Theory of spin excitations in undoped and underdoped cuprates

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Abstract

We consider the magnetic properties of high \(T_c\) cuprates from a gauge theory point of view, with emphasis on the underdoped regime. Underdoped cuprates possess certain antiferromagnetic correlations, as evidenced, for example, by different temperature dependence of the Cu and O site NMR relaxation rates, that are not captured well by slave boson mean field theories of the \(t\)-\(J\) model. We show that the inclusion of gauge fluctuations will remedy the deficiencies of the mean field theories. As a concrete illustration of the gauge-fluctuation restoration of the antiferromagnetic correlation and the feasibility of the \(1/N\) perturbation theory, the Heisenberg spin chain is analyzed in terms of a 1+1D U(1) gauge theory with massless Dirac fermions. The \(1/N\)-perturbative treatment of the same gauge theory in 2+1D (which can be motivated from the mean field \(\pi\)-flux phase of the Heisenberg model) leads to a dynamical mass generation corresponding to an antiferromagnetic ordering. On the other hand, it is argued that in a similar gauge theory with an additional coupling to a Bose (holon) field, symmetry breaking does not occur, but antiferromagnetic correlations are enhanced, which is the situation in the underdoped cuprates.

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I. INTRODUCTION

The essence of the physics of high $T_c$ cuprates boils down to the problem of how to treat the dual nature of the electrons which form local moments in the insulator, and yet make up a Fermi surface when doped with $\sim 15\%$ holes. This problem was brought into sharp focus with the discovery of anomalous properties in the physics of the “underdoped” region which lies between the antiferromagnetic insulator and the optimally doped superconductor. How does the Fermi surface evolve from small hole pockets near $k = (\pm \pi/2, \pm \pi/2)$ in a slightly doped antiferromagnet to the full Fermi surface obeying Luttinger’s theorem in the optimally doped materials? What are the magnetic properties in this intermediate doping region? Experimentalists have already answered a substantial part of these questions. In particular, the angle-resolved photoemission spectroscopy (ARPES) has shown the existence in the normal state of a gap with the same anisotropy as the d-wave gap of the superconducting state [1]. Low-lying excitations are observed along a patch near $(\pm \pi/2, \pm \pi/2)$ [“Fermi surface segments”], but the Fermi surface, in the veritable sense of the word, does not exist. The ARPES results might have been accepted without much grudge simply as a plausible interpolation between the antiferromagnet and the optimally doped superconductor, had our understanding of metals not been so entrenched in the Fermi liquid theory; the notion of a metal without a Fermi surface is a serious embarrassment. At the same time, gap-like suppression of spin excitations are seen in NMR [2]: the Knight shift and spin-lattice relaxation rates all decrease with decreasing temperature below certain temperatures.

Gaplike features in the underdoped cuprates might remind us of the spin liquids — the liquid of spin singlets. Yet the devil is in the details, and the underdoped cuprates deviate significantly from “gapped spin liquids” like spin 1/2 ladders and integer spin chains. The latter have an even number of spins per unit cell and the ground state is a spin singlet. They are characterized by a clear gap to the lowest triplet excitation that is inversely related to the correlation length; this gap can be seen in inelastic neutron scattering. The magnetic responses like uniform susceptibility and the NMR relaxation rate as a function of temperature have activated behaviors. Many of them can be satisfactorily described in terms of the “quantum disordered” phase of the nonlinear sigma model $L = \frac{1}{g} (\partial \phi)^2$, $g > g_c$ [3] which can be also understood in terms of the $CP^{N-1}$ model involving bosonic “spinons” ($z$ fields) — these spin 1/2 excitations are said to be confined, as they do not appear in the basic physical spectra [4]. In other words, a description of the excitations in terms of fluctuating spin 1 objects is most natural for them.

On the other hand, in the underdoped cuprates, inelastic neutron scattering does not show a “full gap” [4]. The decrease of the Knight shift with decreasing temperature looks more like a power-law. On the whole, the magnetic excitation spectrum of the cuprates seems to display a curious mixture of singlet and antiferromagnetic correlations. There are evidences for antiferromagnetic correlations from the Q-space scan of neutron scattering cross section, and from the difference in the temperature dependence of the NMR relaxation rates: the Oxygen $1/T_1 T$ (which has little contribution from spin excitations near wave vector $Q = (\pi, \pi)$) monotonically decreases with decreasing temperature, while the Copper $1/T_1 T$ (which weighs $(\pi, \pi)$ spin excitation strongly) increases with decreasing temperature until around 150K and then falls [3].

The antiferromaget-singlet debate has enormous ramifications for theories of high $T_c$
cuprates (for a succinct review, see Ref. [6]). Some theorists [7] have advocated the picture of Fermi liquid quasiparticles exchanging “antiparamagnons” for understanding the anomalous normal state properties and the superconductive pairing. Such a view is not shared here. Instead of viewing the antiferromagnetic fluctuations as the cause of superconductivity in a BCS-like scenario, in this paper we would rather regard them as a residual but important consequence of local repulsive interactions that lead to superconductivity in the presence of doped holes, a part of Nature’s conspiracy to find a compromise between a magnetic ground state and an itinerant metallic state. This line of thinking goes back to Anderson’s seminal 1987 paper [8] on the resonating valence bond (RVB) theory, in which he reasoned that doped holes may propagate coherently in a liquid of spin singlets.

Theoretical attempts to realize Anderson’s RVB picture are based on strong coupling models, such as the one band Hubbard model or the $t$-$J$ model [9]. These models are considered to contain some essential physics of the cuprates at appropriate parameter values $U/t$ or $J/t$. The $t$-$J$ model, the simpler of the two, captures in a transparent way what is believed to be the basic physics, namely the competition between the magnetic exchange and the delocalization energy of holes. The no-double-occupancy constraint in the $t$-$J$ model can be taken care of by writing the electron operator as a composite of a neutral fermion (spinon) and a spinless boson (holon) $[c^\dagger_{i\sigma} = f^\dagger_{i\sigma} b_i]$ and demanding each site be occupied by either a fermion or a boson ($\sum_{\sigma} f^\dagger_{i\sigma} f_{i\sigma} + b_i b^\dagger_i = 1$). The theory then contains four-particle interactions which can be decoupled by introducing “mean fields” like $\chi_{ij} = \langle f^\dagger_{i\sigma} f_{j\sigma} \rangle$, $\Delta_{ij} = \langle f^\dagger_{i\uparrow} f_{j\downarrow} - f^\dagger_{i\downarrow} f_{j\uparrow} \rangle$, and $n_{ij} = \langle b^\dagger_i b_j \rangle$. Within the mean field approach, Kotliar and Liu, and Fukuyama and coworkers [10] have studied the phase diagram of the $t$-$J$ model. At low doping (and below some temperature scale), it was found that the phases in which the fermions are paired into d-wave singlets ($\Delta_{i,i+x} = -\Delta_{i,i+y} \neq 0$) are favored. Depending on whether the bosons are condensed, they could be superconducting (SC phase) or normal (“d-wave RVB phase”).

As noted by Rice [11] and others, the fermionic mean field theory captures some important features of the spin gap phenomena in the underdoped cuprates that refuse clear-cut characterization in terms of a well-defined correlation length and relaxation times. The theory describes some kind of quantum spin liquid, but unlike gapped spin liquids, there is a particle-hole (spinon-antispinon) continuum, which would create some spectral weight for magnetic excitations at arbitrarily low energy. More specifically, the Dirac spectrum ($\epsilon(k) = v|k|$) of the fermionic quasiparticles in the d-wave RVB phase gives the Knight shift $K \sim T$ and the Oxygen site NMR relaxation rate $1/T_1 \sim T^3$, in rough agreement with experiments in the underdoped cuprates. Moreover, the absence of the gap in the charge response (for example, the in-plane optical conductivity) could be explained simply, since the spin and charge degrees of freedom are separated, i.e. the spin is carried by fermionic spinons while the charge is carried by bosonic holons.

A Dirac-type spectrum as in the d-wave RVB phase was also found by Affleck and Marston who considered the $\pi$-flux phase [12] as a possible spin liquid ground state of the cuprates. It turned out that at half filling the d-wave phase with $|\Delta_{ij}| = |\chi_{ij}|$ is equivalent to the flux phase, due to a local SU(2) symmetry [13]. Wen and one of us reasoned that this symmetry might still be a pretty good (and important) symmetry at small dopings, and came up with a slave boson theory that respects the SU(2) symmetry even away from half filling by introducing an SU(2) doublet of slave bosons, hoping to get a better description...
of underdoped cuprates \[14\]. In this theory, the mean field corresponding to the “spin gap” phase of the underdoped cuprates was identified as the “sFlux phase” which can be considered a combination of the d-wave RVB phase and the staggered flux phase \[15\] of the U(1) theory. This phase also has fermions with a Dirac spectrum, but \textit{in contrast to the d-wave RVB phase of the U(1) mean field theory \[16\], the fluctuations around this mean field include a massless gauge field which is expected to affect strongly the magnetic and transport properties of the system \[17\]. With the inclusion of a residual attraction between bosons and fermions, the sFlux phase was shown to reproduce the gross features of the ARPES, such as the Fermi surface segments near \((\pm \pi/2, \pm \pi/2)\) and a large gap at \((\pi, 0)\).

Despite the successes, the mean field treatments (both the SU(2) theory and its predecessors) are unsatisfactory in several respects. For example, it is not clear how the spin gap phase is connected to the Néel ordered phase at zero doping. \textit{The mean field ansatz loses a lot of antiferromagnetic correlation}; within the mean field theory, the Copper site \(1/T_1T\) has a similar behavior as the Oxygen site \(1/T_1T\), in disagreement with experiments \[2\]. Attempts to fix the problem by some kind of RPA cannot produce, in a natural manner, different temperature scales which mark the decrease of the Copper and Oxygen site relaxation rates \[18\]. Another serious question is the role of gauge fluctuations around the mean field solution which had not been studied carefully so far. In fact, a strong gauge fluctuation might destroy the mean field picture altogether, in which case we have to re-identify the elementary excitations of the theory.

In this paper, we look into these questions. The basic point is that the gauge fluctuations ignored at the mean field level strongly enhance antiferromagnetic correlations. The gauge fluctuation could be so strong that the elementary excitations of the mean field theory disappear completely from the low energy spectrum. This is believed to be the case with the “U(1) \(\pi\)-flux phase” \[12\] description of the undoped cuprates (antiferromagnet), which is a theory of massless Dirac fermions coupled to a U(1) gauge field (massless QED3). The QED3 can be treated in the \(1/N\) perturbation theory. For physical \(N(\approx 2)\), a dynamic mass generation and spontaneous symmetry breaking corresponding to Néel ordering would occur \[19\], while for large enough \(N\), the theory would still describe some kind of a spin liquid. The true low energy excitations of the symmetry-broken case are recognized as the Goldstone bosons—“mesons” which are a bound state of a particle and an antiparticle (spinon & antispinon).

We note that the local SU(2) symmetry at half filling can be utilized to write a theory of the undoped system in terms of massless Dirac fermions coupled to nonabelian (SU(2)) gauge fields \[13\] \[17\]. Since the U(1) theory (at least initially) has one massless gauge field while the SU(2) theory has three, they look quite different. Nonetheless, as long as the gauge fluctuations are treated exactly, the two theories should lead to the same physics, namely the Néel ordering.

As regards the \textit{underdoped} cuprates, we believe that the most promising starting point is the sFlux phase of the SU(2) theory \[17\], which has massless Dirac fermions coupled to a massless U(1) gauge field as in the Affleck-Marston flux phase \[12\], but also has nonrelativistic bosons(holons) coupled to the same gauge field. We shall argue that, due to this additional coupling, the gauge fluctuations will not destroy the essential validity of the mean field picture, though the picture of antiferromagnetic spin excitations will be modified (improved). The coupling to the bosons would result in the screening of the time component of
the gauge field which will prevent Néel ordering. The gauge field will nevertheless mediate an attraction between spinons and antispinons and try to create a bound state with momentum \( \sim (\pi, \pi) \), but due to the particle-hole continuum, this will appear only as a broad resonance. This can be viewed as a Goldstone boson precursor mode that comes down in energy as the transition is approached (as the boson density is reduced). The recent neutron scattering in underdoped cuprates which sees a broad peak in \( Q \approx (\pi, \pi) \) magnetic response whose energy scale is roughly proportional to doping might be consistent with this point of view \cite{22}. We shall also discuss the issue of confinement, as there are lingering questions about the fate of “spinons” in the case of strong coupling gauge theories.

In order to illustrate some aspects of the foregoing ideas more concretely (in particular the gauge-fluctuation restoration of antiferromagnetic correlation and the feasibility of a U(1) gauge theory description of quantum antiferromagnets), we’ll first reexamine the well known spin half chain from the point of view of the Schwinger model.

II. LESSONS FROM SPIN CHAIN

We begin with a discussion of the 1d Heisenberg model as an example where the idea of mean field theory plus gauge fluctuation can be applied to an exactly soluble model. A number of authors have noted that what emerges is the Schwinger model with two flavors \cite{23–25}, and abelian \cite{24} and nonabelian \cite{25} bosonization methods have been applied to solve for the continuum limit of this model. Here we solve the model using a slightly different technique, i.e., chiral rotation, which focuses on the role of gauge fluctuations as a way of correcting the underestimate of the antiferromagnetic correlation in mean field theory. We then show that qualitatively similar results are achieved in \( 1/N \) perturbation theory of the gauge fluctuations. This gives us encouragement that a similar approach may be reasonable in the two-dimensional case.

A. RVB theory of spin one-half chain

The Heisenberg model (\( H = J \sum_{<ij>} S_i \cdot S_j \)) in the fermion representation of the spin can be written

\[
H = -\frac{J}{2} \sum_{<ij>} f_{i\alpha}^\dagger f_{i\beta}^\dagger f_{j\alpha} f_{j\beta} + \text{h.c.}
\]

with the constraint \( \sum_{\alpha} f_{i\alpha}^\dagger f_{i\alpha} = 1 \). The 4-fermion interactions and the constraint can be handled by the introduction of a Hubbard-Stratonovich field and a Lagrange multiplier, which gives

\[
H = -\frac{J}{2} \sum_{<ij>} (\chi_{ij} f_{i\alpha}^\dagger f_{j\alpha} + \text{h.c.}) + i \sum_i \lambda_i (f_{i\alpha}^\dagger f_{i\alpha} - 1).
\]

Within the mean field theory, \( \chi_{ij} = \chi, \lambda = 0 \), hence the mean field hamiltonian in the k-space is
\[ H_{mf} = -\chi J \sum_k \cos(k) f_{\alpha k}^\dagger f_{\alpha k} \]  \hspace{1cm} (3)

\[ = -\chi J \sum_k \cos(k) (f_{\alpha k}^\dagger f_{\alpha k} - f_{\alpha k-\pi}^\dagger f_{\alpha k-\pi}) \]  \hspace{1cm} (4)

\[ = -\chi J \sum_k' \cos(k) (f_{\alpha k}^\dagger f_{\alpha k} + f_{\alpha k}^\dagger f_{\alpha k}). \]  \hspace{1cm} (5)

Here \( \Sigma_k' \) denotes sum over the magnetic BZ, say \( 0 < k < \pi \), and \( f_{\alpha k}, f_{\alpha k} \) are even and odd site operators \( (f_{\alpha k} = \sqrt{2}(f_k + f_{-k}) = \sqrt{\frac{2}{L^2}} \sum_{j \text{ even}} e^{ikj} f_j, f_{\alpha k} = \sqrt{\frac{2}{L^2}} \sum_{j \text{ odd}} e^{ikj} f_j). \)

Linearizing around \( k = \pi/2 + k' \), we arrive at the continuum hamiltonian

\[ H = -\int dk' \psi_{\alpha}^\dagger(k') \sigma_1 k' \psi_{\alpha}(k'), \]  \hspace{1cm} (6)

where \( \psi_{\alpha} = \begin{pmatrix} f_{\alpha e} \\ f_{\alpha o} \end{pmatrix} \) and \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is a Pauli matrix. This is just the hamiltonian of free Dirac fermions (we have set the velocity of the fermions =1). The corresponding (Euclidean space) lagrangian is

\[ L = \bar{\psi}_{\alpha} \gamma_{\mu} \partial_{\mu} \psi_{\alpha}, \]  \hspace{1cm} (7)

where \( \bar{\psi} = \psi^\dagger \gamma_0 \), and \( \mu = 0, 1 \), and the \( \gamma \) matrices are

\[ \gamma_0 = \sigma_3, \quad \gamma_1 = -\sigma_2. \]  \hspace{1cm} (8)

In 1+1D, we define \( \gamma_5 \) matrix as \( \gamma_5 = -i \gamma_0 \gamma_1 = \sigma_1 \), which has the property

\[ \{\gamma_5, \gamma_\mu\} = 0, \quad \epsilon_{\mu\nu} \gamma_\nu = i \gamma_\mu \gamma_5, \]  \hspace{1cm} (9)

(\( \epsilon_{\mu\nu} \) is the antisymmetric tensor with \( \epsilon_{01} = 1 \).)

Including the fluctuations around the mean field (the fluctuations of \( \lambda_i \) and the phase of \( \chi_{ij} \)) amounts to coupling the fermions to a U(1) gauge field by the minimal prescription. Hence the continuum version of \{the mean field + fluctuations\} is the two flavor Schwinger model

\[ L = \bar{\psi}_{\alpha} \gamma_{\mu} (\partial_{\mu} - ia_\mu) \psi_{\alpha}. \]  \hspace{1cm} (10)

The apparent gauge coupling (bare coupling) is infinitely strong, as there is no kinetic term for the gauge field.

Above lagrangian is obviously invariant under the global SU(2) transform (spin rotation symmetry)

\[ \psi_{\alpha} \rightarrow (\exp(i \phi^l \tau^l))_{\alpha \beta} \psi_{\beta} \]  \hspace{1cm} (11)

where \( \tau^l, l = 1, 2, 3 \) are Pauli matrices (belonging to a space different from that of \( \sigma_l \)). The lagrangian is also invariant under the “chiral transformation”

\[ \psi_{\alpha} \rightarrow \exp(-i \theta \gamma_5) \psi_{\alpha}. \]  \hspace{1cm} (12)

This “chiral symmetry” is explicitly broken by higher derivative terms ignored in taking the continuum limit, like
\[ L' = \bar{\psi}_\alpha \gamma_5 \gamma_\mu (\partial_\mu - ia_\mu)^2 \psi_\alpha. \]  \hspace{1cm} (13)

If we label the states by the left and right movers, \( f_R \approx f_k \), \( f_L \approx f_{k-\pi} \), where \( 0 < k < \pi \), the chiral transformation corresponds to \( f_R \to f_R e^{-i\theta} \), \( f_L \to f_L e^{i\theta} \).

We now consider spin correlation functions at the mean field level (i.e. ignoring gauge fields). The spin operators in the continuum have two contributions (uniform & staggered):

\[ S'(x_1) \approx [f^\dagger_{\alpha\epsilon}(x_1) f_{\beta\epsilon}(x_1) + f^\dagger_{\alpha\epsilon}(x_1) f_{\beta\epsilon}(x_1)] \]

\[ + (-1)^{x_1}[f^\dagger_{\alpha\epsilon}(x_1) f_{\beta\epsilon}(x_1) - f^\dagger_{\alpha\epsilon}(x_1) f_{\beta\epsilon}(x_1)] \]

\[ = \bar{\psi}_\alpha(x_1) \gamma_0 \gamma_\tau \psi_\beta(x_1) + (-1)^{x_1} \bar{\psi}_\alpha(x_1) \gamma_0 \gamma_\tau \psi_\beta(x_1). \]  \hspace{1cm} (14)

To evaluate the spin correlation function

\[ \langle S^+(x) S^-(0) \rangle = \langle \bar{\psi}_\gamma \gamma_\tau \psi(x) \bar{\psi}_\gamma \gamma_\tau \psi(0) \rangle + (-1)^{x_1} \langle \bar{\psi}_\gamma \gamma_\tau \psi(x) \bar{\psi}_\gamma \gamma_\tau \psi(0) \rangle \]

\[ = \langle \bar{\psi}_1 \gamma_0 \psi_2(x) \bar{\psi}_2 \gamma_0 \psi_1(0) \rangle + (-1)^{x_1} \langle \bar{\psi}_1 \psi_2(x) \bar{\psi}_2 \psi_1(0) \rangle \]  \hspace{1cm} (15)

\((\tau^\pm = (\tau^1 \pm i\tau^2)/2)\), we need the fermion Green’s function

\[ G_{\alpha\beta}(x) = \langle \bar{\psi}_\alpha(x) \bar{\psi}_\beta(0) \rangle = G(x) \delta_{\alpha\beta} \]  \hspace{1cm} (16)

which can be obtained from the momentum space Green function \( G(k) = -ik\gamma/k^2 \) as

\[ G(x) = \gamma_\mu \frac{\partial}{\partial x_\mu} \int \frac{d^2k}{(2\pi)^2} \frac{e^{-ik\cdot x}}{k^2} = -\frac{x\gamma}{2\pi x^2}. \]  \hspace{1cm} (17)

Here and from now on, unless otherwise specified, we use the usual field theory notation: \( k = (k_0, k_1), x = (x_0, x_1) \) [italics denote space time vectors]; \( x_\gamma \equiv x_\mu \gamma_\mu \), \( x^2 = x_0^2 + x_1^2 \), etc. Using Wick’s theorem, we have

\[ \langle S^+(x) S^-(0) \rangle = -\text{tr}_{\gamma, \tau} [G(x) \gamma_0 \gamma_\tau + G(-x) \gamma_0 \gamma_\tau] - (-1)^{x_1} \text{tr}_{\gamma_1, \tau} [G(x) \gamma_\tau G(-x) \gamma_\tau] \]

\[ = \frac{1}{2\pi^2} \left[ \frac{x_0^2 - x_1^2}{(x_0^2 + x_1^2)^2} + (-1)^{x_1} \frac{1}{x_0^2 + x_1^2} \right] \]

\[ = \frac{1}{4\pi^2} \left[ \frac{1}{x_-^2} + \frac{1}{x_+^2} + (-1)^{x_1} \frac{2}{x_- x_+} \right], \]  \hspace{1cm} (18)

\((x_\pm \equiv x_0 \pm ix_1; \text{tr}_{\gamma, \tau} \) denotes trace over both the \( \gamma \) and \( \tau \) spaces (spinor and spin spaces)), which does (and should) equal the \( \langle S_\gamma(x) S_\gamma(0) \rangle \) correlation function in the XY model \cite{26}.

The equal time correlation function \( \langle S(x_1) \cdot S(0) \rangle \) behaves as

\[ \langle S(x_1) \cdot S(0) \rangle = \frac{3}{4\pi^2 x_1^2} (-1)^{x_1} - 1, \]  \hspace{1cm} (19)

(the spins on the same sublattice are not correlated at all, while the correlation among spins on different sublattices are decaying algebraically as \( 1/x_1^2 \)). This peculiar behavior, which was derived by Arovas and Auerbach in the lattice version of the fermionic mean field theory \cite{27} and agrees with Bulaevskii’s Hartree-Fock treatment of the Jordan-Wigner fermionized Heisenberg model \cite{28}, is viewed as a pathology of the mean field theory: \textit{we have lost a substantial amount of antiferromagnetic correlation.}
B. Schwinger model

We now consider the effect of gauge fluctuations. It is natural to expect that the inclusion of gauge fluctuations will improve the mean field picture. The time component of the gauge field can be regarded to originate from the Lagrange multiplier field (for no-double-occupancy); this corresponds to Gutzwiller-projected (half-filled tight binding) Fermi surface, which is known to be a pretty good description of 1d antiferromagnet [29,30].

As mentioned earlier, our theory with fluctuations is a Schwinger model. For reasons that will become clear shortly, we consider a slightly more general case of $N$-flavors:

$$Z = \int D\bar{\psi}D\psi Da_\mu \exp(-S),$$

$$S = \int d^2x \sum_{\alpha=1}^N \bar{\psi}_\alpha \gamma_\mu (\partial_\mu - ia_\mu) \psi_\alpha + \frac{1}{4e^2} F_{\mu\nu}^2, \quad (e^2 = \infty). \quad (20)$$

The physical case is $N=2$; general (even) $N$ corresponds to an $SU(N)$ antiferromagnet.

Integrating out the fermions gives

$$Z = \int Da_\mu \exp(N \text{Tr} \ln(1 - iG_{\mu\alpha}a_\mu))$$

where $G(x,x') = (\gamma_\mu \partial_\mu)^{-1}\delta(x-x')$, and Tr denotes traces over the spinor space and the position space ($\text{Tr} = \text{tr}\int d^2x d^2x'...$). The logarithm can be expanded, giving

$$Z = \int Da_\mu \exp \left(-\frac{1}{2} \int d^2x d^2x' a_\mu(x)\Pi_{\mu\nu}(x-x')a_\nu(x') \right), \quad (22)$$

where $\Pi_{\mu\nu}(x) = -N \text{tr}[G(x)\gamma_\mu G(-x)\gamma_\nu]$. Note that the beyond-Gaussian terms (like $\Gamma_{\mu\nu\rho\delta}a_\mu a_\nu a_\rho a_\delta$) are all zero; the proof can be found, for example, in Ref. [31].

The polarization function $\Pi_{\mu\nu}(q)$ (in the momentum space) contains a divergence that has to be regulated using gauge invariant schemes, like the dimensional regularization or the Pauli-Villars regularization. Relegating the details to Appendix A, we have

$$\Pi_{\mu\nu}(k) = \frac{N}{\pi} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right), \quad (23)$$

which means that the gauge boson acquires an infinite mass ($= e\sqrt{N/\pi}$). The transversality of Eq.23 guarantees the conservation of the current $j_\mu = \bar{\psi}_\alpha \gamma_\mu \psi_\alpha$:

$$q_\mu j_\mu(q) = q_\mu(i\Pi_{\mu\nu}(q)a_\nu(q)) = 0 \quad \rightarrow \quad \partial_\mu j_\mu = 0. \quad (24)$$

On the other hand the current $j_5\mu = i\bar{\psi}_\alpha \gamma_5 \gamma_\mu \psi_\alpha = -\epsilon_{\mu\nu} j_\nu$ associated with the chiral symmetry (Eq.12) is not conserved:

$$q_\mu j_5\mu = -q_\mu \epsilon_{\mu\rho} i\Pi_{\nu\rho} a_\nu = -\frac{iN}{\pi} \epsilon_{\mu\nu} q_\nu a_\nu \rightarrow \partial_\mu j_5\mu = -\frac{iN}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}. \quad (25)$$

This result, the so-called axial anomaly, can be regarded as either a consequence of or a condition for gauge invariance. Equation (25) is in fact familiar to the solid state physicists [32]. In an electric field, the equation of motion for the crystal momentum state is
\[ \frac{d \mathbf{k}}{dt} = \mathbf{E}. \]  

In one dimension an electric field causes a shift of the occupation between left moving and right moving states. Thus the left and right movers are in fact connected and their densities are not separately conserved. It is easy to see that Eq.(26) is consistent with Eq.(25).

The exact spin correlation functions can be evaluated with the use of “chiral rotation” \[33–36\]. In this approach, the gauge field is written as the sum of a div-free part and a curl-free part:

\[ a_\mu = \epsilon_{\mu\nu} \partial_\nu \theta_a + \partial_\mu \theta_b. \]  

The transform

\[ \begin{align*}
\psi_\alpha &\rightarrow \psi'_\alpha = \exp(-\gamma_5 \theta_a - i\theta_b) \psi_\alpha \\
\bar{\psi}_\alpha &\rightarrow \bar{\psi}'_\alpha = \bar{\psi}_\alpha \exp(-\gamma_5 \theta_a + i\theta_b)
\end{align*} \]  

(28a)

(28b)

decouples the gauge field from the \(\psi'\) fermions. We note that corresponding to Eq.(28b), we have

\[ \psi^\dagger_\alpha \rightarrow \psi'^\dagger_\alpha \exp(\gamma_5 \theta_a + i\theta_b) \]  

(29)

because \(\gamma_0\) and \(\gamma_5\) anticommute. Thus \(\psi^\dagger_\alpha \psi_\alpha = \psi'^\dagger_\alpha \psi'_\alpha\) and the transformation is unitary. Here \(\psi_\alpha\) and \(\psi'^\dagger_\alpha\) are treated as independent Grassmannian variables and \(\exp(\gamma_5 \theta_a)\) should not be thought of as an amplitude transformation. Alternatively, we point out that in real time (as opposed to imaginary time used here), the transformation to cancel the \(a_\mu\) field will indeed be a phase rotation and takes the form \(\psi_\alpha \rightarrow \psi'_\alpha \exp(-i\gamma_5 \theta_a + i\theta_b)\). That this must be the case is evident from Eq.(27), because \(a_0 = \partial_1 \theta_a\) changes from real to imaginary upon going from Minkowski to Euclidean space. Unlike the usual gauge transformation, \(i a_0 f_L\) is cancelled by \(\partial_1 f_L\) and \(i a_1 f_L\) is cancelled by \(\partial_0 f_L\) under the chiral rotation, so that \(\theta_a\) must also change from real to imaginary.

Even though the chiral rotation is unitary, it does not leave the Grassmann measure invariant, due to the fact that \(f_R\) and \(f_L\) are not truly independent, as noted earlier. The jacobian \(\exp(J)\) for the change of measure

\[ D\psi D\bar{\psi} = D\psi' D\bar{\psi}' \exp(J) \]  

(30)

can be found easily using the axial anomaly condition Eq.25(see Appendix B for details):

\[ J = -\frac{N}{2\pi} \int d^2x (\partial_\mu \theta_a)^2. \]  

(31)

Alternative derivation can be found in Ref. \[36\]. In terms of the new fields, the functional integral is

\[ Z = \int D\psi' D\bar{\psi}' D\theta_a \exp -\int d^2x \left( \bar{\psi}'_\alpha \gamma_\mu \partial_\mu \psi'_\alpha + \frac{N}{2\pi} (\partial_\mu \theta_a)^2 \right). \]  

(32)

These are “free” fields, and now the spin correlation function (Eq.15) can be evaluated easily:
\[ \langle \bar{\psi}_1 \gamma_0 \psi_2(x) \bar{\psi}_2 \gamma_0 \psi_1(0) \rangle = \langle \bar{\psi}_1^\prime \gamma_0 \psi_2^\prime(x) \bar{\psi}_2^\prime \gamma_0 \psi_1^\prime(0) \rangle = \frac{1}{2 \pi^2 (x_0^2 + x_1^2)} \] 

where we have used

\[ \langle (\theta_a(x) - \theta_a(0))^2 \rangle = -\frac{2\pi}{N} \int_\Lambda \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} (e^{ik \cdot x} - 1) = \frac{1}{2N} \ln(x^2 \Lambda^2), \]

(\Lambda is a UV cutoff originating from the lattice theory, and \(C\) is a nonuniversal constant that depends on high energy details \(\Lambda\)). In the physical case \((N = 2)\), we then have

\[ \langle \mathbf{S}(x) \cdot \mathbf{S}(0) \rangle \sim (-1)^{x_1} \frac{1}{\sqrt{x_1^2 + x_0^2}} \]

which agrees with the more accurate result [37] up to a \(\ln^{1/2}(x^2)\) factor. The log factor, not captured by the Schwinger model, must be due to terms ignored in our derivation from the lattice theory, e.g., the amplitude fluctuation of the RVB field; this is analogous to the bosonization theory of Heisenberg model, in which the Umklapp processes give rise to logarithmic correction (prefactor) to the power law [38]. The simple correlation function of Eq.35 is known to capture the low energy (temperature) properties of the Heisenberg spin chains quite well [39].

C. additional remarks

Before moving on to the perturbative treatment of the same theory, we briefly compare the gauge theory approach with other approaches. Perhaps the best known treatment of the Heisenberg spin chain relies on the Jordan-Wigner transformation \((f_i = e^{i\phi_i} S_i^-, \phi_i = \pi \sum_{j=1}^{i-1} S_j^+ S_j^-)\). However, unlike the X-Y model, this does not lead to a theory of free fermions, but to a theory with a 4-fermion interaction with coupling constant of order unity, which is then treated by bosonization methods [38]. A drawback of the Jordan-Wigner approach is that the SU(2) spin symmetry is easily lost, and the correct exponent \((= -1)\) of the staggered spin correlation function has to be determined rather indirectly.

Haldane [40], on the other hand, proposed to analyze the Hubbard model, exploiting the “equivalence” between the spin sector of the 1d Hubbard model

\[ H = -\frac{1}{2} (JU)^{1/2} \sum_{i\sigma} (c_i^\dagger \sigma c_{i+1,\sigma} + \text{h.c.}) + U \sum_i (n_i^\uparrow - \frac{1}{2})(n_i^\downarrow - \frac{1}{2}) \]
and the 1d Heisenberg model, that holds even in the weak coupling limit. The Hubbard model can be treated by a bosonization method \[40,26\] that respects the SU(2) symmetry at all stages, and the correct exponent of the spin correlation function can be obtained directly. We can easily adapt Haldane’s approach to the Schwinger model, which as we point out, is equivalent to the RVB mean field theory plus gauge fluctuations. This line of approach was recently given in ref.[24]. Instead of treating the gauge fluctuation by chiral rotation, we first integrate out the gauge field in Eq.(20), which simply enforces the constraint of no double occupation, i.e., the charge fluctuation is completely suppressed. Now we can treat the fermion problem by bosonization in the standard way. We write

\[
(f_{sR}(x_1), f_{sL}(x_1)) = e^{i\sqrt{\pi}\phi_s(x_1)}(e^{i\sqrt{\pi}\theta_s(x_1)+ik_Fx_1}, e^{-i\sqrt{\pi}\theta_s(x_1)-ik_Fx_1})
\]

for \(s = \text{spin up or down}, \) and \(\theta_\uparrow = \frac{1}{\sqrt{2}}(\theta_\rho + \theta_\sigma), \theta_\downarrow = \frac{1}{\sqrt{2}}(\theta_\rho - \theta_\sigma), \phi_\uparrow = \frac{1}{\sqrt{2}}(\phi_\rho + \phi_\sigma), \phi_\downarrow = \frac{1}{\sqrt{2}}(\phi_\rho - \phi_\sigma), \) where \(\partial_x \theta_\rho(\sigma)\) is proportional to the charge (spin) density and is the conjugate variable to \(\phi_\rho(\sigma)\). In the effective Lagrangian, spin and charge fluctuations separate. For free fermions \(\theta_\rho\) and \(\theta_\sigma\) contribute equally to the \(2k_F\) spin-spin correlation functions, yielding Eq.(19). In our case the charge fluctuation is suppressed and we may ignore the \(\theta_\rho\) degrees of freedom. The \(2k_F\) spin correlation function decays with an exponent exactly one half than that of the free fermion, and we recover Eq.(33). This line of reasoning has a slight advantage over Haldane’s original treatment in that the energy scale \(J\) for the fermion bandwidth is correctly produced.

D. perturbation theory

We now examine the gauge theory in terms of the perturbation theory in \(1/N\). The key point is that the nature of the perturbative correction to the mean field results is very different for the uniform part and the antiferromagnetic (staggered) part of the spin correlation: While \(Q = \pi\) response is strongly affected by perturbative correction, the \(Q = 0\) response receives no correction at all (the absence of any correction is a special feature of the 1d).

The leading \(1/N\) correction to the spin correlation functions can be straightforwardly evaluated. They are represented by the Feynman diagrams in Fig.4 and Fig.5. Within the usual Faddeev-Popov scheme of gauge fixing (the introduction of the \(\frac{1}{2N}(\partial \cdot a)^2\) term to the lagrangian), the gauge propagator (represented by the wiggly line) is given by

\[
D_{\mu\nu} = \langle a_\mu a_\nu \rangle = \frac{\pi}{N} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{\lambda q_\mu q_\nu}{(q^2)^2}.
\]

We choose the Landau gauge (\(\lambda = 0\)) which is a natural choice, since in this gauge no infrared divergence occurs in the perturbation theory. The fermion propagator \(G(p) = (ip\gamma)^{-1}\) is represented by a solid line. The fermion-gauge vertex is simply \(i\gamma_\mu\); the external current vertex is \(\gamma_0\) for the uniform part (represented by a square) and 1 for the staggered part (represented by a circle). Of course a trace is taken over the fermion loops.
In 1+1D, the transverse projector has a special property

$$\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} = \epsilon_{\mu\nu\rho\delta} \frac{q_\rho q_\delta}{q^2}$$  \hspace{1cm} (39)$$

which, together with the “Ward identity”

$$G(p + q)q\gamma G(p) = i(G(p) - G(p + q)),$$  \hspace{1cm} (40)$$
simplifies the algebra substantially (Note $\gamma_\mu\epsilon_{\mu\rho}p_\rho = -ip\gamma_5$). For example, the diagram is

$$\frac{\pi}{N} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} \text{tr}[G(p + q)q'\gamma_5G(p + q + q')\gamma_0G(p + q')q'\gamma_5G(p)\gamma_0]/q'^2$$

$$= -\frac{\pi}{N} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} \text{tr}[(G(p + q) - G(p + q + q'))\gamma_0(G(p + q') - G(p))\gamma_0]/q'^2.$$  \hspace{1cm} (41)$$

It’s straightforward to show that sum of the diagrams $[a+c$ is the same as Eq.(41), except for a minus sign. Therefore in the uniform channel, the vertex correction and the self energy correction cancel. Similar cancellation is expected at all orders of perturbation theory; the nonrenormalization of uniform part of the spin correlation function is quite natural, since in our theory

$$\langle \bar{\psi}_1\gamma_0\psi_2\bar{\psi}_2\gamma_0\psi_1 \rangle \propto \langle j_0j_0 \rangle = \Pi_{00},$$  \hspace{1cm} (42)$$

and Eq.(22) is an exact result.
On the other hand, the diagrams in the staggered channel do not cancel. The sum of the diagrams 2a and 2c are equal to 2b, which is given by

\[
\frac{\pi}{N} \int \frac{d^2p \ d^2q'}{(2\pi)^2 (2\pi)^2} \text{tr}[(G(p + q) - G(p + q'))(G(p + q') - G(p))] / q'^2 \\
= \frac{2\pi}{N} \int \frac{d^2p \ d^2q'}{(2\pi)^2 (2\pi)^2} \text{tr}[G(p + q')G(p + q) - G(p + q)G(p)] / q'^2.
\]

Therefore, in the coordinate space, the 1/N correction is

\[
\text{tr}[G(x)G(-x)] \frac{4\pi}{N} \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2} (e^{iq \cdot x} - 1) = \frac{1}{2\pi^2 N x^2} \ln(x^2).
\]

Similar (but a lot more tedious) calculation would show that 1/N^2 correction is given by

\[
\frac{1}{4\pi^2 N x^2} \ln(x^2).
\]

In other words, the perturbation series exponentiates:

\[
\langle \bar{\psi}_1 \psi_2(x) \bar{\psi}_2 \psi_1(0) \rangle = \frac{1}{2\pi^2 x^2} (1 + (1/N) \ln(x^2) + \frac{1}{2}(1/N)^2 \ln^2(x^2) + ...) \propto \frac{1}{(x^2)^{1-1/N}},
\]

giving the same result obtained from the chiral rotation approach.

III. TWO-DIMENSIONAL UNDOPED CUPRATES

The success of 1+1D gauge theory with Dirac fermions in describing the Heisenberg spin chain tempts us that a similar theory of massless Dirac fermions strongly coupled to a U(1) gauge field might describe a 2d quantum antiferromagnet. In fact, it is known from lattice gauge theories \[22\] that this is indeed so. In connection with high T_c cuprates, Marston \[21\], Laughlin \[20\], and others have noted that the “chiral symmetry breaking” in 2+1D U(1) gauge theory, discussed by Pisarski \[43\] and Appelquist et al. \[44\] in particle physics context, corresponds to the Néel ordering. In this section, we shall discuss this matter carefully, and clarify some issues related to the pattern of symmetry breaking and Goldstone bosons (the particle physics literature \[43,44\] envisions symmetry breaking pattern U(4) → U(2) ⊗ U(2), hence 16−2⋅4= 8 Goldstone bosons, while the symmetry breaking pattern for Néel ordering is SU(2) → U(1) which gives 3 − 1 = 2 Goldstone bosons). SU(2) gauge theories with massless Dirac fermions may also describe the quantum antiferromagnet \[13,45\], but we shall not consider this possibility because of the greater complexity of the nonabelian gauge theories.

A. Dirac fermions and the 2d Heisenberg antiferromagnet

A 2+1D theory of Dirac fermions

\[
L = \bar{\psi}_\alpha \partial_\mu \gamma_\mu \psi_\alpha,
\]

(\(\mu=0,1,2\)) contains fermions whose density of states behaves as \(\sim |\epsilon|\) (in a general \(D = d + 1\) dimensions, the density of states will be \(\sim |\epsilon|^{d-1}\)). In condensed matter context, the paramagnetism of such fermions will result in the uniform susceptibility behaving as \(T^{d-1}\).
This seems to have little in common with the 2d antiferromagnet, but we shall see that a gauge field coupled to fermions can produce the correspondence with the physics of 2d antiferromagnet. Such a picture can be motivated from the “$\pi$-flux” phase mean field ansatz of the Heisenberg Hamiltonian. The ansatz is so named since the phase of the product of the mean field parameter $\chi_{ij}$ around each plaquette ($\text{Im} \ln(\chi_{12}\chi_{23}\chi_{34}\chi_{41})$) is $\pi$. Despite the “flux,” the mean field does not break the parity and time reversal symmetry since the flux of $\pi$ is equal to $-\pi$. This phase has a lower energy than the BZA phase (with a large Fermi surface) [20, 27]; it also has the fermion spectrum
\[ \epsilon(k) = v\sqrt{\cos^2(k_x) + \cos^2(k_y)} \]
that roughly captures the high energy features of the undoped cuprates and the dispersion of a single hole in the antiferromagnet [48, 49]. The low energy fermionic excitations of the flux phase reside near two “Fermi points”, $k_{1,2} = (\pi/2, \pm \pi/2)$. Linearizing around these points gives the continuum theory [12]
\[ L = \bar{\psi}_{aa} \partial_\mu \gamma_\mu \psi_{aa}, \]
where $\psi_{a1} = \left( f_{a1e} \ f_{a1o} \right)$, $\psi_{a2} = \left( f_{a2e} \ f_{a2o} \right)$, ($a = 1, 2$ labels the two Fermi points; $e,o$ denote even and odd sites.) Organizing the $\psi_{a1}, \psi_{a2}$ fields into a single spinor $\psi_\alpha \equiv \left( \psi_{a1} \ psi_{a2} \right)$, we have a theory described by the lagrangian of Eq.46, with $4 \times 4$ $\gamma$-matrices:
\[ \gamma_0 = \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{array} \right), \quad \gamma_1 = \left( \begin{array}{cc} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{array} \right), \quad \gamma_2 = \left( \begin{array}{cc} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{array} \right). \]
Similarly to the 1d case, the uniform spin $S_\alpha^{l} = 0$ is given by $\tilde{\psi}_\alpha \gamma_0 \tau_\alpha^l \psi_\beta$, while the staggered spin $S_\alpha^{Q=(\pi,\pi)}$ is given by $\tilde{\psi}_\alpha \tau_\alpha^l \psi_\beta$. The spin correlation function $\langle S^+(x) \cdot S^-(0) \rangle$ can be evaluated easily at the mean field level using the Green’s function
\[ G_{\alpha\beta}(x) = \delta_{\alpha\beta} G(x) = \delta_{\alpha\beta} \gamma_\mu \frac{\partial}{\partial x_\mu} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i q \cdot x}}{q^2} = \delta_{\alpha\beta} \frac{x_\gamma}{4\pi(x^2)^{3/2}} \]
(as in 1+1D, the italics like $x, q$ denote space-time vectors, and $q^2 = q_0^2 + q^2, x^2 = x_0^2 + x^2$.) We have,
\[ \langle S^+(x)S^-(0) \rangle = -\text{tr} [G(x)\gamma_0 G(-x)\gamma_0] - (-1)^{x_1 + x_2} \text{tr} [G(x)G(-x)] \]
\[ = \frac{1}{4\pi^2} \left[ \frac{x_0^2 - x^2}{x^6} + (-1)^{x_1 + x_2} \frac{1}{x^4} \right]. \]
The equal time correlation function is then
\[ \langle S^+(0, x)S^-(0, 0) \rangle = \frac{(-1)^{(x_1 + x_2)} - 1}{4\pi^2 x^4}. \]
The mean field spin correlation function falls off as $1/x^4$, and again as in 1d, the spins on the same sublattice are not correlated [20, 27]. Note, however, that due to our neglect of $(\pi, 0)$ contributions, Eq.52 differs somewhat from the more accurate expressions of Refs. [20, 27].
To go beyond the mean field level, gauge fluctuation is included by the minimal coupling scheme, leading to the following theory:

$$Z = \int D\bar{\psi}D\psi Da_\mu \exp\left(-\int d^3x \sum_{\alpha=1}^{N} \bar{\psi}_\alpha (\partial_\mu - ia_\mu)\gamma_\mu \psi_\alpha \right)$$

(53)

where $N$ is a general number (in the physical case, $N = 2$). The absence of the $\frac{1}{4\pi^2}F^2_{\mu\nu}$ term means the bare coupling is infinite, but the theory is still sensible: the infrared behavior of the theory is well-behaved (within $1/N$ expansion, as we shall see), and the original lattice theory sets a ultraviolet cutoff. (Ultraviolet divergence can be also regulated by the kinetic term $\frac{1}{4\pi^2}\tilde{F}^2_{\mu\nu}$ with $\tilde{c}^2 < \infty$.) As in 1+1D, the lagrangian contains more symmetries than the Heisenberg model. For example the theory is invariant under the transform $\psi_\alpha \to \exp(i\gamma_{4,5}\theta)\psi_\alpha$ (where $\gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$; $\gamma_5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$; $I$ is a $2 \times 2$ unit matrix). Again, these symmetries are broken by higher order terms (lattice effects) which have been ignored.

**B. content of the gauge theory**

We now explore the physical content of the theory with massless Dirac fermions. Integrating out the fermions generates the dynamics for the gauge field [46] (see Appendix A for details)

$$Z = \int Da_\mu \exp\left(-\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} a_\mu(q)\Pi_{\mu\nu}(q)a_\nu(q) \right),$$

$$\Pi_{\mu\nu} = \frac{N}{8} \sqrt{q^2} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right).$$

(54)

The form of $\Pi_{\mu\nu}$ indicates that, unlike the 1+1D case, the gauge field is massless, though the infrared behavior is not as singular as the free gauge field (i.e., $L = \frac{1}{4} F^2_{\mu\nu}$). The effect of the fermion-gauge field interaction can be analyzed perturbatively in $1/N$. The gauge propagator in the Faddeev-Popov prescription is

$$D_{\mu\nu}(q) = \frac{8}{N\sqrt{q^2}} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \lambda \frac{q_\mu q_\nu}{q^4}.$$  

(55)

As in 1+1D (Sec.II), we choose the Landau gauge ($\lambda = 0$) to avoid spurious infrared divergences.

The fermion self-energy at the leading order in $1/N$ is given by

$$\Sigma(k) = i \int \frac{d^3q}{(2\pi)^3} \gamma^\mu \frac{(k+q)\gamma_\nu D_{\mu\nu}(q)}{(k+q)^2} \sim ik\gamma \ln \left( \frac{\Lambda^2}{k^2} \right).$$

(56)

This log divergence does not occur in the $1/N$ correction to the polarization function represented by the diagrams in Fig.3. These diagrams have been calculated explicitly in a different context by Chen et al. [50]. They found that the logarithmic divergences in the self
energy correction (Figs. 3 a+c) are cancelled by the vertex correction (Fig. 3 b); the diagrams sum to

$$\Pi^{1/N}(q) = \frac{3}{4\pi^2} \sqrt{q^2} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right),$$  \hspace{1cm} (57)

which is of the same form as the zeroth order result (Eq. 54) but down by some factor involving $1/N$. Thus, although we do not have a complete cancellation (like 1d), the gauge field is essentially unrenormalized, except for some modification of the effective coupling constant ($1/N \rightarrow 1/N' \approx 1/N$). In other words, $Z_3$ (charge renormalization) $\approx 1$. Weak correction to the gauge propagator (despite strong self energy correction) was seen in several other contexts, including the half-filled Landau level (fermion Chern-Simons theory) [51], bosonic Chern Simons theories [52], and the uRVB gauge theory [53]. This robustness is due to the fact that $\Pi_{\mu\nu}$ is a correlation function of conserved currents, and conserved currents have no anomalous dimensions (as a consequence of the Ward identity) [54]. The foregoing argument provides some optimism for perturbation theory in $1/N$.

![FIG. 3. Leading 1/N correction to vacuum polearization. The diagrams d) and e) are zero due to Furry’s theorem.](image)

Because the uniform spin correlation $\Pi_u(x) = \langle \bar{\psi}_1 \gamma_0 \psi_2(x) \bar{\psi}_2 \gamma_0 \psi_1(0) \rangle \sim \Pi_{00}(x)$, it wouldn’t be renormalized significantly. On the other hand, the staggered part $\Pi_s(x)$ of the spin correlation function does not involve conserved currents, and is expected to be strongly affected by gauge fluctuations. This difference can be more or less seen in perturbation theory. The diagramatic representation of $\Pi_u$ is the same as in the 1+1D case (Fig. 1). The external vertices in this case are a fermion-gauge field vertex $(\gamma_0)$. Using the Ward identity

$$\frac{\partial}{\partial p_\mu} \Sigma(p) = i\Gamma_\mu(p, p),$$  \hspace{1cm} (58)

we can see that the 2 overlapping divergences of Fig. 1d are cancelled by self-energy bubbles of Fig. 1a+c. For $\Pi_u$, the external vertices are 1 (unit matrix in the spinor space). In that case, the cancellation does not occur, and divergences develop in $\Pi_s$.

The question is, what would be ultimately the behavior of $\Pi_s$? Would $\Pi_s(x)$ be characterized by simple power law correlations like $\Pi_s(x) \sim 1/(x^2)^{\alpha}$, $(\alpha < 2)$ like the 1d case?
This might be a realization of Anderson’s 2d Luttinger liquid scenario \[55\], but we feel that \textit{a priori} there is no reason to expect so; after all, 1+1D is rather special, with all sort of fascinating features like the infinite dimension of the conformal group \[26\] and Coleman’s theorem \[56\] which prohibits spontaneous symmetry breaking. More plausibly, we would expect a symmetry-broken phase (Néel order) or a symmetric phase with a more complicated magnetic correlations.

C. spontaneous symmetry breaking

The previous section has identified a possible antiferromagnetic instability, which corresponds to the staggered magnetization \(\langle \bar{\psi} \tau^l \psi \rangle\) acquiring a definite orientation — an SU(2) symmetry breaking. We now examine this possibility more closely. In the symmetry-broken case, the fermions acquire a mass (dynamic mass generation).

Without loss of generality, we assume that the rotation symmetry is broken in the z-direction. Then the fermion Green’s function \(G(k)\) becomes \(1/(ik\gamma + m(k)\tau^3)\). Note that \(G\) is a matrix Green’s function in both the spinor space and the spin space. The “mass” \(m(k)\) is related to the sublattice magnetization \(M\) by

\[
M \sim \int \frac{d^3q}{(2\pi)^3} \text{tr}_{\gamma,\tau}[G(q)\tau^3].
\]

Self-consistent equation for \(m(k)\) can be obtained from the Schwinger-Dyson equation. Expressed in terms of matrix (both in \(\tau\) and \(\gamma\) space) Green’s function and (matrix) self energy (pictorially represented by Fig.4a), the S-D equation is

\[
\Sigma(k) = -m(k)\tau^3 = -\int \frac{d^3q}{(2\pi)^3} \tilde{\Gamma}_\mu(k,q)G(k-q)\gamma_\nu \tilde{D}_{\mu\nu}(q)
\]

where \(\tilde{\Gamma}_\mu, \tilde{D}_{\mu\nu}\) are fully renormalized vertex and gauge propagator.

Within the approximation of replacing \(\tilde{\Gamma}_\mu\) by \(\gamma_\mu\) and \(\tilde{D}_{\mu\nu}\) by \(D_{\mu\nu}\), we have

\[
m(p) = \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_\mu m(k)\gamma_\nu}{k^2 + m^2(k)} D_{\mu\nu}(p-k).
\]

Still this is a nonperturbative theory; it is easy to check that a finite order perturbation theory cannot generate a mass term in the fermion Green’s function. Diagrams that contribute are shown in Fig.4b.

![FIG. 4. Schematic representation of the Schwinger-Dyson equation. In part a) the solid line with shaded blob is the self-consistent Green’s function of the fermions \(G\), the thick wiggly line is the dressed gauge field (which incorporates the changes in the vacuum polarization due to changes in fermion Green’s function), and the shaded triangle is the dressed vertex. Part b) is a representation of the contribution to the self energy.](image-url)
The context in which the self-consistent equation arises is similar to the SDW (spin density wave) problem and the superconductivity \[^{57}\]. However, in our case, the mass \(m(p)\) is dependent upon 3-momentum. Eq.\[^{61}\] has been already analyzed by Appelquist and coworkers in a different context (“chiral symmetry breaking”) \[^{44}\]. Some of the steps are sketched below.

From the result

\[
\gamma_\mu \gamma_\nu \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) = \gamma_\mu \gamma_\mu - \frac{q^2}{q^2} = 3 - 1 = 2
\]

we have, after some angular integrals,

\[
m(p) = \frac{4}{N\pi^2 p} \int_0^\Lambda dk \frac{k m(k)}{k^2 + m^2(k)}(k + p - |k - p|).
\]

Note that the lattice origin of our theory sets the UV cutoff scale \(\Lambda\), while in the theory of Appelquist et al. which retains the kinetic term \(\sim \frac{1}{4e^2 F^2_{\mu\nu}}\), the coupling constant \(e^2\) sets the scale (QED3 is a superrenormalizable theory). The integral equation (Eq.\[^{63}\]) is equivalent to the differential equation

\[
\frac{d}{dp} \left( p^2 \frac{dm(p)}{dp} \right) = -\frac{8}{\pi^2 N p^2 + m^2(p)}
\]

with boundary conditions

\[
\Lambda m'(\Lambda) + m(\Lambda) = 0
\]

and

\[
0 \leq m(0) < \infty.
\]

It turns out that this nonlinear differential equation has a nontrivial solution only for \(N < N_c = 32/\pi^2\) \[^{58}\]. For the physical case of SU(2) antiferromagnet, \(N\) equals 2; therefore, the dynamical mass generation occurs, and the Néel-vector rotation symmetry is spontaneously broken. Thus, provided that we include the gauge fluctuations, we have a Néel order.

The foregoing argument, however, should be taken with a grain of salt. In principle, the gauge propagator that enters the Schwinger-Dyson equation must be a fully dressed one, and so should be the vertex. In the symmetry-broken phase, the gauge propagator is different from the symmetric phase, as the polarization function \(\Pi_{\mu\nu}\) is different. At the crudest level, if we assume that the fermions acquire a constant mass \(m(p) = m\), then the polarization function would be (See Appendix A).

\[
\Pi_{\mu\nu}(q) \sim \frac{1}{m} q^2 \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right),
\]

which means that a kinetic term \(\sim \frac{1}{m} F^2_{\mu\nu}\) is generated. This results in the Coulomb potential for the fermions which in 2+1D is

\[
V(x) = -m \int d^2 q \frac{e^{i q \cdot x}}{q^2} = m \ln \left( \frac{|x|}{R} \right), \quad (R \sim 1/m).
\]
Since the potential increases at large distances, it is a confining potential (Sec. V.A), and a
different physical picture for the fermions might be expected in that case.

Arguments can be made that these considerations do not affect seriously the conclusion
as far as the issue of determining whether dynamical mass generation occurs and (in the case
of occurrence) the value of $N_c$ [59]. (The point is that very near $N_c$, the polarization of the
fermions in the symmetry-broken state must be pretty close to that of the massless fermions.
On the other hand, above treatment is too crude to study the behaviors of quantities like
$m(p)$. Lattice gauge theory simulation [60] does find that the symmetry breaking occurs,
and $N_c \approx 3.5$, which is close to above analytical results.

Having seen the symmetry breaking, we can identify the elementary excitations — the
Goldstone bosons. We all know that the Goldstone bosons in an ordered antiferromagnet are
spin waves. In the fermion picture, the Goldstone bosons are a collective mode, and appear
as a pole in the two-particle Green’s function, the scattering amplitude in the appropriate
channel, and in the related vertex [61]. Here we show this by considering the SU(2) vertex.
The Bethe-Salpeter equation for the vertex is given by

$$\Lambda^l(p; q) = 1 \cdot \tau^l - \int \frac{d^3k}{(2\pi)^3} \gamma_\mu G(k)\Lambda^l(k; q)G(k + q)\gamma_\nu D_{\mu\nu}(p - k),$$

which is represented by the ladder diagrams of Fig. 5 (In Eq. 69, the 1 in the first term on
the RHS is the unit matrix in the spinor space.)

FIG. 5. Bethe-Salpeter equation for the isovector vertex in the $Q = (\pi, \pi)$ channel. The fermion
lines willed a shaded blob represent the renormalized (self-consistent) Green’s function.

If there is a Goldstone boson, then $\Lambda^l(p; q)$ has a pole at $q^2 = 0$, in which case the
homogeneous equation

$$\Lambda^l(p; 0) = -\int \frac{d^3k}{(2\pi)^3} \gamma_\mu G(k)\Lambda^l(k; 0)G(k)\gamma_\nu D_{\mu\nu}(p - k)$$

has a nontrivial solution. It is easy to see that

$$\Lambda^l(p; 0) = [m(p)\tau^3, \tau^l]$$

with $m(p)$ given by Eq. 61 is the solution. Therefore, if there is a dynamical mass generation
($m(p) \neq 0$), then we have two Goldstone bosons (SU(2) symmetry breaking): $\tau^l = \tau^1, \tau^2$
(or $\tau^+, \tau^-$) in Eq. 71 gives a nonvanishing commutator. The Lorentz invariance restricts the
mesons to have a linear dispersion $q^2_0 = q^2$ (Minkowski space), which is indeed the case with
antiferromagnetic spin waves.
IV. TWO-DIMENSIONAL UNDERDOPED CUPRATES

A. antiferromagnetic correlations

The foregoing may appear to be an unnecessarily roundabout way of looking at 2d quantum antiferromagnet, yet we believe it is not without certain value. The qualitative idea that a strong attraction between spinons and antispinons via gauge field can result in the formation of a vector condensate with $Q=(\pi, \pi)$ (the antiferromagnetic channel) may shed some lights on the underdoped cuprates. In the underdoped cuprates, an effective theory based on the sFlux ansatz of the SU(2) mean field theory \cite{64,14,17} consists of 2 flavors of massless Dirac fermions and a U(1) gauge field just like above, but now the gauge field is also coupled to the bosons (holons). In other words, schematically,

$$L = \bar{\psi}_{\alpha} \gamma_\mu \left( \partial_\mu - ia_\mu \right) \psi_{\alpha} - ia_\mu J_b^B + L_B. \quad (72)$$

This additional coupling to the bosons will weaken the gauge field in the sense that it will screen the time component of the gauge field ($\lim_{q \to 0} \Pi_{00} \neq 0$). The gauge propagator then would take the form (in the Coulomb gauge)

$$D_{\mu\nu} \sim \frac{1}{\sqrt{q^2}} \delta_{\mu i} \delta_{\nu j} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) + \frac{1}{xJ + \sqrt{q^2}} \delta_{\mu 0} \delta_{\nu 0} \frac{q^2}{q^2}, \quad (73)$$

where $i, j = 1, 2$ are spatial indices and $x$ is the concentration of doped holes. In the simplest approximation, we ignore the massive part, and consider only the spatial component

$$D_{\mu\nu}(q) = \frac{8}{N\sqrt{q^2}} \delta_{\mu i} \delta_{\nu j} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right), \quad (74)$$

and examine the self-consistent equation for $m(p)$ (Eq.61). Since

$$\gamma_\mu \gamma_\nu D_{\mu\nu} = \gamma_i \gamma_i - \frac{(q_i \gamma_i)(q_i \gamma_i)}{q^2} = 2 - 1 = 1, \quad (75)$$

we have

$$m(p) = \frac{2}{N\pi^2 p} \int_0^\Lambda dk \frac{k m(k)}{k^2 + m^2(k)} (k + p - |k - p|). \quad (76)$$

This is identical to the Eq.61 except for a factor of 2 difference in the prefactor. This factor 2 reduction can be understood from that fact that the gauge field in 2d has one transverse mode and one longitudinal mode the latter of which becomes massive. From the analysis following Eq.61, we know immediately that there will be a symmetry breaking only for $N < N_c' = N_c/2 = 16/\pi^2$. Thus, for the physical case of $N = 2$, the spontaneous symmetry breaking would not occur!—This is what was hoped for our mean field theory.

The attraction in the $Q = (\pi, \pi)$ channel mediated by the gauge field, although not strong enough to generate a condensate, will nevertheless have a strong effect on the spectrum of antiferromagnetic excitation. The fluctuation of the order parameter associated with the transition (staggered moment) can be examined by looking at the staggered-channel spin
correlation function in the ladder approximation, similar to the more familiar problems like the superconducting fluctuations or the ferromagnetic spin fluctuations [65] (See Fig.6).

FIG. 6. Ladder diagrams. a) Our case: staggered spin correlation. The wiggly lines are interactions mediated by the gauge field. Structurally similar examples: b) ferromagnetic spin correlation. The dashed lines are short-range repulsive interactions. c) superconducting correlation. The dotted lines are some kind of attractive interaction causing pairing.

In the problem of nearly ferromagnetic Fermi liquids, short range interaction between fermions are often modelled in terms of an on-site repulsion \( U \) \((U \equiv (E_F) is the dimensionless coupling constant corresponding to our \( 1/N \)). This problem is a lot simpler, as the ladder series sums immediately to the RPA form \( \chi = \chi_0/(1-U\chi_0) \). The pole of the RPA propagator gives the diffusive mode ("paramagnons") associated with a conserved order parameter. This mode (more accurately the peak in \( \chi''(\omega) \)) comes down in energy and becomes sharper as \( U \) approaches \( U_c \).

Unfortunately, in our problem the interaction is retarded and long-ranged, hence the diagrams are not easy to evaluate. Nevertheless, on physical grounds, it is quite reasonable to expect that the same gauge field which caused the antiferromagnetic instability in the absence of holons will try to create a (massive) mode (particle-hole bound state in the \( Q = (\pi, \pi) \) channel) in this case. Because the symmetry is unbroken, there is a particle-hole continuum, as a result of which a sharp mode cannot exist, but a "broad resonance", i.e. a very unstable meson with a Minkowski-space dispersion

\[
q^2 = \tilde{m}^2 - i\tilde{m}\Gamma,
\]

is expected. The mass of the mode \( \tilde{m} \) would come down as the transition is approached ("soft mode"). A physical consequence will be that the dynamic susceptibility \( \chi''_{Q=(\pi,\pi)}(\omega) \) (which can be probed by neutron scattering) has a broad peak, with a substantial rearrangement of the spectral weight compared to the mean field prediction. This heuristic picture is consistent with experiments of Keimer and collaborators [22] that find in the normal state (and in the superconducting state) of underdoped cuprates a magnetic scattering with a broad peak at some frequency scale that comes down in energy as the doping is reduced.
In principle, we should be able to study this by looking into the Bethe-Salpeter equation for spinon-antispinon scattering amplitude or the associated vertex. The analysis, however, entails a number of practical and technical difficulties. We can get around the problem of the complicated Lorentz-noninvariant gauge propagator for the doped case (Eq. [73]) by convincing ourselves that the we would obtain a similar physics by considering a Lorentz invariant one

$$D_{\mu\nu} = \frac{8}{\bar{N}\sqrt{q^2}} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)$$

(78)

with $\bar{N} > N_c$ (Reducing the doping would correspond to reducing $\bar{N}$). Still, the fact that we are investigating a mode that is massive and damped causes complications unseen in the B-S equation for the Goldstone bosons of the ordered antiferromagnet. Generally, the scattering amplitude and the vertex have many components ("invariant amplitudes"), e.g.,

$$\Lambda^I(p; q) = f(p; q)\tau^I + g(p; q)p_\gamma \tau^I + \cdots.$$  

(79)

The Goldstone modes are massless, hence we can focus on $q = 0$ in which case $f(p; 0)$ decouples from other amplitudes and has the same equation as that of the dynamical mass, as we have seen in Sec.III.C. The decoupling doesn’t occur for $q \neq 0$. Appelquist et al. have considered the scalar component of the Euclidean scattering amplitude in the ladder approximation, ignoring the coupling to other components; this corresponds to considering $f(p; q)$ only. They claim to find no light meson whose mass comes down to zero as the transition is approached, and hence the transition must be a novel one, i.e. neither a first order nor a second order transition. However, this conclusion is questionable, as a well-defined mode (pole in the scattering amplitude) in the symmetric phase is too stringent a requirement for a second order phase transition; more generally what happens in the
symmetry-unbroken phase as the second order transition is approached is a build-up of the low energy spectral weight for the response related to the order parameter. In any case, the analysis of Ref. [66] is not appropriate for finding an unstable meson which resides in the second Riemann sheet of the Minkowski space.

We now discuss briefly the NMR relaxation rates, which measure the very low energy limit of the spin excitation spectrum. As noted earlier, the mean field theories predict similar behavior of the Oxygen and Copper site relaxation rates, while experiments show marked difference. It would be difficult to calculate finite temperature properties within our gauge theory, but we believe that what has been discussed so far throws some light on this issue: once the gauge fluctuation is included, there is no reason why the two relaxation rates should have similar behaviors, since the effect of the gauge field on the \( \mathbf{Q} = 0 \) and \( \mathbf{Q} = (\pi, \pi) \) response is very different. It is plausible that at high temperature side the Cu \( 1/T_1T \) will be increasing with decreasing temperature, as the gauge fluctuation restores certain amount of antiferromagnetic correlations (note that above some energy scale, the screening effect of bosons won’t be too important and the propagator would look like that of the undoped cuprates), while at low temperatures (below some scale) it will go down with decreasing temperature, since we still have the d-wave gap if the mean field picture is going to survive.

FIG. 8. Possible contribution from boson-fermion coupling. a) The inner loop (dashed line) and the outer loop (solid line) denote bosons and fermions, respectively, and the dotted lines denote boson-fermion interaction b) “electron” propagator c) schematic representation of the same diagram

Finally, we note that in our discussion of antiferromagnetic excitations in underdoped cuprates we have not considered certain class of diagrams that may be important. An example, shown in Fig.8, involves a boson loop and a fermion loop exchanging a residual attractive interaction. The set of a boson and a fermion line connected by ladder type interaction can be viewed as a electronlike propagator (Bethe-Salpeter equation for electron propagator). This type of contribution may be related to pieces of the Fermi surface near \( (\pm \pi/2, \pm \pi/2) \) [14], and might provide some clue to the origin of low-frequency incommensurate features observed in the neutron scattering.

B. thermodynamic properties

In the underdoped cuprates the gauge field interaction also affects the uniform part of the spin response (the thermodynamic properties) though the effects are subtler. More specifically, the coupling of the gauge field to nonrelativistic bosons results in the logarithmic renormalization of the velocity of the fermions, which has the effect of enhancing the specific heat and the uniform susceptibility [64]. This seems to be in accordance with experiments
on underdoped cuprates. Although one might alternatively view the enhancement features in the normal state as having to do with the “Fermi surface segments”, these are not really quasiparticle states in the strict sense ($z=0$) and there are likely to be theoretical complexity and possibility of overcounting. It seems a lot simpler to view that the fermions account for most of the entropy and spin response. A curious feature \cite{67} of our theory is that the Wilson ratio $W = C/T\chi_u$ has a value quite close to that of the “quantum critical” phase of the nonlinear sigma model \cite{68}: $W$ in our theory is 0.128, while the $O(N)$ nonlinear sigma model gives $W = 0.124$ at zeroth order, and $W = 0.116$ with the inclusion of $1/N$ correction. Could this be a coincidence?

\section*{V. CONCLUDING REMARKS}

We now conclude with a discussion of some difficult matters, hoping not so much as to resolve them but to content ourselves with placing them in perspective.

\subsection*{A. confinement, spinons, and all that}

The term “spinon” has been used rather carelessly in this paper. In the strict sense, it refers to well-defined neutral elementary excitations carrying spin $1/2$. A well-known (and perhaps the only widely-accepted) example is the solitonic (topological) objects in spin chains that are usually understood within the bosonization framework \cite{41,26}.

The “spinons” in this paper refer most of the times to spin $1/2$ fermions strongly coupled to fluctuating gauge fields — the fermions that arise from the RVB mean field theories. These spinons are far from being well-established. In 1d, the fact that they are different from topological spinons (“true spinons”) has been emphasized by Mudry and Fradkin \cite{45}. Nevertheless, we have seen that in the 1d Heisenberg model, the mean field fermions give a reasonable description of the specific heat and uniform susceptibility \cite{69}; the Zeeman splitting of the spin up and down fermions in a magnetic field and the associated incommensurate magnetic excitations \cite{70} can be easily understood in terms of the mean field fermions. Furthermore, we showed in Section II that the perturbative treatment of the gauge fluctuations systematically improves the antiferromagnetic correlation. Thus the mean field fermions are not as bad a starting point as some would regard.

In 2d and 3d, pre-high $T_c$ empiricism and experiences provide strong resistance to the notion of spinons as derived from RVB mean field theory. After all, in most metals (2 and 3d), the basic elementary excitation is quasiparticles that carry both spin $1/2$ and charge $e$, and in most insulating antiferromagnets or in spin liquids (like spin ladders), the elementary excitations are spin 1 objects (spin waves or magnons). In other words, spinons and holons do not exist on their own, but are “confined” in spinon-antispinon composite objects (spin waves: $f^*f$) or in spinon-holon composite objects (quasiparticles: $f^*b$).

The sense in which this “confinement” is discussed is similar to the quark confinement in particle physics — the absence outside the nucleus of the quarks that make up hadrons (baryons and mesons). In fact, the strong interaction physics appears to share quite a few parallels with our problem \cite{71}. Quantum chromodynamics (QCD), which is widely believed to be the correct theory of strong interactions, is a gauge theory whose low energy
physics is as poorly understood as the high $T_c$ cuprates. At a more substantial level, the phenomenology of strong interactions indicates that the (approximate) chiral symmetry is spontaneously broken, giving rise to mesons (such as pions) which are Goldstone bosons, in analogy with spin waves in a Néel ordered system. Just as the nonlinear sigma model gives an excellent description of the low energy spin dynamics of the undoped cuprates [2], in particle physics it has been well known that similar effective lagrangians (sigma models, chiral perturbation theory [3], etc.) give a very good description of the hadronic physics. However, attempts to “derive” the parameters of the effective theories, such as the pion decay constant, from first principles have not been entirely successful. The basic difficulty is that the same asymptotic freedom that led to the confirmation of the quark picture at the high energy side causes grave difficulties in analyzing the low energy physics. At present, no consensus exists as regards the “mechanism” of chiral symmetry breaking in QCD, but at least it seems clear that the problem is intimately connected to the general issue of confinement.

It is the basic underlying idea of this paper that the confinement does not occur (at least) for the normal state of the superconducting cuprates, or more loosely, spinons and holons are “meaningful” objects. On the other hand, some sort of confinement is relevant to the discussion of the Néel ordered state in undoped cuprates, since we know that there the low energy excitations are not the spinons but spin waves. In considering the confinement, we are (helped but also) burdened by previous studies in particle physics regarding the issue. The confinement motivated by the strong interaction phenomenology — no color nonsinglet particles exist in the physical spectrum — is a strong statement, and it is not clear to what extent the confinement in our case should match the quark confinement.

For example, we can ask the following questions in 2d undoped systems. 1) On the low energy side, are the spin waves the only massless degrees of freedom, or could there be an additional mode hiding? 2) On the high energy side, do the spinon states exist, or have they disappeared completely? In other words, can we somehow “see” spinons at high energies?

To address these questions, we need to discuss one possibly important aspect of our gauge theory, the compactness, that has been ignored so far. The models that we have examined in this paper originate from the lattice, hence are compact gauge theories. The usual assumption is that the compact theory can be replaced by a more amenable noncompact theory in the continuum, but this is not always justified. A representative and scary example is the pure gauge theory in 2+1D. Polyakov [4] has shown that the compact pure gauge theory (2+1D electrodynamics) differs from the noncompact one due to instantons — the topologically nontrivial, extended classical solutions of Euclidean gauge field equations, that can be viewed as tunneling events between topologically inequivalent vacua. Instantons cause the Wilson loop to follow the area law $\langle \exp(\oint dx_\mu a_\mu) \rangle \sim \exp(-Area)$, which means the presence of a linear potential $V(x^a, x^b) \sim |x^a - x^b|$ between static sources, a sign of (strong) confinement.

When matter fields are present as in our case and in QCD, the situation is not so simple. In QCD, for example, despite theoretical attempts [7] the relevance of instantons to quark confinement remains uncertain. Generally, the fluctuations of matter fields (especially the massless fields) are adverse to instantons. One specific scenario by which instantons are suppressed is fermion zero modes. For example, in the massless Schwinger model, the fermion determinant $\text{det}[(\partial_\mu - ia_\mu)\gamma_\mu]$ in a gauge configuration $a_\mu$ with an instanton vanishes, so only
the topologically trivial sector contributes to the functional integral (partition function). Similar mechanism turned out to account for the famous U(1) problem \[78\] in QCD. In both cases (1+1D and 3+1D), the zero mode is connected to (axial) anomalies, whose analogue in odd space-time dimensions (2+1D) is nonexistent.

In the context of our problem (compact 2+1D gauge theory with Dirac fermions), Marston \[21\] studied the possibility of fermion zero modes, and concluded that zero modes do not exist, suggesting a possible relevance of instantons. Marston has also calculated the action of an instanton, and found that it is logarithmically divergent with a prefactor proportional to the number of flavors \((S \propto N \ln R)\). Kosterlitz-Thouless type argument then indicates that below critical \(N_c\) which turns out to be 0.9, instantons may proliferate, while for \(N > N_c\) which includes the physical case \((N = 2)\), the instantons are suppressed. Ioffe & Larkin \[46\] have also calculated the action of an instanton, considering only the bilinear part of the gauge field action obtained by integrating out the fermions, and found logarithmic divergence, but \(N_c\) in this case turns out to be 24. It has been noted that this treatment may be too simple, and the question of instantons for physical \(N\) remains unclear \[79\].

The suppression of instantons may no longer be the case if the fermion masses are dynamically generated by spontaneous symmetry breaking, which is indeed our situation. We now have the possibility of symmetry-breaking-induced instantons. These instantons here cannot induce the dimerization (valence bond solid order) \[80\], unlike those of the bosonic spinon theory \[51\]. *Probably the most notable consequence of the instantons is that the gauge field will become massive \[74\], hence the spin waves will be the only low energy excitations of the ordered antiferromagnet.*

On the other hand, in the (simpler) scenario without the instantons (Sec.III.C), there is a massless gauge field in addition to the spin waves in the ordered antiferromagnet. This is rather bothersome, since it is commonly believed that the spin waves deplete the low energy excitations in 2d quantum antiferromagnet. Practically, it may be difficult to determine the absence or the existence of the massless gauge field, as it does not couple to external probes in a simple way. The gauge field may have a \(\sim T^2\) contribution to the specific heat like the spin waves, but the modification of the prefactor due to gauge field might be considered a spin-wave renormalization effect.

The pictures with or without instantons may also have different implications for the spinon confinement. If the instantons are relevant, the Polykov-type nonperturbative mechanism might lead to the final picture in which the spinons have vanished completely, and traces of them may not be present even at high energies. In the picture without instantons, spinons may be still confined, due to the long range nature of the gauge field. A situation in which this works out \[75\] is the 1+1D \(CP^{N-1}\) model \((L = |(\partial_\mu - ia_\mu)z|^2 + m^2 z^\dagger z)\), in which integrating out the matter field \((z\text{-field})\) generates a kinetic term (self energy) for the gauge field \(\frac{1}{m} F_{\mu\nu}^2\) that gives rise to the linearly confining Coulomb potential \((i.e. V(x_1) = -\int dq_1 \exp(iq_1 x_1)/q_1^2 \sim |x_1|)\) between the \(z\)-charges. In our 2+1D problem (Sec.III.C), at a crude level, the dynamical mass generation will have a feedback effect on the gauge field, softening the gauge propagator from \(1/\sqrt{q^2}\) to \(m/q^2\), making it look like a genuine electromagnetic field. In 2+1D, the Coulomb potential associated with electromagnetic field is weakly (marginally) confining \((V(x) \sim \log(\|x\|/R))\) \[74\], and for distances \{energies\} shorter \{higher\} than some scale \(R \sim 1/m \{m\}\), the spinon might be “visible”. Within the framework of Sec.III.C, to see if there are well-defined spinons, we can study
whether the Minkowski-space Green’s function of the spinons has a pole at some mass scale. If the dynamical symmetry breaking occurred in such a way that the spinons acquire a constant mass as in the NJL model [62], then the spinons should exist at high energies, and we might be able to “see” them [82] by a method analogous to the 1d example [70]. If the symmetry is broken by (long-range, retarded) gauge field interactions as in our case, the problem is quite complicated; we need to continue the mass function $m(p)$ from the Euclidean space to the Minkowski space ($p^2 \rightarrow -p^2$), and examine whether $p^2 - m(-p^2) = 0$ has a solution near the real axis. The results from the literature [59] indicate that when the “feedback” effect of the spinon mass generation on the vacuum polarization is taken into account (in which case the gauge field dynamics looks like $F^2_{\mu\nu}$), no poles are found near the real axis.

At present, it is not clear which picture (i.e., with or without instantons) of confinement is realized for the undoped cuprates. In the normal state of the underdoped cuprates, instantons are probably not relevant, as we have massless fermions and there are additional massless fluctuations due to the introduction of holes. In any case, the unusual phenomenology of high $T_c$ cuprates points to that the spinons and holons are deconfined, as emphasized earlier. It is beyond the scope of this paper to discuss the issue of possible confinement in the superconducting state.

B. loose ends

In this paper we have examined the magnetism of undoped and underdoped cuprates from the point of view of neutral fermions with spin 1/2 [83]. Admittedly, the theory as it stands is far from rigorous. The philosophy has been to analyze possibly the simplest effective field theory of massless Dirac fermions, bosons and gauge fields, motivated by the sFlux phase that appears as a saddle point solution of the SU(2) mean field theory. In reality, the situation is a lot more complicated. The mean field fermion spectrum is anisotropic: $\epsilon(k) = \sqrt{v_F^2 k_1^2 + v_2^2 k_2^2}$, and the velocities $v_F, v_2$ have some doping and temperature dependences. These dependences, however, do not account for the puzzling properties of the cuprates that we have discussed, and the photoemission does indicate that the quasiparticle gap remains large in the normal state of the underdoped cuprates (i.e., superconducting transition is not a gap-closing transition) and the gap is only weakly doping dependent. Therefore, the effective theory may be quite sensible for studying qualitative features not captured by mean field theories. We have made a number of approximations to treat the complicated dynamics of the gauge fields, but we hope it’s not too optimistic to view that the qualitative conclusions are correct. In any case, the idea that the holon coupling to the gauge field prevents spontaneous symmetry breaking in the fermion system is an attractive one, and is very much in the spirit of the empirical fact that moving holes quickly destroy antiferromagetic order. Unfortunately, even with simplifications, the calculations quickly become rather intractable.

Eventually the “spin gap” has to close up as we go to the optimally doped regime, which in the mean field theory is modelled by the uRVB saddle point [53]. The details of this crossover is certainly beyond our hopes. This will severely limit us in considering some very interesting issues, like the relation between the neutron scattering peaks in the underdoped cuprates and the sharp 41meV peak in the superconducting phase of the optimally doped YBCO$_7$. Anyway, the present theory does not say much about the spin excitations in the
superconducting state of the underdoped cuprates, since the dynamics of gauge field in the superconducting state is different from that considered here. Another drawback is that in our framework it is not easy to consider possible incommensurate features in spin excitations.

Granting these limitations, we still feel that the picture of spin excitations in underdoped cuprates in terms of deconfined fermionic spinons is reasonable and perhaps more natural than other descriptions like those based on fluctuating staggered moments. Features like the linear-in-$T$ behavior of uniform susceptibility and the specific heat coefficient in underdoped cuprates might be also explainable in terms of the “quantum critical” regime of the nonlinear sigma model, but it is not clear in that approach how to account for the strange behavior of the Copper $1/T_1$ in the same temperature range. Attempts to explain the Copper $1/T_1$ in terms of the “quantum disordered” regime then has to explain why activated behaviors are not seen in quantities like uniform susceptibility. Again, this seems to point to the difficulty of achieving a theoretical description of a system that involves a mysterious combination of gaplike (short range) correlations and critical correlations.

VI. ACKNOWLEDGMENTS

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APPENDIX A: FERMION POLARIZATION

In this appendix, fermion vacuum-polarizations in 1+1D and 2+1D are worked out using “dimensional regularization”.

For massless Dirac fermions, the polarization function $\Pi_{\mu\nu}$ is given by

$$\Pi_{\mu\nu}(q) = -N \int \frac{d^Dk}{(2\pi)^D} \text{tr}[G(k)\gamma_\mu G(k+q)\gamma_\nu]$$

$$= N \int \frac{d^Dk}{(2\pi)^D} \text{tr}\left[\frac{k\gamma_\mu (k+q)\gamma_\nu}{k^2 \gamma_\mu (k+q)^2 \gamma_\nu}\right]$$

$$= N \int_0^1 dx \frac{d^Dk}{(2\pi)^D} \text{tr}\left[(k' + (1-x)q)\gamma_\mu (k' - xq)\gamma_\nu\right] \frac{1}{(k'^2 + q^2x(1-x))^2},$$

where $N$ is the number of flavors of fermions. Using the trace identity

$$\text{tr}[\gamma_\rho \gamma_\mu \gamma_\sigma \gamma_\nu] = (\text{tr}1)(\delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\mu\sigma} + \delta_{\rho\sigma}\delta_{\mu\nu}),$$

The numerator of the integrand is found as

$$2k'_\mu k'_\nu - 2x(1-x)(q_\mu q_\nu - \delta_{\mu\nu} q^2) - \delta_{\mu\nu}[k'^2 + q^2x(1-x)].$$

Substituting, we get
\[ \Pi_{\mu\nu}(q) = N(\text{tr} 1) \int_0^1 dx \int \frac{d^Dk}{(2\pi)^D} \left[ \frac{2k_{\mu}k_{\nu}}{(k^2 + q^2x(1-x))^2} \right. \\
\left. - \frac{\delta_{\mu\nu}}{k^2 + q^2x(1-x)} + \frac{2x(1-x)(\delta_{\mu\nu}q^2 - q_{\mu}q_{\nu})}{(k^2 + q^2x(1-x))^2} \right]. \tag{A4} \]

The first two terms cancel, and we have

\[ \Pi_{\mu\nu}(q) = 2N(\text{tr} 1) \Gamma(2 - D/2) (q^2\delta_{\mu\nu} - q_{\mu}q_{\nu}) \int_0^1 dx \, x(1-x)(q^2x(1-x))^{D/2-2} \]

\[ = \begin{cases} \frac{N(\text{tr} 1)}{32} \sqrt{q_2} (\delta_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q_2}) & D = 2 + 1 \\
\frac{N(\text{tr} 1)}{2\pi} (\delta_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q_2}) & D = 1 + 1. \end{cases} \tag{A5} \]

For the situations discussed in the main text, we have tr1=4 for the 2+1D (Secs. III & IV) as we have 4 component fermions, while tr1=2 for the 1+1D (Sec. II).

If the fermions are massive, i.e., \( L = \bar{\psi}(\partial_{\mu} - ia_{\mu})\gamma_{\mu}\psi + m\bar{\psi}\psi \), similar calculation shows that the polarization function is given by

\[ \Pi_{\mu\nu}(q) = 2N(\text{tr} 1) \Gamma(2 - D/2) (q^2\delta_{\mu\nu} - q_{\mu}q_{\nu}) \int_0^1 dx \, x(1-x)(m^2 + q^2x(1-x))^{D/2-2} \]

\[ = \frac{(\text{tr} 1)N}{2\pi} (q^2\delta_{\mu\nu} - q_{\mu}q_{\nu}) \left( \frac{1}{q^2} - \frac{4m^2}{q^3\sqrt{4m^2 + q^2}} \tanh^{-1}(\frac{q}{4m^2 + q^2}) \right) \quad [D = 1 + 1], \]

\[ = \frac{(\text{tr} 1)N}{4\pi} (q^2\delta_{\mu\nu} - q_{\mu}q_{\nu}) \left( \frac{m}{2q^2} + \frac{q^2 - 4m^2}{4q^4} \sin^{-1}(\frac{q}{\sqrt{4m^2 + q^2}}) \right) \quad [D = 2 + 1]. \]

For small \( q \) (i.e. \( q^2 \ll m^2 \)), we would have

\[ D = 2 + 1 : \Pi_{\mu\nu}(q) \approx \frac{(\text{tr} 1)N}{4\pi} (q^2\delta_{\mu\nu} - q_{\mu}q_{\nu}) \left( \frac{1}{6m} - \frac{q^2}{60m^3 + \ldots} \right) \]

\[ D = 1 + 1 : \Pi_{\mu\nu}(q) \approx \frac{(\text{tr} 1)N}{2\pi} (q^2\delta_{\mu\nu} - q_{\mu}q_{\nu}) \left( \frac{1}{6m^2} - \frac{q^2}{30m^4 + \ldots} \right). \]

Note that the massive fermions give rise to a gauge field dynamics of the form \( L \sim a\Pi a \sim (F_{\mu\nu})^2 \) like the real electromagnetic field.

**APPENDIX B: JACOBIAN FOR CHIRAL ROTATION**

In this appendix, we derive the jacobian \( \exp(J) = \frac{D\bar{\psi}D\psi}{D\psi D\psi} \) for chiral rotation, along the line of Stone [35]. Basically we require the functional integral \( Z = \int D\bar{\psi}D\psi \exp(-L) \) be invariant under the chiral transform. For an infinitesimal chiral transform \( \psi \rightarrow \psi' = \exp(-\gamma_5\theta_a)\psi \), the measure changes as

\[ D\bar{\psi}(x)D\psi(x) \rightarrow D\bar{\psi}(x)D\psi(x) \left( 1 - \int d^2x \frac{\delta J(x)}{\delta \theta_a(x)} \theta_a(x) \right), \tag{B1} \]

while the lagrangian changes as
\[ L(\psi) \rightarrow L(\psi) - (\partial_\mu \theta_a) \bar{\psi} \gamma^5 \gamma^{\mu} \psi = L(\psi) + i \theta_a \partial_\mu j^5_{\mu}. \quad (B2) \]

Setting the functional derivative
\[
\left. \frac{\delta Z}{\delta \theta_a} = \int d^2 x \left( \bar{\psi} \gamma^a \gamma^{\mu} \partial_\mu \psi \right) \theta_a \right) \quad (B3)
\]
to zero gives
\[
\frac{\delta J}{\delta \theta_a} = -\frac{N}{2 \pi} \epsilon_{\mu \nu} F^{\mu \nu} + \frac{N}{\pi} \partial_\mu \theta_a, \quad (B4)
\]
where we have used the axial anomaly condition \( \partial_\mu j^5_{\mu} = -\frac{N}{2 \pi} \epsilon_{\mu \nu} F^{\mu \nu} \) with \( F^{\mu \nu} = \partial_\mu a_{\nu} - \partial_\nu a_{\mu} \), \( a_{\mu} = \epsilon_{\mu \nu} \partial_\nu \theta_a + \partial_\mu \theta_b \). Eq.\(B4\) then implies
\[
J = -\int d^2 x \frac{N}{2 \pi} (\partial_\mu \theta_a)^2. \quad (B5)\]
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