Minimal number of edges in hypergraph guaranteeing perfect fractional matching and MMS conjecture

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Abstract

In this paper we prove Ahlswede-Khachatrian conjecture [1] up to finite number of cases, which can be checked using modern computers. From this conjecture follows conjecture from [2] and Manickam-Miklós-Singhi conjecture.

I. Introduction

For ground set $[n]$ (uniform) hypergraph $H(n,k) = (n,E)$ is $[n]$ together with the subset of edges $E \subset \binom{[n]}{k}$. Perfect fractional matching of $H$ is the set of nonnegative real numbers $(\alpha_1, \ldots, \alpha_{|E|}) : \alpha_j \geq 0, \sum_{j=1}^{|E|} \alpha_j = 1, \sum_{e \in E} \alpha_e e = \left(\frac{k}{n}, \ldots, \frac{k}{n}\right)$. A hypergraph can not have a perfect fractional matching. In [1] R.Ahlswede and L.Khachatrian impose the following

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Conjecture 1 Let $H_f$ be the set of hypergraphs that do not have perfect fractional matching. Then

$$p(n, k) \triangleq \max_{H \in H_f} |E| = \max_{k \geq s \geq 1} \sum_{i=0}^{k-s} \binom{n_s}{i+s} \binom{n-n_s}{k-s-i},$$

where $n_s = \lceil ns/k \rceil - 1$.

In [1] this conjecture was given in other less standard terms (cone dependence) and this drew less attention of the specialists to this problem.

We relax the conditions from this conjecture a little and rewrite them as

$$p(n, k) = \max_{n-1 \geq a \geq 1} \sum_{i > ka/n} \binom{a}{i} \binom{n-a}{k-i}.$$ \hspace{1cm} (2)

To see that (1) is equivalent to (2), one can mention that the sum above decreases when $a$ decreases from $n_s$ to $n_s-1$. Now we direct our attention to another problem. Let

$$(\beta_1, \ldots, \beta_n), \beta_j \in R^1, \sum_{j=1}^{n} \beta_j = 0$$

and

$$U(\{\beta_j\}) = \left\{ e \in \binom{[n]}{k} : \sum_{j \in e} \beta_j \geq 0 \right\}.$$ \hspace{1cm} (3)

Define

$$q(n, k) = \min_{\{\beta_j\}} |U(\{\beta_j\})|.$$ 

Next conjecture was imposed in [2].

Conjecture 2 The following relation is valid

$$q(n, k) = \min_{n-1 \geq a \geq 1} \sum_{i \geq ka/n} \binom{a}{i} \binom{n-a}{k-i}.$$ \hspace{1cm} (3)

The close relation between the problem of determining $p(n, k)$ and the problem of determining $q(n, k)$ was also shown in [2]. In particular it was shown that from Conjecture 1 follows Conjecture 2. Actually, these two problems are the same. We will show this later.
At last we come to Manickam-Miklós-Singhi (MMS) conjecture.

\[ q(n, k) = \binom{n-1}{k-1}, \quad n \geq 4k. \tag{4} \]

From the inequality above it follows that to prove MMS conjecture (assuming that Conjecture 2 is true) we should show that when \( n \geq 4k \), then

\[ \min_{n-1 \geq a \geq 1} \sum_{i \geq ka/n} \left( a^i \binom{n-a}{k-i} \right) = \binom{n-1}{k-1}. \tag{5} \]

We will do this in the Appendix.

Using natural bijection between subsets of \([n]\) and binary \(n\)-tuples, we will not make a difference between them.

Now we prove that both problems are equivalent. From this follows that the proof of Conjecture 1 follows from the proof of the Conjecture 2.

Consider \( A \subset \binom{[n]}{k} \) as the set of vertices of hypersimplex \( \Gamma(n, k) \subset \mathbb{R}^n \) s.t. convex hull \( K(A) \not\in (k/n, \ldots, k/n) \). Then there exists hyperplane

\[ \sum_{j=1}^{n} \omega_j y_j = 0, \]

s.t. \( \sum_{j=1}^{n} \omega_j = 0 \) (it contains the center \((k/n, \ldots, k/n)\)) and \( K(A) \) belongs to one of the opened half spaces on which hyperplane separates \( \mathbb{R}^n \). Vice versa, if there exists a hyperplane with the above properties, then \( K(A) \not\in (k/n, \ldots, k/n) \). If \( A \) is maximal, then

\[ M = \left\{ x \in \binom{[n]}{k} : \sum_{j=1}^{n} \omega_j x_j \geq 0 \right\} \]

is minimal.

This shows the equivalence of the problems.

Assume that \( \omega_1 \geq \omega_2 \geq \ldots \geq \omega_n \), \( \sum_{j=1}^{n} \omega_j = 0 \). Then such space of omega’s has basis \( \{z_j, j = 1, \ldots, n-1\} \), \( z_j = (n-j, \ldots, n-j, -j, \ldots, -j) \). Every \( y \) from this space has representation

\[ y = \sum_{j=1}^{n-1} y_j z_j \]
with nonnegative coefficients \( y_j \geq 0 \). Let’s fix \( x \in \left( \begin{array}{c} n \\ k \end{array} \right) \). We have \( y_i = \sum_{j=1}^{n-1} z_{ji} y_j \):

\[
(x, y) = \sum_{j=1}^{n} x_i y_i = \sum_{j=1}^{n} x_i \sum_{j=1}^{n-1} z_{ji} y_j = \sum_{j=1}^{n-1} y_j \sum_{i=1}^{n} x_i \left( n \sum_{i=1}^{j} x_i - jk \right) = n \sum_{j=1}^{n-1} y_j \sum_{i=1}^{j} x_i - k \sum_{j=1}^{n-1} jy_j.
\]

Once more using the fact that the conditions above are homogeneous and dividing right hand side of the last chain of equations by \( \sum_{j=1}^{n-1} jy_j \) we obtain the condition

\[
\sum_{j=1}^{n-1} \frac{y_j}{\sum_{j=1}^{n-1} jy_j} \sum_{i=1}^{j} x_i > \frac{k}{n}.
\]

Last inequality is equivalent to

\[
\sum_{j=1}^{n-1} \gamma_j x_j \geq \frac{k}{n}, \gamma_j \geq 0, \sum_{j=1}^{n-1} \alpha_j = 1.
\]

If we consider only the strict inequality

\[
\sum_{j=1}^{n-1} \gamma_j x_j > \frac{k}{n}, \gamma_j \geq 0, \sum_{j=1}^{n-1} \gamma_j = 1
\]

and try to find the maximal number of its solutions for \( x \in \left( \begin{array}{c} n \\ k \end{array} \right) \), then it would be exactly problem (6).

From this it follows that

\[
p(n, k) = \max_{\{\gamma_j\}} \left\{ x \in \left( \begin{array}{c} n \\ k \end{array} \right) : \sum_{j=1}^{n-1} x_j \gamma_j > \frac{k}{n} \right\}.
\] (6)

**History**
As already noticed, the problem of determining $p(n, k)$ was first imposed by Ahlswede and Khachatrian \[1\]. From their results, using considerations from \[2\], it can be easily shown that

$$q(n, k) = \binom{n-1}{k-1}, \quad n \geq 2k^3.$$  

In \[8\] this equality was proved for $n \geq \min\{2k^3, 33k^2\}$. At last, the validness of this equality for $n \geq 10^{46}k$ was proved in \[9\].

In \[2\] the connection between two problems was stated and useful facts allowing the reductions of the MMS problem to the problem from paper \[1\] were found.

II Proof of Conjecture 1.

Assume next that $k < n$. We use equality (6). Consider the following function

$$f(\{\gamma_1, \ldots, \gamma_{n-1}\}) = \frac{1}{\sqrt{2\pi}} \sum_{x \in \binom{[n]}{k}} \int_{-\infty}^{\sum_{j=1}^{n-1} \gamma_j x_j - \frac{k}{n}} e^{-\frac{z^2}{2}} dz$$

Define

$$N(\gamma_1, \ldots, \gamma_{n-1}) = \left| x \in \binom{[n]}{k} : \sum_{j=1}^{n-1} \gamma_j x_j > \frac{k}{n} \right|.$$  

Then

$$|N(\{\gamma_j\}) - f(\{\gamma_j\})| < \epsilon(\sigma), \quad \epsilon(\sigma) \overset{\sigma \to 0}{\to} 0$$

uniformly over $\{\gamma_j\}$ s.t.

$$\left| \sum_{j=1}^{n-1} \gamma_j x_j - \frac{k}{n} \right| > \delta, \quad \forall x \in \binom{[n]}{k}.$$  

(7)

For extremal $\gamma$, when $N(\gamma) = p(n, k)$, it is easy to see that $\gamma$ satisfies condition (7), for some $\delta > 0$, because, otherwise, if $\sum_{j=1}^{n-1} \tilde{\gamma}_j x_j^0 = \frac{k}{n}$ for some $x^0 \in \binom{[n]}{k}$, then a little deviation $\gamma'$ of $\tilde{\gamma}$ does not violate the conditions

$$\gamma'_j \geq 0, \quad \sum_{j=1}^{n-1} \gamma'_j = 1; \quad \sum_{j=1}^{n-1} \gamma'_j x_j > \frac{k}{n}, \quad \forall x \in \binom{[n]}{k} : \sum_{j=1}^{n-1} \tilde{\gamma}_j x_j > \frac{k}{n}.$$  


and $\sum_{j=1}^{n-1} \gamma_j x_j^0 > \frac{k}{n}$.

Hence, assuming that we are interested in extremal $\gamma$, we can suppose that (7) is satisfied.

Assume next w.l.o.g. that $\gamma_1 \geq \ldots \geq \gamma_{n-1}$. Since we have the restrictions $\gamma_j \geq 0$, we should look for the extremum among $\gamma$ such that

$$\gamma_{a+1} = \ldots, \gamma_{n-1} = 0, a = 1, \ldots n - 1.$$  

($a = n - 1$ means that we are not imposing any zero condition on $\gamma$.) Assume that this condition is valid for some $a \geq 5$. Case when $a \leq 4$ is easy. Then, because $\gamma_a = 1 - \sum_{j=1}^{a-1} \gamma_j$, we have

$$f'_{\gamma_j} = \frac{1}{\sqrt{2\pi\sigma}} \sum_{x \in \binom{[n]}{k}} e^{-\left(\frac{\sum_{j=1}^{a} \gamma_j x_j - \frac{k}{n}}{2\sigma^2}\right)^2}$$  

and

$$f'_{\gamma_j} = \frac{1}{\sqrt{2\pi\sigma}} \sum_{x \in \binom{[n]}{k}} e^{-\left(\frac{\sum_{j=1}^{a} \gamma_j x_j - \frac{k}{n}}{2\sigma^2}\right)^2}.$$

Next we show that that we can assume that these equalities can be valid together on step functions $\gamma_j = 1/a_i$ for $j \in [a]$. Indeed let’s choose parameter $\sigma$ sufficiently small and then fix them. We see that to satisfy equations (8) we should assume that the following equalities are valid

$$\sum_{x \in \binom{[n]}{k}: j \in x, a \notin x} e^{-\left(\frac{(\gamma_j - \frac{k}{n})}{2\sigma^2}\right)^2} = \sum_{x \in \binom{[n]}{k}: a \in x, j \notin x} e^{-\left(\frac{(\gamma_j - \frac{k}{n})}{2\sigma^2}\right)^2}.$$

To satisfy these equalities we should assume that the exponents form the left sum are equal to the corresponding exponents from the right sum i.e. for each given $j \in [a - 1]$

$$\left((\gamma, x) - \frac{k}{n}\right)^2 = \left((\gamma, y) - \frac{k}{n}\right)^2$$  

where $x \in \binom{[n]}{k}, j \in x, y \in \binom{[n]}{k}, a \in y$ and $x \setminus j, y \setminus a$ run over all sets of cardinality $k - 1$ from $[n - j - a]$. We rewrite equalities (9) as follows:

$$\gamma_j^2 + (\gamma_{j_1} + \ldots, \gamma_{j_{k-1}})^2 - 2\frac{k}{n} \gamma_j - 2\frac{k}{n} (\gamma_{j_1} + \ldots, \gamma_{j_{k-1}})$$
\[
+ \gamma_j(\gamma_{j_1} + \ldots, \gamma_{j_{k-1}}) = \\
\gamma_a^2 + (\gamma_{m_1} + \ldots, \gamma_{m_{k-1}})^2 - 2\frac{k}{n}\gamma_a - 2\frac{k}{n}(\gamma_{m_1} + \ldots, \gamma_{m_{k-1}}) \\
+ \gamma_a(\gamma_{m_1} + \ldots, \gamma_{m_{k-1}})
\]

Summing both sides of these equality over all permissible choices of \(j_1, \ldots, j_{k-1}\) and \(m_1, \ldots, m_{k-1}\) leads to the equality

\[
\begin{align*}
\left(\begin{array}{c} n - 2 \\ k - 1 \end{array}\right) \left(\begin{array}{c} \gamma_j^2 - 2\frac{k}{n}\gamma_j \\ k - 2 \end{array}\right) & - 2\frac{k}{n}R + 2\gamma_jR \\
= \left(\begin{array}{c} n - 2 \\ k - 1 \end{array}\right) \left(\begin{array}{c} \gamma_a^2 - 2\frac{k}{n}\gamma_a \\ k - 2 \end{array}\right) & - 2\frac{k}{n}R + 2\gamma_aR
\end{align*}
\]

where

\[
R = \sum_{x \in \binom{[n]}{k-1} \setminus \{j, a\}} (\gamma, x) = \left(\begin{array}{c} n - 3 \\ k - 2 \end{array}\right) \sum_{m \neq j, a} \gamma_m = \left(\begin{array}{c} n - 3 \\ k - 2 \end{array}\right)(1 - \gamma_j - \gamma_a).
\]

From (11) follows, that \(\gamma_j\) can take at most two values:

\[
\gamma_j = \gamma_a, \quad (11)
\]

\[
\gamma_j + \gamma_a = \lambda \overset{\Delta}{=} 2\frac{n - 2}{1 - 2\frac{k}{n-2}}.
\]

Next we show how we can skip the possibility that \(\gamma_j\) takes second value. Assume at first that to each \(x\) such that \(|x\cap[a]| = p\) corresponds some \(y\) such that \(|y\cap[a]| = p\) for all \(x \in \binom{[n]}{k}\) and \(p\). For given \(p\) we sum left and right sides of the relation (9) over \(x\) and corresponding \(y\) such that \(|x\cap[a]| = p\). Then similar to the case of summation over all \(x\), we obtain two possibilities:

\[
\gamma_j = \gamma_a
\]

or

\[
\gamma_j + \gamma_a = 2\frac{n - 2}{1 - 2\frac{k}{n-2}}
\]

Because we can vary \(p\) it follows, that last equality for some \(p\) contradicts to the second equality from (11).
Now assume that for some \( b \)
\[
\gamma_j = \begin{cases} 
\lambda - \gamma_a, & j \leq b, \\
\gamma_a, & j \in [b + 1, a].
\end{cases}
\]  
(13)

Because \( \sum_j \gamma_j = 1 \) we have the following condition on \( \gamma_a \) and \( \frac{k}{n} \):
\[
b\lambda + (a - 2b)\gamma_a = 1.
\]  
(14)

Let’s \( \gamma_j = \lambda - \gamma_a \). Assume also that for some \( x \) such that \( |x \cap [a]| = p \) corresponds some \( y \) such that \( |y \cap [a]| = q \) for some \( p \neq q \). From (10) follows that there exists two possibilities
\[
(\gamma, x) = (\gamma, y)
\]
or
\[
(\gamma, x) + (\gamma, y) = 2\frac{k}{n}.
\]  
(15)

Each of these equalities impose the condition- first equality the condition (for some integers \( p_1, p_2 \))
\[
p_1\gamma_a + p_2\lambda = 0
\]
which can be inconsistent with equality (14) or together with equality (14) determine the value \( \frac{k}{n} \).

From other side if equality (15) impose the condition (for some integers \( p_3, p_4 \))
\[
p_3\gamma_a + p_4\lambda = 2\frac{k}{n}.
\]  
(16)

It is possible that equality (14) together with equality (16) does not determine the value \( \frac{k}{n} \). In this case we there can be next two possibilities. First possibility that there exists \( x : |x \cap [a]| = m \) (where \( m \) can be equal to \( p \) or \( q \)) sand there exists corresponding \( y : |y \cap [a]| = v \) where \( v \neq p, q \)

Second possibility is that each \( x : |x \cap [a]| = m \), here \( m \neq p, q \) correspond to \( y : |y \cap [a]| = m \). In this, second case we return to the case which leads to the equalities (12) (because when \( a \geq 5 \) the number of such \( m \neq p, q \) is greater that 1.

If we have the first possibility, then we have one additional equation
\[
q_3\gamma_a + q_4\lambda = 2\frac{k}{n}
\]  
(17)
which together with (14) and (16) are inconsistent or determine unique value of \( \frac{k}{n} \).

We see, that if \( b > 1 \), and \( \gamma_j = \gamma - \gamma_a > \gamma_a \) when \( j \leq b \), then value of \( \beta \) can take values only from some discrete finite set. Making small variation of the value \( \frac{k}{n} \) we can achieve the situation that neither of values of these functions are equal with true value of \( \frac{k}{n} \). Once more we mention that such varying we can do always without violation the relation (7).

Let \( N(\bar{\gamma}) \) achieve its extremum on \( \bar{\gamma} \) and \( f(\bar{\gamma}) \) on \( \tilde{\gamma} \). We have

\[
|N(\bar{\gamma}) - f(\bar{\gamma})| < \epsilon,
|N(\bar{\gamma}) - f(\tilde{\gamma})| < \epsilon.
\]

Then

\[
N(\bar{\gamma}) < f(\bar{\gamma}) + \epsilon < f(\tilde{\gamma}) + \epsilon < N(\tilde{\gamma}) + 2\epsilon.
\]

Now \( N(\gamma) \) is a positive integer and the last inequalities mean that

\[
N(\bar{\gamma}) = N(\tilde{\gamma}).
\]

From this follows Conjecture 1.

**Appendix**

Let’s fist mention the fact that if \( k|n \), then \( q(n, k) = \binom{n-1}{k-1} \). This easily follows from the Lemma from [7].

Next we will use the fact that if

\[
p(n, k) = \binom{n-1}{k},
\]

then

\[
p(n + k, k) = \binom{n + k - 1}{k}.
\]

The proof using double counting argument can be found in [10] (only \( q(n, k) \) was considered but the problem for \( p(n, k) \) is equivalent). Thus we can assume that \( k \in (n/5, n/4) \).

Next we assume that \( n \not| ka \). We estimate the probability \( P(k \leq ka/n) \) in several steps. First we use Berry-Esseen inequality for Hypergeometric distribution. We will use considerations from [12]. The problem is that in [12] they
did not calculate the constant $C$ in inequality
\[ P \left( a < \frac{i - ka/n}{\sigma} \leq b \right) - (\Phi(b) - \Phi(a)) < \frac{C}{\sigma}, \]
where
\[ \sigma^2 = \frac{ka}{n} \left( 1 - \frac{a}{n} \right) \left( 1 - \frac{k}{n} \right). \] (18)
Hence we have to repeat those considerations in a way that allows us to obtain the proper upper bound for $C$. We refer the reader to the paper [12] for details.

Let $\delta = 1/20$ and $\sigma > 55$, $n > 12 \cdot 10^4$, $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$.

\[ K_2 = \max \{ i \in \mathbb{Z}_+ : \tilde{x}(i) \geq -\delta \sigma \}, \]
\[ K_1 = \min \{ i \in \mathbb{Z}_+ : \tilde{x}(i) \geq -1 \}, \]
\[ K_0 = \max \{ i \in \mathbb{Z}_+ : \tilde{x} \leq 0 \}, \]

where
\[ \tilde{x}(i) = \frac{i - ka/n}{\sigma}. \]

Then (formula (3.15) in [12])
\[ \Delta \triangleq P(i < K) + \sum_{i=K}^{\lfloor ka/n \rfloor} \left| P(i) - \frac{1}{\sigma} \phi(\tilde{x}(i)) \right| + \sum_{j=K}^{\lfloor ka/n \rfloor} \frac{1}{\sigma} \phi(\tilde{x}(j)) - \Phi(x) = I_1 + I_2 + I_3. \]

Since $K_2 - 1 < ka/n - \delta \sigma^2 \leq K_2$ and using Chebyshev inequality we obtain the following
\[ I_1 = P(i \leq K_2 - 1) \leq P \left( \left| \frac{i - ka/n}{\sigma} \right| \geq \frac{K_2 - ka/n - 1}{\sigma} \right) \]
\[ \leq \frac{\text{Var}(i)}{(K_2 - 1 - ka/n)^2} \leq \frac{n\sigma^2}{n-1} (\delta \sigma^2)^{-2} < 0.1323. \]

Next we have (inequality (3.20) in [12], we take into account that $k \in (n/5, n/4)$)
\[ \sum_{j=K_2}^{\lfloor ka/n \rfloor} \left| P(i) - \frac{1}{\sigma} \phi(\tilde{x}(i)) \right| \]
\[
\begin{align*}
&\leq \frac{17}{3\sqrt{2\pi}\sigma^2} \exp\{\sigma^{-2}\} \left( 2 \int_0^\infty x^3 \exp(-0.07x^2) + \left( \frac{4(3/(2\cdot0.07))^{3/2}}{\sigma} \right) \exp((-3/2)^2) \right) \\
&< \frac{17}{3\sqrt{2\pi}\sigma^2} \exp\{\sigma^{-2}\} \cdot 16 < \frac{60}{\sigma^2} \exp\{\sigma^{-2}\}.
\end{align*}
\]

This inequality differs from (3.20) because we transform it with the usage of inequality

\[
\sum_{i=-\infty}^{\infty} g(ih) = 2 \sum_{i=0}^{\infty} g(ih) \leq 2 \int_0^\infty g(x)dx + 4 hg(x_0),
\]

which is true for symmetric nonnegative unimodal on \([0, \infty)\) function \(g\) with (one of two) maximum in \(x_0\) and \(h > 0\). We use it to approximate the sum in (3.20) as follows

\[
\sum_{j=K_2}^{[ka/n]} |\tilde{x}(j)|^3 \exp(-0.07\tilde{x}^2(i)) \leq \int_{-\infty}^{\infty} |x|^3 \exp((-0.07)x^2)dx + \frac{4(3/(2\cdot0.07))^{(3/2)}}{\sigma} \exp(-(3/2)) < 16.
\]

Next (relations (3.22) in [12]):

\[
\sum_{i=K_1}^{K_0} \left| P(i) - \frac{1}{\sigma} \phi(\tilde{x}(i)) \right| \leq \frac{85}{12\sqrt{2\pi}\sigma} \exp\{\sigma^{-1}\} < \frac{3}{\sigma} \exp\{\sigma^{-1}\}.
\]

Thus

\[
I_2 \leq \frac{60}{\sigma^2} \exp\{\sigma^{-2}\} + \frac{3}{\sigma} \exp\{\sigma^{-1}\} < 0.077.
\]

At last, for \(I_3\) we have the estimation (inequality (3.22) in [12]):

\[
\begin{align*}
I_3 &\leq \frac{1}{12\sigma^2} \left[ \frac{1}{\sqrt{\pi}} + 1 + \frac{10}{\sqrt{2\pi}} \exp\{-1/(8\sigma^2)\} \right] \\
&+ \Phi(1/(2\sigma)) - \Phi(-1/(2\sigma)) + \Phi(-\delta\sigma + 1/(2\sigma)) \\
&\leq \frac{1}{12\sigma^2} \left[ \frac{1}{\sqrt{\pi}} + 1 + \frac{10}{\sqrt{2\pi}} \exp\{-1/(8\sigma^2)\} \right] \\
&+ \frac{1}{\sqrt{2\pi}\sigma} + \frac{\exp(-(\delta\sigma - 1/(2\sigma))^2/2)}{\sqrt{2\pi}(\delta\sigma - 1/(2\sigma))} < 0.011.
\end{align*}
\]
Provided that $\sigma > 55, n > 12 \cdot 10^4$, we obtain from the last considerations the inequalities

$$I_1 + I_2 + I_3 < 0.2203$$

Hence Berry-Esseen inequality looks like

$$|P(i \leq ka/n) - \frac{1}{2}| = |P(i > ka/n) - \frac{1}{2}| < 0.2203 < 1/4.$$ 

Thus in this case it follows that for $k \in (n/5, n/4)$

$$P(i > ka/n) \binom{n}{k} \leq \binom{n-1}{k-1}.$$ 

Using formula (18) for $\sigma$ it can be easily seen that the condition $\sigma \geq 55$ is satisfied when $a \in [n/5, n - n/5]$ and $n > 12 \cdot 10^4$. Next consideration helps us to reduce the possible values of $a$ for which (19) is valid to $a \in [C, n - C]$, for some constant $C$. Using Stirling formula it is easy to see that when $i \leq ka/n$, the following inequality is valid

$$\frac{(n-a)}{(k-i)} \binom{n}{k} < 2^{(n-a)}H\left(\frac{k-i}{n-a}\right) - nH\left(\frac{k}{n}\right)$$

$$\frac{\sqrt{\frac{k}{n} \left(1 - \frac{k}{n}\right)}}{\sqrt{\left(\frac{k-i}{n-a}\right) \left(1 - \frac{k-i}{n-a}\right) \left(1 - \frac{a}{n}\right)}} e^{\frac{1}{12n} + \frac{1}{12(k-i)} + \frac{1}{12(n-a-k+i)}}.$$

Since we can assume that $i < ka/n$ and that $n > 12 \cdot 10^4, a \leq n/5$ we obtain the estimation

$$\frac{(n-a)}{(k-i)} \binom{n}{k} \leq \left(\frac{k}{n}\right)^i \left(1 - \frac{k}{n}\right)^{a-i} \left(1 - \frac{a}{n}\right)^{-1/2} e^{10^{-3}}$$

$$\leq 1.1203 \left(\frac{k}{n}\right)^i \left(1 - \frac{k}{n}\right)^{a-i}.$$

This is true, because when $i \leq ka/n$ we have

$$(n-a)H\left(\frac{k-i}{n-a}\right) - H\left(\frac{k}{n}\right) \leq i \ln \frac{k}{n} + (a-i) \ln \left(1 - \frac{k}{n}\right).$$
Hence
\[
\sum_{i<ka/n} \binom{a}{i} \binom{n-a}{k-i} \leq 1.1203 \sum_{i<ka/n} \binom{a}{i} \binom{k}{i} \left(1 - \frac{k}{n}\right)^{a-i} \binom{n}{k}.
\] (20)

To estimate the sum
\[
\sum_{i<ka/n} \binom{a}{i} \binom{k}{i} \left(1 - \frac{k}{n}\right)^{a-i}
\]
we note that this is the probability $P_0$ that the sum of $a$ i.i.d. Bernulli variables exceeds the average and we use Berry-Esseen inequality [11] (we relax the coefficients a little):
\[
\left|P_0 - \frac{1}{2}\right| < \frac{\rho + 0.43\sigma_1^3}{3\sigma_1^2\sqrt{a}},
\]
where $\sigma_1^2 = \frac{k}{n} \left(1 - \frac{k}{n}\right) \geq \frac{5}{25}$ and $\rho = \sigma_1^2(1 - 2\sigma_1^2)$. Thus
\[
\left|P_0 - \frac{1}{2}\right| < \frac{0.71}{\sqrt{a}}
\]
and
\[
\sum_{i<ka/n} \binom{a}{i} \binom{n-a}{k-i} \leq 1.026 \sum_{i>ka/n} \binom{a}{i} \binom{k}{i} \left(1 - \frac{k}{n}\right)^{a-i} \binom{n}{k}
< 1.1203 \left(\frac{1}{2} + \frac{0.71}{\sqrt{a}}\right) \binom{n}{k}.
\]

R.h.s. of the last inequality is less than $\binom{n-1}{k}$ when
\[
1.1203 \left(\frac{1}{2} + \frac{0.71}{\sqrt{a}}\right) < \frac{3}{4}
\]
or when $a > 14$. Next if $a = n - b > n - n/5$, then we can rewrite the sum in the l.h.s of (20) as
\[
\sum_{i<kb/n} \binom{b}{i} \binom{n-b}{k-i}.
\]
To estimate the ratio
\[
\frac{\binom{n-b}{i}}{\binom{n}{k}},
\]
repeat the previous arguments for \( b \) instead of \( a \) to obtain

\[
\sum_{i < kb/n} \binom{b}{i} \binom{n-b}{k-i} \leq 3/4 \binom{n}{k}
\]

when \( b = n - a < 9 \).

Remark

For \( k \leq n/4, n < 12 \cdot 10^4 \) the inequality

\[
\sum_{i > ka/n} \binom{a}{i} \binom{n-a}{k-i} \leq \binom{n-1}{k}
\]

can be checked using the software Wolfram Mathematica, but it needs fast computer. For \( a \leq 14, a \geq n - 14, k \leq n/4 \) it can be done by hand.

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