Computation and Analysis of Distinguished Hyperbolic Trajectories in Time-Dependent Vector Fields

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Abstract

In this paper, a developed measure function and an optimized numerical algorithm are proposed to calculate distinguished hyperbolic trajectory of time-dependent vector fields. Based on the rapid growth of high-power function, the developed measure function, which is gained by adjusting the parameters of the existing measure function, is more sensitive to difference between measures of trajectories. Through halving the grid in each loop, the optimization algorithm could reduce convergence time, especially in low accuracy region of initial estimation. Besides, we give a discussion and confirmation that the approximate point on the local stable manifold near to special trajectory of time-dependent vector fields converges to unstable manifold in time. Two-dimensional Duffing system is taken as example and verifies that the new measure function and algorithm is available and effective.

Keywords: distinguished hyperbolic trajectory; non-automatic system; time-dependent vector fields

1. Introduction

The development of scientific computing approaches and algorithms for the study of non-linear dynamical system behavior has long been the subject of intense research since the 1970s[1]. After several decades of development frame work of non-linear system theory has been basically achieved. Behaviors of system, such as character of attractors[2], stability[3,4] and asymptotic characteristics etc, have been made in-depth research. At present structuration theory of time-dependent aperiodic vector fields has been building, like bifurcation theory[5,6], ghosting theory[7], chaos theory[8] etc. However, the mathematical
theory for both aperiodic time-dependent flows and finite time aperiodic flows is far from being an integrated system. The study of distinguished hyperbolic trajectory in time-dependent vector fields is very important in aspects of stability analysis and manifold computation\[7\]. So far there are mainly two methods to search for distinguished hyperbolic trajectory. K. Ide\[8\] proposed a method which provided an approximation to distinguished hyperbolic trajectory for entire time-interval of the data set by developing the notion of distinguished hyperbolic trajectory. And J. A. Jiménez Madrid introduced a new definition of distinguished trajectories that generalized the concepts of fixed point and periodic orbit to time dependent aperiodic systems and presented numerical approach to find distinguished trajectories. In our developed measure function and optimized algorithm there are main three innovations: first the rapid growth of power function is considered and it is used to adjust the parameters of measure function for computation of distinguished hyperbolic trajectory. Second the algorithm is changed from fixed step mode to variable step mode. Third adopt the combination of existing methods, initial region is chosen by instantaneous stagnation trajectory. The method and the algorithm are test in two-dimensional Duffing system.

2. The Discussion of Methods

This section first introduces two existing methods and its achievements, and then gives our methods on the basis of analysis of the two existing methods. K. Ide studied the relationship between distinguished hyperbolic trajectory and instantaneous stagnation trajectory and proposed methods to compute distinguished hyperbolic trajectory by linearization and harmonic transformation. J. A. Jiménez Madrid provided a measure function that is simple and convenient for the calculation of distinguished trajectory.

2.1. Existing measure function

Assuming the components of $x(t, x_0, t_0)$ can be represented by 

$$x(t, x_0, t_0) = (x^1(t), x^2(t), \cdots, x^n(t))$$

For any initial value $x_i = x(t_i)$, and $x_i \in \mathbb{R}^n$, a map function: \[12\] $M : \mathbb{R}^n \rightarrow \mathbb{R}^+.$

$$M(x_i) = \int_{t_i-\tau}^{t_i+\tau} \left( \sum_{j=1}^{n} \left( \frac{dx'_j(t)}{dt} \right)^2 \right) dt \quad (1)$$

A trajectory $\gamma(t)$ of non-autonomous system in time $t^{*}$ is point $\tau - DT$, if there is an open set $\mathbb{N}$ such that for any $x \in \mathbb{N}$ content:

$$M(\gamma(t^{*}))_{t_i-\tau} = \min(M(x), t_i \tau) \quad x \in \mathbb{N} \quad (2)$$

2.2. Developed measure function

Distinguished trajectory can be roughly interpreted as curve formed by intersection of stable manifold and unstable manifolds. The idea of Eq.(1) and Eq.(3) to select points of distinguished trajectory by comparing the measures of flows in a limited small range. Physical significance of Eq.(1) is metric function of trajectory length in the phase space. Fundamentally speaking, what we need to compute the distinguished trajectory is a measure function to distinguish the points in a grid. Eqn.(3) shows a developed measure function. The definition of Eq.(1) and Eq.(3) don’t influence the physical significance of the original dynamic system. Assuming right hand items of system can be presented as
\( v = (v_1, v_2, \ldots, v_n) \). For any initial time \( t^* \) corresponding to \( x^* = x(t^*) \) in set \( \mathbb{N} \). Developed measure function can be given below.  
\[
\dot{L}(t^*) = \sum_{i=1}^{n} v_i^i + 1
\]  
(3) 
Assuming computing time \( t^* \) and initial value \( t_0 = 0 \) the measure value can be calculated by \( J(t^*, \tau) = M|_{t^*-\tau} \). And then \( \tau \) distinguished trajectory could be got by Eq.(2).  

2.3. Comparison of methods  
Through equ.(1) and equ.(3), equ(2) can gain the measures of trajectories and elect the optimal trajectory to proximate analytic distinguished hyperbolic trajectory. So sensitivity of equ.(1) and equ.(3) to the measures of trajectories has direct influence in the efficiency of algorithm. The advantage of equ.(3) is shown distinctly in the following instance.  

![Fig. 1(a). Results of comparison from 0 to 1 in time: solid line represents equ.(3) and dotted line stands for equ.(1); Fig. 1(b). Results of comparison from 0 to 3 in time: solid line represents equ.(3) and dotted line stands for equ.(1)](image)

Consider the vector field  
\[
\frac{dx}{dt} = -0.5x + t
\]  
(4) 
We choose \( x_1(t_0) = 0 \) and \( x_2(t_0) = 1 \) as the experimental subjects. Fig.1(a) shows the comparison among \( t \in [0, 1] \); Fig.1(b) gives out the comparison in \( t \in [0, 3] \). Obviously equ.(3) is more outperformed than equ.(1).  

3. Simulation  
In this section, optimized algorithm and simulation are provided, and time complexity of algorithms is analyzed and compared roughly.  

3.1. Optimized algorithm  
The realization of the algorithm process models on the original algorithm. 
Step 1: According to instantaneous stagnation trajectory choose initial region. Set the initial accuracy \( \delta_0 \) and the initial integral length \( \tau_0 \), build grid, and set initial point \( P_0 \) at time \( t_0 \).
Step 2: Change the calculation accuracy, if $\delta_{ij} > \delta$, make $\delta_{i+1,j} = \delta_j / 2$. If $\delta_{ij} < \delta$ jump into step 7. Otherwise, go to next step.

Step3: Assume $P$ as the center and $\delta_{i+1,j}$ as accuracy of the grid to calculate the point $Q$. If $Q \neq P$ set $P = Q$ and continue step 3, otherwise, go to next step.

Step4: Consider $P$ as the center and $\tau_1$, $\tau_2$ as integral length to calculate $W_1$, $W_2$ separately, in that $\tau_1 = \tau_0 + \Delta T$, $\tau_2 = \tau_0 + 2 \times \Delta T$. If $P \neq W_1$ go to next step. If $P = W_1$ and $P \neq W_2$ jump into step 6. If $P = W_1 = W_2$ return step 2.

Step5: Set $\tau_0 = \tau_1$, return step 3.

Step6: Set $\tau_0 = \tau_2$, return step 3.

Step7: Record the result $x_0 = P$ of time $t_0$.

Compared with existing convergence algorithm, optimized algorithm takes variable precision to improve the convergence efficiency. Suppose the distance between initial point and ideal point is $S$, the number that the original algorithm need to rebuild grid is about: $\left\lceil S / \sqrt{2\delta} \right\rceil \sim \left\lfloor S / \delta \right\rfloor$, but optimized algorithm only need $\left\lceil \log_2 S - \log_2 \delta \right\rceil \sim \left\lceil (S - \delta_0) / \delta_0 \right\rceil + \left\lceil \log_2 S - \log_2 \delta \right\rceil$. Due to the randomness of the ideal point and the uncertainty of the integral length, the number of grid reconstruction is just a rough estimation to algorithm’s time efficiency.

3.2. Realization
The periodically forced Duffing system is
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3 + \varepsilon \sin t
\]
(5)

Where \( \varepsilon = 0.01 \). In order to compare between numerical results and theoretical value, approximative distinguished hyperbolic trajectory is obtained by using perturbation theory.

\[
\eta_{DHT}(t) = -\frac{\varepsilon}{2} \left( \sin(t) \right) + O(\varepsilon^3)
\]
(6)

The points corresponding to instantaneous stationary trajectory at \( t = 0 \) can be gained readily: \((0, 0)\), \((1, 0)\) and \((-1, 0)\). We take \((0, 0)\) to estimate the region where the point of distinguished hyperboloid trajectory of equ.(5) may locate. Choose grid \([-0.1, 0.1] \times [-0.1, 0.1]\) as the initial region and set parameters as: \( t \in [0, 12], \ \delta_0 = 0.1, \ \delta = 10^{-6}, \ \tau_0 = 2, \ \Delta T = 0.5 \). Fig.3(a) shows numerical distinguished hyperbolic trajectory of equ.(5); Fig.3(b) reveals the error between numerical results and theoretical value.

4. Conclusion

The main work, in this paper, concludes the following three aspects: ① an improved method which is more sensitive to difference of measures between trajectories is proposed to compute the distinguished hyperbolic trajectory. To some extent, the convergence efficiency is raised. ② The means that halving the grid in each loop is adopted and its advantages are analyzed by theory. ③ The situation that the approximate point on the local stable manifold near to special trajectory of time-dependent vector fields converges to unstable manifold in time is verified by two-dimensional Duffing system. Simulations are mainly achieved in two-dimensional Duffing system and results show in pictures.

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