The enclosure method for the heat equation using
time-reversal invariance for a wave equation

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Abstract

The heat equation does not have time-reversal invariance. However, using a
solution of an associated wave equation which has time-reversal invariance, one can
establish an explicit extraction formula of the minimum sphere that is centered at
an arbitrary given point and encloses an unknown cavity inside a heat conductive
body. The data employed in the formula consist of a special heat flux depending on
a large parameter prescribed on the surface of the body over an arbitrary fixed finite
time interval and the corresponding temperature field. The heat flux never blows up
as the parameter tends to infinity. This is different from a previous formula for the
heat equation which also yields the minimum sphere. In this sense, the prescribed
heat flux is moderate.

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wave equation, Kirchhoff’s formula, time-reversal invariance, time-reversal opera-
tion, non-destructive testing.

1 Introduction

This paper is concerned with the methodology for so-called inverse obstacle problems
which aims at reconstructing or extracting an unknown discontinuity such as an obstacle,
inclusion, cavity, etc. embedded or occurred in a reference medium for various observation
data. As one of direct analytical methods which have an exact mathematical base the
author has introduced the enclosure method in [10] and [9] using infinitely many data and
a single set of data, respectively. The enclosure method enables us to find a domain that
encloses the unknown discontinuity.

Originally the method treats a signal which does not depend on a time variable.
Clearly, this restricts the possible application of the enclosure method since there are a
lot of inverse problems which employs various data depending on a time variable. However,
in [11] the author found an idea how to treat a signal which depends on a time variable
in a bounded interval. Now we have various realizations of this time domain enclosure
method in three-space dimensions for the signals governed by the heat and wave equations

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[12, 13, 14, 15, 16, 24, 25, 26], a system of equations in a linearized viscoelastic body [23], the Maxwell system [17, 18] and references therein.

Quite recently, in [21] the author considered an inverse obstacle problem for the wave governed by the wave equation in a bounded domain and introduced an idea which combines the time-reversal invariance of the wave equation with the newest version of enclosure method [19] which employs a single point on the graph of the response operator associated with the inverse obstacle problem. Note that the idea itself in [19] has been applied also to a system of equations in the linear theory of thermoelasticity in [20].

The key point of the idea introduced in [21] is: making a time-reversal operation on the solution of the free space wave equation supported on an arbitrary given ball at the initial time. This gives us an explicit input Neumann data depending on the ball such that the corresponding Dirichlet data as the output over a finite time interval yields the minimum sphere having the same center point as the ball and enclosing the obstacle.

It is curious to know whether there is an analogous approach to inverse obstacle problems for the signal governed by the heat equation. However, one can easily understand that the heat equation does not have time-reversal invariance and thus everyone may think that there is no hope to establish any corresponding formula.

The aim of this paper is to show that the hope can be realized partially with the help of time-reversal invariance of an associated wave equation with a growing speed. One can establish an explicit extraction formula of the minimum sphere that is centered at an arbitrary given point and encloses an unknown cavity inside a heat conductive body.

The data employed in the formula consists of a special heat flux depending on a large parameter prescribed on the surface of the body over an arbitrary fixed finite time interval and the corresponding temperature field. The heat flux is given by taking the time-reversal operation of the normal derivative of a solution of the associated wave equation. This idea, that is, a combination of the enclosure method and prescribing a heat flux coming from a wave equation was initiated in [22].

Remarkably enough, the flux never blows up as the parameter tends to infinity. This is different from a previous formula established in [25] for the heat equation with a discontinuity coefficient, which also yields the minimum sphere. The flux used there depends on a large parameter, this is a common point, however, blows up on the whole surface of the body as the parameter tends to infinity. In this sense, the prescribed heat flux in this paper is moderate. In fairness, however, it should be pointed out that the formula using the special heat flux in this paper is not a complete answer to the original question itself since the flux depends on a parameter. This is the meaning of the partial realization of the hope mentioned above.

Note that the time-reversal operation appears in the BC-method [1, 2], related works [3, 5, 27] and a numerical approach [4]. However, these are concerned with inverse problems for the wave equation. To the author’s best knowledge, no one considers any application of time-reversal invariance to inverse obstacle problem for the heat equation.

Now let us describe the inverse obstacle problem to be considered. We consider the problem for the signal governed by the heat equation formulated as below. Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with $C^2$-boundary. Let $D$ be a nonempty bounded open set of $\mathbb{R}^3$ with $C^2$-boundary such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. The set $D$ is our target which is a mathematical model of an unknown cavity in a heat conductive body $\Omega$.

Given an arbitrary positive number $T$ and $f = f(x,t), (x,t) \in \partial \Omega \times [0, T]$, let
\[ u = u_f(x, t), (x, t) \in (\Omega \setminus \overline{D}) \times [0, T] \]

denote the solution of the following initial value problem for the heat equation
\[
\begin{aligned}
\left( \partial_t - \Delta \right) u &= 0 \quad \text{in } (\Omega \setminus \overline{D}) \times ]0, T[,
\quad \\
u(x, 0) &= 0 \quad \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial D \times ]0, T[,
\quad \\
\frac{\partial u}{\partial \nu} &= f(x, t) \quad \text{on } \partial \Omega \times ]0, T[.
\end{aligned}
\] (1.1)

The \( f, u \) and \( \partial_t u \) belong to \( L^2(0, T; H^{-1/2}(\partial \Omega)) \), \( L^2(0, T; H^1(\Omega \setminus \overline{D})) \) and \( L^2(0, T; H^1(\Omega \setminus \overline{D}')) \), respectively. The \( u \) satisfies, for a positive constant \( C_T \) being independent of \( f \)
\[
\|u\|_{L^2(0,T;H^1(\Omega\setminus\overline{D}))} + \|\partial_t u\|_{L^2(0,T;H^1(\Omega\setminus\overline{D}'))} + \|u(\cdot, T)\|_{L^2(\Omega\setminus\overline{D})} \leq C_T \|f\|_{L^2(0,T;H^{-1/2}(\partial\Omega))}. \tag{1.2}
\]

See [6] for more information.

The inverse obstacle problem to be considered here is: find an extraction formula of information about the geometry of \( D \) from \( u_f(x, t) \) for \( (x, t) \in \partial \Omega \times ]0, T[ \) corresponding to finitely or infinitely many \( f \).

Let \( B \) be an open ball centered at an arbitrary point \( p \in \mathbb{R}^3 \) with radius \( \eta \). The aim of this paper is to show that: there exists a suitable Neumann data \( f \) depending on \( B \) and a large parameter \( \tau \) such that

- \( f = O(1) \) in \( L^2(0, T; H^{-1/2}(\partial \Omega)) \) as \( \tau \to \infty \);
- the \( u_f \) on \( \partial \Omega \times ]0, T[ \) with an arbitrary fixed \( T \) as \( \tau \to \infty \) yields \( R_D(p) \), where
\[
R_D(p) = \sup_{x \in D} |x - p|.
\]

Note that the quantity \( R_D(p) \) gives the radius of the minimum sphere centered at \( p \) that encloses the unknown cavity \( D \).

From (1.2) we have, for such \( f \) as \( \tau \to \infty \)
\[
\|u_f\|_{L^2(0,T;H^1(\Omega\setminus\overline{D}))} + \|\partial_t u_f\|_{L^2(0,T;H^1(\Omega\setminus\overline{D}'))} + \|u_f(\cdot, T)\|_{L^2(\Omega\setminus\overline{D})} = O(1). \tag{1.3}
\]

In this sense, the temperature field \( u_f \) never blows up inside the body as \( \tau \to \infty \).

Note that from a combination of [24] and [25], one can easily show that: if the heat flux \( f \) takes the form
\[
f(x, t) = \varphi(t) \frac{\partial}{\partial \nu} v_\tau(x; p), \quad x \in \partial \Omega, \quad 0 < t < T,
\]

where
\[
v_\tau(x; p) = \begin{cases} \sinh \sqrt{\tau} |x - p| / |x - p| & \text{if } x \in \mathbb{R}^3 \setminus \{p\}, \\ \sqrt{\tau} & \text{if } x = p, \end{cases}
\]

and \( \varphi \in L^2(0, T) \) satisfies, say \( \varphi(t) \sim t^m \) with an integer \( m \geq 0 \) as \( t \downarrow 0 \), then one can extract \( R_D(p) \) from \( u_f \) on \( \partial \Omega \) over time interval \( ]0, T[ \) as \( \tau \to \infty \). Note that the function
$v_\tau(x; p)$ of $x \in \mathbb{R}^3$ is an entire solution of the modified Helmholtz equation $(\Delta - \tau)v = 0$. However, the $f$ given above blows up as $\tau \to \infty$ in the order of exponential everywhere on $\partial \Omega$ if $p \in \Omega$, which is the most interesting case. Thus, this choice does not satisfy the requirement of moderateness of the state as $\tau \to \infty$.

2 Statement of the result and its proof

The construction of $f$ is the following.

First solve the initial value problem for the wave equation:

$$
\begin{cases}
(\partial_s^2 - \Delta)v = 0 & \text{in } \mathbb{R}^3 \times ]0, \infty[, \\
v(x, 0) = 0 & \text{in } \mathbb{R}^3, \\
\partial_s v(x, 0) = \Psi_B(x) & \text{in } \mathbb{R}^3,
\end{cases}
$$

where

$$
\Psi_B(x) = (\eta - |x - p|)\chi_B(x), \quad x \in \mathbb{R}^3
$$

and the function $\chi_B(x)$ denotes the characteristic function of $B$. Note that one can find the $v$ in the class

$$
C^2([0, \infty[, L^2(\mathbb{R}^3)) \cap C^1([0, \infty[, H^1(\mathbb{R}^3)) \cap C([0, \infty[, H^2(\mathbb{R}^3)).
$$

This is an application of the theory of $C_0$-semigroups [28]. Note that the $v$ has an explicit form which is a consequence of Kirchhoff’s formula [7], that is

$$
v(x, s) = \frac{1}{4\pi s} \int_{\partial B_s(x)} \Psi_B(y)dS_y,
$$

where $B_s(x) = \{y \in \mathbb{R}^3 | |y - x| < s\}$.

Let $\tau > 0$ and define

$$
f_{B, T, \tau}(x, t) = \frac{1}{\sqrt{\tau}} \cdot \frac{\partial}{\partial \nu} v(x, \sqrt{\tau}(T - t)), \quad x \in \partial\Omega, \ 0 < t < T.
$$

Note that the function $v_\tau(x, t), x \in \mathbb{R}^3, t \in [0, T]$ defined by

$$
v_\tau(x, t) = \frac{1}{\sqrt{\tau}} v(x, \sqrt{\tau}t),
$$

satisfies

$$
\begin{cases}
(\partial_t^2 - \tau\Delta)v_\tau = 0 & \text{in } \mathbb{R}^3 \times ]0, T[, \\
v_\tau(x, 0) = 0 & \text{in } \mathbb{R}^3, \\
\partial_t v_\tau(x, 0) = \Psi_B(x) & \text{in } \mathbb{R}^3.
\end{cases}
$$

Thus one has the expression

$$
f_{B, T, \tau}(x, t) = \frac{\partial}{\partial \nu} v_\tau(x, T - t), \quad x \in \partial\Omega, \ 0 < t < T.
$$
This means that $f_{B,T,T}$ is the time-reversal mirror [8] on $\partial \Omega$ of the wave $v_\tau(x,t)$.

As can be seen in [22] it is not difficult to show that

$$\|f_{B,T,T}(\cdot,t)\|_{H^{1/2}(\partial \Omega)} \leq C_{\Omega,B} \sqrt{(T-t)^2 + \frac{3}{\tau}}$$

and hence $f_{B,T,T} = O(1)$ in $L^2(0,T;H^{-1/2}(\partial \Omega))$ as $\tau \to \infty$. Therefore from (1.3) we have,

$$\|u_f(\cdot,T)\|_{L^2(\Omega,\mathcal{D})} = O(1).$$

Using $u_f$ with $f = f_{B,T,T}$, we define the indicator function by the formula

$$I_{\partial \Omega}(\tau;B,T) = \int_{\partial \Omega} (w_f - w^*_f) \frac{\partial w^*_f}{\partial \nu} dS,$$

where

$$w_f(x,\tau) = \int_0^T e^{-\tau t} u_f(x,t) dt, \quad x \in \Omega \setminus \mathcal{D},$$

$$w^*_f(x,\tau) = \int_0^T e^{-\tau t} v_\tau(x,T-t) dt, \quad x \in \mathbb{R}^3.$$

Now we state the result of this paper.

**Theorem 2.1.** Let $\eta$ satisfy

$$\eta + 2R_D(p) > R_\Omega(p).$$

(i) There exists a positive number $\tau_0$ such that $I_{\partial \Omega}(\tau;B,T) > 0$ for all $\tau \geq \tau_0$ and we have

$$\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} (\log I_{\partial \Omega}(\tau;B,T) + 2\tau T) = 2(\eta + R_D(p)).$$

(ii) We have

$$\lim_{\tau \to \infty} e^{-\sqrt{\tau}T}e^{2\tau T}I_{\partial \Omega}(\tau;B,T) = \begin{cases} 0 & \text{if } T > 2(\eta + R_D(p)), \\
\infty & \text{if } T < 2(\eta + R_D(p)). \end{cases}$$

**Remark 2.2.** The $\eta$ which is the radius of ball $B$ can not be arbitrary small since we have the constraint (2.4) and it is really constraint if $2R_D(p) < R_\Omega(p)$. If $2R_D(p) \geq R_\Omega(p)$, then $\eta$ can be arbitrary small. To cover the both cases, one can choose an arbitrary $\eta$ such that $\eta \geq R_\Omega(p)$. This is an advantage of using the initial data $\Psi_B$ in (2.1) with finitely extended support $\mathcal{D}$ not like $\{p\}$.

**Proof.** Note that the assertion (ii) is a direct consequence from (i) since there is no restriction on $T$ in (i). Let $w^* = w^*_f$. One can write

$$w^*(x,\tau) = e^{-\tau T} w_0(x,\tau),$$

where

$$w_0(x,\tau) = \int_0^T e^{\tau s} v_\tau(x,s) ds.$$
From (2.2) we have
\[(\Delta - \tau)w_0 + \frac{1}{\tau}\Psi_B(x) = \frac{e^{\tau T}}{\tau}(\partial_t v_\tau(x, T) - \tau v_\tau(x, T)).\]
Thus, \(w^*\) satisfies
\[(\Delta - \tau)w^* + \frac{1}{\tau}(\sqrt{\tau}v(x, \sqrt{\tau} T) - \partial_s v(x, \sqrt{\tau} T)) = e^{-\tau T}F_0(x), \ x \in \mathbb{R}^3 \tag{2.6}\]
where
\[F_0(x) = -\frac{\Psi_B(x)}{\tau}. \tag{2.7}\]
One the other hand, from (1.1) we see that \(w = w_f\) satisfies
\[
\begin{cases}
(\Delta - \tau)w = e^{-\tau T}F(x) & \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial w}{\partial \nu} = \frac{\partial w^*}{\partial \nu} & \text{on } \partial \Omega, \\
\frac{\partial w}{\partial \nu} = 0 & \text{on } \partial D,
\end{cases} \tag{2.8}
\]
where
\[F(x) = u_f(x, T), \ x \in \Omega \setminus \overline{D}. \tag{2.9}\]
Hereafter let \(\tau\) satisfy
\[\sqrt{\tau} T - \eta \geq R_\Omega(p), \]
that is
\[\tau \geq \left(\frac{\eta + R_\Omega(p)}{T}\right)^2. \]
Then, we have \(\Omega \subset B_{\sqrt{T} - \eta}(p)\). Since we have, for all \(s > \eta\)
\[\text{supp } v(\cdot, s) \cup \text{supp } \partial_s v(\cdot, s) \subset \mathbb{R}^3 \setminus B_{s-\eta}(p), \tag{2.10}\]
from (2.6) we have
\[(\Delta - \tau)w^* = e^{-\tau T}F_0(x), \ x \in \Omega. \tag{2.11}\]
Note that (2.10) is a consequence of Kirchhoff’s formula.
Set \(R = w - w^*\). Then from (2.8) and (2.11) we have
\[
\begin{cases}
(\Delta - \tau)R = e^{-\tau T}(F(x) - F_0(x)) & \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial R}{\partial \nu} = 0 & \text{on } \partial \Omega, \\
\frac{\partial R}{\partial \nu} = -\frac{\partial w^*}{\partial \nu} & \text{on } \partial D.
\end{cases}
\]
Then, we obtain the following decomposition formula which corresponds to (2.6) in [22]:
\[I_{\partial \Omega}(\tau; B) = J_h(\tau) + E_h(\tau) + \mathcal{R}_h(\tau), \tag{2.12}\]
where
\[
\begin{align*}
J_h(\tau) &= \int_D (|\nabla w^*|^2 + \tau |w^*|^2) \, dx, \\
E_h(\tau) &= \int_{\Omega \setminus D} (|\nabla R|^2 + \tau |R|^2) \, dx
\end{align*}
\]
and
\[
R_h(\tau) = e^{-rT} \left\{ \int_D F_0 w^* \, dx + \int_{\Omega \setminus D} FR \, dx + \int_{\Omega \setminus D} (F_0 - F) w^* \, dx \right\}.
\]
And also similarly to Lemma 2.2 in [22] one can derive the estimate
\[
E_h(\tau) = O(\tau J_h(\tau) + e^{-2\tau T}).
\]
From (2.6) and (2.7) we have the expression
\[
w^* = \frac{1}{\tau} \left( w^*_1 + e^{-\tau T} w^*_R \right),
\]
where
\[
w^*_1(x, \tau) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\tau}|x-y|}}{|x-y|} (\sqrt{\tau} v(y, \sqrt{\tau} T) - \partial_s v(y, \sqrt{\tau} T)) dy, \quad x \in \mathbb{R}^3
\]
and \( w^*_R = w^*_R(x, \tau) \) satisfies
\[(\Delta - \tau) w^*_R + \Psi_B = 0, \quad x \in \mathbb{R}^3.\]
By integration by parts we have immediately, as \( \tau \to \infty \),
\[
\sqrt{\tau} ||w^*_R||_{L^2(\mathbb{R}^3)} + ||\nabla w^*_R||_{L^2(\mathbb{R}^3)} = O(1).
\]
Here we note that \( v \) satisfies
\[
supp v(\cdot, s) \cup supp \partial_s v(\cdot, s) \subset B_{s+\eta}(p)
\]
for all \( s \geq 0 \), which is also a consequence of Kirchhoff’s formula. From (2.10) and (2.17) together with Propositions 3.3 and 3.4 in [21] we have the explicit expression of the right-hand side on (2.15):
\[
w^*_1(x, \tau) = \frac{1}{\tau} (\mathcal{H}_+ (\sqrt{\tau}; \sqrt{\tau} T, \eta) + \mathcal{H}_- (\sqrt{\tau}; \sqrt{\tau} T, \eta)) \frac{\sinh \sqrt{\tau}|x-p|}{|x-p|},
\]
for all \( x \in B_{\sqrt{\tau} T - \eta}(p) \setminus \{p\} \), where \( \mathcal{H}_+(\sqrt{\tau}; \sqrt{\tau} T, \eta) \) and \( \mathcal{H}_-(\sqrt{\tau}; \sqrt{\tau} T, \eta) \) are given by the following \( \mathcal{H}_+(\tau; T, \eta) \) and \( \mathcal{H}_-(\tau; T, \eta) \), respectively in which \( \tau \) and \( T \) are replaced with \( \sqrt{\tau} \) and \( \sqrt{\tau} T \), respectively:
\[
\begin{align*}
\mathcal{H}_+(\tau; T, \eta) &= f_\tau(T)e^{-\tau T} - f_\tau(T + \eta)e^{-\tau(T + \eta)}, \\
\mathcal{H}_-(\tau; T, \eta) &= g_\tau(T - \eta)e^{-\tau(T - \eta)} - g_\tau(T)e^{-\tau T}.
\end{align*}
\]
Here $f_{\tau}(\cdot)$ and $g_{\tau}(\cdot)$ are cubic polynomials given by

\[
f_{\tau}(\xi) = \frac{\tau}{6} \xi^3 + \left\{ 1 - \frac{\tau}{4} (\eta + 2T) \right\} \xi^2 + \left\{ \frac{1}{2} \tau T (\eta + T) - (\eta + 2T) + \frac{2}{\tau} \right\} \xi
\]

\[
+ \left\{ \frac{1}{12} \tau (\eta - 2T)(\eta + T)^2 + T(\eta + T) - \frac{\eta + 2T}{\tau} + \frac{2}{\tau^2} \right\}
\]

and

\[
g_{\tau}(\xi) = -\frac{\tau}{6} \xi^3 - \left\{ 1 + \frac{\tau}{4} (\eta - 2T) \right\} \xi^2 + \left\{ \frac{1}{2} \tau T (\eta - T) - (\eta - 2T) - \frac{2}{\tau} \right\} \xi
\]

\[
+ \left\{ \frac{1}{12} \tau (\eta + 2T)(\eta - T)^2 + T(\eta - T) - \frac{\eta - 2T}{\tau} - \frac{2}{\tau^2} \right\}.
\]

Note that, in the derivation of (2.18) we fully made use of Kirchhoff’s formula which gives the solution form of (2.1) in the $(x, t)$-space. Furthermore we have

\[
g_{\tau}(T - \eta) = \frac{1}{\tau} \left( \eta - \frac{2}{\tau} \right).
\]

See (3.8) in [21]. Using these, we have

\[
e^{\sqrt{\tau}(\sqrt{T - \eta})} (\mathcal{H}_+ (\sqrt{T}; \sqrt{T}, \eta) + \mathcal{H}_- (\sqrt{T}; \sqrt{T}, \eta)) = \frac{1}{\sqrt{T}} \left( \eta + O(\tau^{-1/2}) \right).
\]

Then, applying Lemma 2.4 in [21] to the right-hand side on (2.18), we obtain, as $\tau \to \infty$

\[
e^{2rT-2\sqrt{\eta}} \left( \| u^* \|_{L^2(U)}^2 + \| \nabla u^* \|_{L^2(U)}^2 \right) = O(\tau^{\mu_1} e^{2\sqrt{\eta} R_U(p)})
\]

and

\[
e^{2rT-2\sqrt{\eta}} \tau^{-\mu_2} e^{-2\sqrt{\eta} R_U(p)} \| u^* \|_{L^2(U)}^2 \geq C,
\]

where $U$ is an arbitrary bounded open subset of $\mathbb{R}^3$; $p$ an arbitrary point in $\mathbb{R}^3$; $\mu_1$ and $\mu_2$ are real numbers and $C$ a positive constant; it is assumed that $\partial U$ is Lipschitz for (2.21).

From (2.14), (2.16) and (2.20) we obtain, as $\tau \to \infty$

\[
e^{2rT-2\sqrt{\eta}} \left( \| u^* \|_{L^2(U)}^2 + \| \nabla u^* \|_{L^2(U)}^2 \right)
\]

\[
= O(\tau^{-2} \| u^* \|_{L^2(U)}^2 + \| \nabla u^* \|_{L^2(U)}^2)
\]

\[
= O(\tau^{-2} e^{2\sqrt{\eta} R_U(p)}) + O(\tau^{-2} e^{2rT-2\sqrt{\eta} e^{-2rT}})
\]

\[
= O(\tau^{-1} e^{2\sqrt{\eta} R_U(p)}) (1 + \tau^{-\mu_1} e^{-2\sqrt{\eta} e^{-2\sqrt{\eta} R_U(p)})}
\]

\[
= O(\tau^{-1} e^{2\sqrt{\eta} R_U(p)}).
\]
From (2.14), (2.16) and (2.21) we obtain
\[ e^{2rT - 2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_U(p)} \|w^*\|_{L^2(U)}^2 \]
\[ \geq \tau^{-2} \left( \frac{1}{2} e^{2rT - 2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_U(p)} \|w^*_1\|_{L^2(U)}^2 - e^{2rT - 2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_U(p)} e^{-2rT} \|w^*_R\|_{L^2(U)}^2 \right) \]
\[ \geq \tau^{-2} \left( \frac{C}{2} \tau^{-\mu_2} - C_1 e^{-2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_U(p)} \tau^{-1} \right) \]
\[ \geq C_2 \tau^{-(2+\mu_2)}, \]  
where \( C_1, C_2 \) are positive constants being independent of all \( \tau \geq \tau_0 \) and \( \tau_0 \) is a large positive constant.

From (2.22) with \( U = D \) we have
\[ e^{2rT - 2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_D(p)} J_h(\tau) = O(\tau^{\mu_1}). \]  
(2.24)

Then from (2.13) we have
\[ e^{2rT - 2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_D(p)} E_h(\tau) = O(\tau^{\mu_1+1}) + O(e^{-2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_D(p)}) \]
\[ = O(\tau^{\mu_1+1}). \]  
(2.25)

Moreover, applying (2.3) to (2.9) and using (2.7), we have
\[ |R_h(\tau)| \]
\[ \leq e^{-rT} \left( \|F_0\|_{L^2(D)} \|w^*\|_{L^2(D)} + \|F\|_{L^2(\Omega \setminus D)} \|R\|_{L^2(\Omega \setminus D)} + \|F_0 - F\|_{L^2(\Omega \setminus D)} \|w^*\|_{L^2(\Omega \setminus D)} \right) \]
\[ \leq C e^{-rT} \left( \tau^{-1} \|w^*\|_{L^2(D)} + \|R\|_{L^2(\Omega \setminus D)} + \|w^*\|_{L^2(\Omega \setminus D)} \right). \]

Then, from (2.24), (2.25) and (2.22) with \( U = \Omega \setminus D \) we obtain
\[ e^{rT - \sqrt{\tau} \eta} |R_h(\tau)| \]
\[ = C e^{-rT} \left( O(\tau^{\frac{\mu_1}{2} - 2} e^{\sqrt{\tau} R_D(p)}) + O(\tau^{\frac{\mu_1}{2} - 1} e^{\sqrt{\tau} R_D(p)}) + O(\tau^{\frac{\mu_1}{2} - 1} e^{\sqrt{\tau} R_D(p)}) \right) \]
\[ = C e^{-rT} \left( O(\tau^{\frac{\mu_1}{2} - 1} e^{\sqrt{\tau} R_D(p)}) + O(\tau^{\frac{\mu_1}{2} - 1} e^{\sqrt{\tau} R_D(p)}) \right). \]

This yields
\[ e^{2rT - 2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_D(p)} |R_h(\tau)| \]
\[ = O(\tau^{\frac{\mu_1}{2} - 1} e^{-\sqrt{\tau} \eta} e^{-\sqrt{\tau} R_D(p)}) + O(\tau^{\frac{\mu_1}{2} - 1} e^{-\sqrt{\tau} (\eta + 2R_D(p) - R_D(p))}). \]  
(2.26)

Now we make use of the assumption (2.4). Then, from (2.26) one concludes
\[ e^{2rT - 2\sqrt{\tau} \eta} e^{-2\sqrt{\tau} R_D(p)} R_h(\tau) = O(\tau^{-\infty}). \]  
(2.27)
Applying this together with (2.24) and (2.25) to (2.12), we obtain, as $\tau \to \infty$

$$e^{2\tau T-2\sqrt{\tau}}e^{-2\sqrt{\tau} R_D(p)} I_{\partial \Omega}(\tau; B, T) = O(\tau^{\mu_1+1}) \quad (2.28).$$

From (2.23) with $U = D$ we have

$$e^{2\tau T-2\sqrt{\tau}}e^{-2\sqrt{\tau} R_D(p)} \tau^{\mu_2} J_h(\tau) \geq e^{2\tau T-2\sqrt{\tau}}e^{-2\sqrt{\tau} R_D(p)} \tau^{2+\mu_2} \|w^*\|^2_{L^2(D)} \geq C_2.$$

Thus this together with (2.12) and (2.27) gives, for all $\tau \geq \tau_0$ for a sufficiently large $\tau_0 > 0$

$$e^{2\tau T-2\sqrt{\tau}}e^{-2\sqrt{\tau} R_D(p)} \tau^{\mu_2} I_{\partial \Omega}(\tau; B, T) \geq \frac{C_2}{2}. \quad (2.29).$$

Now the assertion (i) follows from (2.28) and (2.29).

\[\square\]

**Remark 2.3.** Let $T = 2(\eta + R_D(p))$. This is the excluded case in Theorem 2.1 (ii). It follows from (2.28) that, as $\tau \to \infty$

$$e^{-\sqrt{\tau}T} e^{2\tau T} I_{\partial \Omega}(\tau; B, T) = O(\tau^{\mu_1+1}).$$

The point is: this right-hand side is at most *algebraic*.

**Remark 2.4.** It follows from (2.28) and (2.29) that, as $\tau \to \infty$

$$\frac{1}{\tau} \log I_{\partial \Omega}(\tau; B, T) = -2T + \frac{2}{\sqrt{\tau}} (\eta + R_D(p)) + O\left(\frac{\log \tau}{\tau}\right). \quad (2.30)$$

This yields the convergence rate for (2.5):

$$\frac{1}{\sqrt{\tau}} \left( \log I_{\partial \Omega}(\tau; B, T) + 2\tau T \right) = 2(\eta + R_D(p)) + O\left(\frac{\log \tau}{\sqrt{\tau}}\right).$$

**Remark 2.5.** In [22] instead of $f_{B, T, \tau}$ we have made use of the heat flux $f = f_{B, \tau}$ given by

$$f_{B, \tau}(x, t) = \frac{\partial}{\partial \nu} v_{\tau}(x, t), \quad x \in \partial \Omega, \quad 0 < t < T,$$

where $B$ is the same open ball as above, however, satisfies the constraint $\overline{B} \cap \overline{\Omega} = \emptyset$. Note that there is no additional constraint on $B$ like (2.4). Therein it is shown that, replacing $w_f^*$ in $I_{\partial \Omega}(\tau; B, T)$ with $w_f^0$ given by

$$w_f^0(x, \tau) = \int_0^T e^{-\tau t} v_{\tau}(x, t) dt, \quad x \in \mathbb{R}^3,$$

one can extract $\text{dist} (D, B)$ by the formula

$$\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \log I_{\partial \Omega}(\tau; B, T) = -2 \text{dist} (D, B).$$

Comparing this with (2.30), one may think that the information $R_D(p)$ is hidden deeply more than dist $(D, B)$. 

10
3 An alternative proof of (2.18)

In Remark 3.5 of [21] we have already pointed out that: together with (2.19) the formula

\[
\begin{align*}
  f_\tau(T) &= \frac{\tau}{12} \eta^3 - \frac{\eta}{\tau} + \frac{2}{\tau^2}, \\
  f_\tau(T + \eta) &= \frac{1}{\tau} \left( \eta + \frac{2}{\tau} \right), \\
  g_\tau(T) &= \frac{\tau}{12} \eta^3 - \frac{\eta}{\tau} - \frac{2}{\tau^2}
\end{align*}
\]

is valid. Therefore we have

\[
H^+(\tau, T, \eta) + H^-(\tau; T, \eta) = 4 \tau^2 e^{-\tau T} \left( \frac{1}{\tau} \left( \eta + \frac{2}{\tau} \right) e^{-\tau(T + \eta)} + \frac{1}{\tau} \left( \eta - \frac{2}{\tau} \right) e^{-\tau(T - \eta)} \right).
\] (3.1)

We do not need this explicit formula (3.1) for the present purpose. However, for future purpose which aims at extending the method presented in this paper to other partial differential equations, including systems, we present here an another proof which is based on the solution formula on (2.1) in \((\xi, t)\)-space not \((x, t)\)-space. It is a combination of Parseval’s identity and the residue theorem and completely different from the original proof given in [21] which is a combination of Kirchhoff’s formula and explicit computation formula of some volume potentials.

Let us formulate what we give a proof.

**Proposition 3.1.** Let \(v\) be the solution of (2.1). Let \(T > \eta\). We have

\[
\frac{\tau^2}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\tau|x-y|}}{|x-y|} (\tau v(y, T) - \partial_t v(y, T)) \, dy = (H^+(\tau, T, \eta) + H^-(\tau; T, \eta)) \frac{\sinh \tau|x-p|}{|x-p|}
\]

for all \(x \in B_{T-\eta}(p) \setminus \{p\}\), where \(H^+(\tau, T, \eta) + H^-(\tau; T, \eta)\) is given by the right-hand side on (3.1).

**Proof.** It is known that

\[
\hat{v}(\xi, T) = \frac{\sin |\xi| T}{|\xi|} \hat{\Psi}_B(\xi)
\]

and

\[
(\hat{\partial_t} v)(\xi, T) = \cos |\xi| T \hat{\Psi}_B(\xi),
\]

where we denote by \(\hat{h}\) the Fourier transform of a function \(h(x)\), that is

\[
\hat{h}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} h(x) \, dx.
\]

We have also

\[
\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\tau|x|}}{|x|} e^{-ix \cdot \xi} \, dx = \frac{1}{|\xi|^2 + \tau^2}
\]
and thus
\[ \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\tau|x-y|}}{|x-y|} e^{-iy\cdot\xi} dy = \frac{1}{|\xi|^2 + \tau^2} e^{-ix\cdot\xi}. \]

By Parseval’s identity, we obtain
\[ I_1(\tau) \equiv \frac{\tau^2}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\tau|x-y|}}{|x-y|} \tau v(y, T) dy = \frac{\tau^2}{(2\pi)^3} \int_{\mathbb{R}^3} \sin |\xi| T \hat{\Psi}_B(\xi) \frac{1}{|\xi|^2 + \tau^2} e^{ix\cdot\xi} d\xi \]
and
\[ I_2(\tau) \equiv \frac{\tau^2}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\tau|x-y|}}{|x-y|} \partial_t v(y, T) dy = \frac{\tau^2}{(2\pi)^3} \int_{\mathbb{R}^3} \cos |\xi| T \hat{\Psi}_B(\xi) \frac{1}{|\xi|^2 + \tau^2} e^{ix\cdot\xi} d\xi. \]

A change of variables yields
\[ \hat{\Psi}_B(\xi) = \frac{4\pi\eta^3 e^{-i\xi\cdot p}}{|\xi|} \int_0^1 r (1 - r) \sin \eta |\xi| r dr. \]

Here we have
\[ \int_0^1 r (1 - r) \sin \eta |\xi| r dr = \frac{1}{(\eta |\xi|)^2} \left\{ \frac{2}{\eta |\xi|} (1 - \cos \eta |\xi|) - \sin \eta |\xi| \right\}. \]

Thus we obtain
\[ \hat{\Psi}_B(\xi) = \frac{4\pi\eta^3 e^{-i\xi\cdot p}}{|\xi|} \cdot \frac{1}{(\eta |\xi|)^2} \left\{ \frac{2}{\eta |\xi|} (1 - \cos \eta |\xi|) - \sin \eta |\xi| \right\} \]
\[ = 4\pi\eta e^{-i\xi\cdot p} g(|\xi|; \eta), \]
where
\[ g(z; \eta) = \frac{1}{z^3} \left\{ \frac{2}{\eta z} (1 - \cos \eta z) - \sin \eta z \right\}. \]

Note that \( g(z; \eta) \) is an even function and that \( z = 0 \) is a removable singularity of \( g(z; \eta) \).
Therefore we obtain the expression

\[
I_1(\tau) = \frac{\tau^2}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\sin |\xi|}{|\xi|} \cdot 4\pi \eta e^{-ix_\xi \cdot p} g(|\xi|; \eta) \frac{1}{|\xi|^2 + \tau^2} e^{ix_\xi \cdot \xi} d\xi
\]

\[
= \frac{\tau^3 \eta}{2\pi^2} \int_{\mathbb{R}^3} \frac{\sin |\xi|}{|\xi|} \cdot g(|\xi|; \eta) \frac{1}{|\xi|^2 + \tau^2} e^{i(x-p) \cdot \xi} d\xi
\]

\[
= \frac{\tau^2 \eta}{2\pi^2} \int_{\mathbb{R}^3} \frac{\sin |\xi|}{|\xi|} \cdot g(|\xi|; \eta) \frac{1}{|\xi|^2 + \tau^2} e^{i(x-p) \cdot \xi^3} d\xi
\]

\[
\int_0^\infty r^2 dr \int_0^{2\pi} \int_0^\pi \sin \varphi d\varphi d\theta \frac{\sin rT}{r} g(r; \eta) \frac{e^{i|x-p| r \cos \varphi}}{r^2 + \tau^2} d\varphi
\]

\[
= \frac{\tau^2 \eta}{\pi |x-p|} \int_0^\infty \frac{r}{r^2 + \tau^2} \sin rT \frac{e^{i|x-p| r \cos \varphi} \sin \varphi d\varphi}{r} d\varphi
\]

\[
= \frac{2\tau^2 \eta}{\pi |x-p|} \int_0^\infty \frac{r}{r^2 + \tau^2} \sin rT g(r; \eta) \sin |x-p| r dr
\]

\[
= \frac{\tau^2 \eta}{\pi |x-p|} \int_{-\infty}^\infty \frac{\sin rT}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr.
\]

Similarly, we have

\[
I_2(\tau) = \frac{\tau^2 \eta}{\pi |x-p|} \int_{-\infty}^\infty \frac{\cos rT}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr.
\]

Since \( g(-r; \eta) = g(r; \eta) \), one can rewrite

\[
\left\{ \begin{array}{l}
\int_{-\infty}^\infty \frac{\sin rT}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr = \frac{1}{i} \int_{-\infty}^\infty \frac{e^{irT}}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr,
\
\int_{-\infty}^\infty \frac{\cos rT}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr = \int_{-\infty}^\infty \frac{re^{irT}}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr.
\end{array} \right.
\]

Therefore we have

\[
\left\{ \begin{array}{l}
I_1(\tau) = -i \frac{\tau^3 \eta}{\pi |x-p|} \int_{-\infty}^\infty \frac{e^{irT}}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr,
\
I_2(\tau) = \frac{\tau^2 \eta}{\pi |x-p|} \int_{-\infty}^\infty \frac{re^{irT}}{r^2 + \tau^2} g(r; \eta) \sin |x-p| r dr.
\end{array} \right.
\]

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Let \( z = \text{Re}^{i\eta} \) with \( 0 < \theta < \pi, \ R > 0 \). We have, as \( R \to \infty \)

\[
\begin{align*}
|g(z; \eta)| &= O(R^{-3} e^{\eta R \sin \theta}), \\
|\sin |x - p|z| &= O(e^{|x - p|R \sin \theta}), \\
|e^{izT}| &= O(e^{-TR \sin \theta}).
\end{align*}
\]

Since we have \( T > \eta + |x - p| \), one can apply the residue theorem by taking a standard closed contour \(-R \to R \to Re^{i\theta} \) \((0 \leq \theta \leq \pi) \to -R \) with \( R > \tau \) and letting \( R \to \infty \), one gets

\[
\int_{-\infty}^{\infty} \frac{e^{irT}}{r^2 + \tau^2} g(r; \eta) \sin |x - p|r \ dr = 2\pi i \text{Res}_{z = i\tau} \left( \frac{e^{izT}}{z^2 + \tau^2} g(z; \eta) \sin |x - p|z \right)
\]

\[
= 2\pi i \left( e^{-\tau T} g(i\tau; \eta) \sin |x - p|i\tau \right)
\]

\[
= -\frac{\pi}{i\tau} e^{-\tau T} g(i\tau; \eta) \sinh \tau |x - p|.
\]

Similarly, we have

\[
\int_{-\infty}^{\infty} \frac{re^{irT}}{r^2 + \tau^2} g(r; \eta) \sin |x - p|r \ dr = 2\pi i \frac{i\tau e^{-\tau T}}{2i\tau} g(i\tau; \eta) \sin |x - p|i\tau
\]

\[
= -\pi e^{-\tau T} g(i\tau; \eta) \sinh \tau |x - p|.
\]

From these one gets

\[
I_1(\tau) = \tau^2 \eta e^{-\tau T} g(i\tau; \eta) \frac{\sinh \tau |x - p|}{|x - p|}
\]

and \( I_2(\tau) = -I_1(\tau) \).

Therefore we obtain

\[
I(\tau) = 2I_1(\tau) = 2\tau^2 \eta e^{-\tau T} g(i\tau; \eta) \frac{\sinh \tau |x - p|}{|x - p|}.
\]

Here we have

\[
g(i\tau; \eta) = \frac{1}{(i\tau)^3} \left\{ \frac{2}{\eta i\tau} (1 - \cos i\eta \tau) - \sin i\eta \tau \right\}
\]

\[
= \frac{2}{\eta \tau^4} \left( 1 - \frac{e^{\tau \eta} + e^{-\tau \eta}}{2} \right) - \frac{1}{(i\tau)^3} \frac{e^{-\tau \eta} - e^{\tau \eta}}{2i}
\]

\[
= \frac{2}{\eta \tau^4} \left( 1 - \frac{e^{\tau \eta} + e^{-\tau \eta}}{2} \right) - \frac{1}{\tau^3} \frac{e^{-\tau \eta} - e^{\tau \eta}}{2}
\]

\[
= \frac{2}{\eta \tau^4} + \left( \frac{1}{2\tau^3} - \frac{1}{\eta \tau^4} \right) e^{\tau \eta} - \left( \frac{1}{2\tau^3} + \frac{1}{\eta \tau^4} \right) e^{-\tau \eta}.
\]
This completes the proof.

\[ \square \]

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