Totally symmetrized spinors and null rotation invariance

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Abstract

In the existing implementations of the Cartan–Karlhede procedure for characterization and classification of spacetimes, a prominent rôle is played by multi-index two-componentspinors symmetrized over both types of index. This paper considers the conditions for, and detection of, null rotational invariance of such spinors, and corrects a previous discussion.

Keywords: spacetime classification, null rotation, spacetime symmetry

1. Introduction

The Cartan–Karlhede procedure for characterizing spacetimes is set out in several places, e.g. Stephani et al (2003), chapter 9. It relies on computing and interpreting Cartan invariants, which are components of the Riemann tensor and its covariant derivatives expressed relative to a canonically-chosen frame. To apply this method the present implementations (MacCallum and Skea 1994, Pollney et al 2000a, Pollney et al 2000b, Pollney et al 2000c) use two-component spinors in the Newman–Penrose formalism (see Stephani et al (2003), chapters 3–7 for a summary of relevant concepts and formulae).

MacCallum and Åman (1986) defined a minimal set of Cartan invariants, taking into account their interrelations, and sufficient for the Cartan–Karlhede procedure. This set of quantities consists of totally symmetrized spinors. A shorthand notation for these will be used here: it first appeared in print in Karlhede and Åman (1980). It is an extension of the Newman–Penrose notation, in which, for example, the completely symmetric Weyl spinor \( \Psi_{ABCD} = \Psi_{(ABCD)} \) is represented by 5 components \( \Psi_A, A = 0, \ldots, 4 \), formed by contractions with the basis spinors \( \sigma^A, \iota^A \). The index \( A \) on \( \Psi_A \) counts the number of contractions with \( \iota^A \); thus, for example, \( \Psi_3 = \Psi_{(ABCD)} \sigma^A \iota^B \iota^C \iota^D \).
This notation is extended as follows. Suppose $Q^{BCD...}_{E'F'...}$ is a completely symmetric multi-index spinor, so that $Q^{BCD...}_{E'F'...} = Q^{BCD...}_{(E'F'...)}$. (Here the unprimed indices have been raised to enable clarity in the use of the standard notation for symmetrization.) Such a spinor is said to have valence $(m, n)$ if it has $m$ unprimed and $n$ primed indices. Then $Q_{ab}$ denotes the component in which $a$ of the unprimed indices and $b$ of the primed indices are contracted with the basis spinors $i^a$ and $i^b$ respectively (and the other indices with $o^A$ and $\partial^X$). (Lower case indices will be used here in $Q_{ab}$ for consistency with the earlier paper on the present topic, Pollney et al (2000a). Normally the same upper case letters are used for both spinor indices and shorthand notation indices where no confusion can arise). As an example, consider the totally symmetrized part of the second covariant derivative of the Weyl spinor, $\nabla AB \nabla BB \Psi_{CDEF}$. It has valence $(6, 2)$. The component $\nabla^2 \Psi_{14} = \nabla^{\iota4}(\chi^{\iota4567})\Psi_{CDEF}Q_{ABCD}E_{EF}Q_{\iota4567}$.

To identify when a spacetime admits isometries, the software CLASSI applies a number of tests to the calculated symmetrized spinor Cartan invariants. These tests are incompletely set out in the published description (MacCallum and Skea 1994) and in the manual (Åman 1987), although the code is publicly available. A more extended description of the process, with (as compared with CLASSI’s version) modified, corrected and extended details, was given by Pollney et al (2000a), who also developed its implementation in Maple.

In their work Pollney et al gave conditions for symmetric spinors $\chi_{ab'}$ of valence $(m, n)$ to be invariant under a group of null rotations preserving the Newman–Penrose basis vector $k$. For invariance under a two-parameter group of null rotations, they showed that only $\chi_{mm0}$ can be non-zero.

They also gave conditions for invariance under a one-parameter group of null rotations about $k$, for brevity referred to here as IGNR, specialized to the cases where the parameter $B$ of Stephani et al (2003), equation (3.15), [$\alpha$ in Pollney et al (2000a)] is either pure real or pure imaginary. Such a null rotation can be expressed in the spinor basis as

$$o^A \rightarrow o^A, \quad \iota^A \rightarrow \iota^A + B o^A,$$

for a complex number $B \neq 0$. Note that considering only these null rotations, rather than the ones preserving $\iota$ and $l$, amounts to preferring, for example, the standard form for the Weyl tensor in which for Petrov type N only $\Psi_4 \neq 0$. In practice an alternate canonical form with, e.g., only $\Psi_0 \neq 0$ may arise or be preferred: swapping $o$ and $\iota$ has the effect that in the arguments and statements below $\chi_{ab'}$ is swapped with $\chi_{(m-a)(n-b)}$, and $B$ with the parameter $E$ in the null rotation $\iota^A \rightarrow \iota^A, \quad o^A \rightarrow o^A + E \iota^A$.

For both $B$ real and $B$ pure imaginary, Pollney et al state that one has

$$\chi_{ab'} = 0, \quad \forall \text{ pairs } a + b < m + n - 2.$$

Unfortunately their argument for (2), and (2) itself, are false. They overlooked the fact that their equations (11), (14), and (16)–(19) are not independent. Specifically (18) and (19) imply (16), (17) and (18) imply (14), and (17)–(19) imply (11). Thus they have only 3 independent conditions for the four $\chi_{ab'}$, $a + b = m + n - 3$, and setting all those $\chi_{ab'}$ to zero is not the only solution.

The issue is thus re-examined here, and a correct set of conditions for totally symmetrized spinors to be invariant under a one-parameter group of null rotations is given. Procedures for identifying when this occurs are suggested.

2. Conditions for null rotation invariance

In the following the rôles of primed and unprimed indices (and thus of $m$ and $n$) can be interchanged, so without loss of generality it can be assumed that $m \geq n$ (this is anyway the case
for the particular spinors defined in the minimal set of MacCallum and Åman (1986). For $m = 0 = n$ IGNR necessarily applies, so we can assume $m \geq 1$.

First consider $n = 0$. Then under (1)

$$\chi_{m}^{*} = \sum_{r=0}^{m} \binom{m}{r} B^{r} \chi_{m-r},$$

(3)

where * denotes the transformed value, so by equating the coefficients of this polynomial in $B$ to zero we find that all terms $\chi_{p} = 0$ for $0 \leq p \leq (m - 1)$. For IGNR, only $\chi_{m}$ can be nonzero.

Now suppose $n \geq 1$. The general formula for the transform under (1) of the component $\chi_{ab}$ of $\chi$ is

$$\chi_{ab}^{*} = \sum_{r=0}^{a} \sum_{s=0}^{b} \binom{a}{r} \binom{b}{s} B^{r} \bar{B}^{s} \chi_{a-r}(b-s).$$

(4)

The set of values $\chi_{rs}$ for $r + s = k$ will be referred to as the line $k$. For each $k \geq 1$ the terms in (4) homogeneous of degree $k$ in $B$ and $\bar{B}$, which will be denoted by $\chi_{ab}^{(k)}$, must vanish in order for $\chi_{ab}$ to be IGNR. This can be shown by writing $B = |B|e^{i\theta}$ and equating coefficients of $|B|^{k}$ in (4) to zero. If $\chi_{ab}$ lies on the line $j$, only values $\chi_{pq}$ on the line $j - k$ contribute to the terms of order $k$ in (4). In particular the linear term in the expansion of $\chi_{pq}^{*}$ must vanish for all $p$ and $q$, if $\chi$ is IGNR, i.e. for $p$ and $q$ at least 1,

$$pB_{\chi_{(p-1)q}^{*}} + q\bar{B}_{\chi_{pq-1}^{*}} = 0.$$

(5)

This is therefore a necessary condition for IGNR. It will be used below to obtain sufficient conditions.

The following lemma excludes the case $m = 1 = n$. In this case we will have (by the argument in the proof below) $\chi_{00}^{*} = 0$ and then (5) for $p + q = 1$ would ensure IGNR.

**Lemma 1.** If $\chi_{ab}$ is IGNR and $m \geq 2$ then all values on the lines $k$, $0 \leq k \leq (m - 1)$ are zero.

**Proof.** We have already dealt with $n = 0$ for all $m$. For $a = m$, $b = 0$, (4) gives the modified form of (3) in which an index $0^{*}$ is added to each occurrence of $\chi$, i.e.

$$\chi_{a0}^{*} = \sum_{r=0}^{m} \binom{m}{r} B^{r} \chi_{m-r0}^{*}.$$

By the same argument as for (3) $\chi_{a0}^{*} = 0$ for $0 \leq a \leq m$. (Similarly if $n \geq 1 \chi_{a0}^{*} = 0$ for $0 \leq b < n$.) Thus $\chi_{ab}^{*} = 0$ for $a + b = 0$. If $m \geq 2$ and $n \geq 1$, $\chi_{0r} = 0$ and (5) implies $\chi_{01} = 0$ thus proving the result for $a + b = 1$.

For $2 \leq k \leq m - 1$ we can prove $\chi_{ab}^{*} = 0$ on the line $k$ by induction. Suppose $\chi_{ab}^{*} = 0$ on all lines $i$, $i \leq j$ and consider (4) for $k = j + 2 \leq m + 1$. Terms of order 2 or more there vanish by the induction hypothesis and we have only (5) with $a + b = j + 1 < m$ as a condition for IGNR. For $a = j$, $b = 1$ this implies $\chi_{j0}^{*} = 0$ (since $\chi_{j+1}0^{*} = 0$) and incrementing $b$ (and decrementing $a$) along the line $j + 1$ we similarly obtain $\chi_{ab}^{*} = 0$ for all values on the line $j + 1$.

The induction step fails at $j + 1 = m$ because we have no equation giving $\chi_{a0}$. We now have to consider IGNR for values on the lines $k \in (m, m + n)$.

**Lemma 2.** If (5) is satisfied for all values on the lines $a + b = k \in (m, m + n - 1)$, then $\chi_{ab}$ is IGNR.
Proof. For an entry $\chi_{ab}$ on a line $\ell > m$ the term homogeneous in $B$ and $\bar{B}$ of degree $k > m - \ell$ is zero as a consequence of lemma 1. For the terms homogeneous of degree $k \leq m - \ell$ in $\chi_{ab}'$, (4) gives the value

$$\chi_{ab}' = \sum_{s=0}^{k} \binom{a}{k-s} \binom{b}{s} B^{k-s} \bar{B}^s \chi_{(a-k+s)(b-s)'}.$$  

A direct check shows that this is equal to

$$\frac{1}{k} \sum_{s=0}^{(k-1)} \binom{a}{k-s-1} \binom{b}{s} B^{(k-s-1)} \bar{B}^s \{ (a-k+1+s)B\chi_{(a-k+s)(b-s)'} + (b-s)\bar{B}\chi_{(a-k+s+1)(b-s-1)'} \}.$$  

Thus $\chi_{ab}'$ is zero if (5) holds for all pairs $(p, q)$ on the line $p + q = a + b - k$. One may note that apart from the factor $1/k$ the coefficients of the brackets containing the $\chi$ terms in (7) are those which appear in the expansion of $\chi_{ab}(k-1)$, suggesting a recursive or iterative proof could also be given.

The equality between (6) and (7) provides in a general form the relationships that Pollney et al (2000a) overlooked.

Combining the two preceding lemmas and the remark concerning $m = n = 1$ we see that the following holds.

**Theorem 1.** A totally symmetrized spinor $\chi_{ab'}$ with valence $(m, n)$ such that $m \geq 1$ and $m \geq n$ is invariant under a one-dimensional group of null rotations which preserves the $o$ basis spinor and has parameter $B$ in (1) if and only if

$$\chi_{ab} = 0 \quad \forall \text{ pairs } a + b \leq m - 1,$$

$$0 = (a + 1)B\chi_{ab} + b\bar{B}\chi_{(a+1)(b-1)'}, \quad \forall \text{ pairs } a + b \in (m, m + n - 1).$$

One may note that, in accordance with the result stated by Pollney et al (2000a), the conditions (8) and (9) imply that a two-parameter group of null rotations is only possible if all $\chi_{ab}' = 0$ except $\chi_{mb'}$.

The conditions (8) and (9) are relatively easy to check, provided one has aligned the frame so that $\mathbf{k}$ is the vector preserved by the null rotation. Nonzero Weyl tensors of Petrov type N can be ignored, and then this means taking a frame in which only $\Psi_4 \neq 0$ (or taking the alternative choice with only $\Psi_0 \neq 0$, with consequent swapped values as above). This is generally easy to achieve.

In conformally flat spacetimes, for the Segre types with non-zero Ricci tensor which admit a null rotation isotropy the canonical forms in table 3 of Pollney et al (2000a) have $\Phi_{22'} \neq 0$, and for types $[(1,1,2), (1,3)]$, and $[(1,1,2), (1,1,2)]$ these are also the forms satisfying the tests above. However, for Segre type [111, 1, 1] the canonical form in which the above conditions for null rotation isotropy (about both $\mathbf{k}$ and $\mathbf{l}$) are manifest is not the one in that table. Instead one has to take a form in which only $\Phi_{11'}$ and $\Phi_{02}$ are non-zero and $2|\Phi_{11'}| = |\Phi_{02}|$: in that form the conditions are satisfied for both null rotations with the same parameter sets for $B$ and $\bar{B}$. For a complete procedure for checking for null rotation invariance in conformally flat spacetime one needs a process for identifying Segre types that may have IGNR, and then aligning the frame to the appropriate one of these canonical forms.
Assuming one has the frame well aligned, checking (8) is straightforward. Checking (9) for a one-parameter group with $B = |B|e^{i\theta}$ could be done in at least two ways. One could first apply a rotation to obtain a frame in which $B$ became pure imaginary (or, if preferred, real), obtaining the required rotation angle from one of the relations (9). Since the requirements imply that values along a line are simple multiples of one another, this should not entail nontrivial division of polynomials or other functions. After applying the rotation found, one could then check that the values on each line $a + b = k$, for $m + n - 1 \geq k \geq m$, satisfy (9).

An alternative is to take any nonzero pair of values of $\chi$ satisfying (5) and then for any other pair $(p, q)$ construct a quadratic in components which will vanish if the $(p, q)$ entries also satisfy (9). For example if the line $k = m + n - 1$ contains nonzero terms one could check that

$$mq\chi_{p(q-1)'}\chi_{(m-1)w'} = np\chi_{(p-1)q'}\chi_{m(n-1)'}$$

for all other $p + q = k$, $m + n - 1 > k \geq m$.

In both ways of proceeding, one has to allow for the possibility that for some metrics not all lines with $k > m$ are populated with nonzero expressions for the components. This was pointed out to me by Jan Åman. An example is provided by $\nabla^4\Psi_{ab'}$ for the metric (12.36) in Stephani et al. (2003) where the lines $k = 9$ and 11 contain only zeroes, although there are nonzero expressions in lines $k = 8, 10$ and 12 (here $m = 8$ and $n = 4$).

To carry out those checks one first has to locate the nonzero entries in $\chi_{ab'}$. Jan Åman has proposed doing so by finding the nonzero entry in $\chi_{ab'}$ which is first in numerical order. He is preparing a CLASSI module for making the required checks beginning with that strategy.

In applying the above theorem when checking for null rotation isotropy while classifying a spacetime metric, it may be useful to note that if there is such isotropy then the same isotropy group must apply to all the symmetrized spinors studied. So one could use the ratios found in one such spinor to construct the quadratic test for another such spinor.

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1 One then has to bear in mind, if using computer algebra software, that the software may be unable to determine whether or not a particular expression is equivalent to zero.
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