Gradient Estimates for the Nonlinear Parabolic Equation with Two Exponents on Riemannian Manifolds

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Abstract. In this paper, we study the nonlinear parabolic equation with two exponents on complete noncompact Riemannian manifolds. The special types of such equation include the Fisher-KPP equation, the parabolic Allen-Cahn equation and the Newell-Whitehead equation. We get the Souplet-Zhang’s gradient estimates for the positive solutions to such equation. We also obtain the Liouville theorem for positive ancient solutions. Our results extend those of Souplet-Zhang (Bull. London. Math. Soc. 38 (2006), 1045–1053) and Zhu (Acta Math. Sci. Ser. B 36 (2016), no. 2, 514–526).

1. Introduction

Let $M$ be a complete noncompact Riemannian manifold. In this paper, we consider the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u(x,t) + \lambda(x,t)u^p + \eta(x,t)u^q$$

on $M$, where the functions $\lambda$ and $\eta$ are $C^1$ in $x$ and $C^0$ in $t$, $p$ and $q$ are positive constants with $p \geq 1$, $q \geq 1$. If $\lambda = -\eta = c$, $p = 1$ and $q = 2$, where $c$ is a positive constant, then the equation (1.1) becomes

$$\frac{\partial u}{\partial t} = \Delta u + cu(1-u)$$

which is called the Fisher-KPP equation [6, 11]. It describes the propagation of an evolutionarily advantageous gene in a population and has many applications. Cao, Liu, Pendleton and Ward [4] derived some differential Harnack estimates for positive solutions to (1.2) on Riemannian manifolds. Geng and the author [7] extended the result of [4]. If $\lambda = 1$, $\eta = -1$, $p = 1$ and $q = 3$, then (1.1) becomes

$$\frac{\partial u}{\partial t} = \Delta u - (u^3 - u)$$

which is called the parabolic Allen-Cahn equation. A Harnack inequality for this equation was studied in [1]. The gradient estimates for the elliptic Allen-Cahn equation on
Riemannian manifolds were obtained by the author in [9]. The special type of (1.1) also includes the Newell-Whitehead equation [16]

\[
\frac{\partial u}{\partial t} = \Delta u + au - bu^3
\]

where \(a\) and \(b\) are positive constants. It is used to model the change of concentration of a substance. The reader may refer to [2] for the recent results for such equation.

The gradient estimate is an important method in study on parabolic and elliptic equations. It was first proved by Yau [19] and Cheng-Yau [5], and was further developed by Li-Yau [13], Li [12], Hamilton [8], Negrin [15], Souplet and Zhang [17], Ma [14], Yang [18], etc. In [17], Souplet and Zhang considered the heat equation

(1.3) \[
\frac{\partial u}{\partial t} = \Delta u
\]

and proved the following result.

**Theorem 1.1.** Let \(M\) be an \(n\)-dimensional Riemannian manifold with \(n \geq 2\) and \(\text{Ricci}(M) \geq -k, k \geq 0\). If \(u\) is any positive solution to (1.3) in \(Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, +\infty)\) and \(u \leq N\) in \(Q_{R,T}\), then there holds

\[
\frac{|\nabla u(x,t)|}{u(x,t)} \leq c \left( \frac{1}{R} + \frac{1}{T^{1/2}} + \sqrt{k} \right) \left( 1 + \log \frac{N}{u(x,t)} \right)
\]

in \(Q_{R/2,T/2}\), where \(c = c(n)\).

Later, using the method of Souplet and Zhang, Zhu [20] studied the equation

(1.4) \[
\left( \Delta - \frac{\partial}{\partial t} \right) u(x,t) + h(x,t)u^p(x,t) = 0, \quad p > 1
\]
on compete noncompact Riemannian manifolds, where the function \(h(x,t)\) is assumed to be \(C^1\) in the first variable and \(C^0\) in the second variable. He proved the following result.

**Theorem 1.2.** Let \(M\) be an \(n\)-dimensional Riemannian manifold with \(n \geq 2\) and \(\text{Ricci}(M) \geq -k, k \geq 0\). If \(u\) is any positive solution to (1.4) in \(Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, +\infty)\) and \(u \leq N\) in \(Q_{R,T}\), then for any \(\beta \in (0, 2)\), there exists a constant \(c = c(n,p,\beta)\) such that

(1.5) \[
\frac{|\nabla u(x,t)|^2}{u(x,t)^\beta} \leq cN^{2-\beta} \left( \frac{1}{R^2} + \frac{1}{T} + k + N^{p-1}\|h^+\|_{L^\infty(Q_{R,T})} + N^{2(p-1)}\|\nabla h\|^2_{L^\infty(Q_{R,T})} \right)^{\frac{2}{3}}
\]
in \(Q_{R/2,T/2}\), where \(h^+ = \max\{h,0\}\).
The same method was also used by Huang and Ma [10] to obtain gradient estimates for the equations
\[ \frac{\partial u}{\partial t} = \Delta u + \lambda u^\alpha \quad \text{and} \quad \frac{\partial u}{\partial t} = \Delta u + au \log u + bu \]
under the Ricci flow, where \( \lambda, \alpha, a \) and \( b \) are constants.

In this paper, we get the following result.

**Theorem 1.3.** Let \( M \) be an \( n \)-dimensional Riemannian manifold with \( n \geq 2 \) and \( \text{Ricci}(M) \geq -k, \ k \geq 0 \). Suppose that \( \lambda(x,t) \) and \( \eta(x,t) \) are \( C^1 \) in \( x \) and \( C^0 \) in \( t \), \( p \) and \( q \) are positive constants with \( p \geq 1 \), \( q \geq 1 \). If \( u \) is any positive solution to (1.1) in \( Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, +\infty) \) and \( u \leq N \) in \( Q_{R,T} \), then there exists a constant \( c = c(n, p, q) \) such that

\[
|\nabla u(x,t)| \leq c \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2}\|\lambda^+\|_{L^\infty(Q_{R,T})} + N^{(q-1)/2}\|\eta^+\|_{L^\infty(Q_{R,T})}ight)
\]

\[
+ N^{1/4(p-1)}\|\nabla \lambda\|_{L^{1/3}(Q_{R,T})} + N^{1/4(q-1)}\|\nabla \eta\|_{L^{1/3}(Q_{R,T})} \left( 1 + \log \frac{N}{u} \right)
\]

in \( Q_{R/2,T/2} \), where \( \lambda^+ = \max\{\lambda, 0\} \), \( \eta^+ = \max\{\eta, 0\} \).

Note that the estimate (1.5) is equivalent to
\[
\frac{|\nabla u(x,t)|}{u(x,t)} \leq c \left( \frac{N}{u} \right)^{1-\beta/2} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2}\|h^+\|_{L^\infty(Q_{R,T})} + N^{1/4(p-1)}\|\nabla h\|_{L^{1/3}(Q_{R,T})} \right).
\]

Applying Theorem 1.3 to (1.4) yields
\[
\frac{|\nabla u(x,t)|}{u(x,t)} \leq c \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2}\|h^+\|_{L^\infty(Q_{R,T})} + N^{1/4(p-1)}\|\nabla h\|_{L^{1/3}(Q_{R,T})} \right) \left( 1 + \log \frac{N}{u} \right).
\]

Since \( \lim_{x \to +\infty} \frac{\log x}{x^{1-\beta/2}} = 0 \), if \( N/u \) is large enough, then we have
\[
1 + \log \frac{N}{u} \leq \left( \frac{N}{u} \right)^{1-\beta/2}.
\]

So in this sense, the estimate (1.6) improves (1.5).

We also get the Liouville type theorem.

**Theorem 1.4.** Let \( M \) be an \( n \)-dimensional Riemannian manifold with nonnegative Ricci curvature. Suppose that \( \lambda, \eta \) are nonpositive constants and one of them is negative, then (1.1) does not admit any positive ancient solution with \( u(x,t) = e^{o(d(x) + \sqrt{|t|})} \) near infinity.

The method of the proofs of main theorems comes from [10,17,20].
2. Proofs of main theorems

2.1. Proof of Theorem 1.3

Let \( \tilde{u} = u/N \). Then \( \tilde{u} \) satisfies

\[
\frac{\partial \tilde{u}}{\partial t} = \Delta \tilde{u} + \tilde{\lambda}u^p + \tilde{\eta}u^q
\]

where \( \tilde{\lambda} = \lambda N^{p-1} \), \( \tilde{\eta} = \eta N^{q-1} \). Noting \( \tilde{u} \leq 1 \), we let

\[
f = \log \tilde{u}, \quad \omega = |\nabla \ln(1 - f)|^2.
\]

In view of (2.1), we have

\[
f_t = 2u/N = \lambda + \eta N = \frac{\partial u}{\partial t} + \Delta u + \tilde{\lambda}u^p + \tilde{\eta}u^q = 0.
\]

By (2.2) and (2.3), we have

\[
\omega_t = \frac{2f_i(f_t)_i}{(1 - f)^2} + \frac{2f_j f_t}{(1 - f)^3} = 2f_i(f_{jj} + 2f_j f_{ji} + \tilde{\lambda}e^{(p-1)f} + \tilde{\eta}(p-1)e^{(p-1)f} f_i + \tilde{\eta}e^{(q-1)f} + \tilde{\eta}(q-1)e^{(q-1)f} f_i)}{(1 - f)^2} + \frac{2f_j^2 (f_{ii} + \tilde{\lambda}e^{(p-1)f} + \tilde{\eta}e^{(q-1)f})}{(1 - f)^3}.
\]

It follows from the similar calculation that

\[
\Delta \omega = \frac{2f_{ij}^2 + 2f_{j} f_{jii} + 8f_i f_j f_{ij} + 2f_{j}^2 f_{ii}}{(1 - f)^2} + \frac{6f_j^2 f_j^2}{(1 - f)^4} = 2f_{ij}^2 + 2f_{j} f_{jii} + 2R_{ij} f_i f_j + \frac{8f_i f_j f_{ij} + 2f_{j}^2 f_{ii}}{(1 - f)^3} + \frac{6f_j^2 f_j^2}{(1 - f)^4}
\]

where Bochner’s identity is used. Noting that \( R_{ij} f_i f_j \geq -k f_i^2 \), we have

\[
\Delta \omega - \omega_t \geq \frac{2f_{ij}^2 - 4f_i f_j f_{ij} - 2e^{(p-1)f} f_i \tilde{\lambda}_i - 2\tilde{\lambda}(p-1)e^{(p-1)f} f_i^2}{(1 - f)^2} - \frac{2e^{(q-1)f} f_i \tilde{\eta}_i + 2\tilde{\eta}(q-1)e^{(q-1)f} f_i^2 + 2k f_i^2}{(1 - f)^2} + \frac{8f_i f_j f_{ij} - 2f_{j}^2 f_i^2 - 2\tilde{\lambda}e^{(p-1)f} f_j^2 - 2\tilde{\eta}e^{(q-1)f} f_j^2}{(1 - f)^3} + \frac{6f_j^2 f_j^2}{(1 - f)^4}.
\]

From (2.2), we deduce that

\[
- \frac{2f}{1 - f} \nabla f \nabla \omega = \frac{4f_i f_j f_{ij}}{(1 - f)^2} + \frac{4f_j^2 f_j^2 - 4f_i f_j f_{ij}}{(1 - f)^3} - \frac{4f_j^2 f_j^2}{(1 - f)^4}.
\]
Combining (2.4) and (2.5), we have
\[
\Delta \omega - \omega_t - \frac{2f}{1-f} \nabla f \nabla \omega \geq \frac{2f_{ij}^2 - 2e^{(p-1)}f_{ij}\tilde{\lambda}_i - 2\tilde{\lambda}(p-1)e^{(p-1)}f_{i}^2}{(1-f)^2} + \frac{2e^{(q-1)}f_{ij}\tilde{\eta}_i + 2\tilde{\eta}(q-1)e^{(q-1)}f_{i}^2 + 2k_1f_{i}^2}{(1-f)^2} \\
+ \frac{4f_{ij}f_{jk}f_{j} + 2f_{j}^2f_{i} - 2\tilde{\lambda}e^{(p-1)}f_{j}^2 - 2\tilde{\eta}e^{(q-1)}f_{j}^2}{(1-f)^3} + \frac{2f_{i}^2f_{j}^2}{(1-f)^4}.
\]

Hölder's inequality implies that
\[
\left|\frac{4f_{ij}f_{jk}f_{j}}{(1-f)^3}\right| \leq \frac{2f_{ij}^2}{(1-f)^2} + \frac{2f_{i}^2f_{j}^2}{(1-f)^4}.
\]
Thus we have
\[
\Delta \omega - \omega_t - \frac{2f}{1-f} \nabla f \nabla \omega \geq -\frac{2e^{(p-1)}f_{ij}\tilde{\lambda}_i + 2e^{(q-1)}f_{ij}\tilde{\eta}_i}{(1-f)^2} - \frac{2\tilde{\lambda}(p-1)e^{(p-1)}f_{i}^2 + 2\tilde{\eta}(q-1)e^{(q-1)}f_{i}^2 + 2k_1f_{i}^2}{(1-f)^2} \\
+ \frac{2f_{j}^2e^{(p-1)}f_{j}^2 - 2\tilde{\lambda}e^{(p-1)}f_{j}^2 - 2\tilde{\eta}e^{(q-1)}f_{j}^2}{(1-f)^3} \\
= 2(1-f)\omega^2 - 2\tilde{\lambda}\left(p - 1 + \frac{1}{1-f}\right)e^{(p-1)}f \omega - 2\tilde{\eta}\left(q - 1 + \frac{1}{1-f}\right)e^{(q-1)}f \omega \\
- \frac{2e^{(p-1)}f_{ij}\tilde{\lambda}_i + 2e^{(q-1)}f_{ij}\tilde{\eta}_i}{(1-f)^2} - 2k_1\omega.
\]

Now we choose a smooth cut-off function \(\psi = \psi(x,t)\) with compact support in \(Q_{R,T}\) such that

1. \(\psi = \psi(r,t), \ 0 \leq \psi \leq 1\) with \(\psi = 1\) in \(Q_{R/2,T/2}\), where \(r = d(x,x_0)\);
2. \(\psi\) is decreasing with respect to \(r\);
3. for any \(0 < \alpha < 1\), \(|\partial_r \psi|/\psi^\alpha \leq C_\alpha/R\), \(|\partial^2_r \psi|/\psi^\alpha \leq C_\alpha/R^2\);
4. \(|\partial_r \psi|/\psi^{1/2} \leq C/T\).

Using (2.6), we get
\[
\Delta(\psi \omega) - \frac{2\nabla \psi}{\psi} \cdot \nabla (\psi \omega) - (\psi \omega)_t \\
\geq 2(1-f)\psi^2 - 2\tilde{\lambda}\left(p - 1 + \frac{1}{1-f}\right)e^{(p-1)}f \psi \omega - 2\tilde{\eta}\left(q - 1 + \frac{1}{1-f}\right)e^{(q-1)}f \psi \omega \\
- 2k_1\psi \omega - \frac{2e^{(p-1)}f_{ij}\tilde{\lambda}_i + 2e^{(q-1)}f_{ij}\tilde{\eta}_i}{(1-f)^2} \psi + \frac{2f}{1-f} \nabla f \nabla (\psi \omega) - \frac{2f_{i}^2 \omega}{1-f} \nabla f \nabla (\psi).
\[-\frac{2|\nabla \psi|^2}{\psi} \omega + (\Delta \psi) \omega - \psi_t \omega.\]

Suppose that $\psi \omega$ attains the maximum at $(x_1, t_1)$. The argument in [3] implies that we can assume $x_1$ is not in the cut-locus of $M$. Then we have $\Delta (\psi \omega) \leq 0$, $(\psi \omega)_t \geq 0$ and $\nabla (\psi \omega) = 0$ at $(x_1, t_1)$. It follows that

\[
2(1 - f)\psi \omega^2 \leq 2\tilde{\lambda} \left( p - 1 + \frac{1}{1 - f} \right) e^{(p-1)\psi \omega} + 2\tilde{\eta} \left( q - 1 + \frac{1}{1 - f} \right) e^{(q-1)\psi \omega}
\]

\[
2k\psi \omega + \frac{2e^{(p-1)\psi \omega} - f \tilde{\lambda} + 2e^{(q-1)\psi \omega} f \tilde{\eta}}{(1 - f)^2}
\]

\[
+ \frac{2f \omega}{1 - f} \nabla f \nabla \psi + \frac{2|\nabla \psi|^2}{\psi} \omega - (\Delta \psi) \omega + \psi_t \omega.
\]

In view of $p \geq 1$, $q \geq 1$ and $f \leq 0$, we have

\[
2\tilde{\lambda} \left( p - 1 + \frac{1}{1 - f} \right) e^{(p-1)\psi \omega} + 2\tilde{\eta} \left( q - 1 + \frac{1}{1 - f} \right) e^{(q-1)\psi \omega}
\]

\[
\leq 2\tilde{\lambda}^+ p \psi \omega + 2\tilde{\eta}^+ \psi \omega
\]

\[
\leq \frac{1}{16} \psi \omega^2 + 16\psi (\tilde{\lambda}^+ p)^2 + \frac{1}{16} \psi \omega^2 + 16\psi (\tilde{\eta}^+ q)^2
\]

\[
\leq \frac{1}{8} \psi \omega^2 + 16(\tilde{\lambda}^+ p)^2 + 16(\tilde{\eta}^+ q)^2
\]

where $\tilde{\lambda}^+ = \max\{\tilde{\lambda}, 0\}$, $\tilde{\eta}^+ = \max\{\tilde{\eta}, 0\}$. Straightforward calculations show

\[
\frac{2e^{(p-1)\psi \omega} f_i \tilde{\lambda}_i + 2e^{(q-1)\psi \omega} f_i \tilde{\eta}_i \psi}{(1 - f)^2}
\]

\[
\leq \frac{f_i^4}{2(1 - f)^4} \psi + \frac{3|\nabla \tilde{\lambda}|^{4/3} \psi}{2(1 - f)^{4/3} \psi} + \frac{f_i^4}{2(1 - f)^4} \psi + \frac{3|\nabla \tilde{\eta}|^{4/3} \psi}{2(1 - f)^{4/3} \psi}
\]

\[
\leq \frac{f_i^4}{(1 - f)^4} \psi + \frac{3}{2} (|\nabla \tilde{\lambda}|^{4/3} + |\nabla \tilde{\eta}|^{4/3})
\]

\[
\leq (1 - f) \psi \omega^2 + \frac{3}{2} (|\nabla \tilde{\lambda}|^{4/3} + |\nabla \tilde{\eta}|^{4/3}),
\]

\[
\left| \frac{2f \omega}{1 - f} \nabla f \nabla \psi \right| \leq 2\omega^{3/2} |\psi||\nabla \psi| = 2[\psi(1 - f)\omega]^{3/4} \frac{|f||\nabla \psi|}{[\psi(1 - f)]^{3/4}}
\]

\[
\leq \frac{1}{8} (1 - f) \psi \omega^2 + c \frac{(f|\nabla \psi|)^4}{[\psi(1 - f)]^3}
\]

\[
\leq \frac{1}{8} (1 - f) \psi \omega^2 + c \frac{f^4}{R^4(1 - f)^3},
\]

\[
2k\psi \omega \leq \frac{1}{8} (1 - f) \psi \omega^2 + ck^2.
\]
By the estimates of Souplet and Zhang [17], we have

\begin{equation}
\frac{|\nabla \psi|^2}{\psi} \omega \leq \frac{1}{8} \psi \omega^2 + c \frac{1}{R^4} \leq \frac{1}{8} (1 - f) \psi \omega^2 + c \frac{1}{R^4},
\end{equation}

(2.12)

\begin{equation}
-(\Delta \psi) \omega \leq \frac{1}{8} \psi \omega^2 + c \frac{1}{R^4} + c k \frac{1}{R^2} \leq \frac{1}{8} (1 - f) \psi \omega^2 + c \frac{1}{R^4} + c k \frac{1}{T^2},
\end{equation}

(2.13)

\begin{equation}
|\psi|\omega \leq \frac{1}{8} \psi \omega^2 + c \frac{1}{T^2} \leq \frac{1}{8} (1 - f) \psi \omega^2 + c \frac{1}{T^2}.
\end{equation}

(2.14)

Combining (2.7)–(2.14), we obtain

\begin{equation}
\frac{1}{8} (1 - f) \psi \omega^2 \leq 16(\tilde{\lambda}^+ p)^2 + 16(\tilde{\eta}^+ q)^2 + \frac{3}{2} (|\nabla \tilde{\lambda}|^{4/3} + |\nabla \tilde{\eta}|^{4/3})
\end{equation}

\begin{equation}
+ c \frac{f^4}{R^4(1 - f)^3} + c k^2 + c \frac{1}{R^4} + c \frac{k}{R^2} + c \frac{1}{T^2}.
\end{equation}

Hence

\begin{align*}
\psi \omega^2(x_1, t_1) &\leq c N^{2p-2} \|\lambda^+\|_{L^\infty(Q_{R,T})}^2 + c N^{q-2} \|\eta^+\|_{L^\infty(Q_{R,T})}^2 + c N^{\frac{4}{3}(p-1)} \|\nabla \lambda\|_{L^\infty(Q_{R,T})}^{4/3} \\
&+ c N^{\frac{4}{3}(q-1)} \|\nabla \eta\|_{L^\infty(Q_{R,T})}^{4/3} + c \frac{f^4}{R^4(1 - f)^3} + c k^2 + c \frac{1}{R^4} + c \frac{1}{T^2}.
\end{align*}

By above estimate, there holds for all \((x, t)\) in \(Q_{R,T},\)

\begin{align*}
\psi \omega^2(x, t) &\leq c N^{2p-2} \|\lambda^+\|_{L^\infty(Q_{R,T})}^2 + c N^{q-2} \|\eta^+\|_{L^\infty(Q_{R,T})}^2 + c N^{\frac{4}{3}(p-1)} \|\nabla \lambda\|_{L^\infty(Q_{R,T})}^{4/3} \\
&+ c N^{\frac{4}{3}(q-1)} \|\nabla \eta\|_{L^\infty(Q_{R,T})}^{4/3} + c \frac{1}{R^4} + c \frac{1}{T^2} + c k^2.
\end{align*}

Noting that \(\psi(x, t) = 1\) in \(Q_{R/2,T/2},\) we get

\begin{align*}
\left|\frac{\nabla f(x, t)}{1 - f(x, t)}\right| &\leq c \frac{R}{\sqrt{T}} + c \frac{1}{\sqrt{T}} + c \sqrt{k} + c N^{(p-1)/2} \|\lambda^+\|_{L^\infty(Q_{R,T})}^{1/2} + c N^{(q-1)/2} \|\eta^+\|_{L^\infty(Q_{R,T})}^{1/2} \\
&+ c N^{\frac{4}{3}(p-1)} \|\nabla \lambda\|_{L^\infty(Q_{R,T})}^{1/3} + c N^{\frac{4}{3}(q-1)} \|\nabla \eta\|_{L^\infty(Q_{R,T})}^{1/3},
\end{align*}

Finally we have

\begin{align*}
\left|\frac{\nabla u(x, t)}{u(x, t)}\right| &\leq c \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2} \|\lambda^+\|_{L^\infty(Q_{R,T})}^{1/2} + N^{(q-1)/2} \|\eta^+\|_{L^\infty(Q_{R,T})}^{1/2} \\
&+ N^{\frac{4}{3}(p-1)} \|\nabla \lambda\|_{L^\infty(Q_{R,T})}^{1/3} + N^{\frac{4}{3}(q-1)} \|\nabla \eta\|_{L^\infty(Q_{R,T})}^{1/3} \right) \left( 1 + \log \frac{N}{u} \right).
\end{align*}

2.2. Proof of Theorem 1.4

We prove it by contradiction. Suppose that \(u\) is a positive solution to (1.1). Noting that \(\lambda\) and \(\eta\) are nonpositive constants, it follows from Theorem 1.3 that

\begin{equation}
\frac{|\nabla u(x, t)|}{u(x, t)} \leq c \left( \frac{1}{R} + \frac{1}{\sqrt{T}} \right) \left( 1 + \log \frac{N}{u} \right).
\end{equation}

(2.15)
By the same argument as in the proof of Theorem 1.2 in [17] and Theorem 1.8 in [20], fixing \((x_0, t_0)\) and applying (2.15) to \(u\) on \(B(x_0, R) \times [t_0 - R^2, t_0]\), we get
\[
\frac{|\nabla u(x_0, t_0)|}{u(x_0, t_0)} \leq \frac{C}{R}[1 + o(R)].
\]

It follows that \(|\nabla u(x_0, t_0)| = 0\) by letting \(R \to \infty\). Noting \((x_0, t_0)\) is arbitrary, we have \(u(x, t) = u(t)\). Then by (1.1), we get \(\frac{du}{dt} = \lambda u^p + \eta u^q\). Without loss of generality, we assume that \(\lambda < 0\).

If \(p > 1\), integrating \(\frac{du}{dt}\) on \([t, 0]\) with \(t < 0\) implies that
\[
\frac{1}{1 - p} \left( u^{1-p}(0) - u^{1-p}(t) \right) \leq -\lambda t.
\]
Then
\[
u^{p-1}(t) \leq u^{p-1}(0) + (1 - p)\lambda t.
\]
This yields that if \(t\) is large enough, \(u^{p-1}(t) < 0\) which contradicts that \(u\) is positive.

If \(p = 1\), we get for \(t < 0\)
\[
\log u(0) - \log u(t) \leq -\lambda t.
\]
Hence \(u(t) \geq u(0)e^{\lambda t}\), which contradicts \(u(x, t) = e^{o(d(x) + \sqrt{|t|})}\) near infinity. We finish the proof.

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