DISTRIBUTIONS OF POLYNOMIALS IN GAUSSIAN RANDOM VARIABLES UNDER STRUCTURAL CONSTRAINTS

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Abstract. We study the regularity of densities of distributions that are polynomial images of the standard Gaussian measure on \( \mathbb{R}^n \). We assume that the degree of a polynomial is fixed and that each variable enters to a power bounded by another fixed number.

Keywords: Distribution of a polynomial, Distribution density, Kantorovich distance, Fortet–Mourier distance, Total variation distance

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1. Introduction

In this note we study the regularity properties of distributional densities of random variables

\[
f(X) := \sum_{j_1=0}^{m} \cdots \sum_{j_n=0}^{m} a_{j_1, \ldots, j_n} X_1^{j_1} \cdots X_n^{j_n},
\]

where \( X_1, \ldots, X_n \) are i.i.d. \( \mathcal{N}(0,1) \) random variables. The regularity properties of general polynomial images of measures of different classes have been extensively studied in recent years (see [7], [9], [10], [11], [14], and [15]). This research has been motivated by the paper [17] about the connection between the total variation and Kantorovich–Rubinstein (Fortet–Mourier) distances on the set of distributions of polynomials (see also [1], [2], [3], [4], [12], and [18]). It turns out that the regularity of distributions plays a crucial role in estimates between these two distances (see [11], [14], and [15]). In particular, the following theorem was proved in [14].

The distribution density \( \varrho_g \) of any non-constant random variable of the form

\[
g(X) := \sum_{j_1+\ldots+j_n \leq d} a_{j_1, \ldots, j_n} X_1^{j_1} \cdots X_n^{j_n},
\]

where \( X := (X_1, \ldots, X_n) \) is a Gaussian random vector, belongs to the Nikolskii–Besov space \( B_{1,\infty}^{1/d} \). Moreover, for any fixed \( d \in \mathbb{N} \) there is a constant \( C(d) \) such that

\[
\omega(\varrho_g, \varepsilon) := \sup_{|h| \leq \varepsilon} \int_{\mathbb{R}} |\varrho_g(s + h) - \varrho_g(s)| \, ds \leq C(d)[\mathbb{D}g(X)]^{-1/2d} \varepsilon^{1/d}
\]

for any random variable \( g(X) \) of the form (1.2), where \( \mathbb{D}g(X) := \mathbb{E}(g(X) - \mathbb{E}g(X))^2 \).

We recall (see [5], [19]) that the Nikolskii–Besov space \( B_{1,\infty}^\alpha \), with \( \alpha \in (0,1) \), consists of all functions \( \varrho \in L^1(\mathbb{R}) \) such that there is a constant \( C > 0 \) for which

\[
\int_{\mathbb{R}} |\varrho(t + h) - \varrho(t)| \, dt \leq C |h|^\alpha \quad \forall h \in \mathbb{R}.
\]

In this note we study bounds for the modulus of continuity \( \omega(\varrho_f, \cdot) \) for polynomials \( f \) of the form (1.1), i.e., we consider polynomial images of the standard Gaussian measure on \( \mathbb{R}^n \) when the polynomial \( f \) is of degree \( d \) and satisfies the following additional constraint on its structure: each variable enters (1.1) only to a power not greater than some fixed number \( m \leq d \).

Our main result is as follows.
Theorem 1.1. Let \( m, d \in \mathbb{N}, d \geq m \), let \( X := (X_1, \ldots, X_n) \) be the standard Gaussian \( n \)-dimensional random vector with independent coordinates. Then there is a constant \( C(m, d) \) depending only on \( d \) and \( m \), such that, for any non-constant polynomial

\[
f(x) := \sum_{j_1=0}^{m} \cdots \sum_{j_n=0}^{m} a_{j_1, \ldots, j_m} x_1^{j_1} \cdots x_n^{j_n}
\]

with the degree \( \deg[f] := \max\{j_1 + \ldots + j_n : a_{j_1, \ldots, j_m} \neq 0\} \leq d \), one has

\[
\omega(g_f, \varepsilon) \leq C(m, d)(\varepsilon/a[f])^{1/m} [\ln(\varepsilon/a[f])]^{d-m} + 1
\]

where \( a[f] := \max_{j_1+\ldots+j_n=d[f]} |a_{j_1, \ldots, j_n}| \) and where \( g_f \) is the density of the random variable \( f(X) \).

This paper has been partially motivated by the papers [13] and [20], where some bounds for the characteristic functions of random variables of type (1.1) are obtained, including the following estimate (see [13, Theorem 5]):

\[
|\mathbb{E}\exp(itf(X))| \leq C(n, m, d[f])|a[f]|^t |t|^{-1/m} \ln(2 + |a[f]| t)^\alpha
\]

where \( \alpha = \frac{1}{2}(3n - \frac{d[f]}{m}) - 1 \), \( d[f] := \max\{j_1 + \ldots + j_n : a_{j_1, \ldots, j_m} \neq 0\} \), \( a[f] := \max_{j_1+\ldots+j_n=d[f]} |a_{j_1, \ldots, j_n}| \).

As a corollary of Theorem 1.1 we deduce a somewhat sharper bound for the characteristic function, independent of the number of variables \( n \), with \( \alpha = d[f] - m \).

2. Definitions, notation, and known results

In this section we introduce the definitions and notation used throughout the paper. We also formulate several known results which will be important in the proof of the main result.

Definition 2.1. For a function \( \varphi \in L^1(\mathbb{R}) \) and \( \varepsilon > 0 \), we set

\[
\omega(\varphi, \varepsilon) := \sup_{|h| \leq \varepsilon} \int_{\mathbb{R}} |\varphi(s + h) - \varphi(s)| \, ds.
\]

This is the usual \( L^1 \) modulus of continuity.

We use the notation

\[
\|\varphi\|_\infty := \sup_{x \in \mathbb{R}} |\varphi(x)|
\]

for a function \( \varphi \in C^\infty_0(\mathbb{R}) \), where \( C^\infty_0(\mathbb{R}) \) is the class of all infinitely differentiable compactly supported functions.

In [16] (see also [15]), the following modulus of continuity was introduced.

Definition 2.2. For a function \( \varphi \in L^1(\mathbb{R}) \) and \( \varepsilon > 0 \), we set

\[
\sigma(\varphi, \varepsilon) := \sup \left\{ \int \varphi'(s) \varphi(s) \, ds : \|\varphi\|_\infty \leq \varepsilon, \|\varphi'\|_\infty \leq 1 \right\},
\]

where the supremum is taken over all functions \( \varphi \in C^\infty_0(\mathbb{R}) \).

We note that \( \sigma(\varphi, \cdot) \) is a monotone concave modulus of continuity (see [16, Lemma 2.1]).

According to [16, Theorem 2.1], one has the following equivalence of these two moduli of continuity.

Proposition 2.3. For any function \( \varphi \in L^1(\mathbb{R}) \), we have

\[
2^{-1} \omega(\varphi, 2\varepsilon) \leq \sigma(\varphi, \varepsilon) \leq 6 \omega(\varphi, \varepsilon).
\]
We note that for the density \( \varrho \) of a random variable \( W \) the modulus of continuity \( \sigma(\varrho, \cdot) \) is calculated as follows:

\[
\sigma(\varrho, \varepsilon) := \sup \left\{ \mathbb{E} \varphi'(W) : \|\varphi\|_\infty \leq \varepsilon, \|\varphi'\|_\infty \leq 1 \right\},
\]

where the supremum is taken over all functions \( \varphi \in C_0^\infty(\mathbb{R}) \).

In [16, Corollary 2.2], the following result is also obtained.

**Proposition 2.4.** Assume that the distribution of some random variable \( W \) is absolutely continuous and has density \( \varrho \). Then

\[
P(W \in A) \leq \sigma(\varrho, \lambda(A))
\]

for each Borel set \( A \subset \mathbb{R} \). Here \( \lambda \) denotes the standard Lebesgue measure on \( \mathbb{R} \).

We will need the following theorem from [14] (see also [11] and [15]).

**Theorem 2.5.** Let \( d \in \mathbb{N} \). Then there is a constant \( C(d) \) depending only on \( d \) such that, for any polynomial \( g \) of degree at most \( d \), any Gaussian random vector \( X \), and any function \( \varphi \in C_0^\infty(\mathbb{R}) \), one has

\[
[Dg(X)]^{1/2d} \mathbb{E} \varphi'(g(X)) \leq C(d) \|\varphi\|_{\infty}^{1/d} \|\varphi'\|_{\infty}^{1-1/d}
\]

where \( Dg(X) \) is the variance of \( g(X) \). In particular,

\[
\sigma(\varrho_g, \varepsilon) \leq C(d)[Dg(X)]^{-1/2d} \varepsilon^{1/d}.
\]

3. **Proof of Theorem 1.1**

We first note that by Proposition 2.3 we can work with \( \sigma(\varrho_f, \cdot) \) in place of \( \omega(\varrho_f, \cdot) \). Since \( \sigma(\varrho_{af}, t) = \sigma(\varrho_f, t/\alpha) \), it is sufficient to prove the bound for polynomials \( f \) with \( a[f] = 1 \).

We now assume that \( a[f] = 1 \). In this case the proof will be done by induction on \( d \). We note that the number \( a[f] \) is a coefficient of some monomial \( x_1^{j_1} \cdots x_n^{j_n} \).

Let us assume that \( d = m \). Due to the equivalence of the \( L^2 \)-norm and the Sobolev norm of the Gaussian Sobolev space \( W^{2,d} \) on the space of all polynomials of degree not greater than \( d \) (see [6, Corollary 5.5.5 and Theorem 5.7.2]) one has

\[
[Df(X)]^{1/2} \geq c_1(d) \left[ \mathbb{E} \left| \frac{\partial^{j_1} \cdots \partial^{j_n} f}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(X) \right|^2 \right]^{1/2} = c_1(d) \cdot j_1! \cdots j_n! \cdot a[f] \geq c_1(d).
\]

By Theorem 2.4 we have

\[
\sigma(\gamma_n \circ f^{-1}, t) \leq C(d)[c_1(d)]^{-1/dt^{1/d}}.
\]

Thus, the base case \( d = m \) of induction is proved.

We now make the inductive step. Let \( Z \) be the standard Gaussian random variable independent of the random vector \( X \). We note that for any \( \varepsilon > 0 \) one has

\[
\mathbb{E} \varphi'(f(X)) = \mathbb{E} [\varphi'(f(X)) - \varphi'(f(X) + \varepsilon Z)] + \mathbb{E} \varphi'(f(X) + \varepsilon Z).
\]

For the first term we have

\[
\mathbb{E} [\varphi'(f(X)) - \varphi'(f(X) + \varepsilon Z)] \leq \|\varphi'\|_\infty \mathbb{E} Z \int_\mathbb{R} |\varrho_f(t) - \varrho_f(t - \varepsilon Z)| dt \
\leq 2\|\varphi'\|_\infty \mathbb{E} Z \sigma(\varrho_f, \varepsilon |Z|) \leq 2\|\varphi'\|_\infty \sigma(\varrho_f, \varepsilon, \mathbb{E} |Z|) \leq 2\|\varphi'\|_\infty \sigma(\varrho_f, \varepsilon),
\]

where we have used Proposition 2.3, the concavity and the monotonicity of the function \( \sigma(\varrho_f, \cdot) \).
We now estimate the second term. As we have already mentioned, the number \( a[f] \) is a coefficient of some monomial \( x_1^{j_1} \cdots x_n^{j_n} \). Without loss of generality we can assume that \( j_n \neq 0 \). Consider the polynomial \( f \) as a polynomial of the \( n \)-th variable \( x_n \):

\[
f(x_1, \ldots, x_{n-1}, x_n) = \sum_{j=0}^{m} f_j(x_1, \ldots, x_{n-1}, x_n).
\]

We apply Theorem 2.1 to the random variable \( f(x_1, \ldots, x_{n-1}, X_n, Z) \), which gives the bound

\[
\mathbb{E} \varphi'(f(X) + \varepsilon Z) = \mathbb{E}_{X_1, \ldots, X_{n-1}} \mathbb{E}_{X_n, Z} \varphi'(f(X_1, \ldots, X_{n-1}, X_n) + \varepsilon Z) \\
\leq C(m)\|\varphi\|_{\infty}^{1/m} \|\varphi'\|_{\infty}^{1-1/m} \mathbb{E}_{X_1, \ldots, X_{n-1}} (\mathbb{D}_{X_n} f(X_1, \ldots, X_{n-1}, X_n) + \varepsilon)^{-1/2m}.
\]

We recall that for any polynomial \( g(s) = \sum_{j=0}^{m} a_j s^j \), by the Hermite polynomial expansion, one has

\[
\mathbb{D} g(X_n) \geq \frac{1}{m} \mathbb{E} |g'(X_n)|^2 \geq c_2(m) \max_{1 \leq j \leq m} |a_j|^2,
\]

where the last inequality follows from the equivalence of any two norms on a finite-dimensional space. Without loss of generality we assume that \( c_2(m) \leq 1 \). Thus,

\[
\mathbb{E} \varphi'(f(X) + \varepsilon Z) \leq C(m)\|\varphi\|_{\infty}^{1/m} \|\varphi'\|_{\infty}^{1-1/m} \mathbb{E}_{X_1, \ldots, X_{n-1}} (c_2(m)^2 |f_{j_0}^0(X_1, \ldots, X_{n-1})|^2 + \varepsilon^2)^{-1/2m}
\]

\[
\leq C(m)c_2(m)^{-1/m} \|\varphi\|_{\infty}^{1/m} \|\varphi'\|_{\infty}^{1-1/m} \mathbb{E}_{X_1, \ldots, X_{n-1}} (|f_{j_0}^0(X_1, \ldots, X_{n-1})|^2 + \varepsilon^2)^{-1/2m}.
\]

Moreover, one has

\[
\mathbb{E}_{X_1, \ldots, X_{n-1}} (|f_{j_0}^0(X_1, \ldots, X_{n-1})|^2 + \varepsilon^2)^{-1/2m}
\]

\[
= \int_{0}^{\epsilon^{-1/m}} P\left(\left|f_{j_0}^0(X_1, \ldots, X_{n-1})\right|^2 + \varepsilon^2\right)^{-1/2m} \leq \frac{1}{m} \int_{0}^{\epsilon^{-1/m}} \int_{0}^{\infty} \frac{s}{(s^2 + \varepsilon^2)^{1+1/m}} P\left(\left|f_{j_0}^0(X_1, \ldots, X_{n-1})\right| \leq s\right) ds \leq \frac{8}{m} \int_{0}^{\epsilon^{-1/m}} \int_{0}^{\infty} \frac{s}{(s + \varepsilon)^{2+1/m}} \sigma(\varphi_{f_{j_0}^0}, s) ds \leq \frac{2}{m} \int_{0}^{\epsilon^{-1/m}} \int_{0}^{\infty} \frac{s}{(s + \varepsilon)^{2+1/m}} \sigma(\varphi_{f_{j_0}^0}, s) ds.
\]

We now note that \( d[f_{j_0}^0] \leq d - 1 \) and \( a[f_{j_0}^0] = a[f] = 1 \). Thus, by the inductive hypothesis, one has

\[
\sigma(\varphi_{f_{j_0}^0}, s) \leq C(m, d - 1)(s^t)^{1/m} \left[\frac{\ln s t \left|d_{1-m} - 1\right|}{t + 1}\right] \leq 2^{d_{1-m}} C(m, d - 1)(s^t)^{1/m} \left[\frac{\ln s t \left|d_{1-m} - 1\right|}{t + 1}\right],
\]

which implies that

\[
\varepsilon^{-1/m} \int_{0}^{\infty} \frac{t}{(t + 1)^{2+1/m}} \sigma(\varphi_{f_{j_0}^0}, s) dt \leq \varepsilon^{-1/m} \int_{0}^{\infty} \frac{1}{(t + 1)^{2+1/m}} \sigma(\varphi_{f_{j_0}^0}, s) dt \leq 2^{d_{1-m}} C(m, d - 1) \int_{0}^{\infty} \frac{\ln s t \left|d_{1-m} - 1\right|}{t + 1} dt + \varepsilon^{-1/m} \int_{0}^{\infty} \frac{1}{(t + 1)^{2+1/m}} dt \leq 2^{d_{1-m}} C(m, d - 1) \int_{0}^{\infty} \frac{\ln s t \left|d_{1-m} - 1\right|}{t + 1} dt + (1 + \epsilon)^{-1/m}.
\]
We now assume that \( \varepsilon \in (0, e^{-1}) \). In this case
\[
\int_0^{e^{-1}} \frac{[|\ln t|^{d-1-m} + |\ln \varepsilon|^{d-1-m} + 1]}{t + 1} dt
= \int_0^1 \frac{[|\ln t|^{d-1-m}]}{t + 1} dt + \int_1^{e^{-1}} \frac{[|\ln t|^{d-1-m}]}{t + 1} dt + [|\ln \varepsilon|^{d-1-m} + 1] \ln(1 + \varepsilon^{-1})
\leq c(d, m) + |\ln \varepsilon|^{d-1-m} \ln(1 + \varepsilon^{-1}) + [|\ln \varepsilon|^{d-1-m} + 1] \ln(1 + \varepsilon^{-1}) \leq 10|\ln \varepsilon|^{d-m} + c(d, m).
\]
Thus, for \( \varepsilon \in (0, e^{-1}) \) we have
\[
\mathbb{E} \varphi'(f(X) + \varepsilon Z) \leq C_1(m, d) \|\varphi\|^{1/m}_{\infty} \|\varphi'\|^{1-1/m}_{\infty} [|\ln \varepsilon|^{d-m} + 1]
\]
and
\[
\mathbb{E} \varphi'(f(X)) \leq 2\|\varphi\|_{\infty} \sigma(\gamma_n \circ f^{-1}, \varepsilon) + C_1(m, d) \|\varphi\|^{1/m}_{\infty} \|\varphi'\|^{1-1/m}_{\infty} [|\ln \varepsilon|^{d-m} + 1],
\]
which implies that
\[
\sigma(\varrho_f, t) \leq 2\sigma(\varrho_f, \varepsilon) + C_1(m, d) t^{1/m} [|\ln \varepsilon|^{d-m} + 1]
\]
for any \( t > 0 \) and \( \varepsilon \in (0, e^{-1}) \). We first consider the case \( t \in (0, e^{-1}) \). In this case,
\[
\sigma(\varrho_f, t) = \sum_{k=0}^{\infty} 2^k \left[ \sigma(\varrho_f, t2^{-2d(k+1)}) - 2\sigma(\varrho_f, t2^{-2d(k+1)}) \right],
\]
since, by Theorem 2.5, one has
\[
2^{k+1} \sigma(\varrho_f, t2^{-2d(k+1)}) \leq C(d)[\mathbb{D} f(X)]^{-1/2d} t^{1/2d} t^{-k-1} \rightarrow 0.
\]
Hence
\[
\sigma(\varrho_f, t) \leq C_1(m, d) t^{1/m} \sum_{k=0}^{\infty} 2^{k(1-2d/m)} [\ln t2^{-2d(k+1)}]^{d-m} + 1.
\]
Note that
\[
\sum_{k=0}^{\infty} 2^{k(1-2d/m)} [\ln t2^{-2d(k+1)}]^{d-m} + 1
\leq 2^{d-m} [\ln t]^{d-m} + 1 \sum_{k=0}^{\infty} 2^{k(1-2d/m)} + 4^{d-m} d^{d-m} \sum_{k=0}^{\infty} (k+1)^{d-m} 2^{k(1-2d/m)}.
\]
Since \( 1 - 2d/m \leq -d/m \leq -1 \), both series above converge. Therefore,
\[
\sigma(\varrho_f, t) \leq C(m, d) t^{1/m} [\ln t]^{d-m} + 1
\]
for \( t \in (0, e^{-1}) \).

For \( t \geq e^{-1} \) we have
\[
\sigma(\varrho_f, t) \leq 1 \leq e^{1/m} t^{1/m} [\ln t]^{d-m} + 1.
\]
The theorem is proved.

4. Applications

In this section we discuss two applications of the obtained result.

Firstly, we apply Theorem 4.11 to obtain bounds for characteristic functions.
Corollary 4.1. Let $m, d \in \mathbb{N}$, $d \geq m$, let $X := (X_1, \ldots, X_n)$ be the standard Gaussian $n$-dimensional random vector with independent coordinates. Then there is a constant $C(m, d)$, depending only on $d$ and $m$, such that, for any non-constant polynomial

$$f(x) := \sum_{j_1=0}^m \cdots \sum_{j_n=0}^m a_{j_1, \ldots, j_n} x_1^{j_1} \cdots x_n^{j_n}$$

of degree $d[f] := \max\{j_1 + \ldots + j_n : a_{j_1, \ldots, j_n} \neq 0\} \leq d$, one has

$$|\mathbb{E} \exp\{itf(X)\}| \leq C(n, m)|a[f] \cdot t|^{-1/m}\left[|\ln |a[f] \cdot t|\right]^{d-m} + 1$$

where $a[f] := \max_{j_1 + \ldots + j_n = d[f]} |a_{j_1, \ldots, j_n}|$.

Proof. As we have already proved, one has

$$\sigma(\varphi, \varepsilon) \leq C(m, d)(\varepsilon/a[f])^{1/m}\left[|\ln(\varepsilon/a[f])|\right]^{d-m} + 1$$

which means that for any function $\varphi \in C_0^\infty(\mathbb{R})$ with $||\varphi'||_\infty \leq 1$ and $||\varphi||_\infty \leq |t|^{-1}$ one has

$$\mathbb{E} \varphi'(f(X)) \leq C(m, d)|a[f] \cdot t|^{-1/m}\left[|\ln |a[f] \cdot t|\right]^{d-m} + 1].$$

We now take $\varphi(s) = \pm t^{-1}\cos(ts)$ and $\varphi(s) = \pm t^{-1}\sin(ts)$ and get the announced bound. \hfill \Box

Secondly, we use Theorem [L.1] to obtain bounds between the total variation and the Kantorovich–Rubinstein distances.

Let $X, Y$ be two random variables. The total variation distance is defined by the equality

$$d_{TV}(X, Y) := \sup\left\{\mathbb{E}[\varphi(X) - \varphi(Y)] : \varphi \in C_0^\infty(\mathbb{R}), \ ||\varphi||_\infty \leq 1\right\}.$$

The Kantorovich–Rubinstein distance is defined by the formula

$$d_{KR}(X, Y) := \sup\left\{\mathbb{E}[\varphi(X) - \varphi(Y)] : \varphi \in C_0^\infty(\mathbb{R}), \ ||\varphi||_\infty \leq 1, \ ||\varphi'||_\infty \leq 1\right\}.$$

We recall (see [8]) that convergence in Kantorovich–Rubinstein distance is equivalent to convergence in distribution (weak convergence of distributions).

In [15, Lemma 3.1], the following bound is proved.

Proposition 4.2. Let $X$ and $Y$ be random variables. Then for any $\varepsilon \in (0, 1)$ one has

$$d_{TV}(X, Y) \leq 6 \max\{\sigma(\varphi_X, \varepsilon), \sigma(\varphi_Y, \varepsilon)\} + \varepsilon^{-1}d_{KR}(X, Y)$$

where $\varphi_X$ and $\varphi_Y$ are distribution densities of $X$ and $Y$, respectively.

Corollary 4.3. Let $d, m \in \mathbb{N}$ and let $a \in \mathbb{R}$ be a positive number. Let $f$ and $g$ be two polynomials of the form (1.1) and let $X := (X_1, \ldots, X_n)$ be the standard Gaussian $n$-dimensional random vector with independent coordinates. Assume that $d[f] \leq d$, $d[g] \leq d$, $a[f] \geq a$, and $a[g] \geq a$.

Then

$$d_{TV}(f(X), g(X)) \leq C(m, d, a)[d_{KR}(f(X), g(X))^{(d-m)m} + 1].$$

Proof. We have

$$\sigma(\varphi_f, \varepsilon) \leq C(m, d)(\varepsilon/a[f])^{1/m}\left[|\ln(\varepsilon/a[f])|\right]^{d-m} + 1] \leq C_1(m, d, a)\varepsilon^{1/m}\left[|\ln \varepsilon|^{d-m} + 1\right]$$

and the same bound is true for $\sigma(\varphi_g, \cdot)$. Thus, by Proposition 1.2 one has

$$d_{TV}(f(X), g(X)) \leq 6C_1(m, d, a)\varepsilon^{1/m}\left[|\ln \varepsilon|^{d-m} + 1\right] + \varepsilon^{-1}d_{KR}(f(X), g(X))$$

for any $\varepsilon \in (0, 1)$. Since $d_{KR}(f(X), g(X)) \leq 2$, we can take

$$\varepsilon = \left[\frac{1}{2}d_{KR}(f(X), g(X))^{m+1}\left[|\ln \left(\frac{1}{2}d_{KR}(f(X), g(X))\right)|^{(m-d)m/m+1}\right]\right]^{1/(m+1)}$$

which implies the announced bound. \hfill \Box
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REFERENCES

[1] Bally, V. and Caramellino, L. (2014). On the distances between probability density functions. *Electron. J. Probab.* 19, 1–33.

[2] Bally, V. and Caramellino, L. (2017). Convergence and regularity of probability laws by using an interpolation method. *Ann. Probab.* 45, 1110–1159.

[3] Bally, V. and Caramellino, L. (2019). Total variation distance between stochastic polynomials and invariance principles. *Ann. Probab.* 47, 3762–3811.

[4] Bally, V., Caramellino, L., Poly, G. (2020). Regularization lemmas and convergence in total variation. *Electron. J. Probab.* 25, 1–20.

[5] Besov, O.V., Il’in, V.P. and Nikolskii, S. M. (1978, 1979). *Integral representations of functions and imbedding theorems. V. I, II.* Washington: Winston & Sons.

[6] Bogachev, V.I. (1998). *Gaussian Measures. Mathematical Surveys and Monographs* 62. Providence, RI: Amer. Math. Soc.

[7] Bogachev, V.I. (2016). Distributions of polynomials on multidimensional and infinite-dimensional spaces with measures. *Uspehi Mat. Nauk* 71, 107–154 (in Russian); English transl.: *Russian Math. Surveys* 71, 703–749.

[8] Bogachev, V.I. (2018). *Weak Convergence of Measures. Mathematical Surveys and Monographs* 234. Providence, RI: Amer. Math. Soc.

[9] Bogachev, V.I. (2019). Distributions of polynomials in many variables and Nikol’skii-Besov spaces. *Real Anal. Exchange* 44, 49–64.

[10] Bogachev, V.I., Zelenov, G.I. and Kosov, E.D. (2016). Membership of distributions of polynomials in Nikol’skii-Besov classes. *Dokl. Akad. Nauk* 469, 651–655 (in Russian); English transl.: *Dokl. Math.* 94, 453–457.

[11] Bogachev, V.I., Kosov, E.D. and Zelenov, G.I. (2018). Fractional smoothness of distributions of polynomials and a fractional analog of the Hardy–Landau–Littlewood inequality. *Trans. Amer. Math. Soc.* 370, No 6 (2018), 4401–4432.

[12] Bogachev, V.I. and Zelenov, G.I. (2015). On convergence in variation of weakly convergent multidimensional distributions. *Dokl. Akad. Nauk* 461, 14–17 (in Russian). English transl.: *Dokl. Math.* 91, 138–141.

[13] Götze, F., Prokhorov, Y.V. and Ulyanov, V.V. (1996). Bounds for characteristic functions of polynomials in asymptotically normal random variables. *Uspehi Mat. Nauk* 51, 3–26 (in Russian); English transl.: *Russian Math. Surveys* 51, 181–204.

[14] Kosov, E.D. (2018). Fractional smoothness of images of logarithmically concave measures under polynomials. *J. Math. Anal. Appl.* 462, 390–406.

[15] Kosov, E.D. (2019). On fractional regularity of distributions of functions in Gaussian random variables. *Fract. Calc. Appl. Anal.* 22, 1249–1268.

[16] Kosov, E.D. (2019). Besov classes on finite and infinite dimensional spaces. *Math. Sb.* 210, 41–71 (in Russian); English transl.: *Sb. Math.* 210, 663–692.

[17] Nourdin, I. and Poly, G. (2013). Convergence in total variation on Wiener chaos. *Stochastic Process. Appl.* 123, 651–674.

[18] Nourdin, I., Nualart, D. and Poly, G. (2013). Absolute continuity and convergence of densities for random vectors on Wiener chaos. *Electron. J. Probab.* 18, 1–19.

[19] Stein, E. (1970). *Singular integrals and differentiability properties of functions.* Princeton: Princeton University Press.

[20] Ulyanov, V.V. (2016). On properties of polynomials in random elements. *Theory Probab. Appl.* 60, 325–336.