Reposition time in probabilistic imperfect memories

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Abstract. We present the analysis of behavior of $N$ identical finite time memories with the imperfections characterized by a step-function model, based on a study of independent copies of the geometric distribution. We show a step-by-step derivation of a formula for the average waiting time for obtaining $N$ successes within a given time span $\tau$, which was used recently in the analysis of quantum repeater rates.

1. Introduction

The problem we are solving in this paper is motivated by a stochastic behavior of physical objects studied in the context of quantum information processing or, more precisely, in the quantum repeater rates analysis. Although in next paragraphs a general description of the system that inspired questions posed and answered here is given, familiarity with quantum information theory is not needed to follow the main part of this paper. Presented results are completely independent of the physical context and were obtained employing minimal mathematical tools: geometric series, multinomial coefficients and elementary probability theory (i.e., knowledge on the level of two first chapters of [1]). A slight complication comes from the fact that subsequently defined probability spaces are constructed from ones introduced previously. As a consequence, similar but connected with different spaces terms and expressions appear in the text. To avoid confusion, we employ rather excessive notation in order to always keep track of random variables and sample spaces. Basic information on the notation can be found in Appendix A.1. Throughout the paper term ‘game’ is used to describe sequences of motions guided by earlier defined rules, which is not the same meaning as the word usually has in the sense of game theory. Readers interested mainly in mathematical aspects of this work might want to skip the rest of this introduction and go directly to section 2.

When dealing with quantum systems, a large part of experimental challenges comes from the relation between the time scale of coherence stability and the time needed for effective, practically useful manipulations on the system. Difficulty with maintaining coherence is directly connected with the problem of quality of quantum memories – and approaching the subject from this angle one can apply many classical intuitions. On the most elementary level, memory can be treated as a function of time with only two possible values: ‘success’, when the state is faithfully stored, or ‘failure’ corresponding
to the unsuccessful storage attempt. In this paper, we use such a simple step-function model assuming that memory is perfect for a specific time $\tau$, but after that time the stored state is completely and irreversibly lost and memory returns to its initial state. Although in the description of noise and decoherence exponential decay is more common, it can always be reduced to the step-function model by adding a threshold and a cut-off, to mimic the assumption that a perfect to some point memory rapidly loses the ability to store the state.

As mentioned before, the initial motivation for this study was the need to develop analytical tools for the rates analysis of quantum repeaters, i.e., devices designed to distribute entanglement over large distances in a more efficient way than the direct (quantum) communication between two remote locations. The main idea of quantum repeater is to divide initial distance into many segments, establish entanglement over these shorter distances, then connect them by entanglement swapping, usually adding also steps of entanglement purification [2]. Depending on a repeater model number of segments, or nodes per segments, etc. changes; depending on a chosen strategy procedures of purification and swapping might appear in different order. Nevertheless, in every repeater model, no mater of its construction strategy, a successful creation of the initial ‘small-distance’ entanglement is always a starting point and a necessary condition for establishing the long-distance entanglement. Discussion in this paper is restricted to this initial process and focuses on effects of memory imperfections.

We will consider the following problem: Imagine, that a single attempt of entanglement creation in one segment of quantum repeater ends successfully with probability $p$. Once created, entanglement is stored perfectly for a specific time $\tau$ and after that time it gets irreversibly lost. We are interested in finding the average waiting times for creation of entanglement in $N$ identical and independent segments. To be on a save side and rather underestimate the rates then overestimate them, we assume that if entanglement was lost in one segment before it was created in all the others, the whole system is reset to its initial state. If this happens, a ‘residual’ entanglement stored in any segment is discarded, and attempt of creating it in all of the segments starts from the beginning. Defined in this way, it is a purely probabilistic problem that has an exact analytical solution. For the perfect memories (infinite $\tau$) the solution was shown in [3]; for finite memory times $\tau$ we present it in this paper.

We have put some effort to obtain analytical solutions in order to be able to establish bounds, calculate accurate limits, and easily compare different systems. Comparison between two analytical formulas is unimpeachable, whereas the same can not be said unconditionally about numerical results. Analytical solutions build intuitions and understanding of the system behavior; they allow for formal proofs – in contrast to educated guesses. As stated before, results presented here are independent from physical settings. They were developed in the context of repeater analysis [4]; but describe, as well, any other $N$-components system with the characteristic as given above. Formulas obtained are universal and their simple structure makes them convenient for implementations.
The paper is structured as follows: In section 2, general rules of the games considered here are defined; a detailed description of the system studied and the problem we are solving is given. Sections 3–5 are devoted to the systematic presentation of the solution of this problem. Most of the details concerning calculations and notation are moved to the Appendixes.

2. System description

Consider a classical bit with values (states) denoted by $|0\rangle$ and $|1\rangle$. Let $T$ be a process that with a probability $p$ ($0 < p < 1$) transfers a bit from a state $|0\rangle$ to $|1\rangle$, but acting on a bit in a state $|1\rangle$ always leaves it untouched. A pair constructed from a sample space $A = \{s_A, f_A\}$ and a map $\alpha : 2^{\{s_A, f_A\}} \rightarrow [0, 1]$, characterized by the properties:

$$
\alpha(\{\emptyset\}) := 0 \quad \alpha(\{s_A, f_A\}) := 1 \quad \alpha(s_A) := p \quad \alpha(f_A) := 1 - p = q,
$$

defines the probability space, which describes formally action of $T$ on $|0\rangle$. The event $s_A$ occurs with the probability $p$, and corresponds to the successful bit-flip from $|0\rangle$ to $|1\rangle$. The event $f_A$ happens with the probability $q = 1 - p$, and corresponds to the failed attempt of a state transfer.

In this paper, we consider systems composed of $N$ classical bits, all initiated in states $|0\rangle$, and probabilistic ‘games’ during which one tries to transfer these bits into states $|1\rangle$, repeatedly applying the procedure $T$. The action of $T$ on the $N$-bit system is defined as the simultaneous and independent action of $T$ on every bit:

$$
T(bit_1, bit_2, ..., bit_N) := (T(bit_1), T(bit_2), ..., T(bit_N)).
$$

2.1. Game $G_N$, perfect memories

Let $G_N$ be a game, in which the goal is to transfer $N$ bits from states $|0\rangle$ into $|1\rangle$ by repeating applications of the procedure $T$. A question of an average number of repetition needed to obtain this goal was answered in [3] (rediscovering result from [5] and [6]). For fixed $N$ and $q = 1 - p$, it is expressed by

$$
\langle K_{G_N} \rangle = \sum_{k=1}^{N} \binom{N}{k} \frac{(-1)^{k+1}}{1 - q^k}.
$$

This result can be generalized: solving recursive formulas, obtained similarly as in [3], we find that an average number of applications of $T$ needed to transfer at least $m$ from $N$ bits of the system from $|0\rangle$ into $|1\rangle$ is given by

$$
\langle K_{m,N} \rangle = \frac{1}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{N!}{(N-m)!} q^k (m-1)^{m-k} \left( \frac{N}{k} \right) \left( \frac{q}{N-q} \right).
$$

It is worth noting, that for $0 < p < 1$, after sufficiently many applications of the procedure $T$, all $N$ bits are transferred to the state $|1\rangle$. Because of the implied in the definition of $G_N$ (and quite reasonable for classical systems) assumption that the bits in state $|1\rangle$ stay undisturbed, no matter how many repetitions of $T$ followed – we say it is a perfect memory case.
2.2. Games $G_{(N,\tau)}$, imperfect memories (finite memory time $\tau$)

A more physical (at least for quantum systems) model should take into account the possibility that bits in state $|1\rangle$ are left undisturbed only for some finite time $\tau$. For simplicity, we will use here discrete time – measured in steps – with the number of steps equal to the number of the subsequent applications of procedure $T$. A memory time $\tau$ means that the system stability can be guaranteed only for $\tau$ repetitions of $T$, counting from the first successful transfer to state $|1\rangle$. After that time the bit (or bits), which flipped to $|1\rangle$ as first, is no longer in a well defined state. This is the reason why only those events, when all $N$ bits flipped to $|1\rangle$ within the time span $\tau$, will be considered as successes in the modified version of game $G_N$.

Obviously, an addition of the assumption of imperfect memories to the game $G_N$ changes its rules and results. Let us fix parameters $N$ and $\tau$. A modified game $G_{(N,\tau)}$ is divided in rounds, which always start with $N$ bits prepared in states $|0\rangle$. During a single round, procedure $T$ is applied to the whole system until all bits are transferred to states $|1\rangle$ but not more then $\tau$ times, counting from the first successful transfer of at least one bit into state $|1\rangle$. If time $\tau$ was not enough to transfer all bits into states $|1\rangle$, the round ends unsuccessfully; the system is reset to its initial state $|0\rangle$ and the next round starts, automatically. The game ends when all bits are finally transferred to $|1\rangle$, i.e., in a moment when some round ended successfully. Note that, for $\tau \geq 1$, game $G_{(N,\tau)}$ does not form a Markov chain. We would like to know, how many times on average procedure $T$ is repeated in game $G_{(N,\tau)}$. Answer, given by the formula

$$
\langle K_{G_{(N,\tau)}} \rangle = \frac{1 - (1 - q^\tau)^N + (1 - q^N) \left[ \tau - \sum_{j=1}^{\tau-1} (1 - q^j)^N \right]}{(1 - q^{\tau+1})^N - q^N (1 - q^\tau)^N},
$$

and its derivation are the main results of this paper.

When considering a single round of game $G_{(N,\tau)}$, it is convenient to divide it into two separate parts. Initial $k \geq 0$ steps ($k$ applications of procedure $T$), during which all $N$ bits remain in state $|0\rangle$, constitute the first part. The second part starts at step $k+1$, when at least one bit gets transferred to $|1\rangle$. A course of the second part is completely independent from a duration of the first. A general description of the second part of an arbitrary round is presented in section 3. In section 4, both of these parts are combined and the general description of single rounds of $G_{(N,\tau)}$ is addressed. Section 5 is devoted to the results that characterize the whole game $G_{(N,\tau)}$.

3. Auxiliary spaces $\Gamma(N, \tau)$

In this section, we shall focus on the description of a second part of an arbitrary round of $G_{(N,\tau)}$, i.e., the part that starts with the first successful bit-flip. To keep track of all possible events, we introduce auxiliary spaces $\Gamma(N, \tau)$ defining them, for natural numbers $\tau \geq 0$ and $N \geq 2$, by

$$
\Gamma(N, \tau) := \left\{ \gamma \in \text{Map}((0, \tau), (0, N)) : \gamma(0) > 0, \sum_{j=0}^{\tau} \gamma(j) \leq N \right\}.
$$
A map $\gamma \in \Gamma(N, \tau)$ corresponds to these rounds of $G_{(N, \tau)}$ during which all $N$ bits stayed in state $|0\rangle$ for initial $k$ steps, then at step $k + 1$ at least one bit ($\gamma(0) > 0$ bits) flipped from state $|0\rangle$ to $|1\rangle$, at step $k + 2$ another $\gamma(1)$ bits changed to $|1\rangle$, etc., up to step $k + 1 + \tau$, when $\gamma(\tau)$ bits flipped to $|1\rangle$.

**Remark 1:** Note, that elements of $\Gamma(N, \tau)$ do not depend on $k$. The same $\gamma$ describes many different rounds of $G_{(N, \tau)}$, as they might correspond to different values of $k$. □

3.1. Notation

Every set $\Gamma(N, \tau)$ can be divided into subsets collecting rounds during which the same number of bits changed from $|0\rangle$ to $|1\rangle$. Therefore, we introduce:

$$\forall M \in (1, N) \quad \Gamma_M(N, \tau) := \left\{ \gamma \in \Gamma(N, \tau) : \sum_{j=0}^{\tau} \gamma(j) = M \right\}.$$ 

Let us fix the parameters $(N, \tau)$ and $M$. Set $\Gamma_M(N, \tau)$ consists of all these elements of $\Gamma(N, \tau)$ in which between steps $k + 1$ and $k + 1 + \tau$ exactly $M$ bits flipped to state $|1\rangle$.

Obviously,

$$\Gamma(N, \tau) = \bigcup_{M=1}^{N} \Gamma_M(N, \tau) = \Gamma_N(N, \tau) \cup \left( \bigcup_{M=1}^{N-1} \Gamma_M(N, \tau) \right).$$

To explicitly distinguish between successful and unsuccessful events, we define:

$$S(N, \tau) := \Gamma_N(N, \tau) \quad \text{and} \quad F(N, \tau) := \bigcup_{M=1}^{N-1} \Gamma_M(N, \tau). \quad (2)$$

The set of successful events, $S(N, \tau)$, consists of these events in which, between steps $k + 1$ and $k + 1 + \tau$, all $N$ bits flipped to $|1\rangle$. The complementary set, $F(N, \tau)$, consists of all events in which after step $k + 1 + \tau$ at least one bit was still left in state $|0\rangle$. These events will be referred to as unsuccessful, or resulting in a failure. For further convenience, we single out subsets of these elements of $\Gamma(N, \tau)$ for which the last from the bit-flips occurred exactly at step $k + 1 + \sigma$:

$$\forall \sigma \in (0, \tau) \quad \Gamma^\sigma(N, \tau) := \left\{ \gamma \in \Gamma(N, \tau) : \max\{l \in (0, \tau) : \gamma(l) > 0\} = \sigma \right\}.$$

Similarly, we define sets $\Gamma_M^\sigma(N, \tau)$ as the intersection of $\Gamma_M(N, \tau)$ and $\Gamma^\sigma(N, \tau)$:

$$\forall (M, \sigma) \in (1, N) \times (0, \tau) \quad \Gamma_M^\sigma(N, \tau) := \Gamma_M(N, \tau) \cap \Gamma^\sigma(N, \tau).$$

In particular, a set of these successful events for which the last bit-flip occurred at step $k + 1 + \sigma$ is denoted by $\Gamma^\sigma_M(N, \tau)$. Note, that the sum of $\Gamma_M^\sigma(N, \tau)$ over all $M \in (1, N)$ and $\sigma \in (0, \tau)$ equals to $\Gamma(N, \tau)$. 

3.2. Measures \( g_{(N,\tau)} \), numbers \( sp(n,\nu) \)

To be able to write down calculations in more condensed form, we introduce the maps: 

\[ \tilde{\gamma}(l) := N - \sum_{j=0}^{l-1} \gamma(j) \quad \text{and} \quad \bar{\gamma}(l) := \tilde{\gamma}(l) - \gamma(l), \]

for \( l \in (0,\tau) \). For a given \( \gamma \in \Gamma(N,\tau) \), values \( \tilde{\gamma}(l) \) and \( \bar{\gamma}(l) \) are equal to the number of bits in states \( |0\rangle \) at steps \( k + l \) and \( k + l + 1 \), respectively. It is easy to check that \( \tilde{\gamma}(\tau) = N - \sum_{j=0}^{\tau} \gamma(j) \) and that, for \( l \in (0,\tau-1) \), the identity \( \bar{\gamma}(l) = \tilde{\gamma}(l + 1) \) holds. Using this notation, we introduce maps \( g_{(N,\tau)} \) as follows:

\[ g_{(N,\tau)} : \Gamma(N,\tau) \ni \gamma \mapsto g_{(N,\tau)}(\gamma) := \prod_{l=0}^{\tau} \left( \frac{\tilde{\gamma}(l)}{\gamma(l)} \right) p^{\gamma(l)} q^{\tilde{\gamma}(l)} \in [0,1]. \tag{3} \]

Measures on \( \Gamma(N,\tau) \), defined as the extensions of (3), are not normalized to 1; we have

\[ \sum_{\gamma \in \Gamma(N,\tau)} g_{(N,\tau)}(\gamma) = 1 - q^{N}, \]

which reflects the fact that events ‘at step \( k + 1 \) all bits remain in state \( |0\rangle \)’ do not belong to \( \Gamma(N,\tau) \).

Remark 2: Values of \( g_{(N,\tau)}(\gamma) \) cannot be interpreted as probabilities characterizing \( \gamma \); pairs \( (\Gamma(N,\tau), g_{(N,\tau)}) \) do not define probability spaces. □

For clarity of notation, we introduce functions \( sp(n,\nu) \) defined, for \( n \geq 1 \) and \( \nu \geq 0 \), by

\[ sp(n,\nu) := (1 - q^{\nu+1})^n - q^n (1 - q^{\nu})^n. \tag{4} \]

Additionally, we assume that \( sp(n,-1) := 0 \). Various combinations of numbers \( sp(n,\nu) \) appear repeatedly in the calculations of measures \( g_{(N,\tau)} \) for the important subsets of \( \Gamma(N,\tau) \). In Appendix B, it is shown that:

\[ g_{(N,\tau)}(S(N,\tau)) = sp(N,\tau), \tag{5} \]
\[ g_{(N,\tau)}(F(N,\tau)) = 1 - q^{N} - sp(N,\tau), \tag{6} \]
\[ g_{(N,\tau)}(\Gamma^*_N(N,\tau)) = sp(N,\sigma) - sp(N,\sigma - 1), \tag{7} \]

for any \( \sigma \in (0,\tau) \). Comparing (5) and (7) we see that, as expected, values of measures corresponding to successes that occurred exactly at a step \( k + 1 + \sigma \) is equal to the values of measures corresponding to the sum of all successes up to the step \( k + 1 + \sigma \) minus the values of measures corresponding to sum of all successes up to the step \( k + \sigma \).

3.3. Expected sum values \( \langle \langle \lambda_{(N,\tau)} \rangle \rangle \)

Let functions \( \lambda_{(N,\tau)} \) be equal to \( \tau \) for all unsuccessful events, and equal to \( \sigma \) for events that ended successfully at step \( k + 1 + \sigma \), i.e., let \( \lambda_{(N,\tau)} \in \text{Map}(\Gamma(N,\tau), \mathbb{R}) \) be defined by \( \lambda_{(N,\tau)}(F(N,\tau)) := \{\tau\} \) and \( \lambda_{(N,\tau)}(\Gamma^*_N(N,\tau)) := \{\sigma\} \), for \( \sigma \in (0,\tau) \). We want to
calculate expected sum values $\langle\langle \lambda_{(N,\tau)} \rangle\rangle$ over the sets of unsuccessful events $F(N,\tau)$ and the sets of successful events $S(N,\tau)$. To shorten the formulas, from now on, we will write $g, \lambda$ instead of $g_{(N,\tau)}, \lambda_{(N,\tau)}$, whenever omission of indexes does not lead to confusion. From (6), it is clear that:

$$\langle\langle \lambda \rangle\rangle_{(F(N,\tau),g)} := \tau g(F(N,\tau)) = \tau (1 - q^N - sp(N,\tau)).$$

(8)

Similar relation for sets $S(N,\tau)$, given by

$$\langle\langle \lambda \rangle\rangle_{(S(N,\tau),g)} := \tau \sum_{\sigma=0}^{\tau} \sigma g(\Gamma_\sigma^\sigma(N,\tau)) = \tau sp(N,\tau) - \tau \sum_{j=0}^{\tau-1} sp(N,j),$$

(9)

is derived in Appendix B.4. Combining (8) and (9) we obtain the expected sum values corresponding to $\Gamma(N,\tau)$:

$$\langle\langle \lambda \rangle\rangle_{(\Gamma(N,\tau),g)} := \sum_{\gamma \in \Gamma(N,\tau)} \lambda(\gamma) g(\gamma) = \sum_{\gamma \in F(N,\tau)} \lambda(\gamma) g(\gamma) + \sum_{\gamma \in S(N,\tau)} \lambda(\gamma) g(\gamma) =$$

$$= \tau g(F(N,\tau)) + \tau \sum_{\sigma=0}^{\tau} \sigma g(\Gamma_\sigma^\sigma(N,\tau)) = \tau (1 - q^N) - \tau \sum_{j=0}^{\tau-1} sp(N,j).$$

**Remark 3:** Because pairs $(\Gamma(N,\tau),g)$ are not probability spaces, functions $\lambda$ are not random variables, and $\langle\langle \lambda \rangle\rangle_{(\Gamma(N,\tau),g)}$ cannot be interpreted as the average values. (See Remark 2 and Appendix A.1.) □

4. Single rounds of $\mathcal{G}_{(N,\tau)}$

4.1. Sample spaces $\mathbb{B}(N,\tau)$

A description characterizing single rounds of $\mathcal{G}_{(N,\tau)}$ from the beginning – not only the second parts as it was done in previous section – has to include information about all steps preceding the first bit-flips. In order to do so, we define sets $\mathbb{B}(N,\tau) := \mathbb{N} \times \Gamma(N,\tau)$. Let us fix parameters $k, \tau, N \in \mathbb{N}$, for $N \geq 2$, and choose some $\gamma \in \Gamma(N,\tau)$. The element $(k,\gamma) \in \mathbb{B}(N,\tau)$ corresponds to the round in which $N$ bits remained in states $|0\rangle$ for initial $k$ steps, then $\gamma(0)$ bits was transferred to $|1\rangle$ at step $k+1$, $\gamma(1)$ bits flipped to $|1\rangle$ at step $k+2$, etc.. Additionally, we define maps $\beta_{(N,\tau)}$

$$\beta_{(N,\tau)} : \mathbb{B}(N,\tau) \ni (k,\gamma) \rightarrow \beta_{(N,\tau)}(k,\gamma) := q^{Nk} g_{(N,\tau)}(\gamma) \in [0,1].$$

(10)

Extensions of (10) introduce measures on the corresponding sets $\mathbb{B}(N,\tau)$. It is easy to check the normalization:

$$\beta_{(N,\tau)}(\mathbb{B}(N,\tau)) = \sum_{(k,\gamma) \in \mathbb{B}(N,\tau)} \beta_{(N,\tau)}(k,\gamma) = \sum_{k=0}^{\infty} q^{Nk} \sum_{\gamma \in \Gamma(N,\tau)} g_{(N,\tau)}(\gamma) = 1,$$

(11)

and conclude that pairs $(\mathbb{B}(N,\tau),\beta_{(N,\tau)})$ define probability spaces.

Every set $\mathbb{B}(N,\tau)$ splits into a subset of rounds that ended successfully, i.e., all $N$ bits were transferred to states $|1\rangle$, and a complementary set of rounds that ended
Reposition time in probabilistic imperfect memories

in failure. We denote these subsets \( S_B(N, \tau) \) and \( F_B(N, \tau) \), respectively, and using (2) write them as
\[
S_B(N, \tau) := \mathbb{N} \times S(N, \tau) \quad \text{and} \quad F_B(N, \tau) := \mathbb{N} \times F(N, \tau).
\]

Taking relations (4), (5) and (6) into account we obtain after a brief calculation:
\[
\beta_{(N, \gamma)}(S_B(N, \tau)) = sp(N, \tau)/(1 - q^N) =: \mathcal{P}(N, \tau),
\]
\[
\beta_{(N, \gamma)}(F_B(N, \tau)) = 1 - sp(N, \tau)/(1 - q^N) =: \mathcal{Q}(N, \tau).
\]

Definitions of \( \mathcal{P}(N, \tau) \) and \( \mathcal{Q}(N, \tau) \), introduced above, will be used to shorten formulas in the sections that follow.

4.2. Average number of steps in a single round of \( \mathcal{G}_{(N, \tau)} \), i.e., \( \langle \Lambda \rangle_{(B(N, \tau), \beta)} \)

The number of steps in any given round \((k, \gamma)\) is equal to the sum of \( k \) steps before the first bit-flip plus 1 (the step at which this first transfer from \( |0\rangle \) to \( |1\rangle \) occurred) plus \( \lambda(\gamma) \) steps after the first bit-flip. The corresponding random variables on sets \( B(N, \tau) \) are defined as
\[
\Lambda_{(N, \tau)} : B(N, \tau) \ni (k, \gamma) \mapsto \Lambda_{(N, \tau)}(k, \gamma) = k + 1 + \lambda_{(N, \tau)}(\gamma) \in \mathbb{R}.
\]

Using (8), (9), and dropping indexes \((N, \tau)\), we calculate the expected sum values \( \langle \langle \Lambda \rangle \rangle \) over sets \( S_B(N, \tau) \) and \( F_B(N, \tau) \) and obtain:
\[
\langle \langle \Lambda \rangle \rangle_{(S_B(N, \tau), \beta)} := \sum_{(k, \gamma) \in S_B(N, \tau)} \Lambda(k, \gamma) \beta(k, \gamma) =
\]
\[
g(S(N, \tau))(1 - q^N)^{-2} + \langle \langle \Lambda \rangle \rangle_{(S(N, \tau), \beta)}(1 - q^N)^{-1}
\]
\[
= \left( [(1 - q^N)^{-1} + \tau] \cdot sp(N, \tau) - \sum_{j=0}^{\tau-1} sp(N, j) \right) (1 - q^N)^{-1}
\]
and
\[
\langle \langle \Lambda \rangle \rangle_{(F_B(N, \tau), \beta)} := \sum_{(k, \gamma) \in F_B(N, \tau)} \Lambda(k, \gamma) \beta(k, \gamma) =
\]
\[
g(F(N, \tau))(1 - q^N)^{-2} + \tau g(F(N, \tau))(1 - q^N)^{-1}
\]
\[
= g(F(N, \tau)) \left[ \tau + (1 - q^N)^{-1} \right] (1 - q^N)^{-1}
\]
\[
= [\tau + (1 - q^N)^{-1}] \mathcal{Q}(N, \tau),
\]
respectively. The average number of steps in single rounds of \( \mathcal{G}_{(N, \tau)} \) can be calculated as the sum of (14) and (15), which yields:
\[
\langle \Lambda \rangle_{(B(N, \tau), \beta)} = \langle \langle \Lambda \rangle \rangle_{(S_B(N, \tau), \beta)} + \langle \langle \Lambda \rangle \rangle_{(F_B(N, \tau), \beta)}
\]
\[
= \tau + \left( 1 - \sum_{j=0}^{\tau-1} sp(N, j) \right) (1 - q^N)^{-1}
\]
\[
= \tau + \left( 1 - (1 - q^\tau)^N \right) (1 - q^N)^{-1} - \sum_{j=1}^{\tau-1} (1 - q^j)^N.
\]
Let \( N \) sets \( Y_i \) and every \( Y_j \).

5.1. Sample spaces

For details of the notation see Appendix A.1.

5. Games \( G(N, \tau) \)

In this section, a general description of the whole course of an arbitrary game \( G(N, \tau) \) (starting with all \( N \) bits of the system in state \(|0\rangle\) and ending with all bits in state \(|1\rangle\)) is presented. To shorten the formulas, elements of a multi-indexes notation are used; for details of the notation see Appendix A.1.

4.3. Order in sets \( S(N, \tau) \) and \( F(N, \tau) \)

It is convenient to label elements of a given \( \Gamma(N, \tau) \), indexing separately elements in the subsets of successfully and unsuccessfully ended rounds, i.e., the elements in \( S(N, \tau) \) and \( F(N, \tau) \). Because \( |\Gamma(N, \tau)| \leq N^{\tau+1} \), all subsets of \( \Gamma(N, \tau) \) are finite. Consequently, there exist natural numbers \( I \) and \( J \) such that

\[
|S(N, \tau)| =: I(N, \tau) = I \quad \text{and} \quad |F(N, \tau)| =: J(N, \tau) = J.
\]

We choose two arbitrary (but fixed for further considerations) bijective functions:

\( \varphi(N, \tau) \in \text{Map}(\overline{(1, I)}, S(N, \tau)) \) and \( \psi(N, \tau) \in \text{Map}(\overline{(1, J)}, F(N, \tau)) \). Then, for a given \((N, \tau)\) and every \( i \in (1, I) \), \( j \in (1, J) \), we denote \( \varphi(N, \tau)(i) =: s_i \) and \( \psi(N, \tau)(j) =: f_j \), which allows to express the sets \( S(N, \tau) \) and \( F(N, \tau) \) as lists of the ordered elements:

\[
S(N, \tau) = \{s_1, s_2, \ldots, s_I\} \quad \text{and} \quad F(N, \tau) = \{f_1, f_2, \ldots, f_J\}.
\]

This notation is easily extended to the following subsets of \( \mathbb{B}(N, \tau) \)

\[
\mathcal{S}_i = \mathcal{S}_i(N, \tau) := \mathbb{N} \times \{s_i\} \quad \text{and} \quad \mathcal{F}_j = \mathcal{F}_j(N, \tau) := \mathbb{N} \times \{f_j\},
\]

where, again, \( i \in (1, I) \) and \( j \in (1, J) \). The probabilities corresponding to \( \mathcal{S}_i \) and \( \mathcal{F}_j \) are equal to:

\[
\beta(\mathcal{S}_i) = g(s_i)(1 - q^N)^{-1} \quad \text{and} \quad \beta(\mathcal{F}_j) = g(f_j)(1 - q^N)^{-1}.
\]

5. Games \( G_{(N, \tau)} \)

In this section, a general description of the whole course of an arbitrary game \( G_{(N, \tau)} \) (starting with all \( N \) bits of the system in state \(|0\rangle\) and ending with all bits in state \(|1\rangle\)) is presented. To shorten the formulas, elements of a multi-indexes notation are used; for details of the notation see Appendix A.1.

5.1. Sample spaces \( D(N, \tau) \)

Let \( N \geq 2, \tau \geq 0 \) be fixed natural numbers, and \( J := |F(N, \tau)| \). We define a family of sets \( Y_M(N, \tau) \) specified, for \( M \in \mathbb{N} \), by the formula:

\[
Y_M(N, \tau) := \left[ \text{Map}(\overline{(0, M)}, \mathbb{N}) \times \left\{ \zeta \in \text{Map}(\overline{(1, J)}, \mathbb{N}) : |\zeta| = M \right\} \right] \times S(N, \tau).
\]

Next, with every family of sets \( Y_M(N, \tau) \) we associate the corresponding sample space \( D(N, \tau) := \bigcup_{M=0}^{\infty} Y_M(N, \tau) \). Elements of \( D(N, \tau) \) correspond to all possible courses of game \( G_{(N, \tau)} \). To be more specific: \( (\xi, \zeta, \gamma) \in D(N, \tau) \) denotes a course of game \( G_{(N, \tau)} \), which started with \( |\zeta| \) unsuccessful rounds and ended with successful round characterized by \( (k, \gamma) \), where \( k = \xi(0) \). Values of \( \zeta(j) \), for \( j \in (1, J) \), match number of times that
unsuccessful rounds of type \( f_j \) were repeated. Number of steps during first parts of rounds that ended unsuccessfully are given by \( \xi \). We introduced also measures on sets \( \mathbb{D}(N, \tau) \), defining them as extensions of maps \( \delta_{(N, \tau)} \in \text{Map}(\mathbb{D}(N, \tau), [0, 1]) \) given by

\[
\delta(\xi, \zeta, \gamma) := g(\gamma)\frac{|\xi|!}{\zeta!} \prod_{l=0}^{\mid\xi\mid} q^{Nl(\xi)} \prod_{j=1}^{J} (g(f_j)^{\zeta(j)}).
\]

As before, we dropped indexes \( (N, \tau) \) referring to measures \( g, \delta \) or \( \beta \). It is relatively easy to show that:

\[
\delta(\mathbb{D}(N, \tau)) = \sum_{(\xi, \zeta, \gamma) \in \mathbb{D}(N, \tau)} \delta(\xi, \zeta, \gamma) = 1,
\]

which means that pairs \((N, \tau)\) define probability spaces. Derivation of (17) is presented in Appendix C.2

5.2. Average number of steps \( \langle K \rangle_{\mathbb{D}(N, \tau), \delta} \)

We introduce random variables \( K_{(N, \tau)} \in \text{Map}(\mathbb{D}(N, \tau), \mathbb{R}) \), defined by

\[
K_{(N, \tau)}(\xi, \zeta, \gamma) := |\xi| + (\tau + 1)|\zeta| + \lambda(\gamma) + 1.
\]

For fixed \((N, \tau)\), the value \( K_{(N, \tau)}(\xi, \zeta, \gamma) \) equals to the number of steps in the course of game \( G_{(N, \tau)} \) characterized by \((\xi, \zeta, \gamma)\). Terms building (18) can be easily interpreted: all \( N \) bits are in initial state \( |0\rangle \) for \(|\xi|\) steps; a length of the second part of successful round equals to \( \lambda(\gamma) + 1 \); a length of the second part of every unsuccessful round is given by \((\tau + 1)\), and \(|\zeta|\) equals to the number of unsuccessful rounds.

In Appendix C.3 it is shown that average number of steps \( \langle K \rangle_{\mathbb{D}(N, \tau)} \) is equal to:

\[
\langle K \rangle_{\mathbb{D}(N, \tau), \delta} := \sum_{(\xi, \zeta, \gamma) \in \mathbb{D}(N, \tau)} K_{(N, \tau)}(\xi, \zeta, \gamma) \delta(\xi, \zeta, \gamma) = \langle A \rangle_{\left(\mathbb{D}(N, \tau), \beta\right)} P(N, \tau)^{-1}.
\]

Using (12) and (16) we finally obtain:

\[
\langle K \rangle_{\mathbb{D}(N, \tau), \delta} = \left(1 - q^{\tau}\right)^N - 1 - \tau(1 - q^N) + (1 - q^N) \sum_{j=1}^{\tau-1} (1 - q^j)^N \left[q^N(1 - q^{\tau})^N - (1 - q^{\tau+1})^N\right]^{-1},
\]

i.e., exactly the formula proclaimed earlier in section 2.2. Note, that although the right-hand side of (20) is well defined for \( N \in \mathbb{R} \), the left-hand side was derived with the assumption that \( N \geq 2 \) is a natural number. Similarly, \( \langle K \rangle_{\mathbb{D}(N, \tau), \delta} \) rewritten as in D.3 has a form of the function well defined for \( \tau \in \mathbb{R} \), however, it was derived only for natural \( \tau \).

For \( \tau = 0 \), the average number of steps is equal to \( \langle K \rangle_{\mathbb{D}(N, 0), \delta} = (1 - q)^{-N} = p^{-N} \).

The limit of \( \langle K \rangle_{\mathbb{D}(N, \tau), \delta} \) for \( \tau \to \infty \) is calculated in Appendix D it is given by:

\[
\lim_{\tau \to \infty} \langle K \rangle_{\mathbb{D}(N, \tau), \delta} = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \frac{1}{1 - q^N}.
\]
Reposition time in probabilistic imperfect memories

As expected, (21) is identical to (1) because this limit corresponds to the perfect memory case. We should also note that (20) can be used to approximate values of (21), as for large $N$ its form is far more convenient for numerical calculations. Analysis of accuracy of this approximation and comparison with results from [6] is an interesting problem, which we plan to address in the near future.

5.3. Intuitive interpretation of (19) and a non-zero reset time

A formal derivation of (19) involved some calculations, but the result itself is not surprising. In the simplest case ($N = 1$) of game $G_{N}$, described in section 2.1, the average number of steps needed to obtain the success is equal to

$$\langle K_{G_{1}} \rangle = \frac{1}{p}.$$  

If procedure $T$ with probability $p$ transfers a bit from state $|0\rangle$ to $|1\rangle$, then to actually flip this bit from $|0\rangle$ to $|1\rangle$ on average $1/p$ applications of procedure $T$ are need. If time needed for a single application of $T$ is equal to $\langle t_{1} \rangle$, then the average waiting time for the successful transfer is given by

$$\left( \text{time needed for } \langle K_{G_{1}} \rangle \right) = \frac{\langle t_{1} \rangle}{p}.$$  

(22)

Note, that (19) has exactly the same structure as (22). Average number of steps needed for the successful end of the total game $G_{(N,\tau)}$ equals to the duration (average number of steps) of one round divided by the probability of success in a single round. Another simple model connected to this intuition is presented in Appendix E.

Up to now, we were assuming that after an unsuccessfully ended round the system is immediately reset to its initial state, but it is easy to release this assumption. To include a non-zero reset time $\Omega$, where $\Omega$ is also measured in steps defined by a time needed for application of procedure $T$ to the system considered, we just define the new random variables: $K_{\Omega}(\xi, \zeta, \gamma) := K(\xi, \zeta, \gamma) + \Omega|\zeta|$. Using (C.1), we obtain the formula:

$$\langle K_{\Omega} \rangle_{(D(N,\tau),\delta)} = \langle K \rangle_{(D(N,\tau),\delta)} + \Omega Q(N, \tau)P(N, \tau)^{-1}.$$  

Again, this result has very simple interpretation: on average, there are $Q(N, \tau)P(N, \tau)^{-1}$ unsuccessfully ended rounds for one round that ended successfully. Thus, the term $\Omega Q(N, \tau)P(N, \tau)^{-1}$ that has to be added to $\langle K \rangle_{(D(N,\tau),\delta)}$ to obtain $\langle K_{\Omega} \rangle_{(D(N,\tau),\delta)}$.

Summary

We have studied the system of $N$ independent identical bits (memories) and derived the formula for the average waiting time for obtaining $N$ successful bit-flips under the condition that the time difference between the first and the last from the bit-flips is not longer than $\tau$. Using simple mathematical tools we have provided an accurate, analytical description of this system and presented the results in a form allowing for effective numerical calculations for arbitrary parameters $N$ and $\tau$. Connecting the result
obtained with a quantum repeater rate analysis in the case of imperfect memories, we have shown that it can be also used as an approximation of identity characterizing the perfect memory case – a problem that was studied not only in the context of quantum repeater [3], but also in bioinformatics [6].

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Appendix A. Notation and Conventions.

Appendix A.1. Basic notation

\( \mathbb{N} = \{0, 1, 2, \ldots \} \) denotes set of natural numbers including 0. For \( a, b \in \mathbb{N} \) and \( a \leq b \) we define \( (a, b) := \{ n \in \mathbb{N} : a \leq n \leq b \} \). Set of real numbers is denoted by \( \mathbb{R} \). For \( r, s \in \mathbb{R} \) and \( r \leq s \) we define \( [r, s] := \{ z \in \mathbb{R} : r \leq z \leq s \} \) and \( ]r, s[ := \{ z \in \mathbb{R} : r < z < s \} \).

We assume that for \( a \in \mathbb{N} \), \( f : \mathbb{N} \rightarrow \mathbb{R} \), and an integer number \( b \) from inequality \( a > b \) follows that

\[
\sum_{n=a}^{b} f(n) := 0 \quad \text{and} \quad \prod_{n=a}^{b} f(n) := 1.
\]

For sets \( B \) and \( A \neq \emptyset \), the set of all mappings of \( A \) into \( B \) is denoted by \( \text{Map}(A, B) \). For any map \( \zeta \in \text{Map}(\{(a, b), \mathbb{N}\}) \) we define \( |\zeta| \) and \( \zeta! \) as

\[
|\zeta| := \sum_{n=a}^{b} \zeta(n) \quad \text{and} \quad \zeta! := \prod_{n=a}^{b} (\zeta(n))!.
\]

In this notation

\[
\frac{|\zeta|!}{\zeta!} = \frac{(\zeta(a) + \zeta(a+1) + \ldots + \zeta(b))!}{(\zeta(a)!) (\zeta(a+1)!) \ldots (\zeta(b)!)}. \]

Let \( \mathbb{M} \) be a finite (or countable) set and \( \mu \) a map \( \mu : \mathbb{M} \rightarrow [0, 1] ; \) \( (\mathbb{M}, \mu) \) denotes the space \( \mathbb{M} \) with measure \( \mu \) defined on a \( \sigma \)-algebra \( 2^\mathbb{M} \) of all subsets of \( \mathbb{M} \):

\[
\mu : 2^\mathbb{M} \ni A \rightarrow \mu(A) := \begin{cases} 0 & A = \emptyset \\ \sum_{a \in A} \mu(a) & A \neq \emptyset \end{cases} \in (\mathbb{R} \cup \{\infty\}).
\]

All pairs \( (\mathbb{M}, \mu) \) considered in this paper fulfil condition \( \mu(\mathbb{M}) \leq 1 \). For clarity, cases \( \mu(A) < 1 \) and \( \mu(A) = 1 \), for \( A \subseteq \mathbb{M} \), are treated separately. To distinguish between them, for a given map \( f : \mathbb{M} \rightarrow \mathbb{R} \), a value of the sum \( \sum_{a \in A} f(a) \mu(a) \) is denoted in this paper by

\[
\langle\langle f \rangle\rangle_{(A, \mu)} := \sum_{a \in A} f(a) \mu(a) \quad \text{when} \quad \mu(A) < 1
\]
Reposition time in probabilistic imperfect memories

and by
\[
\langle f \rangle_{(A,\mu)} := \sum_{a \in A} f(a)\mu(a) \quad \text{when} \quad \mu(A) = 1.
\]

We refer to \(\langle \langle f \rangle \rangle_{(A,\mu)}\) as the ‘expected sum value’ of \(f\), and call \(\langle f \rangle_{(A,\mu)}\) the ‘mean (average) value’ of \(f\).

To emphasize distinction between spaces with measures normalized to 1 and those with not normalized measures, we use the following convention: uppercase blackboard bold Latin letters denote sample spaces and the corresponding probability measures are denoted by lowercase Greek letters, e.g. \((\mathbb{A},\alpha)\). For the spaces with measures not normalized to 1 we use the opposite convention and we write, e.g. \((\Gamma,\gamma)\).

Appendix A.2. Some Useful Formulas

It is well known (see 0.231.2 in [7]) that for any \(a, b \in \mathbb{R}\) and any \(q \in ]-1,1[\) holds
\[
\sum_{k=0}^{\infty} (a + bk)q^k = \frac{a}{1-q} + \frac{bq}{(1-q)^2}.
\]
Combining that with the Newton’s multinomial formula it is easy to obtain the following relation: for any \(a, b \in \mathbb{R}\), \(J \in (\mathbb{N} \setminus \{0\})\), \(F_1, ..., F_J \in ]0,1[\) and \(\sum_{j=1}^{J} F_j < 1\)
\[
\sum_{\zeta(1)=0}^{\infty} \sum_{\zeta(2)=0}^{\infty} \sum_{\zeta(3)=0}^{\infty} |\zeta|^j (a + b|\zeta|) \prod_{j=1}^{J} F_{\zeta(j)} = \frac{a}{1-\sum_{j=1}^{J} F_j} + \frac{b\sum_{j=1}^{J} F_j}{(1-\sum_{j=1}^{J} F_j)^2}.
\]
For any \(q \in ]-1,1[\) and any \(M \in (\mathbb{N} \setminus \{0\})\) holds
\[
\sum_{\xi(1)=0}^{\infty} \sum_{\xi(2)=0}^{\infty} |\xi|^j q^{\xi} = \frac{M \cdot q}{(1-q)(M+1)} = \frac{M \cdot q}{(1-q)} \sum_{\xi(1)=0}^{\infty} \sum_{\xi(2)=0}^{\infty} q^{\xi}.
\]
\((A.1)\) can be easily verified by mathematical induction.

Appendix B. Calculations related to \(\Gamma(N,\tau)\)

In this appendix derivations of (5), (6) and (7) from section 3 are presented.

Appendix B.1. Calculation of \(g_{(N,\tau)}(S(N,\tau))\)

To derive (5) we recall that a round ends successfully if and only if between the first and the \(N\)th successful bit flip not more then \(\tau\) application of procedure \(T\) were needed
\[
(\gamma \in S(N,\tau)) \iff \left(\gamma(0) \in (1,N) \quad \text{and} \quad \gamma(\tau) = N - \sum_{l=0}^{\tau-1} \gamma(l) = \bar{\gamma}(\tau)\right).
\]
Combining this with definition (3), we obtain
\[
g_{(N,\tau)}(S(N,\tau)) = \sum_{\gamma(0)=1}^{\tau-1} \left(\frac{\gamma(0)}{\gamma(0)}\right) p^{\gamma(0)} q^{\bar{\gamma}(0)} \prod_{l=1}^{\tau-1} \sum_{\gamma(l)=0}^{\tau-1} \left(\frac{\gamma(l)}{\gamma(l)}\right) p^{\gamma(l)} q^{\bar{\gamma}(l)} p^{\bar{\gamma}(\tau)}
\]
We have shown that \( g_{(N,\tau)}(S(N,\tau)) = (1 - q^{\tau+1})^N - q^N(1 - q^\tau)^N \). Comparing formula above with (4) we see that \( g_{(N,\tau)}(S(N,\tau)) = \text{sp}(N,\tau) \), which ends derivation of (5).

Appendix B.2. Calculation of \( g_{(N,\tau)}(F(N,\tau)) \)

To derive (4), we recall that according to definition a round ends in failure if and only if \( \tau \) application of procedure \( T \) after the first successful bit flip still at least one bit remains in state \( |0\rangle \).

\[
(\gamma \in F(N,\tau)) \iff (\gamma(0) \in (1,N-1) \quad \text{and} \quad \forall l \in (1,\tau) \quad \gamma(l) \in \{0,\tilde{\gamma(l)}-1\}).
\]

Inserting that into (3), we calculate \( g_{(N,\tau)}(F(N,\tau)) \) and obtain, successively,

\[
g_{(N,\tau)}(F(N,\tau)) = \sum_{\gamma(0)=1}^{\tilde{\gamma}(0)-1} \left( \tilde{\gamma}(0) \right) p^{\gamma(0)} q^{\tilde{\gamma}(0)} \prod_{l=1}^{\tau} \sum_{\gamma(l)=0}^{\tilde{\gamma}(l)-1} \left( \tilde{\gamma}(l) \right) p^{\gamma(l)} q^{\tilde{\gamma}(l)}
\]

\[
= \sum_{\gamma(0)=1}^{\tilde{\gamma}(0)-1} \left( \tilde{\gamma}(0) \right) p^{\gamma(0)} q^{\tilde{\gamma}(0)} \prod_{l=1}^{\tau-1} \sum_{\gamma(l)=0}^{\tilde{\gamma}(l)-1} \left( \tilde{\gamma}(l) \right) p^{\gamma(l)} q^{\tilde{\gamma}(l)} \left( (q + p)\tilde{\gamma}(\tau) - p^{\tilde{\gamma}(\tau)} \right)
\]

\[
= \sum_{\gamma(0)=1}^{\tilde{\gamma}(0)-1} \left( \tilde{\gamma}(0) \right) p^{\gamma(0)} q^{\tilde{\gamma}(0)} \prod_{l=1}^{\tau-1} \sum_{\gamma(l)=0}^{\tilde{\gamma}(l)-1} \left( \tilde{\gamma}(l) \right) p^{\gamma(l)} q^{\tilde{\gamma}(l)} \left[ 1 - \left( p \sum_{j=0}^{0} q^j \right) \tilde{\gamma}(\tau) \right]
\]
Using (11), we rewrite this formula in a form: 

\[ g_{(N, \tau)}(F(N, \tau)) = 1 - q^N - sp(N, \tau) \]

which shows that (9) holds.

**Appendix B.3. Calculation of** \( g_{(N, \tau)}(\Gamma_N^\tau(N, \tau)) \)

For derivation of (7), it is convenient to separate the following three cases:

1) When \( \sigma = 0 \), which means that a round ended successfully at step \( k + 1 \) and, accordingly, \( g_{(N, \tau)}((\Gamma_N^\tau(N, \tau)) = p^N = sp(N, 0) - sp(N, -1). \)

2) When \( \sigma = 1 \), then at least one elementary success has to appear at step \( k + 1 \) and at least one at step \( k + 2 \), thus,

\[
g_{(N, \tau)}(\Gamma_N^1(N, \tau)) = \sum_{\gamma(0) = 1}^{\tilde{\gamma}(0) - 1} \left( \frac{\tilde{\gamma}(0)}{\gamma(0)} \right) p^{\gamma(0)} q^{\tilde{\gamma}(0)} \prod_{l=1}^{\tilde{\gamma}(l) - 1} \sum_{\gamma(l) = 0}^{\tilde{\gamma}(l)} \left( \frac{\tilde{\gamma}(l)}{\gamma(l)} \right) p^{\gamma(l)} q^{\tilde{\gamma}(l)} \times \\
\times \sum_{\gamma(0) = 1}^{\tilde{\gamma}(\tau - 1) - 1} \left( \frac{\tilde{\gamma}(\tau - 1)}{\gamma(\tau - 1)} \right) p^{\gamma(\tau - 1)} q^{\tilde{\gamma}(\tau - 1)} \left[ 1 - \left( p \sum_{j=0}^{p} q^j \right)^{\tilde{\gamma}(\tau - 1)} \right] \\
= \sum_{\gamma(0) = 1}^{\tilde{\gamma}(0) - 1} \left( \frac{\tilde{\gamma}(0)}{\gamma(0)} \right) p^{\gamma(0)} q^{\tilde{\gamma}(0)} \prod_{l=1}^{\tilde{\gamma}(l) - 1} \sum_{\gamma(l) = 0}^{\tilde{\gamma}(l)} \left( \frac{\tilde{\gamma}(l)}{\gamma(l)} \right) p^{\gamma(l)} q^{\tilde{\gamma}(l)} \times \\
\times \left[ 1 - p^{\tilde{\gamma}(\tau - 1)} - \left( p + pq \sum_{j=0}^{pq} q^j \right) p^{\tilde{\gamma}(\tau - 1)} + p^{\tilde{\gamma}(\tau - 1)} \right] \\
= \sum_{\gamma(0) = 1}^{\tilde{\gamma}(0) - 1} \left( \frac{\tilde{\gamma}(0)}{\gamma(0)} \right) p^{\gamma(0)} q^{\tilde{\gamma}(0)} \left[ 1 - \left( p \sum_{j=0}^{pq} q^j \right)^{\tilde{\gamma}(1)} \right] \\
= 1 - p^N - q^N - \left[ \left( p + pq \sum_{j=0}^{pq} q^j \right)^N - p^N - \left( pq \sum_{j=0}^{pq} q^j \right)^N \right] \\
= (1 - q^N) - p^N \left[ \left( \sum_{j=0}^{\tau} q^j \right)^N - \left( \sum_{j=1}^{\tau} q^j \right)^N \right] \\
= (1 - q^N) - \left( 1 - q^{\tau + 1} \right)^N - q^N \left( 1 - q^{\tau} \right)^N.
\]

Using (11), we rewrite this formula in a form: 

\[ g_{(N, \tau)}(F(N, \tau)) = 1 - q^N - sp(N, \tau) \]

which shows that (9) holds.
3) Finally, for $\sigma \in (\overline{2}, \tau)$ we obtain, successively,

$$g_{(N, \tau)}(\Gamma_N^\sigma(N, \tau)) = \sum_{\gamma(0) = 1}^{\gamma(0) - 1} \left( \gamma(0) \right) p^{\gamma(0)} q^{\gamma(0)} \left[ \prod_{l = 1}^{\sigma - 1} \sum_{\gamma(l) = 0}^{\gamma(l) - 1} \left( \gamma(l) \right) p^{\gamma(l)} q^{\gamma(l)} \right] \times$$

$$\times \left[ \left( p \sum_{j = 0}^{1} q^j \right)^{\sigma - 1} - \left( p \sum_{j = 0}^{0} q^j \right)^{\sigma - 1} \right]$$

$$= \sum_{\gamma(0) = 1}^{\gamma(0) - 1} \left( \gamma(0) \right) p^{\gamma(0)} q^{\gamma(0)} \left[ \prod_{l = 1}^{\sigma - 2} \sum_{\gamma(l) = 0}^{\gamma(l) - 1} \left( \gamma(l) \right) p^{\gamma(l)} q^{\gamma(l)} \right] \times$$

$$\times \left[ \left( p \sum_{j = 0}^{2} q^j \right)^{\sigma - 2} - \left( p \sum_{j = 0}^{1} q^j \right)^{\sigma - 2} \right]$$

$$= \ldots$$

Comparing these three cases we see that, for $\sigma \in (0, \tau)$,

$$g_{(N, \tau)}(\Gamma_N^\sigma(N, \tau)) = sp(N, \sigma) - sp(N, \sigma - 1)$$

$$= \left[ (1 - q^{\sigma + 1})^N - (1 + q^N)(1 - q^\sigma)^N + q^N(1 - q^{\sigma - 1})^N \right]. \quad (B.1)$$

It is also easy to check that:

$$\sum_{\sigma = 0}^{\tau} g_{(N, \tau)}(\Gamma_N^\sigma(N, \tau)) = \sum_{\sigma = 0}^{\tau}(sp(N, \sigma) - sp(N, \sigma - 1))$$

$$= sp(N, \tau) - sp(N, -1) = sp(N, \tau) = g_{(N, \tau)}(S(N, \tau)), \quad (B.2)$$

and more generally, for $\tau' < \tau$,

$$\sum_{\sigma = 0}^{\tau'} g_{(N, \tau')}(\Gamma_N^\sigma(N, \tau)) = sp(N, \tau') = g_{(N, \tau')}(S(N, \tau')). \quad (B.3)$$

(B.3) suggests that there exists a natural correspondence between measures $g_{(N, \tau)}$ and $g_{(N, \tau')}$ and a simple embedding of set $\Gamma(N, \tau')$ into $\Gamma(N, \tau)$. For details see Appendix B.5.
Appendix B.4. Calculation of $\langle \langle \lambda \rangle \rangle_{(S(N,\tau),g)}$

To derive (9) we combine (B.1) and (B.2) and obtain:

$$\langle \langle \lambda \rangle \rangle_{(S(N,\tau),g)} = \tau \sum_{\sigma=0}^{\tau} \sum_{j=1}^{\tau} \left( sp(N,\sigma) - sp(N,\sigma - 1) \right) = \tau \sum_{j=1}^{\tau-1} sp(N,j) - \sum_{j=0}^{\tau-1} sp(N,j).$$

It means that:

$$\langle \langle \lambda \rangle \rangle_{(S(N,\tau),g)} = \tau \left[ (1 - q^{\tau+1})^N - q^N(1 - q^\tau)^N \right] - \sum_{j=0}^{\tau-1} \left[ (1 - q^{j+1})^N - q^N(1 - q^j)^N \right].$$

Appendix B.5. Correspondence between measures $g_{(N,\tau')}^\sigma$ and $g_{(N,\tau)}$

For fixed natural numbers $N, \tau, \tau'$ fulfilling conditions $N \geq 2$ and $\tau > \tau'$, there exists a natural embedding of set $\Gamma(N,\tau')$ into $\Gamma(N,\tau)$

$$z_{\tau\tau'} : \Gamma(N,\tau') \ni \gamma \rightarrow z_{\tau\tau'}(\gamma) \in \Gamma(N,\tau)$$

defined as

$$z_{\tau\tau'}(\gamma)(l) := \begin{cases} \gamma(l) & \text{for } l \in (0,\tau) \\ 0 & \text{for } l \in (\tau'+1,\tau). \end{cases}$$

From this definition follows that, for $1 \leq M \leq N$ and $0 \leq \sigma \leq \tau'$,

$$z_{\tau\tau'}(\Gamma_M^\sigma(N,\tau')) = \Gamma_M^\sigma(N,\tau).$$

Thus, we can write the following relation between measures $g_{(N,\tau')}^\sigma$ and $g_{(N,\tau)}$

$$g_{(N,\tau)}(\Gamma_M^\sigma(N,\tau)) = g_{(N,\tau)}(z_{\tau\tau'}(\Gamma_M^\sigma(N,\tau'))) = q^{(N-M)(\tau-\tau')} g_{(N,\tau')}(\Gamma_M^\sigma(N,\tau')),$$

which is valid for $0 \leq \sigma \leq \tau'$. In particular, for $M = N$, identity

$$g_{(N,\tau)}(\Gamma_N^\sigma(N,\tau)) = g_{(N,\tau')}(\Gamma_N^\sigma(N,\tau'))$$

is obtained. From that follows:

$$g_{(N,\tau)} \left( \bigcup_{\sigma=0}^{\tau'} \Gamma_N^\sigma(N,\tau) \right) = g_{(N,\tau')} (S(N,\tau')) = sp(N,\tau'),$$

which gives exactly (B.3).

Appendix C. Calculations related to $(D(N,\tau),\delta)$

Appendix C.1. Additional conventions used for a compact notation

Note, that set $(0,|\zeta|)$ might be rewritten as the following sum of disjoint sets:

$$\{0\} \cup (1,\zeta(1)) \cup (\zeta(1) + 1,\zeta(2)) \cup \ldots \cup (\zeta(J-1) + 1,\zeta(J)).$$
For every map \( \xi \in \text{Map}(\{0, |\xi|\), N) \) we define:
\[
\xi_0 := \xi \bigg|_{\{0\}} \quad \xi_1 := \xi \bigg|_{\{1, \xi(1)\}} \quad \xi_2 := \xi \bigg|_{\{\xi(1)+1, \xi(2)\}} \ldots \quad \xi_j := \xi \bigg|_{\{\xi(j-1)+1, \xi(j)\}},
\]
where \( \xi \bigg|_{(a,b)} \) denotes restriction of a map \( \xi \) to the set \((a, b)\). This notation allows to write the following, valid for \( j \in (1, J - 1) \), abbreviations for sums over \( \xi \)
\[
\sum_{\xi(j+1) = 0}^{\infty} \sum_{\xi(j) = 0}^{\infty} \sum_{\xi(\xi(1)) = 0}^{\infty} \ldots \sum_{\xi(1) = 0}^{\infty}
\]
Accordingly,
\[
\sum_{\xi(j+1) = 0}^{\infty} q^{N|\xi|} := \sum_{\xi(j+1) = 0}^{\infty} q^{N\xi(j+1)} \sum_{\xi(j) = 0}^{\infty} q^{N\xi(j)} \ldots \sum_{\xi(1) = 0}^{\infty} q^{N\xi(1)}
\]
For \( j = 0 \), the corresponding formulas are given by
\[
\sum_{\xi_0 = 0}^{\infty} := \sum_{\xi(1) = 0}^{\infty} \sum_{\xi(2) = 0}^{\infty} \ldots \sum_{\xi(\xi(1)) = 0}^{\infty}
\]
and
\[
\sum_{\xi_0 = 0}^{\infty} q^{N|\xi|} := \sum_{\xi(1) = 0}^{\infty} q^{N\xi(1)} \sum_{\xi(2) = 0}^{\infty} q^{N\xi(2)} \ldots \sum_{\xi(\xi(1)) = 0}^{\infty} q^{N\xi(\xi(1))}
\]
These definitions are introduced to help presenting calculations in a relatively compact form. For the same reason we define \( \xi := \xi \bigg|_{\{1, \xi(J)\}} \) and note that \( |\xi| \) can be expressed as
\[
|\xi| = \xi(0) + |\xi| = \xi(0) + |\xi_1| + |\xi_2| + \ldots + |\xi_J|.
\]

**Appendix C.2. Probabilities \( \delta(\mathbb{D}(N, \tau)) \)**

To calculate (17), we use formulas and conventions from Appendix A and the notation introduced in Appendix C.1. We obtain:
\[
\delta(\mathbb{D}(N, \tau)) = \sum_{(\xi, \zeta, \gamma) \in \mathbb{D}(N, \tau)} \delta(\xi, \zeta, \gamma) = \\
= \sum_{\zeta(1) = 0}^{\infty} \ldots \sum_{\zeta(J) = 0}^{\infty} \sum_{\xi(0) = 0}^{\infty} \sum_{\gamma \in S(N, \tau)} g(\gamma) q^{N\xi(0)} q^{N|\xi|} \ldots q^{N|\xi|} \times \\
\times \frac{|\xi|!}{\prod \xi(j)} g(f_1)^{\zeta(1)} \ldots g(f_J)^{\zeta(J)} \\
= \sum_{\zeta(1) = 0}^{\infty} \ldots \sum_{\zeta(J) = 0}^{\infty} \frac{|\xi|!}{\prod \xi(j)} \left( \sum_{\xi(0) = 0}^{\infty} q^{N\xi(0)} \sum_{\gamma \in S(N, \tau)} g(\gamma) \right) \left( \sum_{\xi_1 = 0}^{\infty} q^{N|\xi_1|} g(f_1)^{\zeta(1)} \right) \times \\
\times \left( \sum_{\xi_2 = 0}^{\infty} q^{N|\xi_2|} g(f_2)^{\zeta(2)} \right) \ldots \left( \sum_{\xi_J = 0}^{\infty} q^{N|\xi_J|} g(f_J)^{\zeta(J)} \right)
\]
The second sum is given by Appendix C.3. Average values \( \langle K \rangle \beta = \text{Represents time in probabilistic imperfect memories} \)

where \( P = \text{is equal to conventions from Appendix A and notation introduced in Appendix C.1} \). The first sum for clarity, we calculate each of these sums separately. As before, we use formulas and conventions from Appendix A and notation introduced in Appendix C.1. The first sum is equal to

\[
\sum_{\zeta} |\xi|^\delta(\xi, \zeta, \gamma) = \sum_{\zeta(1)}^\infty \ldots \sum_{\zeta(J)}^\infty \frac{|\xi|!}{\zeta!} \left( \sum_{\zeta(0)}^\infty q^{N\zeta(0)} \sum_{\gamma \in S(N, \tau)} g(\gamma) \sum_{\xi_1 = 0}^\infty q^{N\xi_1} \ldots \sum_{\xi_J = 0}^\infty q^{N\xi_J} |\xi|^J \prod_{j=1}^J g(f_j)^{C(j)} \right)
\]

\[
= P(N, \tau) \sum_{\zeta(1)}^\infty \ldots \sum_{\zeta(J)}^\infty \frac{|\xi|!}{\zeta!} \left( \prod_{j=1}^J g(f_j)^{C(j)} \right) \sum_{\zeta(0)}^\infty q^{N\zeta(0)} \sum_{\gamma \in S(N, \tau)} g(\gamma) \sum_{\xi_1 = 0}^\infty q^{N\xi_1} \ldots \sum_{\xi_J = 0}^\infty q^{N\xi_J} |\xi|^J
\]

\[
= P(N, \tau) \sum_{\zeta(1)}^\infty \ldots \sum_{\zeta(J)}^\infty \frac{|\xi|!}{\zeta!} \left( \prod_{j=1}^J g(f_j)^{C(j)} \right) |\xi|^N (1 - q^N)^{-|\xi|+1}
\]

\[
= q^N (1 - q^N)^{-1} \sum_{\zeta(1)}^\infty \ldots \sum_{\zeta(J)}^\infty \frac{|\xi|!}{\zeta!} |\xi|^J \beta(F_1)^{C(1)} \ldots \beta(F_J)^{C(J)}
\]

The second sum is given by

\[(\tau + 1) \sum_{\zeta(1)}^\infty \ldots \sum_{\zeta(J)}^\infty \frac{|\xi|!}{\zeta!} |\xi| \times \left( \sum_{\zeta(0)}^\infty q^{N^{\zeta(0)}} \sum_{\gamma \in S(N, \tau)} g(\gamma) \sum_{\xi_1 = 0}^\infty q^{N\xi_1} g(f_1)^{C(1)} \ldots \sum_{\xi_J = 0}^\infty q^{N\xi_J} g(f_J)^{C(J)} \right)
\]

\[(\tau + 1) P(N, \tau) \sum_{\zeta(1)}^\infty \ldots \sum_{\zeta(J)}^\infty \frac{|\xi|!}{\zeta!} |\xi|^J \beta(F_1)^{C(1)} \ldots \beta(F_J)^{C(J)}
\]

\[= (\tau + 1) Q(N, \tau) P(N, \tau)^{-1} \quad (C.1)\]
The third sum can be calculated as
\[
\sum_{(\xi, \zeta, \gamma) \in \mathbb{D}(N, \tau)} (\xi(0) + 1 + \lambda(\gamma))\delta(\xi, \zeta, \gamma) = \sum_{\xi(1)=0}^{\infty} \cdots \sum_{\xi(j)=0}^{\infty} \frac{|\xi|!}{\zeta!} \times 
\times \left( \sum_{\xi(0)=0}^{\infty} q^{N\xi(0)} \sum_{\gamma \in S(N, \tau)} g(\gamma)[\xi(0) + 1 + \lambda(\gamma)] \right) \sum_{\xi_1=0}^{\infty} \cdots \sum_{\xi_j=0}^{\infty} q^{N\xi_j} \prod_{j=1}^{j} g(f_j)\zeta^{(j)} 
= \langle \langle \Lambda \rangle \rangle_{(\mathbb{B}(N, \tau), \beta)} \sum_{\xi(1)=0}^{\infty} \cdots \sum_{\xi(j)=0}^{\infty} \frac{|\xi|!}{\zeta!} \beta(\zeta_1)\zeta^{(1)} \cdots \beta(\zeta_j)\zeta^{(j)} 
= \langle \langle \Lambda \rangle \rangle_{(\mathbb{B}(N, \tau), \beta)} \mathcal{P}(N, \tau)^{-1}.
\]

Finally, combining these three expressions we obtain
\[
\langle \mathcal{K} \rangle_{(\mathbb{D}(N, \tau), \delta)} = \langle \langle \Lambda \rangle \rangle_{(\mathbb{B}(N, \tau), \beta)} \mathcal{P}(N, \tau)^{-1} + \left( \tau + (1 - q^N)^{-1} \right) \mathcal{Q}(N, \tau)\mathcal{P}(N, \tau)^{-1} 
= \langle \langle \Lambda \rangle \rangle_{(\mathbb{B}(N, \tau), \beta)} \mathcal{P}(N, \tau)^{-1} + \langle \langle \Lambda \rangle \rangle_{(\mathbb{F}(N, \tau), \beta)} \mathcal{P}(N, \tau)^{-1} 
= \langle \Lambda \rangle_{(\mathbb{B}(N, \tau), \beta)} \mathcal{P}(N, \tau)^{-1}.
\]

Explicit formulas for \( \langle \Lambda \rangle_{(\mathbb{B}(N, \tau), \beta)} \) and \( \mathcal{P}(N, \tau) \) can be found in section 4.

Appendix D. The Limit \( \lim_{\tau \to \infty} \langle \mathcal{K} \rangle_{(\mathbb{D}(N, \tau), \delta)} \) for \( N \geq 2 \)

To derive (21), it is convenient to treat nominator and denominator of \( \langle \mathcal{K} \rangle_{(\mathbb{D}(N, \tau), \delta)} \) separately. A denominator of \( \langle \mathcal{K} \rangle_{(\mathbb{D}(N, \tau), \delta)} \) equals to \( sp(N, \tau) \); it can be rewritten as
\[
sp(N, \tau) = (1 - q^{\tau+1})^N - q^N(1 - q^\tau)^N 
= \sum_{n=0}^{N} (-1)^n \binom{N}{n} q^{(\tau+1)n} - q^N \sum_{n=0}^{N} (-1)^n \binom{N}{n} q^{\tau n} 
= \sum_{n=0}^{N} (-1)^n \binom{N}{n} q^{\tau n} (q^n - q^N) 
= 1 - q^N + \sum_{n=1}^{N} (-1)^n \binom{N}{n} q^{\tau n} (q^n - q^N).
\]

Because \( 1 > q > 0 \),
\[
\lim_{\tau \to \infty} sp(N, \tau) = 1 - q^N. \tag{D.1}
\]

Let \( L(N, \tau) \) denotes a nominator of \( \langle \mathcal{K} \rangle_{(\mathbb{D}(N, \tau), \delta)} \), i.e.,
\[
L(N, \tau) := 1 - (1 - q^\tau)^N + (1 - q^N) \left[ \tau - \sum_{j=1}^{\tau-1} (1 - q^j)^N \right].
\]

To calculate \( \lim_{\tau \to \infty} L(N, \tau) \), we rewrite formula above using relations
\[
1 - (1 - q^\tau)^N = \sum_{n=1}^{N} (-1)^{n+1} \binom{N}{n} q^{\tau n}
\]
and
\[ \tau - \sum_{j=1}^{\tau-1} (1 - q^j)^N = \tau - \sum_{j=1}^{\tau-1} \sum_{n=0}^{N} (-1)^n \binom{N}{n} q^j^n = \tau - \sum_{j=1}^{\tau-1} \left[ 1 + \sum_{n=1}^{N} (-1)^n \binom{N}{n} q^j^n \right] \]

\[ = 1 - \sum_{n=1}^{N} (-1)^n \binom{N}{n} \sum_{j=1}^{\tau-1} q^j^n = 1 - \sum_{n=1}^{N} (-1)^n \binom{N}{n} q^n \frac{1 - q^\tau n}{1 - q^n}, \]
valid for \( \tau \geq 2 \). (When calculating limit \( \tau \to \infty \), this condition is fulfilled.) We obtain:
\[ L(N, \tau) = \sum_{n=1}^{N} (-1)^{n+1} \binom{N}{n} q^\tau n + (1 - q^N) \left[ 1 - \sum_{n=1}^{N} (-1)^n \binom{N}{n} q^n \frac{1 - q^\tau n}{1 - q^n} \right] \]
and the limit of \( L(N, \tau) \) is given by
\[ \lim_{\tau \to \infty} L(N, \tau) = (1 - q^N) \sum_{n=1}^{N} (-1)^{(n+1)} \binom{N}{n} \frac{1}{1 - q^n}. \quad (D.2) \]

From \((D.1)\) and \((D.2)\) follows that:
\[ \lim_{\tau \to \infty} \langle K \rangle_{(D(N,\tau),\delta)} = \sum_{n=1}^{N} (-1)^{(n+1)} \binom{N}{n} \frac{1}{1 - q^n}. \]

**Remark 4:** Calculations presented above allow to write \([20]\) in a different form:
\[ \langle K \rangle_{(D(N,\tau),\delta)} = \frac{(1 - q^N) - \sum_{n=1}^{N} (-1)^n \binom{N}{n} \left[ q^\tau n + q^n (1 - q^N) (1 - q^\tau n) (1 - q^n)^{-1} \right]}{1 - q^N + \sum_{n=1}^{N} (-1)^n \binom{N}{n} q^\tau n (q^n - q^N)}. \quad (D.3) \]
Right-hand-side of this formula is well defined for \( \tau \in \mathbb{R} \), but we should remember that \( \langle K \rangle_{(D(N,\tau),\delta)} \) was derived for \( N, \tau \in (\mathbb{N} \setminus \{0, 1\}). \quad \square \]

**Appendix E. Toy model**

In order to associate a more intuitive meaning to \([19]\), let us consider the following: for any two real numbers \( 0 < P < 1, Q = 1 - P \) and a map \( \varrho_\rho \) defined by
\[ \varrho_\rho : \mathbb{N} \ni n \longrightarrow \varrho_\rho(n) := Q^n P \in [0, 1], \]
pair \((\mathbb{N}, \varrho_\rho)\) defines a probability space. Introducing also two real parameters \( A, B \) and a random variable \( \mathcal{L}_{AB} : \mathbb{N} \ni n \rightarrow \mathcal{L}_{AB}(n) := A + nB \in \mathbb{R} \), we can easily check that:
\[ \langle \mathcal{L}_{AB} \rangle_{(N,\varrho_\rho)} = A + BQ/P \]
For \( P = \mathcal{P}(N, \tau), Q = \mathcal{Q}(N, \tau) \) defined by \([12]\) and \([13]\) and \( A, B \) equal to:
\[ A = \langle \mathbf{\Lambda} \rangle_{(Q(N,\tau),\beta)} \quad \text{and} \quad B = \langle \mathbf{\Lambda} \rangle_{(Q(N,\tau),\beta)} / \mathcal{Q}(N, \tau), \]
we obtain:
\[ \langle \mathcal{L}_{AB} \rangle_{(N,\varrho_\rho)} = \langle \mathbf{\Lambda} \rangle_{(Q(N,\tau),\beta)} / \mathcal{P}(N, \tau) + \langle \mathbf{\Lambda} \rangle_{(Q(N,\tau),\beta)} / \mathcal{P}(N, \tau) \]
\[ = \langle \mathbf{\Lambda} \rangle_{(Q(N,\tau),\beta)} / \mathcal{P}(N, \tau) = \langle K \rangle_{(D(N,\tau),\delta)}. \]
\( A \) denotes an average number of steps in a round that ended successfully, \( B \) is an average number of steps in a round that ended unsuccessfully, \( Q/P \) is an average number of rounds that ended unsuccessfully for a round that ended successfully:

Average number of steps = (average number of steps in a round that ended successfully) + (average number of steps in a round that ended unsuccessfully) \times (average number of rounds that ended unsuccessfully for a round that ended successfully)

To include a non-zero reset time \( \Omega \), we insert \( B_\Omega = B + \Omega \) instead of \( B \) in the relations above and reproduce results from section 5.3.

Bibliography

[1] Feller W 1968 An Introduction to Probability Theory and Its Applications vol 1 (New York London Sydney: John Wiley & Sons, Inc.)
[2] Dür W, Briegel H-J and Zoller P 2007 Lectures on Quantum Information ed D Bruß and G Leuchs (WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim) pp 505–14
[3] Bernardes N K, Praxmeyer L and van Loock P 2011 Phys. Rev. A 83 012323
[4] Bernardes N K, Praxmeyer L and van Loock P 2013 Memory requirements on distillation-free quantum repeaters with deterministic swapping (to be submitted soon)
[5] Szpankowski W and Rego V 1990 Computing 43 401–10
[6] Eisenberg B 2008 Statistics & Probability Letters 78 135–43
[7] Gradshteyn I S and Ryzhik I M 2007 Table of Integrals, Series, and Products (Elsevier Inc., London 7th edition)