Method for Analyzing the Problem of Determining the Dynamics of Changes in the Structures of Temporal Directed Tree

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Abstract. Problem determining the dynamics of changes in the structures of temporal directed tree is one of the central problems of graph dynamics, i.e. the changes in the similarity of temporal directed tree structures, which is the subject of this research. Two types of problems associated with determining the distance or similarity index for a pair of structures of temporal directed tree are identified. A method for solving these problems is proposed, and example of solution is given.

Keywords: Temporal directed tree, Maximum common fragment, digraph, Tree, Directed tree, Similarity of directed tree, Isomorphism, Automorphism, Graph dynamics.

1. Introduction
Graph theory concepts are of great importance in mathematics (discrete mathematics), because of their wide applications in different fields of sciences, for example, chemistry, physics, biology, engineering, geology, medicine ... etc. The importance of these concepts is evidenced by the possibility of using them to solve problems that can be represented in the form of graphs and then treat them to provide appropriate suggestions for solutions. Recently, this importance has motivated researchers in this field to significantly continue to find and develop methods for solving the problems of graphs, to be more effective in the application [1–5].

Traditional "static" digraphs are not applicable for modeling of structures that change with time. As highlighted in [6–12] at the present time, the actual direction is the development of methods for analyzing digraphs with a changeable structure with time (temporal digraphs (T-digraphs)). The first results of studies on graph dynamics, i.e. on the dynamic description of structures, are presented in [13], in this study, the basic classes of problems are identified, related to the definition of: 1) the equilibrium state of temporal digraph and the domain of convergence to this state; 2) distances (similarities) between the structures of one or two temporal digraphs; 3) trends of change the structures of the temporal digraph.

Temporal directed tree have many applications in other fields of science, for example in economics and management science [13–14], it also has an effective role in solving problems related to the analysis of directed trees, for example, 1) determining isomorphism of directed trees; 2) determining the similarity of directed trees; 3) determining the automorphism group of directed tree [15–18].
2. Basic Definitions and Concepts

A graph is a collection of two finite sets, a set of points (or nodes) called vertices, and a set of pairs of vertices called edges, and it can be formalized as a pair of sets $G = (V, E)$, where $V(G)$ is the set of vertices, $E(G)$ is the set of edges, where $E \subseteq (V \times V)$ and $V \cap E = \emptyset$, and the identifiers (numbers) of vertices and edges of natural numbers, $V = \{v_1, v_2, ..., v_i, ..., v_n\}$, where $v_i$ is any vertex ($v_i \in V, i \in \mathbb{N}$), and $p(G) = |V|$ - the number of vertices, $E = \{e_1, e_2, ..., e_i, ..., e_q\}$, where $e_i$ is any edge ($e_i \in E, i \in \mathbb{N}$), and $q(G) = |E|$ - the number of edges. If a pair of vertices is not ordered, then it is called an edge, but if it is ordered, then it is called an arc (directed edge). The edge $\{v_1, v_2\}$ connects the vertices $v_1$ and $v_2$, and the arc $(v_1, v_2)$ starts at the vertex $v_1$ and ends at the vertex $v_2$. A graph containing only edges is called an undirected graph Fig. 1-a, and a graph containing only arcs is called a directed graph (digraph) Fig. 1-b. If a graph contains both edges and arcs, then it is called a mixed graph Fig. 1-c. An arc (or edge) can start and end at the same vertex, such an arc (edge) is called a loop. Vertices connected by an edge or arc are called adjacent vertices, and edges with a common vertex, also called adjacent edges. An edge (arc) and any of its two vertices are called incident. The degree of a vertex is the number of edges incident to it and it is denoted by $deg(v)$, where $v \in V(G)$. A vertex of degree 1 is called a hanging vertex, and a vertex of degree 0 is called an isolated vertex. Path in a graph is an alternating sequence $(a = v_0, e_1, v_1, e_2, ..., v_{i-1}, e_i, v_i, ..., v_n, e_n, v_n = b)$ vertices and edges of the graph such that, $e_i = \{v_{i-1}, v_i\}, 1 \leq i \leq n, a$ and $b$ are the end vertices of the path. A path is called a chain if all of its edges are different, and the number of edges represents the length of the chain. If the chain vertices are different, then the chain is called simple, otherwise it is composite. A cycle (circuit) is a chain in which the first and last vertices coincide. A cycle is called simple if it does not pass through any vertex of the graph more than once. A graph $G$ is called connected graph, if for any two of its vertices there is a path connecting them Fig. 2-a, otherwise the graph $G$ is called disconnected graph Fig. 2-b. Any disconnected graph $G$ is a collection of connected graphs, each of these graphs is called a component of the graph $G$. A connected graph (consists of one component), that has no cycles Fig. 3-a is called a tree [19–23]. In general, a connected digraph (with $n$ vertices), that does not contain cycles (acyclic digraph), is called a directed tree Fig. 3-b, if the number of arcs is $n - 1$. Every directed tree is anacyclic digraph, but not every acyclic digraph is a directed tree. $\hat{G} = (V^{(t)}, E^{(t)}, T)$, is called a temporal directed tree ($T$-directed tree), where $V^{(t)}(\hat{G})$ is the set of vertices of $\hat{G}$ at the time $t$, $T = \{t_1, ..., t_i, ..., t_n\}$ - is the set of natural numbers defining (discrete) time, $E^{(t)}(\hat{G})$ is the set of arcs of $\hat{G}$, $E^{(t)} \subseteq (V^{(t)} \times V^{(t)})$, i.e. (for $t \in T$) $\exists$ a mapping $\Gamma_t$: $V^{(t)} \rightarrow V^{(t)}$ and $\Gamma_t \in E^{(t)}$ [24]. We adopt $t\hat{G}$ to denote a $T$-directed tree ($\hat{G}$-directed tree) at time $t$, with the number of vertices $|V^{(t)}(\hat{G})| = p^{(t)}$ and the number of arcs $|E^{(t)}(\hat{G})| = q^{(t)}$. Two directed trees $t_1\hat{G} = (V^{(t_1)}, E^{(t_1)}, T)$ and $t_2\hat{G} = (V^{(t_2)}, E^{(t_2)}, T)$ are isomorphic ($t_1\hat{G} \approx t_2\hat{G}$), if they differ only in the numbering of their vertices, more precisely, if there is a mapping $\varphi$ (bijection) of their vertex sets onto each other, that preserves the adjacency relation: $t_1\hat{G} \approx t_2\hat{G}$ $\iff$ $\exists \varphi: V^{(t_1)} \leftrightarrow V^{(t_2)}$, such that $\forall v_i, v_j \in V^{(t_1)}, (v_i, v_j) \in E^{(t_1)} \leftrightarrow (\varphi(v_i), \varphi(v_j)) \in E^{(t_2)}$, where $\varphi(v_i), \varphi(v_j) \in V^{(t_2)}$.

The bijection $\varphi$ of two sets of vertices itself is called an isomorphism.

We denote by $t_1\hat{G} \cong t_2\hat{G}$ for isomorphically embedding of the directed tree $t_1\hat{G} = (V^{(t_1)}, E^{(t_1)}, T)$ into another directed tree $t_2\hat{G} = (V^{(t_2)}, E^{(t_2)}, T)$, i.e. $t_1\hat{G} \cong t_2\hat{G}$ if $t_2\hat{G}$ contains a fragment $\hat{f} = t_2\hat{G}_* = (V^{(t_2)}, E^{(t_2)}, T)$ isomorphic to $t_1\hat{G}$ ($t_2\hat{G}_* \approx t_1\hat{G}$), more precisely, if there is an injection ($h$) of their vertex sets, that preserves the adjacency relation: $t_1\hat{G} \cong t_2\hat{G} \iff \exists h: V^{(t_1)} \rightarrow V^{(t_2)}$, such that $\forall v_i, v_j \in V^{(t_1)}, (v_i, v_j) \in E^{(t_1)} \rightarrow (h(v_i), h(v_j)) \in E^{(t_2)}$, where $h(v_i), h(v_j) \in V^{(t_2)}$. An injection $h$ of two sets of vertices is called an isomorphic embedding. Removing arcs, or vertices and arcs in any $t\hat{G}$, we obtain its fragment at time $t$. Under the maximum common fragment (MCF) for $t_1\hat{G}, t_2\hat{G}$ we understand the fragment $\hat{f}_1 = t_{1,2}\hat{G}^{(+)} = (V^{(t_{1,2})}, E^{(t_{1,2})}, T)$,
for which two conditions are valid: 1) respectively, $t_{1,2} \bar{G}^1 \subseteq t_1 \bar{G}$ and $t_{1,2} \bar{G}^2 \subseteq t_2 \bar{G}$, 2) there is no fragment in $t_1 \bar{G}$ greater than $t_{1,2} \bar{G}^2$ by the number of vertices, for which condition (1) is satisfied.

An automorphism of a directed tree $t \bar{G} = \langle V(t), E(t), T \rangle$ is an isomorphism between $t \bar{G}$ and itself. Thus, an automorphism of $t \bar{G}$ is a mapping $\varphi: V(t) \leftrightarrow V(t)$, such that $\forall v_i, v_j \in V(t), \langle v_i, v_j \rangle \in E(t) \leftrightarrow \langle \varphi(v_i), \varphi(v_j) \rangle \in E(t)$, where $\varphi(v_i), \varphi(v_j) \in V(t)$. The set of all automorphisms of $t \bar{G}$ forms a permutation group, which is called the automorphism group of $t \bar{G}$ and is denoted by $Aut(t \bar{G})$ and the order of this group is denoted by $|Aut(t \bar{G})|$. By $W(t \bar{G}^*, t \bar{G})$ we denote the number of all isomorphic embeddings of $t \bar{G}^*$ into $t \bar{G}$, thus the number of canonical isomorphic embeddings of $t \bar{G}^*$ into $t \bar{G}$ is defined as follows: $w(t \bar{G}^*, t \bar{G}) = W(t \bar{G}^*, t \bar{G}) / |Aut(t \bar{G}^*)|$, where $|Aut(t \bar{G}^*)|$ is the order of the automorphism group of $t \bar{G}^*$ [25].

**Figure 1:** Undirected graph, Directed graph and Mixed graph

**Figure 2:** Connected graph and Disconnected graph
3. Problems of determining the type of dynamics of changes in the structures of Temporal directed tree

Problem 1. If $\tilde{G} = < \tilde{V}(t), \tilde{E}(t), \tilde{T}>$ is a $T$-directed tree, $T = \{t_1, ..., t_i, ..., t_n\}$. For $i = 1...n$, we need to calculate the values, and plot the function $f_1$, which characterizes the changes in the values of the distances between the structures in $\tilde{G}$, then we determine the type of change in the values of the function $f_1$ (monotonic, non-monotonic).

Problem 2. If $\tilde{G} = < \tilde{V}(t), \tilde{E}(t), \tilde{T}>$ is a $T$-directed tree, $T = \{t_1, ..., t_i, ..., t_n\}$. For $i = 1...n$, we need to calculate the values, and plot the function $f_2$, which characterizes the changes in the values of similarity between structures in $\tilde{G}$, then we determine the type of change in the values of the function $f_2$ (monotonic, non-monotonic). To solve problem 1, we will use a function to calculate the distance $D$, where

$$D(\tilde{G}_i, \tilde{G}_j) = p(\tilde{G}_i) + p(\tilde{G}_j) - 2p(MCF(\tilde{G}_i, \tilde{G}_j)),$$

and to solve problem 2, we will use a function to calculate the similarity index value $SI$, where

$$SI(\tilde{G}_i, \tilde{G}_j) = p(MCF(\tilde{G}_i, \tilde{G}_j))^2 / p(t_i \tilde{G})^2 p(t_j \tilde{G}) \quad [26].$$

Note, that distance $D$ is a metrics. In [27], for the first time, a class of problems for analyzing the similarity of $T$-digraph structures based on the determination of the distances $D$, was distinguished. Methods for solving problems were proposed using the following schemes for analyzing $T$-digraph structures: 1) $\forall(t_i, t_{i+1})$, where $i = 1, ..., (n - 1)$, we must determine $D(\tilde{G}_i, \tilde{G}_{i+1})$, and plot the function $f_1$ of changing the values of the calculated distances; 2) $\forall(t_i, t_j)$, where $j = 2, ..., n$, we must determine $D(\tilde{G}_i, \tilde{G}_j)$, and plot the function $f_2$ of changing the values of the calculated distances.

4. The proposed method for solving problems

To solve problem 1 and 2, we define a new solution scheme:

1) We determine $MCF(t_1 \tilde{G}, ..., t_n \tilde{G})$, if it is not predetermined;

2) For $i = 1, ..., n$ we calculate the values of the function $f_1(i) = D(MCF(t_1 \tilde{G}, ..., t_n \tilde{G}), t_i \tilde{G}) = p(MCF(t_1 \tilde{G}, ..., t_n \tilde{G})) + p(t_i \tilde{G}) - 2p(MCF(MCF(t_1 \tilde{G}, ..., t_n \tilde{G}), t_i \tilde{G})).$
It is clear that \( \text{MCF}(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}), t_i \vec{G}) = \text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}) \), so for simplicity, we can write the function \( f_1(i) \) as follows:

\[
 f_1(i) = D(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}), t_i \vec{G}) = p(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G})) + p(t_i \vec{G}) - 2p(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}));
\]

3) For \( i = 1, \ldots, n \) we calculate the values of the function \( f_2(i) = \text{SI}(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}), t_i \vec{G}) = p(\text{MCF}(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}), t_i \vec{G})) / p(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G})) p(t_i \vec{G}). \)

Also, because \( \text{MCF}(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}), t_i \vec{G}) = \text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}) \), so similarly as in 2, we can write the function \( f_2(i) \) as follows:

\[
 f_2(i) = \text{SI}(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}), t_i \vec{G}) = p\left(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G})\right)^2 / p(\text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G})) p(t_i \vec{G}).
\]

4) We plot the function \( f_1 \) and \( f_2 \), then determine the type of change in its values (monotonic, non-monotonic).

We note that the problem of determining \( \text{MCF}(t_1 \vec{G}, \ldots, t_n \vec{G}) \) is one of the independent basic problems of graph dynamics [13]. In [24] for the first time a method was proposed that included four basic steps for determining \( \text{MCF} \) of \( T \)-directed tree or \( T \)-tree, as this method contributes to implementing the basic step in the solution method proposed in our study, by determining the maximum common fragment (\( \text{MCF} \)) between \( T \)-directed tree structures, if it is not predetermined.

Figure 4 shows the temporal directed tree (\( T \)-directed tree \( \vec{G} \)) consisting of five different structures \((t_1 \vec{G}, \ldots, t_5 \vec{G})\) with \( \text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}) \), such that \( p(t_1 \vec{G}) = 8 \), for any structure of \( \vec{G} \), and \( p(\text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G})) = 4 \).

Table 1 shows the results of calculations for \( i = 1, \ldots, n \) values of the function \( f_1(i) = D_t(i) = D(\text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}), t_i \vec{G}) \) when solving problem 1, and values of the function \( f_2(i) = SI_t(i) = SI(\text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}), t_i \vec{G}) \) for \( \vec{G} \) (figure 4).

**Table 1:** The results of calculations for \( i = 1, \ldots, n \) values of the function \( f_1(i) \) and \( f_2(i) \)

| The analyzed pair | \( D_t \) | \( SI_t \) |
|-------------------|----------|----------|
| \( t_1 \vec{G}, \text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}) \) | 4 | 0.5 |
| \( t_2 \vec{G}, \text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}) \) | 4 | 0.5 |
| \( t_3 \vec{G}, \text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}) \) | 4 | 0.5 |
| \( t_4 \vec{G}, \text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}) \) | 4 | 0.5 |
| \( t_5 \vec{G}, \text{MCF}(t_1 \vec{G}, \ldots, t_5 \vec{G}) \) | 4 | 0.5 |
Calculations were performed using the author's software package [18]. Figure 5 shows the results of solving problem 1 and 2.

Thus, we conclude that the functions $f_1$ and $f_2$ for $\tilde{G}$ (figure 4) have a monotonic form and a constant value of the distance $D_1(i)$ and similarity index $SI_1(i)$.

5. The effectiveness of the proposed method

An effective method for determining $MCF$ of $T$-directed tree $\tilde{G}$, was proposed by the author in [24], the basis of this method is the implementation of four basic steps. Experimental estimation of the computational complexity ($EECC$) of this method showed its high effectiveness for determining $MCF$ of $T$-directed tree with number of vertices up to 500. Therefore, we can say that the proposed method in our study is an effective method for determining the dynamics of changes in the structures of $T$-directed tree, because the analysis of its results depends mainly on determining $MCF$ of $T$-directed tree.
6. Conclusion

A new method for solving problems of determining the tendency of similarity’s change between the structures of the temporal directed tree is proposed. Determining MCF of T-directed tree is the main step in the proposed solution method. The type of changes in the structures of the temporal directed tree were determined, the results of solving problems 1 and 2 showed that the function f1, which characterizes the changes in the values of the distances between the structures in ̃G, and the function f2, which characterizes the changes in the values of similarity between structures in ̃G, have a monotonic form and a constant value (distance D1(i) and similarity index S1(i)). In general, it is possible to propose different other methods to determine the dynamics of changes in the structures of the T-directed tree or in the structures of the T-graph (T-digraph) using other properties, for example, the use of the complexity index of graph, and this is a proposal for a future study to find new methods that determine the dynamics of changes in the structures of the T-directed tree or in structures of T-graph (T-digraph).

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