QUASI-LOG VARIETIES

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Abstract. We extend the Cone and Contraction Theorems of the Log Minimal Model Program to log varieties with arbitrary singularities.

0. Introduction

The starting point of the Minimal Model Program is the Cone and Contraction Theorems of S. Mori: the $K_X$-negative part of the cone of effective curves of a non-singular projective 3-fold $X$ is locally rationally polyhedral, with contractible faces. One hopes that by replacing the original variety with the target space of the contraction associated to a negative face, or a small modification of it (a flip), we reach a minimal model or a Mori-Fano fiber space, after finitely many steps. These intermediate varieties have singularities in dimension at least three, so it became clear that one must consider varieties with some mild singularities in order to find minimal models.

In characteristic zero, Y. Kawamata, X. Benveniste, M. Reid, V. V. Shokurov and J. Kollár proved the Cone and Contraction theorems for varieties with Kawamata log terminal singularities. This part of the Log Minimal Model Program was expected to work for log varieties with arbitrary singularities, under certain assumptions on rays or their contractions. This is our main result, and before we state it we make the following definition:

Definition 0.1. A generalized log variety $(X, B)$ is a pair consisting of a normal variety $X$ and an effective Weil $R$-divisor $B$ such that $K + B$ is $R$-Cartier. We denote by $(X, B)_{-\infty}$ the locus were $(X, B)$ does not have log canonical singularities (it has a natural subscheme structure). A log variety is a generalized log variety which has log canonical singularities, i.e. $(X, B)_{-\infty} = \emptyset$.

Theorem 0.2. Let $(X, B)$ be a projective generalized log variety defined over a field of characteristic zero. Let $\overline{NE}(X)$ be the closure of the cone of effective curves of $X$, and set

$$\overline{NE}(X)_{-\infty} = \text{Im}(\overline{NE}((X, B)_{-\infty}) \to \overline{NE}(X))$$
(i) Let $F$ be a face of the cone $\overline{NE}(X)$ such that $F \cap (\overline{NE}(X)_{-\infty} + \overline{NE}(X)_{K+B \geq 0}) = \{0\}$.

Then there exists a projective contraction $\varphi_F : X \to Y$ which contracts exactly the curves belonging to $F$. Furthermore, $\varphi_F$ restricted to $(X, B)_{-\infty}$ is a closed embedding.

(ii) $\overline{NE}(X) = \overline{NE}(X)_{K+B \geq 0} + \overline{NE}(X/S)_{-\infty} + \sum R_j$, where the $R_j$'s are the one dimensional faces satisfying the assumption in (i). Furthermore, the $R_j$'s are discrete in the half space $N_1(X)_{K+B<0}$.

This result is a special case of Theorem 5.10. As a corollary, we generalize a result of J. Kollár [Ko2] (in characteristic zero): if $(X, B)$ has log canonical singularities outside a finite set of points, the Cone Theorem holds exactly as in the Kawamata log terminal case. In particular, this holds for a normal surface with $\mathbb{Q}$-Gorenstein singularities (cf. [Sa4]). See also [Sh2] for applications.

We also establish the Base Point Free Theorem for generalized log varieties, including the log big case (Theorems 5.1, 7.2). Another application is the uniqueness of minimal lc centers of (quasi-) log Fano varieties (Theorem 6.6).

For the proof, it turns out to be easier to work in a larger class of varieties that we call quasi-log varieties. Their definition is motivated by Y. Kawamata’s X-method, which produces global sections of adjoint line bundles $L$: we first create singularities which are not Kawamata log terminal inside $X$, i.e. $LCS(X) \neq \emptyset$. By adjunction, we expect that $L|_{LCS(X)}$ is still an adjoint line bundle, hence if it has a global section (by induction, for instance), we can lift it to a global section of $L$ by the Kawamata-Viehweg vanishing. Unlike the given variety, its LCS locus is no longer normal, not even irreducible or equi-dimensional, and its log canonical class in the usual sense does not make sense either. However, by definition, the LCS locus is the target space of a 0-log contraction (cf. [Sh2, 3.27(2)]) from a variety with only embedded normal crossings singularities. We call quasi-log varieties those varieties appearing as the target space of such contractions. Examples are varieties with embedded normal crossings singularities, generalized log varieties and their LCS loci (see 4.3).

A quasi-log variety $X$ is endowed with an $\mathbb{R}$-Cartier divisor $\omega$, the descent of the log canonical class of the total space of the 0-log contraction, a closed proper subscheme $X_{-\infty} \subset X$, and a finite family $\{C\}$ of reduced and irreducible subvarieties of $X$. We say that $\omega$ is the quasi-log canonical class of $X$, $X_{-\infty}$ is the locus where $X$ does not have qlog canonical singularities, and the $C$'s are the qlc centres of $X$. The open subset $X \setminus X_{-\infty}$ is reduced, with seminormal singularities. We note
here that singularities appearing on special LCS loci have been called *semi-log canonical* in the literature.

The adjunction and vanishing for quasi-log varieties are proved in Theorem 4.4. The former holds by the very definition, while the latter is an extension to normal crossings pairs of the vanishing and torsion freeness theorems of J. Kollár, based on previous work by Y. Kawamata, H. Esnault and E. Viehweg. Applied to log varieties, our vanishing theorem is stronger than Kawamata-Viehweg (or Nadel) vanishing.

We expect that normal quasi-log varieties are equivalent (cf. 4.3.1) to generalized log varieties, according to the Adjunction Conjecture. We only have partial results in this direction (cf. 4.7, 4.9, 4.10). One should also note that if the Adjunction Conjecture holds, the X-method works inductively in the category of log varieties, as long as we restrict to normal lc centers.

Finally, for technical reasons, we require that our varieties with normal crossings singularities are globally embedded as hypersurfaces. This is enough for applications to generalized log varieties, but we expect that this extra assumption is not necessary (see 2.9).

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1. Preliminary

A variety is a scheme of finite type, defined over an algebraically closed field $k$ of characteristic zero. We denote by $\text{Div}(X)$ the abelian group of Cartier divisors of $X$. A *$K$-Cartier divisor* on $X$ is an element of $\text{Div}(X)_K := \text{Div}(X) \otimes_{\mathbb{Z}} K$, for $K \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

Let $\pi : X \to S$ be a proper morphism of varieties. We denote by $Z_1(X/S)$ the abelian group generated by proper integral curves in $X$ mapped to points by $\pi$. The natural pairing $\text{Div}(X) \times Z_1(X/S) \to \mathbb{Z}$ induces, via numerical equivalence and tensoring with $\mathbb{R}$, a perfect pairing of finite dimensional $\mathbb{R}$-vector spaces $N_1(X/S) \times N_1(X/S) \to \mathbb{R}$. We denote by $NE(X/S) \subset N_1(X/S)$ the cone generated by proper integral curves in $X$ mapped to points by $\pi$, and by $\overline{NE}(X/S)$ its closure in the real topology. The dual of $NE(X/S)$ in $N_1(X/S)$ is
called the relatively nef cone. The relatively ample cone \( \text{Amp}(X/S) \) is the cone of \( N^1(X/S) \) generated by classes of relatively ample Cartier divisors (if any). A \( K \)-Cartier divisor \( D \) is relatively nef (ample) if its class in \( N^1(X/S) \) belongs to the relatively nef (ample) cone. If \( X/S \) is projective, S. Kleiman proved that the relatively ample cone is the interior of the relatively nef cone. In particular, a \( K \)-Cartier divisor \( D \) is relatively ample if and only if \( (D \cdot z) > 0 \) for all \( z \in NE(X/S) \setminus \{0\} \).

An \( R \)-Cartier divisor \( D \) is relatively semi-ample if \( D \sim_R f^*H \), where \( f : X/S \to Y/S \) is a proper morphism, and \( H \) is a relatively ample \( R \)-Cartier divisor. If \( D \in \text{Div}(X)_Q \), this is equivalent to the surjectivity of the natural map \( \pi^*\pi_!\mathcal{O}_X(mD) \to \mathcal{O}_X(mD) \) for some large and divisible positive integer \( m \).

An open subset \( U \subseteq X \) is called big if \( X \setminus U \) has codimension at least two in \( X \).

2. Normal crossings pairs

Definition 2.1. A variety \( X \) has multicrossings singularities if for every closed point \( x \in X \), there exist integers \( N, l \), subsets \( I_1, \ldots, I_l \) of \( \{0, \ldots, N\} \), and an isomorphism of complete local rings

\[
\mathcal{O}_{X,x} \sim \frac{k[[x_0, \ldots, x_N]]}{(\prod_{i \in I_1} x_i, \ldots, \prod_{i \in I_l} x_i)}
\]

If \( l = 1 \) for every \( x \in X \), we say that \( X \) has normal crossings singularities. Furthermore, if each irreducible component of \( X \) is non-singular, we say that \( X \) is a simple multicrossings (normal crossings) variety.

For a scheme \( X \), we denote by \( \epsilon : X_\bullet \to X \) the associated simplicial scheme \( ((X_0/X)^{\Delta_n} \to X)_{n \geq 0} \). Here \( \epsilon = \{\epsilon_n\} \), where \( \epsilon_0 : X_0 \to X \) is the normalization, and \( \epsilon_n \) is the natural projection. The simplicial maps are \( \delta_i : X_{n+1} \to X_n, x_0 \times \cdots \times x_{n+1} \mapsto x_0 \times \cdots \hat{x_i} \cdots \times x_n \) and \( s_i : X_n \to X_{n+1}, x_0 \times \cdots \times x_n \mapsto x_0 \times \cdots \times x_i \times x_i \times x_{i+1} \cdots \times x_n \). This is a proper hypercovering \([\text{De}, \text{Ka2}]\). A strata of \( X \) is by definition the image on \( X \) of some irreducible component of \( X_\bullet \).

Lemma 2.2. The following hold for a variety \( X \) with multicrossings singularities:

(i) The associated hypercovering \( \epsilon : X_\bullet \to X \) is proper, smooth and of cohomological descent with respect to locally free sheaves on \( X \).
(ii) We have an isomorphism of functors \( \text{Hom}(X, \cdot) \sim \text{Hom}(X_\bullet, \cdot) \).
(iii) \( X \) has seminormal singularities.
(iv) If \( X \) is a simple multicrossings variety, each strata of \( X \) is non-singular.
Finally, cohomological descent for a locally free sheaf $F$ on $X$ means that $F \sim R^\ast \epsilon_\ast (\epsilon^\ast F)$. Since $\epsilon$ is finite, it is enough to show that the natural map $O_X \to \epsilon_\ast O_{X_0}$ is an isomorphism. This is a local statement, and it can be checked as in [Ka2, 4.1].

(ii) A morphism $f : X \to Y$ induces $f : X_0 \to Y$ with components $f_n = f \circ \epsilon_n$. Conversely, let $f : X_0 \to Y$ be a morphism. The induced map $f : X \to Y$ is defined set-theoretically by $f(x) := f_0(\epsilon_0^{-1}(x))$. This map is well defined since any two points in the fiber of $x_0$ are the images of some point on some $X_n$ under different compositions of $\delta_i$'s. Moreover, $f$ is a morphism since for every $h \in O_Y$, $f_0(h) \in O_{X_0}$ takes the same value on the glueing data, thus belongs to $O_X \subset O_{X_0}$.

(iii) See [A2].

(iv) The normalization $\epsilon_0$ is a disjoint union of embeddings. Therefore the same holds for $\epsilon_n$, $n \geq 1$. Each $X_n$ is smooth since $X$ has multicrossings singularities, hence all strata are smooth. The strata are the components of the intersections of irreducible components of $X$, in this case.

Let $X$ be a variety with multicrossings singularities. A Cartier divisor $D$ on $X$ is called permissible if it induces a Cartier divisor $D_0$ on $X_0$, i.e. $D_0 = \epsilon_0^\ast D$ is a Cartier divisor on $X_0$, for every $n$ (equivalently, $D$ contains no strata of $X$ in its support). We say that $D$ is a multicrossings divisor on $X$ if, in the notations of Definition 2.1, we have

$$O_{D,x} \sim \frac{k[[x_0, \ldots, x_N]]}{(\prod_{i \in I_1} x_i, \ldots, \prod_{i \in I_1} x_i, \prod_{i \in I_1} x_i)},$$

where $I' \subset \Delta_N$ and $I' \cap \bigcup_{j=1}^d I_j = \emptyset$. We denote by $\text{Div}_0(X)$ the free abelian group generated by all permissible Cartier divisors on $X$. A permissible $K$-divisor on $X$ is an element of $\text{Div}_0(X) \otimes_K K$, for $K \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. For a permissible $K$-divisor $D = \sum d_i D_i$, its reduced part is $\sum_{d_i = 1} D_i$. We denote $D^{>1} = \sum_{d_i > 1} d_i D_i$ and $D^{<1} = \sum_{d_i < 1} d_i D_i$. We say that $D$ is a boundary (sub-boundary) if $0 \leq d_i \leq 1 \forall i$ ($d_i \leq 1 \forall i$).
Definition 2.3. A multicrossings pair \((X, B)\) is a multicrossings variety \(X\) endowed with a permissible \(\mathbb{R}\)-divisor \(B\), whose support is a multicrossings divisor on \(X\). If \(X\) has normal crossings singularities, we say that \((X, B)\) is a normal crossings pair.

A strata of \((X, B)\) is a strata of either \(X\), or the reduced part of \(B\). Equivalently, the strata are the images of strata of the log-nonsingular pairs \(\{(X^n, B^n)\}_{n \geq 0}\). For instance, the maximal strata of \((X, B)\) are the irreducible components of \(X\).

Remark 2.4. Compared with the generalized normal crossings varieties introduced by Y. Kawamata \([\text{Ka}2]\), the ambient space \(X\) of a normal crossings pair has generalized normal crossings singularities, but \(B\) has arbitrary coefficients in our case.

Lemma 2.5. The following properties hold for a multicrossings pair \((X, B)\):

(i) Each strata is irreducible, with multicrossings singularities. A strata which is minimal (with respect to inclusion) is non-singular.
(ii) There are only finitely many strata.
(iii) The non-empty intersection of any two strata is a union of strata.

In particular, minimal strata are mutually disjoint.

We say that a permissible divisor \(D\) has multicrossings support on \((X, B)\) if it contains no strata of \((X, B)\) and both \(D\) and its restriction to the reduced part of \(B\) have multicrossings support. A variety with normal crossings \(X\) is locally complete intersection, so it has an invertible dualizing sheaf \(\mathcal{O}_X(K)\). The canonical divisor \(K \in \text{Div}(X)\) is well defined up to linear equivalence.

Remark 2.6 (Dévissage). Let \((X, B)\) be a normal crossings pair, and let \(Y\) be a union of irreducible components of \(X\). Denote by \(X'\) the union of the other irreducible components of \(X\), and write \(B_Y = B|_Y + X'|_Y\), \(B_{X'} = Y|_{X'} + B|_{X'}\). Then the following hold:

(i) \((Y, B_Y)\) and \((X', B_{X'})\) are normal crossings pairs.
(ii) \((K + B)|_Y = K_Y + B_Y\) and \((K + B)|_{X'} = K_{X'} + B_{X'}\).
(iii) \(\mathcal{I}_{Y,X} \approx j_* \mathcal{O}_{X'}(-Y|_{X'})\), where \(j : X' \to X\) is the inclusion.

In particular, let \(L\) be a Cartier divisor on \(X\) such that \(L = K + B + H\). Denote \(L' = L|_{X'} - Y|_{X'}\), so that \(L' = K_{X'} + B|_{X'} + H|_{X'}\). Then we have a short exact sequence \(0 \to j_* \mathcal{O}_{X'}(L') \to \mathcal{O}_X(L) \to \mathcal{O}_Y(L|_Y) \to 0\).

Definition 2.7. We say that a normal crossings pair \((X, B)\) is embedded if there exists a closed embedding \(j : X \to M\), where \(M\) is a non-singular variety of dimension \(\dim(X) + 1\).
Let \((X, B)\) be an embedded normal crossings pair, and let \(C\) be a nonsingular strata. The \textit{embedded log transformation of} \((X, B)\) \textit{in} \(C\), denoted \(\sigma: (Y, B_Y) \rightarrow (X, B)\), is defined as follows: let \(X \subset M\) be an embedding of \(X\) as a hypersurface in a nonsingular ambient space \(M\). We denote by \(Y\) the reduced structure of the total transform of \(X\) in the blow-up of \(M\) in \(C\). The morphism \(\sigma: Y \rightarrow X\) is projective, \(Y\) has normal crossings singularities, the formula \(\sigma^*(K + B) = K_Y + B_Y\) defines a divisor \(B_Y\) on \(Y\), and the following properties hold:

(i) \((Y, B_Y)\) is an embedded normal crossings pair.

(ii) The strata of \((X, B)\) are exactly the images of the strata of \((Y, B_Y)\).

(iii) \(\mathcal{O}_X \sim \rightarrow R^\bullet \sigma_* \mathcal{O}_Y\).

(iv) \(\sigma^{-1}(C)\) is a maximal strata of \((Y, B_Y)\).

**Proposition 2.8.** Let \(X' \subset X\) be the union of some strata of an embedded normal crossings pair \((X, B)\). Then there exists an embedded normal crossings pair \((Y, B_Y)\), and a projective morphism \(f: Y \rightarrow X\) such that:

(i) \(\mathcal{O}_X \sim \rightarrow R^\bullet f_* \mathcal{O}_Y\).

(ii) \(f^*(K + B) = K_Y + B_Y\).

(iii) The strata of \((X, B)\) are exactly the images of the strata of \((Y, B_Y)\).

(iv) \(f^{-1}(X')\) is a union of maximal strata of \((Y, B_Y)\).

**Proof.** First, we may assume that each strata of \((X, B)\) is nonsingular. Indeed, after a finite number of embedded log transformations of \(X\) in its minimal strata, each irreducible component of \(X\) is nonsingular in the minimal strata of \(X\), i.e. \(X\) has simple normal crossings. Similarly, the reduced part of \(B\) becomes simple multicrossings after a finite sequence of embedded log transformations of \((X, B)\) in minimal strata of \(B\).

Once each strata of \((X, B)\) is nonsingular, we reach the conclusion after a finite number of embedded log transformations of \((X, B)\) in the irreducible components of \(X'\). \(\square\)

**Remark 2.9.** The embedded hypothesis is used to prove 2.8, and to resolve singularities of permissible subvarieties of a variety with normal crossings. Once the latter has been established, we expect our results to work for abstract normal crossings pairs.

### 3. Vanishing theorems

We extend the vanishing and torsion freeness theorems of J. Kollár [Ko1] to normal crossings pairs. The proof is based on logarithmic De Rham complexes, and we follow closely the presentation of [EV]. See also [Ka2].
Theorem 3.1. Assume \((X, B)\) is an embedded normal crossings pair such that \(X\) is a proper variety and \(B\) is a boundary. Let \(L\) be a Cartier divisor on \(X\) and let \(D\) be an effective Cartier divisor, permissible with respect to \((X, B)\), with the following properties:

(i) \(L \sim_\mathbb{R} K + B + H\).
(ii) \(H \in \text{Div}(X)_\mathbb{R}\) is semi-ample.
(iii) \(tH \sim_\mathbb{R} D + D'\) for some positive real number \(t\), and for some effective \(\mathbb{R}\)-Cartier divisor \(D'\), permissible with respect to \(X, B\).

Then the natural maps \(H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))\) are injective for all \(q\).

Proof. Blowing up \(X\) and incorporating the negative part of \(B\) into the pullback of \(L\), we may assume that both \((X, B)\) and \(D + D'\) have normal crossings support. Furthermore, we may assume \(H = aD + a'D'\), where \(a > 0, a' \geq 0\), and \(B' = B + aD + a'D'\) is a boundary with \(\lfloor B' \rfloor = \lfloor B \rfloor\).

We have \(L \sim_\mathbb{R} K + B'\). Since \(L, K\) are integral divisors, the set of boundaries having the same support and reduced part as \(B'\) and satisfying the above equality, form a rational polyhedra. After a perturbation of its fractional part, we may assume that \(B'\) is rational. In particular, \(T = -L + K + B'\) is a \(\mathbb{Q}\)-Cartier divisor and \(\nu T \sim 0\) for some positive integer \(\nu\). Assume that \(\nu\) is minimal with this property. Denote \(\mathcal{E} = \mathcal{O}_X(-L + K)\), and let \(R\) be the support of \(B'\).

Let \(X_\bullet \to X\) be the associated smooth, proper hypercovering. By Serre duality and cohomological descent, we have to check the surjectivity of the maps

\[ H^q(X_\bullet, \mathcal{E}^*(-D^*)) \to H^q(X_\bullet, \mathcal{E}^*). \]

We use the following commutative diagram:

\[
\begin{array}{ccc}
H^q(X_\bullet, \mathcal{E}^*(-D^*)) & \longrightarrow & H^q(X_\bullet, \mathcal{E}^*) \\
\downarrow & & \downarrow \beta \\
H^q(X_\bullet, \Omega^*_{X_\bullet}(\log R^*) \otimes \mathcal{E}^*(-D^*)) & \longrightarrow & H^q(X_\bullet, \Omega^*_{X_\bullet}(\log R^*) \otimes \mathcal{E}^*)
\end{array}
\]

Since \(-L + K = |T| - |B|\), the restriction of \(\mathcal{E}^*\) to each component of \(X_\bullet\) admits a logarithmic connection with poles along \(R^*\), whose residues along the components of \(D^*\) belong to the interval \((0, 1)\) [EV, 3.2]. By [EV, 4.3], the map

\[ \Omega^*_{X_\bullet}(\log R^*) \otimes \mathcal{E}^*(-D^*) \to \Omega^*_{X_\bullet}(\log R^*) \otimes \mathcal{E}^* \]

is a quasi-isomorphism componentwise, thus it is a quasi-isomorphism of simplicial complexes. Therefore \(\alpha\) is an isomorphism.
Let \( \pi : Y_\bullet \to X_\bullet \) be the cyclic cover of degree \( \nu \) corresponding to the torsion divisor \( T^\bullet \). By [De], the spectral sequence

\[
E_1^{p,q} = H^q(Y_\bullet, \Omega^p_{Y_\bullet}(\log R^\bullet)) \implies H^{p+q}(Y_\bullet, \Omega^*(\log R^*))
\]
degenerates. Since \( E^* \) is a direct summand of \( \pi_*\Omega^*_{*}(\log R^*) \), the spectral sequence

\[
E_1^{p,q} = H^q(X_\bullet, \Omega^p_{X_\bullet}(\log R^*) \otimes E^*) \implies H^{p+q}(X_\bullet, \Omega^*(\log R^*) \otimes E^*)
\]
degenerates as well. Therefore \( \beta \) is surjective.

**Theorem 3.2.** Let \((Y, B)\) be an embedded normal crossings pair, and assume that \( B \) is a boundary. Let \( f : Y \to X \) be a proper morphism, and let \( L \) be a Cartier divisor on \( Y \) such that \( H = L - (K + B) \) is \( f \)-semi-ample. Then:

(i) Every non-zero local section of \( R^q f_* \mathcal{O}_Y(L) \) contains in its support the \( f \)-image of some strata of \((Y, B)\).

(ii) Let \( \pi : X \to S \) be a projective morphism and assume \( H \sim_{\mathbb{R}} f^* H' \) for some \( \pi \)-ample \( \mathbb{R} \)-Cartier divisor \( H' \) on \( X \). Then \( R^q f_* \mathcal{O}_Y(L) \) is \( \pi_* \)-acyclic.

**Proof.** (i) The conclusion is local, so we may shrink \( X \) to an affine open subset and compactify it afterwards, so that \( X \) is projective, \( Y \) is proper and \( H \) is semi-ample. If \( R^q f_* \mathcal{O}_Y(L) \) admits a local section whose support does not contain any image of the \((Y, B)\) strata, one can find a very ample divisor \( A \) such that:

- \( H^0(X, R^q f_* \mathcal{O}_Y(L)) \to H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \) is not injective.
- \( f^* A \) is a permissible multicrossing divisor on \((Y, B)\).
- The Leray spectral sequence of \( L + f^* A \) with respect to \( f \) degenerates.

Replacing \( L \) by \( L + f^* A \) if necessary, we may also assume that \( H - f^* A \) is semiample. The degeneration of the Leray spectral sequence implies that the map \( H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + f^* A)) \) is not injective, which contradicts Theorem 3.1.

(ii) Assume \( \dim S = 0 \), and let \( H = f^* H_X \). If \( X \) has positive dimension, one can find a divisor \( A \) in some large, divisible multiple of \( H \), such that its pullback \( A' = f^* A \) is a permissible multicrossings divisor on \((Y, B)\), and \( R^q f_* \mathcal{O}_Y(L + A') \) is \( \pi_* \)-acyclic for all \( q \). By (i), we have short exact sequences:

\[
0 \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L + A') \to R^q f_* \mathcal{O}_{A'}(L + A') \to 0
\]

\( R^q f_* \mathcal{O}_Y(L + A') \) is \( \pi_* \)-acyclic by assumption, while \( R^q f_* \mathcal{O}_{A'}(L + A') \) is \( \pi_* \)-acyclic by induction on \( X \). Therefore \( E_2^{p,q} = 0 \) for \( p \geq 2 \) in the
following commutative diagram of spectral sequences:

\[
\begin{array}{ccc}
E_2^{p,q} = R^p\pi_*R^qf_*O_Y(L) & \longrightarrow & R^{p+q}(\pi \circ f)_*O_Y(L) \\
\varphi^{p,q} & & \varphi^{p+q} \\
\bar{E}_2^{p,q} = R^p\pi_*R^qf_*O_Y(L+A') & \longrightarrow & R^{p+q}(\pi \circ f)_*O_Y(L+A')
\end{array}
\]

Since \( E_2^{1,q} \to R^{1+q}(\pi \circ f)_*O_Y(L) \) is injective, \( \varphi^{1+q} \) is injective by Theorem 3.1, and \( \bar{E}_2^{1,q} = 0 \) by assumption, we obtain \( E_2^{1,q} = 0 \).

Assume now the \( S \) is affine of positive dimension, and \( \pi \circ f \) surjects \( Y \) onto \( S \). We use induction on the dimension of \( S \).

a) Assume that each strata of \((Y, B)\) dominates a generic point of \( S \). From the case \( \dim S = 0 \), \( R^p\pi_*R^qf_*O_Y(L)(p > 0) \) does not contain any generic point of \( S \) in its support. Therefore there exists a general hyperplane section \( A \) of \( S \), containing the support of all these sheaves, such that its pullback \( A' \) on \( Y \) is a multicrossings divisor on \((Y, B)\). The argument in (i) shows that \( R^qf_*O_Y(L) \) is \( \pi_* \)-acyclic, except that \( \varphi^{p+q} \) is injective by (i) now, and \( R^p\pi_*R^qf_*O_Y(L) \otimes O_S(A) \) is zero by the choice of \( A \).

b) Let \( Y' \) be the union of all strata of \((Y, B)\) which is not mapped onto generic points of \( S \). After a sequence of embedded log transformations, we may assume that \( Y' \) is a union of irreducible components of \( Y \). By (i), we have exact sequences

\[
0 \to R^qf_*I_{Y'}(L) \to R^qf_*O_Y(L) \to R^qf_*O_{Y'}(L) \to 0.
\]

From Remark 2.3, \( R^qf_*I_{Y'}(L) \to \gamma R^qf_*O_{Y''}(L'') \), where \( L'' = K_{Y''} + B|_{Y''} + f^*H \). The pair \((Y'', B|_{Y''})\) satisfies the hypothesis in (a), hence the first term is \( \pi_* \)-acyclic. The third is \( \pi_* \)-acyclic by induction, thus \( R^qf_*O_Y(L) \) is \( \pi_* \)-acyclic.

\[\square\]

4. QUASI-LOG VARIETIES

**Definition 4.1.** A quasi-log variety is a scheme \( X \) endowed with an \( \mathbb{R} \)-Cartier divisor \( \omega \), a proper closed subscheme \( X_{-\infty} \subset X \), and a finite collection \( \{C\} \) of reduced and irreducible subvarieties of \( X \) such that there exists a proper morphism \( f : (Y, B_Y) \to X \) from an embedded normal crossings pair satisfying the following properties:

1. \( f^*\omega \sim_{\mathbb{R}} K_Y + B_Y \).
2. The natural map \( O_X \to f_*O_Y([-B_Y^{\leq 1}]) \) induces an isomorphism \( I_{X_{-\infty}} \to f_*O_Y([-B_Y^{\leq 1}]) - [B_Y^{\leq 1}] \).
3. The collection of subvarieties \( \{C\} \) coincides with the images of \((X, B)\)-strata which are not included in \( X_{-\infty} \).
We use the following terminology: the subvarieties $C$ are the qlc centres of $X$, $X_{-\infty}$ is the non-qlog canonical locus of $X$, and $f : (Y, B) \to X$ is a quasi-log resolution of $X$. We say that $X$ has qlog canonical singularities if $X_{-\infty} = \emptyset$. Note that a quasi-log variety $X$ is the union of its qlc centers and $X_{-\infty}$. A relative quasi-log variety $X/S$ is a quasi-log variety $X$ endowed with a proper morphism $\pi : X \to S$.

For simplicity, we will refer to a quasi-log variety as $X$ or $(X, \omega)$.

**Remark 4.2.**

(i) $X$ has qlog canonical singularities if and only if $B$ is a sub-boundary. Indeed, the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & f_* I_N & \to & f_* O_Y & \to & f_* O_N \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & I_{X_{-\infty}} & \to & O_X & \to & O_{X_{-\infty}} & \to & 0
\end{array}
$$

implies that $X_{-\infty} \cap f(Y) = f(N)$, where $N = \lfloor B_Y^{<1} \rfloor$. Note that $X_{-\infty} \subsetneq X$ by assumption, but $X_{-\infty}$ may contain irreducible components of $X$. Also, $f$ may not be surjective (cf. 4.3.4).

(ii) If $B$ is a sub-boundary, 4.1.2 says that the natural morphism $O_X \to f_* O_Y(\lceil -(B_Y^{<1}) \rceil)$ is an isomorphism. In particular, $f$ is a surjective map with connected fibers. Furthermore, $X$ is seminormal by [A2]. In general, the same holds over the open subset of qlog-canonical singularities $U = X \setminus X_{-\infty}$.

(iii) The quasi-log canonical class $\omega$ is defined up to $\mathbb{R}$-linear equivalence. This is more general than the case of generalized log varieties, where the log canonical class $K + B$ is defined up to linear equivalence.

(iv) The quasi-log resolution plays a role similar to a log resolution. Embedded log transformations of $(Y, B_Y)$, or blow-ups of $Y$ in centers which contain no $(Y, B_Y)$-strata, leave the quasi-log structure on $X$ invariant. Furthermore, we may slightly perturb the non-reduced components of $B$. In particular, if $\omega$ is a $\mathbb{Q}$-divisor, we may assume that $B$ is a $\mathbb{Q}$-divisor.

**Proof.** We check the invariance of the structure under permissible blow-ups (for embedded log transformations is easier). The blow-ups do not introduce new $(Y, B)$-strata so we only need to check the invariance of the ideal sheaf in 4.1.2. By cohomological descend, we may assume $Y$ is non-singular and $B_Y$ is a divisor with normal crossings support. Assume $\sigma : (Y', B_{Y'}) \to (Y, B_Y)$ is a crepant log non-singular model. Denote $\Delta = B_Y - \lfloor B_Y \rfloor$, let $R$ be the reduced part of $B_Y$, and define $\Delta'$ and $R'$ similarly.
Note the identity
\[ \lceil -B_Y \rceil - \lfloor B_Y \rfloor = \lceil -B_Y \rceil + R. \]

We have \((\lceil -B_Y \rceil + R) - \sigma^* (\lceil -B_Y \rceil + R) = K_Y + \Delta' + R' - f^* (K_Y + \Delta + R)\). It is enough to show that the right hand side is effective. Assume that is negative in some divisor \(E\). Its coefficient \(\text{mult}_E(\Delta' + R') + a(E; \Delta + R) - 1\) is integral, hence \(\text{mult}_E(\Delta' + R') + a(E; \Delta + R) \leq 0\) (here \(a(E; \Delta + R)\) is the log discrepancy of \(E\) with respect to \((Y, \Delta + R)\)). Therefore \(\text{mult}_E(R') = 0\) and \(a(E; B_Y) = 0\). The latter implies that \(c_Y(E)\) is a strata of \(R\), hence we also have \(a(E; B_Y) = 0\) by the normal crossings assumption. Equivalently, \(\text{mult}_E(R') = 1\). Contradiction.

Example 4.3. 1. Any generalized log variety \((X, B)\) is a a quasi-log variety: let \(\omega\) be any \(\mathbb{R}\)-Cartier divisor such that \(\omega \sim_{\mathbb{R}} K + B\), and let \(X_{-\infty}\) be the locus where \((X, B)\) does not have log canonical singularities (with the induced closed subscheme structure). A quasi-log resolution is a log resolution. The qlc centers are exactly the subvarieties \(C\) of \(X\) such that \((X, B)\) has zero log discrepancy in the generic point of \(C\). With the exception of \(X\) (which is a qlc centre), the qlc centres of \((X, \omega)\) are exactly the lc centres of \(Y\). Kawamata [Ka3] which are not included in \((X, B)_{-\infty}\). This is natural, since we do not expect any adjunction on lc centres along which \((X, B)\) does not have log canonical singularities.

Conversely, if \(Y\) is non-singular, \(f\) is birational and \(X\) is normal, then \(X\) is associated (equivalent) to a generalized log variety as above. Indeed, the corresponding generalized log variety is \((X, f_* B_Y)\).

2. Let \((Y, B_Y)\) be a proper log variety such that \(K_Y + B_Y\) is nef. The Abundance Conjecture predicts the existence of a proper morphism \(f : Y \to X\) to a projective variety \(X\) such that \(K_Y + B_Y \sim_{\mathbb{R}} f^* H\) for some ample divisor \(H \in \text{Div}(X)_{\mathbb{R}}\). Then \(X\) is a quasi-log variety with qlc canonical singularities, with \(\omega \sim_{\mathbb{R}} H\) and quasi-log resolution \(f\).

3. Let \((X, B)\) be a generalized log variety, and assume that \(X = LCS(X, B)\) intersects the open subset on which \((X, B)\) has log canonical singularities. Then \(X\) is a quasi-log variety, where \(\omega \sim_{\mathbb{R}} (K_X + B)_{|X}\) and \(X_{-\infty} = (X, B)_{-\infty}\). A quasi-log resolution of \(X\) is induced by restricting to the reduced part of the boundary on
a log resolution of \((\bar{X}, \bar{B})\):

\[
\begin{array}{ccc}
(Y, B_Y) & \longrightarrow & (\bar{Y}, \bar{B}) \\
\downarrow f & & \downarrow \mu \\
X & \longrightarrow & (\bar{X}, B)
\end{array}
\]

Here \(K_{\bar{Y}} + \bar{B} = \mu^*(K_{\bar{X}} + \bar{B})\), \(Y\) is the reduced part of \(\bar{B}\), and \(B_Y = (\bar{B} - Y)|_Y\).

4. Let \(X\) be a divisor with normal crossings support in a non-singular variety \(\bar{X}\), and assume that \(Y\), the reduced part of \(X\), is non-empty. Then \(X\) is a quasi-log variety, where \(\omega \sim_R (K_{\bar{X}} + X)|_X\), and \(X - \infty\) is the union of non-reduced components of \(X\). A quasi-log resolution is \(f : (Y, B_Y) \rightarrow X\), where \(B_Y\) is defined by the adjunction formula \(K_Y + B_Y = (K_{\bar{X}} + X)|_Y\).

**Theorem 4.4** (Adjunction & Vanishing). Let \(X\) be a quasi-log variety, and let \(X'\) be the union of \(X_{-\infty}\) with a (possibly empty) union of some qlc centers of \(X\).

(i) Assume \(X' \neq X_{-\infty}\). Then \(X'\) is a quasi-log variety, with \(\omega' = \omega|_{X'}\) and \(X'_{-\infty} = X_{-\infty}\). Moreover, the qlc centers of \(X'\) are exactly the qlc centers of \(X\) which are included in \(X'\).

(ii) Assume \(X/S\) is projective and let \(L \in \text{Div}(X)\) such that \(L - \omega\) is \(\pi\)-ample. Then \(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)\) is \(\pi_*\)-acyclic.

Proof. (i) After embedded log transformations, we may assume that the union of all strata of \((Y, B_Y)\) mapped into \(X'\), which we denote \(Y'\), is a union of irreducible components of \(Y\). Define \(B_Y\), by \((K_Y + B_Y)|_{Y'} = K_{Y'} + B_{Y'}\). We claim that \(f : (Y', B_{Y'}) \rightarrow X'\) is a quasi-log resolution.

The adjunction formula is clear, so we just check the second property. Denote \(A = \lceil -(B_Y^{<1}) \rceil\) and \(N = \lfloor B_Y^{<1} \rfloor\). Let \(Y''\) be the subscheme of \(Y\) whose ideal sheaf \(\mathcal{I}\) is defined by the exact sequence

\[0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_Y(-N) \rightarrow \mathcal{O}_{Y''}(-N) \rightarrow 0\]

The ideal of the subscheme \(X'\) is the unique ideal sheaf \(\mathcal{I}_{X'} \subset \mathcal{I}_{X_{-\infty}}\) for which the induced map \(\mathcal{I}_{X'} \rightarrow f_*\mathcal{I}(A)\) is an isomorphism. Consider
the following commutative diagram:

\[
\begin{array}{cccccccccc}
0 & \rightarrow & f_*\mathcal{I}(A) & \rightarrow & f_*\mathcal{O}_Y(A-N) & \rightarrow & f_*\mathcal{O}_{Y'}(A-N) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & f_*\mathcal{I}(A) & \rightarrow & f_*\mathcal{O}_Y(A) & \rightarrow & f_*\mathcal{O}_{Y''}(A) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{I}_X & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_X' & \rightarrow & 0
\end{array}
\]

The map \( f_*\mathcal{O}_{Y''}(A-N) \rightarrow f_*\mathcal{O}_{Y''}(A) \) is injective by the definition of \( \mathcal{I} \). Moreover, \( \mathcal{I}(A) \simeq \mathcal{I}_{Y'} \otimes \mathcal{O}_Y(A-N) \) and \( K_Y + B_Y \sim_{\mathbb{R}} 0/X \). From the choice of \( Y' \), we deduce from Theorem 3.2(i) that any local section of \( R^1f_*\mathcal{I}(A) \) which is supported by \( f(Y') \) is zero. Therefore the top row is exact. It is easy to see that \( \mathcal{I}_{X'_{-\infty}} := \mathcal{I}_X/\mathcal{I} \rightarrow f_*\mathcal{O}_{Y'}(A-N) \) is an isomorphism. Finally, the characterization of the qlc centers of \( X' \) follows from the choice of \( Y' \), and the corresponding statement for \((Y',B_{Y'})\) and \((Y,B_Y)\).

(ii) As in the proof of Theorem 3.2(ii.b), it follows from Theorem 3.2(ii) that \( f_*\mathcal{I}(A) \otimes \mathcal{O}_X(L) \) is \( \pi_* \)-acyclic.

\[\square\]

**Remark 4.5.** The above proof gives a commutative diagram of short exact sequences:

\[
\begin{array}{cccccccccc}
0 & \rightarrow & \mathcal{I}_{X'} & \rightarrow & \mathcal{I}_{X_{-\infty}} & \rightarrow & \mathcal{I}_{X'_{-\infty}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{I}_X & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_{X'} & \rightarrow & 0
\end{array}
\]

Therefore we can lift global sections of \( \mathcal{O}_X(L) \) or \( \mathcal{I}_{X_{-\infty}} \otimes \mathcal{O}_X(L) \) from \( X'/S \) to \( X/S \).

**Definition 4.6.** The LCS locus of a quasi-log variety \( X \), denoted \( LCS(X) \), is \( X_{-\infty} \) union with all qlc centers of \( X \) which are not maximal with respect to the inclusion. The subscheme structure is defined as above, and we have a natural embedding \( X_{-\infty} \subseteq LCS(X) \).

**Proposition 4.7.** Let \( X \) be a quasi-log variety whose LCS locus is empty. Then \( X \) is normal.

**Proof.** We may assume that \( X \) is connected. Let \( f : (Y,B_Y) \rightarrow X \) be a quasi-log resolution of \( X \). By assumption, \( B_Y \) is a sub-boundary, \( f \) is surjective with connected fibers and each strata of \((Y,B_Y)\) dominates some irreducible component of \( X \). We first show that \( X \) is irreducible. Indeed, let \( \{X_i\} \) be the irreducible components of \( X \), and let \( Y_i \) be
the union of strata of $Y$ which dominate $X_i$. A non-empty intersection of two strata mapped on different components cannot dominate some component of $X$, thus $Y$ is the disjoint union of the closed subsets $Y_i$. But $Y$ is connected since $f$ has connected fibers, thus $X$ is irreducible.

Let $f_n : Y_n \to X$ be the induced morphisms. Then $Y_n = \sqcup_j Y^j_n$ is the disjoint union of its irreducible components, and $f_n = \sqcup f^j_n$. Each $f^j_n : Y^j_n \to X$ is dominant, thus factors through the normalization: $f^j_n = \nu \circ g^j_n$. The maps $\{g_n = \sqcup g^j_n\}$ glue to a morphism $g : Y_\bullet \to X^\nu$ which factors $f : Y_\bullet \to X$. This map extends to $Y$ according to Lemma \ref{lem:glue}(ii).

Therefore $f$ factors through the normalization of $X$. Since $f$ has connected fibers and $X$ is seminormal, the normalization is an isomorphism. \hfill $\Box$

The following properties of qlc centers generalize \cite[1.5, 1.6]{Ka3} (in particular, minimal lc centers of log varieties have normal singularities):

**Proposition 4.8.** Assume $X$ is a quasi-log variety with qlog canonical singularities. The following hold:

(i) The intersection of two qlc centers is a union of qlc centers.

(ii) For any point $P \in X$, the set of all qlc centers passing through $P$ has a unique minimal element $W$. Moreover, $W$ is normal at $P$.

**Proof.** (i) Let $C_1, C_2$ be two qlc centers of $X$. Fixing $P \in C_1 \cap C_2$, is enough to find an qlc center $C$ such that $P \in C \subset C_1 \cap C_2$. $X' = C_1 \cup C_2$ is a quasi-log variety with two irreducible components, hence it is not normal at $P$. By Proposition \ref{prop:qlc}, $P \in LCS(X')$. Therefore there exists a qlc center $C \subset C_1$ with $\dim C < \dim C_1$ such that $P \in C \cap C_2$. If $C \subset C_2$, we are done. Otherwise we repeat the argument with $C_1 := C$, and reach the conclusion in a finite number of steps.

(ii) The uniqueness follows from (i), and the normality from Proposition \ref{prop:qlc}.

\hfill $\Box$

**Theorem 4.9.** (cf. \cite{Ka3}) Let $(X/S, B)$ be a relative generalized log variety. Let $\nu : W \to X$ be the normalization of an irreducible component of $LCS(X, B)$, and assume $\nu(W)$ is an exceptional lc centre. The following hold:

(i) There exists a quasi-log structure on $W$ such that $\omega \sim_R \nu^*(K + B)$ and $LCS(W, \omega) \subseteq \nu^{-1}((X, B)_{-\infty} \cup \bigcup\{C \text{ lc centre } \neq \nu(W)\})$.

(ii) Assume $H$ be a nef and big $\mathbb{R}$-divisor on $W/S$. Then there exists a generalized log variety structure $(W, B_W)$ on $W$ such that $\omega + H \sim_R K_W + B_W$ and $LCS(W, B_W) \subseteq LCS(W, \omega)$.
**Remark 4.10.** 1. This is a weak form of adjunction. We expect that the inclusion in (i) is an equality (we prove this on a big open subset of $W$). Furthermore, (ii) should hold in a stronger form: the quasi-log structure of $(W, \omega)$ is equivalent to the log structure of $(W, B_W)$.

2. $(X, B)$ induces a natural $\mathbb{R}$-b-divisor $B_{\div}$ of $W$, called the divisorial part of adjunction (cf. [A1, §3]), and the following inequality is expected to hold:

$$A(W, B_W) \leq -B_{\div}$$

If $\dim(X) \leq 4$, this follows from [PSh2]: there exists a birational model $W'/W$ such that $-B_{\div} = A(W', (B_{\div})_{W'})$. This implies the desired inequality if we choose a high enough model $W'$ in Step (ii) of the proof.

**Proof.** (i) The lc centre being exceptional means that among the valuations centered at $\nu(W)$ on $X$, there exists a unique valuation $E$ with zero log discrepancy with respect to $(X, B)$. Let $\mu : (Y, B_Y) \to (X, B)$ be a crepant log resolution such that $E$ is a divisor on $Y$. We can write $B_Y = E + B'$, and set $B_E = B'|_E$ and $\omega = \nu^*(K + B)$. Since $f : E \to \nu(W)$ has connected general fibre, its Stein factorization is $g : E \to W$:

$$
\begin{array}{ccc}
(Y, B_Y) & \leftarrow & (E, B_E) \\
\mu \downarrow & & \downarrow g \\
(X, B) & \leftarrow & (W, \omega)
\end{array}
$$

We claim that $g$ defines a quasi-log structure on $W$. Indeed, the crepant hypothesis is satisfied since $g^*\omega \sim \mathbb{R} K_E + B_E$. For the second hypothesis, it suffices to show the following equality:

$$\mathcal{O}_W = g_*\mathcal{O}_E([-B_E^{<1}])$$

We have a natural inclusion $j : \mathcal{O}_W \to g_*\mathcal{O}_E([-B_E^{<1}])$ which is an isomorphism in the generic point of $W$. Since $\mathcal{O}_W$ is reflexive and $g_*\mathcal{O}_E([-B_E^{<1}])$ is torsion free, it is enough to check surjectivity in codimension one points of $W$ (cf. [Re, 2.iv]). For this, we may assume that $W$ is a curve and $X$ is a germ at a closed point $P \in \nu(W)$. If $[-B']$ is effective, then $\nu(W)$ is normal at $P$ and the desired equality holds. If $[-B']$ is not effective, then $f_*\mathcal{O}_Y([-B']) \subseteq m_{P,X}$. On the other hand, $R^1\mu_*\mathcal{O}_Y([-B_Y])$ is torsion free by [3.2](i). Therefore we have a surjection

$$\mu_*\mathcal{O}_Y([-B']) \to g_*\mathcal{O}_E([-B_E]) \to 0.$$
Theorem 4.11. \([\text{[Ka3, Theorem 1]}]\)

In particular, \(g_*\mathcal{O}_E([-B_E]) \subset m_{Q,W}\) for every point \(Q \in \nu^{-1}(P)\).
This implies that \([-(-(B_E^{\leq 1})])\) contains none of the fibers \(g^{-1}(Q)\) in its support. Consequently, \(\mathcal{O}_W = g_*\mathcal{O}_E([--(B_E^{\leq 1})])\) at \(P\).

By construction, \(\nu(LCS(W, \omega))\) is contained in the union of \((X, B)_{-\infty}\) and all lc centers of \((X, B)\) different than \(\nu(W)\) (this is the subscheme of \(X\) with ideal sheaf \(\mu_*\mathcal{O}_Y([-B'])\)).

(ii) We may assume \(g\) factors as \(g = \sigma \circ h\), where \(\sigma : W' \to W\) is a resolution such that \((E, P) \xrightarrow{h} (W', Q) \to S\) satisfies the assumptions in [4,11]. \(B_E\) is supported by \(P\), \(\text{Supp}(B_E^h)\) has relative normal crossings over \(W' \setminus Q\) and \(h(\text{Supp}(B_E^w)) \subseteq Q\).

Define \(B_{W'} = \sum b_iQ_i\) by the formulas \(1 - b_i = \min_{P_j/Q_i} \frac{1 - b_j}{m_{P_j/Q_i}}\), and let \(M\) be an \(\mathbb{R}\)-divisor on \(W'\) such that

\[
K_E + B_E \sim h^*(K_{W'} + B_{W'} + M)
\]

Since \(g_*\mathcal{O}_E([-B_E]) \subset \mathcal{O}_W\), the negative part of \(B_{W'}\) is exceptional over \(W\). Also, \(LCS(W', B_{W'}) \subset \sigma^{-1}(LCS(W, \omega))\): if \(b_i \geq 1\), there exists \(P_j/Q_i\) such that \(b_j \geq 1\), hence \(\sigma(Q_i) = g(P_j) \subset LCS(W, \omega)\). Note that \(B_{\text{div}} = \sigma_*B_{W'}\) is the divisorial part of adjunction induced by \((X, B)\) on \(W\) (cf. [A1, §3]).

Since \(D = B_E - h^*B_{W'}\) satisfies the hypothesis of [4,11], \(M\) is nef/\(S\). In particular, \(M + \sigma^*H\) is nef and big/\(S\), so there exists an effective \(\mathbb{R}\)-divisor \(\Delta\) with arbitrary small coefficients such that \(M + \sigma^*H \sim \mathbb{R} \Delta\). We set \(B_W = \sigma_*(B_{W'} + \Delta) = B_{\text{div}} + \sigma_*\Delta\). Then \(\sigma : (W', B_{W'} + \Delta) \to (W, B_W)\) is a crepant birational contraction, hence the claim.

\[\square\]

Theorem 4.11. \([\text{[Ka3, Theorem 1]}]\) Let \(h : (Y, P) \to (X, Q)\) be a projective contraction of non-singular varieties endowed with simple normal crossings boundaries, \(Q = \sum Q_i\) and \(P = \sum P_j\), such that \(h^{-1}(Q) \subset P\) and \(h\) is smooth over \(X \setminus Q\). Assume \(X/S\) is a projective morphism, and \(D\) is an \(\mathbb{R}\)-divisor on \(Y\) with the following properties:

\begin{itemize}
  \item[(0)] There exists a non-singular projective variety \(\bar{X}\) endowed with a simple normal crossings divisor \(\bar{Q}\) such that \(\bar{X} \setminus X\) is a simple normal crossings divisor in \(\bar{X}\) having simple normal crossings with \(\bar{Q}\), and \(Q = \bar{Q} \cap X\).
  \item[(1)] \(D = \sum d_iQ_i\) is supported by \(P\), and if we decompose \(D\) into horizontal and vertical components \(D = D^h + D^v\), then \(h(\text{Supp}(D^v)) \subset Q\) and \(\text{Supp}(D^h)\) has relative normal crossings over \(X \setminus Q\).
  \item[(2)] For each \(j\), \(d_j \leq 1 - \text{mult}_{P_j} h^*Q_j\) if \(h(P_j) = Q_j\), and equality holds for some \(i\).
\end{itemize}
(3) $[-D^h]$ is effective and $O_{X,x} \simeq (h_* O_X([-D]))_{x}.$

(4) $K_Y + D \sim_{\mathbb{R}} h^*(K_X + M)$ for some $\mathbb{R}$-divisor $M$ on $X$.

Then $M$ is nef/$S$.

**Proof.** We show that $D$ is a $\mathbb{Q}$-divisor. Since $K_Y + D \equiv 0$ over the generic point of $X$, $D^h$ is rational. Over the generic point of each $Q_j$, $K_Y + D$ is relatively numerically trivial and at least one component of $D$ has rational coefficients by (2). The fibers of $h$ are connected hence $D$ is rational over the generic points of $Q$. In particular, $K_Y + D$ is a rational divisor over a big open subset of $X$, and is $\mathbb{R}$-linearly equivalent to a pull back from $X$. Therefore $D$ is rational (cf. [Sh2, 3.25]).

We may choose a $\mathbb{Q}$-divisor $M'$ such that $M' \sim_{\mathbb{R}} M$ and $K_Y + D \sim_{\mathbb{Q}} h^*(K_X + M)$. Then the statement is just a non-compact version of [Ka3, Theorem 1]. The same argument works, since the semi-positivity is a local analytic statement (the covering trick holds by our assumption (0)). Note that [Ka3, Theorem 1] is stated under the extra assumption $[-D] \geq 0$, which is however not used during the proof (cf. [Al, 3.5]).

---

5. **The cone theorem**

We follow the arguments of [KMM, 2-4] and [KoM], which we also refer to for references.

**Theorem 5.1** (Base Point Free Theorem). Assume $X/S$ is a projective quasi-log variety. Let $L$ be a $\pi$-nef Cartier divisor on $X$ such that:

(i) $qL - \omega$ is a $\pi$-ample for some $q \in \mathbb{R}$.

(ii) $O_{X,-\infty}(mL)$ is $\pi|_{X,-\infty}$-generated for $m \gg 0$.

Then $O_X(mL)$ is $\pi$-generated for $m \gg 0$.

**Proof.** We may shrink $S$ to an affine open subset without further notice.

1. $O_X(mL)$ is $\pi$-generated on $LCS(X)$ for $m \gg 0$. Set $X' = LCS(X)$. The vanishing $R^1 \pi_* \mathcal{I}_{X'} \otimes O_X(mL) = 0$ $(m \geq q)$ implies the surjectivity of the top horizontal map in the diagram below:

$$
\begin{array}{ccc}
\pi^* \pi_* O_X(mL) & \longrightarrow & \pi^* \pi_* O_{X'}(mL) \\
\downarrow \alpha & & \downarrow \alpha' \\
O_X(mL) & \longrightarrow & O_{X'}(mL)
\end{array}
$$

If $X' = X_{-\infty}$, $\alpha'$ is surjective for $m \gg 0$ by assumption. If $X' \neq X_{-\infty}$, then $X'$ is a quasi-log variety, hence $\alpha'$ is surjective for $m \gg 0$ by induction. Therefore $\alpha$ is surjective on $X'$ for $m \gg 0$.

2. $O_X(mL)$ is $\pi$-generated on a non-empty set for $m \gg 0$. According to step (1), we may assume $LCS(X) = \emptyset$. In particular, $X$ is normal.
(a) Assume $L$ is $\pi$-numerically trivial. Vanishing implies that $\pi_*\mathcal{O}_X(L)$ and $\pi_*\mathcal{O}_X(-L)$ are non-zero [Sh1]. Therefore $L$ is trivial, hence $\pi$-generated.

(b) Assume $L$ is not $\pi$-numerically trivial. Denote $H = qL - \omega$. Using a quasi-log resolution of $X$, we can find an $\mathbb{R}$-divisor $D$ on $X$ such that $D \sim_{\mathbb{Q}} c(H + mL)$, $0 < c < 1$, and $(X, \omega + D)$ has qlog canonical singularities, with non-empty LCS locus [Sh1].

Setting $q' = q + cm$, we are reduced to Step 1.

3. Assume $\mathcal{O}_X(mL)$ is $\pi$-generated on a non-empty subset containing $\text{LCS}(X)$, and denote by $B_{sl}|mL|$ the locus $X$ where $\mathcal{O}_X(mL)$ is not $\pi$-generated. Then $B_{sl}|mL|$ is not contained in $B_{sl}|m'L|$ for $m' \gg 0$.

Let $f : (Y, B) \to X$ be a quasi-log resolution. For $D \in |mL|$ general, we may assume that $f^*D = F + M$ has multicrossings support with respect to $(Y, B_Y)$, where $F$ is the $\pi$-fixed part and $M$ is reduced. Let $c$ be maximal such that $B_Y' = B_Y + cf^*D$ is a sub-boundary above $X \setminus X_{-\infty}$. Then $f : (Y, B_Y') \to (X, \omega')$ is a quasi-log resolution of a quasi-log variety, with $\omega' = \omega + cD$ and $X'_{-\infty} = X_{-\infty}$. Moreover, $(X, \omega')$ has a qlc center $C$ included in $B_{sl}|mL|$. Applying Step 1 with $q' = q + cm$, we infer that $\mathcal{O}_X(m'L)$ is $\pi$-generated on $C$ for $m' \gg 0$.

4. The above steps imply that $\mathcal{O}_X(aL)$ and $\mathcal{O}_X(bL)$ are $\pi$-generated if $a$ and $b$ are very high powers of two prime numbers. Since $a$ and $b$ are relatively prime, they generate the semigroup $\mathbb{Z}_{\geq N}$ for some $N$. Therefore $\mathcal{O}_X(mL)$ is $\pi$-generated for $m \geq N$.

\textbf{Definition 5.2.} Let $(X/S, \omega)$ be a quasi-log variety, with non qlog canonical locus $X_{-\infty}$. Set

$$\overline{\text{NE}}(X/S)_{-\infty} := \text{Im}(\overline{\text{NE}}(X_{-\infty}/S) \to \overline{\text{NE}}(X/S)).$$

For $D \in \text{Div}(X)_{\mathbb{R}}$, set $D_{\geq 0} := \{z \in N_1(X/S); D \cdot z \geq 0\}$ (similarly for $D_{> 0}, z > 0, < 0$) and $D^\perp := \{z \in N_1(X/S); D \cdot z = 0\}$. We also use the notation

$$\overline{\text{NE}}(X/S)_{D_{\geq 0}} := \overline{\text{NE}}(X/S) \cap D_{\geq 0}$$

and similarly for $D_{> 0}, z > 0, < 0$.

\textbf{Definition 5.3.} An extremal face of $\overline{\text{NE}}(X/S)$ is a non-zero subcone $F \subseteq \overline{\text{NE}}(X/S)$ such that $z, z' \in \overline{\text{NE}}(X/S), z + z' \in \overline{\text{NE}}(X/S)$ imply that $z, z' \in F$. Equivalently, $F = \overline{\text{NE}}(X/S) \cap H^\perp$ for some $\pi$-nef $\mathbb{R}$-divisor $H \in \text{Div}(X)_{\mathbb{R}}$ (called supporting function of $F$). An extremal ray is a 1-dimensional extremal face.

(i) An extremal face $F$ is called $\omega$-negative if $F \cap \overline{\text{NE}}(X/S)_{\omega_{\geq 0}} = \{0\}$. 

(ii) An extremal face $F$ is called relatively ample at infinity if $F \cap \overline{NE}(X/S)_{-\infty} = \{0\}$. Equivalently, $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample for any supporting function $H \in \text{Div}(X)_{\mathbb{R}}$ of $F$.

(iii) An extremal face $F$ is called contractible at infinity if it has a rational supporting function $H \in \text{Div}(X)_{\mathbb{Q}}$ such that $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-semi-ample.

**Remark 5.4.** - Let $F$ be an extremal face which is ample at infinity. Then $F$ is contractible at infinity if and only if $F$ is rational, i.e. it has a supporting function given by a rational divisor. We will show in the Cone Theorem that if an $\omega$-negative extremal face is ample at infinity, then it is contractible at infinity.

- Any $\omega$-negative extremal face is relatively ample at infinity if $\omega$ is relatively nef on $X_{-\infty}$ (in particular, if $X_{-\infty}$ is empty).

**Definition 5.5.** Let $F$ be an extremal face of $\overline{NE}(X/S)$. The contraction of $F$ is a projective morphism onto a projective variety $Y/S$

\[
\begin{array}{c}
X \\
\downarrow \pi \\
S \\
\downarrow \sigma \\
Y
\end{array}
\]

satisfying the following properties:

1. Let $C$ be an irreducible curve of $X$ such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.
2. $\mathcal{O}_Y = (\varphi_F)_* \mathcal{O}_X$.

By Zariski’s Main Theorem, such a morphism is unique if it exists.

**Theorem 5.6 (Contraction Theorem).** Let $X/S$ be a projective quasilog variety. Let $F$ be an $\omega$-negative extremal face of $\overline{NE}(X/S)$ which is contractible at infinity. Then the contraction of the face $F$ exists.

**Proof.** Let $H \in \text{Div}(X)$ be a $\pi$-nef divisor such that $H|_{X_{-\infty}}$ is relatively semi-ample and $F = \overline{NE}(X/S) \cap H^\perp$. By Kleiman’s ampleness criteria, $aH - \omega$ is $\pi$-ample for some positive integer $a$. Scaling $H$, we may assume that its restriction at infinity is relatively free. According to the Base Point Free Theorem, some multiple of $H$ if relatively free. The Stein factorization $\varphi : X/S \to Y/S$ of the associated morphism satisfies the following properties:

1. $H \sim_\mathbb{Q} \varphi^*(A)$ for some relatively ample $A \in \text{Div}(Y)_{\mathbb{Q}}$.
2. $\mathcal{O}_Y = \varphi_* \mathcal{O}_X$. 

Since $A$ is relatively ample, it is clear that $\varphi$ is the contraction of the face $F$.

**Remark 5.7.** Let $F$ be an $\omega$-negative extremal face which is contractible at infinity. Then $F$ is relatively ample at infinity if and only if the associated contraction $\varphi_F : X \to Y$ embeds $X_{-\infty}$ into $Y$.

**Lemma 5.8.** Let $P(x, y)$ be a non-trivial polynomial of degree at most $d$, let $a$ be a positive integer and let $r$ be either an irrational number, or a rational number such that, in reduced form, $ra$ has numerator bigger $(d + 1)a$. Then $P(x, y) \neq 0$ for all sufficiently large integral points in the strip $\{rax - r < y < rax\}$.

**Proof.** If $r$ is not rational, there are integral points of the strip which are infinitely close to the line $\{y = rax\}$. If $r$ is rational, let $ra = \frac{u}{v}$ be the reduced form decomposition. The line $\{y = rax - \frac{1}{v}\}$ has infinitely many integral points, and it is included in the strip $\{rax - \frac{r}{d+1} < y < rax\}$ if $u > a(d + 1)$.

In both cases, there are infinitely many rays through the origin having at least $d + 1$ integral points common with the strip $\{rax - r < y < rax\}$. Since $P$ is non-trivial, it cannot vanish on more than a finite number of them. \qed

**Theorem 5.9** (Rationality Theorem). Assume $X/S$ is a projective quasi-log variety such that $\omega \in \text{Div}(X)_{\mathbb{Q}}$. Let $H$ be a $\pi$-ample Cartier divisor on $X$, and let $r$ be a positive number such that

(i) $\omega + rH$ is $\pi$-nef, but not $\pi$-ample.

(ii) $(\omega + rH)|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample.

Then $r$ is a rational number, and in reduced form, $ra$ has numerator at most $a(d \dim X/S + 1)$, where $a$ is the index of $\omega$.

**Proof.** Assume, by contradiction, that $r$ does not satisfy the required properties. In particular, the strip

$$S = \{(x, y) \in \mathbb{N}^2; rax - r < y < rax, (x, y) \text{ large}\}$$

has infinitely many points. Set $L(x, y) = x\omega + yH$. The family of Cartier divisors $\{L(x, y)\}_{(x, y) \in S}$ has the following properties with respect to $(X, \omega)$:

(1) The locus $\text{Bsl}_{\pi} |L(x, y)|$, where $\mathcal{O}_X(L(x, y))$ is not $\pi$-generated, is independent of $(x, y) \in S$. We denote this base locus by $\Lambda$.

**Proof.** Note first that if $(x, y)$ is a given point of $S$ and $(kx, ky)$ is a large multiple which does not lie in $S$, then $L(x', y') - L(kx, ky)$
is $\pi$-ample and $\pi$-generated for $(x', y') \in S$ large. In particular, for $(x, y)$ given, $\text{Bs}_\pi |L(x, y)|$ contains $\text{Bs}_\pi |L(x', y')|$ for $(x', y') \in S$ large. The claim follows by Noetherian induction.

(2) $L(x, y)$ is an adjoint divisor with respect to $\omega$ for all $(x, y)$.

Proof. $L(x, y) - \omega = (xa - 1)(\omega + rH) + (y - rax + r)H$ is $\pi$-ample for $y > rax - r$. Note that $L(x, y)$ is $\pi$-ample for $y > rax$.

(3) $\Lambda \cap (X, \omega)_{-\infty} = \emptyset$ and for each qlc center $C$ of $(X, \omega)$, there exists $(x, y)$ such that $\mathcal{O}_C(L(x, y))$ is $\pi_C$-generated on some non-empty subset.

Proof. Since $L(x, y)$ are adjoint with respect to $\omega$, we can lift global sections of $\mathcal{O}_X(L(x, y))$ from $X_{-\infty}$. Therefore $\Lambda$ does not intersect the non-qlog canonical locus if $\mathcal{O}_{X_{-\infty}}(L(x, y))$ is relatively generated for infinitely many values in $S$. The line $y = rax$ is relatively ample on $X_{-\infty}$, hence Lemma 5.8 implies the existence of infinitely many points $(x, y)$ of $S$ for which $L(x, y)|_{X_{-\infty}}$ is relatively ample. The same argument as in (1) shows that $\mathcal{O}_{X_{-\infty}}(L(x, y))$ are relatively generated for large values.

For the latter part, let $C$ be a qlc center of $X$. We may assume that $C$ does not intersect $X_{-\infty}$, and $S$ is a point. By adjunction, $L(x, y)|_C$ are adjoint, hence

$$P(x, y) = \dim H^0(C, \mathcal{O}_C(L(x, y))) = \chi(C, \mathcal{O}_C(L(x, y)))$$

is a polynomial of degree at most $\dim C \leq \dim X/S$. It is non-trivial polynomial, hence $P(x, y) \neq 0$ for $(x, y) \in S$ by Lemma 5.8 again.

By adjunction, for any family $L(x, y)$ satisfying (1) – (3) above, the common base locus $\Lambda$ does not intersect $X_{-\infty}$ and does not contain any qlc center of $X$.

If $\Lambda = \emptyset$, then $\mathcal{O}_X(L(x, y))$ is $\pi$-generated, in particular $\pi$-nef. This is a contradiction. Therefore $\Lambda$ is non-empty. Let $D$ be a general member of $|L(x, y)|$, and choose $0 < c \leq 1$ maximal such that $\omega' := \omega + cD$ has qlog canonical singularities outside $X_{-\infty}$. Note that $(X, \omega')$ and $(X, \omega)$ have the same non-qlog canonical locus, and $(X, \omega')$ has a qlc center contained in $\Lambda$. But $\{L(x, y)\}_{(x, y) \in S}$ has the same properties (1) – (3) with respect to $(X, \omega')$, hence $\Lambda$ cannot contain any qlc center of $(X, \omega')$. Contradiction.

**Theorem 5.10** (Cone Theorem). Let $(X/S, \omega)$ be a projective quasi-log variety. Let $\{R_j\}$ be the $\omega$-negative extremal rays of $\overline{NE}(X/S)$ which are relatively ample at infinity. Then
i) \( \overline{NE}(X/S) = \overline{NE}(X/S)_{\omega \geq 0} + \overline{NE}(X/S)_{-\infty} + \sum R_j \)

(ii) There are only finitely many \( R_j \)'s included in \( (\omega + H)_{< 0} \). For any relatively ample \( H \in \text{Div}(X)_{\mathbb{R}} \). In particular, the \( R_j \)'s are discrete in the half space \( \omega_{< 0} \).

(iii) Let \( F \) be an \( \omega \)-negative extremal face of \( \overline{NE}(X/S) \) which is relatively ample at infinity. Then \( F \) is a rational face (in particular, contractible at infinity).

Proof. Assume first that \( \omega \in \text{Div}(X)_{\mathbb{Q}} \).

1) If \( \dim_{\mathbb{R}} N_1(X/S) \geq 2 \), then

\[
\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega \geq 0} + \overline{NE}(X/S)_{-\infty} + \overline{\sum F},
\]

where the \( F \)'s vary among all rational proper \( \omega \)-negative extremal faces which are relatively ample at infinity, and the overline denotes the closure with respect to the real topology.

Proof. Denote by \( B \) the right hand side. If equality does not hold, there exists a separating function \( M \in \text{Div}(X) \setminus \{0\} \), which is not a multiple of \( \omega \) in \( N^1(X/S) \), such that \( M \) is positive on \( B \setminus \{0\} \), but is not relatively nef. Since \( M \) belongs to the interior of the dual cone of \( \overline{NE}(X/S)_{\omega \geq 0} \), we can scale it so that \( M = \omega + H \) for a relatively ample \( \mathbb{Q} \)-Cartier divisor \( H \).

Let \( r > 1 \) be the largest real number such that \( \omega + rH \) is relatively nef, but not ample. In particular, \( \omega + rH \) is relatively ample on \( X_{-\infty} \). By the Rationality and Contraction Theorems, \( r \) is a rational number and the extremal face \( F \neq \{0\} \), with supporting function \( \omega + rH \), can be contracted. If \( F \) is proper, it is contained in \( B \), hence \( M \) is relatively ample on \( F \). This contradicts \( r > 1 \). Otherwise \( \omega + rH \) is trivial and \( M = \frac{r-1}{r}\omega \) in \( N^1(X/S) \), which contradicts the choice of \( M \).

2) We may take only proper rays in (1):

Proof. Let \( F \) be a rational proper \( \omega \)-negative extremal face which is relatively ample at infinity, and assume \( \dim(F) \geq 2 \). Let \( \varphi_F : X \to W \) be the associated contraction, so that \( -\omega \) is \( \varphi_F \)-ample. Applying (1) to \( X/W \) we obtain

\[
F = \overline{NE}(X/W) \setminus \{0\} = (\overline{NE}(X/W)_{-\infty} + \overline{\sum G}) \setminus \{0\},
\]

where the \( G \)'s are the rational proper \( \omega \)-negative extremal faces of \( \overline{NE}(X/W) \) which are relatively ample at infinity. Since \( \varphi_F \)
embeds $X_{-\infty}$ into $W$, $\overline{\mathcal{N}E}(X/W)_{-\infty} = 0$. The $G$'s are also $\omega$-negative extremal faces of $\overline{\mathcal{N}E}(X/S)$ which are contractible at infinity, and $\dim G < \dim F$. We obtain by induction
\[
\overline{\mathcal{N}E}(X/S) = \overline{\mathcal{N}E}(X/S)_{\omega \geq 0} + \overline{\mathcal{N}E}(X/S)_{-\infty} + \sum R_j.
\]
Note that each $R_j$ does not intersect $\overline{\mathcal{N}E}(X/S)_{-\infty}$.

(3) Let $A$ be a relatively ample Cartier divisor on $X$. Then each $R_j$ is generated by an irreducible reduced curve $C_j$, $r_j = \frac{A \cdot C_j}{\omega \cdot C_j}$ is a rational number, and the denominator of $\frac{r_j}{a}$, written in reduced form, is at most $a(d + 1)$. Indeed, each $R_j$ is contractible, and the statement follows from the Rationality Theorem applied to the contraction $\varphi_{R_j}$.

(4) Let $\{H_i\}_{i=1}^{e-1}$ be relatively ample Cartier divisors on $X$, which together with $\omega$, form a basis over $\mathbb{R}$ of $N^1(X/S)$. By (3), $R_j \cap \{z; -a\omega \cdot z = 1\}$ is included in the lattice
\[
\{z; -a\omega \cdot z = 1, H_i \cdot z \in (a(d + 1))^{-1} \mathbb{Z}\}.
\]
Therefore the extremal rays are discrete in the half-space $\omega_{<0}$, and the real closure can be omitted. We have obtained (i).

(5) We show (ii). Let $H \in \text{Div}(X)_{\mathbb{R}}$ be relatively ample. Since $H - \sum_{i=1}^{e-1} \epsilon_i H_i$ is ample for $0 < \epsilon_i \ll 1$, the $R_j$'s included in $(\omega + H)_{<0}$ correspond to some elements of the above lattice for which $\sum_{i=1}^{e-1} \epsilon_i H_i \cdot z < \frac{1}{a}$. They are finite.

(6) We show (iii). The vector space $V = F^\perp \subset N^1(X)$ is defined over $\mathbb{Q}$, since $F$ is generated by some of the $R_j$'s. There exists a relatively ample divisor $H \in \text{Div}(X)$ such that $F \subset (\omega + H)_{<0}$. Let $<F>$ be the vector space spanned by $F$, and set
\[
W_F = \overline{\mathcal{N}E}(X/S)_{\omega + H \geq 0} + \overline{\mathcal{N}E}(X/S)_{-\infty} + \sum_{R_j \not\in F} R_j.
\]
Then $W_F$ is a closed cone, $\overline{\mathcal{N}E}(X/S) = W_F + F$, $W_F \cap <F> = \{0\}$, and the supporting functions of $F$ are the elements of $V$ which are positive on $W_F \setminus \{0\}$. This is a non-empty open set, thus contains a rational element which, after scaling, gives a relatively nef Cartier divisor $L$ such that $F = L^\perp \cap \overline{\mathcal{N}E}(X/S)$. Therefore $F$ is rational.

The general case when $\omega \in \text{Div}(X)_{\mathbb{R}}$ can be reduced to the rational case via the following trick: if $H \in \text{Div}(X)_{\mathbb{R}}$ is relatively ample and $\omega + H \in \text{Div}(X)_{\mathbb{Q}}$, we can write $H = E + H'$ such that $H' \in \text{Div}(X)_{\mathbb{R}}$ is a relatively ample and $(X, \omega' := \omega + E)$ is a quasi-log variety with
the same qlc centers and non-qlog canonical locus as \((X, \omega)\). Therefore 
\[ \omega + H = \omega' + H', \omega' \in \text{Div}(X)_\mathbb{Q} \text{ and } (X, \omega)_{-\infty} = (X, \omega')_{-\infty}. \]
In (ii) we may assume that \(\omega + H \in \text{Div}(X)_\mathbb{Q}\), and in (iii) we may replace \(\omega\) by \(\omega + H \in \text{Div}(X)_\mathbb{Q}\). As for (i), we have
\[
\overline{\text{NE}}(X/S) = \overline{\text{NE}}(X/S)_{\omega+H \geq 0} + \overline{\text{NE}}(X/S)_{-\infty} + \sum_{(\omega+H) \cdot R_j < 0} R_j
\]
since the same holds for \(\omega' + H' = \omega + H\). Letting \(H\) converge to 0, we obtain (i) using (ii).

\[\square\]

**Corollary 5.11.** Let \(X/S\) be a projective quasi-log variety such that \(\omega\) is relatively nef on \(X_{-\infty}\). If \(\omega\) is not relatively nef, there exists an \(\omega\)-negative extremal ray which is relatively ample at infinity.

### 6. Quasi-log Fano contractions

We specialize the results of the previous section to the equivalent of Fano contractions in our category:

**Definition 6.1.** A quasi-log Fano contraction \(X/S\) is a relative projective quasi-log variety \(X/S\) such that \(-\omega\) is relatively ample and \(\mathcal{O}_S = \pi_* \mathcal{O}_X\).

**Theorem 6.2.** A projective quasi-log Fano contraction \(X/S\) has only finitely many \(\omega\)-negative extremal rays \(R_j\) which are relatively ample at infinity, and 
\[
\overline{\text{NE}}(X/S) = \overline{\text{NE}}(X/S)_{-\infty} + \sum R_j.
\]
Furthermore, \(\text{NE}(X/S)\) is a closed rational polyhedral cone spanned by the \(R_j\)’s, if \(X_{-\infty}/S\) has at most finite fibers.

**Lemma 6.3.** Assume \(X/T \to S/T\) is a diagram of projective morphisms such that \(X/S\) is a quasi-log Fano contraction.

(i) There exists an \(\omega\)-negative extremal face \(F\) of \(\overline{\text{NE}}(X/T)\) which is contractible at infinity such that \(X/T \to S/T\) is the contraction of the face \(F\).

(ii) Let \(L \in \text{Div}(X)_K\) such that \(L \equiv 0/S\). Then there exists \(H \in \text{Div}(S)_K\) such that \(L \sim_K \pi^* H\), if one of the following hold:
- \(K = \mathbb{Z}\): \(mL|_{X_{-\infty}}\) is relatively base point free for \(m \gg 0\).
- \(K = \mathbb{Q}\): \(L|_{X_{-\infty}}\) is relatively semi-ample.
- \(K = \mathbb{R}\): \(X_{-\infty}/S\) has at most finite fibers.

**Corollary 6.4.** Let \(X/S\) be a quasi-log Fano contraction.

(i) Assume \(L \in \text{Div}(X)_\mathbb{Q}\) is relatively nef, and \(L|_{X_{-\infty}}\) is relatively semi-ample. Then \(L\) is relatively semi-ample.
(ii) Assume \( L \in \text{Div}(X)_{\mathbb{R}} \) is relatively nef, and \( L|_{X_{-\infty}} \) is relatively ample. Then \( L \) is relatively semi-ample.

**Proof.** The first statement follows from the Base Point Free Theorem. For (ii), assume \( L \in \text{Div}(X)_{\mathbb{R}} \) is relatively nef, and \( L|_{X_{-\infty}} \) is relatively ample. If \([L] = 0 \in N^1(X/S)\) we just apply 6.3.ii.

If \([L] \neq 0 \in N^1(X/S)\), \( F := L^\perp \cap \text{NE}(X/S) \) is a non-trivial face. By assumption, \( F \cap (\text{NE}(X/S)_{\omega \geq 0} + \text{NE}(X/S)_{-\infty}) = \{0\} \). Theorem 5.10.(iii) and the Contraction Theorem imply that \( F \) is an \( \omega \)-negative extremal face contractible at infinity, and the contraction \( \varphi_F : X/S \to T/S \) exists. We have \( L \equiv 0/T \) and \( X_{-\infty}/T \) is an embedding. By 6.3.ii, \( L \sim_{\mathbb{R}} \pi^*H \) for some relatively ample \( H \in \text{Div}(T)_{\mathbb{R}} \), i.e. \( L \) is relatively semi-ample.

**Remark 6.5.** (cf. Artin’s numerical criteria) Let \( \pi : X \to S \) be a projective birational morphism of normal varieties, and let \( D \) be an effective \( \mathbb{Q} \)-Cartier divisor on \( X \) such that the following hold:
- \( (X, B) \) is a log variety.
- \(-D\) is \( \pi \)-ample.
- For every subscheme \( Y \subset X \) supported by \( \text{Supp}(D) \), any \( \pi \)-nef \( \text{Cartier divisor} \) \( L \in \text{Div}(Y) \) is \( \pi \)-semi-ample.

Then any \( \pi \)-nef Cartier divisor \( L \) on \( X \) is \( \pi \)-semi-ample. Indeed, \( (X/S, B + rD) \) is a quasi-log Fano contraction for \( r \gg 0 \), with non-log canonical locus supported by \( \text{Supp}(D) \). The claim follows from 6.4(i).

**Theorem 6.6.** Let \( \pi : X \to S \) be a quasi-log Fano contraction, and let \( P \in S \) be a closed point.

(i) Assume \( X_{-\infty} \cap \pi^{-1}(P) \neq \emptyset \) and \( C \) is a qlc centre such that \( C \cap \pi^{-1}(P) \neq \emptyset \). Then \( C \cap X_{-\infty} \cap \pi^{-1}(P) \neq \emptyset \).

(ii) Assume \( X \) has qlc canonical singularities. Then the set of all qlc centres intersecting \( \pi^{-1}(P) \) has a unique minimal element with respect to inclusion.

**Proof.** Let \( C \) be a qlc center of \( X \) such that \( P \in \pi(C) \cap \pi(X_{-\infty}) \). By Theorem 4.4 (with \( L = 0 \), \( X' := C \cup X_{-\infty} \) is a quasi-log variety and the restriction map \( \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'} \) is surjective. Since \( \mathcal{O}_S = \pi_*\mathcal{O}_X \), \( X_{-\infty} \) and \( C \) intersect over a neighborhood of \( P \).

Assume now that \( X_{-\infty} = \emptyset \), and let \( C_1, C_2 \) be two qlc centers of \( X \) such that \( P \in \pi(C_1) \cap \pi(C_2) \). The union \( X' = C_1 \cup C_2 \) is a quasi-log variety, and the same argument implies the surjectivity of the restriction map \( \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'} \). Therefore \( C_1 \) and \( C_2 \) intersect over \( P \). Furthermore, the intersection \( C_1 \cap C_2 \) is a union of qlc centres by Proposition 4.8. By induction, there exists a unique qlc centre \( C_P \) over
a neighborhood of $P$ such that $C_P \subseteq C$ for every qlc centre $C$ with $P \in \pi(C)$.

7. The log big case

For certain applications, we need to weaken the projectivity assumption in the Base Point Free Theorem.

Definition 7.1 (M. Reid). Let $X/S$ be a proper quasi-log variety. A relatively nef $\mathbb{R}$-Cartier divisor $H$ on $X$ is called log big if $H|_C$ is relatively big for every qlc centre $C$ of $X$.

Theorem 7.2 (cf. [Fk, Fj]). Let $X/S$ be a proper quasi-log variety, and let $L$ be a relatively nef Cartier divisor on $X$ with the following properties:

(i) $qL - \omega$ is relatively nef and log big for some $q \in \mathbb{R}$.
(ii) $\mathcal{O}_{X,-\infty}(mL)$ is relatively generated for $m \gg 0$.

Then $\mathcal{O}_X(mL)$ is $\pi$-generated for $m \gg 0$.

The proof is parallel to Theorem 5.1. We just need the appropriate equivalent of Theorem 4.4:

Theorem 7.3. Let $X/S$ be a proper quasi-log variety, and let $X'$ be the union of $X_{-\infty}$ with a union of some qlc centers of $X$. Let $L$ be a Cartier divisor on $X$ such that $L - \omega$ is relatively nef and log big. Then $I_{X'} \otimes \mathcal{O}_X(L)$ is $\pi_*$-acyclic.

This is a formal consequence of the log big extension of Theorem 3.2, which we prove below by reduction to the ample case:

Theorem 7.4. Let $f : (Y, B) \to X$ be a proper morphism from an embedded normal crossings pair, such that $B$ is a boundary. Let $L \in \text{Div}(Y)$, let $\pi : X \to S$ be a proper morphism, and assume that $L \sim_{\mathbb{R}} K + B + f^*H$ for a nef and log big $S\mathbb{R}$-Cartier divisor $H$ on $X$. Then:

(i) Every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the $f$-image of some strata of $(Y, B)$.
(ii) $R^q f_* \mathcal{O}_Y(L)$ is $\pi_*$-acyclic.

Proof. (1) Assume first that each strata of $(Y, B)$ dominates some irreducible component of $X$. Taking the Stein factorization, we may assume that $f$ has connected fibers. Assume then that $X$ is connected, which implies that $X$ is irreducible and each strata of $(Y, B)$ dominates $X$. By Chow’s lemma, there exists a proper birational morphism $\mu : X'/S \to X/S$ such that $X'/S$ is projective. Replacing $Y$ by some blow-up, we may assume that $f$ factors through $\mu$: $f = \mu \circ g$. Set
$\mathcal{F} = R^q g_* \mathcal{O}_Y(L)$. Since $\mu^* H$ is nef and big over $S$, and $X'/S$ is projective, we may write $\mu^* H = E + A$, where $E$ is an effective $\mathbb{R}$-divisor such that $B + g^* E$ has multicrossings support and $\lfloor B \rfloor = \lfloor B + g^* E \rfloor$, and $A \in \text{Div}(X')$ is ample over $S$. From the ample case, we infer that $\mathcal{F}$ is $\mu_*$- and $(\pi \circ \mu)_*$-acyclic, and satisfies (i). Therefore $R^q f_* \mathcal{O}_Y(L) \simeq \mu_* \mathcal{F}$ satisfies (i) and (ii).

(2) We treat the general case by induction on $\dim X$. We may assume that $Y = Y'' \cup Y'''$ is a decomposition of $Y$ such that $Y''$ is the union of all strata of $(Y, B)$ which are not mapped to irreducible components of $X$. Since $f : (Y''', B''') \to X$ and $L''$ satisfy the assumption in (1), the long exact sequence of $0 \to f_* \mathcal{O}_{Y'''}(L'') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y''}(L) \to 0$ with respect to $f_*$, breaks up into short exact sequences

$$0 \to R^q f_* \mathcal{O}_{Y''}(L'') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y'}(L) \to 0.$$ 

Since (i) and (ii) hold for the first and third member by case (1) and by induction on dimension respectively, they hold for $R^q f_* \mathcal{O}_Y(L)$ also.  

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