Physical States for Non-linear $SO(N)$ Superstrings

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Abstract

We study some low-lying physical states in a superstring theory based on the quadratically non-linear $SO(N)$–extended superconformal algebra. In the realisation of the algebra that we use, all the physical states are discrete, analogous to the situation in a one-scalar bosonic string. The BRST operator for the $N = 3$ case needs to be treated separately, and its construction is given here.

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1 Introduction

Superstring theories based on \( N = 1 \) superconformal algebras are the most realistic and the most commonly studied string theories so far. It is natural, however, to look for string theories with higher worldsheet symmetries, and to investigate their properties. If one insists on supersymmetry and linearity of the underlying conformal algebra, the list of possibilities is fairly short, namely the \( N \)–extended superconformal algebras with \( N \leq 4 \), and their twisted and truncated versions. However, if one relaxes the requirement of linearity, then there exists two kinds of superconformal algebras that go beyond \( N = 4 \). These are the super–\( W \) type algebras, which include generators with spin higher than two, and the higher-extended (i.e. \( N > 4 \)) superconformal algebras which are also nonlinear, but they do not contain generators with spin higher than two \([1]\).

String theories based on \( W \)-type algebras are very complicated, and so far significant progress has been made only in the study of bosonic \( W \)-strings. A nice feature of the \( W \)-algebras is that their realisation allows a Minkowskian spacetime interpretation. In the case of higher-extended superalgebras, the situation is different, in that while they are algebraically simpler, because they do not contain higher-spin generators, their known realisations do not seem to allow a Minkowskian spacetime interpretation. Instead, they seem to live on group manifolds. Nonetheless, studies of strings on manifolds other than Minkowski spacetime have been very useful in the past, and we expect that the situation may be similar here.

With these considerations in mind, in this paper we take a first look at the properties of a string theory based on an \( SO(N) \)-extended superconformal algebra, where our main interest is for \( N > 2 \). The quantum BRST operator for this algebra has already been constructed \([2, 3]\). The nilpotency of the BRST operator imposes conditions on the central extensions occurring in the algebra. The critical central charge of the energy-momentum tensor for a string theory based on \( SO(N) \)-extended superconformal algebra is \( \text{dim}SO(N) + 1 \). Realisations of the algebra are not easy to come by. So far, there exists only one realisation that satisfies the conditions imposed on the central extensions \([5]\). It makes use of a real scalar with a nonvanishing background charge, \( N \) real fermions and Kac-Moody currents of \( SO(N) \) which may be constructed out of any fields.

In this paper, we shall look at the spectrum of physical states. An analysis of the complete spectrum is a highly nontrivial and complicated task which goes beyond the scope of the present paper. Instead, we shall investigate the spectrum of physical vertex operators at the standard ghost sector. We shall also look for examples of vertex operators in the nonstandard ghost sector.

We also investigate the case of \( N = 3 \), which turns out to require special treatment. This is because the Kac-Moody level must vanish for nilpotence of the BRST operator. A suitable realisation of the algebra is not known. However, in this paper, we construct the abstract BRST operator for this case.

2 The \( SO(N) \)-Extended Superconformal Algebra and Its Realisation

The \( SO(N) \) nonlinear superconformal algebra is generated by the energy-momentum tensor \( T(z) \), the conformal dimension 3/2 fermionic supercurrents \( G_i(z), i = 1, \ldots, N \) and the di-
In the OPE language we have
\[ G_i(z)G_j(w) \sim \frac{C \delta^{ij}}{(z-w)^2} + \frac{\sigma \lambda_a^i J^a(w)}{(z-w)} + \frac{\frac{1}{2} \sigma \lambda_a^i \partial J^a(w)}{(z-w)} + \frac{2 \delta^{ij} T(w)}{(z-w)} + \frac{\mu P_{ab} J^a J^b(w)}{(z-w)}, \]
\[ J^a_z G_i(w) \sim -\lambda_a^i G_i(w), \]
\[ J^a_z J^b(w) \sim -\frac{\delta_{ab}}{2} k \epsilon \delta_{ab} + f_{abc} J^c(w), \]

where the generators \( \lambda_a^i \) and the structure constants \( f_{abc} \) satisfy
\[ [\lambda_a^i, \lambda_b^j] = f_{abc}^{\phantom{abc}} \lambda_c^k, \quad \text{tr} (\lambda_a^i \lambda_b^j) = -\epsilon \delta_{ab}, \]
\[ \lambda_a^i \lambda_b^j \lambda_c^k = \frac{\epsilon}{2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}), \quad f_{acd}^{\phantom{acd}} f_{bde} = -c_{\gamma} \delta_{ab} = -\epsilon (N-2) \delta_{ab}, \]

where \( \epsilon \) is the square of the longest root (in our conventions, \( \epsilon = 2 \) for even \( N \), and \( \epsilon = 4 \) for odd \( N \)). The normal ordering in the nonlinear term is defined by
\[ (J J)(w) = \frac{1}{2 \pi i} \oint d\zeta J^a(\zeta) J^a(w)(\zeta-w), \]

and furthermore the tensor \( P_{ij}^{ab} \) is given by
\[ P_{ij}^{ab} = (\lambda_a^i \lambda_b^j)_{ij} + (\lambda_b^j \lambda_a^i)_{ij} + 2 \delta_{j} \delta_{ab}. \]

The closure requirement of the OPE algebra imposes the following conditions on the parameters occurring in the algebra:
\[ \mu = \frac{2}{\epsilon^2 (k+N-3)}, \quad \sigma = \mu \epsilon (2k + N - 4), \quad C = \frac{k \epsilon}{2} \sigma. \]

The remaining part of the algebra involves the energy momentum tensor. It has the standard form with central extension given by
\[ c = \frac{k(6k + N^2 - 10)}{2(k+N-3)}. \]

The algebra can be realised in terms of a real scalar \( \phi \) with a background charge \( \alpha \) and \( N \) real fermions \( \psi^i \) as follows:
\[ T = -\alpha \partial^2 \phi - \frac{1}{2} \partial \phi \partial \phi - \frac{1}{2} \psi^i \partial \psi^i - \frac{1}{\epsilon (k+N-3)} K^a K_a, \]
\[ G_i = 2i \alpha \partial \psi_i + i \partial \phi \psi_i - \frac{4i \alpha}{\ell \epsilon} \lambda_a^i K_a \psi^j, \]
\[ J^a = K^a + \frac{4}{\ell} \lambda_a^i \psi^i \psi^j. \]

Here the currents \( K^a \) obey a Kac-Moody algebra with level \( \ell \), and we must require the relations
\[ \ell = k-1, \quad \alpha^2 = \frac{\ell^2}{4(k+N-3)}. \]
The free fields $\phi$ and $\psi$ obey the OPE's:
\[
\phi(z)\phi(w) \sim -\log |z - w|, \quad \psi_i(z)\psi_j(w) \sim \delta_{ij}(z - w)^{-1}.
\] (8)

Note that the realisation (6) has a divergent coefficient when $k + N - 3 = 0$, in which case the extended superconformal algebra (1) has singular structure constants. The algebra becomes degenerate since after rescaling the fermionic currents $G^i$ to achieve non-singular structure constants, the central extension for the fermionic currents becomes zero. As we shall see in the next section, for the case of $N = 3$ the criticality condition precisely requires that $k = 0$, and hence the BRST analysis for this case has to be treated separately.

For $N = 2$, the realisation (6) reduces to the one discussed in [6], in the special case of one complex scalar and one complex fermion. The real part of the complex scalar corresponds to $\phi$, while its imaginary part is used to realise the $U(1)$ current $K$. In fact, the $N = 2$ case allows a multi complex scalar and fermion realisation, for which the physical states have been studied in [6].

3 The BRST Operator

It turns out that the BRST operator for the algebra (2.1) is nilpotent provided that
\[
k = 6 - 2N. \tag{9}
\]

Note that $k = 0$ for $N = 3$, in which case the algebra (1) becomes singular. Thus the case of $N = 3$ has to be discussed separately, and will be given at the end of this subsection. For now we shall proceed with the general discussion for $N \geq 4$. With the criticality requirement of the level number $k$ given in (9), the background charge becomes
\[
\alpha = \frac{2N - 5}{2\sqrt{3} - N}. \tag{10}
\]

One can check that the matter contribution to the central extension is given by
\[
c = \left( \frac{N}{2} \right) + (1 + 12\alpha^2) + \left( \frac{\ell \dim G}{\ell + C_V} \right),
\] (11)

where the parantheses in the first line indicate the contributions of $\psi^i, \phi$ and the $K^a$ respectively. The parameter $\alpha$ assumes the value given in (10), and $C_V$ is defined in (2). This value of $c$ is exactly cancelled by the total ghost contributions of spin 2, 3, and 1, which is given by $(-1)^{2s+1}(12s^2 - 12s + 2)$ for each field of spin $s$.

The BRST operator is given by
\[
Q = Q_0 + Q_1 + Q_2 + Q_3,
\]
\[
Q_0 = c \left( -\frac{1}{2} \partial^2 \phi - \alpha \partial^2 \phi - \frac{1}{2} \psi^i \partial \psi^i - \frac{1}{\epsilon(3-N)} K^a K_a - \beta_a \partial r^a - \frac{3}{2} \partial s^i \right),
\]
\[
Q_1 = s^i \left( 2\alpha \partial \psi_i + \partial \phi \psi - \frac{4\alpha}{\ell \epsilon} \lambda^a_{ij} K_a \psi^j \right) - b s_i s_j + \frac{2}{\epsilon} \lambda^a_{ij} \beta_a s_i \partial s_j,
\]
\[
Q_2 = \gamma_a \left( K^a + \frac{1}{2} \lambda^a_{ij} \psi^i \psi^j - \frac{1}{\epsilon} f_{bc} \beta_b \gamma^c + \lambda^a_{ij} r_i s_j \right),
\]
\[
Q_3 = -\frac{3}{2} \mu P^{ab}_{ij} J_a r_b s^i s^j - \frac{3}{4} \mu^2 P^{abc}_{ij} P^{def}_{kl} J_{ac} \beta_b \beta_d \beta_e s^i s^j s^k s^\ell. \tag{12}
\]
The ghost-antighost pairs \((c, b), (s, r, (\gamma, \beta))\) correspond to the generators \(T, G\) and \(J\), respectively. They satisfy the following OPEs:

\[
c(z) b(w) \sim (z - w)^{-1}, \quad s_i(z) r_j(w) \sim \delta^i_j (z - w)^{-1}, \quad \gamma^a(z) \beta^b(w) \sim \delta^{ab}(z - w)^{-1}. \quad (13)
\]

Denoting the spin-\(s\) generators by \(T^{(s)}\), and the corresponding ghost fields by \((c^{(s)}, b^{(s)})\), with \(s = 1, \frac{3}{2}, 2\), we can define ghost generators \(T^{(s)}_{gh}\) via the equations \([Q, b^{(s)}] \equiv T^{(s)}_{tot} = T^{(s)} + T^{(s)}_{gh}\). For linear algebras, one can then write \(Q = \sum c^{(s)} (T^{(s)} + \frac{i}{2} T^{(s)}_{gh})\). However, in our case while \(T_{gh}\) and \(J^a_{gh}\) obey the same algebra as \(T\) and \(J\) do, \(G_{gh}\) does not obey the same algebra as \(G\).

Therefore, it is not as useful to write the ghostly contributions in terms of \(T^{(s)}_{gh}\). In fact, one can check they cannot all be expressed in terms of these currents since there are seven-ghost terms in the BRST operator. Nonetheless, it is true here that \(Q_0 = c T_{tot}\) and \(Q_2 = \gamma a J^a_{tot}\).

The problem of determining the spectrum of physical states amounts to finding all nontrivially BRST invariant vertex operators built from the matter and ghost fields. A physical state corresponding to a physical vertex operator \(V(z)\) is expressed as \(V(0)|0\rangle >\), where \(|0\rangle >\) is the \(SL(2, C)\) invariant vacuum. Denoting the modes of spin-\(s\) ghost system by \((c^{(s)}_n, b^{(s)}_n)\), we recall that the \(SL(2, C)\) invariant vacuum has the property \(c^{(s)}_n|0\rangle = 0\) for \(n \geq s\) and \(b^{(s)}_n|0\rangle = 0\) for \(n \geq (1 - s)\).

Like the \(N = 1\) NSR superstring, it is necessary to bosonise the ghost fields \((s, r)\) for the fermionic currents \(G^i\):

\[
s_i = \eta_i e^{\sigma_i}, \quad r^i = \partial \xi^i e^{-\sigma_i}, \quad (14)
\]

where \(\sigma_i\) are scalar fields, and \((\eta_i, \xi_i)\) are anticommuting spin \((1, 0)\) fields. They obey the OPEs

\[
\sigma_i(z) \sigma_j(w) \sim -\delta_{ij} \log (z - w), \quad \eta_i(z) \xi_j(w) \sim -\frac{\delta^i_j}{z - w}. \quad (15)
\]

In terms of the bosonised fields the BRST operator can be written as

\[
Q = Q_0 + Q_1 + Q_2 + Q_3,
\]

\[
Q_0 = c \left(-\frac{\partial \phi \partial \phi}{\epsilon} - \frac{\partial^2 \phi}{\epsilon} - \frac{\partial \psi^i \partial \psi^j}{\epsilon} \right)\left(K^a \lambda^a K_a - \beta_a \partial \gamma^a - b \partial c\right)
- \frac{1}{2} \left(\partial \sigma_i \right)^2 - \partial^2 \sigma_i - \eta_i \partial \xi_i
\]

\[
Q_1 = \eta_i e^{\sigma_i} \left(2 \alpha \partial \psi_i + \partial \phi \psi_i - \frac{4 \alpha}{\epsilon} \lambda^a \lambda^a K_a \psi_j\right) - b \partial \eta_i \eta_j e^{2\sigma_i} + \frac{2}{\epsilon} \lambda^a \beta_a \eta_i \eta_j \gamma^i \partial \gamma^j + \frac{2}{\epsilon} \lambda^a \beta_a \eta_i e^{\sigma_i} \partial \gamma^j \xi^j
\]

\[
Q_2 = \gamma a \left(K^a + \frac{1}{2} \lambda^a \psi^j \psi^j - \frac{1}{2} f_{bc}^a \beta^b \gamma^c - \lambda^a \partial \xi_i \eta_j e^{2\sigma_i}\right)
\]

\[
Q_3 = \frac{1}{2} \mu^2 \left(\partial \sigma_i \right)^2 - \partial^2 \sigma_i - \eta_i \partial \xi_i
\]

\[
+ \left(\eta^a \partial \eta^a \right) \partial \eta^a \partial \eta^a + \frac{1}{2} P_{ii}^{ab, cd} \partial^2 \eta^a \partial \eta^b \partial \eta^c \partial \eta^d + \frac{1}{2} P_{kk}^{ab, cd} \partial^2 \eta^a \partial \eta^b \partial \eta^c \partial \eta^d
\]

\[
+ P_{ik}^{ab, cd} \partial \eta^a \partial \eta^b \partial \eta^c \partial \eta^d + P_{ik}^{ab, cd} \partial \eta^a \partial \eta^b \partial \eta^c \partial \eta^d + P_{ik}^{ab, cd} \partial \eta^a \partial \eta^b \partial \eta^c \partial \eta^d
\]

\[
+ P_{ik}^{ab, cd} \partial \eta^a \partial \eta^b \partial \eta^c \partial \eta^d + P_{ik}^{ab, cd} \partial \eta^a \partial \eta^b \partial \eta^c \partial \eta^d
\]

\[
\left(16\right)
\]

where summation over repeated indices is understood, and

\[
P_{ij,kl}^{ab, cd} := P_{ij}^{ab} P_{kl}^{cd}.
\]

\[
\left(17\right)
\]
It is to be understood that an expression such as $e^{a_1 + a_2}$ really means $e^{a_1} : e^{a_2} :$, which equals $- : e^{a_2} : e^{a_1} :$ since both of these exponentials are fermions. Thus we have $e^{a_1 + a_2} = - e^{a_2 + a_1}$ in this rather elliptical notation. The total energy-momentum tensor $T_{\text{tot}} = \{Q, b\}$ and total spin–1 current $J_{\text{tot}}^a = \{Q, \beta^a\}$ are given by

$$T_{\text{tot}} = -\frac{1}{\epsilon (3 - N)} K^a K^a,$$

$$-2b \partial c - \partial b c - \beta_\alpha \partial \gamma^\alpha - \frac{1}{4} (\partial \delta_1)^2 - \partial^2 \sigma_i - \eta_i \partial \xi_i,$$

$$J_{\text{tot}}^a = K^a + \frac{1}{4} \lambda^{ab} \lambda^c \gamma^d - f_{bc}^a \beta^b \gamma^c - \lambda^a_{ij} \partial \xi_j \epsilon^{-\sigma_1 + \sigma_2}.$$

The non-vanishing inner product for the ghost fields is given by

$$<0| \partial^2 c \partial c \gamma^1 \gamma^2 \cdots \gamma^{N(N-1)/2} e^{-2\sigma_1 - 2\sigma_2 - \cdots - 2\sigma_N} |0> = 1.$$

In analysing the spectrum of BRST nontrivial physical states, it is useful to keep in mind the picture changing operators and the notion of conjugate states. The only picture changing operators are defined as follows. Given a state characterised by $Z$, the picture changing operators and the notion of conjugate states. The total energy-momentum tensor $T$ together with the energy-momentum tensor $J$ can be set to the critical value $c$ to check that its conjugate state is BRST invariant.

$$G^i(z)G^j(w) \sim \frac{k - 1}{(z - w)^3} + \frac{2(1 - k)}{k} \frac{\varepsilon^{ijk}J^k}{(z - w)^2} + \frac{2\delta_{ij}T + \frac{1 - k}{k}(J^iJ^j + J^jJ^i)}{z - w},$$

$$J^i(z)G^j(w) \sim \frac{\varepsilon^{ijk}G^k}{z - w}, \quad J^i(z)J^j(w) \sim -\frac{1}{(z - w)^2} + \frac{\varepsilon^{ijk}J^k}{z - w},$$

$$G^i(z)G^j(w) \sim \frac{2\varepsilon^{ijk}J^k}{(z - w)^2} + \frac{\varepsilon^{ijk}\partial J^k + J^iJ^j + J^jJ^i}{z - w},$$

$$J^i(z)J^j(w) \sim \frac{\varepsilon^{ijk}G^k}{z - w}, \quad J^i(z)J^j(w) \sim \frac{\varepsilon^{ijk}J^k}{(z - w)},$$

The criticality condition $c_\text{tot} = c_\text{mat} + c_\text{gh} = 0$ implies that $k = -\frac{1}{3}$; however, the condition that $J^j_\text{tot}$ have zero central extension requires that $k = 0$. Thus one cannot build a nilpotent BRST operator for the algebra (21). However, when $k = 0$ the algebra (21) becomes singular. We can rescale the fermionic currents, $G^i \to G^i/\sqrt{k}$, to obtain a contracted algebra for $k = 0$. In terms of the rescaled currents, the algebra (21) becomes:

$$Q = c(T - b \partial c + \frac{1}{2}\varepsilon^{i} \partial s^i + \frac{1}{2}\partial r^i s^i - \beta^i \partial \gamma^i) + s^i \gamma^i G^i + \gamma^i J^i + \varepsilon^{ijk}\gamma^j s^k - \frac{1}{2}\varepsilon^{ijk}\gamma^j \beta^i \gamma^k - J^i \beta^i s^i s_j.$$

$$Q = c(T - b \partial c + \frac{1}{2}\varepsilon^{i} \partial s^i + \frac{1}{2}\partial r^i s^i - \beta^i \partial \gamma^i) + s^i \gamma^i G^i + \gamma^i J^i + \varepsilon^{ijk}\gamma^j s^k - \frac{1}{2}\varepsilon^{ijk}\gamma^j \beta^i \gamma^k - J^i \beta^i s^i s_j.$$
4 Physical States

Starting from the BRST operator (16) for the $SO(N)$-extended superconformal algebra for $N \geq 4$, we shall now study some of the physical states in this section. The simplest BRST invariant operator is clearly the unit operator, corresponding to the $SL(2,C)$ vacuum state. The next natural physical operator to consider is the one which corresponds to the lowest energy state. It is given by

$$V_0 = ce^{-\sigma_1 - \sigma_2 \cdots - \sigma_N} \xi^A \Phi_A e^{p\phi},$$  \hspace{1cm} (24)

where $A$ labels an arbitrary representation of $SO(N)$, $\xi^A$ is a polarisation vector, and $\Phi_A$ is the primary field under the Sugawara energy-momentum tensor $T_{SGW}(z) = -\frac{1}{\epsilon(3-N)} K^a K_a$. The field $\Phi_A$ satisfies the following OPEs:

$$K^a(z)\Phi_A(w) = -\frac{\tau^a_{AB}\Phi_B(w)}{(z-w)},$$  \hspace{1cm} (25)

$$T_{SGW}(z)\Phi_A(w) = \frac{C_R}{\epsilon(3-N)} \frac{\Phi_A(w)}{(z-w)^2} + \frac{\partial \Phi_A}{(z-w)},$$  \hspace{1cm} (26)

where $\tau^a_{AB}$ are the $SO(N)$ generators in the representation $R$ and $C_R$ is the eigenvalue of the second Casimir in this representation: $(\tau^a \tau^a)_{AB} = -C_R \delta_{AB}$. In particular, for the $r$-th rank totally antisymmetric representation we have $C_R(r) = \frac{r}{2r} (N - r)$.

The physical operator $V_0$ (24) has standard ghost structure, in that it is built with the standard ghost vacuum vertex operator $ce^{-\sigma_1 - \sigma_2 \cdots - \sigma_N}$. For such states, the physical-state condition can be summarised as follows:

$$L_0|\text{Phys}_{\text{mat}}\rangle = (1 - \frac{2}{N})|\text{Phys}_{\text{mat}}\rangle,$$

$$L_n|\text{Phys}_{\text{mat}}\rangle = 0 \quad n \geq 1,$$

$$G_{n+\frac{1}{2}}|\text{Phys}_{\text{mat}}\rangle = 0,$$

$$J_n^a|\text{Phys}_{\text{mat}}\rangle = 0 \quad n \geq 0,$$  \hspace{1cm} (27)

where the currents are constructed from matter only. The matter part of the physical operator $V_0$, being tachyonic, is already annihilated by the positive modes of the currents. However, the $J_0$ condition implies that $\Phi_A$ has to be a singlet under the Kac-Moody currents $K_a$, and the remaining mass-shell condition determines that the momentum for the scalar field $\phi$ is given by

$$p_{\pm} = \frac{1}{2\sqrt{3-N}} \left( -2N + 5 \pm 1 \right).$$  \hspace{1cm} (28)

In checking this, and in further calculations, it is useful to note the dimension formulae $\Delta (e^{p\phi}) = -\frac{1}{2} p(p+2\alpha)$, and $\Delta (e^{q\phi}) = -\frac{1}{2} q(q+2)$. Furthermore, for a scalar field $\phi$ satisfying the OPE given in (8), the following OPE holds: $e^{a\phi(z)} e^{b\phi(w)} = (z-w)^{-ab} e^{a\phi(z)+b\phi(w)}$.

Higher-level physical states with standard ghost structure can be obtained by acting with excitations of the basic fields on the lowest-energy state $V_0$. Thus the matter part of the vertex operators take the form

$$V = R^A(\partial \phi, \psi^i, K^a) \Phi_A e^{p\phi},$$  \hspace{1cm} (29)

where $R^A$ is a polynomial in the basic fields $\partial \phi, \psi^i$ and $K^a$. $R^A$ can be characterised by its conformal dimension, i.e. its level number. For a physical operator with level number $n$, the mass-shell condition implies that

$$-\frac{p^2}{2} + \frac{C_R}{\epsilon(3-N)} - 1 + \frac{N}{2} + n = 0.$$  \hspace{1cm} (30)
To obtain the explicit form of a higher level physical operator (29), it is necessary to solve the physical-state conditions (27), which become very complicated with increasing level number. However, we can discuss certain general features of the physical states with standard ghost structure. Since the basic fields and the currents have a one-to-one correspondence, i.e.

\[ T \leftrightarrow \partial \phi, \quad G^i \leftrightarrow \psi^i, \quad J^a \leftrightarrow K^a, \]

(31)

it follows that the matter excitations \( R_A(\partial \phi, \psi^i, K^a) \) can be re-expressed as \( R_A(T_n, G^m_i, J^k) \). Physical states of this form are BRST trivial. Thus we expect that all excited physical states with standard ghost structure are BRST trivial. Such a phenomenon also occurs in the one-scalar string theory and the two-scalar \( W \) string, where all the higher-level physical states with standard ghost structure are trivial.

However, this does not imply that the BRST cohomology of the system is simple. In fact it can have a very rich structure, since there can be many physical states with non-standard ghost structure. Studying the physical states with standard ghost structure, namely the BRST trivial states, can unveil the physical states with non-standard ghost structure. To see this, we note that a higher-level physical state (29) can be written as \( Q\chi \) for generic on-shell momentum since it is BRST trivial. However, if for certain a special value of on-shell momentum the state becomes zero, then it implies that the operator \( \chi \) becomes BRST invariant for this special value of momentum. Obviously the operator \( \chi \) has a non-standard ghost structure, and it corresponds to a BRST-non-trivial physical state.

First we shall look at the physical states with level number \( \frac{1}{2} \), which have the form

\[ V = \xi^A G^i_{-\frac{1}{2}} \Phi_A \varepsilon^{p\phi} = -\xi^A \left( p \psi^i \Phi_A + \frac{2}{\sqrt{3-N}} \lambda^c_{ij} \tau^c_{AB} \psi^i \Phi^B \right) \varepsilon^{p\phi}. \]

(32)

The only non-trivial physical-state conditions are the \( J^a_0 \) and \( L_0 \) conditions. The former implies that \( \Phi_A \) is a singlet under the Kac-Moody currents \( K^a \); the latter implies that the momentum \( p \) satisfies the mass-shell condition (30) with \( C_R = 0 \). This BRST trivial state will not vanish for any on-shell momentum, and hence we do not expect that there exists a BRST non-trivial state in the lower ghost number.

At level \( n = 1 \), we consider \( SO(N) \)-singlet physical states of standard ghost structure, with matter-dependent factors of the form

\[ V = \left( \lambda^a_{ij} G^i_{-\frac{1}{2}} G^j_{-\frac{1}{2}} + x J^a_1 \right) \Phi_a \varepsilon^{p\phi}, \]

(33)

where \( x \) is a constant. It is straightforward to show that these states can be rewritten as

\[ V = \left[ \left( x - \frac{2(q + N - 2)}{3-N} \right) K^a + \frac{1}{2} \left( x + \frac{2(q + N - 2)^2}{3-N} \right) \lambda^a_{ij} \psi^i \psi^j \right] \Phi_a \varepsilon^{p\phi}, \]

(34)

where \( q \equiv p \sqrt{3-N} \). The physical-state conditions (27) then require that the mass-shell condition

\[ (q + N - 1)(q + N - 4) = 0 \]

(35)

be satisfied, and then also give rise to a polarisation condition which determines the constant \( x \):

\[ x = \frac{(q + N - 2)(q + N - 3)}{N-3}. \]

(36)
Thus it is easy to see that if the momentum satisfies the condition $q = 1 - N$, corresponding to the first root of (35), the physical state described by $V$ vanishes identically. This is the situation where a physical state that is generically null becomes identically zero at a special value of the on-shell momentum. It signals the occurrence of a BRST non-trivial physical operator at ghost-number one less than that of the standard physical states, since at this momentum the null operator $Q\chi$ vanishes, implying that $\chi$ itself becomes physical.

For the case of $N = 2$, we can easily find the explicit form of the physical operator $\chi$. It is given by

$$\chi = c \left( \psi_1 \partial_2 e^{-\sigma_1 - 2\sigma_2} + \psi_2 \partial_1 e^{-2\sigma_1 - \sigma_2} + 2\beta e^{-\sigma_1 - \sigma_2} \right) e^{-\phi}. \quad (37)$$

One can check, by constructing the operator conjugate to this, that $\chi$ is BRST non-trivial. For $N > 3$, the analogous operator is given by

$$\chi = c \left( \psi_i \partial_j e^{-\sigma_j - \sum \sigma} \lambda_{ij}^{\alpha} + \frac{2}{\sqrt{3 - N}} \beta^a e^{-\sum \sigma} \Phi_{a} e^{p\phi} \right), \quad (38)$$

where $\exp(-\sum \sigma) = \exp(-\sigma_1 - \cdots - \sigma_N)$ and $\exp(-\sigma_j - \sum \sigma) = \exp(-\sigma_1 - \sigma_2 - \cdots - 2\sigma_j - \cdots - \sigma_N)$. The momentum $p = q/\sqrt{3 - N}$ is given by $q = 1 - N$.

5 Comments

With the realisation (6) for the $SO(N)$–extended superconformal algebra, one expects that in the sector of physical states with standard ghost structure, there should be no states with excitations of the matter fields, and that the only non-trivial physical state should be the tachyon. This is because the number of matter fields is the same as the number of constraints, and thus there are no transverse spacetime directions. In this paper we have seen that this is indeed true, but that in addition there are further physical states with non-standard ghost structure; i.e. with ghost as well as matter excitations. Multi-scalar realisations, which might allow the possibility of obtaining a spacetime interpretation, are unknown for the cases $N > 2$. On the other hand, for $N \leq 2$, where the algebras are linear, multi-scalar realisations are well known, and they indeed lead to string theories with target spacetimes that allow for transverse excitations. Although we have only exhibited a small number of physical states in this paper, we expect that there will exist infinite numbers of such states at arbitrary ghost numbers, in much the same way as one finds in the one-scalar bosonic string.

The case of $N = 3$ is special, because the Kac-Moody level required by nilpotency of the BRST operator is zero, which conflicts in the full $N = 3$ algebra with the value of the central charge that is required by nilpotency. This conflict can be circumvented by performing a rescaling of the fermionic generators $G^i$ that becomes singular in the limit where the Kac-Moody level tends to zero. By this means, we have been able to construct a nilpotent BRST operator for a contraction of the full $N = 3$ algebra.

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