Optimal Actuator Location of the Minimum Norm Controls for Stochastic Heat Equations

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Abstract

In this paper, we study the approximate controllability for the stochastic heat equation over measurable sets, and the optimal actuator location of the minimum norm controls. We formulate a relaxed optimization problem for both actuator location and its corresponding minimum norm control into a two-person zero sum game problem and develop a sufficient and necessary condition for the optimal solution via Nash equilibrium. At last, we prove that the relaxed optimal solution is an optimal actuator location for the classical problem.

Keywords: stochastic heat equation, minimal norm control, optimal actuator location, Nash equilibrium.

AMS subject classifications: 35K05, 49J20, 93B05, 93B07, 93E20.

1 Introduction

Let $\mathcal{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a stochastic basis with usual conditions. On $\mathcal{F}$, we define a standard scalar Wiener process $W = \{w(t)\}_{t \geq 0}$. For simplicity, we assume that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is generated by $W$.

Fix $T > 0$.

Given a Hilbert space $H$, we denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all $H$-valued $\mathcal{F}_t$-adapted processes $X$ such that the square of the canonical norm $\mathbb{E}\|X(\cdot)\|^2_{L^2(0, T; H)} < \infty$; denote by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all $H$-valued $\mathcal{F}_t$-adapted bounded processes, with the essential supremum norm; and denote by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all $H$-valued $\mathcal{F}_t$-adapted continuous processes $X$ such that the square of the canonical norm $\mathbb{E}\|X(\cdot)\|^2_{C([0, T]; H)} < \infty$. For any $t \in [0, T]$, the space $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H)$ consists of all $H$-valued $\mathcal{F}_t$-measurable random variables with finite second moments.

Let $D$ be a bounded domain in $\mathbb{R}^d$ with a $C^2$ boundary $\partial D$. Let $G$ be measurable subset with positive measures of $D$.

We denote by $(\cdot, \cdot)$ the inner product in $L^2(D)$, and denote by $\| \cdot \|$ the norm induced by $(\cdot, \cdot)$. We also use the notations $(\cdot, \cdot)_G$ and $\| \cdot \|_G$ for the inner product and the norm defined on $L^2(G)$, respectively. We denote by $| \cdot |$ the Lebesgue measure on $\mathbb{R}^d$.

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Let $A$ be an unbounded linear operator on $L^2(D)$:

$$\mathcal{D}(A) = H^2(D) \cap H^1_0(D), \quad Av = \Delta v, \quad \forall v \in \mathcal{D}(A).$$

Consider the following stochastic heat equation

$$\begin{cases}
dy = Ay dt + \chi_G u(t) dt + a(t)y dw(t), & t \in (0, T), \\
y(0) = y_0,
\end{cases} \tag{1.1}$$

where $a \in L^\infty(0, T; \mathbb{R})$.

We say the system (1.1) is approximately controllable at time $T$, if for any initial data $y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2((D)))$, and any final state $y_1 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$ and any $\varepsilon > 0$, there exists a control $u$ in the space $L^2_F(0, T; L^2(D))$ such that the solution $y$ of the system (1.1) with initial data $y_0$ and control $u$ satisfies

$$\mathbb{E}\|y(T) - y_1\|^2 \leq \varepsilon.$$ 

Without of generality, we simply choose $y_1 = 0$.

In the sequel, fix $\varepsilon > 0$.

Our first result is to confirm the approximate controllability for system (1.1). Moreover, we solve the following minimum norm control problem

$$N_\varepsilon(G) = \inf \{\mathbb{E}\|u\|^2_{L^2((0,T) \times D)} \mid \mathbb{E}\|y(T; G, u)\|^2 \leq \varepsilon\}, \tag{1.2}$$

where $y(\cdot; G, u)$ is the solution of system (1.1). In the problem (1.2), we say $u$ is an admissible control if $u \in L^2_F(0, T; L^2(D))$ and $\mathbb{E}\|y(T; G, u)\|^2 \leq \varepsilon$; we say $u^*$ is a minimal norm control if $u^*$ is an admissible control such that $N_\varepsilon(G)$ is achieved.

To present our result, let us introduce the following backward stochastic heat equation

$$\begin{cases}
dz = -Az dt - a(t)Z dt + Z dw(t), & t \in (0, T), \\
z(T) = \eta. \tag{1.3}
\end{cases}$$

For each $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$, it is known (see, for example [10, 6]) that the equation (1.3) admits a unique solution $(z, Z)$ in the space of $(L^2_F(\Omega; C(0, T; L^2(D))) \cap L^2_{2F}(0, T; H^1_0(D))) \times L^2_F(0, T; L^2(D))$.

We also define a functional on $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$:

$$J_\varepsilon(\eta) = \frac{1}{2} \int_0^T \mathbb{E}\|z(t; \eta)\|^2_{2F} dt + \varepsilon(\mathbb{E}\|\eta\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta)). \tag{1.4}$$

**Theorem 1.1.** The system (1.1) is approximately controllable at time $T$. Moreover, denote by $\eta^*$ the minimizer of the functional defined in (1.4), $u^* = z(\cdot; \eta^*) \mid_G$ is a minimal norm control and

$$\int_0^T \mathbb{E}\|u^*(t)\|^2_{2F} dt \leq C(\mathbb{E}\|y_0\|^2)^2,$$

where $z(\cdot; \eta)$ is the solution of equation (1.3) with $z(T) = \eta^*$.

Controllability problems of deterministic partial differential equations are extensive studied in literature; see the survey articles [20, 21] and references therein. Recently, there are also some results obtained for the stochastic counterpart [4, 7, 11, 12], to name a few. In particular, authors
in [12] pointed out a different observation from the deterministic equations that null controllability does not necessarily imply approximate controllability in the stochastic system, and showed the later by duality argument and a Riesz-type representation theorem for general stochastic processes [13]. Unlike the method employed in [12], we use a variational technique and provide a constructive proof of approximate controllability, and furthermore, our method leads to the existence of the minimal norm control.

Thanks to the observability inequality developed in [19] (see [16, 2] for the deterministic version), our approximate controllability result is more robust and can allow the control domain \( G \) to be any measurable set with positive Lebesgue measure, compared to works mentioned earlier with the internal control living on an open set. This generalization facilitates the study of the optimal actuator location problem for a wider class of equations. For example, authors in [8] investigate the optimal actuator location of the minimum norm controls for deterministic heat equations in arbitrary dimensions, while [1] studied the one dimensional case and [9] considered a special class of controlled domains. For other actuator location problems, see [5, 17], and related numerical research [14, 15, 18].

The second part of our paper is devoted to the optimal actuator location of the minimum norm control problem for internal approximate controllable stochastic heat equations.

Given \( \alpha \in (0, 1) \), let

\[
W = \{ G \subseteq D \mid G \text{ is Lebesgue measurable with } |G| = \alpha |D| \},
\]

where \( |\cdot| \) is the Lebesgue measure on \( \mathbb{R}^d \).

A classical optimal actuator location of the minimal norm control problem is to seek a set \( G^* \in W \) such that

\[
N_\varepsilon(G^*) = \inf_{G \in W} N_\varepsilon(G).
\]

If such a \( G^* \) exists, we say that \( G^* \) is an optimal actuator location of the minimum norm controls. Any minimum norm control \( u^* \) satisfying

\[
\mathbb{E} \left\| u^* \chi_{G^*} \right\|_{L^2((0,T) \times D)}^2 = N_\varepsilon(G^*),
\]

is called a minimum norm control with respect to the optimal actuator location \( G^* \).

The existence of the optimal actuator location \( G^* \) is generally not guaranteed because of the absence of the compactness of \( W \). For this reason, we extend the feasible set \( W \) to a relaxed set \( B \) (see (3.1)), and solve a relaxed optimal actuator location problem. We prove the existence of the relaxed optimal actuator location and characterize the solution of the relaxed problem via a Nash equilibrium; see Section 3 for problem formulation and details.

The key contribution of this paper is that we are able to recover the solution from a relaxed problem to a solution of the classical optimal actuator location problem.

**Theorem 1.2.** There exists a solution to problem (1.6).

To the best of our knowledge, this is the first result on the existence of optimal actuator location problem in a stochastic system, and it can easily be applied to the null controllability of stochastic heat equations discussed in [19]. In a related work [17], authors studied a deterministic heat equation with random initial data, and minimized the constant in the observability inequality, instead of the norm of admissible controls.
The rest of the paper is organized as follows. In Section 2, we show the existence of the minimizer of $J_\varepsilon$ and construct the minimum norm control to prove Theorem 1.1. In Section 3, we solve a relaxed optimal actuator location problem and provide a sufficient and necessary condition for the relaxed optimal solution via Nash equilibrium. In Section 4, we show the existence of the a classical optimal actuator location problem.

2 Characterization of the minimum norm control

In this section, we study the approximate controllability of the system (1.1). We provide a constructive proof of the existence of an admissible control, and then show that this control is indeed a minimum norm control that solves problem (1.2).

First, we prove the existence of the minimizer of $J_\varepsilon$ defined in (1.3).

Lemma 2.1. Suppose $y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(D))$. Then there exists $\eta^* \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$ such that

$$J_\varepsilon(\eta^*) = \min_{\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))} J_\varepsilon(\eta).$$

Proof. We first show that $J_\varepsilon(\cdot)$ is convex. To this end, fix $\theta \in (0, 1)$ and let $\eta_1, \eta_2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$. By linearity, we have

$$z(\cdot; \theta \eta_1 + (1 - \theta) \eta_2) = \theta z(\cdot; \eta_1) + (1 - \theta) z(\cdot; \eta_2).$$

Thus,

$$J_\varepsilon(\theta \eta_1 + (1 - \theta) \eta_2) = \frac{1}{2} \int_0^T \mathbb{E}\|\theta z(\cdot; \eta_1) + (1 - \theta) z(\cdot; \eta_2)\|_G^2 \, dt + \varepsilon \mathbb{E}\|\theta \eta_1 + (1 - \theta) \eta_2\|^2 \|_{L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))}^{1/2}$$

$$+ \mathbb{E}(y_0, \theta z(0; \eta_1) + (1 - \theta) z(0; \eta_2))$$

$$\leq \frac{\theta}{2} \int_0^T \mathbb{E}\|z(\cdot; \eta_1)\|_G^2 \, dt + \frac{1 - \theta}{2} \int_0^T \mathbb{E}\|z(\cdot; \eta_2)\|_G^2 \, dt$$

$$+ \varepsilon \theta \mathbb{E}\|\eta_1\|^2 \|_{L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))}^{1/2} + \varepsilon (1 - \theta) \mathbb{E}\|\eta_2\|^2 \|_{L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))}^{1/2} + \mathbb{E}(y_0, \theta z(0; \eta_1) + (1 - \theta) z(0; \eta_2)).$$

On the other hand,

$$\theta J_\varepsilon(\eta_1) + (1 - \theta) J_\varepsilon(\eta_2) = \theta \left[ \frac{1}{2} \int_0^T \mathbb{E}\|z(\cdot; \eta_1)\|_G^2 \, dt + \varepsilon \mathbb{E}\|\eta_1\| + \mathbb{E}(y_0, z(0; \eta_1)) \right]$$

$$+ (1 - \theta) \left[ \frac{1}{2} \int_0^T \mathbb{E}\|z(\cdot; \eta_2)\|_G^2 \, dt + \varepsilon \mathbb{E}\|\eta_2\| + \mathbb{E}(y_0, z(0; \eta_2)) \right]$$

$$= \frac{\theta}{2} \int_0^T \mathbb{E}\|z(\cdot; \eta_1)\|_G^2 \, dt + \frac{1 - \theta}{2} \int_0^T \mathbb{E}\|z(\cdot; \eta_2)\|_G^2 \, dt$$

$$+ \varepsilon \theta \mathbb{E}\|\eta_1\|^2 \|_{L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))}^{1/2} + \varepsilon (1 - \theta) \mathbb{E}\|\eta_2\|^2 \|_{L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))}^{1/2} + \mathbb{E}(y_0, \theta z(0; \eta_1) + (1 - \theta) z(0; \eta_2)).$$

Therefore, we obtain

$$J_\varepsilon(\theta \eta_1 + (1 - \theta) \eta_2) \leq \theta J_\varepsilon(\eta_1) + (1 - \theta) J_\varepsilon(\eta_2).$$
and so $\mathcal{J}_\varepsilon(\cdot)$ is convex.

Next, we prove that the functional $\mathcal{J}_\varepsilon(\cdot)$ is continuous. In fact, assume $\eta, \varphi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$, and $h \in \mathbb{R}$, then

$$
\mathcal{J}_\varepsilon(\eta + h\varphi) - \mathcal{J}_\varepsilon(\eta) \\
= \frac{1}{2} \int_0^T \mathbb{E}\|z(t; \eta + h\varphi)\|^2_G dt + \varepsilon(\mathbb{E}\|\eta + h\varphi\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta + h\varphi)) \\
- \frac{1}{2} \int_0^T \mathbb{E}\|z(t; \eta)\|^2_G dt - \varepsilon(\mathbb{E}\|\eta\|^2)^{1/2} - \mathbb{E}(y_0, z(0; \eta)) \\
h \int_0^T \mathbb{E}(z(t; \eta), z(t; \varphi))_G dt + \frac{1}{2} h^2 \int_0^T \mathbb{E}\|z(t; \varphi)\|^2_G dt \\
+ \varepsilon((\mathbb{E}\|\eta + h\varphi\|^2)^{1/2} - (\mathbb{E}\|\eta\|^2)^{1/2}) + h\mathbb{E}(y_0, z(0; \varphi)) \to 0,
$$

as $h \to 0$, since $|\mathbb{E}\|\eta + h\varphi\|^2 - (\mathbb{E}\|\eta\|^2)| \leq |h|(|\mathbb{E}\|\eta\|^2|^{1/2}.

To prove the coercivity, let $\{\eta_n\} \subseteq L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$ such that $\mathbb{E}\|\eta_n\| \to \infty$ as $n \to \infty$, and let $\tilde{\eta}_n = \eta_n/(\mathbb{E}\|\eta_n\|^2)^{1/2}$, so that $(\mathbb{E}\|\tilde{\eta}_n\|^2)^{1/2} = 1$. Then

$$
\mathcal{J}_\varepsilon(\eta_n) = \frac{1}{2}(\mathbb{E}\|\eta_n\|^2)^{1/2} \int_0^T \mathbb{E}\|z(t; \tilde{\eta}_n)\|^2_G dt + \varepsilon + \mathbb{E}(y_0, z(0; \tilde{\eta}_n)).
$$

Note that

$$
|\mathbb{E}(y_0, z(0; \tilde{\eta}_n))| \leq (\mathbb{E}\|y_0\|^2)^{1/2}(\mathbb{E}\|z(0, \tilde{\eta}_n)\|^2)^{1/2} \leq C(\mathbb{E}\|y_0\|^2)^{1/2}.
$$

If $\liminf_{n \to \infty} \int_0^T \mathbb{E}\|z(t; \tilde{\eta}_n)\|^2_G dt > 0$, we have

$$
\mathcal{J}_\varepsilon(\eta_n) \to \infty \quad \text{as} \quad (\mathbb{E}\|\eta_n\|^2)^{1/2} \to \infty,
$$

which implies the coercivity of $\mathcal{J}_\varepsilon$.

If $\liminf_{n \to \infty} \int_0^T \mathbb{E}\|z(t; \tilde{\eta}_n)\|^2_G dt = 0$, then it follows from the observability inequality in [10, Theorem 1.1] that $\tilde{\eta}_n$ is bounded in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$. Thus, we can extract a subsequence $\{\tilde{\eta}_{n_j}\} \subseteq \{\tilde{\eta}_n\}$ such that $\tilde{\eta}_{n_j} \to \tilde{\eta}$ weakly in $L^2(D)$ and $z(\cdot; \tilde{\eta}_{n_j}) \to z(\cdot; \tilde{\eta})$ weakly in $L^2(0, T; H^1(D)) \cap H^1(0, T; H^{-1}(D))$, $\mathbb{P}$-a.s. Moreover, by lower semi-continuity we obtain

$$
\int_0^T \mathbb{E}\|z(t; \tilde{\eta})\|^2_G dt \leq \liminf_{j \to \infty} \int_0^T \mathbb{E}\|z(t; \tilde{\eta}_{n_j})\|^2_G dt = 0.
$$

Then by the observability inequality again, we get $z(0; \tilde{\eta}) = 0$, and thus

$$
\liminf_{n \to \infty} \frac{\mathcal{J}_\varepsilon(\eta_n)}{(\mathbb{E}\|\eta_n\|^2)^{1/2}} \geq \liminf_{j \to \infty} \frac{\mathcal{J}_\varepsilon(\eta_{n_j})}{(\mathbb{E}\|\eta_{n_j}\|^2)^{1/2}} \geq \liminf_{j \to \infty}[\varepsilon + \mathbb{E}(y_0, z(0; \tilde{\eta}_{n_j}))] \\
= \varepsilon + \mathbb{E}(y_0, z(0, \tilde{\eta})) = \varepsilon,
$$

which implies (2.1), and so $\mathcal{J}_\varepsilon$ is coercive.

To sum up, we showed that $\mathcal{J}_\varepsilon$ is convex, continuous, and coercive, and thus the minimizer of $\mathcal{J}_\varepsilon$ exists.
Now we are ready to prove Theorem 1.1 and characterize the minimum norm control via the minimizer $\eta^*$ of $J_\varepsilon$ obtained in Lemma 2.1.

**Proof of Theorem 1.1.** Since $J_\varepsilon$ attains its minimum value at $\eta^*$, for any $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$ and $h \in \mathbb{R}$ we have

$$J_\varepsilon(\eta^*) \leq J_\varepsilon(\eta^* + h\eta).$$

Thus,

$$J_\varepsilon(\eta^* + h\eta) - J_\varepsilon(\eta^*) = \frac{1}{2} \int_0^T \mathbb{E}\|z(t; \eta^* + h\eta)\|^2_G dt + \varepsilon(\mathbb{E}\|\eta^* + h\eta\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^* + h\eta)) - \frac{1}{2} \int_0^T \mathbb{E}\|z(t; \eta^*)\|^2_G dt - \varepsilon(\mathbb{E}\|\eta^*\|^2)^{1/2} - \mathbb{E}(y_0, z(0; \eta^*))
$$

$$= \frac{h^2}{2} \int_0^T \mathbb{E}\|z(t; \eta^*)\|^2_G dt + h \int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt + \varepsilon((\mathbb{E}\|\eta^* + h\eta\|^2)^{1/2} - (\mathbb{E}\|\eta^*\|^2)^{1/2}) + h\mathbb{E}(y_0, z(0; \eta)) \geq 0. \quad (2.3)$$

Also note that

$$|\mathbb{E}\|\eta^* + h\eta\|^2 - (\mathbb{E}\|\eta^*\|^2)| \leq |h|(|\mathbb{E}\|\eta\|^2)|^{1/2},$$

then we obtain

$$\frac{h^2}{2} \int_0^T \mathbb{E}\|z(t; \eta^*)\|^2_G dt + h \int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt + \varepsilon(|\mathbb{E}\|\eta\|^2)|^{1/2} + h\mathbb{E}(y_0, z(0; \eta)) \geq 0.$$  

If $h > 0$, then dividing $h$ and sending $h \to 0^+$ yield

$$\int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt + \varepsilon(\mathbb{E}\|\eta\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta)) \geq 0.$$  

If $h < 0$, then dividing $h$ and sending $h \to 0^-$ yield

$$\int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt - \varepsilon(\mathbb{E}\|\eta\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta)) \leq 0.$$  

Consequently,

$$\left| \int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt + \mathbb{E}(y_0, z(0; \eta)) \right| \leq \varepsilon(\mathbb{E}\|\eta\|^2)^{1/2}. \quad (2.4)$$

Now applying Itô’s formula to $(y, z)$ yields

$$d(y, z) = (y, dz) + (dy, z) + a(y, Z) dt$$

$$= -(y, Az) dt - a(y, Z) dt + (y, Z) dw(t)$$

$$+ (Ay, z) dt + (u, z)_G dt + a(y, z) dw(t) + a(y, Z) dt$$

$$= (u, z)_G dt + (y, Z) dw(t) + a(y, z) dw(t).$$
After we rewrite it in the integral form and take the expectation on both sides, we get

$$\mathbb{E}(y_0, z(0; \eta)) = \mathbb{E}(y(T; u), \eta) - \int_0^T \mathbb{E}(z(t; \eta), u). \tag{2.5}$$

Plugging (2.5) into (2.4) and letting $u = u^* = z(\cdot; \eta^*) \mid_G$, we obtain

$$|\mathbb{E}(y(T; u^*), \eta)| \leq \varepsilon(\|\eta\|^2)^{1/2}.$$  

Since $\eta$ is an arbitrary element in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))$, we conclude that

$$(\mathbb{E}\|y(T; u^*)\|^2)^{1/2} \leq \varepsilon.$$  

Therefore, $u^* = z(\cdot; \eta^*) \mid_G$ is an admissible control.

Next, we show that $u^*$ is indeed the minimum norm control. To this end, let’s go back to the inequality (2.2). Note that

$$\lim_{h \to 0} \frac{(\mathbb{E}\|\eta^* + h\eta\|^2)^{1/2} - (\mathbb{E}\|\eta^*\|^2)^{1/2}}{h} = \lim_{h \to 0} \frac{\mathbb{E}\|\eta^* + h\eta\|^2 - \mathbb{E}\|\eta^*\|^2}{h} \frac{1}{2(\mathbb{E}\|\eta^*\|^2)^{1/2}} = \lim_{h \to 0} \frac{\mathbb{E}\|\eta^* + h\eta\|^2 - \mathbb{E}\|\eta^*\|^2}{\mathbb{E}\|\eta^*\|^2} \frac{1}{2h} \mathbb{E}(\eta^*, \eta) + h^2 \mathbb{E}\|\eta\|^2$$

If $h > 0$, then dividing $h$ and sending $h \to 0^+$ in (2.2) yields

$$\int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt + \varepsilon \cdot \frac{\mathbb{E}(\eta^*, \eta)}{(\mathbb{E}\|\eta^*\|^2)^{1/2}} \mathbb{E}(y_0, z(0; \eta)) \geq 0.$$  

If $h < 0$, then dividing $h$ and sending $h \to 0^-$ in (2.2) yields

$$\int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt + \varepsilon \cdot \frac{\mathbb{E}(\eta^*, \eta)}{(\mathbb{E}\|\eta^*\|^2)^{1/2}} \mathbb{E}(y_0, z(0; \eta)) \leq 0.$$  

To sum up, we have the following Euler-Lagrange equation

$$\int_0^T \mathbb{E}(z(t; \eta^*), z(t; \eta))_G dt + \varepsilon \cdot \frac{\mathbb{E}(\eta^*, \eta)}{(\mathbb{E}\|\eta^*\|^2)^{1/2}} \mathbb{E}(y_0, z(0; \eta)) = 0. \tag{2.6}$$

Plugging (2.5) into (2.6) and using $u^* = z(\cdot; \eta^*) \mid_G$, we get

$$\mathbb{E}(y(T; u^*), \eta) + \varepsilon \cdot \frac{\mathbb{E}(\eta^*, \eta)}{(\mathbb{E}\|\eta^*\|^2)^{1/2}} = 0.$$
If particular, by choosing $\eta = \eta^*$,
\[ E(y(T; u^*), \eta^*) = -\varepsilon (E \| \eta^* \|^2)^{1/2}. \tag{2.7} \]

Suppose now there is another admissible control $\tilde{u}$ such that
\[ (E \| y(T; \eta^*) \|^2)^{1/2} \leq \varepsilon. \]
Then we have by (2.7)
\[
E(y(T; \tilde{u}), \eta^*) \geq -E(y(T; \tilde{u}), \eta^*) \geq -E(y(T; u^*), \eta^*) = (E \| y(T; u^*) \|^2)^{1/2}.
\]

Using (2.5) again, we arrive at
\[
\int_0^T E(z(t; \eta^*), \tilde{u})_G dt \geq \int_0^T E(z(t; \eta^*), u^*)_G dt.
\]
Since $u^* = z(\cdot; \eta^*) \mid_G$, we obtain
\[
\int_0^T E \| u^*(t) \|^2_G dt \leq \int_0^T E(u^*(t), \tilde{u}(t))_G dt,
\]
and by Cauchy-Schwarz inequality,
\[
\int_0^T E \| u^*(t) \|^2_G dt \leq \int_0^T E \| \tilde{u}(t) \|^2_G dt,
\]
which implies the optimality of $u^*$.

To show the boundedness for $u^*$, let’s replace $\eta$ in (2.6), and then
\[
\int_0^T E \| z(t; \eta^*) \|^2_G dt + \varepsilon (E \| \eta^* \|^2)^{1/2} + E(y_0, z(0; \eta^*)) = 0.
\]
By the observability inequality (see [19] Lemma 3.2]) for the (relaxed) system (3.2) that
\[
E \| z(0; \eta^*) \|^2 \leq C \int_0^T E \| z(t; \eta^*) \|^2_G dt,
\]
for some $C > 0$, independent of $\beta$, and thus
\[
\frac{1}{C} E \| z(0; \eta^*) \|^2 + \varepsilon (E \| \eta^* \|^2)^{1/2} + E(y_0, z(0; \eta^*)) \leq 0. \tag{2.8}
\]
Consequently,
\[
\frac{1}{C} E \| z(0; \eta^*) \|^2 \leq -E(y_0, z(0; \eta^*)) \leq (E \| y_0 \|^2)^{1/2} (E \| z(0; \eta^*) \|^2)^{1/2},
\]
or
\[
(E \| z(0; \eta^*) \|^2)^{1/2} \leq C (E \| y_0 \|^2)^{1/2}.
\]
Using (2.8) again, we have
\[ \varepsilon (\| \eta^* \|^2)^{1/2} \leq (\mathbb{E} \| y_0 \|^2)^{1/2} (\mathbb{E} \| z(0; \eta^*) \|^2)^{1/2} \leq C \mathbb{E} \| y_0 \|^2, \]
or equivalently,
\[ (\mathbb{E} \| \eta^* \|^2)^{1/2} \leq \frac{C}{\varepsilon} \mathbb{E} \| y_0 \|^2. \]
Therefore,
\[ \int_0^T \mathbb{E} \| u^*(t) \|_G^2 \, dt = \int_0^T \mathbb{E} \| z(t; \eta^*) \|_G^2 \, dt \leq \int_0^T \mathbb{E} \| z(t; \eta^*) \|^2 \, dt \leq C \mathbb{E} \| \eta^* \|^2 \leq C(\mathbb{E} \| y_0 \|^2)^2, \]
which completes the proof.

3 A relaxed optimal actuator location problem

Without the compactness of \( W \) defined in (1.5), it seems very difficult to solve the classical optimal actuator location problem directly. Instead, we study a relaxed problem and provide a solution in the framework of a two-person zero sum game via Nash equilibrium. To this end, define
\[ B = \{ \beta \in L^\infty(D; [0,1]) \mid \| \beta \|^2 = \alpha \| D \| \}. \] (3.1)

Note that the set \( B \) is a relaxation of the set \( \{ \chi_G \mid G \in W \} \).

For any \( \beta \in B \), consider the following equation
\[ \begin{cases} dy = Ay \, dt + \beta u(t) \, dt + a(t)y \, dw(t), & t \in (0, T), \\ y(0) = y_0. \end{cases} \] (3.2)

We denote by \( y(\cdot; \beta, u) \) the solution of equation (3.2), and say the system (3.2) approximately controllable at time \( T \) if for any \( \varepsilon > 0 \), there exists \( u \in L^2(0,T; L^2(D)) \) such that \( \mathbb{E} \| y(T; \beta, u) \|^2 \leq \varepsilon \).

Accordingly, the problem (1.2) is replaced by
\[ N_\varepsilon(\beta) = \inf \{ \mathbb{E} \| u \|_{L^2((0,T) \times D)}^2 \mid \mathbb{E} \| y(T; \beta, u) \|^2 \leq \varepsilon \}, \] (3.3)
and the classical optimal actuator location problem (1.6) is changed into the following relaxed problem
\[ N_\varepsilon(\beta^*) = \inf_{\beta \in B} N_\varepsilon(\beta). \] (3.4)

Any solution \( \beta^* \) to the problem (3.4) is called a relaxed optimal actuator location of the minimal norm controls.

Theorem 3.1. There exists at least one solution of the problem (3.4). In addition, \( \beta^* \) is a relaxed optimal actuator location of the minimal norm controls if and only if there exists \( \eta^* \in U_M \) such
that the pair \((\beta^*, \eta^*)\) is a Nash equilibrium of the following two-person zero-sum game problem: to find \((\beta^*, \eta^*) \in \mathcal{B} \times \mathcal{U}_M\) such that

\[
\frac{1}{2} \int_0^T \mathbb{E}\|\beta^* z(t; \eta^*)\|^2 dt + \varepsilon (\mathbb{E}\|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*))
\]

\[
\leq \sup_{\beta \in \mathcal{B}} \left\{ \frac{1}{2} \int_0^T \mathbb{E}\|\beta z(t; \eta^*)\|^2 dt + \varepsilon (\mathbb{E}\|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*)) \right\},
\]

\[
\frac{1}{2} \int_0^T \mathbb{E}\|\beta^* z(t; \eta^*)\|^2 dt + \varepsilon (\mathbb{E}\|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*))
\]

\[
= \inf_{\eta \in \mathcal{U}_M} \left\{ \frac{1}{2} \int_0^T \mathbb{E}\|\beta^* z(t; \eta)\|^2 dt + \varepsilon (\mathbb{E}\|\eta\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta)) \right\},
\]

where \(\mathcal{U}_M = \{ \eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D)) \mid \mathbb{E}\|\eta\|^2 \leq M \}\), for some constant \(M > 0\) (see Remark 3.3).

To prove this theorem, we first study a variational problem

\[
\mathcal{J}_c(\beta) = \inf_{\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))} \mathcal{J}_c(\eta; \beta),
\]

where

\[
\mathcal{J}_c(\eta; \beta) = \frac{1}{2} \int_0^T \mathbb{E}\|\beta z(t; \eta)\|^2 dt + \varepsilon (\mathbb{E}\|\eta\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta)).
\]

**Lemma 3.2.** Fix \(\beta \in \mathcal{B}\). Suppose \(y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(D))\). Then

1. Problem \((3.5)\) admits a unique solution \(\eta^*\) in the space \(L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))\), and there exists a positive constant \(C\), independent of \(\beta\) such that

\[
(\mathbb{E}\|\eta^*\|^2)^{1/2} \leq C \mathbb{E}\|y_0\|^2.
\]

2. The control defined by

\[
u^* = \beta z(\cdot; \eta^*)
\]

is the minimal norm control to the problem \((3.3)\), where \(z(\cdot; \eta^*)\) is the solution of the adjoint system \((3.1)\) with \(z(T) = \eta^*\). Moreover,

\[
N_c(\beta) = -2 \mathcal{J}_c(\beta).
\]

**Proof.** In the same spirit of Lemma 2.1 and Theorem 1.1, the existence of \(\eta^*\) and optimality of \(\nu^*\) can be similarly verified.

Moreover, we can also obtain the following Euler-Lagrange equation

\[
\int_0^T \mathbb{E}(\beta^2 z(t; \eta^*), z(t; \eta)) dt + \varepsilon \cdot \frac{\mathbb{E}(\eta^*, \eta)}{(\mathbb{E}\|\eta^*\|^2)^{1/2}} + \mathbb{E}(y_0, z(0; \eta)) = 0,
\]

for any \(\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))\). In particular, choosing \(\eta = \eta^*\) yields

\[
\int_0^T \mathbb{E}\|\beta^* z(t; \eta^*)\|^2 dt + \varepsilon (\mathbb{E}\|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*)) = 0.
\]
It follows from the observability inequality (see [10, Lemma 3.2]) for the (relaxed) system (3.2) that
\[
\frac{1}{C} \mathbb{E} \|z(0; \eta^*)\|^2 \leq \int_0^T \mathbb{E} \|\beta z(t; \eta^*)\|^2 \, dt,
\]
for some \(C > 0\), independent of \(\beta\), and thus
\[
\frac{1}{C} \mathbb{E} \|z(0; \eta^*)\|^2 + \varepsilon (\mathbb{E} \|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*)) \leq 0. \tag{3.11}
\]
Consequently,
\[
\frac{1}{C} \mathbb{E} \|z(0; \eta^*)\|^2 \leq -\mathbb{E}(y_0, z(0; \eta^*)) \leq (\mathbb{E} \|y_0\|^2)^{1/2}(\mathbb{E} \|z(0; \eta^*)\|^2)^{1/2},
\]
or
\[
(\mathbb{E} \|z(0; \eta^*)\|^2)^{1/2} \leq C(\mathbb{E} \|y_0\|^2)^{1/2}.
\]
Using (3.11) again, we have
\[
\varepsilon (\mathbb{E} \|\eta^*\|^2)^{1/2} \leq (\mathbb{E} \|y_0\|^2)^{1/2}(\mathbb{E} \|z(0; \eta^*)\|^2)^{1/2} \leq C\mathbb{E} \|y_0\|^2,
\]
or equivalently,
\[
(\mathbb{E} \|\eta^*\|^2)^{1/2} \leq \frac{C}{\varepsilon} \mathbb{E} \|y_0\|^2,
\]
which proves (3.7).

Setting \(z(\cdot; \eta^*) = u^*/\beta\) in (3.10) yield
\[
\int_0^T \mathbb{E} \|u^*(t)\|^2 \, dt = -\varepsilon (\mathbb{E} \|\eta^*\|^2)^{1/2} - \mathbb{E}(y_0, z(0; \eta^*)), \tag{3.12}
\]
or equivalently
\[
N_{\varepsilon}(\beta) = -\varepsilon (\mathbb{E} \|\eta^*\|^2)^{1/2} - \mathbb{E}(y_0, z(0; \eta^*)).
\]
On the other hand, we have by (3.12)
\[
\mathcal{J}_{\varepsilon}(\beta) = \mathcal{J}_{\varepsilon}(\eta^*; \beta) = \frac{1}{2} \int_0^T \mathbb{E} \|\beta z(t; \eta^*)\|^2 \, dt + \varepsilon (\mathbb{E} \|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*))
\]
\[
= \frac{1}{2} \int_0^T \mathbb{E} \|u^*(t)\|^2 \, dt + \varepsilon (\mathbb{E} \|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*))
\]
\[
= \frac{1}{2} \left[ \varepsilon (\mathbb{E} \|\eta^*\|^2)^{1/2} + \mathbb{E}(y_0, z(0; \eta^*)) \right] \]
\[
= -\frac{1}{2} N_{\varepsilon}(\beta),
\]
which concludes the proof.

Remark 3.3. It follows from (3.7) that we can restrict our search of the minimizer of \(\mathcal{J}_{\varepsilon}(\cdot; \beta)\) within a closed ball of \(L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D))\), and we denote it by \(U_M\) for some \(M > 0\).
Now let’s formulate an equivalent problem to (3.4), which is a two-person zero sum game. Define
\[
\Theta = \left\{ \theta \in L^\infty(D; [0, 1]) \mid \int_D \theta(x) dx = \alpha |D| \right\}. \tag{3.13}
\]
It is clear that \( \beta^2 \in \Theta \) for any \( \beta \in B \), and \( \theta^{1/2} \in B \) for all \( \theta \in \Theta \). \( \tag{3.14} \)
We also define a functional \( f : \Theta \times U_M \to \mathbb{R} \) by
\[
f(\theta, \eta) = -\frac{1}{2} \int_0^T \mathbb{E} \| \theta^{1/2} z(t; \eta) \|^2 dt - \varepsilon (\mathbb{E} \| \eta \|^2)^{1/2} - \mathbb{E}(y_0, z(0; \eta)). \tag{3.15}\]
Then it follows from the relation (3.9) that
\[
\inf_{\beta \in B} \frac{1}{2} N_\varepsilon(\beta) = \inf_{\beta \in B} -\mathcal{J}_\varepsilon(\beta) = \inf_{\theta \in \Theta} \sup_{\eta \in U_M} f(\theta, \eta).
\]
Therefore, seeking a minimizer \( \beta^* \in B \) for \( N(\beta) \) amounts to finding a minimizer \( \theta^* \) for \( \sup_{\eta \in U_M} f(\theta, \eta) \).

**Proof of Theorem 3.1**  
In the following, we will solve both the problems
\[
\inf_{\theta \in \Theta} \sup_{\eta \in U_M} f(\theta, \eta) \text{ and } \sup_{\eta \in U_M} \inf_{\theta \in \Theta} f(\theta, \eta),
\]
at the same time.

(1) Let us equip \( L^\infty(D) \) with the weak* topology. Then \( \Theta \) is compact and convex in \( L^\infty(D) \).

(2) Since \( U_M \) is a closed ball, it is weak compact and convex in \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(D)) \).

(3) Apparently, for each \( \eta \in U_M \), the function \( \theta \mapsto f(\theta, \eta) \) is convex and continuous.

(4) Fix \( \theta \in \Theta \), assume \( \eta_n \) converges weakly in \( U_M \) to \( \eta \). Then \( z(\cdot; \eta_n) \) converges weakly to \( z(\cdot; \eta) \).

In other words, the function \( \eta \mapsto f(\theta, \eta) \) is continuous. It is also easy to see that the function is concave.

It follows from (1)-(4), and the Von Neumann minimax theorem (see [3, Theorem 2.7.2]) that there exist \( \theta^* \in \Theta \) and \( \eta \in U_M \) that
\[
f(\theta^*, \eta) \leq f(\theta^*, \eta^*) \leq f(\theta, \eta^*), \quad \forall \theta \in \Theta, \eta \in U_M.
\]
Note that \( \mathcal{J}_\varepsilon(\eta; \beta) = -f(\theta, \eta) \), where \( \theta = \beta^2 \), and thus
\[
\mathcal{J}_\varepsilon(\eta^*; \beta) \leq \mathcal{J}_\varepsilon(\eta^*; \beta^*) \leq \mathcal{J}_\varepsilon(\eta; \beta^*), \quad \forall \beta \in B, \eta \in U_M,
\]

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where $\beta^* = (\theta^*)^{1/2}$.

By definition, the pair $(\beta^*, \eta^*)$ is a Nash equilibrium of the two-person zero sum game problem: to find $(\beta^*, \eta^*) \in B \times U_M$ such that

$$J_\varepsilon(\eta^*; \beta^*) = \sup_{\beta \in B} J_\varepsilon(\eta^*; \beta),$$

and

$$J_\varepsilon(\eta^*; \beta^*) = \inf_{\theta \in U_M} J_\varepsilon(\eta; \beta^*).$$

This completes the proof.

4 Existence of classical optimal actuator location

In this section, we will prove our main result on the existence of classical optimal actuator location, that is, we solve the problem (1.6). To this end, let’s take a closer look at the corresponding relaxed problem. It follows from Theorem 3.1 that

$$J_\varepsilon(\eta^*; \beta^*) = \sup_{\beta \in B} J_\varepsilon(\eta^*; \beta)$$

$$= \sup_{\beta \in B} \left[ \frac{1}{2} \int_0^T E\|\beta z(t; \eta^*)\|^2 dt + \varepsilon(E\|\eta^*\|^2)^{1/2} + E(y_0, z(0; \eta^*)) \right]$$

$$= \frac{1}{2} \sup_{\beta \in B} \left[ \int_0^T E\|\beta z(t; \eta^*)\|^2 dt + \frac{1}{2} \left[ \varepsilon(E\|\eta^*\|^2)^{1/2} + E(y_0, z(0; \eta^*)) \right] \right]$$

$$= \frac{1}{2} \sup_{\theta \in \Theta} \int_D \theta(x) \int_0^T E|z(t; \eta^*)|^2 dt dx + \frac{1}{2} \left[ \varepsilon(E\|\eta^*\|^2)^{1/2} + E(y_0, z(0; \eta^*)) \right].$$

To wit, if $\beta^*$ is a relaxed optimal actuator location, then $\theta^* = (\beta^*)^2$ solves the following problem

$$\sup_{\theta \in \Theta} \int_D \theta(x) \int_0^T E|z(t; \eta^*)|^2 dt dx.$$ Let

$$H(x) := \int_0^T E|z(t, x; \eta^*)|^2 dt,$$

then $H(x) \geq 0$ for $x \in D$-a.e.

Next, we show that

$$\theta^* = \arg\max_{\theta \in \Theta} \int_D \theta(x) H(x) dx$$

is an indicator function. Therefore, $\beta^* = (\theta^*)^{1/2}$ is also an indicator function, and thus $\beta^* \in B$, i.e., the relaxed optimal actuator location is indeed a classical optimal actuator location.

Define

$$c(\alpha) = \sup\{c \geq 0 : |\{H \geq c\}| \geq \alpha |D|\}.$$

Let $\{c_n\}$ be an increasing sequence that converges to $c(\alpha)$. Then for any $n \in \mathbb{N}$, we have $|\{H \geq c_n\}| \geq \alpha |D|$. But

$$\{H \geq c(\alpha)\} = \bigcap_{n=1}^\infty \{H \geq c_n\},$$

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and thus
\[ |\{H \geq c(\alpha)\}| = \lim_{n \to \infty} |\{H \geq c_n\}| \geq \alpha|D|. \tag{4.1} \]

On the other hand, for any \(\varepsilon > 0\), we have
\[ |\{H \geq c(\alpha) + \varepsilon\}| < \alpha|D|. \]

Note that
\[ \{H > c(\alpha)\} = \bigcup_{\varepsilon > 0} \{H \geq c(\alpha) + \varepsilon\}, \]
and so
\[ |\{H > c(\alpha)\}| = \lim_{\varepsilon \to 0} \{H \geq c(\alpha) + \varepsilon\} \leq \alpha|D|. \tag{4.2} \]

We claim that
\[ \theta^*(x) = 1 \text{ for } x \in \{H > c(\alpha)\} \text{ a.e.} \]
\[ \theta^*(x) = 0 \text{ for } x \in \{H < c(\alpha)\} \text{ a.e.} \tag{4.3} \]

Otherwise, let
\[ A = \{H > c(\alpha)\} \cap \{\theta^* < 1\} \quad \text{and} \quad B = \{H < c(\alpha)\} \cap \{\theta^* > 0\}, \]

then
\[ |A| > 0 \text{ or } |B| > 0. \]

Let’s first assume that \(|A| > 0\). In this case, we define
\[ \bar{\theta}(x) = \begin{cases} 1, & \{H > c(\alpha)\} \cup E; \\ 0, & \text{otherwise}, \end{cases} \tag{4.4} \]

where
\[ E \subseteq \{H = c(\alpha)\} \quad \text{and} \quad |E| = \alpha|D| - |\{H > c(\alpha)\}|. \]

It is noted that \(\bar{\theta}\) is well defined by (4.1) and (4.2), and \(\bar{\theta} \in \Theta\). Thus
\[ \int_{\theta^* > \bar{\theta}} (\theta^* - \bar{\theta}) \, dx + \int_{\theta^* < \bar{\theta}} (\theta^* - \bar{\theta}) \, dx = \int_D (\theta^* - \bar{\theta}) \, dx = 0. \]

Since
\[ \{\bar{\theta} > \theta^*\} = \{\bar{\theta} = 1\} \cap \{\theta^* < 1\} = (\{H > c(\alpha)\} \cup E) \cap \{\theta^* < 1\} \]
\[ = A \cup (E \cap \{\theta^* < 1\}), \]

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we obtain
\[
\int_{\theta^* < \tilde{\theta}} (\theta^* - \tilde{\theta}) H(x) \, dx = \int_{\theta^* < \tilde{\theta}} \chi_{A \cup (E \cap \{\theta^* < 1\})} (\theta^* - \tilde{\theta}) H(x) \, dx \\
= \int_{\theta^* < \tilde{\theta}} \chi_A (\theta^* - \tilde{\theta}) H(x) \, dx \\
+ \int_{\theta^* < \tilde{\theta}} \chi_{E \cap \{\theta^* < 1\}} (\theta^* - \tilde{\theta}) H(x) \, dx \\
= \int_{\theta^* < \tilde{\theta}} \chi_A (\theta^* - \tilde{\theta}) H(x) \, dx \\
+ \int_{\theta^* < \tilde{\theta}} \chi_{E \cap \{\theta^* < 1\}} (\theta^* - \tilde{\theta}) c(\alpha) \, dx \\
< \int_{\theta^* < \tilde{\theta}} \chi_A (\theta^* - \tilde{\theta}) c(\alpha) \, dx \\
+ \int_{\theta^* < \tilde{\theta}} \chi_{E \cap \{\theta^* < 1\}} (\theta^* - \tilde{\theta}) c(\alpha) \, dx \\
= c(\alpha) \int_{\theta^* < \tilde{\theta}} (\chi_A + \chi_{E \cap \{\theta^* < 1\}}) (\theta^* - \tilde{\theta}) \, dx \\
= c(\alpha) \int_{\theta^* < \tilde{\theta}} (\theta^* - \tilde{\theta}) \, dx.
\]

Also, we have
\[
\{\tilde{\theta} < \theta^*\} \subseteq \{\tilde{\theta} > \theta^*\}^c \subseteq A^c \subseteq \{H \leq c(\alpha)\},
\]
and thus
\[
\int_{\theta^* > \tilde{\theta}} (\theta^* - \tilde{\theta}) H(x) \, dx = \int_{\theta^* > \tilde{\theta}} \chi_{\{H \leq c(\alpha)\}} (\theta^* - \tilde{\theta}) H(x) \, dx \\
\leq c(\alpha) \int_{\theta^* > \tilde{\theta}} (\theta^* - \tilde{\theta}) \, dx.
\]
Consequently,
\[
\int_D (\theta^* - \tilde{\theta}) H(x) \, dx = \int_{\theta^* > \tilde{\theta}} (\theta^* - \tilde{\theta}) H(x) \, dx + \int_{\theta^* < \tilde{\theta}} (\theta^* - \tilde{\theta}) H(x) \, dx \\
< c(\alpha) \left[ \int_{\theta^* > \tilde{\theta}} (\theta^* - \tilde{\theta}) \, dx + \int_{\theta^* < \tilde{\theta}} (\theta^* - \tilde{\theta}) \, dx \right] \\
= 0,
\]
or
\[
\int_D \theta^*(x) H(x) \, dx < \int_D \tilde{\theta}(x) H(x) \, dx
\]
which contradicts with the fact that
\[
\int_D \theta^*(x) H(x) \, dx = \sup_{\theta \in \Theta} \int_D \theta(x) H(x) \, dx.
\]
In the case that $|B| > 0$, we use the same $\tilde{\theta}$ defined in (4.4), and note that
\[
\{\theta^* > \tilde{\theta}\} = \{\theta^* > 0\} \cap \{\tilde{\theta} = 0\} \\
= \{\theta^* > 0\} \cap \{H \leq c(\alpha)\} \cap E^c \\
= \{\theta^* > 0\} \cap \{H < c(\alpha)\} \\
= B.
\]
By the similar argument, we also get the (strict) inequality (4.5), and thus a contradiction.
To sum up, if $\beta^*$ is a relaxed optimal actuator location, then
\[
\theta^* = (\beta^*)^2 = \chi_{\{H > c(\alpha)\}} + \chi_E,
\]
where $E \subseteq \{H = c(\alpha)\}$ and $|E| = \alpha|D| - |\{H > c(\alpha)\}|$.
Therefore, by Theorem 3.1 we find an optimal actuator location $\beta^*$ to our classical problem (1.6), and thus we finished the proof of Theorem 1.2.

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