ON THE CAUCHY PROBLEM OF 3D NONHOMOGENEOUS INCOMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS WITH VACUUM

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Abstract. This paper deals with the Cauchy problem of three-dimensional (3D) nonhomogeneous incompressible nematic liquid crystal flows. The global well-posedness of strong solutions with large velocity is established provided that \( \|\rho_0\|_{L^\infty} + \|\nabla d_0\|_{L^3} \) is suitably small. In particular, the initial density may contain vacuum states and even have compact support. Furthermore, the large time behavior of the solution is also obtained.

1. Introduction. The motion of nonhomogeneous incompressible nematic liquid crystal flows is described by the following simplified version of the Ericksen-Leslie equations:

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \rho (u \cdot \nabla) u - \mu \Delta u + \nabla P &= -\lambda \text{div} (\nabla d \circ \nabla d), \\
\text{div} u &= 0, \quad |d| = 1, \\
d_t + u \cdot \nabla d &= \theta (\Delta d + |\nabla d|^2 d),
\end{aligned}
\]

where \( \rho(x,t) : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R} \) denotes the density of the fluid, \( u(x,t) : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3 \) denotes the velocity of the fluid, \( d(x,t) : \mathbb{R}^3 \times (0,\infty) \to \mathbb{S}^2 \) (the unit sphere in \( \mathbb{R}^3 \)) denotes the unit-vector field that represents the macroscopic molecular orientation of the liquid crystal material, \( P(x,t) : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R} \) is the scalar functions representing the pressure. The positive constants \( \mu, \lambda, \theta \) represent viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time for the molecular orientation field respectively. The notation \( \nabla d \circ \nabla d \) denotes the \( 3 \times 3 \) matrix whose \((i,j)\)-th entry is given by \( \partial_i d \cdot \partial_j d \) for \( 1 \leq i, j \leq 3 \).

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We consider the Cauchy problem for (1.1) with \((\rho, u)\) vanishing at infinity (in some weak sense) and the initial conditions
\[
\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0(x), \quad d(x, 0) = d_0(x), \quad |d_0(x)| = 1, \quad x \in \mathbb{R}^3,
\]
for given initial data \(\rho_0, m_0, \) and \(d_0\).

The nematic liquid crystal flows are regarded as slow-moving particles where the fluid velocity and the alignment of the particles affect each other. The hydrodynamics of nematic liquid crystals developed by Ericksen [2] and Leslie [6] in the 1960s, but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen-Leslie model. Mathematically, the system (1.1) is a strongly coupled system between the nonhomogeneous incompressible Navier-Stokes equations and the transported heat flows of harmonic map, and thus, its mathematical analysis is full of challenges.

In the homogeneous case, i.e., \(\rho \equiv \text{constant}\), Lin et al. [12] established that there exist global Leray-Hopf type weak solutions of the initial boundary value problem for system (1.1) on bounded domains in two space dimensions (see also [3, 5]). The approach used in [12] and that used in [3, 5] are different, where the global existence is proven directly to the liquid crystal system with term \(|\nabla d|^2 d\), while in [3, 5], the strategy is to show the convergence of the solutions to the approximate system with penalty term \(1 - |d|^2\varepsilon^2\) as \(\varepsilon \to 0\). The uniqueness of such weak solutions is proved by Lin and Wang [13]. When the space dimension is three, Lin and Wang [14] obtained the existence of global weak solutions when the initial data \((u_0, d_0) \in L^2 \times H^1\) with the initial director field \(d_0\) maps to the upper hemisphere \(S^2_+\). Hong and Mei [4] proved existence of solutions to the Ericksen-Leslie system with initial data having small \(L^3_{uloc}\)-norm. Recently, Liu et al. [17] investigated the problem of optimal boundary control to a simplified Ericksen-Leslie system and showed both the existence and a necessary condition of an optimal boundary control.

In the nonhomogeneous case, i.e., the density dependent case, when the initial data is away from vacuum, Li [7] established the global existence of strong and weak solutions of the two-dimensional system (1.1) provided that the initial orientation \(d_0 = (d_{01}, d_{02}, d_{03})\) satisfies a geometric condition
\[
d_{03} \geq \epsilon_0 \quad \text{for some positive } \epsilon_0 > 0.
\]
Wen and Ding [22] established the global existence and uniqueness of solutions for the two dimensional case if the initial data is of small norm. In the presence of vacuum, if the initial data satisfy the following compatibility condition
\[
- \mu \Delta u_0 - \nabla P_0 - \lambda \text{div}(\nabla d_0 \otimes \nabla d_0) = \sqrt{\rho_0}g
\]
in a bounded smooth domain or whole space, and \((P_0, g) \in H^1(\Omega) \times L^2(\Omega), \Omega \subseteq \mathbb{R}^3\), Li [8] and Ding et al. [1] respectively obtained the global existence of strong solutions when the initial data is suitably small. Li [10] considered the case on the bounded domains in \(\mathbb{R}^2\) (see also [8]). Recently, Liu [18] and Liu et al. [16] independently extended the result of [8, 10] to \(\mathbb{R}^2\) with large initial data, provided that the initial orientation \(d_0 = (d_{01}, d_{02}, d_{03})\) satisfies a geometric condition (1.3). Li et al. [11] got the same result under small initial data without the additional geometric condition (1.3) (see also [24]).

Before formulating our main result, we first explain the notations and conventions used throughout this paper. For simplicity, in what follows, we set
\[
\int dx = \int_{\mathbb{R}^3} dx, \quad \mu = \lambda = \theta = 1.
\]
For $1 \leq r \leq \infty$, $k \geq 1$, and $\beta > 0$, the Sobolev spaces are defined in a standard way,
\[
\begin{align*}
L^r(\mathbb{R}^3), W^{k,p}(\mathbb{R}^3), \quad H^k = W^{k,2},
D^1 = \{v \in L^6(\mathbb{R}^3) | \nabla v \in L^2(\mathbb{R}^3)\},
D^{k,r} = D^{k,r}(\mathbb{R}^3) = \{v \in L^1_{\text{loc}}(\mathbb{R}^3) | D^k v \in L^r(\mathbb{R}^3)\},
C^\infty_{0,\sigma} = \{f \in C^\infty_0 | \text{div} f = 0\},
\hat{D}_{0,\sigma} = \overline{C^\infty_{0,\sigma}} \text{ closure in the norm of } D^1,
\hat{\dot{H}}^\beta = \left\{ f : \mathbb{R}^3 \to \mathbb{R} | \| f \|^2_{\dot{H}^\beta} = \int |\xi|^{2\beta} |\hat{f}(\xi)|^2 d\xi < \infty \right\},
\end{align*}
\]
where $\hat{f}$ is the Fourier transform of $f$.

Motivated by [8, 23], where they established global regularity in smooth bounded domains provided the initial energy $\| \sqrt{\rho_0} u_0 \|^2_{L^2} + \| \nabla d_0 \|^2_{L^2}$ is small, the main goal of this paper is to obtain global strong solutions of the Cauchy problem (1.1)–(1.2) under the assumption that the initial data in some norm is small enough, and don’t need any smallness on the initial velocity, which reads as follows:

**Theorem 1.1.** For $q \in (3, 6)$, assume that the initial data $(\rho_0, m_0, d_0)$ satisfy
\[
\begin{align*}
0 \leq \rho_0(x) \leq \tilde{\rho} < \infty, \quad & \rho_0 \in H^1 \cap W^{1,q}, \quad m_0 = \rho_0 u_0, \\
u_0 \in D^1_{0,\sigma}, \quad & \nabla d_0 \in H^1, \quad |d_0| = 1, \\
\rho + \| \nabla d_0 \|^2_{L^3} = M_0^2, \quad & \| \sqrt{\rho_0} u_0 \|^2_{L^2} + \| \nabla d_0 \|^2_{L^2} = M_1, \\
\| \nabla u_0 \|^2_{L^2} + \| \Delta d_0 \|^2_{L^2} = M_2, & \\
\end{align*}
\]
then there exists a small positive constant $\epsilon_0$ depending only on $q, M_1$, and $M_2$ such that if
\[
M_0 \leq \epsilon_0,
\]
the problem (1.1)–(1.2) admits a unique global strong solution $(\rho, u, d, P)$ satisfying that for any $0 < T < \infty$,
\[
\begin{align*}
0 \leq \rho(x,t) \leq \tilde{\rho}, \quad & \| d(x,t) \| = 1, \quad \forall (x,t) \in \mathbb{R}^3 \times [0, T], \\
\rho \in C([0, T]; W^{1,q}), \quad & (\nabla u, P) \in C([0, T]; H^1) \cap L^1(0, T; W^{1,q}), \\
\rho \in C([0, T]; L^q), \quad & \sqrt{\rho} u \in C([0, T]; L^q), \quad u \in L^2(0, T; H^1), \\
\nabla d \in C([0, T]; H^2) \cap L^2(0, T; H^3), \quad d_t \in C([0, T]; H^1) \cap L^2(0, T; H^2). & \\
\end{align*}
\]
Moreover, $(\rho, u, d, P)$ has the following decay rates, that is, for $t \geq 1$,
\[
\begin{align*}
\| \nabla u(\cdot, t) \|^2_{L^2} + \| \nabla^2 d(\cdot, t) \|^2_{L^2} \leq C t^{-\frac{1}{4}}, \\
\| \nabla^2 u(\cdot, t) \|^2_{L^2} + \| \nabla P(\cdot, t) \|^2_{L^2} + \| \nabla d_t \|^2_{L^2} \leq C t^{-1}, & \\
\end{align*}
\]
where $C$ depends only on initial datum.

**Remark 1.** Compared with [1, 8], where the authors require that the initial data satisfies
\[
- \mu \Delta u_0 - \nabla P_0 - \lambda \text{div}(\nabla d_0 \otimes \nabla d_0) = \sqrt{\rho_0} g
\]
for $(P_0, g) \in H^1 \times L^2$, there is no need to impose the compatibility condition (1.9), and thus we weaken the regularity of the initial data for obtaining the global existence of the strong solution. Indeed, this is achieved by deriving the time weighted estimates on the solution, see Lemma 3.26. Moreover, Theorem 1.1 also extends the result of Li [8] to the whole space, which depends heavily on the boundedness of the domains.
Remark 2. To the best of our knowledge, the smallness condition (1.6) of Theorem 1.1 is new, and the known results [8, 23] require the initial energy \( \|\sqrt{\bar{\rho}}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \) to be suitably small. Furthermore, the large time asymptotic decay with rates of the global strong solution (1.8) is also established.

We now sketch the main idea used in the proof of Theorem 1.1. Note that for initial data satisfying (1.5), the local existence and uniqueness theorem of strong solutions whose proof can be performed by using a semi-Galerkin’s scheme (see [9] for example). Thus, to extend the local strong solution to be a global one, one needs to obtain global a priori estimates on strong solutions to (1.1)–(1.2) in suitable higher norms. It should be pointed out that the crucial techniques of proofs in [8, 23] cannot be adapted directly to the situation treated here, since their arguments depend crucially on the boundedness of the domains. Thus, some new ideas are needed. First we assume that \( \bar{\rho} + \|\nabla d\|_{L^3} \) is less than \( M_0 \) and \( \|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \) is less than \( 4(M_1 + M_2) \) on \( [0, T] \), then we prove that in fact \( \bar{\rho} + \|\nabla d\|_{L^3} \) is less than \( \frac{M_0}{2} \) and \( \|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \) is less than \( 2(M_1 + M_2) \) on \( [0, T] \), provided that \( M_0 \) is suitably small. To this end, motivated by [19], multiplying (1.1) by the material derivatives \( \dot{u} := u_t + u \cdot \nabla u \) instead of the usual \( u_t \), we find one of the key points to obtain the estimate on the \( L^\infty(0, T; L^2(\mathbb{R}^2)) \)-norm of the gradient of the velocity is to bound the term

\[
I_2 := - \int \nabla P \cdot \dot{u} \, dx.
\]

However, the term \( I_2 \) can not be bounded by the technology of [19]. Integrating by parts, we succeed in bounding the term \( I_2 \) by regularity properties of Stokes system and Gagliardo-Nirenberg inequality (see (2.1)). Next, to obtain the estimates on the gradient of the density, motivated by [11, 16, 19], which are crucial in deriving the bound of \( \|\sqrt{\bar{\rho}}\dot{u}\|_{L^2}^2 \). However, it prevents us to achieve this goal due to the absence of the compatibility condition (1.9) for the initial velocity. To overcome this difficulty, we derive the following crucial time-weighted estimate (see (3.28)):

\[
\sup_{0 \leq t \leq T} (t' \|\sqrt{\bar{\rho}}\dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T t' (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2) \, dt \leq C,
\]

where the positive constant \( C \) is independent of \( T \) and the initial data of \( \sqrt{\bar{\rho}}\dot{u} \). As a matter of fact, all these time-weighted estimate (1.10) play an important role in obtain the desired bound of \( \int_0^T \|\nabla u\|_{L^\infty} \, dt \) (see (3.51) for details), which in particular implies the time-dependent estimates on the gradient of density. Finally, with these a priori estimates stated above in hand, we can finally establish the global existence of strong solution to the system (1.1) in Theorem 1.1.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proofs of Theorem 1.1.

2. Preliminaries. In this section, we shall enumerate some auxiliary lemmas used in this paper.

First of all, the following local existence of strong solutions can be shown by similar strategies as in [9].

Lemma 2.1. Assume that \((\rho_0, u_0, d_0)\) satisfies (1.5). Then there exists a small time \( T > 0 \) and a unique strong solution \((\rho, u, d, P)\) to the problem (1.1) and (1.2) in \( \mathbb{R}^3 \times T \) satisfying (1.7).
The following well-known Gagliardo-Nirenberg inequality (see [20, Theorem]) will be used later.

**Lemma 2.2.** Let $u$ belong to $L^q(\mathbb{R}^n)$ and its derivatives of order $m, \nabla^mu$, belong to $L^r(\mathbb{R}^n)$, $1 \leq q, r \leq \infty$. Then for the derivatives $\nabla^j u, 0 \leq j < m$, the following inequality holds

$$\|\nabla^j u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla^m u\|^p_{L^r(\mathbb{R}^n)} \|u\|_{L^q(\mathbb{R}^n)}^{1-\alpha}, \quad (2.1)$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha \left(\frac{1}{r} - \frac{m}{n}\right) + (1 - \alpha)\frac{1}{q} \quad (2.2)$$

for all $\alpha$ in the interval

$$\frac{j}{m} \leq \alpha \leq 1 \quad (2.3)$$

(the constant $C$ depends only on $n, m, j, q, r, \alpha$), with the following exceptional cases:

1. If $j = 0, rm < n$ and $q = \infty$, then we take the additional assumption that either $u$ tends to zero at infinity or $u \in L^q(\mathbb{R}^n)$ for some finite $q > 0$.

2. If $1 < r < \infty$, and $m - j - \frac{n}{q}$ is a nonnegative integer, then (2.1) holds only for $\alpha$ satisfying $\frac{j}{m} \leq \alpha < 1$.

3. **A priori estimates.** In this section, we will establish some necessary a priori bounds of local strong solutions $(\rho, u, d, P)$ to the problem (1.1)–(1.2) whose existence is guaranteed by Lemma 2.2. Thus, let $T > 0$ be a fixed time and $(\rho, u, d, P)$ be the smooth solution of (1.1)–(1.2) on $\mathbb{R}^3 \times (0, T]$ with initial data $(\rho_0, u_0, d_0)$ satisfying (1.5). For simplicity, we shall use the letters $C$ (except Lemmas 3.3–3.4) and $C_i$ ($i = 1, 2, \ldots$) to denote the generic constants which may be dependent on $M_1$ and $M_2$, but independent of $T$ and $M_0$, while the generic constants $C$ may also depend on $T$ in Lemmas 3.3–3.4.

We first aim to get the following key a priori estimates on $(\rho, u, d, P)$.

**Proposition 3.1.** Assume that $(\rho, u, d, P)$ is a smooth solution of (1.1)–(1.2) on $\mathbb{R}^3 \times (0, T]$ satisfying

$$\sup_{0 \leq t \leq T} (\bar{\rho} + \|\nabla d\|_{L^3}) \leq M_0, \quad \sup_{0 \leq t \leq T} (\|\nabla u\|^2_{L^2} + \|\nabla^2 d\|^2_{L^2}) \leq 4(M_1 + M_2). \quad (3.1)$$

Then there exists some small positive constant $\epsilon'_0$ depending only on $q, M_1,$ and $M_2$ such that

$$\sup_{0 \leq t \leq T} (\bar{\rho} + \|\nabla d\|_{L^3}) \leq \frac{M_0}{2}, \quad \sup_{0 \leq t \leq T} (\|\nabla u\|^2_{L^2} + \|\nabla^2 d\|^2_{L^2}) \leq 2(M_1 + M_2), \quad (3.2)$$

provided that

$$M_0 \leq \epsilon'_0, \quad (3.3)$$

where

$$\epsilon'_0 := \min \left\{ \frac{1}{\sqrt{2C_1}}, \frac{1}{\sqrt{2C_2}}, \frac{1}{C_2}, \frac{1}{\sqrt{4C_4}}, \frac{1}{4C_5}, \frac{1}{4} \right\}.$$

Before proving Proposition 3.1, we need to establish the upper bound of density and the basic energy inequality.

**Lemma 3.1.** Let $(\rho, u, d, P)$ be a smooth solution to (1.1)–(1.2). Then it holds that

$$0 \leq \rho(x, t) \leq \sup_{x \in \mathbb{R}^3} \rho_0(x) = \bar{\rho}, \quad (3.4)$$
\[
\sup_{0 \leq t \leq T} (\sum_{j=1}^{3} \rho_j u_j^2 + \sum_{j=1}^{3} \nabla d_j^2) + \int_0^T (\sum_{j=1}^{3} \nabla u_j^2 + \sum_{j=1}^{3} \nabla^2 d_j^2) dt \leq M_1, \tag{3.5}
\]
provided that
\[
\sup_{0 \leq t \leq T} \sum_{j=1}^{3} \nabla d_j^2 \leq \epsilon_1 := \min \left\{ 1, \frac{1}{\sqrt{2C_1}} \right\}.
\]

**Proof.** Noting that (3.4) follows from (1.1)_1 and (1.1)_3 (see Lions [15, Theorem 2.1]). To prove (3.5), multiplying equation (1.1)_2 by \( u \) and integrating by parts, we find
\[
\frac{1}{2} \frac{d}{dt} \int \rho|u|^2 dx + \int |\nabla u|^2 dx = - \int (u \cdot \nabla) d \cdot \Delta u dx. \tag{3.6}
\]

Multiplying equation (1.1)_4 by \( \Delta d \) and integrating over \( \mathbb{R}^3 \), one arrives at
\[
\int (\Delta_t + u \cdot \nabla) d \cdot \Delta u dx = \int |\Delta d|^2 dx + \int |\nabla d|^2 dx. \tag{3.7}
\]

By virtue of the basic fact \(|d| = 1\), we find \(-d \cdot \Delta d = |\nabla d|^2\). Then, integration by parts together with the boundary condition (1.2) implies that
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 + \int |\Delta d|^2 dx = \int (u \cdot \nabla) d \cdot \Delta u dx + \int |\nabla d|^4 dx,
\]
which along with (3.6) and Gagliardo-Nirenberg inequality yields that
\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{j=1}^{3} \rho_j u_j^2 + \sum_{j=1}^{3} \nabla d_j^2 \right) + \int (|\nabla u|^2 + |\nabla^2 d|^2) dx \leq C_1 \sum_{j=1}^{3} \nabla d_j^2 \sum_{j=1}^{3} \nabla^2 d_j^2. \tag{3.8}
\]
This gives rise to
\[
\frac{d}{dt} \left( \sum_{j=1}^{3} \rho_j u_j^2 + \sum_{j=1}^{3} \nabla d_j^2 \right) + \int (|\nabla u|^2 + |\nabla^2 d|^2) dx \leq 0, \tag{3.9}
\]
provided
\[
M_0 \leq \epsilon_1 := \min \left\{ 1, \frac{1}{\sqrt{2C_1}} \right\}.
\]

Then, integrating (3.9) over \( (0,T) \), we arrive at
\[
\sup_{0 \leq t \leq T} \left( \sum_{j=1}^{3} \rho_j u_j^2 + \sum_{j=1}^{3} \nabla d_j^2 \right) + \int_0^T (|\nabla u|^2 + |\nabla^2 d|^2) dt \leq M_1. \tag{3.10}
\]

One completes the proof of this lemma. \(\square\)

**Proof of Proposition 3.1.** Step 1. Multiplying (1.1)_2 by \( \dot{u} \) and integrating over \( \mathbb{R}^3 \), it follows that
\[
\int \rho \dot{u}^2 dx = \int \Delta u \cdot \dot{u} dx - \int \nabla P \cdot \dot{u} dx - \int \div(\nabla d \odot \nabla d) \cdot \dot{u} dx =: \sum_{i=1}^{3} I_i. \tag{3.11}
\]

By the definition of \( \dot{u} \) and Gagliardo-Nirenberg inequality, we have for \( \delta > 0 \),
\[
I_1 = \int \Delta u \cdot (u_t + u \cdot \nabla u) dx
\]
\[
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \sum_{i,j,k=1}^{3} \int \frac{1}{2} \partial_i u^j \partial_j u^k \partial_k u^2 dx
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + C \|\nabla u\|^3.
\]

Therefore, by (3.8) and (3.10), we have
\[
\sup_{0 \leq t \leq T} \left( \sum_{j=1}^{3} \rho_j u_j^2 + \sum_{j=1}^{3} \nabla d_j^2 \right) + \int_0^T (|\nabla u|^2 + |\nabla^2 d|^2) dt \leq M_1.
\]
where we have used $\delta$-Young’s inequality:

$$ab \leq \delta a^p + (\delta p)^{-\frac{2}{p}} q^{-1} b^q$$

for any $\delta > 0$, $p, q > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Integrating by parts together with $\text{div}(\nabla d \otimes \nabla d) = \nabla d \cdot \Delta d + \frac{1}{2} \nabla (|\nabla d|^2)$, regularity properties of Stokes system, and Gagliardo-Nirenberg inequality implies that

$$I_2 = - \int \nabla P \cdot (u_t + u \cdot \nabla u) dx$$

$$= - \int \nabla P \cdot (u \cdot \nabla u) dx$$

$$\leq C\|\nabla P\|_{L^2} \|u\| \|\nabla u\|_{L^2}$$

$$\leq C\|\mu u_t + \rho u \cdot \nabla u + \nabla d \cdot \Delta d + \nabla (|\nabla d|^2)\|_{L^2} \|u\| \|\nabla u\|_{L^2}$$

$$\leq C\|u\|^2_{L^6} \|\nabla u\|^2_{L^6} + C\|\nabla d\|_{L^2}^2 \|\Delta d\|_{L^2}^2$$

$$\leq C\|\nabla u\|^2_{L^2} \|\nabla^2 u\|_{L^2}$$

$$\leq C\delta^{-1} \|\nabla^2 u\|^2_{L^2} + \delta \|\nabla u\|^6_{L^2} + C\|\nabla d\|^2_{L^2} \|\nabla^3 d\|_{L^2}^2$$

Integrating by parts together with $(1.1)_4$, Hölder’s, Young’s and Gagliardo-Nirenberg inequalities gives

$$I_3 = \int (\nabla d \otimes \nabla d) : \nabla u_t dx - \int \text{div}(\nabla d \otimes \nabla d) \cdot (u \cdot \nabla u) dx$$

$$= \frac{d}{dt} \int (\nabla d \otimes \nabla d) : \nabla u dx - \int (\nabla d_t \otimes \nabla d) \cdot \nabla u dx - \int (\nabla d \otimes \nabla d_t) \cdot \nabla u dx$$

$$- \int \text{div}(\nabla d \otimes \nabla d) \cdot (u \cdot \nabla u) dx$$

$$= \frac{d}{dt} \int (\nabla d \otimes \nabla d) : \nabla u dx - \int [\Delta \nabla d - \nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d)] \otimes \nabla d \cdot \nabla u dx$$

$$- \int (\nabla d \otimes (\Delta \nabla d - \nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d))) \cdot \nabla u dx - \int \text{div}(\nabla d \otimes \nabla d) \cdot (u \cdot \nabla u) dx$$

$$= \frac{d}{dt} \int (\nabla d \otimes \nabla d) : \nabla u dx - \int [\Delta \nabla d - \nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d)] \otimes \nabla d \cdot \nabla u dx$$

$$+ \sum_{i,j,k=1}^3 \int u_k \partial_k \partial_i d \cdot \partial_j d \partial_j u_i dx - \int [\nabla d \otimes (\Delta \nabla d - \nabla u \cdot \nabla d + \nabla (|\nabla d|^2 d))] \cdot \nabla u dx$$

$$+ \sum_{i,j,k=1}^3 \int u_k \partial_i d \cdot \partial_k \partial_j d \partial_j u_i dx + \sum_{i,j,k=1}^3 \int \partial_i d \cdot \partial_j d \partial_j (u_k \partial_k u_i) dx$$

$$= \frac{d}{dt} \int (\nabla d \otimes \nabla d) : \nabla u dx - \int [\Delta \nabla d - \nabla u \cdot \nabla d + \nabla (|\nabla d|^2 d)] \otimes \nabla d \cdot \nabla u dx$$

$$- \int [\nabla d \otimes (\Delta \nabla d - \nabla u \cdot \nabla d + \nabla (|\nabla d|^2 d))] \cdot \nabla u dx + \sum_{i,j,k=1}^3 \int \partial_i d \cdot \partial_j d \partial_j u_k \partial_k u_i dx$$
\[
\begin{align*}
&\leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u dx + C \int |\nabla d|^2 |\nabla u|^2 dx + C \int |\nabla u| |\nabla d| |\Delta \nabla d| dx \\
&\quad + C \int |\nabla^2 d| |\nabla u| dx + C \int |\nabla d|^4 |\nabla u| dx \\
&\leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u dx + C \||\nabla u||^2_{L^2} \||\nabla d||^2_{L^3} + \frac{1}{32} |\nabla^3 d|^2_{L^2} + C |\nabla d|^6_{L^6} \\
&\quad + C |\nabla^3 d||^2_{L^2} \||\nabla d||^2_{L^3} \||\nabla u||_{L^2} \\
&\leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u dx + C |\nabla^2 u|^2_{L^2} |\nabla d|^2_{L^3} + \frac{1}{16} |\nabla^3 d|^2_{L^2} \\
&\quad + C |\nabla d|^2_{L^2} |\nabla u|^2_{L^2} + C |\nabla d|^2_{L^3} |\nabla^3 d|^2_{L^2} \\
&\quad + C |\nabla d|^2_{L^2} |\nabla u|^2_{L^2} + C |\nabla d|^2_{L^3} |\nabla^3 d|^2_{L^2}, \\
&\quad \text{(3.14)}
\end{align*}
\]

where we have used the following fact
\[
|\nabla d|_{L^\infty} \leq C |\nabla d|^2_{L^3} |\nabla^2 d|^2_{L^6} \leq C |\nabla d|^2_{L^3} |\nabla^3 d|^2_{L^2}. 
\]

Inserting the estimates of \( I_i \) (\( i = 1, 2, 3 \)) into (3.11), we get
\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 + \int \rho |u|^2 dx \\
&\leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u dx + C |\nabla^2 u|^2_{L^2} |\nabla d|^2_{L^3} + \frac{1}{16} |\nabla^3 d|^2_{L^2} \\
&\quad + C |\nabla d|^2_{L^2} |\nabla u|^2_{L^2} + C |\nabla d|^2_{L^3} |\nabla^3 d|^2_{L^2} + C |\nabla u|^6_{L^6} \\
&\quad + \delta |\nabla u|^2_{L^2} + C \rho |\nabla u|^2_{L^2} + C |\nabla d|^2_{L^3} |\nabla^3 d|^2_{L^2}, \\
&\quad \text{(3.16)}
\end{align*}
\]

Step 2. Multiplying (1.11) by \(-\nabla \Delta d\) and then integrating by parts over \(\mathbb{R}^3\), it follows from Hölder’s and Gagliardo-Nirenberg inequalities, and (3.15) that
\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \int |\nabla^2 d|^2 dx + \int |\nabla^3 d|^2 dx \\
&= \int (\nabla u \cdot \nabla d) \cdot \nabla \Delta d dx - \int \nabla((|\nabla^2 d|^2) \cdot \nabla \Delta dx \\
&= \int (\nabla u \cdot \nabla d) \cdot \nabla \Delta dx + \sum_{i,j,k=1}^3 \int u_i \partial_i \partial_j \partial_k d \cdot \partial_i \partial_j \partial_k dx - \int \nabla(|\nabla^2 d|^2) \cdot \nabla \Delta dx \\
&\leq \frac{1}{32} |\nabla^3 d|^2_{L^2} + C |\nabla d|^3_{L^\infty} |\nabla d|^2_{L^3} + C |\nabla d|^2_{L^3} |\nabla^2 d|^2_{L^6} \\
&\quad + C |\nabla u|^2_{L^6} |\nabla d|^2_{L^3} + C |\nabla u|^2_{L^6} |\nabla^2 d|^2_{L^3} \\
&\leq \left( \frac{1}{32} + \frac{1}{|\nabla d|^4_{L^2} + |\nabla u|^2_{L^2}} \right) |\nabla^3 d|^2_{L^2} + C |\nabla u|^2_{L^2} |\nabla d|^2_{L^3} |\nabla^3 d|^2_{L^2} \\
&\quad + \delta |\nabla u|^2_{L^2} + C |\nabla d|^2_{L^3} |\nabla^2 u|^2_{L^2} \\
&\leq \left( \frac{1}{16} + \frac{1}{|\nabla u|^4_{L^2} + |\nabla u|^2_{L^2}} \right) |\nabla^3 d|^2_{L^2} + C |\nabla u|^6_{L^6} |\nabla d|^2_{L^3} \\
&\quad + C |\nabla d|^2_{L^3} |\nabla^2 u|^2_{L^2}, \\
&\quad \text{(3.17)}
\end{align*}
\]
Thus, we obtain after adding (3.16) and (3.17), and choosing \( \delta = M_0^{\frac{1}{2}} \) suitably small that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2) + \| \sqrt{\rho_{\delta}} \|_{L^2}^2 + \| \nabla^3 d \|_{L^2}^2 \\
\leq \frac{d}{dt} \int (\nabla d \odot \nabla d) : \nabla u dx + C \| \nabla u \|_{L^2}^2 \| \nabla d \|_{L^2}^2 + \frac{1}{16} \| \nabla^3 d \|_{L^2}^2 \\
+ C \| \nabla d \|_{L^3}^2 \| \nabla u \|_{L^2}^2 + C \| \nabla d \|_{L^3}^2 \| \nabla^3 d \|_{L^2}^2 + C M_0^{-\frac{1}{2}} \| \nabla^2 u \|_{L^2}^2 \\
+ M_0^{\frac{1}{2}} \| \nabla u \|_{L^2}^5 + C \rho \| \sqrt{\rho_{\delta}} \|_{L^2}^2 + C \| \nabla d \|_{L^2} \| \nabla^3 d \|_{L^2}^2 + C \| \nabla d \|_{L^3}^2 \| \nabla^2 u \|_{L^2}^2 \\
+ \left( \frac{1}{16} + \| \nabla d \|_{L^3}^2 + \| \nabla d \|_{L^2}^2 \right) \| \nabla^3 d \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla u \|_{L^2}^2 \\
\leq \frac{d}{dt} \int (\nabla d \odot \nabla d) : \nabla u dx + \left( \frac{1}{8} + \| \nabla d \|_{L^3}^2 + \| \nabla d \|_{L^2}^2 \right) \| \nabla^3 d \|_{L^2}^2 + C \rho \| \sqrt{\rho_{\delta}} \|_{L^2}^2 \\
+ C (\| \nabla d \|_{L^2}^2 + M_0^{\frac{1}{2}}) \| \nabla^2 u \|_{L^2}^2 + C (\| \nabla d \|_{L^2}^2 + M_0^{\frac{1}{2}}) \| \nabla u \|_{L^2}^6. \tag{3.18}
\end{align*}
\]

Step 3. Note that \((\rho, u, d, P)\) satisfies the following Stokes system

\[
\begin{align*}
\begin{cases}
-\Delta u + \nabla P &= -\rho \dot{u} - \text{div}(\nabla d \odot \nabla d), &x \in \mathbb{R}^3, \\
\text{div} u &= 0, &x \in \mathbb{R}^3, \\
u(x) &\to 0, &|x| \to \infty.
\end{cases}
\end{align*}
\]

Thus applying the standard \(L^p\)-estimate to the above system (see [21]) ensures that for any \( p \in [2, \infty) \),

\[
\| \nabla^2 u \|_{L^p} + \| \nabla P \|_{L^p} \leq C\| \rho \dot{u} \|_{L^p} + C \| \text{div}(\nabla d \odot \nabla d) \|_{L^p} \\
\leq C \| \sqrt{\rho_{\delta}} \|_{L^p} + C \| \| \nabla d \| \| \nabla^2 d \|_{L^p}. \tag{3.20}
\]

Particularly, choosing \( p = 2 \), we get

\[
\| \nabla^2 u \|_{L^2} + \| \nabla P \|_{L^2} \leq C \rho \| \nabla^2 u \|_{L^2} + C \| \nabla d \|_{L^2} \| \nabla^3 d \|_{L^2}. \tag{3.21}
\]

It thus follows from (3.21), (3.18), and (3.1) that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2) + \| \sqrt{\rho_{\delta}} \|_{L^2}^2 + \| \nabla^3 d \|_{L^2}^2 \\
\leq \frac{d}{dt} \int (\nabla d \odot \nabla d) : \nabla u dx + \left( \frac{1}{8} + CM_0^{\frac{1}{2}} \right) \| \nabla^3 d \|_{L^2}^2 + CM_0^{\frac{1}{2}} \| \sqrt{\rho_{\delta}} \|_{L^2}^2 + CM_0^{\frac{1}{2}} \| \nabla u \|_{L^2}^6,
\end{align*}
\]

which implies that

\[
\begin{align*}
\frac{d}{dt} (\| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2) + \| \sqrt{\rho_{\delta}} \|_{L^2}^2 + \| \nabla^3 d \|_{L^2}^2 \\
\leq 2 \frac{d}{dt} \int (\nabla d \odot \nabla d) : \nabla u dx + CM_0^{\frac{1}{2}} \| \nabla u \|_{L^2}^6. \tag{3.22}
\end{align*}
\]

Integrating (3.22) over \([0, T]\), we then derive from (3.1) and (3.5) that

\[
\begin{align*}
\sup_{0 \leq t \leq T} (\| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2) + \int_0^T (\| \sqrt{\rho_{\delta}} \|_{L^2}^2 + \| \nabla^3 d \|_{L^2}^2) dt \\
\leq \frac{1}{2} \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2 + C_2 \sup_{0 \leq t \leq T} \| \nabla d \|_{L^2}^2 \| \nabla^2 d \|_{L^2}^2 + C_3 M_0^{\frac{1}{2}} \int_0^T \| \nabla u \|_{L^2}^6 dt + M_2,
\end{align*}
\]
that is
\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) + \int_0^T (\|\sqrt{p} \hat{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) dt \leq 2(M_1 + M_2), \tag{3.23}
\]
provided
\[
M_0 \leq \epsilon_2 := \min \left\{ \epsilon_1, \frac{1}{\sqrt{2C_2}}, \frac{1}{C_2^2} \right\}.
\]

Step 4. It remains to estimate \( \sup_{0 \leq t \leq T} \|\nabla d\|_{L^2} \) in order to prove Proposition 3.1. In fact, multiplying (1.1) \(_4\) by \(3|\nabla d|\nabla d\) and integrating by parts over \(\mathbb{R}^3\), by Cauchy-Schwarz and Gagliardo-Nirenberg inequalities, we obtain that
\[
\frac{d}{dt}\|\nabla d\|_{L^2}^2 + \int (|\nabla d|\|\nabla^2 d\| dx + |\nabla d| |\nabla (|\nabla d|)^2|) dx \leq C \int (|\nabla d|^5 + |u|^2|\nabla d|^3) dx
\]
\[
\leq C\|\nabla d\|_{L^3}^2 \|\nabla d\|_{L^3}^2 + C\|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^2}^3
\]
\[
\leq C_4 \|\nabla d\|_{L^3}^2 \|\nabla d\|_{L^3}^2 + C\|\nabla d\|_{L^3}^2 \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^3}^2
\]
\[
\leq (C_4 \|\nabla d\|_{L^3}^2 + \frac{1}{4}) \|\nabla d\|_{L^2}^2 |\nabla^2 d|_{L^2}^2 + C\|\nabla d\|_{L^3}^2 \|\nabla u\|_{L^2}^4, \tag{3.24}
\]
where we have used the following inequality
\[
\|\nabla d\|_{L^2}^2 \leq C\|\nabla d\|_{L^3}^2 \|\nabla d\|_{L^2}^2 \leq C\|\nabla d\|_{L^3}^2 \|\nabla d\|_{L^2}^2 \|\nabla d\|_{L^2}^2. \tag{3.25}
\]
Thus if \(M_0\) is chosen to be sufficiently small such that
\[
M_0 \leq \epsilon_3 := \min \left\{ \epsilon_2, \frac{1}{\sqrt{4C_4}}, \frac{1}{4C_5} \right\},
\]
we deduce from (3.24) and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\nabla d\|_{L^2} \leq \exp \left\{ \int_0^T \|\nabla u\|_{L^2}^2 dt \right\} \frac{1}{2} \|\nabla d_0\|_{L^2} \leq C_5 M_0^2 \leq \frac{M_0}{4}. \tag{3.26}
\]
As a consequence, choosing \(\epsilon_0' := \min \{\epsilon_3, \frac{1}{4}\}\), we directly obtain (3.2). The proof of Proposition 3.1 is finished. \( \square \)

**Lemma 3.2.** There exists a positive constant \(C\) depending only on \(M_1\) and \(M_2\) such that for \(i = 1, 2,\)
\[
\sup_{0 \leq t \leq T} t^{i-1} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) + \int_0^T t^{i-1} (\|\sqrt{p} \hat{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) dt \leq C, \tag{3.27}
\]
\[
\sup_{0 \leq t \leq T} (t^{i}(\|\sqrt{p} \hat{u}\|_{L^2}^2 + \|\nabla d_i\|_{L^2}^2)) + \int_0^T t^{i} (\|\nabla \hat{u}\|_{L^2}^2 + \|\nabla^2 d_i\|_{L^2}^2) dt \leq C, \tag{3.28}
\]
\[
\sup_{0 \leq t \leq T} t^{i}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2) + \int_0^T t^{i-1}\|\nabla d_i\|_{L^2}^2 dt \leq C. \tag{3.29}
\]

**Proof.** Step 1. Multiplying (3.22) by \(t\) leads to
\[
\frac{d}{dt} \left[ t (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \right] + t (\|\sqrt{p} \hat{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2)
\]
\[
\leq 2 \frac{d}{dt} \left[ t \int (\nabla d \circ \nabla d) : \nabla u dx \right] - 2 \int (\nabla d \circ \nabla d) : \nabla u dx + CM_0^{\frac{1}{2}} t \|\nabla u\|_{L^2}^2
\]
which together with (3.5) and Hölder’s inequality yields that

\[
\sup_{0 \leq t \leq T} t(\| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2}) + \int_0^T t(\| \nabla \tilde{\rho} u \|^2_{L^2} + \| \nabla^3 d \|^2_{L^2}) dt \\
\leq \frac{1}{4} \sup_{0 \leq t \leq T} t \| \nabla u \|^2_{L^2} + C \sup_{0 \leq t \leq T} \| \nabla d \|^2_{L^3} \sup_{0 \leq t \leq T} (t \| \nabla^3 d \|^2_{L^2}) \\
+ \int_0^T \| \nabla u \|_{L^2} \| \nabla d \|_{L^3} \| \nabla d \|_{L^6} dt + CM_0^\frac{1}{2} \sup_{0 \leq t \leq T} t \| \nabla u \|^2_{L^2} \int_0^T \| \nabla u \|^2_{L^2} dt \\
+ \int_0^T (\| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2}) dt \\
\leq \frac{1}{4} \sup_{0 \leq t \leq T} t \| \nabla u \|^2_{L^2} + C_6 \sup_{0 \leq t \leq T} \| \nabla d \|^2_{L^3} \sup_{0 \leq t \leq T} (t \| \nabla^3 d \|^2_{L^2}) \\
+ C(1 + \| \nabla d \|^2_{L^4}) \int_0^T (\| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2}) dt + C_7M_0^\frac{1}{2} \sup_{0 \leq t \leq T} t \| \nabla u \|^2_{L^2},
\]

hence

\[
\sup_{0 \leq t \leq T} t(\| \nabla u \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2}) + \int_0^T t(\| \nabla \tilde{\rho} u \|^2_{L^2} + \| \nabla^3 d \|^2_{L^2}) dt \leq C,
\]

provided that

\[
M_0 \leq \epsilon_4 \overset{\text{def}}{=} \min \left\{ \epsilon_0, \frac{1}{2C_6}, \frac{1}{16C_7^2} \right\}.
\]

This combined with (3.23) ensures (3.27).

Step 2. By applying \( \nabla \) on (1.1)_4, we have

\[
\nabla d_t - \nabla \Delta d = -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d).
\]

Thus, due to \( \tilde{\rho} + \sup_{0 \leq t \leq T} \| \nabla d \|_{L^3} \leq 1 \), it follows from (3.21), (3.33), and (3.15) that

\[
\| \nabla d_t \|^2_{L^2} \leq C\| \nabla^3 d \|^2_{L^2} + C\| \nabla (u \cdot \nabla d) \|^2_{L^2} + C\| \nabla (|\nabla d|^2 d) \|^2_{L^2} \\
\leq C\| \nabla d \|^2_{L^2} \| \nabla u \|^2_{L^6} + C\| \nabla u \|^2_{L^6} \| \nabla^2 d \|_{L^2}^2 + C\| \nabla d \|^2_{L^3} \| \nabla d \|^2_{L^3} \\
+ C\| \nabla d \|^2_{L^3} \| \nabla^2 d \|^2_{L^2} + C\| \nabla^3 d \|^2_{L^2} \\
\leq C\| \nabla^2 u \|^2_{L^2} + C\| \nabla u \|^2_{L^2} \| \nabla^3 d \|^2_{L^2} + C\| \nabla^3 d \|^2_{L^2} \\
\leq C\| \sqrt{\rho} \tilde{\rho} u \|^2_{L^2} + C\| \nabla^3 d \|^2_{L^2} + C\| \nabla u \|^2_{L^2},
\]

which together with (3.5), (3.23), and (3.32) leads to

\[
\int_0^T t^{-1} \| \nabla d_t \|^2_{L^2} dt \leq C.
\]

Step 3. Operating \( \partial_t + u \cdot \nabla \) to (1.1)_j (\( j = 1, 2, 3 \)), one obtains by some calculations that

\[
\partial_t (\rho \tilde{u}^j) + \text{div}(\rho u \tilde{w}^j) - \Delta \tilde{w}^j \\
= -\partial_t (\partial_i u \cdot \nabla u^j) - \text{div}(\partial_i u \partial_k u^j) - \partial_i \partial_j P - u \cdot \nabla \partial_j P \\
- \partial_k \partial_i (\partial_i d \cdot \partial_j d) - u \cdot \nabla \partial_i (\partial_i d \cdot \partial_j d).
\]
Multiplying the above equality by $\dot{u}^j$, and then integrating by parts over $\mathbb{R}^3$, it follows that

$$
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 \, dx + \int |\nabla \dot{u}|^2 \, dx = - \sum_{i,j=1}^3 \int \left( \partial_i (\partial_j u \cdot \nabla u^j) + \text{div}(\partial_i u \partial_j u^j) \right) \dot{u}^j \, dx
$$

$$
- \sum_{i,j=1}^3 \int \left( \int \partial_i \partial_j (\partial_i d \cdot \partial_j d) \cdot \dot{u}^j \, dx + u \cdot \nabla \partial_i (\partial_i d \cdot \partial_j d) \dot{u}^j \, dx \right)
$$

$$
- \sum_{j=1}^3 \int (\partial_j \partial_j P + u \cdot \nabla \partial_j P) \dot{u}^j \, dx =: \sum_{i=1}^3 J_i. \quad (3.35)
$$

In what follows, we shall estimate each term on the right-hand side of (3.35) term by term. We first notice that exactly by the similar arguments as [19, Lemma 3.3], one has

$$
J_1 + J_3 \leq \frac{d}{dt} \int \sum_{i,j=1}^3 P \partial_j u^i \partial_i u^j \, dx + C(\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4) + \frac{1}{8} \|\nabla \dot{u}\|_{L^2}^2.
$$

For the term $J_2$, by using Hölder’s and Gagliardo-Nirenberg inequalities, we get

$$
J_2 \leq C\|\dot{u}\|_{L^2} \|\nabla d_i\|_{L^2} \|\nabla d\|_{L^6} + C\|\dot{u}\|_{L^6} \|\nabla d\|_{L^6} \|\nabla^2 d\|_{L^6} \|\nabla u\|_{L^2}
$$

$$
+ C\|\dot{u}\|_{L^6} \|\nabla d\|_{L^6} \|\nabla^2 d\|_{L^6} \|\nabla \dot{u}\|_{L^2}
$$

$$
\leq C\|\dot{u}\|_{L^2} \|\nabla d_i\|_{L^2}^\frac{1}{2} \|\nabla^2 d_i\|_{L^2}^\frac{1}{2} \|\nabla^2 d\|_{L^2} + C\|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^2} \|\nabla^3 u\|_{L^2} \|\nabla \dot{u}\|_{L^2}
$$

$$
\leq \frac{1}{8} \|\nabla \dot{u}\|_{L^2}^2 + \frac{1}{16} \|\nabla^2 d_i\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^2 \|\nabla d_i\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 \|\nabla^3 u\|_{L^2}^2.
$$

Inserting the estimates of $J_i$ ($i = 1, 2, 3$) into (3.35), one obtains

$$
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 \, dx + \frac{3}{4} \int |\nabla \dot{u}|^2 \, dx
$$

$$
\leq \frac{d}{dt} \int \sum_{i,j=1}^3 P \partial_j u^i \partial_i u^j \, dx + \frac{1}{12} \|\nabla^2 d_i\|_{L^2}^2 + C(\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4)
$$

$$
+ C\|\nabla^2 d\|_{L^2}^2 \|\nabla d_i\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2. \quad (3.36)
$$

Step 4. From (1.1)4, we have

$$
\nabla d_i - \nabla \Delta d_i = -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2) d_i,
$$

which together with the fact $u_t = \dot{u} - u \cdot \nabla u$ gives that

$$
\frac{1}{2} \frac{d}{dt} \int |\nabla d_i|^2 \, dx + \int |\nabla^2 d_i|^2 \, dx
$$

$$
= - \int \nabla (\dot{u} \cdot \nabla d) : \nabla d_i dx - \int \nabla (u \cdot \nabla d) : \nabla d_i dx
$$

$$
+ 2 \int \nabla (\nabla d : \nabla d_i d) : \nabla d_i dx + \int \nabla (|\nabla d|^2) d_i : \nabla d_i dx
$$

$$
+ \int \nabla (u \cdot \nabla u \cdot \nabla d_i) dx =: \sum_{i=1}^5 S_i. \quad (3.37)
$$
We estimate all the terms $S_i$ ($i = 1, 2, \cdots, 5$) on the right hand side of (3.37) term by term as follows,

$$S_1 \leq C\|\nabla u\|_{L^2} \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^3} + C\|\nabla u\|_{L^6} \|\nabla^2 d\|_{L^2} \|\nabla d_t\|_{L^3}$$

$$\leq C\|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^2} \|\nabla d_t\|_{L^3} \|\nabla^2 d_t\|_{L^2}$$

$$\leq \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{12} \|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2} \|\nabla d_t\|_{L^2}^2,$$

$$S_2 \leq C\|\nabla u\|_{L^2} \|\nabla d_t\|_{L^2} \|\nabla d_t\|_{L^2} + C\|u\|_{L^6} \|\nabla^2 d_t\|_{L^2} \|\nabla d_t\|_{L^3}$$

$$\leq C\|\nabla u\|_{L^2} \|\nabla^2 d_t\|_{L^2} \|\nabla d_t\|_{L^2}^2 \|\nabla d_t\|_{L^2}$$

$$\leq \frac{1}{12} \|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2,$$

$$S_3 \leq C\|\nabla d_t\|_{L^2}^2 \|\nabla d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2} \|\nabla d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2} \|\nabla^2 d_t\|_{L^2} \|\nabla d_t\|_{L^3}$$

$$\leq C\|\nabla^2 d\|_{L^2}^2 \|\nabla d_t\|_{L^2} \|\nabla^2 d_t\|_{L^2} \|\nabla d_t\|_{L^2}^2 \|\nabla d_t\|_{L^2}$$

$$\leq \frac{1}{2} \|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2,$$

$$S_4 \leq C\|d_t\|_{L^6} \|\nabla d_t\|_{L^2} \|\nabla^2 d\|_{L^2} + C\|\nabla d\|_{L^6} \|\nabla d_t\|_{L^2}^2$$

$$\leq C\|d_t\|_{L^2} \|\nabla d_t\|_{L^2} \|\nabla^2 d\|_{L^2}$$

$$\leq \frac{1}{12} \|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla^2 d\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2,$$

$$S_5 \leq C\|\nabla d\|_{L^2} \|\nabla u\|_{L^6} + C\|u\|_{L^6} \|\nabla^2 d\|_{L^2} + C\|\nabla u\|_{L^6} \|\nabla d\|_{L^3}$$

$$\leq \frac{1}{12} \|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla d\|_{L^6} \|\nabla^3 d\|_{L^2} + C\|\nabla^2 d\|_{L^2} \|\nabla d_t\|_{L^3}$$

$$\leq \frac{1}{12} \|\nabla^2 d_t\|_{L^2}^2 + C\|\nabla u\|_{L^6} + C\|\nabla^2 d\|_{L^2}^4 (\|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2),$$

where we have used the following fact

$$\|\nabla d\|_{L^2}^2 \leq C\|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2} \leq C\|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^6} \leq C\|\nabla d\|_{L^2} \|\nabla^3 d\|_{L^2}.$$
\[
\begin{aligned}
&\leq C\|\nabla d\|_{L^2}^2 (\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) + C\|\nabla d\|_{L^2}^2 \|\nabla u\|_{L^2}^6 + C\|\nabla d_t\|_{L^2}^6 \\
&\quad + C\left(\frac{1}{6} + \|\nabla d_t\|_{L^2}^4 + \|\nabla d\|_{L^2}^6\right) \|\nabla^3 d\|_{L^2}^2 \\
&\leq C(\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + C\|\nabla d\|_{L^2}^6 \|\nabla u\|_{L^2}^6 \\
&\quad + C_8\left(\frac{1}{6} + \|\nabla d_t\|_{L^2}^4 + \|\nabla d\|_{L^2}^6\right) \|\nabla^3 d\|_{L^2}^2,
\end{aligned}
\]

hence
\[
\|\nabla^3 d\|_{L^2}^2 \leq C\|\nabla d_t\|_{L^2}^2 + C\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^6,
\tag{3.39}
\]

provided that
\[
M_0 \leq \epsilon_5 := \min\left\{\epsilon_4, \frac{1}{\sqrt{6C_8}}\right\}.
\]

Thus, (3.39) combined with (3.36) and (3.38) ensures that
\[
\begin{aligned}
\frac{d}{dt}F(t) + &\frac{1}{2} \|\nabla \hat{u}\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 d_t\|_{L^2}^2 \\
\leq C(\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4) + C(\|\nabla u\|_{L^4}^4 + \|\nabla^2 d_t\|_{L^2}^2)(\|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2),\tag{3.40}
\end{aligned}
\]

where
\[
F(t) := \frac{1}{2} \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \frac{1}{2} \|\nabla d_t\|_{L^2}^2 - \int \sum_{i,j=1}^3 \rho \partial_i u \partial_j u_i dx
\]
satisfies
\[
\frac{1}{4} \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \frac{1}{4} \|\nabla d_t\|_{L^2}^2 - C\|\nabla u\|_{L^2}^6 \leq F(t) \leq \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 \tag{3.41}
\]

owing to the following estimates
\[
\begin{aligned}
\sum_{i,j=1}^3 \int \rho \partial_i u \partial_j u_i dx \\
= - \int \nabla P \cdot (u \cdot \nabla u) dx &\leq C\|\nabla P\|_{L^2} \|u\| \|\nabla u\|_{L^2} \\
\leq C\|\rho \hat{u}_t + \rho u \cdot \nabla u + \nabla d \cdot \Delta d\|_{L^2} \|u\| \|\nabla u\|_{L^2} \\
\leq C\delta^{-1} \|\nabla^2 u\|_{L^2}^2 + \delta \|\nabla u\|_{L^2}^6 + C\rho \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + C\|\nabla d\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2 \\
\leq C_9 M_0^{\frac{1}{2}} (\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + C\|\nabla u\|_{L^2}^6, \tag{3.42}
\end{aligned}
\]

provided that
\[
M_0 \leq \epsilon_6 := \min\left\{\epsilon_5, \frac{1}{16C_9^2}\right\}.
\]

Step 5. We shall estimate the terms on the right-hand side of (3.40). To bound the terms \(\|P\|_{L^4}\) and \(\|\nabla u\|_{L^4}\), it follows from Sobolev embedding, (3.20), and Hölder’s inequality that
\[
\begin{aligned}
\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4 &\leq C \left(\|\nabla P\|_{L^{\frac{4}{3}}}^4 + \|\nabla^2 u\|_{L^{\frac{4}{3}}}^4\right) \\
&\leq C \left(\|\rho \hat{u}\|_{L^{\frac{4}{3}}}^4 + \|\nabla d\|_{L^{\frac{4}{3}}}^4 \|\nabla^2 d\|_{L^{\frac{4}{3}}}^4\right) \\
&\leq C\|\rho \hat{u}\|_{L^{\frac{4}{3}}}^4 \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + C\|\nabla d\|_{L^6}^2 \|\nabla^2 d\|_{L^2}^2 \\
&\leq C\|\sqrt{\rho} \hat{u}\|_{L^2}^2 F(t) + \|\nabla u\|_{L^6}^2 + C\|\nabla^2 d\|_{L^2}^6. \tag{3.43}
\end{aligned}
\]
Hence, inserting (3.43) and (3.39) into (3.40), one obtains
\[
\frac{d}{dt} F(t) + \frac{1}{2} \|
abla \hat{u} \|^2_{L^2} + \frac{1}{2} \|
abla^2 d_t \|^2_{L^2} \\
\leq C \langle \sqrt{\rho} \hat{u} \|^2_{L^2} (F(t) + \|\nabla u \|^2_{L^2}) + C \|
abla^2 d \|^2_{L^2} \\
+ C (\|\nabla u \|^2_{L^2} + \|\nabla^2 d_t \|^2_{L^2}) (\|\nabla d_t \|^2_{L^2} + \|\nabla^3 d \|^2_{L^2}) \\
\leq C (\sqrt{\rho} \hat{u} \|^2_{L^2} + \|\nabla u \|^4_{L^2} + \|\nabla^2 d \|^4_{L^2}) (F(t) + \|\nabla u \|^3_{L^2}) \\
+ C (\|\nabla u \|^4_{L^2} + \|\nabla^2 d \|^4_{L^2}) \|\nabla u \|^6_{L^2}.
\] (3.44)

Multiplying (3.44) by \( t^i \) \((i = 1, 2)\), and then applying Gronwall’s inequality, it follows from (3.5), (3.23), (3.32), and (3.34) that
\[
\sup_{0 \leq t \leq T} (t^i F(t)) + \int_0^T t^i (\|\nabla \hat{u} \|^2_{L^2} + \|\nabla^2 d_t \|^2_{L^2}) dt \\
\leq C \int_0^T t^i F(t) dt + C \int_0^T t^i \|\nabla u \|^6_{L^2} (\sqrt{\rho} \hat{u} \|^2_{L^2} + \|\nabla u \|^4_{L^2} + \|\nabla^2 d \|^4_{L^2}) dt \\
\leq C \int_0^T t^i (\|\nabla \hat{u} \|^2_{L^2} + \|\nabla^2 d_t \|^2_{L^2}) dt \\
+ C \sup_{0 \leq t \leq T} (t^i \|\nabla u \|^2_{L^2}) \sup_{0 \leq t \leq T} \|\nabla u \|^2_{L^2} \int_0^T \|\nabla u \|^2_{L^2} dt \\
+ C \sup_{0 \leq t \leq T} (t^i \|\nabla u \|^2_{L^2}) \sup_{0 \leq t \leq T} \|\nabla u \|^2_{L^2} \int_0^T \sqrt{\rho} \hat{u} \|^2_{L^2} dt \\
+ C \sup_{0 \leq t \leq T} (t^i (\|\nabla u \|^4_{L^2} + \|\nabla^2 d \|^4_{L^2})) \sup_{0 \leq t \leq T} \|\nabla u \|^4_{L^2} \int_0^T \|\nabla u \|^2_{L^2} dt \leq C. \] (3.45)

This together with (3.41) ensures (3.28), while the desired (3.29) follows from (3.28), (3.21), and (3.34). This completes the proof of Lemma 3.2.

**Lemma 3.3.** There exists a positive constant \( C \) depending on \( T \) such that for any \( q \in (3, 6) \)
\[
\sup_{0 \leq t \leq T} \|\rho\|_{H^{1 \cap W^{1,q}}} + \int_0^T \left( \|\nabla^2 u \|^2_{L^2} + \|\nabla^2 u \|^p_{L^p} + t \|\nabla^2 u \|^2_{L^2 \cap L^q} \right) dt \\
+ \int_0^T \left( \|\nabla P \|^2_{L^2} + \|\nabla P \|^p_{L^p} + t \|\nabla P \|^2_{L^2 \cap L^q} \right) dt \leq C(T), \] (3.46)

where
\[
1 \leq p < \frac{4q}{5q - 6}.
\]

**Proof.** Step 1. Differentiating the continuity equation (1.1) with respect to \( x_i \) gives rise to
\[
(\rho_x)_t + \nabla \rho \cdot u + \nabla \rho \cdot u_{x_i} = 0. \] (3.47)

Multiplying (3.47) by \(|r| \rho_x |r-2 | \rho_x | (r \geq 2)\) and integrating the resulting equation over \( \mathbb{R}^3 \) indicate that
\[
\frac{d}{dt} \|\nabla \rho \|_{L^r} \leq C \|\nabla u \|_{L^\infty} \|\nabla \rho \|_{L^r}. \] (3.48)
One gets from Gagliardo-Nirenberg inequality and (3.27) that for \( q \in (3, 6) \) as in Theorem 1.1,
\[
\|\nabla u\|_{L^\infty} \leq C \|\nabla u\|_{L^2}^{\frac{2q-6}{2q}} \|\nabla^2 u\|_{L^q}^{\frac{3q}{2q}} \leq C \|\nabla^2 u\|_{L^q}^{\frac{3q}{2q}}. \tag{3.49}
\]
By using (3.4), Sobolev’s and Gagliardo-Nirenberg inequalities, one has
\[
\|\rho u\|_{L^q} \leq C \|\rho u\|_{L^2}^{\frac{6-q}{2q}} \|\rho u\|_{L^q}^{\frac{3q-6}{2q}} \leq C \|\nabla \rho u\|_{L^q}^{\frac{3q-6}{2q}}. \tag{3.50}
\]
This combined with (3.20), (3.27), and (3.28) leads to for \( 1 \leq p < \frac{4q}{5q-6} \),
\[
\int_0^T \|\nabla^2 u\|_{L^p}^p dt \leq C \int_0^T \|\sqrt{\rho u}\|_{L^2}^{\frac{p(6-q)}{4q}} \|\nabla \rho u\|_{L^2}^{\frac{p(3q-6)}{4q}} dt + C \int_0^T \|\nabla d\|_{L^p}^p \|\nabla^2 d\|_{L^q}^p dt
\]
\[
\leq C \sup_{0 \leq t \leq T} \left( \int_0^T \|\sqrt{\rho u}\|_{L^2}^2 dt \right)^\frac{p(6-q)}{4q} \int_0^T (t \|\nabla \rho u\|_{L^2}^2) \frac{p(3q-6)}{4q} dt + C
\]
\[
\leq C \left( \int_0^T t^{-\frac{4q}{4q+6q-3pq}} \frac{4q+6q-3pq}{4q} dt \right)^\frac{4q+6q-3pq}{4q} + C,
\tag{3.51}
\]
which along with (3.49) and \( 0 < \frac{3q}{5q-6} < 1 \) implies
\[
\int_0^T \|\nabla u\|_{L^\infty} dt \leq C. \tag{3.52}
\]
Thus, applying Gronwall’s inequality to (3.48) gives
\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^2 \cap L^q} \leq C. \tag{3.53}
\]

Step 2. It is easy to deduce from (3.5), (3.21), (3.28), and (3.50) that
\[
\int_0^T \left( \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^p t + t \|\nabla^2 u\|_{L^2}^2 \right) dt
\]
\[
+ \int_0^T \left( \|\nabla P\|_{L^2}^2 + \|\nabla P\|_{L^2}^p t + t \|\nabla P\|_{L^2}^2 \right) dt \leq C.
\]
This together with (3.52) ensures (3.46).

\[\square\]

**Lemma 3.4.** There exists a positive constant \( C \) depending on \( T \) such that
\[
\sup_{0 \leq t \leq T} t \|\sqrt{\rho u}\|_{L^2}^2 + \int_0^T t \|\nabla u_t\|_{L^2}^2 dt \leq C(T). \tag{3.54}
\]

**Proof.** Step 1. Differentiating (1.1) with respect to \( t \) leads to
\[
\rho u_{tt} + \rho u \cdot \nabla u_t - \Delta u_t + \nabla P_t = -\rho (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \text{div}(\nabla d \otimes \nabla d)_t. \tag{3.55}
\]
Multiplying (3.54) by \( u_t \) and integrating the resulting equality over \( \mathbb{R}^3 \), we deduce after using (1.1) that
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int |\nabla u_t|^2 dx
\]
\[
= \int [(\nabla d \otimes \nabla d)_t : \nabla u_t - \rho (u_t + \nabla u) u_t + \text{div}(\rho u)(u_t + u \cdot \nabla u) \cdot u_t] dx
\]
By virtue of (3.4), Gagliardo-Nirenberg inequality, and Hölder’s inequality, each
which leads to
Inserting (3.39) into (3.57), we derive that
Substituting the above estimates on
\[ \int \rho|\nabla u|^2 \, dx \]

Step 2. Notice that
\[ \| \nabla^2 u \|_{L^2} \leq C \bar{\rho} \frac{2}{3} \| \sqrt{\rho u_t} \|_{L^2} + C \bar{\rho} \| u \|_{L^6} \| \nabla u \|_{L^3} + C \| \nabla d \|_{L^3} \| \nabla^3 d \|_{L^2} \]

Inserting (3.39) into (3.57), we derive that
\[ \| \nabla^2 u \|_{L^2} \leq \left( \frac{1}{4} + C_{10} M_0^2 \right) \| \nabla^2 u \|_{L^2} + C_{10} M_0^2 \| \nabla^2 u \|_{L^3}^2 \]

which leads to
\[ \| \nabla^2 u \|_{L^2} \leq C (\| \sqrt{\rho u_t} \|_{L^2} + \| \nabla d_t \|_{L^2} ) + C \| \nabla u \|_{L^2}^2, \]

provided that
\[ M_0 \leq \epsilon_0 = \min \left\{ \epsilon_6, \frac{1}{\sqrt{4C_{10}}} \right\}. \]
Inserting (3.58) into (3.56), we have
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \frac{1}{2} \int \mathcal{L}_t u_t^2 dx
\leq C \rho^2 \left( \frac{1}{2} \int \rho u_t^2 dx + \int \nabla d_t \nabla u_{t} dx \right) + C \|u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \|
abla^2 u\|_{L^2}^2 + C \|
abla^3 u\|_{L^2}^2 + C \|
abla^4 u\|_{L^2}^2 + C \|
abla^5 u\|_{L^2}^2
\]
\[+ C \|
abla^6 u\|_{L^2}^2 + C \|
abla^7 u\|_{L^2}^2 + C \|
abla^8 u\|_{L^2}^2 + C \|
abla^9 u\|_{L^2}^2 + C \|
abla^{10} u\|_{L^2}^2.
\]

Step 3. It deduces from (3.21), (3.27), (3.5), and (3.29) that
\[
\int_0^T \int \rho u_{t}^2 dx dt \leq C \int_0^T \left( \int \rho u_{t}^2 dx + \int \rho u_{t}^2 dx \right) dt
\leq C \int_0^T \rho u_{t}^2 dx + C \int_0^T \rho u_{t}^2 dx dt
\leq C \int_0^T \nabla u_{t}^2 dx + C \int_0^T \nabla^2 u dx dt + C
\leq \frac{1}{2} \int_0^T \rho u_{t}^2 dx + C \int_0^T \nabla d_t \nabla u_{t} dx dt + C \int_0^T \nabla u_{t}^2 dx dt + C,
\]
which implies that
\[
\int_0^T \rho u_{t}^2 dx dt \leq C.
\]
Then, multiplying (3.59) by \( t \), one derives (3.53) from Gronwall’s inequality, (3.60), and (3.28).

4. Proof of Theorem 1.1. With the a priori estimates in section 3 at hand, we are now in a position to prove Theorem 1.1.

First, by Lemma 2.2, there exists a \( T_* > 0 \) such that the Cauchy problem (1.1)–(1.2) has a unique local strong solution \((\rho, u, d, P)\) on \( \mathbb{R}^3 \times (0, T_*) \). It follows from (1.5) that there exists a \( T_1 \in (0, T_*) \) such that (3.1) holds for \( T = T_1 \). Set
\[
T^* := \text{sup} \{ T | (\rho, u, d, P) \text{ is a strong solution on } \mathbb{R}^3 \times (0, T) \text{ and (3.1) holds} \}.
\]
Then \( T^* \geq T_1 > 0. \) Hence, for any \( 0 < \tau < T \leq T^* \) with \( T \) finite, one deduces from (3.5), (3.27), (3.28), (3.29), and (4.46) that
\[
\nabla u, \nabla u, \nabla^2 u, \nabla^2 d, P \in C([\tau, T]; L^2 \cap L^q),
\]
where one has used the standard embedding
\[
L^\infty(\tau, T; H^1) \cap H^1(\tau, T; L^2) \hookrightarrow C(\tau, T; L^q) \text{ for any } q \in (3, 6).
\]
Moreover, it follows (4.46) and [15, Lemma 2.3] that
\[
\rho \in C([0, T]; H^1 \cap W^{1,q}).
\]
Now, we claim that
\[
T^* = \infty.
\]
Otherwise, \( T^* < \infty. \) Proposition (3.1) implies that (3.2) holds at \( T = T^* \). It follows from (4.2) and (4.3) that
\[
(p^*, u^*, d^*)(x) := (\rho, u, d)(x, T^*) = \lim_{t \to T^*} (\rho, u, d)(x, t)
\]
(4.5)
satisfies
\[ \rho^* \in H^1 \cap W^{1,p}, \quad u^*, \nabla d^* \in D^1 \cap D^2. \]  
(4.6)
Thus, one can take \((\rho^*, \rho^* u^*, d^*)\) as the initial data and apply Lemma 2.2 to extend the local strong solution beyond \(T^*\). This contradicts the assumption of \(T^*\) in (4.1). Hence, (4.4) holds. The proof of Theorem 1.1 is complete.

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