RH-Dependent Estimates of Remainder in Modified Mertens Formula

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Abstract: Assuming the validity of Riemann Hypothesis (RH), we derive the explicit bilateral estimates ("narrow passage") for the remainder in the modified Mertens asymptotic formula for the sums of primes’ reciprocals. These results are reversible, thus yielding some new criteria for RH.

Keywords: Mertens formula, Ingham method, Riemann Hypothesis.

Bibliography: 7 items

1. Notations, brief history and main results

As usually, let $N, j, k, m, n$ (perhaps with indices) run the set $\mathbb{N}$ of all positive integers, $N_0 := \mathbb{N} \cup \{0\}$, $p$ run the set $\mathbb{P} := \{p_1, p_2, \ldots\}$, $p_j < p_{j+1}$, of all primes, $\varepsilon$ be an arbitrary positive number, $\delta_k$ denote sequences, which → $+0$ (perhaps different even within one and the same formula); $C(a)$ stand for positive constants which may depend only on a parameter $a$; symbols $\triangleright$ and $\square$ denote the proof’s beginning and end; $\log x$ and $\gamma$ stand (resp.) for the natural logarithm of a positive $x$ and the Euler-Masceroni constant:

$$\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577215 \ldots$$ \hspace{1cm} (1.1)

In 1874 F. Mertens [1] proved his famous asymptotic formula

$$S(x) := \sum_{p \leq x} \log \frac{p}{p-1} = \log \log x + \gamma + R(x) \text{ with } R(x) = O \left( \frac{1}{\log x} \right). \hspace{1cm} (1.2)$$

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1. This work was partly supported by the grant of Russian Foundation of Fundamental Research (project # 14−01−00684.)

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In 1984 assuming RH G. Robin [2, Th. 3] has come to the fundamentally stronger estimate:

$$|R(x)| \leq \frac{\log x}{8\pi \sqrt{x}}, \quad x > X_0.$$  \hspace{1cm} (1.3)

J.-L. Nicolas (1983) has considered the modified Mertens formula:\(^3\):

$$S(x) = \log \log \theta(x) + \gamma + Q(x), \quad x \geq 3.$$  \hspace{1cm} (1.4)

which differs from (1.2) by replacing $x$ in log log by the first Chebyshev function:

$$\theta(x) := \sum \{\log p : p \leq x\},$$

cf. [5, S. 3.3], and the following assertion was established [3, th. 3]:

**Proposition 1.** Each of the two conditions is sufficient for RH:

(i) \hspace{1cm} $\forall \varepsilon > 0 : \ B^+_\varepsilon := \lim \sup Q(x) x^{0.5-\varepsilon} < +\infty$,

(ii) \hspace{1cm} $\forall \varepsilon > 0 : \ B^-_\varepsilon := \lim \inf Q(x) x^{0.5-\varepsilon} > -\infty$.  \hspace{1cm} (1.5)

Later the author, basing on the connection between (1.4) and the Ramanujan inequality for Gronwall numbers, has proved in [4]:

**Proposition 2.** The relationship

$$A^+ := \lim \sup Q(x) \sqrt{x} \log x < \infty,$$  \hspace{1cm} (1.6)

is necessary for RH, and this being the case, then necessarily $A^+ \leq 2\sqrt{2}$.

The aim of this paper is to strengthen these results (using different approach) by conditional (RH-dependent) narrow estimates for $A^+$ and for the quantity:

$$A^- := \lim \inf Q(x) \sqrt{x} \log x.$$  \hspace{1cm} (1.7)

**Theorem.** Assume RH. Then one has:

(i) \hspace{1cm} $A^+ \leq 2.5$, \hspace{1cm} (ii) \hspace{1cm} $A^- \geq 1.5$.  \hspace{1cm} (1.8)

**Remark 1.** Directly from definitions, Propositions 1, 2 and the Theorem it follows that

$$\text{RH} \Rightarrow (1.8)(\text{ii}) \Rightarrow (1.5)(\text{ii}) \Rightarrow \text{RH} \Rightarrow (1.8)(\text{i}) \Rightarrow (1.6) \Rightarrow (1.5)(\text{i}) \Rightarrow \text{RH}.$$  \hspace{1cm}

Therefore the Theorem demonstrates, that all six relationships in this chain are equivalent.

Some other criteria for RH in diverse terms will be adduced in Sect. 3.

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\(^3\) For $x < 3$ log log $\theta(x)$ cannot be defined as a real number
2. Proof of the Theorem

We begin with some preliminary assertions.
Recall that RH may be reformulated in terms of the deviation of Chebyshev
function \( \theta(x) \) from \( x \): \( \Delta(x) := \theta(x) - x \), which is unconditionally \( O(x \exp(-c \sqrt{\log x}) \)
by virtue of PNT, cf [5], th. 5.19.
Therefore (cf. also (1.2), (1.4)) for some \( \tilde{x} \) in between of \( x \) and \( \theta(x) \) one has:

\[
Q(x) = R(x) + \log \log x - \log \log \theta(x) = R(x) - \frac{\Delta(x)}{\tilde{x} \log \tilde{x}} = O \left( \frac{1}{\log x} \right). \tag{2.1}
\]

The proposition below is well known classical result by H. von Koch (1901),
cf., e.g., [5], th. 5.21.

**Proposition 3.** The following assertions are equivalent:

(i) RH holds true; (ii) \( |\Delta(x)| < x^{0.5 + \varepsilon} \), \( \forall \varepsilon > 0 \), \( \forall x > X_\varepsilon \);

(iii) \( |\Delta(x)| \leq \frac{\sqrt{x} \log^2 x}{8\pi} \), \( x > X_0 \). \tag{2.2}

In the Ingham’s monograph [6], S. V.10, form. (35), the helpful conditional
estimate for the primitive of \( (\psi(x) - x) \) is adduced.

**Proposition 4.** Assume RH; then \( \left| \int_0^x (\psi(t) - t) \, dt \right| < 0.1x^{3/2}, \ x < X_0 \).

Whence, taking into account the relationship \( \theta(x) = \psi(x) - \sqrt{x} + O(x^{1/3}) \),
one obtains:

**Proposition 5.** The validity of RH implies the following bilateral estimates
for the \( (\theta(x) - x) \)-primitive:

\[
\Phi(x) := \int_0^x \Delta(t) \, dt = (-2/3 + b(x))x^{3/2}; \quad |b(x)| < 0.1, \ x > X_0. \tag{2.3}
\]

The main point of the Theorem’s proof is the following corollary of RH.

**Lemma 1.** Assume RH. Introduce the quantities \( 3 \leq x < y < \infty \):

\[
H(x, y) := \sum_{x < p \leq y} \frac{1}{p} - \log \log \theta(y^+) + \log \log \theta(x); \quad H(x) := \lim_{y \to \infty} H(x, y). \tag{2.4}
\]

Then:

(i) \( -2.5 \leq \liminf H(x) \sqrt{x} \log x \); (ii) \( \limsup H(x) \sqrt{x} \log x \leq -1.5. \tag{2.5} \)
First let us note that the Taylor formula implies:

\[
\log \log \theta(x) - \log \log x = \frac{\Delta(x)}{x \log x} - \frac{\log \hat{x} + 1}{2\hat{x}^2 \log^2 \hat{x}} \Delta^2(x),
\]

(2.6)

where \( \hat{x} \) is a certain number in between of \( x \) and \( \theta(x) \).

On the other hand, integrating twice by parts, one obtains (cf. also (2.3)):

\[
\sum_{x < p \leq y} \frac{1}{p} - \log \log y + \log \log x = \int_x^{y^+} \frac{d\Delta(t)}{t \log t} = \frac{\Delta(t)}{t \log t} \bigg|_x^{y^+} - \int_x^{y^+} \Delta(x) \left( \frac{1}{t \log t} \right)' \, dt
\]

\[
= \frac{\Delta(y^+)}{y \log y} - \frac{\Delta(x)}{x \log x} - \Phi(t) \left( \frac{1}{t \log t} \right)' \bigg|_x^{y^+} + \int_x^{y^+} \Phi(t) \left( \frac{1}{t \log t} \right)'' \, dt.
\]

(2.7)

Making here \( y \) tend to \( \infty \), one comes to

\[
\lim_{y \to \infty} \left( \sum_{x < p \leq y} \frac{1}{p} - \log \log y + \log \log x \right) = -\frac{\Delta(x)}{x \log x} + D(x) + E(x) + F(x),
\]

(2.8)

where the notations are used:

\[
D(x) := -\frac{\Phi(x) (\log x + 1)}{x^2 \log^2 x}, \quad E(x) := \int_x^{+\infty} \frac{\Phi(t)}{t^3 \log t} \left( 2 + \frac{3}{\log t} + \frac{2}{\log^2 t} \right) \, dt,
\]

\[
F(x) := -\frac{\log \hat{x} + 1}{2\hat{x}^2 \log^2 \hat{x}} \Delta^2(x).
\]

(2.9)

Summing this with (2.6) (the terms, involving \( \Delta(x) \), underlined in (2.6), (2.8), mutually reduce), and taking into account (2.2)(iii) and (2.3), one comes to: \( H(x) = D(x) + E(x) + O(\log^3 x/x) \). But by virtue of (2.3) one has:

\[
\frac{17}{30} \leq \lim \inf D(x) \sqrt{x} \log x \leq \lim \sup D(x) \sqrt{x} \log x \leq \frac{23}{30};
\]

\[
-\frac{92}{30} \leq \lim \inf E(x) \sqrt{x} \log x \leq \lim \sup E(x) \sqrt{x} \log x \leq -\frac{68}{30},
\]

(2.10)

and thus \( -2.5 - \varepsilon < H(x) \sqrt{x} \log x < -1.5 + \varepsilon \), for all \( x > X_\varepsilon \), which coincides with (2.5) \( \Box \).
Remark 2. It is important to emphasize that without any a priori estimates for $\Delta(x)$, i.e. unconditionally, from (2.8), (2.9) the inequality follows:

$$H(x) \leq D(x) + E(x), \quad \forall x > 3,$$

(2.11)

because the quantity $F(x)$ is always non-positive.

Also it’s easy to check that if the function $|b(x)|$ in (2.3) would be bounded by some $\delta_0 > 0$ (instead of 0.1), then the boundaries in (2.5) would be $-2 \pm 5\delta_0$.

Proceeding to the proof of the Theorem itself, let us note that defining formula (1.4) immediately implies:

$$\gamma = \lim_{y \to \infty} \left( \sum_{p \leq y} \log \frac{p}{p-1} - \log \log \theta(y) \right).$$

(2.12)

and taking into account definitions (1.4) and (2.4), one obtains unconditionally:

$$Q(x) + H(x) = \sum_{p \leq x} \log \frac{p}{p-1} - \gamma + \lim_{y \to \infty} \left( \sum_{x < p \leq y} \frac{1}{p} - \log \log \theta(y) \right)$$

$$= \lim_{y \to \infty} \left( \sum_{p \leq y} \log \frac{p}{p-1} - \log \log \theta(y) \right) - \gamma + \sum_{p > x} \left( \frac{1}{p} - \log \frac{p}{p-1} \right)$$

$$= \sum_{p > x} \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \right) = - \sum_{p > x} \sum_{k=2}^{\infty} \frac{1}{kp^k} = O \left( \frac{1}{x} \right),$$

(2.13)

and thus (2.5)(i)(ii) imply (1.8)(i)(ii) (respectively) $\Box$.

This completes the Theorem’s proof.

3. Some corollaries and conclusive remarks

The Theorem allows to deduce some new conditions equivalent to RH in terms of the function $\Phi(x)$ and of the primitive of $(\psi(x) - x)^2$.

Corollary 1. In order RH hold true it is necessary and sufficient that at least one (and then all) of the three conditions be fulfilled:

$$\begin{align*}
(i) \quad & \Phi(x) = O(x^{1.5+\varepsilon}), \\
(ii) \quad & \Phi(x) = O(x^{1.5}); \\
(iii) \quad & \int_0^x (\psi(t) - t)^2 dt = O(x^2).
\end{align*}$$

(3.1)

$\triangleright$ Necessity. The implication $\text{RH} \Rightarrow (iii)$ was established by H. Cramér (1921), cf. [5], th. 13.5. Now it remains to notice that in (3.1) (iii)$\Rightarrow (ii) \Rightarrow (i) \Box$. 

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Sufficiency. Let (3.1)(i) be fulfilled, i.e. \(|\Phi(x)| \leq C(\varepsilon)x^{1.5+\varepsilon}, \forall x, \varepsilon > 0\). then from (2.13), (2.11) and (2.9) one obtains for all \(x > X_\varepsilon\):

\[-Q(x) = H(x) + O\left(\frac{1}{x}\right) \leq D(x) + E(x) + O\left(\frac{1}{x}\right)\]

\[\leq C_\varepsilon \left(\frac{x^{1.5+\varepsilon}}{x^2 \log^2 x} + 2 \int \frac{t^{1.5+\varepsilon}}{x^3 \log t} dt\right) \leq C_\varepsilon x^{-0.5+\varepsilon}, \quad (3.2)\]

whence \(Q(x) \geq -C_\varepsilon x^{-0.5+\varepsilon}, \quad x > X_\varepsilon\); but this coincides with (1.5)(ii), which in turn (by virtue of Proposition 1) implies RH □.

The reasonings in the proofs of the Lemma and the Theorem show that

both conditions (2.5)(i)(ii) are (separately) equivalent to RH. \quad (3.3)

This allows to deduce the RH-criteria in terms of the consequences:

\[U_k := \sum_{j>k} \left(\frac{1}{p_j} - \frac{1}{\theta_j}\right), \quad \text{where} \quad \theta_j := \theta(p_j); \quad V_k := \sum_{j>k} \left|\frac{1}{p_j} - \frac{1}{\theta_j}\right|. \quad (3.4)\]

Corollary 2. RH is valid if and only if at least one (and then all) of the following six conditions is fulfilled for any \(\varepsilon > 0, \quad k > K_\varepsilon\):

(i) \(U_k < k^{-0.5+\varepsilon}\); \quad \text{(ii) \(U_k > -k^{-0.5+\varepsilon}\);}

(iii) \(U_k \sqrt{\log p_k} < -1.5 + \varepsilon; \quad \text{(iv) \(U_k \sqrt{\log p_k} > -2.5 - \varepsilon;\)}

(v) \(V_k < k^{-0.5+\varepsilon}\); \quad \text{(vi) \(V_k \sqrt{p_k / \log p_k} < \frac{1 + \varepsilon}{4\pi}. \quad (3.5)\)

First we note that (cp. (2.6)) for certain \(\tau_j \in (\theta_{j-1}, \theta_j)\) one has:

\[\log \log \theta_j - \log \log \theta_{j-1} = \frac{\log p_j}{\theta_j \log \theta_j} + \frac{\log \tau_j + 1}{2\tau_j^2 \log^2 \tau_j} \log^2 p_j\]

\[= \frac{1}{\theta_j} - \frac{\log \theta_j - \log p_j}{\theta_j \log \theta_j} + O\left(\frac{1}{p_j^2 \log p_j}\right) = \frac{1}{\theta_j} + O\left(\frac{1}{p_j^2 \log p_j}\right). \quad (3.6)\]

Here we have also taken into account that \(\log \theta_j - \log p_j = (\theta_j - p_j) / \tilde{x}_j^*,\)

where \(x_j^*\) is some number in between of \(p_j\) and \(\theta_j\), and \(\theta_j \approx p_j\).

Therefore, for \(H_j := H(p_j) \quad \text{(cf. (2.4)) one has:}\)
\[ H_{j-1} - H_j = \frac{1}{p_j} - \frac{1}{\theta_j} + O\left(\frac{1}{p_j^2 \log p_j}\right). \]  \hspace{1cm} (3.7)

Summing these relations from \( j = k \) to infinity, one obtains

\[ H_k = U_k + O(1/k) = -Q(x) + O(1/k), \quad x \in (p_{k-1}, p_k]. \]  \hspace{1cm} (3.8)

and thus (3.5)(i)(ii)(iii)(iv) are equivalent (resp.) to (1.5)(i)(ii), (1.8)(i)(ii), each of which in turn (cf. Sect. 1) \( \iff \) RH.

At last, taking into account that

\[ p_j \approx j \log j, \quad j \to \infty, \]

one obtains by virtue of (2.2)(iii) under assumption of RH:

\[
V_k = \sum_{j=k}^{\infty} \left| \frac{\theta_j - p_j}{\theta_j p_j} \right| \leq \frac{1 + \delta_k}{8\pi} \sum_{j=k}^{\infty} \frac{\log^2 p_j}{p_j \sqrt{p_j}}
\]

\[ \approx \frac{1}{8\pi} \sum_{j=k}^{\infty} \frac{\sqrt{\log j}}{j \sqrt{j}} \approx \frac{\sqrt{\log k}}{4\pi \sqrt{k}} \approx \frac{\log p_k}{\sqrt{p_k}}, \quad \text{where} \ \delta_k \to 0. \]  \hspace{1cm} (3.9)

Hence, \( RH \Rightarrow (3.5)(vi) \Rightarrow (3.5) (v) \Rightarrow (3.5)(i) \Rightarrow RH \) \( \square \)

**Remark 3.** If RH holds true, then Theorem shows not only the fast, but also quasi-monotonic decrease of the remainder \( Q(x) \) in (1.4), in the sense that relations \( y > ax, a > 25/9, x > X_a, \) imply \( Q(x) > Q(y) > 0. \)

The oscillation properties of the remainder \( R(x) \) in the original Mertens formula (1.2), studied by H. Diamond and J. Pintz [7], may be easily derived from the Theorem for the case, when RH is valid.

Indeed, according to (2.1) one has unconditionally:

\[ R(x) = Q(x) + \frac{\Delta(x)}{x \log x}, \quad \text{where} \ \tilde{x} \approx x. \]  \hspace{1cm} (3.10)

Therefore assuming RH, which according to the Theorem (cf. (1.8)) implies \( Q(x) = O(1/\sqrt{x} \log x) \), one obtains that for any positive function \( \eta(x) \to +\infty: \)

\[ \begin{align*}
(i) \quad & \Delta(x) = \Omega_{\pm}(\sqrt{x} \eta(x)) \iff
(ii) \quad & R(x) = \Omega_{\pm}\left(\frac{\eta(x)}{\sqrt{x} \log(x)}\right)
\end{align*} \]  \hspace{1cm} (3.11)

By virtue of the J. Littlewood result (1914), (cf., e. g. [5], th. 6.20), this holds true for \( \eta(x) = \log \log \log x. \)

The same arguments allow to conclude that the estimate \( R(x) = O(\log x/\sqrt{x}) \), cf. (1.3), is not only necessary, but also sufficient for RH.
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