ON COHOMOLOGY OF THE SQUARE OF AN IDEAL SHEAF

JONATHAN WAHL
University of North Carolina

Abstract. For a smooth subvariety $X \subset \mathbb{P}^N$, consider (analogously to projective normality) the vanishing condition $H^1(\mathbb{P}^N, \mathcal{I}_X^2(k)) = 0$, $k \geq 3$. This condition is shown to be satisfied for all sufficiently large embeddings of a given $X$, and for a Veronese embedding of $\mathbb{P}^n$.

For $C \subset \mathbb{P}^{g-1}$, the canonical embedding of a non-hyperelliptic curve, this condition guarantees the vanishing of some obstruction groups to deformations of the cone. Recall that the tangents to deformations are dual to the cokernel of the Gaussian-Wahl map.

Theorem. Suppose the Gaussian-Wahl map of $C$ is not surjective and the vanishing condition is fulfilled. Then $C$ is extendable: it is a hyperplane section of a surface in $\mathbb{P}^g$ not the cone over $C$.

Such a surface is a K3 if smooth, but it could have serious singularities.

Theorem. For a general curve of genus $\geq 3$, this vanishing holds.

Conjecture. If the Clifford index is $\geq 3$, this vanishing holds.

0. Introduction

Let $L$ be a very ample line bundle on a smooth complex projective variety $X$, giving an embedding $X \subset \mathbb{P}^N$. It is well-known that projective normality of the embedding (or normal generation of $L$) is equivalent to the vanishing

$$H^1(\mathbb{P}^N, \mathcal{I}_X(k)) = 0, \quad \text{all } k,$$

where $\mathcal{I}_X$ is the ideal sheaf defining $X$; further, all sufficiently high powers of $L$ are normally generated. In this paper, we shall be concerned with the condition on $L$ (or its embedding)

$$(*) \quad H^1(\mathbb{P}^N, \mathcal{I}_X^2(k)) = 0, \quad \text{all } k \neq 2.$$

(For $k=2$, Proposition 1.8 shows this group is frequently the kernel of the Gaussian map of $L$, hence is rarely 0.) This question arises naturally because these cohomology groups give the torsion submodule of the Kähler differentials of the affine cone over $X$ (Proposition 1.4). Our first main results are:
Theorem 2.1. \((*)\) holds for \(X = \mathbb{P}^n\) and any very ample \(L\).

Corollary 3.3. Given any \(X\) and very ample \(L\), all sufficiently high powers of \(L\) satisfy \((*)\).

Corollary 5.7. Let \(C \subset \mathbb{P}^{g-1}\) be the canonical embedding of a general non-hyperelliptic curve of genus \(g \geq 3\). Then \((*)\) holds.

Verification of \((*)\) for \(\mathbb{P}^n\) surprisingly turns out to be non-trivial; even the \(n = 1\) case requires some serious thought! One must deduce via representation theory of \(G = SL(n+1)\) and the methods of [W4] the surjectivity of

\[
\Gamma(I(k)) \rightarrow \Gamma(I/I^2(k)), \quad k \geq 3.
\]

The difficulty is that the second space is a reducible \(G\)-module, and only half of its irreducible constituents are obviously in the image. Corollary 3.3 is deduced from Theorem 2.1. In general, \((*)\) is difficult to verify; it holds for the Plücker embedding of a Grassmanian \(G(2,n+1)\), and for an embedding of a curve via a line bundle of degree \(\geq 2g + 3\) (J. Rathmann [R], unpublished).

The main point of Corollary 5.7 is to calculate for \(C\) which is pentagonal, i.e., with a base-point-free \(g_5\) (Theorem 5.3). Such a \(C\) sits naturally on a 4-dimensional rational scroll \(X\), for which \((*)\) must be proved. It is then a delicate calculation to “descend” this result to \(C\); the key point is that \((*)\) holds for 5 points in general position in \(\mathbb{P}^3\). As \((*)\) fails for the canonical embedding of a tetragonal curve or plane sextic, one can optimistically hope to prove the following

**Conjecture.** Let \(C\) be a curve of Clifford index \(\geq 3\). Then the canonical embedding of \(C\) satisfies \((*)\).

This paper is motivated by the question of whether a canonical curve \(C \subset \mathbb{P}^{g-1}\) is extendable, i.e., is a linear section of an \(X \subset \mathbb{P}^g\) which is not the cone over \(C\). Such an \(X\) is canonically trivial (c.t.): normal, \(\omega_X \cong O_X\), and \(h^1(O_X) = 0\) ([E], [W5]). If smooth, \(X\) is a K-3; but any normal quartic in \(\mathbb{P}^3\) is c.t. In [W2], extendability of \(C\) is studied via the deformation theory of the affine cone \(A\) over \(C\). Then \(T_A^1\), the module of first-order deformations, is non-trivial in degree \(-1\); by duality, this means that the Gaussian map

\[
\Phi_K : \wedge^2 \Gamma(C, K) \rightarrow \Gamma(C, K^{\otimes 3})
\]

is not surjective. Lifting a first-order deformation of \(A\) to a higher order requires control of the obstruction space \(T_A^2\) which, by local duality, is dual to torsion in the Kähler differentials. Noting the grading on these modules, and combining (1.6) with (6.4) yields the

**Theorem.** Let \(C \subset \mathbb{P}^{g-1}\) be a canonical curve satisfying \((*)\), and let \(A\) be the affine cone.

(a) If \(3 \leq g \leq 10, g \neq 9\), then \(T_A^2 = 0\).
(b) If \( g = 9 \) or \( g \geq 11 \), then \( T_A^2 \) is concentrated in degree \(-1\).

(The \( g = 9 \) exception is the non-injectivity of \( \Phi_K \), shown in [C-M] and reproved in (6.5.2) using work of S. Mukai). We use this to prove a partial converse to the non-surjectivity of the Gaussian for a K-3 curve:

**Theorem 7.1.** Let \( C \subset \mathbb{P}^{g-1} \) be a canonical curve satisfying (*). Then \( C \) is extendable iff the Gaussian \( \Phi_K \) is not surjective.

The canonical embedding of a smooth plane curve \( C \) of degree \( \geq 7 \) satisfies (*), with corank \( \Phi_K = 10 \), hence is extendable; we give an explicit description in (7.3), where a typical extension \( X \) has a non-smoothable simple elliptic singularity. Such a \( C \) sits on no K-3 surface, by [G-L] or [W5]. Because of the special geometry of c.t. surfaces, we offer the

**Conjecture.** A Brill-Noether-Petri general curve of genus \( \geq 8 \) sits on a K-3 surface if and only if the Gaussian is not surjective.

By (1.13), condition (*) is a natural one to consider for a line bundle already known to be normally presented (i.e., satisfying M. Green’s condition \( \langle N \rangle \) — projectively normal, with homogeneous ideal generated by quadrics). Cubics vanishing twice on \( C \) vanish on each secant line, hence on the secant variety \( \text{Sec}(C) \); so (*) should be related to \( \text{Sec}(C) \) being defined by cubics. Following work of Aaron Bertram [B] and Michael Thaddeus [T], one should study more generally for projectively normal \( C \subset \mathbb{P}^N \) the spaces

\[
H^0(\mathbb{P}^N, I_C^n(n + m)).
\]

The paper is organized as follows: Section 1 introduces the groups \( \text{H}^1(I^2(k)) \), relating them to cones and to Gaussian mappings; and (*) is compared with other nice properties of high powers of ample line bundles, specifically normal presentation. In Section 2, (*) is proved for all embeddings of \( \mathbb{P}^n \) and discussed for complex homogeneous spaces. How to “descend” the property (*) from a variety to a subvariety is discussed in §3: products and scrolls are considered in §4, allowing one to prove (*) for linear sections of a Segre embedding. Calculations for pentagonal and tetragonal curves are done in Section 5, using results about the scrolls on which they naturally sit (as described in [S]). What does or might occur for other canonical curves is discussed in §6, where the work of Mukai on curves with \( \geq 9 \) gives another approach to the condition (*). Section 7 describes the application to extendability of canonical curves.

This paper has benefitted from conversations with Rob Lazarsfeld and Michel Brion, as well as from careful comments from the referee. Research was partially supported by DMS-9302717.
1. A cohomological vanishing condition

(1.1) Let $X \subset \mathbb{P}^N$ be a (non-degenerate) projectively normal subvariety, with ideal sheaf $\mathcal{I} = \mathcal{I}_X$; thus, $H^1(\mathcal{I}(k)) = 0$, all $k$. There is an exact sequence

\begin{equation}
\Gamma(\mathcal{I}(k)) \to \Gamma(\mathcal{I}/\mathcal{I}^2(k)) \to H^1(\mathcal{I}^2(k)) \to 0.
\end{equation}

We study the additional condition

\begin{equation}
(\ast) \quad H^1(\mathcal{I}^2(k)) = 0, \quad k \geq 3.
\end{equation}

By (1.3.2), $H^1(\mathcal{I}^2(k)) = 0$ is nearly automatic for $k \leq 1$, but unusual for $k = 2$. One has the easy

**Proposition 1.2.** If $X \subset \mathbb{P}^N$ is a complete intersection, then

\[ H^1(\mathcal{I}^2(k)) = 0, \quad \text{all } k. \]

**Proof.** If the degrees of the defining equations are \{\(d_i\}\}, one has a surjection

\[ \bigoplus \mathcal{O}_\mathbb{P}(-d_i) \to \mathcal{I} \to 0; \]

tensoring with $\mathcal{O}_X$ induces an isomorphism

\[ \bigoplus \mathcal{O}_X(-d_i) \simeq \mathcal{I}/\mathcal{I}^2. \]

Since $X$ is projectively normal, $\Gamma(\mathcal{O}_\mathbb{P}(k-d_i)) \to \Gamma(\mathcal{O}_X(k-d_i))$ is surjective for all $i$. Now use (1.1.1).

(1.3) If $X \subset \mathbb{P}^N$ is non-singular and linearly normal, $L = \mathcal{O}_X(1)$, and $V = \Gamma(\mathcal{O}_\mathbb{P}(1)) = \Gamma(\mathcal{O}_X(1))$, there is a basic diagram

\[
\begin{array}{ccccccc}
0 & \to & \Omega^1_\mathbb{P} \otimes \mathcal{O}_X & \to & V \otimes \mathcal{O}_X(-1) & \to & \mathcal{O}_X & \to & 0 \\
\downarrow & & \downarrow & & \| & & \| \\
\mathcal{I}/\mathcal{I}^2 & \simeq & \mathcal{I}/\mathcal{I}^2 & \downarrow & \downarrow & \to & \mathcal{E} & \to & \mathcal{O}_X & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega^1_X & \to & \mathcal{E} & \to & \mathcal{O}_X & \to & 0
\end{array}
\]
The middle row is the Euler sequence of $\mathbb{P}^N$ restricted to $X$; it may be described intrinsically as a twist of
$$0 \to M_L \to \Gamma(L) \otimes \mathcal{O}_X \to L \to 0,$$
where the second map is evaluation; thus, $M_L = \Omega^1_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$ (cf. [L]). The bottom row is the extension corresponding to the class of the line bundle $L$ in $H^1(\Omega^1_X)$. It is clear that $\Gamma(\Omega^1_{\mathbb{P}^N}(k) \otimes \mathcal{O}_X) = \Gamma(M_L \otimes L^{k-1}) = 0$, $k \leq 1$, whence $\Gamma(I/I^2(k)) = 0$ in that range. From (1.1.1),

$$(1.3.2) \quad X \subset \mathbb{P}^N \text{ linearly normal and nonsingular implies } H^1(I^2(k)) = 0, \quad \text{all } k \leq 1.$$

**Proposition 1.4.** Let $X \subset \mathbb{P}^N$ be a non-singular projectively normal variety, with affine cone $A = \oplus \Gamma(X, \mathcal{O}_X(n))$. Let $\Omega^1_A = \oplus (\Omega^1_A)_k$ be the graded module of Kähler differentials of $A$. Then for all $k$,

$$(1.4.1) \quad (\text{Tors } \Omega^1_A)_k \cong H^1(I^2(k)).$$

**Proof.** Write $P = \oplus \Gamma(\mathcal{O}_P(k)), I = \oplus I_k = \oplus \Gamma(I(k))$; then $A = P/I = \oplus \Gamma(\mathcal{O}_X(k))$. The Kähler differentials of $A$ satisfy

$$I/I^2 \to \Omega^1_P \otimes A \to \Omega^1_A \to 0;$$

the first map is induced from exterior differentiation $d : I \to \Omega^1_P$, and its kernel is a torsion $A$-module ($A$ has an isolated singularity). The graded module $\Omega^1_P$ is canonically $V \otimes_{\mathbb{C}} P(-1)$, so

$$(\Omega^1_P \otimes A)_k \cong V \otimes_{\mathbb{C}} A_{k-1}.$$ 

The kth graded piece of $d(I/I^2)$ is the image of the composition

$$I_k = \Gamma(I(k)) \to \Gamma(I/I^2(k)) \subset \Gamma(\Omega^1_P \otimes \mathcal{O}_X(k)) \subset V \otimes A_{k-1}.$$ 

Letting $J_k$ denote the image of $I_k$ in $\Gamma(I/I^2(k))$, we thus have

$$(1.4.2) \quad 0 \to J_k \to \Gamma(I/I^2(k)) \to H^1(I^2(k)) \to 0.$$ 

Spec $A$ is obtained by collapsing the 0-section $X$ of the geometric line bundle associated to $L^{-1}$, $\pi : V \to X$. Write $U = V - X = \text{Spec}(A) - \{0\}$. Since $\pi^*\mathcal{E} \cong \Omega^1_U(\log X)$, (e.g., [W1], 3.3), one has

$$\Gamma(U, \Omega^1_A) = \Gamma(U, \Omega^1_U) = \oplus \Gamma(X, \mathcal{E}(k)), \quad \text{all } k \in \mathbb{Z}.$$
The local cohomology sequence for \( U \subseteq \text{Spec } A \) gives

\[
(T_{\text{tors }} \Omega^1_A)_k = \text{Ker}\{V \otimes \Gamma(O_X(k - 1))/J_k \to \Gamma(\mathcal{E}(k))\}.
\]

Thus, the inclusion

\[ V \otimes \Gamma(O_X(k - 1))/\Gamma(I/I^2(k)) \hookrightarrow \Gamma(\mathcal{E}(k)) \]

given by the middle column of (1.3.1), plus the sequence (1.4.2), produces (1.4.1).

**Remark (1.5.1).** Proposition (1.4) remains true for singular \( X \) if \( \Omega^1_X \) is reflexive and \( I/I^2 \) is torsion-free (e.g., \( X \) is a normal local complete intersection).

(1.5.2) Higher local cohomology of \( \Omega^1_A \) is computed more easily:

\[
H^i_{\{0\}}(\Omega^1_A)_k \cong \begin{cases} 
\text{Coker } \Phi(L, L^{k-1}) : \Gamma(\Omega^1_{\mathbb{P}^g} \otimes O_X(k)) \to \Gamma(\Omega^1_X(k)) & i = 1 \\
H^{i-1}(X, \mathcal{E}(k)) & i \geq 2
\end{cases}
\]

For the first assertion, see [W3],(1.3.5), plus (1.7) below for the definition of the Gaussian \( \Phi \); the second follows from computing \( H^{i-1}(U, \Omega^1_Y) \) as above. This yields the local cohomology of the \( A \)-module \( I/I^2 \), except for its torsion. Since the kernel of \( I_k \to J_k \) is \( \Gamma(\mathcal{I}^2(k)) \),

\[
\text{Tors } (I/I^2)_k = \Gamma(\mathcal{I}^2(k))/(I^2)_k,
\]

whence

\[
\text{Tors } (I/I^2)_3 = \Gamma(\mathcal{I}^2(3)).
\]

**Corollary 1.6.** Let \( C \subseteq \mathbb{P}^{g-1} \) be the canonical embedding of a non-hyperelliptic curve of genus \( g \geq 3 \), with \( A \) the coordinate ring of the affine cone. Then the graded pieces of the cotangent modules \( T^1_A \) and \( T^2_A \) are given by

(1.6.1) \( (T^1_A)_k \cong \text{Coker } \Phi(K, K^{-k})^* \)

(1.6.2) \( (T^2_A)_k \cong H^1(\mathcal{I}^2(1 - k))^* \).

In particular, if \( H^1(\mathcal{I}^2(k)) = 0, k \neq 2 \), then \( T^2 \) is concentrated in degree \(-1\), and is dual to the kernel of the Gaussian \( \Phi_K : \wedge^2 \Gamma(K) \to \Gamma(K^3) \).

**Proof.** The cone is a Gorenstein surface singularity, with dualizing module \( \omega_A = A(1) \). By local duality, one has ([W2], (3.8))

\[
(T^1_{1-k})^* \cong H^1_{\{0\}}(\Omega^1_A)_k
\]

\[
(T^2_{1-k})^* \cong H^0_{\{0\}}(\Omega^1_A)_k.
\]
Now use Proposition 1.4 above and Proposition 1.8 below.

(1.7). Let $L$ and $L'$ be two line bundles on an arbitrary smooth projective $X$. The diagonal filtration $\mathcal{R}_i(L, L')$ of $\Gamma(L) \otimes \Gamma(L')$ is defined by

$$\mathcal{R}_i(L, L') \equiv \Gamma(X \times X, \mathcal{I}_\triangle^i(L \boxtimes L'));$$

where $\mathcal{I}_\triangle$ is the ideal sheaf of the diagonal on $X \times X$. Since $\mathcal{I}_\triangle^i/\mathcal{I}_\triangle^{i+1} \cong S^i\Omega^1_X$, $\mathcal{R}_{i+1}$ is the kernel of a Gaussian map

$$\Phi_i(L, L') : \mathcal{R}_i(L, L') \to \Gamma(X, S^i\Omega^1_X \otimes L \otimes L'),$$

given by restriction to the diagonal on $X \times X$. Writing multiplication as $\mu \equiv \Phi_0$, and $\Phi \equiv \Phi_1$, one has

$$\mathcal{R}(L, L') \equiv \mathcal{R}_1(L, L') = \text{Ker } (\Phi(L, L') : \Gamma(L) \otimes \Gamma(L') \to \Gamma(L \otimes L')).$$

If $L$ is very ample (or more generally, generated by global sections), one has

$$\mathcal{R}(L, L') = \Gamma(X, M_L \otimes L'),$$

and the first Gaussian map is given by

$$\Gamma(X, M_L \otimes L') \to \Gamma(X, \Omega^1_X \otimes L \otimes L').$$

One concludes:

(1.7.1) $0 \to \text{Coker } \mu(L, L^{k-1}) \to H^1(M_L \otimes L^{k-1}) \to \Gamma(L) \otimes H^1(L^{k-1})$

(1.7.2) $\Gamma(I/I^2(k)) = \text{Ker } \Phi(L, L^{k-1}) = \mathcal{R}_2(L, L^{k-1})$

(1.7.3) $0 \to \text{Coker } \Phi(L, L^{k-1}) \to H^1(I/I^2(k)) \to H^1(M_L \otimes L^{k-1})$.

When $L = L'$, the diagonal filtration is stable by the natural involution of $\Gamma(L) \otimes \Gamma(L)$, so induces a filtration of both $S^2\Gamma(L)$ and $\wedge^2\Gamma(L)$ (viewed as subspaces); we set

$$\mathcal{R}_i^s = S^2\Gamma(L) \cap \mathcal{R}_i(L, L), \quad \mathcal{R}_i^a = \wedge^2\Gamma(L) \cap \mathcal{R}_i(L, L).$$

The Gaussians $\Phi_i$ are $(-1)^i$-symmetric, hence for odd $i$ vanish on symmetric tensors, and for even $i$ vanish on the alternating ones; therefore,

$$\mathcal{R}_{2i}^s = \mathcal{R}_{2i-1}^s \quad \text{and} \quad \mathcal{R}_{2i}^a = \mathcal{R}_{2i+1}^a.$$

In particular,

$$\mathcal{R}(L, L) = \mathcal{R}_1^s \oplus \mathcal{R}_1^a = I_2 \oplus \wedge^2\Gamma(L)$$

$$\mathcal{R}_2(L, L) = \mathcal{R}_2^s \oplus \mathcal{R}_2^a = I_2 \oplus \text{Ker } \Phi_L,$$

where $\Phi_L : \wedge^2\Gamma(L) \to \Gamma(\Omega^1 \otimes L^2)$ is the usual Gaussian map ($s \wedge t \mapsto sdt - tds$). Comparing the last decomposition with (1.7.2) for $k=2$, one identifies the cokernel of $I_2 \to \Gamma(I/I^2(2))$ with $\text{Ker } (\Phi_L)$. Thus,
Proposition 1.8. Suppose $X \subset \mathbb{P}^N$ is a nonsingular projectively normal subvariety, with $L = \mathcal{O}_X(1)$. Then

$$H^1(T^2_X(2)) \cong \ker (\Phi_L: \wedge^2 \Gamma(L) \to \Gamma(\Omega^1_X \otimes L^2)).$$

Remark (1.9). It is thus clear from a dimension count using Riemann-Roch that for $L$ sufficiently ample on a given $X$, $H^1(T^2(2)) \neq 0$.

(1.10) Recall there are natural functorial maps

$$\mathcal{R}_i(L, L') \otimes \Gamma(L'') \to \mathcal{R}_i(L, L' \otimes L''),$$

which commute with the Gaussians. We consider for a very ample $L$ the surjectivity of the induced maps

$$(1.10.1) \quad \mathcal{R}_i^p \otimes \Gamma(L^k) \to \mathcal{R}_i(L, L^{k+1}),$$

where $p = s$ (respectively $p = a$) if $p$ is even (resp. $p$ is odd). Since $\mathcal{R}_0^0 = S^2 \Gamma(L)$ and $\mathcal{R}_1^a = \wedge^2 \Gamma(L)$, the first two cases of (1.10.1) are covered by

Proposition 1.11. Let $L$ be a very ample line bundle on $X$. Then

(a) $L$ is normally generated iff

$$S^2 \Gamma(L) \otimes \Gamma(L^k) \to \Gamma(L) \otimes \Gamma(L^{k+1})$$

is surjective, $k \geq 1$.

(b) If $L$ is normally generated, then it is normally presented iff

$$\wedge^2 \Gamma(L) \otimes \Gamma(L^k) \to \mathcal{R}_1(L, L^{k+1})$$

is surjective, $k \geq 1$.

Proof. The map in (a) factors through $\Gamma(L) \otimes \Gamma(L) \otimes \Gamma(L^k)$, so the surjectivity of the map in (a) easily implies normal generation. For the converse, one reduces by restriction to the case $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$; and, the result for $k = 1$ will imply that for general $k$. One must therefore prove for a vector space $V$ the surjectivity of the composition

$$(1.11.1) \quad S^2 V \otimes V \subset V \otimes V \otimes V \rightarrow V \otimes S^2 V.$$ 

But an element of the kernel is symmetric with respect to the first two entries, antisymmetric with respect to the last two. The signature being the only non-trivial character of $S_3$, there could be no such vectors. The map in (1.11.1) is injective and thus surjective.

The proof for (b) is slightly easier. For $X \subset \mathbb{P}$, $L = \mathcal{O}_X(1)$, one examines the diagram

\[
\begin{array}{ccc}
\wedge^2 \Gamma(\mathcal{O}_\mathbb{P}(1)) \otimes \Gamma(\mathcal{O}_\mathbb{P}(k)) & \longrightarrow & \Gamma(\Omega^1_\mathbb{P}(k + 2)) \\
\downarrow & & \downarrow \\
\wedge^2 \Gamma(\mathcal{O}_X(1)) \otimes \Gamma(\mathcal{O}_X(k)) & \longrightarrow & \Gamma(\Omega^1_X \otimes \mathcal{O}_X(k + 2)).
\end{array}
\]
By (1.3) and (1.6), the horizontal maps are the maps $R^q_1(L) \otimes \Gamma(L^k) \to R^1(L, L^k)$. The top map is surjective for $k \geq 1$ (e.g. it is a non-0 equivariant map into an irreducible $G$-bundle, where $G = \text{Aut}(\mathbb{P})$). The left map is clearly surjective. The cokernel of the right vertical map is (by tensoring the Euler sequence)

$$H^1(\mathbb{P}, \Omega^1_2(k+2) \otimes L_X) \cong \text{Coker} \{ \Gamma(I_X(k+1)) \otimes \Gamma(O_\mathbb{P}(1)) \to \Gamma(I_X(k+2)) \}.$$

This cokernel vanishes for $k \geq 1$ iff $X \subset \mathbb{P}$ is ideal-theoretically generated by quadrics.

Remarks (1.12.1) For $i = 0$ or $1$, the maps in (1.10.1) are surjective iff they are surjective without the “p”. For $i = 1$, this condition for normal presentation is already found in [Mu].

(1.12.2) The previous result for $i = 0$ or 1 motivates the following

Proposition 1.13. Suppose $X \subset \mathbb{P}^N$ is smooth and normally presented. Then for $k \geq 1$,

$$H^1(I^2(k+2)) \cong \text{Coker} \{ R^q_2 \otimes \Gamma(L^k) \to R_2(L, L^{k+1}) \}.$$

Proof. Use (1.1.1), (1.7.2), and the maps

$$I_2 \otimes \Gamma(L^k) \to I_{k+2} \to \Gamma(I/I^2(k+2)).$$

(1.14) We state without proof an unpublished theorem of Jürgen Rathmann which allows calculation of $H^1(I^2(k))$ in the case of a curve $C$. On $C \times C \times C$, let $I_{12}$, $I_{13}$, $I_{23}$ be the ideal sheaves of the “partial diagonals”, i.e., the divisors for which the corresponding coordinates are equal; also, $\pi_1 : C \times C \times C \to C$ is projection.

Theorem 1.15 (Rathmann [R]). Let $C \subset \mathbb{P}^N$ be a projectively normal embedding of a smooth projective curve, ideal-theoretically defined by quadrics. Write $L = O_C(1)$. Then there is an exact sequence on $C$

$$0 \to \wedge^2 M_L \otimes L \to \pi_1^*(I_{12}I_{23}I_{13}(L \otimes L \otimes L)) \to I_2 \otimes L \to I/I^2(3) \to 0.$$

Corollary 1.16. Suppose $\deg L \geq 2g + 3$. Then the embedding defined by $L$ satisfies $H^1(I^2(k)) = 0$, all $k \geq 3$.

Proof. Use (1.13), (1.15), and a vanishing result

$$H^1(C \times C \times C, I_{12}I_{23}I_{13}(L_1 \otimes L_2 \otimes L_3)) = 0,$$

if $\deg L_i \geq 2g + 3$, proved along the general lines of [L].
2. Projective space and other $G/P$

**Theorem 2.1.** Let $X = \mathbb{P}^n \subset \mathbb{P}^N$ be the Veronese embedding of projective space defined by $L = \mathcal{O}_{\mathbb{P}^n}(r)$, where $r \geq 1$. Then

$$H^1(\mathbb{P}^N, \mathcal{T}^2(k)) = 0, \quad \text{for } k \neq 2.$$  

**Proof.** Bypass the trivial case $r = 1$. It is standard that these embeddings are projectively normal and normally presented. By (1.3.2), one may assume $k \geq 3$. By (1.13), it suffices to prove

$$\mathcal{R}_2^s \otimes \Gamma(L^k) \twoheadrightarrow \mathcal{R}_2(L, L^{k+1}), \quad \text{all } k \geq 1.$$  

In fact, we will prove by a descending induction on $i$ that

(2.2.1)  

$$\mathcal{R}_{2i}^s \otimes \Gamma(L^k) \twoheadrightarrow \mathcal{R}_{2i}(L, L^{k+1}), \quad \text{all } i, k \geq 1.$$  

(2.3) $\mathbb{P}^n$ is a homogeneous space for $G = SL(n+1)$; $G$ acts on all relevant spaces, and all natural maps are $G$-equivariant. One deduces, from known results or from ([W4], §3.4):

(2.3.1) $S^i\Omega^1_\mathbb{P}(j) = \pi_* L_{i,j}$, where $L_{i,j}$ is a line bundle on the flag manifold $G/B$, and $\pi$ is the projection $G/B \to \mathbb{P}^n$.

(2.3.2) $\Gamma(S^i\Omega^1_\mathbb{P}(j))$ is an irreducible $G$-module, non-0 iff $j \geq 2i$.

(2.3.3) For $j \geq 2i$, $j' \geq 2i'$, one has a surjection

$$\Gamma(S^i\Omega^1_\mathbb{P}(j)) \otimes \Gamma(S^{i'}\Omega^1_\mathbb{P}(j')) \twoheadrightarrow \Gamma(S^i\Omega^1_\mathbb{P} \otimes S^{i'}\Omega^1_\mathbb{P}(j+j'))$$

(2.3.4) $\mathcal{R}_i(L, L^k) = 0$ iff $i > r$ (recall $L = \mathcal{O}(r)$).

(2.3.5) For all $i \leq r$, $\Phi_i : \mathcal{R}_i(L, L^k)/\mathcal{R}_{i+1}(L, L^k) \cong \Gamma(S^i\Omega^1 \otimes L^{k+1})$ is an isomorphism of irreducible $G$-modules. In particular,

$$\mathcal{R}_2^s = I_2 \cong \bigoplus_{i \geq 1} \Gamma(S^{2i}\Omega^1 \otimes L^2)$$

$$\mathcal{R}_2(L, L^{k+1}) = \bigoplus_{i \geq 2} \Gamma(S^i\Omega^1 \otimes L^{k+2}).$$

While it is clear that

$$\Gamma(S^{2i}\Omega^1 \otimes L^2) \otimes \Gamma(L^k) \twoheadrightarrow \Gamma(S^{2i}\Omega^1 \otimes L^{k+2})$$

is surjective for $2i \leq r$ (2.3.2), the heart of the theorem is to prove that the “odd part” of $\mathcal{R}_2(L, L^{k+1})$ is also in the image.
(2.4) We consider the diagram

\[
\begin{array}{ccccccccc}
0 & \to & R_{2i+1}^s \otimes \Gamma(L^k) & \to & R_{2i}^s \otimes \Gamma(L^k) & \to & \Gamma(S^{2i} \Omega^1 \otimes L^2) \otimes \Gamma(L^k) & \to & 0 \\
\downarrow \alpha_i & & \downarrow \beta_i & & \downarrow \gamma_i & & & & \\
0 & \to & R_{2i+1}(L, L^{k+1}) & \to & R_{2i}(L, L^{k+1}) & \to & \Gamma(S^{2i} \Omega^1 \otimes L^{k+1}) & \to & 0.
\end{array}
\]

The top row is exact by (2.3.5) and the equality \( R_{2i+1}^s = R_{2i}^s \); the bottom, by (2.3.5). \( \gamma_i \) is surjective by (2.3.3). If \( 2i \geq r \), then \( R_{2i+1} = 0 \), so \( \beta_i \) is surjective.

Now assume that \( 2i < r \) and \( \beta_{i+1} \) is surjective; we prove \( \beta_i \) is surjective. By the snake lemma, it suffices to check the

Claim (2.5). \( \text{Ker}(\gamma_i) \to \text{Coker}(\alpha_i) \) is surjective for \( 0 < 2i < r \).

(2.6) First, since \( R_{2i+1}^s = R_{2i+2}^s \), \( \text{Im}(\alpha_i) = \text{Im}(\beta_{i+1}) = R_{2i+2}(L, L^{k+1}) \) (by induction), one has

\[
\text{Coker}(\alpha_i) \cong R_{2i+1}(L, L^{k+1})/R_{2i+2}(L, L^{k+1}) \cong \Gamma(S^{2i+1} \Omega^1 \otimes L^{k+2}).
\]

To identify the map in (2.5), we first define bundles \( M_k \) by

\[
0 \to M_k \to \Gamma(L^k) \otimes \mathcal{O}_X \to L^k \to 0.
\]

Then

\[
\text{Ker}(\gamma_i) = \Gamma(S^{2i} \Omega^1 \otimes L^2 \otimes M_k).
\]

Via the surjection \( M_k \to \Omega^1_X \otimes L^k \) (1.3.1), the map in (2.5) is the composition

\[
\Gamma(S^{2i} \Omega^1 \otimes L^2 \otimes M_k) \to \Gamma(S^{2i} \Omega^1 \otimes L^2 \otimes \Omega^1 \otimes L^k) \to \Gamma(S^{2i+1} \Omega^1 \otimes L^{k+2}).
\]

We assert that one has a surjection after composing this map at the beginning with

\[
\Gamma(S^{2i} \Omega^1 \otimes L^2(-1)) \otimes \Gamma(M_k(1)) \to \Gamma(S^{2i} \Omega^1 \otimes L^2 \otimes M_k).
\]

To accomplish this, note first that

\[
\Gamma(M_k(1)) \to \Gamma(\Omega^1 \otimes L^k(1))
\]

is the Gaussian on \( \mathbb{P}^n \) for \( \Phi(L^k, \mathcal{O}_P(1)) \), hence is surjective. Next, \( 2i < r \) implies \( 4i < 2r - 1 \); by (2.3.2),

\[
\Gamma(S^{2i} \Omega^1 \otimes L^2(-1)) \neq 0.
\]

Finally,

\[
\Gamma(S^{2i} \Omega^1 \otimes L^2(-1)) \otimes \Gamma(\Omega^1 \otimes L^k(1)) \to \Gamma(S^{2i+1} \Omega^1 \otimes L^{k+2})
\]

is surjective by (2.3.3). This completes the proof of the claim, and hence of Theorem 2.1.
Problem 2.7. Let $L$ be a very ample line bundle on a complex homogeneous space $G/P$. Does the corresponding projective embedding have the property of Theorem 2.1?

(2.8) This seems a difficult question in general. It is a theorem of B. Kostant that the embedding defined by $L$ is normally presented, and (3.3) below gives the desired property for $L^k$, where $k$ is large. There is one class of “easy” cases.

Proposition 2.9. Let $X = G/P \subset \mathbb{P}^N$ be an embedding of a complex homogeneous space for which $\mathcal{I}/\mathcal{I}^2$ is an irreducible $G$-bundle. Then

$$H^1(\mathcal{I}_X^2(k)) = 0, \quad \text{all } k.$$ 

Proof. For $k \geq 2$, the non-0 $G$-linear map $\Gamma(\mathcal{I}(k)) \to \Gamma(\mathcal{I}/\mathcal{I}^2(k))$ must be surjective, as the target space is an irreducible $G$-module.

(2.10) Recall ([W4], §4) that if $\lambda$ is a dominant weight for $G$, $L = L_\lambda$ the corresponding ample line bundle on $G/P$, there is a natural $P$-filtration $\{F_k\}$ of $\Gamma(G/P, L_\lambda)$, of length $\ell(\lambda)$. Now, $\mathcal{I}/\mathcal{I}^2$ as a $G$-bundle corresponds to the $P$-module $F_2$, the second piece of the filtration; its irreducibility therefore implies that $\ell(\lambda) = 2$. (We exclude the case $\ell(\lambda) = 1$, which occurs only for $(\mathbb{P}^n, \mathcal{O}(1))$.) It would appear that the following is a complete list of weights with $\ell(\lambda) = 2$ (cf. [W4], 4.7, although the entries for $B_n$ and $C_n$ were there incorrectly interchanged):

$$\begin{align*}
A_n &: \omega_2, \omega_{n-1}, \omega_1 + \omega_n, 2\omega_1, 2\omega_n \\
B_n &: \omega_1 \\
D_n &: \omega_1; \omega_3, \omega_4 (n = 4); \omega_4, \omega_5 (n = 5) \\
E_6 &: \omega_1, \omega_6
\end{align*}$$

Now, for the adjoint representation $\omega_1 + \omega_n$ of $A_n$, the space $F_2$ is not irreducible, although (2.9) still holds (via (4.5) below). One can check in every other case that $F_2$ is indeed an irreducible $P$-module; for instance, $\omega_1$ for $B_n$ and $D_n$ correspond to quadric hypersurfaces. We record carefully one other case.

Theorem 2.11. Let $X = G(2, n + 1) \subset \mathbb{P}^N$ be the Plücker embedding of the Grassmannian of lines in $\mathbb{P}^n$. Then $H^1(\mathcal{I}_X^2(k)) = 0, \text{ all } k$.

Proof. We show that $\mathcal{I}/\mathcal{I}^2$ is an irreducible $G$-bundle. One considers the $G$-module $\Gamma(L_\lambda) = \wedge^2(\mathbb{C}^{n+1})$ as a $P$-module, where $P$ is the parabolic which stabilizes a particular 2-dimensional subspace. As a module over a Levi component $L \cong SL(2) \times SL(n-1) \subset P$, it decomposes as a sum of three $L$-irreducibles

$$\wedge^2(\mathbb{C}^2 \oplus \mathbb{C}^{n-1}) \cong \wedge^2 \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^{n-1} \oplus \wedge^2 \mathbb{C}^{n-1}.$$ 

The third term must be $F_2$, so it is irreducible as a $P$-module.
Example 2.12. Let $X^6 = G(2, 5) \subset \mathbb{P}^9$ be the Grassmannian of lines in $\mathbb{P}^4$, which is defined by the Pfaffians of a generic skew-symmetric $5 \times 5$ matrix. For the affine cone $A = P/I$, $I/I^2$ is a Cohen-Macaulay $A$-module, as can be proved by the methods of this paper; but J. Herzog [H] has proved $I/I^2$ is always Cohen-Macaulay when $A$ is Gorenstein of codimension 3. That $\Gamma(I^2(3))$ is 0 (1.5.2) is also seen because such cubics vanish on the secant lines of $X$, which span $\mathbb{P}^9$.

(2.13) An unusual example in commutative algebra is also furnished from (2.10).

Theorem 2.14. Let $X \subset \mathbb{P}^{15}$ be the 10-dimensional homogeneous space corresponding to a half-spin representation $\omega_5$ of Spin(10) = $D_5$. Let $A = P/I$ be the homogeneous coordinate ring, which is Gorenstein of codimension 5. Then

(a) $I/I^2$ is a Cohen-Macaulay $A$-module.

(b) $\Omega^1_A$ is a Cohen-Macaulay $A$-module.

Proof. (Outline) The half-spin representation $\Gamma(O_X(1))$ is a Spin(10) representation arising from a 10-dimensional complex vector space with a non-degenerate symmetric form. If $V$ is a maximal isotropic subspace, the stabilizer has Levi component $SL(5)$, and the representation space decomposes as

$$\Lambda^0 V \oplus \Lambda^2 V \oplus \Lambda^4 V.$$ 

The conormal bundle $I/I^2$ corresponds to $\Lambda^4 V$, i.e., to $\omega_4$ of $SL(5)$; in particular, it is an irreducible $G$-bundle. But the $n$th symmetric power of $\Lambda^4 V$ is still an irreducible $SL(5)$-module, whence $S^n(I/I^2) \cong I^n/I^{n+1}$ is also an irreducible $G$-bundle. It is now a straightforward matter to prove the assertions, using the usual Bott vanishing theorems.

3. Descending to a subvariety

(3.1) Suppose $X \subset \mathbb{P}^N$ is smooth and projectively normal with property $(\ast)$: when does a subvariety $Y \subset X$ inherit this property?

Proposition 3.2. Let $X \subset \mathbb{P}^N$ be smooth and projectively normal, $k \geq 2$ an integer for which

(a) $\Gamma(I_X(k)) \twoheadrightarrow \Gamma(I_X/I_X^2(k))$ is surjective.

Let $Y \subset X$ be a subvariety, with ideal sheaf $J$. Assume that

(b) $H^1(X, J(1)) = 0$

(c) $H^1(X, J^2(k)) = 0$

(d) $H^1(X, I_X/I_X^2(k) \otimes J) = 0$.

Then for the embedding of $Y$ defined by the line bundle $O_Y(1)$,

$$\Gamma(I_Y(k)) \twoheadrightarrow \Gamma(I_Y/I_Y^2(k))$$ is surjective.
Proof. Let $I' = \text{Ker}(O_{\mathbb{P}^N} \to O_Y)$ be the ideal sheaf of $Y \subset \mathbb{P}^N$; then one has exact sequences

$$0 \to I_X \to I' \to J \to 0$$

and (tensoring with $O_Y$ and noting $I_X/I_X^2$ is locally free)

$$0 \to I_X/I_X^2 \otimes O_Y \to I'/I'^2 \to J/J^2 \to 0.$$ 

Consider now for each $k$ the diagram

$$
\begin{array}{cccccc}
0 & \to & \Gamma(I_X(k)) & \to & \Gamma(I'(k)) & \to & \Gamma(J(k)) & \to & 0 \\
& & \downarrow \alpha_k & & \downarrow \gamma_k & & \downarrow \delta_k \\
& & \Gamma(I_X/I_X^2(k)) & \to & \Gamma(I'/I'^2(k)) & \to & \Gamma(J/J^2(k)). \\
0 & \to & \Gamma(I_X/I_X^2(k) \otimes O_Y) & \to & \Gamma(I'/I'^2(k)) & \to & \Gamma(J/J^2(k)). \\
\end{array}
$$

The top row is exact by projective normality of $X \subset \mathbb{P}^N$. $\alpha_k$, $\beta_k$, and $\delta_k$ are surjective by hypotheses, hence so is $\gamma_k$.

By (b), $O_Y(1)$ embeds $Y$ as a non-degenerate subvariety of its linear span $H \subset \mathbb{P}^N$; let $I_Y = \text{Ker}(O_H \to O_Y)$. By projection of $\mathbb{P}^N$ off a subspace onto $H$, one can split the surjection $I' \to I_Y \to 0$, deducing that $I_Y/I_Y^2$ is a direct summand of $I'/I'^2$. One has thus a surjection

$$\Gamma(I'/I'^2(k)) \to \Gamma(I_Y/I_Y^2(k)).$$

Composing with $\gamma_k$ gives a surjective map which factors through $\Gamma(I_Y(k)) \to \Gamma(I_Y/I_Y^2(k))$, so this last map is also surjective.

**Corollary 3.3.** Let $Y$ be a (possibly singular) projective variety, $A$ a very ample line bundle. Then there is an $m_0$ so that $m \geq m_0$ implies $A^m$ satisfies condition $(\ast)$; i.e., for the corresponding embeddings into projective space,

$$H^1(I_Y^2(k)) = 0, \quad \text{all } k \geq 3.$$ 

Proof. Fix the embedding $Y \subset \mathbb{P}^n = X$ defined by $A$, with ideal sheaf $J$. Let $L = O_X(1)$, and $L' = L^m = O_X(m)$. We claim that there is an integer $m_0$ so that for $m \geq m_0$, the embedding of $X$ into $\mathbb{P}^N$ defined by $L'$ satisfies the conditions of Theorem 3.2, for all $k \geq 3$. By Theorem 2.1, (a) is automatically true for all $m$ and all $k \geq 3$. (b) and (c) are certainly true for all $m \geq m_1$ and all $k \geq 3$, since $J$ does not depend on $m$. 
To verify (d), denote by \( \pi_i : X \times X \rightarrow X \) the projection maps \( (i = 1, 2) \). Tensor the sequences

\[
0 \rightarrow \mathcal{I}_\Delta \rightarrow \mathcal{O}_{X \times X} \rightarrow \mathcal{O}_\Delta \rightarrow 0
\]

and

\[
0 \rightarrow \mathcal{I}_\Delta^2 \rightarrow \mathcal{I}_\Delta \rightarrow \mathcal{I}_\Delta / \mathcal{I}_\Delta^2 \rightarrow 0
\]

with \( \pi_2^*L' \), then apply \( \pi_1^* \); this yields

\[
M_{L'} \cong \pi_1^*(\pi_2^*L' \otimes \mathcal{I}_\Delta)
\]

\[
\mathcal{I}_X / \mathcal{I}_X^2(1) \cong \pi_1^*(\pi_2^*L' \otimes \mathcal{I}_\Delta^2)
\]

for the embedding of \( X \) given by \( L' \). Thus

\[
\mathcal{I}_X / \mathcal{I}_X^2(k) \cong \pi_1^*(\mathcal{I}_\Delta^2 \otimes (L'^{(k-1)} \boxtimes L')).
\]

Fix \( E \), a locally free sheaf on \( X \). We claim there is an \( m_2 \) so that \( m \geq m_2 \) implies

\[
H^i(X, \mathcal{I}_X / \mathcal{I}_X^2(k) \otimes E) = 0 \quad \text{for all } i > 0, \ k \geq 2, \ L' = L^m.
\]

By the projection formula, it suffices to compute

\[
H^i(X, \pi_1^*\{\mathcal{I}_\Delta^2(L^{m(k-1)} \boxtimes L^m) \otimes \pi_1^*E}\}).
\]

But the \( R^q\pi_1^* \) of the bracketed term vanishes for \( q > 0 \) and \( m \) large, so we may calculate the cohomology of the bracketed term on \( X \times X \). Thus, it suffices to show that for \( L \) ample and \( F \) coherent on \( X \times X \), there is a \( j_0 \) so that

\[
H^i(X \times X, F \boxtimes (L^j \boxtimes L^{j'})) = 0, \quad \text{all } i > 0, j, j' \geq j_0.
\]

This is proved in the usual way by induction on \( i \), writing \( F \) as a quotient of a direct sum of terms of the form \( L^s \boxtimes L^s \) and using the Künneth formulas plus standard vanishing theorems.

To prove a comparable vanishing for \( \mathcal{I}_X / \mathcal{I}_X^2(k) \otimes \mathcal{J} \), take a projective resolution of the ideal sheaf \( \mathcal{J} \) by locally free \( \mathcal{O}_X \)-modules.

**Corollary 3.4.** Let \( Y^n \subset \mathbb{P}^m \) be a (not necessarily smooth) complete intersection, defined by hypersurfaces of degrees \( 2 \leq d_1 \leq d_2 \leq \cdots \leq d_{m-n} \), where \( n > 0 \). Consider the embedding of \( Y \) defined by \( \mathcal{O}_Y(r) \), where \( 2r > d_{m-n} \). Then \( H^1(\mathcal{I}_Y^2(k)) = 0, \ k \geq 3 \).

**Proof.** Embed \( X = \mathbb{P}^m \) projectively normally via \( L = \mathcal{O}_\mathbb{P}(r) \) into some \( \mathbb{P}^N \). We verify the conditions of Proposition 3.2, noting the difference between \( L^k \) and
\( \mathcal{O}_X(k) = \mathcal{O}_{p_m}(k) \). (a) is fulfilled by Theorem 2.1, (b) is easy, and (c) is (1.2). For (d), we first deduce from (1.3.1) and from standard vanishing theorems on \( X = \mathbb{P}^m \) that

\[
H^i(X, \mathcal{I}_X/\mathcal{I}_X^2) = 0, \quad 2 \leq j < m, \quad \text{all } j;
\]

one must note that \( H^2(X, \mathcal{I}_X/\mathcal{I}_X^2) = 0 \) even though \( H^1(X, \Omega^1_X) \neq 0 \). Thus, by the Koszul resolution of \( \mathcal{J} = \text{Ker} (\mathcal{O}_X \to \mathcal{O}_Y) \), there is a surjection

\[
\oplus H^1(X, \mathcal{I}_X/\mathcal{I}_X^2 \otimes \mathcal{L}^k(-d_i)) \twoheadrightarrow H^1(X, \mathcal{I}_X/\mathcal{I}_X^2 \otimes \mathcal{L}^k \otimes \mathcal{J}).
\]

By (1.7.3), each summand on the left is of the form \( \text{Coker } \Phi(L, \mathcal{L}^k) \). Since the Gaussian of 2 positive line bundles on \( \mathbb{P}^m \) is surjective, we have the desired vanishing as long as \( \mathcal{L}^k \) is always positive, i.e., if \((k-1)r - d_i > 0\). In particular, this is true for all \( k \geq 3 \) and all \( i \) once \( 2r > d_m \).

(3.5) One may argue more directly when \( Y = X \cap H \) is a linear section by a codimension \( r \) subspace \( H \subset \mathbb{P}^N \). Thus, \( Y \) is an l.c.i. (local complete intersection) in \( X \) of codimension \( r \), and \( \Gamma(\mathcal{O}_X(1) \twoheadrightarrow \Gamma(\mathcal{O}_Y(1)) \).

**Proposition 3.6.** Let \( X^m \subset \mathbb{P}^N \) be smooth, projectively normal, and arithmetically Cohen-Macaulay. Suppose \( 0 < r < n \). Assume

(3.6.1) The Gaussians \( \Phi(\mathcal{O}_X(1), \mathcal{O}_X(k)) \) are surjective, all \( k \geq 1 \).

(3.6.2) \( H^i(\Omega^1_X(k-i)) = 0, \quad 1 \leq i \leq r-1, \quad k \geq 2 \).

(3.6.3) \( H^1(\mathcal{I}_X^2) = 0, \quad k \geq 3 \).

Let \( Y = X \cap H \) be a codimension \( r \) linear section. Then \( Y \) is projectively normal, and

\[
H^1(\mathcal{I}_Y^2) = 0, \quad k \geq 3.
\]

**Proof:** By hypothesis, \( H^i(\mathcal{O}_X(j)) = 0, \quad i \neq 0, n \). A resolution of \( \mathcal{J} = \text{Ker} (\mathcal{O}_X \to \mathcal{O}_Y) \) by \( \mathcal{O}_X \)-modules is given by restriction of the Koszul complex defining \( H \):

(3.6.4) \[
0 \to \mathcal{O}_X(-r) \to \cdots \to \mathcal{O}_X(-1)^r \to \mathcal{J} \to 0.
\]

Thus, \( H^1(\mathcal{J}(k)) = 0, \quad \text{all } k \), whence

\[
\Gamma(\mathcal{O}_X(k)) \twoheadrightarrow \Gamma(\mathcal{O}_Y(k)).
\]

It follows easily that \( Y \subset H \) is projectively normal.

Let \( \mathcal{I}_Y = \text{Ker} (\mathcal{O}_H \to \mathcal{O}_Y) \). \( Y \) is an l.c.i. in \( X, H, \) and \( \mathbb{P}^N \), and \( \mathcal{I}_X/\mathcal{I}_X^2 \) is locally free. So,

\[
\mathcal{I}_X \otimes \mathcal{O}_H \cong \mathcal{I}_Y \quad \mathcal{I}_X/\mathcal{I}_X^2 \otimes \mathcal{O}_Y \cong \mathcal{I}_Y/\mathcal{I}_Y^2.
\]
We thus have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{I}_X \otimes \mathcal{J} & \rightarrow & \mathcal{I}_X & \rightarrow & \mathcal{I}_Y & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{I}_X/\mathcal{I}_X^2 \otimes \mathcal{J} & \rightarrow & \mathcal{I}_X/\mathcal{I}_X^2 & \rightarrow & \mathcal{I}_Y/\mathcal{I}_Y^2 & \rightarrow & 0.
\end{array}
\]

Taking cohomology and using (3.6.3), the conclusion would follow from

\[H^1(X, \mathcal{I}_X/\mathcal{I}_X^2(k) \otimes \mathcal{J}) = 0, \ k \geq 3.\]

Via (3.6.4), it suffices to prove

\[(3.6.5) \quad H^i(X, \mathcal{I}_X/\mathcal{I}_X^2(k-i)) = 0, \ 1 \leq i \leq r, \ k \geq 3.\]

For \(i = 1\), this condition is

\[H^1(\mathcal{I}_X/\mathcal{I}_X^2(k)) = 0, \ k \geq 2,\]

which is a consequence of (3.6.1) via (1.7.3), (1.7.1), and the vanishing of \(H^1(\mathcal{O}_X(j))\).

For \(i \geq 2\), compute using the conormal sequence of \(X \subset \mathbb{P}^N\) and the Euler sequence for \(\Omega^1_{\mathbb{P}^N}\), restricted to \(X\). Then (3.6.5) is a consequence of (3.6.2).

**Corollary 3.7.** Let \(Z \subset \mathbb{P}^3\) be a 0-dimensional, length 5, arithmetically Gorenstein subscheme. Then

\[H^1(\mathcal{I}_Z^2(k)) = 0, \ k \geq 3.\]

**Proof.** By ([S], 4.2), \(Z\) is Pfaffian, with ideal sheaf satisfying

\[0 \rightarrow \mathcal{O}_\mathbb{P}(-5) \rightarrow \mathcal{O}_\mathbb{P}(-3)^5 \rightarrow \mathcal{O}_\mathbb{P}(-2)^5 \rightarrow \mathcal{I}_Z \rightarrow 0;\]

so, \(H^1(\mathcal{I}_Z(k)) = 0, \ k \geq 2.\) \(Z\) is a codimension 6 linear section of the Grassmannian \(X = G(2, 5) \subset \mathbb{P}^9\). Imitating the proof of Proposition (3.6), one may verify the desired property for \(Z\) by checking (3.6.1), (3.6.2), and (3.6.3) for \(r = n(= 6)\), plus also

\[(3.7.1) \quad H^n(\mathcal{O}_X(k-n)) = 0, \quad k \geq 2.\]

But (3.6.1) is in [W4], (3.6.3) is (2.11); (3.6.2) and (3.7.1) use standard calculations (note \(K_X \cong \mathcal{O}_X(-5))\).
4. Products and scrolls

(4.1) If $L_i$ and $L'_i$ are line bundles on projective varieties $X_i$, $i = 1, 2$, one has bundles $N = L_1 \boxtimes L_2$ and $N' = L'_1 \boxtimes L'_2$ on $X_1 \times X_2$. A simple computation shows

(4.1.1) $\mathcal{R}_1(N, N') = \mathcal{R}_1(L_1, L'_1) \otimes \Gamma(L_2) \otimes \Gamma(L'_2) + \Gamma(L_1) \otimes \Gamma(L'_1) \otimes \mathcal{R}_1(L_2, L'_2)$,

where the two spaces on the right have intersection

$$\mathcal{R}_1(L_1, L'_1) \otimes \mathcal{R}_1(L_2, L'_2).$$

If $L_1 = L'_1$ and $L_2 = L'_2$, so that $N = N'$, one has

(4.1.2) $\mathcal{R}_1^*(N, N) = \mathcal{R}_1^*(L_1, L_1) \otimes S^2 \Gamma(L_2) + S^2 \Gamma(L_1) \otimes \mathcal{R}_1^*(L_2, L_2)$

$$+ \wedge^2 \Gamma(L_1) \otimes \wedge^2 \Gamma(L_2).$$

This says that the quadratic equations of the embedding defined by $N$ come from the quadratic equations of $X_1$ and $X_2$ and from the quadratic equations of the Segre embedding (the third term).

(4.2) Returning to the general case, the Gaussian map on $X_1 \times X_2$ maps into the space

$$\Gamma(\Omega^1_{X_1} \otimes L_1 \otimes L'_1) \otimes \Gamma(L_2 \otimes L'_2) \oplus \Gamma(L_1 \otimes L'_1) \otimes \Gamma(\Omega^1_{X_2} \otimes L_2 \otimes L'_2).$$

Therefore, $\mathcal{R}_2(N, N')$ is spanned by three subspaces:

(4.2.1) $\mathcal{R}_2(N, N') = \mathcal{R}_2(L_1, L'_1) \otimes \Gamma(L_2 \otimes L'_2) + \Gamma(L_1 \otimes L'_1) \otimes \mathcal{R}_2(L_2, L'_2)$

$$+ \mathcal{R}_1(L_1, L'_1) \otimes \mathcal{R}_1(L_2, L'_2)$$

(the last space goes to 0 on each component of the image of the Gaussian).

**Proposition 4.3.** Suppose $L_i$ is normally presented on $X_i$ ($i = 1, 2$), and that

$$\mathcal{R}_2^*(L_i, L_i) \otimes \Gamma(L_i^k) \rightarrow \mathcal{R}_2(L_i, L_i^{k+1})$$

is surjective, $k \geq 1$.

Then the same properties hold for $N = L_1 \boxtimes L_2$ on $X_1 \times X_2$.

**Proof.** It is clear that $N$ is very ample and projectively normal; normal presentation is an easy exercise using (4.1). We check surjectivity of

$$\mathcal{R}_2^*(N, N) \otimes \Gamma(N^k) \rightarrow \mathcal{R}_2(N, N^{k+1}), \quad \text{all } k \geq 1.$$

But $\mathcal{R}_1^* = \mathcal{R}_2^*$ is given by (4.1.2), while (4.2.1) gives $\mathcal{R}_2$; it suffices to check surjectivity for each of the three types of terms which appear:

$$\mathcal{R}_1^*(L_1, L_1) \otimes S^2 \Gamma(L_2) \otimes \Gamma(L_1^k) \otimes \Gamma(L_2^k) \rightarrow \mathcal{R}_2(L_1, L_1^{k+1}) \otimes \Gamma(L_2^{k+2})$$

$$S^2 \Gamma(L_1) \otimes \mathcal{R}_1^*(L_2, L_2) \otimes \Gamma(L_1^k) \otimes \Gamma(L_2^k) \rightarrow \Gamma(L_1^{k+1}) \otimes \mathcal{R}_2(L_2, L_2^{k+1})$$

$$\wedge^2 \Gamma(L_1) \otimes \wedge^2 \Gamma(L_2) \otimes \Gamma(L_1^k) \otimes \Gamma(L_2^k) \rightarrow \mathcal{R}_1(L_1, L_1^{k+1}) \otimes \mathcal{R}_1(L_2, L_2^{k+1}).$$

The surjectivity of the first two maps follows easily from the hypotheses. The last map is surjective by (1.11.b).
Proposition 4.4. Let \( X = \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N \) be the Segre embedding, and \( Y = X \cap H \) a linear section of dimension \( > 0 \). Then \( Y \) is normally presented, and

\[ H^1(J_Y^2(k)) = 0, \quad \text{all } k \geq 3. \]

Proof. We verify the hypotheses of Proposition 3.6. \( X \subset \mathbb{P}^N \) is well-known to be projectively normal, arithmetically Cohen-Macaulay, and normally presented. The surjectivity of the Gaussians on \( X \) follows from that on each factor ([W3], 4.12). The vanishing of (3.6.2) is an exercise using the Künneth formula; note e.g. that

\[ H^i(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(a)) \otimes H^j(\mathbb{P}^m, \mathcal{O}(a)) = 0 \text{ for } 0 < i + j < m, \text{ unless } i = 1, j = 0, a = 0. \]

Finally, the vanishing \( H^1(J_Y^2(k)) = 0 \) for \( k \geq 3 \) is guaranteed by Proposition 4.3.

(4.5) Let \( e_1 \geq e_2 \geq \cdots \geq e_d \geq 0 \) be integers, \( \mathcal{E} = \oplus \mathcal{O}_\mathbb{P}(e_i) \) a vector bundle on \( \mathbb{P}^1 \), and \( \pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \) the corresponding \( \mathbb{P}^{d-1} \)-bundle. Write \( f = \sum e_i \geq 2 \). The normal scroll \( X(e_1, e_2, \ldots, e_d) \) is the image of \( \mathbb{P}(\mathcal{E}) \) in \( \mathbb{P}^r \) given by \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \equiv H; \) here \( r = f + d - 1 \). Assume that \( \mathbb{P}(\mathcal{E}) \cong X \subset \mathbb{P}^r \), i.e., all \( e_i > 0 \). \( \text{Pic}(X) \) is generated by \( H \) and \( R \), a fibre of \( \pi \).

It is well-known that \( X \subset \mathbb{P}^r \) is defined determinantly as

\[ rk\begin{pmatrix} x_{10} \cdots x_{1e_1 - 1} & x_{20} \cdots x_{de_d - 1} \\ x_{11} \cdots x_{1e_1} & x_{21} \cdots x_{de_d} \end{pmatrix} \leq 1, \]

where the \( x_{ij} \) are the \( \sum(e_i + 1) = f + d \) coordinates in \( \mathbb{P}^r \) (we use that \( e_i > 0 \)). But the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^{f-1} \) is similarly defined, as

\[ rk\begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1f} \\ z_{21} & z_{22} & \cdots & z_{2f} \end{pmatrix} \leq 1. \]

Therefore, \( X \subset \mathbb{P}^r \) may be viewed as a linear section of the Segre embedding. From the inclusion \( X \subset \mathbb{P}^1 \times \mathbb{P}^{f-1} \) followed by the projection maps, one has that \( H \) is the restriction of \( \mathcal{O}(1) \boxtimes \mathcal{O}(1) \) and \( R \) is the restriction of \( \pi_1^* \mathcal{O}(1) \). Proposition 4.4 implies

Proposition 4.6. Let \( X = X(e_1, \ldots, e_d) \subset \mathbb{P}^r \) be a smooth rational normal scroll. Then \( X \subset \mathbb{P}^r \) is normally presented, and

\[ H^1(J_X^2(k)) = 0, \quad k \geq 3. \]
5. Pentagonal and tetragonal curves

(5.1) Recall the results of F. Schreyer ([S], 6.7). Let $C \subset \mathbb{P}^{g-1}$ be a smooth canonical curve of genus $\geq 7$ with a base-point free $g^1_5$, i.e., a pentagonal curve. The union of the linear spans of divisors in the $g^1_5$ form a four-dimensional scroll $X(e_1, e_2, e_3, e_4) \subset \mathbb{P}^{g-1}$. Write $f = \sum e_i = g - 4$, and assume again that $e_4 > 0$, so that $X = \mathbb{P}(\mathcal{E})$. Note $\mathcal{O}_C(H) = K_C$, while $\mathcal{O}_C(R)$ is the $g^1_5$. Let $J = \text{Ker}(\mathcal{O}_X \to \mathcal{O}_C)$. Then there are integers $a_i, b_i$, at least $-1$, with $a_i + b_i = f - 2$ for $i = 1, 2, \ldots, 5$, and $\sum a_i = 2g - 12$, so that a resolution of $J$ as $\mathcal{O}_X$-module is given by

(5.1.1)

\[ 0 \to \mathcal{O}_X(-5H + (f - 2)R) \to \bigoplus \mathcal{O}_X(-3H + b_iR) \to \bigoplus \mathcal{O}_X(-2H + a_iR) \to J \to 0. \]

In fact, $C$ is the Pfaffian locus of a skew-symmetric map

(5.1.2)

\[ E^* \otimes E^* \to \mathcal{O}_X(-5H + (f - 2)R), \]

\[ E = \bigoplus_{i=1}^5 \mathcal{O}_X(3H - b_iR). \]

(5.2) Straightforward calculations give

(5.2.1)

\[ H^i(\mathcal{O}(jH + kR)) = \begin{cases} 0 & j = -1, -2, -3 \text{ or } i = 2 \\ \mathcal{H}^i(\mathbb{P}^1, S^j \mathcal{E}(k)) & j \geq 0 \\ \mathcal{H}^{4-i}(\mathcal{O}(-4 - j)H + (f - 2 - k)R) & j \leq -4. \end{cases} \]

(5.2.2)\[ H^0(J(jH + kR)) = 0 \text{ for } j \leq 1. \]

(5.2.3)\[ H^1(J(jH + kR)) = 0 \text{ for } j \geq 2, k \geq 0, \text{ or } j = 1, k \leq 1. \]

**Theorem 5.3.** Let $C \subset X(e_1, e_2, e_3, e_4)$ be a pentagonal curve as above, with all $e_i, a_i, b_i > 0$. Then for the canonical embedding $C \subset \mathbb{P}^{g-1}$, one has

\[ H^1(\mathcal{I}_C^g(k)) = 0, \quad k \geq 3. \]

**Proof.** We consider the embedding $C \subset X \subset \mathbb{P}^{g-1}$, and verify the hypotheses of Proposition 3.2. (a) is given by Proposition 4.6, and (b) is contained in (5.2.3).

To check condition (d),

(5.3.1)

\[ H^1(\mathcal{I}_X/\mathcal{I}_X^2(k) \otimes J) = 0, \quad \text{for } k \geq 3, \]

one needs via (5.1.1) the three vanishings

(5.3.2)

\[ H^1(\mathcal{I}_X/\mathcal{I}_X^2(jH + a_iR)) = 0, \quad j \geq 1 \]

\[ H^2(\mathcal{I}_X/\mathcal{I}_X^2(jH + b_iR)) = 0, \quad j \geq 0 \]

\[ H^3(\mathcal{I}_X/\mathcal{I}_X^2(jH + (f - 2)R)) = 0, \quad j \geq -2. \]
Recall the standard short exact sequences on $X$:

\[
\begin{align*}
(5.3.3) & \quad 0 \to \mathcal{I}_X/I_X^2 \to \Omega^1_I | X \to \Omega^1_X \to 0 \\
(5.3.4) & \quad 0 \to \Omega^1_I | X \to \Gamma(H) \otimes \mathcal{O}_X(-H) \to \mathcal{O}_X \to 0 \\
(5.3.5) & \quad 0 \to \pi^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_X \to \Omega^1_{X/\mathbb{P}^1} \to 0 \\
(5.3.6) & \quad 0 \to \Omega^1_{X/\mathbb{P}^1} \to \pi^*\mathcal{E}(-H) \to \mathcal{O}_X \to 0.
\end{align*}
\]

These 4 sequences and the vanishing consequences of (5.2) are used to attack (5.3.2). One also needs a few other easily verified facts:

(5.3.7) The multiplication map

\[
\mu(H, jH + kR) : \Gamma(H) \otimes \Gamma(jH + kR) \to \Gamma((j + 1)H + kR)
\]

is surjective for $j \geq 0, k \geq 0$. (Proof: push to $\mathbb{P}^1$.)

(5.3.8) The following map is surjective for $j \geq 0$, $k > 0$:

\[
\Gamma(S^j \mathcal{E} \otimes \mathcal{E}(k)) \to \Gamma(S^{j+1} \mathcal{E}(k)).
\]

(5.3.9) The Gaussian map

\[
\Phi(H, jH + kR) : \mathcal{R}(H, jH + kR) \to \Gamma(\Omega^1_X((j + 1)H + kR))
\]

is surjective for $j \geq 0, k > 0$. (Proof: push to $\mathbb{P}^1$.)

(5.3.10) One has natural isomorphisms on $\mathbb{P}^1$:

\[
\mathcal{O}_{\mathbb{P}^1} \cong R^1\pi_*\Omega^1_X \cong R^1\pi_*\mathcal{O}_X \cong R^1\pi_*\Omega^1_{X/\mathbb{P}^1}.
\]

(5.3.7)-(5.3.9) give surjectivity of some maps on global sections in twists of the sequences (5.3.4), (5.3.6), and (5.3.3). (5.3.10) is used to show that in (5.3.3),

\[
H^1(\Omega^1_E | X(kR)) \cong H^1(\Omega^1_X(kR)), \quad \text{all } k > 0.
\]

From these ingredients, the desired vanishing of (5.3.2) follows in a straightforward manner.

It remains to verify condition (c) of Proposition 3.2, namely

\[
H^1(X, J^2(k)) = 0, \quad \text{all } k \geq 3.
\]

First, a straightforward calculation using (5.1.1) and the conormal sequence for $C \subset X$ gives

\[
\chi(J(3H)) = \chi(J/J^2(3H)) = 10g - 35.
\]
whence
\[(5.3.12) \quad \chi(J^2(3H)) = 0.\]

One has a surjection from (5.1.1)
\[\bigoplus \mathcal{O}_C((k - 2)H + a_i R) \to J/J^2(kH) \to 0;\]
since \(a_i > 0\) and \(H\) restricted to \(C\) is \(K_C\), one has that
\[H^1(J/J^2(kH)) = 0, \quad k \geq 3.\]
The vanishing of \(H^i(J(kH)) = 0\) for \(i > 0, k \geq 3\) (use (5.1.1)), gives that
\[H^i(J^2(kH)) = 0, \quad i \geq 2, k \geq 3.\]
Combining with (5.3.12) yields
\[(5.3.13) \quad h^0(J^2(3H)) = h^1(J^2(3H)).\]
A fibre \(R \cong \mathbb{P}^3\) of \(\pi\) is a smooth divisor not containing \(C\), so
\[\text{Tor}_i(\mathcal{O}_C, \mathcal{O}_R) = 0, \quad i > 0,\]
and (5.1.1) restricted to \(R\) gives a projective resolution of \(\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^3}\), the ideal sheaf of a 0-dimensional subscheme \(Z\). Since \(\mathcal{J}/\mathcal{J}^2\) is locally free, one shows similarly that \(\mathcal{J}^2\) restricted to a fibre is \(\mathcal{J}^2\). A simple count shows
\[h^0(\mathcal{J}^2(3)) = h^1(\mathcal{J}^2(3));\]
but the second space is 0, by ([S], 4.2), and Corollary 3.7. Since this is true for any fibre, \(\Gamma(J^2(3H)) = 0\). By (5.3.13), we conclude
\[(5.3.14) \quad H^1(J^2(3H)) = 0.\]

Finally, we consider for \(k \geq 4\) the commutative diagram
\[
\begin{array}{ccc}
\bigoplus \Gamma(\mathcal{O}_X((k - 2)H + a_i R)) & \longrightarrow & \Gamma(J(kH)) \\
\downarrow & & \downarrow \\
\bigoplus \Gamma(\mathcal{O}_C((k - 2)H + a_i R)) & \longrightarrow & \Gamma(J/J^2(kH)).
\end{array}
\]
(5.3.15)
The top and left vertical maps are surjective for \(k \geq 4\) by easy calculation using (5.1.1) and (5.2). Tensoring (5.1.1) with \(\mathcal{O}_C\) gives an exact sequence
\[\bigoplus \mathcal{O}_C((k - 3)H + b_i R) \to \bigoplus \mathcal{O}_C((k - 2)H + a_i R) \to J/J^2(kH) \to 0.\]
Since \(H^1\) of the first term is 0 for degree reasons (\(b_i > 0\)), one concludes surjectivity in the bottom row of (5.3.15). Thus, the right vertical map is surjective, whence
\[(5.3.16) \quad H^1(J^2(kH)) = 0, \quad k \geq 4.\]
This completes the proof of the Theorem.
Corollary 5.4. For a general pentagonal curve $C$ of genus $g \geq 8$, the canonical embedding satisfies

$$H^1(\mathcal{I}_C^2(k)) = 0, \quad k \geq 3.$$  

Proof. For $g = 8$, the general curve lies on a scroll with all $e_i = 1$, $4$ $a_i$'s = 1, and one = 0 ([S], 7.1). But it is easy to check that all the needed vanishing arguments of Theorem 5.3 still hold in that case (basically because $\Gamma(E(-2)) = 0$).

Each $g \geq 9$ determines ordered sets of integers $\{e_i\}, \{a_i\},$ and $\{b_i\}$ which satisfy the appropriate equalities, are positive, and are balanced (the numbers in a set differ by at most 1). We claim there is a smooth connected pentagonal curve with these invariants; Theorem 5.3 then implies the Corollary. Let $X = \mathbb{P}(E)$. Following (5.1.2), let

$$E^* = \bigoplus_{i=1}^5 \mathcal{O}_X(-3H + b_iR), \quad L = \mathcal{O}_X(-5H + (f - 2)R);$$

we show that a general skew-symmetric map

$$\Psi : E^* \otimes E^* \to L$$

drops rank by 3 along a smooth connected curve. We apply the following useful observation of R. Lazarsfeld, but postpone the proof.

Lemma 5.4.1. Let $\mathbb{F} \to X$ be a globally generated rank $r$ vector bundle on a variety $X$, and $Z \subset \mathbb{F}$ a closed subvariety, with

$$\dim(\text{Sing } Z) < r.$$

Then for a general section $s \in \Gamma(\mathbb{F})$, the set

$$\{x \in X | s(x) \in Z\}$$

is either empty, or smooth of dimension $\dim(Z) - r$.

Returning to the proof of the Corollary, the map $\Psi$ is a section of

$$F = \wedge^2 E \otimes L = \bigoplus_{j \neq k} \mathcal{O}_X(H - (b_j - a_k)R).$$

Since $\mathcal{O}_X(H - kR)$ is globally generated if all $e_i \geq k$, one has global generation of $F$ if for all $i, j, k$ ($j \neq k$), one has

$$(5.4.2) \quad e_i \geq b_j - a_k.$$  

It is an exercise to show that the balanced invariants of $g \geq 9$ satisfy (5.4.2), except when $g = 10$ or 15 (adding 5 to $g$ increases the maximum of $b_j - a_k$ by 1, while the minimum $e_i$ goes up by at least 1). We exclude henceforth the special calculation
needed to handle \( g = 10 \) or 15. So, apply Lemma 5.4.1 to the rank 10 vector bundle associated to \( F \), where \( Z = \text{Pfaffian locus in } \mathbb{F} \) (where the rank of the skew-form drops by more than 1). \( \text{Sing}(Z) \) is the 0-section of \( \mathbb{F} \), hence has dimension 4. We conclude that for a generic skew-form \( \Psi \), either the rank drops by more than 1 along a smooth curve, or nowhere. The second case would give an exact sequence

\[
0 \to A \to E^* \to E \otimes L \to B \to 0,
\]

where \( A \) and \( B \) are line bundles on \( X \). Restricting to a fibre of \( \pi : \mathbb{P}(E) \to \mathbb{P}^1 \) would give an exact sequence on \( \mathbb{P}^3 \) (cf. (5.1.1))

\[
0 \to \mathcal{O}(-5) \to \mathcal{O}(-3) \oplus 5 \to \mathcal{O}(-2) \oplus 5 \to \mathcal{O} \to 0,
\]

which is clearly impossible (apply \( \Gamma \)). Thus, the cokernel of \( E^* \to E \otimes L \) is the ideal sheaf \( I_C \) of a smooth curve on \( X \); its connectedness follows because \( H^1(I_C) = 0 \), using (5.1.1).

**Proof of Lemma 5.4.1.** Let \( \rho : X \times \Gamma(\mathbb{F}) \to \mathbb{F} \), a surjection of vector bundles; thus, \( \bar{Z} \equiv \rho^{-1}(Z) \) and \( \text{Sing}(\bar{Z}) = \rho^{-1}(\text{Sing}(Z)) \) have predicted dimensions. Let \( \pi : \bar{Z} \to \Gamma(\mathbb{F}) \) be the projection. The hypothesis implies \( \dim \text{Sing}(\bar{Z}) < \dim \Gamma(\mathbb{F}) \); so for some open \( U \subset \Gamma(\mathbb{F}) \), \( \pi^{-1}(U) \cap \text{Sing}(\bar{Z}) = \emptyset \). Apply the theorem on generic smoothness to \( \pi : \pi^{-1}(U) \to U \). A fibre over \( s \in \Gamma(\mathbb{F}) \) is

\[
\{(x, s) \mid s(x) \in Z\} \cong s(X) \cap Z.
\]

If \( \pi \) is dominant, the general fibre is smooth, of dimension

\[
\dim \bar{Z} - \dim \Gamma(\mathbb{F}) = \dim Z - r.
\]

If \( \pi \) is not dominant, the general fibre is empty.

(5.5) Better known than pentagonal curves are the tetragonal ones, with a base-point-free \( g^1_4 \) ([S], 6.2). In this case, there is a 3-dimensional scroll \( X \) defined by \( e_i \)'s which are positive unless the curve is bi-elliptic (i.e., a double cover of an elliptic curve); \( C \) is the complete intersection on \( X \) of divisors of the form \( 2H - b_iR \), where \( 0 \leq b_1 \leq b_2 \), and \( b_1 + b_2 = f - 2 = \sum e_i - 2 = g - 5 \). Unless \( C \) is bielliptic or lies on a del Pezzo surface, one has \( b_1 > 0 \). An argument nearly identical to the one in Theorem 5.3 yields the following, whose proof is omitted:

**Theorem 5.6.** Let \( C \) be a tetragonal curve, with no \( g^1_2 \), \( g^1_3 \), or \( g^5_2 \), and which is not bielliptic (i.e., for which the scroll invariants \( e_i > 0 \)), with \( g \geq 6 \). Then for the canonical embedding of \( C \),

\[
H^1(I_C^2(k)) = 0, \quad k \geq 4
\]

and

\[
\dim H^1(I_C^2(3)) = \begin{cases} 
  g - 7 & b_1 > 0 \\
  2(g - 6) & b_1 = 0.
\end{cases}
\]
Corollary 5.7. Let \( C \subset \mathbb{P}^{g-1} \) be a general canonical curve of genus \( \geq 3 \). Then

\[
H^1(I_C^2(k)) = 0, \quad k \geq 3.
\]

Proof. Corollary 5.4 covers \( g \geq 8 \); Theorem 5.6 covers \( g = 6 \) and \( 7 \); and the general canonical curve of genus \( 3, 4, \) or \( 5 \) is a complete intersection, whence (1.2) applies.

6. Vanishing for other canonical curves

(6.1) We are interested in determining which canonical curves \( C \subset \mathbb{P}^{g-1} \) have the property

\[
(*) \quad H^1(I_C^2(k)) = 0, \quad k \geq 3.
\]

It follows from Theorem 5.3 that the general pentagonal curve of genus \( \geq 8 \) has this property. On the other hand, by Theorem 5.6 this condition fails for every tetragonal curve with \( g \geq 8 \). It seems reasonable to make the following

Conjecture 6.2. Every canonical curve \( C \subset \mathbb{P}^{g-1} \) of Clifford index \( \geq 3 \) satisfies

\[
H^1(I_C^2(k)) = 0, \quad k \geq 3.
\]

Remarks (6.3.1) A hyperplane section of a \( K3 \) surface is a canonical curve, and any sufficiently positive section has the property \((*)\), by (3.3).

(6.3.2) Any complete intersection curve in \( \mathbb{P}^n \) satisfies \((*)\), except for the following cases (3.4): a plane curve of degree \( \leq 6 \); or a space curve with \( d_1 = 2, \ d_2=2, \ 3, \) or \( 4 \) (all of which have a \( g^1_4 \)).

(6.3.3) According to a theorem of Schreyer and Voisin (e.g., [V]), a curve of Clifford index \( \geq 3 \) is normally presented, with first syzygies generated by linear ones. It follows easily (cf. [W2], 2.9.2) that \((T^2)_{-k} = 0 \) for \( k \geq 4 \). By (1.6.2), one concludes \( H^1(I_C^2(k)) = 0, \) all \( k \geq 5 \). Conjecturally, \( \text{Cliff}(C) \geq 4 \) implies the syzygies are linear one further step; [W2] would then imply also that \( H^1(I_C^2(4)) = 0 \).

(6.3.4) This vanishing property is not as delicate as the question of the corank of the Gaussian \( \Phi_K : \wedge^2 \Gamma(K) \to \Gamma(K^{\otimes 3}) \), which is not surjective for any \( K3 \) curve [W2].

Theorem 6.4. Let \( C \subset \mathbb{P}^{g-1} \) be a general canonical curve, \( 3 \leq g \leq 10, \ g \neq 9 \). Let \( A \) be the affine cone over \( C \). Then \( T_A^2 = 0 \).

Proof. For \( 3 \leq g \leq 5 \), \( C \) (and hence \( A \)) is a complete intersection, so \( T^2 = 0 \); so assume \( g \geq 6 \). By Corollary 1.6, one must show \( H^1(I_C^2(k)) = 0, \) all \( k \geq 2 \). Now, vanishing for \( k \geq 3 \) follows from Corollary 5.10. By (1.8), it remains to show that the Gaussian \( \Phi_K \) is injective for a general curve of genus \( 6, 7, 8, \) or \( 10 \). But this result has been observed in [C-M].

Remarks (6.5.1) The above argument implies that for \( C \) of genus \( 9 \) or \( \geq 11 \), \( T^2 \) must be non-zero in weight \(-1\).
(6.5.2) Mukai proves [Mk] that a general curve $C$ of genus 6, 7, 8, or 9 is cut out in a simple way from an appropriate homogeneous space $X_g = G/P \subset \mathbb{P}^{n(g)}$. In the first 3 cases, one checks $\mathcal{I}_X/\mathcal{I}_X^2$ is irreducible (2.10), whence $H^1(\mathcal{I}_X^2(k)) = 0$, all $k$. As in §3, one deduces $H^1(\mathcal{I}_C^2(k)) = 0$, all $k$, $C$ general, of genus 6, 7, or 8. When $g = 9$, $C$ is a linear section of $X_9 \subset \mathbb{P}^{13}$, corresponding to the weight $\omega_3$ of $Sp(3) = C_3$. The Gaussian of $X$ is surjective by [K], so a dimension count shows it has a one-dimensional kernel; since

$$H^1(\mathcal{I}_X^2(2)) \subset H^1(\mathcal{I}_Y^2(2))$$

for any linear section $Y$,

the Gaussian of a genus 9 curve cannot be injective (as proved in [C-M]).

7. Extendability of canonical curves

**Theorem 7.1.** Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of genus $g \geq 8$, with $\text{Cliff}(C) \geq 3$. Suppose

$$(*) \quad H^1(\mathbb{P}^{g-1}, \mathcal{I}_C^2(k)) = 0, \; k \geq 3.$$ 

Then $C$ is extendable if and only if the Gaussian $\Phi_K$ is not surjective.

**Proof.** One implication is well-known [W2], even without $(*)$; we prove the converse. The affine ring of the cone is $A = \bigoplus \Gamma(C, K_C^{\otimes n}) = P/I$, where $P = \mathbb{C}[z_1, \ldots, z_g]$; since $\text{Cliff}(C) \geq 3$, $I$ is generated by quadratic equations, with relations generated by linear ones (6.3.3). By the non-surjectivity of $\Phi_K$, one has a non-trivial first-order deformation of weight -1 of $A$; we show it lifts to a deformation over $\mathbb{C}[t]$, i.e., a flat graded map

$$\mathbb{C}[t] \to A = P[t]/\mathcal{I},$$

where $t$ has degree 1. Then $X = \text{Proj}(A) \subset \mathbb{P}^g$ is an extension of $X \cap \{t = 0\} = C$; it is not the cone over $C$ since the deformation is non-trivial to first-order.

We follow the approach of [St], §2. Write a minimal presentation of $I$:

$$P^\ell \xrightarrow{r} P^k \xrightarrow{f} P \to P/I = A \to 0,$$

where $r$ and $f$ correspond to matrices whose entries are homogeneous of degree respectively 1 and 2. The module of relation is $\mathcal{R} = \text{Im}(r) \cong P^\ell/\text{Ker}(r)$; let $\mathcal{R}_0 \subset \mathcal{R}$ be the submodule of “trivial” (or Koszul) relations. One has

$$T_A^2 = \text{Coker}(r^t : \text{Hom}((P/I)^k, P/I) \to \text{Hom}(\mathcal{R}/\mathcal{R}_0, P/I)),$$

which we assume by $(*)$ and (1.6) to be 0 in degree $\leq -2$. This means (cf [W2], (2.9.2)) that

$$(**) \quad \text{a map } \mathcal{R}/\mathcal{R}_0 \to A \text{ sending homogeneous generators of } \mathcal{R} \text{ to homogeneous elements of degree} \leq 1 \text{ lifts to } (P/I)^k \to A.$$
A deformation of weight $-1$ over $\mathbb{C}[\epsilon]/\epsilon^2$ is given by

$$F = f + \epsilon f^{(1)},$$

where $f^{(1)} : P^k \to P$ is a matrix of linear forms in $P$. By flatness, there is a lifting of relations

$$R = r + \epsilon r^{(1)},$$

where $r^{(1)} : P^\ell \to P^k$, a matrix of constants, satisfies $F \cdot R = 0 \mod \epsilon^2$. Since $f \cdot r = 0$, (7.1.2)

$$f^{(1)}r + fr^{(1)} = 0.$$

This simply means that $f^{(1)} : P^k \to P$ vanishes mod $I$ on $\text{Im}(r)$, inducing a map $I \to P/I$. Try to lift the deformation and relation to $\mathbb{C}[\epsilon]/\epsilon^3$:

(7.1.3)

$$F = f + \epsilon f^{(1)} + \epsilon^2 f^{(2)}$$

$$R = r + \epsilon r^{(1)}.$$

One seeks a matrix $f^{(2)}$ of constants; for homogeneity reasons, one could not perturb further the relations. That $F \cdot R = 0 \mod \epsilon^3$ requires (7.1.4)

$$f^{(1)}r^{(1)} + f^{(2)}r = 0.$$

Now, $f^{(1)}r^{(1)} : P^\ell \to P$ induces

$$P^\ell \to P \to P/I,$$

which we claim vanishes on $\text{Ker}(r)$: If $r(\alpha) = 0$, then by (7.1.2) $f(r^{(1)}\alpha) = 0$. As $\ker(f) = \text{Im}(r)$, there’s a $\beta \in P^\ell$ with

$$r^{(1)}\alpha = r\beta.$$

Thus,

$$f^{(1)}r^{(1)}\alpha = f^{(1)}r\beta = -fr^{(1)}\beta \in I,$$

verifying the claim. Thus, $f^{(1)}r^{(1)}$ induces a homomorphism

$$P^\ell/\text{Ker}(r) \cong \mathcal{R} \to P/I.$$

The map vanishes on $\mathcal{R}_0$, as these relations involve the entries of $f$; one thus has a homomorphism

$$\mathcal{R}/\mathcal{R}_0 \to P/I,$$
sending generators to elements of degree 1 (=deg of entries of \(f^{(1)}r^{(1)}\)). By (**), this lifts to a map
\[ g : (P/I)^k \to P/I, \]
satisfying
\[ gr \equiv f^{(1)}r^{(1)} \mod I. \]
Since these matrices have linear entries and \(I\) is generated by quadrics, one may lift \(-g\) to a map
\[ f^{(2)} : P^k \to P, \]
satisfying (7.1.4).
We assert that the equations in (7.1.3) satisfy \(F \cdot R = 0 \mod \text{any power of } \epsilon\), i.e., define a one-parameter flat deformation. It suffices to show
\[ f^{(2)}r^{(1)} = 0. \]

As before, \(f^{(2)}r^{(1)}\) induces a map
\[ P^\ell \to P/I, \]
which we claim vanishes on \(\text{Ker}(r)\): If \(r(\alpha) = 0\), as above there is a \(\beta \in P^\ell\) with
\[ r^{(1)}(\alpha) = r(\beta). \]
By (7.1.2) and (7.1.4), \(0 = f^{(2)}r(\alpha) = -f^{(1)}r^{(1)}(\alpha) = -f^{(1)}r(\beta) = fr^{(1)}(\beta),\)
whence there is a \(\gamma \in P^\ell\) with
\[ r^{(1)}(\beta) = r(\gamma). \]
So,
\[ f^{(2)}r^{(1)}(\alpha) = f^{(2)}r(\beta) = -f^{(1)}r^{(1)}(\beta) \text{ (by (7.1.4))} = -f^{(1)}r(\gamma) = fr^{(1)}(\gamma) \in I. \]
Again, \(f^{(2)}r^{(1)}\) induces a map
\[ R/R_0 \to P/I, \]
sending generators to constants. By hypothesis, this lifts to a map \(g : P^k \to P/I\) satisfying
\[ gr \equiv f^{(2)}r^{(1)} \mod I. \]
As the entries of \(r\) have degree 1, while \(f^{(2)}r^{(1)}\) has degree 0, one concludes that
\[ f^{(2)}r^{(1)} = 0, \]
as desired to complete the proof.

Remark 7.2: One has actually proved a general statement about deformations: Suppose \(A = P/I\) is a graded \(\mathbb{C}\)-algebra, where \(I\) is generated by quadrics, and the relations are generated by linear ones. Suppose \(T^2_A\) is 0 in degree \(\leq -2\). Then any first-order deformation of \(A\) of negative weight lifts to a graded negative weight deformation over \(\mathbb{C}[t]\).
Example 7.3. Consider the canonical embedding of a smooth plane curve $C$ of degree $d \geq 7$. Then $(\ast)$ is satisfied, $\text{corank}(\Phi_K) = 10$, $C$ sits on no K-3 surface, but $C$ is extendable to a singular rational surface with a non-smoothable simple elliptic singularity of degree $(3d - 9)$.

Proof. $(\ast)$ is given by Corollary 3.4, since $2(d - 3) > d$. The corank of $\Phi$ is well known to be $\dim \Gamma(-K_P) = 10$. That $C$ does not lie on a K-3 was proved in [G-L], but also follows from the deformation point of view [W5]. By Theorem 7.1, $C \subset \mathbb{P}^{g-1}$ is extendable, which one sees explicitly as follows:

Choose a smooth cubic $D$ intersecting $C$ transversally in $Z$, a set of $3d$ points. Consider the linear system $|I_Z(d)|$ of degree $d$ curves through $Z$; it contains $C$ and $D + E$, where $E$ is any degree $(d - 3)$ curve, hence has dimension $g = (d - 1)(d - 2)/2$. One easily checks that the linear system has no base-points off $Z$, and off $D$ separates points and tangent vectors. It thus defines a morphism from $B = Bl_Z(\mathbb{P}^2)$ into $\mathbb{P}^g$, with image a surface $X$ for which $C$ is a hyperplane section. Since $C$ is projectively normal, $X$ is normal and in fact is canonically trivial. The proper transform $\tilde{D}$ of $D$ in $B$ is an elliptic curve of self-intersection $-(3d - 9)$, which gets collapsed to a point in $X$; this point is a simple elliptic singularity, known to be non-smoothable when the degree $3d - 9 > 9$ ([P]). In fact this non-smoothability for $d > 6$ is used in [W5] to prove that $C$ is not in the closure of curves lying on a K-3 surface.

Bibliography

[B] A. Bertram, Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space, J. Diff. Geom. 35 (1992), 429–469.
[C-M] C. Ciliberto and H. P. Miranda, On the Gaussian map for canonical curves of low genus, Duke Math. J. 61 (1990), 417–443.
[E] D. H. J. Epema, Surfaces with canonical hyperplane sections, Indagationes Math. 45 (1983), 173–184.
[G-L] M. Green and R. Lazarsfeld, Special divisors on curves on a K3 surface, Invent. Math. 89 (1987), 357–370.
[H] J. Herzog, Ein Cohen-Macaulay Kriterium mit Anwendungen auf den Konormalenmodul und den Differentialmodul, Math Z. 163 (1978), no. 2, 149–162.
[K] S. Kumar, Proof of Wahl's conjecture on surjectivity of the Gaussian map for flag varieties, Amer. J. Math. 114 (1992), 1201–1220.
[L] R. Lazarsfeld, A sampling of vector bundle techniques in the study of linear series, Lectures on Riemann Surfaces, (M. Cornalba et al, eds.), World Scientific Press, Singapore, 1989, pp. 500–559.
[Mk] S. Mukai, Curves, K3 surfaces, and Fano 3-folds of genus $\leq 10$, Algebraic Geometry and Commutative Algebra, Vol. I, Kinokuniya, Tokyo, 1988, pp. 357–377.
[Mu] D. Mumford, Varieties defined by quadratic equations, Questions On Algebraic Varieties, (C.I.M.E. 1969), Corso, Rome, 1970, pp. 30–100.
[P] H. Pinkham, Deformations of algebraic varieties with $G_m$-action, Astérisque 20 (1974), 1–131.
[R] J. Rathmann, An infinitesimal approach to a conjecture of Eisenbud and Harris, preprint.
[S] F.-O. Schreyer, Syzygies of canonical curves and special linear series, Math. Ann. 275 (1986), 105–137.
[St] J. Stevens, Computing versal deformations, preprint (1995).
[T] M. Thaddeus, Stable pairs, linear systems, and the Verlinde formula, Invent. Math. 117 (1994), 317–353.

[V] C. Voisin, Courbes tetragonales et cohomologie de Koszul, J. Reine Angew. Math. 387 (1988), 111–121.

[W1] J. Wahl, Equisingular deformations of normal surface singularities, I, Ann. Math. 104 (1976), 325–356.

[W2] ———, The Jacobian algebra of a graded Gorenstein singularity, Duke Math. J. 55 (1987), 843–871.

[W3] ———, Gaussian maps on algebraic curves, J. Diff. Geom. 3 (1990), 77–98.

[W4] ———, Gaussian maps and tensor products of irreducible representations, Manuscripta Math. 73 (1991), 229–260.

[W5] ———, Curves on canonically trivial surfaces, to appear.

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250

E-mail address: jw@math.unc.edu