The structure of fixed-point tensor network states characterizes patterns of long-range entanglement

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The algebraic structure of representation theory naturally arises from 2D fixed-point tensor network states, which conceptually formulates the pattern of long-range entanglement realized in such states. In 3D, the same underlying structure is also shared by Turaev-Viro state-sum topological quantum field theory (TQFT). We show that a 2D fixed-point tensor network state arises naturally on the boundary of the 3D manifold on which the TQFT is defined, and the fact that exactly the same information is needed to construct either the tensor network or the TQFT is made explicit in a form of holography. Furthermore, the entanglement of the fixed-point states leads to an emergence of pre-geometry in the 3D TQFT bulk. We further extend these ideas to the case where an additional global onsite unitary symmetry is imposed on the tensor network states.

I. INTRODUCTION

By now it is widely accepted that topological phases originate from the long-range entanglement existing in the condense matter system. Tensor networks,2–3, which focus on the wave functions of the system instead of the Hamiltonian, are generally considered as a natural tool to capture the behavior of long-range properties in a local way. The most successful examples of tensor network states include the Matrix Product States (MPS)4–6 in 1D and the related Projected Entangled Pair States7 in 2D, both of which serve as an efficient ansatz for ground states of topological phases in their respective dimensions.

Besides its popularity in studying strongly-correlated systems, entanglement and tensor networks have also attracted increasing attention from the high energy theory community, in various attempts of realizing8–13 the holographic14,15 AdS/CFT correspondence16, and serving as a framework for Loop Quantum Gravity17–20, both under the spirit of “geometry from entanglement”. It is thus of theoretical interest to better understand the structure underlying tensor networks of topological phases, so as to formulate a more definitive theoretical framework for describing quantum entanglement. This is the subject we are concerned about in this paper, exemplified with a description of entanglement patterns in 2D topological phases.

A tensor network is built from graphs consisting of interconnected tensors, imitating the structure of discrete lattices. The geometry of the network is generated by the pattern of interactions, namely, two sites in the network are close to each other if and only if they are entangled. Every tensor living on the sites of the network can be understood as a building block of entanglement.

To illustrate, consider for example the celebrated AKLT states21 in a spin-1 chain. Such a state can be obtained from a parton construction. As in Fig.1 one regards every spin-1 degrees of freedom on site \( n \) as a composite object consisting of two spin-1/2’s at \( n_L \) and \( n_R \), and links each spin-1/2 spin on the site \( n_L(n_R) \) to its nearest neighbor on \( (n-1)_R \) \((n+1)_L \) with a singlet bond. One then projects into the physical subspace with a spin-1 degree of freedom at each site. From the perspective of representation theory, the two spin-1/2’s \( n_L \) and \( n_R \) can be combined as \( \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \). The operator \( P \) which projects into the physical subspace annihilates the first term on the right hand side and keeps only the spin-1 representation:

\[
P : \frac{1}{2} \otimes \frac{1}{2} \rightarrow 1.
\]

FIG. 1: A parton construction of the AKLT state (see text for details).

The tensor network representation of the AKLT state consists of tensors \( T_{ij} \) at every site of the lattice, where the index \( i \) \( \in \{0, \pm 1\} \) labels the physical spin-1 degrees of freedom on site \( n \), while \( \mu, \nu \in \{1, \downarrow\} \) label the auxiliary degrees of freedom associated with spin-1/2 partons at \( n_L \) and \( n_R \). The tensor \( T \) can thus be understood as an adjoint map \( T = P^\dagger \).

The fusion algebra \( \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \) is a realization of entanglement, in the sense that after the fusion, the quantum states can no longer be factored as a product state of its local constituents (the spin-1/2 partons). This is manifested in the tensor network states in two ways: (i) when viewed as the adjoint of the projection \( P \), the tensor \( T_{ij}^\dagger \) provides a way to encode the entanglement between \( n_L \) and \( n_R \); (ii) contraction of tensors on neighboring sites introduces maximal entanglement across the adjacent sites \( n_L \) and \( (n-1)_R \) (or \( n_R \) and \( (n+1)_L \)).
In two dimensions, there exist intrinsic topological orders not protected by any symmetry. Following the discussion above, one would expect that in order to encode the entanglement of the topological phase in a tensor network, the structure of the tensors should be similar to that of a representation theory. Namely, it should incorporate the fusion the fusion algebra. For an intrinsic topological phase we further require the entanglement to be long-range. The pattern of long-range entanglement is captured by the fixed-point tensor network states that are invariant under renormalization group (RG) transformations of the tensor network.

One natural question thus arises: for a general tensor network state to capture the long-range physics and to be a RG fixed-point state, what are the constraints that need to be satisfied? It turns out that in 2D, the input data for the tensor network indeed form a representation theory, in the form of a unitary fusion category (UFC). In Sec. II we briefly review the algebraic definition of a unitary fusion category (UFC). In Sec. III we discuss how the UFC structure arises from the fixed-point properties of general triple-line tensor network states, and conversely, how to generate fixed-point tensor network states from a UFC.

In Sec. IV we provide a geometrical point-of-view on the structure of fixed-point tensor network states by appealing to 3D state-sum topological quantum field theory (TQFT). The long-range physics of a phase with topological order is described by a TQFT. The state-sum construction of TQFTs discretizes the underlying manifold into a “lattice” or “graph”, making explicit the locality of the theory.

Recently the correspondence between 1D fixed-point tensor network states and 2D state-sum TQFT has been formulated rigorously. In one dimension higher, the state-sum construction of 3D TQFT was proposed by Turaev and Viro, and later generalized by Barrett and Westbury, which requires a UFC as the input data. Given the UFC input data and a triangulation of the 3D manifold Σ, one can combinatorially define a topological invariant \( \tau_{\Sigma} \) that is independent of the specific triangulation.

When the underlying manifold Σ contains boundaries, the TQFT can be viewed as a holographic map from its 3D bulk to the 2D boundary. Upon taking the Poincaré duality, this map produces the desired fixed-point tensor network state. Conversely, starting from a 2D fixed-point tensor network state, we show the generation of pre-geometries for the 3D bulk. It is a pre-geometry in the tensor network state, we show the generation of pre-geometries for the 3D bulk to the 2D boundary. Upon taking the Poincaré duality, this map produces the desired fixed-point tensor network state.

Conversely, starting from a 2D fixed-point tensor network state, we show the generation of pre-geometries for the 3D bulk. It is a pre-geometry in the tensor network state, we show the generation of pre-geometries for the 3D bulk to the 2D boundary. Upon taking the Poincaré duality, this map produces the desired fixed-point tensor network state. Conversely, starting from a 2D fixed-point tensor network state, we show the generation of pre-geometries for the 3D bulk. It is a pre-geometry in the tensor network state, we show the generation of pre-geometries for the 3D bulk to the 2D boundary. Upon taking the Poincaré duality, this map produces the desired fixed-point tensor network state.

In Sec. V we extend the framework to symmetric fixed-point tensor network states, which possesses a global onsite, finite, and unitary symmetry \( G \). The algebraic structure of these theories is given by \( G \)-extension of the UFC \( C \), while the pre-geometric structure is closely related to 3D Homotopy Quantum Field Theory [29, 31]. The construction is parallel to that of symmetry-enriched string-net models [32, 33].

II. REVIEW OF UNITARY FUSION CATEGORIES

To prepare for the discussion of the algebraic structure of fixed-point tensor network states, in this section we briefly review the concept of a unitary fusion category (UFC).

A UFC \( C \) is a set of data \( \{ I, d, N, G \} \) subject to some consistency conditions. \( I \) is the set of (isomorphism classes of) simple objects in \( C \). We require a trivial object \( 0 \in I \). For every \( j \in I \) there is a number \( d_j \in \mathbb{R} \) called the quantum dimension of \( j \), with \( d_0 = 1 \). The rank-3 tensor \( N_{ijk} \) is a non-negative integer and describes the fusion rules between the objects \( i, j \) and \( k \). More specifically, the direct sum decomposition of the tensor product \( i \otimes j \otimes k \) will include \( N_{ijk} \) times the trivial object \( 0 \). It is this feature of tensor product decomposition that gives UFC the interpretation of a representation theory. We assume multiplicity-free fusion rules throughout the paper, which means that we restrict to the case of \( N_{ijk} \in \{0, 1\} \) for \( \forall i, j, k \in I \). We are also led to define the dual object \( j^* \) as the only object that realizes \( N_{0,j^*} = N_{j^*,0} = N_{j^*,j} = 1 \). It satisfies \( j^{**} = j \) and \( d_j = d_{j^*} \). Finally, to every six objects \( i, j, k, l, m, n \in I \) we assign a quantum 6j-symbol, which is a rank-6 tensor \( G_{ikln}^{jlm} \in C \) (Relaxation of the multiplicity-free assumption would lead to four additional indices for the \( G \)-symbols). We will assume full tetrahedral symmetry of the \( G \)-tensors:

\[
G_{ikln}^{mij} = G_{ikn}^{mj} = G_{ikln}^{jlm} = \alpha_m \alpha_n G_{i^*k^*n^*}^{j^*l^*m^*}.
\] (2)

The number \( \alpha_j \) is the Frobenius-Schur indicator: \( \alpha_j = sgn(d_j) \). The three equal signs correspond to the three generators of the \( S_3 \) symmetric (or tetrahedral) group, thus the name tetrahedral symmetry. Relaxing this condition leads to additional phase factors in the above equation that are the second or third roots of unity. In the case where the input data are finite groups, the relaxation of tetrahedral symmetry can lead to the Dijkgraaf-Witten construction [31] and the twisted quantum double model [35] based on three-cocycles, where time-reversal or(anti) parity symmetry can generically be broken. Dropping the multiplicity-free condition and relaxing tetrahedral symmetry would complicate the problem, but we expect the main features of the correspondence to remain qualitatively the same.

For \( C \) to be a UFC, the above tensors need to satisfy certain consistency conditions, including:

(UFC1) Compatibility of \( d_j \) and \( N_{ijk} \):

\[
d_j d_k = \sum_k N_{ijk} d_k.
\] (3)
(UFC2) Pentagon equation:
\[ \sum_n d_n G_{kp^n} G_{mj^n} G_{lk^n} = G_{pq^*} G_{rm^*}. \]

(UFC3) Orthogonality:
\[ \sum_n d_n G_{kp^n} G_{pt^*} = \delta_{pq} N_{mkp}. \]

A useful identity that can be derived from above axioms is
\[ G_{ij^*k} v_j v_k = N_{ijk}. \]

A UFC naturally arises from the representation theory of a finite group \( G \), i.e. \( C = \text{Rep}(G) \). The elements in the label set \( I \) correspond to irreducible representations of \( G \), and the \( N_{ijk} \)-tensor corresponds to the multiplicity of the representation \( k^* \) in the direct sum decomposition of the tensor product \( i \otimes j \). The \( G \)-tensors are simply the Racah \( G \)-symbols of the group representation. More generally, a UFC is the representation category of a \( C^* \)-weak Hopf algebra.

### III. ALGEBRAIC STRUCTURE OF FIXED-POINT TENSOR NETWORK STATES

In this section, we demonstrate the correspondence between fixed-point tensor network states and UFCs from an algebraic point of view. Part \( \text{III A} \) derives the structure of the category from the fixed-point property of tensor network states, while part \( \text{III B} \) deals with the converse.

#### A. Fixed-Point Tensor Network States Give Rise to a UFC

An example of a general 2D tensor network is displayed in Fig. 2. The tensors living on the vertices of the graph have one physical index \( M \) that extends into the third dimension (out of the paper), as well as \( 4 + 4 \) indices that correspond to internal degrees of freedom living on the links \( (j's) \) and plaquettes \( (\mu's) \) of the graph, which are auxiliary and are to be summed over. Generally the vertices in the network can be of valence \( n \), with each tensor possessing \( 2n + 1 \) total indices.

The tensor network state of Fig. 2 is
\[ |\Psi\rangle = \sum_{\{M_k\}} \text{tTr} \left[(T^{M_1})_{j_1 j_2 j_3} \otimes (T^{M_2})_{j_4 j_5} \otimes \cdots \right] |M_1, M_2, M_3, \cdots\rangle, \]

where the tensor trace \( \text{tTr} \) indicates that all the internal indices \( \{j_i\} \) and \( \{\mu_i\} \) are contracted. Note that the tensor network commonly used is the special case where all auxiliary degrees of freedom \( \mu \)’s that live on the plaquettes are taken to be trivial.

To discuss the properties of fixed-point tensor network states, we work on a trivalent graph, or more specifically a honeycomb lattice which is bipartite and has \( A,B \) sublattices. More general graphs can be easily obtained from trivalent graphs.

Assign labels \( i,j,k, \cdots \in I \) to every oriented link of the tensor network graph. For every \( j \in I \) labeling some link, reversing the orientation of the link replaces \( j \) with a dual label \( j^* \in I \). We require the existence of an identity label \( 0 = 0^* \in I \). Associate labels \( \mu, \nu, \lambda, \cdots \in I \) to each plaquette of the graph. These degrees of freedom are “nonlocal”, in the sense that they can only be seen when looking at entire plaquettes. To encapsulate them in a strictly local way, we expand the above construction into a triple-line structure, following a procedure similar to that in Refs. [36,37]. As shown in Fig. 3 for each of the three links that originally connected to some vertex, we sandwich it between two additional links (the physical indices \( \{M_i\} \) are suppressed for simplicity). Upon projecting to the configurations that satisfy \( \mu = \mu' = \cdots, \nu = \nu' = \cdots \), etc., one can see the plaquette degrees of freedom are restored.

To construct a tensor network state, we assign a phys-
In 1D, RG flow corresponds to performing scale transformations by combining two or more adjacent tensors into one composite tensor. In 2D, RG transformations for tensor networks have been worked out in Refs. 59 11 in an approximate way. Exact invariance of tensor network states under RG flow in 2D can be regarded as an invariance of the tensor network state under 2D dual Pachner moves (see Fig. 4 below). These moves are discrete versions of diffeomorphisms of the underlying manifold. In Ref. 11, the authors discussed similar properties of fixed-point wave functions where the degrees of freedom live on the links of a network. The general situation where plaquette degrees of freedom are taken into consideration follows in a parallel way.

Each Pachner move induces a linear transformation between the Hilbert spaces of different graphs, characterized by the coefficients $f_1$, $f_2$, $f_3$. The top move, denoted as $O_1$, is the 2-2 recoupling move, while the second and third ones are the 1-3 and 3-1 moves. Since the dimension of Hilbert spaces is generically changed during the moves, we do not combine the latter two as is usually done in mathematical literature. The physical indices $\{M_a\}$ are understood to be attached to the vertices and thus suppressed in the figure.

The physical motivation for considering these moves comes from the fact that we are interested only in long-range physics. The two diagrams involved in the 2-2 recoupling move, when viewed from far away, both appear as a single four-valent vertex. If our tensor-network state is a fixed-point one, the two ways of decomposing this four-valent vertex into two three-valent vertices (by singular value decomposition) should be essentially the same, differing from one another only by a unitary transformation. The latter two 3+1 moves correspond to usual local scale transformations of the graph, which allows us to take a zoomed-out view of the tensor network.

Note that in the 2–2 move, the plaquettes degree of freedom (colored cyan) are not changed. However, in the 1→3 (3→1) move, an additional closed string $\mu$ is added (removed) from the configuration. Consequently, while $f_1$ have no dependence on the plaquette strings, $f_2$ and $f_3$ do include $\mu$ as a nontrivial parameter.

In order for a tensor network state to be invariant under the Pachner moves, we require the following two necessary conditions:

**(C1)** The moves should be norm-preserving in the ground-state subspace. If $|\Psi'| = O_1 |\Psi\rangle$, then $\langle \Psi' | \Psi' \rangle = \langle \Psi | O_1 O_1 | \Psi \rangle = \langle \Psi | \Psi \rangle$. We emphasize that $\Psi'$ and $\Psi$ are not in the same Hilbert space, and that $O_1$ may not square matrices, i.e., the inverse matrices are not defined. If one rotates the graph by 90 degrees, then $O_1^T$ can again be viewed as a $O_1$. The norm-preserving constraint then reads

\[
(OO_1^T OO_1) = 1, \quad (OO_2 OO_3) = 1, \quad (OO_3 OO_2) = 1, \quad (8)
\]

where the $1$ are identity matrices (of different dimensions).

**(C2)** Two sequences of moves that result in the final tensor network configuration should be equivalent. If a final graph labeling $\{j_1', j_2', \cdots\}$ is obtained from some initial labeling $\{j_1, j_2, \cdots\}$ through two (or more different sequences of Pachner moves, then we require the set of tensors $T_1(j_1', j_2', \cdots)$ and $T_2(j_1', j_2', \cdots)$ on each final graph configuration to be the same.

\[
OO_{\alpha_1} OO_{\beta_1} OO_{\gamma_1} \cdots = OO_{\alpha_2} OO_{\beta_2} \cdots. \quad (9)
\]

These two conditions constrain the form of the functions $f_1$, $f_2$, $f_3$ in above Fig. 4. From the first equation in (8), one can derive, in terms of components,

\[
\delta_{j_5 j_5'} = \sum_{j_3 j_4} f_1(j_4, j_1, j_2, j_3, j_5', j_5) f_1(j_3, j_1, j_2, j_4, j_5, j_5'). \quad (10)
\]

Similar formulas can be obtained for the other two equations in (8).

Now we turn to condition (P2), and construct commutative diagrams from sequences of $O_\alpha$ operators that
result in identical tensor network configurations. Requiring these diagrams to commute will allow us to place various consistency conditions on the \( f_i \) matrix elements.

For tensor network configurations with two and three uncontracted legs, there are no nontrivial commutative diagrams. For tensor network configurations with four uncontracted legs, the only operations we are allowed to do are already fully captured by \( OO_2 \) and \( OO_3 \). But constraints do arise for commutative diagrams involving tensor networks with five uncontracted legs. Indeed, choose the two sequences below in Fig.

\[ \text{FIG. 5: The two different sequences of Pachner moves that share the same initial and final configurations.} \]

The constraint that the above diagram must commute leads to

\[
\sum_{j_5} f_1(j_1, j_2, j_3, j_4, j_5, j_6) f_1(j_5^{*}, j_2, j_7, j_8, j_3, j_4)
\times f_1(j_3, j_1, j_7, j_5, j_6^{*}, j_8^{*}) = f_1(j_4, j_5, j_7, j_6, j_3, j_8) f_1(j_1, j_2, j_8^{*}, j_3^{*}, j_5, j_6^{*}).
\]

(11)

Below we show that the functions \( f_1, f_2, f_3 \) are closely related to the \( 6j \)-symbols (\( G \)-tensors) introduced in the previous section and quantum dimensions (\( d \)'s), thereby arriving at a UFC.

We introduce a new set of symbols with six parameters by

\[
\frac{G_{\mu\nu}^{i\sigma}}{G_{\mu}'G_{\nu}'G_{\sigma}'} = f_i(i, j, k, l, m, n),
\]

(12)

the above conditions (C1),(C2) reduce to

\[ (P1): \delta_{j_5 j_5''} = \sum_{j_6} d_{j_5} v_{j_5} v_{j_5''} G^{j_1' j_2 j_5}_{j_3 j_4 j_5} G^{j_2 j_3 j_5''}_{j_1 j_5 j_5'}, \]

(13)

\[ (P2): \sum_{j_5} d_{j_5} G^{j_1' j_2 j_5''}_{j_3 j_4 j_5} G^{j_2 j_3 j_5''}_{j_1 j_5 j_5'} = G^{j_1' j_2 j_5''}_{j_3 j_4 j_5}. \]

Here we have defined

\[
v_{j_5} := \frac{1}{G_{000}}, \quad d_{j_5} := u_{j_5^2}.
\]

(14)

Comparing the above equations with (11) and (12), we recognize that the norm-preservation condition on Pachner transformations recovers the orthogonality condition, while the path-independence of Pachner transformations recovers the pentagon condition in the definition of a UFC. By appealing to the coherence theorem, any commutative diagram involving two ways of relating two tensor network configurations with \( n > 5 \) uncontracted legs to one another will commute as long as the pentagon identity holds, and so the above conditions exhaust the constraints we can put on the \( G \) tensors.

The two sequences in Fig. involve only the \( OO_1 \) move. One can choose other sequences involving \( 1 \leftrightarrow 3 \) moves and derive the relationship between \( f_1, f_2 \) and \( f_3 \). They differ in prefactors by the product of powers of \( d \)'s and

\[ D = \sum d_{j_5}^{j_5^2}. \]

We rewrite the equation for these Pachner moves in Eq. (15), in which the plaqquette labels that do not change during the moves are suppressed.

\[ (P1) : (T^{M_1})^{j_1 j_2 j_5} (T^{M_2})^{j_3 j_4 j_5} = \sum_{j_6} v_{j_6} v_{j_6'} G^{j_1' j_2 j_5}_{j_3 j_4 j_5} (T^{M_5})^{j_3 j_4 j_5} (T^{M_4})^{j_4' j_2 j_5}, \]

(15)

\[ (P2) : (T^{M})^{j_1 j_2 j_5} = \sum_{j_4, j_5, j_6, \mu} v_{j_4} v_{j_5} v_{j_6} G^{j_2 j_3 j_5}_{j_4 j_6 j_5} (T^{M_1})^{j_1 j_2 j_5}_{\mu} (T^{M_2})^{j_3 j_4 j_5}_{\mu} (T^{M_3})^{j_4 j_2 j_5}_{\mu}, \]

\[ (P3) : \sum_{\mu} (T^{M_1})^{j_1 j_2 j_5}_{\mu} (T^{M_2})^{j_3 j_4 j_5}_{\mu} = v_{j_4} v_{j_5} v_{j_6} G^{j_2 j_3 j_5}_{j_1 j_4 j_5} (T^{M_1})^{j_1 j_2 j_5}_{\mu} = \frac{v_{j_4} v_{j_5} v_{j_6}}{D} G^{j_2 j_3 j_5}_{j_1 j_4 j_5} (T^{M_1})^{j_1 j_2 j_5}_{\mu}. \]

Tetrahedral symmetry is guaranteed by the rotational invariance of the graph, or equivalently by the permutation symmetry of the tensors \( T_{ijk} = T_{kij} = T_{jki} \).

To see the physical meaning of the definition in Eq. (14), we can take \( j_1 = j_2 = j_3 = 0 \) for the 3-1 move in Fig. The constraint \( j_4 = j_5 = j_6 \) must be satisfied, and so (suppressing the irrelevant plaquette degrees of freedom) the move simplifies as in Fig.

Using tetrahedral symmetry and Eq. (14), we see that

\[ v_{j_5}^3 G_{j_5 j_5''}^{000} = v_{j_5}^3 G_{00j}^{j_5 0} = d_{j_5}, \]

consistent with the physical
meaning of the quantum dimensions.

The fusion rules are also encoded in the $G$-symbols, as can be observed already from Eq. (6). When one takes $j_2 = 0$ in the 2-2 move, then $j_1 = j_5$ and $j_3 = j_5$ must be satisfied. Rewriting $i = j_3, j = j_4$ and $k = j_5$, the move reduces to Fig. 7.

![FIG. 7: Taking $j_2 = 0$ in the 2-2 move in Fig. 4 one recovers the fusion rules of $N_{ijk\ast}$.

Using Eq. (6), we see that $v_i v_k G_{ijk\ast} = N_{ijk\ast}$. Consequently, the tensor network configuration on the right is only allowed if $N_{ijk\ast}$ is nonzero, i.e. if the branching rules are satisfied.

Combining the results above, we see that the fixed-point requirement of a tensor network state leads naturally to a set of data $\{I, d, N, G\}$ that satisfy the axioms of UFC $C$.

**B. Construction of a Fixed-Point Tensor Network State from a UFC**

Having shown how a fixed-point tensor network state contains the data of a UFC, we now show how one can begin with a UFC $C$ and construct a fixed-point tensor network state. We will use a triple-line tensor network construction, and will color the triple-line structure by assigning the labels $i, j, k, \ldots \in I$ to the central (blue) links in Fig. 3 and the labels $\mu, \nu, \lambda, \ldots \in I$ to the adjacent black links as before. We organize the labellings in a way so that for any three central links $i, j, k$ that point to a common vertex, $N_{ijk} \neq 0$.

The next step is to import the $\{G\}$-tensors from the UFC into the tensor network. For Fig. 3 we associate to every vertex (small triangle) a tensor $T$ on the $A$ and $B$ sublattices in the following way (parallel to Refs. [36,37]):

A: $(T^M)^{ijk}_{\mu \nu \lambda} = \frac{(v_i v_j v_k)^{1/3}}{\sqrt{D}} \sqrt{v_i v_j v_k} G^0_{\lambda \mu \nu} \delta_{\mu \nu} \delta_{\nu \lambda}$;

B: $(T^M)^{ijk}_{\mu \nu \lambda} = \frac{(v_i v_j v_k)^{1/3}}{\sqrt{D}} \sqrt{v_i v_j v_k} G^0_{\lambda \mu \nu} \delta_{\mu \nu} \delta_{\nu \lambda}$.

where we have denoted $v_j = \sqrt{d_j}$ for $j \in I$. The physical index $M$ is defined as the triple $(i, j, k)$. Upon contracting the internal indices as demonstrated in Fig. 8, one arrives at a tensor network state.

![FIG. 8: (Color online.) An illustration of the tensor network with tensors given by Eq. (16). The upward-pointing (downward-pointing) triangles are located on the $A$ ($B$) sublattice.

It was proved in Ref. [36] that these states are fixed-point states under renormalization transformations of the tensor network. More precisely, they are invariant under the 2D dual Pachner moves of Fig. 4. In the Hamiltonian language, these fixed-point states are the ground states of string-net models [43].

Note that if the tensors appearing in Eq. (16) are to be non-zero, we must have

A: $N_{\mu \ast \nu} = N_{\lambda \ast j} = N_{\lambda \ast k} = N_{ij \ast k} = 1$,

B: $N_{\mu \ast \nu} = N_{i \lambda \ast j} = N_{\lambda \ast k} = N_{i \ast jk} = 1$.

(17)

Although the two sets $\mu, \nu, \lambda, \cdots$ and $i, j, k, \cdots$ both take values in the label set $I$ of the UFC $C$, they are not on the same physical footing. The origin of the above form [16] of tensors is the following.

Denote $B^p_p$ for plaquette $p$ of the graph as the operator that adds a closed loop $\mu$ inside $p$. Further define

$$B_p = \sum_{\mu} \frac{d_{\mu}}{D} B^p_p.$$

(18)

$B_p$ is the composition of the elementary $1 \rightarrow 3$ and $3 \rightarrow 1$ moves in $\mu$ moves in $\mu$.

It changes the labellings of the links the surround the plaquette $p$ while keeping all the other labellings in the graph untouched. Since it is a composition of the elementary moves, it keeps the fixed-point tensor network state invariant.
The tensor network state can actually be constructed from this operator as

\[ |\Psi\rangle = \prod_p B_p |0\rangle = \sum_{\mu,\nu,\lambda,\ldots} d_\mu d_\nu d_\lambda \cdots |\mu,\nu,\lambda,\ldots\rangle_{coh}, \]  

(20)

where the state |0\rangle means the graph is empty, i.e., we assign the vacuum string 0 to every link to the graph, and |\mu,\nu,\lambda,\ldots\rangle_{coh} denotes the state with \mu, \nu, \lambda, \ldots as plaquette degrees of freedom and with all links carrying the label 0. \[ |\mu,\nu,\lambda,\ldots\rangle_{coh} = B_\mu B_\nu B_\lambda \cdots |0\rangle \]

demonstrated in the Fig.9. The factor \(d_\mu\) attributes to the fact that every closed string \(\mu\) has an amplitude of \(d_\mu\). Notice that the state |\mu,\nu,\lambda,\ldots\rangle_{coh} are coherent states; they are not necessarily orthogonal. Furthermore, all the closed loops appearing in the state |\mu,\nu,\lambda,\ldots\rangle_{coh} are independent of each other, i.e., mutually un-entangled.

FIG. 9: Construction of fixed-point tensor network state using loops.

Since they are fixed-point states, one can further translate the degrees of freedom from the loops to the links using dual 2D Pachner moves described in Fig.4. This gives

\[ |\Psi\rangle = \sum_{M_1,M_2,\ldots} \text{Tr} \left[ \otimes_v T^{M_v}_v \right] |M_1,M_2,\ldots\rangle. \]  

(21)

where the \(\{M_i\}\) are physical indices. The \(T\) tensors appearing in the tensor trace take values exactly as in Eq.(16).

The above procedure presents an analogy to the 1D AKLT example discussed in the introduction. The tensor network representation of the AKLT state encodes entanglement in two ways: (i) the spin-1/2 partons \(\mu, \nu, \cdots\) on neighboring sites (e.g., \((n-1)R\) and \(n_L\)) are entangled as singlets, and (ii) the partons on the same site (e.g., \(n_L\) and \(n_R\)) as entangled as triplets. While the former is realized by the contraction of tensors, the latter entanglement is carried by every single tensor in the network.

Similarly in two dimensions, entanglement is created in several steps. (i) In Eq.(21), one first uses the \(B_p\) operators to generate plaquette degrees of freedom \(\mu, \nu, \lambda, \cdots\). In the language of triple-line structure, this corresponds to taking all the \(i, j, k, \cdots = 0\) in Fig.6, and contracting all the the partons \(\mu = \mu', \nu = \nu'\) etc. The latter creates entanglement inside every plaquette. (ii) The next step is to project onto the physical degrees of freedom \(i, j, k, \cdots\), which are defined on the links and correspond to the spin-1 degrees of freedom in the AKLT analogy. Entanglement is created when this projection takes place, i.e. when one uses dual Pachner moves to fuse the loops \(\mu, \nu, \lambda, \cdots\) and rearrange the degrees of freedom from the plaquettes to the links. Mathematically, this is realized by the fusion \(\delta\)-tensors in the UFC. (iii) Finally, the \(i, j, k, \cdots\) are contracted, resulting in the entanglement between different sites.

The pattern of entanglement in the second step manifests itself as 6j G-symbols in the coefficients generated by the Pachner moves, which become encoded in the tensors \(T\) in Eq.(16). These local tensors record the history of the projection in step (ii) by representing the initial mutually un-entangled parton degrees of freedom in terms of the entangled physical degrees of freedom. If the label set of the UFC contains only one trivial object \(I = \{0\}\) (and thus \(D = 1\)), then the entanglement is short-range. Generally if one starts from a nontrivial UFC, the constructed fixed-point tensor networks state will be long-range entangled. We conclude that the local \(T\)-tensors are the building blocks of long-range entanglement in the corresponding topological phase.

IV. GEOMETRICAL PERSPECTIVE OF THE CORRESPONDENCE

In this section we discuss the geometric structure of fixed-point tensor network states and its relationship to a 3D Turaev-Viro state-sum TQFT. We briefly review a few basic TQFT facts. On a three dimensional manifold \(\Sigma\), a full-extended unitary 3D TQFT is a symmetric monoidal functor from the category of three-cobordisms to the category of vector spaces over \(\mathbb{C}\):

\[ \mathcal{F} : 3\text{Cob} \to \text{Vect}_\mathbb{C}. \]  

(22)

Specifically, we assign a Hilbert space of states \(\mathcal{H}\) to each spatial slice (2D manifold) of a three-cobordism. If the spatial slice contains a disjoint union of \(n\) 2D manifolds, the corresponding Hilbert space splits through the tensor product as \(\mathcal{H}^{\otimes n}\). A 3D TQFT associates to \(\Sigma\) a linear map from \(\mathcal{H}^{\otimes n}\) to \(\mathcal{H}^{\otimes n}\), where \(n_i\) is the number of disjoint parts of the incoming spatial slice, and \(n_o\) the number for outgoing spatial slice.

The cylinder map is the identity \(id : \mathcal{H} \to \mathcal{H}\). If the 3D manifold \(\Sigma\) is closed, then the map is a partition function \(Z(\Sigma) : \mathbb{C} \to \mathbb{C}\). Other simple examples include the cap cobordism, where the map is \(Tr : \mathcal{H} \to \mathbb{C}\); the cup cobordism \(\eta : \mathbb{C} \to \mathcal{H}\); the product bordism (a pair of pants) \(m : \mathcal{H}^{\otimes 2} \to \mathcal{H}\); and the coproduct bordism (an inverted pair of pants) \(\Delta : \mathcal{H} \to \mathcal{H}^{\otimes 2}\).
A. 3D State Sum TQFT

A state-sum construction of a TQFT is a discretization of the above formalism. The algebraic data needed to define a 3D state-sum TQFT form a UFC \( \mathcal{C} \) in the following way.

We start from a closed three dimensional manifold \( \Sigma \), and define on it a triangulation \( \mathcal{T}(\Sigma) \). An oriented coloring of the triangulation refers to the assignment of a label \( j \in I \) to every 1-simplex (edge) of the triangulation. Substituting \( j \) by \( j^* \) and reversing the arrow leaves the oriented coloring invariant. Then we associate \( G_{ijm}^{kl} \) to each tetrahedron with edges labeled by \( \{i, j, m, k, l, n\} \), as indicated in Fig. 10. The tetrahedral symmetry condition (2) can be understood geometrically as the requirement that viewing the tetrahedron from four different directions give rise to the same tensor.

\[
\tau_{\mathcal{C}}(\Sigma) = \sum_{\text{labellings}} \prod_{\text{vertices}} \frac{1}{D} \prod_{\text{tetrahedron}} G \prod_{\text{edges}} d_j, \quad (23)
\]

where the total quantum dimension \( D = \sum_{j \in I} d_j^2 \).

Independence of the invariant \( \tau_{\mathcal{C}}(\Sigma) \) with respect to triangulations of \( \mathcal{T}(\Sigma) \) can be shown by following a standard procedure. Any two different triangulations in 3D can be related by a sequence of 3D Pachner moves, depicted in Fig. 11 and 12. Invariance of the state-sum under these moves corresponds exactly to the consistency condition (UFC2) and (UFC3) above, namely, the Pentagon equation and the Orthogonality condition. We demonstrate this correspondence in detail in Appendix A. Consequently, the input category \( \mathcal{C} \) being a UFC automatically guarantees this topological invariance.

B. Manifolds With Boundary

The above discussion can be generalized to the case where \( \Sigma \) has 2D boundaries \( \partial \Sigma \). Following the notation of Ref. [25], we call the initial and final spatial slices of the cobordism as cut boundaries, and all others as brane boundaries. Cobordisms are composed along cut boundary, while boundary conditions need to be imposed on brane boundaries.

Consider the special case where \( \partial \Sigma \) consists of one single component of both cut boundary and brane boundary. One tetrahedron \([i, j, k, \mu, \nu, \lambda]\) in the triangulation near the brane boundary is depicted in Fig. 13. The faces \((\mu, i, \nu)\), \((\nu, j, \lambda)\) and \((\lambda, k, \mu)\) lie on the brane boundary, while the face \((i, j, k)\) is in the bulk. There can be a large number of tetrahedra between the \((i, j, k)\) plane and the cut boundary(initial spatial slice), but one can use Pachner moves in Fig. 11,12 to reduce the number of tetrahedra in the bulk and effectively arrive at a single “layer” of tetrahedra that looks like Fig. 13. In other words, without loss of generality, one can view \((i, j, k)\) as living on the cut boundary.

Applying Poincaré duality, we can associate 3-simplices (tetrahedra) with 0-simplices (vertices) of the dual graph, and 2-simplices (faces) with 1-simplices (edges) of the dual graph. In the tetrahedron
\[i, j, k, \mu, \nu, \lambda\], \(\alpha\) is dual to the triangle bounded by the three links \((\mu, i, \nu)\), \(\beta\) is dual to the triangle bounded by \((\nu, j, \lambda)\), \(\gamma\) is dual to the triangle \((\lambda, k, \mu)\) and the index \(M\) is the collection \((i, j, k)\).

The links \(\alpha\) must match up with another link \(\alpha'\) in the triangulation, which comes from the dual of another tetrahedron that shares the face \((\mu, i, \nu)\) with the above tetrahedron. A similar identification occurs for \(\beta, \gamma\), etc.. Consequently, links carrying the indices \(\alpha, \beta, \gamma, \ldots\) form a 2D trivalent graph, with the extra links like \(M\) dangling in the third dimension of this graph.

The graph generated by Poincaré duality in this way coincides exactly with the setup of a 2D tensor network. We see that the original edges \(\mu, \nu, \lambda\) of the triangulation map to the plaquette degrees of freedom in the dual picture. This precisely gives rise to the triple-line structure depicted in Fig. 3.

The mapping from internal indices \(\alpha, \beta, \gamma\) to the physical index \(M\) can be interpreted as a boundary-to-bulk map in the TQFT context. The factors of \(\delta^{\mu\mu'}\delta^{\nu\nu'}\delta^{\lambda\lambda'}\) in Eq. (16) can now be understood as well: these are the constraints that ensure the plaquette degrees of freedom in the dual graph are associated to links in the original triangulation in a well-defined way. In other words, these constraints entangle the 2-simplices in the same tetrahedron.

Pachner moves in the original triangulation picture Fig. 11 and 12 map to the dual Pachner moves of Fig. 4. This is related to the fact we mentioned above: both moves correspond algebraically to the Pentagon and Orthogonality axioms of 3D state-sum TQFTs.

Consider the situation with three tetrahedra are glued together as in Fig. 14. In the tensor network picture (dual to the triangulation picture), this corresponds to the triple-line structure near a triangular plaquette (Fig. 15).

Gluing another three tetrahedra to the above picture, as depicted in Fig. 15, corresponds to fusing another loop \(\sigma\) into the triangular plaquette. In the tensor network picture, \(i, j, k\) remains the same, while \(l, m, n, \rho\) change into \(l', m', n', \rho'\). This entanglement-producing procedure of fusion can be identified as the operator \(B_p^\sigma\) with matrix elements

\[
\left\langle i' \atop i \right\rangle \left\langle j' \atop j \right\rangle \left\langle k' \atop k \right\rangle B_p^\sigma = v_i v_m v_n v_{l'} v_{n'} v_{l} G^{n'n}_{\sigma m'i'} G^{l'l}_{\sigma l'm'} G^{i'n}_{\sigma n'm'},
\]

(24)

This is exactly the operator that appears in Eq. (18). Since it is a composition of the elementary \(1 \to 3\) and \(3 \to 1\) moves, the tensor network state (21) built from the UFC is an eigenstate of the \(B_p\) operator with eigenvalue one:

\[
B_p |\Psi\rangle = |\Psi\rangle, \quad \forall \text{ plaquette } p.
\]

(25)

Consequently, one can act the \(B_p\) operators multiple times while keeping the fixed-point tensor network states invariant. As discussed above, action of such \(B_p\) operators on the tensor network states corresponds to gluing tetrahedra in the third dimension, thus the action of multiple \(B_p\) operators on the same plaquette would correspond to the growth of a “tower”.

Generally we can have plaquettes surrounded by more than \(n \geq 3\) links, as depicted in Fig. 17. The action of \(B_p\) operators on such a plaquette would correspond to the growth of \(n\) tetrahedra.

To illustrate the consequence of action of \(B_{\sigma L}\) and \(B_{\sigma R}\) operators on neighboring plaquettes in the ten-
in the third dimension. Unfilled, i.e., we have obtained an emergent pre-geometry again generate six tetrahedra BCQQ and the last line to connect would be P to Q. One still connects Q’ to the six vertices of the hexagon ABQFGH and thus generating six tetrahedra. Then the action of B_{\sigma_R} follows, dragging Q to Q’. One still connects Q’ to the five vertices B, C, D, E, F, but not P, for P has already been dragged to P’ by the previous action of B_{\sigma_L}. Therefore, the last line to connect would be P’Q’. In this way, we again generate six tetrahedra BCQQ’, CDQQ’, DEQQ’, EFQQ’, FPP’Q’ and P’BQQ’ and there is no space left unfilled, i.e., we have obtained an emergent pre-geometry in the third dimension.

This is exactly the implication of the Holographic Principle [14–16], where the information in the 3D bulk is fully stored in the 2D tensor network.

We have thus shown the correspondence between fixed-point tensor network states and TQFTs in a higher dimension. If the 2D manifold where the tensor network lives in is closed, i.e., has no boundary, then degeneracy of the ground states in the topological phase described by the fixed-point tensor network state can be expressed in terms of B_p operators [17]. To be specific, we have

\[ GSD_{\mathcal{C}}(\Sigma) = tr \left( \prod_p \sum_\sigma \frac{d_\sigma}{d_p} B_p^\sigma \right) = \tau_C \left( \Sigma \times S^1 \right), \]

once the action of B_p operators is identified with the gluing of three new tetrahedra that share an edge \( \sigma \).

A side-remark: The universality classes of state-sum TQFTs are characterized by the Drinfeld center [45] \( \mathcal{Z}(\mathcal{C}) \) of the UFC \( \mathcal{C} \). In the context of tensor networks, the fixed-point states constructed from two different sets of data \( C_1 \) and \( C_2 \) which satisfy \( \mathcal{Z}(C_1) \sim \mathcal{Z}(C_2) \) as braided tensor categories (i.e. \( C_1 \) is Morita equivalent to \( C_2 \)) are ground states of the same physical phase.

V. SYMMETRY ENRICHED CASE

In this section we provide an extension of the above framework when a global symmetry \( G \) is present. For simplicity, we take \( G \) to be finite, onsite, and unitary.

To start with, we review some mathematical terminology. Given the input data of category \( \mathcal{C} \), one can follow the procedure of previous sections III and IV to construct a tensor network state. This state is the ground state of a topological phase described by the Drinfeld center \( \mathcal{Z}(\mathcal{C}) \) of \( \mathcal{C} \).

It is known [18] that a large subset of \( G \)-symmetry enriched topological phases (SETs) can be described by a braided \( G \)-crossed extension of \( \mathcal{Z}(\mathcal{C}) \). To obtain such a phase, we need to use another UFC \( \mathcal{D} \) as the input data of the tensor network instead of \( \mathcal{C} \). This \( \mathcal{D} \) is called a “\( G \)-extension of \( \mathcal{C} \)” [49]. It is endowed with a \( G \)-graded structure in the following way:

\[ \mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g. \]

Writing \( e \) as the identity element of \( G \), we require \( \mathcal{D}_e = \mathcal{C} \). In other words, if the symmetry group \( G \) is trivial, i.e. has only one single element \( e \), \( \mathcal{D} \) reduces to the original category \( \mathcal{C} \). Furthermore, we require the fusion rules in \( \mathcal{D} \) to be compatible with the group structure of \( G \). This amounts to requiring

\[ \mathcal{D}_g \otimes \mathcal{D}_h \subset \mathcal{D}_{gh}, \quad \mathcal{D}_e = \mathcal{C}, \quad a_g \otimes b_h = \bigoplus_c N_{abc} c_{gh}. \]

If we demand \( \mathcal{D} \) to be the input data for the tensor network, namely, if we require the labels \( i, j, k, \mu, \nu, \lambda, \cdots \) in Fig. 5 to all belong to the label set \( I_\mathcal{D} \) of \( \mathcal{D} \), then the tensor network state will be the ground state of a
“gauged” model of the $\mathcal{G}$-SET in question. In such model the global symmetry $\mathcal{G}$ is promoted to a gauge symmetry and the $g \in \mathcal{G}$ fluxes become deconfined excitations of the “gauged” model.

To return to the $\mathcal{G}$-SET, one has to go through an “ungauging” procedure\cite{32,33}. Since all the labels $i, j, k, \cdots, \mu, \nu, \lambda, \cdots$ belong to some $\mathcal{D}_g$, they are related by construction to a group element of $\mathcal{G}$ obtained by the map $\mathcal{D}_g \mapsto g$. For convenience, we recall the triple-line structure Fig. 19 of tensor network below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig19.png}
\caption{(Color online) Triple-line structure of vertices that belong to $A$ (left) and $B$ (right) sublattices.}
\end{figure}

We then define another group element $\tilde{g}$ for the degrees of freedom on the blue links in the left figure above:

$$
\tilde{g}_i = g^{-1}_\mu g'_\nu, \quad \tilde{g}_j = g^{-1}_\lambda g'_\nu, \quad \tilde{g}_k = g^{-1}_\mu g_\lambda.
$$

The conventions are fixed in the following way: starting from the blue link $i$ in the left figure above, rotate it 90 degrees counterclockwise. The head of the rotated $i$ link points to $\mu$ and the tail of the $i$ link joins to the head of the $\nu'$ arrow. Now invert $g_\nu$, but keep $g_\nu'$ unchanged. Similarly, rotate the link $j$ by 90 degrees counterclockwise. The head of the $j$ link points to $\lambda'$ and its tail to $\nu$, and so we invert $g_\nu'$ but keep $g_\nu$ unchanged. One observes that there is a gauge degree of freedom in the above definition: if an arbitrary group element $g \in \mathcal{G}$ is left-multiplied to all $\mu, \mu', \nu, \nu', \lambda, \lambda', \cdots$, the definition of $\tilde{g}_i, \tilde{g}_j, \tilde{g}_k$ will remain exactly the same.

To complete the “ungauging” procedure, we set

$$
g_i = \tilde{g}_i
$$

for all blue-link degrees of freedom $i, j, k$, etc.. The corresponding tensor network state is the ground state of a $\mathcal{G}$-SET on a sphere, with the tensors on each of the sublattices of the honeycomb lattice given by

$$
A: (T^M)^{ijk}_{\mu\nu'\lambda'} = \frac{(v_\mu v_\nu v_\lambda)^{1/3}}{\sqrt{D}} \sqrt{v_i v_j v_k} G^{\delta \mu \nu' \lambda'}_{\delta \mu' \nu' \lambda'} \delta_{g_i g_j g_k},
$$

$$
B: (T^M)^{ijk}_{\mu'\nu'\lambda'} = \frac{(v_\mu v_\nu v_\lambda)^{1/3}}{\sqrt{D}} \sqrt{v_i v_j v_k} G^{\delta \mu' \nu' \lambda'}_{\delta \mu \nu \lambda'} \delta_{g_i g_j g_k}.
$$

borne in mind that all the labels belong to the label set of $\mathcal{D}$, not of $\mathcal{C}$. Except for this, we notice that the form of tensors in Eq. (31) are the same as the previous Eq. (16), only with an additional flatness constraint on the $\mathcal{G}$ gauge field. The $T^M$ tensors take such a simple form because $\mathcal{G}$ is onsite and unitary. A formulation for more general symmetries is possible and is related to the work in Ref. [33].

The symmetry $\mathcal{G}$ manifests itself as the invariance of the tensor network state under a global action of $U^g$, where $U^g$ is defined as

$$
U^g : g_\mu \rightarrow g_\mu g \quad \forall \mu, \quad g_i \rightarrow g^{-1} g_i g \quad \forall i.
$$

That is, $U^g$ acts as right-multiplication by $g$ for all the group elements associated with the plaquette degrees of freedom, and acts as conjugation by $g$ for all the group elements associated with the links.

On the mathematical side, the TQFT that incorporates the $\mathcal{G}$-symmetry is known as Homotopy Quantum Field Theory (HQFT), which was proposed by Turaev\cite{29,31}. HQFT is a version of TQFT defined on some $\mathcal{G}$-manifold $\Sigma$, which is a manifold endowed with a $\mathcal{G}$ gauge field, i.e. a homotopy class of maps $\Sigma \rightarrow BG$ from the manifold to the classifying space $BG$.

For connected manifolds $\Sigma$, homotopy classes of such maps correspond bijectively to the set of homomorphisms $\text{Hom}(\pi_1(\Sigma), G)$, which in turn completely determines a principle $\mathcal{G}$-bundles over $\Sigma$.

From an algebraic perspective, any braided $\mathcal{G}$-crossed extension of $\mathcal{Z}(\mathcal{C})$ gives rise to an HQFT with target space $BG$\cite{30}. Physically, every realization of symmetry $\mathcal{G}$-enriched topological phase is described by an HQFT with target space $BG$.

The related symmetry-enriched TV-invariant can be constructed following Refs. [29,30]. The formulation is exactly parallel to that of the tensor network above. For a triangulation of a 3D manifold with boundary, we first assign oriented labels in $I_\mathcal{D}$ to the 1-simplices of the triangulation. For a given homotopy class of maps $\Sigma \rightarrow BG$, we then choose a representative map $\mathcal{g}$ that sends all the vertices of the triangulation to a base point of $BG$. We then assign to each 1-simplex a group element in $\mathcal{G}$: $\mu \rightarrow g_\mu$, $\mu' \rightarrow g^{-1}_\mu$.

Similar to the constraint of Eq. (30),
we then impose the flatness condition for all 2-simplices in the bulk of the triangulation. In other words, we require there to be no local $G$-symmetry fluxes. Importantly, we further require the assignments $\mu \mapsto g_\mu$ to be compatible with the $G$-grading structure of $D$. Namely, we require the group element $g_\mu$ assigned to edge $\mu$ to be such that $\mu \in I_{D_{\mu \nu}}$.

After performing this construction, one obtains the TV-invariant $\tau(\Sigma)$ of the $G$-manifold $\Sigma$ in a way similar to the case without symmetry [24]. Since Pachner moves can be extended naturally to the symmetry-enriched case, one can readily prove that $\tau(\Sigma)$ is independent of the chosen triangulation. Furthermore, $\tau(\Sigma)$ is also independent of the choice of representative $g$ in the homotopy class of classifying maps [30].

VI. SUMMARY AND OUTLOOK

We have identified the algebraic structure of 2D fixed-point tensor network states as unitary fusion categories, which are also known as representation theories of $C^*$-weak Hopf algebras. We illustrated how the pattern of long-range entanglement of fixed-point tensor network states arises in such a picture.

Geometrically, we demonstrated how to construct a 2D fixed-point tensor network state from a 3D state-sum topological quantum field theory. The long-range entangled fixed-point tensor network state lives on the 2D boundary of the 3D TQFT, and encodes the same amount of information as the latter, which is a characteristic of holography. Furthermore, we showed how the emergence of bulk pre-geometry arises from the long-range entanglement of the fixed-point tensor network states on the boundary. We further extended the correspondence when a finite unitary symmetry is present.

The correspondence between the data of fixed-point tensor network states, 3D state-sum TQFT and unitary fusion category is summarized in the following table.

| State Sum TQFT          | Tensor Network                        | Unitary Fusion Category          | Entanglement                        |
|-------------------------|---------------------------------------|----------------------------------|-------------------------------------|
| Edges on the brane boundary | Interal d.o.f.s in the plaquettes     | $\mu, \nu, \lambda, \ldots \in I$. | Mutually un-entangled partons       |
| Edges on the cut boundary | Physical d.o.f.s on the links          | $i, j, k, \ldots \in I$          | Entangled physical d.o.f.s          |
| Faces (triangles)       | Triple-line structure                 | $\langle \mu, i, \nu \rangle, \langle \nu, j, \lambda \rangle, \langle \lambda, k, \mu \rangle, \ldots$ | Projections $\mu \otimes \nu \to i$ etc. |
| Tetrahedra              | Vertices and tensors $(T^M)^{ijk}_{\mu \nu \lambda}$ | $G_{\mu \nu \lambda}^{ijk}, \ldots$ | Carriers of entanglement            |
| Invariance under Pachner move | Invariance under RG                   | Pentagon & Orthogonality         | Long-range Entanglement             |

One future direction would be to define the fixed-point tensor network states in terms of the more familiar language of algebras, rather than categories. Namely, instead of isomorphism classes of simple objects in $\mathcal{C}$, we could use basis elements of a $C^*$-weak Hopf algebra $\mathcal{W}$ as the link labels of the trivalent graph. This requires application of the Tannakian duality $\mathcal{C} \simeq \text{Rep}(\mathcal{W})$, see for example Ref. [50]. In the simplest case of finite groups, this duality has a simple interpretation as a generalized “Fourier transformation” [51]. The more interesting quantum group cases, however, requires additional care. This idea is closely related to the work [52], where the authors constructed tensor network states using matrix product operators and a $C^*$ algebra. Topological phases are then described by the central idempotents of the corresponding $C^*$ algebra.

Another extension would be to relax the Tetrahedral symmetry of the 6j-symbols in our formulation. This could lead to interesting physics, and in the finite group case may allow us to obtain a tensor-network representation of the Dijkgraaf-Witten model.

One can go beyond the ground state subspace as well. The structure of fixed-point tensor network states that are excited states of a topological phase is expected to be characterized by a TQFT with marked surfaces.

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Appendix A: 3D Pachner Moves and Consistency Conditions for the $G$-tensors

In this appendix, we sketch how the Pentagon equation [4] and Orthogonality condition [5] are related to the three dimensional Pachner moves.

Every 3-simplex in the 3D triangulation can be mapped [53] to a two dimensional categorical diagram. Here faces of the tetrahedron are mapped to the vertices of the dual diagram, edges are mapped to links, and vertices are mapped to triangles. When two tetrahedra in
FIG. 20: Faces of the tetrahedron are mapped to the vertices of the dual diagram, edges are mapped to links, and vertices are mapped to triangles.

The triangulation share a face, the dual diagram possesses two triangles which share a vertex. In these cases we draw the diagrams separately and connect the common vertex with a dashed line.

These 2D diagrams naturally inherit an action of the Pachner moves from the 3D triangulation. Algebraically, since every 3-simplex is directly related to a \( 6j \)-symbol (as illustrated in Sec. [IV.A]), so is every 2D categorical diagram. Below we show that algebraic expressions for the invariance under Pachner moves in the categorical diagrams are exactly the Pentagon and Orthogonality constraints.

FIG. 21: 3D Pachner 3-2 move.

FIG. 22: 3D Pachner 4-1 move.
I. Affleck, T. Kennedy, E. H. Lieb and H. Tasaki, 21
C. Rovelli, C. Rovelli and L. Smolin, 17
J. Maldacena, M.-X. Han and L.-Y. Hung, 20
F. Pastawski, B. Yoshida, D. Harlow and J. Preskill, 11
X.-L. Qi, 23
M. F. Atiyah, P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter and Z. Yang, P. Hayden and X.-L. Qi, 12
B. Swingle, 6
M. Fannes, B. Nachtergaele and R. F. Werner, 9
G. Evenbly and G. Vidal, 4
B. Zeng, X. Chen, D.-L. Zhou and X.-G. Wen, 3
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