Corrections to the $ns$-levels of hydrogen atom in deformed space with minimal length

M. M. Stetsko*

Department of Theoretical Physics, Ivan Franko National University of Lviv,
12 Drahomanov Str., Lviv, UA-79005, Ukraine

April 1, 2022

Abstract

We investigated the hydrogen atom problem with deformed Heisenberg algebra leading to the existence of minimal length. Using modified perturbation theory developed in our previous work [M. M. Stetsko and V. M. Tkachuk, Phys. Rev. A 74, 012101 (2006)] we calculated the corrections to the arbitrary $s$-levels for hydrogen atom. We received a simple relation for the estimation of minimal length. We also compared the estimation of minimal length obtained here with the results obtained in the preceding investigations.

1 Introduction

In recent years there has been a growing interest in quantum mechanical systems with deformed (generalized) commutation relations. Deformed commutation relations arose as a natural generalization of the canonical ones. So it is interesting to consider the quantum mechanical problems where the position and the momentum operators obey the generalized commutation relations. Deformed commutation relations appeared in the quantum gravity and string theory, where it was expected that generalized commutation relations might eliminate some disadvantages of these theories. String theory and quantum gravity implied on the existence of minimal observable length [1] [2] [3]. Such a suggestion

*E-mail: mykola@ktf.franko.lviv.ua
leads to the appropriate deformation of commutation relations between the momentum and position operators \[4, 5, 6, 7, 8\]. In the \(D\)-dimensional case deformed Heisenberg algebra introducing minimal length takes the tensorial form:

\[
[X_i, P_j] = i\hbar(\delta_{ij}(1 + \beta P^2) + \beta' P_i P_j), \quad [P_i, P_j] = 0, \\
[X_i, X_j] = i\hbar\frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2}(P_j X_i - P_i X_j).
\]

(1)

The hydrogen atom is one of the simplest quantum mechanical systems allowing not only a highly accurate theoretical prediction but also having a well investigated experimental spectrum \[9, 10\]. So it is interestingly to examine the hydrogen atom problem when the position and momentum operators obey deformed commutation relations (1). Such a problem was considered for the first time by Brau \[11\] in a particular case when \(\beta' = 2\beta\). In \[12\] hydrogen atom was investigated in a general case when \(\beta' \neq 2\beta\). The authors developed the perturbation theory for calculating corrections to the energy spectrum. But that perturbation theory gave a possibility to calculate corrections to the energy levels only if \(l \neq 0\). Then for calculating corrections to the \(s\)-levels the authors used a numerical method and the cutoff procedure.

In our previous work \[13\] we developed a modified perturbation theory that gives a possibility to calculate corrections for arbitrary energy levels. This problem was considered in the general case when \(\beta' \neq 2\beta\). We also received an analytical expression for the corrections to the 1s and 2s energy levels. As was shown these results can be reduced to the results obtained in \[11\] when \(\beta' = 2\beta\) and are in good agreement with the results obtained in \[12\] when \(\beta' \neq 2\beta\) and \(l \neq 0\).

In the present work we continue to investigate the hydrogen atom problem in deformed space that leads to the existence of minimal length. We will consider a general case when \(\beta' \neq 2\beta\). We will calculate the corrections to the arbitrary \(s\)-levels.

This paper is organized as follows. In the second section we calculate corrections to the arbitrary \(s\)-levels using the perturbation theory proposed in our previous work \[13\]. In the third section we get a simple relation that gives a possibility to estimate minimal length. We also compare the estimation obtained here with the estimations received in the preceding works \[12, 13\]. And finally the fourth section contains the discussion.
2 Corrections to the energy of $n.s$-levels for hydrogen atom

In this section we consider the eigenvalue problem for hydrogen atom in a three-dimensional case

$$\left( \frac{p^2}{2m} - \frac{e^2}{R} \right) \Psi = E \Psi,$$

(2)

where the operators of position $X_i$ and momentum $P_i$ obey the deformed commutation relation (1) and $R = \sqrt{\sum_{i=1}^{3} x_i^2}$.

As was shown in our preceding investigation [13] the following representation satisfies the algebra in the first order over the parameters $\beta, \beta'$.

$$\begin{align*}
X_i &= x_i + \frac{2\beta - \beta'}{4} (x_i p^2 + p^2 x_i), \\
P_i &= p_i + \frac{\beta'}{\mathcal{B}} p_i p^2;
\end{align*}$$

(3)

where $p^2 = \sum_{k=1}^{3} p_k^2$ and operators $x_i, p_i$ obey canonical commutation relations $[x_i, p_j] = i\hbar \delta_{ij}$. For undeformed Heisenberg algebra the position representation may be taken:

$x_i = x_i, p_i = i\hbar \frac{\partial}{\partial x_i}$.

In paper [13] it was shown that Hamiltonian of hydrogen atom (2) can be written in the linear approximation over the deformation parameters

$$H = \frac{p^2}{2m} + \frac{\beta' p^4}{2m} - \frac{e^2}{2} \left[ \frac{1}{\sqrt{r^2 + b^2}} - \frac{2\beta - \beta'}{4} \left( \frac{1}{r} p^2 + \frac{p^2}{r} \right) \right].$$

(4)

where $r = \sqrt{\sum_{i=1}^{3} x_i^2}$.

As was noted in [13] one can calculate corrections to the arbitrary energy levels of hydrogen atom having Hamiltonian (4). It is necessary to say that for calculation corrections to the energy levels with nonzero angular momentum one can use a somewhat different approach [12, 13].

We rewrite Hamiltonian (4) in the form:

$$H = H_0 + V$$

(5)

where $H_0$ is Hamiltonian of ordinary hydrogen atom and $V$ is the perturbation caused by deformation.

$$V = \frac{\beta' p^4}{2m} - \frac{e^2}{2} \left[ \frac{1}{\sqrt{r^2 + b^2}} - \frac{1}{r} - \frac{2\beta - \beta'}{4} \left( \frac{1}{r} p^2 + \frac{p^2}{r} \right) \right].$$

(6)

where $b = \hbar \sqrt{\alpha}$, and $\alpha = 2\beta - \beta'$. 

3
So having the eigenfunctions for excited s-levels one can calculate corrections to the energy spectrum. At first we consider the correction caused by the term \( \frac{1}{\sqrt{r^2 + b^2}} \). We have

\[
\left\langle \psi_{ns} \left| \frac{1}{\sqrt{r^2 + b^2}} \right| \psi_{ns} \right\rangle = \frac{\pi e^2}{2a} \frac{(n-1)!}{[n!]^3} \sum_{i,j=0}^{n-1} \zeta^{2n-i-j} C_{n-1}^i C_n^i C_{n-1}^j C_n^j \partial^{2n-i-j} \zeta \left[ H_0(\zeta) - Y_0(\zeta) \right]
\]

where \( \zeta = \frac{2a}{na} \), \( a \) is the Bohr radius and \( H, Y \) are the Struve and the Bessel functions, respectively \[14\].

Hamiltonian \[4\] contains the terms linear over the deformation parameters. So we write corrections up to the first order over \( \alpha \) (or \( b^2 \)). As one can see for such an approximation the leading contribution is given by the derivatives from the Bessel functions and, conversely, we can neglect the derivatives from the Struve functions.

So we write the expression ignoring the derivatives from the Struve functions

\[
\left\langle \psi_{ns} \left| \frac{1}{\sqrt{r^2 + b^2}} \right| \psi_{ns} \right\rangle = -\frac{\pi e^2}{2a} \frac{(n-1)!}{[n!]^3} \left( n^2[(n-1)!]^2 \zeta^2 \partial^2 \partial \zeta^2 Y_0(\zeta) + \right.
\]

\[
\left. + \sum_{i,j=0}^{n-1} \zeta^{2n-i-j} C_{n-1}^i C_n^i C_{n-1}^j C_n^j \partial^{2n-i-j} \zeta \left[ H_0(\zeta) - Y_0(\zeta) \right] \right)
\]

The derivatives from the Bessel functions can be represented in the form:

\[
\frac{\partial^k}{\partial \zeta^k} Y_0(\zeta) = \frac{1}{2^k} \sum_{l=0}^{k} (-1)^l C_k^l Y_{2l-k}(\zeta)
\]

We use this representation for derivatives of the Bessel functions and substitute relation \[10\] in expression \[9\]

\[
\left\langle \psi_{ns} \left| \frac{1}{\sqrt{r^2 + b^2}} \right| \psi_{ns} \right\rangle = -\frac{\pi e^2}{2a} \frac{(n-1)!}{[n!]^3} \left( n^2[(n-1)!]^2 \zeta^2 [Y_2(\zeta) - Y_0(\zeta)] + \right.
\]

\[
\left. + \sum_{i,j=0}^{n-1} C_{n-1}^i C_n^i C_{n-1}^j C_n^j \partial^{2n-i-j} \zeta \left( \frac{1}{2} \sum_{l=0}^{2n-i-j} (-1)^l C_{2n-i-j}^l Y_{2l-2n+i+j}(\zeta) \right) \right)
\]

We rewrite the expression \[11\] so that it should take into account only two leading terms
in the last sum,
\[ \left\langle \Psi_{ns} \left| \frac{1}{\sqrt{r^2 + b^2}} \right| \Psi_{ns} \right\rangle = -\frac{\pi^2}{2} \frac{(n-1)!}{[n!]^2} \left( \frac{n^2[(n-1)!]^2}{2} \xi^2 [Y_2(\xi) - Y_0(\xi)] + \right. \\
\sum_{i,j=0}^{n-1} C_{n-1}^i C_n^i C_{n-1}^j C_n^j i! j! \left( \frac{\xi}{2} \right)^{2n-i-j} 2(-1)^{2n-i-j} \left\{ Y_{2n-i-j}(\xi) - (2n - i - j)Y_{2n-i-j-2}(\xi) \right\} \] \tag{12}

Then we develop the Bessel functions in the series and take into consideration only the terms that give the contributions linear in deformation parameters. So we have
\[ \left\langle \Psi_{ns} \left| \frac{1}{\sqrt{r^2 + b^2}} \right| \Psi_{ns} \right\rangle = \frac{e^2}{a} \frac{(n-1)!}{[n!]^2} \left( \frac{n^2[(n-1)!]^2}{2} \xi^2 \left[ \ln \left( \frac{\xi}{2} \right) + \gamma + \frac{1}{2} \right] + 2 \right) + \right. \\
\sum_{i,j=0}^{n-1} C_{n-1}^i C_n^i C_{n-1}^j C_n^j i! j! (-1)^{2n-i-j} ((2n - i - j)!! + (2n - i - j)(2n - i - j - 3)!! \xi^2) \tag{13} \]

At last one can calculate the contributions into the energy spectrum caused by the terms \( \frac{1}{4} p^2 + \frac{1}{4} p^2 \hat{p}^2 \) and \( \hat{p}^4 \). Since these calculations are very simple we can write corrections to the hydrogen atom spectrum in linear approximation over the deformation parameters.
\[ \Delta E^{(1)}_{ns} = \left\langle \Psi_{ns} \left| V \right| \Psi_{ns} \right\rangle = \frac{e^2 \hbar^2}{a^3 n^3} \left[ \frac{2\beta + \beta'}{l + \frac{1}{2}} - \frac{\beta + \beta'}{n} \right] + \frac{e^2}{n^2 a} - \frac{e^2}{n^3 a^3} (2\beta - \beta') \times \]
\[ \left( \ln \left( \frac{\hbar^2 (2\beta - \beta')}{n^2 a^2} \right) + 2\gamma + 1 \right) - \frac{e^2}{a} \frac{(n-1)!}{[n!]^2} \left( \sum_{i,j=0}^{n-1} C_{n-1}^i C_n^i C_{n-1}^j C_n^j i! j! (-1)^{2n-i-j} \times \right) \tag{14} \]
\[ \left[ (2n - i - j - 1)! + (2n - i - j - 2)! + (2n - i - j)(2n - i - j - 3)! \hbar^2 (2\beta - \beta') \right] \] \]

It is easy to verify that in the special case \( n = 1 \) and \( n = 2 \) one can obtain the same corrections as calculated in our previous work for the 1s and 2s levels, respectively.

## 3 Estimation of minimal length

Finally we can proceed to the estimation of minimal length. In \cite{12, 13} it was supposed that minimal length effects were hidden in the discrepancy between theoretical and experimental values of Lamb shift for the \( s \)-levels of hydrogen atom. But calculations of Lamb shift corrections for hydrogen atom contain some inaccuracies. These inaccuracies in the determination of the Lamb shift for arbitrary \( s \)-levels are caused by the contributions
that take into account proton charge distribution and yet uncalculated state-independent corrections \[9\]. As was noted in \[9, 15\] for the determination of the Lamb for arbitrary s-states it is useful to utilize a special difference

\[
\Delta_n = n^3 \Delta E_L(ns) - \Delta E_L(1s).
\]  

Using such a difference one can avoid the problems noticed above. Having this difference the minimal length can be estimated in the similar way as it was shown in previous investigations \[12, 13\]. So, for the estimation of the minimal length we suppose that a shift of energy levels caused by the deformation of the commutation relations does not exceed the difference between theoretical and experimental values for specially introduced expression \[15\].

As was shown in the recent work \[15\] the calculated value of the difference \[15\] for the 1s and 2s states is in the units of frequency \[\Delta_2^{\text{theor}} = 187225.70 (5) \text{ kHz}\]. The experimentally measured values of the Lamb shift are \[L(1s_{1/2}) = 8172840 (22) \text{ kHz}\] and \[L(2s_{1/2}) = 1045009.4 (65) \text{ kHz}\] for the 1s and 2s levels, respectively \[16\]. So we have \[\Delta_2^{\text{expt}} = 187235.2 \text{ kHz}\]. As one can see the inaccuracy of theoretical prediction is much smaller than the experimental one.

For a more accurate estimation of the minimal length it is necessary to use the experimental data of the Lamb shift with uncertainty of the same order as the theoretical one. As is known to obtain the experimental values of the Lamb shift for the 1s and 2s levels a few different transitions such as \[1s - 2s, 1s - 3s, 2s - 6s/d, 2s - 8s/d\] and \[2s - 12d\] transitions \[16\] were used. The accuracy of the latter transitions with the exception of the \[1s - 2s\] transition \[17\] is not so high as for the theoretical one. So, having more precisely measured frequencies of the above mentioned transitions one can obtain more accurate values for the Lamb shift of 1s and 2s levels in hydrogen atom and, as a consequence, this leads to a more precise estimation of minimal length.

It was already mentioned that the estimation of the minimal length can be received if we make use of the assumption:

\[
\Delta_2^{\text{ml}} \leq \Delta_2^{\text{expt}} - \Delta_2^{\text{theor}}
\]  

where \[\Delta_2^{\text{ml}} = 8\Delta E_{2s}^{(1)} - \Delta E_{1s}^{(1)}\] is a special difference constructed similarly to \[15\] and \[\Delta E_{1s}^{(1)}, \Delta E_{2s}^{(1)}\] are the corrections to the 1s and 2s energy levels caused by deformation of commutation relations.
The difference $\Delta_{2}^{\text{ml}}$ can be represented in the following form:

$$ \Delta_{2}^{\text{ml}} = \frac{e^{2} \hbar^{2}}{a^{2}} \left( \frac{1}{2}(\beta + \beta') - (2\beta - \beta') \left( \frac{3}{2} - \ln(4) \right) \right). \quad (17) $$

Similarly to [12, 13] we introduce two dimensionless parameters $\xi = \frac{\Delta_{x_{\text{min}}}}{a}$ and $\eta = \frac{\beta}{\beta + \beta'}$ instead of $\beta$ and $\beta'$, where the minimal length $\Delta_{x_{\text{min}}} = \hbar \sqrt{\beta + \beta'}$. As was noted in [13] our calculations take place if $2\beta - \beta' \geq 0$ and $\beta$, $\beta'$ are nonnegative constants. So we have the constraints on the domain of variation for the dimensionless parameter $\eta$: $\frac{1}{3} \leq \eta \leq 1$. We rewrite the right hand side of expression (17) using the parameters $\eta$ and $\xi$:

$$ \Delta_{2}^{\text{ml}} = \frac{e^{2}}{a} \xi^{2} \left( \frac{1}{2} - (3\eta - 1) \left( \frac{3}{2} - \ln(4) \right) \right). \quad (18) $$

It is easy to obtain the simple expression for the estimation of the minimal length using the relation (18):

$$ \Delta_{x_{\text{min}}} = \xi a = a \sqrt{\frac{2\Delta_{2}^{\text{ml}}}{e^{2} \left( 1 - (3\eta - 1)(3 - 2\ln(4)) \right)}}. \quad (19) $$

So if we suppose that $\Delta_{2}^{\text{ml}}$ is equal to the difference $\Delta_{2}^{\text{exp}} - \Delta_{2}^{\text{theor}}$ we can numerically calculate the minimal length for an arbitrary parameter $\eta$ on its domain of variation.

We also compare the constraints on the minimal length with the results obtained in [12, 13]. This comparison is represented in Fig.1. As it is easy to see the behavior of the minimal length obtained with using expression (19) is qualitatively different from the preceding estimations. So if we enlarge the parameter $\eta$ the minimal length increases. In the previous investigations the minimal length decreased with increasing parameter $\eta$.

It is necessary to say that for the evaluation of minimal length we introduced a special expression $\Delta_{2} = 8\Delta_{L}(2s) - \Delta_{L}(1s)$ in contrast to the previous works [12, 13] where for the estimation of minimal length the Lamb shift of $1s$-level was used. So, it is not strange that dependence of minimal length on the parameter $\eta$ obtained in this paper is somewhat different from the behavior of minimal length received in [12, 13]. But we want to stress that the behavior of the minimal length as a function of the parameter $\eta$ is not so important as the order of magnitude. We see that estimations of the minimal length obtained with using two different approaches give us the minimal length of the same order as it has to be.
We investigated the hydrogen atom problem with the deformed Heisenberg algebra leading to the existence of minimal length. In our previous work [13] we put forward an effective perturbation theory giving a possibility to calculate corrections for arbitrary levels of hydrogen atom including $s$-levels. We used this perturbation theory and calculated corrections to $ns$-level for arbitrary $n$. It is necessary to note that in [13] only the corrections to the $1s$ and $2s$ levels were calculated.

For the estimation of the minimal length we introduced a special difference $8\Delta_L(2s) - \Delta_L(1s)$. Such a difference gave us the possibility to obtain a simple relation for the estimation of minimal length. We stress that our evaluation of the minimal length does not give stringent result but rather the upper bound for the minimal length. Comparison of the results obtained here with experimental data from the precision hydrogen spectroscopy shows that the upper bound for the minimal length is of the order $10^{-16}$ m. The behavior

![Graph](image.png)

Figure 1: The comparison of the estimations for the minimal length obtained by using different approaches. The solid and the dotted lines represent the estimations obtained in the works [13] and [12] respectively by using the same experimental and theoretical data for the $1s$ Lamb shift. The dashed line shows the constraints on the minimal length obtained in the work [13] by using more recent data [16, 18] for the $1s$ Lamb shift. The dash-dotted line shows the estimation for the minimal length obtained by using the data [16, 18] for the $2s$ Lamb shift. At last the dash-dot-dotted line represents the constraints on the minimal length that were obtained by using the expression (19).
of our estimation of minimal length is somewhat different from the previous one [12, 13]. As it was noted above the behavior of minimal length is not important for our estimation. The results obtained here show that the minimal length has the same order as in the previous works [12, 13] and this is most important.

5 Acknowledgments

The author is grateful to Prof. V. M. Tkachuk for many useful discussions and comments. I would also like to thank Dr. A. A. Rovenchak for a careful reading of the manuscript.

References

[1] D. J. Gross and P. F. Mende, Nucl. Phys. B 303, 407 (1988).
[2] M. Maggiore, Phys. Lett. B 304, 65 (1993).
[3] E. Witten, Phys. Today 49, 24 (1996).
[4] A. Kempf, J. Math. Phys. 35, 4483 (1994).
[5] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52, 1108 (1995).
[6] H. Hinrichsen and A. Kempf, J. Math. Phys. 37, 2121 (1996).
[7] A. Kempf, J. Math. Phys. 38, 1347 (1997).
[8] A. Kempf, J. Phys. A 30, 2093 (1997).
[9] M. I. Eides, H. Grotch and V. A. Shelyuto, Phys. Rep. 342, 63 (2001).
[10] S. G. Karshenboim, Phys. Rep. 422, 1 (2005).
[11] F. Brau, J. Phys. A 32, 7691 (1999).
[12] S. Benczik, L.N. Chang, D. Minic and T. Takeuchi, Phys.Rev. A 72, 012104 (2005).
[13] M. M. Stetsko and V. M. Tkachuk, Phys. Rev. A 74, 012101 (2006).
[14] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (New York, Dover, 1965).
[15] A. Czarnecki, U. D. Jentschura and K. Pachucki, Phys. Rev. Lett. 95, 180404 (2005).
[16] B. de Beauvoir, C. Schwob et al., Eur. Phys. J. D 12, 61 (2000).
[17] M. Niering, R. Holzwarth et al., Phys. Rev. Lett. 84, 5496 (2000).
[18] K. Pachucki and U. D. Jentschura, Phys. Rev. Lett. 91, 113005 (2005).