The signless Laplacian spectral radius of subgraphs of regular graphs

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Abstract

Let $q(H)$ be the signless Laplacian spectral radius of a graph $H$. In this paper, we prove that

1. Let $H$ be a proper subgraph of a $\Delta$-regular graph $G$ with $n$ vertices and diameter $D$. Then
   \[ 2\Delta - q(H) > \frac{1}{n(D - \frac{4}{4})}. \]

2. Let $H$ be a proper subgraph of a $k$-connected $\Delta$-regular graph $G$ with $n$ vertices, where $k \geq 2$. Then
   \[ 2\Delta - q(H) > \frac{2(k - 1)^2}{2(n - \Delta)(n - \Delta + 2k - 4) + (n + 1)(k - 1)^2}. \]

Finally, we compare the two bounds. We obtain that when $k > 2\sqrt{\frac{(n-\Delta)(n+\Delta-4)}{n(4D-3)-2}} + 1$, the second bound is always better than the first. On the other hand, when $k < \frac{2(n-\Delta)}{\sqrt{n(4D-3)-2}} + 1$, the first bound is always better than the second.

Key Words: Irregular graph, $k$-connected graph, Signless Laplacian spectral radius, Maximum degree.

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1 Introduction

As usual, let $G = (V(G), E(G))$ be a finite, undirected and simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Set $N_G(v_i) = \{v | v_i v \in E(G)\}$ and $d_G(v_i) = |N_G(v_i)|$, or simply $N(v_i)$ and $d_i = d(v_i)$, respectively. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote

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the minimum degree and maximum degree of the graph $G$, respectively. If $\Delta = \delta$, then $G$ is regular. Let $P = x_1x_2\ldots x_l$ be a path in $G$ with a given orientation. We denote by $x_ix_j$ the path $x_ix_{i+1}\ldots x_{j-1}x_j$ for $i < j$. The distance between any two vertices $v_i$ and $v_j$ in $G$ is the number of edges in a shortest path connecting $v_i$ and $v_j$, denoted by $d_G(v_i, v_j)$. The diameter $D = D_G$ of $G$ is the maximum distance between any two vertices of $G$. The (vertex) connectivity $\kappa(G)$ of $G$ is the minimum number of vertices whose removal disconnects $G$ or reduces it to a single vertex. For an integer $k \geq 1$, $G$ is called $k$-connected if $\kappa(G) \geq k$. For terminologies and notations of graphs undefined here, we refer the reader to [1].

Let $A(G)$ be the adjacency matrix of $G$ and $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees of $G$. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of $G$, and the matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of $G$. The largest eigenvalue of $A(G)$, $L(G)$ and $Q(G)$ are called spectral radius, Laplacian spectral radius and signless Laplacian spectral radius of $G$, and denoted by $\rho(G)$, $\mu(G)$ and $q(G)$, respectively. Since $A(G)$, $L(G)$ and $Q(G)$ are real symmetric matrices, their eigenvalues are real numbers.

If $G$ is a simple connected graph, then the matrix $A(G)$ (or $Q(G)$) is a nonnegative irreducible matrix and the largest eigenvalues of $A(G)$ (or $Q(G)$) is nonnegative. By Perron-Frobenius Theorem, $\rho(G)$ (or $q(G)$) is simple and has a unique positive unit eigenvector.

We all know that $\rho(G) \leq \Delta(G)$ with equality if and only if $G$ is regular. Some good bounds on the spectral radius $\rho(G)$ of connected irregular graphs have been obtained by various authors in [2, 3, 5, 7, 11, 12, 13]. Moreover, if $H$ is a proper subgraph of a connected graph $G$, then $\rho(G) > \rho(H)$. Then Nikiforov in [9] gave a bound of $\rho(G) - \rho(H)$. So combining the above two famous results, the authors in [5, 8, 9, 12] obtained some bounds of $\rho(G) - \rho(H)$ when $H$ is the proper subgraph of connected regular graph $G$.

Also, as we all know that $q(G) \leq 2\Delta(G)$ with equality if and only if $G$ is regular [4]. In fact, in 2013, Ning et al. [10] gave a bound on the signless Laplacian spectral radius of irregular graph $G$ with $n$ vertices, maximum degree $\Delta$ and diameter $D$:

$$2\Delta - q(G) > \frac{1}{n(D - \frac{4}{3})}. \quad (1)$$

And in 2015, Chen and Hou [6] obtained a bound on the signless Laplacian spectral radius of $k$-connected irregular graph $G$:

$$2\Delta - q(G) > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)[n^2 - 2(n - k)] + nk^2}. \quad (2)$$

In [6], they also obtained when $k \geq \sqrt{n}$, bound (2) is always better than bound (1).

We also know that $q(H) \leq q(G)$ whenever $H$ is a subgraph of $G$. So we can arise the following question:

How small $q(G) - q(H)$ can be when $H$ is a subgraph of a regular graph $G$?

In this paper, we give two bounds of $q(G) - q(H)$ when $H$ is a subgraph of a regular graph $G$.

**Theorem 1.1.** Let $H$ be a proper subgraph of a $\Delta$-regular graph $G$ with $n$ vertices and diameter $D$. Then

$$2\Delta - q(H) > \frac{1}{n(D - \frac{4}{3})}. \quad (3)$$

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By taking connectivity parameter into account, we establish the following theorem.

**Theorem 1.2.** Let $H$ be a proper subgraph of a $k$-connected $\Delta$-regular graph $G$ with $n$ vertices. If $k \geq 2$, then

$$2\Delta - q(H) > \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4)+(n+1)(k-1)^2}. \quad (4)$$

Finally, we compare the two bounds. We also obtain when $k > 2\sqrt{\frac{\Delta+\Delta-4}{n(4D-3)^2}} + 1$, bound (3) is always better than bound (4). On the other hand, when $k < \frac{2(n-\Delta)}{\sqrt{n(4D-3)^2}} + 1$, bound (3) is always better than bound (4).

Moreover, we notice that $\mu(G) \leq q(G)$ when $G$ is a graph, and if $G$ is connected, then the equality holds if and only if $G$ is a bipartite graph [14]. Then we can give two upper bound of Laplacian spectral radius of subgraphs of regular graphs.

**Corollary 1.3.** Let $H$ be a proper subgraph of a $\Delta$-regular graph $G$ with $n$ vertices and diameter $D$. Then

$$2\Delta - \mu(H) > \frac{1}{n(D-\frac{1}{4})}.$$ 

**Corollary 1.4.** Let $H$ be a proper subgraph of a $k$-connected $\Delta$-regular graph $G$ with $n$ vertices. If $k \geq 2$, then

$$2\Delta - \mu(H) > \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4)+(n+1)(k-1)^2}.$$ 

2 The proofs of Theorems 1.1 and 1.2

In this section, we begin to prove Theorems 1.1 and 1.2. Before our proofs we give a lemma which is used in the proofs. It is an immediate consequence of the Cauchy-Schwarz inequality (or see [12]).

**Lemma 2.1.** ([12]) If $a, b > 0$, then $a(x-y)^2 + by^2 \geq abx^2/(a+b)$ with equality if and only if $y = ax/(a+b)$.

**Proof of Theorem 1.1.** Let $G$ be a $\Delta$-regular graph. And we suppose that $H$ is a maximal proper subgraph of $G$, i.e., $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $uv$, i.e., $H = G - uv$. Then $d_H(u) = d_H(v) = \Delta - 1$.

Let $x = (x_1, x_2, \cdots, x_n)^T$ be the unique unit positive eigenvector of $Q(H)$ corresponding to $q(H)$. Clearly, $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Let $w$ be a vertex such that $x_w = \max_{1 \leq i \leq n} x_i$. Thus we have $x_w > \frac{1}{\sqrt{n}}$.

We will prove that $u \neq w$ and $v \neq w$. Indeed, if $u = w$, then

$$q(H)x_u = (\Delta - 1)x_u + \sum_{uv \in E(H)} x_i \leq 2(\Delta - 1)x_u,$$

and thus $q(H) \leq 2(\Delta - 1)$, contradicting the fact that $q(H) > 2\delta = 2(\Delta - 1)$ since $H$ is an irregular graph (see [3]). Hence, $u \neq w$. Similarly, $v \neq w$. 

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We also find that
\[ 2\Delta - q(H) = 2\Delta \cdot 1 - x^T Q(H)x \]
\[ = 2\Delta \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} d_i x_i^2 - 2 \sum_{v_i v_j \in E(H)} x_i x_j \]
\[ = 2(x_u^2 + x_v^2) + \sum_{i=1}^{n} d_i x_i^2 - 2 \sum_{v_i v_j \in E(H)} x_i x_j \]
\[ = 2(x_u^2 + x_v^2) + \sum_{v_i v_j \in E(H)} (x_i - x_j)^2. \]

Next we consider the following two cases.

Case 1. \(d_H(w, u) \leq D - 1\).

Select a shortest path \(u = u_0, u_1, \ldots, u_l = w\) joining \(u\) to \(w\) in \(H\), i.e., \(l \leq D - 1\). By Lemma 2.1 and the Cauchy-Schwarz inequality, we have
\[ 2\Delta - q(H) = 2(x_u^2 + x_v^2) + \sum_{v_i v_j \in E(H)} (x_i - x_j)^2 \]
\[ > \sum_{i=0}^{l-1} (x_{u_i} - x_{u_{i+1}})^2 + 2x_u^2 \]
\[ \geq \frac{1}{l} \left[ \sum_{i=0}^{l-1} (x_{u_i} - x_{u_{i+1}}) \right]^2 + 2x_u^2 \]
\[ = \frac{1}{l} (x_w - x_u)^2 + 2x_u^2 \]
\[ \geq \frac{2}{2l + 1} x_w^2 > \frac{2}{2(D - 1) + 1} \cdot \frac{1}{n} \]
\[ = \frac{1}{n(D - \frac{1}{2})} > \frac{1}{n(D - \frac{1}{4})}. \]

Case 2. \(d_H(w, u) \geq D\).

In this case, by symmetry, \(d_H(w, v) \geq D\). Let \(P : u = u_0, u_1, \ldots, u_l = w\) and \(Q\) be shortest paths joining \(u\) to \(w\) and \(v\) to \(w\) in \(G\), respectively. Next we will prove that \(u \notin Q\) and \(v \notin P\).

If \(u \in Q\), then there exists a path of length at most \(D - 1\) joining \(w\) to \(u\) in \(G\), and thus in \(H\), a contradiction. Hence, \(u \notin Q\). By symmetry, \(v \notin P\).

Thus the paths \(P\) and \(Q\) belong to \(H\), and we have
\[ d_H(w, u) = d_H(w, v) = D. \]

Then we have \(l = D\). Let \(t\) be the smallest index \(j\) such that \(u_j\) is on \(Q\), then \(t \geq 1\). Obviously, \(uP\) and \(vQ\) have the same length. Using Lemma 2.1 and the Cauchy-Schwarz inequality,
it follows that
\[
2\Delta - q(H) = 2(x_u^2 + x_v^2) + \sum_{v, v_j \in E(H)} (x_i - x_j)^2 \\
\geq 2(x_u^2 + x_v^2) + \sum_{i=0}^{t-1} (x_{u_i} - x_{u_{i+1}})^2 + \sum_{i, j \in E(vQu_i)} (x_i - x_j)^2 \\
\geq \frac{1}{t} (x_{u_1} - x_u)^2 + \frac{1}{t} (x_{u_1} - x_v)^2 + 2x_v^2 + \frac{1}{D-t} (x_w - x_{u_1})^2 \\
\geq \frac{4}{2t+1} x_{u_1}^2 + \frac{1}{D-t} (x_w - x_{u_1})^2 \\
\geq \frac{4}{4D-2t+1} x_w^2 \geq \frac{4}{4D-1} x_w^2 \\
\geq \frac{1}{n(D-\frac{1}{4})}.
\]

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let \( G \) be a \( k \)-connected \( \Delta \)-regular graph. And we suppose that \( H \) is a maximal proper subgraph of \( G \), i.e., \( V(H) = V(G) \) and \( H \) differs from \( G \) in a single edge \( uv \), i.e., \( H = G - uv \). Then \( d_H(u) = d_H(v) = \Delta - 1 \). Note that \( \Delta \geq k \geq 2 \). We consider the following two cases:

Case 1. \( \Delta = 2 \).

In this case, \( G \) must be the cycle \( C_n \) on \( n \) vertices, and thus \( H \) is the path \( P_n \) on \( n \) vertices. Further, noticing that \( q(P_n) = 2 + 2\cos\frac{\pi}{n} \) and \( \sin x > x - x^3/6 \), one check that
\[
2\Delta - q(H) = 2(1 - \cos\frac{\pi}{n}) = 4\sin^2\frac{\pi}{2n} > \frac{2}{2n^2 - 7n + 9},
\]
as desired, completing the proof of Case 1.

Case 2. \( \Delta \geq 3 \).

In this case, note that \( H \) is connected since \( k \geq 2 \). Then let \( x = (x_1, x_2, \ldots, x_n)^T \) be the unique unit positive eigenvector of \( Q(H) \) corresponding to \( q(H) \). Clearly, \( x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \). And let \( w \) be a vertex such that \( x_w = \max_{1 \leq i \leq n} x_i \). By similar arguments as the proof of Theorem 1.1, we have that \( u \neq w \) and \( v \neq w \). We also find that
\[
2\Delta - q(H) = 2\Delta \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} d_i x_i^2 - 2 \sum_{v, v_j \in E(H)} x_i x_j \\
= 2(x_u^2 + x_v^2) + \sum_{i=1}^{n} d_i x_i^2 - 2 \sum_{v, v_j \in E(H)} x_i x_j \\
= 2(x_u^2 + x_v^2) + \sum_{v, v_j \in E(H)} (x_i - x_j)^2.
\]

Since \( \kappa(H - v) \geq k - 1 \), again by Menger’s Theorem, there are (at least) \( k - 1 \) vertex-disjoint paths joining \( w \) and \( u \) in \( H - v \), say \( P_1, P_2, \ldots, P_{k-1} \), which are as short as possible. Clearly,
each of these paths contains only one vertex in \( N_H(u) \), and then \( \sum_{t=1}^{k-1} |V(P_t)| \leq n - \Delta + 3k - 5 \). Thus by the Cauchy-Schwarz inequality we get

\[
\sum_{v_i,v_j \in E(G)} (x_i - x_j)^2 \geq \sum_{t=1}^{k-1} \sum_{v_iv_j \in E(P_t)} (x_i - x_j)^2 \\
\geq \sum_{t=1}^{k-1} \frac{1}{|V(P_t)| - 1} \left( \sum_{v_iv_j \in E(P_t)} (x_i - x_j)^2 \right) \\
= \sum_{t=1}^{k-1} \frac{1}{|V(P_t)| - 1} (x_w - x_u)^2 \\
\geq \frac{(k-1)^2}{\sum_{t=1}^{k-1} |V(P_t)| - 1} (x_w - x_u)^2 \\
\geq \frac{(k-1)^2}{n - \Delta + 2k - 4} (x_w - x_u)^2. \tag{6}
\]

Combining (5) and (6), and using Lemma 2.1 we have

\[
2\Delta - q(H) > 2x_u^2 + \frac{(k-1)^2}{n - \Delta + 2k - 4} (x_w - x_u)^2 \\
\geq \frac{2(k-1)^2}{2(n - \Delta + 2k - 4) + (k-1)^2} x_w^2. \tag{7}
\]

Let

\[
B = \frac{2(k-1)^2}{2(n - \Delta)(n - \Delta + 2k - 4) + (n+1)(k-1)^2}.
\]

Next we will show that \( 2\Delta - q(H) > B \).

Suppose that \( N_H(u) = \{u_1, u_2, \ldots, u_{\Delta-1}\} \). Here \( w \) may be \( u_t \) for some \( t \in \{1, 2, \ldots, \Delta - 1\} \), if this is the case, for convenience, we assume \( w = u_{\Delta-1} \).

**Subcase 2.1.** \( x_u^2 + x_w^2 \geq B/2 \).

In this case, from (5), we can get

\[
2\Delta - q(H) > 2x_u^2 + x_w^2 > B.
\]

**Subcase 2.2.** \( \sum_{t=1}^{\Delta-2} x_{u_t}^2 \geq \frac{\Delta}{2} B \)

In this case, for avoiding the possible case of \( w = u_{\Delta-1} \), then using (5) and Lemma 2.1 we obtain

\[
2\Delta - q(H) \geq 2x_u^2 + \sum_{t=1}^{\Delta-2} (x_{u_t} - x_u)^2 \\
= \sum_{t=1}^{\Delta-2} \left( \frac{2}{\Delta-2} x_u^2 + (x_{u_t} - x_u)^2 \right) \\
\geq \frac{2}{\Delta} \sum_{t=1}^{\Delta-2} x_{u_t}^2 \geq B.
\]
Subcase 2.3. $x_u^2 + x_v^2 < B/2$ and $\sum_{t=1}^{\Delta-2} x_u^2 < \frac{\Delta}{2} B$.

In this case, noticing that

$$x_w^2 \geq (1 - x_u^2 - x_v^2 - \sum_{t=1}^{\Delta-2} x_u(t)^2) / (n - \Delta) > (1 - \frac{\Delta + 1}{2} B) / (n - \Delta),$$

and from (7) again, we have

$$2\Delta - q(H) > \frac{2(k-1)^2}{2(n-\Delta+2k-4) + (k-1)^2 x_w^2}$$
$$> \frac{2(k-1)^2}{2(n-\Delta+2k-4) + (k-1)^2 (n-\Delta)(1 - \frac{\Delta + 1}{2} B)}$$
$$= \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta + 2k-4) + (n+1)(k-1)^2} = B.$$

This completes the proof of Theorem 1.2. □

3 Final remarks

It is easy to prove that when $k > 2\sqrt{\frac{(n-\Delta)(n+\Delta-4)}{n(4D-3)-2}} + 1$, bound (1) is always better than bound (3). Indeed, when $k > 2\sqrt{\frac{(n-\Delta)(n+\Delta-4)}{n(4D-3)-2}} + 1$, from (4), then we have

$$2\Delta - q(H) > \frac{2(k-1)^2}{2(n-\Delta)(n+\Delta-4) + (n+1)(k-1)^2}$$
$$= \frac{1}{(n-\Delta)(n+\Delta-4)/(k-1)^2 + (n+1)/2}$$
$$> \frac{1}{n(D-1/4)}.$$

On the other hand, when $k < \frac{2(n-\Delta)}{\sqrt{n(4D-3)-2}} + 1$, bound (3) is always better than bound (4). Indeed, when $k < \frac{2(n-\Delta)}{\sqrt{n(4D-3)-2}} + 1$, from (4), we have

$$2\Delta - q(H) > \frac{1}{n(D-1/4)}$$
$$> \frac{1}{(n-\Delta)^2/(k-1)^2 + (n+1)/2}$$
$$= \frac{2(k-1)^2}{2(n-\Delta)^2 + (n+1)(k-1)^2}$$
$$> \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta + 2k-4) + (n+1)(k-1)^2}.$$

To provide some preliminary evidence, we here list some values of bounds (3) and (4), as shown in Table 1. Graphs $G_1$ and $G_2$ are the 3-regular graphs, as shown in Figure 1. And $G_{11}$ and $G_{12}$ are the maximal subgraphs of $G_1$ and $G_{21}$ is the maximal subgraph of $G_2$, respectively, as shown in Figure 2.
Table 1: Bounds (3) and (4) of maximal subgraphs $H$ for regular graphs

| Graph $H$ | Maximal subgraph $H_i$ | $2\Delta - q(H)$ | (3) | (4) |
|-----------|------------------------|------------------|-----|-----|
| $C_6$     | $P_6$                  | 0.268            | 0.0606 | 0.05128 |
| $C_{12}$  | $P_{12}$               | 0.0682           | 0.0159 | 0.0094 |
| $K_6$     | $K_6 - e$              | 0.5359           | 0.2222 | 0.25397 |
| $K_{12}$  | $K_{12} - e$           | 0.2918           | 0.1111 | 0.14948 |
| $G_1$     | $G_{11}$               | 0.4384           | 0.0952 | 0.1379 |
| $G_1$     | $G_{12}$               | 0.4113           | 0.0952 | 0.2069 |
| $G_2$     | $G_{21}$               | 0.2907           | 0.0714 | 0.0816 |

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