Functions on Okounkov bodies coming from geometric valuations

with an appendix by Sébastien Boucksom

Alex Küronya, Catriona Maclean, Tomasz Szemberg

May 3, 2014

Abstract

We study topological properties of functions on Okounkov bodies as introduced by Boucksom and Chen [3], and Witt-Nyström [23] in the case when they come from geometric valuations, and establish their continuity over the whole Okounkov body whenever the body is polyhedral. At the same time, we exhibit an example that shows that continuity along the boundary does not hold in general. In addition, we study formal properties of such functions and the variation of their integrals in the Néron–Severi space. An appendix by Sébastien Boucksom adds a general subadditivity result.

Contents

1 Introduction 2

2 Definitions and examples 4
  2.1 Okounkov bodies ........................................ 4
  2.2 Density of valuation vectors .......................... 7

3 Filtrations 9
  3.1 Filtrations on vector spaces ............................ 9
  3.2 Filtrations on graded algebras ........................ 10

4 Functions on Okounkov bodies 13
  4.1 Okounkov functions as concave envelopes .......... 13
  4.2 Okounkov functions via graded linear series ......... 15
  4.3 An example of a non-continuous Okounkov function .. 16
  4.4 Examples .................................................. 19
  4.5 Invariants of Okounkov functions ..................... 22

5 Integrals of Okounkov functions 24

6 Appendix: a general ’Fekete lemma’ (by Sébastien Boucksom) 27
  6.1 Facts on semigroups ..................................... 27
  6.2 A ’Fekete lemma’ for subadditive functions on semigroups .......... 27
1 Introduction

We aim here to study certain functions on Newton–Okounkov bodies associated to Cartier divisors which arise from geometric valuations of the function field of the underlying variety. We investigate their formal properties, and show how to describe them explicitly in favourable cases by explicit computations using the geometry of the underlying varieties.

Following the pioneering work of Okounkov [19], Lazarsfeld–Mustaţă [18] and Kaveh–Khovanskii [13] showed how to associate a convex body to a big Cartier divisor $D$ via studying the vanishing behaviour of global sections along a complete flag of subvarieties. This body was then called the Newton–Okounkov body of the divisor, and it soon proved to be a fundamental asymptotic invariant of $D$. Subsequent applications of the theory of Newton–Okounkov bodies (Okounkov bodies for short) outside complex geometry include connections to representation theory [12] and Schubert calculus [14].

Okounkov bodies can be considered as generalizations of moment polytopes in symplectic geometry; on smooth toric varieties moment polytopes are special cases of Newton–Okounkov bodies. Philosophically speaking, Newton–Okounkov bodies replace the volume of a divisor $\text{vol}_X(D)$, which is just a number, by a convex body, thus providing it with extra structure. Arguably the most interesting application of this theory so far is related to the moment polytope point of view: in a recent seminal paper, Harada and Kaveh [10] construct completely integrable systems on certain smooth projective varieties that map onto certain Okounkov bodies.

Coming from ideas in complex analytic geometry, Witt-Nyström [23] and Boucksom–Chen [3] present ways to obtain continuous functions on Okounkov bodies given a multiplicative filtration of the associated section ring. As explained by Witt-Nyström in [24], some of these functions are closely related to Donaldson’s test configurations [6, 7, 21, 22] and K-stability.

In this paper we consider functions arising from filtrations that carry significant geometric information, more specifically, we look at filtrations coming from geometric valuations of the function field. As a rough approximation, the value of a function associated to a valuation $\nu$ at a point of the Okounkov body is the supremum over the values of $\nu$ at sections with the same vanishing vector. The most important property of the functions associated to filtrations is that their image measure describes the asymptotic behaviour of the jumping values of the filtration.

To be more specific, given a geometric valuation $\nu$ on a projective variety $X$ over the complex number field, we obtain a filtration $\mathcal{F}_\bullet \mathbb{C}(X)$ of the function field $\mathbb{C}(X)$ by setting

$$\mathcal{F}_t \mathbb{C}(X) \overset{\text{def}}{=} \{ f \in \mathbb{C}(X) : \nu(f) \geq t \} \quad \text{for } t \in \mathbb{R}.$$  

For a big Cartier divisor $D$ on $X$, this induces a multiplicative filtration on the section ring $R(X, D) = \oplus_{m=0}^\infty H^0(X, \mathcal{O}_X(mD))$. This filtration has at most linear growth. By the method of Boucksom and Chen [3] or Witt-Nyström [23], $\mathcal{F}_\bullet$ then gives rise to a non-negative concave function

$$\varphi_{\mathcal{F}_\bullet} : \Delta_{\mathcal{Y}_\bullet}(D) \rightarrow \mathbb{R},$$

which we refer to as an Okounkov function on $\Delta_{\mathcal{Y}_\bullet}(D)$.

By concavity, these functions are always continuous in the interior of the underlying Okounkov bodies, nevertheless, since continuous function on compact spaces have particularly good properties, it is important to be able to control their behaviour on the boundary. Our main result concerns exactly this question.
Theorem 1.1. 1. Let $X$ be a projective variety, $Y_\bullet$ and admissible flag, $D$ a $\mathbb{Q}$-effective Cartier divisor on $X$, $V_\bullet$ a graded linear series associated to $D$. Pick a geometric valuation $\nu$ of $\mathbb{C}(X)$. If the Newton–Okounkov body $\Delta_{Y_\bullet}(V_\bullet)$ is a polytope (not necessarily rational), then the Okounkov function $\varphi_\nu : \Delta_{Y_\bullet}(V_\bullet) \to \mathbb{R}$ is continuous on the whole $\Delta_{Y_\bullet}(V_\bullet)$.

2. On the other hand, there exists a variety $X$, equipped with a flag $Y_1, \ldots, Y_n$, and a geometric valuation on $X$, $\nu$, and an ample divisor $D$ on $X$ such that the Okounkov function $\varphi_\nu$ on $\Delta(D)$ is not continuous.

The Theorem will be proven in subsections 4.1 and 4.3. Coupled with the fact that on surfaces Okounkov bodies of divisors are polygons [15], we obtain the following.

Corollary 1.2. Let $X$ be a smooth projective surface over the complex numbers, $Y_\bullet$ an admissible flag, $L$ a big Cartier divisor, $\nu$ a geometric valuation on $X$. Then the function $\varphi_\nu : \Delta(Y_\bullet)(L) \to \mathbb{R}$ is continuous.

The functions $\varphi_{\mathcal{F}_\bullet}$ have many interesting formal properties, one of them is an interesting reduction property. More precisely, we observe that given a judicious choice of a flag, the computation of $\varphi_{\mathcal{F}_\bullet}$ can be reduced to the boundary of the Newton–Okounkov body.

Theorem 1.3. Assume that $V_\bullet$ is a graded linear series associated to the line bundle $L$ such that there is an irreducible divisor $Y_1 \in |L|$. We take a flag $Y_\bullet$ whose divisorial part is $Y_1$. Let $\mathcal{F}_\bullet$ be a filtration on $V_\bullet$ defined by a geometric valuation $\nu$. Then for $x = (x_1, \ldots, x_n) \in \Delta_{Y_\bullet}(V_\bullet)$ we have

$$\varphi_{\mathcal{F}_\bullet}(x_1, \ldots, x_n) = (1 - x_1)\varphi_{\mathcal{F}_\bullet}(0, \frac{x_2}{1 - x_1}, \ldots, \frac{x_n}{1 - x_1}) + x_1 \cdot \nu(Y_1).$$

We verify this claim in Theorem 4.15 below.

The Okounkov functions we define are without exception integrable. Boucksom and Chen show in passing that the integral of Okounkov functions is independent of the choice of the flag. In a sequel [16] to our current paper we establish that the maximum of an Okounkov function is independent of the chosen flag as well.

The integrals $I(D; \nu)$ give rise to new invariants of divisors or graded linear series. For functions associated to test configurations Witt-Nyström observes in [24] that their normalized integral equals the Futaki invariant $F_0$, nevertheless, the geometric meaning of $I(D; \nu)$ is quite unclear.

Let $\nu$ be a geometric valuation, $D$ a Cartier divisor on $X$. We define

$$I(D; \nu) \overset{\text{def}}{=} \frac{1}{\text{vol}_X(D)} \int_{\Delta_{Y_\bullet}(D)} \varphi_\nu$$

for an arbitrary admissible flag $Y_\bullet$ on $X$. Then one can interpret [3, Theorem 1.11] as saying that $I(D; \nu)$ is the limit of normalized sums of jumping values of the underlying filtration.

We summarize fundamental properties of $I(D; \nu)$ in the following statement.

Theorem 1.4. With notation as above, the invariant $I_\nu$ has the following properties.

1. If $D \equiv D'$, then $I_\nu(D) = I_\nu(D')$.

2. For a positive integer $a$, one has $I_\nu(aD) = a \cdot I_\nu(D)$.
3. There is a unique extension of $I_\nu$ to a continuous function

$$I_\nu : \text{Big}(X) \longrightarrow \mathbb{R}_{\geq 0}.$$ 

The claims above will be shown in Proposition 5.6, Remark 5.8, and Proposition 5.7.

A few words about the organization of this paper: we start, in Section 2, by recalling the definitions of Okounkov bodies, giving some examples of calculations, and proving some technical lemmas which will be needed in the rest of the paper. Section 3 contains definitions and technical preliminaries on filtrations of algebras. In Section 4, we then present Witt-Nyström and Boucksom-Chen’s definitions of Okounkov functions, deal with the important issue of continuity, and calculate several explicit examples of Okounkov functions before turning to the question of invariants of Okounkov functions. We treat integrals of Okounkov functions in Section 5. One of essential tools used repeatedly in the present paper is Fekete Lemma [9]. Section 6 is an appendix by Sébastien Boucksom presenting a general Fekete-type lemma originating from [23], and which can be used to construct the Okounkov function of a filtration.

Acknowledgments. We first heard about the possibility of defining interesting functions on Okounkov bodies from Bo Berndtsson at a workshop in Oberwolfach. We are grateful to Sébastien Boucksom, Lawrence Ein, Patrick Graf, Daniel Greb, and Rob Lazarsfeld for helpful discussions.

During this project Alex Küronya was supported in part by the DFG-Forscher-gruppe 790 “Classification of Algebraic Surfaces and Compact Complex Manifolds”, and the OTKA Grants 77476 and 81203 by the Hungarian Academy of Sciences. Tomasz Szemberg’s research was partly supported by NCN grant UMO-2011/01/B/ST1/04875. Part of this work was done while the second author was visiting the Uniwersytet Pedagogiczny in Cracow. We would like to use this opportunity to thank the Uniwersytet Pedagogiczny for the excellent working conditions.

2 Definitions and examples

2.1 Okounkov bodies

We recall here some basic notions and properties of Okounkov bodies and establish notation. A systematic development of the theory in the geometric setting has been initiated in [18] and [13], we refer to these articles for motivation and additional details. The phrases ‘Okounkov body’ and ‘Newton–Okounkov body’ will be used interchangeably throughout the text.

Let $X$ be an irreducible projective variety of dimension $n$ and

$$Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_{n-1} \supset Y_n = \{p\}$$

be a flag of irreducible subvarieties of $X$ such that $\text{codim}_X(Y_i) = i$ and $p$ is a smooth point of each $Y_i$.

Let $D$ be a Cartier divisor on $X$ and let $V_\bullet$ be a graded linear series associated to $D$ (see [17, Definition 2.4.1]).

The flag $Y_\bullet$ defines a rank-$n$ valuation

$$\nu_{Y_\bullet} : V_k \setminus \{0\} \rightarrow \mathbb{Z}^n.$$
in the following way. Given a section $0 \neq s \in V_k \subset H^0(X, kD)$ we set
\[ \nu_1 = (\nu Y)_{1}(s) := \text{ord}_{Y_1}(s). \]
This determines a section
\[ \bar{s} \in H^0(X, kD - \nu_1 Y_1), \] (1)
which does not vanish identically along $Y_1$, and thus restricts to a non-zero section
\[ s_1 \in H^0(Y_1, (D - \nu_1 Y_1)|_{Y_1}). \]
We repeat the above construction for $s_1$ and so on. In this way we produce a valuation vector
\[ \nu Y_{\bullet}(s) = ((\nu Y)_{1}(s), \ldots, (\nu Y)_{n}(s)) \in \mathbb{Z}^n \]
and an element
\[ (\nu Y_{\bullet}(s), k) \in \Gamma Y_{\bullet}(V_{\bullet}) \subset \mathbb{Z}^{n+1} \] (2)
in the graded semigroup of the linear series $V_{\bullet}$. Let $\text{Val}_{Y_{\bullet}}(V_{\bullet}) \subset \mathbb{R}^n$ be the set of all normalized valuation vectors obtained as above i.e.
\[ \text{Val}_{Y_{\bullet}}(V_{\bullet}) = \left\{ \frac{1}{k} \nu Y_{\bullet}(s) : s \in V_k, k = 1, 2, 3, \ldots \right\} \subset \mathbb{R}^n. \]
We write simply $\text{Val}_{Y_{\bullet}}(D)$ if $V_{\bullet}$ is the complete linear series of $D$. For a given element $v \in \text{Val}_{Y_{\bullet}}(V_{\bullet})$, we define
\[ S_v \overset{\text{def}}{=} \{ k \in \mathbb{N} \mid \exists s \in V_k : \nu Y_{\bullet}(s) = kv \}. \]
Clearly $S_v$ is an additive subsemigroup in $\mathbb{N}$.

**Definition 2.1** (Okounkov body of a graded linear series). The **Okounkov body** $\Delta_{Y_{\bullet}}(V_{\bullet})$ of $V_{\bullet}$ is the closed convex hull of the set $\text{Val}_{Y_{\bullet}}(V_{\bullet})$.

**Remark 2.2.** Note that we abuse notation slightly, since $\Delta_{Y_{\bullet}}(V_{\bullet})$ is in general a convex compact set only. The above definition of a Newton–Okounkov body works fine for any arbitrary graded linear series $V_{\bullet}$. In fact, an interesting topic of Okounkov bodies for non-big pseudo-effective divisors has been taken on recently by Di Biagio and Pazienza [5].

As explained in [18, Lemma 2.6], if $V_{\bullet}$ is big, then the corresponding Okounkov body will indeed contain an open ball. By a big graded linear series we mean one satisfying Condition (C) of [18, Definition 2.9].

We will see below that in fact taking the closure is enough as the normalized valuation vectors are dense in the convex hull. Again, if $V_{\bullet}$ is the complete linear series associated to a Cartier divisor $D$ on $X$, then we write $\Delta_{Y_{\bullet}}(D)$ for its Okounkov body.

**Example 2.3** (Okounkov bodies of $\mathbb{P}^2$ and its blow up). Let $\ell$ be a line in $X_0 = \mathbb{P}^2$ and $P_0 \in \ell$ a point. In what follows we operate with a fixed flag
\[ Y_{\bullet} : X_0 \supset \ell \supset \{ P_0 \}. \]
a). Let $D_0 = \mathcal{O}_{\mathbb{P}^2}(2)$. Then $\Delta_{Y_{\bullet}}(D_0)$ is twice the standard simplex in $\mathbb{R}^2$. 

b). Let $P_1$ be a point in the plane not lying on the line $\ell$ and let $f_1 : X_1 = \text{Bl}_{P_1} X_0 \to X_0$ be the blow up of $P_1$ with exceptional divisor $E_1$. For $D_1 = f_1^*\mathcal{O}_{\mathbb{P}^2}(2) - E_1$, we have

\[
\Delta_Y(D_0)
\]

\[
\Delta_Y(D_1)
\]

\[
\Delta_Y(D_2)
\]

\[
\Delta_Y(D_2)
\]

\[
(1,1)
\]

\[
(1,1)
\]

\[
(1,1)
\]

\[
(1,1)
\]

c). Let $P_1, P_2$ be points in the plane not lying on the line $\ell$ and such that $P_0, P_1, P_2$ are not collinear. Let $f_2 : X_2 = \text{Bl}_{P_1,P_2} X_0 \to X_0$ be the blow up of $P_1,P_2$ with exceptional divisors $E_1,E_2$. For a big and nef line bundle $D_2 = f_2^*\mathcal{O}_{\mathbb{P}^2}(2) - E_1 - E_2$, we have then the Okounkov body as in the picture c1) below. The picture c2) shows the Okounkov body of the same line bundle under assumption that $P_0,P_1,P_2$ are collinear.
2.2 Density of valuation vectors

Here we verify that the points in \( \text{Val}_{\bullet}(V_\bullet) \) are dense in the convex hull of \( \text{Val}_{\bullet}(V_\bullet) \), hence also in \( \Delta_{Y_\bullet}(V_\bullet) \). This means in particular that the closure of \( \text{Val}_{\bullet}(D) \) is convex.

We first treat the case of a complete linear series \( |L| \) on a curve \( C \), because it is particularly transparent and constructive.

Fix a flag \( Y_\bullet: C = Y_0 \supset Y_1 = \{p\} \), and recall that \( \Delta_{Y_\bullet}(L) = [0, \deg L] \), see [18, Example 1.3]. For a given point \( q \in C \) (which might or might not be equal to \( p \)), we write

\[
S_{v,k}(q) \overset{\text{def}}{=} \{ t \in \mathbb{R} \mid \exists s \in V_k: \text{ord}_q(s) \geq t, \nu_{Y_\bullet}(s) = kv \}.
\]

By definition \( S_{v,k}(q) \neq \emptyset \) if and only if \( k \in S_v \).

**Lemma 2.4** (Complete linear series on a curve). With notation as above we have the following claims.

1. \( \text{Val}_{\bullet}(L) \setminus \deg L = [0, \deg L] \cap \mathbb{Q} \), in particular, the set of normalized vanishing vectors is dense in \( \Delta_{Y_\bullet}(L) \).

2. For \( v \in \text{Val}_{Y_\bullet}(V_\bullet) \) the set \( S_v \subseteq \mathbb{N} \) is an additive subsemigroup with the exponent \( e(S_v) = d \), where \( d \) equals the denominator of the rational number \( v \) in its reduced form if \( v < \deg L \), and \( d \) is the order of \( L - (\deg L)p \) in \( \text{Pic}^0 \) otherwise.

3. For given \( v \in \text{Val}_{Y_\bullet}(L) \) and \( q \in C \), the sequence

\[
a_k \overset{\text{def}}{=} \frac{1}{dk} \sup S_{v,dk}(q)
\]

is convergent.

**Remark 2.5.** Let us discuss the possibility of \( \deg L \in \text{Val}_{Y_\bullet}(L) \). By definition, this happens precisely if \( H^0(C, \mathcal{O}_C(mL - m(\deg L)p)) \neq 0 \) for some \( m \geq 0 \). This is equivalent to asking that

\[
L - (\deg L)p
\]

is a torsion point in \( \text{Jac}(C) \). This is certainly not the case for most line bundles \( L \) on a non-rational curve \( C \).

**Proof of Lemma 2.4.** (1) By construction all elements \( v \) in \( \text{Val}_{Y_\bullet}(L) \) are rational numbers, and they sit inside \( \Delta_{Y_\bullet}(L) = [0, \deg L] \), in particular, \( v \leq \deg L \).

In the other direction, let \( v \in \mathbb{Q} \cap [0, \deg L) \). Let \( m = kd \) be so large that

\[
h^1(C, \mathcal{O}_C(mL - mv \cdot p)) = 0 \quad \text{and} \quad h^1(C, \mathcal{O}_C(mL - (mv + 1) \cdot p)) = 0.
\]

We want to show that \( m \in S_v \), i.e. that there exists a section of \( \mathcal{O}_C(mL) \) vanishing at \( p \) to order exactly \( mv \). The vanishing in (3) implies

\[
h^0(C, \mathcal{O}_C((mL - mv \cdot p))) > h^0(C, \mathcal{O}_C((mL - (mv + 1) \cdot p)))
\]

via Riemann–Roch applied on \( C \) to both systems. They are non-empty by the same token. It follows that there is a section in \( mL \) whose vanishing order at \( p \) is exactly \( mv \). Hence \( m \in S_v \).

(2) The claim that \( S_v \) is an additive subsemigroup of \( \mathbb{N} \) is a consequence of the fact that \( \nu_{Y_\bullet} \) behaves logarithm–like on global sections. It must be \( d|e(S_v) \), since
$mv$ is an integer for every $m \in S_v$. In order to show the equality, we need to check that $S_v$ contains all natural numbers $kd$ for $k \gg 0$.

This follows again from a Riemann–Roch computation. Let $v \in \text{Val}_* V_v(L)$ be fixed with $v < \text{deg}(L)$. Since $L - vp$ is an ample $\mathbb{Q}$-divisor, there exists then $m_0$ such that for all $m \geq m_0$ one has the vanishing (3) whenever $mv$ is an integer.

Let $k$ be so that $kd$ is an integer satisfying $kd > m_0$. Then Riemann-Roch together with the vanishing implies as above

$$h^0 \left( C, \mathcal{O}_C (kdL - kd(1 + p)) \right) > h^0 \left( C, \mathcal{O}_C (kdL - (kdv + 1) \cdot p) \right),$$

which in turn means that $kd \in S_v$.

The case of $v = \text{deg} L$ is immediate from Remark 2.5.

(3) Part (2) implies that $\delta_{v,dk}(q) \neq 0$ for $k \gg 0$, hence $b_{dk} := \sup \delta_{v,dk}(q)$ forms a super-additive sequence of rational numbers (that is, different from $-\infty$) in $k$. Consequently, the limit of the sequence $a_k := \frac{1}{kd} b_{dk}$ exists by [9].

We now move on to the general case when the underlying variety $X$ is allowed to have arbitrary dimension, and $V_*$ is a graded linear series.

**Lemma 2.6.** Let $X$ be a projective variety, $V_*$ a graded linear series (not necessarily big) associated to a $\mathbb{Q}$-effective Cartier divisor $D$. Then

(1.) The set $\text{Val}_*(V_*)$ is dense in $\Delta_{Y_*}(V_*)$.

(2.) For $v \in \text{Val}_*(V_*)$ the set $S_v \subseteq \mathbb{N}$ is an additive subsemigroup.

(3.) For given $v \in \text{Val}_*(V_*)$ and $q \in C$, the sequence

$$a_k \overset{\text{def}}{=} \frac{1}{k} \sup \delta_{v,k}(q)$$

with $k$ running through the elements of $S_v$ is convergent.

**Proof.** (1.) The argument now is less constructive than in the case of curves, on the other hand it explains why the closure of the set of normalized valuation vectors is a convex set. Let $v_1, v_2 \in \text{Val}_*(V_*)$, and let $m_i \in \mathbb{N}, s_i \in V_{m_i} \subseteq H^0 (X, \mathcal{O}_X (m_i D))$ for $i = 1, 2$ be such that

$$\nu_{V_*}(s_i) = m_i v_i \quad \text{for } i = 1, 2.$$

Then $s_1^{m_1^2} s_2^{m_1^2} \in V_{2m_1 m_2}$, and

$$\nu_{V_*}(s_1^{m_2^2} s_2^{m_2^2}) = m_2 \cdot \nu_{V_*}(s_1) + m_1 \cdot \nu_{V_*}(s_2) = m_2 m_1 v_1 + m_1 m_2 v_2 = m_1 m_2 (v_1 + v_2),$$

hence

$$\frac{1}{2m_1 m_2} \Gamma_{V_*}(V_{2m_1 m_2}) \supseteq \frac{1}{2m_1 m_2} \cdot \nu_{V_*}(s_1^{m_2^2} s_2^{m_2^2}) = \frac{1}{2} (v_1 + v_2).$$

This shows that the midpoint between two normalized valuation vectors is again a normalized valuation vector, hence density follows.

The above argument shows also that for $v_1, v_2 \in \text{Val}_*(V_*)$ the segment $\overline{v_1 v_2}$ is contained in the closure $\Delta_{Y_*}(V_*)$, therefore the closure is a convex set.

(2.) The fact that $S_v$ is an additive subsemigroup follows from the valuation-like behavior of $\nu_{V_*}$ and the property that $V_k \cdot V_m \subseteq V_{k+m}$.

(3.) The proof is the same as in the case of curves. \qed

**Remark 2.7.** Note that the property (1.) in Lemma 2.6 has been silently used in the proof of [18, Proposition 2.1]. We include a proof here for the lack of a direct reference.
3 Filtrations

Filtrations of vector spaces and graded algebras are used by Boucksom and Chen [3] to define functions on Okounkov bodies. Here we recall the notions we will need, and look at situations that are interesting from the geometric point of view. The formal considerations come from [3] for the most part.

3.1 Filtrations on vector spaces

We begin by making it precise what we mean by a filtration in this article.

Definition 3.1 (Filtration). Let $E$ be a finite dimensional complex vector space. We call a family $\mathcal{F}_t E$ of linear subspaces of $E$ indexed by real numbers $t \in \mathbb{R}$ a filtration on $E$ if

1. for all real numbers $t \in \mathbb{R}$, $\mathcal{F}_t E \subset E$ is a vector subspace;
2. $\mathcal{F}_\cdot$ is non-increasing i.e. from $t_1 \leq t_2$ follows $\mathcal{F}_{t_1} E \supset \mathcal{F}_{t_2} E$;
3. $\mathcal{F}_\cdot$ is left continuous i.e. $\lim_{t \to t_0^-} \mathcal{F}_t E = \mathcal{F}_{t_0} E$;
4. $\mathcal{F}_\cdot$ is left and right bounded i.e. there exist real numbers $t_l$ and $t_r$ such that $\mathcal{F}_{t_l} E = E$ and $\mathcal{F}_{t_r} E = 0$.

A standard situation for this article is the following.

Example 3.2 (Filtration defined by a valuation). Let $X$ be an irreducible projective variety and let $E \subset \mathbb{C}(X)$ be a finite dimensional complex vector subspace of the function field of $X$. Let $\nu : \mathbb{C}(X) \to \mathbb{Z}$ be a rank 1 valuation. Then

$$\mathcal{F}_t E := \{ f \in E : \nu(f) \geq t \}$$

is a filtration on $E$.

The sort of valuation we are mostly interested are geometric valuations, that is, orders of vanish along a subvariety.

Given a filtration we define jumping numbers.

Definition 3.3 (Jumping numbers). Let $\mathcal{F}_\cdot$ be a filtration on a finite dimensional vector space $E$. The numbers

$$e_j(E, \mathcal{F}_\cdot) := \sup \{ t \in \mathbb{R} : \dim \mathcal{F}_t E \geq j \}$$

for $j = 1, \ldots, \dim E$ are the jumping numbers of the filtration $\mathcal{F}_\cdot$. We suppress $E$ and $\mathcal{F}_\cdot$ if the vector space and the filtration are clear from the context and write simply $e_j$ in such a case.
Note that we have the following monotonicity
\[ e_{\min}(E, \mathcal{F}_\bullet) := e_{\dim(E, \mathcal{F}_\bullet)} \leq \cdots \leq e_1(E, \mathcal{F}_\bullet) =: e_{\max}(E, \mathcal{F}_\bullet). \]

In particular,
\[ e_{\min}(E, \mathcal{F}_\bullet) = \inf \{ t \in \mathbb{R} : \mathcal{F}_t E \neq E \} \quad \text{and} \quad e_{\max}(E, \mathcal{F}_\bullet) = \sup \{ t \in \mathbb{R} : \mathcal{F}_t E \neq 0 \}. \]

Following Boucksom and Chen, we define the \textit{mass} of \((E, \mathcal{F}_\bullet)\) as
\[ \text{mass}(E, \mathcal{F}_\bullet) := \sum_{j=1}^{\dim E} e_j(E, \mathcal{F}_\bullet). \]

**Remark 3.4.** Once the functions associated to filtrations will have been defined, the mass of a filtration will be related to the integral of the corresponding function over Newton–Okounkov bodies.

**Example 3.5 (Jumping numbers on homogeneous polynomials).** Let \( X = \mathbb{P}^2 \) and \( E = H^0(\mathcal{O}_{\mathbb{P}^2}(1)). \) We consider the filtration \( \mathcal{F}_\bullet \) on \( E \) introduced by a geometric valuation \( \nu \) given by the order of vanishing \( \text{ord}_p \) at a fixed point \( p \in \mathbb{P}^2 \) as in Example 3.2. Then
\[ e_{\min} = e_3 = 0, \quad e_2 = e_1 = e_{\max} = 1 \quad \text{and} \quad \text{mass} = 2. \]

### 3.2 Filtrations on graded algebras

The constructions from the previous part extend to the setting of graded \( \mathbb{C} \)-algebras.

**Definition 3.6 (A filtration on a graded object).** Let
\[ E_\bullet = \bigoplus_{k \geq 0} E_k \]
be a graded \( \mathbb{C} \)-algebra with \( E_0 = \mathbb{C} \) and \( \dim E_k \) finite for all \( k \). A family \( \mathcal{F}_\bullet E_\bullet \) of subspaces of \( E_\bullet \) is a \textit{filtration of the graded algebra} \( E_\bullet \) if \( \mathcal{F}_\bullet E_k \) is a filtration on the vector space \( E_k \) for all \( k \).

We say that \( \mathcal{F}_\bullet \) is \textit{multiplicative} if for all \( s, t \in \mathbb{R} \) and all \( m, n \) we have
\[ (\mathcal{F}_t E_m) \cdot (\mathcal{F}_s E_n) \subset \mathcal{F}_{t+s} E_{m+n}. \]

**Example 3.7 (A filtration given by a valuation).** Let \( X \) be an irreducible projective variety. Let \( E_\bullet = \bigoplus_{k \geq 0} E_k T^k \subset \mathbb{C}(X)[T] \) be a graded subalgebra which is connected (i.e. \( E_0 = \mathbb{C} \)) and locally finite (that is, \( \dim E_k < \infty \) for all \( k \)).

Let \( \nu \) be a geometric valuation on \( \mathbb{C}(X) \) i.e. a valuation defined by the order of vanishing along a subscheme \( Z \) in \( X \). Since
\[ \nu(f_1 \cdot f_2) = \nu(f_1) + \nu(f_2), \]
the expression
\[ \mathcal{F}_t E_k = \{ f \in E_k : \nu(f) \geq t \} \]
defines a multiplicative filtration.
Definition 3.8 (Linearly bounded filtrations). In the setup of Definition 3.6, we say that the filtration $\mathcal{F}_*E_*$ is linearly left bounded, if there exists a constant $C > 0$ such that for all $k$ we have

$$e_{\min}(E_k, \mathcal{F}_*) \geq -C \cdot k.$$ 

Similarly, $\mathcal{F}_*E_*$ is linearly right bounded, if

$$e_{\max}(E_k, \mathcal{F}_*) \leq C \cdot k$$

for a fixed constant $C > 0$ and all $k$.

We can generalize jumping numbers to the graded setting.

Definition 3.9 (Asymptotic jumping numbers). With notation as in Definition 3.6 we set

$$e_{\min}(V_k, \mathcal{F}_*) := \liminf \frac{1}{k} e_{\min}(E_k, \mathcal{F}_*) \quad \text{and} \quad e_{\max}(V_k, \mathcal{F}_*) := \limsup \frac{1}{k} e_{\max}(E_k, \mathcal{F}_*).$$

Note that a filtration $\mathcal{F}_*E_*$ of the graded $\mathbb{C}$-algebra $E_*$ is linearly left bounded if and only if $e_{\min}(E_k, \mathcal{F}_*) > -\infty$ and similarly, it is linearly right bounded if and only if $e_{\max}(E_k, \mathcal{F}_*) < \infty$.

Proposition 3.10 (Filtration on a graded linear series). Let $X$ be an irreducible normal projective variety of dimension $n$, $D$ a Cartier divisor on $X$ and $V_*$ a graded linear series defined by $D$. Furthermore let $Z$ be a subvariety in $X$, $\nu = \text{ord}_Z$ be the geometric valuation defined by $Z$, and let $\mathcal{F}_*V_*$ be the filtration given by $\nu$ as in Example 3.7. Then $\mathcal{F}_*$ is linearly left and right bounded.

Proof. The valuation $\text{ord}_Z$ is left bounded as $\text{ord}_Z(s) \geq 0$ for all $s \neq 0$, hence also

$$e_{\min}(V_k, \mathcal{F}_*) \geq 0 \quad \text{for all} \quad k.$$ 

For the right boundedness we claim that there exists a positive constant $C$ such that

$$\max \{\text{ord}_Z(s) : s \in V_k\} \leq C \cdot k$$

for all $k$. It is enough to prove this claim for the complete linear series $V_k = H^0(X, kD)$. To this end let $\pi : Y \to X$ be the blowing up along $Z$. There exists a unique irreducible component $E$ of the exceptional locus of $\pi$ mapping surjectively onto $Z$. For this component we have

$$\text{ord}_Z(s) = \text{ord}_E(\pi^*s) \quad \text{for all} \quad s \in H^0(X, kD).$$

Let $H$ be an ample line bundle on $Y$. There exists $C > 0$ such that

$$(\pi^*D - CE) \cdot H^{n-1} < 0.$$ 

This implies that $\text{ord}_Z(s) = \text{ord}_E(\pi^*s) \leq C \cdot k$ for all $s \in H^0(X, kD)$. Thus we have

$$e_{\max}(H^0(X, kD), \mathcal{F}_*) = \max \{\text{ord}_Z(s) : s \in H^0(X, kD)\} \leq C \cdot k.$$ 

$\square$
Example 3.11 (Asymptotic order of vanishing). Let $X$ be a normal projective variety and $V_*$ a graded linear series on $X$. For a geometric valuation $\nu$ we define a filtration $F_*$ on $V_*$ as in Proposition 3.10 and we set

$$\nu(V_k) := \min \{ \nu(s) : s \in V_k \setminus \{0\} \}.$$  

Then

$$e_{\min}(V_k, F_*) = \nu(V_k)$$

and

$$e_{\min}(V_*, F_*) = \lim \frac{1}{k} \nu(V_k) = \inf \frac{1}{k} \nu(V_k)$$

recovers the asymptotic order of vanishing along the center of $\nu$ as defined in [8, Definition 2.2]. The fact that we can write inf and lim instead of lim sup is accounted for by the subadditivity of the sequence $\nu(V_k)$:

$$\nu(V_{k+m}) \leq \nu(V_k) + \nu(V_m)$$

as explained in [8, Lemma 2.1] and Fekete's Lemma [9].

The number $e_{\max}$ behaves similarly under mild additional assumption.

Lemma 3.12 ($e_{\max}$ for graded linear series). Let $V_*$ be a graded linear series such that $V_k \neq 0$ for all $k$. Then

$$e_{\max}(V_*, F_*) = \lim \frac{1}{k} e_{\max}(V_k, F_*) = \sup \frac{1}{k} e_{\max}(V_k, F_*)$$

for an arbitrary filtration on $V_*$. 

Proof. This follows by the superadditivity of the sequence $\{e_{\max}(V_k, F_*)\}$ and again Fekete’s Lemma, see also [3, Lemma 1.4].

Corollary 3.13 (Jumping numbers of Veronese algebra). Let $X$ be a normal projective variety and $V_*$ a graded linear series. Fixing a positive integer $m$, the Veronese algebra $V_m*$ is a graded linear series as well. For a filtration $F_*$ defined on $V_*$ by a geometric valuation $\mu$ on $X$ and the corresponding filtration $F_m*$ on $V_m*$, we have

$$e_{\min}(V_m*, F_m*) = me_{\min}(V_*, F_*) \text{ and } e_{\max}(V_m*, F_m*) = me_{\max}(V_*, F_*)$$

Proof. It follows from Example 3.11 and Lemma 3.12 that $e_{\min}$ and $e_{\max}$ scale well for graded subalgebras.

We get the following characterization of the maximal jumping number in case of a complete linear series.

Remark 3.14 (Maximal jumping number of a complete linear series). Let $X$ be a normal projective variety, $Z$ an irreducible smooth subvariety of $X$. Let $D$ be a Cartier divisor on $X$ and $V_* = R(X, D) = \oplus_{k \geq 0} H^0(X, kD)$ be the section ring of $D$. Moreover let $\pi : Y \to X$ be the normalized blowing up of $Z$ with the exceptional divisor $E$. Then for $s \in H^0(X, kD)$ we have

$$\text{ord}_Z(s) = \text{ord}_E(\pi^* s) = \max \{ m \in \mathbb{N} : \text{div}(\pi^* s) - mE \text{ is effective} \}.$$  

Let $F_*$ be the filtration on $V_*$ induced by the order of vanishing along $Z$. Then it follows from Example 3.11 that

$$e_{\max}(R(X, D), F_*) = \sup \frac{1}{k} \max \{ \text{ord}_Z(s) : s \in H^0(X, kD) \} =$$
Thus we see that \( e_{\text{max}} \) is in this situation closely related to the geometry of the big cone on \( Y \). Namely, it is the non-negative value of \( t \) at which the ray \( \pi^*(L) - tE \) intersects the boundary of the big cone.

4 Functions on Okounkov bodies

Functions on Okounkov bodies have been studied by Boucksom and Chen [3] and Witt-Nyström [23]. As their approaches differ, we present here briefly both of them, keeping in mind that we will be interested later on in continuous functions on Okounkov bodies. As Proposition 4.10 shows, this is a quite delicate issue.

We fix for duration of this section a projective variety \( X \) together with an admissible flag of subvarieties \( Y_\bullet : X = Y_0 \supset \cdots \supset Y_n \).

4.1 Okounkov functions as concave envelopes

We begin with describing Witt-Nyström’s construction, in a slightly different way from [23]. We recall first an auxiliary notion, see [20, Section 7].

Definition 4.1 (Closed concave envelope). Let \( \Delta \subset \mathbb{R}^n \) be a compact, convex set, and let \( f : \Delta \rightarrow \mathbb{R} \) be a bounded real valued function on \( \Delta \). The closed concave envelope \( f_c \) of \( f \) on \( \Delta \) is defined by

\[
 f_c(x) = \inf \{ g(x) | g \geq f, g \text{ concave and upper semi-continuous} \}. 
\]

The closed concave envelope of a bounded function \( f \) can be constructed as follows. Let \( H \) be the hypograph of \( f \) in \( \Delta \times \mathbb{R} \), let \( H^c \) be the closed convex hull of \( H \) and define \( f^c \) to be the unique function on \( \Delta \) having \( H^c \) as its hypograph, cf [20].

Remark 4.2. The function \( f^c \) is concave and upper semi-continuous (since its hypograph is closed). From its concavity it follows that \( f^c \) is continuous in the interior of \( \Delta \). Being concave and upper-semi-continuous, it is continuous along any line segment lying in \( \Delta \).

From now on we work with a linearly bounded filtration \( \mathcal{F}_\bullet \) on \( V_\bullet \) (typically defined by a geometric valuation \( \nu \) on the function field \( \mathbb{C}(X) \)).

We define a function \( \widetilde{\phi}_\mathcal{F}_\bullet \) at points \( v \in \Delta_{\nu_\bullet}(V_\bullet) \) which are normalized valuation vectors by

\[
 \widetilde{\phi}_\mathcal{F}_\bullet(v) := \lim_{k \to \infty} \frac{1}{k} \sup \{ t \in \mathbb{R} : \exists s \in \mathcal{F}_t V_k : \nu_\bullet(s) = k \cdot v \}. \tag{6}
\]

This limit exists because the sequence

\[
 a_k := \sup \{ t \in \mathbb{R} : \exists s \in \mathcal{F}_t V_k : \nu_\bullet(s) = k \cdot v \}
\]

is superadditive, i.e. \( a_k + a_l \leq a_{k+l} \) for all \( k, l \geq 1 \). Indeed, let \( \varepsilon > 0 \) be fixed. There exist sections

\[
 s_1 \in \mathcal{F}_{a_k-\varepsilon/2} V_k \quad \text{and} \quad s_2 \in \mathcal{F}_{a_l-\varepsilon/2} V_l
\]

such that \( \nu_\bullet(s_1) = kv \) and \( \nu_\bullet(s_2) = lv \), so \( (s_1 s_2) \in \mathcal{F}_{a_k+a_l-\varepsilon} V_{k+l} \) by the multiplicativity of the filtration and \( \nu_\bullet(s_1 s_2) = (k+l)v \). The existence of the limit now follows from Fekete’s Lemma [9].
In points \( x \) which are not valuation vectors (in particular in such points that do not belong to \( \Delta_{Y_*}(V_*) \)) we set \( \tilde{\varphi}_{F_*}(x) := 0 \). Thus the mapping \( \tilde{\varphi}_{F_*} \) is defined on the whole space \( \mathbb{R}^n \). Now we are in a position to define the Okounkov function.

**Definition 4.3** (Okounkov function 1). Using the above notation, we set

\[
\varphi_{F_*}(x) := \tilde{\varphi}_{F_*}(x)
\]

for all \( x \in \Delta_{Y_*}(V_*) \). We call this concave function the Okounkov function associated to \( F_* \).

If \( F_* \) is the filtration associated to a geometric valuation \( \nu \) of the function field of \( X \), then we will also use the notation \( \varphi_* \) for \( \varphi_{F_*} \).

We observe now that taking concave envelope leaves the values of the underlying function \( \tilde{\varphi}_{F_*} \) in normalized valuation vectors untouched.

**Lemma 4.4.** For an arbitrary normalized valuation vector \( v \) there is the equality

\[
\varphi_{F_*}(v) = \tilde{\varphi}_{F_*}(v).
\]

**Proof.** It suffices to show that the function \( \tilde{\varphi}_{F_*} \) is "concave" on the normalized valuation vectors. To this end, it suffices to show

\[
\frac{1}{2} \varphi_{F_*}(v) + \frac{1}{2} \varphi_{F_*}(u) \leq \varphi_{F_*}\left(\frac{1}{2} u + \frac{1}{2} v\right)
\]

for all normalized valuation vectors \( u \) and \( v \). Note that it follows from the proof of Lemma 2.6 that \( \frac{1}{2}(u + v) \) is again a normalized valuation vector.

Let \( \varepsilon > 0 \) be fixed. It follows from the discussion right after (6) that the limit in (6) is actually a supremum. Hence there exist numbers \( k, l \in \mathbb{N} \) and \( t_1, t_2 \in \mathbb{R} \), as well as sections \( s_1 \in \mathcal{F}_t V_k \), \( s_2 \in \mathcal{F}_t V_l \) such that \( \nu_{Y_*}(s_1) = k u \), \( \nu_{Y_*}(s_2) = k v \) and

\[
\frac{t_1}{k} > \varphi_{F_*}(u) - \varepsilon \quad \text{and} \quad \frac{t_2}{l} > \varphi_{F_*}(v) - \varepsilon.
\]

Then for \( s = s_1^l s_2^k \) we have

\[
s \in V_{2kl} \quad \text{and} \quad \nu_{Y_*}(s) = 2lk\left(\frac{1}{2} u + \frac{1}{2} v\right).
\]

Moreover \( s \in \mathcal{F}_{t_1 + kl_2} V_{2kl} \) by the multiplicity of \( \mathcal{F}_* \). Hence

\[
\varphi_{F_*}\left(\frac{1}{2} u + \frac{1}{2} v\right) \geq \frac{lt_1 + kl_2}{2lk} > \frac{1}{2} \varphi_{F_*}(u) + \frac{1}{2} \varphi_{F_*}(v) - \varepsilon
\]

which implies (7). \( \square \)

**Remark 4.5.** In [23], Witt-Nyström actually uses the following version of the above construction. For \( (v, k) \) in the graded semigroup \( \Gamma_{Y_*}(V_*) \) he sets

\[
f(v, k) := \sup \{ t \in \mathbb{R} : \exists s \in \mathcal{F}_t V_k : \nu_{Y_*}(s) = v \},
\]

which defines a super-additive function on \( \Gamma_{Y_*}(V_*) \). Writing each \( v \in \hat{\Delta}_{Y_*}(V_*) \) as the limit of a sequence of the form \( \varepsilon_k v_k \) with \( \varepsilon_k \to 0^+ \) and \( (k, v_k) \in \Gamma_{Y_*}(V_*) \), he then proves that

\[
f(v) := \lim_{k \to \infty} \varepsilon_k f(v_k, k)
\]
exists in $\mathbb{R}$, only depends on $v$, and defines a concave function on $\Delta_{Y_*}(V_*).$ His arguments provide a several variable version of the classical 'Fekete lemma', and are presented in the Appendix for the convenience of the reader.

When $v \in \Delta_{Y_*}(V_*)$ is a valuation vector, the definitions combined with the above lemma yield

$$\hat{f}(v) = \varphi_{\mathcal{F}_*}(v) = \varphi_{\mathcal{T}_*}(v),$$

and it follows by density that $\hat{f}$ and $\varphi_{\mathcal{T}_*}$ coincide on $\Delta_{Y_*}(V_*).$

**Remark 4.6.** Keeping the notation from above, let $\mathcal{F}_*$ be the valuation obtained from a geometric valuation $\nu$, let $D$ be a big divisor. Then

$$\inf_{\Delta_{Y_*}(D)} \varphi_{\nu} \geq \nu(\|D\|),$$

where $\nu(\|D\|)$ denotes the asymptotic value of $\nu$ on $D$ as defined in [8, Section 2].

**Remark 4.7.** It is an obvious but important consequence of Remark 4.2 that Okounkov functions are upper-semi-continuous.

**Proof of Theorem 1.1, Part (i).** According to [11, Proposition 3], all non-negative concave upper-semi-continuous functions are continuous on locally polyhedral subsets of $\mathbb{R}^n$. In particular, if $\Delta_{Y_*}(V_*)$ is a polytope (no matter whether it is rational or not), then all Okounkov non-negative Okounkov functions are automatically continuous on the whole of $\Delta_{Y_*}(V_*).$

This latter statement includes in particular all Okounkov functions coming from geometric valuations. $\square$

### 4.2 Okounkov functions via graded linear series

Here we recall the construction by Boucksom and Chen. Let $\mathcal{F}_*$ be a multiplicative filtration on the graded linear series $V_*$. Then, for any $t \in \mathbb{R}$, we can define a new graded linear series $V_*^{(t)}$ via

$$V_k^{(t)} := \mathcal{F}_t k V_k$$

for all $k$. The Okounkov bodies $\Delta_{Y_*} \left( V_*^{(t)} \right)$ form a non-increasing family of compact convex subsets of $\Delta_{Y_*}(V_*)$ and they have been used by Boucksom and Chen [3, Definition 1.8] in order to define functions on Okounkov bodies.

**Definition 4.8 (Okounkov function 2).** With notation as above, put

$$\psi_{\mathcal{F}_*}(x) \overset{\text{def}}{=} \sup \left\{ t \in \mathbb{R} : x \in \Delta_{Y_*} \left( V_*^{(t)} \right) \right\}.$$

for all $x \in \Delta_{Y_*}(V_*)$ and call this function also the Okounkov function associated to $\mathcal{F}_*$.

The following lemma states that these two definitions are equivalent.

**Lemma 4.9.** The definitions 4.3 and 4.8 are equivalent on $\Delta_{Y_*}(V_*)$, i.e.

$$\varphi_{\mathcal{F}_*}(x) = \psi_{\mathcal{F}_*}(x)$$

for all $x \in \Delta_{Y_*}(V_*)$. 
Proof. In what follows, we will denote the closed convex closure of a subset of $S \subseteq \mathbb{R}^n$ by $\clconv(S)$. Consider the set

$$H_1 = \{(x,y)|x \text{ a normalised valuation vector}, \exists v \text{ s.t. } \nu(v) = x, \val(v) \geq y\} \subseteq \Delta \times \mathbb{R}.$$ 

Note that by definition

$$\Delta_t \overset{\text{def}}{=} \text{closed convex hull } (H_1 \cap \{\Delta \times t\}).$$

In particular, if we consider the set $H_2 \subset \Delta \times \mathbb{R}$ defined by

$$H_2 \cap \{\Delta \times t\} = \text{closed convex hull}(H_1 \cap \{\Delta \times t\})$$

then we have that

$$\psi_{\mathcal{T}_s}(x) = \sup \{t \mid (x,t) \in H_2\}.$$ 

Observe that $H_1 \subset H_2 \subset \clconv(H_1)$. Let $H_3$ be the hypograph of $\psi_{\mathcal{T}_s}$; we then have that $H_2 \subset H_3 \subset \cl(H_2)$. Moreover, $H_3$, as the hypograph of an upper semi-continuous concave function is automatically closed and convex, so $H_3 = \cl(H_2)$, and this closure is a convex set. In particular, we have that $H_3 = \cl(H_2) = \clconv(H_1)$. Let $H$ be the hypograph of $\varphi_{\mathcal{T}_s}$, so that $\clconv(H)$ is the hypograph of $\varphi_{\mathcal{T}_s}$. It is immediate from the definition that

$$H_1 \subset H \subset \cl(H_1)$$

and hence $\clconv(H) = \clconv(H_1) = H_3$. The hypograph $\varphi_{\mathcal{T}_s}$ is therefore equal to $H_3$, which is the hypograph of $\psi_{\mathcal{T}_s}$. These two functions are therefore equal. \hfill \Box

### 4.3 An example of a non-continuous Okounkov function

Concave envelopes are in general only upper semicontinuous on the boundary. In the absence of good geometric properties of $\Delta_{Y_s}(V_s)$ (cf. Theorem 1.1 (i)), there is no guarantee that Okounkov functions defined on $\Delta_{Y_s}(V_s)$ will be continuous. Here we show by example that such a situation indeed can occur.

First, the following Proposition gives a sufficient condition for non-continuous behavior of an Okounkov function. After its proof we present an example where the circumstances described do happen.

**Proposition 4.10** (A non-continuity criterion). Let $X$ be a variety, $Y_s : X = Y_0 \supset Y_1 \supset \ldots \supset Y_n$ a flag on $X$ and $D$ a divisor on $X$. Let $\Delta_{Y_s}(D)$ be the Okounkov body of $D$ with respect to this flag and let $p$ be a point in the boundary of $\Delta(D)$ such that $p = \nu(s)$, where $s$ is a section in $H^0(X,D)$ defining a reduced irreducible divisor $Y$ and $\nu_{Y_s}$ is the multivaluation associated to the flag $Y_s$. Let $v$ be the valuation associated to $Y$, i.e. $v = \ord$. If $\Delta_{Y_s}(D)$ is not locally a cone around $p$, then the Okounkov function $\varphi_v$ associated to the valuation $v$ is not continuous at the point $p$.

**Proof.** Let us consider the Okounkov bodies $\Delta_t(D)$ associated to the filtration given by the valuation $v$. We have that $\Delta_t(D) = t\nu_{Y_s}(s) + (1-t)\Delta_{Y_s}(D)$ for $t \in [0,1]$ and $\Delta_t(D) = \emptyset$ if $t > 1$. In other words, if $t \in [0,1]$ then $\Delta_t(D)$ is produced from $\Delta_{Y_s}(D)$ by performing on $\Delta_{Y_s}(D)$ a homothety of ratio $(1-t)$ centered at the point $p = \nu_{Y_s}(s)$. 


In particular, \( p \in \Delta_t(D) \) for all \( t \in [0,1] \) and hence

\[
\varphi_v(p) = \sup \{ t : p \in \Delta_t(D) \} = 1. 
\]

Since \( \Delta_{Y*}(D) \) is not locally a cone around \( p \) we can find a sequence of points \( p_i \) contained in the boundary \( \partial \Delta_{Y*}(D) \) such that

1. \( \lim_{i \to \infty} p_i = p \);

2. For all integers \( i \) and \( t > 0 \) we have that \( p + (1 + t)(p_i - p) \notin \Delta_{Y*}(D) \).

In other words, the line passing through \( p \) and \( p_i \) leaves the Okounkov body \( \Delta_{Y*}(D) \) exactly at the point \( p_i \). In particular, it follows that \( p_i \notin \Delta_t(D) \) for any \( t > 0 \) so that \( \varphi_v(p_i) = 0 \). It follows that \( \varphi_v \) is not continuous at the point \( p \). This completes the proof of Proposition 4.10. \( \square \)

We will now produce a threefold \( X \) along with an admissible flag \( X = Y_0 \supset Y_1 \supset Y_2 \supset Y_3 \), a divisor \( D \) on \( X \) and a section \( s \) of \( D \), such that \( \nu(s) \) lies in the round part of the boundary of \( \Delta(D) \).

Our example comes from [15], which is in turn heavily based on earlier work of Cutkosky [4]. The first part of the discussion is taken from [15] almost verbatim.

In [4], Cutkosky constructs a quartic surface \( S \subseteq \mathbb{P}^3 \) whose Néron-Severi space \( N^1(S) \) is isomorphic to \( \mathbb{R}^3 \) with the lattice \( \mathbb{Z}^3 \) and the intersection form \( q(x,y,z) = 4x^2 - 4y^2 - 4z^2 \). He shows that

1. The divisor class \( (1,0,0) \) on \( S \) corresponds to a very ample divisor class \( [L] \) and the projective embedding corresponding to \( L \) realizes \( S \) as a quartic surface in \( \mathbb{P}^3 \).

2. The nef and effective cones of \( S \) coincide, and are given by the conditions

\[
v^2 \geq 0, \quad ([L] \cdot v) > 0.
\]

Now, take the nef class \( \alpha = (1,1,0) \in N^1(S) \), and let \( C \) be a curve with class \( [C] = \alpha \). We note that since the effective cone of \( S \) has no polyhedral part, any curve \( C \) on \( S \) such that \( C^2 = 0 \) and \( \frac{1}{k}[C] \) is not integral for any \( k > 1 \), is automatically irreducible. In this case, all members of the linear series of \( C \) are irreducible.

Since \( C^2 = 0 \), Riemann-Roch implies that \( \chi(C) = 2 \) hence \( h^0(C) + h^2(C) = h^0(C) + h^0(-C) \geq 2 \). As \( (L \cdot (-C)) = -4 \), we know that \( h^0(-C) = 0 \) and it follows that \( h^0(C) \geq 2 \). There is therefore a pencil of curves on \( S \) with the class \( \alpha \), no two different elements of this pencil meet because \( \alpha^2 = 0 \) and all members of the pencil are irreducible. This pencil is hence base point free and its general element is smooth by Bertini theorem. A general element \( C \subset \mathbb{P}^3 \) of this pencil is then a smooth elliptic curve of degree 4.

Let \( X \) be the blow-up of \( \mathbb{P}^3 \) along the curve \( C \). We denote by \( Y_1 \subset X \) the proper transform of \( S \) in \( X \). We note that \( Y_1 \) is isomorphic under projection to \( S \).

We now choose a sufficiently positive ample divisor \( D \) on \( X \), such that \( D|Y_1 \) and \( Y_1|Y_1 \) are independent in the Picard group of \( Y_1 \). Moreover we can assume that all of the following divisors are ample:

\[
D, \quad D - Y_1, \quad D - Y_1 - K_X, \quad D - 2Y_1 - K_X. \quad (9)
\]
Furthermore, we choose a curve $C'$ on $Y_1 = S$ such that $[C']$ is a primitive integral member of the boundary of $\text{Eff}(Y_1)$. (The class $[C']$ is effective by the Riemann-Roch argument given above). Moreover, we assume that $C'$ is not contained in the image of the restriction map from $\text{Pic}(X)$ to $\text{Pic}(Y_1)$. We note that this implies that

$$[D|_{Y_1}], [Y_1|_{Y_1}] \text{ and } [C'] \text{ are independent in } NS(Y_1).$$

Finally, we pick $Y_2$ to be a smooth curve contained in the class $D|_{Y_1} - C'$ and pick $Y_3$ to be a general point on $Y_2$.

**Proof of Theorem 1.1, (ii).** With $X$, $Y_1$, $Y_2$, $Y_3$ and $D$ as above, we now show that there is a reduced and irreducible divisor $Z$ on $X$, linearly equivalent to $D$, such that close to the point $\nu(Z) \in \Delta(D)$ the set $\Delta(D)$ is not locally a cone. Here $\nu$ denotes the 3–valuation determined by the flag $Y_\bullet$.

Since $D$ and $D - Y_1 - K_X$ are both ample by (9), the restriction map on global sections $H^0(D) \rightarrow H^0(D|_{Y_1})$ is surjective, and indeed so is $H^0(kD) \rightarrow H^0(kD|_{Y_1})$ for any $k$.

We can therefore choose a section of $D$ determining a divisor $Z$ not vanishing along $Y_1$ and such that $Z|_{Y_1} = Y_2 \cup C'$. By generality of $Y_3$ we then have

$$\nu(Z) = (0,1,0).$$

Let us show now that the divisor $Z$ is reduced and irreducible. If not then we can write $Z$ as a sum of non-zero effective divisors

$$Z = Z' + Z''.$$ 

Then $Z'|_{Y_1}$ and $Z''|_{Y_1}$ are non-zero effective divisors and $(Z' + Z'')|_{Y_1} = C' + Y_2,$ where $C'$ and $Y_2$ are both irreducible. Without loss of generality $Z'|_{Y_1} = C'$, but this contradicts our assumption that $C'$ is not a restriction of a divisor on $X$.

Let us now show that $\Delta(D)$ is not locally a cone at $(0,1,0)$. We consider

$$\Delta'(D) = \{(a,b,c)|0 < a < 1, (a,b,c) \in \Delta(D)\}$$

i.e. we consider a part of $\Delta(D)$ with $a$ sufficiently small. From (9) it follows that for any $k$ and any $a \in [0,1]$ such that $ka \in \mathbb{N}$ the mapping

$$H^0(k(D - aY_1)) \rightarrow H^0(k(D - aY_1)|_{Y_1})$$

is surjective. It follows that for any $a \in [0,1]$

$$\Delta(D) \cap \{(a,-,-)\} = \{(a,b,c)|(b,c) \in \Delta((D - aY_1)|_{Y_1})\}.$$ 

In other words, the slice of the Okounkov body $\Delta(D)$ with the plane $(a,-,-)$ is just the Okounkov body of $(D - aY_1)|_{Y_1}$ on $Y_1$.

As $Y_1$ is a surface with no negative curves, the description of its Okounkov bodies given in [18, Theorem 6.4] is then very simply

$$\Delta(D - aY_1)|_{Y_1} = \{(b,c)|(D - aY_1)|_{Y_1} - bY_2 \text{ effective }, 0 < c < (D - aY_1 - bY_2) \cdot Y_2\}$$

or in other words

$$\Delta'(D - aY_1) = \{(a,b,c)|0 < a < 1, f_1 > 0, f_2 > 0, 0 < c < f_3\},$$
where
\[ f_1 = (D|Y_1 - aY_1|Y_1 - bY_2)^2, \]
\[ f_2 = (D|Y_1 - aY_1|Y_1 - bY_2) \cdot L, \]
\[ f_3 = (D|Y_1 - aY_1|Y_1 - bY_2) \cdot Y_2. \]

For simplicity, let us now consider the slice
\[ \Delta''(D) = \{(a, b)| (a, b, 0) \in \Delta_\epsilon(D)\} \]
obtained by intersecting \( \Delta'(D) \) and the plane \( c = 0 \). Alternatively, we can write
\[ \Delta''(D) = \{(a, b)| 0 \leq a \leq 1, (D|Y_1 - aY_1|Y_1 - bY_2)^2 \geq 0 \text{ and } (D|Y_1 - aY_1|Y_1 - bY_2) \cdot L > 0\}. \]

It will be enough to show that \( \Delta''(D) \) is not locally a cone around the point \((0, 1)\). Recall that any cone in \( \mathbb{R}^2 \) is either the whole of \( \mathbb{R}^2 \) or is bounded by two straight half-lines. \((0, 1)\) is not an interior point of \( \Delta''(D) \) so the first possibility is excluded.

The set \( \Delta''(D) \) is bounded by the following curves:

1. the \( x \)-axis,
2. the \( y \)-axis,
3. the line \( x = \epsilon \),
4. the branch of the conic section defined by the equation
\[ (D|Y_1 - aY_1|Y_1 - bY_2)^2 = 0. \]

passing through \((0, 1)\).

The point \((0, 1)\) lies at the intersection of the \( y \)-axis and the conic section defined by the equation \((D|Y_1 - aY_1|Y_1 - bY_2)^2 = 0\).

To establish that \( \Delta''(D) \) is not locally a cone around \( b \) it will be enough to show that the conic section given by the equation \((D|Y_1 - aY_1|Y_1 - bY_2)^2 = 0\) does not contain a straight line. This conic section is the intersection in \( \text{Pic}(Y_1) \) of the nef cone \( x^2 = y^2 + z^2 \) with the plane passing through the points \( D|Y_1 \), \((D - Y_1)|Y_1 \) and \( D|Y_1 - Y_2 \). By (10) this plane does not pass through \( 0 \) so the resulting conic section is not a union of straight lines. This completes the proof of Theorem 1.1. \( \square \)

### 4.4 Examples

We devote this section to several examples where functions associated to various geometric valuations are determined explicitly. First we deal with the one-dimensional case, where Okounkov functions associated to complete linear series can be computed in general.

**Example 4.11** (Okounkov function of a valuation on a curve). Let \( C \) be a smooth curve, \( \mathcal{V}_\bullet \) a big graded linear system associated to a line bundle \( L \) of positive degree, and let
\[ Y_\bullet : C \ni \{p\} \]
be a fixed flag.

a) Consider the filtration \( F_p = \text{ord}_p \) on \( V_p \) defined by the order of vanishing at the point \( p \) in the flag.

Let \( x \in \Delta_{Y_p}(V_p) \) be arbitrary, and write it as a limit of normalized valuation vectors \( x = \lim_{k \to \infty} \frac{\alpha_k}{k} \).

Then
\[
\varphi_{\text{ord}_p}(x) = \lim_{k \to \infty} \frac{1}{k} \sup \{ t \in \mathbb{R} : \exists s \in V_k : \text{ord}_p(s) \geq t \text{ and } \text{ord}_p(s) = k\alpha_k \} = \lim_{k \to \infty} \frac{1}{k} \alpha_k = x.
\]

It turns out that in this case the Okounkov function is the identity.

b) Now consider the filtration \( F_q = \text{ord}_q \) defined by the order of vanishing in a point \( q \) not in the flag. In this case, we take \( V_q \) to be the complete graded linear series associated to the divisor \( L \). At a point \( x = \lim_{k \to \infty} \frac{\alpha_k}{k} \) as above, we have
\[
\varphi_{\text{ord}_q}(x) = \lim_{k \to \infty} \frac{1}{k} \sup \{ t \in \mathbb{R} : \exists s \in V_k : \text{ord}_q(s) \geq t \text{ and } \text{ord}_p(s) = k\alpha_k \} = \lim_{k \to \infty} \frac{1}{k}(k \deg(L) - \alpha_k) = \deg(L) - x.
\]

Next, we move on to the surfaces, where calculations become very difficult very soon. This is not surprising, since invariants of Okounkov functions on surfaces already carry deep geometric information (see [16]).

**Example 4.12** (Okounkov function of a valuation on the projective plane). Set \( X_0 = \mathbb{P}^2 \), \( D_0 = \mathcal{O}_{\mathbb{P}^2}(1) \), and let \( P_0 \in \ell \subset X_0 \) be a flag as in Example 2.3.

a). First, we handle the case \( \nu = \text{ord}_{P_0} \). In the rational points \( (a,b) \in \Delta(D_0) \) the Okounkov function \( \varphi^0 \) is then
\[
\varphi^0(a,b) = \lim_{k \to \infty} \frac{1}{k} \sup \{ t \in \mathbb{R} : \exists s \in |kD_0| : \text{ord}_{\ell}(s) = ka, \text{ord}_{P_0}(s_1) = kb, \text{ord}_{P_0}(s) \geq t \} = \lim_{k \to \infty} \frac{1}{k}(k(a + b)) = (a + b),
\]
where \( s_1 \) is defined as in (1). As the Okounkov body \( \Delta(D_0) \) is a polytope, \( \varphi^0 \) is continuous by Theorem 1.1, hence \( \varphi^0(a,b) = a + b \) for all \( (a,b) \in \Delta(D_0) \). We point out that using the definition of Boucksom and Chen, one can obtain the result without referring to the continuity of \( \varphi^0 \).

b) Now we consider \( \nu = \text{ord}_{P_1} \) for a point \( P_1 \) not on the line \( \ell \). For the rational points \( (a,b) \in \Delta(D_0) \) we have
\[
\varphi^1(a,b) = \lim_{k \to \infty} \frac{1}{k} \sup \{ t \in \mathbb{R} : \exists s \in |kD_0| : \text{ord}_{\ell}(s) = ka, \text{ord}_{P_0}(s_1) = kb, \text{ord}_{P_1}(s) \geq t \} = \lim_{k \to \infty} \frac{1}{k}(k(1 - a)) = 1 - a.
\]

Again, the same formula holds over the whole of \( \Delta(D_0) \) by a continuity argument.
Note that the analogous calculations can be carried out on a projective space of arbitrary dimension.

**Example 4.13** (Okounkov function on a blow up of the projective plane). Keeping the notation of the Example 4.12, let $f : X_1 = \text{Bl}_{P_1} X_0 \to \mathbb{P}^2$ be the blow up of the projective plane in a point $P_1$ not contained in the flag line $\ell$ with exceptional divisor $E_1$. We work now with a $\mathbb{Q}$–divisor $D_\lambda = f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \lambda E_1$, for some fixed $\lambda \in [0, 1]$. A direct computation using [18, Theorem 6.2] gives that the Okounkov body has the shape

![Diagram](attachment:image.png)

a). For the valuation $\nu = \text{ord}_{P_0}$, we get exactly as above

$$\varphi^0(a, b) = a + b.$$  

b). For the valuation $\nu = \text{ord}_{P_2}$, where $P_2$ is a point in $X_1$ not on the exceptional divisor $E_1$ (hence $P_2$ can be considered also as a point on $\mathbb{P}^2$) and not on the line through $P_0$ and $P_1$. We have now for $(a, b) \in \Delta(D_\lambda)$

$$\varphi^1(a, b) = \begin{cases} 
1 - a & \text{for } a + b \leq 1 - \lambda \\
2 - 2a - b - \lambda & \text{for } 1 - \lambda \leq a + b \leq 1
\end{cases}$$

This can be seen as follows. $\varphi^1(a, b)$ is the maximal order of vanishing at $P_2$ among all $\mathbb{Q}$–sections vanishing

a) along $\ell$ to order $a$;

b) in $P_1$ to order $\lambda$;

c) in $P_0$ to order $b$ after dividing by the equation of $\ell$ in power $a$ and after restricting to $\ell$.

Condition a) "costs" $aH$, so we are left with $(1-a)H - \lambda E_1$ to take care of conditions b) and c). If $b \leq 1 - a - \lambda$, then we take a line through the points $P_2$ and $P_1$ with multiplicity $\lambda$ and the line through $P_2$ and $P_0$ with multiplicity $1 - a - \lambda$. Their union has multiplicity $\lambda + (1-a-\lambda) = 1-a$ at $P_2$ and satisfies b) and c). Moreover, there is no $\mathbb{Q}$–divisor equivalent to $(1-a)H - \lambda E_1$ with higher multiplicity at $P_2$, which follows easily from Bézout’s theorem intersecting with both lines.
The argument in the remaining case $b > 1 - a - \lambda$ is similar. We want to split the divisor so that it produces a high vanishing order towards condition c) first and then, after arriving to the threshold

\[ b' = 1 - a' - \lambda', \quad (11) \]

we take again the union of two lines as above. Thus, we start with the conic through $P_1$ and $P_2$ tangent to $\ell$ at $P_0$. We take this conic with multiplicity $\alpha$ subject to condition that

\[ b - 2\alpha = 1 - a - 2\alpha - (\lambda - \alpha), \]

which means that the divisor $(1-a-2\alpha)H - (\lambda - \alpha)E_1$ satisfies (11) with $b' = b - 2\alpha$, $a' = a + 2\alpha$ and $\lambda' = \lambda - \alpha$. The constructed $\mathbb{Q}$-divisor, consisting of the conic and two lines has then multiplicity

\[ a + b + \lambda - 1 + (1 - a - 2(a + b + \lambda - 1)) = 2 - 2a - b - \lambda. \]

Bézout’s theorem shows then that there is no divisor of higher multiplicity.

4.5 Invariants of Okounkov functions

We treat various properties of Okounkov functions.

Given a linearly bounded filtration $\mathcal{F}_*$ on a graded linear series $V_*$, we can restrict it to $\mathcal{F}_{m*}$ on the Veronese subseries

\[ V_{m*} := \bigoplus_{k=1}^{\infty} V_{mk} \]

for $m \geq 1$. The index $m$ in $\mathcal{F}_{m*}$ helps us to keep track to which graded linear series the valuation is applied in the given moment. The corresponding Okounkov bodies scale well by [18, Proposition 4.1]

\[ \Delta_{Y*}(V_{m*}) = m\Delta_{Y*}(V_*), \]

so that it makes sense to compare the corresponding Okounkov functions. It turns out that they scale as well.

**Theorem 4.14** (Veronese homogenity of Okounkov functions). *Let $X$ be an irreducible projective variety and let $\mathcal{F}_*$ be a linearly bounded valuation on the graded linear series $V_*$. Then*

\[ \varphi_{\mathcal{F}_{m*}}(mx) = m \cdot \varphi_{\mathcal{F}_*}(x) \quad (12) \]

*for all $x \in \Delta_{Y*}(V_*)$.*

**Proof.** To begin with let $x \in \Delta_{Y*}(V_*)$ be a normalized valuation vector. Then

\[ \varphi_{\mathcal{F}_{m*}}(mx) = \sup \{ t \in \mathbb{R} : \exists s \in \mathcal{F}_{mt}V_k(mL) : \nu_{Y*}(s) = mx \} \]

\[ = \sup \{ t \in \mathbb{R} : \exists s \in \mathcal{F}_{mt}V_{mk}(L) : \nu_{Y*}(s) = mx \} \]

\[ = m \sup \{ t \in \mathbb{R} : \exists s \in \mathcal{F}_tV_k(L) : \nu_{Y*}(s) = x \} = m \varphi_{\mathcal{F}_*}(x). \]

The equality of both functions follows then from the density statement 2.6 (1.) and the fact that the closed concave envelope is unique. \qed
Using the above result we show that working with an appropriate flag, the Okounkov function can be recovered out of its values on the boundary of $\Delta_{Y_\bullet}(V_\bullet)$. More precisely, we establish the following fact.

**Theorem 4.15** (Reading off Okounkov functions from the boundary). Assume that $V_\bullet$ is a graded linear series associated to the line bundle $L$ such that there is an irreducible divisor $Y_1 \in |L|$. We take a flag $Y_\bullet$ whose divisorial part is $Y_1$. Let $\mathcal{F}_\bullet$ be a filtration on $V_\bullet$ defined by a geometric valuation $\nu$. Then for $x = (x_1, \ldots, x_n) \in \Delta_{Y_\bullet}(V_\bullet)$ we have

$$
\varphi_{\mathcal{F}_\bullet}(x_1, \ldots, x_n) = (1 - x_1) \varphi_{\mathcal{F}_\bullet}\left(0, \frac{x_2}{1 - x_1}, \ldots, \frac{x_n}{1 - x_1}\right) + x_1 \nu(Y_1). 
$$

**Proof.** It suffices to establish (13) in case $x = v$ is a normalized valuation vector. For $m$ large enough and divisible all coordinates $m x_1, \ldots, m x_n$ are integers and we have by (12)

$$
\varphi_{\mathcal{F}_\bullet}(x) = \frac{1}{m} \varphi_{\mathcal{F}_{m \bullet}}(mx) = \frac{1}{m} \lim_{k \to \infty} \frac{1}{k} \sup \{t : \exists s \in V_{mk} : \nu(s) \geq t \text{ and } \nu_{Y_\bullet}(s) = mk \} \ .
$$

A section $s$ with $\nu_{Y_\bullet}(s) = mkx$ can be written as $s = s' \cdot s_1^{mkx_1}$, where $s_1 \in H^0(L)$ is the section defining $Y_1$ and we have

$$
\nu(s) = \nu(s') + mkx_1 \cdot \nu(s_1),
$$

since $\nu$ is a geometric valuation. For the Okounkov valuation $\nu_{Y_\bullet}$ we have

$$
\nu_{Y_\bullet}(s_1^{mkx_1}) = (mkx_1, 0, \ldots, 0) \text{ and } \nu_{Y_\bullet}(s') = (0, mkx_2, \ldots, mkx_n) =: x'.
$$

Note that $s'$ is a section in $V_{(1-x_1)mk}$. Thus, continuing (14) we establish

$$
\varphi_{\mathcal{F}_\bullet}(x) \ := \ \frac{1}{m} \lim_{k \to \infty} \frac{1}{k} \left[ mkx_1 \nu(s_1) + \sup \{t : \exists s' \in V_{(1-x_1)mk} : \nu(s') \geq t \text{ and } \nu_{Y_\bullet}(s') = mkx' \} \right] .
$$

With $x'' := \frac{1}{1-x_1} \cdot x'$ we have

$$
mkx' = (1 - x_1) mk \cdot x''
$$

and thus continuing (15) we have

$$
\varphi_{\mathcal{F}_\bullet}(x) \ = \ x_1 \nu(s_1) + (1 - x_1) \frac{1}{m(1 - x_1)} \times
$$

$$
\times \lim_{k \to \infty} \frac{1}{k} \sup \{t : \exists s' \in V_{(1-x_1)mk} : \nu(s') \geq t \text{ and } \nu_{Y_\bullet}(s') = (1 - x_1) mkx'' \} 
$$

$$
= \ x_1 \nu(s_1) + (1 - x_1) \frac{1}{m(1 - x_1)} \varphi_{\mathcal{F}_{m(1-x_1) \bullet}}(m(1 - x_1) x'')
$$

$$
= \ x_1 \nu(s_1) + (1 - x_1) \varphi_{\mathcal{F}_\bullet}(x'') .
$$

$\square$
Remark 4.16. Repeated applications of Theorem 4.15 reduce the computation of \( \varphi_{\tau_\bullet} \) to the situation where we consider only those global sections of \( L \) that vanish at the point \( Y_n \). If the restriction map

\[
H^0(X, \mathcal{O}_X(mL)) \to H^0(Y_{n-2}, \mathcal{O}_{Y_{n-1}}(mL))
\]

is surjective for \( m \gg 0 \), then this amounts to a calculation on the curve \( Y_{n-1} \). Consequently, the computation of \( \varphi_{\tau_\bullet} \) for very ample divisors can be essentially reduced to the curve case.

At last we check that the functions \( \varphi_\nu \) are continuous when considered as functions on the interior of the global Okounkov body of \( X \).

Proposition 4.17. Let \( X \) be an irreducible projective variety, \( \nu \) a geometric valuation of \( \mathbb{C}(X) \), \( \varphi_\nu : \Delta_{Y_\bullet}(X) \to \mathbb{R}_{\geq 0} \) the associated Okounkov function. Then \( \varphi_\nu \) is continuous on the open subset

\[
U \overset{\text{def}}{=} \bigcup_{\alpha \in \text{Big}(X)} \Delta^\circ_{Y_\bullet}(\alpha) \subseteq \mathbb{N}^1(X)_{\mathbb{R}}.
\]

Proof. Let \( D_1, \ldots, D_\rho \) be integral divisors on \( X \) whose numerical equivalence classes form a \( \mathbb{Z} \)-basis of \( \mathbb{N}^1(X)_{\mathbb{R}} \); assume in addition that every effective divisor on \( X \) is a non-negative integral linear combination of the \( D_i \)'s up to numerical equivalence. This can be arranged by [18, p.30]. For an element \( \mathbf{m} \in \mathbb{N}^\rho \), we set as usual

\[
\mathbf{m} \cdot \mathcal{T} \overset{\text{def}}{=} \sum_{i=1}^\rho m_i D_i.
\]

The multigraded semigroup of \( X \) (with respect to the choices of the \( D_i \)'s and an admissible flag) is

\[
\Gamma_{Y_\bullet}(X) = \{(\mathbf{m}, \nu_{Y_\bullet}(s)) \mid 0 \not= s \in H^0(X, \mathcal{O}_X(\mathbf{m} \cdot \mathcal{T})) \} \subseteq \mathbb{N}^{n+\rho}.
\]

The global Okounkov body of \( X \) is then the closure of the convex hull of the set of normalized multigraded valuation vectors

\[
\bigcup_{\mathbf{q} \in \mathbb{Q}^\rho_{>0}} \bigcup_{k \in \mathbb{N}, k \mathbf{q} \in \mathbb{N}^\rho} \left\{ \left( \mathbf{q}, \frac{1}{k} \nu_{Y_\bullet}(s) \right) : 0 \not= s \in H^0(X, \mathcal{O}_X(k \cdot \mathbf{q} \cdot \mathcal{T})) \right\} \subseteq \mathbb{R}^{n+\rho}.
\]

If \((\mathbf{q}, \alpha)\) is such a vector, then we define

\[
\tilde{\varphi}_\nu(\mathbf{q}, \alpha) \overset{\text{def}}{=} \lim_{k \to \infty} \frac{1}{k} \sup \left\{ t \in \mathbb{R} : \exists s \in \mathcal{F}_t H^0(X, \mathcal{O}_X(k \cdot \mathbf{q} \cdot \mathcal{T})) : \nu_{Y_\bullet}(s) = k \cdot \alpha \right\}.
\]

For all other points of \( \mathbb{R}^{n+\rho} \) we set \( \tilde{\varphi}_\nu \) to be equal to zero. The concave transform of \( \tilde{\varphi}_\nu \) is then a continuous function on \( \Delta_{Y_\bullet}(X) \), which agrees over all classes \( \xi \in \mathbb{N}^1(X)_{\mathbb{Q}} \) with \( \varphi_\nu \) defined on the Okounkov body \( \Delta_{Y_\bullet}(\xi) \). This proves the claim.

5 Integrals of Okounkov functions

In this section we point out a new way of constructing invariants of numerical equivalence classes of Cartier divisors via integrating functions on Okounkov bodies. Let \( X \) be an irreducible projective variety of dimension \( n \) as so far, \( Y_\bullet \) an admissible flag.
Definition 5.1. Let $V_\bullet$ be a big graded linear series, $\nu$ a geometric valuation of $\mathbb{C}(X)$. We set

$$I(V_\bullet; \nu) \overset{\text{def}}{=} \int_{\Delta_{Y_\bullet}(V_\bullet)} \varphi_\nu.$$ 

As usual, we write $I(D; \nu)$, whenever $V_\bullet$ is the complete graded linear series associated to a Cartier divisor $D$ on $X$.

Remark 5.2. The function $\varphi_\nu$ is a bounded upper-semicontinuous concave function on the compact subset $\Delta_{Y_\bullet}(V_\bullet)$, therefore it is Lebesgue integrable. Being non-negative as well, its integral is non-negative, and so $0 \leq I(D; \nu) < \infty$.

It follows from results of [3] that $I(D; \nu)$ is in fact independent of the flag $Y_\bullet$ as the notation suggests.

Proposition 5.3. With notation as above,

$$I(D; \nu) \overset{\text{def}}{=} \text{vol}_{\mathbb{R}^n}\left(\hat{\Delta}(V_\bullet, F_\nu)\right) = \int_{t=0}^{+\infty} \text{vol}_{\mathbb{R}^n}\left(\Delta(V(t))\right) dt = \lim_{k \to +\infty} \frac{\text{mass}(V_k, F_\nu)}{k^{n+1}}.$$ 

Proof. This is the content of [3, Corollary 1.11]. Observe that the right-hand side expression is by its definition independent of the flag $Y_\bullet$. \qed

Example 5.4. Let $f : X_1 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ in a point $P_1$ with the exceptional divisor $E_1$, as in Example 4.13. A divisor

$$D = \alpha f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \beta E_1$$

is big on $X_1$ iff $\alpha > 0$ and $\beta < \alpha$.

For $\beta < 0$ we have

$$\Delta(\alpha f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \beta E_1) = \Delta(\alpha f^*(\mathcal{O}_{\mathbb{P}^2}(1))),$$

so it is enough to consider $0 \leq \beta < \alpha$. Furthermore, we can divide by $\alpha$, see Remark 5.8. Then with $\lambda = \frac{\beta}{\alpha}$, it follows from Example 4.13 that

$$I(f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \lambda E_1, \text{ord}_P) = \frac{1}{3} - \frac{1}{2} \lambda^2 + \frac{1}{6} \lambda^3$$

for $P$ as in case a). or b). of that example.

Remark 5.5. Witt-Nyström points out in [24, Section 6] that test configurations of a pair $(X, L)$ defined by Donaldson [6] (where $X$ is a projective variety, $L$ an ample Cartier divisor on $X$) give rise to filtrations of the section ring $R(X, L)$, and therefore to functions on $\Delta_{Y_\bullet}(L)$ for some flag $Y_\bullet$.

Witt-Nyström also observes (see [24, Corollary 6.6]) that the integral of such a function (properly normalized) recovers the Futaki invariant $F_0$ of the test configuration one starts out with.

The next statement fits in well with the philosophy that asymptotic invariants of line bundles tend to respect numerical equivalence.

Proposition 5.6 (Numerical invariance of Okounkov functions). Let $v$ be a discrete valuation, $D$ a big integral Cartier divisor on $X$. Then the function

$$\varphi_v : \Delta_{Y_\bullet}(D) \to \mathbb{R}$$

depends only on the numerical equivalence class of $D$. 
Proof. Fix an arbitrary numerically trivial divisor \( P \) on \( X \). First of all, as observed in [18, Proposition 4.1], Okounkov bodies are invariant with respect to numerical equivalence of divisors,

\[
\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D + P)
\]

whence the respective domains of the functions \( \varphi_{D,\nu} \) and \( \varphi_{D+P,\nu} \) agree.

Next, following the train of thought of the proof of [18, Proposition 4.1 (i)], we show that

\[
\Delta(|\bullet(D + P)|^{(t)}) = \Delta(|\bullet D|^{(t)})
\]

holds for every \( t \in \mathbb{R} \).

We recall that \( \Delta(|\bullet D|^{(t)}) \) is the Newton–Okounkov body attached to the graded linear series

\[
A_k \overset{\text{def}}{=} \{ s \in H^0(X, \mathcal{O}_X(kD)) \mid v(s) \geq tk \},
\]

while \( \Delta(|\bullet(D + P)|^{(t)}) \) is the convex body associated to the graded linear series

\[
B_k \overset{\text{def}}{=} \{ s' \in H^0(X, \mathcal{O}_X(k(D + P))) \mid v(s') \geq tk \}.
\]

It follows from a Castelnuovo–Mumford regularity argument (see [17, Lemma 2.2.42]) that there exists a divisor \( B \) on \( X \) such that \( B + lP \) is very ample for all \( l \in \mathbb{Z} \). Let \( a \gg 0 \) be such that \( |aD - B| \neq \emptyset \), and let \( s \in H^0(X, \mathcal{O}_X(aD - B)) \) be the section corresponding to an effective divisor. We write

\[
(k + a)(D + P) \sim kD + (aD - B) + (B + (k + a)P).
\]

If we represent \( B + (k + a)P \) by a section not going through the elements of \( Y_\bullet \), then we obtain

\[
A_k \cdot s \subseteq B_k,
\]

hence

\[
\Gamma(A_k) + \nu(s) \subseteq \Gamma(B_k).
\]

By taking limits we obtain

\[
\Delta_{Y_\bullet}(|\bullet D|^{(t)}) = \Delta_{Y_\bullet}(A_\bullet) \subseteq \Delta_{Y_\bullet}(B_\bullet) = \Delta_{Y_\bullet}(|\bullet(D + P)|^{(t)})
\]

Replacing \( D \) by \( D + p \) and \( P \) by \( -P \) in the above argument yields the reverse inclusion. \( \square \)

**Proposition 5.7.** Let \( X \) be an irreducible projective variety, \( \nu \) a geometric valuation of its function field. Then both

\[
I(\cdot, \nu) : \text{Big}(X) \rightarrow \mathbb{R}_{\geq 0} \quad \text{and} \quad \frac{1}{\text{vol}_X(\cdot)} \cdot I(\cdot, \nu) : \text{Big}(X) \rightarrow \mathbb{R}_{\geq 0}
\]

are continuous functions.

**Proof.** The first claim is a consequence of Lebesgue’s dominated convergence theorem and the convexity properties of Okounkov functions. For the second claim, note that the volume function is continuous and non-zero on \( \text{Big}(X) \). \( \square \)

**Remark 5.8.** Change of variables in the integral and homogeneity of Okounkov functions yield

\[
I(mD; \nu) = m^{n+1} \cdot I(D; \nu) \quad \text{and} \quad \frac{1}{\text{vol}_X(mD)} \cdot I(mD, \nu) = m \cdot \frac{1}{\text{vol}_X(D)} \cdot I(D, \nu).
\]
6 Appendix: a general 'Fekete lemma' (by Sébastien Boucksom)

6.1 Facts on semigroups

Let $V$ be a finite dimensional $\mathbb{R}$-vector space, and $S \subset V$ be a subsemigroup, i.e. a non-empty subset stable under taking sums. We denote by:

- $\mathbb{Z}S = \{ s - s' \mid s, s' \in S \}$ the subgroup spanned by $S$,
- $\mathbb{R}S \subset V$ the $\mathbb{R}$-vector space spanned by $S$,
- $C(S) \subset \mathbb{R}S$ the convex cone spanned by $S$,
- $\overline{C}(S)$ its closure, and $\mathring{C}(S)$ its relative interior, i.e. its interior in $\mathbb{R}S$.

We say that $S$ is a discrete semigroup if $\mathbb{Z}S$ is discrete. The regularization of $S$ is then defined as the semigroup

$$S^{\text{reg}} := \mathbb{Z}S \cap \mathring{C}(S).$$

We rely on the following result, which may be attributed to Khovanskii and appears in [13] (see also [2]).

**Proposition 6.1.** Let $S \subset V$ be a discrete semigroup.

(i) For every convex cone $\sigma \subset \mathring{C}(S)$ with compact basis, there exists a finitely generated subsemigroup $T \subset S$ such that $S^{\text{reg}} \cap \sigma = T^{\text{reg}} \cap \sigma$.

(ii) If $T \subset V$ is a discrete semigroup of finite type, then there exists a finite set $F \subset T^{\text{reg}}$ such that $T^{\text{reg}} = T + F$. As a result, $T^{\text{reg}} \setminus T$ meets each cone $\sigma \subset \mathring{C}(T)$ with compact basis in a finite set.

The first point directly follows from the elementary fact that

$$\mathring{C}(S) = \bigcup_{T \subset S} \mathring{C}(T),$$

where $T$ ranges over all finitely generated subsemigroups of $S$. The second point is what the usual proof of Gordan’s lemma yields.

6.2 A 'Fekete lemma' for subadditive functions on semigroups

If $(a_k)_{k \in \mathbb{N}}$ is a subadditive sequence of real numbers, then $a_k/k$ admits a limit in $\mathbb{R} \cup \{-\infty\}$. This elementary result, sometimes known as "Fekete’s subadditivity lemma", admits the following generalization.

**Theorem 6.2.** Let $S \subset V$ be a discrete semigroup and $f : S \to \mathbb{R}$ a subadditive function, so that $f(u + v) \leq f(u) + f(v)$ for all $u, v \in S$. Then we have:

(i) For all $x \in \mathring{C}(S)$ and all sequences $\varepsilon_k u_k$ with $\varepsilon_k > 0$, $u_k \in S$, $\varepsilon_k \to 0$ and $\varepsilon_k u_k \to x$, the limit

$$\hat{f}(x) = \lim_{k \to \infty} \varepsilon_k f(x_k)$$

exists in $\mathbb{R} \cup \{-\infty\}$ and only depends on $x$. 

(ii) We either have \( \hat{f} \equiv -\infty \) on \( \hat{C}(S) \), or \( \hat{f} : \hat{C}(S) \to \mathbb{R} \) is finite valued, homogeneous and subadditive (and hence convex and continuous). In the latter case, we have \( \hat{f} \leq f \) on \( S \cap \hat{C}(S) \), and \( \hat{f} \) is characterized as the largest subadditive and homogeneous function on \( C(S) \) with this property.

As observed in [1, §2], such a result is implicit in [25] for \( S = \mathbb{N}^n \subset V = \mathbb{R}^n \). The general case is due to Witt-Nyström [23], and we will basically follow his strategy of proof.

Proof. Let \( \lambda \in V^* \) be a non-zero linear form, and consider the affine hyperplane
\[ H := \{ \lambda = 1 \}. \]
For all \( x \in V \) with \( \lambda(x) \neq 0 \), set \( \tilde{x} := \lambda(x)^{-1}x \), which belongs to \( H \).

Similarly, for \( u \in S \) with \( \lambda(u) \neq 0 \) set \( \tilde{f}(u) := \lambda(u)^{-1}f(u) \).

Let
\[ K \in K' \subseteq K'' \subseteq \hat{C}(S) \cap H \]
be fixed compact convex sets, and denote by \( \sigma \), \( \sigma' \) and \( \sigma'' \) the corresponding cones. Note that \( \lambda > 0 \) on \( \sigma'' \setminus \{0\} \). To prove (i), it is enough to show that for each \( x \in K \) and each sequence \( u_k \in S \) with \( \lambda(u_k) \to +\infty \) and \( \bar{u}_k \to x \), \( \tilde{f}(u_k) \) has a limit which only depends on \( x \).

Step 1. We first prove that \( \tilde{f} \) is bounded above on \( S \cap \sigma \). Applying Proposition 6.1 to the discrete semigroup \( S \cap \sigma' \), we find finitely many points \( u_i \in S \cap \sigma' \) such that
\[ T := \sum_i n_i u_i \text{ satisfies} \]
\[ S^{\text{reg}} \cap \sigma = T^{\text{reg}} \cap \sigma, \]
and \( T^{\text{reg}} \setminus T \) meets \( \sigma \) in a finite set, say \( A \). It is thus enough to show that \( \tilde{f} \) is bounded above on \( (S \cap \sigma) \setminus A \). Now each \( u \) in the latter set belongs to \( T \), hence writes \( u = \sum_i n_i u_i \) with \( n_i \in \mathbb{N} \). By subadditivity of \( f \), we get
\[ f(u) \leq \sum_i n_i f(u_i) \leq C \sum_i n_i \lambda(u_i) = C \lambda(u) \]
with \( C > 0 \) larger than \( \max_i \lambda(u_i)^{-1} f(u_i) \), and we thus see that \( \tilde{f} \leq C \) on \( S \cap \sigma \).

Step 2. We prove the existence of \( C > 0 \) such that for all \( x \in K \) written as the limit of \( \bar{u}_k \) with \( u_k \in S \) and \( \lambda(u_k) \to +\infty \), and for all \( u' \in S \cap \sigma \), we have
\[ \lim_{k \to \infty} \tilde{f}(u_k) \leq \tilde{f}(u') + C \|x - u'\|. \quad (16) \]
Given \( x \) and \( u' \) as above, let \( z \in \partial K' \) be the unique point such that \( x \in [\bar{u}', z] \). Since \( z \) is in particular in \( \hat{C}(S) \), there exist finitely many points \( w_i \in S \cap \sigma'' \) such that \( \tau := \sum_i \mathbb{R}_+ w_i \) is a neighborhood of \( z \) with
\[ d(\tau, K) \geq \frac{1}{2} d(\partial K', K) > 0. \quad (17) \]
As a result, \( \mathbb{R}_+ u' + \sum_i \mathbb{R}_+ w_i \) is a neighborhood of \( x \) contained in \( \sigma'' \), and we thus have \( u_k \in \mathbb{N} u' + \sum_i \mathbb{N} w_i \) for all \( k \gg 1 \) by (ii) of Proposition 6.1. We may thus write in particular \( u_k = m_k u' + r_k \) with \( m_k \in \mathbb{N} \) and \( r_k \in S \cap \tau \). As a consequence,
\[ t_k := \frac{\lambda(r_k)}{\lambda(u_k)} \]
belongs to \([0, 1]\), and we have
\[ \bar{u}_k = (1 - t_k)\bar{u}' + t_k \bar{r}_k \]
and
\[ f(u_k) \leq (1 - t_k) \bar{f}(u') + t_k \bar{f}(r_k) \]
bym subadditivity of \( f \).Applying Step 1 to \( K'' \) in place of \( K \) yields \( C > 0 \) such that \( \bar{f} \leq C \) on \( S \cap \sigma'' \), and we get
\[ \bar{f}(u_k) - \bar{f}(u') \leq C t_k = C \frac{\|\bar{u}_k - \bar{v}\|}{\|\bar{r}_k - \bar{v}\|} \leq 2C d(\partial K', K)^{-1} \|\bar{u}_k - \bar{v}\| \]
for all \( k \gg 1 \). This proves (16).

Step 3. Let \( x \in K \) and let \( u_k, u'_k \in S \) be two sequences such that \( \lambda(u_k), \lambda(u'_k) \to +\infty \)
and \( \bar{u}_k, \bar{u}'_k \to x \). By (16) we get \( \limsup_k \bar{f}(u_k) \leq \liminf_k \bar{f}(u'_k) \), which proves that \( \hat{f}(x) := \lim_k \bar{f}(u_k) \) exists in \( \mathbb{R} \cup \{-\infty\} \) and only depends on \( x \). Another application of (16) shows that \( |\bar{f}(x) - \bar{f}(x')| \leq C \|x - x'\| \) for all \( x, x' \in K \), which proves that \( \hat{f} \) is finite valued and continuous on \( K \) as soon as there exists \( x \in K \) with \( \hat{f}(x) > -\infty \).

In that case, subadditivity of \( \hat{f} \) easily follows from that of \( f \), and homogeneity of \( \hat{f} \) is automatic, so that \( \hat{f} \) is convex. Given \( u \in S \in \bar{C}(S) \) we have
\[ \hat{f}(u) = \lim_{k \to \infty} \frac{1}{k} f(ku) \leq f(u) \]
by subadditivity of \( f \). Conversely, if \( g \) is a convex and homogeneous function on \( \bar{C}(S) \) such that \( g \leq f \) on \( S \cap \bar{C}(S) \), writing \( x \in \bar{C}(S) \) as the limit of \( \varepsilon_k u_k \) with \( \varepsilon_k \to 0 \) and \( u_k \in S \in \bar{C}(S) \) yields
\[ g(u) = \lim_{k \to \infty} \varepsilon_k g(u_k) \leq \lim_{k \to \infty} \varepsilon_k f(u_k) = \hat{f}(x), \]
and Theorem 6.2 is proved.

\[ \square \]

References

[1] Bloom, T., Levenberg, N.: Transfinite diameter notions in \( \mathbb{C}^N \) and integrals of Vandermonde determinants. Ark. Mat. 48 (2010), no. 1,
[2] Boucksom, S.: Corps d’Okounkov. Séminaire Bourbaki 1059 (2012).
[3] Boucksom, S., Chen, H.: Okounkov bodies of filtered linear series Compositio Math. 147 (2011), 1205–1229.
[4] Cutkosky, S. D.: Irrational asymptotic behaviour of Castelnuovo-Mumford regularity. J. Reine Angew. Math. 522 (2000), 93–103.
[5] Di Biagio, L., Pazienza, G.: Restricted volumes of effective divisors. preprint, arXiv:1207.1204.
[6] Donaldson, S. K.: Scalar curvature and projective embeddings. I. J. Differential Geom. 59 (2001), no. 3, 479–522.
[7] Donaldson, S. K.: Scalar curvature and stability of toric varieties. J. Differential Geom. 62 (2002), no. 2, 289–349.
[8] Ein, L., Lazarsfeld, R., Mustaţă, M., Nakamaye, M., Popa, M.: Asymptotic invariants of base loci. Ann. Inst. Fourier (Grenoble) 56 (2006), 1701–1734.
[9] Fekete, M.: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. Math. Z. 17 (1923), 228–249.
[10] Harada, M., Kaveh, K.: Integrable systems, toric degenerations and Okounkov bodies. preprint, arXiv:1205.5249.

[11] Howe, R.: Automatic continuity of concave functions. Proc. AMS 103 (1988), 1196–1200.

[12] Kaveh, K.: Crystal bases and Newton–Okounkov bodies. preprint, arXiv:1101.1687v1.

[13] Kaveh, K., Khovanskii, A.: Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. Annals of Mathematics 176 (2012), 1–54

[14] Kiritchenko, V., Smirnov, E., Timorin, V.: Schubert calculus and Gelfand–Zetlin polytopes. preprint, arXiv:1101.0278v2.

[15] Küronya, A., Lozovanu, V., Maclean, C.: Convex bodies appearing as Okounkov bodies of divisors, Advances in Mathematics 229 (2012), 2622–2639

[16] Küronya, A., Maclean, C., Szemberg, T.: Okounkov functions and Seshadri constants, in preparation.

[17] Lazarsfeld, R.: Positivity in Algebraic Geometry. I.-II. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48–49., Springer Verlag, Berlin, 2004.

[18] Lazarsfeld, R., Mustaţă, M.: Convex bodies associated to linear series, Ann. Scient. Éc. Norm. Sup., 4 série, t. 42. (2009), 783–835.

[19] Okounkov, A.: Brunn-Minkowski inequalities for multiplicities, Invent. Math. 125 (1996), 405–411.

[20] Rockafeller, R.T., Vol. 28 of Princeton Math. Series, Princeton Univ. Press, 1970

[21] Ross, J., Thomas, R.: An obstruction to the existence of constant scalar curvature Kähler metrics, J. Diff. Geom. 72 (2006), 429–466.

[22] Székelyhidi, G.: Filtrations and test configurations, preprint, arXiv:1111.4986.

[23] Witt-Nyström, D.: Transforming metrics on a line bundle to the Okounkov body. preprint, arXiv:0903.5167v1

[24] Witt-Nyström, D.: Test configurations and Okounkov bodies. preprint, arXiv:1001.3286v1

[25] Zaharjuta, V.: Transfinite diameter, Cebyshev constants and capacity for a compactum in $\mathbb{C}^n$. Mat. Sb. (N.S.) 96 (138) (1975), 374–389, 503.

Sébastien Boucksom, CNRS–Université Paris 6, Institut de Mathématiques, F-75251 Paris Cedex 05, France
E-mail address: boucksom@math.jussieu.fr

Alex Küronya, Budapest University of Technology and Economics, Mathematical Institute, Department of Algebra, Pf. 91, H-1521 Budapest, Hungary.
E-mail address: alex.kuronya@mathe.bme.hu

Current address: Alex Küronya, Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Eckerstraße 1, D-79104 Freiburg, Germany.

Catriona Maclean, Institut Fourier, CNRS UMR 5582 Université de Grenoble, 100 rue des Maths, F-38402 Saint-Martin d’Hères cedex, France
E-mail address: catriona.maclean@ujf-grenoble.fr

Tomasz Szemberg, Instytut Matematyki UP, Podchorążych 2, PL-30-084 Kraków, Poland.
E-mail address: tomasz.szemberg@uni-due.de