Tree spanners of small diameter

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March 23, 2015

Abstract

A graph that contains a spanning tree of diameter at most $t$ clearly admits a tree $t$-spanner, since a tree $t$-spanner of a graph $G$ is a sub tree of $G$ such that the distance between pairs of vertices in the tree is at most $t$ times their distance in $G$. In this paper, graphs that admit a tree $t$-spanner of diameter at most $t+1$ are studied. For $t$ equal to 1 or 2 the problem has been solved. For $t = 3$ we present an algorithm that determines if a graph admits a tree 3-spanner of diameter at most 4. For $t \geq 4$ it is proved that it is an NP-complete problem to decide whether a graph admits a tree $t$-spanner of diameter at most $t+1$.

1 Introduction

There are applications of spanners in a variety of areas, such as distributed computing [2, 18], communication networks [16, 17], motion planning and robotics [1, 9] and phylogenetic analysis [3]. Furthermore, spanners are used in embedding finite metric spaces in graphs approximately [19].

On one hand, in [3, 8, 7] an efficient algorithm to decide tree 2-spanner admissible graphs is presented, where a method to construct all the tree 2-spanners of a graph is also given. On the other hand, in [8, 7] it is proven that for each $t \geq 4$ the problem to decide graphs that admit a tree $t$-spanner is an NP-complete problem. The complexity status of the tree 3-spanner problem is unresolved. In this paper it is shown that the problem to determine whether a graph admits a tree $t$-spanner of diameter at most $t+1$ is tractable, when $t \leq 3$, while it is an NP-complete problem, when $t \geq 4$.

Tree $t$-spanners ($t \geq 3$) have been studied for various families of graphs. If a connected graph is a cograph or a split graph or the complement of a bipartite graph, then it admits a tree 3-spanner [7]. Also, all convex bipartite graphs have a tree 3-spanner, which can be constructed in linear time [20].
Efficient algorithms to recognize graphs that admit a tree 3-spanner have been developed for interval, permutation and regular bipartite graphs [13], planar graphs [10], directed path graphs [12], very strongly chordal graphs, 1-split graphs and chordal graphs of diameter at most 2 [6]. Moreover, every strongly chordal graph admits a tree 4-spanner, which can be constructed in linear time [5]; note that, for each \( t \), there is a connected chordal graph that does not admit any tree \( t \)-spanner. In [6] it is also presented a linear time algorithm that finds a tree spanner in a small diameter chordal graph. In [14] the tree \( t \)-spanner problem is studied for diametrically uniform graphs.

There are NP-completeness results for the tree \( t \)-spanner problem for families of graphs. In [10], it is shown that it is NP-hard to determine the minimum \( t \) for which a planar graph admits a tree \( t \)-spanner. For any \( t \geq 4 \), the tree \( t \)-spanner problem is NP-complete on chordal graphs of diameter at most \( t + 1 \), when \( t \) is even, and of diameter at most \( t + 2 \), when \( t \) is odd [6]; note that this refers to the diameter of the graph not to the diameter of the spanner.

The tree 3-spanner problem is very interesting, since its complexity status is unresolved. In [15] it is shown that only for \( t=3 \) the union of any two tree \( t \)-spanners of any given graph may contain big induced cycles but never an odd induced cycle (other than a triangle); such unions are proved to be perfect graphs. The tree 3-spanner problem can be formulated as an integer programming optimization problem. Constraints for such a formulation appear in [15], providing certificates for non tree 3-spanner admissibility.

## 2 Definitions and lemmas

The definition of tree a \( t \)-spanner follows, while, in general, terminology of [21] is used.

**Definition 1** A graph \( T \) is a tree \( t \)-spanner of a graph \( G \) if and only if \( T \) is a subgraph of \( G \) that is a tree and, for every pair \( u \) and \( v \) of vertices of \( G \), if \( u \) and \( v \) are at distance \( d \) from each other in \( G \), then \( u \) and \( v \) are at distance at most \( t \cdot d \) from each other in \( T \).

Note that in order to check that a spanning tree of a graph \( G \) is a tree \( t \)-spanner of \( G \), it suffices to examine pairs of adjacent in \( G \) vertices. A path of even length has a central vertex, while a path of odd length has a central edge. We take into account this parity fact to define a class of trivially tree \( t \)-spanner admissible graphs.

**Definition 2** A \( t \)-center \( K \) of a graph \( G \) is a subgraph of \( G \) consisting exactly of either a vertex when \( t \) is even, or a pair of adjacent in \( K \) vertices when \( t \) is odd, such that for all \( u \) in \( G \), \( d_G(K,u) \leq \frac{t}{2} \).

Clearly, for any \( t \)-center \( K \), we see that \( |K| = |E(K)| = t \mod 2 \). Assume that a graph \( G \) contains a \( t \)-center \( K \). Any Breadth-First Search tree \( T \) of \( G \) starting from \( K \) has the property that \( d_T(K,u) = d_G(K,u) \), for every vertex \( u \) of \( G \) [21]. Therefore, since \( K \) is a \( t \)-center of \( G \), \( T \) is a tree \( t \)-spanner of \( G \); observe
that the distance in $T$ between any pair of vertices $u$ and $v$ is at most equal to the distance from $u$ to $K$ plus $|K|$ plus the distance from $K$ to $v$. Graphs that admit a $t$-center are defined to be $t$-stars:

**Definition 3** A graph $G$ is a $t$-star if and only if $G$ is the one vertex graph or $G$ admits a $t$-center.

Note that the one vertex graph is a $(2t+1)$-star for any $t$, but it does not admit a $(2t+1)$-center. Also, observe that the only 0-star is the one vertex graph and the only 1-stars are the one vertex graph and the one edge graph. If $k \leq t$, then a $k$-star is also a $t$-star. Moreover, a $t$-star is a connected graph. Also, every connected graph is a $t$-star, for some $t$. For example, a path of length $t$ is a $t$-star and has a unique $t$-center. If a graph is a $t$-star, then at least one spanning tree of the graph has diameter at most $t$:

**Lemma 1** A graph $G$ is a $t$-star if and only if $G$ admits a spanning tree of diameter at most $t$.

**Proof.** On one hand, assume that $G$ is a $t$-star. If $G$ is the one vertex graph, then the graph itself is such a spanning tree. Otherwise, let $K$ be a $t$-center of $G$. Then, as we mentioned earlier, any Breadth-First Search tree of $G$ starting from $K$ has diameter at most $t$.

On the other hand, assume that $G$ is not a $t$-star for some $t$. We prove that the diameter of any spanning tree of $G$ is strictly greater than $t$. Towards a contradiction, assume that $G$ admits a spanning tree $T$ of diameter $d$ such that $d \leq t$. Consider a longest path $P$ of $T$, then $d = |P|$. Also, $P$ itself is a $d$-star with a unique $d$-center $K$. Let $P_1$ and $P_2$ be the subpaths of $P$ from $K$ to $u$ and from $K$ to $v$, respectively, where $u$ and $v$ are the endpoints of $P$. Then, $|P_1| = |P_2| = \left\lfloor \frac{d}{2} \right\rfloor$.

Let $x$ be an arbitrary vertex of $G$. Since $T$ is a spanning tree of $G$, there is a path $P'$ from $x$ to $K$ in $T$. Path $P'$ cannot intersect (out of $K$) with both of $P_1$ and $P_2$, because otherwise $T$ contains a cycle. So, we may assume that $P'$, $P_1$ and $K$ form a path and $d_T(x,u) = |P'| + |P_1| + |K|$. But $d_T(x,u) \leq d$, since $d$ is the diameter of $T$. Therefore, $d_T(K,x) = |P'| = d_T(x,u) - |P_1| - |K| \leq \left\lfloor \frac{d}{2} \right\rfloor$.

But for every vertex $x \in G$, $d_G(K,x) \leq d_T(K,x)$. Also, $\left\lfloor \frac{d}{2} \right\rfloor \leq \left\lfloor \frac{d}{2} \right\rfloor$, since $d \leq t$. So $K$ is a $t$-center of $G$, which is a contradiction. □

As a corollary, a tree is a $t$-star if and only if it has diameter at most $t$, because a tree has only one spanning tree. Now, for general graphs, if a graph is a $t$-star, then the diameter of the graph is at most $t$ but there are graphs of diameter $d$ that are not $d$-stars; for example, a cycle on eight vertices has diameter 4, but it is not a 4-star.

When we examine a graph which is not a $t$-star, we actually face the tree $t$-spanner problem. Before presenting a frequently used lemma, we give a definition to handle long paths:

**Definition 4** A $t$-midst $M(P,t)$ of a path $P$ from $u$ to $v$ is a subpath of $P$ consisting exactly of either one vertex when $t$ is odd, or a pair of adjacent in $M$ vertices when $t$ is even, such that $d_P(M,u) > \left\lfloor \frac{t+1}{2} \right\rfloor$ and $d_P(M,v) > \left\lfloor \frac{t+1}{2} \right\rfloor$. 

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Obviously, a path $P$ has a $t$-midst if and only if $|P| \geq t + 1$. There may be many $t$-midsts in a path but only if $|P| = t + 1$, does $P$ have a unique $t$-midst. Clearly, for any $t$-midst $M$, we see that $|M| = |E(M)| = (t + 1) \mod 2$.

Figure 1: Let $W$ (solid circle) be the vertices of $G$ within distance $\lfloor \frac{t-1}{2} \rfloor$ from t-midst $M$ in $T$. How are the components of $T \setminus M$ (dashed lines) related to the components of $G \setminus W$? Here, $t$ is odd.

**Lemma 2** Let $G$ be a graph and $T$ a tree $t$-spanner of $G$. If $M$ is a $t$-midst of a $u, v$-path $P$ of $T$, then every $u, v$-path $P'$ of $G$ contains a vertex whose distance from $M$ in $T$ is at most $\lfloor \frac{t-1}{2} \rfloor$.

**Proof.** Since $G$ contains a path that admits a $t$-midst, $t$ is not zero. Let $x = M$, when $t$ is odd, and $x = E(M)$ when $t$ is even. Consider the components of $T \setminus x$. Note that when $t$ is even, only two components are formed. Obviously, vertices $u$ and $v$ belong to different such components. Therefore, for any $u, v$-path $P'$ of $G$ there is an edge $ww'$ in $P'$ such that $w$ is in a different component than $w'$ (see figure). Since all the tree paths connecting vertices of different such components pass through $x$, it holds that $d_T(w, w') = d_T(w, M) + d_T(M, w') + |M|$. But the tree distance between $w$ and $w'$ can be at most $t$, therefore at least one of $w$ or $w'$ is within distance $\lfloor \frac{t-1}{2} \rfloor$ from $M$ (consider different cases when $t$ is odd or even; note that when $t$ is even the edge of $M$ participates in the tree path between vertices $w$ and $w'$).

## 3 Overview

Graphs that admit a tree $t$-spanner of small diameter are the subject of this paper. First, for a $t$-star $G$, recall that any shortest paths spanning tree to a $t$-center of $G$ is a tree $t$-spanner of $G$. Second, according to theorem if a graph admits a tree $t$-spanner of diameter at most $t + 1$, then at least one of its shortest paths spanning trees is a tree $t$-spanner of the graph. Third, for each $t \geq 2$, there are graphs which admit a tree $t$-spanner of diameter $t + 2$ but none of their tree $t$-spanners is a shortest paths spanning tree (section). Therefore,
should we expect that for each \( t \) there is an efficient algorithm to determine if a graph admits a tree \( t \)-spanner of diameter at most \( t + 1 \)?

Theorem 2 settles this question. Consider a graph \( G \) that admits a tree \( t \)-spanner of diameter \( t + 1 \) and let \( K \) be the \((t+1)\)-center of the tree. Then, from theorem 1 we know that \( G \) admits a tree \( 3 \)-spanner \( T \) which is a shortest paths to \( K \) spanning tree of \( G \). When \( t \) is odd, \( K \) is just a vertex, so there is a vertex \( u \) in \( G \) such that all the edges of \( G \) incident to \( u \) are in \( T \). For \( t = 3 \), finding the remaining edges of such a tree \( 3 \)-spanner can be done efficiently (proposition 1).

Though, for each \( t \geq 4 \), the problem of determining if a graph admits a tree \( t \)-spanner of diameter at most \( t + 1 \) is an NP-complete problem, where we use a reduction from 3-SAT problem. Note that the situation for this problem from the complexity point of view is the same as the situation for the standard tree \( t \)-spanner problem, except for the \( t = 3 \) case for which the complexity status of the standard problem is unresolved.

4 Shortest paths spanning trees

Let \( K \) be a subtree of a graph \( G \); in this paper \( K \) is just a vertex or a pair of adjacent vertices. A shortest paths to \( K \) spanning tree \( T \) of a graph \( G \) is a spanning tree of \( G \), such that for every vertex \( x \) in \( G \) the unique path of \( T \) between \( x \) and \( K \) is one of the shortest paths of \( G \) between \( x \) and \( K \). For every \( d > 0 \), recall that a tree has a \( d \)-center if and only if it is of diameter at most \( d \) (lemma 1).

**Theorem 1** If a graph \( G \) admits a tree \( t \)-spanner which has a \((t+1)\)-center \( K \), then \( G \) admits a tree \( t \)-spanner \( T \), such that \( d_G(K, x) = d_T(K, x) \), for every vertex \( x \) in \( G \).

**Proof.** Let \( K \) be a \((t+1)\)-center of a tree \( t \)-spanner of a graph \( G \). Amongst all the tree \( t \)-spanners of \( G \) with \((t+1)\)-center \( K \) (we know that there is at least one such tree \( t \)-spanner of \( G \)), let \( T \) be one such that the number of vertices of \( G \) for which the condition of the theorem does not hold is minimized and let \( X \) be this set of vertices, i.e. \( X \) is the set of vertices \( x \) of \( G \) for which \( d_G(K, x) < d_T(K, x) \). Towards a contradiction assume that \( X \) is nonempty. Since \( G \) is connected, there is a shortest path \( P \) in \( G \) from a vertex in \( X \) to \( K \). Since \( K \) does not have any vertex in common with \( X \), path \( P \) contains two consecutive in \( P \) vertices \( u \) and \( v \), such that \( u \) is in \( X \) and \( v \) is not in \( X \). As \( P \) is a shortest path to \( K \), \( d_G(K, u) = d_T(K, v) + 1 \), since \( d_G(K, v) = d_T(K, v) \).

Also, edge \( uv \) is not in \( T \), because otherwise \( u \) would not be in \( X \). Let \( e \) be the edge of \( T \) incident to \( u \) towards \( K \). The tree path between \( u \) and \( v \) contains edge \( e \), because otherwise the tree path from \( v \) to \( K \) would contain \( u \). So, if we replace edge \( e \) of \( T \) with edge \( uv \), the result is another spanning tree \( T' \) of \( G \). Note that \( K \) is a \((t+1)\)-center of both \( T' \) and \( T \), since \( d_{T'}(K, x) \leq d_T(K, x) \leq \lceil \frac{t+1}{2} \rceil \) for every vertex \( x \) of \( T' \).

We prove that \( T' \) is a tree \( t \)-spanner of \( G \). It suffices to examine the vertices in the component \( Q \) of \( T \setminus \{ e \} \) that contains vertex \( u \), because the distances in
$T'$ amongst the remaining vertices are the same as in $T$. So, let $q$ be an arbitrary vertex of $Q$. Vertex $q$ is within distance $\left\lfloor \frac{t+1}{2} \right\rfloor$ from $K$ in $T$, because $K$ is a $(t+1)$-center of $T$. So, $d_{T'}(q, K) \leq \left\lfloor \frac{t+1}{2} \right\rfloor - 1$, since the $T'$ path from $q$ to $K$ contains vertex $u$ and $u$ is strictly closer to $K$ in $T'$ than it is in $T$. Now, for every vertex $p$ of $G$, $d_{T'}(q, p) \leq d_T(q, K) + |K| + d_T(K, p) \leq \left\lfloor \frac{t+1}{2} \right\rfloor - 1 + |K| + \left\lfloor \frac{t+1}{2} \right\rfloor = t$

(note that $|K| = 1$, when $t+1$ is odd, and $|K| = 0$, otherwise; also, $d_{T'}(K, p) \leq \left\lfloor \frac{t+1}{2} \right\rfloor$, since $K$ is a $(t+1)$-center of $T'$). Therefore, $T'$ is a tree $t$-spanner of $G$ with $(t+1)$-center $K$. The fact that $d_{T'}(K, u) = d_G(K, u)$ for vertex $u$ is a contradiction to the minimality of set $X$, since all the vertices of $T$ that are not in $X$ satisfy the condition of the theorem for tree $T'$. Hence, $X$ is the empty set, i.e. $d_G(K, x) = d_t(K, x)$ for every vertex $x$ of $G$.

When we consider tree $t$-spanners with a $(t+2)$-center, this theorem does not hold in general, except for the trivial case where $t \leq 1$. For each $t \geq 2$, we now present an example of a 2-connected graph that admits a tree $t$-spanner with a $(t+2)$-center and there is no tree $t$-spanner of the graph which satisfies the condition of the theorem.

When $t$ is an odd number greater or equal to 3, consider a graph $G$ consisting of (1) two cycles $C_1 = v, u, x_1, \ldots, x_t$ and $C_2 = u, v, y_1, \ldots, y_t$ each of length $t + 2$, such that $C_1 \cap C_2 = \{u, v\}$, (2) two edges $x_1x_t, y_1y_t$ and (3) two paths $u, w_1, \ldots, w_{t-1} = x_1$ and $v, z_1, \ldots, z_{t-1} = y_1$ each of length $t - 1$ and vertex disjoint (but their endpoints) from each other and from the two cycles (see figure 2 (a)). If we remove edges $vx_1, uy_1, x_{\lceil t/2 \rceil}x_{\lceil t/2 \rceil + 1}, y_{\lfloor t/2 \rfloor}y_{\lfloor t/2 \rfloor + 1}, w_{\lceil t/2 \rceil}w_{\lceil t/2 \rceil}, z_{\lfloor t/2 \rfloor}z_{\lfloor t/2 \rfloor}$ from $G$, then the result is a tree $t$-spanner $T$ of $G$, where the pair $u$ and $v$ induces a $(t+2)$-center $K$ of $T$. Now, edges $x_1x_t$ and $ux_1$ have to be in every tree $t$-spanner of $G$, because, otherwise, if edge $x_1x_t$, for example, is not in a tree $t$-spanner of $G$, then we are left with the edges of cycle $C_1$ which has length $t + 2$. So, vertex $x_1$ is at distance 2 from $K$ in all the tree $t$-spanners of $G$ having $K$ as a $(t+2)$-center, although $x_1$ is adjacent to $K$ in $G$, i.e. $x_1$ violates the condition of the theorem. Note that $G$ does not contain any tree $t$-spanner with a $(t+1)$-center and $G$ is 2-connected. Also, for any arbitrary tree $t$-spanner of $G$ with a $(t+2)$-center different than $K$ there is a vertex that violates the condition of the theorem (for example, when $t \geq 5$, the pair $v$ and $x_t$ can be a $(t+2)$-center of a tree $t$-spanner of $G$ and, then, vertex $u$ violates the condition of the theorem).

When $t$ is an even number greater or equal to 2, consider a graph $G$ consisting of (1) two cycles $C_1 = u, x_1, \ldots, x_{t+1}$ and $C_2 = u, y_1, \ldots, y_{t+1}$ each of length $t + 2$, such that $C_1 \cap C_2 = \{u\}$, (2) two edges $x_1x_{t+1}, y_1y_{t+1}$ and (3) a path $x_1 = w_1, \ldots, w_t = y_1$ of length $t - 1$ (see figure 2 (b)). If we remove edges $ux_{t+1}, uy_{t+1}, x_{t/2+1}x_{t/2+2}, y_{t/2+1}y_{t/2+2}$ and $w_{t/2}w_{t/2+1}$ from $G$, then the result is a tree $t$-spanner $T$ of $G$, where vertex $u$ is a $(t+2)$-center of $T$. Since edges $x_1x_{t+1}$ and $ux_1$ are $t$-forced, for every tree $t$-spanner of $G$, vertex $x_{t+1}$ is at distance 2 from $u$, although $x_{t+1}$ is adjacent to $u$ in $G$, i.e. vertex $x_{t+1}$ violates the condition of the theorem. Here, as well as when $t$ is odd, $G$ is a 2-connected graph and $G$ does not contain any tree $t$-spanner with a $(t+1)$-center.
Figure 2: (a) For each odd $t \geq 3$, a graph that admits a tree $t$-spanner with a $(t + 2)$-center, where the dashed lines represent paths of length $t - 2$; when $t = 3$, all the vertices of the graph are shown in the figure and the bold edges form a tree 3-spanner of the graph with 5-center the pair $u$ and $v$. (b) For each even $t \geq 2$, a graph that admits a tree $t$-spanner with a $(t + 2)$-center, where the dashed lines represent paths of length $t - 1$; when $t = 2$, all the vertices of the graph are shown in the figure and the bold edges form a tree 2-spanner of the graph with 4-center $u$.

5 Tree 3-spanners of diameter 4

When $t = 2$, all the tree 2-spanners of a graph can be generated using the skeleton tree of the graph [3, 7]; because of that, one can efficiently decide if a graph admits a tree 2-spanner of diameter at most $d$, for any $d$ (in case that the graph admits a skeleton tree). For $t \geq 3$, there is no tool known analogous to the skeleton tree. According to theorem 1 if a graph admits a tree $t$-spanner with a $(t + 1)$-center, then the graph admits a tree $t$-spanner which is a shortest paths to this center spanning tree of the graph. Because of this property, when $t = 3$, there is a short characterization of graphs that admit a tree 3-spanner of diameter at most 4.

**Proposition 1** A graph $G$ admits a tree 3-spanner of diameter at most 4 if and only if $G$ contains a vertex $u$, such that each component of $G \setminus N[u]$ is included in the neighborhood of a neighbor of $u$.

**Proof.** Assume that a graph $G$ contains a vertex $u$, such that, for every component $Q$ of $G \setminus N[u]$, there is a vertex $v_Q$ in $N(u)$ for which $V(Q) \subseteq N(v_Q)$. Consider the following set of edges $E(T) = \{ux : x \in N(u)\} \cup \bigcup_{Q \in C} \{v_Qx : x \in Q\}$, where $C$ is the set of components of $G \setminus N[u]$. Since every edge in $E(T)$ belongs to $E(G)$, $E(T)$ induces a subgraph $T$ of $G$. Every vertex of $G$ other than $u$ contributes exactly one edge (in the direction to $u$) to $E(T)$ and $E(T)$ does not contain any other edges; so, $E(T)$ contains $n - 1$ edges. Also, every vertex of $G$ other than $u$ is adjacent in $T$ either to $u$ or to a neighbor of $u$ in $T$ and,
therefore, $T$ is connected and has diameter at most 4. Thus, it suffices to prove that $T$ is a 3-spanner of $G$. First, if a vertex is in $N_T(u)$, then it is within distance 3 in $T$ from every vertex of $G$, since $u$ is within distance 2 in $T$ from every vertex of $G$. Second, for the remaining vertices, a vertex $x$ in a component $Q$ of $G \setminus N[u]$ is adjacent in $G$ either to a vertex in $N_T(u)$, which we examined in the first case, or to a vertex $y$ which is also in $Q$ and, therefore, $x$ and $y$ are at distance 2 from each other in $T$ (edges $v_Qx$ and $v_Qy$ are both in $T$).

Assume now that a nonempty graph $G$ admits a tree 3-spanner $T'$ of diameter at most 4. Then, $T'$ has a 4-center $u$ (even if $T'$ is the one vertex graph). Because of theorem 1, $G$ admits a tree 3-spanner $T$, such that $d_G(u, x) = d_T(u, x)$, for every vertex $x$ of $G$. Therefore, $N_G(u) = N_T(u)$. First, let $Q$ be any component of $G \setminus N_G[u]$. Since $u$ is a 4-center of $T$, every vertex in $Q$ is adjacent in $T$ to some vertex in $N_G(u)$. Assume that two distinct vertices $x$ and $y$ of $Q$ are adjacent in $T$ to two distinct vertices $v_x$ and $v_y$ of $N_G(u)$, respectively. Then, $u$ is a 3-midst of path $x, v_x, u, v_y, y$ of $T$. Since $x$ and $y$ are in the same component $Q$, there is a path in $Q$ between $x$ and $y$. But, none of the vertices of $Q$ is at distance 1 from $u$ in $T$, which is a contradiction, because of lemma 2. Hence, all the vertices in $Q$ are adjacent in $T$ to the same vertex, say $v_Q$, in $N_G(u)$. So, for every component $Q$ of $G \setminus N_G[u]$, there is a vertex $v_Q$ in $N_G(u)$ for which $Q \subseteq N_T(v_Q) \subseteq N_G(v_Q)$. Second, if $G \setminus N_G[u]$ is the empty graph, then there is no component to examine and, therefore, the statement holds vacuously. 

This proposition gives rise to an efficient algorithm to determine if a graph admits such a tree 3-spanner. Assume we are given a graph $G$. For every vertex $u$ of $G$, we can find efficiently the components of $G \setminus N[u]$, using a breadth first search algorithm, for example. It remains to check if for each such component $Q$ there is a neighbor $v$ of $u$ for which $V(Q) \subseteq N(v)$, which can be easily performed. The following program illustrates such an algorithm.

```plaintext
input(G);
flag=1; /*In case G is the empty graph
for u in V(G) {
    Let C be the set of components of G-N[u];
    flag=1; /*In case G=N[u] and therefore C is empty
    for X in C {
        flag=0;
        for v in N(u)
            if (X is a subset of N(v)) {flag=1; break;}
        if (flag==0) break;
    }
    if (flag) break; else continue;
}
if (flag)
    output(G admits a tree 3-spanner of diameter at most 4);
else
    output(G does not admit a tree 3-spanner of diameter at most 4);
```
Of course, if a graph admits a tree 3-spanner of diameter at most 4, i.e. the tree 3-spanner is a 4-star, then the graph itself is a 4-star, as well. In contrast, it can be the case that a 4-star graph is tree 3-spanner admissible but none of its tree 3-spanners has diameter at most 4. Figure 2 (a) for \( t = 3 \) depicts such a graph: vertex \( u \) is a 4-center of the graph but all the tree 3-spanners of this graph have diameter greater or equal to 5.

6 Tree \( t \)-spanners of diameter \( t + 1 \) for \( t \geq 4 \)

As shown in [8, 7] using a reduction from 3-SAT, for each \( t \geq 4 \), the problem of determining if a graph admits a tree \( t \)-spanner is an NP-complete problem. In this reduction, for each instance of 3-SAT a graph is generated and it is shown that an instance of 3-SAT is satisfiable if and only if the corresponding graph is tree \( t \)-spanner admissible. It turns out that if an instance of 3-SAT is satisfiable, then its corresponding graph admits a tree \( t \)-spanner of diameter at most \( 2(t + \lceil \frac{t}{2} \rceil - 1) \). It is possible to alter slightly this reduction to prove that the problem of determining if a graph admits a tree \( t \)-spanner of diameter at most \( 2(t + \lceil \frac{t}{2} \rceil - 3) \) is an NP-complete problem. For example, the problem of determining if a graph admits a tree 4-spanner of diameter at most 6 is an NP-complete problem. Unfortunately, it seems as though the reduction used in [8, 7] cannot be employed for lower diameters and we need to employ a much different reduction.

For each \( t \geq 4 \), we consider the following problem. Given a graph \( G \), does \( G \) admit a tree \( t \)-spanner of diameter at most \( t + 1 \)? Each of these problems belongs to NP, since, given graphs \( G \) and \( T \), we can verify efficiently if \( T \) is a tree \( t \)-spanner of \( G \) of diameter at most \( t + 1 \). To see this, it suffices to verify that \( T \) is a spanning tree of \( G \) of diameter at most \( t + 1 \) and no pair of vertices being at distance \( t + 1 \) apart in \( T \) are adjacent in \( G \).

We prove that each of these problems is an NP-complete problem, using a reduction from 3-SAT. An instance of 3-SAT consists of a set of clauses, where each clause is the disjunction of exactly three distinct literals (see for example [11]). A literal is a variable or the negation of a variable. An instance of 3-SAT is satisfiable if and only if there is a truth assignment to the variables that participate in the clauses of the instance, such that all clauses of the instance become true. The 3-SAT problem is to determine if such an instance is satisfiable.

6.1 Stretch factor equals 4

We present an algorithm \( f \) which receives as input an instance \( I \) of 3-SAT and outputs a graph \( f(I) \). The basic steps of this algorithm are the following.

(I) Given \( I \), let \( X(I) \) be the set of variables involved in \( I \). Each variable in \( X(I) \) becomes a vertex of the output graph \( f(I) \). Also, consider a set of 6 vertices \( H = \{ u, v, h_u, h'_u, h_v, h'_v \} \), such that \( H \) does not have any element in common with \( X(I) \).
(II) Let $C(I)$ be the set of clauses of $I$ which do not contain both a variable and its negation, i.e. each clause in $C(I)$ contains exactly 3 distinct variables. For every clause $c$ in $C(I)$, we generate a set of vertices $V^c$, such that $V^c$ does not have any element in common with $X(I)$ or $H$. Also, for every pair of clauses $c$ and $p$ in $C(I)$, sets $V^c$ and $V^p$ do not have any vertex in common. To make this clear we use $c$ as a superscript on each vertex of $V^c$. Let $c$ be a clause in $C(I)$, which involves variables $x_1$, $x_2$ and $x_3$, say; then, $V^c = \{x_1^c, x_2^c, x_3^c\} \cup \{q_1^c, \ldots, q_8^c\}$. Let $Q(I) = \bigcup_{c \in C(I)} V^c$. Upon input $I$ consider the vertex set $V(I) = H \cup X(I) \cup Q(I)$.

(III) We continue with the edge set of the output. Consider the following matrix:

$$M = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}$$

Matrix $M$ has two main properties. On one hand, it consists of three distinct pairs of complementary to each other rows (for example the first row is the complement of the second). On the other hand, if a sub-matrix of $M$ consisting of whole rows of $M$ contains at least one 1 in each column, then the sub-matrix must contain at least one pair of complementary to each other rows. Let $c$ be a clause of $C(I)$ that contains variables $x_1$, $x_2$ and $x_3$, say, where $x_1$ appears first in $c$, $x_2$ appears second and $x_3$ appears third. Consider the ordering (vector) $g^c=\{x_1^c, x_1^c, x_2^c, x_2^c, x_3^c, x_3^c\}$ of vertices of $f(I)$. We generate a set of edges $E^c$ between vertices in $V(I)$ as follows. First, for $1 \leq i \leq 6$ and $1 \leq j \leq 8$, if $M_{i,j} = 1$, then edge $g^c_i g^c_j$ belongs to $E^c$, i.e. matrix $M$ becomes the adjacency matrix between vertices in $g^c$ and vertices in $\{q_1^c, \ldots, q_8^c\}$. Second, for $1 \leq i \leq 3$, if variable $x_i$ appears positive in $c$, then edge $x_i^c u$ belongs to $E^c$; else if the negation of $x_i$ appears in $c$, then edge $x_i^c v$ belongs to $E^c$ (i.e. we use vertex $x_i^c$ to indicate the standing of variable $x_i$ in $c$). Finally, for $1 \leq i \leq 7$, edge $q_i q_{i+1}$ belongs to $E^c$ and $E^c$ does not contain any other edges. We consider the union of these edge sets over all clauses $c$ of $C(I)$; so, let $E(Q) = \bigcup_{c \in C(I)} E^c$.

(IV) For vertices in $X(I)$ we consider the following set of edges, $E(X)$. For every $x$ in $X(I)$, edges $xu$ and $xv$ belong to $E(X)$, and $E(X)$ does not contain any other edges. Also, let $E(H) = \{h_u, h'_u, h'_v, uv, vh'_v, h'_v h_v\}$, i.e. vertices in $H$ form an $h_u, h_v$-path of length 5. Finally, let $E(I) = E(Q) \cup E(X) \cup E(H)$.

(V) Algorithm $f$ outputs $f(I) = (V(I), E(I))$.

For every instance $I$ of 3-SAT, output $f(I)$ of algorithm $f$ is a graph, since $E(I)$ contains edges with endpoints in $V(I)$. Note that $f$ runs in polynomial time. Figure 3 shows part of such an output. Note that clauses that contain both a variable and its negation are disregarded in the construction of $f(I)$.

**Proposition 2** For every instance $I$ of 3-SAT, $I$ is satisfiable if and only if graph $f(I)$ admits a tree 4-spanner of diameter at most 5.
Then, for every clause $c$ edges shown in the figure. $M$ by matrix rectangle, i.e. in vector $g^c$ ordered from left to right, where $g^c_1$ is the leftmost vertex in the rectangle, i.e. $g^c_1$ is vertex $x_1$. The dashed lines represent the edges determined by matrix $M$. Also, vertices in $V^c$ (white vertices) are incident in $f(I)$ only to edges shown in the figure.

**Proof.** Let $A$ be a truth assignment that satisfies an instance $I$ of 3-SAT. Then, for every clause $c$ of $I$, let $x(c)$ be one of the variables that make clause $c$ true with respect to $A$.

Consider the following set of edges, $T^X$. For every variable $x$ of $X(I)$, if $A(x)$ is true, then edge $xu$ belongs to $T^X$, otherwise ($A(x)$ is false) edge $xv$ belongs to $T^X$. Also, let $T^H = \{h_u, h'_u, h_v, uv, vh'_v, h'_v, h_v\} = E(H)$.

Let $C(I)$ be the set of clauses of $I$ that contain 3 distinct variables. For each clause $c$ of $C(I)$ which contains variables $x_1$, $x_2$ and $x_3$, say, we consider the following set of edges $T^c$. For $1 \leq j \leq 3$, the edge of graph $f(I)$ from $x^c_j$ to $u$ or $v$ belongs to $T^c$ (note that each of $x^c_j$ is adjacent to exactly one of $u$ or $v$ in $f(I)$). Also, $x(c) = x_i$ for some $i$ from 1 to 3. Then, all the edges of graph $f(I)$ between $\{x_i, x^c_j\}$ and $\{q^c_1, \ldots, q^c_8\}$ belong to $T^c$. Finally, let $T^Q = \bigcup_{c \in C(I)} T^c$.

We prove that the set of edges $T^X \cup T^H \cup T^Q$ induces a tree 4-spanner $T$ of $f(I)$ of diameter at most 5. Obviously, $T^H \subseteq E(I)$, since $T^H = E(H)$. Moreover, since every vertex in $X(I)$ is adjacent to both $u$ and $v$ in $f(I)$, $T^X \subseteq E(I)$. Finally, for every clause $c$ in $C(I)$, edge set $T^c$ is defined as a subset of $E(I)$, so $T^Q \subseteq E(I)$. Therefore, $T$ is a subgraph of $f(I)$; part of such a graph $T$ is shown in figure [a]. Edge $uv$ is in $T$ and let $K$ be the subgraph of $T$ induced by the pair $u$ and $v$.

Let $c$ be an arbitrary clause in $C(I)$, which contains variables $x_1$, $x_2$ and $x_3$, say. Then, $x(c) = x_i$ for some $i$ from 1 to 3. Vertices $q^c_1, \ldots, q^c_8$ of $V^c$ are adjacent in $T$ to $x_i$ and $x^c_j$ only, because edges of $T^c$ incident to $q^c_1, \ldots, q^c_8$ are exactly the edges of $f(I)$ from $\{x_i, x^c_j\}$ to $\{q^c_1, \ldots, q^c_8\}$. Moreover, for $1 \leq j \leq 8$, vertex $q^c_j$ has degree 1 in $T$, because exactly one of $M_{2i−1, j}$ and $M_{2i, j}$ is 1; observe that the $(2i−1)$ row of $M$ is the complement of the $2i$ row of $M$. On
one hand, if variable $x_i$ appears positive in $c$, then $x_i$ is adjacent to $u$ in $T$ and $x_i$ is adjacent to $v$ in $T$, since $x_i$ is a variable of $c$ that makes $c$ true with respect to truth assignment $A$. On the other hand, if the negation of $x_i$ appears in $c$ then $x_i$ is adjacent to $v$ in $T$ and $x_i$ is adjacent to $u$ in $T$, since $x_i$ is a variable of $c$ that makes $c$ true with respect to truth assignment $A$. So, both $x_i$ and $x_i$ are adjacent to the same vertex of $K$ in $T$. Therefore, the distance in $T$ between any pair of vertices in $\{q_1, \ldots, q_8\}$ is at most 4 and all the vertices in $\{q_1, \ldots, q_8\}$ are at distance 2 in $T$ from $K$. The remaining vertices of $V^c$, i.e. vertices $x_j$ where $j \neq i$ and $1 \leq j \leq 3$, are adjacent in $T$ to exactly one vertex of $K$ and have degree 1 in $T$.

Each vertex in $X(I)$ is adjacent to exactly one vertex of $K$ in $T$. Vertices in $H$ form a path in $T$ and every vertex in $H$ is at distance at most 2 in $T$ from $K$. So, each vertex in $V(I) = Q(I) \cup X(I) \cup H$ is either adjacent to some vertex in $K$ or at distance 2 from $K$, thus $T$ is connected and $K$ is a 5-center of $T$, i.e. $T$ has diameter at most 5 (lemma $I$).

Next, we prove that $T$ does not contain any cycles. A vertex in $T$ but not in $K$ has degree more than 1 in $T$ only if it is in $X(I)$ or it is vertex $x(c)$ for some clause $c$ in $C(I)$ or it is vertex $h_u'$ or $h_v'$; but all these vertices are adjacent in $T$ to exactly one vertex of $K$ and there is no edge of $T$ between them. So, the vertices in $T$ of degree more than 1 form a subtree of $T$, which implies that $T$ does not have any cycles. Hence, $T$ is a tree.

Finally, in order to prove that $T$ is a 4-spanner of $f(I)$, it suffices to examine vertices at distance 2 from $K$ in $T$, since the remaining vertices of $f(I)$ are adjacent to $K$ in $T$ and, therefore, they are within distance 4 in $T$ from every vertex of $f(I)$, because $K$ is a 5-center of $T$. The vertices of $f(I)$ that are at distance 2 from $K$ in $T$ are vertices $h_u, h_v$ and vertices $q_1, \ldots, q_8$, for each clause $c$ of $C(I)$. Obviously, each of $h_u$ or $h_v$ is adjacent to $T$ to its only neighbor in $f(I)$, namely $h_u'$ or $h_v'$, respectively. For every pair of clauses $c$ and $p$ of $C(I)$, there is no edge of $f(I)$ between a vertex in $\{q_1, \ldots, q_8\}$ and a vertex in $\{q_1, \ldots, q_8\}$. Also, for every clause $c$ in $C(I)$, the vertices in $\{q_1, \ldots, q_8\}$ are at distance at most 4 in $T$ from each other, as we pointed out earlier in the proof. Hence, $T$ is a tree 4-spanner of $f(I)$ of diameter at most 5.

Let $T'$ be a tree 4-spanner of $f(I)$ of diameter at most 5. Only vertices $h_u'$ and $u$ are at distance at most 2 from $h_u$ and only vertices $h_v'$ and $v$ are at distance at most 2 from $h_v$, so pair $u$ and $v$ is the only 5-center $K$ of $T'$. Since $f(I)$ admits a tree 4-spanner with 5-center $K$, $f(I)$ admits a tree 4-spanner $T$ with 5-center $K$, such that $d_{f(I)}(x, K) = d_T(x, K)$, for every vertex $x$ in $f(I)$, because of theorem $I$. Every vertex in $X(I)$ is adjacent to both $u$ and $v$ in $f(I)$, so, for every $x$ in $X(I)$, at least one of edges $xu$ or $xv$ is in $T$, since $T$ is a shortest paths to $K$ spanning tree of $f(I)$. Edge $uv$ is in $T$ ($K$ is a 5-center of $T$), so not both edges $xu$ and $xv$ are in $T$, because $T$ does not contain any triangles. Hence, for every vertex $x$ in $X(I)$, exactly one of edges $xu$ or $xv$ is in $T$. Therefore, the following definition of $A$ is a truth assignment to variables in $X(I)$: for every $x$ in $X(I)$, $A(x)$ is true, when edge $ux$ is in $T$, and $A(x)$ is false, when edge $vx$ is in $T$. We prove that $A(x)$ satisfies $I$.

Let $c$ be an arbitrary clause in $C(I)$, which contains variables $x_1, x_2$ and $x_3$,
Figure 4: Two copies of the subgraph of \( f(I) \) induced by vertices in \( H \cup V^c \cup \{x_1, x_2, x_3\} \), where \( c \) is clause \((x_1 \lor x_2 \lor \neg x_3)\) of \( I \). On the left, bold edges represent edges of \( T \), where \( T \) is constructed upon a truth assignment \( A \) that satisfies \( I \). Here, \( A(x_1) \) is false, \( A(x_2) \) is true and \( A(x_3) \) is true. There is only one choice for \( x(c) \), namely \( x(c) = x_2 \). Observe that both of \( x_2 \) and \( x_3^c \) are adjacent in \( T \) to \( u \) and each vertex in \( \{q_1^c, \ldots, q_8^c\} \) is adjacent in \( T \) to exactly one of \( x_2 \) or \( x_3^c \). On the right, bold edges are edges of a tree 4-spanner \( T \) of \( f(I) \) which is a shortest paths to \( K \) spanning tree of \( f(I) \). Given \( T \), we can generate a truth assignment \( A \) for which \( A(x_1) \) is true, \( A(x_2) \) is false and \( A(x_3) \) is true. Since both of \( x_1 \) and \( x_1^c \) are adjacent to \( u \) in \( T \), variable \( x_1 \) makes \( c \) true. Observe that the first and second rows of \( M \) were needed to cover all the vertices in \( \{q_1^c, \ldots, q_8^c\} \).

Say. Every vertex in \( \{q_1^c, \ldots, q_8^c\} \) is at distance exactly 2 from \( K \) in \( T \), because \( K \) is a 5-center of \( T \) and none of \( q_1^c, \ldots, q_8^c \) is adjacent to \( K \) in \( f(I) \). So, each of \( q_1^c, \ldots, q_8^c \) has to be adjacent to at least one of the vertices in \( g^c \), because vertices in \( g^c \) are the only neighbors in \( f(I) \) of vertices in \( \{q_1^c, \ldots, q_8^c\} \) that are adjacent to a vertex in \( K \). If we pick one vertex from each of the following three pairs \( x_1 \) and \( x_1^c \), \( x_2 \) and \( x_2^c \), \( x_3 \) and \( x_3^c \), then the neighborhood in \( f(I) \) of this triplet of vertices does not include all the vertices in \( \{q_1^c, \ldots, q_8^c\} \); observe that, because of the structure of matrix \( M \), unless we pick two complementary rows of \( M \) we cannot have a (proper) submatrix of \( M \) consisting of rows of \( M \), such that the submatrix has at least one 1 in each column (see figure 4(b) for an example). Therefore, for at least one \( i \) from 1 to 3, both of \( x_i \) and \( x_i^c \) are adjacent to some vertices in \( \{q_1^c, \ldots, q_8^c\} \). Also, towards a contradiction, assume that \( x_i^c \) is adjacent to \( u \) and \( x_i \) to \( v \) in \( T \) or \( x_i^c \) to \( v \) and \( x_i \) to \( u \), i.e. assume that vertices \( x_i \) and \( x_i^c \) are adjacent to different vertices of \( K \) in \( T \). Then, \( K \) is the 4-midst of a path \( P \) in \( T \) with endpoints in \( \{q_1^c, \ldots, q_8^c\} \). But, there is another path in \( f(I) \) that avoids the neighborhood of \( K \) in \( f(I) \) between the two endpoints of \( P \) (recall that the vertices in \( \{q_1^c, \ldots, q_8^c\} \) induce a path in \( f(I) \)), which is a contradiction, because of lemma 2. Thus, both of \( x_i \) and \( x_i^c \) are adjacent in \( T \) to the same vertex of \( K \).

On one hand, if variable \( x_i \) appears positive in \( c \), then \( x_i^c \) is adjacent to \( u \), because of the construction of graph \( f(I) \). So, since \( x_i \) and \( x_i^c \) are adjacent to the same vertex of \( K \) in \( T \), vertex \( x_i \) is also adjacent to \( u \) in \( T \). So, in this case, \( A(x_i) \) is true by the definition of \( A \). Therefore, clause \( c \) is satisfied by
truth assignment $A$. On the other hand, if the negation of $x_i$ appears in $c$, then, similarly, $A(x_i)$ is false; so, clause $c$ is satisfied by truth assignment $A$. In both cases $A$ satisfies the arbitrary clause $c$ in $C(I)$. Also, if $c$ is not in $C(I)$ then $c$ contains 3 literals but 2 variables, i.e. $c$ contains both a variable and its negation, so $A$ satisfies $c$. Hence, truth assignment $A$ satisfies all clauses of $I$.

Therefore, the problem of determining if a graph admits a tree 4-spanner of diameter at most 5 is an NP-complete problem.

6.2 The remaining values

In this section, $t \geq 5$. Given graph $f(I)$, where $I$ is an instance of 3-SAT, we describe a second graph $h(f(I), t)$ and, furthermore, we show that the second graph admits a tree 4-spanner of diameter at most $t + 1$ if and only if the first graph admits a tree 4-spanner of diameter at most 5. Therefore, for each $t \geq 5$ and for every instance $I$ of 3-SAT, graph $h(f(I), t)$ admits a tree 4-spanner of diameter at most $t + 1$ if and only if $I$ is satisfiable, because of proposition $2$. Hence, for each $t \geq 5$, the problem of determining if a graph admits a tree $t$-spanner of diameter at most $t + 1$ is an NP-complete problem. The main gadget for this reduction is to add to $f(I)$ a path of length $t - 3$ with endpoints the two vertices of the 5-center of $f(I)$ so that paths of length 5 in a tree 4-spanner of $f(I)$ become paths of length $t + 1$ in a tree $t$-spanner of graph $h(f(I), t)$.

More formally, consider such a graph $f(I)$. For each $t \geq 5$, consider the following graph $R(t)$, for which $f(I) \cap R(t) = \{u, v\}$. First, let $P$ be a path $u = r_1, \ldots, r_{t-2} = v$ of length $t - 3$. Second, let $P_1$ be a path of length $\lceil \frac{t+1}{2} \rceil$ having vertex $r_{\lceil (t-1)/2 \rceil}$ of $P$ as an endpoint and no other vertex in common with $P$. Third, let $P_2$ be a path of length $\lfloor \frac{t+1}{2} \rfloor$ having vertex $r_{\lfloor (t-1)/2 \rfloor}$ of $P$ as an endpoint and no other vertex in common with $P$ or $P_1$. Note that, when $t$ is odd, paths $P_1$ and $P_2$ share an endpoint, namely vertex $r_{\lfloor (t-1)/2 \rfloor} = r_{\lceil (t-1)/2 \rceil}$. Now, $R(t)$ is the union of $P$, $P_1$ and $P_2$. Graph $h(f(I), t)$ is defined as $R(t) \cup f(I)$ (figure 5). For each $t \geq 5$, note that, given $I$, graph $h(f(I), t)$ can be constructed in polynomial time.

**Proposition 3** For each $t \geq 5$, for every instance $I$ of 3-SAT, graph $f(I)$ admits a tree 4-spanner of diameter at most 5 if and only if graph $h(f(I), t)$ admits a tree $t$-spanner of diameter at most $t + 1$.

**Proof.** Assume that graph $f(I)$ admits a tree 4-spanner $T$ of diameter 5. Then, the pair of vertices $u$ and $v$ is the only 5-center of $T$. We prove that the subgraph $T'$ of $h(f(I), t)$ being the union of $R(t)$ and $(T \setminus \{uv\})$ is a tree $t$-spanner of $h(f(I), t)$ of diameter at most $t + 1$. Obviously, $T'$ is a spanning tree of $h(f(I), t)$, since $R(t)$ is a spanning tree of itself, $T$ is a spanning tree of $f(I)$ and edge $uv$ is not in $T'$. Let $K'$ be the subgraph of $T'$ induced by the pair of vertices $r_{\lfloor (t-1)/2 \rfloor}$ and $r_{\lceil (t-1)/2 \rceil}$. Note that, when $t$ is odd, $K'$ consists of one vertex and, when $t$ is even, $K'$ consists of a pair of adjacent in $T'$ vertices. Every vertex of $R(t)$ is within distance $\lceil \frac{t+1}{2} \rceil$ from $K'$ in $T'$, since paths $P_1$ and $P_2$ have length $\lceil \frac{t+1}{2} \rceil$. Therefore, the paths $u = r_1, \ldots, r_{t-2} = v$ of $R(t)$ are within distance $\lceil \frac{t+1}{2} \rceil$ from $K'$ in $T'$, and $T'$ is a tree $t$-spanner of $h(f(I), t)$.
and $K'$ lies in the middle of path $P$ which has length $t - 3$. Also, every vertex of $f(I)$ is within distance 2 in $T'$ from at least one of $u$ and $v$, since $u$ and $v$ induce a 5-center of $T$ (note that edge $uv$ is not in $T'$), where each of $u$ and $v$ is at distance $\left\lfloor \frac{t-1}{2} \right\rfloor$ from $K'$ in $T'$; so, every vertex of $f(I)$ is within distance $\left\lceil \frac{t+1}{2} \right\rceil$ from $K'$ in $T'$. Therefore, $K'$ is a $(t+1)$-center of $T'$, i.e. $T'$ has diameter at most $t+1$ (lemma 1).

Next, we prove that $T'$ is a tree $t$-spanner of $h(f(I), t)$. Since all edges of $R(t)$ are in $T'$, it suffices to examine the adjacencies between vertices in $f(I)$. If two vertices of $f(I)$ are adjacent in $h(f(I), t)$, then they are within distance 4 from each other in $T$, since $T$ is a tree 4-spanner of $f(I)$. But, $d_{T'}(x, y) \leq d_T(x, y) - 1 + t - 3 = d_T(x, y) + t - 4$, for every pair of vertices $x$ and $y$ in $f(I)$, since $T'$ contains path $P$ of length $t - 3$ instead of edge $uv$ of $T$ (of course, when the $x, y$-path of $T$ does not contain edge $uv$, $d_{T'}(x, y) = d_T(x, y)$).

So, every pair of adjacent in $f(I)$ vertices are within distance $t$ from each other in $T'$ (note that if $x$ and $y$ are adjacent in $f(I)$, then $d_T(x, y) \leq 4$). Hence, $T'$ is a tree $t$-spanner of $h(f(I), t)$ of diameter at most $t+1$.

Assume that graph $h(f(I), t)$ admits a tree $t$-spanner of diameter at most $t+1$. Vertices $r_{\lfloor (t-1)/2 \rfloor}$ and $r_{\lceil (t-1)/2 \rceil}$ of $R(t)$ induce the only $(t+1)$-center $K'$ of this tree $t$-spanner of $h(f(I), t)$, because of paths $P_1$ and $P_2$ of $R(t)$. Thus, because of theorem 4, graph $h(f(I), t)$ admits a tree $t$-spanner $T'$ which is a shortest paths to $K'$ spanning tree of $h(f(I), t)$. For each of $u$ and $v$ there is a unique shortest path in $h(f(I), t)$ to $K';$ so path $P$ of $R(t)$ is a subpath of $T'$ and, therefore, edge $uv$ is not in $T'$. We prove that $T = T'[f(I)] \cup \{uv\}$ is a tree $4$-spanner of $f(I)$ of diameter at most 5, where $T'[f(I)]$ is the subgraph of $T'$ induced by the vertices of $f(I)$. Obviously, $T$ is a spanning tree of $f(I)$, since $T$ does contain edge $uv$, instead of path $P$ of $T'$.

Let $x$ be an arbitrary vertex of $f(I)$. The shortest path in $T'$ from $x$ to $K'$ consists of either the $x$, $u$-path of $T$ and the $u$, $K'$-path of $T'$ or the $x$, $v$-path of $T$ and the $v$, $K'$-path of $T'$. But, each of the $u$, $K'$-path and the $v$, $K'$-path has length $\left\lfloor \frac{t-3}{2} \right\rfloor$ and, furthermore, $x$ is within distance $\left\lceil \frac{t+1}{2} \right\rceil$ from $K'$ in $T'$. So, vertex $x$ of $f(I)$ is within distance 2 in $T$ from at least one of $u$ or $v$. Therefore,
since edge $uv$ is in $T$, the pair $u$ and $v$ induces a 5-center $K$ of $T$; i.e. $T$ has diameter at most 5 (lemma\[1\]).

In order to prove that $T$ is a tree 4-spanner of $f(I)$, it suffices to examine vertices at distance 5 from each other in $T$, since $T$ has diameter at most 5. For any two vertices $x$ and $y$ of $f(I)$ at distance 5 from each other in $T$, the path of $T$ between $x$ and $y$ contains edge $uv$, since $K$ is a 5-center of $T$. Therefore, $d_{T'}(x,y) = 5 - 1 + t - 3 = t + 1$, since $T'$ does not contain edge $uv$ but contains path $P$ of $R(t)$ which has length $t - 3$. Thus, $x$ and $y$ are not adjacent to each other in $f(I)$, since $T'$ is a tree $t$-spanner of $h(f(I), t)$. Hence, $T$ is a tree 4-spanner of $f(I)$.

\[\Box\]

7 The situation for each value of the stretch factor

When $t \leq 1$, the tree $t$-spanner problem is trivial. Also, one can prove that there is an efficient algorithm to determine if a graph admits a tree 2-spanner of diameter at most 3. The following theorem summarizes the results in this chapter.

**Theorem 2** For each $t$, the problem of deciding if a graph admits a tree $t$-spanner of diameter at most $t + 1$ can be solved efficiently, when $t \leq 3$, and it is an NP-complete problem, when $t \geq 4$.

8 Acknowledgments

Results in this paper appear in my PhD thesis [15] at the department of computer science of the university of Toronto; I would like to thank my thesis supervisor, professor Derek Corneil.

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