Multilinear Superhedging of Lookback Options

Alex Garivaltis*

October 8, 2018

Abstract

In a pathbreaking paper, Cover and Ordentlich (1998) solved a max-min portfolio game between a trader (who picks an entire trading algorithm, $\theta(\cdot)$) and “nature,” who picks the matrix $X$ of gross-returns of all stocks in all periods. Their (zero-sum) game has the payoff kernel $W_\theta(X)/D(X)$, where $W_\theta(X)$ is the trader’s final wealth and $D(X)$ is the final wealth that would have accrued to a $1$ deposit into the best constant-rebalanced portfolio (or fixed-fraction betting scheme) determined in hindsight. The resulting “universal portfolio” compounds its money at the same asymptotic rate as the best rebalancing rule in hindsight, thereby beating the market asymptotically under extremely general conditions.

Smitten with this (1998) result, the present paper solves the most general tractable version of Cover and Ordentlich’s (1998) max-min game. This obtains for performance benchmarks (read: derivatives) that are separately convex and homogeneous in each period’s gross-return vector. For completely arbitrary (even non-measurable) performance benchmarks, we show how the axiom of choice can be used to “find” an exact maximin strategy for the trader.

Keywords: Super-replication, Lookback options, Minimax, On-line portfolio selection, Constant-rebalanced portfolios, Kelly criterion, Universal portfolios, Variational preferences, Axiom of choice

JEL Classification: C44, C61, C72, D52, D81, G11, G13

*Assistant Professor of Economics, Northern Illinois University, 514 Zulauf Hall, DeKalb IL 60115. E-mail: agarivaltis1@niu.edu. ORCID: 0000-0003-0944-8517.
1 Introduction: volatility harvesting

Any buy-and-hold portfolio or passive index that holds shares of all stocks will compound its money at the same asymptotic rate as the best performing stock in the market. Since the portfolio is never rebalanced, the fraction of wealth held in the stock with the highest growth rate approaches 100% in the long run. Thus, from the standpoint of asymptotic growth, all buy-and-hold strategies (that have full support) are equivalent.

Under mild conditions on the price process of the stock market, there will exist trading strategies that guarantee to compound one’s money at a higher asymptotic rate than every buy-and-hold strategy. For markets with iid returns of known distribution, the asymptotically dominant strategy is called the Kelly Rule (1956). In Kelly betting, the gambler acts each period so as to maximize the expected log of his capital. Literally, he maximizes the expected continuously-compounded capital growth rate over the period throughout which the portfolio is held. By the law of large numbers, the realized growth rate converges to the expected growth rate almost surely.

A Kelly gambler will rebalance the portfolio each period, maintaining a certain growth-optimal fraction of wealth in each stock. Swensen (2005) gives an excellent practical discussion of allocation drift and rebalancing to the target allocation, especially within the context of institutional endowments. Even for agents whose utility is not measured by the asymptotic growth rate, rebalancing serves to continuously maintain the desired risk/return profile. For one thing, the Yale endowment has exploited its tax-advantaged status to earn substantial profits from frequent (even intraday) rebalancing.

The log-optimal rebalancing rule manufactures excess asymptotic growth through
a phenomenon called volatility harvesting. Luenberger (1998) calls it “volatility pumping.” Poundstone (2010) uses the term “Shannon’s Demon” on account of Shannon’s canonical example, given in a lecture at MIT.

Consider a world with cash (that pays no interest) and a single, volatile stock. Each period, there is a 50% chance that the stock price doubles, and a 50% chance that the price gets cut in half. Imagine an investor who starts with a dollar and keeps 100% of his wealth in the stock. After $t$ periods, his expected wealth is $1.25^t$, but his expected log-wealth is zero. His asymptotic capital growth rate will almost surely be zero.

By contrast, the growth-optimal policy is to keep half of wealth in the stock and half in cash at all times. Whenever the stock doubles, the Kelly gambler must sell some shares to restore the target allocation. Likewise, when the stock price gets cut in half, he must summon the courage to buy additional shares. Through this rebalancing, the gambler is able to grow his capital an asymptotic rate of 5.9% per-period. Although the buy-and-hold investor has a higher expected final wealth ($1.25^t$ versus $1.125^t$), he will almost surely underperform the Kelly gambler as $t \to \infty$ (by an exponential factor). The Kelly gambler achieves this outperformance in spite of the fact that he is taking only half the risk of the buy-and-hold investor.

Breiman (1961) proved that the Kelly rule asymptotically outperforms any “essentially different strategy,” and that it has a shorter mean waiting time to reach a distant wealth goal. Inspired by this paper, Edward Thorp (of card-counting fame) used the Kelly criterion to properly size his bets at the Nevada blackjack tables. He went on to use log-optimal portfolios as a professional fund manager, in direct competition with Harry Markowitz, who used his own (1952) theory. This is all discussed in Thorp (2006). Thorp (1969) critiques the mean-variance theory and gives an example of a log-optimal portfolio that is not on the mean-variance frontier.
The stock market is different from a game of blackjack in the sense that the distribution of payoffs (returns) is never actually known *a priori*. Whereas the distribution of payoffs on a given hand of blackjack can in principle be computed explicitly from the rules and structure of the game, the stock market practitioner must specify a return process and estimate the parameters on the fly.

In an important paper, Cover and Gluss (1986) gave the first trading strategy that could guarantee, despite ignorance of the return process, to compound its money at the Kelly asymptotic growth rate. Restricting the return vector to a finite set, they apply Blackwell’s (1956) approachability theorem to show that the Kelly growth rate is uniformly asymptotically approachable. In fact, the trader can guarantee to compound his wealth at the same asymptotic rate as the best rebalancing rule in hindsight, for all possible market behavior. Their “Empirical Bayes Stock Portfolio” has three main defects. First, the convergence is slow. Second, the cardinality of the finite set must be known (or at least bounded) beforehand. Third, the practitioner is required to solve optimization problems in as many dimensions as there are possible values of the return vector. Cover’s (1987) survey of log-optimal portfolios gives a better explanation of the intuition underlying the (1986) paper.

Cover’s aptly-named (1991) “universal portfolio” remedies all these problems without having to restrict the return vector to a finite set. The basic idea is so simple that its veracity hardly needs a technical proof. Imagine that we have a dollar, and that we would prefer that this dollar be managed by the Kelly rule, which is some (unknown) point of the portfolio simplex. In our ignorance, we distribute the dollar uniformly over all the rebalancing rules in the simplex. The amount of money managed by the portfolios in any region of the simplex is now proportional to the volume of that region. On account of allocation drift, the portfolios in the vicinity of the Kelly rule will hold, asymptotically, 100% of the aggregate wealth. Thus, the
overall wealth (that accrues to the initial one dollar investment) will grow at the same asymptotic rate as the Kelly rule. Jamshidian (1992) extends the technique to continuous-time, where stock prices are assumed to follow an Itô process with unknown drift and diffusion. Helmbold, Schapire, Singer, and Warmuth (1998) give a finite-memory universal strategy that only needs to remember the portfolio vector used in the preceding period.

Cover and Ordentlich (1998) give a universal portfolio that is optimized for a specific investment horizon. They solve a two-person, simultaneous move, zero-sum game between a trader and nature. The trader picks an entire trading algorithm, and nature picks the returns of all stocks in all periods. The payoff is the ratio of the trader’s final wealth to that of the best rebalancing rule in hindsight. Remarkably, the game is an outgrowth of Shtarkov’s (1987) theory of universal data compression, which solves a discrete version of the problem. Shtarkov’s theory implies that it is possible to compress a stream of discrete symbols to the Shannon (entropy) limit, even when lacking an \textit{a priori} statistical model of the data. Taking a somewhat ad hoc (guess-and-verify) approach, Cover and Ordentlich provide a trading strategy that exactly achieves the lower value of the game. Cover and Thomas’ (2006) textbook on information theory gives an exposition that is slightly more intelligible than the (1998) paper.

The aim of the present paper is to put universal portfolios in their most natural and general setting, as superhedges of a specific type of lookback option, namely, the final wealth of the best rebalancing rule in hindsight. A superhedge for a derivative is a self-financing trading strategy, together with an initial deposit of money, that guarantees to generate final wealth at least as large as the derivative, for all possible paths of stock prices. Thus, the trading strategy is said to “super-replicate” the option. The concept is due to Bensaid, Lesne, Pages, and Scheinkman (1991). They
show that in the Cox-Ross-Rubinstein (1979) model, under proportional transaction costs, it can be cheaper to super-replicate an option than to hedge it exactly.

Any reasonably efficient superhedge of the hindsight-optimized final wealth (vis-à-vis rebalancing rules) will “beat the market asymptotically” for completely arbitrary price behavior, provided that the best rebalancing rule in hindsight is able to sustain a higher asymptotic growth rate than the best performing stock in the market. Accordingly, this paper constructs a theory of “multilinear superhedging” that generalizes Cover’s universal portfolios. A multilinear trading strategy is one whose final wealth is linear separately in each period’s gross-return vector. Since every multilinear final wealth function corresponds to a unique replicating strategy, one can directly choose the multilinear final wealth that most efficiently super-replicates a given derivative. For derivatives that are convex and homogeneous in each period’s return vector, there will exist a unique multilinear superhedge that is cheaper than any other (multilinear or not). More generally, the paper uses the axiom of choice to show that minimum-cost superhedges (not necessarily multilinear) exist for all possible derivatives.

2 Stock Market

We take up the general market with \( m \) underlying assets (“stocks”) and \( T \) discrete trading sessions, called \( t \in \{1, ..., T\} \). Let \( S_{tj} \) be the price of stock \( j \) at the close of session \( t \), where \( S_{0j} \) is the initial price of stock \( j \). Let \( x_{tj} \geq 0 \) be the gross-return of a $1 investment in stock \( j \) in session \( t \). The net return is \( x_{tj} - 1 \). Thus \( x_{tj} = S_{tj}/S_{t-1,j} \) and \( S_{tj} = S_{0j}x_{1j}x_{2j} \cdots x_{tj} \), assuming that stock \( j \) pays no dividends. More generally, if each common share of stock \( j \) receives a dividend of \( \delta_{tj} \) dollars to the bearer of record in session \( t \), then \( x_{tj} = (S_{tj} + \delta_{tj})/S_{t-1,j} \).

We assume that, at the start of session \( t \), it is possible to buy any number of shares
of stock $j$ at the opening price $S_{t-1,j}$ (so long as we can afford it), and thereby participate fully in the gross return $x_{tj}$. The portfolio that we buy at the open of session $t$ (the close of session $t-1$) must be held until the open of session $t+1$, at which time our holdings in the various stocks will be adjusted. We imagine a completely frictionless situation with no market impact, taxes, transaction costs, or bid/ask spreads. The gross-return vector in session $t$ is denoted $x_t = (x_{t1},...,x_{tj},...,x_{tm}) \in \mathbb{R}_+^m - \{0\}$. The return history after session $t$ is $x^t = (x_1,...,x_t)$, with transition law $x^{t+1} = (x^t,x_{t+1})$.

The empty history is called $h^0$. The set of all return histories (state space) is $\mathcal{H} = \{h^0\} \cup \bigcup_{t=1}^T (\mathbb{R}_+^m)^t$. In accordance with limited liability, all prices and gross returns are nonnegative, with the proviso that $x_t$ is never the zero vector.

### 3 Trading strategies

We consider self-financing trading strategies, generally called $\theta(\cdot)$. Literally, the strategy finances its asset purchases internally, via the sales of other assets. A self-financing strategy is not subject to any deposits or withdrawals other than the initial deposit of money into the strategy.

Thus, a trader deposits $1$ into $\theta(\cdot)$ at $t = 0$ and just “lets it ride.” Let $\theta_{tj} \geq 0$ be the fraction of wealth that the trader puts into stock $j$ at the start of session $t$, where $\sum_{j=1}^m \theta_{tj} = 1$. Thus $\theta_{tj} = \theta_{tj}(x_1,...,x_{t-1}) = \theta_{tj}(x^{t-1})$. The trader’s portfolio vector in session $t$ is denoted $\theta_t = (\theta_{t1},...,\theta_{tm})$. For simplicity, we will merely write $\theta(x_1,...,x_t) \in \Delta$ for the trader’s portfolio in session $t+1$. A trading strategy is a mapping $\theta : \mathcal{H} \rightarrow \Delta$ of the return histories into the portfolio simplex, where $\Delta = \{b \in \mathbb{R}_+^m : \sum_{j=1}^m b_j = 1\}$. Thus, $\theta(h^0)$ is the initial portfolio. A common choice is $\theta(h^0) = (1/m,...,1/m)$. The set of all trading strategies is denoted $\Theta = \Delta^\mathcal{H}$. This set is compact with respect to the product topology.
In session $t$, $\theta(\cdot)$ multiplies the trader’s wealth by the factor \[ \sum_{j=1}^{m} \theta_{tj} (x_{t-1}) x_{tj} = \langle \theta_t(x_{t-1}), x_t \rangle, \] the dot product of the portfolio vector and the return vector. After $T$ sessions, the trader’s initial dollar has grown into \[ W_{\theta}(x_1, \ldots, x_T) = \langle \theta(h^0), x_1 \rangle \langle \theta(x_1), x_2 \rangle \cdots \langle \theta(x_1, \ldots, x_{T-1}), x_T \rangle. \] (1)

This equation formalizes the fact that $\theta(\cdot)$ is self-financing. $W_{\theta}(\cdot)$ is called the final wealth function induced by $\theta$.

**Proposition 1.** If $W(x_1, \ldots, x_T) = W_{\theta}$ is a feasible final wealth function, then \[ \sum_{(j_1, \ldots, j_T) \in \{1, \ldots, m\}^T} W(e_{j_1}, \ldots, e_{j_T}) = 1, \] (2) where $e_k$ denotes the $k^{th}$ unit basis vector for $\mathbb{R}^m$, and the sum is taken over all possible unit basis vectors $e_{j_1}, \ldots, e_{j_T}$.

**Proof.** We start by writing $W(e_{j_1}, \ldots, e_{j_T}) = W(e_{j_1}, \ldots, e_{j_{T-1}}) \langle \theta(e_{j_1}, \ldots, e_{j_{T-1}}), e_{j_T} \rangle$ and summing both sides over $j_T = 1, \ldots, m$. Since the coordinates of any portfolio vector sum to 1, we get \[ \sum_{j_T=1}^{m} W(e_{j_1}, \ldots, e_{j_T}) = W(e_{j_1}, \ldots, e_{j_{T-1}}). \] By induction on $T$, when we sum both sides of this last equation over all $(j_1, \ldots, j_{T-1}) \in \{1, \ldots, m\}^{T-1}$, we get 1, which is the desired result. The base case states that \[ \sum_{j_1=1}^{m} \langle \theta(h^0), e_{j_1} \rangle = 1, \] which is true since the coordinates of the initial portfolio vector sum to 1. \[ \square \]

**Proposition 2.** The mapping $\theta \mapsto W_{\theta}$ is continuous with respect to the product topology. Thus, the set of all feasible final wealth functions (that accrue to an initial $\$1$ deposit) is a compact subset of $\mathbb{R}_+^{mT}$.

**Proof.** Suppose that $\theta(\cdot)$ is a given trading strategy, $x^T = (x_1, \ldots, x_T)$ is a given (fixed) return history, and $\epsilon$ is a given positive number. We must find a neighborhood $U$ of...
θ such that for all trading strategies $\psi(\cdot)$ in this neighborhood, $|W_\psi(x^T) - W_\theta(x^T)| < \epsilon$. Since polynomials are continuous, there exists a neighborhood $G$ of the vector $(\theta(h^0), \theta(x^1), \ldots, \theta(x^{T-1}))$ in $\Delta^T$ such that, for all $(b_1, b_2, \ldots, b_T) \in G$, we have $|\langle b_1, x_1 \rangle \cdots \langle b_T, x_T \rangle - W_\theta(x^T)| < \epsilon$. Setting $U = \{ \psi \in \Theta : (\psi(h^0), \psi(x^1), \ldots, \psi(x^{T-1})) \in G \}$ gives the desired result. \hfill \Box

**Example 1.** A buy-and-hold strategy makes some initial distribution of wealth among the stocks and then “lets it ride.” It induces a final wealth function of the form

$$W_\theta(x_1, \ldots, x_T) = \sum_{j=1}^m \left\{ c_j \prod_{t=1}^T x_{tj} \right\},$$

(3)

where $c_j$ is the initial fraction of wealth put into stock $j$.

**Definition 1.** A **constant rebalancing rule** is a strategy $\theta(x^t) = c = (c_1, \ldots, c_m) \in \Delta$ that keeps a constant fraction of wealth $c_j$ in each stock.

The rebalancing rule $c$ multiplies the trader’s wealth by $\langle c, x_t \rangle$ in session $t$, and induces the final wealth function

$$W_c(x_1, \ldots, x_T) = \langle c, x_1 \rangle \langle c, x_2 \rangle \cdots \langle c, x_T \rangle.$$  

(4)

A rebalancing rule is not a buy-and-hold strategy. At the start of session $t$, the trader puts the fraction $c_j$ of his wealth into each stock $j$, and his portfolio is balanced. At the end of session $t$, just after $x_t$ has been realized, he suddenly has the fraction $c_j x_{tj}/\langle c, x_t \rangle$ of his wealth in stock $j$. The trader is now overweight stock $j$ if $x_{tj} > \langle c, x_t \rangle$, otherwise he is underweight stock $j$. The portfolio must be rebalanced again at the start of session $t + 1$. After each fluctuation, the rebalancing rule $c$ dictates that he sell some shares of the stocks that outperformed the portfolio, and put the
proceeds into the stocks that underperformed the portfolio. Thus, a rebalancing rule will generally trade every period. A rebalancing rule \( c \) is called degenerate if \( c = e_j \) is some unit basis vector. A degenerate rule keeps 100% of wealth in stock \( j \) at all times, and so it just amounts to buying and holding stock \( j \).

**Definition 2.** If the \( x_t \) are independent and identically distributed, then the *Kelly rebalancing rule* is defined by

\[
c^* = \arg\max_{c \in \Delta} E[\log \langle c, x_t \rangle].
\]

(5)

The number \( \gamma^* = \max_{c \in \Delta} E[\log \langle c, x_t \rangle] \) is called the *Kelly asymptotic growth rate*.

By its very definition, the Kelly rule maximizes the expected per-period continuously compounded capital growth rate. The Law of Large Numbers implies that the Kelly gambler’s realized per-period growth rate converges to \( \gamma^* \) almost surely. Asymptotically, the Kelly gambler has (exponentially) more wealth than any gambler that follows an “essentially different” strategy, and he has the shortest mean waiting time to reach a distant wealth goal (Breiman 1961). More generally, we have the *conditionally log-optimal trading strategy* \( \theta(x^{t-1}) = \arg\max_{c \in \Delta} E[\log \langle c, x_t \rangle | x^{t-1}] \), which is asymptotically dominant against a (known) ergodic stationary return process (Cover and Thomas 2006).

The set of all trading strategies is convex, it being the product of convex sets. The convex combination \( \lambda \theta(x^t) + (1 - \lambda) \psi(x^t) \) amounts to maintaining a fixed fraction of wealth in each of the two trading strategies \( \theta \) and \( \psi \). For, the gross-return of the strategy in session \( t + 1 \) is \( \langle \lambda \theta(x^t) + (1 - \lambda) \psi(x^t), x_{t+1} \rangle = \lambda \langle \theta(x^t), x_{t+1} \rangle + (1 - \lambda) \langle \psi(x^t), x_{t+1} \rangle \), which amounts to handing the fraction \( \lambda \) of wealth over to \( \theta(\cdot) \) and \( (1 - \lambda) \) over to \( \psi(\cdot) \) at the start of each trading session.
4 Derivatives

We take up the most general derivative security, which pays off an amount \( D(x_1,\ldots,x_T) \) at the close of session \( T \). The derivative is written (created and sold) by a primary-dealer at \( t = 0 \). At each date thereafter, \( D \) is traded on the secondary market alongside the \( m \) stocks. Note that in each session we have a continuum of possible outcomes \( x_t \in \mathbb{R}^m_+ \), so that \( D \) is not generally a redundant asset.

**Definition 3.** A derivative \( D(x_1,\ldots,x_T) \) is called **multilinear** if it is linear separately in each vector \( x_t \). It is called **multiconvex** if it is convex separately in each \( x_t \).

**Proposition 3.** If \( D \) is convex and positively homogeneous separately in each vector \( x_t \) (equivalently, subadditive and positively homogeneous in each \( x_t \)), then \( D \) is majorized by the multilinear derivative

\[
D \leq \sum_{(j_1,\ldots,j_T) \in \{1,\ldots,m\}^T} D(e_{j_1},\ldots,e_{j_T})x_{1j_1}x_{2j_2}\cdots x_{Tj_T} \tag{6}
\]

**Proof.** By subadditivity and homogeneity, we have

\[
D(x_1,\ldots,x_T) = D(x_1,\ldots,x_{T-1},e_1 + \cdots + x_T e_m) \leq \sum_{j_T=1}^m D(x_1,\ldots,x_{T-1},e_{j_T})x_{Tj_T}.
\]

By induction, we can majorize \( D(x_1,\ldots,x_{T-1},e_{j_T}) \) by

\[
\sum_{(j_1,\ldots,j_{T-1}) \in \{1,\ldots,m\}^{T-1}} D(e_{j_1},\ldots,e_{j_{T-1}})x_{1j_1}x_{2j_2}\cdots x_{(T-1)j_{T-1}},
\]

and the result follows. For \( T = 1 \), the proposition says that \( D(x_1) \leq \sum_{j_1=1}^m D(e_{j_1})x_{1j_1} \), which is true by subadditivity and homogeneity. \( \square \)

**Example 2.** The final wealth function of every rebalancing rule and of every buy-and-hold strategy is a multilinear derivative.

**Example 3.** Consider the strategy that follows the \( c \)-rebalancing rule, but only rebalances the portfolio every \( \tau \) periods. The final wealth function of this strategy is a multilinear derivative.
Multilinear stock indexes

Example 4. The general price-weighted (e.g. Dow Jones) index has the form $D = \lambda(S_{t_1} + S_{t_2} + \cdots + S_{t_m})$, where $\lambda$ is a scale factor chosen for convenience. This is multilinear, since $S_{t_j}$ is a multilinear function of the return data. To replicate the index, simply buy $\lambda$ shares of each stock, and hold. The replicating strategy is the price-weighted portfolio vector $\theta(x^t) = (S_{t_1}, \ldots, S_{t_m})/\sum_{j=1}^{m} S_{t_j}$.

Example 5. The general capitalization-weighted index (e.g. S&P 500) has the form $D = \lambda(n_1S_{t_1} + n_2S_{t_2} + \cdots + n_mS_{t_m})$, where $n_j$ is the number of shares firm $j$ has outstanding. To replicate the index, buy $\lambda n_j$ shares of each firm $j$, and hold. The replicating strategy is the capitalization-weighted portfolio vector $\theta(x^t) = (n_1S_{t_1}, \ldots, n_mS_{t_m})/\sum_{j=1}^{m} n_jS_{t_j}$.

Example 6. An equal-weight index\(^1\) is defined by the uniform rebalancing rule $c = (1/m, \ldots, 1/m)$. It has the form

$$D = \lambda \prod_{t=1}^{T} \left\{ \frac{1}{m} \sum_{j=1}^{m} x_{tj} \right\} = \lambda m^{-T} \langle 1, x_1 \rangle \langle 1, x_2 \rangle \cdots \langle 1, x_T \rangle. \quad (7)$$

Lookback options

Example 7 (Perfect Trader). Suppose you knew the future price charts of all stocks in advance. Each period, you would put all your money into the stock with the highest gross-return. Your growth factor in session $t$ would be $\text{Max}_{1 \leq j \leq m} x_{tj} = ||x_t||_{\infty}$. Your final wealth would be $D = ||x_1||_{\infty}||x_2||_{\infty} \cdots ||x_T||_{\infty}$, where $|| \cdot ||_{\infty}$ is the infinity norm.

Example 8 (Perfect Buy-and-Hold Investor). Suppose you knew the single best performing stock in advance. You would put all your money into that stock, and hold.

\(^1\)The leading equal-weight ETFs (from Guggenheim) are only rebalanced quarterly. These indexes are still multilinear if, say, the length of a trading session is one second, minute, hour, day, week, month, etc.
Your final wealth would be \( D = \max_{1 \leq j \leq m} \prod_{t=1}^{T} x_{tj} \).

**Example 9.** Suppose you know the entire price chart of a certain stock \( j \) in advance. You know the single best period \( s \) to buy and the best time \( t \) to sell. The profit from the single shrewdest trade on \( 1 \leq t \leq T \) is \( \max_{1 \leq s \leq t \leq T} \{ S_{tj} - S_{sj} \} \).

**Example 10 (Cover’s Derivative).** For the return path \( x_1, ..., x_T \), the best rebalancing rule in hindsight is \( \arg\max_{c \in \Delta} \langle c, x_1 \rangle \langle c, x_2 \rangle \cdots \langle c, x_T \rangle \). The hindsight-optimized final wealth is \( D(x_1, ..., x_T) = \max_{c \in \Delta} \langle c, x_1 \rangle \langle c, x_2 \rangle \cdots \langle c, x_T \rangle \).

**Definition 4.** A derivative \( D(x_1, ..., x_T) \) is said to be **perfectly hedgeable** (or replicable) iff there is a self-financing trading strategy \( \theta(\cdot) \) and an initial deposit \( p \) such that \( D = p \cdot W_{\theta} \) for all \( x_1, ..., x_T \). This (necessarily unique) \( \theta \) is called the hedging (or replicating) strategy corresponding to \( D(\cdot) \). The (unique) initial deposit \( p = p^*[D] \) is called the **hedging cost**.

**Proposition 4.** If \( D(\cdot) \) is exactly hedgeable, then the hedging cost is given by the linear functional
\[
p^*[D] = \sum_{(j_1, ..., j_T) \in \{1, ..., m\}^T} D(e_{j_1}, ..., e_{j_T})
\]
The unique replicating strategy is
\[
\theta_k(x^t) = \frac{\sum_{(j_{t+1}, ..., j_T) \in \{1, ..., m\}^{T-t-1}} D(x^t, e_k, e_{j_{t+2}}, ..., e_{j_T})}{\sum_{(j_{t+1}, ..., j_T) \in \{1, ..., m\}^{T-t-1}} D(x^t, e_{j_{t+1}}, ..., e_{j_T})}, \quad (8)
\]
where \( \theta_k(x^t) \) denotes the \( k^{th} \) coordinate of \( \theta(x^t) \).

Assuming that \( D = p \cdot W_{\theta} \) can be hedged perfectly, the unique replicating strategy is derived as follows. Start with
\[
\langle \theta(x^t), x_{t+1} \rangle = W_{\theta}(x^{t+1})/W_{\theta}(x^t), \quad (9)
\]
and substitute \( x_{t+1} = e_j \). This gives \( \theta_j(x^t) = W_{\theta}(x^t, e_j)/W_{\theta}(x^t) \). Summing over \( j \),
we get \( W_\theta(x^t) = \sum_{j=1}^{m} W_\theta(x^t, e_j) \). Applying this last formula repeatedly, one gets the formulas

\[
W_\theta(x^t) = \sum_{(j_{t+1}, ..., j_T) \in \{1, ..., m\}^{T-t}} W_\theta(x^t, e_{j_{t+1}}, ..., e_{j_T})
\]  
(10)

\[
\theta_k(x^t) = \frac{\sum_{(j_{t+2}, ..., j_T) \in \{1, ..., m\}^{T-t-1}} D(x^t, e_{k}, e_{j_{t+2}}, ..., e_{j_T})}{\sum_{(j_{t+1}, ..., j_T) \in \{1, ..., m\}^{T-t}} D(x^t, e_{j_{t+1}}, ..., e_{j_T})}
\]  
(11)

In general, the above formula for \( \theta(\cdot) \) in terms of \( D(\cdot) \) may well be an extraneous solution of the functional equation \( D/p = W_\theta \). Of course, one must substitute the strategy \( \theta \) so obtained back into the equation \( D/p = W_\theta \), and verify that it is a solution. This is illustrated below.

**Proposition 5.** A derivative \( D(x_1, ..., x_T) \) can be exactly dynamically replicated if and only if it satisfies the following functional equation, identically for all \( x_1, ..., x_T \):

\[
\prod_{t=1}^{T} \left( \frac{\sum_{(j_t, ..., j_T) \in \{1, ..., m\}^{T-t+1}} D(x^{t-1}, e_{j_t}, ..., e_{j_T}) x_{tj_t}}{\sum_{(j_t, ..., j_T) \in \{1, ..., m\}^{T-t+1}} D(x^{t-1}, e_{j_t}, ..., e_{j_T})} \right) \equiv \frac{D(x_1, ..., x_T)}{\sum_{(j_1, ..., j_T) \in \{1, ..., m\}^{T}} D(e_{j_1}, ..., e_{j_T})}
\]

(12)

**Corollary 1.** If \( D(x_1, ..., x_T) \geq 0 \) is a multilinear form, e.g. it is linear separately in each vector argument \( x_t \), then \( D \) can be replicated exactly.

To see this, we can write \( D(x^{t-1}, e_{j_t}, ..., e_{j_T}) x_{tj_t} = D(x^{t-1}, x_{tj_t} e_{j_t}, ..., e_{j_T}) \) on account of the fact that \( D \) is homogeneous separately in each argument. We then sum this equation over the indices \( j_t = 1, 2, ..., m \), and get \( D(x^t, e_{j_{t+1}}, ..., e_{j_T}) \) on account of the fact that \( D \) is additive separately in each vector argument. The product on the left-hand side of the functional equation is now seen to be telescopic; it collapses exactly to the ratio given on the right-hand side of the functional equation.

**Corollary 2.** When restricted to the set of all strictly positive trading strategies \( (\theta(x^t) >> 0) \), the mapping \( \theta \mapsto W_\theta \) is a homeomorphism.
Proposition 6. The set of all exactly hedgeable derivatives is a closed, convex cone.

If $D^1$ is replicated by $\theta(\cdot)$ and $D^2$ is replicated by $\psi(\cdot)$, then the nonnegative combination $\lambda D^1 + (1 - \lambda)D^2$ can be replicated by depositing $\lambda p^*[D^1]$ dollars into $\theta(\cdot)$, and $(1 - \lambda)p^*[D^2]$ dollars into $\psi(\cdot)$, and “letting it ride.” Let $\eta(\cdot)$ be the strategy that replicates $\lambda D^1 + (1 - \lambda)D^2$. Then we have

$$\eta_k(x^t) = \frac{\lambda p^*[D^1]W_\theta(x^t)\theta_k(x^t) + (1 - \lambda)p^*[D^2]W_\psi(x^t)\psi_k(x^t)}{\lambda p^*[D^1]W_\theta(x^t) + (1 - \lambda)p^*[D^2]W_\psi(x^t)}.$$  (13)

This is a valid trading strategy, since $\eta_k(x^t) \geq 0$ and $\sum_{k=1}^m \eta_k(x^t) = 1$. A direct calculation verifies that $W_\eta = \lambda D^1 + (1 - \lambda)D^2$.

5 Multilinear derivatives

Definition 5. A multilinear trading strategy is one that induces a multilinear final wealth function.

The general multilinear derivative has the form

$$D(x_1, \ldots, x_T) = p^*[D] \sum_{j_1, \ldots, j_T} \alpha(j_1, \ldots, j_T) x_{1j_1} \cdots x_{Tj_T},$$  (14)

where the coefficients $\alpha(j_1, \ldots, j_T)$ are nonnegative and sum to 1. To specify a multilinear trading strategy, we just pick the coefficients $\alpha(j_1, \ldots, j_T) = D(e_{j_1}, \ldots, e_{j_T})/p^*[D]$ and work out the implied portfolio vectors $\theta(x^t)$. 
Proposition 7. If $D(\cdot)$ is a multilinear derivative, then the replicating strategy is

$$
\theta_k(x^t) = \frac{\sum_{(j_1, \ldots, j_{t-1}) \in \{1, \ldots, m\}^{T-1}} D(e_{j_1}, \ldots, e_{j_{t-1}}, e_k, e_{j_{t+2}}, \ldots, e_{j_T}) x_{1j_1} x_{2j_2} \cdots x_{lj_l}}{\sum_{(j_1, \ldots, j_T) \in \{1, \ldots, m\}^T} D(e_{j_1}, \ldots, e_{j_T}) x_{1j_1} x_{2j_2} \cdots x_{lj_l}}.
$$

(15)

The denominator is just the sum of the numerators for $k = 1, \ldots, m$. The numerators are multilinear functions of the return data $(x_1, \ldots, x_t)$. In the $k^{th}$ numerator, the coefficient of the product $x_{1j_1} \cdots x_{lj_l}$ is given by

$$
\alpha(j_1, \ldots, j_l) = \sum_{j_{t+2}, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_l}, e_k, e_{j_{t+2}}, \ldots, e_{j_T}).
$$

(16)

5.1 Extremal strategies

Definition 6. A trading strategy is extreme (or an extreme point) iff $\theta(x^t)$ is always some unit basis vector. A strategy is memoryless iff each period’s portfolio vector depends only on the time (and not the history). A strategy that is both memoryless and extreme is called an extremal strategy.

Each period, an extreme strategy puts all of its wealth into some stock $j^*(x^t)$, which represents the strategy’s guess as to what the best performing stock will be. An extreme trading strategy induces a final wealth function of the form $\prod_{t=1}^T x_{tj^*(x^{t-1})}$. We will restrict our attention to memoryless extreme strategies. A memoryless extreme strategy is characterized by a tuple $j^T = (j_1, \ldots, j_T) \in \{1, \ldots, m\}^T$. In session $t$, the strategy puts all of its money into stock $j_t$, and has final wealth $\prod_{t=1}^T x_{tj_t}$.

Proposition 8. The set of multilinear derivatives is the conic hull of the set of $(m^T)$ extreme, memoryless strategies.
There is a very intuitive way to understand the replication of the general multilinear payoff \( \sum_{j} \alpha(j^T)x_{1j_1} \cdots x_{Tj_T} \). We take the initial dollar and distribute it among the \( m^T \) memoryless, extreme strategies, and just “let it ride.” We put the fraction \( \alpha(j^T) \) of wealth into strategy \( j^T \). After \( T \) periods, the initial deposit into \( j^T \) has grown to \( \alpha(j^T)x_{1j_1} \cdots x_{Tj_T} \), and thus the aggregate wealth is \( \sum_{j} \alpha(j^T)x_{1j_1} \cdots x_{Tj_T} \).

With this idea in mind, it becomes simple to recover the replicating strategy, without having to memorize any formulas. After \( t \) periods, the overall wealth of the strategy is

\[
\sum_{j^T} \alpha(j^T)x_{1j_1} \cdots x_{tj_t}. \tag{17}
\]

How much money does the composite strategy put into stock \( k \) in session \( t + 1 \)? We must look at the extremals of the form \((j_1, \ldots, j_t, k, j_{t+2}, \ldots, j_T)\) that put all money into stock \( k \) in session \( t + 1 \). In aggregate, these strategies have put

\[
\sum_{j_1, \ldots, j_t, j_{t+2}, \ldots, j_T} \alpha(j_1, \ldots, j_t, k, j_{t+2}, \ldots, j_T)x_{1j_1} \cdots x_{tj_t} \tag{18}
\]
dollars into stock \( k \) in session \( t + 1 \). Thus, the total fraction of wealth put into stock \( k \) in session \( t + 1 \) is

\[
\theta_k(x^T) = \frac{\sum_{j_1, \ldots, j_t, j_{t+2}, \ldots, j_T} \alpha(j_1, \ldots, j_t, k, j_{t+2}, \ldots, j_T)x_{1j_1} \cdots x_{tj_t}}{\sum_{j^T} \alpha(j^T)x_{1j_1} \cdots x_{tj_t}}. \tag{19}
\]

### 5.2 Symmetric multilinear strategies

We can think of the extremal strategy \( j^T = (j_1, \ldots, j_T) \) as a certain “expert” who in period \( t \) recommends that we put all our money into stock \( j_t \). Arbitrary multilinear strategies can be difficult to compute in practice, since there are so many experts to account for (e.g. the sums have an exponential number of terms).
Definition 7. A multilinear trading strategy is called \textit{symmetric} if the induced final wealth function $W(x_1, ..., x_T)$ is symmetric with respect to the vector arguments $x_1, ..., x_T$, e.g. $W(x_1, ..., x_T) = W(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(T)})$ for any permutation $\sigma$ of the indices $1, 2, ..., T$.

A symmetric multilinear strategy is characterized by the fact that the numerical values of the coefficients $\alpha(j_1, ..., j_T)$ do not depend on the order of the indices $j_1, ..., j_T$.

Definition 8. Let $n_k$ be the number of $k$'s that appear among the indices $j_1, ..., j_T$. The counts $(n_1, ..., n_m)$ constitute the \textbf{type} of the expert $j^T = (j_1, ..., j_T)$, where $n_k \geq 0$ and $\sum_{k=1}^{m} n_k = T$.

For a symmetric multilinear strategy, the coefficients $\alpha(j^T)$ depend only on the type of $j^T$, and we can write $\alpha(n_1, ..., n_m)$. Two experts $j^T$ and $k^T$ are called \textit{equivalent}, and we write $j^T \sim k^T$, if they have the same type. This relation decomposes the set of experts into \textit{type classes}. The type class of $(n_1, ..., n_m)$ has $\binom{T}{n_1, n_2, ..., n_m} = T!/(n_1! n_2! \cdots n_m!)$ experts. There are as many type classes as there are solutions of the equation $n_1 + n_2 + \cdots + n_m = T$ in nonnegative integers, namely, $\binom{m+T-1}{m-1}$.

Note that in addition to being simple, the symmetric multilinear strategies have a certain robustness in the sense that a mere reordering of the return vectors $x_1, ..., x_T$ cannot change the final wealth.

In the initial distribution of money, a total of $\binom{T}{n_1, n_2, ..., n_m} \alpha(n_1, ..., n_m)$ is given to the experts in type class $(n_1, ..., n_m)$. Thus, we must have

\[
\sum_{n_1 + \cdots + n_m = T} \binom{T}{n_1, n_2, ..., n_m} \alpha(n_1, ..., n_m) = 1.
\]
Example 11. \( \alpha(n_1, ..., n_m) = \lambda n_1^{n_1} n_2^{n_2} \cdots n_m^{n_m}, \) where \( \lambda = \left\{ \sum_{n_1 + \cdots + n_m = T} T \binom{T}{n_1, ..., n_m} \right\}^{-1}. \) This is the prior distribution in Cover and Ordentlich’s (1998) universal portfolio.

Example 12. \( \alpha(n_1, ..., n_m) = \lambda n_1! n_2! \cdots n_m!, \) where \( \lambda = \left\{ \sum_{n_1 + \cdots + n_m = T} T \binom{T}{n_1, ..., n_m} \right\}^{-1}. \) Thus \( \alpha(n_1, ..., n_m) = \left\{ \frac{(m+T-1)!}{(n_1 + \cdots + n_m)!} \right\}^{-1}. \) This is the prior distribution in Cover’s (1991) universal portfolio. This multilinear strategy characterized by the fact that it distributes an equal amount of money into each type class.

5.2.1 Simplification of \( \theta_k(x^t) \)

For a symmetric multilinear trading strategy, the numerator of \( \theta_k(x^t) \) can be simplified as follows. Let \( \alpha(j^t, k) = \sum_{j_{t+1} \cdots j_T} \alpha(j^t, k, j_{t+1}, ..., j_T) \) be the marginal pmf obtained from \( \alpha \) by summing over the coordinates \( j_{t+1}, ..., j_T. \) This number depends only on \( k \) and the type \( (N_1, ..., N_m) \) of \( j^t, \) where \( N_1 + \cdots + N_m = t. \) In fact, if \( j^t \) has type \( N, \) then \( \alpha(j^t, k) \) is equal to

\[
\sum_{n_1 + \cdots + n_m = T - t - 1} T \binom{T - t - 1}{n_1, ..., n_m} \alpha(N_1 + n_1, ..., N_k + n_k + 1, ..., N_m + n_m). \tag{21}
\]

Denote this number by \( \alpha_{tk}(N_1, ..., N_m). \) We then have

\[
\sum_{j^t} \alpha(j^t, k) x_{1j_1} \cdots x_{lj_t} = \sum_{N_1 + \cdots + N_m = t} \alpha_{tk}(N) \sum_{j^t \text{ has type } N} x_{1j_1} \cdots x_{lj_t}. \tag{22}
\]

Let \( \sigma(N_1, ..., N_m; x^t) \) denote the number \( \sum_{j^t \text{ has type } N} x_{1j_1} \cdots x_{lj_t}. \) Effective calculation of the numerator of \( \theta_k(x^t) \) thus can be broken into three parts:

1. Calculate \( \sigma(N_1, ..., N_m; x^t) \) by a recursive method
2. Calculate $\alpha_{tk}(N)$ by a recursive method (if $\alpha(\cdot)$ allows) or else by direct summation.

3. Explicitly add all the terms in

$$\sum_{N_1+\cdots+N_m=t} \alpha_{tk}(N)\sigma(N;x^t)$$

(23)

A recurrence for $\sigma(N;x^t)$ is derived as follows. $\sigma(N;x^t) = \sum_{k=1}^{m} \left\{ \sum_{j=1}^{t-1 \text{ has type } (N_1,\ldots,N_k-1,\ldots,N_m)} x_1 j_1 \cdots x_{t-1} j_{t-1} \right\} x_{tk} = \sum_{k=1}^{m} \sigma(N_1,\ldots,N_k-1,\ldots,N_m)x_{tk} = \sum_{k=1}^{m} \sigma(N_1,\ldots,N_k-1,\ldots,N_m;x^{t-1})x_{tk}.$$

(24)

The recursion gradually reduces the numbers $N_1,\ldots,N_m$ until one of them (say, the $k^{th}$) is 1 and the rest are 0. The boundary conditions are then

$$\sigma(0,\ldots,1_k,\ldots,0;x_1) = x_{1k}.$$

(25)

Thus, calculating $\sigma(N;x^t)$ requires $m$ recursive calls, and the recursion tree is $t-1$ levels deep. One is required to calculate all the numbers $\sigma(r_1,\ldots,r_N;x^t)$ for which $r_k \geq 0$ and $1 \leq r_1 + \cdots + r_m \leq t-1$. This amounts to calculating and storing $\sum_{s=1}^{t-1 \choose m-1} = O(t^m)$ numbers, which is possible for small values of $m$. A direct recursive implementation should not be attempted, as the recursion tree will involve enormous duplication. Rather, the numbers $\sigma(r_1,\ldots,r_N;x^t)$ should be tabulated according to the “bottom up” approach. At step $s$, we tabulate all the numbers $\sigma(r_1,\ldots,r_N;x^t)$ for which $r_1 + \cdots + r_N = s$, making use of all the numbers tabulated in step $s-1$. Once step $s$ is completed, the numbers tabulated in step $s-1$ no longer need to be stored.
6 Superhedging

Definition 9. A superhedge for $D$ is a pair $(p, \theta)$, where $\theta(\cdot)$ is a self-financing trading strategy and $p$ is an initial deposit, such that $p \cdot W_\theta(x_1, \ldots, x_T) \geq D(x_1, \ldots, x_T)$ for all $x_1, \ldots, x_T$.

Definition 10. The superhedging price of $D$ is
\[
\inf \{ p : (p, \theta) \text{ is a superhedge for some } \theta \}. 
\]
If no superhedge exists, then the superhedging price is $+\infty$.

Proposition 9. The superhedging price of any derivative $D$ is at least $\sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T})$. If $D$ is exactly hedgeable, then the superhedging price is equal to the hedging cost.

Proof. In the defining inequality $p \cdot W_\theta(x_1, \ldots, x_T) \geq D(x_1, \ldots, x_T)$, substitute $x_t = e_{j_t}$ and sum both sides over all possible unit basis vectors $e_{j_1}, \ldots, e_{j_T}$. Using Proposition 1, we see that $\sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T})$ is a lower bound for the set of all monetary deposits $p$ that form a part of some superhedge $(p, \theta)$. \hfill \Box

Thus, it is appropriate to write $p^*[D]$ for the superhedging price. We have extended the functional $p^*[D]$ from the set of all exactly hedgeable derivatives to the set of all derivatives.

Proposition 10. The functional $p^*[D] \geq 0$ is subadditive, positively homogeneous (hence convex), and increasing in $D$. If $\theta(\cdot)$ is a trading strategy, then $p^*[W_\theta] = 1$.

Theorem 1. If a superhedge exists at all, then there is a superhedge that costs exactly $p^*[D]$. Thus, “$\inf$” can be replaced with “$\min$” in the definition of superhedging price.

Proof. Suppose a superhedge exists that costs $\overline{p}$. By the definition of $p^* = p^*[D]$, there is a sequence of superhedges $(p_n, \theta^{(n)})$ such that $p_n \leq \overline{p}$ and $\lim_{n \to \infty} p_n = p^*[D]$.\hfill 20
Define a trading strategy $\psi(\cdot)$ as follows. Fix a particular return path $x_1, \ldots, x_T$. Then

$$
(p_n, \theta^{(n)}(h^0), \theta^{(n)}(x_1), \theta^{(n)}(x_1, x_2), \ldots, \theta^{(n)}(x_1, \ldots, x_T))
$$

(26)
is a sequence of points from the compact set $[0, \bar{p}] \times \Delta^{T+1}$. It has a convergent subsequence. Pick any subsequential limit. It has the form

$$
(p^*[D], \psi(h^0), \psi(x_1), \psi(x_1, x_2), \ldots, \psi(x_1, \ldots, x_T)).
$$

(27)

This serves to define $\psi(\cdot)$ along the particular path $x_1, \ldots, x_T$. Taking the chosen subsequential limit of the inequality

$$
p_n \prod_{t=1}^{T} (\theta^{(n)}(x_{t-1}), x_t) \geq D(x_1, \ldots, x_T),
$$

(28)

we get $p^*[D] \cdot W_\psi(x_1, \ldots, x_T) \geq D(x_1, \ldots, x_T)$. This holds good on any particular path $x_1, \ldots, x_T$.

Remark 1. Notice how the axiom of choice entered into the above proof. For each particular return path $x_1, \ldots, x_T$, we had to make an arbitrary choice from the (nonempty) set of subsequential limits of a certain sequence that depended on $x_1, \ldots, x_T$.

This theorem can be proved another way. We consider the problem of minimizing $p$ over the set of all $(p, \theta) \in [0, \bar{p}] \times \Theta$ such that $(p, \theta)$ is a superhedge for $D$. This amounts to minimizing a continuous function over a compact set. The function $(p, \theta) \mapsto p$ is of course continuous with respect to the product topology. We need only show that the domain is a closed subset of the (compact) set $[0, \bar{p}] \times \Theta$. The domain is defined by a continuum of inequalities, $p \cdot W_\theta(x_1, \ldots, x_T) \geq D(x_1, \ldots, x_T)$ for all paths $(x_1, \ldots, x_T)$. Since $\theta \mapsto W_\theta$ is continuous, each such inequality defines a certain closed
subset of $[0, \mathcal{P}] \times \Theta$. The intersection of any number of closed sets is a closed set.

**Theorem 2.** If $D$ is subadditive and positively homogeneous separately in each $x_t$, then its superhedging price is exactly $\sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T})$. There is a unique multilinear superhedge that achieves the minimum cost.

**Proof.** In the inequality

$$D \leq \sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T})x_{1j_1} \cdots x_{Tj_T},$$

the right-hand side is a multilinear superhedge whose cost achieves the lower bound $\sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T})$. This proves that $p^*[D] = \sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T})$. To show uniqueness, assume that $p^*[D] \sum_{j_T} \alpha(j^T)x_{1j_1} \cdots x_{Tj_T}$ is a minimum-cost multilinear superhedge.

In the definition of a superhedge, make the substitutions $x_1 = e_{j_1}, x_2 = e_{j_2}, \ldots, x_T = e_{j_T}$. We get

$$p^*[D]\alpha(j^T) \geq D(e_{j_1}, \ldots, e_{j_T}),$$

so that $\alpha(j^T) \geq D(e_{j_1}, \ldots, e_{j_T})/p^*[D]$ for all $j^T$. On account of the fact that $\sum_{j_T} \alpha(j^T) = 1$, this forces $\alpha(j^T) = D(e_{j_1}, \ldots, e_{j_T})/p^*[D]$ for all $j^T$. \hfill \Box

### 7 Solution of the generalized max-min game

Cover and Ordentlich (1998) used the concrete performance benchmark $D(x_1, \ldots, x_T) = \max_{c \in \Delta} \prod_{t=1}^T \langle c, x_t \rangle$, which is the final wealth of the best rebalancing rule in hindsight. They formulated a two-person, simultaneous-move, zero-sum game between the trader and “nature,” whereby the trader picks a strategy $\theta(\cdot)$ and nature picks the return path $(x_1, \ldots, x_T)$. In what follows, we will use the convenient notation $X = (x_1, \ldots, x_T)$ to denote the matrix of gross-returns of all stocks in all periods.
We take up the payoff kernel \((\theta(\cdot), X) \mapsto W_\theta(X)/D(X)\). That is, the trader picks \(\theta(\cdot)\) so as to maximize the relative performance measure \(W_\theta(X)/D(X)\), and nature picks \(X\) so as to minimize it. In this section, we solve the game for any multiconvex benchmark \(D(\cdot)\) that is positively homogeneous separately in each \(x_t\). This means that the benchmark is also subadditive separately in each \(x_t\).

**Theorem 3.** For any positive derivative \(D(x_1, \ldots, x_T)\), the lower value \(v[D]\) (in pure strategies) of the generalized max-min game is

\[
v[D] = 1/p^*[D] = \max_{\theta(\cdot)} \inf_{x_1, \ldots, x_T \in \mathbb{R}_+^m \setminus \{0\}} \frac{W_\theta(X)}{D(X)}.
\]

(31)

The max is always achieved exactly (by any minimum-cost superhedge). The upper value in pure strategies is

\[
v[D] = \inf_{x_1, \ldots, x_T} \frac{||x_1||_\infty \cdots ||x_T||_\infty}{D(X)}.
\]

(32)

where the numerator is the payoff of the perfect-trader option.

**Proof.** Let \(\theta(\cdot)\) be a minimum-cost superhedge for \(D(\cdot)\). First, we show that \(\theta\) guarantees a payoff \(\geq 1/p^*[D]\) for all \(X\). By definition, \(p^*[D]W_\theta \geq D\) for all possible return paths \(x_1, \ldots, x_T\). Thus, \(W_\theta(X)/D(X) \geq 1/p^*[D]\) regardless of nature’s choice. Next, we show that \(1/p^*[D]\) is the best possible guarantee. Let \(g\) be a payoff guarantee corresponding to some trading strategy \(\psi(\cdot)\). Then, since \(W_\psi(X)/D(X) \geq g\) for all \(X\), we have \((1/g)W_\psi \geq D\) on all possible return paths. Since \((1/g, \psi)\) is a superhedge for \(D\), by definition we must have \(p^*[D] \leq 1/g\). Thus, since \(g \leq 1/p^*[D]\), \(1/p^*[D]\) is indeed the best possible payoff the trader can guarantee.

For the upper value, just note that when calculating \(\inf_x \max_\theta W_\theta(X)/D(X)\), the trading strategy \(\theta\) is selected with full prior knowledge of all returns. Given this
knowledge, the trader can cherry-pick a strategy that, in every period \( t \), puts all its money into the best performing stock \( j^*(X, t) = \arg\max_{1 \leq j \leq m} x_{ij} \). Thus, the best response to a given \( X \) is the trading strategy \( \theta(x^{t-1}) = e_{j^*(X, t)} \). This yields a final wealth of \( \prod_{t=1}^{T} ||x_t||_{\infty} \).

**Corollary 3.** If \( D(X) \leq \prod_{t=1}^{T} ||x_t||_{\infty} \) is majorized by the perfect-trader option, and \( p^*[D] > 1 \), then there is a duality gap \( \bar{v}[D] < \bar{v}[D] \), and thus there is no pure-strategy equilibrium.

Note that any derivative \( D \) that deserves to be called a “lookback” must obviously have a super-replicating cost higher than 1 (on account of the hindsight-optimization) and have a payoff no greater than that of the perfect-trader (who hindsight-optimizes over all possible trading strategies). Since there will be a duality gap whenever \( D \) is a lookback option, we must resort to randomized strategies to solve the game.

**Theorem 4.** If \( D(\cdot) \) is multiconvex and homogeneous separately in each \( x_t \), then in the mixed-strategy Nash equilibrium, the trader uses the unique minimum-cost multilinear superhedge and nature randomizes over the set of Kelly sequences \( \{ (e_{j_1}, \ldots, e_{j_T}) : (j_1, \ldots, j_T) \in \{1, \ldots, m\}^T \} \) according to the probabilities \( P\{X = (e_{j_1}, \ldots, e_{j_T})\} = D(e_{j_1}, \ldots, e_{j_T})/p^*[D] \). The value of the game is \( 1/p^*[D] \).

**Proof.** First, we note that this is a legitimate assignment of probabilities, since \( \sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T}) = p^*[D] \), on account of the multiconvexity and multi-homogeneity. Next, let \( \theta \) be a minimum-cost superhedge for \( D \). Since \( \theta \) achieves the lower value of the game in pure strategies, we have \( W_\theta(X)/D(X) \geq 1/p^*[D] \) for all \( X \). Thus, we have \( E[W_\theta(X)/D(X)] \geq 1/p^*[D] \), where the expectation is taken with respect to the...
distribution of the random matrix $X$. Hence, $\theta$ guarantees that the expected payoff is $\geq 1/p^*[D]$.

Finally, we show that nature’s randomization guarantees an expected payoff $\leq 1/p^*[D]$. In fact, it guarantees that the expected payoff is exactly $1/p^*[D]$, for all $\theta$:

$$
E[W_\theta(X)/D(X)] = \sum_{j_1,\ldots,j_T} \frac{D(e_{j_1}^{T},\ldots,e_{j_T}^{T})}{p^*[D]} \times \frac{W_\theta(e_{j_1}^{T},\ldots,e_{j_T}^{T})}{D(e_{j_1}^{T},\ldots,e_{j_T}^{T})} = \frac{1}{p^*[D]} \sum_{j_1,\ldots,j_T} W_\theta(e_{j_1}^{T},\ldots,e_{j_T}^{T}) = \frac{1}{p^*[D]}.
$$

(33)

7.1 Cover-Ordentlich preferences for general benchmarks

The remarks in this subsection do not require that $D(\cdot)$ be multi-convex or multi-homogeneous. Note that any minimum-cost superhedge $\theta$ for $D$ will saturate a certain type of variational preferences over trading strategies. On account of the homeomorphism between (strictly positive) trading strategies and final wealth functions, we can just as well select the optimum feasible final wealth function $W(x_1,\ldots,x_T)$, and then recover the (unique) implied trading strategy. Define the utility

$$
U[W] = \inf_{(x_1,\ldots,x_T) \in \mathcal{P}} \frac{W(x_1,\ldots,x_T)}{D(x_1,\ldots,x_T)},
$$

(34)

where $\mathcal{P}$ is some set of return paths.

**Proposition 11.** For all (e.g. even non-measurable) positive performance benchmarks $D(\cdot)$, the Cover-Ordentlich functional $W \mapsto U[W]$ is increasing, concave, and upper semi-continuous. It thus has a maximum over the (compact) set of feasible
final wealth functions, and the set of maximizers is convex. In particular, the set of
minimum-cost superhedges for a given derivative is nonempty and convex.

Proof. For monotonicity, suppose that $W^1 \leq W^2$. Then, taking inf of both sides
of the inequality $W^1(X)/D(X) \leq W^2(X)/D(X)$, we obtain $U[W^1] \leq U[W^2]$. For
concavity, we have

$$U[\lambda W^1 + (1 - \lambda)W^2] = \inf_{X \in \mathcal{P}} \left\{ \frac{\lambda W^1(X)}{D(X)} + (1 - \lambda)\frac{W^2(X)}{D(X)} \right\} \geq \inf_{X \in \mathcal{P}} \left\{ \frac{\lambda W^1(X)}{D(X)} \right\} + \inf_{X \in \mathcal{P}} \left\{ (1 - \lambda)\frac{W^2(X)}{D(X)} \right\} = \lambda U[W^1] + (1 - \lambda)U[W^2].$$

(35)

As to upper-semicontinuity, we will show that the upper contour sets
$C_\alpha = \{ W \in \mathcal{W} : U[W] \geq \alpha \}$ are all closed. Here $\mathcal{W}$ denotes the set of feasible final
wealth functions and $\alpha$ is any real number. We have

$$C_\alpha = \left\{ W \in \mathcal{W} : \inf_{X \in \mathcal{P}} \frac{W(X)}{D(X)} \geq \alpha \right\} = \bigcap_{X \in \mathcal{P}} \{ W \in \mathcal{W} : W(X) \geq \alpha D(X) \}. \quad (36)$$

Thus, $C_\alpha$ is closed because it is an intersection of closed sets. For any given $X$, the
set $\{ W \in \mathcal{W} : W(X) \geq \alpha D(X) \}$ is closed because it is the preimage of the closed
set $[\alpha D(X), \infty)$ under the continuous (projection) mapping $W \mapsto W(X)$. \qed

8 Specialization to Cover and Ordentlich (1998)

We now apply the foregoing theory to Cover’s Derivative,

$$D(x_1, ..., x_T) = \max_{c \in \Delta} \langle c, x_1 \rangle \langle c, x_2 \rangle \cdots \langle c, x_T \rangle. \quad (37)$$
Proposition 12. Cover’s Derivative is symmetric, multiconvex, and positively homogeneous separately in each $x_t$. It is increasing in each variable $x_{ij}$. If $1$ is a vector of ones, then $D(x_1, ..., x_{T-1}, 1) = D(x_1, ..., x_{T-1})$.

Proof. These properties all follow easily from the definition of Cover’s Derivative. Note that $D(x_t, x_{t-1})$ is convex in $x_t$ because it is the maximum of a family of functions that are all linear in $x_t$.

Proposition 13. The final wealth of the best rebalancing rule in hindsight beats a Kelly gambler and, in fact, all rebalancing rules, all buy-and-hold strategies, and all passive indexes.

This is just the plain English manifestation of the following inequalities:

$$\sum_{j=1}^{m} c_j \left\{ \prod_{t=1}^{T} x_{tj} \right\} \leq \max_{1 \leq j \leq m} \left\{ \prod_{t=1}^{T} x_{tj} \right\} \leq D(x_1, ..., x_T). \quad (38)$$

If the $x_t$ are drawn iid from some cumulative distribution function $F(\cdot)$, then the Kelly rebalancing rule $c^*[F] \in \arg\min_{c \in \Delta} E_F[\log\langle c, x_t \rangle]$ will obviously (by definition) yield less final wealth than the best rebalancing rule in hindsight.

Proposition 14. For Cover’s Derivative, the special values $D(e_{j_1}, ..., e_{j_T})$ can be evaluated in closed form. Let $n_k$ be the number of times $j_t = k$ among the indices $(j_1, ..., j_T)$. Then

$$D(e_{j_1}, ..., e_{j_T}) = \prod_{k=1}^{m} (n_k/T)^{n_k}, \quad (39)$$

where we use the convention $0^0 = 1$.

Proof. This is a standard Cobb-Douglas optimization problem over the unit simplex:

$$D(e_{j_1}, ..., e_{j_T}) = \max_{c \in \Delta} e_1^{n_1} e_2^{n_2} \cdots e_m^{n_m}. \quad (40)$$
The solution is $c_k^* = n_k/\sum_{j=1}^m n_j = n_k/T$.

**Proposition 15.** If $p(T, m)$ denotes the superhedging cost of Cover’s Derivative, then

$$p(T, m) = \sum_{n_1 + \ldots + n_m = T} \binom{T}{n_1, \ldots, n_m} \prod_{k=1}^m (n_k/T)^{n_k}. \quad (41)$$

**Example 13.** For $m = 2$ stocks, the superhedging cost is

$$p(T, 2) = \sum_{j=0}^{T} \binom{T}{j} (j/T)^j ((T - j)/T)^{T-j} \quad (42)$$

$$= 2 \sum_{j=0}^{\lfloor T/2 \rfloor} \binom{T}{j} (j/T)^j ((T - j)/T)^{T-j} + 1_{T \text{ is even}} (T/T/2)^{2-T}. \quad (43)$$

**Proposition 16.** The superhedging price $p(T, m)$ of Cover’s Derivative is increasing in both the horizon $T$ and the number of stocks, $m$.

**Proof.** We can decompose the superhedging cost into $p(T, m) = \sum(\text{terms for which } n_m = 0) + \sum(\text{terms for which } n_m > 0) = p(T, m - 1) + \sum(\text{terms for which } n_m > 0)$. Thus, $p(T, m) > p(T, m - 1)$. To show that the superhedging cost is increasing in $T$, we use the fact that $D$ is subadditive separately in each $x_t$. We have

$$p(T, m) = \sum_{j_1, \ldots, j_T} D(e_{j_1}, \ldots, e_{j_T}, 1) \leq \sum_{j_1, \ldots, j_T} \sum_{k=1}^m D(e_{j_1}, \ldots, e_{j_T}, e_k) = p(T + 1, m), \quad (44)$$

where we have substituted $1 = \sum_{k=1}^m e_k$ and used subadditivity.

**Proposition 17.** On account of the simple bound $p(T, m) \leq \binom{T+m-1}{m-1} = O(T^{m-1})$, we have $\lim_{T \to \infty} \log(p(T, m))/T = 0$.  

28
8.1 Excess growth rate of the hindsight-optimized rebalancing rule

The quantity \( \frac{\log(D(x^T))}{T} \) is the per-period continuously-compounded capital growth rate achieved by the best rebalancing rule in hindsight. We compare this with \( \frac{\log(W_\theta(x^T))}{T} \), the growth rate achieved by the trading strategy \( \theta(\cdot) \).

**Definition 11.** A family of horizon-\( T \) trading strategies \( \theta(x^t; T) \) is called **universal** iff for every \( \epsilon > 0 \), there is a sufficiently long horizon \( T \) on which

\[
\log D(x_1, ..., x_T) - \log W_\theta(x_1, ..., x_T) \leq T\epsilon
\]

for all \( x_1, ..., x_T \).

This means that for every spread tolerance \( \epsilon \), on a long enough horizon, the excess per-period continuously-compounded capital growth rate of the best rebalancing rule in hindsight over and above that of the algorithmic trader is at most \( \epsilon \), regardless of the return path of the stock market. Thus, a universal strategy compounds its money at the same asymptotic rate as the best rebalancing rule in hindsight.

**Proposition 18.** If \( \theta(x^t; T) \) is a minimum-cost superhedge for Cover’s Derivative on a \( T \)-period horizon, then the family of strategies \( \theta(x^t; T) \) is universal.

To prove this, we just take logs of the defining inequality \( p(T,m) \cdot W_\theta \geq D \), and use the fact that \( \lim_{T \to \infty} \log(p(T,m))/T = 0 \). In fact, even superhedges more costly than \( p(T,m) \) can be universal, provided that the required initial deposit grows to infinity at a sub-exponential rate. This is true of Cover’s (1991) horizon-free universal portfolio.

**Corollary 4.** If the \( x_t \) are drawn iid from some distribution known to the Kelly gambler (but not to the universal trader), then the Kelly gambler’s excess per-period growth rate can be made arbitrarily small, given a long enough time horizon.
This follows from the fact that a Kelly gambler uses a certain rebalancing rule (the Kelly rule), which by definition yields no more final wealth than the best rebalancing rule in hindsight. Thus, the excess growth rate of the Kelly bettor is no more than the excess growth rate of the best rebalancing rule in hindsight, which can be made arbitrarily small.

### 8.2 Beating the market asymptotically

On the one hand, a universal trading strategy compounds its money at the same asymptotic rate as the best rebalancing rule in hindsight. On the other hand, the best rebalancing rule in hindsight beats the market. However, it would be ridiculous to claim that a universal strategy yields more final wealth than the S&P 500 on all return paths. To be quite correct, the realized path must be such that the best rebalancing rule in hindsight sustains a growth rate that is at least $\epsilon$ higher than that of the S&P 500. For such a return path, on a long enough horizon, a universal trading strategy will beat the market.

**Proposition 19.** Any passive index (or any buy-and-hold strategy that owns every stock in the market) will compound its money at the same asymptotic rate as the best performing stock in the market, e.g.

$$\lim_{T \to \infty} \frac{\log \left( \max_{1 \leq j \leq m} \prod_{t=1}^{T} x_{tj} \right) - \log \left( \sum_{j=1}^{m} c_j \prod_{t=1}^{T} x_{tj} \right)}{T} = 0,$$

where the index puts the initial fraction $c_j > 0$ of wealth into stock $j$, and lets it ride.

**Proof.** The above quantity is equal to the excess growth rate of the best performing stock in the market over and above that of the given buy-and-hold portfolio on the timeframe $1 \leq t \leq T$. Denoting this quantity by $\epsilon(X)$, we obviously have $\epsilon(X) \geq 0$.
for all $X$. Now, let $c = \text{Min}_{1 \leq j \leq m} c_j$. We have $c > 0$, since the index is supposed to hold shares of all stocks. Thus, we have the inequalities $0 \leq \epsilon(X) \leq -\frac{\log(c)}{T}$ for all $X$ and all $T$. Since $\log(c)/T \to 0$ as $T \to \infty$, the result follows.

\begin{proposition}
A universal trading strategy will beat the market asymptotically by an exponential factor, provided that the realized returns $(x_i)_{i=1}^\infty$ satisfy the following (mild) condition:

$$
\liminf_{T \to \infty} \frac{\log D(x_1, \ldots, x_T) - \log \left(\max_{1 \leq j \leq m} \prod_{t=1}^T x_{tj}\right)}{T} > 0 \quad (47)
$$

\end{proposition}

\begin{corollary}
If the returns $(x_i)_{i=1}^\infty$ are iid and $\max_{c \in \Delta} E[\log \langle c, x_i \rangle] > \max_{1 \leq j \leq m} E[\log x_{tj}]$, then every universal trading strategy beats the market asymptotically almost surely.

\end{corollary}

\begin{example}
In a pairs trading ($m = 2$) strategy that rebalances annually, on a horizon of $T = 30$ years one can guarantee to achieve within 6.7\% of the continuously-compounded annual growth rate of the hindsight-optimized rebalancing rule for the relevant pair of stocks.

\end{example}

\subsection{8.3 Practical determination of the horizon}

For a given number of stocks $m$, and a tolerance $\epsilon$ for the excess growth rate of the best rebalancing rule in hindsight, a practitioner is advised to select the smallest horizon $T_\epsilon$ such that $\log(p(T_\epsilon, m))/T \leq \epsilon$. One must try the successive values $T = 1, 2, 3, \ldots$, stopping as soon as $\log(p(T, m))/T \leq \epsilon$.

\begin{proposition}
The superhedging cost $p(T, m)$ can be computed via the recurrence

\begin{equation}
p(T, m) = 1 + \sum_{n=0}^{T-1} \binom{T}{n} (n/T)^{n}((T-n)/T)^{T-n} p(T-n, m-1)\quad (48)
\end{equation}

\end{proposition}
together with the boundary conditions \( p(1, m) = m \) and \( p(T, 1) = 1 \).

This allows us to tabulate exact values of \( p(T, m) \), even for large \( T \) and \( m \). To avoid numerical overflow on the computer, the powers and factorials should be calculated in log-space, and then exponentiated: \( \exp \left\{ \log \left( \binom{T}{n} \right) + n \log n + (T - n) \log (T - n) - T \log T \right\} \). For large \( T \) and \( m \) this will require the calculation of an enormous number of logarithms. Instead, the logs \( L_n = \log n \) should be precomputed and stored for \( 1 \leq n \leq T \), along with the log-factorials \( LF_n = L_n + LF_{n-1} \). The numbers \( \log \left( \binom{T}{n} \right) \) are then calculated easily by \( LF_T - LF_n - LF_{T-n} \). This procedure was used to generate Figure \[1\]

For extremely large values of \( T \) and \( m \), direct calculation of \( p(T, m) \) becomes fairly slow, even with the aid of the above recurrence. Fortunately, we can replace \( p(T, m) \) with Shtarkov’s simple (1987) upper bound, which is very accurate. This is shown in Figure \[2\]

**Shtarkov’s bound:**

\[
p(T, m) \leq \sum_{j=1}^{m} a_j T^{(j-1)/2} = \mathcal{O}(T^{(m-1)/2}),
\]

(49)

where

\[
a_j = \sqrt{\pi} \binom{m}{j} / \left\{ \Gamma(j/2) \cdot 2^{(j-1)/2} \right\}.
\]

(50)

Thus, the practical method for (approximately) solving \( \log(p(T, m))/T < \epsilon \) is to carry out the fixed-point iteration:

\[
T = g(T) = (1/\epsilon) \log \left( \sum_{j=1}^{m} a_j T^{(j-1)/2} \right).
\]

(51)

In the long-run, since \( \log p(T, m) \) grows at a negligible rate, we will need to roughly double the horizon in order to cut \( \epsilon \) in half.
Figure 1: Worst-case excess per-period continuously-compounded growth rate (%) of the best rebalancing rule in hindsight over and above that of the superhedging trader.
Figure 2: The accuracy of Shtarkov’s bound for 2 stocks
8.4 Failure of high-frequency trading to shorten the required horizon

It is tempting to think that one can cheat the situation by trading at a higher frequency. However, getting within 1% of the compound annual growth rate of the best rebalancing rule in hindsight under annual trading (which takes $T = 320$ years if there are 2 stocks) is not the same thing as getting within 1% of the hindsight-optimized growth rate under, say, daily trading, which takes 320 days. Rather, the comparable requirement is that one must get within $(1/252)%$ of the daily growth rate, assuming there are 252 trading days per year. For 2 stocks this will take (in the worst case) 156,500 days, or 621 years.

In general, increasing the trading frequency only serves to lengthen the number of years required to get within $\epsilon$ of the hindsight-optimized compound annual growth rate. This is illustrated in Figure 3. However, we get some compensation from the fact that the hindsight-optimized rebalancing rule may achieve a higher compound-annual growth rate under more frequent trading.

In general, let $f$ be the frequency (number of rebalancings per year). It takes $(1/f)T_{\epsilon/f}$ years to guarantee to get within $\epsilon$ of the compound-annual growth rate of the hindsight-optimized rebalancing rule that trades $f$ times per year.

**Proposition 22.** $\lim_{f \to \infty} (1/f)T_{\epsilon/f} = +\infty$, where $T_\delta$ denotes the shortest horizon that solves the inequality $\log(p(T, m))/T \leq \delta$.

**Proof.** We bound the number $T_{\epsilon/f}$ from below, as follows. From the asymptotic expansion of $p(T, m)$ (Cover and Ordentlich 1998), there is a constant $A$ such that $p(T, m) \geq A \cdot T^{m-1}$, so that $\log(p(T, m))/T \geq (\log A + \frac{m-1}{2} \log T)/T$. Thus,

$\{T \in \mathbb{N} : \log(p(T, m))/T \leq \epsilon/f\} \subseteq \{T \geq 1 : (\log A + \frac{m-1}{2} \log T)/T \leq \epsilon/f\}$. Let
Figure 3: Number of years needed (in the worst case) to get within 1% of the annualized growth rate of the hindsight-optimized (2-stock) rebalancing rule that trades at a given frequency.
\( T^*(f) \) denote the min of this latter set, so that

\[
\frac{1}{f} T_{\epsilon/f} \geq T^*(f)/f,
\]

(52)

where \((\log A+\frac{m-1}{2} \log T^*(f))/T^*(f) = \epsilon/f\). Thus, \( T^*(f)/f = (\log A+\frac{m-1}{2} \log T^*(f))/\epsilon \), which tends to \( \infty \) as \( f \to \infty \), since \( T^*(f) \to \infty \). This gives the desired result.

9 Conclusion

In these pages, we solved the most general tractable version of Cover and Ordentlich’s (1998) on-line portfolio selection game. This obtains for performance benchmarks (or exotic options) that are separately convex and homogeneous in each period’s gross-return vector.

First, we constructed a general theory of “multilinear superhedging” that extends the Cover-Ordentlich (1998) techniques. A multilinear trading strategy is one whose final wealth is linear separately in each period’s gross-return vector. A multilinear superhedge for a given financial derivative is a multilinear trading strategy that guarantees to generate more final wealth than the derivative payoff in any outcome.

Since every multilinear final wealth function corresponds to a unique (and formulaic) replicating strategy, one has the convenience of optimizing directly over the (finite-dimensional) set of multilinear final wealths. If a derivative is separately convex and homogeneous in each period’s return vector, then there is a unique multilinear superhedge that is cheaper than any other (multilinear or not). This is precisely the trading strategy that solves the game (and saturates the corresponding variational preferences). More generally, we showed how the axiom of choice can be used to “find” exact maximin trading strategies for completely arbitrary (even non-measurable) per-
formance benchmarks.

This being done, we re-derived the basic results on universal portfolios in a facile and intelligible manner. We improved Cover and Ordentlich’s somewhat ad-hoc (1998) approach, which is based on a discrete version of the problem from Shtarkov’s (1987) theory of universal data compression. Any sufficiently cost-effective super-hedge of the final wealth of the best rebalancing rule in hindsight will beat the market asymptotically (by an exponential factor) provided that the best rebalancing rule in hindsight compounds its money at a higher rate than the best-performing stock in the market. Basically, this condition is satisfied whenever the realized volatility of stock prices is sufficiently high. Finally, we showed that higher-frequency trading does not necessarily guarantee (in the worst case) that the trader will beat the market any faster.

References

[1] Bensaid, B., Lesne, J.P. and Scheinkman, J., 1992. Derivative Asset Pricing with Transaction Costs. Mathematical Finance, 2(2), pp.63-86.

[2] Blackwell, D., 1956. An Analog of the Minimax Theorem for Vector Payoffs. Pacific Journal of Mathematics, 6(1), pp.1-8.

[3] Breiman, L., 1961. Optimal Gambling Systems for Favorable Games. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics. The Regents of the University of California.

[4] Cover, T.M., 1987. Log Optimal Portfolios. Chapter in “Gambling Research: Gambling and Risk Taking,” Seventh International Conference (Vol. 4).
[5] Cover, T.M., 1991. Universal Portfolios. Mathematical Finance, 1(1), pp.1-29.

[6] Cover, T.M. and Gluss, D.H., 1986. Empirical Bayes Stock Market Portfolios. Advances in Applied Mathematics, 7(2), pp.170-181.

[7] Cover, T.M. and Ordentlich, E., 1996. Universal Portfolios with Side Information. IEEE Transactions on Information Theory, 42(2), pp.348-363.

[8] Cover, T.M. and Thomas, J.A., 2006. Elements of Information Theory. John Wiley & Sons.

[9] Cox, J.C., Ross, S.A. and Rubinstein, M., 1979. Option Pricing: A Simplified Approach. Journal of Financial Economics, 7(3), pp.229-263.

[10] Helmbold, D.P., Schapire, R.E., Singer, Y. and Warmuth, M.K., 1998. On-Line Portfolio Selection Using Multiplicative Updates. Mathematical Finance, 8(4), pp.325-347.

[11] Jamshidian, F., 1992. Asymptotically Optimal Portfolios. Mathematical Finance, 2(2), pp.131-150.

[12] Kelly, J., 1956. A New Interpretation of Information Rate. IRE Transactions on Information Theory, 2(3), pp.185-189.

[13] Luenberger, D.G., 1998. Investment Science. Oxford University Press.

[14] Markowitz, H., 1952. Portfolio Selection. The Journal of Finance, 7(1), pp.77-91.

[15] Ordentlich, E. and Cover, T.M., 1998. The Cost of Achieving the Best Portfolio in Hindsight. Mathematics of Operations Research, 23(4), pp.960-982.

[16] Poundstone, W., 2010. Fortune’s Formula: The Untold Story of the Scientific Betting System that Beat the Casinos and Wall Street. Hill and Wang.
[17] Shtar’kov, Y.M., 1987. Universal Sequential Coding of Single Messages. Problemy Peredachi Informatsii, 23(3), pp.3-17.

[18] Swensen, D.F., 2005. Unconventional Success: A Fundamental Approach to Personal Investment. Simon and Schuster.

[19] Thorp, E.O., 1969. Optimal Gambling Systems for Favorable Games. Review of the International Statistical Institute, 37(3), pp.273-293.

[20] Thorp, E.O., 2006. The Kelly Criterion in Blackjack, Sports Betting and the Stock Market. Handbook of Asset and Liability Management, 1, pp.385-428.