent paper on brane inflation [15] Lidsey and Seery have derived a rather general relation involving the nonlinearity parameter often used to describe primordial non-gaussianity in certain simple kinematical configurations. This note applies the same approach to power law inflation in DBI scalar field theories. The result is a relation between the inflationary observables involving the running of
the nonlinearity parameter. The observational prospects for actually measuring this quantity are remote, but perhaps not hopeless.

2. Inflationary Observables

Inflationary observables related to the primordial perturbation spectra have been calculated (to leading order in the Hubble slow roll parameters) by Garriga and Mukhanov [24] for a wide class of scalar field theories, which can be described by the action

\[ S = \int d^4x \sqrt{-g} (R + P(X, \phi)) , \] (2.1)

where \( X \equiv -\frac{1}{2}(\partial \phi)^2 \). Their results can be written in terms of Hubble slow roll parameters

\[ \epsilon_H = -\frac{1}{H} \frac{d}{dt} \ln H \] (2.2)

\[ \eta_H = -\frac{1}{H} \frac{d}{dt} \ln \epsilon_H \] (2.3)

\[ \sigma_H = -\frac{1}{H} \frac{d}{dt} \ln c_s \] (2.4)

where \( c_s^2 = p_{,X}/\rho_{,X} \) is the speed of sound:

\[ c_s^2 = \frac{P_{,X}}{P_{,X} + 2X P_{,XX}} . \] (2.5)

The spectral indices are then given by

\[ n_S - 1 = -2\epsilon_H + \eta_H + \sigma_H \] (2.6)

\[ n_T = -2\epsilon_H . \] (2.7)

These expressions are valid in the leading order in Hubble slow roll parameters (2.2)-(2.3), which are assumed to be small during the observable phase of inflation.

For the sequel one also needs to recall the notion of the “nonlinearity” parameter \( f_{NL} \), which is an often used measure of non-gaussianity\(^1\). A simple and explicit formula, valid in a wide range of scalar field theories defined by (2.1), has recently been obtained in [17]:

\[ f_{NL} = \frac{35}{108} \left( \frac{1}{c_s^2} - 1 \right) - \frac{5}{81} \left( \frac{1}{c_s^2} - 1 - 2\Lambda \right) \] (2.8)

\(^1\)See for example [23, 17] for the precise definition.
where
\[ \Lambda \equiv \frac{X^2 P_{XX} + \frac{2}{3} X^3 P_{XXX}}{XP, X + 2X^2 P, XX}. \] (2.9)

As emphasized by Chen [21], deviations from scale invariance should manifest themselves also in the running of the non-gaussianity. A measure of it is the index
\[ n_{NL} \equiv \frac{d \ln f_{NL}}{d \ln k} \] (2.10)
defined in [21].

3. DBI scalar field theories

The inflaton in brane inflation scenarios is an open string mode, which implies that its dynamics are described by the Dirac-Born-Infeld action. For spatially homogeneous inflaton configurations the action takes the form [13, 5]
\[ S = -\int d^4x \, a(t)^3 \left\{ f(\phi)^{-1}(\sqrt{1 - f(\phi)\dot{\phi}^2} - 1) + V(\phi) \right\}. \] (3.1)

The function \( f \) appearing here can be expressed in terms of the warp factor in the metric and the \( D3 \)-brane tension. The function \( f \) appearing here is positive by construction\(^2\).

It is convenient to use the Hamilton-Jacobi formalism [29]–[32], which makes use of the Hubble parameter expressed as function of the scalar field\(^3\). The basic point is to eliminate the field derivative using the relation
\[ \dot{\phi} = -\frac{2M_P^2}{\gamma} H', \] (3.2)
where \( \gamma \) is given as a function of \( \phi \) by
\[ \gamma(\phi) = \sqrt{1 + 4M_P^4 f(\phi)H'(\phi)^2}. \] (3.3)

Using this one can calculate the Hubble slow roll parameters
\[ \epsilon_H = \frac{2M_P^2 (H')^2}{\gamma \left(\frac{H}{H'}\right)^2}. \] (3.4)

\(^2\)Similar actions with negative \( f \) have also been discussed in the literature [26]–[28].

\(^3\)In the context of DBI scalar field theories the Hamilton-Jacobi formalism was introduced in [33] and was recently discussed in [34].
\[ \sigma_H = -\frac{2M_P^2 H' \gamma'}{\gamma H' \gamma} \tag{3.5} \]
\[ \eta_H = \frac{4M_P^2 H''}{\gamma H} - 2\epsilon_H + \sigma_H. \tag{3.6} \]

The formula for the nonlinearity parameter (2.8) in the present case simplifies, since one has \[ c_s = \gamma^{-1} \] and \( \Lambda = 0 \). This leads to the simple result \[ \eta \], \[ \gamma \]:
\[ f_{NL} = \frac{35}{108}(\gamma^2 - 1). \tag{3.7} \]

This shows that non-gaussianity in DBI models becomes large in the “ultra-relativistic” regime \( \gamma \gg 1 \) \[ \gamma \].

Lidsey and Seery \[ \gamma \] have noted that using (2.7) and (3.7) one can turn the expression for the tensor to scalar ratio
\[ r = \frac{16\epsilon_H}{\gamma} \tag{3.8} \]
into a consistency relation involving only observable parameters:
\[ 8n_T = -r \sqrt{1 + \frac{108}{35} f_{NL}}, \tag{3.9} \]
which is valid for any DBI scalar field theory, and generalizes the usual consistency relation appearing in \[ \gamma \]. It is a very interesting, testable, prediction of the brane inflation scenario.

The authors of \[ \gamma \] also considered a special case of inflation near the bottom of a warped throat \[ \gamma \] to derive further relations between observable parameters in that situation. In a similar spirit, the following section turns to power law inflation in DBI scalar field theories, where one can obtain another consistency relation of this type, involving the running non-gaussianity parameter (2.10), which can easily be calculated in this class of models:
\[ n_{NL} = -4M_P^2(1 + \frac{35}{108 f_{NL}}) \frac{\gamma' H'}{\gamma H}. \tag{3.10} \]

This is valid to leading order in the Hubble slow roll parameters. As explained in the following section, if one assumes power law inflation then using (3.10) it is possible to rewrite (2.6) as another consistency relation.
4. Power Law Inflation

It was found by Silverstein and Tong [13] that for the case of an AdS throat (where \( f(\phi) = \lambda/\phi^4 \)) a quadratic potential with a suitably high inflaton mass leads to power law inflation\(^4\) in the “ultra-relativistic” regime \( \gamma \gg 1 \). It was subsequently pointed out that power law inflationary solutions exist in DBI scalar field theories even when \( \gamma \) is not large [20]. Furthermore, for any throat geometry there is a potential which leads to power law inflation for some range of parameters. This generalizes the well known fact that in the case of canonical kinetic terms exponential potentials lead to power law inflation [36].

Power law inflation occurs when the parameter \( w \) in the baryotropic equation of state \( p = w\rho \) is constant and \( w < -1/3 \). As shown in [20], power law inflationary solutions will exist if the potential is of the form

\[
V(\phi) = 3M_P^2 H(\phi)^2 - \frac{\gamma(\phi) - 1}{f(\phi)},
\]

where \( H(\phi) \) satisfies the differential equation

\[
4M_P^2 H'^2 = 3(w + 1)H^2 \sqrt{1 + 4M_P^4 f H'^2}
\]

with \( w < -1/3 \).

The essential property of power law inflation is that the parameter \( \epsilon_H \) is constant. Indeed, from (3.4) and (4.2) one concludes that

\[
\epsilon_H = \frac{3}{2}(w + 1).
\]

One immediate consequence is that the tensor spectral index does not run, since by virtue of (2.7) it is constant. Furthermore, a measurement of the tensor spectral index would determine the parameter \( w \), which in the model of [13] is related to the inflaton mass [20].

Since \( \epsilon_H \) is constant it also follows that \( \eta_H \) (defined in (2.3)) vanishes\(^5\). This makes it possible to derive another consistency relation involving the spectral indices,

\(^4\)Other realizations of power law inflation in string theory are described in [34] and [35].

\(^5\)Some authors (e.g. [5]) define a different “\( \eta \)” parameter in this context, related to \( \eta_H \) by \( 2\eta_D = \eta_H + 2\epsilon_H - \sigma_H \). In that language power law inflation implies the relation \( \eta_D = \epsilon_H - \sigma_H/2 \).
the non-gaussianity, and running of the nonlinearity parameter \((2.10)\). Indeed, from (3.5) and (3.10) it follows that
\[
\sigma_H = \frac{1}{2} n_{NL} (1 + \frac{35}{108} \frac{1}{f_{NL}})^{-1}.
\] (4.4)

Using this and \(\eta_H = 0\) in (2.6), (2.7) one finds
\[
n_S - n_T = 1 + \frac{1}{2} n_{NL} (1 + \frac{35}{108} \frac{1}{f_{NL}})^{-1},
\] (4.5)

which is a relation between observable parameters. It is valid for any DBI scalar field theory solution describing power law inflation. In particular, it does not assume simplifications which occur in the “ultra-relativistic” limit, so one can also consider the case of \(f_{NL}\) small or zero in this expression. This implies, in particular, that power law inflation with canonical kinetic terms has \(n_S - n_T = 1\).

In the “ultra-relativistic” limit, when the non-gaussianity is large, this relation can be further simplified to
\[
n_S - n_T = 1 + \frac{1}{2} n_{NL}.
\] (4.6)

While the prospect of measuring \(n_{NL}\) seems distant today, these relations may be tested at some point in the future.

5. Conclusions

Single field inflation with canonical kinetic energy terms leads to negligible non-gaussianity \([37, 38]\). While it is too early to tell whether observation will require more general models of inflation, a lot of attention has been devoted to models where large non-gaussianity may naturally occur. One possibility is models with multiple scalars \([25]\). Another option is DBI inflation, which can generate significant non-gaussianity during a power law inflationary stage.

In field theoretical models power law inflation is realized by an exponential scalar potential, so there is no natural mechanism for inflation to end. One needs to supplement the exponential potential by some external agent which terminates inflation. In the context of brane inflation this role is played by a tachyon which appears when
the mobile $D3$-brane gets within a warped string length of the anti-brane at the bottom of the throat. The process of brane annihilation ends inflation and the energy released is (hopefully \cite{39}, \cite{40}) transferred to standard-model degrees of freedom\footnote{This process could in fact be quite complex, as recently emphasised in \cite{41}, \cite{42}.} localized in another throat in the compactification manifold. One may thus argue that power law inflation finds a very natural place in the brane inflation scheme.

The consistency relation (4.5) is a consequence of assuming power law inflation, but it is valid in DBI scalar field theories without necessarily assuming the “ultra-relativistic” limit $\gamma \gg 1$. It is also worth stressing that it is not restricted to the specific realization of power law inflation discussed in \cite{13}, i.e. a quadratic potential and an anti-de-Sitter throat\footnote{As discussed in reference \cite{18}, this case is probably already ruled out by observation.}. This is rather important, in that there are many contributions to the scalar potential, which are at the moment hard to control. There are also various possibilities for warped throats in type IIB compactifications, and different opinions as to which section of the throat is relevant for inflation\footnote{\cite{8}}, as well as to the direction of the $D$-brane motion\footnote{\cite{6}, \cite{7}}. The consistency relation derived here does not assume a specific choice in these matters; it should be valid whenever the resulting inflationary stage has power law character.

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