GLOBAL ATTRACTOR OF MULTI-VALUED OPERATORS WITH APPLICATIONS TO A STRONGLY DAMPED NONLINEAR WAVE EQUATION WITHOUT UNIQUENESS

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Abstract. The paper investigates the existence of global attractors for a few classes of multi-valued operators. We establish some criteria and give their applications to a strongly damped wave equation with fully supercritical nonlinearities and without the uniqueness of solutions. Moreover, the geometrical structure of the global attractors of the corresponding multi-valued operators is shown.

1. Introduction. In this paper, we are concerned with the existence of global attractors for a few classes of multi-valued operators. We devote to establish some abstract criteria and give their applications to a strongly damped wave equation with fully supercritical nonlinearities and without the uniqueness of solutions. Moreover, the geometrical structure of the global attractors of the corresponding multi-valued operators is shown.

It is well known that for a nonlinear evolution equation, the estimates available for the solutions in supercritical nonlinearity case are evidently too weak to prove their uniqueness (or the uniqueness fails), which leads to the fact that one can not define the solution semigroup according to traditional manner and all the standard criteria to guarantee the existence of the global attractor cease to be effective. In order to overcome this difficulty, Babin and Vishik \cite{1} proposed the concepts of generalized semigroup (see Def. 2.1 below) and its generalized attractor and established some criteria on the existence of weak generalized attractor. They realized that the semigroup identity \( S(t + \tau) = S(t)S(\tau) \) holds only in very exceptional cases and that this identity should be replaced by the inclusion \( S(t + \tau) \subset S(t)S(\tau), \forall t, \tau \geq 0 \).

To the best of our knowledge it is the first paper dedicated to attractors of multi-valued semigroups (see last section in \cite{1}). For the related extensions later on this topic, one can see \cite{2, 3, 23, 24, 25, 26, 31} and references therein. As an application, Babin and Vishik used the abstract criteria in \cite{1} to the IBVP of the weakly damped wave equation.

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semi-linear wave equation
\[ u_{tt} - \Delta u + \epsilon u_t + f(u) = g, \]  
(1)
\[ u|_{\partial \Omega} = 0, \ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \]  
(2)
where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with the smooth boundary. Assume that the nonlinearity \( f(u) \) is of growth exponent \( r \), i.e.,
\[ |f(s)| \leq C(1 + |s|^r), \quad s \in \mathbb{R}. \]  
(3)
It is well-known that \( \hat{p} = 3 \) is the critical exponent of \( f(u) \) because the uniqueness of weak solutions of problem (1)-(2) fails when \( r > 3 \). Under the assumption:
\[ |f(u_1) - f(u_2)| \leq C(1 + |u_1|^{6-3\gamma} + |u_2|^{6-3\gamma})|u_1 - u_2|^{\gamma}, \quad \gamma > 0, \]  
(4)
which implies that the restriction condition (3) holds for \( r < 6 \) (supercritical nonlinearity), they showed that the related generalized semigroup of the multi-valued (solution) operators has a weak global attractor (i.e. both the compactness and the attractiveness of the attractor are in the sense of weak topology).

Instead of introducing the generalized semigroup of multi-valued operators, Carvalho, Cholewa and Dlotko [10] proved the similar results as in [1] by proposing the concept of the subclass \( \mathcal{L}_S \) of limit solutions. And they directly constructed the desired weak global attractor of the subclass \( \mathcal{L}_S \).

Ball [4] proposed the notion of generalized semiflow \( \mathcal{G} \), which is a subclass of all weak solutions, and further [5, 6] established some criteria on the global attractor of \( \mathcal{G} \). He used them (see [6]) to study the existence of global attractor of problem (1)-(2), with \( \Omega \subset \mathbb{R}^N \). In the supercritical nonlinearity case: \( r > \hat{p} \equiv \frac{N}{N-2} \), he proved that the related generalized semiflow \( \mathcal{G} \) has a global attractor.

Recently, by utilizing Strichartz type estimates for the linear wave equation in a bounded domain \( \Omega \subset \mathbb{R}^3 \), Kalantarov, Savostianov and Zelik [22], Savostianov and Zelik [28] established for the critical quintic nonlinearity (\( r = p^* = 5 \)) (see (3)) the existence and uniqueness of Shatah-Struwe solutions (i.e., the weak solution belonging to space \( L^4(0,T;L^{12}(\Omega)) \)) and showed that the (Shatah-Struwe) solution semigroup \( S(t) \) associated with problem (1)-(2) (or say: the subclass of Shatah-Struwe solutions) possesses a smooth global attractor in natural phase space.

Supercritical nonlinearity may lead that the weak solutions of the related evolution equation lose their uniqueness (more accurately, one can not prove their uniqueness), in this case, it is natural to consider the longtime behavior of the class of all weak solutions. Chepyzhov and Vishik [11, 12, 13] proposed the notion of trajectory attractor to study this case. Indeed, one approach to deal with the asymptotic behavior of weak solutions without the requirement of their uniqueness is by using the theory on trajectory attractor (cf. [11, 12, 13]). For example, in the supercritical nonlinearity case: \( r > \hat{p} = 3 \) (\( N = 3 \), Zelik [36] proved that the class of weak solutions of Eq. (1) possesses a trajectory attractor. But the trajectory attractor is in the trajectory phase space rather than in the original phase space. Although the space contains all weak solutions, it is equipped with the weak topology (rather than the strong topology in natural energy space) and the trajectory attractor loses the compactness and attractiveness in the strong topology.

In order to avoid the shortage of the trajectory attractor, mathematicians try to find other approaches to deal with the longtime behavior of the subclass of weak solutions in the supercritical nonlinearity case. They proposed some notions of the subclass of weak solutions satisfying some properties (for example, the generalized
semiflow (cf. [6, 34]), the subclass $\mathcal{L}$ of limit solutions (cf. [10, 32, 33]), the subclass of Shatah-Struwe solutions (cf. [22, 27, 28, 29]), and so on (cf. [3]), and investigated the asymptotic behavior of these subclasses in the original phase space and established the existence of global attractors in the sense of strong or weak topology.

There exists now in literature a lot of nonlinear evolution equations of physical interest for which the uniqueness of solutions fails (or one can not prove the uniqueness due to technical problem) in supercritical nonlinearity case. In such cases, the solution operators become multi-valued maps, which results in that the classical attractor theory based on the semigroup of operators becomes useless. Hence the theory on the attractor of multi-valued operators is important for it allow us to continue to study the longtime behavior of the solutions no matter we have uniqueness or not, and this topic has been extensively investigated over the last one and a half decades, one can see [3, 6, 9, 23, 24, 25, 26, 31] and references therein.

The purpose of the present paper is to establish a few criteria on the existence of global attractors for a few classes of multi-valued operators which may constitute neither the generalized semigroup nor the multi-valued semiflow (cf. [3, 23, 24, 25, 26]), and give their applications to a model equation without uniqueness.

The paper is arranged as follows. In Section 2, we introduce some preliminaries. In Section 3, we establish some abstract criteria on the existence of global attractors. In Section 4, we apply these criteria to a strongly damped wave equation with fully supercritical nonlinearities and without the uniqueness of solutions to establish the existence of its global attractors in natural energy space. In Section 5, we further investigate the geometrical structure of the global attractors.

2. Preliminary. Let $X$ be a complete metric space, $2^X$ be the space of all subsets of $X$, $T(t) : 2^X \to 2^X$ be a multi-valued map for each $t > 0$, i.e., for every $x \in X, T(t)x$ may not be a single point but a subset of $X, T(0) \supset I$ (the identity operator), and

$$T(t)B = \{T(t)x | x \in B\}, \ \forall B \in 2^X, \ t \geq 0.$$  

**Definition 2.1.** The family of (multi-valued) operators $\{T(t)\}_{t \geq 0}$ is said to be a generalized semigroup acting on $X$ if it is of the properties

$$T(0) = I, \ T(t+s)x \subset T(t)T(s)x, \ \forall x \in X, \ t, s \geq 0. \quad (5)$$

**Definition 2.2.** A set $\mathcal{A} \subset X$ is said to be a global attractor of the family of multi-valued operators $\{T(t)\}_{t \geq 0}$ if

(i) $\mathcal{A}$ is a compact set in $X$;

(ii) $\mathcal{A}$ is an invariant set, i.e., $T(t)\mathcal{A} = \mathcal{A}, \ t \geq 0$;

(iii) $\mathcal{A}$ is an attracting set in $X$, i.e., for any bounded set $B$ in $X$,

$$\lim_{t \to +\infty} \text{dist}_X \{T(t)B, \mathcal{A}\} = 0.$$  

Here, $\text{dist}_X \{\cdot, \cdot\}$ is the Hausdorff semi-distance in $X$, i.e.,

$$\text{dist}_X \{A, B\} = \sup_{x \in A} \inf_{y \in B} \text{dist}_X \{x, y\}, \ A, B \subset X.$$

**Definition 2.3.** A set $\mathcal{A} \subset X$ is said to be a generalized global attractor of the family of multi-valued operators $\{T(t)\}_{t \geq 0}$ if

(i) $\mathcal{A}$ is a compact set in $X$;

(ii) $\mathcal{A}$ is an attracting set in $X$;

(iii) (Minimality) $\mathcal{A}$ is contained in any closed attracting set.
Remark 1. (i) In Def. 2.3, both the compactness and the attractiveness of the generalized global attractor \( \mathcal{A} \) are in the sense of strong topology of the phase space \( X \), while the minimality guarantees the uniqueness of \( \mathcal{A} \).
(ii) The global attractors in Def. 2.2 and Def. 2.3 are more general because they are for a family of multi-valued operators rather than only for generalized semigroup or for the multi-valued semiflows as before.
(iii) We use minimality instead of negative invariance (cf. [1, 2]) in Def. 2.3 for the latter may fail in applications. Then we eliminate in our criteria (see Th. 3.3 and Th. 3.5 below) the assumptions of upper semi-continuity or closed graph for the family of multi-valued operators (cf. [23, 24, 25, 26]).
(iv) The concept to define “generalized attractor” as the smallest compact attracting set, without invariance is not new. It can be found, for single-valued autonomous case in [13] and non-autonomous case in [17]. Then, for multi-valued case in [18, 35].

Lemma 2.4. If a set \( \mathcal{A} \) is a global attractor of the family of multi-valued operators \( \{T(t)\}_{t \geq 0} \), then \( \mathcal{A} \) is a generalized global attractor of \( \{T(t)\}_{t \geq 0} \).

Proof. It is enough to prove the minimality. Since \( \mathcal{A} = T(t)\mathcal{A} \) is a compact set in \( X \), we have, for any closed attracting set \( P \) of \( \{T(t)\}_{t \geq 0} \),

\[
\mathrm{dist}_X \{\mathcal{A}, P\} = \mathrm{dist}_X \{T(t)\mathcal{A}, P\} \to 0 \quad \text{as} \quad t \to \infty.
\]

Hence \( \mathcal{A} \subset [P]_X = P \).

Definition 2.5. (i) The family of operators \( \{T(t)\}_{t \geq 0} \) is said to be dissipative if it has a bounded absorbing set in \( X \), i.e., there exists a bounded set \( \mathcal{B}_R \subset X \) such that for every bounded set \( B \subset X, T(t)B \subset \mathcal{B}_R \) as \( t \geq t(B) \).
(ii) The family of operators \( \{T(t)\}_{t \geq 0} \) is said to be asymptotically compact on \( \mathcal{B}_R \) (or \( \mathcal{B}_R \) is asymptotically compact for short) if for any sequence \( \{\varphi_n\} \subset \mathcal{B}_R, t_n \to +\infty \), the sequence \( \{\varphi_n(t_n)\} \), with \( \varphi_n(t_n) \in T(t_n)\varphi_n \), is precompact in \( X \).
(iii) The set

\[
\omega_T(\mathcal{B}_R) = \{\psi|\psi = \lim_{n \to \infty} \varphi_n, \varphi_n \in T(t_n)\psi_n, \text{ with } \psi_n \in \mathcal{B}_R \text{ and } t_n \to +\infty\}
\]
is said to be the \( \omega \)-limit set of \( \mathcal{B}_R \).

Obviously, if the set \( \mathcal{B}_R \) is asymptotically compact, i.e., for any sequence \( \{\phi_n\} \), \( \phi_n \in T(t_n)\varphi_0^n \), with \( \varphi_0^n \in \mathcal{B}_R \) and \( t_n \to +\infty \), there exists a convergent subsequence \( \{\phi_{n_k}\} \), then \( \omega_T(\mathcal{B}_R) \neq 0 \).

3. Some criteria on global attractors.

Theorem 3.1. ([18, 35]) Assume that the generalized semigroup \( \{T(t)\}_{t \geq 0} \) (see Def. 2.1) is dissipative and asymptotically compact on the bounded absorbing set \( \mathcal{B}_R \). Then it has a generalized global attractor \( \mathcal{A} = \omega_T(\mathcal{B}_R) \).

Definition 3.2. Assume that the generalized semigroup \( \{T(t)\}_{t \geq 0} \) (see Def. 2.1) is dissipative and asymptotically compact on the bounded absorbing set \( \mathcal{B}_R \). Define the operator

\[
T_1(t) : 2^X \to 2^X, \quad T_1(t)B = \{T_1(t)\varphi_0|\varphi_0 \in B\}, \quad \forall B \in 2^X,
\]

(i) if \( \varphi_0 \in \omega_T(\mathcal{B}_R) \), i.e., \( \varphi_0 = \lim_{n \to \infty} \varphi_n(0), \varphi_n(0) \in T(t_n)\varphi_0^n \), with \( \varphi_0^n \in \mathcal{B}_R \) and \( t_n \to +\infty \), then

\[
T_1(t)\varphi_0 = \{\varphi^*(t) = \lim_{k \to \infty} \varphi_{n_k}(t)|\varphi_{n_k}(t) \in T(t+t_n)\varphi_0^n\}, \quad t \geq 0;
\]

(ii) if \( \varphi_0 \notin \omega_T(\mathcal{B}_R) \), then

\[
T_1(t)\varphi_0 = \{\varphi^*(t) = \lim_{k \to \infty} \varphi_{n_k}(t)|\varphi_{n_k}(t) \in T(t+t_n)\varphi_0^n\}, \quad t \geq 0.
\]
(ii) if \( \varphi_0 \in X \setminus \omega_T(B_R), T_1(t)\varphi_0 = T(t)\varphi_0. \)

**Remark 2.** (i) Formula (6) is well defined for \( \{T(t)\}_{t \geq 0} \) is asymptotically compact on \( B_R. \)

(ii) Obviously, \( T_1(0) \supset I, \) and the second property in (5) does not hold, hence the family of operators \( \{T_1(t)\}_{t \geq 0} \) does not constitute a generalized semigroup. So does \( \{T_2(t)\}_{t \geq 0} \) below (see Def. 3.4 and Def. 4.6). However, we can also prove the existence of their global attractors (see Th. 3.3 and Th. 3.5 below).

(iii) It is well known that the invariance is one of the essential properties of the global attractor. Under the assumptions of Theorem 3.1, \( A = \omega_T(B_R) \) is the generalized global attractor of the generalized semigroup \( T(t) \) (see Theorem 3.1) rather than the global one for we can not prove its invariance. In order to make up this defect, we define the multi-valued operator \( T_1(t) \) based on \( T(t) \) and show that \( A = \omega_T(B_R) \) is exactly the global attractor of the family of operators \( \{T_1(t)\}_{t \geq 0}, \) which means that under the action of \( T_1(t) \) the attractor \( A \) becomes invariant (see Theorem 3.3). The multi-valued operator \( T_1(t) \) will play an important role in applications in Section 4 to both the strongly damped wave equation (see the definition of \( T_1(t) \) there (after Lemma 4.5) and Theorem 4.8) and the heat equation (see Another example).

**Theorem 3.3.** Under the assumptions of Theorem 3.1, the set \( \omega_T(B_R), \) which is the generalized global attractor of the generalized semigroup \( \{T(t)\}_{t \geq 0}, \) is the global attractor of the family of multi-valued operators \( \{T_1(t)\}_{t \geq 0}. \)

**Proof.** (i) \( \omega_T(B_R) \) is an invariant set, i.e., \( T_1(t)\omega_T(B_R) = \omega_T(B_R), t \geq 0. \) Indeed, for any \( \varphi_0 \in \omega_T(B_R), \varphi_0 = \lim_{n \to \infty} \varphi_n(0), \varphi_n(0) \in T(t_n)\varphi_0^n, \) with \( \varphi_0^n \in B_R \) and \( t_n \to +\infty, \) and for any \( \varphi^*(t) \in T_1(t)\varphi_0 \subset T_1(t)\omega_T(B_R), \)

by formula (6),

\[ \varphi^*(t) = \lim_{k \to \infty} \varphi_{n_k}(t), \quad \varphi_{n_k}(t) \in T(t + t_n)\varphi_0^n, \]

which means \( \varphi^*(t) \in \omega_T(B_R). \) By the arbitrariness of \( \varphi_0 \) and \( \varphi^*(t) \in T_1(t)\varphi_0, \)

\( T_1(t)\omega_T(B_R) \supset \omega_T(B_R). \)

On the other hand, for any \( \varphi_0 \in \omega_T(B_R), \)

\[ \varphi_0 = \lim_{n \to \infty} \varphi_n, \quad \varphi_n \in T(t_n)\varphi_0^n = T(t + t'_n)\varphi_0^n, \]

with \( \varphi_0^n \in B_R \) and \( t'_n = t_n - t \to +\infty. \)

Let \( \phi_n(0) \in T(t'_n)\varphi_0^n. \) By the asymptotic compactness of \( T(t) \) on \( B_R, \) there exists a subsequence \( \{n_k\} \subset \{n\} \) such that

\[ \psi = \lim_{k \to \infty} \phi_{n_k}(0) \in \omega_T(B_R), \quad \varphi_n = \phi_n(t) \in T(t + t'_n)\varphi_0^n, \]

\[ \varphi_0 = \lim_{n \to \infty} \varphi_n = \lim_{n \to \infty} \phi_n(t) \in T_1(t)\psi \subset T_1(t)\omega_T(B_R), \]

which means \( \omega_T(B_R) \subset T_1(t)\omega_T(B_R). \) Therefore, \( T_1(t)\omega_T(B_R) = \omega_T(B_R), t \geq 0. \)

(ii) \( \omega_T(B_R) \) is an attracting set of \( \{T_1(t)\}_{t \geq 0} \) in \( X. \) For any bounded set \( B \subset X, \)

since \( T_1(t)\omega_T(B_R) = \omega_T(B_R) \) and \( \omega_T(B_R) \) attracts \( B, \) i.e.,

\[ \lim_{t \to +\infty} \text{dist}_X\{T(t)B, \omega_T(B_R)\} = 0, \]
we have
\[
\text{dist}_X \{ T_1(t)B, \omega_T(B_R) \} \\
\leq \text{dist}_X \{ T_1(t)(B \cap \omega_T(B_R)), \omega_T(B_R) \} + \text{dist}_X \{ T_1(t)(B \setminus \omega_T(B_R)), \omega_T(B_R) \} \\
= \text{dist}_X \{ T_1(t)(B \setminus \omega_T(B_R)), \omega_T(B_R) \} \\
= \text{dist}_X \{ T(t)(B \setminus \omega_T(B_R)), \omega_T(B_R) \} \to 0 \quad \text{as} \quad t \to +\infty.
\]

(iii) Theorem 3.1 shows that \( \omega_T(B_R) \) is a compact set in \( X \).

Therefore, \( \omega_T(B_R) \) is the global attractor of the family of multi-valued operators \( \{ T_1(t) \}_{t \geq 0} \). \hfill \Box

**Definition 3.4.** Define the operator \( T_2(t) : 2^X \to 2^X \),
\[
T_2(0) = I, \quad T_2(t)\varphi_0 \subset T(t)\varphi_0, \quad \forall \varphi_0 \subset X, \quad t > 0.
\]

**Theorem 3.5.** Under the assumptions of Theorem 3.1, the family of multi-valued operators \( \{ T_2(t) \}_{t \geq 0} \) has in \( X \) a generalized global attractor \( \mathfrak{A} \), and
\[
\mathfrak{A} = \left[ \bigcup_{m \in \mathbb{N}} \omega_{T_2}(B_m) \right] \subset \omega_{T}(B_R),
\]
where \( B_m = \{ x \in X | \text{dist}_X \{ x, 0 \} \leq m \} \) and
\[
\omega_{T_2}(B_m) = \{ \psi | \psi = \lim_{n \to \infty} \varphi_n, \quad \varphi_n \in T_2(t_n)\psi_n, \quad \text{with} \quad \psi_n \in B_m \quad \text{and} \quad t_n \to +\infty \}.
\]

**Proof.** (i) The family of operators \( \{ T_2(t) \}_{t \geq 0} \) is asymptotically compact in \( X \), i.e.,
for any bounded set \( B \subset X \), any sequence \( \{ \varphi_n \} \subset B \), \( t_n \to +\infty \), the sequence \( \{ \varphi_n(t_n) \} \), with \( \varphi_n(t_n) \in T_2(t_n)\varphi_n \), is precompact in \( X \).

Indeed, due to \( T(t)B \subset B_R \) for \( t \geq t_B \) and Def. 3.4, we have
\[
\varphi_n(t_n) \in T_2(t_n)\varphi_n \subset T(t_n-t_B)\varphi_n \subset T(t_n-t_B)T(t_B)\varphi_n \subset T(t_n-t_B)B_R,
\]
which means that \( \varphi_n(t_n) \in T(t_n-t_B)\varphi' \) for some \( \varphi' \in B_R \). Therefore, the sequence \( \{ \varphi_n(t_n) \} \) is precompact in \( X \) for \( \{ T(t) \}_{t \geq 0} \) is asymptotically compact on \( B_R \).

(ii) \( \mathfrak{A} \) is an attracting set in \( X \).

We claim that \( \omega_{T_2}(B_m) \) attracts \( B_m \) for each \( m \in \mathbb{N} \), i.e.,
\[
\lim_{t \to +\infty} \text{dist}_X \{ T_2(t)B_m, \omega_{T_2}(B_m) \} = 0.
\]
Or else, there exist a sequence \( \{ \varphi_n \} \), \( \varphi_n \in T_2(t_n)\psi_n \), with \( \psi_n \in B_m \) and \( t_n \to +\infty \), and a \( \epsilon_0 > 0 \) such that
\[
\inf_{\psi \in \omega_{T_2}(B_m)} \text{dist}_X \{ \varphi_n, \psi \} > \epsilon_0. \quad (7)
\]
By the asymptotic compactness of \( \{ T_2(t) \}_{t \geq 0} \) on \( B_m \) and the definition of \( \omega_{T_2}(B_m) \),
there exist a subsequence \( \{ \varphi_{n_k} \} \) and an element \( \psi \in \omega_{T_2}(B_m) \) such that
\[
\lim_{k \to \infty} \text{dist}_X \{ \varphi_{n_k}, \psi \} = 0,
\]
which violates (7). So the claim is valid.

For any bounded set \( B \subset X \), there exists a \( m \in \mathbb{N} \) such that \( B \subset B_m \), and
\[
\text{dist}_X \{ T_2(t)B, \mathfrak{A} \} \leq \text{dist}_X \{ T_2(t)B_m, \omega_{T_2}(B_m) \} \to 0 \quad \text{as} \quad t \to +\infty.
\]

(iii) \( \mathfrak{A} \) is contained in any closed attracting set.

Indeed, for any \( \varphi \in \bigcup_{m \in \mathbb{N}}\omega_{T_2}(B_m) \), there exists a \( m \) such that \( \varphi \in \omega_{T_2}(B_m) \), then
\[
\varphi = \lim_{n \to \infty} \varphi_n, \quad \varphi_n \in T_2(t_n)\psi_n, \quad \text{with} \quad \psi_n \in B_m \quad \text{and} \quad t_n \to +\infty.
\]
For any closed attracting set $P$, since
\[
\text{dist}_X \{ \varphi, P \} \leq \text{dist}_X \{ T_2(t_n)\psi_n, P \} \leq \text{dist}_X \{ T_2(t_n)B_m, P \} \to 0 \quad \text{as} \quad n \to \infty,
\]
we have
\[
\text{dist}_X \{ \varphi, P \} \leq \text{dist}_X \{ \varphi, \varphi_n \} + \text{dist}_X \{ \varphi_n, P \} \to 0 \quad \text{as} \quad n \to \infty,
\]
i.e., \( \text{dist}_X \{ \varphi, P \} = 0, \varphi \in [P]_X = P \). Therefore,
\[
\bigcup_{m \in \mathbb{N}} \omega_{T_2(B_m)} \subset P, \quad \mathcal{A} \subset [P]_X = P. \tag{8}
\]

(iv) \( \mathcal{A} \) is a compact set.

Indeed, the combination of Theorem 3.1 and Def. 3.4 shows that for any bounded set \( B \subset X \),
\[
\text{dist}_X \{ T_2(t)B, \omega_T(B_R) \} \leq \text{dist}_X \{ T(t)B, \omega_T(B_R) \} \to 0 \quad \text{as} \quad t \to +\infty,
\]
which means that \( \omega_T(B_R) \) is a compact attracting set of \( T_2(t) \). By (8), \( \mathcal{A} \subset \omega_T(B_R)(=A) \) is a compact set.

The combination of (ii)-(iv) gives the desired conclusion. \( \square \)

**Remark 3.** Obviously, \( \omega_T(B_R) \subset \left[ \bigcup_{m \in \mathbb{N}} \omega_T(B_m) \right]_X \). Hence, if \( T_2(t) = T(t), t \geq 0 \), then by Theorem 3.5,
\[
\omega_T(B_R) = \left[ \bigcup_{m \in \mathbb{N}} \omega_T(B_m) \right]_X. \tag{9}
\]

4. **Application to a strongly damped wave equation without uniqueness.**

We consider the strongly damped nonlinear wave equation:
\[
\begin{align*}
&u_{tt} - \Delta u - \Delta u_t + h(u_t) + g(u) = f(x) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\
&u_{|\partial\Omega} = 0, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{10}
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N(N \geq 3) \) with the smooth boundary \( \partial\Omega \), \( h(s) \) and \( g(u) \) are nonlinear functions with fully supercritical growth.

Eq. (9) rules the thermal evolution in a homogeneous isotropic rigid body, where \( h(u_t) \) is a nonlinearly temperature-dependent internal source term, and \( g(u) \) is a source term depending nonlinearly on displacement. For the derivation of the physical model one can see Appendix B in [20] in detail.

When \( \Omega \subset \mathbb{R}^3 \), the growth exponent \( q \) and \( p \) of the nonlinearities \( h(u_t) \) and \( g(u) \) are at most critical, that is, \( 1 \leq q, p \leq 5 \), a complete analysis on the existence of global and exponential attractors of optimal regularity for problem (9)-(10) has been done by Dell’Oro and Pata in [19, 20, 21].

We now pay attention to the case: for general bounded smooth domain \( \Omega \subset \mathbb{R}^N \), with \( N \geq 3 \), the growth exponents \( q \) and \( p \) are fully supercritical, that is, \( q = p > p^* \equiv \frac{N+2}{N-2} \) (in particular, \( p^* = 5 \) if \( N = 3 \)), which leads to Eq. (9) without uniqueness (more accurately, one cannot prove its uniqueness). We mention that mathematicians have obtained some uniqueness of the weak solutions in some supercritical cases: for example, in Eq. (9) either \( p^* < p \leq \frac{(p^*+1)q}{q+1} \) (\( q < p^* \)) and without strong damping \( -\Delta u_t \) (cf. [7, 8, 30]) or \( h(u_t) \equiv 0 \) and \( p^* < p < p^{**} = \frac{N+4}{(N-4)^2} \) (cf. [16]). However, the appearance of both nonlinearities \( h(u_t) \) and \( g(u) \) and the fully super-criticality destroy this uniqueness, and they make the techniques used to prove the uniqueness of weak solutions in the above literatures fail. In fact, it is challenging to show the true non-uniqueness of weak solutions.
solutions by finding two different solutions because one cannot obtain the analysis expression of any weak solutions of problem (9)-(10).

In this case, we investigate the longtime behavior of Eq. (9) by the following approach: we have established in [34] the existence of limit solutions (see Lemma 4.2 below), and discussed the asymptotic behavior of the subclass of limit solutions (see Lemma 4.5 below). This section is a continuation of research from [34]. We first define the multi-valued maps $T(t), t \geq 0$, and apply Theorem 3.1 to prove that the generalized semigroup $\{T(t)\}_{t \geq 0}$ has a generalized global attractor in natural energy space. Then we define multi-valued maps $T_1(t)$ and $T_2(t)$ according to Def. 3.2 and Def. 3.4, and apply Theorem 3.3 and Theorem 3.5 to show that the related family of multi-valued operators $\{T_1(t)\}_{t \geq 0}$ and $\{T_2(t)\}_{t \geq 0}$, which form neither the generalized semigroup nor the multi-valued semiflow, have a global attractor and a generalized one in natural energy space, respectively.

We begin with the following notations:

$$L^p = L^p(\Omega), \quad H^k = W^{k,2}, \quad V_1 = H^1 \cap H^1_0, \quad H = L^2,$$

$$V_1 = H^{-1}, \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \quad \| \cdot \| = \| \cdot \|_{L^2},$$

with $p \geq 1$. The notation $(\cdot, \cdot)$ for the $H$-inner product will also be used for the notation of duality pairing between dual spaces. Define the operator $A : V_1 \to V_1$,

$$(Au, v) = (\nabla u, \nabla v) \quad \text{for any} \quad u, v \in V_1.$$ 

$D(A) = \{u \in H| Au \in H \} = H^2 \cap H^1_0$. Obviously, $A$ is self-adjoint in $H$ and strictly positive on $V_1$, and we can define the power $A^p$ of $A$ ($s \in \mathbb{R}$), and the spaces $V_s = D(A^{\frac{s}{2}})$ are Hilbert spaces with scalar products and norms

$$(u, v)_s = (A^{\frac{s}{2}}u, A^{\frac{s}{2}}v), \quad \| u \|_{V_s} = \| A^{\frac{s}{2}}u \|,$$

respectively. Let $\lambda_1(> 0)$ be the first eigenvalue of $A, \lambda < \lambda_1$, the operator

$$B = A - \lambda J \quad \text{with} \quad D(B) = D(A).$$

Obviously, $B$ is a positive operator commuting with $A$, and

$$(1 - \lambda/\lambda_1)\| u \| \leq \| Bu \|_{V_{-2}} = \| A^{-1} Bu \| \leq (1 + \lambda/\lambda_1)\| u \|. $$

We define the phase space $X = (V_1 \cap L^{p+1}) \times H, X_1 = V_2 \times H$, which are equipped with usual graph norms, for instance,

$$\| (u, v) \|^2_X = \| u \|^2_{V_1} + \| u \|^2_{p+1} + \| v \|^2.$$ 

Obviously,

$$X_1 \hookrightarrow X \quad \text{and} \quad X_1 \text{ is dense in } X \text{ when } 1 \leq p \leq p^* = \frac{N+4}{(N-4)^+}.$$ 

**Assumption 1.** Let

$$h(s) = -\lambda s + \phi(s),$$

where $\phi \in C^1(\mathbb{R}), \phi(0) = 0$, and there exist a constant $K_0 > 0$ such that

$$K_0|s|^{q-1} \leq \phi'(s) \leq C(1 + |s|^{q-1}), \quad s \in \mathbb{R}, \quad \text{with} \quad q > p^* = \frac{N+2}{N-2}.$$ 

Rewriting Eq. (9) at an abstract level, we have

$$u_{tt} + Au + Bu_t + \phi(u) + g(u) = f, \quad (11)$$

$$u(0) = u_0, \quad u_t(0) = u_1. \quad (12)$$
Lemma 4.1. ([34]) Let Assumption 1 be valid, and
\((H_2)\) \(g \in C^2(\mathbb{R})\), and there exist constants \(K_1, K_2 > 0\) such that
\[ g'(s) \geq K_1 |s|^{p-1} - K_2, \quad |g''(s)| \leq C(1 + |s|^{p-2}), \quad s \in \mathbb{R}, \quad \text{with} \quad p = q > p^* .\]
\((H_3)\) \(f \in H, \ (u_0, u_1) \in X, \|u_0, u_1\|_X \leq R\).
Then for every \(\eta > 0\), problem (13)-(12) possesses a weak solution \(u^n\), with \((w^n, u^n) \in C_0(\mathbb{R}^+; X), u^n \in L^2(\mathbb{R}^+; V_3) \cap L^{q+1}(\mathbb{R}^+; L^{q+1})\). Furthermore, the solution possesses the following properties:
(i) (Uniform boundedness)
\[ \|u^n(t), w^n(t)\|_X^2 + \int_0^T \left( \|u^n_2(\tau)\|_V^2 + \eta \|u^n_1(\tau)\|_V^2 + \|u^n_1(\tau)\|_q^{q+1} \right) d\tau \leq C(R) + C_0, \quad t \geq 0, \quad (14)\]
where \(C(R)\) and \(C_0\) are positive constants independent of \(\eta\).
(ii) (Lipschitz stability) In particular, when \((u_0, u_1) \in X_1(= V_2 \times H), p^* < q = p \leq p^* = \frac{2q+4}{q+4}\), problem (13)-(12) admits a unique weak solution \(u^*\), with \((u^*, u^*_1) \in L^\infty(\mathbb{R}^+; X_1) \cap C_w(\mathbb{R}^+; X_1)\), and the solution is of the Lipschitz stability:
\[ \|u(t), w(t)\|_X^2 \leq C(\eta, R_1, T) \|u(0), w(0)\|_X^2, \quad t \in [0, T], \]
where \(w = u^n - v^n, u^*, v^n\) are two weak solutions of problem (13)-(12) corresponding to initial data \((u_0, u_1), (v_0, v_1) \in X_1, \) with \(\|u_0, u_1\|_X + \|v_0, v_1\|_X \leq R_1\).

Let the assumptions of Lemma 4.1 be valid, with \(p^* < q = p \leq p^*\), we define the solution operator
\[ T_\eta(t) : X_1 \rightarrow X_1, \quad T_\eta(t) \varphi_0 = \varphi_\eta(t) = (u^n(t), u^n_1(t)), \quad \forall \varphi_0 \in X_1, \quad t \geq 0, \]
with \(\eta : 0 < \eta \leq 1\), where \(u^n\) is a weak solution of problem (13)-(12), with \(\varphi_\eta(0) = \varphi_0\). Lemma 4.1 shows that \(\{T_\eta(t)\}_{t \geq 0}\) constitutes a continuous semigroup on \(X_1\) for each \(\eta \in (0, 1]\). Moreover, estimate (14) implies that for any bounded set \(B \subset X\), there exists a bounded set \(B_1 \subset X\) such that
\[ \bigcup_{\eta \in (0, 1]} \bigcup_{t \geq 0} T_\eta(t)B \subset B_1. \]
(15)

Lemma 4.2. ([34]) (Existence of limit solutions) Let the assumptions of Lemma 4.1 be valid, with \(p^* < q = p \leq p^*\), and let the sequence \(\{\varphi^n_0\} \subset X_1, \varphi^n_0 \rightarrow \varphi_0 = (u_0, u_1)\) in \(X\) and the sequence \(\eta_n \rightarrow 0^+\). Then there exists a subsequence \(\{\eta_k\}\) such that
\[ T_{\eta_k}(t) \varphi_0^{\eta_k} \rightarrow \varphi^*(t) = (u(t), u_1(t)), \quad k \rightarrow \infty, \quad (16)\]
where \(u\) is a weak solution of problem (11)-(12) (we call it limit solution). Moreover,
\[ \|u(t), u_1(t)\|_X^2 + \int_0^T \left( \|u_1(\tau)\|_V^2 + \|u_1(\tau)\|_q^{q+1} + (\varphi(u_1), u_1) \right) d\tau \leq C(R)e^{-\kappa t} + C_0, \quad t \geq 0, \quad (17)\]
where \(C_0 = C\|f\|_{V_2}\), \(\kappa\) denotes a small positive constant.
Remark 4. The estimate (17) is obtained by directly using the multiplier \( u_t + \epsilon u \) \((0 < \epsilon \ll 1)\) to Eq. (11) rather than by the lower semicontinuity of the weak limit, where \( u \) is any a weak solution of problem (11)-(12), so it holds for all the weak solutions and certainly holds for the limit solutions.

Let \( 2^X \) be the space of all subsets of \( X = (V_1 \cap L^{p+1}) \times H \), with \( p^* < p \leq p^{**} \) (the restriction \( p \leq p^{**} \) shall be removed at last in Remark 5.12 in [34]).

Definition 4.3. Define the operator \( T(t) : 2^X \rightarrow 2^X \), for any \( \varphi_0 \in X \),

\[
T(t)\varphi_0 = \{ \varphi^*(t) = (w) \lim_{k \to \infty} T_{\eta_{n_k}}(t)\varphi^n_{0_k} \mid \{ \varphi^n_{0_k} \} \subset X_1, \varphi^n_0 \rightharpoonup \varphi_0, \eta_n \to 0^+ \}, \tag{18}
\]

\[
T(t)B = \{ T(t)\varphi_0 \mid \varphi_0 \in B \}, \quad \forall B \in 2^X, \ t \geq 0,
\]

where the sign "\((w)\lim\)" denotes the weak limit in space \( X \), \( \{ n_k \} \subset \{ n \} \). We call each limit function \( \varphi^*(\cdot) \in T(\cdot)\varphi_0 \) a limit solution.

Remark 5. (i) Lemma 4.2 shows that for every \( \varphi_0 \in X \), problem (11)-(12) possesses at least one limit solution \( \varphi^*(t) \), with \( \varphi^*(0) = \varphi_0 \). Then the solution operator \( T(t) \) is well defined. But \( T(t)\varphi_0 \) may be multi-valued because there may be lots of subsequences of \( \{ \varphi^n_{0_k} \} \) such that the weak limit in (18) exists and the set of all these weak limits is denoted by \( T(t)\varphi_0 \).

(ii) Obviously, \( T(0) = I \), and the range of the multi-valued operator \( T(t)(t > 0) \) constitutes a class of limit solutions.

Lemma 4.4. The family of operators \( \{ T(t) \}_{t \geq 0} \), with \( T(t) \) defined in Def. 4.3, constitutes a generalized semigroup.

Proof. It is enough to show that for every \( \varphi_0 \in X \),

\[
T(t + s)\varphi_0 \subset T(t)T(s)\varphi_0, \quad \forall t, s \geq 0. \tag{19}
\]

For any \( \varphi_0 \in X, \varphi^*(t + s) \in T(t + s)\varphi_0 \), there exist sequence \( \{ \varphi^n_{0_k} \} \subset X_1, \varphi^n_0 \rightharpoonup \varphi_0 \) in \( X \) and subsequence \( \{ n_k \} \) such that

\[
\varphi^*(t + s) = (w) \lim_{k \to \infty} T_{\eta_{n_k}}(t + s)\varphi^n_{0_k} = (w) \lim_{k \to \infty} T_{\eta_{n_k}}(t)T_{\eta_{n_k}}(s)\varphi^n_{0_k}.
\]

Formula (15) implies that the sequence \( \{ T_{\eta_{n_k}}(s)\varphi^n_{0_k} \} \) is bounded in \( X \), then there exists an element \( \varphi_s \in T(s)\varphi_0 \) such that (subsequence if necessary)

\[
\varphi_s = (w) \lim_{k \to \infty} T_{\eta_{n_k}}(s)\varphi^n_{0_k}.
\]

By the definition of \( T(t)\varphi_s \),

\[
\varphi^*(t + s) \in T(t)\varphi_s \subset T(t)T(s)\varphi_0,
\]

which means \( T(t + s)\varphi_0 \subset T(t)T(s)\varphi_0 \). \( \square \)

Lemma 4.5. ([34]) Let the assumptions of Lemma 4.1 be valid, with \( p^* < q = p \leq p^{**} \). Then generalized semigroup \( \{ T(t) \}_{t \geq 0} \) has a bounded absorbing set \( B_R \), and it is asymptotically compact on \( B_R \) (or say: \( B_R \) is asymptotically compact for short).

Under the assumptions of Lemma 4.1, by Lemma 4.5, we can define the multi-valued operator \( T_1(t) \) according to Def. 3.2, where \( T(t)\varphi_0 \) is the set of limit solutions as shown in (18).

We define the concrete multi-valued operator \( T_2(t) \) as follows.
Definition 4.6. Let the set
\[ \mathcal{D} = \{ \psi | \psi = (w) \lim_{n \to \infty} T_{\eta_n}(t_n)\psi_n, \text{ with } \psi_n \in \mathcal{B}_R \cap X_1, \ t_n \to +\infty \text{ and } \eta_n \to 0^+ \}. \]

Define the operator
\[ T_2(t) : 2^X \to 2^X, \ T_2(t)B = \{ T_2(t)\varphi_0 | \varphi_0 \in B \}, \ \forall B \in 2^X, \]
(i) if \( \varphi_0 \in \mathcal{D}, \) i.e., \( \varphi_0 = (w) \lim_{n \to \infty} T_{\eta_n}(t_n)\varphi_0^n, \) with \( \varphi_0^n \in \mathcal{B}_R \cap X_1, \ t_n \to +\infty \) and \( \eta_n \to 0^+, \) then
\[ T_2(t)\varphi_0 = \{ \varphi^*(t)|\varphi^*(t) = (w) \lim_{k \to \infty} T_{\eta_n}(t + t_m)\varphi_0^k \}, \ t \geq 0; \]
(ii) if \( \varphi_0 \in X \setminus \mathcal{D}, \) \( \varphi_0 = (w) \lim_{n \to \infty} \varphi_0^n, \) with \( \varphi_0^n \in X_1, \) then
\[ T_2(t)\varphi_0 = T(t)\varphi_0 = \{ \varphi^*(t)|\varphi^*(t) = (w) \lim_{k \to \infty} T_{\eta_n}(t)\varphi_0^k \}, \ t \geq 0. \]

Obviously, \( T_2(0) = I, \) and we easily see from Def. 4.6 the relation between \( T_2(t) \) and \( T(t): \)

Lemma 4.7.
\[ T_2(t)\varphi \subset T(t)\varphi, \ \forall \varphi \in X, \ t > 0. \]

Remark 6. (i) Obviously, the range of the operator \( T_1(t) \) \( (t > 0) \) is a subclass of all weak solutions of problem (11)-(12) for both the limit solution and the limit of the sequence of the limit solutions are weak solutions of problem (11)-(12).

(ii) The combination of Remark 5 and Lemma 4.7 shows that the range of the operator \( T_2(t)(t > 0) \) is a subclass of limit solutions.

It follows from Lemma 4.5, Th. 3.1, Th. 3.3 and Th. 3.5 the following results:

Theorem 4.8. Let the assumptions of Lemma 4.1 be valid, with \( p^* < q = p \leq p^{**}. \) Then
(i) the generalized semigroup \( \{ T(t) \}_{t \geq 0} \) possesses in \( X \) a generalized global attractor \( A_1 = \omega_T(\mathcal{B}_R), \) which is the global attractor of the family of multi-valued operators \( \{ T_1(t) \}_{t \geq 0}; \)
(ii) the family of multi-valued operators \( \{ T_2(t) \}_{t \geq 0} \) possesses in \( X \) a generalized global attractor \( A_2, \) and
\[ A_2 = \left[ \bigcup_{m \in \mathbb{N}} \omega_{T_2}(\mathcal{B}_m) \right] \subset A_1. \]

Another example. Now we take another example coming from [35] to illustrate how \( \{ T_1(t) \}_{t \geq 0} \) could be related to \( \{ T(t) \}_{t \geq 0}. \) Let \( \Omega = (0, \pi), v^1(x) = \sin x, x \in \Omega, \)
\[ \alpha(u) = \frac{2}{\pi} (u, v^1), \quad H_+ = \{ u \in L^2|\alpha(u) > 0 \}, \quad H_- = \{ u \in L^2|\alpha(u) \leq 0 \}. \]

Define the operator
\[ T_+(t) : L^2 \to L^2, \ T_+(t)u_0 = u(t), \ t \geq 0, \]
where \( u \in L^2_{\text{loc}}(\mathbb{R}^+; V_1), \) with \( u_t \in L^2_{\text{loc}}(\mathbb{R}^+; V_{-1}), \) is a weak solution of the problem (+):
\[ u_t = u_{xx}, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad u|_{\partial \Omega} = 0, \quad u(x, 0) = u_0(x). \]

Define the operator
\[ T_-(t) : L^2 \to L^2, \ T_-(t)u_0 = u(t), \ t \geq 0, \]
where \( u \in L^2_{loc}(\mathbb{R}^+; V_1) \), with \( u_t \in L^2_{loc}(\mathbb{R}^+; V_{-1}) \), is a weak solution of the problem (−):

\[
\begin{align*}
  u_t &= u_{xx} + v_{txx}, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0(x).
\end{align*}
\]

Then \( T_+(t)H_+ \subset H_+, \quad T_-(t)H_- \subset H_-, \quad t \geq 0 \) (cf. [35]). Define the operator \( T(t) : L^2 \rightarrow L^2 \), for every \( u_0 \in L^2 \),

\[
T(t)u_0 = u(t) = \begin{cases} 
T_+(t)u_0 & \text{for } u_0 \in H_+, \\
T_-(t)u_0 & \text{for } u_0 \in H_-, 
\end{cases}
\]

where \( u \) is the weak solution of the problem

\[
u_t = u_{xx} + v^1_{xx} \chi_{H_-}(u), \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0(x),
\]

where \( \chi_{H_-}(u) \) is the characteristic function of the set \( H_- \), i.e.,

\[
\chi_{H_-}(u) = \begin{cases} 
1 & \text{for } u \in H_-, \\
0 & \text{otherwise},
\end{cases}
\]

Then the family of operators \( \{T(t)\}_{t \geq 0} \) constitutes a semigroup on \( L^2 \), which is single-valued but not continuous. Moreover, it is dissipative and asymptotically compact. For any given \( \epsilon > 0 \), the set

\[
B_\epsilon = B_{L^2}(0, \epsilon) \cup B_{L^2}(-v^1, \epsilon)
\]

is an absorbing set of \( T(t) \) in \( L^2 \). And \( T(t) \) possesses a generalized global attractor

\[
\mathcal{A} = \omega_T(B_\epsilon) = \{0, -v^1\},
\]

which is neither positively nor negatively invariant for \( T(t) \) can not be \( t_+ \)-closed (cf. [35]). In this case, we define the operator \( T_1(t) \) according to Def. 3.2, i.e., for every \( \varphi_0 \in L^2 \),

\[
T_1(t)\varphi_0 = \begin{cases} 
T(t)\varphi_0 & \text{for } \varphi_0 \in L^2 \setminus \mathcal{A}, \\
\{\varphi^*_n(t) = \lim_{k \to \infty} T(t + t_{nk})\varphi^{nk}_0 \} & \text{for } \varphi_0 \in \mathcal{A},
\end{cases}
\]

where

\[
\varphi_0 \in \mathcal{A} \iff \varphi_0 = \lim_{k \to \infty} T(t_{nk})\varphi^{nk}_0, \quad \text{with } \{\varphi^{nk}_0 \} \subset \{\varphi^n_0 \} \subset B_\epsilon.
\]

More accurately,

\[
T_1(t)0 = 0, \quad T_1(t)(-v^1) = -v^1, \quad t \geq 0,
\]

i.e., \( \mathcal{A} \) is the set of the fixed points of \( T_1(t) \). Indeed, we know from [35] that all trajectories of problem (−) converge to zero and

\[
\|T_+(t)u\| \leq \|u\|e^{-t}, \quad \forall u \in L^2,
\]

while all trajectories of problem (−) converge to \( -v^1 \) and

\[
\|T_-(t)u + v^1\| \leq \|u + v^1\|e^{-t}, \quad \forall u \in L^2.
\]

When \( \varphi_0 = 0 = \lim_{k \to \infty} T(t_{nk})\varphi^{nk}_0 \), there must exist a \( K > 0 \) such that \( \varphi^{nk}_0 \in H_+ \) as \( k \to K \) (Or else, one can extract a subsequence \( \{\varphi^{nk}_0\} \subset H_-; T(t)\varphi^{nk}_0 = T_-(t)\varphi^{nk}_0 \in H_- \) and

\[
\|T(t_{ni})\varphi^{ni}_0 + v^1\| \leq \|\varphi^{ni}_0 + v^1\|e^{-t_{ni}} \to 0 \ \text{as} \ l \to \infty,
\]

which violates \( \lim_{n \to \infty} T(t_{nk})\varphi^{nk}_0 = 0 \). Hence,

\[
\lim_{k \to \infty} T(t + t_{nk})\varphi^{nk}_0 = \lim_{k \to \infty} T_+(t + t_{nk})\varphi^{nk}_0 = 0,
\]

where \( u \in L^2_{loc}(\mathbb{R}^+; V_1) \), with \( u_t \in L^2_{loc}(\mathbb{R}^+; V_{-1}) \), is a weak solution of the problem (−):
i.e., $T_1(t)0 = 0, t \geq 0$. Similarly, $T_1(t)(-v^1) = -v^1$. Then by Theorem 3.3, the set $A$ is the global attractor of the family of operators $\{T_1(t)\}_{t \geq 0}$, that is, $A$ is a compact attracting set of $T_1(t)$ in $L^2$ and $T_1(t)A = A, t \geq 0$.

5. **Geometrical structure of the global attractor.** The study on the structure of global attractors is an important problem from the point of view of applications. In the classical single-valued semigroup case, one can give some geometrical descriptions of global attractors for the dynamical systems possessing Lyapunov function (cf. [15]). However, not much is known about the structure of global attractors for the dynamical systems possessing Lyapunov function (cf. [15]). In this section, we further investigate the structure of global attractors and give the following result on the geometrical structure of the global attractor $A_1$.

**Theorem 5.1.** Let the assumptions of Lemma 4.1 be valid, with $p^* < q = p \leq p^{**}$. Then

(i) the global attractor

$$A_1 = M_+(N, A_1),$$

where $N$ is the set of all the stationary points of Eq. (11) and $M_+(N, A_1)$ is the unstable set originating from $N$ in $A_1$, i.e.,

$$N = \{(u, 0) \in X | Au + g(u) = f\},$$

$$M_+(N, A_1) = \{y \in X | \text{there exists full trajectory } \gamma = \{y(t) | t \in \mathbb{R}\} \subset A_1, \text{with } y(0) = y \text{ and } \lim_{t \to -\infty} \text{dist}_X \{y(t), N\} = 0\};$$

(ii) for every full trajectory $\gamma = \{\xi(t) = (u(t), u_t(t)) | t \in \mathbb{R}\}$ lying in $A_1$,

$$\lim_{t \to -\infty} \text{dist}_X (\xi(t), N) = 0, \quad \lim_{t \to +\infty} \text{dist}_X (\xi(t), N) = 0. \quad (20)$$

**Proof:** (i) Obviously, $M_+(N, A_1) \subset A_1$. We show $A_1 \subset M_+(N, A_1)$ as follows.

Due to $A_1 = T_1(0)A_1, t \geq 0$, for any $y_0 \in A_1$ and any $\tau > 0$ there exists a $y_\tau \in A_1$ such that $y_0 \in T_1(\tau) y_\tau$, which means that problem (11)-(12) admits a (weak) solution $u^\tau(t), t \geq 0$ satisfying $(u^\tau(0), u^\tau_t(0)) = y_\tau$ and $(u^\tau(\tau), u^\tau_t(\tau)) = y_0$. Let $v^\tau(t) = u^\tau(t+\tau), \ t \geq -\tau$. Obviously, $v^\tau(t)$ solves Eq. (11), with $(v^\tau(0), v^\tau_t(0)) = y_0$. Let $\tau = \tau_j \to +\infty$ ($j \to \infty$). By (17) and Remark 4,

$$\sup_{t \in [-T, T]} \|v^\tau, v^\tau_t(t)\|_X^2 + \int_{-T}^{+T} \left( \|v^\tau_t(s)\|_{V_1}^2 + \|v^\tau_t(s)\|_q^{q+1} \right) ds \leq C(A_1) \quad (21)$$

for $j$ sufficiently large. By using the standard diagonal process and limiting process, we can extract a subsequence $\{\tau_k\}$ such that $v^{\tau_k}(t)$ converges on each interval $[-T, T]$ to a weak solution $u(t)$ of Eq. (11), with $(u, u_t)(0) = y_0, (u, u_t)(t) \in A_1$ and

$$\sup_{t \in \mathbb{R}} \|u(t), u_t(t)\|_X^2 + \int_{-\infty}^{+\infty} \left( \|u_t(s)\|_{V_1}^2 + \|u_t(s)\|_q^{q+1} \right) ds \leq C(A_1). \quad (22)$$

By the compactness of $A_1$, for any sequence $\{\tau_j\} : \tau_j \to -\infty$, (subsequence if necessary)

$$(u(\tau_j), u_t(\tau_j)) \to (u_0, p_0) \quad \text{in } X = (V_1 \cap L^{p+1}) \times H. \quad (23)$$
By (22), \( p_0 = 0 \). Now we prove \( Au_0 + g(u_0) = f \), i.e., \((u_0, 0) \in \mathcal{N}\). Let
\[
\varphi(t) = \begin{cases} \frac{1}{2} \exp\left\{ \frac{1}{|t|^{p-1}} \right\}, & |t| < 1, \\
0, & |t| \geq 1, \end{cases}
\]
where \( \beta = \int_{-1}^{1} \exp\left\{ \frac{1}{|t|^{p-1}} \right\} dt \). Obviously,
\[
\varphi \in C^\infty_0(\mathbb{R}), \quad \text{supp} \varphi(t) = [-1, 1], \quad \varphi(t) \geq 0, \quad \int_{\mathbb{R}} \varphi(t) dt = \int_{-1}^{1} \varphi(t) dt = 1.
\]
Multiplying Eq. (11) by \( \varphi(t) \), with \( s \geq 1 \) to be determined, and integrate the resulting expression over \((-1, 1)\), we have
\[
- \int_{-1}^{1} \varphi'(t) u_t(t + t_j) dt + \int_{-1}^{1} \varphi_s(t) Bu_t(t + t_j) dt + \int_{-1}^{1} \varphi_s(t) (Au(t + t_j) - Au(t_j)) dt + \int_{-1}^{1} \varphi_s(t) (g(u(t + t_j)) - g(u(t))) dt + Au(t) + g(u(t)) - f = 0 \tag{24}
\]
We denote the first five terms in formula (24) by \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \), and \( \Gamma_5 \), respectively. A simple calculation shows that
\[
\| \Gamma_1 \| \leq \int_{-1}^{1} s^2 |\varphi'(st)||u_t(t + t_j)|| dt \leq \left( \int_{-1}^{1} s^4 |\varphi'(st)|^2 dt \right)^{\frac{1}{2}} \left( \int_{-1}^{1} \|u_t(t + t_j)\|^2 dt \right)^{\frac{1}{2}} \leq \delta_1(t_j)s^{\frac{1}{2}},
\]
where \( \delta_1(t_j) = C(f^1 \|u_t(t + t_j)\|^2 dt)^{\frac{1}{2}} \) and \( \lim_{j \to \infty} \delta_1(t_j) = 0 \);
\[
\| \Gamma_2 \|_{V^{-2}} \leq \int_{-1}^{1} |s\varphi(st)||Bu_t(t + t_j)||_{V^{-2}} dt \leq \left( \int_{-1}^{1} s^2 |\varphi(st)|^2 dt \right)^{\frac{1}{2}} \left( \int_{-1}^{1} \|Bu_t(t + t_j)\|^2_{V^{-2}} dt \right)^{\frac{1}{2}} \leq \delta_2(t_j)s^{\frac{1}{2}},
\]
where \( \delta_2(t_j) = C(f^1 \|Bu_t(t + t_j)\|^2_{V^{-2}} dt)^{\frac{1}{2}} \leq C\delta_1(t_j) \) and \( \lim_{j \to \infty} \delta_2(t_j) = 0 \); 
\[
\| \Gamma_3 \|_{1+\frac{1}{q}} \leq \int_{-1}^{1} |s\varphi(st)||\phi(u_t(t + t_j)) - \phi(0)||_{1+\frac{1}{q}} dt \leq \left( \int_{-1}^{1} s^{q+1} |\varphi(st)|^{q+1} dt \right)^{\frac{1}{q+1}} \left( \int_{-1}^{1} \|\phi'(\lambda u_t(t + t_j))d\lambda u_t(t + t_j)||_{1+\frac{1}{q}} dt \right)^{\frac{1}{q+1}} \leq C\delta_3(t_j)s^{\frac{1}{q+1}},
\]
where \( \delta_3(t_j) = C(f^1 \|u_t(t + t_j)||_{q+1}^{q+1} dt + \int_{-1}^{1} \|u_t(t + t_j)||_{1+\frac{1}{q}}^{1+\frac{1}{q}} dt)^{\frac{1}{q+1}} \to 0 \);
Taking $\delta_j$ we have $j \to \infty$ as
\[
\sup_{|t| \leq \frac{1}{2}} \|u(t + t_j) - u(t_j)\|
\]
Therefore, taking account of the Sobolev embedding:
\[
\parallel \Gamma_4 \parallel_{V_\infty} \leq \int_{-1}^{1} |s\varphi(st)||Au(t + t_j) - Au(t_j)|_{V_\infty} dt
\]
\[
\leq C \sup_{|t| \leq \frac{1}{2}} \|u(t + t_j) - u(t_j)\|
\]
\[
\leq C \sup_{|t| \leq \frac{1}{2}} \int_{t_j}^{t+t_j} \|u_\tau(\tau)\| d\tau
\]
\[
\leq C \sup_{|t| \leq \frac{1}{2}} \left( \int_{t_j}^{t+t_j} \|u_\tau(\tau)\|^2 d\tau \right)^{1/2} t^{1/2}
\]
\[
\leq Cs^{-\frac{1}{2}};
\]
\[
\parallel \Gamma_5 \parallel_{1+1/p}
\]
\[
\leq \int_{-1}^{1} |s\varphi(st)||g(u(t + t_j)) - g(u(t_j))|_{1+1/p} dt
\]
\[
\leq \int_{-1}^{1} |s\varphi(st)|(1 + \|u(t + t_j)\|_{p+1}^{-1} + \|u(t_j)\|_{p+1}^{-1}) \|u(t + t_j) - u(t_j)\|_{p+1} dt
\]
\[
\leq C \sup_{|t| \leq \frac{1}{2}} \|u(t + t_j) - u(t_j)\|_{p+1}
\]
\[
\leq C \sup_{|t| \leq \frac{1}{2}} \left( \int_{t_j}^{t+t_j} \|u_\tau(\tau)\|_{p+1}^2 d\tau \right)^{1/(p+1)} t^{\frac{2}{p+1}}
\]
\[
\leq Cs^{-\frac{1}{p+1}}.
\]
Taking $\delta(t_j) = \delta_1(t_j) + \delta_2(t_j) + \delta_3(t_j)$, $s = s_j = (\delta(t_j))^{-\frac{1}{2}}$, then $s_j \to +\infty$ and
\[
\delta_1(t_j)s_j^2 + \delta_2(t_j)s_j^2 + \delta_3(t_j)s_j = \delta(t_j)s_j^2 + \delta(t_j)s_j^2 + \delta(t_j)s_j^2 \to 0
\]
as $j \to \infty$. Therefore, taking account of the Sobolev embedding:
\[
v_2 \hookrightarrow L^{p+1} = L^{q+1}, L^{1+1/p} = L^{1+1/q} \hookrightarrow V_\infty
\]
we have
\[
\|Au(t_j) + g(u(t_j)) - f\|_{V_\infty}
\]
\[
\leq C \left( \parallel \Gamma_1 \parallel + \parallel \Gamma_2 \parallel_{V_\infty} + \parallel \Gamma_3 \parallel_{1+1/q} + \parallel \Gamma_4 \parallel_{V_\infty} + \parallel \Gamma_5 \parallel_{1+1/p} \right)
\]
\[
\to 0
\]
as $j \to \infty$. It follows from (23) that
\[
\|Au(t_j) - Au_0\|_{V_\infty} = \|u(t_j) - u_0\| \to 0,
\]
and
\[
\|g(u(t_j)) - g(u_0)\|_{V_\infty} \leq \|g(u(t_j)) - g(u_0)\|_{1+1/p}
\]
\[
\leq C(1 + \|u(t_j)\|_{p+1}^{-1} + \|u_0\|_{p+1}^{-1}) \|u(t_j) - u_0\|_{p+1}
\]
\[
\to 0.
\]
Therefore,
\[
Au_0 + g(u_0) = f.
\]
We claim that
\[ \lim_{t \to -\infty} \text{dist}_X \{(u(t), u_t(t)) \}, \mathcal{N} \} = 0. \tag{25} \]
Or else, there exist a \( \epsilon_0 > 0 \) and a sequence \( t_j \to -\infty \) such that
\[ \text{dist}_X \{(u(t_j), u_t(t_j)) \}, \mathcal{N} \} > \epsilon_0, \ \forall j = 1, 2, \cdots. \tag{26} \]
On the other hand, (subsequence if necessary)
\[ (u(t_j), u_t(t_j)) \to (u_0, 0) \in \mathcal{N}, \]
which violates (26). Therefore, \( y_0 \in M_+(\mathcal{N}, A_1) \). By the arbitrariness of \( y_0 \), \( A_1 \subset M_+(\mathcal{N}, A_1) \).

(ii) For every full trajectory \( \gamma = \{\xi(t) = (u(t), u_t(t))| t \in \mathbb{R} \} \) lying in \( A_1 \), \( (u(t), u_t(t)) \) satisfies estimates (17) and (22). Repeating the same proof as above we obtain the first formula in (20). Similarly, multiplying Eq. (11) by \( \varphi_s(t) \), with \( s \geq 1 \), and integrating the resulting expression over \( t \in (-1, 1) \). Let \( t_j \to +\infty \) in (24), we also have \( \delta_i(t_j) \to 0 \) \((i = 1, 2, 3)\) by (22). Then \( (u(t_j), u_t(t_j)) \) tends to \( (u_0, 0) \in \mathcal{N} \) as \( t_j \to +\infty \). Therefore,
\[ \lim_{t \to +\infty} \text{dist}_X (\xi(t), \mathcal{N}) = 0. \]

\[ \square \]

Corollary 1. Under the assumptions of Theorem 5.1,
\[ A_1 = M_+(\mathcal{N}) \equiv \{y \in X| \text{there exists full trajectory } \gamma = \{y(t)| t \in \mathbb{R} \}, \]
which is bounded in \( X \), with \( y(0) = y \) and \( \lim_{t \to -\infty} \text{dist}_X \{y(t), \mathcal{N} \} = 0 \};

i.e., \( A_1 = M_+(\mathcal{N}) \) is the unstable set originating from \( \mathcal{N} \).

Proof. Obviously, by Theorem 5.1, \( A_1 = M_+(\mathcal{N}, A_1) \subset M_+(\mathcal{N}) \).

On the other hand, for any \( y = y(0) \in M_+(\mathcal{N}) \), let
\[ B_\tau = \{y(s)| s \leq -\tau \}, \ \forall \tau \in \mathbb{R}. \]
Then the set \( B_\tau \) is bounded in \( X \), \( y(0) \in \{y(t + s)| t + s \leq t - \tau \} \subset T_1(t)B_\tau \) for \( t \geq \tau \), and
\[ \text{dist}_X \{y, A_1\} \leq \text{dist}_X \{T_1(t)B_\tau, A_1\} \to 0 \ as \ t \to +\infty, \]
which means that \( y \in A_1 \). Therefore, \( A_1 = M_+(\mathcal{N}). \)

\[ \square \]

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REFERENCES

[1] A. V. Babin and M. I. Vishik, Maximal attractor of the semigroups corresponding to evolution differential equations, (Russian) Mat. Sb. (N.S.) 126 (1985), 397–419, 432.

[2] A. V. Babin, Attractor of the generalized semi-group generated by an elliptic equation in a cylindrical domain, Russian Acad. Sci. Izv. Math., 44 (1995), 207–223.

[3] F. Balibrea, T. Caraballo, P. E. Kloeden and J. Valero, Recent developments in dynamical systems: Three perspectives, Int. J. Bifurcat. Chaos, 20 (2010), 2591–2636.

[4] J. M. Ball, On the asymptotic behavior of generalized processes with applications to nonlinear evolution equations, J. Differential Equations, 27 (1978), 224–265.

[5] J. M. Ball, Continuity properties and attractors of generalized semiflows and the Navier-Stokes equations, Nonlinear Science, 7 (1997), 475–502.
[6] J. M. Ball, Global attractors for damped semilinear wave equations, *Discrete Cont. Dyn. Syst.*, 10 (2004), 31–52.
[7] L. Bociu and I. Lasiecka, Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping, *Discrete Cont. Dyn. Syst.*, 22 (2008), 835–860.
[8] L. Bociu and I. Lasiecka, Local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping, *J. Differential Equations*, 249 (2010), 654–683.
[9] T. Caraballo, P. Marín-Rubio and J. C. Robinson, A comparison between two theories for multi-valued semiflows and their asymptotic behaviour, *Set-Valued Anal.*, 11 (2003), 297–322.
[10] A. N. Carvalho, J. W. Cholewa and T. Dlotko, Damped wave equations with fast growing dissipative nonlinearities, *Discrete Contin. Dyn. Syst.*, 24 (2009), 1147–1165.
[11] V. V. Chepyzhov and M. I. Vishik, Trajectory attractors for reaction-diffusion systems, *Topological Methods in Nonlinear Analysis*, 7 (1996), 49–76.
[12] V. V. Chepyzhov and M. I. Vishik, Evolution equations and their trajectory attractors, *J. Math. Pures Appl.*, 76 (1997), 913–964.
[13] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, Providence, RI, 2002.
[14] V. V. Chepyzhov, M. Conti and V. Pata, A minimal approach to the theory of global attractors, *Discrete Cont. Dyn. Syst.*, 32 (2012), 2079–2088.
[15] I. Chueshov, Long-time behavior of second order evolution equations with nonlinear damping, *Memoirs of AMS*, Amer. Math. Soc. Providence, RI, 195 (2008), viii+183 pp.
[16] I. Chueshov, Long-time dynamics of Kirchhoff wave models with strong nonlinear damping, *J. Differential Equations*, 252 (2012), 1229–1262.
[17] H. Cui, Y. Li and J. Yin, Existence and upper semicontinuity of bi-spatial pullback attractors for smoothing cocycles, *Nonlinear Analysis*, 128 (2015), 303–324.
[18] S. Dashkovskiy, P. Feketa, O. Kapustyan and I. Romaniuk, Invariance and stability of global attractors for multi-valued impulsive dynamical systems, *J. Math. Anal. Appl.*, 458 (2018), 193–218.
[19] F. Dell’Oro, Global attractors for strongly damped wave equations with subcritical-critical nonlinearities, *Communications on Pure and Applied Analysis*, 12 (2013), 1015–1027.
[20] F. Dell’Oro and V. Pata, Long-term analysis of strongly damped nonlinear wave equations, *Nonlinearity*, 24 (2011), 3413–3435.
[21] F. Dell’Oro and V. Pata, Strongly damped wave equations with critical nonlinearities, *Nonlinear Analysis*, 75 (2012), 5723–5735.
[22] V. Kalantarov, A. Savostianov and S. Zelik, Attractors for damped quintic wave equations in bounded domains, *Ann. Henri Poincaré*, 17 (2016), 2555–2584.
[23] P. Kalita and G. Lukaszewicz, Global attractors for multi-valued semiflows with weak continuity properties, *Nonlinear Analysis*, 101 (2014), 124–143.
[24] T. Caraballo and P. Marín-Rubio, On global attractors of multi-valued semiflows generated by the 3D Benard system, *Set-Valued Var. Anal.*, 20 (2012), 445–465.
[25] V. S. Melnik, Multi-valued dynamics of nonlinear infinite dimensional systems, *Preprint of NAS of Ukraine, Institute of Cybernetics, Kyiv*, 94 (1994).
[26] V. S. Melnik and J. Valero, On attractors of multi-valued semi-flows and differential inclusions, *Set-Valued Analysis*, 6 (1998), 83–111.
[27] A. Savostianov, Strichartz estimates and smooth attractors for a sub-quintic wave equation with fractional damping in bounded domains, *Adv. Differential Equations*, 20 (2015), 495–530.
[28] A. Savostianov and S. Zelik, Recent progress in attractors for quantic wave equations, *Mathematica Bohemica*, 139 (2014), 657–665.
[29] A. Savostianov, *Strichartz Estimates and Smooth Attractors of Dissipative Hyperbolic Equations*, Doctoral dissertation, University of Surrey, 2015.
[30] E. Vitillaro, On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources, *J. Differential Equations*, 265 (2018), 4873–4941.
[31] Y. J. Wang and L. Yang, Global exponential attraction for multi-valued semidynamical systems with application to delay differential equations without uniqueness, *Discrete Cont. Dyn. Syst. B*, 24 (2019), 1961–1987.
[32] Z. J. Yang, N. Feng and T. F. Ma, Global attractor for the generalized double dispersion equation, *Nonlinear Analysis*, **115** (2015), 103–116.

[33] Z. J. Yang, Z. M. Liu and N. Feng, Longtime behavior of the semilinear wave equation with gentle dissipation, *Discrete Cont. Dyn. Sys. A*, **36** (2016), 6557–6580.

[34] Z. J. Yang and Z. M. Liu, Global attractor for a strongly damped wave equation with fully supercritical nonlinearities, *Discrete Cont. Dyn. Sys. A*, **37** (2017), 2181–2205.

[35] M. C. Zelati and P. Kalita, Minimality properties of set-valued processes and their pullback attractors, *SIAM Journal on Mathematical Analysis*, **47** (2015), 1530–1561.

[36] S. Zelik, Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities, *Discrete Cont. Dyn. Sys.*, **11** (2004), 351–392.

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