A CLASS OF MONOTONIC QUANTITIES ALONG THE RICCI FLOW

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ABSTRACT. We construct a class of monotonic quantities along the normalized Ricci flow on closed $n$-dimensional manifolds.

1. Introduction

The invariants and properties preserved along the Ricci flow often play key roles in the study of geometry and topology of manifolds. Among many of such examples are, positivity of scalar curvature [1] in the study of three-manifolds with positive Ricci curvature, positivity of curvature operators in the study of high dimensional manifolds [2], Hamilton’s entropy and its monotonicity [3] in the study of two-dimensional manifolds, and Perelman’s entropy and the monotonicity of the first eigenvalue of $-4\Delta + R$ along the Ricci flow [7] in the study of high dimensional manifolds, and etc. There has been increasing attention on the last two examples. The monotonicity of the Hamilton’s entropy was a key to get an upper bound of scalar curvature $R$. With such a bound and the Harnack inequality of LYH type, Hamilton [3] was able to get a crucial lower of $R$ and prove the exponential convergence of the the normalized Ricci flow for surfaces with positive Euler characteristic. With his entropy and the monotonicity of the first eigenvalue, Perelman [7] was able to rule out nontrivial steady or expanding breathers on compact manifolds. Hamilton and Perelman’s work on the monotonicity stimulated the research on the topic. Many studies on the topics appeared. For the Laplacian operator, Li Ma [6] studied monotonicity of the first eigenvalue on a domain $D$ in the manifold $M$ with Dirichlet boundary condition, along the unnormalized Ricci flow. The author [4] recently studied the first nonzero eigenvalue of Laplacian under the normalized Ricci flow and gave a Faber-Krahn type of comparison theorem and a sharp bound. In [5], the author studied some asymptotic behavior of the first nonzero eigenvalue of
the Lalacian along the normalized Ricci flow and gave a direct short proof for an asymptotic upper limit estimate.

In this paper, we construct a class of monotonic quantities along the normalized Ricci flow. We show that though the eigenvalues themselves are not monotonic along the normalized Ricci flow in general, the appropriate multiples are. We first introduce some notations we are going to use in this paper.

We let $M$ be a closed $n$-dimensional manifold, $g(t)$ the solution to the normalized Ricci flow equation

\[
\frac{\partial}{\partial t} g = -2Rc + \frac{2r}{n} g \quad \text{for} \quad 0 \leq t < T \leq \infty.
\]

Let $Rc$ be the Ricci tensor, $R$ the scalar curvature, $d\mu$ the volume element, $\Delta$ the Laplacian, of the Riemannian manifold $(M, g(t))$, respectively. Let

\[
r = r(t) = \int_M Rd\mu / \int_M d\mu, \quad \sigma(t) = \int_0^t r(\tau)d\tau,
\]

\[
\rho_0 = \min_M R|_{t=0}, \quad \delta_0 = \max_M R|_{t=0}.
\]

We drop the integral domain $M$ in integrals sometimes.

We present the non-decreasing quantities in the next section and increasing ones in the last section.

2. Non-decreasing quantities

We have the following results on non-decreasing quantities along the normalized Ricci flow.

**Theorem 2.1.** Let $g(t)$ be the solution to the normalized Ricci flow equation (1.1), $\lambda = \lambda(t)$ be any eigenvalue of the Laplacian of the metric $g(t)$ on a closed $n$-dimensional manifold $M$. If the Einstein tensor $E = Rc - \frac{1}{2}Rg$ is non-negative, then the quantity

\[
e^\int_0^t [\frac{2r(\tau) - \varphi(\tau)}{n}] d\tau \lambda(t)
\]

is non-decreasing along the flow, where

\[
\varphi(t) = 1 / \left( e^{\frac{2}{n}\sigma(t)} \left( \frac{1}{\rho_0} - \frac{2}{n} \int_0^t e^{-\frac{2}{n}\sigma(\tau)} d\tau \right) \right).
\]

**Proof.** Let $u$ be an eigenfunction of the eigenvalue of the Laplacian, $-\Delta u = \lambda u$.

Take derivatives with respect to $t$,

\[-(\frac{\partial}{\partial t}\Delta)u - \Delta \frac{\partial}{\partial t}u = (\frac{d}{dt}\lambda)u + \lambda \frac{\partial}{\partial t}u.
\]

Multiply the equation by $u$ and integrate,

\[-\int u(\frac{\partial}{\partial t}\Delta)u - \int u\Delta \frac{\partial}{\partial t}u = (\frac{d}{dt}\lambda) \int u^2 + \lambda \int u \frac{\partial}{\partial t}u.
\]
Noticing that
\[- \int u \Delta \frac{\partial}{\partial t} u = - \int \Delta u \frac{\partial}{\partial t} u = \lambda \int u \frac{\partial}{\partial t} u,\]
we have
\[
\left( \frac{d}{dt} \lambda \right) \int u^2 d\mu
= - \int u (\frac{\partial}{\partial t} \Delta) u d\mu
= - \int (2R^{ij} u \nabla_i \nabla_j u - \frac{2r}{n} u \Delta u) d\mu
= - \int 2R^{ij} u \nabla_i \nabla_j u d\mu - \int \frac{2r}{n} \lambda u^2 d\mu,
\]
where in the second equality we used the equation
\[
(2.3) \quad \frac{\partial}{\partial t} (\Delta) = 2R^{ij} \nabla_i \nabla_j - \frac{2r}{n} \Delta.
\]

This equation is true is due to the following. By (1.1), for a smooth function \(v\) in a local chart \(\{x^i\}\) on \(M\),
\[
\frac{\partial}{\partial t} (\Delta g_{ij}(t)v) = \frac{\partial}{\partial t} (g^{ij} \nabla_i \nabla_j v)
= (\frac{\partial}{\partial t} (g^{ij} \nabla_i \nabla_j v) + g^{ij} \left[ \frac{\partial^2}{\partial x^i \partial x^j} v - \Gamma^k_{ij} \frac{\partial}{\partial x^k} v - \frac{\partial}{\partial t} (\Gamma^k_{ij}) \frac{\partial}{\partial x^k} v \right]
= - g^{ik} g^{jl} (-2R_{kl} + \frac{2r}{n} g_{kl}) \nabla_i \nabla_j v + \Delta \frac{\partial}{\partial t} v - g^{ij} \frac{\partial}{\partial t} (\Gamma^k_{ij}) \frac{\partial}{\partial x^k} v
= 2R^{ij} \nabla_i \nabla_j v - \frac{2r}{n} \Delta v - g^{ij} \frac{\partial}{\partial t} (\Gamma^k_{ij}) \frac{\partial}{\partial x^k} v + \Delta \frac{\partial}{\partial t} v,
\]
and at point \(x\) and in the local normal chart about \(x\),
\[
g^{ij} \frac{\partial}{\partial t} (\Gamma^k_{ij}) = \frac{1}{2} g^{ij} g^{kl} \left[ \frac{\partial}{\partial x^i} g_{lj} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right]
= \frac{1}{2} g^{i[k} \nabla_l \frac{\partial}{\partial t} g_{ij} + g^{ij} \nabla_l \frac{\partial}{\partial t} g_{il} - g^{ij} \nabla_l \frac{\partial}{\partial t} g_{ij} = 0,
\]
where in the last equality we used (1.1) and the contracted second Bianchi identity. Therefore (2.3) holds.

Now the first term in (2.2) is
\[
- \int 2R^{ij} u \nabla_i \nabla_j u d\mu = \int (2 \nabla_i R^{ij}) u \nabla_j u d\mu + \int 2R^{ij} \nabla_i u \nabla_j u d\mu.
\]
The contracted second Bianchi identity implies that
\[
\int (2\nabla_i R^i) u \nabla_j u d\mu = \int u (\nabla^j R) \nabla_j u d\mu = -\int Ru \Delta u d\mu - \int R|\nabla u|^2 d\mu = \lambda \int Ra^2 d\mu - \int R|\nabla u|^2 d\mu.
\]

The above two equations give
\[
-\int 2R^i u \nabla_i \nabla_j u d\mu = \lambda \int Ru^2 d\mu - \int R|\nabla u|^2 d\mu + \int 2Rc(\nabla u, \nabla u) d\mu.
\]

Therefore (2.2) becomes
\[
\frac{d}{dt} \lambda \int u^2 d\mu = \lambda \int Ru^2 - \int R|\nabla u|^2 + \int 2Rc(\nabla u, \nabla u) - \frac{2r}{n} \lambda \int u^2 d\mu
\]
that is,
\[
(2.4) \quad \frac{d}{dt} \lambda = \frac{\int_M [R - \frac{2}{n} r] u^2 d\mu}{\int_M u^2 d\mu} \lambda + \frac{2 \int_M Rc(\nabla u, \nabla u) - \frac{1}{2} R|\nabla u|^2] d\mu}{\int_M u^2 d\mu}.
\]

On the other hand, the evolution equation of \( R \)
\[
\frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2 - \frac{2}{n} R.
\]

and the inequality
\[
|Rc|^2 \geq \frac{1}{n} R^2
\]
imply that
\[
(2.5) \quad \frac{\partial}{\partial t} R \geq \Delta R + \frac{2}{n} R(R - r).
\]

Therefore by the maximum principle, we have
\[
R \geq \varphi(t),
\]
where \( \varphi \), as defined in (2.1), is the solution to the ODE initial value problem
\[
\frac{d}{dt} \varphi = \frac{2}{n} \varphi(\varphi - r), \quad \varphi(0) = \rho_0.
\]

Taking this time-dependent lower bound of \( R \) into (2.4) and using the non-negativity of the Einstein tensor, we get
\[
\frac{d}{dt} \lambda \geq [\varphi(t) - \frac{2}{n} r(t)] \lambda
\]
and
\[ \frac{d}{dt} \left\{ e^{\int_0^td\tau(\tau) - \varphi(\tau)\lambda(t)} \right\} \geq 0. \]

**Theorem 2.2.** Let \( g(t) \) be a solution to the normalized Ricci flow \( (1.1) \) on a closed two-dimensional manifold, \( \lambda = \lambda(t) \) any eigenvalue of the Laplacian \( \Delta \) of \( (M,g(t)) \). Then the quantity
\[ \left| \frac{\rho_0}{r_0} - \frac{\rho_0}{r_0} e^{r_0 t} + e^{r_0 t} \right| \lambda(t) \]
is non-decreasing if the Euler characteristic \( \chi \neq 0 \); and the quantity
\[ (1 - \rho_0 t)\lambda(t) \]
is non-decreasing if the Euler characteristic \( \chi = 0 \), along the normalized Ricci flow \( g(t) \).

**Proof.** By \( (1.1) \) we have
\[ \frac{d}{dt}(d\mu) = (r - R)d\mu \]
and
\[ \frac{d}{dt} \int_M d\mu = \int_M (r - R)d\mu = 0. \]
So the volume of \( M \) remains constant in \( t \) along the flow.

Now in dimension \( n = 2 \), the Gauss-Bonnet Theorem implies that
\[ r = 4\pi \chi / \int_M d\mu. \tag{2.6} \]
Therefore \( r \) is a constant, \( r \equiv r_0 \).

Note that in dimension two the evolution of \( R \) is
\[ \frac{\partial}{\partial t} R = \Delta R + R(R - r). \]
Therefore we have by the maximum principle
\[ R \geq \varphi(t), \]
where \( \varphi \) is the solution to the ODE initial value problem
\[ \frac{d}{dt} \varphi = \varphi(\varphi - r), \quad \varphi(0) = \rho_0 =: \min_M R|_{t=0}. \]
Note that
\[ \varphi(t) = \frac{r_0}{1 - \left(1 - \frac{\rho_0}{r_0}\right) e^{r_0 t}} \]
in the case of \( \chi \neq 0 \);
\[ \varphi(t) = \frac{\rho_0}{1 - \rho_0 t} \]
in the case \( \chi = 0 \).
Taking the above time-dependent lower bound \( \varphi(t) \) into (2.4), noticing that the second term is zero since \( Rc = \frac{1}{2}Rg \) in dimension two, and integrating, we get the theorem. \( \square \)

3. Non-increasing Quantities

We have the following results on non-increasing quantities along the normalized Ricci flow.

**Theorem 3.1.** Let \( g(t) \) be the solution to the normalized Ricci flow equation (1.1), \( \lambda = \lambda(t) \) be any eigenvalue of the Laplacian of the metric \( g(t) \) on a closed \( n \)-dimensional manifold \( M \). If the Ricci tensor \( Rc \) is non-negative and the Einstein tensor \( E = Rc - \frac{1}{2}Rg \) is non-positive, that is,

\[
0 \leq Rc \leq \frac{1}{2}Rg,
\]

then the quantity

\[
e^{\frac{2}{n}\sigma(t) - \int_0^t \psi(\tau)d\tau} \lambda(t)
\]

is non-increasing along the normalized Ricci flow, where

\[
(3.1) \quad \psi(t) = 1 / \left\{ e^{\frac{2}{n}\sigma(t)} \left( \frac{1}{\delta_0} - 2 \int_0^t e^{-\frac{2}{n}\sigma(\tau)d\tau} \right) \right\}.
\]

**Proof.** That \( Rc \geq 0 \) implies \( |Rc|^2 \leq R^2 \). Taking this into the evolution equation of \( R \)

\[
\frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2 - 2 \frac{r}{n} R
\]

we get

\[
\frac{\partial}{\partial t} R \leq \Delta R + 2R(R - \frac{1}{n} r).
\]

Compare the above inequality with the ODE

\[
\frac{d}{dt} \psi = 2\psi(\psi - \frac{1}{n} r).
\]

Let \( \psi(t) \) be the solution of the ODE with the initial value

\[
\psi(0) = \delta_0 =: \max_M R|_{t=0}.
\]

It is easy to see that \( \psi \) is the function defined in (3.1). The maximum principle implies that

\[
R \leq \psi(t).
\]

Taking this into (2.4), and noticing the non-positivity of the Einstein tensor, we get

\[
\frac{d}{dt} \lambda \leq \frac{\iint [R - \frac{2}{n} r] u^2 d\mu}{\iint u^2 d\mu} \lambda \leq (\psi(t) - \frac{2}{n} r) \lambda.
\]

and

\[
\frac{d}{dt} \left( e^{\frac{2}{n}\sigma(t) - \int_0^t \psi(\tau)d\tau} \lambda(t) \right) \leq 0.
\]

\( \square \)
Theorem 3.2. Let \( g(t) \) be a solution to the normalized Ricci flow (1.1) on a closed two-dimensional manifold, \( \lambda = \lambda(t) \) any eigenvalue of the Laplacian \( \Delta \) of \( (M, g(t)) \). Then the quantity
\[
\left| \frac{\delta_0}{r_0} - \frac{\delta_0}{r_0} e^{r_0 t} + e^{r_0 t} \right| \lambda(t)
\]

is non-increasing if the Euler characteristic \( \chi \neq 0 \); and the quantity
\[
(1 - \delta_0 t) \lambda(t)
\]
is non-decreasing if the Euler characteristic \( \chi = 0 \), along the normalized Ricci flow \( g(t) \).

Proof. The same argument as in the proof of Theorem 2.2 shows that \( r \) is a constant independent of \( t \), \( r \equiv r_0 \).

By the evolution equation of \( R \) in dimension two
\[
\frac{\partial}{\partial t} R = \Delta R + R(R - r)
\]
and the maximum principle, we have
\[
R \leq \psi(t),
\]
where \( \psi(t) \) is the solution to the ODE initial value problem
\[
\frac{d}{dt} \psi = \psi(\psi - r), \quad \psi(0) = \delta_0 =: \max_M R|_{t=0}.
\]
It is easy to know that
\[
\psi(t) = \frac{r_0}{1 - \left(1 - \frac{r_0}{\delta_0} \right) e^{r_0 t}}
\]
in the case of \( \chi \neq 0 \), and
\[
\psi(t) = \frac{\delta_0}{1 - \delta_0 t},
\]
in the case \( \chi = 0 \).

Taking this into (2.4) and noticing \( Rc = \frac{1}{2} Rg \) in dimension two, we get the theorem.

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