Resonances and bifurcations in axisymmetric scale-free potentials

Giuseppe Pucacco

Abstract

We investigate an analytical treatment of bifurcations of families of resonant ‘thin’ tubes in axisymmetric galactic potentials. We verify that the most relevant bifurcations are due to the (1:1) resonance producing the ‘inclined’ orbits through two different mechanisms: from the disc orbit and from the ‘thin’ tube associated with the vertical oscillation. The closest resonances occurring after these are the (4:3) resonance in the oblate case and the (2:1) resonance in the prolate case. The (1:1) resonances are treated in a straightforward way using a second-order truncated normal form. The higher order resonances are instead cumbersome to investigate, because the normal form has to be truncated to a high degree and the number of terms grows very rapidly. We therefore adopt a further simplification giving analytical formulae for the values of the parameters at which bifurcations ensue and compare them with selected numerical results. Thanks to the asymptotic nature of the series involved, the predictions are reliable well beyond the convergence radius of the original series.

Key words: methods: analytical – galaxies: kinematics and dynamics.

1 Introduction

Axisymmetric galactic potentials admit only tube orbits around the symmetry axis (Binney & Tremaine 2008). However, departing from the two linear approximations around the circular orbit, epicyclic motion in the symmetry plane and the oscillation perpendicular to this plane, non-linear dynamics are characterized by a hierarchy of resonances. To these the bifurcations of different kinds of ‘thin’ tubes which parent the corresponding ‘thick’ families are associated (Contopoulos 2004). The occurrence of this phenomenon may heavily affect the kinematics of stars with, for example, sudden removal from the disc (levitation) and/or velocity redistribution that can be viewed as a heating mechanism of the disc (Sridhar & Touma 1996). Resonant families may also play a relevant role in self-consistent models of spheroidal halo or disc/halo systems.

Scale-free potentials are often adopted as realistic galactic models (Richstone 1982; Evans 1994; Touma & Tremaine 1997; Sridhar & Touma 1999). The orbit structure is the same at each energy level, resulting in a much simpler description of dynamics. Their axisymmetric version is parametrized by only two parameters: α, the exponent of the power law (which can be taken zero in the limit case of the logarithmic potential), and q, the ellipticity of the isopotentials. Adding to these L, the conserved angular momentum around the symmetry axis, completely specifies the orbit structure. The ensuing two degrees of freedom (d.o.f.) dynamical system is, in general, non-integrable. An analytical approach can be attempted by expanding the effective potential around the critical point corresponding to the circular orbit and pushing the expansion beyond quadratic terms. The Hamiltonian series is afterwards replaced by a ‘normal form’ which plays the role of an integrable approximation of the original non-integrable system (Contopoulos 1963; Gustavson 1966; Gerhard & Saha 1991; Yanguas 2001). The dynamics of the normal form are amenable to a totally analytical treatment that allows us, among other things, to find the bifurcation thresholds of periodic orbits in terms of the parameters of the effective potential (Belmonte, Boccaletti & Pucacco 2006, 2007) and approximate expressions of the solutions of the equations of motion (Pucacco, Boccaletti & Belmonte 2008b).

In the present work, we investigate the properties of the simplest non-trivial series expansion of axisymmetric scale-free potentials beyond the epicyclic approximation. We verify that the most relevant bifurcations are due to the (1:1) resonance producing the ‘inclined’ orbits through two different mechanisms (Hunter et al. 1998): from the disc orbit (the ‘horizontal’ normal mode) and from the ‘thin’ tube associated with the z oscillation (the ‘vertical’ normal mode). The closest resonances occurring after these are the (4:3) resonance in the oblate case and the (2:1) resonance in the prolate case. Other high-order resonances can be present at very low values of the angular momentum.

The (1:1) resonances are treated in a straightforward way using a second-order truncated normal form. The higher order resonances are instead cumbersome to investigate, because the normal form has to be truncated to a high degree and the number of terms grows very rapidly. We therefore adopt a further simplification suggested by the approach of Sanders, Verhulst & Murdock (2007). We give therefore
analytical formulae for the values of \( \alpha, q, L \) at which bifurcations ensue and compare them with selected numerical results. Thanks to the asymptotic nature of the series involved (Pucacco, Boccaletti & Belmonte 2008a), the predictions are reliable well beyond the convergence radius of the original series.

The plan of this paper is as follows: in Section 2, we describe the family of systems under investigation and how they are prepared to undergo a perturbative analysis; in Section 3, we recall the method based on Hamiltonian normal forms; in Section 4, we apply the theory to compute bifurcation thresholds of the main resonant families; in Section 5, we conclude with a discussion of the results.

## 2 THE POTENTIALS AND THEIR SERIES EXPANSIONS

We investigate the dynamics of the family of potentials,

\[
\Phi_\alpha(R, z; q) = \begin{cases} 
\frac{1}{|q|} \left( R^2 + \frac{z^2}{q^2} \right)^{\alpha/2}, & \alpha \neq 0, \\
\frac{1}{2} \log \left( R^2 + \frac{z^2}{q^2} \right), & \alpha = 0.
\end{cases}
\]

(1)

The ellipticity of the equipotentials is determined by the parameter \( q \); we have an ‘oblate’ figure when \( q < 1 \) and a ‘prolate’ figure when \( q > 1 \). The slope of the power law will be restricted to the range \(-1 < \alpha \leq 2\).

The Hamiltonian of the system in cylindrical coordinates is

\[
H = \frac{1}{2} \left( p_R^2 + \frac{p_z^2}{R^2} + p_\phi^2 \right) + \Phi_\alpha,
\]

(3)

that, exploiting the conservation of the axial angular momentum

\[ p_\phi = L, \]

(4)

is effectively the Hamiltonian of a two d.o.f. system in the family of potentials

\[
V_\alpha(R, z; L, q) = \frac{L^2}{2R^2} + \Phi_\alpha(R, z; q).
\]

(5)

These potentials have a unique absolute minimum in

\[ R = R_c(\alpha) = L^{\frac{2}{1+\alpha}}, \quad z = 0, \]

(6)

This is a stable equilibrium on the meridional plane \( \phi = \) constant, corresponding to a circular orbit of radius \( R_c(\alpha) \) of the full three-dimensional problem. Since the dynamics are scale-free, we may fix energy once and for all

\[
E \equiv E_\alpha = \begin{cases} 
\left( \frac{1}{2} + \frac{1}{\alpha} \right) c^{-\frac{2}{\alpha}}, & \alpha \neq 0, \\
0, & \alpha = 0.
\end{cases}
\]

(7)

This implies that the radius of the circular orbit at this energy is

\[ R_c(\alpha) = c^{-\frac{2}{\alpha}}, \quad -1 < \alpha \leq 2, \]

(8)

and we can investigate the dynamics at

\[ E = E_\alpha, \forall \alpha \in (-1, 2], \]

(9)

by varying \( L \) in the range

\[ 0 < L \leq L_{\text{max}} = R_c^{\frac{2\alpha}{\alpha^2}} = \frac{L}{\sqrt{c}}, \]

(10)

without any loss of generality (Hunter et al. 1998).

In order to implement the perturbation method, we expand the effective potential around the minimum (6). A common attitude in the normal form theory holds up the reliability of its predictions well beyond the convergence radius of the expansion. Although a rigorous proof of this statement is still lacking, it is confirmed by several results obtained in analytical and celestial mechanics (Scuflaire 1995; Efthymiopoulos, Giorgilli & Contopoulos 2004; Pucacco et al. 2008a). One of the aims of the present work is to test the reliability of these asymptotic estimates.

We introduce rescaled coordinates according to

\[
x \equiv \frac{R - R_c}{R_c}, \quad y \equiv \frac{z}{R_c} \quad (11)
\]

with origin in the equilibrium point (6). The potential (5) is expanded as a truncated series (in the coordinates \( x, y \)) of the form

\[
V_\alpha^{(N)}(x, y; L, q) = \sum_{k=0}^N \sum_{j=0}^k C_{(j,k-j)}(L, \alpha, q)x^j y^{k-j},
\]

(12)

where the truncation order \( N \) is determined by the resonance under study and is discussed below. From (6) and the rescaling (11), the constant term of the expansion is

\[
C_{(0,0)}(L, \alpha) = \begin{cases} 
\left( \frac{1}{2} + \frac{2}{\alpha} \right) L^{\frac{2}{1+\alpha}}, & \alpha \neq 0, \\
\frac{1}{2} + L, & \alpha = 0,
\end{cases}
\]

(13)

and the other coefficients have the form

\[
C_{(j,k-j)}(L, \alpha, q) = L^{\frac{2}{1+\alpha}} c_{(j,k-j)}(\alpha, q).
\]

(14)

In order to simplify formulae, we introduce the new parameter

\[
\beta = -\frac{2\alpha}{2 + \alpha}, \quad -1 < \beta \leq 2,
\]

(15)

with the same range of \( \alpha \) in view of (2).

The orbit structure of the original family of potentials (5) at the energy level fixed by (7) will be approximated by the orbit structure of the rescaled Hamiltonian

\[
\tilde{H} = \frac{1}{2} \left( p_\phi^2 + p_x^2 \right) + \tilde{V}_\alpha^{(N)}(x, y; q),
\]

(16)

where

\[
\tilde{V}_\alpha^{(N)}(x, y; q) = L^\beta V_\alpha^{(N)}(x, y; q).
\]

(17)

The dynamics given by Hamiltonian (16) take place in the rescaled time

\[ \tau = t/L^{\beta + 1} \]

(18)

at the new ‘energy’

\[
\tilde{E} = L^\beta \left( E_\alpha - C_{(0,0)}(L, \alpha) \right) = \frac{1}{\beta} \left[ 1 - (L/L_{\text{max}})^\beta \right].
\]

(19)

According to (10), the singular value \( L = 0 \) is excluded from the analysis implying that the fictitious energy (19) is always finite and that the expansion around the equilibrium point (6) makes sense.

The non-vanishing coefficients of the expansion of \( \tilde{V}_\alpha^{(N)} \) up to order \( N = 4 \) are the following:

\[
c_{(2,0)} = \frac{2 + \alpha}{2},
\]

(20)

\[
c_{(0,2)} = \frac{1}{2q^2},
\]

(21)

\[
c_{(3,0)} = -\frac{10 + 3\alpha - \alpha^2}{6},
\]

(22)

\[
c_{(1,2)} = \frac{2 - \alpha}{2q^2}.
\]

(23)
The first two of them provide the frequencies of the epicyclic motions. Recalling the time rescaling in (18), the radial and vertical harmonic frequencies are, respectively,

\[ \kappa = \frac{\sqrt{\alpha + \alpha^2}}{L^{\beta+1}} \]  

and

\[ v = \frac{1}{qL^{\beta+1}}. \]  

### 3 Normal Forms

The Hamiltonian (16) is in the form of a power series and is therefore naturally apt to be treated in a perturbative way as a non-linear oscillator system. We construct a ‘normal form’ (Boccaletti & Pucacco 1999; Sanders et al. 2007), namely a new Hamiltonian series which is an integrable approximation of the original one, suitably devised to catch its most relevant orbital features.

The normal form is ‘non-resonant’ when the two harmonic frequencies (27) and (28) are generically non-commensurable: in this case, we get explicit formulae for actions and frequencies of the box orbits parented by the radial (disc) and by the vertical (thin-tube) orbits. A ‘resonant’ normal form is instead assembled by assuming from the start an integer value for the ratio of the harmonic frequencies and including in the new Hamiltonian terms depending on the corresponding resonant combination of the angles. This possibility might be considered an exception: it is instead the rule because, even if the unperturbed system is non-resonant with a certain real value

\[ \rho = \kappa/v \]  

of the frequency ratio, the non-linear coupling between the d.o.f. induced by the perturbation determines a ‘passage through resonance’ with a commensurability ratio, say \( m_1/m_2 \), corresponding to the local ratio of oscillations in the two d.o.f. This in turn is responsible for the birth of new orbit families bifurcating from the normal modes or from lower order resonances.

To evaluate the most relevant resonances in our case, let us come back to the epicyclic frequencies (27) and (28). Their ratio is

\[ \rho = q \sqrt{2 + \alpha}. \]  

We then approximate the frequencies with a rational number plus a small ‘detuning’ (Contopoulos & Moutsoulas 1966; de Zeeuw & Merritt 1983)

\[ \rho = \frac{m_1}{m_2} + \delta \]  

and proceed like the unperturbed harmonic part would be in exact \( m_1/m_2 \) resonance, putting the remaining part inside the ‘perturbation’. We speak of a detuned \((m_1/m_2)\) resonance, with

\[ N_{\text{max}} = m_1 + m_2 \]  

the order of the resonance. In general, a normal form truncated at \( N_{\text{max}} \) includes the first resonant term. With \( \alpha \) in the range (2) and an interval

\[ 0.5 < q < 1.5 \]  

of reasonable values of \( q \), we see that the most relevant resonance values with low commensurability are

\[ \frac{1}{2}, \frac{3}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \]  

In our analysis, we will mainly concentrate on the ‘central’ value 1:1 and on the higher order cases 2:1 and 4:3. In fact, periodic orbits with these frequency ratios and associated quasi-periodic families are usually the most prominent in numerical investigations (Hunter et al. 1998; Contopoulos 2004). The procedure can be easily extended to other cases.

Normal forms for the Hamiltonian system corresponding to (16) are constructed with standard methods (Boccaletti & Pucacco 1999) and were used to determine the main features of the orbit structure of the logarithmic potential (Belmonte et al. 2007; Pucacco et al. 2008b). We briefly resume the procedure in order to fix the notations.

After a scaling transformation

\[ x \rightarrow \sqrt{2 + \alpha} x, \quad p_x \rightarrow p_x/\sqrt{2 + \alpha}, \]  

\[ y \rightarrow y/\sqrt{q}, \quad p_y \rightarrow \sqrt{q} p_y, \]  

the original Hamiltonian (16) undergoes a canonical transformation to new variables \( P_x, P_y, X \) and \( Y \), such that

\[ K(P_x, P_y, X, Y) = \sum_{n=0}^{N} K_n, \]  

generated by a function of the form

\[ G = G_1 + G_2 + \cdots \]  

with the prescription (\( K \) in ‘normal form’)

\[ \{K_0, K\} = 0. \]  

In these and subsequent formulae, we adopt the convention of labelling the first term in the expansion with the index zero: in general, the ‘zero-order’ terms are quadratic homogeneous polynomials and terms of the order of \( n \) are polynomials of degree \( n + 2 \). The zero-order (unperturbed) Hamiltonian,

\[ K_0 = H_0 = \frac{1}{2} \left[ \alpha_1 (P_x^2 + X^2) + \alpha_2 (P_y^2 + Y^2) \right], \]  

with ‘unperturbed’ frequencies

\[ \omega_1 = \kappa L^{1+\beta} = \sqrt{2 + \alpha}, \quad \omega_2 = \nu L^{1+\beta} = 1/q, \]  

plays, by means of the fundamental equation (39), the double role of determining the specific form of the transformation and assuming the status of the second integral of motion. The lowest order term of the generating function, \( G_1 \), is a cubic polynomial.

Using ‘action-angle’-like variables, \( J, \theta \), defined through the transformation

\[ X = \sqrt{2J_1} \cos \theta_1, \quad Y = \sqrt{2J_2} \cos \theta_2, \]  

\[ P_X = \sqrt{2J_1} \sin \theta_1, \quad P_Y = \sqrt{2J_2} \sin \theta_2, \]  

the typical structure of the resonant normal form (truncated when the first resonant term appears) is (Twuankotta & Verhulst 2000; Contopoulos 2004; Sanders et al. 2007)

\[ K = m_1 J_1 + m_2 J_2 + \sum_{k=2}^{m_1+m_2} P^{(k)}(J_1, J_2) \]  

\[ + m_1 m_2 J_1^{m_1} J_2^{m_2} \cos[2(m_1 \theta_1 - m_2 \theta_2)]. \]  

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where $\mathcal{P}^{(k)}$ are homogeneous polynomials of degree $k$ whose coefficients may depend on $\delta$ and the constant $a_{mn2}(q, \alpha)$ is the only marker of the resonance. In these variables, the second integral is
\[
E = m_1 J_1 + m_2 J_2,
\]
and the angles appear only in the resonant combination
\[
\psi = m_2 \theta_1 - m_1 \theta_2.
\]
For a given resonance, these two statements remain true for arbitrary $N > N_{\min}$. Introducing the variable conjugate to $\psi$,
\[
\mathcal{R} = m_2 J_1 - m_1 J_2,
\]
the new Hamiltonian can be expressed in the reduced form $K(\mathcal{R}, \psi; E, q, \alpha)$, that is a family of one d.o.f. systems parametrized by $E$ (and $q, \alpha$).

In the applications below, we are interested in the global structure of phase space, but the explicit solution of the equations of motion is also of great relevance. For a non-resonant normal form, the problem is easily solved: the coefficient $a_{mn2}$ vanishes and $K$ no longer has a term containing angles. Therefore, $J$ are ‘true’ conserved actions and the solutions are
\[
X(\tau) = \sqrt{2J_1} \cos k_1 \tau, \quad Y(\tau) = \sqrt{2J_2} \cos (k_2 \tau + \theta_0),
\]
where
\[
\kappa = \nabla_f K
\]
is the frequency vector and $\theta_0$ is a suitable phase shift.

In the resonant case instead, it is not possible to write the solutions in closed form. It is true that the dynamics described by the one d.o.f. Hamiltonian $K(\mathcal{R}, \psi)$ are always integrable, but, in general, the solutions cannot be written in terms of elementary functions. However, solutions can still be written down in the case of the main periodic orbits, for which $J, \theta$ are true action-angle variables. There are two types of periodic orbits that can be easily identified.

(i) The normal modes for which one of the $J$ vanishes.

(ii) The periodic orbits in general position characterized by a fixed relation between the two angles, $m_2 \theta_1 - m_1 \theta_2 = \theta_0$.

In both cases, it is straightforward to check that the solutions retain a form analogous to equation (48) with known expressions of the actions and frequencies in terms of $E$ and the parameters $q$ and $\alpha$ such that $k_1/k_2 = m_1/m_2$. By using the generating function equation (38), the solutions in terms of standard ‘physical’ coordinates can be recovered (apart from possible scaling factors) inverting the canonical transformation. As discussed in Pucacco et al. (2008b), the transformation back to the physical coordinates is expressed as a series of the form
\[
x(\tau) = x_1 + x_2 + x_3 + \cdots,
\]
and is given explicitly by
\[
x_1 = X,
\]
\[
x_2 = \{G_1, X\},
\]
\[
x_3 = \{G_2, X\} + \frac{1}{2} \{G_1, \{G_1, X\}\}
\]
and so forth. From the knowledge of the normalized solutions (equation 48), we can therefore construct power series approximate solutions of the equations of motion of the original system
\[
d^2x\big|_{r^2} = -\nabla_x V^{(N)}(x, y; q).
\]

As a rule, normal modes exist on each ‘energy’ surface $K = \bar{E}$. Periodic orbits in general position exist instead only beyond a certain threshold, and we speak of a bifurcation ensuing from a detuned resonance. The bifurcation is usually described by a series expansion of the form
\[
\bar{E} = \sum_{k} a_k \delta^k,
\]
where the $a_k$ are coefficients depending on the order $N_{\min}$ and the parameters $q, \alpha$. The order of truncation of the series is itself related to that of truncation of the normal form (Pucacco et al. 2008a). Equation (55) implies that at exact resonance (vanishing detuning) the bifurcation is intrinsic in the system and that, going away from the initial exact ratio of unperturbed frequencies, gradually increases the threshold value for the bifurcation. We will see that already a linear relation given by the first-order truncation provides a reliable estimate and will examine some example of a second-order truncation.

4 APPLICATIONS

We apply the theory resumed in the previous section to a set of typical bifurcation problems related to the potentials of Section 2. In the following, we adopt a ‘taxonomy’ of the main resonant families that is coherent with our previous work and other standard references in the field (Miralda-Escudé & Schwarzschild 1989; Belmonte et al. 2007). In case, we mention alternative denominations adopted by other authors.

4.1 Non-resonant box orbits

We recall that we are studying the dynamics given by Hamiltonian (16) at ‘energy’ (19): a small value of it corresponds to a value of the angular momentum close to its maximum amount. This is the quasi-harmonic regime in which the dynamics in the $x$-$y$ plane are characterized by box orbits oscillating around the two normal modes, $J_1 = 0$ and $J_2 = 0$. The first one is the $y$-axis periodic orbit and, coming back to the unscaled variables, it is the ‘vertical’ $z$ oscillation: in three dimensions, it gives the thin-tube orbits. The second one is the $x$-axis periodic orbit that in the unscaled variables is the horizontal or ‘equatorial’ $R$ oscillation and gives the disc orbits in three dimensions.

A preliminary step is that of exploiting the non-resonant normal form. It can readily be used to get the post-epicyclic solution; more interestingly, we will exploit it later to attempt an approximate description of the higher order resonances. The non-resonant normal form of the Hamiltonian up to the second order takes the following form:
\[
K = J_0 J_1 + a_2 J_2 + a J_1^2 + b J_1^2 + J_1 J_2,
\]
with
\[
a = \frac{3}{4a_0^2} \left[ 2c_{(4,0)} a_0^2 - 5 c_{(3,0)}^2 \right],
\]
\[
b = \frac{6c_{(2,0)} a_0^2}{4a_0^2} \left[ \omega_1^2 - 4 \omega_2^2 \right] - c_{(1,2)} \left( 4 a_0^2 - 8 a_2^2 \right),
\]
\[
c = \frac{2}{a_0^2} \left[ 3c_{(1,2)} c_{(3,0)} - c_{(2,0)} a_0^2 \right] \left( \omega_1^2 - 4 \omega_2^2 \right).
\]
In the coefficients $b$ and $c$ it is appreciable the appearance of ‘small’ denominators related to the 2:1 resonance, which is the first to
appear in view of the lowest order coupling term $xy^2$. We also give the first term of the generating function (38)

$$G_1 = \frac{c_{1,0}P_X (2 P_X^2 + 3 X^2)}{\omega_1^{3/2}} + \frac{c_{1,2}}{\omega_1^{3/2} \omega_2} (\omega_1^2 - 4 \omega_2^2) \right) P_Y (2 \omega_1^2 + \omega_2^2) + \frac{2 \epsilon_{1,2}}{\omega_1^{3/2}} (\omega_1^2 - 4 \omega_2^2),$$

Using expressions (22–26) and (41), the coefficients in the normal form can be written explicitly in terms of the parameters of the potential

$$a = -22 + 13 \alpha - \alpha^2,$$

$$b = \frac{(2 - \alpha)[10 + \alpha - 3 \alpha^2(2 + \alpha)]}{4 \alpha(2 + \alpha)[1 - 4 + \alpha^2(2 + \alpha)]},$$

$$c = \frac{(2 - \alpha)[6 - \alpha - \alpha(2 + \alpha)]}{2 \sqrt{2 + \alpha(1 - 4 + \alpha^2(2 + \alpha))}}.$$

The second-order post-epicyclic solution is then given by the upgraded expression of the two frequencies of the radial and vertical oscillations

$$\kappa = \frac{1}{L_{\beta+1}} \left( \sqrt{2 + \alpha + 2 a J_1 + c J_2} \right)$$

and

$$\nu = \frac{1}{L_{\beta+1}} \left( \frac{1}{q} + 2 b J_2 + c J_1 \right)$$

and by the orbit approximations (51 and 52) computed by means of the generating function (60). These results are the basis for attempting an accurate tracking of normal modes (Contopoulos & Seimenis 1990) and box orbits (Kent & de Zeeuw 1991).

### 4.2 Bifurcations from the disc and from the thin-tube orbit

We now start to illustrate the main body of results concerning the orbit structure as determined by the main bifurcations.

Lowering the angular momentum (namely increasing the fictitious energy $E$), both normal modes may lose their stability through a 1:1 resonance. We denote the bifurcating family as the inclined orbit in view of its natural interpretation as the in phase ($\psi = 0$) 1:1 resonance of the two oscillations (de Zeeuw & Merritt 1983; Belmonte et al. 2007). The antiphase ($\psi = \pi$) 1:1 resonant loops never appear in these systems, at least as a stable family (see Section 4.3). The inclined periodic orbits parent two families of inclined boxes that may arrive quite far from the equatorial plane both above and below the disc: this phenomenon is called levitation (Sridhar & Touma 1996). We recall that our inclined orbits have also been referred to as reflected banana by Lees & Schwarzschild (1992) and Evans (1994) and simply as banana by Hunter et al. (1998): we prefer to leave this term as the standard denomination (Miralda-Escudé & Schwarzschild 1989) for the 2:1 resonance.

To describe the bifurcation of the inclined orbit, the normal form is computed by a small ‘detuning’ (Contopoulos & Moutsoulas 1966; de Zeeuw & Merritt 1983) of the 1:1 resonance so that, from (31) with $m_1 = m_2 = 1$,

$$\rho = q \sqrt{2 + \alpha} = 1 + \delta.$$

The normal form truncated to the first non-zero resonant term is

$$\tilde{K} = q K = J_1 + J_2 + \delta J_1 + q \left( a J_1^2 + b J_2^2 + c J_1 J_2 \right) + d J_1 J_2 \cos(2(\theta_1 - \theta_2)),$$

with $a, b, c$ as in (63–65) and

$$d = \frac{q^2}{12} (1 + \alpha)(2 - \alpha).$$

The rescaling by $q$ lets to embody the detuned resonance (68) in the neatest form. Introducing the resonant combinations

$$\psi = 2(\theta_1 - \theta_2),$$

and

$$\mathcal{R} = J_1 - J_2,$$

the new Hamiltonian can be expressed in the reduced form

$$\tilde{K} = \mathcal{E} + \frac{1}{2} \delta (\mathcal{E} + \mathcal{R}) + A (\mathcal{E}^2 + \mathcal{R}^2) + B \mathcal{E} \mathcal{R}$$

$$+ \frac{1}{4} (\mathcal{E}^2 - \mathcal{R}^2) (qc + d \cos \psi),$$

with

$$A = \frac{q^2}{4} (x + b),$$

$$B = \frac{q^2}{2} (a - b),$$

and

$$\mathcal{E} = J_1 + J_2$$

is the second integral as in (45). Considering the dynamics at a fixed value of $\mathcal{E}$, we have that $\tilde{K}$ defines a one d.o.f. $(\psi, \mathcal{R})$ system with the following equations of motion:

$$\dot{\psi} = 2(\theta_1 - \theta_2)$$

$$\dot{\mathcal{R}} = \mathcal{K} = \frac{d}{4} (\mathcal{E}^2 - \mathcal{R}^2) \sin \psi.$$
(2007), it can be proven that, at the bifurcation, one of the normal modes suffers a stability-instability transition. The more common situation (for models ranging from sensibly oblate to prolate) is that in which the $y$-axis becomes unstable and the inclined appears as a pitchfork bifurcation from the disc orbit. The passage to instability of the $y$-axis is possible only for strongly oblate models and gives rise to a pitchfork bifurcation from the thin tube.

The bifurcation equations (83) and (84) determine critical values of $\dot{E}$ in terms of the parameters $q, \alpha$. To make a quantitative prediction, we want an expression for the corresponding critical angular momentum. The approach we have followed so far is altogether a perturbation approach truncated to the first non-trivial order. Therefore, it is natural to look for expansions truncated to the first order in the detuning parameter. Taking into account the rescaling in $\delta$ and the expressions (63–65) and (71), the first-order expansions of the critical values of the fictitious energy (19) are

$$\dot{E} = \begin{cases} \frac{12(2+\alpha)}{5(2-\alpha+q^2)}\delta, & \delta < 0, \\ \frac{6(2+\alpha)}{5(2-\alpha+q^2)}\delta, & \delta > 0. \end{cases}$$

The first solution corresponds to the bifurcation from the thin tube and the second one corresponds to the bifurcation from the disc. These are examples of series of the form (55) truncated to the first order.

Using the relation between $\dot{E}$ and $L$ established by (19), we get the following expressions for the critical values of the angular momentum below which inclined orbits exist:

$$L_{\text{crit}} = \frac{1}{\sqrt{\dot{E}}} \times \begin{cases} \left[ 1 - \frac{2\alpha(q\sqrt{q^2-1} - 1)}{5(2-\alpha+q^2)} \right]^{-1/2}, & q < \frac{1}{\sqrt{2}}, \\ \left[ 1 + \frac{\alpha(2\sqrt{q^2+1} - 1)}{5(2-\alpha+q^2)} \right]^{-1/2}, & q > \frac{1}{\sqrt{2}}. \end{cases}$$

It is also useful to write the limiting case of the logarithmic potential ($\alpha = 0$):

$$L_{\text{crit}} = \begin{cases} e^{\frac{q^2}{2} + \frac{1}{2}\sqrt{q^2}}, & q < \frac{1}{\sqrt{2}}, \\ e^{-\frac{q^2}{2} - 6\sqrt{q^2}}, & q > \frac{1}{\sqrt{2}}. \end{cases}$$

A comparison with the outcome of numerical determinations of the bifurcation threshold allows us to evaluate the accuracy of these analytical predictions. In Table 1, the critical value of the angular momentum for the bifurcation of the inclined orbits, computed with equation (86) for general $\alpha$ and with equation (87) for $\alpha = 0$, is compared with the numerical data obtained either from published works (Lees & Schwarzschild 1992; Evans 1994; Hunter et al. 1998) or by numerical computations made for this paper. In this case, the bifurcation has been detected tracing the instability threshold of the normal mode by means of the Floquet method (Bender & Orszag 1978).

The accuracy is particularly good when the model is close to the exact resonance. Overall, the discrepancy linearly grows with detuning, as can be expected in this first-order approach. The first two lines represent two strongly oblate models with the thin tube to become unstable: in the rather extreme case with $q = 0.4, \alpha = 0.5$ when the detuning is $\delta = -0.37$ and the relative error in the prediction is 18 per cent. In all other cases, the disc becomes unstable and the quality of the prediction can be represented by the case with $q = 0.8, \alpha = 0.1$ when the detuning is $\delta = 0.16$ and the relative error in the prediction is 8 per cent. We may guess a prediction error

$$\frac{L_{\text{crit,true}} - L_{\text{crit}}}{L_{\text{crit,true}}} \approx \frac{1}{2} \delta = \frac{1}{2} (q\sqrt{q^2+1} - 1),$$

which can be used to further improve the accuracy of (86).

### 4.3 Return to stability of the thin tube

A pitchfork or period-doubling bifurcation is usually followed by a second stability change when the second resonant family appears (Miralda-Escudé & Schwarzschild 1989). In this setting, this possibility occurs if the loops appear. Using the bifurcation equations (83) and (84) with the ‘+’ sign in front of $\delta$ and the explicit expressions of (63–65), the inequality can be satisfied only with values of the parameters corresponding to rather extreme oblate models. This implies a negative value of the detuning and a critical fictitious energy

$$\dot{E} = -\frac{4(2+\alpha)}{2 + \alpha - \alpha^2}\delta.$$

We get the following expression for the critical value of the angular momentum below which loops bifurcate from the thin tube:

$$L_{\text{crit}} = \frac{1}{\sqrt{\dot{E}}} \left[ 1 - \frac{8\alpha(q\sqrt{q^2+1} - 1)}{2 + \alpha - \alpha^2} \right]^{-1/2}.$$

This is again a pitchfork bifurcation so that the thin tube regains its stability. The loops are unstable and, lowering the angular momentum below the critical value, remain unstable for every reasonable combination of the parameters. On the same ground, the inclined tend to occupy an even larger fraction of phase space. Referring again to the models in Table 1, in the case with $q = 0.4, \alpha = 0.5$ the critical value for the return to stability is 0.32 (Evans 1994) and (90) predicts 0.18.

### 4.4 Bifurcation of the inner thin tube

A different phenomenon is that of the bifurcation of the inner thin tube. This happens only in prolate models and is due to a period-doubling bifurcation from the equatorial orbit which becomes unstable at the bifurcation. In our terminology, this is a banana orbit actually related to the 2:1 resonance. In the full three-dimensional problem, it gives rise to a thin tube that is always closer to the symmetry axis than the ‘outer’ thin tube examined above which remains always stable (de Zeeuw 1985).

In this case, the detuning, from (30) and (31) with $m_1 = 2, m_2 = 1$, is

$$\delta = q\sqrt{q^2+\alpha} - 2.$$

\[\text{(91)}\]
The normal form truncated to the first non-zero resonant term is
\[
\tilde{K} = 2J_1 + J_2 + \delta J_1 + D \sqrt{J_1 J_2} \cos(\theta_1 - 2\theta_2)
\]  
(92)

with
\[
D = \frac{q^2 \Gamma_{(1,2)}}{\sqrt{2(2 + \alpha)^3}} = \frac{\alpha - 2}{2\sqrt{2(2 + \alpha)^3}}.
\]  
(93)

Introducing the resonant combinations
\[
\psi = \theta_1 - 2\theta_2
\]  
(94)

and
\[
\mathcal{R} = J_1 - 2J_2,
\]  
(95)

the new Hamiltonian can be expressed in the reduced form
\[
\tilde{K} = \mathcal{E} + \frac{1}{5} \delta(2\mathcal{E} + \mathcal{R}) + \frac{D}{\sqrt{5}} \sqrt{2\mathcal{E} + \mathcal{R}}(\mathcal{E} - 2\mathcal{R}) \cos \psi,
\]  
(96)

where now
\[
\mathcal{E} = 2J_1 + J_2.
\]  
(97)

The fixed points corresponding to the periodic orbits in general position are given by the solutions of
\[
D(7\mathcal{E} + 6\mathcal{R}) - 2\sqrt{3}\sqrt{2\mathcal{E} + \mathcal{R}}\delta = 0, \quad \psi = 0,
\]  
(98)

\[
D(7\mathcal{E} + 6\mathcal{R}) + 2\sqrt{3}\sqrt{2\mathcal{E} + \mathcal{R}}\delta = 0, \quad \psi = \pi.
\]  
(99)

Combining these with the existence conditions
\[
0 \leq J_1 \leq \mathcal{E}/2, \quad 0 \leq J_2 \leq \mathcal{E},
\]  
(100)

gives the critical value
\[
\mathcal{E}_{\text{crit}} = \frac{1}{2D} \delta^2
\]  
(101)

and, taking into account the rescaling implicit in the normal form,
\[
\tilde{\mathcal{E}} = \frac{2(2 + \alpha)}{(2 - \alpha)^2} \delta^2.
\]  
(102)

We now have an example of a series of the form (55) in which the first coefficient vanishes and is therefore truncated to the second order. Using (19), the prediction of the value of the angular momentum below which the inner thin tube bifurcates from the disc is
\[
L_{\text{crit}} = \frac{1}{\sqrt{\mathcal{E}}} \left[ 1 + 4\alpha(q\sqrt{2 + \alpha} - 2)^2 \right]^{\frac{1}{2q^2}}.
\]  
(103)

The pure logarithmic \(\alpha = 0\) limit is given by
\[
L_{\text{crit}} = e^{\frac{1}{4q^2} + \frac{1}{2q} - 2q^2}.
\]  
(104)

In Table 2, the critical value of the angular momentum for the bifurcation of the inner thin-tube orbits, computed with equation (103), is compared with the numerical data. We may guess a prediction error
\[
\frac{L_{\text{crit, true}} - L_{\text{crit, true}}}{L_{\text{crit, true}}} \simeq 2\delta^2 = 2(q\sqrt{2 + \alpha} - 2)^2,
\]  
(105)

which can be used to further improve the accuracy of (103).

| \(q\) | \(\alpha\) | Analytical | Numerical | Source |
|-----|-----|-----------|-----------|-------|
| 1.1 | 0.25 | 0.51      | 0.41      | E     |
| 1.1 | 0    | 0.50      | 0.37      | T     |

4.5 Bifurcations of higher order resonant orbits

As an application of the approach adopted by Sanders et al. (2007), we compute the appearance of the pretzel as a very high order (4:3) resonance. In the literature, it is also denoted as a ‘reflected’ fish (Lees & Schwarzschild 1992; Evans 1994). In spite of the high order, this family happens to play a relevant role in shaping the phase space of these Systems, occupying a substantial fraction of surfaces of section for a wide range of parameters in oblate models (Lees & Schwarzschild 1992; Evans 1994; Hunter et al. 1998).

The implementation of the general procedure followed in all cases treated above would require, in the present instance, a normal form truncated to \(N_{\text{min}} = 7\) in order to include at least the first resonant terms. The algebraic manipulators available nowadays have no problem in accomplishing this endeavour without excessive CPU times. However, this implies a huge number of terms which hinder a completely general algebraic approach. The main troubles come from the solution of the equations for the fixed points of the reduced system, giving in turn the periodic orbits in general position. Being polynomials of degree one less than the normal form in the action variables, these give rise to algebraic equations of which it is very difficult to write the solutions in a manageable way. A straightforward approach would be that of examining specific cases and numerically solving for their roots. However, this spoils this general method of all its appeal. An alternative approach is to give up on a detailed knowledge of the periodic orbits in general position and try an approximate location of their ‘resonance manifold’ together with its first-order condition of existence (Sanders et al. 2007).

We recall that, for a general reduced Hamiltonian of the form \(K(R, \psi; \mathcal{E}, q, \alpha)\), the fixed points with \(\psi = 0\) and \(+\pi\) correspond to different orbit sets with two different values of \(R(0)\) and \(R(\pi)\). They can be separately identified only if the normal form contains at least the first resonant term. The method adopted by Sanders et al. (2007), by truncating at a lower order term, only allows us to find a ‘average’ value of the two \(R\) coordinates together with an existence condition, a linear or quadratic equation in the simplest instances. However, it involves the detuning and the second approximate integral, and therefore provides an estimate of the critical energy for the bifurcation.

In this case, the detuning, from (30) and (31) with \(m_1 = 4\) and \(m_2 = 3\), is
\[
\delta = q\sqrt{2 + \alpha} - \frac{4}{3}.
\]  
(106)

The truncated normal form is
\[
\tilde{K} = 4J_1 + 3J_2 + 3\delta J_1 + 3q (a J_1^2 + b J_2^2 + c J_1 J_2) + \cdots
\]  
(107)

where the second-order term is essentially the same as in the non-resonant normal form (56) and the dots stand as a remainder of the possible necessity of continuing the expansion in case the quality of the approximation is too low. In this case, one can simply choose
\[
R = -J_1
\]  
(108)

and
\[
\mathcal{E} = 4J_1 + 3J_2.
\]  
(109)

The non-constant terms of the reduced Hamiltonian are
\[
-3\delta R + q \left[ (a + \frac{16b}{3} - 4c) R^2 + \left( \frac{8b}{3} - c \right) R \right] \mathcal{E} R + \cdots
\]  
(110)
The resonance manifold is determined by the condition
\[ \dot{\psi} = K \dot{\psi} = 0, \]
so that we have
\[ R_{\text{RM}} = \frac{27 \delta - 8b \varepsilon q + 3c \varepsilon q}{2(9a + 16b - 12c)q}. \]  
(111)

Combining this with the existence condition
\[ \frac{1}{4} \varepsilon \leq R_{\text{RM}} \leq 0, \]  
(112)

the rescaling \( \tilde{E} = 3q^2 \varepsilon \) and the expressions (63–65), we get the following second-order approximation for the critical 'energy'
\[ \tilde{E}_{\text{RM}} = -\frac{15}{\sqrt{2(2 + \alpha)}} \delta - \frac{21}{2 + \alpha} \delta^2. \]  
(113)

Using (19), the prediction of the value of the angular momentum below which the inner thin tube bifurcates from the disc is
\[ L_{\text{crit}} = \frac{1}{\sqrt{\varepsilon}} \left[ 1 - \beta \tilde{E}_{\text{RM}}(q, \alpha) \right]^{1/2}. \]  
(114)

In Table 3, the critical value of the angular momentum for the bifurcation of the inclined orbits, computed with equation (114), is compared with the numerical data. The accuracy reached is unexpectedly good in view of the rough nature of the approximating technique. This method could also be employed to investigate the properties of resonant boxlets of triaxial ellipsoids (Zhao 1999; Zhao, Carollo & De Zeeuw 1999).

### Table 3

Critical value of the angular momentum for the bifurcation of the 4:3 pretzel (reflected fish). The sources are as in the caption of Table 1.

| q     | \( \alpha \) | \( L_{\text{crit}} \) | Analytical | Numerical | Source |
|-------|-------------|------------------------|------------|-----------|--------|
| 0.75  | 0           | 0.17                   | 0.173      | LS        |
| 0.8   | -0.3        | 0.10                   | 0.15       | H         |
| 0.85  | -0.18       | 0.18                   | 0.16       | E         |
| 0.9   | 0           | 0.40                   | 0.38       | T         |

5 DISCUSSION AND CONCLUSIONS

Motion in the meridional plane of an axisymmetric system is a stumbling block and a case study in dynamical astronomy. Its investigation has a long history and has inspired the development of many ideas and methods (Contopoulos 1963; Hénon & Heiles 1964; Gustavson 1966; Hori 1966; Verhulst 1979). The development of techniques to investigate the regular and chaotic structure of phase space and of approximating integrals of motion (a ‘third integral’ in this case, in addition to \( E \) and \( L \)) pays much to the activity on this topic. In particular, a systematic investigation of the regular dynamics by constructing Hamiltonian ‘normal forms’ started with the work of Hori (1966) and Gustavson (1966) and, suitably traduced in algorithms of normalization, is at the basis of many subsequent works including the present one.

Here, we have shown how a simple first-order truncation of the normal form is able to convey not only qualitative information on the phase-space structure but also quantitative predictions with concrete meaning for applications in galactic dynamics. In specific situations, especially in non-scale-free models, painstakingly extended numerical simulations are usually required to even get a flavour of what is going on. A simple albeit rough analytical recipe can therefore be quite useful to get an average picture. With this approach, in place of numerical simulations quite heavy in view of the large parameter space, simple analytical formulae are presented for the computation of the main bifurcation thresholds of the problem. For each scale-free potential with slope given by \( \alpha \) and ellipticity \( q \), a critical value of the angular momentum of the full three-dimensional system determines the appearance of some new orbit family and, possibly, a change in the stability nature of the ‘parent’ orbit. The approach can clearly be generalized to non-scale-free models at the price of a fourth parameter in addition to \( L \), \( \alpha \) and \( q \).

Overall, these results make us confident that, in order to improve the accuracy or to explore the features of minor families and/or higher order resonances, it can be worth the effort of constructing and investigating complete resonant normal forms. However, it is important to remark that the asymptotic nature of the results implies a trade-off between accuracy and extent of the phase-space region that is mimicked by the normal form. This region is largest at some optimal truncation order (Efthymiopoulos et al. 2004) that depends on the nature of the system and the order of the resonance. The accuracy in the predictions of, for example, the bifurcation thresholds can be further improved with respect to that at the optimal order but only at the expense of reconstruction of dynamics. The choice of the whole strategy is then determined by the needs of the problem at hands.

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