The Enclaveless Competition Game

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Abstract

For a subset $S$ of vertices in a graph $G$, a vertex $v \in S$ is an enclave of $S$ if $v$ and all of its neighbors are in $S$, where a neighbor of $v$ is a vertex adjacent to $v$. A set $S$ is enclaveless if it does not contain any enclaves. The enclaveless number $\Psi(G)$ of $G$ is the maximum cardinality of an enclaveless set in $G$. As first observed in 1997 by Slater [J. Res. Nat. Bur. Standards 82 (1977), 197–202], if $G$ is a graph with $n$ vertices, then $\gamma(G) + \Psi(G) = n$ where $\gamma(G)$ is the well-studied domination number of $G$. In this paper, we continue the study of the competition-enclaveless game introduced in 2001 by Philips and Slater [Graph Theory Notes N. Y. 41 (2001), 37–41] and defined as follows. Two players take turns in constructing a maximal enclaveless set $S$, where one player, Maximizer, tries to maximize $|S|$ and one player, Minimizer, tries to minimize $|S|$. The competition-enclaveless game number $\Psi^+_G(G)$ of $G$ is the number of vertices played when Maximizer starts the game and both players play optimally. We study among other problems the conjecture that if $G$ is an isolate-free graph of order $n$, then $\Psi^+_G(G) \geq \frac{1}{4}n$. We prove this conjecture for regular graphs and for claw-free graphs.

Keywords: competition-enclaveless game; domination game.

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1 Introduction

A neighbor of a vertex \( v \) in \( G \) is a vertex that is adjacent to \( v \). A vertex dominates itself and its neighbors. A dominating set of a graph \( G \) is a set \( S \) of vertices of \( G \) such that every vertex in \( G \) is dominated by a vertex in \( S \). The domination number of \( G \), denoted \( \gamma(G) \), is the minimum cardinality of a dominating set in \( G \), while the upper domination number of \( G \), denoted \( \Gamma(G) \), is the maximum cardinality of a minimal dominating set in \( G \). A minimal dominating set of cardinality \( \Gamma(G) \) we call a \( \Gamma \)-set of \( G \).

The open neighborhood of a vertex \( v \) in \( G \) is the set of neighbors of \( v \), denoted \( N_G(v) = \{ u \in V \mid uv \in E(G) \} \). The closed neighborhood of \( v \) is the set \( N_G[v] = \{ v \} \cup N_G(v) \). If the graph \( G \) is clear from context, we simply write \( N(v) \) and \( N[v] \) rather than \( N_G(v) \) and \( N_G[v] \), respectively.

As defined by Alan Goldman and introduced in Slater [22], for a subset \( S \) of vertices in a graph \( G \), a vertex \( v \in S \) is an enclave of \( S \) if it and all of its neighbors are also in \( S \); that is, if \( N[v] \subseteq S \). A set \( S \) is enclaveless if it does not contain any enclaves. We note that a set \( S \) is a dominating set of a graph \( G \) if the set \( V(G) \setminus S \) is enclaveless. The enclaveless number of \( G \), denoted \( \Psi(G) \), is the maximum cardinality of an enclaveless set in \( G \), and the lower enclaveless number of \( G \), denoted by \( \psi(G) \), is the minimum cardinality of a maximal enclaveless set. The domination and enclaveless numbers of a graph \( G \) are related by the following equations.

**Observation 1** If \( G \) is a graph of order \( n \), then \( \gamma(G) + \Psi(G) = n = \Gamma(G) + \psi(G) \).

The domination game on a graph \( G \) consists of two players, Dominator and Staller, who take turns choosing a vertex from \( G \). Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. Upon completion of the game, the set of chosen (played) vertices is a dominating set in \( G \). The goal of Dominator is to end the game with a minimum number of vertices chosen, while Staller has the opposite goal and wishes to end the game with as many vertices chosen as possible.

The Dominator-start domination game and the Staller-start domination game is the domination game when Dominator and Staller, respectively, choose the first vertex. We refer to these simply as the D-game and S-game, respectively. The D-game domination number, \( \gamma_g(G) \), of \( G \) is the minimum possible number of moves in a D-game when both players play optimally. The S-game domination number, \( \gamma'_g(G) \), of \( G \) is defined analogously for the S-game. The domination game was introduced by Brešar, Klavžar, and Rall [4] and has been subsequently extensively studied in the literature (see, for example, [2] [3] [11] [12] [13] [14] [20]).

Phelps and Slater [16] [17] introduced what they called the competition-enclaveless game. The game is played by two players, Maximizer and Minimizer, on some graph \( G \). They take turns in constructing a maximal enclaveless set \( S \) of \( G \). That is, in each turn a player plays a vertex \( v \) that is not in the set \( S \) of the vertices already chosen and such that \( S \cup \{ v \} \) does not contain an enclave, until there is no such vertex. We call such a vertex a playable vertex.
The goal of Maximizer is to make the final set $S$ as large as possible and for Minimizer to make the final set $S$ as small as possible.

The competition-enclaveless game number, or simply the enclaveless game number, $\Psi^+_g(G)$ of $G$ is the number of vertices chosen when Maximizer starts the game and both players play an optimal strategy according to the rules. The Minimizer-start competition-enclaveless game number, or simply the Minimizer-start enclaveless game number, $\Psi^-_g(G)$, of $G$ is the number of vertices chosen when Minimizer starts the game and both players play an optimal strategy according to the rules. The competition-enclaveless game, which has been studied for example in [9, 10, 16, 17, 19], has not yet been explored in as much depth as the domination game. In this paper we continue the study of the competition-enclaveless game. Our main motivation for our study are the following conjectures that have yet to be settled, where an isolate-free graph is a graph that does not contain an isolated vertex.

**Conjecture 1** If $G$ is an isolate-free graph of order $n$, then $\Psi^+_g(G) \geq \frac{1}{2} n$.

Conjecture 1 was first posed as a question by Slater [23] to the 2nd author on 8th May 2015, and subsequently posed as a conjecture in [10]. We refer to Conjecture 1 for general isolate-free graphs as the $\frac{1}{2}$-Enclaveless Game Conjecture. We also pose the following conjecture for the Minimizer-start enclaveless game, where $\delta(G)$ denotes the minimum degree of the graph $G$.

**Conjecture 2** If $G$ is a graph of order $n$ with $\delta(G) \geq 2$, then $\Psi^-_g(G) \geq \frac{1}{2} n$.

We proceed as follows. In Section 2, we discuss the domination game versus the enclaveless game, and show that these two games are very different and are not related. In Section 3, we present fundamental bounds on the enclaveless game number and the Minimizer-start enclaveless game number. In Sections 4 and 5, we show that the $\frac{1}{2}$-Enclaveless Game Conjecture holds for regular graphs and claw-free graphs, respectively. We use the standard notation $[k] = \{1, \ldots, k\}$.

### 2 Game domination versus enclaveless game

Although the domination and enclaveless numbers of a graph $G$ are related by the equation $\gamma(G) + \Psi(G) = n$ (see Observation 1), as remarked in [10] the competition-enclaveless game is very different to the domination game. For example, if $k \geq 3$ and $G$ is a tree with exactly two non-leaf vertices both of which have $k$ leaf neighbors, that is, if $G$ is a double star $S(k,k)$, then $\Psi^+_g(G) = \Psi^-_g(G) = k + 1$ and $\gamma^g(G) = 3$ and $\gamma'_g(G) = 4$. If $n \geq 1$, then Košmrlj [14] showed that $\gamma'_g(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and that $\gamma^g(P_n) = \left\lceil \frac{n}{2} \right\rceil - 1$ if $n \equiv 3 \pmod{4}$ and $\gamma^g(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$, otherwise. This is in contrast to the enclaveless game numbers of a path $P_n$ on $n \geq 2$ vertices determined by Phillips and Slater [17].

**Theorem 1** ([17]) If $n \geq 2$, then $\Psi^+_g(P_n) = \left\lceil \frac{3n+1}{5} \right\rceil$ and $\Psi^-_g(P_n) = \left\lfloor \frac{3n}{5} \right\rfloor$. 

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We remark that for the competition-enclaveless game the numbers $\Psi_g^+(G)$ and $\Psi_g^-(G)$ can vary greatly. For example, if $n \geq 1$ and $G$ is a star $K_{1,n}$, then $\Psi_g^+(G) = n$ while $\Psi_g^-(G) = 1$. However, for the domination game the Dominator-start game domination number and the Staller-start game domination number can differ by at most 1. The most significant difference between the domination game and the competition-enclaveless game is that the so-called Continuation Principle holds for the domination game but does not hold for the competition-enclaveless game.

Another significant difference between the domination game and the competition-enclaveless game is that upon completion of the domination game, the set of played vertices is a dominating set although not necessarily a minimal dominating set, while upon completion of the competition-enclaveless game, the set of played vertices is always a maximal enclaveless set. Thus, the enclaveless game numbers of a graph $G$ are always squeezed between the lower enclaveless number $\psi(G)$ of $G$ and the enclaveless number $\Psi(G)$ of $G$. We state this formally as follows.

**Observation 2** If $G$ is a graph of order $n$, then

$$\psi(G) \leq \Psi_g^-(G) \leq \Psi(G) \quad \text{and} \quad \psi(G) \leq \Psi_g^+(G) \leq \Psi(G).$$

A graph $G$ is *well-dominated* if all the minimal dominating sets of $G$ have the same cardinality. Examples of well-dominated graphs include, for example, the complete graph $K_n$, $C_7$, $P_{10}$, the corona of any graph, and the graph formed from two vertex disjoint cycles of order 5 joined by a single edge. Finbow, Hartnell and Nowakowski [7] characterized the well-dominated graphs having no 3-cycle nor 4-cycle. As observed earlier, upon completion of the enclaveless game, the set of played vertices is always a maximal enclaveless set. Hence, any sequence of legal moves by Maximizer and Minimizer (regardless of strategy) in the enclaveless game will always lead to the game played on a graph $G$ of order $n$ ending in $n - \gamma(G)$ moves. Thus as a consequence of Observation 2 we have the following interesting connection between the enclaveless game and the class of well-dominated graphs.

**Observation 3** If $G$ is a well-dominated graph of order $n$, then $\Psi_g^-(G) = \Psi_g^+(G) = n - \gamma(G)$.

It is well-known that if $G$ is an isolate-free graph of order $n$, then $\gamma(G) \leq \frac{1}{2}n$, implying by Observation 1 that $\Psi(G) = n - \gamma(G) \geq \frac{1}{2}n$. Hence one might think that $\gamma_g(G) \leq \Psi_g^+(G)$ for such a graph $G$ with no isolated vertex. We now provide an infinite class of graphs to show that the ratio $\gamma_g/\Psi_g^+$ of these two graphical invariants can be strictly larger than, and bounded away from, 1. The *corona* $\text{cor}(G)$ of a graph $G$, also denoted $G \circ K_1$ in the literature, is the graph obtained from $G$ by adding for each vertex $v$ of $G$ a new vertex $v'$ and the edge $vv'$ (and so, the vertex $v'$ has degree 1 in $\text{cor}(G)$). The edge $vv'$ is called a pendant edge.

**Theorem 2** If $n \geq 2$ is an integer and $\mathcal{G}_n$ denotes the class of all isolate-free graphs $G$ of order $n$, then

$$\sup_n \frac{\gamma_g(G)}{\Psi_g^+(G)} \geq \frac{11}{10}.$$
where the supremum is taken over all graphs $G \in \mathcal{G}_n$.

**Proof.** Let $n = 10q$ for some positive integer $q$ and $G_n$ be the corona of the path $P_n$. That is, the vertex set of $G_n$ is $X_n \cup Y_n$ where $X_n = \{x_i : i \in [n]\}$ and $Y_n = \{y_i : i \in [n]\}$. The edge set of $G_n$ is $\{x_ix_{i+1} : i \in [n-1]\} \cup \{x_iy_i : i \in [n]\}$. For each $k$ such that $0 \leq k \leq q-1$ we let $B_k$ be the subgraph of $G_n$ induced by $\cup_{i=1}^{10} \{x_{10k+i}, y_{10k+i}\}$. The D-game is played on $G_n$. At any point in this game we say that $B_i$ is open if no vertex in $B_i$ has been played by either player; otherwise we say $B_i$ is not open. By the Continuation Principle we may assume that any vertex played by Dominator belongs to $X_n$. We denote by $d_1, d_2, \ldots$ and $s_1, s_2, \ldots$ the sequence of moves played by Dominator and Staller in the domination game. We now provide a strategy for Staller to show that $\gamma_d(G_n) \geq 11q$.

(a) If Dominator plays $d_1 = x_i$ where $10k + 1 \leq i \leq 10k + 5$ for some $k$ such that $0 \leq k \leq q-1$, then Staller plays $s_1 = y_{10k+8}$.

(b) If Dominator plays $d_1 = x_i$ where $10k + 6 \leq i \leq 10k + 10$ for some $k$ such that $0 \leq k \leq q-1$, then Staller plays $s_1 = y_{10k+3}$.

If Dominator plays a vertex $d_j$ in an open $B_i$ for some $j \geq 1$ and $i$ with $0 \leq i \leq q-1$, then Staller plays $s_j$ also in $B_i$ as described in (a) and (b) above. On the other hand, suppose that Dominator plays a vertex $d_j$ in a $B_i$ that is not open. If this move of Dominator is his second move played in $B_i$, then in this case Staller plays the support vertex that is adjacent to the leaf she played earlier in the game when $B_i$ changed from being open to being not open. This support vertex was a legal move for Staller because of the structure of the graph $G_n$. When the game ends at least one vertex for each pair $x_k, y_k$ must have been played by one of the players. Furthermore, the above strategy for Staller shows that she can ensure that at least eleven vertices are played from each of $B_0, \ldots, B_{q-1}$. Therefore, $\gamma_d(G_n) \geq 11q$.

Now, observe that every minimal dominating set of $G_n$ has cardinality $n$, which implies by Observation 1 that every maximal enclaveless set of $G_n$ also has cardinality $n$; that is, $\psi(G) = \Psi(G) = n$ where we recall that $\psi(G)$ denotes the cardinality of the smallest maximal enclaveless set in $G$ and $\Psi(G)$ is the cardinality of a largest enclaveless set in $G$. Hence by Observation 2 $\Psi^+_g(G_n) = n$. Consequently, we have shown

$$\sup_n \frac{\gamma_d(G_n)}{\Psi^+_g(G_n)} \geq \frac{11}{10}.$$ 

The desired result follows noting that $G_n \in \mathcal{G}_{2n}$. □

### 3 Fundamental bounds

In this section, we establish some fundamental bounds on the (Maximizer-start) enclaveless game number and the Minimizer-start enclaveless game number. We establish next an upper bound on the enclaveless number of a graph in terms of the maximum degree and order of the graph.
**Proposition 1** If $G$ is an isolate-free graph of order $n$ with maximum degree $\Delta(G) = \Delta$, then
\[
\left(\frac{1}{\Delta + 1}\right) n \leq \psi(G) \leq \Psi(G) \leq \left(\frac{\Delta}{\Delta + 1}\right) n.
\]

**Proof.** If $G$ is any isolate-free graph of order $n$ and maximum degree $\Delta$, then $\gamma(G) \geq \frac{n}{\Delta + 1}$, with equality precisely when $G$ has a minimum dominating set consisting of vertices of degree $\Delta$ that is a 2-packing, where a 2-packing is a set $S$ of vertices that are pairwise at distance at least 3 apart. Hence, by Observation 1,
\[
\Psi(G) = n - \gamma(G) \leq n - \frac{n}{\Delta + 1} = \left(\frac{\Delta}{\Delta + 1}\right) n.
\]

On the other hand, let $D$ be a minimal dominating set of maximum cardinality, and so $|D| = \Gamma(G)$. Let $\overline{D} = V(G) \setminus D$, and so $|\overline{D}| = n - |D|$. Let $\ell$ be the number of edges between $D$ and $\overline{D}$. Since $D$ is a minimal dominating set, every vertex in $D$ has at least one neighbor in $\overline{D}$, and so $\ell \geq |D|$. Since $G$ has maximum degree $\Delta$, every vertex in $\overline{D}$ has at most $\Delta$ neighbors in $D$, and so $\ell \leq \Delta \cdot |\overline{D}| = \Delta(n - |D|)$. Hence, $|D| \leq \Delta(n - |D|)$, implying that $\Gamma(G) = |D| \leq \Delta n / (\Delta + 1)$. Thus by Observation 1,
\[
\psi(G) = n - \Gamma(G) \geq n - \left(\frac{\Delta}{\Delta + 1}\right) n = \left(\frac{1}{\Delta + 1}\right) n.
\]

This completes the proof of Proposition 1. $\blacksquare$

By Observation 2, the set of played vertices in either the Maximizer-start enclaveless game or the Minimizer-start enclaveless game is an enclaveless set of $G$. Thus as an immediate consequence of Proposition 1, we have the following result.

**Proposition 2** If $G$ is an isolate-free graph of order $n$ with maximum degree $\Delta(G) = \Delta$, then
\[
\left(\frac{1}{\Delta + 1}\right) n \leq \Psi_g^{-}(G) \leq \left(\frac{\Delta}{\Delta + 1}\right) n \quad \text{and} \quad \left(\frac{1}{\Delta + 1}\right) n \leq \Psi_g^{+}(G) \leq \left(\frac{\Delta}{\Delta + 1}\right) n.
\]

The lower bound in Proposition 2 on $\Psi_g^{-}(G)$ is achieved, for example, by taking $G = K_{1, \Delta}$ for any given $\Delta \geq 1$ in which case $\Psi_g^{-}(G) = 1 = \left(\frac{1}{\Delta + 1}\right) n$ where $n = n(G) = \Delta + 1$. We show next that the upper bounds in Proposition 2 are realized for infinitely many connected graphs.

**Proposition 3** There exist infinitely many positive integers $n$ along with a connected graph $G$ of order $n$ satisfying
\[
\Psi_g^{-}(G) = \Psi_g^{+}(G) = \left(\frac{\Delta(G)}{\Delta(G) + 1}\right) n.
\]
Proof. Let \( r \) be an integer such that \( r \geq 4 \) and let \( m \) be any positive integer. For each \( i \in [m] \), let \( H_i \) be a graph obtained from a complete graph of order \( r + 1 \) by removing the edge \( x_i y_i \) for two distinguished vertices \( x_i \) and \( y_i \). The graph \( F_m \) is obtained from the disjoint union of \( H_1, \ldots, H_m \) by adding the edges \( y_ix_{i+1} \) for each \( i \in [m] \) where the subscripts are computed modulo \( m \). The vertices \( x_i \) and \( y_i \) are called connectors in \( F_m \), and each of the \( r - 1 \) vertices in the set \( V(H_i) \setminus \{x_i, y_i\} \) is called a hidden vertex of \( H_i \). Note that \( F_m \) is \( r \)-regular and has order \( n = m(r + 1) \).

We first show that \( \Psi^-(F_m) = \left(\frac{r}{r+1}\right)n \). Suppose the Minimizer-start enclaveless game is played on \( F_m \). We provide a strategy for Maximizer that forces exactly \( rm \) vertices to be played. Maximizer’s strategy is to make sure that all the connector vertices in the graph are played. If he can accomplish this, then exactly \( rm \) vertices will be played when the game ends because of the structure of \( F_m \). Suppose that at some point in the game Minimizer plays a vertex from some \( H_j \). If one of the connector vertices, say \( x_j \), is playable, then Maximizer responds by playing \( x_j \). If both connector vertices have already been played and some hidden vertex, say \( w \), in \( H_j \) is playable, then Maximizer plays \( w \). If no vertex of \( H_j \) is playable, then Maximizer plays a connector vertex from \( H_i \) for some \( i \neq j \) if one is playable and otherwise plays any playable vertex. Since \( H_k \) contains at least 3 hidden vertices for each \( k \in [m] \), it follows that Maximizer can guarantee that all the connector vertices are played by following this strategy. This implies that for each \( i \in [m] \), exactly one hidden vertex of \( H_i \) is not played during the course of the game. That is, the set of played vertices has cardinality

\[
rm = \left(\frac{r}{r+1}\right)m(r + 1) = \left(\frac{\Delta(F_m)}{\Delta(F_m) + 1}\right)n,
\]

where we recall that \( \Delta(F_m) = r \). Thus,

\[
\Psi^-(F_m) \geq \left(\frac{\Delta(F_m)}{\Delta(F_m) + 1}\right)n.
\]

By Proposition 2,

\[
\Psi^-(F_m) \leq \left(\frac{\Delta(F_m)}{\Delta(F_m) + 1}\right)n.
\]

Consequently, \( \Psi^-(F_m) = \left(\frac{\Delta(F_m)}{\Delta(F_m) + 1}\right)n \).

If the Maximizer-start enclaveless game is played on \( F_m \), then the same strategy as above for Maximizer forces \( rm \) vertices to be played (even with the relaxed condition that \( r \) be an integer larger than 2). Thus as before, \( \Psi^+(F_m) = \left(\frac{\Delta(F_m)}{\Delta(F_m) + 1}\right)n \). \( \square \)

4 Regular graphs

In this section, we show that \( \frac{1}{2} \)-Enclaveless Game Conjecture (see Conjecture 1) holds for the class of regular graphs, as does Conjecture 2 for the Minimizer-start enclaveless game. For a set \( S \subset V(G) \) of vertices in a graph \( G \) and a vertex \( v \in S \), we define the \( S \)-external private
neighborhood of a vertex $v$, abbreviated $\text{epn}_G(v, S)$, as the set of all vertices outside $S$ that are adjacent to $v$ but to no other vertex of $S$; that is,

$$\text{epn}_G(v, S) = \{ w \in V(G) \setminus S \mid N_G(w) \cap S = \{v\} \}.$$  

We define an $S$-external private neighbor of $v$ to be a vertex in $\text{epn}_G(v, S)$.

**Theorem 3** If $G$ is a $k$-regular graph of order $n$, then $\Psi^+_g(G) \geq \frac{1}{2} n$ and $\Psi^-_g(G) \geq \frac{1}{2} n$.

**Proof.** Suppose the Maximizer-start enclaveless game is played on $G$. Let $S$ denote the set of all vertices played when the game ends. By definition of the game, the set $S$ is a maximal enclaveless set in $G$. By Observations 1 and 2, we have $|S| = \Psi^+_g(G) \geq \psi(G) = n - \Gamma(G)$. It therefore suffices to establish the proposition by proving that $\Gamma(G) \leq \frac{1}{2} n$.

This inequality is proved in [21], but we prove it here for the sake of completeness. Let $D$ be an arbitrary minimal dominating set of $G$. Denote by $D_1$ the set of vertices in $D$ that have a $D$-external private neighbor. That is, $D_1 = \{x \in D : \text{epn}_G(x, D) \neq \emptyset\}$. In addition, let $D_2 = D \setminus D_1$. Since $D$ is a minimal dominating set, the set $D_2$ consists of those vertices in $D$ that are isolated in the subgraph $G[D]$ of $G$ induced by $D$. Let

$$C_1 = \bigcup_{x \in D_1} \text{epn}_G(x, D) \quad \text{and} \quad C_2 = V(G) \setminus (D \cup C_1).$$

We note that by definition, there are no edges in $G$ joining a vertex in $D_2$ and a vertex in $C_1$. That is, each vertex in $D_2$ has $k$ neighbors in $C_2$. Since every vertex has degree $k$, each vertex of $C_2$ has at most $k$ neighbors in $D_2$. Denote by $\ell$ the number of edges of the form $uv$ where $u \in D_2$ and $v \in C_2$. It now follows that $|D_2| = \ell \leq k|C_2|$. That is, $|D_2| \leq |C_2|$. Now

$$|D| = |D_1| + |D_2| \leq |C_1| + |C_2| = n - |D|,$$

which shows that $\Gamma(G) \leq |D| \leq \frac{1}{2} n$. Similarly, $\Psi^-_g(G) \geq \psi(G) = n - \Gamma(G) \geq \frac{1}{2} n$. $\square$

We remark that the lower bound in Theorem 3 is achieved for $k = 1$ and $k = 2$ as shown by $K_2$ and $C_4$, respectively. However, it remains an open problem to characterize the graphs achieving equality in Theorem 3 for each value of $k \geq 1$.

A similar proof to that of Theorem 3 will establish the same lower bounds by restricting the minimum degree and forbidding induced stars of a certain size.

**Proposition 4** Let $k$ be a positive integer. If $G$ is a graph of order $n$ with minimum degree at least $k$ and with no induced $K_{1, k+1}$, then both $\Psi^+_g(G)$ and $\Psi^-_g(G)$ are at least $\frac{1}{2} n$.

**Proof.** Let $D$ be a minimal dominating set of $G$. The sets $D_1, D_2, C_1$ and $C_2$ as well as $\ell$ are defined as in the proof of Theorem 3. In this case we get $k|D_2| \leq \ell$ and $\ell \leq k|C_2|$. The first of these inequalities follows since $\delta(G) \geq k$ and the second inequality follows from the fact that $D_2$ is independent and the assumption that $G$ is $K_{1, k+1}$-free. Once again we conclude that $|D_2| \leq |C_2|$, and the result follows. $\square$
5 Claw-free graphs

A graph is *claw-free* if it does not contain the star $K_{1,3}$ as an induced subgraph. In this section, we show that the $\frac{1}{2}$-Enclaveless Game Conjecture (see Conjecture [1] holds for the class of claw-free graphs with no isolated vertex, as does Conjecture [2] for the Minimizer-start enclaveless game. For this purpose, we recall the definition of an irredundant set. For a set $S$ of vertices in a graph $G$ and a vertex $v \in S$, the $S$-private neighborhood of $v$ is the set

$$pn_G[v, S] = \{w \in V \mid N_G[w] \cap S = \{v\}\}.$$ 

If the graph $G$ is clear from context, we simply write $pn[v, S]$ rather than $pn_G[v, S]$. We note that if the vertex $v$ is isolated in $G[S]$, then $v \in pn[v, S]$. A vertex in the set $pn[v, S]$ is called an $S$-private neighbor of $v$. The set $S$ is an irredundant set if every vertex of $S$ has an $S$-private neighbor. The upper irredundance number $IR(G)$ is the maximum cardinality of an irredundant set in $G$.

The independence number $\alpha(G)$ of $G$ is the maximal cardinality of an independent set of vertices in $G$. An independent set of vertices of $G$ of cardinality $\alpha(G)$ we call an $\alpha$-set of $G$. Every maximum independent set in a graph is minimal dominating, and every minimal dominating set is irredundant. Hence we have the following inequality chain.

**Observation 4** ([4]) For every graph $G$, we have $\alpha(G) \leq \Gamma(G) \leq IR(G)$.

The inequality chain in Observation [4] is part of the canonical domination chain which was first observed by Cockayne, Hedetniemi, and Miller [4] in 1978. We shall need the following upper bounds on the independence number of a claw-free graph.

**Theorem 4** If $G$ is a connected claw-free graph of order $n$ and minimum degree $\delta \geq 1$, then the following holds.

(a) ([8, 18]) If $\delta = 1$, then $\alpha(G) \leq \frac{1}{2}(n + 1)$.

(b) ([5, 15]) If $\delta \geq 2$, then $\alpha(G) \leq \frac{2n}{\delta + 2}$.

In 2004, Favaron [6] established the following upper bound on the irredundance number of a claw-free graph.

**Theorem 5** ([6]) If $G$ is a connected, claw-free graph of order $n$, then $IR(G) \leq \frac{1}{2}(n + 1)$. Moreover, if $IR(G) = \frac{1}{2}(n + 1)$, then $\alpha(G) = \Gamma(G) = IR(G)$.

If $G$ is a connected, claw-free graph of order $n$ and minimum degree $\delta \geq 2$, then by Theorem [4](b) we have $\alpha(G) \leq \frac{1}{2}n$. In this case when $\delta \geq 2$, if $IR(G) = \frac{1}{2}(n + 1)$ holds, then by Theorem [5] we have $\alpha(G) = \frac{1}{2}(n + 1)$, a contradiction. Hence when $\delta \geq 2$, we must have $IR(G) \leq \frac{1}{2}n$. We state this formally as follows.
Corollary 1 \([6]\) If \(G\) is a connected, claw-free graph of order \(n\) and minimum degree at least 2, then \(\text{IR}(G) \leq \frac{1}{2}n\).

We are now in a position to prove the following result.

**Theorem 6** If \(G\) is a connected claw-free graph of order \(n\) and \(\delta(G) \geq 2\), then

\[
\Psi_g^+(G) \geq \frac{1}{2}n \quad \text{and} \quad \Psi_g^-(G) \geq \frac{1}{2}n.
\]

**Proof.** Suppose the Minimizer-start enclaveless game is played on \(G\). Let \(S\) denote the set of all vertices played when the game ends. By definition of the game, the set \(S\) is a maximal enclaveless set in \(G\). By Observations 1, 2 and 4 and Corollary 1, we have

\[
|S| = \Psi_g^-(G) \geq \psi(G) = n - \Gamma(G) \geq n - \text{IR}(G) \geq n - \frac{1}{2}n = \frac{1}{2}n,
\]

as desired. Similarly, \(\Psi_g^+(G) \geq \psi(G) \geq n - \text{IR}(G) \geq \frac{1}{2}n\). \(\square\)

By Theorem 6, we note that Conjecture 2 holds for connected claw-free graphs. In order to prove that Conjecture 1 holds for connected claw-free graphs, we need the characterization due to Favaron \([6]\) of the graphs achieving equality in the bound of Theorem 5. For this purpose, we recall that a vertex \(v\) of a graph \(G\) is a simplicial vertex if the neighborhood \(N_G(v)\) induces a complete subgraph of \(G\). A clique of a graph \(G\) is a maximal complete subgraph of \(G\). The clique graph of \(G\) has the set of cliques of \(G\) as its vertex set, and two vertices in the clique graph are adjacent if and only if they intersect as cliques of \(G\). A non-trivial tree is a tree of order at least 2.

Favaron \([6]\) defined the family \(\mathcal{F}\) of claw-free graphs \(G\) as follows. Let \(T_1, \ldots, T_q\) be \(q \geq 1\) non-trivial trees. Let \(L_i\) be the line graph of the corona \(\text{cor}(T_i)\) of the tree \(T_i\) for \(i \in [q]\). If \(q = 1\), let \(G = L_1\). If \(q \geq 2\), let \(G\) be the graph constructed from the line graphs \(L_1, L_2, \ldots, L_q\) by choosing \(q - 1\) pairs \(\{x_{ij}, x_{ji}\}\) such that the following holds.

- \(x_{ij}\) and \(x_{ji}\) are simplicial vertices of \(L_i\) and \(L_j\), respectively, where \(i \neq j\).
- The \(2(q - 1)\) vertices from the \(q - 1\) pairs \(\{x_{ij}, x_{ji}\}\) are all distinct vertices.
- Contracting each pair of vertices \(x_{ij}\) and \(x_{ji}\) into one common vertex \(c_{ij}\) results in a graph whose clique graph is a tree.

To illustrate the above construction of a graph \(G\) in the family \(\mathcal{F}\) consider, for example, such a construction when \(q = 3\) and the trees \(T_1, T_2, T_3\) are given in Figure 1.

We note that if \(G\) is an arbitrary graph of order \(n\) in the family \(\mathcal{F}\), then \(n \geq 3\) is odd and the vertex set of \(G\) can be partitioned into two sets \(A\) and \(B\) such that the following holds.

- \(|A| = \frac{1}{2}(n - 1)\) and \(|B| = \frac{1}{2}(n + 1)\).
- The set \(B\) is an independent set.
- Each vertex in \(A\) has exactly two neighbors in \(B\).
Figure 1: An illustration of the construction of a graph $G$ in the family $\mathcal{F}$

We refer to the partition $(A,B)$ as the partition associated with $G$. For the graph $G \in \mathcal{F}$ illustrated in Figure 1, the set $A$ consists of the darkened vertices and the set $B$ consists of the white vertices.

We are now in a position to state the characterization due to Favaron [6] of the graphs achieving equality in the bound of Theorem 5.

**Theorem 7** ([6]) If $G$ is a connected, claw-free graph of order $n \geq 3$, then $\text{IR}(G) \leq \frac{1}{2}(n+1)$, with equality if and only if $G \in \mathcal{F}$.

We prove next the following property of graphs in the family $\mathcal{F}$.

**Lemma 1** If $G \in \mathcal{F}$ and $(A,B)$ is the partition associated with $G$, then the set $B$ is the unique IR-set of $G$.

**Proof.** We proceed by induction on the order $n \geq 3$ of $G \in \mathcal{F}$. If $n = 3$, then $G = P_3$. In this case, the set $B$ consists of the two leaves of $G$, and the desired result is immediate. This establishes the base case. Suppose that $n \geq 5$ and that the result holds for all graphs $G' \in \mathcal{F}$ of order $n'$, where $3 \leq n' < n$. Let $Q$ be an IR-set of $G$.

By construction of the graph $G$, the set $B$ contains at least two vertices of degree 1 in $G$. Let $v$ be an arbitrary vertex in $B$ of degree 1 in $G$, and let $u$ be its neighbor. We note that $u \in A$. Let $G' = G - \{u,v\}$ and let $G'$ have order $n'$, and so $n' = n - 2$. Let $A' = A \setminus \{u\}$ and $B' = B \setminus \{v\}$. By construction of the graph $G$ and our choice of the vertex $v$, we note that $G' \in \mathcal{F}$ and that $(A',B')$ is the partition associated with $G'$. Applying the inductive hypothesis to $G'$, the set $B'$ is the unique IR-set of $G'$. Let $w$ be the second neighbor of $u$.
in $G$ that belongs to the set $B$, and so $N_G(u) \cap B = \{v, w\}$. By the structure of the graph $G \in \mathcal{F}$, we note that $N_G[w] \subset N_G[u]$ and that the subgraph of $G$ induced by $N_G[w]$ is a clique.

Suppose, to the contrary, that $Q \neq B$. Let $Q'$ be the restriction of $Q$ to the graph $G'$, and so $Q' = Q \cap V(G')$. Suppose that $u \in Q$. Since $Q$ is an irredundant set, this implies that $v \notin Q$. If $w \in Q$, then $pn[w, Q] = \emptyset$, contradicting the fact that $Q$ is an irredundant set. Hence, $w \notin Q$, and so $Q' \neq B'$. By the inductive hypothesis, the set $Q'$ is therefore not an IR-set of $G'$, and so $|Q'| < IR(G')$. Thus, $IR(G) = |Q| = |Q'| + 1 \leq (IR(G') - 1) + 1 = \frac{1}{2}(n' + 1) = \frac{1}{2}(n - 1) < IR(G)$, a contradiction. Hence, $u \notin Q$. In this case, $IR(G) = |Q| \leq |Q'| + 1 \leq IR(G') + 1 = \frac{1}{2}(n' + 1) + 1 = \frac{1}{2}(n + 1) = IR(G)$. Hence, we must have equality throughout this inequality chain. This implies that $u \in Q$ and $|Q'| = IR(G')$. By the inductive hypothesis, we therefore have $Q' = B'$. Hence, $Q = Q' \cup \{v\} = B' \cup \{v\} = B$. Thus, the set $B$ is the unique IR-set of $G$. □

**Corollary 2** If $G \in \mathcal{F}$ and $(A, B)$ is the partition associated with $G$, then the set $B$ is the unique $\alpha$-set of $G$ and the unique $\Gamma$-set of $G$.

**Proof.** By Theorem 5, $\alpha(G) = \Gamma(G) = IR(G) = \frac{1}{2}(n + 1)$. By Lemma 11, the set $B$ is the unique IR-set of $G$. Since every $\alpha$-set of $G$ is an IR-set of $G$ and $\alpha(G) = IR(G)$, this implies that $B$ is the unique $\alpha$-set of $G$. Since every $\Gamma$-set of $G$ is an IR-set of $G$ and $\Gamma(G) = IR(G)$, this implies that $B$ is the unique $\Gamma$-set of $G$. □

We show next that Conjecture 11 holds for connected claw-free graphs.

**Theorem 8** If $G$ is a connected, claw-free graph of order $n \geq 2$, then the following holds.

(a) $\Psi^+(G) \geq \frac{1}{2}n$.

(b) If $G \neq P_3$, then $\Psi^-(G) \geq \frac{1}{2}n$.

**Proof.** Let $G$ be a connected, claw-free graph of order $n \geq 2$. Suppose the Maximizer-start enclaveless game is played on $G$. Let $S$ denote the set of all vertices played when the game ends. By definition of the game, the set $S$ is a maximal enclaveless set in $G$. If $IR(G) \leq \frac{1}{2}n$, then analogously as in the proof of Theorem 5 we have $|S| = \Psi^+(G) \geq \psi(G) \geq n - IR(G) \geq \frac{1}{2}n$. Hence, we may assume that $IR(G) > \frac{1}{2}n$, for otherwise the desired result follows. By Theorem 7, $IR(G) = \frac{1}{2}(n + 1)$ and $G \in \mathcal{F}$. Let $(A, B)$ be the partition associated with $G$. We show in this case we have $\Psi^+_g(G) > \psi(G)$.

By Observation 11, $\Gamma(G) + \psi(G) = n$. Moreover, the complement of every $\Gamma$-set of $G$ is a maximal enclaveless set, and the complement of every $\psi$-set of $G$ is a minimal dominating set. By Corollary 2, the set $B$ is the unique $\Gamma$-set of $G$. These observations imply that the complement of the set $B$, namely the set $A$, is the unique $\psi$-set of $G$. Thus every maximal enclaveless set of $G$ of cardinality $\psi(G)$ is precisely the set $A$.

We now return to the Maximizer-start enclaveless game played on $G$. If Maximizer plays as his first move any vertex from the set $B$ and thereafter both players play optimally, then
the resulting set $S^*$ of moves played during the course of the game contain a vertex of $B$ and is therefore different from the set $A$. Since the set $A$ is the unique $\psi$-set of $G$, this implies that $|S^*| > \psi(G)$. We therefore have that the following inequality chain, where the first inequality, namely $Ψ^+(G) ≥ |S^*| ≥ ψ(G) + 1$, is due to the fact that the first move of Maximizer from the set $B$ may not be an optimal move.

$$Ψ^+(G) ≥ |S^*| ≥ ψ(G) + 1 = (n - \Gamma(G)) + 1 = n - \frac{1}{2}(n + 1) + 1 = \frac{1}{2}(n + 1).$$

This shows that $Ψ^+(G) ≥ \frac{1}{2}n$, as desired. Suppose next that $G \neq P_3$ and the Minimizer-start enclaveless game is played on $G$. Let $S$ denote the set of all vertices played when the game ends. By definition of the game, the set $S$ is a maximal enclaveless set in $G$. If $\text{IR}(G) ≤ \frac{1}{2}n$, then analogously as before we have $|S| = Ψ^-(G) ≥ ψ(G) ≥ n - \text{IR}(G) ≥ \frac{1}{2}n$. Hence, we may assume that $\text{IR}(G) > \frac{1}{2}n$, for otherwise the desired result follows. By Theorem 7, $\text{IR}(G) = \frac{1}{2}(n + 1)$ and $G \in F$. Let $(A, B)$ be the partition associated with $G$.

We show in this case we have $Ψ^-(G) > \psi(G)$. Since $G \neq P_3$, we note that there are at least two vertices in the set $B$ at distance at least 3 apart in $G$. Thus, whatever the first move is played by Minimizer, Maximizer can always respond by playing as his first move a vertex chosen from the set $B$. Thus, analogously as before, the resulting set of played vertices in the game is different from the set $A$. Recall that upon completion of the game the resulting set is a maximal enclaveless set. Therefore, Maximizer has a strategy to finish the game in at least $\psi(G) + 1$ moves, implying that $Ψ^-(G) ≥ \frac{1}{2}(n + 1)$. $\blacksquare$

By Theorem 8(a), we note that Conjecture 1 holds for connected claw-free graphs. Moreover by Theorem 8(b), we note that Conjecture 2 holds for connected claw-free graphs even if we relax the minimum degree two condition and replace it with the requirement that the graph is isolate-free and different from the path $P_3$.

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