Bulk viscosity of hot dense Quark matter in PNJL model

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Starting from the Kubo formula and the the QCD low energy theorem, we study the the bulk viscosity of hot dense quark matter in the PNJL model from the equation of state. We show that the bulk viscosity has a sharp peak near the chiral phase transition, and the ratio of bulk viscosity over entropy rises dramatically in the vicinity of the phase transition. These results agrees with that from lattice and other model calculations. In addition, we show that the increase of chemical potential raises the bulk viscosity.

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The study of transport properties of strong interacting matter has been attracting many interests. It is very important for hydrodynamic simulations of heavy-ion collisions and for understanding properties of compact stars[1–5]. Shear viscosity \( \eta \) characterizes how fast a system goes back to equilibrium under a shear mode perturbation. It is believe that the quarkgluon plasma (QGP) found in relativistic heavy-ion collider (RHIC) is strongly coupled, which is contrast to the weak coupling picture expected earlier. This is the so-called sQGP. Lattice Monte Carlo simulation on sQGP demonstrated that the ratio of the shear viscosity to the entropy density is rather small but still probably larger than the universal lower bound \( 1/4\pi \) which is obtained from Ads/CFT duality[6]. The experimental extracted value with viscous hydrodynamics combining with a microscopic transport model lies within the range \( 1 \sim 2.5 \) times of the lower bound[5].

Bulk viscosity describes how fast a system goes back to equilibrium under a uniform expansion, relating to the deviation from the conformal invariance of the system. It vanishes when the system has a conformal equation of state, therefore the sharp peak of the bulk viscosity would strongly affect the physics of the QCD matter near critical temperature and is very important for the study of QCD phase structure. Also bulk viscosity affects the Elliptic flow near QCD phase transition in the Relativistic heavy ion collisions[7, 8]. The study of bulk viscosity is also important for the physics of compact stars[1–4].

Recently lattice QCD calculation shows that the trace of energy-momentum tensor anomaly and the ratio of the bulk viscosity \( \zeta \) over entropy density \( s \) have a sharp peak or diverge near phase transition[11–14]. Such a sharp peak behavior of \( \zeta \) has also been observed in many model calculations[16–19].

At present most of the calculations are for zero baryon density[13, 15], except a few papers trying to estimate the bulk viscosity with finite density[20, 21]. For example, in Ref. [21] the authors study the viscosity at finite \( \mu \) with Nambu-Jona-Lasinio (NJL) model. In Ref. [22] the authors study the viscosity of strange quark matter at finite \( \mu \) with quasi particle model. While the bulk viscosity was studied in[20] with Dyson-Schwinger equations at finite \( \mu \) but zero temperature. Here we promote the calculation of bulk viscosity to both
finite temperature and finite baryon density in PNJL model incorporating both confinement and chiral symmetry in this paper.

The bulk viscosity of hot dense quark matter is related to the retarded Green’s function of the trace of the energy-momentum tensor by Kubo formula. Using low energy theorems at finite temperature and chemical potential, we can extract the bulk viscosity of hot dense quark matter from the small frequency ansatz.

From Kubo formula, we can express the bulk viscosity at Lehmann representation \[\zeta = \frac{1}{9} \lim_{\omega \to 0} \frac{1}{\omega} \int_0^\infty dt \int d^3 \vec{r} \exp (i\omega t) \langle \theta^\mu (x) \theta^\mu (0) \rangle.\] (1)

Where \(\omega\) is the frequency, \(\theta^\mu\) is the trace of the energy-momentum tensor. Using Fourier transform and \(P\)-invariance, the formula is changed as

\[
\zeta = \frac{1}{9} \lim_{\omega \to 0} \frac{1}{\omega} \int_0^\infty dt \int d^3 \vec{r} \exp (i\omega t) iG^R(x) \\
= \frac{1}{9} \lim_{\omega \to 0} \frac{1}{\omega} iG^R(\omega, \vec{0}) \\
= -\frac{1}{9} \lim_{\omega \to 0} \frac{1}{\omega} ImG^R(\omega, \vec{0}).
\] (2)

In Lehmann representation, the Green’s function is related to spectral density \(\rho(\omega, \vec{p}) = -\frac{1}{\pi} ImG^R(\omega, \vec{p})\). For Kramers-Kroning relation, we can obtain

\[
G^R(\omega, \vec{p}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ImG^R(u, \vec{p})}{u - \omega - i\varepsilon} du \\
= \int_{-\infty}^{\infty} \frac{\rho(u, \vec{p})}{\omega - u + i\varepsilon} du.
\] (3)

The Euclidean Green’s function is

\[
G^E(\omega, \vec{p}) = -G^R(i\omega, \vec{p}) , \omega > 0
\]

Using the formula (3) we have

\[
G^E(0, \vec{0}) = 2 \int_0^\infty \frac{\rho(u, \vec{0})}{u} du.
\] (4)

For QCD, the trace of energy-momentum stress tensor reads

\[
\theta^\mu = m_q \bar{q}q + \frac{\beta(g)}{2g} F^a_{\mu\nu} F^{a\mu\nu} + \theta_F + \theta_G,
\] (5)

where \(g\) is the strong coupling constant, \(\theta_F\) and \(\theta_G\) are the contribution of quark fields and of gluon field, respectively, and \(\beta(g)\) is the QCD \(\beta\)-function which determines the running behavior of \(g\). In Eq. (5) \(q\) are quark fields with two flavors (in this letter we will limit ourselves in two flavor case and set the current quark mass \(m_u = m_d = m\)).

From the QCD low-energy theorems at finite temperature \(T\) and \(\mu\) [24], one can find

\[
\left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - d \right) \langle \hat{O} \rangle_T = \int d^4 x \langle T_i(\theta_G(x), \hat{O}(0)) \rangle.
\] (6)
where $d$ is the canonical dimension of the operator $\hat{O}$. Using the above equation, one has

$$
\left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4 \right) \langle \theta_G \rangle_T = \int d^4x \langle T_i \{ \theta_G(x), \theta_G(0) \} \rangle,
$$
(7)

$$
\left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 3 \right) \langle \theta_F \rangle_T = \int d^4x \langle T_i \{ \theta_G(x), \theta_F(0) \} \rangle.
$$
(8)

From the above two relations one obtains

$$
9\zeta_\omega_0 = \int d^4x \langle T_i \{ \theta^\mu(x), \theta^\mu(0) \} \rangle
$$

$$
= \left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4 \right) \langle \theta_G \rangle_T + 2 \left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 3 \right) \langle \theta_F \rangle_T
$$

$$
+ \int d^4x \langle T_i \{ \theta_F(x), \theta_F(0) \} \rangle
$$

$$
\approx \left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4 \right) \langle \theta^\mu \rangle_T + \left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 2 \right) \langle \theta_F \rangle_T
$$

$$
= f_1(T, \mu)(\varepsilon - 3P) + f_2(T, \mu)\langle \theta_F \rangle_T,
$$
(9)

where

$$
f_1(T, \mu) = \left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4 \right),
$$

$$
f_2(T, \mu) = \left( T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 2 \right),
$$
(10)

and $\varepsilon$ is the energy density and $P$ is the pressure density of QCD. Here, because the current quark mass $m$ of the quark is very small, in deriving Eq. (8) we have neglected the term proportional to $m^2$.

The low energy theorems adapt to long distance, low frequency and strong coupling QCD[9][10]. Using the non-perturbation theory, the Euclidean Green’s function can be represented as

$$
G^E(0, \vec{0}) = \int d^4x < T\theta(x), \theta(0) >
$$

$$
= f_1(T, \mu) < \theta >_T + f_2(T, \mu)\langle \theta_F \rangle_T.
$$
(11)

Where $< \theta >_T$ is the trace of the energy-momentum tensor. Its average value in zero temperature is $< \theta >_0 = -4|\varepsilon_v|$, $\varepsilon_v$ is the vacuum energy density, including the quark condensates and the gluon condensates in our work. In the low energy theorems, the difference of energy density and the pressure corresponds to non-zero vacuum expectation value of the energy-momentum tensor $\varepsilon - 3P = < \theta >_T - < \theta >_0$. Analogously, $\langle \theta_F \rangle_T = < \tilde{m}q >_T + < \tilde{m}q >_0$.

Using the PCAC relations, we can express the vacuum expectation value $< \tilde{m}q >_0$ through the Pion and Kaon masses and decay constants $< \tilde{m}q >_0 = -M^2_\pi f^2_\pi - M^2_k f^2_k$. Using
these relations, combining the formula (4) and (5), we obtain \(^{(13)}\):

\[
2 \int_0^\infty \rho(u, \vec{0}) \frac{du}{u} = (T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4) \langle \theta_T \rangle_T + (T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4) \langle \theta_F \rangle_T
\]

\[
= (T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4) (\varepsilon - 3P + 4|\varepsilon_v| + < m\bar{q} q >_0)
\]

\[
+ (T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 2) (< m\bar{q} q >_T + < m\bar{q} q >_0).
\]

This formula don’t include the perturbative contribution as long as we consider the strong coupling situation. So we can use the following ansatz in the small frequency region \(^{(13)}\):

\[
\rho(\omega, \vec{0}) = \frac{9 \zeta \omega^2}{\pi (\omega_0^2 + \omega^2)}
\]

Where \(\zeta\) is the bulk viscosity and \(\omega_0\) is a scale at which the perturbation theory becomes valid, \(\omega_0 \sim T\). Using this ansatz and the formula (6), we extract the bulk viscosity:

\[
\zeta = \frac{1}{9 \omega_0} (T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4) (\varepsilon - 3P + 4|\varepsilon_v| + < m\bar{q} q >_0)
\]

\[
+ \frac{1}{9 \omega_0} (T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 2) (< m\bar{q} q >_T + < m\bar{q} q >_0).
\]

The NJL (Nambu-Jona-Lasino) model is based on an effective lagrangian of relativistic fermions which interact through local current-current couplings. It can illustrate the transmutation of originally light quarks into massive quasi-particles, and the spontaneously broken chiral symmetry. But the quark confinement is missing in the NJL model. The de-confinement phase transition is characterized by spontaneous breaking of the \(Z(3)\) center symmetry of QCD. The corresponding order parameter is the Polyakov loop (p-loop). So the PNJL model introduce both the chiral condensate \(\langle \bar{\Psi} \Psi \rangle\) and the p-loop \(\Phi\) coupling to the quarks to solve the problem of the NJL model \(^{23, 25}\).

The PNJL model is an effective method to deal with the non-perturbative QCD. So the bulk viscosity extracted from the formula in the low energy theorems can be calculated in this model. The Lagrangian of two-flavor PNJL model at finite chemical potential is given by \(^{25}\)

\[
\mathcal{L}_{PNJL} = \overline{q} (i \gamma^\mu D_\mu - \hat{m}) q + g \left[ (\overline{q} q)^2 + (\overline{q} i \gamma_5 T q)^2 \right] - \mathcal{U}(\Phi(A), \overline{\Phi}(A), T)
\]

Where \(D_\mu = \partial_\mu - i A_\mu, A_\mu = \delta_\mu{}^0 A^0\). The effective potential \(\mathcal{U}\) is expressed in terms of the traced p-loop \(\Phi = \frac{Tr L}{N_C}\) and its conjugate \(\overline{\Phi} = \frac{Tr L^\dagger}{N_C}\), where \(L = \exp(i A_4 / T), A_4\) is the gauge field.

\[
\frac{\mathcal{U}(\Phi, \overline{\Phi}, T)}{T^4} = - \frac{b_2(T)}{2} \overline{\Phi} \Phi - \frac{b_3}{6} (\Phi^3 + \Phi^3) + \frac{b_4}{4} (\Phi \Phi^2);
\]

\[
b_2(T) = a_0 + a_1 \frac{T_0}{T} + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3.
\]
The parameters in the effective potential are chosen in the following Table:\[23\].

| \(a_0\) | \(a_1\) | \(a_2\) | \(a_3\) | \(b_3\) | \(b_4\) |
|-------|-------|-------|-------|-------|-------|
| 6.75  | -1.95 | 2.625 | -7.44 | 0.75  | 7.5   |

With the definition of the chiral condensate \(\sigma = \langle \tau q \rangle\) and the constituent quark mass \(M = m - 2g\sigma\) the grand potential density is given by

\[
\Omega(\Phi, \overline{\Phi}, M, T, \mu) = U(\Phi, \overline{\Phi}, T) + g \langle \tau q \rangle^2 - 2N_C N_f \int \frac{d^3 p}{(2\pi)^3} E_p \left[ \ln N_\Phi^+ (E_p) + \ln N_\overline{\Phi}^- (E_p) \right]. \quad (15)
\]

Where

\[
\frac{1}{N_\Phi^+ (E_p)} = 1 + 3(\Phi + \overline{\Phi} \exp (-\beta E_p^+)) \exp (-\beta E_p^+) + \exp (-3\beta E_p^+)
\]

\[
\frac{1}{N_\overline{\Phi}^- (E_p)} = 1 + 3(\overline{\Phi} + \Phi \exp (-\beta E_p^-)) \exp (-\beta E_p^-) + \exp (-3\beta E_p^-)
\]

\(E_p = \sqrt{p^2 + M^2}\) is the quasi-particle energy for the quarks. \(E_p^\pm = E_p \mp \mu, \mu\) is the quark chemical potential. Here we consider the isospin symmetry. Now we introduce the mean-field approach by minimizing \(\Omega\) with respect to \(\sigma, \Phi\) and \(\overline{\Phi}\), the mean-field equations is given by

\[
\sigma = -6N_f \int \frac{d^3 p}{(2\pi)^3} E_p \frac{M}{E_p} [\theta(\Lambda^2 - p^2) - M_\Phi^+ (E_p) N_\Phi^+ (E_p) - M_\overline{\Phi}^- (E_p) N_\overline{\Phi}^- (E_p)]; \quad (16)
\]

\[
0 = \frac{T^4}{2} \left[ -b_2 (T) \Phi - b_3 \Phi^2 + b_4 \Phi \overline{\Phi} \overline{\Phi}^2 \right] - 12T \int \frac{d^3 p}{(2\pi)^3} \left[ \exp (-2\beta E_p^+) N_\Phi^+ (E_p) + \exp (-\beta E_p^-) N_\overline{\Phi}^- (E_p) \right]; \quad (17)
\]

\[
0 = \frac{T^4}{2} \left[ -b_2 (T) \Phi - b_3 \Phi^2 + b_4 \Phi \overline{\Phi} \overline{\Phi}^2 \right] - 12T \int \frac{d^3 p}{(2\pi)^3} \left[ \exp (-2\beta E_p^-) N_\Phi^+ (E_p) + \exp (-\beta E_p^-) N_\overline{\Phi}^- (E_p) \right]; \quad (18)
\]

The limits of integration is \(0 \sim \Lambda\) which is a global cutoff\[23\]. Where

\[
M_\Phi^+ (E_p) = (\Phi + 2\overline{\Phi} \exp (-\beta E_p^+)) \exp (-\beta E_p^+) + \exp (-3\beta E_p^+),
\]

\[
M_\overline{\Phi}^- (E_p) = (\overline{\Phi} + 2\Phi \exp (-\beta E_p^-)) \exp (-\beta E_p^-) + \exp (-3\beta E_p^-).
\]
Solving the three coupled equations above numerically we can obtain a series of $\sigma, \Phi, \overline{\Phi}$ at different temperature and chemical potential. The thermodynamical quantities such as the pressure, the quark number density, the entropy and the energy density can be computed with the thermodynamic relations:

\[
P = -\frac{\Omega}{V}; \quad \rho_q = -\left(\frac{\partial\Omega}{\partial\mu}\right)_T; \quad \Phi = \frac{\partial P}{\partial T}_\mu; \quad \varepsilon = TS + \mu\rho_q - P.
\]

To this end, we can calculate the bulk viscosity from Eq. (13).

In this work we consider two-flavor quark matter. For numerical calculations, we choose the parameters as followings\cite{23}: the global cutoff $\Lambda = 0.651$ GeV, the quark current mass $m=0.0055$ GeV, the coupling constant $g = 5.04$GeV. We also choose $T_0 = 0.27$GeV, the zero temperature quark condensation $|\sigma_0| = 0.251^3$GeV and $\omega_0 = 1$Gev. The vacuum energy density $|\epsilon_0|^{1/4} = 0.25$GeV

The temperature dependences of the order parameters for chiral phase transition and deconfinement phase transition $\sigma/\sigma_0, \overline{\Phi}, \Phi$ are plotted in Fig. (1). It shows that the chiral phase transition temperature is about 0.24GeV with a quark chemical potential $\mu = 0.2$GeV. This phase transition is a cross over. While the deconfinement phase transition might happen at higher temperature, although the Polyakov loops are not exact order parameters for deconfinement phase transition of QCD with quarks included.

The numerical results for bulk viscosity are depicted in Fig. (2) at different quark chemical potentials. One can see that the bulk viscosity has a sharp peak around the chiral phase transition temperature, just as the results of Masashi Mizutani\cite{15}. It indicates that the finite quark chemical potential increases the bulk viscosity with the same temperature.

We also computed the specific bulk viscosity, the ratio of the bulk viscosity and entropy density, at finite temperature and density shown in Fig. (3). We show that this ratio starts to increase rapidly and blows up around the critical temperature. The result is in agreement with the lattice results\cite{13}.

The finite quark chemical potential decreases the specific bulk viscosity though increases the bulk viscosity. This is because the finite chemical potential enhances the entropy density.
FIG. 2: Bulk viscosity at different chemical potential and the increasing chemical potential raise the bulk viscosity.

FIG. 3: The ratio of bulk viscosity to entropy at different chemical potential.

more rapidly than the bulk viscosity.

In summary, We studied the bulk viscosity of hot quark matter at finite temperature and density within PNJL model by making use of the the Kubo formula and the QCD low energy theorem. We show that the bulk viscosity has a sharp peak near the chiral phase transition, and the ratio of bulk viscosity and the entropy density rises dramatically in the vicinity of the chiral phase transition. These results agrees with that from lattice and other model calculations. In addition, we show that the increase of chemical potential raises the bulk viscosity but decreases the ratio of the bulk viscosity and entropy density.
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