Can the Lorentz symmetry supply something that is supplied by the gauge symmetry?

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At the foundation of modern physics lie two symmetries: the Lorentz symmetry and the gauge symmetry, which play quite different roles in the establishment of the standard model. In this paper, it is shown that, different from what is usually expected, the two symmetries, although mathematically independent of each other, have important overlap in their physical effects. Specifically, we find that the interaction Lagrangian of QED can be “derived”, based on the Lorentz symmetry with some auxiliary assumption about vacuum fluctuations, without resorting to the gauge symmetry. In particular, the derivation is based on geometric relations among representation spaces of the SL(2,C) group. In this formulation of the interaction Lagrangian, the origin of the topological equivalence of the eight basic Feynman diagrams in QED can be seen quite clearly.

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I. INTRODUCTION

Modern physics, in particular, the standard model (SM), the most successful quantum field theory that has ever been developed up to now [1, 2], is based on two symmetries: the Lorentz symmetry and the gauge symmetry. The two symmetries play different roles in the establishment of the SM. Specifically, the Lorentz invariance imposes a stringent restriction to possible forms of the Lagrangian; meanwhile, based on certain assumption about the form of the Lagrangian of free fields, the gauge symmetry can be used to infer the existence of fields that mediate interactions and, further, to derive the interaction Lagrangian.

A question of interest is whether the two symmetries, though mathematically independent of each other, may have important overlap in physical effects they may lead to. In other words, is it possible to use one of them to “derive” a physical result, which is usually regarded as a consequence of the other one? Knowledge about this possibility may deepen our understanding of the SM and, further, be useful in the study of theories beyond the SM, a topic which has attracted lots of attention in recent years (see, e.g., Refs. 3–10).

We ask the above question, based on the following interpretation and understanding of the origin of physical symmetry. As is well known, the existence of a symmetry can be understood, if the underlying physics (e.g., that depicted by the Lagrangian) remains unchanged when descriptions for physical entities undergo certain changes. This suggests that it should be certain feature of the underlying physics that leads to a symmetry that may exist. Reversely, from a symmetry, one may trace back to the related feature of the underlying physics and even to derive a description for the underlying physics (usually with the help of some auxiliary assumption). One example is given by the above-discussed derivation of interaction Lagrangian from the gauge symmetry. Since one underlying physics may have more than one features that may lead to symmetries, in principle, it might be possible to derive a description for the physics from different symmetries (with the help of different auxiliary assumptions).

The specific question we are interested in is whether some interaction Lagrangian may be partially derived from the Lorentz symmetry. Since the interaction Lagrangian in the SM has a very complicated structure, at a first step which we intend to take in this paper, we study the problem within the simplest part of the SM, the quantum electrodynamics (QED). Contrary to the usual expectation that the Lorentz symmetry does not contain enough information for this task, in this paper, we show that it can do this.

Related to the Lorentz symmetry, instead of the proper, orthochronous Lorentz group, we consider its covering group, the so-called SL(2,C) group [11–15]. As is well known, Dirac spinors can be decomposed into two-component Weyl spinors, the latter of which span the smallest nontrivial representation spaces for the SL(2,C) group. The SL(2,C) group has an internal structure much simpler than the Lorentz group, and it brings much convenience in the study of relativistic theories [11]. A key point lies in that representation spaces of the SL(2,C) group have geometric relations, which turns out to be useful in the derivation of the interaction Lagrangian.

Specifically, we find that a part of the interaction Lagrangian of QED can be obtained from the above-discussed geometric relation among representation spaces of the SL(2,C) group. Further, we show that the whole interaction Lagrangian of QED can be “derived”, if one adopts the following description of the physical idea of vacuum fluctuation. That is, an electron and a positron adopts the following description of the physical idea of vacuum fluctuation. That is, an electron and a positron with opposite four-momentum and opposite angular momentum may emerge from the vacuum or vanish into the vacuum, where a negative fourth-component of momentum is not related to energy as to be discussed in detail later.

Another motivation of this paper is related to a technique feature of the eight basic Feynman diagrams in QED, which correspond to 4+4 (conjugated) interaction processes. That is, the 4+4 Feynman diagrams are topologically equivalent. An interesting question is whether this topological equivalence may have some deeper rea-
son, e.g., whether there may exist one origin underlying the 4 + 4 basic Feynman diagrams of QED. We show that the 4 + 4 basic Feynman diagrams of QED indeed have a common origin, in view of the above-discussed derivation of the interaction Lagrangian of QED from the Lorentz symmetry.

The paper is organized as follows. In Sec. II, we give preliminary discussions for the strategy to be employed in this paper. In Sec. III, we recall basic properties of two-component spinors and of four-component vectors, and write them in the abstract notation of Dirac’s ket-bra. We also briefly discuss Dirac’s four-component spinors, written in the abstract notation for two-component spinors. In Sec. IV, we derive expressions of the interaction amplitudes of the interaction Lagrangian of QED, written in the abstract notation for spinors. These expressions enable us to give a geometric interpretation to the interaction Lagrangian, which we discuss in Sec. V.

Then, in Sec. VI, we propose a theory, in which there are two fundamental processes only. The fundamental interaction process can be described by geometric relations among the related representation spaces of the SL(2,C) group. We show that the interaction Lagrangian derived from the proposed theory has the same form as that given in the ordinary formulation of QED. Finally, conclusions and discussions are given in Sec. VII.

II. PRELIMINARY DISCUSSIONS

In Sec. II A, we recall a basic structure of the state space and the interaction Lagrangian in QED. Then, in Sec. II B, we briefly discuss some basic ideas and the strategy to be employed in this paper.

A. The state space and interaction Lagrangian in QED

A convenient method of constructing the state space in QED is to employ creation operators for momentum eigenstates \( |n\rangle \). Due to the generic connection between spin and statistics first shown by Pauli \([10, 11]\), the creation operators for electron, positron, and photon, denoted by \( b_r^\dagger (p) \), \( d_s^\dagger (p) \), and \( a_{\lambda}^\dagger (k) \), respectively, satisfy the following (anti-)commutation relations,

\[
\begin{align*}
\{b_r^\dagger (p), b_s^\dagger (q)\} &= 0, \quad \{d_r^\dagger (p), d_s^\dagger (q)\} = 0, \\
\{b_r^\dagger (p), d_s^\dagger (q)\} &= 0, \quad [a_{\lambda}^\dagger (k), a_{\lambda'}^\dagger (k') ] = 0,
\end{align*}
\]

(1)

with \( r, s = 0, 1 \) and \( \lambda = 0, 1, 2, 3 \).

The state spaces for \( n \) electrons and for \( n \) positrons, denoted by \( \mathcal{E}^{(n)} \) and \( \mathcal{E}^{(\bar{n})} \), respectively, can be constructed, making use of a vacuum state denoted by \( |0\rangle \). Specifically, the space \( \mathcal{E}^{(n)} \) is spanned by basis vectors \( |\Omega_n\rangle = b_{r_1}^\dagger (p_1) \cdots b_{r_n}^\dagger (p_n) |0\rangle \), and similar for \( \mathcal{E}^{(\bar{n})} \) with the operators of \( b^\dagger \) replaced by those of \( d^\dagger \). Using \( \mathcal{E}^{(0)} \) to denote the vacuum state \( |0\rangle \), the total state spaces for electron and for positron are written, respectively, as

\[
\mathcal{E}_{\text{electron}} = \bigoplus_{n=0}^{\infty} \mathcal{E}^{(n)}, \quad \mathcal{E}_{\text{positron}} = \bigoplus_{n=0}^{\infty} \mathcal{E}^{(\bar{n})}.
\]

(2)

The total state space for electron and positron can then be written as

\[
\mathcal{E}_{\text{fermion}} = \bigoplus_{n,m=0}^{\infty} \mathcal{E}^{(n,\bar{m})},
\]

(3)

where \( \mathcal{E}^{(n,\bar{m})} = \mathcal{E}^{(n)} \otimes \mathcal{E}^{(\bar{m})} \) for \( n, m \neq 0 \), \( \mathcal{E}^{(0,\bar{m})} = \mathcal{E}^{(\bar{m})} \), and \( \mathcal{E}^{(n,\bar{0})} = \mathcal{E}^{(n)} \). Similarly, one can construct the state space for \( n \) photons, denoted by \( \mathcal{E}_{\text{ph}}^{(n)} \). The total state space for photon, denoted by \( \mathcal{E}_{\text{ph}} \), is then written as

\[
\mathcal{E}_{\text{ph}} = \bigoplus_{n=0}^{\infty} \mathcal{E}_{\text{ph}}^{(n)},
\]

(4)

with \( \mathcal{E}^{(0)}_{\text{ph}} \) indicating the vacuum state \( |0\rangle \).

The interaction Lagrangian in QED is equal to \(-H_{\text{int}}\), where \( H_{\text{int}} \) is the interaction Hamiltonian written as

\[
H_{\text{int}} = \int d^3x \bar{\psi} \gamma^\mu \psi A_\mu.
\]

(5)

Here, for brevity, the electronic charge is not written explicitly, \( \mu = 0, 1, 2, 3 \), and \( \psi = \psi^* \gamma_0 \) indicates the conjugate field of the field \( \psi \). A remark about notation: We write the conjugate field as \( \bar{\psi} \), not in the often-used notation of \( \psi^\dagger \), because we are to write the interaction Hamiltonian in terms of two-component spinors and, by a convention in the theory of spinors, an overline usually indicates complex conjugate. Following this convention, throughout this paper an overline always indicates complex conjugate both for spinors and for numbers.

In terms of creation and annihilation operators, the quantized fields in QED are usually written in the following form,

\[
\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{b_r(p) U_r(p) e^{-ipx} + d_s^\dagger (p) V_r(p) e^{ipx}}{i},
\]

\[
\bar{\psi}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{b_r^\dagger (p) \tilde{U}_r(p) e^{ipx} + d_s(p) \tilde{V}_r(p) e^{-ipx}}{i},
\]

\[
A_\mu (x) = \int \frac{d^3 k}{(2\pi)^3} \frac{a_{\lambda} (k) \tilde{\epsilon}_\mu^\dagger (k) e^{-ikx} + a_{\lambda}^\dagger (k) \tilde{\epsilon}_\mu (k) e^{ikx}}{i},
\]

(6)

with \( \tilde{U}_r(p) = U_r(p) \gamma_0, \quad \tilde{V}_r(p) = V_r(p) \gamma_0 \).

Here and hereafter, unless otherwise stated, we follow the convention that double appearance of a same index,
one in the upper position and one in the lower position, implies a summation over the index.

Corresponding to 8 basic interaction processes depicted by 8 basic Feynman diagrams, the interaction Hamiltonian is divided into eight parts, \( H_{\text{int}} = \sum_{i=1}^{8} H_{i} \), where

\[
H_{1} = \int dpdqdk \, d_{s}(q,t)b_{r}(p,t)\alpha_{\lambda}(k,t)h_{1},
\]

\[
H_{2} = \int dpdqdk \, b_{i}^{\dagger}(q,t)d_{i}(p,t)\alpha_{\lambda}(k,t)h_{2},
\]

\[
H_{3} = \int dpdqdk \, b_{i}^{\dagger}(q,t)b_{r}(p,t)\alpha_{\lambda}(k,t)h_{3},
\]

\[
H_{4} = \int dpdqdk \, b_{i}^{\dagger}(q,t)b_{r}(p,t)\alpha_{\lambda}(k,t)h_{4},
\]

\[
H_{5} = \int dpdqdk \, d_{s}(q,t)d_{i}^{\dagger}(p,t)\alpha_{\lambda}(k,t)h_{5},
\]

\[
H_{6} = \int dpdqdk \, d_{s}(q,t)d_{i}^{\dagger}(p,t)\alpha_{\lambda}(k,t)h_{6},
\]

\[
H_{7} = \int dpdqdk \, b_{i}^{\dagger}(q,t)d_{i}(p,t)\alpha_{\lambda}(k,t)h_{7},
\]

\[
H_{8} = \int dpdqdk \, d_{s}(q,t)b_{r}(p,t)\alpha_{\lambda}(k,t)h_{8},
\]

where \( \alpha_{\lambda}(k,t) = e^{-i\theta_{i}\mu_{\lambda}}(k) \), \( d_{s}(q,t) = e^{-i\varphi_{q}^{t}d_{s}(q)} \), \( b_{r}(p,t) = e^{i\varphi_{p}^{t}b_{r}(p)} \), and \( h_{i} \) of \( i = 1, \ldots, 8 \) are interaction amplitudes. Here, a common prefactor \( (2\pi)^{3} \) of \( H_{i} \) has been absorbed in the unwritten electronic charge. Below, for brevity, we do not write explicitly the time-dependence of the creation and annihilation operators, e.g., we simply write \( b_{r}(p) \) for \( b_{r}(p,t) \). We use \( \Pi_{1}, \ldots, \Pi_{8} \) to indicate the eight interaction processes described by \( H_{i} \) of \( i = 1, \ldots, 8 \), respectively.

B. Preliminary analysis and the strategy to be employed

In order to study the possibility of “deriving” the interaction Hamiltonian in QED from the symmetry of the SL(2,C) group, a convenient strategy is to rewrite its expression in Eq. (5) in a form, from which its geometric meaning with respect to representation spaces of the SL(2,C) group can be seen more clearly. To this end, a meaningful observation is that each of the eight terms \( H_{i} \) in Eq. (5) can be given a geometric interpretation in terms of states of the three particles involved. In particular, the spin part of the interaction amplitude \( h_{i} \) can be regarded as certain type of “overlap” of the related spin states through the \( \gamma^{\mu} \)-matrices.

As an example, let us consider the term \( H_{1} \) that describes a process, in which an electron and a positron “combine” and “form” a photon, as seen in its operator part, \( \alpha_{\lambda}(k)d_{s}(q)b_{r}(p) \). Substituting Eq. (6) into Eq. (5), one gets the following expression of the amplitude \( h_{1} \),

\[
h_{1} = R^{\text{str}}_{1}(q,p)\epsilon^{\lambda}{}_{\mu}(k)s^{3}(k - p - q),
\]

where

\[
R^{\text{str}}_{1}(q,p) = \hat{V}^{s}(q)\gamma^{\mu}U^{t}(p).
\]

Clearly, the spin part of \( h_{1} \), namely, \( R^{\text{str}}_{1}(q,p)\epsilon^{\lambda}{}_{\mu}(k) \), can be regarded as a type of “overlap” of the three spin states of the three particles involved, through the \( \gamma^{\mu} \)-matrices.

The above discussed geometric interpretation of the interaction amplitudes suggests that it may be possible to find some rules related to geometric meanings of the states involved, from which the interaction Hamiltonian of QED can be “derived”. To achieve this goal, an obstacle is that, in view of the Clifford algebra, geometric meaning of the \( \gamma^{\mu} \)-matrices in the aspect of spinor is not clear.

The above-discussed obstacle can be overcome, if one adopts a specific representation of the \( \gamma^{\mu} \)-matrices, namely, the chiral representation. In this representation, these matrices are expressed by means of the so-called Enfeld-van der Waerden symbols, in short EW-symbols [11–15], the latter of which have a quite clear geometric meaning: The EW-symbols map the direct product of a Weyl-spinor space and its complex-conjugate space to a four-component-vector space. This brings a clearer geometric meaning to the \( \gamma^{\mu} \)-matrices. In this chiral representation, Dirac spinors have the previously-discussed property of being decomposed into two two-component Weyl spinors.

Therefore, we are to rewrite the interaction Hamiltonian in terms of two-component spinors. We’ll show that, once this is done, a clearer geometric meaning of the Hamiltonian can be seen. But, before doing this, we need to give some discussions for properties of spinors, which will be done in the next section.

III. SPINORS AND VECTORS IN ABSTRACTION NOTATION

In this section, we recall some basic properties of spinors and vectors [11–13] and discuss their expressions in Dirac’s abstract notation of ket and bra [15]. These expressions prove convenient for discussions to be given in later sections. Specifically, basic properties of two-component spinors in the abstract notation are briefly discussed in Sec. IIIA and those for stationary solutions of the Dirac equation are given in Sec. IIIB. We recall basic properties of four-component vectors in Sec. III C and discuss their abstract expressions in Sec. III D.

A. Basic properties of two-component spinors

Two-component Weyl spinors span a two-dimensional complex linear space, which we denote by \( \mathcal{W} \). Usually, a two-component spinor is represented by its components, say, \( \kappa^{A} \) with an index \( A = 0, 1 \). But, in the study of
quantum theories, it is sometimes more convenient to write two-component spinors in Dirac’s abstract notation, which we briefly present below (see Ref. [13] for more detailed discussions).

Let us consider the space $\mathcal{W}$ a basis denoted by $|S_A\rangle$ with $A = 0, 1$. A generic spinor in this space is expanded as

$$|\kappa\rangle = \kappa^A|S_A\rangle. \quad (11)$$

One may introduce a space that is dual to $\mathcal{W}$, composed of bras with a basis written as $\langle S_A|$. In order to construct a product, which is a scalar under transformations of the SL(2,C) group, the bra dual to the ket $|\kappa\rangle$ should be written as

$$\langle \kappa| = \langle S_A|\kappa^A, \quad (12)$$

which has the same components as $|\kappa\rangle$ in Eq. (11), but not their complex conjugates [12]. (See Appendix A for a brief discussion of basic properties of the SL(2,C) group.)

Scalar products of the basis spinors needs to satisfy

$$\langle S_A|S_B\rangle = \epsilon_{AB}, \quad (13)$$

where

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (14)$$

It proves convenient to introduce a matrix $\epsilon^{AB}$, which has the same elements as $\epsilon_{AB}$. These two matrices can be used to raise and lower indexes of components, say,

$$\kappa^A = \epsilon^{AB}\kappa_B, \quad \kappa_A = \kappa^B\epsilon_{BA}, \quad (15)$$

as well as for the basis spinors, e.g., $|S_A\rangle = \epsilon^{AB}|S_B\rangle$. It is not difficult to verify that (i) $\langle S_A|S_B\rangle = \epsilon^{AB}\langle S_B|S_B\rangle$; (ii) $f\cdots A (g y)\cdots A = -f\cdots A (g y)\cdots A$; (16)

and (iii) the symbols $\epsilon_{BC}^A = \epsilon^{BA}\epsilon_{BC}$ and $\epsilon^A_C = \epsilon^{AB}\epsilon_{BC}$ satisfy the relation

$$\epsilon^A_C = -\epsilon^A_C = \delta^A_C, \quad (17)$$

where $\delta^A_B = 1$ for $A = B$ and $\delta^A_B = 0$ for $A \neq B$.

It is not difficult to verify that the scalar product of two generic spinors $|\chi\rangle$ and $|\kappa\rangle$, written as $\langle \chi|\kappa\rangle$, has the expression

$$\langle \chi|\kappa\rangle = \chi_{AR}\kappa^A. \quad (18)$$

The anti-symmetry of $\epsilon_{AB}$ implies that

$$\langle \chi|\kappa\rangle = -\langle \kappa|\chi\rangle \quad (19)$$

and, as a consequence, $\langle \kappa|\kappa\rangle = 0$ for all $|\kappa\rangle$.

It is easy to check that (i) the identity operator $I$ in the space $\mathcal{W}$ can be written as

$$I = |S_A\rangle\langle S_A|, \quad (20)$$

satisfying $I|\kappa\rangle = |\kappa\rangle$ for all $|\kappa\rangle \in \mathcal{W}$, and (ii) the components of $|\kappa\rangle$ have the following expressions,

$$\kappa^A = -\langle S_A|\kappa\rangle, \quad \kappa_A = -\langle S_A|\kappa\rangle. \quad (21)$$

An operation of complex conjugation can be introduced, which converts a space $\mathcal{W}$ to a space denoted by $\overline{\mathcal{W}}$, with $|\kappa\rangle \in \mathcal{W} \rightarrow \overline{\kappa} \in \overline{\mathcal{W}}$. Corresponding to a basis $|S_A\rangle \in \mathcal{W}$, the space $\overline{\mathcal{W}}$ has a basis denoted by $\langle S_{A'}|$ with a primed label $A'$. In the basis of $\langle S_{A'}|$, $\overline{\kappa}$ is written as

$$\overline{\kappa} = \overline{\kappa}^B\langle S_{A'}|, \quad (22)$$

where

$$\overline{\kappa}^B := (\kappa^B)^*. \quad (23)$$

One can introduce matrices $\epsilon^{A'B'}$ and $\epsilon_{A'B'}$, which have the same elements as the $\epsilon$ matrices discussed above. When a spinor $\kappa^A$ is transformed by a SL(2,C) matrix, the spinor $\overline{\kappa}^{A'}$ is transformed by its complex conjugation matrix (see Appendix A).

B. Dirac spinors in the abstract notation

It is well known that, in the chiral representation of the $\gamma^\mu$-matrices, a four-component Dirac spinor can be decomposed into two two-component spinors, the left-handed (LH) one and the right-handed (RH) one (see, e.g., Refs. [2, 11–13]. In this section, we briefly discuss some properties of Dirac spinors in this decomposition. (See Appendix A of Ref. [13] for more detailed discussions.)

One can write as $\psi(x) = U(p)e^{-ipx}$ a stationary solution of the Dirac equation for a free electron with a mass $m$ and with a momentum $p$, satisfying $p^\mu p_\mu = m^2$ with a positive $p^0 > 0$. Here, $U(p)$ is a four-component spinor, satisfying the stationary Dirac equation,

$$(\gamma^\mu p_\mu - m)U(p) = 0. \quad (24)$$

In terms of LH (upper) and RH (lower) two-component spinors, the spinor $U(p)$ is written as

$$U(p) = \begin{pmatrix} \chi^A(p) \\ \overline{\nu^B(p)} \end{pmatrix}. \quad (25)$$

The two spinors $U^r(p)$ of $r = 0, 1$ in Eq. (24) constitute a basis for the spin states of electron. Written with labels for two-component spinors, the $\gamma^\mu$-matrices have the form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^{\mu A'B'} \\ \overline{\sigma_{A'B'}} & 0 \end{pmatrix}, \quad (26)$$

where $\sigma^{\mu A'B'}$ are the EW-symbols mentioned previously [11–13, 15]. Here, $\overline{\sigma_{A'B'}}$ indicates the complex conjugate.
of $\sigma^\mu_{AB'}$, namely, $\bar{\sigma}^\mu_{AB'} = (\sigma^\mu_{AB'})^\dagger$. An often-used set of explicit expressions for these symbols is given by

$$
\sigma^{0AB'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^{1AB'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

$$
\sigma^{2AB'} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^{3AB'} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(27)

The stationary Dirac equation (24) is, then, split into two equivalent subequations, namely,

$$
u^A(p) = \frac{1}{m} p_\mu \sigma^{\mu AB'} \bar{\sigma}^\mu_{AB'}(p), \quad \bar{\sigma}^\mu_{AB'}(p) = \frac{1}{m} p_\mu \sigma^{\mu AB'} u^A(p).
$$

(28a, 28b)

The stationary solution for a free positron with a definite four-momentum $p (p^0 > 0)$ is usually written as $\psi(x) = V(p)e^{ipx}$, satisfying

$$
\langle \gamma^\mu p_\mu + m \rangle V(p) = 0,
$$

(29)

where

$$
V(p) = \begin{pmatrix} u^A(p) \\ -\bar{\sigma}^\mu_{AB'}(p) \end{pmatrix}.
$$

(30)

The two spinors $V^r(p)$ of $r = 0, 1$ in Eq. (29) constitute a basis for the spin states of positron.

In the abstract notation, the spinors $U(p)$ and $V(p)$ are written as

$$
|U(p)\rangle = \begin{pmatrix} |u(p)\rangle \\ |\bar{\sigma}(p)\rangle \end{pmatrix}, \quad |V(p)\rangle = \begin{pmatrix} |u(p)\rangle \\ -|\bar{\sigma}(p)\rangle \end{pmatrix}.
$$

(31)

We would emphasize that, to be consistent with the expression of bra in Eq. (12) for two-component spinors, the bras corresponding to the above two kets are written as

$$
\langle U(p)| = \langle |u(p)\rangle, |\bar{\sigma}(p)\rangle\rangle, \quad \langle V(p)| = \langle |u(p)\rangle, -|\bar{\sigma}(p)\rangle\rangle,
$$

(32)

respectively, without taking complex conjugate for the two-component spinors. The complex conjugates of $|U(p)\rangle$ and $\langle U(p)|$ are written as

$$
|\bar{U}(p)\rangle = \begin{pmatrix} |\bar{\sigma}(p)\rangle \\ |v(p)\rangle \end{pmatrix}, \quad \langle \bar{U}(p)| = \langle -|\bar{\sigma}(p)\rangle, |v(p)\rangle\rangle.
$$

(33)

Although a product $\langle U'(p)|U(p)\rangle$ is a Lorentz scalar, it is not an inner product, because $\langle U(p)|U(p)\rangle$ is always equal to zero. The well-known inner product used in QED, which is also a Lorentz scalar, has the form

$$
\langle \bar{U}(p)|U_s(p)\rangle = \delta_{rs}
$$

(34)

with Lorentz-invariant labels $r$ and $s$, where

$$
\langle \bar{U}(p)| := \langle \bar{U}(p)|\gamma_c = (|v(p)\rangle, -|\bar{\sigma}(p)\rangle),
$$

(35)

with

$$
\gamma_c = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
$$

(36)

It is straightforward to verify that

$$
\langle \bar{U}(p)|U_s(p)\rangle = \delta_{rs}.
$$

(37)

A remark: In the formulation in terms of two-component spinors, the matrix $\gamma_0$ in the ordinary definition of $\bar{U} = U^\dagger \gamma_0$ used in QED should be replaced by $\gamma_c$ in Eq. (36) as shown in Eq. (35).

C. Basic properties of four-component vectors

In this section, we recall basic properties of four-component vectors as special cases of spinors [11–13]. We use the ordinary notation in this section and will discuss the abstract notation in the next section.

A basic point is a one-to-one mapping, given by the EW-symbols discussed above, between a direct-product space $\mathcal{W} \otimes \mathcal{W}$ and a four-dimensional space denote by $\mathcal{Y}$. For example, a spinor $\phi_{AB'}$ in the space $\mathcal{W} \otimes \mathcal{W}$ can be mapped to a vector $K^\mu$ in the space $\mathcal{Y}$ by the relation

$$
K^\mu = \sigma^{\mu AB'} \phi_{AB'}.
$$

(38)

In the space $\mathcal{Y}$, of particular importance is a symbol denoted by $g^{\mu \nu}$, defined by the following relation, with the $\epsilon$-symbols in the space $\mathcal{W} \otimes \mathcal{W}$,

$$
g^{\mu \nu} = \sigma^{\mu AB'} \sigma^{\nu CD'} \epsilon_{AC} \epsilon_{B'D'}.
$$

(39)

One can introduce a lower-indexed symbol $g_{\mu \nu}$, which has the same matrix elements as $g^{\mu \nu}$, namely, $[g_{\mu \nu}] = [g^{\mu \nu}]$. These two symbols $g_{\mu}$ like the symbols $\epsilon$ in the space $\mathcal{W}$, can be used to raise and lower indexes, e.g.,

$$
K_{\mu} = K^{\nu} g_{\nu \mu}, \quad K^\nu = g^{\mu \nu} K_{\mu}.
$$

(40)

Making use of the anti-symmetry of the symbol $\epsilon$, it is easy to verify that $g^{\mu \nu}$ is symmetric, i.e.,

$$
g^{\mu \nu} = g^{\nu \mu}.
$$

(41)

Due to this symmetry, the upper-lower positions of a pair of identical indexes ($\mu$) are exchangeable, namely

$$
F_{\mu \cdots}^{\mu \cdots} = F_{\mu \cdots}^{\mu \cdots}.
$$

(42)

The EW-symbols have the following properties,

$$
\sigma_{\mu}^{AB'} \sigma^{CD' \nu} = \delta_{CD'}^{AB'} \delta^\nu_\mu, \quad \sigma^{\mu AB'} \sigma_{\nu CD'} = \delta^\mu_\nu,
$$

(43)

where $\delta^\mu_\nu = 1$ for $\mu = \nu$ and $\delta^\mu_\nu = 0$ for $\mu \neq \nu$, and $\delta_{CD'}^{AB'} := \delta_{CD} \delta_{AB'}$. Using the relations in Eq. (43), it is not difficult to check that the map from $\mathcal{W} \otimes \mathcal{W}$ to $\mathcal{Y}$ given in Eq. (38) is reversible. Making use of Eq. (13), and of a previously-mentioned property that $\epsilon_{AB'} = \delta_{B'}^B$, it is not difficult to verify the following relation,

$$
\sigma_{\mu AB'} \sigma^{\mu CD'} = \epsilon_{AC} \epsilon_{B'D'}.
$$

(44)
Then, substituting the definition of \( g^{\mu\nu} \) in Eq.\((43) \) into the product \( g^{\mu\nu} g_{\nu\lambda} \), after simple algebra, one gets

\[
g^{\mu\nu} g_{\nu\lambda} = g^{\mu\lambda} = g_{\lambda\mu} = \delta^{\mu}_{\lambda}. \tag{45}\]

When a SL(2,C) transformation is carried out on the space \( \mathcal{Y} \), a related transformation is applied to the space \( \mathcal{Y} \). Requiring invariance of the EW-symbols, transformations on the space \( \mathcal{Y} \) can be fixed and turn out to constitute a (restricted) Lorentz group (see Appendix A for detailed discussions.). Therefore, the space \( \mathcal{Y} \) is a four-component vector space. In fact, substituting the explicit expressions of the EW-symbols in Eq.\((27) \) into Eq.\((43) \), one gets

\[
g^{\mu\nu} = \sigma^{\mu}_{\alpha\beta} \sigma^{\nu\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{46}\]

which is just the Minkovski’s metric. Furthermore, one can show that both the product

\[
J_{\mu} K^{\nu} = J^{\mu} g_{\mu\nu} K^{\nu} \tag{47}\]

and the product

\[
J_{\mu} K^{\nu} = J^{\mu} g_{\mu\nu} K^{\nu} \tag{48}\]

are scalars under Lorentz transformations (see Appendix A). The product in Eq.\((48) \) is the one used in QED.

**D. Abstract notation for four-component vectors**

In the abstract notation, a basis in the space \( \mathcal{Y} \) is written as \( |T_{\mu}\rangle \) with \( \mu = 0, 1, 2, 3 \). The index of the basis can be raised by \( g^{\mu\nu} \), i.e., \( |T^{\nu}\rangle = g^{\mu\nu}|T_{\mu}\rangle \), and similarly \( |T_{\mu}\rangle = g_{\mu\nu}|T^{\nu}\rangle \). A generic four-component vector \( |K\rangle \) in the space \( \mathcal{Y} \) is expanded as

\[
|K\rangle = K^{\mu}|T_{\mu}\rangle. \tag{49}\]

In consistency with the expression of bra in Eq.\((12) \) for two-component spinors, the bra corresponding to \( |K\rangle \) is written as

\[
\langle K| = (T_{\mu}|K^{\mu}. \tag{50}\]

To write the scalar product in Eq.\((47) \) as \( \langle J|K\rangle \), one needs

\[
\langle T_{\mu}|T_{\nu}\rangle = g_{\mu\nu} \tag{51}\]

similar to the case of two-component spinor in Eq.\((13) \).

It is not difficult to verify the following properties. (i) Making use of Eq.\((45) \), one can verify that the identity operator in the space \( \mathcal{Y} \) can be written as

\[
I = |T_{\mu}\rangle\langle T_{\mu}| = |T^{\nu}\rangle\langle T^{\nu}|. \tag{52}\]

(ii) The components \( K^{\mu} \) and \( K_{\mu} \) have the following expressions,

\[
K^{\mu} = \langle T^{\nu}|K\rangle, \quad K_{\mu} = \langle T_{\mu}|K\rangle. \tag{53}\]

(iii) The symmetry of \( g^{\mu\nu} \) implies that \( \langle T_{\mu}|T_{\nu}\rangle = \langle T_{\nu}|T_{\mu}\rangle \), as a result,

\[
\langle K|J\rangle = \langle J|K\rangle \tag{54}\]

for two arbitrary vectors \( |K\rangle \) and \( |J\rangle \) in the space \( \mathcal{Y} \).

To get further understanding about the relation between the space \( \mathcal{Y} \) and the space \( \mathcal{W} \otimes \mathcal{W} \), let us discuss their basis vectors in the abstract notation. Basis spinors in \( \mathcal{W} \otimes \mathcal{W} \) can be written as \( |S_{AB}\rangle := |S_{A}\rangle|S_{B}\rangle \), or \( |S_{BA}\rangle := |S_{B}\rangle|S_{A}\rangle \). When dealing with direct products of two-component spinors written in the form of kets, one should pay attention to the order of the spinors. The generic connection between spin and statistics [16, 17] suggests that one may assume anticommutability for the order of the kets, i.e.,

\[
|S_{A}\rangle|S_{B}\rangle = -|S_{B}\rangle|S_{A}\rangle, \quad |S_{A}\rangle|S_{B}\rangle = -|S_{B}\rangle|S_{A}\rangle \tag{55}\]

for all values of \( A \) and \( B \).

When writing bras for the basis in the space dual to \( \mathcal{W} \otimes \mathcal{W} \), we follow the same rule as that discussed above for kets, e.g., \( \langle S_{BA}| := \langle S_{B}|\langle S_{A}| \). Besides, it proves convenient to introduce an operation for transforming between kets and bras, which we call transposition, denoted by a superscript \( T \), defined by the following relations,

\[
|\phi\rangle^{T} := \langle \phi|, \quad |\phi\rangle^{T} := |\phi\rangle, \quad (|\phi\rangle\langle \psi|)^{T} := |\psi\rangle\langle \phi|. \tag{56}\]

Note that, e.g., the transposition of \( |S_{AB}\rangle \) is \( \langle S_{BA}| \), but not \( \langle S_{AB}| \).

Since \( |S_{AB}\rangle = |S_{BA}\rangle \), the operation of complex conjugation maps the space \( \mathcal{Y} \) into itself. One can use \( \overline{T_{\mu}} \) to denote the complex conjugate of \( |T_{\mu}\rangle \). Since \( |T_{\mu}\rangle \) and \( |T_{\mu}\rangle \) lie in the same space, it is unnecessary to introduce any change to the label \( \mu \). Hence, \( \overline{T_{\mu}} \) can be written as \( \overline{T_{\mu}} \) with the label \( \mu \) unchanged.

It proves convenient to introduce an operator for the EM-symbols, denoted by \( \sigma \), namely,

\[
\sigma := \sigma^{AB'}|T_{\mu}\rangle\langle S_{BA}|. \tag{57}\]

Using \( \sigma^{T} \) to indicate the complex conjugate of \( \sigma \), Eq.\((57) \) gives

\[
\overline{\sigma} = \overline{\sigma}^{AB'}|T_{\mu}\rangle\langle S_{BA}|, \tag{58}\]

where \( \sigma^{AB'} = (\sigma^{AB'})^{*} \) as discussed previously. Making use of the explicit expressions of the EM-symbols in Eq.\((27) \), it is easy to verify that

\[
\overline{\sigma}^{BA'} = \sigma^{AB'}. \tag{59}\]

There exists some freedom in the determination of the relation between \( |T_{\mu}\rangle \) and \( \overline{T_{\mu}} \). It proves convenient to assume that the basis vectors \( |T_{\mu}\rangle \) are “real” that is,

\[
|T_{\mu}\rangle = |T_{\mu}\rangle. \tag{60}\]
Then, the complex conjugate of $|K\rangle$ is written as

$$|\overline{K}\rangle = K^{\mu *}|T_{\mu}\rangle. \quad (61)$$

Substituting Eqs. (59)-(60) into Eq. (58) and noting Eq. (55), it is easy to verify that

$$\overline{\sigma} = -\sigma. \quad (62)$$

The transposition of the operator $\sigma$ has the form

$$\sigma^T = \sigma^{AB'}|S_{AB'}\rangle\langle T_{\mu}|. \quad (63)$$

Computing the product of $\sigma$ in Eq. (57) and $\sigma^T$ in Eq. (63), noting Eqs. (51), (43), and (20), it is easy to verify that

$$\sigma^T \sigma = \sigma \sigma^T = I. \quad (64)$$

This implies that $\sigma^T$ is the reverse of $\sigma$, denote by $\sigma^{-1}$, i.e.,

$$\sigma^{-1} = \sigma^T. \quad (65)$$

Using Eq. (60), it is straightforward to verify that

$$\overline{\langle J\rangle} = \langle T_{\mu}|J^{\mu *}. \quad (66)$$

Therefore, in the abstract notation, the scalar product in Eq. (48) is written as

$$\overline{\langle J\rangle |K\rangle} = J^*_\mu K^{\mu}. \quad (67)$$

It is easy to verify that

$$\overline{\langle K\rangle |J\rangle^*} = \langle J\rangle |K\rangle. \quad (68)$$

IV. EXPRESSIONS OF INTERACTION AMPLITUDES IN THE ABSTRACT NOTATION

In this section, we derive expressions for the interaction amplitudes $h_1$, written with the abstract notation for two-component spinors and four-component vectors discussed in the previous section. But, before doing that, it proves useful to give a brief discussion for some notations related to spinors and vectors appearing in Eq. (6).

Firstly, to avoid confusion, when writing components of a labelled two-component spinor or a labelled four-component vector, we write the label in a pair of parentheses. For example, the LH part of a spinor $U^r(p)$ with a label $r$ is written as $u^{(r)A}(p)$ [cf. Eq. (25)]. In the abstract notation, $u^{(r)A}(p)$ is simply written as $|u^r(p)\rangle$.

Secondly, Eq. (34) suggests that the index $s$ of $U_s$ can be raised by $\delta^{sr}$ and that of $U^s$ be lowered by $\delta_{sr}$ [cf. Eqs. (13) and (15)], namely,

$$U^r = \delta^{sr}U_s, \quad U_r = U^s\delta_{sr}. \quad (69)$$

Thus, for these spinors, there is no difference between upper-indexed ones and lower-indexed ones. This rule is also applicable to the corresponding creation and annihilation operators.

Thirdly, in the abstraction notation, a vector $\varepsilon^{(\lambda)}_\mu (k) \in \mathcal{V}$ is written as $|\varepsilon^{(\lambda)}(k)\rangle$, with an expansion

$$|\varepsilon^{(\lambda)}(k)\rangle = \varepsilon^{(\lambda)}_\mu (k)|T^{\mu}\rangle \quad (70)$$

in a basis $|T^{\mu}\rangle$. Making use of Eqs. (53), (60), and (68), direct derivation shows that

$$\varepsilon^{(\lambda)}_\mu (k) = \langle T^{\mu}|\varepsilon^{(\lambda)}(k)\rangle, \quad \varepsilon^{(\lambda)*}_\mu (k) = \langle \overline{\varepsilon^{(\lambda)}(k)}|T^{\mu}\rangle. \quad (71)$$

Since $|\varepsilon^{(\lambda)}(k)\rangle$ also constitute a basis in $\mathcal{V}$, similar to Eq. (51), one has

$$\langle \overline{\varepsilon^{(\lambda)}(k)}|\varepsilon^{(\lambda)}(k)\rangle = g^{\lambda\lambda'} \quad (72)$$

(see textbooks, e.g., Ref. [20]). The label $\lambda$ follows the same raising/lowering rule as that for the index $\mu$ of $|T\rangle$ discussed previously.

A. The amplitude $h_1$

In this section, we derive an expression for the interaction amplitude $h_1$. Its expression in terms of Dirac spinors is given in Eq. (9) and, here, we write it in a form using two-component spinors and four-component vectors in the abstract notation.

To this end, we note that the two-component-spinor forms of $U^r(p)$ and $\gamma^{\mu}$ in the quantity $R^{r\mu}(q,p)$ in Eq. (10) are given in Eq. (25) and Eq. (26), respectively. To get the corresponding expression for $\tilde{V}^s(q)$, as discussed previously, the matrix $\gamma_0$ in its ordinary definition used in QED should be replaced by $\gamma_c$ as shown in Eq. (66) in the quantity $\gamma_c\gamma^\mu$. This implies that we need to consider the following product,

$$\gamma_c\gamma^\mu = \left( \begin{array}{cc} -\sigma_B^A & 0 \\ 0 & \sigma_{BA'} \end{array} \right). \quad (73)$$

Then, noting Eqs. (25) and (30), the term $R_1^{r\mu}(q,p)$ in Eq. (10) can be written as

$$R_1^{r\mu}(q,p) = -\sigma^{(s)B'}(q)\sigma_A^u(u^{(r)A}(p)) - \sigma^{(s)}_B(\sigma BA' u^{(r)}_A(p). \quad (74)$$

Making use of Eq. (21), the right-hand side (rhs) of Eq. (74) can be further written in the abstract notation as

$$-\sigma_{BA'}\langle S^A S^{B'}|F(q)u^r(p)\rangle - \sigma^{BA'}\langle S^A S^{B'}|v^s(q)\sigma^r(p)\rangle. \quad (75)$$

Making use of Eqs. (21), (25), and (10), and of the definitions of $\sigma$ and $\sigma$ in Eqs. (57)–(58), direct derivation shows that

$$R_1^{r\mu}(q,p) = -\langle \overline{\varepsilon^{(\lambda)}(k)}|\gamma^r(c)|u^r(p)\rangle - \langle \overline{\varepsilon^{(\lambda)}(k)}|v^s(q)\sigma^r(p)\rangle. \quad (76)$$
Then, noting Eqs. (61) and (62), this gives

$$R^{\sigma \mu}_{1}(q, p) = \langle e^{(\lambda)*}(k) | \sigma \left[ |p\rangle \langle\pi| - |\pi\rangle \langle p| \right] \rangle .$$

(77)

It proves convenient to introduce the following product for two spinors $|U\rangle = \left| \frac{u}{\sigma} \right\rangle$ and $|W\rangle = \left| \frac{w}{\sigma} \right\rangle$;

$$|U \circ W\rangle := -|u\rangle \langle w| + |\pi\rangle \langle z| ,$$

(78)

which we call an $\circ$-product of $|U\rangle$ and $|W\rangle$. It is clear that this $\circ$-product lies in the direct-product space $\mathcal{W} \otimes \mathcal{W}$. As a result of Eq. (78), this product has the property that

$$|U \circ W\rangle = -|W \circ U\rangle .$$

(79)

Substituting Eq. (77) into the expression of $h_1$ in Eq. (8), making use of Eq. (78) and solutions of the Dirac equation discussed in Sec. II B, one gets that

$$h_1 = \langle \pi^{\lambda}(k) | \sigma | U^{\dagger}(p) \circ U^{\dagger}(q) \rangle \delta^{3}(p - q - k).$$

(80)

**B. The amplitudes $h_2, \ldots, h_8$**

Now, we discuss the interaction amplitudes $h_i$ of $i = 2, \ldots, 8$. We first discuss the process $h_2$. Similar to $h_1$ in Eq. (8), one gets

$$h_2 = R^{\sigma \mu}_{2}(q, p) e_{\mu}^{(\lambda)}(k) \delta(k - p - q) .$$

(81)

where $R^{\sigma \mu}_{2}(q, p) = \tilde{U}^{\dagger}(q) \mu V^{\dagger}(p)$. Since $R^{\sigma \mu}_{2}(q, p)$ is in fact equal to $R^{\sigma \mu}_{1}(q, p)$, the only difference between $h_2$ in Eq. (81) and $h_1$ in Eq. (8) lies in that $h_2$ has a term $\epsilon_{\mu}^{(\lambda)}(k)$ while $h_1$ has its complex conjugate. Due to this difference, unlike $R^{\sigma \mu}_{2}(q, p)$ in (81), we write $R^{\sigma \mu}_{2}(q, p)$ in the following form,

$$-\sigma^{\mu BA'}(u^{(p)}|\pi^{(q)}|S^{BA'} - \sigma^{\mu BA'}(\pi^{(p)}|v^{(q)}|S_{B}S_{A'}) .$$

(82)

Then, noting the expression of $e_{\mu}^{(\lambda)}(k)$ in Eq. (11) and the definition of $\epsilon^{(\lambda)}(k)$ in Eq. (33), we get

$$R^{\sigma \mu}_{2}(q, p) e_{\mu}^{(\lambda)}(k) = -\langle u^{(p)}|\pi^{(q)}|\pi^{T} e^{(\lambda)}(k)\rangle - \langle \pi^{(p)}|v^{(q)}|\sigma^{T} e^{(\lambda)}(k)\rangle .$$

(83)

Finally, following arguments similar to those leading to Eq. (80), noting Eqs. (62) and (60), we find that

$$h_2 = \langle \pi^{\dagger}(q) \circ U^{\dagger}(p) \sigma^{T} e^{(\lambda)}(k) \rangle \delta^{3}(k - p - q) .$$

(84)

Next, the interaction amplitude $h_3$,

$$h_3 = R^{\sigma \mu}_{3}(q, p) e^{(\lambda)*}_{\mu}(k) \delta(p - q - k) .$$

(85)

where $R^{\sigma \mu}_{3}(q, p) = \tilde{U}^{\dagger}(q) \gamma^{\mu} U^{\dagger}(p)$. The quantity $R^{\sigma \mu}_{3}(q, p)$ can be written in a form similar to that of $R^{\sigma \mu}_{1}(q, p)$ in Eq. (74), but with a plus sign between the two products in it. Then, following a procedure similar to that leading to Eq. (80), one gets

$$h_3 = \langle \pi^{\dagger}(k) | \sigma | U^{\dagger}(p) \circ U^{\dagger}(q) \rangle \delta^{3}(p - q - k) .$$

(86)

The rhs of Eq. (86) has a form close to that of Eq. (80), but not exactly the same. We find that the two can be written in a same form, if a negative-$p_{0}$ solution of the Dirac equation is used. Let us use $U_{-}(p)$ to denote a solution of Eq. (84) with $p_{0} = -\sqrt{|p|^{2} + m^{2}}$. It is not difficult to verify that the spinor $U_{-}(p)$ has the following relation to a spinor $V$ as a solution of Eq. (84) (see Appendix B for detailed discussions), namely,

$$U_{-}(p) = V(-p) .$$

(87)

Then, the amplitude $h_3$ can be written as

$$h_3 = \langle \pi^{\dagger}(k) | \sigma | U^{\dagger}_{-}(p) \circ U^{\dagger}_{-}(q') \rangle \delta^{3}(p + q' - k) .$$

(88)

with $q' = -q$, having the same form as $h_1$ in Eq. (80).

Following procedures similar to those given above, it is not difficult to show that the amplitudes $h_i$ of $i = 4, \ldots, 8$ can be written in the following forms (see Appendix C for detailed discussions),

$$h_5 = \langle \pi^{\dagger}(k) | \sigma | U^{\dagger}_{-}(p') \circ U^{\dagger}_{-}(q') \rangle \delta^{3}(p + q' - k) ,$$

$$h_7 = \langle \pi^{\dagger}(k) | \sigma | U^{\dagger}_{-}(p') \circ U^{\dagger}_{-}(q') \rangle \delta^{3}(p + q' - k) ,$$

$$h_4 = \langle U^{\dagger}(q') \circ U^{\dagger}_{-}(p') | \sigma^{T} e^{(\lambda)}(k) \rangle \delta^{3}(p + q' - k) ,$$

$$h_6 = \langle U^{\dagger}_{-}(q') \circ U^{\dagger}_{-}(p') | \sigma^{T} e^{(\lambda)}(k) \rangle \delta^{3}(p + q' - k) .$$

(89)

where $p' = -p$ and $q' = -q$ for momenta $p$ and $q$ used in Eq. (88).

**V. A GEOMETRIC INTERPRETATION FOR QED**

In this section, we show that, based on the expressions of the interaction amplitudes $h_i$ derived in the previous section, a geometric interpretation with respect to representation spaces of the $SL(2, C)$ group can be given to QED. Here, we focus on the odd interaction processes ($11, 13, 15, 17$), because results to be obtained for them can be straightforwardly extended to the even interaction processes ($12, 14, 16, 18$).

Specifically, we discuss a geometric interpretation for the interaction amplitudes $h_i$ in Secs. IV A and IV B from which one may get a geometric interpretation for the interaction Hamiltonian in QED. In Sec. IV C we discuss a geometric property of the state space for one electron and one positron, which can be used to introduce photon states.
A. A geometric interpretation for \( h_1 \) and some problems met with \( h_{3,5,7} \)

In this section, we discuss some features of the expressions of the interaction amplitudes \( h_{1,3,5,7} \) given in the previous section, a possible geometric interpretation to them, and also some problems met.

A remarkable feature is seen in the expressions of the amplitudes \( h_{1,3,5,7} \), given in Eqs.\((80), (88), \) and \((89)\): they share a common form. This suggests that these four interaction processes may be reduced to one fundamental interaction process. Clearly, for this to be possible, a same geometric interpretation should be applicable to all the four amplitudes \( h_{1,3,5,7} \).

The operator part of \( H_1 \) in Eq.\((8)\), namely, \( a_1^\dagger(k)d_1(q)b_r(p) \), can be given the geometric interpretation that an electron and a positron “combine” and “form” a photon. If one wants to give the amplitude \( h_1 \) in Eq.\((8)\) a consistent interpretation, the spinor \( U^\dagger(q) \) should be interpreted as representing the spin state of the positron. Note that, according to the geometric meaning of the EW-symbol operator \( \sigma \), \( \sigma|\kappa\rangle \) gives a four-component-vector equivalent of \( |\kappa\rangle \). Then, the spin part of \( h_1 \), namely, \( \langle\mathcal{E}(k)\sigma|U^\dagger(p)\otimes U(q)\rangle \), is just given by the overlap of the spin state of the outgoing photon and the four-component-vector equivalent of the overlap of the spin states of the incoming electron and positron.

When applying the above-discussed interpretation to the amplitudes \( h_{3,5,7} \), two problems are met. The first one is whether, instead of the ordinary description \( |V^+(q)\rangle \), the spinor \( |U^\dagger(q)\rangle \) can be used to describe the spin state of positron in a generic situation. One should note that, in the computation of the scattering matrix in QED, explicit descriptions of states of positron are used only when computing the interaction amplitudes \( h_i \). This implies that, in principle, one should have some freedom in choosing the way of describing spin states of positron, depending on the expressions of \( h_i \) employed, as long as no change of the values of \( h_i \) is induced.

The second problem is that the expected geometric interpretations to \( h_{3,5,7} \) do not directly match the creation and annihilation operators in \( H_{3,5,7} \). For example, on one hand, the amplitude \( h_3 \) in Eq.\((8)\) contains a spinor \( |U^\dagger(q')\rangle \), on the other hand, \( H_3 \) in Eq.\((8)\) contains no operator for positron. In the next section, we show that a solution to these two problems can be obtained, if appropriate properties are assumed for negative-\( p^0 \) states of electron and positron.

B. Vacuum fluctuation and geometric interpretation for \( h_{3,5,7} \)

In this section, we show that, making use of negative-\( p^0 \) solutions of the Dirac equation, such as \( U^-(-p) \) of Eq.\((24)\), it is possible to reduce the four odd interaction processes in QED to one fundamental interaction process. We would like to stress that we do not adopt Dirac’s original interpretation of these negative-\( p^0 \) solutions as corresponding to negative energies, which has been abandoned in modern QED. In our treatment, a negative value of \( p^0 \) is not interpreted as any negative energy, but, is just used as a label to indicate a solution of the Dirac equation.

In order to avoid confliction with experimental observations, we assume that a particle in a negative-\( p^0 \) state can not exist during any finite time interval. One should note that, under this assumption, the physical state space for real electrons and positrons, which can be observed experimentally, remains the same as that discussed in the ordinary formulation of QED.

We observe that an electron in a positive-\( p^0 \) state \( |p\rangle[U^-(p)] \) and a positron in a negative-\( p^0 \) state \( |-p\rangle[U^+(-p)] \) have exactly opposite charge, 3-momentum, and angular momentum. Two such particles as a whole should have no net physical property that can be detected experimentally. (The Heisenberg’s uncertainty principle suggests that the energy aspect should be completely uncertain, at the instant at which the negative-\( p^0 \) positron may exist.) This suggests that it should be reasonable to assume that two such particles may vanish into the vacuum, or emerge from the vacuum as a vacuum fluctuation. We’ll call such a pair of electron and positron a vacuum-fluctuation pair (VP). Clearly, another possibility of VP is composed of a negative-\( p^0 \) electron and a positive-\( p^0 \) positron, in the states \( |p\rangle[U^+(p)] \) and \( |-p\rangle[U^-(p)] \), respectively.

Under the two assumptions discussed above, a negative-\( p^0 \) electron/positron may exist only instantly in a process of interaction or in a process of vacuum fluctuation. For example, a negative-\( p^0 \) electron may be generated from an interaction process and, then, immediately forms a VP with an existing positive-\( p^0 \) positron and vanishes into the vacuum. Below, we show that this supplies a mechanism, by which the problems discussed in the previous section can be solved.

Let us consider the amplitude \( h_3 \) in Eq.\((8)\) and give it the same geometric interpretation as that discussed previously for \( h_1 \). This interpretation suggests an interaction process of the type of II, in which an electron in a positive-\( p^0 \) state \( |U^-(p)| \) and a positron in a negative-\( p^0 \) state \( |U^+(q')| \) combine and form a photon in a state \( \langle\mathcal{E}(k)| \). This negative-\( p^0 \) positron may come from a VP emerging from the vacuum; meanwhile, the VP should contain a positive-\( p^0 \) electron and a negative-\( p^0 \) positron coming from a VP combine and form a photon, leaving the positive-\( p^0 \) electron in the VP as an outgoing particle. Its net effect is that a positive-\( p^0 \) electron in a state \( |U^+(p)| \) changes to a state \( |U^+(q)| \) and emits a photon in a state \( \langle\mathcal{E}(k)| \). This is exactly what is described by the creation and annihilation operators in \( H_3 \), namely, by \( a_3^\dagger(k)b_r^\dagger(q)b_r(p) \). Therefore, the interaction process I3 can be regarded as a combination of an interaction
process of the type I1 and a vacuum fluctuation process.

It is not difficult to see that the interaction processes I5 and I7 can be interpreted in a similar way. Specifically, I5 can be interpreted as a process, in which a positive-\(p^0\) positron combines with a negative-\(p^0\) electron coming from a VP and the two particles form a photon, leaving the positive-\(p^0\) positron in the VP as an outgoing particle; and I7 as a process, in which the two VPs are separated in the same way with the two negative-\(p^0\) fermions in them immediately forming a photon. (Detailed descriptions for these processes will be given in Sec.VI D.)

To summarize, under the two assumptions discussed above for negative-\(p^0\) states of electron and positron, a solution to the problems discussed in the previous section can be obtained. Specifically, (i) when the expressions of the amplitudes \(h_i\) derived in Sec.IV are used, spin states of positron are described by \(|\bar{U}_r(p)\rangle\). (ii) With the help of VPs, the same geometric interpretation is applicable to all the amplitudes \(h_{1,3,5,7}\), without conflicting the operator parts of \(H_{3,5,7}\). (iii) With the help of vacuum-fluctuation processes, the three odd interaction processes I3, I5, and I7 can be reduced to one fundamental interaction process, which is of the type of the interaction process I1. It is straightforward to generalize the above results to the four even interaction processes and, thus, a unified geometric interpretation for the interaction Hamiltonian in QED is obtained.

C. State spaces of single particles

In this section, we show that, based on a geometric property of the state space for one electron and one positron, it is possible to introduce photon states. This strategy of introducing photon states turns out to be in consistency with the fundamental interaction process discussed in the previous section.

When only one positive-\(p^0\) electron is considered, one can decompose, with no ambiguity, the state space \(\mathcal{E}^{(1)}\) discussed in Sec.IIA according to the momentum and spin degrees of freedom, namely,

\[
\mathcal{E}^{(1)} = \bigoplus_p |p\rangle \otimes \mathcal{S}_e(p),
\]

where \(\mathcal{S}_e(p)\) denotes the subspace spanned by \(|U_r(p)\rangle\) of \(r = 0,1\). Similarly, the state space for one positive-\(p^0\) position can be written as

\[
\mathcal{E}^{(1)} = \bigoplus_p |p\rangle \otimes \mathcal{S}_p(p),
\]

where \(\mathcal{S}_p(p)\) is spanned by \(|\bar{U}_r(p)\rangle\) of \(r = 0,1\). These two spaces for single particles have the common feature that the momentum and spin degrees of freedom are written in a separate way. We call this feature a single-particle structure and it can be formally written as

\[
\mathcal{E}_{\text{single-particle}} = \bigoplus_p |P\rangle \otimes \mathcal{S},
\]

where \(|P\rangle\) indicates a state vector related to momentum and \(\mathcal{S}\) represents a linear subspace for the spin degree of freedom.

Due to the identical-particle nature of electron and positron, among the subspaces \(\mathcal{E}^{(\alpha,n,\overline{n})}\) discussed in Sec.IIA besides the two subspaces \(\mathcal{E}^{(1)}\) and \(\mathcal{E}^{(1)}\), there is only one subspace, namely, \(\mathcal{E}^{(1,1)}\), to which the above-discussed treatment of separating the momentum and spin degrees of freedom can be applied in an unambiguous way [19]. This subspace \(\mathcal{E}^{(1,1)}\) can also be written in the form of Eq.(92), with

\[
|P^{(1,1)}\rangle = |p\rangle|q\rangle, \quad \mathcal{S}^{(1,1)} = \mathcal{S}_e(p) \otimes \mathcal{S}_p(q).
\]

To study properties of the subspace \(\mathcal{S}^{(1,1)}\), we note that the LH and RH parts of a four-component spinor \(|U_r(p)|\rangle\) are functions of each other [see Eq.(28)]. Hence, say, the LH part \(|u_r(p)|\rangle\) should be sufficient to represent the whole spinor \(|U_r(p)|\rangle\). This implies that the spinor space \(\mathcal{W}\) for \(|u_r(p)|\rangle\) should be sufficient to represent the subspace \(\mathcal{S}_e(p)\). Since it is usually useful to explicitly indicate the existence of both the LH and the RH parts of \(|U_r(p)|\rangle\), we can write \(\mathcal{S}_e(p)\) as

\[
\mathcal{S}_e(p) = \mathcal{W} \oplus \mathcal{S}_p, \quad \mathcal{S}^{(1,1)} = \mathcal{W} \oplus \mathcal{S}_p q.
\]

Equation (95) shows that the subspace \(\mathcal{S}^{(1,1)}\) is in fact isomorphic to \(\mathcal{W} \otimes \mathcal{W}\) and, hence, to a four-component-vector space \(\mathcal{V}\) discussed in Sec.IIB. Since the spin subspace for a photon is just a four-component-vector space, the spin part of \(\mathcal{E}^{(1,1)}\), namely, \(\mathcal{S}^{(1,1)}\) is isomorphic to the spin subspace for a photon. This property is in consistence with the fundamental interaction process discussed in the previous section.

The above results suggest an alternative way of introducing photon states. That is, based on the single-particle structure of \(\mathcal{E}^{(1,1)}\), one may assume the existence of a particle (a photon), whose state space has the form in Eq.(92) with \(\mathcal{S}\) being a four-component-vector space. (A natural method of determining the momentum part is by the momentum conservation.) This, together with the previously-given geometric interpretation to the interaction Hamiltonian, gives a consistent geometric interpretation to QED.

VI. A THEORY BASED ON THE LORENTZ SYMMETRY

With the results obtained in the previous section, we are now ready to propose a theory for the interaction
among electron, positron, and photon. In this theory, two fundamental interaction processes are introduced, based on geometric relations among representation spaces of the SL(2,C) group. The purpose of discussing such a theory is to show that the interaction Hamiltonian of QED can be derived from the Lorentz symmetry, with some auxiliary assumptions about vacuum fluctuations.

Specifically, we discuss the state space in Sec. VIA and discuss the fundamental interaction processes in Sec. VIB. In Sec. VIDA and Sec. VIE, we show that the interaction Hamiltonian for positive-\( p^0 \) interaction processes in the proposed theory has exactly the same form as the ordinary one in Eq. (3).

### A. State spaces

Both positive-\( p^0 \) and negative-\( p^0 \) electron and positron are considered at the fundamental level of the proposed theory. It proves convenient to introduce a label (say, \( \varrho \) or \( \beta \)) to explicitly indicate these two cases, with \( \varrho = - \) indicating a negative-\( p^0 \) case and \( \varrho = + \) indicating a positive-\( p^0 \) case. When there is no risk of causing confusion, for brevity, we sometimes omit the label \( \varrho \) for \( \varrho = \pm \) (but never omit it for \( \varrho = 0 \)); for example, for \( |U^\varrho_\varrho(p)\rangle \) with \( \varrho = \pm \), we may simply write \( |U^\varrho(p)\rangle \) as done in previous sections.

Discussions given in the previous section suggest that one may introduce the following assumptions for the state spaces of particles, as representation spaces of the SL(2,C) group. We denote the assumptions by \( A_{S1(2,3)} \) with “\( S \)” standing for “state space”.

- \( A_{S1} \). The state space for one electron, denoted by \( \mathcal{E}_e^{(1)} \), is spanned by vectors \( |p\rangle |U^\varrho_\varrho(p)\rangle \).
- \( A_{S2} \). The state space for one positron, denoted by \( \mathcal{E}_e^{(+)} \), is spanned by \( |p\rangle |U^\varrho_\varrho(p)\rangle \).
- \( A_{S3} \). The single-particle structure of the space \( \mathcal{E}(1,\mathcal{T}) \) implies the existence of a particle, whose state space has a form in Eq. (102) wherein \( S \) is isomorphic to \( S(1,\mathcal{T}) \) and \( |P\rangle = |k\rangle \) with \( k = p + q \) for \( p \) and \( q \) given by \( |p| \), \( |q| \) in Eq. (103).

Assuming further that \( k_0 = |k| \), the particle mentioned in the assumption \( A_{S3} \) can be identified with photon. Since \( S(1,\mathcal{T}) \) is isomorphic to \( \mathcal{Y} \), the state space for one photon can be written as

\[
\mathcal{E}_{ph}^{(1)} = \bigoplus_k |k\rangle \otimes \mathcal{Y}.
\]

Using the above-discussed state spaces for single particles, the total state space can be constructed in the ordinary way (cf. discussions given in Sec. II A).

We’ll use the following (shorthand) notations for electron and positron, respectively,

\[
|p_{(\varrho)}\rangle := b_{\varrho}^\dagger(p)|0\rangle = |p\rangle |U^\varrho_{\varrho}(p)\rangle, \tag{97}
\]

\[
|\varrho_{(\varrho)}\rangle := d_{\varrho}^\dagger(p)|0\rangle = |p\rangle |U^\varrho_{\varrho}(p)\rangle. \tag{98}
\]

Correspondingly, for bras we write \( \langle p_{(\varrho)}| := \langle p|U^\varrho_{\varrho}(p)\rangle \) for electron and \( \langle \varrho_{(\varrho)}| := \langle p|U^\varrho_{\varrho}(p)\rangle \) for positron. By definition, we require that the upper and lower positions of \( \varrho \) are identical, e.g.,

\[
|U^\varrho_{\varrho}(p)\rangle = |U^\varrho_{\varrho}(p)\rangle. \tag{99}
\]

For photon states, we use the shorthand notation,

\[
|k_{(\lambda)}\rangle := a_{\lambda}^\dagger(k)|0\rangle = |k\rangle |\mathcal{E}_{\lambda}(k)\rangle \tag{100}
\]

and \( |\mathcal{E}_{(\lambda)}\rangle = |k\rangle |\mathcal{E}_{\lambda}(k)\rangle \).

Regarding negative-\( p^0 \) electron and positron, as discussed in Sec. VIB, we introduce the following assumptions.

- \( A_{neg-p^0} \). A negative-\( p^0 \) electron/positron cannot exist during any finite time period.
- \( A_{VP} \). From the vacuum there may appear a VP as a vacuum fluctuation; reversely, a VP can vanish into the vacuum.

As discussed previously, a negative-\( p^0 \) electron/positron may exist only instantaneously in an interaction process or in a vacuum fluctuation. For this reason, there is no need to assign any property of energy to particles in negative-\( p^0 \) states and, thus, a negative value of \( p^0 \) is only a label for a state of an electron/positron, not representing any negative energy. But, for a positive value of \( p^0 \), one can interpret it as an energy in the same way as done in the ordinary formulation of QED.

### B. Fundamental Interaction processes

In this section, we discuss fundamental interaction processes (FIPs). They may involve both positive-\( p^0 \) and negative-\( p^0 \) states of fermions. Discussions given in Sec. VI suggest that one may introduce the following assumption as a basic assumption of the proposed theory.

- \( A_{F1} \). There may exist an FIP, called F1, in which an electron and a positron combine and form a photon. It is described by an H-operator denoted by \( H_{F1} \), which maps the direct-product state space for one electron and one positron to the state space for one photon.

Here, we use “H-operator” to refer to an operator describing an FIP. Below, we derive an explicit form of \( H_{F1} \).

Let us first consider the case with only positive-\( p^0 \), in which \( H_{F1} \) maps the space \( \mathcal{E}(1,\mathcal{T}) \) to \( \mathcal{E}_{ph}^{(1)} \); other cases can
be treated in the same way. It is natural to assume that, in the H-operator \( H_{F1} \), the momentum degree of freedom and the spin degree of freedom function separately. This implies a product form of \( H_{F1} \), namely,

\[
H_{F1} = G_m G_s,
\]

(101)

where \( G_m \) is for the momentum degree of freedom and \( G_s \) for the spin degree of freedom. Due to the momentum conservation implied by the assumption \( A_{34} \), the simplest form of \( G_m \) is written as

\[
G_m = \int d\vec{p} d\vec{q} d\vec{k}'|k'|^2 \delta^3(k' - p' - q') \langle q' | p' | e \rangle,
\]

(102)

where the subscripts "e" and "p" stand for electron and positron, respectively. The operator \( G_s \) should first project the product \( U_r(p)\langle \bar{U}_s(q) \rangle \) into the space \( \mathcal{W} \otimes \mathcal{W} \), then, project the result into the space \( \mathcal{W}' \). We take the previously-discussed \( \odot \)-product \( U_r(p) \otimes \bar{U}_s(q) \) [cf.Eq.(75)] for the first projection and use \( G_\odot \) to indicate it,

\[
G_\odot U_r(p)\bar{U}_s(q) = U_r(p)\bar{U}_s(q).
\]

(103)

As discussed previously, the operator \( \sigma \) in Eq.(57) maps the space \( \mathcal{W} \otimes \mathcal{W} \) to \( \mathcal{W}' \). Hence, the map \( G_s \) has the form

\[
G_s = \sigma G_\odot.
\]

(104)

To get an explicit expression of \( H_{F1} \), we write it as \( I_{ph}' H_{F1} I_{ph}(\mathcal{W} \otimes \mathcal{W}) \), where \( I_{ph}(\mathcal{W} \otimes \mathcal{W}) \) is identity operators in the subspaces \( \mathcal{E}(1), \mathcal{E}(\mathcal{W}), \) and \( \mathcal{E}(ph) \), respectively. They have the following explicit forms,

\[
I_{ph}(\mathcal{W}) = \int d\vec{k}' |k(\lambda)| \langle k(\lambda) |, \quad I_{ph}(\mathcal{W}) = \int d\vec{q} \langle \sigma| q(s) \rangle, \quad I_{ph}(\mathcal{W}) = \int d\vec{p} \langle \sigma| p(r) \rangle,
\]

(105)

which can be easily verified, utilizing the following scalar products [cf. Eqs.(63) and (72)]

\[
\langle \sigma| p'(r') \rangle = \delta_{s-r} \delta^3(p - p'), \quad \langle k(\lambda)| p(\nu) \rangle = g_{\lambda \nu} \delta^3(k - k'),
\]

(107)

Substituting Eqs.(105) and (106) into \( I_{ph} H_{F1} I_{ph} \) and using Eqs.(105), (102) and (104), one gets an expression of \( H_{F1} \) in terms of ket and bra.

Changing the ket and bra in the above-obtained expression of \( H_{F1} \) to creation and annihilation operators and, then, generalizing the result to the generic case of the sign of \( p^0 \), one gets the following explicit expression of \( H_{F1} \),

\[
H_{F1} = \int d\vec{p} d\vec{q} d\vec{k} \sigma^k \delta^3(p) \delta^3(q) \langle k^{0, s, 3, \lambda} | h_f^{p, s, 3, \lambda} \rangle,
\]

(109)

where

\[
h_f^{p, s, 3, \lambda} = \mathcal{E}(k) | \sigma U^{s, 3}(p) \rangle \langle U^{s, 3}(q) | \delta^3(p, q - k).
\]

(110)

Below, for brevity, we simply write \( h_f^{p, s, 3, \lambda} \) as \( h_f \). Clearly, \( h_{1,3,5,7} \) in Eqs.(50), (52), and (53) are four specific cases of \( h_f \) and, hence, \( h_f \) can be interpreted in the same way as \( h_1 \).

We introduce the following assumption for the reverse of \( F1 \).

- \( F2 \). The reverse of \( F1 \), called \( F2 \), is also an FIP, described by an H-operator \( H_{F2} = H_{F1}^\dagger \).

Then,

\[
H_{F2} = \int d\vec{p} d\vec{q} d\vec{k} \sigma^k \delta^3(p) \delta^3(q) a_\lambda(k) h_f^\dagger.
\]

(111)

A useful expression for \( h_f^\dagger \) can be obtained by substituting Eq.(57) into Eq.(101) and, then, taking complex conjugate. This gives

\[
h_f^\dagger = \mathcal{E}(k) | \sigma T_\mu \rangle \sigma^{\mu A'B'} \langle S_{BA'} | \sigma^{A'B'} T_p | \delta^3(p) \rangle \delta^3(p - q - k).
\]

(112)

Note that \( \langle S_{BA'} | \sigma^{A'B'} T_p | \delta^3(p) \rangle \delta^3(p - q - k) = \langle U^{s, 3}(p) | U^{s, 3}(q) \rangle \delta^3(p - q - k) \).

Finally, noticing the definition of \( \sigma^T \) in Eq.(63) and the relations in Eqs.(65) and (70), we find that

\[
h_f^\dagger = \langle U^{s, 3}(p) | U^{s, 3}(q) \rangle \delta^3(p - q - k).
\]

(113)

The amplitude \( h_f^\dagger \) in Eq.(113) can be interpreted in a way similar to that for \( h_f \). But, here, the bra \( \langle U^{s, 3}(p) \rangle \) with an overline should correspond to the spin state of an electron, while, the bra \( \langle U^{s, 3}(q) \rangle \) to the spin state of a positron.

C. Positive-\( p^0 \) interaction processes

In this section, we discuss descriptions for interaction processes, which can be studied experimentally. For such an interaction process, all the incoming and outgoing particles should be positive-\( p^0 \) particles. Emphasizing the net effect, we call such a process a positive-\( p^0 \) interaction process. We call the finest positive-\( p^0 \) interaction processes, each of which can not be further decomposed into other positive-\( p^0 \) processes, basic positive-\( p^0 \) interaction processes (BPPIPs). An operator describing a BPPIP is also called an H-operator.

A BPPIP should be composed of FIP(s) and VP(s), and it should contain at least one FIP. The construction of a BPPIP depends on properties of the FIP it contains, particularly, on the signs of \( p^0 \) of the two fermions in the FIP. There are totally three possible cases of the construction and, below, we discuss them separately. In the first case, both signs are positive and, hence, the FIP is a positive-\( p^0 \) process. Clearly, this FIP is a BPPIP.
In the second case, one sign is positive and the other is negative, thus, the FIP contains one negative-\(p^0\) fermion. To satisfy the assumption \(A_{\text{neg}-p^0}\), this negative-\(p^0\) fermion must be involved in another process at the same time. If one assumes that the possibility can be neglected for two FIPs to happen exactly at a same instant, which seems reasonable, then, due to the assumption \(A_{\text{neg}-p^0}\), the negative-\(p^0\) fermion can not be involved in another FIP. This implies that the related BPIP should contain one and only one FIP. As a consequence, the above-mentioned negative-\(p^0\) fermion must belong to some VP and participate in a related vacuum-fluctuation process. Specifically, if it is an incoming particle of the FIP, it must come from a VP; while, if it is an outgoing particle, it must immediately forms a VP with an existing particle. Since the other fermion in the VP must be a positive-\(p^0\) particle, a combination of the FIP and the VP gives a positive-\(p^0\) process. This gives a BPIP.

In the third case, both signs are negative, which implies that the FIP contains two negative-\(p^0\) fermions. Following arguments similar to those given above, it is seen that the two fermions must belong to two VPs and the related BPIP can be constructed from the FIP and the two VPs.

Based on discussions given above and also on those given in Sec. 13 we propose to introduce the following assumption for BPIPs.

- **A_{BP1}**: A BPIP contains one and only one FIP and is described by an \(H\)-operator, which is obtained from the \(H\)-operator of the FIP with the following replacements,

\[
\begin{align*}
    b_r(-p) & \to d^\dagger_r(-p), \quad d_r(-p) \to b^\dagger_r(-p), \\
    b^\dagger_r(-p) & \to d_r(-p), \quad d^\dagger_r(-p) \to b_r(-p).
\end{align*}
\] (114)

Two remarks. (i) In Eq. (114) the label \(\dagger\) for positive-\(p^0\) particles has been omitted, e.g., \(d^\dagger_r(-p)\) in fact indicates \(d^\dagger_r(-p)\). (ii) The replacements in Eq. (114) can be summarized by the following rule: A creation/annihilation operator for a negative-\(p^0\) fermion is replaced by an annihilation/creation operator for a positive-\(p^0\) fermion, which can form a VP with the negative-\(p^0\) fermion.

### D. H-operators of BPIPs

In this section, we derive explicit expressions of the \(H\)-operators of BPIPs. Since each BPIP contains one and only one FIP, we can discuss BPIPs according to properties of the FIPs they contain. There are totally eight BPIPs, which are constructed from the four FIPs of F1 with different signs of \(p^0\) of the two fermions involved and from the four related FIPs of F2. We show that these BPIPs correspond exactly to the processes II-18 in the ordinary formulation of QED.

Firstly, let us consider an FIP of F1, which by itself is a positive-\(p^0\) process. Clearly, this FIP is a BPIP. According to the assumptions \(A_{F1}\) and \(A_{BP1}\), the \(H\)-operator for this BPIP is given by \(H_{F1}\) in Eq. (109) with \(\alpha = \beta = +\). Comparing this \(H\)-operator and the operator \(H_1\) in Eq. (3) with \(h_1\) given by Eq. (80), it is seen that the two operators are the same. Hence, this BPIP is equivalent to the interaction process I in the ordinary formulation of QED.

Secondly, consider an FIP of F1, which contains a positive-\(p^0\) incoming electron and a negative-\(p^0\) incoming positron. As discussed previously, this negative-\(p^0\) positron in the FIP must come from a VP emerging from the vacuum. Combination of this FIP and this VP results in a positive-\(p^0\) process, in which a positive-\(p^0\) incoming electron combines with a negative-\(p^0\) positron coming from a VP to form a photon, leaving the positive-\(p^0\) electron in the VP as an outgoing electron. This is a positive-\(p^0\) interaction process and is a BPIP, with the net effect of the emission of a photon by an electron.

To give an explicit description for the BPIP discussed above, let us use \(|p\rangle|U^\dagger(q)\rangle\) to indicate the state of the incoming electron, and use \(|q\rangle|U(q)\rangle\) and \(|q'\rangle|U^{-1}(q')\rangle\) with \(q' = -q\) to indicate the states of the electron and positron in the VP, respectively. The amplitude \(h_f\) in Eq. (110) for this FIP has the form,

\[
h_f^{(2)} = \langle \zeta^\lambda(k)|\sigma|U^\dagger(p)\rangle \langle \zeta^{-\lambda}(k)|\sigma'|U^\dagger(q')\rangle \delta^3(p + q' - k), \tag{115}
\]

where the number “2” in the superscript of \(h_f^{(2)}\) indicates the number of the case we consider here for BPIP. According to the assumption \(A_{F1}\), for this FIP the \(H\)-operator in Eq. (109) takes the form \(H_{F1} = \int d\vec{p}d\vec{q}d\vec{q}'a^\dagger_\lambda(k)d_{-\lambda}(-q')b_r(p)h_f\). Then, according to the assumption \(A_{BP1}\), the \(H\)-operator for the related BPIP is written as

\[
H_{F1}^{(2)} = \int d\vec{p}d\vec{q}d\vec{q}'a^\dagger_\lambda(k)b_\lambda(q)b_r(p)h_f^{(2)}. \tag{116}
\]

Clearly, \(H_{F1}^{(2)}\) in Eq. (116) is equal to \(H_3\) in Eq. (3) with \(h_3\) given by Eq. (80). Therefore, effectively, this BPIP is the process I3 in QED.

Thirdly, consider an FIP of F1, which contains a negative-\(p^0\) incoming electron and a positive-\(p^0\) incoming positron. Following arguments similar to those given above in the second case, it is seen that the related BPIP is the emission of a photon by a positron. Using \(|q\rangle|U(q)\rangle\) to denote the state of the incoming positron, and using \(|p\rangle|U^\dagger(p)\rangle\) and \(|p'\rangle|U^\dagger(p')\rangle\) with \(p' = -p\) to denote the states of the positron and electron in the involved VP, respectively, the amplitude \(h_f\) in Eq. (110) for this FIP is written as

\[
h_f^{(3)} = \langle \zeta^\lambda(k)|\sigma|U^\dagger(p')\rangle \langle \zeta^{-\lambda}(k)|\sigma|U^\dagger(q)\rangle \delta^3(p' + q - k). \tag{117}
\]

According to the assumption \(A_{F1}\), the \(H\)-operator \(H_{F1}\) in Eq. (109) for this FIP has an operator product
a_1^+(k)d_-(q)b_{-}(p)$. According to the assumption $A_{BPPIP}$, this implies that the operator product for the related BPPIP should be given by $a_1^+(k)d_-(q)d_{-}(p)$. Then, it is ready to see that the obtained H-operator is equal to the operator $H_5$ in Eq. (5). Hence, this BPPIP gives the process I5 in QED.

Fourthly, we discuss an FIP of F1, in which both incoming fermions are negative-$p^0$ particles. Since each negative-$p^0$ fermion should come from a VP, the related BPPIP should involve two VPs, one possessing a negative-$p^0$ electron and the other possessing a negative-$p^0$ positron. Thus, the net effect of the BPPIP should be that a photon, an electron, and a positron emerge from the vacuum. To be specific, let us write states of the particles in the first VP as $|p⟩|U^-(p)⟩$ and $|p'⟩|U^-(p')⟩$ with $p' = -p$, and states of the particles in the second VP as $|q⟩|U^+(q)⟩$ and $|q'⟩|U^+(q')⟩$ with $q' = -q$. For this FIP, the amplitude $h_f$ in Eq. (113) takes the form

$$h_f^{(4)} = (\bar{\psi}^-(k)|\sigma|U^-(p')\otimes U^-(q'))\delta^3(p' + q' - k),$$

and the H-operator $H_{F1}$ in Eq. (113) has an operator product $a_1^+(k)d_{-}(q)b_{-}(p')$. The H-operator for the related BPPIP should, then, have an operator product $a_1^+(k)b_1^+(q)d_{-}(p)$. It is easy to check that the H-operator thus obtained for this BPPIP is equal to $H_1$ in Eq. (3). Therefore, this BPPIP gives the process I7 in QED.

Fifthly, similar to the first case discussed above, an FIP of F2, which contains a positive-$p^0$ outgoing electron and a positive-$p^0$ outgoing positron, is a BPPIP. According to the assumption $A_{F2}$, this BPPIP is described by the H-operator $H_{F2} = H_{F1}$. Note that, here, $H_{F1}$ is equal to $H_2$ as discussed in the first case. Then, since $H_2 = H_{F1}$, it is clear that this BPPIP is equivalent to the process I2 in QED.

Sixthly, we discuss an FIP of F2, in which a photon splits into a positive-$p^0$ electron and a negative-$p^0$ positron. Due to the assumption $A_{neg-p^0}$, the negative-$p^0$ positron generated from the FIP must be annihilated immediately. This can be achieved, only if there exists a positive-$p^0$ incoming electron, which can form a VP with the negative-$p^0$ positron. Thus, the process should be that a positive-$p^0$ incoming electron combines with a negative-$p^0$ positron generated from a photon, and the two particles form a VP and vanish into the vacuum, leaving the positive-$p^0$ electron generated from the photon as an outgoing electron. This gives a BPPIP, with a net effect of the absorption of a photon by an electron.

To be specific, let us use $|k⟩|\epsilon^+(k)⟩$ to denote the state of the incoming photon, and use $|p⟩|U^-(p)⟩$ and $|q⟩|U^+(q)⟩$ to denote the states of the negative-$p^0$ positron and positive-$p^0$ electron generated from the photon. The amplitude $h_f^*$ in Eq. (113) for this FIP is then written as

$$h_f^{(6)*} = (\bar{\psi}^+(q)\otimes U^-)(p')|\sigma^T|\epsilon^+(k))\delta^3(p' + q - k).$$

Note that, as discussed previously, the bra $⟨U^+(q)⟩$ indicates a state of electron and the bra $⟨U^-(p')⟩$ is for a positron. According to the assumption $A_{F2}$, the H-operator for this FIP is given by $H_{F2}$ in Eq. (113), which has an operator product $b_1^+(q)d_{-}(p)|\alpha⟩|k⟩$. Consequently, the negative-$p^0$ incoming electron should be in the state $|p⟩|U^-(p)⟩$ with $p = -p'$. Then, according to the assumption $A_{BPPIP}$, the H-operator for this BPPIP has the form

$$H_{F2}^{(6)} = \int d\bar{p}dpqdk b_1^+(q)b_r(p)|\alpha⟩|k⟩.$$

Comparing with the amplitude $h_{F2}$ in Eq. (113) and the operator $H_4$ in Eq. (5), it is seen that this BPPIP gives the process I4 in QED.

Seventhly, the case of an FIP of F2, in which a photon splits into a negative-$p^0$ electron and a positive-$p^0$ positron, can be treated in the same way as that discussed above. The related BPPIP is the absorption of a photon by a positron. Using $|q⟩|U^-(q)⟩$ and $|k⟩|\epsilon^-(k)⟩$ to denote the states of the incoming positron and incoming photon, respectively, the negative-$p^0$ electron generated from the photon should be in the state $|q⟩|U^+(q)⟩$ with $q' = -q$. We use $|p⟩|U^+(p)⟩$ to denote the state of the positron generated from the photon. Then, the amplitude $h_f^*$ in Eq. (113) for the FIP takes the form

$$h_f^{(7)*} = (\bar{\psi}^+(q)\otimes U^+(p)|\sigma^T|\epsilon^-(k))\delta^3(p + q' - k),$$

the same as the amplitude $h_{F2}$ in Eq. (113) for this FIP has an operator product $b_1^+(q)d_{-}(p)|\alpha⟩|k⟩$. As a result, the H-operator for the related BPPIP has an operator product $d_s(q)d_{-}(p)|\alpha⟩|k⟩$, the same as that in $H_6$ in Eq. (5). Hence, this BPPIP gives the process I6 in QED.

Lastly, let us consider an FIP of F2, in which a photon splits into two negative-$p^0$ fermions. In order to annihilate the two negative-$p^0$ fermions by forming VPs, we may consider two positive-$p^0$ fermions as incoming particles. Hence, the BPPIP should be that three incoming particles (an electron, a positron, and a photon) vanish into the vacuum. To be specific, let us use $|p⟩|U^+(p)⟩$, $|q⟩|U^+(q)⟩$, and $|k⟩|\epsilon^-(k)⟩$ to denote states of the three incoming particles, respectively. In order to form the above-discussed two VPs, the negative-$p^0$ positron and negative-$p^0$ electron generated from the photon should be in the states $|p⟩|U^-(p)⟩$ with $p' = -p$ and $|q⟩|U^+(q)⟩$ with $q' = -q$, respectively. Then, the amplitude for the FIP is written as $h_f^{(8)*} = (\bar{\psi}^-(q)\otimes U^-)(p')|\sigma^T|\epsilon^-(k))\delta^3(p' + q' - k)$, which is just the amplitude $h_{F2}$ in Eq. (113). The H-operator $H_{F2}$ for this FIP has an operator product $b_1^+(q)d_{-}(p)|\alpha⟩|k⟩$, as a result, according to the assumption $A_{BPPIP}$, the H-operator for the related BPPIP contains a product $d_s(q)b_r(p)|\alpha⟩|k⟩$, the same as that in $H_8$ in Eq. (5). Hence, this BPPIP gives the process I8 in QED. Thus, we complete the proof that there exist totally eight BPPIPs, which are effectively the eight interaction processes I2-I8 in QED.
E. The interaction Hamiltonian

We introduce the following assumption for the interaction Hamiltonian.

- $A_{\text{int}}$. The interaction Hamiltonian for positive-$p^0$ interaction processes is given by the sum of the $H$-operators of all BPIPs.

Making use of results obtained in the previous section, it is seen that the interaction Hamiltonian in the proposed theory is equal to $\sum_{i=1}^{8} H_i$. Clearly, it is just the interaction Hamiltonian $H_{\text{int}}$ of QED in Eq.\([3]\).

Furthermore, each BPIP corresponds to one of the processes II-I8 in QED and, hence, corresponds to one basic Feynman diagram. Therefore, the topological equivalence of the eight basic Feynman diagrams is a result of the fact that all the BPIPs are derived from two FIPs, each being the reverse of the other.

VII. CONCLUSIONS AND DISCUSSIONS

In this paper, a theory is proposed to describe the interaction among electron, positron, and photon, which is based on the Lorentz symmetry. Specifically, this is based on geometric relations among the (quantum) state spaces of the three particles, as representation spaces of the SL(2,C) group. It is found that the QED interaction Lagrangian can be "derived" in this theory, without resorting to the gauge symmetry. In this formulation of the interaction Lagrangian, the origin of the topological equivalence of the eight basic Feynman diagrams can be seen quite clearly; that is, these diagrams come from two fundamental interaction processes, each being the reverse of the other.

The above results show that the two most important symmetries in Physics, namely, the Lorentz symmetry and the gauge symmetry, although being mathematically independent of each other, have important overlap in their physical effects, in particular, in the understanding or "derivation" of interaction Lagrangian. This paper is a first step in the investigation of this type of overlap of the two symmetries. Investigations along this line should be useful in the study of theories beyond the SM.

Finally, the proposed theory can be regarded as an alternative way of formulating the basic framework of QED. It also suffers from some problems met in the ordinary formulation of QED, such as the ultraviolet divergence, which implies that renormalization is needed. In addition, the scalar product of photon states in Eq.\([48]\) is not definitely positive, which implies that a technique like the method of indefinite metric is needed when treating photon states \([21]\).

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Appendix A: SL(2,C) and Lorentz transformations

In this appendix, we recall the relation between SL(2,C) transformations and Lorentz transformations. Particularly, when SL(2,C) transformations are carried out on a space $\mathcal{W}$, the corresponding transformations on the space $\mathcal{V}$ are Lorentz transformations.

We recall that the group SL(2,C) is composed of $2 \times 2$ complex matrices with unit determinant \([11,12]\), written as

$$h^{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1. \quad (A1)$$

Under a transformation given by $h^{AB}$, a two-component spinor $\kappa^A$ is transformed to

$$\tilde{\kappa}^A = h^{AB} \kappa^B, \quad (A2)$$

where we use tilde to indicate the result of a SL(2,C) transformation.

It is straightforward to verify that $\epsilon^{AB}$ is invariant under SL(2,C) transformations, that is, $\tilde{\epsilon}^{AB} = h^{AC} h^{BD} \epsilon^{CD}$ has the same matrix form as $\epsilon^{AB}$ in Eq.\([14]\). Direct computation can verify the following relations,

$$h^{AC} h^{BD} = \epsilon^{AC} \epsilon^{BD} \quad (A3)$$

$$h^{AC} h^{BD} = \epsilon^{CD} \quad (A4)$$

It is not difficult to verify that the product $\chi_{AK}^A$ is a scalar product, that is, $\chi_{AK}^A = \chi_{AK}^A$.

When $\kappa^A$ is transformed by a matrix $h^{AB}$, $\tilde{\kappa}^A$ is transformed by its complex-conjugate matrix, namely,

$$\tilde{\kappa}^A = \tilde{h}^{AB} \kappa^B, \quad (A5)$$

where

$$\tilde{h}^{AB} \equiv (h^{BA})^* \quad \text{with} \quad A = A', B = B'. \quad (A6)$$

Now, we discuss relation between SL(2,C) transformations and Lorentz transformations. Related to a SL(2,C) transformation $h^{AB}$ performed on a space $\mathcal{W}$, we use $\Lambda^{A'}_B$ to denote the corresponding transformation on the space $\mathcal{V}$,

$$\tilde{K}^A = \Lambda^{A'}_B K^{B'}, \quad (A7)$$

It proves convenient to require invariance of the EM-symbols under SL(2,C) transformations, namely,

$$\tilde{\sigma}^{A'B} = \sigma^{A'B}, \quad (A8)$$
where
\[ \bar{\sigma}^{\mu A B} = \Lambda^\mu_{\rho \nu} \bar{\tau}^{A}_{\rho} h^{B}_{\mu} \sigma^{C D} \sigma^{\nu C' D}. \]  
(A9)

This requirement can fix the form of \( \Lambda^{\mu \nu} \). In fact, substituting Eq. (A9) into Eq. (A8) and rearranging the positions of some labels, one gets
\[ \sigma^{\mu A B} = \Lambda^\mu_{\rho \nu} \bar{\tau}^{A}_{\rho} h^{B}_{\nu} \sigma^{C D}. \]  
(A10)

Multiplying both sides of Eq. (A10) by \( \bar{\tau}^{A}_{\rho} h^{B}_{\nu} \sigma^{C D} \), the rhs gives
\[ \Lambda^\mu_{\rho \nu} \bar{\tau}^{A}_{\rho} h^{B}_{\nu} \sigma^{C D} \eta^{E F} = \Lambda^\mu_{\rho \nu} \bar{\tau}^{A}_{\rho} h^{B}_{\nu} \sigma^{C D} \eta^{E' F'}. \]

Then, it is easy to check that
\[ \lambda^\mu \eta^{\nu E F} \Lambda_{\mu E}^{\nu} \eta^{\nu E F} = \Lambda^\mu_{\rho \nu} \sigma^{\nu C D} \sigma^{\nu C' D}. \]  
(A11)

where Eq. (A13) and Eq. (43) have been used. Then, one gets the following expression for \( \Lambda^\mu_{\nu} \),
\[ \Lambda^\mu_{\nu} = \sigma^\mu_{\rho \nu} \bar{\tau}^{A}_{\rho} h^{B}_{\nu} \sigma^{C D}. \]  
(A12)

Substituting Eq. (A12) into the product \( \Lambda^\mu_{\nu} \eta^{\nu E F} \sigma^{E' F'} \), one gets
\[ \sigma^\mu_{\rho \nu} \bar{\tau}^{A}_{\rho} h^{B}_{\nu} \sigma^{E' F'} \eta^{E F} = \Lambda^\mu_{\rho \nu} \sigma^{E' F'}. \]

Then, noting Eqs. (17), (43), and (10), it is straightforward to verify Eq. (A14).

(ii) Equation (A13) implies that the matrix \( g^{\mu \nu} \) is invariant under the transformation \( \Lambda \), that is,
\[ g^{\mu \nu} = g^{\nu \mu}. \]  
(A15)

(iii) The product \( K^\mu \eta^{\nu} \sigma^{E' F'} = K^\mu J^\nu \) is a scalar under the transformation \( \Lambda \), i.e.,
\[ \Lambda^{\mu} \eta^{\nu} \sigma^{E' F'} = \Lambda^{\mu} \eta^{\nu} \sigma^{E' F'}. \]  
(A16)

which can be readily proved making use of Eq. (A13).

(iv) Making use of Eq. (59), it is straightforward to show that the transformation \( \Lambda \) is real, namely,
\[ \Lambda^{\mu \nu} = (\Lambda^{\mu \nu})^*. \]  
(A17)

Then, it is easy to check that \( K^\mu J^\nu \) is also a scalar product.

Appendix B: Relation between negative-\( p^0 \) and positive-\( p^0 \) solutions of Dirac equation

In this appendix, we discuss a relation between positive-\( p^0 \) and negative-\( p^0 \) stationary solutions of Dirac equation. Let us first consider solutions of the type \( \psi(x) = U(p)e^{-ipx} \) and of the type \( \psi(x) = V(p)e^{ipx} \), both with \( p_0 > 0 \) [cf. Eqs. (24) and (29)]. One can write the spinors \( U(p) \) and \( V(p) \) in the rest frame of reference as
\[ U_0 = \left( \begin{array}{c} \xi^A \\ \eta_{B'} \end{array} \right) \quad \text{and} \quad V_0 = \left( \begin{array}{c} \xi^A \\ -\eta_{B'} \end{array} \right), \]  
(B1)

respectively. Changing to a reference frame in which the particle moves with a momentum \( p \), its four-momentum \((m, 0, 0, 0)\) is changed to \( p = (p^0, p) \), meanwhile, the spinors \( U_0 \) and \( V_0 \) are changed to \( U(p) \) and \( V(p) \), respectively, by a Lorentz transformation \( \Lambda(p) \), with
\[ U(p) = \Lambda(p)U_0 = \left( \begin{array}{c} u^A(p) \\ \eta_{B'}(p) \end{array} \right), \]  
(B2)

\[ V(p) = \Lambda(p)V_0 = \left( \begin{array}{c} u^A(p) \\ -\eta_{B'}(p) \end{array} \right). \]  
(B3)

A negative-\( p^0 \) solution of the first type discussed above is written as \( \psi(x) = U_-(p)e^{-ipx} \) with \( p_0 < 0 \), where \( U_-(p) \) satisfies Eq. (24). Straightforward derivation shows that, in the rest frame in which the particle has a four-momentum \((m, 0, 0, 0)\), the solution takes the form
\[ U_0 = \left( \begin{array}{c} \xi^A \\ \eta_{B'} \end{array} \right). \]  
(B4)

The Lorentz transformation, which brings the four-momentum \((m, 0, 0, 0)\) to \( p = (|p^0|, p) \), should bring \((m, 0, 0, 0)\) to \((-|p^0|, -p)\). Hence, for the spin degree of freedom, this transformation is written as \( \Lambda(-p) \), bringing \( \xi^A \) to \( u^A(-p) \) and \( \eta_{B'} \) to \( \eta_{B'}(-p) \). Then,
\[ U_-(p) = \Lambda(-p)U_0 = \left( \begin{array}{c} u^A(-p) \\ \eta_{B'}(-p) \end{array} \right). \]  
(B5)

Comparing with Eq. (B3), it is seen that
\[ U_-(p) = V(p). \]  
(B6)

Following similar arguments, one can show that
\[ V_-(p) = U(p). \]  
(B7)

In the abstract notation, the above-discussed Dirac spinors are written as
\[ |U_-(p)\rangle = \left( \begin{array}{c} |u(-p)\rangle \\ -|\eta(-p)\rangle \end{array} \right), \quad |V_-(p)\rangle = \left( \begin{array}{c} |u(-p)\rangle \\ -|\eta(-p)\rangle \end{array} \right). \]  
(B8)
Appendix C: Derivation of the amplitudes $h_4, \ldots, h_8$

In this appendix, we derive the expressions of the interaction amplitudes $h_4, \ldots, h_8$ given in Eq. (83). In a process $I_4$, a photon is absorbed by an electron. Making use of Eqs. (85)-(88), one gets

$$h_4 = R_4^{\gamma\mu}(q, p) e_\mu^5(k) \delta^3(k + p - q), \quad (C1)$$

where $R_4^{\gamma\mu}(q, p) = \hat{V}^\gamma(q)\gamma^\mu U^\gamma(p)$. Following arguments similar to those leading to the expression of $h_3$ in Eq. (83) for a process $I_2$, one gets

$$R_4^{\gamma\mu}(q, p) e_\mu^5(k) = -(i \gamma^\mu(p) \pi^T(q) |\sigma^T| \varepsilon^\lambda(k) + i \sigma^T(p) u^\gamma(q) |\sigma^T| \varepsilon^\lambda(k)). \quad (C2)$$

Then, noting Eqs. (82), (85), (86), and (88), direct derivation shows that

$$h_4 = (\overline{U}_\gamma^\mu(q) \circ U_\gamma^\mu(p') |\sigma^T| \varepsilon^\lambda(k)) \delta^3(p' + q - k), \quad (C3)$$

where $p' = -p$.

In a process $I_5$, a photon is generated by a positron. It has an amplitude

$$h_5 = R_5^{\gamma\mu}(q, p) e_\mu^5(k) \delta^3(p - q + k),$$

where $R_5^{\gamma\mu}(q, p) = \hat{V}^\gamma(q)\gamma^\mu V^\gamma(p)$. Note that $R_5^{\gamma\mu}(q, p)$ is, in fact, exactly equal to $R_4^{\gamma\mu}(q, p)$ in Eq. (83). This implies that one may write $R_5^{\gamma\mu}(q, p) e_\mu^5(k)$ in the same way as that in Eq. (83) for $h_3$. But, this would leave a difference in the $\delta$-functions for momenta in the expressions for $h_4$ and $h_5$.

In order to obtain an expression of $h_5$ whose $\delta$-function has the same form as that of $h_3$ in Eq. (83), at the point where $h_3$ is written in the form in Eq. (83), one can write $h_5$ as

$$h_5 = (\overline{U}_\gamma^\mu(q) |\sigma| U_\gamma^\mu(p') \circ \overline{U}^\gamma(q)) \delta^3(p - q + k). \quad (C4)$$

Then, it is not difficult to find that

$$h_5 = (\overline{U}_\gamma^\mu(q) |\sigma| U_\gamma^\mu(p') \circ \overline{U}_\gamma^\mu(q)) \delta^3(p' + q - k), \quad (C5)$$

where $p' = -p$, which is just the expression of $h_5$ in Eq. (83).

For a process $I_6$, in which a photon is absorbed by a positron, one has an amplitude $h_6$,

$$h_6 = R_6^{\gamma\mu}(q, p) e_\mu^5(k) \delta^3(p - q - k) \quad (C6)$$

where $R_6^{\gamma\mu}(q, p) = \hat{V}^\gamma(q)\gamma^\mu U^\gamma(p)$. Note that the relation between this process and a process $I_4$ is similar to that between $I_5$ and $I_3$ discussed above. Then, it is not difficult to verify that $h_6$ can be written as

$$h_6 = (\overline{U}_\gamma^\mu(q') \circ U_\gamma^\mu(p') |\sigma^T| \varepsilon^\lambda(k)) \delta^3(p + q' - k), \quad (C7)$$

where $q' = -q$.

A process $I_7$ has three outgoing particles (an electron, a positron, and a photon), without any incoming particle. It has an amplitude $h_7$,

$$h_7 = R_7^{\gamma\mu}(q, p) e_\mu^5(k) \delta^3(p + q + k), \quad (C8)$$

where $R_7^{\gamma\mu}(q, p) = \hat{V}^\gamma(q)\gamma^\mu V^\gamma(p)$. Note that $R_7^{\gamma\mu}(q, p) = R_4^{\gamma\mu}(q, p)$ and, hence, $h_7$ can be treated in a way similar to that used in the derivation of $h_1$ in Eq. (84). The result is that

$$h_7 = (\varepsilon^\lambda(k) |\sigma| U_\gamma^\mu(p') \circ \overline{U}^\gamma(q')) \delta^3(p' + q' - k), \quad (C9)$$

with $p' = -p$ and $q' = -q$.

Finally, a process $I_8$, in which a photon is annihilated together with an electron and a positron, has an amplitude

$$h_8 = R_8^{\gamma\mu}(q, p) e_\mu^5(k) \delta^3(p + k + q), \quad (C10)$$

where $R_8^{\gamma\mu}(q, p) = \hat{V}^\gamma(q)\gamma^\mu U^\gamma(p)$. Noting that $R_8^{\gamma\mu}(q, p) = R_2^{\gamma\mu}(q, p)$, following arguments similar to those leading to the expression of $h_2$ in Eq. (84), one gets

$$h_8 = (\overline{U}_\gamma^\mu(q) \circ U_\gamma^\mu(p') |\sigma^T| \varepsilon^\lambda(k)) \delta^3(p' + q' - k), \quad (C11)$$

where $p' = -p$ and $q' = -q$.

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[18] The fields $\psi$ and $A_\mu$ may have some common, constant prefactors. We do not write such prefactors explicitly, because they can be absorbed in the unwritten electronic change in Eq.(5). In literature, the factor $\frac{1}{\sqrt{p^0}}$ in $d\vec{p}$ is sometimes written as $\frac{1}{\sqrt{\mathbf{p}}}$.

[19] Even if disregarding the ambiguity of the treatment, those subspaces $\mathcal{E}_{(n,m)}$ with either $n \geq 2$ or $m \geq 2$ still cannot be written in the form in Eq.(12). For example, let us consider the subspace $\mathcal{E}^{(2)}$. Since the momentum and spin degrees of freedom are written in a separate way on the rhs of Eq.(12), basis vectors related to $|\mathbf{p}(r)\rangle|\mathbf{p}(r)\rangle$ should be legitimate for the space in Eq.(12). However, as well-known, they do not belong to $\mathcal{E}^{(2)}$.

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