Modifying horizon thermodynamics by surface tensions

Deyou Chen1 · Xiaoxiong Zeng2

Received: 16 January 2019 / Accepted: 8 October 2019 / Published online: 12 October 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
The modified first laws of thermodynamics at the black hole horizon and the cosmological horizon of the Schwarzschild de Sitter black hole and the apparent horizon of the Friedmann–Robertson–Walker cosmology are derived by the surface tensions, respectively. The corresponding Smarr relations are obeyed. For the black hole, the cosmological constant is first treated as a fixed constant, and then as a variable associated to the pressure. The law at the apparent horizon takes the same form as that at the cosmological horizon, but is different from that at the black hole horizon. The positive temperatures guarantee the appearance of the worked terms in the modified laws at the cosmological and apparent horizons. While they can disappear at the black hole horizon.

Keywords Surface tensions · Horizon thermodynamics · Schwarzschild de Sitter black holes

1 Introduction
Einstein’s equations can be written as a thermodynamic identity. It was put forward by Padmanabhan and first proved in the spherically symmetric spacetime where Einstein’s equations near the horizon are in the form of the first law of thermodynamics

$$\delta E = T \delta S - P \delta V.$$ (1)

In the above equation, $E$ is the energy of the spacetime geometry, $P$ is the pressure provided by the source of the Einstein’s equations and $\delta V$ is the change of the
volume \([1,2]\). Subsequently, Sarkar and Kothawala \([3]\) found that this view is also applicable to the spherically symmetric horizons of various theories of gravity and the tunneling rate \(\Gamma \sim e^{\Delta S}\) is a natural consequence of the first law. Near the evolving spherically symmetric horizons and the stationary axisymmetric horizons, the laws were obtained from the Einstein’s equations in the references \([4–6]\). These researches effectively support the view of Padmanabhan. In the anti de Sitter spacetime, treating the cosmological constant as the pressure \(P = -\frac{\Lambda}{8\pi}\) and its conjugate quantity as the thermodynamic volume, Kubiznak and Mann \([7]\) studied the thermodynamics of the charged AdS black hole in the extended phase space. The first law was gotten as

\[
\delta M = T \delta S + \Phi \delta Q + V \delta P,
\]

which obeys the corresponding Smarr relation, where \(M\) is the black hole mass identified as the enthalpy. They found that the critical exponents of the black hole are full in consistence with those of the Van der Waals system. This work is the further elaboration of Dolan’s work \([8]\) where the critical behaviour of the AdS black holes in the extended phase space was discussed and the analogy with the Van der Waals was found. Based this interesting work, the thermodynamics of various complicate spacetimes were discussed and many significative critical phenomena were found \([9–26]\).

Recently, Hansen et al. found that the radial Einstein equation at the horizon of the Kerr black hole results in the modified first law of thermodynamics

\[
\delta E = T \delta S + \Omega \delta J - \sigma \delta A,
\]

where \(E\) is the Misner–Sharp mass, \(J = Ea\) is the horizon angular momentum, \(\sigma\) denotes the surface tension at the horizon and \(A\) is the horizon area \([27]\). In this equation, the horizon radius \(r_+\) and the rotation parameter \(a\) are two independent variables, which is different from Eq. \((1)\). Using the relation between the area and the volume and ordering \(J = 0\), one can reduce it to Eq. \((1)\). However, the reduced equation doesn’t obey the corresponding Smarr relation.

Our aim in this paper is to investigate the thermodynamics at the black hole horizon and the cosmological horizon of the Schwarzschild de Sitter black hole and at the apparent horizon of the Friedmann–Robertson–Walker (FRW) cosmology by the surface tensions. The modified first laws of thermodynamics are gotten and obey the corresponding Smarr relations. For a de Sitter black hole contains two horizons, the temperatures are different at different horizons. Therefore, the nonequilibrium state exists and it is difficult to discuss the thermodynamics. To overcome this difficulty, several approaches were adopted \([28–37]\). One approach is defining a mass–energy-like quantity at infinity in asymptotically spacetimes \([28,29]\). The second way is treating the black hole horizon and the cosmological horizon as two independent thermodynamical systems \([30–34]\). Besides, the global approach is the construction of the globally effective temperature and other effective thermodynamic quantities \([35,36]\). Here we adopt the second way to investigate the thermodynamics. We first let the cosmological constant be fixed and derive the modified first laws at the black hole horizon and the cosmological horizon. Then treating the constant as a variable associated to
the pressure $P = -\frac{\Lambda}{8\pi}$, we get the laws in the extended phase spaces. Considering the pure de Sitter spacetime is a special case of the FRW spacetime, we formulate the extension to the FRW cosmology and obtain the modified first law at the apparent horizon. The modified law at the apparent horizon has the same form as that at the cosmological horizon. However, it is different from that at the black hole horizon.

The rest is organized as follows. In the next section, we derive the modified first laws of thermodynamics at the cosmological and black hole horizon by the surface tension. The cosmological constant is treated as a fixed constant and a variable, respectively. In Sect. 3, the modified law at the apparent horizon of the FRW cosmology are investigated. Section 4 is devoted to our discussion and conclusion.

2 Modified thermodynamics of the Schwarzschild de Sitter black hole

2.1 Thermodynamics at the black hole horizon

The thermodynamics of the Schwarzschild de Sitter black hole were studied in [28,29,31,32,35]. The conserved quantities defined by integrals were introduced to obtain the first law of thermodynamics. However, the problem of integration constant exists. To overcome this problem, the Iyer–Wald formalism was adopted [35]. The cosmological constant was seen as a fixed constant in the former discussions [28,29,32]. Subsequently, it was treated as an independent variable in [31,35].

The Schwarzschild de Sitter black hole is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$  

where $f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}$, $\Lambda$ is the cosmological constant and $M$ is the physical mass. There are two positive roots and a negative root for $f(r) = 0$. Our discussion is limited to the range $0 < 9\Lambda M < 1$ which guarantees the existence of the black hole horizon $r_+$ and the cosmological horizon $r_C$. We first investigate the thermodynamics at the black hole horizon by the surface tension. The entropy and the temperature at the black hole horizon are

$$S_+ = \pi r_+^2, \quad T_+ = \frac{f'(r_+)}{4\pi} = \frac{1 - \Lambda r_+^2}{4\pi r_+},$$  

respectively, where $f'(r_+) = \frac{\partial f(r)}{\partial r} |_{r=r_+}$. There are several definitions of mass. In this paper, we adopt the definition of the Misner–Sharp mass. Using the definition [38–41], we get the Misner–Sharp mass surrounded by the black hole horizon as

$$E_+ = \frac{r_+}{2}.$$  

To derive the surface tension and the thermodynamical identity, we first calculate the radial Einstein equation at the black hole horizon and get
\[ G'_r |_{r_+} = 8\pi T'_r |_{r_+} = \frac{r_+ f'(r_+) - 1}{r_+^2}. \]  

(7)

Solving the above equation yields

\[ f'(r_+) = 8\pi r_+ T'_r |_{r_+} + \frac{1}{r_+}. \]  

(8)

Moving the first term on the right hand side (rhs) of the above equation to the left hand side (lhs) and multiplying by \( \frac{\delta S_+}{4\pi} \) on the both sides of the moved equation yield

\[ \frac{f'(r_+)}{4\pi} \delta S_+ - 2r_+ T'_r |_{r_+} \delta S_+ = \frac{\delta r_+}{2}. \]  

(9)

Here we have used the differential expression \( \delta S_+ = 2\pi r_+ \delta r_+ \) to derive the term on rhs of Eq. (9). It is clearly that \( \frac{f'(r_+)}{4\pi} \) is the value of the temperature at the black hole horizon. So the first term on lhs is in of the form \( T_+ \delta S_+ \). The term \( \frac{1}{2} \delta r_+ \) equals \( \delta E \) and is identified as the change of the energy. \( 2r_+ T'_r |_{r_+} \delta S_+ \) is written as \( \sigma_+ \delta A_+ \) by using the relation between the entropy and the horizon area \( A_+ \). Therefore, Eq. (9) is written as

\[ \delta E_+ = T_+ \delta S_+ - \sigma_+ \delta A_+, \]  

(10)

which is the modified first law of thermodynamics at the black hole horizon and \( \sigma_+ = \frac{1}{2} r_+ T'_r |_{r_+} = -\frac{\Lambda r_+}{16\pi} \) denotes the surface tension. Clearly, the corresponding Smarr relation

\[ E_+ = 2(T_+ S_+ - \sigma_+ A_+) \]  

(11)

is obeyed by using the expressions of \( T_+, S_+, \sigma_+ \) and \( A_+ \). The Gibbs free energy at the horizon in the de Sitter spacetime is

\[ G_+ = E_+ - T_+ S_+ + \sigma_+ A_+, \]  

(12)

which obeys differential expression \( \delta G_+ = -S_+ \delta T_+ + A_+ \delta \sigma_+ \). After a simple calculation, the energy is gotten as \( G_+ = \frac{r_+}{A_+} \). When Eq. (10) is reduced to \( \delta E_+ = T_+ e \delta S_+ \) by the relation between the entropy and the area, \( T_+ e = T_+ - 4\sigma_+ = \frac{1}{4\pi r_+} \) expresses the effective temperature at the black hole horizon.

Taking into account the cosmological constant as a variable associated to the pressure in the recent work, we make further to explore the thermodynamics and write Eq. (10) as

\[ \delta E_+ = T_+ \delta S_+ - P_+ \delta V_+, \]  

(13)
by using the relation between the area $\sigma_+$ and the volume $V_+$, where $P_+ = \frac{2\sigma_+}{r_+} = -\frac{\Lambda}{8\pi}$ expresses the pressure at the horizon. Equation (13) can be furthermore written as

$$\delta E_{0+} = T_+ \delta S_+ + V_+ \delta P_+, \quad (14)$$

where $E_{0+} = E_+ + P_+ V_+ = M$. The corresponding Smarr relation $E_{0+} = 2(T_+ S_+ - P_+ V_+)$ is also satisfied when the related expressions are introduced. Therefore, Eq. (14) is the first law of thermodynamics at the black hole horizon in the extended phase space. So both of Eqs. (10) and (14) are the first law at the black hole horizon.

### 2.2 Thermodynamics at the cosmological horizon

In this subsection, we investigate the thermodynamics at the cosmological horizon of the Schwarzschild de Sitter black hole by the surface tension. A parallel process as the Sect. 2 is performed. The temperature at this horizon is lower than that at the black hole horizon. The energy of a positive mass at the cosmological horizon is measured as a negative value. The entropy and the temperature at the cosmological horizon are

$$S_C = \pi r_C^2, \quad T_C = \frac{|f'(r_C)|}{4\pi} = -\frac{f'(r_c)}{4\pi}, \quad (15)$$

respectively, where $f'(r_C) = \frac{\partial f(r)}{\partial r} |_{r=r_C} = \frac{1-\Lambda r_C^2}{4\pi r_C} < 0$. The Misner–Sharp mass surrounded by the cosmological horizon is gotten as

$$M_C = \frac{r_C}{2}. \quad (16)$$

To investigate the surface tension and the modified horizon thermodynamics, we first calculate the radial Einstein equation at the cosmological horizon and get

$$G^r_r |_{r_C} = 8\pi T^r_r |_{r_C} = \frac{r_C f'(r_C)}{r_C^2} - 1, \quad (17)$$

which yields

$$f'(r_C) = 8\pi r_C T^r_r |_{r_C} + \frac{1}{r_C}. \quad (18)$$

Multiplying by $-\frac{\delta S_C}{4\pi}$ on the both sides of the above equation and moving the first term of rhs to lhs, we obtain

$$-\frac{f'(r_C)}{4\pi} \delta S_C + 2r_C T^r_r |_{r_C} \delta S_C = -\frac{\delta r_C}{2}, \quad (19)$$

where $-\frac{f'(r_C)}{4\pi}$ is the temperature at the cosmological horizon and the first term on lhs can be written as $T_C dS_C$. The differential expression $\delta S_C = 2\pi r_C \delta r_C$ is introduced
to derive the term on rhs of Eq. (19). \( \delta M_C = \frac{1}{2} \delta r_C \) expresses the change of the Misner–Sharp mass at the cosmological horizon. We identify the negative Misner–Sharp mass as the energy \( E_C \) and then obtain \( \delta E_C = -\frac{\delta r_C}{2} \) which shows the change of energy is related to that of the horizon location \([1]\). The second term \( 2r_C T_r \big|_{r_C} \delta S_C \) is written in the form of \( \sigma_C \delta A_C \) by the relation between the entropy and the horizon area, where \( \sigma_C = \frac{1}{2} r_C T_r \big|_{r_C} = -\frac{\Delta r_C}{16\pi} \). The inner of a black hole horizon is defined as an inaccessible region surrounded by the horizon. The horizon area increases when a particle is absorbed by the black hole, and then the change of the horizon area is positive \( (\delta A_+ > 0) \). While the outer of the cosmological horizon is an inaccessible region. When a particle enter the cosmology, the volume increases with the increase of the horizon area and radius. Thus, Eq. (19) is rewritten as

\[
\delta E_C = T_C \delta S_C + \sigma_C \delta A_C, \tag{20}
\]

which is the modified first law of thermodynamics at the cosmological horizon and \( \sigma \) is the surface tension. The corresponding Smarr relation

\[
E_C = 2(T_C S_C + \sigma_C A_C) \tag{21}
\]

is obeyed. The Gibbs free energy at this horizon is

\[
G_C = E_C - T_C S_C - \sigma_C A_C. \tag{22}
\]

Using the related expressions, we get \( G_C = -\frac{r_C}{4} \). The differential expression \( \delta G_C = -S_C \delta T_C - A_C \delta \sigma_C \) is also satisfied. From the relation between the entropy and the area, Eq. (20) can be written as \( \delta E_C = T_C \delta S_C \) which shows that the change of the energy is completely caused by absorbing (or releasing) heat. \( T_C e = T_C + 4\sigma_C = -\frac{1}{4\pi r_C} \) denotes the effective temperature. However, this process is not allowed due to the appearance of the negative temperature. When the energy changes, both of the following phenomena happen simultaneously, namely, the system is doing work to the external world and the heat is absorbed or released. Therefore, the worked term \( \sigma_C \delta A_C \) must exist in the first law at the cosmological horizon.

Now we treat the cosmological constant as the pressure and reconsider the thermodynamics. Using the relation between the area \( \sigma_C \) and the volume \( V_C \), we write Eq. (20) as

\[
\delta E_C = T_C \delta S_C + P_C \delta V_C, \tag{23}
\]

where \( P_C = -\frac{\Lambda}{8\pi} \) denotes the pressure at the cosmological horizon. Equation (23) can be furthermore written as

\[
\delta E_{0C} = T_C \delta S_C - V_C \delta P_C, \tag{24}
\]

where \( E_{0C} = E_C - P_C V_C = -M \). This formula is the first law of thermodynamics in the extended phase space. The corresponding Smarr relation \( E_{0C} = 2(T_C S_C + P_C V_C) \)
is also satisfied. Therefore, both of the formulae (20) and (24) are the modified first laws of thermodynamics at the cosmological horizon.

Following the work of Kubiznak and Simovic [30], we treat the cosmological and black hole horizons simultaneously. It is readily found that

\[ G_C(r_C) = -G_+(r_+ \to r_C), \quad T_C(r_C) = -T_+(r_+ \to r_C), \]

(25)

where the definitions of \( G_C, G_+, T_C \) and \( T_+ \) are given in Eqs. (22), (12), (15) and (5), respectively. Therefore, the thermodynamics of these two horizons in the Schwarzschild de Sitter black hole is encoded in \( G = G(r, T) \), given by \( G = \frac{r}{4} \) and \( T = \frac{1-\Lambda r^2}{4\pi r} \). This result is fully in accordance with that gotten by Kubiznak and Simovic.

3 Modified thermodynamics at the apparent horizon of the FRW cosmology

The modified thermodynamics at the apparent horizon of the FRW cosmology is derived by the surface tension in this section. The thermodynamics were studied in [42–46]. Cai and Kim [42] derived the Friedmann equations describing the dynamics of the cosmology with any spatial curvature from the first law of thermodynamics at the apparent horizon and the Bekenstein area-entropy formula. The law is expressed as 

\[ -dE = TdS, \]

where \( dE \) is related to the Misner–Sharp mass and expresses the energy crossing the apparent horizon during an infinitesimal time interval. On the other hand, from the Friedmann equation, the first law at the apparent horizon was gotten in [43].

The FRW metric is given by

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr_0^2}{1 - kr_0^2/a^2} + r_0^2 d\Omega_2^2 \right), \]

(26)

where \( r_0 \) is the comoving coordinate and \( a \) is the scale factor. \( d\Omega_2^2 \) expresses the 2-dimensional sphere with unit radius. \( k = 1, 0 \) and \(-1\) corresponds to a closed, flat and open cosmology, respectively. Define \( r = ar_0 \), the metric (26) becomes

\[ ds^2 = -\frac{1 - r^2/r_A^2}{1 - kr^2/a^2} dt^2 - \frac{2Hr}{1 - kr^2/a^2} dt dr + \frac{1}{1 - kr^2/a^2} dr^2 + r^2 d\Omega_2^2. \]

(27)

In the above equation, \( r_A = \frac{1}{\sqrt{H^2 + k/a^2}} \) is the location of the apparent horizon and \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. When \( k = 0 \), the apparent horizon is just the Hubble horizon. In the Sect. 2, the metric (4) describes the pure de Sitter spacetime when \( M = 0 \). If we perform the transformation \( d\tilde{t} = dt + \frac{Hr}{1 - kr^2/a^2} dr \) on the metric (27) and let \( k = 0 \) and \( r_A = H^{-1} = \sqrt{\frac{3}{\Lambda}} \), the metric (27) is reduced to the pure de Sitter metric. Therefore, the pure de Sitter spacetime is a special case of the FRW cosmology and there may exist some similar properties for them.
For convenience of calculation, we order \( F(r) = \frac{1-r^2/r_A^2}{1-k r^2/a^2} \), \( Y(r) = \frac{2H r}{1-k r^2/a^2} \) and \( G(r) = 1 - k r^2/a^2 \). The temperature and the entropy at the apparent horizon are

\[
T = \frac{1}{2 \pi r_A}, \quad S = \frac{A}{4} = \pi r_A^2,
\]

respectively. The Misner–Sharp mass in the apparent horizon is gotten as \( M = \frac{1}{2} r_A \) \[38,44–46\]. To discuss the thermodynamics, we first calculate the radial Einstein’s equation at the apparent horizon. It is gotten as

\[
G'_r |_{r_A} = 8 \pi T'_r |_{r_A} = \frac{4 F'(r_A) - Y^2(r_A)}{r_A^2 Y^2(r_A)}.
\]

Although the temperature has been expressed in Eq. (28), it isn’t convenient for us to discuss the thermodynamics. We write the temperature in the following form

\[
T = -\sqrt{G(r_A)} F'(r_A) \frac{2}{2 \pi Y(r_A)}.
\]

From Eqs. (29) and (30), the temperature is solved as

\[
T = -\sqrt{G(r_A)} Y(r_A) \frac{2}{8 \pi r_A} \left( 8 \pi r_A^2 T'_r |_{r_A} + 1 \right).
\]

Multiplying by \( \delta S \) on the both sides of the above equation yields

\[
T \delta S = -2 r_A T'_r |_{r_A} \delta S - \frac{1}{2} \delta r_A.
\]

Move the first term on rhs to lrs. Identifying the energy as the negative Misner–Sharp mass yields \( \delta E = -\frac{1}{2} \delta r_A \). Using the relation between the entropy and the horizon area, we get the modified first law of thermodynamics at the apparent horizon as

\[
\delta E = T \delta S + \sigma \delta A,
\]

where \( \sigma = \frac{1}{2} r_A T'_r |_{r_A} = -\frac{3}{16 \pi r_A} \) denotes the surface tension at the apparent horizon. This formula is full in consistence with that derived in Eq. (20) and shows the similarity between the de Sitter spacetime and the FRW cosmology. The corresponding Small relation

\[
E = 2(TS + \sigma A),
\]

is obeyed. The Gibbs free energy at the apparent horizon is

\[
G = E - TS - \sigma A.
\]
Using the related expressions yields \( G = -\frac{r_A}{4} = -\frac{1}{8\pi T} \). The specific heat is \( C = \frac{\partial H}{\partial T} |_\sigma = \frac{1}{8\pi T^2} \). It is easily found that there is no criticality, which is different from that got in [30]. Similarly, we can get the differential expression as \( \delta G = -S\delta T - A\delta \sigma \). If Eq. (33) is written as \( \delta E = T_e \delta S \), \( T_e = -\frac{1}{4\pi r_A} \) denotes the effective temperature at the apparent horizon. The negative temperature appears and isn’t allowed here. Therefore, when the heat flux produces at the horizon, the system is doing work on the surroundings. The worked term exists in the first law at the apparent horizon.

We make further to explore the thermodynamics of the FRW cosmology. Replacing the area \( A \) with the volume \( V \) in Eq. (33), we get

\[ \delta E = T \delta S + P \delta V, \]  

(36)

where \( P = \frac{2\sigma}{r_A} = -\frac{3}{8\pi r_A^2} \) is the pressure at the apparent horizon. Now the corresponding Smarr relation isn’t satisfied.

### 4 Discussion and conclusion

When \( M = 0 \), the Schwarzschild de Sitter black hole is reduced to the pure de Sitter spacetime which is the special case of the FRW spacetime. Therefore, the modified first laws of the de Sitter peacetime also satisfy Eqs. (20) and (24). In the FRW spacetime, if the apparent horizon radius \( r_A \) is treated as a variable associated to the pressure \( P = -\frac{3}{8\pi r_A^2} \), the modified law (36) can be further written as \( \delta E_0 = T \delta S - V \delta P \), where \( E_0 = 0 \). The corresponding Smarr relation is satisfied. Furthermore, the laws (10), (20) and (33) can be written as a relation \( \delta E = \delta Q + \delta W \), where \( \delta E \) express the change of the energy of the system, \( \delta Q = T \delta S \) denotes the change of heat and \( \delta W \) is a worked term. At the black hole horizon, the worked term in the modified law can disappear. While at the cosmological and apparent horizons, the worked terms must exist.

In this paper, we derived the modified first laws of thermodynamics at the black hole horizon and the cosmological horizon of the Schwarzschild de Sitter black hole and the apparent horizon of the FRW cosmology by the surface tensions, respectively. In the black hole, the black hole horizon and the cosmological horizon were seen as two independent thermodynamical systems. The modified laws (10) and (20) were gotten when the cosmological constant was fixed. When the constant was seen as a variable associated to the pressure, we obtained the first laws (14) and (24) in the extended phase spaces. The law at the the apparent horizon of the FRW spacetime takes on the same form as that at the cosmological horizon, but is different from that at the black hole horizon.

**Acknowledgements** This work is supported by the National Natural Science Foundation of China (Grant No. 11875095).
References

1. Padmanabhan, T.: Classical and quantum thermodynamics of horizons in spherically symmetric spaces. Class. Quant. Grav. 19, 5387 (2002). arXiv:gr-qc/0204019
2. Cai, R.G.: Connections between gravitational dynamics and thermodynamics. J. Phys. Conf. Ser. 484, 012003 (2014)
3. Sarkar, S., Kothawala, D.: Hawking radiation as tunneling for spherically symmetric black holes: a generalizad treatment. Phys. Lett. B 659, 683 (2008). arXiv:0709.4448 [gr-qc]
4. Padmanabhan, T., Kothawala, D.: Lanczos–Lovelock models of gravity. Phys. Rep. 531, 115 (2013). arXiv:1302.2151 [gr-qc]
5. Kothawala, D., Sarkar, S., Padmanabhan, T.: Einstein’s equations as a thermodynamic identity: the cases of stationary axisymmetric horizons and evolving spherically symmetric horizons. Phys. Lett. B 652, 338 (2007). arXiv:gr-qc/0701002
6. Akbar, M., Siddiqui, A.A.: Charged rotating BTZ black hole and thermodynamic behavior of field equations at its horizon. Phys. Lett. B 656, 217 (2007). arXiv:1302.2151 [gr-qc]
7. Kubiznak, D., Mann, R.B.: P-V criticality of charged AdS black holes. JHEP 1207, 033 (2012).
8. Dolan, B.P.: Pressure and volume in the first law of black hole thermodynamics. Class. Quant. Grav. 28, 235017 (2011). arXiv:1106.6260 [gr-qc]
9. Gunasekaran, S., Kubiznak, D., Mann, R.B.: Extended phase space thermodynamics for charged and rotating black holes and Born–Infeld vacuum polarization. JHEP 1211, 110 (2012). arXiv:1208.6251 [hep-th]
10. Wei, S.W., Liu, Y.X.: Critical phenomena and thermodynamic geometry of charged Gauss–Bonnet AdS black holes. Phys. Rev. D 87, 044014 (2013). arXiv:1209.1707 [gr-qc]
11. Wei, S.W., Liu, Y.X.: Insight into the microscopic structure of an AdS black hole from thermodynamical phase transition. Phys. Rev. Lett. 115, 111302 (2015). arXiv:1502.00386 [gr-qc]
12. Hendi, S.H., Vahidinia, M.H.: Extended phase space thermodynamics and P-V criticality of black holes with nonlinear source. Phys. Rev. D 88, 084045 (2013). arXiv:1212.6128 [hep-th]
13. Cai, R.G., Cao, L.M., Li, L., Yang, R.Q.: P-V criticality in the extended phase space of gauge–Bonnet black holes in AdS space. JHEP 1309, 005 (2013). arXiv:1306.6233 [gr-qc]
14. Altamirano, N., Kubiznak, D., Mann, R.B.: Reentrant phase transitions in rotating AdS black holes. Phys. Rev. D 88, 101502 (2013). arXiv:1306.5756 [hep-th]
15. Zou, D.C., Zhang, S.J., Wang, B.: Critical behavior of Born–Infeld AdS black holes in the extended phase space thermodynamics. Phys. Rev. D 89, 044002 (2014). arXiv:1311.7299 [hep-th]
16. Giri, Y., Kim, W., Yi, S.H.: The first law of thermodynamics in Lifshitz black holes revisited. JHEP 1407, 002 (2014). arXiv:1403.4704 [hep-th]
17. Johnson, C.V.: The extended thermodynamic phase structure of Taub–NUT and Taub–Bolt. Class. Quant. Grav. 31, 225005 (2014). arXiv:1406.4533 [hep-th]
18. Mirza, B., Sherkatghanad, Z.: Phase transitions of hairy black holes in massive gravity and thermodynamic behavior of charged AdS black holes in an extended phase space. Phys. Rev. D 90, 084006 (2014). arXiv:1409.6839 [gr-qc]
19. Suresh, J., Tharanath, R., Kurikose, V.C.: A unified thermodynamic picture of Hoava–Lifshitz black hole in arbitrary space time. JHEP 1501, 019 (2015). arXiv:1408.0911 [gr-qc]
20. Dehghani, M.H., Kamrani, S., Sheykhi, A.: P-V criticality of charged dilatonic black holes. Phys. Rev. D 90, 104020 (2014). arXiv:1505.02386 [hep-th]
21. Xu, W., Zhao, L.: Critical phenomena of static charged AdS black holes in conformal gravity. Phys. Lett. B 736, 214 (2014). arXiv:1405.7665 [gr-qc]
22. Mustapha, A.A., Marques, G.T., Rodrigues, M.E.: Phantom black holes and critical phenomena. JCAP 1407, 036 (2014). arXiv:1405.5745 [gr-qc]
23. Armas, J., Obers, N.A., Sanchioni, M.: Gravitational tension, spacetime pressure and black hole volume in arbitrary space time. JHEP 1609, 124 (2016). arXiv:1512.09106 [hep-th]
24. Zhao, Z.X., Jing, J.L.: Ehrenfest scheme for complex thermodynamic systems in full phase space. JHEP 1411, 037 (2014). arXiv:1405.2640 [gr-qc]
25. Mo, J.X., Liu, W.B.: P-V criticality of topological black holes in Lovelock–Born–Infeld gravity. Eur. Phys. J. C 74, 2836 (2014). arXiv:1401.0785 [gr-qc]
26. Poshteh, M.B.J., Mirza, B., Sherkatghanad, Z.: Phase transition, critical behavior, and critical exponents of Myers–Perry black holes. Phys. Rev. D 88, 024005 (2013). arXiv:1306.4516 [gr-qc]
27. Hansen, D., Kubiznak, D., Mann, R.B.: Criticality and surface tension in rotating horizon thermodynamics. Class. Quant. Grav. 33, 165005 (2016). arXiv:1604.06312 [gr-qc]
28. Ghezelbash, A., Mann, R.B.: Action, mass and entropy of Schwarzschild–de Sitter black holes and the de Sitter/CFT correspondence. JHEP 0201, 005 (2002). arXiv:hep-th/0111217
29. Balasubramanian, V., de Boer, J., Minic, D.: Mass, entropy and holography in asymptotically de Sitter spaces. Phys. Rev. D 65, 123508 (2002). arXiv:hep-th/0110108
30. Kubiznak, D., Simovic, F.: Thermodynamics of horizons: de Sitter black holes. Class. Quant. Grav. 33, 245001 (2016). arXiv:1507.08630 [hep-th]
31. Sekiwa, Y.: Thermodynamics of de Sitter black holes: thermal cosmological constant. Phys. Rev. D 73, 084009 (2006). arXiv:hep-th/0602269
32. Gomberoff, A., Teitelboim, C.: de Sitter black holes with either of the two horizons as a boundary. Phys. Rev. D 67, 104024 (2003)
33. Wang, B.B., Huang, C.G.: Thermodynamics of Reissner–Nordstrom–de Sitter black hole in York’s formalism. Class. Quant. Grav. 19, 2491 (2002)
34. Urano, M., Tomimatsu, A., Saida, H.: Mechanical first law of black hole spacetimes with cosmological constant and its application to Schwarzschild–de Sitter spacetime. Class. Quant. Grav. 26, 105010 (2009). arXiv:0903.4230 [gr-qc]
35. Zhao, H.H., Zhang, L.C., Ma, M.S., Zhao, R.: P–V criticality of higher dimensional charged topological dilaton de Sitter black holes. Phys. Rev. D 90, 064018 (2014)
36. Gibbons, G.W., Hawking, S.W.: Cosmological event horizons, thermodynamics, and particle creation. Phys. Rev. D 15, 2738 (1977)
37. Misner, C.W., Sharp, D.H.: Relativistic equations for adiabatic, spherically symmetric gravitational collapse. Phys. Rev. B 571, 136 (1964)
38. Cahill, M., McVittie, G.: Spherical symmetry and mass–energy in general relativity. I. general theory. J. Math. Phys. 11, 1382 (1970)
39. Hayward, S.A.: Quasi-local gravitational energy. Phys. Rev. D 49, 831 (1994). arXiv:gr-qc/9303030
40. Hu, Y.P., Zhang, H.S.: Misner–Sharp mass and the unified first law in massive gravity. Phys. Rev. D 92, 024006 (2015). arXiv:1502.00069 [hep-th]
41. Cai, R.G., Kim, S.P.: First law of thermodynamics and Friedmann equations of Friedmann–Robertson–Walker universe. JHEP 0502, 050 (2005). arXiv:hep-th/0501055
42. Gong, Y.G., Wang, A.Z.: Friedmann equations and thermodynamics of apparent horizons. Phys. Rev. Lett. 99, 211301 (2007). arXiv:0704.0793 [hep-th]
43. Cai, R.G., Cao, L.M., Hu, Y.P.: Corrected entropy-area relation and modified Friedmann equations. JHEP 0808, 090 (2008). arXiv:0807.1232 [hep-th]
44. Li, L.F., Zhu, J.Y.: Thermodynamics in loop quantum cosmology. Adv. High Energy Phys. 2009, 905705 (2009). arXiv:0812.3544 [gr-qc]
45. Zhu, T., Ren, J.R., Singleton, D.: Hawking-like radiation as tunneling from the apparent horizon in a FRW universe. Int. J. Mod. Phys. D 19, 159 (2010). arXiv:0902.2542 [hep-th]

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.