The Weil-Petersson gradient flow of renormalized volume on a Bers slice has a global attracting fixed point

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Abstract

We show that the flow on a Bers slice given by the Weil-Petersson gradient vector field of renormalized volume is globally attracting to its Fuchsian basepoint.

1 Introduction

We consider the Weil-Petersson gradient flow of renormalized volume on a Bers slice. Renormalized volume arose from work of Graham and Witten ([GW]) in physics to give an alternative notion of volume for conformally compact Einstein manifolds. In the hyperbolic setting, this was described and developed in the papers [TL, ZT, KS1, KS2] of Takhtajan, Zograf, Teo, Krasnov, and Schlenker. The renormalized volume $V_R(M)$ of a hyperbolic manifold $M$ connects many analytic concepts from the deformation theory to the geometry of $M$ and is closely related to classical objects such as the convex core volume $V_C(M)$ and the Weil-Petersson geometry of Teichmüller space.

The flow arises also naturally in the classical theory of Kleinian groups so the renormalized volume perspective will not be needed in our approach. For the description in terms of renormalized volume we refer the reader to the earlier papers [BBB1, BBB2] for this perspective.

In [BBB2] the authors conjecture that the flow on the deformation space of a (relatively) acylindrical hyperbolic manifold limits to the unique structure with minimum convex core volume. In this paper we prove the conjecture for the Bers slice.

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1.1 Flow on a Bers slice

We consider $S$ be a closed surface of genus $g \geq 2$ and let Teichmüller space $\text{Teich}(S)$ be the space of marked conformal structures on $S$. We let quasifuchsian space $\text{QF}(S)$ be the space of convex co-compact hyperbolic structures on the interior of $S \times [0,1]$. If $M \in \text{QF}(S)$ then $M$ has a natural conformal boundary $\partial_c M$ given by a pair $(X,Y) \in \text{Teich}(S) \times \text{Teich}(S)$. Furthermore Bers simultaneous uniformization states that the map $\text{QF}(S) \to \text{Teich}(S) \times \text{Teich}(S)$ given by $M \mapsto \partial_c M = (X,Y)$ is a diffeomorphism. Given $X \in \text{Teich}(S)$ the Bers slice is the subspace $\mathcal{B}_X \subseteq \text{QF}(S)$ corresponding to $\{X\} \times \text{Teich}(S)$. We will identify $\mathcal{B}_X$ with $\text{Teich}(S)$ by mapping $(X,Y) \to Y$. Further the Bers slice $\mathcal{B}_X$ has a natural basepoint given by $Y = X$ corresponding to the unique Fuchsian structure in $\mathcal{B}_X$.

Given $M = (X,Y) \in \mathcal{B}_X$ we define the associated projective structure on $S$ given as follows; If $M = H^3/\Gamma$, then its domain of discontinuity is $\Omega_M = \Omega_X \cup \Omega_Y$ where $X = \Omega_X/\Gamma$ and $Y = \Omega_Y/\Gamma$. We let $f_Y : H^2 \to \Omega_Y$ be a univalent map uniformizing $\Omega_Y$. Then we define a quadratic differential on $H^2$ given by $\phi_Y = S(f_Y)dz^2$ where $S(f_Y)$ is the Schwarzian derivative of $f_Y$. It follows by naturality that $\phi_Y$ descends to a quadratic differential $\phi_Y \in Q(Y)$. Identifying the tangent space $T_M(\mathcal{B}_X) = T_Y(\text{Teich}(S))$ with the space of Beltrami differentials (modulo an equivalence relation), we define the vector field $V$ by

$$V(Y) = -\frac{\phi_Y}{\rho_Y}$$

where $\rho_Y$ is the hyperbolic area form on $Y$. One can easily check that the unique critical point for $V$ is the basepoint $Y = X$. For further background on Teichmüller theory, see [Le].

The flow given by $V$ has also been used to prove a number of interesting results, in particular connections between Weil-Petersson geometry and convex core volume (see [BBB1, BBB2]). In this paper we prove that the flowlines of $V$ limit to the Fuchsian basepoint $X$ in $\mathcal{B}_X$ and therefore $X$ is a global attracting fixed point for $V$.

The Bers embedding gives an embedding of $\mathcal{B}_X$ as a bounded open topological ball in the finite dimensional vector space $Q(X)$. Work of Komori,Sugawa,Wada and Yamashita (see [KSWY] for images) and Dumas (see [Dum1] for images and [Dum2] for software) has shown that this embedding exhibits fractal behaviour. Also in the paper [Miy], Miyachi proved that cusped points are dense in the boundary. Nevertheless we show that the flow $V$ has a global attracting fixed point at its Fuchsian basepoint $X$.

1.2 A toy model for the flow

An interesting toy model for the flow is obtained by considering the space of univalent maps $f : H^2 \to \hat{C}$ of the form $f(z) = c^z$ whose image is a Jordan domain. This space corresponds to the open disk $\mathcal{D} = \{f(z) = c^z \mid |c-1| < 1\}$. Each $f \in \mathcal{D}$
determines a Beltrami differential \( \nu_c \) on \( \mathbb{H}^2 \) where
\[
\nu_c = -\frac{S(f)}{\rho_{\mathbb{H}^2}}.
\]

We can then define a Beltrami differential \( \mu_c \) on \( \hat{C} \) by defining it to be the pushforward by \( f \) of \( \nu_c \) on the image of \( f \) and zero outside.

There is a complete vector field \( v \) on \( D \) with flow \( \phi_t \) such that for every \( c \in D \) there is a family of quasiconformal maps \( g_t : \hat{C} \rightarrow \hat{C} \) such that the Beltrami differential at time \( t \) is \( \mu_{\phi_t(c)} \). In fact the vector field \( v \) is given explicitly by the equation
\[
v(c) = \frac{1}{4} \left( |c|^4 - 2c \text{Re}(c^2) - c^2 + 2c \right).
\]

By elementary analysis the flow on \( D \) has a global attractor at \( c = 1 \) (see Section 5). We note that although this global attractor is also the global minimum for the \( L^\infty \) norm of the Schwarzian, the \( L^\infty \) norm of the Schwarzian is not monotonic along the flowlines of \( v \).

The properties of this model flow play a crucial role in the proof of our main theorem which we now state.

**Theorem 1.1** Let \( X \in \text{Teich}(S) \) and \( M_t \in \mathcal{B}_X \) be a flowline for \( V \), then \( M_t \) converges to \( X \).

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2 Weil-Petersson geometry

We consider \( S \) a closed surface of genus \( g \geq 2 \). Then Teichmüller space \( \text{Teich}(S) \) is the space of marked conformal structures on \( S \). Given \( X \in \text{Teich}(S) \) the cotangent space \( T^*_X(\text{Teich}(S)) \) is \( Q(X) \) the space of holomorphic quadratic differentials on \( X \). There is a pairing between \( Q(X) \) and \( B(X) \) given by

\[
(\phi, \mu) = \int_X \phi \mu.
\]

If we let \( N(X) \subseteq B(X) \) be the annihilator of \( Q(X) \) under this pairing, we obtain the identification \( T^*_X(\text{Teich}(S)) = B(X)/N(X) \).

Given \( \phi \in Q(X) \) and \( z \in X \) then we define the pointwise norm by

\[
\|\phi(z)\| = \frac{|\phi(z)|}{\rho_X(z)}
\]

where \( \rho_X \) is the hyperbolic metric on \( X \). We define the \( L^p \) norm of \( \phi \), denoted \( \|\phi\|_p \), to be the \( L^p \) norm of the function \( \|\phi(z)\| \) with respect to the hyperbolic area form on \( X \). These \( L^p \) norms define Finsler cometrics on the cotangent bundle of \( \text{Teich}(S) \) and dual Finsler metrics on the tangent bundle of \( \text{Teich}(S) \).

When \( p = 2 \) this norm comes from an inner product and therefore determines a Riemannian metric on \( \text{Teich}(S) \) called the Weil-Petersson metric. Classical results are that it is incomplete (see \cite{Chu1, Wol1}) and strictly negatively curved (see \cite{Tro, Wol2}).

The Weil-Petersson completion \( \overline{\text{Teich}(S)} \) has boundary \( \partial \text{Teich}(S) \) given by the space of marked noded surfaces (see \cite{Mas}). Thus if \( \hat{Y} \in \partial \text{Teich}(S) \) then \( \hat{Y} \) has nodes \( \sigma \) consisting of the collection of disjoint simple closed curves whose length is zero on \( \hat{Y} \).

In our analysis, we will make use of Gardiner’s formula which describes the derivative of length functions on the quasiconformal deformation space of a Fuchsian or Kleinian group. Although we will only use it in the quasifuchsian setting, we will state in in full generality. We let \( \Gamma \) be a Kleinian group and let \( QC(\Gamma) \) be the space of quasiconformal deformations of \( \Gamma \). Given \( \gamma \in \Gamma \), loxodromic, let \( \mathcal{L}_\gamma : QC(\Gamma) \to \mathbb{C}/2\pi i \) be its complex length function. Let \( B(\hat{\mathbb{C}}, \Gamma) \) be the space of Beltrami differentials on \( \hat{\mathbb{C}} \) invariant under \( \Gamma \). Then as before the tangent space of \( QC(\Gamma) \) at \( \Gamma \) is naturally a quotient space of \( B(\hat{\mathbb{C}}, \Gamma) \) under an equivalence relation.

**Theorem 2.1** (Gardiner, \cite{Gard}) Let \( \Gamma \) be a Kleinian group with \( \gamma(z) = e^{\mathcal{L}_\gamma}z \in \Gamma \) and \( \text{Re}\mathcal{L}_\gamma > 0 \). For \( \mu \in B(\hat{\mathbb{C}}, \Gamma) \) then

\[
(d\mathcal{L}_\gamma, \mu) = \frac{1}{\pi} \int_{\{1 \leq |z| \leq e^{\mathcal{L}_\gamma}\}} \mu(z) \frac{dxdy}{z^2}.
\]

If \( \Gamma \) is Fuchsian and \( \mu \) is a Fuchsian deformation (i.e. \( \mu(z) = \overline{\mu(z)} \)) then

\[
(d\mathcal{L}_\gamma, \mu) = \frac{2}{\pi} \text{Re} \int_{\mathbb{H} \cap \{1 \leq |z| \leq e^{\mathcal{L}_\gamma}\}} \mu(z) \frac{dxdy}{z^2}.
\]
In his original paper [Gard], Gardiner proved the variational formula only in the Fuchsian case but as noted in [Miy, First proposition, Section 8], the proof gives the above in the case of a general loxodromic element.

3 Limits of the flow

We now define the flow $V$ on the Bers slice $\mathcal{B}_X$. By Bers we have $\mathcal{B}_X \cong \text{Teich}(S)$ (see [Brs]). Therefore if $M = (X, Y) \in \mathcal{B}_X$ then we have isomorphisms $T_M(\mathcal{B}_X) \cong T_Y(\text{Teich}(S))$. We then define the vector field $V$ on $\mathcal{B}_X$ by

$$V_Y = -\frac{\phi_Y}{\rho_Y}.$$

As discussed, the motivation for defining the flow comes from the study of renormalized volume $V_R$, an analytic function on $\mathcal{B}_X$. Although we do not need to discuss renormalized volume to define the flow, we will now give a brief description. By earlier work of a number of authors, renormalized volume has the following variational formula.

**Theorem 3.1 ([ZT, TH, KS2])** Let $\mu \in T_Y \mathcal{B}_X$ then

$$dV_R(\mu) = \text{Re} \int_Y \phi_Y \mu.$$

As the variational formula gives a formula for the differential of renormalized volume, it can be used as a definition of renormalized volume.

Combining this variational formula with the definition of the Weil-Petersson metric, our flow is also given by

$$V = -\nabla_{WP} V_R,$$

the negative of the Weil-Petersson gradient flow of renormalized volume.

By the Nehari bound on the norm of the Schwarzian derivative of a univalent map (see [Neh]), $V$ is bounded with respect to the Teichmüller metric on $\text{Teich}(S)$. Therefore as the Teichmüller metric is complete, the flowlines exist for all time (see [BBB1] for further details).

Also by the gradient description of $V$ it follows that along a flowline a $M_t$

$$V_R(M_0) - V_R(M_T) = \int_0^T \|\phi_Y\|^2 dt.$$

As $V_R \geq 0$ (see [BBB1]), then it follows that

$$\int_0^\infty \|\phi_Y\|^2 dt < \infty.$$

We now describe the further properties of the flow proved in [BBB2] that we will need in our analysis.

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Theorem 3.2 (Bridgeman-Brock-Bromberg, [BBB2]) Let $M_i = (X,Y_i)$ be a flow-line for $V$ on $\mathcal{B}_X$. Then

1. $Y_i \to \hat{Y} \in \overline{\text{Teich}(S)}$ in the Weil-Petersson completion. Thus $\hat{Y}$ is a noded Riemann surface.

2. $\|\phi_i\|_2 \to 0$ as $t \to \infty$.

In order to prove our main theorem, we need to prove that the set of nodes of $\hat{Y}$ is empty or alternately that $\hat{Y} \in \overline{\text{Teich}(S)}$. We will do this by assuming $\hat{Y}$ is noded and consider the limits of the projective structure as we zoom in on the nodes.

4 Taking limits at a node

We will assume the flow hits the boundary and then derive a contradiction. To do this, we will first need to extract limiting maps.

We let $M_t = (X,Y_t)$ be a smooth path in $\mathcal{B}_X$ such that $Y_t$ converges in $\overline{\text{Teich}(S)}$ to $\hat{Y} \in \partial \text{Teich}(S)$ a noded surface with curve $\alpha$ a node of $\hat{Y}$. We let $\ell_\alpha$ be the length function of $\alpha$ on $\text{Teich}(S)$ with $\ell_\alpha(t) = \ell_\alpha(Y_t) = \log r_\alpha(t)$. Further we let $\mathcal{L}_\alpha(t)$ be the complex length function of $\alpha$ in $M_t$.

We define $c_\alpha(t) = \frac{\mathcal{L}_\alpha(t)}{\ell_\alpha(t)}$.

We fix $f_\alpha^t : \mathbb{H}^2 \to \hat{\mathbb{C}}$ to be a uniformization of a connected component of the domain of discontinuity $\Omega_t$ corresponding to $Y_t$ such that $\alpha$ lifts to the imaginary axis in $\mathbb{H}^2$ with $f_\alpha^t(i) = 0$, $(f_\alpha^t)'(i) = 1$. Then $f_\alpha^t$ is a normal family of univalent maps and every sequence contains a subsequence which converges locally uniformly to a univalent map.

Proposition 4.1 Let $M_t = (X,Y_t)$ be a smooth path in $\mathcal{B}_X$ such that $Y_t \to \hat{Y} \in \partial \text{Teich}(S)$ and $\alpha$ a node of $\hat{Y}$. Then for any subsequence $f_\alpha^t$ of $f_\alpha^s$ locally uniformly converging to $f_\alpha : \mathbb{H}^2 \to \Omega$ we have $c_\alpha(t_n) \to c$ with $|c - 1| \leq 1$. Furthermore there exists a unique Möbius transformation $g$ such that

- If $c \neq 0$ then $(g \circ f_\alpha^t)(z) = z^c$.
- If $c = 0$ then $(g \circ f_\alpha^t)(z) = \log(z)$.

Proof: By the Bers inequality (see [Brs]) if $\beta$ is a curve in $S$ with length $\ell_\beta(Z)$ for $Z \in \text{Teich}(S)$ and complex length $\mathcal{L}_\beta(M)$ for the quasifuchsian manifold $M = (X,Y)$ then

$$\frac{1}{\ell_\beta(X)} + \frac{1}{\ell_\beta(Y)} \leq \frac{2 \text{Re} \mathcal{L}_\beta(M)}{|\mathcal{L}_\beta(M)|^2}.$$}

Therefore applying the inequality to $\alpha$ and $M_t = (X,Y_t)$ gives
\[
\frac{1}{\ell_a(t)} \leq \frac{2 \text{Re} \mathcal{L}_a(t)}{|\mathcal{L}_a(t)|^2}
\]

Therefore the inequality becomes
\[
|c_a(t)|^2 = \frac{|\mathcal{L}_a(t)|^2}{\ell_a(t)^2} \leq 2 \text{Re} \left( \frac{\mathcal{L}_a(t)}{\ell_a(t)} \right) = 2 \text{Re}(c_a(t)),
\]
\[
|c_a(t)|^2 - 2 \text{Re}(c_a(t)) \leq 0,
\]
\[
|c_a(t) - 1|^2 - 1 \leq 0.
\]

Thus \(|c_a(t) - 1| \leq 1\).

We consider the maps \(f_n = f_n^\alpha : \mathbb{H}^2 \to \Omega_n\) such that \(f_n \to f\) uniformly on compact subsets. Then \(S(f_n) \to S(f)\) locally uniformly (see [Le]).

Taking the ratio of \(S(f_n)\) and the quadratic differential \(ds^2/\zeta^2\) on \(\mathbb{H}^2\) we have
\[
S(f_n) = \phi_n(z)ds^2/\zeta^2
\]
where \(\phi_n\) is holomorphic. Also \(\phi_n\) is invariant under the map \(z \to e^z\) for all \(s = k\ell_a(t_n), k \in \mathbb{Z}\). Therefore \(S(f) = \phi(z)ds^2/\zeta^2\) where \(\phi\) is holomorphic with \(\phi_n \to \phi\) locally uniformly. Also \(\phi\) is invariant under the action of \(\mathbb{R}\) on \(\mathbb{H}^2\) given by \(k \cdot z = e^kz\). It follows that \(\phi\) is a constant function giving \(S(f) = Kdz^2/\zeta^2\). By the uniqueness of the Schwarzian derivative, up to post-composition by a Möbius transformation (see [Le]), it follows that \((g \circ f)(z) = z^2\) or \((g \circ f)(z) = \log(z)\) for some unique Möbius transformation \(g\). To complete the proof, we will now show that \(c = \lim_{n \to \infty} c_n(t_n)\).

We have that \(f_n\) conjugates a surface subgroup \(\Gamma_n\) acting on \(\mathbb{H}^2\) to a quasifuchsian subgroup \(\hat{\Gamma}_n\) acting on \(\hat{\mathbb{C}}\). The map \(f\) uniformizes the domain \(\Omega = f(\mathbb{H}^2)\).

The curve \(\alpha\) in \(S\) gives geodesic \(\alpha_n\) in the hyperbolic metric on \(Y_n\). We let \(\ell_n = \ell(t_n)\) and \(\mathcal{L}_n = \mathcal{L}(t_n)\). Similarly we can choose lifts with \(\tilde{\alpha}_n = f_n(i\mathbb{R}_+^+)\).

We consider first when \(f(z) = z^c\) for some \(c \neq 0\). We have that
\[
f_n(e^{k\ell_n}z) = M_n^k f_n(z)
\]
where \(M_n\) is a loxodromic Möbius transformation conjugate to \(e^{\ell_n}z\). For simplicity, if \(m(z)\) is a Möbius transformation conjugate to \(e^{\ell_n}z\), we define \(m'(z)\) to be the map conjugate to \(m'(z) = e^{\ell_n}z\) under the same conjugacy.

We choose \(k_n \in \mathbb{Z}\) such that \(k_n \ell_n \to s\) and therefore
\[
f(e^z) = \lim_{n \to \infty} f_n(e^{k_n \ell_n}z) = \lim_{n \to \infty} (M_n^{k_n} f_n(z)) = \lim_{n \to \infty} (m_n f_n(z))
\]
where \(m_n = M_n^{1/(k_n)}\). Then \(m_n\) is conjugate to a loxodromic of the form \(z \to e^{\ell_n}z\). To show \(m_n\) converges in \(\text{PSL}(2, \mathbb{C})\) we note that if \(m_n(z) = (A_n z + B_n)/(C_n z + D_n)\) then evaluating at \(z = i\) we get
\[
f(e.i) = \lim_{n \to \infty} m_n(0) = \frac{B_n}{D_n} \quad f(e^{-1}.i) = \lim_{n \to \infty} m_n^{-1}(0) = \frac{-B_n}{A_n}.
\]

Now calculating derivatives we have
\[
f'(e.i).e = \lim_{n \to \infty} m'_n(0) = \lim_{n \to \infty} \frac{1}{D_n^2}.
\]
Thus $B_n/D_n, A_n/D_n, C_n/D_n$ all converge. It follows that $A_n/D_n$ and $C_n/D_n$ converge. Thus as

$$m_n(z) = \frac{A_nz + B_n}{C_nz + D_n} = \frac{\frac{A_n}{D_n}z + \frac{B_n}{D_n}}{\frac{C_n}{D_n}z + 1}$$

we have $m_n$ converges. We let $m_n \to m$, then taking limits we have

$$f(e^z) = m'f(z).$$

As $f(z) = z^c, c \neq 0$ then $m$ is conjugate to the map $z \to e^cz$. Furthermore as $m_n = M_n^{1/n}$ then equating translation lengths we have

$$\lim_{n \to \infty} \frac{\ell'_n(t_n)}{\ell_n(t_n)} = \frac{1}{4} (|c|^4 - c^2),$$

$$\lim_{n \to \infty} \frac{\ell_n(t_n)}{\ell_n(t_n)} = \frac{1}{2} (\Re(c^2) - 1),$$

$$\lim_{n \to \infty} c'_n(t_n) = \frac{1}{4} (|c|^4 - 2c\Re(c^2) - c^2 + 2c).$$

Furthermore if $c = 0$ then

$$\lim_{n \to \infty} \frac{c'_n(t_n)}{c_n(t_n)} = \frac{1}{2}.$$

**Proof:** For convenience, we drop the script $\alpha$. We consider two cases:

**Case** $c \neq 0$: We have maps $f_t = f_t^\alpha : \mathbb{H}^2 \to \hat{\mathbb{C}}$ such that $f_n$ converges locally uniformly to $f$ where $f$ is equal to the map $z^c$ post-composed with a Möbius transformation. Then by post-composing by Möbius transformations, we assume $f_t$ fix both 0, $\infty$ and $f_n \to z^c$ locally uniformly. We also let $\gamma$ be the hyperbolic element acting on the upper half plane corresponding to $\alpha$ with axis the positive $y-$axis. Then $f_t$ conjugates $\gamma$ to the Möbius transformation $\gamma_t$.

**Lemma 4.2** Let $M_t = (X_t, Y_t)$ be a flowline for $V$ in the Bers slice $B_X$ such that $Y_t \to \hat{Y} \in \partial \Teich(S)$ with $\alpha$ a node of $\hat{Y}$. Let $f_{t_n}^\alpha$ be a subsequence of $f_t^\alpha$ locally uniformly convergent to $f^\alpha$ with $c_{\alpha}(t_n) \to c$. Then the following limits hold,

$$\lim_{n \to \infty} \frac{\ell'_n(t_n)}{\ell_n(t_n)} = \frac{1}{4} (|c|^4 - c^2),$$

$$\lim_{n \to \infty} \frac{\ell_n(t_n)}{\ell_n(t_n)} = \frac{1}{2} (\Re(c^2) - 1),$$

$$\lim_{n \to \infty} c'_n(t_n) = \frac{1}{4} (|c|^4 - 2c\Re(c^2) - c^2 + 2c).$$

Furthermore if $c = 0$ then

$$\lim_{n \to \infty} \frac{c'_n(t_n)}{c_n(t_n)} = \frac{1}{2}.$$
We define $T(\hat{\gamma})$ to be a fundamental domain for $\hat{\gamma}/\langle \hat{\gamma} \rangle$. Then by Gardiner’s formula in Theorem 2.1 we have
\[
\mathcal{L}'(t) = (d\mathcal{L}', V)_{Y_t} = \frac{1}{\pi} \int_{\hat{\gamma}} \frac{d\gamma^2}{z^2}
\]
where $W_t$ is the image of the Beltrami differential $V_t$ on the domain of discontinuity $\Omega_t$ of $M_t$.

As $M_t$ is quasifuchsian, $\Omega_t = \Omega^0_t \cup \Omega^1_t$ where $\Omega^0_t$ is the lift of $\gamma_t$ and $\Omega^1_t$ is the lift of $X$. Then $W_t$ is zero on $\Omega^1_t$ and we have
\[
(d\mathcal{L}', V)_{Y_t} = \frac{1}{\pi} \int_{\hat{\gamma}} W_t \frac{d\gamma^2}{z^2}.
\]

As $W_t = (f_t)_*V_t$ on $\Omega^0_t$, we can pullback by $f_t$ to get the integral supported on $\mathbb{H}^2$. Then
\[
(d\mathcal{L}', V)_{Y_t} = \frac{1}{\pi} \int_{\mathbb{H}^2} V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right).
\]

We let $A(t) = \{ z \in \mathbb{H}^2 | 1 \leq |z| < r \}$. Then the fundamental domain for the action of $\gamma$ is $A(r(t))$ where $r(t) = e^{\ell(t)}$. Therefore
\[
\frac{1}{\ell(t)} (d\mathcal{L}', V)_{Y_t} = \frac{1}{\pi \ell(t)} \int_{A(r(t))} V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right).
\]

We now consider the limit of the above along our subsequence. We define integers $k_n$ such that
\[
r(t_n)^{k_n} \leq 2 < r(t_n)^{k_n+1}.
\]
Then it follows that $\lim_{n \to \infty} r(t_n)^{k_n} = 2$ or equivalently $\lim_{n \to \infty} k_n \ell(t_n) = \log(2)$. We then have
\[
\lim_{n \to \infty} \frac{k_n}{\ell(t)} \int_{A(r(t_n))} V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right) = \lim_{n \to \infty} \frac{k_n}{\log(2)} \int_{A(r(t_n))} V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right).
\]

Integrating over $k_n$ fundamental domains for $\gamma_n$ we have
\[
\lim_{n \to \infty} \frac{k_n}{\log(2)} \int_{A(r(t_n))} V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right) = \lim_{n \to \infty} \frac{1}{\log(2)} \int_{A(r(t_n))} V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right)
\]

We define $g_n$ by
\[
g_n(z)|dz|^2 = V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right) \chi_{A(r(t_n))}(z)
\]
where $\chi_A$ denotes the characteristic function of a set $A$. By the Nehari bound $|g_n(z)| \leq 3/2$ on $A(2)$ and therefore $g_n \in L^1(A(2), |dz|^2)$. Further we have on $A(2)$
\[
g(z)|dz|^2 = \lim_{n \to \infty} g_n(z)|dz|^2 = \lim_{n \to \infty} V_t f_t^* \left( \frac{d\gamma^2}{z^2} \right) \chi_{A(r(t_n))}(z) = -\overline{\phi}(\overline{\phi}(z))^2 f^* \left( \frac{d\gamma^2}{z^2} \right).
\]
As $f(z) = z^c$ then

$$\phi = S(f) = \frac{(1 - c^2)dz^2}{2z^2}$$

and

$$f^*\left(\frac{dz^2}{z^2}\right) = \frac{f'(z)^2}{f(z)^2} dz^2 = \frac{(cz^{-1})^2}{(z^c)^2} dz^2 = c^2 \frac{dz^2}{z^c}.$$ 

Therefore

$$g(z) = -\frac{1 - c^2}{2c^2} (\text{Im} z)^2 \frac{z^2}{2} = \frac{(z^2 - 1)c^2 (\text{Im} z)^2}{2|z|^4} = \frac{|c|^4 - c^2 (\text{Im} z)^2}{2|z|^4}.$$ 

By Lebesgue dominated convergence we have

$$\lim_{n \to \infty} \int g_n(z) |dz|^2 = \int g(z) |dz|^2 = \frac{|c|^4 - c^2 (\text{Im} z)^2}{4}.$$ 

As

$$\int_{A(2)} \frac{(\text{Im} z)^2 |dz|^2}{|z|^4} = \int_0^2 \int_0^\pi \frac{\sin^2 \theta}{r} dr d\theta = \frac{\log(2)\pi}{2}$$

Then

$$\int g(z) |dz|^2 = (|c|^4 - c^2) \frac{\log(2)\pi}{4}.$$ 

Thus

$$\lim_{n \to \infty} \frac{1}{\ell(t_n)}(dL, V)\gamma_n = \frac{2}{\pi} \frac{1}{\log(2)} \int_{A(2)} (\text{Im} z)^2 |dz|^2 = \frac{(|c|^4 - c^2)}{4}.$$ 

Similarly by the Gardiner formula for the Fuchsian case (see Theorem 2.1)

$$(d\ell, V)\gamma = \frac{2}{\pi} \text{Re} \int_{A(r(t))} V_i \frac{dz^2}{z^c}.$$ 

Then as above we obtain a function $h$ such that

$$\lim_{n \to \infty} \frac{1}{\ell(t_n)}(d\ell, V)\gamma_n = \frac{2}{\pi} \frac{1}{\log(2)} \text{Re} \int h(z) |dz|^2$$

where

$$h(z) |dz|^2 = \lim_{n \to \infty} V_n \left(\frac{dz^2}{z^2}\right) \chi_{A(r(t_n))\gamma_n}(z) = -\bar{\phi}(\text{Im}(z))^2 \left(\frac{dz^2}{z^2}\right) \chi_{A(2)}(z).$$

Therefore on $A(2)$

$$h(z) = \left(\frac{z^2 - 1}{2}\right) \frac{(\text{Im} z)^2}{|z|^4}$$

giving

$$\lim_{n \to \infty} \frac{1}{\ell(t_n)}(d\ell, V)\gamma_n = \frac{2}{\pi} \frac{1}{\log(2)} \text{Re} \left(\frac{z^2 - 1}{2}\right) \int_{A(2)} \frac{(\text{Im} z)^2 |dz|^2}{|z|^4} = \frac{1}{2} \left(\text{Re}(c^2) - 1\right).$$

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As $c(t) = \mathcal{L}(t)/\ell(t)$ then the quotient rule gives
\[
c'(t) = \frac{\mathcal{L}'(t)}{\ell(t)} - \frac{\mathcal{L}(t)}{\ell(t)} \cdot \frac{\ell'(t)}{\ell(t)} = \frac{\mathcal{L}'(t)}{\ell(t)} - c(t) \cdot \frac{\ell'(t)}{\ell(t)}.
\]
Applying the formulas above gives
\[
\lim_{n \to \infty} c'(t_n) = \lim_{n \to \infty} \left( \frac{\mathcal{L}'(t_n)}{\ell(t_n)} - c(t_n) \cdot \frac{\ell'(t_n)}{\ell(t_n)} \right)
\]
Therefore
\[
\lim_{n \to \infty} c'(t_n) = \left( \frac{|c|^4 - c^2}{4} - c \left( \frac{1}{2} \text{Re}(c^2) - 1 \right) \right) = \frac{1}{4} \left( |c|^4 - 2c \text{Re}(c^2) - c^2 + 2c \right).
\]

**Case** $c = 0$: We simplify our notation and let $f_n = f_{t_n}, V_n = V_{t_n}, W_n = W_{t_n}, Y_n = Y_{t_n}, c_n = c_{t_n}, \mathcal{L}_n = \mathcal{L}(t_n), \ell_n = \ell(t_n), r_n = r(t_n)$.

For this case, by postcomposition by Möbius transformations we can assume that $f_n(z) \to f(z) = \log(z)$ locally uniformly and that $f_n$ fixes $\infty$. Thus $f_n$ conjugates $y_n(z) = e^{e_n}z$ to a Möbius transformation $M_n$ with fixed points $p_n, \infty \in \hat{\mathbb{C}}$ where $p_n \to \infty$.

We have $f_n$ satisfies
\[
f_n(e^{e_n}z) = M_n(f_n(z)).
\]
We have $f(e^z) = M \cdot f(z)$ where
\[
M(z) = \lim_{n \to \infty} M_n^{1/e_n}(z).
\]
As $f(z) = \log(z)$ then $M(z) = z + 1$.

As $M_n$ has fixed points $p_n, \infty$ and translation length $\mathcal{L}_n$ then
\[
M_n(z) = e^{e_n/2}(z - p_n) + p_n.
\]
Also as $\lim_{n \to \infty} c_n = 0$ then
\[
\lim_{n \to \infty} M_n^{1/e_n}(z) = \lim_{n \to \infty} e^{e_n/2}(z - p_n) + p_n = \lim_{n \to \infty} (e^{e_n}(z - p_n) + p_n) = z + \lim_{n \to \infty} p_n(1 - e^{e_n}).
\]
It follows that
\[
1 = \lim_{n \to \infty} p_n(1 - e^{e_n}) = -\lim_{n \to \infty} p_n c_n.
\]

We modify our analysis of the Gardiner formula by dividing by $c_n^2$ and consider
\[
\lim_{n \to \infty} \frac{1}{c_n^2 \ell_n} (d\mathcal{L}, V)_{y_n}.
\]
We modify the Gardiner formula to have fixed points at $p_n, \infty$. Then we have
\[
(d\mathcal{L}, V)_{y_n} = \frac{1}{\pi} \int_{T(M_n) \cap Q_0} W_n \frac{dz^2}{(z - p_n)^2}.
\]
where $T(M_n)$ is a fundamental domain for the action of $M_n$. Therefore pulling back to $\mathbb{H}^2$ we get

$$(d\mathcal{L}, V)_n = \frac{1}{\ell_n} \int_{\mathbb{H}^2} V_n \cdot f_n^* \left( \frac{dz^2}{(z-p_n)^2} \right).$$

Then we have

$$\lim_{n \to \infty} \frac{1}{c_n \ell_n} (d\mathcal{L}, V)_n = \lim_{n \to \infty} \frac{1}{\log(2)} \int g_n(z) |dz|^2$$

where

$$g_n(z) |dz|^2 = \frac{1}{c_n} V_n f_n^* \left( \frac{dz^2}{(z-p_n)^2} \right) \chi_{A(\ell_n)}(z).$$

We have on $A(2)$

$$\lim_{n \to \infty} g_n(z) |dz|^2 = \lim_{n \to \infty} \frac{1}{c_n} V_n f_n^* \left( \frac{dz^2}{(z-p_n)^2} \right).$$

As $f_n(z) \to f(z) = \log(z)$, then $f_n'(z) \to 1/z$. Further $V_n \to -\Phi(\text{Im}(z))^2$ where $\Phi = S(f) = dz^2/2z^2$. Also as $\lim_{n \to \infty} p_n c_n = -1$ we have

$$\lim_{n \to \infty} g_n(z) |dz|^2 = -\frac{1}{2} \left( \lim_{n \to \infty} c_n^2 \right) \frac{(\text{Im}(z))^2 |dz|^2}{|z|^4}. $$

Therefore as before, integrating we get

$$\lim_{n \to \infty} \frac{1}{c_n \ell_n} (d\mathcal{L}, V)_n = -\frac{1}{4}. $$

Similarly we have

$$\lim_{n \to \infty} \frac{1}{\ell_n} (d\mathcal{L}, V)_n = -\frac{1}{2}. $$

It follows that

$$\lim_{n \to \infty} c_n' = 0. $$

Furthermore we have that

$$\lim_{n \to \infty} \frac{c_n'}{c_n} = \frac{1}{2}. $$

\[ \square \]

5 Vector Field at infinity

We define the vector field $\nu$ on $\mathbb{C}$ given by

$$\nu(z) = \frac{1}{4} \left( |z|^4 - 2z \Re(z^2) - z^2 + 2z \right).$$
By Proposition 4.1 and Lemma 4.2, for any sequence \( f_\alpha(t_n) \) converging locally uniformly, the pair \( (c_\alpha(t_n), c'_\alpha(t_n)) \) limits to a vector in the vector field \( v \). Specifically, \( c_\alpha(t_n) \to c \) and \( c'_\alpha(t_n) \to v(c) \). On the open unit disk \( |z - 1| < 1 \) this is the toy model discussed earlier. See Figure 2 for a plot of the vector field \( v \).

We now describe the dynamics of the flow on \( \mathbb{C} \) defined by \( v \).

**Proposition 5.1** The critical points of the vector field \( v(z) \) are \( z = 0 \) (unstable), \( z = 1 \) (stable), \( z = 2 \) (saddle), \( z = -1 \) (saddle). The basin of attraction of \( z = 1 \) is the disk \( |z - 1| < 1 \). Furthermore the boundary of the circle \( |z - 1| = 1 \) consists of two semi-circular trajectories from 0 to 2.

**Proof:** We first show that \( |z - 1| < 1 \) is the basin of attraction for \( z = 1 \). We let \( r = |z - 1| \) and consider \( r' \) along the flow. Differentiating \( r^2 = (z - 1)(\bar{z} - 1) \) gives

\[
2rr' = 2\text{Re}(z'(\bar{z} - 1)) = 2\text{Re}(v(z)((\bar{z} - 1)).
\]

We have \( z = 1 + re^{i\theta} \). Then

\[
|z|^2 = z\bar{z} = (1 + re^{i\theta})(1 + re^{-i\theta}) = (1 + r^2 + 2r\cos(\theta))
\]

and

\[
\text{Re}(z^2) = \text{Re}((1 + re^{i\theta})^2) = \text{Re}(1 + 2re^{i\theta} + r^2e^{2i\theta}) = (1 - r^2 + 2r\cos(\theta) + 2r^2\cos^2(\theta)).
\]

Therefore

---

**Figure 2:** Vector field \( v(z) = \frac{1}{4}(|z|^4 - 2z\text{Re}(z^2) - z^2 + 2z) \)
Then as \( rr' = \text{Re}(v(z)(z-1)) \) we have

\[
4rr' = \text{Re} \left( \left( (1 + r^2 + 2r \cos(\theta))^2 - 2(1 + r e^{i\theta})(1 - r^2 + 2r \cos(\theta) + 2r^2 \cos^2(\theta)) - r^2 e^{2i\theta} + 1 \right) (r e^{-i\theta}) \right).
\]

Therefore

\[4r' = (1 + r^2 + 2r \cos(\theta))^2 \cos(\theta) - 2(r + \cos(\theta))(1 - r^2 + 2r \cos(\theta) + 2r^2 \cos^2(\theta)) - r^2 \cos(\theta) + \cos(\theta).\]

Expanding and simplifying we get

\[r' = \frac{1}{4} (r^3 - r)(2 + r \cos(\theta)). \quad (5.1)\]

It follows that if \( r = 0, 1 \) then \( r' = 0 \) and therefore the only critical points in \(|z-1| \leq 1\) are either \( z = 1 \) or on the circle \(|z-1| = 1\). Further for \( 0 < r < 1 \) then \( r' < 0 \) giving \( z = 1 \) is a stable critical point with basin of attraction the open disk \(|z-1| = 1\).

Solving for critical points we solve \( v(z) = 0 \)

\[
0 = 4(1 + \cos(\theta))^2(1 - \cos(\theta)) + 1 - \cos(2\theta)
0 = -4\sin(\theta)\cos(\theta)(1 + \cos(\theta)) - \sin(2\theta) = -\sin(2\theta)(3 + 2\cos(\theta))
\]

Solving the second equation gives \( \sin(2\theta) = 0 \), or \( \theta = 0, \pi/2, \pi, 3\pi/2 \). As only \( 0, \pi \) satisfy the first equation, the only critical points on \(|z-1| = 1\) are \( z = 0, 2 \). Furthermore \( z = 0 \) is unstable and \( z = 2 \) is a saddle. Therefore the circle \(|z-1| = 1\) is composed of two semicircular trajectories joining 0 to 2 (see figure 2).

Outside the disk \(|z-1| \leq 1\) we have \( r' > 0 \) for \( 2 + r \cos(\theta) > 0 \) and \( r' < 0 \) for \( 2 + r \cos(\theta) < 0 \). Equivalently at a point \((x,y)\) outside the disk \(|z-1| \leq 1\) we have \( r' > 0 \) for \( x > -1 \) and \( r' < 0 \) for \( x < -1 \).

It follows that any critical points outside the disk \(|z-1| = 1\) must have \( x = -1 \). We let \( z = -1 + it \). Then

\[
4v(z) = (-1 + it)^4 - 2(-1 + it) \text{Re}((-1 + it)^2) - (-1 + it)^2 + 2(-1 + it)
= (1 + t^2)^2 - 2(-1 + it)(1 - t^2) - (1 - 2it - t^2) + 2(-1 + it)
= t^4 + 2it^3 + 2it + t^2 = t(r^2 + 1)(t + 2i)
\]

Thus the only critical point is for \( t = 0 \) giving \( z = -1 \) is a critical point. By calculation it is a saddle point. □
6 Analysis of the accumulation set

We let $M_t = (X_t, Y_t)$ be a flowline of $V$ with $Y_t \to \hat{Y} \in \partial \text{Teich}(S)$. We choose a curve $\alpha$ a node of $\hat{Y}$ and define $\mathcal{C}$ to be the set of accumulation points of $(c_{\alpha}(t), c'_{\alpha}(t))$ as $t$ goes to infinity. Then $\mathcal{C}$ is a subset of the vector field $v$. We note that $|c_{\alpha}(t) - 1| < 1$ and therefore for $(c, c') \in \mathcal{C}$ then $|c - 1| \leq 1$. Therefore as $(c, c')$ is in the vector field $v$ then $c'$ is also bounded. Therefore $\mathcal{C}$ is a bounded set. It follows that $c'$ is bounded also. Therefore $c'(t)$ must be bounded as otherwise we could choose an unbounded sequence which would give a contradiction.

We first show that there is a point in $\mathcal{C}$ contained in a certain subregion.

Lemma 6.1 There is a $c \in \mathcal{C}$ such that $|c^2 - 1/2| \leq 1/2$.

Proof: Under the flow we have both $\mathcal{L}(t), \ell(t)$ tending to zero. Therefore we can choose a subsequence with $\text{Re} \mathcal{L}'(t_n) < 0$. Thus reducing to a subsequence we have

$$\frac{1}{4} \text{Re}(|c|^4 - c^2) = \lim_{n \to \infty} \text{Re} \frac{\mathcal{L}'(t_n)}{\ell(t_n)} \leq 0.$$

Thus $|c|^4 - \text{Re}(c^2) \leq 0$.

Therefore $|c^2 - 1/2|^2 - 1/4 \leq 0$.

We note that the region $|c^2 - 1/2| \leq 1/2$ intersects the boundary of the disk $|c - 1| \leq 1$ precisely at the point $c = 0$ (see figure 3). We will show that this implies that we have a limit with $c = 1$. This will contradict that fact that $\alpha$ was a node.

We now show that $\mathcal{C}$ is invariant under the flow $v$ and therefore is a union of trajectories.

Lemma 6.2 The set $\mathcal{C}$ is invariant by the forward flow $\Phi_s$ (i.e. $\Phi_s(\mathcal{C}) \subseteq \mathcal{C}$ for $s \geq 0$) induced by the vector field $v(z)$.

Proof:

Given $\tau \in \mathbb{R}$ we define $c_\tau(t) := c(t + \tau)$. Since both $c(t), c'(t)$ are bounded, by Arzela-Ascoli for any sequence $\tau_i \to \infty$ (after possibly taking a subsequence), we have a uniform limit

$$\lim_{i \to \infty} c_{\tau_i}(t) = z(t) = x(t) + iy(t).$$

By definition $z(t) \in \mathcal{C}$ and $z$ is continuous. We will see that $z(t)$ is a flow line for the vector field $v$.  

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We know by Lemma 4.2 that
\[ \lim_{i \to \infty} c'_\tau(t) = v(z(t)). \]

What remains to show is that \( z(t) \) is differentiable and \( z'(t) = \lim_{\tau \to \infty} c'_\tau(t) \).

We have for \( \tau_i \) large \( |c_\tau(t) - z(t)| < \varepsilon \) in a compact neighbourhood about \( t \). Then we have the following bound

\[
\frac{x(t+h) - x(t)}{h} \leq \frac{\text{Re}(c_\tau(t+h)) - \text{Re}(c_\tau(t))}{h} + \frac{2\varepsilon}{h}
\]

which by the intermediate value theorem we can replace by

\[
\frac{x(t+h) - x(t)}{h} \leq \text{Re}(c'_\tau(t^*_i)) + \frac{2\varepsilon}{h}
\]

for some \( t \leq t^*_i \leq t + h \). Sending \( \tau_i \to \infty \), after reducing to a subsequence, we have that \( c'_\tau(t^*_i) \to v(z(t_h)) \) for some \( t \leq t_h \leq t + h \) and subsequently

\[
\frac{x(t+h) - x(t)}{h} \leq \text{Re}(v(z(t_h))).
\]

Taking \( \limsup \) as \( h \to 0 \) this becomes

\[
\limsup_{h \to 0} \frac{x(t+h) - x(t)}{h} \leq \text{Re}(v(z(t))).
\]
where we have used the continuity of $z(t)$ and of the vector field $v(z)$. Similarly we can arrive to the inequality
\[
\liminf_{h \to 0} \frac{x(t+h) - x(t)}{h} \geq \Re(v(z(t))).
\]
which concludes $x'(t) = \Re(v(z(t)))$. Analogously we conclude $y'(t) = \Im(v(z(t)))$, which as observed at the beginning of the proof is the last step to show that $z(t)$ is a flow line for the vector field $v(z)$. □

Lemma 6.3 $0 \notin \mathcal{C}$.

**Proof:** We first show that $\mathcal{C} \neq \{0\}$. Assume $\mathcal{C} = \{0\}$, then for $f(t) = \Re(\log(c(t)))$ we have $\lim_{t \to \infty} f(t) = -\infty$. But by Lemma 4.2
\[
\lim_{t \to \infty} \frac{c'(t)}{c(t)} = \lim_{t \to \infty} \log'(c(t)) = \frac{1}{2}.
\]
Therefore as $f'(t) = \Re(c'(t)/c(t))$ it follows that $f(t)$ is increasing for $t$ large which contradicts $\lim_{t \to \infty} f(t) = -\infty$. Thus $\mathcal{C} \neq \{0\}$.

Assume now that $\{z_0,0\} \subset \mathcal{C}$ with $z_0 \neq 0$. We let $r = |z|$, radial coordinates about $z$. As 0 is repelling, we can choose an $r_0 < |z_0|/2$ such that on the boundary of the ball $B(0, r_0) = \{z \mid |z| \leq r_0\}$ the vector field $v$ is pointing out, i.e. there is an $\varepsilon > 0$ such that $dr(v(z)) > \varepsilon$ for $z \in \partial B(0, r_0) = C(0, r_0)$.

As $c(t)$ accumulates at 0 and $z_0$ and $C(0, r_0)$ separates 0 and $z_0$, there is a sequence of disjoint intervals $[a_n, b_n]$ with $a_n < b_n$ such that $c([a_n, b_n]) \subset B(0, r_0)$ with $c(a_n), c(b_n) \in C(0, r_0)$. Therefore we have $dr(c'(a_n)) \leq 0$. Reducing to a convergent subsequence $a_{n_m}$ we can assume $c(a_{n_m}) \to z_1$ with $z_1 \in C(0, r_0)$. Therefore by Lemma 5.1 $c'(a_{n_m}) \to v(z_1)$. Thus as $dr(c'(a_n)) \leq 0$ we have
\[
\lim_{m \to \infty} dr(c'(a_{n_m})) \leq 0.
\]
This contradicts $dr(v(z)) \geq \varepsilon$ on $C(0, r_0)$. Thus $0 \notin \mathcal{C}$. □

By a similar analysis we have,

Lemma 6.4 $\mathcal{C} = \{1\}$

**Proof:** By Corollary 6.1 we can choose $z_0 \in \mathcal{C}$ such that $z_0 \in R = \{z \mid |z^2 - 1/2| \leq 1/2\}$. By Lemma 6.3 we have $0 \notin \mathcal{C}$ and therefore $|z_0 - 1| < 1$ (see figure 3). Thus $z_0$ is in the basin of attraction for the attractor $z = 1$. Therefore as $\mathcal{C}$ is invariant under forward flow (see lemma 6.2) then $1 \in \mathcal{C}$.

Assume that $\mathcal{C} \neq \{1\}$. Then there is a point $z_0 \in \mathcal{C}$ with $z_0 \neq 1$. In particular $|z_0 - 1| = r_0 \neq 1$. We choose the annulus $A = \{z \mid r_0/3 \leq |z - 1| \leq 2r_0/3\}$. As the curve $c(t)$ accumulates on both $1, z_0$, there must be disjoint intervals $[a_n, b_n]$ with $a_n < b_n$ where $c([a_n, b_n]) \subset A$ with $|c(a_n) - 1| = r_0/3$ and $|c(b_n) - 1| \leq 2r_0/3$. We let $r = |z - 1|$. It follows that for the sequence $c(a_n)$ that
\[
dr(c'(a_n)) \geq 0.
\]
By restricting to a convergent subsequence $f_{a_n}$ of $f$, we have $c(a_{nm}) \to z_1$ and $c'(a_{nm}) \to v(z_1)$ with $|z_1 - 1| = r_0/3$. Therefore as $dr(c'(a_n)) > 0$

$$dr(v(z_1)) = \lim_{m \to \infty} dr(c'(a_{nm})) \geq 0.$$  

But by Equation 5.1, $dr(v(z)) < 0$ in the annulus $0 < |z - 1| < 1$. Therefore by compactness there is a $\delta > 0$ such that $dr(v(z)) < -\delta$ on $|z - 1| = r_0/3$. Thus we obtain our contradiction that $dr(v(z_1)) < -\delta$. □

7 Main Theorem

Before we finish the proof of the main theorem, we will need some analysis of quadratic differentials. Let $X \in \text{Teich}(S)$ and $\phi \in Q(X)$ a quadratic differential. We let $r : X \to \mathbb{R}_+$ be the injectivity function. We define the $\epsilon$-thin part of $X$ by

$$X^{<\epsilon} = \{ z \in X \mid r(z) < \epsilon \}.$$  

Similarly we define the thick part of $X^{>\epsilon}$. In [Teo], Teo bounded the norm $\|\phi(z)\|$ in terms of $\|\phi\|_2$ and the injectivity radius $r(z)$. In order to describe the relation, Teo defined the function

$$C(t) = \left( \frac{4\pi}{3} \left(1 - \text{sech}(t/2)\right) \right)^{-\frac{1}{2}}.$$  

We note that $C$ is monotonically decreasing with

$$C(t) = \frac{\pi^{-\frac{1}{2}}}{t} + O(1)$$  

for $t$ tending to 0. Furthermore $\lim_{t \to 0} C(t) = \sqrt{3/4\pi}$.

**Theorem 7.1** (Teo, [Teo]) Let $\phi \in Q(X)$ and $z \in X$. Then

$$\|\phi(z)\| \leq C(r(z))\|\phi\|_2.$$  

By Theorem 3.2, $\|\phi\|_2$ tends to zero along a flowline $Y$. Therefore it follows from the above theorem that $\|\phi\|$ tends to zero on $Y^{>\epsilon}$ for any fixed $\epsilon$. In order to complete our proof, we need to show that $\|\phi\|$ tends to zero on the thin part $Y^{<\epsilon}$ for some fixed $\epsilon$. We now consider the thin part.

Let $\gamma$ be a simple closed geodesic on $X$ of length $\ell = \ell_Y(X)$. We take the lift of $X$ given by $\gamma$ and obtain the annulus $A = \{ z \in \mathbb{C} \mid e^{-\pi^2/\ell} < |z| < e^{\pi^2/\ell} \}$ with $\gamma$ lifting to the circle $|z| = 1$. Then on $A$ we have a holomorphic function $f$ such that the lift of $\phi$ is given by

$$\tilde{\phi}(z) = f(z) \frac{dz^2}{z^{2\pi}}.$$  

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Taking the Laurent series of $f$ we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$  

We then define

$$f_-(z) = \sum_{n<0} a_n z^n \quad f_0(z) = a_0 \quad f_+(z) = \sum_{n>0} a_n z^n.$$  

Similarly we define the quadratic differentials on $A$

$$\tilde{\phi}_\pm(z) = f_\pm(z) \frac{dz^2}{z^2} \quad \tilde{\phi}_0(z) = f_0(z) \frac{dz^2}{z^2} = a_0 \frac{dz^2}{z^2}.$$  

By the collar lemma (see [Bus]), for $\varepsilon < \sinh^{-1}(1)$, the thin part $X^{<\varepsilon}$ is composed of embedded disjoint collars $A_\gamma(\varepsilon)$ about the simple closed curves $\gamma$ of length less than $2\varepsilon$. We will need the following lemmas.

**Lemma 7.2** (Bridgeman-Wu, [BW, Proposition 3.3]) Let $\phi \in Q(X)$ and $z \in X^{<\mathcal{T}_2}$ where $\mathcal{T}_2 = \frac{1}{4} \log(3)$. Then

$$\|\phi(z)\| \leq C(\mathcal{T}_2)\|\phi\|_2.$$  

Using this we prove the following.

**Lemma 7.3** Let $\phi \in Q(X)$ and $z \in X^{<\varepsilon}$ for $\varepsilon \leq \frac{1}{4} \log(3)$. If $z \in A_\gamma(\varepsilon)$ and $z_0 \in \gamma$ then

$$\|\phi(z)\| \leq \|\phi(z_0)\| + 4C(\mathcal{T}_2)\|\phi\|_2.$$  

**Proof:** Let $\tilde{z}$ be a lift of $z$. As $\tilde{\phi} = \tilde{\phi}_- + \tilde{\phi}_0 + \tilde{\phi}_+$ then

$$\|\tilde{\phi}_0(\tilde{z})\| - \|\tilde{\phi}_-(\tilde{z})\| - \|\tilde{\phi}_+(\tilde{z})\| \leq \|\tilde{\phi}(\tilde{z})\| \leq \|\tilde{\phi}_0(\tilde{z})\| + \|\tilde{\phi}_-(\tilde{z})\| + \|\tilde{\phi}_+(\tilde{z})\|.$$  

Also trivially $\|\tilde{\phi}_0(\tilde{z})\|$ is maximized on the axis of $\gamma$. Thus for $\tilde{z}_0$ a lift of point $z_0$ on the curve $\gamma$ we have

$$\|\tilde{\phi}_0(\tilde{z})\| \leq \|\tilde{\phi}_0(\tilde{z}_0)\| \leq \|\tilde{\phi}(\tilde{z}_0)\| + \|\tilde{\phi}_-(\tilde{z}_0)\| + \|\tilde{\phi}_+(\tilde{z}_0)\|.$$  

Combining we have

$$\|\tilde{\phi}(\tilde{z})\| \leq \|\tilde{\phi}(\tilde{z}_0)\| + \|\tilde{\phi}_-(\tilde{z}_0)\| + \|\tilde{\phi}_+(\tilde{z}_0)\| + \|\tilde{\phi}_-(\tilde{z})\| + \|\tilde{\phi}_+(\tilde{z})\|.$$  

Applying the prior lemma we have

$$\|\tilde{\phi}_\pm(\tilde{z})\| \leq C(\mathcal{T}_2)\|\phi\|_2.$$  

Therefore as $\|\phi(z)\| = \|\tilde{\phi}(\tilde{z})\|$ we get

$$\|\phi(z)\| \leq \|\phi(z_0)\| + 4C(\varepsilon)\|\phi\|_2.$$  

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Thus we have that \( \|\phi(z)\| \) tends to zero along the curves corresponding to the limiting nodes. This follows from the analysis of the prior section. We now prove our main theorem, which we first restate.

**Theorem 1.1** Let \( X \in \text{Teich}(S) \) and \( M_t \in \mathcal{B}_X \) be a flowline for \( V \), then \( M_t \) converges to \( X \).

**Proof:** We have that \( M = (X, Y_t) \). We assume that \( Y_t \) does not converge to \( X \) and obtain a contradiction. Then by Theorem 3.2 we have that \( Y_t \to Y_{\sigma} \) a noded surface with nodes \( \sigma \).

We let \( \phi_t \in \mathcal{O}(Y_t) \) be the projective structure on \( Y_t \). As \( Y_t \to Y_{\sigma} \), there is an \( T > 0, \varepsilon > 0 \) such that for \( t > T \) then \( l_\gamma(Y_t) > \varepsilon \) for \( \gamma \notin \sigma \). We further assume \( \varepsilon < \log(3)/2 \).

By Teo we have \( \|\phi_t(z)\| \leq C(\varepsilon)\|\phi_t\|_2 \) for \( z \in Y_{t,\varepsilon} \). Therefore as \( \lim_{t \to \infty} \|\phi_t\|_2 = 0 \) we have

\[
\lim_{t \to \infty} \|\phi_t\|_{Y_{t,\varepsilon}} \| = 0.
\]

We now choose \( z_t \in Y_{t,\varepsilon} \). Then \( z_t \) is in the collar \( A_{Y_t}(\varepsilon) \) where \( \gamma_t \) is in \( \sigma_t \). Without loss of generality, we can assume that \( \gamma_t \) is the same node of \( \sigma_t \) for all \( t \). By the prior lemma,

\[
\|\phi_t(z_t)\| \leq \|\phi_t(w_t)\| + 4C(\varepsilon)\|\phi_t\|_2
\]

where \( w_t \) is any point on \( \gamma_t \). We let \( f_t : \mathbb{H}^2 \to \hat{\mathbb{C}} \) be the univalent map normalized as before so that \( \gamma_t \) lifts to the imaginary axis. Then by Lemma 6.4 we have \( f_t(z_t) \to z \) locally uniformly. Lifting \( \phi_t \) to \( \hat{\phi}_t \) it follows that \( \hat{\phi}_t = S(f_t) \to S(z) = 0 \) locally uniformly. It follows that choosing \( w_t \) so that \( \hat{w}_t = i \) we have

\[
\lim_{t \to \infty} |\phi_t(w_t)| = \lim_{t \to \infty} |\hat{\phi}_t(\hat{w}_t)| = 0.
\]

Thus for any \( z_t \in Y_{t,\varepsilon} \) then

\[
\lim_{t \to \infty} \|\phi_t(z_t)\| = 0.
\]

Thus

\[
\lim_{t \to \infty} \|\phi_t\|_\infty = 0.
\]

If \( \|\phi_t\| < 1/2 \) then by the Ahlfors-Weil quasiconformal reflection theorem (see [Le] Theorem 5.1))

\[
d_{\text{Teich}}(X, Y_t) \leq \frac{1}{2} \log \left( \frac{1 + 2\|\phi_t\|_\infty}{1 - 2\|\phi_t\|_\infty} \right)
\]

where \( d_{\text{Teich}} \) is the Teichmüller metric. Therefore as \( \lim_{t \to \infty} \|\phi_t\|_\infty = 0 \) it follows that \( \lim_{t \to \infty} Y_t = X \) in the Teichmüller metric on \( \text{Teich}(S) \). By the Cauchy-Schwarz inequality we have \( d_{WP} \leq \sqrt{\text{Area}(S)} d_{\text{Teich}} \) (see [Lin]). It follows that \( \lim_{t \to \infty} Y_t = X \) in the Weil-Petersson metric on \( \text{Teich}(S) \), giving us our contradiction. \( \Box \)
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