A \((k + 1)\)-partite entanglement measure of \(N\)-partite quantum states

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Received: 29 November 2022 / Accepted: 15 November 2023
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Abstract
The concept of “the permutationally invariant part of a density matrix” constitutes an important tool for entanglement characterization of multqubit systems. In this paper, we first present \((k + 1)\)-partite entanglement measure of \(N\)-partite quantum system, which possesses desirable properties of an entanglement measure. Moreover, we give strong bounds on this measure by considering the permutationally invariant part of a multipartite state. We give two definitions of efficient measurable degree of \((k + 1)\)-partite entanglement. Finally, several concrete examples are given to illustrate the effectiveness of our results.

1 Introduction
Quantum entanglement is one of the most distinctive features of quantum mechanics as compared to classical theory \([1]\). It is also recognized as a remarkable resource in the various areas of quantum information processing such as quantum computation \([2]\), quantum teleportation \([3, 4]\) and dense coding \([5]\). Therefore, measuring and detection of entanglement are fundamental tasks in the theory of quantum entanglement \([6–8]\).

For bipartite quantum systems, quantum states consist of separable states and entangled states. In recent years, much effort has been devoted to distinguish separable from entangled states in both experiment and theory \([9–12]\). However, the classification of multipartite quantum states becomes much more complex because multipartite quantum systems contain more than two individual subsystems. For example, the \(N\)-partite quantum states can be divided into \(k\)-separable states and \(k\)-nonseparable states \((2 \leq k \leq N)\) \([13]\). The detection of \(k\)-nonseparability has been investigated extensively, many efficient criteria \([13–22]\) and computable measures \([23–27]\) have been presented. The \(N\)-partite quantum states can also be divided into \(k\)-producible states \((1 \leq k \leq N − 1)\) and \((k + 1)\)-partite entangled states \([28]\). It is worth noting that the \((k + 1)\)-partite entanglement and the \(k\)-nonseparability are two different concepts involving the partitions of subsystems in \(N\)-partite quantum systems, and they are equivalent only in some special cases.

In this work, we introduce a quantitative measure of \((k + 1)\)-partite entanglement for general multipartite quantum states and prove explicitly that this measure satisfies many properties of multipartite entanglement measures. Moreover, we obtain strong bounds on this measure by considering the permutationally invariant part of a multipartite state. Furthermore, we show that if the permutationally invariant part of a state is \((k+1)\)-partite entangled, then so is the actual state. We define also two efficient numbers based on some simple and powerful \((k + 1)\)-partite entanglement criteria of \(N\) qubit states. In addition, we discuss the application of these two quantities in several examples.

2 Preliminaries
For an \(N\)-partite quantum system with Hilbert space \(\mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \cdots \otimes \mathcal{H}_{N}\), a pure state \(|\psi\rangle\) is \(k\)-producible \((1 \leq k \leq N − 1)\), if it can be represented as \(|\psi\rangle = \bigotimes_{i=1}^{m} |\psi_{i}\rangle_{A_{t}}\) under the partition \(A = A_{1}|A_{2}|\cdots|A_{m}\) satisfying

\[ \bigcup_{t=1}^{m} A_{t} = \{1, 2, \ldots, N\}, A_{t} \cap A_{t'} = \emptyset \text{ for any } t \neq t', \]

with the number of particles \(|A_{t}|\) in the subset \(A_{t}\) being no more than \(k\) for any \(t\), and the substate \(|\psi_{i}\rangle_{A_{t}} \in \bigotimes_{i \in A_{t}} \mathcal{H}_{i}\) \([28]\). For the \(N\)-partite mixed state \(\rho\), if it can be written as a convex combination of \(k\)-producible pure states, i.e., \(\rho = \sum_{i} p_{i} |\psi_{i}\rangle \langle \psi_{i}|\), then it is

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called $k$-producible, where the pure state $|\psi_i\rangle$ might be $k$-producible in different partitions satisfying (1). If a quantum state is not $k$-producible, it contains $(k+1)$-partite entanglement.

For an $N$-qubit quantum state $\rho$ in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, its permutation-invariant (PI) part is defined as

$$\rho^{(\text{PI})} = \frac{1}{N!} \sum_{i=1}^{N!} \Pi_i \rho \Pi_i^\dagger,$$

where the sum takes all over $N!$ permutations $\{\Pi_i\}$ of $N$ particles. Let $U := U_1 \otimes U_2 \otimes \cdots \otimes U_N$ be the local unitary transformation with unitary operator $U_i$ acting on $i$-th subsystem, and $\rho^{(\text{PI})} := \frac{1}{N!} \sum_{i=1}^{N!} \Pi_i U \rho U^\dagger \Pi_i^\dagger$.

### 3 A measure of $(k+1)$-partite entanglement

Let us now introduce a $(k+1)$-partite entanglement measure called $(k+1)$-PE concurrence for quantum states in $N$-partite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$. For an $N$-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, we define the $(k+1)$-PE concurrence $(1 \leq k \leq N-1)$ as

$$E_k(|\psi\rangle) = \min_A \frac{\sum_{i=1}^{m} \sqrt{2 - \text{Tr}(\rho_{A_i}^2)}}{m},$$

where $\rho_{A_i}$ is the reduced density operator of pure state $|\psi\rangle$ for subsystem $A_i$, and the minimum is taken over all possible partitions satisfying (1).

For the $N$-partite mixed state $\rho$ in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, we define the $(k+1)$-PE concurrence $(1 \leq k \leq N-1)$ as

$$E_k(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_k(|\psi_i\rangle),$$

with the infimum running over all possible pure state ensemble decompositions $\{p_i, |\psi_i\rangle\}$ of the mixed state $\rho$.

The $(k+1)$-PE concurrence $(1 \leq k \leq N-1)$, GME-concurrence $C_{\text{GME}}(\rho)$ defined in Ref. [25] and $k$-ME concurrence $(2 \leq k \leq N)$ $C_{k-\text{ME}}(\rho)$ presented in [26] quantify the amount of $(k+1)$-partite entanglement, genuine multipartite entanglement and $k$-nonseparability in a given state $\rho$, respectively. GME-concurrence [25] is the special case of the $k$-ME concurrence $(2 \leq k \leq N)$ [26] when $k = 2$, that is $C_{\text{GME}}(\rho) = C_{2-\text{ME}}(\rho)$. Since $k$-nonseparability and $k$-partite entanglement are two different multipartite correlations, $k$-ME concurrence and $(k+1)$-PE concurrence can quantify multipartite correlations from different perspectives. Only in the extreme cases, $(k+1)$-PE concurrence $(1 \leq k \leq N-1)$, GME-concurrence, $k$-ME concurrence $(2 \leq k \leq N)$ have relation. The relations of them are as follows: $E_{N-1}(\rho) = C_{\text{GME}}(\rho) = C_{2-\text{ME}}(\rho)$, $C_{N-\text{ME}}(\rho) = E_1(\rho)$.

The $(k+1)$-PE concurrence $E_k(\rho)$ is multipartite entanglement measure satisfying properties:

(P1) vanishing on all $k$-producible states, $E_k(\rho) = 0$ for any $k$-producible state $\rho$.

(P2) $E_k(\rho) > 0$ for any quantum states $\rho$ containing $(k+1)$-partite entanglement.

(P3) invariant under local unitary transformations, $E_k(U_1 \otimes \cdots \otimes U_N \rho U_1^\dagger \otimes \cdots \otimes U_N^\dagger) = E_k(\rho)$.

(P4) convexity, $E_k(\sum_i p_i \rho_i) \leq \sum_i p_i E_k(\rho_i)$.

(P5) nonincreasing under local operation and classical communication, $E_k(A_{\text{LOCC}}(\rho)) \leq E_k(\rho)$.

(P6) subadditivity, $E_k(\rho \otimes \sigma) \leq E_k(\rho) + E_k(\sigma)$.

The details of proof are in the Appendix A. Moreover, we can demonstrate the following result.

**Proposition 1** For any $N$-qubit quantum state $\rho$, a lower bound of $(k+1)$-PE concurrence $E_k(\rho)$ can be expressed in terms of PI part of $\rho$ as follows

$$E_k(\rho) \geq \max_{\rho^{(\text{PI})}} E_k(\rho^{(\text{PI})}),$$

where the maximum takes over all possible local unitary transformations.

**Proof** To derive the above result, let $A_1| \cdots | A_m$ be a partition satisfying (1), then $\Pi_i(A_1) \cdots \Pi_i(A_m)$ is also a partition satisfying (1). For any pure state $|\psi\rangle$, $\Pi_i|\psi\rangle$ is still pure state, so we can obtain

$$E_k(|\psi\rangle) = E_k(\Pi_i|\psi\rangle),$$

by the definition of $(k+1)$-PE concurrence for pure states.

The PI part of pure state $\rho = \langle \psi | \psi \rangle$ is $\rho^{(\text{PI})} = \frac{1}{N!} \sum_{i=1}^{N!} \Pi_i |\psi\rangle \langle \psi | \Pi_i^\dagger$, then one has

$$E_k(\rho^{(\text{PI})}) \leq \frac{1}{N!} \sum_{i=1}^{N!} E_k(\Pi_i |\psi\rangle \langle \psi | \Pi_i^\dagger) = \frac{1}{N!} \sum_{i=1}^{N!} E_k(|\psi\rangle) = E_k(|\psi\rangle),$$

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where the inequality holds by the convexity of \((k + 1)\)-PE concurrence, the equality follows from (3). By the definition of \((k + 1)\)-PE concurrence for mixed states, the convexity of \((k + 1)\)-PE concurrence and inequality (4), we can get

\[ E_k(\rho^{(PI)}) \leq E_k(\rho) \]

for mixed state \(\rho\). Since the \((k + 1)\)-PE concurrence remains unchanged under any local unitary transformation, we have

\[
\max E_k(\rho^{(PI)}_U) \leq E_k(\rho),
\]

where the maximum takes over all possible local unitary transformations \(U = U_1 \otimes U_2 \otimes \cdots \otimes U_N\). That is what we are trying to prove.

By the items (P1) and (P2) of entanglement measure \((k + 1)\)-PE concurrence and Proposition 1, for any \(k\)-producing state \(\rho\), we can get \(E_k(\rho^{(PE)}) = 0\), this means that PI part \(\rho^{(PI)}\) is also \(k\)-producing. All in all, we get the following conclusion as a \((k + 1)\)-partite entanglement criterion.

**Corollary** If \(N\)-qubit quantum state \(\rho\) is \(k\)-producing, PI part \(\rho^{(PI)}\) is also \(k\)-producing. That is, if PI part \(\rho^{(PI)}\) of quantum state \(\rho\) contains \((k + 1)\)-partite entanglement, the original quantum state \(\rho\) also contains \((k + 1)\)-partite entanglement.

### 4 The degree of \((k + 1)\)-partite entanglement

There are some criteria for identifying \((k + 1)\)-partite entanglement such as the methods mentioned in Ref. [28]: If \(N\)-qubit quantum state \(\rho\) is \(k\)-producing, then it must satisfy

\[
(2^k - 2)|\rho_{1,2^k}| \leq \sum_{i=0}^{2^k-1} \sqrt{|\rho_{i,2^k-i+1}^2|} \quad (5)
\]

and let \(|\phi_1\rangle = |0\rangle^{\otimes N}|, |\phi_2\rangle = |1\rangle^{\otimes N}\) for Theorem 1 of Ref. [28], where \(r = \frac{N}{\pi}\) for \(k\) \(|k\) \(\pi\), and

\[
\sum_{0 \leq i, j \leq N-1} |\rho_{2^i+1,2^j+1}| \leq \sum_{0 \leq i, j \leq N-1} \sqrt{|\rho_{i,2^i+2^j+1}|^2 + (k + 1) \sum_{i=0}^{N-1} |\rho_{2^i+1,2^j+1}|} \quad (6)
\]

for any \(|\omega_1, \ldots, \omega_T\rangle = |1\rangle^{\otimes T-1}|0\rangle^{\otimes (N-T)}\) for Theorem 2 of Ref. [28]. According to inequality (5) and inequality (6), we can introduce the following two quantities, both of which reflect the degree of \((k + 1)\)-partite entanglement, namely

\[
D_k(N) = \log_2\left(\frac{B}{A} + 2\right),
\]

\[
\tilde{D}_k(N) = \frac{C - D}{E} + 1,
\]

where

\[
A = |\rho_{1,2^k}|,
\]

\[
B = \sum_{i=2}^{2^k-1} \sqrt{|\rho_{i,2^k-i+1}|^2 + (k + 1) \sum_{i=0}^{N-1} |\rho_{2^i+1,2^j+1}|^2 + (k + 1) \sum_{i=0}^{N-1} |\rho_{2^i+1,2^j+1}|},
\]

\[
C = \sum_{0 \leq i, j \leq N-1} |\rho_{2^i+1,2^j+1}|,
\]

\[
D = \sum_{0 \leq i, j \leq N-1} \sqrt{|\rho_{i,2^i+2^j+1}|^2 + (k + 1) \sum_{i=0}^{N-1} |\rho_{2^i+1,2^j+1}|},
\]

\[
E = \sum_{i=0}^{N-1} |\rho_{2^i+1,2^j+1}|.
\]

If \(D_k(N) < \left\lceil \frac{N}{\pi} \right\rceil\) or \(\tilde{D}_k(N) > k\), then PI part of quantum state \(\rho\) contains \((k + 1)\)-partite entanglement. Here, \(\left\lceil \frac{N}{\pi} \right\rceil\) is the smallest integer bigger than or equal \(\frac{N}{\pi}\). Now, we will give two examples.

**Example 1** For an \(N\)-qubit quantum state \(\rho(p) = p|G_N\rangle\langle G_N| + \frac{1 - p}{2^N} 1\) with \(|G_N\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}}\), we have

\[
D_k(N, p) = \log_2\left(\frac{2^N - 1 + p - 1}{2^N p}\right).
\]
Clearly, \( \rho(p) \) is a permutation invariant quantum state. The quantum state \( \rho \) contains \((k + 1)\)-partite entanglement when \( D_k(N, p) \) is genuine entangled states when \( \frac{7}{15} < p \leq 1 \) for \( N = 4 \). This means that the white noise tolerances \( p_{\text{tol}} = 1 - p \) of our criterion are 0 \( \leq p_{\text{tol}} < \frac{4}{15} \) and \( 0 \leq p_{\text{tol}} < \frac{4}{15} \) for \( N = 3 \) and \( N = 4 \), respectively. Hence, for quantum state \( \rho(p) = p|G_N\rangle\langle G_N| + \frac{1 - p}{2N}\mathbf{I} \), when \( N = 3 \) and \( N = 4 \), the ranges of genuine entanglement identified by our method are the same as Tab. 1 of Ref. [33]. In addition, our method can detect some \((k + 1)\)-partite entanglement, such as 2-partite entanglement. It can be seen from Fig. 1 that the larger the value range of \( p \) corresponding to \( \rho(p) \) containing 2-partite entanglement is. Using \( D_1(N, p) < N \), one has \( p > \frac{1 - 2^{N-1}}{1 - 2^{N-2}} \). That is, \( \rho(p) \) contains 2-partite entanglement when \( \frac{1 - 2^{N-1}}{1 - 2^{N-2}} < p \leq 1 \). Since \( \lim_{N \to +\infty} \frac{1 - 2^{N-1}}{1 - 2^{N-2}} = 0 \), \( \rho(p) \) can be arbitrarily close to white noise when \( N \) increases, but it still contains 2-partite entanglement.

Example 2 Consider the \( N \)-qubit mixture of \( |W_N\rangle = \frac{10 \cdots 0 + |01 \cdots 0 \rangle + \cdots + |00 \cdots 1 \rangle}{\sqrt{N}} \) and white noise, which is \( \rho(p) = (1 - p)|W_N\rangle\langle W_N| + \frac{p}{2N}\mathbf{I} \). After calculation, we get

\[
\tilde{D}_k(N, p) = \frac{N2^N - (N2^N + N^2 - 2N)p}{2^N - (2^N - N)p}.
\]

Obviously, \( \rho(p) \) is a permutation invariant quantum state. The \( \rho(p) \) contains \((k + 1)\)-partite entanglement when \( \tilde{D}_k(N) > k \). The \( \rho(p) \) contains \( N \)-partite entanglement when \( 0 \leq p < \frac{2^N}{2^N + 2N^2 - 3N} \). Figure 2 shows that the larger \( N \) is, the larger the value range of \( p \) corresponding to \( \rho(p) \) containing \( N \)-partite entanglement is. In fact, by \( \lim_{N \to +\infty} \frac{2^N}{2^N + 2N^2 - 3N} = 1 \), we can conclude that a large value of \( p \) can ensure that \( \rho(p) \) contains \( N \)-partite entanglement as \( N \) increases.

5 Conclusions

In conclusion, we presented a measure of \((k + 1)\)-partite entanglement for general multipartite quantum states. The rigorous proof shows that it satisfies the requirements for a proper multipartite entanglement measure. We have derived bounds on this measure by considering the PI part of a multipartite state. As an immediate consequence of these bounds, we showed that whenever the PI part of a state is \((k + 1)\)-partite entangled, then so is the state itself. Based on some \((k + 1)\)-partite entanglement criteria of \( N \)-qubit states, we defined two numbers that can be efficiently measurable and quantify the degree of separability of the state. The results in this paper may shed some new light on the study of entanglement properties of multipartite quantum systems.
Fig. 2 For an N-qubit quantum state 
\[ \rho(p) = (1 - p)|W_N\rangle\langle W_N| + \frac{p}{2^N}I, \]
the green, red and blue lines represent the graphs of \( D_k(6, p), \)
\( D_k(8, p), \) and \( D_k(12, p), \)
respectively. The part of the graphs corresponding to \( D_k(N, \rho) < 1 \) is not drawn.

Acknowledgements This work was supported by the National Natural Science Foundation of China under Grant Nos. 12071110, 62271189, funded by Science and Technology Project of Hebei Education Department under Grant No. ZD2021066, the Hebei Central Guidance on Local Science and Technology Development Foundation of China under Grant No. 236Z7604G, supported by National Pre-research Funds of Hebei GEO University in 2023 (Grant KY202316), PhD Research Startup Foundation of Hebei GEO University (Grant BQ201615) and the Key Scientific Research Project of Henan Higher Education Institutions under Grant No.22B140006.

Data Availability Statement No data associated in the manuscript.

Appendix A: The proof of properties of measure of \((k + 1)\)-partite entanglement

It is easy to verify by definition of \((k + 1)\)-PE concurrence \( E_k(\rho) \) that item (P1) is true for all \( k\)-producelable state, and item (P2) holds for all quantum states containing \((k + 1)\)-partite entanglement. For any subset \( B \) of \( \{1, 2, \ldots, N\} \), \( \text{Tr}(\tilde{\rho}_B^2) = \text{Tr}(\tilde{\rho}_B^2) \), where \( \tilde{\rho}_B \) is reduced density matrix of subsystem \( B \) with \( \tilde{p} = U_1 \otimes \cdots \otimes U_N \rho U_1^\dagger \otimes \cdots \otimes U_N^\dagger \), then item (P3) is valid.

In order to prove item (P4), let \( \sum_i p_i \rho_i = \rho \), and suppose that \( E_k(\rho_i) = \sum_j q_{ij} E_k(\langle \psi_{ij} | \psi_{ij} \rangle) \) with \( E_k(\rho_i) \) being given under the pure state decompositions \( \{ q_{ij}, | \psi_{ij} \rangle \} \) of the state \( \rho_i \). Based on the above, we have \( E_k(\rho) = E_k(\sum_i p_i \rho_i), \rho = \sum_i p_i \rho_i = \sum_{i,j} p_i q_{ij} | \psi_{ij} \rangle \langle \psi_{ij} | \). Then, we can obtain

\[
E_k \left( \sum_i p_i \rho_i \right) = E_k(\rho) \leq \sum_{i,j} p_i q_{ij} E_k(\langle \psi_{ij} | \psi_{ij} \rangle) = \sum_i p_i \left[ \sum_j q_{ij} E_k(\langle \psi_{ij} | \psi_{ij} \rangle) \right] = \sum_i p_i E_k(\rho_i),
\]

where the inequality is valid because of the definition of \((k + 1)\)-PE concurrence. So we’ve proved that \( E_k(\rho) \) is convex.

We first prove that item (P5) holds for the pure state \( |\psi\rangle \). For any the partitions \( A_1| \cdots |A_m \) satisfying (1), if we think of \( |\psi\rangle \) as a bipartite quantum state of \( \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \), the concurrence of bipartite quantum pure states \( C_{A_1|A_2}(|\psi\rangle) \) is nonincreasing under LOCC [29–32], that is, \( C_{A_1|A_2}(A_{\text{LOCC}}(|\psi\rangle)) \leq C_{A_1|A_2}(|\psi\rangle) \). Then, one has

\[
E_k(A_{\text{LOCC}}(|\psi\rangle)) \leq \min_A \frac{1}{m} \sum_{i=1}^m C_{A_1|A_2}(A_{\text{LOCC}}(|\psi\rangle)) \leq \min_A \frac{1}{m} \sum_{i=1}^m C_{A_1|A_2}(|\psi\rangle) = E_k(|\psi\rangle).
\]

According to the convexity of \( E_k(\rho) \) and the fact that \( E_k(\rho) \) does not increase under LOCC for pure states, we can easily conclude that it does not increase under LOCC for mixed states, and thus item (P5) holds.

Let’s check that item (P6) is true for both \( \rho \) and \( \sigma \) are pure states. Suppose that there exist partitions \( A_1| \cdots |A_m \) and \( B_1| \cdots |B_n \) satisfying (1) such that

\[
E_k(\rho) = \sum_{i=1}^m \sqrt{2 \left[ 1 - \text{Tr}(\rho_{A_i}^2) \right]}, \quad E_k(\sigma) = \sum_{j=1}^n \sqrt{2 \left[ 1 - \text{Tr}(\sigma_{B_j}^2) \right]},
\]

respectively. Based on these and the definition of \((k + 1)\)-PE concurrence for pure states, we have
\[ E_k(\rho \otimes \sigma) \leq \frac{\sum_{t=1}^{m} \sqrt{2 \left[ 1 - \text{Tr}(\rho_A^t) \right]} + \sum_{t=1}^{n} \sqrt{2 \left[ 1 - \text{Tr}(\sigma_B^t) \right]}}{m + n} \]
\[ \leq \frac{\sum_{t=1}^{m} \sqrt{2 \left[ 1 - \text{Tr}(\rho_A^t) \right]} + \sum_{t=1}^{n} \sqrt{2 \left[ 1 - \text{Tr}(\sigma_B^t) \right]}}{n} = E_k(\rho) + E_k(\sigma). \]

So, \((k + 1)\)-PE concurrence owns item (P6) when both \(\rho\) and \(\sigma\) are pure states. Using the definition of \((k + 1)\)-PE concurrence for mixed states, convexity of \((k + 1)\)-PE concurrence and the effectiveness of item (P6) for both \(\rho\) and \(\sigma\) are pure states, we can also deduce that the item (P6) holds when \(\rho\) and \(\sigma\) are mixed states.

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