Flux-confinement in Dilatonic Cosmic Strings

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Abstract We study dilaton-electrodynamics in flat spacetime and exhibit a set of global cosmic string like solutions in which the magnetic flux is confined. These solutions continue to exist for a small enough dilaton mass but cease to do so above a critical value depending on the magnetic flux. There also exist domain wall and Dirac monopole solutions. We discuss a mechanism whereby magnetic monopoles might have been confined by dilaton cosmic strings during an epoch in the early universe during which the dilaton was massless.

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1. Introduction

There has been considerable interest recently in 4-dimensional Dilaton-Einstein-Maxwell and Dilaton-Einstein-Yang-Mills theory. In the abelian case single and multi static and stationary black hole solutions with electric and magnetic charges have been extensively studied. Another class of interesting solutions describe magnetic fields with or without black holes. For Einstein-Maxwell theory these are based on the Melvin solution which represents a sort of super-massive cosmic string [1,2]. The Melvin solution may be generalized to include a coupling to the dilaton [3]. Related black hole metrics have been obtained and their relevance as instantons for the Schwinger production of black holes discussed [4]. Recently Maki and Shiraishi [5] have obtained some interesting time-dependent solutions with a dilaton potential.

In this paper, in order to gain some physical insight into dilaton-electrodynamics and its non-abelian generalization, we will study the simpler flat-space version in which the effects of gravity are ignored. The action is

$$\int d^4x \left( -\frac{1}{4} \exp(-2\tilde{\kappa}\phi) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right)$$  \hspace{1cm} (1.1)

where $\eta^{\mu\nu} = \text{diag}(-1,+1,+1,+1)$ and the dimensionful quantity $\tilde{\kappa}$ in the action can be changed by a suitable rescaling of the variables and so its numerical value (as long as it does not vanish) has no physical significance in the purely classical theory which we study here. The action (1.1) can be obtained from the standard gravitational case, in Einstein conformal gauge, in which the Maxwell field is multiplied by $\exp(-g\sqrt{4\pi G \phi})$ by letting Newton’s constant $G$ tend to zero while at the same time $g\sqrt{4\pi G}$ tends to the finite limit $\tilde{\kappa}$ *. The field $F_{\mu\nu}$ may be abelian or non-abelian. In the latter case there remains, again in the classical theory, sufficient freedom to scale the Yang-Mills connection $A$ such that $F = dA + A \wedge A$. Note that had we taken the above limit of the gravitational Lagrangian expressed in string conformal gauge (i.e in terms of the metric $\exp(2\tilde{\kappa}\phi) g_{\mu\nu}$) we would have obtained a different action from (1.1).

The flat space version (1.1) has been studied by Lavrelashvili and Maison [7] and also by Bizon [8] who have obtained sphaleron type solutions in which the Yang-Mills field is confined by the attractive forces exerted by the dilaton which replaces the attractive forces due to gravity in the Bartnik-McKinnon solution [9]. In the abelian case it is easy to obtain the general static spherically symmetric purely magnetic solution representing a Dirac monopole coupled to the dilaton. Bizon has pointed out a special case of the spherically symmetric Dirac monopoles may be generalized to give multi-Dirac-monopole solutions. These are Bogomol’nyi type solutions and may be regarded as a limiting case of the multi black hole solutions as we shall show in section 4. It is also completely straightforward to obtain plane wave solutions in which the dilaton and the photon are travelling parallel to one another.

The main point of this paper is to derive cosmic string like solutions.

* The sign of $g$ we are using in this paper is consistent with the conventions of the majority of string theorists though it differs from that used in our recent preprint [6].
2. Permeabilities and Permittivities

It follows immediately from (1.1) that the equations of motion for the field $F$ in the presence of the dilaton field are those for a medium in which the electric permittivity $\epsilon$ is given by

$$\epsilon = \exp(-2\kappa\phi)$$

and the magnetic permeability $\mu$ is given by

$$\mu = \exp(2\kappa\phi).$$

The product $\epsilon\mu$ is unity and so the velocity of light remains one everywhere. Thus regions of spacetime for which $\phi < 0$ are diamagnetic while regions with $\phi > 0$ are paramagnetic. One does not usually encounter permittivities $\epsilon$ which are less than unity. One exception is the case of vacuum polarization in quantum chromodynamics which is “anti-screening” rather than “screening” as for all conventional material media. In the non-abelian theory $\exp(\kappa\phi)$ plays the role of spacetime dependent gauge coupling “constant” and in string perturbation theory its expectation value plays the role of a variable coupling constant. Thus weak coupling corresponds to a diamagnetic phase and strong coupling to a paramagnetic phase. The action (1.1) is invariant under the simultaneous change of the sign of the dilaton field $\phi$ and the replacement of the Maxwell field $F$ by its Hodge dual. This symmetry therefore interchanges the weak and the strong coupling phases.

The equation for $\phi$ in the static case is

$$\nabla^2 \phi = -\kappa \exp(-2\kappa\phi) \left( B^2 - E^2 \right),$$

where $E$ and $B$ have their usual meaning. It follows from (2.3) that $\phi$ can have no maximum in a purely magnetic field and no minimum in a purely electric field. Thus if $\phi$ is taken to be zero at infinity then magnetic regions tend to be paramagnetically polarized ($\mu > 1$) and electric regions tend to be dielectrically polarized ($\mu < 1$). Intuitively magnetic flux ($\int B \cdot dS$) tends to get self-trapped in strong coupling domains and electric flux ($\int D \cdot dS$) in weak coupling domains, where $D = \epsilon E$ is the electric displacement and (for later use) $H = \mu^{-1}B$ is the magnetic induction. These observations are borne out by the particular solutions mentioned above. Thus for the Maison-Lavrelashvili-Bizon sphalerons $\mu$ has a maximum at the centre and decreases monotonically to unity at infinity. For magnetic black hole solutions $\mu$ increases monotonically inwards from unity at infinity. In the extreme case it becomes infinitely large and positive as one approaches the horizon. From the string point of view the infinitely long throat is a strong coupling region. In the electrically charged case the opposite is true. There is a parallel here with monopoles in Yang-Mills theory and vortices in the abelian Higgs theory. In the case of $SU(2)$ Yang-Mills theory with the Higgs in the adjoint representation one may take the components of the Higgs field as a triplet of permittivities. The parallel is not completely precise but it is the case for the ’t Hooft-Polyakov monopole that the associated permeabilities monotonically increase as one moves radially inwards. Similarly for the Nileson-Olesen vortex one may think of the magnetic flux as being confined inside a core of high permeability where the Higgs field has a smaller magnitude than it does at infinity. We shall see similar features arising for dilaton electrodynamics.
The paramagnetic behaviour described above and the existence of the dilaton-Melvin solution strongly suggest that there may be non-singular static cosmic string type solutions in which a finite amount of flux is trapped. This is indeed the case, as we shall show in the next section.

### 3. Dilaton Cosmic Strings and Domain Walls

In the static case the dilatonic Maxwell equations are readily seen to be satisfied if the magnetic induction \( \mathbf{H} = (0, 0, H) \), where \( H \) is a constant so long as the dilaton satisfies the two-dimensional Liouville equation:

\[
\nabla^2 \phi = -\kappa H^2 \exp(2\kappa \phi).
\]

The general solution of (3.1) is

\[
\mu = \exp(2\kappa \phi) = \frac{4}{\kappa^2 H^2} \frac{f'(\zeta)g'\bar{\zeta}}{(1 + f(\zeta)g(\bar{\zeta}))^2},
\]

where \( f \) and \( g \) are locally holomorphic functions and \( \zeta = x + iy \). If \( \phi \) is real then

\[
g(\bar{\zeta})/f(\zeta)
\]

must be a real valued holomorphic function of \( \zeta \) and hence constant. With no loss of generality this constant may be taken to be unity and therefore the solution we require is

\[
\mu = \exp(2\kappa \phi) = \frac{4}{\kappa^2 H^2} \frac{|f'(\zeta)|^2}{\left(1 + |f(\zeta)|^2\right)^2}.
\]

Note that \( f \) and \( 1/f \) give the same solution \( \phi \).

Choosing different functions \( f \) gives different types of solution. For example choosing \( f(\zeta) = \zeta \) gives a cylindrically symmetric solution with finite total magnetic flux:

\[
\Phi = \int_{\mathbb{R}^2} \mathbf{B} \cdot d\mathbf{S} = \frac{4\pi}{\kappa^2 H}.
\]

This solution may be obtained as limit of the dilaton-Melvin solution.

The magnetic contribution to the total energy per unit length of our solution is

\[
\frac{1}{2} \Phi H = 2\pi / \kappa^2
\]

which is independent of the magnetic field \( H \). The dilaton however contributes a logarithmically divergent energy per unit length because of its logarithmic dependence on the radius. In this respect our solution resembles a global rather than a local string. However it should be pointed out that the solution, in common with its gravitational version, breaks neither electromagnetic gauge invariance nor any compact internal symmetry.
Flux-confinement in Dilatonic Cosmic Strings

In addition to the single string solution there are multi string solutions. Thus choosing a rational function of Brouwer degree \( k \), i.e. the ratio of two polynomials of order \( p \) and \( q \), gives a solution with finite total magnetic flux

\[
\Phi = k \frac{4\pi}{\tilde{\kappa}^2 H} .
\] (3.5)

Note that the permeability decreases to zero as \((x^2 + y^2)^{-(|p-q|+1)}\) at infinity. Thus the weak coupling region at infinity is strongly diamagnetic and confines the magnetic flux \( \Phi \). The multi-string solutions are not axisymmetric. In flat-space Maxwell theory the only regular solution is the uniform magnetic field which is necessarily axisymmetric. When gravity is included this goes over into the Melvin solution which is also has axial symmetry. If one insists that the metric be boost-invariant then the axisymmetry and hence uniqueness, follow by a version of Birkhoff’s theorem [10]. However the proof given in [10] does not go through in the presence of a scalar field. This suggests that there may exist static non-axisymmetric multi-dilaton-Melvin solutions.

If we take \( f(\zeta) = \exp(\zeta) \) then

\[
\mu = \exp(2\tilde{\kappa}\phi) = \frac{1}{\tilde{\kappa}^2 H^2} \frac{1}{\cosh^2 x} .
\] (3.6)

This solution describes a sheet or membrane confining an amount of flux per unit length \( \frac{2}{\tilde{\kappa}^2 H} \). It may be thought of as a sort of domain wall separating two weak coupling domains.

4. Monopoles and Bogomol’nyi Solutions

In this section we shall consider a general static magnetic solution. Since

\[
\nabla \times \mathbf{H} = 0 ,
\] (4.1)

we may locally introduce a magnetic potential \( \chi \) by

\[
\mathbf{H} = \nabla \chi .
\] (4.2)

If we make the ansatz

\[
\frac{1}{\tilde{\kappa}} \exp(-\tilde{\kappa}\phi) = \chi
\] (4.3)

then all the equations will be satisfied if in addition

\[
\nabla^2 \exp(\tilde{\kappa}\phi) = 0 .
\] (4.4)

These solutions are the same as those mentioned by Bizon and may be obtained from the multi black hole solutions by a limiting procedure. Our cosmic string solutions are not a special case.

In addition to these Bogomol’nyi solutions it is easy to find the general spherically symmetric monopole solution. If one insists that \( \phi \) does not blow up at finite non-zero radius or at infinity one finds that

\[
\exp(\tilde{\kappa}\phi) = \frac{1}{\alpha} \sinh \alpha \tilde{\kappa} P \left( \frac{1}{r} + \frac{1}{b} \right)
\] (4.5)
and

\[ B = \frac{P}{r^3} r, \]  

(4.6)

where the constant of integration \( b \) is chosen so that \( \phi = 0 \) at infinity and \( P \) is the total magnetic charge. Just as in the case of magnetic black holes and ’t Hooft-Polyakov monopoles we find that the magnetic permeability increases monotonically inwards. The Bogomol’nyi solution (4.4) is obtained from the general solution (4.5) and (4.6) by letting \( \alpha \) go to zero.

5. Massive Dilatons

It is widely believed by string theorists that the dilaton acquires a mass due to non-perturbative effects connected with the breaking of supersymmetry. It is therefore of interest to ask what the effect of a mass term for the field \( \phi \) might have on our solutions. In this section we shall simply add in a mass term “by hand”. Thus we add to the Lagrangian in (1.1) a term \(-\frac{1}{2}m^2 \phi^2\). One checks that the work above goes through as long as one replaces \( \nabla^2 \) by \( \nabla^2 - m^2 \) in (2.3) and (3.1). However the solution (3.2) is no longer valid and we do not know how to solve the new version of (3.1) exactly. The circularly symmetric solutions may be treated by regarding the radial coordinate \( r \) as a fictitious time variable. The equation for \( \phi \) becomes that of a particle subject to a time dependent frictional force and moving in a potential \( U(\phi) \)

\[-\frac{1}{2}m^2 \phi^2 + \frac{1}{2}H^2 \exp(2\kappa \phi). \]

(5.1)

Regularity at the origin implies that the radial derivative of \( \phi \) vanishes there. A solution exists for each value of \( \phi \) at the origin. If \( m = 0 \) one has those solutions given in section 3 in which \( \phi \) decreases monotonically to minus infinity at infinity as

\[ \kappa \phi \sim -2 \log r. \]

(5.2)

For non-vanishing mass \( m \) the behaviour depends on the ratio \( m^2/(2e\kappa^2H^2) \) where \( e \) is the base of natural logarithms. If \( 0 < m^2/(2e\kappa^2H^2) < 1 \) the potential \( U(\phi) \) is a monotonic increasing function of \( \phi \) and for all values of \( \phi(0) \), \( \phi \) decreases monotonically with \( r \) and tends to minus infinity at infinity

\[ \phi \sim -c \exp(mr) \]

(5.3)

where \( c \) is a positive constant.

However if \( m^2/(2e\kappa^2H^2) \) is greater than unity then the potential \( U(\phi) \) has a local minimum and maximum. The behaviour of the solutions depends upon \( \phi(0) \). If \( \kappa \phi(0) \) is positive and sufficiently large then \( \phi \) decreases monotonically to minus infinity as before. However if \( \kappa \phi(0) \) lies in a finite interval bounded below by the smaller solution \( x \) of the equation:

\[ x = \frac{\kappa^2H^2}{m^2} \exp 2x \]

(5.4)
then the solutions oscillate about the minimum with an amplitude which decreases to zero as $r$ tends to infinity. Finally if $\tilde{\kappa}\phi(0) < x$ the solutions decrease monotonically to minus infinity.

Thus for given magnetic field $H$ there always exist solutions with finite total flux. If however the mass is large enough, one has solutions in which $\phi$ tends a minimum value dependent upon $H$ of the potential $U(\phi)$.

We have repeated these calculations for the monopole solutions of section 4 in the case of a massive dilaton. We find that the qualitative behaviour of the dilaton is very similar to that in the massless case.

One might imagine a cosmological scenario in which the dilaton is initially massless at some high temperature and acquires a mass during a cosmological phase transition at a lower temperature. If cosmic strings of the type we have described confining a finite flux were initially present and the mass were large enough it seems from our calculation that provided $\phi(0)$ took suitable values the flux would become unconfined. If this were true it might have important consequences for magnetic monopoles. If flux was confined by strings at early times then one might expect magnetic monopoles, of the sort described in section 4, to be found at the ends of flux tubes. These flux tubes should pull the monopoles together and cause their rapid annihilation. At late times magnetic fields would become unconfined. In this way one might have a natural solution to the monopole problem. Clearly more work needs to be done to establish whether this picture is really viable.

It is interesting to note that dilaton electrodynamics with an effective mass term has already been invoked [11] to account for a possible primordial magnetic field. It would interesting to investigate the relation between that work and the monopoles and vortices described in this paper.

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Flux-confinement in Dilatonic Cosmic Strings

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