osp(1|2) Off-shell Bethe Ansatz Equations

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Abstract

The semiclassical limit of the algebraic quantum inverse scattering method is used to solve the theory of the Gaudin model. Via Off-shell Bethe ansatz equations of an integrable representation of the graded $osp(1|2)$ vertex model we find the spectrum of the $N - 1$ independents Hamiltonians of Gaudin. Integral representations of the $N$-point correlators are presented as solutions of the Knizhnik-Zamolodchikov equation. These results are extended for highest representations of the $osp(1|2)$ Gaudin algebra.
1 Introduction

In integrable models of statistical mechanics [1], an important object is the $R$-matrix $R(\lambda)$, where $\lambda$ is the spectral parameter. It acts on the tensor product $V^1 \otimes V^2$ for a given vector space $V$ and it is the solution of the Yang-Baxter (YB) equation

$$R_{12}(\lambda)R_{13}(\lambda + \mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda + \mu)R_{12}(\lambda)$$ (1.1)

in $V^1 \otimes V^2 \otimes V^3$, where $R_{12} = R \otimes I$, $R_{23} = I \otimes R$, etc. and $I$ is the identity matrix. If $R$ depends on a Planck-type parameter $\eta$ so that $R(\lambda, \eta) = 1 + \eta r(\lambda) + o(\eta^2)$, as $\eta \to 0$, then the “classical $r$-matrix” obeys the classical YB equation

$$[r_{12}(\lambda), r_{13}(\lambda + \mu) + r_{23}(\mu)] + [r_{13}(\lambda + \mu), r_{23}(\mu)] = 0$$ (1.2)

Nondegenerate solutions of (1.2) in the tensor product of two copies of simple Lie algebra $g$, $r_{ij}(\lambda) \in g_i \otimes g_j$, $i, j = 1, 2, 3$, were classified by Belavin and Drinfeld [2].

The classical YB equation interplays with conformal field theory in the following way: In the skew-symmetric case $r_{ji}(\lambda) + r_{ij}(\lambda) = 0$, it is the compatibility condition for the system of linear differential equations

$$\kappa \frac{\partial}{\partial z_i} \Psi(z_1, ..., z_N) = \sum_{j \neq i} r_{ij}(z_i - z_j) \Psi(z_1, ..., z_N)$$ (1.3)

in $N$ complex variables $z_1, ..., z_N$ for vector-valued functions $\Psi$ with values in the tensor space $V = V^1 \otimes \cdots \otimes V^N$ and $\kappa$ is a coupling constant.

In the rational case [2], very simple skew-symmetric solutions are known: $r(\lambda) = C_2/\lambda$, where $C_2 \in g \otimes g$ is a symmetric invariant tensor of a finite dimensional Lie algebra $g$ acting on a representation space $V$. The corresponding system of linear differential equations is the Knizhnik-Zamolodchikov (KZ) equation for the conformal blocks of the Wess-Zumino-Novikov-Witten (WZNW) model of conformal theory on the sphere [3]. Therefore, one can consider a KZ equation for each solution of the classical YB equation.

The algebraic Bethe ansatz [4] is the powerful method in the analysis of integrable models. Besides describing the spectra of quantum integrable systems, the Bethe ansatz also is used to find reasonably efficient expressions for the correlators [5]. Various representations of correlators in these models were found by Korepin [6], using this method.

Recently, Babujian and Flume [7] developed a method which reveals a link to the algebraic Bethe ansatz for the theory of the Gaudin model. Their method was confirmed by another different approach presented in the work of Feigin, Frenkel and
Reshetikhin \[9\]. In the Babujian-Flume method the wave vectors of the Bethe ansatz equation for inhomogeneous lattice model render in the semiclassical limit solutions of the KZ equation for the case of simple Lie algebras of higher rank. More precisely, the algebraic quantum inverse scattering method permits us write the following equation

\[ t(\lambda|z)\Phi(\lambda_1,\ldots,\lambda_p) = \Lambda(\lambda,\lambda_1,\ldots,\lambda_p|z)\Phi(\lambda_1,\ldots,\lambda_p) - \sum_{\alpha=1}^{p} \frac{F_{\alpha}\Phi_{\alpha}}{\lambda - \lambda_{\alpha}} \]  \hspace{1cm} (1.4)

Here \( t(\lambda|z) \) denotes the transfer matrix of the rational vertex model in an inhomogeneous lattice acting on an \( N \)-fold tensor product of \( SU(2) \) representation spaces. \( \Phi^{\alpha} = \Phi(\lambda_1,\ldots,\lambda_{\alpha-1},\lambda,\lambda_{\alpha+1},\ldots,\lambda_p) \). \( F_{\alpha}(\lambda_1,\ldots,\lambda_p|z) \) and \( \Lambda(\lambda,\lambda_1,\ldots,\lambda_p|z) \) are c numbers. The vanishing of the so-called unwanted terms, \( F_{\alpha} = 0 \), is enforced in the usual procedure of the algebraic Bethe ansatz by choosing the parameters \( \lambda_1,\ldots,\lambda_p \). In this case the wave vector \( \Phi(\lambda_1,\ldots,\lambda_p) \) becomes an eigenvector of the transfer matrix with eigenvalue \( \Lambda(\lambda,\lambda_1,\ldots,\lambda_p|z) \). If we keep all unwanted terms, i.e. \( F_{\alpha} \neq 0 \), then the wave vector \( \Phi \) in general satisfies the equation (1.4), named in \[8\] as off-shell Bethe ansatz equation (OSBAE). There is a neat relationship between the wave vector satisfying the OSBAE (1.4) and the vector-valued solutions of the KZ equation (1.3): The general vector valued solution of the KZ equation for an arbitrary simple Lie algebra was found by Schechtman and Varchenko \[10\]. It can be represented as a multiple contour integral

\[ \Psi(z_1,\ldots,z_N) = \oint \cdots \oint X(\lambda_1,\ldots,\lambda_p|z)\phi(\lambda_1,\ldots,\lambda_p|z)d\lambda_1\cdots d\lambda_p \]  \hspace{1cm} (1.5)

The complex variables \( z_1,\ldots,z_N \) of (1.3) are related with the disorder parameters of the OSBAE. The vector valued function \( \phi(\lambda_1,\ldots,\lambda_p|z) \) is the semiclassical limit of the wave vector \( \Phi(\lambda_1,\ldots,\lambda_p|z) \). In fact, it is the Bethe wave vector for Gaudin magnets \[11\], but off mass shell. The scalar function \( X(\lambda_1,\ldots,\lambda_p|z) \) is constructed from the semiclassical limit of the \( \Lambda(\lambda = z_k;\lambda_1,\ldots,\lambda_p|z) \) and \( F_{\alpha}(\lambda_1,\ldots,\lambda_p|z) \). This representation of the \( N \)-point correlation function shows a deep connection between the inhomogeneous vertex models and the WZNW theory.

In this paper we apply the Babujian-Flume ideas for \( osp(1|2) \) rational solution of the graded version of the YB equation (1.1).

The paper is organized as follows. In Section 2 we present the algebraic structure of the \( osp(1|2) \) vertex model. The inhomogeneous Bethe ansatz is read of from the homogeneous case previously known \[19\]. We derive the Off-shell Bethe ansatz equations for the fundamental representation of the \( osp(1|2) \) algebra. In Section 3, taking into account the semiclassical limit (\( \eta \to 0 \)) of the results presented in the Section 2, we describe the algebraic structure of the corresponding Gaudin model. Using the \( osp(1|2) \) Gaudin algebra, these results are extended for highest representations. In Section 4,
data of the Gaudin off-shell Bethe ansatz equations are used to construct solutions of
the KZ equation. Conclusions are reserved for Section 5.

2  Structure of the osp(1|2) Vertex Model

We recall that the osp(1|2) algebra is the simplest superalgebra and it can be viewed
as the graded version of sl(2). It has three even (bosonic) generators $H, X^{\pm}$ generating
a Lie subalgebra sl(2) and two odd (fermionic) generators $V^{\pm}$, whose non-vanishing
commutation relations in the Cartan-Weyl basis reads as

$$
[H, X^{\pm}] = \pm X^{\pm}, \quad \{X^{\pm}, X^{-}\} = 2H
$$

$$
[H, V^{\pm}] = \pm \frac{1}{2} V^{\pm}, \quad \{X^{\pm}, V^{\mp}\} = V^{\pm}, \quad \{V^{\pm}, V^{\pm}\} = 0
$$

$$
\{V^{\pm}, V^{\pm}\} = \pm \frac{1}{2} X^{\pm}, \quad \{V^{+}, V^{-}\} = -\frac{1}{2} H
$$

The quadratic Casimir operator is

$$
C^{2} = H^{2} + \frac{1}{2} \{X^{+}, X^{-}\} + \{V^{+}, V^{-}\}
$$

where $\{\cdot, \cdot\}$ denotes the anticommutator and $[\cdot, \cdot]$ the commutator.

The irreducible finite-dimensional representations $\rho_{j}$ with the highest weight vector
are parametrized by half-integer $s = j/2$ or by the integer $j \in N$. Their dimension
is $\dim(\rho_{j}) = 2j + 1$ and the corresponding value of $C^{2}$ is $j(j + 1)/4 = s(s + 1/2),
\ s = 0, 1/2, 1, 3/2, ...$

The representation corresponding to $s = 0$ is the trivial one-dimensional representa-
tion. The $s \geq 1/2$ representation contains two isospin multiplets which belong to
isospin $s$ and $s - 1/2$, denoted by $|s, s, m\rangle$ and $|s, s - 1/2, m\rangle$, respectively. The first quantum number characterizes the representation and the second and third quantum
numbers give the isospin and its third component. After a convenient normalization
of the states, a given $s$-representation is defined by

$$
H |s, s, m\rangle = m |s, s, m\rangle,
$$

$$
X^{\pm} |s, s, m\rangle = \sqrt{(s \mp m)(s \pm m + 1)} |s, s, m \pm 1\rangle
$$

$$
V^{\pm} |s, s, m\rangle = \pm \frac{1}{2} \sqrt{(s \mp m)} |s, s - 1/2, m \pm 1/2\rangle
$$

$$
H |s, s - 1/2, m\rangle = m |s, s - 1/2, m\rangle
$$

$$
X^{\pm} |s, s - 1/2, m\rangle = \sqrt{(s - 1/2 \mp m)(s - 1/2 \pm m + 1)} |s, s - 1/2, m \pm 1\rangle
$$

$$
V^{\pm} |s, s - 1/2, m\rangle = \pm \frac{1}{2} \sqrt{(s - 1/2 \pm m + 1)} |s, s, m \pm 1/2\rangle
$$

(2.3)
The fundamental representation has \( s = 1/2 \) and is given by
\[
H = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix},
X^+ = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
X^- = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
\[
V^+ = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
V^- = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]
(2.4)

In (2.4) the basis is \( \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right>, \left| \frac{1}{2}, 0, 0 \right>, \left| \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right> \). The first and third vectors will be considered as even and the second as odd, i.e., our grading is BFB.

In the \( j \)-representation, the odd part has the form [12]:
\[
V^+ = \begin{pmatrix}
0 & V_{j-1} & 0 & \cdots & 0 \\
0 & 0 & V_{j-2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0 & V_{-j} \\
0 & \cdots & 0 & 0 & 0
\end{pmatrix},
V^- = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
W_j & 0 & \cdots & \vdots & \vdots \\
0 & W_{j-1} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & W_{-j+1} & 0
\end{pmatrix}
\]
(2.5)

where
\[
(V_{j-1}, V_{j-2}, V_{j-3}, \ldots, V_{-j}) = \frac{1}{2}(\sqrt{j}, \sqrt{1}, \sqrt{j-1}, \sqrt{2}, \ldots, \sqrt{1}, \sqrt{j}),
\]
\[
(W_j, W_{j-1}, W_{j-2}, \ldots, W_{-j+1}) = \frac{1}{2}(-\sqrt{j}, \sqrt{1}, -\sqrt{j-1}, \sqrt{2}, \ldots, -\sqrt{1}, \sqrt{j}).
\]
(2.6)

For the even part we can see from (2.3) that \( H \) is diagonal and always has eigenvalue 0 due to isospin integer:
\[
H = \frac{1}{2} \text{diag}(j, j - 1, \ldots, 1, 0, -1, \ldots, -j).
\]
(2.7)

Moreover, \( X^\pm \) are given by the \( sl(2) \) composition which results in a clear relation with the odd part: \( X^\pm = \pm 4(V^\pm)^2 \).

### 2.1 Graded Quantum Inverse Scattering Method

Consider \( V = V_0 \oplus V_1 \) a \( \mathbb{Z}_2 \)-graded vector space where 0 and 1 denote the even and odd parts respectively. The components of a linear operator \( A \otimes B \) in the graded tensor product space \( V \otimes V \) result in matrix elements of the form
\[
(A \otimes B)_{\alpha_\beta}^{\gamma_\delta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))} A_{\alpha\gamma} B_{\delta\beta}
\]
(2.8)
and the action of the permutation operator $\mathcal{P}$ on the vector $|\alpha\rangle \otimes |\beta\rangle \in V \otimes V$ is given by
\[
\mathcal{P} \ |\alpha\rangle \otimes |\beta\rangle = (-)^{p(\alpha)p(\beta)} |\beta\rangle \otimes |\alpha\rangle \quad \Rightarrow \quad (\mathcal{P})_{\alpha\beta}^{\gamma\delta} = (-)^{p(\alpha)p(\beta)} \delta_{\alpha\beta} \delta_{\gamma\delta} \quad (2.9)
\]
where $p(\alpha) = 1 \ (0)$ if $|\alpha\rangle$ is an odd (even) element.

Besides $\mathcal{R}$ we have to consider matrices $R = \mathcal{P}\mathcal{R}$ which satisfy
\[
R_{12}(\lambda)R_{23}(\lambda + \mu)R_{12}(\mu) = R_{23}(\mu)R_{12}(\lambda + \mu)R_{23}(\lambda). \quad (2.10)
\]

The rational solution of the graded YB equation for the fundamental representation of $osp(1|2)$ algebra was found by Kulish in [13]. It has the form
\[
R(\lambda, \eta) = \begin{pmatrix}
  x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & x_5 & 0 & x_2 & 0 & 0 & 0 & 0 \\
  0 & 0 & x_7 & 0 & y_6 & 0 & x_3 & 0 \\
  0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 \\
  0 & 0 & y_{-6} & 0 & -x_4 & 0 & -x_6 & 0 \\
  0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_2 \\
  0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\
  0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1
\end{pmatrix} \quad (2.11)
\]
where
\[
x_1(\lambda, \eta) = \lambda + \eta, \quad x_2(\lambda, \eta) = \epsilon_1 \lambda, \quad x_3(\lambda, \eta) = \lambda - \frac{\lambda \eta}{\lambda + \frac{3}{2} \eta} \\
x_4(\lambda, \eta) = \lambda - \eta + \frac{\lambda \eta}{\lambda + \frac{3}{2} \eta}, \quad x_5(\lambda, \eta) = \eta, \quad x_6(\lambda, \eta) = -\epsilon_2 \frac{\lambda \eta}{\lambda + \frac{3}{2} \eta} \\
x_7(\lambda, \eta) = \eta + \frac{\lambda \eta}{\lambda + \frac{3}{2} \eta}, \quad y_6(\lambda, \eta) = -x_6(\lambda, \eta), \quad (2.12)
\]
where $\epsilon_i = \pm 1, \ i = 1, 2$. Here we have assumed that the grading of threefold space is $p(1) = p(3) = 0$ and $p(2) = 1$ and we will choose the solution of (2.12) for which $\epsilon_1 = \epsilon_2 = 1$.

Let us consider the inhomogeneous vertex model, where to each vertex we associate two parameters: the global spectral parameter $\lambda$ and the disorder parameter $z$. In this case, the vertex weight matrix $\mathcal{R}$ depends on $\lambda - z$ and consequently the monodromy matrix will be a function of the disorder parameters $z_i$. From now on we use a compact notation for the arguments with the shifted spectral parameter: $(\lambda | z) = (\lambda - z_1, \ldots, \lambda - z_N)$.
The graded quantum inverse scattering method is characterized by the monodromy matrix \( T(\lambda|z) \) satisfying the equation

\[
R(\lambda - \mu) \left[ T(\lambda|z) \otimes T(\mu|z) \right] = \left[ T(\mu|z) \otimes T(\lambda|z) \right] R(\lambda - \mu),
\]

whose consistency is guaranteed by the graded version of the YB equation (2.10). \( T(\lambda|z) \) is a matrix in the space \( V \) (the graded auxiliary space) whose matrix elements are operators on the states of the quantum system (the quantum space, which will also be the space \( V \)). The monodromy operator \( T(\lambda|z) \) is defined as an ordered product of local operators \( \mathcal{L}_n \) (Lax operator), on all sites of the lattice:

\[
T(\lambda|z) = \mathcal{L}_N(\lambda|z)\mathcal{L}_{N-1}(\lambda|z) \cdots \mathcal{L}_1(\lambda|z). \tag{2.14}
\]

The Lax operator on the \( n^{th} \) quantum space is given the graded permutation of (2.11):

\[
\mathcal{L}_n(\lambda|z) = \begin{pmatrix}
x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 \\
0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 \\
0 & x_5 & 0 & x_2 & 0 & 0 & 0 & 0 \\
0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 \\
0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 \\
0 & 0 & x_7 & 0 & y_6 & 0 & x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
L_{11}^{(n)}(\lambda|z) & L_{12}^{(n)}(\lambda|z) & L_{13}^{(n)}(\lambda|z) \\
L_{21}^{(n)}(\lambda|z) & L_{22}^{(n)}(\lambda|z) & L_{23}^{(n)}(\lambda|z) \\
L_{31}^{(n)}(\lambda|z) & L_{32}^{(n)}(\lambda|z) & L_{33}^{(n)}(\lambda|z)
\end{pmatrix} \tag{2.15}
\]

Note that \( L_{\alpha\beta}^{(n)}(\lambda), \alpha, \beta = 1, 2, 3 \) are 3 by 3 matrices acting on the \( n^{th} \) site of the lattice. It means that the monodromy matrix has the form

\[
T(\lambda|z) = \begin{pmatrix}
A_1(\lambda|z) & B_1(\lambda|z) & B_2(\lambda|z) \\
C_1(\lambda|z) & A_2(\lambda|z) & B_3(\lambda|z) \\
C_2(\lambda|z) & C_3(\lambda|z) & A_3(\lambda|z)
\end{pmatrix} \tag{2.16}
\]

where

\[
T_{ij}(\lambda|z) = \sum_{k_1, \ldots, k_{N-1}=1}^3 L_{ik_1}^{(N)}(\lambda|z) \otimes L_{kj_2}^{(N-1)}(\lambda|z) \otimes \cdots \otimes L_{k_{N-1}j}^{(1)}(\lambda|z)
\]

\[
i, j = 1, 2, 3. \tag{2.17}
\]

The vector in the quantum space of the monodromy matrix \( T(\lambda|z) \) that is annihilated by the operators \( T_{ij}(\lambda|z), i > j \) \((C_i(\lambda|z)\) operators, \( i = 1, 2, 3 \)) and it is also an
eigenvector for the operators $T_{ii}(\lambda|z)$ ($A_i(\lambda|z)$ operators, $i = 1, 2, 3$) is called a highest vector of the monodromy matrix $T(\lambda|z)$.

The transfer matrix $\tau(\lambda|z)$ of the corresponding integrable spin model is given by the supertrace of the monodromy matrix in the space $V$

$$\tau(\lambda|z) = \sum_{i=1}^{3} (-1)^{p(i)} T_{ii}(\lambda|z) = A_1(\lambda|z) - A_2(\lambda|z) + A_3(\lambda|z).$$

(2.18)

A detailed exposition of the graded quantum inverse scattering method can be found in the references [13], [16] and [17].

2.2 Inhomogeneous Bethe Ansatz

Algebraic Bethe ansatz solution for the inhomogeneous $osp(1|2)$ vertex model can be obtained from the homogeneous case. The only modification one have to do is local shifting of the spectral parameter $\lambda \rightarrow \lambda - z_i$. The algebraic Bethe ansatz solution of the homogeneous rational $osp(1|2)$ vertex model was obtained by Martins in [18]. However, we will consider the rational limit of the trigonometric case presented in [19].

In the sequence we will work with some functions which will be now defined:

$$z(\lambda) = \frac{x_1(\lambda) = \frac{\lambda + \eta}{\lambda}}{x_2(\lambda)} = \frac{x_3(\lambda)}{y(\lambda) = \frac{x_3(\lambda)}{y_6(\lambda)}} = \frac{2\lambda + \eta}{2\eta},$$

$$\omega(\lambda) = -\frac{x_1(\lambda)x_3(\lambda)}{x_4(\lambda)x_3(\lambda) - x_6(\lambda)y_6(\lambda)} = \frac{2\lambda + \eta}{2\lambda - \eta},$$

$$Z(\lambda_i - \lambda_j) = \begin{cases} z(\lambda_i - \lambda_j) & \text{if } k > j \\ z(\lambda_i - \lambda_j)\omega(\lambda_j - \lambda_k) & \text{if } k < j \end{cases}$$

(2.19)

We start defining the highest vector of the monodromy matrix $T(\lambda|z)$ in a lattice of $N$ sites as the even (bosonic) completely unoccupied state

$$|0\rangle = \otimes_{a=1}^{N} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(2.20)

Using (2.17) we can compute the action of the monodromy matrix entries on this state

$$A_i(\lambda|z) |0\rangle = X_i(\lambda|z) |0\rangle, \quad C_i(\lambda|z) |0\rangle = 0, \quad B_i(\lambda|z) |0\rangle \neq \{0, |0\rangle \}$$

$$X_i(\lambda|z) = \prod_{a=1}^{N} x_i(\lambda - z_a), \quad i = 1, 2, 3$$

(2.21)

Unlike to the simple Lie algebras cases [14], the off-shell Bethe ansatz equations for the $osp(1|2)$ algebra present in a more complicated form.
\[ \tau(\lambda|z)\Psi_n(\lambda_1, \ldots, \lambda_n) = \Lambda\Psi_n(\lambda_1, \ldots, \lambda_n) - \sum_{j=1}^{n} \mathcal{F}^{(n)}_j \Psi_j^{(n)} + \sum_{j=2}^{n-1} \sum_{l=1}^{n-j} \mathcal{F}_{lj}^{(n)} \Psi_{n-j}^{(n)} \] 

(2.22)

where the Bethe vectors are defined as normal ordered states \( \Psi_n(\lambda_1, \ldots, \lambda_n) \) which can be written with aid of a recurrence formula \([20]\):

\[ \Psi_n(\lambda_1, \ldots, \lambda_n) = B_1(\lambda_1)\Psi_{n-1}(\lambda_2, \ldots, \lambda_n) \]

\[-B_2(\lambda_1) \sum_{j=2}^{n} \frac{X_1(\lambda_j|z)}{y(\lambda_1 - \lambda_j)} \prod_{k=2, k \neq j}^{n} \mathcal{Z}(\lambda_k - \lambda_j)\Psi_{n-2}(\lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n) \] 

(2.23)

with the initial condition \( \Psi_0 = \langle 0 | \), \( \Psi_1(\lambda_1) = B_1(\lambda_1) \langle 0 | \). \( \lambda_j \) denotes that the rapidity \( \lambda_j \) is absent: \( \Psi(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_n) = \Psi(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n) \).

Let us now describe each term which appear in the right hand side of the OSBAE for the \( osp(1|2) \) model (for more details the reader can see \([19]\)): In the first term the Bethe vectors (2.23) are multiplied by \( c \)-numbers \( \Lambda = \Lambda(\lambda, \lambda_1, \ldots, \lambda_n|z) \) given by

\[ \Lambda = X_1(\lambda|z) \prod_{k=1}^{n} \frac{z(\lambda - \lambda_k)}{\omega(\lambda - \lambda_k)} + X_3(\lambda|z) \prod_{k=1}^{n} \frac{x_2(\lambda - \lambda_k)}{x_3(\lambda - \lambda_k)}. \]

(2.24)

The second term is a sum of new vectors defined as

\[ \Psi_j^{(n)} = \left( \frac{x_5(\lambda_j - \lambda)}{x_2(\lambda_j - \lambda)} B_1(\lambda|z) + \frac{1}{y(\lambda - \lambda_j)} B_3(\lambda|z) \right) \Psi_{n-1}(\lambda_j) \]

(2.25)

and \( \mathcal{F}_{lj}^{(n)} \) are scalar functions given by

\[ \mathcal{F}_{lj}^{(n)} = X_1(\lambda_j|z) \prod_{k \neq j}^{n} \mathcal{Z}(\lambda_k - \lambda_j) + (-)^n X_2(\lambda_j|z) \prod_{k \neq j}^{n} \mathcal{Z}(\lambda_j - \lambda_k). \]

(2.26)

Finally, the last term is a coupled sum of a third type of vector-valued functions

\[ \Psi_{n-j}^{(n)} = B_2(\lambda|z)\Psi_{n-2}(\lambda_{t}, \lambda_{j}) \]

(2.27)

with intricate coefficients

\[ \mathcal{F}_{ij}^{(n)} = G_{ji}X_1(\lambda_i|z)X_1(\lambda_j|z) \prod_{k=1, k \neq j, l}^{n} \mathcal{Z}(\lambda_k - \lambda_l) \mathcal{Z}(\lambda_k - \lambda_j) \]

\[-(-)^n Y_{ji}X_1(\lambda_i|z)X_2(\lambda_j|z) \prod_{k=1, k \neq j, l}^{n} \mathcal{Z}(\lambda_k - \lambda_l) \mathcal{Z}(\lambda_j - \lambda_k) \]

\[-(-)^n F_{ji}X_1(\lambda_j|z)X_2(\lambda_l|z) \prod_{k=1, k \neq j, l}^{n} \mathcal{Z}(\lambda_l - \lambda_k) \mathcal{Z}(\lambda_j - \lambda_k) \]

\[+H_{ji}X_2(\lambda_i|z)X_2(\lambda_j|z) \prod_{k=1, k \neq j, l}^{n} \mathcal{Z}(\lambda_j - \lambda_k) \mathcal{Z}(\lambda_l - \lambda_k) \]

(2.28)
where \( G_{jl} \), \( Y_{jl} \), \( F_{jl} \) and \( H_{jl} \) are additional ratio functions defined by

\[
\begin{align*}
G_{jl} &= \frac{x_7(\lambda_l - \lambda)}{x_3(\lambda_l - \lambda) y(\lambda_l - \lambda_j)} \frac{1}{y_5(\lambda_l - \lambda)} + \frac{z(\lambda_l - \lambda)}{\omega(\lambda_l - \lambda) x_2(\lambda_l - \lambda)} \frac{1}{x_3(\lambda_l - \lambda) y(\lambda_l - \lambda_j)} \\
H_{jl} &= \frac{y_7(\lambda_l - \lambda)}{x_3(\lambda_l - \lambda) y(\lambda_l - \lambda_j)} \frac{1}{y(\lambda_l - \lambda)} - \frac{y_5(\lambda_l - \lambda)}{x_3(\lambda_l - \lambda) y(\lambda_l - \lambda_j)} \frac{1}{y(\lambda_l - \lambda)} \\
Y_{jl} &= \frac{1}{y(\lambda_l - \lambda)} \left\{ \frac{z(\lambda_l - \lambda) y_5(\lambda - \lambda_j)}{x_2(\lambda - \lambda_j)} - \frac{y_5(\lambda_l - \lambda)}{x_2(\lambda - \lambda_l) x_2(\lambda_l - \lambda_j)} \right\} \\
F_{jl} &= \frac{y_5(\lambda_l - \lambda)}{x_2(\lambda_l - \lambda)} \left\{ \frac{y_5(\lambda_l - \lambda)}{x_2(\lambda_l - \lambda_j)} \frac{1}{y(\lambda_l - \lambda)} + \frac{z(\lambda_l - \lambda)}{\omega(\lambda_l - \lambda) y(\lambda_l - \lambda_j)} \right\}
\end{align*}
\]

In the usual Bethe ansatz method, the next step consist in impose the vanishing of the so-called unwanted terms of (2.22) in order to get an eigenvalue problem for the transfer matrix:

We impose \( \mathcal{F}_j^{(\eta)} = 0 \), which also implies \( \mathcal{F}_{lj}^{(\eta)} \) vanishing. Hence, \( \Psi_n(\lambda_1, ..., \lambda_n) \) is an eigenstate of \( \tau(\lambda|z) \) with eigenvalue \( \Lambda \), provided the rapidities \( \lambda_j \) are solutions of the inhomogeneous Bethe ansatz equations

\[
\prod_{a=1}^{N} \left( \frac{\lambda_j - z_a + \eta}{\lambda_j - z_a} \right) = (-)^{n+1} \prod_{k \neq j}^{n} (\lambda_j - \lambda_k + \eta)(\lambda_k - \lambda_j + \eta/2)
\]

\[
\eta = 1, 2, ..., n
\]

These equations were derived for the first time in [22].

### 3 Structure of the \( \text{osp}(1|2) \) Gaudin Model

In this section we will consider the theory of the Gaudin model. To do this we need to calculate the semiclassical limit of the results presented in the previous section.

In order to expand the matrix elements of \( T(\lambda|z) \), up to an appropriate order in \( \eta \), we will start by expanding the Lax operator entries defined in (2.13):

\[
\begin{align*}
L_{11}^{(n)}(\lambda|z) &= 1 + \eta \frac{2H_n}{\lambda - z_n} + \frac{3}{2} \eta^2 \frac{2H_n^2 + H_n}{(\lambda - z_n)^2} + o(\eta^3) \\
L_{22}^{(n)}(\lambda|z) &= 1 + \eta \left( 0 + \frac{3}{2} \eta^2 \frac{1}{(\lambda - z_n)^2} - \frac{4H_n^2}{(\lambda - z_n)^2} + o(\eta^3) \right) \\
L_{33}^{(n)}(\lambda|z) &= 1 - \eta \frac{2H_n}{\lambda - z_n} + \frac{3}{2} \eta^2 \frac{2H_n^2 - H_n}{(\lambda - z_n)^2} + o(\eta^3)
\end{align*}
\]
and for off-diagonal elements, we have
\[
L^{(n)}_{12}(\lambda|z) = -\eta \frac{2V^-}{\lambda - z_n} + o(\eta^2), \quad L^{(n)}_{21}(\lambda|z) = \eta \frac{2V^+}{\lambda - z_n} + o(\eta^2), \\
L^{(n)}_{13}(\lambda|z) = \eta \frac{2X^-}{\lambda - z_n} + o(\eta^2), \quad L^{(n)}_{31}(\lambda|z) = \eta \frac{2X^+}{\lambda - z_n} + o(\eta^2), \\
L^{(n)}_{23}(\lambda|z) = \eta \frac{2V^-}{\lambda - z_n} + o(\eta^2), \quad L^{(n)}_{32}(\lambda|z) = \eta \frac{2V^+}{\lambda - z_n} + o(\eta^2). \quad (3.2)
\]

Here \(H, X^\pm, V^\pm\) are matrices 3 by 3 given by (2.4).

Using (3.1) and (3.2) is now easy to derive the semiclassical expansion of the monodromy matrix entries:

\[
A_1(\lambda|z) = \Gamma(\lambda|z) \left\{ 1 + \eta \sum_{a=1}^N \frac{\mathcal{H}_a}{\lambda - z_a} \\
+ \eta^2 \sum_{a<b} \left( \frac{\mathcal{H}_a \otimes \mathcal{H}_b + \mathcal{X}^+_a \otimes \mathcal{X}^+_b - \mathcal{V}^-_a \otimes \mathcal{V}^-_b}{(\lambda - z_a)(\lambda - z_b)} \right) + \frac{3}{4} \sum_{a=1}^N \frac{[\mathcal{H}^2 - \mathcal{H}]^a}{(\lambda - z_a)^2} + o(\eta^3) \right\}
\]

\[
A_2(\lambda|z) = \Gamma(\lambda|z) \left\{ 1 + \eta \left[ \sum_{a=1}^N \frac{\mathcal{H}_a}{\lambda - z_a} \\
+ \eta^2 \sum_{a<b} \left( \frac{\mathcal{V}^-_a \otimes \mathcal{V}^+_b - \mathcal{V}^+_a \otimes \mathcal{V}^-_b}{(\lambda - z_a)(\lambda - z_b)} \right) - \frac{3}{2} \sum_{a=1}^N \frac{[1 - \mathcal{H}]^a}{(\lambda - z_a)^2} \right] + o(\eta^3) \right\}
\]

\[
A_3(\lambda|z) = \Gamma(\lambda|z) \left\{ 1 - \eta \sum_{a=1}^N \frac{\mathcal{H}_a}{\lambda - z_a} \\
+ \eta^2 \sum_{a<b} \left( \frac{\mathcal{H}_a \otimes \mathcal{H}_b + \mathcal{X}^+_a \otimes \mathcal{X}^+_b + \mathcal{V}^+_a \otimes \mathcal{V}^-_b}{(\lambda - z_a)(\lambda - z_b)} \right) + \frac{3}{4} \sum_{a=1}^N \frac{[\mathcal{H}^2 + \mathcal{H}]^a}{(\lambda - z_a)^2} + o(\eta^3) \right\}
\]

\[
B_1(\lambda|z) = -B_3(\lambda|z) = \Gamma(\lambda|z) \left\{ -\eta \sum_{a=1}^N \frac{\mathcal{V}^-_a}{\lambda - z_a} + o(\eta^2) \right\}
\]

\[
C_1(\lambda|z) = C_3(\lambda|z) = \Gamma(\lambda|z) \left\{ \eta \sum_{a=1}^N \frac{\mathcal{V}^+_a}{\lambda - z_a} + o(\eta^2) \right\}
\]

\[
B_2(\lambda|z) = \Gamma(\lambda|z) \left\{ \eta \sum_{a=1}^N \frac{\mathcal{X}^-_a}{\lambda - z_a} + o(\eta^2) \right\}
\]

\[
C_2(\lambda|z) = \Gamma(\lambda|z) \left\{ \eta \sum_{a=1}^N \frac{\mathcal{X}^+_a}{\lambda - z_a} + o(\eta^2) \right\} \quad (3.3)
\]

where \(\mathcal{H} = 2H, \mathcal{X}^\pm = 2X^\pm, \mathcal{V}^\pm = 2V^\pm\) and \(\Gamma(\lambda|z) = \prod_{a=1}^N (\lambda - z_a)\) is a common factor which can be absorbed after a convenient normalization. From the definition (2.18) it follows that the semiclassical expansion of the normalized transfer matrix \(t(\lambda|z) = \tau(\lambda|z)/\Gamma(\lambda|z)\) has the form

\[
t(\lambda|z) = 1 + 2\eta^2 \left\{ \sum_{a=1}^N \frac{G_a}{\lambda - z_a} + \frac{3}{4} \sum_{a=1}^N \frac{[1]^a}{(\lambda - z_a)^2} \right\} + o(\eta^3) \quad (3.4)
\]
where
\[ G_a = \sum_{b \neq a}^N \mathcal{H}_a \otimes \mathcal{H}_b + \frac{1}{2} (\mathcal{X}_a^+ \otimes \mathcal{X}_b^- + \mathcal{X}_a^- \otimes \mathcal{X}_b^+) + \mathcal{V}_a^+ \otimes \mathcal{V}_b^- - \mathcal{V}_a^- \otimes \mathcal{V}_b^+ \] (3.5)

Here we have used the symmetry \( G_{kj} = P_{jk} G_{jk} P_{jk} = G_{jk} \) and the identity
\[ \frac{1}{(\lambda - z_a)(\lambda - z_b)} = \left( \frac{1}{\lambda - z_a} - \frac{1}{\lambda - z_b} \right) \frac{1}{z_a - z_b} \] (3.6)

The Gaudin model is defined by the residue of \( t(\lambda|z) \) (3.4) at the point \( \lambda = z_a \). This results in \( N \) non-local magnets whose Hamiltonians \( G_a, a = 1, 2, ..., N \) are given by (3.5) and satisfied the property \( \sum_{a=1}^N G_a = 0 \).

### 3.1 osp(1|2) Gaudin Algebra

Let us now consider the semiclassical limit of the fundamental commutation relation (2.13). Let us first write the \( R \)-matrix (2.11) in the form [13]:
\[ R(\lambda, \eta) = \eta \mathcal{I} + \lambda \mathcal{P} + \frac{\lambda \eta}{\lambda + \frac{2}{3} \eta} \mathcal{U} \] (3.7)

where \( \mathcal{I} \) is the identity operator, \( \mathcal{P} \) is the graded permutation operator (2.9) and \( \mathcal{U} \) is the rank-one projector \( \mathcal{U}^2 = \mathcal{U} \). Using the normalization of the previous section we can write the semiclassical expansions of \( T \) and \( R \) in the following form
\[ T(\lambda|z) = 1 + \eta \mathcal{L}(\lambda|z) + o(\eta^2) \]
\[ R(\lambda - \mu) = \mathcal{P} \left[ 1 + \eta \mathcal{R}(\lambda - \mu) + o(\eta^2) \right] \] (3.8)

From (3.3) we can see that the ”classical \( L \)-operator” has the form
\[ L(\lambda|z) = \begin{pmatrix} \mathcal{H}(\lambda|z) & -\mathcal{V}^-(\lambda|z) & \mathcal{X}^-(\lambda|z) \\ \mathcal{V}^+(\lambda|z) & 0 & \mathcal{V}^-(\lambda|z) \\ \mathcal{X}^+(\lambda|z) & \mathcal{V}^+(\lambda|z) & \mathcal{H}(\lambda|z) \end{pmatrix} \] (3.9)

where
\[ \mathcal{H}(\lambda|z) = \sum_{a=1}^N \frac{\mathcal{H}_a}{\lambda - z_a}, \quad \mathcal{V}^-(\lambda|z) = \sum_{a=1}^N \frac{\mathcal{V}_a^-}{\lambda - z_a}, \quad \mathcal{X}^-(\lambda|z) = \sum_{a=1}^N \frac{\mathcal{X}_a^-}{\lambda - z_a} \]
\[ \mathcal{V}^+(\lambda|z) = \sum_{a=1}^N \frac{\mathcal{V}_a^+}{\lambda - z_a}, \quad \mathcal{X}^+(\lambda|z) = \sum_{a=1}^N \frac{\mathcal{X}_a^+}{\lambda - z_a} \] (3.10)

and the classical \( r \)-matrix is obtained by the semiclassical expansion of (3.7). It has the form
\[ r(\lambda - \mu) = \frac{1}{\lambda - \mu} (\mathcal{P} - \mathcal{U}) \] (3.11)
Substituting (3.9) and (3.10) into the fundamental relation (2.13) we will get, from the first non trivial identity, the following equation:

\[
P L(\lambda|z) \hat{\otimes} L(\mu|z) + P r(\lambda - \mu) \left[ L(\lambda|z) \hat{\otimes} 1 + 1 \hat{\otimes} L(\mu|z) \right] \\
= L(\mu|z) \hat{\otimes} L(\lambda|z) P + \left[ L(\lambda|z) \hat{\otimes} 1 + 1 \hat{\otimes} L(\mu|z) \right] P r(\lambda - \mu) \tag{3.12}
\]

whose consistence is guaranteed by the graded classical YB equation.

From (3.12) we can derive commutations relations between the matrix elements of the monodromy matrix used in the usual Bethe ansatz method. It follows from (3.12) that the Gaudin algebra is the semiclassical limit of the commutation relations of the matrix elements of the monodromy matrix used in the usual Bethe ansatz method.

A direct consequence of these relations is the commutativity of \( t(\lambda|z) \) (3.4)

\[
[t(\lambda|z), t(\mu|z)] = 0, \quad \forall \lambda, \mu \tag{3.14}
\]

and as consequence it follows the commutativity of the Gaudin Hamiltonians \( G_a, a = 1, ..., N \).

### 3.2 Gaudin Off-shell Bethe Ansatz Equations

In order to get semiclassical limit of the OSBAE (2.22) we first consider the semiclassical expansions of the Bethe vectors defined in (2.23), (2.25) and (2.27):

\[
\Psi_n(\lambda_1, ..., \lambda_n) = (-\eta)^n \Phi_n(\lambda_1, ..., \lambda_n|z) + o(\eta^{n+1})
\]
\[
\begin{align*}
\Psi_{(n)}^j & = 2(-\eta)^{n+1} \frac{\mathcal{V}^-(\lambda|z)}{\lambda - \lambda_j} \Phi_{n-1}(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_n|z) + o(\eta^{n+2}) \\
\Psi_{(n)}^{ji} & = -(-\eta)^{n-1} \mathcal{X}^-(\lambda|z) \Phi_{n-2}(\lambda_1, \ldots, \lambda_i, \lambda_j, \ldots, \lambda_n|z) + o(\eta^n)
\end{align*}
\] (3.15)

where
\[
\Phi_n(\lambda_1, \ldots, \lambda_n|z) = \mathcal{V}^-(\lambda_1|z) \Phi_{n-1}(\lambda_2, \ldots, \lambda_n|z) - \mathcal{X}^-(\lambda_1|z) \sum_{j=2}^{n} \frac{(-)^j}{\lambda_1 - \lambda_j} \Phi_{n-2}(\lambda_2, \ldots, \lambda_j, \ldots, \lambda_n|z)
\] (3.16)

with \(\Phi_0 = |0\rangle\) and \(\Phi_1(\lambda_1|z) = \mathcal{V}^-(\lambda_1|z)|0\rangle\).

Here we would like make a few comments on the structure of these vector-valued functions. In (3.16) they are written in a normal ordered form. Since we are working with fermionic degree of freedom, the function \(\Phi_n(\lambda_1, \ldots, \lambda_n|z)\) is totally antisymmetric.

\[
\Phi_n(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n|z) = -\Phi_n(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_n|z)
\] (3.17)

To see this one can use the Gaudin algebra (3.13). For instance, in its antisymmetric form the Bethe vector \(\Phi_2\) reads as
\[
\Phi_2(\lambda_1, \lambda_2|z) = \frac{1}{2} [\mathcal{V}^-(\lambda_1|z) \mathcal{V}^-(\lambda_2|z) - \mathcal{V}^-(\lambda_2|z) \mathcal{V}^-(\lambda_1|z)] |0\rangle
\]
\[
-\frac{1}{2} \left[ \frac{\mathcal{X}^-(\lambda_1|z) + \mathcal{X}^-(\lambda_2|z)}{\lambda_1 - \lambda_2} \right] |0\rangle
\] (3.18)

Next we consider the semiclassical expansions of the \(c\)-number functions (2.24), (2.26) and (2.28) presents in the OSBAE (2.22)
\[
\Lambda = 1 + 2\eta^2 \Lambda^{(2)} + o(\eta^3)
\]
\[
\mathcal{F}^{(n)}_j = \eta(-)^j f^{(n)}_j + o(\eta^2), \quad \mathcal{F}^{(n)}_{ij} = 2\eta^3(-)^{j+l} F^{(n)}_{ij} + o(\eta^4)
\] (3.19)

where
\[
\Lambda^{(2)} = \sum_{a=1}^{N} \frac{1}{(\lambda - z_a)} \left( \sum_{\substack{b \neq a \times \frac{1}{z_a - z_b}}} \frac{1}{\lambda - \lambda_a} \right) + \sum_{k=1}^{n} \frac{1}{(\lambda - \lambda_k)} \left( \sum_{\substack{j \neq k \times \frac{1}{\lambda_k - \lambda_j}}} \frac{1}{\lambda - \lambda_k} \right)
\]
\[
+ \sum_{a=1}^{N} \sum_{k=1}^{n} \frac{1}{\lambda - z_a} \frac{1}{\lambda_k - \lambda} + \frac{3}{4} \sum_{a=1}^{N} \frac{1}{(\lambda - z_a)^2}
\] (3.20)

and
\[
f^{(n)}_j = \sum_{k \neq j} \frac{1}{\lambda_j - \lambda_k} - \sum_{a=1}^{N} \frac{1}{\lambda_j - z_a}
\]
\[
F^{(n)}_{ij} = \frac{f^{(n)}_j}{\lambda - \lambda_j} \frac{1}{\lambda_j - \lambda_i} - \frac{f^{(n)}_i}{\lambda - \lambda_i} \frac{1}{\lambda_i - \lambda_j}
\] (3.21)
Substituting these expressions into the (2.22) and comparing the coefficients of the terms $2(-\eta)^{n+2}$ we get the first non-trivial consequence for the semiclassical limit of the OSBAE:

$$\sum_{a=1}^{N} \frac{G_a}{\lambda - z_a} \Phi_n(\lambda_1, ..., \lambda_n) = \Lambda^{(2)} \Phi_n(\lambda_1, ..., \lambda_n) + \sum_{j=1}^{n} \frac{(-)^l f_l^{(n)} \Phi^j_n}{\lambda - \lambda_j}$$

(3.22)

where

$$\Phi^j_n = \mathcal{V}^-(\lambda|z) \Phi_{n-1}(\lambda_1, ..., \hat{\lambda_j}, ..., \lambda_n|z)$$

$$-\mathcal{X}^- (\lambda|z) \sum_{l \neq j}^{n} \frac{(-)^{\tilde{l}}}{\lambda_j - \lambda_l} \Phi_{n-2}(\lambda_1, ..., \hat{\lambda_l}, ..., \hat{\lambda_j}, ..., \lambda_n|z)$$

(3.23)

with $\tilde{l} = l + 1$ for $l < j$ or $\tilde{l} = l$ for $l > j$.

Here we observe that the OSBAE (3.22) is similar to the OSBAE presented by Babujian and Flume for simple Lie algebras. This result could be expected since the superalgebra $osp(1|2)$ has many features which make it very close to the Lie algebra $[23]$.

Now, we take the residue of (3.22) at the point $\lambda = z_a$ to get the Gaudin off-shell equations:

$$G_a \Phi_n(\lambda_1, ..., \lambda_n|z) = g_a \Phi_n(\lambda_1, ..., \lambda_n|z) + \sum_{l=1}^{n} \frac{(-)^l f_l^{(n)} \phi^j_n}{z_a - \lambda_l}$$

_for $a = 1, 2, ..., N$$

(3.24)

where $f_l^{(n)}$ is given by (3.21), $g_a$ is the residue of $\Lambda^{(2)}$ (3.20) at $\lambda = z_a$

$$g_a = \sum_{b \neq a}^{N} \frac{1}{z_a - z_b} - \sum_{l=1}^{n} \frac{1}{z_a - \lambda_l}$$

(3.25)

and $\phi^j_n$ is the residue of $\Phi^j_n$ (3.23) at $\lambda = z_a$

$$\phi^j_n = \mathcal{V}^+_j \Phi_{n-1}(\hat{\lambda_j}|z) - \mathcal{X}^-_j \sum_{l \neq j}^{n} \frac{(-)^{\tilde{l}}}{\lambda_j - \lambda_l} \Phi_{n-2}(\hat{\lambda_l}, \hat{\lambda_j}|z)$$

(3.26)

In this way we arrive to one of the main problems solved by the theory of the Gaudin model, i.e., the determination of the eigenvalues and eigenvectors of the commuting Hamiltonians $G_a$ (3.5): $g_a$ is the eigenvalue of $G_a$ with eigenfunction $\Phi_n$ provided $\lambda_l$ are solutions of the Bethe ansatz equations

$$f_l^{(n)} = 0 \Rightarrow \sum_{k \neq l}^{n} \frac{1}{\lambda_l - \lambda_k} = \sum_{a=1}^{N} \frac{1}{\lambda_l - z_a}, \quad l = 1, 2, ..., n.$$

(3.27)

These Bethe ansatz equations are the semiclassical limit of the inhomogeneous Bethe ansatz equations (2.30).
4 Knizhnik-Zamolodchikov Equation

Since the spectra of the \(osp(1|2)\) Gaudin Hamiltonians are now known, the next step is to calculate correlation functions.

Before we present the result of this section, let us observe a very important consequence due to semiclassical approach. While the solutions of the quantum YB equation \((1.1)\) depend essentially on the representation, the corresponding classical solutions \(r_{ij}(\lambda) \in g_i \otimes g_j\), \(g\) being a Lie algebra, can be written in an invariant form, i.e. independent of the representation. Therefore, one may extend the previous semiclassical results beyond the fundamental representation. To do this we first define a \(r\)-matrix and the corresponding \(L\)-operator for the next representation \((s = 1)\) of the \(osp(1|2)\) algebra presented in \((2.3)\). The classical \(r\)-matrix is constructed out of the quadratic Casimir in a standard way \([13]\),

\[
r(\lambda - \mu) = \frac{1}{\lambda - \mu} [\mathcal{H}^s \otimes \mathcal{H} + \frac{1}{2} (\mathcal{X}^+ \otimes \mathcal{X}^- + \mathcal{X}^- \otimes \mathcal{X}^+) + \mathcal{V}^+ \otimes \mathcal{V}^- - \mathcal{V}^- \otimes \mathcal{V}^+] \quad (4.1)
\]

Using the basis \((2.3)\) with the grading BFBFB one can verify that this classical \(r\)-matrix satisfies the graded version of the classical YB equation \((1.2)\). To write \((3.11)\) in the form \((4.1)\), we recall the symmetries of the YB solution \((2.12)\) by considering the case \(\epsilon_1 = 1\) and \(\epsilon_2 = -1\).

The classical \(L\)-operator is obtained from the semiclassical limit of the canonical identification of the Lax operator \(\mathcal{L}\) with \(R\)-matrix \([4, 14]\). In the \(s = 1\) representation it has the form

\[
L(\lambda|z) = \begin{pmatrix}
2\mathcal{H}(\lambda|z) & \sqrt{2}\mathcal{V}^-(\lambda|z) & \sqrt{2}\mathcal{X}^-(\lambda|z) & 0 & 0 \\
-\sqrt{2}\mathcal{V}^+(\lambda|z) & \mathcal{H}(\lambda|z) & \mathcal{V}^-(\lambda|z) & \mathcal{X}^-(\lambda|z) & 0 \\
\sqrt{2}\mathcal{X}^+(\lambda|z) & \mathcal{V}^+(\lambda|z) & 0 & \mathcal{V}^-(\lambda|z) & \sqrt{2}\mathcal{X}^-(\lambda|z) \\
0 & \mathcal{X}^+(\lambda|z) & -\mathcal{V}^+(\lambda|z) & -\mathcal{H}(\lambda|z) & \sqrt{2}\mathcal{V}^-(\lambda|z) \\
0 & 0 & \sqrt{2}\mathcal{X}^+(\lambda|z) & \sqrt{2}\mathcal{V}^+(\lambda|z) & -2\mathcal{H}(\lambda|z)
\end{pmatrix}
\quad (4.2)
\]

Substituting \((4.2)\) and \((4.1)\) into the fundamental relation \((3.12)\) we will get the same defining relations of the \(osp(1|2)\) Gaudin algebra \((3.13)\).

The fundamental property of the operators \((3.10)\) is that they form the highest weight module of the infinite-dimensional Lie superalgebra. As in the \(sl(2)\) case presented by Sklyanin in \([24]\), it is characterized by the vacuum \(|0\rangle\), its dual \(\langle 0|\)

\[
\mathcal{H}(\lambda|z) |0\rangle = h(\lambda|z) |0\rangle, \quad \mathcal{V}^+(\lambda|z) |0\rangle = 0, \quad \mathcal{X}^+(\lambda|z) |0\rangle = 0
\]

\[
\langle 0| \mathcal{H}(\lambda|z) = h(\lambda|z) \langle 0|, \quad \langle 0| \mathcal{V}^-(\lambda|z) = 0, \quad \langle 0| \mathcal{X}^-(\lambda|z) = 0 \quad (4.3)
\]

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and the highest weight functions \( h(\lambda|z) \)

\[
h(\lambda|z) = \sum_{a=1}^{N} \frac{2s_a}{\lambda - z_a}, \quad s_a = \frac{1}{2}, 1, \frac{3}{2}, ... \quad (4.4)
\]

In this way we are going beyond the fundamental representation \( s_a = \frac{1}{2} \).

The semiclassical expansion of \( t(\lambda|z) \) in the \( s \)-representation is easily obtained from (3.4)

\[
t(\lambda|z) = 1 + 2\eta^2 \left\{ \sum_{a=1}^{N} \frac{G_a}{\lambda - z_a} + \sum_{a=1}^{N} \frac{s_a(s_a + 1)}{2(\lambda - z_a)^2} \right\} + o(\eta^3) \quad (4.5)
\]

Here we remark that this result is general. It comes from the structure of the semiclassical method, where \( t(\lambda|z) \) is defined as supertrace of \( \frac{1}{2}L^2(\lambda|z) \) and the second sum in (4.5) is identified with the Casimir of the subalgebra \( sl(2) \). Consequently, the Gaudin off-shell equations in the \( s \)-representation are known. They are again given by (3.24), where the coefficients \( f^{(n)}_l \) and \( g_a \) are

\[
f^{(n)}_l = \sum_{k \neq l} \frac{1}{\lambda_l - \lambda_k} - 2 \sum_{a=1}^{N} \frac{s_a}{\lambda_l - z_a},
\]

\[
g_a = 4 \sum_{b \neq a} \frac{s_a s_b}{z_a - z_b} - 2 \sum_{l=1}^{n} \frac{s_a}{z_a - \lambda_l} \quad (4.6)
\]

and \( \Phi_n(\lambda_1, ..., \lambda_n) \) for \( s \)-representations again has the form as Eq.(3.16).

Let us now define the vector-valued function \( \Psi(z_1, ..., z_N) \) through multiple contour integrals of the Bethe vectors (3.16)

\[
\Psi(z_1, ..., z_N) = \oint \cdots \oint X(\lambda|z) \Phi_n(\lambda|z) d\lambda_1...d\lambda_N \quad (4.7)
\]

where \( X(\lambda|z) = \mathcal{X}(\lambda_1, ..., \lambda_n, z_1, ..., z_N) \) is a scalar function which in this stage is still undefined.

In analogy with the \( sl(2) \) case [4], we assume that \( \Psi(z_1, ..., z_N) \) is a solution of the equations

\[
\kappa \frac{\partial \Psi(z_1, ..., z_N)}{\partial z_a} = G_a \Psi(z_1, ..., z_N), \quad a = 1, 2, ..., N \quad (4.8)
\]

where \( G_a \) are the Gaudin Hamiltonians (3.3) and \( \kappa \) is a constant.

By construction, these equations are the graded version of the KZ equations [25, 26]. Hence, we can interpret \( \Psi(z_1, ..., z_N) \) as an integral representation for the \( N \)-point correlation function in an chiral conformal field theory with \( osp(1|2) \) symmetry [27, 28].

Substituting (1.7) into (1.8) we have

\[
\kappa \frac{\partial \Psi(z_1, ..., z_N)}{\partial z_a} = \oint \cdots \oint \left[ \kappa \frac{\partial \mathcal{X}(\lambda|z)}{\partial z_a} \Phi_n(\lambda|z) + \kappa \mathcal{X}(\lambda|z) \frac{\partial \Phi_n(\lambda|z)}{\partial z_a} \right] d\lambda_1...d\lambda_N \quad (4.9)
\]
From (3.16) and using the Gaudin algebra (3.9) one can derive the following identity
\[ \frac{\partial \Phi_n}{\partial z_a} = \sum_{l=1}^{n} (-)^l \frac{\partial}{\partial \lambda_l} \left( \frac{\phi^l_n}{\lambda_l - z_a} \right) \] (4.10)
which allows us write (4.9) in the form
\[ \kappa \frac{\partial \Psi}{\partial z_a} = \oint \cdot \cdot \cdot \oint \left[ \kappa \frac{\partial \mathcal{X}(\lambda|z)}{\partial z_a} \Phi_n(\lambda|z) - \sum_{l=1}^{n} (-)^l \kappa \frac{\partial \mathcal{X}(\lambda|z)}{\partial \lambda_l} \left( \frac{\phi^l_n}{\lambda_l - z_a} \right) \right] d\lambda_1...d\lambda_n + \kappa \oint \cdot \cdot \cdot \oint \left[ \sum_{l=1}^{n} (-)^l \frac{\partial}{\partial \lambda_l} \left( \mathcal{X}(\lambda|z) \phi^l_n \right) \right] d\lambda_1...d\lambda_n \] (4.11)
It is evident that the last term of (4.11) is vanishes, because the contours are closed. Moreover, if the scalar function \( \mathcal{X}(\lambda|z) \) satisfies the following differential equations
\[ \kappa \frac{\partial \mathcal{X}(\lambda|z)}{\partial z_a} = g_a \mathcal{X}(\lambda|z), \quad \kappa \frac{\partial \mathcal{X}(\lambda|z)}{\partial \lambda_j} = f_j^{(n)} \mathcal{X}(\lambda|z), \] (4.12)
we are recovering the off-shell Bethe ansatz equations for the Gaudin magnets (3.24) from the first term in (4.11). Taking into account (4.9) we can see that the consistency condition of the system (4.12) is insured by the zero curvature conditions \( \partial f_j^{(n)}/\partial z_a = \partial g_a/\partial \lambda_j \).

The solution of (4.12) is
\[ \mathcal{X}(\lambda|z) = \prod_{a<b}^{N} (z_a - z_b)^{4s_a s_b/\kappa} \prod_{j<k}^{n} (\lambda_j - \lambda_k)^{1/\kappa} \prod_{a,j}^{N,n} (z_a - \lambda_j)^{-2s_a/\kappa}. \] (4.13)
This function determines the monodromy of \( \Psi(z_1, ..., z_N) \) as solution of the \( osp(1|2) \) KZ equation (4.8) and these results are in agreement with the Schechtman-Varchenko construction for multiple contour integral as solutions of the KZ equation in an arbitrary simple Lie algebra [10].

5 Conclusion

In the first part of this paper we have considered an integrable representation of the \( osp(1|2) \) algebra to study the algebraic Bethe ansatz for an inhomogeneous vertex model.

In the second part, through the semiclassical limit of the quantum inverse scattering equations we got arrive to the theory of the Gaudin model. Here, with aid of the Gaudin algebra, we extended the results presented in the first part for highest representations and find the spectra of \( N \) commuting Hamiltonians.
We believe that the procedure presented here will work for all models for which the usual algebraic Bethe ansatz is known. Therefore, it should be interesting extend this procedure for all algebraic versions of the Bethe ansatz. In particular, the natural candidate will be the nested algebraic Bethe ansatz [29]. In [30], a system of matrix difference equations was solved by means of a nested version of a off-shell Bethe ansatz.

Regarding our results about the $N$-point correlators in the $osp(1|2)$ conformal field theory, there are several issues left for future works. For instance, we would like to connect this result with those presented in the $osp(1|2)$ current algebra [31, 32]. It should be interesting since it appears in the quantization of two-dimensional supergravity in the light-cone gauge [33]. Moreover, the affine version of the $osp(1|2)$ algebra is the starting point in the construction of the $N = 1$ superconformal minimal models by means of the Hamiltonian reduction procedure [34].

Acknowledgment: This work was supported in part by Fundação de Amparo à Pesquisa do Estado de São Paulo–FAPESP–Brasil and by Conselho Nacional de Desenvolvimento–CNPq–Brasil.

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