I propose that non-Abelian topological order can emerge from the organization of quantum particles into identical indistinguishable copies of the same quantum many-body state. Quantum indistinguishability (symmetrization) of the collectivities leads to topological degeneracy in the subspace of elementary excitations, giving rise to non-Abelian braiding statistics. The non-Abelian hidden order of a symmetrized structure is manifested in its entanglement properties, and the corresponding non-Abelian fusion and braiding rules can be derived by analyzing the set of symmetrized states on a surface with non-trivial topology like a torus. To illustrate the emergence of non-Abelian statistics from symmetrization, I consider the case of two identical copies of the toric code model. The resulting model is shown to be non-Abelian, exhibiting two types (charge and flux) of quasiparticles with non-trivial fusion channels. The symmetrization construction I present here constitutes a framework for the generation of non-Abelian models from known Abelian ones.

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INTRODUCTION

Quantum Hall systems [1–4] taught us that we should not regard quantum statistics as the result of permuting variables in a many-body wave function, but rather as the one of exchanging particles along a physical path [5]. Within this definition, quantum statistics in two dimensions is a representation of the Braid group [6], where a braid describes the physical exchange of the world-lines of particles. One-dimensional Abelian representations of the Braid group correspond to fractional statistics, with statistical phase interpolating between the one of bosons and fermions. Remarkably, non-Abelian representations open the possibility for the existence of quasiparticles with non-Abelian braiding statistics [7–11]. The concept of non-Abelian statistics is profoundly unintuitive, since it implies that the exchange of identical quasiparticles produces a global change in the underlying vacuum, converting it into a topologically different one. The space of topologically distinct vacua’s, connected through braidings of identical quasiparticles, exhibits a highly non-local structure in which quantum information can be stored safely, immune to local interactions with the environment [9, 12, 13].

Non-Abelian anyons have been predicted to occur both in fractional quantum Hall systems [7, 8], as excitations of the so called Pfaffian state [14–16], and as Majorana fermions attached to vortices in superconductors with $p$-wave pairing [17, 18]. Recently, evidence of such Majorana fermions has been found in semiconductor nanowires coupled to superconductors [19]. Towards the fundamental understanding of non-Abelian topological order, the discovery of exactly solvable models exhibiting non-Abelian phases has played a crucial role [12, 20–31]. In this direction, the spin lattice models developed by Kitaev [12, 21], together with quantum loop models [22, 23, 25, 26, 29, 30] and string-net models [20, 21, 24], constitute eminent examples. Non-Abelian anyons have been also proposed to appear when introducing non-trivial scalar products [25] or twists [32] in quantum loop models. Remarkable insight into the underlying properties of non-Abelian topological order has been gained by Wen’s theory [20, 21], in which both Abelian and non-Abelian states are proposed to emerge from the condensation of extended objects dubbed string-nets [21]. String-net condensation reveals that the mathematical framework underlying topological order is tensor network category. Tensor network representations for the ground states of string-net models have been developed in [33, 34]. But, as much as this endeavor has been fruitful, it is also far from being complete. The challenge remains to find descriptions of non-Abelian states that allow us to characterize them in an intuitive way, identi-
fying the features of their underlying Hamiltonians and paving the way towards their experimental realization.

Here, I propose that non-Abelian topological order can arise from the organization of quantum degrees of freedom into indistinguishable copies of the same quantum many-body state (Fig. 1). The non-commutative algebra characterizing non-Abelian braiding statistics originates from the symmetrization of the identical Abelian algebras defining the Abelian statistics of the copies. This approach aims at providing an intuitive picture for the physical mechanism leading to the formation of non-Abelian anyons, giving a simple description of the pattern of many-body entanglement underlying non-Abelian topological order. Such pattern is inspired by the form of the wave function describing non-Abelian fractional quantum Hall liquids [7], which can be constructed from copies of Abelian quantum Hall states [31, 35, 36]. This work is motivated by my previous work on the emergence of non-Abelian braiding properties from merging of identical quantum loop condensates [37].

The theory is presented in three stages: intuition, conjecture and illustration. First, I explain the intuition behind the general idea, arguing how the symmetrization of identical quantum many-body states leads to the origin of non-Abelian quasiparticles. The reasoning I follow is simple. The freedom to assign quasiparticles to the identical copies gives rise to a set of degenerate states that are locally indistinguishable and are connected to each other by braidings of the quasiparticles. This opens the path for non-Abelian braiding statistics. Second, I present a general conjecture on the properties of topological models arising from symmetrization of identical copies of a given topological model. I propose that the quasiparticles of the symmetrized model are constructed as symmetrizations of tensor products of the Abelian quasiparticles of the copied model. The algebra characterizing the fusion and braiding rules is obtained through symmetrization of the tensor product of the algebras defining the copies. Third, to illustrate the theory I analyze the model emerging from the symmetrization of two copies of the toric code model [12], a seminal example of an Abelian topological model. In order to characterize the arising symmetrized model, I use recent results on the description of topological models through the entanglement properties of the corresponding ground states [38–42]. Within this framework, the quasiparticles statistics and braiding can be derived by analyzing the basis of ground states with minimum entanglement entropy on a torus [39–42]. Following this procedure, I analytically obtain the non-trivial fusion and braiding rules of the symmetrized model, proving its non-Abelian character and showing in this case the validity of the conjecture.

SYMMETRIZED STATES AND NON-ABELIAN STATISTICS: INTUITION

Let me consider a lattice system of local degrees of freedom characterized by the Hilbert space $\mathcal{H}$. Let $|\Phi\rangle$ denote a collective state of such a many-body system. I further consider the state resulting from the symmetrization of $k$ identical copies of $|\Phi\rangle$ in the form:

$$|\Psi\rangle \propto P (|\Phi\rangle \otimes |\Phi\rangle \ldots \otimes |\Phi\rangle).$$

Here, the projector $P = \prod_i P_i$ is a product of local projectors that map the tensor product of the $k$ identical local degrees of freedom $\mathcal{H}^\otimes k$ onto a new degree of freedom $\mathcal{H}^S$, which is symmetric under exchange of any two of them:

$$P : \mathcal{H}^\otimes k \rightarrow \mathcal{H}^S, \text{ with } P = \frac{1 + \prod_{i<j} \text{SWAP}_{ij}}{2},$$

and $\text{SWAP}_{ij}$ being the permutation operator between copies $i$ and $j$. The symmetrized state $|\Psi\rangle$ exhibits, by construction, a global hidden order associated to the internal organization of the local degrees of freedom in identical indistinguishable copies of the same many-body state.

Let me intuitively argue how quasiparticles with non-Abelian statistics can emerge from such a pattern of many-body entanglement. If we assume that each copy exhibits localized quasiparticles as elementary excitations, it is reasonable to expect that the elementary excitations of the symmetrized state are constructed by creating quasiparticles in each of the copies and symmetrizing. If the spatial positions of the quasiparticles are fixed, we
still have freedom to assign them to each of the identical copies in different ways. For instance, for the case of two
particles, leading to a non-Abelian algebra of exchanges.

Let me consider an Abelian topological model describing a lattice of local degrees of freedom characterized by
the local Hilbert space \( \mathcal{H} \). Let the ground state manifold on the torus be spanned by the set of states \( \{ |\Phi_j\rangle \} \), which form the basis of quasiparticles of the model. This manifold defines an effective global Hilbert space that I will denote by \( \mathcal{H}_G \). Let me further define a topological model characterized by a set of ground states on the torus that are obtained as symmetrization of two ground states of
the previous Abelian model in the form:

\[
|\Psi(q_i, q_j)\rangle \equiv \mathbf{P} (|\Phi_{q_i}\rangle \otimes |\Phi_{q_j}\rangle). \tag{5}
\]

This model describes a lattice of local degrees of freedom characterized by the Hilbert space \( \mathcal{H}^S \subset \mathcal{H} \otimes \mathcal{H} \), which is symmetric under exchange of the local degrees of freedom \( \mathcal{H} \). I conjecture that this model is a non-Abelian
topological model with the following properties:

i) Global Hilbert space. The global Hilbert space spanned by the many-body symmetrized states \( \{ |\Phi_j\rangle \} \) is isomorphic to \( \mathcal{H}_G^S \subset \mathcal{H}_G \otimes \mathcal{H}_G \), which is the subspace of the tensor product states in \( \mathcal{H}_G \otimes \mathcal{H}_G \) that are symmetric under exchange of the two global degrees of freedom \( \mathcal{H}_G \).

ii) Topological charges. If \( \{ |q\rangle \} \) is a basis of states in \( \mathcal{H}_G \) corresponding to the charges \( \{ |\Phi_j\rangle \} \) of the Abelian model, the topological charges of the symmetrized model are given by:

\[
|q_i \circ q_j\rangle \propto |q_i\rangle \otimes |q_j\rangle + |q_j\rangle \otimes |q_i\rangle. \tag{6}
\]

They constitute a basis for the symmetrized global Hilbert space \( \mathcal{H}_G^S \). In general, there does not exist a one
to one correspondence between the many-body states in \( \mathcal{H}^S \) and the topological charges in \( \mathcal{H}_G \).

iii) Fusion rules. The fusion rules are determined by the algebra of operators that create the charges \( |q_i \circ q_j\rangle \) out of the vacuum \( |1 \circ 1\rangle \), with \( |1\rangle \) being the vacuum of the Abelian model.

iv) \( S \)-matrix. If the Abelian model is characterized by a modular \( S \)-matrix with elements \( S_{ij} = \langle i|S|j\rangle \), with \( |i\rangle \) forming a basis in \( \mathcal{H}_G \), the \( S \)-matrix of the symmetrized model is given by:

\[
S_{ij,i'j'} = \langle i \circ j|S \otimes S|i' \circ j'\rangle, \tag{7}
\]
Abelian charges in the global effective subspace. Logical charges. The topological charges of the symmetrized symmetric degree of freedom $H^G$ globally projects the local tensor product $H \otimes H$. Model are symmetrized through the projector $P$. Metrized many-body states. Two ground states of the Abelian symmetrization is carried out by the global projector $P$. It corresponds between the local symmetrization carried out with $|i \otimes j \rangle \propto |i \rangle |j \rangle + |j \rangle |i \rangle$ the elements of a basis in $H^G_S$.

Points i) and ii) constitute the core of the conjecture. To prove i) one needs to show the existence of a correspondence between the local symmetrization carried out by the projector $P$ on the tensor product of local degrees of freedom $H \otimes H$, and the global symmetrization that projects the tensor product of global spaces $H_G \otimes H_G$ onto the symmetric global subspace $H^S_G$ (see Fig. 4). Point ii) further requires to demonstrate that the quasiparticles of the symmetrized model are obtained from symmetrization in the global subspace of products of charges of the Abelian model that serves as a copy. This can be done using the correspondence i). Points iii) and iv) follow directly from i) and ii).

Here, I will consider the case of two copies of the toric code model and prove that the emerging symmetrized model satisfies the conjecture above. First, I will establish an isomorphism between the manifold of many-body symmetrized states and the symmetric global Hilbert space. Then I will show that the topological charges of the symmetrized model are given by $\Phi$. Finally, I will obtain the fusion rules and the $S$-matrix of the model, demonstrating its non-Abelian character.

CHARGES, FUSION RULES AND S-MATRIX OF THE TORIC CODE MODEL

The toric code model is a seminal example of an Abelian topological model. It describes $\frac{1}{2}$-spins sitting at the edges of a square two-dimensional lattice. The ground state (see Fig. 5) is a superposition with equal weight of all possible spin states in which up-spins are arranged along a closed loop configuration $L$:

$$| \Phi \rangle \propto \sum_{|L\rangle} |L\rangle .$$

Here, $| |L\rangle \rangle = \prod_{\ell \in L} \sigma_\ell^x | \text{vac} \rangle$, and $| \text{vac} \rangle = \bigotimes_{\ell=1}^N | \downarrow \rangle \ell$. The topological character of the model is manifested in the degeneracy of the ground state, which depends on the topology of the lattice. On a torus there are 4 degenerate ground states:

$$| \Phi \rangle , X_1 | \Phi \rangle , X_2 | \Phi \rangle , X_1 X_2 | \Phi \rangle ,$$

where $X_1(2) = \prod_{\ell \in C_\ell(x,y)} \sigma_\ell^x$, and $C_\ell(x,y)$ are the two different non-contractible loops (see Fig. 5).

The properties of the corresponding Abelian anyon model (topological charges, fusion and braiding rules) are encoded in the algebra of this ground state subspace. This corresponds to the one of two effective $\frac{1}{2}$-spins, $H_G = \frac{1}{2} \bigotimes \frac{1}{2}$. If we define dual operators $Z_1(2) = \prod_{\ell \in C_\ell(x,y)} \sigma_\ell^z$, with $C_\ell(x,y)$ being non-contractible loops in the dual lattice, we have:

$$\{ X_1(2) , Z_1(2) \} = 0.$$

The operators $Z_1(2) (X_1(2))$ therefore represent effective $z(x)$-Pauli matrices. The states in Eq. (9) correspond to the computational basis of the two qubits, for which both have well defined $z$-component.

The topological charges of the model are constructed as superpositions of states in Eq. (9) satisfying the condition

FIG. 4: Local and global symmetrizations. (a) Symmetrized many-body states. Two ground states of the Abelian model are symmetrized through the projector $P$, which locally projects the local tensor product $H \otimes H$ onto the local symmetric degree of freedom $H^G$. (b) Symmetrized topological charges. The topological charges of the symmetrized model are obtained as symmetrization of tensor products of Abelian charges in the global effective subspace $H^S_G$. This symmetrization is carried out by the global projector $P_G$.

FIG. 5: Toric code model. The toric code model describes $\frac{1}{2}$-spins sitting at the edges of a square lattice. (a) The ground state is a superposition of closed loops configurations, where the presence or absence of a line segment at a given edge represents, respectively, the state up or down of the $\frac{1}{2}$-spin. (b) On the torus there are four degenerate ground states, corresponding to the two classes of non-contractible loops $C_x, C_y$ ($\tilde{C}_x, \tilde{C}_y$) in the normal (dual) lattice. (c) The string operators $X_1, Z_1, X_2, Z_2$ (see text) define an effective two-qubit algebra in the space of ground states on the torus.
of minimum topological entanglement entropy [40, 41]. Along the x-axis of the torus they are:

\[
|\Phi_1\rangle_{z}, \quad |\Phi_\pi\rangle_{z} = X_2 |\Phi_1\rangle_{z}, \quad |\Phi_{em}\rangle_{z} = X_2 Z_1 |\Phi_1\rangle_{z}, \quad (11)
\]
where \(|\Phi_1\rangle_{z} = \frac{1}{\sqrt{2}} (1 + X_1) |\Phi\rangle_{z}\) is the vacuum state with no charges, and \(|\Phi_{e}\rangle_{z}\) and \(|\Phi_{m}\rangle_{z}\) are the electric and magnetic charges of the model. In the effective two-qubit Hilbert space these states correspond to states for which the first qubit has well defined x-component, whereas the second has well defined z-component (see Fig. 6):

\[
|\Phi_{p}\rangle_{z} = \sum_{\ell \in \mathcal{L}} \left( \prod_{p \in \mathcal{P}_{\ell}} \right) |\Phi_{\pi}\rangle_{z} \otimes |\Phi_{m}\rangle_{z} \quad (12)
\]
Here, \(X |\pm\rangle_{z} = \pm |\pm\rangle_{z}, \quad Z |\pm\rangle_{z} = \pm |\pm\rangle_{z}, \quad H\) is the Hadamard transformation for one-qubit, which converts the x-basis into the z-basis, and \(|i\rangle\) denotes a state of the computational basis of two qubits.

The fusion rules can be obtained from (11) taking into account the properties of the operators \(Z_1\) and \(X_2\):

\[
e \times e = 1 \quad m \times m = m \quad e \times m = em \quad em \times e = m \quad em \times m = e \quad em \times em = 1.
\]

These fusion rules are trivial, with only one channel, indicating that the model is Abelian.

The elements of the S-matrix characterizing the anyon model are given by the scalar products of the charges \(|q\rangle\) along the x-axis of the torus defined in Eq. (12), with the charges \(|\overline{q}\rangle\) along the y-axis [40, 41]. The two basis of charges are connected by the modular transformation that performs a rotation of angle \(\pi\) on the torus, which corresponds to the exchange or swap of the two qubits. Therefore, we have:

\[
S_{ij} = \langle q_i | q_j \rangle = \langle \overline{q_i} | \overline{q_j} \rangle = \langle i | H \otimes 1 \cdot \text{SWAP} \cdot H \otimes 1 | j \rangle = \langle i | \text{SWAP} \cdot H \otimes H | j \rangle. \quad (13)
\]

FIG. 6: Topological charges of the toric code model. The charges along the x-direction correspond to states of two qubits for which the first and the second qubit have well defined x-component and z-component, respectively. The qubit states are represented by vectors in the Bloch sphere. Starting from the vacuum state \(|1\rangle\), an electric charge \(|e\rangle\) is created by flipping the second qubit, whereas flipping the first qubit creates a magnetic charge \(|m\rangle\).

FIG. 7: Spin-1 symmetrized state. (a) Two copies of a toric code ground state \(\Phi\) are merged by the projector \(P\), leading to the spin-1 state \(\Psi\). (b) Local projection of two spin-1\(_{\pi}\) onto a spin-1. Spin-1\(_{\pi}\) and spin-1 states are represented by line segments.

Here, \(|i\rangle, |j\rangle\) are states of the two-qubit computational basis, and SWAP is the unitary operation exchanging the two qubits.

**SYMMETRIZED ANYON MODEL FROM TWO TORIC CODE COPIES**

Spin-1 symmetrized state

Let me consider the many-body state describing a lattice of spin-1 particles that is constructed by symmetrization of two identical copies of the toric code ground state in Eq. (8):

\[
|\Psi\rangle = P \left( |\Phi\rangle \otimes |\Phi\rangle \right). \quad (14)
\]

Here, the projection \(P = \prod_{\ell} P_{\ell}\) is a product of local projectors that map the tensor product of two \(\frac{1}{2}\)-spins onto the symmetric subspace of total spin 1:

\[
P_{\ell} : \mathcal{H}_\ell \otimes \mathcal{H}_\ell \equiv \frac{1}{2} \otimes \frac{1}{2} \rightarrow \mathcal{H}_\ell^S \equiv \text{spin } 1, \quad (15)
\]

In terms of local spin-1 operators, \(S^\alpha_{\ell}, \alpha = x, y, z\), the state (14) has the form:

\[
|\Psi\rangle \propto \prod_{\ell} (1 + B_{1p}) \sum_{\{L\}} |L_{\ell}\rangle. \quad (16)
\]

Here, \(|L_{\ell}\rangle = \prod_{\ell \in \mathcal{L}} S^z_{\ell} |\cdot\rangle\), with \(|\cdot\rangle = \bigotimes_{\ell = 1}^{N} |\cdot\rangle_{\ell}\), and \(|\cdot\rangle_{\ell}\) denotes the state with minimum z-component of the \(\ell\)th spin-1. The operator \(B_{1p} = \prod_{\ell \in \mathcal{P}_{\ell}} (2 |S^z_{\ell}|^2 - 1)\) acts on the four spins within a lattice plaquette \(p\). Using a language of strings, the three orthogonal states of the local spin-1 degree of freedom, \(|-\rangle_{\ell}, S^z_{\ell} |-\rangle_{\ell} = |0\rangle_{\ell}/\sqrt{2}\), and \((1 - 2 |S^z_{\ell}|^2) |+\rangle_{\ell} = |+\rangle_{\ell}\) are mapped, respectively,
onto no-segment, single-line segment, and double-line segment in the corresponding loop configuration (see Fig. 7). Within this language, the states \( |\mathcal{L}_1\rangle \) are closed loop configurations of single lines, and the symmetrized state \( |\mathcal{L}\rangle \) is a superposition of closed loops made of single lines, modulo plaquette moves \( B_{1p} \) that exchange no-lines with double-lines.

For all the discussion that follows, the actual form of the symmetrized state in terms of spin-1 operators is not relevant. The properties of the symmetrized anyon model will be derived only taking into account the form \( (14) \), using both the properties of the projector and the properties of the Abelian anyon model defining the copies.

---

**Symmetrized ground state subspace on the torus**

Let me consider the topological model that results from symmetrization of two identical toric code models. This model is defined through a ground state manifold on the torus spanned by all possible states that are obtained by symmetrizing products of two ground states of the toric code model. If we consider the basis of charges \( (11) \) for each of the copies, the symmetrized states defining the new model have the form:

\[
|\Psi(q_i, q_j)\rangle \equiv \mathbf{P} \left( |\Phi_{q_i}\rangle \otimes |\Phi_{q_j}\rangle \right).
\]  

(17)

The properties of the model are encoded in the algebra of this subspace. In the following I analyze this algebra.

Before symmetrization, there are 16 different states of the form \( |\Phi_{q_1}\rangle \otimes |\Phi_{q_2}\rangle \), corresponding to all possible tensor products of two toric code charges (Fig. 8a). The space of these product states is equivalent to the one of four qubits (Fig. 8b), with the correspondence:

\[
|\Phi_{q_i}\rangle \otimes |\Phi_{q_j}\rangle \leftrightarrow |q_i\rangle \otimes |q_j\rangle.
\]  

(18)

If the qubits of the first (second) copy are denoted by \( \{1, 2\} \) (\( \{3, 4\} \)), the states \( |q_i\rangle \otimes |q_j\rangle \) correspond to the basis in which qubits \( \{1, 3\} \) have well defined \( x \)-component, whereas \( \{2, 4\} \) have well defined \( z \)-component:

\[
|q_i\rangle \otimes |q_j\rangle = |\pm \hat{x}\rangle_1 |\pm \hat{x}\rangle_3 |\pm \hat{z}\rangle_2 |\pm \hat{z}\rangle_4.
\]  

(19)

The symmetrization carried out by the operator \( \mathbf{P} \) makes some of the tensor product states indistinguishable, leaving 10 linearly independent symmetrized states of the form \( (17) \) (Fig. 8b). In the following, I show that the subspace that these 10 symmetrized states generate is isomorphic to the subspace of four qubits that is symmetric under exchange of the two copies:

\[
\{ |\Psi(q_i, q_j)\rangle \}
\]

(20)

\[
\mathcal{H}_G^S \equiv \left\{ 1 \otimes 1 \otimes 0 \otimes 0, 1 \otimes 0 \otimes 1 \otimes 0, 1 \otimes 0 \otimes 0 \otimes 1, 0 \otimes 1 \otimes 1 \otimes 0, 0 \otimes 1 \otimes 0 \otimes 1, 0 \otimes 0 \otimes 1 \otimes 1, 0 \otimes 0 \otimes 0 \otimes 1, 0 \otimes 0 \otimes 1 \otimes 0, 0 \otimes 0 \otimes 0 \otimes 1 \right\} \subset \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \mathcal{H}_G^S \subset \mathcal{H}_G.
\]

(21)

Finally, we will obtain the fusion and braiding rules characterizing the symmetrized model.
Characterization of symmetrized states through spin-1 string operators

Let me start by defining the following spin-1 string operators on the torus:

\[
\mathcal{X}(\mathcal{X}') = \prod_{\ell \in \mathcal{C}_a(\mathcal{C}_b)} (2[S^\ell|^2] - 1),
\]

\[
\mathcal{Z}(\mathcal{Z}') = \prod_{\ell \in \mathcal{C}_a(\mathcal{C}_b)} (2[S^\ell|^2] - 1). \tag{22}
\]

They trivially fulfill \(\mathcal{X}^2(\mathcal{X}')^2 = \mathcal{Z}^2(\mathcal{Z}')^2 = 1\), so that their eigenvalues are ±1. We can write them as projections of tensor products of spin-\(\frac{1}{2}\) string operators in each copy in the form:

\[
\mathcal{X}(\mathcal{X}')\mathbf{P} = \mathbf{P} \prod_{\ell \in \mathcal{C}_a(\mathcal{C}_b)} P_\ell(\sigma_\ell^x \otimes \sigma_\ell^x)P_\ell = \mathbf{P}(X_{1(2)} \otimes X_{3(4)})
\]

\[
\mathcal{Z}(\mathcal{Z}')\mathbf{P} = \mathbf{P} \prod_{\ell \in \mathcal{C}_a(\mathcal{C}_b)} P_\ell(\sigma_\ell^y \otimes \sigma_\ell^y)P_\ell = \mathbf{P}(Z_{2(1)} \otimes Z_{4(3)}).
\]

From the above expressions, we see that the operators \(\mathcal{Z}, \mathcal{X}\) correspond, respectively, to the parity \((\mathcal{Z} \otimes \mathcal{Z})\) and phase \((\mathcal{X} \otimes \mathcal{X})\) of the pair of qubits \(\{1,3\}\). Conversely, the operators \(\mathcal{Z}', \mathcal{X}'\) correspond to the parity and phase operator of the pair of qubits \(\{2,4\}\) (see Fig. 10).

The symmetrized states in Eq. 17 are eigenstates of \(\mathcal{X}\) and \(\mathcal{Z}'\):

\[
\mathcal{X}|\Psi(q_i, q_j)\rangle = \mathbf{P}(X_1|q_i\rangle \otimes X_3|q_j\rangle) = \pm 1,
\]

\[
\mathcal{Z}'|\Psi(q_i, q_j)\rangle = \mathbf{P}(Z_2|q_i\rangle \otimes Z_4|q_j\rangle) = \pm 1,
\]

since the left (right) qubits of each of the copies have well defined \(x(z)\)-spin component. We can classify them according to the different eigenvalues of \(\mathcal{X}\) and \(\mathcal{Z}'\) (see Table I). Within each subspace the states are related to each other by the operators \(\mathcal{X}'\) and \(\mathcal{Z}\). For example, we have

\[
|\Psi(e, e)\rangle = \mathbf{P}(|e\rangle \otimes |e\rangle) = \mathbf{P}(X_2 \otimes X_4)(|1\rangle \otimes |1\rangle) = \mathcal{X}'|\Psi(1, 1)\rangle.
\]

| \(\mathcal{X}\) | \(\mathcal{Z}'\) | Symmetrized State |
|---|---|---|
| 1 | 1 | \(\Psi(1, 1) \xrightarrow{X'} \Psi(e, e)\) |
| 1 | -1 | \(\Psi(1, e) \xrightarrow{Z'} \Psi(m, e)\) |
| -1 | 1 | \(\Psi(1, m) \xrightarrow{X'} \Psi(e, m)\) |
| -1 | -1 | \(\Psi(e, m) \xrightarrow{Z'} \Psi(1, m)\) |

TABLE I: Symmetrized states are classified by the eigenvalues of the spin-1 string operators \(\mathcal{X}\) and \(\mathcal{Z}'\). Within each subspace the states are connected through the operators \(\mathcal{X}'\) and \(\mathcal{Z}\).

**Correspondence between symmetrized many-body spin-1 states and 4-qubit states**

Let me consider a basis for the symmetric subspace of four qubits \(\mathcal{H}_s^4\) in the form:

\[
|q_i \otimes q_j\rangle \propto |q_i\rangle \otimes |q_j\rangle + |q_j\rangle \otimes |q_i\rangle.
\]

These states are eigenstates of the operators \(X_1 \otimes X_3\) and \(Z_2 \otimes Z_4\) and therefore have well defined phase and parity for the pairs of qubits \(\{1,3\}\) and \(\{2,4\}\), respectively. They fulfill:

\[
X_1 \otimes X_3|q_i \otimes q_j\rangle \propto |q_i\rangle \otimes |q_j\rangle + |q_j\rangle \otimes |q_i\rangle
\]

\[
1 + \text{SWAP}_{13}\text{SWAP}_{24}|q_j\rangle \otimes |q_i\rangle = |1 + \text{SWAP}_{13}\text{SWAP}_{24}|X_1|q_j\rangle \otimes X_3|q_i\rangle = \pm |q_i \otimes q_j\rangle,
\]

and, similarly, for \(Z_2 \otimes Z_4\). Within each eigensubspace the states are connected to each other through the operators \(X_2 \otimes X_4\) and \(Z_1 \otimes Z_3\) as shown in Table II.

| \(X_1X_3\) | \(Z_2Z_4\) | 4-qubit Symmetric State |
|---|---|---|
| 1 | 1 | \(\xrightarrow{X_2X_4} e \otimes e\) |
| 1 | -1 | \(\xrightarrow{Z_1Z_3} m \otimes m\) |
| -1 | 1 | \(\xrightarrow{X_2X_4} e \otimes em\) |
| -1 | -1 | \(\xrightarrow{Z_1Z_3} m \otimes em\) |

TABLE II: Symmetric 4-qubit states are classified by the eigenvalues of the operators \(X_1X_3\) and \(Z_2Z_4\). Within each eigensubspace, the states are connected through the operators \(X_2X_4\) and \(Z_1Z_3\).

If we establish the correspondence:

\[
\mathcal{X}(\mathcal{X}') \longleftrightarrow X_{1(2)} \otimes X_{3(4)}
\]

\[
\mathcal{Z}(\mathcal{Z}') \longleftrightarrow Z_{1(2)} \otimes Z_{3(4)},
\]
between string operators and parity and phase operators for pairs of qubits, the two tables I and II are equivalent (see Fig. 11). We can therefore conclude that the ground state subspace of many-body symmetrized states is isomorphic to the symmetric subspace of 4-qubits, \( \mathcal{H}_S^G \). When establishing the exact correspondence between many-body symmetrized states and 4-qubit states, we have to take into account the fact that the symmetrized states \( |\Psi(q_i, q_j)\rangle \) within each eigensubspace (states within the same row in Table I) are not orthogonal to each other. The actual isomorphism is:

\[
\eta_{\pm}^1(1 \pm \lambda^r) \ (1 \pm \mathcal{Z}) \leftrightarrow \begin{cases} \frac{1}{2}(1 \pm X_2X_3) \ (1 \pm Z_1Z_3) \ |1 \circ \ 1) \\
\frac{1}{2}(1 \pm Z_1Z_3) \ (1 \circ \ 1) \\
\frac{1}{2}(1 \pm Z_1Z_3) \ (1 \circ \ 1) \\
\frac{1}{2}(1 \pm Z_1Z_3) \ (1 \circ \ 1) \\
\frac{1}{2}(1 \pm Z_1Z_3) \ (1 \circ \ 1) \ \ .
\end{cases}
\]

Here, the \( \pm \) superposition states are orthogonal to each other, since they correspond to different eigenvalues of the operators \( \lambda^r \) and \( \mathcal{Z} \). The factors \( \eta_{q, q'}^{r, m} \) are such that the states are normalized.

**CHARGES, FUSION RULES AND S-MATRIX**

**The charges of the model**

The charges \( |Q\rangle \) of the symmetrized model are given by the superpositions of the symmetrized many-body states

\[
|\psi(q_i, q_j)\rangle \text{ in Eq. (17) that satisfy the condition of minimum entanglement entropy. Before symmetrizing, the charges of two copies of the toric code model correspond in the effective subspace of 4-qubits to the tensor product states } |q_i\rangle \otimes |q_j\rangle \text{ defined in Eq. (19). Therefore, for two-copies of the toric code model, to satisfy the condition of minimum entanglement entropy is equivalent in the subspace of 4-qubits to guarantee that the pairs \( \{1, 3\} \) and \( \{2, 4\} \) have well defined } x \text{- and } z \text{- components, respectively.}

Let me assume here the following intuitive result: Since the subspace of symmetrized many-body states is isomorphic to the symmetric subspace of 4-qubits \( \mathcal{H}_S^G \), finding the basis of minimum entanglement entropy states (MES’s) for the symmetrized model is equivalent to finding the basis for 4-qubits that better defines the } x \text{- and } z \text{-components for qubits } \{1, 3\} \text{ and } \{2, 4\} \text{ within } \mathcal{H}_S^G \text{. This result can indeed be proved to be true. By considering a partition on the torus, the microscopic basis of MES’s of the symmetrized model can be obtained } \[44]. \text{ This calculation is similar to the one performed to obtain the charges for a single copy of the toric code model } \[40, 41]. \text{ This proof will be presented elsewhere } \[44].

It is straightforward to see that the basis in the symmetric subspace \( \mathcal{H}_S^G \) that allows for a better definition of the } x \text{- and } z \text{-components of qubits } \{1, 3\} \text{ and } \{2, 4\} \text{ is precisely the basis } |q_i \otimes q_j\rangle \text{ introduced above in Eq. (24) (see Appendix).}

The charges of the model are therefore:

\[
1 \circ 1 \quad 1 \circ e \quad e \circ e \\
1 \circ m \quad (\mathcal{E}M)_1, (\mathcal{E}M)_2 \quad e \circ em.
\]

with \( (\mathcal{E}M)_1 = e \circ m \) and \( (\mathcal{E}M)_2 = em \circ 1 \).

It is useful to write them in terms of tensor products of triplets and singlets for the pairs of qubits \( \{1, 3\} \) and
most of the states $|q_i \otimes q_j\rangle$ can be written as products of triplets $|I\rangle \otimes |I\rangle$ (see Fig. 11). The states $|e \otimes m\rangle$ and $|e m \otimes 1\rangle$ are, respectively, symmetric and antisymmetric superpositions of $|I\rangle \otimes |I\rangle$ and $|s\rangle \otimes |s\rangle$.

The fusion rules

The fusion rules [31,32] follow from the algebra of spin operators in the symmetrized subspace $H^S_0$. On the one hand, we have that $\{I,E,E_2\}$ and $\{I,\mathcal{M},\mathcal{M}_2\}$ fulfill a spin-1 algebra:

$$
\begin{align*}
\mathcal{I} & \xrightarrow{X_2X_4(Z_1Z_3)} \mathcal{E}(\mathcal{M}) \rightarrow \mathcal{E}_2(\mathcal{M}_2) \\
\mathcal{I} & \xrightarrow{X_2X_4(Z_1Z_3)} \mathcal{E}_2(\mathcal{M}_2),
\end{align*}
$$

with $X_2X_4(Z_1Z_3)$ corresponding to the $x$-component of an effective global spin-1 operator $S^x_0$, and $X_2X_4(Z_1Z_3)$ corresponding to $1 - 2|S^x_0|^2$. It therefore follows that:

$$
\begin{align*}
\mathcal{E} \times \mathcal{E} = \mathcal{I} + \mathcal{E}_2 & \quad \mathcal{E} \times \mathcal{E}_2 = \mathcal{E} & \quad \mathcal{E}_2 \times \mathcal{E}_2 = \mathcal{I} \\
\mathcal{M} \times \mathcal{M} = \mathcal{I} + \mathcal{M}_2 & \quad \mathcal{M} \times \mathcal{M}_2 = \mathcal{M} & \quad \mathcal{M}_2 \times \mathcal{M}_2 = \mathcal{I}.
\end{align*}
$$

On the other hand, we have that

$$
\begin{align*}
\mathcal{I} & \xrightarrow{Z_1Z_2Z_4} (\mathcal{E}\mathcal{M})_1 \\
\mathcal{I} & \xrightarrow{Z_1Z_2Z_4} (\mathcal{E}\mathcal{M})_2,
\end{align*}
$$

so that

$$
\begin{align*}
\mathcal{E} \times \mathcal{M} = (\mathcal{E}\mathcal{M})_1 + (\mathcal{E}\mathcal{M})_2 & \quad (\mathcal{E}\mathcal{M})_1 \times (\mathcal{E}\mathcal{M})_1 = (\mathcal{E}\mathcal{M})_2 \times (\mathcal{E}\mathcal{M})_2 = 1 + \mathcal{E}_2\mathcal{M}_2 \\
(\mathcal{E}\mathcal{M})_1 \times (\mathcal{E}\mathcal{M})_2 = \mathcal{E}_2 + \mathcal{M}_2.
\end{align*}
$$

The S-matrix

The S-matrix of the model is easily obtained as the modular transformation relating the two basis of charges $Q$ and $Q$ corresponding to the $-x$ and $-y$ direction of the torus [10,32]. For the model I discuss here, we have:

$$
S_{ij,i'j'} = \langle q_i \otimes q_j | \text{SWAP}_{12} | q_{i'} \otimes q_{j'} \rangle = \langle i \otimes j | H_1 H_3 \cdot \text{SWAP}_{12} S_{34} H_3 H_4 | i' \otimes j' \rangle,
$$

(29)

where I have used that $|q_i \otimes q_j\rangle = H_1 H_3 | i \otimes j\rangle$, with $|i \otimes j\rangle \propto |i\rangle \otimes |j\rangle + |j\rangle \otimes |i\rangle$. Therefore, we obtain that:

$$
S_{ij,i'j'} = \langle i \otimes j | \text{SWAP}_{12} S_{34} \cdot H_2 H_4 H_1 H_3 | i' \otimes j' \rangle = \langle i \otimes j | S \otimes S | i' \otimes j' \rangle.
$$

(30)

Thus the S-matrix of the symmetrized model is the projection onto the symmetric subspace of the tensor product of the S-matrices of the copies. It is illuminating to write the S-matrix in the basis of triplets and singlets $\{|t_\alpha\rangle \otimes |t_\beta\rangle, |s\rangle \otimes |s\rangle\}$. In this basis, the matrix is block diagonal in the subspaces $1 \otimes 1$ and $0 \otimes 0$. Taking into account that

$$
|t_\alpha\rangle H \otimes H |t_\beta\rangle = [e^{i\frac{\pi}{4}S^y}]_{\alpha,\beta},
$$

(31)

with $S^y$ the $y$-component of a spin-1 operator, the S-matrix takes the form:

$$
S' = \text{SWAP}_{tt} e^{i\frac{\pi}{4}S^y} \otimes e^{i\frac{\pi}{4}S^y} \otimes 1_{ss}.
$$

(32)

Here, SWAP$_{tt}$ is a unitary operator in the $1 \otimes 1$ subspace that exchanges the two triplets and maps $|t_\alpha\rangle \otimes |t_\beta\rangle$ onto $|t_\beta\rangle \otimes |t_\alpha\rangle$, reordering the elements of the basis. Up to this reordering, within the subspace of triplets the S-matrix is the tensor product of two copies of the matrix

$$
e^{i\frac{\pi}{4}S^y} = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}.
$$

(33)

The matrix [33] is precisely the S-matrix of the Ising (non-Abelian) model [43]. The S-matrix of the symmetrized model in Eq. (30) is obtained as $U S' U$, where $U$ is the unitary transforming the basis of triplets and singlets $\{|t_\alpha\rangle \otimes |t_\beta\rangle, |s\rangle \otimes |s\rangle\}$ into the basis of states $|i \otimes j\rangle$. The latter are given by $|t_\alpha\rangle \otimes |t_\beta\rangle$, for $(\alpha, \beta) \neq (0,0)$, and $\frac{1}{\sqrt{2}}(|t_\alpha\rangle \otimes |t_0\rangle \pm |s\rangle \otimes |s\rangle)$. This change of basis destroys the tensor product structure of $S'$ in Eq. (32) and couples the two Ising models.

CONCLUSIONS AND OUTLOOK

I have proposed a physical mechanism for the emergence of non-Abelian topological order from a set of physical microscopic degrees of freedom. Within my proposal
Catalan liquids arise from the organization of a quantum many-body system in indistinguishable copies of the same collective state. The intuition behind this idea is simple. The freedom to assign quasiparticles to the identical copies gives origin to a set of degenerate states that are locally indistinguishable and are connected to each other by braiding of quasiparticles. This opens the path for non-Abelian braiding statistics.

I have presented a conjecture on topological models that arise from symmetrization of identical copies of an Abelian model. The manifold of many-body symmetrized states on a torus is proposed to be isomorphic to an effective symmetric global Hilbert space. This global space is spanned by symmetrizations of tensor products of Abelian topological charges, which define the quasiparticles of the symmetrized model. Similarly, the modular $S$-matrix is proposed to be obtained as the projection onto the symmetric global subspace of the tensor product of copies of the Abelian $S$-matrix.

To illustrate the theory, I have analyzed the case of two copies of the toric code model. By defining appropriate spin-1 string operators, an isomorphism has been established between the space of microscopic symmetrized many-body ground states and a symmetric global subspace of four qubits. Using this correspondence, I have argued that the topological charges of the model satisfy the general conjecture and are obtained as symmetrization of products of the toric code quasiparticles. The symmetrized model has been shown to be non-Abelian, corresponding to two copies of an Ising model that are linked together in a non-trivial manner. It is interesting to investigate whether this non-Abelian model might be universal for quantum computation [9]. The symmetrized topological charges can be shown to be the states of minimum entanglement entropy by considering a bipartition on the torus. The details of this result will be presented in an upcoming work [14].

The ideas developed here for the case of two copies of the toric code model can be generalized to prove the conjecture in the general case. Starting with many-body operators characterizing the topological order of the Abelian copies, one can construct appropriate string operators for the symmetrized many-body states. The analysis of the algebra of such operators would allow one to identify the isomorphic symmetric global effective subspace. The topological charges would then be obtained through symmetrization of Abelian quasiparticles in such global subspace.

This formalism opens a path for the generation of non-Abelian models from known Abelian ones. It is challenging to investigate what type of non-Abelian models arise as symmetrization of Abelian copies. For instance, interesting many-body states and models might emerge from the symmetrization of copies of chiral spin liquids, resonating valence bond states or topological insulators. Conversely, it is intriguing to explore whether known non-Abelian models could be deconstructed into copies of Abelian ones. This is the case for the seminal example of a non-Abelian state, the Pfaffian state, which is expected to occur in fractional quantum Hall systems. This state can be written as two copies of a Laughlin (Abelian) state [37]. I believe that similar constructions are also possible for other known instances of non-Abelian anyons and states, such as Fibonacci anyons or $p$-wave superconductors.

Finally, the investigation of the parent Hamiltonians associated to these symmetrized non-Abelian models is of crucial importance. My previous work for the case of two copies of the toric code model [37] suggests that the interactions behind this type of order are local and involve a small number of spins. It is challenging to explore whether we can develop a recipe for the construction of non-Abelian parent Hamiltonians based on the Hamiltonians of the Abelian copies. This approach would help us deepen our understanding of the microscopic interactions underlying non-Abelian anyons, serving as a guide for their experimental realization [45].

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Appendix: Basis of Symmetrized Topological Charges

I aim to find the basis of states of four qubits that better defines the individual $x$-components of spins $\{1,3\}$ and the $z$-components of spins $\{2,4\}$, within the subspace of states that is symmetric under the simultaneous exchange of spins 1 and 3, and 2 and 4. Since the projection onto this symmetric subspace commutes with the total spin components $(X_1 + X_3)/2$ and $(Z_2 + Z_4)/2$, the elements of such basis must be eigenstates of these operators. These eigenstates are precisely $\{|t_\alpha\rangle \otimes |t_\beta\rangle, |s\rangle \otimes |s\rangle\}$, with the triplets $|t_\alpha\rangle, |t_\beta\rangle$, and the singlet $|s\rangle$ defined above (see Fig. 12). They are characterized by different eigenvalues of the operators $(X_1 + X_3)/2$ and $(Z_2 + Z_4)/2$, except for the states $|t_0\rangle \otimes |t_0\rangle$ and $|s\rangle \otimes |s\rangle$, for which both spin components are zero. The symmetric and antisymmetric superpositions of these two states are the ones that better define the individual $x$-components of spins $\{1,3\}$ and the $z$-components of spins $\{2,4\}$, since such superpositions have vanishing expectation value for the operators $(Z_1 + Z_3)/2$ and $(X_2 + X_4)/2$. Therefore the basis that we are looking for is:

$$
|t_\alpha\rangle \otimes |t_\beta\rangle, \ (\alpha, \beta) \neq (0,0)
$$

$$
\frac{1}{\sqrt{2}}(|t_0\rangle \otimes |t_0\rangle \pm |s\rangle \otimes |s\rangle).
$$

This basis coincides with the one of symmetrized four qubit states $|q_i \otimes q_j\rangle$ defined in Eq. [24].