First order Quantum Hall Transitions in Hofstadter Butterfly in the Honeycomb Lattice

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We analyze the effects of nearest neighbor repulsive interactions in the Hofstadter system in a honeycomb lattice. At low fillings, we show that, as the interaction strength is increased there are two first order transitions, a Landau transition with translational and rotational symmetries broken, followed by a topological transition with a jump in the quantized Hall conductivity. We therefore predict that in physical realizations where the interaction effects are strong, there would be translation symmetry broken states with quantized Hall conductivities that differ from those predicted by the non-interacting theory.

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I. INTRODUCTION

The discovery of many different exotic quantum phenomena such as Fractional Quantum Hall Effect (FQHE) [1], spin liquids [2–4], topological insulators [5] etc., have given impetus to the theoretical and experimental studies of a new kind of phase of matter- ‘the topological phase’. This new phase defies the established paradigm of Landau’s theory of phases and phase transitions. Attempts to understand the mechanism of the emergence of robust physical properties of gapped quantum systems (like quantized hall conductivity, fractional quantum numbers and statistics of the quasi-particles) have channelized research towards study of the topology of quantum many body states.

The phases of topological orders are identified by universal quantum numbers or topological invariants which are robust against arbitrary perturbations. The phases with same topological invariants can be grouped into a single class and smoothly changed into one another. Observables corresponding to these topological invariants can take discrete set of values for different classes of topological phases. Examples of such observables are the Hall conductivity characterized by a topological invariant called Chern number of the filled bands [6–8], charges and statistics of the quasi-particles in the fractional quantum Hall systems with $Z_2$-like quantum numbers as the topological invariants [9], spin and statistics of quasi-particles in spin systems with $Z_2$ as one of the topological invariants and parity of the number of Dirac cones on the surface of a topological insulators which are described by $Z_2$ topological invariant [5].

Let us consider a physical system with a Hamiltonian that depends on a set of parameters (coupling constants) and we vary these parameters adiabatically along some curve in the parameter space. A transition between phases with different values of the topological invariants is possible under the following conditions:

1. On varying the Hamiltonian parameters adiabatically, the ground state changes continuously and the two phases are connected in the parameter space by a region where the topological invariants are ill-defined. In other words, the two topological phases are separated by a phase where the topology is not defined. The topological observables can take on a continuous set of values that connect the two discrete ones. A system of non-interacting fermion bands with non-zero Chern number is such an example. Here if the Fermi energy $\epsilon_F$ lies in a gap, the topology is well-defined and hence the Hall conductivity is quantized to an integer which is the sum of the Chern numbers of the occupied bands. When $\epsilon_F$ is not in a gap the Chern number is not defined and the Hall conductivity can have a continuous range of real values. Thus, the Chern insulator phases with quantized Hall conductivity are separated by Topological Fermi liquid [10] phases characterized by an anomalous Hall conductivity.

2. There can also be a phase transition between the two topological phases if the region in the parameter space where the topology is ill-defined is a single point. An example is the Haldane model where the electron hops on a honeycomb lattice with real nearest neighbor (NN) hopping value $t$, complex next nearest neighbor (NNN) hopping $t_2 e^{i\phi}$ and an inversion symmetry breaking mass term $M$ [11]. The Hamiltonian is in a 4 dimensional parameter space of $t, t_2, \phi$ and $M$. For $|t_2/t| < 1/3$, Chern number of the lowest band is ±1 for $|M/t_2| < 3\sqrt{3} |\sin \phi|$ and is zero for $|M/t_2| > 3\sqrt{3} |\sin \phi|$. At $|M/t_2| = 3\sqrt{3} |\sin \phi|$ the two bands touch and the Chern number is ill-defined there.

Thus, in the above two cases the gap closes in the transition region (or point).

3. In the cases of symmetry protected topology it has been shown that there are systems where the gap
does not close in the transition region [12, 13]. As described by Ezawa et al. [12], two distinct topological phases can be connected continuously without closing the gap by changing the symmetry of the system during the process. This is possible by taking a detour in the parameter space about the gap closing point where the topological numbers are not defined throughout the detoured path.

4. The ground state changes discontinuously as the Hamiltonian parameters are varied adiabatically. This is exactly what happens in the case of first order transitions. In this case the topology is well-defined throughout and the two ground states are both gapped and have different values of topological invariants. Thus a first order transition between two distinct topological phases is possible without the gap closing. At the point of transition, the topology remains well-defined but the ground state is degenerate. In this paper, we address this issue in the Hofstadter system.

Electrons moving on a lattice in presence of magnetic field exhibit a fractal single particle energy spectrum, otherwise known as the Hofstadter butterfly [14]. The first fractal structure to be discovered in quantum physics, it has intrigued theorists and experimentalists alike. The prolonged search for the butterfly-like energy spectrum has led to experimental realizations in semiconductor superlattices [15–17]. However the complete spectrum has been seen only recently by moiré pattern [18, 19] formed when graphene is placed over hexagonal boron nitride(hBN). The Hofstadter-Harper Hamiltonian has also been realized using ultracold atoms in optical lattices [20, 21]. This Hofstadter problem in lattice realizes the representative example of the quantum Hall insulator. These experiments have motivated us to look into a less explored area of electron-electron interactions within the Hofstadter butterfly system.

We study the Hofstadter system on a honeycomb lattice with a nearest-neighbor repulsive interaction. We concentrate on the cases where the magnetic flux of \( \phi_q = \frac{2\pi k}{q} \) passes through each plaquette of the lattice where \( q \) is an integer. The non-interacting system has \( 2q \) number of bands [22, 23]. We consider the case where the lowest band is completely filled and the others are empty, i.e. a particle density of \( 1/2q \) particles per unit cell.

We set up a mean-field approximation and solve the self-consistency equations to obtain the phases for \( q = 3, 4, \ldots, 8 \). As the strength of the interaction, \( V \), is increased from zero, we find two first order transitions. The first transition, occurring at the critical value \( V = V_{c1} \), is a Landau transition characterized by the breaking of the spatial translation and rotational symmetries. At the next critical value \( V = V_{c2} \geq V_{c1} \), we find a topological phase transition characterized by a change in the quantized Hall conductivity. For \( q = 3 \) and \( 4 \), the two transitions coincide with \( V_{c1} = V_{c2} \). For \( q = 5, \ldots, 8 \) we have \( V_{c2} > V_{c1} \). The translational symmetry breaking phases have uniform density but a non-zero current on the bonds. We show that the symmetry breaking can be characterized by a pattern of circulating currents in each plaquette.

The topological phase transition described in this paper is quite different from the plateau transitions in integer quantum hall effect (IQHE). The transition between states of different Hall conductivities in IQHE are induced by changing the filling factor (which is a one body operator), the magnetic field or the chemical potential. In the system considered in this paper, the transition is induced by changing the inter-particle interaction strength. Further, the physics of the IQHE plateaus and transitions between them comes from disorder and Anderson localization. When the filling factor is in the plateau region, the longitudinal conductivity is zero because of the mobility gap caused by localization of the single particle states. At the transition point between the plateaus, due to the existence of extended states at the center of the band, the mobility gap closes and we see a divergence in the longitudinal conductivity at zero temperature. There is no energy gap in the system since the density of states decays exponentially beyond the band edges but never actually vanishes.

At the mean field level, our model has a first order transition between gapped states which implies that the longitudinal conductivity is always zero. However, at finite temperatures, domains of the two phases could co-exist at the transition point. There are gapless edge states at the boundary of the domains. This could cause a non-zero longitudinal conductivity in the vicinity of the transition point. However, if the first order nature of the transition survives the fluctuations about mean field, then the longitudinal conductivity will go to zero as the temperature goes to zero.

The rest of this paper is organized as follows. Sec. II describes the model and its symmetries. Sec. III discusses the topology of the bands of the non-interacting system. We set up the mean-field theory in Sec. IV. The solutions and the resulting phase diagram is detailed in Sec. IV A. We summarize and discuss our results in Sec. V.

II. THE HOFSTADTER BUTTERFLY IN A HONEYCOMB LATTICE

The Hofstadter system that we consider for spinless fermions on honeycomb lattice is described by the Hamiltonian,

\[
H = t \sum_{\langle ij \rangle} \left( c_i^\dagger e^{i \mathbf{a}_{ij} \cdot \mathbf{r}} c_j + h.c. \right) + V \sum_{\langle ij \rangle} n_i n_j, \tag{1}
\]

where \( i, j \) label the nearest neighbor sites of a honeycomb lattice with sublattices A and B. Fig. 1, \( t \) is the NN hopping parameter, \( \mathbf{a}_{ij} \) are the gauge fields on the links such that the magnetic flux passing through each
plaque is \( \phi_q = \frac{2\pi q}{\tau} \). \( n_i = c_i^\dagger c_i \) is the particle density operator where \( c_i^\dagger \) and \( c_i \) are respectively the fermion creation and annihilation operators at site \( i \). \( V \geq 0 \) is the strength of the nearest neighbor repulsive interaction.

FIG. 1: (Color online) Honeycomb lattice in magnetic field. A and B are the two sublattices. The arrows on the links indicate the direction of hopping corresponding to the phase \( \phi_q \) such that \( \phi_q \) amount of flux passes through each plaque of the lattice. For \( q = 3 \), three original unit cells form a single magnetic unit cell as shown in a box in the figure. \( \hat{e}_1 \) and \( \hat{e}_2 \) represents the basis vectors of the lattice.

We now define the notations we use in the rest of the paper. The links in the three different directions are \( x,y \) and \( z \), as shown in Fig. 1. The \( x \)-links are chosen as the unit cells and are labeled by two integers, \((m,n)\). The gauge fields are non-zero only on the \( x \)-links which connect \((m,n)\) and \((m,n+1)\). On these links, we choose \( \phi_{(m,n),(m,n+1)} = n\phi_q \). Now we discuss the symmetries of the lattice under which the Hamiltonian given by Eq. (1) remains invariant. The fermion operators for the sublattices A and B are denoted as \((a_{m,n}^\dagger, a_{m,n})\) and \((b_{m,n}^\dagger, b_{m,n})\) respectively.

A. Translational symmetry

In the presence of a constant magnetic field, translation symmetry is implemented projectively \([24]\), namely, the translation is accompanied by a gauge transformation. We denote the generators of unit translations in the \( \hat{e}_1 \) and \( \hat{e}_2 \) directions, Fig. 1, as \( \tau_1 \) and \( \tau_2 \).

\[
\tau_1 c_{m,n} \tau_1^\dagger = e^{i\phi_q} c_{m+1,n}, \quad \tau_2 c_{m,n} \tau_2^\dagger = c_{m,n+1}.
\]

\( c_{m,n} \in \{a_{m,n}, b_{m,n}\} \). The translational operators \( \tau_1 \) and \( \tau_2 \) do not commute and satisfy the relation,

\[
(\tau_1 \tau_2 \tau_1^{-1} \tau_2^{-1}) c_{m,n} (\tau_2 \tau_1 \tau_2^{-1} \tau_1^{-1}) = e^{i\phi_q} c_{m,n}.
\]

B. Rotational Symmetries

The honeycomb lattice has a three-fold rotation symmetry about any lattice site. It also has a two-fold rotation (inversion) symmetry about the center of the links. The transformations on the fermion operators corresponding to the three fold rotation symmetry about any sublattice A is,

\[
Ra_{m,n} e^{i\phi_q} R_\dagger = a_{n-m,-m} e^{-i\phi_q},
\]

\[
Rb_{m,n} e^{i\phi_q} R_\dagger = b_{n-m-1,-m} e^{-i\phi_q}.
\]

(4)

The operation of inversion symmetry operators on the fermion operators is,

\[
Ia_{m,n} e^{i\phi_q} I_\dagger = a_{m-n} e^{-i\phi_q},
\]

\[
Ib_{m,n} e^{i\phi_q} I_\dagger = b_{m-n} e^{-i\phi_q}.
\]

(5)

C. Particle-hole symmetry

The model also has a particle-hole symmetry, otherwise called chiral/sublattice symmetry. This is an anti-unitary transformation in the many-particle Hilbert space. The transformation of the fermion operators under this symmetry is

\[
Pa_{m,n} P^{-1} = a_{m,n}^\dagger, \quad Pb_{m,n} P^{-1} = -b_{m,n}^\dagger, \quad Pi P^{-1} = -i.
\]

(6)

As particle hole symmetry exists only for half filling, the Hamiltonian given by Eq. (1) is invariant under particle hole transformation for the same case.

III. THE NON-INTERACTING LIMIT

The non-interacting problem of honeycomb lattice in magnetic field has been studied earlier \([25-28]\). In this section we briefly review some of the features that are relevant to this paper.

As mentioned earlier, the generators of translations do not commute. In the one-particle sector they obey the algebra \( \tau_1 \tau_2 \tau_1^{-1} = e^{i\phi_q} \tau_2 \). This implies that \( \tau_1 \) and \( \tau_2 \) commute with \( \tau_2 \). So, we choose a magnetic unit cell consisting of \( q \) original unit cells in the \( \hat{e}_1 \) direction as shown in Fig. 1 for \( q = 3 \). The magnetic unit cells have \( 2q \) sublattices. We therefore introduce another index \( \alpha = 1, \ldots, q \) and define the fermion operators \( \psi_{m,n,\alpha,A} = a_{m,n,\alpha}^\dagger \) and \( \psi_{m,n,\alpha,B} = b_{m,n,\alpha}^\dagger \). The fourier transforms of the fermion operators are defined as,

\[
\psi_{k_1, k_2, \alpha, r} = \sum_{m,n} e^{ik_r \hat{R}_{m,n}} \psi_{m,n,\alpha, r},
\]

(7)

where \( r \in A, B \), \( \hat{R}_{m,n,\alpha} \) is the position of the original unit cell labeled by \( (m,n,\alpha) \) and \( k_{1(2)} = \hat{k} \cdot \hat{e}_{1(2)} \). We choose the Brillouin zone to be \(-\pi_2 \leq k_2 \leq \pi \) and \(-\pi \leq k_1 \leq \pi \).
The non-interacting Hamiltonian in the momentum space is given by

$$ H_0 = \sum_{k_1, k_2} \psi_{k_1}^\dagger h_k \psi_{k_2}, \quad (8) $$

where \( h_k \) is the single particle \( 2q \times 2q \) Hamiltonian matrix given by

$$ h_k = \begin{pmatrix} 0 & F_k \\ F_k^\dagger & 0 \end{pmatrix}, \quad (9) $$

where 0 denotes a \( q \times q \) zero matrix and \( F_k \) being a \( q \times q \) matrix whose non-zero elements are \( F_k(\alpha, \alpha) = t(1 + e^{i2\pi(q/\epsilon k)}), F_k(\alpha - 1, \alpha) = te^{-i\epsilon k}, F_k(1, q) = te^{-i\epsilon k}. \)

As the single particle Hamiltonian at every \( k \) is a \( 2q \times 2q \) matrix, we have \( 2q \) number of energy bands. The particle-hole symmetry implies the existence of a matrix, \( \beta \), that anti-commutes with the single-particle Hamiltonian. In our notation this matrix is

$$ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (10) $$

where \( I \) and 0 are \( q \times q \) identity and zero matrices respectively. It is easy to verify that \( h_k \beta = -\beta h_k \). This implies that the eigenvalues of \( h_k \) come in pairs, \( \pm \epsilon_k \). So we denote the spectrum of \( h_k \) as,

$$ h_k u_n^k = \pm \epsilon_k^{[n]} u_n^k, \quad (11) $$

where \( \epsilon_k^{[n]} \) and \( u_n^k \) are the single particle eigenfunction and the single particle energy of \( n^{th} \) band respectively. Here \( n = \pm 1, \pm 2, \ldots, \pm q \) and \( \epsilon_k^{[n]} \geq 0 \).

The single particle energy \( \epsilon_k^{[n]} \) is \( 2q \) degenerate as shown below,

$$ \tau_1 h_{k_1, k_2} \tau_1^\dagger = h_{k_1, k_2 + 2\pi/q}, \quad \tau_2 h_{k_1, k_2} \tau_2^\dagger = h_{k_1, k_2}. \quad (12) $$

The non-interacting Hamiltonian \( H_0 \) is invariant under these translational operations. Thus, \( \epsilon_k^{[n]} = |k_1, k_2 + \frac{2\pi}{q}| \).

In terms of these single particle eigenfunctions, the Pancharatnam-Berry (PB) vector potential, \( A^n_i(k) \) and curvature, \( B^n(k) \) for a given \( n^{th} \) band is

$$ A^n_i(k) = i[(u_n^k)]^\dagger \frac{\partial u_n^k}{\partial k_i}, \quad B^n(k) = \epsilon_{ij} \partial_i \epsilon_{ij} A^n_j(k). \quad (13) $$

Due to the periodicity of \( h_{k_1, k_2} \), from Eq. (12), we conclude that \( B^n(k_1, k_2) = B^n(k_1, k_2 + \frac{2\pi}{q}) \). From the PB curvature, the Chern number of the \( n^{th} \) band can be computed as

$$ \nu^n = \frac{1}{2\pi} \int \frac{d^2k}{4\pi} B^n(k). \quad (14) $$

which is related to the Hall conductivity \( \sigma_H \) as \( \sigma_H = -\frac{e^2}{h} \nu \) \([6–8]\), where \( \nu \) is the total Chern number of the filled bands. The non interacting Hamiltonian is invariant under the particle-hole transformation and thus the total Chern number at half filling is zero \([29]\).

Hatsugai et al \([28]\) have analyzed the pattern of Chern numbers in this problem and have shown that there is a crossover from the so called "Dirac" to "Fermi" behavior. Close to half filling, i.e. \( n \ll q \) there are pairs of almost degenerate bands with total Chern number of \( \nu = -2 \) for any of these pairs. These bands become degenerate in the limit \( q \to \infty \). Thus for high values of \( q \), as the Fermi level changes, the Hall conductivity changes in units of \( 2\epsilon^2/h \), which is characteristic of Dirac fermions in a magnetic field. This behavior extends till the energy where the system has a Van-Hove singularity. Around this energy there is a band with very large Chern number with a change of sign. At higher values of \( n \), till \( n = q \), there are non-degenerate bands with \( \nu = -1 \). Thus in this regime the Hall conductivity changes in units of \( \epsilon^2/h \) as is characteristic of non-relativistic fermions in a magnetic field.

As mentioned earlier we will be working in the region where only the lowest band \( (n = -q) \) is filled. We are thus deep inside the "Fermi" region.

### IV. MEAN FIELD THEORY

The interacting Hamiltonian is given as

$$ H = H_0 + V \sum_{\langle ij \rangle} a_i^\dagger a_j b_i^\dagger b_j \text{ where } a \text{ and } b \in \{ A, B \}. $$

The mean field theory (MFT) described in Ref \([30]\) is used here. The expectation value of the gauge invariant link operators are denoted as

$$ \langle c_{i,a}^\dagger e^{i\hat{\pi} A_{ij}} c_{j,b} \rangle = \frac{1}{V} (\epsilon_{ij} + iJ_{ij}), \quad (15) $$

where \( \epsilon_{ij} \) and \( J_{ij} \) respectively are the real and imaginary parts of the gauge invariant link operators. The mean field Hamiltonian is given by

$$ H_{MF} = \sum_{\langle ij \rangle} \left( \langle (c_{i,a}^\dagger e^{i\hat{\pi} A_{ij}} c_{j,b} + h.c.) \rangle + \epsilon_{ij} (c_{i,a}^\dagger e^{i\hat{\pi} A_{ij}} c_{j,b} + h.c.) - iJ_{ij} (c_{i,a}^\dagger e^{i\hat{\pi} A_{ij}} c_{j,b} - h.c.) \right) + \frac{1}{V} (c_{i,j}^2 + J_{ij}^2). \quad (16) $$

By minimizing the energy, we get the self consistency equations,

$$ \epsilon_{ij} = \frac{V}{2} \langle c_{i,a}^\dagger e^{i\hat{\pi} A_{ij}} c_{j,b} + h.c. \rangle, \quad (17) $$

$$ J_{ij} = \frac{V}{2} \langle c_{i,a}^\dagger e^{i\hat{\pi} A_{ij}} c_{j,b} - h.c. \rangle. \quad (18) $$

We define

$$ \chi_{ij} = te^{i\hat{\pi} A_{ij}} (\epsilon_{ij} - iJ_{ij}), \quad (19) $$

such that the the complex mean field hopping parameters are given by

$$ t'_{ij} = te^{i\hat{\pi} A_{ij}} - \chi_{ij}. \quad (20) $$
The pictorial representation of the mean field hopping for $q = 3$ is given in Fig. 2. Thus for any $q$, there are $3q$ complex bond parameters. These are solved by an iterative method using the self consistency equations, Eqs. (17) and (18), for a given $q$ and $V$ at fixed filling. We summarize the method here. (i) We start the iteration with a random initial guess of $\epsilon_{ij}$ and $J_{ij}$, (ii) diagonalize $H_{MF}$ using $\epsilon_{ij}$ and $J_{ij}$, (iii) calculate the expectation value of the gauge invariant link operators and using this we compute $\epsilon_{ij}$ and $J_{ij}$ from Eqs. (17) and (18). (iv) The whole process from step (ii) is repeated until all the quantities converge. We repeat this process for various initial guesses and often find many mean field solutions especially for higher values of $q$ and $V$. Comparing the energies of these solutions, we pick up the lowest energy state as the ground state of the interacting Hamiltonian. This method is repeated for various values of $q$ and $V$ and we get the phase diagram described in the following section.

A. The phase diagram

We obtain the phase diagram by solving the self consistency equations given by Eqs. (17), (18) for the lowest band filled, at a density of $1/2q$, where we choose $q = 3$ to 8. For the cases where $\epsilon_{ij} = \lambda(V)$ and $J_{ij} = 0$ the ground state does not change with $\lambda$ and kinetic energy scales linearly. Therefore, the total energy, $E$, of this Hamiltonian for $1/2q$th filling, can be written as

$$E = E_0(t - \lambda(V)t) + \sum_{\text{bonds}} \frac{\lambda^2(V)t^2}{V} + V.$$  

(21)

Minimizing the energy with respect to $\lambda(V)$, we get $\lambda(V) = \frac{E_0V}{4q}$ where $E_0$ is the total energy of the non-interacting system. For this type of solution the Hall conductivity remains unchanged and thus $\sigma_H = e^2/h$, with $\nu = -1$, which is the Hall conductivity for the non-interacting system with the lowest energy band filled.

We have solved the self consistency equations in the range of $V = 0$ to 8. As $V$ is increased, we cross two transition points $V_{c1}$ and $V_{c2}$. At $V_{c1}$ we get a first order Landau phase transition (LPT) where the spatial and three-fold rotational symmetries are broken with $\sigma_H = e^2/h$. At $V_{c2}$ topological phase transition (TPT) occurs where Hall conductivity jumps to zero. For $q = 3$ and 4, these transitions occur at the same point, $V_{c1} = V_{c2}$.

For $q$ ranging from 5 to 8, we have two distinct critical points $V_{c1}$ and $V_{c2}$. The lowest band is well separated from the next higher band throughout the range of $V$. At transition points, we notice that there is no gap closing. Hence we find a TPT without closing of the energy gap and topological invariants is well-defined throughout.

We have also checked for charge density wave (CDW) solutions and find that there is no CDW for $V = 0$ to 8 for the values of $q$ we have considered.

Fig. 3 is the phase diagram in the $1/q$ and $V$ plane. The phases of the system lie on the blue, green and red lines. The dashed blue line represents the first phase, PH1, where translation symmetry is preserved and $\sigma_H = e^2/h$. The solid green line represents the second phase, PH2, where translation symmetry is broken but $\sigma_H = e^2/h$. The dash dotted red line represents the third phase, PH3, where translation symmetry is broken and the Hall conductivity is zero. The blue circle denotes the critical $V$ for LPT, i.e. $V_{c1}$, for various values of $q$. The red triangle denotes the critical $V$ for TPT, i.e. $V_{c2}$. As mentioned earlier, we can see that for $q = 3$ and 4, $V_{c1}$ and $V_{c2}$ coincide and from $q = 5$ to 8, they are distinct. With the decrease in the magnetic flux per plaquette, $V_{c1}$ decreases whereas $V_{c2}$ increases.

![Phase diagram](image)

FIG. 3: (Color online) Phase diagram: The blue dashed line depicts the PH1 phase, the green solid line represents the PH2 phase and the red dash dotted line depicts the PH3 phase. The blue circle denotes the critical value of $V$ for LPT i.e. $V_{c1}$ for various values of $q$. The red triangle denotes the critical value of $V$ for TPT i.e. $V_{c2}$.

We study the bulk edge correspondence of this model for $q = 3$ by exactly diagonalizing $H_{MF}$ using converged values of $\epsilon_{ij}$ and $J_{ij}$ for the phases PH1 and PH3 in the cylindrical geometry with zigzag edges as shown in Fig. 4a. The corresponding energy spectrum for phase
PH1 and PH3 are given by Figs. 4b and 4c respectively which shows gapless edge states between the lowest two bands for phase PH1 and absence of edge states for PH3.

The characteristic features distinguishing these three phases are described in the following sub-section.

B. Translational symmetry breaking

As mentioned earlier, the PB curvature is periodic in the Brillouin zone (BZ) as a consequence of the translational invariance as seen in Fig. 5a which gives the color plot of PB curvature of the lowest band in BZ for $V = 4.55 = V_{c1}$ with phase PH1. Figs. 5b and 5c give the PB curvature plot for phase PH2 at $V = 4.55 = V_{c1}$ and $V = 4.719 = V_{c2}$. We see that the periodic nature of PB curvature is now absent as the translational symmetry is broken. Fig. 5d shows the PB curvature plot of phase PH3 for $V = 4.719 = V_{c2}$ where also the periodic nature of the PB curvature is lost.

The magnitude and direction of the current on the links $\mathcal{J}_{ij}$, calculated from the imaginary term $J_{ij}$ of Eq. (15) is given as

$$\mathcal{J}_{ij} = \langle c_{i,a}^{\dagger} e^{i \pi \sigma_z} c_{j,b} - h.c \rangle = \frac{2}{V} J_{ij}. \quad (22)$$

The three phases form current patterns which we call ‘ferromagnetic current wave’ in the phase PH1 and ‘staggered ferromagnetic current wave’ in the phases PH2 and PH3. We describe these two current waves for $q = 5$ magnetic unit cell in the following paragraphs.

For phase PH1, $\mathcal{J}_{ij} = 0$ for all the bonds and for the translational symmetry breaking phases, PH2 and PH3, $\mathcal{J}_{ij}$ is non-zero and same on $x$ and $z$ bonds of original unit cell as shown in Figs. 6b, 6c and 6d. $\mathcal{J}_{ij}$ on the $y$ bonds in PH2 and PH3 phase are zero. On the magnetic unit cell we define $x$ and $z$ bond current as $\mathcal{J}_{\alpha}$ on the links $A_{\alpha} - B_{\alpha}$ where $\alpha = 1, \ldots, q$ denotes the original unit cell in the magnetic unit cell. From the mean field solutions of PH2 and PH3 phase, we get $\mathcal{J}_1 = \mathcal{J}_5$, $\mathcal{J}_2 = \mathcal{J}_4$ and $\mathcal{J}_3 = 0$. In Fig. 6a, we show the plot of the link currents $\mathcal{J}_1$ and $\mathcal{J}_2$ as a function of $V$ and find two sharp jumps at $V_{c1} = 4.55$ and $V_{c2} = 4.719$ indicating the first order transition. Fig. 6c and Fig. 6d show the direction of the current on the bonds and the color of the bonds corresponds to magnitude of the current on the links represented by the same color in Fig. 6a.

Let us assume that current flowing in the plaquettes in a magnetic unit cell is denoted by $\mathcal{J}_\alpha$, as shown in Figs. 6c and 6d, so that $\mathcal{J}_\alpha = \mathcal{J}_\alpha + \mathcal{J}_{\alpha+1}$. $\mathcal{J}_6 = \mathcal{J}_1$ from periodic boundary conditions. For phase PH1, as $\mathcal{J}_\alpha = 0$, $\mathcal{J}_\alpha = \mathcal{J}_{\alpha+1} = \mathcal{J}$ (say). So, we can assume that a circulating current of magnitude $\mathcal{J}$ flows in each plaquette in the same direction, as shown in Fig. 6b, which can be viewed as a ‘ferromagnetic current wave’ with wave-length of lattice constant $a$. For PH2 and PH3 phases, as we have non-zero current on links, so the magnitude of the $\mathcal{J}_\alpha$ is different for different $\alpha$. Thus, in these phases, though the current loop direction in the plaquettes of a magnetic unit cell is same, the magnitudes are different. Therefore, the current loop in the plaquettes of a lattice in phase PH2 and PH3 is not uniform unlike PH1 and is rather staggered. We can picturize this as a ‘staggered ferromagnetic wave’ with $qa$ as the wavelength. Here $\mathcal{J}_2 = \mathcal{J}_5 = \mathcal{J}$ and $\mathcal{J}_3 = \mathcal{J}_4$. 

![Image 4](image1.png)

FIG. 4: (Color online) (a) A finite sample with zigzag edge along $\hat{e}_2$ direction and infinite along $\hat{e}_1$ direction. (b) Energy spectrum for PH1 phase and (c) Energy spectrum for PH3 phase for finite sample shown in Fig. 4a.

![Image 2](image2.png)

FIG. 5: (Color online) Contour plot of PB curvature in the momentum space in BZ for $q = 5$ at (a) $V = V_{c1} = 4.55$ for the phase PH1, (b) $V = V_{c1}4.55$ for PH2, (c) $V = V_{c2} = 4.719$ for PH2 and (d) $V = V_{c2} = 4.719$ for PH3 phase.
A phase transition of first order nature occurs when the ground state energy of a system $E$ has two local minima. The mean field approximation gives more than one solution for an arbitrary $q$ at higher values of $V$. This means that the system has two different equilibrium states which coexist together. These minima are divided by an energy barrier, and we illustrate this for $q = 3$ solution.

As shown in the phase diagram in Fig. 3, for $q = 3$, there are two phases: PH1 and PH3. The mean field hopping parameters, $t_{ij}'$ from Eq. (20) can be labeled as $t_{ij,1}'$ and $t_{ij,2}'$ for the phase PH1 and PH3 respectively. We parameterize the hopping amplitude as

$$\rho_{ij}(s, V) = t'_{ij,1} + s(t'_{ij,2} - t'_{ij,1}),$$

where $s$ is a parameter. Ground state energy $E$ can be computed for a given value of $V$ as a function of $s$ by substituting the values of $\rho_{ij}$ in Eq. (16). In Fig. 7 we plot $E$ vs $s$ for three values of $V$. Two local minima can be seen at $s = 0$ and $s = 1$ for all three values of $V$. The minima (M1) at $s = 0$ corresponds to phase PH1 while the minima (M2) at $s = 1$ corresponds to phase PH3. From the plot, we see that for region $V \leq V_c$, the energy of M1 is less than M2, at $V = V_c = 4.274$, both the minima have same energy and for $V \geq V_c$ regime, M2 has energy lower than M1. This plot hence signals the first order nature of the transition.

C. Effective potential

We give an example of a first order TPT to non-zero Hall conductivity. In this subsection we give an example of a first order TPT to non-zero Hall conductivity. We have investigated the case where flux per plaquette is $e\phi_0 = 6\pi/7$. Here we have 14 bands. The non-interacting Hamiltonian with two bands filled has Hall conductivity $\sigma_H = 3e^2/h$. On increasing the interaction strength $V$, we get a first order transition from $\sigma_H = 3e^2/h$ to $\sigma_H = e^2/h$ at $V = V_{c1} = 0.7068$. On further increasing $V$ the Hall conductivity becomes zero at $V_{c2} = 2.385$. Hence, we give an example of a system where the Hall conductivity changes from a non-zero value to another non-zero value by a first order phase transition without any gap closing with the application of nearest neighbor interaction $V$. More systematic investigations are under progress.

D. Other quantum Hall transitions

Until now the flux per plaquette we have considered gives the topological phase transition from non-zero Hall conductivity to zero Hall conductivity. In this subsection we give an example of a first order TPT to non-zero Hall conductivity. We have investigated the case where flux per plaquette is $e\phi_0 = 6\pi/7$. Here we have 14 bands. The non-interacting Hamiltonian with two bands filled has Hall conductivity $\sigma_H = 3e^2/h$. On increasing the interaction strength $V$, we get a first order transition from $\sigma_H = 3e^2/h$ to $\sigma_H = e^2/h$ at $V = V_{c1} = 0.7068$. On further increasing $V$ the Hall conductivity becomes zero at $V_{c2} = 2.385$. Hence, we give an example of a system where the Hall conductivity changes from a non-zero value to another non-zero value by a first order phase transition without any gap closing with the application of nearest neighbor interaction $V$. More systematic investigations are under progress.

V. CONCLUSION

The topological invariants are defined for gapped systems and remain unchanged when Hamiltonian is varied adiabatically in the parameter space. A transition between phases with different values of the topological invariants usually occurs by the gap closing at the transition point that makes the topological invariants ill-defined. But recent works [12, 13] have illustrated that...
the topological phase transition can occur without closing of the energy gap. This is possible by moving along a path in the parameter space avoiding the gap closing point. Topological invariant is not defined along this path. In this work we presented a topological transition without the gap closing where topological invariants are well-defined throughout the parameter space.

To summarize, we have considered Hofstadter problem on honeycomb lattice with nearest neighbor repulsive interactions at \((1/2q)\)th filling within mean-field approach and have shown that a first order TPT occurs without the gap closing. With increase in the strength of the interaction, for \(q = 3\) and 4, LPT and TPT occur at the same critical value \(V_c\). But for \(q = 5\) to 8 both TPT and LPT occur at two distinct critical values of \(V\). \(V_{c1}\) is the Landau transition point while at \(V_{c2}\) a first order topological transition occurs from non-zero Hall conductivity state to zero Hall conductivity state.

For the filled band case the Fermi energy lies in the gap, the mean field approximation works well for this case. The effect of fluctuations on the nature of phase transitions is an interesting problem in itself. At finite temperatures, there can be coexistence of the domains of two phases at transition point and due to the presence of gapless edge states at the domain walls, we can have non-zero longitudinal conductivity. In the presence of fluctuations, if the first order nature of transition survives, then the longitudinal conductivity will go to zero as temperature goes to zero since in that case the Fermi level will be in the energy gap.

Here we describe briefly the experimental methods that can detect and distinguish the three phases, PH1, PH2 and PH3, of the model described in this paper.

Recent demonstration of the Hofstadter Harper Hamiltonian in an optical lattice [20, 21] has opened a new path in exploring the effect of electron-electron interactions on the topological phases of the complex Hofstadter system. This can be by done by measuring the PB curvature and the topological invariants in future experiments. The Hofstadter Harper Hamiltonian on honeycomb lattice can be realized with the above experimental implementation. PB curvature can be mapped out over the Brillouin zone (BZ) by measuring the velocity of the wavepacket of atoms at a particular momentum in the BZ in presence of an external force as suggested in [31]. Hence, the three phases of the system considered in this work can be distinguished. The Chern number can be calculated by integrating the Berry curvature and thus the topological nature of the phases can be inferred. The Chern number can also be calculated using the time of flight experiments as suggested in [32, 33].

The Hofstadter butterfly system was recently observed on a moiré pattern [18, 19] formed when graphene is placed on hexagonal boron nitride (hBN). In this experimental construction, the ability to tune magnetic length scale and lattice constant make it possible to study topological phases of the Hofstadter pattern with electron-electron interaction. The Chern number of the topological phases can be obtained directly by measuring the Hall conductivity. Thus this experimental approach can probe topologically different phases and their phase transitions described in this paper.

We have therefore, presented a class of systems as an example where we have topological transition without the gap closing and where the topological invariant is well-defined in the whole region of the parameter space of the Hamiltonian. It is an interesting problem to explore the phase transitions in the presence of interactions for other values of flux i.e. \(2\pi p/q\) per plaquette, where \(p\) is an integer and \(p\) and \(q\) are co-prime. We expect that our theoretical predictions will be of interest to the experimental community.

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