Small $x$ resummations confronted with $F_2(x, Q^2)$ data

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Abstract

It has been observed recently that a consistent LO BFKL gluon evolution leads to a steep growth of $F_2(x, Q^2)$ for $x \to 0$ almost independently of $Q^2$. We show that current data from the DESY HERA collider are precise enough to finally rule out a pure BFKL behaviour in the accessible small $x$ region. Several attempts have been made by other groups to treat the BFKL type small $x$ resummations instead as additions to the conventional anomalous dimensions of the successful renormalization group “Altarelli-Parisi” equations. We demonstrate that all presently available $F_2$ data, in particular at lower values of $Q^2$, can not be described using the presently known NLO (two-loop consistent) small $x$ resummations. Finally we comment on the common reason for the failure of these BFKL inspired methods which result, in general, in too steep $x$-dependencies as $x \to 0$.

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In spite of the success [1, 2, 3] of the conventional renormalization group “Altarelli-Parisi” equations (RGE) [4] much work has focused recently on replacing or modifying them by small $x$ resummations. It should be stressed that, so far, this is not required by the experimental data. The rise of $F_2$ at small $x$, for example, has been predicted by the GRV group [3, 4] using the NLO RGE and thus is not a unique sign of the BFKL [7] type of growth. By using the BFKL evolution methods of Askew et al. [8] consistently, we did even show in a previous paper [9] that it is probable that a pure LO BFKL evolution is not compatible with the observed growth in the $F_2(x, Q^2)$ data. We confirm this here using more recent data from the DESY HERA collider [2, 3].

But this does not necessarily prove that the BFKL resummation of leading logarithms in $x$ is inadequate. The original resummation ignores the quarks and uses a fixed strong coupling constant. Thus it is expected to be relevant only in the asymptotic small $x$ limit and it could still be valid at very small values of $x$ that are currently not accessible by experiment. There is also the possibility that the corresponding NLO resummations will change the LO behaviour and that the assumption of gluon dominance limits the region of applicability strongly.

Keeping this in mind the “high energy” ($s \gg Q^2$ leading to small $x$ via $x \sim Q^2/s$) resummations done by Catani et al. [10] seem to offer a way out. They treat the BFKL type resummations as additions to the fixed order anomalous dimensions and Wilson coefficients, hoping to obtain an improved perturbative expansion that way. They also resummed the NLO small $x$ contributions in the quark sector. Furthermore, they calculated all resummations consistently by using the $k_T$-factorization and showed that their methods do not spoil the all-order factorization of collinear singularities. We note that by recalculating the LO resummations they have confirmed the known BFKL results.

The obvious question is then of course if the “improved” expansion can match the success of the conventional RGE. First comparisons with $F_2$ data using these methods were made by Ellis et al. [11]. They stressed the importance of enforcing the fundamental
energy-momentum conservation which is generally broken by the resummations. Starting
the evolution of flat input densities from $Q_0^2 = 4 \text{ GeV}^2$, they can describe the $F_2$ data
fairly well. In this paper we will show that the agreement is spoiled when the starting
scale is lowered to values typical for recent RGE fits, in order to take into account all
presently available HERA data.

First we will present an update of the analysis of $F_2$ data using the LO BFKL evolution
of the gluon. We give now a brief outline of the methods employed in [9], noting that the
references for the formulae and methods used can also be found there. The BFKL equation
concerns the unintegrated gluon density $f(x, k^2)$, because the transverse momentum $k^2$
of the gluons in the corresponding ladder diagram is not strongly ordered as in the LO
RGE case and cannot be integrated out.

The connection between the integrated and unintegrated gluon is given by the equations

$$x g(x, Q^2) = \int_0^{Q^2} \frac{dk^2}{k^2} f(x, k^2), \quad (1a)$$

$$f(x, k^2) = \frac{\partial x g(x, Q^2)}{\partial \ln Q^2} \bigg|_{Q^2=k^2}, \quad (1b)$$

and in terms of the unintegrated gluon one can then write the BFKL equations as an
 evolution equation in $x$

$$-x \frac{\partial f(x, k^2)}{\partial x} = \frac{3 \alpha_s(k^2)}{\pi} k^2 \int \frac{dk'^2}{k'^2} \left[ \frac{f(x, k'^2) - f(x, k^2)}{|k'^2 - k^2|} + \frac{f(x, k^2)}{(4 k^4 + k'^4)^{\frac{3}{2}}} \right]. \quad (2)$$

Predictions for $F_2$ are then made by convoluting the BFKL gluon with the photon-
gluon fusion quark box $F^{(0)}$ using the $k_T$-factorization theorem

$$F_i(x, Q^2) = \int \frac{dk'^2}{k'^4} \int_{x}^{1} \frac{dy}{y} f \left( \frac{x}{y}, k'^2 \right) F^{(0)}_i(y, k'^2, Q^2), \quad (3)$$

with $i = T, L$ denoting the transverse and longitudinal parts, respectively. Since we
have completely ignored the quark sector and non-perturbative effects, we will add a
background fitted to the data. Obviously the background should not dominate over the
BFKL contribution and it also should not have a strong \(x\) dependence that would mask the BFKL behaviour. We use a soft Pomeron ansatz \(C_{IP} x^{-0.08}\), where the constant \(C_{IP}\) is fitted separately for each \(Q^2\) bin. It is well known that the BFKL equation has the same double logarithmic limit as the LO RGE, when its (fixed) coupling constant \(\alpha_s\) is replaced by the standard running one of the LO RGE. Since we want to connect to the LO RGE at large \(x\), this replacement has to be made in (2).

LO RGE gluon distributions are then used at \(x_0 = 10^{-2}\) as input via Eq. (1b). We now have to face the difficulty that (2) requires knowing \(f(x, k^2)\) down to \(k^2 = 0\) GeV\(^2\), but of course even the special LO RGE gluons \(g^{RGE}(x, Q^2)\) used in [9] do not extend far enough down in \(Q^2\) to supply that information. Thus we are forced to make an ansatz for the infrared (IR) region, if we want to avoid cutoffs to the integral. The introduced running coupling also needs to be taken care of in the region of small \(k^2\). Following Askew et al. [8], the IR ansatz of [9] is

\[
f(x, k^2) = \frac{k^2 + k_a^2}{k_c^2 + k_a^2} f(x, k_c^2) \quad \text{for} \quad k^2 < k_c^2.
\]

(4)

At \(x = x_0\) we additionally shift the gluon

\[
f^{RGE}(x_0, k^2) \rightarrow f^{RGE}(x_0, k^2 + k_a^2).
\]

(5)

The strong coupling is frozen: \(\alpha_s(k^2) \rightarrow \alpha_s(k^2 + k_b^2)\). The ansatz is scrutinized closely in [9]. We just want to remind the reader of the salient points: old gluon distributions have to be used, since most of the more recent ones already create a steep growth of \(F_2\) in the low \(x\) region without any additional BFKL boost. The dependence of \(F_2\) on the IR parameters is under control, except for \(k_a^2\) which has to be fixed by forcing the integrated gluon of (3) to match the original gluon \(g^{RGE}\) at medium to high \(Q^2\). For the \(D_0\)-type gluon used below we obtain \(k_a^2 = 0.95\) GeV\(^2\) [9]. The strong BFKL growth of \(F_2\) could be suppressed [8] by increasing \(k_a^2\), but this would of course spoil the consistency with the RGE gluon.

Parametrizations of our results were given in [9]. Using the \(D_0\)-type gluon parametrization and fitting the background to the new data below \(x = 10^{-2}\) from the HERA collab-
orations [2, 3] at DESY as well as to data from the Fermilab E665 experiment [12], we obtain the solid curve in Fig. 1. The background contribution is shown as dashed curve. Note that at $Q^2 = 3.5 \text{ GeV}^2$ and at $Q^2 = 6.5 \text{ GeV}^2$ $C_{IP}$ would be negative and is set to zero by hand. It is obvious that in spite of having the freedom of fitting $C_{IP}$, the data can not be described by the BFKL curves. The steep growth of $F_2 \sim x^{-0.5}$ predicted by the BFKL equation is simply too strong, especially at medium to low $Q^2$. Thanks to the precision of the new $F_2$ data pure BFKL evolution is now definitely ruled out!

Of course immediately the question arises if this can be ameliorated by using modified evolution equations that incorporate the successful RGE. That the NLO RGE are very successful indeed at describing the $F_2$ data is made obvious by the dotted line in Fig. 1 which shows the R1 fit of MRS [13]. We are especially interested in elucidating the question if modified equations can match the success of the NLO RGE in the region of low to medium $Q^2$. The fact that even below $Q^2 = 4 \text{ GeV}^2$ data can be described using those equations has been pioneered by the GRV group [5, 6] and is now becoming universally accepted [2, 3, 13, 14]. This region is also expected to lead to difficulties with the resummed BFKL pole at $\lambda = \frac{3\alpha_s}{\pi} \frac{4}{\ln 2}$, because $\alpha_s$ increases strongly in this region if it runs.

We have tried the method employed by Forshaw et al. [15]. They limit their calculations to the LO small $x$ resummations of the gluon by replacing the standard anomalous dimension in the RGE by the resummed one and by ignoring the quarks. The (unsuccessful) results of trying to extend this method down to lower $Q^2$ will be reported in a forthcoming detailed article [16]. Instead we concentrate here on the method developed by Ellis et al. [11]. Following the work of Catani et al. [10] they add the known LO and NLO resummation corrections to the NLO RGE singlet anomalous dimensions and to the Wilson coefficients. Note that the corrections to the non-singlet anomalous dimensions are less singular than $1/x$ for $x \to 0$ and need not be taken into account. We use the Mellin transformation with respect to $x$ in order to express the results in terms of moments

$$f(n, Q^2) = \int_0^1 dx \ x^n \ f(x, Q^2),$$

(6)
where the Mellin moment is shifted by one compared to the usual convention \(x^{n-1}\), since this will allow a more compact notation in the following.

To the fixed order expansion of the anomalous dimensions

\[
\gamma(n, \alpha_s) = \frac{\alpha_s}{2\pi} \gamma^{(0)}(n) + \left(\frac{\alpha_s}{2\pi}\right)^2 \gamma^{(1)}(n) + \mathcal{O}(\alpha_s^3)
\]  

(7)
towers of small-\(x\) resummation corrections are now added. In LO, that is summing terms of the order \(\alpha_s/n^k\), only corrections to the anomalous dimensions \(\gamma_{g\Sigma}\) and \(\gamma_{gg}\) exist, where \(\Sigma \equiv \sum_q (q + \bar{q})\) denotes the singlet quark density. In NLO the corrections to \(\gamma_{\Sigma\Sigma}\) and to \(\gamma_{\Sigma g}\) are known, but those of the gluon have not yet been completed. The additional resummation corrections can thus compactly be written as

\[
\hat{\gamma}_{\text{res.}} = \begin{pmatrix} \gamma_{\Sigma\Sigma} & \gamma_{\Sigma g} \\ \gamma_{g\Sigma} & \gamma_{gg} \end{pmatrix}_{\text{res.}} = \begin{pmatrix} 0 & 0 \\ C_F\gamma_L & \gamma_L \end{pmatrix} + \left(2n_f\frac{C_F}{C_A} [\gamma_{NL} - \frac{2\alpha_s}{\pi} T_R N_f] \right) + \mathcal{O}\left(\alpha_s^2 \left(\frac{\alpha_s}{n}\right)^k\right),
\]

(8)
with \(k > 0\), and the question marks signify the yet unknown entries.

To obtain the BFKL gluon anomalous dimension \(\gamma_L\) from

\[
n = \bar{\alpha}_s \chi(\gamma_L); \quad \bar{\alpha}_s \equiv \frac{C_A}{\pi} \alpha_s; \quad \chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma),
\]  

(9)
we use a complex Newton method. For the NLO resummation obviously the choice of scheme is important. Since up to now completely resummed expressions are only available in the DIS factorization scheme, we use this scheme in the following.

The DIS scheme is defined by \(C^S_2 = 1\) and \(C^g_2 = 0\), thus all the resummation corrections are shifted into the anomalous dimensions. Then we can write

\[
\gamma_{NL} = \frac{1}{2n_f} h_2(\gamma_L, n) R(\gamma_L),
\]  

(10)
\[
h_2(\gamma, n) = \frac{h_2(\gamma) (1 + \mathcal{O}(n))}{\text{for} \quad n \to 0},
\]  

(11)
\[
h_2(\gamma) = 2n_f\frac{\alpha_s}{2\pi} T_R \frac{2 + 3\gamma - 3\gamma^2}{3 - 2\gamma} \left(\frac{\pi^2 \gamma^2}{1 - 4\gamma^2 \sin(\pi\gamma) \tan(\pi\gamma)}\right),
\]  

(12)
\[ R(\gamma) = \left\{ \frac{\Gamma(1 - \gamma) \chi(\gamma)}{\Gamma(1 + \gamma) [-\gamma \chi'(\gamma)]} \right\}^{\frac{1}{2}} \exp \left\{ \gamma \psi(1) + \int_{0}^{\gamma} d\gamma \frac{\psi'(1) - \psi'(1 - \gamma)}{\chi(\gamma)} \right\}. \tag{13} \]

Here the prime denotes the first derivative with respect to \( \gamma \). The expressions can be directly calculated once \( \gamma_L \) has been determined.

It is a non-trivial check of the work of Catani et al. that by expanding the small \( x \) resummations in \( \overline{\alpha}_s/n \) one obtains the same constant respectively \( 1/n \) singular terms as in the (NLO) RGE in the \( n \to 0 \) limit. To avoid double-counting, these terms have to be subtracted before the additional resummations are added. Effectively, Eq. (8) is then replaced by

\[ \hat{\gamma}_{\text{res.}} = \begin{pmatrix} 2 n_f \frac{C_F}{C_A} \gamma_g \gamma_q \\ 2 n_f \gamma_q \\ \gamma_g \end{pmatrix} \tag{14} \]

\[ \gamma_g = \gamma_L - \frac{\alpha_s}{n} \tag{15} \]

\[ \gamma_q = \left[ h_2(\gamma_L) R(\gamma_L) - \frac{\alpha_s}{2 \pi} T_R \frac{2}{3} \left( 1 + \frac{13}{6} \frac{\alpha_s}{n} \right) \right]. \tag{16} \]

Here we can see clearly that the corrections to the splitting into gluons and quarks just differ by a colour factor. It should also be pointed out, that the first terms in the expansions of the effective corrections are

\[ \gamma_g = 2 \zeta(3) \left( \frac{\alpha_s}{n} \right)^4 + \mathcal{O} \left( \left( \frac{\alpha_s}{n} \right)^6 \right), \tag{17} \]

\[ \gamma_q = \frac{\alpha_s}{2 \pi} T_R \frac{2}{3} \left[ \left( \frac{71}{18} - \zeta(2) \right) \left( \frac{\alpha_s}{n} \right)^2 + \mathcal{O} \left( \left( \frac{\alpha_s}{n} \right)^3 \right) \right] \tag{18} \]

so that the correction in the quark sector leads in terms of powers of \( \alpha_s \) and also starts with the lower power of \( 1/n \). Thus we expect the quark correction to play an important role despite its being formally subleading. The quark correction is also expected to vanish more slowly for \( x \to 1 \), because it does not fall off as steeply when \( n \to \infty \).

It is of course entirely possible that the NLO order corrections in the gluon sector change those expectations, since they depend on the fact that the second and third coefficients in the expansion of \( \gamma_L \) vanish. A further problem is that the added corrections violate the fundamental energy-momentum conservation, as can be easily checked.
by setting $n \to 1$. The calculation of higher resummation corrections will not solve this problem, since the LO corrections cannot be cancelled. Of course the original RGE manifestly obey energy-momentum conversation. This is not only a theoretical problem, since the unconserved growth is very strong and leads to obviously false results. We will look at the simplest ways to restore the conservation of energy and momentum. The “hard” way \cite{11} is simply to subtract the troublesome values of the corrections via $\hat{\gamma}_{\text{res.}}(n) \to \hat{\gamma}_{\text{res.}}(n) - \hat{\gamma}_{\text{res.}}(1)$. As has already been remarked in \cite{11} this does not much reduce the growth and it has turned out to be impossible to fit the experimental $F_2$ data using this approach. Thus we will limit ourselves to the second “soft” method, multiplying the corrections by a factor that goes to zero for $n \to 1$.

This conserving factor, being introduced in an entirely ad hoc way, should go to unity in the limit of very small $n$ which corresponds to small $x$. We test the following factors: $(1-n)$, $(1-n)^2$ and $(1-2n+n^3)$. They have been studied previously in \cite{17} using higher starting scales $Q_0^2$. Obviously these factors will inhibit the growth more strongly, since they weaken the singularities as can be seen by looking at the expansions (17) and (18). It should also be noted that multiplying the quark correction by the third factor leads to a term $\sim n$. This means that the correction times this factor cannot be transformed back into $x$-space on its own. Instead it needs a parton distribution that falls off at least with $1/n$. While this is fulfilled for all reasonable parton distributions, it is still a hint that the parton distributions are not correctly factorized anymore. For this reason we do not examine even higher powers of $n$ in the conserving factor. By checking this range of factors we also get an estimate of the influence of the missing higher order corrections, since they will also be suppressed in powers of $1/n$ versus powers of $\alpha_s$.

The RGE are only known to two loops and can be conveniently solved in $n$-space. We wish to modify the known NLO RGE $n$-space solution \cite{3, 18}, but then it is necessary to check if the corrections can be added to the two-loop solution consistently. Since we only
know the running coupling to two loops, we can write the evolution equation as

\[
\frac{d\vec{q}(n,Q^2)}{d\alpha_s} = \left[ \frac{\alpha_s}{2\pi} \hat{\gamma}^{(0)}(n) + \left( \frac{\alpha_s}{2\pi} \right)^2 \hat{\gamma}^{(1)}(n) + \hat{\gamma}_{\text{res.}} \right] \frac{1}{-\frac{\beta_0}{4\pi} \alpha_s^2 - \frac{\beta_1}{(4\pi)^2} \alpha_s^3} \vec{q}(n,Q^2),
\]

(19)

using for the strong coupling \((n_f)\) is the number of active massless quark flavours)

\[
\frac{\alpha_s(Q^2)}{4\pi} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} - \frac{\beta_1 \ln \ln(Q^2/\Lambda^2)}{\beta_0^3 [\ln(Q^2/\Lambda^2)]^2}, \quad \beta_0 = 11 - \frac{2}{3} n_f, \quad \beta_1 = 102 - \frac{38}{3} n_f.
\]

(20)

In [16] it is shown that a consistent addition of the corrections can be accomplished by replacing

\[
\hat{\gamma}^{(1)}(n) \rightarrow \hat{\gamma}^{(1)}(n) + \frac{\pi \beta_0}{\alpha_s(Q^2) - \alpha_s(Q_0^2)} \int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} d\alpha \frac{\hat{\gamma}_{\text{res.}}(n,\alpha)}{4\pi \alpha^2 + \frac{\beta_1}{(4\pi)^2} \alpha^3}.
\]

(21)

We have tested that by using this method we can reproduce the results shown in [11].

The dashed curve of Fig. 3 shows the strong effects of the added resummations compared to the solid curve of the conventional RGE. For both curves the unchanged R1 input distributions at \(Q_0^2\) were used and the resummations were added using the conserving factor \((1-n)\). Obviously it is necessary to refit the input parton distributions, since the growth in the small \(x\) region is incompatible with the data. It is interesting to note the similarity of that growth to the LO BFKL one shown in Fig. 3 as dotted line. This is a first hint that fitting experimental data with the new method will lead to the same problems at small \(x\). In order to test if the RGE plus corrections can fit the experimental data as well as the pure RGE, we would in principle need to fit to the same range of experiments as is done for example in [13]. Even if the resummation corrections to all those processes were known, the computing time would still be prohibitive due to the complicated calculation of the corrections.

Instead we use the following approach to save computing time: we take the optimal MRS R1 [13] fit as the basis for our fit by using the same starting scale \(Q_0^2\), \(\Lambda_{QCD}(n_f = 4)\) and the same valence distributions for the quarks. Since the R1 fit describes the HERA data very well, we use it with the unmodified RGE to generate \(F_2\) points for our fit. Below \(x = 5 \cdot 10^{-2}\) the errors and \(x\)-range used for the points correspond to those of the HERA data.
data. Above this value of $x$ we force our fit to reproduce the R1 fit by setting an artificial error of one percent. In this large $x$ region not only $F_2$ but also the sea and the gluon are tested. This simulates the numerous experiments constraining the parton distributions at large $x$ which are taken into account by the R1 fit. Since we do our calculations in the DIS scheme, we have to apply the corrections for the change of scheme to the $\overline{\text{MS}}$ partons of R1. This is done at the starting scale $Q_0^2 = 1 \text{ GeV}^2$ and then the RGE in the DIS scheme are used to obtain the constraints. In fact the fitting program uses the same transformation, so that the new fitted parameters are valid in the $\overline{\text{MS}}$ scheme and can be directly compared to those of the MRS R1 fit.

It turns out that the modified evolution equations cannot reproduce well the RGE sea in the large $x$ region. The gluon and $F_2$ are reproduced with reasonable accuracy. But the sea is less and less in agreement when the maximum power of $n$ in the conserving factor is raised. This can be easily understood by noting that the product of the conserving factor with the expansion (18) does not fall off as quickly for $n \to \infty$ as the product with (17) does. Relaxing the strong constraints at large $x$ is not a viable solution to the problem. This is so because, as we will see, the $F_2$ data at small $x$ are not well described by the modified evolution equations. Thus once the tight constraints in the large $x$ regions are removed, the fit program adjusts the parton distributions for the small $x$ region and basically ignores any weak constraints imposed at large $x$. This leads to strong deviations especially of the sea from the original R1 partons. Since here our goal is not to give an alternative description of the small $x$ data, but rather to check if modified evolution equations can match the RGE in describing all data, we cannot avoid imposing the constraints at large $x$.

The results of our fits are displayed in Fig. 3. The curve denoted as “no conservation” is the fit without conservation of energy and momentum, corresponding to the original ideas of Catani et al. [10]. The one described as “shift” is a fit using the conserving factor $(1 - n)$ but also shifting the strong coupling using $\alpha_s(Q^2) \to \alpha_s(Q^2 + 1.5 \text{ GeV}^2)$. We note that the fit using the conserving factor $(1 - 2 n + n^3)$ is not displayed. It leads to
an unacceptable fit which oscillates around the low $x$ data strongly, especially at the low $Q^2$ values. This confirms our earlier suspicions about possible problems with this factor. The “no conservation” curve shows that the unconserved growth is way too strong to be in accord with experimental data.

It is also easy to see that increasing the maximum power of $n$ in the conserving factor leads to the expected decrease of the growth in the small $x$ region. Still it is obvious that the curve corresponding to the $(1 - n)$ factor cannot describe the data except perhaps at the highest $Q^2$ bin. The curve pertaining to the $(1 - n)^2$ factor offers some improvement. This is expected due to the better cancellation of the leading term in the added quark resummation (18) for $n < 1$ which leads to a smaller deviation from the standard RGE. But the small $x$ region below 12 GeV$^2$ is still not fitted well. That this is caused by the strong BFKL like growth at small $x$ is especially obvious at 1.5 GeV$^2$. Here we can see that the fit has adjusted the parton distributions so that the curves at medium to low $x$ are below the data to compensate the growth at very small $x$ somewhat.

Looking at the “dips” in the curves we can also clearly see that the onset of the growth is shifted towards lower $x$ when the power of $n$ in the conserving factor is increased. We can conclude from these fits that the effects of changing the conserving factor are sizable in the small $x$ region and that no definite predictions on $F_2$ at small $x$ can be made for the time being using modified evolution equations. On the other hand the observed strong rise at small $x$ could be taken as an indication that the low $Q^2$ region will not be better described once all NLO terms of the resummations compatible with the two-loop RGE are known\footnote{1}. The BFKL like growth simply is too strong for low $Q^2$ and spoils agreement with data (rather flat in $x$) whenever it becomes dominant. This is shown nicely by the curve labeled “shift” which is generated using a shifted value of $\alpha_s$. Obviously the fit to the data is much improved at low $Q^2$ since the BFKL growth is suppressed due to the artificially lowered value of $\alpha_s$.

1 However higher order terms introduced by a purely kinematic constraint on the gluon ladder suggest that the growth of the gluon could be reduced \footnote{2}. 

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All our attempts to describe experimental data of $F_2$ using methods inspired by the BFKL equations have failed. The direct application of the BFKL equation in the small $x$ region is now finally ruled out by experimental data [2, 3, 12]. Adding the small $x$ resummations to the RGE does not lead to a conflict with data, if the parton distributions are adjusted accordingly and provided the starting scale $Q^2_0$ is kept around 4 GeV$^2$ [11], i. e. if only a fraction of all available data is taken into account. But it is by now widely accepted that it is possible to describe data below this $Q^2_0$ scale using the NLO RGE. The modified evolution equations fail this acid test. This conclusion can be drawn despite the apparent difficulty to obtain any definite predictions using these equations at all.

A further indication of the reliability of our results is provided by a similar analysis of Ball and Forte [20]. They also found that deviations from the standard RGE are not supported by recent HERA data, although their method differs in some respects from ours. Notably they switch on the small $x$ resummations at a fitted $x_0$, they estimate the missing NLO gluon small $x$ contributions by using energy-momentum conservation, and refit only the small $x$ tails of given parton distributions.

There is a common reason for these failures. At low $Q^2$ the BFKL pole $\lambda = \frac{3\alpha_s}{\pi} 4 \ln 2$ dominates over all the other poles since it moves strongly along the positive real axis in the $n$ plane due to the growth of $\alpha_s$. The validity of this argument is nicely demonstrated by the curve labeled “shift” in Fig. [3]. Simply calculating the values for $\lambda$ at $Q^2 = 4$ GeV$^2$ and at $Q^2 = 1$ GeV$^2$ suggests that any future method that includes this pole in the evolution is likely to have trouble to describe the moderate rise in $x$ of $F_2(x, Q^2)$ observed in the experimental data at lower values of $Q^2$.

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Figure Captions

Fig. 1 BFKL predictions using the methods of [9] compared with recent data [2, 3, 12].
The solid curve is the LO BFKL prediction including the fitted background which
is shown separately as dashed curve. Only data with $x \leq 10^{-2}$ is fitted. The dotted
curve shows the conventional RGE fit R1 of MRS [13] for comparison.

FIG. 2 The original MRS-R1 input [13] evolved with the RGE (solid curve) and the
RGE plus resummations (dashed curve). The resummations are added using the
conserving factor $(1 - n)$. For comparison the LO BFKL prediction is shown as
dotted curve. Note that “R1 (res.)” is not fitted to the shown data [3].

Fig. 3 Modified evolution predictions, based on the RGE supplemented by the small $x$
resummations according to Eqs. (14) and (21), compared with experiment (data as
in Fig. 1). The stars show the generated data used for the fit as explained in the
text. The solid and dot-dashed curves are obtained with $(1 - n)$ as conserving factor,
using a shifted $\alpha_s$ in the latter case as discussed in the text, and the dashed curve
is calculated with $(1 - n)^2$. The dotted curve is obtained without any conserving
factor, which corresponds to the original results of [10].
Fig. 1

$Q^2 = 1.5 \text{ GeV}^2$

- H1 '93/'94
- ZEUS '93/'94
- E665

- BFKL
- Bckgr.
- R1 (RGE)
Fig. 2
Fig. 3