ULTRAMETRIC LOGARITHM LAWS I

J. S. ATHREYA
Department of Mathematics, Princeton University
Fine Hall, Washington Road Princeton NJ 08544-1000, USA

ANISH GHOSH
Department of Mathematics, The University of Texas at Austin
1 University Station, C1200 Austin, Texas 78712, USA

AMRITANSHU PRASAD
The Institute of Mathematical Sciences
CIT campus, Taramani, Chennai 600 113, India

Abstract. We announce ultrametric analogues of the results of Kleinbock-Margulis for shrinking target properties of semisimple group actions on symmetric spaces. The main applications are S-arithmetic Diophantine approximation results and logarithm laws for buildings, generalizing the work of Hersonsky-Paulin on trees.

1. Introduction. The notion of shrinking targets for dynamical systems is a much studied and extremely useful concept ([1], [4] and the references therein) especially for geometric and number theoretic applications. Shrinking target properties for group actions on homogeneous spaces were studied in an important paper [17] of D. Kleinbock and G. A. Margulis. We first describe their results. Let $G$ denote a connected semisimple Lie group without compact factors, and $\Gamma$ denote a non-uniform lattice in $G$. Let $K$ be a maximal compact subgroup of $G$, and let $Y = K\backslash G/\Gamma$ denote the associated non-compact irreducible locally symmetric space of finite volume. In [17], D. Kleinbock and G. A. Margulis studied the phenomenon of shrinking target properties for the geodesic flow, generalizing an earlier work of D. Sullivan [32] and established the following theorem, commonly called a logarithm law.

For $x \in Y$, let $T^1_x(Y)$ denote the unit tangent space at $x$. Let $\nu$ denote the Haar measure on $T^1_x(Y)$. For $\theta \in T^1_x(Y)$, and $t \in \mathbb{R}$ let $g_t(x, \theta)$ denote the image of $(x, \theta)$ under geodesic flow for time $t$. Let $d_Y$ denote a metric on $Y$, obtained from a bi-$K$ invariant metric $d$ on $G$. 

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Current Address: Mathematics Department, Yale University, PO Box 208283, New Haven, CT 06520-8283

Current Address: School of Mathematics, University of East Anglia, Norwich, NR4 7TJ UK
Theorem 1.1. (Kleinbock-Margulis [17]) There exists a $k = k(dy) > 0$ such that the following holds: for all $x, y \in Y$, and almost every $\theta \in T^1_x(Y)$,

$$\limsup_{t \to \infty} \frac{dy(g_t(x, \theta), y)}{\log t} = 1/k.$$ 

The Theorem thus studies the statistical properties of geodesic excursions into smaller and smaller cuspidal neighborhoods (i.e. the complements of these neighborhoods are larger and larger compacta) of $Y$. In fact, this phenomenon seems to be prevalent in many dynamical systems, and a very general result to the effect was obtained in [17], which can then be used to obtain Theorem 1.1. To describe this result, we need some more definitions, all taken from loc. cit. Let $(X, \mu)$ be a probability space, $B$ be a family of measurable subsets of $X$ and let $F = \{f_t\}$ denote a sequence of $\mu$-preserving transformations of $X$. Then,

Definition 1.2. $B$ is called Borel-Cantelli for $F$ if for every sequence $\{A_t | t \in \mathbb{N}\}$ of sets from $B$, the following holds:

$$\mu(\{x \in X | f_t(x) \in A_t \text{ for infinitely many } t \in \mathbb{N}\}) = \begin{cases} 0 & \text{if } \sum_{t=1}^{\infty} \mu(A_t) \text{ converges}, \\ 1 & \text{if } \sum_{t=1}^{\infty} \mu(A_t) \text{ diverges}. \end{cases}$$

And for $\kappa > 0$, we say that $\delta$ is $\kappa-DL$ (an abbreviation for $\kappa-$distance like) if it is uniformly continuous and

$$\exists C_1, C_2 > 0, \text{ such that } C_1 e^{-\kappa z} \leq \Phi_\delta(z) \leq C_2 e^{-\kappa z} \forall z \in \mathbb{R},$$

and $DL$ if there exists $\kappa > 0$ such that (1.4) holds. The following is then Theorem 1.8 in [17].

Theorem 1.5. Let $G$, $\Gamma$ and $X$ be as above, $F = \{f_t | t \in \mathbb{N}\}$ be a sequence of elements of $G$ satisfying

$$\sup_{t \in \mathbb{N}} \sum_{s=1}^{\infty} e^{-\beta d(f_s f_t^{-1}, e)} < \infty \forall \beta > 0,$$ (1.6)

and let $\delta$ be a $DL$ function on $X$. Then

$$\mathcal{B}(\delta) \overset{def}{=} \{\{x \in X | \delta(x) \geq r\} | r \in \mathbb{R}\}$$ (1.7)

is Borel-Cantelli for $F$.

Sequences which satisfy (1.6) above are referred to as “exponentially divergent” (abbreviated $ED$) in [17]. To derive Theorem 1.1 from Theorem 1.5, the authors use a philosophy of F.Mautner [21] to realize the geodesic flow on the unit tangent bundle of $Y$ as a one-parameter flow on $G/\Gamma$. One then needs to check the $ED$ condition for this flow, as well as the $DL$ condition for the metric on $G/\Gamma$ described above. In this paper and its sequel [2], we will obtain $S$-arithmetic analogues of Theorem 1.5.

To motivate our results, we start with an analogue of Theorem 1.1. Let $\mathbb{F}_s$ denote the finite field of $s = p^k$ elements, $k$ denote the global function field of...
rational functions with coefficients in \( \mathbb{F}_s \) and \( k \) denote the completion of \( k \) at the infinite place, identified naturally with the field of Laurent series \( \mathbb{F}_s((X^{-1})) \) with coefficients in \( \mathbb{F}_s \). On \( \mathbb{F}_s((X^{-1})) \), we will assume the usual norm and ultrametric, (cf.[33]) for details. We denote this norm \( \| \cdot \| \) and when used in the context of \( k^r \), it will refer to the supremum norm. Let \( G \) denote a simple, isotropic, linear algebraic group defined and split over \( \mathbb{F}_s \), \( \Gamma \) a non-uniform lattice in \( G(k)^3 \), and \( K \) denote a parahoric subgroup of \( G(k) \). We fix a \( K \)-invariant metric on \( G(k) \), and hence also on \( G(k)/\Gamma \) and call this metric \( d(\cdot, \cdot) \). When the field in question is evident, we will sometimes refer to \( G(k) \) simply as \( G \).

A natural geometric object on which \( G \) acts is the Bruhat-Tits building \( Y \) of \( G \). This is a Euclidean building which comes with a natural metric, equipped with which it becomes a CAT-0 space. The stabilizers of vertices in \( Y \) are the maximal parahoric subgroups \( K \) of \( G \). Therefore the quotient \( Y/\Gamma \) can be naturally identified with \( K\backslash G/\Gamma \). Let \( \partial Y \) denote the geodesic boundary of \( Y \) and \( \pi : Y \to Y/\Gamma \) be the natural projection. Let \( \{g_t(x, \theta)\}_{t \geq 0} \) denote the geodesic starting at the vertex \( x \in Y \) in direction \( \theta \in \partial Y \). There is a natural measure class on \( \partial Y \). A function field analogue of Theorem 1.1 would therefore be:

**Theorem 1.8.** There is a \( l = l(Y/\Gamma) \) such that for any \( x \in Y \), \( y \in Y/\Gamma \) and almost all \( \theta \in \partial Y \),

\[
\limsup_{t \to \infty} \frac{\log d_{Y/\Gamma}(\pi(g_t(x, \theta)), y)}{\log t} = 1/l,
\]

where \( d_Y \) is the metric on the quotient \( Y/\Gamma \).

**Remark:** This result was obtained by S. Hersonsky and F. Paulin in [14] in the case of quotients of a regular tree \( Y \) by any non-uniform lattice in \( Aut(Y) \). Their approach is more geometric, since \( Aut(Y) \) is a locally compact group but does not have any obvious algebraic structure, and in some sense is in the spirit of Sullivan [32]. A natural question would be to ask if an analogue of Theorem 1.5 holds in a more general algebraic setting. We answer this in the affirmative.

Let \( k \) be a global field, \( S \) a finite set of places of \( k \) (containing the infinite ones, in case \( k \) is a number field), and for each \( s \in S \), let \( k_s \) denote the completion of \( k \) at the place \( s \). \( G \) be a connected, simple, algebraic \( k \)-group without anisotropic factors, set \( G_s = G(k_s) \) and \( G_S = \prod_{s \in S} G_s \). Let \( Y_s \) denote the Bruhat-Tits building of \( G_s \) (or the symmetric space as the case may be) and \( Y_S = \prod_{s \in S} Y_s \). Let \( \Gamma \) be a non-uniform lattice in \( G_S \) and set \( X_S = G_S/\Gamma \). In [2], we prove:

**Theorem 1.9.** With notation as above, let \( F \) be an exponentially divergent sequence for \( G_S \) and \( \delta \) be a DL function on \( X_S \). Then \( B(\delta) \) is Borel-Cantelli for \( F \).

**Remark 1.** The reader will notice that Theorem 1.5 is stated in somewhat greater generality, i.e. in the context of semisimple groups. However, the principal ideas involved are already present in the case of simple groups. In [2] where the proof of the above theorem will be presented in detail, we will point out the extra details required to extend Theorem 1.9 to semisimple groups.

**Remark 2.** As a corollary, we will obtain generalizations of Theorems 1.1 and 1.8 for S-isotropic symmetric spaces.

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3It is well-known \( G \) contains many such lattices.
We now turn to some examples: Let $O_S$ be the ring of $S$-integers of $k$, i.e.

$$O_S = \{ x \in k \mid |x|_s \leq 1 \text{ for every finite place } s \notin S \}. \quad (1.10)$$

Then, by theorems of Borel-Harish-Chandra and Behr-Harder in the function field case (cf. [20]), $\Gamma \equiv \mathbb{G}(O_S)$ is a lattice in $G$. Let $K_s$ denote a maximal parahoric subgroup (maximal compact if $s$ is an infinite place) of $G_s$, and $d_s$ denote the $K_s$-bi-invariant metric on $G_s$. This gives rise to a product metric $d$ on $X_S$. In [2], using reduction theory we show that $d$ is a $DL$ function on $X_S$ for any non-uniform lattice $\Gamma$, thereby establishing Theorem 1.8 by an application of Theorem 1.9.

For an application to number theory, take $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$, so the space

$$X = SL(n, \mathbb{R})/SL(n, \mathbb{Z}) \quad (1.11)$$

can be identified with the space of unimodular lattices in $\mathbb{R}^n$. In [17], Theorem 1.5 and the function

$$\delta(\Lambda) = \inf_{x \in \Lambda \setminus \{0\}} \|x\| \quad (1.12)$$

which turns out to be $DL$, was used to prove Khintchine’s theorem, a cornerstone of metric Diophantine approximation, and stronger variants of the it. Now let $k = \mathbb{Q}$ and following [18], define

$$GL^1(n, \mathbb{Q}_S) \overset{def}{=} \left\{ g = (g_s)_{s \in S} \in GL(n, \mathbb{Q}_S) \mid \prod_{s \in S} \det(g_s)_{s \in S} = 1 \right\} \quad (1.13)$$

Then $GL^1(n, \mathbb{Q}_S)/GL(n, \mathbb{Z}_S)$ is the space of unimodular lattices in $\mathbb{Q}_S$. In [18], a dynamical interpretation of Diophantine approximation, originally introduced by D. Kleinbock and G. Margulis in [16] was developed in the $S$-arithmetic setting by D. Kleinbock and G. Tomanov to prove $S$-arithmetic analogues of Mahler’s conjectures, see also [11] for function field analogues. In [2], we use Theorem 1.9 in this context with specific, analogous choices of $DL$ functions to establish Khintchine’s theorems and multiplicative versions in the number and function field cases. We illustrate with an example. Let $l$ denote the cardinality of $S$, and assume, for simplicity that $S$ contains the infinite valuation $\bar{v}$. Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a non-increasing function, and let $\text{Mat}(\mathbb{Q}_S)$ denote the set of $m \times n$-matrices with entries in $\mathbb{Q}_S$. Let $W(\psi)$ denote the set of $A \in \text{Mat}(\mathbb{Q}_S)$ for which there exist infinitely many $q \in \mathbb{Z}^n$ such that

$$\left(\|p + AQ\|\right)^m \leq \psi(\|q\|_{\bar{v}})$$

for some $p \in \mathbb{Z}^m$. \quad (1.14)

Here the norms $\|x\|$ and $\|x\|_{\bar{v}}$ are defined as follows:

For $x = (x_1^{(v)}, x_2^{(v)}, \ldots, x_m^{(v)}) \in \mathbb{Q}_v^m$, we define

$$\|x\|_v = \max_i |x_i^{(v)}|_v.$$ \quad (1.15)

and finally, for $x \in \mathbb{Q}_S^m$

$$\|x\| = \max_v \|x\|_v.$$ \quad (1.16)

For a very nice motivation of the theory of $S$-arithmetic Diophantine approximation, and especially the correct normalization, we refer the reader to Section 10.

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4 This, and commensurable lattices are called arithmetic.

5 This is just a convenience because the definitions are slightly more involved cf. [18]. This restriction will be removed in [2].
in [18]. The S-arithmetic analogue of the Khintchine-Groshev theorem would then be:

**Theorem 1.17.** Let \( \mathcal{W}(\psi) \) denote the set of \( \psi \)-approximable matrices as above. Then,

1. Almost every \( A \) belongs to \( \mathcal{W}(\psi) \) if \( \int \psi(x) \, dx \) diverges.

2. Almost no \( A \) belongs to \( \mathcal{W}(\psi) \) if \( \int \psi(x) \, dx \) converges.

This theorem will be proved in [2] by adapting the method of Kleinbock-Margulis to this set-up. In addition, we will prove variations and strengthenings of the above theorem, including:

1. The more general multiplicative case, where the norm \( \| \| \) is replaced by a more general function. This will prove S-arithmetic analogues of a conjecture of Skriganov [30].

2. Function field analogues of Theorem 1.17. This is elaborated upon in the next section.

This paper is intended as an announcement of our results above. In particular, we will not present complete proofs, but rather try to convey the main ideas involved in the proof. To do this, we will focus on the case of a single local field of positive characteristic. Namely, we will outline the proofs of Theorems 1.8 and 1.17 in the case where \( k \) is a local field of positive characteristic. We would like to stress that our strategy is substantially similar to that of Kleinbock-Margulis. The primary content of our work is in generalizing their methods to much wider settings. However, in [2], we will also provide a direct, “non-spectral” proof of Theorem 1.8 in the case where \( G \) is a rank 1 group, which has been explained to us by S. Mozes. In this case, we use a symbolic description of the geodesic flow on trees, developed in [22]. We are indebted to him for sharing his insight.

2. **Borel-Cantelli lemmata and applications.** Theorem 1.9, like Theorem 1.5 are probabilistic in nature-in fact they are strongly reminiscent of \( 0 \)-\( 1 \) laws in probability theory, especially the elementary Borel-Cantelli lemma, which we now recall:

**Lemma 2.1.** (Borel-Cantelli) Let \( \{X_n\}_{n=0}^{\infty} \) be a sequence of \( 0 \)-\( 1 \) random variables, with \( P(X_n = 1) =: p_n \). Then

1. If \( \sum_{n=0}^{\infty} p_n < \infty \), then \( P(\sum_{n=0}^{\infty} X_n = \infty) = 0 \)

2. If the \( X_n \)'s are pairwise independent , i.e.

\[
p_{nm} := P(X_n X_m = 1) = p_n p_m \quad \forall \ m, n,
\]

and \( \sum_{n=0}^{\infty} p_n = \infty \), then \( P(\sum_{n=0}^{\infty} X_n = \infty) = 1 \).

The first part of the above Lemma easily allows one to derive the convergence halves of the various \( 0 \)-\( 1 \) laws we have in mind. Unfortunately, it is typically hopeless to expect that the dynamical random variables (or events) we are interested in are independent, i.e. satisfy (2) in the above theorem. In the context of logarithm laws, for instance, one is only able to show that geodesic excursions to shrinking cusp neighborhoods are relatively (sometimes referred to as “quasi”) independent events. That this suffices for applications is due to the following strengthening of the Borel-Cantelli lemma, abstracted from the works of W. Schmidt, by V. Sprindzhuk.
We first set up some of the notation, taken from [17]. For a function \( f \) on a probability space \((X, \mu)\), we will denote \( \mu(f) \stackrel{def}{=} \int_X f \, d\mu \).

and for \( N \in \mathbb{N} \cup \{\infty\} \), a family of functions \( H = \{h_t \mid t \in \mathbb{N}\} \) on \( X \),

\[
S_{H,N} \stackrel{def}{=} \sum_{t=1}^N h_t(x) \quad \text{and} \quad E_{H,N} \stackrel{def}{=} \sum_{t=1}^N \mu(h_t).
\] (2.2)

**Lemma 2.3.** Let \((X, \mu)\) denote a probability space, and let \( H = \{h_t \mid t \in \mathbb{N}\} \) denote a sequence of functions on \( X \) which satisfy:

\[
\mu(h_t) \leq 1 \quad \text{for every} \quad t \in \mathbb{N}.
\] (2.4)

Assume also that there exists \( C > 0 \) such that

\[
\sum_{s,t=M}^N (\mu(h_s h_t) - \mu(h_s) \mu(h_t)) \leq C \sum_{t=M}^N \mu(h_t) \quad \text{for every} \quad N > M \geq 1.
\] (2.5)

Then for every \( \epsilon > 0 \),

\[
S_{H,N} = E_{H,N} + O\left(E_{H,N}^{1/2} \log^{3/2+\epsilon} E_{H,N}\right)
\] (2.6)

for \( \mu \) almost every \( x \in X \). In particular, for \( \mu \) almost every \( x \),

\[
\lim_{N \to \infty} \frac{S_{H,N}(x)}{E_{H,N}} = 1,
\] (2.7)

whenever \( \sum_{t=1}^\infty \mu(h_t) \) diverges.

So, in order to use Sprindzhuk’s lemma to prove Theorem 1.9, we need to show that the “quasi-independence” condition (2.5) holds in the context of our dynamical systems. We start with the Khintchine-Groshev theorem and its generalizations, focussing on the function field case. Our plan is to first describe a dynamical system to which the above Lemma can be applied to derive the desired number theoretic results, and then to use quantitative mixing bounds for the dynamical system to ensure that condition (2.5) is satisfied.

Accordingly, let us take \( G = \text{SL}(n, \mathbb{F}_s((X^{-1}))) \) and \( \Gamma = \text{SL}(n, \mathbb{F}_s[X]) \). Then \( \Gamma \) is a non-compact lattice in \( G \). Let \( X = G/\Gamma \) and \( \mu \) denote the normalized, projected Haar measure on \( X \). Diophantine approximation in the function field setting is naturally analogous\(^6\) to the real case, namely the role of \( \mathbb{R} \) is played by \( \mathbb{F}_s((X^{-1})) \), while that of \( \mathbb{Z} \) is played by the polynomial ring \( \mathbb{F}_s[X] \), and there is an analogous continued fraction decomposition for every element of \( \mathbb{F}_s((X^{-1})) \) (see, for example, [24]). As would be expected, it is possible to read off Diophantine properties of Laurent series from their continued fraction expansions, and in fact one can provide a proof of Khintchine’s theorem in one dimension using a careful study of these continued fractions.

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\(^6\)This is true to some extent. There are important and much studied distinctions, especially in the approximation of algebraic elements. We will not address this.
More generally, the set \( \mathcal{W}(\psi) \) of \( \psi \)-approximable \((m \times n)\) matrices with entries in \( \mathbb{F}_s((X^{-1})) \) is naturally defined to be those \( A \in \text{Mat}_{m,n}(\mathbb{F}_s((X^{-1}))) \) for which there exist infinitely many \( q \in \mathbb{F}_s[X]^n \) such that

\[
\|Aq + p\|^m < \psi(\|q\|^n) \quad \text{for some } p \in \mathbb{F}_s[X]^m.
\]

And the function field analogue of the Khintchine-Groshev Theorem would precisely be Theorem 1.17 with the above definition of \( \mathcal{W}(\psi) \).

An ingenious scheme, due to Dani [8] in a special case, and developed in full generality by Kleinbock-Margulis [17] translates the above problem into one of shrinking target properties for certain homogeneous flows. Let us briefly describe this correspondence. The group \( \text{SL}_{m+n}(\mathbb{F}_s((X^{-1}))) \) acts transitively on the space of unimodular (i.e., co-volume 1) lattices \( \Omega_{m+n} \) of \( \mathbb{F}_s((X^{-1}))^{m+n} \) and the stabilizer of \( \mathbb{F}_s[X]^{m+n} \) is \( \text{SL}_{m+n}(\mathbb{F}_s[X]) \).

The space \( \text{SL}_{m+n}(\mathbb{F}_s((X^{-1}))) \) acts on the space of \( \Omega_{m+n} \) of \( \mathbb{F}_s((X^{-1}))^{m+n} \) and the stabilizer of \( \mathbb{F}_s[X]^{m+n} \) is \( \text{SL}_{m+n}(\mathbb{F}_s[X]) \).

The space \( \text{SL}_{m+n}(\mathbb{F}_s((X^{-1}))/\text{SL}_{m+n}(\mathbb{F}_s[X])) \) can thus be naturally identified with \( \Omega_{m+n} \). Given \( A \in \text{Mat}_{m,n}(\mathbb{F}_s((X^{-1}))) \), we associate to it the following lattice:

\[
A \sim \Lambda_A \overset{\text{def}}{=} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathbb{F}_s[X]^{m+n}
\]

where \( I_i \) denotes the square identity matrix of dimension \( i \). Good rational approximations to \( A \) can be shown to correspond to small vectors in the lattice \( \Lambda_A \) corresponding to \( A \). Let \( \pi \) denote the uniformizer of \( k \) and set

\[
g_t \overset{\text{def}}{=} \text{diag}(\pi^{nt}, \ldots, \pi^{nt}, \pi^{-mt}, \ldots, \pi^{-mt}),
\]

and let \( \delta \) denote the following function which measures small vectors in lattices:

\[
\delta = \min_{v \in \Lambda_A \setminus \{0\}} \|v\|. \quad \text{(2.10)}
\]

By the above identification of \( \Omega_{m+n} \) with \( \text{SL}(m+n, \mathbb{F}_s((X^{-1}))/\text{SL}(m+n, \mathbb{F}_s[X])) \) and Mahler’s compactness criterion, those \( t \) for which \( \delta(g_t \Lambda_A) \) is small correspond to \( t \) for which the \( g_t \) trajectory of \( \Lambda_A \) has cusp excursions. To define these excursions, we set

\[
\text{Cusp}(t, \delta) \overset{\text{def}}{=} \{ \Lambda \in \Omega_{m+n} \mid \delta(\Lambda) \leq |\pi|^{-t} \}. \quad \text{(2.11)}
\]

The following lemma, proven\(^7\) in [17] then establishes the precise connection between Diophantine properties and cusp excursions.

**Lemma 2.12.** There exists an explicit function \( r(t) \) which depends only on \( \psi, m \) and \( n \) and satisfies

\[
\int \psi(x) \, dx < \infty \iff \int |\pi|^{-(m+n)r(t)} \, dt < \infty.
\]

Moreover, \( A \in \mathcal{W}(\psi) \) if and only if

\[
\exists \text{ infinitely many } t \in \mathbb{N} \text{ such that } g_t \Lambda_A \in \text{Cusp}(r(t), \delta). \quad \text{(2.13)}
\]

So in order to prove Theorem 1.17 for function fields, it clearly suffices to prove that:

\(^7\)In the real case, but the function field proof goes through with obvious modifications. An \( S \)-arithmetic version will be provided in [2].
Theorem 2.14. With notation as above,

1. Almost every $\Lambda_A$ satisfies (2.13) if $\int |\pi|^{-(m+n)r(t)} \, dt$ diverges.

2. Almost no $\Lambda_A$ satisfies (2.13) if $\int |\pi|^{-(m+n)r(t)} \, dt$ converges.

Similarly, in order to prove logarithm laws for $K \backslash G(k)/\Gamma$, where $G$ is a simple group as before, we will use the philosophy of F. Mautner [21] to realize the geodesic flow on the unit tangent bundle of $K \backslash G(k)/\Gamma$ as a one-parameter flow $g_t$ on $G(k)/\Gamma$. Fix $x_0 \in G(k)/\Gamma$, and set

$$ \text{Cusp}(t,d) = \{ y \in G(k)/\Gamma \mid d(x_0, y) > t \}. \quad (2.15) $$

The object of study then becomes the excursions of $g_t$ orbits into $\text{Cusp}(r(t),\delta)$. We now tie these themes up with Sprindzhuk’s lemma, focusing for convenience on Theorem 2.14. Accordingly, let $h_t$ denote the characteristic function of $\text{Cusp}(r(t),\delta)$, and let

$$ H_G = \{ g^{-1}h_t \mid t \in \mathbb{N} \}. \quad (2.16) $$

Then to derive Theorem 2.14 from Sprindzhuk’s Lemma, we need to ensure that

$$ \sum_{s,t=M}^N \left( (g_s^{-1}h_s, g_t^{-1}h_t) - \mu(h_s)\mu(h_t) \right) \quad (2.17) $$

is small, in other words we want to show that the excursions of $g_t$ orbits into $\text{Cusp}(r(t),\delta)$ are quasi-independent. The key to this is the spectral gap of the $G$-action on $G/\Gamma$.

3. Spectral gap. We retain the notation of the previous section, i.e. $G$ is the group of $k$-points of a simple $G$ as before, and $\Gamma$ is a non-uniform lattice in $G$. A natural tool to study the ergodic properties of the action of $G$ on $G/\Gamma$ is the spectral properties of the action of $G$ on $L^2(G/\Gamma)$. Let $L^2_0(G/\Gamma)$ denote the subspace of $G/\Gamma$ orthogonal to constants. Let $K$ denote a maximal compact open subgroup of $G$. We call $\phi \in L^2_0(G/\Gamma)$, $K$-finite if its $K$-span is finite dimensional. On $G$ there is an important function which controls the rate of decay of matrix coefficients, the Harish-Chandra function $\Xi$. Let $G = KAN$ denote the Iwasawa decomposition of $G$. The Harish-Chandra function of $G$ is then defined by:

$$ \Xi(g) = \int_K \delta^{-1/2}(gk) \, dk \quad (3.1) $$

where $\delta$ is the left modular function of $AN$ defined by:

$$ d\mu_G = d\mu_K \delta(\alpha) d\mu_{AN} \quad (3.2) $$

where $\mu_\ast$ denotes Haar measure on the locally compact group $\ast$. The following estimate for decay of matrix coefficients is then known to hold by work of several authors. We refer to Section 5.1 in [13] as a convenient reference.

Theorem 3.3. Let $G, K$ and $\Gamma$ be as above. Then there exist positive constants $C, \chi$ such that for every $K$-finite $\phi, \psi \in L^2_0(G/\Gamma)$ and any $g \in G$,

$$ | <g\phi, \psi> | \leq C \|\phi\| \|\psi\| \Xi(g)^{1/\chi}. \quad (3.4) $$
It turns out that the above estimate, or more precisely a version of this estimate which caters to smooth functions (the $L^2$-norm is then replaced by an appropriate Sobolev norm) is enough to ensure the quasi-independence condition in Sprindzhuk’s lemma. The passage from $L^2$ to smooth functions is quite standard and will be elaborated upon in the $S$-arithmetic setting in [2]. It remains to show that the characteristic functions of $\text{Cusp}(\tau(t), \delta)$ (resp. $\text{Cusp}(t, d)$) can be approximated by appropriate smooth functions, which is precisely showing that the functions in question are $DL$. This “smoothing of cusp neighborhoods” is carried out using reduction theory, which we elaborate upon in the next section.

We return briefly to the proof of Theorem 3.3. It turns out that these bounds are closely related to the notion of spectral gap for the $G$ action. Recall that the $G$ action on a probability space $X$ has spectral gap if the regular representation $\rho_0$ of $G$ on the space $L^2_0(X)$, the subspace of $L^2(X)$ orthogonal to constants, is isolated in the Fell topology from the identity (or trivial) representation. If $G$ is an almost direct product of groups $G_i$, then the $G$ action on $X$ has strong spectral gap if the restriction of $\rho_0$ to any factor $G_i$ is isolated from the trivial representation.

Establishing strong spectral gap turns out to be quite difficult in general. In [17], the authors proved that if $G$ is a connected semisimple, center-free Lie group without compact factors, $\Gamma$ is an irreducible non-uniform lattice in $G$, then the $G$ action on $G/\Gamma$ has strong spectral gap. The analogous question for uniform lattices does not seem to be known, however see the recent work [15]. Of course, if all the factors $G_i$ have Kazhdan’s property $T$, strong spectral gap is immediate.

However, since rank 1-groups defined over non-Archimedean local fields act on their Bruhat-Tits trees without fixed points, they cannot have property $T$. A weaker and very useful property is property $\tau$ as defined by Lubotzky and Zimmer [19], which means that $\rho_0$ is isolated in the Automorphic Spectrum of $G$. In [2], we will gather various tools from representation theory namely the restriction technique of Burger-Sarnak [3], as extended to finite places and function fields by Clozel-Ulmo (cf. [7] and [6]) and property $\tau$ for congruence subgroups of $SL_2$ as established by Selberg [28], Gelbart-Jacquet [10], Clozel [5] and Drinfeld [9] in various contexts, to record a proof of (the $S$-arithmetic generalization) of Theorem 3.3.

We note that in many cases uniform, optimal estimates for decay of matrix coefficients are known due to H.Oh [23] and these have several important applications. While the use of spectral gap is a powerful tool, we remark that very little is known in the context of the automorphism group of a tree or more generally, a building and this poses a potential hurdle to obtain logarithm laws in these settings using spectral methods.

The reader will notice that while the existence of the constant $\chi$ in Theorem 3.3 is crucial as well as sufficient for the purposes of this paper, we do not say anything regarding bounds, i.e. the so-called extent of temperedness of the representation. These bounds are related to bounds towards the Generalized Ramanujan Conjecture and play a crucial role in many Diophantine problems. In [12], the second named author and A. Gorodnik study the connection of Diophantine approximation on symmetric spaces and bounds on temperedness in greater detail. This problem
involves studying shrinking target properties for targets which are balls around a fixed point in the symmetric space, and needs new techniques.

4. Adèlic reduction theory. We now present the tools required to obtain smoothing of the cusp neighborhoods, i.e. to show that the characteristic functions of super-level sets of the various cusp neighborhoods are \( DL \). The main tool is adèlic reduction theory, as developed in \([25]\) (see also \([26, 27]\)) from where we also borrow notation. Let \( k, \mathbb{G} \) etc. be as before and let \( \mathbb{A} \) denote the adèle ring of \( k \). Let \( B \) be a fixed Borel subgroup of \( \mathbb{G}(k) \) containing a maximal split torus \( T \) of \( G \). Let \( X_\ast(T) \) denote the lattice of cocharacters of \( T \). Let \( X_\ast(T)^{++} \) denote the subset of \( X_\ast(T) \) consisting of dominant cocharacters. Let \( W = N_G(T(F_s))/T(F_s) \) denote the Weyl group of \( G \). Define the map \( \phi_\pi : W \times X_\ast(T) \rightarrow \mathbb{G}(\overline{k}) \) by

\[
\phi_\pi(w, \mu) = w \mu(\pi).
\]

\( \phi_\pi \) is the restriction of the map \( \phi \) defined by \( \phi_\pi(w, \mu) = w \mu(\pi) \).

Let \( \rho \) denote the sum of all roots that are positive with respect to \( B \). For \( \tau > 0 \), we denote by \( X_\tau \) the subset of \( \mathbb{G}(\mathbb{A})/\mathbb{G}(k) \) consisting of the union of \( \mathbb{G}(k)\phi_{s-1}(e^\lambda)K \) where \( \lambda \) ranges over the dominant cocharacters of \( T \) for which \( \langle \rho, \lambda \rangle \geq \tau \). Then, the following estimate, which follows from the reduction theory developed in \([25]\) allows us to show the requisite \( DL \) property for \( d(x_0, \cdot) \).

**Theorem 4.2.** Let \( r \) denote the \( k \)-rank of \( \mathbb{G}(k) \). Then, there exist positive real numbers \( C_1 \) and \( C_2 \) such that for every \( T > 0 \),

\[
C_1 < \frac{\mu(X_T)}{\sum_{t \geq T} s^{-t^{r-1}}} < C_2.
\]

We now turn to Khintchine’s theorem, where in light of the previous discussion, we have to develop the reduction theory in a slightly different setting—namely on the space \( \text{SL}(d, F_s((X^{-1})))//\text{SL}(d, F_s[X]) \) and for the function \( \delta \). The key to this, as shown in \([17]\), lies in the so-called Siegel integral formula \([29]\), specifically to a multi-dimensional version of it. We will again proceed in the adèlic setting. Set:

\[
\text{GL}^1(d, \mathbb{A}) \overset{def}{=} \{ g \in \text{GL}(d, \mathbb{A}) \mid \prod_{v \in V(k)} | \det g|_v = 1 \}
\]

where \( V(k) \) denotes the set of inequivalent places of the global field \( k \). Let \( f \in L^1(\mathbb{A}^d), g \in \text{GL}^1(d, \mathbb{A}) \) and define

\[
f(x) = \sum_{x \in \mathbb{Q}^d} f(x)
\]

Then the adèlic version of Siegel’s integral formula, as proved by Weil in \([33]\) states that:

**Theorem 4.6.**

\[
\int_{\mathbb{A}^d} f \ d\mu_\mathbb{A} = C(d) \int_{\text{GL}^1(d, \mathbb{A})//\text{GL}(d, \mathbb{Q})} \tilde{f} \ d\tilde{\mu}.
\]

where \( C(d) \) is a constant depending on the field in question (in this case \( \mathbb{Q} \)). We remark that the above Theorem was proved by Weil for arbitrary number fields as well as function fields. A multidimensional generalization of this theorem is developed in \([2]\) and is shown to imply a local version of this formula which can then be used to show that the function \( \delta \) is \( DL \).
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E-mail address: jayadev.athreya@yale.edu
E-mail address: A.Ghosh@uea.ac.uk
E-mail address: amri@imsc.res.in