UNIQUENESS OF CLOSED SELF-SIMILAR SOLUTIONS TO 
\( \sigma^\alpha_k \)-CURVATURE FLOW

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Abstract. By adapting the test functions introduced by Choi-Daskalopoulos [11] and Brendle-Choi-Daskalopoulos [9] and exploring properties of the \( k \)-th elementary symmetric functions \( \sigma_k \) intensively, we show that for any fixed \( k \) with \( 1 \leq k \leq n - 1 \), any strictly convex closed hypersurface in \( \mathbb{R}^{n+1} \) satisfying \( \sigma^\alpha_k = \langle X, \nu \rangle \), with \( \alpha \geq \frac{1}{k} \), must be a round sphere. In fact, we prove a uniqueness result for any strictly convex closed hypersurface in \( \mathbb{R}^{n+1} \) satisfying \( F + C = \langle X, \nu \rangle \), where \( F \) is a positive homogeneous smooth symmetric function of the principal curvatures and \( C \) is a constant.

1. Introduction

Let \( X : M \to \mathbb{R}^{n+1} \) be a smooth embedding of a closed, orientable hypersurface in \( \mathbb{R}^{n+1} \) with \( n \geq 2 \), satisfying

\[
\sigma^\alpha_k = \langle X, \nu \rangle
\]

where \( \nu \) is the outward unit normal vector field of \( M \), \( \alpha > 0 \), \( 1 \leq k \leq n \) and \( \sigma_k \) is the \( k \)-th elementary symmetric functions of principal curvatures of \( M \).

This type of equation is important for the following curvature flow

\[
\tilde{X}_t = -\sigma^\alpha_k \nu.
\]

Actually, if \( X \) is a solution of (1.1), then

\[
\tilde{X}(x, t) = ((k \alpha + 1)(T - t))^{\frac{1}{1-k\alpha}} X(x)
\]

gives rise to the solution of (1.2) up to a tangential diffeomorphism [22]. So in the same spirit, we call the solutions of (1.1) self-similar solutions of (1.2).

For \( k = 1 \), G. Huisken proved the following famous result:

**Theorem 1.1** (Huisken, [20]). *If \( M \) is a closed hypersurface in \( \mathbb{R}^{n+1} \), with non-negative mean curvature \( \sigma_1 \) and satisfies the equation

\[
\sigma_1 = \langle X, \nu \rangle,
\]

then \( M \) must be a round sphere.*

For \( k = n \), very recently, Choi-Daskalopoulos [11], further, Brendle-Choi-Daskalopoulos [9] proved the following remarkable result:

\[
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\]

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Theorem 1.2 (Choi-Daskalopoulos [11], Brendle-Choi-Daskalopoulos [9]). Let \( M \) be a closed, strictly convex hypersurface in \( \mathbb{R}^{n+1} \) satisfying
\[
\sigma_n^\alpha = \langle X, \nu \rangle.
\]
If \( \alpha > \frac{1}{n+2} \), then \( M \) must be a round sphere; if \( \alpha = \frac{1}{n+2} \), then \( M \) is an ellipsoid.

Remark 1.3. The results of convergence of \( \sigma_n^\alpha \)-curvature flow could imply Theorem 1.2. In case \( \alpha = \frac{1}{n+2} \), Theorem 1.2 was contained in the results of B. Chow in [12]. In case \( n = 2 \), Theorem 1.2 was proved by B. Andrews and X. Chen for \( \frac{1}{2} \leq \alpha \leq 1 \) in [6]. In case \( \alpha = \frac{1}{n+2} \), Theorem 1.2 was proved by B. Andrews in [2]. The more properties of \( \sigma_n^\alpha \)-curvature flow were studied by W. J. Firey [15], B. Chow [12], K. Tso [23], B. Andrews [8], P.-F. Guan and L. Ni [19], B. Andrews, P.-F. Guan and L. Ni [7], etc.

From Theorem 1.1 and Theorem 1.2, the following natural question arises:

**Question.** For any fixed \( k \) with \( 1 \leq k \leq n-1 \), let \( M \) be a closed, strictly convex hypersurface in \( \mathbb{R}^{n+1} \) satisfying (1.1) with \( \alpha \geq \frac{1}{k} \). Can we conclude that \( M \) must be a round sphere?

In this paper, we give an affirmative answer to the above question by proving the following result:

Theorem 1.4. For any fixed \( k \) with \( 1 \leq k \leq n-1 \), let \( M \) be a closed, strictly convex hypersurface in \( \mathbb{R}^{n+1} \) satisfying
\[
\sigma_k^\alpha = \langle X, \nu \rangle
\]
with \( \alpha \geq \frac{1}{k} \). Then \( M \) must be a round sphere.

Remark 1.5. Theorem 1.4 implies Theorem 1.4 for the case \( k = 1 \) and \( \alpha = 1 \). For \( \alpha = \frac{1}{k} \), Theorem 1.4 was contained in the results of B. Chow [12, 13] and B. Andrews [1, 2, 4, 5]. For general \( k \) and \( \alpha \), there are some partial results under certain pinching condition of the principal curvatures of hypersurface, see [22], [8] and [16].

In fact, we prove the following two theorems:

**Theorem A.** For any fixed \( k \) with \( 1 \leq k \leq n \), let \( M \) be a closed, strictly convex hypersurface in \( \mathbb{R}^{n+1} \) satisfying
\[
\sigma_k^\alpha + C = \langle X, \nu \rangle
\]
with constants \( \alpha \) and \( C \). If either \( 1 \leq k \leq n-1 \), \( C \leq 0 \), \( \alpha \geq \frac{1}{k} \), or, \( k = n \), \( C < 0 \), \( \alpha \geq \frac{1}{n+2} \), then \( M \) must be a round sphere.

Remark 1.6. Choose \( C = 0 \), Theorem A reduces to Theorem 1.4. When \( k = \alpha = 1 \), Theorem A implies the uniqueness of closed \( \lambda \)-hypersurfaces introduced by Cheng-Wei [10].

Let \( S_k(\lambda) \) denote the \( k \)-th power sum of the principal curvatures \( \lambda_1, \cdots, \lambda_n \), defined by \( S_k(\lambda) = \sum_{i=1}^{n} \lambda_i^k \).

**Theorem B.** For any fixed \( k \) with \( k \geq 1 \), let \( M \) be a closed, strictly convex hypersurface in \( \mathbb{R}^{n+1} \) satisfying
\[
S_k^\alpha + C = \langle X, \nu \rangle
\]
with constants \( \alpha \) and \( C \). If \( \alpha \geq \frac{1}{k} \) and \( C \leq 0 \), then \( M \) must be a round sphere.
Actually, we consider the following general equation
\begin{equation}
F + C = \langle X, \nu \rangle,
\end{equation}
where $F$ is a homogeneous smooth symmetric function of the principal curvatures of degree $\beta$ and $C$ is a constant, which satisfies the following Condition.

**Condition 1.7.** Suppose $F$ is a smooth function defined on the positive cone $\Gamma_+ = \{ \mu \in \mathbb{R}^n | \mu_1 > 0, \mu_2 > 0, \cdots, \mu_n > 0 \}$ of $\mathbb{R}^n$, and satisfies the following conditions:

i) $F$ is positive and strictly increasing, i.e., $F > 0$ and $\frac{\partial F}{\partial \lambda_i} > 0$ for $1 \leq i \leq n$.

ii) $F$ is homogeneous symmetric function with degree $\beta$, i.e., $F(t\lambda) = t^\beta F(\lambda)$ for all $t \in \mathbb{R}_+$.

iii) For any $i \neq j$, $\frac{\partial F}{\partial \lambda_i} \lambda_i - \frac{\partial F}{\partial \lambda_j} \lambda_j \geq 0$.

iv) For all $(y_1, \ldots, y_n) \in \mathbb{R}^n$,
\begin{equation}
\sum_i \frac{1}{\lambda_i} \frac{\partial \log F}{\partial \lambda_i} y_i^2 + \sum_{i,j} \frac{\partial^2 \log F}{\partial \lambda_i \partial \lambda_j} y_i y_j \geq 0.
\end{equation}

Remark 1.8. By using Lemma 3.2, one can see that iii) and iv) in Condition 1.7 are equivalent to the convexity of the function $F^*(A) = \log F(e^A)$ defined on real $n \times n$ symmetric matrices.

Remark 1.9. We call the inequality (1.6) the key inequality of $F$ in this paper, which plays an important role in our proof. Its $\sigma_k$ version appeared in [18] first, later in [14]. We will give another proof in Lemma 2.5 for $\sigma_k$.

Remark 1.10. Lemma 2.6 and Lemma 2.7 say that both $\sigma_k$ and $S_k^\alpha$ with $\alpha > 0$ satisfy Condition 1.7. In fact, any multiplication combination of such functions satisfies Condition 1.7 such as $\sigma_2 \sigma_3$ and so on.

For such general $F$, we prove

**Theorem 1.11.** Let $M$ be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying
\begin{equation}
F + C = \langle X, \nu \rangle,
\end{equation}
with constant $C$. For $\beta > 1$ and $C \leq 0$, if $F$ satisfies Condition 1.7 then $M$ must be a round sphere.

In our proof, following the idea of Choi-Daskasopoulos [11] and Brendle-Choi-Daskasopoulos [9], we consider the quantities
\begin{align}
Z &= F \, \text{tr} b - \frac{n(\beta - 1)}{2\beta} |X|^2, \\
\tilde{W} &= F \lambda_{\min}^{-1} - \frac{\beta - 1}{2\beta} |X|^2,
\end{align}
where $b = (b^{ij})$ denotes the inverse of the second fundamental form $h = (h_{ij})$ with respect to an orthonormal frame and $\lambda_{\min}$ is the smallest principal curvature of the hypersurface. We find that the techniques in Choi-Daskasopoulos [11] and Brendle-Choi-Daskasopoulos [9] can be carried out effectively on $F$ which satisfies Condition 1.7. First we apply the maximum principle for $W$ (see Section 4 for
definition of $W$) to prove that the maximum point of $\tilde{W}$ is umbilic. Then we use the strong maximum principle of $L = \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j$ for $Z$ to prove Theorem 1.11. In particular, Theorem 1.11 holds for $F = \sigma_k^\alpha$ or $F = S_k^\alpha$ with $\alpha > \frac{1}{k}$. In Theorem 6.3 and Theorem 6.4 we discuss the cases $F = \sigma_k^\alpha$ with $\frac{1}{k} \leq \alpha \leq \frac{1}{2}$ and $F = S_k^\alpha$, respectively.

The structure of this paper is as follows. In Section 2, we give some properties of the elementary symmetric functions $\sigma_k$ and general $F$ satisfying Condition 1.7 and prove that both $\sigma_k^\alpha$ and $S_k^\alpha$ satisfy the key inequality (Lemma 2.7). In Section 3 we derive some fundamental formulas for the closed hypersurfaces which satisfies self-similar equation (1.7) with the general homogeneous symmetric function $F$. In Section 4 we do analysis at the maximum point of $W$. In Section 5 we give a proof of Theorem 1.11. Finally in Section 6, we present the proofs of Theorem A and Theorem B.

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2. SOME PROPERTIES OF ELEMENTARY SYMMETRIC FUNCTIONS AND THE KEY INEQUALITY

We first collect some basic notations, definitions and properties of elementary symmetric functions, which are needed in our investigation of $\sigma_k^\alpha$ self-similar solutions and general $F$ self-similar solutions.

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ denote the principal curvatures of $M$. Throughout this paper, we assume that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Denote

$$\sigma_k(\lambda) = \sigma_k(\lambda(A)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$ 

For convenience, we set $\sigma_0(\lambda) = 1$ and $\sigma_k(\lambda) = 0$ for $k > n$ or $k < 0$. Let $\sigma_{k,1}(\lambda)$ denote the symmetric function $\sigma_k(\lambda)$ with $\lambda_1 = 0$ and $\sigma_{k,i}(\lambda)$, with $i \neq j$, denote the symmetric function $\sigma_k(\lambda)$ with $\lambda_i = \lambda_j = 0$. So $\frac{\partial \sigma_k(\lambda)}{\partial \lambda_i} = \sigma_{k-1, i}$, $\frac{\partial^2 \sigma_k(\lambda)}{\partial \lambda_i \partial \lambda_j} = \sigma_{k-2, i, j}$. Remark that without causing ambiguity we omit $\lambda$ in the notations of $\sigma_k(\lambda)$ for simplicity.

**Definition 2.1.** A hypersurface $M$ is said to be strictly convex if $\lambda \in \Gamma_+ = \{ \mu \in \mathbb{R}^n | \mu_1 > 0, \mu_2 > 0, \cdots, \mu_n > 0 \}$ for any point in $M$.

The following basic properties related to $\sigma_k$ will be used directly.
Proposition 2.2 (See, for example, [21]). For $0 \leq k \leq n$ and $1 \leq i \leq n$, the following equalities hold:
\[
\begin{align*}
\sigma_{k+1} &= \sigma_{k+1;i} + \lambda_i \sigma_{k;i}, \\
\sum_{i=1}^{n} \lambda_i \sigma_{k;i} &= (k+1)\sigma_{k+1}, \\
\sum_{i=1}^{n} \sigma_{k;i} &= (n-k)\sigma_{k}, \\
\sum_{i=1}^{n} \lambda_i^2 \sigma_{k;i} &= \sigma_1 \sigma_{k+1} - (k+2)\sigma_{k+2}.
\end{align*}
\]

We now turn to prove the key inequality for $\sigma_k$. First we show two lemmas. Let $D_n^{(k)}(\lambda) = (d_{ij})$, $i,j = 0, \cdots, m$, denote the following symmetric $(m+1) \times (m+1)$-matrix
\[
\begin{pmatrix}
\sigma_k & \sigma_{k:1} & \sigma_{k:2} & \cdots & \sigma_{k:m} \\
\sigma_{k:1} & \sigma_{k:1:1} & \sigma_{k:1:2} & \cdots & \sigma_{k:1:m} \\
\sigma_{k:2} & \sigma_{k:2:1} & \sigma_{k:2:2} & \cdots & \sigma_{k:2:m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{k:m} & \sigma_{k:m:1} & \sigma_{k:m:2} & \cdots & \sigma_{k:m}
\end{pmatrix},
\]
i.e., $d_{ij} = d_{ji}$ and
\[
d_{ij} = \begin{cases}
\sigma_k(\lambda), & \text{if } i = j = 0, \\
\sigma_{k:j}(\lambda), & \text{if } i = 0, 1 \leq j \leq m, \\
\sigma_{k:i}(\lambda), & \text{if } 1 \leq i = j \leq m, \\
\sigma_{k:i:j}(\lambda), & \text{if } 1 \leq i < j \leq m.
\end{cases}
\]

Lemma 2.3. If $\lambda \in \Gamma_+$ and $n \geq 2$, then $D_n^{(k)}(\lambda)$ is semi-positive definite for $1 \leq k \leq n$.

Proof. First, since $\sigma_{n;i} = \sigma_{n;pq} = 0$ for $1 \leq i, p, q \leq n$, it is clear that $D_n^{(n)}$ is semi-positive definite.

For $1 \leq k \leq n-1$, the statement follows by induction on $n$. In fact, for $n = 2$, the semi-positive-definiteness is proved by direct computation. Now, assume that the statement is true for $n-1$. For $\lambda = (\lambda_1, \ldots, \lambda_n)$, the assumption implies the following matrices are semi-positive definite
\[
D_n^{(k)}(\lambda) = \begin{pmatrix}
\sigma_{k;n} & \sigma_{k;1:n} & \sigma_{k;2:n} & \cdots & \sigma_{k;n-1,n} \\
\sigma_{k;1:n} & \sigma_{k;1:1} & \sigma_{k;1:2} & \cdots & \sigma_{k;1:n-1,n} \\
\sigma_{k;2:n} & \sigma_{k;2:1:n} & \sigma_{k;2:2} & \cdots & \sigma_{k;2:n-1,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{k;n-1,n} & \sigma_{k;n-1:1} & \sigma_{k;n-1:2} & \cdots & \sigma_{k;n-1:n-1}
\end{pmatrix}
\]
for $1 \leq k \leq n-1$. And, using
\[
\sigma_k = \sigma_{k;n} + \lambda_n \sigma_{k-1:n}, \quad \sigma_{k:i} = \sigma_{k;i:n} + \lambda_n \sigma_{k-1;i:n} \quad (1 \leq i \leq n-1),
\]
we obtain
\[
D_n^{(k)}(\lambda) = \lambda_n \left( D_n^{(k-1)}(\lambda) - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + \left( D_n^{(k-1)}(\lambda) \eta \begin{pmatrix} \eta \sigma_{k;n} \end{pmatrix} \right),
\]
where \( \eta^T = (\sigma_{k,n}, \sigma_{k,1:n}, \sigma_{k:2:n}, \ldots, \sigma_{k:n-1,n}) \). For

\[
\begin{pmatrix}
\sigma_{k:n} & \sigma_{k:1:n} & \sigma_{k:2:n} & \cdots & \sigma_{k:n-1,n} & \sigma_{k:n} \\
\sigma_{k:1:n} & \sigma_{k:1:n} & \sigma_{k:1:2:n} & \cdots & \sigma_{k:1:n-1,n} & \sigma_{k:1:n} \\
\sigma_{k:2:n} & \sigma_{k:2:n} & \sigma_{k:2:n} & \cdots & \sigma_{k:2:n-1,n} & \sigma_{k:2:n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{k:n-1,n} & \sigma_{k:n-1,1:n} & \sigma_{k:n-1,2:n} & \cdots & \sigma_{k:n-1,n-1,n} & \sigma_{k:n-1,n} \\
\sigma_{k:n} & \sigma_{k:n,1} & \sigma_{k:n,2} & \cdots & \sigma_{k:n,n-1} & \sigma_{k:n}
\end{pmatrix}
\]

by subtracting the first row from the last row and the first column from the last column, we find that \( \begin{pmatrix} D_{n-1:n}^{(k)} \eta \\ \eta^T \sigma_{k:n} \end{pmatrix} \) is congruent to

\[
\begin{pmatrix} D_{n-1:n}^{(k)} & 0 \\ 0 & 0 \end{pmatrix}
\]

which is semi-positive definite. So \( D_n^{(k)}(\lambda) \) is semi-positive definite. Thus, the proof is completed.

\[\square\]

Lemma 2.4. Let \( \xi^T = (\sigma_{k-1:1}, \sigma_{k-1:2}, \ldots, \sigma_{k-1:n}) \). Then the matrix \( \sigma_k A^{(k)} - \xi \xi^T \) is semi-positive definite.

Proof. Denote \( \sigma_k A^{(k)} - \xi \xi^T = (w_{ij})_{n \times n} \). Thus

\[
w_{ij} = \begin{cases}
\frac{1}{\lambda_i} \sigma_{k-1:i}(\lambda), & \text{for } i = j, \\
\frac{1}{\lambda_i \lambda_j}(\sigma_k \sigma_{k:ij} - \sigma_{k:i} \sigma_{k:j}), & \text{for } i \neq j.
\end{cases}
\]

We divide the proof in three steps.

Step 1. Since the semi-positive-definiteness is preserved under congruent transformation, we multiply \( \lambda_i \) to the \( i \)-th row and the \( i \)-th column of \( \sigma_k A^{(k)} - \xi \xi^T \) for \( 1 \leq i \leq n \). And, let \( \tilde{A}^{(k)} = (\tilde{a}_{ij})_{n \times n} \) denote the new matrix which is defined by

\[
\tilde{a}_{ij} = \begin{cases}
\sigma_{k:i}(\sigma_k - \sigma_{k:i}), & \text{for } i = j, \\
\sigma_k \sigma_{k:ij} - \sigma_{k:i} \sigma_{k:j}, & \text{for } i \neq j.
\end{cases}
\]

We will discuss \( \tilde{A}^{(k)} \) instead of \( \sigma_k A^{(k)} - \xi \xi^T \) in the following.

Step 2. \( \tilde{A}^{(k)} \) is semi-positive definite if and only if its principal minors are all non-negative. Let \( \tilde{A}_m^{(k)} \) denote the upper-left \( m \times m \) sub-matrix of \( \tilde{A}^{(k)} \). For the symmetry of the elemental functions, it suffices to show \( \det \tilde{A}_m^{(k)} \geq 0 \).
Step 3. det $\tilde{A}_m^{(k)}$ can be calculated as follows.

$$
\det \tilde{A}_m^{(k)} = \det \begin{pmatrix}
1 & \sigma_{k;1} & \sigma_{k;2} & \cdots & \sigma_{k;m} \\
0 & \sigma_k \sigma_{k;1} - \sigma_{k;1}^2 & \sigma_k \sigma_{k;12} - \sigma_{k;1} \sigma_{k;2} & \cdots & \sigma_k \sigma_{k;1m} - \sigma_{k;1} \sigma_{k;m} \\
0 & \sigma_k \sigma_{k;12} - \sigma_{k;1} \sigma_{k;2} & \sigma_k \sigma_{k;2} - \sigma_{k;2}^2 & \cdots & \sigma_k \sigma_{k;2m} - \sigma_{k;2} \sigma_{k;m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \sigma_k \sigma_{k;m1} - \sigma_k \sigma_{k;m1} & \sigma_k \sigma_{k;m2} - \sigma_k \sigma_{k;m2} & \cdots & \sigma_k \sigma_{k;m} - \sigma_k^2 \\
\end{pmatrix}
$$

$$
= \sigma_k^{-2} \det \begin{pmatrix}
\sigma_k & \sigma_k \sigma_{k;1} & \sigma_k \sigma_{k;2} & \cdots & \sigma_k \sigma_{k;m} \\
\sigma_k \sigma_{k;1} & \sigma_k \sigma_{k;1} & \sigma_k \sigma_{k;12} & \cdots & \sigma_k \sigma_{k;1m} \\
\sigma_k \sigma_{k;2} & \sigma_k \sigma_{k;12} & \sigma_k \sigma_{k;2} & \cdots & \sigma_k \sigma_{k;2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_k \sigma_{k;m1} & \sigma_k \sigma_{k;m2} & \sigma_k \sigma_{k;m2} & \cdots & \sigma_k \sigma_{k;m} \\
\end{pmatrix}
$$

$$
= \sigma_k^{m-1} \det D_m^{(k)}.
$$

By Lemma 2.3, we know det $D_m^{(k)} \geq 0$. So, det $\tilde{A}_m^{(k)} \geq 0$ which implies $\sigma_k A^{(k)} - \xi \xi^T$ is semi-positive definite.  

With the help of the proceeding two lemmas, we finally obtain the key inequality for $\sigma_k$. It appeared in [13] first, later in [14]. Here we give another proof.

**Lemma 2.5.** For $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, the following inequality holds

$$
\sum_{i=1}^{n} \frac{\sigma_{k-1;i}}{\lambda_i \sigma_k} y_i^2 + \sum_{i \neq j} \frac{\sigma_{k-2;ij}}{\sigma_k} y_i y_j \geq \left( \sum_{i=1}^{n} \frac{\sigma_{k-1;i}}{\sigma_k} y_i \right)^2.
$$

**Proof.** By Lemma 2.3, we know

$$
y^T (\frac{1}{\sigma_k} A^{(k)} - \frac{1}{\sigma_k^2} \xi \xi^T) y \geq 0.
$$

Now we can show that both $\sigma_k^\alpha$ and $S_k^\alpha$ with $\alpha > 0$ satisfy Condition 1.7

**Lemma 2.6.** For $i > j$, for $F = \sigma_k^\alpha$ or $F = S_k^\alpha$ with $\alpha > 0$, Condition 1.7 iii) holds, i.e.,

$$
\frac{\partial F}{\partial \lambda_i} \lambda_i \geq \frac{\partial F}{\partial \lambda_j} \lambda_j.
$$

**Proof.** For $F = S_k^\alpha$, it is clear. For $F = \sigma_k^\alpha$, we have

$$
\frac{\partial F}{\partial \lambda_i} \lambda_i - \frac{\partial F}{\partial \lambda_j} \lambda_j = \alpha \sigma_k^{\alpha-1} (\sigma_{k-1;i} \lambda_i - \sigma_{k-1;j} \lambda_j) = \alpha \sigma_k^{\alpha-1} \sigma_{k-1;ij} (\lambda_i - \lambda_j) \geq 0.
$$

\[\square\]
Lemma 2.7. For all \((y_1, y_2, \ldots, y_n) \in \mathbb{R}^n\), \(F = \sigma_k^\alpha\) or \(F = S_k^\alpha\) with \(\alpha > 0\) satisfies Condition 1.7 iv), i.e.,

\[
\sum_i \frac{1}{\lambda_i} \frac{\partial \log F}{\partial \lambda_i} y_i^2 + \sum_{i,j} \frac{\partial^2 \log F}{\partial \lambda_i \partial \lambda_j} y_i y_j \geq 0.
\]

Proof. For \(F = \sigma_k^\alpha\), it is equivalent to Lemma 2.5.

For \(F = S_k^\alpha\), by the Cauchy-Schwarz inequality, we have

\[
\left(\sum_i \frac{\lambda_i^{k-1}}{S_k} y_i^2\right)^2 \leq \left(\sum_i \frac{\lambda_i^k}{S_k}\right) \left(\sum_i \frac{\lambda_i^{k-2}}{S_k} y_i^2\right) = \sum_i \frac{\lambda_i^{k-2}}{S_k} y_i^2,
\]

which leads to the key inequality for \(S_k^\alpha\). □

Lemma 2.8. If \(F\) satisfies Condition 1.7 \(\lambda \in \Gamma_+\) and \(\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n\), then for \(i > j > 1\), the following equation holds

\[
\frac{\partial F}{\partial \lambda_i} \frac{\lambda_i - \lambda_1}{\lambda_1} > \frac{\partial F}{\partial \lambda_j} \frac{\lambda_j - \lambda_1}{\lambda_1} > \frac{\partial F}{\partial \lambda_j} \frac{\lambda_j - \lambda_1}{\lambda_1} > \frac{\partial F}{\partial \lambda_j} \frac{\lambda_j - \lambda_1}{\lambda_1}.
\]

Proof. For the case \(\lambda_i = \lambda_j\), it is easy to check. Then without loss of generality, we assume \(\lambda_i > \lambda_j\) for \(i > j\). Actually, for \(i > j\), by Condition 1.7 i) and iii), we have

\[
\frac{\partial F}{\partial \lambda_i} \frac{\lambda_i - \lambda_1}{\lambda_1} > \frac{\partial F}{\partial \lambda_j} \frac{\lambda_j - \lambda_1}{\lambda_1} > \frac{\partial F}{\partial \lambda_j} \frac{\lambda_j - \lambda_1}{\lambda_1} > \frac{\partial F}{\partial \lambda_j} \frac{\lambda_j - \lambda_1}{\lambda_1}.
\]

□

3. Fundamental formulas of self-similar solution with general \(F\)

Let \(X : M^n \to \mathbb{R}^{n+1}\) be a closed convex hypersurface. Suppose that \(e_1, e_2, \cdots, e_n\) is an orthonormal frame on \(M\). Let \(h = (h_{ij})\) be the second fundamental form on \(M\) with respect to this given frame. And the principal curvatures are the eigenvalues of the second fundamental form \(h\).

Let us first consider the following general equation

\[
F + C = \langle X, \nu \rangle,
\]

where \(F = F(\lambda(h))\) is a homogeneous symmetric function of the principal curvatures of degree \(\beta\), \(C\) is a constant and \(\nu\) is the outward normal vector field. And, let \(L\) denote the operator \(L = \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j\). We also suppose \(F > 0\) and \((\frac{\partial F}{\partial h_{ij}})\) is positive definite. Inspired by [22], [11] and [9], we have the following proposition.

The summation convention is used unless otherwise stated.

**Proposition 3.1.** Given a smooth function \(F : M \to \mathbb{R}^{n+1}\) described as above, the following equations hold:
(1) \[ \mathcal{L}F = \langle X, \nabla F \rangle + \beta F - \frac{\partial F}{\partial h_{ij}} h_{ij}(F + C), \]

(2) \[ \mathcal{L}h_{kl} = h_{klm} \langle X, e_m \rangle + h_{kl} - Ch_{km} h_{lm} - \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijkl} \]

Then, by Gauss equation we have

(3) \[ \mathcal{L}b^{kl} = \langle X, \nabla b^{kl} \rangle - b^{kl} + C \delta_{kl} + b^{kp} b^{ql} - \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{stq} \]

and

(4) \[ \mathcal{L}(Ftrb) = \langle X, \nabla (Ftrb) \rangle + (\beta - 1) Ftrb - n(\beta - 1) F^2 \]

Then, by Codazzi equation and Ricci identity, we obtain

(5) \[ \mathcal{L} |X|^2_2 = \sum_i \frac{\partial F}{\partial h_{ii}} - \beta F(F + C). \]

**Proof.** (1) Differentiating (1.7) gives

\[ \nabla_i F = h_{ij} \langle X, e_l \rangle \]

and

\[ \nabla_i \nabla_j F = h_{jli} \langle X, e_i \rangle + h_{ij} - h_{jli} \langle X, e_l \rangle \]

Then, by \( \frac{\partial F}{\partial h_{ij}} h_{ij} = \beta F \), we obtain

\[ \mathcal{L}F = \nabla_i F \langle X, e_l \rangle + \beta F - \frac{\partial F}{\partial h_{ij}} h_{ij}(F + C). \]

(2) By Codazzi equation and Ricci identity, we obtain

\[ h_{kli} = h_{kjl} = h_{kjl} + h_{mjk} R_{mkli} + h_{km} R_{mjli}. \]

Then, using Gauss equation we have

\[ \mathcal{L}h_{kl} = \frac{\partial F}{\partial h_{ij}} (h_{kjl} + h_{mjk} R_{mkli} + h_{km} R_{mjli}) \]

\[ = \nabla_l \left( \frac{\partial F}{\partial h_{ij}} h_{ij} \right) - \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijkl} + \frac{\partial F}{\partial h_{ij}} h_{mij}(h_{mi} h_{ki} - h_{mi} h_{kl}) \]

\[ + \frac{\partial F}{\partial h_{ij}} h_{mij}(h_{mi} h_{ij} - h_{mi} h_{jl}) \]

\[ = \nabla_i \nabla_k F - \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijkl} + \frac{\partial F}{\partial h_{ij}} h_{mij}(h_{mi} h_{kl} + h_{km} h_{ml} h_{ij}) \]

\[ = h_{klm} \langle X, e_m \rangle + h_{kl} - h_{km} h_{lm}(F + C) - \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijkl} h_{st} \]
\[ \frac{\partial F}{\partial h_{ij}} f_{mi} h_{kl} + \beta F h_{km} b_{ml}. \]

(3) Since \( h_{km} b_{ml} = \delta_{kl} \), we have

\[ \nabla_j b^{kl} = -b^{kp} b^{lq} \nabla_j h_{pq}. \]

And,

\[ \nabla_i \nabla_j b^{kl} = -\nabla_i (b^{kp} b^{lq} \nabla_j h_{pq}) \]

\[ = -b^{kp} b^{lq} \nabla_i \nabla_j h_{pq} + b^{ks} b^{pt} b^{lq} \nabla_i h_{st} \nabla_j h_{pq} + b^{kp} b^{ls} b^{qt} \nabla_i h_{st} \nabla_j h_{pq} \]

\[ = -b^{kp} b^{lq} \nabla_i \nabla_j h_{pq} + 2b^{ks} b^{pt} b^{lq} \nabla_i h_{st} \nabla_j h_{pq}. \]

Then, we obtain

\[ \mathcal{L} b^{kl} = -b^{kp} b^{lq} \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j h_{pq} + 2b^{ks} b^{pt} b^{lq} \frac{\partial F}{\partial h_{ij}} \nabla_i h_{st} \nabla_j h_{pq} \]

\[ - b^{kp} b^{lq} \frac{\partial F}{\partial h_{ij}} h_{pm} h_{mq} h_{ij} + 2b^{ks} b^{pt} b^{lq} \frac{\partial F}{\partial h_{ij}} h_{stl} h_{pqj} \]

\[ = \langle X, \nabla b^{kl} \rangle - b^{kl} + (F + C) \delta_{kl} + b^{kp} b^{lq} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{sqt} \]

\[ + b^{kl} \frac{\partial F}{\partial h_{ij}} h_{mj} h_{mi} - \beta F \delta_{kl} + 2b^{ks} b^{pt} b^{lq} \frac{\partial F}{\partial h_{ij}} h_{stl} h_{pqj}. \]

(4) From (3), we have

\[ \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j trb = \langle X, \nabla trb \rangle - trb + n(F + C) + b^{kp} b^{lq} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{sqt} \]

\[ + trb \frac{\partial F}{\partial h_{ij}} h_{mj} h_{mi} - n\beta F + 2b^{ks} b^{pt} b^{lq} \frac{\partial F}{\partial h_{ij}} h_{stl} h_{pqj}. \]

Furthermore,

\[ \mathcal{L}(F trb) = 2 \frac{\partial F}{\partial h_{ij}} \nabla_i F \nabla_j trb + trb \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j F + F \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j trb \]

\[ = 2 \frac{\partial F}{\partial h_{ij}} \nabla_i F \nabla_j trb + trb \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j F + F \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j trb \]

\[ - trb \frac{\partial F}{\partial h_{ij}} h_{ij} (F + C) + F \langle X, \nabla trb \rangle - F trb + nF(F + C) \]

\[ + F b^{kp} b^{lq} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{sqt} + F trb B \frac{\partial F}{\partial h_{ij}} h_{mj} h_{mi} - nF \frac{\partial F}{\partial h_{ij}} h_{ij} \]

\[ + 2F b^{ks} b^{pt} b^{lq} \frac{\partial F}{\partial h_{ij}} h_{stl} h_{pqj} \]

\[ = \langle X, \nabla (F trb) \rangle + (\beta - 1) F trb - n(\beta - 1) F^2 + C(nF - trb \frac{\partial F}{\partial h_{ij}} h_{ij}) \]

\[ + 2 \frac{\partial F}{\partial h_{ij}} \nabla_i F \nabla_j trb + F b^{kp} b^{lq} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{sqt} + 2F b^{ks} b^{pt} b^{lq} \frac{\partial F}{\partial h_{ij}} h_{stl} h_{pqj}. \]
(5) By direct computation and (1.7), we have

\[
\mathcal{L} \frac{|X|^2}{2} = \frac{\partial F}{\partial h_{ij}} \nabla_i (\langle X, e_j \rangle) = \sum_i \frac{\partial F}{\partial h_{ii}} - (F + C) \frac{\partial F}{\partial h_{ij}} h_{ij}.
\]

\[\blacksquare\]

To finish this section, we list the following well-known result (See for example [1] and [17]).

**Lemma 3.2.** If \( W = (w_{ij}) \) is a symmetric real matrix and \( \lambda_m = \lambda_m(W) \) is one of its eigenvalues \((m = 1, \ldots, n)\). If \( F = F(W) = F(\lambda(W)) \), then for any real symmetric matrix \( B = (b_{ij}) \), we have the following formulas:

\[(i)\quad \frac{\partial F}{\partial w_{ij}} b_{ij} = \frac{\partial F}{\partial \lambda^p} b_{pp},
\]

\[(ii)\quad \frac{\partial^2 F}{\partial w_{ij} \partial w_{st}} b_{ij} b_{st} = \frac{\partial^2 F}{\partial \lambda^p \partial \lambda^q} b_{pp} b_{qq} + 2 \sum_{p < q} \frac{\partial F}{\partial \lambda^p} \frac{\partial F}{\partial \lambda^q} - \lambda_p - \lambda_q b_{pq}^2.
\]

**Remark 3.3.** In the above lemma, \( \frac{\partial F}{\partial \lambda^p} - \frac{\partial F}{\partial \lambda^q} \lambda_p - \lambda_q \) is interpreted as a limit if \( \lambda_p = \lambda_q \).

4. **Analysis at the maximum points of \( W \)**

In the recent paper [9], S. Brendle, K. Choi and P. Daskalopoulos proved the following powerful lemma.

**Lemma 4.1** ([9]). Let \( \mu \) denote the multiplicity of \( \lambda_1 \) at a point \( x_0 \), i.e., \( \lambda_1(x_0) = \cdots = \lambda_{\mu}(x_0) < \lambda_{\mu+1}(x_0) \). Suppose that \( \varphi \) is a smooth function such that \( \varphi \leq \lambda_1 \) everywhere and \( \varphi(x_0) = \lambda_1(x_0) \). Then, at \( x_0 \), we have

\[i)\quad h_{kl} = \nabla_i \varphi \delta_{kl} \text{ for } 1 \leq k, l \leq \mu.
\]

\[ii)\quad \nabla_i \nabla_i \varphi \leq h_{11i} - 2 \sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} h_{1li}^2.
\]

Let \( \tilde{W} = \frac{F}{\varphi} - \frac{\beta - 1}{2\beta} |X|^2 \) and let \( x_0 \) be an arbitrary point where \( \tilde{W} \) attains its maximum. Then we can choose a smooth function \( \varphi \) such that \( \varphi \leq \lambda_1 \) everywhere, \( \varphi(x_0) = \lambda_1(x_0) \) and \( W = \frac{F}{\varphi} - \frac{\beta - 1}{2\beta} |X|^2 \) attains its maximum at \( x_0 \). Now, we consider \( W \) at \( x_0 \) and apply the previous lemma.

**Lemma 4.2.** At \( x_0 \), \( W = \frac{F}{\varphi} - \frac{\beta - 1}{2\beta} |X|^2 \) and let \( x_0 \) be an arbitrary point where \( \tilde{W} \) attains its maximum. Then we can choose a smooth function \( \varphi \) such that \( \varphi \leq \lambda_1 \) everywhere, \( \varphi(x_0) = \lambda_1(x_0) \) and \( W = \frac{F}{\varphi} - \frac{\beta - 1}{2\beta} |X|^2 \) attains its maximum at \( x_0 \). Now, we consider \( W \) at \( x_0 \) and apply the previous lemma.

\[
\mathcal{L}W \geq \langle X, \nabla (\frac{F}{\varphi}) \rangle + 2 \frac{\partial F}{\partial h_{ij}} \nabla_i F \nabla_j \frac{1}{\varphi} + 2 F \lambda_1^{-3} \frac{\partial F}{\partial \lambda_i} h_{1ii}^2
\]

\[+ F \lambda_1^{-2} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ij} h_{st1} + 2 F \lambda_1^{-2} \frac{\partial F}{\partial \lambda_i} \sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} h_{1li}^2
\]

\[+ \frac{\beta - 1}{\beta} \frac{\partial F}{\partial \lambda_i} (\lambda_i - 1) - C \frac{\partial F}{\partial \lambda_i} (\lambda_i - 1).
\]
Proof. At $x_0$, it follows from Lemma 4.1 and Proposition 3.1 that

\[
\mathcal{L} \varphi \leq \mathcal{L} h_{11} - 2 \frac{\partial F}{\partial \lambda_i} \sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} h_{1ll}
\]

\[
= h_{11m}(X, e_m) + \lambda_1 - \lambda_1^2 C - \lambda_1 \frac{\partial F}{\partial h_{ij}} h_{mj} h_{mi} + \lambda_1^2 (\beta - 1) F - \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ij} h_{st1} - 2 \frac{\partial F}{\partial \lambda_i} \sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} h_{1ll}.
\]

Furthermore, we have

\[
\mathcal{L} \frac{F}{\varphi} = 2 \frac{\partial F}{\partial h_{ij}} \nabla_i F \nabla_j \frac{1}{\varphi} + \frac{1}{\varphi} \mathcal{L} F + F \mathcal{L} \frac{1}{\varphi}
\]

\[
\geq \frac{2 \partial F}{\partial h_{ij}} \nabla_i F \nabla_j \frac{1}{\varphi} + \lambda_1^{-1} \nabla_i F(X, e_i) + \lambda_1^{-1} \frac{\partial F}{\partial h_{ij}} h_{ij} - \lambda_1^{-1} \frac{\partial F}{\partial h_{ij}} h_{ij} h_{ij}(F + C)
\]

\[
+ 2 \lambda_1^{-2} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ij} h_{st1} + F \nabla_m \frac{1}{\varphi} (X, e_m) + (\beta - 1) \lambda_1^{-1} F^2 + FC + F \lambda_1^{-1} \frac{\partial F}{\partial h_{ij}} h_{mj} h_{mi}
\]

According to Proposition 3.1 and the homogeneity of $F$, we have

\[
- \frac{\beta - 1}{\beta} \mathcal{L} \frac{F}{\varphi} + (\beta - 1) \lambda_1^{-1} F^2 + C(F - \lambda_1^{-1} \frac{\partial F}{\partial h_{ij}} h_{ij} h_{ii})
\]

\[
= \frac{\beta - 1}{\beta} \frac{\partial F}{\partial \lambda_i} \lambda_i (\lambda_i - 1) - C \frac{\partial F}{\partial \lambda_i} \lambda_i (\lambda_i - 1),
\]

thus the proof is completed. \qed

Lemma 4.3. At $x_0$, we have the following equalities

\(1\) \( \langle X, \nabla (\frac{F}{\varphi}) \rangle = \frac{\beta - 1}{\beta} \sum_i \lambda_i^{-2} (\nabla_i F)^2 \),

\(2\) \( \lambda_1^{-2} h_{11} = (\lambda_1^{-1} - \frac{\beta - 1}{\beta} \lambda_1^{-1}) \nabla_j \log F, \text{ for } 1 \leq j \leq n \),

\(3\) \( \nabla_m F = 0, \text{ for } 2 \leq m \leq \mu \).

Proof. (1) Using $\nabla W = 0$ and (3.1), we have

\[
\langle X, \nabla (\frac{F}{\varphi}) \rangle = \langle X, \nabla W \rangle + \frac{\beta - 1}{\beta} \sum_m \langle X, e_m \rangle^2
\]

\[
= \frac{\beta - 1}{\beta} \sum_i \lambda_i^{-2} (\nabla_i F)^2.
\]
Lemma 4.4. For Lemma 4.5.

the lemma follows by adding the above two equations.

Proof. Due to
\[0 = F \nabla_j \frac{1}{\varphi} + \frac{1}{\varphi} \nabla_j F - \frac{\beta - 1}{\beta} \lambda_j^{-1} \nabla_j F = -F \lambda_j^{-2} h_{ijj} + (\lambda_j^{-1} - \frac{\beta - 1}{\beta} \lambda_j^{-1}) \nabla_j F.\]

(3) By Lemma 4.1 we have \(h_{11m} = 0\) if \(2 \leq m \leq \mu\). Then, (2) leads to (3).

Lemma 4.4.

\[
\frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijj} h_{st1} + 2 \frac{\partial F}{\partial \lambda_i} \sum_{l>\mu} (\lambda_l - \lambda_1)^{-1} h_{ii}^2 = \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ii1} h_{jj1} + 2 \sum_{i>j} (\lambda_i - \lambda_j)^{-1} \frac{\partial F}{\partial \lambda_i} h_{ii}^2 + 2 \sum_{i>\mu} \frac{\partial F}{\partial \lambda_i} (\lambda_i - \lambda_1)^{-1} h_{11i}^2 \\
+ 2 \sum_{i>j>\mu} (\lambda_i - \lambda_j)^{-1} (\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j}) h_{ijj}^2.
\]

Proof. Due to

\[
\frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijj} h_{st1} = \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ii1} h_{jj1} + 2 \sum_{i>j} (\lambda_i - \lambda_j)^{-1} (\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} h_{ii}^2) \\
+ 2 \sum_{i>j>\mu} (\lambda_i - \lambda_j)^{-1} (\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j}) h_{ijj}^2
\]

and

\[
2 \frac{\partial F}{\partial \lambda_i} \sum_{l>\mu} (\lambda_l - \lambda_1)^{-1} h_{11i}^2 = 2 \frac{\partial F}{\partial \lambda_i} \sum_{l>\mu} (\lambda_l - \lambda_1)^{-1} h_{11i}^2 + 2 \sum_{l>\mu} \frac{\partial F}{\partial \lambda_i} (\lambda_l - \lambda_1)^{-1} h_{11i}^2 \\
+ 2 \sum_{i>i>\mu} \frac{\partial F}{\partial \lambda_i} (\lambda_i - \lambda_l)^{-1} h_{11i}^2 + 2 \sum_{i>i>\mu} \frac{\partial F}{\partial \lambda_i} (\lambda_l - \lambda_1)^{-1} h_{11i}^2,
\]

the lemma follows by adding the above two equations.

Lemma 4.5. For \(\beta \geq 1\), at \(x_0\), \(W\) satisfies the following inequality

\[LW \geq J_1 + J_2 + J_3,
\]

where

\[J_1 = \frac{\beta - 1}{\beta} \frac{\partial F}{\partial \lambda_i} (\frac{\lambda_i}{\lambda_1} - 1) - C \frac{\partial F}{\partial \lambda_i} \lambda_i (\frac{\lambda_i}{\lambda_1} - 1),\]

\[J_2 = 2 F \lambda_1^{-2} \sum_{i>j>\mu} (\frac{\partial F}{\partial \lambda_i} (\lambda_i - \lambda_1)^2 - \frac{\partial F}{\partial \lambda_j} (\lambda_j - \lambda_1)^2) h_{ijj}^2,
\]

\[J_3.
\]
Lemma 4.6. Suppose that $F$ satisfies Condition 1.4. For $\beta > 1$ and $C \leq 0$, the maximum point of $W$ is umbilic.

Proof. For $\frac{\partial F}{\partial \lambda_1} > 0$ and $\frac{\lambda_1}{\lambda_2} \geq 1$, we know $J_1 \geq 0$ and $J_1 = 0$ if and only if $\lambda_1 = \cdots = \lambda_n$. By Lemma 2.8, we have $J_2 \geq 0$.

Observe that

$$J_3 = \frac{\beta - 1}{\beta} F^2 \lambda_1^{-1} \left( \lambda_1^{-1} - \frac{2 \partial \log F}{\partial \lambda_1} \right) (\nabla_1 \log F)^2 + 2 F^2 \lambda_1^{-2} \frac{\partial \log F}{\partial \lambda_1} \sum_{i > \mu} (\lambda_i - \lambda_1)^{-1} h_{1ii}^2$$

$$+ F \lambda_1^{-2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ii1} h_{jj1}. $$

Proof. By Lemma 4.3 we have

$$\langle X, \nabla F \rangle + 2 \frac{\partial F}{\partial h_{ij}} \nabla_i F \nabla_j \frac{1}{F} + 2 F \lambda_1^{-3} \frac{\partial F}{\partial \lambda_1} h_{1ii}^2 \leq \beta F^{-1} \frac{\partial F}{\partial \lambda_1} h_{1ii}^2.$$ 

Furthermore, by Lemma 4.3 and Lemma 4.4, we have

$$\beta \lambda_1^{-1} \left( \lambda_1^{-1} - \frac{2 \partial \log F}{\partial \lambda_1} \right) (\nabla_1 \log F)^2 + 2 F \lambda_1^{-2} \frac{\partial \log F}{\partial \lambda_1} h_{1ii}^2 \leq \frac{2(\beta - 1)}{\beta} F^{-1} \frac{\partial F}{\partial \lambda_1} \lambda_1 \lambda_1^{-1} \left( \lambda_1^{-1} - \frac{1}{\beta} \lambda_1^{-1} \right) (\nabla_1 F)^2.$$ 

Noticing the second term is nonnegative, we finish the proof. 

□

Lemma 4.6. Suppose that $F$ satisfies Condition 1.4. For $\beta > 1$ and $C \leq 0$, the maximum point of $W$ is umbilic.

Proof. For $\frac{\partial F}{\partial \lambda_1} > 0$ and $\frac{\lambda_1}{\lambda_2} \geq 1$, we know $J_1 \geq 0$ and $J_1 = 0$ if and only if $\lambda_1 = \cdots = \lambda_n$. By Lemma 2.8, we have $J_2 \geq 0$.

Observe that

$$J_3 = \frac{\beta - 1}{\beta} F^2 \lambda_1^{-1} \left( \lambda_1^{-1} - \frac{2 \partial \log F}{\partial \lambda_1} \right) (\nabla_1 \log F)^2 + 2 F^2 \lambda_1^{-2} \frac{\partial \log F}{\partial \lambda_1} \sum_{i > \mu} (\lambda_i - \lambda_1)^{-1} h_{1ii}^2$$

$$+ F \lambda_1^{-2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ii1} h_{jj1} + F^2 \lambda_1^{-2} (\nabla_1 \log F)^2 \geq \frac{\beta - 1}{\beta} F^2 \lambda_1^{-1} \left( \lambda_1^{-1} - \frac{2 \partial \log F}{\partial \lambda_1} \right) (\nabla_1 \log F)^2 + 2 F^2 \lambda_1^{-2} \frac{\partial \log F}{\partial \lambda_1} \sum_{i > \mu} (\lambda_i - \lambda_1)^{-1} h_{1ii}^2$$

$$- F^2 \lambda_1^{-2} \frac{\partial \log F}{\partial \lambda_1} h_{ii1}^2 + F^2 \lambda_1^{-2} (\nabla_1 \log F)^2 \geq \frac{\beta - 1}{\beta} F^2 \lambda_1^{-1} \left( \lambda_1^{-1} - \frac{2 \partial \log F}{\partial \lambda_1} \right) (\nabla_1 \log F)^2 - F^2 \lambda_1^{-2} \frac{\partial \log F}{\partial \lambda_1} h_{1ii1}^2.$$
where we use the key inequality in iv) of Condition 1.7 for the above first inequality. Using Lemma 4.3, we have

\[ J_3 \geq \beta - 1 F^{2} \lambda_{1}^{-2} \left( \lambda_{1}^{-1} - \frac{2}{\beta} \frac{\log F}{\partial \lambda_{1}} \right) (\nabla_{1} \log F)^2 - \frac{1}{\beta^2} F^{2} \lambda_{1}^{-1} \frac{\log F}{\partial \lambda_{1}} (\nabla_{1} \log F)^2 + F^{2} \lambda_{1}^{-2} (\nabla_{1} \log F)^2 \]

\[ = \frac{2 \beta - 1}{\beta} F^{2} \lambda_{1}^{-2} \left( F - \frac{F}{\beta} \lambda_{1} \right) (\nabla_{1} \log F)^2. \]

Since \( F = \sum_{i} \frac{1}{\beta} \frac{\partial p}{\partial \lambda_{i}} \lambda_{i} \), we know \( J_3 \geq 0 \). For \( L \) is an elliptic operator, at the maximum point \( x_0 \) of \( W \), we have

\[ 0 \geq LW \geq J_1 + J_2 + J_3 \geq 0. \]

Thus \( J_1 = 0 \), which implies \( \lambda_1 = \cdots = \lambda_n \) at \( x_0 \). Since \( x_0 \) is the maximum point of \( W \), we finish the proof. □

5. Proof of Theorem 1.11

In this section, by considering the quantity

\[ Z = F \text{tr} b - \frac{n(\beta - 1)}{2 \beta} |X|^2, \]

we will prove Theorem 1.11

Lemma 5.1.

\[ \mathcal{L}Z + R(\nabla Z) = L_1 + L_2 + L_3, \]

where \( R(\nabla Z) \) denotes the terms containing \( \nabla Z \),

\[ L_1 = (\beta - 1) F \text{tr} b - \frac{n(\beta - 1)}{\beta} \sum \frac{\partial F}{\partial \lambda_i} + C(n\beta F - \text{tr} b \frac{\partial F}{\partial \lambda_i} \lambda_{1}^{2}), \]

\[ L_2 = \left( \frac{n(\beta - 1)}{\beta} \lambda_{1}^{-1} (2F^{-1} \frac{\partial F}{\partial \lambda_i} + \lambda_{1}^{-1}) - 2F^{-1} \frac{\partial F}{\partial \lambda_i} \text{tr} b \right) (\nabla_i F)^2 \]

and

\[ L_3 = 2F \frac{\partial F}{\partial \lambda_i} \lambda_{p}^{-2} \lambda_{q}^{-1} h_{pij} + F \lambda_{p}^{-2} \frac{\partial F}{\partial \lambda_i} \lambda_{q}^{-1} h_{pij} + F \lambda_{p}^{-2} \sum_{i \neq j} \frac{\partial F}{\partial \lambda_i} (\lambda_i - \lambda_j)^{-1} h_{ijp}. \]

Proof. By Proposition 5.1 we have

\[ \mathcal{L}Z = \langle X, \nabla (F \text{tr} b) \rangle + (\beta - 1) F \text{tr} b - \frac{n(\beta - 1)}{\beta} \sum \frac{\partial F}{\partial h_{ii}} + C(n\beta F - \text{tr} b \frac{\partial F}{\partial h_{ij}} + 2 \frac{\partial F}{\partial h_{ij}} \nabla_j F \nabla_i F) + F b^{kp} b^{k} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{stq} + 2F b^{kp} b^{k} \frac{\partial F}{\partial h_{ij}} h_{st} h_{pqj}. \]
By Condition 1.7 iii), we know

we have

\[ L \]

and

\[ \langle X, \nabla (F \text{trb}) \rangle = \nabla_j (F \text{trb}) (X, e_j) = \nabla_j Z (X, e_j) + \frac{n(\beta - 1)}{\beta} \sum_j (X, e_j)^2 \]

From

\[ \nabla_j Z = \text{trb} \nabla_j F + F \nabla_j \text{trb} - \frac{n(\beta - 1)}{\beta} (X, e_j), \]

we obtain

\[ \nabla_j Z (X, e_j) + \frac{n(\beta - 1)}{\beta} \Delta_j^2 \nabla_j F \]

(5.1)

Then, by Lemma 3.2, we obtain

\[ \mathcal{L} Z + R (\nabla Z) = (\beta - 1) F \text{trb} - \frac{n(\beta - 1)}{\beta} \sum_i \frac{\partial F}{\partial \lambda_i} + C (n\beta F - \text{trb} \frac{\partial F}{\partial \lambda_i})^2 \]

\[ + \left( \frac{n(\beta - 1)}{\beta} \lambda_i^{-1} (2F^{-1} \frac{\partial F}{\partial \lambda_i} + \lambda_i^{-1}) - 2F^{-1} \frac{\partial F}{\partial \lambda_i} \text{trb} \right) (\nabla_i F)^2 \]

\[ + 2F \frac{\partial F}{\partial \lambda_i} \lambda_i^{-2} \lambda_j^{-1} h_i^{2p} + F \lambda_i^{-2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ip} h_{jp} \]

\[ + F \lambda_i^{-2} \sum_{i \neq j} \frac{\partial F}{\partial \lambda_j} (\lambda_i - \lambda_j)^{-1} h_i^{2p}. \]

Lemma 5.2. Suppose that \( F \) satisfies Condition [1.7] For \( \beta \geq 1 \) and \( C \leq 0 \), \( L_1 \geq 0 \).

Proof. It follows from \( \sum \lambda_i \frac{\partial F}{\partial \lambda_i} = \beta F \) that

\[ L_1 = \frac{\beta - 1}{\beta} \sum \lambda_j \frac{\partial F}{\partial \lambda_i} (\lambda_i - 1) - C \sum \lambda_i \frac{\partial F}{\partial \lambda_i} \lambda_i (\lambda_i - 1) \]

\[ = \frac{\beta - 1}{\beta} \sum_{i \neq j} \lambda_i \lambda_j \lambda_i (\lambda_i - \lambda_j) - C \sum \lambda_i \lambda_j (\lambda_i - \lambda_j) \]

\[ = \frac{\beta - 1}{\beta} \sum_{i > j} \lambda_i \lambda_j \lambda_i (\lambda_i - \lambda_j) \]

\[ - C \sum_{i > j} \lambda_i \lambda_j \lambda_i^2 \lambda_j^2 \lambda_i (\lambda_i - \lambda_j). \]

By Condition [1.7 iii), we know \( L_1 \geq 0 \).

Corollary 5.3. For \( F = \sigma_k^\alpha \) with \( 1 \leq k \leq n - 1 \), if \( \alpha > \frac{1}{k} \), \( C \leq 0 \) or \( \alpha \geq \frac{1}{k} \), \( C < 0 \), then \( L_1 = 0 \) is equivalent to \( \lambda_1 = \cdots = \lambda_n \). For \( F = \sigma_n^\alpha \), if \( \alpha > 0 \), \( C < 0 \), then \( L_1 = 0 \) is equivalent to \( \lambda_1 = \cdots = \lambda_n \).
Lemma 5.5. For any $L$ and Lemma 5.5, we have

$$
\sum_i \lambda_i^{-1} \frac{\partial \log F}{\partial \lambda_i} h_{ipp}^2 \geq \frac{1}{\beta} (\nabla_p \log F)^2.
$$

Proof. According to the Cauchy-Schwarz inequality and $\sum_i \lambda_i \frac{\partial F}{\partial \lambda_i} = \beta F$, it follows that

$$(\nabla_p \log F)^2 = \left( \sum_i \frac{\partial \log F}{\partial \lambda_i} h_{ipp} \right)^2
\leq \left( \sum_i \lambda_i \frac{\partial \log F}{\partial \lambda_i} \right) \left( \sum_i \lambda_i^{-1} \frac{\partial \log F}{\partial \lambda_i} h_{ipp}^2 \right)
= \beta \left( \sum_i \lambda_i^{-1} \frac{\partial \log F}{\partial \lambda_i} h_{ipp}^2 \right).$$


Proof of Theorem 1.11. It follows from Lemma 5.1 and Condition 1.7 iv) that

$$L_3 = 2F^2 \frac{\partial \log F}{\partial \lambda_i} \lambda_i^{-2} \lambda_i^{-1} h_{ppi}^2 + F^2 \lambda_i^{-2} \frac{\partial^2 \log F}{\partial \lambda_i \partial \lambda_j} h_{ipp} h_{jp} + F^2 \lambda_i^{-2} (\nabla_p \log F)^2$$

$$+ F \lambda_i^{-2} \sum_{i \neq j} \left( \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \right) (\lambda_i - \lambda_j)^{-1} h_{ijp}$$

$$\geq 2F^2 \frac{\partial \log F}{\partial \lambda_i} \lambda_i^{-2} \lambda_i^{-1} h_{ppi}^2 + F^2 \lambda_i^{-2} \frac{\partial \log F}{\partial \lambda_i} h_{ipp}^2 + F^2 \lambda_i^{-2} (\nabla_p \log F)^2$$

$$+ F \lambda_i^{-2} \sum_{i \neq j} \left( \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \right) (\lambda_i - \lambda_j)^{-1} h_{ijp}.$$

By

$$2F^2 \sum_{i \neq q} \frac{\partial \log F}{\partial \lambda_i} \lambda_i^{-2} \lambda_i^{-1} h_{ppi}^2 + F^2 \lambda_i^{-2} \sum_{i \neq j} \left( \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \right) (\lambda_i - \lambda_j)^{-1} h_{ijp}^2$$

$$= F \lambda_i^{-2} \sum_{i \neq j} \left( \frac{\partial F}{\partial \lambda_i} \lambda_i^2 - \frac{\partial F}{\partial \lambda_j} \lambda_j^2 \right) \lambda_i^{-1} \lambda_j^{-1} (\lambda_i - \lambda_j)^{-1} h_{ijp}$$

$$\geq 0$$

and Lemma 5.5 we have

$$L_3 \geq \frac{\beta + 1}{\beta} F^2 \lambda_i^{-2} (\nabla_p \log F)^2.$$

Since

$$L_2 = F^2 \left( \frac{n(\beta - 1)}{\beta} \lambda_i^{-1} (2 \frac{\partial \log F}{\partial \lambda_i} + \lambda_i^{-1}) - 2 \frac{\partial \log F}{\partial \lambda_i} \text{tr} b \right) (\nabla_i \log F)^2,$$

we have

$$L_2 + L_3 \geq F^2 \left( 2 \frac{\partial \log F}{\partial \lambda_i} (n \lambda_i^{-1} - \text{tr} b) - \frac{2n}{\beta} \lambda_i^{-1} \frac{\partial \log F}{\partial \lambda_i} + \frac{(n + 1) \beta - n}{\beta} \lambda_i^{-2} \right) (\nabla_i \log F)^2.$$
Assume that \( x_0 \) is a maximum point of \( \tilde{W} \). Then it is follows from Lemma 4.6 that \( x_0 \) is an umbilic point. At \( x_0 \), for any fixed \( i \), we have
\[
n\lambda_i^{-1} - \text{tr} b = 0
\]
and
\[
-2n \frac{\lambda_i^{-1}}{\beta} \frac{\partial \log F}{\partial \lambda_i} = -2n \frac{\lambda_i^{-2} F^{1} \lambda_i \frac{\partial F}{\partial \lambda_i}}{\beta} \geq -2\lambda_i^{-2},
\]
thus
\[
L_2 + L_3 \geq F^2 \lambda_i^{-2} \frac{(n-1)(\beta-1)}{\beta} (\nabla_i \log F)^2 \geq 0.
\]
Since \( Z \leq n\tilde{W} \leq n\tilde{W}(x_0) = Z(x_0) \), \( Z \) attains its maximum at \( x_0 \). Hence, there exists a neighborhood of \( x_0 \), denoted by \( U \), such that in \( U \),
\[
L_2 + L_3 + R(\nabla Z) \geq F^2 \lambda_i^{-2} \frac{(n-1)(\beta-1)}{\beta} (\nabla_i \log F)^2 \geq 0.
\]

6. Proofs of Theorem A and Theorem B

In order to prove Theorem A and Theorem B, we use (5.1) to estimate \( L_2 \) and \( L_3 \) in a different way.

**Lemma 6.1.** If \( F \) satisfies i), ii), iii) of Condition 1.7, we have
\[
L_2 + L_3 + R(\nabla Z) \geq F^2 \lambda_i^{-1} \left( -\frac{2n(\beta - 1)}{\beta} \frac{\partial \log F}{\partial \lambda_i} + \frac{\beta(n+1) - n}{\beta} \lambda_i^{-1} (\nabla_i \log F)^2 \right)

+ 2F^2 \frac{\partial \log F}{\partial \lambda_i} \lambda_i^{-1}(\lambda_p^{-1} h_{ppi} - \nabla_i \log F)^2 + F^2 \lambda_p^{-2} \frac{\partial^2 \log F}{\partial \lambda_i \partial \lambda_j} h_{iip} h_{jjp}

+ 2F^2 \sum_{i \neq p} \frac{\partial \log F}{\partial \lambda_i} \lambda_p^{-2} \lambda_i^{-1} h_{pi}^2.
\]

**Proof.** Using (5.1), we have
\[
2F^2 \frac{\partial \log F}{\partial \lambda_i} \left( (-\text{tr} b + \frac{n(\beta - 1)}{\beta} \lambda_i^{-1}) (\nabla_i \log F)^2 + \lambda_p^{-2} \lambda_q^{-1} h_{ppi} \right)

= 2F^2 \frac{\partial \log F}{\partial \lambda_i} \left( \sum_p \lambda_p^{-1} (\lambda_p^{-2} h_{ppi} - (\nabla_i \log F)^2) + \frac{n(\beta - 1)}{\beta} \lambda_i^{-1} (\nabla_i \log F)^2 \right)

+ \sum_{p \neq q} \lambda_p^{-2} \lambda_q^{-1} h_{ppi}^2

= 2F^2 \frac{\partial \log F}{\partial \lambda_i} \left( \sum_p \lambda_p^{-1} (\lambda_p^{-1} h_{ppi} - (\nabla_i \log F)^2) - \frac{n(\beta - 1)}{\beta} \lambda_i^{-1} (\nabla_i \log F)^2 \right)

+ \sum_{p \neq q} \lambda_p^{-2} \lambda_q^{-1} h_{ppi}^2 + R(\nabla Z).
\]
Lemma 6.2. Suppose that we complete the proof.

Proof. In fact, we will estimate the minimum of $F$ and $1 \leq F^2 + \sum_{i \neq j} \lambda_i^2 h_{ijp}^2 + F^2 \sum_{i \neq j} \lambda_i^2 \lambda_j h_{ijp}^2 + F^2 \sum_{i \neq j} \lambda_i h_{ijp}^2 + F^2 \lambda_i^2 \lambda_j h_{ijp}$

Thus,

$$L_2 + L_3 + R(\nabla Z) = \frac{n(\beta - 1)}{\beta} F^2 \lambda_i^{-1}(-2 \frac{\partial \log F}{\partial \lambda_i} + \lambda_i^{-1})(\nabla_i \log F)^2$$

$$+ 2F^2 \frac{\partial \log F}{\partial \lambda_i} \lambda_i^{-1}(\lambda_i^{-1} h_{ppi} - \nabla_i \log F)^2 + F^2 \lambda_i^2 (\nabla_i \log F)^2$$

$$+ 2F^2 \frac{\partial \log F}{\partial \lambda_i} \sum_{i \neq j} \lambda_i^{-2} \lambda_j^{-1} h_{p IQ}^2 + F^2 \lambda_i^{-2} \frac{\partial \log F}{\partial \lambda_i} \frac{\partial h}{\partial \lambda_{ij}} h_{ijp}$$

$$+ F \lambda_i^2 \sum_{i \neq j} \left( \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \right) (\lambda_i - \lambda_j)^{-1} h_{ijp}^2.$$
under the condition \( \sum_i y_i = 1 \). Using Lagrangian multiplier technique, we solve the following equations for \( \hat{f} = f + \tau(\sum_i y_i - 1) \):

\[
0 = \frac{\partial}{\partial y_i} \hat{f} = 2t_i y_i - 4\alpha \delta_{im} + \tau,
\]

\[
0 = \frac{\partial}{\partial \tau} \hat{f} = \sum_i y_i - 1.
\]

And, using \( \sum_{i=1}^n \frac{1}{t_i} = k \), we have \( y_i = \frac{2\alpha \delta_{im}}{t_i} - \frac{2}{kt_m} \) and \( \tau = \frac{4\alpha}{kt_m} - \frac{2}{k} \). Thus, \( y_i = \frac{1}{t_i}(2\alpha \delta_{im} - \frac{2\alpha}{kt_m} + \frac{1}{k}) \). Because \( t_i > 0 \), we know

\[
f_{\text{min}} = \sum_i \frac{1}{t_i} (2\alpha \delta_{im} - \frac{2\alpha}{kt_m} + \frac{1}{k})^2 - 4\alpha (\frac{2\alpha}{kt_m} - \frac{1}{k})^2 = \sum_{i \neq m} \frac{1}{k^2 t_i} (\frac{2\alpha}{t_m} - 1)^2 - \frac{4\alpha^2}{t_m} = \frac{1}{k} (\frac{2\alpha}{t_m} - 1)^2 - \frac{4\alpha^2}{t_m}.
\]

\( \square \)

**Theorem 6.3.** For \( F = \sigma_k^\alpha \) and \( C \leq 0 \), if \( 2 \leq k \leq n-1 \) and \( \frac{1}{\alpha+2} \leq \alpha \leq \frac{1}{\alpha} \), the strictly convex closed solution of (1.7) is a round sphere. For \( F = \sigma_k^\alpha \) and \( C < 0 \), if \( \frac{1}{\alpha+2} \leq \alpha \leq \frac{1}{\alpha} \), the strictly convex closed solution of (1.7) is a round sphere.

**Proof.** Using Lemma 6.1 and Lemma 2.7, we have

\[
\frac{1}{\alpha \sigma_k} (L_2 + L_3) + R(\nabla Z)
\]

\[
\geq \lambda_i^{-1} \left( -\frac{2\alpha ((n-1)k \alpha - n)}{k} \sigma_{k-1;i} \right) \sigma_k + \frac{(n+1)k \alpha - n}{k} \lambda_i^{-1} (\nabla_i \log \sigma_k)^2
\]

\[
+ 2 \frac{\sigma_{k-1;i}}{\sigma_k} \lambda_i^{-1} (\lambda_i^{-1} h_{i;j;j} - \alpha \nabla_i \log \sigma_k)^2 + \lambda_i^{-2} \left( \frac{\partial^2 \log \sigma_k}{\partial \lambda_i \partial \lambda_j} h_{i;j} h_{j;i} \right)
\]

\[
+ 2 \sum_{i \neq j} \frac{\sigma_{k-1;i}}{\sigma_k} \lambda_i^{-2} \lambda_j^{-1} h_{i;i}^2
\]

\[
\geq \lambda_i^{-1} \left( -\frac{2\alpha ((n-1)k \alpha - n)}{k} \sigma_{k-1;i} \right) \sigma_k + \frac{(n+1)k \alpha - n}{k} \lambda_i^{-1} (\nabla_i \log \sigma_k)^2
\]

\[
+ 2 \sum_{i} \frac{\sigma_{k-1;i}}{\sigma_k} \sum_{j \neq i} \lambda_j^{-1} (\lambda_j^{-1} h_{i;j;j} - \alpha \nabla_i \log \sigma_k)^2 + \lambda_i^{-2} \left( \frac{\sigma_{k-1;i}}{\sigma_k} \lambda_j^{-1} h_{i;i}^2 \right)
\]

\[
- 4\alpha \frac{\sigma_{k-1;i}}{\sigma_k} \lambda_i^{-2} h_{i;i} \nabla_i \log \sigma_k.
\]
Let $t_i = \frac{\sigma_k}{\lambda_i \sigma_{k-1}}$ and using Lemma 6.2, we have
\[
\lambda_i^{-2} \frac{\sigma_{k-1+p}}{\sigma_k} \lambda_p^{-1} h_{ppi}^2 - 4\alpha \frac{\sigma_{k-1}}{\sigma_k} \lambda_i^{-2} h_{iii} \nabla_i \log \sigma_k
\]
\[
= \sum_i \lambda_i^{-2} \left\{ \sum_p t_p \left( \frac{\sigma_{k-1+p}}{\sigma_k} \lambda_p^{-1} h_{ppi} \right)^2 - 4\alpha \frac{\sigma_{k-1}}{\sigma_k} \lambda_i^{-2} \left( \sum_p \frac{\sigma_{k-1+p}}{\sigma_k} h_{ppi} \right) \right\}
\]
\[
\geq \sum_i \lambda_i^{-2} \left( \frac{2\alpha}{t_i} - 1 \right)^2 \left( \nabla_i \log \sigma_k \right)^2.
\]
Then, we obtain
\[
\frac{1}{\alpha \sigma_k^2} (L_2 + L_3) + R(\nabla Z)
\]
\[
\geq \sum_i \lambda_i^{-2} \left( \frac{2\alpha}{t_i} \left( \frac{2}{k t_i} - n - 1 \right) + \alpha \left( \frac{2(n-2)}{k t_i} + n + \frac{n-1}{k} \right) \right) \left( \nabla_i \log \sigma_k \right)^2
\]
\[
= \sum_i \lambda_i^{-2} \left( \frac{2\alpha}{t_i} \right) \left( \frac{2}{k t_i} - n - 1 \right) \left( 1 + \frac{n-1}{k} \right) \left( \nabla_i \log \sigma_k \right)^2.
\]
Since $t_i \geq 1$, if $k \geq 2$ and $\alpha \in \left[ \frac{n-1}{k(n+1)-2}, \frac{n}{k} \right]$, then $LZ + R(\nabla Z) \geq 0$. By the strong maximum principle, we know $Z$ is constant. Hence, $L_1 = L_2 + L_3 = 0$. In case $C < 0$ or $\alpha > \frac{1}{k}$, by Corollary 5.3, $L_1 = 0$ implies that $M$ is totally umbilic; in other cases, $L_2 + L_3 = 0$ implies that the second fundamental form is parallel. Either of these implies that the solution is a round sphere. □

For $F = S_k^\alpha$, we have the following theorem.

**Theorem 6.4.** For $F = S_k^\alpha$ and $C \leq 0$, if $k \geq 1$ and $\alpha = \frac{1}{k}$, the solution of (1.7) is a round sphere.

**Proof.** Using Lemma 6.3, we obtain
\[
\frac{1}{\alpha S_k^\alpha} (L_2 + L_3) + R(\nabla Z)
\]
\[
\geq \lambda_i^{-1} \left( -2\alpha \lambda_p (k \alpha - 1) \lambda_i^{-1} S_k + \frac{k \alpha (n+1) - n}{k} \lambda_i^{-1} \nabla_i \log S_k \right)^2
\]
\[
+ 2k \frac{\lambda^{-1}}{S_k} \lambda_p^{-1} \left( \nabla_i \log S_k \right)^2 + \lambda_p^{-2} \frac{\partial^2 \log S_k}{\partial \lambda_i \partial \lambda_j} h_{isip} h_{jjp}
\]
\[
+ 2k \sum_{i \neq p} \lambda_i^{-1} \lambda_p^{-2} \lambda^{-1} h_{ppi}^2.
\]
Since
\[
\frac{\partial^2 \log S_k}{\partial \lambda_i \partial \lambda_j} h_{isip} h_{jjp} = \frac{k(k-1)}{S_k} \frac{\lambda_i^{-2}}{\lambda_j} h_{isip}^2 - \left( \nabla_p \log S_k \right)^2
\]
\[
\geq \frac{k-1}{k} \left( \nabla_p \log S_k \right)^2 - \left( \nabla_p \log S_k \right)^2
\]
\[
= \frac{1}{k} \left( \nabla_p \log S_k \right)^2
\]
where the inequality follows from the Cauchy-Schwarz inequality, we have
\[
\frac{1}{\alpha S_k^2}(L_2 + L_3) + R(\nabla Z) \\
\geq (k\alpha - 1)(-2\alpha n \frac{\lambda_k}{S_k} + \frac{n + 1}{k})\lambda_i^{-2}(\nabla_i \log S_k)^2 \\
+ 2k\frac{\lambda_k - 1}{S_k} \lambda_p^{-1} (\lambda_p^{-1} h_{ppi} - \alpha \nabla_i \log S_k)^2 + 2k \sum_{i \neq p} \frac{\lambda_k - 1}{S_k} \lambda_i^{-2} \lambda_p^{-1} h_{pi}^2 \lambda_i^{-2} h_{ppi}^2 \geq 0.
\]

Thanks to \( L_1 \geq 0 \), by the strong maximum principle, \( Z \) is constant. Hence, \( L_1 = L_2 + L_3 = 0 \). Using the same argument as in the proof of Theorem 6.3, we finish the proof.

\[\square\]

**Proof of Theorem A.** Combining Theorem 1.11, Theorem 6.3 with Theorem 6.4 for \( k = 1 \), we complete the proof of Theorem A.

\[\square\]

**Proof of Theorem B.** Combining Theorem 1.11 with Theorem 6.4, we complete the proof of Theorem B.

\[\square\]

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