Bayesian-Nash-Incentive-Compatible Mechanism for Blockchain Transaction Fee Allocation
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Abstract
In blockchain systems, the design of transaction fee mechanisms is essential for stability and satisfactory for both miners and users. A recent work has proven the impossibility of collusion-proof mechanisms with non-zero miner revenue which is Dominate-Strategy-Incentive-Compatible (DSIC) for users. In our work, we relax the DSIC requirement for users to Bayesian-Nash-Incentive-Compatibility (BNIC), and design a so-called soft second-price mechanism to ensure a form of collusion-proofness with an asymptotic constant-factor approximation of optimal miner revenue. Our result breaks the zero-revenue barrier while preserving reasonable truthfulness and collusion-proof properties. We additionally prove the necessity to either burn transaction fees or have variable block sizes to satisfy both BNIC and collusion-proofness guarantees.

1 Introduction
Blockchains, like Bitcoin and Ethereum, are essentially distributed databases growing over time with history data saved in “blocks” as linked lists created by miners. In each block, users leverage blockchains to either store or verify information, such as money (cryptocurrency) transfers, texts, and modern data such as smart contracts. This on-chain information is usually referred to as “transactions”. Miners, in turn, try to “mine” a block by getting the access to write a block with certain efforts (computational power in PoW or cryptocurrency deposit in PoS) and then putting information into the block.

The seminal work by [Roughgarden, 2021] has modeled the problem of mechanism design on blockchain systems in an organized way. In this modeling, to incentivize the miners to mine the block, the blockchain systems adopt economic mechanisms to get the miners paid in cryptocurrency. Such payments usually consist of a mining reward and an additional reward extract from transaction fees paid by users. Generally, a block has limited size and takes social costs (e.g., PoW/PoS and the storage space) to create, and users can benefit from transactions being confirmed on the blockchain. Thus, it is reasonable to charge transaction fees from the confirmed transactions. As described by [Roughgarden, 2021], the on-chain space is a scarce resource, so to facilitate social efficiency of the system, we want to confirm transactions of high values. Therefore, many blockchains adopt bidding-confirmation transaction fee mechanisms (TFM) such as auctions.

However, due to the online and anonymous properties of blockchains, transaction fee mechanism design faces a major concern of credibility [Daskalakis et al., 2020]. Compared to traditional auctions, the miner has a wider strategy space to conduct dishonest activities, including adding fake transactions, concealing users’ bids, and colluding with users. Therefore, a desirable TFM should prevent these dishonest activities so that it can operate correctly and efficiently.

One advantage of blockchains is as follows. Since the blockchain is public, it is not possible for the miner to behave in a Byzantine manner via commuting different bidding vectors to different users (see the discussion in [Daskalakis et al., 2020]). Besides, the paper by [Daskalakis et al., 2020]) adopts a secure commitment scheme, which uses cryptographic protocols to guarantee that a bid cannot be modified after proposed, thus restricting the strategy space of the miner to merely adding fake transactions and conceal transactions, ruling out strategies for the miner to collude with users and make them change existing bids. However, the security of such cryptographic protocols is being constantly challenged by modern cryptanalysis techniques.
as quantum computation [Anand et al., 2020] and subject to backdoor attacks [Young and Yung, 2005], rendering even currently secure protocols in risk of possible future attacks. Since blockchain systems are designed to operate into the future and there are heavy costs in updating their mechanisms, we are motivated to resolve the collusion issue via economic methods by preventing them from being profitable.

Basically, a desirable TFM should satisfy truthfulness. A strong version of truthfulness of the users can be specified as the User-Dominating-Strategy-Incentive-Compatibility (U-DSIC), which means that any single user will not benefit from deviation from truthful bidding even if she knows all bids of other users (as in Definition 3). In comparison, a weaker version of truthfulness of the users is the User-Bayesian-Nash-Incentive-Compatibility (U-BNIC), which means that when each user only knows the distribution of others’ valuations, it is a Bayesian Nash equilibrium that all users truthfully bid their valuations (as in Definition 4). The truthfulness of the miner can be specified as the Miner-Incentive-Compatibility (MIC), which means that the miner will not benefit from untruthful behavior, e.g. injecting fake transactions or ignoring existing transactions. For the issue of collusion, the paper by [Chung and Shi, 2021] formulates collusion-proofness as $c$-Side-Contract-Proof ($c$-SCP): when the miner colludes with at most $c$ users by asking them to change their bids, the coalition cannot gain increased total payoff by deviations from truthfully bidding their valuations (as in Definition 5).

Unfortunately, the paper by [Chung and Shi, 2021] proved a negative result that any TFM that is U-DSIC and 1-SCP has zero miner revenue, indicating that there does not exist a non-trivial “ideal” TFM which can incentivize everyone to behave honestly. Thus, they relax the payoff function (as “$\gamma$-strict utility”) to make unconfirmed over-bidder still pay a $\gamma$ fraction of the worst-case cost, i.e. if a bidder $i$ has valuation $v_i$ and her bid $b_i > v_i$, even if the transaction is not confirmed, she gets a payoff of $-\gamma (b_i - v_i)$, and a fake transaction is assumed to have zero valuation. Thus, if the confirmation probability is $a_i(b_i, b_{-i})$ ($b_{-i}$ denotes bids of all other users than $i$) and she should pay $p_i(b_i, b_{-i})$ if the transaction gets confirmed, the utility she gets is:

$$u_i(\gamma)(b_i, b_{-i}; v_i) = a_i(b_i, b_{-i})(v_i - p_i(b_i, b_{-i})) - \gamma (1 - a_i(b_i, b_{-i})) \max(b_i - v_i, 0).$$

The above relaxed payoff function is justified by the authors of [Chung and Shi, 2021] by considering the bidding process of more than one blocks in a blockchain: even if an over-bidding (including fake) transaction is not confirmed in the current block, the authors assume that the bid could still be collected and confirmed into future blocks, and in the worst case, the over-bidder would have to pay their full bid and get a utility of $-(b_i - v_i)$. In this setting, the authors have further developed a Burning Second Price TFM that satisfies U-DSIC, MIC and $c$-SCP in the notion of $\gamma$-strict utility.

Intuitively, the setting of $\gamma$-strict utility circumvents the impossibility result essentially by imposing more penalty to deviations in a way that coarsely resemble the nature of blockchains. However, such a relaxation might lead to further issues. For example, the choice of $\gamma$ is essential in the mechanism, but finding an accurate $\gamma$ can be difficult if not impossible: the probability that a currently unconfirmed transaction gets confirmed in future blocks is not a universal constant, as unconfirmed transactions with higher bids are more likely to be confirmed in the future than those with lower bids. In this sense, the authors have proposed an open question: whether there are other reasonable relaxations of the models and incentive compatibility specifications that can also circumvent the impossibility result.

In this work, we address this open question by getting back to the one-block setting where a transaction is only valid for the current block and expires if not confirmed immediately, a setting that is easy to implement via time stamps. In turn, we relax the U-DSIC requirement to U-BNIC, which assumes the users only have information of distributions of other users’ valuations instead of all their bids. This relaxation is reasonable because in the distributed network of blockchain, it is impossible for a user to actually know all other users’ bids, especially those who propose transactions after them but competing the same block. On this basis, we design a TFM that satisfies U-BNIC and 1-SCP for bounded i.i.d. valuation distributions with a constant-factor approximation of the optimal revenue, thus answering the open question of [Chung and Shi, 2021].

More specifically, we first prove an impossibility result that no deterministic U-BNIC + 1-SCP TFM (satisfying mild conditions) can achieve positive miner reward (Theorem 4), and then modify the second-price auction by introducing randomness in the allocation rule, develop our main mechanism for block size 1 via our Auxiliary Mechanism method that essentially establishes connections between BNIC mechanisms and DSIC mechanisms (details in Section 3), and estimate the optimal parameters to achieve an asymptotic constant fraction approximation of the optimal miner revenue (as in Section 4). Then, we extend our mechanism to general block size $k$ with $n \geq \left(\frac{2}{\varepsilon^4} + \Theta(1)\right) k$, and prove that the asymptotic constant fraction approximation also holds in this general case.

Finally, we look into the issue of burning, which means that some of transaction fees are removed from the system instead of paid to the miner. The existence of burning leads to economical inefficiency of the blockchain system, but is shown by [Roughgarden, 2020] to be essential in the Ethereum EIP-1559 mechanism to avoid collusion. As the mechanism proposed by [Chung and Shi, 2021] also incorporates burning in their design, we are curious about the question: is burning generally...
necessary for collusion-proof TFM design? Our research has given a conditional positive answer in the scope of U-BNIC and 1-SCP mechanisms. The detailed discussion is in Section 6.

2 Preliminaries

2.1 The Basic Model

There are n users numbered by 1, 2, . . . , n. For user i, w.l.o.g. we assume her valuation \( v_i \) is in \([0, 1]\) and drawn from an i.i.d. distribution \( V_i = V_0 \) with pdf \( \rho_i(\cdot) = \rho(\cdot) \). By the revelation principle [Durlauf and Blume, 2010], we only need to consider direct mechanisms in which users propose bids, the miner collects the bids and the system decides which transactions to confirm and processes the payments. Formally, we can model any Transaction Fee Mechanism as its allocation, payment and miner revenues, i.e.

Definition 1 (Transaction Fee Mechanism). For a fixed number \( n \) of users, a Transaction Fee Mechanism is modeled as \( M(a, p, r) \):

- The allocation rule \( a : [0, 1]^n \rightarrow [0, 1]^n \) maps the bid vector to the allocation vector indicating the probability each user’s transaction to be confirmed.
- The payment rule \( p : [0, 1]^n \rightarrow \mathbb{R}^n \) maps the bid vector to the payment vector indicating the payment of a user if her transaction is confirmed.
- The miner revenue rule \( r : [0, 1]^n \rightarrow \mathbb{R} \) maps the bid vector to the miner’s revenue.

In the execution of the mechanism, the system just needs to let all users propose their bids \( \{b_i\} \), draw the confirmed transactions\(^1\) according to probabilities from \( \{a_i(b_i, b_{-i})\} \), and then charge transaction fees from confirmed bidders according to \( \{p_i(b_i, b_{-i})\} \) and give the miner revenue \( r(b) \) to the miner. Since the transactions are naturally anonymous and unsorted, in this paper we only consider the mechanisms satisfying the following symmetric condition:

Definition 2 (Symmetry). A TFM is symmetric if the allocation and payment rules do not depend on the order of users, i.e. when we swap any pair of users, each should still have the same allocation probability and payment as in their original positions.

2.2 Incentive Compatibility Conditions

We now discuss the incentive compatibility properties we would like the bidding mechanism to enjoy. When user \( i \) has valuation \( v_i \) and the bidding vector is \( (b_i, b_{-i}) \), user \( i \)’s (expected) utility is \( u_i(b_i, b_{-i}; v_i) = a_i(b_i, b_{-i}) \cdot (v_i - p_i(b_i, b_{-i})) \). We formally define U-DSIC and U-BNIC as:

Definition 3 (User Dominating-Strategy-Incentive-Compatibility (U-DSIC)). For any user \( i \), assuming the miner follows the inclusion rule truthfully, a TFM is U-DSIC if and only if it is a dominating strategy for any user to bid their valuations, i.e.

\[
\forall b_{-i}, v_i \in \arg \max_{b_i} [u(b_i, b_{-i}; v_i)].
\]

Definition 4 (User Bayesian-Nash-Incentive-Compatibility (U-BNIC)). Assume \( \Omega \) is the type space of nature, and there is a public mapping \( B : \Omega \rightarrow \mathbb{R}^n_+ \) that determines the valuation of all users. Thus, the valuation vector \( \nu \sim V \).

For each user \( i \), he only knows her own valuation and the distribution of other users’ valuations conditioned on \( v_i \), denoted as \( V_{-i} = V | v_i \). A TFM is U-BNIC if and only if, when other users all bid their valuations, it maximizes user \( i \)’s expected utility if she bids her valuation too, i.e.

\[
v_i \in \arg \max_{b_i} \mathbb{E}_{b_{-i} \sim V_{-i}} [u(b_i, b_{-i}; v_i)].
\]

To describe the collusion-proofness, we use the notation of \( c \)-SCP in [Chung and Shi, 2021], defined as:

Definition 5 (\( c \)-Side-Contract-Proofness (c-SCP)). We call a TFM \( c \)-SCP when is impossible for the miner to collude with at most \( c \) users to strictly increase their total utility, even if the miner knows all users’ valuations and bids.

Note that a successful collusion only needs the party has increased total utility rather than individual utilities, because the members can make payments among them to make everyone get increased utility.

\(^1\)The drawing scheme for block size one is straightforward, while the drawing scheme for general block size \( k \) can just follow the sampling method discussed in Section 5.1, which ensures exactly \( k \) transactions are confirmed.
2.3 Rationality and Feasibility Requirements

Beside truthfulness and collusion-proofness, the mechanism also needs to satisfy more general properties, e.g. the balance must be feasible, the users should not pay more than their bid, etc. Formally, the following properties should also be satisfied.

**User Individually Rationality (UIR).** Each user gets non-negative utility when truthful bidding, now matter how others bid.

\[ u(v_i, b_{-i}; v_i) \geq 0, \]

equivalently,

\[ a_i(v_i, b_{-i}) > 0 \implies p_i(v_i, b_{-i}) \leq v_i. \]

**Budget Feasibility (BF).** The miner’s revenue \( r(b_i, b_{-i}) \) should not be greater than the (expected) total user payment:

\[ P(b) = \sum_{i=1}^{n} a_i(b_i, b_{-i}) \cdot p_i(b_i, b_{-i}). \]

(1)

In other words, we should have

\[ P(b) \geq r(b). \]

Here we allow the miner revenue to be less than the total fee paid by users, in which the difference will be burnt. The burning can make the feasible space of TFMs wider by partially decoupling payments on miner’s and users’ sides, and has been used in the EIP-1559 TFM of Ethereum. [Leonardos et al., 2021]

2.4 Deterministic and Randomized Mechanisms

We assume generic positions of bids, which means that all bids are distinct. For simplicity, we would want the mechanism to be deterministic, i.e., the same input bidding vector leads to the same allocation outcome, equivalently \( a_i(b_i, b_{-i}) \in \{0, 1\} \).

However, we can prove that even if we relax U-DSIC to U-BNIC, no deterministic U-BNIC and 1-SCP TFM that satisfy mild conditions can achieve positive miner revenue, indicating that the randomness in our main mechanism is necessary. The formal discussion is in Appendix A.

3 Technical Roadmap

3.1 Auxiliary Mechanism Method: Connecting BNIC and DSIC Auctions

While DSIC auctions have been well studied by existing research, especially as Myerson established a famous lemma (Lemma 1, [Myerson, 1981]) showing that a unique payment rule exists for any given allocation rule (with the boundary condition that zero-bidders pay zero), there is a larger space of freedom in the design of payment rules for BNIC mechanisms. Nevertheless, in this paper, we study BNIC auction mechanisms by making essential connections between them and DSIC mechanisms, and further characterize TFMs that simultaneously satisfy U-BNIC and 1-SCP properties in a closed form.

We firstly consider the mechanism on the users’ side, and only consider the allocation rule \( a \) and payment rule \( p \), and the user-side mechanism \((a, p)\) constitutes an auction in the common sense, in which the incentive and behavior of the miner is not yet taken into consideration. Particularly, the Revenue Equivalence Theorem [Tenorio, 1993] indicates that for the same distribution of valuations and given boundary conditions, all BNIC mechanisms with the same allocation rule should have the same expected payment for a bidder with a fixed valuation. Since DSIC mechanisms are also BNIC, we can regard any BNIC mechanism as a “basic” DSIC mechanism with its allocation rule, plus a “perturbation” of the payment rule that has expectations of zero for each bidder. Namely, for any BNIC auction \( \mathcal{A} = (a, p) \), we can construct an auxiliary auction mechanism that has the same allocation rule \( a \), but satisfies DSIC property. From the Myerson’s Lemma (Lemma 1), we can construct the auxiliary payment rule \( p \) as:

\[ p_i(b_i, b_{-i}) = \begin{cases} \int_{b_i}^{\infty} \frac{a_i(t, b_{-i})}{a_i(b_i, b_{-i})} \, dt, & a_i(b_i, b_{-i}) > 0 \\ 0, & a_i(b_i, b_{-i}) = 0 \end{cases}. \]

(2)
Thus, the auxiliary mechanism is $\mathcal{A} = (a, p)$. Then, as we can note that the auxiliary auction mechanism only differs to the original mechanism in the payment rule, we can define a “payment difference” function that denotes the over-payment of each user compared to the auxiliary DSIC mechanism, as $\theta_i(b_i, b_{-i}) = a_i(b_i, b_{-i}) (\tilde{p}_i(b_i, b_{-i}) - p_i(b_i, b_{-i}))$, which has an expectation of zero over the distribution of $b_{-i}$. On the other hand, in auction mechanism $\mathcal{A}$, bidder $i$ gets $\theta_i(b_i, b_{-i})$ less utility than in the auxiliary mechanism $\mathcal{A}$.

Now we take the miner into consideration. It can be immediately observed that when we set the miner revenue as 0, any U-DSIC TFM is also 1-SCP, because the total utility of the party of the miner and any user $i$, equals the utility of user $i$ herself, so since truthfully bidding maximizes the utility of any user, it also maximizes the total utility of any collusion party of the miner and one user. Hence, for any TFM $M = (a, p, \tilde{r})$ that satisfies U-BNIC and 1-SCP, we can also construct an auxiliary TFM $\tilde{M} = (a, p, 0)$ that is U-DSIC and 1-SCP.

### 3.1.1 Relations between payment difference functions $\{\theta_i\}$ and the miner revenue $\tilde{r}$

To characterize the properties of the TFM $\tilde{M}$, we look into the relations between $\tilde{p}$ and $\tilde{r}$. Recall that in our model, users and the miner have different information on the bids: a user only knows the distribution of other users’ valuations (and bids, if the mechanism is U-BNIC), but the miner know all bids accurately. Therefore, in mechanism $\tilde{M}$, for fixed $b_{-i}$, truthfully bidding $v_i$ does not guarantee to maximize user $i$’s utility $\tilde{u}(b_i, b_{-i}; v_i)$, but it must maximize the total utility of her and the miner, as $\tilde{u}(b_i, b_{-i}; v_i) + \tilde{r}(b_i, b_{-i})$, so that the miner is not incentivized to ask her to deviate. Therefore, slightly unrigorously$^2$, we have

$$\left. \frac{\partial}{\partial b_i} (\tilde{u}(b_i, b_{-i}; v_i) + \tilde{r}(b_i, b_{-i})) \right|_{b_i = v_i} = 0. \quad (3)$$

However, in the auxiliary TFM $M$, which is U-DSIC, bidding $b_i = v_i$ does maximize user $i$’s utility, hence

$$\left. \frac{\partial}{\partial b_i} u(b_i, b_{-i}; v_i) \right|_{b_i = v_i} = 0. \quad (4)$$

Since TFMs $\tilde{M}$ and $M$ have the same allocation rule, users’ utilities only differ in payments, we have $u(b_i, b_{-i}; v_i) = \tilde{u}(b_i, b_{-i}; v_i) - \theta_i(b_i, b_{-i})$. Therefore, we get the relation between $\theta_i(b_i, b_{-i})$ and $\tilde{r}(b_i, b_{-i})$ as

$$\left. \frac{\partial}{\partial b_i} \theta_i(b_i, b_{-i}) \right|_{b_i = v_i} = \left. \frac{\partial}{\partial b_i} \tilde{r}(b_i, b_{-i}) \right|_{b_i = v_i}. \quad (5)$$

That is to say, if user $i$ would benefit from an infinitesimal deviation from truthful bidding, the miner would lose the same amount in turn, so that the miner has no incentive to let user $i$ deviate, even though she has additional information about other users’ bids. With the boundary condition $\tilde{p}_i(0, b_{-i}) = 0$ (thus $\theta_i(0, b_{-i}) = 0$), we get

$$\theta_i(b_i, b_{-i}) = \tilde{r}(b_i, b_{-i}) - \tilde{r}(0, b_{-i}). \quad (6)$$

This characterizes the relation between user payments and miner revenue in 1-SCP TFMs.

### 3.1.2 A non-working example: first-price auction

To understand the property of BNIC auctions, we look into the first-price auction as an example. It is well-known that first-price auction is not truthful [Roughgarden, 2021]: bidders would prefer to bid lower than their valuations, which is necessary for them to get any surplus even if they get the item. Nevertheless, there exist Bayesian Nash equilibria for specific settings when distributions of valuations are known. For example, when there are $n$ bidders with i.i.d. uniform valuation distributions in $[0, 1]$, it is a Bayesian Nash equilibrium for each bidder to bid $\frac{n-1}{n} v_i$. By revelation principle, we can actually make each bidder pay $\frac{n}{n-1}$ times her bid. For intuitive discussion, we set $n = 2$, then the following auction mechanism $(a, p)$ is BNIC:

$$a_i(b_i, b_{-i}) = \begin{cases} 1, & b_i > b_{-i} \\ \frac{1}{2}, & b_i = b_{-i} \\ 0, & b_i < b_{-i} \end{cases}, \quad (7)$$

$$\tilde{p}_i(b_i, b_{-i}) = \frac{1}{2} \max \{b_i, b_{-i}\}. \quad (8)$$

$^2$Assuming smoothness. Otherwise, we can “smooth” the non-smooth points to make derivatives well-defined.
Via our auxiliary mechanism method (from (2)), we can get the auxiliary payment rule

\[ p_i(b_i, b_{-i}) = \begin{cases} \min\{b_i, b_{-i}\}, & b_i \geq b_{-i} \\ 0, & b_i < b_{-i} \end{cases} \]

(9)

Then the auxiliary auction, which is actually second-price auction, is indeed DSIC. By definition, we compute the payment difference function

\[ \theta_i(b_i, b_{-i}) = \begin{cases} \frac{1}{2} \max\{b_i, b_{-i}\} - \min\{b_i, b_{-i}\}, & b_i > b_{-i} \\ -\frac{1}{2}b_i, & b_i = b_{-i} \\ 0, & b_i < b_{-i} \end{cases} \]

(10)

It can be observed that \[ \mathbb{E}_{b_{-i} \sim U[0,1]}[\theta_i(0, b_{-i})] = 0 \], indicating that for fixed \( b_i \), bidder \( i \) would pay the same expected amount in the modified first-price auction and the auxiliary (second-price) auction, as the Revenue Equivalence Theorem [Tenorio, 1993] shows. Now that we have derived the formula of \( \theta_i(b_i, b_{-i}) \), it seems that we can construct a U-BNIC and 1-SCP TFM by attaching a miner revenue rule \( \tilde{r} \), which can be computed via (6).

However, it is actually impossible. To get the contradiction, we compute \( \tilde{r}(1, 1) \) in two different ways. Because (6) only constrains the differences of \( \tilde{r}(\cdot) \), and UIR and BF requires that \( \tilde{r}(0) \leq 0 \), we can set \( \tilde{r}(0) = 0 \) for best miner revenue. Then from (6) we have:

\[
\tilde{r}(1, 1) = \tilde{r}(0, 0) + (\tilde{r}(1, 0) - \tilde{r}(0, 0)) + (\tilde{r}(1, 1) - \tilde{r}(1, 0)) \\
= \theta_1(1, 0) + \theta_2(1, 1) \\
= 0.5 - 0.25 \\
= 0.25.
\]

(11) \hspace{1cm} (12) \hspace{1cm} (13) \hspace{1cm} (14)

We can also compute it another way:

\[
\tilde{r}(1, 1) = \tilde{r}(0, 0) + (\tilde{r}(0.5, 0) - \tilde{r}(0, 0)) + (\tilde{r}(0.5, 1) - \tilde{r}(0.5, 0)) + (\tilde{r}(1, 1) - \tilde{r}(0, 1)) - (\tilde{r}(0.5, 1) - \tilde{r}(0, 1)) \\
= \theta_1(0.5, 0) + \theta_2(0.5, 1) + \theta_1(1, 1) - \theta_1(0.5, 1) \\
= 0.25 + 0 - 0.25 - 0 \\
= 0.
\]

(15) \hspace{1cm} (16) \hspace{1cm} (17) \hspace{1cm} (18)

Here the contradiction occurs. Therefore, although we can construct an auxiliary auction mechanism from the modified first-price auction, we actually cannot derive a U-BNIC and 1-SCP TFM from the first-price auction, even for \( n = 2 \).

### 3.1.3 Constraints on payment difference functions \( \{\theta_i\} \)

The failure to construct a desirable TFM from the first-price auction, indicates that the payment difference functions \( \{\theta_i\} \) should satisfy additional constraints, even if the auction is truly BNIC. The problem is that there are actually different paths, as piecewise axis-aligned polylines, all of which can be used to compute the value of \( \tilde{r}(\cdot) \). To successfully construct 1-SCP TFM, the values of \( \{\theta_i(\cdot)\} \) should make it path-independent. Actually, if we define a vector field

\[
D_\theta(b) = \left( \frac{\partial}{\partial b_1}\theta_i(b_1, b_{-1}), \ldots, \frac{\partial}{\partial b_n}\theta_i(b_n, b_{-n}) \right),
\]

then \( D_\theta \) should be a conservative field, i.e. for any closed curve \( C \) (with parametrization \( z \)), the integration

\[
\oint_C D_\theta \cdot dz = 0.
\]

(19) \hspace{1cm} (20)

In this sense, \( \tilde{r} \) is actually the potential of \( D_\theta \).
After discussing the formulation of 1-SCP in terms of \( \{\theta_i(\cdot)\} \) and \( \tilde{r} \) in the section above, now we look into the U-BNIC condition. By definition, \( \tilde{M} \) is U-BNIC if and only if

\[
v_i \in \operatorname{arg} \max_{b_i} \mathbb{E}_{b_{-i} \sim V_{-i}} \left[ u(b_i, b_{-i}; v_i) \right].
\]

Note that \( \tilde{u}(b_i, b_{-i}; v_i) = u(b_i, b_{-i}; v_i) - \theta_i(b_i, b_{-i}) \), and the U-DSIC mechanism \( M \) is also U-BNIC, so

\[
v_i \in \operatorname{arg} \max_{b_i} \mathbb{E}_{b_{-i} \sim V_{-i}} \left[ u(b_i, b_{-i}; v_i) \right].
\]

Therefore we have

\[
\frac{\partial \tilde{u}(b_i, b_{-i}; v_i) - u(b_i, b_{-i}; v_i)}{\partial b_i} \bigg|_{b_i = v_i} = 0,
\]

indicating

\[
\frac{\partial}{\partial b_i} \mathbb{E}_{b_{-i} \sim V_{-i}} \left[ \theta_i(b_i, b_{-i}) \right] = 0.
\]

So with the boundary condition \( \tilde{p}_i(0, b_{-i}) = 0 \), we get a necessary condition for U-BNIC as

\[
\mathbb{E}_{b_{-i} \sim V_{-i}} \left[ \theta_i(b_i, b_{-i}) \right] = 0.
\]

Actually, the condition is also sufficient when the allocation rule is monotone, as shown in Observation 3.

### 3.2 Construction of U-BNIC and 1-SCP TFM via Auxiliary Mechanism Method

Via the argument above, we manage to transform the U-BNIC and 1-SCP conditions into forms of the conditions above. With a slight abuse of notation, we can conceptually denote \( TFM \) as U-BNIC and 1-SCP if:

- \( M \) is U-DSIC and 1-SCP.
- \( T \) is an admissible variation term.

In practice, we would like to construct a TFM that also satisfies UIR and BF conditions. Since the basic TFM \( M \) always satisfies UIR and the zero miner revenue also indicates it satisfies BF, intuitively, if we make the \( T \) “small enough” so that \( \tilde{M} = M + T \) is close to \( M \), \( \tilde{M} \) is also likely to satisfy UIR and BF conditions, although having low miner revenue as well. We also notice that the admissibility condition of \( T \) is scale-free, i.e. if \( T \) is admissible and \( h \in \mathbb{R}, hT \) is also admissible. Therefore, we can construct a desired TFM in the following pipeline:

- Construct a randomized allocation rule \( a \), and derive the basic TFM \( M = (a, p, 0) \) via Myerson’s lemma.
- Find a “good” admissible variation term \( T^* = (\theta^*, \tilde{r}^*) \).
- Compute the (approximately) optimal \( h \) so that \( M + hT^* \) maximizes miner revenue while still obeying UIR and BF conditions.

By our research, we find out that the variation term \( T^* = (\theta^*, \tilde{r}^*) \) works well in both settings of block size one and general block size \( k \):

\[
\theta^*_i(b_i, b_{-i}) = -\frac{1}{2} b_i^2 \left( \frac{\sum_{j \neq i} b_j^2}{c_p(n-1)} - 1 \right),
\]

\[
\tilde{r}^*(b) = \frac{1}{2} \sum_{i=1}^{n} b_i^2 - \frac{\sum_{1 \leq i < j \leq n} b_i^2 b_j^2}{c_p(n-1)}.
\]

Here, \( c_p \) is a parameter determined by the valuation distribution, as defined in (35). Furthermore, we have shown that in both settings, the optimal \( h \) can make the mechanism achieve a constant-fraction approximation of optimal revenue for fixed \( c_p \) and \( n \) large enough.
4 The Proposed Mechanism for Block Size 1

We first consider the case with block size \( k = 1 \), where exactly one transaction is confirmed, to give a simple and intuitive understanding of our mechanism. In Section 5, we further extend our setting to general block size \( k \) for the main mechanism.

4.1 Warm Up: Soft Second-Price Mechanism

The second-price auction mechanism has been widely used in traditional auctions, in which the highest bidder gets confirmed but pays the second highest bid. However, as we prove that any deterministic TFM which is U-BNIC and 1-SCP satisfying mild assumptions has non-positive miner revenue (Appendix A), we try to introduce randomness into the allocation rule.

Here we consider the widely used logit choice model [Benson et al., 2018] in which the choice probability of an item is proportional to the exponential of a parameter \( m \) times its value. If we set the \( m \) to infinity, then the item with highest value is deterministically chosen, coinciding with the allocation rule of second-price auction; if we set \( m \) to zero, then all items are randomly chosen with uniform chances. For \( m \in (0, +\infty) \), the choice is random, but higher valued items are more likely to be chosen.

As a basis of our main mechanism, we firstly develop a U-DSIC and 1-SCP mechanism named soft second-price mechanism. It adopts the logit choice model as the allocation rule. After fixing the allocation rule, the payment rule can be correspondingly fixed by the Myerson’s Lemma (Lemma 1) [Myerson, 1981], and from an impossibility result (1-SCP + U-DSIC \( \Rightarrow \) Zero Miner Revenue) [Chung and Shi, 2021] we set the miner revenue to zero. The resulting mechanism is shown as:

\[
a_i(b_i, b_{-i}) = e^{mb_i} \sum_{j=1}^{n} e^{mb_j} \quad (23)
\]

\[
p_i(b_i, b_{-i}) = b_i - \frac{e^{mb_i}}{me^{mb_i}} \ln \left( \frac{\sum_{j=1}^{n} e^{mb_j}}{1 + \sum_{j \neq i} e^{mb_j}} \right) \quad (24)
\]

\[
r(b) = 0. \quad (25)
\]

Although the soft second-price mechanism has zero miner revenue, we can modify it in a way that preserves U-BNIC and 1-SCP properties and yields positive expected miner revenue, as in Section 4.2.

4.1.1 Connection to second-price auction

We notice that when \( m \to +\infty \), the highest bidder will be confirmed. Actually, we can observe that when \( m \to \infty \), and bids are generic (distinct in \((0, 1)\)), the user payment rule converges to the second price auction, i.e.

**Observation 1.** W.l.o.g. we assume \( b(1) > b(2) > \cdots > b(n) \) is a permutation of \( b \), then

\[
\lim_{m \to +\infty} a_i(b_i, b_{-i}) = \begin{cases} 
1, & b_i = b(1) \\
0, & b_i \neq b(1) 
\end{cases} \quad (26)
\]

\[
\lim_{m \to +\infty} p_i(b_i, b_{-i}) = \min\{b_i, b(2)\}. \quad (27)
\]

Then we consider another extreme case when \( m = 0 \). However, the denominator of \( p_i(b_i, b_{-i}) \) is zero, so we consider its limit of \( m \to 0 \). In this case, the mechanism converges to a random-free-allocation mechanism to confirm a transaction uniformly at random and charge no fee, i.e.

**Observation 2.**

\[
\lim_{m \to 0} a_i(b_i, b_{-i}) = \frac{1}{n} \quad (28)
\]

\[
\lim_{m \to 0} p_i(b_i, b_{-i}) = 0. \quad (29)
\]

The proofs of observations above are deferred to Appendix B.1. In fact, we can observe that when \( m \) increases from 0 to \(+\infty\), the mechanism continuously shifts from random free allocation to all-burn second-price auction, while preserving 1-SCP and U-DSIC properties, as proven in Section 4.1.2. When we consider fixed \( m \in (0, +\infty) \), we can modify the mechanism to make it still U-BNIC but with positive expected miner revenue.
4.1.2 Proof for 1-SCP and U-DSIC properties

Before proving the 1-SCP and U-DSIC properties of the particular mechanism, we firstly introduce sufficient conditions for such properties. Straightforwardly, a mechanism is 1-SCP if it is in the best interest of the party of the miner and any user that the user bids her true valuation no matter what other users bid. For U-DSIC, the criterion is that truthful bidding is in the best interest of any miner alone too.

The famous Myerson’s Lemma [Myerson, 1981] characterizes the sufficient and necessary condition for an auction mechanism to be DSIC. Combined with the IR condition which implies \(a_i(0, b_{-i})p_i(0, b_{-i}) = 0\), it implies that for DSIC auction mechanisms, the payment function can be uniquely determined by the allocation function, as Lemma 1.

**Lemma 1 (Myerson’s Lemma).** The mechanism \(M = (a, p, r)\) is U-DSIC if and only if the following conditions are satisfied:

- **Monotone allocation:** \(a_i(\cdot, b_{-i})\) is monotonic non-decreasing.
- **Constrained payment function:**
  \[
  a_i(b_i, b_{-i})p_i(b_i, b_{-i}) = \int_0^{b_i} t \frac{\partial a_i(t, b_{-i})}{\partial t} dt + a_i(0, b_{-i})p_i(0, b_{-i}).
  \]  

From a reduction to Lemma 1, we can also characterize the sufficient and necessary of a mechanism to be 1-SCP:

**Lemma 2.** The mechanism \(M = (a, p, r)\) is 1-SCP if and only if the following conditions are satisfied:

- **Monotone allocation:** \(a_i(\cdot, b_{-i})\) is monotonic non-decreasing.
- **Constrained payment function:**
  \[
  a_i(b_i, b_{-i})p_i(b_i, b_{-i}) - r(b_i, b_{-i}) = \int_0^{b_i} t \frac{\partial a_i(t, b_{-i})}{\partial t} dt + a_i(0, b_{-i})p_i(0, b_{-i}) - r(0, b_{-i}).
  \]  

**Proof.** Consider another mechanism \(M' = (a, p - \frac{r}{b}, 0)\). Since \(M'\) has zero miner revenue, it is 1-SCP if and only if it is U-DSIC.

From Lemma 1, \(M'\) is U-DSIC if and only if the given conditions hold. So \(M'\) is 1-SCP if and only if the conditions hold. Notice that for the same bidding vector \(b\), the miner and user \(i\) have the same total utilities in mechanisms \(M\) and \(M'\). So \(M\) is 1-SCP if and only if the conditions hold.

Now we have verified that our soft second-price mechanism satisfies conditions in Lemmas 1-2, so it is U-DSIC and 1-SCP.

4.2 Our Proposed Mechanism for Block Size 1

We notice that the Soft Second-Price Mechanism is U-DSIC, and thus it is U-BNIC too. Here, our main idea is that when we perturb the payment function in such a way that the expected payment for any user is preserved, the U-BNIC property is also preserved.

Therefore, if \(M = (a, p, 0)\) is a U-DSIC and 1-SCP mechanism, following the technique in Section 3, we can construct a mechanism \(\tilde{M} = (a, \tilde{p}, \tilde{r})\), in which we denote \(\tilde{p}_i(b_i, b_{-i}) = p_i(b_i, b_{-i}) + \frac{b_i(b_i, b_{-i})}{a_i(b_i, b_{-i})}\). Then we observe that

**Observation 3.** \(\tilde{M}\) is U-BNIC if

\[
\mathbb{E}_{b_{-i} \sim \mathcal{V}_{-i}}[\theta_i(b_i, b_{-i})] = 0.
\]  

**Proof.** Because user \(i\)’s expected utility \(\tilde{u}(b_i, b_{-i}; v_i) = u(b_i, b_{-i}; v_i) - \theta(b_i, b_{-i})\), if \(\mathbb{E}_{b_{-i} \sim \mathcal{V}_{-i}}[\theta_i(b_i, b_{-i})]\), then for any bidding vector \(b\) and \(i\)’s valuation \(v_i\), mechanisms \(M\) and \(\tilde{M}\) have the same expected utility

\[
\mathbb{E}_{b_{-i} \sim \mathcal{V}_{-i}}[u(b_i, b_{-i}; v_i)] = \mathbb{E}_{b_{-i} \sim \mathcal{V}_{-i}}[\tilde{u}(b_i, b_{-i}; v_i)].
\]  

As mechanism \(M\) is U-BNIC, it holds that \(\tilde{M}\) is also U-BNIC.
As we have characterized a sufficient condition for U-BNIC, now we consider the condition for 1-SCP. From Lemma 2 we know that for an 1-SCP mechanism \((a, \hat{p}, \tilde{r})\), if we fix \(b_{-i}\), the difference of \(a_i(\cdot, b_{-i})\hat{p}_i(\cdot, b_{-i})\) and \(\tilde{r}_i(\cdot, b_{-i})\) is a constant. Furthermore, since \(a(\cdot, b_{-i})\) is monotonic increasing, if we want \(\hat{M}\) to be 1-SCP, from Lemma 2 we need and only need:

\[
\theta_i(b_i, b_{-i}) - \theta_i(0, b_{-i}) = \tilde{r}(b_i, b_{-i}) - \tilde{r}(0, b_{-i}).
\]  

(34)

When the distributions of all users’ valuations are i.i.d, i.e. \(V = V_1 \times V_2 \times \cdots \times V_n\) and \(\forall V_i\) has identical pdf \(\rho : [0, 1] \rightarrow [0, +\infty)\), we denote

\[
c_{\rho} = \int_0^1 \rho^2(t)dt,
\]

(35)

then we can construct \((\theta, \tilde{r})\) as following, which satisfies (33-34):

\[
\theta_i(b_i, b_{-i}) = -\frac{1}{2} h b_i^2 \left( \frac{\sum_{j \neq i} b_j^2}{c_{\rho}(n-1)} - 1 \right)
\]

(36)

\[
\tilde{r}(b) = \frac{1}{2} h \left( \sum_{i=1}^{n} b_i^2 - \sum_{1 \leq i < j \leq n} b_i^2 b_j^2 \frac{1}{c_{\rho}(n-1)} \right)
\]

(37)

The corresponding mechanism \(\hat{M} = (a, \hat{p}, \tilde{r})\) can be represented as:

\[
a_i(b_i, b_{-i}) = \frac{e^{mb_i}}{\sum_{j=1}^{n} e^{mb_j}}
\]

\[
\hat{p}_i(b_i, b_{-i}) = b_i - \frac{\sum_{j=1}^{n} e^{mb_j}}{me^{mb_i}} \left( \ln \frac{1}{1 + \sum_{j \neq i} e^{mb_j}} + \frac{1}{2} hmb_i^2 \left( \frac{\sum_{j \neq i} b_j^2}{c_{\rho}(n-1)} - 1 \right) \right)
\]

\[
\tilde{r}(b) = \frac{1}{2} h \left( \sum_{i=1}^{n} b_i^2 - \sum_{1 \leq i < j \leq n} b_i^2 b_j^2 \frac{1}{c_{\rho}(n-1)} \right)
\]

and it is U-BNIC and 1-SCP for \(h \in [0, +\infty)\). We can compute that

\[
E_{b \sim \mathcal{V}}[r(b)] = \frac{1}{4} hnc_{\rho} > 0.
\]

(38)

However, we have to be careful to the value of \(h\). Intuitively, the value of \(h\) describes the extent of perturbation from the original U-DSIC mechanism, and when the perturbation is too large, the individual rationality \((p_i(b_i, b_{-i}) \leq b_i)\) and budget feasibility properties may not hold. For best miner revenue, we want to make \(h\) as large as possible while keeping the mechanism feasible. Fortunately, for fixed \(c_{\rho}\), we have an estimation of optimally feasible \(h\) that enables constant approximation ratio of the optimal revenue while preserving IR and BF constraints, as:

**Theorem 1.** For \(n \geq 1\) and \(M = (a, p, 0)\) given by (23-25), let \(m = 1\), we have \(h_* = h_*(n, c_{\rho}) > 0\) such that \(\forall h \in [0, h_*]\), the corresponding mechanism \(\hat{M} = (a, \hat{p}, \tilde{r})\) is UIR and BF, and for \(n \rightarrow +\infty\),

\[
h_*(n, c_{\rho}) = \Omega(c_{\rho}/n).
\]

(39)
Proof. From (24) we know \( p_i(0, b_{-i}) = 0 \). Then for \( n \to \infty \), from Lemma 1 we get:

\[
\begin{align*}
a_i(b_i, b_{-i}) p_i(b_i, b_{-i}) &= \int_0^{b_i} t \frac{\partial a_i(t, b_{-i})}{\partial t} dt \\
&= \int_0^{b_i} t \cdot \frac{e^t \sum_{j \neq i} e^{b_j} b_j}{(t + \sum_{j \neq i} e^{b_j})^2} dt \\
&= \int_0^{b_i} t \frac{\Theta(n)}{\Theta(n^2)} dt \\
&= \Theta(b_i^2/n).
\end{align*}
\]

Therefore, the difference of total collected fee and miner revenue in \( \tilde{M} \) is

\[
\begin{align*}
\sum_{i=1}^n a_i(b_i, b_{-i}) \tilde{p}_i(b_i, b_{-i}) - r(b) &= \sum_{i=1}^n a_i(b_i, b_{-i}) p_i(b_i, b_{-i}) + \sum_{i=1}^n \theta_i(b_i, b_{-i}) - r(b) \\
&= \Theta \left( \frac{\sum_{i=1}^n b_i^2}{n} \right) - \left( \frac{\sum_{1 \leq i \neq j \leq n} b_i^2 b_j^2}{c_\rho(n-1)} - \frac{h}{2} \sum_{i=1}^n b_i^2 \right) \\
&- \frac{1}{2} h \left( \sum_{i=1}^n b_i^2 - \frac{\sum_{1 \leq i \neq j \leq n} b_i^2 b_j^2}{c_\rho(n-1)} \right) \\
&= \Theta \left( \frac{\sum_{i=1}^n b_i^2}{n} \right) - \frac{h}{2c_\rho(n-1)} \sum_{1 \leq i < j \leq n} b_i^2 b_j^2 \\
&\geq \Theta \left( \frac{\sum_{i=1}^n b_i^2}{n} \right) - \sum_{i=1}^n b_i^2 \left( \frac{h}{4c_\rho(n-1)} \sum_{i=1}^n b_i^2 \right) \\
&\geq \Theta \left( \frac{\sum_{i=1}^n b_i^2}{n} \right) - \sum_{i=1}^n b_i^2 \left( \frac{h}{4c_\rho(n-1)} \Omega(n) \right) \\
&= \Theta \left( \frac{\sum_{i=1}^n b_i^2}{n} \right) - \sum_{i=1}^n b_i^2 \cdot O \left( \frac{h}{c_\rho} \right).
\end{align*}
\]

So there exists \( h = \Omega(c_\rho/n) \) to make \( \tilde{M} \) budget feasible.

For user individual rationality,

\[
\begin{align*}
b_i - \tilde{p}_i(b_i, b_{-i}) &= b_i - p_i(b_i, b_{-i}) - \frac{\theta_i(b_i, b_{-i})}{a_i(b_i, b_{-i})} \\
&= \frac{b_i a_i(b_i, b_{-i}) - a_i(b_i, b_{-i}) p_i(b_i, b_{-i})}{a_i(b_i, b_{-i})} \\
&= \frac{b_i \left( a_i(0, b_{-i}) + \int_0^{b_i} \frac{\partial a_i(t, b_{-i})}{\partial t} dt \right) - \int_0^{b_i} t \frac{\partial a_i(t, b_{-i})}{\partial t} dt}{a_i(b_i, b_{-i})} \\
&= \frac{b_i a_i(0, b_{-i}) + \int_0^{b_i} (b_i - t) \frac{\partial a_i(t, b_{-i})}{\partial t} dt + \frac{1}{2} h b_i^2 \left( \sum_{i \neq i} b_j^2 \right) - 1}{a_i(b_i, b_{-i})} \\
&\geq \Theta(b_i/n) + \Theta(b_i^2/n) - \frac{1}{2} h b_i^2 \Theta(1/n).
\end{align*}
\]

so \( \exists h = \Theta(1/n) = \Omega(c_\rho/n) \) to make \( b_i - \tilde{p}_i(b_i, b_{-i}) \geq 0 \).

Therefore, we have shown \( h_*(n, c_\rho) = \Omega(c_\rho/n) \). 

From this result, when we choose \( k = h_* \), we can get an \( \Omega(c^2_\rho) \) expected miner revenue. As the optimal miner revenue is at most \( \max \{ v_i \} \leq 1 \), for fixed distribution (fixed \( c_\rho \)), our mechanism yields a constant-ratio approximation of the optimal miner revenue for \( n \to \infty \).

5 Mechanism for General Block Size \( k \)

In actual blockchains, a block contains multiple transactions. While the soft second-price mechanism for block size 1 has a simple formulation, it is trickier to develop a randomized auction-like mechanism that is associated to second-price auction for general block sizes, as the softmax function does not have a simple generalization as soft-top-k.

However, from Myerson’s Lemma we know that if we manage to derive a soft-top-k function as the allocation rule, as long as the allocation rule is monotone, a unique payment function can be accordingly found.

The choice of \( k \) highest bids can be understood in another way: in the pool of \( n \) users, we choose the user of the highest bid, and remove her from the pool; if we choose the highest bidding user in the remaining pool, it is the second highest bidder. Therefore, if we do the choose-and-remove \( k \) times, we find the top-k bidders to be confirmed.

Naturally, we can substitute the choice of each step with the logit choice rule, in which way we develop our allocation rule for general block size \( k \).

5.1 Allocation Rule: Weighted Sampling without Replacement

In this section, we assume the bidding vector \( b \) and block size \( k \) are fixed, and for bidder \( i \in B = [n] \), her weight \( w_i = e^{mb_i} \). Now we compute \( a_i \), the probability user \( i \) has her transaction confirmed.

Denote \( \delta_t(i) \) as the probability that user \( i \) in the \( t \)-th round and \( W = \sum_{i=1}^{n} w_i \), then \( \delta_t(i) = \frac{w_i}{W} \). For fixed \( t \geq 2 \), we consider \( j = (j_1, \ldots, j_t) \) as the sampling vector describing the outcome of the weighted sampling without replacement in the first \( t \) rounds, in which \( j_s \) is the user confirmed in the \( s \)-th round, and denote \( \mathcal{J} \) as the distribution of \( j \). Therefore, we get

\[
\delta_t(i) = \Pr_{j \sim \mathcal{J}} [j_t = i].
\]

Denote \( J_t(i) = \{ j \text{ is a sampling vector : } j_t = i \} \), then \( \forall j \in J_t(i) \), denote \( \delta_t(i; j) = \Pr_{u \sim \mathcal{J}} [u = j] \), then we have

\[
\delta_t(i) = \sum_{j \in J_t(i)} \delta_t(i; j).
\]

Note that \( \delta_t(i; j) \) denotes the probability that the sampling outcome is \( (j_1, j_2, \ldots, j_{t-1}, i) \), thus the probability is

\[
\delta_t(i; j) = \frac{w_{j_1}}{W} \cdot \frac{w_{j_2}}{W - w_{j_1}} \cdots \frac{w_{j_{t-1}}}{W - w_{j_1} - \cdots - w_{j_{t-2}}} \cdot \frac{w_i}{W - w_{j_1} - \cdots - w_{j_{t-1}}}.
\]

Since know that

\[
a_t = \sum_{i=1}^{k} \delta_t(i),
\]

The allocation rule can be computed from Equations (56-58), and the payment rule can be computed from the allocation rule \( \tilde{p} \) in the same way as in Section 4.2.

5.2 Estimation of \( h \): How Much Revenue Can Miner Get?

For the case of general block size \( k \), we also have a similar result on the value of \( h \), which additionally requires the number of users \( n \) to be at least \( \left( \frac{\rho}{\Theta(1)} \right) k \approx 1.582k \), but still guarantees that the expected miner revenue is an asymptotic constant-fraction approximation of the optimal revenue. The formal result is shown below:
Theorem 2. For block size $k$ and $\lambda_0 > \frac{c_p}{c_k} \approx 1.582$, there exists $f(\lambda_0) > 0$, and when we let $m = \min \left\{ \frac{1}{2} \ln \frac{1}{\rho k}, 1 \right\}$, then for any $n \geq \lambda_0 k$, there exists

$$h_* = \Omega \left( f(\lambda_0) e^{\frac{k}{\rho m}} \right)$$

s.t. $\forall h \in [0, h_*]$, the mechanism $\tilde{M} = (a, \tilde{p}, \tilde{r})$ is U-BNIC, 1-SCP, UIR and BF, and can yield an expected miner revenue of

$$E_{b \sim V}[\tilde{r}(b)] = \Omega \left( f(\lambda_0) c_k^2 k \right),$$

which is $\Theta(k)$ for fixed $(c_p, \lambda_0)$.

The proof of Theorem 2 is technically more complicated and deferred to Appendix B.2. Because the optimal revenue for block size $k$ is at most $k \max \{v_i\} \leq k$, for fixed $c_p > 0$ and $\lambda_0 > \frac{c_p}{c_k}$, our mechanism yields a constant-factor approximation of the optimal revenue as long as $n \rightarrow \infty$ and $n > \lambda_0 k$.

6 Is Burning Necessary?

In our mechanism, as well as many other TFMs with incentive guarantees such as EIP-1559 [Leonardos et al., 2021] and the paper by [Chung and Shi, 2021], there exists burning in the transaction fee, which means that part of the total transaction fee is not awarded to the miner but “wasted”, removed from the system. While the burning makes the TFM more flexible by decoupling payments on miner’s and users’ sides, it indeed incurs inefficiency in the utilization of funds and should be avoided or minimized if possible. Therefore, while the burning is prevalent in design of blockchain TFMs, we are interested in the theoretical question: is the burning really necessary, even with the minimal incentive (U-BNIC) and collusion-proof (1-SCP) requirements?

Here, we have a result in an interesting form. Informally, the block resource and transaction fees cannot be perfectly utilized simultaneously. In other words, under mild assumptions, TFMs which have constant block size and satisfy U-BNIC and 1-SCP properties must burn some of the collected transaction fees.

While we expect the transaction “fee” a user pays should be non-negative, it is okay as long as the users have a non-negative expected payment to prevent users from submitting transactions to gain money out of nothing, which is guaranteed in our mechanisms for both $k = 1$ and general $k$ (as (32) is satisfied). However, as UIR requires the payment of zero-valuation user to be no greater than zero, we deduce that the payment of zero-valuation user to be always zero (rather than negative). Therefore, we need a condition:

**Definition 6** (No-free-lunch (NFL)). We call a TFM $(a, \tilde{p}, \tilde{r})$ NFL when the payment of a zero-valuation user is always zero, i.e.

$$a_i(0, b_{-i})\tilde{p}_i(0, b_{-i}) = 0.$$  (61)

Besides, while the monotonicity of allocation means the a user would not decrease her allocation if she increases her bid, in a wide scope of auction mechanisms used in practice (first-price, second-price, burning-price-with reserve [Myerson, 1981], EIP-1559, soft-second-price in [Chung and Shi, 2021], etc.), we also expect the auction to be a competition among bidders on a limited number of goods, thus when a bidder increases her bid, other bidders’ allocations would not increase. Therefore, we define Competitiveness of TFMs as:

**Definition 7** (Competitiveness). We call a TFM $(a, \tilde{p}, \tilde{r})$ Competitive when $\forall b_j > b'_j$,

$$a_i(b_i, b_j, b_{-ij}) \leq a_i(b_i, b'_j, b_{-ij}).$$  (62)

The definition of Budget Feasibility (BF) only requires the miner revenue no greater than total transaction fees collected. If the TFM has no burning, then the equality should hold. We call this case Strongly Budget Feasible (StrongBF):

**Definition 8** (Strong Budget Feasibility). We call a TFM $(a, \tilde{p}, \tilde{r})$ Strongly Budget Feasible when

$$\sum_{i=1}^{n} a_i(b_i, b_{-i}) \cdot \tilde{p}_i(b_i, b_{-i}) = \tilde{r}(b).$$  (63)

With assumptions on NFL and Competitiveness, we have proven the necessity of waste in either blocksize or transaction fees for non-trivial TFMs. Formally:
Theorem 3. Any TFM $\tilde{M} = (a, \tilde{p}, \tilde{r})$ which has constant blocksize $k$ and is U-BNIC, 1-SCP, NFL, Competitive, UIR and StrongBF has constant allocation rule and zero expected miner revenue, i.e.

$$a_i(b_i, b_{-i}) = \frac{k}{n}$$

(64)

$$E_{b \sim V}[\tilde{r}(b)] = 0.$$  

(65)

Interpretation. Our theorem only requires constant allocation rule and zero expected miner revenue, but does not require the payments $\tilde{p}(b_i, b_{-i})$ and revenue to be always zero. This is because U-BNIC only requires expectation properties of users’ payments.

Here we construct a slightly unrigorous example: let $\mathcal{H}(b_i, b_{-i})$ be a “hash” function uniformly distributed in $[0, 1]$, and let $\tilde{p}(b_i, b_{-i}) = \frac{n}{i}(2\mathcal{H}(b_i, b_{-i}) - 1)$, while all (positive or negative) payments go to the miner. Then when $n \to \infty$, each individual user on expectation pays approximately zero, so the mechanism is “nearly U-BNIC”, and one user changing her bid does not significantly affect the expected payments of other users, so the mechanism is “nearly 1-SCP”. Nevertheless, whether a mechanism with non-zero payments does exist, it can still be considered trivial because the bids make no effect on allocation or payments, and the miner essentially gets nothing—it is still equivalent to random drawing in the long run.

Proof. Denote $k$ as the constant block size, then from symmetry we know that for $\forall i$,

$$a_i(0|n) = \frac{k}{n}$$

(66)

We construct an auxiliary TFM $M = (a, p, r)$ to be U-DSIC with the same allocation rule as $\tilde{M}$. Here we define

$$p_i(b_i, b_{-i}) = \begin{cases} \int_{\mathcal{H}(b_i, b_{-i})} \frac{\partial b_i}{a_i(b_i, b_{-i})} \, dt, & a_i(b_i, b_{-i}) > 0 \\ 0, & a_i(b_i, b_{-i}) = 0. \end{cases}$$

(67)

$$r(b) = 0.$$  

(68)

From Lemmas 1-2, the mechanism $M = (a, p, r)$ is indeed U-DSIC (thus also U-BNIC) and 1-SCP. Define

$$\theta_i(b_i, b_{-i}) = a_i(b_i, b_{-i})(\tilde{p}_i(b_i, b_{-i}) - p_i(b_i, b_{-i})),$$

(69)

then from NFL we get

$$\theta_i(0, b_{-i}) = 0.$$  

(70)

Recall that over the distribution of $b_{-i} \sim V_{-i}$, the expected utility of user $i$ for bidding $b_i$ is

$$\mathbb{E}_{b_{-i} \sim V_{-i}}[\tilde{u}(b_i, b_{-i}; v_i)] = \mathbb{E}_{b_{-i} \sim V_{-i}}[\tilde{u}(b_i, b_{-i}; v_i) - \theta_i(b_i, b_{-i})].$$

(71)

Therefore,

$$\frac{\partial \mathbb{E}_{b_{-i} \sim V_{-i}}[\tilde{u}(b_i, b_{-i}; v_i)]}{\partial b_i} = \frac{\partial \mathbb{E}_{b_{-i} \sim V_{-i}}[u(b_i, b_{-i}; v_i)]}{\partial b_i} - \frac{\partial \mathbb{E}_{b_{-i} \sim V_{-i}}[\theta_i(b_i, b_{-i})]}{\partial b_i}.\quad (72)$$

Because mechanisms $M$ and $\tilde{M}$ are both U-BNIC, it holds that

$$\frac{\partial \mathbb{E}_{b_{-i} \sim V_{-i}}[\tilde{u}(b_i, b_{-i}; v_i)]}{\partial b_i} \bigg|_{b_i=v_i} = \frac{\partial \mathbb{E}_{b_{-i} \sim V_{-i}}[u(b_i, b_{-i}; v_i)]}{\partial b_i} \bigg|_{b_i=v_i} = 0$$

(73)

We deduce

$$\frac{\partial \mathbb{E}_{b_{-i} \sim V_{-i}}[\theta_i(b_i, b_{-i})]}{\partial b_i} \bigg|_{b_i=v_i} = 0,$$

(74)

i.e.

$$\frac{\partial \mathbb{E}_{b_{-i} \sim V_{-i}}[\theta_i(b_i, b_{-i})]}{\partial b_i} = 0.$$  

(75)
From (70) we get that $\mathbb{E}_{b_{-i} \sim V_{-i}}[\theta_i(0, b_{-i})] = 0$, so
\[
\mathbb{E}_{b_{-i} \sim V_{-i}}[\theta_i(b_i, b_{-i})] = 0. \tag{76}
\]
From Strong Budget Feasibility, we get
\[
\hat{r}(b) = \sum_{i=1}^{n} (a_i(b_i, b_{-i})p_i(b_i, b_{-i}) + \theta_i(b_i, b_{-i})). \tag{77}
\]
Let $b_{-1} = 0$, from NFL we get $\forall i \neq 1$, $a_i(b_i, b_{-i})p_i(b_i, b_{-i}) = 0$ and $\theta_i(b_i, b_{-i}) = 0$. Therefore, we get
\[
\hat{r}(b_1, 0) = a_1(b_1, 0)p_1(b_1, 0) + \theta_1(b_1, 0). \tag{78}
\]
Since $\tilde{M}$ is 1-SCP, from Lemma 2 we have
\[
\hat{r}(b_1, 0) - \hat{r}(0, 0) = \theta_1(b_1, 0) - \theta_1(0, 0). \tag{79}
\]
Let $b_1 = 0$ in (78) we get $\hat{r}(0, 0) = 0$, and from (70) we get $\theta_1(0, 0) = 0$. Therefore,
\[
\hat{r}(b_1, 0) = \theta_1(b_1, 0). \tag{80}
\]
Combine (78) and (80), it holds that
\[
a_1(b_1, 0)p_1(b_1, 0) = 0. \tag{81}
\]
From the definition of $p_i(\cdot)$ in (67), we get:
\[
\int_0^{b_i} \frac{\partial a_i(t, 0)}{\partial t} dt = 0. \tag{82}
\]
Since $\tilde{M}$ is 1-SCP, from Lemma 2, $a_1(\cdot, 0)$ is monotonic non-decreasing, so $\frac{\partial a_i(t, 0)}{\partial t} \geq 0$, and therefore we deduce that $\frac{\partial a_i(t, 0)}{\partial t} = 0$. Hence,
\[
a_1(b_1, 0) = \frac{k}{n}. \tag{83}
\]
Now we use induction. We assume that for given $s = s_0(\leq n - 1)$, $\forall i \leq s$, $a_i(b_1, \cdots, b_s, 0, \cdots, 0) = \frac{k}{n}$, and prove that it also holds for $s = s_0 + 1$.

Actually, from symmetry and block size $k$, we can get
\[
a_{s+1}(b_1, \cdots, b_s, 0, 0, \cdots, 0) = \frac{k}{n}. \tag{84}
\]
Furthermore, from symmetry and (83), we get
\[
a_{s+1}(0, \cdots, 0, b_{s+1}, 0, \cdots, 0) = \frac{k}{n}. \tag{85}
\]
From competitiveness, it holds that $a_{s+1}(b_1, \cdots, b_s, b_{s+1}, 0, \cdots, 0) \leq a_{s+1}(0, \cdots, 0, b_{s+1}, 0, \cdots, 0)$, so
\[
a_{s+1}(b_1, \cdots, b_s, b_{s+1}, 0, \cdots, 0) \leq \frac{k}{n}. \tag{86}
\]
On the other hand, from Lemma 2, $a_{s+1}$ is monotonic non-decreasing with regard to $b_{s+1}$, so $a_{s+1}(b_1, \cdots, b_s, b_{s+1}, 0, \cdots, 0) \geq a_{s+1}(b_1, \cdots, b_s, 0, 0, \cdots, 0)$. Combined with (84), we get
\[
a_{s+1}(b_1, \cdots, b_s, b_{s+1}, 0, \cdots, 0) \geq \frac{k}{n}. \tag{87}
\]
Therefore, $a_{s+1}(b_1, \cdots, b_s, b_{s+1}, 0, \cdots, 0) = \frac{k}{n}$ for $\forall b_{s+1} \in [0, 1]$. By symmetry, it also holds for any index $i \leq s + 1$. 

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By induction, we get that in general,

$$a_i(b_i, b_{-i}) = \frac{k}{n}, \forall b, \forall i. \quad (88)$$

This proves (64).

Further from (67), we get $$p_i(b_i, b_{-i}) \equiv 0.$$

From (69) and by StrongBF and symmetry, we get

$$E_{b \sim V}[\tilde{r}(b)] = E_{b \sim V}\left[\sum_{i=1}^{n} (a_i(b_i, b_{-i})p_i(b_i, b_{-i}) + \theta_i(b_i, b_{-i}))\right] \quad (90)$$

$$= nE_{b \sim V}[a_1(b_1, b_{-1})p_1(b_1, b_{-1}) + \theta_1(b_1, b_{-1})] \quad (91)$$

$$= nE_{b \sim V}[\theta_1(b_1, b_{-1})] \quad (92)$$

According to (76) we get $$E_{b_{-1} \sim V_{-1}}[\theta_1(b_1, b_{-1})] = 0,$$ so

$$E_{b \sim V}[\tilde{r}(b)] = 0. \quad (94)$$

This proves (65).

7 Discussion

This paper has provided a mechanism which satisfies reasonable bidder-side truthfulness and collusion-proof properties and yields a constant fraction of optimal miner revenue for $$n \to \infty$$ under i.i.d. bounded valuation distributions, but it lacks truthfulness guarantees on the miner’s side. As suggested by [Daskalakis et al., 2020], in real-world it may be viable to introduce cryptographic and/or penalty rules to prevent the miner from being dishonest, like the slashing mechanism in Ethereum [Cassez et al., 2022] does. In general, the problem to facilitate truthfulness in blockchain mechanisms is quite open, as blockchain is indeed an interdisciplinary area in which a wide range of methods can be adopted to resolve a practical issue.

Another open problem is that while we set the number of bidders as a fixed number $$n$$, in practical applications it is actually a random variable. It is also an interesting direction in our future research to extend the setting in which the number of users also obeys a distribution.

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Appendix

A Impossibility Result

In this section, we propose an impossibility result that under certain conditions, any deterministic TFM which is U-BNIC and 1-SCP cannot have positive miner revenue. Here we additionally introduce several notions. Although this impossibility does not fully rule out deterministic mechanisms, it does motivate us to introduce randomness into our main mechanism.

**Deterministic.** When bids are distinct, the outcome of the auction is deterministic, i.e. \( a_i \in \{0, 1\} \).

**Symmetric.** When we swap the bids of two users, their allocations and payments are exactly swapped.

**Continuous.** \( p \) and \( r \) are continuous functions of \( b \), and \( V \) has bounded, strictly positive PDF on a simply connected support \( \text{dom}(V) \).

**Strongly Monotone.** If we raise the bid of bidder \( i \) while leave other bids unchanged, \( a_i, p_i \) do not decrease and \( a_j (\forall j \neq i) \) does not increase.

Theorem 4. For all deterministic, symmetric, continuous, strongly monotone, user-individually-rational and budget-feasible TFM, if \( 0 \in V \), then U-BNIC and 1-SCP implies non-positive miner revenue.

A.1 Proof

**Proof sketch.** To prove the non-positive-miner-revenue property of all satisfying mechanisms, we first show that all satisfying mechanisms must obey certain restrictive conditions, as the payment (Sec. A.1.1) and revenue (Sec. A.1.2) rules both must follow corresponding closed-form formulas; then we show that this type of mechanisms have non-positive miner revenue.

In this section, we introduce the \( \delta \)-function with \( \int_{-\epsilon}^{\epsilon} \delta(t) dt = 1, \quad \forall \epsilon > 0. \) (95)

We assume there exists a transaction fee mechanism \( M_0(a, p, r) \) that satisfy all conditions.

A.1.1 Pinning down the payment rule

From definition we know that if \( M_0 \) is BNIC, then

\[
\frac{\partial E_{V_{-i} \sim V_{-i}} [u_i(b_i, v_i, V_{-i})]}{\partial b_i} \bigg|_{b_i = v_i} = 0, \quad \forall v_i
\]

i.e.

\[
\int_{V_{-i}} (v_i - p_i(v_i, V_{-i})) \frac{\partial a_i(v_i, V_{-i})}{\partial v_i} - a_i(v_i, V_{-i}) \frac{\partial p_i(v_i, V_{-i})}{\partial v_i} \rho_{-i}(v_{-i}) dv_{-i} = 0,
\] (97)

in which \( \rho_{-i}(\cdot) \) is the pdf of \( V_{-i} \).

For fixed \( v_{-i} \), since the mechanism is deterministic, we have that \( a_i(\cdot, v_{-i}) \in \{0, 1\} \) almost everywhere. Additionally because \( a_i(\cdot, v_{-i}) \) is monotonic increasing, we have

\[
a_i(v_i, v_{-i}) = \begin{cases} 
0, & v_i < \theta(v_{-i}) \\
1, & v_i > \theta(v_{-i}) 
\end{cases}
\] (98)

in which \( \theta(v_{-i}) \) is a constant for fixed \( v_{-i} \). Therefore,

\[
\frac{\partial a_i(v_i, v_{-i})}{\partial v_i} = \delta(v_i - \theta(v_{-i})).
\] (99)

Now we have a lemma:

**Lemma 3.** For \( \forall v_{-i} \),

\[
p_i(\theta(v_{-i}), v_{-i}) = \theta(v_{-i}).
\] (100)
Lemma 4.\textbf{ }If \( p_i(\theta(v_{-i}), v_{-i}) > \theta(v_{-i}) \), let \( t = p_i(\theta(v_{-i}), v_{-i}) - \theta(v_{-i}) \). Then by continuity, there exists a small \( \epsilon > 0 \) s.t. \( p_i(\theta(v_{-i}) + \epsilon, v_{-i}) > \theta(v_{-i}) + \frac{\epsilon}{2} \) and \( a_i(\theta(v_{-i}) + \epsilon, v_{-i}) = 1 \), and the user \( i \) would have negative utility. In this scenario, the miner would want to collude with user \( i \) and ask him to change his bid to \( \theta(v_{-i}) - \epsilon \), so that user \( i \) would now have 0 utility. But by continuity, the change of the miner’s revenue is arbitrarily small, increasing their total utility. So the 1-SCP property is violated.

If \( p_i(\theta(v_{-i}), v_{-i}) < \theta(v_{-i}) \), similarly there exists a scenario where user \( i \) has valuation \( \theta(v_{-i}) - \epsilon \) but the miner would want to let her bid \( \theta(v_{-i}) + \epsilon \) instead, also violating 1-SCP.

Therefore, it must hold that \( p_i(\theta(v_{-i}), v_{-i}) = \theta(v_{-i}) \).

\[ \Box \]

From Lemma 3 we have

\[
\int_{v_{-i}} \left( (v_i - p_i(v_i, v_{-i})) \frac{\partial a_i(v_i, v_{-i})}{\partial v_i} \right) v_{-i}(v_{-i}) d\nu_{-i} = 0, \tag{101}
\]

so

\[
\int_{v_{-i}} \left( a_i(v_i, v_{-i}) \frac{\partial p_i(v_i, v_{-i})}{\partial v_i} \right) v_{-i}(v_{-i}) d\nu_{-i} = 0. \tag{102}
\]

Since monotonicity implies \( \frac{\partial p_i(v_i, v_{-i})}{\partial v_i} \geq 0 \), we know that \( \forall v_i > \theta(v_{-i}), \frac{\partial p_i(v_i, v_{-i})}{\partial v_i} = 0 \). Therefore,

\[
\forall b_i > \theta(v_{-i}), \epsilon > 0, \quad p_i(b_i, v_{-i}) = p_i(\theta(v_{-i}) + \epsilon, v_{-i}). \tag{103}
\]

Combined with Lemma 3, from continuity we get

\[
\forall b_i \geq \theta(v_{-i}), \quad p_i(b_i, v_{-i}) = \theta(v_{-i}). \tag{104}
\]

A.1.2 Pinning down the miner revenue rule

In this part, we mainly use the 1-SCP property to prove that the miner revenue is a constant with regard to any user. To show this, we prove a lemma:

Lemma 4. \textbf{If } \( v_i \neq \theta(v_{-i}) \), then \( \frac{\partial r(v_i, v_{-i})}{\partial v_i} = 0 \).

\textbf{Proof. } We recall that the total utility of the miner and user \( i \) is

\[
C_i(b_i, v_i, v_{-i}) = a_i(b_i, v_{-i}) \cdot (v_i - p_i(b_i, v_{-i})) + r(b_i, v_{-i}). \tag{105}
\]

From 1-SCP we know that

\[
0 = \left. \frac{\partial C_i(b_i, v_i, v_{-i})}{\partial b_i} \right|_{b_i = v_i} \tag{106}
\]

\[
= \left( (v_i - p_i(v_i, v_{-i})) \frac{\partial a_i(v_i, v_{-i})}{\partial v_i} - a_i(v_i, v_{-i}) \frac{\partial p_i(v_i, v_{-i})}{\partial v_i} \right) + \frac{\partial r(v_i, v_{-i})}{\partial v_i}. \tag{107}
\]

From (104) we know \( a_i(v_i, v_{-i}) \frac{\partial p_i(v_i, v_{-i})}{\partial v_i} \equiv 0 \), and from (99) we know \( v_i \neq \theta(v_{-i}) \Rightarrow \frac{\partial a_i(v_i, v_{-i})}{\partial v_i} = 0 \). So we deduce

\[
v_i \neq \theta(v_{-i}) \Rightarrow \frac{\partial r(v_i, v_{-i})}{\partial v_i} = 0. \tag{108}
\]

\[ \Box \]
Because the continuity condition guarantees \( r(b) \) is a continuous function of \( b \), from Lemma 4 we know that for fixed \( v_{-i} \), \( r(\cdot, v_{-i}) \) is a constant, hence

\[
 r(v_i, v_{-i}) = r(0, v_{-i}).  \tag{109}
\]

By iteratively apply (109) to all components of \( v \), we get

\[
 r(v) = r(0).  \tag{110}
\]

We notice that from UIR,

\[
 r(0) \leq \sum_{i=1}^{n} a_i(0) p_i(0) \leq \sum_{i=1}^{n} a_i(0) \cdot 0 = 0. \tag{111}
\]

Therefore, we have

\[
 r(v) \leq 0, \quad \forall v. \tag{114}
\]

Here we prove Theorem 4.

\[ \textbf{Omitted Proofs} \]

\textit{B.1 Proofs of Observations of Soft Second-Price Mechanism}

\textit{B.1.1 Proof of Observation 1}

We can obviously see (26). Now we prove (27).

We firstly consider the case when \( b_i = b(1) \). Notice that when \( m \to \infty \), \( \sum_{j=1}^{n} e^{mb_j} \sim e^{mb_{(1)}} \), and \( \ln \frac{e^{mb_{(1)}}}{e^{mb_{(2)}}} = m(b_{(1)} - b_{(2)}) \). Therefore,

\[
 \lim_{m \to +\infty} p_i(b_i, b_{-i}) = \lim_{m \to +\infty} \left( b_{(1)} - \frac{1}{m} \cdot m(b_{(1)} - b_{(2)}) \right) = b_{(2)}.  \tag{115}
\]

Then we consider the case of \( 0 < b_i < b(1) \). In this case, we have \( \sum_{j=1}^{n} e^{mb_j} \sim e^{mb_{(1)}} \) and

\[
 \ln \frac{\sum_{j=1}^{n} e^{mb_j}}{1 + \sum_{j \neq i} e^{mb_j}} = \ln \left( 1 + \frac{e^{mb_i} - 1}{1 + \sum_{j \neq i} e^{mb_j}} \right) \sim \ln \left( 1 + \frac{e^{mb_i}}{e^{mb_{(1)}}} \right) \sim \frac{1}{e^{b_{(1)} - b_i}}, \tag{117}
\]

Therefore, \( p_i(b_i, b_{-i}) \sim b_i - \frac{1}{m} \), thus

\[
 \lim_{m \to +\infty} p_i(b_i, b_{-i}) = \lim_{m \to +\infty} \left( b_i - \frac{1}{m} \right) = b_i.  \tag{120}
\]

In conclusion we have (27).
B.1.2 Proof of Observation 2

We also only prove (29). For fixed \( b \) and \( m \to 0 \), we have

\[
p_i(b_i, b_{-i}) = b_i - \frac{\sum_{j=1}^{m} e^{mb_j}}{m} \cdot \ln \frac{\sum_{j=1}^{m} e^{mb_j}}{1 + \sum_{j \neq i} e^{mb_j}}.
\]

(122)

\[
= b_i - \frac{\sum_{j=1}^{m} (1 + mb_j + O(m^2))}{m(1 + mb_i + O(m^2))} \cdot \ln \frac{\sum_{j=1}^{m} (1 + mb_j + O(m^2))}{1 + \sum_{j \neq i} (1 + mb_j + O(m^2))}.
\]

(123)

\[
= b_i - \frac{n + m \sum_{j=1}^{n} b_j + O(m^2)}{m(1 + mb_i + O(m^2))} \cdot \ln \left( 1 + \frac{mb_i + O(m^2)}{n + m \sum_{j \neq i} b_j + O(m^2)} \right).
\]

(124)

\[
= b_i - \frac{n}{m} (1 + o(1)) \cdot \frac{mb_i}{n} (1 + o(1))
\]

(125)

\[
= o(1) b_i.
\]

(126)

Therefore,

\[
\lim_{m \to 0} p_i(b_i, b_{-i}) = 0.
\]

(127)

B.2 Proof of Theorem 2

From Lemma 1, similar to the case of block size 1, we essentially need to derive a lower bound on \( \frac{\partial \delta_i(t; b_{-i})}{\partial t} \), in order to lower bound the total payment. Therefore, we only need to analyze the partial derivative of \( \delta_i(i; j) \) on \( w_i \).

When we fix \( j \) and \( b_{-i} \) (i.e. \( w_{-i} \)), we can regard \( \delta_i(i; j) \) as a function of \( w_i \). Here we make a notation of \( X_s \) for \( 0 \leq s \leq k - 1 \) as

\[
X_s = W - w_i - \sum_{z=1}^{s} w_j_z,
\]

(128)

then \( X_s \) is a constant.

From (57) we get (note that \( W - \sum_{z=1}^{s} w_j_z = X_s + w_i \))

\[
\frac{\partial \delta_i(i; j)}{\partial w_i} = \left( \prod_{s=1}^{t-1} w_{j_s} \right) \cdot \frac{\partial}{\partial w_i} \frac{w_i}{\prod_{s=0}^{t-1} (X_s + w_i)}.
\]

(129)

and

\[
\frac{\partial}{\partial w_i} \frac{w_i}{\prod_{s=0}^{t-1} (X_s + w_i)} = \frac{\partial}{\partial w_i} \left( w_i \cdot \prod_{s=0}^{t-1} \frac{1}{X_s + w_i} \right)
\]

(130)

\[
= \prod_{s=0}^{t-1} \frac{1}{X_s + w_i} + w_i \cdot \frac{\partial}{\partial w_i} \prod_{s=0}^{t-1} \frac{1}{X_s + w_i}
\]

(131)

\[
= \prod_{s=0}^{t-1} \frac{1}{X_s + w_i} - w_i \cdot \left( \sum_{s=0}^{t-1} \frac{1}{X_s + w_i} \right) \cdot \left( \prod_{s=0}^{t-1} \frac{1}{X_s + w_i} \right)
\]

(132)

\[
= \left( \prod_{s=0}^{t-1} \frac{1}{X_s + w_i} \right) \cdot \left( 1 - w_i \sum_{s=0}^{t-1} \frac{1}{X_s + w_i} \right)
\]

(133)

Notice that \( X_s + w_i \) is a sum of \( (n - s) \) weights, each one no less than 1, so \( \frac{1}{X_s + w_i} \leq \frac{1}{n-s} \leq \ln \frac{n-s}{n-s-1} \), and \( w_i = e^{mb_i} \leq e^m \). Therefore,
\[
1 - w_i \sum_{s=0}^{t-1} \frac{1}{X_s + w_i} \geq 1 - e^m \sum_{s=0}^{t-1} \ln \left( \frac{n - s}{n - s - 1} \right) = 1 - e^m \ln \left( \frac{n}{n - t} \right),
\]

Denote
\[
D(m, \lambda) = 1 - e^m \ln \frac{\lambda}{\lambda - 1},
\]
then \( \forall \frac{m}{n} < \frac{e}{e-1}, \exists m > 0 \text{ s.t. } D \left( m, \frac{m}{n} \right) > 0 \).

Therefore, from (129) we have
\[
\frac{\partial \delta_i(j)}{\partial w_i} \geq D \left( m, \frac{m}{n} \right) \left( \prod_{s=1}^{t-1} w_j \right) \left( \prod_{s=0}^{t-1} \frac{1}{X_s + w_i} \right) = D \left( m, \frac{m}{n} \right) \cdot \frac{w_j}{X_0 + w_i} \cdot \frac{w_j}{X_1 + w_i} \cdots \frac{w_{j_{t-1}}}{X_{t-2} + w_i} \cdot \frac{1}{X_{t-1} + w_i}.
\]

We notice that \( \frac{w_j}{X_0 + w_i} \cdot \frac{w_j}{X_1 + w_i} \cdots \frac{w_{j_{t-1}}}{X_{t-2} + w_i} \) is just the probability that the sampling outcome of the first \( t-1 \) rounds are \( (j_1, j_2, \cdots, j_{t-1}) \), denoted as \( P(j_{t-1}) \). Furthermore, from \( X_{t-1} + w_i \leq e^m \cdot n \), we have
\[
\frac{\partial \delta_i(j)}{\partial w_i} \geq \frac{D \left( m, \frac{m}{n} \right)}{e^m n} P(j_{t-1}).
\]

Therefore from (56):
\[
\frac{\partial \delta_i(i)}{\partial w_i} = \sum_{j \in J_i} \frac{\partial \delta_i(j)}{\partial w_i} \geq \frac{D \left( m, \frac{m}{n} \right)}{e^m n} \sum_{j \in J_i} P(j_{t-1}).
\]

For \( j \in J_i \), we observe that \( j_{t-1} \) iterates through all \( (t-1) \)-permutations of \([n]\) that does not contain element \( i \). Therefore, \( \sum_{j \in J_i} P(j_{t-1}) \) is the probability that \( i \) is not chosen in the first \( (t-1) \) rounds.

To compute the probability that \( i \) is not chosen in the first \( (t-1) \) rounds, we consider each round. In each round, there are at least \( (n-k) \) users each with weight at least 1, and user \( i \) has weight at most \( e^m \), so \( i \) is chosen with probability at most \( \frac{e^m}{n-k} \).

Therefore for \( t \) rounds, the probability that \( i \) is not ever chosen is at most \( \left( 1 - \frac{e^m}{n-k} \right)^t \geq \left( 1 - \frac{e^m}{n-k} \right)^k = (1 - o(1)) e^{-\frac{m k}{n-k}} \).

That implies:
\[
\frac{\partial \delta_i(i)}{\partial w_i} \geq (1 - o(1)) \frac{D \left( m, \frac{m}{n} \right)}{e^m n} e^{-\frac{m k}{n-k}},
\]
so
\[
\frac{\partial a_i(b_i, b_{-i})}{\partial t} = \frac{\partial w_i}{\partial b_i} \cdot \frac{\partial a_i(b_i, b_{-i})}{\partial w_i}
\]
(143)

\[
= me^{mb_i} \cdot \sum_{t=1}^{k} \frac{\partial \delta_t(i)}{\partial w_i}
\]
(144)

\[
\geq me^{mb_i} \cdot \sum_{t=1}^{k} \left( 1 - o(1) \right) \frac{D(m, \frac{n}{t})}{e^{m \frac{n}{t}}} e^{-\frac{m}{t-n}}
\]
(145)

\[
= \frac{k}{n} \left( 1 - o(1) \right) me^{mb_i} \frac{D(m, \frac{n}{t})}{e^{m \frac{n}{t}}} e^{-\frac{m}{t-n}}
\]
(146)

For any fixed \( \lambda_0 > \frac{e}{e-1} \), let \( \lambda = \frac{n}{t} \). If \( \lambda \geq \lambda_0 \), let

\[
m = m_{\#}(\lambda_0) = \min \left\{ \frac{1}{2} \ln \frac{1}{\ln \lambda_0 - 1}, 1 \right\}
\]
(147)

be a constant. Because \( m_{\#}(\cdot) \) and \( D(m, \cdot) \) are non-decreasing, we have

\[
\frac{\partial a_i(b_i, b_{-i})}{\partial t} \geq \frac{k}{n} \left( 1 - o(1) \right) me^{mb_i} \frac{D(m, \frac{n}{t})}{e^{m \frac{n}{t}}} e^{-\frac{m}{t-n}}
\]
(148)

\[
\geq \frac{k}{n} \left( 1 - o(1) \right) m \frac{D(m, \lambda_0)}{e} e^{-\frac{m}{t-n}}
\]
(149)

Because \( m, \lambda_0, D(m, \lambda_0) \) are all positive constants, we get

\[
\frac{\partial a_i(b_i, b_{-i})}{\partial t} \geq \frac{k}{n} f(\lambda_0)(1 - o(1)).
\]
(150)

Therefore, from Lemma 1 and \( p_i(0, b_{-i}) = 0 \), we get

\[
a_i(b_i, b_{-i})p_i(b_i, b_{-i}) = \int_0^{b_i} t \frac{\partial a_i(t, b_{-i})}{\partial t} dt
\]
(151)

\[
\geq \int_0^{b_i} t \frac{k}{n} f(\lambda_0)(1 - o(1)) dt
\]
(152)

\[
= f(\lambda_0) \Theta \left( \frac{k}{n} b_i^2 \right).
\]
(153)

Then, when we let \( \tilde{p}_i(b_i, b_{-i}) = p_i(b_i, b_{-i}) + \frac{\theta_i(b_i, b_{-i})}{a_i(b_i, b_{-i})} \) while following the scheme of (36-37), similar to the argument of (44-54), we can get Theorem 2.