Comment Concerning Relativistic Corrections in NRQCD

Ivan Maksymyk

TRIUMF, 4004 Wesbrook Mall, Vancouver, Canada, V6T 2A3

We examine the subtle differences between two possible frameworks for the calculation of quarkonium production. The differences between the two methods are not a concern when one calculates only to leading order in the relativistic expansion, but when relativistic corrections are included, the two formulations seem at first to be at variance with one another. They can be reconciled however via a relativistically corrected mass relation relating the boundstate mass and the quark mass.

I. INTRODUCTION

We present here an observation regarding relativistic corrections in factorization formulas for quarkonium production, as calculated in the non-relativistic QCD framework [1]. We point out that one can conceive of two subtly different frameworks in which calculations of quarkonium production might be carried out. We illustrate the algorithm for both approaches through the detailed presentation of a simple example. We explain that the results of the methods differ on two points: on the question of the definitions of the NRQCD matrix elements; and on the question of the mass parameters that appear in NRQCD factorization formulas. In Method I, factorization formulas are computed in such a manner that only the quark mass $m_c$ appears, while in Method II, the formulation generally entails the mass of the quarkonium boundstate (which we will denote with the upper case $M_H$) as well as $m_c$.

The difference between the two methods is not a concern when one calculates quarkonium production and decay rates only to leading order in the relativistic expansion. In that approximation, one sets $M_H = 2m_c$, and the two methods give compatible results. However, when relativistic corrections are included (and when $M_H$ no longer is equatable to $2m_c$), we find that the two formulations seem at first to be at variance with one another.

The results of the two methods can be reconciled using a relativistically corrected mass relation, which is an equation relating $m_c$, $M_H$ and the NRQCD production matrix elements.

A variant of the relativistically corrected mass relation has recently been discussed in Ref. [2]. The appearance of this work increases the possible interest of the NRQCD community in our observation.

In order to illustrate our observation, we take the simplest possible example, that of charmonium production from the collision of two a ficticious elementary (colorless) scalars, denoted by $φ$:

$$φ + φ → \text{charmonium}.$$ 

The underlying short-distance physics is the partonic process

$$φ + φ → c + \bar{c}$$

which is described by the interaction term

$$\mathcal{L}_{\text{int}} = g (φ(x) φ(x)) \bar{Ψ}(x) γ_5 Ψ(x)$$

(1.1)

where $Ψ(x)$ is the four-spinor field operator for the charm quark (with color index $i$), and where $g$ is a coupling constant of mass dimension $−1$. The reaction $φ + φ → c + \bar{c}$ produces a $c\bar{c}$ pair in a color-singlet state with angular momentum $2S + 1L_J = 1S_0$. This $c\bar{c}$ can hadronize into the $\eta_c$ charmonium particle. We choose this process because of the simplicity of the spin structure and because there are no color-octet contributions which would distract from our main point.

The determination of production factorization formulas in NRQCD is always based, in one way or another, on the Feynman amplitude $\mathcal{M}$ for the production of a free heavy quark and free heavy antiquark. In particular, we desire to know how this quantity depends on $\mathbf{q}$, the relative three momentum of the quark and antiquark in the quark-antiquark restframe.

In our toy example, the Lorentz-invariant Feynman amplitude is given by

$$\mathcal{M}(c\bar{c}(\mathbf{q})) = g \bar{u}v = 2g E_q ξ^1_5 ξ_5^2.$$ 

(1.2)

(Colour indices have been suppressed.) The Dirac four-spinors $u$ and $v$ are normalized so that $\bar{u}u = \bar{v}v = 2m_c$. The passage from the middle piece of Eq. 1.2 to the right-hand side is an exact relation, and not a nonrelativistic approximation. It entails recasting the four-spinors (from the Pauli-Dirac representation) in terms of the two-spinors...
\(\xi\) and \(\zeta\), as described in Ref. [3]. \(\xi\) is the two-spinor for the heavy quark, and \(\zeta\) (called \(\eta\) in Ref. [3]) is the two-spinor for the heavy antiquark. \(\sigma\) and \(\tau\) label heavy quark spins; they are not spinor component indices. The two-spinors are normalized so that \(\xi_{\sigma}^\dagger \xi_{\tau} = \delta_{\sigma\tau}\). As to \(E_q\), this is simply a short form for \(\sqrt{q^2 + m_c^2}\). The invariant mass of the system consisting of the pair of free on-shell heavy quarks is \(2E_q\).

The spinor-structure given in Eq. 1.2 is such that multiplying the expression by the Clebsch-Gordon coefficients \(\langle 1/2, \sigma; 1/2, \tau|0, 0\rangle\) and then summing over spins gives a non-zero result. However, doing the same with the Clebsch-Gordon coefficients \(\langle 1/2, \sigma; 1/2, \tau|1, M\rangle\) gives zero. We therefore see that Eq. 1.2 represents the production of a \(c\bar{c}\) pair in a spin state \(j, M_j = 0, 0\), as was previously stated.

**II. RELATIVISTICALLY CORRECTED FACTORIZATION FORMULAS: METHOD I**

**A. The Concept of Matching**

The NRQCD factorization formalism exploits the equivalence of perturbative QCD and perturbative NRQCD. The computation of the factorization formula for the production of a quarkonium boundstate starts with a computation of the production rate for free on-shell heavy quarks and antiquarks; this computation is performed in some perturbative model of underlying short-distance physics, which could be perturbative QCD, electroweak physics, or (as here) a toy model. In a separate calculation, one uses NRQCD perturbation theory to compute the free-quark matrix elements of the four-fermion operators of the NRQCD effective theory. The perturbative equivalence of NRQCD and the full underlying theory is expressed by the matching condition (Eq. (6.7) of Ref. [1])

\[
\sigma(c\bar{c}(q))|_{\text{pert}} = \sum_n \frac{F_n(m_c)}{m_c^{d_n-4}} \langle 0|O_n^c(q)|0\rangle|_{\text{pert NRQCD}}.
\]

The left-hand side is to be computed in the perturbative model of underlying physics. It is the rate for the production of a heavy quark pair, written expressly as a function of \(q\), the relative three-momentum of the \(c\) and \(\bar{c}\) in the \(c\bar{c}\) restframe. As well as being a function of \(q\), the left-hand side is also a function of whatever other kinematic variables are required for parametrizing the process at hand. As to the right-hand side, it is simply a linear combination NRQCD free-quark matrix elements \(\langle 0|O_n^c(q)|0\rangle\) such as those defined in Appendix I. The sum over \(n\) in the right-hand side is a Taylor series; each of the matrix elements is a function of \(q\) and \(m_c\) only, with the power in \(q^2/m_c^2\) increasing with each higher term. The matching condition is intended to allow a determination of the short-distance coefficients \(F_n\). By definition, these are independent of \(q\). How the matching condition is to be employed will become clear in the example below. The basic idea is that, by calculating the production rate in the full underlying theory on the left-hand side, and by massaging it into a Taylor series of the form of the right-hand side, one can then read off the \(F_n\). \(d_n\) is the mass dimension of the product of the field operators in \(O_n\).

Once the short-distance coefficients \(F_n\) have been calculated for free on-shell heavy quarks in the perturbative theory, the rate for the production of some quarkonium boundstate \(H\) is then known to be (Eq. (6.4) of Ref. [1])

\[
\sigma(H) = \sum_n \frac{F_n(m_c)}{m_c^{d_n-4}} \langle 0|O_n^H|0\rangle.
\]

The above expression is a factorization formula. The NRQCD non-perturbative matrix elements \(\langle 0|O_n^H|0\rangle\) are to be thought of as empirical parameters. A fundamental feature of the NRQCD factorization formalism is that the coefficients \(F_n\) are universal to both free quark and boundstate production. In general, the short-distance coefficients and the hadronic matrix elements also depend on the NRQCD cut-off, but we will not bother to display this dependence, since it is not a point of focus in our discussion.

**B. Manipulation of Phase Space for Method I**

In Method I, we interpret the matching condition, Eq. [2.3], to involve (on both sides of the equation) the rate of production of free heavy quarks; this interpretation defines Method I. In such an approach, the physical meaning of \(\sigma(c\bar{c}(q))\) is fixed via the concept of the total cross-section for free on-shell \(c\bar{c}\) production:

\[
\sigma(c\bar{c}) = \int \frac{d^3q}{(2\pi)^3} \sigma(c\bar{c}(q)) = \int \sigma(c\bar{c}(q)) = \int \frac{d^3q}{(2\pi)^3} \sigma(c\bar{c}(q)).
\]
By construction, $\sigma((c\bar{c})(q))$ depends on $q$, $m_c$ and other kinematic variables of the free $c\bar{c}$ production process, but it logically does not depend on the mass of quarkonium boundstate, $M_H$.

Our goal in this section is to obtain a useful general expression for $\sigma((c\bar{c}(q)))$ for the purposes of matching in Method I. We proceed by first computing the production rate for a pair of free on-shell quarks. The total rate is given by the well-known formula

$$\sigma(c\bar{c}) = \int \frac{d^3p_c}{(2\pi)^3} \frac{d^3p_{\bar{c}}}{(2\pi)^3} \sum_X \frac{1}{\text{flux}} \delta^4(p_i - X - P_e - P_{\bar{e}}) (2\pi)^4 |M(c\bar{c}(q))|^2$$

where, since we are only considering spin-summed rates, we sum over all quantum numbers of the final state free heavy quark and free heavy antiquark. Here, $X$ represents the sum of the momenta of all final state particles other than the two heavy quarks. $\sum_X$ represents the measure of phase space integration over all such momenta.

Let us now define the total and relative four-momenta of the heavy quarks:

$$P \equiv p_c + p_{\bar{c}}$$

$$2Q \equiv p_c - p_{\bar{c}}.$$ 

$P$ is the total four-momentum of the $c\bar{c}$ system in the lab frame. $Q$ is the relative four-momentum of the $c$ and $\bar{c}$ in the lab frame. The lab-frame four-momentum $Q$ is related to the rest-frame three-momentum $q$ by the relation

$$Q^i = \Lambda^i_{\mu} q^\mu$$

where $\Lambda^\mu_{\nu}$ is the Lorentz boost matrix connecting the lab frame with the $c\bar{c}$ restframe. Recalling that the invariant mass of the $c\bar{c}$ system is $2E_q \equiv 2\sqrt{m_c^2 + q^2}$, we see that the components of the total four-momentum of the $c\bar{c}$ system are

$$P = \left\{ \sqrt{4E_q^2 + P^2}, P \right\}$$

We now manipulate the phase space integral in Eq. (2.4) by defining the new variables of integration

$$P = p_c + p_{\bar{c}}$$

$$2Q = p_c - p_{\bar{c}}.$$ 

With these new variables, the total rate is written

$$\sigma(c\bar{c}) = \int \frac{d^3Q}{(2\pi)^6} \frac{d^3P}{2p_c^0 2p_{\bar{c}}^0} \sum_X \frac{1}{\text{flux}} \delta^4(p_i - X - P) (2\pi)^4 |M(c\bar{c}(q))|^2.$$ 

The Jacobian for the above variable redefinition is unity.

We now change variables of integration once more, replacing $Q^i$ in favor of $q^i$. Concerning this transformation, we have the following exact identity:

$$\int d^4Q \left[ \frac{1}{2p_c^0 2p_{\bar{c}}^0} q^0 \right]$$

$$= \int \left[ d^3q \left| \frac{\text{Det}(dQ^i/dq^j)}{2p_c^0 2p_{\bar{c}}^0} \right| \right]$$

$$= \int \left[ d^3q \frac{P^0}{2E_q} \left( 1 - \frac{P^2 q^2}{12(P^0)^2 E_q^2} \right) \right] \left[ \frac{1}{(P^0)^2} \left( 1 - \frac{P^2 q^2}{12(P^0)^2 E_q^2} \right)^{-1} \right],$$

so that the total rate can be written in the form:

$$\sigma(c\bar{c}) = \int \frac{d^3q}{(2\pi)^3} \frac{d^3P}{2E_q P^0} \sum_X \frac{1}{\text{flux}} \delta^4(p_i - X - P) (2\pi)^4 |M(c\bar{c}(q))|^2.$$ 

We are now able to extract $\sigma((c\bar{c}(q)))$, the rate as a function of $q$. This is seen to be

$$\sigma(c\bar{c}(q)) = \int \frac{d^3P}{(2\pi)^3} \frac{d^3P}{2E_q P^0} \sum_X \frac{1}{\text{flux}} \delta^4(p_i - X - P) (2\pi)^4 |M(c\bar{c}(q))|^2. \quad (2.5)$$
C. Matching Condition for Method I

Eq. \( \ref{eq:2.5} \) gives an expression for the left-hand side of the basic matching condition, Eq. \( \ref{eq:2.1} \). The matching condition now becomes

\[
\int \frac{d^3P}{(2\pi)^3 2E_q P^0} \sum_X \frac{1}{\text{flux}} \delta^4(p_i - X - P) (2\pi)^4 \left| \mathcal{M}(c\bar{c}(q)) \right|^2 = \frac{F_n(m_c)}{m_{c_{n-4}}} \langle 0 | \mathcal{O}_n^c(q) | 0 \rangle .
\] (2.6)

We will take this as our matching condition for Method I. It is to be used as follows: Keeping in mind that \( E_q \) and \( P^0 \) are functions of \( q \), one computes the left-hand side in the underlying perturbative theory; one expresses this as a Taylor series in \( q^2/m_c^2 \), and then massages it into the form of the right-hand side; one then reads off the \( F_n \). Because of the universality of the \( F_n \), the \( F_n \), determined in the matching procedure also serve in the factorization formula for the production of boundstate quarkonium, Eq. \( \ref{eq:2.2} \) which is

\[
\sigma(H) = \sum_n \frac{F_n(m_c)}{m_{c_{n-4}}} \langle 0 | \mathcal{O}_n^H | 0 \rangle .
\]

In the next section, we present an illustrative example of the method.

D. Example of Calculation in Method I

We now consider our toy example, \( \phi + \phi \to \eta_c \). The underlying interaction is expressed in Eq. \( \ref{eq:1.1} \). Let us compute the left-hand side of the matching condition Eq. \( \ref{eq:2.6} \). In this example, there is no sum \( \sum_X \). We have

\[
\sigma(c\bar{c}(q)) = \int \frac{d^3P}{(2\pi)^3 2E_q P^0} \frac{1}{\text{flux}} \delta^4(P) \delta(E_f - E_i) (2\pi)^4 \left| \mathcal{M}(c\bar{c}(q)) \right|^2
\]

\[
= \int \frac{d^3P}{(2\pi)^3 (2E_q)^2} \frac{1}{8E_q^2} \delta^4(P) \delta(E_f - E_i) (2\pi)^4 \left| \mathcal{M}(c\bar{c}(q)) \right|^2
\]

\[
= (2\pi) \frac{1}{32E_q^4} \delta(E_f - E_i) \left| \mathcal{M}(c\bar{c}(q)) \right|^2
\]

\[
= (2\pi) \frac{1}{32E_q^4} \delta(E_f - E_i) 4E_q g^2 N_c \sum_{\sigma\tau} \zeta_\sigma^\dagger \xi_\sigma \zeta_\tau^\dagger \
\]

\[
= \frac{\pi g^2}{4m_c^2} \delta(E_f - E_i) \left[ 1 - \frac{q^2}{m_c^2} + \cdots \right] N_c \sum_{\sigma\tau} \zeta_\sigma^\dagger \xi_\sigma \zeta_\tau^\dagger 
\] (2.7)

where we have used the Feynman amplitude given in Eq. \( \ref{eq:1.2} \). This can be written in terms of the free-quark NRQCD matrix elements given in Appendix I:

\[
\text{lhs} = \frac{\pi g^2}{4m_c^2} \delta(E_f - E_i) \left[ \langle 0 | \mathcal{O}_1^{c\bar{c}(q)}(1S_0) | 0 \rangle - \frac{1}{m_c^2} \langle 0 | \mathcal{P}_1^{c\bar{c}(q)}(1S_0) | 0 \rangle + \cdots \right] \] (2.8)

We infer from this that the production rate for the charmonium boundstate \( \eta \) is given by

\[
\sigma(\eta) = \frac{\pi g^2}{4m_c^2} \delta(E_f - E_i) \left[ \langle 0 | \mathcal{O}_1^{c\bar{c}(q)}(1S_0) | 0 \rangle - \frac{1}{m_c^2} \langle 0 | \mathcal{P}_1^{c\bar{c}(q)}(1S_0) | 0 \rangle + \cdots \right] 
\] (2.9)

This is the factorization formula for Method I. It corresponds to Eq. (6.8a) of Ref. [1]. We have included the leading piece and the first relativistic correction.

We must reemphasize that the factorization formulas in this approach make no explicit reference to the boundstate mass \( M_H \). The short-distance coefficients can logically depend only on \( m_c \).
III. RELATIVISTICALLY CORRECTED FACTORIZATION FORMULAS: METHOD II

A. An “Effective Amplitude Squared” for Quarkonium Production

Another approach to carrying out calculations of quarkonium production rates is suggested in Ref. [3]. It is our interpretation that in this work, it is implicitly assumed that there exists a meaningful theoretical quantity — an “effective Feynman amplitude (squared),” |\mathcal{M}(H)|^2 — for the production of a quarkonium boundstate. This assumption is tacitly present in the first equation in Ref. [3], which gives the cross-section for quarkonium boundstate production in terms of |\mathcal{M}(H)|^2:

\[ \sigma(H) = \int \frac{d^3p_H}{(2\pi)^32E_H} \sum_X \frac{1}{\text{flux}} (2\pi)^4 \delta^4(p_i - X - P_H) |\mathcal{M}(H)|^2. \]  

(3.1)

Here, X represents the sum of the momenta of all final state particles other than the quarkonium boundstate. \( \sum_X \) represents the measure of phase space integration over all such momenta. It is important to appreciate that here \( P_H \) represents the four-momentum of the quarkonium boundstate. The components of \( P_H \) are \{\( E_H, P_H \)\}, where \( E_H \) is the mass of the boundstate.

Ref. [3] postulates that the production rate for quarkonia can be written in a factorized form

\[ \sigma(H) = \int \frac{d^3p_H}{(2\pi)^32E_H} \frac{1}{\text{flux}} \sum_n \frac{F_n(m_c,P_H)}{m_{c}^{d_s-4}} \langle 0| \mathcal{O}_n^H |0 \rangle. \]  

(3.2)

We use the symbols \( \langle \rangle \) and \( \langle | \cdot | \rangle \) for the bras and kets of Method II as a reminder that here the matrix elements \( \langle 0| \mathcal{O}_n^H |0 \rangle \) are defined differently from the \( \langle 0| \mathcal{O}_n^{\pi} |0 \rangle \) of Method II. The reader must examine Appendices I and II for the details concerning the different conventions and definitions. The distinction concerns the normalization of states. The \( F_n(m_c,P_H) \) are the short-distance coefficients of Method II, distinct from the \( F_n \) of Method I. With the states being normalized differently, the NRQCD matrix elements (and also the short-distance coefficients) are of different mass dimension in the two methods.

B. Matching Condition for Method II

In order to determine the short-distance coefficients \( F_n \) for a given quarkonium production process, we require a matching condition, analogous to Eq. 2.4. The condition proposed in Refs. [3] and [7] is

\[ \sum_X (2\pi)^4 \delta^4(p_i - X - P(q)) |\mathcal{M}(c\bar{c}(q))|^2 = \sum_n \frac{F_n(m_c,P)}{m_{c}^{d_s-4}} \langle 0| \mathcal{O}_n^{c\bar{c}(q)} |0 \rangle \]  

(3.3)

This matching condition lends itself to being interpreted in the following manner: one computes \( |\mathcal{M}(c\bar{c}(q))|^2 \) in the underlying perturbative theory; this allows one to express the complete left-hand side as a Taylor series in \( q^2/m_c^2 \) (it must be kept in mind that \( P^0 \) depends on \( q \)); one then massages the resulting expression into the form of the right-hand side; finally one reads off the \( F_n \). These \( F_n \) serve in the factorization formula for the production of boundstate quarkonium, Eq. 3.2.

In the next section, we present an illustrative example of the method.

C. Example of Calculation in Method II

We now calculate the factorization formula for our toy example using Method II. The left-hand side of the matching condition, Eq. 3.3, is

\[ \text{lhs} = \delta^3(P)\delta(E_f - E_i) (2\pi)^4 |\mathcal{M}(c\bar{c}(q))|^2 \]

\[ = \delta^3(P)\delta(E_f - E_i) (2\pi)^4 4g^2 E_q^2 N_c \sum_{\sigma \tau} \zeta^\dagger_{\sigma} \xi_\tau \xi^\dagger_{\sigma} \zeta_\tau \]  

(3.4)

This can be written in terms of the free-quark NRQCD matrix elements given in Appendix II.
\[ \text{lhs} = g^2 \delta^3(\mathbf{P}) \delta(E_f - E_i) (2\pi)^4 \langle 0 | \mathcal{O}_1^c(\eta) | 1 S_0 \rangle | 0 \rangle \quad (3.5) \]

Interestingly, the Taylor expansion stops after the first term, but this is not a generic feature of this method. In general, there would be an infinite series of terms in powers of \( q^2 / m_c^2 \). We infer from Eq. (3.5) that the production rate for the charmonium boundstate \( \eta_c \) is given by

\[
\sigma(\eta) = \int \frac{d^3 \mathbf{P}_H}{(2\pi)^3 2E_H} g^2 \delta^3(\mathbf{P}_H) \delta(E_f - E_i) (2\pi)^4 \langle 0 | \mathcal{O}_1^c(1 S_0) | 0 \rangle
\]

\[
= \frac{\pi g^2}{2M_\eta^2} \langle 0 | \mathcal{O}_1^c(1 S_0) | 0 \rangle \delta(E_f - E_i)
\]

(3.6)

This is the factorization formula for Method II.

IV. COMPARISON OF THE TWO METHODS

It is instructive to gather together the matching conditions for the two methods, for the purposes of comparison.

\[
\int \frac{d^3 \mathbf{P}}{(2\pi)^3 2E_q P_0} \sum_X \delta^4(p_i - X - P(\mathbf{q})) \frac{1}{\text{flux}} (2\pi)^4 |M(\mathbf{q})|^2 = \sum_n \frac{F_n(m_c)}{m_{c_n}^{d-4}} \langle 0 | \mathcal{O}_1^c(\mathbf{q}) | 0 \rangle
\]

\[
\sum_X \delta^4(p_i - X - P(\mathbf{q})) \frac{1}{\text{flux}} (2\pi)^4 |M(\mathbf{q})|^2 = \sum_n \frac{F_n(m_c, \mathbf{P})}{m_{c_n}^{d-4}} \langle 0 | \mathcal{O}_1^c(\mathbf{q}) | 0 \rangle
\]

where in both cases, the four-momentum \( P(\mathbf{q}) \) is a function of \( \mathbf{q} \) in the sense that, while \( \mathbf{P} \) is strictly independent of \( \mathbf{q} \), \( P^0_0(\mathbf{q}) \) is given by \( P^0_0(\mathbf{q}) = \sqrt{4E_q + \mathbf{P}^2} \). This last point is not an issue in the toy model, but is an issue for the more general situation in which \( p^0_0 \) (the total initial energy) is fixed (and is independent of \( \mathbf{q} \)), as in the case of \( b \to J/\psi + s \).

Once the short-distance coefficients have been found with these matching conditions, the total production rate is given by

\[
\sigma(\mathcal{H}) = \sum_n \frac{F_n(m_c)}{m_{c_n}^{d-4}} \langle 0 | \mathcal{O}_1^c | 0 \rangle
\]

\[
\sigma(\mathcal{H}) = \int \frac{d^3 \mathbf{P}_H}{(2\pi)^3 2E_H} \frac{1}{\text{flux}} \sum_n \frac{F_n(m_c, \mathbf{P}_H)}{m_{c_n}^{d-4}} \langle 0 | \mathcal{O}_1^c | 0 \rangle
\]

\[
= \sum_n \frac{F_n(m_c, \mathcal{H})}{m_{c_n}^{d-4}} \langle 0 | \mathcal{O}_1^c | 0 \rangle
\]

(4.2)

The reader will appreciate, by trying a few simple examples, that the state normalizations and conventions in Appendix I are ideally suited to Method I, and cannot be naturally employed in Method II. The state normalizations and conventions in Appendix II are ideally suited to Method II, and cannot be naturally employed in Method I.

V. RECONCILIATION OF THE TWO METHODS

We have obtained the following factorization formulas for the charmonium production process \( \phi + \phi \to \eta \):

Method I:  \[
\sigma(\eta) = \frac{\pi g^2}{4m_c^2} \delta(E_f - E_i) \left[ \langle 0 | \mathcal{O}_1^c(1 S_0) | 0 \rangle - \frac{1}{m_c^2} \langle 0 | \mathcal{P}_1^c(1 S_0) | 0 \rangle + \cdots \right]
\]

Method II:  \[
\sigma(\eta) = \frac{\pi g^2}{2M_\eta^2} \delta(E_f - E_i) \langle 0 | \mathcal{O}_1^c(1 S_0) | 0 \rangle
\]

(5.1)

The reader can verify that for charmonium production from the decay of the b-quark, \( b \to s + J/\psi \), one obtains (for the color-singlet contributions only)
Method I: \[ \sigma(J/\psi) = \frac{K}{(24\pi)} \frac{m_b^3}{m_c} \left[ \langle O^{J/\psi}(3S_1) \rangle - \frac{5}{6} \frac{1}{m_c^2} \langle P^{J/\psi}(3S_1) \rangle + \cdots \right] \] (5.2)

Method II: \[ \sigma(J/\psi) = \frac{K}{(96\pi)} \frac{m_b^3}{m_c^2} \left[ \langle\langle O^{J/\psi}(3S_1) \rangle \rangle - \frac{4}{3m_c^2} \langle\langle P^{J/\psi}(3S_1) \rangle \rangle + \cdots \right] \] (5.3)

where \( K \) is some constant of mass dimension \(-4\) involving \( G_F^2 \) and CKM matrix elements. (We have simplified things by assuming, wrongly, that \( m_c \ll m_b \).)

We have checked by explicit calculation that if relativistically correct factorization formulas for color-singlet S-wave production are calculated for any quarkonium production processes, in both methods, one finds that the results can always be reconciled, up to relative order \( v^4 \), using the formulas

\[ 2M_H \langle O^H \rangle = \langle\langle O^H \rangle \rangle \] (5.4)

\[ 2M_H \langle P^H \rangle = \langle\langle P^H \rangle \rangle \] (5.5)

and

\[ M_H = 2m_c \left( 1 + \frac{\langle P^H \rangle}{2m_c^2 \langle O^H \rangle} \right) \left( 1 + O(v^4) \right) = 2m_c \left( 1 + \frac{\langle P^H \rangle}{2m_c^2 \langle O^H \rangle} \right) \left( 1 + O(v^4) \right) \] (5.6)

where \( v \) is the characteristic relative velocity of the heavy quarks in the boundstate.

The first two equations seems reasonable, since one might think that they simply expresses the difference between the nonrelativistic state normalizations of Appendix I and the relativistic state normalizations of Appendix II. As to the second equation, it also seems reasonable, since it expresses the idea that relativistic corrections to the mass of the boundstate are of relative order

\[ \frac{\langle P^H \rangle}{m_c^2 \langle O^H \rangle} \sim v^2 \]

The relativistically corrected mass relation, Eq. (5.6), trades relativistic corrections to the mass of the quarkonium boundstate for relativistic corrections involving the four-fermion production matrix element \( \langle P^H \rangle \). Such an idea has recently been presented in Ref. [2]. These authors use the equations of motion of the heavy quark field operators

\[ \left( iD_t + \frac{D^2}{2m_c} \right) \psi = 0 \quad \left( iD_t - \frac{D^2}{2m_c} \right) \chi = 0 \] (5.7)

to re-express the matrix element \( \langle P^\eta \rangle \) so as to obtain

\[ \langle 0|P^\eta(1S_0)|0 \rangle = m_c \left( M_\eta - 2m_c \right) \langle 0|O^\eta(1S_0)|0 \rangle \] (5.8)

Solving for \( M_\eta \), we find

\[ M_\eta = 2m_c \left( 1 + \frac{\langle P^\eta \rangle}{2m_c^2 \langle O^\eta \rangle} \right) \]

which is exactly our relativistically correct mass relation!

VI. DIFFERENTIAL CROSS-SECTIONS

So far, we have considered only total production cross-sections. For calculating differential (instead of total) cross-sections in Method I, one must calculate

\[ \frac{d\sigma(c\bar{q}(q))}{dX} \]

where \( X \) is a kinematic parameter such as the rapidity \( y \), the Mandelstam variable \( t \), or the transverse momentum squared \( p_T^2 \). The matching condition is
\[
\frac{d\sigma(\mathbf{q})}{dX} = \frac{G_n(m_c, \mathbf{P}, X)}{m_c^{d_n-4}} \langle 0|\mathcal{O}_n^{\mathbf{q}}|0 \rangle \quad (6.1)
\]

and the factorization formula is
\[
\frac{d\sigma(H)}{dX} = \frac{G_n(m_c, \mathbf{P}, X)}{m_c^{d_n-4}} \langle 0|\mathcal{O}_n^H|0 \rangle \quad (6.2)
\]

Let us contrast this situation to that in Method II. There, the matching condition allows one to calculate the “effective Feynman amplitude squared” for charmonium production, and differential cross-sections are obtained in a straightforward manner using that object. There is no need, in Method II, to decide at the level of the matching which sort of (differential) cross-section is ultimately desired.

VII. CONCLUSION

We have pointed out that there exist two different methods for calculating factorization formulas for charmonium production, one suggested by Ref. [1], the other by Ref. [3]. The results of the two methods can be reconciled using the relations
\[
2M_H\langle \mathcal{O}^H \rangle = \langle \mathcal{O}^H \rangle \\
2M_H\langle \mathcal{P}^H \rangle = \langle \mathcal{P}^H \rangle 
\]
and
\[
M_H = 2m_c \left(1 + \frac{\langle \mathcal{P}^H \rangle}{2m_c^2\langle \mathcal{O}^H \rangle} \right) (1 + O(v^4)) = 2m_c \left(1 + \frac{\langle \mathcal{P}^H \rangle}{2m_c^2\langle \mathcal{O}^H \rangle} \right) (1 + O(v^4))
\]

It is interesting to ask: “which method is preferable?” and “Have we truly reconciled the two methods?” Indeed, one might object to the step in which the equations of motion are applied to obtain Eq. 5.8, since the matrix element of the operator \(\psi^\dagger (D^2 \chi)\) is ultraviolet divergent and requires a subtraction that is proportional to \(\psi^\dagger \chi\) [5]:
\[
\langle \mathcal{P}^n \rangle = m_c(M_{\eta} - 2m_c + C) \langle \mathcal{O}^n \rangle
\]
where \(C\) is a subtraction constant that depends on the renormalization scheme.

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APPENDIX A: NRQCD CONVENTIONS FOR METHOD I

In performing the matching procedure, we must evaluate the right-hand side of the matching condition, which is written in terms of objects of the NRQCD effective theory. Therefore we require a set of definitions and conventions for calculating in this theory.

Below is presented a practical set of conventions for performing calculations in Method I. These definitions are compatible with Ref. [1], which employs the standard nonrelativistic normalization of the quark states. In the heavy quark state \(|c(p, \sigma, i)\), \(\sigma\) represents spin and \(i\) represents color. The states and annihilation-creation operators are defined via

\[
\langle c(p, \sigma, i) | c(q, \tau, j) \rangle = (2\pi)^3 \delta(p - q) \delta_{\sigma\tau} \delta_{ij} \quad (A1)
\]

\[
\{a(p, \sigma, i), a^\dagger(q, \tau, j)\} = (2\pi)^3 \delta(p - q) \delta_{\sigma\tau} \delta_{ij} \quad (A2)
\]

\[
a^\dagger(p, \sigma, i) | 0 \rangle = | c(p, \sigma, i) \rangle \quad (A3)
\]

\[
a(p, \sigma, i) | c(q, \tau, j) \rangle = (2\pi)^3 \delta(p - q) \delta_{\sigma\tau} \delta_{ij} | 0 \rangle \quad (A4)
\]

The nonrelativistic heavy quark field operators are

\[
\psi^i(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} a(k, \sigma, i) \xi_{\sigma} e^{-ik \cdot x} \quad (A5)
\]

\[
\psi^i_\dagger(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} a^\dagger(k, \sigma, i) \xi^i_{\sigma} e^{ik \cdot x} \quad (A6)
\]

\[
\chi^i(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \bar{a}^\dagger(k, \sigma, i) \zeta_{\sigma} e^{ik \cdot x} \quad (A7)
\]

\[
\chi^i_\dagger(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \bar{a}(k, \sigma, i) \zeta^i_{\sigma} e^{-ik \cdot x} \quad (A8)
\]

The free-quark NRQCD operators are

\[
O_1^{c\bar{c}}(1S_0) = \chi^\dagger \psi | \langle c\bar{c} \rangle | \chi \quad (A9)
\]

\[
P_1^{c\bar{c}}(1S_0) = \frac{1}{2} \left[ \chi^\dagger \psi | \langle c\bar{c} \rangle | \chi \left( \frac{i \not{D}}{2} \right)^2 + \text{h.c.} \right] \quad (A10)
\]

where

\[
| c\bar{c} \rangle \langle c\bar{c} | \equiv \sum_{\sigma, \tau, i, j} | c(q, \sigma, i) \bar{c}(q, \tau, j) \rangle \langle c(q, \sigma, i) \bar{c}(q, \tau, j) | \quad (A12)
\]

It must be stressed that these operators are intended for use in the calculation of unpolarized production rates, i.e., production rates in which the angular momentum of the final quarkonium state is not specified.

Using Eqs. A1 through A12, one can obtain expressions for the free-quark NRQCD matrix elements, as required by the right-hand side of the matching condition. One obtains

\[
\langle 0 | O_1^{c\bar{c}}(1S_0) | 0 \rangle \equiv \langle 0 | \chi^\dagger \psi | \langle c\bar{c} \rangle | \chi | 0 \rangle = N_c \sum_{\sigma\tau} \xi^\dagger_{\sigma} \zeta_{\tau} \zeta^i_{\tau} \xi_{\sigma} \quad (A13)
\]

\[
\langle 0 | P_1^{c\bar{c}}(1S_0) | 0 \rangle \equiv \langle 0 | \frac{1}{2} \left[ \chi^\dagger \psi | \langle c\bar{c} \rangle | \chi \left( \frac{i \not{D}}{2} \right)^2 + \text{h.c.} \right] | 0 \rangle = N_c \sum_{\sigma\tau} \xi^\dagger_{\sigma} \zeta_{\tau} \zeta^i_{\tau} \xi_{\sigma} q^2 \quad (A14)
\]

These free-quark production matrix elements appear in the right-hand side of the matching condition given in Eq. 2.1 and Eq. 2.6. Eqs. A13 and A14 serve in passing from Eq. 2.3 to Eq. 2.9.

The non-perturbative (analytically inestable) NRQCD matrix elements appearing in the factorization formulas are

\[
\langle 0 | O_1^{\eta}(1S_0) | 0 \rangle \equiv \langle 0 | \chi^\dagger \psi \sum_S | \eta + S \rangle \langle \eta + S | \chi | 0 \rangle \quad (A15)
\]

and similarly for \( \langle 0 | P_1^{\eta}(1S_0) | 0 \rangle \), where \( S \) indexes all soft additions to an \( \eta \) particle, such as one pion, two pions, seven pions and a photon, etc., with total energy less than the NRQCD cut-off.
APPENDIX B: NRQCD CONVENTIONS FOR METHOD II

Below is presented a practical set of conventions for performing calculations in Method II. These definitions are compatible with the formulations in Ref. [3], which employs relativistic normalization of the quark states.

\[ \langle (p, \sigma, i) | c(q, \tau, j) \rangle = 2E_q(2\pi)^3\delta(p - q) \]  
(B1)

\[ \{ a(p, \sigma, i), a^\dagger(bf_q, \tau, j) \} = 2E_q(2\pi)^3\delta(p - q) \]  
(B2)

\[ a^\dagger(p, \sigma, i)|0\rangle = |c(p, \sigma, i)\rangle \]  
(B3)

\[ a(p, \sigma, i)|c(q, \tau, j)\rangle = (2\pi)^32E_q\delta(p - q)|0\rangle \]  
(B4)

The nonrelativistic heavy quark field operators are

\[ \psi^i(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} a(k, \sigma, i)\xi_\sigma e^{-ik\cdot x} \]  
(B5)

\[ \psi^i_\dagger(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} a^\dagger(k, \sigma, i)\xi_\sigma^* e^{ik\cdot x} \]  
(B6)

\[ \chi^i(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} b(k, \sigma, i)\chi_\sigma e^{ik\cdot x} \]  
(B7)

\[ \chi^i_\dagger(x) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} b^\dagger(k, \sigma, i)\chi_\sigma^\dagger e^{-ik\cdot x} \]  
(B8)

The free-quark NRQCD operators are

\[ O_i^{c(q)}(1S_0) \equiv \chi^i|\bar{c}\rangle\langle c|\psi^\dagger \chi \]  
(B9)

\[ P_i^{c(q)}(1S_0) \equiv \frac{1}{2}\left[ \chi^i|\bar{c}\rangle\langle c|\psi^\dagger \left( i\frac{\mathbf{D}}{2} \right)^2 \chi + \text{h.c.} \right] \]  
(B10)

where

\[ |\bar{c}\rangle\langle c| \equiv \sum_{\sigma,\tau,i,j} |c(q, \sigma, i)\bar{c}(-q, \tau, j)\rangle\langle c(q, \sigma, i)\bar{c}(-q, \tau, j)| \]  
(B12)

Using Eqs. (B1) through (B12), one can obtain expressions for the free-quark NRQCD matrix elements, as required by the right-hand side of the matching condition. One obtains

\[ \langle 0|O_i^{c(q)}(1S_0)|0\rangle \equiv \langle 0|\chi^i|\bar{c}\rangle\langle c|\psi^\dagger \chi|0\rangle = 4E_q^2N_c \sum_{\sigma\tau} \xi_\sigma^\dagger\xi_\tau^\dagger \xi_\sigma \]  
(B13)

\[ \langle 0|P_i^{c(q)}(1S_0)|0\rangle \equiv \langle 0|\frac{1}{2}\left[ \chi^i|\bar{c}\rangle\langle c|\psi^\dagger \left( i\frac{\mathbf{D}}{2} \right)^2 \chi + \text{h.c.} \right]|0\rangle = 4E_q^2N_c \sum_{\sigma\tau} \xi_\sigma^\dagger\xi_\tau^\dagger \xi_\sigma \cdot \mathbf{q}^2 \]  
(B14)

These free-quark NRQCD matrix elements appear in the right-hand side of the matching condition given in Eq. (3.3). Eqs. (B13) and (B14) serve in passing from Eq. (3.3) to Eq. (3.6).

The non-perturbative (analytically in calculable) NRQCD matrix elements appearing in the factorization formulas are

\[ \langle 0|O^n(1S_0)|0\rangle \equiv \langle 0|\chi^i \sum_S |\eta + S\rangle\langle \eta + S|\psi^\dagger \chi|0\rangle \]  
(B16)

and similarly for \( \langle 0|P^n(1S_0)|0\rangle \).
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