Algebraic Birkhoff Conjecture for Billiards on Sphere and Hyperbolic Plane

Misha Bialy and Andrey E. Mironov

Abstract. We consider a convex curve $\gamma$ lying on the Sphere or Hyperbolic plane. We study the problem of existence of polynomial in velocities integrals for Birkhoff billiard inside the domain bounded by $\gamma$. We extend the result by S. Bolotin (1992) and get new obstructions on polynomial integrability in terms of the dual curve $\Gamma$. We follow a method which was introduced by S. Tabachnikov for Outer billiards in the plane and was applied later on in our recent paper to Birkhoff billiards with the help of a new the so called Angular billiard.

1. Motivation and the Results

It is an old and difficult problem in theory of billiards going back to Birkhoff to describe those billiard tables $\Omega$ which admit a nonconstant function on the unit tangent bundle $T_1\Omega$ which is polynomial in velocities and keeps constant values along the orbits of the billiard flow. Traditionally such a function is called Polynomial Integral. Under certain condition on the boundary curve, a remarkable result on polynomially integrable billiards in the Euclidean plane was obtained by Bolotin (1990) in [2]. Later on, it was extended [3] also for billiards on constant curvature surfaces.

For Outer billiards the problem about polynomial integrals in the plane was recently considered [6]. In our recent paper [1], the new model of Angular billiard was introduced. Using Angular billiard we extended the method by S. Tabachnikov for Birkhoff billiards in the Euclidean plane. Obstructions to integrability which we obtained by this method are in a sense dual to those obtained by Bolotin.

In this paper we consider Birkhoff billiard inside a convex domain $\Omega$ lying on the surface $\Sigma$ of constant curvature $\pm 1$, i.e on standard Sphere or Hyperbolic plane. This is a particular case of Riemannian billiard. Point moves along geodesic inside $\Omega$ and hitting the boundary reflects according to the law of geometric optics. It turns out, that for the Sphere and Hyperbolic plane it is possible to follow our approach of [1] in order to find the obstructions to polynomial integrability in terms of the dual curve of the boundary, extending the result of [3] for the constant curvature case.

In what follows, for the case of $K = 1$ we realize $\Sigma$ as the standard unit sphere with the induced metric from Euclidean $\mathbb{R}^3$, and as the upper sheet of the hyperboloid $\{x_1^2 + x_2^2 - x_3^2 = -1\}$ in $\mathbb{R}^3$ endowed with $ds^2 = $
Let \( \gamma \) be a smooth regular arc of the boundary of \( \Omega \). We shall denote by \( \hat{\gamma} \) the image of the curve \( \gamma \) under the projection \( \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2 \).

**Example 1.** Consider as an example the domain \( \Omega \) inside the ellipse on \( \Sigma \). In this case the boundary curve \( \gamma \) is the intersection of \( \Sigma \) with a quadratic cone \( \{ax^2 + by^2 + cz^2 = 0\} \). This cone defines a curve \( \hat{\gamma} \) in the projective space. Birkhoff billiard inside \( \Omega \) is known to admit an additional integral which is quadratic in velocities (see [3], [8]).

Let us formulate first the key result of [3], needed for the classification of billiards on \( \Sigma \) admitting polynomial integrals. Everywhere in the paper we shall denote by \( \Lambda \subset \mathbb{C}P^2 \) the absolute, defined by:

\[
\Lambda = \{(x_1 : x_2 : x_3) : x_1^2 + x_2^2 \pm x_3^2 = 0\},
\]

where + sign is taken for the Sphere and – sign for the Hyperboloid.

**Theorem 1.1.** (Bolotin, 1992) Let \( \gamma \) be a smooth non-geodesic arc of the boundary curve of the domain \( \Omega \subset \Sigma \). Suppose that Birkhoff billiard inside \( \Omega \subset \Sigma \) admits a non-constant polynomial integral \( \Phi \) on the energy level \( \{||v|| = 1\} \). It then follows that \( \hat{\gamma} \) is necessarily algebraic curve. Moreover, let \( \tilde{\gamma} \) in \( \mathbb{C}P^2 \) be the irreducible component of \( \hat{\gamma} \). If \( \tilde{\gamma} \) is a smooth curve, such that at least one intersection point of \( \tilde{\gamma} \) with the absolute \( \Lambda \) is transversal, then \( \tilde{\gamma} \) is of degree 2.

It turns out to be a very difficult and open question how to remove the severe assumption in Bolotin’s theorem. For instance, one would expect that the Example 1 exhausts all convex domains with smooth boundaries having additional polynomial integral. We refer to this problem as Algebraic Birkhoff conjecture.

Our approach gives the following result in this direction:

**Theorem 1.2.** Let \( \tilde{\Gamma} \) in \( \mathbb{C}P^2 \) be the dual curve of \( \tilde{\gamma} \). Then the following alternative holds: either \( \tilde{\Gamma} \) is of degree 2, or \( \tilde{\Gamma} \) necessarily contains singular points, so that all the singular points and all inflection points of \( \tilde{\Gamma} \) belong to the absolute \( \Lambda \).

This result is motivated by our approach in [1] for the planar case, where we introduced the so called Angular billiard—the object dual to the Birkhoff billiard near the boundary curve. Also in the present paper the proof of Theorem 1.2 is based on an identity (1) obtained in terms of the dual billiard. Then we extract first nontrivial term of the power series expansion of the identity in the parameter \( \varepsilon \) and get a remarkable equation (2) on \( \tilde{\Gamma} \). The latter is studied in algebro-geometric way.

Let us denote by \( \Gamma \) the dual curve to \( \gamma \). Let us recall that the duality in geometric terms can be defined as follows. For a geodesic on \( \Sigma \) with a velocity vector \( v \in T_S \Sigma \) corresponds the point \( M = r \wedge v \), i.e. the momentum of \( v \) (here and later \( \wedge \) stands for the standard vector product in \( \mathbb{R}^3 \)). Since the momentum is preserved along geodesics, then this correspondence is well defined. Moreover, one can easily see that it extends to the usual projective duality.
By the definition, in the case of the sphere the dual point also lies on
the sphere and for the Hyperboloid the dual point belongs to de Sitter
surface, i.e. to one sheeted Hyperboloid \{x_1^2 + x_2^2 - x_3^2 = 1\}. Choosing
positive orientation and the parametrization by arc-length on \(\gamma\) we get a
parametrization of the dual curve:
\[
\Gamma(s) = \gamma(s) \land \dot{\gamma}(s).
\]

In both cases our approach is based on the following

**Theorem 1.3.** Assume that Birkhoff billiard in \(\Omega\) admits a polynomial in-
tegral \(\Phi\) of even degree \(n\). Then there exists a homogeneous polynomial \(\Psi\) of
degree \(n\) vanishing on \(\Gamma\) which satisfies the following identity. For any
point \(M \in \Gamma\) and a non zero tangent vector \(w \in T_M \Gamma\) one has:
\[
(1) \Psi(M - \varepsilon w) = \Psi(M + \varepsilon w),
\]
for all sufficiently small \(\varepsilon\).

The geometric sense of this result can be expressed by saying that the
dual object to Birkhoff billiard on constant curvature surfa-
cce is the Outer
billiard [7]. In the case of the sphere this is well known to bil-
liard players community but slightly less obvious for the case of Hyperboloid. We shall
make this precise in the remark after the proof of Theorem 1.3.

The algebraic part of our approach is based on the following theorem.
Let us denote by \(F \in \mathbb{C}[x_1, x_2, x_3]\) the irreducible homogenous defining
polynomial of the component \(\tilde{\Gamma} \subset \mathbb{C}P^2\), and let \(d = \deg F, d \geq 2\). As usual
we introduce
\[
\text{Hess}(F)(x_1, x_2, x_3) := \det \begin{pmatrix}
F_{x_1x_1} & F_{x_1x_2} & F_{x_1x_3} \\
F_{x_2x_1} & F_{x_2x_2} & F_{x_2x_3} \\
F_{x_3x_1} & F_{x_3x_2} & F_{x_3x_3}
\end{pmatrix}.
\]

**Theorem 1.4.** Suppose that Birkhoff billiard in \(\Omega\) admits a polynomial in-
tegral \(\Phi\) of even degree \(n\). Let \(F\) be the irreducible homogeneous defining
polynomial of \(\tilde{\Gamma}\), \(d = \deg F\). Then the following identity holds:
\[
(2) \quad Q^k(\text{Hess}(F))^k - c(x_1^2 + x_2^2 + Kx_3^2)^\alpha = F \cdot R, \quad \forall (x_1, x_2, x_3) \in \mathbb{C}^3.
\]
Here \(\alpha\) is a positive integer, when \(d > 2\); \(K\) is the curvature \(\pm 1\) and \(Q, R\) are
homogeneous polynomials, so that \(Q\) does not vanish on \(\tilde{\Gamma}\) identically.

Postponing the proof of Theorem 1.3 and 1.4 let us prove Theorem 1.2.

**Proof.** We follow the idea of Lemma 3 of [6]. Consider the situation in \(\mathbb{C}P^2\).
Any intersection point in \(\mathbb{C}P^2\) between Hessian curve of \(\text{Hess}(\tilde{\Gamma})\) with \(\tilde{\Gamma}\)
is either singular or inflection point of \(\Gamma\). So, if there is a singular or inflection
point \((a : b : c) \in \tilde{\Gamma}\) such that \(a^2 + b^2 + Kc^2 \neq 0\), it then follows from (2)
that \(c = 0\). Therefore, \(\text{Hess}(F)\) must vanish on \(\tilde{\Gamma}\) identically, since \(Q\) does
not vanish identically on \(\tilde{\Gamma}\). This implies that \(\tilde{\Gamma}\) is a line (see [4]), but this
is impossible.

Let us prove now that \(\tilde{\Gamma}\) must have singular points. If on the contrary
\(\tilde{\Gamma}\) is a smooth curve, then it follows from (2) that all inflection points must
belong to the absolute \(\Lambda\) defined by the equations
\[
\Lambda = \{(x_1 : x_2 : x_3) \in \mathbb{C}P^2 : x_1^2 + x_2^2 + Kx_3^2 = 0\}.
\]
Recall, $d$ is the degree of $\hat{\Gamma}$. Then the Hessian curve intersects $\hat{\Gamma}$ exactly in inflection points, and moreover, it is remarkable fact that the intersection multiplicity of such a point of intersection equals exactly the order of inflection point (see [9]), and hence does not exceed $(d - 2)$. Furthermore, the absolute $\Lambda$ intersects $\hat{\Gamma}$ maximum in $2d$ points together. Hence, we have altogether counted with multiplicities not more than $2d(d - 2)$, but on the other hand the Hessian curve has degree $3(d - 2)$ and thus by Bezout theorem the number of intersection points with multiplicities is $3d(d - 2)$. This contradiction shows that $\hat{\Gamma}$ can not be a smooth curve unless $d = 2$.

Theorem 1.2 is proved. □

The plan of the rest of the paper is as follows: in Section 2 we give the geometric proof of Theorem 1.3 and then in Section 3 we treat both cases, of Sphere and Hyperboloid in separate Subsections.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3.

The first observation is the following [2], [3], [5]. Let $\Phi$ is a polynomial integral of Birkhoff billiard inside $\Omega$. Then, one can assume, with no loss of generality, that $\Phi(r, v)$ is homogeneous in velocities of certain even degree $n$. Moreover, since components of the Momentum $M = r \wedge v$ are preserved under the geodesic flow on $\Sigma$ one can show that $\Phi(r, v) \equiv \Psi(M)$, where $\Psi$ is a homogeneous polynomial in the components of $M$ of degree $n$. Furthermore, by a simple modification of $\Phi$ one can achieve that $\Phi$ vanishes on tangent vectors to the boundary curve $\gamma$. This implies that the polynomial $\Psi$ vanishes on the dual curve $\Gamma$, because the latter was constructed by:

$$\Gamma(s) := M(s) = \gamma(s) \wedge \dot{\gamma}(s),$$

with the help of arc-length parameter $s$ on $\gamma$. We remind that Riemannian metric on $\Sigma$ is induced from the Euclidean Space in the case of Sphere, and from Minkowski metric in the case of Hyperboloid.

Fix a point $r = \gamma(s)$ and fix the orthonormal frame $\{v, n\}$ in $T_r \Sigma$, $v = \dot{\gamma}(s)$ and $n$ is the positive unite normal. Denote the dual point by

$$M = \Gamma(s) = \gamma(s) \wedge v = r \wedge v.$$

Let us compute the tangent vector to the dual curve at $M$:

$$\hat{\Gamma}(s) = \dot{M}(s) = \gamma(s) \wedge \dot{\gamma}(s) = \gamma \wedge (\nabla_v \dot{\gamma} + N),$$

where $\nabla$ is the Levi-Civita connection of the induced metric on the surface $\Sigma$, $N$ is a normal vector to $\Sigma$. Recall, that in both geometries $N$ is proportional to $\gamma$ and therefore can be neglected in (3):

$$\hat{\Gamma}(s) = \gamma \wedge \nabla_v \dot{\gamma} = \gamma \wedge (k \cdot n),$$

where $k$ is the geodesic curvature of $\gamma$ at the point $r$.

In order to prove Theorem 1.3 we need to prove the equality of the values of the function $\Psi$ in two points:

$$P_- = \Gamma(s) - \varepsilon \hat{\Gamma}(s), \quad P_+ = \Gamma(s) + \varepsilon \hat{\Gamma}(s).$$

We have:

$$P_- = M - \gamma \wedge (\varepsilon kn) = r \wedge v - r \wedge (\varepsilon kn) = r \wedge (v - \varepsilon kn),$$
\[ P_+ = M + \gamma \wedge (\varepsilon kn) = r \wedge v + r \wedge (\varepsilon kn) = r \wedge (v + \varepsilon kn). \]

The points \( P_-, P_+ \) do not lie on \( \Sigma \) but we can present them as follows:

\[ P_- = r \wedge (v - \varepsilon kn) = (1 + \varepsilon^2 k^2)^{1/2}(r \wedge v_-), \]
\[ P_+ = r \wedge (v + \varepsilon kn) = (1 + \varepsilon^2 k^2)^{1/2}(r \wedge v_+). \]

Here we introduced

\[ v_{\pm} := (1 + \varepsilon^2 k^2)^{-1/2}(v \pm \varepsilon kn), \]

the two unit vectors in \( T_r \Sigma \). Notice, that the vectors \( v_{\pm} \) obviously obey Birkhoff billiard reflection law. Therefore we have:

\[ \Psi(P_-) = \Psi((1 + \varepsilon^2 k^2)^{1/2}(r \wedge v_-)) = (1 + \varepsilon^2 k^2)^{n/2}\Phi(v_-), \]
\[ \Psi(P_+) = \Psi((1 + \varepsilon^2 k^2)^{1/2}(r \wedge v_+)) = (1 + \varepsilon^2 k^2)^{n/2}\Phi(v_+), \]

where we used homogeneity of \( \Psi \). Therefore, the claim follows because by the assumptions \( \Phi \) is an integral of Birkhoff billiard, so that \( \Phi(v_-) = \Phi(v_+) \).

This proves Theorem 1.3.

**Remark 1.** The following fact is the corollary of the proof. Let \( p \) denotes the radial projection onto the unite Sphere for \( K = 1 \) and onto the de Sitter surface for \( K = -1 \). Then we have

\[ \mathbf{p}(P_-) := M_\mathbf{-} = r \wedge v_-; \quad \mathbf{p}(P_+) := M_\mathbf{+} = r \wedge v_+. \]

It then follows that the geodesic segment \([M_-; M_+]\) contains \( M \) as the middle point (in the sense of induced metric on the de Sitter surface), which is equivalent to the fact that \( v_- \) and \( v_+ \) respect the Birkhoff billiard reflection law.

### 3. Proof of Theorem 1.4

In this section we shall treat the equation (1) in the both cases of Sphere and Hyperboloid and derive the equation (2).

Let us denote by \( F \in \mathbb{C}[x_1, x_2, x_3] \) the irreducible homogenous polynomial defining the component \( \tilde{\Gamma} \subset \mathbb{C}P^2 \), and let \( d = \deg F, d \geq 2 \). Since \( \Psi \) vanishes on \( \Gamma \) we can write

\[ \Psi = F^k Q, \]

for some integer \( k \geq 1 \), so that polynomial \( Q \) does not vanish on \( \Gamma \) identically. Notice that \( F, Q \) can be assumed to be real since \( \Gamma \) is real curve. If needed we consider a smaller arc where \( DF \) does not vanish and \( Q > 0 \). We set

\[ G = FQ^k. \]

Let us remark that \( G \) is not a polynomial anymore, but homogeneous function of degree

\[ p = \frac{n}{k} = d + \frac{1}{k} \deg Q \geq 2. \]

Since \( \Psi \) satisfies (1) then also \( G \) does.
3.1. The case of the Sphere. In this case the tangent vector \( w \in T, \Gamma, r = (x_1, x_2, x_3) \) can be taken with the components:
\[
(5) \quad w_1 = x_2 G_{x_3} - x_3 G_{x_2} ; \quad w_2 = x_3 G_{x_1} - x_1 G_{x_3} ; \quad w_3 = x_1 G_{x_2} - x_2 G_{x_1} .
\]
In what follows, we pass from homogeneous functions \( G, f, Q, ... \) to \( g, f, q, ... \) defined by
\[
\begin{align*}
\frac{\partial f}{\partial x} g &= F(x, y, 1), \\
\frac{\partial g}{\partial y} x &= G(x, y, 1), \\
\frac{\partial f}{\partial y} x &= Q(x, y, 1), ...
\end{align*}
\]
so that
\[
g = f q^1.
\]
Here the mapping
\[
(x_1, x_2, x_3) \mapsto (x, y), \quad x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}, \quad x_3 = (x^2 + y^2 + 1)^{-\frac{1}{2}}
\]
is the central projection of the sphere to the plane \( \{ z = 1 \} \).

The derivatives of the functions \( G \) and \( g \) are related in a usual way:
\[
(6) \quad G_{x_3} = x_3^{p-1} g_x = (1 + x^2 + y^2)^{\frac{p-1}{2}} g_x ; \quad G_{x_2} = x_3^{p-1} g_y = (1 + x^2 + y^2)^{\frac{p-1}{2}} g_y ,
\]
Using Euler formula for \( G \), for all \( (x_1, x_2, x_3) \in \Gamma \) so that \( G(x_1, x_2, x_3) = 0 \), we get also:
\[
(7) \quad G_{x_3} = -x_3^{p-1}(x g_x + y g_y) = -(1 + x^2 + y^2)^{\frac{p-1}{2}} (x g_x + y g_y) .
\]
Using (6) and (7) the components (5) of the vector \( w \) take the form:
\[
(8) \quad \frac{w_1}{x_3} = -(1 + x^2 + y^2)^{\frac{p-1}{2}} ((xy) g_x + (1 + y^2) g_y) ,
\]
\[
(9) \quad \frac{w_2}{x_3} = (1 + x^2 + y^2)^{\frac{p-1}{2}} ((1 + x^2) g_x + (xy) g_y) ,
\]
\[
(10) \quad \frac{w_3}{x_3} = (1 + x^2 + y^2)^{\frac{p-1}{2}} (x g_y - y g_x) .
\]

Let us write equation (11) in terms of the function \( g \):
\[
(11) \quad (x_3 - \varepsilon w_3)^p g \left( \frac{x_1 - \varepsilon w_1}{x_3 - \varepsilon w_3}, \frac{x_2 - \varepsilon w_2}{x_3 - \varepsilon w_3} \right) = (x_3 + \varepsilon w_3)^p g \left( \frac{x_1 + \varepsilon w_1}{x_3 + \varepsilon w_3}, \frac{x_2 + \varepsilon w_2}{x_3 + \varepsilon w_3} \right) .
\]
Using formulas (8), (9), (10) equation (11) yields:
\[
(12) \quad (1 - \mu (x g_y - y g_x))^p g \left( \frac{x + \mu ((xy) g_x + (1 + y^2) g_y)}{1 - \mu (x g_y - y g_x)}, \frac{y - \mu ((1 + x^2) g_x + (xy) g_y)}{1 - \mu (x g_y - y g_x)} \right) = (1 + \mu (x g_y - y g_x))^p g \left( \frac{x - \mu ((xy) g_x + (1 + y^2) g_y)}{1 + \mu (x g_y - y g_x)}, \frac{y + \mu ((1 + x^2) g_x + (xy) g_y)}{1 + \mu (x g_y - y g_x)} \right),
\]
where we denoted by \( \mu = \varepsilon (1 + x^2 + y^2)^{\frac{p-1}{2}} \). Our next step is to collect and to equate to zero the terms of order \( \mu^3 \), neglecting the terms containing \( g \), since \( g = 0 \) on the curve. We have:
\[
(13) \quad (1 + x^2 + y^2)^2 (g_{xxx} g_y^2 - 3 g_{xx} g_{yy} g_x + 3 g_{xy} g_{yy} g_x - g_{yy} g_y^3) + 3(2 - p)(g_{xx} g_y^2 - 2 g_{xy} g_x - g_{yy} g_x^2)(x g_y - y g_x) = 0
\]
Denote by \( u = g_y \partial_x - g_x \partial_y \) the vector field tangent to \( \{ f = 0 \} \). Introduce the quantity
\[
H(g) = g_{xx}g_y^2 - 2g_{xy}g_xg_y + g_{yy}g_x^2,
\]
and notice that the following two identities hold
\[
L_u(x^2 + y^2 + 1) = 2(xg_y - yg_x),
\]
\[
L_u H(g) = g_{xxx}g_y^3 - 3g_{xxy}g_y^2g_x + 3g_{xyy}g_yg_x^2 - g_{yy}g_x^3.
\]
Using these identities equation (13) can be rewritten as follows:
\[
(\text{14}) \quad (x^2 + y^2 + 1)L_u H(g) + \frac{3}{2}(2 - p)H(g)L_u(x^2 + y^2 + 1) = 0.
\]
Multiplying (14) by \((x^2 + y^2 + 1)^\frac{3 - 3p}{2}\) we get
\[
(\text{15}) \quad (x^2 + y^2 + 1)^\frac{3 - 3p}{2}L_u H(g) + \frac{3}{2}(2 - p)H(g)(x^2 + y^2 + 1)^\frac{3 - 3p}{2}L_u(x^2 + y^2 + 1) = 0.
\]
But the left hand side of (15) is the complete derivative and thus
\[
L_u \left( H(g)(x^2 + y^2 + 1)^\frac{3 - 3p}{2} \right) = 0.
\]
Since \( u \) is a tangent vector field to \( \{ f = 0 \} \) then the function \( H(g)(x^2 + y^2 + 1)^\frac{3 - 3p}{2} \) must be a constant on \( \{ f = 0 \} \) i.e.
\[
(\text{16}) \quad H(g(x, y)) = c_1(x^2 + y^2 + 1)^\frac{3p - 6}{2}, \quad \forall (x, y) \in \{ f = 0 \},
\]
for some constant \( c_1 \). A direct calculation gives that for any function \( r \)
\[
(\text{17}) \quad H(f(x, y)r(x, y)) = r^3(x, y)H(f(x, y)), \quad \forall (x, y) \in \{ f = 0 \},
\]
and therefore we conclude with the formula
\[
q^3(x, y)H(f(x, y)) = H(g) = c_1(x^2 + y^2 + 1)^\frac{3p - 6}{2}, \quad \forall (x, y) \in \{ f = 0 \}.
\]
Raising to the power \( k \) we get
\[
q^3(x, y)H(f(x, y))^k = c_1^k(x^2 + y^2 + 1)^\frac{k(3p - 6)}{2}, \quad \forall (x, y) \in \{ f = 0 \}.
\]
Therefore the difference is a polynomial divisible by \( f \).
\[
(\text{18}) \quad q^3(x, y)H(f(x, y))^k - c_1^k(x^2 + y^2 + 1)^\frac{k(3p - 6)}{2} = f \cdot r_1, \quad \forall (x, y) \in \mathbb{C}^2,
\]
for some polynomial \( r_1 \). Moreover, using (18) one can compute the degrees of the terms at the left hand side of (13) and to get homogeneous version of (18):
\[
(\text{19}) \quad Q^3(x, y, z)H(F(x, y, z))^k - c_1^k(x^2 + y^2 + z^2)^\frac{k(3p - 6)}{2} z^{2k} = F \cdot R_1,
\]
which is valid for all \((x, y, z) \in \mathbb{C}^3\). Now we use the identities (see [9])
\[
\text{Hess}(F) = \frac{(d - 1)^2}{z^2} \begin{vmatrix}
F_{xx} & F_{xy} & F_x \\
F_{xy} & F_{yy} & F_y \\
F_x & F_y & \frac{d}{d-1}F
\end{vmatrix} = \frac{(d - 1)^2}{z^2} \left( \frac{d}{d-1} F(F_{xx}F_{yy} - F_{xy}^2) - H(F) \right).
\]
Notice that in (20) and in (19) \( H(F(x, y, z)) \) is computed with \( z \) being a parameter. The last step is just to substitute the expression for \( H(F) \) via \( \text{Hess}(F) \) from (20) into (19) in order to get:

\[
Q^3(\text{Hess}(F))^k - c(x^2 + y^2 + z^2)^{\frac{3n-6}{2}} = F \cdot R, \quad \forall (x, y, z) \in \mathbb{C}^3.
\]

But this is exactly what it is claimed in (2).

### 3.2. The case of Hyperboloid

We proceed exactly as in previous Subsection with obvious modifications. We keep the same notation as in the beginning of the Section.

Recall that in the Hyperbolic case \( \Sigma \) is the upper sheet of Hyperboloid, the curve \( \gamma \subset \Sigma \) and the curve \( \Gamma \) lies on the one sheeted Hyperboloid

\[x_1^2 + x_2^2 - x_3^2 = 1.
\]

Therefore, the tangent vector \( w \in T_r \Gamma, r = (x_1, x_2, x_3) \) can be taken with the components:

\[
w_1 = x_2 G_{x_3} + x_3 G_{x_2}; \quad w_2 = -x_3 G_{x_1} - x_1 G_{x_3}; \quad w_3 = x_1 G_{x_2} - x_2 G_{x_1}.
\]

As before we pass from homogeneous functions \( G, F, Q \) to \( g, f, q \) defined above:

\[
f(x, y) = F(x, y, 1), \quad g(x, y) = G(x, y, 1), \quad q(x, y) = Q(x, y, 1), ...
\]

where the mapping

\[(x_1, x_2, x_3) \mapsto (x, y), \quad x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}\]

and \( x_3 = (x^2 + y^2 - 1)^{-\frac{1}{2}} \) is the central projection of the one sheeted hyperboloid to the plane \( \{z = 1\} \). The derivatives of the functions \( G \) and \( g \) are related in this case as follows:

\[
G_{x_1} = x_3^{p-1} g_x = (x^2 + y^2 - 1)^{\frac{1-p}{2}} g_x, \quad G_{x_2} = x_3^{p-1} g_y = (x^2 + y^2 - 1)^{\frac{1-p}{2}} g_y.
\]

Using Euler formula for \( G \), for all \( (x_1, x_2, x_3) \in \Gamma \) so that \( G(x_1, x_2, x_3) = 0 \), we get also:

\[
G_{x_3} = -x_3^{p-1} (x g_x + y g_y) = -(x^2 + y^2 - 1)^{\frac{1-p}{2}} (x g_x + y g_y).
\]

Using (23) and (24) the components (22) of the vector \( w \) in this case take the form:

\[
\frac{w_1}{x_3} = -(x^2 + y^2 - 1)^{\frac{1-p}{2}} ((xy)g_x + (y^2 - 1)g_y),
\]

\[
\frac{w_2}{x_3} = (x^2 + y^2 - 1)^{\frac{1-p}{2}} ((x^2 - 1)g_x + (xy)g_y),
\]

\[
\frac{w_3}{x_3} = (x^2 + y^2 - 1)^{\frac{1-p}{2}} (x g_y - y g_x).
\]

Therefore equation (11) in the case of Hyperboloid can be presented using formulas (25), (26), (27):

\[
(1-\mu (x g_y - y g_x))^p \left( \frac{x + \mu ((xy)g_x + (y^2 - 1)g_y)}{1 - \mu (x g_y - y g_x)}, \frac{y - \mu ((x^2 - 1)g_x + (xy)g_y)}{1 - \mu (x g_y - y g_x)} \right) =
\]
\[ (x^2 + y^2 - 1)(g_{xxx}g_y^3 - 3g_{xxy}g_y^2g_x + 3g_{xyy}g_yg_x^2 - g_{yy}g_x^3) + 3(2 - p)(g_{xx}g_x^2 - 2g_{xxy}g_xg_y + g_{yy}g_x^2)(xg_y - yg_x) = 0. \]

As before we denote by \( u = g_y \partial_x - g_x \partial_y \) be the vector field tangent to \( \{ f = 0 \} \). Recall the notation,

\[ H(g) = g_{xx}g_y^2 - 2g_{xxy}g_xg_y + g_{yy}g_x^2, \]

and use the two identities:

\[ L_u(x^2 + y^2 - 1) = 2(xg_y - yg_x), \]
\[ L_uH(g) = g_{xxx}g_y^3 - 3g_{xxy}g_y^2g_x + 3g_{xyy}g_yg_x^2 - g_{yy}g_x^3. \]

Using these identities equation (29) can be rewritten as follows:

\[ (x^2 + y^2 - 1)L_uH(g) + \frac{3}{2}(2 - p)H(g)L_u(x^2 + y^2 - 1) = 0. \]

Multiplying again (30) by \((x^2 + y^2 - 1)^{\frac{4 - 3p}{2}}\) we get

\[ (x^2 + y^2 - 1)^{\frac{6 - 3p}{2}}L_uH(g) + \frac{3}{2}(2 - p)H(g)(x^2 + y^2 - 1)^{\frac{4 - 3p}{2}}L_u(x^2 + y^2 - 1) = 0. \]

But the left hand side of (31) is the complete derivative and thus

\[ L_u \left( H(g)(x^2 + y^2 - 1)^{\frac{6 - 3p}{2}} \right) = 0. \]

Since \( u \) is a tangent vector field to \( \{ f = 0 \} \), then the function \( H(g)(x^2 + y^2 - 1)^{\frac{6 - 3p}{2}} \) must be a constant on \( \{ f = 0 \} \), i.e.

\[ H(g(x, y)) = c_1(x^2 + y^2 - 1)^{\frac{3p - 6}{2}}, \quad \forall (x, y) \in \{ f = 0 \}, \]

for some constant \( c_1 \). Using again the identity (17) we come to:

\[ q^k(x, y)H(f(x, y)) = c_1(x^2 + y^2 - 1)^{\frac{3p - 6}{2}}, \quad \forall (x, y) \in \{ f = 0 \}. \]

Raising to the power \( k \) we get

\[ q^k(x, y)H(f(x, y))^k = c_1^k(x^2 + y^2 - 1)^k^{\frac{3p - 6}{2}}, \quad \forall (x, y) \in \{ f = 0 \}. \]

Therefore the difference is a polynomial divisible by \( f \).

\[ q^3(x, y)H(f(x, y))^k - c_1^k(x^2 + y^2 - 1)^k^{\frac{3p - 6}{2}} = f \cdot r_1, \quad \forall (x, y) \in \mathbb{C}^2, \]

for some polynomial \( r_1 \). One can compute the degrees using (18) and to get homogeneous version of the last identity:

\[ Q^3(x, y, z)H(F(x, y, z))^k - c_1^k(x^2 + y^2 - z^2)^k^{\frac{3p - 6}{2}}z^{2k} = F \cdot R_1, \]

which is valid for all \((x, y, z) \in \mathbb{C}^3\). Using again the identities (20) we substitute the expression for \( H(F) = \frac{z^2}{(d-1)^2} \text{Hess}(F) \) which is valid on \( \{ F = 0 \} \) into (33). We get:

\[ Q^3(\text{Hess}(F))^k - c(x^2 + y^2 - z^2)^k^{\frac{3p - 6}{2}} = F \cdot R, \quad \forall (x, y, z) \in \mathbb{C}^3. \]
But this is exactly the equation of (2) for the Hyperbolic case. This com-
pletes the proof of Theorem 1.4 in the Hyperbolic case.

Remark 2. It would be very natural to collect terms of $\epsilon^3$ of the equation
(1) written in homogeneous coordinates without going to affine chart. One
would expect that using Euler formula for $F$ and maybe for its derivatives
one would be able to get directly to the equation (2). Surpris ingly, this
attempt leads to very heavy calculations, which we were not able to perform.

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M. BIALY, SCHOOL OF MATHEMATICAL SCIENCES, RAYMOND AND BEVERLY SACKLER
FACULTY OF EXACT SCIENCES, TEL AVIV UNIVERSITY, ISRAEL
E-mail address: bialy@post.tau.ac.il

A.E. MIRONOV, SOBOLEV INSTITUTE OF MATHEMATICS AND
E-mail address: mironov@math.nsc.ru