Aspects of Planckian Scattering beyond the Eikonal

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Abstract

We discuss an approach to compute two-particle scattering amplitudes for spinless particles colliding at Planckian centre-of-mass energies, with increasing momentum transfer away from the eikonal limit. For electrically neutral particles, the amplitude exhibits poles on the imaginary squared cm energy axis at locations that are distinct from those appearing in the eikonal limit. For charged particles, electromagnetic and gravitational effects remain decoupled for the eikonal situation as also the leading order (in momentum transfer, or equivalently, the impact parameter) correction, but mix non-trivially for higher orders.
I. INTRODUCTION

The efficacy of the shock wave picture \cite{1} in the computation of two-particle scattering amplitudes \cite{2} for large $s$ (squared centre-of-mass energy) and small, fixed $t$ (squared momentum transfer), in the eikonal limit $\frac{s}{t} \to \infty$, is now well-established, both for gravitational and electromagnetic interactions \cite{3-8}. The graviton (photon) exchange ladder graphs neatly sum in this kinematical limit to reproduce exactly the semiclassical amplitude of the relatively slower test particle scattering off the gravitational (electromagnetic) shock wave due to the ultrarelativistic ‘source’ particle. Phenomena beyond this highly restrictive kinematical regime (e.g., for higher values of $t$) entail, for their analysis, a calculational scheme for systematic corrections to the eikonal. For electrodynamics, with values of $t$ still sub-Planckian, this is afforded easily by the usual perturbative formulation of quantum electrodynamics. For gravity and electrodynamics of electric and magnetic charges, the lack of a proper local quantum field theory is a major setback to this programme. On the other hand, a determination of corrections to the eikonal is essential to unravel certain features of eikonal scattering itself, like the analytic structure (in complex $s$-plane) of the eikonal amplitude \cite{2}, or the possible interplay between electromagnetic and gravitational effects for charged particle eikonal scattering \cite{9}. One approach which has probed the first of these features with some success is the one based on reggeized string exchange amplitudes with subsequent reduction to the gravitational eikonal limit including the leading order corrections \cite{3}. In this letter, we follow a somewhat different approach \cite{10} : the scattering amplitude is calculated quantum \textit{mechanically} by solving the Klein Gordon equation of the ultrarelativistic particle in the linearized classical Schwarzschild background of the slower ‘target’ particle in the appropriate Lorentz frame. Recall that the role of the scattering particles is the opposite to that in the shock wave picture \cite{2} where the slower particle scatters off the shock wave due to the luminal one. But this switching allows us to investigate leading corrections to the eikonal. The restriction to the linearized Schwarzschild background essentially delineates the inherent limitation in our approach vis-a-vis large (e.g. Planckian) momentum transfers;
the latter situation does indeed require a full quantum theory of gravity, and is therefore not immediately tractable. Admittedly, our approach has been anticipated in analyzing gravitational eikonal scattering in earlier work [7]. Our intention in what follows is to consider implications of this ‘Coulomb scattering’ technique beyond the eikonal.

II. PURELY GRAVITATIONAL SCATTERING

The massless generally covariant Klein Gordon equation for the ultrarelativistic ‘test’ particle is given by

\[ D_\mu D^\mu \phi = 0 \]  
\[ (1) \]

In the classical Schwarzschild background of the slow target particle (of mass \( M \) which is also considered small in comparison with \( \sqrt{s} \))

\[ ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) , \]
\[ (2) \]
we assume a solution of the Klein Gordon wave function of the form

\[ \phi (\vec{r}, t) = \frac{f(r)}{r} e^{iEt} Y_{lm} (\theta, \phi) , \]

where \( E \) is the energy of the test particle as measured by an asymptotic observer. On linearizing the Schwarzschild metric, substituting \( s = 2ME \) and discarding terms proportional to \( (2GM/r)^2 \) or higher powers thereof, we finally obtain the radial part of the wave equation as:

\[ \frac{d^2 f(r)}{dr^2} - \left[ \frac{l(l+1) - G^2 s^2}{r^2} - \frac{2GsE}{r} - E^2 \right] f(r) = 0 . \]
\[ (3) \]
Thus, terms with inverse powers in \( r \) higher than 2 have been dropped. This enables us to solve the resulting eq. (3) without further approximations, while keeping in mind that very small (Planck size) impact parameter scattering cannot be probed thus.

The radial equation (3) above is solved using standard techniques [11], [12] in terms of hypergeometric functions with well-known asymptotic properties. The scattering amplitude
is best expressed in terms of a partial wave expansion, in view of the spherical symmetry of
the ‘potential’ above,

\[ f(\theta) = \frac{1}{2i\sqrt{s}} \sum_{l=0}^{\infty} (2l + 1) \left[ e^{2i\delta_l} - 1 \right] P_l(\cos \theta), \tag{4} \]

where, the phase shift of the partial wave, characterized by a fixed angular momentum
quantum number \( l \gg 1 \), is given by

\[ \delta_l(s) = \arg \Gamma (p_l(s) + 1 - iGs), \tag{5} \]

with \( p_l(s) \) defined by the relation

\[ p_l(s)(p_l(s) + 1) \equiv l(l+1) - G^2 s^2. \tag{6} \]

It is not difficult to show from Eqs. (4) and (6) that, for fixed \( l \), the phase shift has
singularities at cm energies

\[ Gs = \frac{i}{(2N+1)} \left[ l(l+1) - N(N+1) \right], \tag{7} \]

for any non-negative integer \( N \). Although still located on the imaginary axis of the complex
\( s \)-plane, clearly the locations of these poles are quite distinct from those seen in the eikonal
limit \(2\), viz., at \( Gs = -iN \). There is also another distinction: the poles discerned by us
are singularities of the phase shift (for fixed \( l \)) and therefore are physically more appealing
(i.e., they are most likely actual resonances) than the eikonal (large \( l \)) poles which are not
singularities of the phase shift \(3, 4\). Recall that, for eikonal scattering, \( \delta_l \sim \log l \), but it is
perhaps incorrect to suggest that the amplitude has an \( s \)-wave pole because such a low range
of \( l \) cannot be probed within the eikonal approximation. In contrast, while we also cannot
probe very low values of \( l \), the poles do arise for intermediate impact parameter ranges, in
the phase shifts themselves. We do not claim a full understanding of the origin of these
poles, but still feel it useful to point out their existence outside the eikonal limit.

The formulas above also permit us to extract the leading order corrections to the eikonal
limit \( l \to \infty \), by using the asymptotic expansion of the argument of the gamma function
\(13\) in increasing inverse powers of \( l \). We obtain
\[ \delta_t \approx -Gs \left( \log l - \frac{1}{2l} \right) + \frac{(Gs)^3}{2l^2} + O \left( \frac{1}{l^3} \right). \quad (8) \]

The first term in eq. (8) obviously corresponds to the eikonal result, and the sub-leading corrections have been anticipated from reggeized string exchange diagrams \[3\]. The leading correction above to the eikonal phase shift behaves \(Gs/l\). This is somewhat different from the leading correction as seen in the string theory based approach, which is proportional to \(\frac{(Gs)^2}{l} \log s\). In our quantum mechanical approach we do not expect to obtain \(\log s\) corrections; one needs the formalism of quantum field theory for that purpose. However, it is a bit surprising that a \(1/l\) type correction is not obtained in the approach of \[3\]. One possible explanation could be that the string theory in question is quantized around a flat, rather than a Schwarzschild, background and therefore misses this effect. Even if this were true, it is not easy to determine the corresponding spacetime geometry around the target particle beyond the eikonal limit. That is to say, it remains to be seen whether the correction we have found can be interpreted as a contribution to the shift of the appropriate null coordinate found in \[1\] which has a step function discontinuity in the other null coordinate, or is it a smearing of the shock wave found in the eikonal limit, by exchange of transverse gravitons, as seen in \[3\].

The asymptotic behaviour observed in eq. (8) can now be translated easily to calculate the scattering amplitude to incorporate the leading order correction; the partial wave sum is replaced by the integral over the impact parameter \(b \equiv l/E\), with the phase shift being replaced by the first two terms in (8). If, once again, the integral is taken between 0 and \(\infty\) as in \[4\], it can be performed exactly, leading to the result

\[ f(s, t) = f^{(0)}(s, t) + f^{(1)}(s, t) + ..., \]

where \(f^{(0)}(s, t)\) is the eikonal amplitude. The expressions for these amplitudes are

\[ f^{(0)} = \frac{Gs^{3/2}}{2t} \frac{\Gamma(1 - iGs)}{\Gamma(1 + iGs)} \left( \frac{-t}{s} \right)^{iGs} \]
\[ f^{(1)} = -\frac{Gs}{\sqrt{-t}} \frac{\Gamma(1/2 - iGs)}{\Gamma(1/2 + iGs)} \left( \frac{-t}{s} \right)^{iGs}. \quad (9) \]
Thus, the eikonal poles are again manifest at integral values (in Planck units) on the imaginary axis of the complex $s$-plane; in addition, one also observes poles, again originating from the lower limit of the integration ($b = 0$), at half-integral values on the imaginary $s$-axis. As remarked above, our technique cannot illuminate the really small impact parameter regime, and thus these poles are not expected to indicate true resonances because the phase shift, in the large $l$ approximation, has no singularities. Therefore, we have very little to add to the extant wisdom [3], [5] on the issue of singularities of the eikonal amplitude.

### III. INCLUSION OF ELECTROMAGNETISM

Another key issue in Planckian scattering, and one which has not received too much attention, is that of mixing of gravitational and electromagnetic effects in the eikonal approximation. In the earlier literature [2], [8], it was assumed that in the eikonal limit, the gravitational and electromagnetic shock waves acted quite independently, producing a net phase factor in the wave function of the test particle that was a sum of the individual phase factors. Since generically gravity couples to everything including electromagnetism, it becomes important to ascertain whether the assumed independence of the two interactions in the special kinematics of the eikonal limit, really holds. This issue was first addressed in [3] where heuristic arguments were advanced to show that the assumed decoupling did indeed take place, thus vindicating results obtained using this crucial assumption. The present framework provides a less heuristic avenue to re-examine this question, and allows us to establish the earlier conclusions on a sounder footing. In addition, the decoupling of gravitational and electromagnetic effects is seen to persist through the leading order (in inverse powers of the impact parameter) correction to the eikonal.

As in [3], one begins by considering first the scattering of a (luminal) neutral test particle off the Reissner-Nordström metric due to a static point charge. The Klein-Gordon equation of the fast particle can again be written down by replacing the spacetime derivatives by generally covariant derivatives appropriate to the Reissner-Nordström metric.
\[ ds^2 = -(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2})dt^2 + (1 - \frac{2GM}{r} + \frac{GQ^2}{r^2})^{-1}dr^2 + r^2d\Omega^2, \] (10)

where \( d\Omega^2 \) is the metric on the unit two-sphere. Once again, confining ourselves to impact parameters that are large compared to the length scale \( 2GM \) and charges that are of order the electronic charge, the radial equation reduces to

\[
\frac{d^2 f(r)}{dr^2} - \left[ \frac{l(l+1) + 2GQ^2E^2 - G^2s^2}{r^2} - \frac{2GsE}{r} - E^2 \right] f(r) = 0.
\] (11)

The phase shift, for \( l \gg 1 \) are once again given by eq. (5), where, now

\[
p_l(s) (p_l(s) + 1) \equiv l(l+1) - (\zeta Gs)^2
\] (12)

with \( \zeta^2 \equiv 1 - Q^2/2GM^2 \). Clearly, \( \zeta = 1 \) is the reduction to the Schwarzschild case. It is not difficult to show that the phase shift singularities now occur not only on the imaginary axis of the complex \( s \)-plane, but elsewhere in the plane as well:

\[
Gs = \frac{i}{2} \left( \frac{2N + 1}{1 - \zeta^2} \right) \left\{ -1 + \left[ 1 - 4i \left( \frac{1 - \zeta^2}{2N+1} \right) (Gs)_0 \right]^{\frac{1}{2}} \right\},
\] (13)

where, \( (Gs)_0 \) signifies the location of the poles in the Schwarzschild case, given by eq. (7). The \( \zeta \to 1 \) limit to the Schwarzschild case is again obvious. Apart from reporting the existence of these poles as singularities of the phase shift (for intermediate impact parameter ranges), we are unable, at this point, to delve deeper into their true origin or full ramification.

One may expand asymptotically the Gamma function in eq. (5) to extract the eikonal limit and the leading order correction; we obtain

\[
\delta_l(s) = -Gs \left[ \log l - \frac{1}{2l} \right] + \zeta^2 (Gs)^3 \left( \frac{1}{l^3} \right) + O\left( \frac{1}{l^5} \right).
\] (14)

Clearly, the eikonal term and the leading order correction (the two terms in the first pair of square brackets) are completely independent of the charge \( Q \) on the static ‘target’ particle whose gravitational field we have modelled through the metric of a Reissner-Nordström back hole. The subleading corrections (i.e., terms of \( O(l^{-3}) \) or smaller), in contrast, certainly depend on this charge. In other words, the gravitational effect is completely decoupled from
the electromagnetic effect for these first two contributions to the phase shift. The mixing
that one expects to see generically, indeed appears for smaller values of the impact parameter
(smaller $l$). Admittedly, the coefficient of the mixing terms calculated above is not universal
in the sense that one expects corrections to it from transverse graviton exchange \cite{3}; but at
least for the first two terms, we expect our results to be robust.

Further evidence for the decoupling of gravitation and electromagnetism for the eikonal
and leading correction terms, comes from the scattering of a charged particle (of charge
$Q'$ say) off the gravitational and electromagnetic field due to the target. This is seen by
generalizing the generally covariant derivatives in the Klein Gordon equation of the luminal
particle, to be $U(1)$ gauge covariant as well. The radial equation turns out to be a modified
version of eq. (11) :

$$\frac{d^2 f}{dr^2} - \left[\frac{l(l+1) + \lambda(Gs)^2 - \alpha'^2}{r^2} - \frac{2\alpha'E}{r} + E^2\right] f = 0$$

(15)

where $\alpha' \equiv Gs - QQ'$ and $\lambda$ is defined by $\lambda \equiv 1 - \zeta^2$. The asymptotic expansion of the
 corresponding phase shift can now be obtained as before with very little extra work. It has
the form

$$\delta_l(s) = -\alpha' \left[ \log l - \frac{1}{2l} + \frac{\lambda(Gs)^2 - \alpha'^2}{2l^2} + O\left(\frac{1}{l^3}\right) \right].$$

(16)

The replacement $Gs \rightarrow Gs - QQ'$ \cite{4} to account for the electromagnetic effects in the eikonal
limit, is thus clearly correct within our approach. Moreover, such replacement also appears
to be equally valid for the leading order correction to the eikonal. The mixing between
gravity and electromagnetic effects starts from the ‘non-universal’ $O(l^{-2})$ terms\cite{4} where one
expects them to appear in any case.

\footnote{The actual computation of these terms would be sensitive to the precise manner in which graviton
loop ultraviolet divergences are handled, i.e., on a particular proposal (model) for a theory of
quantum gravity.}
IV. CONCLUSIONS

While our (semiclassical) method of computing corrections to the eikonal scattering amplitude appears viable, strictly speaking the predictions from this approach are reliable only for the leading order correction to the eikonal. The subleading terms within our approach are affected nontrivially under true quantum gravitational effects, similar to the inevitable necessity of field theoretic quantum electrodynamics for a proper calculation of the Lamb shift. The difference here is the lack of an appropriate quantum ‘gravidynamics’ which can be reliably used for computation. Since the issue at hand seemingly entails uncontrollable ultraviolet behaviour of a local field theoretic formulation of gravity, starting from Einsteinian general relativity, the use of string theory to tame these divergences is certainly an attractive option. On the other hand, the robustness of the eikonal amplitude may indicate certain non-perturbative aspects of spacetime geometry at short distances which may not be analyzable in terms of perturbative string theory.

It is satisfying to note that our heuristic analysis on non-mixing of electromagnetic and gravitational effects for eikonal scattering [9] can indeed be placed on firmer footing. Likewise, the persistence of this decoupling for the leading corrections leads us to infer that these corrections have a similar degree of universality, not shared by the higher order effects. The regime of validity of the semiclassical approximation for Planckian scattering appears then to have been determined to a reasonable degree of accuracy.

Finally, a word about dilaton gravity. The same heuristic arguments which enable us to show the decoupling of gravity and electromagnetism in general relativity, leads to a nontrivial mixing of these interactions even in the eikonal approximation for the case of dilaton gravity [1]. This also seems to be the case when the technique of this paper is applied to dilaton gravity. One is left with the disturbing possibility that the inclusion of the dilaton might actually make the eikonal limit non-existent! We hope to report on this elsewhere in the near future.

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