ON ASYMPTOTIC PHASE OF DYNAMICAL SYSTEM
HYPERBOLIC ALONG ATTRAJECTING INVARIANT MANIFOLD

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ABSTRACT. We consider a dynamical system which has the hyperbolic structure along an
attracting invariant manifold $M$. The problem is whether every motion starting in a neigh-
borhood of $M$ possesses an asymptotic phase, i.e. eventually approaches a particular motion
on $M$. Earlier, positive solutions to the problem were obtained under the condition that the
decay rate of solutions toward the manifold exceeds the decay rate of the solutions within the
manifold. We show that in our case the above condition is not necessary. To prove that a
neighborhood of $M$ is filled with motions for each of which there exists an asymptotic phase
we apply the Brouwer fixed point theorem. An invariant foliation structure which appears in
the neighborhood of $M$ is discussed.

1. Introduction

It is well known that under quite general conditions, motions of dissipative dynamical system
evolve towards attracting invariant sets. One may reasonably expect that the behavior of system
on attracting set adequately displays main asymptotic properties of system motions in the whole
phase space. It is important to note that in many cases the dimension of attracting set such,
e.g., as fixed point, limit cycle, invariant torus, strange or chaotic attractor, is essentially lower
than the dimension of the total phase space. This circumstance can help us to simplify the
qualitative analysis of the system under consideration.

Nevertheless we should keep in mind that there are cases where no motion starting outside
the attracting invariant set exhibits the same long time behavior as a motion on the set. As an
example consider the planar system

$$
\begin{cases}
\dot{x} = x(1 - x^2 - y^2)^3 - y(1 + x^2 + y^2), \\
\dot{y} = x(1 + x^2 + y^2) + y(1 - x^2 - y^2)^3
\end{cases}
$$

which in polar coordinates $(\varphi \mod 2\pi, r)$ takes the form

$$
\dot{\varphi} = 1 + r^2, \quad \dot{r} = r (1 - r^2)^3.
$$

The limit cycle of the system, $r = 1$, attracts all the orbits except the equilibrium $(0, 0)$. Let
$\varphi(t; \varphi_0, r_0)$ be the $\varphi$-coordinate of the motion starting at point $(r_0 \cos \varphi_0, r_0 \sin \varphi_0)$. Obviously,
$\varphi(t; \varphi_*, 1) = 2t + \varphi_*$, but if $r_0 \notin \{0, 1\}$, then it is not hard to show that

$$
\lim_{t \to \infty} |\varphi(t; \varphi_0, r_0) - \varphi(t; \varphi_*, 1)| = \infty \quad \forall \{\varphi_0, \varphi_*\} \subset [0, 2\pi).
$$

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Let \( \{ g^t : \mathcal{M} \to \mathcal{M} \}_{t \in \mathbb{R}} \) be a flow on a metric space \((\mathcal{M}, \rho(\cdot, \cdot))\) with metric \(\rho(\cdot, \cdot) : \mathcal{M} \to \mathbb{R}_+\), and let there exists an invariant attracting set \( \mathcal{A} \subset \mathcal{M} \) with a basin \( \mathcal{B} \):

\[
\lim_{t \to \infty} \rho \left( g^t(p), \mathcal{A} \right) = 0 \quad \forall p \in \mathcal{B}.
\]

It is said that a motion \( t \mapsto g^t(p), p \in \mathcal{B}, \) has an asymptotic phase if there exists \( p^* \in \mathcal{M} \) such that

\[
\rho \left( g^t(p), g^t(p^*) \right) \to 0, \quad t \to \infty.
\]

The following problem arises: what are the conditions guaranteeing the existence of asymptotic phase? The answer to this problem is rather important, since the existence of asymptotic phase for any \( p \in \mathcal{B} \) ensures that any motion starting in \( \mathcal{B} \) eventually behaves like a corresponding motion on \( \mathcal{A} \), and thus the flow restricted to attractor \( \mathcal{A} \) faithfully describes the long-time behavior of the motions starting in \( \mathcal{B} \).

The above problem was studied in a series of papers. The most complete examination concerns the case where the attracting set is either a cycle or a manifold fibered by cycles \([3, 5, 6, 8, 10]\). N. Fenichel \([9]\) established the existence and uniqueness of asymptotic phase for discrete dynamical system possessing exponentially stable overflowing invariant manifold with, so-called, expanding structure. A. M. Samojlenko \([11]\) and W. A. Coppel \([7]\) studied the problem for the case of exponentially stable invariant torus. B. B. Aulbach \([2]\) proved the existence of asymptotic phase for motions approaching a hyperbolic invariant manifold under assumption that the latter carries a parallel flow. In \([1]\), A. A. Bogolyubov and Yu. A. Il’in established sufficient conditions ensuring the existence of asymptotic phase for stable invariant torus (however the authors did not use the notion of asymptotic phase explicitly). The conditions in \([4]\) admit non-exponential stability of invariant torus but exclude the case of exponential divergence for trajectories within the torus. (See \([2]\) for more comments on the issue).

As was pointed out in \([2]\) the conditions ensuring the existence of an asymptotic phase involve the requirement that the decay rate of solutions toward the manifold is greater than the decay rate of the solutions within the manifold. The aim of the present paper is to show that such a condition is not a necessary one. Like in \([9]\), we consider the case of asymptotically stable hyperbolic invariant manifold, but in contrary to the mentioned article we deal with a flow rather then a cascade, and besides, in our case, the decay rate of solutions toward the manifold need not be greater then the decay rate of the solutions within the manifold. Actually we exploit the maximal of negative Lyapunov exponents characterizing the both rates. Our main observation is that one can weaken the expanding structure condition by abandoning the requirement of asymptotic phase uniqueness. To prove that a neighborhood of stable invariant manifold is filled with motions for each of which there exists an asymptotic phase we apply the Brouwer fixed point theorem, rather than the theorem on invariance of domain for homeomorphisms as in \([9]\).

2. A theorem on the existence asymptotic phase

Let a \( C^2\)-vector field \( v \) generates the flow \( \{ \chi^t(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \}_{t \in \mathbb{R}} \) in space \( \mathbb{R}^n \) endowed with scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| : = \sqrt{\langle \cdot, \cdot \rangle} \). Assume that there is a domain \( \mathcal{D} \subset \mathbb{R}^n \) containing a compact attracting invariant \( C^1\)-sub-manifold \( \mathcal{M} \hookrightarrow \mathcal{D} \) of dimension \( m < n \):

\[
\chi^t(\mathcal{M}) = \mathcal{M} \quad \forall t \in \mathbb{R}; \quad \lim_{t \to \infty} \inf_{\xi \in \mathcal{M}} \| \chi^t(x) - \xi \| = 0 \quad \forall x \in \mathcal{D}.
\]

Consider the autonomous system
\[ \dot{x} = v(x). \] (2.1)

From
\[ \frac{d}{dt} \frac{\partial \chi^t(x)}{\partial x} = v' \left( \chi^t(x) \right) \frac{\partial \chi^t(x)}{\partial x} \]

it follows that
\[ X^t(x) := \frac{\partial \chi^t(x)}{\partial x} \]
is the normed fundamental matrix of variational system
\[ \dot{y} = v' \left( \chi^t(x) \right) y, \]
and the equality \( \chi^{t+s}(x) = \chi^t \circ \chi^s(x) \) implies the co-cycle property of \( X^t(x) \):
\[ X^{t+s}(x) = X^t(\chi^s(x)) X^s(x), \quad X^{-s}(\chi^s(x)) = [X^s(x)]^{-1}. \] (2.2)

We say that the flow \( \{ \chi^t(.) \} \) is hyperbolic along the manifold \( \mathcal{M} \) (equivalently, the manifold \( \mathcal{M} \) is said to be hyperbolic w.r.t. the flow \( \{ \chi^t(.) \} \) if:
1. at every point \( x \in \mathcal{M} \), the tangent space \( T_x \mathbb{R}^n \simeq \mathbb{R}^n \) is decomposed into the direct sum of three sub-spaces:
\[ T_x \mathbb{R}^n = \mathbb{L}_x^- \oplus \mathbb{L}_x^+ \oplus \mathbb{L}_x^0, \quad \dim \mathbb{L}_x^\pm,0 = n_\pm,0, \] (2.3)
where \( \mathbb{L}_x^\pm := \{ \lambda v(x) \}_{\lambda \in \mathbb{R}} \) is a 1-D subspace spanned by the vector \( v(x) := \frac{d}{dt} \big|_{t=0} \chi^t(x) \);
2. the correspondences \( \mathcal{M} \ni x \mapsto \mathbb{L}_x^\pm \) define continuous and \( X^t \)-invariant fields of planes \( \{ \mathbb{L}_x^\pm \}_{x \in \mathcal{M}} \):
\[ X^t(x) \mathbb{L}_x^\pm = \mathbb{L}_x^\pm(\chi^t(x)) \quad \forall (t, x) \in \mathbb{R} \times \mathcal{M}; \]
3. there exist constants \( c \geq 1, \alpha > 0 \) such that
\[ \| X^t(x) \eta \| \leq c e^{-\alpha t} \| \eta \| \quad \forall t \geq 0, \forall x \in \mathcal{M}, \forall \eta \in \mathbb{L}_x^-, \] (2.4)
\[ \| X^t(x) \eta \| \leq c e^{\alpha t} \| \eta \| \quad \forall t \leq 0, \forall x \in \mathcal{M}, \forall \eta \in \mathbb{L}_x^+, \] (2.5)

Observe that
\[ \frac{d}{dt} \chi^t(x) = v' \left( \chi^t(x) \right) \Rightarrow \frac{d}{dt} v \left( \chi^t(x) \right) = v' \left( \chi^t(x) \right) v \left( \chi^t(x) \right) \Rightarrow v \left( \chi^t(x) \right) = X^t(x) v(x). \]
Thus, in addition to (2.4), (2.5), we may consider that
\[ \| X^t(x) \eta \| \leq c \| \eta \| \quad \forall t \in \mathbb{R}, \forall x \in \mathcal{M}, \forall \eta \in \mathbb{L}_x^0. \] (2.6)

In what follows, we will consider the case where the sub-manifold \( \mathcal{M} \) is stable, and thus, \( \mathbb{L}_x^+ \subset T_x \mathcal{M} \) for any \( x \in \mathcal{M} \). The space \( \mathbb{L}_x^- \) admits a decomposition
\[ \mathbb{L}_x^- = \mathbb{J}_x^- \oplus \mathbb{K}_x^- \]
where \( \mathbb{K}_x := T_x \mathcal{M} \cap \mathbb{L}_x^- \) and \( \mathbb{J}_x^- \) is a complement of \( \mathbb{K}_x^- \) in \( \mathbb{L}_x^- \). Hence,
\[ T_x \mathbb{R}^n = T_x \mathcal{M} \oplus \mathbb{J}_x^- , \quad T_x \mathcal{M} = \mathbb{K}_x^- \oplus \mathbb{L}_x^+ \oplus \mathbb{L}_x^0. \]
Let $T^\perp \xi \mathcal{M}$ stands for the orthogonal complement of tangent space $T \xi \mathcal{M}$ considered as a subspace of $\mathbb{R}^n$. Obviously, $\dim T^\perp \xi \mathcal{M} = \dim \mathbb{J}^-_{\xi}$. Since $\mathcal{M}$ is compact, then there is sufficiently small $\delta > 0$ such that for any $x \in \mathcal{M}$ the angle between $\mathbb{J}^-_{\xi}$ and $T_x \mathcal{M}$ exceeds $\delta$. This implies that there is a number $r \in (0, 1)$ such that the set

$$\mathcal{U}_r(\mathcal{M}) := \{(\xi, \zeta) : \xi \in \mathcal{M}, \zeta \in \mathbb{J}^-_{\xi}, \|\zeta\| < r\}$$

forms a tubular neighborhood of $\mathcal{M}$. Obviously,

$$\mathcal{U}_r(\mathcal{M}) \subset \mathcal{N}_r(\mathcal{M}) := \{(\xi, \eta) : \xi \in \mathcal{M}, \eta \in \mathbb{L}^-_{\xi}, \|\eta\| < r\}.$$  

The mapping $\mathcal{N}_r(\mathcal{M}) \ni (\xi, \eta) \mapsto \xi + \eta \in \mathbb{R}^n$ define the natural embedding $\mathcal{N}_r(\mathcal{M}) \hookrightarrow \mathbb{R}^n$, so in what follows we will not distinguish between $(\xi, \eta) \in \mathcal{N}_r(\mathcal{M})$ and $\xi + \eta \in \mathbb{R}^n$, until it leads to confusion.

Since the field of plains $\{T_x \mathcal{M}\}_{x \in \mathcal{M}}$ and $\{\mathbb{L}^-_{x}\}_{x \in \mathcal{M}}$ are $X^t$-invariant, then $\{\mathbb{K}^-_{x}\}_{x \in \mathcal{M}}$ is $X^t$-invariant as well. In such a case, the flow on invariant manifold, $\{\chi^t(\cdot) : \mathcal{M} \hookrightarrow \mathcal{M}\}_{t \in \mathbb{R}}$, has the structure of Anosov dynamical system (ADS) (see, e.g., [1] and Example at the end of this paper). In particular, each of the fields of planes $\{\mathbb{K}^-_{x}\}_{x \in \mathcal{M}}$ and $\{\mathbb{L}^+_{x}\}_{x \in \mathcal{M}}$ are integrable and form, respectively, the contracting and expanding foliations invariant w.r.t. the flow $\{\chi^t(\cdot)\}$. As is well known, ADSs play an important role in the theory of chaos. Yet another circumstance that motivate to study the case under consideration is the structural stability property of ADSs.

Our main result is as follows

**Theorem 1.** *If the flow $\{\chi^t(\cdot)\}$ is hyperbolic along the attracting invariant manifold $\mathcal{M}$, then there is a neighborhood $\mathcal{U}$ of $\mathcal{M}$ such that any motion starting in $\mathcal{U}$ has an asymptotic phase.*

The proof of this theorem is based on construction of local contracting foliation generated by the fields of planes $\{\mathbb{L}^\pm_{x}\}_{x \in \mathcal{M}}$. The existence of such a foliation can be obtained by appropriate interpolation of corresponding results [9] concerning diffeomorphisms. However, the paper [9] does not contain any details on the issue.

### 3. Proof of the main theorem

Let $P_x^{\pm,0} : \mathbb{R}^n \rightarrow L^\pm_{x}^{\pm,0}$ be projections associated with decomposition (2.3). Thus, $P_x^+ + P_x^- + P_x^0 = \text{Id}$ where $\text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the identity map in $\mathbb{R}^n$. Since for each $x \in \mathcal{M}$ the diagram

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{X^t(\cdot)} & \mathbb{R}^n \\
\downarrow P_x^{\pm,0} & & \downarrow P_x^{\pm,0} \\
\mathbb{L}^\pm_{x} & \xrightarrow{\chi^t(\cdot)} & \mathbb{L}^\pm_{x}
\end{array}$$

is commutative, then

$$P_x^{\pm,0} = [X^t(\cdot)]^{-1} P_x^{\pm,0} \chi^t(\cdot).$$  \hspace{1cm} (3.1)

On account of (3.1) and (2.2) we easily obtain

$$X^{t-s}(\chi^s(\cdot)) = X^t(\cdot)X^{s}(\chi^s(\cdot)) = [X^t(\cdot)] [X^s(\cdot)]^{-1}.$$
and
\[ X'(x)P^\pm_0 \left[ X^s(x) \right]^{-1} = \left[ X'(x) \right] \left[ X^s(x) \right]^{-1} P^\pm_0 \chi'(x) = X^{t-s} \left( \chi^s(x) \right) P^\pm_0 \chi'(x), \]

If we define \( K := c \max \left\{ \max_{x \in \mathcal{M}} \left\| P^\pm_0 \right\| \right\} \), then inequalities (3.1), (3.2) yield
\[ \left\| X'(x)P^\pm_0 \left[ X^s(x) \right]^{-1} \right\| \leq Ke^{-\alpha(t-s)}, \quad t \geq s, \]
\[ \left\| X'(x)P^\pm_0 \left[ X^s(x) \right]^{-1} \right\| \leq Ke^{\alpha(t-s)}, \quad t \leq s. \]
\[ \left\| X'(x)P^\pm_0 \left[ X^s(x) \right]^{-1} \right\| \leq K, \quad t, s \in \mathbb{R}. \]

Introduce the new variable \( y \) by
\[ x = \chi^t(\xi) + y \]
where \( \xi \in \mathcal{M} \) is considered as a parameter. If we define
\[ w(t, y, \xi) := v \left( \chi^t(\xi) + y \right) - v \left( \chi^t(\xi) \right) - v' \left( \chi^t(\xi) \right) y, \]
then system (2.1) in new variables takes the form
\[ \dot{y} = v' \left( \chi^t(\xi) \right) y + w(t, y, \xi). \] (3.2)

Since \( v \) is \( C^2 \)-vector field, then there is a constant \( C > 0 \) such that
\[ \|w(t, y, \xi)\| \leq \frac{C}{2} \|y\|^2, \quad \|w'_y(t, y, \xi)\| \leq C \|y\|, \quad \|w''_{yy}(t, y, \xi)\| \leq C \quad \forall (t, \xi) \in \mathbb{R} \times \mathcal{M}, \|y\| \leq 1. \] (3.3)

We consider (3.2) as a family of systems depending on parameter \( \xi \in \mathcal{M} \). It is not hard to show that, for a fixed \( \xi \in \mathcal{M} \), a mapping \( y(\cdot) : \mathbb{R} \mapsto \mathbb{R}^n \) is a solution of (3.2) tending to zero as \( t \to \infty \) if and only if for some \( \eta \in \mathbb{L}^n_{\xi} \) the mapping \( y(\cdot) \) satisfies the integral equation
\[ y(t) = G[y](t, \xi, \eta) := X^t(\xi)\eta + \int_0^\infty G(t, s, \xi)w(s, y(s), \xi)ds \] (3.4)
with kernel (Green function)
\[ G(t, s, \xi) := \begin{cases} -X^t(\xi) \left[ P^+_\xi + P^0_\xi \right] \left[ X^s(\xi) \right]^{-1} & t \leq s \\ X^t(\xi)P^0_\xi \left[ X^s(\xi) \right]^{-1} & t > s. \end{cases} \]

Besides, for given \( \xi \in \mathcal{M} \) and \( \eta \in \mathbb{L}^n_{\xi} \), a solution to (3.4) satisfies \( P^-_\xi y(0) = \eta \).

**Proposition 2.** There exist positive numbers \( \{r, R\} \subset (0, 1) \) such that there hold the following assertions:

(i) For each \( (\xi, \eta) \in \mathcal{N}_r(\mathcal{M}) \) there exists a unique solution to (3.2), \( y_*(\cdot, \xi, \eta) : \mathbb{R}_+ \mapsto \mathbb{R}^n \), such that
\[ \|y_*(t, \xi, \eta)\| \leq Re^{-\alpha t} \quad \forall t \geq 0 \quad \text{and} \quad P^-_\xi y(0) = \eta; \] (3.5)

(ii) The mapping \( y_*(\cdot, \cdot, \cdot) : \mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M}) \mapsto \mathbb{R}^n \) is continuous, and for any \( (t, \xi) \in \mathbb{R}_+ \times \mathcal{M} \) the mapping \( y_*(t, \xi, \cdot) : \mathbb{L}^n_{\xi} \mapsto \mathbb{R}^n \) is twice continuously differentiable;
(iii) there exists a constant $C_0 > 0$ such that
\[ \|y_s(t, \xi, \eta) - X^t(\xi)\eta\| \leq C_0 e^{-\alpha t} \|\eta\|^2 \quad \forall (t, \xi, \eta) \in \mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M}). \quad (3.6) \]

**Proof.** A proof of assertion (i) is obtained in a standard way by means of the Banach contraction principle. For the sake of completeness we present here some essential details. Let $C(\mathbb{R}_+ \mapsto \mathbb{R}^n; \alpha)$ be the subspace of $C(\mathbb{R}_+ \mapsto \mathbb{R}^n)$ endowed with norm
\[ \|\cdot\|_\infty := \sup_{t \geq 0} e^{\alpha t} \|\cdot\|. \]

Define
\[ \mathcal{Y}_{r,R} := \{ y(\cdot) \in C(\mathbb{R}_+ \mapsto \mathbb{R}^n; \alpha) : \|y(t)\| \leq R e^{-\alpha t} \forall t \geq 0 \}. \]

Let us impose conditions on $r, R$ under which $G[y](t, \xi, \eta) : \mathcal{Y}_{r,R} \mapsto \mathcal{Y}_{r,R}$.

On account of (3.3), for each $(\xi, \eta) \in \mathcal{N}_r(\mathcal{M})$ and each $y(\cdot) \in \mathcal{Y}_{r,R}$, we have the following estimates:
\[
\left\| \int_0^t X^t(\xi) \left[ P_{\xi^+}^t + P_{\xi}^0 \right] [X^s(\xi)]^{-1} w(s, y(s), \xi) ds \right\| 
\leq 1 \int_0^t Ke^{\alpha(t-s)} CR^2 e^{-2\alpha s} ds + \frac{1}{2} \int_0^t KCR^2 e^{-2\alpha s} ds
\leq \frac{5}{12\alpha} KCR^2 e^{-2\alpha t},
\]
\[
\left\| \int_0^t X^t(\xi) P_{\xi^-}^- [X^s(\xi)]^{-1} w(s, y(s), \xi) ds \right\| 
\leq 1 \int_0^t Ke^{-\alpha(t-s)} CR^2 e^{-2\alpha s} ds \leq \frac{KCR^2}{2\alpha} e^{-\alpha t}.
\]

Hence, for all $(t, \xi, \eta) \in \mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M})$ and $y(\cdot) \in \mathcal{Y}_{r,R}$, we obtain
\[
\|G[y](t, \xi, \eta)\| \leq \left( cr + \frac{11}{12\alpha} KCR^2 \right) e^{-\alpha t} \leq Re^{-\alpha t} \quad (3.7)
\]
provided that
\[
cr + \frac{11}{12\alpha} KCR^2 \leq R. \quad (3.8)
\]

Now let us find conditions under which $G[\cdot]$ is a contraction mapping in metric space $\mathcal{Y}_{r,R}$ endowed with metric $\rho(\cdot, \cdot) := \|\cdot - \cdot\|_\infty$. On account of
\[
\|w(t, y_1, \xi) - w(t, y_2, \xi)\|
\leq \left[ v' \left( \chi'(\xi) + sy_1 + (1-s)y_2 \right) - v' \left( \chi'(\xi) \right) \right] ds \|y_1 - y_2\|
\]
A metric in \( \tilde{\mathcal{Y}} \) where the space \( \mathcal{N}_r(\mathcal{M}) \) is defined as \( \rho(\mathcal{G}[y_1](\cdot, \xi, \eta), \mathcal{G}[y_2](\cdot, \xi, \eta)) \leq \frac{11}{6\alpha} KCR \rho(y_1(\cdot), y_2(\cdot)) \).

Let
\[
\alpha := 11 KCR (6\alpha) < 1.
\]

Then by the Banach contraction principle, for each \((\xi, \eta) \in \mathcal{N}_r(\mathcal{M})\), equation (3.4) has a unique solution \( y_\ast(\cdot, \xi, \eta) \in \mathcal{Y}_{r,R} \). This completes the proof of assertion (i).

As consequence, we have constructed the mapping \( y_\ast(\cdot, *, *) : \mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M}) \to \mathbb{R}^n \). It is easily seen that this mapping is nothing but the unique fixed point of operator \( \mathcal{G}[\cdot] : \mathcal{Y}_{r,R} \to \mathcal{Y}_{r,R} \) where the space \( \mathcal{Y}_{r,R} \) consists of mappings \( y(\cdot, *, *) \in C(\mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M}) \to \mathbb{R}^n) \) satisfying (3.5).

A metric in \( \mathcal{Y}_{r,R} \) is defined as
\[
\tilde{\rho}(y_1(\cdot, *, *), y_2(\cdot, *, *)) := \sup \left\{ e^{\alpha t} \| y_1(t, \xi, \eta) - y_2(t, \xi, \eta) \| : (t, \xi, \eta) \in \mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M}) \right\}.
\]
Let us prove the differentiability of \( y_*(t, \xi, *) \). We will restrict ourselves to the first order derivatives. Let \( \xi \in \mathcal{M} \) be fixed at will and let \( \mathcal{L}_{r, R}^\prime \) be the space of continuous mappings \( y(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{L}_-^\xi \rightarrow \mathbb{R}^n \) having continuous directional derivatives \( y_\xi(\cdot, \cdot) \) in all directions \( a \in \mathbb{L}_-^\xi \) and satisfying the inequalities

\[
\max \left\{ \| y(t, \eta) \|, \| y_\eta(t, \eta) a \| \right\} \leq Re^{-\alpha t} \quad \forall (t, \eta) \in \mathbb{R}_+ \times \mathbb{L}_-^\xi, \quad \forall a \in \mathbb{L}_-^\xi : \| a \| = r.
\]

For any \((t, \eta) \in \mathbb{R}_+ \times \mathbb{L}_-^\xi\) and \(y(\cdot, \cdot) \in \mathcal{L}_{r, R}^\prime\), we obtain

\[
\begin{align*}
\left\| \int_0^t X^t(\xi) \left[ P_\xi^+ + P_\xi^0 \right] [X^s(\xi)]^{-1} w_y(s, y(s, \eta), \xi) y_\eta(s, \eta) a \text{d}s \right\| &
\leq \int_0^t Ke^{\alpha(t-s)} CR^2 e^{-2\alpha s} \text{d}s + \int_0^t KCR^2 e^{-2\alpha s} \text{d}s \\
&\leq \frac{5}{6\alpha} KCR^2 e^{-2\alpha t},
\end{align*}
\]

\[
\begin{align*}
\left\| \int_0^t X^t(\xi) P_\xi^- [X^s(\xi)]^{-1} w_y(s, y(s, \eta), \xi) y_\eta(s, \eta) a \text{d}s \right\| &
\leq \int_0^t Ke^{-\alpha(t-s)} CR^2 e^{-2\alpha s} \text{d}s \leq \frac{KCR^2}{\alpha} e^{-\alpha t}.
\end{align*}
\]

Thus,

\[
\| \mathcal{G}_\eta[y](t, \eta) a \| \leq \left( cr + \frac{11}{6\alpha} KCR^2 \right) e^{-\alpha t} \leq Re^{-\alpha t}
\]

provided that \( a \in \mathbb{L}_-^\xi, \| a \| = r \) and

\[
cr + \frac{11}{6\alpha} KCR^2 \leq R.
\]

The last inequality together with (3.8) implies that \( \mathcal{G}[\cdot] : \mathcal{L}_{r, R}^\prime \rightarrow \mathcal{L}_{r, R}^\prime \).

Define a metric \( \rho' \) in \( \mathcal{L}_{r, R}^\prime \) as the maximum of

\[
\sup \left\{ e^{\alpha t} \| y_1(t, \eta) - y_2(t, \eta) \| : (t, \eta) \in \mathbb{R}_+ \times \mathbb{L}_-^\xi, \| \eta \| \leq r \right\}
\]

and

\[
\sup \left\{ e^{\alpha t} \left\| \frac{\partial y_1(t, \eta)}{\partial \eta} a - \frac{\partial y_2(t, \eta)}{\partial \eta} a \right\| : (t, \eta, a) \in \mathbb{R}_+ \times \mathbb{L}_-^\xi \times \mathbb{L}_-^\xi, \| \eta \| \leq r, \| a \| = r \right\}.
\]

On account of

\[
\| w_y(t, y_1, \xi) u_1 - w_y(t, y_2, \xi) u_2 \| \leq C \| y_1 \| \| u_1 - u_2 \| + C \| u_2 \| \| y_1 - y_2 \|
\leq C \left( \| y_1 \| + \| u_2 \| \right) \max \left\{ \| y_1 - y_2 \|, \| u_1 - u_2 \| \right\}
\]

\[
\leq C \left( \| y_1 \| + \| u_2 \| \right) \max \left\{ \| y_1 - y_2 \|, \| u_1 - u_2 \| \right\}.
\]
\[ \forall (t, \xi) \in \mathbb{R} \times \mathcal{M}, \|y_1\|, \|y_2\| \leq R, \ u_1, u_2 \in \mathbb{R}^n, \]

in the same way as above we obtain

\[ e^{\alpha t} \|G'_{\eta}[y_1(\cdot)](t, \eta)\alpha - G'_{\eta}[y_2(\cdot)](t, \eta)\alpha \| \leq \frac{11}{3\alpha} KCR \rho \left( y_1(\cdot, \cdot), y_2(\cdot, \cdot) \right), \]

Now to ensure that \( G[\cdot] \) is contracting in \( (Y, \rho') \), it is sufficient to impose the condition

\[ \frac{11}{3\alpha} KCR < 1. \]

(3.10)

Finally let us proceed to assertion (iii). Taking into account the estimates obtained above we have

\[ \|e^{\alpha t} y_*(t, \xi, \eta)\| \leq c \|\eta\| + e^{\alpha t} \int_0^\infty \|G(t, s, \xi)w(t, y(t, s, \xi, \xi))\| \, ds \]

\[ \leq c \|\eta\| + \frac{\alpha}{2} \sup_{t \in \mathbb{R}_+} \|e^{\alpha t} y_*(t, \xi, \eta)\|, \]

and thus,

\[ \|y_*(t, \xi, \eta)\| \leq \frac{2c}{2 - \alpha} e^{-\alpha t} \|\eta\| \quad \forall (t, \xi, \eta) \in \mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M}). \]

In its turn, the last inequality together with (3.3), (3.4) and

\[ \|y_*(t, \xi, \eta) - X^t(\xi)\eta\| \leq \frac{C}{2} \int_0^\infty \|G(t, s, \xi)\| \|y_*(s, \xi, \eta)\|^2 \, ds \]

implies that there exists a constant \( C_0 > 0 \) such that there holds the inequality (3.9). \( \square \)

Now we are in position to prove

**Proposition 3.** Let positive numbers \( r, R \) obey the inequalities (3.9) and (3.10). Then there exists a mapping \( h(\cdot) \in C(\mathcal{N}_r(\mathcal{M}) \rightarrow \mathbb{R}^n) \) such that for each \((\xi, \eta) \in \mathcal{N}_r(\mathcal{M})\) there hold the inequalities \( \|h(\xi, \eta)\| \leq C_0 \|\eta\|^2 \) and

\[ \|\chi^t(\xi + \eta + h(\xi, \eta)) - \chi^t(\xi)\| \leq Re^{-\alpha t} \quad \forall (t, \xi, \eta) \in \mathbb{R}_+ \times \mathcal{N}_r(\mathcal{M}). \]

Besides \( h(\xi, \cdot) \in C^2 \left( L_\xi^- \cap \mathcal{N}_r(\mathcal{M}) \rightarrow L_\xi^+ \oplus L_\xi^0 \right) \) for each \( \xi \in \mathcal{M}. \)

**Proof.** Define \( h(\xi, \eta) := y_*(0, \xi, \eta) - \eta \). Then

\[ h(\xi, \eta) = - \int_0^\infty \left[ P^+_\xi + P^0_\xi \right] [X^t(\xi)]^{-1} w(s, y_*(s, \xi, \eta), \xi) \, ds \in L_\xi^+ \oplus L_\xi^0, \]

and smoothness properties of \( h(\xi, \cdot) \) follow directly from assertion (ii) of Proposition 2. Since \( \chi^t(\xi) + y_*(t, \xi, \eta) \) is the solution of (2.1) taking value \( \xi + \eta + h(\xi, \eta) \) at initial moment of time \( t = 0 \), then

\[ \chi^t(\xi) + y_*(t, \xi, \eta) \equiv \chi^t(\xi + \eta + h(\xi, \eta)). \]

\( \square \)
PROPOSITION 4. If a positive number $\epsilon \in (0, r/2)$ is sufficiently small, then for each $(\xi_0, \zeta_0) \in U_2(\mathcal{M})$ there exists $(\xi_* , \zeta_*) \in U_2(\mathcal{M})$ such that

$$\xi_* + \zeta_* + h(\xi_* , \zeta_*) = \xi_0 + \zeta_0.$$ 

Proof. There exist a domain $Q \subset \mathbb{R}^m$ containing the origin together with its $2\epsilon$-neighborhood and a mapping $\xi(\cdot) \in C^2(Q \to \mathbb{R}^n)$ such that $\xi(0) = \xi_0$ and locally near $\xi_0$ the sub-manifold $\mathcal{M}$ is given by the parametric equation $x = \xi(q)$, $q \in Q$. One can choose coordinates $(q_1, \ldots, q_m)$ in such a way that column vectors $J_q$ base of $\mathcal{M}$ and there is a unique $\zeta$ defined by the parametric equation and a mapping $\xi(\cdot)$ such that column vectors $J_q$ base of $\mathcal{M}$.

After introducing the matrix $A$ with columns $\xi_q^t(0), \ldots, \xi_q^t(n)$, $\nu_1^t(0), \ldots, \nu_{n-m}(0)$, and column vector $(q, p)^T = (q_1, \ldots, q_m, p_1, \ldots, p_{n-m})$, the equation (3.11) is reduced to

$$A^{-1}H(q, p).$$

Since

$$\mu(\epsilon) := \max_{\|q\| \leq \epsilon} \left[ \left( \int_0^1 \|\xi'(s)q - \xi'(0)\| \, ds \right)^2 + \|\nu(q) - \nu(0)\|^2 \right]^{1/2} \to 0, \quad \epsilon \to 0,$$
then for sufficiently small $\epsilon$ and $(q, p) \in \tilde{B}_\epsilon$ we obtain
\[ \|A^{-1}H(q, p)\| \leq \|A^{-1}\| \left[ C_0 \|z_0 + p\|^2 + 2\mu(\epsilon)\epsilon \right] \leq \|A^{-1}\| \left[ 4C_0 \epsilon^2 + 2\mu(\epsilon)\epsilon \right] \leq \epsilon. \]
Hence $A^{-1}H(\ast, \ast) \in C(\tilde{B}_\epsilon \to \tilde{B}_\epsilon)$, and by the Brouwer theorem the mapping $A^{-1}H(\ast, \ast)$ has at least one fixed point $(q_*, p_*) \in \tilde{B}_\epsilon$.

4. IN Variant foliation

For the sake of completeness, not pretending on a novelty, we present here some comments concerning geometric structure which appears in a neighborhood of $\mathcal{M}$ due to its hyperbolicity.

For each $\xi \in \mathcal{M}$ the set
\[ \mathcal{L}_\xi^- := \bigcup_{\eta \in \mathcal{L}_\xi^\perp \cap \mathcal{N}(\mathcal{M})} \{ \xi + \eta + h(\eta, \xi) \} \]
is $C^2$-submanifold of $\mathbb{R}^n$ diffeomorphic to $n_-$-dimensional ball, provided that $\epsilon$ is sufficiently small. Since $T_\xi \mathcal{L}_\xi = \mathcal{L}_\xi^\perp$, then $\mathcal{L}_\xi^\perp$ transversally intersects $\mathcal{M}$ at point $\xi$. The intersection is a $C^1$-sub-manifold $\mathcal{M}_\xi^- := \mathcal{M} \cap \mathcal{L}_\xi^\perp$. A point $\xi^\prime \in \mathcal{M}$ from sufficiently small neighborhood of $\xi$ belongs to $\mathcal{M}_\xi^-$ if $\|\chi^t(\xi) - \chi^t(\xi^\prime)\| = O(\epsilon^{-at})$ as $t \to \infty$. Hence, if $\xi^\prime \notin \mathcal{M}_\xi^-$, then $\mathcal{M}_\xi^- \cap \mathcal{M}_\xi^\prime^- = \emptyset$; and thus, $\mathcal{L}_\xi^\perp \cap \mathcal{L}_{\xi^\prime}^\perp = \emptyset$. In fact, once we suppose that there is $x_0 \in \mathcal{L}_\xi^\perp \cap \mathcal{L}_{\xi^\prime}^\perp$ than
\[ \|\chi^t(x_0) - \chi^t(\xi)\| = O(\epsilon^{-at}) \quad \text{and} \quad \|\chi^t(x_0) - \chi^t(\xi^\prime)\| = O(\epsilon^{-at}) \quad \text{as} \quad t \to \infty \]
\[ \Rightarrow \quad \|\chi^t(\xi) - \chi^t(\xi^\prime)\| = O(\epsilon^{-at}) \quad \text{as} \quad t \to \infty \quad \Rightarrow \quad \xi^\prime \in \mathcal{M}_\xi^- \]
Hence, a neighborhood of invariant manifold $\mathcal{M}$ is fibered with local manifolds $\mathcal{L}_\xi^-$ continuously dependent on $\xi$. Each of such manifolds is formed by initial points of motions associated with common asymptotic phase. Since for each $s \geq 0$, there holds
\[ \|\chi^{t+s}(\xi + \eta + h(\eta, \eta)) - \chi^{t+s}(\xi)\| = O\left(\epsilon^{-\alpha(t+s)}\right), \quad t \to \infty, \]
or, what is the same,
\[ \|\chi^t \circ \chi^s(\xi + \eta + h(\eta, \eta)) - \chi^t \circ \chi^s(\xi)\| = O\left(\epsilon^{-\alpha(t+s)}\right), \quad t \to \infty, \]
then $\chi^s(\xi + \eta + h(\eta, \xi)) \in \mathcal{L}_{\chi^s(\xi)}^-$, and this implies the invariance of fibers: $\chi^t \left(\mathcal{L}_\xi^-\right) \subset \mathcal{L}_{\chi^t(\xi)}^-$ for any $t \geq 0$.

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