Multiple solutions for Kirchhoff equations under the partially sublinear case

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Abstract

In this paper, we prove the infinitely many solutions to a class of sublinear Kirchhoff type equations by using an extension of Clark’s theorem established by Zhaoli Liu and Zhi-Qiang Wang.

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1. Introduction and main results

In this paper we study the existence and multiplicity of solutions for the following Kirchhoff type equations:

\[
(a + \int_{\mathbb{R}^N} |\nabla u|^2 + b \int_{\mathbb{R}^N} u^2) \left[ -\Delta u + bu \right] = K(x)f(x, u), \text{ in } \mathbb{R}^3,
\]

where \( a, b \) are positive constants.

When \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^3 \), the problem

\[
\begin{cases}
- (a + b \int_{\Omega} |\nabla u|^2 \, dx) \triangle u = f(x, u), & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

has been many papers concerned. Perera and Zhang [1] considered the case where \( f(x, \cdot) \) is asymptotically linear at 0 and asymptotically \( 4-\)linear at infinity. They obtained a nontrivial solution of the problems by using the Yang index and critical group. Then, in [1] they considered the cases where \( f(x, \cdot) \) is \( 4-\)sublinear, \( 4-\)superlinear and asymptotically \( 4-\)linear at infinity. By various assumption on \( f(x, \cdot) \) near 0, they obtained multiple and sign changing solutions. Cheng and Wu [3], Ma and Rivera [4] studied the existence of positive solutions of (1.2) and He and Zou [5] obtained the existence of infinitely many positive solutions of (1.2), respectively; Mao and Luan [6] obtained the existence of signed and sign-changing solutions for the problem (1.2) with asymptotically

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4-linear bounded nonlinearity via variational methods and invariant sets of descent flow; Sun and Tang [7] studied the existence and multiplicity results of nontrivial solutions for the problem (1.2) with the weaker monotony and 4-superlinear nonlinearity. For (1.2), Sun and Liu [8] considered the cases where the nonlinearity is superlinear near zero but asymptotically 4-linear at infinity, and the nonlinearity is asymptotically linear near zero but 4-superlinear at infinity. By computing the relevant critical groups, they obtained nontrivial solutions via Morse theory.

Comparing with (1.1) and (1.2), $\mathbb{R}^3$ in place of the bounded domain $\Omega \subset \mathbb{R}^3$. This makes that the study of the problem (1.1) is more difficult and interesting. Wu [11] considered a class of Schrödinger Kirchhoff-type problem in $\mathbb{R}^N$ and a sequence of high energy solutions are obtained by using a symmetric Mountain Pass Theorem. In [12], Alves and Figueiredo study a periodic Kirchhoff equation in $\mathbb{R}^N$, they get the nontrivial solution when the nonlinearity is in subcritical case and critical case. Liu and He [13] get multiplicity of high energy solutions for superlinear Kirchhoff equations in $\mathbb{R}^3$. Li, Li and Shi in [15] proved the existence of a positive solution to a Kirchhoff type problem on $\mathbb{R}^N$ by using variational methods and cut-off functional technique.

In [9], Jin and Wu in consider the following problem:

$$
\begin{cases}
- \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \triangle u + u = f(x, u), & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
$$

(1.3)

where constants $a > 0, b > 0$, $N = 2$ or $3$ and $f \in C(R^N \times R, R)$.

By using the Fountain Theorem, they obtained the following theorem.

**Theorem A** [9] Assume that the following conditions hold:

If the following assumptions are satisfied,

(H$_1$) $f(x, u) = o(|u|)$ as $|u| \to 0$ uniformly for any $x \in \mathbb{R}^N$.

(H$_2$) There are constants $1 < p < 2^* - 1$ and $c > 0$ such that

$$|f(x, u)| \leq c(1 + |u|^p), \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R},$$

where

$$2^* - 1 = \begin{cases}
\frac{N+2}{N-2}, & N \geq 3; \\
+\infty, & N = 1, 2.
\end{cases}$$

(H$_3$) There exists $\mu > 4$ such that

$$\mu F(x, u) = \mu \int_0^u f(x, s) \, ds \leq u f(x, u), \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}.$$

(H$_4$) $\inf_{x \in \mathbb{R}^N, |u| = 1} F(x, u) > 0$

(H$_5$) $f(gx, u) = f(x, u)$ for each $g \in O(N)$ and for each $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, where $O(N)$ is the group of orthogonal transformations on $\mathbb{R}^N$. 

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(H_6) \( f(x, -u) = -f(x, u) \) for any \((x, u) \in \mathbb{R}^N \times \mathbb{R} \).

Then problem (1.3) has a sequence \( \{u_k\} \) of radial solutions.

Recently, the authors obtained an extension of Clark’s theorem as follows.

**Theorem B** [10] Let \( X \) be a Banach space, \( \Phi \in C^1(X, \mathbb{R}) \). Assume \( \Phi \) is even and satisfies the (PS) condition, bounded from below, and \( \Phi(0) = 0 \). If for any \( k \in \mathbb{N} \), there exists a \( k \)-dimensional subspace \( X^k \) of \( X \) and \( \rho_k > 0 \) such that \( \sup_{X^k \cap S_{\rho_k}} \Phi < 0 \), where \( S_{\rho} = \{u \in X \|u\| = \rho\} \), then at least one of the following conclusions holds.

(i) There exists a sequence of critical points \( \{u_k\} \) satisfying \( \Phi(u_k) < 0 \) for all \( k \) and \( \|u_k\| \to 0 \) as \( k \to \infty \).

(ii) There exists \( r > 0 \) such that for any \( 0 < a < r \) there exists a critical point \( u \) such that \( \|u\| = a \) and \( \Phi(u) = 0 \).

In this paper, we consider the multiple solutions for Kirchhoff equations under the partially sublinear case by using the Theorem C. Our main result is as follows.

**Theorem 1.1** Assume that \( f \) satisfies (B3) and the following conditions:

(f1) There exist \( \delta > 0 \), \( 1 \leq \gamma < 2 \), \( C > 0 \) such that \( f \in C(\mathbb{R}^3 \times [-\delta, \delta], \mathbb{R}) \) and \( |f(x, z)| \leq C|z|^{\gamma-1} \);

(f2) \( \lim_{z \to 0} F(x, z)/|z|^2 = +\infty \) uniformly in some ball \( B_r(x_0) \subset \mathbb{R}^3 \), where \( F(x, z) = \int_0^z f(x, s)ds \).

(f3) \( K : \mathbb{R}^3 \to \mathbb{R}^+ \) is a positive continuous function such that \( K \in L^{2/(1-\gamma)}(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3) \).

Then \( (1.1) \) possesses infinitely many solutions \( \{u_k\} \) such that \( \|u_k\|_{L^\infty} \to 0 \) as \( k \to \infty \).

**Remark 1.1.** Throughout the paper we denote by \( C > 0 \) various positive constants which may vary from line to line and are not essential to the problem.

The paper is organized as follows: in Section 2, some preliminary results are presented. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary

In this Section, we will give some notations and Lemma that will be used throughout this paper.

Let \( H^1 = H^1(\mathbb{R}^3) \) be the completion of \( C_0^\infty(\mathbb{R}^3) \) with respect to the inner product and norm

\[
(u, v) = \int_{\mathbb{R}^3} |\nabla u \nabla v + buv|dx, \quad \|u\| = (u, u)^{1/2}.
\]

Moreover, we denote the completion of \( C_0^\infty(\mathbb{R}^3) \) with respect to the norm

\[
\|u\|_{D^1} = \int_{\mathbb{R}^3} |\nabla u|^2dx
\]
by $D^1 = D^1(\mathbb{R}^3)$. To avoid lack of compactness, we need consider the set of radial functions as follows:

$$H = H^1_1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) | u(x) = u(|x|)\}.$$ 

Here we note that the continuous embedding $H \hookrightarrow L^q(\mathbb{R}^3)$ is compact for any $q \in (2, 6)$.

Define a functional

$$J_1(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} \|u\|^4 - \int_{\mathbb{R}^3} K(x) F(x, u), \ u \in H.$$ 

Then we have from (f1) that $J_1$ is well defined on $H$ and is of $C^1$, and

$$(J_1(u), v) = a(u, v) + \|u\|^2(u, v) - \int_{\mathbb{R}^3} K(x) f(x, u) v, \ u, v \in H.$$ 

It is standard to verify that the weak solutions of (1.1) correspond to the critical points of the functional $J_1$.

3. Proofs of the main result

**Proof of Theorem 1.1.** Choose $\hat{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ such that $\hat{f}$ is odd in $u \in \mathbb{R}$, $\hat{f}(x, u) = f(x, u)$ for $x \in \mathbb{R}^N$ and $|u| < \delta/2$, and $\hat{f}(x, u) = 0$ for $x \in \mathbb{R}^N$ and $|u| > \delta$. In order to obtain solutions of (1.1) we consider

$$\left(a + \int_{\mathbb{R}^N} |\nabla u|^2 + b \int_{\mathbb{R}^N} u^2 \right) [-\Delta u + bu] = K(x) \hat{f}(x, u), \text{ in } \mathbb{R}^N, \ (3.1)$$

Moreover, (3.1) is variational and its solutions are the critical points of the functional defined in $H$ by

$$J(u) = \frac{1}{2} a \|u\|^2 + \frac{1}{4} \|u\|^4 - \int_{\mathbb{R}^3} K(x) \hat{F}(x, u) dx.$$ 

From (f1), it is easy to check that $J$ is well defined on $H$ and $J \in C^1(H^1_1(\mathbb{R}^3), \mathbb{R})$ (see [? ] for more detail), and

$$J'(u)v = a(u, v) + \|u\|^2(u, v) - \int_{\mathbb{R}^3} K(x) \hat{f}(x, u) v dx, \ v \in H.$$ 

Note that $J$ is even, and $J(0) = 0$. For $u \in H^1_1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} K(x) |\hat{F}(x, u)| dx \leq C \int_{\mathbb{R}^3} K(x)|u|^\gamma dx \leq C \|K\|_{L^{\frac{2}{2-\gamma}}(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}^\gamma \leq C \|u\|^\gamma.$$ 

Hence, it follows from Lemma 2.1 that

$$J(u) \geq \frac{1}{2} \|u\|^2 - C \|u\|^\gamma, \ u \in H. \ (3.2)$$

We now use the same ideas to prove the (PS) condition. Let $\{u_n\}$ be a sequence in $H$ so that $J(u_n)$ is bounded and $J'(u_n) \to 0$. We shall prove that $\{u_n\}$ converges. By (3.2),
we claim that \( \{u_n\} \) is bounded. Assume without loss of generality that \( \{u_n\} \) converges to \( u \) weakly in \( H \). Observe that

\[
\langle J'(u_n) - J'(u), u_n - u \rangle = a\|u_n - u\|^2 + \|u_n\|^2\|u_n - u\|^2 + (\|u_n\|^2 - \|u\|^2)(u, u_n - u)
\]

\[
- \int_{\mathbb{R}^3} K(x)(\hat{f}(x, u_n) - \hat{f}(x, u))(u_n - u)dx.
\]

Hence, we have

\[
a\|u_n - u\|^2 \leq \langle J'(u_n) - J'(u), u_n - u \rangle - (\|u_n\|^2 - \|u\|^2)(u, u_n - u)
\]

\[
+ \int_{\mathbb{R}^3} K(x)(\hat{f}(x, u_n) - \hat{f}(x, u))(u_n - u)dx.
\]

\[
\equiv I_1 + I_2 + I_3,
\]

It is clear that \( I_1 \to 0 \) and \( I_2 \to 0 \) as \( n \to \infty \). In the following, we will estimate \( I_3 \), by using \((f_3)\), for any \( R > 0 \),

\[
\int_{\mathbb{R}^3} K(x)|\hat{f}(x, u_n) - \hat{f}(x, u)||u_n - u|dx
\]

\[
\leq C \int_{\mathbb{R}^3 \setminus B_R(0)} K(x)(\|u_n\| + |u|)dx + C \int_{B_R(0)} (\|u_n\| + |u|)\|u_n\| - |u|dx
\]

\[
\leq C \left( \|u_n\|_{L^2(\mathbb{R}^3 \setminus B_R(0))} + \|u\|_{L^2(\mathbb{R}^3 \setminus B_R(0))} \right) \|K\|_{L^{\frac{2}{\gamma}}(\mathbb{R}^3 \setminus B_R(0))}
\]

\[
+ C \left( \|u_n\|_{L^\gamma(\mathbb{R}^3 \setminus B_R(0))} + \|u\|_{L^\gamma(\mathbb{R}^3 \setminus B_R(0))} \right) \|u_n - u\|_{L^\gamma(B_R(0))}
\]

\[
\leq C\|K\|_{L^{\frac{2}{\gamma}}(\mathbb{R}^3 \setminus B_R(0))} + C\|u_n - u\|_{L^\gamma(B_R(0))},
\]

which implies

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} K(x)|\hat{f}(x, u_n) - \hat{f}(x, u)||u_n - u|dx = 0.
\]

Therefore, \( \{u_n\} \) converges strongly in \( H \) and the (PS) condition holds for \( J \). By \((f_2)\) and \((f_3)\), for any \( L > 0 \), there exists \( \delta = \delta(L) > 0 \) such that if \( u \in C_0^\infty(B_r(x_0)) \) and \( |u|_\infty < \delta \) then \( K(x)\hat{F}(x, u(x)) \geq L|u(x)|^2 \), and it follows from Lemma 2.1 that

\[
J(u) \leq \frac{a}{2}\|u\|^2 + \frac{1}{4}\|u\|^4 - L\|u\|_{L^2(\mathbb{R}^3)}^2.
\]

This implies, for any \( k \in \mathbb{N} \), if \( X^k \) is a \( k \)-dimensional subspace of \( C_0^\infty(B_r(x_0)) \) and \( \rho_k \) is sufficiently small then \( \sup_{X^k} J(u) < 0 \), where \( S_\rho = \{u \in \mathbb{R}^3 | \|u\| = \rho \} \). Now we apply Theorem C to obtain infinitely many solutions \( \{u_k\} \) for \((3.1)\) such that

\[
\|u_k\|_{L^\infty} \to 0, \ k \to \infty.
\]

Finally we show that \( \|u_k\|_{L^\infty} \to 0 \) as \( k \to \infty \). Let \( u \) be a solution of \((3.1)\) and \( \alpha > 0 \). Let \( M > 0 \) and set \( u^M(x) = \max\{-M, \min\{u(x), M\}\} \). Multiplying both sides of \((3.1)\) with \( |u^M|^\alpha u^M \) implies

\[
\frac{4a}{(\alpha + 2)^2} \int_{\mathbb{R}^3} |\nabla |u|^M|^{\frac{\alpha}{\alpha + 2} + 1}|^2 dx \leq C \int_{\mathbb{R}^3} |u^M|^\alpha + 1 dx.
\]
By using the iterating method in [10], we can get the following estimate

\[ \|u\|_{L^\infty(\mathbb{R}^3)} \leq C_1 \|u\|_{L^6(\mathbb{R}^3)}^{\nu}, \]

where \( \nu \) is a number in \((0, 1)\) and \( C_1 > 0 \) is independent of \( u \) and \( \alpha \). By (3.3) and Sobolev imbedding Theorem [14], we derive that \( \|u_k\|_{L^\infty(\mathbb{R}^3)} \to 0 \) as \( k \to \infty \). Therefore, \( u_k \) are the solutions of (1.1) as \( k \) sufficiently large. The proof is completed.

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