A WAVELET METHOD FOR NONLINEAR VARIABLE-ORDER
TIME FRACTIONAL 2D SCHRÖDINGER EQUATION

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Abstract. In this study, an efficient semi-discrete method based on the two-dimensional Legendre wavelets (2D LWs) is developed to provide approximate solutions for nonlinear variable-order time fractional two-dimensional (2D) Schrödinger equation. First, the variable-order time fractional derivative involved in the considered problem is approximated via the finite difference technique. Then, by help of the finite difference scheme and the theta-weighted method, a recursive algorithm is derived for the problem under examination. After that, the real functions available in the real and imaginary parts of the unknown solution of the problem are expanded via the 2D LWs. Finally, by applying the operational matrices of derivative, the solution of the problem is transformed to the solution of a linear system of algebraic equations in each time step which can simply be solved. In the proposed method, acceptable approximate solutions are achieved by employing only a small number of the basis functions. To illustrate the applicability, validity and accuracy of the wavelet method, some numerical test examples are solved using the suggested method. The achieved numerical results reveal that the method established based on the 2D LWs is very easy to implement, appropriate and accurate in solving the proposed model.

1. Introduction. It is very well-known that the nonlinear Schrödinger equation is highly interested in mathematical physics. This equation describes a wide variety of phenomena, e.g. photonics [24], condensed matter physics [58], optics of nonlinear media [65], plasma [17], quantum mechanics [4], electromagnetic wave propagation [42] and under-water acoustics [63]. Hence, solving this equation is highly demanded specially due to its importance in physics and engineering models. Anyway, there are many numerical and analytical methods proposed by researchers to solve this partial differential equation (PDE), e.g. [19, 16, 49, 18, 44, 60].
During the past three decades, the subject of fractional calculus (that is, calculus of derivatives and integrals of arbitrary orders) has achieved significant popularity and importance according to its applications in numerous fields in science and engineering. For example, it has been successfully applied to problems in biology [69], physics [8, 47, 53], chemistry and biochemistry [3], hydrology [9, 46], and finance [57]. In recent years, the fractional model of the differential equation has been well-investigated in various perspectives, for example see [52, 67, 5, 6, 22, 21].

The investigation of fractional Schrödinger equation has attracted much attention. The fractional model of nonlinear Schrödinger equation is a very important pervasive models describing diverse nonlinear physical systems. For example, it is used to express the improvement of slowly varying packets corresponding to quasi-monochromatic waves observed in weakly nonlinear media. This fractional PDE has been well-investigated in various perspectives, for example see [41, 25, 23, 10, 7, 2] and references therein. In [41], Hu et al. investigated the properties of existence together with uniqueness of the general solution considering a special type of fractional nonlinear Schrödinger equations in conjunction with the periodic boundary conditions. In [25] the authors used the Adomian decomposition procedure for solving the cubic nonlinear space-time fractional Schrödinger equation. In [23], the authors proposed the homotopy perturbation method and Sumudu transform method to achieve the analytic and approximate solution corresponding to the space-time fractional Schrödinger equation. Bhrawy and Abdelkawy [10] applied a fully spectral method for multi-dimensional fractional Schrödinger equation. The authors of [7] proposed an approximate analytical method for solving two coupled equations of time fractional nonlinear Schrödinger type. In [2], the authors proposed the fractional mapping based expansion method for obtaining analytical solution of the space–time fractional cubic nonlinear Schrödinger equation.

Variable-order fractional calculus theory investigates derivative and integral operators of variable-order [56, 54, 55]. In this generalization, orders of operators are given functions with respect to the space and/or time meanwhile in the classic fractional calculus, orders of operators are given constant values. The practical systems modelled by this new disciplinary in engineering, physics, biology and finance show more sensitivity and accuracy according to variable-order fractional operators [51, 62, 61, 15, 45]. Note that finding analytical solutions of systems involved with variable-order fractional operators are too complex and then, numerical approaches are more applicable for obtaining their approximate solutions. For instance, several numerical approximation methods proposed for solving different types of variable-order fractional differential equations (V-OFDEs) can be found in [12, 70, 43, 1, 11, 66, 50, 48, 32, 31, 28, 29, 30, 27, 39, 38, 40, 64].

The LWs as special type of orthonormal wavelets possess the orthogonality and spectrally accuracy properties of the Legendre polynomials as well as suitable properties of wavelets. Therefore, they have been successfully applied to solve diverse differential equations. In [35], Heydari et al. used the LWs to provide approximate solutions of the fractional Poisson equation. A computational method based on these basis functions is proposed in [36] to solve fractional partial differential equations subject to Dirichlet boundary conditions. In [37], a wavelet algorithm is proposed to achieve numerical solutions for time-fractional telegraph equation. In [33], an accurate and efficient computational method is introduced based on the LWs to numerical solving the time fractional diffusion-wave equation. The authors
of [34] applied an accurate numerical technique using the LWs for fractional optimal control problems. Heydari et al. [26] suggested a LWs Galerkin method for numerical solving fractional sub-diffusion equation. The authors of [28] proposed an optimization LWs method dealing with variable-order fractional Poisson equation. Recently, Hosseiniinia et al. [39] established a semi-discrete method based on the 2D LWs for solving a special class of variable-order fractional nonlinear advection-diffusion equation.

The present work considers nonlinear variable-order time fractional Schrödinger equation

\[
 i c_0^D_\alpha(x,t) \Theta(x,t) + \beta \Delta \Theta(x,t) + \sigma |\Theta(x,t)|^2 \Theta(x,t) + w(x) \Theta(x,t) = f(x,t),
\]

where \( x = (x, y) \in \Omega, \ t \in [0,1] \) with those initial and boundary conditions

\[
 \Theta(x,0) = g(x), \quad x \in \Omega, \tag{2}
\]

\[
 \Theta(x,t) = h(x,t), \quad x \in \partial \Omega, \quad t \in [0,1], \tag{3}
\]

where \( i = \sqrt{-1} \), \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) indicates the Laplacian operator, \( \Theta(x,t) \) represents an undetermined complex function, \( \Omega = [0,1] \times [0,1], \partial \Omega \) is the boundary of \( \Omega, \ g(x) \) and \( h(x,t) \) are complex functions and \( w(x) \) is a real value function and \( \beta, \sigma \) are known constants, \( c_0^D_\alpha(x,t) \) indicates the variable-order time fractional derivative operator with order \( \alpha(x,t) \in (0,1] \) in the Caputo type that will be further described in the next section.

To the best of our knowledge, there is not yet any wavelet method to solve nonlinear variable-order time fractional two-dimensional Schrödinger equation. So, we establish an efficient and accurate semi-discrete procedure based on the 2D LWs to achieve approximate solutions of this problem. In the offered wavelet algorithm, we first approximate the variable-order fractional operator involved in the problem via the finite difference technique. Then, by applying the finite difference scheme and the theta-weighted method, we derive a recursive formula for the problem. Afterwards, we approximate the real functions available in the real and imaginary parts of the unknown solution via the 2D LWs. Thereafter, by help of the operational matrices of derivative, solution of the problem is transformed to solution of a linear system of algebraic equations in each time step. Finally, by solving these yielded systems we obtain approximate solutions in various times for the problem. The obtained numerical results show that we can achieve acceptable approximate solutions by employing only a small number of the LWs.

The lineament of this study is as follows: In Section 2, we briefly review the variable-order fractional calculus theory. In Section 3, we present the 2D LWs and express some of their corresponding properties. Moreover, we completely explain the process of the suggested semi-discrete wavelet method in Section 4. The finite difference scheme and 2D LWs are applied on the mentioned nonlinear Schrödinger equation in Section 4. In Section 5, several numerical examples are solved to examine the accuracy and reliability of the offered method. The article is ended with a concise conclusion in Section 6.

2. Variable-order fractional calculus. This section deals with an introduction of the Caputo fractional derivative with variable-order and the Mittag-Leffler function together with theirs properties.
Definition 2.1. ([59, 14]). The Caputo type variable-order fractional derivative with order $\alpha(x, t) \in (q - 1, q)$, $q \in \mathbb{N}$ of the function $\Theta(x, t)$ toward $t$ is defined as follows
\[
\left(\frac{\partial}{\partial t}^\alpha \Theta(x, t)\right)(x, t) = \begin{cases} 
\frac{1}{\Gamma(q - \alpha(x, t))} \int_0^t (t - \tau)^{q-\alpha(x, t)-1} \frac{\partial^n \Theta(x, \tau)}{\partial \tau^n} \, d\tau, & \alpha(x, t) \in (q - 1, q), \\
\frac{\partial\Theta(x, t)}{\partial t}, & \alpha(x, t) = q,
\end{cases}
\]
where $\Gamma(.)$ is the Gamma function.

Definition 2.2. ([24]). The Mittag-Leffler function $E_{\vartheta, \mu}(z)$ considering different parameters $\vartheta$ and $\mu$ is expressed as follows
\[
E_{\vartheta, \mu}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\vartheta + \mu)}, \quad \vartheta, \mu > 0, \quad z \in \mathbb{C}.
\]

Corollary 1. Based on the above definitions, whenever $\alpha(x, t) \in (0, 1]$, we obtain
\[
\begin{align*}
\frac{\partial}{\partial t}^\alpha \exp(t) &= t^{1-\alpha(x, t)} E_{1, 2-\alpha(x, t)}(t), \\
\frac{\partial}{\partial t}^\alpha \exp(-t) &= -t^{1-\alpha(x, t)} E_{1, 2-\alpha(x, t)}(-t), \\
\frac{\partial}{\partial t}^\alpha \sin(t) &= t^{1-\alpha(x, t)} E_{2, 2-\alpha(x, t)}(-t^2), \\
\frac{\partial}{\partial t}^\alpha \cos(t) &= -t^{2-\alpha(x, t)} E_{2, 3-\alpha(x, t)}(-t^2).
\end{align*}
\]

3. Two-dimensional Legendre wavelets (2D LWs) and their properties.

The 2D LWs are often considered on the unit square $[0, 1] \times [0, 1]$ and defined by the help of the one-dimensional Legendre wavelets (1D LWs) as follows [39]
\[
\psi_{nm, \bar{n}\bar{m}}(x) = \begin{cases} 
\psi_{nm}(x)\psi_{\bar{n}\bar{m}}(y), & x \in I_{nk, \bar{n}k}, \\
0, & \text{otherwise},
\end{cases}
\]
where $x = (x, y)$, $I_{nk, \bar{n}k} = \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right] \times \left[\frac{\bar{n} - 1}{2^k}, \frac{\bar{n}}{2^k}\right]$, $\psi_{nm}(x) = \sqrt{2m + 1} 2^{k+1} L_m(2^{k+1}x - 2n + 1)$ and $\psi_{\bar{n}\bar{m}}(y)$ is defined as $\psi_{nm}(x)$. Here, $k, \bar{k} \in \mathbb{Z}^+ \cup \{0\}$; $M, \bar{M} \in \mathbb{N}$; $n = 1, 2, \ldots, 2^k$, $\bar{n} = 1, 2, \ldots, 2^\overline{k}$, $m = 0, 1, \ldots, M - 1$ and $\bar{m} = 0, 1, \ldots, \bar{M} - 1$. Moreover, $L_m$ and $L_{\bar{m}}$ are the Legendre polynomials of degree $m$ and $\bar{m}$, respectively that are defined over the interval $[-1, 1]$ (to see more details about these polynomials refer to [13]).

It is worth noting that the 2D LWs result in orthonormal basis for the space $L^2([0, 1] \times [0, 1])$. Then any function $\Theta(x) \in L^2([0, 1] \times [0, 1])$ can be approximated by these basis functions as follows
\[
\Theta(x) \simeq \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} \sum_{\bar{n}=0}^{\overline{k}} \sum_{\bar{m}=0}^{\bar{M}-1} \theta_{nm, \bar{n}\bar{m}} \psi_{nm, \bar{n}\bar{m}}(x) = \Theta^T \Psi(x),
\]
where $\theta_{nm, \bar{n}\bar{m}} = \langle \Theta(x), \psi_{nm, \bar{n}\bar{m}}(x) \rangle$ and $\langle \cdot, \cdot \rangle$ indicates the inner product in the space $L^2([0, 1] \times [0, 1])$. Moreover, $\Theta$ and $\Psi(x)$ are $NN = (2^kM)(2^\overline{k}\bar{M})$ column
vectors. For simplicity, Eq. (7) also can be rewritten as follows

$$\Theta(x) \simeq \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{\theta}_{ij} \hat{\psi}_{ij}(x) = \Theta^T \Psi(x) = \Psi(x)^T \Theta,$$

(8)

where \( \hat{\theta}_{ij} = \theta_{nm,\bar{n}m} \) and \( \hat{\psi}_{ij}(x) = \psi_{nm,\bar{n}m}(x) \), and the indexes \( i \) and \( j \) are determined by the relations \( i = M(n-1) + m + 1 \) and \( j = \bar{M}(\bar{n}-1) + \bar{m} + 1 \), respectively. Hence, we have

$$\Theta = \begin{bmatrix} \hat{\theta}_{11} & \hat{\theta}_{12} & \cdots & \hat{\theta}_{1N} \end{bmatrix} \begin{bmatrix} \hat{\theta}_{21} & \hat{\theta}_{22} & \cdots & \hat{\theta}_{2N} \end{bmatrix} \cdots \begin{bmatrix} \hat{\theta}_{N1} & \hat{\theta}_{N2} & \cdots & \hat{\theta}_{NN} \end{bmatrix}^T,$$

and

$$\Psi(x) = \begin{bmatrix} \hat{\psi}_{11}(x) & \hat{\psi}_{12}(x) & \cdots & \hat{\psi}_{1N}(x) \end{bmatrix} \begin{bmatrix} \hat{\psi}_{22}(x) & \hat{\psi}_{22}(x) & \cdots & \hat{\psi}_{2N}(x) \end{bmatrix} \cdots \begin{bmatrix} \hat{\psi}_{N1}(x) & \hat{\psi}_{N2}(x) & \cdots & \hat{\psi}_{NN}(x) \end{bmatrix}^T.$$

(9)

The partial derivatives of the vector \( \Psi(x) \) in Eq. (9) with respect to \( x \) and \( y \) variables can be expressed accordingly

$$\frac{\partial \Psi(x)}{\partial x} = D_x \Psi(x), \quad \frac{\partial \Psi(x)}{\partial y} = D_y \Psi(x),$$

(10)

where \( D_x \) and \( D_y \) are \( N\bar{N} \times N\bar{N} \) matrices, and referred as the operational matrices of derivative with respect to \( x \) and \( y \) variables of the 2D LWs, respectively. For observing more details about the structure of these matrices, the interested reader is referred to [68]. It should be noted that using Eq. (10), the following operational matrices of partial derivatives can be simply derived

$$\frac{\partial^r \Psi(x)}{\partial x^r} = D_x^{(r)} \Psi(x), \quad \frac{\partial^s \Psi(x)}{\partial y^s} = D_y^{(s)} \Psi(x),$$

(11)

where \( D_x^{(r)} \) and \( D_y^{(s)} \) are the \( r \)- and \( s \)-th power of the matrices \( D_x \) and \( D_y \), respectively.

4. **Explanation of the proposed method.** Herein, we establish a semi-discrete wavelet method to solve the variable-order time fractional differential equation introduced in Eq. (1).

4.1. **Variable-order fractional derivative approximation.** By employing the finite difference technique and substituting \( t^{n+1} \) into Eq. (4), we can express the variable-order time fractional derivative available in Eq. (1) as follows [for more details see [39, 64]]:

$$\frac{c}{0} \mathcal{D}^{\alpha}_{t^{n+1}} \Theta(x, t^{n+1}) = a_{\alpha}^{n+1}(x) \left[ \Theta^{n+1}(x) - \Theta^n(x) + \sum_{j=1}^{n} b_{\alpha j}^{n+1}(x) \right].$$

(12)

where \( t^n = n\delta t, \Theta^n(x) = \Theta(x, t^n) \) for \( n = 0, 1, \ldots, \bar{N} \), \( \delta t = T/\bar{N} \), \( a_{\alpha}^{n+1}(x) = \frac{(\delta t)^{-\alpha^{n+1}}(x)}{\Gamma(2-\alpha^{n+1}(x))} \) and \( b_{\alpha j}^{n+1}(x) = (j + 1)^{1-\alpha^{n+1}(x)} - (j)^{1-\alpha^{n+1}(x)}. \)
4.2. Constructing the recursive formula using the theta-weighted method.

By help of the forward finite difference scheme and the theta-weighted method (with the parameter $\theta \in [0, 1]$), we achieve the following relation

$$i_0 D_t^{\alpha(x,t^{n+1})} \Theta(x,t^{n+1}) + \bar{\theta} [\beta \Delta \Theta^{n+1}(x) + w(x)\Theta^{n+1}(x)] + (1 - \bar{\theta}) [\beta \Delta \Theta^n(x) + w(x)\Theta^n(x)] + \sigma |\Theta^n(x)|^2 \Theta^n(x) = f^{n+1}(x), \quad (13)$$

where $f^{n+1}(x) = f(x,t^{n+1})$. Substituting Eq. (12) into Eq. (13) yields

$$i a^{n+1}_\alpha(x) \Theta^{n+1}(x) + \bar{\theta} [\beta \Delta \Theta^{n+1}(x) + w(x)\Theta^{n+1}(x)] = i a^{n+1}_\alpha(x) \Theta^n(x) - (1 - \bar{\theta}) [\beta \Delta \Theta^n(x) + w(x)\Theta^n(x)] - \sigma |\Theta^n(x)|^2 \Theta^n(x)$$

$$- i a^{n+1}_\alpha \sum_{j=1}^n b^{n+1}_\alpha j(x) (\Theta^{n-j}(x) - \Theta^{n-j}(x)) + f^{n+1}(x). \quad (14)$$

The unknown complex function $\Theta(x,t)$ can be considered in its real and imaginary components as follows

$$\Theta(x,t) = u(x,t) + iv(x,t), \quad (15)$$

where $u(x,t)$ and $v(x,t)$ are real functions. We express the known complex function $f(x,t)$ in its real and imaginary parts as follows

$$f(x,t) = f_1(x,t) + if_2(x,t), \quad (16)$$

where $f_1(x,t)$ and $f_2(x,t)$ are known real functions. Also, the known complex functions $g(x)$ and $h(x,t)$ introduced in Eqs. (2) and (3) can be expressed as

$$g(x) = g_1(x) + ig_2(x),$$

$$h(x,t) = h_1(x,t) + ih_2(x,t), \quad (17)$$

where $g_1$, $g_2$, $h_1$ and $h_2$ are real given functions.

Utilizing the substitution of Eqs. (15) and (16) into Eq. (14), the following relations can be achieved

$$a^{n+1}_\alpha(x) v^{n+1}(x) - \bar{\theta} [\beta \Delta u^{n+1}(x) + w(x)u^{n+1}(x)] = a^{n+1}_\alpha(x) v^n(x) + (1 - \bar{\theta}) [\beta \Delta u^n(x) + w(x)u^n(x)] + \sigma G_1(u^n(x), v^n(x))$$

$$- a^{n+1}_\alpha \sum_{j=1}^n b^{n+1}_\alpha j(x) (v^{n-j}(x) - v^{n-j}(x)) - f_1^{n+1}(x), \quad (18)$$

and

$$a^{n+1}_\alpha(x) u^{n+1}(x) + \bar{\theta} [\beta \Delta v^{n+1}(x) + w(x)v^{n+1}(x)] = a^{n+1}_\alpha(x) u^n(x) - (1 - \bar{\theta}) [\beta \Delta v^n(x) + w(x)v^n(x)] - \sigma G_2(u^n(x), v^n(x))$$

$$- a^{n+1}_\alpha \sum_{j=1}^n b^{n+1}_\alpha j(x) (u^{n-j}(x) - u^{n-j}(x)) + f_2^{n+1}(x), \quad (19)$$

where $G_1(u^n(x), v^n(x)) = (u^n(x))^2 + v^n(x)^2) u^n(x)$, $G_2(u^n(x), v^n(x)) = (u^n(x))^2 + v^n(x)^2) v^n(x)$, $f_1^{n+1}(x) = f_1(x,t^{n+1})$ and $f_2^{n+1}(x) = f_2(x,t^{n+1})$. We also notice that the initial and boundary conditions in Eqs. (15) and (17) can be presented as

$$u(x,0) = g_1(x), \quad v(x,0) = g_2(x), \quad x \in \Omega, \quad (20)$$
and

\[ u(x, t) = h_1(x, t), \quad v(x, t) = h_2(x, t), \quad x \in \partial \Omega, \quad t \in [0, T]. \]  \tag{21}

4.3. The 2D LWs approximation. We approximate \(u^n(x)\) and \(v^n(x)\) by the 2D LWs as follows

\[
\begin{align*}
    u^n(x) & \simeq \sum_{l=1}^{N} \tilde{\lambda}_l^n \tilde{\psi}_l(x) = \Psi(x)^T [A]^n, \\
    v^n(x) & \simeq \sum_{l=1}^{N} \tilde{\gamma}_l^n \tilde{\psi}_l(x) = \Psi(x)^T [\Gamma]^n,
\end{align*}
\]  \tag{22}

where \(\tilde{\lambda}_l^n = \tilde{\lambda}_{ij}\), \(\tilde{\gamma}_l^n = \tilde{\gamma}_{ij}\), \(\tilde{\psi}_l(x) = \tilde{\psi}_{ij}(x)\) and \(l = N(i-1) + j\) for \(i = 1, 2, \ldots, N\) and \(j = 1, 2, \ldots, N\). We must compute the unknown coefficients \(\tilde{\lambda}_l^n\) and \(\tilde{\gamma}_l^n\) for \(l = 1, 2, \ldots, N^2\). Hence, we evaluate the functions \(u^n(x)\) and \(v^n(x)\) at the collocation points \(x_{ij} = (x_i, y_j)\) of the domain \(\Omega\) as follows

\[
\begin{align*}
    u^n(x_{ij}) & \simeq \sum_{l=1}^{N} \tilde{\lambda}_l^n \tilde{\psi}_l(x_{ij}) = \Psi(x_{ij})^T [A]^n, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N, \\
    v^n(x_{ij}) & \simeq \sum_{l=1}^{N} \tilde{\gamma}_l^n \tilde{\psi}_l(x_{ij}) = \Psi(x_{ij})^T [\Gamma]^n, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N. \quad \tag{23}
\end{align*}
\]

We introduce the column vectors \([U]^n\), \([A]^n\), \([V]^n\) and \([\Gamma]^n\), and also the \(N^2 \times N^2\) matrix \(A\) as follows

\[
\begin{align*}
    [U]^n & = [u_1^n \ u_2^n \ \ldots \ u_{N^2}^n]^T, \quad [A]^n = \begin{bmatrix} \tilde{\lambda}_1^n & \tilde{\lambda}_2^n & \ldots & \tilde{\lambda}_{N^2}^n \end{bmatrix}^T, \\
    [V]^n & = [v_1^n \ v_2^n \ \ldots \ v_{N^2}^n]^T, \quad [\Gamma]^n = \begin{bmatrix} \tilde{\gamma}_1^n & \tilde{\gamma}_2^n & \ldots & \tilde{\gamma}_{N^2}^n \end{bmatrix}^T, \quad \tag{24}
\end{align*}
\]

and

\[
A = [a_{ij}] = \\
\begin{bmatrix}
\Psi(x_{11}) & \Psi(x_{12}) & \ldots & \Psi(x_{1N}) & \Psi(x_{21}) & \ldots & \Psi(x_{2N}) & \ldots & \Psi(x_{N^2-1}) & \Psi(x_{N^2})
\end{bmatrix}^T. \quad \tag{25}
\]

where \(u_l^n = u^n(x_l)\) and \(v_l^n = v^n(x_l)\) for \(l = 1, 2, \ldots, N^2\). From Eqs. (24) and (25), we can rewrite Eq. (23) in the following matrix form

\[
\begin{align*}
    [U]^n & = A [A]^n, \\
    [V]^n & = A [\Gamma]^n. \quad \tag{26}
\end{align*}
\]

The coefficients matrix \(A\) can be expressed as \(A = A_d + A_b\), where

\[
\begin{align*}
    A_d = [a_{d_{ij}}] = \begin{cases} a_{ij}, & x_{ij} \in \Omega, \\
                         0, & x_{ij} \in \partial \Omega \end{cases}, \\
    A_b = [a_{b_{ij}}] = \begin{cases} 0, & x_{ij} \in \Omega, \\
                          a_{ij}, & x_{ij} \in \partial \Omega. \end{cases}
\end{align*} \tag{27}
\]

Using Eqs. (10) and (11), the functions \(u^n_{xx}(x)\), \(u^n_{yy}(x)\), \(v^n_{xx}(x)\), and \(v^n_{yy}(x)\) can be expressed by the 2D LWs as follows

\[
\begin{align*}
    u^n_{xx}(x) & = \Psi(x)^T \left( \mathbf{D}^{(2)}_x \right)^T [A]^n, \quad u^n_{yy}(x) = \Psi(x)^T \left( \mathbf{D}^{(2)}_y \right)^T [A]^n, \\
    v^n_{xx}(x) & = \Psi(x)^T \left( \mathbf{D}^{(2)}_x \right)^T [\Gamma]^n, \quad v^n_{yy}(x) = \Psi(x)^T \left( \mathbf{D}^{(2)}_y \right)^T [\Gamma]^n. \quad \tag{28}
\end{align*}
\]
Finally, by substituting Eqs. (26) and (28) into Eqs. (18) and (19) for all interior points together with considering the boundary conditions in Eq. (21), we achieve the below system of recursive formulae

\[
\begin{align*}
B_1[\Gamma]^{n+1} + B_2[\Lambda]^{n+1} = & E_1[\Gamma]^n + E_2[\Lambda]^n + [R_1]^{n-1} + \sigma G_1 ([U_d]^n, [V_d]^n) \\
& - [F_1]^{n+1} + [H_1]^{n+1}, \\
B_3[\Gamma]^{n+1} + B_4[\Lambda]^{n+1} = & E_3[\Gamma]^n + E_4[\Lambda]^n + [R_2]^{n-1} - \sigma G_2 ([U_d]^n, [V_d]^n) \\
& + [F_2]^{n+1} + [H_2]^{n+1},
\end{align*}
\]

(29)

where

\[
\begin{align*}
B_1 = B_4 = E_1 = E_4 = [a_o]^{n+1} + A_d, \\
B_2 = -B_3 = -\bar{\theta} \left\{ \beta A_d \left( \left( D_x^2 \right)^T + \left( D_y^2 \right)^T \right) + w * A_d \right\} + A_b, \\
E_2 = -E_3 = (1 - \bar{\theta}) \left\{ \beta A_d \left( \left( D_x^2 \right)^T + \left( D_y^2 \right)^T \right) + w * A_d \right\}, \\
[R_1]^{n-1} = -[b_o]^{n+1} \sum_{j=1}^{n} [b_j]^{n+1} * (V^{n-j+1} - V^{n-j}), \\
[R_2]^{n-1} = -[a_o]^{n+1} \sum_{j=1}^{n} [b_j]^{n+1} * (U^{n-j+1} - U^{n-j}),
\end{align*}
\]

(30)

and the \( N \bar{N} \)-column vectors \([a_o]^{n+1}, [b_j]^{n+1}, [F_1]^{n+1}, [F_2]^{n+1}, [H_1]^{n+1}, [H_2]^{n+1}, G_1 ([U_d]^n, [V_d]^n), G_2 ([U_d]^n, [V_d]^n) \) and \([w] \) are computed using the collocation points \( x_{ij} \) as

\[
\begin{align*}
[a_o]^{n+1} = \left\{ \begin{array}{ll} a_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \Omega, \\
0, & \text{if } x_{ij} \in \partial \Omega, \end{array} \right.
& [b_j]^{n+1} = \left\{ \begin{array}{ll} b_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \Omega, \\
0, & \text{if } x_{ij} \in \partial \Omega, \end{array} \right.
\end{align*}
\]

\[
\begin{align*}
[F_1]^{n+1} = \left\{ \begin{array}{ll} f_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \Omega, \\
0, & \text{if } x_{ij} \in \partial \Omega, \end{array} \right.
& [F_2]^{n+1} = \left\{ \begin{array}{ll} f_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \Omega, \\
0, & \text{if } x_{ij} \in \partial \Omega, \end{array} \right.
\end{align*}
\]

\[
\begin{align*}
[H_1]^{n+1} = \left\{ \begin{array}{ll} h_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \Omega, \\
h_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \partial \Omega, \end{array} \right.
& [H_2]^{n+1} = \left\{ \begin{array}{ll} h_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \Omega, \\
h_i^{n+1}(x_{ij}), & \text{if } x_{ij} \in \partial \Omega, \end{array} \right.
\end{align*}
\]

\[
[w] = \left\{ \begin{array}{ll} w(x_{ij}), & \text{if } x_{ij} \in \Omega, \\
0, & \text{if } x_{ij} \in \partial \Omega, \end{array} \right.
\]

(31)

so that according to system (29) we can easily get \([U]^{n+1} \) and \([V]^{n+1} \) for \( n = 0,1, \ldots, \bar{N} - 1 \). In the system (29) for \( n = 0 \), we have

\[
\begin{align*}
B_1[\Gamma]^1 + B_2[\Lambda]^0 = & E_1[\Gamma]^0 + E_2[\Lambda]^0 + \sigma G_1 ([U_d]^0, [V_d]^0) + [F_1]^1 + [H_1]^1, \\
B_3[\Gamma]^1 + B_4[\Lambda]^0 = & E_3[\Gamma]^0 + E_4[\Lambda]^0 - \sigma G_2 ([U_d]^0, [V_d]^0) + [F_2]^1 + [H_2]^1, \\
\end{align*}
\]

(32)
Obviously, we have \( [U]^{0} = [g_{1}(x_{11}) g_{1}(x_{12}) \ldots g_{1}(x_{N,N})]^{T} \), \( [V]^{0} = [g_{2}(x_{11}) g_{2}(x_{12}) \ldots g_{2}(x_{N,N})]^{T} \), \( [\Lambda]^{0} = \Lambda^{-1}[U]^{0} \) and \( [\Gamma]^{0} = \Lambda^{-1}[V]^{0} \).

**Remark 1.** Howbeit the system (29) is valid for any value of parameter \( \tilde{\theta} \in [0,1] \), but in the sequel, we take \( \theta = 1/2 \) and use the Crank-Nicholson scheme.

5. **Numerical examples.** In the present section, the numerical results of the expressed method are provided for several test examples. To show the accuracy and validity of the presented approach, we report the relative maximum errors for both real and imaginary parts and also the modulus of error \(|\varepsilon|\) as follows

\[
\varepsilon_{\text{real}} = \max_{1 \leq i \leq N} \max_{1 \leq j \leq N} \left| u(x_{ij}) - \tilde{u}(x_{ij}) \right|, \quad \varepsilon_{\text{image}} = \max_{1 \leq i \leq N} \max_{1 \leq j \leq N} \left| v(x_{ij}) - \tilde{v}(x_{ij}) \right|,
\]

\[
|\varepsilon| = \sqrt{\varepsilon_{\text{real}}^{2} + \varepsilon_{\text{image}}^{2}}.
\]

where \( u(x_{ij}), \tilde{u}(x_{ij}), v(x_{ij}) \) and \( \tilde{v}(x_{ij}) \) are the exact and approximate solutions in the collocation points \( x_{ij} \), respectively. Also, without loss of generality, we consider \((M, k) = (M, \tilde{k})\) which follows \( N = \tilde{N} \). Moreover, we use the shifted Gauss-Legendre points as the collocation points \( x_{ij} \).

**Problem.** First, consider the nonlinear variable-order time fractional Schrödinger equation [20]

\[
i_{0}^{\alpha}D_{t}^{\alpha}(x,t)\Theta(x,t) - \Delta \Theta(x,t) - |\Theta(x,t)|^{2}\Theta(x,t) = f(x,t).
\]

The corresponding analytical solution in the case of \( \alpha(x,t) = 1 \) is

\[
\Theta(x,t) = \frac{1}{2} \exp \left(i \left(x + y - \frac{7}{4}t \right) \right).
\]

The right-hand side function and the corresponding required conditions can be extracted from the analytic solution. The problem is examined by the present approach with \( N = 10(M = 10, k = 0) \) and three different values of \( \delta t \) for some selected \( \alpha(x,t) \). Approximate solutions for real and imaginary parts for the case \( \delta t = 0.1 \) are shown in Figs. 1 and 2, respectively. Graphs of the modulus of the approximate solutions in \( x = 0.2 \) (up) and \( y = 0.4 \) (down) at final computed time are shown in Fig. 3. To investigate the precision of the present approach, we listed the obtained error values for the case \( \alpha(x,t) = 1 \) in Table 1. The achieved results demonstrate the accuracy of the present method in solving this problem.

| Table 1. | The obtained error values by the presented wavelet method in case of \( \alpha(x,t) = 1 \) with three values of \( \delta t. \) |
|-----------------|-----------------|-----------------|-----------------|
| \( \delta t = 0.1 \) | \( \delta t = 0.01 \) | \( \delta t = 0.005 \) |
| \( t \) | \( \varepsilon_{\text{real}} \) | \( \varepsilon_{\text{image}} \) | \( |\varepsilon| \) | \( \varepsilon_{\text{real}} \) | \( \varepsilon_{\text{image}} \) | \( |\varepsilon| \) | \( \varepsilon_{\text{real}} \) | \( \varepsilon_{\text{image}} \) | \( |\varepsilon| \) |
| 0.1 | 1.0947E-4 | 2.7398E-4 | 2.9109E-4 | 5.1840E-5 | 1.7040E-4 | 5.2912E-4 | 2.7150E-5 | 6.2235E-5 | 7.3515E-5 |
| 0.3 | 1.4097E-4 | 2.9682E-4 | 3.2831E-4 | 7.5040E-5 | 1.4841E-4 | 7.1729E-4 | 5.5009E-5 | 7.4200E-5 | 8.2779E-5 |
| 0.5 | 7.2304E-4 | 1.2798E-3 | 1.5050E-3 | 3.6200E-4 | 6.3540E-4 | 7.3198E-4 | 1.8100E-4 | 3.1770E-4 | 3.6560E-4 |
| 0.7 | 2.1254E-4 | 3.2793E-3 | 3.5090E-3 | 1.0600E-4 | 1.6367E-3 | 1.6000E-4 | 5.3200E-5 | 8.1835E-5 | 9.7607E-4 |
| 0.9 | 1.5291E-3 | 1.0866E-3 | 1.8000E-3 | 7.6500E-4 | 5.1130E-4 | 9.1715E-4 | 3.8090E-4 | 2.5656E-4 | 4.5857E-4 |
| 1.0 | 8.3072E-4 | 4.0087E-3 | 1.0000E-3 | 4.1536E-4 | 5.0390E-4 | 6.5392E-4 | 2.0768E-4 | 2.5159E-4 | 3.8651E-4 |
Problem. In this problem, we examine the problem presented in Eq. (1) with $\beta = 1$, $\sigma = 1$, $w(\mathbf{x}) = 0$, $\alpha(\mathbf{x}, t) = 0.2 - 0.1(\cos(xt) \sin(x) + \cos(yt) \sin(y))$ and the analytic solution

$$\Theta(\mathbf{x}, t) = t^2(\sin(x) \sin(y) + i \cos(x) \cos(y)).$$

The non-homogeneous term is achieved as

$$f(\mathbf{x}, t) = \left( \frac{2t^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))} - 2t^2 + t^6(\sin^2(x) \sin^2(y) + \cos^2(x) \cos^2(y)) \right) (\sin(x) \sin(y) + i \cos(x) \cos(y)).$$

The corresponding required conditions can be achieved from the analytic solution.

We solved this problem on the unit square domain by the wavelet method proposed in Section 4 with $N = 8(M = 8, k = 0)$. Table 2 shows the obtained error values with different time step $\delta t$ where $T = 1$. Fig. 4 shows the surface plot of the obtained solution (up) and the corresponding absolute error (AE) function (down) for the real part whenever $\delta t = 0.0025$ at $T = 1$. Also, the obtained solution (up)
The behavior of the real part of the wavelet solutions in the spaces \((0.2, y)\) (up) and \((x, 0.4)\) (down) at \(t = 1\) for some selections \(\alpha(x, t)\).

and the corresponding AE function (down) are illustrated for the imaginary part in Fig. 5. Modulus of the evaluated solution (up) and the corresponding error (down) are presented in Fig. 6. The obtained results confirm the acceptable accuracy of the proposed wavelet method.

**Table 2.** The obtained error values by the presented wavelet method with three different values of \(\delta t\).

| \(\delta t\) | \(t\) | \(\varepsilon_{\text{real}}\) | \(\varepsilon_{\text{image}}\) | \(|\varepsilon|\) | \(\varepsilon_{\text{real}}\) | \(\varepsilon_{\text{image}}\) | \(|\varepsilon|\) | \(\varepsilon_{\text{real}}\) | \(\varepsilon_{\text{image}}\) | \(|\varepsilon|\) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 2.9620E-5 | 1.1656E-4 | 1.1981E-4 | 1.4703E-5 | 5.4867E-5 | 5.6803E-5 | 7.3241E-6 | 2.7443E-5 | 2.8404E-5 |
| 0.3 | 8.850E-5 | 3.2366E-4 | 3.3563E-4 | 4.4328E-5 | 1.6168E-4 | 1.6765E-4 | 2.2127E-5 | 8.0924E-5 | 8.3895E-5 |
| 0.5 | 1.3329E-4 | 4.7970E-4 | 4.9758E-4 | 6.6145E-5 | 2.3966E-4 | 2.4899E-4 | 3.3150E-5 | 1.1985E-4 | 1.2375E-4 |
| 0.7 | 1.1358E-4 | 4.9724E-4 | 5.0944E-4 | 5.5570E-5 | 1.9680E-4 | 2.0450E-4 | 2.7615E-5 | 9.8391E-5 | 1.0210E-4 |
| 0.9 | 9.3184E-5 | 3.9372E-4 | 4.0944E-4 | 4.8381E-5 | 1.9775E-4 | 2.0358E-4 | 2.4638E-5 | 9.9143E-5 | 1.0216E-4 |
| 1.0 | 3.3547E-4 | 1.3286E-3 | 1.4606E-3 | 1.7083E-4 | 6.6725E-4 | 6.8877E-4 | 8.6191E-5 | 3.3438E-4 | 3.4591E-4 |
Problem. Consider Eq. (1) where $\beta = \sigma = 1$, $w(x) = 1 - \frac{2}{x^2} - \frac{2}{y^2} - x^4y^4$ and
\[
f(x, t) = ix^2y^2 \left( -t^{2-\alpha(x,t)} E_{2,3-\alpha(x,t)} \left( -t^2 \right) + it^{1-\alpha(x,t)} E_{2,2-\alpha(x,t)} \left( -t^2 \right) \right) + 2 (x^2 + y^2) \exp(it) + x^6y^6 \exp(it) + (x^2y^2 - 2 (x^2 + y^2) - x^6y^6) \exp(it) .
\]
The corresponding analytic solution is
\[
\Theta(x, t) = x^2y^2 \exp(it).
\]
The corresponding required conditions are in an agreement with the analytic solution. The achieved results of the introduced wavelet procedure with $k = 0$ and some values of $M$ are displayed in Table 3 where $\alpha(x, t) = 1 - 0.75 \exp(-(xyt)^2)$ and $\delta t = 0.01$. According to the information of this table, we observe the computed results are in a good agreement with those of exact ones. The obtained solutions and the corresponding AE functions for the real and imaginary parts whenever $N = 12$ ($M = 12, k = 0$) are shown in Figs. 7 and 8, respectively. Fig. 9 simulates the modulus of the obtained solution and the corresponding error function.
Figure 4. The behavior of the real part of the wavelet solution and the corresponding AE function (up and down, respectively) at the final time where $\delta = 0.0025$.

Table 3. The obtained error values by the presented wavelet method with $k = 0$ and three different values of $M$ with $\delta t = 0.01$.

| $t$ | $\varepsilon_{\text{real}}$ | $\varepsilon_{\text{image}}$ | $|\varepsilon|$ | $\varepsilon_{\text{real}}$ | $\varepsilon_{\text{image}}$ | $|\varepsilon|$ | $\varepsilon_{\text{real}}$ | $\varepsilon_{\text{image}}$ | $|\varepsilon|$ |
|-----|-----------------|----------------|-------------|-----------------|----------------|-------------|-----------------|----------------|-------------|
| 0.1 | 1.5390E-5       | 2.9309E-5     | 3.3085E-5   | 1.5040E-5       | 3.1348E-5     | 3.4769E-5   | 1.5840E-5       | 3.0785E-5     | 3.4621E-5   |
| 0.3 | 6.3197E-6       | 5.3790E-5     | 5.8953E-5   | 5.3587E-5       | 5.3953E-5     | 5.3411E-5   | 5.3121E-5       | 5.3141E-5     | 5.3141E-5   |
| 0.5 | 3.1900E-5       | 5.8084E-5     | 5.8359E-5   | 3.4016E-5       | 5.8860E-5     | 5.8018E-5   | 3.4113E-5       | 5.7111E-5     | 5.8158E-5   |
| 0.7 | 2.4309E-5       | 4.0514E-5     | 4.1831E-5   | 2.4347E-5       | 4.0151E-5     | 4.1931E-5   | 2.5324E-5       | 3.6798E-5     | 4.1951E-5   |
| 0.9 | 3.6034E-5       | 4.0237E-5     | 5.1417E-5   | 3.6060E-5       | 4.0415E-5     | 5.1330E-5   | 3.1359E-5       | 3.9263E-5     | 5.0248E-5   |
| 1.0 | 4.7832E-5       | 5.7988E-5     | 6.1678E-5   | 4.6654E-5       | 5.8373E-4     | 5.8616E-4   | 4.4804E-4       | 5.6276E-4     | 4.4951E-4   |

Problem. Finally, a nonlinear variable-order time fractional Schrödinger equation is considered as

$$i_0^\alpha D_t^{\alpha(t)}\Theta(x, t) + \Delta \Theta(x, t) + 2|\Theta(x, t)|^2\Theta(x, t) + \left(1 - 2\cos^2(x)\cos^2(y)\right)\Theta(x, t) = f(x, t),$$
with the non-homogeneous term
\[
f(x, t) = i \cos(x) \cos(y) \left( -t^{2-\alpha(x,y,t)} E_{2,3-\alpha(x,y,t)}(-t^2) \right.
- \left. it^{1-\alpha(x,y,t)} E_{2,2-\alpha(x,y,t)}(-t^2) \right)
- 2 \cos(x) \cos(y) \exp(-it) + 2 \cos^3(x) \cos^3(y) \exp(-it)
+ (1 - 2 \cos^2(x) \cos^2(y)) \cos(x) \cos(y) \exp(-it),
\]
and the corresponding analytic solution
\[
\Theta(x, t) = \cos(x) \cos(y) \exp(-it).
\]

The corresponding required conditions can be extracted from the analytic solution.
We have examined this example by the introduced wavelet method with \( k = 1 \) and two different values of \( M \) whenever \( \alpha(x, t) = 0.5 - 0.45 \sin(x + y + t) \) and \( \delta t = 0.005 \). Table 4 provides the numerical results produced by the suggested wavelet method. The achieved results demonstrate the numerical solutions tend to the exact ones as \( M \)
increases. Figs. 10 and 11 display derived solutions and the corresponding AE functions for the real and imaginary parts, respectively whenever $N = 10(M = 5, k = 1)$. Fig. 12 indicates the modulus of the obtained solution (up) and the corresponding error function (down).

**Table 4.** The obtained error values by the presented wavelet method with $k = 1$ and two different values of $M$ with $\delta t = 0.005$.

| $t$ | $\varepsilon_{\text{real}}$ | $\varepsilon_{\text{image}}$ | $|\varepsilon|$ | $\varepsilon_{\text{real}}$ | $\varepsilon_{\text{image}}$ | $|\varepsilon|$ |
|-----|-----------------|-----------------|-----------|-----------------|-----------------|-----------|
| 0.2 | 7.0513E-5       | 4.6682E-4       | 4.7212E-4 | 7.0460E-5       | 4.6155E-4       | 4.6690E-4 |
| 0.4 | 2.5190E-5       | 4.0126E-4       | 4.0205E-4 | 2.4703E-5       | 3.9546E-4       | 3.9623E-4 |
| 0.6 | 7.0915E-5       | 2.8288E-4       | 2.9163E-4 | 6.5807E-5       | 2.8002E-4       | 2.8765E-4 |
| 0.8 | 1.5620E-4       | 1.6841E-4       | 2.2970E-4 | 1.5549E-4       | 1.6815E-4       | 2.2902E-4 |
| 1.0 | 2.4920E-4       | 8.7739E-5       | 2.6419E-4 | 2.5251E-4       | 8.8278E-5       | 2.6750E-4 |
6. Conclusion. In the current paper, we employed the two-dimensional Legendre wavelets (2D LWs) to obtain approximate solutions of the nonlinear variable-order time fractional two-dimensional Schrödinger equation. The achieved results confirmed the 2D LWs method is very suitable for solving the considered problem. Moreover, comparing the obtained numerical results with the analytical solutions reveals the appropriate features of applicability and accuracy of the proposed scheme.

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Figure 8. The behavior of the imaginary part of the wavelet solution and the corresponding AE function (up and down, respectively) at the final time where $\delta = 0.01$. 

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