An hybrid deterministic-stochastic iterative procedure to solve the heat equation

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Abstract

Our goal in this paper is to solve the 1-D heat equation by an hybrid deterministic-
stochastic iterative procedure. The deterministic side consists in discretizing the equation
by the Crank-Nicolson method and the stochastic side consists of applying
Robbins Monro procedure to solve the resulting matrix system. The almost complete
convergence and the rate of convergence of our procedure are established.

Keywords: Robbins-Monro, heat equation, Crank-Nicolson, Almost complete convergence.

Mathematics Subject Classification: 65N21, 65C50.

Introduction

Consider the heat equation problem on an interval $I$ in $\mathbb{R}$,

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} - D \frac{\partial^2 u(x,t)}{\partial x^2} &= 0, \\
   u(x,0) &= f(x).
\end{align*}
\]

Many researchers have worked on the one dimensional heat conduction equation using
various numerical methods and finite difference methods are the mostly used of all the
numerical methods [12, 13]. Recently, O. Nikan et al [7] proposed an efficient coupling of
the Crank–Nicolson scheme and localized meshless technique for viscoelastic wave model in
fluid flow, C.Chen et al [1] present a second-order accurate Crank-Nicolson scheme for the
two-grid finite elements for nonlinear Sobolev equations and M. Ran et al [10] proposed
a linearized Crank–Nicolson scheme for the nonlinear time space fractional Schrödinger
equations.

Many other authors proposed iterative methods to solve the heat conduction equation.
For example, Newell–Whitehead–Segel equations of fractional order are solved by fractional
variational iteration method in [9] and Landweber iterative regularization method

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to identify the initial value problem of the time-space fractional diffusion-wave equation was studied in [16].

Stochastic algorithms are part of modern techniques for numerical solution of many practical problems: signal processing and adaptive control [2 4], inverse problems [6], communication and system identification [15, 5, 3].

Let $\left( \Omega, \mathcal{F}, P \right)$ be a probability space and assume that $\inf \{ r \in \lambda ; \lambda \in \sigma (A) \} > 0$ with $\sigma (A)$ is a spectrum of a matrix $A$.

Our goal is to solve the heat equation (1) by a hybrid iterative procedure (deterministic and stochastic). The deterministic side consists in discretizing equation (1) by the Crank-Nicolson method and the stochastic side consists of applying Robbins Monro procedure [11], which uses the full forward model when the noise on the right hand side is stochastic, to solve the resulting matrix system.

By Hoeffding exponential inequalities, the almost complete convergence and the rate of convergence of the iterative procedure to solve the heat equation (1) are established.

1 Methodology

The crank-Nicolson method combines the stability of an implicit method with the accuracy of a second-order method in both space and time. Simply from the average of the explicit and implicit FTCS schemes (left and right sides are centred at time step $m + 0.5$).

The following approximation expression holds

$$\frac{\partial u(x,t)}{\partial t} - D \frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u_{n,m+1} - u_{n,m}}{\Delta t} - \frac{D}{2} \left( \frac{u_{n+1,m+1} - 2u_{n,m+1} + u_{n-1,m+1} + u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{\Delta x^2} \right).$$

The scheme can be written as follow

$$au_{n-1,m} + (2 - 2a)u_{n,m} + au_{n+1,m} = -au_{n-1,m+1} + (2 + 2a)u_{n,m+1} - au_{n+1,m+1}.$$ 

These equations holds for $1 \leq n \leq N - 1$, and the boundary conditions supply the two missing equations. The Crank-Nicolson method will be written in the following matrix form

$$
\begin{pmatrix}
-a & 2 + 2a & -a & 0 & . & . & . \\
0 & -a & 2 + 2a & -a & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
\end{pmatrix}
\begin{pmatrix}
u_{0,m+1} \\
u_{1,m+1} \\
. \\
. \\
. \\
. \\
u_{N-1,m+1} \\
u_{N,m+1}
\end{pmatrix}
= \begin{pmatrix}\end{pmatrix}
\begin{pmatrix}
u_{0,m+1} \\
u_{1,m+1} \\
. \\
. \\
. \\
. \\
u_{N-1,m+1} \\
u_{N,m+1}
\end{pmatrix}
$$
The matricial equation above is a representation of \( N-1 \) equations with \( N+1 \) unknown, the matrices have \( N-1 \) rows and \( N+1 \) columns and the two missing equations come from the boundary conditions which we use to convert this matricial equation into a system of equations involving a square matrix of the form:

\[
A_{m+1}u_{m+1} + r_{m+1} = B_m u_m + w_m,
\]

with

\[
A_{m+1} = \begin{pmatrix}
2 + 2a & -a & 0 & \ldots & . \\
-a & 2 + 2a & . & \ldots & . \\
0 & . & . & \ldots & . \\
. & . & . & \ldots & -a \\
. & . & . & \ldots & 0 & -a & 2 + 2a & -a \\
. & . & . & \ldots & 0 & -a & 2 + 2a \\
\end{pmatrix},
\]

\[
B_m = \begin{pmatrix}
2 - 2a & a & 0 & \ldots & . \\
-a & 2 - 2a & a & \ldots & . \\
0 & . & . & \ldots & . \\
. & . & . & \ldots & a \\
. & . & . & \ldots & 0 & a & 2 - 2a & a \\
. & . & . & \ldots & 0 & a & 2 - 2a \\
\end{pmatrix},
\]

The equation \((2)\) can be written in following form

\[
A_{m+1}u_{m+1} = B_m u_m + w_m - r_{m+1}.
\]

The matrix inversion method to find the solution \( u_{m+1} \) is very time consuming and computationally inefficient. In this paper, we propose the following iterative procedure to solve the equation \((3)\).
\[
\begin{align*}
\{ X_{(k+1)} &= X_{(k)} - \frac{1}{k} \left[ A_{m+1} X_{(k)} - (B_m u_m + w_m - r_{m+1}) - \xi_{(k)} \right], \quad k = 1, 2, \ldots, \\
X_{(0)} &\in \mathbb{R}^{N-1}
\end{align*}
\]

where \((X_{(k)})_{k \in \mathbb{N}}\) are vectors of \(\mathbb{R}^n\) and \((\xi_{(k)})_{k \in \mathbb{N}}\) is a sequence of bounded and i.i.d random variables with values in \(\mathbb{R}^n\) satisfying
\[
\left\| \xi_{(k)} \right\| < b, \quad b \in \mathbb{R}
\]

2 Preliminary results

According to [6, lemma 1] and after successive iterations, the following relation is obtained
\[
X_{(k+1)} - X_{(ex)} = \prod_{i=1}^{k} (I - \frac{1}{i} A_{m+1}) (X_{1} - X_{ex}) + \sum_{i=1}^{k} \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \frac{1}{i} \xi_{i},
\]

where, \(X_{(ex)}\) is the exact solution and \(\prod_{j=k+1}^{k} (I - \frac{1}{j} A_{m+1}) = I\) for \(1 \leq i, j \leq k\), with \(I\) is the unit matrix.

**Lemma 1** Suppose that \(\inf \{ \text{re} \lambda; \lambda \in \sigma(A_{m+1}) \} > 0\). The following expression holds.
\[
\exists \gamma > 0, \exists \ p > 0, \forall 1 \leq i \leq k : \left\| \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \right\| \leq \gamma \left( \frac{i+1}{k+1} \right)^p .
\]

**Proof.** Under the condition \(\inf \{ \text{re} \lambda; \lambda \in \sigma(A_{m+1}) \} > 0\), H. Walk obtained the following result [14, Lemma 3.b]
\[
\exists \gamma > 0, \exists \ p > 0, \forall 1 \leq i \leq k : \left\| \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \right\| \leq \gamma \left( \prod_{j=i+1}^{k} \left( 1 - \frac{1}{j} \right) \right)^p .
\]

Then, taking into account that: \(\ln (1 + x) \leq x\), for \(x > -1\), we have
\[
p \ln \left( 1 - \frac{1}{j^p} \right) \leq - \frac{p}{j^p}
\]
\[
\sum_{j=i+1}^{k} \ln \left( 1 - \frac{1}{j^p} \right)^p \leq -p \int_{i+1}^{k+1} \frac{1}{x^p} dx.
\]
Then
\[ \sum_{j=i+1}^{k} \ln \left(1 - \frac{1}{j^\theta} \right)^p \leq p \ln \left(\frac{i+1}{k+1} \right). \]

This implies that
\[ \exp \left( \sum_{j=i+1}^{k} \ln \left(1 - \frac{1}{j^\theta} \right)^p \right) \leq \exp \left( p \ln \left(\frac{i+1}{k+1} \right) \right). \]

So,
\[ \gamma \left( \prod_{j=i+1}^{k} \left(1 - \frac{1}{j^\theta} \right) \right)^p \leq \gamma \left(\frac{i+1}{k+1} \right)^p. \]  \hspace{1cm} (9)

From (9) we deduce that
\[ \lim_{k \to \infty} \left\| \prod_{j=1}^{k} (I - \frac{1}{j} A_{m+1}) \right\| \leq \lim_{k \to \infty} \gamma \left( \prod_{j=1}^{k} \left(1 - \frac{1}{j^\theta} \right) \right)^p = 0. \]  \hspace{1cm} (10)

\[ \boxed{\text{Lemma 2} \quad \text{Under assumptions of Lemma 1, the following expression holds.}} \]

\[ \exists \gamma > 0, \exists p > 0, \forall 1 \leq i \leq k : \sum_{i=1}^{k} \left\| \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \right\| \leq C \frac{\gamma^2 (i+1)^{2p}}{(k+1)^{2p}}, \]  \hspace{1cm} (11)

with $C$ is a constant.

\[ \text{Proof.} \quad \text{By virtue of the relation (7) one has} \]

\[ \exists \gamma > 0, \exists p > 0, \forall 1 \leq i \leq k : \left\| \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \right\|^2 \leq \gamma^2 \frac{(i+1)^{2p}}{(k+1)^{2p}}, \]

Then
\[ \exists \gamma > 0, \exists p > 0, \forall 1 \leq i \leq k : \sum_{i=1}^{k} \left\| \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \right\|^2 \leq \frac{\gamma^2 (i+1)^{2p}}{(k+1)^{2p}} \sum_{i=1}^{k} \frac{(i+1)^{2p}}{i^2}. \]  \hspace{1cm} (12)
By Kronecker’s lemma, \( \frac{n^2}{(k+1)^{2p}} \sum_{i=1}^{k} \frac{(i+1)^{2p}}{i^2} \) tends to 0 when \( k \) tends to infinity. In fact, \( (\frac{1}{i})_{i \in \mathbb{N}^*} \) is a convergent sequence and \( \lim_{i \to +\infty} (i+1)^{2p} = +\infty. \) So,
\[
\lim_{i \to +\infty} \frac{1}{(k+1)^{2p}} \sum_{i=1}^{k} \frac{(i+1)^{2p}}{i^2} = 0. \tag{13}
\]
From the relation (13) one deduces that: \( \exists (N^* \in \mathbb{N}^*) \) such that
\[
\lim_{k \to +\infty} \frac{1}{(k+1)^{2p}} \sum_{i=N^*+1}^{k} \frac{(i+1)^{2p}}{i^2} = 0. \tag{14}
\]
We have the following relationship
\[
\frac{1}{(k+1)^{2p}} \sum_{i=1}^{k} \frac{(i+1)^{2p}}{i^2} = \frac{1}{(k+1)^{2p}} \sum_{i=1}^{N^*} \frac{(i+1)^{2p}}{i^2} + \frac{1}{(k+1)^{2p}} \sum_{i=N^*+1}^{k} \frac{(i+1)^{2p}}{i^2} \leq \frac{1}{(k+1)^{2p}} \sum_{i=1}^{N^*} \frac{(i+1)^{2p}}{i^2} = C \frac{1}{(k+1)^{2p}},
\]
with, \( \sum_{i=1}^{N} \frac{(i+1)^{2p}}{i^2} = C. \)
Replacing in (12) we find (11).

\[\Box\]

3 Exponential inequalities and convergence results

In this section, exponential inequalities of the Hoeffding type are established. These allow us to establish the almost complete convergence and the rate of convergence of the iterative procedure (4) to solve the heat equation (1).

Definition 1 The sequence of random variables \( (X_{(k)})_{k \in \mathbb{N}^*} \) converges almost completely (a.co) to a random variable \( X \), when \( k \) tends to infinity, if and only if: \( \forall \varepsilon > 0, \lim_{k \to +\infty} \mathbb{P} \left( \|X_{(k)} - X_{(ex)}\| > \varepsilon \right) = 0. \)

Theorem 2 Let \( A \in L(H) \). Under the condition \( \inf \{ \text{re } \lambda; \lambda \in \sigma(A) \} > 0 \), the following exponential inequalities hold.
\[
\mathbb{P} \left( \|X_{(k+1)} - X_{(ex)}\| > \varepsilon \right) \leq 2 \exp \left( -\frac{(k+1)^{2p}\varepsilon^2}{\alpha} \right), \alpha \in \mathbb{R}. \tag{15}
\]
Proof. By virtue of the relation (6), one has the following expression

\[
P \left( \| X_{(k+1)} - X_{(ex)} \| > \varepsilon \right) = P \left( \left\| \prod_{i=1}^{k} (I - \frac{1}{j} A_{m+1}) (X_{(1)} - X_{(ex)}) + \sum_{i=1}^{k} \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \frac{1}{i} \xi_i \right\| > \varepsilon \right)
\]

\[
\leq P \left( \left\| \sum_{i=1}^{k} \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \frac{1}{i} \xi_i \right\| > \varepsilon - \left\| \prod_{i=1}^{k} (I - \frac{1}{j} A_{m+1}) (X_{(1)} - X_{(ex)}) \right\| \right).
\]

The relation (10) proves that:

\[
\exists \varepsilon > 0, \left\| \prod_{i=1}^{k} (I - \frac{1}{j} A_{m+1}) (X_{(1)} - X_{(ex)}) \right\| \leq \frac{\varepsilon}{2}. \tag{16}
\]

Then,

\[
P \left( \| X_{(k+1)} - X_{(ex)} \| > \varepsilon \right) \leq P \left( \left\| \sum_{i=1}^{k} \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \frac{1}{i} \xi_i \right\| > \varepsilon - \frac{\varepsilon}{2} \right)
\]

\[
\leq P \left( \left\| \sum_{i=1}^{k} \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \frac{1}{i} \xi_i \right\| > \frac{\varepsilon}{2} \right).
\]

We pose

\[
\eta_i = \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \frac{1}{i} \xi_i. \tag{17}
\]

\((\eta_i)_{i \in \mathbb{N}^*}\) is a sequence of bounded and i.i.d random variables in a Hilbert space such that

\[
\| \eta_i \| < \left\| \prod_{j=i+1}^{k} (I - \frac{1}{j} A_{m+1}) \frac{1}{i} b = d_i. \tag{18}\]

Thus

\[
P \left( \| X_{(k+1)} - X_{(ex)} \| > \varepsilon \right) \leq P \left( \left\| \sum_{i=1}^{k} \eta_i \right\| > \frac{\varepsilon}{2} \right). \tag{19}\]
We give the Pinelis-Hoeffding inequality for the sequence \((\eta_i)_{i\in\mathbb{N}}\) such that \(\|\eta_i\| < d_i\) in a Hilbert space \(H\) (\[8\]).

\[
P \left( \left\| \sum_{i=1}^{k} \eta_i \right\| > \varepsilon \right) \leq 2 \exp \left( -\frac{\varepsilon^2}{2 \sum_{i=1}^{k} d_i^2} \right).
\]  

(20)

Then, we deduce from (18) and (20) the following relation.

\[
P \left( \left\| \sum_{i=1}^{k} \eta_i \right\| > \frac{\varepsilon}{2} \right) \leq 2 \exp \left( -\frac{\varepsilon^2}{8b^2 \sum_{i=1}^{k} \prod_{j=i+1}^{k} (I - \frac{1}{2}A_{m+1}) (\frac{1}{2})^2} \right).
\]  

(21)

Finally, by virtue of the relation (19) and (11) and from the relation (21), we obtain

\[
P \left( \|X_{(k+1)} - X_{(ex)}\| > \varepsilon \right) \leq 2 \exp \left( -\frac{\varepsilon^2}{8C(\gamma b)^2 (k+1)^{2p}} \right).
\]

(22)

By putting \(\alpha = 8C(\gamma b)^2\) we will find (15).

In the next corollary, we give the proof that the iterative procedure (4) converges almost completely to the solution of the equation (1).

**Corollary 1** Under the conditions of Theorem 1:

The recursive procedure (4) converges almost completely (a.co) to the solution of the equation (1):

\[
\forall \varepsilon > 0, \sum_{k=1}^{+\infty} P \left( \|X_{(k+1)} - X_{(ex)}\| > \varepsilon \right) < +\infty.
\]  

(22)

Additionally,

\[
\|X_{(k+1)} - X_{(ex)}\| = O(k^{-2p}), \quad p > \frac{1}{2}.
\]  

(23)

**Proof.** 1) Let us pose

\[
v_k = 2 \exp \left( -\frac{(k+1)^{2p}\varepsilon^2}{\alpha} \right) \leq 2 \exp \left( -\frac{(k+1)^{2p}\varepsilon^2}{\alpha} \right),
\]

(24)
Applying Cauchy’s rule to the positive term series $v_k$ it follows that:
When $p > \frac{1}{2}$, $\sum_{k=1}^{+\infty} v_k$ is a convergent series.
This implies that
$$\forall \varepsilon > 0, \sum_{k=1}^{+\infty} P \left( \|X(k+1) - X(\varepsilon x)\| > \varepsilon \right) \leq \sum_{k=1}^{+\infty} 2 \exp \left( - (k + 1)^2 \varepsilon^2 \right) < +\infty. \quad (24)$$

2) To obtain (23), it is sufficient to choose $A = \varepsilon k^{2p}$ in (24) to have
$$\sum_{k=1}^{+\infty} P \left( \|X(k+1) - X(\varepsilon x)\| > A k^{-2p} \right) < +\infty.$$ 

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