Monotonicity of the value function for a two-dimensional optimal stopping problem

Adriana Ocejo

Supervisors: Dr Sigurd Assing and Professor Saul Jacka
Department of Statistics, University of Warwick, UK

Introduction

Setting: consider a two-dimensional strong Markov process \((X, Y)\) on \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})\) with \(S \subseteq (0, \infty)\), such that
\[
dx = a(X)\,YdB
\]
y and \(Y\) is any of the two classes:
Regime-switching: \(Y\) is a continuous-time MC, independent of \(B\).
Diffusion: \(Y\) solves a stochastic differential equation of the type
\[
ey = \gamma(y)\,dt + \sigma(Y)\,dW
\]
where \(\gamma\) is a standard Brownian motion such that \(\gamma(t) \geq \delta\), \(\gamma(t) \geq \delta\), \(\delta > 0\), for some real parameter \(\delta \in [-1, 1]\).

Aim: our main focus is on proving the monotonicity of \(y \mapsto v(x, y)\), \(v(x, y) = \sup \mathbb{E}_x[e^{-\gamma t}g(X_t)], (x, y) \in R \times S\),
\[
\text{where } \gamma > 0, t \in [0, \infty), \text{ the gain function } g: R \rightarrow \mathbb{R}\text{ is a measurable function and the supremum is taken over all finite stopping times with respect to the filtration generated by } (X, Y).
\]

As a consequence of the monotonicity, we are also able to show continuity of \(y \mapsto v(x, y)\), in both finite and infinite horizon.

Application: pairs \((X, Y)\) as above are common in mathematical finance:
\(X\) discounted price of an asset
\(Y\) stochastic volatility.

The results can be applied to prices of American options in popular models such as Heston or Hull-White.

Some background: the monotonicity property was investigated for American options with convex gain function by Hobson [2] with (1)-(2), American options by Ekstrom [1] in the case that \(Y = 1\) and \(a = a(t, r)\).

Note: in this presentation we concentrate on the diffusion case and \(g \geq 0\).

[1] Ekstrom, E., Properties of American options prices, Stoch. Proc. Appl. 114, 265-278 (2004),
[2] Hobson, D., Comparison results for stochastic volatility models via coupling, Finance and Stochastics 14, 129-152 (2010).

Theorem 1 (Monotonicity)
Assumptions: let \((X, Y)\) be as in (1)-(2) and assume that
\(a\) \(Y\) takes values in \(S = (0, \infty)\) and \(X \in R \times S\).
\[
P_{\mathcal{F}_t} \left( \int_0^t \mathcal{Y}_s ds < \infty, \forall t \geq 0 \right) = 1, \quad P_{\mathcal{F}_t} \left( \int_0^t \mathcal{Y}_s ds = \infty \right) = 1.
\]
\(b\) The measurable functions \(a, \gamma, \eta, \theta\) are such that the system
\[
\begin{align*}
\frac{\partial}{\partial t} & \mathcal{G}(t, x, \eta(t), \theta(t)) = a(t, x)\mathcal{G}(t, x, \eta(t), \theta(t)), \\
\frac{\partial}{\partial \mathcal{G}} & \mathcal{G}(t, x, \eta(t), \theta(t)) = \gamma(t, x)\mathcal{G}(t, x, \eta(t), \theta(t)) + \theta(t, x)^2\mathcal{G}(t, x, \eta(t), \theta(t)),
\end{align*}
\]
has a unique non-explosive strong solution with \(\eta \in \mathcal{T} \times S, t \geq 0, \theta \geq 0\).

Result: for every \(x \in R\) and \(y, y' \in S\) with \(y \leq y'\), \(v(x, y) \leq v(x, y')\).

Idea of proof: we sketch the proof in three steps (see middle column):
1. Construct a weak solution \((\hat{X}, \hat{Y})\) of (1)-(2) by time-changing (\(\cdot\)).
2. Formulate an equivalent optimal stopping problem in the time-changed setting.
3. Use coupling to make path comparison of processes.

Theorem 2 (Continuity)
Assumptions: let the assumptions of Theorem 1 hold and let the \(\xi\) component of \((x)\) result in a Feller process with state space \(S = (0, \infty)\).

Result:
\(i\) If the horizon \(T = \infty\) then \(v(x, \cdot)\) is continuous (although for right-continuity one has to assume a stronger integrability condition).
\(ii\) If the horizon \(T < \infty\) AND \(g\) is continuous then \(v(x, \cdot)\) is continuous.

Idea of proof:
\(i\) Fix \(x = x_0 \in S\) such that \(y_0 \leq y < y_0 \in S\),
\(ii\) Let \((\hat{X}, \hat{Y})\) be the solution of \((\cdot)\) starting from \((x, y_0)\), \(n = 1, 2, \ldots\), and construct \(\hat{X}_n, \hat{Y}_n, (\hat{X}_n, \hat{Y}_n)\) as in Step 1.

\(iii\) If the sequence \(\hat{y}_n\) is monotone, then the coupling argument and the Feller property of \(\xi\) imply \(\hat{y}_n \to \hat{y}\) and \(\hat{A}_n \to A\) if \(n \to \infty\) for all \(t \geq 0, a.s\).

\(iv\) The monotonicity and the fact of \(v(x, \cdot)\) are the key tools to show left-continuity and right-continuity of \(v(x, \cdot)\).

Example: Hull and White model
\[
\begin{align*}
dx &= \sigma^2 x \,dt + \eta \,dB_X + \theta \,dB_Y, \\
\eta &= \theta = \text{positive constants},
\end{align*}
\]
\[
\text{The diffusion process } Z_t = \frac{Z_t - \mathbb{E}[Z_0]}{\mathbb{E}[Z_0]} \text{ is a Bessel process of dimension } \phi = 1 + 2\theta/\sigma^2.
\]

Some extensions
\(i\) All the previous results can be stated for a general measurable gain function \(g\) which may take negative values, in the case \(T = \infty\).

We assume \(\left[ g(0) = 0 \right] \) and \(P_{\mathcal{F}_t} (\mathbb{E}_x[\mathbb{E}_x[e^{-\gamma t}g(X_t)]]) = 1\), which is sufficient for \(v(x, y) = \sup_{\mathcal{F}_t} \mathbb{E}_x[e^{-\gamma t}g(X_t)]\), where \(\mathbb{E}_x\) consists of all the finite stopping times such that \(g(X_t) \geq 0\).

\(ii\) In the context of American option pricing, the payoff takes the form \(g(e^{\phi t}X_t)\) if exercised at \(\tau\). The setup becomes
\[
P(X, \tau) = \sup_{\mathcal{F}_t} \mathbb{E}_x[e^{-\gamma t}g(e^{\phi t}X_t)].
\]

Monotonicity of \(P(x, \cdot)\) will follow if \(g\) is a non-increasing function, while continuity carries over if we further assume that \(g\) is continuous.

Our results would not change in principle if \(S \subseteq (0, \infty)\) instead. What would change is only that increasing would turn into decreasing.