PERVERSE SHEAVES AND
QUANTUM GROTHENDIECK RINGS

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Abstract. We define a quantum analogue of the Grothendieck ring of finite dimensional modules of a quantum affine algebra of simply laced type via an analogue of Lusztig’s restriction functor on perverse sheaves on a variety related to quivers. We get also a new geometric construction of the tensor category of finite dimensional modules of a finite dimensional simple Lie algebra of type $A-D-E$.

1. Introduction

Finite-dimensional representations of quantum affine algebras, say $U = U_q(Lg)$, have been studied from various viewpoints. However, little is known on the decomposition factors of tensor product of simple modules. From Lusztig’s work the direct sum of the Grothendieck rings of affine Hecke algebras of type $A$ can be identified with the algebra of regular functions of the pro-unipotent group of upper triangular unipotent $Z \times Z$-matrices with finite support, in such a way that simple modules are mapped to the dual canonical basis of $U_q^+(\mathfrak{sl}_\infty)$. It was observe recently that the induction product of simple modules of affine Hecke algebra should be related to conjectural multiplicative properties of the dual canonical basis, see [NLT]. The aim of our paper is to give a similar approach for tensor product of simple modules for all simply laced types, using the geometric realization of quantum affine algebras in [N2], see also [GV], [Va] for type $A$. In order to do this we give a geometric construction of a flat deformation, denoted by $GR$, of the Grothendieck ring of $U$ in terms of perverse sheaves on a singular variety related to quivers. The product is defined via an analogue of Lusztig’s restriction functor. It is not commutative in general, and $GR$ affords a canonical basis. Note that $GR$ and its canonical basis appeared already in [N3] in a different form. It was also observed, there, that the elements of the canonical basis could be identified to simple $U$-modules with a prescribed, conjectural, filtration. However, the construction in [N3] does not give the positivity statement in Theorem 4.3. There is no geometric construction of the tensor category of finite-dimensional $U$-modules. The positivity in Theorem 4.3 suggests that a large number of information on tensor products of $U$-modules can be captured from the ring $GR$. In particular, we formulate a generalization of a conjecture of Berenstein-Zelevinsky, see [BZ].

A similar construction gives a new geometric interpretation of the tensor category of finite dimensional $g$-modules, see §5. It would be interesting to relate it with

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the tensor category of perverse sheaves on the affine Grassmanian of the Langlands dual group. This question appeared independently in [M].

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2. The Grothendieck rings

2.1. Let $\mathfrak{g}$ be a simple complex Lie algebra with Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let $d_i \in \{1,2,3\}$ be the coprime positive integers such that the matrix with entries $b_{ij} = d_i a_{ij}$ is symmetric. Let $\alpha_i$ and $\omega_i$ be the simple roots and the fundamental weights of $\mathfrak{g}$. Set $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i$ and let $P^+, Q^+$ be the semi-groups generated by $\{\alpha_i\}$ and $\{\omega_i\}$. Recall that $Q$ is embedded in $P$ by the linear map such that $\alpha_i \mapsto \sum_j a_{ij} \omega_j$. For any $\lambda \in P$, $\alpha, \beta \in Q$ we write $\beta \geq \alpha$ if $\beta - \alpha \in Q^+$ (resp. we write $\lambda \geq \alpha$ if $\lambda - \alpha \in P^+$). If $\lambda \in P^+$ let $V(\lambda)$ be the simple $\mathfrak{g}$-module with highest weight $\lambda$. If $\lambda \in P$ and $V$ is an integrable $\mathfrak{g}$-module, let $V_\lambda \subseteq V$ be the corresponding weight subspace in $V$. We put

$$\text{\Lambda}(\lambda) = \{\alpha \in Q^+ \mid V(\lambda)_{\lambda - \alpha} \neq \{0\}\}, \quad \text{\Lambda}^+(\lambda) = \{\alpha \in \text{\Lambda}(\lambda) \mid \lambda \geq \alpha\}.$$  

Let $R(\mathfrak{g})$ be the ring of finite dimensional representations of $\mathfrak{g}$. In this paper, except in §5.2, we consider only simply laced Lie algebras.

2.2. The quantum loop algebra associated to $\mathfrak{g}$ is the $\mathbb{C}(q)$-algebra $U$ generated by $x_{ir}^\pm, k_{i,\pm s}^\pm, k_i^\pm = k_{i0}^\pm$ ($i \in I$, $r \in \mathbb{Z}$, $s \in \mathbb{N}$) modulo the following defining relations

$$k_i k_i^{-1} = 1 = k_i^{-1}k_i, \quad [k_i^\pm, k_j^\pm] = 0,$$

$$k_i x_{jr}^\pm k_i^{-1} = q^{\pm a_{ij}} x_{jr}^\pm,$$

$$(z - q^{\pm a_{ij}} w) k_j^\pm (z) x_{jr}^\pm (w) = (q^{\pm a_{ij}} z - w) x_{jr}^\pm (w) k_j^\pm (z)$$

$$(z - q^{\pm a_{ij}} w) x_{jr}^\pm (z) x_{jr}^\pm (w) = (q^{\pm a_{ij}} z - w) x_{jr}^\pm (w) x_{jr}^\pm (z)$$

$$[x_{ir}^+, x_{js}^-] = \delta_{ij} k_{i, r+s}^+ - k_{i, r+s}^-$$

$$\sum_{w} \sum_{p=0}^{1-a_{ij}} (-1)^p [1 - a_{ij}]_p x_{ir}^\pm (1) \cdots x_{ir}^\pm (p) x_{jr}^\pm (1) \cdots x_{jr}^\pm (p+1) = 0$$

where $i \neq j, r_1, \ldots, r_{1-a_{ij}} \in \mathbb{Z}$ and $w \in S_{1-a_{ij}}$. Here we have set $\varepsilon = +$ or $-$,

$$[n] = (q^n - q^{-n})/(q - q^{-1}), \quad [n]! = [n][n-1]\cdots[2], \quad \left[\begin{array}{c} m \\ p \end{array}\right] = \frac{|m|!}{|p|!(m-p)!},$$

$$k_i^\pm (z) = \sum_{r \geq 0} k_{i, \pm r}^\pm z^{\mp r} \quad \text{and} \quad x_i^\pm (z) = \sum_{r \in \mathbb{Z}} x_{ir}^\pm z^{\mp r}.$$  

Let $U^\pm \subseteq U$ be the subalgebra generated by the elements $x_{ir}^\pm$ with $i \in I$, $r \in \mathbb{Z}$. For a future use, we also introduce the elements $h_{irs} \in U$, $s \neq 0$, such that

$$k_i^\pm (z) = k_i^\pm \exp \left( \pm (q - q^{-1}) \sum_{s \geq 1} h_{irs} \pm z^{\mp s} \right).$$  

Let $\Delta$ be the coproduct defined in terms of the Kac-Moody generators $e_i, f_i, k_i^\pm$, $i \in I \cup \{0\}$, of $U$ as follows

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i.$$.  

2.3. Fix $\lambda = \sum_i \ell_i \omega_i \in P^+$, $\alpha = \sum_i a_i \alpha_i \in Q^+$, $G_\alpha = \prod_i GL_{a_i}$, $G_\lambda = \prod_i GL_{\ell_i}$. For any algebraic group $G$ let $G^\vee$ be the set of cocharacters of $G$, and let $G^{\vee,\text{ad}}$ be the set of conjugacy classes in $G^\vee$. Thus $(\mathbb{C}^\times)^\vee = \{q^k; k \in \mathbb{Z}\}$. The direct sum $GL_m^\vee \times GL_n^\vee \to GL_{m+n}^\vee$ gives a semigroup structure on the sets $X^+ = \bigsqcup_{\lambda \in P^+} G_{\lambda}^{\vee,\text{ad}}$, $Y^+ = \bigsqcup_{\alpha \in Q^+} G_\alpha^{\vee,\text{ad}}$. The Abelian groups $X, Y$, associated to $X^+, Y^+$, are identified with the groups $\mathbb{Z}[q^{-1}, q] \otimes \mathbb{Z} P, \mathbb{Z}[q^{-1}, q] \otimes \mathbb{Z} Q$ via the maps

$$\gamma \mapsto \sum_i (\text{tr } \gamma_i) \otimes \omega_i, \quad \eta \mapsto \sum_i (\text{tr } \eta_i) \otimes \alpha_i,$$

where $\gamma_i, \eta_i$ are the $i$-th components of the elements $\gamma \in G_{\lambda}^{\vee,\text{ad}}, \eta \in G_\alpha^{\vee,\text{ad}}$. Hereafter we may omit the symbol $\otimes$ and write simply $q^n \lambda$ instead of $q^n \otimes \lambda$. Consider the $\mathbb{Z}[q, q^{-1}]$-linear map

$$\Omega : Y \to X, \alpha_i \mapsto [2]\omega_i - \sum_{i,j} -1 \omega_j.$$

Hereafter, let $\gamma + \eta$ denote the element $\gamma + \Omega(\eta) \in X$. We write $\eta \geq \delta$ if $\eta, \delta \in Y$ are such that $\eta - \delta \in Y^+$ (resp. we write $\gamma \geq \eta$ if $\gamma \in X, \eta \in Y$ are such that $\gamma - \eta \in X^+$). We have $q^{-1} \Omega = A + q^{-1} B + q^{-2} A$, where $A, B$ are $\mathbb{Z}[q, q^{-1}]$-linear operators such that $A(\alpha_i) = \omega_i$ for all $i \in I$. Let

$$(\varepsilon, \varepsilon) : (\mathbb{Z}((q^{-1})) \otimes \mathbb{Z} Q) \times X \to \mathbb{Z}((q^{-1}))$$

be the $\mathbb{Z}((q^{-1}))$-bilinear form such that $(\alpha_i | \omega_j) = \delta_{ij}$. Let $\Omega^{-1} : X \to \mathbb{Z}((q^{-1})) \otimes Q$ be the inverse of $\Omega$. For any $\gamma, \gamma' \in X$, we put

$$\varepsilon_{\gamma \gamma'} = (q^{-1} \Omega^{-1}(\gamma) | \gamma')_0, \quad \langle \gamma, \gamma' \rangle = \varepsilon_{\gamma \gamma'} - \varepsilon_{\gamma \gamma'},$$

where $f_0$ is the constant term of a formal series $f$, and $-$ is the $\mathbb{Z}$-linear involution such that $\bar{q} = q^{-1}$. It is easy to see that

$$\varepsilon_{\gamma + \gamma', \gamma''} + \varepsilon_{\gamma \gamma'} = \varepsilon_{\gamma, \gamma' + \gamma''} + \varepsilon_{\gamma', \gamma''},$$

for all $\gamma, \gamma', \gamma'' \in X^+$. Put $A = \mathbb{Z}[v, v^{-1}]$. Let $A_X$ be the $A$-algebra linearly spanned by elements $e^\gamma, \gamma \in X$, such that

$$(1) \quad e^\gamma \cdot e^{\gamma'} = v^{(\gamma, \gamma')} e^{\gamma + \gamma'}.$$

2.4. The simple finite dimensional $U$-modules are labelled by $I$-uples of monic polynomials in $\mathbb{C}(q)[t]$ with nonzero constant terms, called the Drinfeld polynomials. If $\gamma = \sum_k \gamma_k \in X^+$ with $\gamma_k = q^{n_k} \omega_{i_k}$ and $n_k \in \mathbb{Z}$, let $V(\gamma)$ be the simple finite dimensional $U$-module whose $i$-th Drinfeld polynomial is $F^{(i)}(z) = \prod_{i_k = i} (z - q^{n_k})$. For any $U$-module $V$ and any $I$-uple of formal series $\psi^\pm = (\psi^\pm_i) \in \mathbb{C}(q)[[\mathbb{Z}+1]]^I$, set $V_\psi = \bigcap V \cap \ker (k(z) - \psi^1_i Id)^\vee \subseteq V$. If $\gamma = \gamma^+ - \gamma^-$ with $\gamma^\pm \in X^+$, we put $V_\gamma = V_\psi$ where $\psi^\pm_i$ is the expansion at $\infty$ or $0$ of the rational function

$$q^{(\gamma^+(1) | \alpha_i)} \cdot P^{(i)}(1/qz) \cdot P^{(i)}(q/z)^{-1},$$
and \( P_{\gamma}^{(i)} = P_{\gamma}^{(i)}/P_{\gamma}^{(i)} \). Let \( \mathcal{C}_q \) be the category of pairs \((V,F)\) where \( V \) is a finite dimensional \( U \)-module such that \( V = \bigoplus_{\gamma} V_{\gamma} \), and \( F \) is a decreasing \( \mathbb{Z} \)-filtration on \( V \) compatible with the weight decomposition, i.e. \( F_{\ell}V = \bigoplus_{\gamma} (V_{\gamma} \cap F_{\ell}V) \) for all \( \ell \).

Let \( K(\mathcal{C}_q) \) be the \( A \)-module with one generator for each \((V,F) \in \text{Ob}(\mathcal{C}_q)\) modulo the relations

- \((V,F) = (V',F') + (V'',F'')\) if there is an exact sequence
  \[ 0 \to (V',F') \cap (V,F) \to (V'',F'') \to 0, \]
- \((V,F) = v(V',F')\) if \((V,F)\) is isomorphic to \((V',F'[1])\).

Fix an element \( \gamma = \sum_{k=1}^r \gamma_k \in X^+ \), with \( \gamma_k = q^{n_k} \omega_k \), such that \( n_1 \geq n_2 \geq \cdots \geq n_r \). The \( U \)-module \( V(\gamma_1) \otimes V(\gamma_2) \otimes \cdots \otimes V(\gamma_r) \) does not depend on the choice of such a decomposition of \( \gamma \), since

\[ V(q^{n_1} \omega_1) \otimes V(q^{n_2} \omega_2) \simeq V(q^{n_1} \omega_1) \otimes V(q^{n_2} \omega_2) \]

for all \( n,i,j \), see [Ka]. Let denote it by \( W(\gamma) \). Fix a highest weight vector \( v_\gamma \in V(\gamma) \). It is known that \( W(\gamma) \) is a cyclic \( U^- \)-module generated by the monomial \( w_\gamma = v_{\gamma_1} \otimes v_{\gamma_2} \otimes \cdots \otimes v_{\gamma_r} \), see [Ka], [VV]. The geometric construction in [N2] implies that

\[ W(\gamma) = \bigoplus_{\gamma'} W(\gamma)_{\gamma'}, \quad x_{ir}^- (W(\gamma)_{\gamma'}) \subseteq \bigoplus_{\gamma''} W(\gamma)_{\gamma''}, \]

where the sum is over all elements \( \gamma'' \in \gamma' - q^2 \alpha_i \), see also [FM]. Note that the element \( x_{ir}^- \) is not homogeneous for the weight decomposition above. Let \( x_{ir}^{(t)} \) be its component in

\[ \bigoplus_{\gamma'} \text{Hom} \left( W(\gamma)_{\gamma'}, W(\gamma)_{\gamma' - q^2 \alpha_i} \right). \]

Set also

\[ \phi_{ir}^{(t)} = \sum_{s=0}^{r} \binom{r}{s} (-1)^r - s q^{-st} x_{is}^{(t)}. \]

We endow \( W(\gamma) \) with the decreasing \( \mathbb{Z} \)-filtration such that

- \( \{0\} = F_1 W(\gamma)_{\gamma} \subset F_0 W(\gamma)_{\gamma} = W(\gamma), \)
- \( F_k W(\gamma)_{\gamma''} = \sum_{i,r,t} \phi_{ir}^{(t)} (F_t W(\gamma)_{\gamma'}), \)

where \( \ell, \gamma' \) are such that

\[ \gamma' = \gamma'' + q^i \alpha_i, \quad \ell = k - 2r - 1 - g_{i,t+1}'' + g_{i,t-1}'', \quad \gamma'' = \sum_{i,k} g_{ik}'' q^k \omega_i. \]

We have \((W(\gamma), F) \in \text{Ob}(\mathcal{C}_q)\). There is a unique surjective homomorphism of \( U \)-modules \( W(\gamma) \to V(\gamma) \) such that \( w_\gamma \mapsto v_\gamma \). The module \( V(\gamma) \) is endowed with the quotient filtration. Hereafter, the classes of the pairs \((V(\gamma), F), (W(\gamma), F)\) in \( K(\mathcal{C}_q) \) are simply denoted by \( V(\gamma), W(\gamma) \). Let \( GR \subset K(\mathcal{C}_q) \) be the \( A \)-submodule spanned by the elements \( V(\gamma) \). The tensor product of two objects in \( \mathcal{C}_q \) is endowed with the filtration such that

\[ F_k (V_\gamma \otimes V_\gamma') = \sum_{\ell + \ell' = k + \langle \gamma, \gamma' \rangle} F_\ell V_\gamma \otimes F_{\ell'} V_\gamma'. \]

Put

\[ \text{gdm}(V_\gamma, F) = \sum_{\ell} \text{dim}(Gr^\ell_\gamma V_\gamma) \cdot v^\ell, \quad \text{gch}(V, F) = \sum_{\gamma} \text{gdm}(V_\gamma, F) \cdot e^\gamma, \]

where \( Gr^F \) is the associated graded space.
Proposition. (a) The map \( \text{gch} : \mathbf{K}(\mathcal{C}_q) \to \mathbf{A}_X \) is a ring homomorphism.

(b) The map \( \text{gch} : \mathbf{GR} \to \mathbf{A}_X \) is injective.

(c) \( \mathbf{GR} \) is a subring of \( \mathbf{K}(\mathcal{C}_q) \).

Proof: Put \( \gamma = \gamma^+ - \gamma^- \), where \( \gamma^\pm = \sum_k q^{n_k} \omega_k \in X^+ \). The eigenvalue of \( h_{ir} \) on \( V_\gamma \) is \( r^{-1}[r] \sum_i \delta_{ir} (q^{n_r} - q^{-n_r}) \), see [FM, (2.11)] for instance. It is known that \( \Delta(h_{ir}) = h_{ir} \otimes 1 + 1 \otimes h_{ir} \), modulo the linear span of elements \( m_0 m^{-} m_0 \otimes n^0 n^+ \), where \( m_0 \) (resp. \( m^{-}, m^0, n^0, n^+ \)) is a monomial in the generators \( k^\pm_i \) (resp. \( x^+_i, h_i, x^-_i, x^0_i \)) such that \( m^-, n^+ \) have a non-zero degree, see [D]. Thus, the weight \( \gamma \) subspace in \( V'. \otimes V'' \) is

\[
\bigoplus_{\gamma = \gamma' + \gamma''} (V'_\gamma \otimes V''_{\gamma'}). 
\]

Then, Claim (a) follows from (2.3.1) and (2.4.1). Claim (b) is obvious. Claim (c) is proved in Theorem 4.3.

For a future use we introduce the following sets

\[
\Lambda(\gamma) = \{ \eta \in Y^+ \mid W(\gamma) \gamma - \eta \neq 0 \}, \quad \Lambda^+(\gamma) = \{ \eta \in \Lambda(\gamma) \mid \gamma \geq \eta \}.
\]

Remark. The map \( \text{gch} \) appeared first in [N3]. By [Vr], the same construction holds for Yangians. The specialization of \( \text{gch} \) at \( v = 1 \) first appeared in [Kn], for Yangians. The case of quantum affine algebras was done in [FR].

Example. We give a few computations in the case \( g = \mathfrak{sl}_2 \). To simplify we omit \( \omega_1 \) : we write \( q^n \) instead of \( q^n \omega_1 \). We get

\[
e^q m \cdot e^q n = v^t e^{q^m + q^n} \in \mathbf{A}_X,
\]

where \( t = 0 \) if \( n - m \) is zero or odd, and \( t = (-1)^\ell \) if \( n - m = 2\ell \) with \( \ell < 0 \). We have

\[
\text{gch} V(q^n) = e^{q^n} + e^{-q^{n+2}} \in \mathbf{A}_X,
\]

and

\[
W(q^n + q^{n-2}) = v^{-1} (q^n, q^{n-2}) V(q^n) \otimes V(q^{n-2}), \quad W(kq^n) = V(q^n) \otimes k \in \mathbf{GR}.
\]

Thus,

\[
\text{gch} W(q^n + q^{n-2}) = e^{q^n + q^{n-2}} + e^{q^{n-2} - q^{n+2}} + e^{-q^n - q^{n+2}} + v,
\]

\[
\text{gch} W(kq^n) = \sum_{i=0}^{k} \binom{k}{i}_v e^{i(q^n - (k-i)q^{n+2}},
\]

where \( \binom{k}{i}_v \) is the \( v \)-binomial coefficient. Note that our normalizations are different from [N3] (we use \( v = t^{-1} \)).
3. Reminder on quiver varieties

3.1. Consider the graph such that: \( I \) is the set of vertices, and there are \( 2\delta_{ij} - a_{ij} \) edges between \( i, j \in I \). Each edge is endowed with the two possible orientations. The corresponding set of arrows is denoted by \( H \). If \( h \in H \) let \( h' \) and \( h'' \) the incoming and the outcoming vertex of \( h \). Let \( \overline{h} \in H \) denote the arrow opposite to \( h \). Fix two \( I \)-graded finite dimensional complex vector spaces \( V, W \) of graded dimension \( (a_i), (\ell_i) \). Let us fix once for all the following convention: the dimension of the graded vector space \( V \) is identified with the root \( \alpha = \sum_{i \in I} a_i\alpha_i \in Q^+ \) while the dimension of \( W \) is identified with the weight \( \lambda = \sum_i \ell_i\omega_i \in P^+ \). Set

\[
E(V,W) = \bigoplus_{h \in H} M_{h',a_n}(\mathbb{C}), \quad L(V,W) = \bigoplus_{i \in I} M_{i,a_i}(\mathbb{C}),
\]

\[
M_{\lambda\alpha} = E(V,V) \oplus L(W,V) \oplus L(V,W).
\]

For any \( (B,p,q) \in M_{\lambda\alpha} \) let \( B_h \) be the component of \( B \) in \( \text{Hom}(V_{h''}, V_{h'}) \) and set

\[
m_{\lambda\alpha}(B,p,q) = \sum_h \varepsilon(h) B_h B_{\overline{h}} pq \in L(V,V),
\]

where \( \varepsilon \) is a function \( \varepsilon : H \to \mathbb{C}^\times \) such that \( \varepsilon(h) + \varepsilon(\overline{h}) = 0 \). A triple \( (B,p,q) \in m_{\lambda\alpha}^{-1}(0) \) is \( \spadesuit \)-stable if there is no nontrivial \( B \)-invariant subspace of \( \text{Ker} \ q \). Let \( m_{\lambda\alpha}^{-1}(0) \spadesuit \) be the subset of \( \spadesuit \)-stable triples. The group \( \mathbb{C}^\times \times G_\lambda \times G_\alpha \) acts on \( M_{\lambda\alpha} \) by

\[
(z,g_\lambda,g_\alpha) \cdot (B,p,q) = (zg_\alpha Bg_\alpha^{-1}, zg_\alpha pg_\lambda^{-1}, zg_\lambda qg_\alpha^{-1}).
\]

The action of \( G_\alpha \) on the subset \( m_{\lambda\alpha}^{-1}(0) \spadesuit \) is free. Consider the varieties

\[
Q_{\lambda\alpha} = \text{Proj} (\bigoplus_{n \geq 0} A_n) \quad \text{and} \quad N_{\lambda\alpha} = m_{\lambda\alpha}^{-1}(0)/G_\alpha,
\]

where \( \bigoplus \) is the categorical quotient, and

\[
A_n = \{ f \in \mathbb{C}[m_{\lambda\alpha}^{-1}(0)] \mid f(g_\alpha \cdot (B,p,q)) = (\det g_\alpha)^{-n} f(B,p,q) \}.
\]

The variety \( Q_{\lambda\alpha} \) is smooth and there is a bijection \( Q_{\lambda\alpha} \simeq m_{\lambda\alpha}^{-1}(0) \spadesuit /G_\alpha \).

3.2. Let \( \pi_{\lambda\alpha} : Q_{\lambda\alpha} \to N_{\lambda\alpha} \) be the affinization map. It is a proper map. Put \( d_{\lambda\alpha} = \dim Q_{\lambda\alpha} \). It is known that \( d_{\lambda\alpha} = (a|2\lambda - \alpha) \). If \( \alpha \geq \beta \) the extension by zero of representations of the quiver gives a closed embedding \( N_{\lambda\beta} \hookrightarrow N_{\lambda\alpha} \). Set \( N_\lambda = \bigcup_{\alpha} N_{\lambda\alpha}, Q_\lambda = \bigcup_{\alpha} Q_{\lambda\alpha}, F_\lambda = \bigcup_{\alpha} F_{\lambda\alpha}, \) where \( F_{\lambda\alpha} = \pi_{\lambda\alpha}^{-1}(0) \). A triple \( (B,p,q) \in m_{\lambda\alpha}^{-1}(0) \) is regular if it is \( \spadesuit \)-stable and its \( G_\alpha \)-orbit is closed. Let \( m_{\lambda\alpha}^{-1}(0) \heartsuit \subseteq m_{\lambda\alpha}^{-1}(0) \spadesuit \) be the subset of regular triples. Let \( Q_{\lambda\alpha}^{\heartsuit} = m_{\lambda\alpha}^{-1}(0) \heartsuit /G_\alpha \) and \( N_{\lambda\alpha}^{\heartsuit} = m_{\lambda\alpha}^{-1}(0) \heartsuit /G_\alpha \) be the corresponding open subsets in \( Q_{\lambda\alpha}, N_{\lambda\alpha} \). The map \( \pi_{\lambda\alpha} \) gives an isomorphism \( Q_{\lambda\alpha}^{\heartsuit} \xrightarrow{\sim} N_{\lambda\alpha}^{\heartsuit} \). It is proved in [N1], [N2] that

- \( N_{\lambda\alpha}^{\heartsuit} \neq \emptyset \iff \alpha \in \bigwedge^+(\lambda), \) and \( Q_{\lambda\alpha} \neq \emptyset \iff \alpha \in \bigwedge(\lambda), \)
- \( N_\lambda = \bigcup_{\alpha} N_{\lambda\alpha}^{\heartsuit}, \) and \( N_{\lambda\beta}^{\heartsuit} \subseteq N_{\lambda\alpha}^{\heartsuit} \iff \alpha \geq \beta. \)
3.3. The fixedpoint set of a bijection $\phi : X \rightarrow X$ is denoted by $X^{\phi}$. The group $\mathbb{C}^* \times G_\alpha$ acts on $Q_{\lambda^\alpha}$, $N_{\lambda^\alpha}$. For any $k \in \mathbb{Z}$ and $(\gamma, \eta) \in G^\vee \times G^\vee_\alpha$ we set

$$Q_{\gamma \eta, k} = (G_\alpha \cdot m^{-1}_{\lambda^\alpha}(0) \cdot (q^k \gamma, \eta)) / G_\alpha, \quad Q_{\gamma, k} = Q_{\lambda^\alpha}^{(q^k \gamma)}, \quad N_{\gamma, k} = N_{\lambda^\alpha}^{(q^k \gamma)}.$$ 

It is known that $Q_{\gamma \eta, k}$ is either empty or a connected component of $Q_{\gamma, k}$. Let $\pi_{\gamma, k} : Q_{\gamma, k} \rightarrow N_{\gamma, k}$ be the restriction of the map $\pi_\lambda$. We set $F_{\gamma, k} = F_{\gamma \eta, k}$, $Q_{\gamma \eta, k} = Q_{\gamma, k} \cap Q_{\lambda^\alpha}$, $N_{\gamma \eta, k} = \pi_\gamma(Q_{\gamma \eta, k})$. The restriction of $\pi_{\gamma, k}$ to $Q_{\gamma \eta, k}$ is an isomorphism onto $N_{\gamma \eta, k}$. It is proved in [N2] that

- $Q_{\gamma, k} = \bigcup_\eta Q_{\gamma \eta, k}$, $N_{\gamma, k} = \bigcup_\eta N_{\gamma \eta, k}$, and $Q_{\gamma \eta, k}$ is connected (or empty),
- the set $N_{\gamma \eta, k}$ depends only on the conjugacy classes of $\gamma, \eta$,
- $N_{\gamma \eta, 1} \neq \emptyset \iff \eta \in \Lambda^+(\gamma)$ and $Q_{\gamma \eta, 1} \neq \emptyset \iff \eta \in \Lambda(\gamma)$.

To simplify, hereafter we set $Q_{\gamma \eta} = Q_{\gamma \eta, 1}$, $Q_{\gamma} = Q_{\gamma, 1}$, $N_{\gamma} = N_{\gamma, 1}$, etc. Put

$$d_{\gamma \eta} = (\bar{\eta} | 2 [\gamma - q \eta])_0.$$ 

If $Q_{\gamma \eta} \neq \emptyset$ then $d_{\gamma \eta} = \dim Q_{\gamma \eta}$, see [N2, (4.1.6)].

3.4. If an algebraic group $G$ acts on a variety $X$, and if $\phi \in G^\vee$, we put

$$X^{+\phi} = \{x \in X \mid \lim_{z \rightarrow 0} \phi(z) \cdot x \in X^{\phi}\}, \quad X^{-\phi} = \{x \in X \mid \lim_{z \rightarrow \infty} \phi(z) \cdot x \in X^{\phi}\}.$$ 

For any $k \in \mathbb{Z}$, $\gamma \in G^\vee_\alpha$, $\tau \in (\mathbb{C}^* \times G_\alpha)^\vee$ we have the commutative diagram

\[
\begin{array}{ccc}
Q_{\gamma, k} & \overset{i_+}{\leftarrow} & Q^+_\gamma \, k \, \overset{\kappa_+}{\rightarrow} & Q^\tau \, k \\
N_{\gamma, k} & \overset{i_+}{\leftarrow} & N_{\gamma, k}^{+\tau} & \overset{\kappa_+}{\rightarrow} & N_{\gamma, k}^\tau,
\end{array}
\]

where $i_\pm, \kappa_\pm$ are the embeddings, and $\kappa_\pm, \kappa_\pm$ are the obvious projections. Since the map $\pi_{\gamma, k}$ is proper the left square is Cartesian.

Remark. The maps $i_\pm, \kappa_\pm$ are closed embeddings. We have $Q^\pm \gamma \, k = \pi^{-1}_{\gamma, k}(N^{\pm \tau}_{\gamma, k})$ since $\pi_{\gamma, k}$ is a proper map. Thus, it is sufficient to consider the case of $i_\pm$. From [L2], we can fix a finite set of generators of the ring $\mathbb{C}[m^{-1}_{\lambda^\alpha}(0)]^{G_\alpha}$ consisting of eigenvectors of the group $\tau(\mathbb{C}^*) \times \gamma(\mathbb{C}^*) \subset \mathbb{C}^* \times G_\alpha$. These generators give a $\tau(\mathbb{C}^*)$-equivariant closed embedding of the variety $N_{\gamma, k}$ in a finite dimensional representation of $\tau(\mathbb{C}^*)$. But $X^{\tau \phi}$ is a closed subset of $X$ in the particular case where $X$ is a representation of the one-parameter subgroup $\phi$. Thus $N^\tau_{\gamma, k}$ is a closed subset of $N_{\gamma, k}$.

3.5. Fix $\lambda', \lambda'' \in P^+$, fix $I$-graded vector spaces $W', W''$ of dimension $\lambda', \lambda''$, and fix $\gamma' \in G^\vee_\alpha$, $\gamma'' \in G^\vee_\alpha$. Put $\gamma = \gamma' + \gamma''$, $\lambda = \lambda' + \lambda''$, $W = W' \oplus W''$ and $\tau = q \cdot \text{Id}_{W'} \oplus \text{Id}_{W''}$. 

Lemma 1. (a) The direct sum of representations of the quiver gives an isomorphism $Q_{\gamma', k} \times Q_{\gamma'', k} \simeq Q_{\gamma, k}$, and a map $\phi : N_{\gamma', k} \times N_{\gamma'', k} \rightarrow N_{\gamma, k}$.

(b) The map $\phi$ is finite, bijective and is compatible with the stratifications.

Proof: The first claim is well-known, see [VV, Lemma 4.4] for instance. Let $\phi : m_{\lambda' \alpha'}^{-1}(0) \times m_{\lambda'' \alpha''}^{-1}(0) \rightarrow m_{\lambda \alpha}^{-1}(0)$ be the direct sum of representations of the quiver in
§3.1. The induced map \( N_{\lambda'\alpha'} \times N_{\lambda''\alpha''} \to N_{\lambda\alpha} \) is a morphism of algebraic varieties.

We have
\[
\phi(m_{\lambda'\alpha'}^{-1}(0)^\vee \times m_{\lambda''\alpha''}^{-1}(0)^\vee) \subset m_{\lambda\alpha}^{-1}(0)^\vee,
\]
since a triple \((B,p,q) \in m_{\lambda\alpha}^{-1}(0)\) is regular if and only if it is stable and costable (i.e. there is no proper \(B\)-invariant subspace of \(V\) containing \(\text{Im}p\)), see [L2]. Fix \(\eta' \in G_{\lambda'\alpha'}, \eta'' \in G_{\lambda''\alpha''}\) such that \(\eta = \eta' + \eta''\). By the first part, \(\phi\) gives an isomorphism \(N_{\eta',\eta'} \times N_{\eta'',\eta''} \to (N_{\eta,\eta})^\tau\). In particular it induces a bijection
\[
N_{\eta',\eta'} \times N_{\eta'',\eta''} \to \bigsqcup_{\eta'} N_{\eta',\eta'} \times N_{\eta'',\eta''} \to \bigsqcup_{\eta} (N_{\eta,\eta})^\tau = N_{\eta,\eta},
\]
which is compatible with the stratifications. This map is clearly affine, since \(N_{\lambda\alpha}\)
is an affine variety. Thus it is finite.

If \(k = 1\) we get

\[
\begin{array}{ccc}
Q_{\gamma'} & \xleftarrow{\varepsilon} & Q_{\gamma''} \\
\pi_{\gamma'} & \downarrow & \downarrow \\
N_{\gamma} & \xleftarrow{\kappa} & N_{\gamma''}
\end{array}
\]

Fix \(\eta' \in G_{\lambda'\alpha'}, \eta'' \in G_{\lambda''\alpha''}\). Let \(\kappa_{\eta',\eta''}^\pm\) be the relative dimension of the map \(\kappa_{\eta',\eta''}\) above the component \(Q_{\gamma'} \times Q_{\gamma''}\). Set \(\eta = \eta' + \eta''\).

**Lemma 2.** We have

(a) \(\kappa_{\eta',\eta''}^+ + \kappa_{\eta',\eta''}^- = d_{\gamma} = d_{\gamma'} - d_{\gamma''},\)

(b) \(\kappa_{\eta',\eta''}^+ = \kappa_{\eta''\gamma'}^+\),

(c) If \(\delta' \in \Lambda^+(\gamma'), \delta'' \in \Lambda^+(\gamma'')\) are such that \(\eta' \succeq \delta', \eta'' \succeq \delta''\), then
\[
\varepsilon_{\gamma'-\gamma''} - \varepsilon_{\eta'\delta',\gamma''-\delta''} = \kappa_{\eta',\eta''}^+ - \kappa_{\eta''\gamma'}^+ = \kappa_{\eta',\eta''}^+.
\]

**Proof:** Part (a) is immediate. Let us check Part (b). The one-parameter subgroup \(q \cdot \text{Id}_{\gamma'} \oplus \text{Id}_{\gamma''}\) acts fiberwise on the normal bundle to \(Q_{\gamma'} \times Q_{\gamma''}\) in \(Q_{\gamma}\). By definition \(\kappa_{\eta',\eta''}^+\) is the dimension of the attracting (resp. repulsing) subbundle. The class in equivariant \(K\)-theory of the tangent bundle to \(Q_{\gamma}\) is given in [N1, §4.1]. We get

\[
\kappa_{\eta',\eta''}^+ = (\eta' | q^{-1} \gamma'')_0 + (\eta'' | q \gamma')_0 - (\eta' | q \Omega(\eta'))_0,
\]

and \(\kappa_{\eta',\eta''}^- = \kappa_{\eta'\gamma'}^+\). Observe that
\[
(\Omega^{-1}(\gamma') | \Omega(\eta)) = (\eta | \gamma), \quad \forall \gamma \in X, \eta \in Y.
\]

Part (c) is proved by a direct computation using (3.5.1) and
\[
\varepsilon_{\gamma'-\gamma''} - \varepsilon_{\gamma'-\delta',\gamma''-\delta''} = (q^{-1} \Omega^{-1}(\gamma') | \gamma'')_0 - (q^{-1} \Omega^{-1}(\gamma') | q^{-1} \delta' | \gamma'' - \Omega(\delta''))_0
\]
\[
= (q^{-1} \Omega^{-1}(\gamma') | \Omega(\delta''))_0 + (q^{-1} \delta' | \gamma''')_0 - (q^{-1} \delta' | \Omega(\delta''))_0
\]
\[
= (q\delta'' | \gamma')_0 + (q^{-1} \delta' | \gamma''')_0 - (q\delta'' | \Omega(\delta''))_0.
\]

\(\Box\)
4. The product

4.1. For any complex algebraic variety $X$, let $\mathcal{D}(X)$ be the bounded derived category of complexes of constructible sheaves of $\mathbb{C}$-vector spaces on $X$. For any irreducible local system $\phi$ on a locally closed set $Y \subset X$, let $IC(Y, \phi)$ be the corresponding intersection cohomology complex. Let $\mathbb{C}_Y$ be the constant sheaf on $Y$. We set $IC(Y) = IC(Y, \mathbb{C}_Y)$. Recall that the direct image of a simple perverse sheaf by a finite bijective map is still a simple perverse sheaf. Let $\mathbb{D}$ denote the Verdier duality.

Fix $\gamma, \gamma' \in X^+$, $\eta, \eta' \in Y^+$. Fix $\lambda, \alpha$ such that $\gamma \in G^\vee, \alpha$, $\eta \in G^\vee, \alpha$. Hereafter we may identify a cocharacter in $G^\vee, G^\vee, \alpha$, and its conjugacy class in $X^+, Y^+$. Let $\mathcal{D}(N_{\gamma})^\vee$, $\mathcal{D}(N_{\gamma} \times N_{\gamma'})^\vee$ be the full subcategories of $\mathcal{D}(N_{\gamma})$, $\mathcal{D}(N_{\gamma} \times N_{\gamma'})$ consisting of all complexes which are constructible with respect to the stratification in $\mathbb{S}3.3$. Set $IC_{\gamma\eta} = IC(N_{\gamma})^\vee$, $\mathbb{C}_{\gamma\eta} = \mathbb{C}_{N_{\gamma}[d_{\gamma\eta}]}$ for any $\eta \in \Lambda^+(\gamma)$, and $L_{\gamma\eta} = \pi_{\gamma!\mathbb{C}_{\gamma\eta}[d_{\gamma\eta}]}$ for any $\eta \in \Lambda(\gamma)$. Let $Q_{\gamma}, Q_{\gamma\gamma'}$ be the full subcategories of $\mathcal{D}(N_{\gamma})^\vee$, $\mathcal{D}(N_{\gamma} \times N_{\gamma'})^\vee$ consisting of all complexes which are isomorphic to finite direct sums of the sheaves $IC_{\gamma\eta}[k], IC_{\eta\gamma} \otimes IC_{\gamma\eta}[k'], k, k' \in \mathbb{Z}$. The complex $L_{\gamma\eta}$ belongs to $\mathbb{O}_{\mathbb{O}}(Q_{\gamma})$, see [N2, Theorem 14.3.2]. If $\gamma', \gamma''$, $\iota, \kappa, \tau$ are as in $\mathbb{S}3.5$, we have the functor $\mathbb{D}(N_{\gamma})^\vee \rightarrow \mathcal{D}(N_{\gamma})^\vee$.

Lemma. We have

(a) $\mathbb{D}(IC_{\gamma\eta})(L_{\gamma\eta}) = \bigoplus_{\eta' \leq \eta, \eta''} \pi_{\gamma!\mathbb{C}_{\gamma\eta}[d_{\gamma\eta}]}(L_{\gamma\eta}[\eta'] \otimes L_{\gamma\eta''}([\kappa_{\eta'\eta''} - \kappa_{\eta''\eta'}]),$

(b) $\mathbb{D} \circ \mathbb{D}(IC_{\gamma\eta}) = \mathbb{D}(IC_{\gamma\eta}) \circ \mathbb{D}$, and $\mathbb{D}(IC_{\gamma\eta''}'') = \mathbb{D}(IC_{\gamma\gamma''}').$

(c) For any complex $P \in \mathbb{O}_{\mathbb{O}}(Q_{\gamma})$ there is a complex $P' \in \mathbb{O}_{\mathbb{O}}(Q_{\gamma''})$ such that $\mathbb{D}(IC_{\gamma\eta})$ is unique up to isomorphism.

Proof: By base change, the diagram in $\mathbb{S}3.5$ gives

$$\mathbb{D}(IC_{\gamma\eta})(L_{\gamma\eta}) = \pi_{\gamma!\mathbb{C}_{\gamma\eta}[d_{\gamma\eta}].}$$

From [L1, 8.1.6] the complex $\pi_{\gamma!\mathbb{C}_{\gamma\eta}[d_{\gamma\eta}]}$ is semi-simple, and there are short exact sequences of perverse sheaves

$$0 \rightarrow pH^n(f_j); i_{\pm\gamma}^+C_{\gamma\eta} \rightarrow pH^n(f_{\leq j}); i_{\pm\gamma}^+C_{\gamma\eta} \rightarrow pH^n(f_{\leq j-1}); i_{\pm\gamma}^+C_{\gamma\eta} \rightarrow 0,$$

where $pH^n$ is the perverse cohomology, and $f_j$ (resp. $f_{\leq j}$) is the restriction of the map $\pi_{\gamma}K_{\pm\gamma}$ to the union of all subvarieties

$$K_{\pm\gamma}^{-1}(Q_{\gamma\eta'} \times Q_{\gamma''}) \subset Q_{\gamma}^\gamma$$

of dimension $j$ (resp. $\leq j$). We have also

$$\pi_{\gamma!\mathbb{C}_{\gamma\eta}[d_{\gamma\eta}]} = \phi_{\gamma!}(L_{\gamma\eta'} \otimes L_{\gamma''})[d_{\gamma\eta} - 2\kappa_{\eta'\eta''}].$$

Thus, Claim (a) follows from Lemma 3.5.2.(a). Claim (b) is due to the auto-duality of $L_{\gamma\eta}$, since the map $\pi_{\gamma}$ is proper, and Lemma 3.5.2.(b). The first part of Claim (c) follows from Claim (a), since a direct summand of a complex in $Q_{\gamma}$ belongs to $Q_{\gamma}$. The second part of Claim (c) is due to Lemma 3.5.1.(b). \qed
4.2. Let $\mathcal{K}_\gamma$ be the $\mathbb{A}$-module with one generator for each isomorphism class of object of $Q_{\gamma}$, with relations $P + P' = P''$ if the complex $P''$ is isomorphic to $P \oplus P'$, and $P = v P'$ if the complex $P$ is isomorphic to $P'[1]$. The elements $IC_{\gamma\eta}$, with $\eta \in \Lambda^+(\gamma)$, form a $\mathbb{A}$-basis of $\mathcal{K}_\gamma$. Let $\text{res}_{\gamma'\gamma''}^{\gamma}$ be the $\mathbb{A}$-linear map $\mathcal{K}_\gamma \to \mathcal{K}_{\gamma'} \otimes \mathcal{K}_{\gamma''}$ such that

$$\text{res}_{\gamma'\gamma''}^{\gamma}(P) = v^{(\gamma',\gamma'')} \sum_i P'_i \otimes P''_i$$

where $\text{res}_{\gamma'\gamma''}^{\gamma}(P) = \bigoplus_i \phi_i (P'_i \otimes P''_i)$. It is well-defined and unique by Lemma 4.1.(c).

**Lemma 1.** (a) In $\mathcal{K}_\gamma$ we have

$$L_{\gamma\eta} = \sum_{\delta} \text{gdm}V(\gamma - \delta)_{\gamma - \eta} IC_{\gamma\delta}.$$

In particular, the elements $L_{\gamma\eta}$, with $\eta \in \Lambda^+(\gamma)$, form a $\mathbb{A}$-basis of $\mathcal{K}_\gamma$.

(b) If $\delta \in \Lambda^+(\gamma)$ there is a unique surjective map $\mathcal{K}_\gamma \to \mathcal{K}_{\gamma - \delta}$ such that $L_{\gamma\eta} \mapsto L_{\gamma\eta - \delta}$ if $\eta \in \Lambda(\gamma)$, $\eta \geq \delta$, and $L_{\gamma\eta} \mapsto 0$ else.

(c) If $\delta \in \Lambda^+(\gamma)$, $\delta' \in \Lambda^+(\gamma')$, $\delta'' \in \Lambda^+(\gamma'')$, $\delta = \delta' + \delta''$, the square

$$\begin{array}{ccc}
\mathcal{K}_\gamma & \xrightarrow{\text{res}} & \mathcal{K}_{\gamma'} \otimes \mathcal{K}_{\gamma''} \\
\downarrow & & \downarrow \\
\mathcal{K}_{\gamma - \delta} & \xrightarrow{\text{res}} & \mathcal{K}_{\gamma' - \delta'} \otimes \mathcal{K}_{\gamma'' - \delta''}
\end{array}$$

is commutative.

**Proof:** Fix $\delta \in \Lambda^+(\gamma) \cap G^V_{\beta,\text{ad}}$ such that $\eta \geq \delta$, and fix $x_\delta \in N^S_{\gamma\delta}$. We have an isomorphism

$$W(\gamma - \delta)_{\gamma - \eta} \simeq \bigoplus_k H_k(F_{\gamma - \delta,\eta - \delta}) \simeq \bigoplus_k H_k(\mathcal{Q}_{\gamma\eta} \cap \pi^{-1}_\gamma(x_\delta))$$

such that $w_{\gamma - \delta} \in H_0(F_{\gamma - \delta,0})$, see [VV, Theorem 7.12], [N2, Theorems 3.3.2 and 7.4.1]. We first check that

$$\text{gdm}W(\gamma - \delta)_{\gamma - \eta} = \sum_k v^{d_{\gamma - \delta,\eta - \delta} - k} \dim H_k(\mathcal{Q}_{\gamma\eta} \cap \pi^{-1}_\gamma(x_\delta)).$$

To simplify the notations, we may assume that $\delta = 0$, without loss of generalities. Let $C_{\lambda,\alpha + a_i,\alpha} \subseteq Q_{\lambda,\alpha + a_i} \times Q_{\lambda,\alpha}$ be the set of pairs $(x',x)$ such that $x$ is a subrepresentation of $x'$. For any $\eta, \eta'$ put

$$C_{\eta'\eta} = C_{\lambda,\alpha + a_i,\alpha} \cap (Q_{\gamma\eta'} \times Q_{\gamma\eta}).$$

If $C_{\eta'\eta} \neq \emptyset$ then $\eta' = \eta + q^t\alpha_i$ for some $t \in \mathbb{Z}$. Set

$$d_{\eta'\eta} = \dim C_{\eta'\eta}, \quad e_{\eta'\eta} = d_{\gamma\eta} + d_{\gamma\eta'} - 2d_{\eta'\eta}.$$

Let $\ast$ be the convolution product in Borel-Moore homology, see [CG]. By definition, we have

$$H^B_{d_{\gamma\eta'} + d_{\gamma\eta} - e}(C_{\eta'\eta}) \ast H_{d_{\gamma\eta} - \epsilon}(F_{\gamma\eta}) \subseteq H_{d_{\gamma\eta'} - k}(F_{\gamma\eta'}), \quad k = \ell + e,$$
see [CG, Lemma 8.9.5]. Recall that \( x_{ir}^{(t)} \) acts on \( H_*(F_\gamma) \) by the \(*\)-product by an element of the form
\[
\sum_{\eta} (\theta_{\eta'}} \cup q^r e^{r\omega_{\eta'}}) \cap \lfloor C_{\eta'}} \rceil \in H^{BM}_*(C_{\eta'}).
\]
where \( \eta' = \eta + q^t \alpha_t \), \([C_{\eta'}]\) is the fundamental class, \( \omega_{\eta'}} \), \( \theta_{\eta'}} \in H^{2s}(Q_{\gamma' \times Q_{\eta}}) \), \( \text{deg} \omega_{\eta'}} = 2 \), and \( \theta_{\eta'}} \) is invertible. Moreover, \( \omega_{\eta'}} \), \( \theta_{\eta'}} \) do not depend on \( r \). More precisely, from [N2, (9.3.2), §13.4], we have
\[
(2) \quad \theta_{\eta'}} = e^{k\omega_{\eta'}} \cup (1 \otimes \nu_{\eta'}}
\]
where \( k \in \mathbb{Z} \) and \( \nu_{\eta'}} \in H^{2s}(Q_{\gamma}) \) is invertible. Fix a non-zero \( v \in H_0(F_{\gamma_0}) \). The space \( H_*(F_\gamma) \) is spanned by the elements \( x_{ir_1} \cdots x_{ir_s}(v) \). Thus, for any \( \eta' \in Y^+ \backslash 0 \), we get
\[
H_*(F_{\eta'}) = \sum_{i,t,r} x_{ir}^{(t)} \ast H_*(F_{\eta}),
\]
where \( \eta = \eta' - q^t \alpha_t \). Set \( \psi_{ir}^{(t)} = \sum_{\eta} \omega_{\eta'}} \cap \lfloor C_{\eta'} \rceil \). Using (4.2.2) we get
\[
H_*(F_{\eta'}) = \sum_{i,t,r} \psi_{ir}^{(t)} \ast H_*(F_{\eta}).
\]
The \(*\)-product by \( \psi_{ir}^{(t)} \) on \( H_*(F_{\eta}) \) is a homogeneous operator of degree \( e_{\eta'}} + 2r \in \mathbb{Z} \).
Thus,
\[
H_{d_{\eta'}}-k(F_{\eta'}) = \sum_{i,t,r} \psi_{ir}^{(t)} \ast H_{d_{\eta'}}-\ell(F_{\eta}),
\]
where \( k = e_{\eta'}} + \ell + 2r \). Set
\[
F_{k} H_*(F_{\eta}) = \bigoplus_{\ell \geq 0} H_{d_{\eta'}}-\ell(F_{\eta}).
\]
A direct computation gives
\[
\phi_{ir}^{(t)} = \sum_{\eta} \theta_{\eta'}} \cap (e^{\omega_{\eta'}} - 1)^r \cap \lfloor C_{\eta'} \rceil,
\]
where \( \eta' = \eta + q^t \alpha_t \). Thus
\[
(3) \quad F_k H_*(F_{\eta'}) = \sum_{i,t,r} \phi_{ir}^{(t)} \ast F_k H_*(F_{\eta}),
\]
where \( k = e_{\eta'}} + 1 + 2r \).

The \( \gamma \)-fixed part of the complex \([N2, (5.1.1)]\) is the normal bundle of \( C_{\eta'}} \) in \( Q_{\gamma' \times Q_{\gamma}} \). Thus
\[
d_{\gamma} + d_{\gamma'} - d_{\eta'}} = (q\eta' + q^{-1}\eta' \mid \gamma)_0 - (q\eta' \mid \Omega(\eta'))_0.
\]
From \( \eta' = \eta + q^t \alpha_t \), we get
\[
e_{\eta'}} = (\eta' - \eta \mid q^{-1}(\gamma - \eta) - q(\gamma - \eta'))_0 = 1 + (\alpha_t \mid q^{-1}(q^{-1} - q)(\gamma - \eta')).
\]
Using (4.2.3) and §2.4 we get
\[ F_\ell W(\gamma)_{\gamma-\eta} = F_\ell H_*(F_{\gamma\eta}). \]

The identity (4.2.1) follows.

To prove Lemma 4.2.1. set \( L_{\gamma\eta} = \bigoplus_{k, \delta \leq \eta} M_{\delta k} \otimes IC_{\gamma\delta}[k]. \) If \( \gamma, \eta \geq \delta, \) let \( \phi_{\delta k} : H_{d_{\gamma-\delta, \eta-\delta}}(F_{\gamma-\delta, \eta-\delta}) \to H^{d_{\gamma-\delta, \eta-\delta}+k}(F_{\gamma-\delta, \eta-\delta}) \)
be the composition of the maps
\[ H_{*k}(F_{\gamma-\delta, \eta-\delta}) \to H_{*k}^B(Q_{\gamma-\delta, \eta-\delta}) \to H^{*+k}(Q_{\gamma-\delta, \eta-\delta}) \to H^{*+k}(F_{\gamma-\delta, \eta-\delta}). \]

A detailed analysis of the gradings in [N2, §14], [CG, §8] shows that \( M_{\delta k} = \text{Im } \phi_{\delta k}. \)
Since \( V(\gamma - \delta)_{\gamma-\eta} \simeq \bigoplus_k M_{\delta k}, \) we get \( \text{gdm} V(\gamma - \delta)_{\gamma-\eta} = \sum_k v^k \dim M_{\delta k}. \)
Let us prove part (b). By [N2, Theorem 3.3.2] we have for any \( \delta \in \bigwedge^+(\gamma) \)
\[ N_{\gamma-\delta, \eta-\delta} = \emptyset \iff N_{\gamma\eta} = \emptyset, \quad Q_{\gamma-\delta, \eta-\delta} = \emptyset \iff Q_{\gamma\eta} = \emptyset. \]
Thus, using §3.3 we get
\[ \eta - \delta \in \bigwedge^+(\gamma - \delta) \iff \eta \in \bigwedge^+(\gamma), \quad \eta \geq \delta, \]
\[ \eta - \delta \in \bigwedge(\gamma - \delta) \iff \eta \in \bigwedge(\gamma), \quad \eta \geq \delta. \]

By (4.2.4) there is a unique surjective map \( K_{\gamma} \to K_{\gamma-\delta} \) such that
\[ IC_{\gamma\eta} \mapsto IC_{\gamma\eta} \text{ if } \eta \geq \delta, \quad \text{and } IC_{\gamma\eta} \mapsto 0 \text{ else.} \]

Using (4.2.4) again and Claim (a) of the lemma, we see that this map satisfies the requirements in Claim (b).
Set
\[ A = \kappa_{\gamma', \gamma''} - \kappa_{\gamma'}^{+} + (\gamma', \gamma''), \]
\[ B = \kappa_{\gamma'-\delta', \gamma''-\delta''}^{+} - \kappa_{\gamma'-\delta', \gamma''-\delta''} - (\gamma'-\delta', \gamma''-\delta''). \]

Using Lemma 3.5.2. and (c) we get \( A = B. \) Thus, Claim (c) follows from Claim (b) and Lemma 4.1.(a).

4.3. Let \( (b_{\gamma\eta}), (c_{\gamma\eta}) \) be the bases of \( GA_{\gamma} = \text{Hom}_{\mathfrak{A}}(K_{\gamma}, \mathfrak{A}) \) dual to \( (IC_{\gamma\eta}), (L_{\gamma\eta}). \)
Let \( \otimes : GA_{\gamma'} \otimes GA_{\gamma''} \to GA_{\gamma'+\gamma''} \) and \( \theta : GA_{\gamma} \to GA_{\gamma} \) be the maps dual to \( \text{res}_{\gamma', \gamma''} \) and \( \mathbb{D}. \) We consider the inductive system of \( \mathfrak{A}\)-modules \( (GA_{\gamma}) \) such that \( b_{\gamma\eta} \mapsto b_{\gamma+\delta, \eta+\delta}. \) Let \( GA = \lim_{\rightarrow} GA_{\gamma} \) be the limit. Let \( b_{\gamma}, c_{\gamma} \in GA \) be the images of the elements \( b_{\gamma0}, c_{\gamma0} \in GA_{\gamma}. \)

**Theorem.** The \( \mathfrak{A}\)-module GR is a subring of \( K(C_{\eta}). \) The linear map such that \( b_{\gamma} \mapsto V(\gamma) \) is an algebra isomorphism \( GA \cong GR. \) The map \( \theta \) is a skew-linear antihomomorphism of \( GA \) fixing the bases \( B = (b_{\gamma}), C = (c_{\gamma}). \) For any \( \gamma, \gamma' \) we have
\[ b_{\gamma} \otimes b_{\gamma'} \in \bigoplus_{\gamma''} \mathbb{N}[v^{-1}, v] \cdot b_{\gamma''}. \]

**Proof:** The maps \( \mathbb{D}, \text{res}_{\gamma', \gamma''} \) are compatible with the projective system \( (K_{\gamma}). \) The limit, denoted \( (K, \text{res}), \) is a co-algebra with a skew-linear involution \( \mathbb{D}. \) By Lemma
4.2.1.(a), (b), the projective system maps $IC_{\gamma \eta}$ to $IC_{\gamma - \delta, \eta - \delta}$, for any $\eta \in \Lambda^+ (\gamma)$ such that $\eta \succeq \delta$. In $\mathcal{K}$ we consider the elements $IC_{\gamma} = (IC_{\gamma + \delta, \delta}),$ with $\gamma \in X^+$, and $L_{\gamma} = (L_{\gamma + \delta, \delta}),$ with $\gamma \in X$. We have

$$\text{res}(L_{\gamma}) = \sum_{\gamma' = \gamma + \gamma''} v^{(\gamma', \gamma'')} L_{\gamma'} \otimes L_{\gamma''}.$$ 

Let $A^\flat_X$ be the $A$-coalgebra with the $A$-basis $(a_\gamma)$, $\gamma \in X$, and the coproduct $a_\gamma \mapsto \sum_{\gamma' = \gamma + \gamma''} v^{(\gamma', \gamma'')} a_{\gamma'} \otimes a_{\gamma''}$. The $A$-linear map $A^\flat_X \to (\mathcal{K}, \text{res})$ such that $a_\gamma \mapsto L_{\gamma}$ is a surjective co-algebra homomorphism. By Lemma 4.2.1.(a) we have

$$L_{\gamma} = \sum_{\gamma' \in X^+} \text{gdm} V(\gamma') \cdot IC_{\gamma'}, \forall \gamma \in X.$$ 

The elements $IC_{\gamma'}, \gamma' \in X^+$, form a $A$-basis of $\mathcal{K}$. Thus, the linear map

$$\psi : GA \rightarrow A_X, \quad b_{\gamma'} \mapsto \sum_{\gamma \in X} \text{gdm} V(\gamma') \cdot e^\gamma, \forall \gamma' \in X^+,$$ 

is an injective ring homomorphism. Consider the linear map $\phi : GR \rightarrow GA$ such that $V(\gamma) \mapsto b_\gamma$ for all $\gamma \in X^+$. We get the commutative square of linear maps

$$\begin{array}{ccc}
GR & \xrightarrow{\phi} & GA \\
\downarrow & & \downarrow \psi \\
K(C_q) & \xrightarrow{\text{gch}} & A_X
\end{array}$$

where $\psi, gch$ are ring homomorphisms, see Proposition 2.4.(a), the vertical maps are injective, and $\phi$ is invertible. Thus, $GR$ is a subring of $K(C_q)$ and $\phi$ is a ring homomorphism. If $\gamma' + \gamma'' = \gamma$ in $X^+$, then

$$(\mathcal{D} \otimes \mathcal{D}) \circ \text{res}_{\gamma', \gamma''} \circ \mathcal{D} = \text{res}_{\gamma', \gamma''}.$$ 

Thus $\theta$ is an antihomomorphism. □

If $b_\gamma \otimes b_{\gamma'} = v^{(\gamma, \gamma')} b_{\gamma + \gamma'}$, then the $U$-module $V(\gamma) \otimes V(\gamma')$ is simple and isomorphic to $V(\gamma + \gamma')$. Conversely, if $V(\gamma) \otimes V(\gamma')$ is a simple $U$-module it is isomorphic to $V(\gamma + \gamma')$. Then, the positivity in Theorem 4.3 implies that $b_\gamma \otimes b_{\gamma'} \in v^\mathbb{Z} b_{\gamma + \gamma'}$. Then, by (2.3.1) we get $b_\gamma \otimes b_{\gamma'} = v^{(\gamma', \gamma')} b_{\gamma + \gamma'}$. The following conjecture generalizes to all simply laced types the conjecture in [BZ] (for type $A$).

**Conjecture.** The following statements are equivalent:

$b_\gamma \otimes b_{\gamma'} \in v^\mathbb{Z} B$, $b_\gamma \otimes b_{\gamma'} \in v^\mathbb{Z} b_{\gamma'} \otimes b_\gamma$, and $b_\gamma \otimes b_{\gamma'} = v^{(\gamma', \gamma')} b_{\gamma + \gamma'}$.

5. **The classical case**

5.1. Fix $\lambda, \lambda' \in P^+$. Let $\mathcal{D}(N_{\lambda})^\circ, \mathcal{D}(N_{\lambda} \times N_{\lambda'})^\circ$ be the full subcategories of $\mathcal{D}(N_{\lambda}), \mathcal{D}(N_{\lambda} \times N_{\lambda'})$ consisting of all complexes which are constructible with respect to the stratification in §3.2. Set $IC_{\lambda \alpha} = IC(N_{\lambda \alpha}^\circ), C_{\lambda \alpha} = C_{N_{\lambda \alpha}^\circ}[d_{\lambda \alpha}]$ if $\alpha \in \Lambda^+(\lambda)$, and set $L_{\lambda \alpha} = \pi_{\lambda \alpha} C_{Q_{\lambda \alpha}}[d_{\lambda \alpha}]$ if $\alpha \in \Lambda(\lambda)$. Let $\mathcal{P}_{\lambda}, \mathcal{P}_{\lambda \lambda'}$ be the full subcategories of
\[ \mathcal{D}(N_{\lambda})^\circ, \mathcal{D}(N_{\lambda} \times N_{\lambda'})^\circ \] consisting of all complexes which are isomorphic to finite direct sums of complexes of the form \( IC_{\lambda \alpha}, IC_{\lambda \alpha} \otimes IC_{\lambda' \alpha'} \).

Assume that \( \lambda = \lambda' + \lambda'' \). Setting \( k = 0, \gamma = \text{Id}_W \) in \( \S 3.5 \) we get the commutative diagram
\[
\begin{array}{ccc}
Q_{\lambda} \overset{\iota_\pm}{\leftarrow} Q_{\lambda}^{\pm \tau} \overset{\tilde{\kappa}_\pm}{\rightarrow} Q_{\lambda}^\pm & \cong & Q_{\lambda'} \times Q_{\lambda''} \\
\downarrow \quad \square \quad \downarrow \quad \downarrow \quad \downarrow \\
N_{\lambda} \overset{\iota_\pm}{\leftarrow} N_{\lambda}^{\pm \tau} \overset{\kappa_\pm}{\rightarrow} N_{\lambda}^\pm & \phi & N_{\lambda'} \times N_{\lambda''}.
\end{array}
\]

The restriction of the map \( \tilde{\kappa}_\pm \) to \( \tilde{\kappa}_\pm^{-1}(Q_{\lambda' \alpha'} \times Q_{\lambda'' \alpha''}) \) is a vector bundle of rank
\[
(1) \quad (d_{\lambda\alpha} - d_{\lambda' \alpha'} - d_{\lambda'' \alpha''})/2,
\]
where \( \alpha = \alpha' + \alpha'' \). Indeed, let \( T_{\lambda \tau} \) be the normal bundle to \( Q_{\lambda}^\pm \) in \( Q_{\lambda} \), and let \( T_{\lambda \tau}^\pm \) be the restriction to \( Q_{\lambda}^\pm \) of the relative tangent bundle to the map \( \tilde{\kappa}_\pm \). The cocharacter \( \tau \) acts on \( T_{\lambda \tau} \) with non zero weights, and \( T_{\lambda \tau}^\pm \) is the subbundle consisting of the positive (resp. negative) weights subspaces. Recall that \( Q_{\lambda} \) has a \( G_{\lambda} \)-invariant holomorphic symplectic form, see \([N1, (3.3)] \). Thus, the subvariety \( Q_{\lambda}^\pm \) is symplectic, and the rank of \( T_{\lambda \tau}^\pm \) is twice the rank of \( T_{\lambda \tau} \).

Consider the functor
\[
\text{res}_{\lambda' \lambda''}^\pm = \kappa_\pm/\iota_\pm^* : \mathcal{D}(N_{\lambda})^\circ \rightarrow \mathcal{D}(N_{\lambda'}^\circ)^\circ.
\]

For any \( \mu \in P^+ \) we set
\[
V(\lambda', \lambda'')_{\mu} = \text{Hom}_G(V(\mu), V(\lambda') \otimes V(\lambda'')).
\]

**Lemma 1.** For any \( \alpha \in \bigwedge (\lambda) \) we have
1. \( \text{res}_{\lambda' \lambda''}^+(L_{\lambda \alpha}) = \text{res}_{\lambda' \lambda''}^+(L_{\lambda \alpha}) = \bigoplus_{\alpha' + \alpha'' = \alpha} \phi_!(L_{\lambda' \alpha'} \otimes L_{\lambda'' \alpha''}) \),
2. \( \text{res}_{\lambda' \lambda''}^+(IC_{\lambda \alpha}) \cong \bigoplus_{\alpha', \alpha''} V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda - \alpha} \otimes \phi_!(IC_{\lambda' \alpha'} \otimes IC_{\lambda'' \alpha''}) \),
3. \( \text{res}_{\lambda' \lambda''}^+ \) commutes to the Verdier duality,
4. For any complex \( P \in \text{Ob}(\mathcal{P}_\lambda) \) there is a complex \( P' \in \text{Ob}(\mathcal{P}_{\lambda' \lambda''}) \) such that \( \text{res}_{\lambda' \lambda''}^+(P) \cong \phi_!(P') \).

**Proof:** Claim (a) is proved as Lemma 4.1, using (5.1.1). Using \([N2, \text{Theorem 15.3.2}] \) we get an isomorphism
\[
(2) \quad L_{\lambda \alpha} \cong \bigoplus_{\beta \in Q^+} H_{\text{top}}(F_{\lambda - \beta, \alpha - \beta}) \otimes IC_{\lambda \beta}.
\]

Using Part (a) and (5.1.2) we get
\[
\bigoplus_{\alpha \geq \beta} V(\lambda - \beta)_{\lambda - \alpha} \otimes \text{res}_{\lambda' \lambda''}^+(IC_{\lambda \beta}) \cong
\]
\[
\bigoplus_{\alpha \geq \beta} \bigoplus_{\beta', \beta''} V(\lambda - \beta)_{\lambda - \alpha} \otimes V(\lambda' - \beta', \lambda'' - \beta'')_{\lambda - \beta} \otimes \phi_!(IC_{\lambda' \beta'} \otimes IC_{\lambda'' \beta''}),
\]
where the sum is over all \( \beta', \beta'' \in Q^+ \). An induction on \( \beta \) gives
\[
\text{res}_{\lambda' \lambda''}^+(IC_{\lambda \beta}) \cong \bigoplus_{\beta', \beta''} V(\lambda' - \beta', \lambda'' - \beta'')_{\lambda - \beta} \otimes \phi_!(IC_{\lambda' \beta'} \otimes IC_{\lambda'' \beta''}).
\]

\( \square \)
By Lemma 3.5.1.(b), the functor $\phi_!$ is an equivalence from $\mathcal{P}_{\lambda',\lambda''}$ to a full subcategory of $\mathcal{D}(N^+_\lambda)^\vee$. Composing $\text{res}_{\lambda',\lambda''}$ with a quasi-inverse to $\phi_!$ we get a functor $\text{res}_{\lambda',\lambda''}: \mathcal{P}_\lambda \to \mathcal{P}_{\lambda',\lambda''}$. Let $\text{Vec}$ be the category of finite dimensional complex vector spaces, and let $\mathcal{P}_\lambda^\circ$ be the category dual to $\mathcal{P}_\lambda$. We consider the following functors

$\odot: \mathcal{P}_\lambda^\circ \times \mathcal{P}_\lambda^\circ \to \mathcal{P}_\lambda^\circ$, $(P', P'') \mapsto \bigoplus_\alpha \text{Hom}_{\mathcal{P}_{\lambda',\lambda''}}(\text{res}_{\lambda',\lambda''}(\text{IC}_{\lambda\alpha}), P' \otimes P'') \otimes \text{IC}_{\lambda\alpha}$,

$\Phi_\lambda: \mathcal{P}_\lambda^\circ \to \text{Vec}$, \quad $P \mapsto \text{Hom}_\mathcal{P}_\lambda(P, \bigoplus_\alpha L_{\lambda\alpha})$,

$p_{\lambda\beta}: \mathcal{P}_\lambda^\circ \to \mathcal{P}_{\lambda'-\beta}^\circ$, \quad $P \mapsto \bigoplus_{\alpha \geq \beta} \text{Hom}_\mathcal{P}_\lambda(\text{IC}_{\lambda\alpha}, P) \otimes \text{IC}_{\lambda'-\beta,\alpha-\beta}$,

where $\beta \in \bigwedge^+(\lambda)$. Note that [N2, Theorem 3.3.2] and §3.2 give

$$\alpha - \beta \in \bigwedge^+(\lambda - \beta) \iff \alpha \in \bigwedge^+(\lambda), \quad \alpha \geq \beta,$$

and similarly with $\bigwedge(\lambda)$. By (5.1.2) we have

$$p_{\lambda\beta}(L_{\lambda\alpha}) \simeq \begin{cases} L_{\lambda'-\beta,\alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{else.} \end{cases}$$

We define a new category $\mathcal{P}^\circ$ as follows. Objects of $\mathcal{P}^\circ$ are collections $P = (P_\lambda, \gamma_{\lambda\beta})$, where $\lambda \in P^+$, $\beta \in \bigwedge^+(\lambda) \setminus \{0\}$, $P_\lambda \in \text{Ob}(\mathcal{P}_\lambda)$ and $\gamma_{\lambda\beta} \in \text{Isom}_{\mathcal{P}_{\lambda'-\beta}}(P_{\lambda'-\beta}, p_{\lambda\beta}(P_\lambda))$

are isomorphisms satisfying the obvious chain condition. Morphisms $P' \to P''$ are collections $(\phi_\lambda) \in \prod_\lambda \text{Hom}_\mathcal{P}_\lambda(P''_\lambda, P'_\lambda)$ such that

$$\gamma_{\lambda\beta}' \circ \phi_{\lambda'-\beta} = p_{\lambda\beta}(\phi_\lambda) \circ \gamma_{\lambda\beta}'' \in \text{Hom}_{\mathcal{P}_{\lambda'-\beta}}(P''_{\lambda'-\beta}, p_{\lambda\beta}(P'_\lambda)).$$

Lemma 2. Fix $\beta \in \bigwedge^+(\lambda)$, $\beta' \in \bigwedge^+(\lambda')$, $\beta'' \in \bigwedge^+(\lambda'')$ such that $\beta = \beta' + \beta''$. For any $P, P', P'' \in \text{Ob}(\mathcal{P}^\circ)$ we have natural embeddings

$$\Phi_{\lambda'-\beta}(P_{\lambda'-\beta}) \subset \Phi_\lambda(P_\lambda), \quad P'_{\lambda'-\beta} \otimes P''_{\lambda'-\beta} \subset p_{\lambda\beta}(P'_\lambda \otimes P''_\lambda).$$

Moreover we have $\sum_{\beta', \beta''} P'_{\lambda'-\beta'} \otimes P''_{\lambda'-\beta''} = p_{\lambda\beta}(P'_\lambda \otimes P''_\lambda)$.

Proof: Fix an isomorphism as in (5.1.2) for each $\alpha \in \bigwedge(\lambda)$. For any such $\alpha$ we get a morphism of functors

$$\bigoplus_\alpha \text{Hom}(\cdot, \text{IC}_{\alpha\lambda\alpha'}) \otimes \text{Hom}(\text{IC}_{\lambda'-\beta,\alpha'-\beta}, L_{\lambda'-\beta,\alpha'-\beta}) \to \text{Hom}(\cdot, L_{\lambda\alpha}).$$

By definition of $\Phi_\lambda$, $p_{\lambda\beta}$ this morphism gives a morphism of functors $\Phi_{\lambda'-\beta} \circ p_{\lambda\beta} \to \Phi_\lambda$. The morphism $\Phi_{\lambda'-\beta}(P_{\lambda'-\beta}) \to \Phi_\lambda(P_\lambda)$ is the composition of the isomorphism $\Phi_{\lambda'-\beta}(\gamma_{\lambda\beta})$ and the morphism $\Phi_{\lambda'-\beta} \circ p_{\lambda\beta} \to \Phi_\lambda$ above. Using (5.1.2) we get

$$\Phi_\lambda(\text{IC}_{\lambda\alpha}) \simeq V(\lambda - \alpha), \quad \Phi_{\lambda'-\beta} \circ p_{\lambda\beta}(\text{IC}_{\lambda\alpha}) \simeq \begin{cases} V(\lambda - \alpha) & \text{if } \alpha \geq \beta, \\ 0 & \text{else.} \end{cases}$$
This proves Claim one. For any \( \alpha' \in \bigwedge^+(\lambda') \), \( \alpha'' \in \bigwedge^+(\lambda'') \) Lemma 5.1.1 gives an isomorphism of complexes

\[
\text{(3)} \quad IC_{\lambda', \alpha'} \odot IC_{\lambda'', \alpha''} \simeq \bigoplus_{\alpha} V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda - \alpha} \otimes IC_{\lambda, \alpha},
\]

where the sum is over all \( \alpha \in \bigwedge^+(\lambda) \) such that \( V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda - \alpha} \neq \{0\} \). Fix such a family of isomorphisms. It gives a morphism of functors \( p_{\lambda', \beta'}(-) \odot p_{\lambda'', \beta''}(-) \to p_{\lambda}\beta(- \odot -) \). The morphism \( P'_{\lambda', \beta'} \odot P''_{\lambda'', \beta''} \to p_{\lambda}\beta(P'_{\lambda'} \odot P''_{\lambda''}) \) is the composition of the isomorphism \( \gamma_{\lambda', \beta'} \odot \gamma_{\lambda'', \beta''} \), and the morphism of functors \( p_{\lambda, \beta'}(-) \odot p_{\lambda', \beta''}(-) \to p_{\lambda\beta(- \odot -)} \) above. Then, Claim two and three are consequences of the following identities. If \( V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda - \alpha} \neq \{0\} \), then \( \alpha \geq \alpha' + \alpha'' \), and thus

\[
\alpha \geq \beta \iff \alpha' \geq \beta', \alpha'' \geq \beta'',
\]

\[
\alpha \geq \beta \implies \exists \beta', \beta'' \text{ s.t. } \alpha' \geq \beta', \alpha'' \geq \beta'', \beta = \beta' + \beta''.
\]

We are done. \( \square \)

By Lemma 5.1.2 the category \( \mathcal{P}^\circ \) is endowed with the functors \( \Phi : \mathcal{P}^\circ \to \text{Vec} \), \( \odot : \mathcal{P}^\circ \times \mathcal{P}^\circ \to \mathcal{P}^\circ \) such that

\[
\Phi(P) = \lim_{\longrightarrow \lambda} \Phi_\lambda(P_\lambda), \quad (P' \odot P'')_\lambda = \sum_{\lambda = \lambda' + \lambda''} P'_{\lambda'} \odot P''_{\lambda''}.
\]

Then, (5.1.3) gives the following.

**Lemma 3.** \( (\mathcal{P}^\circ, \odot) \) is a tensor category, and \( \Phi \) is a tensor functor.

Let \( A \) be the Grothendieck group of \( \mathcal{P}^\circ \). The functor \( \odot \) gives a product \( A \otimes A \to A \). Let \( b_\lambda, c_\lambda \) be the classes in \( A \) of the objects of \( \mathcal{P}^\circ \) associated to the families \( (IC_{\lambda + \beta}, L_{\lambda + \beta}) \), \( (\lambda + \beta, \beta) \). Then \( (b_\lambda), (c_\lambda) \) are bases of \( A \). Let \( (R(\mathfrak{g}), \odot) \) be the tensor category of finite dimensional \( \mathfrak{g} \)-modules. We have proved the following theorem.

**Theorem.** The tensor categories \( (\mathcal{P}^\circ, \odot) \), \( (R(\mathfrak{g}), \odot) \) are equivalent. The group homomorphism such that \( b_\lambda \mapsto V(\lambda) \) is a ring isomorphism \( A \xrightarrow{\sim} R(\mathfrak{g}) \). Moreover, we have \( b_\lambda = \sum_\mu \dim V(\lambda)_{\mu} \cdot c_\mu \).

**5.2.** In this subsection we consider the non simply laced case. Our construction is based on [L1, §11]. Assume that \( \mathfrak{g} \) is a non simply laced, simple, complex Lie algebra. Fix a simply laced simple Lie algebra \( \mathfrak{g} \) and a diagram automorphism \( a \) of \( \mathfrak{g} \) such that the Dynkin graph of \( \mathfrak{g} \) is deduced from the Dynkin graph of \( \mathfrak{g} \) as in [L1, §14]. Let \( n \) be the order of the automorphism \( a \) \((n = 2 \text{ for types } B_k, C_k, F_4, \text{ and } n = 3 \text{ for type } G_2)\). The automorphism \( a \) is identified with a permutation of the set \( I \times H \), see §3.1, such that

\[
a(h') = a(h'), \quad a(h'') = a(h''), \quad a(h) = a(h).
\]

Let \( \langle a \rangle \) be the cyclic group of automorphisms of \( (I, H) \) generated by \( a \). Let \( I \) be the set of \( \langle a \rangle \)-orbits in \( I \), and let \( P^+_a = (P^+)^a, Q^+_a = (Q^+)^a \) be the corresponding sub-semigroups of \( P^+, Q^+ \). The simple root \( \alpha_i \) and the fundamental weight \( \omega_i \) of \( \mathfrak{g} \) are identified with the sums \( \sum_{i \in I} \alpha_i \in Q^+, \sum_{i \in I} \omega_i \in P^+ \). For any \( \lambda \in P^+, \alpha \in \mathbb{N} \),
$Q^+$, the diagram automorphism induces natural isomorphisms $Q_{\lambda a} \sim Q_{a(\lambda),a(\alpha)}$, $N_{\lambda a} \sim N_{a(\lambda),a(\alpha)}$. Let denote them by $a$ again.

To avoid confusions, finite dimensional representations of $g$, $\mathfrak{g}$ are denoted by $V(\lambda)$, $\mathcal{V}(\lambda)$ respectively. The subsets of $Q^+$, $\overline{Q}^+$ defined in §2.1 are denoted by $\Lambda(\lambda)$, $\Lambda^+(\lambda)$ and $\Delta(\lambda)$, $\Delta^+(\lambda)$ respectively.

Fix $\lambda, \lambda' \in P^+$ and $\alpha \in Q^+$. Following [L1, §11] we consider new categories $a^{P\lambda}$, $a^{P\lambda'}$. An object of $a^{P\lambda}$ is a pair $(P,\theta)$, where $P \in \text{Ob}(P\lambda)$ and $\theta : a^*P \sim P$ is an isomorphism such that the composition

$$a^*P \rightarrow \cdots \rightarrow a^{n*}P \xrightarrow{a^*\theta} a^*P \xrightarrow{\theta} P$$

is the identity. A morphism $(P,\theta) \rightarrow (P',\theta')$ is a morphism $f : P \rightarrow P'$ such that $f\theta = \theta'(a^*f)$. The category $a^{P\lambda'}$ is constructed in the same way. Both categories are Abelian. For any functor $F : P\lambda \rightarrow P\lambda'$ and for any isomorphism of functor $a^*F \sim Fa^*$ there is the functor $a^*F : a^{P\lambda} \rightarrow a^{P\lambda'}$ such that $a^*F(P,\theta) = (F(P),\theta^F)$ where $\theta^F$ is the composition of the chain of maps

$$a^*F(P) \rightarrow F(a^*P) \xrightarrow{F(\theta)} F(P).$$

The functor $a^*$ on $P\lambda$ has the order $n$, where $n = 2$ or 3. Let $a^I\lambda$ be the full subcategory of $a^{P\lambda}$ whose objects are the pairs $(P,\theta)$ such that $P \sim P' \oplus a^*P' \oplus \cdots \oplus (a^*)^{n-1}P'$ for some $P' \in P\lambda$, and $\theta$ is an isomorphism carrying the direct summand $(a^*)^jP' \subset a^*P$ onto the direct summand $(a^*)^jP' \subset P$. The objects of $a^I\lambda$ are said to be traceless.

The automorphism $a$ preserves the stratification of $N_{\lambda}$. Since $IC_{\lambda a}$ is canonically attached to $N_{\lambda a}^\circ$, there is a canonical isomorphism $a^*IC_{\lambda,a(\alpha)} \sim IC_{\lambda a}$. If $\alpha \in Q^+$ the corresponding object in $a^{P\lambda}$ is denoted by $a^*IC_{\lambda a}$. Let $\mu_n \subset \mathbb{C}^\times$ be the set of $n$-th roots of unity. For any $\zeta \in \mu_n$ and any $Q = (P,\theta) \in \text{Ob}(a^{P\lambda})$ we put $Q(\zeta) = (P,\zeta \theta)$. If $\alpha \notin Q^+$ and $\zeta_1,\ldots,\zeta_n \in \mu_n$, let $a^*IC_{\lambda a}(\zeta_1,\ldots,\zeta_n)$ be the object of $a^{P\lambda}$ associated to the perverse sheaf

$$P = IC_{\lambda a} \oplus IC_{\lambda,a(\alpha)} \oplus \cdots IC_{\lambda,a^{n-1}(\alpha)}$$

and the isomorphism $a^*P \sim P$ which maps the summand $a^*IC_{\lambda,a^j(\alpha)}$ onto the summand $IC_{\lambda,a^{j-1}(\alpha)}$ by $\zeta_{j+1}$ times the canonical isomorphism. A simple object in $a^{P\lambda}$ is isomorphic either to $a^*IC_{\lambda a}(\zeta)$ for some $\alpha \in Q^+$ and $\zeta \in \mu_n$, or to $a^*IC_{\lambda a}(\zeta_1,\ldots,\zeta_n)$ for some $\alpha \in Q^+ \setminus Q^+$ and $\zeta_1,\ldots,\zeta_n \in \mu_n$. Let $1^{P\lambda}$ be the full subcategory of $a^{P\lambda}$ whose objects are isomorphic to finite direct sums of the objects $a^*IC_{\lambda a}$.

The image by the functor $\pi_{\lambda a!}$ of the obvious isomorphism $a^*C_{Q\lambda,a(\alpha)} \sim C_{Q\lambda a}$ is an isomorphism $a^*L_{\lambda,a(\alpha)} \sim L_{\lambda a}$. If $\alpha \in Q^+$ the corresponding object in $a^{P\lambda}$ is denoted by $a^*L_{\lambda a}$. Assume that $\beta \in Q^+$ is such that $\alpha \geq \beta$ and $N_{\lambda\beta}^\circ \neq \emptyset$. Fix an element $x_\beta \in N_{\lambda\beta}^\circ$. One proves as in [N2, Theorem 3.3.2] that there are $\langle a \rangle$-invariant open sets

$$U_{\alpha} \subset \langle a \rangle(N_{\lambda a}), \quad U_{\beta}^\circ \subset \langle a \rangle(N_{\lambda\beta}^\circ), \quad U_{\alpha-\beta} \subset \langle a \rangle(N_{\lambda-\beta,a-\beta})$$
containing \( x_\beta, x_\beta, 0 \) respectively, and a commutative square

\[
\begin{array}{ccc}
U_\alpha & \xrightarrow{\sim} & U_\beta^\vee \times U_{\alpha-\beta} \\
\pi \uparrow & & \uparrow \text{id} \times \pi \\
\pi^{-1}(U_\alpha) & \xrightarrow{\sim} & U_\beta^\vee \times \pi^{-1}(U_{\alpha-\beta}),
\end{array}
\]

where \( \pi \) denotes either \( \pi_{\lambda \alpha} \) or \( \pi_{\lambda-\beta, \alpha-\beta} \). The horizontal maps are analytic \((a)\)-equivariant isomorphisms carrying the element \( x_\beta \in U_\alpha \) to \( (x_\beta, 0) \in U_\beta^\vee \times U_{\alpha-\beta} \). By \((5.1.2)\) we have

\[
L_{\lambda \alpha} \simeq \bigoplus_{\beta \in Q^+} H_{\text{top}}(F_{\lambda-\beta, \alpha-\beta}) \otimes IC_{\lambda \beta},
\]

and the isomorphism \( a^*L_{\lambda,a(\alpha)} \xrightarrow{\sim} L_{\lambda \alpha} \) maps the direct summand

\[
H_{\text{top}}(a(F_{\lambda-\beta, \alpha-\beta})) \otimes a^*IC_{\lambda,a(\beta)} \quad \text{onto} \quad H_{\text{top}}(F_{\lambda-\beta, \alpha-\beta}) \otimes IC_{\lambda \beta}
\]

in the obvious way. By \([X, \text{Theorem 3.2.1}]\), if \( \alpha, \beta \in Q^+ \) the number of irreducible components of \( F_{\lambda-\beta, \alpha-\beta} \) which are mapped to themselves by \( a \) is the multiplicity \( \dim V(\lambda-\beta)_{\lambda-\alpha} \). Thus \( aL_{\lambda \alpha} = 1L_{\lambda \alpha} \oplus I_{\lambda \alpha} \) where

\[
(1) \quad 1L_{\lambda \alpha} \simeq \bigoplus_{\beta \in Q^+} V(\lambda-\beta)_{\lambda-\alpha} \otimes aIC_{\lambda \beta} \in \text{Ob}(^{1}\mathcal{P}_{\lambda}), \quad I_{\lambda \alpha} \in \text{Ob}(a\mathcal{I}_{\lambda}).
\]

Assume that \( \lambda = \lambda' + \lambda'' \in P^+ \). The maps \( \iota_\pm, \kappa_\pm, \phi \) commute to the automorphism \( a \) of \( N_\lambda \). Thus, there is a natural isomorphism \( a^* \text{res}_{\lambda , \lambda'} \sim \text{res}_{\lambda , \lambda''} a^* \). We get the functor \( a^*\text{res}_{\lambda , \lambda'} : a\mathcal{P}_{\lambda} \rightarrow a\mathcal{P}_{\lambda , \lambda''} \). Lemma 5.1.1 implies the following.

**Lemma.** For any \( \alpha \in Q^+ \) there are traceless objects \( I, I' \) such that

(a) \( a^*\text{res}_{\lambda , \lambda''}(aL_{\lambda \alpha}) = I + \bigoplus_{\alpha=\alpha'+\alpha''} aL_{\lambda , \lambda'} \otimes aL_{\lambda', \alpha''}, \)

(b) \( a^*\text{res}_{\lambda , \lambda''}(aIC_{\lambda \alpha}) = I' + \bigoplus_{\alpha=\alpha'+\alpha''} V(\lambda-\alpha', \lambda''-\alpha'')_{\lambda-\alpha} \otimes (aIC_{\lambda', \alpha'} \otimes aIC_{\lambda'', \alpha''}). \)

For any \( \beta \in \bigwedge^+ (\lambda) \cap Q^+ \), there is an obvious isomorphism of functors \( a^*p_{\lambda \beta} \sim p_{\lambda \beta} a^* \).

The corresponding functor \( a^*p_{\lambda \beta} : a\mathcal{P}_{\lambda} \rightarrow a\mathcal{P}_{\lambda-\beta} \) is exact and satisfies

\[
a^*p_{\lambda \beta}(aIC_{\lambda \alpha}) = \begin{cases} aIC_{\lambda-\beta, \alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{else.} \end{cases}
\]

We have also, see \((5.2.1)\),

\[
a^*p_{\lambda \beta}(aL_{\lambda \alpha}) \simeq \begin{cases} aL_{\lambda-\beta, \alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{else.} \end{cases}
\]

Let \( K(a\mathcal{P}_{\lambda}) \) be the Grothendieck group of \( a\mathcal{P}_{\lambda} \). The class of an object \( P \) is still denoted by \( P \). Let \( k \subseteq \mathbb{C} \) be subring generated by \( \mu_n \). Let \( K'_{\lambda} \) be the quotient of \( K(a\mathcal{P}_{\lambda}) \otimes k \) by the relations:

- \( P(\zeta) = P \otimes \zeta \) for any \( \zeta \in \mu_n \),
- the class of a traceless object is zero.

Let \( K_{\lambda} \subseteq K'_{\lambda} \) be the subgroup spanned by the classes of objects in \( 1\mathcal{P}_{\lambda} \), and let \( A_{\lambda} = K'_{\lambda} \) be the dual group. Using the maps \( a^*p_{\lambda \beta} \) we construct, as in \( \S 4.3 \), an inductive system of groups \( (A_{\lambda}) \). The limit, denoted by \( A_{\lambda} \), is endowed with a product \( A \otimes A \rightarrow A \), and two distinguished bases \( (b_{\lambda}), (c_{\lambda}) \) associated to the families \( (aIC_{\lambda+\beta, \beta}), (aL_{\lambda+\beta, \beta}) \).
The group homomorphism such that $b_\lambda \mapsto V(\lambda)$ is a ring isomorphism $
abla \rightarrow R(g)$. Moreover, we have $b_\lambda = \sum_\mu \dim V(\lambda)_\mu \cdot c_\mu$.

5.3. In this subsection we explain how a similar construction gives a natural restriction map $GR \rightarrow R(g) \otimes A$. Consider the diagram

$$
Q_\lambda \xleftarrow{\iota}\ Q_\gamma = \nabla \rightarrow Q_\gamma \\
\downarrow \quad \downarrow \quad \downarrow \\
N_\lambda \xleftarrow{\iota}\ N_\gamma = \nabla \rightarrow N_\gamma.
$$

Set $\varepsilon_\gamma = \varepsilon_\gamma\gamma$. Let $\kappa_{\eta}^\pm$ be the relative dimension of $\tilde{\kappa}_\pm$ above the component $Q_{\gamma\eta}$. The same computations as in Lemma 3.5.2 or in (5.1.1) give

$$
\kappa^-_\eta = d_{\lambda\alpha}/2 - d_{\gamma\eta}, \quad \kappa^+_\eta = d_{\lambda\alpha}/2,
$$

$$
\varepsilon_\gamma - \varepsilon_\gamma - \eta = d_\gamma\delta, \quad d_\gamma\eta - d_{\gamma-\eta\delta} = d_\gamma\delta,
$$

for any $\delta \in \Lambda^+(\gamma), \eta \geq \delta$. Consider the functor

$$
\text{res}_\gamma^\pm = \kappa_\pm!\iota^* : D(N\lambda)^\vee \rightarrow D(N\gamma)^\vee.
$$

By base change we get, for any $\eta \in \Lambda^+(\gamma)$,

$$
\text{res}_\gamma^\pm L_{\lambda\alpha} = \pi_\gamma!\tilde{\kappa}_\pm!\iota^*\mathbb{C}_{Q_{\lambda\alpha}}[d_{\lambda\alpha}] = \bigoplus \eta \pi_\gamma!\mathbb{C}_{Q_{\gamma\eta}}[d_{\lambda\alpha} - 2\kappa^-_\eta] = \bigoplus \eta L_{\gamma\eta}[\pm d_{\gamma\eta}].
$$

Lemma. (a) For any complex $P \in \text{Ob}(P_{\lambda})$ the complex $\text{res}_\gamma^\pm(P)$ belongs to $\text{Ob}(P_{\gamma})$.

(b) We have $\mathcal{D} \circ \text{res}_\gamma^\pm = \text{res}_\gamma^- \circ \mathcal{D}$.

The corresponding group homomorphism $v^\varepsilon\text{res}_\gamma^+ : K(P_{\lambda}) \rightarrow K_{\gamma}$ is compatible with the projective systems in §5.1, §4.2. Let $\text{res} : GA \rightarrow A \otimes A$ be the inductive limit of the system of maps dual to $v^\varepsilon\text{res}_\gamma^+$.

Proposition. The element $\text{res}(b_\gamma)$ belongs to $\bigoplus \lambda \mathbb{N}[v^{-1}, v] \cdot b_\lambda$ for all $\gamma \in X^+$. If $\gamma \in G^\vee_{\lambda\alpha}$ then $\text{res}(c_\gamma) = v^\varepsilon\gamma \cdot c_\lambda$.

References

[BZ] Berenstein, A., Zelevinsky, A., String bases for quantum groups of type $A_r$, Advances in Soviet Math. 16 (1993), 51-89.

[CG] Chriss, N., Ginzburg, V., Representation theory and complex geometry, Birkhäuser, Boston-Basel-Berlin, 1997.

[D] Damiani, I., La R-matrice pour les algèbres quantiques de type affine non tordu, Ann. Sci. École Norm. Sup. (4) 31 (1998), 493-523.

[FM] Frenkel, E., Mukhin, E., Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras, math.QA/9911112.

[FR] Frenkel, E., Reshetikhin, N., The q-characters of representations of quantum affine algebras and deformations of W-algebras, Contemporary Math. 248 (2000), 163-205.

[GV] Ginzburg, V., Vasserot, E., Langlands reciprocity fo affine quantum groups of type $A_n$, Internat. Math. Res. Notices 3 (1993), 67-85.

[Ka] Kashiwara, M., On level zero representations of quantized affine algebras, math.QA/0010293.

[Kn] Knight, H., Spectra of tensor products of finite-dimensional representations of Yangians, J. Algebra 174 (1994), 187-196.
[L1] Lusztig, G., Introduction to quantum groups, Birkhäuser, Boston-Basel-Berlin, 1994.
[L2] Lusztig, G., On quiver varieties, Adv. in Math. 136 (1998), 141-182.
[LNT] Leclerc, B., Nazarov, M., Thibon, J.-Y., Induced representations of affine Hecke algebras and canonical bases of quantum groups, math.QA/0011074.
[M] Malkin, A., Tensor product varieties and crystals. ADE case, math.AG/0103025.
[N1] Nakajima, H., Quiver varieties and Kac-Moody algebras, Duke Math. J. 91 (1998), 515-560.
[N2] Nakajima, H., Quiver varieties and finite dimensional representations of quantum affine algebras, Jour. A. M. S. 14 (2001), 145-238.
[N3] Nakajima, H., $t$-analogue of the $q$-character of finite dimensional representations of quantum affine algebras, math.QA/0009231.
[Vr] Varagnolo, M., Quiver varieties and Yangians, Letters in Math. Phys. 53 (2000), 273-283.
[Va] Vasserot, E., Affine quantum groups and equivariant K-theory, Transformation groups 3 (1998), 269-299.
[VV] Varagnolo, M., Vasserot, E., Standard modules of quantum affine algebras, math.QA/0006084.
[X] Xu, F., A note on quivers with symmetries, math.QA/9707003.

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