Generalized Taub-NUT metrics and
Killing-Yano tensors

Mihai Visinescu *

Department of Theoretical Physics,
National Institute for Physics and Nuclear Engineering,
P.O.Box M.G.-6, Magurele, Bucharest, Romania

Abstract

A necessary condition that a Stäckel-Killing tensor of valence 2 be the contracted product of a Killing-Yano tensor of valence 2 with itself is re-derived for a Riemannian manifold. This condition is applied to the generalized Euclidean Taub-NUT metrics which admit a Kepler type symmetry. It is shown that in general the Stäckel-Killing tensors involved in the Runge-Lenz vector cannot be expressed as a product of Killing-Yano tensors. The only exception is the original Taub-NUT metric.

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1 Introduction

The Euclidean Taub-NUT metric is involved in many modern studies in physics. Hawking [1] has suggested that the Euclidean Taub-NUT metric might give rise to the gravitational analog of the Yang-Mills instanton. In this case Einstein’s equations are satisfied with zero cosmological constant and the manifold is $\mathbb{R}^4$ with a boundary which is a twisted three-sphere $S^3$ possessing a distorted metric. The Kaluza-Klein monopole was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein

*E-mail: mvisin@theor1.theory.nipne.ro
theory. On the other hand, in the long-distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space \[2, 3\].

From the symmetry viewpoint, the geodesic motion in Taub-NUT space admits a “hidden” symmetry of the Kepler type if a cyclic variable is gotten rid of \[4, 5, 6, 7\]. In general the “hidden” symmetries of the manifold manifest themselves as Stäckel-Killing tensors of valence \(r > 1\) \[8\]. The conserved quantities along geodesics are homogeneous functions in momentum \(p_\mu\) of degree \(r\), and which commute with the Hamiltonian

\[
H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu
\]

in the sense of Poisson brackets.

In the Taub-NUT geometry there are four Killing-Yano tensors \[9\]. Three of these are complex structure realizing the quaternionic algebra and the Taub-NUT manifold is hyper-Kähler \[5\]. In addition to these three vector-like Killing-Yano tensors, there is a scalar one which has a non-vanishing field strength and it exists by virtue of the metric being type \(D\).

For the geodesic motions in the Taub-NUT space, the conserved vector analogous to the Runge-Lenz vector of the Kepler type problem is quadratic in 4-velocities, its components are Stäckel-Killing tensors and they can be expressed as symmetrized products of Killing-Yano tensors \[5, 10, 11, 12\].

The Killing-Yano tensors play an important role in the models for relativistic spin one half particles involving anti-commuting vectorial degrees of freedom, usually called the spinning particles \[13, 14, 15, 16\]. The configuration space of spinning particles (spinning space) is an extension of an ordinary Riemannian manifold, parametrized by local coordinates \(\{x^\mu\}\), to a graded manifold parametrized by local coordinates \(\{x^\mu, \psi^\mu\}\), with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anti-commuting). In the spinning case the generalized Killing equations are more involved and new procedures have been conceived \[15, 12\]. In particular, if the Killing tensors can be written in terms of Killing-Yano tensors (and that is the case of the Taub-NUT space), the generalized Killing equations can be solved explicitly in a simple, closed form.

In the last time, Iwai and Katayama \[17, 18, 19, 20\] extended the Taub-NUT metric so that it still admits a Kepler-type symmetry. This class of metrics, of course, includes the original Taub-NUT metric.
The aim of this paper is to investigate if the Stäckel-Killing tensors involved in the conserved Runge-Lenz vector of the extended Taub-NUT metrics can also be expressed in terms of Killing-Yano tensors.

The relationship between Killing tensors and Killing-Yano tensors has been investigated to the purpose of the Lorentzian geometry used in general relativity [21, 22]. In the next section we re-examine the conditions that a Killing tensor of valence 2 be the contracted product of a Killing-Yano tensor of valence 2 with itself. The procedure is quite simple and devoted to the Riemannian geometry appropriate to Euclidean Taub-NUT metrics.

In Section 3 we show that in general the Killing tensors involved in the Runge-Lenz vector cannot be expressed as a product of Killing-Yano tensors. The only exception is the original Taub-NUT metric.

Our comments and concluding remarks are presented in Section 4.

2 The relationship between Killing tensors and Killing-Yano tensors

We consider a 4–dimensional Riemannian manifold $M$ and a metric $g_{\mu\nu}(x)$ on $M$ in local coordinates $x^\mu$. We write the metric in terms of the local orthonormal vierbein frame $e^a_\mu$

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = \sum_{a=0,1,2,3} (e^a)^2$$

where $e^a = e^a_\mu dx^\mu$. Greek indices $\mu, \nu, ...$ are raised and lowered with $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$, while Latin indices $a, b, ...$ are raised and lowered by the flat metric $\delta_{ab}, a, b = 0, 1, 2, 3$. Vierbeins and inverse vierbeins inter-convert Latin and Greek indices when necessary.

The following two generalization of the Killing vector equation have become of interest in physics [22]:

(a) A tensor $f_{\mu_1...\mu_r}$ is called a Killing-Yano tensor of valence $r$ if it is totally anti-symmetric and it satisfies the equation

$$f_{\mu_1...(\mu_r;\lambda)} = 0.$$  \hspace{1cm} (3)

(b) A symmetric tensor field $K_{\mu_1...\mu_r}$ is called a Stäckel-Killing tensor of valence $r$ iff

$$K_{(\mu_1...\mu_r;\lambda)} = 0.$$  \hspace{1cm} (4)
Let $\Lambda^2$ be the space of two-forms $\Lambda^2 := \Lambda^2 T^*(\mathbb{R}^4 - \{0\})$. We define self-dual and anti-self dual bases for $\Lambda^2$ using the vierbein one-forms $e^a$:

\[
\text{basis of } \Lambda^2_\pm = \begin{cases} 
\lambda_1^\pm = e^0 \wedge e^1 \pm e^2 \wedge e^3 \\
\lambda_2^\pm = e^0 \wedge e^2 \pm e^3 \wedge e^1 \\
\lambda_3^\pm = e^0 \wedge e^3 \pm e^1 \wedge e^2
\end{cases}
\]  

\[\ast \lambda_i^\pm = \pm \ast \lambda_i^\pm \]  

(5)

Let $f$ be a Killing-Yano tensor of valence 2 and $\ast f$ its dual. The symmetric combination of $f$ and $\ast f$ is a self-dual two-form

\[f + \ast f = \sum_{i=1,2,3} y_i \lambda_i^+ \]  

(6)

while their difference is an anti-self-dual two-form

\[f - \ast f = \sum_{i=1,2,3} z_i \lambda_i^- \]  

(7)

An explicit evaluation shows that

\[(f + \ast f)^2 = - \sum_{i=1,2,3} (y_i)^2 \cdot \mathbb{1}, \]  

(8)

\[(f - \ast f)^2 = - \sum_{i=1,2,3} (z_i)^2 \cdot \mathbb{1} \]  

(9)

where $\mathbb{1}$ is 4x4 identity matrix.

Let us suppose that a Stäckel-Killing tensor $K_{\mu\nu}$ can be written as the contracted product of a Killing-Yano tensor $f_{\mu\nu}$ with itself:

\[K_{\mu\nu} = f_{\mu\lambda} \cdot f_{\nu}^\lambda = (f^2)_{\mu\nu} , \quad \mu, \nu = 0, 1, 2, 3. \]  

(10)

We infer from the last equations that:

\[K + \frac{1}{16} \left[ \sum_i (y_i^2 - z_i^2) \right]^2 K^{-1} + \frac{1}{2} \sum_i (y_i^2 + z_i^2) \cdot \mathbb{1} = 0. \]  

(11)

On the other hand the Killing tensor $K$ is symmetric and it can be diagonalized with the aid of an orthogonal matrix. Its eigenvalues satisfy an equation of the second degree:

\[\lambda_\alpha^2 + \frac{1}{2} \sum_i (y_i^2 + z_i^2) \lambda_\alpha + \frac{1}{16} \left[ \sum_i (y_i^2 - z_i^2) \right]^2 = 0 \]  

(12)

with at most two distinct roots.

In conclusion a Stäckel-Killing tensor $K$ which can be written as the square of a Killing-Yano tensor has at the most two distinct eigenvalues.
3 Generalized Taub-NUT metrics

For a special choice of coordinates the generalized Euclidean Taub-NUT metric considered by Iwai and Katayama [17, 18, 19, 20] takes the form:

$$ds_G^2 = f(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2] + g(r)[d\chi + \cos \theta d\varphi]^2$$  \hspace{1cm} (13)

where $r > 0$ is the radial coordinate of $\mathbb{R}^4 - \{0\}$, the angle variables $(\theta, \varphi, \chi), (0 \leq \theta < \pi, 0 \leq \varphi < 2\pi, 0 \leq \chi < 4\pi)$ parameterize the unit sphere $S^3$, and $f(r)$ and $g(r)$ are arbitrary functions of $r$.

We decompose the metric (13) into the orthogonal vierbein basis:

$$e^0 = g(r)^{\frac{1}{2}}(d\chi + \cos \theta d\varphi),$$
$$e^1 = r f(r)^{\frac{1}{2}}(\sin \chi d\theta - \sin \theta \cos \chi d\varphi),$$
$$e^2 = r f(r)^{\frac{1}{2}}(-\cos \chi d\theta - \sin \theta \sin \chi d\varphi),$$
$$e^3 = f(r)^{\frac{1}{2}}dr.$$  \hspace{1cm} (14)

Spaces with a metric of the form above have an isometry group $SU(2) \times U(1)$. The four Killing vectors are

$$D_A = R_A^\mu \partial_\mu, \quad A = 0, 1, 2, 3,$$  \hspace{1cm} (15)

where

$$D_0 = \frac{\partial}{\partial \chi},$$
$$D_1 = -\sin \varphi \frac{\partial}{\partial \theta} - \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \chi},$$
$$D_2 = \cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \chi},$$
$$D_3 = \frac{\partial}{\partial \varphi}.$$  \hspace{1cm} (16)

$D_0$ which generates the $U(1)$ of $\chi$ translations, commutes with the other Killing vectors. In turn the remaining three vectors, corresponding to the invariance of the metric (13) under spatial rotations ($A = 1, 2, 3$), obey an $SU(2)$ algebra with

$$[D_1, D_2] = -D_3, \text{ etc.} ....$$  \hspace{1cm} (17)
Let us consider geodesic flows of the generalized Taub-NUT metric which has the Lagrangian \( L \) on the tangent bundle \( T(\mathbb{R}^4 - \{0\}) \)

\[
L = \frac{1}{2} f(r)(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)) + \frac{1}{2} g(r)(\dot{\chi} + \cos \theta \dot{\varphi})^2
\]  

(18)

where \((\dot{r}, \dot{\theta}, \dot{\varphi}, \dot{\chi}, r, \theta, \varphi, \chi)\) stand for coordinates in the tangent bundle. Since \(\chi\) is a cyclic variable

\[
q = g(r)(\dot{\theta} + \cos \theta \dot{\varphi})
\]

(19)

is a conserved quantity. This is known in the literature as the “relative electric charge”.

Taking into account this cyclic variable, the dynamical system for the geodesic flow on \( T(\mathbb{R}^4 - \{0\}) \) can be reduced to a system on \( T(\mathbb{R}^3 - \{0\}) \). The reduced system admits manifest rotational invariance, and hence has a conserved angular momentum:

\[
\vec{J} = \vec{r} \times \vec{p} + q \frac{\vec{r}}{r}
\]

(20)

where \(\vec{r}\) denotes the three-vector \(\vec{r} = (r, \theta, \varphi)\) and \(\vec{p} = f(r)\dot{\vec{r}}\) is the mechanical momentum.

If \(f(r)\) and \(g(r)\) are taken to be

\[
f(r) = \frac{4m + r}{r}, \quad g(r) = \frac{16m^2 r}{4m + r}
\]

(21)

the metric \(ds_G^2\) becomes the original Euclidean Taub-NUT metric. As observed in [5], the Taub-NUT geometry also possesses four Killing-Yano tensors of valence 2. The first three are rather special: they are covariantly constant (with vanishing field strength)

\[
f_i = 8m(dx_i + \cos \theta d\varphi) \wedge dx - \epsilon_{ijk}(1 + \frac{4m}{r})dx_j \wedge dx_k,
\]

\[
D_\mu f_i^\nu = 0, \quad i, j, k = 1, 2, 3.
\]

(22)

They are mutually anti-commuting and square the minus unity:

\[
f_if_j + f_jf_i = -2\delta_{ij}.
\]

(23)

Thus they are complex structures realizing the quaternion algebra. Indeed, the Taub-NUT manifold defined by (13) and (21) is hyper-Kähler.
In addition to the above vector-like Killing-Yano tensors there also is a scalar one

\[ f_Y = 8m(d\chi + \cos \theta d\phi) \land dr + 4r(r + 2m)(1 + \frac{r}{4m}) \sin \theta d\theta \land d\phi \]  

(24)

which has a non-vanishing component of the field strength

\[ f_{Y\theta\phi} = 2(1 + \frac{r}{4m})r \sin \theta. \]  

(25)

In the original Taub-NUT case there is a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problem:

\[ \vec{K} = \frac{1}{2} \vec{K}_{\mu\nu} \hat{x}^\mu \hat{x}^\nu = \vec{p} \times \vec{j} + \left( \frac{q^2}{4m} - 4mE \right) \frac{\vec{r}}{r} \]  

(26)

where the conserved energy \( E \), from eq. (1), is

\[ E = \frac{\vec{p}^2}{2 f(r)} + \frac{q^2}{2 g(r)}. \]  

(27)

The components \( K_{i\mu\nu} \) involved with the Runge-Lenz type vector (26) are Killing tensors and they can be expressed as symmetrized products of the Killing-Yano tensors \( f_i \) (22) and \( f_Y \) (24) [11, 12]:

\[ K_{i\mu\nu} - \frac{1}{8m}(R_{0\mu}R_{i\nu} + R_{0\nu}R_{i\mu}) = m \left( f_{Y\mu\lambda} f_{i\nu}^{\lambda} + f_{Y\nu\lambda} f_{i\mu}^{\lambda} \right). \]  

(28)

Returning to the generalized Taub-NUT metric, on the analogy of eq. (26), Iwai and Katayama [17, 18, 19, 20] assumed that in addition to the angular momentum vector there exist a conserved vector \( \vec{S} \) of the following form:

\[ \vec{S} = \vec{p} \times \vec{j} + \kappa \frac{\vec{r}}{r} \]  

(29)

with an unknown constant \( \kappa \).

It was found that the metric [13] still admits a Kepler type symmetry (29) if the functions \( f(r) \) and \( g(r) \) take, respectively, the form

\[ f(r) = \frac{a + br}{r}, \quad g(r) = \frac{ar + bn^2}{1 + cr + dr^2} \]  

(30)
where \(a, b, c, d\) are constants. The constant \(\kappa\) involved in the Runge-Lenz vector (29) is

\[
\kappa = -aE + \frac{1}{2} c q^2.
\]  

(31)

If \(ab > 0\) and \(c^2 - 4d < 0\) or \(c > 0, d > 0\), no singularity of the metric appears in \(\mathbb{R}^4 - \{0\}\). On the other hand, if \(ab < 0\) a manifest singularity appears at \(r = -a/b\) [13].

It is straightforward to verify that the components of the vector \(\vec{S}\) are Stäckel-Killing tensors in the extended Taub-NUT space (13) with the function \(f(r)\) and \(g(r)\) given by (30). Moreover the Poisson brackets between the components of \(\vec{J}\) and \(\vec{S}\) are [17]:

\[
\{J_i, J_j\} = \epsilon_{ijk} J_k,
\]

\[
\{J_i, S_j\} = \epsilon_{ijk} S_k,
\]

\[
\{S_i, S_j\} = (dq^2 - 2bE)\epsilon_{ijk} J_k
\]

(32)
as it is expected from the same relations known for the original Taub-NUT metric.

Our task is to investigate if the components of the Runge-Lenz vector \(\vec{S}\) can be the contracted product of Killing-Yano tensors of valence 2. On the model of eq.(28) from the original Taub-NUT case it is not required that a component \(S_i\) of the Runge-Lenz vector (29) to be directly expressed as a symmetrized product of Killing-Yano tensors. Taking into account that \(\vec{S}\) transforms as a vector under rotations generated by \(\vec{J}\), eq.(32), the components \(S_{i\mu\nu}\) can be combined with trivial Stäckel-Killing tensors of the form \((R_{0\mu} R_{i\nu} + R_{0\nu} R_{i\mu})\) to get the appropriate tensor which has to be decomposed in a product of Killing-Yano tensors.

In order to use the results from the previous section, we shall write the symmetrized product of two different Killing-Yano tensors \(f'\) and \(f''\) as a contracted product of \(f' + f''\) with itself, extracting adequately the contribution of \(f'^2\) and \(f''^2\). Since the generalized Taub-NUT space (13) does not admit any other non-trivial Stäckel-Killing tensor besides the metric \(g_{\mu\nu}\) and the components \(S_{i\mu\nu}\) of (29), \(f'^2\) and \(f''^2\) should be connected with the scalar conserved quantities \(E, J, q^2\) through the tensors \(g_{\mu\nu}, \sum_{A=1,2,3} R_{A\mu} R_{A\nu}\) and \(R_{0\mu} R_{0\nu}\).

In conclusion we shall consider a general linear combination between a component \(S_i\) of the Runge-Lenz vector (29) and symmetrized pairs of Killing
vectors of the form

\[ S_{iab} + \alpha_1 \sum_{A=1}^{3} R_{Aa} R_{Ab} + \alpha_2 R_{0a} R_{0b} + \alpha_3 (R_{0a} R_{i0} + R_{ia} R_{0b}) \]  \quad (33)

where \( \alpha_i \) are constants. We are looking for the conditions the above tensor be the contracted product of a Killing-Yano tensor with itself. For this purpose we evaluate the eigenvalues of the matrix (33) and we get that it has at the most two distinct eigenvalues if and only if

\[
\begin{align*}
\alpha_1 + \alpha_2 &= 0, \\
\alpha_3 &= -\frac{c}{4}, \\
d &= \frac{c^2}{4}.
\end{align*}
\]  \quad (34)

For example, if the above conditions are satisfied, the eigenvalues of the matrix (33) for the third component \( S_3 \) of the Runge-Lenz vector (29) are

\[
\lambda_1 = \frac{1}{2} \left( br \cos \theta + (a + br) \left( r\alpha_1 + \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta} \right) \right)
\]  \quad (35)

with the eigenvectors

\[
\begin{align*}
\tan \chi, \left( r\alpha_1 + \cos \theta + \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta} \right) \csc \theta \sec \chi, 0, 1 \\
\left( (r\alpha_1 + \cos \theta - \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta}) \csc \theta \sec \chi, -\tan \chi, 1, 0 \right)
\end{align*}
\]  \quad (36)

and

\[
\lambda_2 = \frac{1}{2} \left( br \cos \theta + (a + br) \left( r\alpha_1 - \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta} \right) \right)
\]  \quad (37)

with the eigenvectors

\[
\begin{align*}
\tan \chi, \left( r\alpha_1 + \cos \theta - \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta} \right) \csc \theta \sec \chi, 0, 1 \\
\left( (r\alpha_1 + \cos \theta + \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta}) \csc \theta \sec \chi, -\tan \chi, 1, 0 \right)
\end{align*}
\]  \quad (38)
Hence the constants involved in the functions \( f, g \) are constrained, restricting accordingly their expressions. It is worth to mention that if relation (34) between the constants \( c \) and \( d \) is satisfied, the metric is conformally self-dual or anti-self-dual depending upon the sign of the quantity \( 2 + cr \). More precisely, for the Weyl curvature tensor

\[
C_{ijkl} = R_{ijkl} - \frac{1}{2}(\delta_k R_{jl} - \delta_l R_{jk} + \delta_i R_{kl} - \delta_l R_{ik}) + \frac{1}{6} R(\delta_k \delta_l - \delta_i \delta_j)
\]  

(39)

one can define a two-form

\[
W_{ij} = \frac{1}{2} \sum_{k,l} C_{ijkl} e^k \wedge e^l.
\]  

(40)

With respect to basis (3) the representation matrix \( W \) of (40) takes the block diagonal form

\[
W = \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix}
\]  

(41)

where \( W^+ \) and \( W^- \) are \( 3 \times 3 \) matrices representing the induced linear transformation of the invariant subspaces \( \Lambda^2_+ \) and \( \Lambda^2_- \) respectively. If the constants \( c \) and \( d \) satisfy (34), the extended Taub-NUT metric (30) with \( 2 + cr > 0 \) is conformally self-dual and one has

\[
W^+ = \frac{c}{2(a + br)(1 + cr/2)^2} W_0, \quad W^- = 0
\]  

(42)

where \( W_0 \) is a diagonal matrix

\[
W_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 0 \end{pmatrix}
\]  

(43)

For \( 2 + cr < 0 \), the metric is conformally anti-self-dual and the expressions of \( W^+ \) and \( W^- \) are interchanged.

Finally the condition stated for a Stäckel-Killing tensor to be written as the square of a skew symmetric tensor in the form (10) must be supplemented with eq.(3) which defines a Killing-Yano tensor. To verify this last condition we shall use the Newman-Penrose formalism for Euclidean signature [24]. We introduce a tetrad which will be given as an isotropic complex dyad defined
by the vectors $l, m$ together with their complex conjugates subject to the normalization conditions

$$l_\mu \bar{l}^\mu = 1, \quad m_\mu \bar{m}^\mu = 1 \quad (44)$$

with all others vanishing and the metric is expressed in the form

$$ds^2 = l \otimes \bar{l} + \bar{l} \otimes l + m \otimes \bar{m} + \bar{m} \otimes m. \quad (45)$$

For a Stäckel-Killing tensor $K$ with two distinct eigenvalues one can choose the tetrad in such that

$$K_{\mu\nu} = 2\lambda_1^2 l_\mu \bar{l}_\nu + 2\lambda_2^2 m_\mu \bar{m}_\nu. \quad (46)$$

The skew symmetric tensor $f_{\mu\nu}$ which enter decomposition (10) has the form

$$f_{\mu\nu} = 2\lambda_1 l_\mu \bar{l}_\nu + 2\lambda_2 m_\mu \bar{m}_\nu. \quad (47)$$

Taking again the example of the third component $S_3$, the eigenvalues $\lambda_1$ and $\lambda_2$ are given by (35) and (37) and the tetrad (14) can be inferred from the eigenvectors (36) and (38) through a standard orthonormalization procedure. Finally, imposing eq.(3), we get that (47) is a Killing-Yano tensor only if

$$c = \frac{2b}{a}. \quad (48)$$

With this constraint, together with (34), the extended metric (13) coincides, up to a constant factor, with the original Taub-NUT metric on setting $a/b = 4m$. Note that in eqs.(35)-(38) the constant $\alpha_1$ is not fixed. In fact, the product of two Killing-Yano tensors $f', f''$ is invariant under the rescaling $f' \to \alpha f', f'' \to \frac{1}{\alpha} f''$. Choosing adequately the normalization of the Killing-Yano tensors, for $\alpha_1 = -\frac{1}{4m}$ we recover precisely the original Taub-NUT decomposition (28) with $f' = f_i$ and $f'' = f_Y$ normalized as in (22) and (24).

### 4 Concluding remarks

The aim of this paper is to show that the extensions of the Taub-NUT geometry do not admit a Killing-Yano tensor, even if they possess Stäckel-Killing tensors.
This result is not unexpected. The conserved quantities $K_{i\mu\nu}$ which enter eq. (28) are the components of the Runge-Lenz vector $\vec{K}$ given in (20). In the original Taub-NUT case these components $K_{i\mu\nu}$ are related to the symmetrized products between the Killing-Yano tensors $f_i$ (22) and $f_Y$ (24). Adequately the three Killing-Yano tensors $f_i$ transform as vectors under rotations generated by $\vec{J}$ like the Runge-Lenz vector (32), while $f_Y$ is a scalar.

The extended Taub-NUT metrics are not Ricci flat and, consequently, not hyper-Kähler. On the other hand the existence of the Killing-Yano tensors $f_i$ is correlated with the hyper-Kähler, self-dual structure of the metric.

The in-existence of the Killing-Yano tensors makes more laborious the study of ”hidden” symmetries in models of relativistic particles with spin involving anti-commuting vectorial degrees of freedom. In general the conserved quantities from the scalar case receive a spin contribution involving an even number of Grassmann variables $\psi^\mu$. For example, starting with a Killing vector $K_\mu$, the conserved quantity in the spinning case is

$$J(x, \dot{x}, \psi) = K^\mu \dot{x}_\mu + \frac{i}{2} K_{[\mu\nu]} \psi^\mu \psi^\nu. \tag{49}$$

The first term in the r.h.s. is the conserved quantity in the scalar case, while the last term represents the contribution of the spin.

A “hidden” symmetry is encapsulated in a Stäckel-Killing tensor of valence $r > 1$. The generalized Killing equations on spinning spaces including a Stäckel-Killing tensor are more involved. Unfortunately it is not possible to write closed, analytic expressions of the solutions of these equations using directly the components of the Stäckel-Killing tensors. However, assuming that the Stäckel-Killing tensors can be written as symmetrized products of pairs of Killing-Yano tensors, the evaluation of the spin corrections is feasible [15, 11, 12, 16].

If the Killing-Yano tensors are missing, to take up the question of the existence of extra supersymmetries and the relation with the constants of motion we are forced to enlarge the approach to Killing equations (3), (4). In fact, in ref. [13], supersymmetries are shown to depend on the existence of a tensor field $f_{\mu\nu}$ satisfying eq. (3) which will be referred to as the $f$-symbol. The general conditions for constants of motion were derived, and it was shown that one can have new supercharges which do not commute with the original supercharge $Q = \dot{x}_\mu \psi^\mu$ if one allows the $f$-symbols to have a symmetric part. It was shown that in this case the anti-symmetric part does
not satisfy the Killing-Yano condition \( \mathbb{3} \). We should like to remark that the general conditions of ref.\( \mathbb{14} \) allow more possibilities than Killing-Yano tensors for the construction of supercharges.

Summing up, we believe that the relation between the \( f \)-symbols and the Killing-Yano tensors could be fruitful and that it should deserve further studies. An analysis of the \( f \)-symbols in the generalized Taub-NUT geometry is under way \( \mathbb{25} \).

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