ON THE GENERALIZED ZAKHAROV-KUZNETSOV EQUATION AT CRITICAL REGULARITY

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Abstract. The Cauchy problem for the generalized Zakharov-Kuznetsov equation
\[ \partial_t u + \partial_x \Delta u = \partial_x u^{k+1}, \quad u(0) = u_0 \]
is considered in space dimensions \( n = 2 \) and \( n = 3 \) for integer exponents \( k \geq 3 \).
For data \( u_0 \in \dot{B}^s_{q,\infty} \), where \( 1 \leq q \leq \infty \) and \( s_c = \frac{n}{2} - \frac{2}{k} \) is the critical Sobolev regularity, it is shown, that this problem is locally well-posed and globally well-posed, if the data are sufficiently small. The proof follows ideas of Kenig, Ponce, and Vega [14] and uses estimates for the corresponding linear equation, such as local smoothing effect, Strichartz estimates, and maximal function inequalities. These are inserted into the framework of the function spaces \( U^p \) and \( V^p \) introduced by Koch and Tataru [17, 18].

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1. INTRODUCTION

The Zakharov-Kuznetsov equation (ZK)

\[ \partial_t \Delta u + \partial_x \Delta u = \partial_x u^2 \]

with \( \Delta = \partial_x^2 + \sum_{i=1}^{n-1} \partial_{y_i}^2 \), \((x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}\), is a generalization of the famous Korteweg-de Vries equation (KdV) to arbitrary higher dimensions. In 1974, Zakharov and Kuznetsov derived [11] as a model describing the unidirectional wave propagation in a magnetized plasma in three space dimensions [33 equation (6)]. For two dimensions, a derivation of [11] from the basic hydrodynamic equations is due to Laedke and Spatschek [19 Appendix B]. We also refer to the paper [20] by

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Lannes, Linares, and Saut for a rigorous justification of ZK valid for \( n \in \{2, 3\} \). Both, the Cauchy problem as well as several initial boundary value problems connected with \( \text{(1)} \) have attracted considerable interest in recent years; we mention [2], [3], [5], [11], [24], [29], [30]. This list is by no means exhaustive. Similar as for KdV and - to the author’s knowledge - beginning with the work [2] of Biagioni and Linares on the modified equation, generalizations of ZK with higher power nonlinearities

\[
\partial_t u + \partial_x \Delta u = \partial_x u^{k+1} \quad \text{with} \quad u(0) = u_0
\]

are considered, too. We call \( \text{(2)} \) the \( k \)-th generalized ZK equation, for short \( \text{gZK-}k \).

In this paper we are concerned with local and small data global well-posedness of the Cauchy problem for \( \text{gZK-}k \) in two and three space dimensions for integers \( k \geq 3 \). (Unfortunately our arguments break down for the modified equation, i.e. for \( k = 2 \).) In the 2-D case the following results are known for data in the classical Sobolev spaces \( H^s \).

- In 2011 Linares and Pastor [23] showed that the Cauchy problem for \( \text{gZK-}k \) is locally well-posed in \( H^s \), if \( k \geq 2 \) and \( s > \max(\frac{3}{4}, 1 - \frac{2}{n - k}) \). If the data are sufficiently small in \( H^1 \), then the corresponding solutions extend globally in time.
- For \( k > 8 \) the lower bound on \( s \) was pushed down to \( s > 1 - \frac{2}{n} \) by Farah, Linares, and Pastor [7] in 2012. Since \( s_c = 1 - \frac{2}{n} \) is the critical regularity by scaling considerations, this result covers the whole subcritical range.
- Further progress on the local problem was reached by Ribaud and Vento [29] in 2012. Their results almost reached \( s_c \) for all \( k \geq 4 \), while for the quartic nonlinearity they assume \( s > \frac{5}{12} \).

Further results on \( \text{gZK-}k \) in two dimensions with data in weighted spaces were recently obtained by Fonseca and Pachon [8]. The author is not aware of any comparable results for \( k \geq 3 \) in the three dimensional case, where the critical regularity is \( s_c = \frac{n}{2} - \frac{1}{k} \). More generally we have

\[
s_c = s_c(n, k) = \frac{n}{2} - \frac{2}{k}.
\]

Roughly speaking, the method of proof is the same in all three papers [23], [7], and [29]. The authors adapt the strategy developed by Kenig, Ponce, and Vega in [14] in the KdV-context and apply a combination of local smoothing estimate, Strichartz inequality, and maximal function estimate in a contraction mapping argument. Here we shall pick up these ideas, push them down to the critical regularity and extend the arguments to the three dimensional case. Following Molinet-Ribaud [27] and especially in our method of proof Koch-Marzuola [16] in their works on gKdV, we consider data in the homogeneous Besov spaces

\[
\dot{B}^s_{2,q} = \{ u_0 \in Z' : \|u_0\|_{\dot{B}^s_{2,q}} \},
\]

where \( Z' \) is the dual space of

\[
Z = \{ f \in S : (D^\alpha F f)(0) = 0 \quad \text{for every multi-index} \quad \alpha \}.
\]

Here and below \( F \) denotes the Fourier transform. For \( q < \infty \) the Besov-norm is in general given by

\[
\|u_0\|_{\dot{B}^s_{p,q}} = \left( \sum_{N \in 2^n} \|P_N u_0\|_{L^p}^q \right)^{\frac{1}{q}},
\]

where \( P_N = F^{-1} \chi_{[|\xi| \sim N]} F \) are the Littlewood-Paley projections. A case of special interest is \( q = 2 \), where \( \dot{B}^s_{2,2} = \dot{H}^s \), the homogeneous Sobolev (or Riesz-potential) space.
For $q = \infty$ one has the usual modification $\|u_0\|_{\dot{B}^s_{p,\infty}} = \sup_{N \in 2^\mathbb{N}} \|P_N u_0\|_{L^p}$, and in this case (with $p = 2$) we will in addition assume for our data, that
\[
\lim_{N \to \infty} \|P_N u_0\|_{L^2} = \lim_{N \to 0} \|P_N u_0\|_{L^2} = 0.
\]
With $\dot{B}^s_{2,\infty}$ we will denote the closed subspace of all $u_0 \in \dot{B}^s_{2,\infty}$, for which these limits vanish. Then $Z$ is dense in $\dot{B}^s_{2,\infty}$. Without this additional assumption, several of our arguments break down, e.g. we loose the persistence property of the solution. Observe for $1 \leq \tilde{q} \leq 2 \leq q < \infty$ the inclusions
\[
\dot{B}^{s,\tilde{q}}_{2,1} \subset \dot{B}^{s,q}_{2,\tilde{q}} \subset \dot{B}^{s,\tilde{q}}_{2,q} \subset \dot{B}^{s,o}_{2,\tilde{q}} \subset \dot{B}^{s,o}_{2,\infty} \subset \dot{B}^{s,o}_{2,\infty},
\]
so for fixed $s_c$, on the fine scale of the $q$’s, the $\dot{B}^{s,o}_{2,\infty}$ is the largest data space we cope with. After these preparations we can state our

**Main result:** Let $n \in \{2,3\}$ and $k \geq 3$ an integer. Then the Cauchy problem (2) is locally well-posed for data in $\dot{B}^{s,c}_{2,q}$, if $q < \infty$, and in $\dot{B}^{s,o}_{2,\infty}$. Moreover, we have global well-posedness for small data in these spaces.

A more precise statement will follow at the end of Section 2. We remark already, that no smallness assumption is needed for the local part, but that - as usual in a critical case - the lifespan of the solutions cannot be controlled by the size of the data in their natural norm. To obtain the result, two main difficulties have to be overcome. The first is to prove a sharp global maximal function estimate or to find a substitute for this. In $2D$ we can solve this problem by symmetrizing the equation, see Section 3.1, especially Proposition 1 below, while in $3D$ a surprisingly soft argument allows us to circumvent this obstacle, see Section 5.1. The second problem is the missing generalized Leibniz rule in higher dimensional mixed Lebesgue spaces of type $L^p_x L^q_t$. This is solved by using the spaces $U^p(\mathbb{R}_x^2)$ and $V^p(\mathbb{R}_x^2)$ of $L^2$-valued functions of the time variable, which were introduced by Koch and Tataru in [17], [18], see also the exposition by Hadac, Herr, and Koch in [12] and Koch’s lecture [15]. Since the norms of these spaces depend on the size of the spatial Fourier transform, the “distribution” of derivatives on various factors can easily be handled. Some basics about these spaces, as far as needed here, are gathered in Section 2.

In proving the result, we can restrict ourselves to apply linear estimates for free solutions - no bilinear refinement of a Strichartz type inequality is used. For $k = 3$ this is astonishing, if we compare our results here with the theory for gKdV. Using linear estimates only, Kenig, Ponce, and Vega obtained well-posedness for gKdV-3 in $H^s(\mathbb{R})$ for $s \geq \frac{1}{12}$. To push this down to the critical regularity, a bilinear estimate for free solutions is needed, see the result in [9] by the author, which was later on improved by Tao [31] to the endpoint and by Koch-Marzuola [16] to critical Besov spaces. As our calculations show, linear estimates are sufficient in higher dimensions. Furthermore we remark that for the quartic nonlinearity in $2D$ our result closes a gap of $\frac{1}{12}$ derivatives between the existing LWP theory and the scaling heuristic.

**Acknowledgement:** The author is indebted to Herbert Koch and Sebastian Herr for numerous explanations about the function spaces $U^p$ and $V^p$. 
2. Function Spaces and Precise Statement of Results

Here we collect the necessary facts about the function spaces \( U^p \) and \( V^p \), respectively \( U^p_\omega \) and \( V^p_\omega \). For proofs and detailed descriptions we refer to the works \cite{12} and \cite{15}. We begin with the functions of bounded \( p \)-variation, which were (in the real valued case) introduced by Wiener in \cite{32}. Let \( I \subset \mathbb{R} \) be an interval and \( \mathcal{P}_I \) denote the system of all finite partitions \( P = \{t_0 < \cdots < t_K \} \subset I \) of \( I \). Here \( t_K = \infty \) is admitted, if \( I \) is unbounded to the right. For a function \( v : I \to L^2 \) the \( p \)-variation \( \omega_p(I,v) \) is defined by

\[
\omega_p(I,v) := \sup_{P \in \mathcal{P}_I} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2} \right)^{\frac{1}{p}},
\]

which, for \( 1 \leq p < \infty \), is a seminorm. Setting

\[
\|v\|_{V^p} := \max(\|v\|_{L^p(L^2)}, \omega_p(I,v))
\]

we get a norm on the linear space

\[
V^p(L^2) := \{v : I \to L^2 : \omega_p(I,v) < \infty \},
\]

which thereby becomes a B-space. Functions in \( V^p(L^2) \) are not necessarily continuous, but one sided limits always exist. The closed subspace of all right continuous functions in \( V^p(L^2) \) is denoted by \( V^p_\text{rc}(L^2) \). For \( 1 \leq p < q < \infty \) the embeddings

\[
V^p(L^2) \subset V^q(L^2) \subset L^\infty(L^2)
\]

are continuous. Closely related are the function spaces \( U^p(L^2) \), where again \( 1 \leq p < \infty \). Let \( P = \{t_0 < \cdots < t_K \} \) be a partition as above and \( \psi_1, \ldots, \psi_K \in L^2 \). Then the step function

\[
a = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)} \psi_k
\]

is called a \( U^p \)-atom, if \( \sum_{k=1}^{K} \|\psi_k\|_{L^p}^p = 1 \). One says that \( u \in U^p(L^2) \), if there exist sequences \( (\lambda_j)_{j \in \mathbb{N}} \in \ell^1(\mathbb{N}) \) and \( (a_j)_{j \in \mathbb{N}} \) of \( U^p \)-atoms, so that \( u = \sum_{j=1}^{\infty} \lambda_j a_j \). These functions constitute a linear space, which endowed with the norm

\[
\|u\|_{U^p} := \inf \{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j \}
\]

becomes a B-space. If \( 1 \leq p < q < \infty \) the embeddings

\[
U^p(L^2) \subset U^q(L^2) \subset L^\infty(L^2)
\]

are continuous. \( U^p \)-functions are continuous from the right. These two scales of function spaces are tied by continuous embeddings. Assume once more \( 1 \leq p < q < \infty \). Then we have

\[
U^p(L^2) \subset V^p(L^2) \quad \text{and} \quad V^p_\text{rc}(L^2) \subset U^q(L^2).
\]

Comparing with Besov-norms (of \( L^2 \)-valued functions) we have the inequalities

\[
\|v\|_{B^p_{p,\infty}} \lesssim \|v\|_{V^p} \quad \text{and} \quad \|u\|_{U^p} \lesssim \|u\|_{B^p_{p,1}}^{\frac{1}{p}}.
\]

Apart from the embeddings above, the \( U^p \)'s and \( V^p \)'s are connected by duality. In fact for \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) we can identify

\[
(U^p(L^2))' \simeq V^q(L^2),
\]
where the dual pairing \( B : U^p(L^2_x) \times V^p(L^2_x) \to \mathbb{C} \) is given by a generalized Stieltjes integral
\[
B(u, v) = \int_t (u, dv) = \int_t (u, \frac{dv}{dt}) dt,
\]
the latter, if \( v \) has a locally integrable weak derivative. We refer to Sections 3.5, 3.8, and 3.10 of [15] for details on the duality between \( U^p \) and \( V^p \).

Now Bourgain’s construction of the \( X^{s,b} \)-spaces is repeated. Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be a phase function and \( (U_\varphi(t))_{t \in \mathbb{R}} \) the unitary group associated with the linear equation \( u_t = i\varphi(-i\nabla)u \). Then one defines
\[
V^p_\varphi = U_\varphi V^p(L^2_x) \quad \text{and} \quad U^p_\varphi = U_\varphi U^p(L^2_x)
\]
with norms \( \|v\|_{V^p} = \|U_\varphi(-\cdot)v\|_{V^p} \) and \( \|u\|_{U^p} = \|U_\varphi(-\cdot)u\|_{U^p} \). As for Bourgain’s spaces, an immediate consequence of this definition is the equality \( \|U_\varphi u_0\|_{V^p} = \|u_0\|_{L^2_x} \), which we shall frequently use. The duality [14] gives us the estimate
\[
\| \int_0^t U_\varphi(t-s)F(s)ds \|_{V^p_q} = \sup_{\|w\|_{L^p} \leq 1} \left| \int \int F(x,t)w(x,t)dxdt \right|
\]
for the solution of the inhomogeneous linear equation, cf. Lemma 3.33 in [15]. As the \( X^{s,b} \)-spaces, the spaces \( U^p_\varphi \) admit a transfer principle. A Strichartz type estimate
\[
\|U_\varphi u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2_x} \quad \text{implies} \quad \|u\|_{L^q_t L^r_x} \lesssim \|u\|_{U^p_\varphi},
\]
if the order of integration is reversed, we have
\[
\|U_\varphi u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2_x} \quad \text{implies} \quad \|u\|_{L^q_t L^r_x} \lesssim \|u\|_{U^p_\varphi},
\]
where \( r = \min(p,q) \). The \( U^p_\varphi \)- and \( U^p_\varphi \)-norms on the right can be further estimated in \( V^p_\varphi \), provided \( p > 2 \) and \( r > 2 \), due to the continuous embedding \( V^p_\varphi \subset U^{2+} \). A multilinear version of the transfer principle holds true as well, see [12, Proposition 2.19], but we will not make use of it here.

We now take \( I = \mathbb{R} \) and specify \( \varphi \) to the phase function
\[
\phi : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}, \quad (\xi, \eta) \mapsto \phi(\xi, \eta) = \xi(\xi^2 + |\eta|^2)
\]
corresponding to the linear part of ZK. For \( 1 \leq q < \infty \) we introduce
\[
\|u\|_{X^q_\varphi} := \left( \sum_{N \in \mathbb{Z}} N^{sq} \|P_N u\|_{V^q_\varphi}^q \right)^{\frac{1}{q}},
\]
which we modify for \( q = \infty \) in the usual way, i. e.
\[
\|u\|_{X^\infty_\varphi} := \sup_{N \in \mathbb{Z}} N^s \|P_N u\|_{V^q_\varphi}.
\]
Then we define the \( B \)-spaces
\[
X^q_\varphi := \{ u \in C(\mathbb{R}, \dot{B}^s_{2,q}) : \|u\|_{X^q_\varphi} < \infty \},
\]
if \( 1 \leq q < \infty \), and
\[
\dot{X}^s_\varphi := \{ u \in C(\mathbb{R}, \dot{B}^s_{2,\infty}) : \|u\|_{X^\infty_\varphi} < \infty, \lim_{N \to \infty} N^s \|P_N u\|_{V^q_\varphi} = \lim_{N \to 0} N^s \|P_N u\|_{V^q_\varphi} = 0 \}.
\]
By the limit conditions the latter is adapted to our data space \( \dot{B}^{s,o}_{2,\infty} \). We emphasize that here and below the Littlewood-Paley projections are always applied with respect to all space variables, which we can fix in the form
\[
P_N = F^{-1}_{xy} \chi((\xi,\eta) \sim N) F_{xy}.
\]
If in the above the real axis is replaced by a time interval \( I = [0, T) \), we write \( \mathcal{X}_{q,T}^s \) (instead of \( \mathcal{X}_q^s \)), which are for \( s = s_c \) our solution spaces. Here \( T = \infty \) is admitted for the global result. In the proof, and already to make our local statement precise, we need an auxiliary norm, which depends on \( k \) and on the space dimension. It is motivated by the linear estimates we shall use. Let \( I_x \) and \( I_y \) denote the Riesz potential operators of order \(-1\) with respect to \( x \in \mathbb{R} \) and \( y \in \mathbb{R}^{n-1} \), respectively.

- For \( n = 2 \) we define \( K(I_x, I_y)^q = F_{xy}^{-1} |3\xi^2 - \eta^2|^q F_{xy} \) and set
  \[
  |P_N u|_{(k)} := N^{s_c} |K(I_x, I_y)^q P_N u|_{L_a^4 \times t} + N^{s_c} |P_N u|_{L_a^4 \times t} + N^{s_c} \| P_N u \|_{L^4_y \times t},
  \]
  where \( s_c = 1 - \frac{2}{q} \).

- For \( n = 3 \) we have \( s_c = \frac{2}{3} - \frac{2}{q} \) and define
  \[
  |P_N u|_{(k)} := N^{s_c} |I_a^{4/3} P_N u|_{L_{x\times t}^8} + N^{s_c} |P_N u|_{L_{x\times t}^8} + N^{s_c} \| P_N u \|_{L_x^4 \times t},
  \]

From these we build up the Besov type norms \( |u|_{(k,q)} = (\sum_{N \in \mathbb{Z}} |P_N u|_{(k)}^q)^{1/q} \), with the usual modification for \( q = \infty \). If the time interval is \([0,T)\) instead of \( \mathbb{R} \), we write \( |P_N u|_{(k,T)} \) and \( |u|_{(k,q,T)} \), respectively. The linear estimates in Sections 4.2 and 5.1 imply via the transfer principle that
\[
|P_N u|_{(k)} \lesssim N^{s_c} |P_N u|_{V^2_q} \quad \text{and hence} \quad |u|_{(k,q)} \lesssim |u|_{X^s_q}.
\]

Since all Hölder exponents in \( |\cdot|_{(k)} \) are finite, this has the consequence, that for all \( u \in \mathcal{X}_q^s \) we have \( \lim_{T \to \infty} |u|_{(k,q,T)} = 0 \). For \( q = \infty \) this is due to our assumption \( \lim_{N \to \infty} N^s |P_N u|_{V^2_q} = \lim_{N \to 0} N^s |P_N u|_{V^2_q} = 0 \) in the definition of \( \mathcal{X}_q^s \). Now our main result takes the following shape.

**Theorem 1.** Let \( n \in \{2,3\} \), \( k \geq 3 \) an integer and \( s_c = \frac{2}{q} - \frac{2}{q} \). Assume \( u_0 \in B_{2,q}^{s_c} \), if \( q < \infty \), or \( u_0 \in B_{2,\infty}^{s_c} \). Then

1. there exists a \( T > 0 \) and a unique solution \( u \in \mathcal{X}_{q,T}^s \) of (2) with \( u(0) = u_0 \). Moreover, there exists a constant \( C = C(k,q) > 0 \), so that the lifespan of solutions can be chosen uniformly equal to \( T \) on the subset
   \[
   D_T := \{ u_0 : |\mathcal{U}_s u_0|_{(k,q,T)} \leq (4C)^{-k} \}
   \]
   of the data space, and the map
   \[
   S_T : D_T \to \mathcal{X}_{q,T}^s, \quad u_0 \mapsto S_T u_0 := u
   \]
   (data upon solution) is Lipschitz continuous.

2. there exists \( \varepsilon = \varepsilon(k,q) > 0 \) such that, if \( |u_0|_{B_{2,q}^{s_c}} \leq \varepsilon \), there exists a unique global solution \( u \in \mathcal{X}_{q,\infty}^s \) of (2) with \( u(0) = u_0 \). The solution map \( S_\infty \) is Lipschitz continuous from the ball \( B_2 := \{ u_0 : |u_0|_{B_{2,q}^{s_c}} \leq \varepsilon \} \) (contained in the data space) into \( \mathcal{X}_{q,\infty}^s \).

3. Symmetrization and linear estimates in two space dimensions

In [11] Section 2.1 we observed, that in two space dimensions the Zakharov-Kuznetsov equation can be symmetrized by a linear change of the space variables. For that purpose we fixed \( \mu := 4^{-\frac{1}{4}} \), \( \lambda := \sqrt{3} \mu \) and introduced
\[
R : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (x',y') := (\mu x + \lambda y, \mu x - \lambda y)
\]
as well as \( R_v := v \circ R_0 \). Let \( u = R_v \). Then \( u \) is a solution of the Zakharov-Kuznetsov equation
\[
(5) \quad \partial_t u + \partial_x (\partial_x^2 + \partial_y^2) u = c_0 \partial_x u^{k+1}
\]
with initial condition \( u(0) = u_0 \), if \( v \) solves
\[
(6) \quad \partial_t v + (\partial_x^2 + \partial_y^2)v = \mu c_0 (\partial_x + \partial_y)e^{k+1} \quad \text{and} \quad v(0) = R^{-1}u_0.
\]
For more details, see [11]. The exponent 2 in that paper can be replaced without any other changes by \( k + 1 \). For the study of linear estimates we may choose \( c_0 = 0 \) in (5) and (6). The map \( R \) introduced above defines an isomorphism on any of the spaces \( H^s(\mathbb{R}^2) \), \( \dot{H}^s(\mathbb{R}^2) \), \( B_{p,q}^s(\mathbb{R}^2) \), \( \dot{B}_{p,q}^s(\mathbb{R}^2) \) and on all mixed Lebesgue-spaces of the types \( L_{x,y}^p \), \( L_t^r \) and \( L_{x,y}^p L_t^r \). Thus especially the well-posedness theory remains unchanged if we pass over from (5) to (6).

3.1. Estimates for the linear part of the symmetrized equation, the maximal function estimate.

For the solution of (6) with \( c_0 = 0 \) and initial datum \( v_0 \) we write \( U_{\varphi_{sym}}(t)v_0 \). With familiar notation we have \( U_{\varphi_{sym}}(t) = e^{-t(\partial_x^2 + \partial_y^2)} \) or, by using the Fourier transform in the space variables,
\[
U_{\varphi_{sym}}(t)v_0(x,y) = \int_{\mathbb{R}^2} e^{i(x\xi + y\eta + t\varphi_{sym}(\xi,\eta))} \hat{v}_0(\xi,\eta) d\xi d\eta,
\]
where \( \varphi_{sym}(\xi,\eta) = \xi^3 + \eta^3 \) is the phase function associated with the symmetrized linear problem (i.e. (6) with \( c_0 = 0 \) and initial datum \( v_0 \)). The main advantage of the symmetrization is, that it allows us to obtain the following maximal function estimate, which is global in time and avoids any technical loss of derivatives.

**Proposition 1.** Let \( v_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \) and \( f \in L_{x,y}^{\frac{1}{4}} L_t^1(\mathbb{R}^3) \). Then the estimates
\[
(7) \quad \| (I_x I_y)^{-\frac{1}{2}} \int_0^t U_{\varphi_{sym}}(t-s)f(\cdot,s)ds \|_{L_{x,y}^4 L_t^\infty} \lesssim \| f \|_{L_{x,y}^{\frac{1}{4}} L_t^1}
\]
and
\[
(8) \quad \| U_{\varphi_{sym}}v_0 \|_{L_{x,y}^2 L_t^\infty} \lesssim \| (I_x I_y)^{\frac{1}{4}} v_0 \|_{L_{x,y}^2}
\]
hold true.

**Proof.** We use and follow the arguments in [14] Section 3]. From Lemma 3.6 in that reference we know the estimate
\[
\left| \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^2)} \frac{d\xi}{|\xi|^\frac{1}{2}} \right| \lesssim |x|^{-\frac{1}{4}},
\]
where the oscillatory integral should be understood as
\[
\lim_{\varepsilon \to 0, \varepsilon > 0} \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^2 - \varepsilon^2)} \frac{d\xi}{|\xi|^\frac{1}{2}}.
\]
This interpretation allows us to use Fubini’s theorem for a double integral of this kind, and we obtain
\[
\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x\xi + y\eta + t(\xi^2 + \eta^2))} \frac{d\xi d\eta}{|\xi| |\eta|^\frac{1}{2}} \right| \lesssim |xy|^{-\frac{1}{2}}.
\]
Thus we have
\[
\left| \int_{-\infty}^{\infty} (I_x I_y)^{-\frac{1}{2}} U_{\varphi_{sym}}(t-s)f(x,y,s)ds \right| \lesssim |xy|^{-\frac{1}{2}} \ast \int_{-\infty}^{\infty} |f(x,y,s)| ds.
\]

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\(^1\)We also refer to the systematic treatise on linear transformations in connection with dispersive estimates for third order equations in two space dimensions by Ben-Artzi, Koch, and Saut, in [4].
Corollary 1. Let \( p \) here. Interpolation of (8) and the \( f \)
Replacing (11)
\[
\| (I_s I_y) \frac{1}{2} U_{\varphi_{\text{sym}}} (t - s) f (\cdot, s) ds \|_{L^2_y} \lesssim \| f \|_{L^1_x L^1_t}.
\]
Following this, we obtain as a special case \( (8) \).
\( \square \)

Next we recall the Strichartz estimates with derivative gain. From [13, Theorem 3.1] we obtain as a special case
\[
\| (I_s I_y) \frac{1}{2} U_{\varphi_{\text{sym}}} v_0 \|_{L^2_x L^1_y} \lesssim \| v_0 \|_{L^2_y},
\]
provided \( 2 < p \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{3}{4} \). As usual, the endpoint case \( p = 2 \) is excluded here. Interpolation of \( [3] \) and the \( p = q = 4 \)-case of (10) leads to

**Corollary 1.** Let \( 4 \leq r \leq \infty \) and \( \sigma = \frac{1}{4} - \frac{3}{2r} \). Then the estimate
\[
\| U_{\varphi_{\text{sym}}} v_0 \|_{L^r_x L^\sigma_y} \lesssim \| (I_s I_y) \frac{1}{2} v_0 \|_{L^2_y},
\]
holds true. Moreover we have for \( 6 \leq r \leq \infty \) and \( s = \frac{1}{2} - \frac{3}{r} \)
\[
\| U_{\varphi_{\text{sym}}} v_0 \|_{L^r_x L^s_y} \lesssim \| v_0 \|_{H^2_y}.
\]

In order to control the derivative in the nonlinearity we will use the sharp version of the local smoothing effect, which – by the product structure of \( U_{\varphi_{\text{sym}}} (t) = e^{-it\varphi} e^{-it\tilde{\varphi}} \) – can be easily deduced from the one of the Airy equation. For that purpose we fix both space variables \( x \) and \( y \) and recall from the proof of [14, Theorem 3.5] the identity
\[
\| I_s e^{-it\varphi} v_0 (x, y) \|_{L^2_t L^2_x}^2 = c^2 \| v_0 \|_{L^2_y}^2.
\]
Integration with respect to \( y \) gives
\[
\| I_s e^{-it\varphi} v_0 (x, \cdot) \|_{L^2_t L^2_x}^2 = c \| v_0 \|_{L^2_y}^2,
\]
which combined with the unitarity of \( e^{-it\varphi} \) on \( L^2_y \) leads to
\[
\| I_s e^{-it(\partial_x^3 + \partial_x^3)} v_0 (x, \cdot) \|_{L_y^2 L_x^2}^2 = \| I_s e^{-it\varphi} v_0 (x, \cdot) \|_{L_y^2 L_x^2}^2 = c \| v_0 \|_{L^2_y}^2.
\]
Thus and by symmetry we have shown the identities
\[
\| I_s U_{\varphi_{\text{sym}}} v_0 \|_{L^\infty_x L^2_y} = c \| v_0 \|_{L^2_y},
\]
and
\[
\| I_y U_{\varphi_{\text{sym}}} v_0 \|_{L^\infty_x L^2_y} = c \| v_0 \|_{L^2_y}.
\]
This smoothing effect is relatively weak in so far, as we get control over \( I_s \) or \( I_y \), respectively, but not over the full gradient in both space variables, compare with [15] below. This might be seen as the price to pay for the symmetrization.
3.2. Linear estimates for the original ZK equation in 2 D.

In the subsequent analysis we will apply the linear estimates of the previous section to treat the symmetrized \( \phi \)ZK equation \([9]\). Nonetheless it may be of interest to compare them with known linear estimates for the original ZK equation, and especially to trace back the Strichartz- and maximal function estimates by the aid of the transformation \( R \) from above. So let \( \phi(\xi, \eta) = \xi(\xi^2 + \eta^2) \) be the phase function and \( U_\phi(t) \) the propagator associated with (the linear part of) equation \([9]\).

The local smoothing estimate
\[
\| I U_\phi u_0 \|_{L^3_y L^6_x} \lesssim \| u_0 \|_{L^2_y}
\]
was shown in this case by Faminskii, see \([5\text{, Theorem 2.2}]\). Here \( I \) denotes the Riesz potential operator of order \(-1\) with respect to \( x \) and \( y \), so there is the in comparison with \([13]\) stronger gain of the full gradient in this estimate. (Since \([13]\) is sharp and in view of the transformation \( R \), this difference seems somewhat surprising – at least it was for the author. But it merely reflects the fact that \( R \) is not well-behaved as a mapping on mixed Lebesgue spaces of type \( L^p_x L^q_y \), if \( p \neq q \).)

To convert the Strichartz- and maximal function estimates for \( v \) into estimates for \( u = Rv = v \circ R_0 \) in terms of \( u_0 = v_0 \circ R_0 \), we apply the Fourier transform to the last identity and obtain \( \hat{u}_0 = (\det R_0)^{-1} \hat{v}_0 \circ (R_0^\top)^{-1} \), hence
\[
\hat{v}_0(\xi', \eta') = |(\det R_0)| \hat{u}_0 \circ R_0^\top (\xi', \eta').
\]
We set \( (\xi, \eta) = R_0^\top (\xi', \eta') \) and multiply both sides by \( |\xi'\eta'|^\sigma = c_\sigma |3\xi^2 - \eta^2|^\sigma \). Then
\[
|\xi'\eta'|^\sigma \hat{v}_0(\xi', \eta') = c_\sigma |3\xi^2 - \eta^2|^\sigma \hat{u}_0(\xi, \eta)
\]
which can be squared and integrated with respect to \( d\xi'd\eta' = |(\det R_0)^{-1}d\xi d\eta| \). With the Fourier multiplier
\[
K(I_x, I_y)^\sigma := F_{xy}^{-1} |3\xi^2 - \eta^2|^\sigma F_{xy}
\]
we then have
\[
\| (I_x I_y)^\sigma v_0 \|_{L^2_y} = c(K(I_x, I_y)^\sigma u_0) \|_{L^2_y}.
\]
This gives the following Strichartz type inequality for \( u \).
\[
\| K(I_x, I_y)^{\frac{1}{2}} U_\phi u_0 \|_{L^3_x L^6_y} \lesssim \| u_0 \|_{L^2_y},
\]
provided \( 2 < p \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \). If \( p = q = 4 \), we can recognize this as the special case of (the dual estimate to) Theorem 1.1 in \([14\text{ by Carbery, Kenig, and Ziesler, which has been applied by Molinet and Pilod in their work } 24\text{ on the ZK equation, cf. Proposition 3.5 in that paper. In fact, our simple considerations here give a wider range of validity with a stronger gain of derivatives, if } p \to 2. \text{ Again the endpoint } p = 2 \text{ is excluded.}

Similarly, we have the maximal function estimate
\[
\| U_\phi u_0 \|_{L^p_y L^\infty_x} \lesssim \| K(I_x, I_y)^{\frac{1}{2}} u_0 \|_{L^1_y} \lesssim \| u_0 \|_{H^1_y}^{\frac{1}{2}}.
\]
A Sobolev embedding in the \( y \)-Variable gives
\[
\| U_\phi u_0 \|_{L^2_y L^\infty_x} \lesssim \| K(I_x, I_y)^{\frac{1}{2}} J_y^{\frac{1}{2}} u_0 \|_{L^2_y},
\]
which is comparable with Proposition 1.5 in \([22\text{ by Linares and Pastor. see also Corollary 2.7 of } 23\text{. The advantage here is, that } 18\text{ holds globally in time.} \]
4. Discussion of the 2D Case

Throughout this section we have \( n = 2 \) and hence \( s_c = 1 - \frac{2}{7}. \) \( U^p_{\varphi_{sym}} \) and \( V^p_{\varphi_{sym}} \) will denote the \( U^p \) - and \( V^p \) - spaces associated with the phase function
\[
\varphi_{sym} : \mathbb{R}^2 \to \mathbb{R}, \quad (\xi, \eta) \mapsto \varphi_{sym}(\xi, \eta) = \xi^3 + \eta^3.
\]
The solution space \( X^s_q \) is that one with norm built on \( V^2_{\varphi_{sym}}. \)

4.1. The Central Multilinear Estimate.

We will start with a multilinear estimate on dyadic pieces of functions \( v_1, \ldots, v_{k+1} \in V^2_{\varphi_{sym}}. \) We recall the quantity
\[
|P_N v|_{(k)} := N^{s_c} \| (I_{\xi} I_{\eta}) \frac{1}{2} P_N v \|_{L^2_{\xi \eta}} + N^{s_c} \| P_N v \|_{L^1_{\xi \eta} L^2_{\xi} L^\infty_{\eta}} + N^{s_c} \| P_N v \|_{L^1_{\xi \eta} L^1_{\eta} L^4_{\xi}}.
\]
We remark that by the transfer principle and the linear estimates (11) and (12) all three contributions are bounded by the \( V^2_{\varphi_{sym}} \) - norm, more precisely we have
\[
|P_N v|_{(k)} \lesssim N^{s_c} \| P_N v \|_{V^2_{\varphi_{sym}}}.
\]
If the mixed norms are replaced by \( L^4_{xy} L^\infty_T, \) which means that the integration is restricted to the time interval \((0, T), \) we write \( |P_N v|_{(k, T)} \) for the corresponding composed quantity. Here we may have \( T = \infty. \) Observe that \( \lim_{T \to 0} |P_N v|_{(k, T)} = 0, \) whenever \( v \in V^2_{\varphi_{sym}}. \)

Lemma 1. Let \( v_1, \ldots, v_{k+1} \in V^2_{\varphi_{sym}}, w \in U^p_{\varphi_{sym}} \) with \( \| w \|_{U^p_{\varphi_{sym}}} \leq 1, N, N_1, \ldots, N_{k+1} \) dyadic numbers with \( N_1 \leq N_2 \leq \cdots \leq N_{k+1} \) and \( N \lesssim N_{k+1}. \) Then there exists \( \varepsilon > 0 \) such that
\[
N^{s_c} \int_{\mathbb{R}^3} P_{N_1} v_1 \cdot \ldots \cdot P_{N_{k+1}} v_{k+1} \cdot \partial_x P_N w dx dy dt \lesssim N^{\varepsilon} N_1^{s_c} N_k^{2s_c} \prod_{j=1}^{k+1} |P_{N_j} v_j|_{(k)}.
\]
The same holds true, if \( \partial_x \) is replaced by \( \partial_y. \)

Proof. We consider three cases, depending on the relative sizes of the spatial frequencies \( (\xi_{k+1}, \eta_{k+1}) \) and \( (\xi, \eta) \) of \( v_{k+1} \) and \( w, \) respectively. Observe that by our assumptions \( |(\xi, \eta)| \lesssim |(\xi_{k+1}, \eta_{k+1})|. \) In the sequel, let \( \varepsilon \) be a positive number, which has to be chosen sufficiently small in dependence of \( k. \)

Case 1: \( |\eta_{k+1}| \lesssim |\xi_{k+1}|. \) Here we may replace the factor \( P_{N_{k+1}} v_{k+1} \) by \( N^{-\varepsilon} I^2_{\xi} P_{N_{k+1}} v_{k+1}. \)

Case 2: \( |\xi_{k+1}| \ll |\eta_{k+1}| \) and \( |\xi| \lesssim |\eta|. \) Here we may replace \( (P_{N_{k+1}} v_{k+1}) (\partial_x P_N w) \) by \( (N_{k+1}^{\varepsilon} I^2_{\xi} P_{N_{k+1}} v_{k+1})(\partial_y P_N w)) \) and argue as in Case 1 with the roles of \( x \) and \( y \) (respectively of the \( \xi \)'s and \( \eta \)'s) interchanged.

Case 3: \( |\xi_{k+1}| \ll |\eta_{k+1}| \) and \( |\eta| \lesssim |\xi|. \) Since in this case \( |\xi| \lesssim N_{k+1} \sim |\eta_{k+1}|, \) we have \( |\eta| \ll |\eta_{k+1}|. \) By the convolution constraint \( \sum_{j=1}^{k+1} |\eta_j| = |\eta|, \) there exists at least one \( j \in \{1, \ldots, k\} \) with \( |\eta_j| \sim |\eta_{k+1}|. \) This implies \( N_K \sim N_{k+1} \) and especially \( N \lesssim N_k. \)

Treatment of Case 1: The contribution from this case is bounded by
\[
N^{\varepsilon} N_k^{2s_c} \int_{\mathbb{R}^3} P_{N_1} v_1 \cdot \ldots \cdot P_{N_k} v_k \cdot (I^2_{\xi} P_{N_{k+1}} v_{k+1}) \cdot \partial_x P_N w dx dy dt \lesssim N^{\varepsilon} N_k^{2s_c} \| P_{N_1} v_1 \cdot \ldots \cdot P_{N_k} v_k \cdot (I^2_{\xi} P_{N_{k+1}} v_{k+1}) \|_{L^1_{\xi} L^2_{\eta} L^\infty_{\xi}} \| \partial_x P_N w \|_{L^\infty_{\xi} L^2_{\eta}},
\]
where by the local smoothing effect \((13)\) and the transfer principle the last factor is bounded by \(\|P_N v\|_{L^2_{x,yz}} \leq 1\). For \(\varepsilon > 0\) sufficiently small we choose
\[
\frac{1}{p_j} = \frac{3}{4k}, \quad \frac{1}{q_j} = \frac{1}{4k} + \varepsilon, \quad \frac{1}{r_j} = \frac{1}{3k - \frac{2\varepsilon}{3}},
\]
and, for \(j \in \{2, \ldots, k\}\),
\[
\frac{1}{p_j} = \frac{3}{4k}, \quad \frac{1}{q_j} = \frac{1}{4k}, \quad \frac{1}{r_j} = \frac{1}{3k},
\]
as well as
\[
\frac{1}{p_{k+1}} = \frac{1}{4}, \quad \frac{1}{q_{k+1}} = \frac{1}{4} - \varepsilon, \quad \frac{1}{r_{k+1}} = \frac{1}{6} + \frac{2\varepsilon}{3}.
\]
Since \(k \geq 3\), we have \(p_j \geq 4\) and \(q_j \geq 4\) for all \(j \in \{1, \ldots, k + 1\}\). Moreover
\[
\sum_{j=1}^{k+1} \frac{1}{p_j} = 1 \quad \text{and} \quad \sum_{j=1}^{k+1} \frac{1}{q_j} = \sum_{j=1}^{k+1} \frac{1}{r_j} = \frac{1}{2},
\]
so that Hölder’s inequality gives the upper bound
\[
N^2 N_{k+1}^2 \left( \prod_{j=1}^{k} \|P_N v_j\|_{L_{x,yz}^{p_j} L_{xy}^{q_j} L_{iz}^{r_j}} \right) N_{k+1}^2 \|P_N v_{k+1} \|_{L_{x,yz}^{p_{k+1}} L_{xy}^{q_{k+1}} L_{iz}^{r_{k+1}}}.
\]

For the \(v_1\) - factor we use Sobolev embeddings in the space variables to obtain
\[
\|P_N v_1 \|_{L_{x,yz}^{p_1} L_{xy}^{q_1} L_{iz}^{r_1}} \lesssim N_1^{\frac{1}{2} - \frac{1}{4} - \varepsilon} \|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}} = N_1^{\frac{1}{2} - \frac{1}{4} - 2\varepsilon} \|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}}.
\]
We choose \(\theta\) so that \(\frac{1}{p_1} = \frac{3}{4} + \frac{\theta}{4}\). Then Lyapunov’s inequality gives the bound
\[
N_1^{\frac{1}{2} - \frac{1}{4} - 2\varepsilon} \|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}} \leq \left( N_1^\theta \|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}} \right)^{1-\theta} \left( N_1^{\frac{1}{2} - \frac{1}{4} - 2\varepsilon} \|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}} \right)^{\theta},
\]
which in turn is dominated by
\[
N_1^\theta \|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}} + N_1^{\frac{1}{2} - \frac{1}{4} - 2\varepsilon} \|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}} \lesssim \|P_N v_1 \|_{(k)}.
\]
Collecting terms we obtain
\[
\|P_N v_1 \|_{L_{x,y}^{1} L_{iz}^{1}} \lesssim N_1 \|P_N v_1 \|_{(k)},
\]
and, taking \(\varepsilon = 0\) in this calculation for \(v_1\), we as well have for \(j \in \{2, \ldots, k\}\) that
\[
\|P_N v_j \|_{L_{x,y}^{p_j} L_{xy}^{q_j} L_{iz}^{r_j}} \lesssim \|P_N v_j \|_{(k)}.
\]
For the last factor we use a Sobolev embedding with respect to the \(y\) - variable and a convexity inequality (like Lyapunov’s inequality above) to obtain
\[
\|P_N v_{k+1} \|_{L_{x,y}^{p_{k+1}} L_{xy}^{q_{k+1}} L_{iz}^{r_{k+1}}} \lesssim \left\| (I_{x} I_{y})^\frac{1}{2} P_{N_{k+1}} v_{k+1} \right\|_{L_{x,y}^{q_{k+1}} L_{iz}^{r_{k+1}}},
\]
so that
\[
N_{k+1}^\frac{1}{2} \|P_N v_{k+1} \|_{L_{x,y}^{p_{k+1}} L_{xy}^{q_{k+1}} L_{iz}^{r_{k+1}}} \lesssim \|P_{N_{k+1}} v_{k+1} \|_{(k)}.
\]
Summarizing the estimates for the single factors, we see that \((19)\) is in fact bounded by
\[
N^2 N_{k+1}^2 \prod_{j=1}^{k+1} \|P_N v_j \|_{(k)},
\]
as desired.

**Estimation for Case 3:** Since \(N_k \sim N_{k+1}\) here, the contribution is bounded by
\[
N^2 N_{k+1}^2 N_{k+1}^{-2\varepsilon} \|P_N v_1 \|_{(k)} \cdots \|P_{N_{k+1}} v_{k+1} \|_{L_{x,y}^{1} L_{iz}^{1}}.
\]
For $j \in \{1, \ldots, k-1\}$ we choose Hölder exponents $p_j$, $q_j$, and $r_j$ precisely as in Case 1. Moreover we set

$$\frac{1}{p_k} = \frac{3}{4k}, \quad \frac{1}{q_k} = \frac{1}{4k} - \varepsilon, \quad \frac{1}{r_k} = \frac{1}{3k} + \frac{2\varepsilon}{3},$$

as well as

$$\frac{1}{p_{k+1}} = \frac{1}{4}, \quad \frac{1}{q_{k+1}} = \frac{1}{4}, \quad \frac{1}{r_{k+1}} = \frac{1}{6}.$$

Then again we have $p_j \geq 4$ and $q_j \geq 4$ for all $j \in \{1, \ldots, k+1\}$ and $\sum_{j=1}^{k+1} \frac{1}{p_j} = 1$ as well as $\sum_{j=1}^{k+1} \frac{1}{q_j} = \frac{1}{2}$. Hölder’s inequality gives the only slightly different upper bound

$$(20) \quad N^\varepsilon N_{k+1}^{-2\varepsilon} \left( \prod_{j=1}^{k} \left\| P_{N_j} v_j \right\|_{L_{x}^{p_j} L_{y}^{q_j} L_{t}^{r_j}} \right) N_{k+1}^{\varepsilon} \left\| P_{N_{k+1}} v_{k+1} \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}}.$$

From the estimates concerning Case 1 we already know that

$$(21) \quad \prod_{j=1}^{k-1} \left\| P_{N_j} v_j \right\|_{L_{x}^{p_j} L_{y}^{q_j} L_{t}^{r_j}} \lesssim N_{k} \prod_{j=1}^{k-1} \left| P_{N_j} v_j \right|_{(k)}.$$

Moreover it is clear by our choices, that

$$N_{k+1}^{\varepsilon} \left\| P_{N_{k+1}} v_{k+1} \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}} \leq \left| P_{N_{k+1}} v_{k+1} \right|_{(k)},$$

and it remains to estimate the factor for $j = k$. Sobolev embeddings in $x$ and $y$ give

$$\left\| P_{N_k} v_k \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}} \lesssim N_{k}^{\frac{1}{2} - \frac{1}{p_k} + \varepsilon} \left\| P_{N_k} v_k \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}} = N_{k}^{\frac{1}{2} - \frac{1}{p_k} + 2\varepsilon} \left\| P_{N_k} v_k \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}}.$$

Applying Lyapunov’s inequality again we obtain (replace $\varepsilon$ by $-\varepsilon$ in the corresponding argument for $v_1$ in Case 1)

$$N_{k}^{\frac{1}{2} - \frac{1}{p_k} + 2\varepsilon} \left\| P_{N_k} v_k \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}} \lesssim N_{k}^{\varepsilon} \left\| P_{N_k} v_k \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}} + N_{k}^{-\varepsilon} \left\| P_{N_k} v_k \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}} \lesssim \left| P_{N_k} v_k \right|_{(k)},$$

so that

$$(22) \quad \left\| P_{N_k} v_k \right\|_{L_{x}^{p_k} L_{y}^{q_k} L_{t}^{r_k}} \lesssim N_{k}^{-\varepsilon} \left| P_{N_k} v_k \right|_{(k)}.$$

The comparison of (21) and (22) shows, that we have successfully exchanged the large factor $N_{k}^{\varepsilon}$ by the smaller $N_{k}^{-\varepsilon}$. Alltogether

$$\text{(20)} \quad \lesssim N_{1}^{\varepsilon} N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} \left| P_{N_j} v_j \right|_{(k)}.$$

This completes the estimation in Case 3.

The statement about $\partial_y$ instead of $\partial_x$ is obvious by symmetry. □

The next step is to sum up these estimates on dyadic pieces, which will necessarily involve the auxiliary norms

$$|v|_{(k,q)} = \left( \sum_{N \in \mathbb{Z}} |P_{N}v|_{(k)} \right)^{\frac{1}{q}}.$$
with the usual modification for $q = \infty$. (We write $\|v\|((k,q,T))$, if these norms are assembled from $(|P_N v|((k,T))|_{N \in \mathbb{Z}^d}$.) They remain finite for $v \in X^{s_r}_q$ (or $v \in X^{s_{q,T}}_q$, respectively). For $v_1, \ldots, v_{k+1} \in X^{s_{q,T}}_q$ we introduce

$$F(v_1, \ldots, v_{k+1})(t) := \int_0^t U_{\phi_{sym}}(t - s)(\partial_x)(v_1 \cdot \ldots \cdot v_{k+1})(s)ds,$$

the dependence on $x$ and $y$ was suppressed here.

**Lemma 2.** For $v_1, \ldots, v_{k+1} \in X^{s_{q,T}}_q$ we have $F(v_1, \ldots, v_{k+1}) \in X^{s_{q,T}}_q$ and the estimate

$$\|F(v_1, \ldots, v_{k+1})\|_{X^{s_{q,T}}_q} \lesssim \prod_{j=1}^{k+1} \|v_j\|_{(k,q)}$$

holds true.

Remark: The statement is still correct, if $\dot{X}^{s_{q,T}}_q$ and $\|\cdot\|_{(k,q)}$ are replaced by $\dot{X}^{s_{q,T}}_{q,T}$ and $\|\cdot\|_{(k,q,T)}$, respectively, and if $\partial_x$ is changed into $\partial_y$.

**Proof.** By the duality between $V^2_q$ and $U^2_q$ (cf. [13] and the subsequent remark) we have

$$\|F(v_1, \ldots, v_{k+1})\|_{V^2_{\phi_{sym}}} = \sup_{\|u\|_{V^2_{\phi_{sym}}} \leq 1} \left| \int_{\mathbb{R}^3} v_1 \cdot \ldots \cdot v_{k+1} \partial_x w dx dy dt \right|.$$

Thus Lemma [11] tells us that for $N_1 \leq \ldots \leq N_{k+1}$

$$N^{s_r} \|P_N F(P_{N_1} v_1, \ldots, P_{N_{k+1}} v_{k+1})\|_{V^2_{\phi_{sym}}} \lesssim N^{s_r} N_1^{-2r} \prod_{j=1}^{k+1} \|P_{N_j} v_j\|_{(k)}. \tag{22}$$

(Here, by the convolution constraint $|\xi, \eta| = |\sum_{j=1}^{k+1} (\xi_j, \eta_j)|$ we have only contributions for $N \lesssim N_{k+1}$. We fix $N$ and $N_{k+1} \gtrsim N$ and sum up the geometric series in $N_1 \leq \ldots \leq N_k$. This gives

$$\sum_{N_1 \leq \ldots \leq N_k} N^{s_r} \|P_N F(P_{N_1} v_1, \ldots, P_{N_{k+1}} v_{k+1})\|_{V^2_{\phi_{sym}}} \lesssim N^{s_r} N_{k+1}^{-2r} \left( \prod_{j=1}^{k} \|v_j\|_{(k,\infty)} \right) \|P_{N_{k+1}} v_{k+1}\|_{(k)}. \tag{23}$$

Now we distinguish between $q = \infty$ and $q < \infty$.

*Case 1: $q = \infty$. Here we simply sum up one last geometric series in $N_{k+1} \gtrsim N$, which leads to

$$\sum_{N_1 \leq \ldots \leq N_k} N^{s_r} \|P_N F(P_{N_1} v_1, \ldots, P_{N_{k+1}} v_{k+1})\|_{V^2_{\phi_{sym}}} \lesssim \prod_{j=1}^{k+1} \|v_j\|_{(k,\infty)} \tag{24}$$

Since this works for all orders of $N_1, \ldots, N_{k+1}$, we have for $N$ fixed by the triangle inequality

$$N^{s_r} \|P_N F(v_1, \ldots, v_{k+1})\|_{V^2_{\phi_{sym}}} \lesssim \prod_{j=1}^{k+1} \|v_j\|_{(k,\infty)} .$$

Taking the supremum over all $N \in 2^\mathbb{Z}$ we have achieved the claimed inequality in the case $q = \infty$.\]
Going back to (23), we see that, since the sums over $N_1, \ldots, N_k+1$ exist,

$$\lim_{N \to \infty} \sum_{N \leq N_1 \leq \ldots \leq N_{k+1}} N^{\varepsilon_q} \| P_N F(P_{N_1}, v_1, \ldots, P_{N_{k+1}}, v_{k+1}) \|_{\mathcal{V}_{\varphi, ym}} = 0,$$

which implies $\lim_{N \to \infty} N^{\varepsilon_q} \| P_N F(v_1, \ldots, v_{k+1}) \|_{\mathcal{V}_{\varphi, ym}} = 0$. To see that the limit for $N \to 0$ vanishes, too, let $\delta > 0$ be given. Then there exists $N_\delta \in 2\mathbb{Z}$ such that $|P_{N_k+1} v_{k+1}(k)| \leq \delta$ for all $N_k+1 \leq N_\delta$. Thus (cf. the right hand side of (23))

$$\sum_{N_{k+1} \geq N} N^{\varepsilon_q} \frac{k}{N_{k+1}} \left( \prod_{j=1}^k \| v_j \|_{(k,\infty)} \right) \| P_{N_{k+1}} v_{k+1} \|_{(k)} \lesssim \delta \prod_{j=1}^k \| v_j \|_{(k,\infty)} + \frac{N^{\varepsilon_q} \prod_{j=1}^{k+1} \| v_j \|_{(k,\infty)}}{N_\delta}.$$

Now $N \to 0$, then $\delta \to 0$.

To close the discussion in the case $q = \infty$, we have to show that $F(v_1, \ldots, v_{k+1}) \in C(\mathbb{R}, B_{2,\infty}^{s_0,0})$. For that purpose we fix $t_0 \in \mathbb{R}$ and denote the characteristic function of the $t$ - intervall between $t_0$ and $t_0 + h$ by $\chi_h$ (h may be negative). Then, by the continuous embedding $X_{2,\infty} \subset C(\mathbb{R}, B_{2,\infty}^{s_0,0})$ and the estimate already shown we have

$$\| F(v_1, \ldots, v_{k+1})(t_0 + h) - F(v_1, \ldots, v_{k+1})(t_0) \|_{B_{2,\infty}^{s_0,0}} \leq \sup_{t \in \mathbb{R}} \| F(\chi_h v_1, \ldots, \chi_h v_{k+1})(t) \|_{B_{2,\infty}^{s_0,0}} \leq \| F(\chi_h v_1, \ldots, \chi_h v_{k+1}) \|_{X_{2,\infty}} \lesssim \prod_{j=1}^{k+1} \| \chi_h v_j \|_{(k,\infty)},$$

which tends to zero with $h \to 0$.

Case 2: $q < \infty$. We sum up the right hand side of (23) in $N_{k+1}$ using Hölder’s inequality. This gives

$$\sum_{N_{k+1} \geq N} N^{\varepsilon_q} \frac{k}{N_{k+1}} \left( \prod_{j=1}^k \| v_j \|_{(k,\infty)} \right) \| P_{N_{k+1}} v_{k+1} \|_{(k)} \lesssim \prod_{j=1}^k \| v_j \|_{(k,\infty)} \left( \sum_{N_{k+1} \geq N} N^{\varepsilon_q} N_{k+1}^{-q \varepsilon_q} \| P_{N_{k+1}} v_{k+1} \|_{(k)} \right)^{\frac{1}{q}}.$$

Now we take the $\ell_q(2\mathbb{Z})$ - norm of this and sum up first in $N \lesssim N_{k+1}$ and then in $N_{k+1}$ to obtain

$$\left( \sum_{N \in 2\mathbb{Z}} N^{\varepsilon_q} N_{k+1}^{-q \varepsilon_q} \| P_N F(P_{N_1}, v_1, \ldots, P_{N_{k+1}}, v_{k+1}) \|_{\mathcal{V}_{\varphi, ym}}^q \right)^{\frac{1}{q}} \lesssim \left( \prod_{j=1}^k \| v_j \|_{(k,\infty)} \right) \| v_{k+1} \|_{(k,\infty)} \lesssim \prod_{j=1}^{k+1} \| v_j \|_{(k,\infty)},$$

where in the last step the continuous embedding $\ell_q \subset \ell^\infty$ was used. The same bound holds for all orders of $N_1, \ldots, N_{k+1}$, hence we get the claimed inequality. The continuity follows by the same arguments as in Case 1.

\[\square\]

### 4.2. Well-posedness for the symmetrized equation.

Here we prove the local and global well-posedness of the Cauchy problem for (6) with initial data $v_0$ in Besov spaces of critical regularity. By the discussion about symmetrization at the beginning of Section 3.1 this implies the two dimensional part of Theorem 1.
Theorem 2. Let \( v_0 \in \dot{B}^{s_c}_{2,q} \), if \( q < \infty \), or \( v_0 \in \dot{B}^{s_c,o}_{2,\infty} \). Then

1. there exists a \( T > 0 \) and a unique solution \( v \in \dot{X}^{s_c}_{q,T} \) of (6) with \( v(0) = v_0 \). Moreover, there exists a constant \( C = C(k,q) > 0 \), so that the lifespan of solutions can be chosen uniformly equal to \( T \) on the subset

\[
D_T := \{ v_0 : \| U_{\phi sym} v_0 \|_{(k,q,T)} \leq (4C)^{-k} \}
\]

of the data space, and the map

\[
S_T : D_T \to \dot{X}^{s_c}_{q,T}, \quad v_0 \mapsto S_T v_0 := v
\]

(data upon solution) is Lipschitz continuous.

2. there exists \( \varepsilon = \varepsilon(k,q) > 0 \) such that, if \( \| v_0 \|_{\dot{B}^{s_c}_{2,q}} \leq \varepsilon \), there exists a unique global solution \( v \in \dot{X}^{s_c}_{q,\infty} \) of (6) with \( v(0) = v_0 \). The solution map \( S_\infty \) is Lipschitz continuous from the ball \( B_\varepsilon := \{ v_0 : \| v_0 \|_{\dot{B}^{s_c}_{2,q}} \leq \varepsilon \} \) (contained in the data space) into \( \dot{X}^{s_c}_{q,\infty} \).

Proof. For given \( v_0 \) we search a solution \( v = \psi + w \), where \( \psi = U_{\phi sym} v_0 \) is a solution of the integral equation with \( \psi(0) = v_0 \) and \( w \) solves the integral equation \( w = \Lambda_{\psi} w \) defined by

\[
\Lambda_{\psi} w(t) = \int_0^t U_{\phi sym}(t-s)(\partial_x + \partial_y)(w + \psi)^{k+1}(s) ds.
\]

For \( j \in \{1,2\} \) let \( \psi_j(0) \in \dot{B}^{s_c}_{2,q} \) (respectively \( \psi_j(0) \in \dot{B}^{s_c,o}_{2,\infty} \)) and \( \psi_j = U_{\phi sym} \psi_j(0) \) with \( \| \psi_j \|_{(k,q,T)} \leq R_0 \) as well as \( w_j \in \dot{X}^{s_c}_{q,T} \) with \( \| w_j \|_{\dot{X}^{s_c}_{q,T}} \leq R \). (The relation between \( T, R_0 \) and \( R \) will be specified within the next few lines.) Then by Lemma [2] and some elementary estimates we obtain

\[
\| \Lambda_{\psi_j} w_1 - \Lambda_{\psi_2} w_2 \|_{\dot{X}^{s_c}_{q,T}} \leq C(R^k + R_0^k)(\| w_1 - w_2 \|_{\dot{X}^{s_c}_{q,T}} + \| \psi_1 - \psi_2 \|_{(k,q,T)})
\]

with a constant \( C \), which may only depend on \( k \) and \( q \). Especially for \( w_2 = \psi_2 = 0 \) we see that

\[
\| \Lambda_{\psi_1} w_1 \|_{\dot{X}^{s_c}_{q,T}} \leq C(R^k + R_0^k)(R + R_0),
\]

if we take \( \psi_1 = \psi_2 \) in (25), we get

\[
\| \Lambda_{\psi_1} w_1 - \Lambda_{\psi_2} w_2 \|_{\dot{X}^{s_c}_{q,T}} \leq C(R^k + R_0^k)\| w_1 - w_2 \|_{\dot{X}^{s_c}_{q,T}}.
\]

Now we fix \( R = R_0 \) in that way, that \( CR^k = CR_0^k = \frac{1}{2} \). Since for any \( v_0 \in \dot{B}^{s_c}_{2,q} \) (respectively \( v_0 \in \dot{B}^{s_c,o}_{2,\infty} \)) we have \( \lim_{T \to 0} \| U_{\phi sym} v_0 \|_{(k,q,T)} \leq 0 \), we can reach \( \| \psi_j \|_{(k,q,T)} \leq R_0 \) by choosing \( T \) small enough. With this choice we have

\[
\| \Lambda_{\psi_1} w_1 \|_{\dot{X}^{s_c}_{q,T}} \leq R \quad \text{and} \quad \| \Lambda_{\psi_1} w_1 - \Lambda_{\psi_2} w_2 \|_{\dot{X}^{s_c}_{q,T}} \leq \frac{1}{2}\| w_1 - w_2 \|_{\dot{X}^{s_c}_{q,T}}.
\]

Moreover, we know from Lemma [2] that - for \( w \in \dot{X}^{s_c}_{q,T} \) - \( \Lambda_{\psi_j} w \in \dot{X}^{s_c}_{q,T} \), especially it is a continuous function with values in the data space. Thus for fixed \( \psi_j \) the mapping \( \Lambda_{\psi_j} \) is a contraction of the closed ball of radius \( R \) in \( \dot{X}^{s_c}_{q,T} \) into itself. The contraction mapping principle provides a solution of \( \Lambda_{\psi_j} w = w \), which is unique in this ball. Since for any \( w \in \dot{X}^{s_c}_{q,T} \) we have \( \lim_{T \to 0} \| w \|_{(k,q,T)} = 0 \), we can use a standard argument, to extend the uniqueness property to the whole \( \dot{X}^{s_c}_{q,T} \). The statement about the lifespan merely reflects our choices. These also give, if inserted into (25) the inequality

\[
\| \Lambda_{\psi_1} w_1 - \Lambda_{\psi_2} w_2 \|_{\dot{X}^{s_c}_{q,T}} \leq \frac{1}{2}\| w_1 - w_2 \|_{\dot{X}^{s_c}_{q,T}} + \frac{1}{2}\| \psi_1 - \psi_2 \|_{(k,q,T)},
\]
which for solutions \( w_1 = \Lambda \phi_1 w_1 \) and \( w_2 = \Lambda \phi_2 w_2 \) implies the Lipschitz bound
\[
\| w_1 - w_2 \|_{\dot{X}^s_{q,r}} \leq \| \psi_1 - \psi_2 \|_{(k,q,T)} \lesssim \| \psi_0^{(1)} - \psi_0^{(2)} \|_{B^{s}_{q,r}}.
\]
Clearly, if \( v_j = \psi_j + w_j \), we have the same (up to a factor) upper bound for \( \| v_1 - v_2 \|_{\dot{X}^s_{q,r}} \). Now the local part of the Theorem is shown. The global part is similar; one uses \( \| \psi_j \|_{(k,q,\infty)} \lesssim \| \psi_0^{(j)} \|_{B^{s}_{q,r}} \) and replaces \( R_0 \) by \( \varepsilon \) in the inequalities. We omit further details. \( \square \)

5. Modifications in the 3D case

5.1. Linear estimates in 3D.

The linear part of the ZK equation in 3D is
\[
\dot{u} + \partial_x \Delta u = 0,
\]
where the Laplacian can be written as \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) in order to emphasize the symmetry in the second and third space variable. The phase function corresponding to (28) is
\[
\phi(\xi, \eta) = \xi (\xi^2 + |\eta|^2) \quad \text{with} \quad (\xi, \eta) = (\xi_1, \eta_1, \eta_2) \in \mathbb{R}^3.
\]
Let \( (U_\phi(t))_{t \in \mathbb{R}} \) denote the associated unitary group, so that solutions \( u \) of (28) with initial datum \( u_0 \) become \( u(t, x, y) = U_\phi(t) u_0(x, y) \). Then we can rely on various known linear estimates for such solutions. In order to control the derivative in the multilinear estimates we may use the local smoothing effect of Kato type, i.e.
\[
\| IU_\phi u_0 \|_{L^2_x L^4_y} \lesssim \| u_0 \|_{L^2_x}.
\]
Here \( I \) denotes the Riesz potential operator of order \(-1\) with respect to all space variables. The proof of (29) follows the same lines as in the 2D case, the calculation is carried out by Ribaud and Vento in [28, Proposition 3.1]. On the other hand we have the following Strichartz type estimates due to Linares and Saut.

**Lemma 3.** Let \( \frac{1}{4} \leq \frac{1}{p} < \frac{2}{7} \) and \( s = \frac{6}{p} - \frac{3}{7} \). Then
\[
\| I_\phi^s u_0 \|_{L^p_{t,x} L^q_y} \lesssim \| u_0 \|_{L^2_x}.
\]

The derivative gain here involves only the \( x \)-variable, not the full gradient. For \( p < 4 \) this estimate is the special case of [24, Proposition 3.1], where \( p = q \). The case \( p = q = 4 \), which will play a major role in our considerations, can be obtained by similar arguments. An alternative approach (allowing a bilinear refinement) was sketched in Section 2 of [10].

A problem seems to occur, if we try to prove an appropriate maximal function estimate (global in time and even without an \( \varepsilon \) unnecessary derivative loss), since the symmetrization argument we applied successfully in 2D fails in three space dimensions. Nonetheless, let us for a short heuristic consider the symmetric phase function
\[
\tilde{\phi}(\xi, \eta_1, \eta_2) = \xi^3 + \eta_1^3 + \eta_2^3.
\]
Then the argument in the proof of Proposition 1 gives the bound
\[
\| U_\phi u_0 \|_{L^2_{t,x} L^\infty_y} \lesssim \| I_\phi^\frac{1}{2} I_\phi^\frac{1}{2} u_0 \|_{L^2_{t,x}}.
\]

\( ^2 \)The regularity gain in the \( p = q \)-version written down here is restricted by \( s < \frac{1}{14} \), the nonsymmetric version is stronger and exhibits a gain of up to \( \frac{1}{14} \)-derivatives, see [24]. For our purposes an \( I_\phi^s \) will do, but this \( \varepsilon \) is essential in our treatment of the quartic nonlinearity.
which shows, what we may expect: The loss of $\frac{3}{2}$ derivatives in an $L^4_{xy} L^\infty_t$-estimate. It turns out that a fairly soft argument combined with the Strichartz type estimate (30) will give us an appropriate substitute. This works, since we are in three dimensions and the phase function is cubic.

Lemma 4. Assume $0 < \frac{1}{q} \leq \frac{1}{p} < \frac{2}{r}$ and $\frac{1}{q} + \frac{5}{p} \leq \frac{3}{2}$. Then for $s = 3\left(\frac{1}{2} - \frac{1}{p} - \frac{1}{q}\right)$ we have

\begin{equation}
\|U_0 u_0\|_{L^p_{xy} L^q_t} \lesssim \|u_0\|_{H^s_{xy}}.
\end{equation}

Proof. By Sobolev embedding in the space variables we may assume $p \leq 4$. Let $u = U_0 u_0$. Then for the space-time Fourier transform of $u$ we have

$$F u(\xi, \tau, \eta) = \delta_0(\tau - \phi(\xi, \eta))F_{xy} u_0(\xi, \eta),$$

so that $\tau = \phi(\xi, \eta) = \xi(\xi^2 + |\eta|^2)$ in the support of $F u$. Now if $p \leq q < \infty$ we can apply a Sobolev embedding in the time variable to obtain with $s_p = \frac{6}{p} - \frac{3}{2}$ as in Lemma 3

$$\|U_0 u_0\|_{L^p_{xy} L^q_t} \lesssim \|F^{-1}|\tau|^{\frac{1}{p} - \frac{1}{r} + \frac{1}{\xi}} F u_0\|_{L^r_{xy}},$$

$$\leq \|F^{-1}|\xi|^{\frac{1}{p} - \frac{1}{r} - s_p} (\xi^2 + |\eta|^2)^{\frac{1}{p} - \frac{1}{q} + \frac{1}{2}} F_{xy} u_0\|_{L^q_{xy}},$$

where in the last step we have applied (30). The assumption $\frac{1}{1} + \frac{5}{q} - \frac{3}{2} \leq \frac{3}{2}$ implies that $\frac{1}{p} - \frac{1}{q} - s_p \geq 0$, so that the Fourier multiplier can be estimated by $(\xi^2 + |\eta|^2)^{\frac{1}{2}}$. \[\square\]

5.2. The multilinear estimate on dyadic pieces in $3D$.

Here we prove the estimate on dyadic pieces in three dimensions, which corresponds to Lemma 1 in Section 4.1. This will look like a copy, but there are differences. We fix $s_c = \frac{1}{2} - \frac{2}{5}$ for the remaining section and recall that for the $3D$-case we have chosen the auxiliary quantity as

$$|P_N u|_{(k)} := N^{s_c} \|P_N u\|_{L^4_{xy} L^\infty_t} + N^{s_c} \|P_N u\|_{L^4_{xy}} + N^{s_c} \|P_N u\|_{L^4_{xy}},$$

By the linear estimates (30) and (31) the three contributions are controlled by

$$|P_N u|_{(k)} \lesssim N^{s_c} \|P_N u\|_{V^2}.$$

Again, if the time interval is taken $(0, T)$ in the involved norms, we write $|P_N u|_{(k, T)}$ instead of $|P_N u|_{(k)}$ and then we can rely on $\lim_{T \to 0} |P_N u|_{(k)} = 0$ for all $u \in V_0^2$.

Lemma 5. Let $u_1, \ldots, u_{k+1} \in V_0^2$, $w \in U_0^2$ with $\|w\|_{U^2} \leq 1$, $N, N_1, \ldots, N_{k+1}$ dyadic numbers with $N_1 \leq N_2 \leq \cdots \leq N_{k+1}$ and $N \geq N_{k+1}$. Then there exists $\varepsilon > 0$ such that

$$N^{s_c} \left| \int_{\mathbb{R}^5} P_{N_1} u_1 \cdots P_{N_{k+1}} u_{k+1} \cdot \partial_{x} P_N wx dx dy dt \right| \lesssim N^\varepsilon N_1^{2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} u_j|_{(k)}.$$

Proof. We consider two cases.

Case 1: $N_{k+1}^{s_c} \left| \xi \right| \lesssim \left| \xi_{k+1} \right|^\varepsilon N$.

Case 2: $N_{k+1}^{s_c} \left| \xi \right| \lesssim N_{k+1}^{s_c} N$. 
In fact there is no further alternative. Clearly, we are in Case 2, if \( N_{k+1} \lesssim N_k \). Otherwise we have \( N_{k+1} \lesssim N \) and hence \( N^{\xi}_{k+1} \lesssim N \). Now, since \( \xi = \sum_{j=1}^{k+1} \xi_j \), we have

(i) \( |\xi| \lesssim |\xi_{k+1}| \), hence \( |\xi|^p \lesssim |\xi_{k+1}|^p \) and we are in Case 1, or

(ii) \( |\xi| \lesssim |\xi_j| \) for one \( j \in \{1, \ldots, k\} \), hence \( |\xi|^p \lesssim N^p_k \) and we are in Case 2 again.

*Estimation for Case 1:* We use \( N^{\xi}_{k+1} |\xi| \lesssim |\xi_{k+1}|^p N_k \approx N_{k+1} \) and the Kato smoothing effect to obtain

\[
N^\xi N_{k+1}^{2\nu} \left\| P_{N_k} u_1 \cdots P_{N_k} u_k \cdot (I_{x}^s P_{N_k+1} u_{k+1}) \right\|_{L^1_x L^\nu_{x,y,t}},
\]

as upper bound for the contribution from this case. We choose Hölder exponents \( p_j \) and \( q_j \) with

\[
\frac{1}{p_j} = \frac{3}{4k} - \frac{\varepsilon}{6}, \quad \frac{1}{q_j} = \frac{1}{4k} - \frac{\varepsilon}{6},
\]

and, for \( j \in \{2, \ldots, k\} \),

\[
\frac{1}{p_j} = \frac{3}{4k}, \quad \frac{1}{q_j} = \frac{1}{4k},
\]

as well as

\[
\frac{1}{p_{k+1}} = \frac{1}{4} + \frac{\varepsilon}{6}, \quad \frac{1}{q_{k+1}} = \frac{1}{4} + \frac{\varepsilon}{6},
\]

so that \( \sum_{j=1}^{k+1} \frac{1}{p_j} = 1 \) and \( \sum_{j=1}^{k+1} \frac{1}{q_j} = \frac{1}{2} \). Hölder’s inequality gives

\[
\left\| P_{N_k} u_1 \cdots P_{N_k} u_k \cdot (I_{x}^s P_{N_k+1} u_{k+1}) \right\|_{L^1_x L^\nu_{x,y,t}} \leq \left( \prod_{j=1}^{k} \left\| P_{N_k} u_j \right\|_{L^{p_j}_{x,y,t}} \right) \left\| I_{x}^s P_{N_k+1} u_{k+1} \right\|_{L^{q_{k+1}}_{x,y,t}}.
\]

For the first factor we use Sobolev embeddings in the space variables to obtain

\[
\left\| P_{N_k} u_1 \right\|_{L^{p_j}_{x,y,t}} = N_k^{\frac{1}{p_j} - \frac{\varepsilon}{4k}} \left\| P_{N_k} u_1 \right\|_{L^{p_j}_{x,y,t}} \lesssim N_k^{1 - \frac{\varepsilon}{4k}} \left\| P_{N_k} u_1 \right\|_{L^{q_j}_{x,y,t}}.
\]

Using a convexity inequality we can control \( N_k^{1 - \frac{\varepsilon}{4k}} \left\| P_{N_k} u_1 \right\|_{L^{q_j}_{x,y,t}} \) by the second and third term in the auxiliary quantity \( |\cdot|_{(k)} \) and we arrive at

\[
\left\| P_{N_k} u_1 \right\|_{L^{p_j}_{x,y,t}} \lesssim N_k^{1 - \frac{\varepsilon}{4k}} \left\| P_{N_k} u_1 \right\|_{(k)}.
\]

In this calculation for \( u_1 \) we may take \( \varepsilon = 0 \) and have for \( j \in \{2, \ldots, k\} \) the bound

\[
\left\| P_{N_k} u_j \right\|_{L^{p_j}_{x,y,t}} \lesssim \left\| P_{N_k} u_j \right\|_{(k)}.
\]

Finally for \( u_{k+1} \) we have

\[
\left\| I_{x}^s u \right\|_{L^{q_{k+1}}_{x,y,t}} \lesssim \left\| I_{x}^s u \right\|_{L^{q_{k+1}}_{x,y,t}} + \left\| u \right\|_{L^{q_{k+1}}_{x,y,t}}
\]

and hence

\[
N_{k+1}^{\nu} \left\| I_{x}^s P_{N_k+1} u_{k+1} \right\|_{L^{q_{k+1}}_{x,y,t}} \lesssim \left\| P_{N_k+1} u_{k+1} \right\|_{(k)}.
\]

Summarizing we get

\[
(32) \lesssim N^\xi N_k^{1 - \frac{\varepsilon}{4k}} \prod_{j=1}^{k+1} \left\| P_{N_k} u_j \right\|_{(k)}.
\]

*Treatment of Case 2:* Here we apply \( |\xi| \lesssim N_{k+1}^{1 - \frac{\varepsilon}{2k}} N \), eliminate the \( N \) by the application of the local smoothing estimate and remain with the task of estimating

\[
N^\xi N_k^{1 - \frac{\varepsilon}{2k}} \left\| P_{N_k} u_1 \cdot \cdots \cdot P_{N_k} u_k \cdot P_{N_k+1} u_{k+1} \right\|_{L^1_x L^{\nu}_{x,y,t}}.
\]
We choose
\[ \frac{1}{p_1} = \frac{3}{4k}, \quad \frac{1}{q_1} = \frac{1}{4k} + \frac{\varepsilon}{2}, \quad \frac{1}{r_1} = \frac{1}{4k}, \]
for \( j \in \{2, \ldots, k-1\}, \)
\[ \frac{1}{p_j} = \frac{3}{4k}, \quad \frac{1}{q_j} = \frac{1}{4k}, \quad \frac{1}{r_j} = \frac{1}{4k}, \]
as well as
\[ \frac{1}{p_k} = \frac{3}{4k}, \quad \frac{1}{q_k} = \frac{1}{4k} + \frac{\varepsilon}{2}, \quad \frac{1}{r_k} = \frac{1}{4k}, \]
and
\[ \frac{1}{p_{k+1}} = \frac{1}{4}, \quad \frac{1}{q_{k+1}} = \frac{1}{4}, \quad \frac{1}{r_{k+1}} = \frac{1}{4}. \]
Hölder’s inequality gives
\[ \|P_N u_1 \ldots u_{k+1}\|_{L^1_t L^p_y} \leq \left( \prod_{j=1}^k \|P_N u_j\|_{L^p_y L^{q_j}_t L^{r_j}_x} \right) \|P_{N_{k+1}} u_{k+1}\|_{L^4_y}. \]
Sobolev inequalities in \( x \) and \( y \) give
\[ \|P_N u_1\|_{L^p_x L^q_y L^{r_1}_t} \lesssim N_1^\varepsilon N_2^{\frac{3}{5} - \frac{2}{5}} \|P_N u_1\|_{L_x^1 L_y^{r_1}_t} \lesssim N_1^{-\varepsilon} N_2^{\frac{3}{5} + \frac{1}{5}} \|P_N u_1\|_{L_x^2 L_y^{r_1}_t} \lesssim N_1^{-\varepsilon} |P_N u_1|_{(k)}, \]
the latter by earlier calculation. Similarly we have for \( j \in \{2, \ldots, k-1\} \) that
\[ \|P_N u_j\|_{L^p_y L^{q_j}_t L^{r_j}_x} \lesssim |P_N u_j|_{(k)}, \]
and for the \( k \)th factor by almost the same Sobolev embeddings
\[ \|P_N u_k\|_{L^p_x L^{q_k}_y L^{r_k}_t} \lesssim N_k^{-\varepsilon} N_k^{\frac{3}{5} + \frac{1}{5}} \|P_N u_k\|_{L_x^1 L_y^{r_k}_t} \lesssim N_k^{-\varepsilon} |P_N u_k|_{(k)}. \]
The estimate for \( u_{k+1} \) is clear, since the \( L^4_y \) - norm is a part of \( |\cdot|_{(k)} \). Collecting terms we arrive at
\[ 33 \lesssim N_k^{-\varepsilon} N_1^{-\varepsilon} \prod_{j=1}^{k+1} |P_N u_j|_{(k)}, \]
which completes the calculation. \( \square \)

The further procedure is now the same as for the symmetrized equation in 2 \( D \). From the quantities \( |P_N u|_{(k)} \) and \( |P_N u|_{(k,T)} \), respectively, one builds the auxiliary norms \( \|u\|_{(k,q)} \) and \( \|u\|_{(k,q,T)} \) as norms of Besov type. Since we avoided to use an \( L^\infty_T \) - norm, we have \( \lim_{T \to 0} \|u\|_{(k,q,T)} = 0 \), whenever \( u \) belongs to our solution space. For \( u_1, \ldots, u_{k+1} \in \dot{X}^s_q \), one defines
\[ F(u_1, \ldots, u_{k+1})(t) := \int_0^t U_\phi(t-s) \partial_x (u_1 \cdots u_{k+1})(s) ds. \]
Summation of the dyadic pieces as in Lemma \( \text{[2]} \) gives \( F(u_1, \ldots, u_{k+1}) \in \dot{X}^s_q \) and the estimate
\[ \|F(u_1, \ldots, u_{k+1})\|_{\dot{X}^s_q} \lesssim \prod_{j=1}^{k+1} \|u_j\|_{(k,q)}, \]
which, if inserted into the proof of Theorem \( \text{[2]} \), leads to the claimed local and global well-posedness result in 3 \( D \). No further argument comes in, which is specific for the 3 \( D \) - case.
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