On the equivalence between two problems of asymmetry on convex bodies

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Abstract

The simplex was conjectured to be the extremal convex body for the two following “problems of asymmetry”:

P1) What is the minimal possible value of the quantity \( \max_{K'} |K'|/|K| \)? Here, \( K' \) ranges over all symmetric convex bodies contained in \( K \).

P2) What is the maximal possible volume of the Blaschke-body of a convex body of volume 1?

Our main result states that (P1) and (P2) admit precisely the same solutions. This complements a result from [K. Böröczky, I. Bárány, E. Makai Jr. and J. Pach, Maximal volume enclosed by plates and proof of the chessboard conjecture], stating that if the simplex solves (P1) then the simplex solves (P2) as well.

1 Introduction

Let \( K \) be a convex body in \( \mathbb{R}^n \). The goal of this paper is to study some properties of the extremal convex bodies for the following problem.

Problem 1.1. Among all convex bodies \( K \), find the one that minimizes the quantity

\[
m(K) = \max_{K'} \frac{|K'|}{|K|},
\]

where \( K' \) ranges over all symmetric convex bodies contained in \( K \).

Here, \( | \cdot |_n = | \cdot | \) is the volume functional in \( \mathbb{R}^n \). However, the determination of these extremals is a difficult and long standing problem. Besicovitch [1] proved that triangles are extremals in the plane (see also [5]). The problem remains open in higher dimensions. However, an asymptotically sharp estimate due to Stein [26] is valid: \( m(K) > 2^{-n} \). Improvements of this result (with the same quantitative estimate) were established in [5] and [19].

A remarkable result concerning this problem is due to Fáry and Rédei [7]. They proved that there exists a unique symmetric convex body \( K' \subseteq K \) of maximal volume, called “the symmetric kernel of \( K \)”. The center of \( K' \) is often called “the pseudo-center of \( K \)” (see e.g. [20]). We will denote this by \( Ps(K) \).

Define the quantity

\[
q(K) := \max_{x \in \mathbb{R}^n} |(K + x) \cap -K| = \max_{x \in \mathbb{R}^n} |(K + x) \cap -(K + x)|.
\]
Then,
\[ m(K) = \frac{q(K)}{|K|}. \]

Note that by the uniqueness result of Fáry and Rédei (mentioned previously), there exists a unique point \( x_0 \in \mathbb{R}^n \), such that \( |(K+x_0) \cap -K| = \max_{x \in \mathbb{R}^n} |(K+x) \cap -K|. \) Since \( (K+x_0) \cap -K = [(K+x_0/2) \cap -(K+x_0/2)] + x_0/2 \), it follows that \( x_0/2 = Ps(-K) \). Set \( Q(K) = 2Ps(-K) \). Clearly, the mapping \( K \mapsto Q(K) \) is well defined and it is continuous with respect to the Hausdorff metric.

For the symmetric Kernel \( K' \) of \( K \) we have
\[ K' = (K + Q(K)) \cap -K. \]  (1)

As noted by Besicovitch \([1]\), \( m(K) \) measures the asymmetry of \( K \). We refer to the book of Grünbaum \([11]\) for an extensive discussion on the topic of measures of asymmetry (for recent developments, see e.g. \([9]\) \([10]\) \([15]\)). Let us consider another measure of asymmetry here. The Blaschke-body \( \nabla K \) of \( K \) is the unique origin-symmetric convex body surface area measure is given by:
\[ S_{\nabla K}(\cdot) = \frac{1}{2} [S_K(\cdot) + S_{-K}(\cdot)] , \]
where \( S_K \) is the surface area measure of \( K \) as defined on \( S^{n-1} \) (see the next section). The existence and uniqueness of this body are ensured by the Minkowski existence Theorem, stating that any measure on \( S^{n-1} \), whose centroid is 0 and the affine hull of its support is full dimensional, is the surface area measure of a unique (up to translation) convex body. It is true (see \([16]\)) that \( |\nabla K| \geq |K| \), with equality if and only if \( K \) is symmetric. Thus, the quantity \( |\nabla K|/|K| \) is indeed a measure of asymmetry. The following problem arises naturally.

**Problem 1.2.** Among all convex bodies \( K \), find the one that maximizes the quantity \( |\nabla K|/|K| \).

Problem 1.2 seems (see \([8]\)) to be related with Nakajima’s problem \([21]\), asking if a convex body of constant width and constant brightness has to be a ball. See \([12]\) \([13]\) \([14]\) for newer results on this problem.

Introduce the quantities
\[ m_n := \inf \{ m(K) : K \text{ is a convex body in } \mathbb{R}^n \} , \]
\[ M_n := \sup \{ |\nabla K|/|K| : K \text{ is a convex body in } \mathbb{R}^n \} . \]

Clearly, the functionals \( m(K) \) and \( |\nabla K|/|K| \) are affine invariant. Thus, the existence of convex bodies for which the quantities \( m_n \) and \( M_n \) are attained (i.e. the existence of solutions to Problems 1.1 and 1.2) follows easily by the Blaschke selection theorem. It has repeatedly been conjectured (see e.g. \([6]\) \([7]\) \([4]\)) that Problems 1.1 and 1.2 admit only one solution: the simplex. Problem 1.2 is open as well; in two dimensions it is confirmed \([4]\) that the simplex is the only solution. In addition, in \([4]\), the following was established: If the simplex is a solution for Problem 1.1 then it solves Problem 1.2 as well. Moreover, \( M_n \leq m_n^{-1} \). We are now ready to state our main results.

**Theorem 1.3.** Let \( K \) be a convex body in \( \mathbb{R}^n \). \( K \) is a solution for Problem 1.1 if and only if \( K \) is a solution for Problem 1.2.
Theorem 1.4. If $K$ is a solution for Problem 1.1 or for Problem 1.2 and $|K \cap -K| = q(K)$, then

$$K \cap -K = m \sqrt{n} \nabla K.$$  \hspace{1cm} (2)

Corollary 1.5. $M_n = m \sqrt{n}^{-1}$. 

Proof. Take $K$ of volume 1 to be a solution of Problem 1.1, such that $|K \cap -K| = q(K)$. Then, by Theorem 1.3, $|\nabla K| = M_n$ and the assertion follows immediately by taking volumes in (2). $\square$

Corollary 1.6. [1] (The planar case) The triangle is the only solution for Problem 1.1.

Proof. Let $K$ be a two-dimensional convex body. It is well known that in $\mathbb{R}^2$, $\nabla K = \frac{1}{2}(K - K)$, so by the Rogers-Shephard inequality [22], $\frac{\nabla K}{|K|}$ is maximal if and only if $K$ is a simplex. This was remarked in [4]. Thus, the triangle is the only solution for Problem 1.2 in two dimensions and (by Theorem 1.3) it is the only solution for Problem 1.1 as well. $\square$

2 Background

We will need some basic results about convex bodies. We refer to [25] an extensive discussion, proofs and references concerning the facts that will be mentioned in this section.

Let $K$ be a convex body in $\mathbb{R}^n$. The support function of $K$ at $x \in \mathbb{R}^n$ is defined as

$$h_K(x) = \max_{y \in K} \langle x, y \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the (usual) inner product in $\mathbb{R}^n$. Note that if $K$ contains 0 in its interior, $F$ is a facet of $K$ and $u$ is the outer unit normal vector of $F$, then $h_K(u)$ is exactly the distance of $F$ from the origin. It should be remarked that any convex and positively homogeneous function $h : \mathbb{R}^n \to \mathbb{R}$ is a support function of a unique convex body.

The surface area measure of $K$ (viewed as a measure on the unit sphere $S^{n-1}$) is defined as

$$S_K(\omega) = \left\{ x \in \text{bd}(K) : \exists (u, t) \in \omega \times \mathbb{R}, \text{ so that } (tu + u^\perp) \cap K = \{x\} \right\}_{n-1}.$$

For instance, if $K$ is a polytope the support of $S_K$ is exactly the set of the outer unit normal vectors of the facets of $K$. Using this fact, one can easily see that if $K$ contains the origin,

$$|K| = \frac{1}{n} \int_{S^{n-1}} h_K(x)dS_K(x).$$ \hspace{1cm} (3)

Let $L$ be another convex body. The mixed volume $V(K, L \ldots L)$ of $K$ and $L$ is defined as the derivative of the quantity $|tK + L|$, as $t \to 0^+$. Here $A + B = \{x + y : x \in A, y \in B\}$ is the Minkowski sum of the sets $A, B$. It can be proven that if $K$ contains 0, then

$$V(K, L \ldots L) = \frac{1}{n} \int_{S^{n-1}} h_K(x)dS_L(x).$$ \hspace{1cm} (4)

Let us state two fundamental facts about mixed volumes: The first is monotonicity: it is true that if $K \subseteq K'$, then $V(K, L \ldots L) \leq V(K', L \ldots L)$. The second is the Minkowski inequality

$$V(K, L \ldots L) \geq |K|^{\frac{1}{n}} |L|^{\frac{n-1}{n}}.$$ \hspace{1cm} (5)
Equality here holds if and only if $K$ and $L$ are homothetic.

The projection body $\Pi K$ of $K$ is defined as the convex body whose support function along the direction $u \in S^{n-1}$ equals the projection of $K$ in the same direction. It is true that

$$h_{\Pi K}(u) = \frac{1}{2} \int_{S^{n-1}} \langle x, u \rangle dS_K(x).$$

This, together with Theorem 1.4, shows immediately the following:

**Corollary 2.1.** If $|K \cap -K| = q(K) = m_n|K|$, the projection bodies of $K \cap -K$ and $\nabla K$ are homothetic.

It is natural to ask the following:

**Problem 2.2.** For which (non-symmetric) convex bodies $K$, such that $Q(K) = 0$, the projection bodies of $K$ and of $K \cap -K$ are homothetic?

It is true that the simplex is such a convex body, however certainly not the only one as the examples of the regular polygons show.

# 3 Proofs

This section is devoted to the proof of Theorems 1.3 and 1.4. We will prove the two Theorems simultaneously; a few lemmas are required. However, the key technical fact towards this direction is the next lemma. Let us first introduce the following quantity:

$$m_n^N := \min\{m(K) : K \text{ is a convex body in } \mathbb{R}^n \text{ with at most } N \text{-facets}\}.$$

Since the limit (in the sense of the Hausdorff metric) of a sequence of polytopes with at most $N$-facets is also a polytope with at most $N$-facets, a simple compactness argument shows that $m_n^N$ is meaningful and attained by a polytope with at most $N$-facets.

**Lemma 3.1.** Let $K$ be a polytope with at most $N$-facets. Suppose, furthermore, that $|K \cap -K| = q(K) = m_n^N|K|$. If $F$ is a facet of $K$, then

$$|F \cap -K|_{n-1} = \frac{1}{2} m_n^N |F|_{n-1}.$$

Proof. Let

$$H = \{x \in \mathbb{R}^n : \langle x, u \rangle = c\}$$

be the corresponding supporting hyperplane of $F$, i.e. $F \subseteq H$ and

$$K \subseteq G := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq c\}$$

(the corresponding half-space), where $u$ is some vector in $\mathbb{R}^n$. Since $K$ is a polytope of at most $N$ facets, it can be written as the intersection of $N$-half-spaces:

$$K = \bigcap_{i=1}^{N-1} G_i \cap G.$$
Set
\[ H_t := \{ x \in \mathbb{R}^n : \langle x, u \rangle = c + t \} , \quad G_t := \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq c + t \} \]
and
\[ K_t := \bigcap_{i=1}^{N-1} G_i \cap G_t , \quad t \in \mathbb{R} . \]

Clearly, \( K_0 = K \). Assume, now, that \( |F \cap -K|_{n-1} \neq \frac{1}{2}m_n^N |F| \). We distinguish two cases.

Case I: \( |F \cap -K|_{n-1} > \frac{1}{2}m_n^N |F| \). Then,
\[
|H \cap K \cap -K|_{n-1} = |F \cap -K|_{n-1} > \left( \frac{1}{2}m_n^N + \delta \right)|F|_{n-1} = \left( \frac{1}{2}m_n^N + \delta \right)|H \cap K|_{n-1} ,
\]
for some \( \delta > 0 \). By continuity, one has that
\[
|H_t \cap K \cap -K|_{n-1} > \left( \frac{1}{2}m_n^N + \delta \right)|H_t \cap K| ,
\]
for negative values of \( t \) sufficiently close to 0. Set
\[ A_t := K \cap (G \setminus G_t) . \]

Then, for negative values of \( t \) sufficiently close to 0 and by Fubini’s Theorem we obtain:
\[
|A_t \cap -K|_n = \int_t^0 |H_s \cap K \cap -K|_{n-1} ds > \frac{1}{2} \left( m_n^N + \delta \right) \int_t^0 |H_s \cap K|_{n-1} ds = \left( \frac{1}{2}m_n^N + \delta \right)|A_t|_n .
\]

Hence, by another continuity argument, there exists a \( t_0 < 0 \), so that for all \( 0 < t < t_0 \) and for all \( x \in \mathbb{R}^n \) with \( |x| < t_0 \),
\[
|(A_t + x) \cap -K|_n > \left( \frac{1}{2}m_n^N + \frac{\delta}{2} \right)|A_t|_n \geq \frac{m_n^N}{2} |A_t|_n .
\]

Note that for \( t < 0 \), \( K = A_t \cup K_t \) and \( K_t \cap A_t = 0 \). Since \( Q(K_t) \to Q(K) = 0 \) as \( t \to 0^- \), there exists a \( t_0 < t_1 < 0 \), so that \( |Q(K_{t_1})| < t_0 \). Also, one may clearly assume that \( [A_{t_1} + Q(K_{t_1})] \cap -A_{t_1} = \emptyset \). Write
\[
K + Q(K_{t_1}) = [K_{t_1} + Q(K_{t_1})] \cup [A_{t_1} + Q(K_{t_1})] ,
\]

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Since all four sets that appear are pairwise disjoint (and the last one is empty). Therefore, taking $m$ for all $0 < t < t$

Thus, 

$$\left| [K + Q(K_{t_1})] \cap -K \right| = \left| [(K_{t_1} + Q(K_{t_1})) \cap -K_{t_1}] \cup [(K_{t_1} + Q(K_{t_1})) \cap -A_{t_1}] \cup [(A_{t_1} + Q(K_{t_1})) \cap -K_{t_1}] \cup [(A_{t_1} + Q(K_{t_1})) \cap -A_{t_1}] \right|$$

$$= \left| (K_{t_1} + Q(K_{t_1})) \cap -K_{t_1} \right| + \left| (K_{t_1} + Q(K_{t_1})) \cap -A_{t_1} \right| + \left| (A_{t_1} + Q(K_{t_1})) \cap -K_{t_1} \right| + \left| (A_{t_1} + Q(K_{t_1})) \cap -A_{t_1} \right|$$

since all four sets that appear are pairwise disjoint (and the last one is empty). Therefore, taking $m$ into account, we obtain:

$$\left| [K + Q(K_{t_1})] \cap -K \right| = m_n^N \left| K_{t_1} \right| + 2 \left| (K_{t_1} + Q(K_{t_1})) \cap -K_{t_1} \right|$$

which is impossible so Case I cannot occur.

**Case II:** $|F \cap -K|_{n-1} < \frac{1}{2} m_n^N |F|$. In order to exclude Case II, we will use a similar argument as in Case I. For $t > 0$, set

$$B_t := K_t \cap (G_t \setminus G) .$$

Then, if $t$ is small enough,

$$|B_t \cap -K| < \left( \frac{1}{2} m_n^N - \delta \right) |B_t| ,$$

for some $\delta > 0$ and by continuity we get as before:

$$\left| (B_t + x) \cap -K \right| < \frac{1}{2} m_n^N |B_t| ,$$

for all $0 < t < t_0$ and for all $x \in \mathbb{R}^n$ with $|x| < t_0$, for some $t_0 > 0$. Since $K_t = K \cup B_t$, $|K \cap B_t| = 0$ and $Q(K_t) \to Q(K) = 0$, as $t \to 0^+$, we can find a $0 < t_1 < t_0$ with $|Q(K_{t_1})| < t_0$. Also, we may assume that $|B_{t_1} + Q(K_{t_1})| \cap B_{t_1} = \emptyset$. Consequently, as before one gets:

$$\left| (K_{t_1} + Q(K_{t_1})) \cap -K_{t_1} \right|$$

$$\leq \left| (K + Q(K_{t_1})) \cap -K \right| + \left| (K + Q(K_{t_1})) \cap -B_{t_1} \right| + \left| (B_{t_1} + Q(K_{t_1})) \cap -K \right| + \left| (B_{t_1} + Q(K_{t_1})) \cap -B_{t_1} \right|$$

$$< m_n^N |K| + 2 \frac{1}{2} m_n^N |B_{t_1}|$$

$$= m_n^N (|K| + |B_{t_1}|) = m_n^N |K_{t_1}| .$$

We conclude that $m(K_{t_1}) < m_{\alpha}$, which is a contradiction. $\square$

We are now ready to prove part of Theorem 1.4.
Lemma 3.2. If $|K \cap -K| = q(K) = m_n |K|$, then $K \cap -K = m_n^{1\over n-1} \nabla K$.

Proof. Assume that $|K \cap -K| = m_n |K|$. Then, there exists a sequence of polytopes $K_N$ with at most $N$-facets, such that $|K_N \cap -K_N| = m_n^N |K_N|$ and $K_N \to K$ in the sense of the Hausdorff distance, as $N \to \infty$. The previous lemma shows exactly that

$$K_N \cap -K_N = (m_n^N)^{1\over n-1} \nabla K_N$$

and taking limits we get the desired result. $\square$

Lemma 3.3. Let $K$ be a convex body such that $|K \cap -K| = q(K) = m_n |K|$. Then,

$$V(K \cap -K, K, \ldots, K) = |K| .$$

Proof. Let $N \geq n + 1$ be an integer and $K_N$ be a polytope with at most $N$-facets, such that $|K_N \cap -K_N| = q(K_N)$ and $|K_N \cap -K_N| = m_n^N |K_N|$. Since by Lemma 3.1 for every facet $F$ of $K_N$ the intersection $F \cap -K_N$ is non-empty, it follows for all $x$ in the support of $S_{K_N}$, $h_{K_N \cap -K_N}(x) = h_{K_N}$.

This, together with (3) shows that

$$V(K_N \cap -K_N, K_N, \ldots, K_N) = \frac{1}{n} \int_{S^{n-1}} h_{K_N \cap -K_N}(x) dS_{K_N}(x)$$

$$= \frac{1}{n} \int_{S^{n-1}} h_{K_N}(x) dS_{K_N}(x)$$

$$= |K| .$$

Now, as in the proof of Lemma 3.2 the rest of the proof of the lemma follows by approximation. $\square$

The following lemma will make use of the Minkowski inequality in a similar way as in [4, Theorem 5'].

Lemma 3.4. Let $K$ be a convex body in $\mathbb{R}^n$. Then,

$$|K \cap -K|^{1/n} |\nabla K|^{n-1/n} \leq |K| .$$

Equality holds if $|K \cap -K| = q(K)$ and $K$ is a solution for Problem 1.1. In that case, $K$ is a solution for Problem 1.2 as well and also,

$$|\nabla K| = m_n^{1\over n-1} |K| .$$

Proof. Suppose that $K$ is a solution for Problem 1.1. We may clearly assume that $|K \cap -K| = q(K)$ and also that $|K| = 1$. Let $P$ be any convex body of volume 1, such that $|P \cap -P| = q(P)$. The Minkowski inequality (5) (together with (4) and the definition of the Blaschke-body) implies

$$|P \cap -P|^{1/n} |\nabla P|^{n-1/n} \leq V(P \cap -P, \nabla P, \ldots, \nabla P)$$

$$= \frac{1}{2} V(P \cap -P, P, \ldots, P) + \frac{1}{2} V(P \cap -P, -P, \ldots, -P)$$

$$\leq |P| = 1 .$$
If $P = K$, it follows by Lemma 3.2 that equality must hold in the Minkowski inequality and also, by Lemma 3.3 equality must hold in the last inequality as well. In other words, equality holds everywhere in the previous inequalities if $P = K$. Moreover, it is a trivial fact that $|P \cap -P| \geq m_n$, with equality if $P = K$. Combining these inequalities together with their equality cases we obtain $|\nabla P|^{n-1/n} \leq (1/m_n)^{1/n}$, with equality if $P = K$, proving our last two claims.\[\square\]

Proof of Theorem 1.3
The fact that if $K$ is a solution for Problem 1.1 then it also solves Problem 1.2 is established in Lemma 3.4. On the other hand, as (7) shows, if $K$ is a solution for Problem 1.2 then $|\nabla K| = m_n^{-\frac{1}{n-1}}|K|$. Assume, furthermore, that $|K \cap -K| = q(K)$ and that $K$ is not a solution for Problem 1.1. Then, $|K \cap -K| > m_n|K|$ and by (6) it follows that $|\nabla K| < m_n^{-\frac{1}{n-1}}|K|$. This is a contradiction and the assertion follows.\[\square\]

Proof of Theorem 1.4:
Immediate by Lemma 3.2 and Theorem 1.3.\[\square\]

Remark 3.5. Using similar variational arguments as in Lemma 3.7, one can prove the following: Suppose that $K \cap -K = q(K)$ and $K$ is an extremal body for Problem 1.1 or equivalently Problem 1.2. Then, $K \cap -K$ contains no extreme points of $K$ in its interior. This shows for example that the extremal bodies cannot have smooth boundary.

4 The projection body of the Blaschke-body of the simplex

Schneider (1982) \[24\] asked for the maximizers of the affine invariant

$$P(K) := \frac{|\Pi(K)|}{|K|^{n-1}}.$$ 

His original conjecture stated that the maximum among centrally symmetric convex bodies is attained if $K$ is the n-dimensional cube $C_n$; in this case, $P(C_n) = 2^n$. Counterexamples were discovered by Brannen \[2\] \[3\]. He conjectured that the simplex is the only maximizer in the general case and the centrally symmetric convex body of maximal volume contained in the simplex in the centrally symmetric case (see also \[23\] for the proof of some other conjectures of Brannen concerning Schneider’s problem). The latest is (as observed in \[4\]) homothetic to the Blaschke-body of the simplex, $\nabla \Delta_n$. Below, we mention some observations about the role of Blaschke-body of the simplex (which is, in some sense, conjectured to be the extremal body for Problems 1.1 and 1.2) in the study of Schneider’s problem.

Fact 4.1. Suppose that $\nabla \Delta_n$ is the only maximizer of $P(\cdot)$ in the symmetric case and that the simplex is the only solution for Problem 1.2. Then $\Delta_n$ is the only maximizer of $P(\cdot)$ in the general case.
Proof. It is true that

\[
P(K) = \frac{|\Pi \nabla K|}{|K|^{n-1}}
\]

\[
= \frac{|\Pi \nabla K|}{|\nabla K|^{n-1}} \cdot \frac{|\nabla K|^{n-1}}{|K|^{n-1}}
\]

\[
\leq \frac{|\Pi \nabla \Delta_n|}{|\nabla \Delta_n|^{n-1}} \cdot \frac{|\nabla \Delta_n|^{n-1}}{|\Delta_n|^{n-1}}
\]

\[
= P(\Delta_n),
\]

with equality if and only if \(K\) is a simplex. \(\Box\)

Let us discuss another question concerning Schneider’s problem. As mentioned earlier, there exists a convex body \(K\) with \(P(K) > P(C_n)\). It is natural to ask however if Schneider’s conjecture is in some sense “almost” correct. It is well known that

\[P(K) < A^n,\]

for all convex bodies \(K\), where \(A > 0\) is some absolute constant. What appears to be unknown is the following

**Problem 4.2.** Is it true that the ratio

\[
\left( \frac{\max_K P(K)}{P(C_n)} \right)^{\frac{1}{n}}
\]

tends to 1, as \(n\) tends to infinity? Here \(K\) runs over all symmetric convex bodies.

**Fact 4.3.** Let \(\Delta_n\) be an \(n\)-dimensional simplex. Then,

\[
\frac{P(\nabla \Delta_n)}{c^{\sqrt{n}} P(C_n)} \to 1, \text{ as } n \to \infty,
\]

where \(c > 0\) is some absolute constant. Thus, if the conjecture of Brannen in the symmetric case of Schneider’s problem is correct, then the previous question would have a strong affirmative answer.

To see that (8) is correct, use the asymptotic formula (proven in [4] [7])

\[
\frac{|\nabla \Delta_n|}{|\Delta_n|} \sim \sqrt{\frac{3}{2}} \left(\frac{\epsilon}{2}\right)^n
\]

(9)

and also note that it is not too difficult to compute

\[
P(\Delta_n) = \frac{n^n(n+1)}{n!}.
\]
Remark 4.4. As (8) shows, for large dimensions, \( P(\nabla \Delta_n) > 2^n = P(C_n) \) (thus, the Blaschke-body of the simplex indeed provides a counterexample to the original conjecture of Schneider). To prove that the same is true in any dimension, one can work similarly as in [23, Lemma 3.3] to show that \( P(\Pi K) = 2^n \), for any symmetric convex body \( K \) with at most \( 2(n+1) \)-facets. Then, one can use Schneider’s trick (see again [23]) that \( P(K) > P(\Pi K) \), with equality if and only if the bodies \( K \) and \( \Pi \Pi K \) are homothetic. Since \( \nabla \Delta_n \) has \( 2(n+1) \)-facets and it is well known that it is not the projection body of any convex body, our claim follows.

Finally, take the polar body \( \Pi^* K \) of \( \Pi K \) (i.e. the unit ball of the dual of the normed space that has \( \Pi K \) as its unit ball) and the affine invariant \( R(K) := ||\Pi^* K|| |K|^{n-1} \). It has been conjectured that \( C_n \) minimizes \( R(K) \) among all centrally symmetric convex bodies (see e.g. [17]). The non-symmetric version of the previous conjectured inequality is indeed true [27] (see also [25] for related results). One may consider the following analogue of Problem 4.2.

Problem 4.5. Is it true that the ratio
\[
\left( \frac{\min_K R(K)}{P(C_n)} \right)^{\frac{1}{n}}
\]
tends to 1, as \( n \) tends to infinity? Again \( K \) runs over all symmetric convex bodies.

Using (9) and the exact value of \( R(\Delta_n) \) (see again [25]) one easily obtains:

Fact 4.6. There exists a constant \( C > 1 \), such that
\[
\frac{R(\Delta_n)}{R(C_n)} \to C \quad \text{as} \quad n \to \infty .
\]

References

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