WIGNER-VILLE DISTRIBUTION ASSOCIATED WITH THE QUATERNION OFFSET LINEAR CANONICAL TRANSFORMS

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Abstract. The Wigner-Ville distribution (WVD) and quaternion offset linear canonical transform (QOLCT) are a useful tools in signal analysis and image processing. The purpose of this paper is to define the Wigner-Ville distribution associated with quaternionic offset linear canonical transform (WVD-QOLCT). Actually, this transform combines both the results and flexibility of the two transform WVD and QOLCT. We derive some important properties of this transform such as inversion and Plancherel formulas, we establish a version of Heisenberg inequality, Lieb’s theorem and we give the Poisson summation formula for the WVD-QOLCT.

keywords: Wigner-Ville distribution, Offset linear canonical transform, linear canonical transform, quaternionic transform, Heisenberg uncertainty.

1. Introduction

The Fourier transformation used for a simple description of the input-output relationships of the filters linear, occupies a privileged place in the theory and signal processing. However, this transformation can not give a temporal signal, it only gives a global frequency information: its natural field of application is analysis stationary signals. So, as soon as we consider modulated signals or non-process stationary the Fourier transform becomes insufficient to study this type of signal. One solution to this problem is to associating to directly search a tool adapted to the study of non-stationary signal, without direct reference to the methods resulting from the stationary case. In this case, a particular axis of interest has been manifested for many years to a proposed transformation in Quantum Mechanics by E. P. Wigner [27] in 1932. This transformation allows to define what we will call the distribution of Wigner-Ville (WVD) in reference and tribute to J. City which first introduced this same notion in Signal Theory. In recent years, this distribution has served as a useful analysis tool in many fields as diverse as optics, biomedical engineering, signal processing and image processing. Due to the large applications of the linear canonical transform (LCT) [28] in several area including radar analysis, signal processing and optics [22, 23, 25]. The LCT has received attention since 1970 is introduced integral transform with four parameters (a,b,c,d) [8, 21]. A lot of authors were interested to study LCT. This transform is also known under the affine Fourier transform [1], and the generalized Fresnel Fourier transform [17]. Moreover the Fourier transform [5] and the Fresnel transform [12] are all special cases of the LCT. In [23], the LCT is generalized by introducing two extra parameters, one corresponding to time shift and an other to frequency modulation. This generalized of LCT is called offset LCT (OLCT) [24, 29], and it is known under six parameters linear
transform. These two parameters make the OLCT more general and flexible than LCT, in consequence the OLCT can apply to most electrical and optical signal systems. The two-sided quaternionic Fourier transform (QFT) was introduced in [9]. The QFT has many application in large domains, in [9] the QFT used in analysis of 2D linear time invariant dynamic systems, In [4] the authors used the QFT to design a digital color image water marking scheme, in [26] the QFT is used for filtering color images.

The main objective of this work is the combination between the WVD, QFT and the OLCT, in order to get the Quaternion Offset Wigner-Ville distribution associated to linear canonical transforms (WVD-QOLCT). The paper is organized as follows, in section 2, we recall the main results about the quaternion algebra and harmonic analysis related to QFT, QLCT and QOLCT. In section 3, we introduce the WVD-QOLCT, and establish its important properties. The section 4 is devoted to give the analogue of Heisenberg’s inequality, Poisson summation formula, and Lieb’s theorem for the WVD-QOLCT. In section 5, we conclude this paper.

2. Preliminaries

2.1. The quaternion algebra.

In the present section we collect some basic facts about quaternions, which will be needed throughout the paper. For all what follows, let $\mathbb{H}$ be the Hamiltonian skew field of quaternions:

$$\mathbb{H} = \{ q = q_0 + iq_1 + jq_2 + kq_3; \ q_0, q_1, q_2, q_3 \in \mathbb{R} \},$$

which is an associative noncommutative four-dimensional algebra.

where the elements $i, j, k$ satisfy the Hamilton’s multiplication rules:

$$ij = -ji = k; \ jq = -kj = i; \ ki = -ik = j; \ i^2 = j^2 = k^2 = -1.$$ 

In this way the Quaternionic algebra can be seen as an extension of the complex field $\mathbb{C}$.

Quaternions are isomorphic to the Clifford algebra $Cl_{(0,2)}$ of $\mathbb{R}^{(0,2)}$:

$$\mathbb{H} \cong Cl_{(0,2)}. \quad (2.1)$$

The scalar part of a quaternion $q \in \mathbb{H}$ is $q_0$ denoted by $Sc(q)$, the non scalar part (or pure quaternion) of $q$ is $iq_1 + jq_2 + kq_3$ denoted by $Vec(q)$.

The quaternion conjugate of $q \in \mathbb{H}$, given by

$$\overline{q} = q_0 - iq_1 - jq_2 - kq_3,$$

is an anti-involution, namely,

$$\overline{qp} = \overline{p} \ \overline{q}, \ \overline{p + q} = \overline{p} + \overline{q}, \ \overline{p} = p.$$

The norm or modulus of $q \in \mathbb{H}$ is defined by

$$|q|_Q = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$ 

Then, we have
\[ |pq|_Q = |p|_Q|q|_Q. \]

In particular, when \( q = q_0 \) is a real number, the module \( |q|_Q \) reduces to the ordinary Euclidean module \( |q| = \sqrt{q_0^2} \).

It is easy to verify that \( 0 \neq q \in \mathbb{H} \) implies:

\[ q^{-1} = \frac{\bar{q}}{|q|_Q^2}. \]

Any quaternion \( q \) can be written as \( q = |q|_Qe^{i\theta} \), where \( e^{i\theta} \) is understood in accordance with Euler’s formula

\[ e^{i\theta} = \cos (\theta) + \mu \sin (\theta), \]

where \( \theta = \arctan \frac{|\text{Vec}(q)|}{S_{c}(q)} \), \( 0 \leq \theta \leq \pi \) and \( \mu := \frac{\text{Vec}(q)}{|\text{Vec}(q)|} \)

verifying \( \mu^2 = -1 \).

Let \( \lambda \) be a pure unit quaternion, \( \lambda^2 = -1 \), clearly, we have for all \( x \in \mathbb{R}^2 \),

\[ |e^{\lambda x}|_Q = 1. \quad (2.2) \]

In this paper, we will study the quaternion-valued signal \( f : \mathbb{R}^2 \to \mathbb{H} \), \( f \) which can be expressed as

\[ f = f_0 + if_1 + jf_2 + kf_3, \]

with \( f_m : \mathbb{R}^2 \to \mathbb{R} \) for \( m = 0, 1, 2, 3 \). Let us introduce the canonical inner product for quaternion valued functions \( f, g : \mathbb{R}^2 \to \mathbb{H} \), as follows:

\[ < f, g > = \int_{\mathbb{R}^2} f(t) \overline{g(t)} dt, \ dt = dt_1 dt_2. \quad (2.3) \]

Hence, the natural norm is given by

\[ |f|_{2,Q} = \sqrt{< f, f >} = (\int_{\mathbb{R}^2} |f(t)|^2_Q dt)^{\frac{1}{2}}, \]

and the quaternion module \( L^2(\mathbb{R}^2, \mathbb{H}) \), is given by

\[ L^2(\mathbb{R}^2, \mathbb{H}) = \{ f : \mathbb{R}^2 \to \mathbb{H}, \ |f|_{2,Q} < \infty \}. \]

Furthermore, for \( 2 < p < \infty \), we introduce the quaternion modules \( L^p(\mathbb{R}^2, \mathbb{H}) \), as

\[ L^p(\mathbb{R}^2, \mathbb{H}) = \{ f : \mathbb{R}^2 \to \mathbb{H}, \ |f|_{p,Q}^p = \int_{\mathbb{R}^2} |f(x)|^p_Q dx < \infty \}. \]

From (2.3), we obtain the quaternion Schwartz’s inequality

\[ \forall f, g \in L^2(\mathbb{R}^2, \mathbb{H}) : \quad \left| \int_{\mathbb{R}^2} f(x)\overline{g(x)}dx \right|^2_Q \leq \int_{\mathbb{R}^2} |f(x)|^2_Q dx \int_{\mathbb{R}^2} |g(x)|^2_Q dx. \]

Besides the quaternion units \( i, j, k \), we will use the following real vector notation: \( t = (t_1, t_2) \in \mathbb{R}^2, \ |t|^2 = t_1^2 + t_2^2, \ f(t) = f(t_1, t_2), \ dt = dt_1 dt_2. \)
2.2. The general two-sided quaternion Fourier transform.

In this subsection, we begin by defining the two-sided QFT, and reminder some properties for this transform.

Let us define the two-sided QFT and provide some properties used in the sequel.

**Definition 2.1** (\([14]\)).

Let \(\lambda, \mu \in \mathbb{H}\), be any two pure unit quaternions, i.e., \(\lambda^2 = \mu^2 = -1\). For \(f \in L^1(\mathbb{R}^2, \mathbb{H})\), the two-sided QFT with respect to \(\lambda; \mu\) is

\[
F_{\lambda, \mu} \{ f \}(u) = \int_{\mathbb{R}^2} e^{-\lambda u_1 t_1} f(t) e^{-\mu u_2 t_2} dt, \quad \text{where } t, u \in \mathbb{R}^2.
\]

We define a new module of \(F \{ f \}_{\lambda, \mu}\) as follows:

\[
\| F_{\lambda, \mu} \{ f \} \|_Q := \sqrt{\sum_{m=0}^{m=3} |F_{\lambda, \mu} \{ f_m \}|^2}.
\]

Furthermore, we define a new \(L^2\)-norm of \(F \{ f \}\) as follows:

\[
\| F_{\lambda, \mu} \{ f \} \|_{2,Q} := \sqrt{\int_{\mathbb{R}^2} \| F_{\lambda, \mu} \{ f \}(y) \|_Q^2 dy}.
\]

It is interesting to observe that \(\| F_{\lambda, \mu} \{ f \} \|_Q\) is not equivalent to \(|F_{\lambda, \mu} \{ f \}|_Q\) unless \(f\) is real valued.

**Lemma 2.2** (Dilation property). (see page 50 in [6])

Let \(k_1, k_2\) be a positive scalar constants, we have

\[
F_{\lambda, \mu} \{ f(t_1, t_2) \} \left( \frac{u_1}{k_1}, \frac{u_2}{k_2} \right) = k_1 k_2 F_{\lambda, \mu} \{ f(k_1 t_1, k_2 t_2) \} (u_1, u_2).
\]

By following the proof of Theorem (3.2) in [7], and replacing \(i\) by \(\lambda, j\) by \(\mu\) we obtain the next lemma.

**Lemma 2.3.** (QFT Plancherel)

Let \(f \in L^2(\mathbb{R}^2, \mathbb{H})\), then

\[
\int_{\mathbb{R}^2} \| F_{\lambda, \mu} \{ f \}(u) \|_Q^2 du = 4\pi^2 \int_{\mathbb{R}^2} |f(t)|_Q^2 dt.
\]

**Lemma 2.4.** If \(f \in L^2(\mathbb{R}^2, \mathbb{H}), \frac{\partial^m \partial^n}{\partial t_1^m \partial t_2^n} f\) exist and are in \(L^2(\mathbb{R}^2, \mathbb{H})\) for \(m, n \in \mathbb{N}_0\) then

\[
F_{\lambda, \mu} \left\{ \frac{\partial^m \partial^n}{\partial t_1^m \partial t_2^n} f \right\} (u) = (\lambda u_1)^m F_{\lambda, \mu} \{ f \} (u) (\mu u_2)^n.
\]

Proof. See ([6], Thm. 2.10).

**Lemma 2.5.** [Inverse QFT] (see [16])
If \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), and \( F^\lambda \mu \{ f \} \in L^1(\mathbb{R}^2, \mathbb{H}) \), then the two-sided QFT is an invertible transform and its inverse is given by

\[
f(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\lambda u_1 t} F^\lambda \mu \{ f(t) \}(u) e^{\mu u_2 t} du.
\] (2.10)

3. The Offset Quaternionic Linear Canonical Transform

Morais et al [18] introduce the quaternion linear canonical transform (QLCT). They consider two real matrices

\[
A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \mathbb{R}^{2 \times 2}.
\]

with \( a_1 d_1 - b_1 c_1 = 1 \), \( a_2 d_2 - b_2 c_2 = 1 \),

Eckhard Hitzer [15] generalize the definitions of [18] to be: the two-sided QLCT of signals \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), is defined by

\[
\mathcal{L}_{A_1, A_2}^\lambda \mu \{ f \}(u) = \begin{cases} \int_{\mathbb{R}^2} K_{A_1}^\lambda (t_1, u_1) f(t) K_{A_2}^\mu (t_2, u_2) dt, & b_1, b_2 \neq 0; \\
\sqrt{d_1} e^{\lambda \frac{a_1 d_1}{2} u_1^2} f(d_1 u_1, t_2) K_{A_2}^\mu (t_2, u_2), & b_1 = 0, b_2 \neq 0; \\
\sqrt{d_2} K_{A_1}^\lambda (t_1, u_1) f(t_1, d_2 u_2) e^{\mu \frac{c_2 d_2}{2} u_2^2}, & b_1 \neq 0, b_2 = 0; \\
\sqrt{d_1 d_2} e^{\lambda \frac{a_1 d_1}{2} u_1^2} f(d_1 u_1, d_2 u_2) e^{\mu \frac{c_2 d_2}{2} u_2^2}, & b_1 = b_2 = 0. \end{cases}
\] (3.1)

with \( \lambda, \mu \in \mathbb{H} \), denote two pure unit quaternions, \( \lambda^2 = \mu^2 = -1 \), including the cases \( \lambda = ±\mu \),

\[
K_{A_1}^\lambda (t_1, u_1) = \frac{1}{\sqrt{\lambda 2 \pi b_1}} e^{\lambda (a_1 t_1^2 - 2t_1 u_1 + d_1 u_1^2)/2b_1}, \quad K_{A_2}^\mu (t_2, u_2) = \frac{1}{\sqrt{\mu 2 \pi b_2}} e^{\mu (a_2 t_2^2 - 2t_2 u_2 + d_2 u_2^2)/2b_2},
\]

In [18], the properties of the right-sided QLCT and its uncertainty principles are studied in detail. El Haoui et al [11] introduced and studied the QOLCT, and established its properties and uncertainty principles. Let’s give the definitions of Quaternionic offset linear canonical transform as follows:

**Definition 3.1.** Let \( A_l = \begin{bmatrix} a_l & b_l \\ c_l & d_l \end{bmatrix} \tau_l \),

the parameters \( a_l, b_l, c_l, d_l, \tau_l, \eta_l \in \mathbb{R} \) such that \( a_l d_l - b_l c_l = 1 \), for \( l = 1, 2 \)

the two-sided quaternionic offset linear canonical transform (QOLCT) of a signal \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), is given by
\( \mathcal{O}_{A_1, A_2}^\lambda \{ f(t) \} (u) = \begin{cases} 
abla_{A_1} (t_1, u_1) f(t) K_{A_2}^\mu (t_2, u_2) dt, & b_1, b_2 \neq 0; \\
abla_{A_1} (t_1, u_1) f (t_1, d_2 (u_2 - \tau_2)) e^{\mu (\sqrt{\lambda} (u_2 - \tau_2)^2 + u_2 \tau_2)}, & b_1 \neq 0, b_2 = 0; \\
abla_{A_1} (t_1, u_1) f (d_1 (u_1 - \tau_1), d_2 (u_2 - \tau_2)) e^{\mu (\sqrt{\lambda} (u_2 - \tau_2)^2 + u_2 \tau_2)}, & b_1 = 0, b_2 \neq 0; \\
abla_{A_1} (t_1, u_1) f (d_1 (u_1 - \tau_1), d_2 (u_2 - \tau_2)) e^{\mu (\sqrt{\lambda} (u_2 - \tau_2)^2 + u_2 \tau_2)}, & b_1 = b_2 = 0. 
\end{cases} \)

Where

\( K_{A_1}^\lambda (t_1, u_1) = \frac{1}{\sqrt{\lambda} 2 \pi b_1} e^{\lambda (a_1 t_1^2 - 2 t_1 (u_1 - \tau_1) - 2 u_1 (d_1 \tau_1 - b_1 \eta_1) + d_1 (u_1^2 + \tau_1^2)) \frac{1}{b_1}}, \)

for \( b_1 \neq 0, \) \hspace{1cm} (3.2)

and

\( K_{A_2}^\mu (t_2, u_2) = \frac{1}{\sqrt{\mu} 2 \pi b_2} e^{\mu (a_2 t_2^2 - 2 t_2 (u_2 - \tau_2) - 2 u_2 (d_2 \tau_2 - b_2 \eta_2) + d_2 (u_2^2 + \tau_2^2)) \frac{1}{b_2}}, \)

for \( b_2 \neq 0, \) \hspace{1cm} (3.3)

with

\( \frac{1}{\sqrt{\lambda}} = e^{-\lambda \frac{\tau_1}{2}}, \quad \frac{1}{\sqrt{\mu}} = e^{-\mu \frac{\tau_2}{2}}. \)

The left-sided and right-sided QOLCTs can be defined by placing the two kernel factors both on the left or on the right, respectively.

We remark that, when \( \tau_1 = \tau_2 = \eta_1 = \eta_2 = 0, \) the two-sided QOLCT reduces to the QLCT.

Also, when \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \) the conventional two-sided QFT is recovered. Namely,

\[ \mathcal{O}_{A_1, A_2}^\lambda \{ f(t) \} (u) = \frac{1}{\sqrt{\lambda} 2 \pi} \int_{\mathbb{R}^2} e^{-\lambda t_1 u_1} f(t) e^{-\mu t_2 u_2} dt \frac{1}{\sqrt{\mu} 2 \pi} e^{-\lambda \frac{\tau_1}{2} - \mu \frac{\tau_2}{2}}, \]

where \( \mathcal{F}^{\lambda, \mu} \{ f \} \) is the QFT of \( f \) given by (2.4).

The following lemma gives the relationships of two-sided QOLCTs and two-sided QFTs of 2D quaternion-valued signals.

**Lemma 3.2.** The QOLCT of a signal \( f \in L^1 (\mathbb{R}^2, \mathbb{H}) \) can be reduced to the QFT

\[ \mathcal{O}_{A_1, A_2}^\lambda \{ f(t) \} (u_1, u_2) = \mathcal{F}^{\lambda, \mu} \{ h(t) \} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right), \]

with
$$h(t) = \frac{1}{\sqrt{2\pi\lambda b_1}} e^{\frac{\lambda}{2} u_1 (d_1 t_1 - b_1 \eta_1) + \frac{\lambda}{2} (u_1^2 + \tau_1^2) + \frac{1}{2} t_1 \tau_1 + \frac{u_1^2}{2}} f(t)$$

$$\times e^{\mu\left[-\frac{\lambda}{2} u_2 (d_2 t_2 - b_2 \eta_2) + \frac{\mu}{2} (u_2^2 + \tau_2^2) + \frac{1}{2} t_2 \tau_2 + \frac{u_2^2}{2}\right]} \frac{1}{\sqrt{2\pi\mu b_2}}.$$

By using lemma 2.3 and (3.3), we get the inversion formula for the QOLCT,

**Theorem 3.3.** If $f$ and $O_{A_1,A_2}^{\lambda,\mu}\{f\}$ are in $L^1(\mathbb{R}^2, \mathbb{H})$, then the inverse transform of the QOLCT can be derived from that of the QFT, and we have

$$f(t) = \int_{\mathbb{R}^2} K_{A_1}^\lambda (t_1, u_1) O_{A_1,A_2}^{\lambda,\mu}\{f(t)\} (u_1, u_2) K_{A_2}^\mu (t_2, u_2) du.$$ 

**Theorem 3.4.** (Plancherel’s theorem of the QOLCT)

Every 2D quaternion-valued signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and its QOLCT are related to the Plancherel identity in the following way:

$$\|O_{A_1,A_2}^{\lambda,\mu}\{f\}\|_{2,Q} = |f|_{2,Q}.$$  \hspace{1cm} (3.5)

4. **Wigner-Ville distribution associated with quaternionic offset linear canonical transform**

The Fourier transform is a powerful tool to study the stationary signals, but it has become not sufficient for characterize the non-stationary signals. However, in practice, most natural signals are non stationary. In order to study a non stationary signal the Wigner-Ville distribution has become a suite tool for the analysis of the non stationary signals.

In this section, we are going to give the definition of Wigner-Ville distribution associated with the quaternionic offset linear canonical transform WVD-QOLCT, then, we will investigate its important properties, and establish the Heisenberg uncertainty principle, Poisson summation formula and Lieb’s theorem related for the WVD-QOLCT.

**Definition 4.1.**

Let $A_l = \begin{bmatrix} a_l & b_l \\ c_l & d_l \end{bmatrix}$, with $a_l, b_l, c_l, d_l, \tau_l, \eta_l \in \mathbb{R}$ such that $a_l d_l - b_l c_l = 1$, for $l = 1, 2$.

The Wigner-Ville distribution associated with the two-sided quaternionic offset linear canonical transform (WVD-QOLCT) of a signal $f \in L^1(\mathbb{R}^2, \mathbb{H})$, is given by
By the equation (4.1)

Proof.

WVD-QLCT given by Lemma 4.3.

we have,

\[
\mathcal{W}^{\lambda_1, \lambda_2}_{f,g}(t, u) = \begin{cases} 
    f_{R^2} K_{\lambda_1}^{\lambda}(s_1, u_1) f(t + \frac{\varepsilon_1}{2}) \mathcal{G}(t - \frac{\varepsilon_1}{2}) K_{\lambda_2}^{\mu}(s_2, u_2) ds, & b_1, b_2 \neq 0, \\
    \sqrt{d_1} e^{\lambda \frac{d_1}{2} (u_1 - \tau_1)^2 + u_1 \tau_1} f(t_1 + \frac{d_1 (u_1 - \tau_1)}{2}, t_2 + \frac{u_2}{2}) \\
    \times \mathcal{G}(t_1 - \frac{d_1 (u_1 - \tau_1)}{2}, t_2 - \frac{u_2}{2}) K_{\lambda_2}^{\mu}(s_2, u_2), & b_1 = 0, b_2 \neq 0; \\
    \sqrt{d_2} K_{\lambda_1}^{\lambda}(s_1, u_1) f(t_1 + \frac{\varepsilon_2}{2}, t_2 + \frac{d_2 (u_2 - \tau_2)}{2}) \\
    \times \mathcal{G}(t_1 - \frac{\varepsilon_2}{2}, t_2 - \frac{d_2 (u_2 - \tau_2)}{2}) e^{\mu \frac{d_2}{2} (u_2 - \tau_2)^2 + u_2 \tau_2}, & b_1 \neq 0, b_2 = 0; \\
    \sqrt{d_1 d_2} e^{\lambda \frac{d_1}{2} (u_1 - \tau_1)^2 + u_1 \tau_1} f(t_1 + \frac{d_1 (u_1 - \tau_1)}{2}, t_2 + \frac{d_2 (u_2 - \tau_2)}{2}) \\
    \times \mathcal{G}(t_1 - \frac{d_1 (u_1 - \tau_1)}{2}, t_2 - \frac{d_2 (u_2 - \tau_2)}{2}) e^{\mu \frac{d_2}{2} (u_2 - \tau_2)^2 + u_2 \tau_2}, & b_1 = b_2 = 0.
\end{cases}
\]

where \(K_{\lambda_1}^{\lambda}(s_1, u_1)\), and \(K_{\lambda_2}^{\mu}(s_2, u_2)\), are given respectively by (3.2), and (3.3).

Remark 4.2. It’s clear that if we take \(h_{f,g}(t, s) = f(t + \frac{\varepsilon_1}{2}) \mathcal{G}(t - \frac{\varepsilon_1}{2})\) for all \(t, s \in \mathbb{R}^2\), we have,

\[
\mathcal{W}^{\lambda_1, \lambda_2}_{f,g}(t, u) = \mathcal{O}^{\lambda_1 \lambda_2}_{A_1, A_2} \{h_{f,g}(t, s)\}(u). 
\] (4.1)

We note that when we take \(\tau_1 = \eta_1 = 0, l = 1, 2\) the WVD-QLCT reduces to the WVD-QOLCT\[2].

And by using (3.3), we obtain the relation between WVD-QOLCT and QFT:

Lemma 4.3.

\[
\mathcal{W}^{\lambda_1, \lambda_2}_{f,g}(t, u) = \mathcal{F}^{\mu \lambda} \{k_{f,g}(t, s)\} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right), 
\] (4.2)

where \(k_{f,g}(t, s) = \frac{1}{\sqrt{2 \pi b_1 b_2}} e^{\lambda - \frac{d_1}{2} (u_1 - \tau_1)^2 + u_1 \tau_1} + \frac{d_1 (u_1 - \tau_1)}{2} + \frac{d_2 (u_2 - \tau_2)}{2} \frac{1}{\sqrt{2 \pi b_1 b_2}} h_{f,g}(t, s)\)

\[
\times e^{\mu - \frac{d_2}{2} (u_2 - \tau_2)^2 + u_2 \tau_2} + \frac{d_2 (u_2 - \tau_2)}{2} + \frac{d_2 (u_2 - \tau_2)}{2} \frac{1}{\sqrt{2 \pi b_1 b_2}}.
\]

Now, we give the inversion formula for the WVD-QOCLT

Theorem 4.4.

If \(f, g\) and \(\mathcal{W}^{\lambda_1, \lambda_2}_{f,g}\) are in \(L^2(\mathbb{R}^2, \mathbb{H})\), then, the inverse transform of QWVD-OCLT is given by

\[
f(v) = \frac{1}{|g|^{2, Q}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^{\lambda_1} \left( \frac{u_1 + \varepsilon_1}{2}, u_1 \right) \mathcal{W}^{\lambda_1, \lambda_2}_{f,g} \left( v + \frac{\varepsilon_1}{2}, u \right) K_{A_2}^{\mu} \left( \frac{u_2 + \varepsilon_2}{2}, u_2 \right) g(\varepsilon) dud\varepsilon.
\] (4.3)

Proof. By the equation (4.1)

\[
\mathcal{W}^{\lambda_1, \lambda_2}_{f,g}(t, u) = \mathcal{O}^{\lambda_1 \lambda_2}_{A_1, A_2} \{f(t + \frac{\varepsilon_1}{2}) \mathcal{G}(t - \frac{\varepsilon_1}{2})\}(u).
\]

then, by theorem 3.3, we obtain
\[ f(t + \frac{s}{2})g(t - \frac{s}{2}) = \int_{\mathbb{R}^2} K_{A_1}^\lambda (t, u_1) f_{f,g}^{A_1,A_2} (t, u) K_{A_2}^\mu (t, u) du. \]

By taking \( v = t + \frac{s}{2} \) and \( \varepsilon = t - \frac{s}{2} \) we get \( t = \frac{v + \varepsilon}{2} \) and
\[ f(v)g(\varepsilon) = \int_{\mathbb{R}^2} K_{A_1}^\lambda \left( \frac{v_1 + \varepsilon_1}{2}, u_1 \right) f_{f,g}^{A_1,A_2} \left( \frac{v + \varepsilon}{2}, u \right) K_{A_2}^\mu \left( \frac{v_2 + \varepsilon_2}{2}, u_2 \right) du. \] (4.4)

Multiplying both sides of (4.4) from the right by \( g \) and integrating with respect to \( d\varepsilon \) we get
\[ f(v) \int_{\mathbb{R}^2} |g(\varepsilon)|^2 d\varepsilon = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^\lambda \left( \frac{v_1 + \varepsilon_1}{2}, u_1 \right) f_{f,g}^{A_1,A_2} \left( \frac{v + \varepsilon}{2}, u \right) K_{A_2}^\mu \left( \frac{v_2 + \varepsilon_2}{2}, u_2 \right) g(\varepsilon) dud\varepsilon. \] (4.5)

Consequently,
\[ f(v) = \frac{1}{|g|^2_{2,Q}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^\lambda \left( \frac{v_1 + \varepsilon_1}{2}, u_1 \right) f_{f,g}^{A_1,A_2} \left( \frac{v + \varepsilon}{2}, u \right) K_{A_2}^\mu \left( \frac{v_2 + \varepsilon_2}{2}, u_2 \right) g(\varepsilon) dud\varepsilon. \] (4.6)

The following theorem gives the Plancherel’s identity for the WVD-QOLCT,

**Theorem 4.5** (Plancherel’s theorem for WVD-QOLCT).

Let \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \), then we have,
\[ \| f_{f,g}^{A_1,A_2} \|_{2,Q}^2 = |f|_{2,Q}^2 |g|_{2,Q}^2. \] (4.7)

**Proof.** We have by the equality (4.1)
\[ f_{f,g}^{A_1,A_2} (t, u) = O_{A_1,A_2}^{\lambda,\mu} \{ h_{f,g}(t, .) \}(u), \]
and the Plancherel formula for the QOLCT (3.5)
\[ \| O_{A_1,A_2}^{\lambda,\mu} \{ f \} \|_{2,Q} = |f|_{2,Q}. \]

So
\[ \| f_{f,g}^{A_1,A_2} \|_{2,Q} = \| O_{A_1,A_2}^{\lambda,\mu} \{ h_{f,g} \} \|_{2,Q} = |h_{f,g}|_{2,Q} \]
\[ = \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(t + \frac{s}{2})g(t - \frac{s}{2})|^2 dtds \right)^{\frac{1}{2}} \]
\[ = \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(u)g(v)|^2 dudv \right)^{\frac{1}{2}} \]
\[ = \int_{\mathbb{R}^2} |f(u)|^2 du \int_{\mathbb{R}^2} |g(v)|^2 dv \]
\[ = |f|_{2,Q}^2 |g|_{2,Q}^2. \]

□
**Theorem 4.7.** (Heisenberg QOLCT)  

Let \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \) for \( k = 1, 2 \), then

\[
|s_k f(t)|^2 \geq \frac{1}{16\pi^2} |f(s)|^2, \tag{4.8}
\]

The next theorem states the Heisenberg’s uncertainty principle for the WVD-QOLCT.

**Theorem 4.8.**

Let \( f, g \in \mathcal{S}(\mathbb{R}^2, \mathbb{H}) \). We have the following inequality

\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |s_k f(t) + \frac{s}{2} g(t - \frac{s}{2})|^2 dt ds \right) \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \| \frac{\xi_k}{2\pi b_k} \mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{ f(s) \} (\xi) \|_{2,Q}^2 dt ds \right) \geq \frac{1}{16\pi^2} |f(s)|^2 \tag{4.9}
\]

**Proof.** Let \( h_{f,g} \), be rewritten as in remark [12].

As \( f, g \in \mathcal{S}(\mathbb{R}^2, \mathbb{H}) \), we obtain that \( h_{f,g}(\cdot,s) \in L^2(\mathbb{R}^2, \mathbb{H}) \). Therefore by applying (4.9), we get

\[
|s_k h_{f,g}(\cdot,s)|^2 \geq \frac{1}{16\pi^2} |h_{f,g}(\cdot,s)|^2. \tag{4.10}
\]

By taking the square root on both sides of (4.10) and integrating both sides with respect to \( dt \), we get
\[
\int_{\mathbb{R}^2} \left( \left( \int_{\mathbb{R}^2} |s_k f(t + \frac{s}{2})|^2 d\tau(t - \frac{s}{2}) \right)^\frac{1}{2} \left( \int_{\mathbb{R}^2} \frac{\xi_k}{2\pi b_k} W^{A_1,A_2}_{f,g}(t, \xi) \|Q\| d\xi \right)^\frac{1}{2} \right) dt \\
\geq \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|h_{f,g}(t, s)\|^2_Q d\tau s dt. \quad (4.11)
\]

Now, by applying the Schwartz’s inequality to the left hand side of (4.11), and using (4.7), we obtain

\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |s_k f(t + \frac{s}{2})| d\tau(t - \frac{s}{2}) d\tau s dt \right)^\frac{1}{2} \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\xi_k}{2\pi b_k} W^{A_1,A_2}_{f,g}(t, \xi) \|Q\| d\xi dt \right)^\frac{1}{2} \geq \frac{1}{4\pi} |f|^2_{L^2,Q} |g|^2_{L^2,Q}.
\]

Therefore, the proof is complete. \(\square\)

4.1. Poisson summation formula.

It is well known that, the Poisson summation formula play an important role in mathematics, due to its various applications in signal processing. In this section we generalize the above mentioned formula into WVD-QOLCT domain.

**Proposition 4.9. (see [7])** Let \(f \in L^1(\mathbb{R}^2, \mathbb{H})\), then

\[
\sum_{(k_1,k_2) \in \mathbb{Z}^2} f(s_1 + k_2, s_2 + k_2) = \sum_{(k_1,k_2) \in \mathbb{Z}^2} e^{2\pi i k_1 s_1} \hat{f}(k_1, k_2) e^{2\pi j k_2 s_2} \quad (4.12)
\]

where \(\hat{f}\) is the QFT of \(f\) defined by \(\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi is_1 \xi_1} f(s) e^{-2\pi js_2 \xi_2} ds\).

Now, we give a version of Poisson summation formula for the WVD-QOLCT,

**Theorem 4.10.** Let \(f, g \in L^2(\mathbb{R}^2, \mathbb{H})\), then

\[
\sum_{(k_1,k_2) \in \mathbb{Z}^2} e^{\frac{i}{\pi}(s_1 + k_1) \tau_1 + \frac{a}{\pi b_1} k_1^2} f(t + \frac{s}{2}) \overline{g}(t - \frac{s}{2}) e^{\frac{i}{\pi}(s_2 + k_2) \tau_2 + \frac{a}{\pi b_2} k_2^2} = \\
\sqrt{2\pi i b_1} \sum_{(k_1,k_2) \in \mathbb{Z}^2} e^{2\pi i k_1 s_1} e^{2\pi i k_1 (d_1 \tau_1 - b_1 \gamma_1) - \frac{a}{\pi b_1} (4\pi^2 k_1^2 + \gamma_1^2)} W^{A_1,A_2}_{f,g}(t, (2\pi b_1 k_1, 2\pi b_2 k_2))
\]

\[
\times e^{2\pi j k_2 s_2} e^{2\pi j k_2 (d_2 \tau_2 - b_2 \gamma_2) - \frac{a}{\pi b_2} (4\pi^2 k_2^2 + \gamma_2^2)} \sqrt{2\pi j b_2}.
\]

**Proof.** Let \(\omega_{f,g}(t, s) = e^{\frac{i}{\pi}(s_1 + k_1) \tau_1 + \frac{a}{\pi b_1} k_1^2} f(t + \frac{s}{2}) \overline{g}(t - \frac{s}{2}) e^{\frac{i}{\pi}(s_2 + k_2) \tau_2 + \frac{a}{\pi b_2} k_2^2} \).

As \(f, g \in L^2(\mathbb{R}^2, \mathbb{H})\), we have by Hölder’s inequality \(\omega_{f,g} \in L^1(\mathbb{R}^2, \mathbb{H})\), then by proposition 4.9 we have

\[
\sum_{(k_1,k_2) \in \mathbb{Z}^2} \omega_{f,g}(t, s_1 + k_2, s_2 + k_2) = \sum_{(k_1,k_2) \in \mathbb{Z}^2} e^{2\pi i k_1 s_1} \mathcal{F}^{i,j} \{\omega_{f,g}(t, (s_1, s_2))\}(2\pi k_1, 2\pi k_2) e^{2\pi j k_2 s_2}.
\]

Applying (4.2) leads to

\[
\sum_{(k_1,k_2) \in \mathbb{Z}^2} e^{\frac{i}{\pi}(s_1 + k_1) \tau_1 + \frac{a}{\pi b_1} k_1^2} f(t + \frac{s}{2}) \overline{g}(t - \frac{s}{2}) e^{\frac{i}{\pi}(s_2 + k_2) \tau_2 + \frac{a}{\pi b_2} k_2^2} = 
\]
\[
\sqrt{2\pi i b_1} \sum_{(k_1, k_2) \in \mathbb{Z}^2} e^{2\pi ik_1s_1} e^{2\pi i k_1(d_1 \tau_1 - b_1 \eta_1) - \frac{d_1^2}{2\pi^2} (4\pi^2 b_1^2 k_1^2 + \tau_1^2)} W_{f,g}^{A_1,A_2}(t, (2\pi b_1 k_1, 2\pi b_2 k_2)) \\
\times e^{2\pi i k_2 s_2} e^{2\pi i k_2(d_2 \tau_2 - b_2 \eta_2) - \frac{d_2^2}{2\pi^2} (4\pi^2 b_2^2 k_2^2 + \tau_2^2)} \sqrt{2\pi j b_2}.
\]

\[\square\]

4.2. Lieb’s theorem.

In this part of this paper, we are going to give a version of Lieb’s theorem for the WVD-QOLCT.

In the following theorem \cite{3}, we state Lieb’s theorem related to the QLCT.

**Theorem 4.11.** If \(1 \leq p \leq 2\) and let \(q\) be such that \(\frac{1}{p} + \frac{1}{q} = 1\), then, for all \(f \in L^p(\mathbb{R}^2, \mathbb{H})\), it holds that

\[
|\mathcal{L}_{A_1,A_2}^{i,j} \{f\}|_{q,Q} \leq \frac{|b_1 b_2|^{\frac{1}{p} + \frac{1}{q}}}{2\pi} |f|_{p,Q}.
\]

**Proof.** For the proof see \cite{3}. \[\square\]

**Theorem 4.12** (Lieb’s theorem associated with the WVD-QOLCT).

Let \(2 \leq p < \infty\) and \(f, g \in L^2(\mathbb{R}^2, \mathbb{H})\). Then

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W_{f,g}^{A_1,A_2}(t, u)|^p du dt \leq C \frac{|b_1 b_2|^{\frac{p}{2} + \frac{1}{p}}}{(2\pi)^p} |f|_{2,Q}^p |g|_{2,Q}^p,
\]

where \(C\) is a positive constant.

Before proving this theorem, we need the following lemma,

**Lemma 4.13.** Let

\[
A_l = \begin{bmatrix}
  a_l & b_l \\
  c_l & d_l
\end{bmatrix}
\]

and

\[
B_l = \begin{bmatrix}
  a_l & b_l \\
  c_l & d_l
\end{bmatrix},
\]

with \(a_l d_l - b_l c_l = 1\) for \(l = 1, 2\).

For \(f \in L^1(\mathbb{R}^2, \mathbb{H})\), we have the relation:

\[
\mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{f\}(u) = e^{\lambda(2t_1 \tau_1 - 2u_1(d_1 \tau_1 - b_1 \eta_1))} e^{\mu d_1 \tau_1} L_{B_1,B_2} \{f\}(u) e^{\mu d_2 \tau_2} e^{\mu(2t_2 \tau_2 - 2u_2(d_2 \tau_2 - b_2 \eta_2))}.
\]

**Proof.** To prove this lemma we just use the definitions of the QOLCT and QLCT to obtain the result. \[\square\]

Now we give a demonstration of the theorem \ref{4.12}.

**Proof.**
We have by the equation (4.14),

\[
\left( \int_{\mathbb{R}^2} |\mathcal{W}^{A_1,A_2}_{f,g}(t,u)|_Q^p \, du \right)^\frac{1}{p} = \left( \int_{\mathbb{R}^2} |\mathcal{O}^{A_1,A_2}_{A_1,A_2} (f(t + \frac{s}{2})\mathcal{F}(t - \frac{s}{2}) (u))|_Q^p \, du \right)^\frac{1}{p} = \left( \int_{\mathbb{R}^2} |\mathcal{L}^{A_1,A_2}_{B_1,B_2} (f(t + \frac{s}{2})\mathcal{F}(t - \frac{s}{2}) (u))|_Q^p \, du \right)^\frac{1}{p} \leq \frac{|b_1b_2|^{\frac{1}{2} - \frac{1}{p}}}{(2\pi)^{\frac{1}{p}}} \left( \int_{\mathbb{R}^2} |f(t + \frac{s}{2})\mathcal{F}(t - \frac{s}{2})|_Q^q ds \right)^\frac{q}{p}.
\]

In the last equality we used (4.13).

Furthermore,

\[
\int_{\mathbb{R}^2} |\mathcal{W}^{A_1,A_2}_{f,g}(t,u)|_Q^p \, du \leq \frac{|b_1b_2|^{\frac{1}{2} - \frac{1}{p}}}{(2\pi)^{\frac{1}{p}}} \left( \int_{\mathbb{R}^2} |f(t + \frac{s}{2})\mathcal{F}(t - \frac{s}{2})|_Q^q ds \right)^\frac{q}{p}
\]

integrating both sides of the last equality with respect to \(dt\) yields

\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |\mathcal{W}^{A_1,A_2}_{f,g}(t,u)|_Q^p \, du \right) \, dt \leq \frac{|b_1b_2|^{\frac{1}{2} - \frac{1}{p}}}{(2\pi)^{\frac{1}{p}}} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(t + \frac{s}{2})\mathcal{F}(t - \frac{s}{2})|_Q^q ds \right)^\frac{q}{p} \, dt.
\]

Using relation (3.3) in the proof of theorem 1 in [20], we have

\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(t + \frac{s}{2})\mathcal{F}(t - \frac{s}{2})|_Q^q ds \right)^\frac{q}{p} \, dt \leq C \|[f]_{2,Q}|g|_{2,Q}\|^p,
\]

where \(C\) is a positive constant.

Consequently, we obtain

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathcal{W}^{A_1,A_2}_{f,g}(t,u)|_Q^p \, dudt \leq C \frac{|b_1b_2|^{\frac{1}{2} - \frac{1}{p}}}{(2\pi)^{\frac{1}{p}}} \|[f]_{2,Q}|g|_{2,Q}\|^p.
\]

\[\square\]

5. Conclusion

Firstly, we introduced an extension of the Winger-Ville distribution to the quaternion algebra by means of the quaternionic offset linear canonical Fourier transform (QOLCT), namely the WVD-QOLCT transform. Secondly, the Plancherel theorem and the inversion formula have been demonstrated. Thirdly, Heisenberg’s uncertainty principle and Poisson summation formula associated with WVD-QOLCT were established by using the theorems obtained for the QFT and QOLCT. Finally the Lieb’s theorem related to the WVD-QOLCT transform was formulated by applying the Lieb’s theorem for the QLCT.

References

[1] S. Abe and J. T. Sheridan, *Optical operations on wave functions as the Abelian subgroups of the special affine Fourier transformation*, Opt. Lett.,(1994) vol. 19, no. 22, pp. 1801-1803.
[2] M. Bahri and F. M. Saleh Arif, Relation between Quaternion Fourier Transform and Quaternion Wigner-Ville Distribution Associated with Linear Canonical Transform, Journal of Applied Mathematics, vol. 2017, Article ID 3247364.

[3] M. Bahri, Resnawati, S. Musdalifah, A Version of Uncertainty Principle for Quaternion Linear Canonical Transform, Abstract and Applied Analysis, vol. 2018, Article ID 8732457, 7 pages, 2018.

[4] P. Bas, N. Le Bihan, Chassery J.M., Color image watermarking using quaternion Fourier transform, in: Proceedings of the IEEE International Conference on Acoustics Speech and Signal Processing, ICASSP, Hong-Kong, (2003), pp. 521-5

[5] R. Bracewell, The Fourier transform and its applications, third ed., New York: McGraw-Hill Book Co., (1986)

[6] T. Bülow, Hypercomplex spectral signal representations for the processing and analysis of images. Ph.D. Thesis, Institut für Informatik und Praktische Mathematik, University of Kiel, Germany,(1999).

[7] L.P. Chen, K.I. Kou, M.S. Liu, Pitt’s inequality and the uncertainty principle associated with the quaternion Fourier transform, J. Math. Anal. Appl., vol. 423, no. 1, pp. 681-700, 2015.

[8] S. A. Collins, Lens-system diffraction integral written in term of matrix optics, J. Opt. Soc. Amer., (1970), vol. 60, no. 9, pp. 1168-1177.

[9] T.A. Ell, Quaternion-Fourier transformations for analysis of two-dimensional linear time-invariant partial differential systems. In: Proceeding of the 32nd Conference on Decision and Control, San Antonio, Texas, (1993), pp. 1830-1841.

[10] Y. El Haoui and S. Fahlaoui, Benedicks-Amrein-Berthier type theorem related to the two-sided Quaternion Fourier transform, https://arxiv.org/abs/1807.04079.

[11] Y. El Haoui S. and S. Fahlaoui, Generalized Uncertainty Principles associated with the Quaternionic Offset Linear Canonical Transform, https://arxiv.org/abs/1807.04068v1

[12] Goodman J. W., Introduction to Fourier optics, 2nd ed., New York: McGraw-Hill, 1988

[13] W. Heisenberg, Über den anschaulichen inhalt der quanten theoretischen kinematik und mechanik. Zeitschrift fûr Physik (1927), 43, 172-198.

[14] E. Hitzer, S. J. Sangwine, The Orthogonal 2D Planes Split of Quaternions and Steerable Quaternion Fourier Transformations, in E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier transforms and wavelets", Trends in Mathematics 27, Birkhauser, Basel, 2013, pp. 15-39. DOI: 10.1007/978-3-0348-0603-9_2, Preprint: http://arxiv.org/abs/1306.2157

[15] E. Hitzer, New Developments in Clifford Fourier Transforms, Adv. in Appl. and Pure Math., Proc. of the 2014 Int. Conf. on Pure Math., Appl. Math., Comp. Methods (PMAMCM 2014), Santorini, Greece, July 2014, Math. Comp. in Sci. and Eng., Vol. 29.

[16] E. Hitzer, Two-Sided Clifford Fourier Transform with Two Square Roots of -1 in Cl(p; q). Adv. Appl. Clifford Algebras, (2014),24, pp. 313-332, DOI:10.1007/s00006-014-0441-9.
[17] D. F. V. James and G. S. Agarwal, The generalized Fresnel transform and its applications to optics, Opt. Commun., (1996) vol. 126, no. 5, pp. 207-212.

[18] K. I. Kou, J. Morais, Y. Zhang, Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis, Mathematical Methods in the Applied Sciences 36 (9) (2013), pp. 1028-1041. doi:10.1002/mma.2657.

[19] K. I. Kou and J. Morais, Asymptotic behaviour of the quaternion linear canonical transform and the Bochner-Minlos theorem, Applied Mathematics and Computation, vol. 247, no. 15, pp. 675-688, 2014.

[20] E. H. Lieb, Integral bounds for radar ambiguity functions and Wigner distributions, Journal of Mathematical Physics 31, 594 (1990).

[21] M. Moshinsky and C. Quesne, Linear canonical transform and their unitary representations, J. Math. Phys., (1971), vol. 12, pp. 1772-1783.

[22] H.M. Ozaktas, M.A. Kutay, Z. Zalevsky, The Fractional Fourier Transform with Applications in Optics and Signal Processing, Wiley, New York, (2000).

[23] S.C. Pei, J.J. Ding, Eigenfunctions of the offset Fourier, fractional Fourier, and linear canonical transforms, J. Opt. Soc. Am. A20 (2003) 522-532.

[24] A. Stern, Sampling of compact signals in offset linear canonical transform domains, Signal Image Video Process (2007) 359-367.

[25] R. Tao, B. Deng, Y. Wang, Fractional Fourier Transform and its Applications, Tsinghua University Press, Beijing, (2009).

[26] Viksas R. Dubey, Quaternion Fourier transform for colour images, International Journal of Computer Science and Information Technologies, Vol. 5 (3), (2014), 4411-4416.

[27] WIGNER E. P., On the quantum correction for thermodynamic equilibrium, Phys. Rev., (1932), 40., p. 749-759.

[28] Xu T. Z. and Li B. Z., Linear canonical transform and its application. Beijing: Science Press, (2013).

[29] Zhi X., Wei D., Zhang W., A generalized convolution theorem for the special affine Fourier transform and its application to filtering, Optik 127 (5) (2016) 2613-2616.

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