The norm of pre-Schwarzian derivative on subclasses of bi-univalent functions

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Abstract: In the present paper, we give the best estimates for the norm of pre-Schwarzian derivatives
\[ ||T_f(z)|| = \sup_{|z| < 1} \left| \frac{f''(z)}{f'(z)} \right| \] for subclasses of bi-univalent functions.

Keywords: bi-univalent functions; bi-starlike functions; subordination; pre-Schwarzian derivatives

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1. Introduction and definitions

Let \( \mathcal{A} \) be the class of functions \( f \) of the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \] (1.1)
which are analytic in the open unit disk \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). An analytic function in a domain \( D \) is said to be univalent in \( D \) if it does not take the same value twice i.e, \( f(z_1) \neq f(z_2) \) for all pairs of distinct points \( z_1 \) and \( z_2 \) in \( D \).

The Koebe one-quarter theorem et al [3] ensures that the image of \( \Delta \) under every univalent function \( f \in \mathcal{A} \) contains the disk with the center at origin and of the radius \( 1/4 \). Thus, every univalent function \( f \in \mathcal{A} \) has an inverse \( f^{-1} : f(\Delta) \to \Delta \), satisfying \( f^{-1}(f(z)) = z, \ (z \in \Delta) \) and
\[ f \left( f^{-1}(w) \right) = w \quad \left( |w| < r_0(f); \ r_0(f) \geq \frac{1}{4} \right). \]

Moreover, it is easy to see that the inverse function has the series expansion of the form
\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ...; \ w \in \Delta, \]
which implies that $f^{-1}$ is analytic. The derivative of $f^{-1}$ (see pp. 1038 [4]) is given by

$$\frac{d}{dw} (f^{-1}(w)) = \frac{1}{f'(z)}.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. We denote the class of bi-univalent functions by $\sigma$. (see [2])

The function $f$ in class $\mathcal{A}$ is said to be starlike of order $\alpha$ where $0 \leq \alpha < 1$ in $\Delta$ if it satisfies the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,$$

where $z \in \Delta$. We denote the class of starlike functions of order $\alpha$ by $S^*(\alpha)$. The function $f$ of the form (1) is said to be bi-starlike function of order $\alpha$ where $0 \leq \alpha < 1$ if each of the following conditions are satisfied

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$$

and

$$\text{Re}\left\{\frac{wg'(w)}{g(w)}\right\} > \alpha,$$

where $f \in \sigma$, $g = f^{-1}$ and $w = f(z)$. We denote the class of bi-starlike functions of order $\alpha$ by $S^* \sigma[\alpha]$. If $f$ and $g$ are analytic functions in $\Delta$, we say that $f$ is subordinate to $g$, written as $f \prec g$, if there exists a Schwarz function $w$ analytic in $\Delta$, with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \Delta$), such that $f(z) = g(w(z))$. In particular, when $g$ is univalent then the above definition reduces to $f(0) = 0$ and $f(\Delta) \subseteq g(\Delta)$.

The pre-Schwarzian derivative of $f$ is denoted by

$$T_f(z) = \frac{f''(z)}{f'(z)}$$

and its norm is given by

$$\|T_f\| = \sup_{|z|<1} (1 - |z|^2) \left|\frac{f''(z)}{f'(z)}\right|.$$

This norm have a significant meaning in the theory of Teichmuller spaces. For a univalent function $f$ it is well known that $\|T_f\| < 6$. This is the best possible estimation.

Defining two subclasses for bi-univalent functions as follows

**Definition 1.1.** A function $f$ given by (1) is said to be in the class $S^* \sigma[\alpha, A, B]$, if the following conditions are satisfied

$$\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz},$$

$$\frac{wg'(w)}{g(w)} < \frac{1 + Aw}{1 + Bw},$$

where $f \in \sigma$, $g = f^{-1}$, $w = f(z)$, $w \in \Delta$ and $-1 \leq B < A \leq 1$.

**Remark 1.1.** If we take $A = (1 - 2\alpha)$ and $B = -1$ in the above Definition 1.1 where $0 \leq \alpha < 1$, the class becomes $S^* \sigma[(1 - 2\alpha), -1] \equiv S^* \sigma[\alpha]$.
Remark 1.2. If we take $A = 1$ and $B = -1$ in the above Definition 1.1, the class becomes $S^*_\sigma[1, -1] \equiv S^*_\sigma$.

Definition 1.2. A function $f$ given by (1) is said to be in the class $V^*_\sigma[A, B]$, if the following conditions are satisfied

$$
\left( \frac{z}{f(z)} \right)^2 f'(z) < \frac{1 + Az}{1 + Bz},
$$

$$
\left( \frac{w}{g(w)} \right)^2 g'(w) < \frac{1 + Aw}{1 + Bw},
$$

where $f \in \sigma$, $g = f^{-1}$, $w = f(z)$, $w \in \Delta$ and $-1 \leq B < A \leq 1$.

Remark 1.3. If we take $A = (1 - 2\alpha)$ and $B = -1$ in the above Definition 1.2 where $0 \leq \alpha < 1$, the class becomes $V^*_\sigma[(1 - 2\alpha), -1] \equiv V^*_\sigma(\alpha)$.

Remark 1.4. If we take $A = 1$ and $B = -1$ in the above Definition 1.2, the class becomes $V^*_\sigma[1, -1] \equiv V^*_\sigma$.

In this paper, we shall give the best norm estimation for the classes $S^*_\sigma[A, B]$ and $V^*_\sigma[A, B]$.

2. Main result

Theorem 2.1. Let the function $f$ given by (1) be in the class $f \in S^*_\sigma[A, B]$, then

$$
\|T_f\| \leq \min \left\{ \frac{2(A - B)(A + 2)}{(A + 1)}, \frac{2(A - B)|A|}{(A + 1)} \right\}.
$$

Proof. Since $f \in S^*_\sigma[A, B]$, let us assume that

$$
h(z) = \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} = p(z).
$$

Using the definition of subordination, there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, such that

$$
h(z) = p \circ \phi(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}.
$$

Hence, $h(z)$ becomes

$$
h(z) = \frac{zf'(z)}{f(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)}.
$$

By logarithmic differentiation of (3), we get

$$
\frac{1}{z} + \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} = \frac{A}{(1 + A\phi(z))} - \frac{B}{(1 + B\phi(z))}.
$$

Above equation gives us the pre-Schwarzian derivative of $f$, i.e,

$$
T_f(z) = \frac{f''(z)}{f'(z)} = \frac{(A^2 - AB)\phi(z) + 2(A - B)}{(1 + A\phi(z))(1 + B\phi(z))}.
$$
Setting \( \phi(z) = id_\Delta \) (as \( \phi \) belongs to the class of Schwarz functions and \( \phi(z) < z \) on \( \Delta \)) and rearranging the terms, we get
\[
(1 - |z|^2) |T_f(z)| = (1 - |z|^2) \left| \frac{(A^2 - AB)z + 2(A - B)}{(1 + A|z|)(1 + B|z|)} \right|.
\]
Taking the supremum value both sides in the unit disc, the above equation becomes
\[
\sup_{|z| < 1} (1 - |z|^2) |T_f(z)| \leq \sup_{|z| < 1} \left(1 - |z|^2\right) \left| \frac{(A^2 - AB)|z| + 2(A - B)}{(1 + A|z|)(1 + B|z|)} \right|.
\]
As \(-1 \leq B\), we get \((1 - |z|) \leq (1 + B|z|)\), therefore the above inequality becomes
\[
\sup_{|z| < 1} (1 - |z|^2) |T_f(z)| \leq \sup_{|z| < 1} (1 + |z|) \left| \frac{(A^2 - AB)|z| + 2(A - B)}{(1 + A|z|)} \right|.
\]
The above inequality gives us the norm of pre-Schwarzian derivative of \( f \), denoted by \( \|T_f\| \). To estimate the upper bound of \( \|T_f\| \) in the unit disc \( \Delta \), \( z \) must lead to 1 and therefore
\[
\lim_{z \to 1} (1 + |z|) \left| \frac{(A^2 - AB)|z| + 2(A - B)}{(1 + A|z|)} \right| = \frac{2(A - B)(A + 2)}{(A + 1)}.
\]
Finally we get
\[
\|T_f\| \leq \frac{2(A - B)(A + 2)}{(A + 1)}.
\] (2.2)
For the second part of the proof, let us assume that
\[
k(z) = \frac{w g'(w)}{g(w)} < \frac{1 + Aw}{1 + Bw} = p(w)
\]
where \( z = f^{-1}(w) = g(w) \). By definition of subordination, there exists a Schwarz function \( \phi : \Delta \to \Delta \) with \( \phi(0) = 0 \) and \( |\phi(z)| < 1 \), such that
\[
k(z) = p \circ \phi(z) = \frac{1 + A \phi(z)}{1 + B \phi(z)}.
\]
Since \( f \in \sigma \), both \( f \) and \( f^{-1} \) are analytic and univalent in \( \Delta \). The derivative of \( f^{-1} \) is given by
\[
\frac{d(f^{-1}(w))}{w} = \frac{1}{f'(z)}.
\]
Therefore, \( k(z) \) can be expressed as
\[
\frac{w g'(w)}{g(w)} = k(z) = \frac{f(z)}{zf'(z)} = \frac{1 + A \phi(z)}{1 + B \phi(z)}.
\] (2.3)
Taking logarithmic differentiation of (5), we get
\[
\frac{f'(z)}{f(z)} - \frac{1}{z} - \frac{f''(z)}{f'(z)} = \frac{A \phi'(z)}{(1 + A \phi(z))} - \frac{B \phi'(z)}{(1 + B \phi(z))}.
\]
Setting $\phi(z) = id_\Delta$ (as $\phi$ belongs to the class of Schwarz functions and $\phi(z) < z$ on $\Delta$), we have

$$\frac{f''(z)}{f'(z)} = \frac{Az(A-B)}{(1+Az)(1+Bz)}.$$  

Following the previous steps and using $(1-|z|) \leq (1+B|z|)$, we get

$$\sup_{|z|<1} (1-|z|^2)|T_f(z)| \leq \sup_{|z|<1} (1+|z|) \left[ \frac{|A|(A-B)|z|}{(1+A|z|)} \right].$$

For upper bound of $\|T_f\|$, $z$ must lead to 1, i.e,

$$\lim_{z \to 1} (1+|z|) \frac{|A|(A-B)|z|}{(1+A|z|)} = \frac{2(A-B)|A|}{(A+1)}.$$  

Therefore

$$\|T_f\| \leq \frac{2(A-B)|A|}{(A+1)}. \quad (2.4)$$

Combining (4) and (6), the proof is complete. 

Let $A = (1-2\alpha)$ and $B = -1$ in the above theorem where $0 \leq \alpha < 1$, the class becomes $S^*_\alpha[(1-2\alpha), -1] \equiv S^*_\alpha(\alpha)$.

**Corollary 2.1.** If $f \in S^*_\alpha(\alpha)$, then $\|T_f\| \leq \min\{6-4\alpha, |2-4\alpha|\}$. 

If $f$ is analytic and locally univalent in $\Delta$ such that $f \in S^*_\alpha(\alpha)$, where $0 \leq \alpha < 1$ then $\|T_f\| \leq (6-4\alpha)$, which is due to Yamashita [1]. The above corollary generalizes the result for bi-univalent functions.

Let $A = 1$ and $B = -1$ in the above theorem, the class becomes $S^*_\alpha[1, -1] \equiv S^*_\alpha$.

**Corollary 2.2.** If $f \in S^*_\alpha$, then $\|T_f\| \leq 6$.

The above corollary generalizes the norm estimation for bi-univalent functions.

**Theorem 2.2.** Let the function $f(z)$ given by (1) be in the class $V^*_\alpha[A, B]$, then

$$\|T_f\| \leq \min \left\{ \frac{2(3+2A)(A-B)}{(A+1)}, \frac{2(A-B)(1+2A)}{(A+1)} \right\}.$$

**Proof.** Since $f \in V^*_\alpha[A, B]$, let us assume that

$$k(z) = \left( \frac{z}{f(z)} \right)^2 f'(z) = \frac{1+Az}{1+Bz} = p(z).$$

Therefore, there exists a Schwarz function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$k(z) = p \circ \phi(z) = \frac{1+A\phi(z)}{1+B\phi(z)}.$$

Therefore, $k(z)$ can be expressed as

$$k(z) = \left( \frac{z}{f(z)} \right)^2 f'(z) = \frac{1+A\phi(z)}{1+B\phi(z)}. \quad (2.5)$$
By logarithmic differentiation on (7), we get
\[
\frac{2}{z} - \frac{2f'(z)}{f(z)} + \frac{f''(z)}{f'(z)} = \frac{A\phi'(z)}{(1 + A\phi(z))} - \frac{B\phi'(z)}{(1 + B\phi(z))}.
\]
\[
\frac{f''(z)}{f'(z)} = \frac{2f'(z)}{f(z)} - \frac{2}{z} + \frac{A\phi'(z)}{(1 + A\phi(z))} - \frac{B\phi'(z)}{(1 + B\phi(z))}.
\]
Hence, the pre-Schwarzian derivative of \( f \) becomes
\[
T_f(z) = \frac{f''(z)}{f'(z)} = \frac{2(1 + A\phi(z))}{z(1 + B\phi(z))} - \frac{2}{z} + \frac{A\phi'(z)}{(1 + A\phi(z))} - \frac{B\phi'(z)}{(1 + B\phi(z))}.
\]
Setting \( \phi(z) = id_{\Delta} \), we get
\[
T_f(z) = \frac{f''(z)}{f'(z)} = \frac{2(A - B) + 2Az(A - B)}{(1 + Az)(1 + Bz)} = \frac{(A - B) (3 + 2Az)}{(1 + Az)(1 + Bz)}.
\]
Therefore
\[
\sup_{|z|<1} \left(1 - |z|^2\right) |T_f(z)| \leq \sup_{|z|<1} (1 + |z|) \left[\frac{(3 + 2A |z|)(A - B)}{(1 + A |z|)}\right].
\]
Again, to estimate the upper bound of \( ||T_f|| \), \( z \) must lead to \( 1 \) and hence we get
\[
\lim_{z \to 1} (1 + |z|) \frac{(A - B) (3 + 2A |z|)}{(1 + A |z|)} = \frac{2(A - B)(3 + 2A)}{(A + 1)}.
\]
Finally
\[
\left\|T_f\right\| \leq \frac{2(A - B)(3 + 2A)}{(A + 1)}. \tag{2.6}
\]
For the second part of the proof, its given \( f \in V^*_n [A, B] \) and therefore we assume
\[
k(z) = \left(\frac{w}{g(w)}\right)^2 \frac{g'(w)}{1 + Aw} = p(w)
\]
where \( w = f(z), g = f^{-1} \) and \( w \in \Delta \). Since \( f \in \sigma \) (as explained in the second part of previous theorem) we see,
\[
g' = \frac{d}{dw} \left(f^{-1}(w)\right) = \frac{1}{f'(z)}.
\]
Using above equation, \( k(z) \) can be expressed as
\[
k(z) = \left(\frac{f(z)}{z}\right)^2 \frac{1}{f'(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)}.
\]
By logarithmic differentiation of above equation, we get
\[
\frac{f''(z)}{f'(z)} = \frac{2(1 + A\phi(z))}{z(1 + B\phi(z))} - \frac{2}{z} - \frac{A\phi'(z)}{(1 + A\phi(z))} + \frac{B\phi'(z)}{(1 + B\phi(z))}.
\]
Setting $\phi(z) = id_{\Delta}$, the pre-Schwarzian derivative of $f$ becomes

$$T_f(z) = \frac{f''(z)}{f'(z)} = \frac{(A - B)(1 + 2Az)}{(1 + Az)(1 + Bz)}$$

Following the similar steps as in the first part of this theorem, we get

$$\|T_f\| \leq \sup_{|z|<1} \frac{(A - B)(1 + 2|z|)}{(1 + A|z|)}.$$

Finally,

$$\|T_f\| \leq \frac{2(A - B)(1 + 2A)}{(A + 1)}.$$  \hspace{1cm} (2.7)

Combining (8) and (9), the proof is complete. \hspace{1cm} \square

Let $A = (1 - 2\alpha)$ and $B = -1$ in the above theorem where $0 \leq \alpha < 1$, the class becomes $V^*_{\sigma}[(1 - 2\alpha), -1] \equiv V^*_{\sigma}(\alpha)$.

**Corollary 2.3.** If $f \in V^*_{\sigma}(\alpha)$, then $\|T_f\| \leq \min\{10 - 8\alpha, 6 - 8\alpha\}$.

The above corollary deduces to the exact same norm estimation for analytic and bi-univalent functions in $\Delta$ which lies in a similar class denoted by $V^*_{\sigma}(\alpha)$ and is studied by Rahmatan [4].

Let $A = 1$ and $B = -1$ in the above theorem, the class becomes $V^*_{\sigma}[1, -1] \equiv V^*_{\sigma}$.

**Corollary 2.4.** If $f \in V^*_{\sigma}$, then $\|T_f\| \leq 6$.

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**Conflict of Interest**

Authors declare that there is no conflict of interest.

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