On Adiabatic Oscillations of a Stratified Atmosphere on the Flat Earth

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Abstract
We consider the oscillations of the atmosphere around a stratified background density and entropy distribution under the gravitation on the flat Earth. The atmosphere is supposed to be an ideal gas and the motion is supposed to be governed by the compressible Euler equations. The density distribution of the back ground equilibrium is supposed to touch the vacuum at the finite height of the stratosphere. Considering the linearized approximation for small perturbations, we show that time periodic oscillations with a sequence of time periods which accumulate to infinity, say, slow and slow oscillations, so called ‘g-modes’, can appear when the square of the Brunt-Väisälä frequency is positive everywhere for the considered back ground equilibrium.

Key Words and Phrases. Euler equations, Atmospheric Oscillations, Vacuum boundary, Brunt-Väisälä frequency, Gravity modes. Eigenvalue problem of Strum-Liouville type.

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1 Introduction
We investigate the motion of an atmosphere on the flat earth under the constant gravitational force. We suppose that the atmosphere consists of an ideal gas with the most simple equation of state and the motion is adiabatic motion governed by the compressible Euler equations.

The pioneering mathematical investigation of the oscillations of an atmosphere can be found as the paper ‘On the Vibrations of an Atmosphere’ by Lord Rayleigh, 1890, [10]. He wrote

In order to introduce greater precision into our ideas respecting the behavior of the Earth’s Atmosphere, it seems advisable to solve any problems that may present themselves, even though the search for

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simplicity may lead us to stray rather far from the actual question. It is supposed here to consider the case of an atmosphere composed of gas which obeys Boyle’s law, viz. such that the pressure is always proportional to the density. And in the first instance we shall neglect the curvature and rotation of the Earth, supposing that the strata of equal density are parallel planes perpendicular to the direction in which gravity acts.

Our investigation in this article will be done under the same spirit as that of Lord Rayleigh quoted above, except for his starting point that the back ground state is supposed to be that of the isothermal stratified gas, say, \( \rho = \rho_0 \exp(-gz/A) \), where \( \rho \) is the density and \( z \) is the height, \( \rho_0, g, A \) being positive constants. Instead we are interested in a back ground equilibrium like \( \rho = C(z_+ - z)^\nu \), where \( z_+ \) is the height of the stratosphere, \( C, \nu(>1) \) being positive constants, which touches the vacuum \( \rho = 0 \) on \( z > z_+ \) at the height \( z = z_+ \). Then the vacuum boundary at which the gas touches the vacuum should be treated as a free boundary, and requires a delicate treatise in the mathematically rigorous view point.

First of all we consider the problem under the linearized approximation for small perturbations around the back ground density and entropy distributions. The aim of the study should be to clarify the spectral property of the differential operator which governs the perturbations around the back ground stratified density distribution in a suitable functional space in view of the linearized approximation, but, especially we are interested in the existence of a sequence of eigenvalues which accumulate to 0, namely, the existence of time periodic oscillations whose time periods are long and long. This kind of modes are called ‘g-modes’ in the context of Helioseismology, and generally of astroseismology, by astrophysicists, and mathematically rigorous justification of their existence for self-gravitating gaseous star model governed by the Euler-Poisson equations has not yet be done. This article tries to prove the existence of such a sequence of modes in the simple situation of the motion under the constant gravitation over the flat Earth. Such a sequence of eigenvalues accumulating to 0 cannot appear if the background state is isentrooic and the Brunt-Väisälä frequency vanishes everywhere. So the existence of g-modes is an effect of buoyancy.

Let us describe the problem more precisely.

We consider the motions of the atmosphere on the flat earth governed by
the Euler equations
\[ \frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}(\rho v^k) = 0, \quad (1.1a) \]
\[ \rho \left( \frac{\partial v^j}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial v^j}{\partial x^k} \right) + \frac{\partial P}{\partial x^j} + \rho \frac{\partial \Phi}{\partial x^j} = 0, \quad j = 1, 2, 3, \quad (1.1b) \]
\[ \rho \left( \frac{\partial S}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial S}{\partial x^k} \right) = 0, \quad (1.1c) \]
\[ \Phi = g x^3. \quad (1.1d) \]

Here \( t \geq 0, x = (x^1, x^2, x^3) \in \Omega := \{ x \in \mathbb{R}^3 | x^3 > 0 \} \). The unknowns \( \rho \geq 0, P, S \) are mass density, pressure, entropy density, and \( v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3} \) is the velocity field. \( g \) is a positive constant. The boundary condition is
\[ \rho v^3 = 0 \quad \text{on} \quad x^3 = 0, \quad (1.2) \]
and the initial condition is
\[ \rho = \hat{\rho}(x), \quad v = \hat{v}(x) = (\hat{v}^1(x), \hat{v}^2(x), \hat{v}^3(x)) \quad \text{at} \quad t = 0. \quad (1.3) \]

We assume that \( P \) is a function of \( (\rho, S) \), and put the following

**Assumption 1** \( P \) as the function of \( (\rho, S) \) is given by
\[ P = \rho^\gamma \exp \left( \frac{S}{C_V} \right) \quad \text{for} \quad \rho \geq 0, \quad (1.4) \]

where \( \gamma \) and \( C_V \) are positive constant such that \( 1 < \gamma < 2 \).

In this article we denote
\[ \nu := \frac{1}{\gamma - 1}. \quad (1.5) \]

Let us fix a stratified equilibrium \( \rho = \bar{\rho}, P = \bar{P}, S = \bar{S}, \) which are functions of \( x^3 \) only such that
\[ \{ \bar{\rho} > 0 \} = \Pi := \{ 0 \leq z < z_+ \}. \quad (1.6) \]

Here and hereafter we denote \( x = x^1, y = x^2, z = x^3 \).

We consider the Eulerian perturbations
\[ \xi = \sum \xi^k \frac{\partial}{\partial x^k} = \delta x = \sum \delta x^k \frac{\partial}{\partial x^k}, \]
\( \delta \rho, \delta P, \delta S \) at this fixed equilibrium. Here we use the Lagrangian co-ordinates which will be denoted by the diversion of the letter \( x = (x^1, x^2, x^3) = (x, y, z) \)
of the Eulerian co-ordinates.

Here let us recall the definition of the Euler perturbation $\delta Q$ and the Lagrange perturbation $\Delta Q$ of a quantity $Q$:

$$
\Delta Q(t, x) = Q(t, \varphi(t, x)) - \bar{Q}(x),
\delta Q(t, x) = Q(t, \varphi(t, x)) - \bar{Q}(\varphi(t, x)),
$$

where $x = \varphi(t, x) = x + \xi(t, x)$ is the steam line given by

$$
\frac{\partial}{\partial t}\varphi(t, x) = v(t, \varphi(t, x)), \quad \varphi(0, x) = x.
$$

We assume that \( \bar{\rho} = \bar{\rho} \), or, the initial Lagrangian perturbation $\Delta \rho |_{t=0} = 0$, so that

$$
\rho(t, x + \xi) = \bar{\rho}(x) + \Delta \rho(t, x) = \frac{\bar{\rho}(x)}{\det J(t, x)},
$$

where

$$
J(t, x) := \left( \delta^j_k + \frac{\partial \xi^j}{\partial x^k} \right)_{j,k}.
$$

We suppose $\Delta S |_{t=0} = 0$, so that $S(t, x + \xi(t, x)) = \bar{S}(x)$. So $x$ runs over the fixed domain $\Pi = \{0 \leq z < z_+\}$.

The linearized approximation of the equation which governs the perturbations turns out to be

$$
\frac{\partial^2 \xi}{\partial t^2} + L \xi = 0,
$$

where

$$
L \xi = \frac{1}{\bar{\rho}} \text{grad} \delta P + \frac{g}{\bar{\rho}} \delta \rho e_3,
$$

$$
\delta \rho = -\text{div}(\bar{\rho} \xi),
$$

$$
\delta P = \frac{\gamma P}{\rho} \delta \rho + \gamma A \bar{P} \xi^3.
$$

Here $e_3$ means the unit vector $\partial / \partial x^3 = \partial / \partial z$ and we define

$$
A := \frac{1}{\rho} \frac{d\rho}{dz} - \frac{1}{\gamma \bar{P}} \frac{d\bar{P}}{dz} = -\frac{1}{\gamma C_V} \frac{dS}{dz}.
$$

We shall denote

$$
c^2 := \frac{\gamma P}{\rho},
$$

$$
\lambda^2 := -gA = -g \left( \frac{1}{\rho} \frac{d\rho}{dz} + \frac{g}{c^2} \right).
$$
In the physical context the quantity \(c^2\) is the square of the sound speed and \(N^2\) is the square of the Brunt-Väisälä frequency.

Then we see

\[\gamma A \bar{P} = c^2 \frac{d \rho}{dz} + g \bar{\rho},\]  

(1.16)

therefore

\[\delta P = c^2 \delta \rho + \left( c^2 \frac{d \rho}{dz} + g \bar{\rho} \right) \xi^3.\]  

(1.17)

The boundary condition is

\[\xi^3 (\xi|e_3) = 0 \text{ on } z = 0.\]  

(1.18)

The initial condition is

\[\xi = 0, \quad \frac{\partial \xi}{\partial t} = \circ \text{ at } t = 0.\]  

(1.19)

In this article we investigate time periodic solutions of the equation (1.9). This is the eigenvalue problem associated with the operator \(L\). In the viewpoint of the functional analysis a clarification of the spectral property of the operator \(L\) is desired, but it is not yet done completely. We shall prove the existence of a sequence of eigenvalues which accumulate to 0 under an assumption (Assumption 2) on the value distributions of the square of the Brunt-Väisälä frequency \(N^2\).

## 2 Equilibrium for a prescribed entropy distribution

In this section we establish the existence of equilibria which enjoy good properties used in the following consideration on \(L\).

We put the following

**Definition 1** A stratified equilibrium \((\bar{\rho}, \bar{S})\) is said to be admissible if

1) \(\{ \bar{\rho} > 0 \} = \Pi := \{ 0 \leq z < z_+ \}.\)  

(2.1)

2) \(z \mapsto \bar{\rho}, \bar{S} \in C^\infty(\Pi);\)

3) \(\frac{d \rho}{dz} < 0 \) for \(0 \leq z < z_+;\)

4) \(\bar{\rho} = C \rho (z_+ - z)^\circ (1 + [z_+ - z]) \) for \(0 < z_+ - z \ll 1\)  

(2.2)
and \( z \mapsto \bar{S} \) is analytic at \( z_+ \), that is,

\[
\bar{S} = [z_+ - z]_0 \quad \text{for} \quad |z_+ - z| \ll 1. \tag{2.3}
\]

Here \( C_\rho \) is a positive constant and we use the following notation.

**Notation 1** \( [X]_K \) stands for various convergent power series of the form \( \sum_{k \geq K} a_k X^k \)

We claim

**Theorem 1** Let a smooth function \( \Sigma \) on \( \mathbb{R} \) and a positive number \( z_+ \) be given. Assume that it holds, for \( \eta > 0 \), that

\[
\gamma + \frac{\gamma - 1}{C_V} \eta \frac{d}{d \eta} \Sigma(\eta) > 0. \tag{2.4}
\]

Then there exists an admissible equilibrium \((\bar{\rho}, \bar{S})\) such that \( \bar{S} = \Sigma(\bar{\rho}^{\gamma - 1}) \).

Proof. Consider the functions \( f^P, f^u \) defined by

\[
f^P(\rho) := \rho^\gamma \exp \left[ \frac{\Sigma(\rho^{\gamma - 1})}{C_V} \right], \tag{2.5}
\]

\[
f^u(\rho) := \int_0^\rho \frac{Df^P(\rho')}{\rho'} d\rho' \tag{2.6}
\]

for \( \rho > 0 \). Thanks to the assumption (2.4) we have

\[
Df^P(\rho) > 0
\]

for \( \rho > 0 \), and there exists a smooth function \( \Lambda \) on \( \mathbb{R} \) such that \( \Lambda(0) = 0 \) and

\[
f^P(\rho) = \Lambda \rho^\gamma (1 + \Lambda(\rho^{\gamma - 1})) \tag{2.7}
\]

for \( \rho > 0 \). Here \( \Lambda := \exp(\Sigma(0)/C_V) \) is a positive constant. Then we have

\[
u = f^u(\rho) = \frac{\gamma A}{\gamma - 1} \rho^{\gamma - 1}(1 + \Lambda_u(\rho^{\gamma - 1})) \tag{2.8}
\]

for \( \rho > 0 \), where \( \Lambda_u \) is a smooth function on \( \mathbb{R} \) such that \( \Lambda_u(0) = 0 \), and the inverse function \( f^\rho \) of \( f^u \)

\[
f^\rho(u) = \left( \frac{\gamma - 1}{\gamma A} \right)^{\frac{1}{\gamma - 1}} (u \lor 0)^{\frac{1}{\gamma - 1}} (1 + \Lambda^\rho(u)) \tag{2.9}
\]

is given so that \( \rho = f^\rho(u) \iff u = f^u(\rho) \) for \( u > 0 (\rho > 0) \). Here \( u \lor 0 \) stands for \( \max(u, 0) \) and \( \Lambda^\rho \) are smooth functions on \( \mathbb{R} \) such that \( \Lambda^\rho(0) = 0 \).

Therefore the problem is reduced to that for barotropic case to solve

\[
\frac{du}{dz} + g = 0,
\]
which has the solution
\[ u = g(z_+ - z). \]
Of course \( u > 0 \iff z < z_+ \) and we are putting
\[ \rho = f^p(g(z_+ - z)) \]
\[ = \left( \frac{1 - \gamma}{\gamma A} \right)^{1/\gamma} g((z_+ - z) \vee 0)^{-1/\gamma} (1 + \Lambda_\rho(g(z_+ - z))) \]  \hspace{1cm} (2.10)
This is the required admissible equilibrium. \( \square \)

Hereafter in this article we fix an admissible equilibrium \( (\bar{\rho}, \bar{S})). \)

3 Self-adjoint realization of \( L \)

In this article we consider perturbations which are periodic in \( x \)- and \( y \)-coordinates.
Therefore \( \Pi \) will denotes the space \( (\mathbb{R}/x+\mathbb{Z}) \times (\mathbb{R}/y+\mathbb{Z}) \times [0, z_+] \), that is, a function \( f \) on \( \Pi \) is such that
\[ f(x + lx_+, y + my_+, z) = f(x, y, z) \text{ for } \forall l, m \in \mathbb{Z}. \]
Here the periods \( x_+, y_+ \) are arbitrarily fixed to be positive numbers.

We are considering the differential operator
\[ L_\xi = \frac{1}{\rho} \text{grad}\delta P + \frac{g}{\rho} \delta \rho e_3 \]  \hspace{1cm} (3.1)
where
\[ \delta \rho = -\text{div}(\rho \xi), \]  \hspace{1cm} (3.2a)
\[ \delta P = \frac{\gamma P}{\rho} \delta \rho + \gamma A P \xi. \]  \hspace{1cm} (3.2b)

Here and hereafter the bars to denote the quantities evaluated at the fixed equilibrium are omitted, that is, \( \rho, P \) etc stand for \( \bar{\rho}, \bar{P} \) etc

Let us consider the operator \( L \) in the Hilbert space \( \mathcal{H} = L^2(\Pi, \rho dx), \mathbb{C}^3) \) endowed with the norm \( \|\xi\|_\mathcal{H} \) defined by
\[ \|\xi\|_\mathcal{H}^2 = \int_\Pi |\xi(x)|^2 \rho(x)dx = \int_0^{x_+} \int_0^{y_+} \int_0^{z_+} |\xi(x)|^2 \rho(z)dzdydx. \]  \hspace{1cm} (3.3)

We shall use

\textbf{Notation 2} For complex number \( z = x + \sqrt{-1}y, x, y \in \mathbb{R} \), the complex conjugate is denoted by \( z^* = x - \sqrt{-1}y \). Thus, for \( \xi = \sum \xi^k \partial/\partial x^k, \xi^k \in \mathbb{C} \), we denote
\[ \xi^* = \sum (\xi^k)^* \frac{\partial}{\partial x^k}. \]
First we observe \( \mathbf{L} \) restricted on \( C^\infty_0(\Pi^o, \mathbb{C}^3) \). Here and hereafter we denote
\[
\Pi^o := (\mathbb{R}/x_+ \mathbb{Z}) \times (\mathbb{R}/y_+ \mathbb{Z}) \times [0, z_+] = \Pi \setminus \{z = 0\}. \tag{3.4}
\]
We look at
\[
\mathbf{L} \xi = \text{grad} \left( -\frac{\gamma P}{\rho^2} \text{div}(\rho \xi) + \frac{\gamma AP \xi}{\rho} \right) + \frac{\gamma AP}{\rho^2} \left( -\text{div}(\rho \xi) + \frac{d\rho}{dz} \xi_3 \right) e_3. \tag{3.5}
\]
Using this expression for \( \xi_{(\mu)} \in C^\infty_0(\Pi^o), \mu = 1, 2 \), we have the following formula by integration by parts:
\[
(\mathbf{L} \xi_{(1)} | \xi_{(2)})_{\mathcal{D}} = \int \frac{\gamma P}{\rho^2} \text{div}(\rho \xi_{(1)}) \text{div}(\rho \xi_{(2)}) + \int \frac{\gamma AP}{\rho} \left[ (\xi_{(1)} | e_3) \cdot \text{div}(\rho \xi_{(2)}) - \text{div}(\rho \xi_{(1)}) \cdot (\xi_{(2)} | e_3)^* \right] + \int \frac{\gamma AP d\rho}{dz} (\xi_{(1)} | e_3)(\xi_{(2)} | e_3)^*.
\]
We see that \( \mathbf{L} \) restricted on \( C^\infty_0(\Pi^o) \) is a symmetric operator. Of course \( C^\infty_0(\Pi^o) \) is dense in \( \mathcal{D} \).

Moreover we have
\[
(\mathbf{L} \xi | \xi) = \int \frac{\gamma P}{\rho^2} |\text{div}(\rho \xi)|^2 + 2\sqrt{-1} \text{Im} \left[ \int \frac{\gamma AP}{\rho} \xi_3 \cdot \text{div}(\rho \xi^*) \right] + \int \frac{\gamma AP d\rho}{dz} |\xi_3|^2.
\]
Since \( \mathcal{A} \in C^\infty(\overline{\Pi}) \), we have
\[
|\mathcal{A}| \sqrt{\frac{\gamma P}{\rho}} \leq C_1
\]
on \( 0 < z < z_+ \), for \( P/\rho = O(z_+ - z) \). Therefore
\[
\left| \int \frac{\gamma AP}{\rho} \xi_3 \text{div}(\rho \xi)^* \right| \leq C_1 \int \sqrt{\frac{\gamma P}{\rho}} |\xi_3| |\text{div}(\rho \xi)| \leq \frac{C_1}{2} \left[ \frac{1}{\epsilon} \int \rho |\xi_3|^2 + \epsilon \int \frac{\gamma P}{\rho^2} |\text{div}(\rho \xi)|^2 \right] \leq \frac{C_1}{2} \left[ \frac{1}{\epsilon} \|\xi_3\|_{\mathcal{D}}^2 + \epsilon \int \frac{\gamma P}{\rho^2} |\text{div}(\rho \xi)|^2 \right].
\]
Since \( \frac{P d\rho}{\rho dz} = O(\rho) \), we have
\[
\left| \frac{\gamma AP d\rho}{\rho dz} \right| \leq C\rho.
\]
Therefore we have

\[ \left| \int \frac{\gamma A P}{\rho} \frac{d\rho}{dz} |\xi|^2 \right| \leq C_2 \|\xi\|_{H^2}. \]

Thus

\[ (L\xi,\xi)_{\tilde{\mathcal{H}}} \geq \left( 1 - \frac{\epsilon C_1}{2} \right) \int \frac{\gamma P}{\rho^2} (\text{div}(\rho\xi))^2 - \left( \frac{C_1}{2\epsilon} + C_2 \right) \|\xi\|_{H^2}^2. \]

Taking \( \epsilon \) so small that \( 1 - \frac{\epsilon C_1}{2} \geq 0 \), we get

\[ (L\xi,\xi)_{\tilde{\mathcal{H}}} \geq - \left( \frac{C_1}{2\epsilon} + C_2 \right) \|\xi\|_{H^2}^2. \]

Summing up, \( L \) is bounded from below in \( \tilde{\mathcal{H}} \). Therefore, thanks to [6, Chapter VI, Section 2.3], we have

**Theorem 2** The differential operator \( L \) on \( C_0^\infty(\Pi^0, C^3) \) admits the Friedrichs extension, which is a self-adjoint operator, in \( \tilde{\mathcal{H}} \).

We want to clarify the spectral property of the self-adjoint operator \( L \).

When \( \tilde{S} \) is constant, say, the equilibrium is isentropic, so that \( A = 0 \) and \( N^2 = 0 \) everywhere, the spectral property of \( L \) is clear in some sense. Actually in this case the operator \( L \) turns out to be

\[ L\xi = -\text{grad} \left( \frac{c^2}{\rho} g \right) \quad \text{with} \quad g := \text{div}(\rho\xi). \]

Therefore the vector wave equation

\[ \frac{\partial^2 \xi}{\partial t^2} + L\xi = 0 \]

can be reduced to the scalar wave equation

\[ \frac{\partial^2 g}{\partial t^2} - \text{div} \left( \rho \text{grad} \left( \frac{c^2}{\rho} g \right) \right) = 0. \]

Then by the argument developed in [5] we can claim the following

**Theorem 3** Suppose that \( \tilde{S} \) is constant so that \( N^2 = 0 \) everywhere. Then \( L \) can be considered as a self-adjoint operator on the Hilbert space

\[ \tilde{\mathcal{F}} := \tilde{\mathcal{H}} \cap \left\{ g = \text{div}(\rho\xi) \in L^2(\Pi, \frac{c^2}{\rho} dx) \left| \int_\Pi g = 0 \right. \right\}, \quad (3.6) \]

which is dense in \( \tilde{\mathcal{H}} \), and the spectrum of the operator \( L \) in \( \tilde{\mathcal{F}} \) consists of the eigenvalue \( 0 \) of infinite multiplicity and sequence of eigenvalues \( \lambda_n \) of finite multiplicity such that \( \lambda_n \to +\infty \) as \( n \to \infty \).
But we are interested in the case in which \( N^2 \) does not vanish identically. We shall show that if \( N^2 > 0 \) everywhere, there can appear sequence of eigenvalues which accumulates to 0.

We see that for \( \xi \in D(\mathbf{L}) \) the quantity

\[
\int \frac{\gamma P}{\rho^2} |\text{div}(\rho \xi)|^2 \, dx
\]

is finite. This implies that

\[
|\xi^3| \leq C\sqrt{z}
\]

holds for \( 0 \leq z \ll 1 \). ( Recall that \( \frac{\rho}{c^2} = \frac{\rho^2}{\gamma P} \in C^{\infty}(0 \leq z < z_+) \). ) So the boundary condition (1.18) at \( z = 0 \) is satisfied for \( \xi \in D(\mathbf{L}) \).

## 4 A class of particular solutions

Let us consider perturbations \( \xi \) of solutions of the particular form

\[
\xi = u(z) \sin lx \mathbf{e}_x + w(z) \cos lx \mathbf{e}_x.
\]  

(4.1)

Here the frequency \( l \) in \( x \) is such that \( lx \in 2\pi \mathbb{Z} \). Then the eigenvalue problem

\[
\mathbf{L}\xi = \lambda \xi
\]

turns out to be

\[
L_l^u = \lambda u, \quad L_l^w = \lambda w,
\]

(4.2)

where

\[
L_l^u = -\frac{l}{\rho} \delta \check{P}, \\
L_l^w = \frac{1}{\rho} \frac{d}{dz} \delta \check{P} + \frac{g}{\rho} \delta \check{P}.
\]

(4.3a)

(4.3b)

where

\[
\delta \check{P} = -\rho \dot{u} - \frac{d}{dz}(\rho w),
\]

(4.4)

\[
\delta \check{P} = c^2 \delta \check{P} + \left(c^2 \frac{dp}{dz} + g \rho \right) w
\]

\[
= -c^2 \rho \left( lu + \frac{dw}{dz} \right) + g \rho w
\]

(4.5)
4.1 Case $l = 0$

Let us consider the case $l = 0$, namely, let us consider merely vertical perturbations.

By neglecting $u$ and eliminating $\delta \rho$, the problem is reduced to the system

\[
\begin{align*}
\frac{dw}{dz} - \frac{g}{c^2} w + \delta \hat{P} = 0, \\
\frac{d}{dz} \delta \hat{P} + \frac{g}{c^2} \delta \hat{P} + (N^2 - \lambda) \rho w &= 0,
\end{align*}
\]

and it is reduced to the single equation

\[
- \frac{d}{dz} \left( c^2 \frac{dw}{dz} \right) + \left( \frac{1}{\rho} \frac{d\rho}{dz} + \frac{g^2}{c^2} + N^2 \right) \rho w = \lambda \rho w.
\]

Let us perform the Liouville transformation. See, e.g., \[1, \text{p.275, Theorem 6}\] or \[12, \text{p.110}\]. We put

\[
\begin{align*}
\zeta &= \int_0^z \sqrt{\frac{a}{\kappa}} \, dz, \quad W = (a\kappa)^{\frac{1}{4}} w, \\
q &= \frac{b}{\kappa} + \frac{1}{4} a \left[ \frac{d^2}{dz^2} \log(a\kappa) - \frac{1}{4} \left( \log(a\kappa) \right)^2 \right] + \\
&\quad \left( \frac{d}{dz} \log(a) \right) \left( \frac{d}{dz} \log(a\kappa) \right)
\end{align*}
\]

for

\[
a = c^2 \rho, \quad b = \left( \frac{g}{\rho} \frac{d\rho}{dz} + \frac{g^2}{c^2} + N^2 \right) \rho, \quad \kappa = \rho,
\]

which transform (4.7) to

\[
- \frac{d^2 W}{d\zeta^2} + qW = \lambda W.
\]

We see

\[
\zeta = \int_0^z \frac{1}{c} \, dz
\]

so that

\[
\begin{align*}
\zeta_+ := \int_0^{z_+} \frac{1}{c} \, dz &< \infty, \\
\zeta_+ - \zeta &= 2 \sqrt{\frac{\nu}{\xi}} \sqrt{z_+ - z(1 + [z_+ - z])} \quad \text{for} \quad 0 < z_+ - z \ll 1,
\end{align*}
\]

since

\[
c^2 = \frac{\xi}{\nu} (z_+ - z)(1 + [z_+ - z])
\]
By a tedious calculation we see
\[ q = \frac{(2\nu + 1)(2\nu - 1)}{4} \frac{1}{(\zeta_+ - \zeta)^2(1 + [(\zeta_+ - \zeta)^2]_1)}. \]  
(4.13)

Note that \((2\nu + 1)(2\nu - 1)/4 > 3/4\) for \(\nu > 1\). Therefore we can claim

The operator \(-\frac{d^2}{d\zeta^2} + q\) defined on \(C_0^\infty([0, \zeta_+])\) admits the Friedrichs extension, which is a self-adjoint operator, in the Hilbert space \(L^2([0, \zeta_+])\). Its spectrum consists of simple eigenvalues

\[ \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to +\infty. \]

Clearly these eigenvalues are those of \(L\). Therefore we can claim

**Theorem 4** Let \(l = 0\). Then there exists a sequence of eigenvalues \((\lambda'_n)_{n \in \mathbb{N}}\) such that

\[ \lambda'_1 < \lambda'_2 < \cdots < \lambda'_n < \cdots \to +\infty. \]

**4.2 Case** \(l > 0\)

Let us consider the equations when \(l > 0\). Moreover we try to find eigenvalue \(\lambda \neq 0\). We want to give a mathematically rigorous justification of the existence of ‘g-modes’, namely, eigenvalues accumulating to 0 and ‘p-modes’, namely, eigenvalues accumulating to \(+\infty\)

Corresponding to \(\mathcal{H}\), we are considering the Hilbert space \(X_l = L^2([0, \zeta_+], \rho(z)dz; \mathbb{C}^2)\) of functions \(\vec{w} = (u, w)^\top\) endowed with the norm \(\| \cdot \|_{X_l}\) given by

\[ \| \vec{w} \|^2_{X_l} = \int_0^{\zeta_+} (l^2|u|^2 + |w|^2)\rho(z)dz. \]

The operators \(\vec{L}_l = (L^u_l, L^w_l)^\top\) is considered as a self-adjoint operator in \(X_l\) and the domain \(D(\vec{L}_l)\) is

\[ D(\vec{L}_l) = \mathfrak{W}_l \cap \{ \vec{w} | \vec{L}_l \vec{w} \in X_l \}. \]

Here \(\mathfrak{W}_l\) is the closure of \(C_0^\infty([0, \zeta_+])\) in the Hilbert space \(\mathfrak{W}_l\) endowed with norm \(\| \cdot \|_{\mathfrak{W}_l}\) defined by

\[ \| \vec{w} \|^2_{\mathfrak{W}_l} = \| \vec{w} \|^2_{X_l} + \| \delta \vec{\beta} \|^2_{L^2(\frac{1}{\sqrt{\rho}}dz)}. \]

Of course this domain corresponds to the domain of the Friedrichs extension \(L\), say, \(\vec{w} \in D(\vec{L}_l)\) if and only if \(\xi\) specified by \(4.1\) belongs to \(D(L)\).

We can claim
Proposition 1 Let $\vec{w} = (u, w) \in \mathcal{M}_1$. Then $\vec{w} \in \mathcal{W}_l$ if and only if

$$w = 0 \text{ at } z = 0, \quad \text{and} \quad w = O(1) \text{ as } z \to z_+ - 0.$$  

By eliminating $u$ and $\delta \rho$, the problem is reduced to

$$\begin{align*}
\frac{dw}{dz} - \frac{g}{c^2} w + \left(1 - \frac{l^2 c^2}{\lambda}ight) \frac{\delta \rho}{c^2} &= 0, \quad (4.14a) \\
\frac{d}{dz} \delta P + \frac{g}{c^2} \delta \rho + (N^2 - \lambda) \rho w &= 0. \quad (4.14b)
\end{align*}$$

It would be natural to eliminate $\delta \rho$ to deduce a single equation for $w$ as for the case $l = 0$. The result would be

$$- \frac{d}{dz} \left(\frac{c^2 \rho}{Q} \frac{dw}{dz}\right) + \left[\frac{g}{\rho} \frac{d}{dz} \rho + \frac{g^2}{c^2} (N^2 - \lambda)\right] \frac{\rho}{Q} w = 0, \quad (4.15)$$

where

$$Q := 1 - \frac{l^2 c^2}{\lambda}. \quad (4.16)$$

But, since $l^2 \neq 0$ and $c^2 > 0$,

$$c^2 \sim \frac{\bar{g}}{\nu} (z_+ - z) \quad \text{as } z \to z_+ - 0,$$

for each $\lambda \in [0, \lambda_0]$, $\lambda_0$ being a small positive number, there exists a unique $z = z(\lambda) \in [0, z_+]$, namely, the ‘turning point’, such that

$$Q(z) < 0 \quad \text{for } 0 < z < z(\lambda)$$
$$Q(z(\lambda)) = 0$$
$$Q(z) > 0 \quad \text{for } z(\lambda) < z < z_+.$$

Therefore the coefficients of the equation $\text{(4.15)}$ have poles at the turning point. This may cause trouble during the analysis. So, we should seek another reduction to a single second order equation which has no singularity in the interior of the interval $[0, z_+]$ and we can treat without trouble.

Remark 1 We note that the equation $\text{(4.15)}$ is equivalent to \cite[p. 23, (22)]{11}, which is nothing but

$$- \lambda Q c^2 \frac{d}{dz} \left(\rho \frac{dw}{dz}\right) - \lambda Q^2 \frac{d}{dz} \left(\frac{c^2 \rho}{Q} \frac{dw}{dz}\right) + \lambda \left[\cdots\right] Q \rho c^2 w = 0,$$

given by multiplying $\text{(4.15)}$ by $\lambda Q^2$, while we can verify

$$Q^2 \frac{d}{dz} \frac{c^2 \rho}{Q} = \frac{dc^2}{dz} \rho.$$
Actually the symbols $\alpha^2, c^2, \sigma^2, (\cdot)'$ in [11] read $l^2, c^2, \lambda, \frac{d}{dx}(\cdot)$ respectively in this article. This equation could be said to be a natural generalization of the so called ‘Taylor-Goldstein equation’ which is often used in the study of stratified fluid motions successfully, but it seems not suitable for mathematical investigations here.

In order to avoid the above mentioned trouble, let us introduce the variable $\eta$ instead of $\delta P$, suggested by the argument by D. O. Gough in [3], by

$$\eta := \delta P + \frac{dP}{dz}w = \delta P - g\rho w,$$

which is $\Delta \delta P$ in the linearized approximation. Let us note the following

**Proposition 2** $\vec{w} = (u, w)^\top \in \mathfrak{W}_l$ if and only if $w \in L^2(\rho dz)$ and $\eta \in L^2\left(\frac{1}{\rho}dz\right)$.

Then the set of equations (4.14a), (4.14b) is equivalent to

$$\frac{dw}{dz} + A_{11}w + A_{12}\eta = 0,$$  \hspace{1cm} (4.18a)

$$\frac{d\eta}{dz} + A_{21}w + A_{22}\eta = 0,$$  \hspace{1cm} (4.18b)

where

$$A_{11} = -\frac{l^2g}{\lambda}, \quad A_{12} = \left(1 - \frac{l^2c^2}{\lambda}\right)\frac{1}{c^2\rho},$$

$$A_{21} = \frac{l^2g^2}{\lambda}\left(1 - \frac{\lambda^2}{l^2g^2}\right)\rho, \quad A_{22} = \frac{l^2g}{\lambda}. \hspace{1cm} (4.19)$$

We see that the set of equations (4.18a), (4.18b) reduces to the single equation

$$\frac{d^2\eta}{dz^2} + \frac{1}{H[\rho]} \frac{d\eta}{dz} + \left(\frac{\lambda}{c^2} - l^2\left(1 - \frac{\lambda^2}{\lambda}\right)\right)\eta = 0.$$  \hspace{1cm} (4.20)

Here and hereafter we use the notation

$$\frac{1}{H[Q]} := -\frac{d}{dz} \log Q \hspace{1cm} (4.21)$$

for any quantity $Q$ which is a function of $z$. $H[Q]$ is so called the scale height of $Q$.

In fact, (4.20) can be derived by eliminating $w$ from (4.18a), (4.18b), provided that $\lambda \neq l\gamma$, when $A_{21} \neq 0$, and can be verified directly when $\lambda = l\gamma$. 

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Note that
\[ N^2 = -\frac{g^2}{c^2} + \frac{g}{H[\rho]} \]  
(4.22)

5 Existence of ‘g-modes’ and ‘p-modes’

5.1 g-modes

We introduce

Assumption 2 It holds
\[ N^2(z) > 0 \quad \text{for} \quad 0 \leq z \leq z_+ . \]  
(5.1)

Under this assumption we are going to prove the following

Theorem 5 Let \( l > 0 \). Suppose Assumption 2. There exists a sequence of positive eigenvalues \((\lambda_{-n})_{n \in \mathbb{N}}\) such that \( \lambda_{-n} \to 0 \) as \( n \to \infty \).

Proof. We rewrite the equation (4.20) as
\[ -\frac{d}{dz} \left( \frac{1}{\rho} \frac{d \eta}{dz} \right) = \left[ \frac{\lambda}{c^2} - l^2 \left( 1 - \frac{N^2}{\lambda} \right) \right] \frac{1}{\rho} \eta . \]  
(5.2)

We consider the eigenvalue problem for the eigenvalue \( \Lambda \):
\[ -\frac{d}{dz} \left( \frac{1}{\rho} \frac{d \eta}{dz} \right) + \left( l^2 - \frac{\lambda}{c^2} \right) \frac{1}{\rho} \eta = \Lambda \frac{l^2N^2}{\rho} \eta , \]  
(5.3)
in which \( \lambda \) in the left-hand side is considered as a parameter. If \( \frac{1}{\lambda} = \Lambda \), then (5.3) coincides with (5.2).

Taking \( \lambda_0 < lg \), we consider \( 0 \leq \lambda \leq \lambda_0 \). Then for each fixed \( \lambda \in [0, \lambda_0] \), we have a sequence of eigenvalues \( \Lambda_n(\lambda) \) such that
\[ \Lambda_1(\lambda) < \Lambda_2(\lambda) < \cdots < \Lambda_n(\lambda) < \cdots \to +\infty . \]

In fact we perform the Liouville transformation
\[ \zeta = \int_0^z \sqrt{\frac{\kappa}{a}} dz , \quad v = (ak)^{3/4} \eta , \]
\[ q = \frac{b}{\kappa} + \frac{1}{4 \kappa} \left( \frac{d}{dz} \log(ak) \right)^2 - \frac{1}{4} \left( \frac{d}{dz} \log(ak) \right)^2 + \left( \frac{d}{dz} \log a \right) \left( \frac{d}{dz} \log(ak) \right) \]  
(5.4)
for
\[ a = \frac{1}{\rho}, \quad b = (l^2 - \frac{\lambda}{c^2}) \frac{1}{\rho}, \quad \kappa = \frac{l^2 N^2}{\rho}, \]
which transform (5.3) to
\[ -\frac{d^2 v}{d \zeta^2} + q v = \Lambda v. \quad (5.6) \]
We put
\[ q = q(\zeta; \lambda) = -\frac{1}{l^2 c^2 N^2} \lambda + q_0(\zeta) \quad (5.7) \]
with
\[ q_0(\zeta) = \frac{1}{N^2} + \frac{1}{4} \frac{1}{l^2 c^2 N^2} \left[ \frac{d^2}{dz^2} \log \left( \frac{N^2}{\rho^2} \right) + \frac{1}{4} \log \left( \frac{N^2}{\rho^2} \right) \right] + \left( \frac{d}{dz} \log \frac{1}{\rho} \right) \left( \frac{d}{dz} \log \left( \frac{N^2}{\rho^2} \right) \right). \quad (5.8) \]
We see that
\[ \zeta_+ := \int_0^{z_+} \sqrt{\frac{\kappa}{a}} d \zeta = \int_0^{z_+} N d \zeta < \infty \]
and
\[ \zeta_+ - \zeta = lN(z_+)(z_+ - z)(1 + [z_+ - z]_1). \]
Thus the interval \([0, z_+]\) is mapped onto \([0, \zeta_+]\). We see
\[ q = \frac{\nu(\nu + 2)}{4} \frac{1}{(\zeta_+ - \zeta)^2} (1 + [\zeta_+ - \zeta]_1). \]
Therefore for each fixed \(\lambda \in [0, \lambda_0]\) we have simple eigenvalues
\[ \Lambda_1(\lambda) < \Lambda_2(\lambda) < \cdots < \Lambda_n(\lambda) < \cdots \to +\infty. \]
The \(n\)-th eigenvalue is given by the Max-Min principle as follows:
For any set \(Y = \{y_1, \cdots, y_n\} \subset \mathcal{E}\) we put
\[ d(Y; \lambda) = \inf \{ Q[v; \lambda] \mid (v|y)_\mathcal{E} = 0 \quad \text{for} \quad y \in Y, v \in \mathcal{E}_1, \|v\|_\mathcal{E} = 1 \}. \]
Here \(\mathcal{E} = L^2(0, \zeta_+), \)
\[ Q[v; \lambda] = \int_0^{\zeta_+} \left( \left| \frac{dv}{d\zeta} \right|^2 + q(\zeta; \lambda) + K_0 \right) |v|^2 d\zeta, \]
for
\[ q(\zeta; \lambda) + K_0 \geq 1 + \frac{K_1}{(\zeta_+ - \zeta)^2}, \quad \frac{3}{4} < K_1 < \frac{\nu(\nu + 2)}{4}. \]
We take \(K_0\) independent of \(\lambda \in [0, \lambda_0]\). We put
\[ \mathcal{E}_1 = \{v\mid \int_0^{\zeta_+} \left( \left| \frac{dv}{d\zeta} \right|^2 + q_0(\zeta) + K_0 \right) |v|^2 d\zeta < \infty \}. \]
Here note that \( q(\zeta; \lambda) \leq q_0(\zeta) \), for \( \lambda \geq 0 \), so that \( Q[v; \lambda] \leq Q[v; 0] \). The space \( \mathcal{E}_1 \) is independent of \( \lambda \). Then it is known that

\[
\Lambda_n(\lambda) = \sup \{ \mathcal{D}(Y; \lambda) \mid Y \subset \mathcal{E}, \sharp Y = n \} - K_0.
\]

We claim

**Proposition 3** The function \( \lambda \mapsto \Lambda_n(\lambda) \) is continuous on \([0, \lambda_0]\).

Proof. Let \( 0 \leq \lambda + \delta \lambda, \lambda \leq \lambda_0 \). Then

\[
Q[v; \lambda + \delta \lambda] - Q[v; \lambda] = \left( - \int_0^{\zeta} \frac{1}{l^2 c^2 N^2} |v|^2 d\zeta \right) \cdot \delta \lambda.
\]

But

\[
\int_0^{\zeta} \frac{1}{l^2 c^2 N^2} |v|^2 d\zeta \leq M \int_0^{\zeta} \frac{1}{\zeta + \zeta} |v|^2 d\zeta,
\]

since \( c^2 \sim \frac{\xi}{\nu}(z+\bar{z}) \) and \( c^2 > 0 \). On the other hand

\[
Q[v; \lambda] = \int_0^{\zeta} \left( \left| \frac{dv}{d\zeta} \right|^2 + \left( - \frac{1}{l^2 c^2 N^2} \lambda + q_0(\zeta) + K_0 \right) |v|^2 d\zeta
\]

\[
\geq \frac{1}{M'} \int_0^{\zeta} \frac{1}{(\zeta + \zeta)^2} |v|^2 d\zeta,
\]

where \( M' \) can be independent of \( \lambda \). Therefore

\[
\int_0^{\zeta} \frac{1}{l^2 c^2 N^2} |v|^2 d\zeta \leq \zeta M' Q[v; \lambda],
\]

and

\[
|Q[v; \lambda + \delta \lambda] - Q[v; \lambda]| \leq \zeta M' Q[v; \lambda] |\delta \lambda|.
\]

Thus we have

\[
- \zeta M' |\delta \lambda| \mathcal{D}(Y; \lambda_0) \leq - \zeta M' |\delta \lambda| \mathcal{D}(Y; \lambda + \delta \lambda) \leq \mathcal{D}(Y; \lambda + \delta \lambda) - \mathcal{D}(Y; \lambda)
\]

\[
\leq \zeta M' |\delta \lambda| \mathcal{D}(Y; \lambda) \leq \zeta M' |\delta \lambda| \mathcal{D}(Y; 0).
\]

This estimate implies the Lipschitz continuity of \( \Lambda_n(\lambda) = \sup \mathcal{D}(Y; \lambda) \). \( \square \)

Now, since \( \Lambda_n(\lambda_0) \to +\infty \), we can find \( n_0 \) such that \( \Lambda_n(\lambda_0) > \frac{1}{\lambda_0} \) for \( n \geq n_0 \).

Note that \( \Lambda_n(\lambda) \geq \Lambda_n(\lambda_0) \) for \( \lambda \in [0, \lambda_0] \). By Proposition 3, the function

\[
f : \lambda \mapsto \lambda - \frac{1}{\Lambda_n(\lambda)}
\]
is continuous on \([0, \lambda_0]\) and

\[
f(0) = -\frac{1}{\Lambda(0)} < 0, \quad f(\lambda_0) = \lambda_0 - \frac{1}{\Lambda(\lambda_0)} > 0.
\]

Therefore there exists at least one \(\lambda \in [0, \lambda_0]\) such that \(f(\lambda) = 0\), that is, \(\Lambda_n(\lambda) = \frac{1}{\lambda}\). Although we cannot claim that the solution is unique, we denote by \(\lambda_{-n}\) one of the solutions. Then

\[
\lambda_{-n} = \frac{1}{\Lambda_n(\lambda_{-n})} \leq \frac{1}{\Lambda_n(\lambda_0)} \to 0.
\]

Thus completes the proof of Theorem 5.

**5.2 p-modes**

In the same way we can prove the following

**Theorem 6** Let \(l > 0\). There exists a sequence of positive eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) such that \(\lambda_n \to +\infty\) as \(n \to \infty\).

Proof. In order to deal with the equation (5.2) we consider the eigenvalue problem for the eigenvalue \(\Lambda:\)

\[
-\frac{d}{dz} \left( \frac{1}{\rho} \frac{d\eta}{dz} \right) + l^2 (1 - \mu \bar{N}^2) \frac{1}{\rho} \eta = \Lambda \frac{1}{c^2 \rho} \eta, \quad (5.9)
\]

in which \(\mu\) is a parameter, which stands for \(1/\lambda\). If \(\mu = 1/\lambda = 1/\Lambda\), then (5.9) coincides with (5.2) and this \(\lambda\) gives a solution.

Taking \(\mu_0 = 1/\lambda_0 < 1/l\), we consider \(0 \leq \mu \leq \mu_0\). Then for each fixed \(\mu \in [0, \mu_0]\), we have a sequence of eigenvalues \((\Lambda_n(\mu))_{n \in \mathbb{N}}\) such that

\[
\Lambda_1(\mu) < \Lambda_2(\mu) < \cdots < \Lambda_n(\mu) < \cdots \to +\infty.
\]

In fact, we perform the Liouville transformation

\[
\zeta = \int_0^z \sqrt{\frac{\kappa}{a}} dz, \quad v = (ak)^{1/4} \eta, \quad q = \frac{b}{\kappa} + \frac{1}{4} \frac{a}{\kappa} \left( \frac{d^2}{dz^2} \log(ak) - \frac{1}{4} \left( \log(ak) \right)^2 + \left( \frac{d}{dz} \log a \right) \left( \frac{d}{dz} \log(ak) \right) \right),
\]

for

\[
a = \frac{1}{\rho}, \quad b = l^2 (1 - \bar{N}^2 \mu) \frac{1}{\rho}, \quad \kappa = \frac{1}{c^2 \rho}, \quad (5.10)
\]

which transform (5.9) to

\[
-\frac{d^2 v}{d\zeta^2} + qv = \Lambda v. \quad (5.12)
\]
We put
\[ q = q(\zeta; \mu) = -l^2 c^2 N^2 \mu + q_0(\zeta) \] (5.13)
\[ q_0(\zeta) = l^2 c^2 + \frac{1}{4} c^2 \left[ \frac{d^2}{dz^2} \log \left( \frac{N^2}{\rho^2} \right) - \frac{1}{4} \log \left( \frac{N^2}{\rho^2} \right) + \left( \frac{d}{dz} \log \frac{1}{\rho} \right) \left( \frac{d}{dz} \log \left( \frac{N^2}{\rho^2} \right) \right) \right]. \] (5.14)

We see that
\[ \zeta_+ := \int_0^{z_+} \sqrt{\frac{k}{a}} dz = \int_0^{z_+} \frac{1}{c} dz < \infty \]
and
\[ \zeta_+ - \zeta = 2 \sqrt{\frac{g}{\nu}} \sqrt{z_+ - z(1 + [z_+ - z]_1)}, \]
since
\[ c^2 = \frac{g}{\nu} (z_+ - z)(1 + [z_+ - z]_1). \]
Thus the interval \([0, z_+]\) is mapped onto \([0, \zeta_+]\) We see
\[ q = \frac{(2\nu + 1)(2\nu + 3)}{4} \frac{1}{(\zeta_+ - \zeta)^2} (1 + [(\zeta_+ - \zeta)^2]_1). \]

Therefore for each fixed \( \mu \in [0, \mu_0] \) we have simple eigenvalues
\[ \Lambda_1(\mu) < \Lambda_2(\mu) < \cdots < \Lambda_n(\mu) < \cdots \to +\infty. \]
The n-th eigenvalue is given by the Max-Min principle as follows:
For any set \( Y = \{y_1, \ldots, y_n\} \subset \mathcal{E} \) we put
\[ \vartheta(Y; \mu) = \min \{ Q[v; \mu] | (v|y)_\mathcal{E} = 0 \text{ for } y \in Y, v \in \mathcal{E}_1, \|v\|_\mathcal{E} = 1 \}. \]
Here \( \mathcal{X}_t = L^2(0, \zeta_+) \),
\[ Q[v; \mu] = \int_0^{\zeta_+} \left( \left| \frac{dv}{d\zeta} \right|^2 + q(\zeta; \mu) + K_0 \right) |v|^2 d\zeta, \]
for
\[ q(\zeta; \mu) + K_0 \geq 1 + \frac{K_1}{(\zeta_+ - \zeta)^2}, \quad \frac{3}{4} < K_1 < \frac{2\nu + 1)(2\nu + 3)}{4}. \]
We take \( K_0 \) independent of \( \mu \in [0, \mu_0] \). We put
\[ \mathcal{E}_1 = \{v| \int_0^{\zeta_+} \left( \left| \frac{dv}{d\zeta} \right|^2 + q_0(\zeta) + K_0 \right) |v|^2 d\zeta < \infty \}. \]
Here note that, since \( |l^2 c^2 N^2| \leq M \) uniformly, \( M \) being a sufficiently large constant, we have
\[ |q(\zeta; \lambda) - q_0(\zeta)| \leq M \mu_0 \]
so that
\[ |Q[v; \mu] - Q[v; 0]| \leq M \mu_0 \|v\|_E^2 \]
for \(0 \leq \mu \leq \mu_0\). The space \(E_1\) is independent of \(\mu\). Then it is known that
\[ \Lambda_n(\mu) = \sup\{d(Y; \mu) | Y \subset E_1, \#Y = n\} - K_0. \]

We claim

**Proposition 4** The function \(\mu \mapsto \Lambda_n(\mu)\) is continuous on \([0, \mu_0]\).

**Proof.** Let \(0 \leq \mu, \mu + \delta \mu \leq \mu_0\). Then
\[ Q[v; \mu + \delta \mu] - Q[v; \mu] = (\int_0^{\zeta+} t^2 c^2 N^2 |v|^2 d\zeta) \cdot \delta \mu. \]
But
\[ \int_0^{\zeta+} t^2 c^2 N^2 |v|^2 d\zeta \leq M \int_0^{\zeta+} |v|^2 d\zeta = M \]
for \(\|v\|_{X_1} = 1\). Hence
\[ |Q[v; \mu + \delta \mu] - Q[v; \mu]| \leq M |\delta \mu|. \]
Easily this implies
\[ |\Lambda_n(\mu + \delta \mu) - \Lambda_n(\mu)| \leq M |\delta \mu|. \]
\(\Box\)

Now, since \(\Lambda_n(\mu_0) \to +\infty\) as \(n \to \infty\), we can find \(n_0\) such that \(\Lambda_n(\mu_0) > \frac{1}{\mu_0}\)
for \(n \geq n_0\). Note that we can suppose that \(\Lambda_n(\mu) \geq \Lambda_n(\mu_0) - M \mu_0 > \frac{1}{\mu_0} - M \mu_0 > 0\) for \(0 \leq \mu \leq \mu_0\), provided that \(\mu_0\) is sufficiently small. Then the function
\[ f : \mu \mapsto \mu - \frac{1}{\Lambda_n(\mu)} \]
is continuous on \([0, \mu_0]\) thanks to Proposition 4 and
\[ f(0) = -\frac{1}{\Lambda_n(0)} < 0, \quad f(\mu_0) = \mu_0 - \frac{1}{\Lambda_n(\mu_0)} > 0. \]
Therefore there exists at least one \(\mu\), say \(\mu_n\), in \([0, \mu_0]\) such that \(f(\mu_n) = 0\), that is, \(\mu_n = \frac{1}{\Lambda_n(\mu_n)}\); or \(\lambda_n := \frac{1}{\mu_n} = \Lambda_n(\frac{1}{\lambda_n})\), say, \(\lambda = \lambda_n\) satisfies (5.2), and
\[ \lambda_n = \Lambda_n(\frac{1}{\lambda_n}) \geq \Lambda_n(\frac{1}{\lambda_0}) - M \mu_0 \to +\infty. \]
This completes the proof of the Theorem. \(\Box\)

Note that we need not Assumption 2 in order to prove the existence of p-modes.
6 Absence of continuous spectrum

Let us suppose \( l > 0 \) and consider the operator \( \vec{L}_l \): 

\[
\vec{L}_l \vec{w} = \begin{bmatrix}
L_u^l \\
L_w^l 
\end{bmatrix} = \begin{bmatrix}
\frac{l}{\rho} \delta \tilde{P} \\
\frac{1}{\rho} \frac{d}{dz} \delta \tilde{P} + \frac{g}{\rho} \delta \tilde{P}
\end{bmatrix}.
\]  

(6.1)

for \( \vec{w} = (u, w)^\top \).

Suppose \( \lambda \neq 0 \). Then the equation

\[
\vec{L}_l \vec{w} = \lambda \vec{w}
\]  

(6.2)

turns out to be the system (4.18a) (4.18b), say,

\[
\frac{dw}{dz} + A_{11} w + A_{12} \eta = 0, \\
\frac{d\eta}{dz} + A_{21} w + A_{22} \eta = 0,
\]  

(6.3a) (6.3b)

where

\[
A_{11} = -\frac{l^2 g^2}{\lambda}, \quad A_{12} = \left(1 - \frac{l^2 c^2}{\lambda^2}\right) \frac{1}{c^2 \rho}, \\
A_{21} = \frac{l^2 g^2}{\lambda} \left(1 - \frac{\lambda^2}{l^2 g^2}\right) \rho, \quad A_{22} = \frac{l^2 g}{\lambda}.
\]  

(6.4)

Since \( z = 0 \) is a regular boundary point, we can consider the solutions \( \vec{\eta}_j = (\eta_j, \eta_j \eta_j) \top \), \( j = 1, 2 \), such that

\[
\vec{\eta}_1 = \begin{bmatrix}
w_1 \\
\eta_1
\end{bmatrix} = \begin{bmatrix}
0 \\
1
\end{bmatrix} \text{ at } z = 0
\]

and

\[
\vec{\eta}_2 = \begin{bmatrix}
w_2 \\
\eta_2
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix} \text{ at } z = 0.
\]

Let us denote

\[
\varphi_{Oj} = \vec{\eta}_j, \quad j = 1, 2.
\]

Then

\[
\Phi_O = [\varphi_{O1} \varphi_{O2}]
\]

is a fundamental matrix of solutions of (6.3a) (6.3b). Of course only \( \varphi_{O1} \) satisfies the boundary condition

\[
w = 0 \quad \text{at } z = 0.
\]
Let us specify another fundamental matrix \( \Phi_S = [\varphi_{S1} \ \varphi_{S2}] \) in view of asymptotic behaviors at the singular boundary \( z = z_+ \). In order to do so, we suppose

**Assumption 3** The index \( \nu = \frac{1}{\gamma - 1} \) is a rational number.

Let \( \nu = N/D \), where \( N, D \) are mutual prime positive natural numbers. Of course \( N > D \), since \( \nu > 1 \). Let us introduce the variable \( s \) defined by

\[
s = (z_+ - z)^{1/D} \quad \Leftrightarrow \quad z_+ - z = s^D, \tag{6.5}\]

\( s \) runs over the interval \( [0, s_+] \), where \( s_+ := z_+^{1/D} \). We are supposing

\[
\rho = C_{\rho} s^N (1 + [s^D]_1) .
\]

The system of equations (6.3a)(6.3b) reads

\[
\begin{align*}
s \frac{dw}{ds} - Ds^D A_{11} w - Ds^D A_{12} \eta &= 0, \tag{6.6a} \\
s \frac{d\eta}{ds} - Ds^D A_{21} w - Ds^D A_{22} \eta &= 0. \tag{6.6b}
\end{align*}
\]

Put

\[
\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w \\ s^{-N} \eta \end{bmatrix} \quad \Leftrightarrow \quad \vec{\eta} = \begin{bmatrix} w \\ \eta \end{bmatrix} = \begin{bmatrix} y_1 \\ s^N y_2 \end{bmatrix} . \tag{6.7}
\]

Then the system of equations reduces to

\[
\begin{align*}
s \frac{dy_1}{ds} + B_{11} y_1 + B_{12} y_2 &= 0, \tag{6.8a} \\
s \frac{dy_2}{ds} + B_{21} y_1 + B_{22} y_2 &= 0, \tag{6.8b}
\end{align*}
\]

where

\[
\begin{align*}
B_{11} &= -Ds^D A_{11} = \frac{DI^2 g}{\lambda} s^D, \\
B_{12} &= -Ds^{D+N} A_{12} = -\frac{N}{gC_\rho} \gamma (1 + [s^D]_1), \\
B_{21} &= Ds^{D-N} A_{21} = \frac{DC_{\rho}I^2 g^2}{\lambda} (1 - \frac{\lambda^2}{I^2 g^2}) s^D (1 + [s^D]_1), \\
B_{22} &= -Ds^D A_{22} + N = N - \frac{DI^2 g}{\lambda} s^D . \tag{6.9}
\end{align*}
\]
Hence we see
\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{12} & B_{22}
\end{bmatrix} = \begin{bmatrix}
0 & -\frac{N}{gC_\rho} \\
0 & N
\end{bmatrix} + [s^D]_1.
\] (6.10)

Therefore, by an application of the recipe prescribed in the proof of [2, Chapter 4, Theorem 4.2], we have solutions \( \vec{y}_1, \vec{y}_2 \) of the form
\[
\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + O(s),
\] (6.11a)
\[
\vec{y}_2 = \begin{bmatrix} \frac{1}{gC_\rho} \\ 1 \end{bmatrix} s^{-N}(1 + O(s \log s)).
\] (6.11b)

Correspondingly we put
\[
\varphi_{S1} = \begin{bmatrix} y_{(1)1} \\ s^Ny_{(1)2} \end{bmatrix} = \begin{bmatrix} 1 + O(s) \\ s^NO(s) \end{bmatrix},
\] (6.12)
\[
\varphi_{S2} = \begin{bmatrix} y_{(2)1} \\ s^Ny_{(2)2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{gC_\rho} s^{-N}(1 + O(s \log s)) \\ 1 + O(s \log s) \end{bmatrix}.
\] (6.13)

Thus we have a fundamental system of solutions:
\[
\Phi_S = [\varphi_{S1} \varphi_{S2}].
\] (6.14)

Only \( \varphi_{S1} \) is admissible, since \( \eta_{S2} = 1 + O(s \log s) \) does not belong to \( L^2(dz/\rho) \).

Let
\[
\Phi_S = \Phi_O C(\lambda), \quad C(\lambda) = \begin{bmatrix} c_{11}(\lambda) & c_{12}(\lambda) \\
c_{21}(\lambda) & c_{22}(\lambda) \end{bmatrix}.
\] (6.15)

Here \( C(\lambda) \) is a non-singular matrix which is a holomorphic function of \( \lambda \in \mathbb{C} \setminus \{0\} \). See [2, Chapter 1, Theorem 7.3].

Now we can claim that \( \lambda \) is an eigenvalue if and only if
\[
D(z, \lambda) := \det[\varphi_{O1}(z, \lambda) \varphi_{S1}(z, \lambda)] = 0.
\] (6.16)

But we see
\[
D(z, \lambda) = c_{21}(\lambda) \det \Phi_O(z, \lambda).
\]

Since \( \det \Phi_O \neq 0 \), we see that \( \lambda \) is an eigenvalue if and only if \( c_{21}(\lambda) = 0 \). Since \( c_{21}(\lambda) \) is a holomorphic function of \( \lambda \in \mathbb{C} \setminus \{0\} \), we can claim
Theorem 7  Let $l > 0$. Suppose Assumption 3. The eigenvalues of (6.2) cannot accumulate to a value $\neq 0$.

In fact, since $\tilde{L}_l$ is a self-adjoint operator in $X_l$, its resolvent set contains $\mathbb{C} \setminus \mathbb{R} = \{ \lambda \mid \text{Im} [\lambda] \neq 0 \}$ so that $c_{12}$ cannot vanish identically.

The spectrum of $\tilde{L}_l$ is a subset of $[C, +\infty[ \subset \mathbb{R}$. But does it contain continuous spectra, or, does it consist of eigenvalues? In order to examine this question, we consider $\lambda \neq 0$ which is not an eigenvalue and try to solve the equation

$$(\tilde{L}_l - \lambda)\vec{w} = \vec{f}, \quad (6.17)$$

where

$$\vec{f} = \begin{bmatrix} f^u \\ f^w \end{bmatrix} \in X_l. \quad (6.18)$$

The equation reads

$$\begin{align*}
\frac{dw}{dz} + A_{11}w + A_{12}\eta &= h_1, \\
\frac{d\eta}{dz} + A_{21}w + A_{22}\eta &= h_2,
\end{align*} \quad (6.19a, 6.19b)$$

where

$$\vec{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \left[ \begin{array}{c} \frac{l}{\lambda} f^u \\ \rho (f^w - \frac{l}{\lambda} f^u) \end{array} \right]. \quad (6.20)$$

The solution of (6.19a)(6.19b) should be given by

$$\vec{\eta}(z) = \Phi(z) \int_{z'}^z \Phi(z')^{-1}\vec{h}(z')dz' + \Phi(z)c, \quad (6.21)$$

where

$$\Phi = [\varphi_{O1} \varphi_{S1}], \quad (6.22)$$

which is invertible since $\lambda$ is not an eigenvalue, and $c = (c_1, c_2)^T$ is a constant vector which should be chosen so that $\vec{w}$ corresponding to $\vec{\eta}$ given by (6.21) belong to $D(\tilde{L}_l)$.

Actually it is possible by taking

$$\begin{align*}
c_1 &= -\int_0^z [\Phi(z')^{-1}\vec{h}(z')]dz', \\
c_2 &= 0, \quad (6.23a,b)
\end{align*}$$

provided that $\vec{f} \in \mathfrak{M}_l$. Let us show it.
Hereafter we suppose \( \| f \|_{L^1} \leq 1 \) so that
\[
\| h_1 \|_{L^2(\rho dz)} \leq M_0, \quad \| h_2 \|_{L^2(dz/\rho)} \leq M_0.
\]

Let
\[
C^{-1} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}.
\] (6.24)

Since
\[
\det \Phi = c_{21} \det \Phi_O,
\]
we have \( c_{21} \neq 0 \) for \( \lambda \) is not an eigenvalue. Therefore \( \gamma_{21} = -c_{21}/\det C \neq 0 \), too. This implies that
\[
\varphi_{O_1} = \gamma_{11} \varphi_{S_1} + \gamma_{21} \varphi_{S_2} = \gamma_{21} \begin{bmatrix} -\frac{1}{gC_\rho} s^{-N} (1 + O(s \log s)) \\ 1 + O(s \log s) \end{bmatrix}.
\]

Therefore we have
\[
\Phi = \begin{bmatrix} \frac{\gamma_{21}}{gC_\rho} s^{-N} (1 + O(s \log s)) & 1 + O(s) \\ \gamma_{21} (1 + O(s \log s)) & s^N O(s) \end{bmatrix}
\] (6.25)

and
\[
\Phi^{-1} = \begin{bmatrix} O(s^{N+1}) (1 + O(s \log s)) & -\frac{1}{\gamma_{21}} (1 + O(s \log s)) \\ -(1 + O(s \log s)) & \frac{1}{gC_\rho} s^N (1 + O(s \log s)) \end{bmatrix}
\] (6.26)

Let us denote
\[
s' = (z_+ - z')^{1/D} \iff z_+ - z' = (s')^D.
\]

Then, thanks to (6.20), we see
\[
[\Phi(z')^{-1} \Xi(z')] dz' =
= \left( [\Phi(z')^{-1}]_{11} h_1(z') + [\Phi(z')^{-1}]_{12} h_2(z') \right) dz'
= \left( O((s')^{N+1}) h_1(z') + O(1) h_2(z') \right) (s')^{D-1} ds'.
\]

Since
\[
\int_0^{s^+} s^{N+D} |h_1| ds \leq \sqrt{\int_0^{s^+} s^{N+D+1} ds} \sqrt{\int_0^{s^+} |h_1|^2 s^{N+D-1} ds} \leq CM_0,
\]
\[
\int_0^{s^+} s^{D-1} |h_2| ds \leq \sqrt{\int_0^{s^+} s^{N+D-1} ds} \sqrt{\int_0^{s^+} |h_2|^2 s^{-N+D-1} ds} \leq CM_0,
\]

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we have
\[ c_1 = \int_0^{+\infty} \Phi(z')^{-1} \tilde{h}(z') dz' < \infty. \]

Moreover, using (6.25), (6.26), we can show that
\[ w(z) = -\int_z^{+\infty} \Phi_{11}(z)[\Phi(z')^{-1}]_{11} h_1(z') dz' + \]
\[ + \int_0^z \Phi_{12}(z)[\Phi(z')^{-1}]_{21} h_1(z') + [\Phi(z')^{-1}]_{12} h_2(z') dz', \quad (6.27) \]
\[ \eta(z) = -\int_z^{+\infty} \Phi_{21}(z)[\Phi(z')^{-1}]_{11} h_1(z') + [\Phi(z')^{-1}]_{12} h_2(z') dz' + \]
\[ + \int_0^z \Phi_{22}(z)[\Phi(z')^{-1}]_{21} h_1(z') + [\Phi(z')^{-1}]_{12} h_2(z') dz' \quad (6.28) \]
satisfy
\[ w \in L^2(\rho dz), \quad \eta \in L^2\left(\frac{dz}{\rho}\right) \]
and boundary conditions.

Summing up, we can claim

**Theorem 8** Let \( l > 0. \) Suppose Assumption 3. Let \( \lambda \neq 0, \) and \( \lambda \) is not an eigenvalue. Then for any \( f \in \mathcal{M}_l \) the equation
\[ (\bar{L}_l - \lambda) \bar{w} = \bar{f} \]
admits a solution \( \bar{w} \in D(\bar{L}_l) \) and
\[ \|\bar{w}\|_{\mathcal{M}_l} \leq C \|\bar{f}\|_{\mathcal{M}_l}, \]
that is, \( (\bar{L}_l - \lambda)^{-1} \) is bounded in \( \mathcal{M}_l, \) or, \( \lambda \) belongs to the resolvent set of the operator \( \bar{L}_l \) in \( \mathcal{M}_l. \) Therefore the spectrum of \( \bar{L}_l \) in \( \mathcal{M}_l \) consist of a countable many eigenvalues, whose eigenfunctions form a complete system of \( \mathcal{W}_l. \)

**Remark 2** Actually \( \mathcal{M}_l \) is dense in \( \mathcal{X}_l, \) but we do not claim that \( (\bar{L}_l - \lambda)^{-1} \) is bounded in the \( \mathcal{X}_l\)-norm. It is an open problem to answer to the question : Does there exist continuous spectra for the self-adjoint operator \( \bar{L}_l \) considered in \( \mathcal{X}_l? \)

**7 Concluding remarks**

Under the Assumption 2 that \( N^2 \geq 1/C > 0, \) we have given a mathematically rigorous proof of the existence of a sequence of eigenvalues which accumulates to 0, say, the existence of the g-modes. However the problem to clarify the structure of the spectrum of the operator \( L \) for this case cannot be said to be completely solved, say, the question whether the spectrum of the self-adjoint operator \( L \) considered in a suitable dense subspace \( \mathcal{S} \) of \( \mathcal{H} \) can be exhausted by
eigenvalues, which accumulate to 0 and $+\infty$ or not is still open. In this sense the present result has not yet caught up that, say Theorem 5 for the case in which $N^2 = 0$ everywhere given by [5].

On the other hand our argument on the existence of g-modes cannot work if $N^2$ takes negative values somewhere. However in the theory of astroseismology by astrophysicists the square of the Brunt-Väisälä frequency $N^2$ turns out to be negative near the surface in many realistic stellar models. (See, e.g., [7].) In this sense we should continue the study for the case in which the Assumption 2 does not hold.

Of course we should not forget that this article concerns the solutions of the linearized approximation. The ultimate aim of the mathematical study is to construct true solutions of the original non-linear equations for which the constructed time-periodic oscillations of the linearized equations turn out to be the first approximations of the true solutions. As for barotropic and spherically symmetric evolution of the structure of the atmosphere or the self-gravitating gaseous stars, this task has been done successfully in [8], [9], [4], but there remain open problems of this task for not barotropic, or, not spherically symmetric perturbations.

Therefore even if we consider the problem of gaseous adiabatic oscillations under gravitation in the most simple situation on the flat Earth around a stratified back ground density distribution, there remain many interesting mathematical problems still open. Mathematical difficulty arises from the treatise of the free boundary of the gas which touches the vacuum.

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