Research Article

On Some Algebraic and Operator-Theoretic Properties of \( \lambda \)-Toeplitz Operators

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Based on a spectral problem raised by Barria and Halmos, a new class of Hardy-Hilbert space operators, containing the classical Toeplitz operators, is introduced, and some of their Toeplitz-like algebraic and operator-theoretic properties are studied and explored.

1. Introduction

All of the work I am about to describe takes place in the Hardy-Hilbert space of the unit circle \( \mathbb{T} \), denoted by \( H^2(\mathbb{T}) \), and consists of all square-integrable (with respect to the normalized arc-length measure \( d\theta/2\pi \)) functions, on \( \mathbb{T} \), whose negative Fourier coefficients all vanish; that is,

\[
H^2 = \left\{ f \in L^2(\mathbb{T}) \mid \hat{f}(n) := \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta = 0, \quad n \in \mathbb{Z}^+ \right\}.
\]

For more details and basic properties of Hardy spaces, the reader is referred to [1, Chapters 1 and 2] or [2, Chapter 17].

Two of the most intensely studied classes of bounded operators on \( H^2 \) are Toeplitz and Hankel operators. Originally, an infinite matrix is called Toeplitz (resp., Hankel) if its entries depend just on the difference (resp., the sum) of their indices. Hence, Toeplitz matrices are the ones with constant diagonals, and Hankel matrices are those with constant skew-diagonals. They both play a decisive role in a very wide circle of problems in operator theory, \( C^* \)-algebras, moment problems, interpolation by holomorphic or meromorphic functions, inverse spectral problems, orthogonal polynomials, prediction theory, Wiener-Hopf equations, boundary problems of function theory, the extension theory of symmetric operators, singular integral equations, models of statistical physics, and many others. Also, there exists a vast literature on the theory of Toeplitz and Hankel operators; see, for example, [3–8].

Maybe a naïve reason also for their importance is the fact that Toeplitz and Hankel operators are compressions of (bounded) multiplication operators and their flipped, respectively, to \( H^2 \). Indeed, any essentially bounded function \( \phi \) on \( \mathbb{T} \) induces, in a natural way, three bounded operators, one on \( L^2(\mathbb{T}) \) and the two others on \( H^2 \), as follows.

(i) The Multiplication operator \( M_\phi \) is given by \( M_\phi f = \phi f \), for \( f \in L^2(\mathbb{T}) \).

(ii) The Toeplitz operator \( T_\phi \) is defined, in terms of the orthogonal projection \( P \) from \( L^2(\mathbb{T}) \) onto \( H^2(\mathbb{T}) \), as the compression of \( M_\phi \) to \( H^2 \); \( T_\phi f = PM_\phi f \), for \( f \in H^2(\mathbb{T}) \).

(iii) The Hankel operator \( H_\phi \) is defined as the compression of the “flipped” \( M_\phi \) onto \( H^2(\mathbb{T}) \); \( H_\phi f = P|J|M_\phi f \), for \( f \in H^2(\mathbb{T}) \), where \( J \) is the unitary self-adjoint operator on \( L^2(\mathbb{T}) \) (the so-called flip operator) defined by \( (Jf)(\xi) := \overline{\xi} f(\overline{\xi}) \), for \( \xi \in \mathbb{T} \), mapping \( H^2 \) onto \( (H^2)^\perp \) (the orthogonal complement of \( H^2 \)) and \( (H^2)^\perp \) onto \( H^2 \).

In each case, \( \phi \) is called the symbol of the operator. These classes of operators can also be considered as solutions to some linear operator-equations involving the Toeplitz
operator $T_{e^\sigma}$, known as the *unilateral forward shift*, and its Hilbert-adjoint $T_{e^{-\sigma}}$, usually called the *unilateral backward shift*. Indeed, it is well known that an operator $H$ is Hankel if and only if $T_{e^{-\sigma}}H = HT_{e^\sigma}$ (Hankel equation) and that an operator $T$ is Toeplitz if and only if $T_{e^{-\sigma}}TT_{e^\sigma} = T$ (Toeplitz equation).

Generalizations of such operator-equations have been studied and explored for some time. For instance, in [9] the operator-equation $S^*X T = X$, for arbitrary contractions $S$ and $T$ acting on different Hilbert spaces, has been studied. Pták in [10] studied the solutions to the operator-equation $S^*X = XT$, where $S$ and $T$ are contractions.

Here we study an operator-equation, on $\mathcal{B}(H^2)$, which is a slight modification to the Toeplitz equation; namely, $T_{e^{-\sigma}}XT_{e^\sigma} = \lambda X$, for an arbitrary complex number $\lambda$. This operator-equation appeared in [11] and it was asked what its operator-solutions could be, what algebraic and operator-theoretic properties those solutions had, and how these operator-solutions relate to the case $\lambda = 1$ (Toeplitz operators). Fortunately, this problem is a spectral one; that is, its solutions are the eigen-operators of a bounded operator-$\sigma$-Toeplitz operator. For a fixed complex number $\lambda$, a typical example of a $\lambda$-Toeplitz matrix is

$$
\begin{pmatrix}
\gamma & \lambda & \lambda^2 & \lambda^3 & \ldots \\
\lambda & \lambda^2 & \lambda^3 & \lambda^4 & \ldots \\
\lambda^2 & \lambda^3 & \lambda^4 & \lambda^5 & \ldots \\
\lambda^3 & \lambda^4 & \lambda^5 & \lambda^6 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
$$

In [13], it is shown that Toeplitz operators, on $H^2$, are the solutions to the equation $T_{e^{-\sigma}}XT_{e^\sigma} = X$, on $\mathcal{B}(H^2)$. Equivalently, this means that they are the eigen-operators of the following operator-valued linear transformation (let us call it the **Toeplitz mapping** on $\mathcal{B}(H^2)$):

$$
\Gamma : \mathcal{B}(H^2) \rightarrow \mathcal{B}(H^2)
$$

$$
X \mapsto T_{e^{-\sigma}}XT_{e^\sigma},
$$

**2. $\lambda$-Toeplitzness**

Here we give two approaches for defining the concept of “$\lambda$-Toeplitzness”: matricial and operator-theoretic approaches.

**Definition 1.** One calls a singly infinite matrix a $\lambda$-Toeplitz matrix if, on each diagonal (parallel to the main diagonal), the entries are in continued proportion; that is, the matrix $(a_{mn})_{m,n=0}^\infty$ is a $\lambda$-Toeplitz matrix if there exists a $\lambda \in \mathbb{C}$ such that $a_{m+1,n+1} = \lambda a_{mn}$, for $m, n = 0, 1, 2, \ldots$.

For a fixed complex number $\lambda$, a typical example of a $\lambda$-Toeplitz matrix is

$$
\begin{pmatrix}
a_0 & a_{-1} & a_{-2} & a_{-3} & \ldots \\
a_1 & \lambda a_0 & \lambda a_{-1} & \lambda a_{-2} & \ldots \\
a_2 & \lambda a_1 & \lambda^2 a_0 & \lambda^2 a_{-1} & \ldots \\
a_3 & \lambda a_2 & \lambda^2 a_1 & \lambda^3 a_0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
$$

In [13], it is shown that Toeplitz operators, on $H^2$, are the solutions of the operator-equation $T_{e^{-\sigma}}XT_{e^\sigma} = X$, on $\mathcal{B}(H^2)$. Equivalently, this means that they are the eigen-operators of the following operator-valued linear transformation (let us call it the **Toeplitz mapping** on $\mathcal{B}(H^2)$):

$$
\Gamma : \mathcal{B}(H^2) \rightarrow \mathcal{B}(H^2)
$$

$$
X \mapsto T_{e^{-\sigma}}XT_{e^\sigma},
$$

corresponding to the eigenvalue $1$. This suggests a general context in which Toeplitz operators can be embedded.

**Definition 2.** One calls an operator, in $\mathcal{B}(H^2)$, a $\lambda$-Toeplitz operator if it is an eigen-operator of the Toeplitz mapping $\Gamma$ corresponding to one of its eigenvalues.
More precisely, for \( \lambda \in \sigma_p(\Gamma) \), the set \( \mathcal{T}_\lambda := \ker(\lambda I - \Gamma) \) consists of \( \lambda \)-Toeplitz operators corresponding to \( \lambda \), or, equivalently,

\[
\mathcal{T}_\lambda = \{ X \in \mathcal{B}(\mathcal{H}^2) \mid \Gamma(X) = T_{\lambda_X} X \} = \{ \lambda X \}.
\]

(4)

It can be easily checked that the Toeplitz mapping is a contraction; moreover, since \( T_{\lambda_X} T_{\mu_X} \equiv 1 \), we should have \( \|\Gamma\| = 1 \). Thus, there is no \( \lambda \)-Toeplitz operator for \( |\lambda| > 1 \); that is, \( \sigma(\Gamma) \subseteq \mathbb{U} \). But every diagonal operator with diagonal \((\lambda^n)_{n=0}^\infty\), for \( \lambda \in \mathbb{U} \), is a solution for \( [\lambda I - \Gamma](X) = 0 \). Thus, \( \mathbb{U} \subseteq \sigma_p(\Gamma) \subseteq \sigma(\Gamma) \). Therefore, \( \sigma_p(\Gamma) = \sigma(\Gamma) = \mathbb{U} \); that is, the only eigen-operators for \( \Gamma \) are the ones corresponding to the eigenvalues living in \( \mathbb{U} \).

Observation 1. Bounded operator-solutions to \( T_{\lambda_X} X \equiv \lambda X \), on \( \mathcal{B}(\mathcal{H}^2) \), exist if and only if \( \lambda \in \mathbb{U} \). More details on the form of a \( \lambda \)-Toeplitz operator, see [12].

Theorem 3 (Sun [12]). Let \( \lambda \in \mathbb{C} \). The operator-equation \( T_{\lambda_X} X = \lambda X \) has bounded solutions if and only if \( |\lambda| \leq 1 \). One then has the following.

(i) If \( |\lambda| = 1 \), all solutions are of the form \( D_T \lambda \), where \( T \) is a Toeplitz operator and \( D \) is the diagonal unitary operator defined as \( D e_n = \lambda^n e_n \) for all \( n \).

(ii) If \( |\lambda| < 1 \), all solutions are compact operators of the form

\[
\sum_{n=0}^{\infty} \lambda^n (T_{\lambda_X} f) \otimes e_n + e_n \otimes (T_{\lambda_X} g)
\]

(5)

for some \( f \) and \( g \in \mathcal{H}^2 \).

For convenience, let us divide \( \lambda \)-Toeplitz operators into two main classes: unimodular \( \lambda \)-Toeplitz operators and nonunimodular \( \lambda \)-Toeplitz operators, which are the ones corresponding to the eigenvalues of the Toeplitz mapping \( \Gamma \) on the unit circle and the unit disk, respectively.

Remark 4. Some immediate consequences of Sun’s Theorem are that the nonunimodular \( \lambda \)-Toeplitz operators are compact (so are not invertible) and unimodular \( \lambda \)-Toeplitz operators are not. Indeed, in the latter case, the only compact unimodular \( \lambda \)-Toeplitz operator is the zero operator.

Remark 5. By Definition 2, if \( X \in \mathcal{T}_\lambda \), for some \( \lambda \in \mathbb{U} \), then \( T_{\lambda_X} X \equiv \lambda X \). Hence, the entries of its matrix representation \((a_{mn})_{m,n=0}^{\infty}\), with respect to the monomial basis for \( \mathcal{H}^2 \), satisfy

\[
a_{m+1,n+1} = \langle X e_{m+1}, e_{n+1} \rangle_{\mathcal{H}^2} = \langle T_{\lambda_X} X e_{m+1}, T_{\lambda_X} e_{n+1} \rangle_{\mathcal{H}^2} = \langle T_{\lambda_X} e_{m+1}, T_{\lambda_X} e_{n+1} \rangle_{\mathcal{H}^2} = \langle \lambda X e_m, e_n \rangle_{\mathcal{H}^2} = \lambda a_{m,n}, \text{ for } m, n = 0, 1, 2, \ldots.
\]

(6)

This yields

\[
(a_{mn})_{m,n=0}^{\infty} = \begin{bmatrix}
0 & 0 & 0 & \cdots \\
0 & a_{0,1} & a_{0,2} & \cdots \\
\lambda a_{0,0} & 0 & a_{0,2} & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix},
\]

(7)

where

\[
X e_0 = \sum_{n=0}^{\infty} a_{n,0} e^{n\theta}, \quad X^* e_0 = \sum_{n=0}^{\infty} \overline{a_{0,n}} e^{n\theta}.
\]

(8)

3. Basic Properties of \( \lambda \)-Toeplitz Operators

Recall that \( \mathcal{T}_1 \) consists of all (classical) Toeplitz operators, and it turns out, as we will see later, that other \( \lambda \)-Toeplitz operators behave like them. Also, notice that, for each \( \lambda \in \mathbb{U} \), \( \mathcal{T}_\lambda \) forms a complex vector subspace of \( \mathcal{B}(\mathcal{H}^2) \).

Here we look at the following straightforward properties of the \( \lambda \)-Toeplitz operators. First, from Definition 2, we observe that, for each \( \lambda \in \mathbb{U} \), \( \mathcal{T}_{\lambda} \) is topologically well behaved. Indeed, for \( X \in \mathcal{T}_\lambda \), since \( \Gamma(X) = T_{\lambda_X} X \) is weakly continuous in its middle factor, \( \mathcal{T}_{\lambda} \) is weakly closed, and, therefore, a fortiori, it is strongly and uniformly closed.

The next result, inspired by [14, Theorem 4.5], states that self-adjointness only exists among real \( \lambda \)-Toeplitz operators; that is, \( \lambda \)-Toeplitz operators correspond to real eigenvalues for \( \Gamma \).

Proposition 6. For \( \lambda \in \mathbb{U} \) and \( X \in \mathcal{T}_\lambda \), one has the following.

(i) \( X^* \in \mathcal{T}_\lambda \).

(ii) If \( X \neq 0 \) and \( X = X^* \), then \( \lambda \in \mathbb{R} \).

Proof. (i) Since \( X \in \mathcal{T}_\lambda \), we have \( T_{\lambda_X} X = \lambda X \), from which, by taking adjoints, we get \( T_{\lambda_X} X^* = \overline{\lambda} X^* \). So \( X^* \in \mathcal{T}_\lambda \).

(ii) If \( X \) is a nonzero self-adjoint element of \( \mathcal{T}_\lambda \), then \( X \in \mathcal{T}_\lambda \cap \mathcal{T}_{\overline{\lambda}} \). But this means that

\[
\lambda X = T_{\lambda_X} X = T_{\lambda_X} X^* = \overline{\lambda} X^* = \overline{\lambda} X,
\]

(9)

which implies \( \lambda = \overline{\lambda} \), or, equivalently, \( \lambda \in \mathbb{R} \).

Remark 7. As a consequence of Sun’s Theorem, every nonunimodular \( \lambda \)-Toeplitz operator is compact. This, in turn, states that they are not only noninvertible, but also nonessentially invertible. But the situation is different for unimodular ones. Indeed, the only compact unimodular one is the zero operator: for \( \lambda \in \mathbb{U} \), letting \( X \in \mathcal{T}_\lambda \), \( n \) and \( n + k \) be nonnegative integers, we have

\[
\langle X e_n, e_{nk} \rangle_{\mathcal{H}^2(\partial \mathbb{D})} = \lambda^k \langle X e_0, e_k \rangle_{\mathcal{H}^2(\partial \mathbb{D})}.
\]

(10)

Now, if \( X \) is a compact operator, then \( \|X e_n\|_{\mathcal{H}^2(\partial \mathbb{D})} \to 0 \), as \( n \to \infty \); it follows that \( \langle X e_0, e_k \rangle_{\mathcal{H}^2(\partial \mathbb{D})} = 0 \), for all nonnegative \( k \).

And, if we apply the same procedure for \( X^* \), we obtain \( \langle X^* e_0, e_k \rangle_{\mathcal{H}^2(\partial \mathbb{D})} = 0 \), for all nonnegative \( k \). Therefore, \( X = 0 \).
4. Analyticity and Coanalyticity of \( \lambda \)-Toeplitz Operators

In [13, p. 96], analyticity and coanalyticity of a (classical) Toeplitz operator are defined and characterized in terms of its commutativity with \( T_{e^\phi} \) and \( T_{e^{-\phi}} \), respectively [13, Theorem 7]. Here, we define and give an analogous characterization of these two properties in the \( \lambda \)-Toeplitz operators’ setting. But let us first assign them a “symbol” (similar to one in the classical case) as a generating function.

**Definition 9.** For \( \lambda \in \overline{U} \), let \( X \in \mathcal{F}_\lambda \). The symbol of \( X \) is defined to be

\[
X_{\phi} + X^*_{\phi} e_0 - \langle X_{\phi} e_0, e_0 \rangle_{H^2}^\perp
\]

and is denoted by \( \text{sym}(X) \), in which \( X_{\phi} \) is called the analytic symbol and \( X^*_{\phi} e_0 \) is called the coanalytic symbol of \( X \).

**Observation 2.** For a \( \lambda \)-Toeplitz operator \( X \), \( \text{sym}(X) \) is the function whose nonnegative Fourier coefficients are the terms of the 0th-column of its matrix representation, with respect to the monomial basis for \( H^2 \), and whose nonpositive Fourier coefficients are the terms of the 0th-row of that matrix.

**Definition 8** determines two \( H^2 \)-functions, namely, \( X_{\phi} \) and \( X^*_{\phi} e_0 \), by which we may characterize the properties of analyticity and coanalyticity, for such operators, as follows.

**Definition 9.** A \( \lambda \)-Toeplitz operator, \( X \), is called

(i) **analytic** if \( \text{sym}(X) \) is an analytic function (i.e., \( X^* e_0 \) is the constant function \( \langle e_0, X e_0 \rangle_{H^2}^\perp \));

(ii) **coanalytic** if \( \text{sym}(X) \) is a coanalytic function (i.e., \( X_{\phi} \) is the constant function \( \langle X_{\phi} e_0, e_0 \rangle_{H^2}^\perp \)).

This definition makes the following remark obvious.

**Remark 10.** For \( \lambda \in \overline{U} \), letting \( X \in \mathcal{F}_\lambda \)

(i) \( X \) is analytic if and only if \( \langle X^* e_0, e_n \rangle_{H^2} = 0 \), for all \( n > 0 \) (i.e., \( T_{e^{-\phi}} X^* e_0 = 0 \)),

(ii) \( X \) is coanalytic if and only if \( \langle X_{\phi} e_0, e_n \rangle_{H^2} = 0 \), for all \( n > 0 \) (i.e., \( T_{e^\phi} X_{\phi} e_0 = 0 \)).

Hence, \( X \) is analytic if and only if \( X^* \) is coanalytic.

**Observation 3.** Notice that for \( \lambda \in \overline{U} \) and \( \phi \in L^\infty(\partial U) \), \( D\lambda T_{\phi} \in \mathcal{F}_\lambda \), where \( D\lambda \) is the diagonal operator with diagonal \( (\lambda^n)_{n=0}^\infty \). Indeed,

\[
T_{e^\phi} D\lambda T_{e^\phi} T_{e^\phi} = T_{e^\phi} D\lambda \left( T_{e^\phi} e_0 \otimes e_0 \right) T_{e^\phi} T_{e^\phi} = (T_{e^\phi} D\lambda T_{e^\phi}) (T_{e^\phi} e_0 \otimes e_0)
\]

\[
+ T_{e^\phi} D\lambda e_0 \otimes e_0
\]

\[
= \lambda D\lambda T_{e^\phi} + T_{e^{-\phi}} e_0 \otimes e_0
\]

\[
= \lambda D\lambda T_{e^\phi}.
\]

This observation provides us with two classes of typical examples of analytic and coanalytic \( \lambda \)-Toeplitz operators.

(1) For \( \lambda \in \overline{U} \) and \( \phi \in H^\infty(\partial U) \), \( D\lambda T_{\phi} \) is an analytic \( \lambda \)-Toeplitz operator. Indeed for \( n = 1, 2, \ldots \),

\[
\left\langle \left( T_{e^\phi} D\lambda \right)^n e_0, e_n \right\rangle_{H^2} = \left\langle \left( D\lambda T_{\phi} \right)^n e_0, \phi e_n \right\rangle
\]

\[
= \left\langle e_0, \phi e_n \right\rangle = \overline{\phi}(n) = 0.
\]

(2) For \( \lambda \in \overline{U} \) and \( \phi \in H^\infty(\partial U) \), \( D\lambda T_{\phi} \) is a coanalytic \( \lambda \)-Toeplitz operator. Indeed for \( n = 1, 2, \ldots \),

\[
\left\langle D\lambda T_{\phi} e_0, \phi e_n \right\rangle_{H^2} = \left\langle T_{\phi} e_0, \phi D\lambda e_n \right\rangle
\]

\[
= \left\langle T_{\phi} e_0, \phi \right\rangle = \overline{\phi}(n) = 0.
\]

Before stating the first result of this section, we need to introduce some terms. For a Hilbert space bounded operator \( A \), consider the operator-equation

\[
AX = \lambda XA
\]

for some complex number \( \lambda \). If there is a nonzero (bounded) operator \( X \) and a scalar \( \lambda \) as above that satisfy (16), according to [15], it is said that \( A \) \( \lambda \)-commutes with \( X \) and that \( \lambda \) is an extended eigenvalue and \( X \) is an extended eigen-operator of \( A \).

Equation (16) has been studied in [16] and, independently, in [17]. These works provided extensions of Lomonosov’s classic result [18].

Now, we use these terms to state our next result which characterizes analyticity and coanalyticity of \( \lambda \)-Toeplitz operators in terms of their \( \lambda \)-commutativity with \( T_{e^\phi} \) and \( T_{e^{-\phi}} \).

**Theorem 11.** Let \( \lambda \in \overline{U} \setminus \{0\} \) and \( X \in \mathcal{B}(H^2) \). A necessary and sufficient condition that \( X \) is an analytic (coanalytic) \( \lambda \)-Toeplitz operator in \( \mathcal{F}_\lambda \) is that \( \lambda \)-commutes with \( T_{e^\phi} (\lambda^{-1} \text{-commutes with } T_{e^{-\phi}}) \); that is, \( XT_{e^\phi} = \lambda T_{e^\phi} X \) (\( T_{e^{-\phi}} X = \lambda X T_{e^{-\phi}} \)).

**Proof.** (i) Let \( X \) be an analytic \( \lambda \)-Toeplitz operator in \( \mathcal{F}_\lambda \); that is, \( T_{e^{-\phi}} X^* e_0 = 0 \). Hence,

\[
\lambda T_{e^\phi} X = T_{e^\phi} (\lambda X) = T_{e^\phi} (T_{e^{-\phi}} X^* e_0)
\]

\[
= (I - e_0 \otimes e_0) XT_{e^\phi}
\]

\[
= X T_{e^\phi} - (e_0 \otimes e_0) e_0 = XT_{e^\phi}.
\]

\[
\lambda T_{e^\phi} = \lambda (\lambda X) = \lambda T_{e^\phi} (T_{e^{-\phi}} X^* e_0)
\]
Now, let $X \in \mathcal{B}(H^2)$ be such that $XT_e^\omega = \lambda XT_e^\omega X$. Just by multiplying both sides by $T_{e^\omega}$ from the left, one can easily see that $X \in \mathcal{F}_\lambda$. To show it is analytic, we need to prove $(X^*e_0, e_n)_{H^2} = 0$ for $n = 1, 2, \ldots$. So,

$$
(X^*e_0, e_n)_{H^2} = (e_0, XT_e^\omega e_{n-1})_{H^2} = (e_0, \lambda T_e^\omega Xe_{n-1})_{H^2} = \lambda(XT_e^\omega e_0, e_n)_{H^2} = 0,
$$

which means $X$ is an analytic $\lambda$-Toeplitz operator.

(ii) If $X$ is a coanalytic $\lambda$-Toeplitz operator in $\mathcal{F}_\lambda$, that is, $T_{e^\omega}Xe_0 = 0$, then

$$
\lambda XT_{e^\omega} = (T_{e^\omega}X e_0)_{H^2} = T_{e^\omega}X(I - e_0 \otimes e_0)
$$

$$
= T_{e^\omega}X - (T_{e^\omega}X e_0 \otimes e_0) = T_{e^\omega}X.
$$

Now, let $X \in \mathcal{B}(H^2)$ $\lambda^{-1}$-commute with $T_{e^\omega}$; that is, $T_{e^\omega}X = \lambda XT_{e^\omega}$. Multiplying both sides by $T_{e^\omega}$ from the right shows that $X \in \mathcal{F}_\lambda$. To prove coanalyticity, we need to show $T_{e^\omega}Xe_0 = 0$. So, we have

$$
(T_{e^\omega}X)e_0 = (\lambda XT_{e^\omega})e_0 = 0. \quad \square
$$

**Remark 12.** Using the terms aforementioned, Theorem 11 can also be restated as follows.

**Theorem 11’**

(i) Let $\lambda \in \mathbb{U} \setminus \{0\}$. A necessary and sufficient condition in which $X \in \mathcal{F}_\lambda$ is analytic is that $T_{e^\omega}$ is an extended eigen-operator of $A$ corresponding to the extended eigenvalue $\lambda$.

(ii) Let $\lambda \in \mathbb{U} \setminus \{0\}$. A necessary and sufficient condition in which $X \in \mathcal{F}_\lambda$ is coanalytic is that $T_{e^\omega}$ is an extended eigen-operator of $A$ corresponding to the extended eigenvalue $\lambda^{-1}$.

**Remark 13.** Notice that if $X \in \mathcal{F}_\mu$, it can be represented by the finite-rank operator $[(Xe_0 - (Xe_0, e_0)_{H^2}e_0) \otimes e_0] + e_0 \otimes X^*e_0$. Thus,

(1) $X$ is analytic if and only if $X^*e_0$ is the constant function $(e_0, Xe_0)_{H^2}$ and, in this case, $X = Xe_0 \otimes e_0$.

(2) And $X$ is coanalytic if and only if $Xe_0$ is the constant function $(Xe_0, e_0)_{H^2}$ and, in this case, $X = e_0 \otimes X^*e_0$.

**5. Multiplicative Properties of $\lambda$-Toeplitz Operators**

Although, for a fixed $\lambda \in \mathbb{U}$, $\mathcal{F}_\lambda$ is closed under finite summation of its elements, the corresponding result rarely holds for products. As an application of Theorem 11, we will see that $\lambda$-Toeplitzness is preserved under multiplication, on the right, by analytic $\lambda$-Toeplitz operators and, on the left, by coanalytic ones.

**Theorem 14.** For $\lambda_1, \lambda_2 \in \mathbb{U}$, let $X_1 \in \mathcal{F}_{\lambda_1}$ and $X_2 \in \mathcal{F}_{\lambda_2}$.

A necessary and sufficient condition that the product $X_1X_2$ is a $\lambda$-Toeplitz operator in $\mathcal{F}_{\lambda_1\lambda_2}$ is that either $X_1$ is coanalytic or $X_2$ is analytic.

**Proof.** Let us first assume $X_1X_2 \in \mathcal{F}_{\lambda_1\lambda_2}$. Hence,

$$
\lambda_1\lambda_2X_1X_2 = T_{e^\omega}X_1X_2T_{e^\omega}
$$

$$
= T_{e^\omega}X_1(X_{e^\omega}T_{e^\omega} + e_0 \otimes e_0)X_2T_{e^\omega}
$$

$$
= (T_{e^\omega}X_1T_{e^\omega})(T_{e^\omega}X_2T_{e^\omega})
$$

$$
+ (T_{e^\omega}X_1e_0 \otimes T_{e^\omega}X_2^*e_0)
$$

$$
= \lambda_1\lambda_2X_1X_2 + (T_{e^\omega}X_1e_0 \otimes T_{e^\omega}X_2^*e_0).
$$

So, for holding the equality, we should have either $T_{e^\omega}X_1e_0 = 0$ or $T_{e^\omega}X_2^*e_0 = 0$; that is, either $X_1$ is coanalytic or $X_2$ is analytic.

Let us suppose, for now, $X_1$ is a coanalytic $\lambda$-Toeplitz operator in $\mathcal{F}_{\lambda_1}$ and $X_2 \in \mathcal{F}_{\lambda_2}$. To show that $X_1X_2 \in \mathcal{F}_{\lambda_1\lambda_2}$, we apply Theorem 11 to write

$$
T_{e^\omega}X_1X_2T_{e^\omega} = \lambda_1X_1T_{e^\omega}X_2T_{e^\omega}
$$

$$
= (\lambda_1X_1)(\lambda_2X_2) = \lambda_1\lambda_2X_1X_2,
$$

which proves $X_1X_2$ is a $\lambda$-Toeplitz operator in $\mathcal{F}_{\lambda_1\lambda_2}$.

And if $X_2$ is an analytic $\lambda$-Toeplitz operator in $\mathcal{F}_{\lambda_2}$ and $X_1 \in \mathcal{F}_{\lambda_1}$, again using Theorem 11 results in

$$
T_{e^\omega}X_1X_2T_{e^\omega} = T_{e^\omega}X_1(\lambda_2T_{e^\omega}X_2)
$$

$$
= \lambda_2(T_{e^\omega}X_1T_{e^\omega})X_2 = \lambda_1\lambda_2X_1X_2,
$$

which proves the same thing. \square

From the proof of Theorem 14, along with considering Remark 7, one may deduce the following property for unimodular $\lambda$-Toeplitz operators.

**Proposition 15.** For $\lambda_1, \lambda_2 \in \mathbb{U}$, let $X_1 \in \mathcal{F}_{\lambda_1}$ and $X_2 \in \mathcal{F}_{\lambda_2}$. If $X_1X_2 \in \mathcal{F}_{\mu} \setminus \{0\}$ for some $\mu \in \mathbb{U}$, then $\mu = \lambda_1\lambda_2$. Moreover, either $X_1$ is coanalytic or $X_2$ is analytic.

**Proof.** Considering the assumptions, we have

$$
\mu X_1X_2 = T_{e^\omega}X_1X_2T_{e^\omega}
$$

$$
= T_{e^\omega}X_1[(T_{e^\omega}T_{e^\omega} + e_0 \otimes e_0)X_2T_{e^\omega}]
$$

$$
= (T_{e^\omega}X_1T_{e^\omega})(T_{e^\omega}X_2T_{e^\omega})
$$

$$
+ (T_{e^\omega}X_1e_0 \otimes T_{e^\omega}X_2^*e_0)
$$

$$
= \lambda_1\lambda_2X_1X_2 + (T_{e^\omega}X_1e_0 \otimes T_{e^\omega}X_2^*e_0),
$$

which implies

$$
(\mu - \lambda_1\lambda_2) X_1X_2 = T_{e^\omega}X_1e_0 \otimes T_{e^\omega}X_2^*e_0.
$$

Since $\mu \neq 0$ and $X_1X_2$ is a nonzero $\lambda$-Toeplitz operator in $\mathcal{F}_{\mu}$, $X_1X_2$ cannot be of finite rank (see Remark 7). Hence, both sides in (25) should be zero. Therefore, $\mu = \lambda_1\lambda_2$, and this in turn implies that either $X_1$ should be coanalytic or $X_2$ is analytic. \square
Recall that if the symbols of two (classical) Toeplitz operators are either analytic or coanalytic, they necessarily commute [13, Theorem 9]. But, surprisingly, this is not the case among $\lambda$-Toeplitz operators; for, let us look at an example.

**Example 16.** For some $\lambda \in \mathbb{U} \setminus \{1\}$,

1. let $X_1, X_2 \in \mathcal{T}_\lambda$ be analytic such that $X_2$ is arbitrary and $X_1$ is given by
   \[ \langle X_1 e^j, e_i \rangle_{\mathcal{H}^2} = \lambda^i \delta_{i,j+1} \quad i, j = 0, 1, 2, \ldots, \quad (26) \]
   where $\delta_{i,j+1}$ is the Kronecker delta. Note that $X_2$ can also be represented as
   \[ \langle X_2 e^j, e_i \rangle_{\mathcal{H}^2} = \begin{cases} 0 & \text{if } i < j \\ \lambda^i \langle X_2 e^0, e_{i-j} \rangle_{\mathcal{H}^2} & \text{if } i \geq j. \end{cases} \quad (27) \]
   Hence, we have
   \[ \langle X_1 X_2 e^j, e_i \rangle_{\mathcal{H}^2} = \begin{cases} 0 & \text{if } i \leq j \\ \lambda^{i-j} \langle X_2 e^0, e_{i-j} \rangle_{\mathcal{H}^2} & \text{if } i > j. \end{cases} \quad (28) \]
   Therefore, as analytic $\lambda$-Toeplitz operators, $X_1$ and $X_2$ do not commute.

2. In the other direction, consider two coanalytic $\lambda$-Toeplitz operators $X_3$, $X_4 \in \mathcal{T}_\lambda$, such that $X_4$ is arbitrary and $X_3 = X_4^*$, the Hilbert space adjoint of $X_1$ in the previous case; that is,
   \[ \langle X_3 e^j, e_i \rangle_{\mathcal{H}^2} = \hat{\lambda}^i \delta_{i,j+1} \quad i, j = 0, 1, 2, \ldots. \quad (29) \]
   Again, note that $X_4$ can also be represented as
   \[ \langle X_4 e^j, e_i \rangle_{\mathcal{H}^2} = \begin{cases} 0 & \text{if } i > j \\ \lambda^{-i} \langle X_4 e^0, e_{j-i} \rangle_{\mathcal{H}^2} & \text{if } i \leq j. \end{cases} \quad (30) \]
   Hence, we have
   \[ \langle X_3 X_4 e^j, e_i \rangle_{\mathcal{H}^2} = \begin{cases} 0 & \text{if } j < i \\ \lambda^{-j} \langle X_4 e^0, e_{j-i} \rangle_{\mathcal{H}^2} & \text{if } j > i. \end{cases} \quad (31) \]
   Therefore, as coanalytic $\lambda$-Toeplitz operators, $X_3$ and $X_4$ do not commute.

Though, we can still obtain some necessary and sufficient conditions for pairs of (co-)analytic $\lambda$-Toeplitz operators to commute.

**Theorem 17.** For $\lambda_1, \lambda_2 \in \mathbb{U}$, let $X_1 \in \mathcal{T}_{\lambda_1}$ and $X_2 \in \mathcal{T}_{\lambda_2}$.

(i) If both $X_1$ and $X_2$ are analytic, then $X_1 X_2 = X_2 X_1$ if and only if $(X_2 e^0)(\lambda_1 \zeta)(X_1 e^0)(\zeta) = (X_1 e^0)(\lambda_2 \zeta)(X_2 e^0)(\zeta)$, for almost all $\zeta \in \mathbb{D}$.

(ii) If both $X_1$ and $X_2$ are coanalytic, then $X_1 X_2 = X_2 X_1$ if and only if $(X_2^* e^0)(\lambda_2 \zeta)(X_1^* e^0)(\zeta) = (X_1^* e^0)(\lambda_1 \zeta)(X_2^* e^0)(\zeta)$, for almost all $\zeta \in \mathbb{D}$.

**Proof.** (i) Assume that $X_1 \in \mathcal{T}_{\lambda_1}$ and $X_2 \in \mathcal{T}_{\lambda_2}$ are both analytic $\lambda$-Toeplitz operators such that $X_2 e^0(\lambda_1 \zeta)X_1 e^0(\zeta) = X_1 e^0(\lambda_2 \zeta)X_2 e^0(\zeta)$, for almost all $\zeta \in \mathbb{D}$, so, by Theorem 14, $X_1 X_2, X_2 X_1 \in \mathcal{T}_{\lambda_1 \lambda_2}$. Also, analyticity of $X_1$ and $X_2$ reveals that
   \[ \langle X_1 e^j, e_i \rangle_{\mathcal{H}^2} = \begin{cases} 0 & \text{if } i < j \\ \lambda_1^i \langle X_1 e^0, e_{i-j} \rangle_{\mathcal{H}^2} & \text{if } i \geq j. \end{cases} \quad (32) \]
   \[ \langle X_2 e^j, e_i \rangle_{\mathcal{H}^2} = \begin{cases} 0 & \text{if } i < j \\ \lambda_2^i \langle X_2 e^0, e_{i-j} \rangle_{\mathcal{H}^2} & \text{if } i \geq j. \end{cases} \quad (33) \]
   which in turn implies
   \[ \langle X_1 X_2 e^j, e_i \rangle_{\mathcal{H}^2} = \begin{cases} 0 & \text{if } i < j \\ \sum_{k=j}^{i} \lambda_1^k \lambda_2^{i-k} \langle X_1 e^0, e_{k} \rangle \langle X_2 e^0, e_{i-k} \rangle_{\mathcal{H}^2} & \text{if } i \geq j. \end{cases} \quad (34) \]
   But since $X_1 X_2$ is an analytic $\lambda$-Toeplitz operator, we just need to consider the Fourier coefficients of $X_1 X_2 e^0$, which can be obtained from the finite sum in (33) by letting $j = 0$, which gives us the $i$th-Fourier coefficient of $X_1 X_2 e^0$, that is,
   \[ \sum_{k=0}^{i} \lambda_1^k \lambda_2^{i-k} \langle X_1 e^0, e_{k} \rangle \langle X_2 e^0, e_{i-k} \rangle_{\mathcal{H}^2}, \]
   which is nothing but the $i$th-Fourier coefficient of $(X_2 e^0)(\lambda_1 \zeta)(X_1 e^0)(\zeta)$, for almost all $\zeta \in \mathbb{D}$, since
   \[ (X_2 e^0)(\lambda_1 \zeta)(X_1 e^0)(\zeta) = \left( \sum_{j=0}^{\infty} \tilde{X}_2(e^0(i)\lambda_1^{j}) \right) \left( \sum_{k=0}^{\infty} \tilde{X}_1(e^0(i)\zeta^{k}) \right) \]
   \[ = \sum_{j,k} \lambda_1^j \lambda_2^{k} \tilde{X}_2(e^0(k)\lambda_1^{-j}) \tilde{X}_1(e^0(i-k)) \zeta^j, \]
   where $\tilde{X}_2(e^0(i)) = \langle X_2 e^0, e_{i} \rangle_{\mathcal{H}^2}$.

By the assumption $(X_2 e^0)(\lambda_1 \zeta)(X_1 e^0)(\zeta) = (X_1 e^0)(\lambda_2 \zeta)(X_2 e^0)(\zeta)$, hence the finite sum in (34) is also equal to the $i$th-Fourier coefficient of $(X_2 e^0)(\lambda_2 \zeta)(X_1 e^0)(\zeta)$; that is,
   \[ \sum_{k=0}^{i} \lambda_2^k \langle X_1 e^0, e_{k} \rangle_{\mathcal{H}^2} \langle X_2 e^0, e_{i-k} \rangle_{\mathcal{H}^2}, \]
   which is the $i$th-Fourier coefficient of $X_2 X_1 e^0$. What we already showed is that the $i$th-Fourier coefficients of
sym(𝑋₁𝑋₂) and sym(𝑋₂𝑋₁) are equal. Therefore, 𝑋₁𝑋₂ = 𝑋₂𝑋₁. This proves the sufficiency condition in (37).

Let us assume that 𝑋₁𝑋₂ = 𝑋₂𝑋₁. This assumption, along with analyticity of 𝑋₁ and 𝑋₂, implies

$$\sum_{k,j} \lambda_{k,j}^i \langle X_1 e_0, e_{-k} \rangle \langle X_2 e_0, e_{k-j} \rangle$$

for some \( i = 0, 1, 2, \ldots \), such that \( i \geq j \). Now, letting \( j = 0 \) in (37), we obtain

$$(X_2 e_0) (\lambda_j \zeta) (X_1 e_0) (\zeta) = (X_1 e_0) (\lambda_j \zeta) (X_2 e_0) (\zeta)$$

for almost all \( \zeta \in \partial \mathbb{U} \). This proves the necessity condition in (37).

(ii) Assume that \( X_1 \in \mathcal{T}_{\lambda_1} \) and \( X_2 \in \mathcal{T}_{\lambda_2} \) are both coanalytic \( \lambda \)-Toeplitz operators such that

$$(X_2^* e_0) (\overline{\lambda_j \zeta}) (X_1^* e_0) (\zeta) = (X_1^* e_0) (\overline{\lambda_j \zeta}) (X_2^* e_0) (\zeta),$$

for almost all \( \zeta \in \partial \mathbb{U} \). Their coanalyticity implies that \( X_1^* \in \mathcal{T}_{\lambda_1} \) and \( X_2^* \in \mathcal{T}_{\lambda_2} \) are analytic, which satisfies (39). Therefore, by the necessity condition in (37), they commute. This in turn implies that \( X_1 \) and \( X_2 \) commute. This proves the necessity condition in (37).

Now, suppose coanalytic \( \lambda \)-Toeplitz operators \( X_1 \) and \( X_2 \) commute, which means analytic \( \lambda \)-Toeplitz operators \( X_1^* \in \mathcal{T}_{\lambda_1} \) and \( X_2^* \in \mathcal{T}_{\lambda_2} \) commute. Hence, by the necessity condition in (37), we should have

$$(X_2^* e_0) (\overline{\lambda_j \zeta}) (X_1^* e_0) (\zeta) = (X_1^* e_0) (\overline{\lambda_j \zeta}) (X_2^* e_0) (\zeta),$$

which proves the sufficiency condition in (37).

Another consequence of Theorem 11 characterizes the \( \lambda \)-Toeplitz operators having \( \lambda \)-Toeplitz operator inverses.

**Corollary 18.** For \( \lambda \in \partial \mathbb{U} \), let \( X \in \mathcal{T}_\lambda \). If \( X \) is invertible, then a necessary and sufficient condition that \( X^{-1} \) is a \( \lambda \)-Toeplitz operator is that \( X \) is either analytic or coanalytic.

**Proof.** Suppose that \( X \) is invertible. If \( X \) is analytic, then, by Theorem 11, \( X \lambda \)-commutes with \( T_{\lambda, \alpha} \); that is,

$$XT_{\lambda, \alpha} = \lambda T_{\lambda, \alpha} X,$$

from which follows

$$X^{-1} T_{\lambda, \alpha} = \lambda T_{\lambda, \alpha} X^{-1}.$$  \hspace{1cm} (42)

But, on one hand, (42) implies that \( X^{-1} \) is a \( \lambda \)-Toeplitz operator in \( \mathcal{T}_\lambda \) and, on the other hand, that it is an analytic \( \lambda \)-Toeplitz operator, using Theorem 11.

The case for coanalyticity walks through the same steps as the latter case.

Suppose now that \( X^{-1} \) is known to be a \( \lambda \)-Toeplitz operator in \( \mathcal{T}_{\mu} \), for some \( \mu \in \partial \mathbb{U} \). Having the following operator-equations,

$$T_{\lambda, \alpha} X T_{\lambda, \alpha} = \lambda X,$$

$$T_{\lambda, \alpha} X^{-1} T_{\lambda, \alpha} = \mu X^{-1},$$

we obtain

$$T_{\lambda, \alpha} X e_0 \otimes T_{\lambda, \alpha} (X^{-1})^* e_0 = (1 - \lambda \mu) I,$$

$$T_{\lambda, \alpha} X^{-1} e_0 \otimes T_{\lambda, \alpha} X^* e_0 = (1 - \lambda \mu) I,$$

each of which implies \( \mu = \frac{1}{\lambda} \); that is, \( X^{-1} \in \mathcal{T}_{\lambda^{-1}} \) and, in this case, from the first equation follows that \( X \) is coanalytic or \( X^{-1} \) is analytic. And the second one also implies either \( X^{-1} \) is coanalytic or \( X \) is analytic.

If \( X \) is not coanalytic, then \( X^{-1} \) is analytic and not constant; this implies that \( X^{-1} \) is not coanalytic and hence that \( X \) is analytic. The same reasoning also works when it is assumed that \( X \) is not analytic.

\[ \square \]

### 6. \( \lambda \)-Toeplitzness versus Hankelness

One of the properties of Hankelness is that it is preserved under multiplication, on the right, by analytic Toeplitz operators or, on the left, by coanalytic Toeplitz operators. Indeed, if \( H, T \in \mathcal{B}(H^2) \) such that \( H \) is a Hankel operator and \( T \) is an analytic Toeplitz operator, then

$$T_{\lambda, \alpha} H T = H T_{\lambda, \alpha} T = H T T_{\lambda, \alpha},$$

which states that \( H T \) satisfies the Hankel equation, so is a Hankel operator. A similar way shows that \( H T \) is a Hankel operator, where \( T \) is a coanalytic Toeplitz operator.

It turns out that Hankel operators behave in a similar manner when they meet analytic/coanalytic \( \lambda \)-Toeplitz operators.

**Theorem 19.** Let \( H \in \mathcal{B}(H^2) \) be a Hankel operator and \( \lambda \in \mathbb{U} \). If \( X_1 \) is analytic and \( X_2 \) is a coanalytic \( \lambda \)-Toeplitz operator in \( \mathcal{T}_\lambda \), then \( H X_1 \) and \( X_2 X \) satisfy the Hankel equation in the sense that \( \lambda T_{\lambda, \alpha} H X_1 = H X_1 T_{\lambda, \alpha} \) and \( T_{\lambda, \alpha} X_2 H = \lambda X_2 T_{\lambda, \alpha} \).

**Proof.** As it is well known, \( H \) is a Hankel operator if and only if \( T_{\lambda, \alpha} H = H T_{\lambda, \alpha} \). Then, simply, we have

$$\lambda T_{\lambda, \alpha} H X_1 = H T_{\lambda, \alpha} X_1 \Rightarrow H X_1 T_{\lambda, \alpha},$$

$$T_{\lambda, \alpha} X_2 H = \lambda X_2 T_{\lambda, \alpha} \Rightarrow \lambda X_2 T_{\lambda, \alpha} H = \lambda X_2 H T_{\lambda, \alpha},$$

proving the assertion.

\[ \square \]

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.
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