BLENDER-HORSESHOES
IN CENTER-UNSTABLE HÉNON-LIKE FAMILIES

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ABSTRACT. A blender-horseshoe is a locally maximal transitive hyperbolic set that appears in dimension at least three carrying a distinctive geometrical property: its local stable manifold “behaves” as a manifold of topological dimension greater than the expected one (the dimension of the stable bundle). This property persists under perturbations turning this kind of dynamics an important piece in the global description of robust non-hyperbolic systems. In this paper, we consider a parameterized family of center-unstable Hénon-like of endomorphisms in dimension three and show how blender-horseshoes naturally occur in a specific parameter range.

To Welington de Melo, in memoriam

1. INTRODUCTION

Naively, a blender is a transitive hyperbolic set that appears in dimension at least three and whose special geometrical configuration implies that the “dimension” of its stable set is larger than the “expected” one. To be a bit more precise, recall that the index of a transitive hyperbolic set Λ, denoted by ind(Λ), is the dimension of its stable bundle (by transitivity, the index is well defined). The leaves of the (local) stable sets of points in Λ have dimension ind(Λ), however the (local) stable set of the blender Λ behaves as a set of dimension ind(Λ) + 1 (or greater). In practical terms and applications, blenders are dynamical “local plugs” which in some (semi-local or global) configurations carry further important properties of the dynamics (see the next paragraph). For an informal presentation of blenders and a discussion on their role in smooth dynamical systems we refer to [5] and [10, Chapter 6.2]. Blenders were introduced in [6] as a formalisation of the constructions in [11] in the context of bifurcations via heterodimensional cycles. In [6], blenders were used to construct new classes of robustly transitive diffeomorphisms. Later, blenders were used in several dynamical contexts: Generation of robust heterodimensional cycles and homoclinic tangencies, stable ergodicity, Arnold diffusion, and construction of nonhyperbolic measures, among others. Each of these applications involves a specific type of blender such as blender-horseshoes [8], symbolic blenders [20, 2], dynamical blenders [1] and super-blenders [1].

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In the original definition in [6] the main emphasis is placed on the persistence of its geometrical configuration that was key to guarantee the robust transitivity of non-hyperbolic sets, see the discussion in [10, Chapter 6]. Although in many contexts the “original” blenders in [6] are shown to be very useful, a major con of them is that they fail to be locally maximal sets, this deficiency carries some constraints in their use and applications. This weakness was bypassed in [8] by introducing a special type of blenders, called \textit{blender-horseshoes}, which are locally maximal and also conjugate to the standard Smale horseshoe, see Definition 2.3. These two additional useful properties can be explored to get additional relevant properties: blender-horseshoes are the key local plugs to get \textit{robust heterodimensional cycles} and \textit{robust homoclinic tangencies} in the $C^1$-topology, see [7] and [8]. In some cases, one can also get some extra “fractal-like” information about these blenders, see [12] and also [19]. Considering these aspects and also the use of blenders to get robust cycles in bifurcation theory, one can think of blender-horseshoes as a version of the so-called \textit{thick horseshoes} introduced by Newhouse in the construction of robust homoclinic tangencies of surface diffeomorphisms, see [21].

In what follows, for simplicity and also considering the scope of this paper, our discussion is restricted to the three-dimensional case (adjustments to higher dimensions are straightforward). There are some settings where blender-horseshoes appear in a natural way. A first one is the bifurcation of \textit{heterodimensional cycles} (i.e., there are a pair of saddles having indices one and two whose invariant manifolds meet cyclically). In this context, the occurrence of blender-horseshoes is related to the existence of some non-normally hyperbolic dynamics that can be illustrated as follows. Think of a standard horseshoe defined on a “square” and “multiply” this dynamics by a “weak expansion” in the normal direction (to the square), see Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Non-normally hyperbolic dynamics.}
\end{figure}
In this way, one gets a hyperbolic set (of index one) contained in a non-normally hyperbolic (local) manifold. Persistence of hyperbolicity implies that this horseshoe has continuations for small perturbations of the dynamics. However, since the horseshoe is contained in a non-normally hyperbolic square, the new horseshoes are in general not contained in a local surface. It turns out that appropriate perturbations of the initial dynamics provide blender-horseshoes. For a complete discussion of this construction (and also with explicit formulae) we refer to [9] (note that in [9] the term blender is not used).

An interesting question is to provide explicit examples of maps (with an explicit analytic formula) exhibiting blender-horseshoes. This leads to the second ingredient of this paper, a family of endomorphisms so-called center-unstable Hénon-like families, see equation (1.1). We recall that in the two-dimensional case, Hénon-like maps are a fundamental ingredient in the study of homoclinic bifurcations which provide a “limit dynamics”: there exists a sequence of bifurcation parameters providing a sequence of return maps at the homoclinic tangency converging to a Hénon-like map in suitable rescaled coordinates. This construction, known as renormalisation scheme, when performed at homoclinic tangencies allows to translate (robust) properties of the Hénon-like family to the dynamics of diffeomorphisms nearby the bifurcating one, for details see [22] Chapter 3. Two remarkable examples of such portable properties are the persistence of homoclinic tangencies [22] Chapter 3 and the existence on strange attractors [17].

In view of the above discussion, it is natural to ask about renormalisation schemes and limit dynamics in heterodimensional settings. In this direction, in [13] it is considered a heterodimensional cycle (associated to a pair of saddles of indices one and two) involving a heteroclinic orbit corresponding to the tangential contact of the two-dimensional invariant manifolds of the saddles. This heteroclinic orbit is called a heterodimensional tangency, see [14]. In [13] it is provided a renormalisation scheme whose limit dynamics is a center-unstable Hénon-like family. This discussion justifies the following technical remark. On the one hand, the theory of homoclinic bifurcations and renormalisation schemes requires at least $C^2$-regularity of the diffeomorphisms. On the other hand, the construction of robustly non-hyperbolic dynamics (robust cycles and tangencies) associated to heterodimensional cycles is mostly developed in the $C^1$-case. Thus, an interesting problem is to develop these theories in higher regularity.

First, for direct approach dealing with perturbation of product dynamics (a hyperbolic part times the identity) we refer to [3]. On the other hand, bifurcations of heterodimensional tangencies seem to be an appropriate setting for obtaining robustly non-hyperbolic dynamics in high regularity, see for instance [16] where $C^2$-robust heterodimensional tangencies and $C^2$-robust heterodimensional cycles involving heterodimensional tangencies are obtained using blenders and the results of [23]. Our results are motivated by the ideas of [13], where blenders are generated at the bifurcation of heterodimensional cycles in high regularity topologies.

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1 Besides the regularity of the maps, necessary for the convergence of the renormalisation scheme, another key fractal-like ingredient is the thickness of a hyperbolic set, which has a radically different behaviour in the $C^1$ and $C^2$-topologies, see [25] and [15].

2 The starting point of this progress is due to the development of a series of typically $C^1$-tools (started with Pugh’s $C^1$ closing lemma and with Franks derivative perturbation lemma) that to the current date have no equivalents in $C^r$-topologies with $r > 1$. On the other hand, $C^1$-regularity is not sufficient to some results requiring control of the distortion.
More precisely, in [13] blender are obtained for some (open) range of parameters of the center-unstable Hénon-like family and some applications (involving a renormalisation scheme) are given for the bifurcation of heterodimensional cycles in high regularity (in the spirit of [22]). In this paper, we prove that the blenders obtained in [13] are indeed blender-horseshoes. This step will allow (in further applications) to improve versions of [13, Theorem 1.4], getting robust cycles and robust tangencies in higher regularity (in the same spirit as in [7, 8]). In a forthcoming paper (see also [24]) we will introduce a renormalisation scheme for some non-transverse heterodimensional cycles (cycles with heterodimensional tangencies) converging to the center-unstable Hénon-like family and state the persistence of cycles and tangencies (in higher regularity) after its bifurcation.

Finally, let us observe that [15] provides a quite complete numerical analysis of the center-unstable Hénon family in (1.1), showing strong numerical evidences of the occurrence of blenders in a parameter range wider than the one in [13] and illustrates the vanishing of these blenders beyond this range. We believe that the blenders detected in [15] are indeed blender-horseshoes.

It follows the main result of this paper.

**Theorem 1.** Consider the center-unstable Hénon-like family of endomorphisms

\[
G(\xi, \mu, \kappa, \eta)(x, y, z) \overset{\text{def}}{=} (y, \mu + y^2 + \kappa y z + \eta z^2, \xi z + y), \quad \xi > 1.
\]

Then there is \(\varepsilon > 0\) such that for every

\[\bar{\nu} = (\xi, \mu, \kappa, \eta) \in \mathcal{O}_\varepsilon \overset{\text{def}}{=} (1.18, 1, 19) \times (-10, -9) \times (-\varepsilon, \varepsilon)^2\]

the endomorphism \(G_{\bar{\nu}}\) has a blender-horseshoe in the cube \(\Delta \overset{\text{def}}{=} [-4, 4]^2 \times [-40, 22]\).

As a consequence, every diffeomorphism or endomorphism sufficiently \(C^1\)-close to \(G_{\bar{\nu}}\) has a blender-horseshoe in \(\Delta\).

The consequence pointed out in the theorem arises from the \(C^1\)-persistence of blenders, see Remarks 2.4 and 2.8. Let us observe that this result is a version of [13, Theorem 1.1] where blenders are replaced by blender-horseshoes in a similar range of parameters.

This paper is organised as follows. In Section 2, we introduce the definitions of blender and blender-horseshoe and state the distinctive property of a blender-horseshoe (Lemmas 2.5 and 2.6). In Section 3, we prove Theorem 1.

### 2. Blenders and Blender-horseshoes

#### 2.1. Blenders

The notion of a cu-blender (or simply blender) was introduced in [6], where were used to generate \(C^1\)-robust transitivity in the non-hyperbolic setting. The main virtue of a blender comes from its special internal geometry: a cu-blender is a transitive hyperbolic set whose (local) stable set robustly behaves as manifold of topological dimension larger than the dimension of its stable bundle. We now discuss the (axiomatic) definition of blenders in the three-dimensional case.

**Definition 2.1.** (cu-Blender, Definition 3.1 in [8]) Let \(f : M \to M\) be a three-dimensional diffeomorphism. A transitive hyperbolic compact set \(\Lambda\) of index two of \(f\) is a cu-blender if there are a \(C^1\)-neighbourhood \(U\) of \(f\) and a \(C^1\)-open set \(\mathcal{D}\) of embeddings of one-dimensional discs \(D\) into \(M\) such that for every \(g \in U\) and every disc \(D \in \mathcal{D}\) the local stable manifold \(W^s_{\text{loc}}(\Lambda_g)\) of the continuation \(\Lambda_g\) intersects \(D\). The set \(\mathcal{D}\) is called the region of superposition of the blender.
2.2. Blender-horseshoes. This kind of blenders was introduced in [8] as a mechanism for the generation of $C^1$-robust tangencies in dimension equal to or greater than three. Comparing with the standard blenders, blender-horseshoes satisfy the following additional property: they are locally maximal invariant sets conjugate to a complete shift of two symbols. These properties provide a complete description of its local stable manifold as well as a nice geometrical structure: the local stable manifold of a blender-horseshoe is the Cartesian product of a “fat Cantor set” by an “interval”, see Remark 2.2. We now give the definition of a blender-horseshoe following [8, Section 3.2], for further details we refer to that paper. As the construction is local, we assume that the ambient space is $\mathbb{R}^3$. We start with some preliminary definitions.

For $a > 0$ consider the interval $I_a \overset{\text{def}}{=} [-a, +a]$ and for $x, y, z \in \mathbb{R}^+$ the cube $\Delta_{x,y,z} \overset{\text{def}}{=} I_x \times I_y \times I_z \subset \mathbb{R}^3$. We divide the boundary $\partial \Delta$ of $\Delta$ into three parts as follows:

$$\partial^0 \Delta \overset{\text{def}}{=} \partial I_x \times I_y \times I_z, \quad \partial^u \Delta \overset{\text{def}}{=} I_x \times \partial I_y \times I_z, \quad \partial^\circ \Delta \overset{\text{def}}{=} I_x \times \partial(I_y \times I_z).$$

Note that $\partial \Delta = \partial^0 \Delta \cup \partial^u \Delta$ and $\partial^u \Delta \subset \partial^0 \Delta$.

Given $\theta > 0$ and $p \in \mathbb{R}^3$, define the s-, uu- and u-cone fields of size $\theta$ as follows:

$$C^0_\theta(p) \overset{\text{def}}{=} \{(u, v, w) \in \mathbb{R}^3 : \sqrt{v^2 + w^2} < \theta |u|\},$$

$$C^{uu}_\theta(p) \overset{\text{def}}{=} \{(u, v, w) \in \mathbb{R}^3 : \sqrt{u^2 + w^2} < \theta |v|\},$$

$$C^u_\theta(p) \overset{\text{def}}{=} \{(u, v, w) \in \mathbb{R}^3 : |u| < \theta \sqrt{v^2 + w^2}\}.$$

Note that $C^{uu}_\theta(p) \subset C^u_\theta(p)$. Related to these cone fields, we define $s_\theta$- and uu-$\theta$-discs and uu-$\theta$-strips as follows:

- Let $L$ be a regular curve. We say that $L$ is an $s_\theta$-disc if it is contained in $\Delta$, $T_p L \subset C^0_\theta(p)$ for each $p \in L$, and its end-points are contained in different connected components $\partial^u \Delta$. Similarly, we say that $L$ is a uu-$\theta$-disc if $L \subset \mathbb{R} \times I_y \times \mathbb{R}$, $T_p L \subset C^{uu}_\theta(p)$ for each $p \in L$, and its end-points are contained in different connected components of $\mathbb{R} \times \partial I_y \times \mathbb{R}$.
- A surface $S \subset \Delta$ is a uu-$\theta$-strip if $T_p S \subset C^{uu}_\theta(p)$ for every $p \in S$ and there exists a $C^1$-embedding $E : I_y \times J \to \Delta$ (where $J$ is a subinterval of $I_z$) such that $E(I_y \times J) = S$ and $L(z) \overset{\text{def}}{=} E(I_y \times \{z\})$ is a uu-$\theta$-disc for every $z \in J$. The width of $S$, denoted by $w(S)$, is the infimum of the length of the curves in $S$ which are transverse to $C^{uu}_\theta$ and join the two components of $E(I_y \times \partial J)$

Remark 2.2 (Right and left classes of uu-discs). In what follows, we fix $\theta, \vartheta > 0$. Note that every $s_\theta$-disc $W$ such that $(W \setminus \partial W)$ is contained in the interior of $\Delta$ defines two different (free) homotopy classes of uu-discs disjoint from $W$. This allows us to consider uu-discs at the left and at the right of $W$ (corresponding to the two different homotopy classes), denoted by $U^r_W$ and $U^l_W$, respectively. The right class $U^r_W$ (resp., left class $U^l_W$) is the class containing the uu-disc $\{0\} \times I_y \times \{+\}$ (resp., containing the $\{0\} \times I_y \times \{-\}$). With a slight abuse of notation, we also denote by $U^r_W$ the union of the uu-discs in $U^r_W$, $i = r, l$.

Similarly, a uu-strip $S$ through $\Delta$ is at the right (resp. at the left) of $W$ if it is foliated by uu-discs at the right (resp. at the left) of $W$.

We are now ready to recall the definition of a blender-horseshoe in [8].
Figure 2. (a) s-legs of the blender-horseshoes. (b) Projection of $F(\Delta) \cap (\mathbb{R} \times I_y \times \mathbb{R})$ in the plane $XY$.

**Definition 2.3 (Blender-Horseshoe).** The maximal invariant $\Lambda_F \defeq \cap_{i \in \mathbb{Z}} F^i(\Delta) \subset \text{int}(\Delta)$ of a (local) diffeomorphism $F : \Delta \to F(\Delta) \subset \mathbb{R}^3$ is a blender-horseshoe if conditions (BH1)-(BH6) below hold:

(BH1) s- and u-legs : There are a connected subsets $A$ and $B$ of $\Delta$, called s-legs of the blender, with $A \cap B = \emptyset$ and $(A \cup B) \cap \partial uu \Delta = \emptyset$ such that $F(\Delta) \cap (\mathbb{R} \times I_y \times \mathbb{R}) = F(A) \cup F(B) \subset (x^-, x^+) \times I_y \times \mathbb{R}$.

Note that the sets $F(A)$ and $F(B)$ are the connected components of $F(\Delta) \cap (\mathbb{R} \times I_y \times \mathbb{R})$, they are called the u-legs of the blender. See Figure 2.

(BH2) Contracting and expanding invariant cone fields. There exist $\theta, \vartheta > 0$, $\ell \in \mathbb{N}$, $c > 1$, and cone fields $C^s_\vartheta$, $C^u_\theta$, and $C^{uu}_\vartheta$ such that:

(i) **Strict invariance**: for every $p \in A \cup B$ we have that $DF^\ell_p(C^s_\vartheta(p)) \supset C^s_\vartheta(F^\ell(p))$, $DF^\ell_p(C^u_\theta(p)) \subset C^u_\theta(F^\ell(p))$, and $DF^\ell_p(C^{uu}_\vartheta(p)) \subset C^{uu}_\vartheta(F^\ell(p))$.

(ii) **Expansion/Contraction.** For every $v \in C^s_\vartheta(p)$ and every $w \in C^u_\theta(p)$ we have that $|DF^\ell_p v| \leq c^{-1} |v|$ and $|DF^\ell_p w| \geq c |w|$.

Conditions (BH1) and (BH2) imply the existence of two fixed saddles $P \in A$ and $Q \in B$, called the reference saddles of $\Lambda_F$. We define the local stable manifolds of $P$ and $Q$ by

\[(2.2) \quad W^s_{\text{loc}}(R) \defeq \text{connected component of } W^s(R) \cap \Delta \text{ containing } R,\]

where $R = P, Q$. These local stable manifolds are s-discs (in what follows we omit the dependence of $\theta$ and $\vartheta$). Thus, either $\mathcal{U}^w_{W^s_{\text{loc}}(P)} \cap \mathcal{U}^r_{W^s_{\text{loc}}(Q)} \neq \emptyset$ or $\mathcal{U}^r_{W^s_{\text{loc}}(P)} \cap \mathcal{U}^w_{W^s_{\text{loc}}(Q)} \neq \emptyset$. We assume that the first case holds and denote by $\mathcal{U}^b \defeq \mathcal{U}^r_{W^s_{\text{loc}}(P)} \cap$\[\text{...}]}
The family of discs $\mathcal{U}^b$ is called the superposition region of the blender-horseshoe. We say that a uu-disc is in between if it is contained $\mathcal{U}^b$. Similarly, a u-strip is in between if it is foliated by uu-discs in between.

(BH3) Markov partition. The connected components of $F^{-1}(\Delta) \cap \Delta$ are the sets

$$A \overset{\text{def}}{=} F^{-1}(F(A) \cap \Delta) \quad \text{and} \quad B \overset{\text{def}}{=} F^{-1}(F(B) \cap \Delta),$$

which satisfy

$$A \cup B \subset I_x \times (y^-, y^+) \times (z^-, z^+), \quad F(A) \cup F(B) \subset (x^-, x^+) \times I_y \times \mathbb{R}.$$

(BH4) uu-discs through the local stable manifolds of $P$ and $Q$: Let $L$ and $L'$ be uu-discs such that $L \cap W^s_{\text{loc}}(P) \neq \emptyset$ and $L' \cap W^s_{\text{loc}}(Q) \neq \emptyset$. Then

$$L \cap (\partial^s \Delta \setminus \partial^{uu} \Delta) = \emptyset, \quad L' \cap (\partial^s \Delta \setminus \partial^{uu} \Delta) = \emptyset.$$

(BH5) Positions of images of uu-discs: Let $L$ be a uu-disc in $\Delta$ and consider

$$L_C \overset{\text{def}}{=} L \cap C, \quad C = A, B.$$

By (BH1) and (BH2), $F(L_C)$ is a uu-discs in $I_x \times I_y \times \mathbb{R}$. The relative position of $F(L_C)$ obeys the following rules:

1. if $L \in \mathcal{U}^b_{W^s_{\text{loc}}(P)}$ then $F(L_A) \in \mathcal{U}^b_{W^s_{\text{loc}}(P)}$,
2. if $L \in \mathcal{U}^b_{W^u_{\text{loc}}(P)}$ then $F(L_A) \in \mathcal{U}^b_{W^u_{\text{loc}}(P)}$,
3. if $L \in \mathcal{U}^b_{W^s_{\text{loc}}(Q)}$ then $F(L_B) \in \mathcal{U}^b_{W^s_{\text{loc}}(Q)}$,
4. if $L \in \mathcal{U}^b_{W^u_{\text{loc}}(Q)}$ then $F(L_B) \in \mathcal{U}^b_{W^u_{\text{loc}}(Q)}$,
5. if $L \in \mathcal{U}^b_{W^{su}_{\text{loc}}(P)}$ or $L \cap W^s_{\text{loc}}(P) \neq \emptyset$ then $F(L_B) \in \mathcal{U}^b_{W^s_{\text{loc}}(P)}$, and
6. if $L \in \mathcal{U}^b_{W^{su}_{\text{loc}}(Q)}$ or $L \cap W^s_{\text{loc}}(Q) \neq \emptyset$ then $F(L_A) \in \mathcal{U}^b_{W^s_{\text{loc}}(Q)}$.

(BH6) Positions of images of uu-discs in $\mathcal{U}^b$: Let $L$ be a uu-disc in $\Delta$ such that $L \in \mathcal{U}^b$, then either $F(L_A)$ or $F(L_B)$ is contained in $\mathcal{U}^b$.

Figure 3 illustrates a prototypical blender-horseshoe.

We now pointed out some consequences of conditions (BH1)-(BH6), see [8, Section 3.2.4] for more details.

Remark 2.4.

- The existence of the invariant (contracting or expanding) cone fields in (BH2) implies the hyperbolicity (and partial hyperbolicity) of the set $\Lambda_F$: the set $\Lambda_F$ is hyperbolic and partially hyperbolic with a dominated splitting

$$T_{\Lambda_F}(\mathbb{R}^3) = E^s \oplus E^u \oplus E^{uu},$$

where $E^s$ and $E^u = E^{cu} \oplus E^{uu}$ are the stable and unstable bundles of $\Lambda_F$, respectively.

- From (BH1)-(BH2), one gets that $\{A, B\}$ is a Markov partition generating $\Lambda_F$. Therefore, the dynamics of $F$ in $\Lambda_F$ is hyperbolic and conjugate to the full shift of two symbols. In particular, the set $\Lambda_F$ contains exactly two fixed points of $F$, $P \in A$ and $Q \in B$.

- Since $\Lambda_F$ is locally maximal, we have that

$$W^s_{\text{loc}}(\Lambda_F) \overset{\text{def}}{=} \bigcap_{n \in \mathbb{N}} F^{-n}(\Delta) = \bigcup_{x \in \Lambda_F} W^s_{\text{loc}}(x) \subset W^s(\Lambda_F),$$

for $\mathcal{U}^b$. The family of discs $\mathcal{U}^b$ is called the superposition region of the blender-horseshoe. We say that a uu-disc is in between if it is contained $\mathcal{U}^b$. Similarly, a u-strip is in between if it is foliated by uu-discs in between.
where $W^u_{loc}(x)$ is the connected component of $W^u(x) \cap \Delta$ containing $x$. We can write the local stable manifold $W^u_{loc}(\Lambda_F)$ as the Cartesian product of a Cantor set, say $C$, by an interval. This Cantor set is “fat” in the following sense: the projection of $C$ in the center-unstable direction contains (open) intervals. See Figure 3-(b).

- Conditions (BH1)-(BH6) are $C^1$-open. Hence if $\Lambda_F$ is a blender-horseshoe of $F$ then the continuation $\Lambda_G$ of $\Lambda_F$ is a blender-horseshoe for every $G$ sufficiently $C^1$-close to $F$ (with the same reference cube $\Delta$).

The next lemma states the distinctive property of a blender-horseshoe.

**Lemma 2.5** (Lemma 3.13 in [4]). For every $L \in U^b$ it holds $L \cap W^u_{loc}(\Lambda_F) \neq \emptyset$.

**Proof.** Consider $L = L'_0 \in U^b$. By condition (BH6), $F(L)$ contains a disc $L'_1 \in U^b$. We let $F^{-1}(L'_1) = L_1 \subset L$. We inductively define $L_n \subset L$ and $L'_n \in U^b$ for $n > 1$ as follows. Assuming defined $L'_{n-1} \in U^b$ and $L_{n-1} \subset L_0$ with $F^{-1}(L'_{n-2})$ and $F^{-n+1}(L'_{n-1}) = L_{n-1}$, we consider $L'_n \in U^b$ contained in $F(L'_{n-1})$ and let $F^{-n}(L'_n) = L_n \subset L$. The sequence $(L_n)$ is nested and hence $\emptyset \neq \bigcap_{n} L_n \subset L$. By construction, $\bigcap_{n} L_n \subset W^u_{loc}(\Lambda_F)$. $\square$

We also have the following refinement of the above lemma.

**Lemma 2.6.** Every u-strip in between intersects transversely $W^u(P)$.

**Proof.** Note that $F^{-1}(W^u_{loc}(P)) \cap \Delta$ consists of two connected components. We denote by $W^u_0$ the connected component that does not contain $P$. Note that this set is an s-disc. Observe that there is $\alpha > 0$ such that every u-strip $S$ with $w(S) > \alpha$ intersects $W^u_0$ transversely. Conditions (BH2) and (BH6) imply that the width of a u-strip $S \subset \Delta$ in between grows exponentially after iterations by $F$ (for simplicity let us assume that $\ell$ in (BH2) is $\ell = 1$): there is $c' > 1$ (independent of the strip) such that there are two possibilities, either $F(S)$ intersects (transversely) $W^u_{loc}(P)$ or $F(S)$ contains a u-strip $S'$ in between such that $w(S') > c'w(S)$.
Take now a u-strip $S = S_0$ in between. If $S \cap W^s_{0} \neq \emptyset$ we are done. Otherwise we consider $F(S)$. If $F(S)$ intersects either $W^s_{0}$ or $W^u_{\text{loc}}(P)$ we are also done. Otherwise we get a new u-strip $S_1$ in between contained in $F(S_0)$ with $w(S_1) > c'w(S_0)$. We now argue inductively, at some step we get a first $n$ such that either $F(S_n)$ intersects $W^s_{0}$ or $W^u_{\text{loc}}(P)$ or $w(S_n) > \alpha$ and hence $S_n$ intersects $W^u_{0}$. In both cases, we are done. This proves the lemma.

2.2.1. Blender-horseshoes for endomorphisms. For endomorphisms the blender horseshoe are defined as in the case of diffeomorphisms.

**Definition 2.7** (Blender-horseshoes for endomorphisms). The maximal invariant set $\Lambda_G := \bigcap_{i \in \mathbb{Z}} G^i(\Delta) \subset \text{int}(\Delta)$ of an endomorphism $G : \Delta \to \mathbb{R}^3$ is a **blender-horseshoe** if $G$ satisfies the conditions (BH1)-(BH6).

**Remark 2.8** (Continuations of blender-horseshoes for endomorphisms). Assume that the endomorphism $G$ has a blender-horseshoe in $\Delta$. Then every diffeomorphism or endomorphism $F$ such that $F|_{\Delta}$ is sufficiently close to $G|_{\Delta}$ has a blender-horseshoe in $\Delta$.

3. Proof of Theorem 1

Theorem 1 is a consequence of following result and Remark 1.

**Theorem 3.1.** For every $(\xi, \mu) \in \mathcal{P} \overset{\text{def}}{=} (1.18, 1.19) \times (-10, -9)$, the endomorphism

$G(\xi, \mu, 0, 0)(x, y, z) = (y, \mu + y^2, \xi z + y)$

has a blender-horseshoe in $\Delta = [-4, 4]^2 \times [-40, 22]$.

The proof of this theorem involves some preliminary steps. First, for the endomorphisms $G_{\xi, \mu} \overset{\text{def}}{=} G(\xi, \mu, 0, 0)$, where $(\xi, \mu) \in \mathcal{P}$, we study their hyperbolic fixed points and their invariant manifolds. As we will see, these fixed points will be the reference saddles of the blender-horseshoe of $G_{\xi, \mu}$ in $\Delta$.

3.1. Hyperbolic fixed points of $G_{\xi, \mu}$. We calculate the hyperbolic fixed points of $G_{\xi, \mu}$ and their invariant manifolds.

**Lemma 3.2.** For every $(\xi, \mu) \in \mathcal{P}$, the endomorphism $G_{\xi, \mu}$ has two hyperbolic fixed saddles $P_{\xi, \mu} = (p_{\xi, \mu}, p_{\xi, \mu}, q_{\xi, \mu}, q_{\xi, \mu})$ and $Q_{\xi, \mu} = (q_{\xi, \mu}, q_{\xi, \mu}, p_{\xi, \mu}, p_{\xi, \mu})$ in $\Delta$, where

$$
\begin{align*}
\quad
p_{\xi, \mu} &= \mu + (p_{\xi, \mu})^2 = (1 - \xi) \tilde{p}_{\xi, \mu}, \\
p_{\xi, \mu} &= \frac{1 - (1 - 4 \mu)^{1/2}}{2},
\end{align*}
$$

$$
\begin{align*}
\quad
q_{\xi, \mu} &= \mu + (q_{\xi, \mu})^2 = (1 - \xi) \tilde{q}_{\xi, \mu}, \\
q_{\xi, \mu} &= \frac{1 + (1 - 4 \mu)^{1/2}}{2}.
\end{align*}
$$

Proof. A simple calculation shows that $P_{\xi, \mu} = (p_{\mu}, p_{\mu}, \tilde{p}_{\xi, \mu})$ and $Q_{\xi, \mu} = (q_{\mu}, q_{\mu}, \tilde{q}_{\xi, \mu})$ are the two solutions of $G_{\xi, \mu}(x, y, z) = (x, y, z)$. Using equation (3.1) and that $(\xi, \mu) \in \mathcal{P}$, we get the following estimates for the coordinates of $P_{\xi, \mu}$ and $Q_{\xi, \mu}$:

$$
\begin{align*}
-2.7 < p_{\mu} < -2.5, \quad 13 < \tilde{p}_{\xi, \mu} < 15, \\
3.5 < q_{\mu} < 3.71, \quad -20.6 < \tilde{q}_{\xi, \mu} < -18.4.
\end{align*}
$$

Thus, $P_{\xi, \mu}, Q_{\xi, \mu} \in \Delta$. We observe that the eigenvalues of $DG_{\xi, \mu}(P_{\xi, \mu}),$ and $DG_{\xi, \mu}(Q_{\xi, \mu})$ are, respectively,

$$
\begin{align*}
\lambda^s(P_{\xi, \mu}) &= 0, \quad \lambda^u(P_{\xi, \mu}) = \xi, \quad \lambda^u(P_{\xi, \mu}) = 2p_{\mu}, \\
\lambda^s(Q_{\xi, \mu}) &= 0, \quad \lambda^u(Q_{\xi, \mu}) = 2q_{\mu}.
\end{align*}
$$
Consider the intervals $I^\gamma(3.7)$ with respective eigenvectors

$$v^s(P_{\xi,\mu}) = (1, 0, 0), \ v^u(P_{\xi,\mu}) = (0, 0, 1), \ v^{uu}(P_{\xi,\mu}) = (2p_{\mu} - \xi, 2p_{\mu}(p_{\mu} - \xi), 2p_{\mu}),$$

$$v^s(Q_{\xi,\mu}) = (1, 0, 0), \ v^u(Q_{\xi,\mu}) = (0, 0, 1), \ v^{uu}(Q_{\xi,\mu}) = (2q_{\mu} - \xi, 2q_{\mu}(q_{\mu} - \xi), 2q_{\mu}).$$

As $\xi > 1$ and $|\lambda^{uu}(P_{\xi,\mu})| = 2|p_{\mu}| > 5$ and $|\lambda^{uu}(Q_{\xi,\mu})| = 2|q_{\mu}| > 7$, we have that $P_{\xi,\mu}$ and $Q_{\xi,\mu}$ are hyperbolic fixed points of $G_{\xi,\mu}$ for every $(\xi, \mu) \in P$, ending the proof of the lemma.

\[\square\]

**Remark 3.3** (Invariant directions and foliations). For $R = P, Q$ consider the eigenspaces

$$E^s(R_{\xi,\mu}) \triangleq \mathbb{R} \times \{(0, 0)\} \quad \text{and} \quad E^u(R_{\xi,\mu}) \triangleq \{(0, 0)\} \times \mathbb{R},$$

associated to the eigenvalues $\lambda^s(R_{\xi,\mu}) = 0$ and $\lambda^u(R_{\xi,\mu}) = \xi > 1$, and consider the straight lines through $R_{\xi,\mu}$:

$$\{R_{\xi,\mu} + (t, 0, 0) : t \in \mathbb{R}\} \quad \text{and} \quad \{R_{\xi,\mu} + (0, 0, t) : t \in \mathbb{R}\}.$$

These lines are, respectively, tangent to the eigenspaces $E^s(R_{\xi,\mu})$ and $E^u(R_{\xi,\mu})$ at $R_{\xi,\mu}$, and invariant by $G_{\xi,\mu}$:

$$G_{\xi,\mu}(R_{\xi,\mu} + (t, 0, 0)) = R_{\xi,\mu}, \quad G_{\xi,\mu}(R_{\xi,\mu} + (0, 0, t)) = R_{\xi,\mu} + (0, 0, \xi t),$$

for every $t \in \mathbb{R}$. Moreover,

$$W^s(R_{\xi,\mu}) = \{R_{\xi,\mu} + (t, 0, 0) : t \in \mathbb{R}\}, \quad R = P, Q.$$

We define the **center unstable manifold** of $R_{\xi,\mu}$ by

$$W^c(R_{\xi,\mu}) \triangleq \{R_{\xi,\mu} + (0, 0, t) : t \in \mathbb{R}\}, \quad R = P, Q.$$

Consider the endomorphism of $\mathbb{R}^2$ obtained by projecting $G_{\xi,\mu}$ into the $YZ$-plane,

$$g_{\xi,\mu} : \mathbb{R}^2 \to \mathbb{R}^2, \quad g_{\xi,\mu}(y, z) \triangleq (\mu + y^2, \xi z + y).$$

This endomorphism preserves the foliation $\mathcal{F} = \{(y) \times \mathbb{R} : y \in \mathbb{R}\}$. In particular, for $r = p, q$, the leaves

$$W^c_{\xi,\mu}(r_{\mu}, \tilde{r}_{\xi,\mu}) \triangleq \{(r_{\mu}, \tilde{r}_{\xi,\mu} + t) : t \in \mathbb{R}\},$$

are invariant by $g_{\xi,\mu}$.

### 3.2. The legs of the blender-horseshoe.

In this section, we will concentrate on property (BH1) of blender-horseshoes. The definitions of s- and u-legs involve some preliminary constructions that we describe below.

For $\mu \in (-10, -9)$, consider the points

$$a_{\mu} \triangleq -\sqrt{4 - \mu}, \quad b_{\mu} \triangleq -\sqrt{-4 - \mu}, \quad c_{\mu} \triangleq \sqrt{-4 - \mu}, \quad d_{\mu} \triangleq \sqrt{4 - \mu}.$$

Note that if $\mu \in (-10, -9)$ it holds

$$-\sqrt{14} < a_{\mu} = -d_{\mu} < -\sqrt{13}, \quad -\sqrt{6} < b_{\mu} = -c_{\mu} < -\sqrt{5}.$$ 

Consider the intervals $I_{\mu} \triangleq [a_{\mu}, b_{\mu}]$ and $J_{\mu} \triangleq [c_{\mu}, d_{\mu}]$. The choice of the parameter $\mu$ and the estimates in (3.7) imply that

$$I_{\mu} = [a_{\mu}, b_{\mu}] \subset (-4, 0) \quad \text{and} \quad J_{\mu} = [c_{\mu}, d_{\mu}] \subset (0, 4).$$

Consider the sub-cubes of $\Delta$ defined by

$$A_{\xi,\mu} \triangleq [-4, 4] \times I_{\mu} \times [-40, 22], \quad B_{\xi,\mu} \triangleq [-4, 4] \times J_{\mu} \times [-40, 22].$$
From (3.8) it follows

$$A_{\xi,\mu} \cap B_{\xi,\mu} = \emptyset \quad \text{and} \quad (A_{\xi,\mu} \cup B_{\xi,\mu}) \cap \partial^u \Delta = \emptyset.$$  

**Remark 3.4.** If \( \mu \in (-10, -9) \) then \( p_\mu \in (a_\mu, b_\mu) \), \( q_\mu \in (c_\mu, d_\mu) \), and thus \( P_{\xi,\mu} \in \text{interior}(A_{\xi,\mu}) \) and \( Q_{\xi,\mu} \in \text{interior}(B_{\xi,\mu}) \).

Hence the sets \( A_{\xi,\mu} \) and \( B_{\xi,\mu} \) satisfy the first part of condition (BH1). To prove that \( G_{\xi,\mu}(A_{\xi,\mu}) \) and \( G_{\xi,\mu}(B_{\xi,\mu}) \) satisfy the second part of (BH1), as in the case of the boundary of \( \Delta \), we split the boundary of \( A_{\xi,\mu} \) as follows. Let

\[
\partial^u A_{\xi,\mu} \overset{\text{def}}{=} [-4, 4] \times \partial I_\mu \times [-40, 22], \quad \partial^a A_{\xi,\mu} \overset{\text{def}}{=} [-4, 4] \times \partial (I_\mu \times [-40, 22]), \quad \partial^s A_{\xi,\mu} \overset{\text{def}}{=} \partial([-4, 4]) \times I_\mu \times [-40, 22].
\]

Note that \( \partial A_{\xi,\mu} = \partial^u A_{\xi,\mu} \cup \partial^a A_{\xi,\mu} \), \( \text{and} \ \partial^u A_{\xi,\mu} \subset \partial^a A_{\xi,\mu} \). Analogously, we split the boundary of \( B_{\xi,\mu} \).

**Remark 3.5.** We observe that for \( C = A, B \) it holds that

\[
\partial^C_{\xi,\mu} \setminus (\partial^u C_{\xi,\mu} \cup \partial^a C_{\xi,\mu}) \subset \partial^u \Delta \setminus \partial^a \Delta, \quad (\xi, \mu) \in \mathcal{P}.
\]

Roughly, these relations between the boundaries say that the “front” and “rear cover” of \( A_{\xi,\mu} \) and \( B_{\xi,\mu} \) are contained in the “front” and “rear cover” of \( \Delta \), respectively, (see Figure 4).

**Lemma 3.6.** For every \( (\xi, \mu) \in \mathcal{P} \) it holds

\[
a) \ G_{\xi,\mu}(\Delta) \cap (\mathbb{R} \times [-4, 4] \times \mathbb{R}) = G_{\xi,\mu}(A_{\xi,\mu}) \cup G_{\xi,\mu}(B_{\xi,\mu}), \\
b) \ G_{\xi,\mu}(A_{\xi,\mu}) \cup G_{\xi,\mu}(B_{\xi,\mu}) \subset (-4, 4) \times [-4, 4] \times \mathbb{R}.
\]

**Proof.** We begin showing the equality of the item a). Keeping in mind Remark 3.5 the inclusion “\( \subset \)" is obtained from the relations (see Figure 4):

\[
(3.10) \ G_{\xi,\mu}(A_{\xi,\mu}) \cap G_{\xi,\mu}(B_{\xi,\mu}) = \emptyset, \quad G_{\xi,\mu}(\Delta \setminus (A_{\xi,\mu} \cup B_{\xi,\mu})) \cap \Delta = \emptyset, \quad (\xi, \mu) \in \mathcal{P}.
\]

The reciprocal inclusion “\( \supset \)" follows from the relation:

\[
(3.11) \ G_{\xi,\mu}(A_{\xi,\mu}) \cup G_{\xi,\mu}(B_{\xi,\mu}) \subset \{ y \mid |y| = 4 \}, \quad (\xi, \mu) \in \mathcal{P}.
\]

To get the first relation in (3.10), it is sufficient to study the projections of \( G_{\xi,\mu}(A_{\xi,\mu}) \) and \( G_{\xi,\mu}(B_{\xi,\mu}) \) in the plane \( xy \). We denote such projection by \( \Pi_3 \).

**Claim 3.7.** For every \( (\xi, \mu) \in \mathcal{P} \) it holds \( \Pi_3(G_{\xi,\mu}(A_{\xi,\mu})) \cap \Pi_3(G_{\xi,\mu}(B_{\xi,\mu})) = \emptyset. \)

**Proof.** Let \( (\xi, \mu) \in \mathcal{P} \), then we have that

\[
\Pi_3(G_{\xi,\mu}(A_{\xi,\mu})) = \{ (y, \mu + y^2) : y \in I_\mu \}, \quad \Pi_3(G_{\xi,\mu}(B_{\xi,\mu})) = \{ (y, \mu + y^2) : y \in J_\mu \}.
\]

From \( I_\mu \cap J_\mu = \emptyset \) it follows that \( \Pi_3(G_{\xi,\mu}(A_{\xi,\mu})) \cap \Pi_3(G_{\xi,\mu}(B_{\xi,\mu})) = \emptyset, \) ending the proof of the claim.

**Remark 3.8.** Equation (3.8) and the proof of the claim above also imply that

\[
\Pi_3(G_{\xi,\mu}(A_{\xi,\mu}) \cup G_{\xi,\mu}(B_{\xi,\mu})) \subset (-4, 4) \times [-4, 4], \quad \text{for every} \ (\xi, \mu) \in \mathcal{P}.
\]
Figure 4. The blender-horseshoe of $G_{ξ,µ}$

We now prove (3.11) and the second part of (3.10). Since the endomorphisms $G_{ξ,µ}$ collapse the $X$-direction, it is sufficient to study the corresponding projections in the plane $YZ$. For this, consider the sets

\[ \Gamma_µ \overset{\text{def}}{=} \left(-4, a_µ\right) \cup \left(b_µ, c_µ\right) \cup \left(d_µ, 4\right) \times [-40, 22], \]
\[ C^1_µ \overset{\text{def}}{=} \{a_µ\} \times [-40, 22], \quad C^2_µ \overset{\text{def}}{=} \{b_µ\} \times [-40, 22], \]
\[ C^3_µ \overset{\text{def}}{=} \{c_µ\} \times [-40, 22], \quad C^4_µ \overset{\text{def}}{=} \{d_µ\} \times [-40, 22]. \]

Note that $Γ_µ$, $C^1_µ$, $C^2_µ$, $C^3_µ$, and $C^4_µ$ are, respectively, the projections on the plane $YZ$ of the sets

\[ \Delta \setminus (A_{ξ,µ} \cup B_{ξ,µ}), \]
\[ ∂^{mu}A_{ξ,µ} \cap \{y = a_µ\}, \quad ∂^{mu}A_{ξ,µ} \cap \{y = b_µ\}, \]
\[ ∂^{mu}B_{ξ,µ} \cap \{y = c_µ\}, \quad ∂^{mu}B_{ξ,µ} \cap \{y = d_µ\}. \]

Recall the definition of the endomorphism $g_{ξ,µ}$ in (3.5).

Claim 3.9. For every $(ξ, µ) \in P$ it holds that

- a') $g_{ξ,µ}(Γ_µ) \cap ([-4, 4] \times [-40, 22]) = \emptyset$,
- b') $g_{ξ,µ}(C^1_µ \cup C^4_µ) \subset \{y = 4\}$, and
- c') $g_{ξ,µ}(C^2_µ \cup C^3_µ) \subset \{y = -4\}$.

Proof. Consider the projection $Π_{13}(x, y, z) \overset{\text{def}}{=} y$. It is easy to check the following equalities:

\[ Π_{13}\left(g_{ξ,µ}([-4, a_µ] \times [-40, 22])\right) = (4, µ + 16), \]
\[ Π_{13}\left(g_{ξ,µ}(b_µ, c_µ] \times [-40, 22]\right) = [µ, -4), \]
\[ Π_{13}\left(g_{ξ,µ}((d_µ, 4] \times [-40, 22]\right) = (4, µ + 16). \]

Recalling that $µ \in (-10, -9)$ we get item a'). From Remark 3.3 and equation (3.6) it follows

- $g_{ξ,µ}$ preserves the foliation $F = \{\{y\} \times \mathbb{R} : y ∈ \mathbb{R}\}$, and
Therefore, note that
\[ \mu + a^2_\mu = \mu + d^2_\mu = -(\mu + b^2_\mu) = -(\mu + c^2_\mu) = 4. \]
These two facts imply items b') and c'). This ends the proof of the claim. \(\square\)

The proof of item a) of the lemma is now complete. Finally, item b) follows directly from Remark 3.8. The proof of the lemma is now complete. \(\square\)

### 3.3. Contracting/expanding invariant cone fields
In this section, we study the condition (BH2) of a blender-horseshoe involving invariance, contraction, and expansion of the cone fields in (2.1). This condition is a consequence of the following lemma.

**Lemma 3.10.** Let \( \vartheta > 0 \) and \( \theta = 1/2 \). Then, for every \((\xi, \mu) \in \mathcal{P}\) and every \( p \in \mathcal{A}_{\xi, \mu} \cup \mathcal{B}_{\xi, \mu} \) the following holds:

(i) \( C^\vartheta_\vartheta(G_{\xi, \mu}(p)) \subset D(G_{\xi, \mu})_p(C^\vartheta_\vartheta(p)) \),

(ii) \( D(G_{\xi, \mu})_p(C^\vartheta_\vartheta(p)) \subset C^\vartheta_\vartheta(G_{\xi, \mu}(p)) \),

(iii) \( D(G_{\xi, \mu})_p(C^\vartheta_\vartheta(p)) \subset C^\vartheta_\vartheta(G_{\xi, \mu}(p)) \),

(iv) \( DF|_{\vartheta^\vartheta} \) is uniformly expanding and \( DF|_{\vartheta^\vartheta} \) is uniformly contracting for every \( \vartheta \) sufficiently small.

**Proof.** Consider \( p = (x, y, z) \in \mathcal{A}_{\xi, \mu} \cup \mathcal{B}_{\xi, \mu} \) and \( v = (u, v, w) \in T_p \Delta \), write
\[
(u_1, v_1, w_1) \overset{\text{def}}{=} D(G_{\xi, \mu})_p(v) = (v, yv, v + \xi w).
\]
Recalling (3.9) and (3.7), we have that if \((x, y, z) \in \mathcal{A}_{\xi, \mu} \cup \mathcal{B}_{\xi, \mu}\) then \(y \in I_\mu \cup J_\mu\) and thus \(|y| > \sqrt{5}\), for every \(\mu \in (-10, -9)\).

The items of the lemma are proved in the following claims.

**Claim 3.11** (Item (i)). Let \( \vartheta > 0 \). For every \( v \in \partial C^\vartheta_\vartheta(p) \setminus \{0\} \) we have \( D(G_{\xi, \mu})_p(v) \in (C^\vartheta_\vartheta(G_{\xi, \mu}(p)))^\vartheta \).

**Proof.** If \( v \in \partial C^\vartheta_\vartheta(p) \setminus \{0\} \) then \( \vartheta (\sqrt{u^2 + w^2}) = |v| \). Since \(|y| > \sqrt{5}\), we get that
\[
\vartheta (\sqrt{u^2_1 + w^2_1}) \geq \vartheta |v_1| > 2|y||v| > 2 \sqrt{5}|v| = 2 \sqrt{5}|u_1| > |u_1|.
\]
Therefore \( D(G_{\xi, \mu})_p(v) \notin C^\vartheta_\vartheta(G_{\xi, \mu}(p)) \), proving the claim. \(\square\)

**Claim 3.12** (Item (ii)). For every \( v \in C^\vartheta_\vartheta(p) \) it holds \( D(G_{\xi, \mu})_p(v) \in C^\vartheta_\vartheta(G_{\xi, \mu}(p)) \).

**Proof.** Since \(|y| > \sqrt{5}\), we have that
\[
\sqrt{u^2_1 + w^2_1} \geq |v_1| = 2|y||v| > 2 \sqrt{5}|v| > 2|u_1|,
\]
proving the claim. \(\square\)

**Claim 3.13** (Item (iii)). For every \( v \in C^\vartheta_\vartheta(p) \) it holds \( D(G_{\xi, \mu})_p(v) \in C^\vartheta_\vartheta(G_{\xi, \mu}(p)) \).

**Proof.** We need to check that
\[
\sqrt{u^2 + w^2} < \frac{1}{2} |v| \quad \Rightarrow \quad \sqrt{u^2_1 + w^2_1} < \frac{1}{2} |v_1|.
\]
Note that \( \sqrt{u^2 + w^2} < \frac{1}{2} |v| \) implies that \(|w| < \frac{1}{2} |v|\), and hence
\[
u^2 + w^2 = v^2 + (v + \xi w)^2 \leq 2v^2 + 2\xi |v||w| + \xi^2 |w|^2 \leq \left(2 + \xi + \left(\frac{\xi}{2}\right)^2\right)v^2.
\]
Now \( \xi \in (1.18, 1.19) \) implies that
\[
\left( 2 + \frac{\xi}{2} + \left( \frac{\xi}{2} \right)^2 \right) < 4
\]
and hence
\[
u^2_1 + w^2_1 < 4v^2.
\]
Thus, since \( p = (x, y, z) \in A_{\xi, \mu} \cup B_{\xi, \mu} \) implies that \( |y| > \sqrt{5} \), it follows
\[
2 \sqrt{u^2_1 + w^2_1} < 4|v| < 2|y||v| = |v_1|,
\]
proving the claim. \( \square \)

**Claim 3.14 (Item (iv)).** \( DG_{\xi, \mu}|c_1^{1/2} \) is uniformly expanding and, if \( \vartheta \) is small enough, \( DG_{\xi, \mu}|c_o^* \) is uniformly contracting.

**Proof.** The uniform contraction of the cone field \( C_o^* \) for small \( \vartheta \) follows from the fact that \( D(G_{\xi, \mu})p \) is an endomorphism whose eigenspace associated the eigenvalue 0 is spanned by \( (1, 0, 0) \).

To see that \( D(G_{\xi, \mu}) \) uniformly expands the vectors in \( C_1^{1/2} \) consider the norm
\[
|(u, v, w)_*| \overset{\text{def}}{=} \max \left\{ |u|, \sqrt{v^2 + w^2} \right\}.
\]
Take \( v = (u, v, w) \in C_1^{1/2}(p) \) and write \( D(G_{\xi, \mu})p(v) = (u_1, v_1, w_1) = (v_2 y, v, \xi w) \). We claim that if \( v \in C_1^{1/2}(p) \) then \( |(D(G_{\xi, \mu})_p v)_*| > |v|_* \). By compactness, this implies that \( |(D(G_{\xi, \mu})_p v)_*| > c_0 |v|_* \), for some uniform \( c_0 > 1 \). Note that the Euclidean norm \( || \cdot || \) and \( | \cdot |_* \) are equivalent, hence there is \( \kappa > 1 \) such that \( \kappa^{-1} |\cdot| \leq |\cdot|_* \leq \kappa |\cdot| \). The number \( \ell \) in (BH2) is the first \( \ell_0 \) with \( \kappa^\ell_0 > \kappa \).

We now prove that \( |(D(G_{\xi, \mu})_p v)_*| > |v|_* \). Note that for \( v = (u, v, w) \in C_1^{1/2}(p) \) we have \( |v|_* = \sqrt{v^2 + w^2} \) and
\[
v^2_1 + w^2_1 = 4v^2 y^2 + (v + \xi w)^2 \geq 4v^2 y^2 + v^2 - 2 \xi |v| |w| + \xi^2 w^2.
\]
We divide the proof into two cases: (6.5) \( |v| \geq |w| \) and (6.5) \( |v| \leq |w| \). If (6.5) \( |v| \geq |w| \), using that \( \xi \in (1.18, 1.19) \) and \( |y| > \sqrt{5} \), we get that
\[
4v^2 y^2 - 2 \xi |v| |w| \geq (20 - 13 \xi) v^2 > 4v^2 \geq 0.
\]
Equations (3.12) and (3.13) immediately imply that
\[
v^2_1 + w^2_1 > 5v^2 + \xi^2 w^2 > v^2 + w^2.
\]
Hence, \( |(D(G_{\xi, \mu})_p v)_*| > |v|_* \), proving the first case. Similarly, if (6.5) \( |v| \leq |w| \) then
\[
v^2_1 + w^2_1 \geq 4y^2 v^2 + \xi^2 w^2 - 2 \xi |v| |w| + v^2 > 4y^2 v^2 + \xi^2 w^2 - 2 \xi (6.5)^{-1} w^2 + v^2.
\]
Condition \( \xi \in (1.18, 1.19) \) implies that
\[
\xi^2 - 2 \xi (6.5)^{-1} > 1.
\]
Thus
\[
v^2_1 + w^2_1 \geq v^2 + w^2.
\]
Hence, \( |(D(G_{\xi, \mu})_p v)_*| > |v|_* \). This ends the proof of the claim. \( \square \)

The proof of the lemma is now complete.
Remark 3.15. For each \( p = (x, y, z) \in \mathbb{R}^3 \) we identify \( T_p \mathbb{R}^3 \) with \( \mathbb{R}^3 \) and consider the canonical basis \( \{i, j, k\} \). Note that \( D(G_{\xi,\mu})_p(i) = 0 \), \( D(G_{\xi,\mu})_p(j) = i + 2yj_2 + k \), and \( D(G_{\xi,\mu})_p(k) = \xi k \). In particular, \( \langle D(G_{\xi,\mu})_p(j), j \rangle < 0 \) (resp. > 0) if \( y < 0 \) (resp. \( y > 0 \)). As a consequence, for every \( \theta > 0 \) and every \( p \in A_{\xi,\mu} \), the derivative \( D(G_{\xi,\mu})_p \) maps the semi-positive cone \( C^u_\theta(p) \cap \{ y > 0 \} \) (resp. semi-negative cone) into the semi-positive cone \( \{ y < 0 \} \). When \( p \in B_{\xi,\mu} \) the derivative \( D(G_{\xi,\mu})_p \) maps the semi-negative cone \( C^u_\theta(p) \cap \{ y > 0 \} \) (resp. semi-positive cone) into \( \{ y > 0 \} \) (resp. \( y < 0 \)).

3.4. **The Markov partition.** To define the Markov partition in Condition (BH3) we need some preliminary constructions.

For \( (\xi,\mu) \in \mathcal{P} \) consider the auxiliary straight lines \( R^{1}_{\xi,\mu}, R^{2}_{\xi,\mu} \) in the plane \( YZ \) defined by the equations and depicted in Figure 5:

\[
R^{1}_{\xi,\mu} \overset{\text{def}}{=} \{(y, z^1_\xi(y)) : z^1_\xi(y) = \xi^{-1}(22 - y), y \in \mathbb{R}\},
R^{2}_{\xi,\mu} \overset{\text{def}}{=} \{(y, z^2_\xi(y)) : z^2_\xi(y) = \xi^{-1}(-40 - y), y \in \mathbb{R}\}.
\]

Recall the definition of the intervals \( I_\mu = [a_\mu, b_\mu] \) and \( J_\mu = [c_\mu, d_\mu] \) in (3.8). Consider the auxiliary parallelogram \( A_{\xi,\mu} \) in the plane \( YZ \) whose boundary consists of the following segments (see Figure 5):

\[
\mathcal{S}^{1}_{\xi,\mu} \overset{\text{def}}{=} \{(y, z^1_\xi(y)) : y \in I_\mu\}, \quad \mathcal{S}^{2}_{\xi,\mu} \overset{\text{def}}{=} \{a_\mu\} \times [z^2_\xi(a_\mu), z^1_\xi(a_\mu)],
\]

\[
\mathcal{S}^{3}_{\xi,\mu} \overset{\text{def}}{=} \{(y, z^2_\xi(y)) : y \in I_\mu\}, \quad \mathcal{S}^{4}_{\xi,\mu} \overset{\text{def}}{=} \{b_\mu\} \times [z^2_\xi(b_\mu), z^1_\xi(b_\mu)].
\]

Analogously, consider the parallelogram \( B_{\xi,\mu} \) in the plane \( YZ \) bounded by

\[
\tilde{\mathcal{S}}^{1}_{\xi,\mu} \overset{\text{def}}{=} \{(y, z^1_\xi(y)) : y \in J_\mu\}, \quad \tilde{\mathcal{S}}^{2}_{\xi,\mu} \overset{\text{def}}{=} \{c_\mu\} \times [z^2_\xi(c_\mu), z^1_\xi(c_\mu)],
\]

\[
\tilde{\mathcal{S}}^{3}_{\xi,\mu} \overset{\text{def}}{=} \{(y, z^2_\xi(y)) : y \in J_\mu\}, \quad \tilde{\mathcal{S}}^{4}_{\xi,\mu} \overset{\text{def}}{=} \{d_\mu\} \times [z^2_\xi(d_\mu), z^1_\xi(d_\mu)].
\]

Remark 3.16. Since \((\xi,\mu) \in \mathcal{P}\), it follows that \( A_{\xi,\mu} \) and \( B_{\xi,\mu} \) are contained in \((-4, 0) \times (-40, 22)\). By the definitions of \( A_{\xi,\mu} \) and \( B_{\xi,\mu} \), it holds that

\[
g_{\xi,\mu}(\partial A_{\xi,\mu}) = g_{\xi,\mu}(\bigcup_{i=1}^{4} \mathcal{S}^{i}_{\xi,\mu}) = \partial([-4, 4] \times [-40, 22]),
\]
and thus
\[ L \] \( BH3 \) and thus
\[ \Pi \] \( G \) for every \( \xi, \mu \ ∈ P \) the following holds
a) \( A_{\xi, \mu} \cup B_{\xi, \mu} \ ⊆ [-4, 4] \times (-4, 4) \times (-40, 22) \),
b) \( G_{\xi, \mu}(A_{\xi, \mu}) \cup G_{\xi, \mu}(B_{\xi, \mu}) \ ⊆ (-4, 4) \times [-4, 4] \times \mathbb{R} \).

**Proof.** Item a) follows from Remark 3.16. For item b), note that Lemma 3.6 implies that
\[ G_{\xi, \mu}(A_{\xi, \mu}) \cup G_{\xi, \mu}(B_{\xi, \mu}) \ ⊆ (-4, 4) \times [-4, 4] \times \mathbb{R} \], completing of proof of lemma.

3.5. **u"-discs through the local stable manifolds.** We study Condition (BH4) of blender horseshoes about the relative position of the \( u" \)-discs through the local stable manifolds of \( P_{\xi, \mu} = (p_\mu, p_\mu, \tilde{p}_{\xi, \mu}) \) and \( Q_{\mu} = (q_\mu, q_\mu, \tilde{q}_{\xi, \mu}) \) with respect to the boundary of \( \Delta \). We reduce this analysis to the two dimensional case by projecting these discs on the plane \( \mathbb{R}^2 \). Consider the projection
\[ \Pi_1 : \mathbb{R}^3 \to \mathbb{R}^2, \quad \Pi_1(x, y, z) \stackrel{\text{def}}{=} (y, z). \]

Recalling the formulae for the stable manifolds \( W^s(P_{\xi, \mu}) \) and \( W^s(Q_{\xi, \mu}) \) in (3.3), we get \( \Pi_1(W^s(P_{\xi, \mu})) = (p_\mu, \tilde{p}_{\xi, \mu}) \) and \( \Pi_1(W^s(Q_{\xi, \mu})) = (q_\mu, \tilde{q}_{\xi, \mu}) \).

Consider the auxiliary straight lines in the plane \( \mathbb{R}^2 \) through \( (p_\mu, \tilde{p}_{\xi, \mu}) \) and \( (q_\mu, \tilde{q}_{\xi, \mu}) \):
\[ L_{1, \mu} \stackrel{\text{def}}{=} \{ (y, z_{1, \mu}(y)) : z_{1, \mu}(y) = \frac{1}{2}(y - p_\mu) + \tilde{p}_{\xi, \mu}, \ y \ ∈ \mathbb{R} \}, \]
\[ L_{2, \mu} \stackrel{\text{def}}{=} \{ (y, z_{2, \mu}(y)) : z_{2, \mu}(y) = \frac{1}{2}(y - q_\mu) + \tilde{q}_{\xi, \mu}, \ y \ ∈ \mathbb{R} \}. \]

Note that \( L_{1, \mu} \) and \( L_{2, \mu} \) are contained in the boundary of \( \Pi_1(C_{1/2}^u(P_{\xi, \mu})) \) and of \( \Pi_1(C_{1/2}^u(Q_{\xi, \mu})) \), respectively. These conditions are depicted in Figure 6. Thus (BH4) follows now from the next lemma.

**Lemma 3.18.** For every \( (\xi, \mu) ∈ P \) it holds that
\[ L_{1, \mu} \cap (\Pi_1(\Delta) \cap \{ z = 22 \}) = \emptyset, \ L_{2, \mu} \cap (\Pi_1(\Delta) \cap \{ z = -40 \}) = \emptyset. \]

**Proof.** To prove the lemma it is enough to check that
\[ z_{1, \mu}(4) < 22 \quad \text{and} \quad z_{2, \mu}(-4) > -40, \quad \text{for every} \ (\xi, \mu) \ ∈ P. \]

The choice of parameters \( (\xi, \mu) \) and the estimates of \( p_\mu, q_\mu, \tilde{p}_{\xi, \mu}, \tilde{q}_{\xi, \mu} \) in (3.2), lead directly to these inequalities. □
3.6. Position of images of uu-discs. We now study the relative positions of the images of uu-discs contained in \( \Delta \) in Condition (BH5). We see that this condition follows from the one-dimensional dynamics on the unstable center manifolds of the saddles \( P_{\xi,\mu} \) and \( Q_{\xi,\mu} \), recall (3.4).

3.6.1. One-dimensional associated dynamics. Recall that \( P_{\xi,\mu} = (p_\mu, p_\mu, \bar{p}_{\xi,\mu}) \) and \( Q_\mu = (q_\mu, q_\mu, \bar{q}_{\xi,\mu}) \) and that the restriction of \( G_{\xi,\mu} \) to the one-dimensional center unstable manifolds \( W^{cu}(P_{\xi,\mu}) \), \( W^{cu}(Q_{\xi,\mu}) \) in (3.4) is just and affine multiplication by \( \xi > 1 \), see Remark 3.3. Denote by \( \phi_{\xi,\mu}^r \) the restriction map \( G_{\xi,\mu}|_{W^{cu}(R_{\xi,\mu}) \cap \Delta} \), \( r = p, q \) and \( R = P, Q \), that is given by

\[
\phi_{\xi,\mu}^r : [-4, 22] \rightarrow \mathbb{R}, \quad \phi_{\xi,\mu}^r(z) \overset{\text{def}}{=} \xi z + r_\mu = \xi z + (1 - \xi) \bar{r}_{\xi,\mu}, \quad r = p, q,
\]

where we use the relation \( r_\mu = (1 - \xi) r_{\xi,\mu} \). For \( r = p, q \), consider the interval \( \Gamma_{\xi,\mu}^r \overset{\text{def}}{=} [\alpha_{\xi,\mu}^r, \beta_{\xi,\mu}^r] \), where

\[
\alpha_{\xi,\mu}^r \overset{\text{def}}{=} \xi^{-1}(-40 - (1 - \xi) \bar{r}_{\xi,\mu}), \quad \beta_{\xi,\mu}^r \overset{\text{def}}{=} \xi^{-1}(22 - (1 - \xi) \bar{r}_{\xi,\mu}).
\]

Note that \( \phi_{\xi,\mu}^r(\Gamma_{\xi,\mu}^r) = [-40, 22] \) and \( \phi_{\xi,\mu}^r(\bar{r}_{\xi,\mu}) = \bar{r}_{\xi,\mu} \in \Gamma_{\xi,\mu}^r \).

Lemma 3.19. Given a uu-disc \( L \) contained in \( \Delta \) let \( L_{C_{\xi,\mu}} \overset{\text{def}}{=} L \cap C_{\xi,\mu} \), with \( C = A, S. \) Then \( G_{\xi,\mu}(L_{C_{\xi,\mu}}) \) satisfies (BH5).

Proof. We first show item (1) of (BH5). Items (2), (3), and (4) are obtained similarly and their proofs will be omitted.

From (2.2) and (3.3), the local stable manifolds of \( P_{\xi,\mu} \) and \( Q_{\xi,\mu} \) are given by

\[
(3.14) \quad W_{\text{loc}}(P_{\xi,\mu}) = \left\{ (t + p_{\xi,\mu}, p_{\xi,\mu}, \bar{p}_{\xi,\mu}) : -4 < p_{\xi,\mu} \leq t \leq 4 - p_{\xi,\mu} \right\},
\]

\[
W_{\text{loc}}(Q_{\xi,\mu}) = \left\{ (t + q_{\xi,\mu}, q_{\xi,\mu}, \bar{q}_{\xi,\mu}) : -4 < q_{\xi,\mu} \leq t \leq 4 - q_{\xi,\mu} \right\}.
\]

Given a uu-disc \( L \subset \Delta \) consider the intersections

\[
X_{\mu}^L \overset{\text{def}}{=} L \cap (\Delta \cap \{ y = p_\mu \}) = L_{A_{\xi,\mu}} \cap (\Delta \cap \{ y = p_\mu \}) = (x_\mu, p_\mu, z_\mu),
\]

\[
\bar{X}_{\mu}^L \overset{\text{def}}{=} L \cap (\Delta \cap \{ y = \bar{q}_\mu \}) = L_{B_{\xi,\mu}} \cap (\Delta \cap \{ y = \bar{q}_\mu \}) = (\bar{x}_\mu, q_\mu, \bar{z}_\mu).
\]
Remark 3.20. Recall the definitions of the right and left classes of uu-discs \( U^R \) and \( U^L \), respectively, in Remark 2.2. Using \((3.14)\) we have the following:

- \( L \in U^L_{W^p_{loc}(P^a)} \) if \( z_\mu < \bar{\mu} \xi, \mu \) and \( L \in U^R_{W^p_{loc}(P^a)} \) if \( z_\mu > \bar{\mu} \xi, \mu \).
- \( L \in U^L_{W^r_{loc}(Q^a)} \) if \( z_\mu < \bar{\mu} \xi, \mu \) and \( L \in U^R_{W^r_{loc}(Q^a)} \) if \( z_\mu > \bar{\mu} \xi, \mu \).

To prove (1) in (BH5), take any \( L \in U^R_{W^p_{loc}(P^a)} \). We will see that \( G(\xi, \mu)(L_{A^a_{\xi, \mu}}) \in U^R_{W^p_{loc}(P^a)} \). By Remark 3.20 the point \( x^{L^*}_\mu = (x_\mu, p_\mu, z_\mu) \) satisfies \( z_\mu > \bar{\mu} \xi, \mu \). Note that

\[
G(\xi, \mu)(X^L_\mu) = (p_\mu, p_\mu, \phi^p_{\xi, \mu}(z_\mu)) = (p_\mu, p_\mu, \xi z_\mu + (1 - \xi) \bar{\mu} \xi, \mu).
\]

Since \( z_\mu > \bar{\mu} \xi, \mu \) it follows that \( \phi^p_{\xi, \mu}(z_\mu) > \bar{\mu} \xi, \mu \). Remark 3.20 now implies that \( G(\xi, \mu)(L_{A^a_{\xi, \mu}}) \in U^R_{W^p_{loc}(P^a)} \).

Since items (5) and (6) of (BH5) are analogous we just prove item (5). We just need to check that if \( L \in U^R_{W^p_{loc}(P^a)} \) or \( L \cap W^r_{loc}(P^a) \neq \emptyset \) then \( G(\xi, \mu)(L_{B^a_{\xi, \mu}}) \in U^R_{W^p_{loc}(P^a)} \).

Remark 3.21. Consider the projection \( \Pi_1(x, y, z) = (y, z) \) and note that

\[
\Pi_1(L \cap \{y \geq p_\mu\}) \subset \Gamma_{\xi, \mu} \overset{\text{def}}{=} \{(y, z) : z \geq z^*_\xi, \mu(y)\},
\]

see Figure 7. Moreover, \( \Pi_1(L_{B^a_{\xi, \mu}}) \subset \Gamma_{\xi, \mu} \cap \Pi_1(B_{\xi, \mu}) \).

Note that the worst case to prove (5) in (BH5) occurs when \( L \) is contained in the plane \( YZ \) and equal to the straight line \( L^*_{\xi, \mu} \) in the plane \( YZ \) through \( (p_\mu, \bar{\mu} \xi, \mu) \) given by

\[
L^*_{\xi, \mu} \overset{\text{def}}{=} \{(y, z^*_\xi, \mu(y)) : z^*_\xi, \mu(y) = -\frac{1}{2}(y - p_\mu) + \bar{\mu} \xi, \mu, y \in \mathbb{R}\}.
\]

Consider the segment of \( L^*_{\xi, \mu} \) given by (see Figure 7)

\[
\gamma_{\xi, \mu} \overset{\text{def}}{=} \{(y, z^*_\xi, \mu(y)) : y \in J_{\mu}\} \subset L^*_{\xi, \mu} \cap \Pi_1(B_{\xi, \mu})
\]

and the point \( \bar{z}_{\xi, \mu} \) defined by

\[
g_{\xi, \mu}(\gamma_{\xi, \mu}) \cap \{y = p_\mu\} = \{(p_\mu, \bar{z}_{\xi, \mu})\},
\]

where the endomorphism \( g_{\xi, \mu} \) obtained by projecting \( G_{\xi, \mu} \) into the plane \( YZ \) defined in (3.5). By Remark 3.20 to get \( G_{\xi, \mu}(L_{B^a_{\xi, \mu}}) \in U^R_{W^p_{loc}(P^a)} \) it is sufficient to show that \( \bar{z}_{\xi, \mu} > \bar{\mu} \xi, \mu \).

Claim 3.22. It holds \( \bar{z}_{\xi, \mu} > \bar{\mu} \xi, \mu \) for every \( (\xi, \mu) \in \mathcal{P} \).

Proof. The intersection (3.16) is defined by the conditions

\[
(p_\mu, \bar{z}_{\xi, \mu}) = (y^2 + \mu, \xi z^*_\xi, \mu(y) + y), \quad y > 0.
\]

Recalling the definition of \( z^*_\xi, \mu(y) \) in (3.15), we get

\[
\bar{z}_{\xi, \mu} = \xi \bar{z}_{\xi, \mu}(\sqrt{p_\mu - \mu}) + \sqrt{p_\mu - \mu} = \frac{\xi}{2} p_\mu + \left(1 - \frac{\xi}{2}\right)\sqrt{p_\mu - \mu} + \xi \bar{\mu} \xi, \mu.
\]

Hence

\[
\bar{z}_{\xi, \mu} - \bar{\mu} \xi, \mu = \frac{\xi}{2} p_\mu + \left(1 - \frac{\xi}{2}\right)\sqrt{p_\mu - \mu} + (\xi - 1) \bar{\mu} \xi, \mu.
\]

The estimates in (3.2) and the choice of \((\xi, \mu) \in \mathcal{P}\) imply that

\[
\frac{\xi}{2} p_\mu > -1.6065, \quad \left(1 - \frac{\xi}{2}\right)\sqrt{p_\mu - \mu} > 1.014, \quad (\xi - 1) \bar{\mu} \xi, \mu > 2.34.
\]
These inequalities imply that $\tilde{z}_{\xi,\mu} - \tilde{p}_{\xi,\mu} > 0$, proving the claim. □

The proof of the lemma is now complete. □

3.7. Position of images of uu-discs in between. Condition (BH6) is given by Lemma 3.23 below. First, recall the definition of the family of disks in between $U_b \triangleq U^\ell_{W_{loc}}(P_{\xi,\mu}) \cap U^r_{W_{loc}}(Q_{\xi,\mu})$.

**Lemma 3.23.** Consider any $L \in U^b$. Then either $G_{\mu,\xi}(L_{A_{\xi,\mu}})$ or $G_{\mu,\xi}(L_{B_{\xi,\mu}})$ contains a uu-disc in $U^b$.

**Proof.** Consider $L \in U^b$. By item (2) in (BH5), if $G_{\xi,\mu}(L_{A_{\xi,\mu}}) \in U^r_{W_{loc}}(Q_{\xi,\mu})$ then $G_{\xi,\mu}(L_{A_{\xi,\mu}}) \in U^b$ and we are done. Similarly, by item (3) in (BH5), if $G_{\xi,\mu}(L_{B_{\xi,\mu}}) \in U^r_{W_{loc}}(P_{\xi,\mu})$ then $G_{\xi,\mu}(L_{B_{\xi,\mu}}) \in U^b$ and we are done. Thus in what follows we argue by contradiction assuming that:

a) $G_{\xi,\mu}(L_{A_{\xi,\mu}}) \in U^r_{W_{loc}}(Q_{\xi,\mu})$ or intersects $W^s_{loc}(Q_{\xi,\mu})$ and

b) $G_{\xi,\mu}(L_{B_{\xi,\mu}}) \in U^r_{W_{loc}}(P_{\xi,\mu})$ or intersects $W^s_{loc}(P_{\xi,\mu})$.

To prove the lemma we need some auxiliary constructions. Consider the point $Y_{\mu}^L = (x_{\mu}, a_{\mu}, z_{\mu}) \triangleq L \cap \{ y = a_{\mu} \}$, where $a_{\mu}$ is defined in \([3.6]\). In the plane $YZ$, take the auxiliary straight line $\hat{L}_{\mu}$ through $(a_{\mu}, z_{\mu})$ given by (see Figure 8)

$$\hat{L}_{\mu} \triangleq \{ (y, z_{\mu}(y)) : z_{\mu}(y) = \frac{1}{2}(y - a_{\mu}) + z_{\mu}, \ y \in \mathbb{R} \}.$$  

Observe that $\hat{L}_{\mu} \subset \partial \Pi_1(C^w_{1/2}(Y_{\mu}^L))$. Consider the sub segments of $\hat{L}_{\mu}$ given by (see Figure 8)

$$\hat{L}^I_{\mu} \triangleq \{ (y, z_{\mu}^I(y)) : y \in I_{\mu} \} \quad \text{and} \quad \hat{L}^J_{\mu} \triangleq \{ (y, z_{\mu}^J(y)) : y \in J_{\mu} \}.$$
Recall that \( P_{\xi,\mu} = (p_{\mu}, p_{\mu}, \tilde{p}_{\xi,\mu}) \) and \( Q_{\mu} = (q_{\mu}, q_{\mu}, \tilde{q}_{\xi,\mu}) \) and consider the straight lines \( \hat{L}_{I}^{p}_{\mu} \) and \( \hat{L}_{J}^{q}_{\mu} \) contained in \( \partial \Pi_{1}(C_{uu}^{1/2}(P_{\xi,\mu})) \) and \( \partial \Pi_{1}(C_{uu}^{1/2}(Q_{\xi,\mu})) \), respectively, given by

\[
\hat{L}_{I}^{p}_{\mu} \overset{\text{def}}{=} \{(y, z_{p}(y)) : z_{p}(y) = \frac{1}{2}(y - p_{\mu}) + \tilde{p}_{\xi,\mu}, \ y \in \mathbb{R}\},
\]
\[
\hat{L}_{J}^{q}_{\mu} \overset{\text{def}}{=} \{(y, z_{q}(y)) : z_{q}(y) = -\frac{1}{2}(y - q_{\mu}) + \tilde{q}_{\xi,\mu}, \ y \in \mathbb{R}\}.
\]

Finally, consider the following subsets of \( \Delta \)

\[
\Sigma_{\xi,\mu}^{p} = \left([-4, 4] \times \hat{L}_{I}^{p}_{\mu}\right) \cap \Delta, \quad \Sigma_{\xi,\mu}^{q} = \left([-4, 4] \times \hat{L}_{J}^{q}_{\mu}\right) \cap \Delta.
\]

Observe that \( \Delta \setminus \Sigma_{\xi,\mu}^{r} \), \( r = p, q \), consists of two connected components. We let \( \Delta_{\xi,\mu,\text{right}}^{Q} \) the connected component of \( \Delta \setminus \Sigma_{\xi,\mu}^{q} \) containing \( P_{\xi,\mu} \) and by \( \Delta_{\xi,\mu,\text{left}}^{Q} \) the other component. Similarly, we let \( \Delta_{\xi,\mu,\text{left}}^{P} \) the connected component of \( \Delta \setminus \Sigma_{\xi,\mu}^{p} \) containing \( Q_{\xi,\mu} \) and by \( \Delta_{\xi,\mu,\text{right}}^{P} \) the other component.

After these preliminary constructions, we are now ready to prove the lemma. Note that by Remark 3.15 “\( G_{\xi,\mu}([-4, 4] \times \hat{L}_{I}^{f}_{\mu}) \) is at the left of \( G_{\xi,\mu}(L_{A_{\xi,\mu}})\)” and “\( G_{\xi,\mu}([-4, 4] \times \hat{L}_{J}^{f}_{\mu}) \) is at the right of \( G_{\xi,\mu}(L_{B_{\xi,\mu}})\)”.

Therefore

- condition (a) implies that \( G_{\xi,\mu}([-4, 4] \times \hat{L}_{I}^{f}_{\mu}) \subset \text{closure}(\Delta_{\xi,\mu,\text{left}}^{Q}) \),
- condition (b) implies that \( G_{\xi,\mu}([-4, 4] \times \hat{L}_{J}^{f}_{\mu}) \subset \text{closure}(\Delta_{\xi,\mu,\text{right}}^{P}) \).

We now see that these two conditions cannot hold simultaneously. Consider \( \omega_{I,\mu}^{f}, \omega_{J,\mu}^{f} \in \mathbb{Z} \) given by

\[
g_{\xi,\mu}(\hat{L}_{I}^{f}_{\mu}) \cap \{y = q_{\mu}\} = (q_{\mu}, \omega_{I,\mu}^{f}) \quad \text{and} \quad g_{\xi,\mu}(\hat{L}_{J}^{f}_{\mu}) \cap \{y = p_{\mu}\} = (p_{\mu}, \omega_{J,\mu}^{f}).
\]

Arguing as in Claim 3.22 we get

\[
\omega_{I,\mu}^{f} = \frac{\xi}{2}(q_{\mu} - a_{\mu}) + \xi z_{\mu} + q_{\mu} \quad \text{and} \quad \omega_{J,\mu}^{f} = \xi z_{\mu} + a_{\mu}.
\]
On the other hand, our assumptions and Remark 3.20 imply that $\omega^{I}_{\xi, \mu} \leq \tilde{q}_{\xi, \mu}$ and $\omega^{J}_{\xi, \mu} \geq \tilde{p}_{\xi, \mu}$. Thus

$$|\tilde{q}_{\xi, \mu} - \tilde{p}_{\xi, \mu}| \leq |\omega^{I}_{\xi, \mu} - \omega^{J}_{\xi, \mu}| \leq \left(\frac{\xi}{2} + 1\right) |q_{\mu} - a_{\mu}| \leq 12.16,$$

where the last inequality follows from the estimates in (3.2) and (3.7). Since, also by (3.2), we have that $|\tilde{q}_{\xi, \mu} - \tilde{p}_{\xi, \mu}| \in [31.4, 35.6]$ we derive a contradiction, completing the proof of the lemma. □

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