1 Introduction

Consider a polynomial ordinary differential equation (ODE)

\[
\frac{dx}{dt} = f(x)
\]  

In this paper we address the question

Q1: What rational conserved integral(s) and (inverse) polynomial volume form (if any) does the ODE possess?

Since finding rational integrals generally requires solving a nonlinear problem we propose a three step program, that, using a certain ansatz, only requires the solution of linear problems:

Step 1: Discretise the ODE using a “suitable” method. In this paper we will use Kahan’s method (but much of the following also holds for certain other birational integration methods given in the references). Compute the Jacobian determinant \( J \) of the discretisation, and factorise \( J \).

Step 2: Use the factors of \( J \) as candidate discrete cofactors for finding discrete Darboux polynomials (DPs).

Step 3: Take the continuum limits of the discrete cofactors and DPs found in step 2. If possible, use these DPs as building blocks for time-dependent/time-independent first integrals and preserved measure of the ODE. If one is very lucky, it may even be possible to use them to derive the exact solution to the initial value problem for the ODE.

2 Step 1

Our ongoing example in this paper will be the ODE

\[
\begin{align*}
\dot{x} &= 2 - 2x + xz \\
\dot{y} &= -y + yz \\
\dot{z} &= -y - 3z + z^2
\end{align*}
\]  

(2)
For any quadratic ODE:
\[
\frac{dx_i}{dt} = \sum_{j,k} a_{ijk} x_j x_k + \sum_j b_{ij} x_j + c_i
\]
(3)

Kahan’s “unconventional” method is defined by
\[
\frac{x_i' - x_i}{h} = \sum_{j,k} a_{ijk} \frac{x_j' x_k + x_j x_k'}{2} + \sum_j b_{ij} \frac{x_j + x_j'}{2} + c_i
\]
(4)

here \( x_i := x_i(nh), \ x_i' := x_i((n+1)h), \) and \( h \) is the timestep.

It is not hard to show that eq(4) can be rearranged as follows:
\[
\frac{x_i' - x_i}{h} = \left( I - \frac{h}{2} \frac{f'(x)}{f(x)} \right)^{-1} f(x),
\]
(5)

This defines the Kahan map \( x_{n+1} = \phi(x_n) \) [1].

Next we compute the Jacobian determinant \( J \) of \( \phi \):
\[
J(x) = \left| \frac{\partial \phi_i(x)}{\partial x_j} \right|,
\]
(6)

and use an algebraic manipulation package to factorise \( J \).

For our example
\[
J(x) = \frac{K_1 K_2 K_3 K_4}{D_1 D_2^4},
\]
(7)

where \( K_i, D_j(i = 1, \ldots, 4; j = 1, 2) \) are given in the appendix.

3 Step 2

Given a map \( x_{n+1} = \phi(x_n) \), a polynomial \( P(x) \) is a (discrete) DP of \( \phi \) if there exists a (non-tautological) rational function \( C \) s.t.
\[
P(x_{n+1}) = C(x_n) P(x_n),
\]
(8)

where \( P(x_{n+1}) = P(\phi(x_n)) \) and \( C \) is called the (discrete) cofactor of \( P \) [2, 3].
### Table 1:

| i | $C_i$ | $P_i$ | $C_i$ |
|---|---|---|---|
| 1 | $\frac{K_1}{D_1}$ | $z - y - 3$ | $z$ |
| 2 | $\frac{K_2}{D_2}$ | $2z + y$ | $z - 3$ |
| 3 | $\frac{K_3}{D_3}$ | $y$ | $z - 1$ |
| 4 | $\frac{K_4}{D_4}$ | $x + y + z - 1$ | $z - 2$ |

Ansatz: Given a rational map $\phi$ with Jacobian determinant

$$J(x) = \frac{\prod_{i=1}^{l} K_i^{f_i}(x)}{\prod_{j=1}^{m} D_j^{g_j}(x)},$$

we try all cofactors (up to a certain polynomial degree $d$) of the form

$$C(x) = \pm \frac{\prod_{i=1}^{l} K_i^{f_i}(x)}{\prod_{j=1}^{m} D_j^{g_j}(x)},$$

where $f_i, g_j \in \mathbb{N}_0$.

NOTE:

(a) There is a finite number of these co-factors up to a certain degree. For each of this finite number of co-factors, we only need to solve a linear problem (up to a chosen degree)!

(b) If $C(x) = J(x)$, the corresponding Darboux polynomials are inverse densities of preserved measures.

The discrete cofactors $C_i$ and corresponding DPs $P_i$ for our example are given in the first two columns of Table 1:

### 4 Step 3

The continuum limits $\bar{P}_i$, $\bar{C}_i$ are given by $\lim_{h \to 0} P_i$ resp. $\lim_{h \to 0} \frac{C_i - 1}{h}$, and satisfy the ODEs

$$\dot{\bar{P}}_i = \bar{C}_i \bar{P}_i,$$  \hspace{1cm} (11)

A useful property of the cofactor $\bar{C}_i$ is

$$\dot{\bar{P}}_i = \bar{C}_i \bar{P}_i \rightarrow \dot{\bar{P}} = \bar{C} \bar{P}$$  \hspace{1cm} (12)
where
\[
P := \prod_i \tilde{P}_i^{\alpha_i}, C := \sum_i \alpha_i \tilde{C}_i
\] (13)

This has the following implications:

(a) \( C(x) = 0 \rightarrow \dot{P} = 0 \rightarrow P \) is a first integral

(b) \( C(x) = C \rightarrow \dot{P} = CP \rightarrow P(x(t)) = P(x(0))e^{Ct} \)

(c) \( C(x) = div f(x) \rightarrow \dot{P} = CP \rightarrow f \) preserves the measure \( \frac{dx}{P(x)} \)

For our problem, the \( \tilde{C}_i \) are given in the last column of Table 1. (For affine DPs, \( \tilde{P}_i = P_i \). For two theorems regarding affine DPs, cf [2]).

Note that
\[
\tilde{C}_1 - \tilde{C}_2 = 3, \tilde{C}_1 - \tilde{C}_3 = 1, \tilde{C}_1 - \tilde{C}_4 = 2
\] (14)

Hence
\[
\begin{align*}
P_1 & = \frac{z - y - 3}{2z + y} = k_2 e^{3t} \\
P_2 & = \frac{z - y - 3}{y} = k_3 e^t \\
P_3 & = \frac{z - y - 3}{x + y + z - 1} = k_4 e^{2t}
\end{align*}
\] (15)

and this yields 2 time-independent first integrals:
\[
\begin{align*}
I_1 & = \frac{P_2^2}{P_1 P_4} = \frac{y^2}{(z - y - 3)(x + y + z - 1)} \\
I_2 & = \frac{P_3 P_4}{P_1 P_2} = \frac{y(x + y + z - 1)}{(z - y - 3)(2z + y)}
\end{align*}
\] (18)

Hence \( f \) is integrable. Moreover \( J = C_1 C_2 C_3 C_4 \) implies that \( f \) preserves the measure
\[
\frac{dx dy dz}{P_1 P_2 P_3 P_4} = \frac{dx dy dz}{y(2z + y)(z - y - 3)(x + y + z - 1)}
\] (20)

Equations (15), (16), & (17) can be combined to give the explicit solution of ODE (2)
\[
\begin{align*}
x & = \frac{6e^{2t}k_2 k_2 + 2 (-3e^t k_2 + (1 + k_2 e^{3t}) k_4) k_3}{k_4 (2e^{3t} k_2 k_3 + 3e^{2t} k_2 - k_3)} \\
y & = \frac{6k_2 e^{2t}}{-2e^{3t} k_2 k_3 - 3e^{2t} k_2 + k_3} \\
z & = \frac{-3k_3 + 3e^{2t} k_2}{2e^{3t} k_2 k_3 + 3e^{2t} k_2 - k_3}
\end{align*}
\] (21)

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APPENDIX

The explicit factors of $J(x)$ in eq (7) are:

$$
K_1 = -\frac{1}{4} h^2 x_2 + \frac{3}{4} h^2 x_3 - \frac{3}{4} h^2 - \frac{1}{2} h x_3 - h + 1
$$

$$
K_2 = -\frac{1}{4} h^2 x_2 - \frac{1}{4} h^2 x_3 - \frac{3}{4} h^2 - \frac{1}{2} h x_3 + h + 1
$$

$$
K_3 = -\frac{1}{4} h^2 x_2 - 3/4 h^2 x_3 + 3/4 h^2 - 1/2 h x_3 + 2 h + 1
$$

$$
K_4 = 1/8 h^3 x_2 x_3 - 1/8 h^3 x_3^2 - 1/4 h^3 x_2 + \frac{7}{8} h^3 x_3 + 1/4 h^2 x_3^2 - 3/4 h^3
$$

$$
-1/4 h^2 x_2 - 1/4 h^2 x_3 - 5/4 h^2 - h x_3 + h + 1
$$

$$
D_1 = -\frac{1}{2} h x_3 + h + 1
$$

$$
D_2 = 1/2 h^2 x_3^2 + 1/4 h^2 x_2 - 5/4 h^2 x_3 + 3/4 h^2 - 3/2 h x_3 + 2 h + 1
$$

References

[1] Celledoni E, McLachlan RI, Owren B, Quispel, GRW 2013, Geometric properties of Kahan’s method J. Phys. A 46 12 025201

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[4] Goriely A, Integrability and Nonintegrability of Dynamical Systems, World Scientific Publishing, Singapore (2001), Section 2.5