Operator representations for a class of quantum entanglement measures and criterions

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We find that a class of entanglement measures for bipartite pure state can be expressed by the average values of quantum operators, which are related to any complete basis of one partite operator space. Two specific examples are given based on two different ways to generalize Pauli matrices to d dimensional Hilbert space and the case for identical particle system is also considered. In addition, applying our measure to mixed state case will give a sufficient condition for entanglement.

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I. INTRODUCTION

Quantum entanglement is an essential physical resource to process quantum information and computation, which enables us to complete the tasks intractable in classical domain, such as quantum teleportation, quantum cryptography, Shor’s algorithm of factoring large numbers, and Grover’s quantum searching algorithm [1].

In order to use such kind of resources efficiently, it is necessary to qualify the properties and quantify the degrees of quantum entanglement for a given quantum state. In this direction, continuous progresses have been made. To clarify the meaning and qualify the properties of entanglement, Werner defined separate state from whether being able to prepare the state classically, whose definition has become the standard mathematical basis of entanglement state [2]. Next, Peres proposed a famous necessary condition for separity —positive of partial transpose density operator [3], then Horodeckies prove this criteria is also sufficient in the cases of $\mathcal{H}^2 \otimes \mathcal{H}^2$ and $\mathcal{H}^2 \otimes \mathcal{H}^3$ [4].

In order to quantify this property, many entanglement measures have been proposed in the past years, both for pure states and mixed states [5, 6, 7]. However, only for bipartite
pure states the quantitative theory of entanglement satisfies with all the priori axioms of a good entanglement measure, which is mainly due to the existence of the celebrated Schmidt decomposition for these states. It is well known that the von Neumann entropy of the reduced density matrix $S_E$ is the unique measure for bipartite pure states in the sense that $S_E$ can be concentrated and diluted with unit asymptotic efficiency. However, Vidal developed the concept of entanglement monotone and shows that to characterize the non-local property of finite number bipartite pure states, indeed $d - 1$ independent measure is needed in the sense that there is $d - 1$ Schmidt coefficients. Thus it is known that although the entanglement monotones have different asymptotic properties than $S_E$, they are important for characterizing non-local properties under LOCC transformations. For mixed state case, recently Wooters gives out an explicit expression of entanglement of formation in $\mathcal{H}^2 \otimes \mathcal{H}^2$. However, there are still many open questions, especially for many partite system and mixed states.

As we know, entanglement measures are functionals of density operator. However, quantities in traditional quantum physics are quantum observables. In this sense, entanglement measures are not standard physical quantum observables, and they are also not the average values of some entanglement measure operators. In this article, we attempt to establish the relations between entanglement measures and quantum observables. However, we achieve this end only for a specific class of entanglement measures, which will be analyzed in Section 2, where the case for identical system is also considered. In sec. 3, two specific examples are given based on different generations of Pauli matrixes, which preserve Hermitian and Unitary respectively. Finally, we apply those results to form a criterion of mixed state entanglement and a short summary is also given in sec. 4.

II. OPERATOR SPACE REPRESENTATIONS FOR A CLASS OF QUANTUM ENTANGLEMENT MEASURES

For a bipartite pure state $|\psi_{AB}\rangle$ in Hilbert space $\mathcal{H}_A^d \otimes \mathcal{H}_B^{d'}$ ($d \leq d'$), a class of functions of reduced density operator $\rho_A$ can be defined as

$$M_e(n) = 1 - \text{Tr}\rho_A^n, \quad (n \in \mathbb{N} \text{ and } n \geq 2)$$  (1)
where the reduced density operator $\hat{\rho}_A = \text{Tr}_B (|\psi_{AB}\rangle \langle \psi_{AB}|)$. It is easy to show that the above class of functions are entanglement monotones, or entanglement measures due to the fact that they only depend on the eigenvalues the reduced density matrix $\rho_A$, or equivalently the Schmidt numbers of the state $|\psi_{AB}\rangle$.

Denote the linear space of operators act on the Hilbert space $\mathcal{H}_A^d$ as $\mathcal{M}_A^d$, which is a linear space of $d \times d$ dimensions, and denote the arbitrary operator $P \in \mathcal{M}_A^d$ as $|P\rangle$ and $P^\dagger \in \mathcal{M}_A^d$ as $\langle P|$. Define the inner product of $\mathcal{H}_A^d$ as

$$\langle P|Q \rangle = \text{Tr}(P^\dagger Q), \quad \forall P, Q \in \mathcal{M}_A^d. \quad (2)$$

Then we can rewrite the class of entanglement measures in Eq. (1) as

$$M_e(n) = 1 - \langle \rho_A | \rho_A^{n-2} | \rho_A \rangle. \quad (3)$$

For each entanglement measure $M_e(n)$, we take $n - 1$ sets of complete operators $S_C^m = \{O_i^m\}$ $(m = 1, 2, \cdots, n - 1)$, which satisfy

$$\sum_i |O_i^m\rangle \langle O_i^m| = 1. \quad (4)$$

Using the above relations, we rewrite the entanglement measures as

$$M_e(n) = 1 - \sum_{i_1,i_2,\cdots,i_{n-1}} \langle \rho_A | O_{i_1}^1 \rangle \langle O_{i_1}^1 | \rho_A | O_{i_2}^2 \rangle \cdots \langle O_{i_{n-2}}^{n-2} | \rho_A | O_{i_{n-1}}^{n-1} \rangle \langle O_{i_{n-1}}^{n-1} | \rho_A \rangle$$

$$= 1 - \sum_{i_1,i_2,\cdots,i_{n-1}} \langle O_{i_1}^1 \rangle \langle O_{i_2}^2 | O_{i_1}^1 \rangle \cdots \langle O_{i_{n-1}}^{n-1} | O_{i_{n-2}}^{n-2} \rangle \langle O_{i_{n-1}}^{n-1} \rangle, \quad (5)$$

where

$$\langle O \rangle = \text{Tr}(\rho_A O), \quad (6)$$

and obviously it is the also expected value of operator $O$ in state $|\psi_{AB}\rangle$.

Eq. (5) is the main result of this paper and it relates the entanglement measures with physical observables, i.e., it tells us the following information: If we obtain a serious of expected values for some complete operators, the degree of entanglement can be evaluated by Eq. (5). In other words, we can measure entanglement by measuring some physical observables. It is worthy to note that physical observables can be represented by unitary operators besides Hermitian ones, in the sense that for any unitary operator we can always find such a Hermitian operator that might be mapped to the unitary operator by exponential functions.
In the case of $n = 2$, Eq. (5) takes a much simpler form:

$$M_e(2) = 1 - \sum_i |\langle O_i \rangle|^2 = \frac{1}{2} C_I^2. \quad (7)$$

Where $C_I$ is the generalized concurrence, or $I$-concurrence for two qudits. It is well known that among all the entanglement monotones, concurrence is important since it is related to the entanglement of formation for two qubits. It is also found that there is many ways to define concurrence for bipartite pure states, which reveals different physical meanings. Very recently, the concept of concurrence is generalized to higher dimensions based on the “universal inverter” and the mathematical point of view, although almost all the ways of defining concurrence for two qubits can not be generalized to higher dimensions. It is found that the generalized $I$-concurrence $C_I$ with its mixed state counterpart is useful in characterizing the non-local properties for bipartite states, both pure and mixed. Due to these reasons, we will concentrate our attention to this specific case and give explicit examples in the following section.

Before going to concrete example, we first consider a special case, i.e., entanglement of identical particle systems. Although the theory of entanglement is widely developed in the systems of distinguishable particles, only very recently the entanglement properties in identical particle systems began to attract much attention in the fields of quantum information and quantum computation. It is also shown that for any $N$ identical particle pure state $|\Psi_N\rangle$, all the information of their quantum correlation between one particle and the others are contained in the single particle density matrix. Therefore our entanglement measure is not only suitable for bipartite case here, but also a measure (to see this is indeed an entanglement measure here, see ref [26]) for $N$ identical particle entanglement, i.e.,

$$M_e(2) = 1 - \sum_{i=0}^{d^2-1} |\langle \Psi_N | O_i | \Psi_N \rangle|^2. \quad (8)$$

III. REALIZATION OF $M_e(2)$ WITH PAULI OPERATORS AND ITS HIGH DIMENSIONAL GENERALIZATIONS

In this section, we will give examples of realization of $M_e(2)$ with Pauli operators and the two different generations of Pauli matrixes to higher dimensional Hilbert space, which
preserve Hermitian and Unitary respectively. We know that an arbitrary state of two qubits in the Hilbert space $H = H_A \otimes H_B$ (where $H_A = H_B = C^2$) can be written as

$$\Psi = \alpha_1 |00\rangle + \alpha_2 |01\rangle + \alpha_3 |10\rangle + \alpha_4 |11\rangle.$$  \hfill (9)

where $\sum_i |\alpha_i|^2 = 1$.

Let $s_i = \frac{1}{\sqrt{2}} \sigma_i$, where $\sigma_0 = I$ and $\sigma_i$ ($i = 1, 2, 3$) are usual Pauli operators. Obviously, $\{s_i\}$ form a basis for $2 \times 2$ operator and thus

$$M_e(2) = 1 - \sum_{i=0}^{3} \langle s_i \rangle^2 = 1 - \frac{1}{2} \sum_{i=0}^{3} \langle \sigma_i \rangle^2$$

$$= \frac{1}{2} \left( 1 - \sum_{i=0}^{3} \langle \sigma_i \rangle^2 \right) = \frac{1}{2} C^2,$$ \hfill (10)

where

$$C = 2 |\alpha_1 \alpha_4 - \alpha_2 \alpha_3|$$ \hfill (11)

is the usual concurrence.

For qudits case, we demonstrate two kinds of commonly used “generalized” Pauli operators. The first kind is so-called Gell-mann matrices $\lambda_i$, which are Hermitian generators of $SU(d)$. From the completeness relation of $\lambda_i$

$$\sum_{i=1}^{d^2-1} (\lambda_i)_{k\ell} (\lambda_i)_{pq} = 2 \left( \delta_{kp} \delta_{l\ell} - \frac{1}{d} \delta_{kp} \delta_{l\ell} \right),$$ \hfill (12)

it is easy to show that

$$M_e(2) = \frac{(d-1)}{d} - \frac{1}{d} \sum_{i=1}^{d^2-1} (\Psi | \lambda_i | \Psi)^2.$$ \hfill (13)

It is noticed that this result is in fact already gotten in Ref. \cite{27}.

Another kind of generalized Pauli operators are $Z^m X^n$, which are all unitary matrices. Here $Z$ and $X$ are the generators of quantum plane algebra with $q^d = 1$ [28]. The $Z$-diagonal representation of $Z$ and $X$ given by

$$Z \equiv \sum_{k_0}^{d-1} |k\rangle q_{d}^{k} |k\rangle,$$ \hfill (14)
\[ X \equiv \sum_{k=0}^{d-1} |k\rangle\langle k + 1|, \quad (15) \]

for \( q_d = e^{i \frac{2\pi}{d}} \).

From the completeness relation of \( Z^m X^n \)

\[ \frac{1}{d} \sum_{m,n=0}^{d-1} |Z^m X^n\rangle\langle Z^m X^n| = 1, \quad (16) \]

it is easy to show that

\[ M_e(2) = 1 - \frac{1}{d} \sum_{m,n=0}^{d-1} |\langle \Psi |Z^m X^n|\Psi \rangle|^2. \quad (17) \]

**IV. APPLICATIONS TO MIXED STATE ENTANGLEMENT**

Apparently the entanglement measure defined in Eq. (1) cannot be an entanglement measure for mixed state case. However, the technique developed above can help us to derive some criterion for mixed state entanglement.

The completeness relation Eq. (4) is equivalent to

\[ \sum_{i=1}^{d^2} O_i^\dagger Y O_i = trY \quad (18) \]

for arbitrary \( d \times d \) operator \( Y \). Therefore, if \( Y = I \) and \( O_m^i \) are hermitian, we have

\[ \sum_{i=1}^{d^2} O_i^2 = d. \quad (19) \]

So the sum of uncertainty of \( O_m \) gives that

\[ \sum_{i=1}^{d^2} (\delta O_i)^2 = \sum_{i=1}^{d^2} tr(\rho O_i^2) - (tr(\rho O_i))^2 = d - \sum_{i=1}^{d^2} (tr(\rho O_i))^2 = d - \sum_{i=1}^{d^2} \langle O_i \rangle^2 \]

\[ = d - tr(\rho^2) \geq d - 1. \quad (20) \]

Then we can get a non-trivial sum uncertainty relation
\[ \sum_{i=1}^{d^2} \left( \delta (O_i A - O_i B) \right)^2 \geq 2(d - 1) \]  

(21)

to result in a sufficient condition for entanglement if the above inequality is violated. This entanglement criterion may be stronger than Peres-Horodecki criterion for it is shown that some PPT state violate this criterion\[30\].

This idea is also useful in \( N \)-identical particle case, which will lead to an entanglement criterion based on the sum uncertainty of collective operators for many identical particles. For \( N \) identical particles, the collective operator is defined as

\[ O_i = \sum_{K=1}^{N} O_{iK}, \quad (K = 1, 2, ..., N). \]  

(22)

Correspondingly the sufficient condition for a \( N \) identical particles state to be entangled is

\[ \sum_{i=1}^{d^2} (\delta O_i)^2 < N(d - 1). \]  

(23)

Usually, for \( N \)-identical qubits we choose \( O_{iK} (i = 0, 1, 2, 3) \) as \( I, s_1, s_2, s_3 \), then \( O_i (i = 0, 1, 2, 3) \) will be the total spin of the system apart from a constant multiplier \( \frac{1}{\sqrt{2}} \). This criterion is analogous to the criterions defined by the squeezing parameters in the literatures \[31, 32, 33\].

In summary, we showed that a class of entanglement measures for bipartite pure state can be expressed by the average values of quantum operators, which are related to any complete basis of one partite operator space with two specific examples given based on two different ways to generalize Pauli matrices to \( d \) dimensional Hilbert space. In addition, applying our measure to mixed state case gave a sufficient condition for entanglement and the case for identical particle systems was also considered.

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[1] M. A. Nielsen and I. S. Chuang, *Quantum computation and quantum information*, Cambridge University Press (2000).

[2] R. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A 40, 4277 (1989).

[3] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[4] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[5] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin et. al, Phys. Rev. **A54**, 3824 (1996).

[6] W. K. Wootters, Phys. Rev. Lett. **80**, 2245, 1998.

[7] M. Horodecki, P. Horodecki, and R. Horodecki, Springer Tr. Mod. Phys. **173**, 151 (2001)

[8] M. A. Nielsen, Phys. Rev. **A61**, 064301 (2000).

[9] G. Vidal, J. Mod. Opt. **47**, 355 (2000).

[10] A. F. Abouraddy, B. E. A. Saleh, and A. V. Sergienko et. al, Phys. Rev. **A64**, 050101 (2001).

[11] J. L. Chen, L. Fu, A. A. Ungar and X. G. Zhao, Phys. Rev. **A65**, 044303 (2002).

[12] J. Schliemann, D. Loss and A. H. MacDonald, Phys. Rev. **B63**, 085311 (2001).

[13] J. Schliemann, J. I. Cirac, M. Kus, M. Lewenstein and D. Loss, Phys. Rev. **A64**, 022303 (2001).

[14] P. Paskauskas and L. You, Phys. Rev. **A64**, 042310 (2001).

[15] Y. S. Li, B. Zeng, X. S. Liu and G. L. Long, Phys. Rev. **A64**, 054302 (2001).

[16] J. R. Gittings and A. J. Fisher, quant-ph/0202051

[17] K. Eckert, J. Schliemann, D. Bruž, and M. Lewenstein, Annals of Physics (New York) **299**, 88 (2002).

[18] A. Fang and Y. C. Zhang, Phys. Lett. **A311**, 443 (2003).

[19] C.H. Bennett, D. P. DiVincenzo and J.A. Smolin et al, Mixed State Entanglement and Quantum Error Correction, Phys. Rev. A 54, 3824 (1996).

[20] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. 80, 2245 (1998).

[21] P. Rungta, V. Bužek, and C. M. Caves et. al, Phys. Rev. **A64**, 042315 (2001).

[22] S. Albeverio and S. M. Fei, J. Opt. **B3**, 223 (2001).
[23] K. G. H. Vollbrecht and R. F. Werner, J. Math. Phys. 41, 6772 (2000).

[24] P. Rungta and C. M. Caves, quant-ph/0208002

[25] A. Delgado and T. Tessier, quant-ph/0210153

[26] G. K. Brennen, quant-ph/0305094

[27] G. Mahler, V. A. Weverruß, Quantum Networks, dynamics of open nanostructures, Springer-Verlag Berlin Herdelberg (1995).

[28] C. P. Sun, in “Quantum Group and Quantum Integrable Systems”, ed by M. L. Ge, World Scientific, 1992, p.133; M. L. Ge, X. F. Liu, C. P. Sun, J. Phys A-Math. Gen 25 (10): 2907, (1992).

[29] Holger F. Hofmann, and Shigeki Takeuchi, quant-ph/0305002

[30] Holger F. Hofmann, quant-ph/0305003

[31] A. S. Sørensen, L.M. Duan, J.I. Cirac, and P. Zoller, Nature 409, 63 (2001).

[32] A. Messikh, Z. Ficek, and M.R.B. Wahiddin, quant-ph/0305166

[33] J.K. Stochton, J.M. Geremia, A.C. Doherty, and H. Mabuchi, Phys. Rev. A67, 022112 (2003).