Maximization of recursive utilities under convex portfolio constraints

Anis MATOUSSI
Université du Maine
Risk and Insurance Institut
Laboratoire Manceau de Mathématiques
e-mail: anis.matoussi@univ-lemans.fr

Hanen Mezghani
University of Tunis El Manar
Laboratoire de Modélisation Mathématique et Numérique
dans les Sciences de l’Ingénieur, ENIT
e-mail: hanen.mezghani@lamsin.rnu.tn

Mohamed MNIF
University of Tunis El Manar
Laboratoire de Modélisation Mathématique et Numérique
dans les Sciences de l’Ingénieur, ENIT
e-mail: mohamed.mnif@enit.rnu.tn

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Abstract: We study a robust maximization problem from terminal wealth and consumption under a convex constraints on the portfolio. We state the existence and the uniqueness of the consumption-investment strategy by studying the associated quadratic backward stochastic differential equation (BSDE in short). We characterize the optimal control by using the duality method and deriving a dynamic maximum principle.
1. Introduction

The utility maximization is a basic problem in mathematical finance. It was introduced by Merton [15]. Using stochastic control methods, he exhibits a closed formula for the value function and the optimal proportion-portfolio when the risky assets follow a geometric Brownian motion and the utility function is of CRRA type.

In the literature, many works assume that the underlying model is exactly known. In this paper we consider a problem of utility maximization under uncertainty. The objective of the investor is to determine the optimal consumption-investment strategy when the model is not exactly known. Such problem is known as the robust utility maximization and is formulated as

$$\text{find } \sup_{\pi} \inf_{Q} U(\pi, Q)$$

(1.1)

where $U(\pi, Q)$ is the $Q$-expected utility. The investor has to solve a sup inf problem. He considers the worst scenario by minimizing over a set of probability measures and then he maximizes his utility. In the literature there are two approaches to solve the robust utility maximization problems. The first one relies on duality methods such as Quenez [17] or Shi and Wu [19]. They considered a set of probability measures called priors and they minimized over this set. The second approach, which is followed in this paper, is based on the penalisation method and the minimization is taken over all possible models such as in Anderson, Hansen and Sargent [1]. Moreover Skiadas [20] followed the same point of view and he gave the dynamics of the control problem via BSDE in the Markovian context. In our case, the $Q$-expected utility is the sum of a classical utility function and a penalization term based on a relative entropy. In Bordigoni et al. [4], they proved the existence of a unique $Q^*$ optimal model which minimizes our cost function. They used the stochastic control techniques to study the dynamic value of the minimization problem. In the case of continuous filtration, they showed that the value function is the unique solution of a generalized BSDE with a quadratic driver.

In Faidi, Matoussi and Mnif [10], they studied the maximization part of the problem (1.1) in a complete market by using the BSDE approach as in Duffie and Skiadas [7] and El Karoui et al. [8].

In our paper, we assume that the portfolio is constrained to take values in a given closed convex non-empty subset $K$ of $\mathbb{R}^d$. Such problem was studied when the underlying model is known by Karatzas, Lehoczky, Shreve and Xu [11] in the incomplete market case and then by Cvitanic and Karatzas [5] for convex constraints on the portfolio. In our context, we study the robust formulation of the same problem and we add a constraint on the state process. Moreover, using change of measures and optional decomposition under constraints, we state an existence result to the optimization problem where the criterion is the solution at time 0 of a quadratic BSDE with unbounded terminal condition. By using the duality method, we derive a maximum principle which shows a necessary and sufficient conditions of optimality. Thanks to this result, we give an implicit expression of the optimal terminal wealth and the optimal consumption rate depending optimal probability measure solution of the robust problem and the probability measure solution of the dual problem. This later result is a generalization of Cvitanic and Karatzas [5] and Faidi, Matoussi and Mnif [10] works.

The paper is organized as follows. Section 2 describes the model and the stochastic control problem. Section 3 is devoted to the existence and the uniqueness of an optimal strategy. In section 4, we characterize the optimal consumption strategy and the optimal terminal wealth by using duality techniques. In section 5, we relate the optimal control to the solution of a
forward-backward system and we study an example in the case of incomplete market.

2. Problem formulation

We consider a probability space \((\Omega, \mathcal{F}, P)\) supporting a \(d\)-dimensional standard Brownian motion \(W = (W^1, \ldots, W^d)\), over the finite time horizon \([0, T]\). We shall denote by \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) the \(P\)-augmentation of the filtration \(\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)\) generated by \(W\). We also assume that \(\mathcal{F} = \mathcal{F}_T\).

For any probability measure \(Q \ll P\) on \(\mathcal{F}_T\), the density process of \(Q\) with respect to \(P\) is the RCLL \(P\)-martingale \(Z^Q = (Z^Q_t)_{0 \leq t \leq T}\) with

\[
Z^Q_t = \left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = EP\left[ \frac{dQ}{dP} \middle| \mathcal{F}_t \right].
\]

Bordigoni et al. [4] study a robust control problem with a dynamic value process of the form

\[
Y_t = \text{ess inf}_{Q \in \mathcal{Q}_t} \left( \frac{1}{S^0_t} E_Q \left[ \int_t^T \alpha S^d_s \hat{U}_s ds + \bar{\alpha} S^d_T \hat{U}_T \middle| \mathcal{F}_t \right] + \beta E_Q \left[ R^\delta_{t, T}(Q) \middle| \mathcal{F}_t \right] \right),
\]

where

\[
\mathcal{Q}_t = \left\{ Q | Q \ll P, Q = P \text{ on } \mathcal{F}_0 \text{ and } H(Q|P) := E_Q[\log \frac{dQ}{dP}] < \infty \right\},
\]

\(\alpha\) and \(\bar{\alpha}\) are non negative constants, \(\beta \in (0, \infty)\), \(\delta = (\delta_t)_{0 \leq t \leq T}\) and \(\hat{U} = (\hat{U}_t)_{0 \leq t \leq T}\) are \(\mathcal{F}\) progressively measurable processes, \(\hat{U}_T\) is a \(\mathcal{F}_T\)-measurable random variable, \(S^d_t = e^{-\int_0^t \delta_s ds}\) is the the discounting factor and \(R^\delta_{t, T}\) is the penalization term which is the sum of the entropy rate and the terminal entropy:

\[
R^\delta_{t, T} = \frac{1}{S^0_t} \int_t^T \delta_s S^d_s \log \frac{Z^Q_s}{Z^Q_t} ds + \frac{S^d_T}{S^d_t} \log \frac{Z^Q_T}{Z^Q_t}.
\]

We define the following spaces:

\(L^0_d(\mathcal{F}_T)\) is the set of non-negative \(\mathcal{F}_T\)-measurable random variables.

\(L^{\exp}\) is the space of all \(\mathcal{F}_T\)-measurable random variables \(X\) with

\[
EP\left[ \exp (\gamma |X|) \right] < \infty \quad \text{for all } \gamma > 0.
\]

\(D^\exp_0\) is the space of all progressively measurable processes \(X = (X_t)_{0 \leq t \leq T}\) with

\[
EP\left[ \exp (\gamma \text{ess sup}_{0 \leq t \leq T} |X_t|) \right] < \infty \quad \text{for all } \gamma > 0.
\]

\(D^\exp_1\) is the space of all progressively measurable processes \(X = (X_t)_{0 \leq t \leq T}\) such that

\[
EP\left[ \int_0^T \exp (\gamma |X_s|) ds \right] < \infty \quad \text{for all } \gamma > 0.
\]

\(H^2_F(\mathbb{R}^d)\) is the set of progressively measurable processes \(\mathbb{R}^d\) valued \(Z = (Z_t)_{0 \leq t \leq T}\) such that

\[
||Z||^2_{H^2} := EP\left[ \int_0^T |Z_t|^2 dt \right] < \infty.
\]

We shall assume that:

\((H1)\) \(0 \leq \delta \leq ||\delta||_\infty\) for some constant \(||\delta||_\infty\).
(H2) $\tilde{U} \in D^\exp_1$ and $\tilde{U}_T \in L^\exp$.

Under (H1)-(H2), Bordigoni et al. [4] (Theorem 12 and Proposition 16) prove the existence and the uniqueness of a unique optimal probability measure $Q^*$ of the problem (2.1). They showed that the dynamics of $(Y_t)_{t \in [0,T]}$ satisfies the following BSDE

$$dY_t = (\delta_t Y_t - \alpha \tilde{U}_t)dt + \frac{1}{2\beta} |Z_t|^2 dt + Z_t dW_t, \quad (2.2)$$

$$Y_T = \bar{\alpha} \tilde{U}_T, \quad (2.3)$$

where $|.|$ stands the euclidean norm. They established for $Y$ the recursive relation

$$Y_t = - \beta \log \mathbb{E}^P \left[ \exp \left( \frac{1}{\beta} \int_t^T (\delta_s Y_s - \alpha \tilde{U}_s) ds - \frac{1}{\beta} \bar{\alpha} \tilde{U}_T \right) \big| \mathcal{F}_t \right], \quad (2.4)$$

They proved that there exists a unique pair $(Y, Z) \in D^\exp_0 \times H^2_T (\mathbb{R}^d)$ that solves (2.2)-(2.3).

Moreover, they showed that the density of the probability measure $Q^*$ is a true martingale and is given by

$$Z^*_t = \mathcal{E}(- \frac{1}{\beta} \mathcal{M}^Y_t), \quad (2.5)$$

where $\mathcal{E}$ denotes the stochastic exponential.

From now, we are interested in the problem of utility maximization. Let us consider an investor who can consume between time 0 and time $T$. We denote by $c = (c_t)_{0 \leq t \leq T}$ the consumption rate. We consider a financial market consisting of a bond and $d$ risky assets. Without loss of generality, we assume that the bond is constant. The risky assets $S := (S^1, ..., S^d)$ evolve according to the stochastic differential equations

$$dS^i_t = S^i_t (b^i_t dt + \sum_{j=1}^d \sigma^i_j dW^j_t), \quad S^i_0 = 1, \ i = 1 ... d.$$  

We assume that the process $b = (b^1_t, ..., b^d_t)_{t \in [0,T]}$ (vector of instantaneous yield) and the process $\sigma = \left( \sigma^i_j \right)_{1 \leq i,j \leq d}$ (volatility matrix) are $\mathbb{F}$ adapted. We shall assume throughout that the relative risk process

$$\theta_t := \sigma^{-1} b_t,$$

satisfies the integrability condition

$$E[\int_0^T ||\theta_t||^2 dt] < \infty.$$  

We denote by $H = ((H^1_t, ..., H^d_t)_{t \in [0,T]})^*$ the investment strategy representing the amount of each asset invested in the portfolio. The notation $*$ denotes the transposition operator. We shall fix throughout a nonempty, closed, convex set $K$ in $\mathbb{R}^d$, and denote by

$$\delta(x)^{supp} := \delta^{supp}(x|K) := \sup_{H \in K} (-H^* x) : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$$

the support function of the convex set $-K$. This is a closed, positively homogeneous, proper convex function on $\mathbb{R}^d$ finite on its effective domain (Rockafellar [18] p. 114)

$$\tilde{K} := \{ x \in \mathbb{R}^d, \ \delta^{supp}(x|K) < \infty \}$$

$$= \{ x \in \mathbb{R}^d, \ \text{there exists } \beta \in \mathbb{R} \text{ s.t. } -H^* x \leq \beta, \ \forall H \in K \}.$$
which is a convex cone (called the barrier cone of $-K$).

We assume that the function $\delta_{\text{supp}(\cdot | K)}$ is continuous on $\tilde{K}$.

The investment strategy is constrained to remain in the convex set $K$. We denote by $\tilde{C}$ and $\tilde{H}$ the following sets

$$\tilde{C} = \{c = (c_t)_{t \in [0,T]} \mid \mathbb{F} - \text{adapted}, c_t \geq 0 \text{ dt} \otimes dP \text{ a.e. and } \int_0^T c_t dt < \infty\},$$

$$\tilde{H} = \{H = (H_t)_{t \in [0,T]} \mid \mathbb{F} - \text{adapted, } \mathbb{R}^d \text{ valued and } H^* \text{diag}(S)^{-1} \in L(S) \text{ and } H_t \in K\text{ dt} \otimes dP \text{ a.e.}\},$$

where $L(S)$ denotes the set of $\mathbb{F}$-adapted processes, $\mathbb{R}^d$ valued such that the stochastic integral with respect to $S$ is well-defined. Given an initial wealth $x \geq 0$ and a policy $(c, H) \in \tilde{C} \times \tilde{H}$, the wealth process at time $t$ follows the dynamics given by:

$$dX^{x,c,H}_t = H_t \text{diag}(S_t)^{-1}dS_t - c_t dt, \quad X^{x,c,H}_0 = x. \quad (2.6)$$

We impose the following constraint on the wealth process

$$X^{x,c,H}_t \geq d \text{ a.s. } \forall t \in [0,T) \text{ for some } d \in \mathbb{R}, \quad (2.7)$$

$$X^{x,c,H}_T \geq 0 \text{ a.s.} \quad (2.8)$$

The investor has preferences modelled by the utility functions $U$ and $\bar{U}$ satisfying the following assumption

(iii) $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\bar{U} : \mathbb{R}_+ \rightarrow \mathbb{R}$ are $C^1$ on the sets $\{U < \infty\}$ and $\{\bar{U} < \infty\}$ respectively, strictly increasing and concave.

(ii) $U$ and $\bar{U}$ satisfy the usual Inada conditions i.e. $U'(\infty) = \bar{U}'(\infty) = 0$ and $U'(0) = \bar{U}'(0) = \infty$.

Definition 2.1 (i) We define the set $\mathcal{A}$ as the largest convex in $\tilde{C} \times L$ where $\tilde{C} \times L$ consists of all processes $(c, \xi) \in \tilde{C} \times L^0_+(\mathbb{F}_T)$ such that there exists $H \in \tilde{H}$ and an initial wealth $x$ satisfying $X^{x,c,H}_T = \xi$ and constraints (2.7)-(2.8) hold as well as the families

$$\left\{ \int_0^T \exp(\gamma \vert U(c_s) \vert) \text{ds : } c \in \tilde{C} \right\}, \quad (2.9)$$

$$\left\{ \int_0^T \exp(\gamma \vert U'(c_s) \vert) \text{ds : } c \in \tilde{C} \right\}, \quad (2.10)$$

$$\{ \exp(\gamma \vert \bar{U}(\xi) \vert) : \xi \in L^0_+(\mathbb{F}_T) \}, \quad (2.11)$$

$$\{ \exp(\gamma \vert \bar{U}'(\xi) \vert) : \xi \in L^0_+(\mathbb{F}_T) \} \quad (2.12)$$

are uniformly integrable for all $\gamma > 0$.

(ii) $\tilde{H}$ consists of all processes $H \in \tilde{H}$ such that (2.11) and (2.12) are checked.

We add the following assumption on the set of consumption terminal wealth.

(H4) If $(c, \xi) \in \mathcal{A}$, then $(c + \alpha, \xi + \alpha') \in \mathcal{A}$ for any $\alpha > 0$ and $\alpha' > 0$.

Remark 2.1 The set $\tilde{C} \times L$ is a convex one when the utilities are power functions.
Remark 2.2 Assumption (H4) is satisfied if the utility functions are subadditive.

By the martingale representation theorem for Brownian motion (see e.g. Karatzas and Shreve [12]), any probability measure equivalent to $P$ has a density process in the form:

$$Z^\nu = \mathcal{E} \left( -\int (\theta + \sigma^{-1}\nu) \, dW \right), \quad \text{(2.13)}$$

where $\nu$ lies in the set $\Theta_{ad}$ of $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process such that $\int_0^T |\sigma_t^{-1}\nu_t|^2 \, dt < \infty$ $P$ a.s. and $E[Z_0^\nu] = 1$. We define the process $(V_{t}^{c,H})_{t \geq 0}$ by

$$V_{t}^{c,H} := \int_0^t H_s \text{diag}(S_t)^{-1} \, dS_s - \int_0^t c_s \, ds. \quad \text{(2.14)}$$

By Girsanov’s theorem, the Doob-Meyer decomposition of $(V_{t}^{c,H})_{t \geq 0}$ under $P^\nu = Z_t^\nu \, P$, for $\nu \in \mathcal{N}$ and $(c,H) \in \tilde{C} \times \tilde{H}$ satisfying (2.7)-(2.8), is given by:

$$V_{t}^{c,H} = \int_0^t H_s^* \sigma_s dW_s^\nu - \int_0^t c_s \, ds + A_t^{\nu,H}, \quad t \in [0,T],$$

where $W^\nu$ is a $P^\nu$-Brownian motion and $A_t^{\nu,H} = \int_0^t (-H_s^t \nu_s) \, ds$ is the $P^\nu$-predictable compensator.

It follows that there is an upper bound for $\{A^{\nu,H}, H \in \tilde{H}, H_t \in K dt \otimes dP, t \in [0,T]\}$ if and only if $\nu$ is valued in $\hat{K}$. Therefore, by Follmer and Kramkov (9), Lemma 2.1) the probability measure $Q$ belongs to $\mathcal{P}^0$ if and only if there is an upper bound for all predictable processes arising in the Doob-Meyer decomposition of the semimartingale $V_{t}^{c,H}$ under $Q$. In this case the upper variation is equal to this upper bound. Thanks again to Lemma 2.1 in [9], the set $\mathcal{P}^0$ consists of all probability measures $P^\nu$ for $\nu \in \mathcal{N}(\hat{K})$ where

$$\mathcal{N}(\hat{K}) := \{\nu \in \Theta_{ad} : \nu \in \hat{K} \quad \text{and} \quad \int_0^T \delta^{\text{supp}}(\nu_t) \, dt \quad \text{is bounded} \}. \quad \text{(2.15)}$$

The upper variation process is given by:

$$A_t(P^\nu) = \int_0^t \delta^{\text{supp}}(\nu_s) \, ds, \quad t \in [0,T].$$

Moreover, for any nondecreasing predictable process $A$, such that

$$\int_0^t H_s^* \sigma_s dW_s^\nu - \int_0^t c_s \, ds + A_t, \quad t \in [0,T],$$

is a $P^\nu$-local supermartingale for any $H \in \hat{H}$, we have

$$(A_t - A_t(P^\nu))_{t \in [0,T]} \text{ is nondecreasing.} \quad \text{(2.16)}$$

We first state a dual characterization of the admissible control set $\mathcal{A}$.

**Proposition 2.1** Let $x \in \mathbb{R}_+$. Then there exists $(c,H) \in \tilde{C} \times \tilde{H}$ such that $(c,X_T^{x,c,H}) \in \mathcal{A}$ and $\xi \leq X_T^{x,c,H}$ if and only if properties (2.9)-(2.12) are satisfied and

$$v(c, \xi) := \sup_{P^\nu \in \mathcal{P}^0} E^{P^\nu} \left[ \xi + \int_0^T c_t \, dt - A_T(P^\nu) \right] \leq x. \quad \text{(2.16)}$$
**Proof.** Necessary condition. Let \((c, H) \in \tilde{C} \times \tilde{H}\) such that \((c, X^{x,c,H}_T) \in \mathcal{A}\) and \(\xi \leq X^{x,c,H}_T\). From the definition of \(\mathcal{P}^0\), \(X^{x,c,H}_T + \int_0^T c_t dt - A_t(P^\nu)\) is bounded from below by a \(P^\nu\)-integrable random variable. By Fatou’s lemma the \(P^\nu\)-local supermartingale \((X^{x,c,H}_t + \int_0^t c_s ds - A_t(P^\nu))\) is a \(P^\nu\)-supermartingale. We deduce that:

\[
E^{P^\nu}\left[\xi + \int_0^T c_t dt - A_T(P^\nu)\right] \leq E^{P^\nu}\left[X^{x,c,H}_T + \int_0^T c_t dt - A_T(P^\nu)\right] \leq x,
\]

for all \(P^\nu \in \mathcal{P}^0\). This shows that \(v(c, \xi) \leq x\).

**Sufficient condition.** Consider the random variable from below \(g = \xi + \int_0^T c_t dt\). Since

\[
v(c, \xi) = \sup_{P^\nu \in \mathcal{P}^0} E^{P^\nu}\left[g - A_0^\nu(P^\nu)\right] \leq x < \infty,
\]

then by the stochastic control Lemma A.1 of Föllmer and Kramkov [13], there exists a RCLL version of the process:

\[
V_t = \text{ess sup}_{P^\nu \in \mathcal{P}^0} E^{P^\nu}\left[g - A_T(P^\nu) + A_t(P^\nu)|F_t\right] 0 \leq t \leq T. \tag{2.17}
\]

Moreover, for any \(P^\nu \in \mathcal{P}^0\), the process \((V_t - A_t(P^\nu))_{t \in [0,T]}\) is a \(P^\nu\)-local supermartingale. By the optional decomposition under constraints of Föllmer and Kramkov (see their Theorem 3.1), the process \(V\) admits a decomposition:

\[
V_t = v(c, X) + U_t - C_t, \ t \in [0, T]
\]

where \(U \in \hat{S} := \{X^{x,c,H} + \int c_t dt - x\}\) and \(C\) is an (optional) nondecreasing process with \(C_0 = 0\).

Hence there exists \((c, H) \in \tilde{C} \times \tilde{H}\) and \((c, X^{x,c,H}_t) \in \mathcal{A}\) such that:

\[
V_t \leq X^{x,c,H}_T + \int_0^T c_s ds, \ \text{a.s.} \ 0 \leq t \leq T. \tag{2.18}
\]

Since \(v(c, \xi) \leq x\) and \((V_t)_{t \in [0,T]}\) is bounded from below, by (2.17), inequality (2.18) implies that for \(t = T\)

\[
V_T := \xi + \int_0^T c_t dt \leq X^{x,c,H}_T + \int_0^T c_s ds, \ \text{0 \leq t \leq T},
\]

and so \(\xi \leq X^{x,c,H}_T\) where \((c, H) \in \tilde{C} \times \tilde{H}\) such that \((c, X^{x,c,H}_T) \in \mathcal{A}\) and the proof is ended. \(\Box\)

We say that the strategy \((c, \xi)\) is admissible if \((c, \xi) \in \mathcal{A}\) and \(v(c, \xi) \leq x\). We denote by \(\bar{\mathcal{A}}(x)\) the set of admissible strategies.

The problem of optimal consumption-investment is formulated as

\[
V(x) = \sup_{(c, \xi) \in \bar{\mathcal{A}}(x)} Y^{x,c,\xi}_0, \ x \in \mathbb{R}_+,
\]

where the dynamics of \(Y^{x,c,\xi}_t = (Y^{x,c,\xi}_t)_{0 \leq t \leq T}\) is given by

\[
\begin{align*}
dY^{x,c,\xi}_t &= (\delta_t Y^{x,c,\xi}_t - \alpha U(c_t))dt + \frac{1}{2\beta} |Z^{x,c,\xi}_t|^2 dt + Z^{x,c,\xi}_t dW_t \\
Y^{x,c,\xi}_T &= \tilde{a}U(\xi)
\end{align*}
\]

The following result is the comparison theorem for the BSDE (2.2)-(2.3). The proof is given in Faidi et al [10] (Theorem 3.1 pp. 1022) and for sake of completeness, we give the proof in the Appendix.
Theorem 2.1 We consider \((Y^1, Z^1)\) and \((Y^2, Z^2)\) two solutions of the BSDE (2.2)-(2.3) associated to \((U^1, \tilde{U}^1)\) and \((U^2, \tilde{U}^2)\) respectively. We assume that the Assumptions (H1)-(H2) hold and that
\[
\begin{align*}
\tilde{U}_t^1 &\leq \tilde{U}_t^2 \ dt \otimes dP \ a.e. \ , t \in [0, T], \\
\tilde{U}_t^1 &\leq \tilde{U}_T^2 \ dP \ a.s.
\end{align*}
\] (2.22) (2.23)
Then, we have
\[
Y_t^1 \leq Y_t^2 \ dt \otimes dP \ a.e. \ , t \in [0, T].
\]

Also, we have a continuity result for the solution of the BSDE (2.20)-(2.21) which will be useful later. The proof is given in Faidi et al [10] (Proposition 3.2 pp. 1024).

Proposition 2.2 We assume (H1) and (H3). Let \((c, \xi) \in \mathcal{A}\) and \((c^n, \xi^n)_{n \in \mathbb{N}}\) a sequence of admissible strategies.

(i) If \(\xi^n \searrow \xi\) \(dP\) a.s. and \(c^n_t \searrow c_t\), \(0 \leq t \leq T\), \(dt \otimes dP\) a.e. when \(n\) goes to infinity, then \(Y_{t}^{x, c^n, \xi^n} \searrow Y_{t}^{x, c, \xi}\), \(0 \leq t \leq T\), \(dt \otimes dP\) a.s. when \(n\) goes to infinity.

(ii) If \(\xi^n \nearrow \xi\) \(dP\) a.s. and \(c^n_t \nearrow c_t\), \(0 \leq t \leq T\), \(dt \otimes dP\) a.e. when \(n\) goes to infinity, then \(Y_{t}^{x, c^n, \xi^n} \nearrow Y_{t}^{x, c, \xi}\), \(0 \leq t \leq T\), \(dt \otimes dP\) a.s. when \(n\) goes to infinity.

3. Optimum Strategy Plan

In this section, we will study the existence of an optimal consumption-investment strategy.

Lemma 3.1 The sets \(\mathcal{A}\) and \(\hat{\mathcal{A}}(x)\) are convex and closed for the topology of convergence in measure.

Proof. The convexity of \(\hat{\mathcal{A}}(x)\) follows from the convexity of \(\mathcal{A}\) and the Proposition 2.1. Pham [16] studied in Lemma 7.1 the closedness of set \(\{X^{x,c,H} = x + \int H_t \, dS_t - \int c_t \, dt \ s.t. H \in \hat{\mathcal{H}}, c \in \hat{\mathcal{C}}\}\) in the semimartingale topology associated to the Emery distance. He proved that for every sequence \((\xi^n)_{n \in \mathbb{N}}\),
\[
\xi^n = x + \int H^n_t \, dS_t - \int c^n_t \, dt,
\]
which converges to \(\xi\) in the semimartingale topology, there exist \(H \in \hat{\mathcal{H}}\) and \(c \in \hat{\mathcal{C}}\) such that
\[
H^n_t \rightarrow H_t \ dt \otimes dP \ a.e. \ c^n_t \rightarrow c_t \ dt \otimes dP \ a.e.
\]
and \(\xi = x + \int H_t \, dS_t - \int c_t \, dt\). By Fatou’s lemma, we have
\[
E^{P^\nu} \left[\xi + \int_0^T c_t \, dt - A_T(P^\nu)\right] \leq \liminf_{n \to \infty} E^{P^\nu} \left[\xi^n + \int_0^T c^n_t \, dt - A_T(P^\nu)\right] \leq x,
\]
and so if \((c^n, \xi^n) \in \hat{\mathcal{A}}(x)\) and using the uniform integrability of the families (2.9), (2.10), (2.11) and (2.12), we have \((c, \xi) \in \hat{\mathcal{A}}(x)\). This proves that \(v(c, \xi) \leq x\) and so from Proposition 2.1, the closedness property of \(\hat{\mathcal{A}}(x)\) is checked.

Lemma 3.2 The density \(Z^*\) of the probability measure \(Q^*\) is in \(L^p(P)\) for all \(p \geq 1\).
Proof. From the dynamics of $Y^{x,c,ξ}$ given by the equation (2.20), we have

$$\frac{1}{\beta}(Y^{x,c,ξ} - Y_0^{x,c,ξ}) + \frac{1}{\beta} \int_0^t (\delta_t Y^{x,c,ξ} - \alpha U(c_s)) ds = -\frac{1}{2\beta^2} \int_0^t |Z^{x,c,ξ}_s|^2 ds - \frac{1}{\beta} \int_0^t Z^{x,c,ξ}_s dW_s,$$

and so for all $p \geq 1$, we obtain

$$E_P \left[ \exp \left( p \left( \frac{1}{\beta}(Y^{x,c,ξ} - Y_0^{x,c,ξ}) + \frac{1}{\beta} \int_0^t (\delta_t Y^{x,c,ξ} - \alpha U(c_s)) ds \right) \right) \right] = E_P \left[ (Z^*_t)^p \right].$$

Since $Y \in D^{exp}_0$ and $(U(c_t))_{0 \leq t \leq T} \in D^{exp}_1$, the result follows. \qed

Lemma 3.3 The functional $(c, ξ) \rightarrow Y^{x,c,ξ}_0$ is strictly concave and upper-semicontinuous.

Proof. Bodigoni in [3], Chapter 3, Lemma 3.9, proved that the functional $(c, ξ) \rightarrow Y^{x,c,ξ}_0$ is strictly concave.

Let $λ$ be a fixed real. We fix $ν \in (0, \infty)$. We consider the following set

$$A_λ := \{(c, ξ) \in \tilde{A}(x) \mid Y^{x,c,ξ}_0 ≥ λ\}.$$

Let $(c^n, ξ^n)_{n \in \mathbb{N}}$ be a sequence of admissible strategies such that $(c^n, ξ^n) \rightarrow (c, ξ)$ when $n$ goes to infinity in $H^2_T(\mathbb{R}) \times L^2_T(\mathbb{R})$, where

$$L^2_T(\mathbb{R}) := \{ y \mathcal{F}_T \text{-measurable s.t. } ||y||^2_{L^2} = E_P[|y|^2] < \infty \}.$$

There exists a subsequence, denoted also by $(c^n, ξ^n)$ such that $(c^n, ξ^n) \rightarrow (c, ξ)$ a.s. when $n$ goes to infinity. We assume that $(c^n, ξ^n) \in A_λ$ for all $n \in \mathbb{N}$. From Lemma 3.1, we have $(c, ξ) \in \tilde{A}(x)$. From the definition of the reward function and using the concavity property of the utility functions, we have

$$Y^{x,c^n,ξ^n}_0 - Y^{x,c,ξ}_0 \leq E_Q^* \left[ \int_0^T \alpha S^*_t U(c^n_t) - U(c_s) ds + \alpha S^*_t (\bar{U}(ξ^n) - \bar{U}(ξ)) \right],$$

where the probability measure $Q^*$ has a density given by the $P$-martingale $Z^* = (Z^*_t)_{0 \leq t \leq T} = (\mathcal{E}(\frac{1}{\beta} \int_0^T Z^{x,c,ξ}_s dW_s))_{0 \leq t \leq T}$. From Assumption (H3)(i), we have

$$U(c^n_t) - U(c_s) \leq U'(c_s)(c^n_t - c_s) \quad \text{and} \quad \bar{U}(ξ^n) - \bar{U}(ξ) \leq \bar{U}'(ξ)(ξ^n - ξ),$$

which implies

$$Y^{x,c^n,ξ^n}_0 - Y^{x,c,ξ}_0 \leq E_P \left[ α \int_0^T Z^*_s S^*_t U'(c_s)(c^n_t - c_s) ds \right.

+ \left. Z^*_T \alpha S^*_T \bar{U}'(ξ)(ξ^n - ξ) \right].$$

Using the Cauchy-Schwarz inequality and since $(c, ξ) \in \tilde{A}(x)$, we obtain

$$Y^{x,c^n,ξ^n}_0 - Y^{x,c,ξ}_0 \leq \sqrt{E_P \left[ \int_0^T (αZ^*_s S^*_t U'(c_s))^2 ds \right]} \sqrt{E_P \left[ \int_0^T (c^n_t - c_s)^2 ds \right]} \quad \text{and} \quad \sqrt{E_P \left[ (\bar{α}Z^*_s S^*_T \bar{U}'(ξ))^2 \right]} \sqrt{E_P \left[ (ξ^n - ξ)^2 \right]}.$$
Since $U'(c_1) \in D_1^{\exp}$ and $\bar{U}'(\xi) \in \bar{D}_1^{\exp}$, by the Cauchy-Schwarz inequality and from Lemma 3.2, we have
\begin{equation}
Y_0^{x,c^n,\xi^n} - Y_0^{x,c,\xi} \leq M \left( \sqrt{E P \left[ \int_0^T (c^n_s - c_s)^2 ds \right]} + \sqrt{E P \left[ (\xi^n_s - \xi_s)^2 \right]} \right),
\end{equation}
(3.2)
where $M$ is a positive constant independent of $n$. Sending $n$ to infinity in inequality (3.2), we deduce that
\begin{equation}
Y_0^{x,c,\xi} \geq \lambda.
\end{equation}
(3.3)
This proves that $Y_0^{x,c,\xi}$ is upper-semicontinuous.

**Lemma 3.4** Under Assumption (H1), we have
\begin{equation}
\sup_{(c,\xi) \in \bar{A}(x)} Y_0^{x,c,\xi} < \infty.
\end{equation}
(3.4)
**Proof.** For all $x \geq 0$, we have $|\bar{U}(x)| \leq \exp |\bar{U}(x)|$ and $|U(x)| \leq \exp |U(x)|$. From the definition of $Y_0^{x,c,\xi}$, and using Assumption (H1), we have
\begin{equation}
Y_0^{x,c,\xi} \leq C E P \left[ \int_0^T |U(c_s)| ds + |\bar{U}(\xi)| \right],
\end{equation}
\begin{equation}
\leq C \left( \sup_{(c,\xi) \in \bar{A}(x)} E P \left[ \int_0^T |U(c_s)| ds + E P [\bar{U}(\xi)] \right] \right)
\end{equation}
From the uniform integrability of the families (2.9) and (2.11), the result follows. \qed

Our next result is the existence of a unique solution to the problem (4.8). The uniqueness follows since $J$ is strictly concave. We shall assume

(H5) There exist $\gamma_1 > 0$ and $\gamma_2 > 0$ such that $z \leq \exp (\gamma_1 U(z))$ and $z \leq \exp (\gamma_2 \bar{U}(z))$ for all $z \geq 0$.

**Remark 3.1** Assumption (H5) holds if $U(z) = \bar{U}(z) = \log(z)$ or $U(z) = \bar{U}(z) = \frac{z^n}{\eta}$, $\eta \in (0, 1)$.

**Theorem 3.1** Let the Assumptions (H1), (H3) and (H5) hold. There exists a unique solution $(c^*, \xi^*) \in \bar{A}(x)$ of (2.19).

**Proof.** Let $(c^n, \xi^n)_{n \in \mathbb{N}} \in \bar{A}(x)$ be a maximizing sequence of the problem (2.19) i.e.
\begin{equation}
\lim_{n \to \infty} Y_0^{x,c^n,\xi^n} = \sup_{(c,\xi) \in \bar{A}(x)} Y_0^{x,c,\xi} < \infty,
\end{equation}
(3.5)
which is finite by Lemma 3.4.
Since $\xi^n \geq 0$ $dP - a.s$ and $c^n_t \geq 0$ $dt \otimes dP - a.s$, then by Lemma A.1.1 of Delbaen and Schachermeyer [6], there exists a sequence $(\hat{c}^n, \hat{\xi}^n) \in \text{conv} \{(c^n, \xi^n), (c^{n+1}, \xi^{n+1}), \ldots\}$ such that $(\hat{c}^n, \hat{\xi}^n)$ converges almost surely to $(c^*, \xi^*) \in L_0^T [0, T] \times \mathcal{F}_T$ $\times L_0^T (\mathcal{F}_T)$. By Lemma 3.1, we have $(c^*, \xi^*) \in \bar{A}(x)$. By the concavity and the upper-semicontinuity of the function $(c, \xi) \mapsto J(x,c,\xi)$, we have
\begin{equation}
\sup_{(c,\xi) \in \bar{A}(x)} Y_0^{x,c,\xi} \leq \limsup_{n \to \infty} Y_0^{x,c^n,\xi^n}.
\end{equation}
Let $\epsilon$ be a positive constant, we have the following inequality
\[
E_P[(\hat{\xi}^n - \xi^*)^2] \leq E_P[(\hat{\xi}^n - \xi^*)^2 1_{|\hat{\xi}^n - \xi^*| \leq \epsilon}] + E_P[(\hat{\xi}^n - \xi^*)^2 1_{|\hat{\xi}^n - \xi^*| > \epsilon}]
\]
\[
\leq \epsilon^2 + 2E_P[(\hat{\xi}^n)^2 1_{|\hat{\xi}^n - \xi^*| \leq \epsilon}] + 2E_P[(\xi^*)^2 1_{|\hat{\xi}^n - \xi^*| < \epsilon}]
\]
\[
\leq \epsilon^2 + \sqrt{E_P[(\hat{\xi}^n)^4]} \sqrt{P(|\hat{\xi}^n - \xi^*| > \epsilon)}
\]
\[
+ \sqrt{E_P[(\hat{\xi}^n)^4]} \sqrt{P(|\hat{\xi}^n - \xi^*| > \epsilon)}.
\]

From Assumption (H5) and the uniform integrability of the family (2.11), the sequence $((\hat{\xi}^n)^4)$ is uniformly integrable and so converges in $L^1(P)$. Sending $n$ to infinity and $\epsilon$ to 0, we have $E_P[(\hat{\xi}^n - \xi^*)^2] \to 0$ and so $\hat{\xi}^n$ converges to $\xi^*$ in $L^2(P)$ when $n$ goes to infinity. Similarly $\hat{c}_n$ converges to $c^*$ in $H^2(P)$ when $n$ goes to infinity. From Proposition 3.3, The functional $(c, \xi) \mapsto Y_0^{x,c,\xi}$ is upper-semicontinuous and so
\[
\limsup_{n \to \infty} Y_0^{x,c,\xi^n} \leq Y_0^{x,c,\xi^*}.
\]
Therefore $(c^*, \xi^*)$ solves (2.19).

4. Duality

The aim of this section is to provide a description of the solution structure to problem (2.19) by means of the dual formulation. Usually, when the model is known, the criterion is taken under the historical probability measure. The dual approach is based on the conjugate function of $U$ and $\bar{U}$. In our case, the criterion is taken under $Q^*$ which depends on the optimal consumption-investment strategy. The use of the conjugate function is not appropriate in our case. The following result shows the existence of an optimal probability measure solution of the dual problem. To prove such result, we need to use convex duality arguments. We will show that the probability measure $P^{\nu^*}$ solution of the dual problem
\[
v(c^*, \xi^*) = \sup_{P^\nu \in \mathcal{P}^0} E^{P^\nu}\left[\xi^* + \int_0^T c^*_t dt - A_T(P^{\nu})\right],
\]
(4.1)
lives in a subset of $\mathcal{P}^0$ denoted by $\mathcal{P}^{aux}$ which consists of all probability measure $P^{\nu} \in \mathcal{P}^0$ such that $(Z^{\nu}_T)^\eta$ is integrable under $P$ for a fixed constant $\eta > 1$. Then, we will show that the budget constraint is satisfied with equality which is a consequence from the strict concavity of the utility functions. As in Pham [16], we start with the following lemma.

**Lemma 4.1.** The set of probability measures $\mathcal{P}^0$ is convex and the function
\[
\mathcal{P}^0 \rightarrow \mathbb{R}_+
\]
\[
Q \rightarrow E^Q[A_T(Q)]
\]
is convex.

**Proof.**

Let $P^{\nu_1}, P^{\nu_2} \in \mathcal{P}^0$, $Z^{\nu_1}, Z^{\nu_2}$ their density process, $\alpha \in [0, 1]$ and denote by $P^{\nu} \sim P$ the probability measure $P^{\nu} = \alpha P^{\nu_1} + (1 - \alpha) P^{\nu_2}$ and by $Z^{\nu}$ its density process. Consider the process $A^{P^{\nu}}$ defined by
\[
A^{P^{\nu}}_t = \alpha \int_0^t \frac{Z^{\nu_1}_u}{Z^{\nu}_u} dA_u(P^{\nu_1}) + (1 - \alpha) \int_0^t \frac{Z^{\nu_2}_u}{Z^{\nu}_u} dA_u(P^{\nu_2}) \quad 0 \leq t \leq T.
\]
Fix $0 \leq u \leq t \leq T$. We have, for $i = 1, 2$,

\[
\mathbb{E}\left[Z_t \int_0^t \frac{Z_s}{Z_u} dA_u(P^{\nu i}) \mid \mathcal{F}_u\right] = Z_u \int_0^u \frac{Z_s}{Z_u} dA_s(P^{\nu i}) + \mathbb{E}\left[Z_t \int_0^t \frac{Z_s}{Z_u} dA_s(P^{\nu i}) \mid \mathcal{F}_u\right]
\]

\[
= Z_u \int_0^u \frac{Z_s}{Z_u} dA_s(P^{\nu i}) + \mathbb{E}\left[Z_t \int_0^t \frac{Z_s}{Z_u} dA_s(P^{\nu i}) \mid \mathcal{F}_u\right]
\]

\[
= Z_u \int_0^u \frac{Z_s}{Z_u} dA_s(P^{\nu i}) + \mathbb{E}\left[Z_t \int_0^t \frac{Z_s}{Z_u} dA_s(P^{\nu i}) \mid \mathcal{F}_u\right]
\]

\[
= Z_u \int_0^u \frac{Z_s}{Z_u} dA_s(P^{\nu i}) - Z_u A_u(P^{\nu i}) + \mathbb{E}\left[Z_t A_t(P^{\nu i}) \mid \mathcal{F}_u\right]
\]

(4.2)

where we used the properties that $Z$ is a $P$-martingale, Bayes formula and law of iterated conditional expectations. So, we have

\[
Z_t(X_t^{x,c',H^*} + \int_0^t c_s^i ds - A_t^{P^{\nu i}}) = \alpha \left[Z_t^{\nu 1}(X_t^{x,c',H^*} + \int_0^t c_s^{\nu 1} ds) - Z_t \int_0^t \frac{Z_s^{\nu 1}}{Z_u} dA_u(P^{\nu 1})\right]
\]

\[
+ (1 - \alpha) \left[Z_t^{\nu 2}(X_t^{x,c',H^*} + \int_0^t c_s^{\nu 2} ds) - Z_t \int_0^t \frac{Z_s^{\nu 2}}{Z_u} dA_u(P^{\nu 2})\right]
\]

and using relations (4.2) for $i = 1, 2$, we obtain by the supermartingale property of $Z_t(X_t^{x,c',H^*} + \int_0^t c_s^i ds - A(P^{\nu i}))$ under $P$,

\[
\mathbb{E}\left[Z_t(X_t^{x,c',H^*} + \int_0^t c_s^i ds - A_t^{P^{\nu i}}) \mid \mathcal{F}_u\right] = \alpha \mathbb{E}\left[Z_t^{\nu 1}(X_t^{x,c',H^*} + \int_0^t c_s^{\nu 1} ds) - Z_t \int_0^t \frac{Z_s^{\nu 1}}{Z_u} dA_u(P^{\nu 1}) \mid \mathcal{F}_u\right]
\]

\[
+ (1 - \alpha) \mathbb{E}\left[Z_t^{\nu 2}(X_t^{x,c',H^*} + \int_0^t c_s^{\nu 2} ds) - Z_t \int_0^t \frac{Z_s^{\nu 2}}{Z_u} dA_u(P^{\nu 2}) \mid \mathcal{F}_u\right]
\]

\[
- \alpha Z_u \int_0^u \frac{Z_s^{\nu 1}}{Z_u} dA_u(P^{\nu 1}) - (1 - \alpha) Z_u \int_0^u \frac{Z_s^{\nu 2}}{Z_u} dA_u(P^{\nu 2})
\]

\[
+ \alpha Z_u A_u(P^{\nu 1}) + (1 - \alpha) Z_u A_u(P^{\nu 2})
\]

\[
\leq \alpha Z_u^{\nu 1}(X_t^{x,c',H^*} + \int_0^t c_s^{\nu 1} ds - A_u(P^{\nu 1}))
\]

\[
+ (1 - \alpha) Z_u^{\nu 2}(X_t^{x,c',H^*} + \int_0^t c_s^{\nu 2} ds - A_u(P^{\nu 2}))
\]

\[
- Z_u A_u^{P^{\nu 1}} + \alpha Z_u A_u(P^{\nu 1}) + (1 - \alpha) Z_u A_u(P^{\nu 2})
\]

\[
= Z_u(X_t^{x,c',H^*} + \int_0^t c_s^i ds - A_u^{P^{\nu i}})
\]

This proves the supermartingale property under $P$ of $Z_t(X_t^{x,c',H^*} + \int_0^t c_s^i ds - A^{P^{\nu i}})$ and so the supermartingale property under $P^{\nu}$ of $X_t^{x,c',H^*} + \int_0^t c_s^i ds - A^{P^{\nu}}$, so, $P^{\nu} \in \mathcal{P}^0$. This shows the convexity of $\mathcal{P}^0$. From (2.15), we have $A_T(P^{\nu}) \leq A_T^{P^{\nu}}$. Moreover, applying (4.2) for $u = 0$ and
\[ t = T, \text{ we have} \]
\[
\mathbb{E}^P \left[ A_T(P') \right] \leq \mathbb{E}^P \left[ Z^\nu_T A_T^\nu \right] \\
= \alpha \mathbb{E}^P \left[ Z^\nu_T A_T(P'^1) \right] + (1 - \alpha) \mathbb{E}^P \left[ Z^\nu_T A_T(P'^2) \right] \\
= \alpha \mathbb{E}^P \left[ A_T(P'^1) \right] + (1 - \alpha) \mathbb{E}^P \left[ A_T(P'^2) \right]
\]
which proves the convexity of the function \( P' \in \mathcal{P} \to \mathbb{E}^P \left[ A_T(P') \right] \).

\[ \square \]

**Theorem 4.1** Under Assumptions (H1)-(H3)-(H4)-(H5), there exists a probability measure \( \bar{P} \in \mathcal{P}^0 \) equivalent to \( P \) s.t.
\[
\sup_{P' \in \mathcal{P}^0} \mathbb{E}^{\bar{P}} \left[ \xi^* + \int_0^T c_t^* dt - A_T(P') \right] = \sup_{P' \in \mathcal{P}^{\text{aux}}} \mathbb{E}^{\bar{P}} \left[ \xi^* + \int_0^T c_t^* dt - A_T(\bar{P}^*) \right], \tag{4.3}
\]
where \( \xi^* := X_T^{*,c^*,H^*} \), and the budget constraint is satisfied with equality i.e.
\[
\mathbb{E}^{\bar{P}} \left[ \xi^* + \int_0^T c_t^* dt - A_T(\bar{P}^*) \right] = x, \tag{4.4}
\]

**Proof.**

\[ \star \text{ First step: Let } F(P') \text{ defined by the following functional:}\]
\[
F(P') = \xi^* + \int_0^T c_t^* dt - A_T(P')
\]
and
\[
G(P') = \mathbb{E}^{\bar{P}} \left[ F(P') \right]
\]
the main result of this theorem is that of maximizing \( G(P') \) over \( P' \in \mathcal{P}^0 \) has a solution \( \bar{P}^* \in \mathcal{P}^0 \) is even equivalent to \( P \).

Let \( (P^{\nu_n})_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{P}^{\text{aux}} \) such that:
\[
\lim_{n \to +\infty} G(P^{\nu_n}) = \sup_{P' \in \mathcal{P}^0} G(P') < \infty.
\]
and denote by \( Z^n = Z^{P^{\nu_n}} \) the corresponding density process. Since each \( Z^n_T \geq 0 \), it’s follows from Komlos’ theorem that there exists a sequence \( (\bar{Z}^n_T)_{n \in \mathbb{N}} \) with \( (\bar{Z}^n_T)_{n \in \mathbb{N}} \in \text{conv}(Z^n_T, Z^{n+1}_T, \ldots) \) for each \( n \in \mathbb{N} \) and such that \( (\bar{Z}^n_T) \text{ cv } P.a.s \text{ to some random variable } (\bar{Z}^\infty_T) \), which is then also nonnegative but may take value \(+\infty\).

Because \( \mathcal{P}^{\text{aux}} \) is convex, each \( \bar{Z}^n_T \) is again associated to some \( \bar{P}^{\nu_n} \in \mathcal{P}^{\text{aux}} \). By de la Vallée-Poussin’s criterion, \( (\bar{Z}^n_T)_{n \in \mathbb{N}} \) is uniformly integrable and therefore converges in \( L^1(P) \). This implies that \( \lim_{n \to +\infty} \mathbb{E}^P [\bar{Z}^n_T] = \mathbb{E}^P [\bar{Z}^\infty_T] = 1 \) and so \( d\bar{P}^{\nu_n} = \bar{Z}^{\nu_n}_T \text{ dP} \) defines a probability measure which is equivalent to \( P \).

\[ \star \text{ Second step: We claim } \bar{P}^{\nu_n} \in \mathcal{P}^0 \text{. For this, we will show that } \bar{P}^{\nu_n} \text{ attains the supremum of } P' \to G(P') \in \mathcal{P}^0 \text{ and therefore, we examine } G(\bar{P}^{\nu}) \text{ more closely.} \]
Since we know that \( (\bar{Z}^n_T) \) converges to \( \bar{Z}^{\nu_n} \) in \( L^1(P) \), the Doob’s maximal inequality
\[
P \left[ \sup_{0 \leq t \leq T} | \bar{Z}^{\nu_n}_t - \bar{Z}^n_t | \geq \varepsilon \right] \leq \frac{1}{\varepsilon} \mathbb{E}_P [ \bar{Z}^{\nu_n}_T - \bar{Z}^n_T ]
\]
implies that \((\sup_{0 \leq t \leq T} | \hat{Z}_t^n - \bar{Z}_t^n|)_{n \in \mathbb{N}}\) converges to 0 in \(P\)-probability.

Going to a subsequence, still denoted by \((\hat{Z}_n)_{n \in \mathbb{N}}\), we can assume that 
\((\sup_{0 \leq t \leq T} | \hat{Z}_t^n - \bar{Z}_t^n|)_{n \in \mathbb{N}}\) converges to 0 \(P\)-a.s.

By the Burkholder Davis Gundy inequality, there is a constant \(C\) such that
\[
E[(\hat{Z}_T^n - \bar{Z}_T^n)^2] \leq CE[\sup_{0 \leq t \leq T} | \hat{Z}_t^n - \bar{Z}_t^n|].
\]

Let \(M^n_t := \sup_{0 \leq s \leq t} | \hat{Z}_s^n - \bar{Z}_s^n |\) and \((\tau_n)\) a sequence of stopping time defined by
\[
\tau_n = \begin{cases} 
\inf \{ t \in [0, T] : M^n_t \geq 1 \} & \text{if } \{ t \in [0, T] : M^n_t \geq 1 \} \neq \emptyset \\
T & \text{otherwise}
\end{cases}
\]

Since \(M^n_{\tau_n}\) is bounded by \(M^n_T \wedge 1\) then \(M^n_{\tau_n}\) converges almost surely to 0 and by the dominated convergence theorem converges to 0 in \(L^1(P)\). Then, using Burkholder Davis Gundy inequality \((\hat{Z}_T^n - \bar{Z}_T^n)^2\) converges to 0 in \(L^1(P)\) and a fortiori in probability.

As, \((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n} = (\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n}1_{\{\tau_n = T\}} + (\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n}1_{\{\tau_n < T\}}\), then for all \(\varepsilon > 0\),
\[
P((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n} \geq \varepsilon) \leq P((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n}1_{\{\tau_n = T\}} \geq \varepsilon) + P((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n}1_{\{\tau_n < T\}} \geq \varepsilon)
\]
\[
\leq P((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n} \geq \varepsilon) + P(\tau_n < T)
\]

From the convergence in probability of \((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n}\), we have \(\lim_{n \to +\infty} P((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n} \geq \varepsilon) = 0\). Since \(M^n\) is an increasing process, we have
\[
P(\tau_n < T) = P(\{ \exists t \in [0, T] : M^n_t \geq 1 \}) = P(\{ M^n_T \geq 1 \}).
\]

Since \(M^n_T\) converges in probability to 0, we have \(P(\{ M^n_T \geq 1 \}) \to 0\). Then \(\lim_{n \to +\infty} P(\tau_n < T) = 0\), and consequently \(\lim_{n \to +\infty} P((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n} \geq \varepsilon) = 0\).

We can extract a subsequence denoted also by \(\hat{Z}_n\) such that \((\hat{Z}_T^n - \bar{Z}_T^n)_{\tau_n}\) converges almost surely to 0.

On the other hand we have
\[
\langle \hat{Z}_T^n - \bar{Z}_T^n \rangle_T = \int_0^T (\hat{Z}_u^n(\theta_u + \sigma_u^{-1}\hat{\nu}_u^n) - \bar{Z}_u^n(\theta_u + \sigma_u^{-1}\hat{\nu}_u^n))^2 du.
\]

Since \(\hat{Z}_n \to \hat{Z}_T^n dt \otimes dP\text{-a.e.}\), we have \(\hat{\nu}_n\) converges to \(\hat{\nu}_T\) \(dt \otimes dP\text{-a.e.}\).

From the continuity of the support function \(\delta^{supp}\), we deduce that \(\delta^{supp}(\hat{\nu}_n)\) converges to \(\delta^{supp}(\hat{\nu}_T)\) \(dt \otimes dP\text{-a.e.}\). We denote by \(E_0\) the following set \(E_0 := \{ t \in [0, T] \text{ s.t. } \delta^{supp}(\hat{\nu}_n) \to \delta^{supp}(\hat{\nu}_T) \}\).

By the definition of \(E_0\), we have \(E_0 = [0, T]\). Let \(g_n\) be defined by \(g_n(t) = |\delta^{supp}(\hat{\nu}_n) - \delta^{supp}(\hat{\nu}_T)|, t \in [0, T]\). We fix \(p \in \mathbb{N}\) and we consider the following set \(E_{p_m} = \{ t \in E_0 \text{ s.t. } \sup_{m \geq n \geq 1} g_m(t) \leq \frac{1}{p} \}\). For \(p \geq 2\), we have \(\bigcup_{n \geq 1} E_{p_m} = E_0\). The set \(E_{p_m}\) is measurable, the sequence \((E_{p_m})_n\) is increasing for a fixed \(p\), and so \(\lim_{n \to +\infty} E_{p_m} = E_0\). From the monotounous convergence theorem, we deduce that \(\lambda(E_{p_m}) = \lambda(E_0)\), where \(\lambda\) denotes the Lebesgue measure. We fix \(\varepsilon > 0\), there exists \(n_p \in \mathbb{N}\) such that \(\lambda(E_{p_m}) > T - \frac{\varepsilon}{2p}\). We consider \(E_\varepsilon = \cap_{m=1}^\infty E_{p_m}\).

For all \(t \in E_\varepsilon\) and \(n \geq n_p\), we have \(g_n(t) \leq \frac{1}{p}\), which implies that for a fixed \(w\), \(\delta^{supp}(\hat{\nu}_n)\) converges uniformly in time to \(\delta^{supp}(\hat{\nu}_T)\) on \(E_\varepsilon\). On the other hand, we have
\[
\lambda([0, T] \setminus E_\varepsilon) = \lambda(E_0 \setminus E_\varepsilon) \leq \sum_{p=1}^\infty \lambda(E_0 \setminus E_{p_m}) \leq \sum_{p=1}^\infty \frac{\varepsilon}{2p} = \varepsilon
\]
which implies \(T - \varepsilon \leq \lambda(E_\varepsilon) \leq T\) and
so $\delta^{\text{supp}}(\tilde{\nu}_n)$ converges uniformly in time to $\delta^{\text{supp}}(\tilde{\nu}_\infty)$ on $[0, T]$. From the uniform convergence and when $n$ goes to infinity, we obtain

$$\int_0^T \delta^{\text{supp}}(\tilde{\nu}_n^t) dt \longrightarrow \int_0^T \delta^{\text{supp}}(\tilde{\nu}_\infty^t) dt \text{ P-a.s} \quad (4.5)$$

and so we deduce that, when $n$ goes to infinity,

$$\tilde{Z}_{T_n}^{\nu_n} F(\tilde{P}^{\nu_n}) \longrightarrow \tilde{Z}_{T}^{\nu} F(\tilde{P}^{\nu}) \: P - \text{a.s.} \quad (4.6)$$

From de la Vallée Poussin’s criterion, we deduce the uniform integrability of the family $(\tilde{Z}_{T_n}^{\nu_n} F(\tilde{P}^{\nu_n}))_n$. In fact, for a fixed $\eta$ satisfying $\eta > \eta' > 1$, by the Cauchy Shwartz inequality, there exists $\eta''$ such that for all $n$, we have

$$E_P \left[ (\tilde{Z}_{T_n}^{\nu_n} F(\tilde{P}^{\nu_n}))^{\eta''} \right] \leq E_P \left[ (\tilde{Z}_{T_n}^{\nu_n})^{\eta} \right] E_P \left[ (\xi^* + \int_0^T c_t^* dt)^{\eta''} \right]. \quad (4.7)$$

From (4.6) and (4.7), we obtain the convergence in $L^1(P)$ of the sequence $(\tilde{Z}_{T_n}^{\nu_n} F(\tilde{P}^{\nu_n}))_n$, which yields

$$\mathbb{E}_P \left[ \tilde{Z}_{T}^{\nu_n} F(\tilde{P}^{\nu_n}) \right] = \lim_{n \to \infty} \mathbb{E}_P \left[ \tilde{Z}_{T_n}^{\nu_n} F(\tilde{P}^{\nu_n}) \right].$$

This shows that

$$G(\tilde{P}_\infty^\nu) = \mathbb{E}_P \left[ \tilde{Z}_{T}^{\nu_n} F(\tilde{P}^{\nu_n}) \right] = \lim_{n \to \infty} G(\tilde{P}^{\nu_n}) = \sup_{P \in \mathcal{P}^{\text{aux}}_\infty} G(P^\nu)$$

which proves that $\tilde{P}_{\infty}^\nu$ is indeed optimal.

* Third step: we show that the budget constraint is satisfied with equality. We denote by $\nu^* = \nu_\infty$ and $P^*$ the probability measure associated with $Z_T^{\nu^*}$.

We assume that

$$E_P \left[ Z_T^{\nu^*} \xi^* + \int_0^T Z_T^{\nu^*} c_t^* dt - Z_T^{\nu^*} A_T(P^*) \right] = l < x.$$  

Following the same arguments as in the characterisation (2.16), we have

$$\sup_{P \in \mathcal{P}^{\text{aux}}_\infty} E_P \left[ \xi^* + \int_0^T c_t^* dt - A_T(P^\nu) \right] \leq l,$$

and so, we deduce that $(c^\nu, \xi^* \in A$, which implies that, there exists $H^* \in \mathcal{H}$ such that

$$X_t^{1,c^\nu,H^*} = l + \int_0^t H_s^* dS_s - \int_0^t c_s^\nu ds, \: dt \otimes dP \: a.e., \: t \in [0, T],$$

and $\xi^* = X_T^{1,c^\nu,H^*}$. We denote by $\tilde{c}_t = c_t^\nu + \frac{x-l}{T} dt \otimes dP \: a.e., \: t \in [0, T]$. Then

$$X_t^{\tilde{c},H^*} = x + \int_0^t H_s^* dS_s - \int_0^t \tilde{c}_s ds
= l + \int_0^t H_s^* dS_s - \int_0^t c_s^\nu ds
= X_t^{1,c^\nu,H^*},$$
which implies $X_T^{\hat{c},\hat{\xi}^*} = X_T^{c^*,\xi^*} = \xi^*$. Under Assumption (H4), $(\hat{c},\xi^*)$ satisfies (2.9)-(2.12) and
\[
E^P \left[ Z_T^* \xi^* + \int_0^T Z_t^* \hat{c}_t dt - Z_T^* A_T(\hat{P}^*) \right] = x,
\]
which implies $(\hat{c},\xi^*) \in \hat{A}(x)$. From the strict concavity of the utility function $U$, we obtain $Y_0^{x,\hat{c},\hat{\xi}^*} > Y_0^{x,c^*,\xi^*}$ which contradicts the optimality of the strategy $(c^*,\xi^*)$ and so $l = x$ and the equality (4.4) holds and from Proposition 2.1, we deduce that $\hat{P}^*$ solves the dual problem (4.1). \qed

Our aim is to derive a necessary and sufficient condition of optimality of $(c^*,\xi^*)$. We follow the approach of Duffie and Skiadas [7] and El Karoui et al. [8] by studying an auxiliary optimization problem without constraints. Let $\lambda$ be a positive constant, we consider the following consumption-investment problem

\[
\sup_{(c,\xi) \in \mathcal{A}} J(x, c, \xi, \hat{P}^*, \lambda),
\]

where the functional $J$ is defined on $\mathcal{A}$ by

\[
J(x, c, \xi, \hat{P}^*, \lambda) = Y_0^{x,\hat{c},\hat{\xi}^*} + \lambda (x - E^{\hat{P}^*}[\xi + \int_0^T c_t dt - A_T(\hat{P}^*)])
\]

We recall the following classical result of convex analysis (see e.g. Luenberger [14], Theorem 1 page 217 and Theorem 2 page 221) which relates the solutions of the problems (2.19) and (4.8).

**Proposition 4.1** We assume that (H1)-(H3)-(H4)-(H5) hold.

(i) There exists a constant $\lambda^*$ such that

\[
V(x) = \sup_{(c,\xi) \in \mathcal{A}} J(x, c, \xi, \hat{P}^*, \lambda^*).
\]

(ii) The maximum is attained in (4.8) by $(c^*,\xi^*)$ with $E^{\hat{P}^*}[\xi^* + \int_0^T c_t^* dt - A_T(\hat{P}^*)] = x$.

(iii) Conversely, If there exist a constant $\lambda^*$ and $(c^*,\xi^*) \in \mathcal{A}$ that achieve the maximum in (4.8) with $E^{\hat{P}^*}[\xi^* + \int_0^T c_t^* dt - A_T(\hat{P}^*)] = x$, then the maximum is attained in (2.19) by $(c^*,\xi^*)$.

**Proof.** (i) and (ii): The set $\mathcal{A}$ is convex. The Slater condition for the optimization problem (4.8) holds since the strategy $(\hat{c},\hat{\xi})$ defined by $\hat{\xi} = \frac{\xi}{\lambda}$ and $\hat{c}_t = \frac{c_t}{\lambda}$, $0 \leq t \leq T$ is admissible (i.e. $(\hat{c},\hat{\xi}) \in \mathcal{A})$. From Lemma 3.4, the value function (4.10) is finite. From Luenberger [14], Theorem 1 page 217, there exists a positive constant $\lambda^*$ such that equality (4.10) and the assertion (ii) hold.

(iii) From the definition of $\lambda^*$ and $(c^*,\xi^*)$, we have

\[
Y_0^{x,\hat{c},\hat{\xi}^*} + \lambda^* (x - E^{\hat{P}^*}[\xi + \int_0^T c_t dt - A_T(\hat{P}^*)])
\]

\[
\leq Y_0^{x,c^*,\xi^*} + \lambda^* (x - E^{\hat{P}^*}[\xi^* + \int_0^T c_t^* dt - A_T(\hat{P}^*)])
\]

\[
\leq Y_0^{x,c^*,\xi^*} + \lambda (x - E^{\hat{P}^*}[\xi^* + \int_0^T c_t^* dt - A_T(\hat{P}^*)])
\]
for all $\lambda \geq 0$, $(c, \xi) \in A$ and so $(\lambda^*, (c^*, \xi^*))$ is a saddle point. Then from Luenberger [14] Theorem 2 page 221, the assertion (iii) holds.

We shall introduce the dual function $\tilde{V}$ defined on $(0, \infty)$ by

$$\tilde{V}(\lambda) = \sup_{(c, \xi) \in A} \left\{ Y_0^{x,c,\xi} - \lambda E^{\tilde{P}^*}[\xi + \int_0^T c_t dt - A_T(\tilde{P}^*)] \right\},$$

and we have the following result

**Theorem 4.2** Assume that (H1)-(H3)-(H4)-(H5) hold.

1. We have the conjugate duality relation
   $$V(x) = \min_{\lambda > 0} \left\{ \tilde{V}(\lambda) + \lambda x \right\}, \forall x > 0$$

2. Let $\lambda^*$ be such that equality (4.10) holds. Let $(c^*, \xi^*)$ be the solution of the optimization problem (4.8), then the dual function $\tilde{V}$ is differentiable at $\lambda^*$ and
   $$\tilde{V}'(\lambda^*) = -E^{\tilde{P}^*}[\xi^* + \int_0^T c_t dt - A_T(\tilde{P}^*)]$$

3. The consumption-investment control $(c^*, \xi^*)$ is the unique solution of (2.19).

**Proof.**

1. From the definition of the value function, we have for all $\lambda > 0$
   $$V(x) = \sup_{(c, \xi) \in A} Y_0^{x,c,\xi}$$
   $$\leq \sup_{(c, \xi) \in A} \left\{ Y_0^{x,c,\xi} - \lambda (E^{\tilde{P}^*}[\xi + \int_0^T c_t dt - A_T(\tilde{P}^*)] - x) \right\}$$
   $$\leq \sup_{(c, \xi) \in A} \left\{ Y_0^{x,c,\xi} - \lambda E^{\tilde{P}^*}[\xi + \int_0^T c_t dt - A_T(\tilde{P}^*)] \right\} + \nu x.$$

   From the last inequality, we deduce that
   $$V(x) \leq \inf_{\lambda > 0} \left\{ \tilde{V}(\lambda) + \lambda x \right\}.$$

   From equality (4.10), we have
   $$V(x) = \tilde{V}(\lambda^*) + \lambda^* x,$$

   and the result follows.

2. From the definition, $\tilde{V}$ is the upper envelope of affine function with a non positive slope. Since $\xi \geq 0$ dP a.s and $c_t \geq 0, 0 \leq t \leq T$ dP, we have $\tilde{V}$ is concave and non-increasing and so the right-hand derivative (respectively the left-hand derivative) of $\tilde{V}$ denoted by $\tilde{V}'_r$ (respectively $\tilde{V}'_l$) is well-defined. Let $\epsilon$ be a positive constant. From the definition of $(c^*, \xi^*)$, we have
   $$\frac{\tilde{V}(\lambda^* + \epsilon) - \tilde{V}(\lambda^*)}{\epsilon} \geq -E^{\tilde{P}^*}[\xi^* + \int_0^T c_t dt - A_T(\tilde{P}^*)].$$

   Sending $\epsilon \rightarrow 0^+$, we have
   $$\tilde{V}'_r(\lambda^*) \geq -E^{\tilde{P}^*}[\xi^* + \int_0^T c_t dt - A_T(\tilde{P}^*)].$$
Similarly, we have
\[ \tilde{V}'(\lambda^*) \leq -E^{\hat{P}^*}[\xi^* + \int_0^T c^*_t dt - A_T(\hat{P}^*)]. \]

From the monotonicity property of \( \tilde{V} \), we have \( \tilde{V}'(\lambda^*) \leq \tilde{V}'(\lambda^*) \), \( \tilde{V} \) is differentiable at \( \lambda^* \) and the equality (4.11) holds.

(3) Since the minimum of \( \lambda \mapsto \tilde{V}(\lambda) + \lambda x \) is achieved in \( \lambda^* > 0 \), we have \( \tilde{V}'(\nu^*) = -x. \)

The next result is a dynamic maximum principle. It relates the utility derivatives of the consumption and the terminal wealth to the density of the risk neutral measure and the density of the probability measure representing the worst case. The proof is technical and is postponed in the appendix.

**Theorem 4.3** let Assumptions (H1)-(H3)-(H4)-(H5) hold. Let \((c^*, \xi^*) \in A\) be the optimal consumption and the optimal terminal wealth for (4.8) with \( \lambda = \lambda^* \) given in Proposition 4.1. Let \((Y^{x,c^*}, Z^{x,c^*})\) be the solution for the BSDE (2.20)-(2.21). Then the following maximum principle holds:
\[
\begin{align*}
\alpha Z^*_t S^*_t \tilde{U}'(\xi^*) &= \lambda^* \tilde{Z}^*_t \ dP \ a.s. \\
\alpha Z^*_t S^*_t \tilde{U}'(c^*_t) &= \lambda^* \tilde{Z}^*_t, \ 0 \leq t \leq T dt \otimes dP \ a.e.
\end{align*}
\]

where \( Z^*_t = \mathcal{E}(-\frac{1}{\beta} \int_0^t Z^{x,c^*} dW_s), \ 0 \leq t \leq T. \)

Theorem 4.3 provides a characterization of the solution of the primal problem in terms of \( \tilde{Z}^* \) the density of \( P^{\nu^*} \) and \( Z^* \) the density of the probability measure associated with the worst case. It is a generalization of the result of Cvitanic and Karatzas [5] (Section 12) when \( \alpha = \bar{\alpha} = 1, Z^*_t = 1, S^*_t = 1 dt \otimes dP \ a.e. \) for all \( t \in [0, T] \). By using Legendre-Fenchel transformation, they showed that
\[
\begin{align*}
\tilde{U}'(\xi^*) &= \lambda^* \tilde{Z}^*_T \ dP \ a.s. \\
U'(c^*_t) &= \lambda^* \tilde{Z}^*_t, \ 0 \leq t \leq T dt \otimes dP \ a.e.
\end{align*}
\]

5. **Forward-Backward System**

In this section, we characterize the optimal consumption-investment strategy as a solution of a forward-backward system. This characterization is a consequence of the maximum principle. In fact, from Theorem 4.3, The optimal terminal wealth \( \xi^* \) and the optimal consumption \( c^*_t \) are given by
\[
\begin{align*}
c^*_t &= I_1 \left( \frac{\lambda^*_t}{\alpha} \exp \left( \int_0^t \delta_u du \right) \tilde{Z}^*_t Z^*_t \right), \ 0 \leq t \leq T dt \otimes dP \ a.s. \quad (5.1) \\
\xi^* &= I_2 \left( \frac{\lambda^*_T}{\alpha} \exp \left( \int_0^T \delta_u du \right) \tilde{Z}^*_T Z^*_T \right) \ dP \ a.s. \quad (5.2)
\end{align*}
\]

where \( I_1 \) (resp. \( I_2 \)) is the inverse of the derivative function of \( U \) (resp. \( \tilde{U} \)). The following result is a direct consequence of Theorem 4.3 and Theorem 4.2.
Theorem 5.1 Assume that (H1), (H3), (H4) and (H5) hold. We consider \( Y \in \mathcal{D}_0^{c,F} \), \( Z = (Z_t)_{t \in [0,T]} \) \( \mathbb{R}^d \)-valued adapted process satisfying \( E[\int_0^T |Z_t|^2 dt] < \infty \), \((c^*, \xi^*) \in \hat{A}(x)\) and \( Z^Y \) a density of a probability measure. Then \( Y \) coincides with the optimal value process given by \( Y^{x,c^*,\xi^*} \), \((c, X_T)\) coincide with \((c^*, \xi^*)\) given by (5.1)-(5.2), \( Z^Y \) coincides with the density of the minimizing measure \( Z^* \) and \( Z^Y \) coincides with \( Z^* \) if and only if there exists a unique solution of the following forward-backward system

\[
\begin{align*}
    dX_t &= H_t dS_t - c_t dt, \quad X_0 = x \\
    dY_t &= (\delta_t Y_t - \alpha U(c_t) + \frac{1}{2\beta}|Z_t|^2) dt + Z_t dW_t, \quad Y_T = \bar{U}(X_T) \\
    dZ_t &= -\frac{1}{2} Z_t Z_t dW_t, \quad Z_0 = 1 \\
    dZ^*_t &= Z^*_t(b_t + \sigma_t^{-1} \nu_t) dW_t \\
    Z^*_0 &= 1.
\end{align*}
\]

Proof. If \( Y \) is given by (2.1) and \( Z^Y \) is given by (2.5), then, from Bordigoni et al. [4], \((Y, Z)\) is the unique solution of BSDE (2.2) with the terminal condition (2.3) and \( Z^Y \) is the unique solution of the third equation in our forward-backward system. From Theorem 4.1, the density \( Z^* \) solves the problem 4.1. From Theorem 4.3 and Theorem 4.2, the couple \((c^*, \xi^*)\) is the unique solution of (2.19) i.e. \( V(x) = Y^{x,c^*,\xi^*} \).

Example 5.1 We consider a financial market consisting of two risky assets \( S = (S^1_t, S^2_t)_{0 \leq t \leq T} \), where the price is governed by

\[
\begin{align*}
    dS^1_t &= b^1 S^1_t dt + \sigma^1 S^1_t dW^1_t, \\
    dS^2_t &= b^2 S^2_t dt + \sigma^2 S^2_t dW^2_t,
\end{align*}
\]

where \((W^1, W^2)\) is a \( P \)-\( \mathbb{F} \) Brownian motion, \( b^1, b^2, \sigma^1, \sigma^2 \) are constants. We take \( K = \{ H \in \mathbb{R}^2; H_2 = 0 \} \) where \( H = (H^1, H^2) \) is the investment strategy representing the amount of each asset invested in the portfolio. This is the incomplete market case studied by Karatzas et al [11] where the investment is restricted only to the first risky asset. It follows that the support function of the convex set \(-K\) is given:

\[
\begin{align*}
    \delta^\text{supp}(x) &= 0 \text{ if } x_1 = 0 \\
    \delta^\text{supp}(x) &= \infty \text{ otherwise },
\end{align*}
\]

and

\( \hat{K} = \{ x \in \mathbb{R}^2; x_1 = 0 \} \).

We know that the density of the risk neutral measure is given by \( Z^*_T = \mathcal{E}(-\theta^1 W^1_T - \int_0^T (\theta^2 + \frac{\nu^2_t}{\sigma^2}) dW^2_t) \) where \( \theta^i = \frac{b^i}{\sigma^i}, i = 1, 2 \) and by the Girsanov theorem,

\[
\begin{align*}
    \tilde{W}^1_t &= W^1_t + \theta^1 t \\
    \tilde{W}^2_t &= W^2_t + \theta^2 t + \int_0^t \frac{\nu^2_s}{\sigma^2} ds
\end{align*}
\]

is a \( \tilde{P}^* \)-\( \mathbb{F} \) Brownian motion.

If \( \delta \equiv 0, \alpha = 0, \bar{\alpha} = 1 \) and \( \bar{U}(z) = \log(z) \), then from the recursive relation, we obtain

\[
Y^x_{0, \xi} = -\beta \log E_P \left[ \exp \left( -\frac{1}{\beta} \bar{U}(\xi) \right) \right],
\]

which is a typical example in the dynamic entropic risk measure. We refer to Barrieu and El Karoui [2] for more details about risk measures. The stochastic control problem (2.19) is related to the problem

\[
V^\text{rm}(x) := \sup_{\xi \in \mathcal{X}(x)} E_P \left[ -\exp \left( -\frac{1}{\beta} \bar{U}(\xi) \right) \right],
\]
The utility function $U^m(z) = -\exp\left(-\frac{1}{\theta}U(z)\right)$ is strictly concave and increasing. It satisfies the Inada condition. From Kramkov and Schachermayer [13], if the dual problem admits a solution i.e. the process $\tilde{Z}_T$ exists, the optimal terminal wealth is given by

$$\xi^* = I^m(y\tilde{Z}_T^\beta) \ a.s. \quad (5.3)$$

where $I^m(z) = ((U^m)^{-1})(z) = \left(\frac{1}{\beta}\right)^{\frac{\beta}{\beta+\sigma^2}} z - \frac{\beta}{\beta+\sigma^2}$ and

$$\tilde{Z}_T^\beta = \mathcal{E}(\theta^1\tilde{W}_T^1 - \int_0^T (\theta^2 + \frac{\nu^2}{\sigma^2})d\tilde{W}_T^2).$$

We know by classical results in duality theory, that

$$X_t^{x,H^1*} = E_{\tilde{P}}[I^m(y\tilde{Z}_T^\beta)|\mathcal{F}_t].$$

Since the market is incomplete, the variation of the process $(X_t^{x,H^1*})_t$ is independent of the Brownian motion $W^2$ which implies that $\theta^2 + \frac{\nu^2}{\sigma^2} = 0 \ dt \otimes dP$, $t \in [0,T]$. It yields that

$$X_t^{x,H^1*} = \left(\frac{1}{\beta}\right)^{\frac{\beta}{\beta+\sigma^2}} y - \frac{\beta}{\beta+\sigma^2} \exp\left(-\frac{\beta}{(1+\beta)^2} \frac{(b^1)^2}{2(\sigma^1)^2} T\right) Z_t^\beta,$$  

where $Z_t^\beta = \mathcal{E}(\frac{b^1}{1+\beta} \tilde{W}_t^1)$. Since $X_0^{x,H^1*} = x$, we have $\left(\frac{1}{\beta}\right)^{\frac{\beta}{\beta+\sigma^2}} y - \frac{\beta}{\beta+\sigma^2} x = x$. From equation (5.4) and using Itô’s formula, we have

$$dX_t^{x,H^1*} = x \exp\left(-\frac{\beta}{(1+\beta)^2} \frac{(b^1)^2}{2(\sigma^1)^2} T\right) \frac{\beta}{1 + \beta (\sigma^1)^2} \tilde{Z}_t^\beta \frac{dS_t^1}{S_t^1}.$$  

Since $dX_t^{x,H^1*} = H^1_t \frac{dS_t^1}{S_t^1}$, we have by identification that

$$H^1_t = x \exp\left(-\frac{\beta}{(1+\beta)^2} \frac{(b^1)^2}{2(\sigma^1)^2} T\right) \frac{\beta}{1 + \beta (\sigma^1)^2} Z_t^\beta, \quad a.s., \forall t \in [0,T],$$

and so the number of shares denoted by $(\tilde{\theta}^\beta)_t \in [0,T]$ invested in the risky asset $S^1$ is given by

$$\tilde{\theta}^\beta_t = x \exp\left(-\frac{\beta}{(1+\beta)^2} \frac{(b^1)^2}{2(\sigma^1)^2} T\right) \frac{\beta}{1 + \beta (\sigma^1)^2} S_t^1, \quad a.s., \forall t \in [0,T].$$

If we send $\beta$ to infinity, we obtain

$$\tilde{\theta}^\infty_t = x \frac{b^1}{(\sigma^1)^2} Z_t^{\nu^* S_t^1}, \quad a.s., \forall t \in [0,T], \quad (5.5)$$

where $Z^{\nu^*}$ is given by the expression (2.13) for the control $\nu^* = (\nu^*_1, \nu^*_2) \equiv (0, -b^2)$. Such result is coherent with the intuition since when $\beta$ goes to infinity, we force the penalty term which appears in the dynamic value process (see equation (2.1)) to be equal to zero and so our model of utility maximization under uncertainty converges to a classical utility maximization problem when the underlying model is known. The optimal strategy of investment in the first risky asset given in (5.5) corresponds to the solution of utility maximization problem in incomplete market when the utility function $U(x) = \log(x)$. Such result could be interpreted as a stability result. In the context of robust maximization problem, the coefficient $\frac{b^1}{1 + \beta (\sigma^1)^2}$ could be interpreted as a modified relative risk. Also, one could see such coefficient as a change of the level of the volatility. The volatility increases from the level $\sigma^1$ to $\frac{b^1}{1 + \beta (\sigma^1)^2} \sigma^1$ since the volatility is unobservable parameter in the market. If $\beta$ is close to 0, then the modified relative risk is small enough and the number of shares invested in the first risky asset decreases which is consistent with the intuition since we maximize the worst case.
6. Appendix

6.1. Proof of the comparison theorem

Denote by $\Delta Y_t = Y^1_t - Y^2_t$, $\Delta U_t = \bar{U}^1_t - \bar{U}^2_t$ and $\Delta \bar{U}_T = \bar{U}^1_T - \bar{U}^2_T$. The pair $(\Delta Y, \int (Z^1 - Z^2)dW)$ is the solution of the following equation

$$
d\Delta Y_t = (\delta \Delta Y_t - \alpha \Delta U_t)dt + \frac{1}{2\beta} |Z^1_t|^2dt - \frac{1}{2\beta} |Z^2_t|^2dt + (Z^1_t - Z^2_t)dW_t
$$

$$
\Delta Y_T = \bar{\alpha} \Delta \bar{U}_T.
$$

which implies for any stopping time $T \geq t$, we have

$$
\Delta Y_{\tau} - \Delta Y_t = \int_t^\tau (\delta_s \Delta Y_s - \alpha \Delta U_s)ds + \frac{1}{2\beta} \int_t^\tau |Z^1_s|^2ds
$$

$$
- \frac{1}{2\beta} \int_t^\tau |Z^2_s|^2ds + \int_t^\tau (Z^1_s - Z^2_s)dW_s.
$$

From the inequality $\int_0^t |Z^1_s|^2ds - \int_0^t |Z^2_s|^2ds - 2\int_0^t |Z^1_s|^2ds - 2\int_0^t |Z^2_s|^2ds = \int_0^t |Z^1_s - Z^2_s|^2ds \geq 0$, where $<. , >$ denotes the inner product associated with the euclidean norm. We deduce that

$$
\Delta Y_t \leq \Delta Y_{\tau} - \int_t^\tau (\delta_s \Delta Y_s - \alpha \Delta U_s)ds + \frac{1}{\beta} \int_t^\tau Z^1_s - Z^2_s \geq ds - \int_t^\tau (Z^1_s - Z^2_s)dW_s.
$$

We define the probability measure $Q^{*,2} \sim P$ where its density is the $P$-martingale $Z^{*,2}_T$ with

$$
Z^{*,2}_T = \mathcal{E}(\frac{1}{\beta} \int_0^T Z^2_s dW_s).
$$

Since $\int_0^t (Z^1_s - Z^2_s)dW_s$ is a $P$-martingale, then $\int_0^t (Z^1_s - Z^2_s)dW_s - \frac{1}{\beta} \int_0^t < Z^1_s - Z^2_s, Z^2_s > ds$ is a $Q^{*,2}$-local martingale. Let $(T_n)_n$ be a reducing sequence for

$$
\int_0^t (Z^1_s - Z^2_s)dW_s - \frac{1}{\beta} \int_0^t < Z^1_s - Z^2_s, Z^2_s > ds
$$

then, for $n$ large enough, we have $T_n \geq t$ and so

$$
\int_0^t (Z^1_s - Z^2_s)dW_s - \frac{1}{\beta} \int_0^t < Z^1_s - Z^2_s, Z^2_s > ds
$$

$$
= E_{Q^{*,2}} [\int_0^{\tau \wedge T_n} (Z^1_s - Z^2_s)dW_s - \frac{1}{\beta} \int_0^{\tau \wedge T_n} < Z^1_s - Z^2_s, Z^2_s > ds | F_t] \quad \text{on} \quad \{T \geq \tau \wedge T_n \geq t\},
$$

which implies

$$
\Delta Y_t \leq E_{Q^{*,2}} [\Delta Y_{\tau \wedge T_n} - \int_t^{\tau \wedge T_n} (\delta_s \Delta Y_s - \alpha \Delta U_s)ds | F_t] \quad \text{on} \quad \{T \geq \tau \wedge T_n \geq t\}.
$$

Sending $n$ to infinity, we have $\tau \wedge T_n \rightarrow \tau$, $Q^{*,2}$ a.s. and $\Delta Y_{\tau \wedge T_n} \rightarrow \Delta Y_{\tau}$ $Q^{*,2}$ a.s. Since $Y^1$ (resp. $Y^2$) is in $D_0^{exp}$ and $\bar{U}^1$ (resp. $\bar{U}^2$) is in $D_1^{exp}$, by the dominated convergence theorem, we have

$$
\Delta Y_t \leq E_{Q^{*,2}} [\Delta Y_{\tau} - \int_t^\tau (\delta_s \Delta Y_s - \alpha \Delta U_s)ds | F_t] \quad \text{on} \quad \{T \geq \tau \geq t\}.
$$

From the stochastic Gronwall-Bellman inequality (see Appendix C, Skiadas and Schroeder [?]), we have

$$
\Delta Y_t \leq E_{Q^{*,2}} [\int_t^\tau e^{-\int_s^\tau \delta_u du} \Delta U_s ds + \bar{\alpha} e^{-\int_t^\tau \delta_u du} \Delta \bar{U}_t | F_t].
$$

(6.1)

From inequalities (2.22)-(2.23), we have $\Delta Y_t \leq 0$, $0 \leq t \leq T$, $dt \otimes dP$ a.s. and the result follows. □
6.2. Proof of the maximum principle

We fix $\epsilon > 0$ and $\eta > 0$ such that $\epsilon < \eta$.

**First step:** We prove that

\[
\tilde{a}Z_T^* S_T^T \tilde{U}'(\xi^*) \leq \lambda^* \tilde{Z}_T \, \text{d}P \, \text{a.s.} \tag{6.2}
\]

We consider the following set

\[ A_{\epsilon, \eta} := \left\{ Z_T^* S_T^T \tilde{a}\tilde{U}'(\xi^*) - \lambda^* \tilde{Z}_T > 0, \epsilon < \xi^* < \eta \right\}. \]

We define $\xi_n$ as follows: $\xi_n = \xi^* + \frac{1}{n} \mathbf{1}_{A_{\epsilon, \eta}}$.

* We prove that $(c^*, \xi_n) \in \mathcal{A}$: From the representation theorem under $\tilde{P}^0$, there exists a process $H_n \in \mathcal{H}$ such that

\[
\frac{1}{n} \mathbf{1}_{A_{\epsilon, \eta}} = E^{\tilde{P}^0} \left[ \frac{1}{n} \mathbf{1}_{A_{\epsilon, \eta}} \right] + \int_0^T H_{n_\lambda}(\text{diag}S_s)^{-1} \text{d}S_s,
\]

which implies that

\[
\xi_n = x + E^{\tilde{P}^0} \left[ \frac{1}{n} \mathbf{1}_{A_{\epsilon, \eta}} \right] + \int_0^T H_{n_\lambda}(\text{diag}S_s)^{-1} \text{d}S_s - \int_0^T c^*_s \text{d}s,
\]

where $H_n = H^* + H_n$. For $n$ large enough, we have $0 \leq \frac{1}{n} \leq \frac{\epsilon}{2}$ and so

\[
\epsilon \leq \xi_n \leq \eta + \frac{\epsilon}{2}
\]

on the set \{ $\epsilon < \xi^* < \eta$ \}.

From Assumption (H3), we have

\[
\tilde{U}(\epsilon) \leq \tilde{U}(\xi_n) \leq \tilde{U}(\eta + \frac{\epsilon}{2})\text{ on the set } \{ \epsilon < \xi^* < \eta \}
\]

and

\[
\tilde{U}'(\eta + \frac{\epsilon}{2}) \leq \tilde{U}'(\xi_n) \leq \tilde{U}'(\epsilon)\text{ on the set } \{ \epsilon < \xi^* < \eta \}
\]

and so for $n$ large enough, $E[\exp(\gamma|\tilde{U}(\xi_n)|)]$ and $E[\exp(\gamma|\tilde{U}'(\xi_n)|)]$ are finite, which implies that $(c^*, \xi_n) \in \mathcal{A}$.

* We prove that $P(A_{\epsilon, \eta}) = 0$: From the definition of $J$ (see (4.9)) and the optimality of the strategy $(c^*, \xi^*)$, we have

\[
0 \geq n(J(x, c^*, \xi^n, \tilde{P}^*, \nu^*) - J(x, c^*, \xi^*, \tilde{P}^*, \nu^*))
\]

\[
= n(Y_{0\xi}^x - Y_{0,0x,0,0x}^x) - \nu^* E_{\tilde{P}^*}[\mathbf{1}_{A_{\epsilon, \eta}}]
\]

\[
= nE_{Q^n}[\tilde{a}S_0^T(\tilde{U}(\xi^n) - \tilde{U}(\xi^*)) - \nu^* E_{\tilde{P}^*}[\mathbf{1}_{A_{\epsilon, \eta}}]
\]

\[
= nE_{P'}[Z_{T}^* \tilde{a}S_0^T(\tilde{U}(\xi^n) - \tilde{U}(\xi^*))\mathbf{1}_{A_{\epsilon, \eta}}] - \nu^* E_{\tilde{P}^*}[\mathbf{1}_{A_{\epsilon, \eta}}],
\]

where the probability measure $Q^n$ has a density given by the $P$-martingale $Z_{T}^n = (Z_{t}^Q)_{0 \leq t \leq T} = (\mathcal{E}(\frac{1}{\tilde{P}^*}M_{t}^{\epsilon,c^*,\xi^n}))_{0 \leq t \leq T}$ and $M_{t}^{\epsilon,c^*,\xi^n} = \int_0^t Z_{s}^\epsilon c^* \xi^n \text{d}W_s$.

Since there exists $\theta^n$ between $\xi^n$ and $\xi^*$ such that $\tilde{U}(\xi^n) - \tilde{U}(\xi^*) = \tilde{U}'(\theta^n)(\xi^n - \xi^*)$, we deduce that

\[
n(\tilde{U}(\xi^n) - \tilde{U}(\xi^*))\mathbf{1}_{A_{\epsilon, \eta}} \rightarrow \tilde{U}'(\xi^*)\mathbf{1}_{A_{\epsilon, \eta}} \, \text{d}P \, \text{a.s.} \tag{6.4}
\]
and
\[
|n(U(\xi^n) - \bar{U}(\xi^*))1_{\{t < \xi^* < \eta\}}| \leq \bar{U}'(\epsilon) \, dP \text{ a.s.} \tag{6.5}
\]

From the definition of \(Z_t^{Q^n}\), we have
\[
Z_t^{Q^n} = \exp\left(-\frac{1}{\beta} M_t Z_x.c.\xi^n - \frac{1}{2\beta^2} < M_t x.c.\xi^n >_t\right). \tag{6.6}
\]

From the BSDE (2.20), we obtain
\[
Y_t^{x,c.\xi^n} - Y_0^{x,c.\xi^n} = \int_0^t (\delta_s Y_s^{x,c.\xi^n} - \alpha U(c_s^*)) ds + \frac{1}{2\beta} (M_t x.c.\xi^n)_t + M_t x.c.\xi^n. \tag{6.7}
\]

Plugging (6.7) into (6.6), we obtain
\[
Z_t^{Q^n} = \exp\left(\int_0^t \frac{1}{\beta} (\delta_s Y_s^{x,c.\xi^n} - \alpha U(c_s^*)) ds - \frac{1}{\beta} (Y_t^{x,c.\xi^n} - Y_0^{x,c.\xi^n})\right).
\]

From Proposition 2.2 (i), we have
\[
\lim_{n \to \infty} Z_t^{Q^n} = \exp\left(\int_0^t \frac{1}{\beta} (\delta_s Y_s^{x,c.\xi^n} - \alpha U(c_s^*)) ds - \frac{1}{\beta} (Y_t^{x,c.\xi^n} - Y_0^{x,c.\xi^n})\right) = Z_t^* dt \otimes dP \text{ a.s.} \tag{6.8}
\]

Under Assumption (H1) and since \((Y_t^{x,c.\xi^n})_{0 \leq t \leq T} \in D_0^{exp}\), we have
\[
|Z_t^{Q^n}| \leq \exp\left(\frac{T}{\beta} ||\delta||_{\infty} \sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^n}| + \frac{\alpha}{\beta} \int_0^T |U(c_s^*)| ds + \frac{2}{\beta} \sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^n}|\right). \tag{6.9}
\]

From Proposition 2.2 (i), we have \(Y_t^{x,c.\xi^1} \geq Y_t^{x,c.\xi^n} \geq Y_t^{x,c.\xi^*}\) and so
\[
\sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^n}| \leq \sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^1}| + \sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^*}|. \tag{6.10}
\]

Using the inequalities (6.5), (6.9) and (6.10), we have
\[
|n(U(\xi^n) - \bar{U}(\xi^*))1_{\{t < \xi^* < \eta\}}| Z_t^{Q^n} \leq \bar{U}'(\epsilon) \exp\left(\frac{T}{\beta} ||\delta||_{\infty} (\sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^1}| + \sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^*}|) \right)
\]
\[
+ \frac{\alpha}{\beta} \int_0^T |U(c_s^*)| ds + \frac{2}{\beta} (\sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^1}| + \sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^*}|) := g_T.
\]

From Cauchy Schwarz inequality, we have
\[
E_P[|g_T|] \leq E_P\left[\exp\left(\frac{2\alpha}{\beta} \int_0^T |U(c_s^*)| ds\right)^\frac{1}{2}\right] \tag{6.11}
\]
\[
E_P\left[\exp\left(\frac{2(2 + ||\delta||_{\infty} T)}{\beta} (\sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^1}| + \sup_{0 \leq t \leq T} |Y_t^{x,c.\xi^*}|)\right)^\frac{1}{2}\right].
\]

From Assumption (H1) and since \((c^*, \xi^*) \in A, (c^*, \xi^1) \in A, Y^{x,c.\xi^*} \in D_0^{exp}\) and \(Y^{x,c.\xi^1} \in D_0^{exp}\), we have \(g_T \in L^1(P)\). By the dominated convergence theorem and substituting inequalities (6.4) and (6.8) into (6.3), we have
\[
0 \geq \lim_{n \to \infty} E_{Q^n}\left[\tilde{a} S_T^n (\bar{U}(\xi^*) - \bar{U}(\xi(n)))1_{\{t < \xi^* < \eta\}}\right] - \nu^* E_{P^n}\left[1_{A_n}\right] = E_Q\left[\tilde{a} S_T^n \bar{U}(\xi^*)1_{A_n}\right] - \nu^* E_{P^n}\left[1_{A_n}\right].
\]
which implies \(P(A_{c,\eta}) = 0\) for all \(0 < \epsilon < \eta < \infty\). Sending \(\epsilon \rightarrow 0\) and \(\eta \rightarrow \infty\), we have

\[
A_{c,\eta} \cap \left\{ Z_T^* S_T^c \bar{\alpha} U'(\xi^*) - \lambda^* \bar{Z}_T > 0 \right\}
\]

and so inequality (6.2) is proved.

**Second step:** We prove that

\[
\bar{\alpha} Z_T^* S_T^c \bar{U}'(\xi^*) \geq \lambda^* \bar{Z}_T^* \, dP \; \text{a.s.}
\] (6.12)

We consider the following set

\[
B_{c,\eta} := \left\{ Z_T^* \bar{\alpha} S_T^c \bar{U}'(\xi^*) - \lambda^* \bar{Z}_T < 0, \epsilon < \xi^* < \eta \right\}.
\]

We define \(\xi_n^*\) as follows:

\[
\xi^n_\epsilon := \xi^* - \frac{1}{n} 1_{B_{c,\eta}}.
\]

*We prove that \((c^*, \xi^*_n) \in A\): As in the first step, for \(n\) large enough, we have \(0 < \frac{1}{n} < \frac{\epsilon}{2}\) and so

\[
\frac{\epsilon}{2} \leq \xi^*_n \leq \eta \quad \text{on the set} \{\epsilon < \xi^* < \eta\}.
\]

From Assumption (H3), we have

\[
\bar{U}(\frac{\xi}{2}) \leq \bar{U}(\xi_n^*) \leq \bar{U}(\eta) \quad \text{on the set} \{\epsilon < \xi^* < \eta\}
\]

and

\[
\bar{U}'(\eta) \leq \bar{U}'(\xi_n^*) \leq \bar{U}'(\frac{\xi}{2}) \quad \text{on the set} \{\epsilon < \xi^* < \eta\}.
\]

This shows that for \(n\) large enough, \(E[\exp(\gamma|\bar{U}(\xi_n^*)|)]\) and \(E[\exp(\gamma|\bar{U}'(\xi_n^*)|)]\) are finite and so \((c^*, \xi^*_n) \in A\).

*We prove that \(P(B_{c,\eta}) = 0\): From the definition of \(J\) (see (4.9)) and the optimality of the strategy \((c^*, \xi^*)\), we have

\[
0 \geq n(J(x, c^*, \xi^n, \bar{P}^*, \nu^*) - J(x, c^*, \xi^*, \bar{P}^*, \nu^*))
\]

\[
= n(Y_0^x, c^*, \xi^n - Y_0^x, c^*, \xi^*) + \nu^* E_{\bar{P}^*}[1_{B_{c,\eta}}]
\]

\[
\geq n E_{Q^n}[\bar{\alpha} S_T^c (\bar{U}(\xi^n) - \bar{U}(\xi^*))] + \nu^* E_{\bar{P}^*}[1_{B_{c,\eta}}]
\]

\[
= n E_{P}[Z_T^c \bar{\alpha} S_T^c (\bar{U}(\xi^n) - \bar{U}(\xi^*))1_{B_{c,\eta}}] + \nu^* E_{\bar{P}^*}[1_{B_{c,\eta}}],
\]

where the probability measure \(Q^n\) has a density given by the \(P\)-martingale \(Z_t^{Q^n} = (Z_t^{Q^n})_{0 \leq t \leq T} = (E(-\frac{1}{\beta} M_t^{x, c^*, \xi^n}))_{0 \leq t \leq T} \) and \(M_t^{x, c^*, \xi^n} = \int_0^t Z_s^{x, c^*, \xi^n} dW_s\).

Since there exists \(\theta^*\) between \(\xi^n\) and \(\xi^*\) such that \(\bar{U}(\xi^n) - \bar{U}(\xi^*) = \bar{U}'(\theta^*)(\xi^n - \xi^*)\), we deduce that

\[
n(\bar{U}(\xi^n) - \bar{U}(\xi^*))1_{B_{c,\eta}} \longrightarrow -\bar{U}'(\xi^*)1_{B_{c,\eta}} \, dP \; \text{a.s.}
\] (6.14)

and

\[
|n(\bar{U}(\xi^n) - \bar{U}(\xi^*))1_{\{\epsilon < \xi^* < \eta\}}| \leq \bar{U}'(\epsilon) \, dP \; \text{a.s.}
\] (6.15)

From the definition of \(Z_t^{Q^n}\), we have

\[
Z_t^{Q^n} = \exp(-\frac{1}{\beta} M_t^{x, c^*, \xi^n} - \frac{1}{2 \beta^2} < M_t^{x, c^*, \xi^n} > t).
\] (6.16)

From the BSDE (2.20), we obtain

\[
Y_t^{x, c^*, \xi^n} - Y_0^{x, c^*, \xi^n} = \int_0^t (\delta X_s^{x, c^*, \xi^n} - \alpha U(c_s^*)) ds + \frac{1}{2 \beta^2} (M_t^{x, c^*, \xi^n})_t + M_t^{x, c^*, \xi^n}.
\] (6.17)
Plugging (6.17) into (6.16), we obtain

$$Z_t^{Q^n} = \exp \left( \int_0^t \frac{1}{\beta} (\delta_s Y_s x^c, \xi^n - \alpha U(c_s) + \frac{1}{\beta} (Y_t x^c, \xi^n - Y_0 x^c, \xi^n)) \right).$$

From Proposition 2.2 (ii), we have

$$\lim_{n \to \infty} Z_t^{Q^n} = \exp \left( \int_0^t \frac{1}{\beta} (\delta_s Y_s x^c, \xi^n - \alpha U(c_s))ds - \frac{1}{\beta} (Y_t x^c, \xi^n - Y_0 x^c, \xi^n) \right) = Z_t^* dt \otimes dP \ a.s. \quad (6.18)$$

Under Assumption (H1) and since $(Y_t x^c, \xi^n)_{0 \leq t \leq T} \in D_0^{exp}$, we have

$$|Z_t^{Q^n}| \leq \exp \left( \frac{T}{\beta} ||\delta|| \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| + \frac{\alpha}{\beta} \int_0^T |U(c_s)|ds + \frac{2}{\beta} \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| \right). \quad (6.19)$$

From Proposition 2.2 (ii), we have $Y_t x^c, \xi^n \leq Y_t x^c, \xi^*$ and so

$$\sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| \leq \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| + \sup_{0 \leq t \leq T} |Y_t x^c, \xi^*|. \quad (6.20)$$

Using the inequalities (6.15), (6.19) and (6.20), we have

$$|n(\bar{U}(\xi^n) - \bar{U}(\xi^*))1_{(\epsilon < \xi^* < \eta)}| |Z_t^{Q^n}| \leq \bar{U}'(\epsilon) \exp \left( \frac{T}{\beta} ||\delta|| \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| + \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| \right) + \frac{\alpha}{\beta} \int_0^T |U(c_s)|ds + \frac{2}{\beta} \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| + \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| \right) := \tilde{g}_T.$$

From Cauchy Schwarz inequality, we have

$$E_P[|\tilde{g}_T|] \leq E_P \left[ \left( \frac{2(2 + ||\delta|| T)}{\beta} \left( \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| + \sup_{0 \leq t \leq T} |Y_t x^c, \xi^n| \right) \right) \right]. \quad (6.21)$$

From Assumption (H1) and since $(x^c, \xi^*) \in \mathcal{A}$, $(x^c, \xi^*) \in \mathcal{A}$, $Y_t x^c, \xi^n \in D_0^{exp}$ and $Y_t x^c, \xi^* \in D_0^{exp}$, we have $\tilde{g}_T \in L^1(P)$. By the dominated convergence theorem and substituting inequalities (6.14) and (6.18) into (6.13), we have

$$0 \geq \lim_{n \to \infty} E_P \left[ \left( aS_t^n n(\bar{U}(\xi^n) - \bar{U}(\xi^*)) 1_{(\epsilon < \xi^* < \eta)} + \nu^* E_P \left[ 1_{B_{\epsilon}, \eta} \right] \right) \right] + \nu^* E_P \left[ 1_{B_{\epsilon}, \eta} \right].$$

which implies $P(B_{\epsilon}, \eta) = 0$ for all $0 < \epsilon < \eta < \infty$. Sending $\epsilon \to 0$ and $\eta \to \infty$, we obtain

$$Z_T^{Q^n} \exp\left( - \int_0^T \delta_a du \right) \bar{U}'(\xi^*) \geq \lambda^* \tilde{Z}_T^* \quad \text{on the set } \{ \xi^* > 0 \} \ a.s. \quad (6.22)$$

Since the utility function satisfies the Inada conditions (Assumption (H3)), we have $P(\xi^*) = 0 = 0$ and so $P(B_{\epsilon}, \eta) = 0$ for all $0 < \epsilon < \eta < \infty$.

* We prove inequality (6.12): Sending $\epsilon \to 0$ and $\eta \to \infty$ we have $B_{\epsilon}, \eta \not\supset \left\{ Z_T^{Q^n} \delta_t n(\bar{U}'(\xi^*) - \lambda^* \tilde{Z}_T^* < 0 \right\}$ and so inequality (6.12) is proved.

The result follows from (6.2) and (6.12). The same argument holds for the consumption process. \qed
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