Poincaré inequalities and Newtonian Sobolev functions on noncomplete metric spaces

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Abstract. Let $X$ be a noncomplete metric space satisfying the usual (local) assumptions of a doubling property and a Poincaré inequality. We study extensions of Newtonian Sobolev functions to the completion $\hat{X}$ of $X$ and use them to obtain several results on $X$ itself, in particular concerning minimal weak upper gradients, Lebesgue points, quasicontinuity, regularity properties of the capacity and better Poincaré inequalities. We also provide a discussion about possible applications of the completions and extension results to $p$-harmonic functions on noncomplete spaces and show by examples that this is a rather delicate issue opening for various interpretations and new investigations.

Key words and phrases: Lebesgue point, local doubling, locally compact metric space, noncomplete metric space, Newtonian space, nonlinear potential theory, $p$-harmonic function, Poincaré inequality, quasicontinuity, quasiminimizer, semilocal doubling, Sobolev space.

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1. Introduction

Our aim in this paper is to study Poincaré inequalities and Newtonian (Sobolev) functions on noncomplete metric spaces, and primarily to do so using their completion. This turns out to be a rather fruitful approach which, however, has certain subtleties and limitations, in particular when dealing with $p$-harmonic functions.

Let $X = (X,d,\mu)$ be a metric measure spaces, where $\mu$ is a positive complete Borel measure $\mu$ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We let $\hat{X}$ be the completion of $X$ with respect to the metric $d$, and extend $d$ and $\mu$ to $\hat{X}$ so that $\mu(\hat{X} \setminus X) = 0$. Also let $1 < p < \infty$.

Much of analysis on metric spaces has been done assuming global doubling and global Poincaré inequalities, which for instance are assumed in the monographs Hajłasz–Koskela [18], Björn–Björn [3] and Heinonen–Koskela–Shanmugalingam–Tyson [22]. For wider applicability we study properties that hold under more local assumptions. Such assumptions have earlier been considered e.g. by Cheeger [13], Danielli–Garofalo–Marola [14], Garofalo–Marola [17] and Holopainen–Shanmugalingam [23]. In the following definition we follow the recent terminology from Björn–Björn [6], where a more extensive discussion of these assumptions can be found.
Definition 1.1. The measure \( \mu \) is doubling within a ball \( B(x_0, r_0) \) if there is \( C > 0 \) (depending on \( x_0 \) and \( r_0 \)) such that \( \mu(2B) \leq C \mu(B) \) for all balls \( B \subset B(x_0, r_0) \).

Similarly, the \( p \)-Poincaré inequality holds within a ball \( B(x_0, r_0) \) if there are constants \( C > 0 \) and \( \lambda \geq 1 \) (depending on \( x_0 \) and \( r_0 \)) such that for all balls \( B \subset B(x_0, r_0) \), all integrable functions \( u \) on \( \Lambda B \), and all upper gradients \( g \) of \( u \),

\[
\int_B |u - u_B| \, d\mu \leq C r_B \left( \int_{\Lambda B} g^p \, d\mu \right)^{1/p},
\]

where \( u_B := \int_B u \, d\mu := \int_B u \, d\mu/\mu(B) \). These properties are called local if for every \( x_0 \in X \) there is \( r_0 > 0 \) (depending on \( x_0 \)) such that the doubling property or the \( p \)-Poincaré inequality holds within \( B(x_0, r_0) \). They are called semilocal if they hold within every ball in \( X \).

Our first observation is that if \( \mu \) is doubling (resp. supports a \( p \)-Poincaré inequality) within a ball \( B(x_0, r_0) \) in \( X \) then its zero extension is also doubling (resp. supports a \( p \)-Poincaré inequality) within the corresponding ball \( B(x_0, r_0) \) in \( \hat{X} \). In particular, this means that the semilocal assumptions extend from \( X \) to \( \hat{X} \), see Corollaries 3.4 and 3.7. On the other hand, local doubling (resp. a local \( p \)-Poincaré inequality) on \( X \) does not extend to \( \hat{X} \), even though it does extend to a locally compact open subset of \( \hat{X} \) containing \( X \) (see Lemma 4.6), which may be sufficient for many applications.

The following extension result is one of the main results in this paper. (See Theorem 4.1 for a more extensive version.) For an open set \( \Omega \) in \( X \), we let

\[
\Omega^\wedge = \hat{X} \setminus \overline{X \setminus \Omega},
\]

where the closure is taken in \( \hat{X} \). This makes \( \Omega^\wedge \) into the largest open set in \( \hat{X} \) such that \( \Omega = \Omega^\wedge \cap X \).

Theorem 1.2. Assume that the doubling property and the \( p \)-Poincaré inequality hold within the ball \( B_0 \) in the sense of Definition 1.1. Let \( \Omega \subset B_0 \) be open and \( u \in N^{1,p}(\Omega) \). Then the function

\[
\hat{u}(x) = \limsup_{\rho \to 0} \int_{B(x, \rho) \cap X} u \, d\mu, \quad x \in \Omega^\wedge,
\]

belongs to \( N^{1,p}(\Omega^\wedge) \) and is a pointwise extension of a representative of \( u \) to \( \Omega^\wedge \). Moreover, the minimal \( p \)-weak upper gradients \( g_{\hat{u}} \) and \( g_u \) of \( \hat{u} \) and \( u \) with respect to \( \hat{X} \) and \( X \) satisfy

\[
g_{\hat{u}} \leq A_0 g_u \quad \text{a.e. in} \ \Omega,
\]

where \( A_0 \) is a constant only depending on \( p \), the doubling constant and both constants in the \( p \)-Poincaré inequality within \( B_0 \).

For \( \Omega = X \), with \( X \) locally compact and under global assumptions, similar extension results appear in Aikawa–Shanmugalingam [1, Proposition 7.1] and Heinonen–Koskela–Shanmugalingam–Tyson [22, Lemma 8.2.3]. Theorem 1.2 makes it possible to study functions \( u \) on \( X \) using properties known to hold for their extensions \( \hat{u} \) on \( \hat{X} \). We use this to obtain some \( L^p \)-Lebesgue point and quasicontinuity results for Newtonian Sobolev functions in noncomplete spaces.

When \( X \) is complete and \( \mu \) is globally doubling, a deep result due to Keith–Zhong [24, Theorem 1.0.1] shows that the Poincaré inequality is an open-ended property, in the sense that if \( \mu \) supports a global \( p \)-Poincaré inequality then it also supports a global \( q \)-Poincaré inequality for some \( q < p \). Counterexamples due to Koskela [29] show that this is false for locally compact \( X \). Nevertheless, by localizing
the arguments in [24] local versions of this self-improvement result were obtained in [6] for locally compact spaces. In Section 5, we further generalize these results to non-locally compact spaces, using our extension theorem as the key tool.

We end the paper with a discussion on \( p \)-harmonic functions (and more generally quasiminimizers and quasisuperminimizers) on noncomplete spaces with particular emphasis on locally compact spaces. It turns out that the choice of the test function space and the local Newtonian space for \( p \)-harmonic functions plays an important role for the validity of several of the fundamental properties of \( p \)-harmonic functions, such as various Harnack inequalities and maximum principles. There are several different natural choices of these spaces, which all coincide in the complete case.

Thus, it is the intended applications and particular results, which essentially determine the “right definition” of \( p \)-harmonic functions for various purposes in noncomplete spaces. The continuity of \( p \)-harmonic functions is, however, possible to obtain under most of these definitions, see Theorem 6.2.

Some of this versatility is demonstrated in Example 6.3 and it is for instance possible to treat mixed and Neumann boundary data as special cases of Dirichlet data. In complete spaces, most of the suggested definitions reduce to the usual definition of \( p \)-harmonic functions.

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2. Upper gradients and Newtonian spaces

We assume throughout the paper that \( X = (X, d, \mu) \) is a metric space equipped with a metric \( d \) and a positive complete Borel measure \( \mu \) such that \( 0 < \mu(B) < \infty \) for all balls \( B \subset X \). It follows that \( X \) is separable and Lindelöf. We also assume that \( 1 < p < \infty \), although the results in Sections 2 and 3 also hold if \( p = 1 \). Proofs of the results in this section can be found in the monographs Björn –Björn [3] and Heinonen–Koskela–Shanmugalingam–Tyson [22].

A curve is a continuous mapping from an interval, and a rectifiable curve is a curve with finite length. Unless said otherwise, we will only consider curves which are nonconstant, compact and rectifiable, and thus each curve can be parameterized by its arc length \( ds \). A property is said to hold for \( p \)-almost every curve if it fails only for a curve family \( \Gamma \) with zero \( p \)-modulus, i.e. there exists \( 0 \leq \rho \in L^p(X) \) such that \( \int_\gamma \rho \, ds = \infty \) for every curve \( \gamma \in \Gamma \).

Following Heinonen–Koskela [21], we introduce upper gradients as follows (they called them very weak gradients).

**Definition 2.1.** A Borel function \( g : X \to [0, \infty] \) is an upper gradient of a function \( f : X \to \mathbb{R} := [-\infty, \infty] \) if for all curves \( \gamma : [0, l_\gamma] \to X \),

\[
|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \tag{2.1}
\]

where the left-hand side is considered to be \( \infty \) whenever at least one of the terms therein is infinite. If \( g : X \to [0, \infty] \) is measurable and (2.1) holds for \( p \)-almost every curve, then \( g \) is a \( p \)-weak upper gradient of \( f \).

The \( p \)-weak upper gradients were introduced in Koskela–MacManus [30]. It was also shown therein that if \( g \in L^p_{\text{loc}}(X) \) is a \( p \)-weak upper gradient of \( f \), then one can find a sequence \( \{g_j\}_{j=1}^\infty \) of upper gradients of \( f \) such that \( \|g_j - g\|_{L^p(X)} \to 0 \). If \( f \) has an upper gradient in \( L^p_{\text{loc}}(X) \), then it has an a.e. unique minimal \( p \)-weak upper gradient \( g_f \in L^p_{\text{loc}}(X) \) in the sense that for every \( p \)-weak upper gradient \( g \in L^p_{\text{loc}}(X) \)
of $f$ we have $g_f \leq g$ a.e., see Shanmugalingam [32]. Following Shanmugalingam [33], we define a version of Sobolev spaces on the metric space $X$.

**Definition 2.2.** For a measurable function $f : X \to \mathbb{R}$, let

$$\|f\|_{N^{1,p}(X)} = \left( \int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients $g$ of $f$. The Newtonian space on $X$ is

$$N^{1,p}(X) = \{ f : \|f\|_{N^{1,p}(X)} < \infty \}.$$

The quotient space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [33]. We also define

$$D^p(X) = \{ f : f \text{ is measurable and has an upper gradient in } L^p(X) \}.$$

In this paper we assume that functions in $N^{1,p}(X)$ and $D^p(X)$ are defined everywhere (with values in $\mathbb{R}$), not just up to an equivalence class in the corresponding function space. This is important for upper gradients to make sense.

For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space in its own right. We say that $f \in N^{1,p}_{\text{loc}}(E)$ if for every $x \in E$ there exists a ball $B_x \ni x$ such that $f \in N^{1,p}(B_x \cap E)$. The spaces $D^p(E)$ and $D^p_{\text{loc}}(E)$ are defined similarly. If $f, h \in D^p_{\text{loc}}(X)$, then $g_f = g_h$ a.e. in $\{ x \in X : f(x) = h(x) \}$, in particular for $c \in \mathbb{R}$ we have $g_{\min(f,c)} = g_f 1_{\{f < c\}}$ a.e.

**Definition 2.3.** The (Sobolev) capacity of a set $E \subset X$ is the number

$$C^X_p(E) = C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on $E$.

We say that a property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. Namely, if $u \in N^{1,p}(X)$ then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in D^p_{\text{loc}}(X)$ and $u = v$ a.e., then $u = v$ q.e.

We let $B = B(x, r) = \{ y \in X : d(x, y) < r \}$ denote the ball with centre $x$ and radius $r$, and let $\lambda B = B(x, \lambda r)$. We assume throughout the paper that balls are open. In metric spaces it can happen that balls with different centres and/or radii denote the same set. We will however make the convention that a ball $B$ comes with a predetermined centre and radius $r_B$. Note that it can happen that $B(x_0, r_0) \subset B(x_1, r_1)$ even when $r_0 > r_1$. In disconnected spaces this can happen also when $r_0 > 2 r_1$. If $X$ is connected, then $B(x_0, r_0) \subset B(x_1, r_1)$ with $r_0 > 2 r_1$ is possible only when $B(x_0, r_0) = B(x_1, r_1) = X$.

### 3. Local doubling and Poincaré inequalities

Our aim in this paper is to study noncomplete spaces $X$ and primarily to do so using their completion $\hat{X}$. The completion is taken with respect to the metric $d$, whose extension to $\hat{X}$ is also denoted $d$. The measure $\mu$ is extended so that $\mu(\hat{X} \setminus X) = 0$ and so that

$$\hat{\mathcal{M}} = \{ E \subset \hat{X} : E \cap X \in \mathcal{M} \}$$

is the $\sigma$-algebra of measurable sets on $\hat{X}$, where $\mathcal{M}$ is the $\sigma$-algebra of measurable sets on $X$. 
Lemma 3.1. \(\mu\) is a complete Borel regular measure on \(\hat{X}\). Moreover, if \(B\) and \(\hat{B}\) are the Borel \(\sigma\)-algebras on \(X\) and \(\hat{X}\), respectively, then
\[
B = \{E \cap X : E \in \hat{B}\}.
\] (3.1)

Proof. We start by proving (3.1). As \(\hat{B}\) is a \(\sigma\)-algebra it follows directly that \(B' := \{E \cap X : E \in \hat{B}\}\) is a \(\sigma\)-algebra, and since it contains all open sets on \(X\) it must contain \(B\). Conversely, \(\{E \subset \hat{X} : E \cap X \in B\}\) is a \(\sigma\)-algebra which contains all open subsets of \(\hat{X}\) and hence \(\hat{B}\), from which it follows that \(B' \subset B\). Thus (3.1) holds.

Since \(E \subset \hat{X}\) has zero outer measure if and only if \(E \cap X\) has zero measure, it follows that \(\mu\) is a complete Borel regular measure on \(\hat{X}\) with the \(\sigma\)-algebra \(\hat{M}\). \(\Box\)

Recall from the introduction that for an open set \(\Omega \subset X\),
\[
\Omega^\wedge = \hat{X} \setminus \overline{\hat{X} \setminus \Omega},
\]
with the closure taken in \(\hat{X}\), is the largest open set in \(\hat{X}\) such that \(\Omega = \Omega^\wedge \cap X\). Note that \(X^\wedge = \hat{X}\). We denote balls with respect to \(\hat{X}\) by \(\hat{B}\) or \(\hat{B}(x, r) = \{y \in \hat{X} : d(x, y) < r\}\), and balls with respect to \(X\) by \(B\), as before. Note that we do not assume any general connection between \(B\) and \(\hat{B}\), and in particular they may have different centres and radii. The inclusion \(\hat{B}(x, r) \subset B(x, r)^\wedge\) can be strict, but the difference of the two sets is always of measure zero.

Much of analysis on metric spaces has been done assuming global doubling and global Poincaré inequalities. Here, we study properties that hold under (semi)local assumptions.

Definition 3.2. We say that \(\mu\) is locally doubling (on \(X\)) if for every \(x_0 \in X\) there is \(r_0 > 0\) (depending on \(x_0\)) such that \(\mu\) is doubling within \(B(x_0, r_0)\) in the sense of Definition 1.1.

If \(\mu\) is doubling within every ball \(B(x_0, r_0)\) then it is semilocally doubling (on \(X\)), and if moreover the doubling constant within \(B(x_0, r_0)\) is independent of \(x_0\) and \(r_0\), then \(\mu\) is globally doubling (on \(X\)).

See Heinonen [20] for more on doubling measures. If \(\mu\) is locally doubling on \(X\) and \(\Omega \subset X\) is open, then \(\mu\) is also locally doubling on \(\Omega\). A similar restriction property fails for semilocal and global doubling, see [6, Example 4.3].

Proposition 3.3. The measure \(\mu\) on \(X\) is doubling within \(B(x_0, r_0)\) in the sense of Definition 1.1 if and only if its zero extension to \(\hat{X}\) is doubling within \(\hat{B}(x_0, r_0)\), with the same doubling constant \(C_0\).

For a corresponding result with global assumptions see Aikawa–Shanmugalingam [1, Proposition 7.1] and Heinonen–Koskela–Shanmugalingam–Tyson [22, Lemma 8.2.3].

Proof. The sufficiency follows directly from the fact that \(\mu(B(x, r)) = \mu(\hat{B}(x, r))\) for all \(x \in X\) and \(r > 0\).

For the necessity, let \(\hat{B}(\hat{x}, r) \subset \hat{B}(x_0, r_0)\) and \(0 < \varepsilon < \frac{r}{2}\) be arbitrary. Find \(x_\varepsilon \in X\) such that \(d(x_\varepsilon, \hat{x}) < \varepsilon\). Then
\[
\mu(\hat{B}(\hat{x}, 2r - 3\varepsilon)) \leq \mu(B(x_\varepsilon, 2(r - \varepsilon))) \leq C_0\mu(B(x_\varepsilon, r - \varepsilon)) \leq C_0\mu(\hat{B}(\hat{x}, r)),
\]
since \(B(x_\varepsilon, r - \varepsilon) \subset B(x_0, r_0)\). Letting \(\varepsilon \to 0\) in the left-hand side shows that \(\mu(\hat{B}(\hat{x}, 2r)) \leq C_0\mu(\hat{B}(\hat{x}, r))\). \(\square\)

Corollary 3.4. The measure \(\mu\) is semilocally doubling on \(X\) if and only if it is semilocally doubling on \(\hat{X}\).
Definition 3.5. Let $1 \leq q < \infty$. We say that the $(q,p)$-Poincaré inequality holds within $B(x_0, r_0)$ if there are constants $C > 0$ and $\lambda \geq 1$ (depending on $x_0$ and $r_0$) such that for all balls $B \subset B(x_0, r_0)$, all integrable functions $u$ on $\lambda B$, and all upper gradients $g$ of $u$,
\[
\left( \frac{1}{|B|} \int_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C r_B \left( \frac{1}{|\lambda B|} \int_{\lambda B} g^p \, d\mu \right)^{1/p}.
\]  
(3.2)

We also say that $X$ (or $\mu$) supports a local $(q,p)$-Poincaré inequality (on $X$) if for every $x_0 \in X$ there is $r_0$ (depending on $x_0$) such that the $(q,p)$-Poincaré inequality holds within $B(x_0, r_0)$.

If the $(q,p)$-Poincaré inequality holds within every ball $B(x_0, r_0)$ then $X$ supports a semilocal $(q,p)$-Poincaré inequality, and if moreover $C$ and $\lambda$ are independent of $x_0$ and $r_0$, then $X$ supports a global $(q,p)$-Poincaré inequality.

If $q = 1$ we usually just write $p$-Poincaré inequality.

The Poincaré inequality (3.2) can equivalently be required to hold for all measurable $u$ on $\lambda B$ and all $p$-weak upper gradients $g$ of $u$, where the left-hand side is interpreted as $\infty$ if $u_B$ is not defined. This follows from the proof of Proposition 4.13 in [3]. However, the use of the dominated convergence at the end of that proof should perhaps be explained more carefully by replacing the last inequality therein by
\[
\infty \leq \left( \frac{1}{|B|} \int_B |u - u_B|^q \, d\mu \right)^{1/q} = \lim_{j \to \infty} \left( \frac{1}{|B|} \min\{j, |u - u_B|^q\} \, d\mu \right)^{1/q}
= \lim_{j \to \infty} \lim_{k \to \infty} \left( \frac{1}{|B|} \min\{j, |u_k - (u_k)_B|^q\} \, d\mu \right)^{1/q} \leq C \text{diam}(B) \left( \frac{1}{|\lambda B|} \int_{\lambda B} g^p \, d\mu \right)^{1/p}.
\]

Alternatively Fatou’s lemma can be used.

As in the case of the doubling condition, local Poincaré inequalities are inherited by open subsets, i.e. if $\Omega \subset X$ is open and $X$ supports a local $(q,p)$-Poincaré inequality, then so does $\Omega$. This fails for semilocal and global Poincaré inequalities, see [6, Example 4.3].

Proposition 3.6. If $X$ supports a $(q,p)$-Poincaré inequality within $B(x_0, r_0)$ in the sense of Definition 3.5, with constants $C_{P1}$ and $\lambda$. Then $\hat{X}$ supports a $(q,p)$-Poincaré inequality within $B(\hat{x}, r_0)$, with the same constants.

Proof. Let $\hat{B} = \hat{B}(\hat{x}, r) \subset B(x_0, r_0)$ and $0 < \varepsilon < \frac{r}{4}$ be arbitrary. Let $u$ be integrable on $\lambda \hat{B}$ and let $\lambda g$ be an upper gradient of $u$ with respect to $\hat{X}$. Then $\lambda g|_{\hat{X}}$ is an upper gradient of $u$ also with respect to $X$. By the proof of Proposition 4.13 in [3], we can assume that $u$ is bounded. Find $x_\varepsilon \in X$ such that $d(x_\varepsilon, \hat{x}) < \varepsilon$ and let $\hat{B}_\varepsilon := B(x_\varepsilon, r - \varepsilon)$. Then
\[
\hat{B}(\hat{x}, r - 2\varepsilon) \cap X \subset B_\varepsilon \subset \hat{B} \quad \text{and} \quad \hat{B}(\hat{x}, \lambda(r - 2\varepsilon)) \cap X \subset \lambda B \subset \lambda \hat{B},
\]
which implies that
\[
\mu(B_\varepsilon) \to \mu(\hat{B}), \quad \mu(\lambda B) \to \mu(\lambda \hat{B}) \quad \text{and} \quad u_{B_\varepsilon} \to u_{\hat{B}}, \quad \text{as} \ \varepsilon \to 0.
\]
Since $B_\varepsilon \subset B(x_0, r_0)$, the $(q,p)$-Poincaré inequality on $X$ implies that
\[
\left( \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} |u - u_{B_\varepsilon}|^q \, d\mu \right)^{1/q} \leq C_{P1} r \left( \frac{1}{|\lambda B_\varepsilon|} \int_{\lambda B_\varepsilon} \lambda g|_{\hat{X}}|^p \, d\mu \right)^{1/p} \leq C_{P1} r \left( \frac{\mu(\lambda \hat{B})}{\mu(\lambda B_\varepsilon)} \int_{\lambda B} \lambda g^p \, d\mu \right)^{1/p}
\]
and letting $\varepsilon \to 0$ concludes the proof, by dominated convergence. □
Corollary 3.7. If $X$ supports a semilocal $(q,p)$-Poincaré inequality, then so does $\hat{X}$.

There is no equivalence in Proposition 3.6 or Corollary 3.7, as is easily seen by considering $X = \mathbb{R} \setminus Q$. For corresponding results with global assumptions see Aikawa–Shanmugalingam [1, Proposition 7.1] and Heinonen–Koskela–Shanmugalingam–Tyson [22, Lemma 8.2.3].

Note that, in spite of Propositions 3.3 and 3.6, neither local doubling nor local Poincaré inequalities extend to $\hat{X}$. Indeed, the Lebesgue measure on any open set $X \subset \mathbb{R}^n$ is locally doubling and supports a local 1-Poincaré inequality, whereas for the completion $\hat{X} \subset \mathbb{R}^n$ these properties hold only in special cases. (A typical example where the local doubling property fails is the closed outer cusp of exponential type, while Poincaré inequalities usually fail on disconnected (or essentially disconnected) sets, such as the bow-tie in [3, Example A.23]. See also [6, Example 4.3].)

In fact, for the completion $\hat{X}$, the local and semilocal properties are essentially equivalent. Indeed, this follows from the following proposition.

Proposition 3.8. ([6, Proposition 1.2 and Theorem 4.4]) If $X$ is proper then $\mu$ is locally doubling if and only if it is semilocally doubling.

If $X$ is, in addition, connected then also the local and semilocal $(q,p)$-Poincaré inequalities are equivalent.

The space $X$ is proper if all closed and bounded sets are compact. Properness always implies completeness, and the following special case of [6, Proposition 3.4] shows that the converse holds if $\mu$ is semilocally doubling. It is also shown therein that the constant $\frac{2}{3}$ is sharp.

Proposition 3.9. If $\mu$ is doubling within $B(x_0,r_0)$ in the sense of Definition 1.1 then $B(x_0,\delta r_0)$ is totally bounded for every $\delta < \frac{2}{3}$.

In particular, if $\mu$ is semilocally doubling then $X$ is proper if and only if it is complete.

Thus, under semilocal doubling, $\hat{X}$ is always proper and a local $(q,p)$-Poincaré inequality on $\hat{X}$ implies a semilocal one, whenever $\hat{X}$ is connected.

4. Extensions of Newtonian functions to $\hat{X}$

Recall that from now on it is required that $p > 1$.

If a function $u : \hat{X} \to \mathbb{R}$ has a $(p)$ upper gradient $g$ on $\hat{X}$, then clearly $g|_X$ is a $(p)$ upper gradient of $u|_X$. The converse is not true in general, as seen e.g. in $X = \mathbb{R} \setminus Q \subset \mathbb{R} = \hat{X}$, but we will prove the following extension result.

Theorem 4.1. Assume that the doubling property and the p-Poincaré inequality hold within the ball $B_0$ in the sense of Definitions 1.1 and 3.5. Let $\Omega \subset B_0$ be open and $u \in D^p(\Omega)$.

Then there is $\tilde{u} \in D^p(\Omega^\wedge)$ such that $\tilde{u} = u \, C_\mu^X$-q.e. in $\Omega$ and the minimal $p$-weak upper gradient $g_\mu$ of $\tilde{u}$ with respect to $\hat{X}$ satisfies

$$g_\mu \leq A_0 g_u \quad \text{a.e. in } \Omega,$$

(4.1)

where $A_0$ is a constant depending only on $p$, the doubling constant and both constants in the p-Poincaré inequality within $B_0$.

If $\mu$ is semilocally doubling and supports a semilocal p-Poincaré inequality, then the conclusion of the theorem holds for all bounded open $\Omega \subset X$. Under global assumptions, the conclusion holds also for unbounded $\Omega$ and $A_0$ depends only on $p$, the global doubling constant and both constants in the global p-Poincaré inequality.
Moreover, if $\Omega$ is $p$-path open in $\tilde{X}$ then we can, in the above conclusions, take $\hat{u} = u$ in $\Omega$ and $g_u = g_u$ a.e. in $\Omega$. When $\mu$ is semilocally doubling and supports a semilocal $p$-Poincaré inequality, the extension result holds also for unbounded open $\Omega \subset X$, which are $p$-path open in $\tilde{X}$.

A set $\Omega \subset \tilde{X}$ is $p$-path open in $\tilde{X}$ if for $p$-almost every curve $\gamma : [0, l_\gamma] \to \tilde{X}$, the set $\gamma^{-1}(\Omega)$ is relatively open in $[0, l_\gamma]$. By Shanmugalingam [32, Remark 3.5], $\Omega$ is $p$-path open in $\tilde{X}$ if it is quasiopen in $\tilde{X}$; see also Björn–Björn–Malý [9] for the converse implication under certain assumptions. (The set $\Omega$ is quasiopen in $\tilde{X}$ if for every $\varepsilon > 0$ there is an open set $G \subset \tilde{X}$ such that $C_p^X(G) < \varepsilon$ and $\Omega \cup G$ is open.) Note that if $\mu$ is locally doubling, then $X$ (and thus $\Omega$) is open in $\tilde{X}$ if and only if it is locally compact.

For locally compact $X$ with global assumptions, the extension result $\hat{u} = u$ in $X$ with $g_u = g_u$ a.e. in $X$ was for $u \in N^{1, p}(X)$ proved in Lemma 8.2.3 in Heinonen–Koskela–Shanmugalingam–Tyson [22]. A similar result in Aikawa–Shanmugalingam [1, Proposition 7.1] relies (via Cheeger [13, Theorems 6.1 and 17.1]) on Cheeger’s results, which assume that $X$ is complete.

**Remark 4.2.** By Proposition 4.8 below, $\hat{u}$ in Theorem 4.1 may be defined by

$$\hat{u}(x) = \limsup_{r \to 0} \int_{B(x, r) \cap \Omega} u \, d\mu, \quad x \in \Omega^c.$$  

The simple example $X = \mathbb{R} \setminus \{0\}$ with $u = \chi_{(0, \infty)}$ shows that the requirement $\Omega \subset B_0$ in Theorem 4.1 cannot be omitted. It also demonstrates that in general, under local assumptions, functions in $D_p^\Omega(\Omega)$ may fail to have extensions even to $D_p^{loc}(\Omega^c)$. A partial remedy for this situation is provided by Lemma 4.6 below for functions from $D_p^{loc}(X)$.

The following example shows that it is essential to require a Poincaré inequality on $X$ in Theorem 4.1.

**Example 4.3.** Let $X = \mathbb{R} \setminus \mathbb{Q}$ equipped with the Lebesgue measure $\mu$, which is globally doubling on $X$. As $X$ is totally disconnected, $g_u = 0$ a.e. for every $u \in L^p(X)$ and hence $N^{1, p}(X) = L^p(X)$. Thus no Poincaré inequality is supported on $X$, and there is no extension result to $\tilde{X}$ similar to Theorem 4.1.

A natural question is whether the constant $A_0$ in Theorem 4.1 can be chosen equal to one when $\Omega$ is not $p$-path open in $\tilde{X}$. Example 4.3 shows that it can happen that $g_u \chi = 0 < g_u \tilde{X}$ a.e., but such $X$ does not support any Poincaré inequality, even though $\mu$ is globally doubling on $X$. On the other hand, the usage of Proposition 3.5 from Björn–Björn [5] at the end of the proof of Theorem 4.1 shows that $A_0 = 1$ also when $\Omega$ is only $p$-path almost open in $\tilde{X}$, i.e. when for $p$-almost every curve $\gamma : [0, l_\gamma] \to \tilde{X}$, the set $\gamma^{-1}(\Omega)$ is a union of a relatively open set in $[0, l_\gamma]$ and a set of $1$-dimensional Hausdorff measure zero.

Note, however, that this relaxed assumption is not enough to guarantee that $\hat{u}$ can be chosen equal to $u$ everywhere in $\Omega$. This is because in $p$-path almost open $\Omega$, it can happen that there are much fewer zero sets for the capacity $C_p^\tilde{X}$ than for the smaller capacity $C_p^\Omega$. For example, every $U \subset \mathbb{R}$ with zero $1$-dimensional Hausdorff measure is $p$-path almost open in $\mathbb{R}$ but, as it is totally disconnected, we see that $C_p^\Omega$ is trivial while $C_p^{\mathbb{R}}(U)$ can be positive.

**Open problem 4.4.** Under the assumptions in Theorem 4.1 (and without assuming that $\Omega$ is $p$-path almost open in $\tilde{X}$) can it happen that it is not true that $g_u = g_u$ a.e.?
Proof of Theorem 4.1. In this proof, $C$ will denote various constants which only depend on the constants in the local assumptions, and which may change even within the same line. Assume to start with that $u$ is bounded. Let $\varepsilon_k$ be a sequence decreasing to 0 as $k \to \infty$. Lemmas 5.1 and 5.2 in Heikkinen–Koskela–Tuominen [19] (or a standard Whitney type construction) provide us, for each $k$, with a cover $\{B_{ik}\}_i$ of $\Omega$ by balls $B_{ik}$ of radii $r_{ik} \leq \varepsilon_k$ and a subordinate Lipschitz partition of unity $\{\varphi_{ik}\}_i$ so that

- $10\lambda B_{ik} \subset \Omega$ for all $i$ and $k$;
- each $10\lambda B_{ik}$ meets at most $M$ balls $10\lambda B_{jk}$, and in that case $r_{jk} \leq 2r_{ik}$;
- each $\varphi_{ik}$ is $C/r_{ik}$-Lipschitz and vanishes outside $2B_{ik}$;
- $\sum_i \varphi_{ik} = 1$ in $\Omega$.

Here $\lambda$ denotes the dilation constant in the local $p$-Poincaré inequality within $B_0$. Lemma 5.3 in [19] and its proof (note that $B_{jk} \subset 10B_{ik}$ whenever $2B_{jk} \cap 2B_{ik} \neq \emptyset$) then show that the functions

$$ u_k := \sum_i u_{B_{ik}} \varphi_{ik} $$

satisfy $u_k \to u$ in $L^p(\Omega)$, as $k \to \infty$, and moreover

$$ |u_k(x) - u_k(y)| \leq \frac{Cd(x, y)}{r_{ik}} \int_{10B_{ik}} |u - u_{10B_{ik}}| \, d\mu \leq C d(x, y) \left( \int_{10B_{ik}} g_u^p \, d\mu \right)^{1/p} $$

for all $x, y \in B_{ik}$.

By passing to a subsequence, we can in addition assume that $u_k \to u$ a.e. in $\Omega$.

Strictly speaking, $u_k$ are to start with only defined on $\Omega$, but the functions $\varphi_{ik}$, being Lipschitz, extend uniquely to $\Omega^\lambda$ and thus, so do $u_k$. Call these extensions $\hat{u}_k$. Then (4.2) holds for $\hat{u}_k$ and all $x, y \in B_{ik}$ as well. Let

$$ \text{Lip} \hat{u}_k(x) := \limsup_{r \to 0} \sup_{y \in B(x, r)} \frac{\hat{u}_k(y) - \hat{u}_k(x)}{r} $$

be the upper pointwise dilation of $\hat{u}_k$ (also called the local upper Lipschitz constant).

It follows from (4.2) that the minimal $p$-weak upper gradient $g_{\hat{u}_k}$ (with respect to $\hat{X}$) satisfies

$$ g_{\hat{u}_k}(x) \leq \text{Lip} \hat{u}_k(x) \leq C \left( \int_{10\lambda B(x)} g_u^p \, d\mu \right)^{1/p} \text{ for a.e. } x \in B_{ik}^\lambda, \tag{4.3} $$

see Proposition 1.14 in [3]. Since $\mu(\hat{X} \setminus X) = 0$ and the balls $10\lambda B_{ik}$ have bounded overlap, this implies that

$$ \int_{\Omega^\lambda} g_{\hat{u}_k}^p \, d\mu \leq C \sum_i \int_{B_{ik}} \left( \int_{10\lambda B_{ik}} g_u^p \, d\mu \right) \, d\mu \leq C \int_{\Omega} g_u^p \, d\mu. $$

We can therefore conclude from Lemma 6.2 in [3] that there is a subsequence of $\hat{u}_k$ (also denoted $\hat{u}_k$) converging weakly in $L^p(\Omega^\lambda)$ to some $\hat{u} \in L^p(\Omega^\lambda)$ and such that $g_{\hat{u}_k} \to g$ weakly in $L^p(\Omega^\lambda)$, where $g \in L^p(\Omega^\lambda)$ is a $p$-weak upper gradient (with respect to $\hat{X}$) of $\hat{u}$. Moreover, $\hat{u} \in N^1,p(\Omega^\lambda)$ and $\hat{u} = u$ a.e. in $\Omega$. Hence also $\hat{u} = u \in C^\lambda_\rho \text{-a.e. in } \Omega$, since $u, \hat{u} \in N^1,p(\Omega)$.

The Lebesgue differentiation theorem holds in $B_0$, cf. [6, Theorem 3.9]. Let $x \in \Omega$ be a Lebesgue point of both $g$ and $g_u$. Then for each $\varepsilon > 0$ there exists $\rho_0 > 0$ such that for every $B = B(x, \rho)$ with $0 < \rho < \rho_0$,

$$ |g(x) - g_B| < \varepsilon \quad \text{and} \quad |g_u^p(x) - (g_u^p)_B| < \varepsilon. $$
We then have by (4.3) and the weak convergence of \( g_{u_k} \) that
\[
g(x) - \varepsilon < \liminf_{k \to \infty} \frac{C}{\mu(B)} \left( \sum_{B_i \cap B \neq \emptyset} \int_{10\lambda B_i} g_{u_k}^p \, dm \right)^{1/p} \leq C \liminf_{k \to \infty} \left( \int_{B(x, \rho + 20\lambda B_{ik})} g_{u_k}^p \, dm \right)^{1/p} \leq C(g_u(x) + \varepsilon).
\]

Letting \( \varepsilon \to 0 \) proves the first part of the theorem for bounded \( u \) and \( \Omega \). For unbounded \( u \), use the truncations \( \min\{k, \max\{u, -k\}\} \) of \( u \) at \( \pm k \).

If \( \Omega \) is unbounded and \( \mu \) is globally doubling and supporting a global \( p \)-Poincaré inequality, then we apply the above arguments to the sets \( \Omega_k = \Omega \cap B(x_0, k) \). More precisely, by the above we can find \( \hat{u}_1 \in D^p(\Omega_k^1) \) such that \( \hat{u}_1 = u \) \( C^X_p \)-q.e. on \( \Omega_1 \). We can also find \( \hat{u}_2 \in D^p(\Omega_k^2) \) such that \( \hat{u}_2 = u \) \( C^X_p \)-q.e. in \( \Omega_2 \). As the set \( \{ y \in \Omega_k^1 : \hat{u}_1(y) \neq \hat{u}_2(y) \} \) has measure zero, it must be of zero \( C^X_p \)-capacity, and we are thus free to choose \( \hat{u}_2 = \hat{u}_1 \) in \( \Omega_k^1 \). Proceeding in this way, we can construct \( \hat{u} \in D^p(\Omega^\infty) \) so that \( \hat{u} = u \) \( C^X_p \)-q.e. on \( X \). Moreover, \( g_u \leq A g_u \) a.e. in \( \Omega \), where \( A \) only depends on \( p \), the global doubling constant and the constants in the global \( p \)-Poincaré inequality.

If \( \Omega \) is \( p \)-path open in \( \bar{X} \) then the capacities \( C^{\Omega}_p \) and \( C^X_p \) have the same zero sets in \( \Omega \), by Proposition 4.2 in Björn–Björn–Malý [9]. By Lemma 2.24 in [3] the zero sets are also the same for \( C^{\Omega}_p \) and \( C^X_p \) for sets in \( \Omega \). This shows that we may choose \( \hat{u} = u \) in \( \Omega \). That the minimal \( p \)-weak upper gradients with respect to \( X \) and \( \hat{X} \) are equal follows from Proposition 3.5 in Björn–Björn [5]. In this case, the argument above for unbounded \( \hat{\Omega} \) also holds under semilocal assumptions, since \( A = 1 \).

The extension Theorem 4.1 makes it possible to obtain several qualitative results about Newtonian functions on noncomplete spaces under local assumptions. For this, it is even enough that the functions belong to the local spaces. The following two lemmas will therefore be useful.

**Lemma 4.5.** If \( X \) supports a local \((p, p)\)-Poincaré inequality (or if \( \mu \) is locally doubling and supports a local \( p \)-Poincaré inequality) then \( N^{1,p}_{\text{loc}}(X) = D^p_{\text{loc}}(X) \).

*Proof.* By Theorem 5.1 in [6] we can assume that \( X \) supports a local \((p, p)\)-Poincaré inequality, from which the result now follows as in the proof of [3, Proposition 4.14].

**Lemma 4.6.** Assume that \( \mu \) is locally doubling and supports a local \( p \)-Poincaré inequality. Then for every \( u \in N^{1,p}_{\text{loc}}(X) \) there is an open set \( \hat{G} \supset X \) in \( \hat{X} \) and
a function \( \hat{u} \in N_{\text{loc}}^{1,p}(\hat{G}) \) such that \( u = \hat{u} \) \( C_p^X \)-q.e. on \( X \). Moreover, \( \hat{G} \) is locally compact and \( \mu|_{\hat{G}} \) is locally doubling and supports a local \( p \)-Poincaré inequality.

If \( X \) is \( p \)-path open in \( \hat{X} \), then one can choose \( \hat{u} = u \) and \( g_\hat{u} = g_u \) everywhere in \( X \).

**Proof.** For each \( x \in X \) we can find a ball \( B(x, r_x) \) such that the \( p \)-Poincaré inequality and the doubling property for \( \mu \) hold within \( B(x, r_x) \), and such that \( u \in N^{1,p}(B(x, r_x)) \). As \( X \) is Lindelöf, we can find a countable cover \( \{B_j\}_{j=1}^\infty \) of \( X \), where \( B_j = B(x_j, r_{x_j}) \). Let \( \hat{B}_j = \hat{B}(x_j, r_{x_j}) \) and \( \hat{G} = \bigcup_{j=1}^\infty \hat{B}_j \).

By Theorem 4.1, we can find \( \hat{u}_1 \in N^{1,p}(\hat{B}_1) \) such that \( \hat{u}_1 = u \) \( C_p^X \)-q.e. on \( B_1 \). We can also find \( \hat{u}_2 \in N^{1,p}(\hat{B}_1 \cup \hat{B}_2) \) such that \( \hat{u}_2 = u \) \( C_p^X \)-q.e. on \( B_1 \cup B_2 \). As the set \( \{y \in \hat{B}_1 : \hat{u}_1(y) \neq \hat{u}_2(y)\} \) has measure zero, it must be of zero \( C_p^X \)-capacity, and thus we are free to choose \( \hat{u}_2 = \hat{u}_1 \) on \( \hat{B}_1 \). Proceeding in this way, we can construct \( \hat{u} \in N_{\text{loc}}^{1,p}(\hat{G}) \) so that \( \hat{u} = u \) \( C_p^X \)-q.e. on \( X \). Note that, by construction,

\[
\int_{\hat{B}_j} g_\hat{u}^p \, d\mu \leq A_j \int_{\hat{B}_j} g_u^p \, d\mu, \tag{4.4}
\]

where \( A_j \) is the constant provided by Theorem 4.1 on \( B_j \). If \( X \) is \( p \)-path open in \( \hat{X} \), then it follows from the last part of Theorem 4.1 that we can choose \( \hat{u} = u \) everywhere in \( X \) and \( A_j \equiv 1 \), i.e. \( g_\hat{u} = g_u \).

The local doubling property and the local \( p \)-Poincaré inequality for \( \mu|_{\hat{G}} \) follow from Propositions 3.3 and 3.6. Consequently, each \( \hat{B}_j \) (and thus also \( \hat{G} \)) is locally compact, by Proposition 3.9.

As the local assumptions are inherited by open subsets of \( X \), Lemma 4.6 can be directly applied to them as well. Note that the set \( \hat{G} \) depends on \( u \). The following example shows that this drawback cannot be avoided.

**Example 4.7.** Let \( B \) be a ball in \( \mathbb{R}^n \) and \( Z = \{z_j\}_{j=1}^\infty \) a dense subset of \( B \). Set \( X = B \setminus Z \), equipped with the Lebesgue measure. Note that \( \hat{X} = \overline{B} \).

If \( p < n \) then \( C_p(Z) = 0 \) and hence there are \( p \)-almost no curves in \( \mathbb{R}^n \) passing through \( Z \). It follows that \( p \)-weak upper gradients with respect to \( X \) and \( \mathbb{R}^n \) are the same for every measurable function \( u : X \to \overline{\mathbb{R}} \) (extended arbitrarily on \( Z \)). Thus, \( X \) supports a global \( p \)-Poincaré inequality (and is, of course, globally doubling).

Now, for each \( j = 1, 2, \ldots, \), the function \( u_j(x) = |x - z_j|^\alpha, \alpha \in \mathbb{R} \), belongs to \( N_{\text{loc}}^{1,p}(X) \). However, for \( \alpha \leq 1 - n/p \) it can only extend to a function in \( N_{\text{loc}}^{1,p}(\hat{X} \setminus \{z_j\}) \), not in \( N_{\text{loc}}^{1,p}(\hat{X}) \). This shows that the set \( \hat{G} \) in Lemma 4.6 indeed must depend on \( u \).

The following two results are now relatively easy consequences of the above extensions to \( \hat{X} \) and the corresponding results in complete spaces from [6].

**Proposition 4.8.** Assume that \( \mu \) is locally doubling and supports a local \( p \)-Poincaré inequality. Then every \( u \in N_{\text{loc}}^{1,p}(X) \) has \( L^p \)-Lebesgue points \( C_p^X \)-q.e., and moreover the extension \( \hat{u} \) in Lemma 4.6 can be given by

\[
\hat{u}(x) = \limsup_{r \to 0} \frac{1}{\mu(B(x, r) \cap X)} \int_{B(x, r) \cap X} u \, d\mu, \quad x \in \hat{G}. \tag{4.5}
\]

See Remark 7.2 in [6] for further discussion on \( L^q \)-Lebesgue points for \( q > p \). Note that the proof below shows that the limit

\[
\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu
\]

actually exists for \( C_p^X \)-q.e. \( x \in X \), even though it only equals \( u(x) \) for \( C_p^X \)-q.e. \( x \).
Proof. Find \( \tilde{G} \) and \( \tilde{u} \) as in Lemma 4.6. It then follows from [6, Theorem 7.1] that
\( \tilde{u} \) has \( L^p \)-Lebesgue points \( C_p^X \)-q.e. in \( \tilde{G} \). As \( u = \tilde{u} \) \( C_p^X \)-q.e. in \( X \), we conclude that
\( u \) has \( L^p \)-Lebesgue points \( C_p^X \)-q.e. in \( X \).

Finally, if \( \tilde{u} \) is given by \( (4.5) \), then \( \tilde{u} = \tilde{u} \) at all \( L^1 \)-Lebesgue points of \( \tilde{u} \), i.e. \( C_p^X \)-q.e. in \( \tilde{G} \). Hence, \( \tilde{u} \) may also be chosen so that it satisfies \( (4.5) \).

Proof of Theorem 1.2. This result follows directly from Theorem 4.1 and Proposition 4.8.

Proposition 4.9. Assume that \( \mu \) is locally doubling and supports a local \( p \)-Poincaré inequality, and that \( X \) is \( p \)-path open in \( \tilde{X} \). Then every \( u \in N_{\text{loc}}^{1,p}(X) \) is quasicontinuous.

A function \( u \) is quasicontinuous on \( X \) if for every \( \varepsilon > 0 \) there is an open \( G \subset X \) such that \( C_p^X(G) < \varepsilon \) and \( u|_{X\setminus G} \) is real-valued and continuous.

Quasicontinuity has earlier been established for Newtonian functions under various assumptions in Björn–Björn–Shanmugalingam [11], Heinonen–Koskela–Shanmugalingam–Tyson [22], Björn–Björn–Lehrbäck [8] and in [6] for open sets in locally compact spaces. Existence of quasicontinuous representatives under global assumptions was obtained already in Shanmugalingam [33]. Assuming completeness and global assumptions, quasicontinuity can be proved also on quasiopen sets, see Björn–Björn–Latvala [7] and Björn–Björn–Malý [9].

Proof. Find \( \tilde{G} \) and \( \tilde{u} \) as in Lemma 4.6, with \( \tilde{u} = u \) in \( X \). It then follows from [6, Theorem 9.1] that \( \tilde{u} \) is quasicontinuous on \( \tilde{G} \), which immediately yields that \( u \) is quasicontinuous on \( X \), since \( C_p^X \) is dominated by \( C_p^X \).

As a direct consequence of Proposition 4.9 we can also conclude from [3, Theorem 5.31] that \( C_p^X \) is an outer (and Choquet) capacity on \( X \). Moreover, by [6, Theorem 8.4 and Proposition 9.3], if \( K \subset X \) is compact, then
\[ C_p^X(K) = \inf \|u\|_{N_{\text{loc}}^{1,p}(X)}, \]
where the infimum is taken over all locally Lipschitz \( u \) such that \( u \geq 1 \) on \( K \).

Remark 4.10. Even for \( u \in N^{1,p}(X) \), Lemma 4.6 only guarantees an extension in the local Newtonian space \( N_{\text{loc}}^{1,p}(\tilde{G}) \). The set \( \tilde{G} \) can, however, be chosen independently of \( u \), since the covering balls can be chosen so, when \( u \in N^{1,p}(X) \). In general, we do not know if it is possible to find an extension in \( N^{1,p}(\tilde{G}) \), since we lack a uniform control of the constant \( A_0 \) in Theorem 4.1, and thus in (4.4). However, this can be achieved in the following situations (which can also be combined on different parts of \( X \)):
(a) One can find a finite cover by balls \( B_j \) as in the proof of Lemma 4.6.
(b) Each ball \( B_j \) is \( p \)-path almost open in \( \tilde{X} \), which guarantees that \( A_0 \equiv 1 \).
(c) \( \mu \) is both locally doubling and supports a local \( p \)-Poincaré inequality with uniform constants independent of \( x_0 \) and \( r_0 \), which guarantees that \( A_0 \) is uniformly bounded.

As a matter of fact, as discussed just before Open problem 4.4, it is not known if the constant \( A_0 \) in Theorem 4.1 ever needs to be larger than 1.

5. Self-improvement of Poincaré inequalities

A deep result due to Keith–Zhong [24, Theorem 1.0.1] shows that the Poincaré inequality is an open-ended property. See also Heinonen–Koskela–Shanmugalingam–Tyson [22, Theorem 12.3.9] and Eriksson–Bique [15]. By localizing the arguments
in [22], the following local version of the self-improvement result was obtained in [6, Theorem 5.3].

**Theorem 5.1.** Let $B_0 = B(x_0, r_0)$ be a ball such that $\overline{B}_0$ is compact and the $p$-Poincaré inequality and the doubling property for $\mu$ hold within $B_0$ in the sense of Definitions 1.1 and 3.5.  

Then there exist constants $C$, $\lambda$ and $q < p$, depending only on $p$, the doubling constant and both constants in the $p$-Poincaré inequality within $B_0$, such that for all balls $B$ with $3\lambda B \subset B_0$, all integrable functions $u$ on $\lambda B$, and all $q$-weak upper gradients $g$ of $u$,

$$
\int_B |u - u_B| \, d\mu \leq C r_B \left( \int_{\lambda B} g^q \, d\mu \right)^{1/q}.
$$

(5.1)

Theorem 5.1 relatively easily leads to a local self-improvement under local assumptions. With a little more work it also yields a semilocal conclusion, cf. [6, Theorem 5.4].

In Heinonen–Koskela–Shanmugalingam–Tyson [22, Theorem 12.3.10] it is explained how (under global assumptions) the properness of $X$ in Keith–Zhong [24] can be relaxed to local compactness, with somewhat weaker global conclusions (namely that the weak upper gradients considered therein are required to be $L^p$-integrable). In fact, using extension Theorem 4.1, even local compactness can be disposed of, as we shall now see.

**Theorem 5.2.** Let $B_0 = B(x_0, r_0)$ be a ball such that the $p$-Poincaré inequality holds within $B_0$, while the doubling property for $\mu$ holds within $\tau B_0$ for some $\tau > \frac{3}{2}$, in the sense of Definitions 1.1 and 3.5.

Then there exist constants $C$, $\lambda$ and $q < p$, depending only on $p$, the doubling constant and both constants in the $p$-Poincaré inequality within $B_0$, such that for all balls $B$ with $3\lambda B \subset B_0$, all integrable functions $u \in D^p(\lambda B)$ and all $p$-weak upper gradients $g \in L^p(\lambda B)$ of $u$,

$$
\int_B |u - u_B| \, d\mu \leq C r_B \left( \int_{\lambda B} g^q \, d\mu \right)^{1/q}.
$$

(5.2)

If $B_0$ is, in addition, $p$-path almost open in $\widehat{X}$, which in particular holds if $X$ is locally compact, then (5.2) holds for all $q$-weak upper gradients $g$ of $u \in D^p(\lambda B)$ in $\lambda B$.

Note that, since $u$ is assumed to have an $L^p$-integrable upper gradient, the latter part of this result does not show that $X$ supports a (semi)local $q$-Poincaré inequality. Neither does [22, Proposition 12.3.10] imply that $X$ supports a global $q$-Poincaré inequality. Koskela [29] has given counterexamples showing that this cannot be concluded without completeness.

The proof shows that if it is known that $B_0$ is totally bounded, then it is enough to require doubling only within $B_0$.

**Proof.** Proposition 3.9 implies that the ball $B_0$ is totally bounded and hence the $\widehat{X}$-closure of $\overline{B}_0 := \overline{B}(x_0, r_0)$ is compact. Propositions 3.3 and 3.6 imply that the doubling property and the $p$-Poincaré inequality hold within $\overline{B}_0$, with the same constants.

It then follows from Theorem 5.1 that there exist constants $C$, $\lambda$ and $q < p$ such that the following variant of (5.1) holds for all balls $\widehat{B}$ with $3\lambda \widehat{B} \subset \overline{B}_0$, all integrable functions $\hat{u}$ on $\lambda \widehat{B}$ and all $q$-weak upper gradients $\hat{g}$ of $\hat{u}$ with respect to $\lambda \widehat{B}$,

$$
\int_{\widehat{B}} |\hat{u} - \hat{u}_{\widehat{B}}| \, d\mu \leq C r_{\widehat{B}} \left( \int_{\lambda \widehat{B}} \hat{g}^q \, d\mu \right)^{1/q}.
$$

(5.3)
Now, let \( B = B(x,r) \) with \( 3\lambda B \subset B_0 \) be arbitrary and set \( \tilde{B} = \tilde{B}(x,r) \). Using Theorem 4.1, we can for every \( u \in D^p(\lambda B) \) find \( \tilde{u} \in D^p(\tilde{\lambda} \tilde{B}) \), which is an extension of a representative of \( u \) and such that the minimal \( p \)-weak upper gradients of \( g_u \) and \( g_{\tilde{u}} \) of \( \tilde{u} \) and \( u \) (with respect to \( \tilde{\lambda} \tilde{B} \) and \( \lambda B \), respectively) satisfy \( g_{\tilde{u}} \leq A_0 g_u \) a.e. in \( \lambda B \), where the constant \( A_0 \) depends only on \( p \) and the doubling and Poincaré constants within \( B_0 \). Since \( g_u \) and \( g_{\tilde{u}} \) are also \( q \)-weak upper gradients (by [3, Proposition 2.45]), we conclude from (5.3) that

\[
\frac{1}{B} |u - u_B| \, d\mu = \frac{1}{\tilde{B}} |\tilde{u} - \tilde{u}_{\tilde{B}}| \, d\mu \leq C\alpha r \left( \frac{1}{\lambda B} g_u^q \, d\mu \right)^{1/q} \leq C\alpha r \left( \frac{1}{\lambda B} g_{\tilde{u}}^q \, d\mu \right)^{1/q},
\]

whenever \( q \in L^p(\lambda B) \) is a \( p \)-weak upper gradient of \( u \) (although not necessarily for upper gradients \( q \in L^q(\lambda B) \), since they need not extend to \( \lambda \tilde{B} \)). This proves (5.2).

For the last part, assume that \( B_0 \) is, in addition, \( p \)-path almost open in \( \tilde{X} \), and that \( q \) is a \( q \)-weak upper gradient of \( u \) in \( \lambda B \) such that the right-hand side in (5.2) is finite. Then \( q \geq g_{u,\lambda B,q} \) a.e., where \( g_{u,\lambda B,q} \) is the minimal \( q \)-weak upper gradient of \( u \) in \( \lambda B \). Since \( B_0 \) is \( p \)-path almost open in \( \tilde{X} \), it is easily verified that \( \lambda B \) is \( p \)-path almost open in \( \tilde{X} \), and hence also \( q \)-path almost open, by [3, Proposition 2.45]. Proposition 3.5 in Björn–Björn [5] then shows that

\[
g_{u,\lambda B,q} = g_{\tilde{u},\lambda \tilde{B},q} \quad \text{a.e. in } \lambda B.
\]

Thus, by (5.3) again,

\[
\frac{1}{B} |u - u_B| \, d\mu = \frac{1}{\tilde{B}} |\tilde{u} - \tilde{u}_{\tilde{B}}| \, d\mu \leq C\alpha r \left( \frac{1}{\lambda B} g_{\tilde{u}}^q \, d\mu \right)^{1/q} \leq C\alpha r \left( \frac{1}{\lambda B} g_{\tilde{u}}^q \, d\mu \right)^{1/q}.
\]

\[\square\]

**Corollary 5.3.** If \( \mu \) is locally doubling and supports a local \( p \)-Poincaré inequality, then for every \( x_0 \in X \) there is a ball \( B'_0 \ni x_0 \), together with constants \( C, \lambda \) and \( q < p \), such that (5.2) holds for all balls \( B' \subset B'_0 \) (not just for \( 3\lambda B \subset B'_0 \)), all integrable functions \( u \in D^p(\lambda B) \) and all \( p \)-weak upper gradients \( q \in L^p(\lambda B) \) of \( u \).

If the assumptions about doubling and \( p \)-Poincaré inequality are semilocal, then the conclusion of the theorem is also semilocal, i.e. it holds for all balls \( B'_0 \subset X \). Under global assumptions, the constants \( C, \lambda \) and \( q \) are independent of \( B_0 \), i.e. the conclusion is global.

If \( X \) is, in addition, \( p \)-path almost open in \( \tilde{X} \), which in particular holds if \( X \) is locally compact, then (5.2) holds for all \( q \)-weak upper gradients \( g \) of \( u \in D^p(\lambda B) \) in \( \lambda B \).

**Proof.** Let \( x_0 \in X \) be arbitrary and find \( r_0 > 0 \) so that the assumptions of Theorem 5.1 hold for \( B_0 = B(x_0,r_0) \). Then choose a radius \( 0 < r'_0 \leq (7\lambda)^{-1} r_0 \) so that \( B'_{r'_0} := B(x_0,r'_0) \neq X \) and \( \text{dist}(x_0,X \setminus B'_{r'_0}) = r'_0 \). For \( B \subset B'_{r'_0} \) it then follows that \( \tau_B \leq 2r'_0 \) and hence \( 3\lambda B \subset B(x_0,r_0) \). The first statement then follows from Theorem 5.2.

Since \( \lambda \) in Theorem 5.2 depends on \( B_0 \), we cannot directly obtain a semilocal conclusion (under semilocal assumptions) from it. However, under global assumptions, the constants \( C, \lambda, q < p \) and \( A_0 \) will be independent of \( B_0 \), which yields the global result.

To reach a semilocal conclusion under semilocal assumptions, we instead note that \( \tilde{X} \) is proper and connected, by Proposition 3.9 and the proof of [3, Proposition 4.2]. Theorem 5.4 in [6] then implies that for every ball \( B'_0 = B(x_0,r'_0) \subset X \)
there exist constants $C$, $\lambda$ and $q < p$, such that (5.3) holds for all balls $\tilde{B} \subset \tilde{B}(x_0, r'_0)$, all integrable functions $\tilde{u}$ on $\lambda \tilde{B}$ and all $q$-weak upper gradients $\tilde{g}$ of $\tilde{u}$ with respect to $\lambda \tilde{B}$. By enlarging $r'_0$ if necessary, we may assume that $\text{dist}(x_0, X \setminus B'_0) = r'_0$. (If $B'_0 = X$, we instead note that $X$ is bounded and thus semilocal assumptions are the same as global assumptions, which were handled above.) If now $B \subset B'_0$ is a ball then $r'_B \leq 2r'_0$ and hence $\lambda B \subset (1 + 2\lambda)B'_0 =: B_0$. Theorem 4.1, applied to $B_0$ and followed by (5.4), then yields (5.2).

The last part about $q$-weak upper gradients in the $p$-path almost open case follows as in the (last part of the) proof of Theorem 5.2. \hfill $\square$

### 6. $p$-harmonic functions in noncomplete spaces

In this section we conclude the paper with a discussion on possible directions for developing the theory of $p$-harmonic functions and quasiminimizers on noncomplete spaces.

Let $\Omega \subset X$ be open throughout this section. Traditionally, e.g. in $\mathbb{R}^n$ and other complete spaces, $p$-harmonic functions on $\Omega$ are required to belong to the local space $N^{1,p}_{\text{loc}}(\Omega)$ and their $p$-harmonicity is tested by sufficiently smooth (e.g. Lipschitz or Sobolev) functions $\varphi$ with compact support in $\Omega$ (or with zero boundary values) as follows:

$$
\int_{\varphi \neq 0} \hat{g}^p \, d\mu \leq \int_{\varphi \neq 0} \hat{g}^{p+q} \, d\mu. \tag{6.1}
$$

For practical applications it can then often be shown that the $p$-harmonicity can equivalently be tested by other classes of test functions as well. Let us have a closer look at these spaces. In Section 2, we defined $N^{1,p}_{\text{loc}}(\Omega)$ as the space of all functions $u$ such that

for every $x \in \Omega$ there exists a ball $B_x \ni x$ such that $u \in N^{1,p}(B_x)$.

It is an easy exercise to see that if $\Omega$ is locally compact then this definition is equivalent to the requirement that

$$
\forall G \text{ open \ inside } \Omega, \quad u \in N^{1,p}(G) \text{ for all open sets } G \text{ such that } \overline{G} \text{ is a compact subset of } \Omega.
$$

Note that $\Omega$, being an open subset of $X$, is always locally compact if $X$ is proper. Also recall that, by Proposition 3.9, if $\mu$ is semilocally doubling then $X$ is proper if and only if it is complete.

In noncomplete spaces, defining $N^{1,p}_{\text{loc}}(\Omega)$ through compact subsets of $\Omega$ might not be so useful, since there may be no (or very few) nonempty open sets with compact closures. The same applies to the definitions of the space of test functions in (6.1). We therefore consider the following families of bounded open subsets of $\Omega$:

$$
\mathcal{G}_{\text{cpt}} = \{ \text{bounded open } G \subset \Omega : \overline{G} \subset \Omega \text{ is compact} \},
$$

$$
\mathcal{G}_{\text{dist}} = \{ \text{bounded open } G \subset \Omega : \text{dist}(G, X \setminus \Omega) > 0 \},
$$

$$
\mathcal{G}_{\text{bdy}} = \{ \text{bounded open } G \subset \Omega : \text{dist}(G, \partial \Omega) > 0 \},
$$

$$
\mathcal{G}_{\text{clus}} = \{ \text{bounded open } G \subset \Omega : \overline{G} \subset \Omega \}.
$$

(Here we consider $\text{dist}(G, \emptyset) > 0$.)

It is easily verified that

$$
\mathcal{G}_{\text{cpt}} \subset \mathcal{G}_{\text{dist}} \subset \mathcal{G}_{\text{bdy}} \subset \mathcal{G}_{\text{clus}}.
$$

Hence, if the local Newtonian space $N^{1,p}_{\text{loc},x}(\Omega)$, with $x \in \{ \text{cpt, dist, bdy, clus} \}$, is defined by

$$
N^{1,p}_{\text{loc},x}(\Omega) = \{ u : \Omega \to \mathbb{R} : u \in N^{1,p}(G) \text{ for all } G \in \mathcal{G}_x \},
$$

then $\mathcal{G}_{\text{clus}}$ is semilocal for the space $N^{1,p}_{\text{loc},x}(\Omega)$ and followed by (6.1), then yields (5.2).
then
\[ N^{1,p}_{\text{loc,cl}}(\Omega) \subset N^{1,p}_{\text{loc,bdy}}(\Omega) \subset N^{1,p}_{\text{loc,dist}}(\Omega) \subset N^{1,p}_{\text{loc}}(\Omega) \subset N^{1,p}_{\text{loc,cpt}}(\Omega), \]
where the last two inclusions follow from the fact that every ball \( B(x, r_x) \) with \( x \in \Omega \) and \( r_x \) small enough belongs to \( G_{\text{dist}} \) and that every compact set can be covered by finitely many such balls.

If \( X \) is proper then clearly \( G_{\text{cpt}} = G_{\text{cl}} \) and all the above five local Newtonian spaces coincide, while the last two spaces always coincide if \( X \) is locally compact. Depending on \( X \) and \( \Omega \), some partial equalities are possible also in noncomplete spaces, see Example 6.3.

Now we turn our attention to the spaces of test functions in (6.1) and define:
\[
\begin{align*}
N^{1,p}_0(E) &= \{ \varphi \in N^{1,p}(X) \text{ and } \varphi = 0 \text{ in } X \setminus E \} \quad \text{for } E \subset X, \\
N^{1,p}_{0,\text{cl}}(\Omega) &= \{ \varphi : \Omega \to \overline{\mathbb{R}} : \varphi \in N^{1,p}(G) \text{ for some } G \in G_{\text{cl}} \},
\end{align*}
\]
where the closure is taken in \( N^{1,p}(X) \) and functions in \( N^{1,p}_0(G) \) are regarded as extended by zero outside of \( G \). Alternatively, only the noncomplete spaces
\[
\{ \varphi : \Omega \to \overline{\mathbb{R}} : \varphi \in N^{1,p}_0(G) \text{ for some } G \in G_{\text{cl}} \}
\]
could be considered. Since \( N^{1,p}_0(\Omega) \) is closed in \( N^{1,p}(X) \) (by [3, Theorem 2.36]), we immediately see that
\[
N^{1,p}_{0,\text{cl}}(\Omega) \subset N^{1,p}_{0,\text{dist}}(\Omega) \subset N^{1,p}_{0,\text{bdy}}(\Omega) \subset N^{1,p}_{0,\text{clo}}(\Omega) \subset N^{1,p}_0(\Omega). \tag{6.2}
\]
As before, if \( X \) is proper then the first four spaces of test functions coincide. If, in addition, all functions in \( N^{1,p}(X) \) are quasicontinuous then [3, Lemma 5.43] implies that \( N^{1,p}_0(\Omega) = N^{1,p}_{0,\text{clo}}(\Omega) \), i.e. all the above spaces of test functions coincide. This in particular holds if \( X \) is locally compact and \( \mu \) is locally doubling and supporting a local \( p \)-Poincaré inequality, by [6, Theorem 9.1].

There are also other classes of test functions that one can consider, e.g.
\[
N^{1,p}_c(\Omega) = \{ \varphi \in N^{1,p}(X) : \text{supp } \varphi \text{ is a compact subset of } \Omega \},
\]
i.e. \( N^{1,p}_c(\Omega) \) is defined as \( N^{1,p}_{0,\text{clo}}(\Omega) \) omitting the word “open” in \( G_{\text{c}} \). If \( X \) (or \( \Omega \)) is locally compact, then \( N^{1,p}_c(\Omega) = N^{1,p}_{0,\text{clo}}(\Omega) \), but this is not true in general. On the other hand, since \( X \) is a normal topological space, we have
\[
N^{1,p}_{0,\text{dist}}(\Omega) = \{ \varphi : \Omega \to \overline{\mathbb{R}} : \varphi \in N^{1,p}_0(E) \text{ for some } E \in \mathcal{E}_x \} \quad \text{if } x \in \{ \text{dist, bdy, clos} \},
\]
where \( \mathcal{E}_x \) is defined as \( G_{\text{cl}} \) but omitting the word “open”. Test function classes based on Lipschitz functions similarly to any of the above classes are also possible, see Björn–Marola [12, Section 4]. We will not discuss these classes of test functions further.

Each of the local Newtonian spaces, defined above, can appear in the definition of \( p \)-harmonic functions, together with one of the above spaces of test functions.

**Definition 6.1.** A function \( u \in N^{1,p}_{\text{loc,c}}(\Omega) \) is a **\( Q \)-quasi(super)minimizer** in \( \Omega \) if
\[
\int_{\varphi \neq 0} g_p^u \, d\mu \leq Q \int_{\varphi \neq 0} g_{p+\varphi}^u \, d\mu \tag{6.3}
\]
for all (nonnegative/nonpositive) \( \varphi \in N^{1,p}_0(\Omega) \). If \( Q = 1 \) in (6.3) then \( u \) is a **(super)minimizer**. A **\( p \)-harmonic function** is a continuous minimizer.
Here each of $\mathbf{x}$ and $\mathbf{y}$ stands for one of the above defined subscripts, or the absence of such a subscript in the case of $N_{\text{loc}}^{1,p}$ and $N_0^{1,p}$. Naturally, different choices of $\mathbf{x}$ and $\mathbf{y}$ in Definition 6.1 lead to different classes of quasiminimizers and $p$-harmonic functions, which may have advantages and disadvantages, depending on the situation and the intended applications. In Example 6.3 below we demonstrate some of these differences, but first we show that interior regularity can be obtained for most of these definitions.

The largest class of (quasi)minimizers is obtained when allowing for a large local Newtonian space and by testing with as few test functions as possible. This is reflected in the choice of function spaces in the following regularity result. To cover also non-locally compact spaces, we exclude the definitions involving $G_{\text{cpt}}(\Omega)$, as well as $N_1^{1,p}(\Omega)$. We say that a function $u : \Omega \to \mathbb{R}$ is lsc-regularized if

$$u(x) = \operatorname{ess} \lim_{y \to x} u(y) \quad \text{for all } x \in \Omega.$$ 

**Theorem 6.2.** Assume that $\mu$ is locally doubling and supports a local $p$-Poincaré inequality in $\Omega$. Let $u \in N_{\text{loc}}^{1,p}(\Omega)$ be a $Q$-quasi(super)minimizer in $\Omega$, tested by $\varphi \in N_{0,\text{dist}}^{1,p}(\Omega)$. Then $u$ has a representative $\tilde{u}$ which is continuous (resp. lsc-regularized).

Moreover, the strong minimum principle holds for $\tilde{u}$: if $\Omega$ is connected and $\tilde{u}$ attains its minimum in $\Omega$ then it must be constant.

A bit surprising, perhaps, is that the weak minimum principle, which compares infima on sets and their boundaries, does not follow, see Example 6.3 below.

Note that the local assumptions on $\mu$ in Theorem 6.2 are required only in $\Omega$, but the ambient space $X$ plays an implicit role in the definition of quasi(super)-minimizers through the range of test functions $\varphi \in N_{0,\text{dist}}^{1,p}(\Omega) \subset N^{1,p}(X)$. Under global assumptions, (Hölder) continuity of (quasi)minimizers has been deduced on metric spaces in Kinnunen–Shanmugalingam [28], Kinnunen–Martio [26, 27] and Björn–Marola [12]. In [26] and [27] completeness was assumed but not used for these results, although it certainly influenced their formulation of the definition of quasi(super)minimizers (using $G_{\text{cpt}}$ in our notation).

In addition to having stronger assumptions on $X$, these papers also use more restrictive definitions of $N_{\text{loc}}^{1,p}(\Omega)$ and/or a larger class of test functions than here. The local space $N_{\text{loc}}^{1,p}(\Omega)$ in [26] and [27] coincides with our $N_{\text{loc,cpt}}^{1,p}(\Omega)$ and their test functions belong to $N_{0,\text{dist}}^{1,p}(\Omega)$ (which equals $N_0^{1,p}(\Omega)$ because of the assumed completeness), while [12] uses $N_{\text{loc,cpt}}^{1,p}(\Omega)$ and $N_{0,\text{dist}}^{1,p}(\Omega)$. In [28], the test functions belong to $N_{0,\text{loc}}^{1,p}(\Omega)$, while the definition of $N_{\text{loc}}^{1,p}(\Omega)$ is through bounded subsets and imposes integrability conditions also near the boundary $\partial \Omega$, so that it coincides with $N^{1,p}(\Omega)$ for bounded $\Omega$.

The smallest test space $N_{0,\text{loc}}^{1,p}(\Omega)$ is used in [26] and [27], as well as in Holopainen–Shanmugalingam [23] (in locally compact spaces). Such a definition guarantees that $p$-harmonicity in each $\Omega_j$ for an increasing exhaustion $\Omega_1 \subset \Omega_2 \subset ...$ implies $p$-harmonicity in $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. This need not be true with the other classes of test functions, as seen in Example 6.3 below. At the same time, in general spaces there may be no nonempty open sets with compact closures.

**Proof of Theorem 6.2.** For each $x \in \Omega$ there are a ball $B_x$ and $\lambda_x$ such that $u \in N^{1,p}(B_x)$ and the doubling property and the $p$-Poincaré inequality hold within $2B_x \subset \Omega$ with dilation constant $\lambda_x$. As $X$ is Lindelöf we can find countably many balls $\{B_j\}_{j=1}^{\infty}$ such that $B_j = (50\lambda_x)^{-1}B_{2j-1}$ and $\Omega \subset \bigcup_{j=1}^{\infty} B_j$.

To deduce continuity, we need to obtain suitable weak Harnack inequalities for $u$ within each ball $B_j$, in the sense that they hold with fixed constants (depending
on \( j \) for each ball \( B \subset B_j \). The arguments for proving such weak Harnack inequalities in Kinnunen–Shanmugalingam [28] and Kinnunen–Martio [27, Section 5] are all local (and do not use completeness), so local assumptions are enough for them. They do rely on a better \( q \)-Poincaré inequality for some \( q < p \) but it is only applied to \( L^p \)-integrable \( p \)-weak upper gradients and thus inequality (5.2) provided by Theorem 5.2 is sufficient.

If \( u \) is a quasiminimizer then it is a standard procedure using these weak Harnack inequalities to deduce continuity for a representative of \( u \) in \( \Omega \), see [28] for the details. This even gives local Hölder continuity, but without uniform control of the Hölder exponent, since it locally depends on the constants within each \( B_j \), and the \( B_j \) in turn depend both on \( \Omega \) and \( u \).

If \( u \) is a quasisuperminimizer then also the Lebesgue point result provided by Proposition 4.8 is needed. The lac-regularity for a representative of \( H \) is sufficient.

Example 6.3. Let \( X = \mathbb{R}^2 \setminus (\{(-1, \infty) \times \{0\}\}) \) be the slit plane, i.e. \( \mathbb{R}^2 \) with a ray removed. Equip \( X \) with the Euclidean metric and the Lebesgue measure. Note that \( X \) is locally compact and \( \mu \) is globally doubling and supports a local 1-Poincaré inequality. Also let \( \Omega = (-1, 1) \times (0, 2) \subset X \).

Since \( \text{dist}(G, \Omega) = \text{dist}(G, \mathbb{R}^2 \setminus \Omega) \) for every open \( G \subset \Omega \), it is easily seen that \( \mathcal{G}_{\text{cpt}}(\Omega) = \mathcal{G}_{\text{dist}}(\Omega) \). On the other hand,

\[
G = \left(-\frac{1}{2}, \frac{1}{2} \right) \times (0, 1) \in \mathcal{G}_{\text{bdy}}(\Omega) \setminus \mathcal{G}_{\text{dist}}(\Omega)
\]

shows that \( \mathcal{G}_{\text{dist}}(\Omega) \subset \mathcal{G}_{\text{bdy}}(\Omega) \). Similarly, the closure of

\[
H = \{(x_1, x_2) \in \Omega : |x_1| + x_2 < 1\},
\]

taken with respect to \( X \), satisfies \( \overline{H} \subset \Omega \) and hence \( H \in \mathcal{G}_{\text{clos}}(\Omega) \setminus \mathcal{G}_{\text{bdy}}(\Omega) \). We thus conclude that

\[
\mathcal{G}_{\text{cpt}}(\Omega) = \mathcal{G}_{\text{dist}}(\Omega) \subset \mathcal{G}_{\text{bdy}}(\Omega) \subset \mathcal{G}_{\text{clos}}(\Omega), \tag{6.4}
\]

which immediately implies that

\[
\mathcal{N}^{1,p}_{\text{loc}, \text{dist}}(\Omega) = \mathcal{N}^{1,p}_{\text{loc}}(\Omega) = \mathcal{N}^{1,p}_{\text{loc}, \text{cpt}}(\Omega).
\]

On the other hand, the functions \(|x - (1, 1)|^{-1}, |x - (1, 0)|^{-1}\) and \(|x|^{-1}\) show that

\[
\mathcal{N}^{1,p}(\Omega) \subset \mathcal{N}^{1,p}_{\text{loc},\text{clos}}(\Omega) \subset \mathcal{N}^{1,p}_{\text{loc},\text{bdy}}(\Omega) \subset \mathcal{N}^{1,p}_{\text{loc},\text{dist}}(\Omega).
\]

For the zero spaces, it follows from (6.4) that

\[
\mathcal{N}^{1,p}(\Omega) = \mathcal{N}^{1,p}_{0, \text{cpt}}(\Omega) = \mathcal{N}^{1,p}_{0, \text{dist}}(\Omega).
\]

At the opposite end of the chain (6.2) of zero spaces, it follows from [6, Theorem 9.1] that all functions in \( \mathcal{N}^{1,p}(X) \) are quasicontinuous and hence [3, Lemma 5.43] implies that

\[
\mathcal{N}^{1,p}_{0, \text{clos}}(\Omega) = \mathcal{N}^{1,p}_{0}(\Omega).
\]

Furthermore, by regarding \( \Omega \) as a subset of \( \mathbb{R}^2 \), we can conclude that every function in \( \mathcal{N}^{1,p}_{0, \text{dist}}(\Omega) \) extends by zero to a function in \( \mathcal{N}^{1,p}(\mathbb{R}^2) \) and hence has boundary values 0 q.e. on \([-1, 1] \times \{0\}\). Since it is easily verified that \( \text{dist}(\cdot, \partial \Omega) \in \mathcal{N}^{1,p}_{0, \text{bdy}}(\Omega) \), this implies that

\[
\mathcal{N}^{1,p}_{0, \text{dist}}(\Omega) \subset \mathcal{N}^{1,p}_{0, \text{bdy}}(\Omega).
\]
To investigate the remaining inclusion $N_{0,\text{bdy}}^{1,p}(\Omega) \subset N_{0,\text{clo}}^{1,p}(\Omega)$, assume that $\varphi \in N_{0}^{1,p}(G)$ for some $G \in \mathcal{G}_{\text{clo}}(\Omega)$ with $\text{dist}(G, \partial \Omega) = 0$. The only limit points (with respect to $\mathbb{R}^2$) that $G$ and $\partial \Omega$ can share, are $z_\pm = (\pm 1, 0)$.

Next, we distinguish between $p \leq 2$ and $p > 2$. Since singletons in $\mathbb{R}^2$ have zero $p$-capacity, when $p \leq 2$, there exist Lipschitz cut-off functions $\eta_j$ supported in $B(z_+, 1/j) \cup B(z_-, 1/j)$ such that

$$\eta_j = 1 \text{ in } B(z_+, 1/j) \cup B(z_-, 1/j) \quad \text{and} \quad \|\eta_j\|_{N^{1,p}(\mathbb{R}^2)} \to 0 \text{ as } j \to \infty.$$ 

The functions $(1 - \eta_j)\varphi$ then belong to $N_{0,\text{bdy}}^{1,p}(\Omega)$ and approximate any bounded $\varphi$ in the $N^{1,p}(X)$-norm. As unbounded functions can be approximated by their truncations, this shows that

$$N_{0,\text{bdy}}^{1,p}(\Omega) = N_{0,\text{clo}}^{1,p}(\Omega) \text{ for } p \leq 2.$$

For $p > 2$, we proceed as follows. Since $\mu$ is globally doubling and supports a global 1-Poincaré inequality on the upper half-plane, Theorem 4.1 implies that $\varphi$ extends to $\hat{\varphi} \in N^{1,p}(\mathbb{R} \times [0, \infty))$, with the same norm and minimal $p$-weak upper gradient. Moreover, $\hat{\varphi}$ is continuous and $\hat{\varphi}(z_\pm) = 0$. This implies that the functions $(\varphi - 1/j)_+ \in N_{0,\text{bdy}}^{1,p}(\Omega)$ approximate $\varphi_+$ in the $N^{1,p}(X)$-norm and thus

$$N_{0,\text{bdy}}^{1,p}(\Omega) = N_{0,\text{clo}}^{1,p}(\Omega) \text{ also for } p > 2.$$

By varying both the local Newtonian space for (quasi)minimizers and the class of test functions in (6.3) one obtains different definitions. Let us have a closer look at some extreme cases:

1. Assume that $u \in N_{0,\text{loc}}^{1,p}(\Omega)$ and test with $\varphi \in N_{0,\text{Mir}}^{1,p}(\Omega) = N_{0,\text{dist}}^{1,p}(\Omega)$. This gives the most general definition and the largest class of (quasi)minimizers. Since $\Omega$ is open in $\mathbb{R}^n$, we have $N_{0,\text{loc}}^{1,p}(\Omega) = W_{0,\text{loc}}^{1,p}(\Omega)$, the usual local Sobolev space on Euclidean domains. It follows that this definition provides us with the usual $p$-harmonic functions and quasiminimizers on the Euclidean domain $\Omega \subset \mathbb{R}^n$.

However, since the boundary $\partial \Omega$ with respect to $X$ does not include the segment $[-1, 1] \times \{0\}$, it is easily verified that uniqueness is lost in the Dirichlet problem for $p$-harmonic functions when the boundary data are only prescribed on $\partial \Omega$, e.g. by requiring that $u - f \in N_{0}^{1,p}(\Omega)$. A remedy of this problem is achieved by a larger class of test functions below.

Furthermore, the weak maximum principle is violated, as well as certain (weak) Harnack inequalities with respect to balls in $\Omega \subset X$. To see this, consider e.g. the usual fundamental solution

$$u(x) = \begin{cases} |x|^{(p-2)/(p-1)}, & p \neq 2, \\ -\log |x|, & p = 2, \end{cases} \quad (6.5)$$

for the $p$-Laplacian $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$.

This suggests that testing only with $\varphi \in N_{0,\text{clo}}^{1,p}(\Omega) = N_{0,\text{dist}}^{1,p}(\Omega)$ and allowing (quasi)minimizers to belong to $N_{0,\text{clo}}^{1,p}(\Omega)$ may be too generous and that $N_{0,\text{clo}}^{1,p}(\Omega)$ or $N_{0,\text{clo}}^{1,p}(\Omega)$, together with larger classes of test functions, might be better choices. On the other hand, the strong maximum principle, stating that a nonconstant $p$-harmonic function cannot attain its maximum in $\Omega$, as well as (weak) Harnack inequalities with respect to compact subsets of $\Omega$ remain true even in this situation.

2. The space $N_{0,\text{clo}}^{1,p}(\Omega) = N_{0,\text{clo}}^{1,p}(\Omega) = N_{0,\text{clo}}^{1,p}(\Omega)$ allows for test functions which need not vanish on the real axis. This indirectly forces the (quasi)minimizers on $\Omega \subset X$ to have zero Neumann boundary values on $(-1, 1) \times \{0\}$, i.e. on the “missing”
boundary segment. This fails e.g. for the linear function \((x_1, x_2) \mapsto x_2\), which is thus not \(p\)-harmonic with this more restrictive definition.

As mentioned above, the zero Neumann condition on the “missing” boundary may restore uniqueness in the Dirichlet problem (with respect to \(X\)). Note, however, that for \(p \leq 2\) the fundamental solution (6.5) still satisfies (6.3) with \(Q = 1\) for all test functions \(\varphi \in N_{0,\text{bdy}}(\Omega) = N_{0,\text{clo}}(\Omega) = N_{0}^{1,p}(\Omega)\). (Indeed, as singletons have zero \(p\)-capacity, test functions in \(N_{0,\text{bdy}}^{1,p}(\Omega)\) can be approximated therein by test functions vanishing near the singularity \((0,0)\).)

A rather general existence and uniqueness result for the Dirichlet problem for \(p\)-harmonic functions with Sobolev boundary values was given by Björn–Björn [5, Theorem 4.2], which covers the case considered here. There the functions are required to belong to \(D^p(\Omega)\) and the test function space is \(N_{0}^{1,p}(\Omega)\).

3. For \(p \leq 2\), the fundamental solution \(u\) in (6.5) would be excluded (and the uniqueness in the Dirichlet problem restored) if \(p\)-harmonic functions were required to belong to \(N_{\text{loc, clo}}^{1,p}(\Omega)\) or \(N_{\text{loc, bdy}}^{1,p}(\Omega)\). However, the translated fundamental solutions satisfy
\[
u(x - (0, 1)) \in N_{\text{loc, bdy}}^{1,p}(\Omega) \setminus N_{\text{loc, clo}}^{1,p}(\Omega) \quad \text{and} \quad \nu(x - (1, 1)) \in N_{\text{loc, clo}}^{1,p}(\Omega) \setminus N_{\text{bdy}}^{1,p}(\Omega)
\]
and both can be tested by \(\varphi \in N_{0}^{1,p}(\Omega)\).

For \(p > 2\), the fundamental solution (6.5) belongs to \(N^{1,p}(\Omega)\), but testing (6.3) with
\[
\varphi = (1 - u)_+ \in N_{0,\text{bdy}}^{1,p}(\Omega) = N_{0,\text{clo}}^{1,p}(\Omega) = N_{0}^{1,p}(\Omega)
\]
shows that it is not \(p\)-harmonic in \(\Omega \subset X\) with such a definition. It is, however, a subminimizer with this class of test functions.

The above observations concerning the fundamental solutions hold also for the power functions \(x \mapsto |x|^\alpha\) with \(\alpha < 1 - 2/p < 0\) and \(\alpha > 1 - 2/p > 0\), which are quasiminimizers in \(\mathbb{R}^2 \setminus \{0\}\) for \(p < 2\) and \(p > 2\), respectively, in view of Björn–Björn [4, Theorems 5.1 and 6.1].

We have thus seen that the different possible definitions have various pros and cons, and that the “correct” definition depends on the particular applications or results one has in mind. For example, suitable choices of spaces of test functions in noncomplete spaces also make it possible to treat certain mixed boundary value problems within the scope of Dirichlet problems.

A seemingly simple way of treating (quasi)minimizers on noncomplete spaces might be to use the completion \(\hat{X}\) of \(X\) together with our main extension theorem (Theorem 4.1): Starting with a quasiminimizer \(u\) on some open subset \(\Omega\) of \(X\) one would like to extend (a representative of) \(u\) to a function \(\hat{u}\) on some open subset \(\hat{\Omega}\) of \(\hat{X}\), which can be achieved using Lemma 4.6. The next step would be to show that \(\hat{u}\) is a quasiminimizer in \(\hat{G}\), and then apply the potential theory for quasiminimizers on \(\hat{X}\).

There are several conditions that need to be fulfilled for such an approach to be fruitful. First of all, in order to have a useful potential theory on \(\hat{X}\), we need to assume that \(\mu\) is locally doubling and supports a local \(p\)-Poincaré inequality on \(\hat{X}\). In view of Corollaries 3.4 and 3.7, it seems that the most natural condition to impose on \(X\) to achieve this is requiring that \(\mu\) is semilocally doubling and supporting a semilocal \(p\)-Poincaré inequality on \(X\), which ensures that these semilocal conditions also hold on \(\hat{X}\). By [6, Theorem 4.4 and the discussion following it], it follows that \(\hat{X}\) is also proper and connected. Hence, most of the nonlinear potential theory on metric spaces is available for \(\hat{X}\), see [6, Section 10].

So assume that \(\mu\) is semilocally doubling and supports a semilocal \(p\)-Poincaré inequality on \(X\). The proof of Lemma 4.6 shows that (a representative of) any
u ∈ N^{1,p}_{\text{loc,x}}(\Omega) extends to a function ˆu ∈ N^{1,p}_{\text{loc,x}}(\Omega^\Lambda) if x ∈ \{\text{dist, bdy, clos}\}. For u ∈ N^{1,p}_{\text{loc}}(\Omega) there is only an extension to some open \(\hat{G} \subset \hat{X}\) which may depend on u, see Example 4.7.

Now, if (4.1) in Theorem 4.1 holds for all such extensions with a uniform \(A_0 \geq 1\) (e.g. if \(X\) is locally compact or \(p\)-path open in \(\hat{X}\), in which case \(A_0 ≡ 1\)) then (6.3) implies that also

\[
\int_{\varphi \neq 0} g_{\varphi}^p d\mu \leq QA_0^p \int_{\varphi \neq 0} g_{\varphi + \varphi}^p d\mu
\]

for all \(\varphi \in N^{1,p}_{0,\varphi}(\hat{G})\) since \(\varphi|_X \in N^{1,p}_{0,\varphi}(\Omega)\). In other words, ˆu is a \(QA_0^p\)-quasi(super)-minimizer in ˆ\(G\), where ˆ\(G = \Omega^\Lambda\) if \(x \notin \{\text{dist, bdy, clos}\}\). (As before we omit the cases when \(x = \text{cpt}\) or \(y = \text{cpt}\).) This, in particular, implies that various local properties, such as the Hölder continuity, (weak) Harnack inequalities and maximum principles, hold for ˆu in ˆ\(G\), and thus also for u in \(\Omega\). When \(A_0 ≡ 1\), even more can be said.

Another point is whether there is a one-to-one correspondence between the (equivalence classes of) \(Q\)-quasiminimizers on \(\Omega\) and on \(\Omega^\Lambda\), when \(A_0 ≡ 1\). From

\[G_{\text{dist}}(\Omega^\Lambda) = \{\hat{G} \subset \Omega^\Lambda : \hat{G} \cap X \in G_{\text{dist}}(\Omega)\}\]

it follows that \(N^{1,p}_{\text{loc, dist}}(\Omega^\Lambda) = N^{1,p}_{\text{loc, dist}}(\Omega)\) and \(N^{1,p}_{\text{loc, dist}}(\Omega^\Lambda) = N^{1,p}_{0,\text{dist}}(\Omega)\) (or more precisely there is a one-to-one correspondence between the equivalence classes in these spaces), which shows that such an equivalence holds if \(x = y = \text{dist}\) (under the assumptions above). On the other hand, this fails for the other families \(G_{\text{cpt}}, G_{\text{bdy}}\) and \(G_{\text{clos}}\), and thus such a correspondence is unlikely to hold in any other case. Example 6.3 gives several counterexamples when applied to \(X_+ = X \cap (R \times [0, \infty))\) on which global assumptions hold.

Using the test function class \(N^{1,p}_{0}(\Omega)\), the Dirichlet and obstacle problems on bounded (not necessarily open) sets in noncomplete spaces with very weak assumptions were studied in Björn–Björn [5]. Functions considered therein belong to \(N^{1,p}(\Omega)\), so the different types of local spaces do not play a role in that discussion. The space \(N^{1,p}_{0}(\Omega)\) of test functions therein is large enough to give a sufficiently restrictive definition of \(p\)-harmonic functions which, under rather mild assumptions, guarantees uniqueness in the Dirichlet problem, cf. Parts 1–3 in our Example 6.3.

It was also shown in [5] that in complete spaces (with global assumptions) Dirichlet and obstacle problems are naturally studied on quasiparentheses (or finely open) sets, but not really beyond that. In our setting it could therefore be interesting to know what happens when \(X\) (and thus \(\Omega\)) is quasiparentheses in \(\hat{X}\), which is closely related to its \(p\)-path openness, see the comments after Theorem 4.1.

For a fruitful nonlinear potential theory in noncomplete spaces it may be worth to consider how the fine potential theory on quasiparentheses (and finely open) sets has been developed in \(R^n\) and in metric spaces, and in particular the role of so-called \(p\)-strict subsets, see Kilpeläinen–Malý [25], Latvala [31] and Björn–Björn–Latvala [7].

In connection with the Dirichlet problem it would be interesting to develop a suitable theory for Perron solutions in noncomplete spaces. A major obstacle may however be the comparison principle (as in [3, Theorem 9.39]), since even in the complete case (and with global assumptions) it is not known whether the boundary condition (for bounded functions) can be omitted even at a single point with zero capacity. Perhaps a suitable theory could be developed if one assumes that the boundary is compact, even though the underlying space may be noncomplete.

On the other hand, it would be interesting to study how continuous boundary data should be treated on noncompact boundaries, in which case there are at least three natural counterparts to the usual space of continuous boundary
data: continuous functions, bounded continuous functions and uniformly continuous functions. See Björn [2] for one study, in a very special case, treating Perron solutions for $p$-harmonic functions with a noncompact boundary; and also Estep–Shanmugalingam [16].

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