Stationary convection in a rotating binary magnetic fluid

J. Martínez-Mardones\textsuperscript{a}, D. Laroze\textsuperscript{a}, J. Bragard\textsuperscript{b}

\textsuperscript{a}Instituto de Física, Universidad Católica de Valparaíso, Casilla 4059, Valparaíso, Chile
\textsuperscript{b}Departamento de Física y Matemática Aplicada, Universidad de Navarra, E-31080 Pamplona, España

E-mail: jmartine@ucv.cl

Abstract. In this work we report theoretical and numerical results on convection for a binary magnetic mixture under rotation. We focus in the stationary convection for idealized boundary conditions and we obtain explicit expressions of convective thresholds in terms of the control parameters of the system. The effect of the magnetophoresis and Kelvin force are emphasized. Finally, we analyze the stabilizing effect of rotation on instability thresholds for aqueous suspensions.

1. INTRODUCTION

 Ferrofluids are superparamagnetic fluids formed by a stable colloidal suspension of ferromagnetic nanoparticles dispersed in a carrier liquid, and there are two main features that distinguish ferrofluids from ordinary fluids, the polarization force and the body couple. Beside, the study of the phenomenon of convective mechanism has found an application in high-power capacity transformer system where the ferrofluid is used as a material in the core as well as a coolant in the transform. To activate convective cooling, knowledge of concentration gradient, which will induce convection is required.

The first continuum description of the magnetic fluids was due to Neuringer et al.\cite{1}. Later, Finlayson \cite{2} studied the convective instability of a magnetic fluid for a fluid layer heated from below in the presence of a uniform vertical magnetic field. He discussed both shear free and rigid horizontal boundaries using the linear stability method. Convective instability for a rotating layer of a magnetic fluid have been studied by Gupta et al. \cite{3}, and Venkatasubramanian et al. \cite{4}. Auernhammer et al. \cite{5} formulated the Küppers-Lortz instability for a magnetic fluid. In addition, Rysking et al. \cite{6} using the nonequilibrium thermodynamics have derived a complete set of equations to describe ferrofluids in an external magnetic field. They did so in terms of a binary mixture where the magnetophoretic effect, as well as magnetic stresses, have been taken into account in the static and dynamic parts of the ferrofluid equations. Recently, when the magnetophoretic effect can be neglected, we analyzed the thermal convection for rotating ferrofluid for idealized boundary condition for the typical conductive state in the stationary case and was found that an analytical expression for the Rayleigh number as function of control parameters \cite{7}.

The purpose of this paper is to communicate our analysis of the influence of magnetophoretic effect and the coupling between the concentration and magnetic effect in the rotating convective
thresholds of the binary magnetic mixtures for idealized boundary conditions. To this aim, a binary mixture of ferrofluids heated from below and rotated around the vertical axis is considered. The linear stability analysis of the conduction state is performed in the framework of Navier-Stokes equations and we obtain explicit expressions of convective thresholds in terms of the control parameters of the system. The paper is organized as follows: In Sec. II, the basic hydrodynamic equations for binary magnetic mixture convection are presented and the linear stability analysis of the conduction state is performed. Finally, conclusions are presented in Sec. III.

2. THEORETICAL MODEL AND RESULTS
We consider a layer of incompressible binary magnetic fluid, of thickness \( d \), parallel to the xy-plane, with very large horizontal extension, in a gravitational field \( g \) and submitted to a vertical temperature gradient. The layer is rotating uniformly about the vertical with uniform angular velocity \( \omega \). The magnetic fluid properties can be model as electrically nonconducting superparamagnets, and it is assumed to be placed in a magnetic field \( H \) parallel to \( \hat{z} \), which would be homogeneous if the magnetic fluid were absent. Let us choose the z-axis such that \( g = -g\hat{z} \) and that the layer has its interfaces at \( z = -d/2 \) and \( z = d/2 \). A static temperature difference across the layer is imposed, \( T(z = -d/2) = T_0 + \Delta T \) and \( T(z = d/2) = T_0 \). Under the Boussinesq approximation, the dimensionless balance perturbation equations of the conduction states and the Maxwell equations read as

\[
\nabla \cdot \mathbf{v} = 0, \quad (1)
\]

\[
(\partial_t + \mathbf{v} \cdot \nabla)(\theta - M_1 \partial_z \phi) = (1 - M_1)w + \nabla^2 \theta + F \nabla^2 (c - M_2 \partial_z \phi) \quad (2)
\]

\[
\begin{align*}
\nabla \cdot \mathbf{v} & = 0, \\
(\partial_t + \mathbf{v} \cdot \nabla)c & = L \nabla^2 (c + \theta) - M_2 \partial_z \phi) \quad (3)
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot \mathbf{v} & = 0, \\
(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} & = -\nabla p_{eff} + \nabla^2 \mathbf{v} + T^{1/2} \mathbf{v} \times \hat{z} + \\
+ R_a \{ (1 + M_1) \theta - (\psi + M_1 \psi_M) c & - (M_1 - M_5) \partial_z \phi \} \hat{z} + \\
+ R_a M_1 (\theta - \psi_M c) \nabla (\partial_z \phi) \quad (4)
\end{align*}
\]

\[
\begin{align*}
(\partial_{zz} + M_5 \nabla^2) \phi - \partial_z (\theta - \psi_M c) & = 0 \quad (5)
\end{align*}
\]

\[
\nabla^2 \phi_{ext} = 0 \quad (6)
\]

where \( \mathbf{v} = (u, v, w)^T \) is the velocity field, \( \theta \) is the temperature, \( p_{eff} \) is the effective pressure, \( \phi \) and \( \phi_{ext} \) are the internal and the external scalar magnetic potential, respectively. Also the following groups of dimensionless numbers have been introduced: a) (pure fluids) the Rayleigh number \( R_a = \alpha_T g \Delta T d^3 / \kappa \nu^2 \) accounting for buoyancy effects and the Prandtl number \( P = \nu / \kappa \), relating viscous and thermal effects; b) (rotation in pure fluids) the Taylor number \( T_a = (2 \nu \chi^2 / \nu^2) \); c) (binary mixtures) the Lewis number \( L = D_c / \kappa \), relating diffusion with thermal diffusivity, the separation ratio \( \psi = \alpha_c D_T / (\alpha_T D_c) \) and the Dufuor number \( F = D^2_T / (D \kappa) \); and d) (magnetic fluid) the strength of magnetic force relative to buoyancy \( M_1 = \beta \chi^2 \hat{H}_0^2 / (\rho_0 g \alpha_T (1 + \chi)) \), the magnetophoretic number which gives rise to a field dependence of heat and concentration currents \( M_2 = D \chi \chi_T \hat{H}_0^2 / (\rho_0 g \alpha_T (1 + \chi)) \), the nonlinearity of magnetization \( M_3 = 1 - (\chi_T \hat{H}_0^2) / (1 + \chi) \) is a measure of the deviation of the magnetization curve from the linear behavior \( M_0 = \chi \hat{H}_0 \), the relative strength of temperature dependence of the magnetic susceptibility \( M_4 = \chi^2 \hat{H}_0^2 T_0 / \rho_0 c_0 (1 + \chi) \), the he ratio of magnetic to thermal
buoyancy $M_5 = \alpha_H \chi_T H_0^2 / (\omega_T (1 + \chi))$ and the magnetic separation ratio $\psi_M = -\chi_L D_T / (\chi_T D_c)$. Let us comment about of the numerical values of the parameters, the parameters $R_a$ and $T_a$ may be changed in several orders of magnitude. For aqueous suspensions of ferrofluid, we assume the separation ratio $\psi$ positive, so there is only the stationary convection case. The magnetic numbers have the following order of magnitude $M_1 = 10^{-4} - 10^1$, $M_3 \approx 1.1$, $M_4 \approx M_5 \approx 10^{-6}$ [7]. The values of $M_4$, $M_5$ and $F$ are very small, so in our calculation we will not take them into account.

For calculating the linear stability analysis, we only need the linear parts of equations (1)-(5), the effective pressure and two components of the velocity field could easily be eliminated by applying the rotor and double rotor operetator in the Navier-Stokes equation and then considering (5), the effective pressure and two components of the velocity field could easily be eliminated by applying the usual normal mode expansion

\[(\partial_t \theta, c, \zeta, w, \phi)(r,t) = (\Theta, \Psi, Z, W, \Phi)(z) \exp[i\delta t] \tag{7}\]

with $\varphi = k \cdot \mathbf{r}_\perp + \sigma t$, being $\zeta$ the z-component of the vorticity, and $k$ the horizontal wavenumber vector and where $\sigma$ denote the growth factor of a perturbation. Therefore, for the stationary state, this leads to the following coupled ordinary differential equations:

\[(D^2 - k^2)\Theta + W = 0, \tag{8}\]
\[(D^2 - k^2)Z + T_a^{1/2}DW = 0, \tag{9}\]
\[(D^2 - k^2)^2W - T_a^{1/2}DZ = \]

\[R_a k^2 \left[ (\psi + M_1 \psi_M) \Psi + (1 + M_1) \Theta - M_1 D\Phi \right] \]
\[(D^2 - k^2)(\Psi + \Theta - M_2 D\Phi) = 0, \tag{10}\]
\[D^2\Phi - M_3 k^2 \Phi - D\Theta + \psi_M D\Psi = 0 \tag{11}\]
\[D^2 - k^2)(\Psi + \Theta - M_2 D\Phi) = 0, \tag{11}\]
\[D^2\Phi - M_3 k^2 \Phi - D\Theta + \psi_M D\Psi = 0 \tag{12}\]

where $D^m f = \partial_z^m f$. In order to solve the set of differential equations (8)-(12) and then to find analytical expressions we impose the following boundary conditions $W = D^2W = DZ = \Theta = D\Psi = 0 = \Phi = 0$ are imposed in $z = \pm 1$. So, with them the eigenvalue problem gives the Rayleigh number. The stationary bifurcation is obtained for $\sigma = 0$ and for the fundamental mode, gives the marginal curve for stationary convection:

\[R_a = \frac{(k^2 M_3 + \eta) \left( \pi^2 T_a + (\pi^2 + k^2)^2 \right)}{k^2 \left[ \pi(\psi + 2 M_1 \psi_M)(\pi - M_2) + k^2 M_3 \lambda + \eta \right]} \tag{13}\]

where $\eta = \pi(\pi - M_2 \psi_M))$ and $\lambda = 1 + \psi + M_1(1 + \psi_M)$.

Note that, the equation (13) reproduce all the know result, for example if we take $\psi = 0$ we obtain the expression of $R_a$ for a simple ferrofluid [4].The minimum of the marginal curve $(\partial_k R_a = 0)$ gives the critical wavenumber $k_c$ and, the critical Rayleigh number, $R_{ac}$, can be obtained by replacing it in equation (13).

The main results are displayed in Figures 1 and 2, where we plot the critical Rayleigh number, $R_{ac}$, as function of the Taylor number, $T_a$, at $M_3 = 1.1$, $M_2 = 1$ and $\psi = 1$ for different values of the strength of magnetic force relative to buoyancy $M_1$ and different values of magnetic separation ratio $\psi_M$, respectively. We remarks that, when the $M_1$ and $\psi_M$ increases the critical Rayleigh number decreases while its increases when the rotation ratio increases. Therefore, the magnetic and binary effect are destabilize effect and the rotation stabilize the system.

3. CONCLUSIONS
In the present work, Rayleigh-Benard convection in a binary magnetic fluid liquid mixture under rotation is studied taken into account the magnetophoretic effect. We determine the stability
Figure 1. Critical Rayleigh number $R_{ac}$ as function $T_a$ at $M_2 = 1, M_3 = 1.1, \psi = 10$ and $\psi_M = 1$ for different values of $M_1$.

Figure 2. Critical Rayleigh number $R_{ac}$ as function $T_a$ at $M_1 = 10, M_2 = 1, M_3 = 1.1$ and $\psi = 0.1$ for different values of $\psi_M$. 
thresholds for stationary convection for idealized boundary conditions. We shown that, the binary effects drastically reduce the critical Rayleigh number. Beside, our results suggest that a rotation can be used as a tool to enhance the range of the laminar regime of convective in aqueous solutions.

Acknowledgments
This research received financial support from MECESUP FSM-9901, MECESUP USA-0108.

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