Supertrace divergence terms for the Witten Laplacian

P. Gilkey, K. Kirsten, and D. Vassilevich

Abstract. We use invariance theory to compute the divergence term $a_{m+2,m}^{d+\delta}$ in the super trace for the twisted de Rham complex for a closed Riemannian manifold.

1. Introduction

Let $(M, g)$ be a compact $m$ dimensional Riemannian manifold without boundary. The fundamental solution of the heat equation $e^{-tD}$ for an operator of Laplace type $D$ on $M$ is an infinitely smoothing operator. Let $f \in C^\infty(M)$ be an auxiliary smooth ‘smearing’ function. Work of Seeley [15] shows the smeared heat trace has a complete asymptotic expansion as $t \downarrow 0$ of the form:

$$
\text{Tr}_{L^2}(fe^{-tD}) \sim \sum_{n \geq 0} a_{n,m}(f, D)t^{(n-m)/2}.
$$

The heat trace invariants $a_{n,m}$ vanish if $n$ is odd; if $n$ is even, there are local invariants $a_{n,m}(x, D)$ so that

$$
a_{n,m}(f, D) = \int_M f(x)a_{n,m}(x, D)\,d\text{vol}_g(x).
$$

The function $f$ localizes the problem and permits us to recover divergence terms which would otherwise not be detected.

Let $\phi$ be an auxiliary smooth function called the dilaton. We twist the exterior derivative $d$ and the coderivative $\delta$ to define

$$
d_\phi := e^{-\phi}d e^{\phi} \quad \text{and} \quad \delta_{\phi,g} := e^{\phi}\delta g e^{-\phi}.
$$

We denote the associated Laplacian by $\Delta_{\phi,g}$ on $C^\infty(\Lambda^p(M))$. It appears in supersymmetric quantum mechanics [3] and in the study of Morse theory [19]. It also is used to study quantum $p$ form fields interacting with a background dilaton [11, 17].

Let $\chi(M) := \sum_p (-1)^p \dim H^p(M; \mathbb{R})$ be the Euler-Poincaré characteristic of $M$. Arguments of McKean and Singer [12] extend to the twisted setting to show

$$
\sum_p (-1)^p\text{Tr}_{L^2}(e^{-t\Delta_{\phi,g}}) = \chi(M).
$$

We define the local supertrace asymptotics by setting:

$$
a_{n,m}^{d+\delta}(\phi, g)(x) := \sum_p (-1)^p a_{n,m}(x, \Delta_{\phi,g}^p).
$$

We expand the left hand side of equation (1.a) and then equate powers of $t$ to see:

$$
\int_M a_{n,m}^{d+\delta}(\phi, g)(x)\,d\text{vol}_g(x) = \begin{cases} 
\chi(M) & \text{if } n = m, \\
0 & \text{if } n \neq m.
\end{cases}
$$

Let $R_{ijkl}$ be the components of the Riemann curvature tensor relative to a local orthonormal frame for the tangent bundle with the sign convention that $R_{1221} = +1$ on the unit sphere $S^2 \subset \mathbb{R}^3$. We adopt the Einstein convention and sum over repeated indices. If $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_m)$ are $m$ tuples of indices, let

$$
\varepsilon_I^J := g(e_{i_1} \wedge \ldots \wedge e_{i_m}, e_{j_1} \wedge \ldots \wedge e_{j_m})
$$

2000 Mathematics Subject Classification: 58J50.

Key words: Heat trace asymptotics, twisted de Rham complex, invariance theory.
be the totally anti-symmetric tensor and let

$$\mathcal{R}^t_{J,s} := R_{i_{s+1}i_{s+2}...i_1}...R_{i_{s-1}i_s}$$

we set $\mathcal{R}^t_{J,s} = 1$ for $t < s$.

**Theorem 1.1.**  
1. If $n < m$ or if $n$ is odd, then $a^{d+\delta}_{m,n}(\phi, g)(x) = 0$.
2. If $m = 2\bar{m}$, then $a^{d+\delta}_{m,n}(\phi, g)(x) = \frac{1}{8^n m} \varepsilon^{j} \mathcal{R}^{m}_{J,1}$.
3. If $m = 2\bar{m} + 1$, then $a^{d+\delta}_{m+1,n}(\phi, g)(x) = \frac{1}{\sqrt{8^n m}} \varepsilon^{j} \phi_{i_{i+1}j} \mathcal{R}^{m}_{J,2}$.

Assertions (1) and (2) were proved in the untwisted case $(\phi = 0)$ by Atiyah, Bott, and Patodi [4], by Gilkey [5], and by Patodi [13]. This provided a heat equation proof of the classical Chern-Gauss-Bonnet [5] theorem. Assertions (1) and (2) in the twisted setting were established in [1] and the divergence term $a^{d+\delta}_{m+1,n}$ was identified in [10]. Our previous paper [10] dealt with the odd dimensional case for manifolds with boundary. The present paper computes $a^{d+\delta}_{m+2,n}$ for closed even dimensional manifolds. This requires significantly different techniques. We note that some information concerning $a^{d+\delta}_{m+2,n}$ was derived earlier in [8] using an entirely different approach.

The main new result of this paper is the following:

**Theorem 1.2.** Let $M$ be a closed Riemannian manifold of dimension $m = 2\bar{m}$.

$$a^{d+\delta}_{m+2,m} = \frac{1}{8^n m} \varepsilon^{j} (4\bar{m}\phi_{i_{ij}} \phi_{i_{j}} \mathcal{R}^{m}_{J,3})_{j;j} + \frac{1}{12} (\mathcal{R}^{m}_{J,1};kk)$$

In Section 2 we recall combinatorial formulas for the invariants $a_{n,m}(x, D)$ for $n = 0, 2, 4, 6$ [8]; formulas for $a_{8}$, and for $a_{10}$ are available [1, 8, 16]. These formulas become very complicated as $n$ increases and it seems hopeless to try to establish Theorem 1.2 via direct computation even for $m = 6$ and $m = 8$. There are also closed formulas available due to Polterovich [4]. However, it does not seem possible to make a direct use of these formulas to derive Theorem 1.2. Instead, we proceed indirectly. In Section 3 we establish some functorial properties of these invariants and use invariance theory to prove the following result:

**Lemma 1.3.** If $m$ is even, then there exist universal constants so that

$$a^{d+\delta}_{m+2,m}(\phi, g) = c_{m+2,m}^{1} \varepsilon^{j} \phi_{i_{ij}} \phi_{i_{j}} \mathcal{R}^{m}_{J,3})_{j;j} + c_{m+2,m}^{2} (\varepsilon^{j} \mathcal{R}^{m}_{J,1}kk)$$

We shall complete the proof of Theorem 1.2 in Section 4 by evaluating these normalizing constants. The new features of this investigation are that both H. Weyl’s first and second main theorems of invariance theory [13] play a crucial role as does the analysis of the formal cohomology groups of spaces of $p$ form valued invariants [6]. Thus we expect that the techniques presented in this paper will be useful in other similar investigations of this type.

2. **Local Formulae for the Heat Trace Invariants**

If $D$ is an operator of Laplace type, then there is a canonical connection $\nabla$ on the underlying vector bundle that, together with the Levi-Civita connection, we use to covariantly differentiate tensors of all types - we denote multiple covariant differentiation by ‘;’. There is also a canonical endomorphism $E$ so that

$$D = -(\text{Tr}(\nabla^2) + E) \quad \text{i.e.} \quad Du = -(u_{;ii} + Eu).$$

Let $\Omega_{ij}$ be the curvature of the connection $\nabla$. Let $\rho_{ij} = R_{ikkj}$ be the Ricci tensor, and let $\tau = \rho_{ij}$.

The heat trace invariants $a_{n,m}$ for an operator of Laplace type can be expressed in this formalism [15].
Theorem 2.1.
(1) \(a_{0,m} = (4\pi)^{-m/2} \text{Tr}\{\text{Id}\}\).
(2) \(a_{2,m} = (4\pi)^{-m/2} \frac{1}{60} \text{Tr}\{6E + \tau \text{Id}\}\).
(3) \(a_{4,m} = (4\pi)^{-m/2} \frac{1}{5040} \text{Tr}\{60E_{kk} + 60\tau E + 180E^2 + (12\tau_{kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2) \text{Id} + 30\Omega_{ij}\Omega_{ij}\}\).
(4) \(a_{6,m} = (4\pi)^{-m/2} \text{Tr}\{(\frac{14}{72}\tau_{iijj} + \frac{7}{24}\tau_{kk}^2 - \frac{1}{3}\rho_{ij;ij} - \frac{1}{6}\rho_{ij}^2 - \frac{1}{4}\rho_{ij}\rho_{ij};nn + \frac{24}{77}\tau^2) \text{Id} + 2\Omega_{ij}\Omega_{ij}\}\).

We refer to [10] for the proof of the following results:

Lemma 2.2. On the circle, \(a_{2,1}^d = \frac{1}{\sqrt{\pi}}\phi_{11}\).

Lemma 2.3. Let \(M = (M_1 \times M_2, \phi_1 + \phi_2, g_1 + g_2)\) decouple as a product. Then
\[a_{n,m}^d(\phi, g)(x_1, x_2) = \sum_{n_1+n_2=n, m_1+n_2=m} a_{n_1,m_1}^d(\phi_1, g_1)(x_1) \cdot a_{n_2,m_2}(\phi_2, g_2)(x_2).\]

3. Spaces of Invariants

Let \(Q_m\) be the space of \(O(m)\) invariant polynomials in the components of the tensors \(\{R, \nabla R, \nabla^2 R, \ldots, \phi, \nabla \phi, \nabla^2 \phi, \ldots\}\). Define a grading on \(Q_m\) by setting:
weight \(R_{ijkl}\) is \(|\beta| + 2\) and weight \((\phi, g)\) is \(|\beta|\).

An element \(Q \in Q_m\) is homogeneous of weight \(n\) if and only if
\[Q(\phi, e^{-2}g) = c^n Q(\phi, g).\]

Let \(Q_{n,m} \subset Q_m\) be the set of all \(O(m)\) invariant polynomials which are homogeneous of weight \(n\); we then have a direct sum decomposition:
\[Q_m = \oplus \subset Q_{n,m}..\]

We may use the \(\mathbb{Z}_2\) action \(\phi \to -\phi\) to decompose \(Q_{n,m} = Q_{n,m}^+ \subset Q_{n,m}\) where
\[Q_{n,m}^+ := \{Q \in Q_{n,m} : Q(-\phi, g) = \pm Q(\phi, g)\}.\]

The following natural restriction map \(r : Q_{n,m} \to Q_{n,m-1}\) will play a crucial role. If \((N, g_N, \phi_N)\) are structures in dimension \(m - 1\), then we can define corresponding structures in dimension \(m\) by setting
\[(M, \phi_M, g_M) := (N \times S^1, \phi_N, g_N + d\theta^2)\].

If \(x \in N\) is the point of evaluation, we take the corresponding point \((x, 1) \in M\) for evaluation; which point on the circle chosen is, of course, irrelevant as \(S^1\) has a rotational symmetry. The restriction map is characterized dually by the formula:
\[r(Q)(\phi_N, g_N)(x) = Q(\phi_N, g_N + d\theta^2)(x, 1)\).

We can also describe the restriction map \(r\) in classical terms. H. Weyl’s [8] first theorem of invariance theory implies orthogonal invariants are built by contracting indices in pairs, where the indices range from 1 through \(m\). If \(P\) is given in terms of such a Weyl spanning set, then \(r(P)\) is given in terms of the same Weyl spanning set by restricting the range of summation to be from 1 through \(m-1\). Thus necessarily \(r\) is surjective. We refer to [18] for the proof of:
Lemma 3.1. If $m$ is even, then $a_{m+2,m}^{d+\delta} \in Q_{m+2,m}^+ \cap \ker(r)$.

We use H. Weyl’s second theorem to see $Q_{m+2,m}^+ \cap \ker(r)$ is generated by invariants where we contract $2m$ indices using the $\varepsilon$ tensor and contract the remaining indices in pairs, we refer to the discussion in [11] for details. A direct calculation shows, after some additional work to eliminate dependencies, that:

$$Q_{m+2,m}^+ \cap \ker(r) = \text{Span}\{ \varepsilon^I_1, \phi_{i_1 j_1}, \phi_{i_2 j_2}, R_{j_2 j_3}^I, \varepsilon^I_1, \phi_{i_1 j_1}, R_{j_1 j_2}^I, \varepsilon^I_1, R_{k_1 j_2 j_3}, R_{j_2 j_3}^I, \varepsilon^I_1, R_{k_1 j_2}, R_{j_2 j_3 k}, R_{j_3 j_4}^I, \varepsilon^I_1, R_{k_1 j_2 j_3}, R_{j_2 j_3 k}, R_{j_3 j_4}^I, \varepsilon^I_1 R_{k_1 j_2 j_3}, R_{j_2 j_3 k}, R_{j_3 j_4 k} \}$$

Although this is some gain in simplifying the question, the list of invariants is still quite long. We shall use equation (1.b) to further reduce the number of invariants to be considered and complete the proof of Lemma 3.1.

Let $Q_{n,m}^+$ be the space of $p$ form valued invariants in the curvature tensor, the covariant derivatives of the curvature tensor, and the covariant derivatives of the $\phi$ which are even in $\phi$. The exterior co-derivative $\delta_g$ induces a natural map

$$\delta_g : Q_{n,m}^+ \to Q_{n+1,m}^{p-1}.$$

Let $i : N \to N \times \{1\} \subset N \times S^1$. The analysis of $[8]$ shows that $p$ form valued invariants are constructed by alternating $p$ indices and by contracting the remaining indices in pairs. The restriction map

$$r : Q_{n,m}^p \to Q_{n,m}^{p-1}$$

is defined by restricting the range of summation of the indices involved; it is characterized by the identity:

$$r(Q) (\phi N, g N) = i^* Q(\phi N, g N + d\theta^2).$$

One verifies that $r$ is surjective and that

$$r \circ \delta_g = \delta_{g N} \circ r.$$

The analysis of the formal cohomology groups of the spaces of invariants of Riemannian manifolds, which was given in [9], then extends immediately to this more general setting to yield:

Lemma 3.2. (1) If $Q \in Q_{n,m}^+$, if $\int_M Q(\phi, g) = 0$ for all $(\phi, g)$, and if $n \neq m$, then there exists $Q^1 \in Q_{n-1,m}^{1-p}$ so that $\delta_g Q^1 = Q$.

(2) If $Q^1 \in Q_{n,m}^{1-p}$, if $\delta_g Q^1 = 0$, and if $n \neq m-1$, then there exists $Q^2 \in Q_{n+1,m}^{2-p}$ so that $Q^1 = \delta_g Q^2$.

The first assertion shows that any scalar invariant which always integrates to zero is canonically a divergence and that any 1 form valued invariant which is co-closed is canonically co-exact. The restriction on the weight is a technical one which plays no role as we shall take $n = m + 2$ in assertion (1) and $n = m + 1$ in assertion (2).

We use this result to show

Lemma 3.3. If $m$ is even, then there exists a 1 form valued invariant $Q_{m+1,m}^1$ in $Q_{m+2,m}^{1+p} \cap \ker(r)$ so that $\delta_g Q_{m+1,m}^1 = a_{m+2,m}^{d+\delta}$.

Proof. By equation (1.b) and Lemma 3.2 (1), there exists $Q_{m+1,m}^1 \in Q_{m+1,m}^{1+p}$ so

$$\delta_g Q_{m+1,m}^1 (\phi, g) = a_{m+2,m}^{d+\delta} (\phi, g).$$

Unfortunately, $r(Q_{m+1,m}^1)$ need not be zero and we must correct for this. Since $r(a_{m+2,m}^{d+\delta}) = 0$,

$$0 = r(\delta_g Q_{m+1,m}^1) = \delta_g r(Q_{m+1,m}^1).$$
Thus by Lemma 3.3 (2), there exists $Q_{m,m-1}^2 \in Q_{m,m-1}^2$ so that
\[ r(Q_{m+1,m}^1) = \delta_\theta(Q_{m,m-1}^2). \]
We observed above that $r$ is surjective. Thus we can find $Q_{m,m}^2 \in Q_{m,m}^2$ so that
\[ r(Q_{m,m}^2) = Q_{m,m-1}^2. \]
We complete the proof by setting $Q_{m+1,m}^1 := Q_{m+1,m}^1 - \delta_\theta(Q_{m,m}^2)$ and computing:
\[ \delta_\theta(Q_{m+1,m}^1) = \delta_\theta(Q_{m+1,m}^1) - \delta_\theta(Q_{m,m}^2) = \delta_\theta(Q_{m+1,m}^1) = \delta_\theta(-Q_{m,m}^2), \]
\[ r(Q_{m+1,m}^1) = r(Q_{m+1,m}^1) - r(\delta_\theta(Q_{m,m}^2)) = r(Q_{m+1,m}^1) - \delta_\theta(r(Q_{m,m}^2)) \]
\[ = r(Q_{m+1,m}^1) - \delta_\theta(2Q_{m,m-1}^2) = 0. \]

For $m$ even, we define elements of $Q_{m+1,m}^1 \cap \ker(r)$ by setting:
\[ \Xi_{m+1,m}^{1,\ell} := \varepsilon^{j} \phi^{\ell} \phi_{i} R_{i j k l} R_{J, k}^{l} e_{J}, \]
\[ \Xi_{m+1,m}^{2,\ell} := \varepsilon^{j} \phi^{\ell} R_{i j k l} R_{J, k}^{l} e_{J}, \]
\[ \Xi_{m+1,m}^{3,\ell} := \varepsilon^{j} \phi^{\ell} R_{i j k l} R_{J, k}^{l} e_{J}, \]
\[ \Xi_{m+1,m}^{4,\ell} := \varepsilon^{j} \phi^{\ell} \phi_{i} R_{i j k l} R_{J, k}^{l} e_{J}, \]
\[ \Xi_{m+1,m}^{5,\ell} := \varepsilon^{j} \phi^{\ell} \phi_{i} R_{i j k l} R_{J, k}^{l} e_{J}. \]

**Lemma 3.4.** If $m = 2m$ is even, then \( Q_{m+1,m}^1 \cap \ker(r) = \text{Span}\{\Xi_{m+1,m}^{1,\ell}\}_{i,\ell}. \)

**Proof.** We use H. Weyl’s theorem on the invariants of the orthogonal group. Let
\[ A = \phi^{\ell} \phi_{i}, \ldots, \phi_{i}, \phi_{i}, R_{i j k l} R_{i j k l} R_{J, k}^{l} e_{J} \]
be a typical 1 form valued monomial where $|\alpha_{\nu}| \geq 1$ and \( \ell + u \) is even. Note that:
\[ n = \sum_{\mu} |\alpha_{\mu}| + \sum_{\eta} (|\beta_{\eta}| + 2). \]
We must contract $2m$ indices using the $\varepsilon$ tensor and contract the remaining indices in pairs; we refer to [10] where this was discussed in some detail for scalar invariants – the extension to 1 form valued invariants is similar. We may estimate:
\[ 2m \leq \text{number of indices in } A \]
\[ = \sum_{\mu} |\alpha_{\mu}| + \sum_{\eta} (|\beta_{\eta}| + 4) + 1 = n + 2v + 1 \]
\[ = 2n + 1 - \sum_{\mu} |\alpha_{\mu}| - \sum_{\eta} |\beta_{\eta}| \leq 2n + 1 \]
We set $n = m + 1$. Since $2m$ and $m + 1 + 2v + 1$ are both even, the inequality in equation (3.3) must be strict and represents an increase either of 1 or of 3.

Suppose first that equation (3.11) is an equality. Then all the $2m$ indices present in $A$ are contracted using the $\varepsilon$ tensor. We can commute covariant derivatives at the cost of introducing additional curvature terms. Thus since all indices are to be contracted using the $\varepsilon$ tensor, we may assume $|\alpha_{\mu}| \leq 2$ for all $\mu$. Furthermore, by the first and second Bianchi identity, at most 2 indices can be alternated in $R_{ijkl}$. Thus $|\beta_{\eta}| = 0$ for all $\eta$ so $\sum_{\mu} |\alpha_{\mu}| = 3$. This leads to the invariants $\Xi_{m+1,m}^{1,\ell}$. Suppose next that equation (3.11) is not an equality. Then there are $2m + 2$ indices and one explicit covariant derivative present in $A$; $2m$ indices are contracted using the $\varepsilon$ tensor and two indices are contracted as a pair. This yields the invariants $\Xi_{m+1,m}^{i,\ell}$ for $i = 2, 3, 4, 5$ and the additional invariants:
\[ \Theta_{m+1,m}^{1,\ell} := \varepsilon^{j} \phi^{\ell} \phi_{i} R_{i j k l} R_{J, k}^{l} e_{J}, \]
\[ \Theta_{m+1,m}^{2,\ell} := \varepsilon^{j} \phi^{\ell} R_{i j k l} R_{J, k}^{l} e_{J}, \]
\[ \Theta_{m+1,m}^{3,\ell} := \varepsilon^{j} \phi^{\ell} R_{i j k l} R_{J, k}^{l} e_{J}, \]
\[ \Theta_{m+1,m}^{4,\ell} := \varepsilon^{j} \phi^{\ell} R_{i j k l} R_{J, k}^{l} e_{J}. \]

Invariance theory 5
To complete the proof, we must show the invariants $\Theta_{m+1,m}^{i,\ell}$ play no role. Let $U$ and $V$ be collections of $m + 1$ indices. Since $\varepsilon_V = 0$, we have

$$0 = \varepsilon_V^{U,\ell} \phi_{U_1} R_{V_2}^{U,m+1} e_{v_1}.$$

We set $u_1 = k$ and then set $v_1 = k, v_2 = k, \ldots$, and $v_{m+1} = k$ in turn to see:

$$0 = \varepsilon_{j,l}^f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{f} - m \varepsilon_f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{j} - m \Theta_{m+1,m}^{i,\ell}.$$

We set $u_2 = k$ and expand in $v$ to see:

$$0 = \varepsilon_{j,l}^f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{f} - m \varepsilon_f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{j} - (m - 2) \Theta_{m+1,m}^{i,\ell}.$$

Next, we set $v_1 = k$ and expand in $u$ to see:

$$0 = \varepsilon_{j,l}^f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{f} + m \varepsilon_f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{j} + \Theta_{m+1,m}^{i,\ell} - 2 \Xi_{m+1,m}^{3,\ell}.$$

Finally, we set $v_2 = k$ and expand in $u$ to see:

$$0 = \varepsilon_{j,l}^f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{f} + m \varepsilon_f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{j} + \Theta_{m+1,m}^{i,\ell} - 2 \Xi_{m+1,m}^{3,\ell}.$$

We can show that

$$\{\Theta_{m+1,m}^{1,\ell}, \Theta_{m+1,m}^{2,\ell}, \Theta_{m+1,m}^{3,\ell}\} \subset \text{Span}\{\Xi_{m+1,m}^{i,\ell}\}_{i,\ell}$$

by computing:

$$\Theta_{m+1,m}^{1,\ell} = \frac{1}{m} \Xi_{m+1,m}^{3,\ell},$$

$$\Theta_{m+1,m}^{2,\ell} = \frac{1}{m} \Xi_{m+1,m}^{4,\ell} = \frac{1}{m} \Xi_{m+1,m}^{5,\ell},$$

$$\Theta_{m+1,m}^{3,\ell} = \frac{1}{m} \Xi_{m+1,m}^{3,\ell} + (m - 2) \Theta_{m+1,m}^{3,\ell}.$$ 

Finally, we put $u_1 = k$ in the identity

$$0 = \varepsilon_{j,l}^{U,\ell} R_{u_2i_3j} \varepsilon_{v_1} u_1 R_{v_2}^{U,m+1} e_{v_1}$$

and expand in $v$ to show

$$0 = \varepsilon_{j,l}^f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{f} - m \varepsilon_f \phi_{i_1} R_{i_2i_3j} \epsilon_{k}^{j} - (m - 2) \Theta_{m+1,m}^{i,\ell}.$$

This establishes the lemma. \qed

We now prove Lemma 3. Let $m = 2 \ell$ be even. We apply Lemma 3.3 and Lemma 3.4 to see there exist universal constants so

$$a_{m+2,m}^{d+\ell} \phi, g = \sum_{i,\ell} a_{m+1,m}^{d+\ell} \phi, g \Xi_{m+1,m}^{i,\ell},$$

where $\ell$ is chosen so $\phi$ appears an even number of times in each expression. Terms which are linear in the $2$ jets of $\phi$ and which are of total weight $2$ in $\phi$ or $g$ arise only from $i = 4$ and $i = 5$. Consequently we have

$$a_{m+2,m}^{d+\ell} \phi, g = - \sum_{\ell} \phi \varepsilon_f \{ c_{m+1,m}^{4,\ell} \phi_{i_1j_1} R_{i_2j_1}^{f} + c_{m+1,m}^{5,\ell} \phi_{i_1j_1} R_{i_2j_1}^{f} \} + \ldots$$

where $\ell$ is odd. Replacing $\phi$ by $\phi + c$ does not change $d_\phi$ and $\delta_\phi$. Thus $\phi^\mu$ does not appear in the formula for $a_{m+2,m}^{d+\ell}$ for $\mu > 0$. Consequently,

$$0 = c_{m+1,m}^{4,\ell} \phi_{i_1j_1} R_{i_2j_1}^{f} + c_{m+1,m}^{5,\ell} \phi_{i_1j_1} R_{i_2j_1}^{f}.$$ 

(3.d)
We consider the expressions:

\[ A_1 := \phi_{11} R_{1221} R_{3443} \ldots R_{m-1,m,m,m} \quad \text{and} \]
\[ A_2 := \phi_{12} R_{1332} R_{3443} \ldots R_{m-1,m,m,m} \]

We may then expand

\[ \phi_{kk} R_{J_1}^{l,m} = 4^\bar{m} \bar{m}! A_1 + 0 A_2 + \ldots, \]
\[ \phi_{ij} R_{kjk} R_{J_3}^{l,m} = 4^{\bar{m}-1}(\bar{m}-1)! A_1 - 4 \cdot 4^{\bar{m}-1}(\bar{m}-1)! A_2 + \ldots \]

Consequently equation (3.d) implies \( c_{m+1,m}^{4\ell} = 0 \) and \( c_{m+1,m}^{5\ell} = 0 \).

We argue similarly to show that if \( \ell > 0 \), then

\[ 0 = c_{m+1,m}^{1\ell} \epsilon_j f_{ij} \phi_{ij} \phi_{ij} R_{J_3}^{l,m}, \quad \text{and} \]
\[ 0 = c_{m+1,m}^{2\ell} \epsilon_j f_{ij} \phi_{ij} R_{J_3}^{l,m} + c_{m+1,m}^{3\ell} \epsilon_j f_{ij} \phi_{ij} R_{J_3}^{l,m}. \]

This shows \( c_{m+1,m}^{1\ell} = 0 \) for \( \ell > 0 \). We consider the expressions:

\[ B_1 := \phi_{11} R_{1221} R_{3443} \ldots R_{m-1,m,m,m} \quad \text{and} \]
\[ B_2 := \phi_{33} R_{1332} R_{3443} \ldots R_{m-1,m,m,m} \]

and expand

\[ \epsilon_j f_{ij} \phi_{ij} R_{J_3}^{l,m} = 4^{\bar{m}}(\bar{m}-1)! B_1 + 4^{\bar{m}}(\bar{m}-1)! B_2 + \ldots, \]
\[ \epsilon_j f_{ij} \phi_{ij} R_{J_3}^{l,m} = 2 \cdot 4^{\bar{m}-1}(\bar{m}-1)! B_1 + \ldots \]

to see \( c_{m+1,m}^{2\ell} = 0 \) and \( c_{m+1,m}^{3\ell} = 0 \) for \( \ell > 0 \); Lemma 3 now follows. \( \square \)

4. Determining the Normalizing Constants

We complete the proof of Theorem 1.2 by evaluating the normalizing constants of Lemma 3.

**Lemma 4.1.** Let \( m = 2\bar{m} \). Then

\[ (1) \quad c_{m+2,m}^1 = \frac{4\bar{m}}{\pi^2 8^m \bar{m}!}, \]
\[ (2) \quad c_{m+2,m}^2 = \frac{1}{12 \pi^2 8^m \bar{m}!}, \]
\[ (3) \quad c_{m+2,m}^3 = \frac{1}{6 \pi^2 8^m \bar{m}!}. \]

**Proof.** We shall apply Theorem 1.1, Theorem 2.3, Lemma 2.2, and Lemma 2.3. We use the method of universal examples. Give \( M := S^{m-2} \times S^1 \times S^1 \) the product metric. Let \( \phi = \phi_1(\theta_1) + \phi_2(\theta_2) \). Then:

\[ a_{m+2}^{d+\delta} (\phi, g) = 2 c_{m+2,m}^1 2^{\bar{m}-1}(\bar{m}-2)! \phi_{m-1,m-1} \phi_{m,m} \]
\[ = a_{m-2}^{d+\delta} (0, g_{S^{m-2}}) \cdot a_{2,1}^{d+\delta} (\phi, d\theta_1^2) \cdot a_{2,1}^{d+\delta} (\phi, d\theta_2^2) \]
\[ = \frac{1}{8m-4} (\bar{m}-1)! \cdot 2^{\bar{m}-1}(\bar{m}-2)! \bar{m}! \phi_{1,m-1,m-1} \phi_{2,m}. \]

We solve this equation for \( c_{m+2,m}^1 \) to establish assertion (1).

For the remainder of the proof of the Lemma, we set \( \phi = 0 \) to consider only metric invariants. We express

\[ a_{m+2}^{d+\delta} = \lambda_{m,m} c_{m,m} \quad \text{for} \quad \lambda_{m,m} = \frac{\epsilon_j f_{ij} R_{J_1}^{l,m}}{8^m \bar{m}!}, \quad \text{and} \quad c_{m,m} = \frac{1}{8^m \bar{m}!}. \]

If \( m = 2 \), then the invariants \( (\epsilon_j f_{ij} R_{J_1}^{l,m})_{kk} \) and \( (\epsilon_j f_{ij} R_{kjk} R_{J_3}^{l,m})_{kk} \) are not linearly independent. If \( (N, g_N) \) is a Riemann surface, then we may establish assertions (2) and (3) for \( m = 2 \) by computing:

\[ a_{2,2}^{d+\delta} (g_N) = \frac{1}{4 \pi^2} \sum_p (1) \text{Tr}(E^p) \]
\[ a_{2,2}^{d+\delta} (g_N) = \frac{1}{4 \pi^2} \left( \sum_p (1) \text{Tr}(E^p) \right)_{kk} + O(R^2) \]
\[ = \frac{1}{6} (a_{2,2}^{d+\delta})_{kk} + O(R^2) = \frac{1}{4 \pi^2} R_{i j j} + O(R^2). \]
Suppose now that \( m = 4 \). Since \( \sum p_{r}(-1)^{p_r}\text{Tr}(\dim(A_v)) = 0 \), we compute:

\[
0 = a_{4,4}^{d+\delta}(0, g) = \frac{1}{8 \pi^2} \sum p_{r}(-1)^{p_r}\text{Tr}(E_p),
\]

\[
a_{4,4}^{d+\delta} = \frac{1}{15 \pi^2} \sum p_{r}(-1)^{p_r}\text{Tr}\left(\frac{1}{2}E_p E_p + \frac{1}{12}\text{Tr}(O_p^{ij}O_p^{ij})\right),
\]

\[
a_{6,4}^{d+\delta} = \frac{1}{3 \pi^2} \sum p_{r}(-1)^{p_r}\text{Tr}\left(\frac{1}{4}O_p^{ij}O_p^{ij} + \frac{1}{2}\text{Tr}(O_p^{ij}O_p^{ij})\right) + \frac{1}{60}O_p^{ij}O_p^{ij} + \frac{1}{12}E_p^{p_{r}E_p^{p_{r}}} + O(R^3).
\]

We study the expressions \( C_1 := R_{1221}R_{3443} \) and \( C_2 := R_{1221,2}R_{3443,2} \) and suppress other terms. Only the term \( E_p^{p_{r}E_p^{p_{r}}} \) can give rise to the expression \( A_2 \) and only the term \( E_p^{p_{r}E_p^{p_{r}}} \) can give rise to the expression \( A_3 \). We prove assertion (2) if \( m = 4 \) by computing:

\[
a_{4,4}^{d+\delta} = \frac{1}{2 \cdot 4 \pi^2} \sum p_{r}(-1)^{p_r}\text{Tr}(E_p E_p) + \ldots = \frac{32}{8 \pi^2}R_{1221}R_{3443} + \ldots
\]

\[
a_{6,4}^{d+\delta} = \frac{1}{3 \pi^2} \sum p_{r}(-1)^{p_r}\text{Tr}(E_p E_p) + \ldots = \frac{1}{2 \pi^2} \frac{1}{12} \sum p_{r}(-1)^{p_r}\text{Tr}(E_p E_p) + \ldots
\]

\[
= \frac{1}{12} (a_{4,4}^{d+\delta})_{kk} + \ldots = \frac{32}{6 \pi^2}R_{1221,2}R_{3443,2} + \ldots
\]

\[
= 2c_{6,4}^{2}\varepsilon^{ij}R_{1i2j3k}R_{1i4j3k}R_{1i4j3k} + \ldots = 64c_{6,4}^{2}R_{1221,2}R_{3443,2} + \ldots,
\]

so

\[
c_{6,4}^{2} = \frac{1}{12} \left( \frac{1}{8 \pi^2} \right) R_{1221}R_{3443} + \ldots
\]

If \( m > 4 \), let \( (M, g) := (N^4 \times S^{m-4}, g_N + g_0) \). Assertion (2) follows in general from:

\[
a_{m+2, m}^{d+\delta}(g) = \tilde{m}(\tilde{m}-1)\varepsilon_{m+2, m}^{c_{m+2, m}}(R_{1i2j3k}R_{1i4j3k}R_{1i4j3k})(g_N)\varepsilon_{m+2, m}(g_0) + \ldots
\]

\[
= a_{m,4}^{d+\delta}(g_N) c_{m+2, m}^{d+\delta}(g_0) + \ldots
\]

\[
= 2c_{6,4}^{2}m_{m+4, m-4}(g_0) + \ldots
\]

\[
c_{m+2, m}^{2} = \frac{2}{m(m-1)} \frac{1}{12 \pi^2} \frac{1}{8 \pi^2} R_{1i2j3k}R_{1i4j3k}R_{1i4j3k}(g_N)\varepsilon_{m+2, m}(g_0) + \ldots
\]

Let \( (M, g) := (N^4 \times S^{m-2}, g_N + g_0) \). We derive a relation between the invariants \( c_{m+2, m}^{2} \) and \( c_{m+2, m}^{3} \) to complete the proof of assertion (3):

\[
a_{m+2, m}^{d+\delta}(g_0) = (2\tilde{m}c_{m+2, m}^{2} + c_{m+2, m}^{3})R_{ijjkkk}(g_N) \varepsilon_{m+2, m}(g_0)
\]

\[
= a_{4,2}^{d+\delta}(g_N) c_{m+2, m}^{d+\delta} = \frac{1}{12 \pi^2} \frac{1}{60} R_{1i2j3k}R_{1i4j3k}(g_N) c_{m+2, m}(g_0)
\]

\[
2\tilde{m}c_{m+2, m}^{2} + c_{m+2, m}^{3} = \frac{1}{12 \pi^2} \frac{1}{60} \frac{1}{8 \pi^2} = \frac{\tilde{m}}{3 \pi^3 m^4 m!}.
\]

**Acknowledgments**

Research of PG partially supported by the NSF (USA), the MPI (Leipzig, Germany), and the Mittag-Leffler (Stockholm, Sweden). Research of KK and DV supported by the MPI (Leipzig, Germany).

**References**

[1] P. Amsterdamski, A. Berkin, and D. O’Connor, *Hamidew coefficient for a scalar field*, Classical Quantum Grav., 6 (1989), 1981–1991.

[2] H. A. Andrianov, N. V. Borisov and M. V. Iofe, *Factorization Method And Darboux Transformation For Multidimensional Hamiltonians*, Theor. Math. Phys. 61 (1984) 1078 [Teor. Mat. Fiz. 61 (1983) 183].

[3] I. G. Avramidi, *The covariant technique for the calculation of the heat kernel asymptotic expansion*, Physics Letters B., 238 (1990), 92–97.

[4] M. F. Atiyah, R. H. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. 19 (1973), 279–330; Errata 28 (1975), 277–280.

[5] S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. 45 (1944), 741–752.

[6] P. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Advances in Math., 10 (1973), 344–382.

[7] ———, *Local invariants of an embedded Riemannian manifold*, Annals of Math., 102 (1971), 187–203.
[8] —, Curvature and the heat equation for the DeRham complex, in Geometry and Analysis (Papers dedicated to the memory of V.K. Patodi), Indian Academy of Sciences (1979), 47–80.

[9] —, Invariance Theory, the heat equation, and the Atiyah-Singer index theorem 2nd ed., CRC Press ISBN 0-8493-7874-4 (1994), 516pp.

[10] P. Gilkey, K. Kirsten, and D. Vassilevich, Divergence terms in the supertrace heat asymptotics for the de Rham complex on a manifold with boundary, math-ph/0211021.

[11] P. Gilkey, K. Kirsten, D. Vassilevich, and A. Zelnikov, Duality symmetry of the p-form effective action and super trace of the twisted de Rham complex, hep-th/0209125, to appear in Nucl. Phys. B.

[12] H. P. McKean and I. M. Singer, Curvature and the eigenvalues of the Laplacian, J. Diff. Geo., 1 (1967), 43–69.

[13] V. K. Patodi, Curvature and the fundamental solution of the heat operator, J. Indian Math. Soc. 34 (1970), 269–285.

[14] I. Polterovich, Heat invariants of Riemannian manifolds, Isr. J. Math. 119 (2000), 239-252.

[15] R. Seeley, Complex powers of an elliptic operator, in Amer. Math. Soc. Proc. Symp. Pure Math, 10 (1968), 288–307.

[16] A. E. van de Ven, Index-free heat kernel coefficients, Class. Quant. Grav., 15 (1998), 2311–2344, hep-th/9708152.

[17] D. Vassilevich and A. Zelnikov, Discrete symmetries of functional determinants, Nucl. Phys. B 594 (2000), 501–517.

[18] H. Weyl, The Classical Groups, Princeton Univ. Press, Princeton, 1946.

[19] E. Witten, Supersymmetry and Morse Theory, J. Diff. Geom. 17 (1982), 661–692.

PG: Mathematics Department, University of Oregon, Eugene OR 97403 USA

E-mail address: gilkey@darkwing.uoregon.edu

KK: Department of Mathematics, Baylor University, Waco, TX 76798 USA and Max-Planck-Institute for Mathematics in the Sciences, Inselstrasse 22-26, 04103 Leipzig Germany

E-mail address: klaus.kirsten@mis.mpg.de and Klaus_Kirsten@baylor.edu

DV: Max-Planck-Institute for Mathematics in the Sciences, Inselstrasse 22-26, 04103 Leipzig Germany

E-mail address: vassil@itp.uni-leipzig.de