EFFECTIVE IDENTIFIABILITY CRITERIA FOR TENSORS AND POLYNOMIALS

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Abstract. A tensor $T$, in a given tensor space, is said to be $h$-identifiable if it admits a unique decomposition as a sum of $h$ rank one tensors. A criterion for $h$-identifiability is called effective if it is satisfied in a dense, open subset of the set of rank $h$ tensors. In this paper we give effective $h$-identifiability criteria for a large class of tensors. We then improve these criteria for some symmetric tensors. For instance, this allows us to give a complete set of effective identifiability criteria for ternary quintic polynomial. Finally, we implement our identifiability algorithms in Macaulay2.

Contents

1. Introduction 1
2. Tensors and flattenings 3
3. Effective identifiability 5
References 11

1. Introduction

A tensor rank-1 decomposition of a tensor $T$, lying in a given tensor space over a field $k$, is an expression of the type

$$T = \lambda_1 U_1 + \ldots + \lambda_h U_h$$

(1.1)

where the $U_i$’s are linearly independent rank one tensors, $\lambda_i \in k^*$, and $k$ is either the real or complex field. The rank of $T$, denoted by $\text{rank}(T)$, is the minimal positive integer $h$ such that $T$ admits a decomposition as in (1.1).

Tensor decomposition problems and techniques are of relevance in both pure and applied mathematics. For instance, tensor decomposition algorithms have applications in psycho-metrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience and graph analysis [KB09, CM96, CGLM08, LO15, MR13]. In pure mathematics tensor decomposition issues naturally arise in constructing and studying moduli spaces of all possible additive decompositions of a general tensor into a given number of rank one tensors [Dol04, DK93, MM13, Mas16, RS00, TZ11].

We say that a tensor rank-1 decomposition has the generic identifiability property if the expression (1.1) is unique, up to permutations and scaling of the factors, on a dense open

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subset of the set of tensors admitting an expression as in (1.1). This uniqueness property is useful in several application, we refer to [COV16] for an account.

Given a tensor rank-1 decomposition of length \( h \) as in (1.1) the problem of specific identifiability consists in proving that such a decomposition is unique. Following [COV16] we call an algorithm for specific identifiability effective if it is sufficient to prove identifiability on a dense open subset of the set of tensors admitting a decomposition as in (1.1). Therefore, an algorithm is effective if its constraints are satisfied generically, in other words if the same algorithm proves generic identifiability as well.

In this paper we consider symmetric tensors, mixed skew-symmetric tensors, and mixed symmetric tensors. The corresponding rank-1 tensors are parametrized by Veronese varieties, Segre-Grassmann varieties, and Segre-Veronese varieties respectively. We provide \( h \)-identifiability effective criteria for these spaces, under suitable numerical assumptions on \( h \). Our algorithm are based on the existence of suitable flattenings of a given tensor admitting a decomposition as in (1.1). We would like to stress that we do not need to know an explicit decomposition but just the fact that such a decomposition exists.

Recall that the border rank \( \rank(T) \) of a tensor \( T \) is the smallest integer \( r > 0 \) such that \( T \) is in the Zariski closure, in the tensor space where \( T \) belongs, of the set of tensors of rank \( r \). In particular \( \rank(T) \leq \rank(T) \). Roughly speaking, our methods require that suitable linear spaces, defined in terms of flattenings, intersect the relevant varieties parametrizing rank one tensors in a zero dimension scheme of a given length. Such a zero dimensional scheme is not required to be reduced and then our criteria can be applied also in border rank identifiability problems, see Remark 3.7.

Symmetric tensors can also be interpreted as homogeneous polynomials. By rephrasing (1.1) in the symmetric case we say that a polynomial rank-1 decomposition of a homogeneous degree \( d \) polynomial \( F \in \mathbb{k}[x_0, \ldots, x_n]_d \) is an expression of the type

\( F = \lambda_1 L_1^d + \cdots + \lambda_h L_h^d \)

where \( L_i \) are linearly independent degree 1 polynomials, \( \lambda_i \in \mathbb{k}^* \), and \( k \) is either the real or complex field. Let \( h(n, d) \) be the minimum integer such that a general \( F \in \mathbb{k}[x_0, \ldots, x_n]_d \) admits a decomposition as in (1.2). The number \( h(n, d) \) has been determined in [AH95] and \( h(n, d) \)-identifiability very seldom holds [Mel06], [Mel09], [GM16]. Indeed, by [GM16, Theorem 1] a general polynomial \( F \in \mathbb{k}[x_0, \ldots, x_n]_d \) is \( h(n, d) \)-identifiable only in the following cases:

- \( n = 1, d = 2m + 1, h(n, d) = m \) [Syl04],
- \( n = d = 3, h(3, 3) = 5 \) [Syl04],
- \( n = 2, d = 5, h(2, 5) = 7 \) [Hil88].

In Theorem 3.8 we provide effective \( h \)-identifiability criteria for these polynomials and combined with the previous results this furnishes a complete set of identifiability criteria for these, and few more, polynomials. We would like to stress that the identifiability criteria in Theorem 3.8 give new proves of the uniqueness of the decomposition for the general polynomial in the three cases listed above. Finally, in Section 3.9 we implemented our identifiability algorithms in Macaulay2 [Mac92].
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2. Tensors and flattenings

Let \( \underline{n} = (n_1, \ldots, n_p) \) and \( \underline{d} = (d_1, \ldots, d_p) \) be two \( p \)-uples of positive integers. Set

\[
d = d_1 + \cdots + d_p, \quad n = n_1 + \cdots + n_p, \quad \text{and} \quad N(\underline{n}, \underline{d}) = \prod_{i=1}^{p} \left( \frac{n_i + d_i}{n_i} \right).
\]

Let \( V_1, \ldots, V_p \) be vector spaces of dimensions \( n_1 + 1 \leq n_2 + 1 \leq \cdots \leq n_p + 1 \), and consider the product

\[
\mathbb{P}^n = \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*).
\]

The line bundle

\[
O_{\mathbb{P}^n}(d_1, \ldots, d_p) = O_{\mathbb{P}(V_1^*)}(d_1) \boxtimes \cdots \boxtimes O_{\mathbb{P}(V_p^*)}(d_p)
\]

induces an embedding

\[
\sigma_{\underline{n}} : \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*) \longrightarrow \mathbb{P}(\text{Sym}^{d_1} V_1^* \otimes \cdots \otimes \text{Sym}^{d_p} V_p^*) = \mathbb{P}^{N(\underline{n}, \underline{d})-1},
\]

\[
([v_1], \ldots, [v_p]) \longmapsto [v_1^{d_1} \otimes \cdots \otimes v_p^{d_p}]
\]

where \( v_i \in V_i \). We call the image

\[
SV_{\underline{n}} = \sigma_{\underline{n}}(\mathbb{P}^n) \subset \mathbb{P}^{N(\underline{n}, \underline{d})-1}
\]

a Segre-Veronese variety. It is a smooth variety of dimension \( n \) and degree \( \frac{(n_1+\cdots+n_p)!}{n_1! \cdots n_p!} d_1^{n_1} \cdots d_p^{n_p} \) in \( \mathbb{P}^{N(\underline{n}, \underline{d})-1} \).

When \( p = 1 \), \( SV_{\underline{d}}^n \) is a Veronese variety. In this case we write \( V_1^n \) for \( SV_{\underline{d}}^n \), and \( v_1^n \) for the Veronese embedding. When \( d_1 = \cdots = d_p = 1 \), \( SV_{1,\ldots,1}^n \) is a Segre variety. In this case we write \( S \underline{n} \) for \( SV_{1,\ldots,1}^n \), and \( \sigma_{\underline{n}} \) for the Segre embedding. Note that

\[
\sigma_{\underline{n}} = \sigma_{\underline{n}'} \circ \left( \nu_{d_1}^{n_1} \times \cdots \times \nu_{d_p}^{n_p} \right),
\]

where \( \underline{n}' = (N(n_1, d_1) - 1, \ldots, N(n_p, d_p) - 1) \).

Similarly, given a \( p \)-uple of \( k \)-vector spaces \( (V_1^{n_1}, \ldots, V_p^{n_p}) \) and \( \underline{d} = (d_1, \ldots, d_p) \) we may consider the Segre-Plücker embedding

\[
\sigma_{\underline{n}} : Gr(d_1, n_1) \times \cdots \times Gr(d_p, n_p) \longrightarrow \mathbb{P}(\wedge^{d_1} V_1^{n_1} \otimes \cdots \otimes \wedge^{d_p} V_p^{n_p}) = \mathbb{P}^{N(\underline{n}, \underline{d})-1},
\]

\[
([H_1], \ldots, [H_p]) \longmapsto [H_1 \otimes \cdots \otimes H_p]
\]

where \( N(\underline{n}, \underline{d}) = \prod_{i=1}^{p} \binom{n_i}{d_i} \). We call the image

\[
SG_{\underline{n}} = \sigma_{\underline{n}}(Gr(d_1, n_1) \times \cdots \times Gr(d_p, n_p)) \subset \mathbb{P}^{N(\underline{n}, \underline{d})}
\]

a Segre-Grassmann variety.

The \( h \)-secant variety \( \text{Sec}_h(X) \), of an irreducible, non-degenerate \( n \)-dimensional variety \( X \subset \mathbb{P}^N \), is the Zariski closure of the union of the linear spaces spanned by collections of \( h \) points on \( X \). The expected dimension of \( \text{Sec}_h(X) \) is

\[
\text{expdim}(\text{Sec}_h(X)) := \min\{nh + h - 1, N\}.
\]
However, the actual dimension of $\text{Sec}_h(X)$ might be smaller than the expected one. Indeed, this happens when through a general point of $\text{Sec}_h(X)$ there are infinitely many $(h-1)$-planes $h$-secant to $X$. We will say that $X$ is $h$-defective if $\dim(\text{Sec}_h(X)) < \exp\dim(\text{Sec}_h(X))$.

The following remark was the starting point of the investigation in [MM13].

**Remark 2.1.** If a polynomial $F \in k[x_0, \ldots, x_n]_d$ admits a decomposition as in (1.2) then $F \in \text{Sec}_h(V^n_1)$, and conversely a general $F \in \text{Sec}_h(V^n_1)$ can be written as in (1.2). If $F = \lambda_1 L_1^d + \cdots + \lambda_h L_h^d$ is a decomposition then the partial derivatives of order $s$ of $F$ can be decomposed as a linear combination of $L_1^{d-s}, \ldots, L_h^{d-s}$ as well.

These partial derivatives are \binom{n+s}{n} homogeneous polynomials of degree $d-s$ spanning a linear space $H_{\partial,s} \subseteq \mathbb{P}(k[x_0, \ldots, x_n]_{d-s})$. Therefore, the linear space $\langle L_1^{d-s}, \ldots, L_h^{d-s} \rangle$ contains $H_{\partial,s}$.

Our first aim is to generalize Remark 2.1 to tensors. The natural tools to replace partial derivatives are flattenings.

**2.1. Flattenings.** Let $V_1, \ldots, V_p$ be $k$-vector spaces of finite dimension, and consider the tensor product $V_1 \otimes \cdots \otimes V_p = (V_{a_1} \otimes \cdots \otimes V_{a_s}) \otimes (V_{b_1} \otimes \cdots \otimes V_{b_{p-s}}) = V_A \otimes V_B$ with $A \cup B = \{1, \ldots, p\}, B = A^c$. Then we may interpret a tensor

$$T \in V_1 \otimes \cdots \otimes V_p = V_A \otimes V_B$$

as a linear map $\tilde{T} : V_A^* \to V_B$. Clearly, if the rank of $T$ is at most $r$ then the rank of $\tilde{T}$ is at most $r$ as well. Indeed, a decomposition of $T$ as a linear combination of $r$ rank one tensors yields a linear subspace of $V_A^*$, generated by the corresponding rank one tensors, containing $\tilde{T}(V_A^*) \subseteq V_A^*$. The matrix associated to the linear map $\tilde{T}$ is called an $(A, B)$-flattening of $T$.

In the case of mixed tensors we can consider the embedding

$$\text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_p} V_p \hookrightarrow V_A \otimes V_B$$

where $V_A = \text{Sym}^{a_1} V_1 \otimes \cdots \otimes \text{Sym}^{a_s} V_p, V_B = \text{Sym}^{b_1} V_1 \otimes \cdots \otimes \text{Sym}^{b_p} V_p$, with $d_i = a_i + b_i$ for any $i = 1, \ldots, p$. In particular, if $n = 1$ we may interpret a tensor $F \in \text{Sym}^{d_1} V_1$ as a degree $d_1$ homogeneous polynomial on $\mathbb{P}(V_1^*)$. In this case the matrix associated to the linear map $\tilde{F} : V_A^* \to V_B$ is nothing but the $a_1$-th catalecticant matrix of $F$, that is the matrix whose lines are the coefficient of the partial derivatives of order $a_1$ of $F$. This identifies the linear space $H_{\partial,s}$ in Remark 2.1 with $\mathbb{P}(\tilde{F}(V_A^*)) \subseteq \mathbb{P}(V_B)$, where $a_1 = s, b_1 = d - a_1 = d - s$.

Similarly, by considering the inclusion

$$\bigwedge^{d_1} V_1 \otimes \cdots \otimes \bigwedge^{d_p} V_p \hookrightarrow V_A \otimes V_B$$

where $V_A = \bigwedge^{a_1} V_1 \otimes \cdots \bigwedge^{a_s} V_p, V_B = \bigwedge^{b_1} V_1 \otimes \cdots \bigwedge^{b_p} V_p$, with $d_i = a_i + b_i$ for any $i = 1, \ldots, p$, we get the so called skew-flattenings. We refer to [Lan12] for details on the subject.
3. Effective identifiability

In this section we give $h$-identifiability criteria for tensors, and we derive effective $h$-identifiability criteria, under some constraints on $h$.

**Proposition 3.1.** Let $T \in \text{Sym}^{d_1} V_1 \otimes \ldots \otimes \text{Sym}^{d_n} V_n$ be a tensor admitting a decomposition $T = \sum_{i=1}^{h} \lambda_i U_i$ as in (1.4). Fix an $(A, B)$-flattening $\bar{T}: V_A^* \to V_B$ of $T$ such that $\dim(V_A^*) \geq h$, and assume that

i) the linear space $\mathbb{P}(\bar{T}(V_A^*))$ has dimension $h - 1$,

ii) $\dim(\mathbb{P}(\bar{T}(V_A^*)) \cap SV_{\mathbb{C}}^{2h}) = 0$,

iii) $\deg(\mathbb{P}(\bar{T}(V_A^*)) \cap SV_{\mathbb{C}}^{2h}) = h$.

where $\mathbb{C} = (b_1, \ldots, b_n)$. Then $T$ is $h$-identifiable and it has rank $h$.

In particular, if $F \in k[x_0, \ldots, x_n]_d$ is a polynomial admitting a decomposition $F = \sum_{i=1}^{h} \lambda_i L_i^n$, $s$ is an integer such that $(n+s) \geq h > (n+s-1)$, and

i) the linear space $H_{\partial, s}$ generated by the partial derivatives of order $s$ of $F$ has dimension $h - 1$,

ii) $\dim(H_{\partial, s} \cap V_{d-s}^n) = 0$,

iii) $\deg(H_{\partial, s} \cap V_{d-s}^n) = h$.

Then $F$ is $h$-identifiable and it has rank $h$.

**Proof.** Assume that $T = \sum_{i=1}^{h} \lambda_i U_i = \sum_{i=1}^{h} \mu_i V_i$ admits two different decompositions. Since $\dim(\mathbb{P}(\bar{T}(V_A^*))) = h - 1$ by Section 2.1 we have $\mathbb{P}(\bar{T}(V_A^*)) = \langle \bar{U}_1, \ldots, \bar{U}_h \rangle = \langle \bar{V}_1, \ldots, \bar{V}_h \rangle$, where $\bar{U}_i, \bar{V}_i$ are the rank one tensors in $\mathbb{P}(V_B)$ induced by $U_i$ and $V_i$ respectively. Hence there are at least $h + 1$ points in the intersection $\mathbb{P}(\bar{T}(V_A^*)) \cap SV_{\mathbb{C}}^{2h}$, contradicting iii). □

Next, we check when the conditions in Proposition 3.1 define effective criteria.

**Proposition 3.2.** The criterion in Proposition 3.1 is effective when $N(\underline{n}, h) > h + \dim(SV_{\mathbb{C}}^{2h})$ in the mixed symmetric case. In particular, in the symmetric case the criterion is effective when $(n + d - s) > h + n$.

**Proof.** Let $[T] \in \text{Sec}_h(SV_{\mathbb{C}}^{2h})$ be a general point. Assume that $\dim(\mathbb{P}(\bar{T}(V_A^*))) \leq h - 2$. This condition forces the $(A, B)$-flattening matrix to have rank at most $h - 1$. On the other hand, by [SU00] Proposition 4.1] these minors do not vanish on $\text{Sec}_h(SV_{\mathbb{C}}^{2h})$ and therefore define a closed subset of $\text{Sec}_h(SV_{\mathbb{C}}^{2h})$. To conclude observe that by the Trisecant Lemma [CC02 Proposition 2.6], the general $h$-secant $(h - 1)$-linear space intersects $SV_{\mathbb{C}}^{2h}$ in $h$ points as long as $N(\underline{n}, h) > h + n$. □

We may slightly improve Proposition 3.2 under suitable numerical assumption.

**Proposition 3.3.** Let $T \in \text{Sym}^{d_1} V_1 \otimes \ldots \otimes \text{Sym}^{d_n} V_p$ be a tensor admitting a decomposition $T = \sum_{i=1}^{h} \lambda_i U_i$. Fix an $(A, B)$-flattening $\bar{T}: V_A^* \to V_B$ of $T$ such that $\dim(V_A^*) \geq h$, and assume that

i) the linear space $\mathbb{P}(\bar{T}(V_A^*))$ has dimension $h - 1$,

ii) $\dim(\mathbb{P}(\bar{T}(V_A^*)) \cap SV_{\mathbb{C}}^{2h}) = 0$,

iii) $h + n = N(\underline{n}, h)$,
iv) \( \deg(SV^n_d) \leq h + 1 \),
v) \( \deg(\langle [U_1], \ldots, [U_h] \rangle \cap SV^n_d) = h \).

Then \( T \) is \( h \)-identifiable and the criterion is effective.

In particular, in the symmetric case we have the following. Let \( F \in k[x_0, \ldots, x_n]_d \) be a polynomial admitting a decomposition \( F = \sum_{i=1}^h \lambda_i L^d_i \). Fix an integer \( s \) such that \( \binom{n+s}{n} \geq h > \binom{n+s-1}{n} \). Assume that:

i) the linear space \( H_{\partial,s} \) generated by the partial derivatives of order \( s \) of \( F \) has dimension \( h - 1 \),

ii) \( \dim(H_{\partial,s} \cap V^n_{d-s}) = 0 \),

iii) \( h + n = \binom{n+d-s}{n} \),

iv) \( (d-s)n \leq h + 1 \),

v) \( \deg(\langle [L^d_1], \ldots, [L^d_h] \rangle \cap V^n_d) = h \).

Then \( F \) is \( h \)-identifiable and the criterion is effective.

Proof. Assume that \( T = \sum_{i=1}^h \lambda_i U_i = \sum_{i=1}^h \mu_i V_i \) admits two different decompositions. Since \( \dim(\mathbb{P}(\mathcal{T}(V^n_d))) = h - 1 \) by Section 2.1 we have \( \mathbb{P}(\mathcal{T}(V^n_d)) = \langle \tilde{U}_1, \ldots, \tilde{U}_h \rangle = \langle \tilde{V}_1, \ldots, \tilde{V}_h \rangle \), where \( \tilde{U}_i, \tilde{V}_i \) are the rank one tensors in \( \mathbb{P}(V^n_d) \) induced by \( U_i \) and \( V_i \) respectively. Assumptions ii), iii), and iv) show that \( \mathbb{P}(\mathcal{T}(V^n_d)) \) intersects \( SV^n_d \) in at most \( h + 1 \) points. Therefore, without loss of generality we may assume that \( U_i = V_i \) for \( i = 1, \ldots, h-1 \). By construction we have

\[
\langle V_1, \ldots, V_h \rangle = \langle V_1, \ldots, V_{h-1}, F \rangle = \langle U_1, \ldots, U_{h-1}, F \rangle = \langle U_1, \ldots, U_h \rangle
\]

hence \( \deg(\langle U_1, \ldots, U_h \rangle \cap SV^n_d) \geq h+1 \) contradicting assumption v). The criterion is effective again by the Trisecant Lemma [CC02, Proposition 2.6]. \( \square \)

Remark 3.4. Propositions 3.1, 3.2, 3.3 can be easily extended to the skew symmetric case, using the skew-flattenings in Section 2.1, and the Segre-Grassmann variety instead of the Segre-Veronese variety. We leave the details to the reader.

Next, we work out our criterion in some interesting cases, for the readers convenience we report also the skew symmetric case.

Corollary 3.5. Let us consider the tensor space \( \text{Sym}^{d_1} V^n_1 \otimes \ldots \otimes \text{Sym}^{d_p} V^n_p \), and set \( m_i = \lfloor \frac{d_i}{2} \rfloor \). If

\[
h < \prod_{i=1}^p \left( \frac{n-1+m_i}{n-1} \right) - p(n-1)
\]

then the criterion in Proposition 3.2 is effective, while for tensors in \( \wedge^{d_1} V^n_1 \otimes \ldots \otimes \wedge^{d_p} V^n_p \) criterion in Proposition 3.2 is effective when

\[
h < \prod_{i=1}^p \left( \frac{n}{m_i} \right) - \prod_{i=1}^p m_i(n-m_i).
\]

Now, consider \( V^n_1 \otimes \ldots \otimes V^n_p \) and set \( m = \lfloor \frac{p}{2} \rfloor \). If

\[
h < n^m - m(n-1)
\]

then the criterion in Proposition 3.2 is effective.
Finally, let $V_1^{n_1} \otimes \cdots \otimes V_p^{n_p}$ be an unbalanced product, that is $n_1 > 1 + \prod_{i=2}^{p} n_i - \sum_{i=2}^{p} (n_i - 1)$. If

$$h < \prod_{i=2}^{p} n_i - \sum_{i=2}^{p} (n_i - 1)$$

then the criterion in Proposition 3.1 is effective.

Proof. In the mixed symmetric case consider the flattening

$$\left( \bigotimes_{i=1}^{p} \text{Sym}^{\lfloor \frac{d_i}{2} \rfloor} V_i^n \right)^* \to \bigotimes_{i=1}^{p} \text{Sym}^{\lfloor \frac{d_i}{2} \rfloor} V_i^n$$

while in the mixed skew-symmetric case it is enough to consider the analogous skew-flatflattening.

Similarly, in the second case we choose the flattening

$$\left( \bigotimes_{i=1}^{\lfloor \frac{p}{2} \rfloor} V_i^n \right)^* \to \bigotimes_{i=1}^{\lfloor \frac{p}{2} \rfloor + 1} V_i^n.$$ 

Finally, we consider the flattening

$$(V_1^{n_1})^* \to \bigotimes_{i=2}^{p} V_i^{n_i}$$

in the unbalanced case. 

\[ \square \]

Remark 3.6. For Veronese varieties our results are equivalent to the identifiability criterion given by A. Iarrobino and V. Kanev in [IK99]. In the d-factor Segre case they are weaker than reshaped Kruskal [COV16 Proposition 16] for $d$ odd but they perform better for $d$ even. While for unbalanced Segre our criteria perform better than [COV16 Proposition 17].

Remark 3.7. The algorithm in Proposition 3.1 works for the border rank as well. Indeed, let $T$ be a tensor, and $P_t = U_{1,t} + \cdots + U_{r,t}$, $Q_t = V_{1,t} + \cdots + V_{r,t}$ be two sequence of rank $r$ tensors such that $\lim_{t \to 0} P_t = \lim_{t \to 0} Q_t = T$, and $\lim_{t \to 0} \{U_{1,t}, \ldots, U_{r,t}\} \neq \lim_{t \to 0} \{V_{1,t}, \ldots, V_{r,t}\}$. Fix an $(A,B)$-flattening $\tilde{T} : V_A^* \to B$ of $T$ such that $\dim(V_A^*) \geq r$, and let us denote by $\tilde{U}_{1,t}, \ldots, \tilde{U}_{r,t}$, $\tilde{P}_t, \tilde{Q}_t$ the corresponding flattenings of $U_{1,t}, \ldots, U_{r,t}, P_t, Q_t$. Then $\mathbb{P}(\tilde{P}_t(V_A^*)) \subseteq \langle \tilde{U}_{1,t}, \ldots, \tilde{U}_{r,t} \rangle$ and $\mathbb{P}(\tilde{Q}_t(V_A^*)) \subseteq \langle \tilde{V}_{1,t}, \ldots, \tilde{V}_{r,t} \rangle$ yield $\lim_{t \to 0} \mathbb{P}(\tilde{P}_t(V_A^*)) \subset \Gamma_U$, $\lim_{t \to 0} \mathbb{P}(\tilde{Q}_t(V_A^*)) \subset \Gamma_V$, where $\Gamma_U = \lim_{t \to 0} \langle \tilde{U}_{1,t}, \ldots, \tilde{U}_{r,t} \rangle$ and $\Gamma_V = \lim_{t \to 0} \langle \tilde{V}_{1,t}, \ldots, \tilde{V}_{r,t} \rangle$.

Now, let $X \subset \mathbb{P}(V_B)$ be the variety parametrizing rank one tensors. Since by hypothesis $\dim(\mathbb{P}(\tilde{T}(V_A^*))) = r - 1$ we have that $\mathbb{P}(\tilde{T}(V_A^*)) = \lim_{t \to 0} \mathbb{P}(\tilde{P}_t(V_A^*)) = \lim_{t \to 0} \mathbb{P}(\tilde{Q}_t(V_A^*))$ forces $\mathbb{P}(\tilde{T}(V_A^*)) = \Gamma_U = \Gamma_V$. Finally, since

$$\lim_{t \to 0} \{\tilde{U}_{1,t}, \ldots, \tilde{U}_{r,t}\} \subseteq X \cap \Gamma_U = X \cap \mathbb{P}(\tilde{T}), \quad \lim_{t \to 0} \{\tilde{V}_{1,t}, \ldots, \tilde{V}_{r,t}\} \subseteq X \cap \Gamma_V = X \cap \mathbb{P}(\tilde{T}(V_A^*))$$

and $\lim_{t \to 0} \{\tilde{U}_{1,t}, \ldots, \tilde{U}_{r,t}\} \neq \lim_{t \to 0} \{\tilde{V}_{1,t}, \ldots, \tilde{V}_{r,t}\}$ we get that $\deg(\mathbb{P}(\tilde{T}(V_A^*)) \cap X) \geq r + 1$, a contradiction with hypothesis iii) of Proposition 3.1.
Finally, we give an effective 7-identifiability criterion for planes quintics, and we extend it to the cases listed in Section 1 when the uniqueness of decomposition holds for a general polynomial.

**Theorem 3.8.** Let $F \in \mathbb{C}[x_0, \ldots, x_n]_d$ be a polynomial, and $H_{\partial,s}$ the linear span of its partial derivatives of order $s$ in $\mathbb{P}(k[x_0, \ldots, x_n]_{d-s})$.

Assume that:

- $(n, d, h, s) \in \{(1, 2h - 1, h, h - 2), (2, 5, 7, 2), (3, 3, 5, 1)\}$,
- $H_{\partial,s}$ has dimension $\binom{n+s}{n} - 1$,
- $H_{\partial,s} \cap V_{d-s}^n$ is empty.

Then $F$ is $h$-identifiable.

**Proof.** Let us consider the case $(n, d, h, s) = (2, 5, 7, 2)$. Assume that $F$ admits two different decompositions $\{L_1, \ldots, L_7\}$ and $\{l_1, \ldots, l_7\}$. Consider the second partial derivatives of $F$ and their span $H_{\partial} \subseteq \mathbb{P}^9$. By Remark 2.1 a decomposition of $F$ induces a decomposition of its partial derivatives, hence we have

$$H_L := \langle L_1^3, \ldots, L_7^3 \rangle \supset H_{\partial} \subset \langle l_1^3, \ldots, l_7^3 \rangle =: H_l.$$  

By hypothesis $\dim H_{\partial} = 5$ and $H_{\partial} \cap V_2^3 = \emptyset$, these yield:

i) $H_{\partial} = H_L \cap H_l$,

ii) $L_i \neq l_j$ for any $i, j \in \{1, \ldots, 7\}$,

iii) $H_L \cap V_3^2$ and $H_l \cap V_3^2$ are zero dimensional and $\sharp(H_L \cap V_3^2) = \sharp(H_l \cap V_3^2) = 7$.

Let $H := \langle H_L, H_l \rangle$ then $H$ intersects $V_2^3$ in at least 14 points and therefore $H \cap V_3^2$ contains a curve $\Gamma$ of degree $3\gamma \leq 6$. Let $\Lambda$ be the pencil of hyperplanes containing $H$. Then $\Lambda V_3^2 = \Gamma + \Sigma$, with $\Sigma$ a pencil of curves. Let $s$ be the degree of the base locus of $\Sigma$.

The hypothesis $H_{\partial} \cap V_2^3 = \emptyset$ and iii) yields

$$s + 6\gamma = 14.$$  

On the other hand we only have the following possibilities:

- $\gamma = 1$ and $s = 4$,
- $\gamma = 2$ and $s = 1$.

This contradiction proves the statement.

For 4-uples $(n, d, h, s) = (1, 2h - 1, h, h - 2), (3, 3, 5, 1)$ we may argue similarly to derive $h$-identifiability criteria, we leave the details to the reader. \hfill $\Box$

For some special values our methods yield a complete set of identifiability criteria.

**Corollary 3.9.** Let $V(n, d) := k[x_0, \ldots, x_n]_d$ be the vector space of homogeneous polynomial of degree $d$, with $k = \mathbb{C}, \mathbb{R}$. Assume that the pair $(n, d)$ is in the following list

$$(1, d), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4).$$

Then there is an effective criteria for specific $s$-identifiability for $V(n, d)$ for every $s$ where generic $s$-identifiability holds.

**Proof.** Let $k = \mathbb{C}$ be the complex field. For pairs $(1, d)$, $d$ odd, $(2, 5), (3, 3)$ we apply the identifiability conditions expressed in Theorem 3.8 for the generic rank and Proposition 3.2 for subgeneric ranks. For $(2, 4)$ Proposition 3.2 applies to ranks less then or equal to 4, and for rank 5 there is not generic identifiability due to defectivity. For $(3, 4)$ Proposition 3.2
applies to ranks less than or equal to 6 and Proposition 3.3 applies to rank 7, while rank 8 is not generically identifiable, \cite{COV15}. For (2, 6) we apply Proposition 3.2 for $s \leq 7$ and Proposition 3.3 for $s = 8$, while rank 9 is not generically identifiable, due to weak defectivity \cite{COV15}.

To conclude we only need to extend the results to the real field. For this let $F = \sum_{i=1}^{k} \lambda_i L_i^d$ be a real polynomial rank-1 decomposition. Then via a field extension we consider it over $\mathbb{C}$ and apply the criterion to prove complex and hence real identifiability.

3.9. Macaulay2 implementation. Finally, we implement our identifiability algorithms in Macaulay2 \cite{Mac92}. The package is in the ancillary file Identifiability.m2. After loading this package in Macaulay2, the main method available is certifyIdentifiability.

The easiest ways to use this method are either by giving in input a mixed symmetric tensor $T$, represented by a multihomogeneous polynomial, and a positive integer $h$, or by inputting one of its decompositions $T = T_1 + \cdots + T_h$ into $h$ rank one mixed symmetric tensors. Then the method returns the boolean value true if the constraints of the corresponding $h$-identifiability criterion are satisfied for $T$. For more details we refer to the documentation (viewHelp certifyIdentifiability). In what follows we show how it works in some cases.

Macaulay2, version 1.9.2
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

```plaintext
i1 : loadPackage "Identifiability";

-- Example 1 -- Random degree 5 polynomial in 3 variables
i2 : P2 = QQ[x,y,z];
i3 : T = for i in 1..7 list (random(1,P2))^5;
i4 : time certifyIdentifiability(sum T,7)
-- got symmetric tensor of dimension 3 and degree 5
-- applying Theorem 3.7 (7-identifiability for 3-forms of degree 5)...
-- 7-identifiability certified
    -- used 0.257789 seconds

o4 = true

i5 : time certifyIdentifiability matrix{T}
-- got symmetric tensor of dimension 3 and degree 5
-- applying Theorem 3.7 (7-identifiability for 3-forms of degree 5)...
-- 7-identifiability certified
    -- used 0.228473 seconds

o5 = true

i6 : -- first 6 summands of T
   T' = T_{0..5};
i7 : time certifyIdentifiability(sum T',6)
-- got symmetric tensor of dimension 3 and degree 5
-- specific 6-identifiability certified
    -- used 0.0363902 seconds

o7 = true
```
i8 : time certifyIdentifiability matrix{T'}
   -- got symmetric tensor of dimension 3 and degree 5
   -- 6-identifiability certified
      -- used 0.0511795 seconds
   o8 = true

   -- Example 2 -- the command below creates a random mixed symmetric
   -- tensor of dimensions {2,5,4}, multidegree {3,2,3}, rank<=5
i9 : T = multirandom({2,5,4},{3,2,3},5);
i10 : -- number terms of the tensor T
      # terms T
   o10 = 1200
i11 : time certifyIdentifiability(T,5)
   -- got mixed symmetric tensor of dimensions {2, 5, 4}
   and multidegree {3, 2, 3}
   -- specific 5-identifiability certified
      -- used 4.54164 seconds
   o11 = true

   -- Example 3 -- Random 1 x 7 matrix of degree 4 polynomials in 4 variables
i12 : decomposition = multirandom'({4},{4},7);
i13 : time certifyIdentifiability decomposition
   -- got symmetric tensor of dimension 4 and degree 4
   -- applying Proposition 3.3...
   -- 7-identifiability certified
      -- used 1.03492 seconds
   o13 = true

   -- Example 4 -- Random 1 x 8 matrix of degree 6 polynomials in 3 variables
i14 : decomposition = multirandom'({3},{6},8);
i15 : time certifyIdentifiability decomposition
   -- got symmetric tensor of dimension 3 and degree 6
   -- applying Proposition 3.3...
   -- 8-identifiability certified
      -- used 0.440192 seconds
   o15 = true

   -- Example 5 -- Random degree 3 polynomial in 4 variables of rank<=5
i16 : F = multirandom({4},{3},5);
i17 : time certifyIdentifiability(F,5)
   -- got symmetric tensor of dimension 4 and degree 3
   -- applying Theorem 3.7 (5-identifiability for 4-forms of degree 3)...
   -- 5-identifiability certified
      -- used 0.098442 seconds
   o18 = true

   -- Example 6 -- Random degree 69 polynomial in 2 variables
i19 : P1 = QQ[x,y];
i20 : F = random(69,P1);
i21 : time certifyIdentifiability(F,35)
   -- got symmetric tensor of dimension 2 and degree 69
EFFECTIVE IDENTIFIABILITY CRITERIA FOR TENSORS AND POLYNOMIALS

applying Theorem 3.7 (35-identifiability for 2-forms of degree 69)...
35-identifiability certified
used 469.406 seconds

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