Idempotent functional analysis:  
An algebraic approach  

G. L. Litvinov, V. P. Maslov, and G. B. Shpiz  

Received May 17, 2000  

Abstract—This paper is devoted to Idempotent Functional Analysis, which is an “abstract” version of Idempotent Analysis developed by V. P. Maslov and his collaborators. We give a brief survey of the basic ideas of Idempotent Analysis. The correspondence between concepts and theorems of the traditional Functional Analysis and its idempotent version is discussed in the spirit of N. Bohr’s correspondence principle in Quantum Theory. We present an algebraic approach to Idempotent Functional Analysis. Basic notions and results are formulated in algebraic terms; the essential point is that the operation of idempotent addition can be defined for arbitrary infinite sets of summands. We study idempotent analogs of the basic principles of linear functional analysis and results on the general form of a linear functional and scalar products in idempotent spaces.  

Published in Math. Notes, vol. 69, no. 5 (2001), p. 696–729.  

Key words: Idempotent Analysis, Functional Analysis, linear functional, scalar product.  

And above all I value Analogies,  
my most reliable teachers.  

J. Kepler  

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1. INTRODUCTION

1.1. Heuristics

Idempotent Functional Analysis is an abstract version of Idempotent Analysis in the sense of [1–8]. Idempotent Analysis is closely related to optimal control theory, optimization theory, convex analysis, the theory of vector lattices and ordered algebraic systems. Ideas, results, and terminology accumulated in these fields are taken into account where possible, but Idempotent Functional Analysis is considered here as a part of Idempotent Mathematics (see below) and presented in the framework of a different paradigm.

This paradigm is expressed by the correspondence principle [9, 10]. This principle is similar to the well-known correspondence principle of N. Bohr in Quantum Theory (and closely related to it). Actually, there exists a heuristic correspondence between important, interesting, and useful constructions and results of traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i.e., semirings and semifields with idempotent addition; for rigorous definitions see below).

A systematic and consistent application of the “idempotent” correspondence principle leads to a variety of results, often quite unexpected. As a result, in parallel with traditional mathematics over rings, its “shadow”, Idempotent Mathematics, appears. This “shadow” stands approximately in the same relation to traditional mathematics as classical physics to Quantum Theory. In many respects Idempotent Mathematics is simpler than traditional mathematics. However, the transition from traditional concepts and results to their idempotent analogs is often nontrivial. A correct formulation of these analogs is sometimes the most difficult step. In this sense, Idempotent Mathematics resembles mathematics over $p$-adic fields.

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}_+$ the semiring of all nonnegative real numbers (with respect to the ordinary sum and product). The change of variables $x \mapsto u = h \ln x$ defines a map
Φ_h: \mathbb{R}_+ \to S = \mathbb{R} \cup \{-\infty\}. Let the addition and multiplication operations be mapped from \mathbb{R} to S by Φ_h, i.e., let
\[ u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h)), \quad u \odot v = u + v, \quad 0 = -\infty = \Phi_h(0), \quad 1 = 0 = \Phi_h(1). \]
It can easily be checked that \( u \oplus_h v \to \max\{u, v\} \) as \( h \to 0 \), and \( S \) forms a semiring with respect to the addition \( u \oplus v = \max\{u, v\} \) and the multiplication \( u \odot v = u + v \) with zero \( 0 = -\infty \) and unit \( 1 = 0 \). Denote this semiring by \( \mathbb{R}_{\max} \); it is idempotent, i.e., \( u \oplus u = u \) for all its elements.\(^1\) The analogy with quantization is obvious; the parameter \( h \) plays the role of the Planck constant, so \( \mathbb{R}_+ \) (or \( \mathbb{R} \)) can be viewed as a “quantum object” and \( \mathbb{R}_{\max} \) as the result of its “dequantization”. A similar procedure gives the semiring \( \mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\} \) with the operations \( \oplus = \min, \odot = + \); in this case \( 0 = +\infty, \quad 1 = 0 \). The semirings \( \mathbb{R}_{\max} \) and \( \mathbb{R}_{\min} \) are isomorphic.

Connections with physics and imaginary values of the Planck constant are discussed below in Sec. 1.3. The idempotent semiring \( \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \) with the operations \( \oplus = \max, \odot = \min \) is obtained as a result of a “second dequantization” of \( \mathbb{R} \) (or \( \mathbb{R}_+ \)). Dozens of interesting examples of nonisomorphic idempotent semirings may be cited, as well as a number of standard methods of deriving new semirings from them (see [1–9, 11–21] and below).

1.2. Idempotent Analysis

Let \( S \) be an arbitrary semiring with idempotent addition \( \oplus \) (which is always assumed to be commutative), multiplication \( \odot \), zero \( 0 \), and unit \( 1 \). The set \( S \) is supplied with the standard partial order \( \preceq \): by definition, \( a \preceq b \), if (and only if) \( a \oplus b = b \). Thus all elements of \( S \) are positive: \( 0 \preceq a \) for all \( a \in S \). Due to the existence of this order, Idempotent Analysis is closely related to lattice theory [22, 23], the theory of vector lattices, and the theory of ordered spaces [21–27]. Moreover, this partial order allows to model a number of basic notions and results of Idempotent Analysis at the purely algebraic level, see [11–14]; in this paper we develop this line of reasoning systematically.

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map \( X \to S \), where \( X \) is an arbitrary set and \( S \) is an idempotent semiring. Functions with values in \( S \) can be added, multiplied by each other, and multiplied by elements of \( S \) pointwise.

The idempotent analog of a linear functional space is a set of \( S \)-valued functions that is closed under addition of functions and multiplication of functions by elements of \( S \), or an \( S \)-semimodule. Consider, e.g., the \( S \)-semimodule \( \mathcal{B}(X, S) \) of functions \( X \to S \) that are bounded in the sense of the standard order on \( S \).

If \( S = \mathbb{R}_{\max} \), then the idempotent analog of integration is defined by the formula
\[ I(\varphi) = \int_X \varphi(x) \, dx = \sup_{x \in X} \varphi(x), \quad (1.1) \]
where \( \varphi \in \mathcal{B}(X, S) \). Indeed, a Riemann sum of the form
\[ \sum_i \varphi(x_i) \cdot \sigma_i \]
corresponds to the expression
\[ \bigoplus_i \varphi(x_i) \odot \sigma_i = \max_i \{\varphi(x_i) + \sigma_i\}, \]
\footnote{The semiring \( \mathbb{R}_{\max} \) is actually a semifield; also, it is often called the (max, +) algebra.}
which tends to the right-hand side of (1.1) as \( \sigma_i \to 0 \). Of course, this is a purely heuristic argument.

Formula (1.1) defines the idempotent integral not only for functions taking values in \( \mathbb{R}_{\text{max}} \), but also in the general case when any bounded (from above) subset of \( S \) has a least upper bound; semirings of this type are called \textit{boundedly complete}.

An idempotent measure on \( X \) is defined by
\[
m_\psi(Y) = \sup_{x \in Y} \psi(x),
\]
where \( \psi \in B(X, S) \). The integral with respect to this measure is defined by
\[
I_\psi(\varphi) = \int_X \varphi(x) \, dm_\psi = \int_X \varphi(x) \odot \psi(x) \, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)). \tag{1.2}
\]

Obviously, if \( S = \mathbb{R}_{\text{min}} \), then the standard order \( \preceq \) is opposite to the usual order \( \leq \), so in this case Eq. (1.2) assumes the form
\[
\int_X \varphi(x) \, dm_\psi = \int_X \varphi(x) \odot \psi(x) \, dx = \inf_{x \in X} (\varphi(x) \odot \psi(x)), \tag{1.3}
\]
where \( \inf \) is understood in the sense of the usual order \( \leq \).

The functionals \( I(\varphi) \) and \( I_\psi(\varphi) \) are linear over \( S \); their values correspond to limits of Lebesgue (or Riemann) sums. The formula for \( I_\psi(\varphi) \) defines the idempotent scalar product of the functions \( \psi \) and \( \varphi \). Various idempotent functional spaces and an idempotent version of the theory of distributions can be constructed on the basis of idempotent integration [1–8]. The analogy between idempotent and probability measures leads to spectacular parallels between optimization theory and probability theory. For example, the Chapman–Kolmogorov equation corresponds to the Bellman equation (see the survey of Del Moral [28] and [29–35, 12, 13]). Many other idempotent analogs may be cited (in particular, for the basic constructions and theorems of functional analysis). For instance, the Legendre transform is nothing but the \( \mathbb{R}_{\text{max}} \) version of the Fourier transform, see [1–8].

Indeed, suppose \( S = \mathbb{R}_{\text{max}} \), \( G = \mathbb{R}^n \); let \( G \) have a topological group structure. The ordinary Fourier–Laplace transform is defined as
\[
\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x) \, dx, \tag{1.4}
\]
where \( e^{i\xi \cdot x} \) is a character of the group \( G \), i.e., a solution of the following functional equation:
\[
f(x + y) = f(x)f(y).
\]
The idempotent analog of this equation is
\[
f(x + y) = f(x) \odot f(y) = f(x) + f(y),
\]
so “continuous idempotent characters” are linear functionals of the form
\[
x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n.
\]
As a result, the transform in (1.4) assumes the form
\[
\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G \xi \cdot x \odot \varphi(x) \, dx = \sup_{x \in G} (\xi \cdot x + \varphi(x)). \tag{1.5}
\]
The transform in (1.5) is nothing but the \textit{Legendre transform} (up to some notation); transforms of this kind establish a correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics.
1.3. The superposition principle and linear problems

The basic equations of Quantum Theory are linear (the superposition principle). The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the usual sense. However it is linear over the semiring $\mathbb{R}_{\min}$. Also, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings (V. P. Maslov’s idempotent superposition principle), compare [1–8]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where $X$, $H$, $F$ are matrices with coefficients in an idempotent semiring $S$ and the unknown matrix $X$ is determined by $H$ and $F$ [16]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbb{R}_{\max}$ and $S = \mathbb{R}_{\min}$, respectively. In [16], it was shown that the main optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman’s shortest path algorithm corresponds to a version of Jacobi’s algorithm, Ford’s algorithm corresponds to the Gauss–Seidel iterative scheme, etc.; see also [1, 5, 7, 8, 16–21, 12, 13, 74].

The linearity of the Hamilton–Jacobi equation over $\mathbb{R}_{\min}$ (and $\mathbb{R}_{\max}$) is closely related to the (ordinary) linearity of the Schrödinger equation. Consider the classical dynamical system specified by the Hamiltonian

$$H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x), \quad (1.6)$$

where $x = (x_1, \ldots, x_N)$ are generalized coordinates, $p = (p_1, \ldots, p_N)$ are generalized momenta, $m_i$ are generalized masses, and $V(x)$ is the potential. In this case the Lagrangian $L(x, \dot{x}, t)$ has the form

$$L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \frac{\dot{x}_i^2}{2} - V(x), \quad (1.7)$$

where $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N)$, $\dot{x} = dx/dt$. The action functional $S(x, t)$ has the form

$$S(x, t) = \int_{t_0}^{t} L(x, \dot{x}, t) \, dt, \quad (1.8)$$

where the integration is performed along a trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or least action principle), see, e.g., [37].

The action functional can be regarded as a function taking the set of curves (trajectories) to the set of real numbers. Assume that its range lies in the semiring $\mathbb{R}_{\min}$. In this case the minimum of the action functional can be viewed as the idempotent integral of this function over the set of trajectories or the idempotent analog of the Feynman path integral. Thus the least action principle can be considered as the idempotent version of the well-known Feynman approach to quantum mechanics (which is presented, e.g., in [38]); here, one should remember that the exponential function involved in the Feynman integral is monotone on the real axis. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Olešník formula for solving the Hamilton–Jacobi equation (see below).

Since

$$\frac{\partial S}{\partial x_i} = p_i, \quad \frac{\partial S}{\partial t} = -H(p, x), \quad$$

the following Hamilton–Jacobi equation holds:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_i}, x_i\right) = 0. \quad (1.9)$$
Quantization (see, e.g., [39]) leads to the Schrödinger equation

\[ -\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \hat{H} \psi = H(\hat{p}_i, \hat{x}_i) \psi, \]  

(1.10)

where \( \psi = \psi(x, t) \) is the wave function, i.e., a time-dependent element of the Hilbert space \( L^2(\mathbb{R}^N) \), and \( \hat{H} \) is the energy operator obtained by substituting the momentum operators

\[ \hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial x_i} \]

and the coordinate operators \( \hat{x}_i: \psi \mapsto x_i \psi \) for the variables \( p_i \) and \( x_i \) in the Hamiltonian function \( H(p, x) \), respectively. This equation is linear in the ordinary sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function:

\[ \psi = a(x, t) e^{(i/\hbar) S(x, t)}, \]

and to expand Eq. (1.10) in powers of \( \hbar \rightarrow 0 \) (the ‘semiclassical’ limit).

Instead of doing this, we switch to imaginary values of the Planck constant \( \hbar \) by the substitution \( \hbar = i \hbar \), assuming \( \hbar > 0 \). Thus the Schrödinger equation (1.10) turns to an analog of the heat equation:

\[ \hbar \frac{\partial u}{\partial t} = H\left(-\hbar \frac{\partial}{\partial x_i}, \hat{x}_i\right) u, \]  

(1.11)

where the real-valued function \( u \) corresponds to the wave function \( \psi \). A similar idea (the switch to imaginary time) is used in Euclidean quantum field theory (see, e.g., [40, 41]); let us remember that time and energy are dual quantities.

The linearity of Eq. (1.10) implies that of Eq. (1.11). Thus if \( u_1 \) and \( u_2 \) are solutions of (1.11), then so is their linear combination

\[ u = \lambda_1 u_1 + \lambda_2 u_2. \]  

(1.12)

Let \( S = -\hbar \ln u \) or \( u = e^{-S/\hbar} \) as in Sec. 1.1 above. It can easily be checked that Eq. (1.11) thus assumes the form

\[ \frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^N \frac{1}{2m_i} \left( \frac{\partial S}{\partial x_i} \right)^2 - \hbar \sum_{i=1}^n \frac{1}{2m_i} \frac{\partial^2 S}{\partial x_i^2}. \]  

(1.13)

This equation is nonlinear in the ordinary sense. However, if \( S_1 \) and \( S_2 \) are its solutions, then so is the function

\[ S = \lambda_1 \circ S_1 \oplus_h \lambda_2 \circ S_2, \]  

(1.14)

obtained from (1.12) by means of our substitution \( S = -\hbar \ln u \). Here the generalized multiplication \( \circ \) coincides with ordinary addition and the generalized addition is the image of ordinary addition under the above change of variables. As \( \hbar \rightarrow 0 \), we obtain the operations of the idempotent semiring \( \mathbb{R}_{\min} \), i.e., \( \circ = + \), \( \oplus = \min \), and Eq. (1.13) turns to the Hamilton–Jacobi equation (1.9), since the third term in the right-hand side of Eq. (1.13) vanishes.

Thus it is natural to consider the limit function \( S = \lambda_1 \circ S_1 \oplus_h \lambda_2 \circ S_2 \) as a solution of the Hamilton–Jacobi equation and to expect that this equation may be treated as linear over \( \mathbb{R}_{\min} \). This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations, see [6–8] and also [3]. Notice that if \( \hbar \) is changed to \( -\hbar \), then the resulting Hamilton–Jacobi equation is linear over \( \mathbb{R}_{\max} \).

The idempotent superposition principle indicates that there exist important problems that are linear over idempotent semirings.
1.4. Idempotent functional analysis

Idempotent Functional Analysis is an analog of traditional Functional Analysis in the framework of Idempotent Mathematics. Here we formulate a number of well-known results of Idempotent Analysis at a new, more abstract level; also, we consider problems that were not discussed in the earlier literature on Idempotent Analysis.

The most important results of Idempotent Analysis obtained so far are fixed point theorems and the spectral theory of linear operators on specific idempotent semimodules. These operators satisfy additional conditions of continuity type (see, e.g., [2–8]) or of algebraic regularity type in the spirit of [14, 6, 11–13] and the theory of $C^*$ and $W^*$ algebras. The known theorems on spectra of operators depend on the idempotent analog of the integral representation of operators (an analog of the L. Schwartz kernel theorem), which in turn employs the integral representation of linear functionals.

At the moment, the best studied idempotent semimodules are semimodules of bounded functions (with values in idempotent semirings), idempotent semimodules of continuous or semicontinuous functions (taking values in the extended real axis and other idempotent semirings) that are defined on open subsets of compacta with various boundary conditions, and free finite-dimensional idempotent semimodules (the idempotent linear algebra). In addition, a study of a class of semimodules that can be regarded as idempotent analogs of Sobolev spaces was carried out in [42, 43, 6] in connection with the study of the Hamilton–Jacobi equations. For these semimodules, neither theorems on the general form of functionals and integral representations of operators nor spectral theorems are proved, but existence and uniqueness theorems for fixed points of linear operators were obtained under some additional conditions.

Note that, as a rule, subsemimodules of semimodules of the classes studied so far do not belong to the same classes (for instance, a subsemimodule of a free semimodule generally is not free). Another important example that cannot be treated by known methods is the semimodule of integrable functions on a measure space. In this case any (idempotent) linear functional that is continuous (with respect to the $L^1$ norm) is constant.

In this paper we present an algebraic approach to Idempotent Functional Analysis: basic notions and results are “simulated” in algebraic terms. Elements of this approach can be traced back to [11–13] and especially [14]. The essential point is that the operation of idempotent addition can be defined for an infinite set of summands if the semiring is complete as an ordered set. In this case, the continuity property of functionals and maps can be simulated by the preservation of infinite sums under a suitable completion. Central to our analysis is a natural class of abstract semimodules that are idempotent analogs of vector spaces. We call semimodules of this kind idempotent spaces (see Sec. 4 below). The class of idempotent spaces contains most of the examples important for Functional Analysis, and subspaces of idempotent spaces belong to the same class. In particular, idempotent subspaces of (topological) vector lattices provide a very important example of idempotent spaces.

Further, in this paper we present idempotent versions of the basic results concerning linear functionals and scalar products, including the theorem on the general form of a linear functional and idempotent analogs of the Hahn–Banach and Riesz–Fischer theorems. We also present analogs of the Banach–Steinhaus and the closed graph theorems. In forthcoming papers we will apply the “topological” approach in the spirit of [1–8] and construct abstract Idempotent Functional Analysis starting from basic notions and results and leading up to analogs of A. Grothendieck’s results on topological tensor products, kernel spaces, and operators.

For additional comments and historical remarks see Sec. 6 (Commentary).
2. IDEMPOTENT SEMIGROUPS AND PARTIAL ORDER

2.1. Recall that a semigroup is a nonempty set endowed with an associative operation called addition (in additive semigroups) or multiplication (in multiplicative semigroups). A semigroup with a neutral element (called its zero or unit, respectively) is called a monoid; see, e.g., [22].

Definition 2.1. An idempotent semigroup is an additive semigroup $S$ with commutative addition $\oplus$ such that $x \oplus x = x$ for all $x \in S$. If this semigroup is a monoid, then its neutral element is denoted by 0 or $0_S$.

For a rich collection of examples of idempotent semigroups see Sec. 2.9 below.

2.2. Any idempotent semigroup is a partially ordered set with respect to the standard order defined below.

Definition 2.2. The standard order on an idempotent semigroup $S$ is the partial order $\preceq$ such that $x \preceq y$ if and only if $x \oplus y = y$.

From now on, we assume that all idempotent semigroups (and semirings) are ordered in this way and the notation $<, >, \succ$ has the obvious meaning. For instance, the relation $x < y$ means that $x \preceq y$ and $x \neq y$.

For basic definitions and results of the theory of ordered sets see, e.g., [23]. It can easily be checked that Definition 2.2 is self-consistent and defines an order relation. It follows from this definition that $0 \preceq x$ for all $x \in S$ if $S$ is a monoid. By $\mathbb{I}$ or $\mathbb{I}_S$ denote the (unique) element of $S$ such that $x \preceq \mathbb{I}$ for all $x \in S$ (if this element exists). It is clear that $0 = \inf S = \sup \emptyset$ and $\mathbb{I} = \sup S = \inf \emptyset$, where $\emptyset$ is the empty subset of $S$ and inf and sup denote the greatest lower bound and the least upper bound, respectively.

Any idempotent semigroup $S$ is a $\lor$-semilattice (or upper semilattice), i.e., for any $x, y \in S$ the set $\{x, y\}$ has the least upper bound $\sup\{x, y\} = x \lor y$. Obviously,

$$\sup\{x, y\} = x \oplus y.$$ 

Thus the class of all idempotent semigroups coincides with the class of all upper semilattices.

Remark 2.1. Although formally the notions of idempotent semigroup and semilattice coincide, classical lattice theory is aimed at the study of lattices of subsets (subspaces, ideals, etc.). Idempotent Analysis deals primarily with sublattices of ordered vector and functional spaces. These two approaches are in a relation similar to that of measure theory and probability theory: the object is common but the sources of interesting problems are different.

2.3. Let an idempotent semigroup $S$ be a lattice with respect to the standard order, i.e., let a greatest lower bound $\inf\{x, y\}$, denoted by $x \land y$, be defined for any two elements $x, y \in S$ along with their least upper bound $x \lor y$. In this case we call the semigroup $S$ a lattice semigroup or simply (by abuse of terminology) a lattice.

Definition 2.3. An idempotent semigroup $S^\circ$ is called ordinally dual (or $\alpha$-dual) to a lattice semigroup $S$ if it consists of the set of all elements of $S$ equipped with the (idempotent) operation $x, y \mapsto x \land y$ as addition.

Note that the operations $\lor = \oplus$ and $\land$ (“union”, or “addition”, and “intersection”) are associative and commutative for arbitrary sets of operands (summands or factors); compare, e.g., [24, Chap. 1, Sec. 6].
2.4. Let $S$ be an arbitrary partially ordered set. This set is called (ordinally) **complete** if any of its subsets, including the empty one, has a greatest lower and a least upper bound. A set $S$ is called **boundedly complete** or **conditionally complete** if each of its nonempty subsets that is bounded from above (from below) has a least upper bound (respectively, a greatest lower bound).

**Lemma 2.1.** If any subset $S_1$ in $S$ that is bounded from above has a least upper bound $\sup S_1$, then any subset $S_2$ in $S$ that is bounded from below has a greatest lower bound $\inf S_2$; if any subset of $S$ that is bounded from below has a greatest lower bound in $S$, then any subset of $S$ that is bounded from above has a least upper bound in $S$.

This lemma is well known (see, e.g., [23, 24]). Note that

\[
\inf S_2 = \sup\{x \mid x \preceq s \text{ for all } s \in S_2\}, \quad \text{(2.1)}
\]
\[
\sup S_1 = \inf\{x \mid s \preceq x \text{ for all } s \in S_1\}. \quad \text{(2.1')}
\]

Recall that a **cut** in $S$ is a subset of the form $I(X) = \{s \in S \mid x \preceq s \text{ for all } x \in X \subseteq S\}$, where $X$ is any subset of $S$. By $\hat{S}$ denote the set of all cuts endowed with the following partial order: $I_1 \preceq I_2$ if and only if $I_1 \supset I_2$; the set $\hat{S}$ is called the normal completion of the ordered set $S$, see [23].

The map $i: x \mapsto I(\{x\})$ is an embedding of $S$ into $\hat{S}$ with the following well-known properties:

1. The embedding $i: S \to \hat{S}$ preserves all greatest lower and least upper bounds that exist in $S$ (thus we can identify $S$ with a subset of $\hat{S}$).

2. The cut $I(X)$ coincides with $\sup(i(X))$ for any subset $X$ of $S$; in particular, any element of the completion $\hat{S}$ is the least upper bound of some subset $X$ of $S$.

3. If $S$ is complete, then $\hat{S} = S$; in particular, $\hat{S} = \hat{S}$ (here we take into account the natural identification of $\hat{S}$ with a subset of $\hat{S}$, see property 1).

4. If $\{X_\alpha\}_{\alpha \in A}$ is a family of subsets of $S$, then

\[
\sup_\alpha I(X_\alpha) = I\left(\bigcup_\alpha X_\alpha\right)
\]

in $\hat{S}$; note that $I(\cup X_\alpha) = \cap I(X_\alpha)$.

5. The normal completion $\hat{S}$ has the structure of idempotent semigroup $(x \oplus y = \sup\{x, y\})$; if $S$ is an idempotent semigroup, then the embedding $S \to \hat{S}$ is a semigroup homomorphism. In this case we call $\hat{S}$ the a-completion of the idempotent semigroup $S$.

The set $\hat{S}_b$ of all cuts of the form $I(X)$, where $X$ runs over all subsets of $S$ that are bounded from above, is a conditionally complete set; the set $\hat{S}_b$ is a subsemigroup of $\hat{S}$. If $S$ is an idempotent semigroup, then we call $\hat{S}_b$ the b-completion of this idempotent semigroup.

It is clear that $\hat{S}_b$ consists of all elements of $\hat{S}$ that are majorized by elements of $S$; in particular, the cut $I(\varnothing) \in \hat{S}_b$ (where $\varnothing$ is the empty set) is the zero of $\hat{S}_b$.

**Definition 2.4.** An idempotent semigroup $S$ is called **a-complete** (or **algebraically complete**) if it is complete as an ordered set. It is called **b-complete** (or **boundedly algebraically complete**) if it is boundedly complete as an ordered set and has the neutral element $0$.

It is easy to show that $\hat{S} = \hat{S}_b \cup \{\mathbb{1}_S\}$; note that $\mathbb{1}_S = \sup\hat{S}$ may belong to $S$. If $S$ is a b-complete idempotent semigroup, then $\hat{S}_b$ coincides with $S$. 
2.5. Important notation. Let $S$ be an idempotent semigroup. By $\oplus X$ and $\wedge X$, respectively, denote $\sup(X)$ and $\inf(X)$ for any subset $X$ of $S$, if these bounds exist, i.e., belong to $S$. Thus in an $a$-complete idempotent semigroup the sum is defined for any subset; in a $b$-complete idempotent semigroup the sum is defined for any subset that is bounded from above, including the empty one.

Let $X$ be a subset of an idempotent semigroup $S$. We introduce the following notation:

$$\text{Up}(X) = I(X) = \{y \in S \mid x \lessgtr y \text{ for all } x \in X\},$$

$$\text{Low}(X) = \{y \in S \mid y \lessgtr x \text{ for all } x \in X\}.$$  

2.6. Let $S$ and $T$ be idempotent semigroups.

Definition 2.5-a. Suppose the semigroups $S$ and $T$ are $a$-complete. We call a homomorphism $g : S \to T$ algebraically continuous, or an $a$-homomorphism for short, if

$$g(\oplus X) = \oplus g(X),$$

i.e.,

$$g\left(\sup_{x \in X} x\right) = \sup_{x \in X} g(x)$$

for any subset $X \subseteq S$.

For arbitrary idempotent semigroups $S$ and $T$ we call a homomorphism $g : S \to T$ an $a$-homomorphism, if it is uniquely extended to $\hat{S}$ by an $a$-homomorphism $\hat{g} : \hat{S} \to \hat{T}$ of the corresponding normal completions.

Many authors have introduced conditions similar to (2.2'); in [44], this condition is called monotonic continuity.

It is natural to consider the following variant of this definition (in the spirit of [14]):

Definition 2.5-b. Let $S$ and $T$ be $b$-complete idempotent semigroups. A homomorphism $g : S \to T$ is called boundedly algebraically continuous, or a $b$-homomorphism for short, if condition (2.2), which coincides with (2.2'), is satisfied for any subset $X \subseteq S$ bounded from above.

For arbitrary idempotent semigroups $S$ and $T$, a homomorphism $g : S \to T$ is called a $b$-homomorphism, if it is uniquely extended to $\hat{S}_b$ by a $b$-homomorphism $\hat{g} : \hat{S}_b \to \hat{T}_b$ of the corresponding $b$-completions.

Applying equality (2.2) to the empty set, we see that $a$-homomorphisms and $b$-homomorphisms take zero to zero provided a zero element exists. Note that in the general case an $a$-homomorphism $S \to T$ does not necessarily take the element $\mathbb{I}_S = \sup S$ to $\mathbb{I}_T = \sup T$, even if both of these elements exist.

Proposition 2.1. The composition (i.e., the product) of $a$-homomorphisms ($b$-homomorphisms) is an $a$-homomorphism (respectively, a $b$-homomorphism).

This statement follows immediately from the definitions.

In fact, we are dealing with three different categories of idempotent semigroups, where the morphisms are homomorphisms, $a$-homomorphisms, and $b$-homomorphisms, respectively. There exist other interesting categories of idempotent semigroups. The notions introduced so far can be extended to the case of idempotent semirings (see below).

By $\text{Hom}(S_1, S_2)$ denote the set of all homomorphisms of an idempotent semigroup $S_1$ to an idempotent semigroup $S_2$; by $\text{Hom}_a(S_1, S_2)$ denote the set of all $a$-homomorphisms. Finally, by $\text{Hom}_b(S_1, S_2)$ denote the set of all $b$-homomorphisms of a $b$-complete idempotent semigroup $S_1$ to a $b$-complete idempotent semigroup $S_2$ (though the notion of $b$-homomorphism can be extended to arbitrary idempotent semigroups using the notion of conditional (bounded) completion [23]).
Proposition 2.2. All three sets $\text{Hom}(S_1, S_2)$, $\text{Hom}_a(S_1, S_2)$, and $\text{Hom}_b(S_1, S_2)$ are idempotent semigroups with respect to the pointwise sum. If $S_2$ is an $a$-complete (a $b$-complete) semigroup, then $\text{Hom}_a(S_1, S_2)$ (respectively, $\text{Hom}_b(S_1, S_2)$) is an $a$-complete (respectively, a $b$-complete) idempotent semigroup.

This statement follows directly from the definitions.

2.7. We borrow the following definition from L. Fuchs' book [27]; the notation was introduced in Sec. 2.5 above.

Definition 2.6. Let $S$ be an idempotent semigroup. By definition, put $X^\cong = \text{Low} (\text{Up}(X))$ for any subset $X$ of $S$; we call $X^\cong$ an $o$-closure of $X$.

Note that the $o$-closure of the empty set may be nonempty.

Definition 2.7. Let $S$ and $T$ be idempotent semigroups. A map $f : S \to T$ is said to be $a$-regular if $f(X^\cong) \subset f(X)^\cong$ for any subset $X$ of $S$; this map is said to be $b$-regular if $f(X^\cong) \subset f(X)^\cong$ for any subset $X$ of $S$ that is bounded from above.

It is easy to prove that $b$-regular (and therefore $a$-regular) maps are homomorphisms of idempotent semigroups. For a stronger statement, see Proposition 2.3 below.

For example, let $\mathbb{R}$ be the set of all real numbers equipped with the structure of idempotent semigroup with respect to the operation $\oplus = \max$. A map $f : \mathbb{R} \to \mathbb{R}$ is a homomorphism of this idempotent semigroup to itself if it is nondecreasing. This homomorphism is $b$-regular (and $a$-regular) if and only if it is lower semicontinuous (for the definition of semicontinuity and some generalizations, see Sec. 2.8 and Sec. 2.9 below).

The following statements can easily be checked.

Proposition 2.3. Let $S$ and $T$ be idempotent semigroups. A map $f : S \to T$ is $a$-regular ($b$-regular) and takes zero to zero if and only if it is an $a$-homomorphism (respectively, a $b$-homomorphism).

Proposition 2.4. A homomorphism $f$ of an idempotent semigroup $S$ to an idempotent semigroup $T$ is an $a$-homomorphism if and only if it is a $b$-homomorphism and $\text{Up}(f(X)) = \text{Up}(f(S))$ for any subset $X$ of $S$ that is not bounded from above.

Definition 2.8. Let $S$, $T$, and $U$ be idempotent semigroups. We say that a map $f : S \times T \to U$ is a separate $a$-homomorphism ($b$-homomorphism) if the maps $f(s, \cdot) : t \mapsto f(s, t)$ and $f(\cdot, t) : s \mapsto f(s, t)$ are $a$-homomorphisms (respectively, $b$-homomorphisms).

In [45], maps of this type were called bimorphisms.

Proposition 2.5. Let $S$, $T$, and $U$ be idempotent semigroups.

1. The normal completion $\hat{S} \times \hat{T}$ of the direct product $S \times T$ of the semigroups $S$ and $T$ coincides with the direct product $\hat{S} \times \hat{T}$ of their normal completions (i.e., there exists a canonical isomorphism that identifies these semigroups). An analogous statement holds for $b$-completions.

2. The following equalities (isomorphisms) hold:

\[
\oplus (X \times Y) = (\oplus X) \times (\oplus Y),
\]
\[
\text{Low}(X \times Y) = \text{Low}(X) \times \text{Low}(Y),
\]
\[
\text{Up}(X \times Y) = \text{Up}(X) \times \text{Up}(Y),
\]

where $X \subset S$, $Y \subset T$.
3. A map \( f: S \times T \to U \) is a separate \( a \)-homomorphism (\( b \)-homomorphism) if and only if it can be (uniquely) extended to \( \hat{S} \) (respectively, to \( \hat{S}_b \)) by a separate \( a \)-homomorphism \( \hat{f}: \hat{S} \times \hat{T} \to \hat{U} \) (respectively, a separate \( b \)-homomorphism \( \hat{f}: \hat{S}_b \times \hat{T}_b \to \hat{U}_b \)).

Let us prove, for instance, Proposition 2.3 for the case in which \( S \) and \( T \) are \( a \)-complete and \( a \)-regularity is considered. Note that in an \( a \)-complete idempotent semigroup we have

\[
X^{\leq} = \text{Low}(\oplus X)
\]

for any subset \( X \). Let \( f \) be an \( a \)-regular map. Using equality (2.6), we get \( f(\oplus X) \in f(X^{\leq}) \); thus \( f(\oplus X) \in f(X^{\leq}) \subset f(X)^{\leq} = \text{Low}(\oplus f(X)) \) and therefore \( f(\oplus X) \preceq \oplus f(X) \). Since any homomorphism preserves the order, we obtain \( f(\oplus X) \succeq \oplus f(X) \). Thus \( f(\oplus X) = \oplus f(X) \), i.e., \( f \) is an \( a \)-homomorphism. Now let \( f \) be an \( a \)-homomorphism; then \( f(\oplus X) = \oplus f(X) \). Since \( f \) preserves the order, we can use formula (2.6) to get

\[
f(X^{\leq}) = f(\text{Low}(\oplus X)) \subset \text{Low}(f(\oplus X)) = \text{Low}(\oplus f(X)) = f(X)^{\leq},
\]

as claimed. The other statements can be proved similarly.

2.8. Semicontinuity. The notion of semicontinuity is involved in the relationship between the “topological” approach, which will be discussed in forthcoming papers, and the “algebraic” one; also, this notion is needed here for the presentation of some important examples.

By \( \mathbb{R} \) denote the extended real axis \( \mathbb{R} \cup \{ -\infty, +\infty \} \) endowed with the usual order \( \leq \) (which coincides with the standard order in \( \mathbb{R}_{\max} \), so \( \mathbb{R} \) may be identified with \( \mathbb{R}_{\max} \)). The set \( \mathbb{R} \) has the standard topology, which coincides with the ordinary topology on \( \mathbb{R} \), so that the space \( \mathbb{R} \) is homeomorphic to the segment \([ -1, 1 ]\) of the real axis.

A function \( f: T \to \mathbb{R} \) defined on an arbitrary topological space \( T \) is said to be lower semicontinuous [46, Chap. IV] if for any finite number \( s \) the set

\[
T_s = \{ t \in T \mid f(t) \leq s \}
\]

is closed in \( T \). This condition is equivalent to the condition that the set \( \{ t \in T \mid f(t) > s \} \) is open in \( T \). An upper semicontinuous function is defined similarly; a function \( f \) is upper semicontinuous if and only if the function \(-f\) is lower semicontinuous. Obviously, a real-valued function \( f \) is continuous if and only if it is both upper and lower semicontinuous. The set

\[
\text{epi}(f) = \{ (t, s) \mid f(t) \leq s \}
\]

in \( T \times \mathbb{R} \) is called the epigraph of the function \( f \). A function \( f \) taking finite values is lower semicontinuous if and only if its epigraph is closed, see, e.g., [47, 48]. Thus a subset in \( T \) is open (closed) if and only if its characteristic function is lower (respectively, upper) semicontinuous.

Suppose \( \{ f_\alpha \}_{\alpha \in A} \) is a family of functions defined on \( T \) and taking values in \( \mathbb{R} \). The lower envelope \( \inf_\alpha f_\alpha \) (the upper envelope \( \sup_\alpha f_\alpha \)) is the function that is defined on \( T \) and assigns the value \( \inf_\alpha (f_\alpha(t)) \) (respectively, \( \sup_\alpha (f_\alpha(t)) \)) to each \( t \in T \), see [46]. It is well known that the upper envelope of a family \( \{ f_\alpha \} \) of lower semicontinuous functions on \( T \) is lower semicontinuous on \( T \). An analogous statement holds for the lower envelope of a family of upper semicontinuous functions, see [46, 47].

The definition of semicontinuity can be extended to the case of maps \( f: T \to S \), where \( T \) is a topological space and \( S \) is a (partially) ordered set. We call this map lower semicontinuous if the set

\[
T_s = \{ t \in T \mid f(t) \preceq s \}
\]

is closed.
is closed in $T$ for all $s \in S$. A map is upper semicontinuous if it is lower semicontinuous with respect to the dual (opposite) order in $S$.

If $S$ is a complete lattice (or, equivalently, an $a$-complete idempotent semigroup), then it can be equipped with the order topology in the sense of [23, Chap. 10]. In [7, 8], the notion of semicontinuity is studied under some additional conditions on this topology. In [46, Chap. IV, Sec. 6], semicontinuous maps were studied for the case in which the order in $S$ is linear (in particular, see exercise 6, where $S$ is compact and open intervals form a base for the topology in $S$). In these cases the classical properties of semicontinuity are fulfilled.

**Definition 2.9.** A topological idempotent semigroup $S$ is an idempotent semigroup $S$ endowed with a topology such that its subsemigroup $S^b = \{s \in S \mid s \preceq b\}$ is closed for any $b \in S$.

**Proposition 2.6-a.** Suppose $T$ and $S$ are b-complete topological idempotent semigroups, $S$ is a-complete, and for any nonempty bounded subsemigroup $X$ in $T$ the element $\oplus X$ lies in the closure $\bar{X}$ of the subset $X$ in $T$. Then a homomorphism $f : T \to S$ taking $0_T$ to $0_S$ is an a-homomorphism if and only if the map $f$ is lower semicontinuous.

**Proof.** First suppose that the map $f$ is lower semicontinuous. For an arbitrary bounded subsemigroup $X \subset T$ let $s = \oplus f(X)$. It is clear that $X \subset T_s$, where $T_s$ is defined by (2.7). Now it follows from the fact that $T_s$ is closed and the assumption of Proposition 2.6-a that $\oplus X \subset T_s$, i.e., $f(\oplus X) \preceq s = \oplus f(X)$. Since $f$ preserves the order, the inequality $f(\oplus X) \preceq f(\oplus f(X))$ holds, so $f(\oplus X) = \oplus f(X)$ if $X$ is a nonempty bounded subsemigroup in $T$. Suppose now that $X$ is an arbitrary nonempty subset of $T$ and denote by $X^\oplus$ the subsemigroup of $T$ generated by $X$ (i.e., consisting of all finite sums of elements of $X$). Since the map $f$ is a homomorphism, $f(X^\oplus) = f(X)^\oplus$; taking into account that $\oplus X = \oplus X^\oplus$, we obtain

$$f(\oplus X) = f(\oplus X^\oplus) = \oplus f(X^\oplus) = \oplus f(f(X)^\oplus) = \oplus f(X).$$

It remains to prove that $f(\oplus \emptyset) = \oplus f(\emptyset)$, but this means that $0_T$ is taken to $0_S$. This completes the proof of the first part of Proposition 2.6-a.

Now suppose that the map $f$ is an a-homomorphism. We must prove that $T_s$ is closed in $T$ for any $s \in S$, i.e., that the limit of any net (or generalized sequence) consisting of elements of $T_s$ belongs to $T_s$. Suppose that $x_\alpha \in T_s$ and $x = \lim_\alpha x_\alpha$; let $t = \bigoplus_\alpha x_\alpha = \sup_\alpha x_\alpha$. Then $f(t) = \bigoplus_\alpha f(x_\alpha) = \sup_\alpha f(x_\alpha)$. Hence $f(t) \preceq s$ (since $f(x_\alpha) \preceq s$), $f(x) \preceq f(t)$ (since $x \preceq t$ and the homomorphism preserves the order), and $x \in T_{f(t)} \subset T_s$, i.e., $x \in T_s$. Thus the homomorphism $f$ is lower semicontinuous. This completes the proof. □

**Proposition 2.6-b.** Suppose $f : T \to S$ is a homomorphism of b-complete topological idempotent semigroups, $f(0_T) = 0_S$, and for any nonempty subsemigroup $X$ in $T$ that is bounded from above the element $\oplus X$ lies in the closure $\bar{X}$ of the set $X$ in $T$. Then $f$ is a b-homomorphism if and only if its restriction to the closed subsemigroup $T^x = \{t \in T \mid t \preceq x\}$ is lower semicontinuous for any $x \in T$.

Since $T^x$ is an $a$-complete idempotent semigroup and $f(T^x)$ belongs to the $a$-complete idempotent semigroup $S^{f(x)} = \{s \in S \mid s \preceq f(x)\}$, Proposition 2.6-b is a straightforward consequence of Proposition 2.6-a. The conditions of Proposition 2.6-b are fulfilled for a wide class of idempotent semigroups; among the examples are all Banach $L$-lattices (see [25, 26]), including the lattice of integrable functions, spaces of semicontinuous real-valued functions with the ‘weak’ topology in the sense of [2, 6–8], etc.

The analogy between the lower semicontinuity property for semirings of the $\mathbb{R}_{\max}$ type and continuity of countably additive measures in $\emptyset$ was observed by A. M. Chebotarev in [5, Sec. 1.1.6] and applied to the construction of the idempotent analog of the integral (1.1) and the “Fourier–Legendre transform” (1.5).
Remark 2.2. Suppose a topological idempotent semigroup $T$ satisfies the conditions of Proposition 2.6-a (2.6-b); then any of its subsemigroups also satisfies these conditions whenever it is closed in the topology in $T$ and is closed with respect to summation of arbitrary (respectively, arbitrary bounded from above) subsets.

2.9. Examples.

2.9.1. The set $\mathbb{R}$ of real numbers is an idempotent semigroup with respect to the operation $x \oplus y = \max\{x, y\}$; here, the standard order $\preceq$ coincides with the usual order $\leq$. By $\mathbb{R}_{\text{max}}$ denote the idempotent monoid $\mathbb{R} \cup \{-\infty\}$ obtained by the addition of the “minimal” element $-\infty$ to $\mathbb{R}$, i.e., $0 = -\infty$. The semigroup $\mathbb{R}_{\text{max}}$ is nothing but the $b$-completion of the idempotent semigroup $\mathbb{R}$, so $\mathbb{R}_{\text{max}}$ is $b$-complete. The normal completion $\hat{\mathbb{R}}_{\text{max}} = \mathbb{R}_{\text{max}} \cup \{+\infty\} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is an $a$-complete idempotent semigroup. The dual order ($\geq$) corresponds to the $o$-dual semigroups with the operation $x \ominus y = \min\{x, y\}$, which include the monoid $\mathbb{R}_{\text{min}} = \mathbb{R} \cup \{+\infty\}$ and its normal completion $\hat{\mathbb{R}}_{\text{min}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$; obviously, the semigroups $\mathbb{R}_{\text{max}}$ and $\mathbb{R}_{\text{min}}$ are isomorphic, as well as their completions.

In the sequel, as in other examples (see below), $\mathbb{R}_{\text{max}}, \mathbb{R}_{\text{min}}$, and their completions will have the structure of idempotent semirings in addition to that of idempotent semigroup.

2.9.2. The completion $\hat{\mathbb{R}}_{\text{max}}$ is isomorphic to the semigroup formed by the segment $[0, 1]$ under the operation $x \oplus y = \max\{x, y\}$.

2.9.3. The set $\mathbb{Z}$ of all integers is an idempotent semigroup with respect to the operation $x \oplus y = \max\{x, y\}$. By $\mathbb{Z}_{\text{max}}$ denote a $b$-complete idempotent semigroup $\mathbb{Z} \cup \{-\infty\}$ obtained by adding the element $0 = -\infty$. After adding the largest element $I = +\infty$ to $\mathbb{Z}_{\text{max}}$, we get its normal completion $\hat{\mathbb{Z}}_{\text{max}}$. These idempotent semigroups are subsemigroups of the idempotent semigroups $\mathbb{R}, \mathbb{R}_{\text{max}}, \hat{\mathbb{R}}_{\text{max}}$, of Example 2.9.1. Similarly are defined the idempotent semigroups $\mathbb{Z}_{\text{min}}$ and $\hat{\mathbb{Z}}_{\text{min}}$.

2.9.4. Let $X$ be an arbitrary set, $S$ be an idempotent semigroup. By $\text{Map}(X, S)$ denote the set of all functions defined on $X$ and taking values in $S$ (i.e., the set of all maps $X \to S$). The standard order on the set $\text{Map}(X, S)$ is defined by the rule

$$f \preceq g, \text{ if and only if } f(x) \preceq g(x) \text{ for all } x \in X.$$  (2.8)

This order corresponds to the operation of pointwise addition $f, g \mapsto f \oplus g$ that endows $\text{Map}(X, S)$ with the structure of idempotent semigroup.

2.9.5. By $\mathcal{B}(X, S)$ denote the subsemigroup in $\text{Map}(X, S)$ formed by bounded functions (i.e., functions with bounded range). This idempotent semigroup is $b$-complete if $S$ is $b$-complete.

2.9.6. By $\mathcal{B}(X)$ denote the semigroup $\mathcal{B}(X, S)$ defined in Example 2.9.5, if $S$ coincides with the idempotent semigroup $\mathbb{R}$ defined in Example 2.9.1.

2.9.7. By $C(X)$ denote the set of all continuous real-valued functions on a topological space $X$. The operation $f \oplus g$ is defined by the equality

$$(f \oplus g)(x) = \max\{f(x), g(x)\}$$  (2.9)

for all $x \in X$; this operation turns $C(X)$ to an idempotent semigroup.
2.9.8. By $USC(X)$ ($LSC(X)$) denote the set of upper (respectively, lower) semicontinuous real-valued functions on a topological space $X$. Recall that a function $f(x)$ is upper semicontinuous if
\[ f(x) = \inf_{\nu} \{ f_{\nu}(x) \} \quad (2.10) \]
for all $x \in X$ (the pointwise inf), where $\{ f_{\nu} \}$ is any family of upper semicontinuous (in particular, continuous) functions (see Sec. 2.8 above). Similarly, $f(x)$ is lower semicontinuous if
\[ f(x) = \sup_{\nu} \{ f_{\nu}(x) \} \quad (2.10') \]
for all $x \in X$, where $\{ f_{\nu} \}$ is a family of lower semicontinuous functions.

Let $\mathbb{R}$ be the idempotent semigroup of real numbers with respect to the operation $\oplus = \max$ (Example 2.9.1). Then $USC(X)$ and $LSC(X)$ are subsets of the partially ordered set $\text{Map}(X, \mathbb{R})$. Thus they are partially ordered by the induced order
\[ f \leq g \iff f(x) \leq g(x) \quad (2.11) \]
for all $x \in X$. The idempotent operation
\[ f \oplus g = \sup \{ f, g \} \quad (2.12) \]
endows both $USC(X)$ and $LSC(X)$ with the structure of idempotent semigroups (subsemigroups of $\text{Map}(X, \mathbb{R})$).

**Remark 2.3.** In general, least upper (greatest lower) bounds of infinite sets in the semigroup $\text{Map}(X, \mathbb{R})$ and its subsemigroups $USC(X)$ and $LSC(X)$ do not coincide. In $LSC(X)$, only least upper bounds coincide with those in $\text{Map}(X, \mathbb{R})$; in $USC(X)$, only greatest lower bounds do. Any set in $USC(X)$ that is bounded from below has a greatest lower bound (the pointwise inf); thus it follows from Lemma 2.1 that the idempotent semigroup $USC(X)$ is a $b$-complete lattice. By the same argument, $LSC(X)$ is a $b$-complete lattice.

2.9.9. The set $\text{Conv}(X, \mathbb{R})$ of all convex real-valued functions defined on a convex subset $X$ of some vector space\(^2\) is an idempotent semigroup (and a $b$-complete lattice) with respect to the operation (2.12). This idempotent semigroup is a subsemigroup of $\text{Map}(X, \mathbb{R})$, and the embedding is an $a$-homomorphism.

2.9.10. The set $\text{Conc}(X, \mathbb{R})$ of all concave real-valued functions defined on a convex subset $X$ of some vector space has the structure of idempotent semigroup under the restriction of the order defined by (2.11). This idempotent semigroup is a lattice but is not a subsemigroup of $\text{Map}(X, \mathbb{R})$.

2.9.11. Let $X$ be a measure space with measure $\mu$ and $L^1(X, \mu)$ be the (Banach) space of $\mu$-integrable functions on this space (i.e., classes of functions on $X$ that differ on a set of zero measure $\mu$). Then $L^1(X, \mu)$ is a boundedly complete idempotent semigroup with respect to the operation (2.12) generated by the standard order (2.11), where the inequality $f(x) \leq g(x)$ is supposed to hold almost everywhere. This result is true for any Banach space $L^p(X, \mu)$ with $1 \leq p \leq \infty$. These spaces provide an example of boundedly complete Banach lattices (for the theory of vector lattices see [25, 26] and [23, 24]).

---

\(^2\)Recall that a function is said to be convex if its epigraph (for the definition see Sec. 2.8 above) is a convex subset of $X \times \mathbb{R}$; a function $f$ is said to be concave if the function $-f$ is convex (see [47, 48]).
2.9.12. Suppose $X$ is a metric space with metric $\rho$; by $\text{lip}(X)$ denote the semigroup of all real-valued functions on $X$ satisfying the Lipschitz condition
\[ |f(x) - f(y)| \leq \rho(x, y). \] (2.13)

The structure of idempotent semigroup is defined by (2.11) and (2.12).

By $\text{lip}(X)$ denote this semigroup supplemented with the element $0$, so that $\text{lip}(X)$ is an idempotent monoid.

Remark 2.4. The semigroups $\text{USC}(X)$, $\text{LSC}(X)$, $\text{Conv}(X, \mathbb{R})$, $\text{lip}(X)$, and $L^p(X, \mu)$ are conditionally complete (but neither $b$-complete nor $a$-complete). Adding $0$ to any of these semigroups, we obtain $b$-complete semigroups.

3. IDEMPOTENT SEMIRINGS, SEMIFIELDS, AND QUASIFIELDS

3.1. Idempotent semirings and semifields

Idempotent semirings and semifields are the main objects of Idempotent Mathematics.

Definition 3.1. An idempotent semiring (or semiring for short) is an idempotent semigroup $K$ (with addition $\oplus$) endowed with an additional associative multiplication operation $\odot$ such that for all $x, y, z \in K$
\[ x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z), \]
\[ (y \oplus z) \odot x = (y \odot x) \oplus (z \odot x). \] (3.1)

Note that $a \odot x \preceq a \odot y$ if $x \preceq y$ for all $x, y, a \in K$.

An element $1 \in K$ is unit of a semiring $K$ if it is neutral with respect to multiplication, i.e., if
\[ 1 \odot x = x \odot 1 = x \] (3.2)
for all $x \in K$. In the sequel we always assume that all idempotent semirings contain a unit, unless otherwise stated.

An element $0 \in K$ is zero of a semiring $K$ if it is zero with respect to addition $\oplus$, i.e., $x \oplus 0 = x$, and
\[ x \odot 0 = 0 \odot x = 0 \] (3.3)
for all $x \in K$. We also denote zero of a semiring $K$ by $0_K$. An idempotent semiring with zero is often called a dioid, see, e.g., [12]. We always assume that $0 \neq 1$, unless otherwise stated. If a zero exists in a semiring, it is also a zero of the additive semigroup of this semiring. The converse is not always true. The set $\{1, 2, 3, \ldots\}$ of natural numbers is a semiring with respect to the operations $\oplus = \max$, $\odot = +$; the element $1$ is a zero of its additive semigroup but not of the whole semiring.

An idempotent semiring $K$ is commutative if the multiplication operation is commutative. There exist different versions of the axiomatics of idempotent semirings; for these, as well as for historical remarks, see [8, 9, 12, 13, 16–21, 49].

Definition 3.2. An idempotent division semiring is an idempotent semiring with unit in which any nonzero element has a multiplicative inverse. An idempotent semifield (or semifield for short) is a commutative idempotent division semiring.

This notion of idempotent semifield does not coincide with the notion of semifield in the sense of [50].
3.2. Examples

3.2.1. All examples listed in Sec. 2.9 above, except Example 2.9.12, become idempotent semirings after a suitable product is defined; a semiring without unit appears in Example 2.9.12 if \( \odot \) is defined as min. In Example 2.9.1, multiplication \( \odot \) is defined as the ordinary addition \( + \) and extended to the additional elements \( \pm \infty \) in the natural way. For example, in \( \hat{\mathbb{R}}_{\text{max}} \), equalities (3.3) and the rules \( x \odot (+\infty) = x + (+\infty) = +\infty = \sup \hat{\mathbb{R}}_{\text{max}} = 1 \) hold for all \( x \neq 0 \). The element 1 coincides with the usual zero. The idempotent semiring \( \mathbb{R} \) with these operations is often denoted by \( \mathbb{R}(\text{max}, +) \). In Example 2.9.2, multiplication is induced by the isomorphism with \( \hat{\mathbb{R}}_{\text{max}} \).

3.2.2. Suppose \( S \) is an arbitrary idempotent semigroup and \( \text{Hom}(S, S) \) is the semigroup of homomorphisms of \( S \) to itself (i.e., endomorphisms). The operation of composition of maps endows \( \text{Hom}(S, S) \) with the structure of an idempotent semiring.

3.3. Complete semirings

3.3.1. We begin with the definition of algebraic completeness.

**Definition 3.3.** An idempotent semiring \( K \) is \( a \)-complete (algebraically complete) if it is an \( a \)-complete idempotent semigroup and for any subset \( \{x_\alpha\} \) of \( K \) and any \( y \in K \),

\[
\left( \bigoplus_\alpha x_\alpha \right) \odot y = \bigoplus_\alpha (x_\alpha \odot y), \quad y \odot \left( \bigoplus_\alpha x_\alpha \right) = \bigoplus_\alpha (y \odot x_\alpha),
\]

i.e.,

\[
\left( \sup_\alpha \{x_\alpha\} \right) \odot y = \sup_\alpha \{x_\alpha \odot y\}, \quad y \odot \left( \sup_\alpha \{x_\alpha\} \right) = \sup_\alpha \{y \odot x_\alpha\}. \tag{3.6'}
\]

This means that the “homotheties” \( x \mapsto x \odot y \) and \( x \mapsto y \odot x \) are \( a \)-homomorphisms of the semigroup \( K \).

A semiring \( K \) with zero and unit is \( a \)-complete if and only if it is a complete dioid in the sense of [12]; the notion of algebraically complete semiring dates back to [51] and is closely related to different versions of the notion of Kleene algebra (see [52]). The notion of \( a \)-complete idempotent semiring is essentially equivalent to that of quantale (see, e.g., [53]), which appears in connection with the foundations of Quantum Theory. Morphisms of the corresponding category \textbf{Quant} preserve infinite sums. A quantale does not necessarily contain a (two-sided) unit element; a quantale with unit is called a uniquantale. Commutative \( a \)-complete idempotent semirings were studied in [54], where they were called commutative monoids in the category of complete sup-lattices.

Note that an \( a \)-complete semiring \( K \) cannot be a semifield (except for the case \( K = \{0, 1\} \)) [49, 14]. For instance, the element \(+\infty\) is not invertible in \( \hat{\mathbb{R}}_{\text{max}} \).

3.3.2. A weaker but more useful version of the notion of \( a \)-completeness is the bounded completeness or \( b \)-completeness of semirings.
Definition 3.4. An idempotent semiring \( K \) is called \( b \)-complete (or boundedly complete) if it is a \( b \)-complete idempotent semigroup and equalities (3.6) hold for any subset \{ \( x_\alpha \) \} of \( K \) that is bounded from above and for any \( y \in K \).

For simplicity, we shall always assume in the sequel that any \( b \)-complete idempotent semiring has a (semiring) zero element, unless otherwise stated. In [14] commutative idempotent semirings with this property are called regular; in [49] they are called boundedly complete (BC) dioids.

Obviously, \( a \)-completeness implies \( b \)-completeness.

3.3.3. We introduce \( b \)-complete semirings with the following important additional property.

Definition 3.5. We call a semiring \( K \) lattice \( b \)-complete if it is a \( b \)-complete idempotent semiring with zero and for any nonempty subset \{ \( x_\alpha \) \} of \( K \) and any \( y \in K \setminus \{ 0 \} \)

\[
(\wedge \alpha x_\alpha) \odot y = \wedge \alpha (x_\alpha \odot y), \quad y \odot (\wedge \alpha x_\alpha) = \wedge \alpha (y \odot x_\alpha).
\] (3.7)

Note that any nonempty subset of \( K \) is bounded from below by zero and therefore has a greatest lower bound (which is a least upper bound of the set of its lower bounds, see Lemma 2.1).

Proposition 3.1. Any idempotent division semiring that is \( b \)-complete as a semigroup (hence, any \( b \)-complete idempotent semifield) is lattice \( b \)-complete.

Proof. It is sufficient to prove equalities (3.6) and (3.7). These equalities hold trivially if \( y = 0 \). If \( y \neq 0 \), then \( y \) is invertible; thus the homotheties \( x \mapsto x \odot y \), \( x \mapsto y \odot x \) are invertible maps that preserve the order and, in particular, least upper and greatest lower bounds. This means that (3.7) and (3.6) hold. \(\square\)

3.3.4. Suppose \( S \) is a semigroup (not necessarily idempotent) with respect to the multiplication \( x, y \mapsto xy \). This semigroup is called lattice ordered (see [23]) if \( S \) is a lattice and the semigroup translations \( x \mapsto xy \), \( x \mapsto yx \) are isotonic (i.e., order-preserving). By \( S_0 \) denote either the set \( S \) itself, if it contains the least element \( 0 \) such that \( 0x = x0 = 0 \) for all \( x \in S \), or the set \( S_0 = S \cup \{ 0 \} \), where \( 0 \) is an additional element, in the converse case. Let \( x \oplus y = \sup \{ x, y \} \) if \( x, y \in S \); also, let \( x \oplus 0 = 0 \oplus x = x \) for all \( x \in S_0 \).

Proposition 3.2. The set \( S_0 \) is an idempotent semiring with respect to the addition \( x \oplus y \) and the multiplication \( x \odot y = xy \), where by definition \( x \odot 0 = 0 \odot x = 0 \) for all \( x \in S_0 \).

Proposition 3.3. If \( G \) is a boundedly complete lattice ordered group, then \( G_0 \) is a \( b \)-complete idempotent division semiring, i.e., any nonzero element of the semiring \( G_0 \) has a multiplicative inverse. Any \( b \)-complete idempotent division semiring has the form \( G_0 \), where \( G \) is a boundedly complete lattice ordered group.

These statements follow directly from the definitions.

Proposition 3.4. Any \( b \)-complete idempotent division semiring is commutative, i.e., is an idempotent semifield.

This statement follows from Proposition 3.3 and [27, Part 1, Chap. 5, Theorem 18].

3.4. Quasifields

Let \( K \) be an idempotent semiring.

Definition 3.6. An element \( x \in K \) is quasi-invertible if there exists a set \( X \) of invertible elements of \( K \) such that \( x = \oplus X \).

We call a semiring \( K \) a division quasiring if any of its nonzero elements is quasi-invertible.
Definition 3.7. A semiring $K$ is integrally closed if $x \preceq 1$ whenever the set $\{x^n \mid n = 1, 2, \ldots\}$ of all powers of an element $x \in K$ is bounded from above (for all $x \in K$).

For division semirings, integral closedness means that the group of all invertible elements is integrally closed in the sense of [23, Chap. XIII, Sec. 2] (sometimes this property is called complete integral closedness).

Definition 3.8. An idempotent division quasiring $K$ is a quasifield if it is integrally closed.

The following statement follows directly from the definitions.

Proposition 3.5. Any integrally closed semifield is a quasifield.

Note that any quasifield is commutative, although this is not mentioned in Definition 3.8 explicitly (see Proposition 3.7 below).

3.5. Completion of semirings

Suppose $K$ is an idempotent semiring. Considering it as an idempotent semigroup, we can construct its $a$-completion $\hat{K}$ and $b$-completion $\hat{K}_b$. In this section we discuss whether it is possible to endow $\hat{K}$ (respectively, $\hat{K}_b$) with the structure of $a$-complete (respectively, $b$-complete) idempotent semiring. By $\mu$ denote the map $\mu : K \times K \to K$, defined by the multiplication operation, i.e.,

$$\mu : (x, y) \mapsto x \odot y.$$  

Definition 3.9. A semiring $K$ is said to be $a$-regular ($b$-regular) if the map $\mu$ has a unique extension $\hat{\mu} : \hat{K} \times \hat{K} \to \hat{K}$ (respectively, $\hat{\mu} : \hat{K}_b \times \hat{K}_b \to \hat{K}_b$) that defines the structure of $a$-complete (respectively, $b$-complete) idempotent semiring in $\hat{K}$ (respectively, $\hat{K}_b$).

Clearly, the $a$-regularity ($b$-regularity) of $K$ implies that the multiplication $\mu$ is a separate $a$-homomorphism (respectively, a separate $b$-homomorphism) in the sense of Definition 2.8.

Definition 3.10. If $K$ is an $a$-regular (a $b$-regular) semiring, then the semiring $\hat{K}$ (respectively, $\hat{K}_b$) is called the $a$-completion (respectively, $b$-completion) of the semiring $K$; we denote $\hat{\mu}(x, y)$ by $x \odot y$ as before.

Proposition 3.6. A semiring $K$ with zero is $a$-regular ($b$-regular) if and only if the homotheties

$$\mu(\cdot, y) : x \mapsto x \odot y, \quad \mu(y, \cdot) : x \mapsto y \odot x$$  

are $a$-homomorphisms (respectively, $b$-homomorphisms) of the additive semigroup of $K$ for all $y \in K$.

Proof. If $K$ is regular, then the homotheties (3.8) are obviously regular. Suppose now that the homotheties (3.8) are $a$-homomorphisms ($b$-homomorphisms); then by Definition 2.5 they can be uniquely extended to the whole set $\hat{K}$ (respectively, $\hat{K}_b$). Thus the product $x \odot y$ is well defined if one of the factors belongs to the semigroup completion of $K$. In the general case, let $x = \sup_{\alpha} x_\alpha$, $y = \sup_{\beta} y_\beta$, where $x_\alpha, y_\beta \in \hat{K}$; let

$$x \odot y = \sup_{\alpha} (x_\alpha \odot y) = \sup_{\beta} (x \odot y_\beta).$$  

(3.9)

It is easily shown (by using Propositions 2.5 and 2.2) that the product (3.9) is well defined. A straightforward calculation shows that this product is associative and endows $\hat{K}$ (or $\hat{K}_b$) with the required structure. □

Recall that the $a$-completion of a semiring $K$ cannot be a semifield (if $K \neq \{0, 1\}$). Obviously, completions of commutative semirings are commutative.
Proposition 3.7. Any quasifield $K$ is commutative and $b$-regular; hence $\hat{K}_b$ is a $b$-complete idempotent semifield.

Proof. First we use Proposition 3.6 to check that the quasifield $K$ is $b$-regular. If an element $y \in K$ is invertible, then it follows from Proposition 3.1 that the homotheties (3.8) are $b$-regular (i.e., are $b$-homomorphisms). For a quasi-invertible $y \in K$ the same statement follows from Proposition 2.2 (the corresponding homothety is the sum, or the least upper bound, of a set of $b$-homomorphisms). Since any nonzero element of $K$ is quasi-invertible, this implies that $K$ is $b$-regular and $\hat{K}_b$ is a $b$-complete idempotent semiring. Now from the results of [27, Chap. V] (see Theorem 19 and its corollaries) it follows immediately that $\hat{K}_b$ is a $b$-complete semifield.

It can easily be checked that any $b$-complete semifield is integrally closed and is a quasifield. Thus $b$-complete semifields are the most important examples of quasifields; on the other hand, any quasifield $K$ can be embedded in a $b$-complete semifield $\hat{K}_b$. □

Remark 3.1. If $K$ is a $b$-complete semifield that does not coincide with $\{0, 1\}$, then $\hat{K} = K \cup \{\mathbb{I}\}$ (where $\mathbb{I} = \sup K$, see Sec. 2.2 above) has the structure of commutative idempotent semiring, where $0 \odot \mathbb{I} = 0$ and $x \odot \mathbb{I} = \mathbb{I}$ for all $x \neq 0$ in $K$. Thus any $b$-regular quasifield is $a$-regular.

3.6. Examples

3.6.1. It is easy to check that the semirings $\mathbb{R}_{\text{max}}$ and $\mathbb{R}_{\text{min}}$ are $b$-complete idempotent semifields; the idempotent semiring $\mathbb{R}$ is a semifield and a quasifield (see Examples 2.9.1 and 3.2.1).

3.6.2. The subsemiring in $\mathbb{R}_{\text{max}}$ formed by all integers and the element 0 is a $b$-complete semifield.

3.6.3. The semiring $K = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ with the operations $\oplus = \max$ and $\odot = \min$, where $0 = -\infty$, $1 = +\infty$, is lattice $b$-complete.

3.6.4. The semiring $\text{Map}(X, \hat{\mathbb{R}}_{\text{max}})$ of all functions taking values in $\hat{\mathbb{R}}_{\text{max}}$ (see Examples 2.9.3 and 3.2.1) is $a$-complete but not lattice $b$-complete.

3.6.5. It follows from Proposition 3.3 that any (boundedly) complete vector lattice over $\mathbb{R}$ generates a $b$-complete idempotent semifield. In particular, $L^p(x, \mu)_0 = L^p(x, \mu) \cup \{0\}$ is a $b$-complete semifield, see Examples 2.9.11, 3.2.1.

3.6.6. Any vector lattice can be considered as a semifield; in particular, the set $C(X)$ of all continuous real-valued functions on a topological space $X$ is a semifield and a quasifield with respect to the standard idempotent operations defined in Examples 2.9.7 and 3.2.1.

4. IDEMPOTENT SEMIMODULES AND SPACES

4.1. Basic definitions

Definition 4.1. Let $V$ be an idempotent semigroup, $K$ be an idempotent semiring, and let a multiplication operation $k, x \mapsto k \odot x$ be defined so that the following equalities hold

\begin{align*}
(k_1 \odot k_2) \odot x &= k_1 \odot (k_2 \odot x), \\
(k_1 \oplus k_2) \odot x &= (k_1 \odot x) \oplus (k_2 \odot x), \\
k \odot (x \oplus y) &= (k \odot x) \oplus (k \odot y), \\
1 \odot x &= x
\end{align*}

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for all \( x, y \in V, \ k, k_1, k_2 \in K \); then the semigroup \( V \) is called a (left) idempotent semimodule over the semiring \( K \) (or simply a semimodule).

**Definition 4.2.** Let \( V \) be an idempotent semimodule over \( K \). An element \( 0_V \in V \) is zero of the semimodule \( V \) if \( k \odot 0_V = 0_V \) and \( 0_V \oplus x = x \) for all \( k \in K, \ x \in V \).

In the sequel, the zero of a semimodule \( V \) is denoted by \( 0 \) if it exists and this notation does not lead to confusion.

As usual, a subsemimodule of a semimodule \( V \) is a subsemigroup of \( V \) that is invariant under multiplication by coefficients from \( K \).

By \( L_k \) denote the operator of “homothety” in \( V \), i.e., \( L_k : x \mapsto k \odot x \), where \( k \in K, \ x \in V \). Evidently, the map \( k \mapsto L_k \) is a homomorphism of \( K \) to \( \text{Hom}(V, V) \), where \( \text{Hom}(V, V) \) is the semiring of all homomorphisms of the semimodule \( V \) (see Example 3.2.2 above).

### 4.2. Complete and standard semimodules

**Definition 4.3.** A semimodule \( V \) over an idempotent semiring \( K \) is called \( a \)-complete if \( V \) is an \( a \)-complete idempotent semigroup, all homotheties \( L_k \) are \( a \)-homomorphisms, the homomorphism \( k \mapsto L_k \) is (uniquely) extended to give a semigroup homomorphism \( \hat{K} \to \text{Hom}(V, V) \) defining a multiplication operation \( \hat{K} \times V \to V \), and the following equalities hold:

\[
(\oplus Q) \odot x = \oplus(Q \odot x), \tag{4.5}
\]

i.e.,

\[
(\sup Q) \odot x = \sup_{k \in Q} \{k \odot x\} \tag{4.5'}
\]

for all \( x \in V, \ Q \subset \hat{K} \). The notion of \( b \)-complete semimodule over \( K \) is defined similarly (but \( Q \) is assumed to be a subset of \( \hat{K}_b \)).

**Remark 4.1.** Assuming that the set \( Q \) is empty in (4.5), we see that for all \( x \in V \)

\[
0_K \odot x = 0_V. \tag{4.6}
\]

Note that idempotent semimodules were considered in many publications, see, e.g., [4–14, 16–21, 42–44, 49]. In [45], \( a \)-complete idempotent semimodules over commutative \( a \)-complete idempotent semirings were considered in different terms. For additional references see Sec. 6 below.

**Definition 4.4.** A semimodule \( V \) over a lattice \( b \)-complete idempotent semiring \( K \) (see Definition 3.5) is called standard if \( V \) is \( b \)-complete (in particular, \( a \)-complete) and for any \( x \in V \) such that \( x \neq I = \sup V \) and any nonempty subset \( Q \subset K \) the following equalities hold:

\[
(\land Q) \odot x = \land(Q \odot x), \tag{4.7}
\]

i.e.,

\[
(\inf Q) \odot x = \inf_{k \in Q} \{k \odot x\}. \tag{4.7'}
\]

Note that \( Q \) is bounded from below since a \( b \)-complete semiring \( K \) has zero \( 0_K \).

**Definition 4.5.** A semimodule \( V \) over an idempotent semiring \( K \) is called \( a \)-regular (\( b \)-regular) if the product \( \mu : K \times V \to V \) is a separate \( a \)-homomorphism (respectively, \( b \)-homomorphism) of idempotent semigroups in the sense of Definition 2.8.

Using Proposition 2.5 and arguing as in Sec. 3.5, we obtain the following statement.
Proposition 4.1. Let \( V \) be an \( a \)-regular (a \( b \)-regular) semimodule over a \( b \)-regular semiring \( K \) and \( \mu: K \times V \to V \) be the corresponding product. Then \( \mu \) has a unique extension \( \hat{\mu}: \hat{K} \times \hat{V} \to \hat{V} \) (respectively, \( \hat{\mu}: \hat{K}_b \times \hat{V}_b \to \hat{V}_b \) ), which defines on \( \hat{V} \) (respectively, on \( \hat{V}_b \) ) the structure of a-complete (respectively, \( b \)-complete) semimodule over \( K \) and \( \hat{K}_b \). If the semiring \( K \) is \( a \)-regular, i.e., if \( \hat{K} \) is a semiring, then \( \hat{V} \) is an \( a \)-complete semimodule over \( \hat{K} \) as well.

We denote \( \hat{\mu}(k, x) \) by \( k \odot x \) as before.

Definition 4.6. If \( V \) is an \( a \)-regular (a \( b \)-regular) semimodule over a \( b \)-regular semiring \( K \), then we call the semimodule \( \hat{V} \) (respectively, \( \hat{V}_b \) ) over \( \hat{K}_b \) the \( a \)-completion (respectively, the \( b \)-completion) of the semimodule \( V \) (see Proposition 4.1).

We stress that the procedure of completion of a semimodule \( V \) requires that the semiring \( K \) be replaced with its completion \( \hat{K}_b \).

4.3. Idempotent spaces

To obtain substantial results on semimodules as well as maps and functionals defined on semimodules, it is appropriate to consider the case when the basic semiring \( K \) is a quasifield or a semifield. This case is particularly important for the problems of calculus.

Definition 4.7. We call a semimodule \( V \) over a quasifield or a semifield \( K \) an idempotent space.

This notion is analogous to the notion of linear (vector) space over a field.

Definition 4.8. We call a semimodule \( V \) over a quasifield \( K \) an idempotent \( a \)-space (\( b \)-space) if it is \( a \)-regular (respectively, \( b \)-regular) and \( \hat{V} \) (respectively, \( \hat{V}_b \) ) is a standard semimodule over the \( b \)-complete semifield \( \hat{K}_b \), so that equality (4.7) of Definition 4.4 holds for \( x \neq I = \text{sup } V \).

A quasifield or semifield is an example of a ("one-dimensional") idempotent space over itself.

The following statement is a straightforward consequence of Propositions 3.1 and 3.7, the definitions, and Remark 3.1.

Proposition 4.2. If \( K \) is a quasifield, then \( \hat{K}_b \) is an idempotent \( b \)-space over \( K \) and \( \hat{K}_b \), and the semiring \( \hat{K} \) considered as a semimodule is an idempotent \( a \)-space over \( K \) and \( \hat{K}_b \) and an \( a \)-complete semimodule over \( \hat{K} \).

Direct sums and products of semimodules, idempotent spaces, \( a \)-spaces, and \( b \)-spaces over the same semiring \( K \) can be defined and described in the usual way and provide a number of new examples. Additional nontrivial examples are generated when subsemimodules and subspaces are considered.

4.4. Linear maps and functionals

Suppose that \( V \) and \( W \) are idempotent semimodules over an idempotent semiring \( K \) and \( P: V \to W \) is a map from \( V \) to \( W \). The following definition is standard.

Definition 4.9. A map \( P: V \to W \) is called additive if

\[
P(x \oplus y) = P(x) \oplus P(y)
\]  

(4.8)

for all \( x, y \in V \). This map is called homogeneous if

\[
P(k \odot x) = k \odot P(x)
\]  

(4.9)

for all \( x \in V, k \in K \). A map \( P: V \to W \) is called linear if it is additive and homogeneous.
**Definition 4.10.** Suppose $V$ and $W$ are idempotent $a$-regular ($b$-regular) semimodules over an idempotent semiring $K$. A linear map $P: V \rightarrow W$ is called $a$-linear (respectively, $b$-linear), if it is an $a$-homomorphism (respectively, a $b$-homomorphism).

Clearly, the notion of $a$-linear ($b$-linear) map provides an algebraic model of the notion of linear (semi)continuous map (see Sec. 2.8 above; note that in the case of ordinary linear operators semicontinuity is equivalent to continuity).

**Proposition 4.3.** Let a map $P: V \rightarrow W$, where $V$ and $W$ are a-regular ($b$-regular) semimodules over a $b$-regular semiring $K$, be $a$-linear (respectively, $b$-linear). Then it is uniquely extended to a map $\hat{P}: \hat{V} \rightarrow \hat{W}$ (respectively, $\hat{P}: \hat{V}_b \rightarrow \hat{W}_b$) that is $a$-linear (respectively, $b$-linear) over $\hat{K}_b$, where $\hat{V}$ and $\hat{W}$ are the semimodule completions in the sense of Definition 4.6.

**Proof.** The proof is by direct calculation. It follows from the definition of $a$-homomorphism (a $b$-homomorphism) of idempotent semigroups that the extension $\hat{P}$ is uniquely defined and additive. If $k \in \hat{K}_b$, then $k = \oplus Q$, where $Q \subset K$. Obviously,

$$\hat{P}(k \circ x) = \hat{P}(\oplus Q \circ x) = \oplus P(Q \circ x) = \oplus Q \circ P(x) = k \circ P(x) = k \circ \hat{P}(x),$$

if $x \in V$; if $x \in \hat{V}$, then $x = \oplus X$, where $X \subset V$, and by the similar argument $\hat{P}(k \circ x) = k \circ \hat{P}(x)$. This concludes the proof. □

**Definition 4.11.** A functional on a semimodule $V$ over an idempotent semiring $K$ is a map $V \rightarrow K$ or a map $V \rightarrow \hat{K}$ from this semimodule to the completion $\hat{K}$ of the semimodule $K$. A functional is called linear if this map is linear. A linear functional is called $a$-linear ($b$-linear) if it is an $a$-homomorphism (respectively, a $b$-homomorphism).

We assume that an $a$-linear (a $b$-linear) functional takes values in the completion $\hat{K}$ (respectively, $\hat{K}_b$) if this completion has the natural structure of semimodule over $K$. This is always true if $K$ is a quasifield (by Proposition 3.7 and Remark 3.1).

A general description of $a$-linear functionals on idempotent spaces is presented in Sec. 5 below.

**Remark 4.2.** It follows from Proposition 2.4 that a $b$-linear functional $f$ on $V$ is $a$-linear if and only if $\text{Up}(f(X)) = \text{Up}(f(V))$ for any subset $X$ of $V$ that is not bounded from above. By definition, $f$ has an extension $\hat{f}: \hat{V}_b \rightarrow \hat{K}_b$. If $V$ contains the element $\hat{1} = \text{sup} V$, then $\hat{f}$ is defined on $\hat{V}$ and is $a$-linear. Otherwise $f$ must be extended to $\hat{V} = \hat{V}_b \cup \{\hat{1}\}$, i.e., defined for the element $\hat{1}$. But $\hat{1} = \text{sup} X$ for any subset $X$ of $V$ that is not bounded from above. Thus $\text{sup} f(X)$ must not depend on the choice of $X$; then we may put $\hat{f}(\hat{1}) = \text{sup} f(X) = \oplus f(X)$.

As an example, suppose $\mathcal{B}(X, \mathbb{R})$ is the set of all bounded functions defined on an arbitrary set $X$ containing more than one point; consider $\mathcal{B}(X, \mathbb{R})$ as an idempotent $b$-space over the semifield $\mathbb{R}(\text{max}, +)$. The linear functional $\delta_a: \varphi \mapsto \varphi(a)$, i.e., the “delta-function”, is $b$-linear but not $a$-linear. However a $b$-linear functional on $\mathcal{B}(X, \mathbb{R})$ always has an $a$-linear extension defined on $\mathcal{B}(X, \hat{\mathbb{R}})$.

### 4.5. Idempotent semimodules and spaces associated with vector lattices

Examples of idempotent semimodules and spaces that are most important for Idempotent Analysis are either subsemimodules of (topological) vector lattices or are dual to these in the sense that they consist of linear functionals (see Definition 4.11) subject to some regularity conditions, e.g., of $a$-linear functionals.
Recall (see [26, 25, 23]) that a vector space $V$ over the field of real numbers $\mathbb{R}$ is called ordered (or semiordered [24]) if $V$ is equipped with a (partial) order $\preceq$ such that all translations $x \mapsto x+y$ and all homotheties $x \mapsto \lambda x$ preserve this order (i.e., are isotonic) whenever $\lambda > 0$ and $x, y \in V$. If the operations $x \vee y = x \oplus y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ are defined, then $V$ is called a vector lattice (in this case the additive group of $V$ is lattice ordered, see Sec. 3.3.4 above). A topological vector lattice is a Hausdorff topological vector space $V$ such that $V$ is a vector lattice, the lattice operations $x, y \mapsto x \oplus y$ and $x, y \mapsto x \wedge y$ are continuous, and the positive cone $\{x \in V \mid x \geq 0\}$ is normal relative to the topology in $V$. In particular, a normed (Banach) lattice is a normed (respectively, Banach) space that is a vector lattice such that the operations $\vee = \oplus$ and $\wedge$ are jointly continuous.

Let $V$ be a topological vector lattice over $\mathbb{R}$, $K$ be a subsemigroup of the additive group of $V$ (with respect to the ordinary sum), and $M$ be a subset of $V$ invariant under translations $x \mapsto x+k$, where $k \in K$. Suppose further that $K$ and $M$ are idempotent subsemigroups in $V$ (with respect to $\vee = \oplus$) and the embeddings $K \to V$ and $M \to V$ are $b$-homomorphisms.

Obviously, $V$ is a semifield with respect to the standard idempotent operations $\oplus = \vee = \sup$ and $\odot = \odot = +$ and $K$ is its subsemifield. Here, the product $K \times M \to M$ is defined by $(k, x) \mapsto k \odot x = k + x$.

**Proposition 4.4.** 1) The semifields $V$ and $K$ are integrally closed (hence, are quasifields) with respect to the standard idempotent operations $\oplus = \sup$ and $\odot = \odot = +$; in this case $\mathbf{1} = 0$.

2) $M$ is an idempotent $b$-space over the quasifield $K$; the product $K \times M \to M$ is defined by $k \odot x = k + x$. In particular, the whole lattice $V$ is an idempotent $b$-space and $M$ is an idempotent $b$-subspace of $V$ (although it need not be a subspace of $V$ in the usual sense).

3) The product $K \times M \to M$ is the restriction of the product in the $b$-complete semifield $\widetilde{V}_b$ to $K \times M$.

**Proof.** We begin with the first statement. Let us prove that the semifield $K$, which contains an (idempotent) inverse $x^{-1} = -x$ for any of its elements, is integrally closed. Let $x, b \in K$ and $x^n = nx \preceq b$ for all $n = 1, 2, \ldots$. Then $x \preceq (1/n)b$; hence $x \oplus ((1/n)b) = (1/n)b$. Using the continuity of the ordinary product and the idempotent sum, we obtain the following sequence of equalities:

$$x \odot 1 = x \odot 0 = x \odot \lim_{n \to \infty} \left((\frac{1}{n})b\right) = \lim_{n \to \infty} \left(x \odot \left(\frac{1}{n}\right)b\right) = 0 = 1,$$

which implies $x \odot 1 = 1$, i.e., $x \preceq 1$, and thus proves the statement. The integral closedness of the semifield $V$ can be shown similarly. The rest of Proposition 4.4 can be checked by a direct calculation. □

**Definition 4.12.** We call any idempotent $b$-space $M$ described in Proposition 4.4 a space of $(V, K)$ type.

**4.6. Examples**

4.6.1. Suppose $K$ is a $b$-complete semifield with zero and $K$ does not coincide with the two-element semifield $\{0, 1\}$.

The semigroup Map($X, \hat{K}$) (see Examples 2.9.4, 3.2.1, 3.6.4) is an idempotent semimodule over $K$ (with respect to the pointwise product of functions by elements of $K$) but this semimodule is not standard if $X$ consists of two or more elements.

The idempotent semimodules $USC(X, \mathbb{R}_{\max})$, and $LSC(X, \mathbb{R}_{\max})$ of semicontinuous functions are defined similarly; generally, these semimodules are not standard if $X$ consists of two or more elements but they are $a$-complete.
4.6.2. The semigroups \( USC(X) \), \( LSC(X) \), \( L^p(X) \), and \( \text{Conv}(X, \mathbb{R}) \) defined in Examples 2.9.8–2.9.11 (see also Examples 3.2.1 and 3.6) are idempotent \( a \)-spaces over the semifield (and quasifield) \( K = \mathbb{R}(\max, +) \) under pointwise multiplication of functions by elements of \( K \). The normal completions of these idempotent semigroups are \( a \)-complete idempotent \( a \)-spaces over the \( b \)-complete semifield \( \mathbb{R}_{\max} \) and \( a \)-complete semimodules over the semiring \( \hat{\mathbb{R}}_{\max} \).

4.6.3. The semiring \( \hat{\mathbb{R}}_{\max} \) is an idempotent \( a \)-space over the semifield \( \mathbb{R}_{\max} \). Similarly, the set \( \hat{\text{Map}}(X, \mathbb{R}) \) is an idempotent \( a \)-space over \( \mathbb{R}_{\max} \). It is natural to call this space \( n \)-dimensional if \( X \) consists of \( n \) elements.

4.6.4. Any \( a \)-complete idempotent semigroup is an idempotent \( a \)-space with respect to the natural action of the Boole algebra, i.e., the semifield \( \{0, 1\} \).

4.6.5. Spaces of \((V, K)\) type.

4.6.5.1. The space \( C(X) \) of continuous real-valued functions on a topological space \( X \) (see Examples 2.9.7, 3.2.1, and 3.6.6 above) is a space of \((V, K)\) type, where \( K \) is the subgroup of constants (i.e., \( \mathbb{R} \)) and \( M = V = C(X) \).

4.6.5.2. Let \( V = C(X) \), \( K \) be the subgroup of integer constants (i.e., \( \mathbb{Z} \), see Example 3.6.2), \( M \) be the subgroup of integer-valued functions.

4.6.5.3. Let \( V = C(\mathbb{R}) \), \( K \) be the subgroup of even functions, \( M = V \). Likewise, let \( V = C(X) \) and \( K \) be the subgroup of functions invariant under some group of continuous transformations of \( X \).

4.6.5.4. Let \( X \) be a measure space with measure \( \mu \). Let \( V = M = L^p(X, \mu) \), \( K \) be the subgroup of functions invariant under a group of measure-preserving transformations of \( X \).

4.6.5.5. Let
\[
V = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2, \quad K = \{(x, x) \mid x \in \mathbb{R}\}, \quad M = \{(x, y) \in V \mid |x - y| \leq 1\}.
\]
Note that \( M \) is an “infinite-dimensional” subspace of a two-dimensional space in the sense that it is not a finitely generated subspace. Clearly, \( M \) is not a subspace of \( V \) in the conventional sense.

5. THE STRUCTURE OF \( a \)-LINEAR FUNCTIONALS ON IDEMPOTENT SEMIRINGS

5.1. The basic construction

Suppose \( V \) is an idempotent \( b \)-space (e.g., an idempotent \( a \)-space, see Definition 4.8) over a quasifield \( K \).

Definition 5.1. Let \( x \in \hat{V} \); by \( x^* \) denote the functional \( V \to \hat{K} \) defined by
\[
q(y) = \wedge\{k \in K \mid y \approx k \odot x\} = \inf_{y \approx k \odot x} k,
\]
for any \( y \in V \); we call this functional the \( x \)-functional on \( V \).

By \( \mathbb{I}_V \) denote the largest element of \( V \) (if it exists); recall that \( \mathbb{I}_V = \sup V = \inf \emptyset \), where \( \emptyset \) is the empty subset of \( V \). Similarly, \( \mathbb{I}_K = \sup K \). Note that \( \mathbb{I}_V^*(y) \equiv 0_K \) for all \( y \in V \) (with the possible exception of \( y = \mathbb{I}_V \); \( 0_V^*(y) = \mathbb{I}_K \), if \( y \neq 0_V \), and \( 0_V^*(0_V) = 0_K \).
Theorem 5.1. Let $V$ be an idempotent $b$-space over a quasifield $K$. Then for any $x \in \hat{V}$ the $x$-functional $x^*: y \mapsto x^*(y)$ is an $a$-linear functional on $V$.

Proof. Replacing $K$ and $V$ by their $b$-completions and using Definition 4.8 and Proposition 4.1, we may assume that $V$ is $b$-complete and $K$ is a $b$-complete semifield. Then it is clear that $\hat{V} = V \cup \{I_V\}$ and $\hat{K} = K \cup \{I_K\}$. For $x = I_V$ the statement of the theorem can be checked directly; consider the case when $x \not\approx I_V$. The fact that $V$ is standard (see Definitions 4.4 and 4.8) implies that $y \not\leq x^*(y) \circ x$, if the set $K(y) = \{k \in K \mid y \not\leq k \circ x\}$ is not empty.

Let $Y$ be an arbitrary subset of the semimodule $V$. Using the construction of the map $y \mapsto x^*(y)$, it is easy to prove that this map preserves the order (i.e., is isotonic); therefore $\oplus x^*(Y) = \sup\{x^*(y) \mid y \in Y\} = x^*(\oplus Y)$.

If the set $K(y) = \{k \in K \mid y \not\leq k \circ x\}$ is empty, then
\[
x^*(\oplus Y) = I_K, \quad \oplus x^*(Y) = \sup\{x^*(y) \mid y \in Y\} = I_K,
\]
where $I_K = \sup K = \sup \hat{K}$; thus in this case $x^*(\oplus Y) = \oplus x^*(Y)$. Otherwise,
\[
(\oplus x^*(Y)) \circ x \succ x^*(y) \circ x \succ y \quad \text{for any} \ y \in Y.
\]
Thus $(\oplus x^*(Y)) \circ x \succ \oplus Y$; therefore, $\oplus x^*(Y) \succ x^*(\oplus Y)$. Since we have already proved that $\oplus x^*(Y) \not\leq x^*(\oplus Y)$, it follows that
\[
x^*(\oplus Y) = \oplus x^*(Y), \tag{5.2}
\]
i.e., the functional $y \mapsto x^*(y)$ is an $a$-homomorphism.

Now let us prove that this functional is homogeneous, i.e., that
\[
k \circ x^*(y) = x^*(k \circ y) \tag{5.3}
\]
for all $k \in K$ and $y \in V$. Suppose that $p$ is an invertible element of $K$ and $y$ is any element of $V$. Since multiplication by $p$, or homothety, is an automorphism of $K$, we see that
\[
p \circ K(y) = p \circ \{k \in K \mid y \not\leq k \circ x\} = \{k \in K \mid p \circ y \not\leq k \circ x\} = K(p \circ y).
\]
Therefore the set $K(p \circ y)$ is empty if the set $K(y)$ is empty; thus $x^*(y) = x^*(p \circ y) = I_K$ and $p \circ x^*(y) = p \circ x^*(p \circ y)$ since $k \circ I_K = I_K$. Now if the set $K(y)$ is not empty, then $K(p \circ y)$ is not empty and
\[
p \circ x^*(y) = p \circ \wedge K(y) = \wedge(p \circ K(y)) = \wedge K(p \circ y) = x^*(p \circ y).
\]
Hence the homogeneity (5.3) is proved for all invertible, i.e., nonzero, elements $p = k \in K$. Finally, if $k = 0$, then $0 \circ x^*(y) = 0 = x^*(0) = x^*(0 \circ y)$. We have proved that the functional is homogeneous; this completes the proof of Theorem 5.1. \qed

5.2. The basic theorem on the structure of functionals

Proposition 5.1. Suppose $V$ is a standard semimodule over a $b$-complete semiring $K$ (e.g., $V$ is a $b$-complete $b$-space, see Definitions 4.4 and 4.8); let $f$ be an $a$-linear functional on $V$ such that $f$ takes the value $1$ and $f(I_V) \succ 1$, where $I_V = \sup V$. Then there exists a unique $x$-functional $y \mapsto x^*(y)$ such that $x^*(y) \not\leq f(y)$ for any $y \in V$ and $x^*(y) = f(y)$ if $f(y)$ is invertible in $K$; here $x = \oplus\{y \in V \mid f(y) \not\leq 1\}$.
Theorem 5.2. concludes the proof of the lemma.

Note that earlier theorems of this type (see, e.g., [7, 8, 11, 14]) were proved using Remark 5.1. Hence the area of application of Theorem 5.2 is to a large degree complementary to that contrary, our proof of Theorem 5.2 is based on the construction of “lower envelopes”. Thus the assumption that the semimodule contains sufficiently many “Dirac delta functions”. On the other hand, since \( x^*(y) \gtrless y \), it follows that \( x^*(y) = x^*(y) \circ f(x) \gtrsim y \). As a result, \( x^*(y) = f(y) \), if \( f(y) \) is invertible. □

Lemma 5.1. Let \( K \) be a \( b \)-complete semifield that does not coincide with the Boolean algebra \( \{0, 1\} \). Then

\[
0 = \wedge(K \setminus \{0\}) = \inf(K \setminus \{0\}).
\]

Proof. Let \( m = \wedge(K \setminus \{0\}) \). It follows from the assumption that there exists an invertible element \( k \neq 1 \); we may assume that \( k \prec 1 \) (in the converse case replace \( k \) by \( (1 \circ k)^{-1} \)). Thus it follows from the definition of \( m \) that \( m \lessdot k \prec 1 \). If \( m \neq 0 \), then the elements \( m \) and \( m \circ m \) are invertible and \( m \circ m \prec m \circ 1 \prec m \), which contradicts the definition of the element \( m \). This concludes the proof of the lemma. □

Theorem 5.2. Suppose \( V \) is an idempotent \( b \)-space over a quasifield \( K \); then any nonzero \( a \)-linear functional \( f \) defined on \( V \) has the form \( f = x^* \) for a unique \( x \in \hat{V}_b \). If \( K \neq \{0, 1\} \), then

\[
x = \oplus\{y \in V \mid f(y) \lessdot 1\}.
\]

In the converse case \( x = \oplus\{y \in V \mid f(y) = 0\} \).

Proof. Replacing \( K \) and \( V \) by their \( b \)-completions and using Definition 4.8 and Proposition 4.1, we may assume that \( V \) is \( b \)-complete and \( K \) is a \( b \)-complete semifield. Suppose \( f \) is an arbitrary nonzero \( a \)-linear functional defined on \( V \) and \( \hat{f}: \hat{V} \to \hat{K} \) is its extension to \( \hat{V} \) according to the definitions of \( a \)-linearity and \( a \)-homomorphism (see Definitions 4.10 and 2.5-a).

First suppose that \( K = \{0, 1\} \); let \( x = \oplus\{y \in V \mid f(y) = 0\} \). It is easily shown that in this case \( f(y) = 0 \) if \( y \lessdot x \) and \( f(y) = 1 \) otherwise; thus \( f = x^* \). If \( x = \hat{1}_V \), then \( \hat{f} (\hat{1}_V) = 0 \), which is impossible if \( f \) is nonzero; thus \( x \in \hat{V} = \hat{V}_b \).

Now let \( K \neq \{0, 1\} \). Lemma 5.1 implies that \( 0 = \wedge(K \setminus \{0\}) \). Let \( x = \oplus\{y \in V \mid f(y) \lessdot 1\} \). If \( x = \hat{1}_V \), then \( \hat{f} (\hat{1}_V) \lessdot 1 \), which is impossible if \( f \) is nonzero since by homogeneity the range of \( f \) must contain all nonzero \( (i.e., \text{invertible}) \) elements of \( K \), including some \( k \gtrsim 1 \). Thus \( x \in V = \hat{V}_b \) and, evidently, \( f(x) = 1 \).

In the proof of Proposition 5.1 it was shown that \( x^* \gtrsim f \) and \( x^*(y) = f(y) \) if \( f(y) \in K \setminus \{0\} \). Thus in order to prove that \( x^* = f \) it is sufficient to check that \( x^*(y) = \hat{0} \) if \( f(y) = \hat{0} \). Indeed, if \( f(y) = \hat{0} \), then \( f(k \circ y) = \hat{0} \prec 1 \) if \( k \neq \hat{0} \). But this implies that \( k \circ y \lessdot x \) if \( k \neq \hat{0} \); therefore \( k \circ x \gtrsim y \) if \( k \neq \hat{0} \), so \( x^*(y) \lessdot \wedge(K \setminus \{0\}) = \hat{0} \lessdot x^*(y) \), which completes the proof. □

Remark 5.1. Note that earlier theorems of this type (see, e.g., [7, 8, 11, 14]) were proved using the assumption that the semimodule contains sufficiently many “Dirac delta functions”. On the contrary, our proof of Theorem 5.2 is based on the construction of “lower envelopes”. Thus Theorem 5.2 is valid in the cases when “delta functions” do not exist (e.g., in spaces of integrable functions). Hence the area of application of Theorem 5.2 is to a large degree complementary to that of the results based on “delta functions”. Moreover, in general the property of having a sufficient collection of “delta functions” is not inherited by subsemimodules.

5.3. Theorems of the Hahn–Banach type

Let \( V \) be a semimodule over an idempotent semiring \( K \). A subsemimodule \( W \) of the semimodule \( V \) is a subsemigroup \( W \) in \( V \) that is closed under multiplication by elements (coefficients) of \( K \). Note that \( W \) itself is a semimodule over \( K \).
Definition 5.2. Let $V$ be an idempotent $a$-space ($b$-space) over a quasifield $K$. A subsemimodule $W$ of $V$ is called an $a$-subspace (respectively, a $b$-subspace) of $V$ if the embedding $i: W \to V$ has a unique $a$-linear (respectively, $b$-linear) extension $\tilde{W} \to \tilde{V}$ (respectively, $\tilde{W}_b \to \tilde{V}_b$) to the completions of the semimodules defined over $\tilde{K}_b$.

The basic definitions immediately imply the following statement.

Proposition 5.2. Let $V$ be an idempotent $a$-space ($b$-space) over a quasifield $K$ and $W$ be its $a$-subspace (respectively, $b$-subspace). Then $W$ and its completion $\tilde{W}$ (respectively, $\tilde{W}_b$) are idempotent $a$-spaces (respectively, $b$-spaces).

Theorem 5.3. Let $V$ be an idempotent $b$-space over a quasifield $K$ and $W$ be its $b$-subspace. Then any $a$-linear functional on $W$ has an $a$-linear extension to $V$.

This result is an immediate consequence of Theorem 5.2 and Proposition 5.2.

Theorem 5.4. Let $V$ be an idempotent $b$-space. If $x, y \in V$ and $x \neq y$, then there exists an $a$-linear functional $f$ on $V$ such that $f(x) \neq f(y)$.

Proof. If $x \succ y$, then $y^*(x) > 1$ and $y^*(y) \leq 1$, so $f = y^*$ has the required property. Indeed, in the converse case we see that $x \prec 1_V$, the inequality $x^*(y) \leq 1$ is not possible, but $x^*(x) \ngeq 1$; thus $f = x^*$ has the required property. \qed

Remark 5.2. It follows from the definition of spaces of $(V, K)$ type (see Sec. 4.6 above) that their $b$-subspaces are spaces of $(V, K)$ type themselves.

Remark 5.3. The subsemimodule $C(X)$ in $USC(X)$ is not a $b$-subspace since the corresponding embedding is not a $b$-homomorphism. Still, $C(X)$ is an idempotent $b$-space (and $a$-space) over $\mathbb{R}(\max, +)$, and $a$-linear functionals on $C(X)$ have $a$-linear extensions to $USC(X)$. It is easy to extend Theorem 5.3 to cases of this kind.

5.4. Analogs of the Banach–Steinhaus theorem and the closed graph theorem

The following statements are straightforward consequences of the definitions; they can be considered as analogs of well-known results of the traditional Functional Analysis (the Banach–Steinhaus theorem and the closed graph theorem). In this section all $a$-spaces are assumed to be $a$-complete. The results can easily be extended to the case of incomplete spaces by means of the completion procedure.\textsuperscript{3}

Proposition 5.3. Suppose $\mathcal{P} = \{P_{\alpha}\}$ is a family of $a$-linear maps of an $a$-space $V$ to an $a$-space $W$ and the map $P: V \to W$ is the pointwise sum of these maps, i.e., $P(x) = \sup\{P_{\alpha}(x) \mid P_{\alpha} \in \mathcal{P}\}$; then the map $P$ is $a$-linear.

Indeed, we have

\begin{equation}
P(k \odot x) = \sup\{P_{\alpha}(k \odot x) \mid P_{\alpha} \in \mathcal{P}\} = \sup\{k \odot P_{\alpha}(x) \mid P_{\alpha} \in \mathcal{P}\}
= k \odot \sup\{P_{\alpha}(x) \mid P_{\alpha} \in \mathcal{P}\} = k \odot P(x),
\end{equation}

for any $x \in V$; thus the map $P$ is homogeneous. If $X \subset V$, then

\begin{equation}
P(\oplus X) = \sup\{P_{\alpha}(\oplus X) \mid P_{\alpha} \in \mathcal{P}\} = \sup\{\oplus P_{\alpha}(X) \mid P_{\alpha} \in \mathcal{P}\}
= \sup\{P_{\alpha}(x) \mid x \in X, P_{\alpha} \in \mathcal{P}\} = \sup\{\sup\{P_{\alpha}(x) \mid x \in X\} \mid P_{\alpha} \in \mathcal{P}\}
= \sup\{P(x) \mid x \in X\} = \oplus P(X);
\end{equation}

\textsuperscript{3}An idempotent analog of the Banach–Steinhaus theorem for spaces of continuous functions that tend to $0 = \infty$ at infinity is formulated in [7, p. 59]; see also [8, p. 52]. These spaces are not $a$-complete but can be completed in the usual way.
this completes the proof.$\square$

**Corollary.** The pointwise sum (i.e., the least upper bound) of a family of $a$-linear functionals is an $a$-linear functional.

**Proposition 5.4.** Let $V$ and $W$ be $a$-spaces. A linear map $P: V \rightarrow W$ is $a$-linear if and only if its graph $\Gamma$ in $V \times W$ is closed under sums (i.e., least upper bounds) of arbitrary subsets.

The hypothesis of Proposition 5.4 implies that the embedding $i: \Gamma \rightarrow V \times W$ is $a$-linear. To complete the proof, it is sufficient to note that $P$ is the composition of three $a$-linear maps: the isomorphism $x \mapsto (x, P(x)) \in \Gamma$, the embedding $i$, and the projection $V \times W \rightarrow W$. $\square$

### 5.5. The scalar product

Let us consider a class of idempotent spaces on which a natural structure of scalar product can be defined.

**Definition 5.3.** An idempotent $b$-space $A$ over a quasifield $K$ is called an idempotent $b$-semialgebra over $K$ if $A$ is equipped with the structure of idempotent semiring consistent with the product $K \times A \rightarrow A$ (in the sense that the product is associative). If the product $A \times A \rightarrow A$ is a separate $a$-homomorphism ($b$-homomorphism), then the $b$-semialgebra $A$ is called $a$-regular (respectively, $b$-regular).

The theory of vector lattices is an important source of examples of idempotent $b$-semialgebras. 

**Proposition 5.5.** For any invertible element $x \in A$, where $A$ is a $b$-semialgebra, and any element $y \in A$, the following equality holds:

$$x^*(y) = 1_A^*(y \circ x^{-1}).$$  \hfill (5.4)

**Proof.** The completion procedure allows to reduce the proof to the case in which $K$ is a $b$-complete semifield. In this case

$$x^*(y) = \inf \{ k \mid k \circ x \succeq y \} = \inf \{ k \mid (k \circ x) \circ x^{-1} \succeq y \circ x^{-1} \} = \inf \{ k \mid k \circ 1_A \succeq y \circ x^{-1} \}.$$  

To conclude the proof, note that $\inf \{ k \mid k \circ 1_A \succeq y \circ x^{-1} \} = 1_A^*(y \circ x^{-1})$. $\square$

**Definition 5.4.** Suppose $A$ is a commutative $b$-semialgebra over a quasifield $K$. The map $A \times A \rightarrow \hat{K}$ defined by the formula $(x, y) \mapsto \langle x, y \rangle = 1^*(x \circ y)$ is called the canonical scalar product (or simply the scalar product) on $A$.

**Proposition 5.6.** The scalar product $\langle x, y \rangle$ on a commutative $a$-regular ($b$-regular) $b$-semialgebra over a quasifield $K$ has the following properties:

1. the map $(x, y) \mapsto \langle x, y \rangle$ is a separate $a$-homomorphism (respectively, $b$-homomorphism);
2. for all $k \in K$, $x, y \in A$ and any (respectively, any bounded from above) subset $X$ of $A$, the following equalities hold:

$$\langle x, y \rangle = \langle y, x \rangle,$$  \hfill (5.5)

$$\langle k \circ x, y \rangle = \langle x, k \circ y \rangle = k \circ \langle x, y \rangle,$$  \hfill (5.6)

$$\langle \oplus X, y \rangle = \bigoplus_{x \in X} \langle x, y \rangle.$$  \hfill (5.7)

Proposition 5.6 follows directly from the definitions; formula (5.7) follows from the first statement.

The following statement is an idempotent analog of the well-known Riesz–Fischer theorem.
Theorem 5.5. Let $A$ be a commutative $b$-semialgebra over a quasifield $K$ such that $A$ itself is a quasifield. Then any nonzero $a$-linear functional $f$ on $A$ has the form

$$f(y) = \langle x, y \rangle,$$

(5.8)

where $x \in \widehat{A}_b$, $x \neq 0$, and $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\widehat{A}_b$.

This theorem follows directly from Remark 3.1, Theorem 5.2, and Proposition 5.5.

Remark 5.4. Evidently, the canonical scalar product can be defined for the case in which the $b$-semialgebra $A$ is not commutative, and Theorem 5.5 can be extended to this case as well.

5.6. The skew-scalar product and elements of duality

5.6.1. Let $V$ be an idempotent $b$-space over a $b$-complete semifield $K$. For any two elements $x, y \in V$, by $[x, y]$ denote the value $x^*(y)$ of the functional $x^*$ at the element $y$ (see Definition 5.1).

Definition 5.5. We say that the map $V \times V \rightarrow \widehat{K}$ defined by $(x, y) \mapsto [x, y] = x^*(y)$ is the canonical skew-scalar product (or simply the skew-scalar product) on $V$.

The following statements are direct consequences of our definitions.

Proposition 5.7. The skew-scalar product has the following properties:

$$[x, x] \preceq 1,$$

(5.9)

$$[x, k_1 \circ y_1 \oplus k_2 \circ y_2] = k_1 \circ [x, y_1] \oplus k_2 \circ [x, y_2],$$

(5.10)

$$[k \circ x, y] = k^{-1} \circ [x, y],$$

(5.11)

$$[x_1 \wedge x_2, y] = [x_1, y] \oplus [x_2, y]$$

(5.12)

for all $x, x_1, x_2 \in \widehat{V}$, $y, y_1, y_2 \in V$, $k, k_1, k_2 \in K$, $k \neq 0$.

Proposition 5.8. If the canonical scalar product $(x, y) \mapsto \langle x, y \rangle$ is defined on a space $V$ (i.e., $V$ is a $b$-semialgebra, see Sec. 5.5 above), then the following equalities hold:

$$\langle x, y \rangle = [y^{-1}, x], \quad [x, y] = \langle x^{-1}, y \rangle,$$

(5.13)

$$[x, y] = [y^{-1}, x^{-1}]$$

(5.14)

for all invertible $x, y \in V$.

5.6.2. Suppose $V$ and $W$ are $a$-complete $a$-spaces over a $b$-complete semifield $K$ that does not coincide with the Boolean algebra $\{0, 1\}$ and $W$ is an $a$-subspace of $V$.

By $V^*$ denote the set of all $a$-linear functionals on $V$; this set forms an idempotent semimodule under pointwise operations. It follows from Theorem 5.2 that the sets of elements of $V$ and $V^*$ are in one-to-one correspondence; however the respective structures of semimodules over $K$ are different.

The following result follows from Definitions 4.3, 4.4, 4.8, and 4.11, Propositions 3.1 and 5.7, and Theorems 5.1 and 5.2.
Theorem 5.6. A functional \( y \mapsto f(y) \) on an \( a \)-complete \( a \)-space \( V \) is \( a \)-linear if and only if it has the form \( f(y) = [x, y] = x^*(y) \), where \( x \in V \). An idempotent semimodule \( V^* \) over \( K \) is an \( a \)-complete \( a \)-space. In addition,

\[
x_1^* \odot x_2^* = (x_1 \wedge x_2)^*, \quad k \odot x^* = (k^{-1} \odot x)^*, \quad 0_V^* = I_{V^*} = \sup V^*, \quad I_V^* = \sup V = 0_V^*
\]

and the canonical order on \( V^* \) is opposite to that on \( V \).

Define a map \( P: V \rightarrow W \) of the space \( V \) to its \( a \)-subspace \( W \) by the formula

\[
P(x) = \inf \{ w \in W \mid w \succ x \}.
\]

(5.15)

Proposition 5.9. The map \( P \) is an \( a \)-linear projection.

The proof of this statement is similar to that of Theorem 5.1.

Proposition 5.10. The subspace \( W \) is the set of all solutions to the system of equations

\[
[y, x] = [y, P(x)],
\]

where \( y \) runs over \( V \) and the projection \( P \) is defined by (5.15).

This statement is a straightforward consequence of Theorem 5.4.

Theorem 5.7. The map \( x \mapsto x^{**} = (x^*)^* \) is an isomorphism of \( a \)-spaces \( V \rightarrow V^{**} = (V^*)^* \).

This theorem follows from Theorem 5.6 since it can easily be checked that \( x^{**}(y^*) = y^*(x) \).

7. Examples

The semifield \( B(X) \) of all bounded real-valued functions on an arbitrary set \( X \) (see Examples 2.9.6 and 3.2.1) is a \( b \)-semialgebra over the semiring \( K = \mathbb{R}(\max, +) \).

In this case,

\[
1^*(\varphi) = \sup_{x \in X} \varphi(x) = \int_X \varphi(x) dx
\]

(5.16)

and the scalar product can be expressed in terms of idempotent integration (see the introduction, Sec. 1.2):

\[
\langle \varphi_1, \varphi_2 \rangle = \sup_{x \in X} \left( \varphi_1(x) \odot \varphi_2(x) \right) = \int_X \left( \varphi_1(x) + \varphi_2(x) \right) dx,
\]

(5.17)

where \( \varphi_1, \varphi_2 \in B(X) \).

Scalar products similar of the (5.17) type have been systematically used in Idempotent Analysis (see, e.g., [1–11, 14, 42–43, 49]) in specific spaces, while \( a \)-linear functionals on idempotent spaces (including spaces of \( (V, K) \) type defined in Sec. 4.5) can easily be described in terms of idempotent measures and integrals by means of Theorems 5.2, 5.5, and 5.6.

For example, the idempotent semiring of convex functions \( \text{Conv}(X, \mathbb{R}) \) is a \( b \)-subspace of the \( b \)-semialgebra \( \text{Map}(X, \mathbb{R}) \) over the semifield (and quasifield) \( K = \mathbb{R}(\max, +) \).

Any nonzero \( a \)-linear functional \( f \) on \( \text{Conv}(X) \) has the form

\[
\varphi \mapsto f(\varphi) = \sup_x \{ \varphi(x) + \psi(x) \} = \int_X \varphi(x) \odot \psi(x) dx,
\]

where \( \psi \) is a concave function, i.e., an element of the idempotent semiring

\[
\text{Conv}(X, \mathbb{R}) = - \text{Conv}(X, \mathbb{R}) \subset \text{Map}(X, \mathbb{R}),
\]

see Examples 2.9.9, 2.9.10, and 3.2.1.
6. COMMENTARY

6.1. When this paper was already finished, V. N. Kolokoltsov kindly called the authors’ attention to several early papers [55–59] and A. N. Sobolevskii pointed to the paper [36]. In [55–58, 36], elements of matrix and linear algebra over idempotent semirings were considered, including the eigenvector and eigenvalue problem [57]. In [36], the matrix calculus over $\mathbb{R}_{\min}$ was applied to optimization problems on graphs. In the remarkable introduction to his paper [57], N. N. Vorob’ev predicted the onset of Idempotent Mathematics as a far-reaching new field of mathematics. N. N. Vorob’ev called Idempotent Mathematics ‘extremal mathematics’ and idempotent semimodules ‘extremal spaces’. Developing N. N. Vorob’ev’s ideas, A. A. Korbut announced in [59] several results including a variant of the Hahn–Banach theorem (on linearity under finite combinations of elements) and a finite-dimensional analog of the Riesz–Fisher theorem. In the same framework, K. Zimmermann [60] presented a very general geometrical variant of the Hahn–Banach theorem. Recently, G. Cohen, S. Gaubert, and J.-P. Quadrat proved a separation theorem of the Hahn–Banach type for a point and a subsemimodule in the case of the finite-dimensional free semimodule $\hat{\mathbb{R}}_{\max}^n$ over the semiring $\hat{\mathbb{R}}_{\max}$ and described analogs of the basic notions of “Euclidean” geometry (see the paper [61], which was kindly sent to us by its authors). Note that the earliest paper on idempotent linear algebra that we know (with applications to the theory of finite automata) belongs to S. Kleene [15]. For a good survey of idempotent linear algebra, see [62]; see also [6–8, 11–13, 16–21].

The first stage of development of Idempotent Analysis is presented in the books [1, 2, 4] and the papers [3, 5]. The next stage, at which important results of Idempotent Analysis were obtained for some specific functional spaces, is represented by the books [6–8]. V. N. Kolokoltsov’s survey [63] describes aspects of applications of Idempotent Analysis.

In many papers (see, e.g., [64–66]), elements of Idempotent Mathematics appeared implicitly. For additional historical information and references see [8, 9, 12, 13, 19–21, 49, 52, 53].

6.2. Idempotent quantization (dequantization), which was discussed in the introduction, is related to logarithmic transformations that date back to the classical papers of E. Schrödinger [67] and E. Hopf [68]. The subsequent progress of E. Hopf’s ideas has culminated in the vanishing viscosity method (the method of viscosity solutions). This method was developed by P.-L. Lions and others for the study of solutions to first and second order PDE and the related problems [69–73]. A different approach, which leads to similar results, was considered in the framework of Idempotent Analysis (see, e.g., [3, 6–8]).

The Lax–Oleinik formula was introduced in [74, 75]; see also an important survey paper by A. I. Subbotin [72].

Recently O. Viro [76] described the result of idempotent dequantization of real algebraic geometry.

6.3. The main results presented in Sec. 5 of this paper were announced in [77], which was preceded by preliminary publications [78, 79]. The algebraic approach was applied in [80] to the construction of an idempotent version of topological tensor products and kernel operators in the spirit of Grothendieck [81]. Papers devoted to idempotent kernel spaces, eigenvector theorems, an idempotent version of the Schauder theorem, and the idempotent representation theory, with applications to the idempotent harmonic analysis, are now in preparation.

ACKNOWLEDGMENTS

The authors wish to express their thanks to V. N. Kolokoltsov for useful comments; they are especially grateful to A. N. Sobolevskii who read the text carefully and made a number of corrections.
The English translation was thoroughly examined and corrected by A. B. Sossinsky.
This research was supported by INTAS and the Russian Foundation for Basic Research (RFBR)
under the joint INTAS–RFBR grant no. 95-91 and the RFBR grant no. 99-01-00196.

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(V. P. Maslov) M. V. Lomonosov Moscow State University
(G. L. Litvinov, G. B. Shpiz) International Sophus Lie Center