Horizontal Path Lifting for General Connections

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23 May 2011

Abstract: I characterize the existence of horizontal path lifts for general connections with a new property that also gives fresh insight into linear and $G$-connections.

MSC(2000): Primary 53C29; Secondary 53C05.
1 Introduction and Preliminaries

A general connection on a manifold $M$ is a subbundle $\mathcal{H}$ of the second tangent bundle $\pi_T : TTM \to TM$ which is complementary to the vertical bundle $\mathcal{V} = \ker \pi_* = \ker T\pi$, so that

$$TTM = \mathcal{H} \oplus \mathcal{V}.$$  

Connections originally were so named because they connected distant tangent spaces by means of parallel transport [9]. Horizontal lifts of paths in $M$ are used to define parallel transport. So what we want is a theorem like the following for any general connection.

Let $\gamma : I \to M$ be a path with $\gamma(0) = p$ and $\gamma(1) = q$. For every $v \in T_pM$ there exists a unique horizontal lift $\overline{\gamma}$ such that $\overline{\gamma}(0) = v$ and $\overline{\gamma}(1) \in T_qM$.

The usual proof for linear connections starts this way. Consider the pullback bundle $\gamma^*TM$ over $I$, let $D = \frac{d}{dt}$ denote the standard vector field on $I$, let $\overline{D}$ denote its horizontal lift to $\gamma^*TM$, and let $c$ denote the unique integral curve of $\overline{D}$ with $c(0) = v$. Then $(\gamma^*\pi)c$ is an integral curve of $D$ with $(\gamma^*\pi)c(t) = t$ on $I$.

Unfortunately, $(\gamma^*\pi)c$ need not extend over all of $I$ in general. Figure 1 shows a simple 1-dimensional example of this.

![Figure 1](image.png)

Figure 1: Horizontal lifts of a path $c$ from $p$ to $q$ in $M$. Infinity of the fibers has been brought into the finite plane by a compression such as $y \mapsto \tanh y$. The solid lines begin in one fiber and go away to infinity. The dotted lines do not intersect either fiber. Note that no horizontal lift of $c$ reaches from over $p$ to over $q$. 

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Definition 1.1  A general connection is uniformly vertically bounded (UVB) if and only if $\mathcal{H}_v$ is bounded away from $\mathcal{V}_v$ in each $T_vTM$, uniformly along the fibers of $TM$.

Assuming that $\mathcal{H}_v$ is bounded away from $\mathcal{V}_v$ in each $T_vTM$, uniformly along the fibers of $TM$, removes this obstacle by allowing the standard argument to go through, preventing horizontal lifts from running “off the edge” [7] or “away to infinity” [3]. All linear connections are UVB; more generally, so are all $G$-connections for a Lie group $G$.

First we need a way to determine how far away $\mathcal{H}_v$ is from $\mathcal{V}_v$. Wong [10] showed that this can be done with $n$ principal angles between any two $n$-dimensional subspaces of $\mathbb{R}^{2n}$. If we let $g$ be an auxiliary Riemannian metric on $TM$, these can be measured between the respective unit vectors of some $g$-orthonormal bases of $\mathcal{H}_v$ and $\mathcal{V}_v$ [2]. We denote these principal angles by $\{\theta_i\}$ for $i = 1, \ldots, n$. Since $\mathcal{H}_v$ is complementary to $\mathcal{V}_v$, each $\theta_i > 0$.

Now recall that linear connections satisfy $\mathcal{H}_{av} = a_* \mathcal{H}_v$ and that $a_*$ is a motion of $T_vTM$, so therefore preserves the principal angles along the line through 0 in $T_pM$ determined by $v$ [10, 2]. Hence the $\theta_i$ are constant along each fiber of $T_pM$ so there is an absolute minimum value among them, say $\theta_m > 0$, and this is uniform along the fiber $T_pM$.

That $G$-connections are UVB will not be used here; a proof may be inferred from [3, Sec. 31].

2 Main Theorem

To my knowledge, the following theorem is the first complete characterization of (necessary and sufficient condition for) the horizontal path lifting property (HPL) as first given by Ehresmann [4, p. 36].

Theorem 2.1 Let $\mathcal{H}$ be a general connection on $M$ and let $\gamma : I \rightarrow M$ be a path with $\gamma(0) = p$ and $\gamma(1) = q$. For every $v \in T_pM$ there exists a unique horizontal lift $\gamma$ such that $\gamma(0) = v$ and $\gamma(1) \in T_qM$ if and only if $\mathcal{H}$ is UVB.

Proof: Consider the pullback bundle $\gamma^*TM$ over $I$. Note that $\gamma^*TM \cong I \times \mathbb{R}_p^n$. The label will be used to distinguish copies of $\mathbb{R}^n$. The pullback $\gamma^*TTM \cong \mathbb{R}^{2n} = \mathbb{R}_B^n \oplus \mathbb{R}_V^n$. The general connection pulls back to the family of horizontal $n$-planes $\mathcal{H}_u$ for each $u \in \mathbb{R}_T^n$. We may assume that $\mathbb{R}_V^n$ is the vertical space, and shall refer to $\mathbb{R}_B^n$ as the basal space.
We seek a curve \( c : I \to \mathbb{R}^n_T \) such that \( c(0) = v \in \mathbb{R}^n_T \) and \( \dot{c}(t) \in \mathcal{H}_{c(t)} \). It is convenient to identify \( \mathbb{R}^n_T \) and \( \mathbb{R}^n_B \). Let \( D \) denote the usual derivative in \( \mathbb{R}^n \) so that \( Dc \) is the Jacobian matrix of \( c \). Then with \( c(t) \in \mathbb{R}^n_B \), we have \( \dot{c}(t) = (c(t), Dc(t)) \); i.e., \( c(t) \) is the basal component and \( Dc(t) \) is the vertical component of \( \dot{c}(t) \).

Now the UVB condition implies the existence of an upper bound on \( \|Dc\| \) which is uniform along the fibers of \( \gamma^*TM \). With \( I \) compact, the bound may also be taken to be uniform along \( I \). Applying an MVT [11, p. 366], this implies that \( \|c\| \) is bounded on the part of \( I \) where it exists. The FEUT [5, pp. 162f, 169] provides the existence of \( c \), and the Extension Theorem [5, p. 171f] shows that \( c \) extends to all of \( I \).

So we may regard \( c \in \Gamma(\gamma^*TM) \) over \( I \). By definition, \( c \) is a horizontal section. It follows that the pushforward \( \gamma \) of \( c \) is a horizontal section of \( TM \) along \( \gamma \) with \( \gamma(0) = v \) and \( \gamma(1) \in T_qM \). Uniqueness of \( \gamma \) follows immediately from that of \( c \).

The converse follows similarly, noting that \( \|c\| \) bounded implies \( \|Dc\| \) is also bounded since \( c \) is smooth. \( \square \)

**Corollary 2.2** Each path \( \gamma \) in \( M \) from \( p \) to \( q \) defines a diffeomorphism \( \mathcal{P}_\gamma : T_pM \to T_qM \) that we call parallel transport along \( \gamma \). Note that

\[
\pi_* \gamma = \gamma \pi
\]

as for vector fields. If \( \gamma \) is not injective, however, this has to be interpreted via the pullback bundle \( \gamma^*TM \).

**Proof:** Clearly, uniqueness and smooth dependence of integral curves on initial conditions [6, p. 80] imply that the set of all horizontal lifts of \( \gamma \) defines such a diffeomorphism. \( \square \)

It is now routine to obtain all the usual properties of parallel transport as in [8], for example, except of course for linearity of the maps \( \mathcal{P}_\gamma \).

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