Rigorous FEM for 1D Burgers equation

Piotr Kalita and Piotr Zgliczyński

Abstract. We propose a method to integrate dissipative PDEs rigorously forward in time with the use of Finite Element Method (FEM). The technique is based on the Galerkin projection on the FEM space and estimates on the residual terms. The technique is illustrated on a periodically forced one-dimensional Burgers equation with Dirichlet conditions. For two particular choices of the forcing we prove the existence of the periodic globally attracting trajectory and give precise bounds on its shape.

1. Introduction

In the study of evolutionary partial differential equations (PDEs) purely analytical methods often appear insufficient to gain full understanding of the behaviour of solutions. While, typically, they allow us to obtain well-posedness, and, in dissipative situation, existence and some basic properties of the global attractor, the precise description of the dynamics on the attractor is often out of reach of such tools [33, 36]. On the other side with the present day computers a lot of interesting PDEs can be numerically investigated and quite often on the heuristic level the dynamics of such systems can be well understood. This however does not constitute a mathematical proof and there are situations when the numerical simulations can be misleading. Our goal in this paper is to take a FEM discretization method and to build rigorous numerics around it. By rigorous integration of dissipative PDE we understand an algorithm which produces a set which is guaranteed to contain all solutions of a given problem originating from any initial datum in a given set. We want these bounds to be of sufficient quality to satisfy assumptions of some abstract theorem which gives us the existence of some interesting dynamical object - in the present paper it is a periodic orbit.

The task of designing algorithms for rigorous numerics for evolution in time of parabolic PDEs is not new. For problems with periodic boundary conditions it was realized (using the Fourier basis) with considerable success,

- for the Kuramoto–Shivashinsky equation on the line where the periodic [1, 22, 41, 42] and, more recently chaotic [38] solutions were verified,
- for the Burgers equation where periodic orbit attracting all solutions was shown [8],
- for the one-dimensional Ohta–Kawasaki model in [7] some heteroclinic orbits between some fixed points has been proven to exist.

The main point in our paper is the use of FEM basis for the algorithm. This choice is motivated by the wide applicability of FEM in numerical solving of PDEs for general boundary condition. Before we outline our FEM-based approach let us highlight first some properties of the Fourier basis, which were crucial in works [1, 7, 8, 22, 38, 41, 42]
• product of two functions from the Fourier basis is a function which is an element of the same Fourier basis,
• each differential operator with constant coefficients is diagonal in the Fourier basis.

With these properties the nonlinear terms can be easily expressed and estimated via convolutions. The diagonality of the leading differential operator greatly helps with getting a priori bounds for short time intervals. These properties make it possible to design an algorithm for rigorous integration of dissipative PDEs with periodic boundary conditions, using the method of self-consistent bounds developed in [22, 42]. In this approach it is possible to obtain rigorous bounds on trajectories for all sufficiently high-dimensional Galerkin projections and then after passing to the limit one gets rigorous bounds for the solutions of the PDE under consideration.

In the present paper as an object to study we chose one dimensional problem governed by the following Burgers equation with Dirichlet boundary conditions

\begin{align}
  u_t - u_{xx} + uu_x &= f(x, t) \quad \text{for} \quad (x, t) \in (0, 1) \times (t_0, \infty), \\
  u(0, t) &= u(1, t) = 0, \\
  u(t_0) &= u_0.
\end{align}

We use the simplest FEM basis of first order Lagrange elements (intervals). We treat this example as a toy model, which contains probably all difficulties coming from the choice of a quite arbitrary basis, hence the experience gained studying it should be transferable to other equations and boundary conditions. Observe that in the FEM basis none of the above listed properties of the Fourier basis are satisfied, moreover some additional problems arise due the following issue

• elements of the FEM basis might not be in the domain of the PDE under consideration.

We demonstrate the correctness of our approach by showing that for problem (1.1)–(1.3) for two particular choices of periodic in time \(f\) there exists the periodic solution \(u\) in some neighborhood of numerical solution found by the standard, nonrigorous, FEM. Let us stress that problem (1.1)–(1.3) with periodic forcing very likely can be treated using the Fourier basis and the technique developed in [8], but the result on attracting periodic solution of (1.1)–(1.3) is not the main goal of the paper. The goal is to outline a FEM-based method for rigorous integration of dissipative PDEs.

One time step of length \(h > 0\) of our method consists of the following two stages.

**STAGE 1** Given a set \(N\) of initial conditions taken at time \(t_0\), which is bounded, closed, and convex in \(H^s\), we construct the set \(W\), such that \(N \subset W\), \(W\) is the bounded, closed, and convex set in \(H^s\), such that all solutions starting from \(N\) are defined for \(t \in [t_0, t_0 + h]\) and are contained in \(W\). The value of \(s\) is dictated by the need to estimate the projection error of the Laplacian operator in the FEM-basis. To this end, we need \(s = 4\) for Burgers equation and Lagrange elements, while for other differential operators and more regular FEM bases \(s\) might be bigger.

Effective realization of this part of the algorithm depends on the particular equation. In the case of the Burgers equation global in time a priori bounds based on the energy estimates (and their local in time versions) are used in our work. These derivations are standard, but we propose several interesting tricks to make the bounds as small as possible, their derivation is presented in Appendix B.

**STAGE 2** We reduce in appropriate way the original PDE to the finite dimensional problem governed by a system of ODEs. The infinite dimensional residual term appears in the reduced problem so we need to estimate it by the size of the mesh with the use of the a priori estimates, \(W\), obtained in the first stage. This was the reason we needed the bounds up to the order \(s\). With these estimates
the problem is reduced to the Ordinary Differential Inclusion. The resultant inclusion is solved rigorously using the algorithm from [18] implemented in CAPD library for rigorous numerics. For the purpose of the proof of the periodic solution existence, we verify if the set obtained after the period of integration is the subset of the set of initial data. Since this is the case, by the Schauder theorem for the mapping of the forward in time translation by the period, we obtain the periodic solution existence.

As the reader familiar with our previous work [22,41,42] might notice, the basic scheme of one step of the method is the same as in the periodic boundary conditions case. Let us highlight the differences.

- In STAGE 1
  - In the periodic case, due to the isolation property we were able to obtain the bounds for each Fourier coefficient. This can be easily automatized and a general enclosure algorithm can be given. This is accomplished by some standard ODE-type reasonings.
  - In the FEM-case, we work with various Sobolev norms, very much in the spirit of the modern theory of PDEs [33,36]. This part of the algorithm relies on various tricks and is technically much more involved than in the periodic case.

- In STAGE 2
  - In the periodic case, it is straightforward to obtain the ordinary differential inclusion.
  - In the FEM-case, obtaining the ordinary differential inclusion requires that our a-priori bounds \( W \) from STAGE 1 are in \( H^s \) for \( s \) sufficiently large. This allows us treat the error contributions coming from the Laplacian and various nonlinear terms.

The computation times for the simple problem (1.1)–(1.3) are rather long (around 1 hr). This is mainly due to the fact that we used the first order FEM-elements. There is no doubt that using a higher order FEM will greatly improve the performance, but this requires local-in-time a-priori estimates in \( H^s \) for \( s > 4 \). In our paper we developed such estimates by hand for \( s = 1, \ldots, 4 \), for each \( s \) separately. It will good to have an algorithm, which will do it for us for any \( s > 0 \).

Other known approaches to rigorous numerics for PDEs. There is an approach called functional analytic as opposed to our, which can be termed topological or geometric.

It is based on the Newton method and some fixed point theorem, the problem under consideration is written as

\[ \mathcal{F}(u) = 0, \]

where \( \mathcal{F} : X \to Y \) is a mapping between Banach spaces \( X \) and \( Y \). The map \( \mathcal{F} \) and spaces \( X, Y \) encode the boundary conditions and when looking for periodic orbits also periodic boundary conditions in time direction.

The method has been used successfully to verify the solutions for elliptic (using FEM or Fourier basis) [10,11,13,19,20,29] and parabolic [21,23–26,35] PDEs (but this mainly for periodic boundary conditions). The up to date information about these techniques can be found in the recent monograph [27]. We stress, however, that the approach for the parabolic case does not involve its forward in time integration. Rather than that, the space time differential operator of the parabolic problem is inverted. The technique can be used to identify the invariant objects belonging to the global attractor [4,9,12]. In particular it has been used recently to verify the existence of time periodic solution for the Taylor–Green problem for the 2D incompressible Navier–Stokes equations [2], where the rigorous calculations using the Fourier basis have been realized.

The plan of the article. We conclude the introduction with the brief presentation of the scheme of the paper: is Section 2 the problem is defined and some of its basic properties such as the existence and uniqueness of the weak and strong solutions, their continuous dependence on the initial data and basic energy estimates
are recalled. Next, Section 4 is devoted to the algorithm of reduction of the original PDE to the Ordinary Differential Inclusion, and the algorithm of the computer assisted proof is described. Some details concerning the implementation, as well as two examples of computations are presented in Section 5. The technical mathematical part of the paper, in particular the derivation of higher order energy estimates, which is simple but cumbersome, is contained in Appendixes A and B.

2. The Burgers equation: problem setting, existence of solutions, estimates, and trapping sets

In this section we provide basic facts and results concerning our model problem: the nonautonomous Burger’s equation. As a space domain we always consider the interval $\Omega = (0, 1)$. We will use the shorthand notation for the spaces of functions defined on the interval $\Omega$, for example we will write simply $L^2$ in place of $L^2(\Omega)$, $H^1_0$ in place of $H^1_0(\Omega)$ and so on. The norm in $H^1_0$ is defined as $\|u\|_{H^1_0} = \|u\|_{H^1_0(\Omega)}$. Scalar product in $L^2$ will be denoted simply by $(\cdot, \cdot)$. We will also denote in a simplified way, by dropping the time variable, the spaces of functions leading from $\mathbb{R}$ or its subinterval to some spaces of space dependent functions, for example $L^\infty(L^2)$ will be the abbreviation for $L^\infty(\mathbb{R}; L^2)$. We stress that all proofs of this section are standard and they use well known techniques based on the energy estimates. We include them only for the exposition completeness. We define the initial time as $t_0 \in \mathbb{R}$. We are interested in solving the following problem governed by the one-dimensional Burgers equation (1.1) with the boundary condition (1.2) and the initial condition (1.3). We will always assume that the forcing term $f$ is defined for every $t \in \mathbb{R}$.

2.1. Weak solution and its properties. We begin by the definition of the weak solution for the considered problem

**Definition 2.1.** The function $u \in L^2_{loc}(t_0, \infty; H^1_0)$ with $u_t \in L^2_{loc}(t_0, \infty; H^{-1})$ is a weak solution of the Burgers equation with the initial data $u(t_0) = u_0$ if the following equation holds

$$
(2.1) \quad (u_t, v)_{H^{-1} \times H^1_0} + (u_x, v_x) + (uu_x, v) = (f(\cdot, t), v) \quad \text{for every} \quad v \in H^1_0 \quad \text{a.e.} \quad t > t_0.
$$

The proof of the following result is standard and it follows by the Galerkin method. It is omitted here, but the details can be found, for example, in [33, 36],

**Theorem 2.2.** Suppose that $f \in L^2_{loc}(L^2)$ and $u_0 \in L^2$. Then the problem given by Definition 2.1 has a unique weak solution.

We derive the energy estimate satisfied by every weak solution of the above problem.

**Lemma 2.3.** Let $u_0 \in L^2$ and $f \in L^2_{loc}(L^2)$ and let $u$ be the weak solution corresponding to initial datum $u_0$ taken at time $t_0$ and $f$. The following bounds are valid

$$
(2.2) \quad \frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\pi^2\|u(t)\|_{L^2}^2 \leq 2\|f(t)\|_{L^2}\|u(t)\|_{L^2} \quad \text{for almost every} \quad t > t_0,
$$

$$
(2.3) \quad \int_{t_0}^t \|u_x(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + \frac{1}{\pi^2} \int_{t_0}^t \|f(s)\|_{L^2}^2 ds \quad \text{for every} \quad t > t_0.
$$

**Proof.** The proof is standard. We test (2.1) by $u$. Note that the regularity of the weak solution guarantees that $(u_t(t), u(t))_{H^{-1} \times H^1_0} = \frac{1}{2} \frac{d}{dt}\|u(t)\|_{L^2}^2$ for almost every $t > 0$, cf. [39, Proposition 23.23]. From Lemma A.5 we obtain

$$
(2.4) \quad \frac{1}{2} \frac{d}{dt}\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 = (f(t), u).
$$
Now, (2.2) follows that by the Schwarz and Poincaré inequalities (see Lemma A.2). On the other hand, integrating (2.4), we obtain

\[ \int_{t_0}^{t} \| u_x(s) \|^2_{L^2} \, ds \leq \frac{1}{2} \| u_0 \|^2_{L^2} + \int_{t_0}^{t} \| f(s) \|_{L^2} \| u(s) \|_{L^2} \, ds \] for a.e. \( t > 0 \).

From this inequality, after using the Poincaré and Cauchy inequalities we obtain

\[ \int_{t_0}^{t} \| u_x(s) \|^2_{L^2} \, ds \leq \| u_0 \|^2_{L^2} + 2 \int_{t_0}^{t} \| f(s) \|_{L^2} \| u(s) \|_{L^2} \, ds - \int_{t_0}^{t} \| u_x(s) \|^2_{L^2} \, ds \]

\[ \leq \| u_0 \|^2_{L^2} + \int_{t_0}^{t} \left( \frac{\| f(s) \|^2_{L^2}}{\pi^2} + \pi^2 \| u(s) \|^2_{L^2} \right) \, ds - \pi^2 \int_{t_0}^{t} \| u(s) \|^2_{L^2} \, ds \]

\[ = \| u_0 \|^2_{L^2} + \frac{1}{\pi^2} \int_{t_0}^{t} \| f(s) \|^2_{L^2} \, ds. \]

The proof is complete. \( \square \)

The mapping that assigns to the initial data taken at time \( t_0 \) the value of the solution at time \( t \geq t_0 \) will be denoted by \( S(t, t_0) : L^2 \to L^2 \). Clearly \( S(t, t_0) \) is a process, i.e. \( S(t, t_1)S(t_1, t_0) = S(t, t_0) \) for every \( t_0 \leq t_1 \leq t \) and \( S(t_0, t_0) = I \).

2.2. Strong solution and its properties. We give the definition of the strong solution for the considered problem.

**Definition 2.4.** The function \( u \in L^2_{loc}(t_0, \infty; H^1 \cap H^2) \) with \( u_t \in L^2_{loc}(t_0, \infty; L^2) \) is the strong solution of the Burgers equation with the initial data \( u(t_0) = u_0 \) if there holds

\[ u_t - u_{xx} + uu_x = f(\cdot, t) \] holds in \( L^2 \) for a.e. \( t > t_0 \).

The proof of the following result is standard and we omit it here [33, 36].

**Theorem 2.5.** Suppose that \( f \in L^2_{loc}(L^2) \) and \( u_0 \in H^1_0 \). Then the problem given by Definition 2.4 has a unique strong solution.

It is clear that a strong solution is also a weak solution, so the process \( S(t, t_0) \) applied to an element of \( u_0 \in H^1_0 \) defines the value of a strong solution at time \( t \) if the initial data \( u_0 \) is taken at time \( t_0 \). The following result provides the energy estimate satisfied by the strong solutions.

**Lemma 2.6.** Let \( u_0 \in H^1_0 \) and \( f \in L^2_{loc}(L^2) \) and let \( u \) be the strong solution corresponding to \( u_0 \) taken at time \( t_0 \) and \( f \). Let \( \alpha, \beta > 0 \) be two constants such that \( \alpha + \beta < 2 \). The following differential inequalities hold for a.e. \( t > t_0 \)

\[ \frac{d}{dt} \| u_x \|^2_{L^2} \leq -2 \| u_{xx} \|_{L^2} \left( \| u_{xx} \|_{L^2} - \| f(t) \|_{L^2} - \| u \|_{L^2}^{5/4} \| u_{xx} \|_{L^2}^{3/4} \right). \]

\[ \frac{d}{dt} \| u_x \|^2_{L^2} + \pi^2 (2 - \alpha - \beta) \| u_x \|^2_{L^2} \leq \frac{1}{\alpha} \| f(t) \|^2_{L^2} + \frac{7}{2^{16} \beta^2} \| u \|^{10}_{L^2}. \]

**Proof.** We multiply (2.5) by \(-u_{xx}\), whence we get the bound

\[ \frac{1}{2} \frac{d}{dt} \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \leq \| f(t) \|_{L^2} \| u_{xx} \|_{L^2} + \int_0^1 |u| |u_x| |u_{xx}| \, dx. \]

It follows that

\[ \frac{1}{2} \frac{d}{dt} \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \leq \| f(t) \|_{L^2} \| u_{xx} \|_{L^2} + \| u \|_{L^\infty} \| u_x \|_{L^2} \| u_{xx} \|_{L^2}. \]
Using Lemma A.4 we deduce that

\[ \frac{1}{2} \frac{d}{dt} \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \leq \| f(t) \|^2_{L^2} \| u_{xx} \|^2_{L^2} + \| u \|^5/4 \| u_{xx} \|^{7/4}_{L^2}. \]

We obtain (2.6).

Let \( \alpha, \beta > 0 \) be two constants. We use Lemma A.1 with \( p = 8, q = q/7, \epsilon = (\beta 4/7)^{7/8} \) to estimate \( \| u \|^{5/4}_{L^2} \| u_{xx} \|^{7/4}_{L^2} \) and with \( p = q = 2, \epsilon = \sqrt{\alpha} \) to estimate \( \| f(t) \|_{L^2} \| u_{xx} \|_{L^2} \). We get

\[ \frac{1}{2} \frac{d}{dt} \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \leq \frac{1}{2\alpha} \| f(t) \|^2_{L^2} + \frac{\alpha}{2} \| u_{xx} \|^2_{L^2} + \frac{77}{21\beta^2} \| u \|^{10}_{L^2} + \beta \| u_{xx} \|^2_{L^2}, \]

whence the following inequality holds

\[ \frac{d}{dt} \| u_x \|^2_{L^2} + (2 - \alpha - \beta) \| u_{xx} \|^2_{L^2} \leq \frac{1}{\alpha} \| f(t) \|^2_{L^2} + \frac{77}{21\beta^2} \| u \|^{10}_{L^2}. \] (2.8)

Using the Poincaré inequality we obtain (2.7). The proof is complete. \( \square \)

We pass to the proof of \( H^1 \) continuity of \( S(t, t_0) \).

**Lemma 2.7.** Let \( f \in L^2_{loc}(L^2) \) and let \( t_0 \in \mathbb{R} \). If \( u_0, v_0 \in H^1_0 \) and \( u, v \) are two strong solutions corresponding to \( u_0, v_0 \) taken at \( t_0 \), respectively, then

\[ \| u(t) - v(t) \|_{H^1_0} \leq e^{\frac{1}{\alpha} \| u_0 \|^2_{L^2} + \| v_0 \|^2_{L^2} + \frac{77}{21\beta^2} \| u \|^{10}_{L^2}} \| u_0 - v_0 \|_{H^1_0} \quad \text{for every} \quad t \geq t_0. \]

**Proof.** Let \( u_0, v_0 \in H^1_0 \) and let \( u, v \) be strong solutions corresponding to \( u_0, v_0 \), respectively. Denoting \( w = u - v \) there holds the following equation

\[ w_t - w_{xx} + uw_x - v w_x = 0 \quad \text{a.e.} \quad t > t_0, x \in (0, 1). \]

Testing this equation by \( -w_{xx} \), we obtain (using \( uu_x - vv_x = uw_x + vw_x \))

\[ \frac{1}{2} \frac{d}{dt} \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \leq \| (uw_x, w_{xx}) \| + \| (v_x w, w_{xx}) \|. \]

Using \( \| w \|_{L^\infty} \leq \| w \|_{L^2} \) it follows that

\[ \frac{1}{2} \frac{d}{dt} \| w \|^2_{H^1_0} + \| w_{xx} \|^2_{L^2} \leq \| u \|_{L^\infty} \| u_x \|_{L^2} \| w_{xx} \|_{L^2} + \| w \|_{L^\infty} \| v_x \|_{L^2} \| w_{xx} \|_{L^2} \]

\[ \leq \| u_x \|_{L^2} \| w_x \|_{L^2} \| w_{xx} \|_{L^2} + \| w_x \|_{L^2} \| v_x \|_{L^2} \| w_{xx} \|_{L^2} \]

whence, as \( \| w \|_{H^1_0} = \| w \|_{L^2} \),

\[ \frac{1}{2} \frac{d}{dt} \| w \|^2_{H^1_0} + \| w_{xx} \|^2_{L^2} \leq (\| u_x \|_{L^2} + \| v_x \|_{L^2}) \| w \|_{H^1_0} \| w_{xx} \|_{L^2}. \]

It follows that

\[ \frac{d}{dt} \| w \|^2_{H^1_0} \leq (\| u_x \|^2_{L^2} + \| v_x \|^2_{L^2}) \| w \|^2_{H^1_0}. \]

The assertion follows by the Gronwall lemma and estimates (2.3). \( \square \)
2.3. Asymptotic behavior of solutions. We start this section from the recollection of the definition of the eternal strong solution.

**Definition 2.8.** The function $u \in C(H^1_0)$ is called the eternal strong solution of the Burgers equation if there exists a constant $D > 0$ such that $\|u_x(t)\|_{L^2} \leq D$ for every $t \in \mathbb{R}$ and for every $t_0 \in \mathbb{R}$ the function $u$ restricted to $[t_0, \infty)$ is the strong solution given by Definition 2.4 with the initial data $u(t_0)$.

We recall the results of [17].

**Theorem 2.9.** (cf. [17, Theorem 4.1]) Let $f \in L^\infty(L^2)$. There exists the unique eternal strong solution in the sense of Definition 2.8.

We use the notation $B(L^2)$ for nonempty and bounded subsets of $L^2$ and $\|B\|_{L^2} = \sup_{x \in B} \|x\|_{L^2}$ for $B \in B(L^2)$.

**Theorem 2.10.** (cf. [17, Theorem 4.7]) Let $f \in L^\infty(L^2)$. Let $v_0 \in B \in B(L^2)$ and let $v$ be a weak solution starting from the initial data $v_0$ at time $t_0$. Let $u$ be a unique eternal solution given by Theorem 2.9. There exists a constant $D = D(\|B\|_{L^2}) > 0$ (depending continuously and monotonically on $\|B\|_{L^2}$) such that for every $t \geq t_0$ there holds

$$\|u(t) - v(t)\|_{L^2} \leq D(\|B\|_{L^2}) e^{-C(t-t_0)}$$

Since the right-hand side of the last estimate depends on $\|B\|_{L^2}$ and not on $v$ or $v_0$ we deduce that the estimate can be replaced with

$$\text{dist}_{L^2}(S(t, t_0)B, \{u(t)\}) \leq C(\|B\|_{L^2}) e^{-C(t-t_0)},$$

where $\text{dist}_{L^2}$ is the Hausdorff semidistance defined as

$$\text{dist}_{L^2}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{L^2}.$$  

It is clear that the right-hand side of the estimate (2.9) tends to zero either as $t_0 \rightarrow -\infty$ for fixed $t$ or as $t \rightarrow \infty$ as $t_0$ is fixed. This means that the unique eternal strong solution attracts all weak (and thus also strong) solutions uniformly with respect to bounded sets of the initial data.

3. Trapping sets.

This section is devoted to construction of a set on which the process $S(t, t_0)$ is positively invariant (i.e. it is, in a sense, trapping as once the trajectory enters it, it can never leave it). The results of this section will be used later when we pass to the numerical part of this article. We will later derive the error estimates of the Galerkin projection of the solution in $H^1_0$ and, to this end, we will need to control up to fourth space derivative of the solution. Hence, the results of this chapter provide the existence of positively invariant set on which we control $L^2$ norms of the space derivatives up to fourth of the solution. Note, that this requires the increased assumptions of the regularity of $f$.

We start from the definition of the positively invariant (trapping) set.

**Definition 3.1.** Let $X$ be a Banach space and let family of maps $\{S(t, t_0)\}_{t \geq t_0}$ be a process on $X$. The set $B \subset X$ is said to be positively invariant (trapping) if for every $t_0 \in \mathbb{R}$ and $t > 0$ there holds $S(t_0 + t, t_0)B \subset B$.

In the following result we demonstrate the existence of a convex trapping sets being a closed and bounded subset of $H^4$. Since the proof of this result is technical, it is postponed until Appendix B.
Theorem 3.2. Define \( Y = \{ u \in H^4 \cap H_0^1 : u_{xx} \in H_0^1 \} \) endowed with the norm of \( H^4 \). Let \( f \in L^\infty(Y) \). There exists the nonempty trapping set \( B_0 \subset Y \) which is convex, closed and bounded in \( Y \). Moreover, it is possible to find explicitly the radii \( R_1, R_2, R_3, R_4, R_5 \) such that if \( u \in B_0 \) then
\[
\|u\|_{L^2} \leq R_1, \quad \|u_x\|_{L^2} \leq R_2, \quad \|u_{xx}\|_{L^2} \leq R_3, \quad \|u_{xxx}\|_{L^2} \leq R_4, \quad \|u_{xxxx}\|_{L^2} \leq R_5.
\]

Remark 3.3. The values of the radii \( R_1 - R_5 \) will be calculated basing on the a priori estimates of the equation and then used in the computer assisted construction of the attracting trajectory. The construction will be based on the splitting of the whole space \( H_0^1 \) into a finite dimensional part and its infinite dimensional remainder which will be estimated basing on these radii. Then, the contribution of the remainder will be incorporated in a multivalued additive term thus leading to the need of the rigorous numerical solution of an ordinary differential inclusion. From numerical reasons it will be crucial that the width of this inclusion is as small as possible. As this width depends on \( R_1 - R_5 \) significant technical effort is put in Appendix B to construct the trapping set with smallest possible radii. In fact for each \( R \) several algorithms originating from different a priori estimates are presented in Appendix B. Then, in computational part for particular function \( f \) we implement all algorithms and choose the smallest obtained value for each radius. For readers convenience all algorithms are summarised in Tab.1.

Remark 3.4. The restriction that \( f \) as well as \( f_{xx} \) should satisfy the Dirichlet condition comes from the fact that we need the homogeneous Dirichlet boundary conditions for \( u_{xx} \) and \( u_{xxxx} \), as we derive the energy estimates for the original equation to which we apply the second and fourth space derivatives. This requirement is only technical and its purpose is to make the estimates simpler. We avoid full generality which could be achieved by translating the second and fourth space derivatives of the solution \( u \) by any function \( a(x,t) \) which satisfies the same boundary conditions as \( f \). Indeed, if \( u \) solves (1.1)-(1.3), then \( v = u_{xx} \) solves
\[
v_t - v_{xx} + 3uv_x + uv_x = f_{xx},
\]
and \( w = v + a = u_{xx} + a \) is guaranteed to satisfy the homogeneous Dirichlet condition if only \( a = f \) in points \( x = 0 \) and \( x = 1 \). The function \( w \) satisfies the following equation
\[
w_t - w_{xx} + 3wu_x + uw_x = f_{xx} + a_t - a_{xx} + 3au_x + a_{xx}.
\]

### Table 1. Methods to derive the radii \( R_1 - R_5 \) of the trapping sets.

| Radius of trapping set | Algorithm |
|------------------------|-----------|
| \( R_1 \)              | Lemma B.1 |
| \( R_2 \)              | Lemma B.2 |
|                        | Formula (B.10) and Corollary A.9 |
| \( R_3 \)              | Lemma B.9 (two methods) |
|                        | Lemma B.10 (three methods) |
| \( R_4 \)              | Lemma B.15 (two methods) |
|                        | Lemma B.16 (four methods) |
| \( R_5 \)              | Lemma B.19 |
|                        | Lemma B.20 |
|                        | Lemma B.22 |

We always calculate the radii by all these methods (using the rigorous interval arithmetics) and choose the smallest obtained value.
One possible choice is $a = f$, provided $f$ is smooth enough. Then the resulting problem with homogeneous conditions has the form

$$w_t - w_{xx} + 3wu_x + uu_x = f_t + 3fu_x + uf_x,$$

but any function which satisfies the same boundary conditions as $f$ can be used. Another possibility would be to pick $a = f(0,t)(1-x) + f(1,t)x$, which would lead us to the equation

$$w_t - w_{xx} + 3wu_x + uu_x = f_x + f(0,t)(1-x) + f(1,t)x + 3(f(0,t)(1-x) + f(1,t)x) u_x + u(f(1,t) - f(0,t)).$$

The argument of finding the trapping sets and local estimates leading to Theorem 3.2 can be realized for the translated function $w$. Now, the trapping balls for $L^2$ norms of $w$ and $w_x$ will be centered at zero. This leads to the trapping balls centered centered at $-a$ and $-a_x$ for the derivatives $u_{xx}$ and $u_{xxx}$ of the solution of the original problem. Similar procedure can be done for $w_{xx}$ this time translating it by any function which satisfies the same boundary conditions as the right-hand side of the equation for $w$, thus leading to the equation with homogeneous Dirichlet conditions for $w_{xx}$. In the sequel, however, to avoid technicalities, we restrict ourselves to the situation where $f(0,t) = f(1,t) = f_{xx}(0,t) = f_{xx}(1,t) = 0$.

Theorem 3.2 provides the estimates which hold on the trapping set for any time and is based on the estimates which use the norms in $L^\infty$ with respect to time $t \in \mathbb{R}$ norms of the forcing term taken over all time. As the non-autonomous term varies with time, and for some $t$ the norms of $f(t)$ or its space derivatives can be very small, in Appendix B we also derive local estimates, which are given in the following lemma.

**Lemma 3.5.** Let $u \in Y$ be such that

$$\|u\|_{L^2} \leq R_1^i, \quad \|u_x\|_{L^2} \leq R_2^i, \quad \|u_{xx}\|_{L^2} \leq R_3^i, \quad \|u_{xxx}\|_{L^2} \leq R_4^i, \quad \|u_{xxxx}\|_{L^2} \leq R_5^i,$$

with $R_1^i, R_2^i > 0$, and let $t_{i+1} > t_i$. There exist positive numbers $M_1^{i+1}, M_2^{i+1}, M_3^{i+1}, M_4^{i+1}, M_5^{i+1}$, which can be calculated explicitly, such that

$$\|S(t,t_i)u\|_{L^2} \leq M_1^{i+1}, \quad \|(S(t,t_i)u)_x\|_{L^2} \leq M_2^{i+1}, \quad \|(S(t,t_i)u)_{xx}\|_{L^2} \leq M_3^{i+1}, \quad \|(S(t,t_i)u)_{xxx}\|_{L^2} \leq M_4^{i+1}, \quad \|(S(t,t_i)u)_{xxxx}\|_{L^2} \leq M_5^{i+1} \quad \text{for every } t \in [t_i, t_{i+1}],$$

and positive numbers $R_1^{i+1}, R_2^{i+1}, R_3^{i+1}, R_4^{i+1}, R_5^{i+1}$, which can be calculated explicitly, such that

$$\|S(t_{i+1},t_i)u\|_{L^2} \leq R_1^{i+1}, \quad \|(S(t_{i+1},t_i)u)_x\|_{L^2} \leq R_2^{i+1}, \quad \|(S(t_{i+1},t_i)u)_{xx}\|_{L^2} \leq R_3^{i+1}, \quad \|(S(t_{i+1},t_i)u)_{xxx}\|_{L^2} \leq R_4^{i+1}, \quad \|(S(t_{i+1},t_i)u)_{xxxx}\|_{L^2} \leq R_5^{i+1}.$$

We stress that the above lemma is valid if we set as $R_k^i, R_k^{i+1}$ and $M_k^{i+1}$ for $k \in \{1, \ldots, 5\}$ the global radii $R_k^i$. The point of the above lemma is the construction, presented in Appendix B of the optimal local radii $R_k^{i+1}$ and $M_k^{i+1}$, which in practice are often much smaller than the corresponding radii on the trapping set. Such localization of the estimates allows to make the width of the constructed differential inclusion more narrow, which is crucial from the numerical point of view.

**Remark 3.6.** Local bounds $M_k^i$ and $R_k^i$ are derived for $k = 1, \ldots, 5$ basing on energy inequalities which can be derived in several possible ways, each of them leading to the different bound. For every timestep of the simulation we calculate those values using all derived estimates and always choose the smallest obtained bound. All energy estimates used to derived the bounds are summarized, for reader’s convenience in Table 2. We stress that the propagation of a priori estimates could depend only on the global trapping radii $R_k$, but incorporating the local in time estimates allows us to make the width of the inclusion more narrow which is crucial for numerical reasons.
Table 2. Methods to derive the local estimates in the time intervals and on their endpoints. Again for all time steps all estimates are calculated (using the rigorous interval arithmetics) and always the smallest one is chosen.

| $k$ in the local estimate | result used in the computation |
|---------------------------|-------------------------------|
| $k = 1$                   | inequality (B.2)             |
| $k = 2$                   | Lemma B.4                    |
|                           | Lemma B.6                    |
| $k = 3$                   | inequalities (B.27) and (B.28) (two methods) |
|                           | Lemma B.14 (two methods)     |
| $k = 4$                   | Lemma B.17 (two methods)     |
|                           | Lemma B.18 (two methods)     |
| $k = 5$                   | Lemma B.21                   |
|                           | Lemma B.23                   |

4. A rigorous integration algorithm based on the Galerkin projection in FEM space.

4.1. Galerkin projector of first order and its basic properties. We define $V_k$ as the subspace of $H^1_0$ of the functions which are linear on intervals $\left(\frac{i}{k}, \frac{i+1}{k}\right)$ for $i \in \{0, \ldots, k-1\}$.

The dimension of the space $V_k$ is equal to $k-1$, we denote the length of the mesh interval as $h_k = \frac{1}{k}$. We split any $u \in H^1_0$ into $P_k u$ (orthoprojection in $H^1_0$ on $V_k$) and $Q_k u = u - P_k u$. Observe, that $P_k$ coincides with the piecewise linear interpolation operator.

There hold the following inequalities

\[
\|Q_k u\|_{H^1_0} \leq \frac{h_k}{\pi} \|u_{xx}\|_{L^2} \quad \text{for every} \quad u \in H^1_0 \cap H^2, \tag{4.1}
\]

\[
\|Q_k u\|_{L^2} \leq \frac{h_k}{\pi} \|Q_k u\|_{H^1_0} \quad \text{for every} \quad u \in H^1_0, \tag{4.2}
\]

\[
\|P_k u\|_{H^1_0} \leq \frac{\sqrt{12}}{h_k} \|P_k u\|_{L^2} \quad \text{for every} \quad u \in H^1_0. \tag{4.3}
\]

The proof of (4.1) can be found in [34, Theorem 2.5], also c.f., [28]. The estimate (4.2) is a direct consequence of the fact that the function $Q_k u$ vanishes in the nodes of the mesh and the Poincaré inequality given in Lemma A.2, also c.f., [34, Theorem 1.2]. Finally, the proof of (4.3) can be found in [34, Theorem 1.5]. Note, that in fact (4.2) and (4.3) hold, respectively, in subspaces $Q_k H^1_0$ and $P_k H^1_0$, i.e.,

\[
\|u\|_{L^2} \leq \frac{h_k}{\pi} \|u\|_{H^1_0} \quad \text{for every} \quad u \in Q_k H^1_0, \tag{4.4}
\]

\[
\|u\|_{H^1_0} \leq \frac{\sqrt{12}}{h_k} \|u\|_{L^2} \quad \text{for every} \quad u \in P_k H^1_0. \tag{4.5}
\]

With estimates (4.1) and (4.2) we deduce a very simple lemma.

**Lemma 4.1.** Suppose that a function $u : [t_1, t_2] \to H^2 \cap H^1_0$ is such that

\[
\|u_{xx}(t)\|_{L^2} \leq R \quad \text{for every} \quad t \in [t_1, t_2].
\]
Then the following estimates hold

\[ \|Q_k u(t)\|_{\dot{H}^0} \leq \frac{h_k}{\pi} R \quad \text{for every} \quad t \in [t_1, t_2], \]

\[ \|Q_k u(t)\|_{L^2} \leq \frac{h_k^2}{\pi^2} R \quad \text{for every} \quad t \in [t_1, t_2]. \]

4.2. Equation satisfied by \( P_k u \). Let \( u \) be the solution of the problem (1.1)–(1.3) confined in the trapping set \( B_0 \). Splitting \( u(t) = P_k u(t) + Q_k u(t) \) we obtain

\[
(P_k u_t, v) + (Q_k u_t, v) + ((P_k u)_x, v_x) + ((Q_k u)_x, v_x) + (P_k u(P_k u)_x, v) + (Q_k uu_x, v)
+ (P_k u(Q_k u)_x, v) = (f(t), v) \quad \text{for every} \quad v \in V_k.
\]

Noting that \( ((Q_k u)_x, v_x) = 0 \), we can rewrite this equation as

\[
(P_k u_t, v) + ((P_k u)_x, v_x) + (P_k u(P_k u)_x, v) = (f(t), v) - \left[ (Q_k u_t, v) + (Q_k uu_x, v) + (P_k u(Q_k u)_x, v) \right] \quad \text{for every} \quad v \in V_k,
\]

The equation

\[
(P_k u_t, v) + ((P_k u)_x, v_x) + (P_k u(P_k u)_x, v) = (f(t), v)
\]

correspond to the Galerkin scheme for the considered problem, while the expression

\[
\left[ (Q_k u_t, v) + (Q_k uu_x, v) + (P_k u(Q_k u)_x, v) \right]
\]

is the residual error which we need to estimate. Denote by \( \{v^i\}_{i=1}^{k-1} \) the basis functions of \( V_k \) defined by the relation \( v^i(jh_k) = \delta_{ij} \). Then representing \( P_k u(t) \) as \( P_k u(t) = \sum_{i=1}^N \alpha_i(t) v^i(x) \), we will formulate a differential inclusion satisfied by the time dependent coefficients \( \alpha_i \). The system takes the form

\[
\sum_{i=1}^N \alpha_i'(t)(v^i, v) + \sum_{i=1}^N \alpha_i(t)(v^i_x, v_x) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i(t) \alpha_j(t)(v^i v^j_x, v) = (f(t), v)
- \left[ (Q_k u_t, v) + (Q_k uu_x, v) + (P_k u(Q_k u)_x, v) \right] \quad \text{for every} \quad v \in V_k.
\]

Let us calculate the \( m \)-th equation of the system \((m \in \{1, \ldots, k-1\})\) obtained by taking as the test function \( v \) the \( m \)-th element of the basis

\[
\sum_{i=1}^N \alpha_i'(t)(v^i, v^m) + \sum_{i=1}^N \alpha_i(t)(v^i_x, v^m_x) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i(t) \alpha_j(t)(v^i v^j_x, v^m_x) = (f(t), v^m)
- \left[ (Q_k u_t, v^m) + (Q_k uu_x, v^m) + (P_k u(Q_k u)_x, v^m) \right].
\]

We only provide the details of further calculations for \( m \neq 1 \) and \( m \neq k-1 \). For a given \( m \) there will be three \( i \)-s for which the contribution to the sums will be nonzero. They associated formula is the following

\[
\alpha_m'(t)(v^m), v^m) + \alpha_{m-1}'(t)(v^{m-1}, v^m) + \alpha_{m+1}'(t)(v^{m+1}, v^m)
+ \alpha_m(t)(v^m_x, v^m_x) + \alpha_{m-1}(t)(v^{m-1}_x, v^m_x) + \alpha_{m+1}(t)(v^{m+1}_x, v^m_x)
+ \sum_{j=1}^N \alpha_m(t) \alpha_j(t)(v^m v^j_x, v^m_x) + \sum_{j=1}^N \alpha_{m-1}(t) \alpha_j(t)(v^{m-1} v^j_x, v^m_x) + \sum_{j=1}^N \alpha_{m+1}(t) \alpha_j(t)(v^{m+1} v^j_x, v^m_x)
= (f(t), v^m) - \left[ (Q_k u_t, v^m) + (Q_k uu_x, v^m) + (P_k u(Q_k u)_x, v^m) \right].
\]
We further elaborate three remaining sums by keeping the nonzero terms
\[
\alpha'_m(t)(v^m, v^m) + \alpha'_{m-1}(t)(v^{m-1}, v^m) + \alpha'_{m+1}(t)(v^{m+1}, v^m) \\
+ \alpha_m(t)(v^m, v^m) + \alpha_{m-1}(t)(v^{m-1}, v^m) + \alpha_{m+1}(t)(v^{m+1}, v^m) \\
+ \alpha^2_m(t)(v^m, v^m) + \alpha_m(t)\alpha_{m-1}(t)(v^{m-1}, v^m) + \alpha_m(t)\alpha_{m+1}(t)(v^{m+1}, v^m) \\
+ \alpha_{m-1}(t)\alpha_m(t)(v^{m-1}v^m, v^m) + \alpha_{m-1}(t)(v^{m-1}v^m, v^m) \\
+ \alpha_{m+1}(t)\alpha_m(t)(v^{m+1}v^m, v^m) + \alpha_{m+1}(t)(v^{m+1}v^m, v^m) \\
= (f(t), v^m) - \left[(Q_k u_t, v^m) + ((Q_k u) u_x, v^m) + (P_k u(Q_k u)_x, v^m)\right].
\]

It is clear that \((v^m v^m, v^m) = 0\). Hence, calculating all integrals in the above formula, we get
\[
\alpha'_m(t) \frac{2h_k}{3} + \alpha'_{m-1}(t) \frac{h_k}{6} + \alpha'_{m+1}(t) \frac{h_k}{6} + \alpha_m(t) \frac{2}{h_k} - \alpha_{m-1}(t) \frac{1}{h_k} - \alpha_{m+1}(t) \frac{1}{h_k} \\
- \alpha_m(t)\alpha_{m-1}(t) \frac{1}{6} + \alpha_m(t)\alpha_{m+1}(t) \frac{1}{6} - \alpha_{m-1}(t) \frac{1}{6} + \alpha_{m+1}(t) \frac{1}{6} \\
= (f(t), v^m) - \left[(Q_k u_t, v^m) + (Q_k uu_x, v^m) + (P_k u(Q_k u)_x, v^m)\right].
\]

We can integrate by parts
\[
(P_k u(Q_k u)_x, v^m) = -((P_k u)u Q_k u, v^m) - (P_k u Q_k u, v^m),
\]
whence the system takes the form
\[
\alpha'_m(t) \frac{2h_k}{3} + \alpha'_{m-1}(t) \frac{h_k}{6} + \alpha'_{m+1}(t) \frac{h_k}{6} + \alpha_m(t) \frac{2}{h_k} - \alpha_{m-1}(t) \frac{1}{h_k} - \alpha_{m+1}(t) \frac{1}{h_k} \\
- \alpha_m(t)\alpha_{m-1}(t) \frac{1}{6} + \alpha_m(t)\alpha_{m+1}(t) \frac{1}{6} - \alpha_{m-1}(t) \frac{1}{6} + \alpha_{m+1}(t) \frac{1}{6} \\
= (f(t), v^m) - \left[(Q_k u_t, v^m) + (Q_k uu_x, v^m) + (P_k u(Q_k u)_x, v^m)\right].
\]

We multiply this equation by \(\frac{6}{h_k}\) to get
\[
4\alpha'_m(t) + \alpha'_{m-1}(t) + \alpha'_{m+1}(t) + \frac{12}{h_k} \alpha_m(t) - \frac{6}{h_k^2} \alpha_{m-1}(t) - \frac{6}{h_k^2} \alpha_{m+1}(t) \\
- \frac{1}{h_k} \alpha_m(t)\alpha_{m-1}(t) + \frac{1}{h_k} \alpha_m(t)\alpha_{m+1}(t) - \frac{1}{h_k} \alpha_{m-1}(t) + \frac{1}{h_k} \alpha_{m+1}(t) \\
= \frac{6}{h_k} (f(t), v^m) - \frac{6}{h_k} \left[(Q_k u_t, v^m) - (Q_k uu_x, v^m) + (P_k u(Q_k u)_x, v^m)\right],
\]
for \(m \in \{2, \ldots, k-2\}\). Together with the equations for \(m = 1\) and \(m = k-1\) (not given here, they are derived analogously) the system can be rewritten in the matrix form as
\[
M\alpha' = S\alpha + N(\alpha) + F(t) + \frac{6}{h_k}((Q_k u Q_k u_x - Q_k u_t, v^m) - (P_k u(Q_k u)_x, v^m))_{m=1}^{k-1},
\]
where \(F(t) = (F_m(t))_{m=1}^k\) is given by \(F_m(t) = \frac{6}{h_k}(v^m, f(t))\), \(M\) is the mass matrix, \(S\) is the stiffness matrix, and \(N\) is the expression coming from the quadratic terms. We multiply this equation by \(M^{-1}\). Then
\[
\alpha' = M^{-1} S\alpha + M^{-1} N(\alpha) + M^{-1} F(t) + M^{-1} \frac{6}{h_k} ((Q_k u Q_k u_x - Q_k u_t, v^m) - (P_k u (Q_k u)_x, v^m))_{m=1}^{k-1}.
\]
Now let $B$ be a nonsingular quadratic matrix and introduce the new variables $\beta$ given by $\alpha = B\beta$. Then $B$ is the change of basis matrix. In new variables the equation takes the form
\[
\beta'(t) = B^{-1}M^{-1}SB\beta(t) + B^{-1}M^{-1}N(B\beta(t)) + B^{-1}M^{-1}F(t)
\]
\[
+ B^{-1}M^{-1}\frac{6}{h_k}((Q_k u(t)Q_k u_x(t) - Q_k u(t), v^m) - (P_k u(t)Q_k u(t), v^m_m))_{m=1}^{k-1}.
\]

The matrix $B$ is found in such a way that the matrix $B^{-1}M^{-1}SB$ is close to diagonal. Now if we denote the coefficients of the matrix $B^{-1}M^{-1}$ by $c_{lm}$ for $l, m \in \{1, \ldots, k-1\}$, and we denote
\[
(4.9) \quad w^l(x) = \sum_{m=1}^{k-1} c_{lm} v^m(x),
\]
then the above equation is equivalent to
\[
\beta'(t) = B^{-1}M^{-1}SB\beta(t) + B^{-1}M^{-1}N(B\beta(t)) + B^{-1}M^{-1}F(t)
\]
\[
+ \frac{6}{h_k}((Q_k u(t)Q_k u_x(t) - Q_k u(t), w^l) - (P_k u(t)Q_k u(t), w^m_x))_{l=1}^{k-1}.
\]

We will solve the differential inclusion
\[
(4.10) \quad \beta'(t) \in B^{-1}M^{-1}SB\beta(t) + B^{-1}M^{-1}N(B\beta(t)) + B^{-1}M^{-1}F(t) + G(t),
\]
where $G : \mathbb{R} \to 2^{k^{-1}}$ with $G(t) = (G_l(t))_{l=1}^{k-1}$ is a multifunction such that
\[
(4.11) \quad \frac{6}{h_k}((Q_k u(t)Q_k u_x(t) - Q_k u(t), w^l) - (P_k u(t)Q_k u(t), w^m_x)) \in G_l(t) \quad \text{for} \quad l = 1, \ldots, k-1.
\]

### 4.3. A numerical algorithm.
We divide the interval $[t, t + T]$ into subintervals
\[
t = t_0 < t_1 < \ldots < t_n = t + T.
\]

In our algorithm we construct a sequence of sets $D_i$
\[
D_i = \{v \in H^1 \cap H^1_0 \mid u_{xx} \in H^1_0, \|v\|_{L^2} \leq R_1, \|v_x\|_{L^2} \leq R_2, \|v_{xx}\|_{L^2} \leq R_3, \|v_{xxx}\|_{L^2} \leq R_4, \|v_{xxxx}\|_{L^2} \leq R_5, P_kv \in P^i_k\},
\]
\[
P^i_k \subset V_k \quad \text{is nonempty, closed, convex and bounded}
\]
such that $u(t_i) \in D_i$ for $i = 0, 1, \ldots, n$. In each step of the algorithm we propagate $D_i$ forward in time through the process $S(t_{i+1}, t_i)$ governing the strong solutions given by Definition 2.4.

Before we present the algorithm, we prove the lemma, which allows us to refine the estimates on the $L^2$ norms of the function and its first derivative.

**Lemmas 4.2.** Let $N_m > 0$ for $m \in \{1, \ldots, 5\}$ and let $S \subset V_k$. Let the set $D$ be given by
\[
D = \{v \in H^1 \cap H^1_0 \mid u_{xx} \in H^1_0, \|v\|_{L^2} \leq N_1, \|v_x\|_{L^2} \leq N_2, \|v_{xx}\|_{L^2} \leq N_3, \|v_{xxx}\|_{L^2} \leq N_4, \|v_{xxxx}\|_{L^2} \leq N_5, P_kv \in S\}.
\]

Define
\[
(4.12) \quad N_1 = \min \left\{ N_1, \frac{N_3 h_k^2}{\pi^2} + \sup_{v \in S} \|v\|_{L^2} \right\},
\]
\[
(4.13) \quad N_2 = \min \left\{ N_2, \frac{N_3 h_k^2}{\pi} + \sup_{v \in S} \|v_x\|_{L^2} \right\}.
\]
Then
\[ D = \{ v \in H^4 \cap H^1_0 : u_{xx} \in H^1_0, \quad \|v\|_{L^2} \leq N_1, \|v_x\|_{L^2} \leq N_2, \|v_{xx}\|_{L^2} \leq N_3, \|v_{xxx}\|_{L^2} \leq N_4, \|v_{xxxx}\|_{L^2} \leq N_5, P_kv \in S \}. \]

**Proof.** Denote
\[ D_1 = \{ v \in H^4 \cap H^1_0 : u_{xx} \in H^1_0, \quad \|v\|_{L^2} \leq N_1, \|v_x\|_{L^2} \leq N_2, \|v_{xx}\|_{L^2} \leq N_3, \|v_{xxx}\|_{L^2} \leq N_4, \|v_{xxxx}\|_{L^2} \leq N_5, P_kv \in S \}. \]

It is clear that \( D_1 \subset D \). Let \( w \in D \). Then from (4.4) and (4.5) we deduce that
\[
\|Q_kw\|_{L^2} \leq \frac{h_k^2}{\pi} N_3, \quad \|(Q_kw)_x\|_{L^2} \leq \frac{h_k}{\pi} N_3,
\]

hence
\[
\|w\|_{L^2} \leq \|P_kw\|_{L^2} + \|Q_kw\|_{L^2} \leq \sup_{v \in S} \|v\|_{L^2} + \frac{h_k^2}{\pi} N_3,
\]

\[
\|w_x\|_{L^2} \leq \|(P_kw)_x\|_{L^2} + \|(Q_kw)_x\|_{L^2} \leq \sup_{v \in S} \|v_x\|_{L^2} + \frac{h_k}{\pi} N_3,
\]

which proves that \( w \in D_1 \) and the proof is complete. \( \square \)

We pass to the presentation of our algorithm.

(i) **Initialization.** Choose \( D_0 \), so that it contains our initial condition. Refine the radii \( R^{0}_0 \) and \( R^{0}_2 \) using Lemma 4.2.

(ii) **For \( i = 0 \ldots , n - 1 \) repeat steps (iii)–(v).**

(iii) **Computation of local integral bounds on interval \([t_i, t_{i+1}]\).** From the local a priori estimates of Lemma 3.5 find constants \( M^{i+1}_k \) such that for every solution \( u \) defined on interval \([t_i, t_{i+1}]\) with the initial data \( u(t_i) = u_i \) such that \( u_i \in D_i \) there holds
\[
\|u(t)\|_{L^2} \leq M^{i+1}_1, \quad \|u_x(t)\|_{L^2} \leq M^{i+1}_2, \quad \|u_{xx}(t)\|_{L^2} \leq M^{i+1}_3, \quad \|u_{xxx}(t)\|_{L^2} \leq M^{i+1}_4,
\]

\[
\|u_{xxxx}(t)\|_{L^2} \leq M^{i+1}_5,
\]

\[
\|Q_ku(t)\|_{H^0_k} \leq \frac{M^{i+1}_3 h_k}{\pi}, \quad \|Q_ku(t)\|_{L^2} \leq \frac{M^{i+1}_3 h_k^2}{\pi^2} \quad \text{for} \quad t \in [t^i, t^{i+1}].
\]

Use these estimates to find the multifunction \( G \) on the interval \([t_i, t_{i+1}]\). The details of the calculation of \( G \) is given in Subsection 4.4.

(iv) **Solving differential inclusion.** Solve rigorously numerically the inclusion (4.10) using the initial data in \( P^i_k \) and the multivalued term, \( G(t) \), calculated in step (iii). The solution of the inclusion gives the set \( P^{i+1}_k \subset V_k \) which contains the projections on \( V_k \) of values at \( t_{i+1} \) of all trajectories such that \( u(t_i) \in D_i \).

(v) **Calculation of bounds at \( t_{i+1} \).** From the local a priori estimates of Lemma 3.5 find constants \( R^{i+1}_1, R^{i+1}_2, R^{i+1}_3, R^{i+1}_4 \), and \( R^{i+1}_5 \) such that for every solution with \( u(t_i) \in D_i \) there holds
\[
\|u(t_{i+1})\|_{L^2} \leq R^{i+1}_1, \quad \|u_x(t_{i+1})\|_{L^2} \leq R^{i+1}_2, \quad \|u_{xx}(t_{i+1})\|_{L^2} \leq R^{i+1}_3,
\]

\[
\|u_{xxx}(t_{i+1})\|_{L^2} \leq R^{i+1}_4, \quad \|u_{xxxx}(t_{i+1})\|_{L^2} \leq R^{i+1}_5.
\]

Refine the radii \( R^{i+1}_1 \) and \( R^{i+1}_2 \) using Lemma 4.2. Obtained radii \( R^{i+1}_1 \)–\( R^{i+1}_5 \) together with the set \( P^{i+1}_k \) from step (iv) define the set \( D_{i+1} \).
In our proofs of periodic orbit existence we initialize \( D_0 \) by taking the global radii from Theorem 3.2, i.e. \( R_{i_0}^0 = R_{m}^0 \) for \( m \in \{1, \ldots, 5\} \) and for \( P_k \) we take some neighborhood of the numerically found periodic point. This choice guarantees us that for every \( t > t_0 \) the bounds
\[
\|w\|_{L^2} \leq R_1, \|w_x\|_{L^2} \leq R_2, \|w_{xx}\|_{L^2} \leq R_3, \|w_{xxx}\| \leq R_4, \|w_{xxxx}\|_{L^2} \leq R_5.
\]
In order to prove the periodic orbit existence we need to verify that \( S(t_n, t_0)D_0 \subset D_n \). As the algorithm is constructed in such a way, that \( S(t_n, t_0)D_0 \subset D_n \), it follows that \( P_k S(t_n, t_0)D_0 \subset P_k^0 \) and it suffices only to verify that \( P_k^0 \subset P_k^0 \).

An alternative approach to obtain the periodic orbit would be to choose any initial radii \( R_{i_0}^0 \) for \( m \in \{1, \ldots, 5\} \), unrelated with the global radii from Theorem 3.2. Then, after the algorithm stops, we would need to verify that \( D_n \subset D_0 \), i.e. both that \( P_k^0 \subset P_k^0 \) and that \( R_{i_0}^0 \subset P_k^0 \) for \( m \in \{1, \ldots, 5\} \).

In principle in the case of the Burgers equation if our set of initial conditions is contained in the trapping set from Theorem 3.2 we could have skipped the stage (iii) and use global bounds throughout the entire simulation, i.e. we will have \( R_i^0 = M_i^0 = R_j^0 \) for \( j = 1, \ldots, 5 \) and all \( i \). In such situation the multivalued term \( G \) could be the same for all time steps. This however will be very inefficient and will require very fine mesh \((k \text{ large})\) to obtain good rigorous bounds. In practice, in all examples which we have run the use of local estimates leads to very significant gain.

A reader might wonder whether the algorithm can be generalized to any PDE of the form
\[
u_t = \Delta u + N(u, Du) + f(t),
\]
equipped with some boundary condition.

Let us briefly discuss the problems one can face
- in stage (iii) it might be impossible to obtain a local integral bounds for solution for a given time step, for example the solution might blow up,
- in stage (iv) some of the solutions of the differential inclusion also could blow up for a given time step.

Both stages involve some heuristics - we need to obtain the a priori bounds, which could depend on the particular form of the nonlinearity and the boundary conditions.

### 4.4. Construction of \( G(t) \) from local a priori bounds.

Following (4.11) we need to construct \( G : \mathbb{R} \to 2^{\mathbb{R}^{k-1}} \) given by \( G(t) = (G_l(t))_{l=1}^{k-1} \) such that
\[
\frac{6}{h_k} \left( (Q_k u(t))Q_k u_x(t) - Q_k u_x(t), w^l \right) - (P_k u(t)Q_k u(t), w^l_x) \right) \in G_l(t) \text{ for } l = 1, \ldots, k - 1.
\]
The multifunction \( G_l(t) = (G_l(t))_{l=1}^{k-1} \) will be constant on every interval \([t_i, t_{i+1})\) and
\[
G_l(t) = [-\epsilon_{i+1,l}, \epsilon_{i+1,l}] \text{ for } t \in [t_i, t_{i+1}).
\]
To find the concrete numerical values \( \epsilon_{i+1,l} \) using the bounds (4.19) and Lemma A.3 we estimate
\[
\|(Q_k u P_k u, w^l_x)\| \leq \|Q_k u\|_{L^\infty} \|w^l_x\|_{L^2} \|P_k u\|_{L^\infty} \leq \frac{M_2^{i+1} h_k^2}{2} \|Q_k u\|_{L^2} \|w^l_x\|_{L^2},
\]
\[
\|Q_k u P_k u\|_{L^2} \leq \frac{M_3^{i+1} h_k^2}{2} \|w^l_x\|_{L^2} \leq \frac{M_3^{i+1} h_k^2}{2} \|w^l_x\|_{L^2}.
\]
Now
\[\|Q_k u(Q_k u)_{x}, w^j\| \leq \|Q_k u\|_{L^\infty} \|Q_k u\|_{L^2} \|w^j\|_{L^2} \leq \frac{(M_3^{i+1})^2 h_k^{5/2}}{\pi^{5/2}} \|w^j\|_{L^2}.\]

Moreover
\[
\|Q_k u, w^j\| = \|Q_k u_{xx}, w^j\| + \|Q_k(u u_x), w^j\| + \|Q_k w, w^j\| \\
\leq (\|Q_k u_{xx}\|_{L^2} + \|Q_k u_{xx}\|_{L^2}) \|w^j\|_{L^2} + \sup_{t \in [t_i, t_{i+1}]} \|Q_k f(t), w^j\| \\
\leq \frac{h_k^2}{\pi^2} (\|u_{xx}\|_{L^2} + \|u_{xxx}\|_{L^2}) \|w^j\|_{L^2} + \sup_{t \in [t_i, t_{i+1}]} \|Q_k f(t), w^j\|. 
\]

On the other hand
\[
\|u_{xxx}\|_{L^2} + \|u_{xxw}\|_{L^2} \\
\leq \|u_{xxx}\|_{L^2} + 3\|u_x\|_{L^\infty} \|u_{xx}\|_{L^2} + \|u\|_{L^\infty} \|u_{xxx}\|_{L^2} \\
\leq \|u_{xxx}\|_{L^2} + 3\sqrt{2} \|u_x\|_{L^2} \|u_{xx}\|_{L^2} \|w^j\|_{L^2} + \|u\|_{L^2} \|u_{x}\|_{L^2} \|u_{xx}\|_{L^2}. 
\]

So on interval \([t_i, t_{i+1}]\) there holds
\[
(\|Q_k u, w^j\|) \\
\leq \frac{h_k^2}{\pi^2} \|w^j\|_{L^2} \left( M_5^{i+1} + 3\sqrt{2} (M_2^{i+1})^{1/2} (M_3^{i+1})^{3/2} + (M_1^{i+1})^{1/2} (M_2^{i+1})^{1/2} M_4^{i+1} \right) \\
+ \sup_{t \in [t_i, t_{i+1}]} \|Q_k f(t), w^j\| \\
:= \frac{h_k^2}{\pi^2} C_{i+1} \|w^j\|_{L^2} + \sup_{t \in [t_i, t_{i+1}]} \|Q_k f(t), w^j\|. 
\]

Summarizing the above three estimates, there holds
\[
(4.21) \quad \epsilon_{i,t} = \frac{6h_k}{\pi^2} \left( \frac{M_5^{i+1} M_3^{i+1}}{2} \|w^j\|_{L^2} + \frac{(M_3^{i+1})^2 h_k^{1/2}}{\pi^{1/2}} \|w^j\|_{L^2} + C_{i+1} \|w^j\|_{L^2} \right) + \frac{6}{h_k} \sup_{t \in [t_i, t_{i+1}]} \|Q_k f(t), w^j\|. 
\]

These values can be calculated effectively in step (iii) of the algorithm leading to the width of the differential inclusion in every time step.

The desired feature of \(\epsilon_{i,t}\) should be that it decreases to zero with the decrease of \(h_k\). The change of basis matrix \(B\) used in the derivation of (4.10) is not uniquely defined, hence neither are vectors \(w^j\). In the numerical examples of Section 6 we have chosen \(B = (b_{ij})_{i,j=1}^{k-1}\) so that its columns are normalized as follows
\[
(4.22) \quad \sum_{i=1}^{k-1} b_{ij}^2 = 1, 
\]

i.e. the euclidean norm of each column of \(B\) is normalized to one.

We perform a brief analysis of the behavior of the width \(\epsilon_{i,t}\) of the inclusion as \(h_k\) decreases. As the functions \(w_t\) are approximately equal to the eigenfunctions of the Laplace operator, after reordering of the
basis $w^j$ so that $|\lambda_l|$ is increasing, we obtain

$$\lambda_l \approx -\pi^2 l^2, \quad w_l(x) = \sum_{m=1}^{k-1} c_{lm} v^m(x) \approx \pm \alpha(k, l) \sin(l \pi x).$$

The normalization choice (4.22) makes the unknowns $(\beta_l)_{l=1}^{k-1}$ scale with the increase of $k$. In order to avoid this, we perform the analysis for such choice of $B$ that the $L^2$ norm of the basis vectors $w_l$ are constant. Then, approximately,

$$\|w^j\|_{L^2} \approx 1, \quad \|w^j_t\|_{L^2} \approx l \pi.$$
5.1. Solving the differential inclusion. We use the method of [18, 42] to rigorously solve the inclusion
\[(5.1) \quad \beta'(t) \in B^{-1} M^{-1} SB \beta(t) + B^{-1} M^{-1} N(B \beta(t)) + B^{-1} M^{-1} F(t) + G(t).\]
We solve this inclusion on the time interval \([t_i, t_{i+1}]\), and we equip it with the initial data \(\beta(t_i) \in P_k^t\). By the rigorous solution we understand finding the set \(P_{k+1}^t\) such that for every absolutely continuous function \(\beta : [t_i, t_{i+1}] \to \mathbb{R}^k\) satisfying (5.1) for almost every \(t \in (t_i, t_{i+1})\) with \(\beta(t_0) \in P_k^t\) there holds \(\beta(t_{i+1}) \in P_{k+1}^t\). Note that on \([t_i, t_{i+1}]\) the set \(G(t)\) is independent of time and equal to \(G(t) = (G_i(t))_{k=1}^{t_{i+1}}\) with
\[G_i(t) = [-\epsilon_{i,j}, \epsilon_{i,j}]\]
where \(\epsilon_{i,j}\) is given by (4.21) and can be effectively calculated.

The set \(P_{k+1}^t\) is found using the algorithm of \([18, Lemma 5.2]\) which is a part of rigorous numerics CAPD library [3]. We briefly recall the algorithm here. For simplicity we rewrite (5.1) as
\[\beta'(t) = f(\beta(t)) + h(t) + y(t),\]
where \(y(t) \in G(t)\) for a.e. \(t \in (t_i, t_{i+1})\). As the multifunction \(G(t)\) is always centered at zero, in the first step, an equation
\[(5.2) \quad \beta'(t) = f(\beta(t)) + h(t),\]
is solved with the initial data \(\beta(t_i) \in P_k^t\). The rigorous numerical solution of this ODE uses the explicit Taylor scheme [3] (in all examples we use the fourth order scheme) and the Lohner algorithm [40] for representation of sets \(P_k^t\) as parallelepipeds and their propagation in time. For a parallelepiped \(P_k^t\), another parallelepiped is found which is guaranteed to contain the values at \(t_{i+1}\) of all solutions of the above ODE with the initial data taken at \(t_i\) in the set \(P_k^t\). Next, a correction is calculated and added to resultant set to guarantee to contain all solutions of the inclusion. This correction [18, Sec. 6.3] is equal to \((-d_j, d_j)_{j=1}^{k-1}\), where
\[d_j = \left| \int_{t_i}^{t_{i+1}} e^J(t_i-s)C \, ds \right|,\]
where the vector \(C\) is given by
\[C = (\epsilon_{i,j})_{l=1}^{k-1}\]
and the matrix \(J\) is given by
\[J_{ij} = \begin{cases} \sup_{\beta \in [W]} |\frac{\partial f_i(\beta)}{\partial \beta_j}| & \text{if } i = j, \\ \sup_{\beta \in [W]} |\frac{\partial f_i(\beta)}{\partial \beta_j}| & \text{otherwise,} \end{cases}\]
where the set \([W] \subset \mathbb{R}^k\) is the so called enclosure, i.e. the set which is guaranteed to contain the values at all \(t \in [t_i, t_{i+1}]\) of all solutions of (5.1) with the initial data \(\beta(t_i) \in P_k^t\). The enclosure is found using the first order rough enclosure algorithm [41, 42].

5.2. Method of dissipative modes. We rewrite the differential inclusion (5.1) as follows
\[(5.3) \quad \beta'(t) \in A \beta(t) + g(\beta(t)) + h(t) + G(t).\]
The linear term \(A = B^{-1} M^{-1} SB\) is given by the interval matrix which is close to the diagonal one, and its diagonal entries, sorted by the increasing moduli of the eigenvalues, are approximately given by \(|A_{ii}| \approx -\pi^2 l^2\) for \(l \in \{1, \ldots, k-1\}\). The algorithm of dissipative modes designed and described in [42] allows us to use to our advantage the fact that these diagonal entries are large and negative for large \(l\). The method allows us to rigorously solve the \(k-1\) dimensional differential inclusion by splitting all unknowns \(\beta = (\beta_i)_{i=1}^{k-1}\) into two groups \(\beta = (\beta_1, \beta_2)\). The variables \(\beta_1\) correspond to those entries \([A_{ii}]\) which have the moduli smaller than some arbitrary cut-off value (in numerical examples we took the 8 first coordinates of \(\beta\) as \(\beta_1\)) while the remaining unknowns are assigned to \(\beta_2\). The rigorous numerical integration which
uses the Taylor scheme and Lohner algorithm is performed only for variables in \( \beta_1 \) while the dissipative modes algorithm allows us to treat the variables belonging to \( \beta_2 \). Each of the inclusions for these variables can be written as

\[
\beta_{2,l}(t) + ||A_U||\beta_{2,l}(t) = f_l(\beta_1(t), \beta_2(t)) + ||A_U||\beta_{2,l}(t) + h_l(t) + G_l(t).
\]

In the first step, an enclosure is found, that is the set \([W] \subset \mathbb{R}^k\) such that \( \beta(t) \in [W] \) for \( t \in [t_i, t_{i+1}] \). In other words all solutions of the inclusion (5.3) are guaranteed to belong to this set for \( t \in [t_i, t_{i+1}] \) if the initial data satisfies \( \beta(t_i) \in P_k \). We note that in the algorithm of finding this enclosure the fact that \( ||A_U|| \) in equation (5.4) plays a crucial role, see \[42, Section 5\] for the details. Once the enclosure is found, it is possible to find the numbers \( N^{-}_{-} \) and \( N^{+}_{-} \) which are bounds from above and from below on the sets \( f_l(\beta_1(t), \beta_2(t)) + |A|\beta_{2,l}(t) + h_l(t) + G_l(t) \) for \( t \in [t_i, t_{i+1}] \), i.e. for every solution \( \beta \) with the initial data \( \beta(t_i) \in P_k \), every \( t \in [t_i, t_{i+1}] \), and every \( y \in G_l(t) \) there holds

\[
N^{-}_{-} \leq f_l(\beta_1(t), \beta_2(t)) + ||A_U||\beta_{2,l}(t) + h_l(t) + y \leq N^{+}_{-}.
\]

Thus all solutions \( \beta_{2,l}(t) \) satisfy the differential inequalities

\[
N^{-}_{-} \leq \beta_{2,l}(t) + ||A_U||\beta_{2,l}(t) \leq N^{+}_{-}.
\]

A simple computation leads us to

\[
(\beta_{2,l}(t_i^-)e^{-||A_U||(t_{i+1} - t_i)}) + \frac{N^{-}_{-}}{||A_U||}(1 - e^{-||A_U||(t_{i+1} - t_i)}) \leq \beta_{2,l}(t_{i+1}) \leq (\beta_{2,l}(t_i^+)e^{-||A_U||(t_{i+1} - t_i)}) + \frac{N^{+}_{-}}{||A_U||}(1 - e^{-||A_U||(t_{i+1} - t_i)}),
\]

whence we obtain the interval which is guaranteed to contain \( \beta_{2,l}(t_{i+1}) \), cf. \[42, Theorem 23\]. Independently, the found enclosure on variables \( \beta_1 \) and \( \beta_2 \) is used to construct the low dimensional inclusion for \( \beta_1 \) which is solved by the rigorous integration algorithm described in Section 5.1.

Thanks to the use of the dissipative modes approach we have the following advantages.

- Since the rigorous integration is performed only for fixed small number of variables in \( \beta_1 \), the contribution from the multivalued term \( G_l(t) \) for these variables, given by \( IO(h_k) \), is effectively equal to \( O(h_k) \) as \( l \) is small.
- In the variables belonging to \( \beta_2 \) the contribution from \( G_l(t) \) could be equal even to \( O(1) \) for \( l \) close to the maximal value \( k - 1 \). This large width of \( G_l(t) \), incorporated into \( N^{+}_{-} \) and \( N^{-}_{-} \) is counteracted by the division by \( ||A_U|| \) in (5.5), making the contribution of the multivalued term effectively equal to \( O(h_k)/l \) in the computation of \( \beta_{2,l}(t_{i+1}) \).
- The high computational cost of the Taylor integration algorithm and Lohner algorithm is significantly reduced, as the algorithm needs to be run only for low-dimensional problems.
- As the Taylor scheme is explicit, the admissible time step length required for its stability is limited by the same value. This value is, approximately, the increasing function of maximal of \( 1/||A_U|| \) for \( l \) assigned to \( \beta_1 \). Hence, as only those \( l \) for which \( ||A_U|| \) is small are assigned to \( \beta_1 \), we can perform the simulation with larger time steps, leading to the significant reduction of the computation time.

The above observations are backed by our numerical examples, the implementation of the approach by the dissipative modes turned out to be absolutely crucial factor to prevent the blow-up of the sets \( P_k \) obtained during the course of the rigorous numerical simulation.
6. Computer assisted verification of the periodic solution existence

In this section we present the theorem on the existence of the periodic orbit for two particular choices of the forcing term. In both examples we rigorously integrate forward in time the inclusion (4.10) for the 1-periodic in time forcing and for the initial data belonging to some set $\mathcal{D}_0$ given by Step (i) of the algorithm described in Section 4.3. After time 1, the period of $f$, all solutions of the inclusion are verified to belong to the set which is a subset of $\mathcal{D}_0$, guaranteeing that all assumptions of the abstract Schauder type fixed point theorem are satisfied.

6.1. Schauder-type theorem. In the result of this subsection we establish the existence of periodic orbit using the following Schauder type theorem with $X = H_0^1$.

**Theorem 6.1.** Let $X$ be a Banach space and let $\mathcal{B} \subset X$ be a nonempty, compact, and convex set. If the mapping $S : \mathcal{B} \to \mathcal{B}$ is continuous, then it has a fixed point $u_0 \in \mathcal{B}$.

Once the time-periodic solution exists, Theorem 2.10 establishes that it exponentially attracts all weak solutions given by Definition 2.1. In fact Theorem 2.10 already establishes the existence of periodic attract-
This function satisfies \( u(t) \in B_0 \), it is \( T \)-periodic, and it is a strong solution of the Burgers equation, in concordance with Definition 2.4. Moreover it must hold that \( P_{f_k} u(t_i) \in P^i_k \), and, by Lemma 4.1
\[
\| Q_k u(t_i) \|_{L^2} \leq \frac{h_k^2}{\pi^2} R_3^i, \| Q_k u(t_i) \|_{H^1_0} \leq \frac{h_k}{\pi} R_3^i \text{ for every } i = 0, \ldots, n.
\]
The proof is complete. \( \square \)

We remark that the set \( P^0_k \) must be found experimentally, based on the numerical simulations. In practice it is found by classical (nonrigorous) FEM simulations and choosing some ball enclosing the found nonrigorous solution at time \( t_0 \). In two examples given in the following subsection we use the condition (ii) of the above theorem, which makes it necessary to numerically verify only the inclusion \( P^i_k \subset P^0_k \).

6.2. Examples. Our numerical experiments show that the key factor which decides if the sets \( P^i_k \) shrink or expand in time, is the width of the multivalued term \( G(t) \).

Our aim was to perform the computer-assisted proofs in the situation when \( G(t) \) is large and hence we chose two particular forms of the forcing term
\[
f(x, t) = 8(\sin(3\pi x) + \sin(4\pi x))(1 + \sin(2\pi t)), \quad f(x, t) = 12 \sin(\pi x) \sin(2\pi t).
\]
Note that the width of \( G(t) \) is determined by the values \( \epsilon_{i,1} \) given by (4.21) which are monotone increasing functions of the constants \( M_{k+1}^i \), for \( k = 1, \ldots, 5 \), given by Lemma 3.5. These constants are in turn determined by the estimates of Appendix B, given in Lemmas B.1, B.4, B.6, B.13, B.14, B.17, B.18, B.21, and B.23. The estimates of these Lemmas depend monotonically on the \( L^2 \) norms of \( f(t), f_x(t), f_{xx}(t), \) and \( f_{xxx}(t) \). We chose for our construction the functions \( f \) with amplitude 8 and 12, and, in the second example, frequencies 3 and 4, to show that we can cope with the situation where these norms are relatively large.

6.2.1. Example 1. We have validated the precise form of the periodic trajectory for the problem (1.1)–(1.3) with the forcing term
\[
f(x, t) = 8(\sin(3\pi x) + \sin(4\pi x))(1 + \sin(2\pi t)),
\]
where the space domain is equal to (0, 1). The simulation was performed for the mesh interval length size equal to $h_k = 1/128$, i.e. the interval (0, 1) was divided into 128 equidistant intervals of length $h_k$. This yields 127 variables, which correspond to the inner mesh points in the space domain. The finite element basis has been diagonalized in such a way that the euclidean norm of each vector of the diagonal basis is equal to one. After diagonalization 8 variables were treated as nondissipative ones, and the remaining 119 variables were treated as dissipative. The constant time step was chosen to be equal to $1/2048$. Simulation took around 55 minutes. First, we have run the standard FEM in order to identify the candidate for the periodic solution, and then, experimentally we found the set of initial data $P_k^{0}$ containing the found solution
RIGOROUS FEM FOR 1D BURGERS EQUATION

| radius | value   | optimal method                               |
|--------|---------|----------------------------------------------|
| $R_1$  | 2.29264 | Lemma B.1                                    |
| $R_2$  | 13.9504 | Lemma B.2 improved with Corollary A.9        |
| $R_3$  | 135.816 | Lemma B.9, inequality (B.19) with interpolation |
| $R_4$  | 1946.47 | Lemma B.15 with inequality (B.36)            |
| $R_5$  | 130542  | Lemma B.20 with interpolation                |

Table 3. Optimal radii of trapping sets for the forcing term (6.1). For each radius several methods to compute it were implemented and optimal radius was chosen.

| coordinate | interval                     |
|------------|------------------------------|
| $\beta_1$  | [-0.011479, 0.00736611]      |
| $\beta_2$  | [-0.00405501, 0.00404116]    |
| $\beta_3$  | [0.667299, 0.672336]         |
| $\beta_4$  | [-0.39088, -0.387498]        |
| $\beta_5$  | [-0.00946911, 0.000953705]   |
| $\beta_6$  | [-0.00140807, -0.0000232083] |
| $\beta_7$  | [-0.00125524, -0.000184778]  |
| $\beta_8$  | [-0.000243156, 0.000611533]  |

Table 4. Values of first 8 modes of found periodic solution at time $t = 1$ for the forcing term (6.1). For all modes higher than 8 zero always belongs to the projection of the set $P_k^i$ onto the given mode.

at time $t = 0$. The simulation showed that the set obtained after time 1 was the subset of $P_k^0$ and hence the periodic solution was found. The radii $R_1$-$R_5$ are given in Tab.3.

First 8 coordinates of the found periodic solution in the orthogonalized basis are given in Tab.4. The corresponding coordinate of the found periodic solution at $t = 0$ must belong to the found interval. Modes 3 and 4 are largest as they are the only two modes contained in the forcing term. It is visible that also modes 6 and 7 are nonzero because of the presence of the nonlinear term which induces the energy transport from modes 3 and 4 to those modes.

Fig. 1 presents the initial datum and set $P_k^0$ after time 1. The set obtained for $t = 1$ is the subset of the initial datum whence all conditions of Theorem 6.2 are satisfied and we obtain the sought periodic solution.

Fig. 2 presents the intervals containing the first 8 coordinates of the evolved set as functions of time. All plots present lower (blue) and upper (orange) bound of corresponding intervals. Apart from variables 3 and 4 which correspond to modes of the forcing term also variables 6 and 7 are nonzero during the time of simulation.

Fig. 3 presents the perturbation (width of the inclusion) as a function of the variable index in the diagonalized basis during one time step, for $t = 0.5$. For the first 8 nondissipative variables the perturbation contains the two components: the one coming from the estimates of the infinite dimensional remainder, and (typically much smaller) contribution from the modes which are treated as dissipative ones. Growth of the perturbation with increasing index of variable $l$ is the effect of the terms $\|w^l\|_{L^2}$ and $\|w^l_x\|_{L^2}$ in (4.21).

Finally, Fig.4 presents the intervals which are guaranteed to contain $L^2$ and $H^1_0$ norm of the solution as the function of time. The sets are constructed by algebraic adding of the sets $P_k^i$ obtained during the simulation (taking into account both dissipative and leading modes which contribute into $P_k^i$) and local $L^2$
and $H_1^0$ estimates for the infinite dimensional remainder $Q_k$. We formulate the result of this simulation as the following theorem.

**Theorem 6.3.** Consider the unique eternal bounded periodic solution $u(t)$ of problem (1.1)–(1.3) with the forcing term (6.1). Let $k = 128$ and let $P_k u(t)$ be the $H_1^0$ orthoprojection on the space $V_k$. In the diagonalized basis of $V_k$, the 8 coordinates of $P_k u(0)$ corresponding to the highest (least negative) 8 eigenvalues, belong to the intervals given in the Tab. 4. The $L^2$ and $H_1^0$ norm of the periodic solution belongs to the intervals depicted in Fig. 4.

6.2.2. Example 2. In the second simulation we have validated the precise form of the periodic trajectory for the problem (1.1)–(1.3) with the forcing

$$f(x,t) = 12 \sin(\pi x) \sin(2\pi t).$$

**PZ: nie ma czasu obliczen w tym przykladzie**

Similar as in the first test, we divided the time interval $(0,1)$ into the equidistant 2048 time steps, and the space interval $(0,1)$ into 128 intervals. We treated 8 modes corresponding to the highest (least negative) eigenvalues as the nondissipative ones. The computation time, as in the first example, was around 55 minutes. Intervals containing the values of first 8 modes of the found periodic solution are presented in Tab. 5. Clearly, the first mode has the highest amplitude, as the energy is inserted into the problem by the forcing term on this mode, but also modes 2 and 3 are nonzero indicating the occurrence of energy transport from the first mode to the higher modes via the nonlinearity. The set of initial data as well as the set obtained for $t = 1$ in the FEM basis are presented in Fig. 5. The plots of intervals containing the first 8 modes of the found periodic solution as functions of time are presented in Fig. 6. The result can be formulated as follows.

**Theorem 6.4.** Consider the unique eternal bounded periodic solution $u(t)$ of problem (1.1)–(1.3) with the forcing term (6.2). Let $k = 128$ and let $P_k u(t)$ be the $H_1^0$ orthoprojection on the space $V_k$. In the diagonalized basis of $V_k$, the 8 coordinates of $P_k u(0)$ corresponding to the highest (least negative) 8 eigenvalues, belong to the intervals given in the Tab. 5.
TABLE 5. Values of nondissipative 8 modes of found periodic solution at time $t = 1$ for the forcing term \((6.2)\).

| coordinate | interval            |
|------------|---------------------|
| $\beta_1$  | [4.33795, 4.41266]  |
| $\beta_2$  | [0.136948, 0.152245]|
| $\beta_3$  | [-0.00751169, -0.00328565]|
| $\beta_4$  | [-0.000773302, 0.00117999]|
| $\beta_5$  | [-0.000581862, 0.00059719]|
| $\beta_6$  | [-0.000409425, 0.000408848]|
| $\beta_7$  | [-0.000304782, 0.00030476]|
| $\beta_8$  | [-0.000238547, 0.000238549]|

**Acknowledgement**

The authors wish to thank D. Wilczak and T. Kapela for stimulating discussions and great help concerning the C++ programming in CAPD. Work was supported by the National Science Center (NCN) of the Republic of Poland by the project no UMO-2016/22/A/ST1/00077. Work of PK has also been partially supported by NCN of the Republic of Poland by the grant no DEC-2017/25/B/ST1/00302.

**References**

[1] G. Arioli, H. Koch, Integration of Dissipative Partial Differential Equations: A Case Study, SIAM J. Appl. Dyn. Syst. 9, (2010), 1119–1133
[2] J. Bouve van den Berg, M. Breden, J.-P. Lessard, L. van Veen, Spontaneous periodic orbits in the Navier–Stokes flow, arxiv.org, arXiv:1902.00384v1.
[3] CAPD - Computer Assisted Proofs in Dynamics, a package for rigorous numeric, http://capd.ii.uj.edu.pl.
[4] R. Castelli, M. Gameiro, J.-P. Lessard, Rigorous numerics for ill-posed PDEs: periodic orbits in the Boussinesq equation, Archive for Rational Mechanics and Analysis, 228 (2018), 129–157.
Figure 6. 8 variables of the solution $\beta$ with the eigenvalues having the smallest modulus for the forcing (6.2) as functions of time. Orange line depicts the upper bound of the interval and the blue one - the lower bound.

[5] V.V. Chepyzhov, V. Pata, and M.I. Vishik, Averaging of nonautonomous damped wave equations with singularly oscillating external forces, Journal de Mathématiques Pures et Appliquées 90 (2008), 469–491.

[6] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, Series “Studies in Mathematics and its Applications”, North-Holland, Amsterdam, 1978.

[7] J Cyranka, T Wanner, Computer-Assisted Proof of Heteroclinic Connections in the One-Dimensional Ohta–Kawasaki Model. SIAM Journal on Applied Dynamical Systems 17 (1), 694–731.

[8] J. Cyranka, P. Zgliczyński, Existence of globally attracting solutions for one-dimensional viscous Burgers equation with nonautonomous forcing - a computer assisted proof, SIAM Journal of Applied Dynamical Systems 14 (2015), 787–821.

[9] J.-L. Figueras, M. Gameiro, J.-P. Lessard, R. de la Llave. A framework for the numerical computation and a posteriori verification of invariant objects of evolution equations. SIAM Journal on Applied Dynamical Systems 16 (2017), 1070–1088.

[10] M. Gameiro, J.-P. Lessard, Analytic estimates and rigorous continuation for equilibria of higher-dimensional PDEs, Journal of Differential Equations 249 (2010), 2237–2268.

[11] M. Gameiro, J.-P. Lessard, Efficient rigorous numerics for higher-dimensional PDEs via one-dimensional estimates, SIAM Journal on Numerical Analysis, 51 (2013), 2063–2087.

[12] M. Gameiro, J.-P. Lessard, A posteriori verification of invariant objects of evolution equations: periodic orbits in the Kuramoto-Sivashinsky PDE, SIAM Journal on Applied Dynamical Systems, 16 (2017), 687–728.

[13] M. Gameiro, J.-P. Lessard, K. Mischaikow, Validated continuation over large parameter ranges for equilibria of PDEs, Mathematics and Computers in Simulation 79 (2008), 1368–1382.

[14] A.T. Hill, E. Süli, Dynamics of a nonlinear convection–diffusion equation in multidimensional bounded domains, Proceedings of the Royal Society of Edinburgh, 125A, 439–448, 1995.

[15] H.R. Jauslin, H.O. Kreiss, J. Moser, On the Forced Burgers Equation with Periodic Boundary Condition, Proceedings of Symposia in Pure Mathematics, Vol. 65, 1999.

[16] H.V. Ly, K.D. Mease, E.S. Titi, Distributed and boundary control of the viscous Burgers equation, Numerical Functional Analysis and Optimization, 18, 143–188, 1997.

[17] P. Kalita, P. Zgliczyński, On non-autonomously forced Burgers equation with periodic and Dirichlet boundary conditions, to appear in Proceedings of the Royal Society of Edinburgh, Section A, Mathematics, DOI: https://doi.org/10.1017/prm.2019.11.

[18] T. Kapela, P. Zgliczyński, A Lohner-type algorithm for control systems and ordinary differential inclusions, Discrete and Continuous Dynamical Systems B, 11 (2009), 365–385.

[19] P.J. McKenna, F. Pacella, M. Plum, and D. Roth, A uniqueness result for a semilinear elliptic problem: a computer-assisted proof, J. Differential Equations 247 (2009), 2140–2162.

[20] P.J. McKenna, F. Pacella, M. Plum, and D. Roth, A computer-assisted uniqueness proof for a semilinear elliptic boundary value problem, Inequalities and Applications 2010, International Series of Numerical Mathematics, Vol. 161, Part 1, 31–52, 2012.
Appendix A. Auxiliary inequalities and lemmas.

A.1. Some basic inequalities. We first recall several basic inequalities which will be used in the paper. We stress that they are well known and we provide the proofs (with optimal constants) only for the exposition completeness. First of all, we will frequently use the following inequality

LEMMA A.1 (Young inequality with $\epsilon$). If $a \geq 0$, $b \geq 0$, $\epsilon > 0$, and $p > 1$ then

\[
ab \leq \frac{a^p}{\epsilon^p} + \frac{\epsilon^{q/p} b^{q}}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]  
(A.1)

Usually in this work (A.1) is applied with $p = q = 2$ (we will say in such case that we use the Cauchy inequality), but when different $p, q$ are used then we will list the values of $p, q, \epsilon$.

We recall the following inequality.
Lemma A.2 (Poincaré inequality). For every \( u \in H^1_0 \) and for every \( u \in H^1 \) with zero mean there holds
\[
\|u\|_{L^2} \leq \frac{1}{\pi} \|u_x\|_{L^2},
\]
where the constant \( 1/\pi \) in is optimal for both classes of functions.

The proofs of the following interpolation inequalities are well known. We recall them only for the completeness of the exposition.

Lemma A.3 (Embedding constant \( H^1_0 \subset L^\infty \)). We have the inequality
\[
\|u\|_{L^\infty} \leq \frac{1}{2} \|u_x\|_{L^2} \quad \text{for} \quad u \in H^1_0.
\]

Proof. For \( x_0 \in [0,1] \) there hold the bounds
\[
|u(x_0)| \leq \int_0^{x_0} |u_x(x)| \ dx,
\]
\[
|u(x_0)| \leq \int_{x_0}^1 |u_x(x)| \ dx,
\]
and the proof follows easily.

Lemma A.4 (Interpolation inequalities). We have the following inequalities.
\[
\|u\|_{L^\infty} \leq \|u_x\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2} \quad \text{for} \quad u \in H^1_0,
\]
\[
\|u\|_{L^\infty} \leq \sqrt{2} \|u_x\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2} \quad \text{for} \quad u \in H^1 \ \int_0^1 u(x) \ dx = 0,
\]
\[
\|u_x\|_{L^2} \leq \|u\|_{L^2} \|u_{xx}\|_{L^2}^{1/2} \quad \text{for} \quad u \in H^2 \cap H^1_0 \quad \text{or} \quad u \in H^2, u_x \in H^1_0.
\]

Proof. We first prove (A.3). For a smooth function \( u : [0,1] \to \mathbb{R} \)
\[
\frac{d}{dx}|u(x)|^2 = 2u(x)u_x(x) \quad \text{for} \quad x \in [0,1].
\]
Assume that \( |u(x_0)| = \sup_{x \in [0,1]} |u(x)|. \) Hence, as \( u(0) = 0, \)
\[
|u(x_0)|^2 = 2 \int_0^{x_0} u(y)u_x(y) \ dy \leq 2 \int_0^{x_0} |u(y)||u_x(y)| \ dy.
\]
In a similar way, as \( u(1) = 0, \)
\[
|u(x_0)|^2 = -2 \int_{x_0}^1 u(y)u_x(y) \ dy \leq 2 \int_{x_0}^1 |u(y)||u_x(y)| \ dy.
\]
This means that
\[
2 \sup_{x \in [0,1]} |u(x)|^2 = 2|u(x_0)|^2 \leq 2 \int_0^1 |u(y)||u_x(y)| \ dy \leq 2\|u\|_{L^2}\|u_x\|_{L^2},
\]
whence we get the assertion. If \( u \) is not smooth the assertion (A.3) follows by density. Inequality (A.4) follows the similar proof that uses the fact that the function which is mean free on \( (0,1) \) must have a root in this interval. We pass to the proof of (A.5). There holds
\[
\|u_x\|_{L^2}^2 = \int_0^1 u_xu_x \ dx = -\int_0^1 u_{xx}u \ dx \leq \|u_{xx}\|_{L^2}\|u\|_{L^2},
\]
and the proof is complete. \( \square \)
Now we prove the important property of the trilinear term which appears in the Burgers equation.

**Lemma A.5.** If $u \in H^1$, satisfies $u(0) = u(1)$ then
\[ \int_0^1 u u_x u \, dx = 0. \]

In particular, the above equality holds for $u \in H^1_0$.

**Proof.** For a smooth function $u$ defined on $[0, 1]$ such that $u(0) = u(1)$ there holds the relation
\[ \int_0^1 u u_x u \, dx = \int_0^1 \frac{d}{dx} \left( \frac{u^3}{3} \right) \, dx = \frac{u(1)^3 - u(0)^3}{3} = 0. \]

By density, this relation holds also for functions from $H^1$, which have the same value on both endpoints of the interval. \(\square\)

**A.2. A polynomial equation.** We will use several times the following lemma.

**Lemma A.6.** Let $d_i > 0$ for $i = 0, \ldots, s$ and $0 < p_i < 1$ for $i = 1, \ldots, s$. The equation
\[ x = h(x) = d_0 + \sum_{k=1}^s d_k x^{p_k}, \]
has a unique positive solution $A$, moreover $x < h(x)$ for $x < A$ and $x > h(x)$ for $x > A$.

**Proof.** Observe that $x < h(x)$ for $x \to 0$ and $x > h(x)$ for $x \to \infty$. Let us set
\[ x_0 = \sup\{x \geq 0 : t < h(t) \text{ for every } t \in (0, x)\}. \]
Obviously $x_0 < \infty$ and it is the smallest solution of (A.7). Observe that $h'(x_0) \leq 1$, because otherwise in the neighborhood of $x_0$ the function $h(x)$ will be growing faster then $x$, hence we will have $h(x_0 - \delta) < x_0 - \delta$ for some small $\delta > 0$. This implies that there exists $\bar{x}_0 < x_0$ such that $\bar{x}_0 = h(\bar{x}_0)$. We obtain a contradiction. Hence $h'(x_0) \leq 1$. Since $h''(x) < 0$ for $x > 0$, it follows that
\[ h'(x) < 1, \quad x > x_0. \]

Now we rule out the existence of other solutions of equation (A.7). Let us take $x > x_0$, then
\[ h(x) = h(x_0) + \int_{x_0}^x h'(t) \, dt < x_0 + \int_{x_0}^x 1 \, dt = x. \]

The proof is complete. \(\square\)

**A.3. A trick of X. Wang.** The following observation inspired by [37] is quite simple, but for the role it plays in our developments we elevate it to the status of the theorem.

**Theorem A.7.** Let $A, B, C, D, E$ be real constants, such that $A \geq 0$ and $C > 0$. Assume that we have absolutely continuous functions $g : [t_0, \infty) \to [0, A]$ (i.e. $g$ is bounded) and $v : [t_0, \infty) \to \mathbb{R}$ such that
\[ \frac{dg}{dt} + C v(t) \leq B \quad \text{for almost every } t \geq t_0, \]
\[ \frac{dv}{dt} \leq D + E v(t) \quad \text{for almost every } t \geq t_0 \]
Then for every $0 \geq \lambda$ such that $\lambda + E \geq 0$ and for every $t \geq t_0$ there holds

\[
(A.11) \quad v(t) + \frac{E + \lambda}{C} g(t) \leq \left( v(t_0) + \frac{E + \lambda}{C} g(t_0) \right) e^{-\lambda(t-t_0)}
\]

\[
+ \left( \frac{D}{\lambda} + \frac{E + \lambda}{C} \left( A + \frac{B}{\lambda} \right) \right) \left( 1 - e^{-\lambda(t-t_0)} \right).
\]

**Proof.** We will show that for sufficiently large $F \geq 0$ the value $v + Fg$ is bounded from above. Indeed, if only $FC - E \geq 0$, then

\[
\frac{d}{dt}(v + Fg) \leq D + E v + F(B - Cv) = -(FC - E)v + (D + FB)
\]

\[
= -(FC - E)(v + Fg) + (D + FB) + (FC - E)Fg
\]

\[
\leq -(FC - E)(v + Fg) + (D + FB) + (FC - E)FA.
\]

From the above inequality it follows that $v + Fg$ is bounded from above and there holds

\[
v(t) + Fg(t) \leq (v(t_0) + Fg(t_0))e^{-(FC-E)(t-t_0)}
\]

\[
+ ((D + FB) + F(FC - E)A) \left( 1 - e^{-(FC-E)(t-t_0)} \right).
\]

Let us set

\[
\lambda = FC - E.
\]

Then $F = \frac{\lambda + E}{C}$ and we require that $\lambda + E \geq 0$ in order to have $F \geq 0$. After this substitution we obtain our assertion. \qed

**Corollary A.8.** Assume that assumptions of Theorem A.7 are satisfied. Then $v$ is bounded from above.

Usually the above corollary and theorem will be applied to function $v$, which is nonnegative, hence the lower bound will be automatic.

**Corollary A.9.** To find $\lambda$ such that in (A.11) the constant in front of $(1 - e^{-\lambda(t-t_0)})$ is minimal consider the following cases

- Either $CD + BE \leq 0$ or $(CD + BE > 0$ and $\sqrt{\frac{CD + BE}{A}} \leq -E$ (note that $E$ must be a negative number). Then one needs to take $\lambda = -E$ to minimize the constant in front of $(1$, whence it follows that

\[
v(t) \leq v(t_0)e^{-\lambda(t-t_0)} - \frac{D}{E} \left( 1 - e^{-\lambda(t-t_0)} \right),
\]

Observe that the above inequality follows directly from (A.10).

- $CD + BE > 0$ and $\sqrt{\frac{CD + BE}{A}} > -E$. Then the minimal value of the constant in front of $(1 - e^{-\lambda(t-t_0)})$ is obtained for

\[
(A.12) \quad \lambda = \lambda = \sqrt{\frac{CD + BE}{A}}.
\]
We get the bound
\[ v(t) + \frac{E + \lambda}{C} g(t) \leq \left( v(t_0) + \frac{E + \lambda}{C} g(t_0) \right) e^{-\frac{t-t_0}{\lambda}} + \frac{1}{C} \left( EA + B + 2\sqrt{A(CD+BE)} \right) \left( 1 - e^{-\frac{t-t_0}{\lambda}} \right), \]

REMARK A.10. One is tempted to think, that the above Corollary can improve the estimate \( g(t) \leq A \). This, however, is not the case. Indeed, after an easy calculation we obtain
\[ g(t) \leq \frac{1}{\sqrt{E}} \left( EA + B + 2\sqrt{A(CD+BE)} \right) = A + \frac{B\sqrt{A} + A\sqrt{CD+BE}}{E\sqrt{A} + \sqrt{CD+BE}}. \]
The last constant is greater than or equal to \( A \).

Appendix B. Trapping sets and local estimates.

This appendix contains five subsections devoted, respectively, to the calculation of the radii of the trapping sets for the \( L^2 \) norm of the function and its space derivatives up to fourth. The results derived here depend heavily on the particular form of the Burgers equations, but we believe that some of the techniques will be also transferable to other problems. Apart from the global estimates also the local estimates are derived. For each quantity several techniques to compute the estimates are presented and in the numerical realization all estimates are computed and the smallest one is always chosen. The computations are rather standard, although sometimes technically cumbersome. We present all derivations for the exposition completeness. Summary of all results of this section used in our computational code can be found in Tables 1 and 2.

B.1. Trapping sets in \( L^2 \). Let \( R_1 > 0 \) be a number. We define
\[ W_{L^2}(R_1) = \{ v \in L^2 : \| v \|_{L^2} \leq R_1 \}. \]
We prove the following lemma on the existence of \( L^2 \)-trapping set. Note that we need the regularity of \( f \) to be only \( L^\infty(L^2) \).

**LEMMA B.1.** Let \( f \in L^\infty(L^2) \). There exists an \( L^2 \)-trapping set which is nonempty and bounded in \( L^2 \). In fact if only
\[ R_1 \geq \frac{\| f \|_{L^\infty(L^2)}}{\pi^2}, \]
then \( W_{L^2}(R_1) \) is \( L^2 \)-trapping. Moreover for every \( t_1 \in \mathbb{R} \) and every \( t > t_1 \) there holds the local estimate
\[ \| u(t) \|_{L^2} \leq \| u_0 \|_{L^2} e^{-\frac{\pi^2}{2}(t-t_1)} + \frac{\| f \|_{L^\infty(t_1,t;L^2)}}{\pi^2} \left( 1 - e^{-\frac{\pi^2}{2}(t-t_1)} \right). \]

**PROOF.** By the comparison of (2.2) with the solution of the ODE
\[ v'(s) = -2\pi^2 v(s) + 2\| f \|_{L^\infty(t_1,t;L^2)} \sqrt{v(t)} \]
we obtain (B.2). Now, to prove that \( W_{L^2}(R_1) \) is trapping let us assume that \( \| u_0 \|_{L^2} \leq R_1 \). Then
\[ \| u(t) \|_{L^2} \leq R_1 e^{-\frac{\pi^2}{2}(t-t_1)} + \frac{\| f \|_{L^\infty(L^2)}}{\pi^2} \left( 1 - e^{-\frac{\pi^2}{2}(t-t_1)} \right) \leq R_1, \]
and the proof is complete. \( \square \)
B.2. Trapping sets in $H^1_0$. In this subsection we will use the notation
\[ W_{H^1}(R_1, R_2) = \{ v \in H^1_0(\Omega) : \| v \|_{L^2} \leq R_1, \| v_x \|_{L^2} \leq R_2 \}. \]

B.3. A priori estimates leading to the trapping set on $\| u_x \|_{L^2}$. The radius of the trapping set for $\| u_x \|_{L^2}$ can be effectively calculated due to the following Lemma.

**Lemma B.2.** Let $f \in L^\infty(L^2)$. There exists an $H^1_0$-trapping set which is nonempty and bounded in $H^1_0$. In fact if only
\[ R_1 \geq \frac{\| f \|_{L^\infty(L^2)}}{\pi^2}, \]
and $R_2 = \min \{ A/\pi, (AR_1)^{1/2} \}$, where $A$ is greater than or equal to the positive root of the equation (whose existence and uniqueness follow from Lemma A.6)
\[ x - \| f \|_{L^\infty(L^2)} - R_1^{5/4}R_2^{3/4} = 0, \]
then $W_{H^1}(R_1, R_2)$ is $H^1_0$-trapping.

**Proof.** Since the ball $W_{L^2}(R_1)$ is $L^2$-trapping we can assume that if $u_0 \in W_{H^1}(R_1, R_2)$ then $u(t) \in W_{L^2}(R_1)$, i.e. $\| u(t) \|_{L^2} \leq R_1$ for every $t \geq t_0$. To obtain the bounds for $\| u_x \|_{L^2}$ we will use (2.6). As the factor $\| u_{xx} \|_{L^2} - \| f(t) \|_{L^2} - \| u \|_{L^2}^{5/4} \| u_{xx} \|_{L^2}^{3/4}$ at the right-hand side of the estimate (2.6) depends on the norm of $\| u_{xx} \|_{L^2}$, we will get the bound in terms of $\| u_x \|_{L^2}$ using the Poincaré inequality or the interpolation inequality (A.5).

By (2.6), it is enough to find $R_2$, such that if $\| u \|_{L^2} \leq R_1$ and $\| u_x \|_{L^2} \geq R_2$, then
\[ \frac{d\| u_x \|_{L^2}^2}{dt} = -2\| u_{xx} \|_{L^2} \left( \| u_{xx} \|_{L^2} - \| f(t) \|_{L^2} - \| u \|_{L^2}^{5/4} \| u_{xx} \|_{L^2}^{3/4} \right) \leq 0. \]

It is easy to see that if $\| u_{xx} \|_{L^2} \geq A$ and $\| u \|_{L^2} \leq R_1$, then
\[ \frac{d\| u_x \|_{L^2}^2}{dt} \leq 0 \]
\[ \frac{d\| u_x \|_{L^2}^2}{dt} \leq 0. \]

Now, from the interpolation inequality (A.5) it follows that
\[ \| u_{xx} \|_{L^2} \geq \frac{\| u_x \|_{L^2}^2}{\| u \|_{L^2}}. \]

Hence in order to have $\| u_{xx} \|_{L^2} \geq A$ it is enough to satisfy
\[ \frac{\| u_x \|_{L^2}^2}{R_1} \geq A. \]

Hence we obtain $\| u_x \|_{L^2} \geq (AR_1)^{1/2}$.

If instead of the interpolation inequality we use the Poincaré inequality $\| u_{xx} \|_{L^2} \geq \pi \| u_x \|_{L^2}$, then in order to have $\| u_{xx} \|_{L^2} \geq A$ it is enough to satisfy
\[ \| u_x \|_{L^2} \geq \frac{A}{\pi}. \]

Therefore $W_{H^1}(R_1, R_2)$ is a trapping region, and the proof is complete. \[ \square \]
**Remark B.3.** We will prove that in the above lemma, if only $R_1$ is optimal, i.e.,

$$ R_1 = \frac{F}{\pi^2}, \quad F = \|f\|_{L^\infty(L^2)}, $$

then the approach by the interpolation inequality yields always the better result then the approach by the Poincaré inequality. Indeed, we will prove that

(B.7) \[(AR_1)^{1/2} < \frac{A}{\pi},\]

which is equivalent to

(B.8) \[A > \pi^2 R_1 = F.\]

The equation (B.4) which defines $A$ is given as follows.

(B.9) \[x - F - \frac{F^5}{\pi^5/2} x^{3/4} = 0,\]

Observe that for $x = F$ there holds $x - F - \frac{F^5}{\pi^5/2} x^{3/4} < 0$, hence indeed the root of the above equation satisfies $A > F$. This establishes (B.7).

| $\|f\|_{L^\infty(L^2)}$ | $\frac{A}{\pi}$ | $(AR_1)^{1/2}$   |
|------------------------|----------------|------------------|
| 0.1                    | 0.032013       | 0.032013         |
| 1                      | 0.33733        | 0.327675         |
| 10                     | 6.17           | 4.43             |
| 100                    | 34117          | 1042             |

**Table 6.** The minimal values of $R_2$ computed for various $\|f\|_{L^\infty(L^2)}$. Observe that always $\frac{A}{\pi} > (AR_1)^{1/2}$, the difference increases with $\|f\|_{L^\infty(L^2)}$.

**B.3.1. Using the Wang’s trick.** The bound of Lemma B.2 can be possibly improved using the ideas of Theorem A.7 and Corollary A.9. Indeed we have two estimates, the first one following from (2.4), and another one from (2.7)

$$ \frac{d}{dt} \|u\|_{L^2}^2 + 2 \|ux\|_{L^2}^2 \leq 2 \|f\|_{L^\infty(L^2)} R_1, $$

$$ \frac{d}{dt} \|ux\|_{L^2}^2 + (2 - \alpha - \beta)\pi^2 \|ux\|_{L^2}^2 \leq \frac{1}{\alpha} \|f\|_{L^\infty(L^2)}^2 + \frac{7}{2^{10} \beta^7} R_1^{10}, $$

Setting

(B.10) \[A = R_1^2, \quad B = 2 \|f\|_{L^\infty(L^2)} R_1, \quad C = 2, \]

$$ D = \frac{\|f\|_{L^\infty(L^2)}^2}{\alpha} + \frac{7 R_1^{10}}{2^{10} \beta^7}, \quad E = -(2 - \alpha - \beta)\pi^2, $$

one can use Corollary A.9 and get (possibly) lower value of $R_2$, the radius of the trapping set for $\|ux\|_{L^2}$. Note that in such case the trapping set would have the form

$$ W_{H^1}(R_1, R_2, S) = \{ v \in H^1_0(\Omega) : \|v\|_{L^2} \leq R_1, \|ux\|_{L^2}^2 + S \|v\|_{L^2}^2 \leq R_2^2 \}. $$

This set is convex and closed and bounded in $H^1$. The values $\alpha$ and $\beta$ can be chosen to minimize $R_2$. In all numerical examples we calculate $R_2$ according to Lemma B.2 and to Corollary A.9 and we choose the
lowest obtained value. We also perform the search over the discrete set of possible values of $\alpha, \beta$ which are positive to get the possibly best estimate for $\|u_x\|_{L^2}$. In the sequel we will simply denote the found trapping set with the smallest radii by $W_{H^1}(R_1, R_2)$, remembering that it is possible that the method based on Corollary A.9 can produce the smaller radius $R_2$, and then the set would depend on the constant $S$.

B.3.2. Local in time a priori estimates of $\|u_x\|_{L^2}$.

**Lemma B.4.** Let $f \in L^\infty((t_1, t); L^2)$ and let $\alpha, \beta > 0$ be such that $\alpha + \beta < 2$. Assume that the solution of the Burgers equation $u : [t_1, t] \to H^1_0$ satisfies the estimate

$$
\|u(s)\|_{L^2} \leq R_1 \quad \text{for} \quad s \in [t_1, t].
$$

Then

$$
\|u_x(t)\|_{L^2}^2 \leq \|(u_0)_x\|_{L^2}^2 e^{-\pi^2(2-\alpha-\beta)(t-t_1)} + \left( \frac{\|f\|_{L^\infty(t_1, t; L^2)}^2}{R_1^{10} 10^7} + \frac{R_1^{10} 7^2}{2 \pi^2 (2-\alpha-\beta)} \right) \left( 1 - e^{-\pi^2(2-\alpha-\beta)(t-t_1)} \right).
$$

The minimal value of the constant in front of $(1 - e^{-\pi^2(2-\alpha-\beta)(t-t_0)})$ is obtained by taking $\alpha$ as the positive root of the equation

$$
\alpha + \sqrt{\frac{R_1^5}{\|f\|_{L^\infty(t_1, t; L^2)}}} \alpha^{1/4} - 1 = 0,
$$

and $\beta = 7/4 - 7\alpha/4$.

**Proof.** Application of the Gronwall lemma in (2.7) implies

$$
\|u_x(t)\|_{L^2}^2 \leq \|(u_0)_x\|_{L^2}^2 e^{-\pi^2(2-\alpha-\beta)(t-t_1)} + \left( \frac{\|f\|_{L^\infty(t_1, t; L^2)}^2}{R_1^{10} 10^7} + \frac{7^2 R_1^{10}}{2 \pi^2 16} \right) \int_{t_1}^t e^{-\pi^2(2-\alpha-\beta)(t-s)} \, ds.
$$

Calculating the integral, we obtain (B.11). We minimize the expression

$$
\left( \frac{\|f\|_{L^\infty(t_1, t; L^2)}^2}{R_1^{10} 10^7} + \frac{7^2 R_1^{10}}{2 \pi^2 16} \right)
$$

over the set $\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0, \beta > 0, \alpha + \beta < 2\}$. A simple but cumbersome computation shows that this expression is minimal, if $\alpha$ is the positive root of the equation

$$
\alpha + \sqrt{\frac{R_1^5}{\|f\|_{L^\infty(t_1, t; L^2)}}} \alpha^{1/4} - 1 = 0,
$$

and

$$
\beta = \frac{7}{4} \sqrt{\frac{R_1^5}{\|f\|_{L^\infty(t_1, t; L^2)}}} \alpha^{1/4} = \frac{7}{4} - \frac{7\alpha}{4},
$$

and the assertion is proved. \hfill \square

**Remark B.5.** It is possible to minimize with respect to $\alpha$ and $\beta$ the whole expression on the right-hand side of (B.11) (including the exponents and not just the constant in the second term). Let us observe that the
righthand side of (B.11) attains its minimum for

$$\beta = \frac{7}{4} \sqrt[4]{\frac{R_1^5}{\|f\|_{L^\infty(t_1,t;L^2)}^{1/4}}}.$$  

Hence it is sufficient to minimize the radius over the set

$$\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0, \beta > 0, \alpha + \beta < 2, \beta = \frac{7}{4} \sqrt[4]{\frac{R_1^5}{\|f\|_{L^\infty(t_1,t;L^2)}^{1/4}}} \},$$

which is a one dimensional numerical procedure. The approximate solution of a minimization problem gives slightly better estimate of the right-hand side of (B.11) than \(\alpha\) and \(\beta\) given in Lemma B.4.

The following Lemma provides the alternative estimate based on the interpolation inequality.

**Lemma B.6.** Let \(f \in L^\infty(t_1, t; L^2)\) and let \(\alpha, \beta > 0\) be such that \(\alpha + \beta < 2\). Assume that the solution of the Burgers equation \(u : [t_1, t] \rightarrow H^1_0\) satisfies the estimate

$$\|u(s)\|_{L^2} \leq R_1$$  

for \(s \in [t_1, t]\).

Then

$$\|u_x(t)\|_{L^2}^2 \leq \frac{D}{C} \frac{\tanh(\sqrt{CD}(t - t_1)) + \sqrt{CD}\|u_x(t_1)\|_{L^2}^2}{\tanh(\sqrt{CD}(t - t_1))\|u_x(t_1)\|_{L^2}^2 + \sqrt{CD}}$$

with

$$C = \frac{2 - \alpha - \beta}{R_1^2} \quad \text{and} \quad D = \frac{\|f\|_{L^2(t_1,t;L^2)}^2}{\alpha} + \frac{7^7 R_1^{10}}{2^{10} \beta^2}.$$  

**Proof.** Using the interpolation inequality (A.5) in (2.8), we obtain

$$\frac{d}{dt}\|u_x\|_{L^2}^2 + \frac{2 - \alpha - \beta}{R_1^2}\|u_x\|_{L^2}^2 \leq \frac{1}{\alpha}\|f(t)\|_{L^2}^2 + \frac{7^7}{2^{10} \beta^2}\|u\|_{L^2}^2.$$  

Hence \(z(t) = \|u_x(t)\|_{L^2}^2\) satisfies the following differential inequality

$$z'(t) \leq -Cz^2(t) + D,$$

with

$$C = \frac{2 - \alpha - \beta}{R_1^2} \quad \text{and} \quad D = \frac{1}{\alpha}\|f\|_{L^2(t_1,t;L^2)}^2 + \frac{7^7}{2^{10} \beta^2}R_1^{10}.$$  

It is easy to check that the solution of equation \(z'(t) = -Cz^2(t) + D\) with initial condition \(y(t_0) = z(t_0)\) is given by

$$y(t) = \frac{D \tanh(\sqrt{CD}(t - t_0)) + \sqrt{CD}z(t_0)}{C \tanh(\sqrt{CD}(t - t_0))z(t_0) + \sqrt{CD}}$$

Hence

$$z(t) \leq y(t),$$

and the proof is complete.

**Remark B.7.** Again, it is possible to minimize the resulting estimate over constants \(\alpha, \beta\) in the set \(\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0, \beta > 0, \alpha + \beta < 2\}\) to obtain the smallest possible upper bound.
B.4. Trapping set for $u_{xx}$ in $L^2$. From now on we assume that $f(0, t) = f(1, t) = 0$. Then, provided $u_{xx} \in H^1$, the boundary condition $u_{xx}(0, t) = u_{xx}(1, t) = 0$ holds. We differentiate the original equation twice with respect to $x$ and denote $v = u_{xx}$. This function satisfies the following system

(B.16)  
$$v_t - v_{xx} + 3u_x v + uv_x = f_{xx},$$

(B.17)  
$$v(0, t) = v(1, t) = 0.$$

If the condition $f(0, t) = f(1, t) = 0$ is not satisfied, the method that we use still works, the assumption is made to avoid technicalities, see Remark 3.4.

The aim of this subsection is to derive the optimal estimates for $\|u_{xx}\|_{L^2}$. The following result follows by the Galerkin method, similar as the existence and uniqueness result for the original problem given in Definitions 2.1 and 2.4, see [33, 36]. We skip the proof for the sake of the article brevity.

Lemma B.8. Assume that $f \in L^\infty(H^1_0)$ (which, in particular, implies that $f(1, t) = f(0, t) = 0$), and that $u \in L^\infty(H^1_0)$. If $v_0 \in L^2$ then the problem governed by (B.16) with the boundary data (B.17) and the initial condition $v_0$ taken at time $t_0$ has a unique weak solution with the regularity $v \in C([t_0, \infty); L^2)$. Moreover, if $u_0 \in H^2 \cap H^1_0$ then, in distributional sense, $v = u_{xx}$, where $u$ is the weak solution of the problem given by Definition 2.1 with the initial data $v_0$ taken at $t_0$. If $v_0 \in H^1_0$ and $f \in L^\infty(H^2 \cap H^1_0)$ then the weak solutions of the problem governed by (B.16) with the boundary data (B.17) and the initial condition $v_0$ taken in time $t_0$ is also its strong solution with the regularity $v \in C([t_0, \infty); H^2_0)$.

Lemma B.9. Let $f \in L^\infty(H^1_0)$. There exists an $H^2 \cap H^1_0$-trapping set which is nonempty and bounded in $H^2$. In fact if only $R_1$ and $R_2$ are given by Lemma B.2 and $R_3 = \min(A/\pi, (AR_2)^{1/2})$, where $A$ is greater than or equal to the smaller number of the positive roots of the equations

(B.18)  
$$x - 5R_1^{1/2}R_2x^{1/2} - \|f_x\|_{L^\infty(L^2)} = 0$$

(B.19)  
$$x - 5\frac{\sqrt{2}}{4}R_2^{1/4}x^{1/4} - \|f_x\|_{L^\infty(L^2)} = 0$$

then the set $W_{H^2}(R_1, R_2, R_3) = \{u \in H^2 \cap H^1_0 : \|u\|_{L^2} \leq R_1, \|u_x\|_{L^2} \leq R_2, \|u_{xx}\|_{L^2} \leq R_3\}$ is $H^2 \cap H^1_0$-trapping.

Proof. We take the scalar product in $L^2$ of (B.16) with $v$, whence

$$\langle v_t, v \rangle - (v_{xx}, v) + (3u_x v + uv_x, v) = -(f_x, v_x).$$

Keeping in mind that $3u_x v + uv_x = (uv + u^2 x)_x$, and both $u$ and $v$ satisfy the Dirichlet condition at the boundary, it follows that

(B.20)  
$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 - (uv + u^2 x, v_x) = -(f_x, v_x).$$

Now note that

$$(u^2 x, v_x) = -((u_x)^2, v) = -2(u_x v, v) = 4(uw, v_x).$$

This means that (B.20) has the following two alternative representations

(B.21)  
$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 - 5(uv, v_x) = -(f_x, v_x).$$

(B.22)  
$$\frac{1}{2} \frac{d}{dt} \|v_x\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 - 5\frac{1}{4}(u^2 x, v_x) = -(f_x, v_x).$$

Now

$$|uv, v_x| \leq \|u\|_{L^\infty} \|v\|_{L^2} \|v_x\|_{L^2}, \quad |u^2 x, v_x| \leq \|u_x\|_{L^\infty} \|u_x\|_{L^2} \|v_x\|_{L^2}$$
By Lemma A.4 we deduce that
\[ |(uv, v_x)| \leq ||u||_{L^2}^{1/2} ||u_x||_{L^2}^{1/2} ||u_{xx}||_{L^2} ||v_x||_{L^2} \leq ||u||_{L^2}^{1/2} ||u_x||_{L^2} ||u_{xxx}||_{L^2}^{1/2} ||v_x||_{L^2} = ||u||_{L^2}^{1/2} ||u_x||_{L^2} ||v_x||_{L^2}^{3/2}. \]
\[ |(u^2, v_x)| \leq \sqrt{2} ||u||_{L^2}^{3/2} ||u_{xx}||_{L^2}^{1/2} ||v_x||_{L^2} \leq \sqrt{2} ||u||_{L^2}^{3/2} ||u_x||_{L^2}^{1/4} ||u_{xxx}||_{L^2}^{1/4} ||v_x||_{L^2} = \sqrt{2} ||u||_{L^2}^{7/4} ||v_x||_{L^2}^{5/4}. \]
We obtain the following two estimates
\[
\frac{d}{dt} ||v||_{L^2}^2 = -2 ||v_x||_{L^2}^2 + 10(uv, v_x) - 2(f_x, v_x)
\leq -2 ||v_x||_{L^2}^2 + 10 ||u||_{L^2}^{1/2} ||u_x||_{L^2} ||v_x||_{L^2}^{1/2} + 2 ||f_x(t)||_{L^2} ||v_x||_{L^2}
= -2 ||v_x||_{L^2} \left( ||v_x||_{L^2} - 5 ||u||_{L^2}^{1/2} ||u_x||_{L^2} ||v_x||_{L^2}^{1/2} - ||f_x(t)||_{L^2} \right).
\]
\[
\frac{d}{dt} ||v||_{L^2}^2 = -2 ||v_x||_{L^2}^2 + \frac{10 \sqrt{2}}{4} ||u||_{L^2}^{7/4} ||v_x||_{L^2}^{5/4} + 2 ||f_x(t)||_{L^2} ||v_x||_{L^2}
= -2 ||v_x||_{L^2} \left( ||v_x||_{L^2} - \frac{5 \sqrt{2}}{4} ||u||_{L^2}^{7/4} ||v_x||_{L^2}^{1/4} - ||f_x(t)||_{L^2} \right).
\]
The rest of the proof follows the same argument as Lemma B.2.

B.4.1. Bounds based on Wang’s trick. In the next argument we use Corollary A.9 to combine the above estimates of \( ||v||_{L^2} = ||u_{xx}||_{L^2} \) with the estimates of \( ||u_x||_{L^2} \) obtained in Lemma 2.6 to get possibly better bound of \( R_3 \). To this end let us first recall the equation (2.7)
\[
\frac{d}{dt} ||u_x||_{L^2}^2 + (2 - \gamma - \delta) ||u_{xx}||_{L^2}^2 \leq \frac{1}{\gamma} ||f(t)||_{L^2}^2 + \frac{7^{7/2}}{216 \delta^7} ||u||_{L^2}^{10},
\]
where \( \gamma, \delta > 0 \) are arbitrary constants such that \( \gamma + \delta < 2 \). As \( u_{xx} = v \) and \( ||u(t)||_{L^2} \leq R_1 \), this leads us to
\[
\frac{d}{dt} ||u_x||_{L^2}^2 + (2 - \gamma - \delta) ||v||_{L^2}^2 \leq \frac{||f||_{L^2}^2}{\gamma} + \frac{7^{7/2}}{216 \delta^7} R_1^{10}.
\]
The above bound together with either of the bounds of Lemma B.13 allow us to use Corollary A.9. We use this result with the following three sets of parameters
\[
A = R_2^2, \quad B = \frac{||f||_{L^\infty(L^2)}}{\gamma} + \frac{7^{7/2}}{216 \delta^7} R_1^{10}, \quad C = 2 - \gamma - \delta,
\]
\[
D = \frac{5^{4/3} R_2^4 R_1^2}{21 \beta^3} + \frac{||f_x||_{L^\infty(L^2)}}{\alpha}, \quad E = -(2 - \alpha - \beta) \pi^2,
\]
\[
A = R_2^2, \quad B = \frac{||f||_{L^\infty(L^2)}}{\gamma} + \frac{7^{7/2}}{216 \delta^7} R_1^{10}, \quad C = 2 - \gamma - \delta,
\]
\[
D = \frac{3 \cdot 5^{1/3} R_2^{14/3}}{2^{28/3} \beta^{5/3}} + \frac{||f_x||_{L^\infty(L^2)}}{\alpha}, \quad E = -(2 - \alpha - \beta) \pi^2,
\]
A = R_2^2, \quad B = \frac{\|f\|_{L^\infty(L^2)}^2}{\gamma} + \frac{7^7 R_1^{10}}{2^{10} \delta^2}, \quad C = 2 - \gamma - \delta,
D = \frac{\|f_x\|_{L^\infty(L^2)}^2}{\alpha}, \quad E = \frac{25 R_1 R_2}{\beta} - \pi^2(2 - \alpha - \beta).

We define the set
\[W_{H^2}(R_1, R_2, R_3, S) = \{v \in H^2 \cap H_0^1 : \|v\|_{L^2} \leq R_1, \|v_x\|_{L^2} \leq R_2, \|v_{xx}\|_{L^2} + S\|v_x\|_{L^2} \leq R_3^{2}\}.\]

Corollary A.9 implies the following result, that states that, for appropriate choice of constants, this set is $H^2 \cap H_0^1$ trapping.

**Lemma B.10.** Assume that $f \in L^\infty(H_0^1)$. If only $R_1$ and $R_2$ are taken as in Lemma B.2, and
\[R_3^2 \geq F(\alpha, \beta, \gamma, \delta), \quad S = G(\alpha, \beta, \gamma, \delta)\]
for some $\alpha, \beta, \gamma, \delta > 0$ such that $\alpha + \beta \leq 2$ and $\gamma + \delta < 2$, where
\[F(\alpha, \beta, \gamma, \delta) = \begin{cases} -\frac{D}{E} & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} \leq -E, \\ \frac{1}{C} \left( EA + B + 2 \sqrt{A(CD + BE)} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} > -E, \end{cases}\]
\[G(\alpha, \beta, \gamma, \delta) = \begin{cases} 0 & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} \leq -E, \\ \frac{1}{C} \left( E + \sqrt{\frac{CD+BE}{A}} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} > -E, \end{cases}\]

and $A, B(\gamma, \delta), C(\gamma, \delta), D(\alpha, \beta), E(\alpha, \beta)$ are given by either of three possibilities (B.24)–(B.26), then the set $W_{H^2}(R_1, R_2, R_3, S)$, is $H^2 \cap H_0^1$ trapping.

**Remark B.11.** For sets of parameters (B.24) and (B.25) the constant $E$ is negative and hence it is sufficient to use Lemma B.9 to get the existence of trapping set with the radius $-D/E$. It turns out, however, that Lemma B.10 can yield better bounds than merely Lemma B.9. If we use (B.26), then $E$ can be positive and then the argument of type as in Lemma B.9 does not work, however, B.10 gives us the trapping set.

**Remark B.12.** The minimization of $F(\alpha, \beta, \gamma, \delta)$ can be performed numerically. The set is trapping for every $\alpha, \beta, \gamma, \delta > 0$ such that $\alpha + \beta \leq 2$ and $\gamma + \delta < 2$. So to obtain the smallest radius of the trapping set, one needs to search the parameter space and use the approximate minimizer in the estimates.

**Summary for global bounds on $\|u_{xx}\|_{L^2}$.** Summarizing, we obtained five possibilities to get trapping set on $\|u_{xx}\|_{L^2}$: two by Lemma B.9 and three by Lemma B.10. In practice we compute all five radii and choose the smallest one. This smallest radius will be denoted by $R_3$ and the resulting trapping set by $W_{H^2}(R_1, R_2, R_3)$. To simplify the notation, the additional constant $S$, which enters the trapping set definition if the method of Lemma B.10 yields the optimal estimate, is neglected in further notation. Note that always the trapping set is convex, and, due to the bound on $\|u_{xx}\|_{L^2}$, compact in $H_0^1$. 

B.4.3. Local estimates on $\|u_{xx}\|_{L^2}$. 

Lemma B.13. Assume that for $t \in [t_1, t_2]$ there hold the bounds $\|u(t)\|_{L^2} \leq R_1$ and $\|u_x(t)\|_{L^2} \leq R_1$. Then for every $\alpha, \beta > 0$ such that $\alpha + \beta \leq 2$ and for a.e. $t \in (t_1, t_2)$ there hold the following estimates:

\[
\begin{align*}
\frac{d}{dt} \|v\|_{L^2}^2 &+ (2 - \alpha - \beta) \pi^2 \|v\|_{L^2}^2 \leq \frac{5^43^3R_1^4R_2^2}{24\beta^3} + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha}, \\
\frac{d}{dt} \|v\|_{L^2}^2 &+ (2 - \alpha - \beta) \pi^2 \|v\|_{L^2}^2 \leq \frac{3 \cdot 5^{13/3}R_2^{14/3}}{2283\beta^{5/3}} + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha}, \\
\frac{d}{dt} \|v\|_{L^2}^2 &\leq \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha} + \left(\frac{25R_1R_2}{\beta} + \pi^2 (\alpha + \beta - 2)\right) \|v\|_{L^2}^2.
\end{align*}
\]

Proof. Estimates (B.21) and (B.22) can be rewritten as

\[
\begin{align*}
\frac{d}{dt} \|v\|_{L^2}^2 + 2\|v_x\|_{L^2}^2 &\leq 10\|u\|_{L^2}^2\|u_x\|_{L^2}\|v_x\|_{L^2}^3/2^3 + 2\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2 \|v_x\|_{L^2}, \\
\frac{d}{dt} \|v\|_{L^2}^2 + 2\|v_x\|_{L^2}^2 &\leq \frac{5\sqrt{2}}{2} \|u\|_{L^2}^4/\|v_x\|_{L^2}^7/4 + 2\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2 \|v_x\|_{L^2}.
\end{align*}
\]

After using the Young inequality they get the form

\[
\begin{align*}
\frac{d}{dt} \|v\|_{L^2}^2 + (2 - \alpha - \beta) \|v_x\|_{L^2}^2 &\leq \|u_x\|_{L^2}^4/\|u\|_{L^2}^2 \|v_x\|_{L^2}^3/2^3 + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha}, \\
\frac{d}{dt} \|v\|_{L^2}^2 + (2 - \alpha - \beta) \|v_x\|_{L^2}^2 &\leq \|u\|_{L^2}^{14/3} \|v_x\|_{L^2}^{14/3} + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha}.
\end{align*}
\]

After using the previously obtained radii of the trapping sets for $\|u\|_{L^2}$ and $\|u_x\|_{L^2}$, and the Poincaré inequality this yields (B.32) and (B.23). On the other hand, multiplying (B.16) by $v$ and integrating over $(0, 1)$ we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 + 3 \int_0^1 u_x v v dx + \int_0^1 u v_x v dx = - \int_0^1 f_x v_x dx.
\]

Integrating by parts and using the Schwarz inequality, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 \leq ||f_x(t)||_{L^2} \|v_x\|_{L^2} + 5 \int_0^1 u v_x v dx.
\]

It follows that

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 \leq ||f_x(t)||_{L^2} \|v_x\|_{L^2} + 5 ||u||_{L^\infty} \|v\|_{L^2} \|v_x\|_{L^2}.
\]

From Lemma A.4, and the Young inequality, we have

\[
\begin{align*}
\|f_x(t)\|_{L^2} \|v_x\|_{L^2} &\leq \frac{\|f_x(t)\|_{L^2}^2}{2\alpha} + \frac{\alpha}{2} \|v_x\|_{L^2}^2, \\
5 ||u||_{L^\infty} \|v\|_{L^2} \|v_x\|_{L^2} &\leq \left(\frac{5 ||u||_{L^2}^{1/2} \|u_x\|_{L^2}^{1/2} \|v\|_{L^2}^2}{2\beta}\right)^2 + \frac{\beta}{2} \|v_x\|_{L^2}^2,
\end{align*}
\]

where $\alpha, \beta$ are positive constants. Hence we deduce

\[
\frac{d}{dt} \|v\|_{L^2}^2 + (2 - \alpha - \beta) \|v_x\|_{L^2}^2 \leq \frac{\|f_x(t)\|_{L^2}^2}{\alpha} + \frac{25 ||u||_{L^2} \|u_x\|_{L^2} \|v\|_{L^2}^2}{\beta}.
\]
Indeed, using the interpolation inequality is analogous to Lemma B.6, namely interpolation inequality is used in place of the Poincaré inequality. Let us define the set

$$ W_t := \{ u \in H^3 : u_{xx} \in L^2 \}. $$

We pass to the result which gives alternative to the local in time estimates on the quantity $\|u_{xx}\|_{L^2}$ of Lemma B.13. The lemma is based on the estimates (B.30) and (B.31), similar as Lemma B.13 and is analogous to Lemma B.6, namely interpolation inequality is used in place of the Poincaré inequality. Indeed, using the interpolation inequality $\|u_{xx}\|_{L^2} \leq \|u_x\|_{L^2} \|u_{xxx}\|_{L^2} \leq R_2 \|u_{xx}\|_{L^2}$ in (B.30) and (B.31) we obtain

\begin{align}
\frac{d}{dt} \|v\|_{L^2}^2 + 2 - \alpha - \beta \|v\|_{L^2}^4 \leq \frac{5^4 3^3 R_1^4 R_2^4}{24 \beta^3} + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha},
\end{align}

\begin{align}
\frac{d}{dt} \|v\|_{L^2}^2 + 2 - \alpha - \beta \|v\|_{L^2}^4 \leq \frac{3 \cdot 5^{13/3} R_2^{14/3}}{228/3 \beta^{5/3}} + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha}.
\end{align}

\textbf{Lemma B.14.} Let \( f \in L^\infty(t_1,t_2;H^1_0) \) and let \( \alpha, \beta > 0 \) be such that \( \alpha + \beta \leq 2 \). Assume that the solution of the Burgers equation \( u : [t_0, t] \to H^1_0 \) satisfies the estimates

$$ \|u(s)\|_{L^2} \leq R_1, \|u_x(s)\|_{L^2} \leq R_2 \quad \text{for} \quad s \in [t_1, t_2]. $$

Then

\begin{align}
\|u_{xx}(t)\|_{L^2}^2 \leq \frac{D \tanh(\sqrt{CD}(t - t_1)) + \sqrt{CD}\|u_{xx}(t_1)\|_{L^2}^2}{C \tanh(\sqrt{CD}(t - t_1))\|u_{xx}(t_1)\|_{L^2}^2 + \sqrt{CD}} \quad \text{for} \quad t \in (t_1, t_2),
\end{align}

with

$$ C = \frac{2 - \alpha - \beta}{R_2^2} \quad \text{and either} \quad D = \frac{5^4 3^3 R_1^4 R_2^4}{24 \beta^3} + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha} $$

or

$$ D = \frac{3 \cdot 5^{13/3} R_2^{14/3}}{228/3 \beta^{5/3}} + \frac{\|f_x\|_{L^\infty(t_1,t_2;L^2)}^2}{\alpha}. $$

\textbf{B.5. Trapping set for } \( u_{xx} \) \textbf{ in } \( L^2 \). In this section we establish global and local estimates for $\|u_{xx}\|_{L^2}$. We will use the notation

$$ X = \{ u \in H^3 \cap H^1_0 : u_{xx} \in H^1_0 \}. $$

Define the set

$$ W_{H^3}(R_1, R_2, R_3, R_4) = \{ v \in X : \|v\|_{L^2} \leq R_1, \|v_x\|_{L^2} \leq R_2, \|v_{xx}\|_{L^2} \leq R_3, \|v_{xxx}\|_{L^2} \leq R_4 \}. $$

\textbf{Lemma B.15.} Assume that \( f \in L^\infty(H^2 \cap H^1_0) \). There exists the $X$-trapping set which is nonempty and bounded in $H^3$. In fact if only $R_1, R_2, R_3$ are as in Section B.4 and

$$ R_4 \geq \min \left( A/\pi, (AR_3)^{1/2} \right), $$

where $A$ is the smaller of the positive roots of two equations

\begin{align}
 x - \frac{7}{2} R_2 R_3^{3/4} x^{1/4} - \|f_{xx}\|_{L^\infty(L^2)} = 0,
\end{align}
$$(B.37) \quad x - 7R_1^{1/2}R_2^{1/2}R_3^{1/2}x^{1/2} - \|f_{xx}\|_{L^\infty(L^2)} = 0,$$

then the set $W_{H^3}(R_1, R_2, R_3, R_4)$, is X-trapping.

**Proof.** Multiplying (B.16) by $-v_{xx}$ and integrating over the space interval $(0, 1)$ we get the bound

$$\frac{1}{2} \frac{d}{dt} \|v_x\|^2_{L^2} + \|v_{xx}\|^2_{L^2} - 3(u_xv', v_{xx}) - (uv, v_{xx}) = -(f_{xx}, v_{xx}).$$

Let us integrate by parts

$$(uv, v_{xx}) = \left( u \frac{1}{2} \frac{d}{dx} v_x^2 \right) - \frac{1}{2} \left( u_xv, v_x \right) = \frac{1}{2} \left( v, v_x \right) + \frac{1}{2} (u_x, v_{xx}).$$

This means that the above equation can be rewritten in the following two possible ways

$$\frac{1}{2} \frac{d}{dt} \|v_x\|^2_{L^2} + \|v_{xx}\|^2_{L^2} - \frac{7}{2} (u_x, v_{xx}) = -(f_{xx}, v_{xx}),$$

(B.38) $$\frac{1}{2} \frac{d}{dt} \|v_x\|^2_{L^2} + \|v_{xx}\|^2_{L^2} - 7(v_x, v_{xx}) = -(f_{xx}, v_{xx}).$$

(B.39)

Using Lemma A.4 we estimate the scalar products above as follows (we want to get rid of $\|v_x\|_{L^2}$, but we are happy with $\|v_{xx}\|^p_{L^p}$ as long as $p < 2$)

$$\left| (u_x, v_{xx}) \right| \leq \|u_x\|_{L^\infty} \|v\|_{L^\infty} \|v_{xx}\|_{L^2} \leq \|u_x\|_{L^2} \|v\|_{L^1/2} \|v_x\|_{L^1} \|v_{xx}\|_{L^2} \leq \|u_x\|_{L^2} \|v\|_{L^{1/2}} \|v_x\|_{L^{2/3}} \|v_{xx}\|_{L^2} = \|u_x\|_{L^2} \|v\|_{L^{1/2}} \|v_x\|_{L^{1/2}} \|v_{xx}\|_{L^2}$$

and

$$\left| (uv, v_{xx}) \right| \leq \|u\|_{L^\infty} \|v_x\|_{L^2} \|v_{xx}\|_{L^2} \leq \|u\|_{L^2} \|v_x\|_{L^{1/2}} \|v\|_{L^{1/2}} \|v_{xx}\|_{L^2} = \|u\|_{L^2} \|v_x\|_{L^{1/2}} \|v\|_{L^{1/2}} \|v_{xx}\|_{L^2}$$

It follows that

$$\frac{1}{2} \frac{d}{dt} \|v_x\|^2_{L^2} + \|v_{xx}\|^2_{L^2} \leq \frac{7}{2} \|u_x\|_{L^2} \|v\|_{L^{1/2}} \|v_x\|_{L^{1/2}} \|v_{xx}\|_{L^2} \leq \frac{7}{2} \|u_x\|_{L^2} \|v\|_{L^{1/2}} \|v_x\|_{L^{1/2}} \|v_{xx}\|_{L^2} + \|f_{xx}\|_{L^2} \|v_{xx}\|_{L^2}.$$
holds. Moreover we have the interpolation inequality \( \|v_x\|_{L^2} \leq \|v\|_{L^2}^{1/2} \|v_{xx}\|_{L^2}^{1/2} \leq R_1^{1/2} \|v_{xx}\|_{L^2}^{1/2} \).
Proceeding exactly as in the proof of Lemma B.2 we obtain the assertion of the Lemma.

B.5.1. Using Wang’s trick. Let us rewrite the equations (B.30) and (B.31) as

\[
\frac{d}{dt} \|v\|_{L^2}^2 + (2 - \delta - \gamma) \|v_x\|_{L^2}^2 \leq \frac{5^4 \delta^3 R_1^2 R_2^2}{24 \delta^3} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\gamma},
\]

\[
\frac{d}{dt} \|v\|_{L^2}^2 + (2 - \delta - \gamma) \|v_x\|_{L^2}^2 \leq \frac{3 \cdot 5^{13/3} R_2^{14/3} R_3^{8/3}}{228/3 \delta^{5/3}} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\gamma}.
\]

We can combine either of the above two estimates with either of estimates (B.46) and (B.47) and use Corollary A.9 to get possibly smaller value of \( R_4 \). This allows us to define the following four sets of parameters

\[
A = R_3^2, \quad B = \frac{5^4 \delta^3 R_2^4 R_3^2}{24 \delta^3} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\gamma}, \quad C = 2 - \delta - \gamma,
\]

\[
D = \frac{3 \cdot 7^8/3 \delta^{5/3} R_2^{8/3} R_3^2}{228/3 \delta^{5/3}} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\beta}, \quad E = -\pi^2 (2 - \alpha - \beta),
\]

\[
A = R_3^2, \quad B = \frac{3 \cdot 5^{13/3} R_2^{14/3}}{228/3 \delta^{5/3}} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\gamma}, \quad C = 2 - \delta - \gamma,
\]

\[
D = \frac{3 \cdot 7^8/3 \delta^{5/3} R_2^{8/3} R_3^2}{228/3 \delta^{5/3}} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\beta}, \quad E = -\pi^2 (2 - \alpha - \beta),
\]

\[
A = R_3^2, \quad B = \frac{5^4 \delta^3 R_2^4 R_3^2}{24 \delta^3} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\gamma}, \quad C = 2 - \delta - \gamma,
\]

\[
D = \frac{7^4 \delta^3 R_2^4 R_3^2}{24 \alpha^3} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\beta}, \quad E = -\pi^2 (2 - \alpha - \beta),
\]

\[
A = R_3^2, \quad B = \frac{3 \cdot 5^{13/3} R_2^{14/3}}{228/3 \delta^{5/3}} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\gamma}, \quad C = 2 - \delta - \gamma,
\]

\[
D = \frac{7^4 \delta^3 R_2^4 R_3^2}{24 \alpha^3} + \frac{\|f_x\|_{L^{\infty}(L^2)}}{\beta}, \quad E = -\pi^2 (2 - \alpha - \beta),
\]

We define the set

\[
W_{H^3}(R_1, R_2, R_3, R_4, S) = \{ v \in X : \|v\|_{L^2} \leq R_1, \|v_x\|_{L^2} \leq R_2, \|v_{xx}\|_{L^2} \leq R_3, \|v_{xxx}\|_{L^2} + S \|v_{xx}\|_{L^2} \leq R_4 \}.
\]

Similar as in the estimate on \( \|u_{x}\|_{L^2} \) Corollary A.9 implies the following result which states that this set is \( X \) trapping for appropriate \( R_4 \) and \( S \).

**Lemma B.16.** Assume that \( f \in L^\infty(H^2 \cap H_1^1) \). There exists the \( X \) trapping set which is nonempty and bounded in \( H^3 \). In fact if only \( R_3, R_2, R_3 \) are taken as in Section B.4, and

\[
R_4^2 \geq F(\alpha, \beta, \gamma, \delta), \quad S = G(\alpha, \beta, \gamma, \delta)
\]
for some $\alpha, \beta, \gamma, \delta > 0$ such that $\alpha + \beta \leq 2$ and $\gamma + \delta < 2$, where

$$F(\alpha, \beta, \gamma, \delta) = \begin{cases} -\frac{D}{\varepsilon} & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} \leq -E, \\ \frac{1}{\varepsilon} \left( EA + B + 2\sqrt{A(CD + BE)} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} > -E, \end{cases}$$

$$G(\alpha, \beta, \gamma, \delta) = \begin{cases} 0 & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} \leq -E, \\ \frac{1}{\varepsilon} \left( E + \sqrt{\frac{CD+BE}{A}} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD+BE}{A}} > -E, \end{cases}$$

and $A, B(\delta, \gamma), C(\delta, \gamma), D(\alpha, \beta), E(\alpha, \beta)$ are given by either of four possibilities (B.42)–(B.45), then the set $W_{E} = (R_1, R_2, R_3, R_4, S)$, is $X$ trapping.

B.5.2. Summary of global bounds. Similar as in the case of $\|u_{xx}\|_{L^2}$ we get six trapping set bounds on $\|u_{xxx}\|_{L^2}$ by Lemmas B.15 and B.16. Since we are interested to know the best possible bound, we calculate all of them, optimizing all four bounds of Lemma B.16 with respect to constants $\alpha, \beta, \gamma, \delta$ and we choose the best obtained bound for further computations.

B.5.3. Local estimates on $\|u_{xxx}\|_{L^2}^2$. In the following Lemma we obtain the differential inequality on $\|v_{xx}\|^2$, which will be useful to get the localized in time estimates.

**Lemma B.17.** Suppose that on interval $(t_1, t_2)$ there hold bounds $\|u(t)\|_{L^2} \leq R_1$, $\|u_{x}(t)\|_{L^2} \leq R_2$, $\|u_{xxx}(t)\|_{L^2} = \|v(t)\|_{L^2} \leq R_3$. Then for every $\alpha, \beta > 0$ such that $\alpha + \beta \leq 2$ and for a.e. $t \in (t_1, t_2)$ there hold the bounds

\begin{align}
\frac{d}{dt} \|v_{xx}\|_{L^2}^2 + \pi^2(2 - \alpha - \beta) \|v_{xx}\|_{L^2}^2 \leq \frac{3 \cdot 78/3 \cdot 5/3 \cdot R_2^8/3}{2 \cdot 3 \cdot 5/3} \|f_{xx}\|_{L^\infty(t_1, t_2; L^2)}^2 + \frac{\|f_{xx}\|_{L^\infty(t_1, t_2; L^2)}^2}{\beta}, \\
\frac{d}{dt} \|v_{xx}\|_{L^2}^2 + \pi^2(2 - \alpha - \beta) \|v_{xx}\|_{L^2}^2 \leq \frac{\|f_{xx}\|_{L^\infty(t_1, t_2; L^2)}^2}{\beta}. \tag{B.47}
\end{align}

**Proof.** Similar as in the proof of Lemma B.15 we multiply (B.16) by $-v_{xx}$ and integrate over the space interval $(0,1)$. This gives us the bounds

$$\frac{d}{dt} \|v_{xx}\|_{L^2}^2 + 2 \|v_{xx}\|_{L^2}^2 \leq 7R_2R_3^{3/4} \|v_{xxx}\|_{L^2}^{5/4} + 2 \|f_{xx}\|_{L^\infty(t_1, t_2; L^2)} \|v_{xx}\|_{L^2} = I_1 + I_3,$n

$$\frac{d}{dt} \|v_{xx}\|_{L^2}^2 + 2 \|v_{xx}\|_{L^2}^2 \leq 14R_1^{1/2}R_2^{1/2}R_3^{3/2} \|v_{xx}\|_{L^2}^{3/2} + 2 \|f_{xx}\|_{L^\infty(t_1, t_2; L^2)} \|v_{xx}\|_{L^2} = I_2 + I_3.$$

Observe that the terms on the right-hand side of both estimates contain $\|v_{xx}\|$ to power lower than 2, therefore we would like use the Young inequality to majorize them by $\|v_{xx}\|^2$. In order to do this, we need to make $\pi^2(2 - \alpha - \beta) \|v_{xx}\|_{L^2}^2$ to power lower than 2, therefore we would like use the Young inequality to majorize them by $\|v_{xx}\|^2$.

(i) *Estimate of $I_1$. We use the Young inequality (A.1) with $a = 7R_2R_3^{3/4}$, $b = \|v_{xx}\|_{L^2}^{5/4}$, $q = 8/5$, $p = 8/3$ and $\epsilon = (\alpha8/5)^{5/8}$. This yields

$$I_1 = 7R_2R_3^{3/4} \|v_{xx}\|_{L^2}^{5/4} \leq \frac{3 \cdot 78/3 \cdot 5/3 \cdot R_2^8/3}{2 \cdot 3 \cdot 5/3} + \alpha \|v_{xx}\|_{L^2}^2. \tag{B.48}$$

(ii) *Estimate of $I_2$. Again we use the Young inequality (A.1) with $a = 14R_1^{1/2}R_2^{1/2}R_3^{3/2}$, $b = \|v_{xx}\|_{L^2}^{3/2}$, $q = 4/3$, $p = 4$ and $\epsilon = (\alpha4/3)^{3/4}$. We obtain

$$I_2 = 14R_1^{1/2}R_2^{1/2}R_3^{3/2} \|v_{xx}\|_{L^2}^{3/2} \leq \frac{\|f_{xx}\|_{L^\infty(t_1, t_2; L^2)}^2}{\beta}.$$
(iii) Estimate of $I_3$. This time we set $p = q = 2$, $\epsilon = \sqrt{27}$ and we obtain

$$I_3 = 2\|f_{xx}\|_{L^\infty(t_1,t_2;L^2)}\|u_{xx}\|_{L^2} \leq \frac{\|f_{xx}\|_{L^\infty(t_1,t_2;L^2)}^2}{\beta} + \|v_{xx}\|_{L^2}^2.$$  

Hence, by the Poincaré inequality we deduce the assertion of the Lemma. □

Analogously to Lemmas B.6 and B.14 we formulate a result which gives local in time estimates on the quantity $\|u_{xxx}\|_{L^2}$ alternative with respect to the ones which follow from Lemma B.17. The lemma is based on the estimates of Lemma B.17 but we estimate the term $\|v_{xx}\|_{L^2} = \|u_{xxx}\|_{L^2}$ from below not by the Poincaré inequality, but by interpolation inequalities. Hence, using the interpolation inequality $\|u_{xx}\|_{L^2} \leq \|u_{xx}\|_{L^2} \|u_{xxx}\|_{L^2} \leq R_3\|u_{xxx}\|_{L^2} = R_3\|v_{xx}\|_{L^2}$ we obtain

$$\frac{d}{dt}\|v_x\|_{L^2}^2 + \frac{2 - \alpha - \beta}{R_3^2}\|v_x\|_{L^2}^4 \leq 3 \cdot \frac{7^{8/3}5^{5/3}R_2^{8/3}R_3^2}{2^{25/3}\alpha^{5/3}} + \frac{\|f_{xx}\|_{L^\infty(t_1,t_2;L^2)}^2}{\beta}.$$  

**Lemma B.18.** Let $f \in L^2(t_1,t_2;H^2 \cap H_0^1)$ and let $\alpha, \beta > 0$ be such that $\alpha + \beta < 2$. Assume that the solution of the Burgers equation $u : [t_1, t_2] \to X$ satisfies the estimates

$$\|u(s)\|_{L^2} \leq R_1, \|u_x(s)\|_{L^2} \leq R_2, \|u_{xx}(s)\|_{L^2} \leq R_3 \text{ for } s \in [t_1, t_2].$$  

Then

$$\|u_{xxx}(s)\|_{L^2}^2 \leq \frac{D\tanh(\sqrt{CD}(t - t_1)) + \sqrt{CD}\|u_{xxx}(t_1)\|_{L^2}}{C\tanh(\sqrt{CD}(t - t_1))\|u_{xxx}(t_1)\|_{L^2} + \sqrt{CD}}$$  

with

$$C = \frac{2 - \alpha - \beta}{R_3^2} \text{ and either } D = \frac{3 \cdot 7^{8/3}5^{5/3}R_2^{8/3}R_3^2}{2^{25/3}\alpha^{5/3}} + \frac{\|f_{xx}\|_{L^\infty(t_1,t_2;L^2)}^2}{\beta}$$  

or

$$D = \frac{7^{43/3}R_1^4R_2^2R_3^2}{2^4\alpha^3} + \frac{\|f_{xx}\|_{L^\infty(t_1,t_2;L^2)}^2}{\beta}.\]  

**B.6. Trapping set for $u_{xxx}$ in $L^2$.** The last estimates will be the ones of $\|u_{xxx}\|_{L^2}$. Similar as in previous situations we will get a local and a global estimates of this quantity. Let us differentiate the original equation four times with respect to the space variable and denote $w = u_{xxx}$ and $v = u_{xx}$. This procedure is valid provided we reinforce the previous assumptions by $f_{xx}(0, t) = 0$ and $f_{xx}(1, t) = 0$. After differentiation we obtain the following equation.

$$w_t - w_{xx} + 5u_{xx}w + 10vw_x + w_{xx} = f_{xxxx}$$  

with the boundary conditions

$$w(0, t) = w(1, t) = 0.$$  

We test this equation with $w$ which yields

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|w_x\|_{L^2}^2 + 5(u_{xx}w, w) + 10(v_x, w) + (uw_x, w) = (f_{xxxx}, w).$$  

Performing integration by parts in the last term on the left-hand side, we deduce

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|w_x\|_{L^2}^2 + \frac{9}{2}(u_xw, w) + 10(v_x, w) = (f_{xxxx}, w).$$
By the Poincaré inequality
\[ \frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \pi^2 \|w\|_{L^2}^2 \leq \frac{9}{2} \|u_x\|_{L^\infty} \|w\|_{L^2}^2 + 10 \|v\|_{L^\infty} \|v_x\|_{L^2} \|w\|_{L^2} + \|f_{xxx}(t)\|_{L^2} \|w\|_{L^2}. \]

Multiplying by 2 and using the Cauchy inequality with \( \epsilon \), we obtain for any \( \beta, \gamma > 0 \)
\[ \frac{d}{dt} \|w\|_{L^2}^2 + (2\pi^2 - \beta - \gamma - 9\|u_x\|_{L^\infty}) \|w\|_{L^2}^2 \leq \frac{1}{\gamma} 100 \|v\|_{L^\infty}^2 \|v_x\|_{L^2}^2 + \frac{1}{\beta} \|f_{xxx}(t)\|^2. \]

On the other hand, (B.38) implies
\[ \frac{1}{2} \frac{d}{dt} \|u_{xxx}\|_{L^2}^2 + \|w\|_{L^2}^2 = \frac{7}{2} (u_{xxx}, u_{x} u_{xxx}) + (f_{xx}, w). \]

After multiplication by two and some simple computations
\[ \frac{d}{dt} \|u_{xxx}\|_{L^2}^2 + (2 - \alpha) \|w\|_{L^2}^2 \leq 7 \|u_x\|_{L^\infty} \|u_{xxx}\|_{L^2}^2 + \frac{1}{\alpha} \|f_{xx}(t)\|^2. \]

Now suppose that evolution is inside the trapping set
\[ W_{H^3}(R_1, R_2, R_3, R_4) = \{ v \in X : \|v\|_{L^2} \leq R_1, \|v_x\|_{L^2} \leq R_2, \|v_{xxx}\|_{L^2} \leq R_3, \|v_{xxxx}\|_{L^2} \leq R_4 \}. \]

Then there hold two differential inequalities
\[ \frac{d}{dt} \|w\|_{L^2}^2 + (2\pi^2 - \beta - \gamma - 9\sqrt{2} R_2 R_3) \|w\|_{L^2}^2 \leq \frac{1}{\gamma} 100 R_3 R_4^3 + \frac{1}{\beta} \|f_{xxx}\|_{L^\infty(L^2)}^2, \]
\[ \frac{d}{dt} \|u_{xxx}\|_{L^2}^2 + (2 - \alpha) \|w\|_{L^2}^2 \leq 7 \sqrt{2} R_2 R_3 R_4^3 + \frac{1}{\alpha} \|f_{xx}\|_{L^\infty(L^2)}^2. \]

These two inequalities, by Corollary A.9 and the earlier obtained bound on \( \|u_{xxx}\|_{L^2} \) allow us to find the trapping set for \( \|w\|_{L^2} \). Let
\[ Y = \{ u \in H^4 \cap H^1_0 : u_{xx} \in H^1_0 \}. \]

Define the set
\[ W_{H^4}(R_1, R_2, R_3, R_4, R_5, S) = \{ v \in Y : \|v\|_{L^2} \leq R_1, \|v_x\|_{L^2} \leq R_2, \|v_{xxx}\|_{L^2} \leq R_3, \|v_{xxxx}\|_{L^2} \leq R_4, \|v_{xxxxx}\|_{L^2} + S \|v_{xxxx}\|_{L^2} \leq R_5^2 \}. \]

Now, we can use Corollary A.9 taking
\[ (B.55) \quad A = R_3^2, \quad B = 7\sqrt{2} R_2 R_3 R_4^3 + \frac{\|f_{xx}\|_{L^\infty(L^2)}}{\alpha}, \quad C = 2 - \alpha, \]
\[ D = \frac{100 R_3 R_4^3}{\gamma} + \frac{\|f_{xxx}\|_{L^\infty(L^2)}}{\beta}, \quad E = \beta + \gamma + 9 \sqrt{2} R_2 R_3 - 2\pi^2, \]

which leads us to the following result

**Lemma B.19.** Assume that \( f \in L^\infty(Y) \). There exists the \( Y \)-trapping set which is nonempty and bounded in \( H^4 \). In fact if only \( R_3 - R_4 \) are taken as in Section B.5, and
\[ R_5^2 \geq F(\alpha, \beta, \gamma), \quad S = G(\alpha, \beta, \gamma) \]
for some $\alpha, \beta, \gamma > 0$, where

$$F(\alpha, \beta, \gamma) = \begin{cases} -\frac{D}{e} & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} \leq -E, \\ \frac{1}{\gamma} \left( EA + B + 2\sqrt{A(CD + BE)} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} > -E, \end{cases}$$

$$G(\alpha, \beta, \gamma) = \begin{cases} 0 & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} \leq -E, \\ \frac{1}{\gamma} \left( E + \sqrt{CD + BE} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} > -E, \end{cases}$$

and $A, B(\alpha), C(\alpha), D(\beta, \gamma), E(\beta, \gamma)$ are given by (B.55), then the set $W_{H \ast}(R_1, R_2, R_3, R_4, R_5, S)$, is $Y$-trapping.

The following results give alternative bounds for the radius of positively invariant set for $\|u_{xxx}\|_{L^2}$.

**Lemma B.20.** If $R_1 – R_4$ are as in Section B.5 then there holds the bound

$$(B.56) \quad \frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 \leq \|w_x\|_{L^2} \left( \|f_{xxx}(t)\|_{L^2} - \|w_x\|_{L^2} + 11R_1^{1/2}R_2^{1/2}R_4^{1/2}\|w_x\|_{L^2} + 10\sqrt{2}R_2^{1/2}R_3^{1/2}R_4 \right).$$

Hence if only $A$ is the positive root of

$$x - \|f_{xxx}\|_{L^\infty(L^2)} = -11R_1^{1/2}R_2^{1/2}R_4^{1/2}x^{1/2} - 10\sqrt{2}R_2^{1/2}R_3^{1/2}R_4 = 0,$$

and

$$R_5 \geq \min \left(\frac{A}{\pi}, (AR_4)^{1/2}\right),$$

then the set

$$W_{H \ast}(R_1, R_2, R_3, R_4, R_5) = \{v \in X : \|v\|_{L^2} \leq R_1, \|v_x\|_{L^2} \leq R_2, \|v_{xxx}\|_{L^2} \leq R_3, \|v_{xxx}\|_{L^2} \leq R_4 \}$$

is $Y$-trapping.

**Proof.** We rewrite (B.54) as

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 = \|w_x\|_{L^2}^2 - 9(ww_x, w) + 10(v_x, w) = -(f_{xxx}, w_x).$$

As

$$(vw_x, w) = -(uw, w) - (wuv_x, w_x) = 2(uw, w_x) - (u_x v_x, w_x),$$

we can rewrite the above equality as

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 \geq \|w_x\|_{L^2}^2 + 11(uw, w_x) - 10(u_x v_x, w_x) = -(f_{xxx}, w_x).$$

We deduce

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 \leq \|f_{xxx}\|_{L^2} \|w_x\|_{L^2}^2 - \|w_x\|_{L^2}^2 + 11\|u\|_{L^\infty} \|w\|_{L^2}^2 \|w_x\|_{L^2} + 10\|u_x\|_{L^\infty} \|v_x\|_{L^2} \|w_x\|_{L^2}.$$

By interpolation inequalities it follows that

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 \leq \|w_x\|_{L^2}^2 \cdot \left( \|f_{xxx}\|_{L^2} - \|w_x\|_{L^2}^2 + 11\|u\|_{L^2}^{1/2} \|w_x\|_{L^2}^{1/2} + 10\sqrt{2}\|u_x\|_{L^2}^{1/2} \|v_x\|_{L^2}^{1/2} \|w_x\|_{L^2}^{1/2} \right),$$

and the proof is complete. $\square$
B.6.1. Local bounds. By the Young inequality applied to (B.56) we deduce the following Lemma which is useful to get the local estimates for \( \|u_{xxxx}\|_{L^2} \)

**Lemma B.21.** Suppose that on interval \([t_1, t_2]\) there hold bounds \( \|u(t)\|_{L^2} \leq R_1, \|u_x(t)\|_{L^2} \leq R_2, \|u_{xx}(t)\|_{L^2} \leq R_3, \|u_{xxx}(t)\|_{L^2} \leq R_4. \) Then for every \( \alpha, \beta, \gamma > 0 \) such that \( \alpha + \beta + \gamma \leq 2 \) there holds the estimate

\[
\frac{d}{dt} \|w\|_{L^2}^2 + (2 - \alpha - \beta - \gamma)\pi^2 \|w\|_{L^2}^2 \leq \frac{\|f_{xxxx}(t)\|_{L^2}^2}{\alpha} + \frac{200R_2R_3R_4^2}{\beta} + \frac{3\pi^4 R_1^2 R_2^2 R_4^2}{2^{4\gamma^3}}.
\]

Now, similar as in Lemma B.17 there hold the following bounds with \( \delta, \epsilon > 0, \) and \( \delta + \epsilon < 2 \)

\[
\frac{d}{dt} \|v_x\|_{L^2}^2 + (2 - \epsilon - \delta)\|v_x\|_{L^2}^2 \leq \frac{3 \cdot 7^{8/3} \pi^{5/3} R_2^8 R_3^6}{2^{25/3} \gamma^{5/3}} + \frac{\|f_{xx}\|_{L^\infty(L^2)}^2}{\epsilon},
\]

\[
\frac{d}{dt} \|v_x\|_{L^2}^2 + (2 - \epsilon - \delta)\|v_x\|_{L^2}^2 \leq \frac{\|f_{xx}\|_{L^\infty(L^2)}^2}{\epsilon}.
\]

Hence, we can derive two more algorithms to find the radius of the positively invariant set for \( \|u_{xxxx}\|_{L^2} = \|w\|_{L^2} \) using Corollary A.9 with the following two sets of parameters

\[
(B.57) \quad A = R_2^4, \quad B = \frac{3 \cdot 7^{8/3} \pi^{5/3} R_2^8 R_3^6}{2^{25/3} \gamma^{5/3}}, \quad C = 2 - \epsilon - \delta,
\]

\[
D = \frac{\|f_{xxxx}\|_{L^\infty(L^2)}^2}{\alpha} + \frac{200R_2R_3R_4^2}{\beta} + \frac{3\pi^4 R_1^2 R_2^2 R_4^2}{2^{4\gamma^3}}, \quad E = (\alpha + \beta + \gamma - 2)\pi^2,
\]

\[
(B.58) \quad A = R_4^2, \quad B = \frac{\|f_{xx}\|_{L^\infty(L^2)}^2}{\alpha} + \frac{200R_2R_3R_4^2}{\beta} + \frac{3\pi^4 R_1^2 R_2^2 R_4^2}{2^{4\gamma^3}}, \quad E = (\alpha + \beta + \gamma - 2)\pi^2.
\]

**Lemma B.22.** Assume that \( f \in L^\infty(H^3 \cap H^1_0) \) satisfies \( f_{xx} \in L^\infty(H^1_0) \). If \( R_1-R_4 \) are taken as in Section B.5, and

\[
R_5^2 \geq F(\alpha, \beta, \gamma, \delta), \quad S = G(\alpha, \beta, \gamma, \delta),
\]

for some \( \alpha, \beta, \gamma, \delta, \epsilon > 0 \) such that \( \alpha + \beta + \gamma \leq 2 \) and \( \epsilon + \delta < 2 \), where

\[
F(\alpha, \beta, \gamma, \delta, \epsilon) = \begin{cases} 
-\frac{D}{E} & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} \leq -E, \\
\frac{1}{6} \left( EA + B + 2 \sqrt{A(CD + BE)} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} > -E,
\end{cases}
\]

\[
G(\alpha, \beta, \gamma, \delta, \epsilon) = \begin{cases} 
0 & \text{when } CD + BE \leq 0 \text{ or } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} \leq -E, \\
\frac{1}{6} \left( E + \sqrt{\frac{CD + BE}{A}} \right) & \text{when } CD + BE > 0 \text{ and } \sqrt{\frac{CD + BE}{A}} > -E,
\end{cases}
\]

and \( A, B(\delta, \epsilon), C(\delta, \epsilon), D(\alpha, \beta, \gamma), E(\alpha, \beta, \gamma) \) are given by either of two possibilities (B.57)–(B.58), then the set \( W_{H^3}(R_1, R_2, R_3, R_4, R_5, S) \), is Y-trapping.

Using (B.56) and arguing the same as in Lemma B.21, by the interpolation inequality \( \|w\|_{L^2}^2 \leq \|u_{xxxx}\|_{L^2} \|w_x\|_{L^2} \leq R_4 \|w_x\|_{L^2} \) we obtain the local estimate alternative to the one of Lemma B.21.

\[
(B.59) \quad \frac{d}{dt} \|w\|_{L^2}^2 + \frac{2 - \alpha - \beta - \gamma}{R_4^2} \|w\|_{L^2}^2 \leq \frac{\|f_{xxxx}(t)\|_{L^2}^2}{\alpha} + \frac{200R_2R_3R_4^2}{\beta} + \frac{3\pi^4 R_1^2 R_2^2 R_4^2}{2^{4\gamma^3}}.
\]

This gives us the following result...
LEMMA B.23. Let $f \in L^\infty(t_1, t_2; H^3 \cap H^1_0)$ satisfy $f_{xx} \in L^\infty(t_1, t_2; H^1_0)$ and let $\alpha, \beta, \gamma > 0$ be such that $\alpha + \beta + \gamma \leq 2$. Assume that the solution of the Burgers equation $u : [t_1, t_2] \to Y$ satisfies the estimates

$$\|u(s)\|_{L^2} \leq R_1, \quad \|u_x(s)\|_{L^2} \leq R_2, \quad \|u_{xx}(s)\|_{L^2} \leq R_3, \quad \|u_{xxx}\|_{L^2} \leq R_4 \quad \text{for} \quad s \in [t_1, t_2].$$

Then

$$\|u_{xxxx}(t)\|_{L^2}^2 \leq \frac{D \tanh(\sqrt{CD}(t - t_1)) + \sqrt{CD} \|u_{xxxx}(t_1)\|_{L^2}^2}{C \tanh(\sqrt{CD}(t - t_1)) \|u_{xxxx}(t_1)\|_{L^2}^2 + \sqrt{CD}}$$

with

$$C = \frac{2 - \alpha - \beta - \gamma}{R_4^2} \quad \text{and} \quad D = \frac{f_{xxx}\|_{L^\infty(t_0, t; L^2)}^2}{\alpha} + \frac{200R_2R_3R_4^2}{\beta} + \frac{3^{11/4}R_1^2R_2^2R_4^2}{2^4\gamma^3}.$$