Restricting positive energy representations of $Diff^+(S^1)$ to the stabilizer of $n$ points

Mihály Weiner

Dipartimento di Matematica,
Università di Roma “Tor Vergata”
Via della Ricerca Scientifica, 1, I-00133, Roma, ITALY
E-mail: weiner@mat.uniroma2.it

Abstract

Let $G_n \subset Diff^+(S^1)$ be the stabilizer of $n$ given points of $S^1$. How much information do we lose if we restrict a positive energy representation $U_{c}^{h}$ associated to an admissible pair $(c, h)$ of the central charge and lowest energy, to the subgroup $G_n$? The question, and a part of the answer originate in chiral conformal QFT.

The value of $c$ can be easily “recovered” from such a restriction; the hard question concerns the value of $h$. If $c \leq 1$, then there is no loss of information, and accordingly, all of these restrictions are irreducible. In this work it is shown that $U_{h}^{c}|_{G_n}$ is always irreducible for $n = 1$ and, if $h = 0$, it is irreducible at least up to $n \leq 3$. Moreover, an example is given for $c > 2$ and certain values of $h \neq \tilde{h}$ such that $U_{\tilde{h}}^{c}|_{G_1} \cong U_{h}^{c}|_{G_1}$. It is also concluded that for these values $U_{h}^{c}|_{G_n}$ cannot be irreducible for $n \geq 2$. For further values of $c, h$ and $n$, the question is left open. Nevertheless, the example already shows that in general, local and global intertwiners in a QFT model may not be equivalent.

1 Introduction

This paper concerns a purely mathematical problem regarding the representation theory of infinite dimensional Lie groups, and it is intended to be

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1
largely self-contained. The actual proofs, apart from those of Prop. 4.2 and Corollary 6.5 will not require any knowledge of chiral conformal QFT. Nevertheless, at least in this introductory section, we shall shortly discuss the physical motivations.

A chiral component of a conformal QFT “lives” on a lightline, but it is often extended to the compactified lightline, that is, to the circle. For several reasons, it is more convenient to study such a theoretical model on the circle than on the lightline. However, keeping in mind the physical motivation, one should always clear the relation between the properties that a model has on the lightline and the properties that it has on the circle.

For example, one may adjust the Doplicher-Haag-Roberts (DHR) theory to describe charged sectors of models given on the circle in the setting of Haag-Kastler nets. It is well known that a model, when restricted to the lightline, may admit new sectors that cannot be obtained by restrictions. These sectors are usually called solitonic. However, so far one may have thought that the restriction from the circle to the lightline is at least injective: each sector restricts to a sector (i.e. to something irreducible, and not to a sum of sectors), and different sectors restrict to different sectors. In fact, under the assumption of strong additivity, this is indeed true. However, there are interesting (i.e. not pathological) models, in which strong additivity fails; most prominently, the Virasoro net with central charge $c > 1$. (The Virasoro nets are fundamental, because each chiral conformal model contains a Virasoro net as a subnet in an irreducible way.)

As it is known, see e.g. the book [KR], for certain values of the central charge $c$ and lowest energy $h$, there exists a unitary lowest energy representation of the Virasoro algebra. By [GW], each of these representations gives rise to a projective unitary representation $U^c_h$ of the group $\text{Diff}^+(S^1)$, that is, of the group of orientation-preserving smooth diffeomorphisms of the circle. These representations are all irreducible, and every positive energy irreducible representation of $\text{Diff}^+(S^1)$ is equivalent with $U^c_h$ for a certain admissible pair $(c, h)$. Moreover, two of these representations are equivalent if and only if both their central charges and their lowest energies coincide.

A representation $U^c_h$ with lowest energy $h = 0$ gives rise to a conformal net (in its vacuum representation) on the circle. This is the so-called Virasoro net at central charge $c$, and it is denoted by $\mathcal{A}_{\text{Vir}_c}$. Every charged sector of $\mathcal{A}_{\text{Vir}_c}$ arises from a positive energy representation of $\text{Diff}^+(S^1)$ with (the same) central charge $c$. Two charged sectors are equivalent if and only if they arise from equivalent positive energy representations of $\text{Diff}^+(S^1)$.

Viewing the circle as the compactified lightline, i.e. the lightline together with the “infinite” point, one has that a diffeomorphism of the circle restricts to a diffeomorphism of the lightline if and only if it stabilizes the chosen
infinite point. So by what was roughly explained, we are motivated to ask
the following questions. Let \( G_n \subset \text{Diff}^+(S^1) \) be the stabilizer subgroup of \( n \) given points of \( S^1 \). Then

- is the restriction of \( U^c_h \) to \( G_1 \) irreducible?
- for what values of \((c, h)\) and \((\tilde{c}, \tilde{h})\) we have \( U^c_h|_{G_1} \simeq U^\tilde{c}_{\tilde{h}}|_{G_1} \)?

Actually, with \( G_1 \) replaced with \( G_n \), there are reasons to consider these questions not only for \( n = 1 \), but in general. (Note that though the actual elements of \( G_n \) depend on the choice of the \( n \) points, different choices result in conjugate subgroups: thus all of these questions are well-posed.) In fact, the (possible) irreducibility of \( U^c_h|_{G_n} \) for \( h = 0 \), is directly related to the (possible) \( n \)-regularity of \( A_{\text{Vir}} \). (See [GLW] for more on the notion of \( n \)-regularity.) The other reason is the relation between the answers regarding different values of \( n \). Of course, we have some trivial relations, since \( G_n \) may be considered to be a subgroup of \( G_m \) whenever \( n \geq m \). However, as it will be proved at Prop. 4.1, we have the further relation:

\[
U^c_h|_{G_{n+1}} \text{ is irred.} \implies U^c_h|_{G_n} \simeq U^\tilde{c}_{\tilde{h}}|_{G_n} \text{ if and only if } (c, h) = (\tilde{c}, \tilde{h}).
\]

As it was mentioned, both the questions, and a part of their answers originate in chiral conformal QFT. For example, Haag-duality is known [BGL, FrG] to hold in the vacuum sector of any chiral conformal net on the circle. This could be used to conclude that \( U^c_0|_{G_2} \) is irreducible for all values of the central charge \( c \). However, we shall not enter into details of this argument, because in any case we shall prove some stronger statements regarding irreducibility. In particular, by considering the problem at the Lie algebra level, it will be shown that the the representation \( U^c_h|_{G_n} \) for \( h = 0 \), is always irreducible (Corollary 3.6), and for \( h = 0 \), we have irreducibility at least up to \( n \leq 3 \) (Prop. 4.3).

Apart from general statements regarding conformal nets, by now we have a detailed knowledge, in particular, of Virasoro nets. For example, it is known, that for \( c \leq 1 \) they are strongly additive, [KL, Xu]. This permits us to conclude (Prop. 4.2), that whenever \( c \leq 1 \), the representation \( U^c_h|_{G_n} \) is irreducible for any positive integer \( n \), and accordingly, \( U^c_h|_{G_n} \simeq U^\tilde{c}_{\tilde{h}}|_{G_n} \) if and only if \((c, h) = (\tilde{c}, \tilde{h})\).

Thus for \( c \leq 1 \), our questions are answered. Let us discuss now the region \( c > 1 \). It is easy to show, that — in general — \( U^c_h|_{G_n} \simeq U^\tilde{c}_{\tilde{h}}|_{G_n} \) implies \( c = \tilde{c} \) (Corollary 3.3). Hence the real problem is to “recover” the value of the lowest energy.

The main result of this paper is an example, showing that already for \( n = 1 \), the value of the lowest energy cannot be always determined by the restriction, since in particular for \( h = \frac{c}{32}, \bar{h} = \frac{c}{32} + \frac{1}{2} \) and \( c > 2 \) we have...
$U_h^c|_{G_1} \simeq U_{\tilde{h}}^c|_{G_1}$. It follows (Corollary 6.5), that the Virasoro net with $c > 2$ is an example for a net in which local and global intertwiners are not equivalent.

The values here exhibited may have more to do with the actual construction, than with the problem itself. (For example, it could turn out that when $c > 1$, all of the representation $U_h^c|_{G_1}$ with $h$ varying over the positive numbers, are equivalent.) The method of showing equivalence is obtained by a combined use of two known tricks:

- the realization (appearing e.g. in [BS]) of the lightline-restriction of the Virasoro net at $c > 1$ as a subnet of the so called $U(1)$-current,
- the observation (appearing e.g. in [LX]) that for any $k = 1, 2, \ldots$, the map $l_n \mapsto \frac{1}{k}l_{kn} + \frac{C}{24}(k - \frac{1}{k})\delta_{n,0}$ gives an endomorphism of the Virasoro algebra.

2 Preliminaries

The Virasoro algebra (Vir) is spanned by the elements $\{l_n : n \in \mathbb{Z}\}$ together with the central element $C$ obeying the commutation relations

\[
[l_n, l_m] = (n - m)l_{n+m} + \frac{C}{12}(n^3 - n)\delta_{-n,m}
\]
\[
[l_n, C] = 0.
\] (1)

For a representation $\pi$ of Vir on a complex vector space $V$, set $L_n = \pi(l_n)$. An eigenvalue of $L_0$ is usually referred as a value of the energy, and the corresponding eigenspace as the energy level associated to that value. If $L_0\Phi = \lambda\Phi$, then by a use of the commutation relations $L_0(L_n\Phi) = (\lambda - n)\Phi$, i.e. the operator $L_n$ decreases the value of the energy by $n$. We say that $\pi$, with representation space $V \neq 0$, is a lowest energy representation with central charge $c \in \mathbb{C}$ and lowest energy $h \in \mathbb{C}$, iff

(1) $h$ is an eigenvalue of $L_0$, and if $\text{Re}(s) < \text{Re}(h)$ then $s$ is not an eigenvalue of $L_0$ (i.e. $h$ is the “lowest energy”),

(2) $\pi(C) = c\mathbb{1}$,

(3) $V$ is spanned by the orbit (under $\pi$) of a vector of $\text{Ker}(L_0 - h\mathbb{1})$.

In this case $\text{Ker}(L_0 - h\mathbb{1})$ is one-dimensional, so up to a multiplicative constant there exists a unique lowest energy vector $\Psi_h^c$ and $L_n\Psi_h^c = 0$ for all $n > 0$. Moreover, $V = \bigoplus_{n=0}^\infty V_{(h+n)}$ where $V_{(h+n)} = \text{Ker}(L_0 - (h + n)\mathbb{1})$ for $n = 0, 1, \ldots$ and actually the dimension of $V_{(h+n)}$ is smaller than or equal to the number of partitions of $n$, as in fact

\[
V_{(h+n)} = \text{Span}\{L_{-n_1} \ldots L_{-n_j} \Psi_h^c \mid j \in \mathbb{N}, n_1 \geq \ldots \geq n_j > 0, \sum_{l=0}^j n_l = n\} \quad (2)
\]
where \( j = 0 \) means that no operator is applied to \( \Psi^c_h \).

A **unitary representation** of the Virasoro algebra is a representation \( \pi \) of Vir on a complex vector space \( V \) endowed with a (skew symmetric, positive definite) scalar product \( \langle \cdot, \cdot \rangle \) satisfying the condition

\[
\langle \pi(l_n)\Phi_1, \Phi_2 \rangle = \langle \Phi_1, \pi(l_{-n})\Phi_2 \rangle \quad (\Phi_1, \Phi_2 \in V, \ n \in \mathbb{Z}),
\]

or in short, that \( \pi(l_n)^+ \equiv \pi(l_n)^*|_V = \pi(l_{-n}) \). (We use the symbol \( \pi^{+} \), keeping \( \pi^{*} \) exclusively for the adjoint defined in the von Neumann sense on a Hilbert space.) Note that the formula

\[
\theta(l_n) = l_{-n}
\]

defines a unique antilinear involution with the property that \( [\theta(x), \theta(y)] = \theta([y, x]) \), and that unitarity means that \( \pi(x)^+ = \pi(\theta(x)) \) for every \( x \in \text{Vir} \).

A pair \((c, h)\) is called **admissible**, if there exists a unitary lowest energy representation with central charge \( c \) and lowest energy \( h \). If \((c, h)\) is admissible, then up to equivalence there exists a unique unitary lowest energy representation with central charge \( c \) and lowest energy \( h \). In this paper this unique representation will be denoted by \( \pi^c_h \), the corresponding representation space by \( V^c_h \), and the (up-to-phase unique) normalized lowest energy vector by \( \Psi^c_h \). As is known, this representation is irreducible (in the algebraic sense) and two such representations are equivalent (in the algebraic sense) if and only if their central charges, as well as their lowest energies, coincide.

Of course \((c, h) = (0, 0)\) is an admissible pair and the corresponding representation is trivial, but a pair \((c, h) \neq (0, 0)\), as is known (see e.g. the book [KR] for further explanations), is admissible if and only if it belongs to either to the **continuous part** \([1, \infty) \times [0, \infty)\) or to the **discrete part** \( \{(c(m), h_{p,q}(m)) | m \in \mathbb{N}, \ p = 1, ..., m + 1; \ q = 1, ..., p\} \)

\[
c(m) = 1 - \frac{6}{(m + 2)(m + 3)}, \quad h_{p,q}(m) = \frac{(m + 3)p - (m + 2)q - 1}{4(m + 2)(m + 3)}. \quad (4)
\]

Let us see now what all this has to do with the so-called positive energy representations of \( \text{Diff}^+(S^1) \), where by the symbol \( \text{Diff}^+(S^1) \) we mean the group of orientation preserving (smooth) diffeomorphisms of the unit circle \( S^1 \equiv \{ z \in \mathbb{C} | \| z \| = 1 \} \). We shall always consider \( \text{Diff}^+(S^1) \) as a continuous group with the usual \( C^\infty \) topology.

We shall often think of a smooth function \( f \in C^\infty(S^1, \mathbb{R}) \) as the vector field symbolically written as \( z = e^{i\theta} \mapsto f(e^{i\theta}) \frac{d}{d\theta} \). The corresponding one-parameter group of diffeomorphisms will be denoted by \( t \mapsto \exp(tf) \).

We shall denote by \( \mathcal{U}(\mathcal{H}) \) the group of unitary operators of a Hilbert space \( \mathcal{H} \). A **projective unitary operator** on \( \mathcal{H} \) is an element of the quotient group \( \mathcal{U}(\mathcal{H})/\{z1 | z \in S^1\} \). A (strongly continuous) projective representation
of a (continuous) group $G$ is a (strongly continuous) homomorphism from $G$ to $U(H)/\{z1 | z \in S^1\}$.

We shall often think of a projective unitary operator $Z$ as a unitary operator. Although there are more than one way of fixing phases, note that expressions like $\text{Ad}(Z)$ or $Z \in M$ for a von Neumann algebra $M \subset B(H)$ are unambiguous. Note also that the self-adjoint generator of a one-parameter group of strongly continuous projective unitaries $t \mapsto Z(t)$ is well defined up to a real additive constant: there exists a self-adjoint operator $A$ such that $\text{Ad}(Z(t)) = \text{Ad}(e^{itA})$ for all $t \in \mathbb{R}$, and if $A'$ is another self-adjoint with the same property then $A' = A + r \mathbb{1}$ for some $r \in \mathbb{R}$.

Let now $(c, h)$ be an admissible pair for Vir, and denote by $H_c^c$ the Hilbert space obtained by the completion of the representation space $V_{c}^c$ of $\pi_{c}^c$. The operator $L_n = \pi_{c}^c(l_n)$ may be viewed as a densely defined operator on this space. By the unitarity of $\pi_{c}^c$, we have that $L_n^* \supset L_{-n}$ (i.e. $L_n^*$ is an extension of $L_{-n}$) and hence $L_n$ is closable.

If $f : S^1 \to \mathbb{C}$ is a smooth function with Fourier coefficients

$$\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta})d\theta \quad (n \in \mathbb{Z}),$$

then the sum $\sum_{n \in \mathbb{Z}} \hat{f}_n L_n$ is strongly convergent on $V_{c}^c$, and the operator given by the sum is closable. Denoting by $T_{c}^c(f)$ the corresponding closed operator, by a use of Nelson’s commutator theorem [GW, Prop. 2], one has that $T_{c}^c(f)^* = T_{c}^c(\overline{f})$ and so in particular that $T_{c}^c(f)$ is self-adjoint whenever $f$ is a real function. By the main result of [GW], there exists a unique projective unitary representation $U_{c}^c$ of Diff$^+(S^1)$ on $\mathcal{H}$ such that

$$U_{c}^c(\text{Exp}(f)) = e^{iT_{c}^c(f)}$$

for every $f \in C^\infty(S^1, \mathbb{R})$. This representation is strongly continuous, and moreover, it is irreducible. Note that this latter property does not follow immediately from the fact that $\pi_{c}^c$ is irreducible (in the algebraic sense). For example, $U_{c}^c$ could have a nontrivial invariant closed subspace which has a trivial intersection with the dense subspace $V_{c}^c$; see also the related remark after Prop. 3.1. However, this is not so. Indeed, if $W$ is bounded operator in the commutant of $U_{c}^c$, then, in particular $W \mathcal{T}_0 \subset \mathcal{T}_0 W$ and hence $W$ preserves each eigenspace of $\mathcal{T}_0$. But by assumption one can form a complete orthonormal system consisting of eigenvectors of $L_0$, and so the eigenspaces of $\mathcal{T}_0$ are exactly the eigenspaces of $L_0$. Thus $W$ preserves each energy space and so also the dense subspace $V_{c}^c$. Similar arguments show that $U_{c}^c \equiv U_{\tilde{c}}^c$ if and only if $c = \tilde{c}$ and $h = \tilde{h}$.
A positive energy representation $U$ of $\text{Diff}^+(S^1)$ on $\mathcal{H}$ is a strongly continuous homomorphism from $\text{Diff}^+(S^1)$ to $\mathcal{U}(\mathcal{H})/\{z \mid z \in S^1\}$ such that the self-adjoint generator of the anticlockwise rotations is bounded from below. (Note that although the generator is defined only up to a real additive constant, the fact whether it is bounded from below is unambiguous.)

Diffeomorphisms of $S^1$ of the form $z \mapsto \frac{az + b}{bz +a}$ with $a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1$ are called Möbius-transformations. The subgroup $\text{Möb} \subset \text{Diff}^+(S^1)$ formed by these transformations is isomorphic to $\text{PSL}(2, \mathbb{R})$, and it is generated by the (real combinations of the complex) vector fields $z \mapsto z^{\pm 1}$ and $z \mapsto 1$. Note that in the representation $U_{ch}$, the three listed complex vector fields correspond to the three operators $L_{\pm 1}$ and $L_0$.

A strongly continuous projective representation of $\text{Möb}$ always lifts to a unique strongly continuous unitary representation of the universal covering group $\tilde{\text{Möb}} \equiv \tilde{\text{PSL}}(2, \mathbb{R})$ of $\text{Möb}$. Through restriction and this lifting, one may fix the additive constant in the definition of the self-adjoint generator of anticlockwise rotations and define the conformal Hamiltonian $L_0$ of a strongly continuous projective representation of $\text{Diff}^+(S^1)$. As is well known, $L_0$ is bounded from below (i.e. the representation is of positive energy type) if and only if $L_0$ is actually bounded by 0. Moreover, each irreducible positive energy representation of $\text{Diff}^+(S^1)$ is equivalent to $U_{ch}$ for a certain admissible pair $(c,h)$; see [Ca, Theorem A.2].

3 The passage to the Lie algebra level

One could view $G_n$ as a Lie subgroup of $\text{Diff}^+(S^1)$, with the corresponding Lie algebra consisting of those vector fields that vanish at the given $n$ points. Without any loss of generality, let us assume that the given points of the unit circle are $e^{i \frac{2\pi k}{n}}$ for $k = 1, \ldots, n$. Then the mentioned (complexified) Lie subalgebra can be identified with the set of functions $\mathfrak{G}_n \equiv \{f \in C^\infty(S^1, \mathbb{C}) : f(e^{\frac{2\pi k}{n}}) = 0 \text{ for } k = 1, \ldots, n\}$. To find a suitable base, consider the function defined by the formula

$$e_{j,r}(z) \equiv z^j - z^{r+j} = z^j(1 - z^r)$$

where $r \in \mathbb{Z}$ and $j \in \mathbb{N}$. Then $\{e_{j,r_n} : r \in \mathbb{Z}, j = 0, \ldots, n - 1\}$ is a set of linearly independent elements of $\mathfrak{G}_n$ whose span is dense in $\mathfrak{G}_n$, where the latter is considered with the usual $C^\infty$ topology.

Omitting the indices of central charge and lowest energy, we have that $T(e_{j,r}) = \frac{\pi}{n}(l_j - l_{r+j})$. So let us set

$$k_{j,r} \equiv l_j - l_{r+j}.$$
We shall often use $k_{0,r}$. To shorten formulae, we shall set $k_r \equiv k_{0,r}$. By direct calculation we find that

$$[k_r, k_m] = r k_r - m k_m - (r - m) k_{r + m} + \frac{C}{12} (r^3 - r) \delta_{r,m},$$

implying that the elements $\{k_r : r \in \mathbb{Z}\}$ together with the central element $C$ span a Lie subalgebra of the Virasoro algebra, which we shall denote by $\mathfrak{R}$. In fact, by a similar straightforward calculation one has that

$$\mathfrak{R}_n \equiv \text{Span}(\{k_{j,m} : r \in \mathbb{Z}, j = 0, \ldots, n - 1\} \cup \{C\})$$

is a Lie subalgebra of $\text{Vir}$ for any positive integer $n$. (Note that for $n = 1$ we get back $\mathfrak{R}$, i.e. $\mathfrak{R} = \mathfrak{R}_1$.) Intuitively, viewing the Virasoro algebra from the point of view of vector fields on the circle, $\mathfrak{R}_n$ corresponds to the algebra of Laurent-polynomial (polynomial in $z$ and $z^{-1}$) vector fields, that are zero at the chosen $n$ points of $S^1$. In what follows, and throughout the rest of this paper, for a densely defined operator $A$ we shall denote its closure by $\overline{A}$.

**Proposition 3.1.** Let $(c, h)$ and $(\tilde{c}, \tilde{h})$ be two admissible pairs for the Virasoro algebra, and assume that $U_{c, \tilde{c}}[G_n] \simeq U_{h, \tilde{h}}[G_n]$. Then there exists a unitary operator $V : \mathcal{H}_h^c \to \mathcal{H}_\tilde{h}^\tilde{c}$ and a linear functional $\phi : \mathfrak{R}_n \to \mathbb{C}$ with $\text{Ker}(\phi) \supset [\mathfrak{R}_n, \mathfrak{R}_n]$ such that for all $x \in \mathfrak{R}_n$ we have

$$V \pi_h^c(x) V^* = \pi_{\tilde{h}}^\tilde{c}(x) + \phi(x) 1.$$

**Proof.** If the real vector field $f$ belongs to $\mathfrak{S}_n$, then $\text{Exp}(tf) \in G_n$ for every $t \in \mathbb{R}$. Using that both the real part and the imaginary part of $x$ is in $\mathfrak{S}_n$, the fact that the finite energy vectors form a core for all operators of the form $T(f)$, and some standard arguments, one can easily show that if the two representations are equivalent, then there exists a unitary $V$ such that $V \pi_h^c(x) V^* = \pi_{\tilde{h}}^\tilde{c}(x) + \text{an additive constant}$, that may depend (linearly) on $x$; say $\phi(x) 1$. (Recall that we are dealing with projective representations.) As $\pi$ is a Lie algebra representation, we have that $[\pi(x), \pi(y)] = \pi([x, y])$. Actually, as $\pi(x) = \pi(\theta(x))^*$, we have that

$$[\pi(x), \pi(y)] = [\pi(\theta(x))^*, \pi(\theta(y))^*] \subset [\pi(\theta(y)), \pi(\theta(y))^*] = \pi(\theta([x, y]))^* = \pi([x, y]).$$

Thus it follows that

$$V \pi_h^c([x, y]) V^* = V[\pi_h^c(x), \pi_h^c(y)] V^* = [V \pi_h^c(x) V^*, V \pi_h^c(y) V^*] \subset ([\pi_h^c(x) + \phi(x) 1], [\pi_h^c(y) + \phi(y) 1]) = [\pi_h^c(x), \pi_h^c(y)] \subset \pi_{\tilde{h}}^\tilde{c}([x, y])$$

and hence $\phi([x, y]) = 0$, which concludes our proof. \qed

8
Remark. Note that even if $\phi = 0$, the unitary operator $V$ appearing in the above proposition does not necessarily make an equivalence between $\pi^c_{\mathfrak{h}}|_{\mathfrak{r}_n}$ and $\pi^c_{\mathfrak{h}}|_{\mathfrak{r}_n}$, since it may not take the dense subspace $V^c_{\mathfrak{h}}$ into $V^c_{\mathfrak{h}}$.

This possibility is not something which is specific to infinite dimensional Lie groups. In fact, consider two of the unitary lowest energy irreducible representations (with lowest energy different from zero), say $\eta_1$ and $\eta_2$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Moreover, consider the base $e_+, e_-$ and $h$ (in the complexified) Lie algebra satisfying the usual commutation relations $[h, e_\pm] = \mp e_\pm$ and $[e_-, e_+] = 2h$. The two elements $t = 2h - (e_- + e_+)$ and $s = i(e_- - e_+)$ span a two-dimensional Lie subalgebra of $\mathfrak{sl}(2, \mathbb{R})$, and it is easy to prove, that if $\eta_1$ and $\eta_2$ are inequivalent, then also their restrictions to this subalgebra are inequivalent (in the algebraic sense). However, the corresponding representations of the corresponding Lie subgroups are in fact equivalent; see for example the remarks in the proof of [GLW, Theorem 2.1].

Proposition 3.2. $C, k_{rn} \in [\mathfrak{r}_n, \mathfrak{r}_n]$ for every $r \in \mathbb{Z}$ and positive integer $n$. In particular, $[\mathfrak{r}, \mathfrak{r}] = \mathfrak{r}$.

Proof. Let $\phi : \mathfrak{r}_n \to \mathbb{C}$ be a linear functional such that $\text{Ker}(\phi) = [\mathfrak{r}_n, \mathfrak{r}_n]$. Our aim is to show that $\phi(C) = \phi(k_{rn}) = 0$. To shorten notations, we shall set $\phi_r \equiv \phi(k_{rn})$. Then by equation (9) one finds that

$$r\phi_r - m\phi_m - (r - m)\phi_{r+m} + \frac{\phi(C)}{12}(n^2r^3 - r)\delta_{r,m} = 0$$

(13)

for all $r, m \in \mathbb{Z}$. Let us now analyze the above relation (together with the fact that $\phi_0 = \phi(k_0) = \phi(0) = 0$). If $r > 1$ and $m = 1$, then by substituting into (13) we obtain the recursive relation $r\phi_r + \phi_1 - (r - 1)\phi_{r-1} = 0$. Resolving the recursive relation we get that for $r > 1$ we have

$$\phi_r = (r - 1)(\phi_2 - \phi_1) + \phi_1.$$  

(14)

Similarly, letting $r < -1$ and $m = -1$ and resolving the resulting recursive relation we get that $\phi_r$ is a (possibly different) first order polynomial of $r$ for the region $r < -1$, too. Then letting $m = -r$ and using that $\phi_0 = 0$, we find by substitution that

$$r\phi_r + r\phi_{-r} + \frac{\phi(C)}{12}(n^2r^3 - r) = 0.$$  

(15)

The expression on the left-hand side — by what was just explained — for the region $r > 1$, is a polynomial of $r$. Thus each coefficient of this polynomial must be zero, and hence, by what was just obtained about degrees, we find
that \( \phi(C) = 0 \) which then by the above equation further implies that \( \phi_r = -\phi_{-r} \) for every \( r \in \mathbb{Z} \). Moreover, returning to (13), we have that
\[
 r\phi_r - m\phi_m - (r - m)\phi_{r+m} = 0
\] 
and also, by exchanging \( m \) with \(-m\) and using that \( \phi_{-m} = -\phi_m \), we have that
\[
 r\phi_r - m\phi_m - (r + m)\phi_{r-m} = 0.
\] 
Taking the difference of these two equations and setting \( m = r - 1 \), we find that \( \phi_{2r-1} = (2r - 1)\phi_1 \). Restricting our attention to the region \( r > 1 \), and confronting what we have just obtained with (14), we get that \( \phi_1 = \phi_2 = 0 \) and hence again by (14) that \( \phi_r = 0 \) for all \( r \geq 0 \) and so actually for all \( r \in \mathbb{Z} \), which concludes our proof. \( \square \)

**Corollary 3.3.** Let \((c, h)\) and \((\hat{c}, \hat{h})\) be two admissible pairs for the Virasoro algebra, and assume that \( U^c_{h}\mid_{G_n} \approx U^\hat{c}_{\hat{h}}\mid_{G_n} \). Then \( c = \hat{c} \).

**Proof.** It follows trivially from Prop. 3.1 and 3.2. \( \square \)

We shall now formulate a useful condition of irreducibility. Fix an admissible pair \((c, h)\) of the Virasoro algebra, and a positive integer \( n \). Recall that we have denoted by \( \Psi^c_h \) the (up to phase) unique normalized lowest energy vector of the representation \( \pi^c_h \). It is clear that the subset of \( \mathfrak{K}_n \)
\[
\mathcal{O}^c_{h,n} \equiv \{ x \in \mathfrak{K}_n : \pi(x)\Psi^c_h = \lambda_x\Psi^c_h \text{ for some } \lambda_x \in \mathbb{C} \} \quad (18)
\]
is in fact a Lie subalgebra. Note that \( \theta(\mathfrak{K}_n) = \mathfrak{K}_n \), but \( \theta(\mathcal{O}^c_{h,n}) \neq \mathcal{O}^c_{h,n} \).

**Proposition 3.4.** Suppose that \( V^c_h \) is spanned by vectors of the form
\[
\pi(\theta(x_1))\ldots\pi(\theta(x_j))\Psi^c_h,
\]
where \( x_1, \ldots, x_j \in \mathcal{O}^c_{h,n} \) and \( j \in \mathbb{N} \) (with \( j = 0 \) meaning the vector \( \Psi^c_h \) itself). Then \( U^c_{h}\mid_{G_n} \) is irreducible.

**Proof.** Simple arguments (similar to those appearing in the proof of Prop. 3.1) show, that if \( V \) is a unitary operator commuting with \( U^c_{h}(G_n) \), then for all \( x \in \mathfrak{K}_n \) we have \( V\pi^c_h(x) = \pi^c_h(x)V \). Let \( B \equiv V - \{\Psi^c_h, V\Psi^c_h\} \mathbb{1} \); then for every \( j \in \mathbb{N} \) and \( x_1, \ldots, x_j \in \mathcal{O}^c_{h,n} \) we have
\[
\langle \pi^c_h(\theta(x_1))\ldots\pi^c_h(\theta(x_j))\Psi^c_h, B\Psi^c_h \rangle = \langle \pi(x_1)^*\ldots\pi(x_j)^*\Psi^c_h, B\Psi^c_h \rangle = \langle \Psi^c_h, B\pi^c_h(x_j)\ldots\pi^c_h(x_1)\Psi^c_h \rangle = \text{multiple of } \langle B\Psi^c_h, \Psi^c_h \rangle = 0 \quad (19)
\]
and hence by the condition of the proposition \( B\Psi^c_h = 0 \). In turn this implies that \( B\pi^c_h(\theta(x_1))\ldots\pi^c_h(\theta(x_j))\Psi^c_h = \pi^c_h(\theta(x_1))\ldots\pi^c_h(\theta(x_j))B\Psi^c_h = 0 \) and hence that \( B = 0 \); i.e. that \( V = \{\Psi^c_h, V\Psi^c_h\} \mathbb{1} \). \( \square \)
In order to use the above proposition, let us fix a certain admissible value of \( c \) and \( h \). To simplify notations, we shall set \( L_n \equiv \pi_h^c(l_n) \) and \( K_n \equiv \pi_h^c(k_n) = L_0 - L_n \). Note that \( K_0 = 0 \) and that \( \pi_h^c(C) = c\mathbb{1} \).

**Lemma 3.5.** The vectors of the form

\[
K_{-n_1} K_{-n_2} \ldots K_{-n_k} \Psi_h^c
\]

where \( k \in \mathbb{N}, n_j \in \mathbb{N}^+(j = 1 \ldots k) \) and \( n_1 \geq n_2 \geq \ldots \geq n_k \) (and where \( k = 0 \) means the vector \( \Psi_h^c \) in itself, without any operator acting on it) span the representation space \( V_h^c \).

**Proof.** The statement with “\( K \)” everywhere replaced by “\( L \)” is true by definition. On the other hand,

\[
L_{-n_1} \Psi_h^c = (h\mathbb{1} - L_0 + L_{-n_1})\Psi_h^c = -K_{-n_1}\Psi_h^c + h\Psi_h^c
\]

Similarly,

\[
L_{-n_1}L_{-n_2}\Psi_h^c = ((h + n_2)\mathbb{1} - L_0 + L_{-n_1})(h\mathbb{1} - L_0 + L_{-n_2})\Psi_h^c
\]

\[
= ((h + n_2)\mathbb{1} - K_{-n_1})(h\mathbb{1} - K_{-n_2})\Psi_h^c
\]

\[
= K_{-n_1}K_{-n_2}\Psi_h^c = (h + n_2)K_{-n_2}\Psi_h^c + h\Psi_h^c;
\]

and it is not too difficult to generalize the above argument, by induction, to show that \( L_{-n_1}L_{-n_2} \ldots L_{-n_k} \Psi_h^c \) is a linear combination of vectors of the discussed form.

The lowest energy vector \( \Psi_h^c \), though (in general) it is not annihilated by the operators \( K_n \) \((n > 0)\), is still a common eigenvector for them:

\[
\forall n > 0 : K_n \Psi_h^c = h\Psi_h^c.
\]

Hence \( \theta(k_{-n}) = k_n \in D_{h,1}^c \) and thus by Lemma 3.5 and Prop. 3.4 we can draw the following conclusion.

**Corollary 3.6.** Let \((c, h)\) be any admissible pair. If \( n = 1 \) then \( U_h^c|G_n \) is irreducible.

By Prop. 3.1 and 3.2, if \( V \) is a unitary operator making an equivalence between \( U_h^c|G_1 \) and \( U_{\tilde{c}}^e|G_1 \), then it also makes an equivalence between \( \pi_h^c|\mathbb{R} \) and \( \pi_{\tilde{c}}^e|\mathbb{R} \). To show the converse, one needs to overcome the following difficulty: we do not know, whether the subgroup of \( G_n \) generated by the exponentials is dense in \( G_n \). In what follows we shall denote this subgroup by \( \tilde{G}_n \).
Proposition 3.7. Let $B$ be a bounded operator from $\mathcal{H}_h^c$ to $\mathcal{H}_h^c$. Then $B$ intertwines $U_h^c|G_1$ with $U_h^c|G_1$ if and only if it intertwines $U_h^c|\tilde{G}_1$ with $U_h^c|\tilde{G}_1$.

Proof. Clearly, we have never used in our proof of irreducibility the whole group $G_1$, but only the subgroup $\tilde{G}_1$ generated by the exponentials. Hence we have that also $U_h^c|\tilde{G}_1$ and $U_h^c|\tilde{G}_1$ are irreducible representations.

If $B$ intertwines $U_h^c|G_1$ with $U_h^c|G_1$ then of course it also intertwines $U_h^c|\tilde{G}_1$ with $U_h^c|\tilde{G}_1$. So let us assume that $B$ is an intertwiner of the latter two. Then, since these representations are unitary and irreducible, it follows that $B$ is a multiple of unitary operator.

So we may assume that $B$ is unitary. Then, using the intertwining property and the fact that the conjugate of an exponential in $G_1$ is still an exponential, it is easy to show that $\text{Ad}(BU_h^c(g)B^*)(U_h^c(\tilde{g})) = U_h^c(g\tilde{g}g^{-1})$ for all $g \in G_1$ and $\tilde{g} \in \tilde{G}_1$. Thus by the irreducibility of $U_h^c|\tilde{G}_1$ it follows that $\text{Ad}(BU_h^c(g)B^*) = \text{Ad}(U_h^c(g))$ and so that in the projective sense $BU_h^c(g)B^* = U_h^c(g)$, which finishes our proof.

Corollary 3.8. Let $(c, h)$ and $(\tilde{c}, \tilde{h})$ be two admissible pairs for the Virasoro algebra. Then $U_h^c|G_1 \simeq U_h^c|G_1$ if and only if there exists a unitary operator $V$ such that $V\pi_h^x V^* = \pi_{\tilde{h}}^x$ for all $x \in \mathbb{R}$.

Proof. The “only if” part follows from Prop. 3.1 and 3.2. As for the “if” part: it is clear, that if the two representations are equivalent at the Lie algebra level, then they are also equivalent on the subgroup generated by the exponentials. Hence the “if” part follows directly from the previous proposition.

4 Further observations

Proposition 4.1. Let $(c, h)$ and $(\tilde{c}, \tilde{h})$ be two admissible pairs, $n$ a positive integer, and suppose that $U_h^c|G_{n+1}$ is irreducible. Then $U_h^c|G_n \simeq U_h^c|G_n$ if and only if $(c, h) = (\tilde{c}, \tilde{h})$.

Proof. The “if” part is trivial; we only need to prove the “only if” part. So suppose the two representations of $G_n$ in question are equivalent. In fact, assume that they actually coincide. (Clearly, we can safely do so.) So we shall fix $n$ (different) points $p_1, ..., p_n$ on the circle, we shall think of $G_n$ as the their stabilizer, and we shall assume that $U_h^c|G_n = U_h^c|G_n$ (so in particular we assume that the two representations of $\text{Diff}^+(S^1)$ are given on the same Hilbert space). To simplify notations, for the rest of the proof we shall further set $U \equiv U_h^c$ and $\tilde{U} \equiv U_h^c$. [Details of proof follow here]
Suppose $\xi \in \text{Diff}^+(S^1)$ is such that it preserves all but one of our $n$ fixed points. Let this point be $p_j$. Set $q \equiv \xi(p_j)$, and let us think of $G_{n+1}$ as the stabilizer of the points $p_1, ..., p_n$ and $q$; then $\xi^{-1}G_{n+1}\xi \subset G_n$. Accordingly, we have that for all $\varphi \in G_{n+1}$

$$\text{Ad} \left( U(\xi)\tilde{U}(\xi^{-1}) \right)(U(\varphi)) = U(\varphi). \tag{23}$$

However, we cannot immediately conclude that $U(\xi)\tilde{U}(\xi^{-1})$ commutes with $U(\varphi)$, since the above equation is meant in the sense of projective unitaries. Nevertheless, it follows that there exists a complex unit number $\lambda(\xi)$, such that in the sense of unitary operators (i.e. not only in the projective sense) we have

$$U(\varphi)^* \text{Ad} \left( U(\xi)\tilde{U}(\xi^{-1}) \right)(U(\varphi)) = \lambda(\xi) \mathbb{1}. \tag{24}$$

Clearly, the value of $\lambda(\xi)$ is independent of the chosen phase of $U(\varphi)$, and moreover, it is largely independent from the diffeomorphism $\xi$. Indeed, if $\xi'$ is another diffeomorphism such that $\xi'(p_k) = p_k$ for $k \neq j$ and $\xi'(p_j) = q$, then $\xi' = \xi \circ \beta$ where $\beta \equiv \xi^{-1} \circ \xi' \in G_n$ and thus $U(\beta) = \tilde{U}(\beta)$ and so in the projective sense

$$U(\xi')\tilde{U}(\xi'^{-1}) = U(\xi)\tilde{U}(\beta)\tilde{U}(\xi'^{-1}) = U(\xi)\tilde{U}(\xi^{-1}) \tag{25}$$

implying that $\lambda(\xi') = \lambda(\xi)$. However, the map $\xi \mapsto \lambda(\xi)$ is clearly continuous, so the above argument actually shows that $\lambda(\xi) = 1$; i.e. that $U(\xi)\tilde{U}(\xi^{-1})$ commutes with $U(\varphi)$. Hence we have shown that $U(\xi)\tilde{U}(\xi^{-1})$ is in the commutant of $U(G_n)$ and so — by the condition of irreducibility — it follows that $U(\xi) = \tilde{U}(\xi)$. This concludes our proof, since $\text{Diff}^+(S^1)$ is evidently generated by the diffeomorphisms that preserve all but one of the points $p_1, ..., p_n$. \hfill \qedsymbol

At this point it is natural to ask: what are the admissible pairs $(c, h)$, for which we can prove the irreducibility of $U^c_h|G_n$ for some $n > 1$? (Recall that for $n = 1$ we have already obtained irreducibility, but in order to use the above proposition, we need $n > 1$.)

Here we shall prove irreducibility for two (overlapping) regions: for $c \leq 1$, and for $h = 0$ (the latter only for $n \leq 3$). The irreducibility in the first of them is an evident consequence of the known properties of the Virasoro nets. Nevertheless, it is worth to state it.

**Proposition 4.2.** Let $(c, h)$ be an admissible pair with $c \leq 1$. Then $U^c_h|G_n$ is irreducible for every positive integer $n$. Moreover, $U^c_h|G_n \simeq U^c_{\tilde{h}}|G_n$ if and only if $h = \tilde{h}$.  

13
As it is known, \( [KL, Xu] \), the Virasoro net with \( c \leq 1 \) is strongly additive. Moreover, it is also known, if \((c, h)\) is an admissible pair with \( c \leq 1 \), then the representation \( U^c_h \) gives rise to a locally normal representation of the conformal net \( \mathcal{A}_{\text{Vir}_c} \); see the discussion before [Ca] Prop. 2.1 explaining for which values of \( c \) and \( h \) it is known (and from where) that \( U^c_h \) gives a locally normal representation of \( \mathcal{A}_{\text{Vir}_c} \). This clearly shows that \( U^c_h |_{G_n} \) is irreducible. The rest of the proposition follows from irreducibility and the previous proposition.

**Proposition 4.3.** Let \((c, h = 0)\) be an admissible pair and \( n \leq 3 \). Then \( U^c_0 |_{G_n} \) is irreducible.

**Proof.** It is enough to show the statement for \( n = 3 \). As usual in case of \( h = 0 \), we shall omit the index of the lowest energy, and we shall denote the lowest energy vector by \( \Omega \) (the “vacuum vector”) instead of \( \Psi_0 \). Moreover, we shall set \( L_k \equiv \pi_c^0(l_k) \) \((k \in \mathbb{Z})\).

The proof relies on the simple fact that in case of \( h = 0 \), the equality \( L_k \Omega = L_{k+1} \Omega = 0 \) is satisfied for 3 different values of \( k \); namely for \( k = 0, \pm 1 \). Using this, we shall show that each energy level of \( V^c_0 \) is in \( S \equiv \text{Span}\{A^+_1 \ldots A^+_j \Omega | j \in \mathbb{N}, A_1, \ldots A_j \in \pi^c_0(\mathfrak{h}_3), A_1 \Omega = \ldots = A_j \Omega = 0\} \) \((26)\) (where \( j = 0 \) means the vector \( \Omega \) itself). This is enough; then the statement follows by Prop. 3.4.

We shall argue by induction on the energy level. The zero energy level \( (V^c_0)_{(0)} \) is in \( S \), since \( (V^c_0)_{(0)} = \mathbb{C} \Omega \). So suppose that \( (V^c_0)_{(k)} \subset S \) for all \( k \leq m \), and consider the case \( k = m + 1 \).

Of course, the energy level \( (V^c_0)_{(m+1)} \) is spanned by vectors of the form \( L_{-n_1} \ldots L_{-n_j} \Omega \) where \( j \) and \( n_1 \leq n_2 \ldots \leq n_j \) are positive integers such that \( n_1 + \ldots + n_j = m + 1 \). However, as it is well known, for \( h = 0 \), these vectors are not independent, and \( (V^c_0)_{(m+1)} \) is already spanned by the vectors of the above form with the further condition that \( 2 \leq n_1 \leq n_2 \ldots \leq n_j \).

So consider one of these vectors, and let \( r \) be the number in \( \{0, \pm 1\} \) such that \( r \equiv n_1 \) modulo 3. Then setting \( A \equiv L_{n_1} - L_r \) we have that \( A \in \pi^c_0(\mathfrak{h}_3) \) and \( A \Omega = 0 \). It follows that \( A^+ S \subset S \). Moreover,

\[
L_{-n_1} \ldots L_{-n_j} \Omega = A^+(L_{-n_2} \ldots L_{-n_j} \Omega) + L_{-r}(L_{-n_2} \ldots L_{-n_j} \Omega)
\]

and of course by the inductive condition both the vector \( L_{-n_2} \ldots L_{-n_j} \Omega \) and the vector \( L_{-r} L_{-n_2} \ldots L_{-n_j} \Omega \) is in \( S \) (as \( r < 2 \leq n_1 \), the energies of both vectors are smaller than \( m + 1 \)). Thus by the above equation \( L_{-n_1} \ldots L_{-n_j} \Omega \in S \) and so \( (V^c_0)_{(m+1)} \subset S \), which concludes the inductive argument and our proof.
5 Constructing representations of $\mathfrak{K}$

Recall that $\theta(k_n) = k_{-n}$ and $\theta(C) = C$ and hence $\theta(\mathfrak{K}) = \mathfrak{K}$. A representation $\eta$ of $\mathfrak{K}$ on complex scalar product space $V$ satisfying $\eta(\theta(x)) = \eta(x)^+$ for every $x \in \mathfrak{K}$ will be said to be unitary. So far, as a concrete example for such representation, we only had the representations $\pi_{\mathfrak{K}|\mathfrak{K}}$ obtained by restriction. We shall now exhibit more examples.

Let us begin now our list of constructions with an abstract one. Suppose that we have a $\gamma : \mathfrak{K} \to \mathfrak{K}$ endomorphism that commutes with the antilinear involution $\theta$. Then it is clear, that for any unitary representation $\eta$, the composition $\eta \circ \gamma$ is still a unitary representation.

For example, following the similar constructions for the Virasoro algebra, cf. [LX], for any $r \in \mathbb{N}^+$ consider the linear map $\gamma_r$ given by

\[ k_n \mapsto \frac{1}{r} k_{rn} + \frac{C}{24}(r - \frac{1}{r}) \quad (n \in \mathbb{Z} \setminus \{0\}) \]
\[ C \mapsto rC. \quad (28) \]

By a straightforward calculation using the commutation relations of the algebra $\mathfrak{K}$, we have that $\gamma_r$ is an endomorphism and it is clear that it commutes with $\theta$. Thus for any unitary representation we can construct a family of new unitary representations by taking compositions with $\gamma_r$.

Just as in the case of the Virasoro algebra, we can also get some interesting constructions considering the $U(1)$ current algebra. As it is well-known, for every $q \in \mathbb{R}$ there exists a linear space $V_q$ with positive scalar product, a unit vector $\Phi_q \in V_q$ and a set of operators $\{J_n \in \text{End}(V_q) | n \in \mathbb{Z}\}$ satisfying the following properties.

- $[J_n, J_m] = n \delta_{n,m} \mathbb{1}$ and $J_{-n} = J_n^+$.
- $J_n \Phi_q = 0$ for all $n > 0$.
- $J_0 = q \mathbb{1}$.
- $V_q$ is the smallest invariant subspace for $\{J_n | n \in \mathbb{Z}\}$, containing $\Phi_q$.

We shall call this representation of the $U(1)$ current algebra the representation with charge $q \in \mathbb{R}$. The formally infinite sum of the normal product of the current with itself

\[
:J^2: = \sum_{k>n} J_{n-k}J_k + \sum_{k\leq n} J_kJ_{n-k} \quad (29)
\]

becomes finite on each vector of $V_q$, thus giving a well-defined linear operator. Setting $L_n \equiv \frac{1}{2} :J^2: n$ one finds that the map $l_n \mapsto L_n$ extends to a unitary
representation of the Virasoro algebra with the central charge represented by $c$. Moreover, one finds that

- $[L_n, J_m] = -m J_{n+m}$,
- $L_n \Phi = 0$ for all $n > 0$,
- $L_0 \Phi = \frac{1}{2} q^{2} \Phi$.

We shall now give a new construction for some unitary representation of $\mathfrak{H}$. The next proposition — although it can be understood and justified even without knowing anything more than what was so far listed — needs some “explanations”. Without making explicit definitions and rigorous arguments, let us mention the following. (In any case, the precise statement and its proof will make no explicit use of this.)

The main idea of [BS] is the fact, that — using their settings and notations, but changing the singular point from $-1$ to $1$ — on the punctured plane $\mathbb{C} \setminus \{1\}$,

$$T_\alpha(z) = T(z) + \alpha \left(J'(z) + iz + 1 + J(z)\right)$$

follows the commutation relations of a stress-energy tensor at central charge $c = 1 + 12\alpha^2$. However, the function $z \mapsto iz + 1$ has a singularity at point $z = 1$, and the power series expansion of $T_\alpha$ will depend on the chosen region. As a consequence, the operators appearing in the expansion will give rise to a representation of the Virasoro algebra, which — on the full representation space $V_0$ — will not satisfy the unitarity condition. On the other hand, with $h(z) \equiv 1 - z^n$, the product $hT_\alpha$ will have an unambiguous expansion, as in fact

$$\frac{z + 1}{z - 1} h(z) = -(z + 1)(1 + z + z^2 \ldots z^{n-1}) = 1 + z^n - 2 \sum_{k=0}^{n} z^k$$

is a polynomial. This suggests the following statement.

**Proposition 5.1.** For any fixed $\alpha \in \mathbb{R}$, setting

$$K_n^\alpha \equiv (L_0 - L_n) + i n \alpha \left(J_n + \frac{1}{n!} (J_0 + J_n - 2 \sum_{k=\min(0,n)}^{\max(0,n)} J_k)\right)$$

the assignment $k_n \mapsto K_n^\alpha$ ($n \in \mathbb{Z} \setminus \{0\}$), $C \mapsto c(\alpha) = 1 + 12\alpha^2$ gives a unitary representation of $\mathfrak{H}$.

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1After a Cayley transformation, the formula on the real lines simplifies to “stress-energy tensor + $\alpha$-times the (real-line) derivative of the current”.

16
Unitarity is manifest, and the rest of the proposition may be justified by a long, but straightforward calculation using what was previously listed about the $U(1)$ current. Note that here the formula is given in a compact way, which is a fine thing for a proposition, but not necessarily the best for actual calculations. (For example, there is a hidden sign factor, appearing as $n/|n|$.)

The reader is encouraged to really check the commutation relations. It might seem something tedious (and boring), but — according to the author’s personal opinion — in fact it is interesting to observe how the apparent contradictions disappear by some “miraculous cancellations” of the terms.

6 Equivalence with $h_1 \neq h_2$

Let $(c, h)$ be an admissible pair and consider the representation $\pi^c_h$ on the representation space $V^c_h$. It is clear, that for any $\Phi \in V^c_h$ we have that $\pi^c_h(k_n)\Phi = (\pi^c_h(l_0) - \pi^c_h(l_n))\Phi$ which, for $n$ sufficiently large, is further equal to $\pi^c_h(l_0)\Phi$. This shows two things: first, that up to phase $\Phi = \Psi^c_h$ is the unique normalized vector in $V^c_h$ such that $\pi^c_h(k_n)\Phi = h\Phi$ for all $n > 0$; second, that by knowing the restriction $\pi^c_h|_R$ one can “recover” the operator $\pi^c_h(l_0)$ and hence the whole representation $\pi^c_h$. It follows that $\pi^c_h|_R$ is equivalent to $\pi^c_h$ if and only if $\pi^c_h$ is equivalent to $\pi^c_h^\prime$; i.e. if and only if $(c, h) = (\tilde{c}, \tilde{h})$.

However, as it was already explained, what we are interested in is not the equivalence of $\pi^c_h|_R$ and $\pi^c_h|_R^\prime$ but the equivalence of $\pi^c_h|_R$ and $\pi^c_h|_R^\prime$. In order to investigate the second kind of equivalence, first we shall consider different ways of exhibiting the representation $\pi^c_h|_R$.

Let $\eta$ be a representation of $R$ on a complex scalar product space $V$. Recall that $\theta(k_n) = k_{-n}, \theta(C) = C$ and so $\theta(R) = R$. Assume that $\eta$ satisfies the following properties:

(A) $\eta$ is unitary: $\eta(\theta(x)) = \eta(x)^+$ for all $x \in R$,

(B) $\eta(C) = c1$,

(C) up to phase there exists a unique normalized vector $\Psi$ with the property $\eta(k_n)\Psi = h\Psi$ for all $n > 0$,

(D) $V$ is the smallest invariant space for $\eta$ containing $\Psi$.

Using the commutation relations and the listed properties it is an exercise to show that
The value of the scalar product
\[ \langle \eta(k_{n_1})\eta(k_{n_2})\ldots\eta(k_{n_r})\Psi, \eta(k_{m_1})\eta(k_{m_2})\ldots\eta(k_{m_s})\Psi \rangle \]
is “universal”: it is completely determined by the values of \( c, h \) and the integers \( n_1, \ldots, n_r \) and \( m_1, \ldots, m_s \). That is, the scalar product can be calculated by knowing these values; even without having the actual form of the representation \( \eta \) or knowing anything more (than just the required properties) about it.

The representation space is spanned by the vectors of the form appearing in Lemma 3.5.

These two consequences imply that the representation, up to equivalence, is uniquely determined by the pair \( (c, h) \). It is worth to state this in a form of a statement.

**Corollary 6.1.** Let everything be as it was explained. Then the map
\[ \eta(k_{n_1})\eta(k_{n_2})\ldots\eta(k_{n_r})\Psi \mapsto \pi_c^h(k_{n_1})\pi_c^h(k_{n_2})\ldots\pi_c^h(k_{n_r})\Psi \]
extends to a unique unitary operator which establishes an isomorphism between \( \eta \) and \( \pi_c^h|K \).

We shall now get to the “main trick” of this paper, which is a combination of the two constructions discussed in the previous section. So consider the unitary representation of \( \mathfrak{g} \) given by Proposition 5.1 for \( q = 0 \), and compose it with the endomorphism \( \gamma_2 \) given by (28). We get that for every \( \alpha \in \mathbb{R} \), the map \( k_n \mapsto K_n^{(\alpha, 2)} \) \((n \in (\mathbb{Z} \setminus \{0\}))\), where
\[ K_n^{(\alpha, 2)} = \frac{1}{2}(L_0 - L_{rn}) + \text{in} \alpha \left( J_{2n} + \frac{1}{2n}(J_{2n} - 2 \sum_{k=\min(0, 2n)}^{\max(0, 2n)} J_k) \right) + \frac{1 + 12\alpha^2}{16} \mathbb{1} \]
(32)
extends to a unitary representation \( \rho_{(\alpha, 2)} \) of \( \mathfrak{g} \) with central charge \( c(\alpha, 2) = 2(1 + 12\alpha^2) \) (i.e. the element C is represented by \( c(\alpha, 2)\mathbb{1} \)). Note that in the above formula we omitted \( J_0 \), since we are in the vacuum representation of the \( U(1) \) current (i.e. \( q = 0 \) and so \( J_0 = 0 \), too). Moreover, as it is usual in the vacuum representation, we shall denote denote the lowest energy vector — corresponding to the “true conformal energy” \( L_0 \) — by \( \Omega \), rather than by \( \Phi_0 \), and we shall call it the vacuum vector.
Lemma 6.2. Let $\Phi \equiv J_{-1}\Omega + i\frac{16\alpha}{1 + 12\alpha^2}\Omega$. Then for every $n > 0$ we have

$$K_n^{(\alpha,2)}\Omega = \frac{1 + 12\alpha^2}{16}\Omega, \quad K_n^{(\alpha,2)}\Phi = \frac{9 + 12\alpha^2}{16}\Phi.$$ 

Proof. Anything which lowers the energy (in the sense of $L_0$) by more than 1, annihilates both the vector $\Omega$ and $\Phi$. Thus $J_k\Phi = L_k\Phi = J_k\Omega = L_k\Omega = 0$ for every $k > 1$ and hence one finds that the operator $K_n^{(\alpha,2)}$, for every $n > 0$, acts exactly like the operator

$$\frac{1}{2}L_0 - i\alpha J_1 + \frac{1 + 12\alpha^2}{16}\mathbb{1}$$

on the mentioned vectors. The rest is trivial calculation. \qed

By the previous lemma and by Corollary 6.1, for $c > 2, h_1 = \frac{c}{12}$ and $h_2 = \frac{1}{2} + \frac{c}{32}$, the representations $\pi_{h_1}^c\mathbb{R}$ and $\pi_{h_2}^c\mathbb{R}$ appear as subrepresentations of a common, non-irreducible representation, namely the representation $\rho_{(\alpha,2)}$, with $\alpha = \sqrt{\frac{c/2 - 1}{12}}$.

Let $V_{h_1}$ be the minimal invariant subspace for $\rho_{(\alpha,2)}$ containing $\Omega$, and $V_{h_2}$ the one containing the previously given vector $\Phi$. These are the subspaces on which the representation is isomorphic to $\pi_{h_1}^c\mathbb{R}$ and $\pi_{h_2}^c\mathbb{R}$, respectively, since — as we have seen — the vectors $\Omega$ and $\Phi$ behave like “lowest energy vectors” for the representation $\rho_{(\alpha,2)}$ of $\mathbb{R}$.

The important observation is that these two vectors, since $\alpha \neq 0$, are not orthogonal. Thus, neither the two subspaces $V_{h_1}$ and $V_{h_2}$ can be so. It should follow therefore, that the corresponding irreducible representations (consisting of closed operators) cannot be inequivalent.

This argument however, is not completely rigorous as we deal with (unbounded) operators rather than unitary representations of groups. So in what follows, we shall find a way to deal with the technical difficulties.

Lemma 6.3. Let $Q_1$ and $Q_2$ be the orthogonal projections onto $\overline{V}_{h_1}$ and $\overline{V}_{h_2}$. Then for every $x \in \mathbb{R}$ such that $\theta(x) = x$, we have that $\rho_{(\alpha,2)}(x)$ is self-adjoint and

$$Q_je^{it\rho_{(\alpha,2)}(x)} = e^{it\rho_{(\alpha,2)}(x)}Q_j \quad (t \in \mathbb{R}, j = 1, 2).$$

Proof. The finite energy vectors of the $U(1)$ current are analytic for every operator which is a finite sum of the “$L$” operators; in particular, for $\rho_{(\alpha,2)}(x)$, too. This shows, that if $\theta(x) = x$ then $\rho_{(\alpha,2)}(x)$ is essentially self-adjoint. Moreover, as the the subspaces $V_{h_j} (j = 1, 2)$ are invariant for $\rho_{(\alpha,2)}(x)$, the analyticity property also shows, that the subspaces $\overline{V}_{h_j} (j = 1, 2)$ are invariant for $e^{it\rho_{(\alpha,2)}(x)} (t \in \mathbb{R})$. \qed
Corollary 6.4. Let $c > 2$, $h_1 = \frac{c}{32}$ and $h_2 = \frac{1}{2} + \frac{c}{32}$. Then the representations $U_{h_1}^c|G_1$ and $U_{h_2}^c|G_1$ are unitary equivalent.

Proof. By Corollary 3.8 the restrictions of $\rho(\alpha, 2)$ onto $V_{h_1}$ and $V_{h_2}$ give rise to two unitary representations (in that corollary, all elements of $\mathbb{R}$ appear in the condition, but it is clear, that the hermitian ones, i.e. the elements invariant under $\theta$, are sufficient for us) of $G_1$ on $\overline{V}_{h_1}$ and $\overline{V}_{h_1}$, respectively, with the first one being unitary equivalent to $U_{h_1}^c|G_1$, while the second one to $U_{h_2}^c|G_1$. By Corollary 3.6 these representations are irreducible, and by the previous lemma and Prop. 3.7 the restriction of $Q_1$ is an intertwiner between them. Hence if $Q_1$, as a map from $\overline{V}_{h_2}$ to $\overline{V}_{h_1}$, is not zero, then the two representations are equivalent. This is indeed the case, as $\langle \Omega, \Phi \rangle = \frac{16i\alpha_1 + 12\alpha_2}{1 + 12\alpha_2} \neq 0$, and $\Omega \in V_{h_1}$ whereas $\Phi \in V_{h_2}$. 

Thus we have managed to give examples for values $h \neq \hat{h}$ such that $U_{\hat{h}}^c|G_1 \simeq U_{h}^c|G_1$. More examples could be generated by i) taking tensor products (and then restrictions), ii) using the endomorphism $\gamma_r$ with $r$ different from 2 (which we have used so far). However, at the moment our aim was just to find some examples.

Corollary 6.5. In the Virasoro net with $c > 2$, local and global intertwiners are not equivalent.

Proof. The representations $U_{\hat{h}}^c$ and $U_{\hat{h}}^c$, where $h = \frac{c}{32}$ and $\hat{h} = \frac{1}{2} + \frac{c}{32}$ give rise to two locally normal, irreducible representations of the conformal net $A_{\text{Vir}_c}$; see the discussion before [Ca, Prop. 2.1] explaining for which values of $c$ and $h$ it is known (and from where) that $U_{h}^c$ gives a locally normal representation of $A_{\text{Vir}_c}$.

In any locally normal irreducible representation of $A_{\text{Vir}_c}$, there is a unique strongly continuous unitary representation of the universal covering group of the M"{o}bius group, which implements the M"{o}bius symmetry in the given locally normal irreducible representation of the net; see [DFK] for the details. This shows that the value of the lowest energy in any locally normal irreducible representation of $A_{\text{Vir}_c}$ is well-determined and hence, globally, the locally normal irreducible representations given by $U_{h}^c$ and $U_{\hat{h}}^c$ are inequivalent. However, by the previous result, locally they are equivalent.

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