Criterion on remote clocks synchronization within a Heisenberg scaling accuracy

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(Dated: February 6, 2014)

PACS numbers: 03.67.Hk, 06.30.Ft, 95.55.Sh, 03.67.Mn

Introduction.—Quantum entanglement, a distinctive feature of quantum mechanics, is at the heart of applications in distributed systems, e.g., quantum key distribution and clock synchronization. Clock synchronization with high precision is a fundamental and an important problem in that it is crucial for many modern technologies and researches, such as global positioning system (GPS), long baseline interferometry, synchronous data transfer, gravitational wave observation (LIGO), tests of theory of general relativity, and distributed computation.

There are two standard methods for synchronizing two spatially separated clocks in the frame of special theory of relativity. One is based on Einstein’s synchronization procedure which uses an operational line-of-sight exchange of light pulses between two spatially separated clocks [1]. The other one is based on the internal time evolution of quantum systems, like Eddington’s infinitesimally slow clock transport [2]. The quantum clock synchronization method based on the sharing of prior entanglement has also been proposed in [3], and has been generalized to several multiparty clock synchronization protocols [4–6]. Independent of the parties’ knowledge of their relative locations or of the properties of the intervening media, these procedures utilize the instantaneous of wavefunction collapsing after the measurement is performed on the shared entangled states. Since the process of distributing entanglement is adiabatic, these protocols are tantamount to Eddington protocol [2].

Quantum entanglement-enhanced parameter estimation that plays a vital role in quantum metrology uses quantum mechanical property to enhance the sensitivity of the measurement of classical quantities. It has been pointed that the standard quantum limit $1/\sqrt{N}$, where $N$ is the number of particles used in the measurement, can be beaten by using the coherent light with squeezed vacuum [8]. In the study of quantum metrology, quantum Fisher information theory and quantum Cramér-Rao bound based on the statistical distance of states have been proposed and developed in [10–17]. The NOON state has been demonstrated to be able to achieve a phase sensitivity saturating the Heisenberg limit $1/N$ [10,11]. Some related strategies have been proposed to perform a high precision in quantum metrology framework [17,18], whilst many experiments have also been performed on this topic [20–27]. One direct and natural idea to apply this technique is in the clock synchronization. Chuang [26] has presented a high efficiency quantum ticking qubits handshake protocol which allows two remote clocks to be synchronized independent of message transport time; and a similar protocol has been proposed to beat the standard quantum limit [27].

In this Letter, we relate the quantum clock synchronization protocol to the problem of estimating an unknown parameter. We investigate the performance of the bipartite maximally entangled spin-zero singlet in the scheme of two-clock synchronization and offer a standard to judge whether two spatially separated clocks have been synchronized in a specific accuracy. This criterion is practical and is consistent to Heisenberg scaling, additionally it does not rely on the unbiased estimation condition which is a fundamental hypotheses in the quantum Fisher information theory.

General framework of quantum clock synchronization.—Suppose two spatially separated parties, Alice and Bob, rested on the same reference frame, both possess high-precision clocks, such as Cs atomic clocks, running at exactly the same rate. They do not agree on a common time at the same readout, for example twelve o’clock. The difference of time $Y$ between their clocks can be expressed as

$$Y = t_B - t_A$$

(1)

In a quantum scheme, in order to eliminate the relative phase that may emerge during qubits transport-
ing to the spatially separated locations, the entangled states should be distributed to Alice and Bob adiabatically. After the entanglement distribution, Alice and Bob respectively perform measurements on all of their qubits simultaneously when their clocks point to the same readout. We choose a normalized entangled state \( |\psi\rangle = \sum_j p_{ij} |i_A\rangle |j_B\rangle \) which is merely changed with an overall unobservable phase under the unitary evolution \( U_{AB}(t) = e^{-iH_A t}/\hbar \otimes e^{-iH_B t}/\hbar \). \( |i_A\rangle \) and \( |j_B\rangle \) are orthonormal basis of measurements which satisfy the completeness \( \sum_{i,j} |i_A\rangle \langle j_B| = I \) I denotes to the identity. Suppose Alice performs the measurement before Bob and obtains a result \( |i_A\rangle \) with probability \( P(i_A) = \sum_j |p_{ij}|^2 \), then the collapsed state evolves as
\[
e^{-iH_{AB}Y/h} \sum_j p_{ij} |j_B\rangle = \sum_{k,j} p_{ij} U_{kj} |k_B\rangle
\]
where \( U_{kj} = \langle k_B|e^{-iH_{AB}Y/h}|j_B\rangle \). Then Bob will obtain the result \( |k_B\rangle \) with probability \( P(k_B|i_A) = |\sum_j p_{ij} U_{kj}|^2 / P(i_A) \).

Fisher information theory and Cramér-Rao bound can be utilized in this clock synchronization situation while the estimation is asymptotically unbiased. By comparing the ratio of observed measurement outcomes with probability distribution that is determined by the parameter \( Y \), two issues may prevent one estimating \( Y \) with a high precision. First, the number of experimental trials is finite, so the ratio of measured outcomes may deviate from the distribution; only when the number \( \nu \) is large enough, could the estimation be unbiased and the Cramér-Rao bound be reached. Second, a one-to-one mapping \( P(\xi|Y) \leftrightarrow Y \) between the probability distributions and parameter is essential. In this Letter, we assume that Alice and Bob respectively perform measurements \( \hat{X} = \langle 0|\langle 0| \otimes \langle 1|\langle 1| \) on all of their own qubits simultaneously when their own clock points to a specific value, where \( \langle \tilde{x} \rangle = \frac{1}{2\nu} \sum_{i=0}^{\nu-1} e^{-i\pi/2} i \langle x \rangle \), and \( |0\rangle, |1\rangle \) are the orthogonal eigenstates of each qubit, which have the identical Hamiltonian \( \hat{H} \) satisfying \( \hat{H} |0\rangle = E_0 |0\rangle \), \( \hat{H} |1\rangle = E_1 |1\rangle \), and \( \omega = (E_1 - E_0)/\hbar > 0 \). One way to implement these ticking qubits in experiment is to put some spin-1/2 particles into the magnetic fields with the same field strength.

Quantum clock synchronization with Bell state and GHZ state.– At first, we suppose that Alice and Bob share \( \nu \) pairs entangled qubits with form \( |\Psi(-)\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \) which is invariant under unitary evolution \( e^{-iH_A t}/\hbar \otimes e^{-iH_B t}/\hbar \). Alice and Bob perform measurements expressed as an operator \( \hat{f} = \hat{X}(t_A) \otimes \hat{X}(t_B) \) in the Heisenberg picture, and it can be described as a set of positive operator valued measurements with element \( \hat{E}(\xi) = |\tilde{x}\rangle \langle \tilde{x}| \otimes |\tilde{y}\rangle \langle \tilde{y}| \) while \( \xi = (\tilde{x}, \tilde{y}) \) where \( x, y = 0, 1 \). Then, the probability distribution of all the measurement results is
\[
P(\xi|Y) = \text{Tr}[\hat{E}(\xi) \hat{\rho}(Y)] = \frac{1}{2} \left( \delta_{x,-y} \cos^2 \frac{\beta}{2} + \delta_{x,y} \sin^2 \frac{\beta}{2} \right)
\]
and the Fisher information is calculated as
\[
\mathcal{F}_Y = \sum_\xi P(\xi|Y) \left( \frac{\partial \ln P(\xi|Y)}{\partial Y} \right)^2 = \omega^2
\]
where \( \hat{\rho}(Y) \) is the density matrix of pure state \( e^{-iH_{AB}(t+y)/\hbar} \otimes e^{-iH_{AB}(t+y)/\hbar} |\Psi(-)\rangle \), \( \beta = \omega|Y| \), and \( \delta_{x,y} \) is Kronecker’s delta. In addition the average of the measurement operator can be calculated as \( \langle f | Y = \sum_\xi g(\xi) P(\xi|Y) = -\cos \theta \) where \( g(\xi) = (-1)^{x+y} \). If \( |Y| < \pi/\omega \) holds, we can obtain the difference of time \( |Y_{est}| \) from the observed expectation value \( \mathcal{F}_Y = \frac{1}{\nu} \sum_{i=1}^{\nu} g(\xi_i) \) after measurements and classical communication. The sign of \( Y_{est} \) can be determined by the outcomes of Alice’s and Bob’s measurement because the one who firstly performed \( \nu \) times measurement would get a duel results with probability \( P(\tilde{0}) = P(\tilde{1}) = 1/2 \). Furthermore, the uncertainty of estimation of \( Y \) could reach the Cramér-Rao bound \( \delta Y_{est} = 1/(\omega \sqrt{\nu}) \mathcal{F}_Y = 1/(\omega \sqrt{\nu}) \) which is the standard quantum limit. Therefore, in this scheme Alice and Bob can synchronize their clocks with accuracy 1/(\omega \sqrt{\nu}).

Some researches have also focused on using the quantum entanglement strategies and employing GHZ like states, \( |\text{GHZ}\rangle = (|0\rangle_A \otimes |1\rangle_B \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B \otimes |0\rangle_B )/\sqrt{2} \) to enhance the precision of parameter estimation. It is easy to verify that GHZ state is merely changed with an overall phase under the unitary evolution \( e^{-iH_{AB}(t+y)/\hbar} \otimes e^{-iH_{AB}(t+y)/\hbar} \). The probability distribution in this protocol takes the form \( P(\xi'|Y) = \frac{1}{2}\text{Tr}[\hat{E}(\xi') \hat{\rho}(\nu \omega Y)] \) where \( g(\xi') = (-1)^{\sum_{i=1}^{\nu} (x_i + y_i)} \) while the symbol is \( \xi' = (\tilde{x}_1, \cdots, \tilde{x}_N, \tilde{y}_1, \cdots, \tilde{y}_N) \), \( x_i, y_j = 0, 1 \); and \( \hat{f}' = \hat{X}(t_A) \otimes \hat{X}(t_B) \) in the measurement operator in the Heisenberg picture. The average of the operator is calculated as \( \cos(\nu \omega Y) \), and Fisher information \( \mathcal{F}_Y = N \omega^2 \). Considering that the probability distributions and expectation value are all function with periodicity \( 2\pi/N \), thus one can unambiguously obtain \( Y_{est} \) from the observed expectation value after measurements and classical communication only when the condition \( |Y| < \pi/(\omega N) \) is satisfied. The sign of \( Y_{est} \) can also be determined by the outcomes of Alice’s and Bob’s measurement because the one who firstly performed \( \nu \) times measurement would get the probability \( P(\tilde{x}_1 \cdots \tilde{x}_N) = 1/2^N \) and \( P(\tilde{x}_i) = 1/2 \). When \( \nu \) is large enough, the uncertainty can attain the Cramér-Rao bound \( \delta Y_{est} = 1/(\omega N \sqrt{\nu}) \) which has a Heisenberg scaling accuracy. Despite of this optimal local distinguishability in the Hilbert space, GHZ states are inappropriate to obtain more advantageous information from any values of parameter \( Y \) in this single
procedure since the condition \( |Y| < \pi/(\omega N) \) is required \[28\].

Quantum clock synchronization with bipartite maximally entangled states. — We next consider a scheme which exploit different entanglement resource. The bipartite maximally entangled spin-zero singlet has been proposed as a resource for quantum-enhanced metrology \[29\], with the following form:

\[
|\chi\rangle = \frac{1}{\sqrt{2J+1}} \sum_{M=-J}^{J} (-1)^{J-M} |J, M\rangle_{z,A} |J, -M\rangle_{z,B}
\]  

where \( J = N/2 \), and \( |J, M\rangle_z \) is a completely symmetric normalized state (Dicke state) with \((J - M)\) qubits being \( |0\rangle \) and \((J + M)\) qubits being \( |1\rangle \). There is an explicit mapping between the two-symmetric entangled state and the direct product of \( N \) maximally entangled states, which is presented in \[30\], then one obtains

\[
|\chi\rangle = \frac{2^{N/2}}{N!\sqrt{N+1}} \sum_{\sigma} |\Psi(-)\rangle_{A_1B_1, \ldots} |\Psi(-)\rangle_{A_NB_N}
\]

where \( \mathbb{I} \) is the identity operator on Hilbert space \( \mathcal{H} = \{0, 1\} \), \( S = \sum_{M=-J}^{J} \mathcal{H}_{z,A} \mathcal{H}_{z,B} \) is the symmetric projector that maps states in \( \mathcal{H}^\otimes N \) onto its symmetric subspace \( \mathcal{H}^{\otimes N}_{\text{sym}} \), and denotes to a permutation. After the adiabatic distribution of \( N \) pairs entanglement \( |\Psi(-)\rangle \), Alice or Bob can perform the symmetric projector \( S \) to obtain \( \nu \) pairs \( |\chi\rangle \). Because \( |\Psi(-)\rangle = (|0\rangle - |1\rangle)/\sqrt{2} \) changes only with an overall unobservable phase under any unitary evolution of form \( U \otimes U \) in two-qubit space, then this singlet has the rotational invariance property under unitary evolution \( U \otimes U \otimes U \otimes U \) and has identical expression in any spin basis, e.g. \( z \rightarrow x \rightarrow y \) when ignoring the overall phase. Another proof has been presented in Ref. \[31\], and this invariance property has been tested in the experiment \[32\]. These bipartite maximally entangled states play an important role in the quantum information distribution and concentration \[33, 34\]. Recently these states used in our scheme have been generated experimentally by using stimulated parametric down-conversion and have been used in the 1 to 3 + 2 information distribution \[32, 33\]. Additionally, some other experiments also produce such entanglement and realize the quantum information distribution \[37\]. Next, we show that these technologies can also be utilized to implement our scheme for quantum clock synchronization.

The pure state evolved as

\[
(e^{-iH_{\text{At}/\hbar}})^{\otimes N} \otimes (e^{-iH_{\text{Bt}/\hbar}})^{\otimes N} |\chi\rangle = (H_{A}^{\otimes N} \otimes H_{B}^{\otimes N})(I_{A}^{\otimes N} \otimes (e^{-iYB_{0}}H_{A}^{\otimes Nh_{B}/\hbar})^{\otimes N})|\chi\rangle = (H_{A}^{\otimes N} \otimes H_{B}^{\otimes N})(I_{A}^{\otimes N} \otimes U_{B}(\pi/2, \beta, -\pi/2)^{\otimes N})|\chi\rangle
\]

where the overall phase is ignored; \( H_{A,B} = \frac{1}{\sqrt{2}} [1 \, 1 \, -1] \) is the Hadmard matrix and the unitary operator \( U(\alpha, \beta, \gamma) \) is expressed by three Euler angles in the basis \( \{|0\rangle, |1\rangle\} \) as the following:

\[
U(\alpha, \beta, \gamma) = \begin{bmatrix} \cos \frac{\alpha}{2} e^{i(\alpha+\gamma)/2} & \sin \frac{\alpha}{2} e^{-i(\alpha-\gamma)/2} \\ -\sin \frac{\alpha}{2} e^{i(\alpha-\gamma)/2} & \cos \frac{\alpha}{2} e^{-i(\alpha+\gamma)/2} \end{bmatrix}
\]

Moreover, according to group theory of the irreducible representation, we obtain an analytical expression of the unitary operator in \( N \)-qubit space \[38\]

\[
U(\alpha, \beta, \gamma)^{\otimes N} |JM\rangle = \sum_{M'} e^{-i(M\alpha + M'\gamma)} d_{MM'}^{\gamma} |JM\rangle
\]

Thus, we can obtain the probability distribution of measurement outcomes \( \xi' = (x_1, \ldots, x_N, y_1, \ldots, y_N) \), with \( x_i, y_j = 0, 1 \):

\[
P(\xi'|Y) = \frac{\left[ d_{M',-M}^{\gamma}(\beta) \right]^2}{(2J+1)C_{2J}^{J-M} C_{2J}^{J-M'}}
\]

where \( (J - M) \) is the number of 0 in \( \{x_1, \ldots, x_N\} \) while \( (J - M') \) is the number of 0 in \( \{y_1, \ldots, y_N\} \). As before the expectation value of measurement operator \( \hat{f}' = \hat{X}(t_A)^{\otimes N} \otimes \hat{X}(t_B)^{\otimes N} \) is calculated as

\[
f(\beta) := \langle \hat{f}' \rangle_Y = \frac{1}{\sum_{M,M'=0}^{J} (-1)^{N-M+M'} \left[ d_{M',-M}^{\gamma}(\beta) \right]^2}{2J+1} = \frac{(-1)^{N} \sin(N+1)\beta}{N+1} \sin \beta.
\]

The function \( f(\beta) \) against its argument \( \beta \) is shown in Fig. 1 with different numbers of qubits. Furthermore, Fisher information reads

\[
\mathcal{F}_Y = \sum_{M,M'=-J}^{J} \frac{4 \left[ d_{M',-M}^{\gamma}(\beta) \right]^2}{2J+1} = \frac{4(J+1)/\omega}{3}
\]

FIG. 1: The functional relation between the expectation value \( f(\beta) \) and the parameter \( \beta \).

\[\text{Fig. 1 with different numbers of qubits.} \]
with which it is straightforward to find that lower bound
\[ \delta Y_{\text{est}} = \sqrt{3}/(\omega \sqrt{N(2+N^2)}) \] obviously breaks the quantum standard limit and performs a Heisenberg scaling accuracy.

Nevertheless, this scheme has some special properties that differ from the previous schemes. Considering that \( f(\beta) \) is clearly “peaked” around \( \beta = 0 \) with width \( \sim \pi/(N+1) \) (see Fig. 1), we can confirm that the uncertainty of \( Y_{\text{est}} \) could reach the Cramér-Rao bound \( 1/(\omega \sqrt{N^2}) \) for a large number \( N \) when the condition \( |Y| < \pi/(N+1) \omega \) is satisfied. Although the bipartite maximally entangled spin-zero singlet fails to gain more advantageous information of the parameter \( Y \) from the expected value of the measurement results by the inequality \( |f_Y| \leq \pi/(N+1) \omega \), with increasing the number \( N \), we can step further to obtain the difference between two clocks with a Heisenberg scaling accuracy in accordance to the expectation of the measurement results by increasing the number \( N \).

In conclusion, we propose a novel quantum scheme for remote clocks synchronization within a specific accuracy. This bound of accuracy scales as the Heisenberg limit which is the ultimate limit of precision measurements under all conditions. With developments in creating experimentally the entanglement resources, this quantum scheme of remote clocks synchronization may be implemented which may possess unprecedented precision.

We would like to thank Guo-Yong Xiang, Xiang-Ru Xiao and Li Jing for useful discussions. This work was supported by 973 program (2010CB922904), NSFC (11175248), NFFTS(J1030310,J1103205), grants from Chinese Academy of Sciences, and the Chun-Tsung scholar fund of Peking University.

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\[d_{M'}^J(\beta) = \sum_{\nu} (-1)^\nu \left[\frac{(J+M')(J-M')(J+M)(J-M)}{(J+M'-\nu)(J-M'-\nu)(\nu+M-M')}\right]^{1/2} \times \left(\cos^2 J+M'-M-2\nu(\sin^2 J)\right)^{\nu-M'-M'} .\]
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