A note on damped wave equations with a nonlinear dissipation in non-cylindrical domains *

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Abstract

In this paper, we study the large time behavior of a class of wave equation with a nonlinear dissipation in non-cylindrical domains. The result we obtained here relaxes the conditions for the nonlinear term coefficients (in precise, that is $\beta(t)|u|^\rho u$) in [1] and [3] (which require $\beta(t)$ to be a constant or $\beta(t)$ to be decreasing with time $t$) and has less restriction for the defined regions.

Key words: Wave equation; stabilization; dissipative nonlinearity; non-cylindrical domain.

1 Introduction and main results

Fix $t \geq 0$. Let $\Omega_t$ be a bounded domain in $\mathbb{R}$. Given $T > 0$. Set $\hat{Q}_T = \Omega_t \times (0, T)$ and denote by $\hat{\Sigma}_T$ the lateral boundary of $\hat{Q}_T$. Consider the following wave equation with a nonlinear dissipation in the non-cylindrical domain $\hat{Q}_T$:

$$
\begin{cases}
  u'' - \Delta u + au' + bu + \beta(t)|u|^\rho u = 0 & (x, t) \in \hat{Q}_T, \\
  u = 0 & (x, t) \in \hat{\Sigma}_T, \\
  u(x, 0) = u_0(x), \ u'(x, 0) = u_1(x) & x \in \Omega_0,
\end{cases}
$$

where $(u_0, u_1)$ is any given initial couple, $(u, u')$ is the state variable and $a, b > 0$.

In order to study the qualitative theory of (1.1), we need the following assumptions on the domain $\hat{Q}_T$:

(A1) $\alpha \in C^2[0, T]$ such that $\alpha(0) = 1$, $\alpha'(t) \geq 0$ and $\sup_{t \in [0, T]} \alpha'(t) < 1$.

(A2) $\beta(t), \beta'(t) \geq 0$, $t \in [0, T]$ and $\beta' \in L^\infty(0, T)$.

(A3) if $n > 2$, then $0 < \rho \leq \frac{2}{n-2}$; if $n = 1$ or $n = 2$, then $0 < \rho < \infty$.

The wellposedness result for (1.1) is stated as follows:

*This work is supported by the NSF of China under grants 11471070, 11771074 and 11371084.

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Theorem 1.1 Let \( u_0 \in H^2_0(0,1) \) and \( u_1 \in H^1_0(0,1) \). If assumptions (A1)-(A3) hold, then there exists a unique strong solution \( u \) of problem (1.1) such that \( u \in L^\infty(0,T;H^1(\Omega_t)) \), \( u_t \in L^\infty(0,T;H^2(\Omega_t)) \), and \( u(0) = u_0, u_t(0) = u_1 \).

The proof of Theorem 1.1 is quite similar to the proof of wellposedness results in [2], so we omit it (but what we need to point out is that since the assumption (A2) is different from \( \beta' \leq 0 \), the result we obtained here just admits the solution to belong to \( L^\infty(0,T;H^1_0(\Omega_t) \cap H^2(\Omega_t)) \), not to \( L^\infty(0,\infty;H^1_0(\Omega_t) \cap H^2(\Omega_t)) \).

Lemma 1.1 ([4]) Suppose that \( \tilde{Q}_T \) has a regular lateral boundary \( \tilde{\Sigma}_T \). If \( u \in C^1(\mathbb{R};L^2(\Omega_t)) \), then we have

\[
\frac{d}{dt} \int_{\Omega_t} u(x,t) dx = \int_{\Omega_t} \frac{d}{dt} u(x,t) dx + \int_{\Gamma_t} u(x,t) \hat{x} n_x d\sigma
\]

where \( \Gamma_t \) is the boundary of \( \Omega_t \), \( \hat{x} \) is the velocity of \( x \in \Gamma_t \), and \( n = (n_x,n_t) \) is the unit exterior normal to \( \tilde{\Sigma}_T \). Moreover, it was observed that for \( u \in H^1(\tilde{Q}_T) \) with \( u = 0 \) on \( \tilde{\Sigma}_T \) (all tangential derivative of \( u \) also vanishes on \( \tilde{\Sigma}_T \)), consequently the full gradient of \( u \) satisfies \( \nabla_{x,t} u = (\partial_n u)n \) which implies that

\[
uu = (\partial_n u)n_t \quad \text{and} \quad \nabla_x u = (\partial_n u)n_x.
\]

The energy of system (1.1) \( \mathcal{E}(t) \) is given by

\[
\mathcal{E}(t) = \int_{\Omega_t} \left[ \frac{1}{2} u_t^2(t) + \frac{1}{2} u_x^2(t) + \frac{1}{2} u^2(t) + \beta(t) \frac{1}{\rho + 2} |u(t)|^{\rho + 2} \right] dx.
\]

Then the main result of this paper is stated as follows.

Theorem 1.2 One can find \( \lambda > 0 \) and \( \beta(t) \) satisfying \( \lambda(\rho + 1)\beta(t) \geq \beta'(t) \), such that the inequality

\[
\mathcal{E}(t) \leq C\mathcal{E}(0)\varphi^{-1}(t), \quad (1.2)
\]

hold, where \( \varphi(t) \) is chosen by \( \varphi(t) = e^{\lambda t} \), \( C \) is some positive constant.

Proof. Firstly, let \( \varphi \) be a unknown continuous function. Secondly, multiplying both sides of the first equation in (1.1) by \( (u_t + \lambda u)\varphi(t) \), where \( \lambda > 0 \), and then integrating it on \((0,T) \times \Omega_t \), we get

\[
\int_0^T \int_{\Omega_t} (u'' - \Delta u + au' + bu + \beta(t)|u|^\rho u)(u_t + \lambda u)\varphi(t) dxdt = 0.
\]
Calculating the above equality, we have

\[
\int_0^T \int_{\Omega_t} u''(u_t + \lambda u) \varphi(t) dx dt
\]

\[
= \int_0^T \int_{\Omega_t} \left[ \frac{1}{2} u_t^2 \varphi(t) \right]_t + (\lambda \varphi(t) u_t)_t - \lambda \varphi(t) u_t^2 - \lambda \varphi'(t) u_t - \frac{1}{2} \varphi'(t) u_t^2 \right] dx dt
\]

\[
= \int_{\Omega_T} \int_{\Omega_t} \left[ \frac{1}{2} u_t^2(T) \varphi(T) + \lambda \varphi(T) u(T) u_t(T) \right] dx - \int_{\Omega_0} \int_{\Omega_t} \left[ \frac{1}{2} u_t^2(0) \varphi(0) + \lambda \varphi(0) u(0) u_t(0) \right] dx
\]

\[
+ \int_0^T \int_{\Gamma_t} \frac{1}{2} u_t^2 \varphi(t) n_t \sigma dt - \int_0^T \int_{\Omega_t} \left[ \lambda \varphi(t) u_t^2 + \lambda \varphi'(t) u_t + \frac{1}{2} \varphi'(t) u_t^2 \right] dx dt,
\]

\[
\int_0^T \int_{\Omega_t} -\Delta u(u_t + \lambda u) \varphi(t) dx dt
\]

\[
= \int_0^T \int_{\Omega_t} \left[ (-u_x u_t \varphi(t))_x + u_x u_t \varphi(t) - (u_x \lambda u \varphi(t))_x - \lambda \varphi(t) u_x^2 \right] dx dt
\]

\[
= \int_0^T \int_{\Omega_t} \left[ (-u_x u_t \varphi(t))_x + \frac{1}{2} u_x^2 \varphi(t) - \frac{1}{2} \varphi'(t) u_x^2 - (\lambda \varphi(t) u u_x)_x + \lambda \varphi(t) u_x^2 \right] dx dt
\]

\[
= \int_0^T \int_{\Omega_t} (-u_x u_t \varphi(t))_x dx dt + \int_{\Omega_T} \frac{1}{2} u_x^2(T) \varphi(T) dx - \int_{\Omega_0} \frac{1}{2} u_x^2(0) \varphi(0) dx
\]

\[
+ \int_0^T \int_{\Gamma_t} \frac{1}{2} u_x^2 \varphi(t) n_t \sigma dt - \int_0^T \int_{\Omega_t} \left[ \frac{1}{2} \varphi'(t) u_x^2 - \lambda \varphi(t) u_x^2 \right] dx dt,
\]

\[
\int_0^T \int_{\Omega_t} a u'(u_t + \lambda u) \varphi(t) dx dt = \int_0^T \int_{\Omega_t} \left[ a \varphi(t) u_t^2 + a \lambda uu_t \varphi(t) \right] dx dt,
\]

\[
\int_0^T \int_{\Omega_t} b u(u_t + \lambda u) \varphi(t) dx dt
\]

\[
= \int_0^T \int_{\Omega_t} \left[ b u u_t \varphi(t) + b \lambda \varphi(t) u_t^2 \right] dx dt
\]

\[
= \int_0^T \int_{\Omega_t} \left[ \frac{1}{2} b u^2 \varphi(t) \right]_t - \frac{b}{2} \varphi'(t) u_t^2 + b \lambda \varphi(t) u_t^2 \right] dx dt
\]

\[
= \int_{\Omega_T} \int_{\Omega_t} \left[ \frac{1}{2} b \varphi(T) u^2(T) - \int_{\Omega_0} \frac{1}{2} b \varphi(0) u^2(0) dx - \int_0^T \int_{\Omega_t} \left[ \frac{b}{2} \varphi'(t) u_t^2 - b \lambda \varphi(t) u_t^2 \right] dx dt,
\]

\[
\int_0^T \int_{\Omega_t} \beta(t) |u|\rho u(u_t + \lambda u) \varphi(t) dx dt
\]
\[
= \int_0^T \int_{\Omega_t} \left[ \beta(t) \left( \frac{1}{\rho + 2} |u|^{\rho+2} \right)_t \varphi(t) + \lambda \beta(t) |u|^{\rho+2} \varphi(t) \right] \, dx \, dt
\]
\[
= \int_0^T \int_{\Omega_t} \left( \frac{1}{\rho + 2} |u|^{\rho+2} \beta(t) \varphi(t) \right)_t - \beta'(t) \varphi(t) \frac{1}{\rho + 2} |u|^{\rho+2} \beta(t) \varphi(t) \left( \frac{1}{\rho + 2} |u|^{\rho+2} \right) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega_t} \lambda \beta(t) |u|^{\rho+2} \varphi(t) \, dx \, dt
\]
\[
= \int_{\Omega_T} \beta(T) \varphi(T) \frac{1}{\rho + 2} |u(T)|^{\rho+2} \, dx - \int_{\Omega_0} \beta(0) \varphi(0) \frac{1}{\rho + 2} |u(0)|^{\rho+2} \, dx
\]
\[
+ \int_0^T \int_{\Omega_t} \left[ \beta'(t) \varphi(t) \frac{1}{\rho + 2} |u|^{\rho+2} + \beta(t) \varphi'(t) \frac{1}{\rho + 2} |u|^{\rho+2} - \lambda \beta(t) |u|^{\rho+2} \varphi(t) \right] \, dx \, dt
\]

Adding (1.3) to (1.7), we obtain
\[
0 = \int_{\Omega_T} \frac{1}{2} u_x^2(T) \varphi(T) + \lambda \varphi(T) u(T) u_t(T) \, dx - \int_{\Omega_0} \frac{1}{2} u_t^2(0) \varphi(0) + \lambda \varphi(0) u(0) u_t(0) \, dx
\]
\[
+ \int_0^T \int_{\Gamma_t} \frac{1}{2} u_x^2(t) n_t \, d\sigma \, dt - \int_0^T \int_{\Omega_t} \left[ \lambda \varphi(t) u_t^2 + \lambda \varphi'(t) uu_t + \frac{1}{2} \varphi'(t) u_t^2 \right] \, dx \, dt
\]
\[
- \int_0^T \int_{\Omega_t} (u_x u_t \varphi(t))_x \, dx \, dt + \int_{\Omega_T} \frac{1}{2} u_x^2(T) \varphi(T) \, dx - \int_{\Omega_0} \frac{1}{2} u_x^2(0) \varphi(0) \, dx
\]
\[
+ \int_0^T \int_{\Gamma_t} \frac{1}{2} u_x^2(t) n_t \, d\sigma \, dt - \int_0^T \int_{\Omega_t} \left[ \frac{1}{2} \varphi'(t) u_x^2 - \lambda \varphi(t) u_x^2 \right] \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega_t} [a \varphi(t) u_x^2 + a \lambda uu_t \varphi(t)] \, dx \, dt
\]
\[
+ \int_{\Omega_T} \frac{1}{2} b \varphi(T) u_x^2(T) \, dx - \int_{\Omega_0} \frac{1}{2} b \varphi(0) u_x^2(0) \, dx - \int_0^T \int_{\Omega_t} \frac{1}{2} \varphi'(t) u_x^2 - b \lambda \varphi(t) u_x^2 \, dx \, dt
\]
\[
+ \int_{\Omega_T} \beta(T) \varphi(T) \frac{1}{\rho + 2} |u(T)|^{\rho+2} \, dx - \int_{\Omega_0} \beta(0) \varphi(0) \frac{1}{\rho + 2} |u(0)|^{\rho+2} \, dx
\]
\[
+ \int_0^T \int_{\Omega_t} \left[ - \beta'(t) \varphi(t) \frac{1}{\rho + 2} |u|^{\rho+2} - \beta(t) \varphi'(t) \frac{1}{\rho + 2} |u|^{\rho+2} + \lambda \beta(t) |u|^{\rho+2} \varphi(t) \right] \, dx \, dt.
\]

Since the assumption (A1) means that

(H1) The domain \( \hat{Q}_T \) is time-like, i.e., \( |n_t| < |n_x| \).

(H2) \( \hat{Q}_T \) is monotone increasing, i.e., \( \Omega_t \) is expanding with respect to \( t \) or \( n_t \leq 0 \).

\[
\int_0^T \int_{\Gamma_t} \left[ \frac{1}{2} u_t^2 \varphi(t) n_t + \frac{1}{2} u_x^2 \varphi(t) n_t \right] \, d\sigma \, dt - \int_0^T \int_{\Omega_t} (u_x u_t \varphi(t))_x \, dx \, dt
\]
Furthermore, (1.8) yields
\[
\int_{\Omega_T} \left[ \frac{1}{2} u_t^2(T) + \lambda u(T) u_t(T) + \frac{1}{2} u_x^2(T) + \frac{1}{2} b u^2(T) + \beta(T) \frac{1}{\rho + 2} |u(T)|^{\rho + 2} \right] \phi(T) dx
\]
\[
\leq \int_{\Omega_0} \left[ \frac{1}{2} u_t^2(0) + \lambda u(0) u_t(0) + \frac{1}{2} u_x^2(0) + \frac{1}{2} b u^2(0) + \beta(0) \frac{1}{\rho + 2} |u(0)|^{\rho + 2} \right] \phi(0) dx
\]
\[
+ \int_0^T \int_{\Omega_t} \left[ \lambda \phi(t) u_t^2 + \lambda \phi(t) u u_t + \frac{1}{2} \phi'(t) u_t^2 \right] dx dt + \int_0^T \int_{\Omega_t} \left[ \frac{1}{2} \phi'(t) u_x^2 - \lambda \phi(t) u_x^2 \right] dx dt
\]
\[
- \int_0^T \int_{\Omega_t} \left[ a \phi(t) u_t^2 + a \lambda u u \phi(t) \right] dx dt + \int_0^T \int_{\Omega_t} \left[ \frac{b}{2} \phi'(t) u^2 - b \lambda \phi(t) u^2 \right] dx dt
\]
\[
+ \int_0^T \int_{\Omega_t} \left[ \beta'(t) \phi(t) \frac{1}{\rho + 2} |u|^{\rho + 2} + \beta(t) \phi'(t) \frac{1}{\rho + 2} |u|^{\rho + 2} - \lambda \beta(t) |u|^{\rho + 2} \phi(t) \right] dx dt. \tag{1.9}
\]

We can choose \( \phi(t) = e^{st}, s > 0 \). In particular, let \( \phi(t) = e^{\lambda t} \) (\( \lambda \) be small) and
\[
\lambda (\rho + 1) \beta(t) \geq \beta'(t). \tag{1.10}
\]

We can put
\[
\beta(t) = e^{\mu t} \quad \text{with} \quad \mu \leq \lambda (\rho + 1),
\]
or
\[
\beta(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0,
\]
with \( a_i > 0 (i = 0, \cdots, n) \) such that (1.10) holds.

Then the last three terms of inequality (1.9) are negative. Hence, we deduce
\[
\int_{\Omega_T} \left[ \frac{1}{2} u_t^2(T) + \lambda u(T) u_t(T) + \frac{1}{2} u_x^2(T) + \frac{1}{2} b u^2(T) + \beta(T) \frac{1}{\rho + 2} |u(T)|^{\rho + 2} \right] \phi(T) dx
\]
\[
\leq \int_{\Omega_0} \left[ \frac{1}{2} u_t^2(0) + \lambda u(0) u_t(0) + \frac{1}{2} u_x^2(0) + \frac{1}{2} b u^2(0) + \beta(0) \frac{1}{\rho + 2} |u(0)|^{\rho + 2} \right] \phi(0) dx.
\]

From the above inequality, we finally derive
\[
\mathcal{E}(t) \leq C \mathcal{E}(0) \phi^{-1}(t),
\]
for some constant \( C > 0 \).
Remark 1.1 If $b = 0$ in (1.7), then use the method before, (1.9) becomes

$$
\int_{\Omega_t} \left[ \frac{1}{2} u_t^2(T) + \lambda u(T) u_t(T) + \frac{1}{2} u_x^2(T) + \frac{1}{2} T \frac{1}{\rho + 2} |u(T)|^{\rho+2} \right] \varphi(T) dx
\leq \int_{\Omega_0} \left[ \frac{1}{2} u_t^2(0) + \lambda u(0) u_t(0) + \frac{1}{2} u_x^2(0) + \frac{1}{2} T \frac{1}{\rho + 2} |u(0)|^{\rho+2} \right] \varphi(0) dx
$$

$$+
\int_0^T \int_{\Omega_t} \left[ \lambda \varphi(t) u_t^2 + \lambda \varphi'(t) u u_t + \frac{1}{2} \varphi'(t) u_x^2 \right] dx dt + \int_0^T \int_{\Omega_t} \left[ \frac{1}{2} \varphi'(t) u_x^2 - \lambda \varphi(t) u_x^2 \right] dx dt
\leq \int_0^T \int_{\Omega_t} \left[ a \varphi(t) u_t^2 + a \lambda u u_t \varphi(t) \right] dx dt
$$

$$+
\int_0^T \int_{\Omega_t} \left[ \beta'(t) \varphi(t) \frac{1}{\rho + 2} |u|^{\rho+2} + \beta(t) \varphi'(t) \frac{1}{\rho + 2} |u|^{\rho+2} - \lambda \beta(t) |u|^{\rho+2} \varphi(t) \right] dx dt.
$$

In this case, in order to absorb the mixed term $\int_0^T \int_{\Omega_t} a \lambda u u_t \varphi(t) dx dt$, we must use Poincaré inequality whose coefficients depend on geometry of the domain. That is

$$\int_{\Omega_t} u^2(x, t) dx \leq |\Omega_t|^2 \int_{\Omega_t} u_x^2(x, t) dx.$$

Thus

$$\int_0^T \int_{\Omega_t} a \lambda u u_t \varphi(t) dx dt \leq \int_0^T \int_{\Omega_t} \frac{1}{2} a \lambda^2 \varphi(t) u^2 dx dt + \int_0^T \int_{\Omega_t} \frac{1}{2} a \varphi(t) u_t^2 dx dt
\leq \int_0^T \int_{\Omega_t} \frac{1}{2} a \lambda^2 |\Omega_t|^2 \varphi(t) u_x^2 dx dt + \int_0^T \int_{\Omega_t} \frac{1}{2} a \varphi(t) u_t^2 dx dt.$$

When $\alpha \in L^\infty(0, \infty)$, and there exist two bounded domains $\Omega_*, \Omega^* \subset \mathbb{R}^1$ such that $\Omega_* \subset \Omega_t \subset \Omega^*, \forall t < t$. Then we have $|\Omega_t| \leq |\Omega^*|, \forall t > 0$. Let $a \lambda |\Omega^*|^2 < 1$. With a similar argument as before, we get

$$\mathcal{E}(t) \leq C \mathcal{E}(0) \varphi^{-1}(t), \quad t > 0,$$

for some constant $C > 0$.

If non-cylindrical domains become unbounded in some $X_1$-direction of space, as the time $t$ goes to infinite, and are bounded in other $X_2$-direction of space. Since the projection of it in $X_2$-direction is a bounded open set, written as $w$, then the Poincaré inequality in $X_2$-direction turns out

$$\int_{\Omega_t} u^2(x, t) dx \leq C_w^2 \int_{\Omega_t} |\nabla u(x, t)|^2 dx \leq C_w^2 \int_{\Omega_t} |\nabla u(x, t)|^2 dx,$$

where $C_w$ is the Poincaré constant.

Therefore, the above conclusion is still valid for this case.
Remark 1.2 For the case of domains becoming unbounded in every spatial direction, as the time $t$ goes to infinite, the condition $b \neq 0$ is needed to make (1.2) true. Otherwise, for any given $T > 0$, let $\lambda = \lambda(T)$ (depending on time $T$) be small and then it follows that

$$E(t) \leq C E(0) \varphi_T^{-1}(t), \quad 0 < t < T,$$

where $\varphi_T^{-1}(t) = e^{-\lambda(T)t}$.

Since Poincaré inequality does not hold for a fixed number in any totally unbounded area, it seems difficult for us to get an estimate (1.2) without compensation ($b = 0$) and this is also an open problem that has been mentioned in some literature such as [3].

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