The Sums of a Double Hypergeometric Series and of the First $m+1$ Terms of $3F_2(a, b, c; (a+b+1)/2, 2c; 1)$ when $c = -m$ is a Negative Integer

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Abstract

A summation formula is derived for the sum of the first $m+1$ terms of the $3F_2(a, b, c; (a+b+1)/2, 2c; 1)$ series when $c = -m$ is a negative integer. This summation formula is used to derive a formula for the sum of a terminating double hypergeometric series that arose in another project by one of us (C.D.)

1 Introduction

In the process of proving the terminating double hypergeometric summation formula in Proposition 2 below, we needed to derive a summation formula for a special case of the sum of the first $m+1$ terms of the $3F_2(a, b, c; (a+b+1)/2, 2c; 1)$ series when $c = -m$ is a negative integer. This $3F_2$ series also appears in Watson’s summation formula

$$3F_2\left(\begin{array}{c}a, b, c \\ a+b+1, 2c\end{array}; 1\right) = \frac{\Gamma \left(\frac{1}{2}\right) \Gamma \left(\frac{1}{2} + c\right) \Gamma \left(\frac{1}{2} + \frac{a}{2} + \frac{b}{2}\right) \Gamma \left(\frac{1}{2} - \frac{a}{2} - \frac{b}{2} + c\right)}{\Gamma \left(\frac{1}{2} + \frac{a}{2}\right) \Gamma \left(\frac{1}{2} + \frac{b}{2}\right) \Gamma \left(\frac{1}{2} - \frac{a}{2} + c\right) \Gamma \left(\frac{1}{2} - \frac{b}{2} + c\right)}, \quad (1)$$

which was published by Watson in 1924 for $a$ being a negative integer, and in 1925 by Whipple for the more general case when Re $\{2c+1-a-b\} > 0$ for convergence and assuming $2c, \frac{a+b+1}{2} \not\in \mathbb{Z}_{\leq 0}$ (the numbers $0, -1, -2 \ldots$) so that the denominators in the terms of the series are never zero. See Bailey’s book [4, Sec. 3.3].

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1.1 Remarks

During the writing of the first version the authors found that, as the saying goes, the journey is more educational than the destination. The starting point was the need to sum a certain terminating truncated $3F_2$ series; the next logical step was to refer to standard sources, especially the easy-to-access on-line Digital Library of Mathematical Functions \[7\], \url{http://dlmf.nist.gov}. Formula (1) is to be found there \[7\, 16.4.6\] but one has to avoid being too casual with applying formulae for hypergeometric series when there are negative integers among the denominator parameters, notably in the formulae for polynomials orthogonal with respect to a finite discrete measure such as the Krawtchouk and Hahn polynomials. As worked out below, formula (1) does indeed contain subtle pitfalls, besides which, one feels that a finite summation formula should be provable without recourse to infinite series. So the next step on the journey is to consult a knowledgeable colleague; in this case G.G. who quickly found a proof using an 85-year-old result of Bailey’s (which can actually be found in an on-line archive - if one knows to look for it!). So then version 1 of this note was written. But as perhaps should be expected for something as widely used as $3F_2$ sums, some strongly related results (besides those of Bailey) had been previously obtained. Tom Koornwinder pointed out that formula (3) can also be deduced from Whipple’s sum \[7\, 16.4.7\] by reversing the order of summation (set $k = m - j$) in the left hand side. Some other technical comments are provided at the end of Section 2. The authors deem it worthwhile to describe these diverse approaches to summation problems as an instructive example of the solution process.

1.2 Limits of Watson’s Formula

Let $k, m = 0, 1, 2, \ldots$. We use the transformations $\Gamma (a - m) = (-1)^m \frac{\Gamma (a)}{(1 - a)_m}$ and $\Gamma \left( \frac{1}{2} + t \right) \Gamma \left( \frac{1}{2} - t - m \right) = \frac{(-1)^m \pi}{(\frac{1}{2} + t)_m \cos \pi t}$ (the Pochhammer symbol is defined by \((t)_0 = 1\), \((t)_{m+1} = (t)_m (t + m)\) for \(t \in \mathbb{C}\)), and

\[
\lim_{c \to -m} \frac{(c)_k}{(2c)_k} = \frac{(-m)_k}{(-2m)_k}, k = 0, 1, \ldots, 2m,
\]

which equals zero for $k = m + 1, \ldots, 2m$. It follows from (1) that under the above convergence conditions

\[
\frac{\cos \frac{\pi a}{2} \cos \frac{\pi b}{2} \left( \frac{a+1}{2} \right)_m \left( \frac{b+1}{2} \right)_m}{\cos \frac{\pi (a+b)}{2}} = \lim_{c \to -m} 3F_2 \left( \frac{a, b, c}{\frac{a+b+1}{2}, 2c, 1} \right) = \sum_{k=0}^{m} \frac{(a)_k (b)_k (-m)_k}{k! (\frac{a+b+1}{2})_k (-2m)_k} + \lim_{c \to -m} \sum_{k=m+2}^{\infty} \frac{(a)_k (b)_k (c)_k}{k! (\frac{a+b+1}{2})_k (2c)_k}.
\]

Thus we see that deriving a summation formula for the first sum in the right hand side of (2) from formula (1) is equivalent to the problem of evaluating the
limit of the infinite series on the right side as \( c \to -m \). In fact the termwise limit of the series is a multiple of \( 3F_2 \left( \frac{a+2m+1, b+2m+1, m+1}{a+b+2m+\frac{3}{2}, 2m+2} ; 1 \right) \), which can be summed by (1), and after some simplification the value of the limiting sum is

\[
\sin \frac{\pi a}{2} \sin \frac{\pi b}{2} \left( \frac{\pi a}{2} \right)_m \left( \frac{\pi b}{2} \right)_m; 1
\]

a careful argument using the dominated convergence theorem is needed to justify these operations. Of course for reasonably small \( m \) the sum can be evaluated by computer algebra systems like Maple\textsuperscript{TM} or Mathematica\textsuperscript{TM} with the result

\[
\sum_{k=0}^{m} \frac{(a)_k (b)_k (-m)_k}{k!} = \frac{(-m)_m}{(\frac{a+1}{2})_m (\frac{b+1}{2})_m},
\]

(3) but this is not a proof. In Section 2 we give our proof of (3) for all nonnegative integer values of \( m \), without using infinite series (and having to justify using the termwise limit of the infinite series on the right side of (2) as \( c \to -m \) to derive (3)). The summation formula for the previously mentioned double hypergeometric sum is considered in Section 3.

### 2 The Single Sum

Explicitly we need to prove (ignoring the trivial case \( m = 0 \) where the sum equals one):

**Proposition 1** The summation formula (3) holds for \( m = 1, 2, 3 \ldots \) when the parameters \( a, b \) satisfy \( a+b+\frac{1}{2} \neq 0, -1, -2, \ldots, 1 - m \).

**Proof.** We start with the transformation formula

\[
3F_2 \left( \frac{-m, 2a, 2b}{a+b+\frac{1}{2}, 2c+1} ; 1 \right) = 4F_3 \left( \frac{a, b, 2c+m, -m}{a+b+\frac{1}{2}, c, c+\frac{1}{2}} ; 1 \right),
\]

(equation (4.31) in Bailey’s 1929 paper [3]) which holds for \( m = 1, 2, 3 \ldots \) and \( a+b+\frac{1}{2}, 2c \notin \mathbb{Z}_{\leq 0} \), and where both sums are over the first \( m+1 \) terms of the series. Since both series in (4) terminate it is permissible to take term-by-term limits as \( c \to -m \) giving:

\[
\sum_{k=0}^{m} \frac{(-m)_k (2a)_k (2b)_k}{k! (a+b+\frac{1}{2})_k (-2m)_k} = \sum_{k=0}^{m} \frac{(a)_k (b)_k (-m)_k (-m)_k}{k! (a+b+\frac{1}{2})_k (-m+\frac{1}{2})_k}
\]

(5)

\[
= 3F_2 \left( \frac{a, b, -m}{a+b+\frac{1}{2}, -m+\frac{1}{2}} ; 1 \right)
\]

\[
= \frac{(a+\frac{1}{2})_m (b+\frac{1}{2})_m}{(a+b+\frac{1}{2})_m (\frac{1}{2})_m}
\]

by the Pfaff-Saalschütz summation formula (see [4, 2.2(1)]). Replacing \( a, b \) in (5) by \( \frac{a}{2}, \frac{b}{2} \) respectively completes the proof. \( \square \)
The series in (3) can also be written in the Bailey notation [4, Sec. 10.4] as \(3_F_2(a, b, -m; \frac{a+b+1}{2}, -2m; 1)\) to \(m + 1\) terms, or as a truncated \(3_F_2(a, b, -m; \frac{a+b+1}{2}, -2m; 1)_m\) series, where the subscript \(m\) denotes the sum of the first \(m + 1\) terms of the \(3_F_2\) series. Before writing the first version of this note, we had searched for the summation formula (3) in many papers and books on special functions, including well-known books by G.E. Andrews and R. Askey and R. Roy [2], R.P. Agarwal, W.N. Bailey, A. Erdélyi (Higher Transcendental Functions), G. Gasper and M. Rahman, I.S. Gradshteyn and I.M. Ryzhik, M.E.H. Ismail, Y.L. Luke, L.J. Slater, G. Szegő, and G.N. Watson (see, e.g., the References in [2] and [3]) without finding it.

We extended the search to basic and elliptic hypergeometric series to see if (3) could be obtained as a limit case of any published summation formulas for such series. Eventually, this led us to the observation that from formula (3.17) in the Jain paper [6] \(4\phi_3\left(\frac{a^2}{b^2}, \frac{q}{b}, -q^{-N}, q^{-N}; \frac{b^2}{ab}\sqrt{q}, -\frac{ab}{\sqrt{q}}, q^{-2N}; q; q\right)_N = \frac{(a^2q; q^2)_N (b^2q; q^2)_N}{(ab^2q; q^2)_N (q; q^2)_N}\),

(6)

where we inserted a missing subscript \(N\) on the right side of the series to indicate that it is a truncated \(4\phi_3\) series, it follows by replacing \(a, b\) in it by \(q^a, q^b\) respectively, and letting \(q \rightarrow 1\) that this formula is a \(q\)-analogue of (3). Analogous to Koornwinder’s observation mentioned in Sec. 1.1, formula (6) can be deduced from Andrews’ \(q\)-analogue of Whipple’s sum in [1] by reversing the order of summation on the left hand side (see the formula for reversing the summation order in terminating \(q\)-series in [3] Ex. 1.4(ii)). For a nonterminating \(q\)-analogue of (1), see Ex. 2.17 in [3].

3 The Double Sum

Another project of one of us (C.D.) deals with evaluating separability probabilities for \(4 \times 4\) so-called \(X\)-density matrices, a continuation of investigations by C. D. and P. Slater, [8]. These matrices form a 7-dimensional subset of the space \(M_4(\mathbb{C})\) and the probability calculation involves a five-fold iterated integral, which leads to the definite integral of a double sum containing \(\int_0^1 x^{i+j} dx\).

This can be phrased as

\[
S(m, n) := \sum_{i=0}^{m} \frac{(-m)_i (n+1)_i}{i! (m+n+2)_i} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{1}{2} - n\right)_j}{j! \left(\frac{1}{2}\right)_j} \frac{1}{i+j+\frac{1}{2}}.
\]

(7)

This series is of hypergeometric type because \(\frac{1}{i+j+\frac{1}{2}} = 2 \left(\frac{i+j}{2}\right)_i\). For \(m = 0\) the formula is easily evaluated with the Chu-Vandermonde \(2F_1\) sum (see [4] Sec. 1.3), and for \(n = 0\) the sum is a special case of a terminating well-poised \(3F_2\) series formula. Using symbolic computation to evaluate \(S(m, n)\) for
$m = 0, 1, 2, 3$ suggested that a closed form does exist; specifically the formula

$$
\frac{S(m, n)}{S(m - 1, n + 1)} = \frac{2m (n + 1)}{(2n + 1) (n + 2m + 1)}
$$

was verified for a few small values of $m, n$. After further exploration (which led to trying to fill in the gap between $i + j + \frac{1}{2}$ and $j - \frac{1}{2}$, the last factor in $(\frac{1}{2})_j$), a proof was found that needed a special case of formula (3).

**Proposition 2** For $m, n = 0, 1, 2, 3, \ldots$

$$
S(m, n) = 2^{2m+2n} \frac{m!(m+n)!(m+n+1)! (\frac{1}{2})_n}{n!(n+2m+1)! (\frac{1}{2})_{m+n+1}}.
$$

**Proof.** Let

$$
A_{ni} := \sum_{j=0}^{n} (-n)_j (\frac{1}{2} - n)_j j! (\frac{1}{2})_j (i + j + \frac{1}{2}),
$$

and

$$
\sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{1}{i+\frac{1}{2}} = \sum_{i=0}^{k} (-k)_i (j + \frac{1}{2})_i = \frac{1}{j + \frac{1}{2}} (j + \frac{1}{2})_k = \frac{k!}{(j + \frac{1}{2})_{k+1}},
$$

by the Chu-Vandermonde sum. Then

$$
\sum_{i=0}^{k} \binom{k}{i} (-1)^i A_{ni} = \sum_{j=0}^{n} (-n)_j (\frac{1}{2} - n)_j j! (\frac{1}{2})_j (j + \frac{1}{2})_{k+1}
$$

$$
= \frac{k!}{(\frac{1}{2})_{k+1}} \sum_{j=0}^{n} (-n)_j (\frac{1}{2} - n)_j j! (\frac{1}{2} + k + 1)_j = \frac{k!}{(\frac{1}{2})_{k+1}} (k + n + 1)_n
$$

$$
= \frac{k! (k + n + 1)_n}{(\frac{1}{2})_{k+n+1}} =: B_{nk}.
$$

The matrix $M$ with $M_{ij} = \binom{i}{j} (-1)^i$ for $i, j \geq 0$ (and $M_{ij} = 0$ for $i < j$) is its own inverse, thus (note $B_{nk} = \sum_{i=0}^{k} M_{ki} A_{ni}$)

$$
A_{ni} = \sum_{k=0}^{i} \binom{i}{k} (-1)^k B_{nk}.
$$
The case $S(0, n) = A_{n,0} = B_{n,0}$ is trivial, so we assume $m \geq 1$ in the following. The expression for $A_{n,k}$ is now used to find that:

\[
S(m, n) = \sum_{i=0}^{m} \frac{(-m)_i (n + 1)_i}{i! (m + n + 2)_i} A_{ni} = \sum_{i=0}^{m} \frac{(-m)_i (n + 1)_i}{i! (m + n + 2)_i} \sum_{k=0}^{i} (-1)^k \frac{i!}{k! (i-k)!} B_{nk}
\]

\[
= \sum_{k=0}^{m} (-1)^k B_{nk} \frac{(-m)_k (n + 1)_k}{k! (m + n + 2)_k} \sum_{j=0}^{m-k} \frac{(k-m)_j (n+1)_j}{j! (m + n + 2 + k)_j}
\]

\[
= \sum_{k=0}^{m} (-1)^k B_{nk} \frac{(-m)_k (n + 1)_k}{k! (m + n + 2)_k} \sum_{j=0}^{m-k} \frac{(m+1-j)_j (n+1)_j}{(m + n + 2 + k)_j}
\]

\[
= \frac{(m+1)_m}{(m + n + 2)_m} \sum_{k=0}^{m} \frac{k! (n+1)_k (-m)_k (n+1)_k}{(\frac{1}{2})_{k+n+1} (-2m)_k}
\]

this used the change of summation index $i = k + j$, so that $0 \leq k \leq m$ and $0 \leq j \leq m - k$, and the identity

\[
(m+1)_{m-k} = \frac{(m+1)_{m-k} (2m+1-k)_k}{(2m+1-k)_k} = \frac{(m+1)_m (-1)^k}{(-2m)_k}.
\]

Finally we set $a = n + \frac{1}{2}, b = \frac{1}{2}$ in (15) to obtain

\[
S(m, n) = \frac{(m+1)_m (n+1)_m}{(m + n + 2)_m} \frac{m! (n+1)_m}{(\frac{1}{2})_m (n + \frac{1}{2})_m}
\]

\[
= \frac{(2m)! (n+1)_m (n+1)_m}{(m + n + 2)_m (\frac{1}{2})_{m+n+1} (\frac{1}{2})_m} = 2^{2m+2n} \frac{m! (n+1)_m}{(n + m + 2)_m (n + \frac{1}{2})_{m+1}}
\]

\[
\text{using } (2m)! = 2^{2m} m! \left(\frac{1}{2}\right)_m \text{ and } n! (n+1)_n = (2n)! = 2^{2n} n! \left(\frac{1}{2}\right)_n. \text{ The last expression is equivalent to the stated formula in the Proposition (typical step: } (n+1)_m = \frac{(m+n)!}{m!}). \]

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