1 Introduction

Let \( F(X, Y) \in \mathbb{Q}[X, Y] \) be a \( \mathbb{Q} \)-irreducible polynomial. In 1929 Skolem [8] proved the following beautiful theorem:

**Theorem 1.1 (Skolem)** Assume that \( F(0, 0) = 0 \).

Then for every non-zero integer \( d \), the equation \( F(X, Y) = 0 \) has only finitely many solutions in integers \( (X, Y) \in \mathbb{Z}^2 \) with \( \gcd(X, Y) = d \).

In the same year, Siegel obtained his celebrated finiteness theorem for integral solutions of Diophantine equations: equation \( F(X, Y) = 0 \) has finitely many solutions in integers unless the corresponding plane curve is of genus 0 and has at most 2 points at infinity. While Siegel’s result is, certainly, deeper and more powerful than Theorem 1.1, the latter has one important advantage. Siegel’s theorem is known to be non-effective: it does not give any bound for the size of integral solutions. On the contrary, Skolem’s method allows one to bound the solutions explicitly in terms of the coefficients of the polynomial \( F \) and the integer \( d \). Indeed, such a bound was obtained by Walsh [9]; see also [4].

In 2008 Abouzaid [1] gave a far-going generalization of Skolem’s theorem. He extended it in two directions.

First, he studied solutions not only in rational integers, but in arbitrary algebraic numbers. To accomplish this, he introduced the notion of logarithmic \( \gcd \) of two algebraic numbers \( \alpha \) and \( \beta \) which coincides with the logarithm of the usual \( \gcd \) when \( \alpha, \beta \in \mathbb{Z} \).

Second, he not only bounded the solution in terms of the logarithmic \( \gcd \), but obtained a sort of asymptotic relation between the heights of the coordinates and their logarithmic \( \gcd \).

Let us state Abouzaid’s principal result (see [1, Theorem 1.3]). In the sequel we assume that \( F(X, Y) \in \bar{\mathbb{Q}}[X, Y] \) is an absolutely irreducible polynomial, and use the notation

\[
m = \deg_X F, \quad n = \deg_Y F, \quad M = \max\{m, n\}.
\]

We denote by \( h(\alpha) \) the absolute logarithmic height of \( \alpha \in \bar{\mathbb{Q}} \) and by \( \lgcd(\alpha, \beta) \) the logarithmic \( \gcd \) of \( \alpha, \beta \in \mathbb{Q} \). We also denote by \( h_p(F) \) the projective height of the polynomial \( F \). For all definitions, see Subsection 3.1.

**Theorem 1.2 (Abouzaid)** Assume that \( (0, 0) \) is a non-singular point of the plane curve \( F(X, Y) = 0 \). Let \( \varepsilon \) satisfy \( 0 < \varepsilon < 1 \). Then for any solution \( (\alpha, \beta) \in \mathbb{Q}^2 \) of \( F(X, Y) = 0 \), we have either

\[
\max\{h(\alpha), h(\beta)\} \leq 56M^8\varepsilon^{-2}h_p(F) + 420M^{10}\varepsilon^{-2}\log(4M),
\]
\[
\max\{|h(\alpha) - n \text{lgcd}(\alpha, \beta)|, |h(\beta) - m \text{lgcd}(\alpha, \beta)|\} \leq \varepsilon \max\{h(\alpha), h(\beta)\} \\
+ 742M^7\varepsilon^{-1}h_p(F) + 5762M^9\varepsilon^{-1}\log(2m + 2n).
\]

Informally speaking,
\[
\frac{h(\alpha)}{n} \sim \frac{h(\beta)}{m} \sim \text{lgcd}(\alpha, \beta)
\]

as \(\max\{h(\alpha), h(\beta)\} \to \infty\).

Unfortunately, Abouzaid’s assumption is slightly more restrictive than Skolem’s (1): he assumes not only that the point \((0, 0)\) belongs to the plane curve \(F(X, Y) = 0\), but that \((0, 0)\) is a non-singular point on this curve. The purpose of the present article is to get rid of this non-singularity hypothesis.

Let us define, first of all, a certain quantity \(r\) which would “measure the singularity” of the point \((0, 0)\). Let \(\mathcal{C}\) be (a non-singular projective model of) the plane algebraic curve defined by \(F(X, Y) = 0\). We denote by \(x, y\) the coordinate functions on the curve \(\mathcal{C}\). We set
\[
r = \sum_{P} \min\{\nu_P(x), \nu_P(y)\},
\]
where the sum runs over the points of \(\mathcal{C}\) with \(x(P) = y(P) = 0\). Clearly, \(r > 0\) if and only if \(F(0, 0) = 0\) and \(r = 1\) if and only \((0, 0)\) is a non-singular point of the plane curve \(F(X, Y) = 0\).

We can now state our principal result.

**Theorem 1.3** Let \(F(X, Y) \in \mathbb{Q}[X, Y]\) be an absolutely irreducible polynomial satisfying \(F(0, 0) = 0\). Let \(\varepsilon\) satisfy \(0 < \varepsilon < 1\). Then, for any \(\alpha, \beta \in \mathbb{Q}\) such that \(F(\alpha, \beta) = 0\), we have either:
\[
h(\alpha) \leq 13\varepsilon^{-1}n^4m h_p(F) + 39\varepsilon^{-1}n^5m,
\]
or
\[
\left| \frac{\text{lgcd}(\alpha, \beta)}{r} - \frac{h(\alpha)}{n} \right| \leq \frac{1}{r} \left( (64n^4\varepsilon^{-1} + 45m)h_p(F) + 13n^3m(19n^2\varepsilon^{-1} + 12m) \right).
\]

By symmetry, the same kind of bound holds true for the difference \(\frac{\text{lgcd}(\alpha, \beta)}{r} - \frac{h(\beta)}{m}\). Informally speaking,
\[
\frac{h(\alpha)}{n} \sim \frac{h(\beta)}{m} \sim \frac{\text{lgcd}(\alpha, \beta)}{r}
\]
as \(\max\{h(\alpha), h(\beta)\} \to \infty\).

## 2 Definitions and results used in the article

In this section, we compile some definitions and results from different sources, which will be required for our proof. We advise the reader interested in the proof to go directly to Section 3 and come back to this Section 2 as required.

We normalize the absolute values on number fields so that they extend standard absolute values on \(\mathbb{Q}\): if \(v \mid p\) (non-Archimedean) then \(|p|_v = p^{-1}\) and if \(v \mid \infty\) (Archimedean) then \(|2014|_v = 2014\). We denote by \(M_k\) the set of places (normalized absolute values) of the number field \(K\).

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1 We reserve capital letters \(X, Y, \ldots\) for independent variables and small letters \(x, y, \ldots\) for algebraic functions.
2.1 Heights and lgcd of algebraic numbers

Let \( d = [K : \mathbb{Q}] \) and \( d_v = [K_v : \mathbb{Q}_v] \). The height of an algebraic number \( \alpha \) is defined as

\[
h(\alpha) = \frac{1}{d} \sum_{v \in M_K} d_v \log^+ |\alpha|_v.
\]

where \( K \) is a number field containing \( \alpha \) and \( \log^+ = \max\{\log, 0\} \). It is well-known that the height does not depend on the particular choice of \( K \), but only on the number \( \alpha \) itself. It is equally well-known that

\[
h(\alpha) = h(\alpha^{-1}),
\]

so that

\[
h(\alpha) = \frac{1}{d} \sum_{v \in M_K} -d_v \log^- |\alpha|_v = \sum_{v \in M_K} h_v(\alpha),
\]

where \( \log^- = \min\{\log, 0\} \) and

\[
h_v(\alpha) = -\frac{d_v}{d} \log^- |\alpha|_v.
\]

The quantities \( h_v(\alpha) \) can be viewed as “local heights”. Clearly, \( h_v(\alpha) \geq 0 \) for any \( v \) and \( \alpha \).

We define the logarithmic gcd of two algebraic numbers \( \alpha \) and \( \beta \), not both 0, as

\[
\lgcd(\alpha, \beta) = \sum_{v \in M_K} \min\{h_v(\alpha), h_v(\beta)\},
\]

where \( K \) is a number field containing both \( \alpha \) and \( \beta \). It again depends only \( \alpha \) and \( \beta \), not on \( K \). A simple verification shows that for \( \alpha, \beta \in \mathbb{Z} \) we have \( \lgcd(\alpha, \beta) = \log \gcd(\alpha, \beta) \).

Now let \( K \) be a number field and \( S \) be a set of places of \( K \). We define the \( S \)-height by

\[
h_S(\alpha) = \sum_{v \in S} h_v(\alpha).
\]

Similarly we define \( \lgcd_S \).

We shall frequently use the inequality \( \lgcd_S(\alpha, \beta) \leq h_S(\alpha) \leq h(\alpha) \) without special reference.

2.2 Affine and projective heights of polynomials

We define the projective and the affine height of a vector \( \mathbf{a} = (a_1, \cdots, a_m) \in \bar{\mathbb{Q}} \) with algebraic entries by

\[
h_p(\mathbf{a}) = \frac{1}{d} \sum_{v \in M_K} d_v \log \max_{1 \leq k \leq m} |a_k|_v, \ a \neq 0,
\]

(7)

\[
h_a(\mathbf{a}) = \frac{1}{d} \sum_{v \in M_K} d_v \log^+ \max_{1 \leq k \leq m} |a_k|_v.
\]

(8)

Here, \( K \) is a number field containing \( a_1, \cdots, a_m \), and \( d, d_v \) are defined as in the previous subsection. We notice that the height of an algebraic number defined in the previous subsection corresponds to the affine height of a projective vector.

We define the projective and affine height of a polynomial as the corresponding heights of the vector of its non-zero coefficients. If \( F \) is a non-zero polynomial, then:

\[
h_p(\alpha F) = h_p(F), \ \alpha \in \bar{\mathbb{Q}}^*,
\]

\[
h_p(F) \leq h_a(F),
\]

and \( h_p(F) = h_a(F) \) if \( F \) has a coefficient equal to 1.

In [6, Lemma 4], Schmidt proves the following lemma:
**Proposition 2.1** Let $F(X, Y) \in \bar{\mathbb{Q}}[X, Y]$ be a polynomial with algebraic coefficients, such that $m = \deg_X F$ and $n = \deg_Y F$. Let $R_F(X) = \text{Res}_Y(F, F'_Y)$ be the resultant of $F$ and its derivative polynomial with respect to $Y$. Then:

$$h_p(R_F) \leq (2n - 1)h_p(F) + (2n - 1)\log((m + 1)(n + 1)\sqrt{n}). \tag{9}$$

Another useful theorem on the height of the resultants is [1, Proposition 2.4]:

**Proposition 2.2** Let $F_1(X, Y)$ and $F_2(X, Y)$ be polynomials with algebraic coefficients, and let $R(X)$ be their resultant with respect to the variable $Y$. Put:

$$m_i = \deg_X F_i, \ n_i = \deg_Y F_i \ (i=1,2).$$

Then

$$h_p(R) \leq n_1 h_p(F_2) + n_2 h_p(F_1) + (m_1 n_2 + m_2 n_1) + (n_1 + n_2) \log(n_1 + n_2).$$

A lemma widely found in literature (for example, [2, Proposition 3.6]) is the following:

**Lemma 2.3** Let $F(X)$ be a polynomial of degree $m$ with algebraic coefficients. Let $\alpha$ be a root of $F$. Then, $h(\alpha) \leq h_p(F) + \log 2$

We will also use [1, Proposition 2.5]:

**Lemma 2.4** Let $F(X, Y) \in \bar{\mathbb{Q}}[X, Y]$ be a polynomial with $m = \deg_X F$ and $n = \deg_Y F$ and let $\alpha, \beta$ be two algebraic numbers. Then

1. We have $h(F(\alpha, \beta)) \leq h_a(F) + mh(\alpha) + nh(\beta) + \log((m + 1)(n + 1))$,

2. If $F(\alpha, \beta) = 0$ with $F(\alpha, Y)$ not vanishing identically, then:

$$h(\beta) \leq h_p(F) + mh(\alpha) + n + \log(m + 1).$$

### 2.3 Coefficients versus roots

In this subsection we establish some simple relations between coefficients and roots of a polynomial over a field with absolute value, needed in the proof of our main result. It will be convenient to use the notion of $v$-Mahler measure of a polynomial.

Let $K$ be a field with absolute value $v$ and $f(X) \in K[X]$ a polynomial of degree $n$. Let $\beta_1, \ldots, \beta_n \in \bar{K}$ be the roots of $f$:

$$f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 = a_n (X - \beta_1) \cdots (X - \beta_n).$$

Define the $v$-Mahler measure of $f$ by

$$M_v(f) = |a_n|_v \prod_{i=1}^{n} \max \{1, |\beta_i|_v\},$$

where we extend $v$ somehow to $\bar{K}$. (Clearly, $M_v(f)$ does not depend on the particular extension of $v$.) It is well-known that $|f|_v = M_v(f)$ for non-archimedean $v$ (“Gauss lemma”) and $M_v(f) \leq (n + 1)|f|_v$ for archimedean $v$ (Mahler).

**Lemma 2.5** Let $\beta_1, \ldots, \beta_{\ell+1}$ be $\ell + 1$ distinct roots of $f(X)$, where $0 \leq \ell \leq n - 1$. Then

$$\max \{|\beta_1|_v, \ldots, |\beta_{\ell+1}|_v\} \geq c_v(n) \frac{|a_\ell|_v}{|f|_v},$$

where $c_v(n) = 1$ for non-archimedean $v$ and $c_v(n) = (n + 1)^{-1}2^{-n}$ for archimedean $v$. 

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4
Proof We have
\[ a_\ell = \pm a_n \sum_{1 \leq i_1 < \cdots < i_{\ell} \leq n} \beta_{i_1} \cdots \beta_{i_{\ell}}, \tag{10} \]
where \( \beta_1, \ldots, \beta_n \) are all roots of \( f(X) \) in \( \bar{K} \) counted with multiplicities. Observe that each term in the sum above contains one of the roots \( \beta_1, \ldots, \beta_{\ell+1} \), and the product of the other roots together with \( a_n \) is \( v \)-bounded by \( M_v(f) \). Hence, denoting \( \mu = \max\{|\beta_1|_v, \ldots, |\beta_{\ell+1}|_v\} \geq 1 \), we obtain \( |a_\ell|_v \leq \mu M_v(f) \) in the non-archimedean case and \( |a_\ell|_v \leq (\ell)! \mu M_v(f) \) in the archimedean case. Since \((\ell)! \leq 2^n\), the result follows. \( \square \)

Lemma 2.6 Let \( \ell \) be the number of roots \( \beta \in \bar{K} \) of \( f \) (counted with multiplicities) satisfying \( |\beta|_v < 1 \). (We again fix an extension of \( v \) to \( \bar{K} \); clearly, \( \ell \) does not depend on the choice of the extension.) Assume that \( v \) is non-archimedean. Then \( |a_\ell|_v = |f|_v \) and \( |a_i|_v < |f|_v \) for \( i = 0, \ldots, \ell - 1 \).

Proof Write the roots of \( f \) as \( \beta_1, \ldots, \beta_n \), where
\[ |\beta_1|_v \leq \cdots \leq |\beta_\ell|_v < 1 \leq |\beta_{\ell+1}|_v \leq \cdots \leq |\beta_n|_v. \]
The second statement follows from Lemma 2.5: applying it with \( i \) instead of \( \ell \), we find
\[ |a_i|_v \leq |\beta_{\ell+1}|_v M_v(f) < M_v(f) = |f|_v, \]
as wanted.

For the first statement, using (10), we find
\[ |a_\ell|_v = |a_n \beta_{\ell+1} \cdots \beta_n|_v = M_v(f) = |f|_v, \]
as wanted. \( \square \)

We will apply Lemma 2.6 in the following situation. Let \( K \) be a field, and let \( F(Y) \in K[X,Y] \) be a polynomial not divisible by \( X \) (that is, the polynomial \( F(0,Y) \) is not identically 0). Write
\[ F(X,Y) = F_n(X)Y^n + \cdots + F_0(X). \]
Viewing \( F(x,Y) \) as a polynomial in \( Y \) over the field \( K((x)) \) of formal power series, it has \( n \) roots in the algebraic closure \( \bar{K}((x)) \). We extend the additive valuation \( \nu_x \) from \( K((x)) \) to \( \bar{K}((x)) \).

Corollary 2.7 In the above set-up, assume that among \( n \) roots \( y(x) \) of \( F(x,Y) \) there are exactly \( \ell \) such that \( \nu_x(y(x)) > 0 \). Then the polynomial \( F(0,Y) \) has at 0 a zero of order \( \ell \).

Proof Lemma 2.6 implies that \( \nu_x(F_i(x)) > 0 \) for \( i = 0, \ldots, \ell - 1 \) and \( \nu_x(F_\ell(x)) = 0 \). This is exactly what is wanted. \( \square \)

2.4 Eisenstein’s theorem

In this subsection, we recall the quantitative Eisentein’s theorem due to work from Dwork, Robba, Schmidt and Van der Poorten, as given in [2]. It will be convenient to use the notion of \( M_K \)-divisor.

An \( M_K \)-divisor is an infinite vector \((A_v)_{v \in M_K}\) of positive real numbers, each \( A_v \) being associated to one \( v \in M_K \) such that for all but finitely many \( v \in M_K \) we have \( A_v = 1 \). An \( M_K \)-divisor is effective if for all \( v \in M_K \), \( A_v \geq 1 \).

We define the height of an \( M_K \)-divisor \( \mathcal{A} = (A_v)_{v \in M_K} \) as
\[ h(\mathcal{A}) = \sum_{v \in M_K} \frac{d_v}{d} \log A_v. \tag{11} \]

In [2, Theorem 7.5], Bilu and Borichev prove the following version of Eisenstein’s theorem:
Theorem 2.8 Let \( F(X,Y) \) be a separable polynomial of degrees \( m = \deg_X F \) and \( n = \deg_Y F \). Further, let \( y(X) = \sum_{k=0}^{\infty} a_k X^{k/e} \in \mathbb{K}[X^{1/e}] \) be a power series satisfying \( F(X,y(X)) = 0 \).

Then, there exists an effective \( M_{K}-\text{divisor} \) \( \mathcal{A} = (A_v)_{v \in M_K} \) such that:
\[
|a_k|_v \leq \max\{1, |a_{\ell}|_v\} A_{k/e}^{1/\ell} (k = 1, 2, \ldots),
\]
for any \( v \in M_K \) anyhow extended to \( \overline{\mathbb{K}} \) and any \( k \geq \kappa \), and such that:
\[
h(\mathcal{A}) \leq (3n + e - 1)h_p(F) + 3n \log(nm) + 10en.
\]

Applying this theorem to our case \((a_k = 0 \text{ for } k \leq 0)\) and setting \( \kappa = 0 \), we obtain that:

Lemma 2.9 Let \( F(X,Y) \) be a separable polynomial of degrees \( m = \deg_X F \) and \( n = \deg_Y F \). Further, let \( y(X) = \sum_{k=0}^{\infty} a_k X^{k/e} \in \mathbb{K}[X^{1/e}] \) be a power series satisfying \( F(X,y(X)) = 0 \).

Then, there exists an effective \( M_{K}-\text{divisor} \) \( \mathcal{A} = (A_v)_{v \in M_K} \) such that:
\[
|a_k|_v \leq A_{k/e} (k = 1, 2, \ldots),
\]
for any \( v \in M_K \) anyhow extended to \( \overline{\mathbb{K}} \) and any \( k \geq 1 \), and such that:
\[
h(\mathcal{A}) \leq (3n + e - 1)h_p(F) + 3n \log(nm) + 10en \leq 4nh_p(F) + 3n \log(nm) + 10en.
\]

Another lemma that will be useful in this paper is [1, Proposition 2.7]:

Lemma 2.10 Let \( \mathbb{K} \) be a number field and let \( y(X) = \sum_{k=1}^{\infty} a_k X^{k/e} \) be a series with coefficients in \( \mathbb{K} \) such that there exists an effective \( M_{K}-\text{divisor} \) \( \mathcal{A} = (A_v)_{v \in M_K} \), such that for all index \( k \geq 1 \) we have \( |a_k|_v \leq A_{k/e}^{(\ell+1)/\ell} \). For all \( \ell \in \mathbb{K} \), let’s put \( y^\ell(X) = \sum_{k=1}^{\infty} a_k^{(\ell)} X^{k/e} \). Then, for any valuation \( v \) and for all index \( k \geq 1 \) we have:
\[
|a_k^{(\ell)}|_v \leq \left\{ \begin{array}{ll}
|A_{k/e}|^{(\ell+1)/\ell} A_{k/e}^{(\ell+1)/\ell}, & \text{if } v|\infty, \\
|A_{k/e}|^{(\ell+1)/\ell}, & \text{if } v < \infty.
\end{array} \right.
\] (12)

2.5 Siegel’s "absolute" lemma

The height of a vector space \( W \) generated by the free set \((w_1, \ldots, w_d)\) is defined as:
\[
h_s(W) = \sum_{v \in M_K} \frac{d_v}{d} \log ||w_1 \wedge \cdots \wedge w_d||_v,
\] (13)
where
\[
||w_i||_v = \left\{ \begin{array}{ll}
(|w_{1i}|^2 + \cdots + |w_{di}|^2)^{1/2}, & \text{if } v|\infty, \\
\max_{1 \leq j \leq d} \{|w_{ij}|_v\}, & \text{if } v < \infty,
\end{array} \right.
\]
is a norm. In [3, Lemma 4.7], David and Philippon give the following version of Siegel’s lemma:

Lemma 2.11 Let \( W \subset \mathbb{Q}^{n+1} \) be a vector subspace of dimension \( d \) and let \((w_1, \ldots, w_d)\) be a basis of \( W \) over a number field \( \mathbb{K} \). Then, for any \( \varepsilon > 0 \), there exists \( y \in W \) such that:
\[
h(y) \leq \frac{h_s(W)}{d} + \frac{1}{d} \sum_{s=1}^{d-1} \sum_{t=1}^{s} \frac{1}{2t} + \varepsilon.
\] (14)

For our purposes, we find the following corollary:
Lemma 2.12 Let $n$ be a positive integer, and $L_i(x) \in \bar{\mathbb{Q}}[x]$ $(1 \leq i \leq M < n)$ a set of $M$ linear forms in $x$. Then, there exists $\alpha \in \bar{\mathbb{Q}}^n$, $\alpha \neq 0$ such that for all $1 \leq i \leq M$, $L_i(\alpha) = 0$ and

$$h(\alpha) \leq \frac{1}{n-M}(h(L_1) + \cdots + h(L_M)) + \frac{M}{2(n-M)} \log M + \frac{1}{2} \log(n-M).$$

(15)

Proof: Let $\mathcal{L}^* = \mathcal{L}(\bar{\mathbb{Q}}^n, \mathbb{K})$ be the $\mathbb{K}$-vector space of linear forms from $\bar{\mathbb{Q}}^n$ into the base number field $\mathbb{K}$. Let $(L_1, \cdots, L_M)$ be a free system in $\mathcal{L}^*$. Let $W = \{ X \in \bar{\mathbb{Q}}^n : L_1(X) = 0, \cdots, L_M(X) = 0 \}$. It is well known that $W$ is a subspace of $\bar{\mathbb{Q}}^n$ (as the intersection of $M'$ hyperplanes).

Then, the orthogonal of $W$, defined by $W^\perp = \{ \text{Linear forms } L \text{ such that } \forall w_i \in W, L(w_i) = 0 \}$, is exactly the $\mathbb{K}$-vector space generated by $(L_1, \cdots, L_M)$. We know that, $\dim W^\perp = n - \dim W = n - d = M'$. We also know that $h_s(W) = h_s(W^\perp)$ (cf. \cite[Chapter 1, Lemma 4C]).

Clearly,

$$\frac{1}{d} \sum_{s=1}^{d-1} \sum_{t=1}^{s} \frac{1}{2t} \leq \frac{1}{2} \log d - \frac{1}{2d}.$$ 

Applying Lemma 2.11, we obtain that there exists $y \in W$ such that:

$$h(y) \leq \frac{h_s(W)}{d} + \frac{1}{d} \sum_{s=1}^{d-1} \sum_{t=1}^{s} \frac{1}{2t} + \varepsilon \leq \frac{h_s(W^\perp)}{d} + \frac{1}{2} \log d - \frac{1}{2d} + \varepsilon.$$

Let’s set $\varepsilon = \frac{1}{2d}$. Let $\mathcal{L}$ be the $M' \times M'$ matrix of $(L_1, \cdots, L_M)$ expressed in $(e_1, \cdots, e_M)$, an orthonormal basis of $W^\perp$.

Hadamard’s inequality applied to $\mathcal{L}$ gives $|\det \mathcal{L}| \leq ||L_1||_2 \cdots ||L_M||_2$, which, in our case, translates into:

$$|\det \mathcal{L}|_v \leq \left\{ \begin{array}{cl} M'^\frac{M'}{2} ||L_1||_v \cdots ||L_M||_v, & \text{if } v|\infty, \\ ||L_1||_v \cdots ||L_M||_v, & \text{if } v < \infty. \end{array} \right.$$ 

Therefore,

$$||L_1 \wedge \cdots \wedge L_M||_v \leq \left\{ \begin{array}{cl} M'^\frac{M'}{2} ||L_1||_v \cdots ||L_M||_v, & \text{if } v|\infty, \\ ||L_1||_v \cdots ||L_M||_v, & \text{if } v < \infty. \end{array} \right.$$ 

By taking logarithms of both sides and adding up, we obtain:

$$h_s(W^\perp) \leq h(L_1) + \cdots + h(L_M) + \frac{M'}{2} \log M'.$$

We notice that if instead of a basis of $W^\perp$ we were working with a system $(L_1, \cdots, L_M)$ $(M \geq M')$ generating $W^\perp$, we would have

$$h_s(W^\perp) \leq h(L_1) + \cdots + h(L_M) + \frac{M'}{2} \log M' \leq h(L_1) + \cdots + h(L_M) + \frac{M}{2} \log M.$$

$\Box$

3 Proof

In the cases $m = 1$ or $n = 1$ the things are much simpler, and one can easily obtain a numerically sharper result. Therefore, we allow ourselves to assume that $m, n \geq 2$.

Let $P_1, \ldots, P_n$ be the common zeros of $x$ and $y$, and $e_i = \nu_{P_i}(x)$, $\kappa_i = \nu_{P_i}(y)$. Then, $y$ has $e_i$ Puisieux expansions in each $P_i$ belonging to the gcd of the common zeros of $x$ and $y$, of the form:
\[ y_{ij}(X) = \sum_{k=\kappa_i}^{\infty} a_{ik} \left( \zeta_i^{j-1} X^{1/e_i} \right)^k = \sum_{k=\kappa_i}^{\infty} a^{(ij)}_k X^{k/e_i}, (j = 1, \ldots, e_i), (i = 1, \ldots, s), \]

where

\[ \kappa_i \geq 1 \text{ and } a^{(ij)}_k > 0. \]

We have \( \kappa_i \geq 1 \) because for all \( i = 1, \ldots, s \), we have \( y(P_i) = 0 \). We define \( E = \sum_{i=1}^{s} e_i \) and \( \Xi = \sum_{i=1}^{s} \kappa_i. \)

Let \( \mathbb{K} \) be a number field containing the coefficients of \( F \), the numbers \( \alpha, \beta \) and the coefficients \( a^{(ij)}_k \) of the power series \( y_{ij} \), as well as the roots of \( F(\alpha, Y) \in \mathbb{K}[Y] \). The series \( y_{ij}(X), i \in \{1, \ldots, s\}, j \in \{1, \ldots, e_i\}, \) satisfy \( F(X, y_{ij}(X)) = 0. \)

According to Lemma 2.9, there exist \( s \) effective \( \mathcal{M}_\mathbb{K} \)-divisors \( A^{(i)}_v \) such that:

\[ |a^{(ij)}_k|_v \leq \left( A^{(i)}_v \right)^{k/e_i}, \]

for any \( v \in \mathcal{M}_\mathbb{K} \) and any \( k \geq 1. \)

Let \( H = \max_{i=1, \ldots, s} A^{(i)} \). Then, \( H \) satisfies:

\[ H \leq 4n\rho(F) + 3n \log(nm) + 10En. \]

We know that \( |a^{(ij)}_k|_v = |\zeta_i^{(1-j)k}|_v |a^{(ii)}_k|_v = |a^{(ii)}_k|_v. \)

Let’s set:

\[ H_v = \begin{cases} \min_{1 \leq i \leq s} \left\{ \min_{1 \leq k \leq s} \left( \min_{\kappa_i} \left( 2^{e_i} A^{(i)}_v \right)^{-(\kappa_i+1)} \right) \right. & \text{if } v \mid \infty, \\
\min_{1 \leq i \leq s} \left\{ \min_{1 \leq k \leq s} \left( \min_{\kappa_i} \left( A^{(i)}_v \right)^{-(\kappa_i+1)} \right) \right. & \text{if } v < \infty. \end{cases} \]

We may partition \( \mathcal{M}_\mathbb{K} \) into three disjoint subsets \( \mathcal{M}_\mathbb{K} = S \cup S' \cup S'' \), where:

\[ S = \left\{ v \in \mathcal{M}_\mathbb{K} \text{ such that } |v|_v \begin{cases} \leq H_v & \text{if } v \mid \infty, \\
< H_v & \text{if } v < \infty \end{cases} \right\}, \]

\[ S' = \left\{ v \in \mathcal{M}_\mathbb{K} \text{ such that } \begin{cases} H_v < |v|_v < 1 & \text{if } v \mid \infty, \\
H_v \leq |v|_v < 1 & \text{if } v < \infty \end{cases} \right\}, \]

\[ S'' = \left\{ v \in \mathcal{M}_\mathbb{K} \text{ such that } |v|_v \geq 1 \right\}, \]

and

\[ \operatorname{lcmd}(\alpha, \beta) = \operatorname{lcmd}_S(\alpha, \beta) + \operatorname{lcmd}_S'(\alpha, \beta) + \operatorname{lcmd}_{S''}(\alpha, \beta). \]

Since \( \operatorname{lcmd}_{S''}(\alpha, \beta) = 0 \), we have \( \operatorname{lcmd}(\alpha, \beta) = \operatorname{lcmd}_S(\alpha, \beta) + \operatorname{lcmd}_S'(\alpha, \beta). \)

For every \( v \in S \), each of the series \( y_{ij}(X) \) converges \( v \)-adically at \( X = \alpha \) (but not necessarily to \( \beta \)). The sum is a root of the polynomial \( F(\alpha, Y) \in \mathbb{K}[Y] \). Let

\[ T_{ij} = \left\{ v \in S : y_{ij}(\alpha) = \beta \right\} \tag{16} \]

be the subset of \( S \) such that the \( v \)-adic sum of \( y_{ij}(x) \) at \( \alpha \) is \( \beta \). We say that \( y_{ij}(\alpha) \equiv \beta \), where \( \equiv \) means that the equality is understood in the sense of \( v \)-adic convergence. We may partition \( S \) as

\[ S = \bigcup_{i=1}^{s} \bigcup_{j=1}^{e_i} T_{ij} \cup T' \]

where

\[ T' = \left\{ v \in S : y_{ij}(\alpha) = \lambda_{ij} \neq \beta, \text{ for all } i \in \{1, \ldots, s\}, j \in \{1, \ldots, e_i\} \right\}. \]

Then:
The first term will be bounded in Proposition 3.3 and the other two in Propositions 3.4 and 3.5 respectively. From Proposition 3.3, we obtain:

\[ |\lgcd_{T'}(\alpha, \beta)| \leq h_p(F) + 3m\mathcal{H} + (s + 2)(m + 1)n. \]

From Proposition 3.4, we obtain:

\[ |\lgcd_{S'}(\alpha, \beta)| \leq 3m\mathcal{H} + s(m + 1)n. \]

Finally, from Proposition 3.5, we obtain:

\[ \left| \frac{\lgcd(S \setminus T)(\alpha, \beta)}{r} - \frac{h(\alpha)}{n} \right| \leq \frac{1}{r} \left( (64n^4\varepsilon^{-1} + 45m)h_p(F) + 13n^3m(19n^2\varepsilon^{-1} + 12m) \right). \]

The rest of the proof consists in bounding these three terms. For this, we start with some preliminaries by reminding some useful definitions and results, and preparing some theorems and properties to be used in our context. Then, we easily bound the first and second term. The rest of the paper leads us to the bound of the third term.

### 3.1 Preliminaries

#### 3.1.1 Technical lemma

This technical lemma which will be used all throughout the rest of the proofs:

**Lemma 3.1** We have:

\[ \sum_{\nu \in M_\mathcal{E}} \frac{d_\nu}{d} \log^+ \left( H_\nu^{-1} \right) \leq 3m\mathcal{H} + s(m + 1)n. \]
Proof:

\[ \sum_{v \in M_K} \frac{d_v}{d} \log^+ (H_v^{-1}) = \sum_{v \in M_{K_v}^\infty} \frac{d_v}{d} \log^+ (H_v^{-1}) + \sum_{v \in M_{K_v}^\infty} \frac{d_v}{d} \log^+ (H_v^{-1}) \]

\[ \leq \sum_{i=1}^{s} \left[ e_i \sum_{v \in M_K} \frac{d_v}{d} \left( \log^+ |a_{\kappa_i}|_{v}^{-1} \right) + (\kappa_i + 1)h \left( \mathcal{A}(i) \right) \right] + s(m + 1)n \log 2 \]

\[ = \sum_{i=1}^{s} \left[ e_i h(a_{\kappa_i}) + (\kappa_i + 1)h \left( \mathcal{A}(i) \right) \right] + s(m + 1)n \]

\[ \leq \sum_{i=1}^{s} \left[ e_i \frac{d_v}{d} \left( \log^+ |a_{\kappa_i}|_{v}^{-1} \right) + (\kappa_i + 1)h \left( \mathcal{A}(i) \right) \right] + s(m + 1)n \log 2 \]

\[ = \sum_{i=1}^{s} \left[ e_i h(a_{\kappa_i}) + (\kappa_i + 1)h \left( \mathcal{A}(i) \right) \right] + s(m + 1)n \]

\[ \leq \mathcal{H} \sum_{i=1}^{s} (2\kappa_i + 1) + s(m + 1)n \leq 3m \mathcal{H} + s(m + 1)n \]

(using Lemma 2.9 we find that \( e_i h(a_{\kappa_i}) \leq \kappa_i h(\mathcal{A}(i)) \), and \( \Xi \leq m \)).

□

3.1.2 Partition of \( S \)

We remind that \( T_{ij} = \{ v \in S : y_{ij}(\alpha) \equiv \beta \} \), is the subset of \( S \) such that the \( v \)-adic sum of \( y_{ij}(x) \) at \( \alpha \) is \( \beta \) (cf. 16), and \( S = \bigcup_{i=1}^{s} \bigcup_{j=1}^{e_i} T_{ij} \cup T' \).

Proposition 3.2 If \( h(\alpha) \geq 2n (h_p(F) + 5 \log(mn)) \), then the sets \( T_{ij} \) are pairwise disjoints

Proof  Let \( Q(Y) = \prod_{i=1}^{s} \prod_{j=1}^{e_i} (Y - y_{ij}(X)) \in \mathbb{K}[[X]][Y] \). Write:

\[ F(X,Y) = F_n(X)Y^{n} + \cdots + F_0(X) \]

We may assume that \( F_n(\alpha) \neq 0 \). Otherwise, by Lemma 2.3, \( h(\alpha) \leq h_p(F_n) + \log 2 \leq h_p(F) + \log 2 < 2n h_p(F) + 10 n \log(mn) \). The polynomial \( Q(Y) \) divides \( F(X,Y) \) in the ring \( \mathbb{K}[[X]][Y] \). Write \( F(X,Y) = Q(Y)U(Y) \), where \( U(Y) = F_n(X)Y^{n-c} + u_{n-c-1}(X)Y^{n-c-1} + \cdots + u_0(X) \). Each of the series \( u_i(X) \), \( 0 \leq i \leq n - r - 1 \) can be expressed as a polynomial in \( y_{i1}(X), \ldots, y_{ir}(X) \). Thus, every series \( u_i(X) \), \( 0 \leq i \leq n - r - 1 \) \( v \)-adically converges at \( X = \alpha \) for all \( v \in S \) and the convergence is absolute when \( v \) is Archimedean. Therefore, for \( v \in S \),

\[ F(\alpha, Y) = (Y - y_{11}(\alpha)) \cdots (Y - y_{sc}(\alpha)) \left( F_n(\alpha)Y^{n-c} + u_{n-c-1}(\alpha)Y^{n-c-1} + \cdots + u_0(\alpha) \right) \]

Now, let’s assume that \( v \in T_{ij} \cap T_{ik} \), that is \( \beta = y_{ij}(\alpha) = y_{ik}(\alpha) \). Hence, \( \beta \) is at least double root of \( F(\alpha, Y) \). It follows that \( R_E(\alpha) = 0 \) (\( R_E \) as defined in Proposition 2.1). We find that:

\[ h(\alpha) \leq h_p(R_E) + \log 2 \]

\[ \leq (2n - 1) h_p(F) + (2n - 1) \log((m + 1)(n + 1)\sqrt{n}) + \log 2 \quad (\text{Lemma 2.3}) \]

\[ < 2n h_p(F) + 10 n \log(mn) \quad (\text{Proposition 2.1}) \]

(17)

From now on, we assume that \( h(\alpha) \geq 2n h_p(F) + 10 n \log(mn) \).
3.2 First bound

To bound \(|\lgcd_{F'}(\alpha, \beta)|\), we will mainly use Lemma 2.5.

**Proposition 3.3** We have:

\[
\lgcd_{F'}(\alpha, \beta) \leq h_p(F) + 3m \mathcal{H} + (s + 2)(m + 1)n.
\]

**Proof** We write \(F(X, Y)\) as

\[
F(X, Y) = F_n(X)Y^n + \cdots + F_0(X).
\]

Since the polynomial \(F(0, Y)\) has at 0 a root of order exactly \(E = e_1 + \ldots + e_s\) (this follows from Corollary 2.7), we have \(F_E(0) \neq 0\), and we may assume in the sequel that \(F_E(0) = 1\); in particular, \(|F|_v \geq 1\) for all \(v\).

We will prove that for every \(v \in T'\) we have either

\[
|\alpha|_v \geq c(m, n, v)|F|_v^{-1}H_v, \quad (18)
\]

or

\[
|\beta|_v \geq c(m, n, v)|F|_v^{-1}, \quad (19)
\]

where

\[
c(m, n, v) = \begin{cases} 
((m + 1)(n + 1)2^{n+1})^{-1}, & \text{if } v|\infty, \\
1, & \text{if } v < \infty.
\end{cases}
\]

In the sequel we fix \(v \in T'\), and prove (19) assuming that (18) is false. When \(v \in T'\) the polynomial

\[
f(Y) = F(\alpha, Y) = F_n(\alpha)Y^n + \cdots + F_0(\alpha),
\]

has \(\sum_{i=1}^s e_i + 1 = E + 1\) distinct roots \(y_{11}(\alpha), \ldots, y_{se}(\alpha, \beta)\), and we want to apply Lemma 2.5. For this purpose, we need to estimate \(|F_E(\alpha)|_v\) from below and \(|f|_v\) from above. Since \(|\alpha|_v \leq 1\), we have

\[
|f|_v \leq \begin{cases} 
(m + 1)|F|_v, & v|\infty, \\
|F|_v, & v < \infty.
\end{cases}
\]

Further, since \(F_E(0) = 1\) and (18) is false, we have

\[
|F_E(\alpha)|_v \geq \begin{cases} 
1/2, & v|\infty, \\
1, & v < \infty;
\end{cases}
\]

After this preparation, we may apply Lemma 2.5. We obtain

\[
\max\{|y_{11}(\alpha)|_v, \ldots, |y_{se}(\alpha)|_v, |\beta|_v\} \geq c(m, n, v)|F|_v^{-1}. \quad (20)
\]

We will now bound \(|y_{ij}(\alpha)|_v\). We know that \(|y_{ij}(\alpha)|_v \leq \sum_{k=1}^\infty (A^{(i)}|\alpha|_v)^{k/e}\). Therefore, because \(A^{(i)}|\alpha|_v \leq 1\), we have that for all \(k \geq 1\), \((A^{(i)}|\alpha|_v)^{1/e} \geq (A^{(i)}|\alpha|_v)^{k/e}\) and thus, if \(v < \infty\), \(|y_{ij}(\alpha)|_v \leq (A^{(i)}|\alpha|_v)^{1/e}\). In the Archimedean case, we find (using the definition of the set \(S\)) that \(|y_{ij}(\alpha)|_v \leq 2 (A^{(i)}|\alpha|_v)^{1/e}\).

Since \(|y_{ij}(\alpha)|_v \leq 2 (A^{(i)}|\alpha|_v)^{1/e}\) and (18) is false, we cannot have \(|y_{ij}(\alpha)|_v \geq c(m, n, v)|F|_v^{-1}\) for \(i = 1, \ldots, r\) and \(j = 1, \ldots, e_i\). Hence \(|\beta|_v \geq c(m, n, v)|F|_v^{-1}\), which is (19). Thus, for \(v \in T'\) either (18) or (19) holds true. It follows that

\[
\min\{h_v(\alpha), h_v(\beta)\} \leq \frac{d_v}{d} (\log |F|_v + \log H_v^{-1} + \log c(m, n, v)^{-1}).
\]
Summing up over \( v \in T' \), we obtain
\[
\lgcd_{T'}(\alpha, \beta) \leq h_p(F) + 3m\mathcal{H} + s(m + 1)n + (n + 1) \log 2 + \log((m + 1)(n + 1)) \\
\leq h_p(F) + 3m\mathcal{H} + (s + 2)(m + 1)n.
\]

\[\square\]

### 3.3 Second bound

To bound \( |\lgcd_{S'}(\alpha, \beta)| \), we will use the technical Lemma 3.1.

**Proposition 3.4** We have:
\[
\lgcd_{S'}(\alpha, \beta) \leq 3m\mathcal{H} + s(m + 1)n.
\]

**Proof:** We have
\[
\lgcd_{S'}(\alpha, \beta) \leq h_{S'}(\alpha) \\
= \frac{1}{d} \sum_{v \in S'} d_v \log^+ (\lfloor |\alpha|_v^{-1} \rfloor) \\
\leq \frac{1}{d} \sum_{v \in M_k} d_v \log^+ (H_v^{-1}) \\
\leq 3m\mathcal{H} + s(m + 1)n.
\]

\[\square\]

### 3.4 Third bound

We now need to bound \( |\lgcd_{S \setminus T'}(\alpha, \beta) - \frac{e}{n} h(\alpha)| \). Remember that \( S \setminus T' = \bigcup_{i=1}^s \cup_{j=1}^{e_i} T_{ij} \) where \( T_{ij} \) has been defined in (16).

**Proposition 3.5** We have:
\[
\left| \lgcd_{S \setminus T'}(\alpha, \beta) - \frac{e}{n} h(\alpha) \right| \leq (2\Xi + 3E(4\varepsilon^{-1}n^2 + m) + 4nE)\mathcal{H} + 44En^2\varepsilon^{-1}.
\]

**Proof:** Below we prove Proposition 3.6 and Proposition 3.7. Combining them, we have:
\[
\left| \lgcd_{T_{ij}}(\alpha, \beta) - \frac{e}{n} h(\alpha) \right| \leq 2\Xi h(\mathcal{A}^{(i)}) + \log 2 + 3(4\varepsilon^{-1}n^2 + m)\mathcal{H} + 42n^2\varepsilon^{-1} + 4n(h(\mathcal{A}^{(i)}) + 1).
\]

Summing up over \( j \in \{1, \ldots, e_i\} \) and over \( i \in \{1, \ldots, s\} \), we obtain the result. \(\square\)

From now on, we focus on a particular \( T_{ij} \). We fix \( i \) and \( j \), and write \( T = T_{ij}, y(X) = y_{ij}(X), \mathcal{A} = \mathcal{A}^{(i)}, A_v = A_v^{(i)}, \kappa = \kappa_i, e = e_i, \theta = \theta_i, a_k = a_k^{(ij)}, a_\kappa = a_\kappa_i, \ldots \)

**Proposition 3.6** We have
\[
\left| \lgcd_T(\alpha, \beta) - \min\{1, \frac{\kappa_i}{e}\} h_T(\alpha) \right| \leq 2\frac{\kappa}{e} h(\mathcal{A}) + \log 2. \tag{21}
\]
Proof: For \( v \in T \), \( y(\alpha) = \beta \). Then, \( \beta = \sum_{k=\kappa}^{\infty} a_k \alpha^{k/e} \).

According to Lemma 2.9, there exists an effective \( M_{k} \)-divisor \( \mathcal{A} \) such that \( |a_k|_v \leq A_v^{k/e} \).

Let's call (A) the assumption \( \gamma \frac{\kappa}{e} \geq 1^{\kappa} \) and (B) the assumption \( \gamma \frac{\kappa}{e} < 1^{\kappa} \).

Case (A) We have:

\[
|a_k \alpha^{k/e}|_v \leq A_v^{k/e} |\alpha|_v^{\kappa/e} |\alpha|_v^{(k-\kappa)/e}.
\]

However, for \( k \geq \kappa \):

\[
\begin{align*}
|a_k \alpha^{k/e}|_v &\leq A_v^{k/e} |\alpha|_v^{\kappa/e} (2^{\kappa} A_v)^{-(k-\kappa)/e} \leq A_v^{k/e} |\alpha|_v^{\kappa/e} 2^{-k+\kappa}, \quad \text{if } v|\infty, \\
|a_k \alpha^{k/e}|_v &\leq A_v^{k/e} |\alpha|_v^{\kappa/e} A_v^{-(k-\kappa)/e} \leq A_v^{k/e} |\alpha|_v^{\kappa/e}, \quad \text{if } v < \infty.
\end{align*}
\]

In the non-Archimedean case, we know that \( v \in S \Rightarrow |a_k \alpha^{k/e}|_v \xrightarrow{k \to \infty} 0 \) and thus that \( \sum_{k=\kappa}^{\infty} a_k \alpha^{k/e} \) converges \( v \)-adically. We just proved that in this case, we also have \( \forall k \geq \kappa, |a_k \alpha|_v^{k/e} \leq A_v^{k/e} |\alpha|_v^{k/e} \), thus \( |\beta|_v = |\sum_{k=\kappa}^{\infty} a_k \alpha^{k/e}|_v \leq A_v^{\kappa/e} |\alpha|_v^{k/e} \).

In the Archimedean case, we have obtained that:

\[
\left| \sum_{k=\kappa}^{\infty} a_k \alpha^{k/e} \right|_v \leq 2^{\kappa} A_v^{k/e} |\alpha|_v^{\kappa/e} \sum_{k=\kappa}^{\infty} 2^{-k} = 2 A_v^{k/e} |\alpha|_v^{\kappa/e}.
\]

We notice that these results remain valid in Case (B).

Thus,

\[
|\alpha|_v \leq \max\{ |\alpha|_v, |\beta|_v \} \leq \begin{cases} 
\max\{ 2 A_v^{k/e} |\alpha|_v^{\kappa/e}, |\alpha|_v \} \leq 2 A_v^{\kappa/e} |\alpha|_v, & \text{if } v|\infty, \\
\max\{ A_v^{\kappa/e} |\alpha|_v^{\kappa/e}, |\alpha|_v \} \leq A_v^{\kappa/e} |\alpha|_v, & \text{if } v < \infty,
\end{cases}
\]

and therefore,

\[
|\lgcd_v(\alpha, \beta) - h_v(\alpha)| \leq \begin{cases} 
\frac{\kappa}{e} \frac{d_v}{d} \log^+ A_v + \frac{d}{e} \log 2, & \text{if } v|\infty, \\
\frac{\kappa}{e} \frac{d_v}{d} \log^+ A_v, & \text{if } v < \infty.
\end{cases}
\]

Summing up over \( v \in T \), we obtain:

\[
|\lgcd_T(\alpha, \beta) - h_T(\alpha)| \leq \frac{\kappa}{e} h(\mathcal{A}) + \log 2.
\]

Case (B) We have:

\[
|a_k \alpha^{k/e}|_v \leq A_v^{k/e} |\alpha|_v^{k/e} |\alpha|_v^{1/e} |\alpha|_v^{(k-(k+1))/e}.
\]

However, for \( k \geq \kappa + 1 \):

\[
\begin{align*}
|a_k \alpha^{k/e}|_v &\leq A_v^{k/e} |\alpha|_v^{\kappa/e} |\alpha|_v^{(2^{\kappa} A_v)^{-(k+1)/e}} (2^{\kappa} A_v)^{-(k-\kappa-1)/e} \leq |a_{\kappa} \alpha^{k/e}|_v 2^{-k}, \quad \text{if } v|\infty, \\
|a_k \alpha^{k/e}|_v &< A_v^{k/e} |\alpha|_v^{\kappa/e} |\alpha|_v A_v^{-(k+1)/e} A_v^{-(k-\kappa-1)/e} \leq |a_{\kappa} \alpha^{k/e}|_v, \quad \text{if } v < \infty.
\end{align*}
\]

In the non-Archimedean case, we know that \( v \in S \Rightarrow |a_k \alpha^{k/e}|_v \xrightarrow{k \to \infty} 0 \) and thus that \( \sum_{k=\kappa+1}^{\infty} a_k \alpha^{k/e} \) converges \( v \)-adically. As we just proved that in this case, we also have \( \forall k > \kappa, |a_k \alpha^{k/e}|_v < |a_{\kappa} \alpha^{k/e}|_v \), thus \( |\sum_{k=\kappa+1}^{\infty} a_k \alpha^{k/e}|_v < \sum_{k=\kappa+1}^{\infty} |a_k \alpha^{k/e}|_v \). Therefore, \( |\beta|_v = |a_{\kappa} \alpha^{k/e} + \sum_{k=\kappa+1}^{\infty} a_k \alpha^{k/e}|_v = |a_{\kappa} \alpha^{k/e}|_v \).

In the Archimedean case, we have obtained that:

\[
\left| \sum_{k=\kappa+1}^{\infty} a_k \alpha^{k/e} \right|_v \leq |a_{\kappa} \alpha^{k/e}|_v \sum_{k=\kappa+1}^{\infty} 2^{-k} = 2^{-\kappa} |a_{\kappa} \alpha^{k/e}|_v \leq \frac{1}{2} |a_{\kappa} \alpha^{k/e}|_v \text{ as } \kappa > 0.
\]
Using the triangular inequality in this case, we have that:

\[ |\beta|_v \geq \left| a_\alpha a^{\kappa/e}_v \right| - \left| \sum_{k=\alpha+1}^{\infty} a_k a^{k/e}_v \right|_v \geq \frac{1}{2} \left| a_\alpha a^{\kappa/e}_v \right|_v. \]

Thus,

\[ \begin{cases} \frac{1}{2} |a_\alpha|_v a^{\kappa/e}_v & \leq |\beta|_v, \text{ if } v|\infty, \\ |a_\alpha|_v a^{\kappa/e}_v & = |\beta|_v, \text{ if } v < \infty; \end{cases} \]

We have hence bounded \(|\beta|_v\) both in the Archimedean and in the non-Archimedean cases:

\[ \begin{cases} \frac{1}{2} |a_\alpha|_v a^{\kappa/e}_v \leq |\beta|_v \leq \max\{|\alpha|_v, |\beta|_v\} \leq \max\{|\alpha|_v, 2A^{\kappa/e}_v |a^{\kappa/e}_v| \} \leq 2A^{\kappa/e}_v |a^{\kappa/e}_v|, & \text{if } v|\infty, \\ |a_\alpha|_v a^{\kappa/e}_v \leq |\beta|_v \leq \max\{|\alpha|_v, |\beta|_v\} \leq \max\{|\alpha|_v, A^{\kappa/e}_v |a^{\kappa/e}_v| \} \leq A^{\kappa/e}_v |a^{\kappa/e}_v|, & \text{if } v < \infty, \end{cases} \]

We can now bound the quantity we are interested in, that is \(|\gcd_v(\alpha, \beta) - \frac{\kappa}{e} h_v(\alpha)|_v|:\n
\[ \begin{cases} |\gcd_v(\alpha, \beta) - \frac{\kappa}{e} h_T(\alpha)| \leq \frac{\kappa}{e} h(\mathcal{A}) + \log 2, & \text{if } v|\infty, \\ |\gcd_v(\alpha, \beta) - \frac{\kappa}{e} h_v(\alpha)| \leq \frac{\kappa}{e} h(\mathcal{A}) + \log 2, & \text{if } v < \infty. \end{cases} \]

Summing up over \(v \in T\), we obtain:

\[ |\gcd_T(\alpha, \beta) - \frac{\kappa}{e} h_T(\alpha)| \leq \frac{\kappa}{e} h(\mathcal{A}) + \log 2. \]

We know that \(h(a_v^{-1}) = h(a_v)\) and using Lemma 2.9, we find that \(h(a_v) \leq \frac{\kappa}{e} h(\mathcal{A})\)

Therefore, in case (B), we obtain:

\[ |\gcd_T(\alpha, \beta) - \frac{\kappa}{e} h_T(\alpha)| \leq 2\frac{\kappa}{e} h(\mathcal{A}) + \log 2. \]

Thus, combining both cases we obtain:

\[ |\gcd_T(\alpha, \beta) - \min\{1, \frac{\kappa}{e}\} h_T(\alpha)| \leq 2\frac{\kappa}{e} h(\mathcal{A}) + \log 2. \]

\[ \square \]

**Proposition 3.7** Let \(0 < \varepsilon \leq \frac{1}{2}\). Then, we have either:

\[ h(\alpha) \leq (n-1)h_p(\mathcal{F}) + 3\varepsilon^{-1}n^3m(\mathcal{H} + 5), \]

or

\[ |h(\alpha) - nh_T(\alpha)| \leq 3n(4\varepsilon^{-1}n^2 + m)\mathcal{H} + 42n^3\varepsilon^{-1} + 4n^2(h(\mathcal{A}) + 1). \quad (22) \]

**Proof** We split the proof of Proposition (3.7) in two Propositions (3.9 and 3.10), proved a little further, to find first an upper bound and then a lower bound. For this, we shall need an auxiliary polynomial with very useful properties (Proposition 3.8). Combining Propositions (3.9) and (3.10), we obtain Proposition (3.7).

Let us fix \(\delta\) satisfying \(0 < \delta \leq \frac{1}{2}\) and \(N > 1\).

**Proposition 3.8** There exists a non-zero \(G(X, Y) \in \mathbb{K}[X, Y]\) such that:

\[ \deg_X G \leq N, \deg_Y G \leq n - 1, \]

and

\[ \eta = \nu_x(G(X, y(X))) \geq (1 - \delta)Nn, \]

and

\[ h_p(G) \leq \delta^{-1}nN(\mathcal{H} + 2). \]
Proof: Let

$$G(X, Y) = \sum_{1 \leq i \leq N} g_{ij}X^iY^j,$$

and

$$y(X) = \sum_{k=1}^{\infty} a_kX^{k/e}.$$  

Condition $\eta = \nu_X(G(X, y(X))) \geq (1 - \delta)Nn$ is equivalent to a system of at most $(1 - \delta)Nn$ equations in the coefficients of $G$. Each coefficient of these equations can be viewed as a linear form with algebraic coefficients $a_{ik}$ (like in Lemma 2.10, $0 \leq \ell \leq n - 1$, $k \leq (1 - \delta)Nn$), the variables being the coefficients of $G$. We write, $G(X, y(X)) = \sum_{k=1}^{\infty} G_kX^{k/e} = \sum_{k=1}^{\infty} G_kX^{k/e}$.  

Lemma 2.10 implies that the height of each linear equation is bounded by

$$nN(h(A) + \log 4).$$

Using Lemma 2.12, there exists $G$ with $\deg G = N(n - 1)$ such that:

$$h_p(G) \leq \delta^{-1}nN(A' + 2).$$

We may assume that at least one coefficient of $G$ is one, thus the affine height is the same as projective height.

For the sake of notation simplification, let $\gamma = G(\alpha, \beta)$.

**Proposition 3.9** Either:

$$h(\alpha) \leq (n - 1)h_p(F) + 3\varepsilon^{-1}n^3m(A' + 5),$$  

or

$$nh_T(\alpha) \leq (1 + \varepsilon)h(\alpha) + 6\varepsilon^{-1}n(2A' + 7) + 4n(h(A) + 1).$$

**Assume that** $\gamma = 0$. In this case, $F(\alpha, \beta) = 0$ and $G(\alpha, \beta) = 0$. Therefore, $F(\alpha, Y)$ and $G(\alpha, Y)$ have at least one common root, namely $\beta$. Thus, their resultant $R(X)$ with respect to $Y$ admits $\alpha$ as root, and using Lemma 2.3, Proposition 2.2 and Proposition 3.8, we find:

$$h(\alpha) \leq (n - 1)h_p(F) + \delta^{-1}n^2N(A' + 2) + m(n - 1) + nN + (2n - 1)\log(2n - 1) + \log 2.$$  

We will assume in the sequel that $N \leq \frac{mn}{3}$. Then, the above equation becomes:

$$h(\alpha) \leq (n - 1)h_p(F) + \delta^{-1}n^3m(A' + 5).$$

We notice that $2nh_p(F) + 10n\log(mn) \leq (n - 1)h_p(F) + \delta^{-1}n^3m(A' + 5)$. Therefore, following Proposition 3.2, when $h(\alpha) > (n - 1)h_p(F) + \delta^{-1}n^3m(A' + 5)$, the sets $T_{ij}$ are disjoints.
Assume that $\gamma \neq 0$. Let us obtain an upper bound for $h(\gamma)$. First of all, we estimate $h(\beta)$:

$$h(\beta) \leq h_p(F) + mh(\alpha) + n + \log(m + 1), \quad (\text{Lemma 2.4, (2))}.$$  

Using Lemma (2.4, (1)) and Proposition 3.8, we find:

$$h(\gamma) \leq (N + mn)h(\alpha) + \delta^{-1}nN(\mathcal{H} + 5).$$

Let us obtain a lower bound for $h(\gamma)$. For that, let us first define a new set:

Let $T^* = \left\{ v \in T : \|\alpha\|_v \leq \left\{ \frac{1}{6} \min\{1, |a_\alpha|_{\nu}\}(2A_{\nu}\epsilon)^{-2(n+1)}, v\|\infty \right\} \right\}$.  

In this case,

$$h_{T^*}(\alpha) \leq h_{M_k \setminus T^*}(\alpha) = \sum_{v \in M_k \setminus T^*} h_v(\alpha) \leq 4nh_p(F) + 3n \log(nm) + 13ne.$$

We know that $G(X, y(X)) = \sum_{k=\eta}^{\infty} G_kX^k$ (where $\eta \geq (1 - \delta)Nn$). Then, for all $v \in T$, we have $G(\alpha, y(\alpha))^{y,\gamma}$.

We observe that each $G_k$ is a sum of at most $nN$ terms in $g_k$ and $a_k$. Applying Lemma 2.10 and using the bound $n(N + 1)(l + k - 1) \leq 6^k$ (because $k \geq \eta \geq \frac{1}{2}nN$), we find:

$$\left\{ \begin{array}{ll} |G_k|_\nu \leq n(N + 1)(l + k - 1)|G_k|_\nu A_k \leq |G_k|_\nu |\alpha|_\nu A^k_\nu (\frac{1}{2})^{k-\eta}, & \text{if } v|\infty, \\ |G_k|_\nu \leq |G_k|_\nu A_k \leq |G_k|_\nu |\alpha|_\nu A^k_\nu, & \text{if } v < \infty. \end{array} \right.$$  

Therefore,

$$\left\{ \begin{array}{ll} |\gamma|_v \leq |G_k|_\nu |\alpha|_\nu A^k_\nu, & \text{if } v|\infty, \\ |\gamma|_v \leq |G_k|_\nu |\alpha|_\nu A^k_\nu, & \text{if } v < \infty, \end{array} \right.$$  

and thus, taking $\log^+$ of both sides and summing over $v \in T^*$,

$$h_{T^*}(\gamma) \geq \eta h_{T^*}(\alpha) - \eta h(\mathcal{A}) - h_p(G) - \eta \log 6.$$  

Using the fact that $h(\gamma) \geq h_{T^*}(\gamma)$, combining this lower bound with the upper bound (3.4) and setting $\delta = \frac{\epsilon}{2}$ and $N \geq \frac{mn}{(2 - 3\delta)}$ (for instance, $N = \lfloor \frac{mn}{(2 - 3\delta)} \rfloor$, which is in agreement with the assumption that $N \leq \frac{mn}{(2 - 3\delta)}$), we find:

$$nh_{T^*}(\alpha) \leq (1 + \epsilon)h(\alpha) + 6\epsilon^{-1}n(2\mathcal{H} + 7) + 4n(h(\mathcal{A}) + 1).$$

Setting $\delta = \frac{\epsilon}{3}$, equation (24) becomes:

$$h(\alpha) \leq (n - 1)h_p(F) + 3\epsilon^{-1}n^3m(\mathcal{H} + 5).$$

\[\square\]

**Proposition 3.10** Either:

$$h(\alpha) \leq (n - 1)h_p(F) + 3\epsilon^{-1}n^3m(\mathcal{H} + 5),$$

or

$$nh_{T^*}(\alpha) \geq (1 - \epsilon)h(\alpha) - 3n(4\epsilon^{-1}n^2 + m)\mathcal{H} - 42n^3\epsilon^{-1} - 4n^2(h(\mathcal{A}) + 1).$$
Proof: Let $R_F$ be as in Proposition 2.2, the resultant of $F$ and its derivative polynomial with respect to $Y$. If $R_F(\alpha) = 0$, using Proposition 2.2 and Lemma 2.3, we find that:

$$h(\alpha) \leq (2n - 1)h_p(F) + (2n - 1)\log[(m + 1)(n + 1)\sqrt{2n}] + \log 2$$

$$< 2n h_p(F) + 10n \log(mn).$$

If $R_F(\alpha) \neq 0$, let us set:

$$F(\alpha, Y) = \prod_{k=1}^{n} (Y - \beta_k),$$

where $\beta_1 = \beta$. Define:

$$T^{(k)} = \{ v \in S : y(\alpha) v = \beta_k \}.$$

Then, the set $\{T^{(k)}|1 \leq k \leq n\}$ forms a partition of $S$. Indeed, for all $v \in S$, $y(\alpha)$ converges $v$-adically to one of the roots of $F(\alpha, Y)$. And if it converges to one root, it cannot converge at the same time to another one. Thus:

$$h_S(\alpha) = \sum_{k=1}^{n} h_{T^{(k)}}(\alpha).$$

Applying Proposition 3.9 with $\varepsilon/n$ instead of $\varepsilon$, we have:

$$nh_{T^{(k)}}(\alpha) \leq (1 + \frac{\varepsilon}{n})h(\alpha) + 6\varepsilon^{-1}n^2(2H + 7) + 4n(h(\alpha) + 1). \quad (25)$$

We notice that the definition of $T = \{ v \in S|y(\alpha) v = \beta \}$ coincides with that of $T^{(1)}$. Then, $h_T(\alpha) = h_{T^{(1)}}(\alpha) = h_S(\alpha) - \sum_{k=2}^{n} h_{T^{(k)}}(\alpha)$, and:

$$h_S(\alpha) = h(\alpha) - h_{M_S \setminus \{\alpha\}}(\alpha) \geq h(\alpha) - 3mH - s(m + 1)n. \quad (26)$$

Combining Equations (25) and (26) we find the result. \qed

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