Position-Momentum Duality and Fractional Quantum Hall Effect in Chern Insulators

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We develop a first quantization description of fractional Chern insulators that is the dual of the conventional fractional quantum Hall (FQH) problem, with the roles of position and momentum interchanged. In this picture, FQH states are described by anisotropic FQH liquids forming in momentum-space Landau levels in a fluctuating magnetic field. The fundamental quantum geometry of the problem emerges from the interplay of single-body and interaction metrics, both of which act as momentum-space duals of the geometrical picture of the anisotropic FQH effect. We then present a novel broad class of ideal Chern insulator lattice models that act as duals of the isotropic FQH effect. The interacting problem is well-captured by Haldane pseudopotentials and affords a detailed microscopic understanding of the interplay of interactions and non-trivial quantum geometry.

PACS numbers: 73.43.-f, 71.10.Fd, 03.65.Vf

The effects of topology and quantum geometry in condensed matter physics have recently garnered immense attention. With topological insulators and the quantum anomalous Hall effect constituting the well-understood case of non-interacting or weakly-correlated electron systems [1–3], the recent theoretical discovery of the fractional quantum Hall (FQH) effect in nearly-flat bands with non-trivial topology [4–8] poses deep questions regarding the confluence of strong interactions and non-trivial quantum geometry. On one hand, an experimental realization of such a fractional Chern insulator (FCI) could conceivably push relevant energy scales by an order of magnitude, paving the way to robust FQH signatures [9–14]. On the other hand, the disparity of conventional Landau levels and flat bands with non-zero Chern number $C$ suggests a rich playground to realize novel states with topological order that cannot be attained in a conventional electron gas in a magnetic field [15, 16], while simultaneously presenting a profound challenge to understand the underlying microscopics of strong interactions.

Most of our current understanding regarding FCIs stems from exact diagonalization (ED) of small clusters for $C = 1$ [17–30] and $C > 1$ [31–35], mutatis-mutandis mappings of the Hilbert space of flat Chern bands to the lowest Landau level (LLL) [15, 36–40] or vice versa [16], and approximate long-wavelength projected density algebra [41–48]. The latter approaches however treat exclusively the universal long-wavelength continuum limit of the FQH problem, whereas the presence and relevance of the lattice manifests itself in the short-wavelength physics. This conundrum is highlighted by the zoo of FCI lattice models established so far, which display strongly varying proclivities to host stable FQH phases that do not correlate well with simple measures such as ”flatness” of band dispersion and Berry curvature.

At its heart, the theoretical challenge stems from the fact that the immense success in describing the microscopics of the conventional FQH effect resists a simple description in second quantization [49] that is essential to describe interacting electrons on the lattice [50–55]. A resolution is crucial to provide a foundation for studies of non-Abelian phases [56–58] and to provide microscopic insight that can ultimately drive experimental discovery.

In this work, we develop a first quantization description of FQH states in FCIs that leads to an effective Hamiltonian that is the dual of the usual FQH problem, with the roles of position and momentum interchanged. In this picture, the FCI analogues of FQH states are described by anisotropic FQH liquids forming in momentum-space Landau levels in a fluctuating magnetic field. In analogy to Haldane’s geometrical description of the anisotropic FQHE [59–61], the effective interacting system is governed by the interplay of a single-body and interaction metric that determines the fundamental guiding center geometry. Guided by these insights, we then present and provide examples of a broad class of ideal FCI host lattice models that constitute the FCI analog of the isotropic FQHE. These models afford a particularly simple description of the interacting low-energy dynamics, acting as FQH parent Hamiltonians with emergent guiding center and SU(C) symmetry. The effects of quantum geometry and Berry curvature fluctuations are analyzed in terms of Haldane pseudopotentials.

Consider a 2D band insulator hosting an isolated fractionally-filled flat band with non-zero $C$, generically described by an $N$-orbital Bloch Hamiltonian $\hat{h}_k$. In band basis, the flat band of interest is spanned by Bloch states $|u_k\rangle$ with dispersion $\hat{h}_k|u_k\rangle = \epsilon_k|u_k\rangle$. If the energy required for interband excitations is larger than intra-band Coulomb interactions, then the kinetic energy within the band is effectively quenched while momentum-dependent orbital mixing of Bloch states gives rise to a non-trivial quantum geometry, expressed by a gauge field and a Riemann metric on $\mathbb{CP}^{N-1}$ for the Bloch band, the Berry curvature $\Omega(k)$ and Fubini-Study metric $g_{\mu\nu}(k)$:

$$\Omega(k) = e^{\mu\nu} \partial_k A_\nu(k) \quad A_\nu(k) = -i \langle u_k | \partial_k u_k \rangle$$  \hspace{1cm} (1)

$$g_{\mu\nu}(k) = \frac{1}{2} \langle \partial_\mu u_k | [1 - |u_k\rangle \langle u_k|] \partial_\nu u_k \rangle + (\mu \leftrightarrow \nu)$$  \hspace{1cm} (2)
Here, $A_μ(\mathbf{k})$ is the Berry connection, and $C = \frac{e}{\pi \hbar} \int_{BZ} d^2 k \ \Omega(\mathbf{k})$. We use lattice constants $a_0 = \hbar = 1$.

The first task at hand is to describe the guiding-center basis. In the case of an isotropic free electron gas in a magnetic field, Laughlin constructed a series of incompressible FQH trial wave functions [62] for odd-fraction filling factors, from a single-body basis of radially-localized symmetric-gauge LLL wave functions $\langle \mathbf{r} | m \rangle \sim z^m e^{-r/\lambda}$, which are uniquely determined by demanding that they are eigenstates of the angular momentum operator. While angular momentum does not readily translate to the lattice, a key observation is that such states are simultaneous eigenstates of a parabolic confinement potential $\hat{V}(\mathbf{r}) = \frac{1}{2} \mathbf{r}^2$ projected to the LLL: $[\sum_m | m \rangle \langle m' | \hat{V}(\mathbf{r}) | m \rangle = \lambda (m+1) | m \rangle$.

A natural way to adapt this construction to an FCI is to consider anisotropic confinement on a lattice, with $\hat{V}(\mathbf{r}) = \frac{1}{2} \lambda x_μ \eta^{μν} x_ν$ (3)

where $\eta^{μν}$ is a unimodular Galilean metric which a priori serves as a variational degree of freedom with the constraint that $\hat{V}(\mathbf{r})$ retains the discrete rotational symmetries of the host lattice. Here, $\mathbf{r} = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2, m_{1,2} \in \mathbb{Z}$ indexes the unit cell, where $\mathbf{a}_{1,2}$ are the (reciprocal) lattice vectors. Placing $\hat{V}$ on an $L \times L$ lattice via appropriate long-distance regularization [63], the low-energy dynamics follow from projection onto the flat band using projector $\hat{P} = \sum_{\mathbf{k}} | \mathbf{k} \rangle \langle \mathbf{k} |$, and taking $L \to \infty$:

$$\hat{V} = \hat{P} \hat{V}(\mathbf{r}) \hat{P}^\dagger + \lambda \sum_{\mathbf{a}_μ} \eta^{μν} \hat{P} \hat{\Pi}_μ + \frac{\lambda}{2} \eta^{μν} g_{μν} \ \mathbf{k} \in BZ \ (4)$$

where $\hat{\Pi}_μ$ are momentum-space projectors, with $μ = x, y$:

$$\hat{\Pi}_μ = -i \partial_{k_μ} + A_μ(\mathbf{k}) \quad [\hat{\Pi}_μ, \hat{\Pi}_ν] = -i \epsilon_{μνλ} \Omega(\mathbf{k}) \ (5)$$

Isotropic case: Physical insight may be gleaned by identification with an electron in a magnetic field but in momentum-space; $η^{μν} = δ^{μν}$ is particularly instructive:

$$\hat{V} = \frac{\lambda}{2} \sum_{\mathbf{a}^μ} [−i \nabla_μ + A_μ(\mathbf{k})]^2 + \lambda \nabla \cdot \mathbf{g}(\mathbf{k}) \ \mathbf{k} \in BZ \ (6)$$

In analogy to the Aharonov-Bohm effect in a magnetic field, electrons scattering at small momenta pick up Berry phase factors. $Ω(\mathbf{k})$ identifies with the magnetic field, and eigenstates are momentum-space Landau levels (MLLs), whereas the underlying lattice is fully accounted for via torus boundary conditions in $\mathbf{k}$.

Unlike the idealized FQH problem, FCIs exhibit inhomogeneous Berry curvature across the BZ. A smooth gauge choice separates $A_μ(\mathbf{k}) = \frac{eC}{\pi R_2} [-k_μ, k_μ]^T + A_{μ\text{rect}}(\mathbf{k})$ into a topological contribution and fluctuations; $C \neq 0$ stipulates that wave functions acquire twisted boundary conditions $ψ_{nm}(\mathbf{k} + m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2) = e^{-i \frac{1}{\pi} (m_1 a_2 - m_2 a_1 \mathbf{k})} ψ_{nm}(\mathbf{k})$. The relevant length scale is the inverse magnetic length $k_B = \frac{\hbar}{\sqrt{2m_0 eB}}$, whereas fluctuations can be quantified by $δ^2_B = \frac{1}{\lambda A_{μ\text{rect}}} \int d^2k | \nabla \times A_{μ\text{rect}}(\mathbf{k})|^2$, with $A_{μ\text{rect}}$ the area of the BZ. For benign fluctuations $δ_B < k_B$, the eigenstates of $H_μ$ resemble the well-known Landau levels on the torus penetrated by flux $C$ [64]. They are indexed by two quantum numbers $m$ and $n$ with eigenspectrum $ε_{mn} = \frac{\lambda C}{\sqrt{A_{μ\text{rect}}}} (m + 1)$, where $m$ is the Landau level index and $n$ indexes a $C$-fold degeneracy per MLL. The identification of the usual guiding center coordinates is thus reversed: the MLL index $m$ plays the role of the FCI guiding center index and can be identified with discrete $C_N$ rotational symmetry if present, whereas $n = 0, ..., C - 1$ acts as a component index in $C > 1$ FCIs [65]. Since (4) serves merely to identify the single-body basis, $λ \to 0$ restores translation symmetry in the infinite system. However $λ \neq 0$ enters as a proper energy scale when considering finite-size droplets [66].

Anisotropic case: The confinement metric $η^{μν}$ can be expressed in terms of a complex vector $ω$ that obeys $η^{μν} = \bar{ω}^{μν} + \bar{ω}^{νμ}$. In the isotropic limit $η^{μν} = δ^{μν}$, $ω = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$. Corresponding guiding center operators $\hat{π} = \hat{\Pi}_μ ω_μ^\dagger$, $\hat{π}^\dagger = \bar{ω}^μ \hat{\Pi}_μ$ (7)

obey commutation relations $[\hat{π}, \hat{π}^\dagger] = Ω$ when $δ_{κμ} ω_μ^\dagger = 0$, which fixes a phase freedom $ω \to ω e^{iφ}$. Substituting (7) in (4) yields the confinement Hamiltonian in guiding-center language that determines the single-body basis:

$$\hat{V} = λ \hat{π} \hat{π}^\dagger + \frac{λ}{2} (η^{μν} g_{μν} - Ω) \ (8)$$

Comparison with Haldane’s construction [59] reveals that $η^{μν}$ is precisely the FCI momentum-dual of the guiding-center metric of the anisotropic FQHE. Given an appropriate choice of $η^{μν}(\mathbf{k})$, any FCI can in principle be captured by many-body trial ground states, constructed from the single-body eigenstates of (8). For instance, given the lowest MLL wave function $Ψ_0(\mathbf{k})$ and ladder operators $\hat{a}_μ^\dagger$ that generate the MLLs of (8), the Laughlin state at $ν$ reads $Ψ_ν \sim \prod_{i<j} (\hat{a}_i^\dagger - \hat{a}_j^\dagger)^{1/ν} Ψ_0(\mathbf{k})$.

With the guiding-center basis established, determining $η^{μν}$ remains the key challenge. Importantly, (8) describes momentum-space dynamics dual to a spin-polarized electron gas in a fluctuating magnetic field [67] if $η^{μν} g_{μν} = Ω$. Above condition can be met if and only if

$$2\sqrt{\text{det} \mathbf{g}(\mathbf{k})} = Ω(\mathbf{k}) \ (9)$$

This is the condition for an ideal FCI droplet. In deriving this result, a positive Chern number was assumed without loss of generality. The above condition is satisfied by every two-band model [63] while placing constraints on models with three or more bands. Conversely, the ideal droplet condition fixes a preferred guiding-center metric:

$$η(k) = \sqrt{\text{det} \mathbf{g}(\mathbf{k})} \ \mathbf{g}^{-1}(\mathbf{k}) \ (10)$$

Interactions: Insight into stabilization of trial states may be gleaned by recasting into first quantization the
dynamics of the interacting many-body problem. In analogy to the FQHE, density interactions may be generalized using a momentum-space interaction metric \( \tilde{\eta}^{\mu\nu} \):

\[
\tilde{H}_I = \sum_{\mathbf{q}} V_\mathbf{q} \hat{\rho}_\mathbf{q} \hat{\beta}_\mathbf{q}, \quad \hat{\rho}_\mathbf{q} = \sum_{\omega_k} \hat{c}^\dagger_{\mathbf{k}\omega} \mathcal{T}_\mathbf{q}[\tilde{\eta}^{\mu\nu}] \hat{c}_{\mathbf{k}\omega}
\]  

(11)

where \( \mathcal{T}_\mathbf{q}[\tilde{\eta}^{\mu\nu}] = e^{iq\cdot\tilde{\eta}^{\mu\nu}} \partial_{\eta\mu} \) enters as a metric-dependent momentum-space translation operator. In contrast to the FQHE on the plane, translation symmetry of \( \tilde{H}_I \) and periodicity in \( \mathbf{q} \) constrain \( \tilde{\eta}^{\mu\nu} \) to SL(2,\( \mathbb{Z} \)). Importantly, the precise guiding center metric is still a variational degree of freedom if the deviation of (10) and \( \tilde{\eta}^{\mu\nu} \) is significant. To analyze the dynamics in detail, note that if the ideal droplet condition (9) is satisfied, then an exact operator identity \( [1 - \sum_k |u_k\rangle\langle u_k|] \tilde{\omega}^{\mu\nu} \partial_{\omega\mu} [\sum_k |u_k\rangle\langle u_k|] = 0 \) entails that any operator of the form \( \hat{O} = \Lambda^-(i\partial_t \omega^\dagger_{\mu}\Lambda^+)(-i\partial_t \omega^\dagger_{\mu}) \) with analytic functions \( \Lambda^\pm \) can be projected to the flat band as \( \hat{P}\hat{O}\hat{P}^\dagger = \hat{\Lambda}^- (\hat{\pi}) \hat{\Lambda}^+ (\hat{\pi}^\dagger) \). Consider thus a decomposition of \( \tilde{\eta}^{\mu\nu} = \hat{\chi}^\mu\omega^\nu + \hat{\chi}^\nu\omega^\mu \) with complex vector \( \hat{\chi} \); if \( \hat{\chi} \) is momentum-independent, then the translation operator is precisely of this form: \( \mathcal{T}_\mathbf{q} = e^{i\hat{q}_\mu\omega^\mu_{\mathbf{q}}} e^{i\hat{q}_\nu\omega^\nu_{\mathbf{q}}} \) with \( \hat{q}_\pm = q_\mu \hat{x}^\mu_{\mathbf{q}}, q_\nu \hat{x}^\nu_{\mathbf{q}} \) — this case is considered in detail below [68]. Note that in the isotropic limit \( \tilde{\eta}^{\mu\nu}, \eta^{\mu\nu} = \delta^{\mu\nu}, \mathcal{T}_\mathbf{q} \) reduces to the conventional isotropic translation operator \( \hat{q} \equiv e^{i\mathbf{q}\cdot\mathbf{x}} \) with \( q_\pm = q_x \pm iq_y \), while \( \hat{H}_I \) is just the usual density interaction. While emphasis has thus far been placed on narrowing down to a suitable class of interactions, substantial progress has been made: projected to the flat band, the joint dynamics of (3), (11) can now be succinctly expressed in guiding-center language:

\[
\hat{H} = \lambda \sum_{i,j} \hat{\pi}_j \hat{\pi}_i \mp \sum_{i,j<\mathbf{q}} V_{\mathbf{q}} e^{iq\cdot(\hat{\pi}_j - \hat{\pi}_i)} e^{-iq\cdot(\hat{\pi}_j^\dagger - \hat{\pi}_i^\dagger)}
\]  

(12)

This Hamiltonian is the central result of this paper - it provides a first-quantized description of the low-energy dynamics of an ideal FCI in terms of the quantum geometry \( \eta^{\mu\nu}, \tilde{\eta}^{\mu\nu} \) of the lattice. Its interaction describes a two-body momentum-space magnetic translation, and acts solely on the relative guiding center degrees of freedom. A key consequence is the approximate conservation of center-of-mass guiding center, quantified by Berry curvature fluctuations averaged over the BZ with

\[
\mathcal{A}_{\hat{\pi}_{\text{cm}}}^{\hat{\pi}_\mu}(\hat{\pi}_{\text{cm}}) = \frac{\hbar}{\mathcal{A}_{\text{BZ}}} \mathcal{T}[\mathcal{O}(\mathbf{k}) - 2\pi \mathcal{C}]^{1/2} \text{, where } \hat{\pi}_{\text{cm}} = (\hat{\pi}_1 + \hat{\pi}_2)/\sqrt{2}, \hat{\pi}_{\text{rel}} = (\hat{\pi}_1 - \hat{\pi}_2)/\sqrt{2} \text{ span the two-body problem in the Chern band [69]. Furthermore (12) does not act on the intra-MLL index } n, \text{ stipulating an emergent SU(3) symmetry for } C > 1 \text{ [70].}

The physics of the above Hamiltonian can be studied via a pseudopotential decomposition of the two-body interaction matrix elements \( V^{MM'}_{m'm'} = \langle m'M| \hat{H} | m'm' \rangle |_{x=0} \) with \( m, M \) relative and center-of-mass guiding center indices, and intra-MLL indices omitted. Approximate conservation of center-of-mass in (12) entails that the two-body repulsion depends only on the relative coordinate, \( V^{MM'}_{m'm'} \approx V_{mm'} \delta_{MM'} \). Since the guiding center index identifies with discrete rotational symmetries, it is tempting to speak of an emergent continuous rotational symmetry in the flat band - however, it persists even for \( V_q \) anisotropic; \( C_N \) symmetry of \( V_q \) instead constrains relative guiding center transitions \( V_{m'm'} \to (m-m') \mod N = 0 \). The dominant matrix elements are thus well-captured by Haldane pseudopotentials [71] \( V_{m'm'} = \frac{1}{\sqrt{M}} \sum_{M=0}^{M} \sqrt{V_{MM'M}} |M|_{M \to \infty} \), which indicate stabilization of FQH trial states. Treating Berry curvature fluctuations as a perturbation, the limit with ladder operators \( \hat{\pi}_j \to \hat{a}_j, |m,M\rangle = (\hat{a}^\dagger_{\text{cm}})^m (\hat{a}^\dagger_{\text{cm}})^M |0,0\rangle / \sqrt{m!M!} \) reads

\[
V_{m'm'} = \int d\mathbf{q} \frac{V_q(iq_{\pm})^{m-m'}! F_1 \left[ \frac{m+1}{m-m'+1}, \frac{q_xq_y}{\hbar^2 k_B^2} \right]}{k_B^{2m-m'} \sqrt{2^m m!(m-m')! (m-m')!}}
\]  

(13)

Here, \( F_1(\cdot) \) is the Kummer confluent hypergeometric function. The well-known pseudopotentials of the conventional FQHE can be readily recovered for \( m = m' \), \( q_{\pm} = q_x \pm iq_y \) with \( V_m = \frac{2\pi}{\mathcal{A}_{\text{BZ}}} \int d^2 \mathbf{q} \ V_q L_m \left( \frac{\mathbf{q}}{\Omega} \right) e^{-2\pi \mathcal{C}} \).

**Isotropic ideal FCI models:** While the above prescription treats the general anisotropic case, a key question concerns finding FCI analogs of the isotropic FQHE. Such models exist indeed: the isotropic case \( \eta^{\mu\nu} = \tilde{\eta}^{\mu\nu} = \delta^{\mu\nu} \) with corresponding Fubini-Study metric \( g_{xz} = g_{yy} = g_{xy} = 0, \text{tr } g(\mathbf{k}) = \Omega(\mathbf{k}) \text{ is uniquely satisfied by any} \)

![FIG. 1. (color online). (a-f) Real-space lattice guiding-center wave functions for the ideal isotropic \( C = 3 \) model, evaluated from (8). Insets depict corresponding conventional LLL wave functions. (g) Berry curvature \( \Omega_k \) (inset) and associated MLL spectra for the FCI models discussed in the main text. m is the guiding center index; dotted lines indicate ideal flat-\( \Omega \) spectrum \( \epsilon_m = \frac{\hbar^2 \Omega}{2m} \). The \( C \)-fold degeneracy of MLLs reflects the \( C \)-component basis for \( C > 1 \) models. The guiding-center structure remains robust in the presence of \( \Omega_k \) fluctuations.](image-url)
Bloch state can be written without normalization as a meromorphic function \( \tilde{u}_k \). The number of poles in the BZ defines the Chern number [72]. Periodic boundary conditions in \( k \) restricts \( \tilde{u}_{k+\pi} \) to elliptic functions, constrained to \( C \geq 2 \). Skew-anisotropic guiding center metrics ensue from distortions of the lattice: for instance, a ‘squeezed’ FQH liquid with \( \eta_{xx} = 1/\zeta, \eta_{yy} = \zeta \) readily follows from a strain deformation of the BZ with \( k_x \to \pi - k_x \).

To illustrate our construction, we consider two multi-orbital toy models on the square lattice for \( C = 2, 3 \) [75], with the guiding-center basis of MLLs depicted in fig. 1:

\[
\tilde{u}_k^{(C=2)} = \begin{bmatrix} 1, \alpha \varphi(k_x + ik_y) \end{bmatrix}^T \tag{14}
\]

\[
\tilde{u}_k^{(C=3)} = \begin{bmatrix} 1, \beta \varphi(k_x + ik_y), \gamma \varphi'(k_x + ik_y) \end{bmatrix}^T \tag{15}
\]

Here, \( \varphi(z) \) is the Weierstrass elliptic function with periods \( 2\pi, 2\pi i \) and a second-order pole at the \( \Gamma \) point, and \( \alpha = 5.77, \beta = -7.64, \gamma = 6.73 \) are band structure parameters, chosen to minimize Berry curvature fluctuations.

The corresponding Bloch Hamiltonians are not unique; a possible definition with a flat band is \( \mathbf{h}_C(k) = 1 - \langle \tilde{u}_k^\dagger \tilde{u}_k \rangle / \langle \tilde{u}_k^\dagger \tilde{u}_k \rangle^2 \). In general, \( \mathbf{h}_C(k) \) exhibits long-ranged but exponentially-decaying hopping terms [72] and acts as an artificial toy lattice model for FCI states with topological order, much like the Hubbard model truncated to nearest-neighbor (NN) hoppings acts as a toy model for strongly-correlated states with conventional order. While a classification of such ideal FCI host lattice models remains an important open task, note that simple physical models can emerge after truncation of irrelevant hoppings. For instance, \( \mathbf{h}_{C=2} \) is well-described by a canonical d-wave lattice model \( \mathbf{h}_{C=2}(k) = d(k) \cdot \sigma \) with \( d(k) = [t(cos k_x + cos k_y), t(cos k_x - cos k_y), t' sin(k_x) sin(k_y)]^T \) with nearest- and next-nearest-neighbor hoppings.

Analysis: We apply the first-quantized interacting formalism to the FCI models discussed above. In addition to the ideal isotropic models (14,15), we study a conventional \( p_x, p_y, d_{x^2-y^2} \) three-orbital \( C = 1 \) model on the square lattice [50] that does not satisfy (9). To discuss the influence of Berry curvature, we furthermore deform this model by adiabatically adding numerically optimized hoppings up to 5\( ^{\text{th}} \) neighbor to attain \( \Omega(k) \) approximately flat over the BZ (var(\( \Omega_k \)) \( \approx 10^{-6} \)) while preserving lattice symmetries and \( C \) (fig. 1g) [63]. Employing a MLL basis constructed on a 160 \times 160-site lattice, two-body interaction matrix elements and corresponding pseudopotentials are evaluated in Fig. 2 for on-site and NN density interactions with \( \tilde{\rho}_{\mu\nu} = \delta_{\mu\nu} \). The \( C = 2, 3 \) ideal lattice models display both emergent guiding-center and SU(\( C \)) symmetries, manifested in a vanishing center-of-mass dependence of pseudopotentials (figs. 2b, 2c), and suppression of symmetry-breaking two-body interaction terms (fig. 2d). As anticipated from (13), on-site repulsion results in a non-zero pseudopotential only for \( V_0 \), acting as an optimal (221)-Halperin state [73] parent Hamiltonian for hardcore bosons, whilst not stabilizing fermionic FQH states [74]. The latter can be remedied by tuning NN repulsion to tune \( V_1, V_2 \). The controlled expansion in pseudopotentials is a key merit of this construction and highlights that, contrary to common perception, the confluence of flat Berry curvature and local interaction does not stabilize a fermionic FQH liquid in an FCI. Conversely, the non-optimal \( C = 1 \) models violate (9) and do not pin the guiding-center metric to \( g \). The guiding-center description (12) is incomplete, broadening the effective interaction range with non-vanishing and decaying \( V_m \forall m \) with substantial center-of-mass deviation even for a purely local interaction whilst breaking guiding center symmetry. This situation is largely independent of the uniformity of Berry curvature, as shown by the optimized \( C = 1 \) model with flat \( \Omega \) (fig. 2a). We stress that this highlights the shortcomings of long-wavelength limit arguments [41–47] in predicting the microscopics of the FQHE on the lattice.

In summary, we introduced a first-quantized description of FCIs, with the FQHE emerging in a picture of anisotropic momentum-space Landau levels in a fluctuating magnetic field. We presented a novel class of ideal FCI droplet host lattice models as duals of the isotropic FQHE and demonstrated their optimality via an expansion in Haldane pseudopotentials. A primary goal of this
work is to establish a deeper microscopic understanding of the stabilization of FQH states in flat Chern bands - the resulting interplay of topology and geometry to determine long- and short-wavelength physics on the lattice serves as a natural application of the formalism of the anisotropic FQHE. The results presented set a foundation for microscopic analysis of non-Abelian phases on the lattice and extension to fractional topological insulators.

We acknowledge support from the U. S. Department of Energy, Office of Basic Energy Science, Division of Materials Science and Engineering under Contract No. DE-AC02-76SF00515. C.H.L. is supported by a fellowship from the Agency of Science, Technology and Research of Singapore. R.T. is supported by the European Research Council through ERC-StG-336012-TOPOLECTRICS. X.L.Q. is supported by the David & Lucile Packard Foundation.

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[64] Residual dispersion in the Chern band can be understood as a momentum-space confinement potential, giving rise to a Darwin-Fock eigenbasis [76] on the torus, with the ratio of confinement $\lambda$ and bandwidth entering as a variational degree of freedom when building the single-body basis.

[65] Both the conventional and dual QH problems can be described in terms of a pair of harmonic oscillators. The FQH 'right-handed' guiding-center oscillators identify with guiding-center MLLs in an FCI. The FCI component index oscillator has no analogue in the conventional FQHE; conversely, the FQH 'left-handed' (Landau level) degrees of freedom have no analogues in an FCI.

[66] The energy penalty of a steep confinement potential is approximately $\langle m | (r/r_0)^n | m' \rangle \sim (C/r_0)^n m^n$, hence a finite geometry with a certain filling $\nu$ may be selected by truncating at a maximum guiding center index $m_{\text{max}}$.

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[68] The general case follows from deformation of $\epsilon^{\mu+\nu} \partial_\mu \epsilon^{\mu-\nu} \partial_\mu$, and functions $q_\pm(q)$ to satisfy $\tilde{T}_q | q \rightarrow 0 \approx 1 - iq_\mu \tilde{\eta}^{\mu\nu}(-i\partial_\nu) \text{ whilst ensuring translation symmetry of (11).}$

[69] Equivalently, two-body eigenstates of $\tilde{\pi}^\dagger_{\text{rel}} \tilde{\pi}_{\text{rel}} | \tilde{m} \rangle = \epsilon_{\tilde{m}} | \tilde{m} \rangle$ display perfect center-of-mass degeneracy $\tilde{\pi}^\dagger_{\text{rel}} \tilde{\pi}_{\text{rel}} | m, M \rangle \sim m | m, M \rangle$ only for $[\tilde{\pi}^\dagger_{\text{rel}}, \tilde{\pi}_{\text{cm}}] \rightarrow 0$.

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Supplementary Material

Momentum-Space Landau Levels

As described in the main text, the guiding-center basis in a FCI can be understood in terms of momentum-space Landau levels (MLLs). We seek the low-energy momentum-space dynamics induced by a real-space confinement potential $V(r) = \frac{1}{2} x_\mu \eta^{\mu\nu} x_\nu$. The confinement metric $\eta^{\mu\nu}$ can be expressed as an $N_d$-fold superposition of reciprocal lattice vectors $\eta^{\mu\nu} = \sum_i N_d \eta_i d_i^\mu d_i^\nu$, where $d_i = m_1 b_1 + m_2 b_2$ and $m_1, m_2 \in \mathbb{Z}$. A minimum of three basis vectors $d_i$ is required to reproduce the three independent components of $\eta$. Regularization of the long-distance behavior of the confinement potential on an $L \times L$ lattice is not unique; a simple choice reads:

$$\hat{V}_{\text{reg}} = \frac{\lambda}{2} \left( \frac{L^2}{(2\pi)^2} \right)^2 \sum_i N_d \eta_i \left[ 1 - \cos(d_i \cdot r/L) \right] \tag{16}$$

The regularized form of the confinement potential is used for numerical results presented in the main text. Choosing a sufficiently large lattice, one can solve for the guiding-center basis as eigenstates of $\hat{H}_0 + \hat{V}_{\text{reg}}$, where care must be taken to ensure that the band gap of $\hat{H}_0$ is much larger than the depth of the confinement potential. As is canonical in numerical studies of FCIs, we assume that the energy scale of interactions in the flat band is much greater than its band width and work with a modified lattice Hamiltonian $\hat{H}_0 = \frac{1}{K} \hat{H}_0$ to eliminate residual dispersion $\epsilon_k$ in the flat band. This step is not necessary for the ideal models of equations (14), (15) in the main text, as they intrinsically exhibit a perfectly flat Chern band. The low-energy dynamics follow from projecting $\hat{V}_{\text{reg}}$ onto the reduced Hilbert space built from the set of Bloch states $|u_k\rangle$ spanning the flat band, and taking the thermodynamic limit:

$$\hat{H} = \lim_{L \to \infty} \frac{\lambda}{2} \sum_{k k' \in \text{BZ}} |k\rangle \langle u_k| \hat{V}_{\text{reg}} |u_k\rangle \langle k'|$$

$$= \lim_{L \to \infty} \frac{\lambda L^2}{4} \sum_{k \in \text{BZ}} |k\rangle \langle u_k| \left( -2 \sum_i \eta_i \right) |u_k\rangle \langle k|$$

$$= \frac{\lambda}{2} \sum_i \eta_i \int_{\text{BZ}} d^2 k \ |k\rangle \langle u_k| \left( -i d_i \cdot \nabla_k \right)^2 |u_k\rangle \langle k|$$

$$= \frac{\lambda}{2} \int d^2 k \left( \sum_i \eta_i d_i^\mu d_i^\nu \right) |k\rangle \left[ -i \partial_\mu \left( -i \partial_\nu \right) + iA_\mu \partial_\nu + iA_\nu \partial_\mu - \left( \langle u_k| \partial_\mu |u_k\rangle \right) \right] |k\rangle$$

$$= \frac{\lambda}{2} \int d^2 k \ |k\rangle \left[ \left( -i \partial_\mu + A_\mu \right) \eta^{\mu\nu} \left( -i \partial_\nu + A_\nu \right) + \eta^{\mu\nu} g_{\mu\nu} \right] |k\rangle \tag{17}$$

where $A_\mu = -i \langle u_k| \partial_\mu |u_k\rangle$ is the Berry connection, $\Omega = \epsilon^{\mu\nu} \partial_\mu A_\nu$ is the Berry curvature, and the Fubini-Study metric $g_{\mu\nu}$ is defined as:

$$g_{\mu\nu} = \frac{1}{2} \left[ \langle \partial_\mu u_k | \partial_\nu u_k \rangle - \langle \partial_\mu u_k | \partial_\nu u_k \rangle + (\mu \leftrightarrow \nu) \right] \tag{18}$$

The last line of equation 17 uses the fact that the metric tensor $\eta^{\mu\nu}$ is symmetric, hence $\eta^{\mu\nu} \epsilon_{\mu\nu} = 0$. In terms of operators $\hat{\Pi}_\mu = -i \partial_\mu + A_\mu$, one finally arrives at the first-quantized Hamiltonian (4) described in the main text.

Ideal Droplet Condition – Two-Band Models

The ideal droplet condition $2 \sqrt{\det \mathbf{g}(\mathbf{k})} = \Omega(\mathbf{k})$ is automatically satisfied by any two-band FCI model. To see this, consider a generic two-band Bloch Hamiltonian $\mathbf{h}(\mathbf{k}) = \epsilon(\mathbf{k}) \mathbf{d}(\mathbf{k}) \cdot \hat{\sigma}$ parameterized by unit vector $\mathbf{d}(\mathbf{k})$ and dispersion $\pm \epsilon(\mathbf{k})$, where $\hat{\sigma}$ are the Pauli matrices in orbital basis. The Berry curvature and Fubini-Study metric read:

$$\Omega(\mathbf{k}) = \frac{1}{2} \mathbf{d}(\mathbf{k}) \cdot \frac{\partial \mathbf{d}(\mathbf{k})}{\partial \mathbf{k}_x} \times \frac{\partial \mathbf{d}(\mathbf{k})}{\partial \mathbf{k}_y} \tag{19}$$

$$g_{\mu\nu}(\mathbf{k}) = \frac{1}{4} \left( \frac{\partial \mathbf{d}(\mathbf{k})}{\partial \mathbf{k}_\mu} \times \mathbf{d}(\mathbf{k}) \right) \cdot \left( \mathbf{d}(\mathbf{k}) \times \frac{\partial \mathbf{d}(\mathbf{k})}{\partial \mathbf{k}_\nu} \right) \tag{20}$$

$$\hat{P}_\mu = -i \partial_\mu + A_\mu$$
The determinant of the Fubini-Study metric follows from application of quadruple-product identities and satisfies
\[ 2\sqrt{\det g(k)} = \Omega(k). \]

**Haldane Pseudopotentials**

This section discusses the derivation of Haldane pseudopotentials for momentum-space Landau levels. The starting point is the two-body interaction in first quantization that follows from equation (12) in the main text. The guiding-center basis can be found via numerical solution of the single-body problem. Given the lowest MLL state \(|\Psi_0\rangle\) and a set of ladder operators \(\hat{a}, \hat{a}^\dagger\), \([\hat{a}_i, \hat{a}^\dagger_j] = 1\) generating the \(m\)-th MLL \(|m\rangle = \frac{(\hat{a}^\dagger)^m}{\sqrt{m!}} |\Psi_0\rangle\), one can define two-body states

\[ |m, M \rangle = \frac{1}{\sqrt{m!M!}} \left( \frac{\hat{a}^\dagger_1 - \hat{a}^\dagger_2}{\sqrt{2}} \right)^m \left( \frac{\hat{a}^\dagger + \hat{a}^\dagger_2}{\sqrt{2}} \right)^M \langle \Psi_0 \rangle \]  

(21)

The Haldane pseudopotentials follow

\[ V_{mM} = \left\langle m, M \left| \sum_q V_q e^{iq\cdot(q_\perp - \vec{q}_\parallel)} e^{iq\cdot(\vec{q}_\perp - \vec{q}_\parallel)} \right| m, M \right\rangle \]  

(22)

and can be evaluated numerically, with the residual center-of-mass dependence studied in the main text. To recover the limit of flat Berry curvature, note that the guiding center operators can be written in terms of the ladder operators \(\vec{q}_\parallel = \hat{a}_\parallel/k_B\) with \(k_B = \sqrt{2\pi/C}\) the magnetic wave vector for a square lattice, which allows for an algebraic evaluation of the two-body matrix elements:

\[ V_{mM}^{m'M'} = \int d^2q \ V_q \left\langle mM' \left| \hat{V} \right| m'M' \right\rangle \]

\[ = \int d^2q \ V_q \frac{1}{\sqrt{m!m'!}} \left\langle 0 \left| \left( \hat{a} - \hat{a}_2 \right)^m e^{iq\cdot(q_\perp - \vec{q}_\parallel)} e^{iq\cdot(\vec{q}_\perp - \vec{q}_\parallel)} \left( \hat{a}^\dagger + \hat{a}^\dagger_2 \right)^{m'} \right| 0 \right\rangle \delta_{MM'} \]

\[ = \int d^2q \ V_q \frac{1}{\sqrt{m!m'!}} \sum_{i} \frac{(iq_+/k_B)^{m'} (iq_-/k_B)^{m}}{i^{i!} (m + i)!} (\hat{a}^\dagger)^{m+i} \delta_{MM'} \]

\[ = \int d^2q \ V_q \frac{1}{\sqrt{m!m'!}} \sum_{i} \frac{(iq_+/k_B)^{m'} (iq_-/k_B)^{m-m'} i^{i!} (m - m' + l)!}{l!(m-l)!} (\hat{a}^\dagger)^{m'} \delta_{MM'} \]

\[ = \int d^2q \ V_q \sqrt{\frac{m!}{m'!}} \frac{iF_1(m+1; m - m' + 1, -\frac{\vec{q}^2}{k_B^2})}{2^{m+m'}(m-m')!} \delta_{MM'} \]  

(23)

where \(iF_1(a; b; z)\) is the Kummer confluent hypergeometric function. Here \(q_+ = \chi^\mu q_\mu, q_- = \chi^\mu q_\mu, \) and \(\vec{q} = q_- - q_+ = q_\mu \chi^\mu \chi^\nu q_\nu. \) While the center of mass guiding center dependence is eliminated, lack of continuous rotational symmetry on the lattice does not forbid subdominant scattering between states with different relative guiding center indices \(m, m'\) unless the parent interaction is purely local \(V_q \rightarrow V. \) Instead, \(C_N\) symmetry can provide weaker selection rules with \(V_{mm'} \neq 0\) only for \(m - m' \mod N = 0\). The conventional Haldane pseudopotentials follow from imposing \(m = m'\), and writing the hypergeometric function in terms of a Laguerre polynomial:

\[ V_m = 2\pi \ 2^{-m} \int d^2q \ V_q L_m \left( \frac{\vec{q}^2}{k_B^2} \right) e^{-\vec{q}^2/k_B^2} \]  

(24)

which is the formula quoted in the main text.

**Elliptic Function Models**

The main text describes a class of ideal FCI models which can be defined from unnormalized Bloch states \(|\hat{u}_{k_x + ik_y}\rangle\) written as elliptic functions. To see this, consider a generic N-band model with unnormalized Bloch state written in terms of complex momenta

\[ |\tilde{u}_k\rangle = [\phi_1(z, \bar{z}), \phi_2(z, \bar{z}), \ldots, \phi_N(z, \bar{z})]^T \]  

(25)
where \( z, \tilde{z} = k_z \pm ik_y \). The corresponding Berry curvature and Fubini-Study metric may be evaluated straightforwardly:

\[
\Omega = 2 \sum_{n>m}^N \left[ \frac{|\phi_m \partial_x \phi_n - \phi_n \partial_x \phi_m|^2}{|\phi|^4} - \frac{|\phi_m \partial_y \phi_n - \phi_n \partial_y \phi_m|^2}{|\phi|^4} \right] \tag{26}
\]

\[
\text{tr } g = 2 \sum_{n>m}^N \left[ \frac{|\phi_m \partial_x \phi_n - \phi_n \partial_x \phi_m|^2}{|\phi|^4} + \frac{|\phi_m \partial_y \phi_n - \phi_n \partial_y \phi_m|^2}{|\phi|^4} \right] \tag{27}
\]

\[
g_{xx} - g_{yy} = 2 \sum_{n>m}^N \frac{(\phi_m \partial_x \phi_n - \phi_n \partial_x \phi_m) (\phi_m \partial_x \phi_n - \phi_n \partial_x \phi_m)^* + \text{c.c.}}{|\phi|^4} \tag{28}
\]

\[
g_{xy} = i \sum_{n>m}^N \frac{(\phi_m \partial_x \phi_n - \phi_n \partial_x \phi_m) (\phi_m \partial_y \phi_n - \phi_n \partial_y \phi_m)^* - \text{c.c.}}{|\phi|^4} \tag{29}
\]

\[
det g = \frac{1}{4} [(\text{tr } g)^2 - (g_{xx} - g_{yy})^2 - 4g_{xy}^2] \tag{30}
\]

If the unnormalized Bloch state is a function only of \( k_x + ik_y \), then \( \partial_x \phi_n = 0 \) and it follows by inspection that \( \Omega = \text{tr } g = 2\sqrt{\det g} \) and \( g_{xx} - g_{yy} = g_{xy} = 0 \). The corresponding droplet confinement metric is just the Euclidean metric as described in the main text. Generally, one can consider an arbitrary static lattice deformation defined by transformation \( k \rightarrow U \cdot k \). As such a transformation preserves both \( \Omega \) and \( \det g \), \( \Omega = 2\sqrt{\det g} \) remains satisfied, whereas the confinement metric changes to deform the shape of the droplet accordingly.

### Three-Orbital \( C = 1 \) Model – Flat Berry Curvature Optimization

To study the effects of Berry curvature fluctuations, we introduced in the main text a \( C = 1 \) model with flat Berry curvature. This model is adiabatically connected to the three-orbital model by Sun et al. [50] by adding longer-ranged hopping terms to minimize Berry curvature fluctuations. We start from the original lattice Hamiltonian

\[
\hat{H} = \sum_k \begin{bmatrix}
\hat{c}_{d,k}^\dagger & \hat{c}_{p_x,k}^\dagger & \hat{c}_{p_y,k}^\dagger
\end{bmatrix}
\begin{bmatrix}
-2td_d (\cos k_x + \cos k_y) + \delta & 2t_{pd} \sin k_x & 2it_{pd} \sin k_y \\
-2t_{pd} \sin k_x & 2tp_x \cos k_x - 2tp_y \cos k_y & i\Delta \\
-2t_{pd} \sin k_y & -i\Delta & 2tp_y \cos k_y - 2tp_x \cos k_x
\end{bmatrix}
\begin{bmatrix}
\hat{c}_{d,k} \\
\hat{c}_{p_x,k} \\
\hat{c}_{p_y,k}
\end{bmatrix}
\tag{31}
\]

for a three-orbital model on the square lattice with \( d_x, d_y, p_x, p_y \) orbitals per site. This model exhibits a \( C_4 \) rotational symmetry and mirror symmetries with symmetry operators

\[
\hat{U}_{C_4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \hat{R}_{\pi/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \hat{I}_x \cdot \hat{K}
\tag{32}
\]

where \( \hat{R}_{\pi/2}, \hat{I}_x \) and \( \hat{K} \) are \( \pi/2 \) rotation, x-reflection and complex conjugation operators, respectively. To proceed, we write down the most general model Hamiltonian with up to 5th neighbor hoppings between orbitals that is invariant under the symmetry operations above. The parameter vector \( v \) of hopping amplitudes serve as the variational parameter of the numerical minimization of Berry curvature fluctuations using cost function

\[
K[v] = \int d^2k \left[ \Omega(k, v) - \frac{1}{2\pi} \right]^2 \tag{33}
\]

The minimization procedure uses the original 3-orbital model parameters as starting values for \( v \). Furthermore, care has to be taken to ensure that the band gap remains open. We quote here the resulting model Hamiltonian that is used in the main text:

\[
\hat{H}_{\text{optimized}} = \begin{bmatrix}
h_{dd} & h_{dx} & h_{dy} \\
h_{dx}^* & h_{xx} & h_{xy} \\
h_{dy}^* & h_{zy} & h_{yy}
\end{bmatrix}
\tag{34}
\]
with

\[
\begin{align*}
    h_{dd} &= 0.203 \cos(k_x - 2k_y) - 0.0443 \cos(2k_x - 2k_y) + 2.75 \cos(k_x - k_y) \\
    &+ 0.203 \cos(2k_x - k_y) + 2.75 \cos(k_x + k_y) + 0.203 \cos(2k_x + k_y) + 0.203 \cos(k_x + 2k_y) - 0.0443 \cos(2k_x + 2k_y) \\
    &+ 4.13 \cos(k_x) - 0.663 \cos(2k_x) + 4.13 \cos(k_y) - 0.663 \cos(2k_y) - 0.76 \\
    h_{dx} &= -(0.0372 - 0.1273i) \sin(k_x - 2k_y) + (0.0391 - 0.0043i) \sin(2k_x - 2k_y) - (0.18 - 1.46i) \sin(k_x - k_y) \\
    &+ (0.0439 - 0.0177i) \sin(2k_x - k_y) + (0.18 + 1.46i) \sin(k_x + k_y) - (0.0439 + 0.0177i) \sin(2k_x + k_y) \\
    &+ (0.0372 + 0.1273i) \sin(k_x + 2k_y) - (0.0391 + 0.0043i) \sin(2k_x + 2k_y) + 1.01i \sin(k_x) - 0.374i \sin(2k_x) \\
    &+ 3.10 \sin(k_y) + 0.0351 \sin(2k_y) \\
    h_{dy} &= (0.0439 + 0.0177i) \sin(k_x - 2k_y) + (0.0391 + 0.0043i) \sin(2k_x - 2k_y) - (0.18 + 1.46i) \sin(k_x - k_y) \\
    &- (0.0372 + 0.1273i) \sin(2k_x - k_y) - (0.18 - 1.46i) \sin(k_x + k_y) + (0.0372 - 0.1273i) \sin(2k_x + k_y) \\
    &+ (0.0439 - 0.0177i) \sin(k_x + 2k_y) + (0.0391 - 0.0043i) \sin(2k_x + 2k_y) - 3.10 \sin(k_x) - 0.0351 \sin(2k_x) \\
    &+ 1.01i \sin(k_y) - 0.374i \sin(2k_y) \\
    h_{xx} &= -0.0912 \cos(k_x - 2k_y) + 0.0912 \cos(2k_x - 1k_y) + 0.0912 \cos(2k_x + k_y) - 0.0912 \cos(k_x + 2k_y) \\
    &+ 0.156 \cos(k_x) + 0.0472 \cos(2k_x) - 0.156 \cos(k_y) - 0.0472 \cos(2k_y) \\
    h_{xy} &= (0.0808 + 0.0062i) \cos(k_x - 2k_y) + (0.0273 + 0.0191i) \cos(2k_x - 2k_y) + (0.627 - 0.847i) \cos(k_x - 1k_y) \\
    &+ (0.0808 - 0.0062i) \cos(2k_x - 1k_y) + (0.627 + 0.847i) \cos(k_x + k_y) \\
    &- (0.0808 - 0.0062i) \cos(2k_x + k_y) - (0.0808 - 0.0062i) \cos(k_x + 2k_y) - (0.0273 - 0.0191i) \cos(2k_x + 2k_y) \\
    &- 0.921i \cos(k_x) - 0.0454i \cos(2k_x) - 0.921i \cos(k_y) - 0.0454i \cos(2k_y) - 1.07i \\
    h_{yy} &= 0.0912 \cos(k_x - 2k_y) - 0.0912 \cos(2k_x - 1k_y) - 0.0912 \cos(2k_x + k_y) + 0.0912 \cos(k_x + 2k_y) \\
    &- 0.156 \cos(k_x) - 0.0472 \cos(2k_x) + 0.156 \cos(k_y) + 0.0472 \cos(2k_y)
\end{align*}
\]

which yields a measure of Berry curvature fluctuations

\[
(2\pi)^2 \int d^2k \left[ \Omega(k, \nu) - \frac{1}{2\pi} \right]^2 = 5.37 \times 10^{-7}
\]