SYSTEMATIC ERRORS IN THE HUBBLE CONSTANT MEASUREMENT FROM THE SUNYAEV-ZEL’DOVICH EFFECT

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ABSTRACT

The Hubble constant estimated from the combined analysis of the Sunyaev-Zel’dovich effect and X-ray observations of galaxy clusters is systematically lower than estimates from other methods by 10%–15%. We examine the origin of the systematic underestimate using an analytic model of the intracluster medium (ICM), and compare the prediction with idealistic triaxial models and with clusters extracted from cosmological hydrodynamic simulations. We identify three important sources for the systematic errors: density and temperature inhomogeneities in the ICM, departures from isothermality, and asphericity. In particular, the combination of the first two leads to the systematic underestimate of the ICM spectroscopic temperature relative to its emission-weighted one. We find that these three systematics reproduce well both the observed bias and the intrinsic dispersions of the Hubble constant estimated from the Sunyaev-Zel’dovich effect.

Subject headings: cosmology: observations — galaxies: clusters: general — X-rays: galaxies

1. INTRODUCTION

Galaxy clusters constitute an important cosmological probe, in particular in determining the Hubble constant $H_0$ through the combined analysis of the Sunyaev-Zel’dovich effect (SZE) (Sunyaev & Zel’dovich 1972) and X-ray observations. Recent high-resolution X-ray and radio observations enable one to construct a statistical sample of clusters for the $H_0$ measurement. Carlstrom et al. (2002) compiled the previous results of 38 distance determinations to 26 different galaxy clusters, and obtained $H_0 = 60 \pm 3$ km s$^{-1}$ Mpc$^{-1}$ (Reese et al. 2002; Uzan et al. 2004; but see Bonamente et al. 2006). Despite its relatively large individual errors, the mean value of $H_0$ estimated from the SZE and X-rays appears systematically lower than those estimated with other methods: e.g., $H_0 = 72 \pm 8$ km s$^{-1}$ Mpc$^{-1}$ from the distance to Cepheids (Freedman et al. 2001) and $H_0 = 73 \pm 3$ km s$^{-1}$ Mpc$^{-1}$ from the cosmic microwave background anisotropy (Spergel et al. 2007).

Possible systematic errors in the $H_0$ measurement from the SZE have been extensively studied by several authors (Inagaki et al. 1995; Kobayashi et al. 1996; Yoshikawa et al. 1998; Hughes & Birkinshaw 1998; Birkinshaw 1999; Wang & Fan 2006); they have addressed a number of physical sources of possible biases including the finite extension, clumpiness, asphericity, and non-isothermality of the intracluster medium (ICM). Nevertheless, they were not able to identify any systematic error that affects the estimate of $H_0$ by 10%–15%. Therefore, it has been generally believed that the reliability of $H_0$ from the SZE is dominated by the statistics. Given that, the 10%–15% underestimate bias mentioned above, if real, needs to be explained in terms of additional ICM physics beyond the simple models used in previous studies.

Recently, Mazzotta et al. (2004) have pointed out that the spectroscopic temperature, $T_{\text{spec}}$, is systematically lower than the emission-weighted temperature, $T_{\text{ew}}$. Kawahara et al. (2007, hereafter Paper I) investigated the origin of the discrepancy and found that both the fluctuation of density and temperature and non-isothermality cause a difference between $T_{\text{spec}}$ and $T_{\text{ew}}$. They also found that the probability density functions (PDFs) of both density and temperature are well approximated by a lognormal function.

The aim of the paper is to revisit the origin of the bias based on an observable quantity, $T_{\text{spec}}$, and a lognormal description for fluctuations of the ICM. The paper is organized as follows. In §2 we briefly review the conventional method for estimating $H_0$ from the spherical isothermal $\beta$-modeling of galaxy clusters. Then we describe several possible sources of the systematic bias based on a lognormal description of the fluctuations and the spectroscopic temperature. We propose an analytical model for the bias in §3. Nonspherical effects are considered in §4 on the basis of triaxial model clusters which include the lognormal fluctuation and the temperature profile. Section 5 explores the validity of our analytic model for the systematic bias using clusters extracted from cosmological hydrodynamic simulations. Finally, we summarize our conclusions in §6. The Appendix describes the semianalytic distribution function of the bias of the estimated Hubble constant due to the asphericity of clusters.

2. ESTIMATING $H_0$ FROM THE SZE IN THE SPHERICAL ISOTHERMAL $\beta$-MODEL

A conventional estimate of $H_0$ from the SZE is based on the assumptions that the gas temperature is isothermal, $T(r) = T_{\text{el}}$ (=constant), and that the gas density follows the spherical $\beta$-model:

$$n(r) = n_0 \left[ 1 + \left( \frac{r}{r_c} \right)^2 \right]^{-3\beta/2},$$  (1)

where $n_0$ is the central density, $r_c$ is the core radius, and $\beta$ is the index characterizing the density profile. These approximations are insufficient to model the full complexity of real galaxy clusters. It has been (implicitly) assumed that the average over a number of clusters should significantly reduce the resulting error in the estimate of $H_0$. While we quantitatively argue below that this is
not the case, we summarize here the commonly adopted estimator for \( H_0 \) in the spherical isothermal \( \beta \)-model (Inagaki et al. 1995; Kobayashi et al. 1996).

In this idealistic model, the X-ray surface brightness and \( \gamma \)-parameter of SZE at an angle \( \theta \) from the center of cluster are given by

\[
S_X(\theta) = \frac{\Lambda_X(T_{cl}) n_0^2 r_c G(\beta)}{4\pi(1+z)^4} \left[ 1 + \left( \frac{\theta}{\theta_c} \right)^2 \right]^{-3\beta+1/2}, \tag{2}
\]

\[
y(\theta) = n_0 \sigma T k T_{cl} r_c G(\beta/2) \left[ 1 + \left( \frac{\theta}{\theta_c} \right)^2 \right]^{(3/2)\beta+1/2}, \tag{3}
\]

where \( m_e \) is the electron mass, \( k \) is the Boltzmann constant, \( c \) is the speed of light, \( \sigma_T \) is the Thomson cross section, \( \Lambda_X(T) \) is the cooling function, \( z \) is the redshift of the cluster, and we define

\[
G(\beta) \equiv \sqrt{\pi} \frac{\Gamma(3\beta-1/2)}{\Gamma(3\beta)}, \tag{4}
\]

with \( \Gamma(x) \) being the gamma function.

Combining equations (2) and (3), one can eliminate \( n_0 \) and estimate the core radius as

\[
r_{c,\text{iso}}(T_{cl}) = \frac{y(0)^2}{S_X(0)} \frac{m_e^2 c^4 \Lambda_X(T_{cl})}{4\pi(\sigma T k T_{cl})^2 (1+z)^4 \left[ G(\beta/2) \right]^2}, \tag{5}
\]

where \( S_X(0) \) and \( y(0) \) denote the values at \( \theta = 0 \), the line of sight through the center of the galaxy cluster. Note that the right-hand side of equation (5) is written entirely in terms of observable quantities.

Equation (5) corresponds to the estimate of the core radius along the line of sight. If the cluster is spherically symmetric, it should be equal to the core radius in the plane of the sky. With the assumption, the measured angular core radius, \( \theta_c^{\text{fit}} \), is related to the physical core radius simply by

\[
r_{c,\text{fit}} = \theta_c^{\text{fit}} d_A(z), \tag{6}
\]

with \( d_A(z) \) being the angular diameter distance of the cluster at \( z \). Equations (5) and (6) may be combined to estimate the angular diameter distance to the cluster (Silk & White 1978):

\[
d_A^{\text{est}}(z) \equiv \frac{r_{c,\text{iso}}}{\theta_c^{\text{fit}}} \tag{7}
\]

If one obtains \( d_A^{\text{est}}(z) \) for a number of clusters at different redshifts, one can estimate cosmological parameters by fitting to the angular diameter distance versus redshift relation, \( d_A(z) \). In what follows, however, we consider the above methodology for the purpose of estimating \( H_0 \). Thus, following Inagaki et al. (1995) we introduce the ratio of the estimated to the true value of \( H_0 \):

\[
f_H = \frac{d_A}{d_A^{\text{est}}} = \frac{H_0^{\text{est}}}{H_0^{\text{true}}} = \frac{r_\parallel}{r_\bot}. \tag{8}
\]

Equations (5) and (6) provide commonly used estimators for the radius of clusters along and perpendicular to the line of sight, \( r_\parallel \) and \( r_\bot \), respectively, but they are model-dependent and ill defined for generic nonspherical clusters. We come back to this issue below (§§ 4 and 5). Note that \( f_H > 1 \) \((<1) \) corresponds to overestimating (underestimating) the true \( H_0 \).

Given the approximations underlying the spherical isothermal \( \beta \)-model, it is not surprising that \( f_H \) for an individual cluster deviates from unity. A more relevant question is whether the average over a number of clusters, \( \langle f_H \rangle \), is still systematically larger or smaller than unity. If such systematic errors exist, can we correct for them by identifying their physical origin? This is what we address in the present paper.

In fact, there have been several previous attempts toward the same goal, mainly utilizing numerically simulated galaxy clusters (Inagaki et al. 1995; Yoshikawa et al. 1998; Sulkkenen 1999).

The conclusion of this analysis is that departure from sphericity and isothermality of clusters results in \( f_H \neq 1 \), but after averaging over a sample of clusters the systematic errors are relatively small, \( \langle f_H - 1 \rangle \approx 5\% \). Our analysis below is different from the previous ones in adopting the spectroscopic temperature, \( T_{\text{spec}} \) for \( T_{cl} \). Indeed, \( T_{cl} \) is a somewhat ambiguous quantity for actual clusters (not isothermal). It has been shown in this context to assume the emission-weighted temperature,

\[
T_{\text{ew}} \equiv \int f n^2 \Lambda_X(T) T^2 \ dV \int f n^2 \Lambda_X(T) \ dV,
\]

is approximately equal to \( T_{\text{spec}} \) (the above integration is carried out over the entire cluster volume). Thus, the previous conclusion is entirely based on the assumption that \( T_{cl} = T_{\text{ew}} \). Recently, however, Mazzotta et al. (2004) and Rasia et al. (2005) pointed out that \( T_{\text{spec}} \) estimated by fitting the thermal continuum and the emission lines of the X-ray spectrum, is systematically lower than \( T_{\text{ew}} \). Furthermore, in Paper I we found that the difference between \( T_{\text{ew}} \) and \( T_{\text{spec}} \) could be explained through an analytic model of the temperature profile and inhomogeneities in the ICM. We evaluate \( f_H \) by applying the model and then comparing the numerical simulations in the subsequent sections.

3. ANALYTIC MODELING OF SYSTEMATIC ERRORS OF \( H_0 \) FOR SPHERICAL CLUSTERS

Identifying possible systematic errors in the estimate of \( H_0 \) for realistic clusters is inevitably complicated. In order to address the problem as analytically as possible, we consider spherical clusters that follow the density profile of equation (1) and a polytropic temperature profile but with lognormal density and temperature fluctuations. While the approach in this section is not entirely generic, it is useful in understanding the physical origin of systematic errors. The present analytic modeling is tested against numerically generated triaxial cluster samples in § 4, and against those from cosmological hydrodynamic simulations in § 5.

Our task here is to derive analytic expressions for more general cases, which correspond to equations (2)–(5) in the case of the isothermal \( \beta \)-model. Let us consider first the effect of inhomogeneities in the ICM. The X-ray surface brightness at the center of the cluster is written as an integral over the line of sight:

\[
S_X(0) = \frac{1}{2\pi(1+z)^5} \int n(r)^2 \Lambda_X(T(r)) \ dr. \tag{10}
\]

Paper I found that the fluctuation fields defined as \( \delta_\parallel \equiv n(r)/n(r) \) and \( \delta_T \equiv T(r)/T_{\text{spec}} \) are approximately independent and follow the \( r \)-independent lognormal PDF, \( P_{\text{LN}}(\delta_\parallel; \sigma_{L,\parallel}) \) and \( P_{\text{LN}}(\delta_T; \sigma_{L,T}) \), where \( \sigma_{L,\parallel} \) and \( \sigma_{L,T} \) denote the standard
deviations of the density and temperature logarithms. The average of equation (10) over many independent lines of sight can then be computed by integrating over the lognormal PDFs. If we further assume that the cooling function, $\Lambda_X(T)$, is dominated by thermal bremsstrahlung (bolometric), $\Lambda_{\text{brems}}(T) \propto \sqrt{T}$, we can rewrite equation (10) as:

$$S_X(0) = \frac{1}{4\pi(1+z)^4} \int \delta_n^2 P_{\text{LN}}(\delta_n) P_{\text{LN}}(\delta_T) d\delta_n d\delta_T$$

$$\times \int n(r)^2 \Lambda_{\text{brems}}[T(r)] dr$$

$$= \exp \left( \frac{\sigma_{\text{LN},n}^2 - \sigma_{\text{LN},T,r}^2}{4(1+z)^4} \right) \int n(r)^2 \Lambda_{\text{brems}}[T(r)] dr. \quad (11)$$

On the contrary, their fluctuations do not affect $y(0)$ because the integrand of the $y$-parameter is a linear function of both temperature and density. Thus, the inhomogeneity effect is well described by the factor

$$\chi_n = \exp \left( \frac{\sigma_{\text{LN},n}^2 - \sigma_{\text{LN},T, r}^2}{8} \right). \quad (12)$$

The polytropic temperature profile is expressed as

$$T(r) = -\frac{T_0}{\gamma} \ln \left( \frac{n(r)}{n_0 r_0^{\gamma-1}} \right), \quad (13)$$

where $T_0$ is the central temperature (at $r = 0$) and $\gamma$ is the polytropic index. Then we obtain

$$S_X(0) = \chi_n \frac{1}{4\pi(1+z)^4} \int n(r)^2 \Lambda_{\text{brems}}[T(r)] dr$$

$$= \frac{\Lambda_{\text{brems}}(T_0) n_0^{\gamma-1} r_0, G(\beta(\gamma + 3)/4)}{4\pi(1+z)^4} \quad (14)$$

$$y(0) = \frac{n_0 r_0, G(\beta/2)}{m_0 c^2}. \quad (15)$$

Therefore, the core radius in this model is written as

$$r_{c, \text{polyLN}} = \chi_n \frac{\gamma(0)^2}{S_X(0)} \frac{m_0^2 c^4 \Lambda_{\text{brems}}(T_0)}{4\pi(\sigma r_0, G(T_0))} \frac{G(\beta(\gamma + 3)/4)}{(1+z)^2} \frac{[G(\beta/2)]^2}{[G(\beta/2)]^2}, \quad (16)$$

If one attempts to fit the X-ray surface brightness profile under the assumption of an isothermal $\beta$-model, the fitted value of the $\beta$-parameter should be

$$\beta_{\text{fit}} = \frac{\beta(\gamma + 3)}{4}, \quad (17)$$

since

$$\Lambda_{\text{brems}}[T(r)] n(r)^2 \propto T(r)^{\gamma-1} n(r)^2 \propto \left[ n(r)^{(\gamma+3)/4} \right]^2.$$  

In addition, the fitted temperature should be equal to the spectroscopic temperature $T_{\text{spec}}$. Thus, the estimated core radius is given by equation (5):

$$r_{c, \text{iso}}(T_{\text{spec}}) = \frac{y(0)^2}{S_X(0)} \frac{m_0^2 c^4 \Lambda_{\text{brems}}(T_{\text{spec}})}{4\pi(\sigma r_0, G(T_{\text{spec}}))} \frac{G(\beta_{\text{fit}})}{(1+z)^2}[G(\beta_{\text{fit}}/2)]^2 \frac{[G(\beta/2)]^2}{[G(\beta/2)]^2}. \quad (18)$$

Therefore, the systematic bias in the estimate of the Hubble constant in this particular model should be

$$f_{H, \Lambda_{\text{polyLN}}} = \frac{r_{c, \text{polyLN}}}{r_{c, \text{iso}}(T_{\text{spec}})}$$

$$= \frac{\Lambda_{\text{brems}}(T_0)}{\Lambda_{\text{brems}}(T_{\text{spec}})} \frac{T_0^2}{T_{\text{spec}}^2} \frac{G(\beta(\gamma + 3)/4)}{[G(\beta/2)]^2} \frac{G(\beta/2)}{G(\beta_{\text{fit}})} \quad (19)$$

where we define $\chi_T$ to express the effect of the temperature structure in the ICM.

It may be more instructive to rewrite equation (19) as

$$f_{H, \Lambda_{\text{polyLN}}} = \frac{\chi_T(T_{\text{ew}})}{\chi_T(T_{\text{spec}})} \frac{\chi_T(T_{\text{ew}})}{\chi_T(T_{\text{spec}})}, \quad (20)$$

since $T_{\text{ew}}$ was often assumed to be equal to $T_{\text{ew}}$. Equation (20) makes it clear that the systematic bias in the estimate of $H_0$ results from three major effects: $\chi_n$, due to inhomogeneities in the ICM, $\chi_T(T_{\text{ew}})$, representing the temperature structure assuming that $T_{\text{ew}} = T_{\text{ew}}$, and finally, $\chi_T(T_{\text{spec}})/\chi_T(T_{\text{ew}})$, coming from the differences between the spectroscopic and the emission-weighted temperatures of the ICM.

Those three factors can be expressed in an approximate but analytic fashion as follows. If we adopt the lognormal PDF for the density and temperature inhomogeneities in the ICM, $\chi_n = \exp(\sigma_{\text{LN},n}^2 - \sigma_{\text{LN},T,r}^2/8)$ (eq. [12]). As shown in Paper I, cosmological hydrodynamic simulations indicate that $\sigma_{\text{LN},n} \approx 0.2--0.5$ and $\sigma_{\text{LN},T} \approx 0.2--0.3$. Thus, $\chi_n \approx 0.5$. The second factor can be estimated by using the analytical relation of $T_0$ and $T_{\text{ew}}$ in the current model:

$$T_{\text{ew}}/T_0 = \exp(\sigma_{\text{LN},T,r}^2/2) F(\beta; \gamma; r_c/r_{\text{vir}}), \quad (21)$$

where we assume that the cluster has a finite extension and $n(r) = 0$ for radius $r$ beyond the virial radius of the cluster, $r_{\text{vir}}$, and we define

$$F(\beta; \gamma; r_c/r_{\text{vir}}) = \frac{zF_3(3/2, 3; \beta[1 + (\gamma - 1)/4]; 5/2; -r_{\text{vir}}^2/r_c^2)}{zF_3(3/2, 3; \beta[1 + (\gamma - 1)/4]; 5/2; -r_{\text{vir}}^2/r_{\text{vir}}^2)}, \quad (22)$$

with $zF_3(\alpha; \beta; \gamma; \zeta)$ being the hypergeometric function (see § 3 of Paper I). Just for simplicity, we neglect the term $\exp(\sigma_{\text{LN},T}^2/2)$, which represents the temperature inhomogeneity, because it is relatively small for $\sigma_{\text{LN},T} \approx 0.2--0.3$. If we further adopt $\Lambda_X(T) = \Lambda_{\text{brems}}(T) \propto \sqrt{T}$, then $\chi_T(T_{\text{ew}})$ reduces to

$$\chi_T(T_{\text{ew}}) \approx \left( \frac{T_{\text{ew}}}{T_0} \right)^{\gamma/4} \frac{G(\beta(\gamma + 3)/4)}{[G(\beta/2)]^2} \quad (23)$$

The result is plotted in Figure 1 for typical values of the parameters, and indicates that $\chi_T(T_{\text{ew}})$ ranges from 0.8 to 1.0 for $\beta = 0.5--0.8$ and $\gamma = 1.1--1.2$. 
The third set corresponds to the model described in simulations (sample sets “model clusters” in order to distinguish them from added asphericity; the lognormal fluctuations, the polytropic temperature profile to the three axis directions by a factor of $\lambda_{er}$, $\lambda_{br}$, and $\lambda_{cr}$, respectively.

In mock observations, we extract the quantities necessary to compute $r_{c,iso}(T_{cl})$ and $r_{c,fit}$ via equations (5) and (6) in the following manner. We first fit the projected profiles of $S_x(\theta)$ with a functional form $S_x(0)[1 + (r/r_{c,fit,x})^3]^2 - h_0^{-1/2}$ from $r = 0$ to $1$ $h^{-1}$ Mpc over 1024 random LOSs toward each cluster. For each LOS, we also compute $\gamma(0)$ and, unless otherwise stated, use it directly in our analysis. We discuss other methods of obtaining $\gamma(0)$ in § 5.3. As will be described below, the gas temperature $T_{cl}$ is obtained by either fitting the mock X-ray spectra or simply using the input temperature, depending on the purpose of the analysis. We use the template of the spectral energy distribution computed using SPEX 2.0 assuming collisional ionization equilibrium, an energy range of $0.5$–$10.0$ keV, and a constant metallicity $0.3$ $Z_{\odot}$. Assuming that $r_\parallel = r_{c,iso}(T_{cl})$ and $r_\perp = r_{c,fit,x}$, we calculate $f_{HI}$ for each LOS.

To quantify the bias due to the projection effect, we also compute the volume-averaged radial profile of the gas density, directly from the grid data within the radius $1$ $h^{-1}$ Mpc. By fitting the profile to the $\beta$ model, we obtain the estimated core radius $r_{c,fit,3D}$, which is independent of LOS. We compare the values of $f_{HI}$ using $r_\perp = r_{c,fit,3D}$ and $r_{c,fit,x}$ in the discussion that follows.

Figure 2a shows the mean and rms values of $f_{HI}$ for spherical clusters with no temperature gradient. We consider two cases for the lognormal density and temperature fluctuations with $\langle \sigma_{1N.L,e} \rangle = (0.5, 0.0)$ and $(0.5, 0.3)$. The latter set corresponds to the typical value for the simulated cluster, as reported in Paper I. To present the bias produced solely by gas inhomogeneities, we here adopt for $T_{cl}$ the volume-averaged temperature instead of fitting the mock X-ray spectra. It is evident that the fluctuations in gas density yield $f_{HI} \sim 1.3$ (i.e., overestimating $H_0$ by $\sim 30\%$), while those in gas temperature do not contribute significantly to the bias. The mean value of $f_{HI}$ is in good agreement with our analytical expectation $\chi_0$ (Fig. 2a, dashed horizontal lines). The bias due to the projection is $\sim 10\%$.

The bias produced by ellipsoidal shapes is displayed in Figure 2b for the two sets of the axis ratio ($\lambda_{er} : \lambda_{br} : \lambda_{cr} = 0.6 : 0.7 : 1$ and $0.9 : 0.9 : 1$). These sets are the typical values of the simulated cluster. Again, to present the bias solely from asphericity, the gas is assumed to be isothermal without any fluctuations, and we adopt $T_{cl} = T_0$ in computing $f_{HI}$. The average bias is relatively small ($\lesssim 15\%$), and that due to the projection is $\sim 3\%$. These results indicate that the bias due to asphericity, after averaging over a statistical sample of clusters, is smaller than that from gas inhomogeneity.

Figure 2c illustrates the bias in a more realistic case; we create ellipsoidal clusters with a polytropic temperature profile and fluctuations. Two sets of axis ratio ($\lambda_{er} : \lambda_{br} : \lambda_{cr} = 0.6 : 0.7 : 1$ and $0.9 : 0.9 : 1$) are chosen, adopting $\langle \sigma_{1N,L,e} \rangle = (0.5, 0.3)$ and $(0.3, 0.2)$, respectively. In this panel, we show the values of $f_{HI}$ based on the following three methods, so as to understand clearly the physical origin of the overall bias in the $H_0$ estimation.

The first method (Fig. 2c, black symbols) corresponds to the most conventional case of using the isothermal $\beta$-model and the spectroscopic temperature $T_{spec}$. To obtain $T_{spec}$ we fit the mock X-ray spectra from the central $(r < 1$ $h^{-1}$ Mpc) region of each cluster using XSPEC assuming a single-temperature MEKAL model. We assume the perfect response, and ignore its effect on
the spectral temperature (Paper I). Clearly, the value of $H_0$ is underestimated by $\sim 10\%$–$20\%$. This is in good agreement with our analytical estimation for $f_{H, \text{poly}LN}|_{\text{iso}} = \chi_\sigma \chi_\ell (T_{\text{spec}})$ for a spherical cluster (Fig. 2c, solid horizontal lines). To obtain $\chi_\ell (T_{\text{spec}})$, we compute the volume-averaged profile of the density and the temperature. These profile are fitted to equation (1) and equation (13), taking $n_0$, $r_{c, \text{th}, 3D}$, $\beta$, $T_0$ and $\gamma$ as free parameters. We use the adopted values ($\sigma_{\text{LN}, n}$, $\sigma_{\text{LN}, T}$) = (0.5, 0.3) and (0.3, 0.2) to compute $\chi_\sigma$.

The second method (Fig. 2c, red symbols) aims to mimic previous numerical studies of the $H_0$ bias (Inagaki et al. 1995; Yoshikawa et al. 1998) and adopts the isothermal $\beta$-model and the emission-weighted temperature $T_{\text{ew}}$. We obtain $T_{\text{ew}}$ by directly summing up the temperature of each grid point from the central ($r < 1 \, h^{-1} \text{Mpc}$) region. Also plotted for comparison is an analytical estimate $\chi_\ell (T_{\text{spec}})$ (dotted lines). The values of $\chi_\sigma$, and $\chi_\ell (T_{\text{ew}})$ are computed as described above. In this case, $\chi_\sigma$ and $\chi_\ell (T_{\text{ew}})$ practically cancel each other, and $f_{H, \text{poly}LN}|_{\text{iso}}$ is close to unity, consistent with the previous findings of Inagaki et al. (1995) and Yoshikawa et al. (1998). This shows that the absence of the bias in previous studies is simply an artifact of using $T_{\text{ew}}$, which is systematically larger than $T_{\text{spec}}$.

The third method (Fig. 2c, blue symbols) attempts to eliminate the bias due to the temperature gradient by using the polytropic profile to estimate the core radius (Ameglio et al. 2006):

$$r_{c, \text{poly} \beta} = \frac{\gamma(0)}{8\pi(0)} \frac{m_c^2 c^4 A_X(T_0) G(3\gamma/4 + 3\beta/4)}{4\pi(\sigma T k T_0)^2(1 + z)^2 \left[ G(3\beta/2) \right]^2}$$

where we adopt $T_0$ and $\gamma$ from fitting the volume-averaged temperature profile of the model clusters. The value of $\beta$ is obtained from equation (17) using $\beta_{\text{fit}}$ and $\gamma$. The value of $f_{H, \text{fit}}$ so obtained should represent the bias arising from sources other than the spectral fitting and the temperature gradient. Given the good agreement with the analytical estimate for $\chi_\sigma$ (Fig. 2c, dashed lines), we conclude that the bias in this case is dominated by the effect from gas inhomogeneities.

In summary, there are three major sources for the bias of $H_0$: the spectral fitting, the temperature gradient, and local density fluctuations. The first two lead to an underestimation, while the latter leads to an overestimation of $H_0$. In every case studied here, the bias due to asphericity is much smaller than any of these three.

5. COMPARISON WITH CLUSTERS FROM COSMOLOGICAL HYDRODYNAMIC SIMULATIONS

5.1. Cosmological Hydrodynamic Simulations

We now compare the bias described in the previous section with simulated clusters. These are extracted from the smoothed particle hydrodynamic (SPH) simulation of the local universe performed by Dolag et al. (2005), assuming $\Lambda$CDM universe with $\Omega_m = 0.3$, $\Omega_k = 0.7$, $\sigma_8 = 0.9$, and $h = 0.7$. The numbers of dark matter and SPH particles are $\sim 20$ million each within a high-resolution sphere of radius $\sim 110$ Mpc, which is embedded in a periodic box $\sim 343$ Mpc on a side that is filled with nearly 7 million low-resolution dark matter particles. The simulation is designed to reproduce the mass distribution of the local universe, adopting initial conditions based on the IRAIS galaxy distribution, smoothed over a scale of $4.9\, h^{-1}$ Mpc. We choose the six massive clusters identified as Coma, Perseus, Virgo, Centaurus, A3627, and Hydra. Figure 3 shows projected surface density maps of these simulated clusters. The values of $\beta$ and $\gamma$ of these clusters are listed in Table 1. A cubic region of $6\, h^{-1}$ Mpc around the center of each cluster is extracted and divided into $512^3$ cells. The density and temperature of each mesh point are calculated from SPH particles using the B-spline smoothing kernel. A detailed description of this procedure is given in Paper I.

We perform mock observations over 1024 LOSs for each simulated cluster in a manner similar to that described in § 4, except for the following points. First, we compute $T_{\text{spec}}$ and $T_{\text{ew}}$ within the virial radius instead of $1\, h^{-1}$ Mpc. Second, we use the fitted value of $\sigma_{\text{LN}, n}$ and $\sigma_{\text{LN}, T}$ in calculating $\chi_\sigma$ of the analytical model.

![Fig. 2.—Average and rms of $f_H$ of the model clusters: (a) spherical clusters with gas inhomogeneities and no temperature gradient, (b) ellipsoidal and isothermal clusters, and (c) tilted ellipsoidal clusters with temperature gradient and gas inhomogeneities. Crosses and plus signs denote $f_H$ adopting $r_{c, \text{th}, 3D}$ and $r_{c, \text{th}, 3D}$, respectively. Thick horizontal lines indicate analytical estimations for $rc$ and $T_{\text{spec}}$ from equation (17) using $\gamma_{\text{iso}}$ and $\gamma_{\text{spec}}$, respectively. In panel (c), black symbols indicate $f_H$ using $r_{c, \text{iso}}(T_{\text{spec}} = T_{\text{ew}})$, red symbols $r_{c, \text{iso}}(T_{\text{spec}} = T_{\text{ew}})$, and blue symbols $r_{c, \text{poly} \beta}$, which correspond to the isothermal fit with $T = T_{\text{spec}}$, the isothermal fit with $T = T_{\text{ew}}$, and the polytropic fit, respectively (see the main text for details).](image-url)
5.2. Results

Figure 4 displays a set of histograms of $f_H$ for the simulated Coma cluster. The same analysis is done for the other five clusters. Histograms in different colors correspond to the symbols of the same color in Figure 2, and indeed show similar trends for each component of the bias. Since the physical length of clusters along the LOS is not symmetrically distributed around its mean, the corresponding histograms of $f_H$ are skewed positively. In Appendix A, we compute the distribution for the two extreme cases, the prolate and the oblate ellipsoids, and find that they yield positively and negatively skewed distributions, respectively. Indeed, this is consistent with the fact that the simulated Coma is nearly prolate (Table 1).

The (simple arithmetic) mean, $\langle f_H \rangle$, is plotted in Figure 5 for six simulated clusters. The quoted error bars indicate 1/$\sqrt{27}$ standard deviation from the mean. Except for the simulated Virgo cluster, $\langle f_H \rangle$ is below unity, i.e., $H_0$ is underestimated. It is remarkable that a simple analytical model for systematic effects (solid, dotted, and dashed horizontal lines) described in §3 can reproduce the bias in the simulated clusters.

We have made sure that the bias from other sources is minor; first, if a cluster has a finite extension and is bounded within the virial radius, the value of $\langle f_H \rangle$ becomes smaller by $\leq 5\%$ (Fig. 5, open circles). Second, we compute the axis ratio ($\lambda_a < \lambda_b < \lambda_c$) of each simulated cluster, basically following the method of Jing & Suto (2002), but using the gas density not the dark matter density. The isodensity surfaces corresponding to the gas densities of $n_e = 3 \times 10^{-3}, 1 \times 10^{-3}, 5 \times 10^{-4}, 3 \times 10^{-4},$ and $1 \times 10^{-4}$ cm$^{-3}$ are shown in Figure 3. After eliminating substructures, the axis ratio is calculated by diagonalizing the inertial tensor of each surface. The averaged axis ratios of the five different density regions (Table 1) are similar to the typical value adopted in §4. Therefore,

![Fig. 3.—Projected surface density maps of the six simulated clusters. Five different ($n_e = 3 \times 10^{-3}, 1 \times 10^{-3}, 5 \times 10^{-4}, 3 \times 10^{-4},$ and $1 \times 10^{-4}$ cm$^{-3}$) isodensity surfaces are indicated with different colors (red, orange, yellow, green, and blue, respectively). The left panels of each cluster indicate the view from our galaxy. The right panels are the projection of each simulated cluster as seen by a distant observer located to the “right” of each panel on the left. The horizontal yellow lines indicate the physical size of $1\, h^{-1}$ Mpc.

![Fig. 4.—Distribution of $f_H$ over 1024 LOSs for the simulated Coma cluster. Black, red and blue histograms indicate the results for the isothermal fit with $T = T_{\text{spec}}$, the isothermal fit with $T = T_{\text{ew}}$, and the polytropic fit, respectively.

### Table 1: Properties of the Six Simulated Clusters

| Cluster   | $\beta_{\text{fit,3D}}$ | $\gamma$ | $\lambda_a/\lambda_c$ | $\lambda_b/\lambda_c$ | $\langle r_{\text{fit,5a}}/r_{\text{fit,3D}} \rangle$ |
|-----------|--------------------------|----------|------------------------|------------------------|--------------------------------------------------|
| Coma      | 0.74                     | 1.17     | 0.59                   | 0.64                   | 1.03 ± 0.14                                      |
| Perseus   | 0.64                     | 1.09     | 0.49                   | 0.61                   | 1.04 ± 0.18                                      |
| Virgo     | 0.60                     | 1.15     | 0.44                   | 0.61                   | 1.16 ± 0.31                                      |
| Centaurus | 0.69                     | 1.17     | 0.68                   | 0.78                   | 1.03 ± 0.13                                      |
| A3627     | 0.69                     | 1.15     | 0.79                   | 0.83                   | 1.08 ± 0.06                                      |
| Hydra     | 0.70                     | 1.22     | 0.84                   | 0.93                   | 1.03 ± 0.05                                      |

The values of $\beta_{\text{fit,3D}}$ and $\gamma$ are slightly changed from those listed in Paper I due to improvement of the fit routine.
we conclude that the spherical approximation itself is not a major source of the bias for the simulated cluster. Finally, the bias due to the projection is small (Fig. 5, crosses and plus signs). We list in Table 1 the average values of \( r_c \) over 1024 LOSs relative to \( r_c; \text{fit} \text{3D} \). The ratio is unity within 10\% (except for Virgo, which has a relatively large dispersion), and basically all are consistent with unity within the uncertainty.

It is interesting to emphasize here that the shape of the distribution of \( f_{H} \) reflects the shape of clusters from the perspective of measuring the three-dimensional shape of clusters (e.g., Sereno et al. 2006). If all clusters had the same shape, the observation of one cluster toward multiple directions might correspond to that of multiple clusters toward each LOS. Of course, the real shapes vary from cluster to cluster. However, if clusters tend to be prolate (oblate) preferentially, the distribution of \( H_{\text{0,est}} \) should be skewed positively (negatively), as shown in Appendix A. Therefore, independently of the knowledge of real value of \( H_{\text{0, true}} \), statistical information about the shape distribution may be obtained in principle from the distribution of \( H_{\text{0,est}} \).

### 5.3. Comparison with Previous Studies

The above results are consistent with the previous results of \( f_{H} \approx 1 \) with \( T_{\text{cl}} = T_{\text{ew}} \) (Inagaki et al. 1995; Yoshikawa et al. 1998). On the other hand, Ameglio et al. (2006) explored the bias of \( d_{A} \) using the cosmological hydrodynamic simulations, and reported that \( H_{0} \) is overestimated by more than a factor of 2 if one adopts the isothermal \( \beta \) model. This is opposite to our conclusion here, and we found that this should be ascribed to the sensitivity of \( f_{H} \) to the adopted values of \( y(0) \), i.e., \( f_{H} \propto d_{A, \text{est}}^{-1} \propto y^{-2}(0) \), as explained below.

Ameglio et al. (2006) obtained \( y(0) \) by fitting the noiseless profile of \( y(\theta) \) up to \( R_{500} \), fixing other \( \beta \)-model parameters from the X-ray profile, while we directly use the projected values of \( y(0) \) in the simulation data. The difference between these two methods is apparent in Figure 5 (right) of Ameglio et al. (2006); their fit (solid line) was affected largely by the data points at large radii and yielded a value of \( y(0) \) smaller by \( \sim 50\% \) than the actual data. This enhances the value of \( f_{H} \) by more than a factor of 2, and indeed accounts for their apparently opposite conclusions. We have checked that other differences between their analysis and ours (the use of mass-weighted temperature for \( T_{\text{cl}} \) and the removal of the cluster central region) do not affect the results significantly.

As far as the bias in the previous SZE observations is concerned, we believe that our method is more relevant to what has been done with the real data, because these observations were not capable of constraining the radial profile of the \( y \)-parameter up to large radii with high S/N (see Komatsu et al. 2001; Kitayama et al. 2004 for the currently highest angular resolution observation of the SZE).

We have further made sure that the effects of the finite spatial resolution of the observations and the central cooling region, which were neglected in our preceding analysis, are minor; first, we have evaluated \( y(0) \) by fitting \( y(\theta) \) within a radius of 100 and 200 \( h^{-1} \) kpc. These approximately correspond to the typical angular resolution of the SZE observation (\( \sim 1 \) minute) at \( z = 0.1 \) and 0.3, respectively. The values of \( \beta \) and \( \theta_{c} \) are fixed from the X-ray profile. For the six simulated clusters, the resulting values of \( f_{H} \) differ from our initial analysis (see Fig. 5) by \(-6\% \) to \(+7\% \) (\(+2\% \) on average) for \( r < 100 \) \( h^{-1} \) kpc, and by \(-3\% \) to \(+10\% \) (\(+5\% \) on average) for \( r < 200 \) \( h^{-1} \) kpc.

Second, we have also performed the fitting separately for the X-ray and SZE profiles. The values of \( S_{X} \), \( \beta \), and \( \theta_{c} \) are evaluated by fitting \( S_{X}(\theta) \) with equation (2), while that of \( y(0) \) is obtained by fitting \( y(\theta) \) with equation (3) independently of the X-ray profile. As a result, the values of \( f_{H} \) differ from those of Figure 5 by \(-6\% \) and \(+1\% \) (\(-3\% \) on average).

Note that observationally there are several different ways to evaluate \( y(0) \), \( S_{X}(0) \), \( \beta \), and \( \theta_{c} \). One conventional method is to fit the X-ray imaging data \( S_{X}(\theta) \) first. Then the SZ image is fitted to obtain \( y(0) \), assuming the values of \( \beta \) and \( \theta_{c} \) from the X-ray data. Our analysis procedure adopted here follows the conventional method. While Reese et al. (2002) have determined \( d_{A} \) from the
joint fit to the X-ray and SZE imaging data, the result is almost equivalent to the conventional method, since the X-ray imaging data have a much higher S/N than the SZE data.

6. CONCLUSIONS

We have considered various possible systematic errors of $H_0$ from a combined analysis of the Sunyaev-Zel’ dovich effect and X-ray observations. In particular, we addressed the validity and limitations of the spherical isothermal model in estimating $H_0$, which has been used widely as a reasonable approximation among averaging over a number of clusters. We introduced the ratio of the estimated to the true Hubble constant, $f_H$, to characterize the systematic errors. We constructed an analytic model for $f_H$, and identified three important sources for the systematic errors: density and temperature inhomogeneities in the ICM, the temperature profile, and departures from sphericity. Except for the nonspherical effect, the most important analytical expression that summarizes our conclusion is equation (20), or equivalently,

$$
\frac{H_{0,\text{ext}}}{H_{0,\text{true}}} = \chi_\sigma \chi_T(T_{\text{ew}}) \chi_{\text{spec-ew}}.
$$

(26)

In our analytic model discussed in §3, the inhomogeneity bias, $\chi_\sigma$, the nonisothermality bias, $\chi_T(T_{\text{ew}})$, and the temperature bias, $\chi_{\text{spec-ew}}$, are given by equations (12), (23), and (24), respectively.

While the above model prediction is fairly general, the net value of $f_H$ sensitively depends on the degree of the inhomogeneity and multiphase temperature structure of real ICM. Our simulated cluster sample implies that $\chi_\sigma \approx 1.1-1.3$, $\chi_T(T_{\text{ew}}) \approx 0.8-1$, and $\chi_{\text{spec-ew}} \approx 0.8-0.9$, and therefore $\langle f_H \rangle \approx 0.8-0.9$. Given the result of Reese et al. (2002) this is certainly indicative, but it may need to be interpreted with caution because the result is critically dependent on the reliability of the adopted numerically simulated clusters as representative samples of clusters observed in the real universe. For this reason, we are attempting more direct (not statistical) comparison of our model prediction against observed cluster samples, which will be presented in a future paper (Reese et al. in preparation).

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APPENDIX

DISTRIBUTION OF $f_H$ FOR PROLATE AND OBLATE ELLIPSOIDS

In this Appendix, we derive the distribution of $f_H$, due to the asphericity of clusters, by considering the following two extreme cases, the prolate ($\lambda_a = \lambda_b < \lambda_c$) and the oblate ($\lambda_a < \lambda_b = \lambda_c$) ellipsoids. We choose $z$-axis as the long (short) axis and $x$- and $y$-axes as the short (long) axes for an axi (prolate) oblate ellipsoid. The direction of the unit vector along the LOS of an observer, $\bfa$, is defined in terms of the spherical coordinate $(\theta, \phi)$. Figure 6 shows a schematic picture of a prolate ellipsoid.

Let us define the quantity $\Lambda \equiv \lambda_a/\lambda_c = \lambda_b/\lambda_c (\Lambda \equiv \lambda_b/\lambda_a = \lambda_c/\lambda_a)$ for the prolate (oblate) ellipsoid. We assume that the gas density follows the prolate (oblate) $\beta$ model:

$$
n(r)_{|\theta_a} = n_0 \left[ 1 + (\tilde{r}/r_c)^2 \right]^{-3/2},
$$

(A1)

$$
\tilde{r} \equiv |\bf r| \left( \sin^2 \theta / \Lambda^2 + \cos^2 \theta \right)^{1/2},
$$

(A2)

![Fig. 6.—Schematic representation of a prolate cluster with an axis ratio $\lambda_a : \lambda_b : \lambda_c = 1 : \Lambda : 1$. Given the symmetry around the z-axis (the long axis), a LOS through the cluster center is specified by an angle $\theta_a$ from the z-axis. An arbitrary position in the cluster $\bf r$ is expressed in terms of $L$ (the projection of $\bf r$ onto the LOS direction), $R$ (the projection of $\bf r$ onto the plane normal to the LOS), and $\zeta$ (the azimuthal angle on the plane normal to the LOS).](image)
and $\theta_r$ is the angle between $z$-axis and $r$. Because the surface brightness profile is independent of $\phi_a$ due to the $z$-axial symmetry, one can express $f_{\mu}$ as function of $\theta_a$.

For an isothermal cluster, the surface brightness averaged over a circle of radius $R$ is proportional to $\int n^2 \, dL$ averaged over the angle $\zeta$ of the circle. We put $\zeta = 0$ where $r$ is located on the same plane defined by the LOS and $z$-axis. To compute the averaged surface brightness, we need an expression of the density $n$ as a function of $L, R$, and $\zeta$. In Cartesian coordinates, $r(\zeta = 0)$ is

$$r(\zeta = 0) = \frac{\sqrt{R^2 + L^2 \cos \phi_a \sin(\arctan R/L + \theta_a)}}{\sqrt{R^2 + L^2 \sin^2(\arctan R/L + \theta_a)}}.$$  

(A3)

Multiplying the rotation matrix around $a$, $M_a(\zeta)$ to $r(\zeta = 0)$, we obtain

$$r(\zeta) = M_a(\zeta)r(\zeta = 0) = \begin{pmatrix} R \cos \zeta \cos \phi_a \cos \theta_a + R \sin \zeta \sin \phi_a + L \cos \phi_a \sin \theta_a \\ R \cos \zeta \sin \phi_a \cos \theta_a - R \sin \zeta \cos \phi_a + L \sin \phi_a \sin \theta_a \\ \cos \theta_a - R \cos \zeta \sin \theta_a \end{pmatrix}. $$

(A4)

where

$$M_a(\zeta) = \begin{pmatrix} x_a^2 + (1 - x_a^2) \cos \zeta & x_a y_a (1 - \cos \zeta) + z_a \sin \zeta & z_a y_a (1 - \cos \zeta) - y_a \sin \zeta \\ x_a y_a (1 - \cos \zeta) - z_a \sin \zeta & y_a^2 + (1 - y_a^2) \cos \zeta & y_a z_a (1 - \cos \zeta) + x_a \sin \zeta \\ z_a y_a (1 - \cos \zeta) + y_a \sin \zeta & y_a z_a (1 - \cos \zeta) - x_a \sin \zeta & z_a^2 + (1 - z_a^2) \cos \zeta \end{pmatrix}. $$

(A5)

Thus, we obtain

$$|r(\zeta)| \cos \theta_r = L \cos \theta_a - R \cos \zeta \sin \theta_a. $$

(A7)

Then, $|r(\zeta)|$ and $\theta_r$ are written as

$$|r(\zeta)| = \sqrt{L^2 + R^2},$$

(A8)

$$\theta_r = \arccos \left( \frac{L \cos \theta_a - R \cos \zeta \sin \theta_a}{\sqrt{L^2 + R^2}} \right).$$

(A9)

Combining with equation (A1), we can write $n(r)_{\theta_a}$ in terms of $L,R$, and $\zeta$ as

$$n(r)_{\theta_a} = n_0 \left\{ 1 + \left[ \tilde{r}(R,L,\zeta)_{\theta_a}/r_c \right]^2 \right\}^{-3\beta/2} \equiv n(R,L,\zeta),$$

(A10)

where

$$\tilde{r}(R,L,\zeta)_{\theta_a} \equiv \tilde{r} = \sqrt{L^2 + R^2 + (\Lambda^2 - 1)(L \cos \theta_a - R \cos \zeta \sin \theta_a)^2} / \Lambda.$$ 

(A11)

Then, the averaged surface brightness at $R$ is

$$S_X(R)_{\theta_a} = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dL \left[ n(R, L, \zeta)_{\theta_a} \right]^2 = \frac{n_0^2 r_c}{2\pi} \int_0^{2\pi} d\zeta \int_{-\infty}^{\infty} dq_L \left[ \frac{q_L^2 + q_R^2 + (\Lambda^2 - 1)(q_L \cos \theta_a - q_R \cos \zeta \sin \theta_a)^2}{\Lambda^2} \right]^{-3\beta/2} \equiv \frac{n_0^2 r_c}{2\pi} I(q_R)_{\theta_a}$$

(A12)

where we define the normalized length by $r_c$, where $q_R \equiv R/r_c$ and $q_L \equiv L/r_c$. We compute $I(q_R)_{\theta_a}$ numerically for $\Lambda = 0.5$ (prolate) and $\Lambda = 2.0$ (oblate), adopting $\beta = 0.65$. We fit $I(q_R)_{\theta_a}$ from $q_R = 0$ to 10.0 by with a functional form of the surface brightness profile assuming the spherical beta model $\propto (1 + (q_R q_c)_{\theta_a}^2)^{-3\beta/2(q_c)}$. Thus, we obtain the counterpart of $r_{\text{fin,0}}, q_{\text{c,fin}}{\theta_a} \equiv r_{\text{fin,0}}, q_{\text{c,fin}}/r_c$ and the fitted value of $\beta, \beta_{\text{fin}}{\theta_a}$. While $q_{\text{c,iso}}{\theta_a} \equiv r_{\text{c,iso}}/r_c$ is written as

$$q_{\text{c,iso}}{\theta_a} = \left( \sin^2 \theta_a / \Lambda^2 + \cos^2 \theta_a \right)^{1/2} G(\beta) G(\beta_{\text{fin}}{\theta_a} / 2)^2 / G(\beta/2) G(\beta_{\text{fin}}{\theta_a}),$$

(A13)
The first term of the right-hand side represents the elongation of the radius toward the LOS. The second term is the correction to the use of $\beta_{\text{los}}$ in observation instead of the true $\beta$. However, the correction is very small (within 0.01% error).

Finally, we obtain the bias of $H_0$ as a function of $\theta_a$.

$$f_{H}(\theta_a) \equiv \frac{q_{c,\text{los}/\theta_a}}{q_{c,\text{iso}/\theta_a}}.$$  \hspace{1cm} (A14)

The probability of $f_{H}$ for the random assignment is proportional to the solid angle $\Omega(f_{H})$. If $f_{H}(\theta_a)$ is a monotonic function, the PDF of $f_{H}$ is obtained as

$$P(f_{H}) = \frac{1}{4\pi} \frac{d\Omega}{df_{H}} = \frac{1}{4\pi} \frac{d\Omega}{d\theta_a} \left| \frac{d\theta_a}{df_{H}} \right| = \frac{\sin \theta_a(f_{H})}{2} \left| \frac{d\theta_a(f_{H})}{df_{H}} \right|,$$  \hspace{1cm} (A15)

where $\theta_a(f_{H}) = f_{H}^{-1}(\theta_a)$.

Dotted lines in Figure 7 (top) show equation (A15) for prolate ($\Lambda = 0.5$) and oblate ($\Lambda = 2.0$) ellipsoids. As shown in the lower panel, the corresponding $\theta_a$ is a monotonically increasing (decreasing) function of $f_{H}$ for the prolate (oblate) ellipsoid. At $\theta_a = 0$, $f_{H}$ is equal to $\Lambda$, which corresponds to the case that the LOS is along the $z$-axis.

The PDF diverges at $\theta_a = \pi/2$. This can be understood as follows. Equations (A13)–(A15) imply that

$$P(f_{H}) \propto \sin \theta_a(f_{H}) \left| \frac{dq_{c,\text{los}/\theta_a}}{d\theta_a(f_{H})} \right|^{-1} \propto \sqrt{\cos^2 \theta_a(f_{H}) + \Lambda^2 \sin^2 \theta_a(f_{H})}.$$ \hspace{1cm} (A16)

where we ignore the $\theta_a$ dependence of $q_{c,\text{los}/\theta_a}$ and $\beta_{\text{los}}$. Thus $\theta_a \approx \pi/2$, $P(f_{H})$ diverges as $1/\cos \theta_a$. Note, however, its integration over a finite size of $f_{H}$ does not diverge (see eq. [A15]). This is plotted in the solid histograms, where the bin size $H = 0.05$ is adopted. The resulting distribution is skewed positively (negatively) for the prolate (oblate) ellipsoid, which is consistent with the results shown in Figure 4.

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