HOMOGENIZATION OF THERMAL-HYDRO-MASS TRANSFER PROCESSES

SHIXIN XU
Department of Mathematics, Soochow University
Suzhou 215006, China
and
Department of Mathematics, University of Science and Technology of China
Hefei 230026, China

XINGYE YUE
Department of Mathematics, Soochow University
Suzhou 215006, China

Abstract. In the repository, multi-physics processes are induced due to the long-time heat-emitting from the nuclear waste, which is modeled as a nonlinear system with oscillating coefficients. In this paper we first derive the homogenized system for the thermal-hydro-mass transfer processes by the technique of two-scale convergence, then present some error estimates for the first order expansions.

1. Introduction. Accompanied by the developing of the nuclear power, more and more nuclear waste is produced. The half-life period of radioactivity of nuclear waste is usual very long, especially for the high radioactive waste, which may be several tens of thousands years long. The safe disposal of nuclear waste is an important problem.

Due to the long-term heat emitted by radioactivity, the rock and the underwater in the repository will be heated up. Since the water and rock have different thermal expansivity, thermal input may cause significant pore pressure change which will induce convective flow in porous media. The temperature builds up to a certain level and then decreases. It will take 15-100 years to attain the peak of temperature near the waste repository and 200-1000 years for the far field. That means the thermal-hydro-mechanical processes will last a very long time. A lot of researches have been done to the thermal-hydro-mechanical processes (see [6, 7, 14, 21, 23, 24, 25]). There also exits the possiblity of the leakage of nuclear waste. If so, the nuclear waste will be dissolved in water and transferred to the far field by the underwater flow. Then the diffusion-convection processes of radioactive nuclear waste must be considered and it leads to the thermal-hydro-mass transfer processes in porous media. In order to describe these processes mathematically, We first show some notations that will be used.

Suppose a cubic domain $\Omega \subset R^3$ is occupied by the porous media which is adjacent to the waste at the left boundary $\Gamma_1$, through which the heat and waste comes into the porous media.

2010 Mathematics Subject Classification. Primary: 35B27, 35B40; Secondary: 35M30.
Key words and phrases. Homogenization, two-scale convergence, first order expansion, thermal-hydro-mass transfer processes.

This work is supported in part by NSF of China under the Grants 10871190 and 11271281.
By the conservation of energy, the thermal process is governed by
\[
\frac{\partial}{\partial t}(T_\varepsilon \bar{\rho} \bar{c}) - \nabla \cdot (\bar{D}_T \nabla T_\varepsilon) = 0,
\]
with \( \bar{\rho} \bar{c} = \rho_s c_s + \phi_c (\rho_f c_f - \rho_s c_s) \).

By the conservation law of mass of fluid, we have
\[
\frac{\partial (\rho_f \phi_c)}{\partial t} + \nabla \cdot (v_\varepsilon \rho_f) = 0.
\]

By the conservation law of mass of nuclear waste, the mass transfer process is controlled by
\[
\frac{\partial (\rho_f C_\varepsilon \phi_c)}{\partial t} - \nabla \cdot (\bar{D}_c \nabla C_\varepsilon) - \nabla \cdot (v_\varepsilon \rho_f C_\varepsilon) = 0,
\]
As a constitution relation, the Darcy’s law is needed:
\[
v_\varepsilon = -K_\varepsilon \nabla p_\varepsilon.
\]

To close the above model, we still need the following state equation by Boussinesq approximation to the density of fluid ([11]):
\[
\rho_f = \rho_0 (1 - \alpha (T_\varepsilon - T_r)),
\]
where \( \rho_0 \) is constant standing for the density of fluid at initial temperature \( T_r \) and \( \alpha \) is the thermal expansion coefficient of water. In reality, \( \alpha \) will be very small, i.e. \( 0 < \alpha \ll 1 \) and the thermal expansion coefficient of rock matrix is even much smaller than \( \alpha \). And note that the small change of the density of fluid only has significant effect on the pressure of fluid, not directly on the temperature and concentration. So we take some assumptions on the density:
1. \( \rho_s \) is a constant in this study;
2. \( \rho_f = \rho_0 \) in equations (1) and (3).

Then the thermal-hydro-mass transport processes can be described as:

\[
\begin{cases}
(1 + \phi_c \gamma) \frac{\partial T_\varepsilon}{\partial t} - \nabla \cdot (\bar{D}_T \nabla T_\varepsilon) = 0, & \text{in } \Omega_T, \\
-\nabla \cdot ((1 + \alpha (T_\varepsilon - T_r)) K_\varepsilon \nabla p_\varepsilon) = \alpha \phi_c \frac{\partial T_\varepsilon}{\partial t}, & \text{in } \Omega_T, \\
\frac{\partial C_\varepsilon}{\partial t} \phi_c - \nabla \cdot (\bar{D}_c \nabla C_\varepsilon) - \nabla \cdot (K_\varepsilon \nabla p_\varepsilon C_\varepsilon) = 0, & \text{in } \Omega_T,
\end{cases}
\]
with \( \Omega_T = \Omega \times (0, T] \), \( \gamma = \frac{\rho_0 c_f - \rho_s c_s}{\rho_s c_s} \), \( D_T^\varepsilon = \frac{\bar{D}_T^\varepsilon}{\rho_s c_s} \) and \( D_c^\varepsilon = \frac{\bar{D}_c^\varepsilon}{\rho_0 c_f} \).
As to the boundary conditions, Robin boundary conditions for $T_\varepsilon$ and $C_\varepsilon$ are assumed on $\Gamma_1$ due to the heat dissipation and leakage of waste. For pressure $p_\varepsilon$, we assume that the fluid is impermeable on $\Gamma_2$. On the right boundary $\Gamma_2$, which is far away from the waste, the Dirichlet boundary conditions are applied. On the other boundaries $\Gamma_3 = \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2)$, impermeable conditions for $T_\varepsilon$, $C_\varepsilon$ and $p_\varepsilon$ are imposed. We also need the initial values of $T_\varepsilon$ and $C_\varepsilon$, which we assume to be zero, the same as the far field data.

\[
\begin{align*}
(D_\varepsilon^T \nabla T_\varepsilon) \cdot \nu & = \beta_1(T_{\text{out}} - T_\varepsilon), \quad (K^\varepsilon \nabla p_\varepsilon) \cdot \nu = 0, \quad \text{on } \Gamma_1 \times (0, T], \\
(D_\varepsilon^T \nabla C_\varepsilon) \cdot \nu & = \beta_2(C_{\text{out}} - C_\varepsilon), \quad \text{on } \Gamma_1 \times (0, T], \\
T_\varepsilon & = p_\varepsilon = C_\varepsilon = 0, \quad \text{on } \Gamma_2 \times (0, T], \\
(D_\varepsilon^T \nabla T_\varepsilon) \cdot \nu & = (K^\varepsilon \nabla p_\varepsilon) \cdot \nu = (D_\varepsilon^T \nabla C_\varepsilon) \cdot \nu = 0, \quad \text{on } \Gamma_3 \times (0, T], \\
T_\varepsilon(x, 0) & = 0, \quad C_\varepsilon(x, 0) = 0, \quad \text{in } \Omega,
\end{align*}
\]

where $\nu$ is the unit outward normal vector.

System (6) is a nonlinear partial differential system with high oscillating coefficients. From a numerical point of view, resolving the microscopic details of (6) using typical numerical methods would require at least a cost of $O(\varepsilon^{-n})$ ($n=3$) or more. This often becomes prohibitively expensive since $\varepsilon \ll 1$. One way of avoiding this is to solve instead the homogenized equation of the problem (6). The general theory of homogenization can be found in [4, 5, 10, 26] for simple model problems. This paper is devoted to establish the homogenization theory of thermal-hydro-mass transfer processes (6). Two-scale convergence method is employed to deal with the coupled nonlinear terms and the weak convergence for $p_\varepsilon$. Two-scale convergence was first introduced by G. Allaire [1, 2] and G. Nguetseng [18]. In [3, 15], two-scale convergence for time dependent problem was considered. At the same time, the error estimate between the solutions of original problem and their first order expansions is also very important in homogeneous theory. The usual error estimate method for elliptic problems can be found in [10] and [26]. In order to reduce the regularity assumptions for the homogenized problems, the skew-symmetric matrix technique [26] and boundary correctors are always used. For parabolic problems, the initial value corrector is also needed [8]. For problem (6)-(7), it is more complex since the mixed Robin-Dirichlet boundary conditions are imposed. Following the idea of [9], we derive the error estimates between the solutions of problem (6)-(7) and their first order expansions. There is one thing also worth pointing out that the homogenized behavior of Equation (3) is different with a single diffusion-convection equation with a prescribed multiscale velocity field. For a single diffusion-convection equation with a prescribed multiscale velocity field, the homogenized velocity is determined not only by micro velocity itself, but also by the micro diffusion coefficient [26]. But here in a system, the homogenized velocity is only determined by the micro pressure equation (2), not related with the micro diffusion coefficient $D_\varepsilon^c$. Throughout the paper, we also use some mathematical notations such as $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ for general Sobolev spaces, $V = \{ v \in H^1(\Omega), v = 0 \text{ on } \Gamma_2 \}$, $W = \{ v \in L^2(0, T; V); \frac{\partial v}{\partial n} \in L^2(0, T; H^{-1}(\Omega)) \}$ and $C^\infty_{\Gamma_1, \Gamma_2}(\Omega) = \{ v \in C^\infty(\Omega); v = 0 \text{ on } \Gamma_2 \}$. $C > 0$ is a general constant independent of $\varepsilon$, which may be different at different occurrences. Einstein summation convention is also applied. The outline of paper is as follows: in §2 we first get some a priori estimates of the system (6);
in §3 we use two-scale method to derive the homogenized system; in §4 we present some error estimates between the solutions of (6) and their first order expansions. As to the multiscale numerical method for problem (6), we will show it in another paper.

2. A priori estimates. In this section, some a priori estimates of the problem (6) will be derived. We need some assumptions on the coefficients in order to ensure the regularity.

- (H0): The exchange coefficients $\beta_1$ and $\beta_2$, the density $\rho_\varepsilon$, the specific heats $c_\varepsilon$ and $c_f$ are positive constants. The conductive coefficients $D_T^\varepsilon$, $D_c^\varepsilon$ and permeability coefficient $K^\varepsilon$ are in the forms of $D_T^\varepsilon(x) = D_T(x, \frac{x}{\varepsilon})$, $D_c(x) = D_c(x, \frac{x}{\varepsilon})$, $K^\varepsilon(x) = K(x, \frac{x}{\varepsilon})$ and $D_T(x, y)$, $K(x, y)$, $D_c(x, y)$ satisfy that
  - symmetric,
  - continuous and periodic with respect to $y \in Y = [0, 1]^n$,
  - uniformly elliptic, i.e. there exist positive constants $\lambda_i$ and $\Lambda_i$ ($i = 1, 2, 3$), such that
  \[
  \begin{cases}
  \lambda_1|\xi|^2 \leq \xi^T D_T(x, y) \xi \leq \Lambda_1|\xi|^2, & \forall \xi \in \mathbb{R}^n, x \in \Omega, y \in Y, \\
  \lambda_2|\xi|^2 \leq \xi^T K(x, y) \xi \leq \Lambda_2|\xi|^2, & \forall \xi \in \mathbb{R}^n, x \in \Omega, y \in Y, \\
  \lambda_3|\xi|^2 \leq \xi^T D_c(x, y) \xi \leq \Lambda_3|\xi|^2, & \forall \xi \in \mathbb{R}^n, x \in \Omega, y \in Y.
  \end{cases}
  \]

For porosity $\phi_\varepsilon$, we give the following assumption.

- (H1): $0 < c \leq \phi^\varepsilon(x) \leq 1$, $\phi^\varepsilon = \phi(x, \frac{x}{\varepsilon})$ and $\phi(x, y)$ is continuous and periodic with respect to $y \in Y = [0, 1]^n$.

So we have $\phi^\varepsilon \rightharpoonup \phi_0(x) = \int_Y \phi(x, y)dy$ weakly * in $L^\infty(\Omega)$.

The system (6) is not fully coupled in the sense that one computes first $T_\varepsilon$, then $p_\varepsilon$ and finally $C_\varepsilon$. By the weak maximum principle [16], there exists a constant $C > 0$ such that
\[
\|T_\varepsilon\|_{L^\infty(\Omega_T)} + \|C_\varepsilon\|_{L^\infty(\Omega_T)} \leq C.
\]

By the basic theory of elliptic and parabolic equations [12, 13, 16], we can get the existence and uniqueness of solutions.

**Theorem 2.1.** Suppose hypotheses (H0), (H1) hold. If $T_{\text{out}}, C_{\text{out}} \in L^2((0, T) \times \Gamma_1)$ and $\alpha$ is sufficiently small, then there exist uniqueness solutions $T_\varepsilon \in W$, $p_\varepsilon \in L^2(0, T; V)$ and $C_\varepsilon \in W$ to system (6) and (7).

Next we show some a priori estimates for the solutions.

**Theorem 2.2.** Let $T_\varepsilon$, $p_\varepsilon$ and $C_\varepsilon$ be the solutions of problem (6) and (7). If (H0) and (H1) hold, $T_{\text{out}} \in L^\infty(0, T; L^2(\Gamma_1))$, $\frac{\partial T_{\text{out}}}{\partial t} \in L^2(0, T; H^\frac{1}{2}(\Gamma_1))$, $C_{\text{out}} \in L^2((0, T) \times \Gamma_1)$ and $\alpha$ is sufficiently small, then there exists a constant $C > 0$ independent of $\varepsilon$, such that
\[
\left| \frac{\partial T_\varepsilon}{\partial t} \right|_{L^2(\Omega_T)} + \sup_{0 \leq t \leq T} \|T_\varepsilon\|_{H^1(\Omega)} \
\leq C \left( \|T_{\text{out}}\|_{L^\infty(0, T; L^2(\Gamma_1))} + \left| \frac{\partial T_{\text{out}}}{\partial t} \right|_{L^2(0, T; H^\frac{1}{2}(\Gamma_1))} \right),
\]
\[
\|p_\varepsilon\|_{L^2(0, T; V)} \leq C \left( \|T_{\text{out}}\|_{L^\infty(0, T; L^2(\Gamma_1))} + \left| \frac{\partial T_{\text{out}}}{\partial t} \right|_{L^2(0, T; H^\frac{1}{2}(\Gamma_1))} \right),
\]
\[ \phi \frac{\partial C}{\partial t} \|_{L^2(0,T;H^{-1}((\Gamma_1)))} + \sup_{0 \leq t \leq T} \| C \|_{L^2(\Omega)} + \| \nabla C \|_{L^2(\Omega)} \leq C \left( \| \text{out} \|_{L^2((0,T) \times \Gamma_1)} + \| T \|_{L^\infty(0,T;L^2(\Gamma_1))} + \| \frac{\partial T_{\text{out}}}{\partial t} \|_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1))} \right). \]

Consequently, we get that \( T_\varepsilon \) is bounded in \( W \). By Aubin-Lions lemma, there exists a compact injection \( W \subset L^2(\Omega_T) \) \cite{10}, so there exists a subsequence of \( T_\varepsilon \) converges strongly in \( L^2(\Omega_T) \). \( C_\varepsilon \) is bounded in \( L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V) \) and \( \phi \frac{\partial C_\varepsilon}{\partial t} \) is bounded in \( L^2(0,T;H^{-1}(\Omega)) \). By a variant of Aubin-Lions lemma \cite{20, 17}, we also have a subsequence of \( C_\varepsilon \) converging strongly in \( L^2(\Omega_T) \). In conclusion, we get the following lemma.

**Lemma 2.3.** Let \( T_\varepsilon \), \( p_\varepsilon \) and \( C_\varepsilon \) be the solutions of problem \( (6) \). Under the same conditions of Theorem 2.2, there exist \( T_0 \), \( C_0 \) and \( p_0 \) \in \( L^2(0,T;V) \) such that up to a subsequence,

\[
\begin{align*}
& (1) \ T_\varepsilon \to T_{0}(x,t) \quad \text{weakly in} \quad L^2(0,T;V), \\
& (2) \ T_\varepsilon \to T_{0} \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\Omega)), \\
& (3) \ T_\varepsilon(t,x) \to T_{0} \quad \text{strongly in} \quad L^2(\Omega_T), \\
& (4) \ C_\varepsilon \to C_{0}(x,t) \quad \text{weakly in} \quad L^2(0,T;V), \\
& (5) \ C_\varepsilon(t,x) \to C_{0} \quad \text{strongly in} \quad L^2(\Omega_T), \\
& (6) \ p_\varepsilon \to p_{0}(x,t) \quad \text{weakly in} \quad L^2(0,T;V). 
\end{align*}
\]

**Remark 1.** Please note that the temperature \( T_\varepsilon \) and concentration \( C_\varepsilon \) converge strongly in \( L^2(\Omega_T) \) while the pressure \( p_\varepsilon \) only converges weakly in \( L^2(0,T;V) \) since there is no information on its derivative with respect to time. This leads to some difficulties to deal with the nonlinear coupled terms, which can be overcome by the method of two-scale convergence.

Before we prove Theorem 2.2, we first present the weak forms of system \( (6) \) and \( (7) \), which will be needed later. Find \( T_\varepsilon \in W \), \( p_\varepsilon \in L^2(0,T;V) \) and \( C_\varepsilon \in W \) such that

\[
\langle (1 + \phi_\varepsilon \gamma) \frac{\partial T_\varepsilon}{\partial t}, v \rangle_{H^{-1}(\Omega),H^1(\Omega)} + \int_\Omega D_\varepsilon T_\varepsilon \nabla v dx + \int_{\Gamma_1} \beta_1 T_\varepsilon v ds \]

\[
= \int_{\Gamma_2} \beta_2 \text{out} v ds, \quad \forall v \in V; \tag{11}
\]

\[
\int_\Omega K_\varepsilon \nabla p_\varepsilon \nabla v dx = \alpha \int_\Omega \frac{\partial T_\varepsilon}{\partial t} \phi_\varepsilon v dx + \alpha \int_\Omega (T_\varepsilon - T_r) K_\varepsilon \nabla p_\varepsilon \nabla v dx, \quad \forall v \in V; \tag{12}
\]

\[
\langle \phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}, v \rangle_{H^{-1}(\Omega),H^1(\Omega)} + \int_\Omega D_\varepsilon C_\varepsilon \nabla v dx + \int_\Omega K_\varepsilon \nabla p_\varepsilon C_\varepsilon \nabla v dx \]

\[
= \beta_2 \int_{\Gamma_1} (\text{out} - C_\varepsilon) v ds, \quad \forall v \in V. \tag{13}
\]

**Proof of Theorem 2.2. Step 1.** Taking \( v = T_\varepsilon(\cdot,t) \) in (11) and using (H0) and (H1), we have

\[
\frac{d}{dt} \left\| \sqrt{1 + \phi_\varepsilon \gamma} T_\varepsilon \right\|_{L^2(\Omega)}^2 + \lambda_1 \| \nabla T_\varepsilon \|_{L^2(\Omega)}^2 + \beta_1 \| T_\varepsilon \|_{L^2(\Gamma_1)}^2 \leq C \| \text{out} \|_{L^2(\Gamma_1)}^2. \tag{14}
\]

Here we have used Trace’s Theorem, Hölder’s inequality and Poincaré’s inequality for \( T_\varepsilon(\cdot,t) \in V \) in the above estimate. Integrating over \((0,t)\) with \( t \in [0,T] \) on both
Multiplying the first equation of system (6) by $\frac{\partial T_\varepsilon}{\partial t}$ and integrating by parts, it yields
\[
\int_\Omega (1 + \phi_\varepsilon \gamma) |\frac{\partial T_\varepsilon}{\partial t}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega D_\varepsilon^\varepsilon |\nabla T_\varepsilon|^2 dx + \frac{\beta_1}{2} \frac{d}{dt} \int_{\Gamma_1} |T_\varepsilon|^2 ds
= \beta_1 \int_{\Gamma_1} T_{out} \frac{\partial T_\varepsilon}{\partial t} ds
= \beta_1 \frac{d}{dt} \left( \int_{\Gamma_1} T_{out} T_\varepsilon ds \right) - \beta_1 \left( \frac{\partial T_{out}}{\partial t}, T_\varepsilon \right)_{H^{-\frac{1}{2}}(\Gamma_1), H^{\frac{1}{2}}(\Gamma_1)}.
\]
Integrating over $(0, t)$ with $t \in [0, T]$ and using the Trace Theorem, it follows
\[
\| \frac{\partial T_\varepsilon}{\partial t} \|_{L^2(\Omega_T)} + \sup_{0 \leq t \leq T} \| T_\varepsilon \|_{H^1(\Omega)}
\leq C \left( \| T_{out} \|_{L^\infty(0, T; L^2(\Gamma_1))} + \| \frac{\partial T_{out}}{\partial t} \|_{L^2(0, T; H^{-\frac{1}{2}}(\Gamma_1))} \right).
\]

**Step 2.** Estimate $\| \nabla p_\varepsilon \|_{L^2(\Omega_T)}$. Multiplying $p_\varepsilon$ on both sides of the second equation of (6), we have by integration by parts
\[
\int_\Omega K^\varepsilon \nabla p_\varepsilon \nabla p_\varepsilon dx = \alpha \int_\Omega \frac{\partial T_\varepsilon}{\partial t} \phi_\varepsilon p_\varepsilon dx + \alpha \int_\Omega (T_\varepsilon - T_\tau) K_\varepsilon \nabla p_\varepsilon \nabla p_\varepsilon dx.
\]
Thanks to the estimate of (H0) and (H1), we obtain
\[
\lambda_2 \| \nabla p_\varepsilon \|_{L^2(\Omega_T)}^2 \leq \alpha \| \frac{\partial T_\varepsilon}{\partial t} \|_{L^2(\Omega_T)} \| p_\varepsilon \|_{L^2(\Omega_T)}
+ \alpha \lambda_2 \left( \| T_\varepsilon \|_{L^\infty(\Omega)} + \| T_\varepsilon \|_{L^2(\Omega_T)} \right) \| \nabla p_\varepsilon \|_{L^2(\Omega_T)}.
\]
For sufficiently small $\alpha > 0$, $\alpha \lambda_2 \left( \| T_\varepsilon \|_{L^\infty(\Omega)} + \| T_\varepsilon \|_{L^2(\Omega_T)} \right)$ and it follows from the Poincare’s inequality and (17) that
\[
\| \nabla p_\varepsilon \|_{L^2(\Omega_T)}^2 \leq C \left( \frac{\alpha}{\lambda_2} \right)^2 \| \frac{\partial T_\varepsilon}{\partial t} \|_{L^2(\Omega_T)}^2
\leq C \left( \| T_{out} \|_{L^\infty(0, T; L^2(\Gamma_1))} + \| \frac{\partial T_{out}}{\partial t} \|_{L^2(0, T; H^{-\frac{1}{2}}(\Gamma_1))} \right).
\]
So the second inequality of Theorem 2.2 is proved.

**Step 3.** Estimate $C_\varepsilon$. Multiplying $C_\varepsilon$ on both sides of the third equation of (6) and integrating by parts, it yields
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \phi_\varepsilon |C_\varepsilon|^2 dx + \int_\Omega D_\varepsilon^\varepsilon |\nabla C_\varepsilon|^2 dx + \beta_2 \int_{\Gamma_1} |C_\varepsilon|^2 ds
= \beta_2 \int_{\Gamma_1} C_{out} C_\varepsilon ds + \int_\Omega K^\varepsilon \nabla p_\varepsilon C_\varepsilon \nabla C_\varepsilon dx
\leq C \| C_{out} \|_{L^2(\Gamma_1)}^2 + \frac{\beta_2}{2} \| C_\varepsilon \|_{L^2(\Omega_T)}^2 + C \| \nabla p_\varepsilon \|_{L^2(\Omega_T)}^2 + \frac{\lambda_3}{2} \| \nabla C_\varepsilon \|_{L^2(\Omega_T)}^2,
\]
where we have used the fact that $\| C_\varepsilon \|_{L^\infty}$ is uniformly bounded from the maximum principle (9). Integrating over $(0, t)$ with $t \in [0, T]$ on both sides and combining (18), we have that
\[
\sup_{0 \leq t \leq T} \| C_\varepsilon \|_{L^2(\Omega_T)} + \| \nabla C_\varepsilon \|_{L^2(\Omega_T)}
\leq C \left( \| T_{out} \|_{L^\infty(0, T; L^2(\Gamma_1))} + \| \frac{\partial T_{out}}{\partial t} \|_{L^2(0, T; H^{-\frac{1}{2}}(\Gamma_1))} + \| C_{out} \|_{L^2(0, T; L^2(\Gamma_1))} \right).
\]
Combining the weak form of $C_\varepsilon(13)$ and estimate (18), we can get the $H^{-1}$ norm estimate of $\phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}$:

$$\|\phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \left( \|T_{\text{out}}\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\frac{\partial T_{\text{out}}}{\partial t}\|_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1))} + \|C_{\text{out}}\|_{L^2((0,T) \times \Gamma_1)} \right).$$

The proof is completed.

3. **Homogenization.** In this section, we will derive the homogenized equations for the limit $T_0$, $C_0$ and $p_0$ by the method of two-scale convergence for time dependent problem. Before stating the main results, we first review the concept on two-scale convergence.

**Definition 3.1.** ([2, 22]) A sequence $u_\varepsilon(t, x) \in L^2(\Omega_T)$ two-scale converges to $u(t, x, y) \in L^2(\Omega_T \times Y)$ and we write $u_\varepsilon \overset{\text{2.s.c.}}{\rightharpoonup} u_0(t, x, y)$, if for all $\varphi(t, x, y) \in L^2(\Omega_T, C_{\text{per}}(Y))$, we have

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u_\varepsilon(t, x) \varphi(t, x, \frac{x}{\varepsilon}) dx dt = \int_0^T \int_\Omega u_0(t, x, y) \varphi(t, x, y) dy dx dt.$$

**Proposition 1.** ([22])

- Let $u_\varepsilon(t, x)$ be a bounded sequence in $L^2(\Omega_T)$. Then there exists a subsequence, still denoted by $u_\varepsilon$, and a function $u_0 \in L^2(\Omega_T \times Y)$ such that

  $$u_\varepsilon \rightharpoonup u_0(t, x, y).$$

Moreover, $u_\varepsilon$ converges weakly in $L^2(\Omega_T)$ to the average of the two-scale limit

$$u_\varepsilon \rightharpoonup v(t, x) = \int_Y u_0(t, x, y) \text{ weakly in } L^2(\Omega_T).$$

- Let $u_\varepsilon$ be a bounded sequence in $L^2(0,T;V)$ such that

  $$u_\varepsilon \rightharpoonup u_0(x,t) \text{ weakly in } L^2(0,T;V).$$

Then $u_\varepsilon$ two-scale converges to $u_0$ in $L^2(\Omega_T)$. In addition there exists a $u_1 \in L^2(\Omega_T; H_{\text{per}}(Y))$, up to a subsequence, still denoted by $u_\varepsilon$, such that

$$\nabla u_\varepsilon \overset{\text{2.s.c.}}{\rightharpoonup} \nabla_x u_0 + \nabla_y u_1.$$

One of our main results on the homogenization of thermal-hydro-mass transfer processes is as follows

**Theorem 3.2.** Let $p_\varepsilon$, $T_\varepsilon$, $C_\varepsilon$ be the solutions of problem (6). If (H0), (H1) hold, $T_{\text{out}} \in L^\infty(0,Y;L^2(\Gamma_1))$, $\frac{\partial T_{\text{out}}}{\partial t} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1))$, $C_{\text{out}} \in L^2((0,T) \times \Gamma_1)$ and $\alpha$ is sufficiently small, then

$$\begin{align*}
p_\varepsilon &\rightharpoonup p_0, \quad \text{in } L^2(\Omega_T \times Y), \\
T_\varepsilon &\rightharpoonup T_0, \quad \text{in } L^2(\Omega_T \times Y), \\
C_\varepsilon &\rightharpoonup C_0, \quad \text{in } L^2(\Omega_T \times Y),
\end{align*}$$

(20)
such that
\[ (1 + \phi_0 \gamma) \frac{\partial p_0}{\partial t} - \nabla \cdot (D^p_k(x) \nabla T_0) = 0, \quad \text{in } \Omega_T, \]
\[ -\nabla \cdot ((1 + \alpha (T_r - T_0)) K^0(x) \nabla p_0) = \alpha \phi_0 \frac{\partial p_0}{\partial t}, \quad \text{in } \Omega_T, \]
\[ \frac{\partial C_0}{\partial t} \phi_0 - \nabla \cdot (D^0_0(x) \nabla C_0) - \nabla \cdot (K^0(x) \nabla p_0 C_0) = 0, \quad \text{in } \Omega_T, \]
\[ (D^p_k \nabla T_0) \cdot \nu = \beta_1 (T_{out} - T_0), \quad (K^0 \nabla p_0) \cdot \nu = 0, \quad \text{on } \Omega_1 \times (0, T], \]
\[ (D^0_0 \nabla C_0) \cdot \nu = \beta_2 (C_{out} - C_0), \quad \text{on } \Gamma_1 \times (0, T], \]
\[ (D^0_k \nabla T_0) \cdot \nu = (K^0 \nabla p_0) \cdot \nu = (D^0_k \nabla C_0) \cdot \nu = 0, \quad \text{on } \Gamma_3 \times (0, T], \]
\[ T_0 = p_0 = C_0 = 0, \quad \text{on } \Gamma_2 \times (0, T], \]
\[ T_0(x, 0) = 0, \quad C_0(x, 0) = 0, \quad \text{in } \Omega, \]
where
\[ (D^p_k(x))_{ij} = \int_Y (D_T)_{ij}(x, y) + (D_T)_{jk}(x, y) \frac{\partial N_j}{\partial y_k} \, dy \]
\[ \triangleq \left\langle (D_T)_{ij}(x, y) + (D_T)_{jk}(x, y) \frac{\partial N_j}{\partial y_k} \right\rangle_Y, \quad \text{(22)} \]
with \( N_j(x, y) \) being the solution of the following cell problem
\[ \begin{cases} 
-\nabla_y \cdot (D_T(x, y) \nabla_y N_j(x, y)) = \nabla_y \cdot (D_T(x, y)e_j), \quad \text{in } Y, \\
N_j(x, y) \text{ is } Y\text{-periodic, and } \langle N_j(x, y) \rangle_Y = 0; 
\end{cases} \quad \text{(23)} \]
\[ (K^0(x))_{ij} = \left\langle K_{ij}(x, y) + K_{ik}(x, y) \frac{\partial \chi}{\partial y_k} \right\rangle_Y \text{ with } \chi^j(x, y) \text{ solving the cell problem} \]
\[ \begin{cases} 
-\nabla_y \cdot (K(x, y) \nabla_y \chi^j(x, y)) = \nabla_y \cdot (K(x, y)e_j), \quad \text{in } Y, \\
\chi^j(x, y) \text{ is } Y\text{-periodic, and } \langle \chi^j(x, y) \rangle_Y = 0, \end{cases} \quad \text{(24)} \]
and \( (D^c_k(x))_{ij} = \left\langle (D_c)^{ij}(x, y) + (D_c)^{ik}(x, y) \frac{\partial \pi}{\partial y_k} \right\rangle_Y \) with \( \pi^j(x, y) \) solving the cell problem
\[ \begin{cases} 
-\nabla_y \cdot (D_c(x, y) \nabla_y \pi^j(x, y)) = \nabla_y \cdot (D_c(x, y)e_j), \quad \text{in } Y, \\
\pi^j(x, y) \text{ is } Y\text{-periodic, and } \langle \pi^j(x, y) \rangle_Y = 0. \end{cases} \quad \text{(25)} \]
Furthermore, there exist \( p_1, T_1 \) and \( C_1 \), which are in form as
\[ p_1 = \chi^e_z \frac{\partial p_0}{\partial x_k}, \quad T_1 = N^e_k \frac{\partial T_0}{\partial x_k}, \quad C_1 = \pi^e_z \frac{\partial C_0}{\partial x_k}, \quad \text{(26)} \]
such that
\[ \nabla p_1 \overset{2}{\rightarrow} \nabla_x p_0(t, x) + \nabla_y p_1(t, x, y) \quad \text{in } [L^2(\Omega_T \times Y)]^n, \]
\[ \nabla T_1 \overset{2}{\rightarrow} \nabla_x T_0(t, x) + \nabla_y T_1(t, x, y) \quad \text{in } [L^2(\Omega_T \times Y)]^n, \]
\[ \nabla C_1 \overset{2}{\rightarrow} \nabla_x C_0(t, x) + \nabla_y C_1(t, x, y) \quad \text{in } [L^2(\Omega_T \times Y)]^n, \]
where \( \chi^e_z := \chi(x, \frac{z}{e}), \quad N^e_k := N^e_k(x, \frac{z}{e}), \quad \text{and } \pi^e_z := \pi^e_z(x, \frac{z}{e}). \)

**Remark 2.** The third equation of (21) is a homogenized equation of a convection-diffusion problem, wherein the homogenized velocity \( v_0 = -K^0 \nabla p_0 \) is totally determined by the micro-scale velocity \( v_\varepsilon = -K^0 \nabla p_\varepsilon \). This is interesting and different with the behavior of the homogenization for a single equation of convection-diffusion.
problem with prescribed multiscale velocity, where the homogenized velocity may be determined by both the micro-scale velocity $v_ε$ and the micro-scale diffusion coefficient $D_ε^{c}$ [26], when the velocity field is not divergence-free.

The weak forms of system (21) are as follows. Find $T_0 \in W$, $p_0 \in L^2(0,T;V)$ and $C_0 \in W$ such that

$$\int_Ω (1 + \phi_0) \frac{∂T_0}{∂t} v dx + \int_Ω D_ε^c \nabla T_0 \nabla v dx + \int_{Γ_1} β_1 T_0 v ds$$

$$= \int_{Γ_1} β_1 T_{out} v ds, ∀v ∈ V; \quad (27)$$

$$\int_Ω K^0 \nabla p_0 \nabla v dx = α \int_Ω \frac{∂T_0}{∂t} φ_0 v dx + α \int_Ω (T_0 - T_ε) K^0 \nabla p_0 \nabla v dx, ∀v ∈ V; \quad (28)$$

$$(\phi_0, \frac{∂C_0}{∂t}, v)_{H^{-1}(Ω), H^1(Ω)} + \int_Ω D_ε^c \nabla C_0 \nabla v dx + \int_Ω K^0 \nabla p_0 C_0 \nabla v dx$$

$$= β_2 \int_{Γ_1} (C_{out} - C_0) v ds, ∀v ∈ V. \quad (29)$$

**Theorem 3.3.** Suppose (H0) and (H1) hold. If $T_{out}, C_{out} ∈ L^2((0,T) \times Γ_1)$ and $α$ is sufficiently small, then there exist uniqueness solutions $T_0 ∈ W$, $p_0 ∈ L^2(0,T;V)$ and $C_0 ∈ W$ for system (21).

By Theorem 2.2, Lemma 2.3 and Proposition 1, we can get the following two-scale convergence results.

**Lemma 3.4.** If $p_ε$, $T_ε$ and $C_ε$ solve problem (6), then up to a subsequence, still denoted by $p_ε$, $T_ε$, $C_ε$, there exist $p_1(t,x,y)$, $T_1(t,x,y)$ and $C_1(t,x,y) ∈ L^2(Ω_T; H_{per}(Y))$ such that

$$\left\{ \begin{array}{l}
p_ε \xrightarrow{ε→0} p_0(t,x) \quad \text{in } L^2(Ω_T × Y), \\
\nabla p_ε \xrightarrow{ε→0} \nabla_x p_0(t,x) + \nabla_y p_1(t,x,y) \quad \text{in } [L^2(Ω_T × Y)]^n, \\
\end{array} \right. \quad (30)$$

$$\left\{ \begin{array}{l}
T_ε \xrightarrow{ε→0} T_0(t,x) \quad \text{in } L^2(Ω_T × Y), \\
\nabla T_ε \xrightarrow{ε→0} \nabla_x T_0(t,x) + \nabla_y T_1(t,x,y) \quad \text{in } [L^2(Ω_T × Y)]^n, \\
\end{array} \right. \quad (31)$$

$$\left\{ \begin{array}{l}
C_ε \xrightarrow{ε→0} C_0(t,x) \quad \text{in } L^2(Ω_T × Y), \\
\nabla C_ε \xrightarrow{ε→0} \nabla_x C_0(t,x) + \nabla_y C_1(t,x,y) \quad \text{in } [L^2(Ω_T × Y)]^n, \\
\end{array} \right. \quad (32)$$

Where $p_0$, $T_0$ and $C_0$ are defined in (10).

Let us check the convergence of the coupled term $K^ε \nabla p_ε C_ε$.

**Lemma 3.5.** If $p_ε$ and $C_ε$ are the solutions of problem (6), then up to a subsequence, still denoted by $p_ε$, $C_ε$, for every $φ_ε(t,x) = φ(t,x) + ε φ_1(t,x,y)$ with $φ(t,x) ∈ C^∞(Ω_T)$ and $φ_1(t,x,y) ∈ C^∞(Γ_2; C^∞_{per}(Y))$, we have

$$\lim_{ε→0} \int_0^T \int_Ω K^ε \nabla p_ε C_ε \cdot \nabla φ_ε(t,x) dx dt$$

$$= \int_0^T \int_Ω K(x,y)(\nabla_x p_0 + \nabla_y p_1)C_0 \cdot (\nabla_x φ + \nabla_y φ_1) dy dx dt,$$

where $p_0$, $C_0$, $p_1$, and $C_1$ are defined in Theorem 3.2.
Proof. For any \( \varphi_\varepsilon(t, x) = \psi(t, x) + \varepsilon \varphi_1(t, x, \frac{x}{\varepsilon}) \) with \( \varphi(t, x) \in C^\infty_T(\Omega_T), \varphi_1(t, x, y) \in C^\infty_T(\Omega_T; C^\infty_{\text{per}}(Y)) \),

\[
\int_0^T \int_\Omega K^\varepsilon \nabla p_\varepsilon \varphi_\varepsilon(t, x) dx dt = \int_0^T \int_\Omega K^\varepsilon \nabla p_\varepsilon (C_\varepsilon - C_0) \cdot \nabla \varphi_\varepsilon(t, x) dx dt + \int_0^T \int_\Omega K^\varepsilon \nabla p_\varepsilon C_0 \cdot \nabla \varphi_\varepsilon(t, x) dx dt.
\]

The first term at the right-hand side tends to 0 since we have from Lemma 2.3 that \( C_\varepsilon(t, x) \to C_0 \) strongly in \( L^2(\Omega_T) \) as \( \varepsilon \to 0 \). So using Lemma 3.4, we obtain

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega K^\varepsilon \nabla p_\varepsilon C_\varepsilon \cdot \nabla \varphi_\varepsilon(t, x) dx dt = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega K^\varepsilon \nabla p_\varepsilon C_0 \cdot \nabla \varphi_\varepsilon(t, x) dx dt = \int_0^T \int_Y K(x, y)(\nabla_x p_0 + \nabla_y p_1)C_0 \cdot (\nabla_x \varphi + \nabla_y \varphi_1) dy dx dt.
\]

\( \square \)

Remark 3. In the above proof, only the two scale convergence result (2.3) is used. So for temperature \( T_\varepsilon \), we also have

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega K^\varepsilon \nabla p_\varepsilon T_\varepsilon \cdot \nabla \varphi_\varepsilon(t, x) dx dt = \int_0^T \int_\Omega \int_Y K(x, y)(\nabla_x p_0 + \nabla_y p_1)T_0 \cdot (\nabla_x \varphi + \nabla_y \varphi_1) dy dx dt,
\]

where \( p_0, T_0, p_1, \) and \( T_1 \) are defined in Theorem 3.2.

Now we give the proof of Theorem 3.2.

Proof of Theorem 3.2. For the temperature equation, it is a direct result of homogenization theory for parabolic equation [10, 20].

We begin with the pressure equation in (6). For any \( \varphi_\varepsilon(x) = \psi(x) + \varepsilon \varphi_1(x, \frac{x}{\varepsilon}) \) with \( \varphi(x) \in C^\infty_T(\Omega), \varphi_1 \in C^\infty_T(\Omega; C^\infty_{\text{per}}(Y)) \) and any \( \psi(t) \in D([0, T]) \),

\[
\int_0^T \int_\Omega K^\varepsilon \nabla p_\varepsilon(t, x) \cdot \nabla \varphi_\varepsilon(x, t) \psi(t) dx dt = -\alpha \int_0^T \int_\Omega \phi_\varepsilon T_\varepsilon \frac{\partial \psi(t)}{\partial t} \varphi_\varepsilon(x) dx dt + \alpha \int_0^T \int_\Omega (T_0 - T_\varepsilon) K(x, y)(\nabla_x p_0 + \nabla_y p_1)(\nabla_x \varphi + \nabla_y \varphi_1) \psi(t) dy dx dt.
\]

Thanks to Lemma 2.3, Lemma 3.4 and Remark 3, up to a subsequence, the above formula converges in two-scale to

\[
\int_0^T \int_Y K(x, y)(\nabla_x p_0 + \nabla_y p_1)(\nabla_x \varphi + \nabla_y \varphi_1(x, y)) \psi(t) dy dx dt
\]

\[
= -\alpha \int_0^T \phi_\varepsilon T_0 \frac{\partial \psi(t)}{\partial t} \varphi_\varepsilon(x) dx dt + \alpha \int_0^T \int_\Omega (T_0 - T_\varepsilon) K(x, y)(\nabla_x p_0 + \nabla_y p_1)(\nabla_x \varphi + \nabla_y \varphi_1(x, y)) \psi(t) dy dx dt.
\]

Setting \( \varphi(x) = 0 \) and \( \varphi_1 = \mu(x) \mu_1(y) \) for any \( \mu \in C^\infty_T(\Omega), \mu_1 \in C^\infty_{\text{per}}(Y) \) in (34), we obtain

\[
\int_0^T \int_Y (1 - \alpha(T_0 - T_\varepsilon)) K(x, y)(\nabla_x p_0 + \nabla_y p_1)\nabla_x \mu_1(y) \mu(x) \psi(t) dy dx dt = 0.
\]
Due to the arbitrariness of \( \mu(x) \) and \( \psi(t) \), \( p_1 = p_1(t,x,y) \) satisfies that

\[
\int_Y K(x,y)(\nabla_x p_0(x) + \nabla_y p_1(y)) \mu_1(y) dy = 0, \quad \forall \mu_1(y) \in C^\infty_{\text{per}}(Y). \tag{35}
\]

Comparing with the cell problem (24), we have \( p_1 = \mu \frac{\partial p_0}{\partial x} \).

Setting \( \varphi_1 = 0 \) in (34), we obtain

\[
\int_0^T \int_\Omega (1 - \alpha(T_0 - T_r)) K(x,y) (\nabla_x p_0 + \nabla_y p_1) \nabla_x \varphi(x) \psi(t) dy dx dt = -\alpha \int_0^T \int_\Omega \phi_0 T_0 \frac{\partial \varphi(t)}{\partial t} \varphi(x) dx dt, \tag{36}
\]

i.e.

\[
\int_0^T \int_\Omega (1 - \alpha(T_0 - T_r)) K^0(x) \nabla_x p_0 \nabla \varphi(t) dy dx dt = \alpha \int_0^T \int_\Omega \phi_0 \frac{\partial T_0(t)}{\partial t} \varphi(t) dy dx dt + \alpha \int_\Omega \phi_0 T_0(0) \psi(0) \varphi(x) dx.
\]

Hence we obtain the homogenized equation for the pressure

\[
- \nabla_x \cdot (1 - \alpha(T_0 - T_r)) K^0(x) \nabla_x p_0 = \alpha \phi_0 \frac{\partial T_0(t)}{\partial t} \tag{37}
\]

with \( K^0(x) \nabla_x p_0 \cdot \nu = 0 \) on \( \partial \Omega \backslash \Gamma_2 \) and \( T_0|_{t=0} = 0 \).

For the concentration equation in (6), choosing the same test functions as above, we also have

\[
- \int_0^T \int_\Gamma_1 C_\varepsilon(t,x) \phi_\varepsilon \varphi_\varepsilon \frac{\partial \psi}{\partial t}(t) dx dt + \int_0^T \int_\Omega D_\varepsilon \nabla C_\varepsilon \nabla \varphi_\varepsilon \psi(t) dx dt + \int_0^T \int_\Omega K_\varepsilon \nabla p_0 \nabla \varphi_\varepsilon(t) dx dt = \beta_2 \int_0^T \int_\Gamma_1 (C_{\text{out}} - C_\varepsilon) \varphi_\varepsilon(t) ds dt.
\]

By Lemmas 3.4 and 3.5, up to a subsequence, the above formula converges in two-scale to

\[
- \int_0^T \int_\Omega C_0(t,x) \phi_0 \varphi(x) \frac{\partial \psi}{\partial t}(t) dx dt + \int_0^T \int_Y D_\varepsilon(x,y) (\nabla_x C_0 + \nabla_y C_1(x,y)) (\nabla_x \varphi + \nabla_y \varphi_1(x,y)) \psi(t) dy dx dt + \int_0^T \int_Y K(x,y) (\nabla_x p_0 + \nabla_y p_1) C_0 (\nabla_x \varphi(x) + \nabla_y \varphi_1(x,y)) \psi(t) dy dx dt = \beta_2 \int_0^T \int_\Gamma_1 (C_{\text{out}} - C_0) \varphi(x) \psi(t) ds dt. \tag{38}
\]

Setting \( \varphi(x) = 0 \) and noting that the cell problem (24) for the pressure, we get

\[
\int_0^T \int_Y D_\varepsilon(x,y) (\nabla_x C_0 + \nabla_y C_1(x,y)) \nabla_y \varphi_1(x,y) \psi(t) dy dx dt = 0.
\]
Comparing with the cell problem (25) for concentration, we have $C_1 = \pi_\varepsilon^2 \frac{\partial C_0}{\partial x_k}$.

Choosing $\varphi_1 = 0$ in (38), we obtain
\[
- \int_0^T \int_0^\Omega C_0(t, x) \phi_0 \varphi(x) \frac{\partial \psi}{\partial t}(t) dx dt + \int_0^T \int_\Omega \beta_2 (C_{out} - C_0) \varphi(t) ds dt
\]
\[= \oint_0^\Gamma (C_{out} - C_0) \varphi(t) ds dt - \oint_\Omega \phi_0 C_0(0, x) \psi(0) \varphi(x) dx.
\]

So we get the homogenized equation for the concentration
\[
\phi_0 \frac{\partial C_0}{\partial t} - \nabla \cdot (D_\varepsilon^p(x) \nabla C_0) - \nabla \cdot (K_\varepsilon^p(x) \nabla p_0 C_0) = 0, \quad \text{in } \Omega_T,
\]
with $C_0|_{t=0} = 0$, $D_\varepsilon^p \nabla C_0 \cdot \nu = \beta_2 (C_{out} - C_0)$ on $\Gamma_1$ and $D_\varepsilon^p \nabla C_0 \cdot \nu = 0$ on $\partial \Omega \setminus (\Gamma_1 \cup \Gamma_2)$.

The whole sequence two-scale convergence comes from the uniqueness of the solution for the homogenized system (21). Hence we complete the proof of Theorem 3.2. □

4. Error estimates for first order expansions. In this section, we will present the error estimates between $p_\varepsilon$, $T_\varepsilon$, $C_\varepsilon$ and their first order expansions
\[
p_1^\varepsilon = p_0 + \varepsilon \chi_\varepsilon \frac{\partial p_0}{\partial x_k}, \quad T_1^\varepsilon = T_0 + \varepsilon \chi_\varepsilon \frac{\partial T_0}{\partial x_k}, \quad C_1^\varepsilon = C_0 + \varepsilon \chi_\varepsilon \frac{\partial C_0}{\partial x_k}.
\]
To this purpose, we need some regularity assumptions on homogenized problem (21)

\[\text{(H2)} : \left\{ \begin{array}{l}
p_0(t, x) \in L^2(0, T; W^{2,\infty}(\Omega)), \\
T_0(t, x), C_0(t, x) \in L^2(0, T; W^{1,\infty}(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
\nabla \left( \frac{\partial p_0}{\partial t} \right), \nabla \left( \frac{\partial C_0}{\partial x_k} \right) \in (L^2(\Omega_T))^n, \\
\chi_1(x, y), N_i(x, y), \pi_i(x, y) \in L^\infty(\Omega, W^{1,\infty}_{per}(Y)) \cap L^\infty(Y; H^1(\Omega)), \\
i = 1, \ldots, n.
\end{array} \right.\]

In the pressure equation of system (6), since $\phi_\varepsilon$ only weakly* converges to $\phi_0$ in $L^\infty(\Omega)$, we can only get that the right hand side $\phi_\varepsilon \frac{\partial C_0}{\partial x_k}$ weakly converges in $L^2(0, T; H^{-1}(\Omega))$. Here due to the special structure of right hand side, we have got the convergence results in Theorem 3.2. In order to get the error estimate for first order expansion, further assumption has to be imposed either on the porosity $\phi_\varepsilon$, wherein we may consider the situation of porosity with only small change in
amplitude \( \phi_\varepsilon(x) = \phi_0 + \varepsilon \phi_1 \left( \frac{x}{\varepsilon} \right) \) or on the temperature field \( T_\varepsilon \), wherein we need more regularity on \( T_\varepsilon \). Here we choose the second case and assume that

\[
(H3) : \begin{cases}
T_{\text{out}} \text{ satisfies some more compatibility conditions such that } \frac{\partial}{\partial t} T_\varepsilon \in C^0([0,T];L^2(\Omega)), \\
\frac{\partial}{\partial n} T_\varepsilon(t,x) \in L^2(0,T;W^{1,\infty}(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\
\nabla(\frac{\partial^2 T_\varepsilon}{\partial x^2}) \in (L^2(\Omega_T))^n.
\end{cases}
\]

The following theorem on the error estimate for the first order expansions is also one of our main results.

**Theorem 4.1.** Let \( p_\varepsilon \), \( T_\varepsilon \), \( C_\varepsilon \) be the solutions of problem (6) and \( p_1^\varepsilon \), \( T_1^\varepsilon \), \( C_1^\varepsilon \) be defined in (40). If (\( H0 \)), (\( H1 \)) and (\( H2 \)) hold, \( T_{\text{out}} \in L^\infty(0,T;L^2(\Gamma_1)), \) \( \frac{\partial T_{\text{out}}}{\partial n} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1)) \) and \( \alpha \) is sufficiently small, then there exists constant \( C > 0 \) independent of \( \varepsilon \), such that

\[
\sup_{0 \leq t \leq T} \| T_{\varepsilon} - T_{1,\varepsilon} \|_{L^2(\Omega)} + \| \nabla(T_{\varepsilon} - T_{1,\varepsilon}) \|_{L^2(\Omega_T)} \leq C \varepsilon^{\frac{1}{2}}.
\]

Furthermore, if (\( H3 \)) also holds and \( C_{\text{out}} \in L^2((0,T) \times \Gamma_1) \), then

\[
\sup_{0 \leq t \leq T} \| p_{\varepsilon} - p_1^\varepsilon \|_{H^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}},
\]

\[
\sup_{0 \leq t \leq T} \| C_{\varepsilon} - C_1^\varepsilon \|_{L^2(\Omega)} + \| \nabla(C_{\varepsilon} - C_1^\varepsilon) \|_{L^2(\Omega_T)} \leq C \varepsilon^{\frac{1}{2}}.
\]

Before proving the theorem, several lemmas are needed.

**Lemma 4.2.** [9] Let \( \theta^\varepsilon \in V \) be the boundary corrector satisfying the following problem

\[
\begin{cases}
- \nabla \cdot (A^\varepsilon \nabla \theta^\varepsilon) = 0, & \text{in } \Omega, \\
(A^\varepsilon \nabla \theta^\varepsilon) \cdot \nu = \frac{\partial}{\partial x_i} (Q_{ik}^\varepsilon \frac{\partial u}{\partial x_j}) \cdot \nu, & \text{on } \partial \Omega \setminus \Gamma_2, \\
\theta^\varepsilon = 0, & \text{on } \Gamma_2,
\end{cases}
\]

where \( A^\varepsilon(x) = A(x, \frac{x}{\varepsilon}) \) is positive definite and uniformly bounded, \( u \in H^2(\Omega) \bigcap W^{1,\infty}(\Omega) \) and \( Q_{ik}^\varepsilon(x, \frac{x}{\varepsilon}) \in L^\infty(\Omega;W^{1,\infty}(Y)) \bigcap L^\infty(Y;H^1(\Omega)) \) is skew-symmetric matrix \( Q_{ik}^\varepsilon = -Q_{ki}^\varepsilon \), then

\[
\| \varepsilon \nabla \theta^\varepsilon \|_{L^2(\Omega)} \leq C \varepsilon (|u|_{H^1(\Omega)} + |u|_{H^2(\Omega)}) + C \varepsilon^{\frac{1}{2}} |u|_{W^{1,\infty}(\Omega)}. \tag{47}
\]

This boundary corrector problem is slightly different with that defined in [9], where the Dirichlet boundary condition is missing and \( Q_{ik}^\varepsilon \) is a problem specific matrix. However, the idea used in [9] still works. One can prove the above lemma similarly.

**Lemma 4.3.** If \( \eta_\varepsilon \in V \) is the boundary corrector satisfying the following problem

\[
\begin{cases}
(1 + \phi_\varepsilon \gamma) \frac{\partial u}{\partial t} - \nabla \cdot (D_\varepsilon \nabla \eta_\varepsilon) = 0, & \text{in } \Omega_T, \\
(D_\varepsilon \nabla \eta_\varepsilon) \cdot \nu = -\beta_1 \eta_\varepsilon, & \text{on } \Gamma_1 \times (0,T], \\
(D_\varepsilon \nabla \eta_\varepsilon) \cdot \nu = 0, & \text{on } \Gamma_3 \times (0,T], \\
\eta_\varepsilon = 0, & \text{on } \Gamma_2 \times (0,T], \\
\eta_\varepsilon \big|_{t=0} = 0, & \text{in } \Omega,
\end{cases}
\]

and (\( H0 \)) - (\( H2 \)) hold, then

\[
\sup_{0 \leq t \leq T} \| \varepsilon \eta^\varepsilon \|_{L^2(\Omega)} \leq C \varepsilon, \quad \| \varepsilon \nabla \eta^\varepsilon \|_{L^2(\Omega_T)} \leq C \varepsilon^{\frac{1}{2}}. \tag{49}
\]
We omit the proof of this lemma since it is similar to Theorem 3.1 in [8].

**Lemma 4.4.** Let $C^*_i$ be defined as in (40) and $C_0$ be the solution of problems (21), respectively. If $(H0)$ and $(H2)$ hold, then there exists a positive constant $C$ independent of $\varepsilon$ such that, for any $\varphi \in V$,

$$\left| \int_{\Omega} (D^*_c \nabla C^*_i - D^0_c \nabla C_0) \nabla \varphi dx \right| \leq C \varepsilon^{1/2} \| \nabla \varphi \|_{L^2(\Omega)}.$$ 

**Proof.** For $\phi \in V$, according to the definition of $C^*_i$, after some simple calculation, we have

$$J_1 = \int_{\Omega} (D^*_c \nabla C^*_i - D^0_c \nabla C_0) \nabla \varphi dx \leq \int_{\Omega} g_i^j \frac{\partial C_0}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \int_{\Omega} R^*_i \nabla \varphi dx, \quad (50)$$

with

$$g_i^j(x, y) = (D^*_{ij}) + \sum_k (D^*_{ik}) \frac{\partial \pi^j_k}{\partial y_k} - (D^0_{ij}),$$

$$R^*_i = \sum_{j,k} \left( \varepsilon(D^*_{ij}) \pi^j_k \frac{\partial^2 C_0}{\partial x_j \partial x_k} + \varepsilon(D^*_{ij}) \frac{\partial \pi^j_k}{\partial x_j} \right). \quad (51)$$

Since $\langle g_i^j \rangle = 0$ and $\nabla_y \cdot g^i = 0$, there exists a skew-symmetric matrix ([26]) $G^j_{ik}(x, y) \in L^\infty(\Omega; W^{1,\infty}(Y)) \cap L^\infty(Y; H^1(\Omega))$ such that $\sum_k \frac{\partial G^j_{ik}}{\partial y_k} = g^i_j$. With this notation, we can rewrite

$$g_i^j \frac{\partial C_0}{\partial x_j} = \sum_k \left( \frac{\partial}{\partial x_k} (\varepsilon G^j_{ik} \frac{\partial C_0}{\partial x_j}) - \varepsilon G^j_{ik} \frac{\partial C_0}{\partial x_j} \frac{\partial \varphi}{\partial x_i} - \varepsilon G^j_{ik} \frac{\partial^2 C_0}{\partial x_j \partial x_k} \right). \quad (52)$$

Then we have

$$J_1 = \int_{\Omega} \frac{\partial}{\partial x_k} (\varepsilon G^j_{ik} \frac{\partial C_0}{\partial x_j}) \frac{\partial \varphi}{\partial x_i} + \int_{\Omega} R^*_i \nabla \varphi dx - \int_{\Omega} R^*_2 \nabla \varphi dx, \quad (53)$$

with $R^*_2 = \sum_{j,k} \left( \varepsilon \frac{\partial G^j_{ik}}{\partial x_k} \frac{\partial C_0}{\partial x_j} + \varepsilon G^j_{ik} \frac{\partial^2 C_0}{\partial x_j \partial x_k} \right)$. Let $\theta^*_\epsilon$ be the boundary corrector as a solution of the following problem:

$$\begin{cases}
-\Delta \theta^*_\epsilon = 0, & \text{in } \Omega, \\
(\nabla \theta^*_\epsilon) \cdot \nu = \frac{\partial}{\partial x_k} (G^j_{ik} \frac{\partial C_0}{\partial x_j}) \cdot \nu_i, & \text{on } \partial \Omega \setminus \Gamma_2, \\
\theta^*_\epsilon = 0, & \text{on } \Gamma_2. \quad (54)
\end{cases}$$

From the skew-symmetry of matrix $G^j_{ik}$, the weak form of problem (54) reads

$$\int_{\Omega} \nabla \theta^*_\epsilon \nabla \varphi dx = \int_{\Omega} \frac{\partial}{\partial x_k} (G^j_{ik} \frac{\partial C_0}{\partial x_j}) \frac{\partial \varphi}{\partial x_i}.$$ 

Then we have

$$J_1 = \varepsilon \int_{\Omega} \nabla \theta^*_\epsilon \nabla \varphi dx + \int_{\Omega} R^*_i \nabla \varphi dx - \int_{\Omega} R^*_2 \nabla \varphi dx, \quad (55)$$
Lemma 4.5. Let \( \varphi \) be the solution of problem \( H \) with \( \epsilon \) as in Lemma 4.4 that for any \( C_\varepsilon \), \( K \), \( \psi \), \( L \) such that for \( \varepsilon \) such that for any \( \varphi \in V \),

\[
\left| \int_\Omega (D^2_T \nabla T_1^\varepsilon - D^2_T \nabla T_0) \nabla \varphi \, dx \right| \leq C\varepsilon^{\frac{5}{2}} \| \nabla \varphi \|_{L^2(\Omega)}.
\]

Lemma 4.5. Let \( C_\varepsilon, p_\varepsilon \) be the solutions of problem (6) and \( C_0, p_0 \) be the solutions of problem (21). If \((H0)\) and \((H2)\) hold, then there exists a positive constant \( C \) independent of \( \varepsilon \) such that for any \( \varphi \in V \),

\[
\left| \int_\Omega [K^\varepsilon(x) \nabla p_\varepsilon C_\varepsilon - K^0(x) \nabla p_\varepsilon C_0] \nabla \varphi \, dx \right| \leq C\varepsilon^{\frac{5}{2}} \| \nabla \varphi \|_{L^2(\Omega)} + \Lambda_2 \| C_\varepsilon \|_{L^\infty(\Omega)} \| \nabla (p_\varepsilon - p_0) \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)}
\]

Proof. First,

\[
\left| \int_\Omega [K^\varepsilon(x) \nabla p_\varepsilon C_\varepsilon - K^0(x) \nabla p_\varepsilon C_0] \nabla \varphi \, dx \right| = J_2 + J_3
\]

with \( J_2 = \int_\Omega [K^\varepsilon(x) \nabla p_\varepsilon C_\varepsilon - K^0(x) \nabla p_\varepsilon C_0] \nabla \varphi \, dx \) and \( J_3 = \int_\Omega [K^\varepsilon(x) \nabla p_\varepsilon C_0 - K^0(x) \nabla p_\varepsilon C_0] \nabla \varphi \, dx \).

For the first term, we have

\[
|J_2| \leq \left| \int_\Omega [K^\varepsilon \nabla (p_\varepsilon - p_0) C_\varepsilon] \nabla \varphi \, dx \right| + \int_\Omega [K^\varepsilon \nabla p_\varepsilon C_\varepsilon - C_0 \nabla \varphi \, dx]
\]

\[
\leq \Lambda_2 \| C_\varepsilon \|_{L^\infty(\Omega)} \| \nabla (p_\varepsilon - p_0) \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} + \Lambda_2 \| \nabla p_\varepsilon \|_{L^2(\Omega)} \| C_\varepsilon - C_0 \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} + C\varepsilon \Lambda_2 \| \nabla p_\varepsilon \|_{L^\infty(\Omega)} \| C_\varepsilon \|_{H^1(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)}.
\]

The term \( J_3 \) can be bounded by a similar way used to treat term \( J_1 \) in Lemma 4.4. But we need a slightly different corrector problem to treat the coupling with \( C_0 \). By simple calculations, we have for \( i = 1, \ldots, n \),

\[
(K^\varepsilon \nabla p_\varepsilon - K^0 \nabla p_0)_i = \sum_j h_{ij}^\varepsilon \frac{\partial p_0}{\partial x_j} + (r_i^\varepsilon)no,
\]

with \( h_{ij}^\varepsilon(x, y) = K_{ij}^\varepsilon + \sum_k K_{ik}^\varepsilon \frac{\partial x_j}{\partial y_k} - K_{ij}^0 \) and \( (r_i^\varepsilon)_i = \sum_{j, k} (\varepsilon K_{ij}^\varepsilon \frac{\partial x_j}{\partial x_k} \frac{\partial p_0}{\partial x_k} + \varepsilon K_{ij}^\varepsilon \frac{\partial x_j}{\partial x_k} \frac{\partial^2 p_0}{\partial x_k \partial x_k}) \). Since \( (h_{ij}^\varepsilon)_{ji} = 0 \) and \( \nabla_y h^\varepsilon = 0 \), there exists a skew-symmetric matrix \( H_{ij}^\varepsilon(x, y) \in L^\infty(\Omega; W^{1,\infty}(Y)) \cap L^\infty(Y; H^1(\Omega)) \) such that \( \sum_k \frac{\partial H_{ij}^\varepsilon}{\partial x_k} = h_{ij}^\varepsilon \).
With this notation, we can rewrite
\[ h_i^2 \frac{\partial p_0}{\partial x_j} = \sum_k \left( \frac{\partial}{\partial x_k} \left( \varepsilon H_{i}^{j} \frac{\partial p_0}{\partial x_j} \right) - \varepsilon \frac{\partial H_{i}^{j}}{\partial x_k} \frac{\partial p_0}{\partial x_j} - \varepsilon H_{i}^{j} \frac{\partial^2 p_0}{\partial x_j \partial x_k} \right) \].

(59)

In summary, we obtain for \( i = 1, \ldots, n \),
\[ (K^i \nabla p_1^i - K^0 \nabla p_0)_i = \sum_{j,k} \frac{\partial}{\partial x_k} \left( \varepsilon H_{i}^{j} \frac{\partial p_0}{\partial x_j} \right) + (r_1^i)_i - (r_2^i)_i, \]

with \( (r_2^i)_i = \sum_{j,k} \left( \varepsilon H_{i}^{j} \frac{\partial p_0}{\partial x_j} + \varepsilon H_{i}^{j} \frac{\partial^2 p_0}{\partial x_j \partial x_k} \right) \).

Let the boundary corrector \( \theta^\varepsilon_p \) satisfying the following problem, which is slightly different with the boundary corrected problem (54),
\[
\begin{aligned}
-\Delta \theta^\varepsilon_p &= 0, \quad \text{in } \Omega, \\
(\nabla \theta^\varepsilon_p) \cdot \nu &= \left( \frac{\partial}{\partial x_k} \left( \varepsilon H_{i}^{j} \frac{\partial p_0}{\partial x_j} C_0 \right) \right) \cdot \nu_i \quad \text{on } \partial \Omega \setminus \Gamma_2, \\
\theta^\varepsilon_p &= 0 \quad \text{on } \Gamma_2.
\end{aligned}
\]

(61)

By the skew-symmetry of matrix \( H_{i}^{j} \), the weak form of the above problem is as follows: Find \( \theta^\varepsilon_p \in V \) such that
\[
\int_{\Omega} \nabla \theta^\varepsilon_p \nabla \varphi dx = \int_{\Omega} \frac{\partial}{\partial x_k} \left( \varepsilon H_{i}^{j} \frac{\partial p_0}{\partial x_j} C_0 \right) \frac{\partial \varphi}{\partial x_i} dx, \quad \forall \varphi \in V.
\]

(62)

By (60), \( J_3 \) can be rewritten as
\[
J_3 = \int_{\Omega} \frac{\partial}{\partial x_k} \left( \varepsilon H_{i}^{j} \frac{\partial p_0}{\partial x_j} C_0 \right) \frac{\partial \varphi}{\partial x_i} dx + \int_{\Omega} r_1^i C_0 \nabla \varphi dx - \int_{\Omega} r_2^i C_0 \nabla \varphi dx
\]
\[ -\varepsilon \int_{\Omega} \frac{\partial C_0}{\partial x_k} H_{i}^{j} \frac{\partial p_0}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx \]
\[ = \varepsilon \int_{\Omega} \nabla \theta^\varepsilon_p \nabla \varphi dx + \int_{\Omega} r_1^i C_0 \nabla \varphi dx - \int_{\Omega} r_2^i C_0 \nabla \varphi dx
\]
\[ -\varepsilon \int_{\Omega} \frac{\partial C_0}{\partial x_k} H_{i}^{j} \frac{\partial p_0}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx. \]

(63)

From Lemma 4.2 and the definitions of \( r_1^i, r_2^i \), we obtain the estimate of \( J_3 \)
\[
|J_3| \leq C \varepsilon^2 |C_0| \| \nabla \varphi \|_{L^2(\Omega)} + C \varepsilon |p_0| \| \nabla \varphi \|_{L^2(\Omega)} + C \varepsilon \| \nabla p_0 \|_{L^2(\Omega)} |C_0| \| \nabla \varphi \|_{L^2(\Omega)}.
\]

(64)

Combining (57) with (64), we complete the proof.

\[ \square \]

**Remark 5.** If (H0) and (H2) hold, then we can get, in a same way as the above Lemma, that for any \( \varphi \in V \),
\[
\left| \int [K^\varepsilon(x) \nabla p_0 T_\varepsilon - K^0(x) \nabla p_0 T_0] \nabla \varphi dx \right| \leq C \varepsilon^2 \| \nabla \varphi \|_{L^2(\Omega)}
\]
\[ + \Lambda_2 \| T_\varepsilon \|_{L^\infty(\Omega)} \| \nabla (p_\varepsilon - p_\varepsilon^0) \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)}
\]
\[ + \Lambda_2 \| \nabla p_0^\varepsilon \|_{L^\infty(\Omega)} \| T_\varepsilon - T_0^\varepsilon \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)}. \]
In the estimates of Lemma 4.5 and Remark 5, $T_\varepsilon$ and $C_\varepsilon$ are uniformly bounded by (2.2) and $\nabla p_1^\varepsilon$ is also uniformly bounded since under the assumption ($\text{H2}$), there exists a constant $C > 0$ such that

$$\|\nabla p_1^\varepsilon\|_{L^\infty(\Omega)} \leq \|\nabla p_0 + \varepsilon \nabla \chi(x,y) \frac{\partial p_0}{\partial x_i} + \nabla g \chi(x,y) \frac{\partial p_0}{\partial x_i} + \varepsilon \chi_i \nabla \frac{\partial p_0}{\partial x_i}\|_{L^\infty(\Omega)} \leq C.$$ 

We also need the following lemma to deal with the terms containing $\phi_\varepsilon$.

**Lemma 4.6.** ([19]) Let $g(x,y) \in L^\infty(\Omega \times R^N)$ be periodic in $Y$ with respect to $y$ and satisfy $\int_V g(x,y)dy = 0$ for any $x \in \Omega$. Then there exists a constant $C > 0$ independent of $\varepsilon$ such that for any $u, v \in H^1(\Omega)$,

$$\left| \int_\Omega u v g(x, \frac{x}{\varepsilon})dx \right| \leq C \varepsilon \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \quad (65)$$

Now we give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** **Step 1.** For $T_\varepsilon - T_1^\varepsilon$, by the weak forms (11) for $T_\varepsilon$ and (27) for $T_0$, we have

$$\langle (1 + \phi_\varepsilon) \frac{\partial(T_\varepsilon - T_1^\varepsilon)}{\partial t}, \varphi \rangle_{H^{-1}, H^1} + \int_\Omega D_\varepsilon \nabla(T_\varepsilon - T_1^\varepsilon) \nabla \varphi dx$$

$$+ \beta_1 \int_{\Gamma_1} (T_\varepsilon - T_1^\varepsilon) \varphi ds = \langle (1 + \phi_\varepsilon) \frac{\partial(T_0 - T_1^\varepsilon)}{\partial t}, \varphi \rangle_{H^{-1}, H^1} + \langle \gamma(\phi_0 - \phi_\varepsilon) \frac{\partial T_0}{\partial t}, \varphi \rangle_{H^{-1}, H^1}$$

$$+ \int_\Omega \langle D_\varepsilon \nabla T_0 - D_\varepsilon \nabla T_1^\varepsilon \rangle \nabla \varphi + \beta_1 \int_{\Gamma_1} (T_0 - T_1^\varepsilon) \varphi ds$$

$$= -\varepsilon \int_\Omega (1 + \phi_\varepsilon) N_\varepsilon^j \frac{\partial T_0}{\partial x_j} \varphi dx + \int_\Omega \langle D_\varepsilon \nabla T_0 - D_\varepsilon \nabla T_1^\varepsilon \rangle \nabla \varphi dx$$

$$+ \int_\Omega \gamma(\phi_0 - \phi_\varepsilon) \frac{\partial T_0}{\partial t} \varphi dx - \beta_1 \varepsilon \int_{\Gamma_1} N_\varepsilon^j \frac{\partial T_0}{\partial x_j} \varphi ds. \quad (66)$$

Introduce the boundary corrector $\eta_\varepsilon$ defined in problem (48). A weak form for problem (48) reads, for every $\varphi \in V$,

$$\int_\Omega (1 + \phi_\varepsilon) \frac{\partial \eta_\varepsilon}{\partial t} \varphi dx + \int_\Omega D_\varepsilon \nabla \eta_\varepsilon \nabla \varphi dx + \int_{\Gamma_1} \beta_1 \eta_\varepsilon \varphi ds = 0. \quad (67)$$

Then $e_\varepsilon = T_\varepsilon - T_1^\varepsilon - \varepsilon \eta_\varepsilon^\varepsilon \in V$. Thanks to (67) and setting $\varphi = e_\varepsilon$ in (66), we obtain

$$\langle (1 + \phi_\varepsilon) \frac{\partial e_\varepsilon}{\partial t}, e_\varepsilon \rangle_{H^{-1}, H^1} + \int_\Omega D_\varepsilon \nabla e_\varepsilon \nabla e_\varepsilon dx + \beta_1 \int_{\Gamma_1} e_\varepsilon e_\varepsilon ds$$

$$= -\varepsilon \int_\Omega (1 + \phi_\varepsilon) N_\varepsilon^j \frac{\partial T_0}{\partial x_j} e_\varepsilon dx + \int_\Omega \langle D_\varepsilon \nabla T_0 - D_\varepsilon \nabla T_1^\varepsilon \rangle \nabla e_\varepsilon dx$$

$$+ \int_\Omega \gamma(\phi_0 - \phi_\varepsilon) \frac{\partial T_0}{\partial t} e_\varepsilon dx - \beta_1 \varepsilon \int_{\Gamma_1} N_\varepsilon^j \frac{\partial T_0}{\partial x_j} e_\varepsilon ds. \quad (68)$$
Using the uniformly elliptic condition, Poincare’s inequality, Remark 4 and Lemma 4.6, we have

\[\frac{1}{2}\frac{d}{dt}\|\sqrt{1+\phi} e_T\|^2_{L^2(\Omega)} + \lambda_1 \|\nabla e_T\|^2_{L^2(\Omega)} + \beta_1 \|e_T\|^2_{L^2(\Gamma_1)}\]

\[\leq C\varepsilon^2 + C\varepsilon + \frac{\lambda_1}{2} \|\nabla e_T\|^2_{L^2(\Omega)} + \frac{\beta_1}{2} \|e_T\|^2_{L^2(\Gamma_1)}\]

\[\leq C\varepsilon + \frac{\lambda_1}{2} \|\nabla e_T\|^2_{L^2(\Omega)} + \frac{\beta_1}{2} \|e_T\|^2_{L^2(\Gamma_1)}\].

Integrating over \((0,t)\) with \(t \in (0,T)\) on both sides of above formula and using Gronwall’s inequality, we obtain

\[\sup_{0 \leq t \leq T} \|e_T\|^2_{L^2(\Omega)} + \|\nabla e_T\|^2_{L^2(\Omega)} + \beta_1 \|e_T\|^2_{L^2((0,T) \times \Gamma_1)} \leq C\varepsilon. \quad (69)\]

From the estimates for \(\eta_k\) in Lemma 4.3, the error estimate for temperature field (43) is obtained.

**Step 2.** Estimate \(p_\varepsilon - p_0^\varepsilon\). By the weak forms (12) for \(p_\varepsilon\) and (28) for \(p_0\), for every \(\varphi \in V\),

\[\int_{\Omega} K^\varepsilon \nabla (p_\varepsilon - p_0^\varepsilon) \nabla \varphi dx = \int_{\Omega} (K^\varepsilon \nabla p^\varepsilon - K^0 \nabla p_0) \nabla \varphi dx\]

\[+ \int_{\Omega} (K^0 \nabla p_0 - K^\varepsilon \nabla p_0^\varepsilon) \nabla \varphi dx = I_1 + I_2. \quad (70)\]

The second term has been treated in the proof of Lemma 4.5. From (60),

\[I_2 = \varepsilon \int_{\Omega} K^\varepsilon \nabla \theta_2^\varepsilon \nabla \varphi dx + \int_{\Omega} r_1^\varepsilon \nabla \varphi dx - \int_{\Omega} r_2^\varepsilon \nabla \varphi dx, \quad (71)\]

where we have used the weak form of the boundary corrector \(\theta_2^\varepsilon\), which is the solution of the following problem

\[
\begin{cases}
-\nabla \cdot (K^\varepsilon \nabla \theta_2^\varepsilon) = 0, & \text{in } \Omega, \\
(K^\varepsilon \nabla \theta_2^\varepsilon) \cdot \nu = \left(\frac{\partial}{\partial x_k} (H^j_{ik} \frac{\partial p_0}{\partial x_j})\right) \cdot \nu, & \text{on } \partial \Omega \setminus \Gamma_2, \\
\theta_2^\varepsilon = 0, & \text{on } \Gamma_2
\end{cases}
\quad (72)\]

and thanks to the skew-symmetry of matrix \(H^j_{ik}\), the weak form is

\[\int_{\Omega} K^\varepsilon \nabla \theta_2^\varepsilon \nabla \varphi dx = \int_{\Omega} \frac{\partial}{\partial x_k} \left(H^j_{ik} \frac{\partial p_0}{\partial x_j}\right) \frac{\partial \varphi}{\partial x_i} dx, \quad \forall \varphi \in V. \quad (73)\]

We also need another boundary corrector \(\theta_2^\varepsilon\) such that

\[
\begin{cases}
-\nabla \cdot (K^\varepsilon \nabla \theta_2^\varepsilon) = 0, & \text{in } \Omega, \\
(K^\varepsilon \nabla \theta_2^\varepsilon) \cdot \nu = 0, & \text{on } \partial \Omega \setminus \Gamma_2, \\
\theta_2^\varepsilon = -\chi^j \frac{\partial p_0}{\partial x_j}, & \text{on } \Gamma_2
\end{cases}
\quad (74)\]

with a weak form as

\[\int_{\Omega} K^\varepsilon \nabla \theta_2^\varepsilon \nabla \varphi dx = 0, \quad \forall \varphi \in V. \]

Denote by \(e_p = p_\varepsilon - p_0^\varepsilon - \varepsilon \theta_1^\varepsilon - \varepsilon \theta_2^\varepsilon \in V\). Insert (71) into (70). Using the weak form of (74), we have

\[\int_{\Omega} K^\varepsilon \nabla e_p \nabla \varphi dx = I_1 + \int_{\Omega} r_1^\varepsilon \nabla \varphi dx - \int_{\Omega} r_2^\varepsilon \nabla \varphi dx. \quad (75)\]
To deal with the first term $I_1$, from problem (6) and homogenized problem (21), we have

$$I_1 = \int_\Omega (K^e \nabla p^e - K^0 \nabla p_0) \nabla \varphi dx$$

$$= \alpha \int_\Omega \left( \phi \frac{\partial T_e}{\partial t} - \phi_0 \frac{\partial T_0}{\partial t} \right) \varphi dx - \alpha \int_\Omega (K^0 \nabla p_0 T_0 - K^e \nabla p_2 T_e) \nabla \varphi dx$$

$$\equiv I_3 + I_4.$$  

The term $I_4$ has been treated in Remark 5. What’s left is to treat term $I_3$. To this purpose, we need further the assumption (H3). Denote by $w_\varepsilon = \frac{\partial T_2}{\partial t}$ and by $w_0 = \frac{\partial T_0}{\partial t}$. We have that $w_\varepsilon$ satisfies that

$$\begin{cases}
(1 + \phi_0 \gamma) \frac{\partial w_\varepsilon}{\partial t} - \nabla \cdot (D_T^0 \nabla w_\varepsilon) = 0, & \text{in } \Omega_T, \\
(D_T^0 \nabla w_\varepsilon) \cdot \nu = \beta_1 (\frac{\partial T_{out}}{\partial n} - w_\varepsilon), & \text{on } \Gamma_1 \times (0, T], \\
w_\varepsilon = 0, & \text{on } \Gamma_2 \times (0, T], \\
(D_T^0 \nabla w_\varepsilon) \cdot \nu = 0, & \text{on } \Gamma_3 \times (0, T], \\
w_\varepsilon(x, 0) = 0, & \text{in } \Omega,
\end{cases}$$

The corresponding homogenized problem is

$$\begin{cases}
(1 + \phi_0 \gamma) \frac{\partial w_0}{\partial t} - \nabla \cdot (D_T^0 \nabla w_0) = 0, & \text{in } \Omega_T, \\
(D_T^0 \nabla w_0) \cdot \nu = \beta_1 (\frac{\partial T_{out}}{\partial n} - w_0), & \text{on } \Gamma_1 \times (0, T], \\
w_0 = 0, & \text{on } \Gamma_2 \times (0, T], \\
(D_T^0 \nabla w_0) \cdot \nu = 0, & \text{on } \Gamma_3 \times (0, T], \\
w_0(x, 0) = 0, & \text{in } \Omega,
\end{cases}$$

By the argument used in Step 1 to derive the error estimate (43), we have that

$$\sup_{0 \leq t \leq T} \|w_\varepsilon - w_0\|_{L^2(\Omega)} \leq C \varepsilon^{\frac{1}{2}}.$$  

Now we can deal with the term $I_3$. By the above result and Lemma 4.6, we have

$$|I_3| = \left| \alpha \int_\Omega (\phi_0 w_\varepsilon - \phi_0 w_0) \varphi dx \right|$$

$$= \left| \alpha \int_\Omega \phi_\varepsilon (w_\varepsilon - w_0) \varphi dx - \alpha \int_\Omega (\phi_\varepsilon - \phi_0) w_0 \varphi dx \right|$$

$$\leq C \alpha \varepsilon \|\varphi\|_{L^2(\Omega)} + C \alpha \varepsilon \|w_0\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}.$$  

Choose $\varphi = e_\varepsilon$ in equation (75). Using (76) and (80) and Remark 5, we obtain

$$\lambda_2 \|\nabla e_\varepsilon\|^2_{L^2(\Omega)} \leq \int_\Omega K^e \nabla e_\varepsilon \nabla e_\varepsilon dx$$

$$\leq C_1 \alpha \left( (\varepsilon + \varepsilon^{1/2}) \|\nabla e_\varepsilon\|_{L^2(\Omega)} + \|T_\varepsilon - T_1\|_{L^2(\Omega)} \|\nabla e_\varepsilon\|_{L^2(\Omega)} \\
+ \|\nabla (p_\varepsilon - p_1)\|_{L^2(\Omega)} \|\nabla e_\varepsilon\|_{L^2(\Omega)} \right)$$

$$+ C(\|p_\varepsilon\|_{L^2(\Omega)} \|\nabla e_\varepsilon\|_{L^2(\Omega)} + \|p_1\|_{L^2(\Omega)} \|\nabla e_\varepsilon\|_{L^2(\Omega)}).$$
By (43),
\[ \|T - T_1\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}. \] (82)

By Lemma 4.2,
\[ \|\nabla(p - p_1)\|_{L^2(\Omega)} \leq \|\nabla e_0\|_{L^2(\Omega)} + \|\varepsilon\nabla\theta_1\|_{L^2(\Omega)} + \|\varepsilon\nabla\theta_2\|_{L^2(\Omega)} \]
\[ \leq \|\nabla e_0\|_{L^2(\Omega)} + C\varepsilon^{1/2}. \] (83)

Inserting (82)-(83) into (81), we obtain
\[ \|\nabla e_0\|_{L^2(\Omega)}^2 \leq C\varepsilon, \] (84)

since \(\alpha\) is small enough to satisfy that \(C_1\alpha \leq \lambda_2/2\).

Thanks to the estimates on \(\theta_1\) and \(\theta_2\) in Lemma 4.2, (84) means that
\[ \sup_{0 \leq t \leq T} \|p - p_1\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}. \] (85)

**Step 3.** Estimate \(C - C_1\). By the weak forms (13) for \(C\) and (29) for \(C_0\), we have for \(\forall v \in V\),
\[ \langle \phi, \frac{\partial}{\partial t}(C - C_1) \rangle_{H^{-1}(\Omega), H(\Omega)} + \int_\Omega D_C^c \nabla(C - C_1) \nabla v dx + \beta_2 \int_{\Gamma_1} (C - C_1)v ds \]
\[ = \langle (\phi - e_0) (C_0 - e_0) \nabla \xi, v \rangle_{H^{-1}(\Omega), H(\Omega)} + \int_\Omega (D_C^e \nabla C_0 - D_C^e \nabla C_1^e) \nabla v dx \]
\[ + \int_\Omega (K^0 \nabla p_0 C_0 - K^e \nabla p e C) \nabla v dx - \varepsilon \beta_2 \int_{\Gamma_1} \pi_x^2 \frac{\partial e_0}{\partial x_j} v ds \] (86)

If we denote \(e_\varepsilon = C - C_1 - \varepsilon \xi\), where \(\xi\) is boundary corrector as the solution of the following problem:
\[
\begin{cases}
\phi \frac{\partial}{\partial t}(D_C^c \nabla \xi) = 0, & \text{in } \Omega_T, \\
(D_C^c \nabla \xi) \cdot \nu = -\beta_2 \xi, & \text{on } \Gamma_1 \times (0, T), \\
(D_C^c \nabla \xi) \cdot \nu = 0, & \text{on } \Gamma_2 \times (0, T), \\
\xi|_{t=0} = 0, & \text{in } \Omega, \\
\end{cases}
\] (87)

then for \(e_\varepsilon = C - C_1 - \varepsilon \xi \in V\) we have,
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega e_\varepsilon dx + \int_\Omega D_C^e \nabla e_\varepsilon \nabla dx + \beta_2 \int_{\Gamma_1} e_\varepsilon^2 dx \]
\[ = -\varepsilon \int_\Omega \phi \pi_x^2 \frac{\partial}{\partial t} \frac{\partial C_0}{\partial x_j} \nabla e_\varepsilon dx + \int_\Omega (D_C^e \nabla C_0 - D_C^e \nabla C_1^e) \nabla e_\varepsilon dx \]
\[ + \int_\Omega (K^0 \nabla p_0 C_0 - K^e \nabla p e C) \nabla e_\varepsilon dx - \varepsilon \beta_2 \int_{\Gamma_1} \pi_x^2 \frac{\partial C_0}{\partial x_j} e_\varepsilon ds \]
\[ - \int_\Omega (\phi - \phi_0) \frac{\partial C_0}{\partial t} e_\varepsilon dx. \] (88)

Thanks to \(\int_Y (\phi(x, y) - \phi_0(x)) dy = 0\), we can get by Lemma 4.6 that
\[ |\int_\Omega (\phi - \phi_0) \frac{\partial C_0}{\partial t} e_\varepsilon dx| \leq C\varepsilon \|\frac{\partial C_0}{\partial t}\|_{H^1(\Omega)} \|\nabla e_\varepsilon\|_{L^2(\Omega)}. \] (89)

The second term on the right hand side of (88) can be bounded by Lemma 4.4. The third term on the right hand side can be treated by Lemma 4.5. The fourth
term on the right hand side can be handled by Trace Theorem. So we can get the estimate of $C_\varepsilon - C_1^\varepsilon$
\[ \sup_{0 \leq t \leq T} \| C_\varepsilon - C_1^\varepsilon \|_{L^2(\Omega)} + \| \nabla (C_\varepsilon - C_1^\varepsilon) \|_{L^2(\Omega_T)} \leq C\varepsilon^{1/2}. \] (90)

Now we complete the proof of Theorem 4.1.

5. Conclusion. In summary, a mathematical model is established for the thermal-hydro-mass transfer processes in porous media. Then the corresponding homogenized system is derived with the help of two-scale convergence. Some error estimates are also presented for the first order expansions under some higher regular assumptions.

Acknowledgments. The authors would like to thank the referees for their valuable suggestions to improve the paper. This work is supported in part by NSF of China under the Grants 10871190 and 11271281.

REFERENCES

[1] G. Allaire, Homogénéisation et convergence à deux échelles, application à un problème de convection-diffusion, C. R. Acad. Sci. Paris, 312 (1991), 581–586.
[2] G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal., 23 (1992), 1482–1518.
[3] G. Allaire, Homogenization of the unsteady Stokes equations in porous media, in Progress in partial differential equations: Calculus of variations, applications (eds. Bandle C, et al.) (Pont-à-Mousson, 1991), Pitman Research Notes in Math. Ser., 267, Longman Higher Education, Harlow, 1992, 109–123.
[4] I. Babuška, Solution of problem with interfaces and singularities, in Mathematical Aspects of Finite Elements in Partial Differential Equations (ed. C. de Boor), Academic Press, New York, 1974, 213–277.
[5] A. Bensoussan, J. L. Lions and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.
[6] G. S. Bodvarsson, W. Boyle, R. Patterson and D. Williams, Overview of scientific investigations at Yucca Mountain-the potential repository for high-level nuclear waste, J. Contam. Hydro., 38 (1999), 3–24.
[7] T. A. Buscheck, N. D. Rosenberg, J. A. Blink, Y. Sun and J. Gansemer, Analysis of thermohydrologic behavior for above-boiling and below-boiling thermal-operating modes for a repository at Yucca Mountain, J. Contam. Hydro., 62-63 (2003), 441–457.
[8] Z. M. Chen, W. B. Deng and H. Ye, A new upscaling method for the solute transport equations, Discrete Contin. Dyn. Syst., Ser. A, 13 (2005), 941–960.
[9] Z. M. Chen and T. Y. Hou, A mixed multiscale finite element method for elliptic problems with oscillating coefficients, Math. Comp., 72 (2003), 541–576.
[10] D. Cioranescu and P. Donato, An Introduction to Homogenization, Oxford Lecture Series in Mathematics and Its Applications, 17, Oxford University Press, 1999.
[11] H. I. Ewe and E. Sanchez-Palencia, Some thermal problems in flow through a periodic model of porous media, Int. J. Eng. Sci., 19 (1981), 117–127.
[12] L. C. Evans, Partial Differential Equations, American Mathematical Society, Providence, 1998.
[13] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Berlin Heidelberg New York, 1983.
[14] L. N. Guimaraes, A. Gens, M. Sanchez and S. Olivella, THM and reactive transport analysis of expansive clay barrier in radioactive waste isolation, Commun. Numer. Meth. Engng., 22 (2006), 849–859.
[15] H. Haller, Verbundwerkstoffe mit Formgedächtnislegierung: Mikromechanische Modellierung und Homogenisierung, Dissertation, TU-München 1997.
[16] G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, 1996.
[17] E. Marusic-Paloka and A. Piatnitski, Homogenization of a nonlinear convection-diffusion equation with rapidly oscillating coefficients and strong convection, *J. Lond. Math. Soc.*, 72 (2005), 391–409.

[18] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.*, 20 (1989), 608–623.

[19] O. A. Oleinik, A. S. Shamaev and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, Studies in Mathematics and its Applications, 26, North-Holland Publishing Co., Amsterdam, 1992.

[20] S. B. Otsmane, G. Francfort and F. Murat, Correctors for the homogenization of the wave and heat equations, *J. Math. Pures Appl.*, 71 (1992), 197–231.

[21] D. Patriarche, E. Ledoux, J. L. Michelot, R. S. Coincon and S. Savoye, Diffusion as the main process for mass transport in very low water argillites: 2. Fluid flow and mass transport modeling, *Water Resource Research*, 40 (2004), W01516.

[22] G. A. Pavliotis and A. M. Stuart, *Multiscale Methods Averaging and Homogenization*, Springer, 2008.

[23] J. Rutqvist and C. F. Tsang, Analysis of thermal-hydrologic-mechanical behavior near an emplacement drift at Yucca Mountial, *J. Contam. Hydro.*, 62-63 (2003), 637–652.

[24] H. R. Thomas, Y. He, M. R. Sansom and C. L. W. Li, On the development of a model of the thermo-mechanical-hydraulic behaviour of unsaturated soils, *Engineering Geology*, 41 (1996), 197–218.

[25] C. F. Tsang, O. Stephansson and J. A. Huds, A discussion of thermo-hydro-mechanical (THM) process associated with nuclear waste repositories, *Int. J. Rock Mech. Min. Sci.*, 37 (2000), 397–402.

[26] V. V. Zhikov, S. M. Kozlov and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer Berlin, 1994.

Received December 2012; revised February 2013.

E-mail address: xsztr@mail.ustc.edu.cn
E-mail address: xyyue@suda.edu.cn