A Dual Approach for Optimal Algorithms in Distributed Optimization over Networks *

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Abstract We study the optimal convergence rates for distributed convex optimization problems over networks, where the objective is to minimize the sum \( \sum_{i=1}^{m} f_i(z) \) of local functions of the nodes in the network. We provide optimal complexity bounds for four different cases, namely: the case when each function \( f_i \) is strongly convex and smooth, the cases when it is either strongly convex or smooth and the case when it is convex but neither strongly convex nor smooth. Our approach is based on the dual of an appropriately formulated primal problem, which includes the underlying static graph that models the communication restrictions. Our results show distributed algorithms that achieve the same optimal rates as their centralized counterparts (up to constant and logarithmic factors), with an additional cost related to the spectral gap of the interaction matrix that captures the local communications of the nodes in the network.

Keywords Distributed Optimization · Convergence Rates · Optimal Algorithms · Optimization over Networks

Mathematics Subject Classification (2000) 90C25 · 90C30 · 90C60 · 90C35

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1 Introduction

The study of distributed algorithms can be traced back to classic papers from the 70s and 80s [2, 3, 4]. The adoption of distributed optimization algorithms on several fronts of applied and theoretical machine learning, robotics, and resource allocation has increased the attention on such methods in recent years [5, 6, 7, 8, 9]. The particular flexibilities induced by the distributed setup make them suitable for large-scale learning problems involving large quantities of data [10, 11, 12, 13, 14].

Initial algorithms for distributed optimization, such as distributed subgradient methods, were shown successful for solving optimization problems in a distributed manner over networks [15, 16, 17, 18]. Nevertheless, these algorithms are particularly slow compared with their centralized counterparts. Recently, distributed methods that achieve linear convergence rates for minimizing a sum of strongly convex and smooth (network) objective functions have been proposed. One can identify three main approaches to the study of distributed algorithms. In [19], a new method was proposed where it was shown that \( O((m^2 + \sqrt{L/\mu}) \log \varepsilon^{-1}) \) iterations are required to find an \( \varepsilon \) solution to the optimization problem when the function is \( \mu \)-strongly convex and \( L \)-smooth, and \( m \) is the number of nodes in the network. In [20], a new analysis technique for the convergence rate of distributed optimization algorithms via a semidefinite programming characterization was proposed. This approach provides an innovative procedure to numerically certify worst-case rates of a plethora of distributed algorithms, which can be useful to fine-tune parameters in existing algorithms based on feasibility conditions of a semidefinite program. In [21], a unifying approach was proposed, that recovers rate results from several existing algorithms such as those in [22, 23]. This newly proposed general method is able to recover existing rates and achieves an \( \varepsilon \) precision in \( O(\sqrt{L/(\mu\lambda_2)} \log \varepsilon^{-1}) \) iterations, where \( \lambda_2 \) is the second largest eigenvalue of the interaction matrix. These results require some minimal information about the topology of the network and provide explicit statements about the dependency of the convergence rate on the problem parameters. Specifically, polynomial scalability is shown with the network parameter for particular choices of small enough step-sizes and even uncoordinated step-sizes are allowed [24]. One particular advantage of this approach is that it can handle time-varying and directed graphs. Nevertheless, optimal dependencies on the problem parameters and tight convergence rate bounds are far less understood. A third approach was recently introduced in [25], where the first optimal algorithm for distributed optimization problems was proposed. This new method achieves an \( \varepsilon \) precision in \( O(\sqrt{L/\mu(1 + \tau/\sqrt{\gamma})} \log \varepsilon^{-1}) \) iterations for \( \mu \)-strongly convex and \( L \)-smooth problems, where \( \tau \) is the diameter of the network and \( \gamma \) is the normalized eigengap of the interaction matrix. Even though extra information about the topology of the network is required, the work in [25] provides a coherent understanding of the optimal convergence rates and its dependencies on the communication network. The work in [25] is based on the representation of the communication structure as an additional set of linear constraints on the distributed problem to guarantee consensus on the solution, from which a primal-dual method can be applied [26, 27, 28, 29, 30, 31]. For example in [32], the authors develop a new primal-dual algorithm that uses the Laplacian of the communication graph as a set of linear constraints to induce coordination. Moreover, with additional metric subregularity conditions a linear convergence rate is shown. Recently in [33], the authors extended the optimality lower bounds from [25] to the non-smooth case using the randomized regularization approach [34].

In this paper, we consider the following optimization problem

\[
\min_{z \in \mathbb{R}^n} \sum_{i=1}^{m} f_i(z),
\]

where the each \( f_i \) is a closed convex function known by an agent \( i \) only, that represents a node in an arbitrary communication network. Problem (1) is to be solved in a distributed manner by repeated interactions of a set of agents over a static network. We follow the approach in [25] by formulating a dual problem and exploit recent results in the study of convex optimization problems with affine constraints [35, 36, 37] to develop
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Table 1: Iteration complexity of distributed optimization algorithms. All estimates are presented up to logarithmic factors, i.e., of the order $\tilde{O}$.

| Approach          | Reference | $\mu$-strongly convex and $L$-smooth | $\mu$-strongly convex and $M$-Lipschitz | $L$-smooth | $M$-Lipschitz |
|-------------------|-----------|-------------------------------------|----------------------------------------|------------|--------------|
| Centralized       | [29]      | $\sqrt{L/\mu}$                      | $M^2/(\mu\varepsilon)$                | $\sqrt{L/\varepsilon}$ | $M^2/\varepsilon^2$ |
| Gradient Computations | [40]      | $(L/\mu)^{\varepsilon^2/m^2}$      | $-\varepsilon^2/m^2$                   | $M/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                   | [19]      | $m^2 + m \sqrt{L/\mu}$              | $-\varepsilon$                         | $1/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                   | [41]      | $-\varepsilon$                       | $-\varepsilon$                         | $1/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                   | [42]      | $(L/\mu)m^2$                         | $-\varepsilon$                         | $M/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                   | [43]      | $-\varepsilon$                       | $-\varepsilon$                         | $M/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                   | [44]      | $(L/\mu)m^4$                         | $-\varepsilon$                         | $1/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                   | [45]      | $\sqrt{L/\mu}m^2$                   | $-\varepsilon$                         | $M/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                   | [21]      | $\sqrt{L/\mu}m$                     | $-\varepsilon$                         | $M/\varepsilon$ | $mM^2/\varepsilon^2$ |
| Communication Rounds | [25]      | $\sqrt{L/\mu}m^2$                   | $-\varepsilon$                         | $M/\varepsilon$ | $mM^2/\varepsilon^2$ |
|                  | [46]      | $\sqrt{L/\mu}m^2$                   | $-\varepsilon$                         | $M/\varepsilon$ | $mM^2/\varepsilon^2$ |

* Additionally, it is assumed functions are proximal friendly. No explicit dependence on $L$, $M$ or $m$ is provided.

A linear dependence on $m$ is achieved if $L$ is sufficiently close to $\mu$. 

Our results match known optimal complexity bounds for centralized convex optimization (obtained by classical methods such as Nesterov’s fast gradient method FGM [38]), with an additional cost induced by the network of communication constraints. This extra cost appears in the form of a multiplicative term proportional to the square root of the spectral gap of the interaction matrix. In summary, our primary results provide an algorithm that achieves relative accuracy on any fixed, connected and undirected graph according to Table 1, where universal constants, logarithmic terms, and dependencies on the initial conditions are hidden for simplicity. The resulting iteration complexities are given both for the optimality of the solution and the violation of the consensus constraints. Note that for distributed algorithms based on primal iterations these estimates translate to computations of gradients of the local functions for each of the agents. On the other side, in dual based algorithms, the complexity refers to computations of the gradients of the Lagrangian dual function, which translates to the number of communication rounds in the network.

This paper is organized as follows: Section 2 introduces the problem of distributed optimization over a network. Section 3 presents a series of preliminary definitions and results. Section 4 provides our main results on the optimal convergence rates for distributed convex minimization problems with a dual friendly structure. Discussion are provided in Sections 5. Section 6 provides our main results on the optimal convergence rates for distributed convex minimization problems where an exact solution to the dual subproblem is not available. Section 7 presents a method to improve the dependency of the convergence rate on the condition number of the function $F$. Finally, Section 8 provides numerical experiments of the proposed algorithms.

**Notation:** We will assume that the nodes in the network, also referred as agents, are indexed from 1 through $m$ (no actual enumeration is needed in the execution of the proposed algorithms; it is only used in our analysis). We use the superscripts $i$ or $j$ to denote agent indices and the subscript $k$ to denote the iteration index of an algorithm. We denote by $[A]_{ij}$ the entry of the matrix $A$ in its $i$-th row and $j$-th column, and write $I_n$ for the identity matrix of size $n$. For a symmetric non-negative matrix $W$, we let $\lambda_{\max}(W)$ be its largest
eigenvalue and $\lambda_{\min}^+(W)$ be its smallest positive eigenvalue, and we denote its condition number by $\chi(W) = \lambda_{\max}(W)/\lambda_{\min}(W)$. Given a matrix $A$, define $\sigma_{\max}(A) \triangleq \lambda_{\max}(A^T A)$ and $\sigma_{\min}^+(A) \triangleq \lambda_{\min}^+(A^T A)$. We use $1$ to denote a vector with all entries equal to 1. We write $\tilde{O}$ to denote a complexity bound that ignores logarithmic factors. We will work in the standard Euclidean norm, denoted by $\| \cdot \|_2$.

2 Problem Statement

Initially, let us introduce the stacked column vector $x = [x_1^T, x_2^T, \ldots, x_m^T]^T \in \mathbb{R}^{mn}$ and rewrite problem (1) in an equivalent form as follows:

$$\min_{x_1 = \ldots = x_m} F(x) \quad \text{where} \quad F(x) \triangleq \sum_{i=1}^{m} f_i(x_i). \quad (2)$$

The distributed optimization framework assumes we want to solve problem (2) in a distributed manner over a network. We model such a network as a fixed connected undirected graph $G = (V, E)$, where $V$ is the set of $m$ nodes, and $E$ is the set of edges. We assume that the graph $G$ does not have self-loops. The network structure imposes information constraints; specifically, each node $i$ has access to the function $f_i$ only and a node can exchange information only with its immediate neighbors, i.e., a node $i$ can communicate with node $j$ if and only if $(i, j) \in E$.

We can represent the communication constraints imposed by the network by introducing a set of equivalent constraints via the Laplacian $\bar{W} \in \mathbb{R}^{m \times m}$ of the graph $G$ defined as

$$[\bar{W}]_{ij} = \begin{cases} -1, & \text{if } (i, j) \in E, \\ \deg(i), & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases}$$

where $\deg(i)$ is the degree of the node $i$, i.e., the number of neighbors of the node. Finally, define the communication matrix (also referred to as an interaction matrix) by $W \triangleq \bar{W} \otimes I_n$, where $\otimes$ indicates the Kronecker product.

Throughout this paper, we assume the graph $G = (V, E)$ is connected and undirected. Under this assumption, the Laplacian matrix $\bar{W}$ is symmetric and positive semi-definite. Furthermore, the vector $1$ is the unique (up to a scaling factor) eigenvector associated with the eigenvalue $\lambda = 0$. Given the definition $W \triangleq \bar{W} \otimes I_n$, one can verify that $W$ inherits all the properties of $\bar{W}$, i.e., it is a symmetric positive semi-definite matrix and it satisfies the following relations:

- $W x = 0$ if and only if $x_1 = \ldots = x_m$.
- $\sqrt{W} x = 0$ if and only if $x_1 = \ldots = x_m$.
- $\sigma_{\max}(\sqrt{W}) = \lambda_{\max}(W)$.

Therefore, one can equivalently rewrite problem (2) as follows:

$$\min_{\sqrt{W} x = 0} F(x) \quad \text{where} \quad F(x) \triangleq \sum_{i=1}^{m} f_i(x_i). \quad (3)$$

Note that the constraint set $\{x \mid \sqrt{W} x = 0\}$ is the same as the set $\{x \mid x_1 = \ldots = x_m\}$, since $\ker(\sqrt{W}) = \text{span}(1)$ due to the connectivity of the graph $G$. We note that a similar idea of writing the Laplacian as a product of a matrix $B$ and its transpose has been employed in [32] using the incidence matrix.
Example 2.1 Consider a network of agents as shown in Figure 1 where the agents in the network seek to cooperatively solve the following regularized linear regression problem

$$\min_{z \in \mathbb{R}^n} \frac{1}{2m} \|Hz - b\|_2^2 + \frac{1}{2}c\|z\|_2^2,$$

(4)

where $b \in \mathbb{R}^{ml}$, $H \in \mathbb{R}^{ml \times n}$ and $c > 0$ is some constant. Furthermore, assume the data in $b$ and $H$ are distributed over the network, where no single agent has full access to the complete information, i.e., each agent has access to a subset of points such that

$$b^T = [b_1^T | b_2^T | \cdots | b_m^T], \quad \text{and} \quad H^T = [H_1^T | H_2^T | \cdots | H_m^T],$$

where $b_i \in \mathbb{R}^l$ and $H_i \in \mathbb{R}^{l \times n}$ for each $i$. In this setup, each agent $i$ has a private local function

$$f_i(x_i) \triangleq \frac{1}{2ml} \|b_i - H_ix_i\|_2^2 + \frac{1}{2}c\|x_i\|_2^2.$$

Therefore, problem (4) is equivalent to

$$\min_{\sqrt{W}x = 0} \sum_{i=1}^{m} \left( \frac{1}{2ml} \|b_i - H_ix_i\|_2^2 + \frac{1}{2}c\|x_i\|_2^2 \right),$$

where $W = \bar{W} \otimes I_n$. Particularly, for the cycle graph network of 5 agents shown in Figure 1(a), agent 1 can share information with agents 2 and 5, agent 5 shares information with agents 1 and 4, and similarly for the other agents. Thus, the corresponding interaction matrix $\bar{W}$ is

$$\bar{W} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$
3 Preliminaries

In this section, we provide some definitions and preliminary information that we will use in the forthcoming sections.

We will refer to a function $f$ as $\mu$-strongly convex with $\mu > 0$, if for any $x, y$ it holds that

$$f(y) \geq f(x) + \langle \hat{\nabla} f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|^2,$$

where $\hat{\nabla} f(x)$ is any subgradient of $f$ at $x$.

We will refer to a function $f$ as having $L$-Lipschitz continuous gradients (or $L$-smooth), if it is differentiable and for any $x$ and $y$ it holds that

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|.$$

Particularly, if each function $f_i$ in (3) is $\mu_i$-strongly convex in $x_i$, then $F$ in (3) is $\mu$-strongly convex in $x$ with $\mu = \min_{1 \leq i \leq m} \mu_i$. Also, if each $f_i$ is $L_i$-smooth, then $F$ is $L$-smooth with $L = \max_{1 \leq i \leq m} L_i$. Later in Section 7 we will explore a method to improve the dependency on the condition number $\mu$ from the worst case parameter to the average strong convexity.

We will build the proposed algorithms based on Nesterov’s fast gradient method (FGM) [47]. For example, a version of the FGM for a $\mu$-strongly convex and $L$-smooth function $f$ is shown in Algorithm (1). Other variants of this method can be found in [47,48,49].

Algorithm 1 Nesterov’s Constant Step Scheme II.

1: Choose $x_0 \in \mathbb{R}^n$ and $\alpha_0 \in (0, 1)$. Set $y_0 = x_0$ and $q = \frac{\mu}{L}$.
2: $k$th iteration ($k \geq 0$).
   (a) Compute $\nabla f(y_k)$. Set $x_{k+1} = y_k - \frac{1}{L} \nabla f(y_k)$.
   (b) Compute $\alpha_{k+1} \in (0, 1)$ from equation $\alpha_{k+1}^2 = (1 - \alpha_{k+1}) \alpha_k^2 + q \alpha_{k+1}$.
      and set $\beta_k = \frac{\alpha_k(1 - \alpha_k)}{\alpha_k^2 + \alpha_{k+1}}$, $y_{k+1} = x_k + \beta_k(x_{k+1} - x_k)$.

Algorithm 1 has the following property:

**Theorem 3.1** (Theorem 2.2.2 in [47]) If in Algorithm 1 $\alpha_0 \geq \frac{\sqrt{\mu}}{L}$, then Algorithm 1 generates a sequence \( \{x_k\}_{k=0}^\infty \) such that

$$f(x_k) - f^* \leq \left(1 - \frac{\mu}{L}\right)^k \|x_0 - x^*\|^2,$$

where $f^*$ denotes the minimum value of the function $f$ over $\mathbb{R}^n$ and $x^*$ is its minimizer. Moreover, Algorithm 1 is optimal for unconstrained minimization of strongly convex and smooth functions.

In what follows, we will consider a generic optimization problem with linear constraints. Then, we will use the convergence results in (5) to obtain some fundamental insights for the distributed optimization problem. Moreover, we will derive the results for a corresponding distributed algorithm for solving problem (3). The main idea of our analysis will be to explore the case when the linear constraints $Ax = 0$ represent the network communication constraints as $\sqrt{W}x = 0$ and the function $f(x)$ corresponds to the network function $F(x)$ as defined in (3).
Initially, consider a \( \mu \)-strongly convex and an \( L \)-smooth function \( f \) to be minimized over a set of linear constraints, i.e.,

\[
\min_{Ax=0} f(x).
\] (6)

Assume that problem (6) is feasible, in which case a unique solution exists, denoted by \( x^* \). However, we will be interested in finding approximate solutions of (6) that attain a function value arbitrarily close to the optimal value and have arbitrarily small feasibility violation of the linear constraints. For this, we introduce the following definition.

**Definition 3.1** \cite{46} A point \( x \in \mathbb{R}^{mn} \) is called an \((\varepsilon, \tilde{\varepsilon})\)-solution of (6) if the following conditions are satisfied

\[
f(x) - f^* \leq \varepsilon, \quad \text{and} \quad \|Ax\|_2 \leq \tilde{\varepsilon},
\]

where \( f^* = f(x^*) \) denotes the optimal value for the primal problem in (6).

The Lagrangian dual of (6) is given by

\[
\min_{Ax=0} f(x) = \max_y \left\{ \min_x \{ f(x) - \langle A^T y, x \rangle \} \right\}.
\] (7)

Moreover, (7) can be re-formulated as an equivalent minimization problem, as follows:

\[
\min_y \varphi(y) \quad \text{where} \quad \varphi(y) \triangleq \max_x \Psi(x, y),
\] (8)

and

\[
\Psi(x, y) \triangleq \langle A^T y, x \rangle - f(x).
\]

The function \( \varphi(y) \) is \( \mu_x \)-strongly convex on \( \ker(A^T)^\perp \) with \( \mu_x = \lambda_{\min}^+ (A^T A)/L \). Moreover, it has \( L_x \)-Lipschitz continuous gradients with \( L_x = \lambda_{\max} (A^T A)/\mu \), see Lemma 3.1 in \cite{50}. Proposition 12.60 in \cite{51}. Theorem 1 in \cite{52}, Theorem 6 in \cite{53}. Additionally, from Demyanov-Danskin’s theorem (see, for example, Proposition 4.5.1 in \cite{54}), it follows that \( \nabla \varphi(y) = Ax^*(A^T y) \) where \( x^*(A^T y) \) denotes the unique solution to the inner maximization problem

\[
x^*(A^T y) = \arg \max_x \Psi(x, y).
\] (9)

Note that we call \( x^* \) the minimizer of (6) with smallest norm. On the other hand, we denote \( x^*(A^T y) \) as the solution of (9) for a given value \( A^T y \). Particularly, note that \( x^*(0) = \arg \max_x \{-f(x)\} \). Moreover, there is no duality gap between the primal problem in (6) and its dual problem in (5), and the dual problem has a solution \( y^* \) (see for example, Proposition 6.4.2 in \cite{54}). Thus, it holds that \( x^* = x^*(A^T y^*) \). In general, the dual problem in (5) can have multiple solutions of the form \( y^* + \ker(A^T) \) when the matrix \( A \) does not have a full row rank, for example when \( A \) is the Laplacian of a graph. If the solution is not unique, then we will use \( y^* \) to denote the smallest norm solution, and we let \( R \) be its norm, i.e. \( R = \|y^*\|_2 \).

In the next Lemma, we provide an auxiliary condition to check whether a point \( x \) is an \((\varepsilon, \tilde{\varepsilon})\)-solution in terms of the properties of the dual function \( \varphi \).

**Lemma 3.1** (Lemma 1 in \cite{53}) Let \( \langle y, \nabla \varphi(y) \rangle \leq \varepsilon \) and \( \|\nabla \varphi(y)\|_2 \leq \tilde{\varepsilon} \). Then, \( x^*(A^T y) \) is an \((\varepsilon, \tilde{\varepsilon})\)-solution of (6).
In what follows, we will apply the bound for the FGM algorithm in (5) on the dual problem (8), which is not strongly convex in the ordinary sense (on the whole space). However, by choosing $y_0 = x_0 = 0$ in Algorithm 1 as the initial condition, the algorithm applied to the dual problem will produce iterates that lie in the linear space of gradients $\nabla \varphi(y)$, which are of the form $Ax$ for $x = x^*(A^T y)$. In this case, the dual function $\varphi(y)$ will be strongly convex when $y$ is restricted to the linear space spanned by the range of the matrix $A$. Line 2(a) of Algorithm 1 applied to the dual problem then specializes to

$$y_{k+1} = \tilde{y}_k - \frac{1}{L_{\varphi}} Ax^*(A^T \tilde{y}_k).$$  

(10)

Our results provide convergence rate estimates for the solution of the problem in (1) for four different cases in terms of the properties of the function $F(x) = \sum_{i=1}^m f_i(x_i)$.

**Assumption 1** Consider a function $F(x) = \sum_{i=1}^m f_i(x_i)$, and assume:

(a) Each $f_i$ is $\mu_i$-strongly convex and $L_i$-smooth, thus $F$ is $\mu$-strongly convex and $L$-smooth;

(b) each $f_i$ is $\mu_i$-strongly convex and $M_i$-Lipschitz on a bounded set, thus $F$ is $\mu$-strongly convex and $M$-Lipschitz on the same bounded set;

(c) each $f_i$ is convex and $L_i$-smooth, thus $F$ is convex and $L$-smooth;

(d) each $f_i$ is convex and $M_i$-Lipschitz on a bounded set, thus $F$ is convex and $M$-Lipschitz on the same bounded set;

where $\mu = \min_{1 \leq i \leq m} \mu_i$, $L = \max_{1 \leq i \leq m} L_i$ and $M = \max_{1 \leq i \leq m} M_i$.

Note that (10) requires an explicit computation of $x^*(A^T y)$, which is the solution of the inner maximization problem (9). Section 4 will present the proposed algorithms and convergence rates, for the different convexity and smoothness assumptions on the functions $f_i$, expressed in Assumption 1, when an explicit solution to the inner maximization problem (9) is available. We will denote this scenario as dual-friendly functions. Later in Section 6, we extend the results of Section 4 to the case where only approximate solutions to (9) can be computed. Particularly, to find $x^*(A^T y)$ one can use optimal (randomized) numerical methods [47, 56, 57].

### 4 Results for Dual Friendly Functions: Algorithms and Iteration Complexity Analysis

In this section, we will assume that we have access to $x^*(A^T y)$ explicitly for any given $y$. Section 6 discusses possible extension when no dual solution is explicitly available.

**Definition 4.1** A function $f(x)$ is dual-friendly if, for any $y$, one has immediate access to an explicit (or efficiently computed) solution $x^*(A^T y)$ to the dual subproblem associated with the optimization problem in (6). Sometimes this function are also called admissible [58, 59].

Examples of optimization problems for which Definition 4.1 holds can be found in the literature, e.g., the entropy-regularized optimal transport problem [60], the entropy linear programming problem [55] or the ridge regression. Note that by definition, finding a solution for the problem (9) corresponds to finding a maximizing point of the Legendre transformation $f^*$ of the function $f$, where $f^*$ is defined as

$$f^*(y) = \max_x \{ \langle x, y \rangle - f(x) \}.$$
Moreover, if the conjugate dual function is available, then the maximizing argument is
\[ x^* (A^T y) = \nabla f^* (y), \]
see Proposition 8.1.1 in [54]. For example, for the ridge regression problem
\[ \min_x \|H x - b\|^2_2, \]
the maximizing argument in (9) can be explicitly computed as
\[ x^* (A^T y) = (H^T H)^{-1} (A^T y + H^T b). \]
Another example where one can find an explicit solution to the auxiliary dual problem is the Entropy Linear Program (ELP) [61], i.e.,
\[ \min_{x \in S_n(1)} \sum_{j=1}^n x_j \log \left( \frac{x_j}{q_j} \right), \]
where
\[ S_n(1) = \{ x \in \mathbb{R}^n : x_j \geq 0 ; j = 1, 2, \ldots, n ; \sum_{j=1}^n x_j = 1 \} \]
is a unit simplex in \( \mathbb{R}^n \) and \( q \in S_n(1) \). The maximizing argument (9) for problem (11) can be explicitly computed as
\[ [x^* (A^T y)]_i = \frac{q_i \exp([A^T y]_i)}{\sum_{j=1}^n q_j \exp([A^T y]_j)}. \]
Additional examples related to optimal transport problems of computation of Wasserstein barycenter can be found in [60,62,63].

Next, we provide a sequence of theorems considering each case in Assumption 1. For each case, we present an algorithm and the minimum number of iterations required for the algorithm to reach an approximate solution of (3).

4.1 Sums of strongly convex and smooth functions
In this subsection, we analyze the distribution optimization problem in (2) when Assumption 1(a) holds. That is, when each of the functions \( f_i \) is strongly convex and smooth. Initially, consider (10), and replace \( A = \sqrt{W} \), thus
\[ y_{k+1} = \tilde{y}_k - \frac{1}{L_{f^*}} \sqrt{W} x^* (\sqrt{W}^T \tilde{y}_k). \]
For simplicity of notation we denote as \( y^* \) as the solution of (3). Unfortunately, (13) cannot be executed over a network in a distributed manner because the sparsity pattern of the matrix \( \sqrt{W} \) need not be compliant with the graph \( \mathcal{G} \) in the same way the matrix \( W \) is. Therefore, we make the following change of variables: \( \sqrt{W} y_k = z_k \) and \( \sqrt{W} \tilde{y}_k = \tilde{z}_k \), resulting in an algorithm that can be executed in a distributed manner. Interaction between agents is dictated by the term \( W x^* (\tilde{z}_k) \) which depends only on local information. As a result, each agent \( i \) in the network has its local variables \( z^i_k \) and \( \tilde{z}^i_k \), and to compute their value at the next iteration, it only requires the information sent by the neighbors defined by the communication graph \( \mathcal{G} \). Additionally, the dual subproblem can be computed in a distributed manner at node \( i \) as
\[ x^*_i (\tilde{z}^i_k) = \arg \max_{z_i} \{ \langle z^i_k, x_i \rangle - f_i(x_i) \} \]
Next, we formally state the FGM algorithm applied to the dual of problem (2) with the change of variable that allow a distributed execution.
Algorithm 2

Distributed FGM for strongly convex and smooth problems

1: All agents set $z^0_i = \tilde{z}^0 = 0 \in \mathbb{R}^n$, $q = \frac{\mu^2}{\lambda_{\max}(W)}$, $\alpha_0$ solves $\frac{\alpha_0^2 - q}{4\alpha_0} = 1$ and $N$.
2: For each agent $i$
3: for $k = 0, 1, 2, \cdots, N - 1$ do
4: $\hat{x}^*_i(z^k_i) = \arg \max \{ \langle z^k_i, x_i \rangle - f_i(x_i) \}$
5: Share $\hat{x}^*_i(z^k_i)$ with neighbors, i.e. $\{ j \mid (i, j) \in E \}$.
6: $\tilde{z}^k_i = \hat{z}^k_i - \frac{\alpha}{\lambda_{\max}(W)} \sum_{j=1}^m W_{ij} \hat{x}^*_j(z^k_i)$
7: Compute $\alpha_{k+1} \in (0, 1)$ from $\alpha_{k+1}^2 = (1 - \alpha_{k+1}) \alpha_k^2 + q \alpha_{k+1}$ and set $\beta_k = \frac{\alpha_k(1 - \alpha_k)}{\alpha_k + \alpha_{k+1}}$.
8: $\tilde{z}^k_{i+1} = \beta_k (\tilde{z}^k_{i+1} - x^k_i)$
9: end for

Note that Algorithm 2 requires the number of iterations $N$, which effectively corresponds to the number of communication rounds. We define a communication round as an iteration of Algorithm 2 where every node shares its local estimates with its neighbors and updates its local variables. Thus, we are interested in finding a lower bound on $N$ such that we can guarantee certain optimality of the local solutions in the sense of Definition 3.1.

Next, we state our main result regarding the number of iterations required by Algorithm 2 to reach an approximate solution of problem (3).

Theorem 4.1

Let $F(x)$ be dual friendly and Assumption 2.(a) hold. For any $\varepsilon > 0$, the output $x^*(z_N)$ of Algorithm 2 is an $(\varepsilon, \varepsilon/R)$-solution of (3) for

$$N \geq 2 \sqrt{\frac{L}{\mu \chi(W)}} \log \left( \frac{2\sqrt{2} \lambda_{\max}(W) R^2}{\mu \varepsilon} \right),$$

where $R = \|y^*\|_2$, and $\chi(W) = \lambda_{\max}(W)/\lambda_{\min}^+(W)$.

Proof. Algorithm 2 follows from the FGM in (1) applied to the dual problem (8) with the change of variables $\sqrt{W} y_k = z_k$ and $\sqrt{W} \tilde{y}_k = \tilde{z}_k$. Therefore, we are going to use the convergence results of the FGM for the dual problem in terms of the dual variables $y_k$ and $\tilde{y}_k$ and provide an estimate of the convergence rate of in terms of the primal variables.

Initially, it follows from Theorem 2.2.2 in [47], Section 2.2.1, that the sequence of estimates generated by the iterations in (13) has the following property:

$$\varphi(y_k) - \varphi^* \leq L_\varphi R^2 \exp \left( -k \sqrt{\frac{\|y_k\|_2^2}{L_\varphi}} \right).$$

Moreover, it holds that

$$\varphi^* \leq \varphi(y_{k+1}) \leq \varphi(\tilde{y}_k) - \frac{1}{2L_\varphi} \|\nabla \varphi(\tilde{y}_k)\|_2^2.$$

Thus

$$\|\nabla \varphi(\tilde{y}_k)\|_2^2 \leq 2L_\varphi (\varphi(y_k) - \varphi^*)$$

$$\|\sqrt{W} x^*(\sqrt{W} \tilde{y}_k)\|_2^2 \leq 2L_\varphi^2 R^2 \exp \left( -k \sqrt{\frac{\|y_k\|_2^2}{L_\varphi}} \right).$$

We can conclude that $\|\sqrt{W} x^*(z_k)\|_2 \leq \varepsilon/R$ if $k \geq 2 \sqrt{\frac{\|y_k\|_2^2}{L_\varphi}} \log \left( \frac{2\sqrt{2} \lambda_{\max}(W) R^2}{\mu \varepsilon} \right)$. 
Now, by using the Cauchy–Schwarz inequality, it follows that
\[
|\langle y_k, \sqrt{W} x^*(\sqrt{W} y_k) \rangle|^2 \leq \|y_k\|_2^2 \|\sqrt{W} x^*(\sqrt{W} y_k)\|_2^2.
\]

We can bound \(\|y_k\|_2\) following ideas from [64], where it was shown that
\[
\|y_k - y^*\|_2 \leq \|y_0 - y^*\|_2.
\]
Thus, since we assume \(y_0 = 0\), it holds that
\[
\|y_k\|_2 \leq 2 \|y^*\|_2 \leq 2R,
\]
then
\[
|\langle y_k, \sqrt{W} x^*(\sqrt{W} y_k) \rangle|^2 \leq 4R^2 \|\sqrt{W} x^*(\sqrt{W} y_k)\|_2^2 \leq 8R^4 L^2 \phi \exp \left(-k \frac{\mu \phi}{L \phi}\right).
\]

Therefore 
\[
f(x^*(z_k)) - f^* \leq \epsilon \text{ if } k \geq 2 \sqrt{\frac{L \mu}{\chi(W)}} \log \left(\frac{\max\{2\sqrt{2} L \phi R^2, \sqrt{2} L \phi R^2\}}{\epsilon}\right).
\]

Finally, based on Lemma 3.1, Algorithm 2 will produce an \((\epsilon, \epsilon/R)\)-solution if
\[
N \geq 2 \sqrt{\frac{L \phi}{\mu \phi}} \log \left(\frac{2\sqrt{2} L \phi R^2}{\epsilon}, \frac{\sqrt{2} L \phi R^2}{\epsilon}\right).
\]

Following the definitions of \(L, \mu, \phi\), and \(\chi(W)\), we obtain the desired result. \(\square\)

Theorem 4.1 states that in order to obtain an \((\epsilon, \epsilon/R)\)-solution of (3), when each function \(f_i\) is strongly convex and smooth, the communication complexity is
\[
O\left(\sqrt{\frac{L \mu \chi(W) \log(1/\epsilon)}{\epsilon}}\right).
\]

4.2 Sums of strongly convex and \(M\)-Lipschitz functions on a bounded set

In this subsection, we propose a distributed algorithm for sum of strongly convex functions that are Lipschitz on a bounded set to be specified later. Moreover, we show the convergence rates of the proposed algorithm.

We will build our results by using Nesterov smoothing [52,64]. Particularly, we will use the following result

**Proposition 4.1** Consider a convex function \(\varphi\), and the strongly convex term \(\frac{\hat{\mu}}{2} \|y\|_2^2\), and define \(\hat{\varphi}(y) = \varphi(y) + \frac{\hat{\mu}}{2} \|y\|_2^2\). Then, \(\hat{\varphi}(y)\) is \(\hat{\mu}\)-strongly convex. Moreover, if \(\hat{\mu} \leq \epsilon/R^2\) and assume that there exists \(y_N\) such that \(\hat{\varphi}(y_N) - \hat{\varphi}^* \leq \epsilon/2\), it holds that \(\varphi(y_N) - \hat{\varphi}^* \leq \epsilon\), where \(\varphi^*\) is the optimal value of the function \(\varphi\). Moreover, if \(\varphi\) is defined in (8), then \(\nabla \hat{\varphi}(y) = Ax^*(A^T y) + \hat{\mu} y\).

Now, we can state the distributed algorithm we proposed for the minimization of sums of convex and Lipschitz (on a bounded set to be specified later) functions.

Note that the main difference between Algorithm 2 and Algorithm 3 is that we have an additional term inside the parenthesis in Line 6. Moreover, the corresponding strong convexity constant of the dual function is \(\epsilon/R^2\) as induced by the regularization term, where \(R\) is an upper bound on the norm of optimal solution of the dual problem \(y^*\), i.e., \(\|y^*\|_2 \leq R\). Next, we state our main result regarding the number of iterations required by Algorithm 3 to reach an approximate solution of problem (2).
Algorithm 3 Distributed FGM for strongly convex and $M$-Lipschitz problems

1. All agents set $z_0^i = z_0 = 0 \in \mathbb{R}^n$, $q = \frac{\nu/(4R^2)}{\lambda_{\max}(W)/(\mu + \nu/(4R^2))}$, $\alpha_0$ solves $\frac{q^2 - 2}{1 - q} = 1$ and $N$.  
2. For each agent $i$  
3. for $k = 0, 1, 2, \ldots, N - 1$ do  
4. $x_k^i = \arg \max \{ (z_k^i, x_i) - f_i(x_i) \}$  
5. Share $x_k^i$ with neighbors, i.e. $\{ j | (i, j) \in E \}$.  
6. $z_k^{i+1} = z_k^i - \frac{1}{\lambda_{\max}(W)/(\mu + \nu/(4R^2))} \left( \sum_{j=1}^m W_{ij} x_j^k(z_k^i) + \frac{\epsilon}{4R^2} z_k^i \right)$  
7. Compute $\alpha_{k+1} \in (0, 1)$ from $\alpha_{k+1}^2 = (1 - \alpha_k) \alpha_k + q \alpha_{k+1}$ and set $\beta_k = \frac{\alpha_k (1 - \alpha_k)}{\alpha_k^2 + \alpha_{k+1}}$  
8. $z_{k+1}^i = z_k^{i+1} + \beta_k (x_k^i - z_k^i)$  
9. end for

Theorem 4.2 Let $F(x)$ be dual friendly and Assumption 1(b) hold. Moreover, assume $F(x)$ is $M$-Lipschitz in the set $\{ x \mid \| x - x^* \|_2 \leq R_x \}$ with $R_x = \| x^*(0) - x^* \|_2$. For any $\epsilon > 0$, the output $x^*(z_N)$ of Algorithm 3 is an $(\epsilon, \epsilon/R)$-solution of $\{ 3 \}$ for

$$N \geq 2\sqrt{4\chi(W) \frac{M^2}{\mu + \epsilon} + 1 \log \left( 4\chi(W) \frac{M^2}{\mu + \epsilon} + 1 \right)},$$

where $\chi(W) = \lambda_{\max}(W)/\lambda_{\min}^*(W)$.

Proof Initially, consider the regularized dual function $\hat{\phi}$ with $\hat{\mu} = \frac{\hat{\mu}}{\lambda_{\min}}$, which is $\mu_{\hat{\phi}}$-strongly convex with $\mu_{\hat{\phi}} = \frac{\lambda_{\max}(W)}{\mu + \nu/(4R^2)}$, and $L_{\hat{\phi}}$-smooth with $L_{\hat{\phi}} = \frac{\lambda_{\max}(W)}{\mu + \nu/(4R^2)}$. Thus, similarly as in (15)

$$\hat{\phi}(y_k) - \hat{\phi}^* \leq L_{\hat{\phi}} \hat{R}^2 \exp \left( -k \sqrt{\frac{\hat{\mu}}{L_{\hat{\phi}}}} \right) \leq L_{\hat{\phi}} \hat{R}^2 \exp \left( -k \sqrt{\frac{\hat{\mu}}{L_{\hat{\phi}}}} \right),$$

where $\hat{R} = \| y^* \|_2$, and $\hat{y}^*$ is the smallest norm solution of the regularized dual problem. Note that by definition $\hat{R} = \| y^* \|_2 \leq \| y^* \|_2 = R$.

Next, we provide a relation between the distance to optimality of the non-regularized primal problem and the regularized dual problem. Note that for any $y$ it holds that

$$\hat{\phi}(y) - \hat{\phi}^* \geq \frac{\| \nabla \phi(y) \|_2^2}{2L_{\hat{\phi}}} = \frac{\| \nabla \phi(y) + \hat{\mu} y \|_2^2}{2L_{\hat{\phi}}} \geq \frac{\hat{\mu}}{L_{\hat{\phi}}},$$

Therefore,

$$\langle y, \nabla \phi(y) \rangle \leq \frac{L_{\hat{\phi}}}{\hat{\mu}} \left( \hat{\phi}(y) - \hat{\phi}^* \right) \leq \frac{2}{\epsilon} L_{\hat{\phi}} \hat{R}^4 \exp \left( -k \sqrt{\frac{\hat{\mu}}{L_{\hat{\phi}}}} \right).$$

Consequently, if $k \geq 2\sqrt{L_{\hat{\phi}}/\hat{\mu}} \log \left( 2L_{\hat{\phi}} R^2 / \epsilon \right)$, then $\langle y, \nabla \phi(y) \rangle \leq \epsilon$.

Moreover, it follows from the definition of the regularized dual function that

$$\| \nabla \phi(y_k) \|_2 \leq \| \nabla \phi(y_k) \|_2 + \hat{\mu} \| y_k \|_2$$

$$\leq \sqrt{2L_{\hat{\phi}} (\hat{\phi}(y_k) - \hat{\phi}^*) + \hat{\mu} \| y_k \|_2}$$

$$\leq \sqrt{2L_{\hat{\phi}}} \hat{R} \exp \left( -k \frac{\hat{\mu}}{2 \sqrt{L_{\hat{\phi}}}} \right) + 2\hat{\mu} \hat{R}$$
\[ \leq \sqrt{2L_\phi} R \exp \left( -\frac{k}{2 \sqrt{\mu_\phi}} \right) + \frac{\varepsilon}{2R}. \]

Using the definition of the gradient of the dual function then we have that
\[ \|\sqrt{W} x^* (\sqrt{W} y_0)\|_2 \leq \varepsilon/R, \text{ for } k \geq 2 \sqrt{L_\phi / \mu_\phi} \log \left( \sqrt{2L_\phi} R^2 / \varepsilon \right). \]

We conclude, from Lemma 3.1, that we will have an \((\varepsilon, \varepsilon/R)\) solution of (3) if
\[
\begin{align*}
k &\geq 2 \sqrt{\frac{L_\phi}{\mu_\phi}} \log \left( \frac{2L_\phi R^2}{\varepsilon}, \frac{\sqrt{2L_\phi} R^2}{\varepsilon} \right) \\
&\geq 2 \sqrt{\frac{L_\phi}{\mu_\phi}} \log \left( \frac{2L_\phi R^2}{\varepsilon} \right) \\
&= 2 \sqrt{\frac{\lambda_{\max}(W)}{\mu} + \frac{\varepsilon}{3R^2}} \log \left( \frac{2R^2 \left( \frac{\lambda_{\max}(W)}{\mu} + \frac{\varepsilon}{3R^2} \right)}{\varepsilon} \right) \\
&= 2 \sqrt{\frac{4R^2 \lambda_{\max}(W)}{\mu \cdot \varepsilon} + \frac{1}{\log \left( \frac{4R^2 \lambda_{\max}(W)}{\mu \cdot \varepsilon} + 1 \right)}}. 
\end{align*}
\]

Now, we focus our attention to find a bound on the value \(R\) such that we can provide an explicit dependency on the minimum non zero eigenvalue of the graph Laplacian. This will allow us to provide an explicit iteration complexity in terms of the condition number of the graph Laplacian.

Theorem 3 in [46] provides a bound that relates \(R\) with the magnitude of the gradient of \(F(x)\) at the optimal point \(x = x^*\). Particularly, it is shown that
\[
R^2 = \|y^*\|_2^2 \leq \frac{\|\nabla F(x^*)\|_2^2}{\sigma^+_\min(A)}. \tag{17}
\]

It was shown in [63] that the iterations generated by the FGM in (1) always lie inside an Euclidean ball around the optimal solution \(y^*\) (\(y^*\) is the optimal solution of the dual problem in this case), with a radius equal to \(\|y_0 - y^*\|_2\) which is effectively equal to \(R\) given our initialization \(z_0 = 0\). The set \(\{y \mid \|y - y^*\| \leq R\}\) is defined in the dual variables. However, we seek to provide a condition on the primal variables, i.e., \(x\). It follows from the definition of the function \(x^*(\sqrt{W} y)\), that the set \(\{y \mid \|y - y^*\| \leq R\}\) is mapped into an Euclidean ball centered at \(x^*\), since the point \(x^*(\sqrt{W} y^*) = x^*\). As for the radius, note that \(x^*(0) = \arg \min_x F(x)\). Thus, given the assumption that \(F(x)\) is \(M\)-Lipschitz in the set \(\{x \mid \|x - x^*\|_2 \leq R_x\}\) with \(R_x = \|x^* - x^*(0)\|\), it holds that
\[
R^2 \leq \frac{M^2}{\sigma^\min(A)}. 
\]

Therefore to have an \((\varepsilon, \varepsilon/R)\)-solution it is necessary that
\[
k \geq 2 \sqrt{\frac{\lambda_{\max}(W)}{\lambda_{\min}(W)} \cdot \frac{M^2}{\mu \cdot \varepsilon} + \frac{1}{\log \left( \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)} \cdot \frac{M^2}{\mu \cdot \varepsilon} + 1 \right)}}.
\]

\[ \sqrt{\lambda_{\max}(W)} \leq \sqrt{\lambda_{\min}(W)} \cdot \chi(W) \leq \sqrt{\lambda_{\max}(W)} \cdot \chi(W). \]

Theorem 4.2 states the communication complexity of Algorithm 3. Particularly, the total number of communication rounds required by each agent to find an \((\varepsilon, \varepsilon/R)\)-solution of (3) can be bounded by \(\tilde{O} \left( \sqrt{\frac{M^2}{\mu \cdot \varepsilon} \cdot \chi(W) \lambda_{\max}(W)} \right)\).
4.3 Sums of smooth and convex functions

In this subsection, we propose a distributed optimization algorithm to find a solution of the problem $\text{2}$ when the function $\hat{F}(x)$ is smooth, i.e., $\hat{F}(x)$ has Lipschitz gradients. We follow the idea of regularization to induce strong convexity, this time on the primal problem. Moreover, we show the minimum number of iterations required to compute an approximate solution to the problem.

**Algorithm 4** Distributed FGM for smooth convex problems

1: All agents set $z^0_i = \hat{z}^0_i = 0 \in \mathbb{R}^n$, $q = \frac{\epsilon/\sqrt{R_w^2}}{L + \frac{\epsilon}{2R_w^2}} \lambda_{\text{min}}(W)$, $n_0$ solves $\frac{n_0^2 - n_0}{4n_0 - 1} = 1$ and $N$.
2: For each agent $i$
3: for $k = 0, 1, 2, \cdots, N - 1$ do
4: Set $\hat{x}^*_k(z^k_i) = \log \max x_i \{ (\hat{z}^k_i, x_i) - f_i(x_i) - \frac{\epsilon}{2R_w^2} \| x_i - x^*_i(0) \|^2 \}$
5: Share $\hat{x}^*_k(z^k_i)$ with neighbors, i.e., $\{ j \mid (i, j) \in E \}$.
6: $z^*_k + 1 \leftarrow z^*_k - \frac{\epsilon/\sqrt{R_w^2}}{\lambda_{\text{max}}(W)} \sum_{j=1}^m W_{ij} \hat{x}^*_k(z^k_i)$
7: Compute $\alpha_{k+1} \in (0, 1)$ from $\alpha_{k+1}^2 = (1 - \alpha_{k+1}) \alpha_k^2 + q \alpha_k$ and set $\beta_k = \alpha_k \alpha_{k+1}$
8: $\hat{z}^k_{k+1} = z^*_{k+1} + \beta_k (z^*_{k+1} - z^k_i)$
9: end for

**Theorem 4.3** Let $F(x)$ be dual friendly and Assumption 7(c) hold. For any $\epsilon > 0$, the output $x^*(z_N)$ of Algorithm 4 is an $(\epsilon, \epsilon/R)$-solution of (3) for

$$N \geq \sqrt{\frac{2LR_w^2}{\epsilon}} + 1 \log \left( \frac{8\sqrt{2}\lambda_{\text{max}}(W)R^2R_w^2}{\epsilon^2} \right),$$

where $\chi(W) = \lambda_{\text{max}}(W)/\lambda_{\text{min}}(W)$ and $R_x = \| x^* - x^*(0) \|_2$.

**Proof** Initially, consider the regularized problem

$$\min_{\sqrt{W}x=0} \hat{F}(x) \quad \text{where} \quad \hat{F}(x) \triangleq F(x) + \frac{\epsilon}{2R_w^2} \| x - x^*(0) \|^2_2, \quad (18)$$

where $F(x)$ is defined in (3). The function $\hat{F}(x)$ is $\bar{\mu}$-strongly convex with $\bar{\mu} = \frac{\epsilon}{2R_w^2}$ and $\bar{L}$-smooth with $\bar{L} = L + \bar{\mu}$. Given that the regularized primal function is strongly convex and smooth, we can use the results from Theorem 4.1. Particularly, in order to have an $(\epsilon/2, \epsilon/(2R))$-solution of problem (18), one can use Algorithm 2 with

$$N \geq 2 \sqrt{\frac{\bar{L} + \frac{\epsilon}{2R_w^2}}{\epsilon}} \chi(W) \log \left( \frac{4\sqrt{2}\lambda_{\text{max}}(W)R^2}{\bar{\mu} \cdot \epsilon} \right)$$

$$= 2 \sqrt{L + \frac{\epsilon}{2R_w^2}} \chi(W) \log \left( \frac{4\sqrt{2}\lambda_{\text{max}}(W)R^2}{L \cdot \epsilon} \right)$$

$$= 2 \sqrt{\left( \frac{2LR_w^2}{\epsilon} + 1 \right) \chi(W) \log \left( \frac{8\sqrt{2}\lambda_{\text{max}}(W)R^2R_w^2}{\epsilon^2} \right)}.$$

Having an $(\epsilon/2, \epsilon/(2R))$-solution of problem (18), guarantees that $\hat{x}_N^*$ is an $(\epsilon, \epsilon/(R))$-solution of problem (3), and the desired result follows. \qed
Theorem 4.3 states the communication complexity of Algorithm 4. Particularly, the total number of communication rounds required by each agent to find an \((\varepsilon, \varepsilon/R)\)-solution of (3) can be bounded by \(\tilde{O}\left(\sqrt{\frac{LR^2}{\varepsilon}}\right)\).

4.4 Sums of convex and \(M\)-Lipschitz functions

In this subsection, we present the distributed algorithm for optimization of convex function when no strong convexity or smoothness is guaranteed. The main idea is to use regularization both in the primal and the dual problem. Therefore, we can build our algorithm and its analysis from the results in Theorem 4.2 and Theorem 4.3. Next, we present the proposed algorithm and their convergence analysis.

**Algorithm 5** Distributed FGM for \(M\)-Lipschitz functions

```plaintext
1: All agents set \(z_0^i = \hat{z}_0^i = 0 \in \mathbb{R}^n, q = \frac{\alpha_0}{\lambda_{\max}(W)/(\varepsilon/R)^2 + 1}, \alpha_0 \text{ solves } \frac{\alpha_0^2 - 2}{1 - \alpha_0} = 1 \text{ and } N.
2: For each agent \(i\)
3: for \(k = 0, 1, 2, \ldots, N - 1\) do
4: Set \(\hat{x}_k^i = \arg\max_{x_i} \{x_i - f_i(x_i) - \frac{\varepsilon}{2R^2} \|x_i - x_i^*(0)\|_2^2\}
5: Share \(\hat{x}_k^i\) with neighbors, i.e. \(\{j \mid (i, j) \in E\}.
6: Compute \(\alpha_{k+1} \in (0, 1)\) from \(\alpha_{k+1}^2 = (1 - \alpha_{k+1})\alpha_k^2 + q\alpha_{k+1}\) and set \(\beta_k = \frac{\alpha_k(1 - \alpha_k)}{\alpha_k^2 + \alpha_{k+1}}
7: \(\tilde{z}_{k+1}^i = z_k^i + \beta_k(z_k^i - z_k^i)
8: \(\hat{z}_{k+1}^i = \hat{z}_k^i + \beta_k(\hat{z}_k^i - \hat{z}_k^i)
9: end for
```

**Theorem 4.4** Let \(F(x)\) be dual friendly and Assumption [4d] hold. For any \(\varepsilon > 0\), the output \(x^*(z_N)\) of Algorithm 5 is an \((\varepsilon, \varepsilon/R)\)-solution of (3) for

\[
N \geq 2\sqrt{16\chi(W)\frac{M^2R^2}{\varepsilon^2}} + 1 \log \left(16\chi(W)\frac{M^2R^2}{\varepsilon^2} + 1\right),
\]

where \(\chi(W) = \lambda_{\max}(W)/\lambda_{\min}^+(W), R = \|y^*\|_2, \text{ and } R_x = \|x^* - x^*(0)\|_2\).

**Proof** Consider again, as in Theorem 4.3, the regularized problem (18) where \(F(x)\) is defined in (3). The function \(\tilde{F}(x)\) is \(\mu\)-strongly convex with \(\mu = \frac{2L}{\varepsilon}\). However, we have assumed now that \(F(x)\) is not smooth. Nevertheless, from Theorem 4.2 we have that Algorithm 3 will generate an \((\varepsilon/2, \varepsilon/(2R))\)-solution of (18), namely \(x_N\), for

\[
N \geq 2\sqrt{16\chi(W)\frac{M^2R^2}{\mu \cdot \varepsilon}} + 1 \log \left(16\chi(W)\frac{M^2}{\mu \cdot \varepsilon} + 1\right) = 2\sqrt{16\chi(W)\frac{M^2}{2R \varepsilon}} + 1 \log \left(16\chi(W)\frac{M^2}{2R \varepsilon} + 1\right) = 2\sqrt{16\chi(W)\frac{M^2R^2}{\varepsilon^2}} + 1 \log \left(16\chi(W)\frac{M^2R^2}{\varepsilon^2} + 1\right).
\]

Therefore, by Proposition 4.1, \(x^*(z_N)\) is an \((\varepsilon, \varepsilon/R)\)-solution for problem (3). \(\square\)
Theorem 4.4 states the communication complexity of Algorithm 5. Particularly, the total number of communication rounds required by each agent to find an \((\varepsilon, \varepsilon/R)\)-solution of (3) can be bounded by \(\tilde{O} \left( \sqrt{\frac{R^2 M^2}{\varepsilon^2}} \chi(W) \right)\).

5 Discussion

Table 2 presents a summary of the results presented in Section 4. In particular, it shows the number of communication rounds required to obtain an \((\varepsilon, \varepsilon/R)\)-solution for each the presented properties of the function \(F(x)\).

| Property of \(F(x)\) | Iterations Required |
|----------------------|---------------------|
| \(\mu\)-strongly convex and \(L\)-smooth | \(\tilde{O} \left( \sqrt{\frac{L}{\mu}} \chi(W) \right)\) |
| \(\mu\)-strongly convex and \(M\)-Lipschitz | \(\tilde{O} \left( \sqrt{\frac{M^2}{\mu\varepsilon}} \chi(W) \right)\) |
| \(L\)-smooth | \(\tilde{O} \left( \sqrt{\frac{LR^2}{\varepsilon}} \chi(W) \right)\) |
| \(M\)-Lipschitz | \(\tilde{O} \left( \sqrt{\frac{M^2 R^2}{\varepsilon^2}} \chi(W) \right)\) |

Table 2: A summary of algorithmic performance

The estimates in Table 2 are optimal up to logarithmic factors. In the smooth cases, where \(L < \infty\), these estimates follow from classical centralized complexity estimation of the FGM algorithm. In the distributed setting, one has to perform \(O\left(\sqrt{\chi(W)} \log(1/\varepsilon)\right)\) additional consensus steps at each iteration. This corresponds to the number of iterations needed to solve the consensus problem

\[
\min_{x} \frac{1}{2} \langle x, W x \rangle ,
\]

where \(W\) is a communication matrix as defined in Section 2. FGM provides a direct estimate on the number of iterations required to reach consensus, given that (19) is \(\sigma_{\min}(\sqrt{W})\)-strongly convex in \(x_0 + \ker(W)\) and has \(\sigma_{\max}(\sqrt{W})\)-Lipschitz continuous gradients, and this estimate cannot be improved up to constant factors.

The specific value of \(\chi(W)\), and its dependency on the number of nodes \(m\) has been extensively studied in the literature of distributed optimization \[14\]. In \[65\], Proposition 5 provides an extensive list of worst-case dependencies of the spectral gap for large classes of graphs. Particularly, for fixed undirected graphs, in the worst case we have \(\chi(W) = O\left(m^2\right) \[41\]. This matches the best upper bound found in the literature of consensus and distributed optimization \[66,67\]. Thus, the consensus set described by the constraint \(\sqrt{W}x = 0\) should be preferred over the description as \(Wx = 0\), even though both representations correctly describe the consensus subspace \(x_1 = \ldots = x_m\). Particularly, when we pick \(A = \sqrt{W}\), we have \(\chi(A^T A) = \chi(W)\) instead of \(\chi(W) = \chi(W^2) \gg \chi(W)\).

The cases when \(F(x)\) is convex or strongly convex can be generalized to \(p\)-norms, with \(p \geq 1\), see \[35\]. The definitions of the condition number \(\chi\) needs to be defined accordingly. Let’s introduce a norm \(\|x\|_p^2 = \|x_1\|^2_p + \ldots + \|x_m\|^2_p\) for \(p \geq 1\) and assume that \(F(x)\) is \(\mu\)-strongly convex and \(L\)-Lipschitz continuous gradient in this (new) norm \(\|\cdot\|_p\) (in \(\mathbb{R}^m\)), see \[68\] (Lemma 1), \[69\] (Lemma 1) and \[52\] (Theorem 1). Thus,

\[
\chi(W) = \max_{\|h\|_1 = 1} \frac{\langle h, W h \rangle}{\mu L} < \min_{\|h\|_1 = 1, h \perp \ker(W)} \frac{\langle h, W h \rangle}{L}.
\]
Note that we typically do not know $R$ or $R_x$. Thus, we require a method to estimate the strong convexity parameter, which is challenging \cite{70,71}. Some recent work have explored restarting techniques to reach optimal convergence rates when the strong convexity parameters are unknown \cite{71,72}. Similarly, a generalization of the FGM algorithm can be proposed when the smoothness parameter is unknown \cite{55}. However, the effect of restarting in the distributed setup requires further study and is out of the scope of this paper.

6 Results: Algorithms and Iteration Complexity Analysis when $F(x)$ is not dual friendly

The results in Section 4 assume $F(x)$ is dual-friendly, see Definition 4.1. In this section, we explore the case when no exact solution to the dual problem is available. We will build on the results in \cite{73,74} on the analysis of first-order methods with inexact oracle, and provide a set of distributed algorithms and their respective iterations complexities. Initially for completeness, we recall the definition of an inexact oracle for a smooth strongly convex function, and the corresponding iteration complexity of FGM with an inexact oracle.

Definition 6.1 (Definition 1 in \cite{73}) Let function $f$ be convex on a convex set $Q$. We say that it is equipped with a first-order $(\delta, L, \mu)$-oracle if for any $y \in Q$ we can compute a pair $(f(\delta, L, \mu)(y), g(\delta, L, \mu)(y)) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$\frac{\mu}{2} \|x - y\|^2 \leq f(x) - (f(\delta, L, \mu)(y) + \langle g(\delta, L, \mu)(y), x - y \rangle) \leq \frac{L}{2} \|x - y\|^2 + \delta,$$

for all $x \in Q$ where $\delta \geq 0$ and $L \geq \mu \geq 0$.

Theorem 6.1 (Theorem 7 in \cite{73}) The FGM in \cite{1} applied to a function $f$ endowed with a $(\delta, L, \mu)$-oracle generates a sequence $\{y_k\}_{k>1}$ satisfying:

$$f(y_k) - f^* \leq LR^2 \exp \left( -\frac{k}{2} \sqrt{\frac{\mu}{L}} \right) + \left( 1 + \sqrt{\frac{L}{\mu}} \right) \delta.$$ 

Now, we recall an auxiliary result that shows that an approximate solution to the auxiliary inner maximization problem \cite{9} defines a $(\delta, L, \mu)$-oracle. Furthermore, in the sequel, we describe the distributed algorithms and their iterations complexities, similarly as in Section 4 when we remove the assumption of the function $F$ being dual friendly.

Theorem 6.2 (Theorem 2 in \cite{73}) Assume that $f$ is $\mu$-strongly convex and $L$-smooth. Let $y \in \mathbb{R}^n$ an assume that instead of computing $x^*(A^Ty)$, the unique solution of the subproblem \cite{9}, we compute $w(A^Ty)$ such that:

$$\Psi(y, x^*(A^Ty)) - \Psi(y, w(A^Ty)) \leq \xi.$$ (20)

Then,

$$\Psi(y, w(A^Ty)) - \xi = f(w(A^Ty)) + \langle Aw(A^Ty), y \rangle - \xi, Aw(A^Ty)$$

is a $(\delta, L_{\varphi, \delta}, \mu_{\varphi, \delta})$-oracle for $\varphi$ with $\delta = 3\xi$, $L_{\varphi, \delta} = 2L_{\varphi}$ and $\mu_{\varphi, \delta} = \frac{1}{2} \mu_{\varphi}$.

We are now ready to state the algorithms and their convergence rates for the distributed optimization of sums of non-dual friendly convex functions.
Algorithm 6 Distributed FGM for non-dual friendly strongly convex and smooth problems

1: All agents set \( z_0^i = \tilde{z}_0^i = 0 \in \mathbb{R}^n \), \( q = \frac{4}{\mu} \), \( \tilde{q} = \tilde{q}_{\max}(W) \), \( \alpha_0 \) solves \( \frac{\alpha_0^2 - q}{4\mu q} = 1 \), \( \tilde{\alpha}_0 \) solves \( \frac{\tilde{\alpha}_0^2 - \tilde{q}}{4\mu \tilde{q}} = 1 \), \( T \) and \( N \).
2: For each agent \( i \)
3: for \( k = 0, 1, 2, \ldots, N - 1 \) do
4: \( w_0^i = w_0^{i+1} = 0 \in \mathbb{R}^n \)
5: for \( t = 0, 1, 2, \ldots, T - 1 \) do
6: \( w_{t+1}^i = w_t^i + \frac{\tilde{q}}{\mu} (R_k - \nabla f_i(u_t^i)) \)
7: Compute \( \tilde{\alpha}_{t+1} \in (0, 1) \) from \( \tilde{\alpha}_{t+1}^2 = (1 - \tilde{\alpha}_{t+1}) \tilde{\alpha}_t^2 + \tilde{\phi}_{t+1} + \tilde{\beta}_t = \frac{\tilde{\alpha}_t (1 - \tilde{\alpha}_t)}{\tilde{\alpha}_t + \tilde{\alpha}_t + \tilde{\alpha}_t + \tilde{\alpha}_t} \)
8: \( \tilde{w}_{t+1}^i = w_{t+1}^i + \tilde{\beta}_t (w_{t+1}^i - w_t^i) \)
9: end for
10: Share \( w_T^i \) with neighbors, i.e. \( \{ j \mid (i, j) \in E \} \).
11: \( z_{t+1}^i = z_t^i - \frac{\mu}{\max(W)} \sum_{m=1}^M W_{ij} w_t^j \)
12: Compute \( \alpha_{t+1} \in (0, 1) \) from \( \alpha_{t+1}^2 = (1 - \alpha_{t+1}) \alpha_t^2 + q \alpha_{t+1} + \beta_t = \frac{\alpha_t (1 - \alpha_t)}{\alpha_t + \alpha_t + \alpha_t + \alpha_t} \)
13: \( \tilde{z}_{t+1}^i = z_{t+1}^i + \beta_t (z_{t+1}^i - z_t^i) \)
14: end for

6.1 Sum of non-dual friendly strongly convex and smooth functions

In this subsection, we introduce a distributed algorithm for the minimization of sums of strongly convex and smooth functions removing the assumption of dual friendliness. Moreover, we provide its iteration complexity.

**Theorem 6.3** Let \( F(x) \) be a function such that Assumption 4(a) hold. For any \( \varepsilon > 0 \), the output \( x^*(z_N) \) of Algorithm 6 is an \( (\varepsilon, \varepsilon/R) \)-solution of (3) for

\[
N \geq 8 \sqrt{\frac{L}{\mu}} \chi(W) \log \left( \frac{2\sqrt{2}\lambda_{\max}(W)R^2}{\mu \cdot \varepsilon} \right)
\]

and

\[
T \geq \sqrt{\frac{L}{\mu}} \log \left( \frac{6LR^2R^2_w}{\varepsilon^2} \right) \sqrt{\frac{L}{\mu}} \chi(W),
\]

where \( R = \|y^*\|^2, R_x = \|x^* - x^*(0)\|^2, R_w = R_x + \|x^*\|^2 \) and \( \chi(W) = \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)} \).

**Proof** Lines 5 – 7 in Algorithm 6 are the FGM on the inner problem. Therefore, \( \Psi(y, x^*(A^T y)) - \Psi(y, w_T(A^T y)) \leq 0 \) for \( T \geq \sqrt{L/\mu} \log (L R^2_w/\varepsilon) \). Note that at the beginning of iteration \( k \), \( R_w = \|w_0 - w^*\| \), \( w_0 = 0 \), and \( w^* = x^*(\tilde{z}_i) \). Therefore, \( R_w = \|x^*(\tilde{z}_k)\| \leq \|x^*(\tilde{z}_k) - x^*\|^2 + \|x^*\|^2 \leq \|x^*(0) - x^*\|^2 + \|x^*\|^2 \).

Moreover, Theorem 6.2 shows us that we have endowed the function \( \varphi \) with an \((3\varepsilon, 2L_{\varphi}, 2\mu_{\varphi})\)-oracle. Thus, from Theorem 6.1 it holds that

\[
\varphi(y_k) - \varphi^* \leq L_{\varphi} R^2 \exp \left( -\frac{k}{4} \sqrt{\frac{R_w}{L_{\varphi}}} \right) + \left( 1 + 2 \sqrt{\frac{L_{\varphi}}{\mu}} \right) 3\xi.
\]

Now, recall from Theorem 4.1 that

\[
\|\sqrt{W}x^*(\sqrt{W}y_k)\|^2 \leq 2L_{\varphi} (\varphi(y_k) - \varphi^*),
\]
\[ \leq 2L_\varphi^2 R^2 \left( \exp \left( -\frac{k}{4} \sqrt{\frac{\mu_\varphi}{L_\varphi}} \right) + \left( 1 + 2 \sqrt{\frac{L_\varphi}{\mu_\varphi}} \right) \frac{3\xi}{2} \right). \]

Therefore, in order to have \( \| \sqrt{W} x^* (\sqrt{W} y_k) \|_2 \leq \varepsilon/R \) it is necessary that

\[ N \geq 8 \sqrt{L_\varphi}{\mu_\varphi} \log \left( \frac{\sqrt{6}L_\varphi R^2}{\varepsilon} \right) \quad \text{and} \quad \xi \leq \frac{\varepsilon^2}{6R^2} \sqrt{\frac{\mu_\varphi}{L_\varphi}}. \]

Moreover,

\[ |\langle y_k, \sqrt{W} x^* (\sqrt{W} y_k) \rangle |^2 \leq 4R^2 \| \sqrt{W} x^* (\sqrt{W} y_k) \|_2^2, \]

\[ \leq 8R^4 L_\varphi^2 \left( \exp \left( -\frac{k}{4} \sqrt{\frac{\mu_\varphi}{L_\varphi}} \right) + \left( 1 + 2 \sqrt{\frac{L_\varphi}{\mu_\varphi}} \right) \frac{3\xi}{2} \right). \]

Therefore, in order to have \( f(z_N) - f^* \leq \varepsilon \) it is necessary that

\[ N \geq 8 \sqrt{L_\varphi}{\mu_\varphi} \log \left( \frac{\sqrt{8}L_\varphi R^2}{\varepsilon} \right) \quad \text{and} \quad \xi \leq \frac{\varepsilon^2}{6} \sqrt{\frac{L_\varphi}{\mu_\varphi}}. \]

Finally, we can conclude that to obtain an \( (\varepsilon, \varepsilon/R) \)-solution of (3) we require

\[ N \geq 8 \sqrt{L_\varphi}{\mu_\varphi} \log \left( \frac{2\sqrt{2}L_\varphi R^2}{\varepsilon} \right) \quad \text{and} \quad T \geq \sqrt{\frac{L}{\mu}} \log \left( \frac{6LR^2 R^2}{\varepsilon^2} \sqrt{\frac{L_\varphi}{\mu_\varphi}} \right). \]

The desired result follows from the definitions of \( L_\varphi \) and \( \mu_\varphi \). \qed

Theorem 6.3 shows that if no-dual solution is explicitly available for (9), then one can use FGM to find an approximate solution. This approximate solution is itself an inexact oracle. Particularly, the number of total number of communication rounds required by each agent to find an \( (\varepsilon, \varepsilon/R) \)-solution of (3) can be bounded by \( O \left( \sqrt{L_\mu} \chi(W) \log (1/\varepsilon) \right) \). Moreover, at each communication round the number of local oracle calls for each agent can be bounded by \( O \left( \sqrt{L_\mu} \log (1/\varepsilon) \right) \). Unfortunately, the total number of oracle calls for each agent at all communication rounds is bounded by \( O \left( \frac{L_\mu}{\mu} \log (1/\varepsilon) \right) \). However, in the centralized case, one oracle call corresponds to the gradient computation of \( F(x) \) which is composed by \( m \) functions \( f_i \) for \( 1 \leq i \leq m \), whereas in the distributed case, local oracle calls are computed in parallel by all agents at the same time. Therefore, one can argue that if the number of agent \( m \) is of the order of \( \sqrt{L_\mu} \), then provided estimates are optimal given that the oracle calls of all agents in the network are done in parallel.

6.2 Sums of non-dual friendly smooth convex functions

In this subsection we propose a distributed algorithm for the distributed minimization of sums of non-dual friendly smooth convex functions and provide its iteration complexity.
Theorem 6.4 Let $F(x)$ be a function such that Assumption 1(c) hold. For any $\varepsilon > 0$, the output $x^*(z_N)$ of Algorithm 7 is an $(\varepsilon, \varepsilon/R)$-solution of (3) for

$$N \geq 8 \sqrt{\left(\frac{2LR^2}{\varepsilon} + 1\right) \chi(W) \log \left(\frac{8\sqrt{2}\lambda_{\max}(W)R^2}{\varepsilon^2}\right)},$$

and

$$T \geq \sqrt{\frac{2LR^2}{\varepsilon} + 1 \log \left(\frac{2\sqrt{6}R^2R^2}{\varepsilon} \left(\frac{L}{\varepsilon} + 1 + \frac{1}{2R^2}\right) \sqrt{\left(\frac{2LR^2}{\varepsilon} + 1\right) \chi(W)}\right)},$$

where $R = \|y^*\|_2$, $R_x = \|x^* - x^*(0)\|_2$, $R_w = R_x + \|x^*\|_2$, and $\chi(W) = \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)}$.

Proof Similarly as in Theorem 6.3, we consider the regularized primal problem (15). Therefore, the auxiliary inner maximization problem seeks to maximize an $\bar{\mu}$-strongly convex function, with $\bar{\mu} = \frac{\varepsilon}{2R^2}$, that is also $\bar{L}$-smooth, with $L = L + \bar{\mu}$. It follows from Theorem 6.3 that Algorithm 7 will generate obtain an $(\varepsilon/2, \varepsilon/(2R))$-solution for problem (18) for

$$N \geq 8 \sqrt{\frac{\bar{L}}{\bar{\mu}} \chi(W) \log \left(\frac{4\sqrt{2}\lambda_{\max}(W)\bar{R}^2}{\bar{\mu} \cdot \varepsilon}\right)},$$

and

$$T \geq \sqrt{\frac{\bar{L}}{\bar{\mu}} \log \left(\frac{2\sqrt{6}\bar{L}\bar{R}^2\bar{R}^2}{\varepsilon^2} \sqrt{\frac{\bar{L}}{\bar{\mu}} \chi(W)}\right)},$$

where $\bar{L} = \frac{L}{\mu}$, $\bar{\mu} = \frac{\varepsilon}{2R^2}$, and $\chi(W) = \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)}$.
\[
\sqrt{\frac{2LR^2}{\varepsilon}} + 1 \log \left( \frac{2\sqrt{6}R^2W}{\varepsilon} \left( \frac{L}{\varepsilon} + \frac{1}{2R^2} \right) \sqrt{\left( \frac{2LR^2}{\varepsilon} + 1 \right) \chi(W)} \right).
\]

Finally, from Proposition 4.1, it holds that an \((\varepsilon/2, \varepsilon/(2R))\)-solution for problem (18) is an \((\varepsilon, \varepsilon/R)\)-solution of (3).

Theorem 6.4 shows that the total number of communication rounds required by each agent to find an \((\varepsilon, \varepsilon/R)\)-solution of (3), when the functions are not strongly convex, can be bounded by \(\tilde{O}\left( \sqrt{\frac{L R^2}{\varepsilon}} \sqrt{\chi(W)} \right)\).

Moreover, at each communication round the number of local oracle calls for each agent can be bounded by \(\tilde{O}\left( \sqrt{\frac{L R^2}{\varepsilon}} \right)\). Similarly as in Theorem 6.3, the total number of oracle calls of each agent can be bounded by \(O\left( \frac{L R^2}{\varepsilon} \sqrt{\chi(W)} \right)\).

6.3 Distributed optimization of sums of non-smooth functions

In this subsection, we present an approach for developing distributed algorithms for non-smooth optimization, i.e., either Assumption 1(b) or Assumption 1(d) hold. These scenarios have been recently studied in [46], where similar convergence rates have been derived. However, our particular selection of \(\sqrt{W}\) instead of \(W\) allows for a better dependency in terms of the graph condition number.

Initially, consider a convex function \(f\) that is also \(M\)-Lipschitz, i.e., Assumption 1(d), and apply Nesterov’s smoothing technique \([52, 75]\) to (7) as follows:

\[
\min_{Ax=0} f(x) = \min_x \left\{ \max_y \{ \langle y, Ax \rangle - f(x) \} \right\},
\]

and add the regularization term as

\[
\min_x \left\{ \max_y \left\{ \langle y, Ax \rangle - \frac{\varepsilon}{2R^2} \|y\|_2^2 - f(x) \right\} \right\}.
\]

Moreover, define

\[
F_\varepsilon(Ax) = \max_y \left\{ \langle y, Ax \rangle - \frac{\varepsilon}{2R^2} \|y\|_2^2 \right\} = \frac{R^2}{2\varepsilon} \|Ax\|_2^2,
\]

and we obtain the following composite smooth/non-smooth optimization problem

\[
\min_{\|x\|_2 \leq R_x} \{ F_\varepsilon(Ax) + f(x) \}
\]

where \(F_\varepsilon\) is \(L_\varepsilon\)-smooth, with \(L_\varepsilon = \frac{2\max(A^T A)R^2}{\varepsilon}\), and \(f\) is \(M\)-Lipschitz. For this class of composite problems, one can use the accelerated gradient sliding method proposed in [76]. As a result, it follows from Corollary 2 in [76], that in order to find an \((\varepsilon)\)-solution for problem (3), the total number of oracle calls for \(F_\varepsilon\) and \(f\) can be bounded by

\[
O \left( \sqrt{\frac{L_\varepsilon R^2}{\varepsilon^2}} \right) = O \left( \sqrt{\frac{M^2 R^2}{\varepsilon^2} \chi(W)} \right),
\]
and
\[ O\left( \frac{M^2 R^2}{\varepsilon^2} + \sqrt{\frac{L_x R^2}{\varepsilon}} \right) = O\left( \frac{M^2 R^2}{\varepsilon^2} + \sqrt{\frac{M^2 R^2}{\varepsilon^2} \chi(W)} \right), \]
respectively.

Similarly, if we additionally assume that the function \( f \) is \( \mu \)-strongly convex, i.e., Assumption 1(b). Then, from Theorem 3 in [76] it follows that the total number of oracle calls for \( F_\varepsilon \) and \( f \) required by the multi-phase gradient sliding algorithm to find an \( (\varepsilon) \)-solution for problem (3) can be bounded by
\[ O\left( \sqrt{\frac{L_x}{\mu}} \log \left( \frac{R_x}{\varepsilon} \right) \right) = O\left( \sqrt{\frac{M^2}{\varepsilon} \chi(W) \log \left( \frac{\mu R^2}{\varepsilon} \right)} \right), \]
and
\[ O\left( \frac{M^2}{\varepsilon \mu} + \sqrt{\frac{L_x}{\mu}} \log \left( \frac{R_x}{\varepsilon} \right) \right) = O\left( \frac{M^2}{\varepsilon \mu} + \sqrt{\frac{M^2}{\varepsilon \mu} \chi(W) \log \left( \frac{\mu R_x^2}{\varepsilon} \right)} \right), \]
respectively.

It follows from [76] that the above estimates are optimal up to logarithmic factors. Moreover, one can extend these results to stochastic optimization problems and the estimations will not change [46].

7 Improving on the Dependence of the Strong Convexity Parameter: Computation-Communication Trade-Off

Considering the general problem in (1), the condition number \( L/\mu \) can be large if one of the \( \mu_i \) is small or even zero. It follows from Sections 4 and 3 that the iteration complexity of the proposed algorithms can be very large, even if only one of the functions has a small strong convexity. In this section, we propose a reformulation of the original problem (1) such that the dependency on the individual strong convexity constants can be improved. However, we will see that the improvement on the dependency of the condition number of the function \( F \) comes at a price in terms of the communication rounds.

Consider the following problem:
\[ \min_{x \in \mathcal{W}} F_\alpha(x) = F(x) + \frac{\alpha}{2} \langle x, Wx \rangle = \sum_{i=1}^m f_i(x_i) + \frac{\alpha}{2} \langle x, Wx \rangle. \tag{22} \]
A solution to (22) is clearly a solution to (1).

The function \( F_\alpha \) is \( \mu_\alpha \)-strongly convex with \( \mu_\alpha = \min \{ \sum_{i=1}^m \mu_i, \alpha \lambda_{\min}(W) \} \) and has \( L_\alpha \)-Lipschitz continuous gradients with \( L_\alpha = L + \alpha \lambda_{\max}(W) \). Choose \( \alpha = \frac{\sum_{i=1}^m \mu_i}{\lambda_{\min}(W)} \) and the function \( F_\alpha \) will have a condition number
\[ \frac{L_\alpha}{\mu_\alpha} = \frac{\max_i L_i}{\sum_{i=1}^m \mu_i} + \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)} = \frac{\max_i L_i}{\sum_{i=1}^m \mu_i} + \chi(W). \]

Unfortunately, the structure of the function \( F_\alpha \) does not allow a decentralized computation of a solution for the inner problem (9) as in (14), i.e., each agent can compute the solution \( x^*_i \) using local information only. Nevertheless, the additional term in (22) has a gradient with a network structure and can be computed in a
determined manner using information shared from the neighbors of each agent. Particularly, consider the auxiliary dual problem

$$\varphi_\alpha(y) = \max_x \psi_\alpha(x, y) \quad \text{where} \quad \psi_\alpha(x, y) = \langle x, \sqrt{W} y \rangle - F(x) - \frac{\alpha}{2} \langle x, Wx \rangle. \quad (23)$$

Then, we have that

$$\nabla_x \psi_\alpha(x, y) = \sqrt{W} y - \nabla F(x) - \alpha Wx.$$  

In this case, we can use the FGM to obtain an approximate solution to the inner problem using the classic result \cite{5}. In Algorithm \cite{8} we propose a modification of Algorithm \cite{6} to take into account the new structure for the solution of the inner auxiliary problem.

**Algorithm 8** Augmented Distributed FGM for strongly convex and smooth problems

1: All agents set \(w_0^i = z_0^i = 0 \in \mathbb{R}^n\), \(\mu_{\alpha} = \sum_{i=1}^{m} \mu_i\), \(\alpha = \mu_\alpha / \lambda_{\min}(W)\), \(L_{\alpha} = L + \alpha \lambda_{\max}(W)\), \(\hat{q} = \frac{\mu_{\alpha}}{\lambda_{\min}(W)}\), \(\alpha_0 \geq 0\)
2: for each agent \(i\)
3: for \(t = 0, 1, 2, \ldots, N - 1\) do
4: \(w_0^i = w_0^i = 0 \in \mathbb{R}^n\)
5: \(t = 0, 1, 2, \ldots, T - 1\) do
6: Share \(w_t^i\) with neighbors, i.e. \(\{j \mid (i, j) \in E\}\).
7: \(w_{t+1}^i = w_t^i + \frac{1}{\mu_{\alpha}} (z_{t+1}^i - \nabla f_i(w_t^i) - \alpha \sum_{j=1}^{m} W_{ij} \tilde{w}_t^j)\)
8: Compute \(\tilde{\alpha}_{t+1} \in (0, 1)\) from \(\tilde{\alpha}_{t+1} = (1 - \tilde{\alpha}_{t+1}) \tilde{\alpha}_t + q \tilde{\alpha}_{t+1}\) and set \(\tilde{\beta}_t = \frac{\tilde{\alpha}_t(1 - \tilde{\alpha}_{t+1}) \tilde{\alpha}_{t+1}}{\tilde{\alpha}_t + \tilde{\alpha}_{t+1}}\)
9: \(\tilde{\omega}_{t+1} = w_{t+1}^i + \tilde{\beta}_t(w_{t+1}^i - w_t^i)\)
10: end for
11: Share \(w_{T}^i\) with neighbors, i.e. \(\{j \mid (i, j) \in E\}\).
12: \(z_{k+1}^i = z_{k}^i - \frac{\mu_{\alpha}}{\lambda_{\max}(W)} \sum_{j=1}^{m} W_{ij} w_{T}^j\)
13: Compute \(\alpha_{k+1} \in (0, 1)\) from \(\alpha_{k+1} = (1 - \alpha_{k+1}) \alpha_k^2 + q \alpha_{k+1}\) and set \(\beta_k = \frac{\alpha_k(1 - \alpha_k)}{\alpha_k + \alpha_{k+1}}\)
14: \(z_{k+1}^i = z_{k+1}^i + \beta_k(z_{k+1}^i - z_k^i)\)
15: end for

**Corollary 7.1** Let \(F(x)\) be defined in \cite{2}, and assume \(f_i\) is \(L_i\)-smooth for \(1 \leq i \leq m\) and \(\bar{\mu} = \sum_{i=1}^{m} \mu_i > 0\). For any \(\varepsilon > 0\), the output \(x^* (z_N)\) of Algorithm 8 is an \((\varepsilon, \varepsilon / R)\)-solution of (3) for

\[
N \geq 8 \sqrt{\frac{L}{\bar{\mu}} + \chi(W)} \chi(W) \log \left( \frac{2 \sqrt{2} \lambda_{\max}(W) R^2}{\bar{\mu} \cdot \varepsilon} \right),
\]

and

\[
T \geq \sqrt{\frac{L}{\bar{\mu}} + \chi(W)} \log \left( \frac{6L_{\alpha} R^2 R_w^2}{\varepsilon^2} \sqrt{\frac{L}{\bar{\mu}} + \chi(W)} \chi(W) \right),
\]

where \(R = \|y^*\|_2\), \(R_x = \|x^* - x^*(0)\|_2\), \(R_w = R_x + \|x^*\|_2\), and \(\chi(W) = \lambda_{\max}(W) / \lambda_{\min}(W)\).
Corollary 7.1 implies that at each of the outer iterations, required to obtain an approximate solution to the inner maximization problem, the number of oracle calls for $f$ and communication rounds between agents can be bounded by

$$
\tilde{O}\left(\sqrt{\frac{L}{\mu} + \chi(W)}\right).
$$

Moreover, the number of outer communication rounds can be bounded by

$$
\tilde{O}\left(\sqrt{\frac{L}{\mu} + \chi(W)}\right).
$$

The total number of communications rounds and local oracle calls, taking into account the inner and outer loops is

$$
\tilde{O}\left(\sqrt{\frac{L}{\mu} + \chi(W)}\right).
$$

This estimate shows that we can replace the smallest strong convexity constant for the sum among all of them, but we have to pay an additive price proportional to the condition number of the graph and additional communication rounds in the inner maximization problem proportional to the number of oracle calls for $f$.

This result can be extended to the case when $F(x)$ is just smooth by using the regularization technique with $\mu_i = \varepsilon/(R^2_i)$. Particularly, consider the regularized function

$$
\hat{F}_\alpha(x) = F(x) + \frac{\varepsilon}{R^2} \|x - x^*(0)\|^2_2 + \frac{\alpha}{2} \langle x, Wx \rangle
$$

Chooses $\alpha = \frac{m\varepsilon}{\lambda_{\text{max}}(W)}$. Under this specific choice of $\alpha$, the function $F_\alpha$ will have a condition number

$$
\hat{L}_\alpha = \frac{L}{m\mu} + \frac{\lambda_{\text{max}}(W)}{\lambda_{\text{min}}(W)} + \frac{R^*_2L}{m\varepsilon} + \chi(W) + 1.
$$

The next Corollary shows the complexity of the proposed distributed augmented algorithm for the solution of sums of smooth convex functions.

**Corollary 7.2** Let $F(x)$ be a function such that Assumption 7(a) hold. Then, for any $\varepsilon > 0$, the output $x^*(z_N)$ of Algorithm 8 is an $(\varepsilon, \varepsilon/R)$-solution of (3) for

$$
N \geq 8\sqrt{\frac{2R^*_2L}{m\varepsilon} + \chi(W) + 1} \chi(W) \log C_1,
$$

and

$$
T \geq \sqrt{\frac{2R^*_2L}{m\varepsilon} + \chi(W) + 1} \log C_2,
$$

where

$$
C_1 = \frac{8\sqrt{2}\lambda_{\text{max}}(W)R^*_2R^2}{m \cdot \varepsilon^2}.
$$
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Algorithm 9 Augmented Distributed FGM for smooth problems

1. All agents set \( z_0^i = \tilde{z}_0^i = 0 \in \mathbb{R}^n, \mu_0 = m \frac{\varepsilon}{R^2}, \alpha = \mu_0 / \lambda_{\text{min}}(W), L_0 = L + \alpha \lambda_{\text{max}}(W) + m \frac{R^2}{\varepsilon} \), \( \tilde{q} = \frac{\mu_0}{L_0}, q = \frac{\lambda^\alpha_{\text{min}}(W)}{\lambda_{\text{min}}(W)} \).

2. For each agent \( i \)
3. for \( t = 0, 1, 2, \ldots, N - 1 \) do
4. \( w_0^i = \tilde{w}_0^i = 0 \in \mathbb{R}^n \)
5. for \( t = 0, 1, 2, \ldots, T - 1 \) do
6. Share \( \tilde{w}_i^t \) with neighbors, i.e. \( \{ j \mid (i, j) \in E \} \).
7. \( w_1^t + 1 = \tilde{w}_1^t + \alpha \sum_{j=1}^{n} W_{ij} \tilde{w}_j^t - \frac{\alpha}{\tilde{q}} (w_i^t - x^*_i(0)) \)
8. Compute \( \alpha_{t+1} \in (0, 1) \) from \( \tilde{q}_{t+1} = (1 - \alpha_{t+1}) \tilde{q}_{t} + \bar{q} \alpha_{t+1} \) and set \( \beta_t = \frac{\alpha_k(1 - \alpha_k)}{\alpha_k + \alpha_{k+1}} \).
9. \( \tilde{w}_{t+1}^i = w_{t+1}^i + \beta_t (w_{t+1}^i - w_i^t) \)
10. end for
11. Share \( w_{t+1}^i \) with neighbors, i.e. \( \{ j \mid (i, j) \in E \} \).
12. \( z_{t+1}^i = \tilde{z}_t^i - \frac{\bar{q}}{\lambda_{\text{max}}(W)} \sum_{j=1}^{n} W_{ij} w_j^t \)
13. Compute \( \alpha_{k+1} \in (0, 1) \) from \( \alpha_{k+1}^2 = (1 - \alpha_{k+1}) \alpha_k^2 + q \alpha_{k+1} \) and set \( \beta_k = \frac{\alpha_k(1 - \alpha_k)}{\alpha_k + \alpha_{k+1}} \).
14. \( \tilde{z}_{k+1}^i = z_{k+1}^i + \beta_k (z_{k+1}^i - z_i^k) \)
15. end for

\[
C_2 = \frac{24(L + \alpha \lambda_{\text{max}}(W) + m \frac{\varepsilon}{R^2}) R^2 R_{\omega}^2}{\varepsilon^2} \left( \left( \frac{2R^2 L}{m \varepsilon} + \chi(W) \right) + 1 \right) \chi(W),
\]

\( R = \| y^* \|_2, R_x = \| x^* - x^*(0) \|_2, R_{\omega} = \| x^* \|_2, \) and \( \chi(W) = \frac{\lambda_{\text{max}}(W)}{\lambda_{\text{min}}(W)} \).

The number of inner communication rounds and local oracle calls required by Algorithm 9 to obtain an \( (\varepsilon, R) \)-solution of (3) can be bounded by \( \tilde{O} \left( \sqrt{\frac{L R^2}{m \varepsilon} + \chi(W)} \right) \). On the other hand the number of outer communication rounds can be bounded by \( \tilde{O} \left( \sqrt{\frac{L R^2}{m \varepsilon} + \chi(W)} \right) \). Therefore, this approach is useful for large but well-connected networks, where \( m \gg 1 \) and \( \chi(W) = O(1) \) or \( \chi(W) = O(\log(m)) \).

A similar method to improve the definition of the global strong convexity parameter was proposed in [125]. In [125], the authors propose to introduce the proxy function \( f_i(x) - (\mu_i - \frac{1}{m} \sum_{i=1}^{m} \mu_i) \| x \|_2^2 \). With this new function, the condition number of \( F \) improves to

\[
\frac{\max_i L_i - \mu_i}{\frac{1}{m} \sum_{i=1}^{m} \mu_i} = 1.
\]

8 Experimental results

In this section we will provide experimental results that show the performance of the optimal distributed algorithm presented in Sections 4 and 6. We will consider two different graph topologies: the for cycle and Erdős-Rényi random graph of various sizes. We choose the cycle graph \( \chi(W) = O(m^2) \) and the Erdős-Rényi random graph \( \chi(W) = O(\log(m)) \) to show the scalability properties of the algorithms.

Initially, consider the ridge regression (strongly convex and smooth) problem

\[
\min_{z \in \mathbb{R}^n} \frac{1}{2m} \| b - H z \|_2^2 + \frac{1}{2} \| z \|_2^2,
\]

(26)
to be solved distributedly over a network. Each entry of the data matrix \( H \in \mathbb{R}^{ml \times n} \) is generated as an independent identically distributed random variable \( H_{ij} \sim \mathcal{N}(0, 1) \), the vector of associated values \( b \in \mathbb{R}^m \) is generated as a vector of random variables where \( b = Hx^* + \epsilon \) for some predefined \( x^* \in \mathbb{R}^n \) and \( \epsilon \sim \mathcal{N}(0, 0.1) \). The columns of the data matrix \( H \) and the output vector \( b \) are evenly distributed among the agents with a total of \( l \) data points per agent. The regularization constant is set to \( \epsilon = 0.1 \).

Figure 2 shows experimental results for the ridge regression problem for a cycle graph and an Erdős-Rényi random graph. For each type of graph we show the distance to optimality as well as the distance to consensus as its corresponding class assignment. We assume there is a total of \( m \) data points distributed evenly among \( n \) agents, where each agent holds \( l \) data points. The regularization constant is set to \( \epsilon = 1 \cdot 10^{-10} \) versus the number of nodes in the graph. We compare the performance of the proposed algorithm with some of the state of the art methods for distributed optimization. DIST-OPT refers to Algorithm 2, NONACC-DIST refers to the non-accelerated version of Algorithm 3, FGM is the centralized FGM. Acc-DNDG-NSC refers to the algorithm proposed in [40] with parameter \( \eta = 0.1 \) and \( \alpha = \sqrt{\bar{\mu}/\eta} \). EXTRA refers to the algorithm proposed in [22] with parameter \( \alpha = 1 \). DIGING refers to the algorithm proposed in [13] with parameter \( \alpha = 0.1 \). Figure 2 shows linear convergence rate with faster performance than other algorithms and linear scalability with respect to the size of the cycle graphs.

Now, consider the Kullback-Leibler (KL) barycenter computation problem (strongly convex and \( M \)-Lipschitz)

\[
\min_{z \in S_n(1)} \sum_{i=1}^{m} D_{KL}(z\| q_i) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} z_i \log (z_i/[q_i]_j),
\]

where \( S_n(1) = \{ z \in \mathbb{R}^n : z_j \geq 0; j = 1, 2, \ldots, n; \sum_{j=1}^{n} z_j = 1 \} \) is a unit simplex in \( \mathbb{R}^n \) and \( q_i \in S_n(1) \) for all \( i \). Each agent has a private probability distribution \( q^i \) and seek to compute the a probability distribution that minimizes the average KL distance to the distributions \( \{ q_i \}_{i=1, \ldots, m} \). Figure 3 shows the results for the KL barycenter problem for a cycle graph with \( m = 100 \), \( n = 10 \) and various values of the regularization parameter when Algorithm 4 is used. We show the distance to optimality as well as the distance to consensus and the scalability of the algorithm.

In (26), if we assume \( c = 0 \) and \( H_i \) is a wide matrix where \( n \gg l \) (i.e., the dimension of the data points is much larger than the number of data points per agent), then the resulting problem is smooth but no longer strongly convex. Figure 4 shows the performance of Algorithm 4 over a cycle graph and an Erdős-Rényi random graph, where \( m = 50 \), \( n = 20 \) and \( l = 10 \), for different values of the regularization parameter. As expected, smaller values of the regularization parameter increase the precision of the algorithm but hinder its convergence rate. We compare the performance of Algorithm 4 with the distributed accelerated method proposed in [40] for non-strongly convex functions (Acc-DNDG-NSC) for a fixed regularization value \( \bar{\mu} = 1 \cdot 10^{-6} \). As presented in Table 1, the algorithms have similar convergence rates, as shown by the intersection of the curves around the accuracy point corresponding to the regularization parameter. Nevertheless, as seen in Figure 3, Acc-DNDG-NSC has a worst scalability with respect to the number of nodes, which is particularly evident for the cycle graph.

Now consider the logistic regression problem for training linear classifiers. We seek to solve the following optimization problem:

\[
\min_{x \in \mathbb{R}^d} \frac{1}{2ml} \sum_{i=1}^{ml} \log (1 + \exp (-y_i \cdot A_i^T x)) + \frac{1}{2} c \| x \|_2^2,
\]

where \( A_i \in \mathbb{R}^{d} \) is a data point with \( y_i \in \{ -1, 1 \} \) as its corresponding class assignment. We assume there is a total of \( m \) data points distributed evenly among \( n \) agents, where each agent holds \( l \) data points. For our experiments, initially we generate a random vector \( x_{true} \in \mathbb{R}^n \) where each entry is chosen uniformly at random.
random on $[-1,1]$, we fixed $c = 0.1$, the data points $A_i$ are generated uniformly at random on $[-1,1]^n$, and the each label is computed as $y_i = \text{sign}(A_i^T x_{true})$. Note that each of the agents in the network will have a local function

\[ f_i(x) = \frac{1}{2ml} \sum_{j=1}^{l} \log \left(1 + \exp \left(-y_i^j \cdot [A_i^j]^T x\right)\right) + \frac{1}{2m} c\|x\|^2, \tag{28} \]

where $A_i^j \in \mathbb{R}^{l\times n}$ and $y_i^j \in \{-1,1\}$ are the data points held by agent $j$ and their corresponding class assignments. Moreover, (28) is not dual friendly. Therefore, we will use Algorithm 6 for our next set of experimental results.

Figure 5 shows the distance to optimality and the distance to consensus of the output of Algorithm 6 for the problem of logistic regression. We use cycle graphs and Erdős-Rényi random graph for a problem with 10000 data points of dimension 10. For each class of graphs, we explore three different scenarios for the...
Fig. 3: Distance to optimality and consensus, and network scalability for a strongly convex and $M$-Lipschitz problem over a cycle graph with $m = 100$, $n = 10$ and various values of the regularization parameter $\hat{\mu}$. The brown line shows the performance for the non-accelerated distributed gradient descent of the dual problem.

Fig. 4: Distance to optimality and consensus for a smooth problem over a Erdős-Rényi random graph with $m = 100$, $n = 50$, $l = 10$ and various values of the regularization parameter $\epsilon$. 
distribution of the data among agents. We present the results for networks of 10, 100, and 1000 agents; where each agent holds 1000, 100 and 10 data points respectively. We compare the results of the ACC-DNGD algorithm in [40] with parameter $\eta = 0.1$ and $\alpha = \sqrt{\mu \eta}$, the EXTRA algorithm in [22] with parameter $\alpha = 1$, and the DIGING algorithm in [13] with parameter $\alpha = 0.1$. Figure 5 shows a faster geometric convergence rate of Algorithm 6 with respect to ACC-DNGD, EXTRA and DIGING. Nonetheless, we point out that those algorithms could be subject to improved convergence rates if the particular parameters of each algorithm are carefully selected. In the presented results, we do not claim to have selected the optimal step sizes for the algorithms we are comparing our proposed method. For the cycle graph in Figure 5(a), as the size of the network increases and the number of points per agent decreases the convergence rates slows down. The EXTRA algorithms seem to have a near-optimal scaling on its convergence rate with respect to the size of the network. The ACC-DNGD and DIGING algorithms rapidly decrease their convergence rate with the size of the network. Due to the better condition number of the Erdős-Rényi random graphs, Figure 5(b) shows a better scaling with the size of the network for all the analyzes algorithms. For this class of networks, the ACC-DNGD algorithms outperforms EXTRA and DIGING.

In Figure 6, we use datasets from the library LibSVM [77] to compare the performance of Algorithm 6 as in Figure 5. We seek to distributedly solve the logistic regression problem over the following datasets: A9A, MUSHROOMS, IJCNN1 and PHISHING. Table 3 shows a brief description of the four datasets used. For each problem, we created an Erdős-Rényi random graph with 100 agents and evenly distributed the data points among all agents. Algorithm 6 outperforms the other compared algorithms where the ACC-DNGD having the second best performance following the same scaling patterns as in Figure 5 for Erdős-Rényi random graphs. The EXTRA and DIGING algorithms have a worst scaling of their convergence rate as the size of the network increases.

| Name      | Classes | Data points | Features |
|-----------|---------|-------------|----------|
| A9A       | 2       | 32561       | 123      |
| MUSHROOMS | 2       | 8214        | 112      |
| IJCNN1    | 2       | 49990       | 22       |
| PHISHING  | 2       | 11055       | 68       |

Table 3: Real Datasets from the LibSVM Library

9 Conclusions

We have provided convergence rate estimates for the solution of convex optimization problems in a distributed manner. The provided complexity bounds depend explicitly on the properties of the function to be optimized. If $F(x)$ is smooth, then our estimates are optimal up to logarithmic factors otherwise our estimates are optimal up to constant factors. The inclusion of the graph properties in terms of $\sqrt{\chi(W)}$ shows the additional price to be paid in contrast with classical (centralized/non-distributed) optimal estimates. The authors recognize that the proposed algorithms required, to some extent, global knowledge about the graph properties and the condition number of the network function. Nevertheless, our aim was to provide a theoretical foundation for the performance limits of the distributed algorithms. The cases where global information is not available require additional study.

One can further extend our results and obtain the same rates of converge when the graphs change with time by using restarting techniques [78,79]. Nevertheless, we require additional assumptions. Particularly, the network changes should not happen often and nodes must be able to detect when these changes occur. The condition number of the sequence of graphs $\chi(W_k)$ then is the worst one among all the graphs in the
Fig. 5: Logistic regression on synthetic data over a cycle graph and an Erdős-Rényi random graph for a total of 10000 data points and various graph sizes evenly distributing the data points among the agents.
Fig. 6: Logistic regression results with data from the LIBSVM Library on an Erdős-Rényi random graph with 100 agents.

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