ON THE SEMISIMPLICITY OF THE CATEGORY $KL_k$ FOR AFFINE LIE SUPERALGEBRAS

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Abstract. We study the semisimplicity of the category $KL_k$ for affine Lie superalgebras and provide a super analog of certain results from [8]. Let $KL_k^{fin}$ be the subcategory of $KL_k$ consisting of ordinary modules on which a Cartan subalgebra acts semisimply. We prove that $KL_k^{fin}$ is semisimple when 1) $k$ is a collapsing level, 2) $W_k(g, \theta)$ is rational, 3) $W_k(g, \theta)$ is semisimple in a certain category. The analysis of the semisimplicity of $KL_k$ is subtler than in the Lie algebra case, since in super case $KL_k$ can contain indecomposable modules. We are able to prove that in many cases when $KL_k^{fin}$ is semisimple we indeed have $KL_k^{fin} = KL_k$, which therefore excludes indecomposable and logarithmic modules in $KL_k$. In these cases we are able to prove that there is a conformal embedding $W \hookrightarrow V_k(g)$ with $W$ semisimple (see Section 10). In particular, we prove the semisimplicity of $KL_k$ for $g = sl(2|1)$ and $k = -\frac{m+1}{m+2}$, $m \in \mathbb{Z}_{>0}$. For $g = sl(m|1)$, we prove that $KL_k$ is semisimple for $k = -1$, but for $k$ a positive integer we show that it is not semisimple by constructing indecomposable highest weight modules in $KL_k^{fin}$.

1. Introduction

The aim of this paper is to extend results from [8] to the super case. In [8] we proved semisimplicity of the category of ordinary modules $KL_k$ for a simple affine vertex algebra $V_k(g)$ when $g$ is a Lie algebra and one of the following assumptions holds:

(L-1) $k$ is collapsing level, i.e., $W_k(g, \theta)$ collapses to its affine subalgebra;
(L-2) $W_k(g, \theta)$ is rational;
(L-3) $W_k(g, \theta)$ is semisimple in a category of ordinary modules.

Here $W_k(g, \theta)$ is the simple affine $W$-algebra attached to $g$ and a minimal nilpotent element. It is natural to look for an extension of these results when $g$ is a Lie superalgebra.

Our paper [8] has attracted a lot of interest in various direction. Here we mention two interesting applications/generalizations:

- T. Creutzig and J. Yang showed in [19] that at all levels investigated in [8] there is a braided tensor category structure. It is interesting that they also use our previous work on the decomposition of conformal embeddings to prove rigidity of tensor categories.
- T. Arakawa, J. van Ekeren and A. Moreau in [15] have constructed a large new family of collapsing levels which are admissible.

In order to apply the results from [8], we need to extend certain structural results on self-extension of irreducible modules in $KL_k$. A first problem is that the category $KL_k$ for Lie superalgebras is more complicated than in the case of Lie algebras. For instance, in the super case, $KL_k$ can have non-semisimple and logarithmic modules. A nice illustration...
for these phenomena in given by the Lie superalgebra \( \mathfrak{gl}(1|1) \) and its affine vertex algebra \( V_1(\mathfrak{gl}(1|1)) \), which admits highest weight modules whose top components are two dimensional indecomposable \( \mathfrak{gl}(1|1) \)-modules (cf. [12], [20]). One can also construct logarithmic \( \mathfrak{gl}(1|1) \)-modules in \( KL_k \).

1.1. **Semisimplicity of \( KL_k^{fin} \).** In order to extend directly our methods from [8], it seems that we have one natural choice. We can consider the smaller categories \( KL_k^{ss} \) (resp. \( KL_k^{fin} \)) consisting of modules from \( KL_k \) on which \( L_g(0) \) acts semisimply (resp. a Cartan subalgebra of the affinization \( \hat{g} \) of \( g \) acts semisimply), cf. Definition 2.1. We prove in Theorem 4.3 a super analog of [8, Theorem 5.5]:

- Let \( g \) be a basic Lie superalgebra. If every highest weight \( V_k(g) \)-module in \( KL_k^{fin} \) is irreducible, then the category \( KL_k^{fin} \) is semisimple.

With this modification we can prove the semisimplicity of \( KL_k^{fin} \) in the following cases

(S-2) \( W_k(g, \theta) \) is rational.
(S-3) \( W_k(g, \theta) \) is semisimple in the category of ordinary modules.

Regarding the super analogue of condition (L-1), i.e., \( k \) is a collapsing level, we introduce the notion of **collapsing chain** (cf. Definition 4.7) and in many cases we reduce to prove semisimplicity of \( KL_k^{fin} \) by looking at conditions (S-2) or (S-3) for explicitly determined subalgebras \( g_n \) of \( g \) (see Theorem 4.9). Collapsing levels for Lie superalgebras were classified in [8]. So our results immediately gives semisimplicity of \( KL_k^{fin} \) for these levels. A comprehensive list of all the cases covered by the above conditions is given in Corollary 4.12.

Let us mention some cases of rationality:

- \( g = osp(1|2) \). Then the minimal \( W \)-algebra \( W_k(g, \theta) \) is the \( N = 1 \) super-conformal algebra which is rational for certain \( k \).
- \( g = sl(2|1) \). Then for \( k = -m/2, m \in \mathbb{Z}_{>0} \), the algebra \( W_k(g, \theta) \) is isomorphic to \( N = 2 \) superconformal vertex algebra at central charge \( c = 3m/(m+2) = -3(2k+1) \), which is rational by [2].

Next we have interesting cases when \( W_k(g, \theta) \) is semisimple in a certain category.

- \( g = psl(2|2) \) and conformal level \( k = 1/2 \). Then \( W_k(g, \theta) \) is the \( N = 4 \) superconformal vertex algebra with central charge \( c = -9 \) [4], which is semisimple in the category of ordinary modules (cf. Theorem 9.2).
- \( g = D(2, 1, \alpha) \). Then for collapsing level \( k \) we can have that \( W_k(g, \theta) = V_{k'}(sl(2)) \). If \( k' \) is a positive integer, or admissible we conclude that \( V_k(g) \) is semisimple in \( KL_k \). In [7] we described a conformal embedding \( V_{k_1}(sl(2)) \otimes V_{k_2}(sl(2)) \hookrightarrow W_k(g, \theta) \). If both \( k_1, k_2 \) are admissible, we expect that then \( KL_k^{fin} \) is semisimple.

More results on the semisimplicity of \( KL_k \), regarding \( V_{n+1}(C(n+1)), V_{-1}(psl(m|m)), m \geq 3 \), are given in Theorems 6.3, 7.2 respectively.

1.2. **When \( KL_k = KL_k^{fin} \)?** We have already observed that the category \( KL_k \) is the most natural choice of category of \( V_k(g) \)-modules. The example of \( V_1(\mathfrak{gl}(1|1)) \) shows that in general \( KL_k \neq KL_k^{fin} \). On the other hand, we can prove equality in some cases. In Proposition 5.1 we give two sufficient conditions for this equality to hold.

- Assume that \( g_{\mathbb{R}} \) is a semisimple Lie algebra. Then \( KL_k^{fin} = KL_k^{ss} \).
- There is a conformal embedding of \( V_{k_1}(g_{\mathbb{R}}) \hookrightarrow V_k(g) \) such that the category \( KL_{k_1} \) for \( V_{k_1}(g_{\mathbb{R}}) \) is semisimple.
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These conformal embeddings were classified in [9].

We believe that when $KL^{fin}_k$ is semisimple, we should have $KL_k = KL^{fin}_k$. Under the assumption that $KL^{fin}_k$ is semisimple, we prove in Theorem 5.5 the following result:

- Assume that for any irreducible $V_k(g)$–module $M$ in $KL_k$ we have

$$\text{Ext}^1(M_{top}, M_{top}) = \{0\}$$

in the category of finite-dimensional $g$–modules. Then $KL_k$ is semisimple and $KL^{fin}_k = KL_k$.

We will prove that $KL_k = KL^{fin}_k$ in the following cases:

- $V_{-\frac{1}{2}}(C(n + 1))$: the proof (see Theorem 6.2) relies on results from [9] and fusion rules arguments.
- $V_{-1}(sl(m|1))$ (see Theorem 7.7).

Since $k = -1$ is a collapsing level, we get that $KL^{fin}_k$ is semisimple. Next we refine the classification of irreducible modules in $KL_k$ and prove that top components of irreducible modules in $KL_k$ are atypical $g$–modules. Then the result of [24] on extensions of finite-dimensional $sl(m|1)$–modules implies that there are no self-extensions among irreducible modules in $KL_k$, so the condition (1.1) is satisfied. Then Theorem 5.5 gives that the larger category $KL_k$ is semisimple.

- $V_{-(m+1)/(m+2)}(sl(2|1))$, $m \in \mathbb{Z}_{\geq 0}$. In this case we prove that the center of $g_0$ belongs to a regular vertex operator algebra $D_{m+1,2}$ from [3]: see Section 10. Then the condition (1.1) is also satisfied, which gives that in this case the category $KL_k$ is semisimple.

1.3. Examples when $KL_k$ is not semisimple. M. Gorelik and V. Serganova in [26, Section 5.6.4] constructed examples of indecomposable weak $V_1(sl(2|1))$–modules on which the Sugawara operator $L(0)$ does not act semisimply. Their construction uses the theory of Zhu’s algebras and a description of the maximal ideal in the universal affine vertex algebra $V^1(sl(2|1))$. In the present paper we use a different approach and apply free-field realisation. In Theorem 8.1 we construct a highest weight $V_1(sl(m|1))$–module $W = V_1(sl(m|1)).(a^+)^{-m} \otimes |m >$ which has a proper submodule isomorphic to $V_1(sl(m|1))$. The module $W$ belongs to $KL^{fin}_k$, and therefore $KL_k$ is not semisimple for $k = 1$. We extend this example by showing that $KL^{fin}_k$ is not semisimple for $k \in \mathbb{Z}_{>0}$. A complete analysis of indecomposable modules will appear in forthcoming papers.

Acknowledgements. We would like to thank to Maria Gorelik, Victor Kac, Ozren Perše, Thomas Creutzig and Veronika Pedić on useful discussions. We thank the referee for his/her careful reading of the paper and some very helpful hints.

D.A. is partially supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

2. Setup

Let $g = g_0 \oplus g_1$ be a basic Lie superalgebra, i.e. a simple Lie superalgebra such that $g_0$ is a reductive Lie algebra and there exists an even invariant supersymmetric bilinear form on it (see [28] for more details and the classification). Choose a Cartan subalgebra $h$ for $g_0$ and let $\Delta = \Delta_0 \cup \Delta_1$ be the set of roots. Fix a positive system $\Delta^+$ in $\Delta$ and choose an even root
θ maximal in $\Delta_0^+ = \Delta^+ \cap \Delta_0$. We may choose root vectors $e_\theta$ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in h, \quad [x, e_{\pm\theta}] = \pm e_{\pm\theta}. $$

Due to the minimality of $-\theta$, the eigenspace decomposition of $ad x$ defines a minimal $\frac{1}{2} \mathbb{Z}$-grading ([33, (5.1)]):

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, $$

where $\mathfrak{g}_{\pm 1} = C e_{\pm\theta}$. Furthermore, one has

$$\mathfrak{g}_0 = h^0 \oplus \mathbb{C} x, \quad h^0 = \{ a \in \mathfrak{g}_0 \mid (a|x) = 0 \},$$

Note that $\mathfrak{g}^0$ is the centralizer of the triple $\{ f_\theta, x, e_\theta \}$. We can choose $h^0 = \{ h \in h \mid (h|x) = 0 \}$, as a Cartan subalgebra of the Lie superalgebra $\mathfrak{g}^0$, so that $h = h^0 \oplus \mathbb{C} x$.

For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot | \cdot)$ on $\mathfrak{g}$ by the condition

$$(\theta | \theta) = 2.$$  

The dual Coxeter number $h^\vee$ of the pair $(\mathfrak{g}, \theta)$ is defined to be half the eigenvalue of the Casimir operator of $\mathfrak{g}$ corresponding to $(\cdot | \cdot)$, normalized by (2.3). Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$, i.e.

$$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C} K \oplus \mathbb{C} d,$$

where $d$ acts as $t \frac{d}{dt}$, $K$ is central and the bracket on $[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$ is defined by

$$[x_{(m)}, y_{(n)}] = [x, y]_{(m+n)} + m \delta_{m+n} (x|y) K,$$

where $x_{(m)} = t^m \otimes x$, $x \in \mathfrak{g}$. Set $\widehat{\mathfrak{h}} = h \oplus \mathbb{C} K \oplus \mathbb{C} d$. Write $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C} \Lambda_0 \oplus \mathbb{C} \delta$ where $\Lambda_0(K) = 1$, $\Lambda_0(h) = 0$, $\delta_0(d) = 0$, $\delta(K) = 0$, $\delta(h) = 0$, $\delta(d) = 1$.

By category $\mathcal{O}$ for $\widehat{\mathfrak{g}}$ we mean the set of the $\widehat{\mathfrak{g}}$-modules which are $\widehat{\mathfrak{h}}$-diagonalizable with finite dimensional weight spaces and a finite number of maximal weights. Let $\mathcal{O}^k$ be the subcategory of $\widehat{\mathfrak{g}}$-modules in $\mathcal{O}$ of level $k$ (i.e., $K$ acts as $k I_d$).

Let $\mathfrak{a}$ be a Lie superalgebra equipped with a nondegenerate invariant supersymmetric bilinear form $B$. The universal affine vertex algebra $V^B(\mathfrak{a})$ is the universal enveloping vertex algebra of the non–linear Lie conformal superalgebra $R = (\mathbb{C}[T] \otimes \mathfrak{a})$ with $\lambda$-bracket given by

$$[a_\lambda b] = [a, b] + \lambda B(a, b), \quad a, b \in \mathfrak{a}.$$

In the following, we shall say that a vertex algebra $V$ is an affine vertex algebra if it is a quotient of some $V^B(\mathfrak{a})$.

If $k \in \mathbb{C}$, we will write simply $V^k(\mathfrak{g})$ for $V^{k(\cdot | \cdot)}(\mathfrak{g})$. We will always assume that $k$ is non–critical, i.e. $k \neq -h^\vee$. With this assumption, it is known that $V^k(\mathfrak{g})$ has a unique simple quotient, denoted by $V_k(\mathfrak{g})$ (see [30, § 4.7 and Example 4.9b]).

The vertex algebras $V^k(\mathfrak{g}), V_k(\mathfrak{g})$ are VOAs with Virasoro vector $L_{\mathfrak{g}}(0)$ given by the Sugawara construction.

If $M$ is a restricted module of level $k$ for $\widehat{\mathfrak{g}}$ then it is a weak module for $V^k(\mathfrak{g})$; conversely, letting $d$ act on weak modules by $-L_0(0)$ yields restricted modules for $\widehat{\mathfrak{g}}$.

**Definition 2.1.** We denote by $KL^B(\mathfrak{g})$ the category of weak modules for $V^B(\mathfrak{g})$, which

1. are locally finite as $\mathfrak{g}$-modules;
2. admit a decomposition into generalized eigenspaces for $L_{\mathfrak{g}}(0)$ whose eigenvalues are bounded below.

We denote by

- $KL^B_{\text{fin}}(\mathfrak{g})$ the full subcategory of modules in $KL^B(\mathfrak{g})$ on which $\widehat{\mathfrak{h}}$ acts semisimply.
• $KL_{ss}^B(g)$ the full subcategory of modules in $KL^B(g)$ on which $L_0(0)$ acts semisimply.

If $B = k(\cdot, \cdot)$ we simply write $KL^B(g)$, $KL^B_{fin}(g)$, $KL^B_{ss}(g)$. We also denote by $KL_k(g)$, $KL^\fin_k(g)$, $KL^\ss_k(g)$ the full subcategories of $KL^k(g), KL^\fin_k(g), KL^\ss_k(g)$ consisting of the $V_k(g)$-modules. If $g$ is clear from the context we omit it in the notation.

Let $V$ be a conformal vertex algebra. Denote by $L$ its conformal vector (with $Y(L,z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$). We say that a $V$–module is ordinary if $L(0)$ acts semisimply with finite dimensional eigenspaces. In our settings, let $C_k$ be the category of ordinary modules.

**Remark 2.2.** Note that (2) is exactly the definition of a logarithmic module for a conformal vertex algebra. Condition (1) says that each generalized eigenspace for $V$ is a direct sum of finite-dimensional modules. In particular, we have $KL^\fin_k = KL^\ss_k$. But when $g$ is a Lie superalgebra, we really have modules in $KL_k$ and $KL^\ss_k$ with non–semisimple action of $g$. One such example is the vertex algebra $V_1(\mathfrak{gl}(1|1))$ and its modules considered in [12]. More examples are given in Section 3.

### 3. $W$-algebras and Collapsing Levels

Denote by $W^k(g, \theta)$ the affine $W$–algebra obtained from $V^k(g)$ by Hamiltonian reduction relative to the minimal nilpotent element $e_{-\theta}$. More precisely, let $M$ be a restricted $V^k(g)$-module. Consider the complex

\[ C^M = M \otimes F(A_{ch}) \otimes F(A_{ne}), \]

where $F(A_{ch}), F(A_{ne})$ are fermionic vertex algebras, defined in [33], attached to the following superspaces. Denote by $A_{ne}$ the vector superspace $g_{1/2}$ with the bilinear form

\[ \langle a, b \rangle_{ne} = (e_{-\theta}[a, b]). \]

Denote by $A$ (resp. $A^*$) the vector superspace $g_{1/2} \oplus g_1$ (resp. $(g_{1/2} \oplus g_1)^*$) with the reversed parity, let $A_{ch} = A \oplus A^*$ and define an even skew-supersymmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{ch}$ on $A_{ch}$ by

\[ \langle A, A \rangle_{ch} = 0 = \langle A^*, A^* \rangle_{ch}, \]

\[ \langle a, b^* \rangle_{ch} = -(-1)^{p(a)p(b)} \langle b^*, a \rangle_{ch} = b^*(a) \text{ for } a \in A, b^* \in A^*. \]

Here and further, $p(a)$ stands for the parity of an (homogeneous) element of a vector superspace. Choose a basis $\{u_\alpha\}_{\alpha \in S_j}$ of each $g_j$ in (2.1), and let $S = \coprod_{j \in \frac{1}{2} \mathbb{Z}} S_j; S_+ = \coprod_{j > 0} S_j$.

Let $p(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ denote the parity of $u_\alpha$, and let $m_\alpha = j$ if $\alpha \in S_j$. Define the structure constants $c^{\gamma}_{\alpha \beta}$ by $[u_\alpha, u_\beta] = \sum_\gamma c^{\gamma}_{\alpha \beta} u_\gamma$ $(\alpha, \beta, \gamma \in S)$. Denote by $\{\varphi_\alpha\}_{\alpha \in S_+}$ the basis of $A$ corresponding to $u_\alpha$ in the identification $A = g_{1/2} \oplus g_1$, and by $\{\varphi^{\alpha}\}_{\alpha \in S_+}$ the basis of $A^*$ such that $\langle \varphi_\alpha, \varphi^{\beta} \rangle_{ch} = \delta_{\alpha \beta}$. Denote by $\{\Phi_\alpha\}_{\alpha \in S_{1/2}}$ the corresponding basis of $A_{ne}$, and by $\{\Phi^\alpha\}_{\alpha \in S_{1/2}}$ the dual basis with respect to $\langle \cdot, \cdot \rangle_{ne}$, i.e., $\langle \Phi_\alpha, \Phi^{\beta} \rangle_{ne} = \delta_{\alpha \beta}$. Define

\[ H(M) = H^0(C^M, d_0) \]

where $d_0$ is defined in [33]. Then

\[ W^k(g, \theta) = H(V^k(g)). \]
Table 1.

| $\mathfrak{g}$ | $p(k)$ |
|----------------|--------|
| $sl(m|n), n \neq m$ | $(k+1)(k+(m-n)/2)$ |
| $psl(m|m)$ | $k(k+1)$ |
| $osp(m|n)$ | $(k+2)(k+(m-n-4)/2)$ |
| $spo(n|m)$ | $(k+1/2)(k+(n-m+4)/4)$ |
| $D(2,1;a)$ | $(k-a)(k+1+a)$ |
| $F(4), \mathfrak{g}^{4}= so(7)$ | $(k+2/3)(k-2/3)$ |
| $F(4), \mathfrak{g}^{4}= D(2,1;2)$ | $(k+3/2)(k+1)$ |
| $G(3), \mathfrak{g}^{3}= G_{2}$ | $(k-1/2)(k+3/4)$ |
| $G(3), \mathfrak{g}^{3}= osp(3|2)$ | $(k+2/3)(k+4/3)$ |

Note: when writing $osp(m|n) = spo(n|m)$ we adopt the conventions of [29, 2.1.2], so that $n$ is even.

Denote by $W_{k}(\mathfrak{g}, \theta)$ the unique simple quotient of $W^{k}(\mathfrak{g}, \theta)$. It is proved in [33] that $W^{k}(\mathfrak{g}, \theta)$ has a vertex subalgebra isomorphic to $V^{\beta_{k}}(\mathfrak{g}^{5})$, where

$$
\beta_{k}(a,b) = \frac{1}{4}((k + h^{\vee}/2)(a|b) - \frac{3}{2}\kappa_{0}(a,b)), \quad a, b \in \mathfrak{g}^{5}.
$$

and $\kappa_{0}$ is the Killing form of $\mathfrak{g}_{0}$. Let $\mathcal{V}_{k}(\mathfrak{g}^{5})$ be the image of $V^{\beta_{k}}(\mathfrak{g}^{5})$ in $W_{k}(\mathfrak{g}, \theta)$.

**Definition 3.1.** If $W_{k}(\mathfrak{g}, \theta) = \mathcal{V}_{k}(\mathfrak{g}^{5})$, we say that $k$ is a collapsing level.

**Theorem 3.2.** [3] Theorem 3.3 Let $p(k)$ be the polynomial listed in Table 1 below. Then $k$ is a collapsing level if and only if $p(k) = 0$.

4. Results in $KL_{k}^{fin}$

For $\lambda \in \widehat{\mathfrak{h}}^{*}$, denote by $L(\lambda)$ the irreducible $\mathfrak{g}$-module of highest weight $\lambda$.

**Lemma 4.1.** $Ext_{\mathfrak{g},k}^{1}(L(\lambda), L(\lambda)) = 0$.

**Proof.** Suppose that there is an extension

$$
0 \to L(\lambda) \xrightarrow{\delta} N \xrightarrow{\rho} L(\lambda) \to 0.
$$

Since $\widehat{\mathfrak{h}}$ acts diagonally, this implies that

$$
0 \to L(\lambda)_{\lambda} \to N_{\lambda} \to L(\lambda)_{\lambda} \to 0
$$

splits. We now prove that [4.1] splits too. Indeed, let $g_{\lambda} : L(\lambda)_{\lambda} \to N_{\lambda}$ be a section. Let $M(\lambda)$ be the Verma module and $\pi : M(\lambda) \to L(\lambda)$ be the canonical projection. Let $\eta : M(\lambda) \to N$ be the unique map such that $\eta(1) = g_{\lambda}(\pi(1))$. Remark that obviously, $f(\eta(Ker \pi)) = 0$, so $\eta(Ker \pi) \subset h(L(\lambda))$. Since $\eta(Ker \pi)_{\lambda} = 0$, we have $\eta(Ker \pi) = 0$. Define $g : L(\lambda) \to N$ by setting $g(\pi(v)) = \eta(v)$. It is easy to verify that $g$ is a well defined section which splits [4.1].

**Proposition 4.2.** Suppose that $M \in KL_{k}^{fin}$ is finitely generated. Then $M \in O^{k}$.

**Proof.** By assumption $\widehat{\mathfrak{h}}$ acts semisimply. Let $\widehat{\mathfrak{g}}_{+} = \mathfrak{g} \oplus t\mathbb{C}[t]\mathfrak{g}$. Let $\{m_{1}, \ldots, m_{k}\}$ be a set of generators for $M$. By the finiteness of the $\mathfrak{g}$-action and since the conformal weights are bounded below, $M' = U(\widehat{\mathfrak{g}}_{+})(\sum_{i=1}^{k} \mathbb{C}m_{i})$ is finite dimensional, in particular, it has only
a finite number of weights. Let \( \widehat{\mathfrak{g}}_- = t^{-1}[t^{-1}]\mathfrak{g} \), so that \( U(\widehat{\mathfrak{g}}) = U(\mathfrak{g}_- \otimes U(\mathfrak{g}_+) \) and \( M = U(\mathfrak{g}_-)M' \), hence \( M \) has a finite number of maximal weights. Since

\[
M_\lambda = \sum_\nu U(\mathfrak{g}_-)_\lambda - \nu M'_\nu ,
\]

we see that the \( \widehat{\mathfrak{h}} \)-eigenspaces are finite dimensional.

**Theorem 4.3.** Let \( \mathfrak{g} \) be a basic Lie superalgebra. If every highest weight \( V_k(\mathfrak{g}) \)-module in \( KL^\text{fin}_k \) is irreducible, then the category \( KL^\text{fin}_k \) is semisimple.

**Proof.** Let \( M \) be a module in \( KL^\text{fin}_k \). If \( M \) is finitely generated, then, by Proposition 4.2, we have that \( M \in \mathcal{O}^k \). Since Lemma 4.1 holds, we can then use the argument in [8, Theorem 5.5] to prove the complete reducibility of \( M \). As in loc. cit., the case when \( M \) is not finitely generated can be reduced to the finitely generated case.

If \( V \) is a vertex algebra, we say that a \( V \)-module is ordinary if \( L(0) \) acts semisimply with finite dimensional eigenspaces. Recall that we denote by \( C_k \) the category of ordinary modules in \( KL_k \).

**Corollary 4.4.** Assume that \( \mathfrak{g}_0 \) is semisimple. If every highest weight \( V_k(\mathfrak{g}) \)-module in \( C_k \) is irreducible, then \( C_k \) is semisimple.

**Proof.** If \( M \) is a highest weight module in \( KL^\text{fin}_k \), then, by Proposition 4.2, \( M \in C_k \), thus \( M \) is irreducible. By Theorem 4.3, \( KL^\text{fin}_k \) is semisimple. Let \( M \) be a module in \( C_k \). Since \( \mathfrak{h} \subset \mathfrak{g}_0 \) is a Cartan subalgebra of the semisimple Lie algebra \( \mathfrak{g}_0 \), and \( \mathfrak{h}, L(0) \) commute, then \( \mathfrak{h} \) acts semisimply on \( M \), thus \( M \in KL^\text{fin}_k \) and therefore \( M \) is completely reducible. □

Recall from [8, Lemma 5.6] the following result.

**Lemma 4.5.** Let \( k \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0} \). Assume that \( H(U) \) is an irreducible, non-zero \( W_k(\mathfrak{g}, \theta) \)-module for every non-zero highest weight \( V_k(\mathfrak{g}) \)-module \( U \) from the category \( KL^\text{fin}_k \). Then every highest weight \( V_k(\mathfrak{g}) \)-module in \( KL^\text{fin}_k \) is irreducible.

### 4.1. Rational case.

**Theorem 4.6.** Let \( \mathfrak{g} \) be a basic Lie superalgebra and \( k \notin \mathbb{Z}_{\geq 0} \). If \( W_k(\mathfrak{g}, \theta) \) is rational then \( KL^\text{fin}_k(\mathfrak{g}) \) is semisimple.

**Proof.** By Theorem 4.3 it suffices to prove that every highest weight \( V_k(\mathfrak{g}) \)-module in \( KL^\text{fin}_k \) is irreducible. Let \( U \neq 0 \) be such a module. Then \( H(U) \) is a highest weight module for \( W_k(\mathfrak{g}, \theta) \), which is nonzero since \( k \notin \mathbb{Z}_{\geq 0} \). If \( W_k(\mathfrak{g}, \theta) \) is rational then \( H(U) \) is irreducible, hence \( U \) is irreducible by Lemma 4.5. □

### 4.2. Collapsing case.

**Definition 4.7.** Write \((\mathfrak{g}_1, k_1) \triangleright (\mathfrak{g}_2, k_2)\) if \( H(V_{k_1}(\mathfrak{g}_1, \theta)) = V_{k_2}(\mathfrak{g}_2) \). We call a sequence \((\mathfrak{g}, k) \triangleright (\mathfrak{g}_1, k_1) \triangleright \ldots \triangleright (\mathfrak{g}_n, k_n)\) a collapsing chain for \((\mathfrak{g}, k)\).

It follows from [13, Main Theorem] that if \((\mathfrak{g}_1, k_1) \triangleright (\mathfrak{g}_2, k_2)\), then \( k_1 \notin \mathbb{Z}_{\geq 0} \).

**Proposition 4.8.** Assume that \( H(V_k(\mathfrak{g})) = V_{k'}(\mathfrak{g}') \). If \( M \) is a highest weight module in \( KL_k \), then \( H(M) \in KL^\text{fin}_{k'}(\mathfrak{g}') \).
Proof. By [33], we know that $H(M)$ is a highest weight module, in particular $\hat{h}^i$ acts semisimply and the $L_\theta^i(0)$-eigenvalues are bounded below. It remains only to show that the action of $g^i$ on the complex $C^M$ locally is locally finite. If $v \in g^i$, then the action of $v$ on the complex is given by $J^{(v)}$, where

$$J^{(v)} = v + \sum_{\alpha, \beta \in S_+} (-1)^{p(\alpha)} c_{\alpha \beta}(v) \cdot \varphi_\alpha \varphi_\beta + (-1)^{p(v)/2} \sum_{\alpha \in S_{1/2}} :\Phi^\alpha \Phi_{[u_\alpha, v]} :.$$  

$J^{(v)}(0)$ acts on the tensor product $M \otimes F$, where $F = F(A_{ch}) \otimes F(A_{ne})$ is a Clifford vertex algebra. The action of $v(0)$ on $M$ is locally finite, since the action of $g$ is locally finite. More precisely, $M$ admits a $\mathbb{Z}_{\geq 0}$ gradation:

$$M = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} M_\ell, \quad g \text{ acts locally finite on } M_\ell.$$  

On the other hand $F$ admits a Virasoro vector $L_F$ and the corresponding eigenvalue decomposition

$$F = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} F_\ell,$$  

has finite-dimensional eigenspaces. By (4.3), $J^{(v)} - v$ is primary of conformal weight 1, hence $J^{(v)}(0) - v(0)$ commutes with $L_F(0)$. In particular it stabilizes the eigenspaces of $L_F$, so that we have a gradation:

$$M \otimes F = \bigoplus_{\ell \in \mathbb{1/2} \mathbb{Z}_{\geq 0}} (M \otimes F)_\ell,$$  

where

$$(M \otimes F)_\ell = \bigoplus_{i=0}^\ell M_i \otimes F_{\ell-i}$$

and each $(M \otimes F)_\ell$ is a $g^i$-module. Since each $M_i$ is $g$-locally finite and $F_j$ is finite-dimensional, we conclude that $(M \otimes F)_\ell$ is $g^i$-locally finite. The claim follows.

Theorem 4.9. Let $(g, k) \triangleright (g_1, k_1) \triangleright \ldots \triangleright (g_n, k_n)$ be a collapsing chain. Assume that $KL_{k_n}(g_n)$ is semisimple. Then $KL_{k_1}^{\text{fin}}$ is semisimple. In particular this happens when $V_{k_n}(g_n)$ is rational or admissible.

Proof. We proceed by induction on the length of the collapsing chain. The base case $n = 0$ is obvious. Now assume $n > 0$. First remark that every highest weight module $U$ in $KL_{k_1}^{\text{fin}}$ is irreducible. Indeed if $U$ is such a module, then $H(U)$ is a $V_{k_1}(g_1)$-module in $KL_{k_1}^{\text{fin}}$ by Proposition 4.8 and non-zero and highest weight by [13], [33]. By induction $KL_{k_1}^{\text{fin}}$ is semisimple, hence $H(U)$ is irreducible. By Lemma 1.5 $U$ is irreducible, so we are in the hypothesis of Theorem 4.3 and therefore we can conclude that $KL_{k_1}^{\text{fin}}$ is semisimple. In particular, if $V_{k_n}(g_n)$ is rational or admissible, then $KL_{k_n}^{\text{fin}}(g_n)$ is semisimple (by definition in the rational case, by Arakawa’s Main Theorem from [13] in the admissible case).
Lemma 4.10. There is a collapsing chain \((\mathfrak{so}(m), \frac{4-m}{2}) \triangleright \ldots \triangleright (\mathfrak{g}', k')\) with

\[
(4.4) \quad (\mathfrak{g}', k') = \begin{cases} 
\mathbb{C} & \text{if } m \equiv 0, 1 \mod 4, \\
M(1) & \text{if } m \equiv 2 \mod 4, \\
(\mathfrak{sl}(2), 1) & \text{if } m \equiv 3 \mod 4.
\end{cases}
\]

Proof. Using Table 2 below, one sees that for \(m \gg 0\)

\[
H(V_{\frac{4-m}{2}}(\mathfrak{so}(m))) = V_{\frac{8-m}{2}}(\mathfrak{so}(m-4)).
\]

Since \(\frac{8-m}{2}\) is again a collapsing level for \(\mathfrak{so}(m-4)\), by induction on \(m\) we are reduced to the cases when \(m = 4, 5, 6, 7\), where one concludes using Table 2 once again. \(\square\)

Proposition 4.11. The following are collapsing chains

\[
(4.5) \quad (F(4), -1) \triangleright (D(2, 1; 2), 1/2) = (D(2, 1; 1/2), 1/2) \triangleright (\mathfrak{sl}(2), -4/3)
\]

\[
(4.6) \quad (F(4), 2/3) \triangleright (\mathfrak{so}(7), -2) \triangleright (\mathfrak{sl}(2), -1/2)
\]

\[
(4.7) \quad (G(3), 1/2) \triangleright (G_2, -5/3) \triangleright \mathbb{C}
\]

\[
(4.8) \quad (\mathfrak{sp}(2|3), -3/4) \triangleright (\mathfrak{sl}(2), 1)
\]

\[
(4.9) \quad (\mathfrak{sp}(2|1), -5/4) \triangleright \mathbb{C}
\]

\[
(4.10) \quad m \neq n + 1, n \geq 2, (\mathfrak{sp}(n|m), -1/2) \triangleright \mathbb{C}
\]

\[
(4.11) \quad m \text{ odd}:
\]

\[
(\mathfrak{sp}(n|m), \frac{m-n-4}{4}) \triangleright (\mathfrak{sp}(n-2|m), \frac{m-n-2}{4}) \triangleright \ldots \triangleright (\mathfrak{sp}(2|m), \frac{m-6}{4}) \triangleright (\mathfrak{so}(m), \frac{4-m}{2}) \triangleright \mathbb{C} \tag{4.12}
\]

\[
(\mathfrak{osp}(n + 8|n), -2) \triangleright \mathbb{C}
\]

\[
(4.13) \quad m \neq n + 4, n + 8, m \geq 4 : (\mathfrak{osp}(m|n), -2) \triangleright (\mathfrak{sl}(2), \frac{m-n-8}{2}) \tag{4.14}
\]

\[
(\mathfrak{osp}(3|n), n + 1) = (\mathfrak{sp}(n|3), -\frac{n+1}{4}) \triangleright \ldots \triangleright (\mathfrak{sp}(2|3), -\frac{3}{4}) \triangleright (\mathfrak{sl}(2), 1)
\]

\[
(4.15) \quad (\mathfrak{osp}(5|n), \frac{n-1}{4}) \triangleright (\mathfrak{osp}(1|n), -\frac{n-3}{4}) = (\mathfrak{sp}(n|1), -\frac{n-3}{4}) \triangleright \ldots \triangleright (\mathfrak{sp}(2|1), -\frac{5}{4}) \triangleright \mathbb{C}
\]

\[
(4.16) \quad (\mathfrak{osp}(7|n), \frac{n-3}{2}) \triangleright (\mathfrak{osp}(3|n), 1 + n) \triangleright \mathbb{C} \tag{4.14}
\]
\( m \ \text{odd}, m \geq 8 : \ (osp(m|n), \frac{n-m^2+4}{2}) \triangleright (osp(m - 4|n), \frac{8-m+n}{2}) \triangleright \ldots \triangleright (4.15) \) or \( (4.16) \)

\( m \ \text{even}, m > n + 4, m \equiv n \mod 4 : \ (osp(m|n), \frac{n-m^4+4}{2}) \triangleright (osp(m - 4|n), \frac{8-m+n}{2}) \triangleright \ldots \triangleright (4.12) \)

\( m \ \text{even}, m > n + 4, m \equiv n + 2 \mod 4 : \)

\( (osp(m|n), \frac{n-m^4+4}{2}) \triangleright (osp(m - 4|n), \frac{8-m+n}{2}) \triangleright \ldots \triangleright (osp(n + 6|n), -1) \triangleright (osp(n + 2|n), 1) \)

\( (4.20) \)

\( (psl(m|m), -1) \triangleright \mathbb{C} \)

\( (4.21) \)

\( m \neq n, n + 1, n + 2, m \geq 2, (sl(m|n), -1) \triangleright (M(1), 1) \)

\( (4.22) \)

\( n \ \text{odd} : \ (sl(2|n), n/2 - 1) \triangleright (sl(n), -n/2) \triangleright \ldots \triangleright (sl(3), -3/2) \triangleright \mathbb{C} \)

\( (4.23) \)

\( n \ \text{even} \)

\( (sl(3|n), \frac{2-n^2}{2}) \triangleright (sl(1|n), \frac{1-n^2}{2}) \triangleright (sl(n - 2|1), \frac{3-n^2}{2}) \triangleright \ldots \triangleright (sl(2|1), -\frac{1}{2}) \triangleright \mathbb{C} \)

\( (4.24) \)

\( 3 < m, n, m \ \text{even}, n \ \text{odd} \)

\( (sl(m|n), \frac{n-m}{2}) \triangleright (sl(m - 2|n), \frac{n-m}{2} + 1) \triangleright \ldots \triangleright (sl(2|n), \frac{n}{2} - 1) \triangleright (4.22) \)

\( (4.25) \)

\( 3 < m, n, m \ \text{odd}, n \ \text{even} \)

\( (sl(m|n), \frac{n-m}{2}) \triangleright (sl(m - 2|n), \frac{n-m}{2} + 1) \triangleright \ldots \triangleright (sl(3|n), \frac{n-3}{2}) \triangleright (4.23) \)

\( (4.26) \)

\( 3 < n < m, m \equiv n \mod 2 \)

\( (sl(m|n), \frac{n-m}{2}) \triangleright (sl(m - 2|n), \frac{n-m}{2} + 1) \triangleright \ldots \triangleright (sl(n + 2|n), -1) = (sl(n|n + 2), 1) \)

\( (4.27) \)

\( (D(2, 1; a), -\frac{1+2a}{1+a}) \triangleright (sl(2), -\frac{1+2a}{1+a}) \)

\( (4.28) \)

\( (D(2, 1; a), -a - 1) \triangleright (sl(2), -\frac{1+2a}{a}) \)

Proof. The proof of the proposition is based on the data in Table 2. This table is built up by using the data computed in [6, Tables 5, 6, 7]. Case (4.5) is proven by looking at lines 19 and 17 in Table 2. A similar direct analysis works in cases (4.6) (line 21, 15), (4.7) (lines 23, 28), (4.8) (line 10), (4.9) (line 11), (4.10) (line 12), (4.11) (line 16), (4.12) (line 15), (4.13) (line 16), (line 13'), (4.14) and (4.15) (line 13), (4.16) (line 7), (4.17) (line 6), (4.18) (line 17) , (4.19) (line 18).

Cases (4.11) Since \( n \) is even and \( m \) is odd, \( (m - n - 2)/4 \) is never an integer, so \( H(V_{(m-n-2)/4}) \neq 0 \); use line 8 up to arriving to \( (spo(2|m), \frac{m-6}{2}) \). Using line 8 prevents to consider the case \( (spo(m - 2, m), -\frac{1}{2}) \): on the other hand in this case we have collapsing to \( \mathbb{C} \) by line 12. If \( m \geq 5 \) by line 9 we arrive at \( (so(m), \frac{4-m}{2}) \) which collapses according to Lemma (4.10). For \( m = 3 \) we have \( (spo(2|3), -\frac{1}{2}) \) which collapses to \( (sl(2), 1) \) by line 10. For \( m = 1 \) we have \( (spo(2|1), -\frac{1}{2}) \) which collapses to \( \mathbb{C} \) by line 11.
Corollary 4.12. In cases (4.15)–(4.22), (4.24) with $m - n \geq 5$, (4.14)–(4.26) the category $KL_{k}^{fin}$ is semisimple. If $a \notin Q$ or $a = \frac{q}{p} - 1, p, q \in \mathbb{Z}_{\geq 0}, p \geq 1$ (resp. $a = -\frac{q}{p}$), the category $KL_{a}^{fin}$ (resp. $KL_{k,a-1}^{fin}$) for $D(2,1;a)$ is semisimple.

Proof. We use Theorem 4.9. In cases (4.5), (4.6), the final level in the collapsing chain is admissible. In case (4.13) the final vertex algebra is rational if $m - n \geq 7$ is even and admissible if it is odd. The cases $m - n = 5, 6$ are covered by [19, Theorem 4.1.1]. In case (4.27), the vertex algebra $V_{k'}(sl(2))$ is

1. rational or admissible if $a = \frac{q}{p} - 1, p, q \in \mathbb{Z}_{\geq 0}, p \geq 2$, and $KL_{k'}$ is semi-simple by [11],
2. isomorphic to $V_{-2+1/q}(sl(2))$ if $p = 1$, and $KL_{k'}$ is semi-simple by [19, Theorem 4.1.1],
3. generic if $a \notin Q$, and $KL_{k'}$ is semi-simple by [30].

Similarly for (4.28). In all other cases except $V_{1}(osp(n+2|m), V_{1}(sl(n|n+2))$ the final vertex algebra in the collapsing chain is rational. In the two above special cases, semisimplicity of $KL_{k}^{fin}$ is given by the following arguments:

- Recall from [9, Proposition 4.8] that the following decomposition holds:

$$V_{1}(osp(2m|n)) = V_{1}(so(2m)) \otimes V_{-1/2}(sp(n)) + V_{1}(\omega_{1}) \otimes V_{-1/2}(\omega_{1}),$$

and the embedding $V_{1}(so(2m)) \otimes V_{-1/2}(sp(n)) \hookrightarrow V_{1}(osp(2m|n))$ is conformal. Since $V_{1}(osp(2m))$ is rational and $V_{-1/2}(sp(2m))$ is admissible, we can apply Proposition 5.1 (2) below to obtain $KL_{k}^{fin} = KL_{k}$.

Assume that $W$ is any highest weight $V_{1}(osp(2m|n))$–module in $KL_{k}$. Then $W$ is a completely reducible as $V_{1}(so(2m)) \otimes V_{-1/2}(sp(n))$ and it contains an irreducible $V_{1}(so(2m)) \otimes V_{-1/2}(sp(n))$–submodule isomorphic to the exactly one of the following modules:

$$M_{0} \cong V_{1}(so(2m)) \otimes V_{-1/2}(sp(n)),$$
$$M_{1} \cong V_{1}(\omega_{1}) \otimes V_{-1/2}(\omega_{1}),$$
$$M_{2} \cong V_{1}(\omega_{1}) \otimes V_{-1/2}(sp(n)),$$
$$M_{3} \cong V_{1}(so(2m)) \otimes V_{-1/2}(\omega_{1}).$$

Using the fusion rules arguments as in the proof of [9, Proposition 4.8] we easily get that $W$ is isomorphic to exactly one of the following two irreducible modules:

$$W_{0} \cong M_{0} \oplus M_{1}, W_{1} = M_{2} \oplus M_{3}.$$
Table 2.

Values of $k$ and $k'$. Assume that $k \notin \mathbb{Z}_{\geq 0}$.

| $g$ | $V_{k'}(g^2)$ | $k$ | $k'$ |
|-----|---------------|-----|-----|
| 1   | $sl(m|n)$, $m \neq n, m > 3, m - 2 \neq n$ | $V_{k'}(sl(m-2|n))$ | $\frac{n-m}{2}$ | $\frac{n-m+2}{2}$ |
| 2   | $sl(3|n)$, $n \neq 3, n \neq 1, n \neq 0$ | $V_{k'}(sl(1|n))$ | $\frac{n-3}{2}$ | $\frac{1-n}{2}$ |
| 3   | $sl(3)$ | $C$ | $-\frac{3}{2}$ | 0 |
| 4   | $sl(2|n)$, $n \neq 2, n \neq 1, n \neq 0$ | $V_{k'}(sl(n))$ | $\frac{n-2}{2}$ | $-\frac{n}{2}$ |
| 5   | $sl(2|1) = spo(2|2)$ | $C$ | $-\frac{1}{2}$ | 0 |
| 6   | $psl(m|m)$, $m \geq 2$ | $M(1)$ | $-1$ | 1 |
| 7   | $spo(n|m)$, $m \neq n, n + 2, m \geq 4$ | $V_{k'}(spo(n-2|m))$ | $\frac{m-n-4}{4}$ | $\frac{m-n-2}{4}$ |
| 8   | $spo(2|m)$, $m \geq 5$ | $V_{k'}(so(m))$ | $\frac{m-6}{4}$ | $\frac{4-m}{2}$ |
| 9   | $spo(2|3)$ | $V_{k'}(sl(2))$ | $-\frac{3}{2}$ | 1 |
| 10  | $spo(2|1)$ | $C$ | $-\frac{5}{4}$ | 0 |
| 11  | $spo(n|m)$, $m \neq n + 1, n \geq 2$ | $C$ | $-1/2$ | 0 |
| 12  | $osp(m|n)$, $m \neq n, m \neq n + 8, m \geq 8$ | $V_{k'}(osp(m-4|n))$ | $\frac{n-m+4}{2}$ | $\frac{8-m+n}{2}$ |
| 13' | $osp(7|n)$ | $V_{k'}(osp(3|n))$ | $\frac{n-3}{2}$ | $1 + n$ |
| 14  | $osp(m|n)$, $n \neq m, 0 \leq m \leq 6$ | $V_{k'}(osp(m-4|n))$ | $\frac{n-m+4}{2}$ | $\frac{m-n-8}{4}$ |
| 15  | $osp(m|n)$, $m \neq n + 4, n + 8; m \geq 4$ | $V_{k'}(sl(2))$ | $-\frac{2}{m}$ | $\frac{m-n-8}{4}$ |
| 16  | $osp(n + 8|n)$, $n \geq 0$ | $C$ | $-2$ | 0 |
| 17  | $D(2, 1; a)$ | $V_{k'}(sl(2))$ | $a$ | $-\frac{1+2a}{1+a}$ |
| 18  | $D(2, 1; a)$ | $V_{k'}(sl(2))$ | $-a - 1$ | $-\frac{1+2a}{a}$ |
| 19  | $F(4)$ | $V_{k'}(D(2, 1; 2))$ | $-1$ | $\frac{1}{2}$ |
| 20  | $F(4)$ | $C$ | $-3/2$ | 0 |
| 21  | $F(4)$ | $V_{k'}(so(7))$ | $\frac{2}{3}$ | $-2$ |
| 22  | $F(4)$ | $C$ | $-\frac{2}{3}$ | 0 |
| 23  | $G(3)$ | $V_{k'}(G_2)$ | $\frac{1}{2}$ | $-\frac{5}{4}$ |
| 24  | $G(3)$ | $C$ | $-\frac{3}{4}$ | 0 |
| 25  | $G(3)$ | $V_{k'}(osp(3|2))$ | $-\frac{2}{3}$ | 1 |
| 26  | $G(3)$ | $C$ | $-\frac{4}{3}$ | 0 |
| 27  | $G_2$ | $V_{k'}(sl(2))$ | $-\frac{4}{3}$ | 1 |
| 28  | $G_2$ | $C$ | $-\frac{5}{3}$ | 0 |
Therefore \( V_1(osp(2m|n)) \) has two irreducible modules in \( KL_k \) and every highest weight module in \( KL_k \) is irreducible. Now using Theorem 4.3 we have that \( KL_{k}^{fin} \) is semisimple. In particular, for \( V_1(osp(n + 2|n)) \), the category \( KL_{k}^{fin} = KL_k \) is semisimple.

• \( V_1(sl(n|n+2)) = V_{-1}(sl(2n+2)) \), and by results of Section 7.2 we have semisimplicity in \( KL_{k}^{fin} \).

\[ \square \]

5. Category \( KL_k \) of \( g \)-locally finite \( V_k(g) \)-modules

We first investigate some sufficient conditions to have either \( KL_k^{ss} = KL_k^{fin} \) or \( KL_k = KL_k^{fin} \).

**Proposition 5.1.**

1. Assume that \( g_{\mathfrak{g}} \) is a semisimple Lie algebra. Then \( KL_k^{fin} = KL_k^{ss} \).

2. Assume that there is a conformal embedding of \( V_{k_1}(g_{\mathfrak{g}}) \hookrightarrow V_k(g) \) and every module \( W \) from \( KL_k \) is semisimple as a \( V_{k_1}(g_{\mathfrak{g}}) \)-module. Then \( KL_k^{fin} = KL_k \).

**Proof.** Consider case (1). Assume that \( W \) is any module from \( KL_k^{ss} \). We need to show that \( h \) acts semisimply on \( W \). Each \( L(0) \)-eigenspace of \( W \) is a sum of finite-dimensional \( g \)-modules, therefore \( W \) is a sum of finite-dimensional \( g \)-module. Since there is an embedding \( V_{k_1}(g_{\mathfrak{g}}) \hookrightarrow V_k(g) \) and \( g_{\mathfrak{g}} \) is semisimple, we conclude that \( W \) is a direct sum of finite-dimensional \( g_{\mathfrak{g}} \)-modules. Since the action of the Cartan subalgebra \( h \) is obtained by the action of operators from \( V_{k_1}(g_{\mathfrak{g}}) \), we conclude that these operators act semisimply and therefore \( W \) is in \( KL_k^{fin} \).

Now we consider the case (2). Let \( W \) be a module from \( KL_k \). We have directly that \( W \) is a semisimple as \( V_{k_1}(g_{\mathfrak{g}}) \)-module. So \( W \) is a direct sum of irreducible \( V_{k_1}(g_{\mathfrak{g}}) \)-modules in \( KL_{k_1} \), which are highest weight modules, and therefore, since the embedding is conformal, we get that \( h \) and \( L(0) \) must act semisimply. The claim follows.

\[ \square \]

We have the following consequence:

**Corollary 5.2.** Assume that the condition (2) of Proposition 5.1 holds and that

3. Any highest weight \( V_k(g) \)-module in \( KL_k \) is irreducible.

Then \( KL_k \) is semisimple.

**Proof.** The assumption (2) implies that the Cartan algebra and the Virasoro element \( L(0) \) acts semisimply. This implies that \( KL_k = KL_k^{fin} \). Then the result follows by applying Theorem 4.3.

\[ \square \]

The conformal embeddings \( g_{\mathfrak{g}} \hookrightarrow g \) were classified in [9], and they include all collapsing levels for Lie superalgebras. Only in some cases when \( g_{\mathfrak{g}} \) is reductive, the semisimplicity of \( V_k(g_{\mathfrak{g}}) \) is still an open problem.

**Lemma 5.3.** Let \( M \) be a non-zero \( V_k(g) \)-module from \( KL_k \). Then there is a non-zero \( V_k(g) \)-submodule \( M^{fin} \subset M \) which belongs to \( KL_k^{fin} \).

**Proof.** Since \( M_{top} \) is a locally finite \( g \)-module, we get that \( h \) and \( L(0) \) acts locally finitely on \( M_{top} \). So there is a common eigenvector \( w \) for the action of \( h \) and \( L(0) \). Therefore \( M^{fin} = V_k(g).w \) is a \( V_k(g) \)-submodule of \( M \) which is in the category \( KL_k^{fin} \).

\[ \square \]

**Lemma 5.4.**

1. Let \( M \) be a logarithmic \( V_k(g) \)-module in \( KL_k \). Then \( L(0) - L_{ss}(0) \) is a \( V_k(g) \)-homomorphism.
(2) For any module $M$ in $KL_k$, the operator $h(0) - h_{ss}(0)$ is a $V_k(g)$–homomorphism for any $h \in \mathfrak{h}$.

Proof. The assertion (1) is already proved in [27] Remark 2.21. For completeness, we present here a version of their proof.

We have

$$\mathcal{M} = \bigoplus_{\alpha \in \mathcal{C}} M_\alpha, \quad M_\alpha = \{ v \in \mathcal{M} \mid (L(0) - \alpha)^{N_\alpha} v = 0 \text{ for some } N_\alpha > 0 \}.$$  

Define $Q \in \text{End}(\mathcal{M})$ by

$$Qv = (L(0) - \alpha)v, \quad v \in M_\alpha.$$  

Therefore $Q = L(0) - L_{ss}(0)$. Take $v \in M_\alpha$ and $n \in \mathbb{Z}$. Then for each $N \in \mathbb{Z}_{>0}$ we have

$$(L(0) - (\alpha - n))^N x(n)v = x(n)(L(0) - \alpha)^N v$$

which gives that $x(n)v \in M_{\alpha - n}$. Therefore

$$x(n)Qv = x(n)L(0)v - \alpha x(n)v = L(0)x(n)v - (\alpha - n)x(n)v = Qx(n)v,$$

which implies that $Q$ is a $\widehat{g}$–homomorphism, hence a $V_k(g)$–homomorphism. This proves (1). The proof of (2) is completely analogous. \hfill \Box

Theorem 5.5. Assume that the category $KL^{\text{fin}}_k$ is semisimple and that for any irreducible $V_k(g)$–module $M$ in $KL_k$ we have

$$(5.1) \quad \text{Ext}^1(M_{\text{top}}, M_{\text{top}}) = \{0\}$$

in the category of finite-dimensional $g$–modules. Then $KL_k$ is semisimple and $KL^{\text{fin}}_k = KL_k$.

Proof. In a view of [25] Lemma 1.3.1 it suffices to show that

1. $\text{Ext}^1(M, N) = \{0\}$ for any two irreducible modules $M, N$ in $KL_k$;
2. Each module $M$ in $KL_k$ contains an irreducible submodule.

Assume that we have a non-split extension

$$(5.2) \quad 0 \to M \to M^{\text{ext}} \to N \to 0$$

for a certain $\mathbb{Z}_{\geq 0}$–gradable module $M^{\text{ext}}$ in $KL_k$. If $M^{\text{ext}}$ is in $KL^{\text{fin}}_k$, then $M^{\text{ext}} \cong M \oplus N$ because $KL^{\text{fin}}_k$ is semisimple. This contradicts the assumption that $M^{\text{ext}}$ is the non-split extension $(5.2)$.

If $M^{\text{ext}}$ does not belong to $KL^{\text{fin}}_k$, then $L(0)$ does not act semisimply or there is $h \in \mathfrak{h}$ such that $h(0)$ does not act semisimply. Define accordingly the operator $Q$ as in Lemma 5.4, i.e. $Q = L(0) - L(0)_{ss}$ or $Q = h(0) - h(0)_{ss}$. Then $Q$ is a non-zero $V_k(g)$–homomorphism and therefore $N \cong QM^{\text{ext}} \cong M$. Indeed, since $M$ is irreducible, we have that $QM = 0$. It follows that $QM^{\text{ext}} \cong M^{\text{ext}}/\text{Ker } Q \subset M^{\text{ext}}/M \cong N$. Since $N \neq 0$, we find $QM^{\text{ext}} \cong N$. If the extension does not split, we must have $QM^{\text{ext}} \cap M \neq 0$, so $QM^{\text{ext}} \cong M$, and at the end $M \cong N$. By applying the Zhu’s functor to $(5.2)$ we get a non-split extension

$$0 \to M_{\text{top}} \to (M^{\text{ext}})_{\text{top}} \to M_{\text{top}} \to 0$$

in the category of finite-dimensional $g$–modules. This contradicts $(5.1)$. Thus (1) holds.

Let us prove (2). From Lemma 5.3 we get that $M$ contains a non-zero submodule $M^{\text{fin}}$ in $KL^{\text{fin}}_k$. Since $KL^{\text{fin}}_k$ is semisimple, we conclude that $M^{\text{fin}}$ contains an irreducible submodule. The claim follows. \hfill \Box
ON THE SEMISIMPACITY OF THE CATEGORY $KL_k$ FOR AFFINE LIE SUPERALGEBRAS

6. The case $\mathfrak{g} = C(n+1)$

6.1. Collapsing level $k = -\frac{1}{2}$.

Lemma 6.1. Let $\mathfrak{g} = C(n+1)$ and $k = -\frac{1}{2}$. Then the unique irreducible modules in $KL_{-\frac{1}{2}}$ are $V_{-\frac{1}{2}}(\mathfrak{g})$ and $L(-\frac{1}{2}\Lambda_0 + \frac{1}{2}\theta)$, where $\theta$ is the highest root of $\mathfrak{sp}(2n)$.

Proof. The discussion preceding [9, Lemma 4.1] applies: if $L(\Lambda)$ is irreducible in $KL_{-\frac{1}{2}}$ then $\Lambda = -\frac{1}{2}\Lambda_0 + \ell\theta$ and $\ell^2 - (k+1)\ell = 0$. □

Theorem 6.2. Let $\mathfrak{g} = C(n+1)$ and $k = -\frac{1}{2}$. Then we have:

- Irreducible modules in $KL_k$ have no self-extensions.
- The category $KL_{k}^{fin}$ is semisimple and $KL_k = KL_{k}^{fin}$.

Proof. It suffices to check that condition (2) of Proposition 5.1 and condition (3) of Corollary 5.2 hold.

Let us first check that:

(*) any $V_k(\mathfrak{g})$–module $W$ in $KL_k$ is completely reducible as $V_k(\mathfrak{g}_0) = V_{-1/2}(\mathfrak{sp}(2n)) \otimes M(1)$–module, where $M(1)$ is the Heisenberg vertex algebra of rank one.

Let $V_D = M(1) \otimes \mathbb{C}[D]$ be the lattice vertex algebra associated to rank one lattice $D = Z\alpha, \langle \alpha, \alpha \rangle = 4$. It has 4 non-isomorphic modules:

$$U_i = V_{D + \frac{1}{4}\alpha}, i = 0, 1, 2, 3,$$

and the following fusion rules:

$$U_i \times U_j = U_{(i+j)} \mod 4.$$

From [9, Proposition 4.15], we have that

$$V_{-\frac{1}{2}}(\mathfrak{g}) = (V_{-1/2}(\mathfrak{sp}(2n)) \otimes U_0) \bigoplus (L_{-1/2}(\omega_1) \otimes U_2)$$

(note that in loc. cit. a different normalization is used, so that the level 1 used there turns into level $-1/2$). The irreducible module $L(-\frac{1}{2}\Lambda_0 + \frac{1}{2}\theta)$ decomposes as

$$L(-\frac{1}{2}\Lambda_0 + \frac{1}{2}\theta) = V_{-1/2}(\mathfrak{sp}(2n)) \otimes U_1 \bigoplus L_{-1/2}(\omega_1) \otimes U_3.$$

By using the regularity of $V_D$ (i.e., complete reducibility in the entire category of weak $V_D$–modules, cf. [21]) and the concept of Heisenberg coset (cf. [13]), we get that

$$W = M_0 \otimes U_0 \bigoplus M_1 \otimes U_1 \bigoplus M_2 \otimes U_2 \bigoplus M_3 \otimes U_3$$

for certain $V_{-1/2}(\mathfrak{sp}(2n))$–modules $M_i$ in $KL_k$, $i = 0, 1, 2, 3$. By using complete reducibility for the admissible vertex algebra $V_{-1/2}(\mathfrak{sp}(2n))$ in the category $\mathcal{O}$ (cf. [1]), we get that the $M_i$ are direct sum of copies of $V_{-1/2}(\mathfrak{sp}(2n))$ or $L_{-1/2}(\omega_1)$. This implies that $W$ is a direct sum of irreducible $V_{-1/2}(\mathfrak{sp}(2n)) \otimes U_0$–modules. Since each irreducible $U_0$–module is a direct sum of irreducible modules for the Heisenberg vertex algebra $M(1)$, we get that $W$ is a direct sum of $V_k(\mathfrak{g}_0)$–modules. So (*) holds.

It remains to prove that any highest weight $V_k(\mathfrak{g})$–module in $KL_k$ is irreducible. By using the same arguments as above and fusion rules, we see that if $W$ is a highest weight module in $KL_k$, it decomposes as

$$V_{-1/2}(\mathfrak{sp}(2n)) \otimes U_0 \bigoplus L_{-1/2}(\omega_1) \otimes U_2 \text{ or } V_{-1/2}(\mathfrak{sp}(2n)) \otimes U_1 \bigoplus L_{-1/2}(\omega_1) \otimes U_3.$$

Therefore it is irreducible. The claim follows. □
6.2. Collapsing level \( k = -\frac{n+1}{2} \).

**Theorem 6.3.** Assume that \( \mathfrak{g} = C(n+1) \), \( k = -\frac{n+1}{2} \). Then \( KL^f_{k} \) is semisimple.

**Proof.** First we consider the case \( n = 1 \), so \( \mathfrak{g} = C(2) \cong sl(2|1) \). Then \( k = -1 \) is the critical level. Recently T. Creutzig and J. Yang in [19] Theorem 6.6) proved that in this case every module in \( KL^f_{k} \) is completely reducible (to match Creutzig-Yang’s result with our setting note that \( V_{-1}(sl(m|n)) = V_{1}(sl(m|n)) \). Therefore we have the collapsing chain

\[
(C(n + 1), -\frac{n+1}{2}) \triangleright (C(n), -\frac{n}{2}) \triangleright \ldots \triangleright (C(2), -1)
\]

implying that \( KL^f_{k} \) is semisimple. \( \square \)

7. Semisimplicity of \( KL_{-1} \) for \( \mathfrak{g} = sl(m|n) \) and \( \mathfrak{g} = psl(n|n) \).

In \( sl(n|m), n \neq m \) set \( \alpha_{i}^{\vee} = E_{ii} - E_{i+1,i+1} \) for \( i \neq n \) and \( \alpha_{n}^{\vee} = E_{mn} + E_{n+1,n+1} \) (\( E_{ij} \) are matrix units). Define \( \omega_{i} \in \mathfrak{h}^* \) by setting \( \omega_{i}(\alpha_{j}^{\vee}) = \delta_{ij} \) and \( \omega_{0} = 0 \). When \( n = m \), we work modulo the identity.

Recall that the defect \( \text{def} \mathfrak{g} \) of a basic classical Lie superalgebra \( \mathfrak{g} \) is the dimension of a maximal isotropic subspace in the real span of roots. When \( \mathfrak{g} = sl(m|n) \) the defect is \( \text{min}\{m, n\} \).

Also recall that the atypicality of a weight \( \lambda \) is the maximal number of linearly independent mutually orthogonal isotropic roots which are also orthogonal to \( \lambda \). The atypicality of an irreducible finite dimensional \( \mathfrak{g} \)-module \( V \) of highest weight \( \lambda \) is the atypicality of \( \lambda + \rho \) (here \( \rho \) is the half sum of positive even roots minus the half sum of positive odd roots).

In [32] it is shown that the atypicality does not depend on the choice of the set of positive roots. Let \( L_{\rho_{0}}(\lambda) \) be the finite dimensional irreducible \( \mathfrak{g}_{0} \)-module of highest weight \( \lambda \). The following conditions are equivalent [28]:

1. \( \lambda + \rho \) is atypical.
2. The Kac module \( Ind_{\rho_{0}}^{\rho}(L_{\rho_{0}}(\lambda)) \) is not irreducible.

It turns out that the atypicality of the trivial module, i.e. that of \( \rho \), is \( \text{def} \mathfrak{g} \).

7.1. The representation theory of \( V_{1}(psl(m|m)) \) via decomposition of conformal embedding. Let us consider first the case \( V = V_{1}(psl(m|m)) \) for \( m \geq 3 \). For \( s \in \mathbb{Z}_{\geq 0} \) consider the following \( V_{-1}(sl(m))-\)modules

\[
\pi_{s} := L_{-1}(sw_{1}), \quad \pi_{-s} := L_{-1}(sw_{m-1}).
\]

In order to prove that \( V \) is semisimple in \( KL_{1} \), we need to prove the following:

**Proposition 7.1.** Assume that \( M \) is any highest weight \( V \)-module in \( KL_{1} \). Then \( M \cong V \).

**Proof.** Since \( V_{1}(sl(m)) \) is rational and \( KL_{-1}(sl(m)) \) is semisimple [3] Theorem 1.2) we conclude that \( M \) is completely reducible as \( U = V_{1}(sl(m)) \otimes V_{-1}(sl(m))-\)module. This implies that \( M \) contains a \( U \)-submodule isomorphic to \( U_{r_{0},s_{0}} = L_{1}(\omega_{r_{0}}) \otimes \pi_{-s_{0}} \), for some \( 0 \leq r_{0} \leq n - 1 \), and \( s_{0} \in \mathbb{Z} \). (For \( r_{0} = 0 \), we set \( L_{1}(\omega_{0}) = V_{1}(sl(m)) \).)

The lowest conformal weight of \( U_{r,s} \) and \( U_{r,-s} \) are

\[
h[r,s] = \frac{(\omega_{r}, \omega_{r} + 2\rho)}{2(m+1)} + \frac{(s\omega_{m-1}, s\omega_{m-1} + 2\rho)}{2(m-1)} = \frac{(m-r)r}{2m} + \frac{s^2 + ms}{2m} = \frac{(m-r)r + s(s+m)}{2m} = \frac{(\omega_{r}, \omega_{r} + 2\rho)}{2(m+1)} + \frac{(s\omega_{1}, s\omega_{1} + 2\rho)}{2(m-1)} = h[r,-s]
\]
We can choose \((r_0, s_0)\) so that the conformal weight of \(U_{r_0,s_0}\) coincides with the conformal weight of the highest weight vector of \(M\).

For \(s \in \mathbb{Z}\), let \(\overline{s} \in \{0, 1, \ldots, m-1\}\) be such that \(s \equiv \overline{s} \mod (m-1)\). For \(0 \leq r_1, r_2 \leq m-1\), we set \(r_3 = \overline{r_1} + \overline{r_2}\). Recall from [9, Theorem 4.4] that

\[
V = \bigoplus_{s \in \mathbb{Z}} U_{\overline{s},s},
\]

and that \(V\) is generated by \(V_1(sl(m)) \otimes V_{-1}(sl(m)) \oplus U_{1,1} \oplus U_{-1,-1}\).

Using the fusion rules:

\[
L_1(\omega_1) \times L_1(\omega_r) = L_1(\omega_{r+1}),
\]

\[
L_1(\omega_{m-1}) \times L_1(\omega_r) = L_1(\omega_{m-1-r}),
\]

\[
\pi_s \times \pi_{s'} = \pi_{s+s'},
\]

we conclude that \(M\) also contains submodules:

\[
U_{r_0+\ell,s_0+\ell} \quad (\ell \in \mathbb{Z}).
\]

Since \(M\) is a highest weight module, it is generated by \(U_{r_0,s_0}\)

\[
M = V.U_{r_0,s_0} = \bigoplus_{\ell \in \mathbb{Z}} U_{r_0+\ell,s_0+\ell}.
\]

We conclude that \(M\) is \(\mathbb{Z}_{\geq 0}\)-graded. By a direct calculation of the lowest conformal weight of \(U_{r_0+\ell,s_0+\ell}\) we get that the following statements are equivalent:

1. \(M\) is \(\mathbb{Z}\)-graded;
2. \(h[r + 1, s + 1] - h[r, s] = \frac{m-r+s}{m} \in \mathbb{Z}\) for \(r \in \{0, 1, \ldots, m-2\}, s \in \mathbb{Z}\);
3. \(h[r - 1, s - 1] - h[r, s] = -\frac{m+r+s}{m} \in \mathbb{Z}\) for \(r \in \{1, 2, \ldots, m-1\}, s \in \mathbb{Z}\);
4. \(r \equiv s \mod n\).

But then \(M\) contains a \(V_1(sl(m)) \otimes V_{-1}(sl(m))\)-module isomorphic to \(U_{r,r}\), implying that \(M = V\). The claim follows. \(\square\)

Since \(g_0 = sl(m) \times sl(m)\) is semisimple, and the categories \(KL_1\) and \(KL_{-1}\) are semisimple for \(sl(m)\), using Proposition 7.1 we conclude:

**Theorem 7.2.** Assume that \(g = psl(m|m)\) for \(m \geq 3\) and \(k = -1\). The category \(KL_k\) is semisimple.

7.2. **The case** \(V_{-1}(sl(m)\ell).\) Let \(g = sl(m|1)\). Recall [9] that there is a conformal embedding \(V_{-1}(sl(m)) \otimes M_c(1) \rightarrow V_{-1}(g)\) with the following decomposition:

\[
V_{-1}(g) = \bigoplus_{q \in \mathbb{Z}} \pi_q \otimes M_c(1, -q \sqrt{\frac{m-1}{m}}).
\]

where we set

\[
c = \frac{1}{\sqrt{m(m-1)}} \begin{pmatrix} I_m & 0 \\ 0 & m \end{pmatrix},
\]

so that \([c,c] = \lambda\).

For \(\ell \in \mathbb{Z}\) and \(r \in \mathbb{C}\), we define the following \(V_{-1}(sl(m)) \otimes M_c(1)\)-module:

\[
L[\ell, r] = \bigoplus_{q \in \mathbb{Z}} \pi_{q+\ell} \otimes M_c(1, -(q + r) \sqrt{\frac{m-1}{m}}).
\]

Note that for each \(s \in \mathbb{Z}\) we have \(L[\ell + s, r + s] = L[\ell, r]\).
Recall that $V_{-1}(sl(m|n))$ is realized as a vertex subalgebra of $M_m \otimes F$, where $M_m$ is the Weyl vertex algebra with generators $a_i^\pm$, $i = 1, \ldots, m$, and $F$ is the Clifford vertex algebra with generators $\Psi_i^\pm$ (cf. [31], [9]).

In this section we set $F = F_1$, $\Psi^\pm = \Psi_1^\pm$. We consider $V_{-1}(g)$ as subalgebra of $M_m \otimes F$.

**Proposition 7.3.** For every $\ell \in \mathbb{Z}$, $L[\ell, -\frac{\ell}{m-1}]$ has the structure of an irreducible $V_{-1}(g)$–module. It is realized as

$$L[\ell, -\frac{\ell}{m-1}] = V_{-1}(g)w_{\ell}$$

where $w_{\ell} =: (a_1^+)_{\ell-1} : \Psi^+$ for $\ell \in \mathbb{Z}_{>0}$ and $w_{\ell} =: (a_m^-)^{\ell} :$ for $\ell \in \mathbb{Z}_{<0}$.

Moreover, $L[\ell, -\frac{\ell}{m-1}]_{\text{top}} = U(g).w_{\ell}$ is an atypical irreducible, finite-dimensional $g$–module.

**Proof.** The results from [31] give that $M_m \otimes F$ is a completely reducible irreducible $V_{-1}(gl(m|1))$–module so that

$$M_m \otimes F = \bigoplus_{\ell \in \mathbb{Z}} V_{-1}(gl(m|1)).w_{\ell}.$$  

This implies that $V_{-1}(g).w_{\ell}$ is an irreducible $V_{-1}(g)$–module. Set $r = -\ell/(m - 1)$.

By identifying the highest weights, we get

$$V_{-1}(sl(m)) \otimes M_c(1).w_{\ell} = \pi_{\ell} \otimes M_c(1, -r \sqrt{\frac{m-1}{m}}).$$

By using fusion rules for $V_{-1}(sl(m)) \otimes M_c(1)$–modules we get:

$$V_{-1}(g).w_{\ell} = V_{-1}(g).\pi_{\ell} \otimes M_c(1, -r \sqrt{\frac{m-1}{m}})$$

$$= \bigoplus_{q \in \mathbb{Z}} (\pi_q \otimes M_c(1, -q \sqrt{\frac{m-1}{m}})).\pi_{\ell} \otimes M_c(1, -r \sqrt{\frac{m-1}{m}})$$

$$= \bigoplus_{q \in \mathbb{Z}} \pi_{q+\ell} \otimes M_c(1, -(q+r) \sqrt{\frac{m-1}{m}}) = L[\ell, r].$$

The top component is then an irreducible $g$–module $U(g).w_{\ell}$ which has all 1–dimensional weight spaces. Therefore $U(g).w_{\ell}$ can not isomorphic to the Kac module obtained from the corresponding $gl(m)$–module. Therefore $U(g).w_{\ell}$ is atypical. (In Remark 7.3 below we check the atypicality by computing explicitly the highest weights). The claim follows. \hfill \square

The same argument of Proposition 7.3 yields:

**Proposition 7.4.** Let $m \geq 3$. Assume that $M$ is any highest weight $V_{-1}(g)$–module in $KL_{-1}$. Then $M$ is irreducible and there is $\ell \in \mathbb{Z}$ such that

$$M \cong L[\ell, -\frac{\ell}{m-1}].$$

In particular, $M_{\text{top}}$ is an atypical $g$–module.

**Proof.** Since $M$ is a highest weight $V_{-1}(g)$–module in $KL_k$, its top component $M_{\text{top}}$ must contain a singular vector $w$ such that

$$V_{-1}(sl(m)) \otimes M_c(1).w \cong \pi_{\ell} \otimes M_c(1, -r \sqrt{\frac{m-1}{m}})$$

for certain $\ell \in \mathbb{Z}$ and $r \in \mathbb{C}$. By using fusion rules for $V_{-1}(sl(m)) \otimes M_c(1)$–modules we get:

$$M = V_{-1}(g).\pi_{\ell} \otimes M_c(1, -r \sqrt{\frac{m-1}{m}})$$

$$= \bigoplus_{q \in \mathbb{Z}} \pi_{q+\ell} \otimes M_c(1, -(q+r) \sqrt{\frac{m-1}{m}}) = L[\ell, r].$$
Let $h[\ell, r]$ denotes the conformal weight of the top component of $\pi_\ell \otimes M_\epsilon(1, -r \sqrt{m-1 \over m})$. It is given by the formula

$$h[\ell, r] = \ell^2 + |\ell| m \over 2m + r^2 m - 1 \over 2m.$$  

Since the embedding $V_{-1}(sl(m) \otimes M_\epsilon(1)) \hookrightarrow V_{-1}(sl(m|1))$ is conformal, the conformal weight of $\pi_{q+\ell} \otimes M_\epsilon(1, -(q + r) \sqrt{m-1 \over m})$ must differ from $h[\ell, r]$ by a positive integer. In particular, we must have the following conditions:

- $h[\ell - 1, r - 1] - h[\ell, r] \in \mathbb{Z}_{\geq 0}$,
- $h[\ell + 1, r + 1] - h[\ell, r] \in \mathbb{Z}_{\geq 0}$.

These relations have the solutions $r = -\ell + m \over m - 1$ for $\ell \geq 0$ and $r = -\ell m \over m - 1$ for $\ell \leq 0$ (the solution is indeed unique if $\ell \neq 0$). Therefore

$$M \cong L[\ell, -\ell + m \over m - 1] = L[\ell + 1, -\ell + 1 \over m - 1] \quad (\ell \geq 0),$$

$$M \cong L[\ell, -\ell m \over m - 1] \quad (\ell \leq 0).$$

The atypicality of $M_{top}$ follows from Proposition 7.3.

\[ \square \]

**Remark 7.5.** Less conceptually, we can check the atypicality by computing explicitly the highest weights. Identify $\mathfrak{h}^*$ with \{ $r_0 \delta_1 + \sum_{i=1}^m r_i \varepsilon_i \mid r_0 + \sum_{i=1}^m r_i = 0$ \}. Then the highest weight of $(a_1^+)^\ell \Psi^+: (\ell > 0)$ is

$$\lambda_\ell^+ := \ell \omega_1 + (m + \ell) \left( - {1 \over m(m-1)} (\varepsilon_1 + \cdots + \varepsilon_m) + {1 \over m-1} \delta_1 \right)$$

$$= \ell (\varepsilon_1 - {1 \over m} \sum_{i=1}^m \varepsilon_i) - {m + \ell \over m(m-1)} \sum_{i=1}^m \varepsilon_i + {m + \ell \over m-1} \delta_1$$

$$= (\ell - {\ell \over m} + {m + \ell \over m - m^2}) \varepsilon_1 + \sum_{i=2}^m \varepsilon_i + {m + \ell \over m-1} \delta_1.$$  

Since $\rho = \sum_{i=1}^m (i \over 2 - i + 1) \varepsilon_i - {m \over 2} \delta_1$, we have

$$\lambda_\ell^+ + \rho = {m \ell - 2 \ell - 1 \over m - 1} \varepsilon_1 + \sum_{i=1}^m (1 - i + {1 + \ell \over 1 - m} + {m \over 2} \varepsilon_i + {m + \ell \over m-1} - {m \over 2}) \delta_1$$

$$= {m \ell - 2 \ell - 1 \over m - 1} \varepsilon_1 + ({m \over 2} - {m + \ell \over m - 1}) \varepsilon_2 + \sum_{i=3}^m (1 - i + {1 + \ell \over 1 - m} + {m \over 2} \varepsilon_i + {m + \ell \over m-1} - {m \over 2}) \delta_1,$$

and $(\lambda_\ell^+ + \rho) | \delta_1 - \varepsilon_2 = 0$.  

The highest weight of \((a_m^-)_{-\ell}\) is:
\[
\lambda_{\ell} := -\ell\omega_{m-1} + \ell \left( -\frac{1}{m(m-1)}(\varepsilon_1 + \cdots + \varepsilon_m) + \frac{1}{m-1}\delta_1 \right)
\]
\[
= \ell \left( -\sum_{i=1}^{m-1} \varepsilon_i + m - 2 \sum_{i=1}^{m-1} \varepsilon_i + \frac{1}{m-1}\delta_1 \right)
\]
\[
= \ell \left( -\sum_{i=1}^{m-1} \varepsilon_i + m - 2 \varepsilon_m + \frac{1}{m-1}\delta_1 \right).
\]
Hence
\[
\lambda_{\ell} + \rho = \ell \left( -\sum_{i=1}^{m-1} \varepsilon_i + m - 2 \varepsilon_m + \frac{1}{m-1}\delta_1 \right) + \rho
\]
\[
= \sum_{i=1}^{m-1} \left( \frac{m}{2} - i + 1 - \frac{\ell}{m-1} \right) \varepsilon_i + \left( 1 + m - 2 \delta_1 \right) \varepsilon_m + \left( \frac{\ell}{m-1} - \frac{m}{2} \right) \delta_1
\]
and \((\lambda_{\ell} + \rho|\delta_1 - \varepsilon_1) = 0\).

In the following theorem we need to use results from [24] and [42] (see also [26]), which we recall in our setting. Let \(\mathcal{L}^{(k)}\) be the category of finite dimensional \(sl(m|1)\)-modules on which the center acts with Jordan blocks of size at most \((1, 2)\). Let \(\lambda, \mu\) be dominant weights, and let \(\rho_1\) be the half sum of the positive odd roots. Combining [24, Proposition 6.1.2. (iii)] and [42, Lemma 6.6] one has

**Proposition 7.6.** If \(\lambda\) has atypicality 1, then
\[
\text{Ext}_{\mathcal{L}^{(k)}}(L(\lambda), L(\mu)) = \begin{cases} 
\mathbb{C} & \text{if } \lambda = \mu \pm 2\rho_1, \\
0 & \text{otherwise.}
\end{cases}
\]

Note that the non-trivial extensions above were explicitly realized inside a Kac module, which is a weight module. Since our category \(KL_{k}^{\text{fin}}\) is semi-simple, such extensions cannot appear for \(V_k(\mathfrak{g})\)-modules. We shall now see that there are no non-trivial extensions in the larger category \(KL_k\).

**Theorem 7.7.** Assume that \(\mathfrak{g} = sl(m|1)\) for \(m \geq 2\) and \(k = -1\). Let \(M\) be an irreducible \(V_{-1}(\mathfrak{g})\)-module in \(KL_{-1}\). Then

1. \(\text{Ext}^1(M_{\text{top}}, M_{\text{top}}) = \{0\}\) in the category of finite-dimensional \(\mathfrak{g}\)-modules.
2. \(\text{Ext}^1(M, M) = \{0\}\) in the category \(KL_{-1}\).
3. The category \(KL_k\) is semi-simple.

**Proof.** The classification of irreducible modules in \(KL_k\) implies that the top component \(M_{\text{top}}\) of an irreducible module in \(KL_k\) is an irreducible highest weight \(\mathfrak{g}\)-module. Since \(\text{def} \, sl(m|1) = 1\), the atypicality of \(M_{\text{top}}\) is at most 1. Then assertion (1) follows from Proposition 7.6. The assertions (2) and (3) follow from (1) and semi-simplicity of \(KL_{k}^{\text{fin}}\) by using Theorem 5.5. \(\square\)

**Remark 7.8.** Theorem 7.7 generalizes to the whole category \(KL_{-1}(sl(m|1))\) the semi-simplicity result of Creutzig and Yang [19], who deal with modules of finite length with semi-simple \(h\)-action in \(KL_{-1}\) for \(V_{-1}(sl(m|n)) = V_{1}(sl(n|m))\) in the more general case \(m \geq 2, n \geq 1\). Both results rely on the classification of irreducible modules in \(KL_{-1}\), that we construct using...
the conformal embedding $V_{-1}(g) \hookrightarrow M_m \otimes F$, whereas Creutzig and Yang use tensor categories and induced modules: see [19, Corollary 6.11].

8. The category $KL_k$ is not semisimple for $g = sl(m|1)$ and $k \in \mathbb{Z}_{>0}$.

Theorem 7.7 shows that indecomposable non-irreducible modules in $KL_k$ do not exist for $k = -1$.

Using Zhu’s algebra theory in [26], the authors construct indecomposable weak $V_k(sl(m|1))$–modules for $k = 1$ on which the element $L(0)$ of the Virasoro algebra does not act semisimply (these modules are also called logarithmic modules). Note that the level $k = 1$ is neither conformal nor collapsing for $g = sl(m|1)$.

In this section we shall first refine the example presented in [26] and show that even smaller category $KL_k^{fin}$ is not semisimple for $k = 1$. Next we shall extend this result for $k \in \mathbb{Z}_{>0}$.

Recall that the vertex algebra $V_1(g)$ is realized as a subalgebra of $M \otimes F_m$, where $M = M_1$ is the Weyl vertex algebra generated by $a^\pm = a_i^\pm$, and $F_m$ the Clifford vertex algebra generated by $\Psi_i^\pm$, $i = 1, \ldots, m$ (cf. 31). Even generators of $V_1(g)$ are realized by

$$E_{i,j} := \Psi_i^+ \Psi_j^- ; \quad i, j = 1, \ldots, m,$$

and odd generators by

$$E_{1,j+1} := a^+ \Psi_j^- ; \quad E_{j+1,1} := a^- \Psi_j^+ ; \quad j = 1, \ldots, m.$$

Define $|m\rangle =: \Psi_1^+ \cdots \Psi_m^+ : \in F_m$. We know from [31] that

$$(8.1) \quad (a^+)^\ell : \otimes |m\rangle >$$

is a singular vector for $V_1(g)$ for each $\ell \in \mathbb{Z}_{>0}$, and it generates an irreducible, highest weight $V_1(g)$–module. In order to construct indecomposable, highest weight modules, we need to allow that $\ell$ in the formula (8.1) is a negative integer in certain sense. In order to achieve this we shall consider a larger vertex algebra containing $M \otimes F_m$ such that formula (8.1) makes sense for $\ell$ negative. Fortunately, there is a nice construction of the vertex algebra $\Pi(0)$ obtained using a localisation of the Weyl vertex algebra $M$. The vertex algebra $\Pi(0)$ was originally constructed in [16], and has appeared recently in realisation of certain vertex algebras and their modules [22, 5, 12, 10].

Let $L = Zc + Zd$ be the rank two lattice such that

$$\langle c, d \rangle = 2, \quad \langle c, c \rangle = \langle d, d \rangle = 0.$$

Let $V_L = M(1) \otimes \mathbb{C}[L]$ be the associated lattice vertex algebra (cf. 30). Then $\Pi(0)$ is realized as the following subalgebra of $V_L$:

$$\Pi(0) = M(1) \otimes \mathbb{C}[Zc].$$

There is an embedding of $M$ into $\Pi(0)$ such that

$$a^+ = e^c, \quad a^- = -\frac{c(-1) + d(-1)}{2} e^{-c}.$$

Then $e^{-c}$ plays the role of the inverse $(a^+)^{-1}$ of $a^+$.

**Theorem 8.1.** Define

$$\bar{w} := (a^+)^{-m} \otimes |m\rangle = e^{-mc} \otimes |m\rangle \in \Pi(0) \otimes F_m.$$

Then we have:

- $\bar{W} = V_1(g)\bar{w}$ is a highest weight $V_1(g)$–module in the category $KL_k^{fin}$. 


• $\tilde{W}$ is reducible and it contains a proper submodule isomorphic to $V_i(\mathfrak{g})$.

In particular, the category $KL_k^{\text{fin}}$ is not semisimple for $k = 1$.

Proof. The proof that $\tilde{w}$ is a singular vector is the same as in the case of the singular vector $s_i$. Therefore $\tilde{W}$ is an highest weight $V_k(\mathfrak{g})$–module. By using the action of $E_{i,j+1}(0)$, $j = 1, \ldots, m$ we see that

$$\tilde{W}_{\text{top}} = \text{span}_\mathbb{C}\{e^{-sc} \otimes \Psi^+_{j_1} \cdots \Psi^+_{j_s} : \},$$

for $0 \leq s \leq m$, $1 \leq j_1 < \cdots < j_s \leq m$. So $\dim \tilde{W}_{\text{top}} = 2^m$. This implies that $\tilde{W}$ is in the category $KL_k^{\text{fin}}$. Moreover $\tilde{w}$ is a highest weight vector for $\tilde{W}$, hence $\tilde{W}$ is indecomposable.

Since $1_{M \otimes F_m} = e^0 \otimes 1_{F_m} \in \tilde{W}_{\text{top}}$, we conclude that $V_1(\mathfrak{g}) \cong V_i(\mathfrak{g}).1_{M \otimes F_m}$ is a proper submodule of $\tilde{W}$. Therefore, $\tilde{W}$ is reducible and indecomposable $V_k(\mathfrak{g})$–module.

Remark 8.2. Note that the building block for a construction of indecomposable $V_k(\mathfrak{g})$–module in $KL_k$ is the indecomposable $M$–module $\Pi(0)$. By using the singular vectors $(a_1^+)^{-m} \otimes |m\rangle$, we get indecomposable, weight $V_i(\mathfrak{sl}(m|n))$–modules for every $n \in \mathbb{Z}_{>0}$. But one can show that these modules are in the category $KL_k$ if and only if $k = 1$. A more detailed analysis of these modules will appear elsewhere.

Let now $k \in \mathbb{Z}_{>0}$ is arbitrary. In [26, Corollary 5.4.3], the authors proved that $V_k(\mathfrak{g}) = V^k(\mathfrak{g})/I$, where $I$ is the ideal in $V^k(\mathfrak{g})$ generated by the singular vector $e_{\mathfrak{g}}(-1)^{k+1}1$. Now we can combine this result with Theorem 8.1 and show that there exist indecomposable $V_k(\mathfrak{g})$–modules.

Corollary 8.3. The category $KL_k^{\text{fin}}$ is not semisimple for any $k \in \mathbb{Z}_{>0}$.

Proof. It is clear that there is a diagonal action of $V^k(\mathfrak{g})$ on $V_1(\mathfrak{g}) \otimes^k$. Using [26], one gets that

$$V_k(\mathfrak{g}) \cong V^k(\mathfrak{g}).(1 \otimes \cdots \otimes 1)^{k \text{ times}} \subset V_1(\mathfrak{g}) \otimes^k.$$ 

As a consequence, we have that $\tilde{W} \otimes V_1(\mathfrak{g})^{\otimes (k-1)}$ is a $V_k(\mathfrak{g})$–module. Define

$$\tilde{w}^{(k)} = \tilde{w} \otimes 1 \otimes \cdots \otimes 1,$$

$$\tilde{W}^{(k)} = V_k(\mathfrak{g}).\tilde{w}^{(k)} \subset \tilde{W} \otimes V_1(\mathfrak{g})^{\otimes (k-1)}.$$ 

One easily sees that:

$$\tilde{W}_{\text{top}}^{(k)} = \text{span}_\mathbb{C}\{(e^{-sc} \otimes \Psi^+_{j_1} \cdots \Psi^+_{j_s} : \otimes 1 \otimes \cdots \otimes 1)^{(k-1) \text{ times}} \},$$

for $0 \leq s \leq m$, $1 \leq j_1 < \cdots < j_s \leq m$. We can then argue as in Theorem 8.1 and conclude that $\tilde{W}^{(k)}$ is indecomposable with $V_k(\mathfrak{g}) \cdot (1 \otimes \cdots \otimes 1)$ a proper submodule. \hfill \Box

Remark 8.4. Assume that for certain non-integral level $k \in \mathbb{C}$, we have

$$V_k(\mathfrak{g}) \hookrightarrow V_{k-1}(\mathfrak{g}) \otimes V_1(\mathfrak{g}).$$

Then the same proof as above implies that $KL_k^{\text{fin}}$ is not semisimple. Beyond integral levels $k \geq 2$, we believe that (8.2) holds for principal admissible levels (cf. [34], [35]). As far as we
know, this is not proved yet. So, one expects that $V_k(\mathfrak{g})$ will be semisimple only for principal admissible levels $k$ such that $k - 1$ is not admissible. In the case $\mathfrak{g} = \mathfrak{sl}(2|1)$, such levels will be described in Section 10.

9. Semisimplicity of $V_{1/2}(\mathfrak{psl}(2|2))$

In this section we will present an example where $KL_k^{\text{fin}}$ is semisimple with $k$ non-collapsing and $W_k(\mathfrak{g}, \theta)$ irrational.

In this section we let $k = 1/2$, $k_0 = -3/2$. Set $V = W_k(\mathfrak{psl}(2|2), \theta)$. By [33] §8.4, $V$ is isomorphic to the simple $N = 4$ superconformal vertex algebra at central charge $c = -9$. A free-field realization of the vertex algebra $V$ was presented in [4], where it was proved that there is a conformal embedding $V_{k_0}(\mathfrak{sl}(2)) \hookrightarrow V$. Next, we consider the category of $V$–modules which belong to $KL_k(\mathfrak{sl}(2))$, which we denote by $KL_N=4$. Since $KL_{k_0}(\mathfrak{sl}(2))$ is semi-simple, $KL_N=4$ coincides with the category of ordinary $V$–modules.

Using the same methods as in Section 4 and the fact that for $\mathfrak{g} = \mathfrak{sl}(2)$ we have $KL_{k_0}(\mathfrak{sl}(2)) = KL_k^{\text{fin}}(\mathfrak{sl}(2))$, we get:

**Lemma 9.1.** In $KL_{N=4}$ we have:

$$\text{Ext}^1_{KL_{N=4}}(V, V) = \{0\}.$$ 

**Theorem 9.2.**

1. The category $KL_{N=4}$ is semisimple. In particular, every highest weight $V$–module in $KL_{N=4}$ is irreducible and isomorphic to $V$.

2. The category $KL_k^{\text{fin}}$ for $V_{1/2}(\mathfrak{psl}(2|2))$ is semisimple.

**Proof.** By [4], $V$ is the unique irreducible $V$–module in $KL_{N=4}$. Assume that $M$ is a highest weight module in $KL_{N=4}$. By using the relation

$$[e](\omega + 1/2) = 0$$

in the Zhu’s algebra $A(V)$ from [4] Proposition 3 (with notation used there), one easily sees that $M$ is irreducible and isomorphic $V$. Now, applying Lemma 9.1 we get that $KL_{N=4}$ is semisimple.

Assume now that $W$ is a non-zero highest weight $V_{1/2}(\mathfrak{psl}(2|2))$–module in $KL_k$. Then $H(W)$ is a non-zero highest weight $V$–module in $KL_{N=4}$ and therefore $H(W) \cong V$. Now, using Lemma 4.5 we get that $KL_k^{\text{fin}}(\mathfrak{psl}(2|2))$ is semisimple. 

10. The category $KL_k$ for $\mathfrak{sl}(2|1)$

In this section let $\mathfrak{g} = \mathfrak{sl}(2|1)$ and $k = -\frac{m+1}{m+2}$, $m \in \mathbb{Z}_{\geq 0}$. Recall that $W_k(\mathfrak{g}, \theta)$ is isomorphic to the $N = 2$ superconformal vertex algebra, which is rational by [2]. Therefore $KL_k^{\text{fin}}$ is semisimple by Theorem 4.6. Let us see that in this case $KL_k = KL_k^{\text{fin}}$.

**Theorem 10.1.** Let $\mathfrak{g} = \mathfrak{sl}(2|1)$ and $k = -\frac{m+1}{m+2}$, $m \in \mathbb{Z}_{\geq 0}$. Then $KL_k$ is semisimple.

**Proof.** Note that $V_k(\mathfrak{g})$ has a subalgebra isomorphic to the affine vertex algebra $\tilde{V}_k(\mathfrak{sl}(2))$, which is a certain quotient of $V^k(\mathfrak{sl}(2))$. Theorem 10.6 below implies that there is a conformal embedding $U = \tilde{V}_k(\mathfrak{sl}(2)) \otimes W \hookrightarrow V_k(\mathfrak{g})$, where $W$ is isomorphic to the regular vertex algebra $D_{m+1,2}$ from [3], also investigated recently in [40, 41]. In the case $m = 0$, the assertion is already proved in Theorem 6.2 and the vertex algebra $D_{1,2}$ is the lattice vertex algebra $V_D$. Since $\mathfrak{g}_0 \cong \mathfrak{sl}(2) \oplus \mathbb{C}H^+$, and the center $\mathbb{C}H^+$ belongs to $W$ (see (10.2)), we have that
V_k(g_0) \subset U. Note that the sl(2) subalgebra of g_0 acts semisimply on any module M from KL_k^s. Since H^+ \in W and W is regular, we get that M is completely reducible as a module for the Heisenberg vertex algebra generated by H^+, which we denote by M_H^+(1). Therefore, M is a sum of V_k(sl(2)) \otimes M_H^+(1)-modules in KL_k^{ss^+} with semisimple action of sl(2). This implies that the Cartan subalgebra of g acts semisimply on any module in KL_k^{ss^+}. Hence KL_k^{ss} = KL_k^{fin}. 

Assume M and N are irreducible V_k(g)-module in KL_k and that we have an extension

(10.1) \quad 0 \to M \to M^{ext} \to N \to 0

for a certain \(Z_{\geq 0}\)-gradable module \(M^{ext}\) in KL_k. We have the following cases:

- If \(M^{ext}\) is in \(KL_k^{ss} = KL_k^{fin}\), then \(M^{ext} \cong M \oplus N\) because \(KL_k^{fin}\) is semisimple.
- If \(M^{ext}\) is logarithmic, then \(Q = L(0) - L_{ss}(0)\) is a \(V_k(g)\)-homomorphism and therefore \(N \cong Q \cdot M \cong M\). So we can assume that \(M = N\). Then applying Zhu’s functor we get an extension

\[0 \to M_{top} \to (M^{ext})_{top} \to M_{top} \to 0.\]

We proved above that \(H^+(0)\) is semisimple. Note that \((M^{ext})_{top}\) is finite-dimensional and therefore the action of sl(2) on \((M^{ext})_{top}\) is semi-simple. Thus the Cartan subalgebra \(h\) of \(g\) acts diagonally on \((M^{ext})_{top}\). Moreover, the action of \(L(0)\) on \((M^{ext})_{top}\) is given by

\[L(0) = L^{sl(2)}(0) + L^W(0)\]

where \(L^{sl(2)}\) is Sugawara Virasoro vector in \(\tilde{V}_k(sl(2))\), and \(L^W\) is Virasoro vector in \(W\). Since \(W\) is regular, we have that the action of \(L^W(0)\) is diagonal. Since the action of \(L^{sl(2)}(0)\) on \((M^{ext})_{top}\) is proportional to the action of Casimir element of \(sl(2)\), we conclude that \(L^{sl(2)}(0)\) also acts diagonally on \((M^{ext})_{top}\). Therefore \(L(0)\) acts diagonally on \((M^{ext})_{top}\). This implies that \(M^{ext}\) is a module from \(KL_k^{fin}\), and therefore (10.1) splits.

\(\square\)

**Remark 10.2.** Using the language of tensor categories and concepts from [19], Theorem 10.1 implies that KL_k is a semisimple braided tensor category.

**Remark 10.3.** Although KL_k(sl(2)) is semi-simple for k admissible, since \(V^k(sl(2))\) is not simple, the category KL_k(sl(2)) is not semi-simple. Moreover, it is expected that KL_k(sl(2)) contains logarithmic modules. The paper [38] presents some conjectural logarithmic modules at admissible level, which to belong KL_k(sl(2)).

We believe that the subalgebra \(\tilde{V}_k(sl(2))\) of \(V_k(g)\) is simple for \(k = \frac{-m+1}{m+2}\). But we don’t need this information for proving semisimplicity of KL_k.

Based on Theorem 10.1 and the arguments presented in Remark 8.4 we expect that the following conjecture holds.

**Conjecture 10.4.** The category KL_k is semisimple if and only if \(k \in \{-1, \frac{-m+1}{m+2} | m \in \mathbb{Z}_{\geq 0}\}\).

\(^1\)We thank the referee for this information.
10.1. The vertex algebra $D_{m+1,2}$. The vertex algebras $D_{m+1,k}$ are defined in [3] for arbitrary $m \in \mathbb{C}$ and $k \in \mathbb{Z}_{>0}$. We shall here assume that $m \in \mathbb{Z}_{>0}$ and $k = 2$.

For $p \in \mathbb{Z}$, let $F_p = M_p(1) \otimes \mathbb{C}[Z\delta]$ be the rank one lattice vertex algebra associated to the lattice $\mathbb{Z}\delta^{(p)}$, $(\delta^{(p)}, \delta^{(p)}) = p$. Here $M_p(1)$ is the Heisenberg vertex algebra generated by $\delta^{(p)}(z) = \sum_{m \in \mathbb{Z}} \delta^{(p)}(z)z^{-n-1}$. Following [3], let $D_{m+1,2}$ be the vertex subalgebra of $V_{m+1}(sl(2)) \otimes F_2$ generated by

$$\overline{X} = e(-1)1 \otimes e^{\delta(2)}, \quad \overline{Y} = f(-1)1 \otimes e^{-\delta(2)}.$$

It was proved in [3] that $D_{m+1,2}$ is a regular vertex operator algebra (i.e., every weak $D_{m+1,2}$-module is completely reducible) and that

$$D_{m+1,2} \otimes F_{-2} = V_{m+1}(sl(2)) \otimes F_{-2(m+2)}.$$

Consider now the vertex algebra $V_k(g)$ for $k = -\frac{m+1}{m+2}$ and $g = sl(2|1)$. It is generated by four even fields $E^{1,2}, H^-, F^{1,2}, H^+$, such that $E^{1,2}, 2H^-, F^{1,2}$ define the homomorphism $\Phi_1 : V^k(sl(2)) \rightarrow V_k(g)$ commuting with the Heisenberg subalgebra generated by $H^+$, and four odd fields $E^1, E^2, F^1, F^2$ (cf. [17]). Denote by

$$X = E^1(-1)F^2(-1)1, \quad Y = F^1(-1)E^2(-1)1.$$

Set $X(n) = X_{n+1}, Y(n) = Y_{n+1}$. Let $W$ be the vertex subalgebra of $V_k(g)$ generated by vectors $X$ and $Y$. A direct computation, which uses the relations displayed in [17, Appendix D], shows that

$$X(1)Y = k1 + 2H^+.$$  

Lemma 10.5. In $V_k(g)$, we have:

$$(10.3) \quad X(-2m-4)X(-2m-2) \cdots X(-2)1 = 0,$$

$$(10.4) \quad X(-2m-5)X(-2m-3) \cdots X(-2)1 = 0.$$  

Proof. The proof the relation (10.3) follows from the fact that there is $\nu \neq 0$ such that

$$X(-2m-4)X(-2m-2) \cdots X(-4)X(-2)1 = \nu F^2(-m-2) \cdots F^2(-1)E^1(-m-2) \cdots E^1(-1)1,$$

and

$$F^2(-m-2) \cdots F^2(-1)E^1(-m-2) \cdots E^1(-1)1$$

is a charged singular vector in $V^k(g)$ (cf. [39]). The relation (10.4) follows by applying derivation on (10.3). 

Consider the vertex algebra $V_k(g) \otimes F_{-2}$. Set $\delta = \delta^{(-2)}$. As shown in [17, (4.2)], there exists another vertex algebra homomorphism $\Phi_2 : V^{m+1}(sl(2)) \rightarrow W \otimes F_{-2} \subset V_k(g) \otimes F_{-2}$ uniquely determined by

$$e \mapsto \frac{1}{k+1} X \otimes e^{\delta}, \quad f \mapsto \frac{1}{k+1} Y \otimes e^{-\delta}, \quad h \mapsto (m+1)\delta + 2(m+2)H^+.$$  

Using Lemma 10.5 we get that $e(-1)^{m+2}1 = 0$ in $W \otimes F_{-2}$, implying that

$$V_{m+1}(sl(2)) \hookrightarrow W \otimes F_{-2}.$$  

Theorem 10.6. We have $W \cong D_{m+1,2}$. 

Proof. Let \( \beta = (m + 2)(\delta + 2H) \). Then \( \beta \in \text{Com}(V_{m+1}(sl(2)), W \otimes F_{-2}) \) and \( \beta(1)\beta = -2(m+2) \). Let \( M_{\beta}(1) \) be the Heisenberg vertex algebra generated by \( \beta \). Let

\[
  u^+ = X(-2m-2) \cdots X(-2)1 \otimes e^{(m+2)\delta}, \quad u^- = Y(-2m-2) \cdots Y(-2)1 \otimes e^{-(m+2)\delta}
\]

Then

\[
  \beta(0)u^+ = (2(m+2)(m+1) - 2(m+2)^2)u^+ = -2(m+2)u^+, \quad \beta(0)u^- = 2(m+2)u^-.
\]

We have:

\[
  (k+1)e(0)u^+ = X(-2m-6)X(-2m-2) \cdots X(-2)1 \otimes e^{2m+1}e^{(m+2)\delta}.
\]

\[
  +X(-2m-5)X(-2m-2) \cdots X(-2)1 \otimes e^{2m+3}e^{(m+2)\delta} = 0
\]

Since \( Y(2m+3+j)X(-2m-2) \cdots X(-2)1 = 0 \) for \( j \geq 0 \), we conclude

\[
  (k+1)f(0)u^+ = Y(2m+3)X(-2m-2) \cdots X(-2)1 \otimes e^{-2m-5e^{(m+2)\delta}} = 0
\]

We prove analogous relations for \( u^- \), which implies that

\[
  u^\pm \in \text{Com}(V_{m+1}(sl(2)), W \otimes F_{-2}).
\]

Let \( Z \) be the subalgebra of \( W \otimes F_{-2} \) generated by \( u^\pm \). So we have that \( V_{m+1}(sl(2)) \otimes Z \subset W \otimes F_{-2} \). By applying the action of \( f(n) \) on \( u^+ \) (resp. \( e(n) \) of \( u^- \)), one gets that \( e^{\pm\delta} \in V_{m+1}(sl(2)) \otimes Z \). From this one gets \( X, Y \in V_{m+1}(sl(2)) \otimes Z \). Therefore:

\[
  W \otimes F_{-2} = V_{m+1}(sl(2)) \otimes Z.
\]

We conclude that there is a conformal embedding \( M_{\beta}(1) \hookrightarrow Z \), so that \( Z \) is generated by singular vectors \( u^\pm \). Using results on the uniqueness of the lattice vertex algebras (cf. [37]), we get that \( Z \cong F_{-2(m+2)} \). The isomorphism is determined by

\[
  e^{\delta(-2(m+2))} \mapsto u^+, \quad e^{-\delta(-2(m+2))} \mapsto au^-,
\]

for a certain \( a \neq 0 \). Therefore we get an isomorphism

\[
  W \otimes F_{-2} \cong V_{m+1}(sl(2)) \otimes F_{-2(m+2)}.
\]

On the other hand, using construction from [33, Section 6] we get an isomorphism

\[
  V_{m+1}(sl(2)) \otimes F_{-2(m+2)} \cong D_{m+1,2} \otimes F_{-2}
\]

such that the subalgebra \( F_{-2} \) is generated by

\[
  f(-1)^{m+1}1 \otimes e^{\delta(-2(m+2))}, \quad e(-1)^{m+1}1 \otimes e^{-\delta(-2(m+2))}.
\]

(This argument is essentially based on the fact that \( f(-1)^{m+1}1, e(-1)^{m+1}1 \) generate a subalgebra of \( V_{m+1}(sl(2)) \) isomorphic to \( F_{2m+2} \).) The isomorphism \( (10.6) \) maps the generators \( (10.7) \) to elements of \( W \otimes F_{-2} \):

\[
  f(-1)^{m+1}u^+, \quad ae(-1)^{m+1}u^-,
\]

which are proportional to \( e^\delta, e^{-\delta} \). Thus, we get an isomorphism \( W \otimes F_{-2} \cong D_{m+1,2} \otimes F_{-2} \) which preserves \( F_{-2} \). This implies that \( W \cong D_{m+1,2} \). □
References

[1] D. Adamović, Some rational vertex algebras, Glas. Mat. Ser. III 29 (1994), 25-40.
[2] D. Adamović, Vertex algebra approach to fusion rules for N=2 superconformal minimal models, Journal of Algebra 239 (2001) 549–572.
[3] D. Adamović, A family of regular vertex operator algebras with two generators, Central European J. Math. 5 (2007), 1-18.
[4] D. Adamović, A realization of certain modules for the $N = 4$ superconformal algebra and the affine Lie algebra $A_2^{(1)}$, Transformation Groups 21, n. 2 (2016) 299-327.
[5] D. Adamović, Realizations of simple affine vertex algebras and their modules: the cases $\widehat{sl}(2)$ and $osp(1,2)$, Comm. Math. Phys. 366 (2019) 1025–1067, arXiv:1711.11342 [math.QA].
[6] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, Conformal embeddings of affine vertex algebras in minimal $W$-algebras I: Structural results Journal of Algebra, 500, (2018), 117–152.
[7] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, Conformal embeddings of affine vertex algebras in minimal $W$-algebras II: decompositions, Japanese Journal of Mathematics, 12, 2, 261–315.
[8] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, An application of collapsing levels to the representation theory of affine vertex algebras, Int. Math. Res. Not., Volume 2020, 13, 2020, 4103–4143.
[9] D. Adamović, P. Möseneder Frajria, P. Papi, O. Perše, Conformal embeddings in affine vertex superalgebras, Advances in Mathematics, 360 (2020) 106918.
[10] D. Adamović, K. Kawasetsu, D. Ridout, A realisation of the Bershadsky-Polyakov algebras and their relaxed modules, Letters in Math. Physics 111, 38 (2021).
[11] D. Adamović and A. Milas, Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$, Math. Res. Lett. 2 (1995), 563–575.
[12] D. Adamović, V. Pedić, On fusion rules and intertwining operators for the Weyl vertex algebra, Journal of Mathematical Physics 60 (2019), 081701, 18 pp., arXiv:1903.10248 [math.QA].
[13] T. Arakawa, Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture, Duke Math. J. 130 (3) (2005), 435–478.
[14] T. Arakawa, Rationality of admissible affine vertex algebras in the category $O$, Duke Math. J. 165 (2016), no. 1, 67–93.
[15] T. Arakawa, J. van Ekeren, A. Moreau, Singularities of nilpotent Slodowy slices and collapsing levels of $W$-algebras, arXiv:2102.13462.
[16] S. Berman, C. Dong, and S. Tan, Representations of a class of lattice type vertex algebras, J. Pure Appl. Algebra, 176, 27–47, 2002.
[17] P. Bowcock, B. L. Feigin, A. M. Semikhatov, A. Taormina, Affine $sl(2|1)$ and affine $D(2|1) : \alpha$ as vertex operator extensions of dual affine $sl(2)$ algebras. Comm. Math. Phys. 214. 495-545 (2000).
[18] T. Creutzig, S. Kanade, A. Linshaw and D. Ridout, Schur-Weyl duality for Heisenberg cosets. Transformation Groups 24 (2019), 301–354.
[19] T. Creutzig, J. Yang, Tensor categories of affine Lie algebras beyond admissible level. Math. Ann. 380 (2021), no. 3-4, 1991–2040.
[20] T. Creutzig, R. McRae, J. Yang, Tensor structure on the Kazhdan–Lusztig category for affine $gl(1|1)$, International Mathematics Research Notices, 2021; rnab080, https://doi.org/10.1093/imrn/rnab080.
[21] C. Dong, H. Li, G. Mason, Regularity of rational vertex operator algebras, Adv. Math. 132, 1997, 148–166.
[22] E. Frenkel, Lectures on Wakimoto modules, opers and the center at the critical level, Adv. Math. 195 (2005) 297-404.
[23] I. Frenkel, Y. Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Am. Math. Soc. 104 (1993)
[24] J. Gervoni, Indecomposable representations of general linear Lie superalgebras, J. of Alg., 209, (1998), 367–401.
[25] M. Gorelik, V. Kac, On complete reducibility for infinite-dimensional Lie algebras, Advances in Mathematics 226 (2011) 1911–1972.
[26] M. Gorelik, V. Serganova, Integrable Modules Over Affine Lie Superalgebras $\mathfrak{sl}(1|n)^{(1)}$, Comm. Math. Phys. 364, 635–654 (2018).
[27] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, Conformal Field Theories and Tensor Categories, 169–248, Math. Lect. Peking Univ., Springer, Heidelberg, 2014.

[28] V. G. Kac. Representation of classical Lie superalgebras, in Differential geometrical methods in Mathematical Physics, II (Proc. Conf. Univ. Bonn, Bonn 1977), LNM 676, Springer, Berlin, 1978, 597–626

[29] V. G. Kac. Lie superalgebras, Adv. Math. 26, 8–96, (1977).

[30] V. G. Kac, Vertex Algebras for Beginners, University Lecture Series, Second Edition, AMS, Vol. 10 (1998).

[31] V. G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and Appell’s function, Comm. Math. Phys. 215 (2001), 631–682

[32] V. G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, Lie theory and geometry, 415–456, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994.

[33] V. G. Kac, M. Wakimoto, Quantum reduction and representation theory of superconformal algebras, Adv. Math. 185 (2004), 400–458.

[34] V. G. Kac, M. Wakimoto, Representations of affine superalgebras and mock theta functions, Trans. Groups 19, no. 2 (2014): 383–455.

[35] V. G. Kac, M. Wakimoto, Representations of superconformal algebras and mock theta functions Trans. Moscow Math. Soc. 2017, 9–74.

[36] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras. I, II, J. Amer. Math. Soc., 6 (1993), 905–947; 949–1011.

[37] H. Li, X. Xu, A characterization of vertex algebras associated to even lattices, J. Algebra 173 (1995) 253–270

[38] J. Rasmussen, Staggered and Kac modules over $A^{(1)}_1$, Nuclear Phys. B 950(2020), 114865, 46 pp.

[39] A. M. Semikhatov, A. Taormina, Twists and singular vectors in $sl(2|1)$ representations. Teor. Mat. Fiz. 128, 474–491 (2001); [Theor. Math. Phys. 128, 1236-251 (2001)]

[40] H. Yamada, H. Yamauchi, Simple Current Extensions of Tensor Products of Vertex Operator Algebras, Int. Math. Res. Not. 2021, no. 16, 12778–12807.

[41] H. Yamada, H. Yamauchi, $Z_{2k}$–code vertex operator algebras, J. of Alg. 573, 451–475 (2021)

[42] J. Van der Jeugt, J. W. B. Hughes, R. C. King, and J. Thierry-Mieg, A character formula for singly atypical modules of the Lie superalgebra $sl(m,n)$., Comm. Alg. 181990., 3454 –3480.

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