Learning Influence-Receptivity Network Structure with Guarantee

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Abstract

Traditional works on community detection from observations of information cascade assume that a single adjacency matrix parametrizes all the observed cascades. However, in reality the connection structure usually does not stay the same across cascades. For example, different people have different topics of interest, therefore the connection structure would depend on the information/topic content of the cascade. In this paper we consider the case where we observe a sequence of noisy adjacency matrices triggered by information/events with different topic distributions. We propose a novel latent model using the intuition that the connection is more likely to exist between two nodes if they are interested in similar topics, which are common with the information/event. Specifically, we endow each node two node-topic vectors: an influence vector that measures how much influential/authoritative they are on each topic; and a receptivity vector that measures how much receptive/susceptible they are to each topic. We show how these two node-topic structures can be estimated from observed adjacency matrices with theoretical guarantee, in cases where the topic distributions of the information/events are known, as well as when they are unknown. Extensive experiments on synthetic and real data demonstrate the effectiveness of our model.

1 Introduction

Uncovering latent network structure is an important research area in network models and has a long history [23, 6]. For a $p$ node network, traditional approaches usually assume a single $p \times p$ adjacency matrix, either binary or real-valued, that quantifies the connection intensity between nodes, and aim to learn the community structure from it. For example in Stochastic Block Model (SBM) [10] we assume that nodes within a group have an edge with each other with probability $p_0$ while nodes across groups have an edge with probability $q_0$ where $p_0 > q_0$. In information diffusion we observe the propagation of information among nodes and aim to recover the underlying connections between nodes [20, 9, 8]. In time-varying networks we allow the connections and parameters to change over time [15, 2].

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In this paper, we consider the case where we have a sequence of information/events with different topics, and we observe a noisy adjacency matrix for each of them. The connection between nodes varies under each topic distribution and this cannot be captured using only one adjacency matrix. For example, each researcher has her own research interests and would collaborate with others only on the areas they are both interested in. Specifically, suppose researcher 1 is interested in computational biology and information theory; researcher 2 is interested in computational biology and nonparametric statistics; researcher 3 is interested in information theory only. Then if researcher 1 wants to work on computational biology, she would collaborate with researcher 2; while if the topic is on information theory, then she would collaborate with researcher 3. As another example, suppose student 1 is interested in music and sports while student 2 is interested in music and chess. If the topic of a University event is music, then there will be an edge between these two students; however, if the topic of the event is sports or chess, then there would not be an edge between them.

Intuitively, for a specific information/event/collaboration, there will be an edge between two nodes if and only if they are both interested in the topic of this information/event/collaboration. In this paper we quantify this intuition by giving each node two node-topic vectors: one influence vector (how authoritative they are on each topic) and one receptivity vector (how susceptible they are on each topic). In addition, each information/event/collaboration is associated with a distribution on topics. The influence and receptivity vectors are fixed but different topic distributions result in different adjacency matrices among nodes. In this paper we consider the cases where the topic distribution is known or unknown, and provide algorithms to estimate the node-topic structure with theoretical guarantees. In particular, we show that our algorithm converges to the true values up to statistical error. Our node-topic structure is much easier to interpret than a large adjacency matrix among nodes, and the result can be used to make targeted advertising or recommendation systems.

**Related Works** There is a vast literature on uncovering latent network structures. The most common and basic model is the Stochastic block model (SBM) [10] where connections are assumed to be dense within group and are sparse across groups. Many practical algorithms have been proposed for SBM [5, 13, 21, 18] and many variants and extensions about SBM have been developed to better fit real world network structures, including Degree-corrected block model (DCBM) [14], Mixed membership stochastic block models (MMSB) [3], Degree Corrected Mixed Membership (DCMM) model [12], etc. Other models include information diffusion [20, 9, 8, 28], time-varying networks [15, 2], graphical models [1, 4, 26], buyer-seller networks [16, 24, 22], etc. However, most of the existing work focuses on a single adjacency matrix and ignores the node-topic structure. In [25] the authors propose a node-topic model for information diffusion problem, but it requires the topic distribution to be known and lacks of theoretical guarantees.

**Notation** In this paper we use $p$ to denote the number of nodes in the network; we assume there are $K$ topics in total, and we observe $n$ adjacency matrices under different topic distributions. We use subscript $i \in \{1, \ldots, n\}$ to index samples/observations; subscript $j, \ell \in \{1, \ldots, p\}$ to index nodes; and subscript $k \in \{1, \ldots, K\}$ to index topic. For any matrix $A$, we use $\|A\|_0 = |(j, k) : A_{jk} \neq 0|$ to denote the number of nonzero elements of $A$. Also,
for any $d$, $I_d$ is the identity matrix with dimension $d$.

## 2 Model

In this section we propose a model to capture the node-topic structure in networks. As stated before, we use the intuition that, for a specific information/event/collaboration, there would be an edge between two nodes if they are interested in similar topics, which are common with the information/event/collaboration. Furthermore, the connection is directed where an edge from node $1$ to node $2$ is more likely to exist if node $1$ is influential/authoritative in the topic, and node $2$ is receptive/susceptible to the topic. For example, a famous professor would have a large influence value (but maybe a small receptivity value) on his/her research area, while a high-producing, young researcher would have a large receptivity value (but maybe a small influence value) on his/her research area.

Our node-topic structure is parametrized by two matrices $B_1, B_2 \in \mathbb{R}^{p \times K}$. The matrix $B_1$ measures how much a node can infect others (the influence matrix) and the matrix $B_2$ measures how much a node can be infected by others (the receptivity matrix). We use $b_{jk}^1$ and $b_{jk}^2$ to denote the elements of $B_1$ and $B_2$, respectively. Specifically, $b_{jk}^1$ measures how influential node $j$ is on topic $k$, and $b_{jk}^2$ measures how receptive node $j$ is on topic $k$. We use $b_k^1$ and $b_k^2$ to denote the columns of $B_1$ and $B_2$, respectively.

Each observation $i$ is associated with a topic distribution $m_i = (m_{i1}, ..., m_{iK})$ on the $K$ topics satisfying $m_{i1} \geq 0$ and $m_{i1} + ... + m_{iK} = 1$. The choice of $K$ can be heuristic and pre-specified or alternatively can be decided by methods such as in [11] which learn the distribution over the number of topics. For each observation $i$, the true adjacency matrix is given by

$$
(x_i^*)_{j\ell} = \sum_{k=1}^{K} b_{jk}^1 \cdot m_{ik} \cdot b_{\ell k}^2,
$$

or in matrix form,

$$
X_i^* = B_1 \cdot M_i \cdot B_2^T,
$$

where $M_i$ is a diagonal matrix

$$
M_i = \text{diag}(m_{i1}, m_{i2}, ..., m_{iK}).
$$

The interpretation of the model is straightforward from (1). For an observation $i$ on topic $k$, there will be an edge $j \to \ell$ if and only if node $j$ tends to infect others on topic $k$ (large $b_{jk}^1$) and node $\ell$ tends to be infected by others on topic $k$ (large $b_{\ell k}^2$). This intuition applies to each topic $k$ and the final value is the summation over all the $K$ topics.

If we do not consider self connections, we can zero out the diagonal elements and get

$$
X_i^* = B_1 M_i B_2^T - \text{diag}(B_1 M_i B_2^T).
$$

For notational simplicity, we still stick to (2) for the definition of $X_i^*$ in the subsequent sections. We are given $n$ observations $\{X_i\}_{i=1}^n$ satisfying

$$
X_i = X_i^* + E_i.
$$
The only requirements on the noise term $E_i$ are that they are mean 0 and independent. They are not necessarily identically distributed and can follow an unstructured distribution. The observations $X_i$ can be either real-valued or binary. For binary observations we are interested in the existence of an connection only, while for real-valued observation we are also interested in how strong the connection is, i.e. larger values indicate stronger connections.

3 Optimization

The loss function is given by

$$f(B_1, B_2) = \frac{1}{2n} \sum_{i=1}^{n} \left\| X_i - B_1 M_i B_2^\top \right\|_F^2. \quad (3)$$

Using the notation $B_1 = [b^1_1, ..., b^1_K]$ and $B_2 = [b^2_1, ..., b^2_K]$, we can rewrite (2) as

$$X^*_i = B_1 M_i B_2^\top = \sum_{k=1}^{K} m_{ik} \cdot b^1_k b^2_k^\top.$$

Denote $\Theta_k = b^1_k b^2_k^\top$; with some abuse of notation we can rewrite the loss function (3) as

$$f(\Theta) = f(\Theta_1, ..., \Theta_K) = \frac{1}{2n} \sum_{i=1}^{n} \left\| X_i - \sum_{k=1}^{K} m_{ik} \cdot \Theta_k \right\|_F^2. \quad (4)$$

From (4) we can see that solving for $B_1, B_2$ is equivalent to solving for rank-1 matrix factorization problem on $\Theta_k$. This model is therefore not identifiable on $B_1$ and $B_2$, since if we multiply column $k$ of $B_1$ by some scalar $\gamma$ and multiply column $k$ of $B_2$ by $1/\gamma$, the matrix $X^*_i$ remains unchanged for any $i$, since $b^1_k b^2_k^\top$ does not change. Hence the loss function also remains unchanged. Therefore we need additional regularization term to ensure unique solution. To address this issue, we propose the following two alternative regularization terms.

1. The first regularization term is an $L_1$ penalty on $B_1$ and $B_2$. We define the following norm

$$\| B_1 + B_2 \|_{1,1} \triangleq \sum_{j,k} b^1_{jk} + b^2_{jk},$$

and minimize the following regularized loss function

$$f_1(B_1, B_2) = \frac{1}{2n} \sum_{i=1}^{n} \left\| X_i - B_1 M_i B_2^\top \right\|_F^2 + \lambda \cdot \| B_1 + B_2 \|_{1,1},$$

where $\lambda$ is a tuning parameter. To see why this penalty ensures unique solution, we focus on column $k$ only, and the term we want to minimize is

$$\gamma \cdot \| b^1_k \|_1 + \frac{1}{\gamma} \| b^2_k \|_1. \quad (5)$$

In order to minimize (5) we should select $\gamma$ such that the two terms in (5) are equal. In other words, the column sums of $B_1$ and $B_2$ are equal.
2. The second regularization term is borrowed from matrix factorization literature defined as
\[ g(B_1, B_2) = \lambda \frac{1}{2} \sum_{k=1}^{K} \left( \| b_k^1 \|_2^2 - \| b_k^2 \|_2^2 \right)^2. \]

We minimize the following regularized loss function
\[ f_2(B_1, B_2) = \frac{1}{2n} \sum_{i=1}^{n} \| X_i - B_1 M_i B_2^\top \|_F^2 + \lambda \frac{1}{2} \sum_{k=1}^{K} \left( \| b_k^1 \|_2^2 - \| b_k^2 \|_2^2 \right)^2. \]

This regularization term forces the 2-norm of each column of \( B_1 \) and \( B_2 \) to be the same. At the minimizer, this regularization term is 0, and therefore we can pick any \( \lambda > 0 \).

Both regularization terms forces the columns of \( B_1 \) and \( B_2 \) to be balanced. Intuitively, this means that, for each topic \( k \), the total magnitudes of “influence” and “receptivity” are the same. This acts like a conservation law that the total amount of output should be equal to the total amount of input.

The first regularization term introduces bias, but it automatically encourages sparse solution; the second regularization term does not introduce bias, but we need an additional hard thresholding step to get sparsity. The loss function (3) is nonconvex in \( B_1 \) and \( B_2 \), hence proving theoretical results is much harder for the first regularization term. Therefore our theoretical results focus on the second alternative proposed above. The final optimization problem is given by

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2n} \sum_{i=1}^{n} \| X_i - B_1 M_i B_2^\top \|_F^2 + \lambda \frac{1}{2} \sum_{k=1}^{K} \left( \| b_k^1 \|_2^2 - \| b_k^2 \|_2^2 \right)^2 \\
\text{subject to} & \quad B_1, B_2 \geq 0
\end{align*}
\]

Initialization. We initialize by solving the convex relaxation problem (4) without the rank-1 constraint on \( \Theta_k \), and apply rank-1 svd on estimated \( \hat{\Theta}_k \), i.e., we keep only the largest singular value: \( [u_k, s_k, v_k] = \text{rank-1 svd of } \hat{\Theta}_k \). The initialization is given by \( B_1^{(0)} = [u_1 s_1^{1/2}, ..., u_K s_K^{1/2}] \) and \( B_2^{(0)} = [v_1 s_1^{1/2}, ..., v_K s_K^{1/2}] \). Being a convex relaxation, we can find the global minimum \( \hat{\Theta}_k \) of problem (4) by using gradient descent algorithm.

Algorithm. With this initialization, we alternatively apply proximal gradient method [19] on \( B_1 \) and \( B_2 \) until convergence. In practice, each node would be interested in only a few topic and hence we would expect \( B_1 \) and \( B_2 \) to be sparse. To encourage sparsity we need an additional hard thresholding step on \( B_1 \) and \( B_2 \). The overall procedure is given in Algorithm 1. The operation \( \text{Hard}(B, s) \) keeps the largest \( s \) elements of \( B \) and zeros out others; the operation \( [\hat{B}]_+ \) keeps all positive values and zeros out others.
Algorithm 1 Alternating proximal gradient descent

Initialize $B_1^{(0)}$, $B_2^{(0)}$

for $t = 1, ..., T$ do
\[ B_1^{(t+0.5)} = \left[ B_1^{(t)} - \eta \cdot \nabla B_1 f(B_1^{(t)}, B_2^{(t)}) - \eta \cdot \nabla B_1 g(B_1^{(t)}, B_2^{(t)}) \right]_+ \]
\[ B_1^{(t+1)} = \text{Hard}(B_1^{(t+0.5)}, s) \]
\[ B_2^{(t+0.5)} = \left[ B_2^{(t)} - \eta \cdot \nabla B_2 f(B_1^{(t)}, B_2^{(t)}) - \eta \cdot \nabla B_2 g(B_1^{(t)}, B_2^{(t)}) \right]_+ \]
\[ B_2^{(t+1)} = \text{Hard}(B_2^{(t+0.5)}, s) \]
end for

4 Theoretical result

In this section we derive the theoretical results for our algorithm. We denote $B_1^*$ and $B_2^*$ as the true value and $\Theta_k^* = b_k^1 b_k^2 + b_k^2 b_k^1$ as the corresponding true rank-1 matrices. In this section we assume the topic distribution $M_i$ is known. The case where $M_i$ is unknown is considered in Section 5. All the detailed proofs are relegated to the Appendix. We start by stating some mild assumptions on the parameters of the problem.

Sparse Condition (SC). Both $B_1^*$ and $B_2^*$ are sparse: $\|B_1^*\|_0 = \|B_2^*\|_0 = s^*$. (We use a single $s^*$ for notational simplicity, but is not required).

Topic Condition (TC). Denote the Hessian matrix on $\Theta$ as
\[ H_\Theta = \frac{1}{n} \begin{bmatrix}
\sum_{i=1}^n m_{i1}^2 & \sum_{i=1}^n m_{i1} m_{i2} & \cdots & \sum_{i=1}^n m_{i1} m_{iK} \\
\sum_{i=1}^n m_{i1} m_{i2} & \sum_{i=1}^n m_{i2}^2 & \cdots & \sum_{i=1}^n m_{i2} m_{iK} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^n m_{i1} m_{iK} & \sum_{i=1}^n m_{i2} m_{iK} & \cdots & \sum_{i=1}^n m_{iK}^2
\end{bmatrix}.\]
We require that $H_\Theta \succeq \mu_\Theta \cdot I_K$ for some constant $\mu_\Theta > 0$.

Intuitively, this condition requires that, the correlation among topic distributions in the $n$ observations cannot be too large. This makes sense because if several topics are highly correlated with each other among the $n$ observations, then clearly we cannot distinguish them. With this condition, the objective function (4) is strongly convex in $\Theta$.

An immediate corollary of this condition is that the diagonal elements of $H_\Theta$ must be at least $\mu_\Theta$, i.e., for each topic $k$, we have $\frac{1}{n} \sum_{i=1}^n m_{ik}^2 \geq \mu_\Theta$. This means that at least a constant proportion of the observed data should focus on this topic. The necessity of this condition is also intuitive: if we only get tiny amount of data on some topic, then we cannot expect to recover the structure for that topic accurately.

Subspace distance. For matrix factorization problems, it is common to measure the subspace distance because the factorization $\Theta_k = b_k^1 b_k^2 + b_k^2 b_k^1$ is not unique. Here since we know that $\Theta_k$ are exactly rank-1 and we have non-negativity constraints on $B_1, B_2$, we would not suffer from rotation issue (the only way to rotate scalar is $\pm 1$, but with non-negativity...
constraint, \(-1\) is impossible). Therefore the subspace distance between \(B = [B_1, B_2]\) and \(B^* = [B^*_1, B^*_2]\) is just defined as
\[
d^2(B, B^*) = \min_{o_k \in \{\pm 1\}} \sum_{k=1}^K \|b^*_k - b^*_k o_k\|^2 + \|b^*_k - b^*_k o_k\|^2 = \|B_1 - B^*_1\|^2_F + \|B_2 - B^*_2\|^2_F. \tag{7}
\]

**Statistical error.** Denote
\[
\Omega = \{ \Delta : \Delta = [\Delta_1, \ldots, \Delta_K] \in \mathbb{R}^{K \times p}, \text{rank}(\Delta_k) = 2, \|\Delta_k\|_0 = s, \|\Delta\|_F = 1 \}.
\]
The statistical error on \(\Theta\) is defined as
\[
e_{\text{stat}, \Theta} = \sup_{\Delta \in \Omega} \langle \nabla f_\Theta(\Theta^*), \Delta \rangle = \sup_{\Delta \in \Omega} \sum_{k=1}^K \left( \frac{1}{n} \sum_{i=1}^n E_{ik} \cdot m_{ik}, \Delta_k \right). \tag{8}
\]

Intuitively, this statistical error measures how much accuracy we can expect for the estimator. Specifically, if we are within \(c \cdot e_{\text{stat}}\) distance with the true value, then we are already optimal.

In this way we transform the original problem to a standard matrix factorization problem with \(K\) rank-1 matrices \(\Theta_1, \ldots, \Theta_K\). A function \(f(\cdot)\) is termed to be strongly convex and smooth if there exist constant \(\mu\) and \(L\) such that
\[
\frac{\mu}{2}\|Y - X\|^2_F \leq f(Y) - f(X) - \langle \nabla f(X), Y - X \rangle \leq \frac{L}{2}\|Y - X\|^2_F.
\]

We next show that the objective function (4) is strongly convex and smooth in \(\Theta\). Since the loss function (4) is quadratic on each \(\Theta_k\), it is easy to see that this condition is equivalent to \(\mu \cdot I_K \preceq H_\Theta \preceq L \cdot I_K\). The lower bound is satisfied according to assumption (TC) with \(\mu = \mu_\Theta\), and the upper bound is trivially satisfied with \(L = L_\Theta = 1\). Therefore we see that the objective function (4) is strongly convex and smooth in \(\Theta\). The following lemma quantifies the accuracy of the initialization.

**Lemma 1.** Suppose \(\hat{\Theta} = (\hat{\Theta}_1, \ldots, \hat{\Theta}_K)\) are the global minimum of the convex relaxation (4), then we have
\[
\sum_{k=1}^K \|\Theta_k^* - \hat{\Theta}_k\|^2_F \leq \frac{2}{\mu_\Theta} \|\nabla f(\Theta^*)\|^2_F.
\]

The bound we obtain from Lemma 1 scales with \(n^{-1/2}\) and therefore can be small as long as we have enough samples. We are then ready for our main theorem. The following Theorem 2 shows that the iterates of Algorithm 1 converge linearly up to statistical error.

**Theorem 2.** Suppose conditions (SC) and (TC) hold. We set the sparsity level \(s = cs^*\). If the step size \(\eta\) satisfies
\[
\eta \leq \frac{1}{16 \|B^{(0)}\|^2_2} \cdot \min \left\{ \frac{1}{2(\mu_\Theta + L_\Theta)}, 1 \right\},
\]
then for large enough \(n\), after \(T\) iterations, we have
\[
d^2(B^{(T)}, B^*) \leq \beta^T d^2(B^{(0)}, B^*) + C \cdot e_{\text{stat}, \Theta}^2, \tag{9}
\]
for some constant \(\beta < 1\) and constant \(C\).
Remark 3. Although in this paper we focus on the simplest loss function \((3)\), our analysis works for any general loss functions \(f(B_1 M B_2^T)\), as long as the initialization is “good” and the (restricted) strongly convex and smoothness conditions are satisfied.

5 Learning network and topic distributions jointly

So far we assume the topic distributions \(m_i\) for each sample \(i\) are given and fixed. However, sometimes we do not have such information. In this case we need to learn the topic distributions and the network structure simultaneously.

We denote \(m^*_i\) as the true topic distribution of observation \(i\) and \(M = [m_1, ..., m_n]\) is the stack of all the topic distributions. The algorithm for joint learning is simply alternating minimization on \(B_1, B_2\) and \(M\). For fixed \(M\), the optimization on \(B_1, B_2\) is the same as before, and can be solved using Algorithm 1. For fixed \(B_1, B_2\), it is straightforward to see that the optimization on \(M\) is separable to each \(i\). For each \(i\), we solve the following optimization problem to estimate \(M_i = \text{diag}(m_i)\):

\[
\begin{align*}
\text{minimize} & \quad ||X_i - B_1 M_i B_2^T||_F^2 \\
\text{subject to} & \quad m_i \geq 0, 1^T \cdot m_i = 1
\end{align*}
\]

This problem is convex in \(M_i\) and can be easily solved using projected gradient descent. Namely in each iteration we do gradient descent on \(M_i\) and then project to the simplex. The overall procedure is summarized in Algorithm 2. With some abuse of notation we write

\[
f(\Theta, M) = \frac{1}{2n} \sum_{i=1}^{n} \left\| X_i - \sum_{k=1}^{K} m_{ik} \cdot \Theta_k \right\|_F^2.
\]

(11)

Besides the scaling issue mentioned in Section 3, the problem now is identifiable only up to permutation of the position of the topics. However we can always permute \(M^*\) to match the permutation obtained in \(M\). From now on we assume that these two permutations match and ignore the permutation issue. The statistical error on \(M\) is defined as

\[
e_{\text{stat}, M}^2 = \sum_{i=1}^{n} \sum_{k=1}^{K} \left( \nabla_{m_{ik}} f(\Theta^*, M^*) \right)^2 = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{K} \langle E_i, \Theta_k^* \rangle^2.
\]

The problem is much harder with unknown topic distribution. Similar to condition (TC), we need the following assumption on the Hessian matrix on \(M\).

Diffusion Condition (DC). Denote the Hessian matrix on \(M\) as

\[
H_M = \begin{bmatrix}
\langle \Theta_1, \Theta_1^* \rangle & \langle \Theta_1, \Theta_2^* \rangle & \ldots & \langle \Theta_1, \Theta_K^* \rangle \\
\langle \Theta_2, \Theta_1^* \rangle & \langle \Theta_2, \Theta_2^* \rangle & \ldots & \langle \Theta_2, \Theta_K^* \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \Theta_K, \Theta_1^* \rangle & \langle \Theta_K, \Theta_2^* \rangle & \ldots & \langle \Theta_K, \Theta_K^* \rangle
\end{bmatrix},
\]
Algorithm 2 Learning network structure and topic distributions jointly

Initialize $B_1, B_2$

while $\text{tolerance} > \epsilon$ do

Optimize $M$ according to (10) using projected gradient descent.

Optimize $B_1, B_2$ according to Algorithm 1

end while

where $\langle A_1, A_2 \rangle = \text{tr}(A_1^\top A_2)$ is the inner product of matrices $A_1, A_2$. We require that $H_M \succeq \mu_M \cdot I_K$ for some constant $\mu_M > 0$.

With this condition, the objective function (4) is strongly convex in $M$. The intuition is similar as in condition (TC). We require that $\Theta_k$ can be distinguished from each other.

**Initialization.** Denote $\bar{X}, \bar{X}^*, \bar{E}$ as the sample mean of $X_i, X_i^*, E_i$, respectively. We have

$$
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i^* + E_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{K} m_{ik}^* \Theta_k^* + \frac{1}{n} \sum_{i=1}^{n} E_i = \sum_{k=1}^{K} \left( \frac{1}{n} \sum_{i=1}^{n} m_{ik}^* \right) \Theta_k^* + \bar{E} = \bar{X}^* + \bar{E}
$$

We do rank-$K$ svd on $\bar{X}$ and obtain $[\widetilde{U}, \widetilde{S}, \widetilde{V}] = \text{rank-K svd of } \bar{X}$. We denote $\widetilde{X} = \widetilde{U} \widetilde{S} \widetilde{V}^\top = \sum_{k=1}^{K} \widetilde{\sigma}_k \widetilde{u}_k \widetilde{v}_k^\top$. It is well known that $\bar{X}$ is the best rank $K$ approximation for $\bar{X}$. We propose the following initialization method

$$
\Theta_k^{(0)} = K \cdot \widetilde{\sigma}_k \widetilde{u}_k \widetilde{v}_k^\top
$$

To see why this initialization works, we first build intuition for the easiest case, where $E_i = 0$ for each $i$, $\frac{1}{n} \sum_{i=1}^{n} m_{ik}^* = \frac{1}{K}$ for each $k$, and the columns of $B_1^*$ and $B_2^*$ are orthogonal. In this case it is easy to see that $\bar{X} = \bar{X}^* = \sum_{k=1}^{K} \frac{1}{K} \Theta_k^*$. Note that this expression is a singular value decomposition of $\bar{X}^*$ since we have $\Theta_k^* = b_k^1 b_k^2^\top$ and the columns $\{b_k^1\}_{k=1}^{K}$ and columns $\{b_k^2\}_{k=1}^{K}$ are orthogonal. Now that since $\bar{X}$ is exactly rank $K$, the best rank $K$ approximation would be itself, i.e., $\bar{X} = \widetilde{X} = \sum_{k=1}^{K} \widetilde{\sigma}_k \widetilde{u}_k \widetilde{v}_k^\top$. By the uniqueness of singular value decomposition, as long as the singular values are distinct, we have (up to permutation) $\frac{1}{K} \Theta_k^* = \widetilde{\sigma}_k \widetilde{u}_k \widetilde{v}_k^\top$ and therefore $\Theta_k^* = K \cdot \widetilde{\sigma}_k \widetilde{u}_k \widetilde{v}_k^\top$. This is exactly what we want to estimate. With this intuition in mind, we relax these restrictions and impose the following conditions.

**Orthogonal Condition (OC).** Let $B_1^* = Q_1 R_1$ and $B_2^* = Q_2 R_2$ be the QR decomposition of $B_1^*$ and $B_2^*$, respectively. Denote $A^*$ as a diagonal matrix with diagonal elements $\frac{1}{n} \sum_{i=1}^{n} m_{ik}^*$. Denote $R_1 A^* R_2^\top = A_{\text{diag}} + A_{\text{off}}$ where $A_{\text{diag}}$ captures the diagonal elements and $A_{\text{off}}$ captures the off-diagonal elements. We require that $\|A_{\text{off}}\|_F \leq \rho_0$ for some constant $\rho_0$. Moreover, we require that $\frac{1}{n} \sum_{i=1}^{n} m_{ik} \leq \eta/K$ for some $\eta$.

This condition requires that $B_1^*$ and $B_2^*$ are not too far away from orthogonal matrix, so that when doing the QR rotation, the off diagonal values of $R_1$ and $R_2$ are not too large. The condition $\frac{1}{n} \sum_{i=1}^{n} m_{ik} \leq \eta/K$ is trivially satisfied with $\eta = K$. However, in general $\eta$ is usually a constant that does not scale with $K$, meaning that the topic distribution among the $n$ observations is more like evenly distributed than several topics dominate.
Finally it is useful to point out that the condition (OC) is for this specific initialization method only. Since we are doing singular value decomposition, we end up with orthogonal vectors so we require that $B_1^*$ and $B_2^*$ are not too far away from orthogonal; since we do not know the value $\frac{1}{n} \sum_{i=1}^n m_{ik}^*$ and use $1/K$ to approximate, we require that topics are not far away from evenly distributed so that this approximation is reasonable. In practice we can also use other initialization methods, for example we can do alternating gradient descent on $B_1, B_2$ and $M$ based on the objective function (11). This method also works reasonably well in practice.

The following lemma shows that $\Theta_k^{(0)}$ is indeed a reasonable initialization for $\Theta_k^*$.

**Lemma 4.** Suppose the condition (OC) is satisfied, then the initialization $\Theta_k^{(0)}$ satisfies

$$\|\Theta_k^{(0)} - \Theta_k^*\|_F \leq 2\tilde{C}K\rho_0 + (\eta - 1)\sigma_{\text{max}},$$

for some constant $\tilde{C}$ where $\sigma_{\text{max}} = \max_k \|\Theta_k^*\|_2$.

The initialization $\Theta_k^{(0)}$ is no longer $\sqrt{n}$-consistent. Nevertheless it is not required. With this initialization, we then follow Algorithm 2 and estimate $B_1, B_2$ and $M$ alternatively. Note that when estimating $B_1$ and $B_2$, we run Algorithm 1 for large enough $T$ so that the first term in (9) is small compared to the second term. This $T$ iterations for Algorithm 1 is one iteration for Algorithm 2 and we use $B^{[t]} = [B_1^{[t]}, B_2^{[t]}]$ and $M^{[t]}$ to denote the iterates we obtained from Algorithm 2. Similar to (7), we denote the distance between an estimation $M$ and true value $M^*$ as

$$d^2(M, M^*) = \frac{1}{n} \sum_{i=1}^n \sum_{k_0=1}^K (m_{ik_0} - m^*_{ik_0})^2.$$ 

The following Theorem 5 shows that the iterates of Algorithm 2 converge linearly up to statistical error.

**Theorem 5.** Suppose the conditions in Theorem 2 hold and suppose condition (DC) and (OC) hold. For large enough $n$, after $T$ iterations of Algorithm 2 we have

$$d^2(B^{[T]}, B^*) + d^2(M^{[T]}, M^*) \leq \frac{C_1\epsilon_{\text{stat},M}^2 + C_2\epsilon_{\text{stat},\Theta}^2}{1 - \beta_0} + \beta_0 T \left[ d^2(B^{[0]}, B^*) + d^2(M^{[0]}, M^*) \right],$$

for some constant $\beta_0 < 1$ and constants $C_1, C_2$.

### 6 Synthetic data

In this section we evaluate our model and algorithms on synthetic datasets. We first consider the setting where the topics are known and we consider $p = 200$ nodes with $K = 10$ topics. The true matrices $B_1^*$ and $B_2^*$ are generated row by row where we randomly select 1-3 topics for each row and set a random value generated from $\text{Uniform}(1,2)$. All the other values are set to be 0. This gives sparsity level $s^* = 2p$ in expectation, and we set $s = 2s^* = 4p$ in
the algorithm as the hard thresholding parameter. For each observation, we randomly select 1-3 topics and assign each selected topic a random value $\text{Uniform}(0, 1)$, and 0 otherwise. We then normalize this vector to get the topic distribution $m_i$. The true value $X_i^*$ is generated according to (2). Note that $X_i^*$ is also a sparse matrix. We consider two types of observation: real valued observation and binary valued observation. For real valued observation, we generate $X_i$ (equivalently, set $E_i$) in the following way: first we randomly select 10% of the nonzero values in $X_i^*$ and set to 0 (miss some edges); second for each of the remaining nonzero values, we generate an independent random number $\text{Uniform}(0, 3)$ and multiply with the original value (observe edges with noise); finally we randomly select 10% of the zero values in $X_i^*$ and set them as $\text{Uniform}(0, 1)$ (false positive edges). For binary observations, we treat the true values in $X_i^*$ as probability of observing an edge, and generate $X_i$ as $X_i = \text{Bernoulli}(X_i^*)$. For those true values greater than 1 we just set $X_i$ to be 1. Finally we again pick 10% false positive edges.

We vary the number of observations $n \in \{20, 30, 50, 80, 120, 200\}$ and compare our model with the following two methods: first ignores the topic information and uses one $p \times p$ matrix to capture the entire dataset (termed “One matrix”). This matrix is given by $X$. The second ignores the node-topic structure and assigns each topic a $p \times p$ matrix (termed “$K$ matrices”). In this way we ignore the rank constraint; and return the matrix $\Theta_k$ given by the initialization procedure. Note that “One matrix” method has $p^2$ parameters, “$K$ matrices” has $p^2 K$ parameters, but our method has only $2pK$ parameters. Since we usually have $K \ll p$, we are able to use much smaller number of parameters to capture the network structure, and would not suffer too much from overfitting. For fair comparison, we also do hard thresholding on each of these $p \times p$ matrices with parameter $4p$. The comparison is done by evaluating the objective function on independent test dataset (prediction error). Figure 1 and Figure 2 show the comparison results for real valued observation and binary observation, respectively. Each result is based on 20 replicates. We can see that our method has the best prediction error since we are able to utilize the topic information and the structure among nodes and topics; “One matrix” method completely ignores the topic information and ends up with bad prediction error; “$K$ matrices” method ignores the structure among nodes and topics and suffers from overfitting. As sample size goes large, “$K$ matrices” method will behave closer to our model in terms of prediction error, since our model is a special case of the $K$ matrices model. However, it still cannot identify the structure among nodes and topics and is hard to interpret.

We then consider the setting where the topics are unknown. We initialize and estimate $B_1, B_2$ and $M$ according to the procedure described in Section 5; for “One matrix” method, the estimator is still given by $X$; for “$K$ matrices” method, we estimate $\Theta$ and $M$ by alternating gradient method on the objective function (11). All the other setups are the same as the previous case. Figure 3 and Figure 4 shows the comparison results for real valued observation and binary observation, respectively. Again we can see that our model behaves the best. These experiment results demonstrate the effectiveness of our model and algorithm.
Figure 1: Prediction error for real-valued observation, with known topics

Figure 2: Prediction error for binary observation, with known topics

Figure 3: Prediction error for real-valued observation, with unknown topics

Figure 4: Prediction error for binary observation, with unknown topics
Table 1: The influence matrix $B_1$ for citation dataset

| Authors                | black hole | quantum model | gauge theory | algebra space | states space | noncommutative space | boundary | string theory |
|------------------------|------------|---------------|--------------|---------------|-------------|----------------------|----------|---------------|
| Christopher Pope       | 0.359      | 0.468         | 0.318        |               |             |                      |          | 0.318         |
| Arkady Tseytlin        | 0.223      | 0.565         | 0.25         |               |             |                      |          |               |
| Emilio Elizalde        |            |               |              |               |             |                      | 0.109    |               |
| Cumrun Vafa            |            |               |              | 0.85          | 0.623       | 0.679                |          | 0.513         |
| Edward Witten          | 0.204      | 0.795         | 0.678        |               |             |                      | 1.07     | 1.87          |
| Ashok Das              | 0.155      | 0.115         |              |               |             |                      |          |               |
| Sergei Odintsov        |            |               |              |               |             |                      |          |               |
| Sergio Ferrara         | 0.297      | 0.889         | 0.345        | 0.457         | 0.453       | 0.249                |          |               |
| Renata Kallosh         | 0.44       | 0.512         |              | 0.326         | 0.382       |                      |          |               |
| Mirjam Cvetic          | 0.339      | 0.173         | 0.338        |               |             |                      |          |               |
| Burt A. Ovrut          | 0.265      | 0.191         | 0.127        | 0.328         | 0.133       |                      | 0.35     | 0.286         |
| Ergin Sezgin           |            |               |              |               |             |                      |          |               |
| Ian I. Kogan           |            |               |              |               |             |                      | 0.193    |               |
| Gregory Moore          | 0.323      | 0.91          | 0.325        | 0.536         |             |                      |          |               |
| I. Antoniadis          | 0.443      | 0.485         | 0.545        | 0.898         |             |                      | 0.342    |               |
| Mirjam Cvetic          | 0.152      | 0.691         | 0.228        | 0.187         |             |                      |          |               |
| Andrew Strominger      | 0.207      | 0.374         | 0.467        | 1.15          |             |                      |          |               |
| Barton Zwiebach        | 0.16       | 0.222         | 0.383        | 0.236         |             |                      |          |               |
| P.K. Townsend          | 0.629      | 0.349         |              |               |             |                      |          | 0.1           |
| Robert C. Myers        | 0.439      | 0.28          |              |               |             |                      |          |               |
| E. Bergshoeff          | 0.357      | 0.371         |              |               |             |                      |          |               |
| Amihay Hanany          | 0.193      | 0.327         |              |               |             |                      | 1.09     |               |
| Ashoke Sen             | 0.319      | 0.523         |              |               |             |                      | 0.571    |               |
### Table 2: The receptivity matrix $B_2$ for citation dataset

| authors            | black hole | quantum model | gauge theory | algebra space | states space | noncommutative | string theory | supergravity |
|--------------------|------------|---------------|--------------|---------------|--------------|----------------|---------------|--------------|
| Christopher Pope   | 0.477      | 0.794         | 0.59         |               |              |                |               |              |
| Arkady Tseytlin    | 0.704      | 1.16          | 0.312        | 0.487         |              |                |               | 0.119        |
| Emilio Elizalde    | 0          |               |              |               |              |                |               |              |
| Cumrun Vafa        | 0.309      | 0.428         | 0.844        | 0.203         |              |                |               | 0.693        |
| Edward Witten      | 0.352      | 0.554         | 0.585        | 0.213         |              |                |               | 0.567        |
| Ashok Das          | 0.494      | 0.339         | 0.172        |               |              |                |               |              |
| Sergei Odintsov    | 0.472      |               |              |               |              |                |               |              |
| Sergio Ferrara     | 0.423      | 0.59          | 0.664        | 0.776         |              |                |               | 0            |
| Renata Kallosh     | 0.123      | 0.625         | 0.638        | 0.484         |              |                |               | 0.347        |
| Mirjam Cvetic      | 0.47       | 0.731         | 0.309        |               |              |                |               |              |
| Burt A. Ovrut      | 0.314      | 0.217         | 0.72         | 0.409         |              |                |               | 0.137        |
| Ergin Sezgin        | 0.108      | 0.161         | 0.358        |               |              |                |               |              |
| Ian I. Kogan       | 0.357      | 0.382         |              |               |              |                |               | 0.546        |
| Gregory Moore      | 0.375      | 0.178         | 0.721        | 0.69          | 0.455        |                |               | 0.517        |
| I. Antoniadis      | 0.461      | 0.699         | 0.532        |               |              |                |               | 0.189        |
| Mirjam Cvetic      | 0.409      | 1.11          | 0.173        | 0.361         |              |                |               |              |
| Andrew Strominger  | 0.718      | 0.248         | 0.196        | 0.133         |              |                |               |              |
| Barton Zwiebach    | 0.308      | 0.204         |              |               |              |                |               | 0.356        |
| P.K. Townsend      | 0.337      | 0.225         | 0.245        | 0.522         |              |                |               | 0            |
| Robert C. Myers    | 0.364      | 0.956         |              | 0.545         |              |                |               | 0.139        |
| E. Bergshoeff       | 0.487      | 0.459         | 0.174        | 0.619         |              |                |               |              |
| Amihay Hanany      | 0.282      | 0.237         | 0.575        |              |              |                |               | 0.732        |
| Ashoke Sen          | 0.214      | 0.18          | 0.37         |               |              |                |               |              |

### Table 3: Comparison of the 3 methods on test cascades for citation dataset

|                | train error | test error | # parameters | # nonzero |
|----------------|-------------|------------|--------------|-----------|
| One matrix     | 7.628       | 8.223      | 40000        | 7695      |
| $K$ matrices   | 5.861       | 8.415      | 240000       | 19431     |
| Our method     | 8.259       | 8.217      | 2400         | 1200      |
7 Real data

In this section we evaluate our model on real dataset. The dataset we use is the ArXiv collaboration and citation network dataset on high energy physics theory [17, 7]. This dataset covers papers uploaded to ArXiv high energy physics theory category in the period from January 1993 to April 2003, and the citation network for each paper. For our experiment we treat each author as a node and each publication as an observation. For each publication $i$, we set the observation matrix $X_i$ in the following way: the component $(x_{ij}) = 1$ if this paper is written by author $j$ and cited by author $\ell$, and $(x_{ij}) = 0$ otherwise. Since each paper has only several authors, we consider a variant of our original model as

$$X_i^* = [B_1 M_i B_2^T] \odot A_i,$$

where operator $\odot$ is component-wise product and $A_i \in \mathbb{R}^{p \times p}$ is an indictor matrix with $(a_{ij}) = 1$ if $j$ is the author of this paper, and $(a_{ij}) = 0$ otherwise. This means that for each paper, we only consider the influence behavior of its authors.

For our experiment we consider the top 200 authors with about top 10000 papers, and split the papers into 8000 training set and 2000 test set. We first do Topic modeling on the abstracts of all the papers and extract $K = 6$ topics as well as the topic distribution on each paper. We then treat this topic information as known and apply our Algorithm 1 to the training set and learn the two node-topic matrices. The estimated $B_1$ and $B_2$ matrices are shown in Table 1 and Table 2. The keywords of the 6 topics are shown at the head of the two tables and the first column of the two tables is the name of the author.

We then compare the node-topic structure to the research interests and publications listed by the authors themselves on their website. The comparison results show that our model is able to capture the research topics accurately. For example, Christopher Pope reports quantum gravity and string theory; Arkady Tseytlin reports quantum field theory; Emilio Elizalde reports quantum physics; Cumrun Vafa reports string theory; Ashoke Sen reports string theory and black holes as their research areas in their webpages. These are all successfully captured by our method.

Finally we compare the result with “One matrix” and “$K$ matrices” methods on test set. The comparison result is given in Table 3 for training error, testing error, number of total parameters, and number of nonzero parameters. Since our model has much fewer parameters, it has the largest training error. However we can see that our model has the best test error, and both the other two methods do not generalize to test set and suffer from overfitting. These results demonstrates that the topic information and node-topic structure do exist, and our model is able to capture them.

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References

[1] Graphical Models, volume 17 of Oxford Statistical Science Series. The Clarendon Press Oxford University Press, New York, 1996. Oxford Science Publications.

[2] Amr Ahmed and Eric P Xing. Recovering time-varying networks of dependencies in social and biological studies. Proceedings of the National Academy of Sciences, 106(29):11878–11883, 2009.

[3] Edoardo M Airoldi, David M Blei, Stephen E Fienberg, and Eric P Xing. Mixed membership stochastic blockmodels. Journal of Machine Learning Research, 9(Sep):1981–2014, 2008.

[4] Rina Foygel Barber and Mladen Kolar. Rocket: Robust confidence intervals via kendall’s tau for transelliptical graphical models. ArXiv e-prints, arXiv:1502.07641, February 2015.

[5] Peter J Bickel and Aiyou Chen. A nonparametric view of network models and newman–girvan and other modularities. Proceedings of the National Academy of Sciences, 106(50):21068–21073, 2009.

[6] Ronald S Burt. The network structure of social capital. Research in organizational behavior, 22:345–423, 2000.

[7] Johannes Gehrke, Paul Ginsparg, and Jon Kleinberg. Overview of the 2003 kdd cup. ACM SIGKDD Explorations Newsletter, 5(2):149–151, 2003.

[8] Manuel Gomez Rodriguez, Jure Leskovec, and Andreas Krause. Inferring networks of diffusion and influence. In Proceedings of the 16th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 1019–1028. ACM, 2010.

[9] Manuel Gomez-Rodriguez, Le Song, Hadi Daneshmand, and Bernhard Schölkopf. Estimating diffusion networks: Recovery conditions, sample complexity & soft-thresholding algorithm. Journal of Machine Learning Research, 2015.

[10] Paul W Holland, Kathryn Blackmond Laskey, and Samuel Leinhardt. Stochastic blockmodels: First steps. Social networks, 5(2):109–137, 1983.

[11] Wei-Shou Hsu and Pascal Poupart. Online bayesian moment matching for topic modeling with unknown number of topics. In NIPS, 2016.

[12] Jiashun Jin, Zheng Tracy Ke, and Shengming Luo. Estimating network memberships by simplex vertex hunting. arXiv preprint arXiv:1708.07852, 2017.

[13] Brian Karrer and Mark EJ Newman. Message passing approach for general epidemic models. Physical Review E, 82(1):016101, 2010.

[14] Brian Karrer and Mark EJ Newman. Stochastic blockmodels and community structure in networks. Physical review E, 83(1):016107, 2011.
[15] Mladen Kolar, Le Song, Amr Ahmed, and Eric P Xing. Estimating time-varying networks. The Annals of Applied Statistics, pages 94–123, 2010.

[16] Rachel E Kranton and Deborah F Minehart. A theory of buyer-seller networks. American economic review, 91(3):485–508, 2001.

[17] Jure Leskovec, Jon Kleinberg, and Christos Faloutsos. Graphs over time: densification laws, shrinking diameters and possible explanations. In Proceedings of the eleventh ACM SIGKDD international conference on Knowledge discovery in data mining, pages 177–187. ACM, 2005.

[18] Krzysztof Nowicki and Tom A B Snijders. Estimation and prediction for stochastic blockstructures. Journal of the American statistical association, 96(455):1077–1087, 2001.

[19] Neal Parikh and Stephen Boyd. Proximal algorithms. Foundations and Trends® in Optimization, 1(3):127–239, 2014.

[20] Manuel Gomez Rodriguez, David Balduzzi, and Bernhard Schölkopf. Uncovering the temporal dynamics of diffusion networks. arXiv preprint arXiv:1105.0697, 2011.

[21] Karl Rohe, Sourav Chatterjee, and Bin Yu. Spectral clustering and the high-dimensional stochastic blockmodel. The Annals of Statistics, pages 1878–1915, 2011.

[22] Achim Walter, Thomas Ritter, and Hans Georg Gemünden. Value creation in buyer–seller relationships: Theoretical considerations and empirical results from a supplier’s perspective. Industrial marketing management, 30(4):365–377, 2001.

[23] Stanley Wasserman and Katherine Faust. Social network analysis: Methods and applications, volume 8. Cambridge university press, 1994.

[24] David T Wilson. An integrated model of buyer-seller relationships. Journal of the academy of marketing science, 23(4):335–345, 1995.

[25] Ming Yu, Varun Gupta, and Mladen Kolar. An influence-receptivity model for topic based information cascades. 2017 IEEE International Conference on Data Mining (ICDM), pages 1141–1146, 2017.

[26] Ming Yu, Mladen Kolar, and Varun Gupta. Statistical inference for pairwise graphical models using score matching. In Advances in Neural Information Processing Systems, pages 2829–2837, 2016.

[27] Ming Yu, Zhaoran Wang, Varun Gupta, and Mladen Kolar. Recovery of simultaneous low rank and two-way sparse coefficient matrices, a nonconvex approach. arXiv preprint arXiv:1802.06967, 2018.

[28] Ke Zhou, Hongyuan Zha, and Le Song. Learning social infectivity in sparse low-rank networks using multi-dimensional hawkes processes. In Artificial Intelligence and Statistics, pages 641–649, 2013.
A Technical proofs

A.1 Proof of Lemma 1.

Proof. Since $\hat{\Theta} = (\hat{\Theta}_1, \ldots, \hat{\Theta}_K)$ are the global minimum of (4), we have

$$0 \geq f(\hat{\Theta}) - f(\Theta^*) \geq \langle \nabla f(\Theta^*), \hat{\Theta} - \Theta^* \rangle + \frac{\mu_{\Theta}}{2} \|\hat{\Theta} - \Theta^*\|_F^2.$$  

We then have

$$\|\hat{\Theta} - \Theta^*\|_F^2 \leq -\frac{2}{\mu_{\Theta}} \langle \nabla f(\Theta^*), \hat{\Theta} - \Theta^* \rangle \leq \frac{2}{\mu_{\Theta}} \|\nabla f(\Theta^*)\|_F \cdot \|\hat{\Theta} - \Theta^*\|_F,$$

and hence

$$\|\hat{\Theta} - \Theta^*\|_F \leq \frac{2}{\mu_{\Theta}} \|\nabla f(\Theta^*)\|_F.$$

$\square$

A.2 Proof of Theorem 2.

Proof. We apply the non-convex optimization result in [27]. Since the initialization condition and (RSC/RSS) are satisfied for our problem according to Lemma 1, we apply Lemma 3 in [27] and obtain

$$d^2(B^{(t+1)}, B^*) \leq \xi^2 \left[ (1 - \eta \cdot \frac{2}{5} \mu_{\min} \sigma_{\max}) \cdot d^2(B^{(t)}, B^*) + \eta \cdot \frac{L_{\Theta} + \mu_{\Theta}}{L_{\Theta} \cdot \mu_{\Theta}} \cdot \epsilon_{\text{stat}, \Theta}^2 \right],$$  

(12)

where $\xi^2 = 1 + \frac{2}{\sqrt{\epsilon_{\text{stat}, \Theta}}}$ and $\sigma_{\max} = \max_k \|\Theta_k^*\|_2$. Define the contraction value

$$\beta = \xi^2 \left( 1 - \eta \cdot \frac{2}{5} \mu_{\min} \sigma_{\max} \right) < 1,$$

we can iteratively apply (12) for each $t = 1, 2, \ldots, T$ and obtain

$$d^2(B^{(T)}, B^*) \leq \beta^T d^2(B^{(0)}, B^*) + \frac{\xi^2 \eta}{1 - \beta} \cdot \frac{L_{\Theta} + \mu_{\Theta}}{L_{\Theta} \cdot \mu_{\Theta}} \cdot \epsilon_{\text{stat}, \Theta}^2,$$

which shows linear convergence up to statistical error. $\square$

A.3 Proof of Lemma 4.

Proof. Since $\tilde{X}$ is the best rank $K$ approximation for $X$ and $X^*$ is also rank $K$, we have

$$\|\tilde{X} - X\|_F \leq \|X^* - X\|_F$$

and hence

$$\|\tilde{X} - X\|_F \leq \|\tilde{X} - X\|_F + \|X^* - X\|_F \leq 2\|X^* - X\|_F = 2\|E\|_F.$$

(13)

By definition we have

$$X^* = \sum_{k=1}^K \left( \frac{1}{n} \sum_{i=1}^n m_{ik}^* \right) \Theta_k^* = B_1^* A^* B_2^\top = Q_1 R_1 A^* R_2^\top Q_2^\top = Q_1 (A_{\text{diag}} + A_{\text{off}}) Q_2^\top.$$
Plug back to (13) we obtain
\[ \| \tilde{X} - Q_1 (A_{\text{diag}} + A_{\text{off}}) Q_2^\top \|_F \leq 2 \| E \|_F, \]
and hence
\[ \left\| \sum_{k=1}^{K} \tilde{\sigma}_k \tilde{u}_k \tilde{v}_k^\top - Q_1 A_{\text{diag}} Q_2^\top \right\|_F \leq 2 \| E \|_F + \| Q_1 A_{\text{off}} Q_2^\top \|_F \leq 2 \| E \|_F + \rho_0. \]  \hspace{1cm} (14)

Since $E$ is the mean value of i.i.d. errors $E_i$, we have that $\| E \|_F \propto n^{-1/2}$ and therefore can be arbitrarily small with large enough $n$. Moreover, the left hand side of (14) is the difference of two singular value decompositions. According to the matrix perturbation theory, for each $k$ we have (up to permutation)
\[ \left\| \tilde{\sigma}_k \tilde{u}_k \tilde{v}_k^\top - q_{1,k} a_{\text{diag},k} \cdot q_{2,k}^\top \right\|_F \leq 2C \rho_0; \]
and hence
\[ \left\| \tilde{\sigma}_k \tilde{u}_k \tilde{v}_k^\top - \frac{1}{n} \sum_{i=1}^{n} m_{ik} \Theta_k^* \right\|_F \leq 2\tilde{C} \rho_0. \]

Finally we obtain
\[ \left\| K \cdot \tilde{\sigma}_k \tilde{u}_k \tilde{v}_k^\top - \Theta_k^* \right\|_F = K \cdot \left\| \tilde{\sigma}_k \tilde{u}_k \tilde{v}_k^\top - \frac{1}{K} \Theta_k^* \right\|_F \leq K \cdot \left( 2\tilde{C} \rho_0 + \left| \frac{1}{n} \sum_{i=1}^{n} m_{ik} - \frac{1}{K} \right| \| \Theta_k^* \|_F \right) \leq 2\tilde{C} K \rho_0 + (\eta-1) \sigma_{\text{max}}. \]

\section*{A.4 Proof of Theorem 5}

We analyze the two estimation step in Algorithm 2.

\textbf{Update on $B_1$ and $B_2$.} The update algorithm on $B_1$ and $B_2$ is the same with known $M$. Besides the statistical error defined in (8), we now have an additional error term due to the error in $M$. Recall that $d^2(M, M^*) = \frac{1}{n} \sum_{i=1}^{n} \sum_{k_0=1}^{K} (m_{ik} - m_{ik}^*)^2$; Lemma 6 quantifies the effect of one estimation step on $B$.

\textbf{Lemma 6.} Suppose the conditions in Theorem 2 hold and suppose condition (DC) and (OC) hold, we have
\[ d^2(B^{[t]}, B^*) \leq C_1 \cdot e_{\text{stat},\Theta}^2 + \beta_1 \cdot d^2(M^{[t]}, M^*), \]
for some constant $C_1$ and $\beta_1$.

\textbf{Update on $M$.} Lemma 7 quantifies the effect of one estimation step on $M$.

\textbf{Lemma 7.} Suppose the condition (TC) holds, we have
\[ d^2(M^{[t]}, M^*) \leq C_2 \cdot e_{\text{stat},M}^2 + \beta_2 \cdot d^2(B^{[t]}, B^*), \]
for some constant $C_2$ and $\beta_2$.

Denote $\beta_0 = \min\{\beta_1, \beta_2\}$, as long as the signal $\sigma_{\text{max}}$ is small and the noise $E_i$ is small enough we can guarantee that $\beta_0 < 1$. Combine Lemma 6 and 7 we complete the proof.
A.5 Proof of Lemma 6.

Proof. The analysis is exactly the same with the case where $M$ is known except that the statistical error is different. Specifically, for each $k$ we have

$$
\nabla_{\Theta_i} f(\Theta^*, M) = -\frac{1}{n} \sum_{i=1}^{n} \left( X_i - \sum_{k_0=1}^{K} m_{ik_0} \Theta_{k_0}^* \right) \cdot m_{ik}
$$

$$= -\frac{1}{n} \sum_{i=1}^{n} \left( E_i + \sum_{k_0=1}^{K} (m_{ik_0}^* - m_{ik_0}) \Theta_{k_0}^* \right) \cdot m_{ik}
$$

$$= -\frac{1}{n} \sum_{i=1}^{n} E_i m_{ik}^* + \frac{1}{n} \sum_{i=1}^{n} E_i (m_{ik}^* - m_{ik}) + \frac{1}{n} \sum_{i=1}^{n} \sum_{k_0=1}^{K} (m_{ik_0}^* - m_{ik_0}) \Theta_{k_0}^* \cdot m_{ik}.
$$

The first term $R_1$ is just the usual statistical error term on $\Theta$. For term $R_2$, denote $e_0 = \frac{1}{n} \sum_{i=1}^{n} \| E_i \|_F^2$, we have

$$
\| R_2 \|_F^2 \leq \frac{1}{n^2} \left( \sum_{i=1}^{n} \| E_i \|_F^2 \right) \cdot \sum_{i=1}^{n} (m_{ik} - m_{ik}^*)^2 \leq \frac{e_0}{n} \sum_{i=1}^{n} (m_{ik} - m_{ik}^*)^2.
$$

For term $R_3$, we have

$$
\| R_3 \|_F^2 \leq \frac{1}{n^2} \left( \sum_{i=1}^{n} \sum_{k_0=1}^{K} (m_{ik_0} - m_{ik_0}^*) \Theta_{k_0}^* \cdot m_{ik} \right)^2 \leq \frac{1}{n^2} \left( \sum_{i=1}^{n} \sum_{k_0=1}^{K} (m_{ik_0} - m_{ik_0}^*)^2 \right) \cdot \left( \sum_{i=1}^{n} \sum_{k_0=1}^{K} \| \Theta_{k_0}^* \|_F^2 \cdot m_{ik}^2 \right)
$$

$$\leq \frac{K \sigma_{\text{max}}^2}{n^2} \left( \sum_{i=1}^{n} m_{ik}^2 \right) \cdot \left( \sum_{i=1}^{n} \sum_{k_0=1}^{K} (m_{ik_0} - m_{ik_0}^*)^2 \right).
$$

Taking summation over all $k$, the first term $R_1$ gives the statistical error as before, the terms $R_2$ and $R_3$ gives

$$
\sum_{k=1}^{K} \| R_2 \|_F^2 + \| R_3 \|_F^2 \leq \frac{e_0 + K \sigma_{\text{max}}^2}{n} \left( \sum_{i=1}^{n} \sum_{k=1}^{K} (m_{ik} - m_{ik}^*)^2 \right).
$$

□

A.6 Proof of Lemma 7.

Proof. The estimation on $M$ is separable with each $m_i$. Denote the objective function on observation $i$ as

$$
f_i(\Theta, m_i) = \| X_i - \sum_{k=1}^{K} m_{ik} \cdot \Theta_k \|_F^2.
$$

(15)

According to condition (DC), the objective function (15) is $\mu_M$-strongly convex in $m_i$. Similar to the proof of Lemma 1, we obtain

$$
\sum_{k=1}^{K} (m_{ik} - m_{ik}^*)^2 \leq \frac{4}{\mu_M^2} \| \nabla_{m_i} f_i(\Theta, m_i^*) \|_F^2 = \frac{4}{\mu_M^2} \sum_{k=1}^{K} \left( \nabla_{m_{ik}} f_i(\Theta, m_i^*) \right)^2.
$$
Moreover, we have
\[
\nabla m_{ik} f_i(\Theta, m_i^*) = -\langle X_i - \sum_{k_0=1}^{K} m_{ik_0}^* \cdot \Theta_{k_0}, \Theta_k \rangle
\]
\[
= -\langle E_i, \Theta_k^* \rangle + \langle E_i, (\Theta_k^* - \Theta_k) \rangle + \langle \sum_{k_0=1}^{K} m_{ik_0}^* (\Theta_{k_0} - \Theta_{k_0}^*), \Theta_k \rangle.
\]

The first term \(T_1\) is just the usual statistical error term on \(M\). For term \(T_2\), we have
\[
\sum_{i=1}^{n} \sum_{k=1}^{K} (T_2)^2 \leq \left( \sum_{i=1}^{n} \|E_i\|^2_F \cdot \|\Theta_k^* - \Theta_k\|^2_F \right) \cdot \left( \sum_{k=1}^{K} \|\Theta_k^* - \Theta_k\|^2_F \right).
\] (16)

For term \(T_3\) we have
\[
\sum_{k=1}^{K} (T_3)^2 \leq K \sigma_{\max}^2 \left( \sum_{k=1}^{K} \|\Theta_k^* - \Theta_k\|^2_F \right) \cdot \left( \sum_{k=1}^{K} m_{ik_k}^2 \right)
\]
\[
\leq K \sigma_{\max}^2 \left( \sum_{k=1}^{K} \|\Theta_k^* - \Theta_k\|^2_F \right).
\] (17)

Moreover, we have
\[
\|\Theta_k^* - \Theta_k\|^2_F = \|b_k^1 b_k^2^T - b_k^1 b_k^2^T\|^2_F = \|b_k^1\|^2_2 \|b_k^2^*\|^2_2 - b_k^1 \|b_k^2^* - b_k^1\|^2_2 \leq 2\sigma_{\max} \left( \|b_k^2^* - b_k^2\|^2_2 + \|b_k^1 - b_k^1\|^2_2 \right),
\]
and hence
\[
\sum_{k=1}^{K} \|\Theta_k^* - \Theta_k\|^2_F \leq 4\sigma_{\max}^2 \sum_{k=1}^{K} \left( \|b_k^2^* - b_k^2\|^2_2 + \|b_k^1 - b_k^1\|^2_2 \right) \leq 8\sigma_{\max}^2 d^2(B, B^*).
\]

Combine (16) and (17), taking summation over \(i\), we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{K} (T_2)^2 + (T_3)^2 \leq \left( e_0 + K \sigma_{\max}^2 \right) \cdot \left( \sum_{k=1}^{K} \|\Theta_k^* - \Theta_k\|^2_F \right) \leq 8\sigma_{\max}^2 \left( e_0 + K \sigma_{\max}^2 \right) \cdot d^2(B, B^*).
\]