FIELD OF MODULI OF GENERALIZED FERMAT CURVES

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Abstract. As a consequence of the Riemann-Roch theorem, a closed Riemann surface $S$ can be described by a non-singular complex projective algebraic curve $C$. A field of definition for $S$ is any subfield $D$ of $C$ so that we may choose $C$ to be defined by polynomials in $D[x_0, \ldots, x_n]$. The field of moduli of $S$ is $\mathbb{R}$ if and only if $S$ admits an anticonformal automorphism. In the case that the field of moduli of $S$ is $\mathbb{R}$, then $S$ can be defined over the field of moduli if and only if $S$ admits an anticonformal involution. It may happen that the field of moduli is not a field of definition.

In this paper, we consider certain class of closed Riemann surfaces, called generalized Fermat curves. These surfaces are the highest Abelian branched cover of certain orbifolds. In this class of Riemann surfaces, we study the problem of deciding when the field of moduli is $\mathbb{R}$ and when, in such a case, it is a field of definition.

1. Introduction

Let $S$ be a closed Riemann surface. As a consequence of the Riemann-Roch theorem, $S$ can be described by a non-singular complex projective algebraic curve $C \subset \mathbb{P}^n(\mathbb{C})$. If $C$ can be chosen to be defined by homogeneous polynomials $f_1, \ldots, f_r \in D[x_0, \ldots, x_n]$, where $D$ is a subfield of $\mathbb{C}$, then we say that $D$ is a field of definition of $S$.

Let $C$ be defined by the homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$, then the complex conjugated curve $\overline{C}$ is the algebraic curve defined by the homogeneous polynomials $\hat{f}_1, \ldots, \hat{f}_r \in \mathbb{C}[x_0, \ldots, x_n]$, where $\hat{f}_j$ is obtained by application of $\sigma(z) = \overline{z}$ to the coefficients of $f_j$. The field of moduli of $S$ is then defined as $M(S) = \mathbb{R}$ if $C$ and $\overline{C}$ are conformally equivalent as closed Riemann surfaces. Note that the above definition of $M(S)$ does not depends on the choice of $C$. If $J_n : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ is the conjugation $J_n([x_0 : \ldots : x_n]) = [\overline{x_0} : \ldots : \overline{x_n}]$, then $J_n : C \to \overline{C}$ defines an anticonformal isomorphism (as closed Riemann surfaces). In this way, the field of moduli of $S$ is $\mathbb{R}$ if and only if $C$ admits an anticonformal automorphism.

If $S$ can be defined over $\mathbb{R}$, we also say that $S$ is a real Riemann surface, then we may chose $C$ defined by polynomials $f_1, \ldots, f_r \in \mathbb{R}[x_0, \ldots, x_n]$. In this case, $J_n$ defines an anticonformal involution on $S$. Conversely, as a consequence of Weil’s theorem, if $S$ admits an anticonformal involution, then $S$ can be defined over $\mathbb{R}$.

We are interested on those closed Riemann surfaces whose field of moduli is $\mathbb{R}$ and which can or cannot be definable over $\mathbb{R}$.

By the uniformization theorem, there is one conformal class of Riemann surfaces of genus 0; this given by the Riemann sphere $\hat{\mathbb{C}}$. This clearly has an anticonformal involution ($J(z) = \overline{z}$); it follows that a genus zero Riemann surface has field of moduli equal to $\mathbb{R}$ and that it is real. A closed Riemann surface of genus one can be described by an algebraic curve of
the form $C_{\lambda} := \{ y^2 z = x(x-z)(x-\lambda z) \} \subset \mathbb{P}^2(\mathbb{C})$, where $\lambda \in \mathbb{C} - \{0,1\}$. If $j(\lambda) = (1-\lambda + \lambda^2)^3/\lambda(\lambda - 1)^2$ is its $j$-invariant and $a(\lambda) = 27j(\lambda)/(j(\lambda) - 1)$, then $C_{\lambda}$ is isomorphic to $D_{\lambda} = \{ y^2 = 4x^3 - a(\lambda)x - a(\lambda) \}$; so $\mathbb{Q}(j(\lambda))$ is a field of definition of $C_{\lambda}$. It can be seen that $C_{\lambda}$ is real if and only if $j(\lambda)$ is real.

If $S$ has genus $g \geq 2$, then the situation gets more complicated. The first examples of closed Riemann surfaces of genus at least two which are not real and whose field of moduli is $\mathbb{R}$ where provided by Shimura [7] and Earle [2] around 1972. These examples where all hyperelliptic Riemann surfaces (that is, there is a two-fold branched cover over Riemann sphere). More recently, in [5] a non-hyperelliptic non-real curve with field of moduli equal to $\mathbb{R}$ was provided. Such a non-hyperelliptic example (depending on two real parameters) turns out to be the homology cover of an orbifold with signature $(0; 2, 2, 2, 2, 2)$, that is, a closed Riemann surface $S$ of genus 17 admitting a group $H \cong \mathbb{Z}_2^5$ as a group of conformal automorphisms so that $S/H$ is the Riemann sphere with exactly 6 cone points, each one of order 2.

In this paper we consider those closed Riemann surfaces $S$ admitting a group $H \cong \mathbb{Z}_p^n$, where $p \geq 2$ is a prime and $n \geq 2$, so that $S/H$ is an orbifold with signature $(0; p, n+1, p)$. We study the problem of deciding when such a surfaces have field of moduli equal to $\mathbb{R}$ and when they are reals.

2. Preliminaries

2.1. Riemann orbifolds and Fuchsian groups. Let $S$ a closed Riemann surface. We will denote by $\text{Aut}(S)$ its full group of conformal automorphisms. For $H$ subgroup of $\text{Aut}(S)$, we denote by $\text{Aut}_H(S)$ the normalizer of $H$ inside $\text{Aut}(S)$ and by $H'$ its commutator subgroup.

A Riemann orbifold $\mathcal{O}$ of signature $s(\mathcal{O}) = (\gamma : m_1, \ldots, m_r)$ is given by a closed Riemann surface $S$ of genus $\gamma$ (called the underlying Riemann surface structure of $\mathcal{O}$), a collection of $r$ different points, say $p_1, ..., p_r \in S$ (called the cone points) and an assignment of an integer $m_j \geq 2$ to the point $p_j$ (called the cone order of the cone point $p_j$). By a conformal automorphism of a Riemann orbifold $\mathcal{O}$ we mean a conformal automorphism of the underlying Riemann surface that preserve the conic points and their orders. We denote by $\text{Aut}_{\text{Orb}}(\mathcal{O})$ the conformal automorphisms group of $\mathcal{O}$.

By a fuchsian group we mean a discrete subgroup of the group $\text{Aut}(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R})$ of conformal automorphisms of the upper half plane. For details see [6]. If $\Gamma$ is a co-compact fuchsian group, then $\Gamma$ has a presentation in terms of $2\gamma$ hyperbolic generators, say, $a_1, b_1, \ldots, a_\gamma, b_\gamma$ and $r$ eliptics, say $x_1, \ldots, x_r$, with the relations

$$[a_1, b_1] \cdots [a_\gamma, b_\gamma] \cdot x_1 \cdots x_r = x_1^{m_1} = \cdots = x_r^{m_r} = 1$$

where $[a, b] = aba^{-1}b^{-1}$. The signature of $\Gamma$ in this case is given by $s(\Gamma) = (\gamma : m_1, \ldots, m_r)$. In this case, the quotient $\mathcal{O} = \mathbb{H}/\Gamma$ is a Riemann orbifold with $s(\mathcal{O}) = s(\Gamma)$.

By the classical uniformization theorem [3], every compact Riemann surface $S$ genus $g \geq 2$, can be realized as a quotient $\mathbb{H}/\Gamma$ of the hyperbolic plane $\mathbb{H}$ under the action of a torsion free co-compact fuchsonian group $\Gamma$. We set $s(\Gamma) = (g : -)$ and say that $\Gamma$ is a surface group.

A finite abstract group $G$ acts as a group of automorphisms of $S = \mathbb{H}/\Gamma$ if and only if, $G \cong \Lambda/\Gamma$ for some fuchsonian group $\Lambda$ that contains $\Gamma$ as a normal subgroup index $|G|$; equivalently, if there exists an epimorphism of groups $\Theta : \Lambda \longrightarrow G$ with $\Gamma = \ker(\Theta)$.
2.2. Generalized Fermat curves. Let \( n, k \geq 2 \) integers. A closed Riemann surface \( S \) is called a generalized Fermat curve of type \((k, n)\) if exists a subgroup \( H < \text{Aut}(S) \), \( H \cong \mathbb{Z}_k^n \) (direct sum of \( n \) copies of \( \mathbb{Z}_k \)) so that \( S/H \) is a Riemann orbifold of signature \((0; k, n+1, k)\). We say that \( H \) is a generalized Fermat group of type \((k, n)\) and the pair \((S, H)\) is a generalized Fermat pair of type \((k, n)\). By the Riemann-Hurwitz formula [3], the genus of a generalized Fermat curve of type \((k, n)\) is
\[
g(k, n) = 1 + \frac{k^{n-1}}{2}((n-1)(k-1) - 2).
\]

Two pairs \((S_1, H_1)\) and \((S_2, H_2)\) of the same type are topologically equivalent (conformally equivalent) if exists an orientation-preserving homeomorphism (conformal homeomorphism) \( \varphi : S_1 \rightarrow S_2 \) so that \( \varphi^{-1}H_2\varphi = H_1 \).

The only non-hyperbolic generalized Fermat pairs are of type \((2, 2)\), \((2, 3)\) and \((3, 2)\). For example, if \((S, H)\) is a generalized Fermat curve of type \((2, 2)\) then \( S \) is genus zero, therefore \( S \) is conformally equivalent to the Riemann sphere and the generalized Fermat group \( H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is generated for the transformations \( z \mapsto -z \) and \( z \mapsto 1/z \).

For hyperbolic cases, classical uniformization theorem asserts that the Riemann orbifold \( S/H \) is uniformized by a fuchsian group \( \Gamma < \text{Aut}(\mathbb{H}^2) \) whose presentation is
\[
\Gamma = \langle x_1, \ldots, x_{n+1} : x_1^k = \cdots = x_{n+1}^k = x_1 \cdots x_{n+1} = 1 \rangle.
\]

**Proposition 1 ([4]).** Let \((S, H)\) a hyperbolic generalized Fermat curve of type \((k, n)\) and let \( \Gamma \) be a orbifold universal cover group of \( S/H \). Then \( S \) is non-hyperelliptic and \((S, H)\) and \((\mathbb{H}^2/\Gamma', \Gamma/\Gamma')\) are conformally equivalent generalized Fermat pairs.

Let us consider a generalized Fermat pair \((S, H)\) of type \((k, n)\). Let us assume, after a Möbius transformation, that the cone locus of the orbifold \( S/H \) is given by
\[
\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}.
\]

Consider the non-singular projective algebraic curve in \( \mathbb{P}^n(\mathbb{C}) \) defined by
\[
C(\lambda_1, \ldots, \lambda_{n-2}; k) : \begin{cases}
x_1^k + x_2^k + x_3^k = 0 \\
\lambda_1 x_1^k + x_2^k + x_4^k = 0 \\
\lambda_2 x_1^k + x_2^k + x_5^k = 0 \\
\vdots \\
\lambda_{n-2} x_1^k + x_2^k + x_{n+1}^k = 0
\end{cases}
\]

As \( \lambda_j \in \mathbb{C} - \{0, 1\}, \lambda_i \neq \lambda_j \) if \( i \neq j \), it can seen that \( C(\lambda_1, \ldots, \lambda_{n-2}; k) \) is a non-singular algebraic curve; so it describes a closed Riemann surface. We note that the projective linear transformations
\[
a_j([x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_{j-1} : \exp\{2\pi i/k\}x_j : x_{j+1} : \cdots : x_n : x_{n+1}],
\]
with \( j \in \{1, \cdots, n\} \) provides a faithful representation
\[
\Theta : \mathbb{Z}_k^n \hookrightarrow \text{Aut}(C(\lambda_1, \ldots, \lambda_{n-2}; k)).
\]

In addition, the \( k^n \) degree conformal map
\[
\pi : C(\lambda_1, \ldots, \lambda_{n-2}; k) \rightarrow \hat{\mathbb{C}}
\]
defined by \( \pi([x_1: \cdots : x_{n+1}]) = -(x_2/x_1)^k \), satisfies \( \pi \circ a_j = \pi \) for every \( 1 \leq j \leq n \). We note that \( H_0 := \Theta(\mathbb{Z}_n^u) \) is a generalized Fermat group of type \((k, n)\) for the closed Riemann surface \( C(\lambda_1, \ldots, \lambda_{n-2}; k) \).

**Proposition 2 ([4]).** The generalized Fermat pairs \((S, H)\) and \((C(\lambda_1, \ldots, \lambda_{n-2}; k), H_0)\) are conformally equivalent.

We have that \( \text{Aut}_{H_0}(C(\lambda_1, \ldots, \lambda_{n-2}; k))/H_0 \) is a group isomorphic to the subgroup of \( \text{PSL}(2, \mathbb{C}) \) that preserves the conic set \( \{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\} \). It follow that, if \( H_0 \) is unique (or normal) inside \( \text{Aut}(S) \), then

\[
\frac{\text{Aut}(C(\lambda_1, \ldots, \lambda_{n-2}; k))}{H_0} \cong \text{Aut}_{\text{Orb}}(S/H_0)
\]

**Remark 3.** We note that if \( H_0 \) is normal, then we can obtain \( \text{Aut}(S) \) lifting \( \text{Aut}_{\text{Orb}}(S/H_0) \). In fact, if \( f \in \text{Aut}_{\text{Orb}}(S/H_0) \), then exists \( \hat{f} \in \text{Aut}(S) \) so that \( \pi \hat{f} = f \pi \). In [4] they note that if \( f \) induces the permutation \( \sigma \in S_{n+1} \) in the set \( \{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n+1} = \lambda_{n-2}\} \), then \( \hat{f} \) is defined by

\[
\hat{f}([x_1: \cdots : x_{n+1}]) = [c_1 x_{\sigma^{-1}(1)} : \cdots : c_{n+1} x_{\sigma^{-1}(n+1)}]
\]

where the complex constants \( c_j \) can be easily computed using the algebraic equations that defines the curve. (for simplicity, we may assume \( c_1 = 1 \)).

The next results will be important in the proofs of the main results of this paper.

**Theorem 4 ([4]).** Let \((S, H)\) a generalized Fermat pair of type \((2, 4)\). Then \( H \) in unique inside \( \text{Aut}(S) \).

**Theorem 5 ([4]).** Let \( S \) a generalized Fermat curve of type \((p, n)\) with \( p \geq 2 \) prime and \( n \geq 2 \) so that \((n-1)(p-1) > 2\). If \( H_1 \) and \( H_2 \) are two generalized Fermat groups of type \((p, n)\) then they are conjugate inside \( \text{Aut}(S) \).

### 2.3. Action of the Galois group.

Let \( F < E \) an extension fields and consider the Galois group \( \text{Gal}(E/F) \) asociated to the extension. \( \text{Gal}(E/F) \) acts in the polynomial ring \( E[x_0, \ldots, x_n] \): if

\[
f(x_0, \ldots, x_n) = \sum a_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n}
\]

then

\[
(\sigma \cdot f)(x_0, \ldots, x_n) = \sum \sigma(a_{i_0 \cdots i_n}) x_0^{i_0} \cdots x_n^{i_n}
\]

Set \( f^\sigma := \sigma \cdot f \). This accion induces an action in the set of projective algebraic varieties. If \( X \) is defined by \( f_1, \ldots, f_r \), then we can consider the polynomials \( f_1^\sigma, \ldots, f_r^\sigma \) that defines a new projective algebraic variety; say \( X^\sigma \).
2.4. Field of Moduli and Fields of Definition. Let $F < E$ an extension fields and let $X \subset \mathbb{P}^n(E)$ be a projective algebraic variety.

The field of moduli $M_{E/F}(X)$ of $X$, associated to the extension $F < E$, is defined as the fixed field of the subgroup

$$E_F(X) = \{ \sigma \in \text{Gal}(E/F) : X \simeq X^\sigma \},$$

where "\simeq" means birational isomorphism. It is clear from the definition that $F < M_{E/F}(X) < E$.

A field of definition for $X$ is any subfield $D$, $F < D < E$, so that exists $Y \simeq X$ defined by polynomials in $D[x_0, \ldots, x_n]$. It is clear from the definition that if $D$ is a field of definition for $X$, then every extension of $D$ (inside $E$) is also a field of definition, nevertheless, it is not clear that there is a smallest field of definition.

If $F < E$ is a general Galois extension (i.e. for every $F < N < E$ holds that $\text{Fix}(\text{Gal}(E/N)) = N$), then is known that the field of moduli is contained in every field of definition. We are interested in the Galois extension $\mathbb{R} < \mathbb{C}$. As a Galois extension is a general Galois extension, the previous result holds. The main result here is a theorem due by A. Weil in 1956, see [9]. We present a simplicated version of this theorem (sufficient for our purpose).

**Theorem 6 (Weil’s Theorem).** Let $F < E$ a finite Galois extension, $X \subset \mathbb{P}^n(E)$ be a projective algebraic variety and $F = M_{E/F}(X)$. Then $F$ is a field of definition for $X$ if and only if for every $\sigma \in E_F(X) = \text{Gal}(E/F)$ there exists a birational isomorphism $f_\sigma : X \to X^\sigma$ so that for every $\sigma, \tau \in \text{Gal}(E/F)$ it holds that $f_{\sigma \tau} = f_\tau \circ f_\sigma$.

2.5. The complex case. We are interested in the complex case, that is, $E = \mathbb{C}$ and $F = \mathbb{R}$; a Galois extension of degree two. In this case, the projective algebraic variety $X$ became in complex algebraic variety. We will interested only in the case when $X$ is a non-singular curve (that is a closed Riemann surface).

We know that the field of moduli is contained in every field of definition, in particular, the intersection of all the fields of definition contains the moduli field. An interesting question is to know when the field of moduli is a field of definition and when it is $\mathbb{R}$ (this is equivalent to define $X$ using real polynomials).

The field of moduli is $\mathbb{R}$ if and only if $X$ and $X^\sigma$ are birational equivalent for every $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R}) = \{id, z \mapsto \overline{z}\}$. It is clear that the field of moduli is $\mathbb{R}$ if and only if $X$ admits an anticonformal automorphism. For the other hand, if $X$ can be defined by real polynomial, then $J_n : \mathbb{P}^n(\mathbb{C}) \longrightarrow \mathbb{P}^n(\mathbb{C})$ defined by $J([x_0, \ldots, x_n]) = [\overline{x_0}, \ldots, \overline{x_n}]$, induces an order two anticonformal automorphism for $X$. Reciprocally, if $X$ admits and order two anticonformal automorphism $\tau$, it is not difficult to prove that $\{f_e = id, f_\sigma = \tau \circ J_n\}$ satisfices the Weil Theorem conditions.

2.6. Examples: Curves. We have already noted the cases when the Riemann surfaces are of genus 0 and 1. For genus $g \geq 2$ the problem is more complicated. Shimura and Earle provided the first examples of algebraic curves with field of moduli $\mathbb{R}$ but which are not definable using real polynomials. We proceed to recall these examples.
(1) **Earle’s example** [2]: Let \( a \in (-\infty, -(3 + \sqrt{2})) \) and \( b \in \mathbb{H}^2 \) with \(|b|^2 = -a\). Consider
\[
p(x, y, z) = y^2 z^3 - x(x - z)(x - a z)(x^2 - b^2 z^2)
\]
and \( X \subset \mathbb{P}^2(\mathbb{C}) \) the hyperelliptic algebraic curve defined by \( p \). Then the field of moduli of \( X \) is \( \mathbb{R} \) and it cannot be defined over \( \mathbb{R} \).

(2) **Shimura’s example** [7]: Let \( a_0 \in \mathbb{R} \), \( a_m = 1, a_1, a_2, \ldots, a_{m-1} \in \mathbb{C} \) and \( m \) an odd positive integer so that the set
\[
C = \{a_0, a_1, \ldots, a_{m-1}, \bar{a}_1, \ldots, \bar{a}_{m-1}\}
\]
will be algebraically independent over \( \mathbb{Q} \). We let consider
\[
p(x, y, z) = y^2 z^{2m-2} - a_0 x^m y^m - \sum_{j=1}^{m} (a_j x^{m+j} y^{m-j} + (-1)^j a_j x^{m-j} y^{m+j})
\]
and \( X \subset \mathbb{P}^2(\mathbb{C}) \) the plane algebraic curve defined by \( p \). Then the field of moduli of \( X \) is \( \mathbb{R} \) and it cannot be defined over \( \mathbb{R} \).

For details and explicit anticonformal automorphisms see [2] and [7]. The Shimura and Earle examples are hyperelliptic algebraic curves. The first non-hyperelliptic example was due by R. Hidalgo en [5].

(3) **Hidalgo’s example** [5]: Let \( \lambda_1 \in \mathbb{R} \), \( \lambda_2 \in \mathbb{C} \) so that \( \lambda_1 < -(3 + 2\sqrt{2}) \), \( \text{Im}(\lambda_2) > 0 \), \( \text{Re}(\lambda_2) < 0 \), \( |\lambda_2|^2 = -\lambda_1 \) and \( X \) the non-singular projective algebraic curve defined by
\[
X : \begin{cases}
x_1^2 + x_2^2 + x_3^2 = 0 \\
\lambda_1 x_1^2 + x_2^2 + x_4^2 = 0 \\
\lambda_2 x_1^2 + x_2^2 + x_5^2 = 0 \\
-\lambda_2 x_1^2 + x_2^2 + x_6^2 = 0
\end{cases}
\]
in \( \mathbb{P}^5(\mathbb{C}) \). Then \( X \) is a non-hyperelliptic closed Riemann surface genus 17 which admits an anticonformal automorphism order 4 but does not admit an anticonformal involution. In particular, the field of moduli of \( X \) is \( \mathbb{R} \) but it cannot be definable over \( \mathbb{R} \).

We note that Hidalgo’s example is a generalized Fermat curve type \((2, 5)\). An important fact in this example is the uniqueness of the generalized Fermat group inside the conformal automorphisms full group.

Next result will be frequently used in the proofs.

**Proposition 7.** Let \( k \geq 3 \) be an odd integer and \((S, H)\) be a generalized Fermat pair of type \((k, n)\), where \( n \geq 2 \). If the orbifold \( S/H \) admits an anticonformal involution, then \( S \) has field of moduli \( \mathbb{R} \) and it is a field of definition.

**Proof.** Let \( \tilde{\tau} : S/H \to S/H \) be an anticonformal involution. Then, as \( S \) is the homology cover of \( S/H \), the involution \( \tilde{\tau} \) lifts as an anticonformal automorphism \( \tau : S \to S \). So the field of moduli of \( S \) is \( \mathbb{R} \). As \( \tilde{\tau} \) has order two, it follows that \( \tau^2 \in H \). As \( H \) has odd order, then \( \tau^2 \) is either the identity or it has odd order, say \( s \). In the last case, \( \tau^s \) is an anticonformal involution. The existence of an anticonformal involution is equivalent for \( S \) to be real. \( \square \)
3. Main results

In this section, we will always consider the extension \( \mathbb{R} < \mathbb{C} \) and we set \( M(X) \) instead of \( M_{\mathbb{C}/\mathbb{R}}(X) \) for simplicity.

Let \( (S, H) \) a generalized Fermat pair of type \((k, n)\) and let \( \hat{\tau} \) be an anticonformal automorphism (as orbifold) of \( S/H \) of even order \( 2M \) with \( M \geq 1 \). Let \( P \subset \hat{\mathbb{C}} \) be the set of cone points of the Riemann orbifold \( S/H \). As \( \hat{\tau} \) must keep invariant \( P \) and \( P \) has cardinality \( n+1 \), we may assume (after conjugation by a suitable M"obius transformation) that \( \hat{\tau}(z) = e^{i\theta}/\bar{z} \) with \( \theta = 2\pi/N \), (rotation and reflection in \( S^3 \)). In this case, if \( N \) is odd, then \( N = M \) and if \( N \) is even, then \( 2M = N \).

We consider the action of the cyclic group \((\hat{\tau}) \cong \mathbb{Z}_{2M}\) over the set \( P \).

(1) If \( N \geq 3 \) is odd, then we have exactly 3 possible types of orbits: \( C \in \{0, 1\} \) orbits of length 2, \( B \) orbits of length \( N \) and \( A \) of length \( 2N \).

(2) If \( N \geq 4 \) is even, then we have exactly 2 possible types of orbits: \( C \in \{0, 1\} \) orbits of length 2, and \( B \) orbits of length \( N \). In this case we set \( A = 0 \).

(3) If \( N = 1 \), then we have exactly 2 possible types of orbits: \( A \) orbits of length 2 and \( B \) orbits of length 1. In this case we set \( C = 0 \).

(4) If \( N = 2 \), then we have \( B \) orbits of length 2. In this case we set \( A = C = 0 \).

It is clear from the definition of \( A, B \) and \( C \) that

\[
(* \quad n + 1 = 2NA + NB + 2C.
\]

We will make use of this in the rest of the paper.

3.1. Our first result concerns Hidalgo’s example.

**Theorem 8.** Let \((S, H)\) a generalized Fermat pair of type \((k, 5)\) with \( k \geq 2 \) of the form

\[
C_k : \begin{cases}
x_1^k + x_2^k + x_3^k = 0 \\
\lambda_1 x_1^k + x_2^k + x_4^k = 0 \\
\lambda_2 x_1^k + x_2^k + x_5^k = 0 \\
-\lambda_2 x_1^k + x_2^k + x_6^k = 0
\end{cases}
\]

with \( \lambda_1 \in \mathbb{R} \) and \( \lambda_2 \in \mathbb{C} \) so that \( \lambda_1 = -|\lambda_2|^2 \). Then the field of moduli of \( C_k \) is \( \mathbb{R} \). Moreover, if \( k \) is odd then \( C_k \) is real.

**Proof.** Let us consider the anticonformal automorphism, of order two, \( f(z) = \lambda_1/z \) of \( S/H \). We note that \( f \) defines the permutation \( \sigma = (1 2)(3 4)(5 6) \) in the cone points locus. It follows that, see Remak \( \square \) the lift of \( f \) is of the form

\[
\hat{f}([x_1 : x_2 : x_3 : x_4 : x_5 : x_6]) = [x_2 : c_2x_1 : c_3x_4 : c_4x_3 : c_5x_6 : c_6x_5],
\]

for suitable values of \( c_j \)'s. Using the algebraic equations that defines \( S \) is not difficult to see that \( c_2^k = \lambda_1, c_3^k = 1, c_4^k = \lambda_1, c_5^k = \lambda_2 \) and \( c_6^k = -\lambda_2 \). We have that \( \hat{f} \) is anticonformal automorphism of \( C_k \), so the field of moduli of \( C_k \) is \( \mathbb{R} \).

Moreover, if \( k \) is odd, then we may choose \( c_2 = c_4, c_3 = 1 \) and \( c_5 = -c_6 \) and then \( \hat{f}^k \) is anticonformal involution of \( C_k \), and then it is real. \( \square \)
3.2. Our second result states that classical Humbert curves with field of moduli $\mathbb{R}$ are necessarily reals.

**Theorem 9.** Let $S$ a generalized Fermat curve of type $(2, 4)$ (a classical Humbert curve). If the field of moduli is $\mathbb{R}$, then it is a field of definition.

**Proof.** Let $H \cong \mathbb{Z}_4$ the generalized Fermat group. Since $H$ is unique inside $\text{Aut}(S)$ (see Theorem 4) we have that $\text{Aut}(S)/H \cong \text{Aut}_{\text{orb}}(S/H)$. Since we are assuming that $M(S) = \mathbb{R}$, there exists an anticonformal automorphism of $S$, say $\tau : S \to S$.

Let us denote by $\hat{\tau} : S/H \to S/H$ the anticonformal automorphisms (as orbifold) induced by $\tau$. We suppose that $|\hat{\tau}| = 2M$ for some $M \geq 1$. As already noted, the set of cone points of $S/H$, say $\mu_1, \mu_2, \mu_3, \mu_4$ and $\mu_5$, should be invariant under $\hat{\tau}(z) = e^{i\theta}/z$ with $\theta = 2\pi/N$, where $N = M$ for $N$ odd and $N = 2M$ for $N$ even.

As $n = 4$, it follows from $(\ast)$ that $2NA + NB + 2C$ is odd, so $N$ is necessarily odd (that is, $N = M$). We claim that $N \leq 5$. In fact, if $N \geq 7$ is odd, then $5 = 2NA + NB + 2C \geq 14A + 7B + 2C$. It follows that $A = B = 0$. But in this case, $5 = 2C \in \{0, 2\}$, a contradiction.

If $N = 1$, then $C = 0$ and $2A + B = 5$; so $(A, B, C) \in \{(0, 5, 0), (1, 3, 0), (2, 1, 0)\}$.

If $N = 3$, then $6A + 3B + 2C = 5$, and this implies $(A, B, C) = (0, 1, 1)$.

If $N = 5$, then $10A + 5B + 2C = 5$, and this implies that $(A, B, C) = (0, 1, 0)$.

We summarize the information in the following table:

| $N$ | $A$ | $B$ | $C$ |
|-----|-----|-----|-----|
| 1   | 0   | 5   | 0   |
| 1   | 1   | 3   | 0   |
| 1   | 2   | 1   | 0   |
| 3   | 0   | 1   | 1   |
| 5   | 0   | 1   | 0   |

The rest of the proof is devoted to analyze separately each case.

(A) Case $(N, A, B, C) = (1, 0, 5, 0)$. In this case, all the cone points belong to the unit circle. Using an appropriate Mobius transformation, we can side the cone points on the real axis; that is, we may assume the cone points to be $\infty, 0, 1$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. In this way, the algebraic equations that defines the curve $S$, say $C(\lambda_1, \lambda_2)$, is real. In this way, $\mathbb{R}$ is a field of definition for $S$.

(B) Case $(N, A, B, C) = (1, 1, 3, 0)$. Conjugation by a Möbius transformation that keeps invariant the unit disc, we may assume that the cone points are given by $\mu_1 = \infty$, $\mu_2 = 0$, $\mu_3 = 1$, $\mu_4$ and $\mu_5$ with $|\mu_4| = |\mu_5| = 1$. We have that $\hat{\tau}(z) = 1/\tau$ induces the permutation $\sigma = (1 2)$ in the conic point set. We will return to this case below.

(C) Case $(N, A, B, C) = (1, 2, 1, 0)$. Conjugation by a Möbius transformation that keeps invariant the unit disc, we may assume that $\mu_1 = \infty$, $\mu_2 = 0$, $\mu_3 = 1$, $\mu_4$ and $\mu_5 =
Now, we analyze the cases (B) and (C).

(D) Case $(N, A, B, C) = (3, 0, 1, 1)$. Conjugation by a Möbius transformation that keeps invariant the unit disc, we may assume that $\mu_1 = \infty$, $\mu_2 = 0$, $\mu_3 = 1$, $\mu_4 = \omega$, and $\mu_5 = \omega^2$ with $\omega = e^{2\pi i/3}$. In this case, $\hat{\tau}(z) = \omega/z$, but this configuration also admits the anticonformal involution $z \mapsto 1/z$, which induces the permutation $\sigma = (1 2)$. Observe that this case is a particular case of (B).

(E) Case $(N, A, B, C) = (5, 0, 1, 0)$. In this case the five cone points belong to the unit circle. As done in the case (A), we have that $\mathbb{R}$ is a field of definition for $S$.

Now, we analyze the cases (B) and (C).

(1) Case (B). As $\tau : S \to S$ is a lifting of $\hat{\tau}(z) = \frac{1}{z} \in \text{Aut}_{\text{orb}}(S/H)$ and the induced permutation is $\sigma = (1 2)$, it follows that $\tau$ must have the following form (see Remark 3)

$$\tau([x_1 : x_2 : x_3 : x_4 : x_5]) = [x_1 : x_2 : c_3 x_3 : c_4 x_4 : c_5 x_5],$$

where $c_3^2 = 1$, $c_4^2 = \lambda_1$ and $c_5^2 = \lambda_2$. We can observe that $\tau$ is an order two anticonformal automorphism of $S$, in particular, that $S$ is real.

(2) Case (C). Again, as $\tau : S \to S$ is a lifting of $\hat{\tau}(z) = \frac{1}{z} \in \text{Aut}_{\text{orb}}(S/H)$ and the induced permutation is $\sigma = (1 2)(4 5)$, it follows that $\tau$ must have the following form (see Remark 3)

$$\tau([x_1 : x_2 : x_3 : x_4 : x_5]) = [x_1 : x_2 : c_3 x_3 : c_4 x_4 : c_5 x_5],$$

where $c_3^2 = 1$, $c_4^2 = \lambda_1$ and $c_5^2 = \lambda_2 = 1/\lambda_1$. We choose $c_4, c_5 \in \mathbb{C}$ so that $c_4 c_5 = 1$. We can observe that $\tau$ is an order two anticonformal automorphism of $S$, in particular, that $S$ is real.

\[\square\]

3.3. If $(S, H)$ is a hyperbolic generalized Fermat pair of type $(p, n)$ with $p, n \geq 2$ and $p$ prime, then the generalized Fermat group $H$ is unique up to conjugation inside $\text{Aut}(S)$ (see Theorem 5). This fact allows us to obtain the following.

**Theorem 10.** Let $S$ a generalized Fermat curve of type $(p, n)$ with $p \geq 3$ prime and $n \geq 2$ an even integer. If the field of moduli of $S$ is $\mathbb{R}$, then $S$ is real.

**Proof.** Let $H \cong \mathbb{Z}_p^n$ be a generalized Fermat group of $S$ of type $(p, n)$. Since $M(S) = \mathbb{R}$, there is an anticonformal automorphism $f : S \to S$. Since $f^{-1} H f$ is also a generalized Fermat group of type $(p, n)$. As this is unique, up to conjugation by Theorem 5 there is some $g \in \text{Aut}(S)$ so that $(gf)^{-1} H (gf) = H$.

If we set $\tau := fg$, then $\tau$ is an anticonformal automorphism of $S$ that normalizes $H$. It follows that $\tau$ induces an anticonformal automorphism $\hat{\tau}$ of the orbifold $\mathcal{O} = S/H$.

Since $\hat{\tau}$ is anticonformal, there exists $M \in \mathbb{N}$ so that $|\hat{\tau}| = 2M$. We set $N = M$ for $N$ odd and $N = 2M$ for $N$ even.
As we are assuming \( n \) even, then \( n + 1 = 2NA + NB + 2C \) is odd, from which we obtain that \( N \) is necessarily odd (that is, \( N = M \)). In this way, \( \hat{\tau}^N \) is an anticonformal involution and, by Proposition 7, the Riemann surface \( S \) is real.

\[ \square \]

3.4. In the proof of the Theorem 10 we strongly uses the parity of \( n \). For \( n \) odd we have the following parcial result.

**Theorem 11.** Let \( (S, H) \) a generalized Fermat pais of type either \((p, 3)\) or \((p, 5)\) with \( p > 2 \) prime. If the field of moduli of \( S \) is \( \mathbb{R} \), then \( S \) is real.

**Proof.** As we are assuming that \( \operatorname{M}(S) = \mathbb{R} \), there exists an anticonformal automorphism \( f : S \to S \). We have that \( f^{-1}Hf \) is other generalized Fermat group of the same type, so by Theorem 5 there is some \( g \in \operatorname{Aut}(S) \) so that \( (fg)^{-1}H(fg) = H \). If we set \( \tau := gh \), then \( \tau \) is an anticonformal automorphism of \( S \) that normalizes \( H \). In particular, \( \tau \) induces an anticonformal automorphism \( \hat{\tau} \) of the orbifold \( S/H \).

As already noted, we may assume that \( \hat{\tau}(z) = e^{i\theta}/z \) with \( \theta = 2\pi/N \), where \( N \geq 1 \). As \( n + 1 = 2NA + NB + 2C \) and \( n \) is odd, then \( NB \) must be even.

If \( N \in \{1, 2\} \), then \( \hat{\tau} \) has order two and, by Proposition 7 it follows that \( S \) is real. Now on, we assume \( N \geq 3 \).

1. Type \((p, 3)\).
   - If \( N \geq 3 \) odd, then \( 4 = 2NA + NB + 2C \geq 6A + 3B + 2C \) and, as \( C \in \{0, 1\} \), this is not possible.
   - If \( N \geq 4 \) is even, then \( 4 = NB + 2C \geq 4B + 2C \). As \( C \in \{0, 1\} \), we must have that \( N = 4, B = 1 \) and \( C = 0 \). Up to conjugation by a Möbius transformation that keeps the unit disc invariant, we may assume that the cone points are \( \mu_1 = 1, +\infty \), \( \mu_2 = it \), where \( t \in (0, 1] \), \( \mu_3 = -\mu_1 \) and \( \mu_4 = -\mu_2 \). In this case, the orbifold \( S/H \) admits the anticonformal involution \( \eta(z) = \frac{1}{z} \). By Proposition 7 we have that \( S \) is real.

2. Type \((p, 5)\).
   - If \( N = 3 \) then \( 6 = 6A + 3B + 2C \), then \( (A, B, C) \in \{(1, 0, 0), (0, 2, 0)\} \).
   - If \( N \geq 4 \) is even, then \( A = 0 \) and \( 6 = NB + 2C \), and this implies that either (a) \( N = 4, B = C = 1 \) or (b) \( N = 6, B = 1 \) and \( C = 0 \).
   - If \( N \geq 5 \) is odd, then \( 6 = 2NA + NB + 2C \geq 10A + 5B + 2C \), and this implies that \( A = 0 \). As \( C \in \{0, 1\} \), the quality \( 6 = NB + 2C \) is not possible.

The following table summarizes the possible cases.

| N | A | B | C |
|---|---|---|---|
| 3 | 0 | 2 | 0 |
| 3 | 1 | 0 | 0 |
| 4 | 0 | 1 | 1 |
| 6 | 0 | 1 | 0 |
(i) If \((N, A, B, C) = (3, 0, 2, 0)\), then \(\hat{\tau}\) has order six and there are two orbits: each one of length three. We can suppose that the orbits are \(\{1, \omega, \omega^2\}\) and \(\{\mu, \mu \omega, \mu \omega^2\}\) with \(\omega = e^{2 \pi i / 3}\) and \(\mu \in S^1 - \{1, \omega, \omega^2\}\). This configuration also admits the reflection in \(S^1\), \(\eta(z) = 1/z\), as an anticonformal involution. A lift of \(\eta\) is an anticonformal automorphism of \(S\) of order either 2 or \(2p\). As before, \(S\) admits an anticonformal involution, so \(S\) is real.

(ii) If \((N, A, B, C) = (3, 1, 0, 0)\), then \(\hat{\tau}\) has order six and there is an unique orbit of six elements. We can suppose that the orbit is \(\{\lambda, \omega/\lambda, \omega^2/\lambda\}\) with \(\lambda > 1\) and \(\omega = e^{2 \pi i / 3}\). It is clear that this points configuration also admits the conjugation as an anticonformal automorphism and, as above, \(S\) is real.

(iii) If \((N, A, B, C) = (4, 0, 1, 1)\), then \(\hat{\tau}\) has order four and there are two orbits: one of length two and one of length four. Without loss of generality, we can suppose that the orbits are \(\{\lambda, i/\lambda, -\lambda, -i/\lambda\}\) with \(\lambda > 1\) and \(\hat{\tau}(z) = i/z\). It is clear that this points configuration also admits the conjugation as an anticonformal involution and, as above, \(S\) is real.

(iv) If \((N, A, B, C) = (6, 0, 1, 0)\), then \(\hat{\tau}\) has order six and there is an unique orbit of six elements. We can suppose that the orbit is \(\{\lambda, \omega \lambda, \omega^2 \lambda, -1/\lambda, -\omega/\lambda, -\omega^2/\lambda\}\) with \(\lambda \neq 1\) and \(\omega = e^{2 \pi i / 3}\). It is clear that this points configuration also admits the conjugation as an anticonformal automorphism and, as above, \(S\) is real.

□

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