Renormalisability of the SU(n) Gauge Theory
with Massive Gauge Bosons

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Abstract

The problem of renormalisability of the SU(n) theory with massive gauge bosons is reinvestigated in the present work. We expound that the quantization under the Lorentz condition caused by the mass term of the gauge fields leads to a ghost action which is the same as that of the usual SU(n) Yang–Mills theory in the Landau gauge. Furthermore, we clarify that the mass term of the gauge fields cause no additional complexity to the Slavnov-Taylor identity of the generating functional for the regular vertex functions and does not change the equations satisfied by the divergent part of this generating functional. Finally, we prove that the renormalisability of the theory can be deduced from the renormalisability of the Yang–Mills theory.

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I. Introduction

As is well known, a negative answer to the question of whether a SU(n) theory with massive gauge bosons is renormalisable was commonly accepted even before knowing the Faddeev-Popov-De Witt method [1–3] to quantize the usual SU(n) Yang-Mills gauge theory. However, for various reasons including the lack of experimental evidence for the Higgs bosons of the SU_L(2)×U_Y(1) electroweak theory, this issue was repeatedly studied (see for example Refs. [4–10]) and several approaches have been developed for finding a positive answer. The authors of Refs. [4–6], studied interesting models where some terms in the action were introduced as an assumption. In Ref. [7] the mass term of the gauge fields was modified to be gauge invariant in such a way that it tends to the original mass term under the Landau gauge. It should be point out with emphasis that since the mass term of the gauge fields make the theory obey the Lorentz condition one can regard the theory as a gauge invariant one and correctly quantize it with the help of such a gauge invariant mass term (see also the reasoning in section 2 of the present paper ). As for the renormalisability, no proof was presented in Ref. [7]. At present the subject can be stated as follows: Can one prove the renormalisability under the original expression of the mass term of the gauge fields with a correct quantisation method ? It will be proven in this paper that the renormalisability of the theory can be deduced from the renormalisability of the SU(n) Yang-Mills theory.

We will use two kinds of path integral of the generating functional for the Green functions. One of them consists of the sources associate to all the variables including the Lagrange multipliers \( \lambda_a \). Another one is the generating functional for the Green functions in the so-called \( \xi \) gauge, which does not involve \( \lambda_a \). It will be shown that the mass term of the gauge fields cause no extra complexity to the Slavnov-Taylor identity of the generating functional \( \Gamma \) for the regular vertex functions and does not change the equations satisfied by the divergent part of \( \Gamma \). Consequently, we will be able to determine the general form of the counterterms order by order based on the renormalisability of the Yang–Mills theory and prove that the mass term of the gauge fields is harmless to the renormalisability of the theory. In this way we will also reveal that the renormalisability of the SU(n) theory with massive gauge bosons is ensured by the renormalisability of the Yang–Mills theory. The scattering matrix will be discussed in a separate paper [11].

The method of quantization will be explained in section 2. Section 3 and section 4 are devoted
to prove the renormalisability of the theory. Concluding remarks will be given in the final section.

II. Quantization and BRST Invariance

With \( A_{\mu} \), \( M \) standing for the SU(n) gauge fields and their mass parameter the Lagrangian including the mass term \( L_{AM} \) of the gauge fields has the form

\[
L = L^{(N)} + L_{AM},
\]

where

\[
L_{AM} = \frac{1}{2} M^2 A_{\mu} A^{\mu},
\]

\( L^{(N)} \) is the Lagrangian of a usual SU(n) gauge theory, namely

\[
L^{(N)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + L_{\psi} + L_{\psi A},
\]

\[
F_{\mu\nu} = \partial_\mu A_{\nu} - \partial_\nu A_{\mu} - g f_{abc} A_b \partial_\mu A_c\rho.
\]

Under the infinitesimal gauge transformation, one has

\[
\delta \int d^4x L(x) = \int d^4x M^2 A_{\mu} \delta A^{\mu} = -\frac{1}{g} M^2 \int d^4x A_{\mu} \partial^\mu \delta \theta^a = \frac{1}{g} M^2 \int d^4x \partial^\mu (A_{\mu}) \delta \theta^a,
\]

where \( \delta A^{\mu} \) stands for the infinitesimal gauge transformation of the gauge fields

\[
\delta A^{\mu} = -\frac{1}{g} \partial^\mu \delta \theta^a (x) - f_{abc} \delta \theta^b (x) A_c^{\mu} (x).
\]

Since the classical equations of motion make the action invariant under an arbitrary infinitesimal transformation of the field functions, they certainly make the mass term of the gauge fields invariant under an arbitrary infinitesimal gauge transformation. This means that when \( M \) is not equal to zero, the classical equations of motion leads to the following Lorentz condition

\[
\partial^\mu A_{\mu} = 0.
\]

It should be noticed with emphasis that the Lorentz condition makes the mass term invariant with respect to the infinitesimal gauge trasformation. Consequently, the combination of the action
and the Lorentz condition is invariant with respect to the infinitesimal gauge trasformation that satisfies the following equations

$$\delta (\partial^\mu A_{a\mu}) = 0.$$  \hfill (2.4)

Since such a residual invariance is not broken by the mass term of the gauge fields it is natural to imagine that the ghost action should be the same as that of the SU(n) Yang–Mills theory in the Lorentz gauge (see for example, Ref. [12]). However, this was often disregarded in the literature. For instance, in the discussion in Ref. [13], concerning the massive gauge fields theory without matter fields, the original form of the generating functional for the Green functions was taken to be

$$\int D[A] \exp \{ i [I + J_\alpha^\mu (x) A_{a\mu} (x)] \},$$

where $J_\alpha^\mu (x) A_{a\mu} (x)$ is the source term and $I$ is the action defined by $L(x)$. In this way, the Lorentz condition (2.3) was ignored. The same drawback was included in Ref. [14].

Taking the Lorentz condition into account one should write the path integral of the Green functions involving only the original fields as

$$\frac{1}{N_0} \int D[A, \bar{\psi}, \psi] \Delta[A] \prod_{a', x'} \delta (\partial^\sigma A_{a'\sigma} (x')) A_{a\mu} (x) A_{b\nu} (y) A_{c\rho} (z) \cdots \exp \{ iI \},$$  \hfill (2.5)

where

$$N_0 = \int D[A, \bar{\psi}, \psi] \Delta[A] \prod_{a', x'} \delta (\partial^\lambda A_{a'\lambda} (x')) \exp \{ iI \} .$$

The problem is to determined the weight factor $\Delta[A]$ and can be solved by modifying the mass term $L_{AM}$ according to the method of Ref. [7]. In fact, only the field functions which satisfy the Lorentz condition can play roles in the integral (2.5) and the value of the Lagrangian can be changed for the field functions which do not satisfy this condition. In view of the fact that the Lorentz condition makes the mass term invariant with respect to the infinitesimal gauge trasformation, we now imagine to replace $L_{AM}$ with a gauge invariant mass term $\tilde{L}_{AM}$ which is equal to $L_{AM}$ when the Lorentz condition is satisfied. Thus, analogous to the case in the Fadeev–Popov method [1-3, 12, 13, 15], $\Delta[A]$ should be gauge invariant and make the following equation valid for an arbitrary gauge invariant quantity $O(A, \bar{\psi}, \psi)$

$$\int D[A, \bar{\psi}, \psi] \Delta[A] \prod_{a', x'} \delta (\partial^\lambda A_{a'\lambda} (x')) O(A, \bar{\psi}, \psi) \exp \{ iI \} \propto \int D[A, \bar{\psi}, \psi] O(A, \bar{\psi}, \psi) \exp \{ i\tilde{I} \} ,$$
where $\tilde{I}$ is a gauge invariant action obtained by replacing $L_{AM}$ with $\tilde{L}_{AM}$. This means that $\Delta[A]$ can be determined according to the Fadeev–Popov equation in the usual Yang–Mills theory and is proportional to $\text{det}[\partial \cdot D]$, where $D$ denotes the covariant derivative in the adjoint representation. Therefore, the ghost Lagrangian or action has the same form as that of the SU(n) Yang-Mills theory in the Landau gauge. Namely

\[
L^{(C)}(x) = \left( - \partial_\mu \ov{C}_a(x) \right) D^\mu_{ab} C_b(x), \quad I^{(C)} = \int d^4x L^{(C)}(x),
\]

(2.6)

where $C_a(x)$ and $\ov{C}_a(x)$ are the F–P ghost fields and

\[
D^\mu_{ab}(x) = \delta_{ab} \partial^\mu + gf_{abc} A^\mu_c(x).
\]

(2.7)

As usual one can further generalized the theory by regarding as new variables the Lagrange multipliers $\lambda_a(x)$ associated with the Lorentz condition. Thus the total effective Lagrangian and action are

\[
\mathcal{L}_{\text{eff}}(x) = \mathcal{L} + \mathcal{L}^{(C)}(x) + \lambda_a(x) \partial^\mu A_{a\mu}(x),
\]

(2.8)

\[
I_{\text{eff}} = \int d^4x \mathcal{L}_{\text{eff}}(x).
\]

(2.9)

Correspondingly, the path integral of the generating functional for the Green functions is

\[
Z[\ov{\eta}, \eta, \ov{\chi}, \chi, J, j] = \frac{1}{N_{\lambda}} \int D[A, \ov{\psi}, \psi, \ov{C}, C, \lambda] \exp \left\{ i \left( I_{\text{eff}} + I_s \right) \right\},
\]

(2.10)

where $N_{\lambda}$ is a constant to make $Z[0, 0, 0, 0, 0, 0]$ equal to 1, $I_s$ is the source term. They are defined by

\[
N_{\lambda} = \int D[A, \ov{\psi}, \psi, \ov{C}, C, \lambda] \exp \left\{ i I_{\text{eff}} \right\},
\]

\[
I_s = \int d^4x \left[ J_a^\mu(x) A_{a\mu}(x) + j_a(x) \lambda_a(x) + \ov{\chi}_a(x) C_a(x) + \ov{\eta}_a(x) \psi_a(x) + \ov{\psi}_a(x) \eta_a(x) \right],
\]

where $J_a^\mu(x)$, $j_a(x)$, $\ov{\chi}_a(x)$, $\chi_a(x)$ and $\ov{\eta}_a$, $\eta_a$ are the sources associate to various fields.

We now check the BRST invariance of the effective action $I_{\text{eff}}$ defined by (2.8) and (2.9). With the gauge fields, the matter fields and the ghost fields transforming as usual, one has

\[
\delta_B A^\mu_a = \delta \zeta D^\mu_{ab} C_b(x),
\]

\[
\delta_B \ov{C}_a(x) = - \delta \zeta \chi_a(x),
\]

\[
\delta_B C_a(x) = \delta \zeta \lambda_a(x),
\]
\[
\delta_B C_a(x) = \delta \zeta \frac{g}{2} f_{abc} C_b(x) C_c(x),
\]
\[
\delta_B I_{eff} = \int d^4x \left\{ (\delta_B \lambda_a(x) - \delta \zeta M^2 C_a(x)) \partial^\mu A_{a\mu} \right\},
\]
where \( \delta \zeta \) is an infinitesimal fermionic parameter independent of \( x \). Obviously, the effective action is invariant when the transformation of \( \lambda_a(x) \) are defined as

\[
\delta_B \lambda_a(x) = \delta \zeta M^2 C_a(x). 
\]

It is also clear that the transformation is no longer nilpotent.

We are also interested in the \( \xi \) gauge Green functions that are defined by replacing the \( \delta \)–functions in the numerator and denominator of (2.5) with the gauge-fixing term

\[
-\frac{1}{2\xi} (\partial^\mu A_{a\mu})^2,
\]

where \( \xi \) is a parameter. The total effective Lagrangian and action including the gauge-fixing term and the ghost term become (in the same notations as used above)

\[
\mathcal{L}_{eff}(x) = \mathcal{L} + \mathcal{L}^{(C)}(x) - \frac{1}{2\xi} (\partial^\mu A_{a\mu})^2, 
\]

\[
I_{eff} = \int d^4x \mathcal{L}_{eff}(x). 
\]

Therefore the generating functional for such Green functions is

\[
Z[\eta, \eta, \chi, \chi, J] = 1 \frac{1}{N_\xi} \int D[A, \bar{\psi}, \psi, \bar{C}, C] A_{a\mu}(x) A_{b\nu}(y) A_{b\rho}(z) \cdots \exp \left\{ iI_{eff} \right\}, 
\]

where \( N_\xi \) is a constant to make \( Z[0, 0, 0, 0, 0] \) equal to 1, \( I_s \) is the source term

\[
I_s = \int d^4x \left[ J^\mu_a(x) A_{a\mu}(x) + \bar{\chi}_a(x) C_a(x) + \bar{C}_a(x) \chi_a(x) + \bar{\eta}_a(x) \psi_a(x) + \bar{\psi}_a(x) \eta_a(x) \right].
\]

It should be noticed that the Lorentz condition will take no effect in the generating functional for the \( \xi \) gauge Green functions unless \( \xi \) tends to zero.

III. Renormalisability

Based on the quantization method explained in last section we will prove that the renormalisability of the SU(n) theory with massive gauge bosons can be deduced from the renormalisability of the Yang–Mills theory. In this section we will start with the the Green function generating
functional which also includes sources associated with $\lambda_a$. The method of reasoning for the theory in the $\xi$ gauge is similar and will be briefly described in section 4.

Assume that $A_{a\mu}(x)$, $C_a(x)$, $\overline{C}_a(x)$ and $\lambda_a(x)$ stand for the renormalized field functions, $g$, $M$ are renormalized parameters. The matter fields $\psi$, $\overline{\psi}$ do not affect the discussion in this section and will be omitted. As usual we define the composite field functions $\Delta A_\mu^a(x)$ and $\Delta C_a(x)$ by

$$
\delta_B A_\mu^a(x) = \delta_{\xi} \Delta A_\mu^a(x), \quad \delta_B C_a(x) = \delta_{\xi} \Delta C_a(x),
$$

where $\Delta A_\mu^a(x)$ is just $D_\mu^a C_b(x)$ and $\Delta C_a(x)$ is $\frac{1}{2} g f_{abc} C_b(x)C_c(x)$. Introducing new sources $K_\mu^a(x)$ and $L_a(x)$ and adding a source term of these composite fields into the effective Lagrangian without counterterm, one gets

$$
\mathcal{L}_{\text{eff}}^{[0]}(x) = -\frac{1}{4} F_{\mu\nu}^a(x) F_\mu^a(x) + \frac{1}{2} M^2 A_{a\mu}(x) A_\mu^a(x) + \lambda_a(x) \partial^\mu A_\mu^a(x)
$$

$+$

$$
- \partial_\mu C_a(x) D_{ab} C_b(x)
$$

$+$

$$
K_\mu^a(x) \Delta A_\mu^a(x) + L_a(x) \Delta C_a(x).
$$

The complete effective Lagrangian is the sum of $\mathcal{L}_{\text{eff}}^{[0]}$ and the counterterm $\mathcal{L}_{\text{count}}$:

$$
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{[0]} + \mathcal{L}_{\text{count}}.
$$

In terms of the action $I_{\text{eff}}^{[0]}$ formed by the effective Lagrangian $\mathcal{L}_{\text{eff}}^{[0]}$, we define the generating functional for Green functions

$$
\mathcal{Z}^{[0]}[J, j, \overline{\chi}, \chi, K, L] = \frac{1}{N} \int \mathcal{D}[A, C, \overline{C}, \lambda, \chi] \exp \left\{ i \{ I_{\text{eff}}^{[0]} + I_s \} \right\},
$$

where $N$ is a constant to make $\mathcal{Z}^{[0]}[0, 0, 0, 0, 0, 0]$ equal to 1, the source term $I_s$ is given by

$$
I_s = \int d^4 x [ J^\mu_a(x) A_{a\mu}(x) + j_a(x) \lambda_a(x) + \overline{\chi}_a(x) C_a(x) + \overline{C}_a(x) \chi_a(x) ].
$$

Correspondingly, the generating functionals $\mathcal{W}^{[0]}$, $\Gamma^{[0]}$ for connected Green functions and regular vertex functions are

$$
\mathcal{Z}^{[0]}[J, j, \overline{\chi}, \chi, K, L] = \exp \left\{ i \mathcal{W}^{[0]}[J, j, \overline{\chi}, \chi, K, L] \right\},
$$

$$
\Gamma^{[0]}[\mathcal{A}, \overline{\mathcal{C}}, \mathcal{C}, \overline{\chi}, \chi, K, L] = \mathcal{W}^{[0]}[J, j, \overline{\chi}, \chi, K, L]
$$

$-$

$$
- \int d^4 x \left[ J^\mu_a(x) \mathcal{A}_{a\mu}(x) + j_a(x) \mathcal{\lambda}_a(x) + \mathcal{\overline{\chi}}_a(x) \mathcal{C}_a(x) + \mathcal{\overline{C}}_a(x) \mathcal{\chi}_a(x) \right].
$$
where $\tilde{A}_{a\mu}(x)$, $\tilde{C}_a(x)$, $\overline{C}_a(x)$, and $\tilde{\lambda}_a(x)$ are the so-called classical fields defined by

$$
\tilde{A}_{a\mu}(x) = \frac{\delta W[0]}{\delta J_a^\mu(x)}, \quad \tilde{\lambda}_a(x) = \frac{\delta W[0]}{\delta J_a(x)}, \quad \tilde{C}_a(x) = \frac{\delta W[0]}{\delta \bar{\chi}_a(x)}, \quad \overline{C}_a(x) = -\frac{\delta W[0]}{\delta \chi_a(x)}.
$$

(3.6)

Therefore one has

$$
J_a^\mu(x) = -\frac{\delta \Gamma[0]}{\delta A_{a\mu}(x)}, \quad J_a(x) = -\frac{\delta \Gamma[0]}{\delta \lambda_a(x)}, \quad \overline{C}_a(x) = \frac{\delta \Gamma[0]}{\delta C_a(x)}, \quad \chi_a(x) = -\frac{\delta \Gamma[0]}{\delta \bar{\chi}_a(x)}.
$$

(3.7)

and

$$
\frac{\delta W[0]}{\delta K^a_\mu(x)} = \frac{\delta \Gamma[0]}{\delta K^a_\mu(x)}, \quad \frac{\delta W[0]}{\delta L_a(x)} = \frac{\delta \Gamma[0]}{\delta L_a(x)}.
$$

(3.8)

In order to find the Slavnov–Taylor identity satisfied by the generating functional for the regular vertex functions, we change the variables in the path integral of $Z[0]$ as follows

$$
A^\mu_a(x) \to A^\mu_a(x) + \delta \zeta \Delta A^\mu_a(x),
$$

$$
C_a(x) \to C_a(x) + \delta \zeta \Delta C_a(x),
$$

$$
\overline{C}_a(x) \to \overline{C}_a(x) - \delta \zeta \lambda_a(x),
$$

$$
\lambda_a(x) \to \lambda_a(x).
$$

The volume element of the path integral does not change and the changes of the source term and the mass term of the gauge fields lead to

$$
\int d^4x \left\{ \frac{\delta \Gamma[0]}{\delta K^a_\mu(x)} \frac{\delta \Gamma[0]}{\delta A^\mu_a(x)} + \frac{\delta \Gamma[0]}{\delta L_a(x)} \frac{\delta \Gamma[0]}{\delta C_a(x)} - \tilde{\lambda}_a(x) \frac{\delta \Gamma[0]}{\delta \overline{C}_a(x)} - M^2 \tilde{A}_{a\mu}(x) \frac{\delta \Gamma[0]}{\delta K^a_\mu(x)} \right\} = 0.
$$

(3.9)

Next, by using the invariance of the path integral of $Z[0]$ with respect to the translation of the integration variables $\overline{C}_a(x)$ and $\lambda_a(x)$, one can get a set of auxiliary identities

$$
\partial_\mu \frac{\delta \Gamma[0]}{\delta K^a_\mu(x)} - \frac{\delta \Gamma[0]}{\delta \overline{C}_a(x)} = 0,
$$

(3.10)

$$
\frac{\delta \Gamma[0]}{\delta \lambda_a(x)} - \partial^\mu \tilde{A}_{a\mu}(x) = 0.
$$

(3.11)

In the following we will denote by $\Gamma[0][A, \overline{C}, C, \lambda, K, L]$ the functional that is obtained from $\Gamma[0][\tilde{A}, \tilde{C}, \tilde{C}, \tilde{\lambda}, K, L]$ by replacing the classical field functions with the usual field functions. Defined $\Gamma[0]$ as

$$
\Gamma[0] = \Gamma[0] - \int d^4x \left\{ \lambda_a(x) \partial^\mu A_{a\mu}(x) \right\} - \int d^4x \left\{ \frac{1}{2} M^2 A^\mu_a(x) A_{a\mu}(x) \right\}.
$$

(3.12)
Thus (3.9)–(3.11) lead to

\[
\int d^4x \left\{ \frac{\delta \Gamma^{[0]}_1}{\delta K_{\mu}^a(x)} \frac{\delta A_{\mu}^a(x)}{\delta A_a^\mu(x)} + \frac{\delta \Gamma^{[0]}_0}{\delta L_a(x)} \frac{\delta \Gamma^{[0]}_0}{\delta C_a(x)} \right\} = 0, \tag{3.13}
\]

\[
\frac{\partial}{\partial \delta \Gamma^{[0]}_0} - \frac{\delta \Gamma^{[0]}_0}{\delta C_a(x)} = 0, \tag{3.14}
\]

\[
\frac{\delta \Gamma^{[0]}_0}{\delta \lambda_a(x)} = 0. \tag{3.15}
\]

Assume that the dimensional regularization method is used and the relations (3.13)–(3.15) are guaranteed. Denote the tree part and one loop part of \( \Gamma^{[0]} \) by \( \Gamma^{[0]}_0 \) and \( \Gamma^{[0]}_1 \) respectively, \( \Gamma^{[0]}_0 \) is thus the modified action \( I^{[0]}_{\text{eff}} \) without the \( \lambda \) term and the mass term of the gauge fields. From (3.13)–(3.15) one has

\[
\Gamma^{[0]}_0 \ast \Gamma^{[0]}_0 = 0, \tag{3.16}
\]

\[
\partial_{\mu} \frac{\delta \Gamma^{[0]}_0}{\delta K_{\mu}^a(x)} - \frac{\delta \Gamma^{[0]}_0}{\delta C_a(x)} = 0, \tag{3.17}
\]

and

\[
\Gamma^{[0]}_0 \ast \Gamma^{[1]}_0 + \Gamma^{[1]}_1 \ast \Gamma^{[0]}_0 = \Lambda_{\text{op}} \Gamma^{[0]}_1 = 0, \tag{3.18}
\]

\[
\frac{\partial}{\partial \delta \Gamma^{[1]}_0} - \frac{\delta \Gamma^{[1]}_0}{\delta C_a(x)} = 0, \tag{3.19}
\]

\[
\frac{\delta \Gamma^{[1]}_0}{\delta \lambda_a(x)} = 0. \tag{3.20}
\]

The notations \( A \ast B, \Lambda_{\text{op}} \) are defined in the usual way, namely

\[
A \ast B = \int d^4x \left\{ \frac{\delta A}{\delta K_{\mu}^a(x)} \frac{\delta B}{\delta A_a^\mu(x)} + \frac{\delta A}{\delta L_a(x)} \frac{\delta B}{\delta C_a(x)} \right\}, \tag{3.21}
\]

\[
\Lambda_{\text{op}} = \int d^4x \left\{ \frac{\delta \Gamma^{[0]}_0}{\delta K_{\mu}^a(x)} \frac{\delta}{\delta A_a^\mu(x)} + \frac{\delta \Gamma^{[0]}_0}{\delta L_a(x)} \frac{\delta}{\delta C_a(x)} \right\}, \tag{3.22}
\]

The pole part of \( \Gamma^{[0]}_1 \) will be denoted by \( \Gamma^{[0]}_{1, \text{div}} \). Of course it also satisfies (3.18)–(3.20), namely

\[
\Lambda_{\text{op}} \Gamma^{[0]}_{1, \text{div}} = 0, \tag{3.23}
\]

\[
\partial_{\mu} \frac{\delta \Gamma^{[0]}_{1, \text{div}}}{\delta K_{\mu}^a(x)} - \frac{\delta \Gamma^{[0]}_{1, \text{div}}}{\delta C_a(x)} = 0, \tag{3.24}
\]

\[
\frac{\delta \Gamma^{[0]}_{1, \text{div}}}{\delta \lambda_a(x)} = 0. \tag{3.25}
\]

This is the same equations as that appearing in the Yang–Mills theory.
If $M = 0$ then it is known from the renormalisability of the theory that $\Gamma^{[0]}_1, \text{div}$ is a combination of the three terms
\[
g \frac{\partial \Gamma^{[0]}_0}{\partial g}, \quad \int d^4x \left\{ A_{\mu\nu}(x) \frac{\partial \Gamma^{[0]}_0}{\partial A_{\mu\nu}(x)} + L_a(x) \frac{\partial \Gamma^{[0]}_0}{\partial L_a(x)} \right\},
\]
\[
\int d^4x \left\{ C_a(x) \frac{\partial \Gamma^{[0]}_0}{\partial C_a(x)} + \overline{C}_a(x) \frac{\partial \Gamma^{[0]}_0}{\partial \overline{C}_a(x)} + K_\mu^a(x) \frac{\partial \Gamma^{[0]}_0}{\partial K_\mu^a(x)} \right\}.
\]
Since each of these satisfies equations (3.23)–(3.25) a new term appearing when $M \neq 0$, if any, should include $M^2$ as a factor and also satisfy (3.23)–(3.25). Now the equations cannot have such a solution. It follows that
\[
\Gamma^{[0]}_1, \text{div} = \alpha_1 \left( g \frac{\partial \Gamma^{[0]}_0}{\partial g} \right) + \beta_1 \int d^4x \left\{ A_{\mu\nu}(x) \frac{\partial \Gamma^{[0]}_0}{\partial A_{\mu\nu}(x)} + L_a(x) \frac{\partial \Gamma^{[0]}_0}{\partial L_a(x)} \right\},
\]
\[
+ \gamma_1 \int d^4x \left\{ C_a(x) \frac{\partial \Gamma^{[0]}_0}{\partial C_a(x)} + \overline{C}_a(x) \frac{\partial \Gamma^{[0]}_0}{\partial \overline{C}_a(x)} + K_\mu^a(x) \frac{\partial \Gamma^{[0]}_0}{\partial K_\mu^a(x)} \right\},
\]
where $\alpha_1$, $\beta_1$ and $\gamma_1$ are constants of order $(\hbar)^1$ and are divergent when the space-time dimension tends to 4.

In order to cancel the one loop divergence the counterterm of order $(\hbar)^1$ in the action should be chosen as
\[
\delta I_\text{count}^{[1]}[A, C, \overline{C}, K, L, g, M] = -\Gamma^{[0]}_{1, \text{div}}[A, C, \overline{C}, K, L, g, M].
\]
Thus the sum of $\Gamma^{[0]}_0$ and $\delta I_\text{count}^{[1]}$, to order $\hbar^1$, can be written as
\[
\Gamma^{[1]}_{\text{eff}}[A, C, \overline{C}, K, L, g]
\]
\[
= \Gamma^{[0]}_0 \left[ (Z_3^{[1]})^{1/2} A, (\overline{Z}_3^{[1]})^{1/2} C, (Z_3^{[1]})^{1/2} \overline{C}, (\overline{Z}_3^{[1]})^{1/2} K, (Z_3^{[1]})^{1/2} L, Z_g^{[1]} g \right],
\]
where
\[
(Z_3^{[1]})^{1/2} = 1 - \beta_1,
\]
\[
(\overline{Z}_3^{[1]})^{1/2} = 1 - \gamma_1,
\]
\[
Z_g^{[1]} = 1 - \alpha_1.
\]
Next by adding the $\lambda$ term and the mass term of the gauge fields into $\Gamma^{[1]}_{\text{eff}}$ and forming
\[
I^{[1]}_{\text{eff}}[A, C, \overline{C}, \lambda, K, L, g, M] = \Gamma^{[1]}_{\text{eff}}[A, C, \overline{C}, K, L, g]
\]
\[
+ \int d^4x \left\{ \lambda_a(x) \partial^\mu A_{\mu\alpha}(x) \right\}
\]
\[
+ \int d^4x \left\{ \frac{1}{2} M^2 A_{\mu\alpha}(x) A_\alpha^\mu(x) \right\},
\]
\[
(3.29)
\]
one has

\[ I^{[1]}_{\text{eff}}[A, C, \overline{C}, \lambda, K, L, g, M] = I^{[0]}_{\text{eff}}[A^{[0]}, C^{[0]}, \overline{C}^{[0]}, \lambda^{[0]}, K^{[0]}, L^{[0]}, g^{[0]}, M^{[0]}], \]  

where \( A^{[0]}, C^{[0]}, \overline{C}^{[0]}, \cdots \), to order \( h^1 \), stand for the bare quantities and are defined by

\[
\begin{align*}
A^{[0]}_{a\mu} &= (Z_3^{[1]})^{1/2} A_{a\mu}, \quad C^{[0]}_a = (\overline{Z}_3^{[1]})^{1/2} C_a, \quad \overline{C}^{[0]}_a = (Z_3^{[1]})^{1/2} \overline{C}_a, \\
K^{[0]}_\mu &= (Z_3^{[1]})^{1/2} K_\mu, \quad L^{[0]}_a = (Z_3^{[1]})^{1/2} L_a, \\
g^{[0]} &= Z_3^{[1]} g, \quad M^{[0]} = (Z_3^{[1]})^{-1/2} M, \quad \lambda^{[0]}_a = (Z_3^{[1]})^{-1/2} \lambda_a.
\end{align*}
\]

Obviously, if the action \( I^{[1]}_{\text{eff}}[A, C, \overline{C}, \lambda, K, L, g, M] \) is used to replace \( I^{[0]}_{\text{eff}}[A, C, \overline{C}, \lambda, K, L, g, M] \) in (3.2) and define the generating functional \( \Gamma^{[1]} \) as well as

\[
\Gamma^{[1]} = \Gamma^{[1]} - \int d^4x \left\{ \lambda_a(x) \partial^\mu A_{a\mu}(x) \right\} - \int d^4x \left\{ \frac{1}{2} A^{[a]}_a(x) A_{a\mu}(x) \right\},
\]

then one has

\[
\Gamma^{[1]}[A, \overline{C}, C, K, L] = \Gamma^{[0]}[A^{[0]}, \overline{C}^{[0]}, C^{[0]}, K^{[0]}, L^{[0]}].
\]

We then expand the right hand side of this equation into the form

\[
\Gamma_0^{[0]}[A^{[0]}, \overline{C}^{[0]}, C^{[0]}, K^{[0]}, L^{[0]}] + \Gamma_1^{[0]}[A^{[0]}, \overline{C}^{[0]}, C^{[0]}, K^{[0]}, L^{[0]}] + \cdots.
\]

In the first term the divergences of order \( h^1 \) are due to \( \delta \Gamma_{\text{count}}^{[1]} \). In the second term the divergences of this order do not contain the contribution of \( \delta \Gamma_{\text{count}}^{[1]} \) and are therefore due to the action of order \( h^0 \). It follows that, to order \( h^1 \), \( \Gamma^{[1]} \) is finite. Moreover from (3.13)–(3.15) and (3.34) one gets

\[
\int d^4x \left\{ \frac{\delta \Gamma^{[1]}_\mu}{\delta (K^{[0]}_\mu)(x) \delta A_{a\mu}(x)} + \frac{\delta \Gamma^{[1]}_\mu}{\delta L^{[0]}_a(x) \delta C_a^{[0]}(x)} \right\} = 0,
\]

\[
\frac{\partial}{\partial \lambda_a^{[0]}(x)} \frac{\delta \Gamma^{[1]}_\mu}{\delta K^{[0]}_\mu(x) \delta A_{a\mu}(x)} = 0,
\]

\[
\frac{\delta \Gamma^{[1]}_\mu}{\delta \lambda_a^{[0]}(x)} = 0.
\]

With the help of (3.31)–(3.33), these equations can be written as

\[
\int d^4x \left\{ \frac{\delta \Gamma^{[1]}_\mu}{\delta K^{[0]}_\mu(x) \delta A_{a\mu}(x)} + \frac{\delta \Gamma^{[1]}_\mu}{\delta L^{[0]}_a(x) \delta C_a^{[0]}(x)} \right\} = 0,
\]

\[
\frac{\partial}{\partial \lambda_a^{[0]}(x)} \frac{\delta \Gamma^{[1]}_\mu}{\delta K^{[0]}_\mu(x) \delta A_{a\mu}(x)} = 0,
\]

\[
\frac{\delta \Gamma^{[1]}_\mu}{\delta \lambda_a^{[0]}(x)} = 0.
\]
We now know well how to prove the renormalisability of the theory by using the Slavnov–Taylor identities and the inductive method. Let us assume that up to \( n \) loop the theory has been proved to be renormalisable by introducing the counterterm

\[
I_{\text{count}}^{[n]} = \sum_{l=1}^{n} \delta I_{\text{count}}^{[l]},
\]

where \( \delta I_{\text{count}}^{[l]} \) is the counterterm of order \( \hbar^l \) and has the form of (3.26),(3.27). This also means that \( \Gamma^{[n]} \) determined by the action

\[
I_{\text{eff}}^{[n]} = I_{\text{eff}}^{[0]} + I_{\text{count}}^{[n]}
\]

satisfies the Slavnov–Taylor identities and is finite to order \( \hbar^n \). We have to proved that by using a counterterm of order \( \hbar^{n+1} \) which also has the form of (3.26),(3.27), \( \Gamma^{[n+1]} \) determined by the action

\[
I_{\text{eff}}^{[n+1]} = I_{\text{eff}}^{[n]} + \delta I_{\text{count}}^{[n+1]}
\]

can be make satisfy the Slavnov–Taylor identities and finite to order \( \hbar^{n+1} \).

Denote by \( \Gamma_k^{[n]} \) the part of order \( \hbar^k \) in \( \Gamma^{[n]} \). For \( k \leq n \), \( \Gamma_k^{[n]} \) is equal to \( \Gamma_k^{[k]} \), because it can not contain the contribution of a counterterm of order \( \hbar^{k+1} \) or higher. Thus on expanding \( \Gamma^{[n]} \) to order \( \hbar^{n+1} \) one has

\[
\Gamma^{[n]} = \sum_{k=0}^{n} \Gamma_k^{[k]} + \Gamma_{n+1}^{[n]} + \cdots.
\]

Using this and extracting the terms of order \( \hbar^{(n+1)} \) in the Slavnov–Taylor identities of \( \Gamma^{[n]} \), we find

\[
\Gamma_0^{[0]} * \Gamma_{n+1}^{[n]} + \Gamma_{n+1}^{[n]} * \Gamma_0^{[0]} = 0,
\]

\[
\frac{\partial}{\partial \delta K_\mu^a(x)} \frac{\delta \Gamma_{n+1}^{[n]}}{\delta \lambda_a(x)} = 0,
\]

\[
\frac{\delta \Gamma_{n+1}^{[n]}}{\delta \lambda_a(x)} = 0.
\]

Let \( \Gamma_{n+1,\text{div}}^{[n]} \) stand for the pole part of \( \Gamma_{n+1}^{[n]} \). By repeating the steps going from (3.23) to (3.26), one can arrive at

\[
\Gamma_{n+1,\text{div}}^{[n]} = \alpha_{n+1} \left( g \frac{\partial \Gamma_0^{[0]}}{\partial g} + \beta_{n+1} \int d^4 x \left\{ A_{a\nu}(x) \frac{\delta \Gamma_0^{[0]}}{\delta A_{a\nu}(x)} + L_a(x) \frac{\delta \Gamma_0^{[0]}}{\delta L_a(x)} \right\} \right) + \gamma_{n+1} \left( C_a(x) \frac{\delta \Gamma_0^{[0]}}{\delta C_a(x)} + \Gamma_a^{[0]}(x) \frac{\delta \Gamma_0^{[0]}}{\delta L_a(x)} \right) + \delta \Gamma_0^{[0]} \frac{\delta \Gamma_0^{[0]}}{\delta \lambda_a(x)}.
\]
where \(\alpha_{n+1}, \beta_{n+1}\) and \(\gamma_{n+1}\) are constants of order \((\hbar)^{n+1}\). Therefore, in order to cancel the \(n+1\) loop divergence the counterterm of order \(\hbar^{n+1}\) should be chosen as

\[
\delta I^{[n+1]}_{\text{count}}[A, C, \overline{C}, K, L, g, M] = -\Gamma^{[n]}_{n+1, \text{div}}[A, C, \overline{C}, K, L, g, M].
\]

(3.45)

After including this counterterm and the gauge fixing term as well as the mass term of the gauge fields \(I^{[n+1]}_{\text{eff}}\), to order \(\hbar^{n+1}\), can be expressed as

\[
I^{[n+1]}_{\text{eff}}[A, C, \overline{C}, K, L, g, M] = I^{[0]}_{\text{eff}}[A^{[0]}, C^{[0]}, \overline{C}^{[0]}, \chi^{[0]}, K^{[0]}, L^{[0]}, g^{[0]}, M^{[0]}],
\]

(3.46)

where \(A^{[0]}, C^{[0]}, \overline{C}^{[0]}, \ldots\), to order \(\hbar^{n+1}\), stand for the bare quantities

\[
A^{[0]}_{a\mu} = (Z_3^{[n+1]})^{1/2} A_{a\mu}, \quad C^{[0]}_a = (\tilde{Z}_3^{[n+1]})^{1/2} C_a, \quad \overline{C}^{[0]}_a = (\overline{Z}_3^{[n+1]})^{1/2} \overline{C}_a,
\]

(3.47)

\[
K^{[0]}_{\mu a} = (\tilde{Z}_3^{[n+1]})^{1/2} K_{\mu a}, \quad L^{[0]}_a = (Z_3^{[n+1]})^{1/2} L_a,
\]

(3.48)

\[
g^{[0]} = Z_g^{[n+1]} g, \quad M^{[0]} = (Z_3^{[n+1]})^{-1/2} M, \quad \chi^{[0]}_a = (Z_3^{[n+1]})^{-1/2} \chi_a,
\]

(3.49)

with

\[
(Z_3^{[n+1]})^{1/2} = (Z_3^{[n]})^{1/2} - \beta_{n+1},
\]

\[
(\tilde{Z}_3^{[n+1]})^{1/2} = (\tilde{Z}_3^{[n]})^{1/2} - \gamma_{n+1},
\]

\[
Z_g^{[n+1]} = Z_g^{[n]} - \alpha_{n+1}.
\]

Therefore, the generating functional \(\Gamma^{[n+1]}\) for proper functions determined by the action \(I^{[n+1]}_{\text{eff}}\) can be found from \(\Gamma^{[0]}\). Namely

\[
\Gamma^{[n+1]}[A, C, \overline{C}, K, L] = \Gamma^{[0]}[A^{[0]}, \overline{C}^{[0]}, C^{[0]}, K^{[0]}, L^{[0]}].
\]

(3.50)

With this, one can verify that \(\Gamma^{[n+1]}\) satisfies (3.38)–(3.40) and is finite to order \(\hbar^{n+1}\). Since the theory can be renormalized to one loop the renormalisability has been proved by the inductive method.

**IV. Renormalisability of the theory in the \(\xi\) gauge**

Similar to section 3, let \(A_{a\mu}(x), C_a(x)\) and \(\overline{C}_a(x)\) stand for the renormalized field functions, \(g, M\) be renormalized parameters, and \(\xi\) is an auxiliary parameter. The matter fields are also
omitted. Now the effective Lagrangian without counterterm is

\[ L_{\text{eff}}^0(x) = -\frac{1}{4} F_{a\mu\nu}(x) F^{a\mu\nu}(x) + \frac{1}{2} M^2 A_{a\mu}(x) A_{\mu a}^a(x) - \frac{1}{2 \xi} \left( \partial^{\nu} A_{\mu a}^a(x) \right)^2 
+ \left( - \partial_\mu \bar{C}_\alpha(x) \right) D_{a\mu}^\alpha C_b(x) 
+ K_{a\mu}^\alpha(x) \Delta A_{\mu a}^a(x) + L_a(x) \Delta C_a(x). \] (4.1)

The generating functional for Green functions is

\[ Z[0][J, \chi, \bar{\chi}, K, L] = \frac{1}{N} \int D[A, C, \bar{C}] \exp \left\{ i \left( L_{\text{eff}}^0 + I_s \right) \right\}, \] (4.2)

where \( N \) is a constant to make \( Z[0][0, 0, 0, 0, 0] \) equal to 1, the source term \( I_s \) is given by

\[ I_s = \int d^4x \left[ J_{a\mu}^\alpha(x) A_{\mu a}^a(x) + \bar{\chi}_a(x) C_a(x) + \bar{C}_a(x) \chi_a(x) \right]. \]

Correspondingly, the generating functionals \( W[0], \Gamma[0] \) for connected Green functions and regular vertex functions are

\[ Z[0][J, \chi, \bar{\chi}, K, L] = \exp \left\{ i W[0][J, \chi, \bar{\chi}, K, L] \right\}, \] (4.3)

\[ \Gamma[0][\bar{A}, \bar{C}, \bar{\bar{C}}, K, L] = W[0][J, \chi, \bar{\chi}, K, L] 
- \int d^4x \left[ J_{a\mu}^\alpha(x) \bar{A}_{\mu a}^a(x) + \bar{\chi}_a(x) \bar{C}_a(x) + \bar{\bar{C}}_a(x) \chi_a(x) \right], \] (4.4)

where the classical fields are defined by

\[ \bar{A}_{\mu a}^a(x) = \frac{\delta W[0]}{\delta J_{a\mu}^\alpha(x)}, \quad \bar{C}_a(x) = \frac{\delta W[0]}{\delta \bar{\chi}_a(x)}, \quad \bar{\bar{C}}_a(x) = -\frac{\delta W[0]}{\delta \chi_a(x)}. \] (4.5)

One therefore has

\[ J_{a\mu}^\alpha(x) = -\frac{\delta \Gamma[0]}{\delta A_{\mu a}^a(x)}, \quad \bar{\chi}_a(x) = \frac{\delta \Gamma[0]}{\delta \bar{C}_a(x)}, \quad \chi_a(x) = -\frac{\delta \Gamma[0]}{\delta \bar{C}_a(x)}. \] (4.6)

and

\[ \frac{\delta W[0]}{\delta K_{a\mu}^\alpha(x)} = -\frac{\delta \Gamma[0]}{\delta K_{a\mu}^\alpha(x)}, \quad \frac{\delta W[0]}{\delta L_a(x)} = \frac{\delta \Gamma[0]}{\delta L_a(x)}. \] (4.7)

In order to find the Slavnov–Taylor identity satisfied by the generating functional for the regular vertex functions, we change the variables in the path integral of \( Z[0] \) as follows

\[ A_{\mu a}^a(x) \rightarrow A_{\mu a}^a(x) + \delta \zeta \Delta A_{\mu a}^a(x), \]

\[ C_a(x) \rightarrow C_a(x) + \delta \zeta \Delta C_a(x), \]

\[ \bar{C}_a(x) \rightarrow \bar{C}_a(x) + \delta \zeta \frac{1}{\xi} \partial_\mu A_{\mu a}^a(x). \]
The volume element of the path integral does not change and the changes of the source term and the mass term of the gauge fields lead to
\[
\int d^4x \left\{ \frac{\delta \Gamma^0[0]}{\delta K^a_\mu(x)} \frac{\delta \Gamma^0[0]}{\delta A^a_\mu(x)} + \frac{\delta \Gamma^0[0]}{\delta L_a(x)} \frac{\delta \Gamma^0[0]}{\delta C_a(x)} \right. \\
\left. + \frac{1}{\xi} \left( \partial_\mu \tilde{A}^a_\mu(x) \right) \frac{\delta \Gamma^0[0]}{\delta C_a(x)} - M^2 \tilde{A}^a_\mu(x) \frac{\delta \Gamma^0[0]}{\delta K^a_\mu(x)} \right\} = 0. \quad (4.8)
\]

Next, by using the invariance of the path integral of \( Z^0 \) with respect to the translation of the integration variables \( \tilde{C}_a(x) \), one can get a set of auxiliary identities
\[
\partial_\mu \frac{\delta \Gamma^0[0]}{\delta K^a_\mu(x)} - \frac{\delta \Gamma^0[0]}{\delta C_a(x)} = 0. \quad (4.9)
\]

Let \( \Gamma^0[A, \tilde{C}, C, K, L] \) be the functional that is obtained from \( \Gamma^0[\tilde{A}, \tilde{C}, \tilde{C}, K, L] \) by replacing the classical field functions with the usual field functions. Defined \( \tilde{\Gamma}^0 \) as
\[
\tilde{\Gamma}^0 = \Gamma^0[0] + \int d^4x \left\{ \frac{1}{2\xi} \left( \partial^\nu A_{a\nu}(x) \right)^2 \right\} - \int d^4x \left\{ \frac{1}{2} M^2 A_{a\mu}(x) A^a_\mu(x) \right\}. \quad (4.10)
\]

Thus from (4.8) and (4.9) one has
\[
\int d^4x \left\{ \frac{\delta \Gamma^0[0]}{\delta K^a_\mu(x)} \frac{\delta \tilde{\Gamma}^0[0]}{\delta A^a_\mu(x)} + \frac{\delta \Gamma^0[0]}{\delta L_a(x)} \frac{\delta \tilde{\Gamma}^0[0]}{\delta C_a(x)} \right\} = 0, \quad (4.11)
\]
\[
\partial_\mu \frac{\delta \Gamma^0[0]}{\delta K^a_\mu(x)} - \frac{\delta \Gamma^0[0]}{\delta C_a(x)} = 0. \quad (4.12)
\]

It is now obvious that the method used in last section can be followed to prove the renormalisability of the theory in the \( \xi \) gauge.

V. Concluding Remarks

We have expounded that the quantization under the Lorentz condition caused by the mass term of the gauge fields leads to a ghost action which is the same as that of the usual SU(n) Yang–Mills theory in the Landau gauge. Furthermore, we have clarified that the mass term of the gauge fields cause no extra complexity to the Slavnov-Taylor identity of the generating functional for the regular vertex functions. In particular, the equations satisfied by the divergent part of this generating functional are independent of \( M \). Consequently, we have been able to determine the general form of the counterterms order by order based on the renormalisability of the Yang–Mills theory and prove that the mass term of the gauge fields is harmless to the renormalisability of the
theory. In this way we have also revealed that the renormalisability of the SU(n) theory with the mass term of the gauge fields is ensured by that of the Yang–Mills theory theory.

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