Ground states of an Ising model on an extended Shastry-Sutherland lattice and the 1/2-magnetization plateau in some rare-earth-metal tetraborides

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(Dated: February 5, 2013)

A complete solution of the ground-state problem for an Ising model on the Shastry-Sutherland lattice with an additional interaction along the diagonals of “empty” squares in an applied magnetic field is presented. A rigorous proof is given that this interaction gives rise to a plateau at one-half of the saturation magnetization. Such a fractional plateau has been observed in some rare-earth-metal tetraborides, in particular, in strong Ising magnets ErB$_4$ (where it is the only one) and TmB$_4$ (where it is the broadest one), but its origin has remained unclear. Our study sheds new light on the solution of this problem.

PACS numbers: 75.60.Ej, 05.50.+q, 75.10.Hk, 75.10.-b

I. INTRODUCTION

Here we consider an Ising model on the Shastry-Sutherland (SS) lattice with an additional diagonal interaction. The SS lattice is topologically equivalent to the Archimedean $3^2$.4.3.4 lattice (Fig. 1). It has been shown that the ground-state problem of the quantum Heisenberg model on the SS lattice in an applied magnetic field can be solved exactly. This model has attracted considerable interest when the magnetic subsystem of some compounds has been shown to consist of weakly coupled layers of magnetic ions arranged on a lattice that is topologically equivalent to the SS one. These compounds are referred to as Shastry-Sutherland magnets. SrCu$_2$(BO$_3$)$_2$ is the most known and the most studied among them. But during the last decade, many other SS magnets have been discovered. In particular, this concerns an entire group of rare-earth-metal tetraborides RB$_4$ ($R = $ La – Lu). Magnetic ions $R^{3+}$ carry spins large enough to be considered as classical Heisenberg ones. If, in addition, the crystal field effects are strong, then the magnets can be described in terms of the effective spin-$1/2$ model under strong Ising anisotropy. This is the case of TmB$_4$ and ErB$_4$.

SS magnets have attracted particular interest, since they exhibit sequences of fractional magnetization plateaus. A single plateau at one-half of the saturation magnetization ($m/m_s = 1/2$) has been observed in ErB$_4$. TmB$_4$, in addition to an extended 1/2-plateau, exhibits a sequence of narrow plateaus with fractional values of $m/m_s = 1/6, 1/7$ up to $1/12$ for temperatures below 4 K (the magnetic field is normal to SS planes).

Some attempts have been made to explain the origin of the fractional magnetization plateaus in terms of the Ising model on the SS lattice, however, this model was shown, both numerically and analytically, to predict a single fractional plateau—at $m/m_s = 1/3$. This suggested that only quantum models and maybe with longer-range interactions could describe the 1/2-plateau and other fractional plateaus observed in the rare-earth-metal tetraborides. In Refs. [1] and [11], a 1/2-plateau was obtained using the quantum SS model (and also its Ising limit) with additional interactions $J_3$ (along the diagonals of “empty” squares) and $J_4$ (the next-nearest neighbors along edges).

In our previous work, we showed that the fractional plateaus in an Ising-type model on the SS lattice can be generated by the long-range RKKY-type interactions (these are present in rare-earth-metal tetraborides since the latter are good metals), and that the 1/2-plateau is given rise by the additional diagonal interaction $J_3$. Recently, a 1/2-plateau (as well as a narrow 2/3-plateau) was shown numerically to be generated by dipole-dipole interactions.

In our opinion, to explain the sequence of the magnetization plateaus in TmB$_4$, it is sufficient to find the ground states of an Ising-type model on an extended SS lattice with long-range interactions. It is difficult to find
a numerical solution but our analytical method for determining the ground states of lattice gas models or equivalent spin ones makes it possible.

Here, we present a complete solution of the ground-state problem for the Ising model on the SS lattice with additional interaction $J_3$ along the diagonals of “empty” squares (in what follows, we refer to this model as the Ising model on the extended SS lattice). To find this solution, we use the method of basic rays and basic sets of cluster configurations which was developed in our previous works.\textsuperscript{13,14} Moreover, we generalize the method, and consider configurations of two (instead of one) different clusters that is quite natural for the SS lattice with additional diagonal bonds.

We rigorously prove the existence of a rather wide 1/2-plateau in this model. It corresponds to three different phases and appears under the condition that the interaction $J_1$ along the edges of squares and at least one of the two remaining interactions—along the SS diagonals ($J_2$) and along the diagonals of the “empty” squares ($J_3$)—is antiferromagnetic. It depends on the signs of the interactions $J_2$ and $J_3$ which of these three phases is realized. A single 1/2-plateau exists, in particular, in the case when the interactions $J_1$ and $J_2$ are antiferromagnetic and $J_3$ is ferromagnetic (under the condition $J_2 < -2J_3$). A single 1/2-plateau has been observed in ErB$_4$, but its origin remains unclear. Hence, our investigation sheds light on this problem.

The paper is organized as follows. In Sec. II, a two-cluster approach to the determination of the ground states of Ising-type models is developed and a solution of the Ising model on the extended SS lattice is found. In Sec. III, the full-dimensional structures are determined and analyzed. In Sec. IV, the structures at the three-dimensional boundaries of the full-dimensional regions are considered and applied to predict the order of phase transitions between full-dimensional phases. These are also important for the investigation of the effects of longer-range interactions. In Sec. V, the ground states at the two-dimensional boundaries of the full-dimensional regions are analyzed. In Sec. VI, the ground-state phase diagrams in the $(h, J_2)$-plane are presented as well as possible sequences of phases for the magnetization processes in ErB$_4$ and TmB$_4$. Section VII gives some conclusions.

II. SOLUTION OF THE GROUND-STATE PROBLEM FOR THE ISING MODEL ON THE EXTENDED SS LATTICE: A TWO-CLUSTER APPROACH

In our previous studies,\textsuperscript{9,13,14} we have constructed the ground-state structures for Ising-type models (or equivalent lattice-gas models) with configurations of some cluster. We used only one kind of cluster. For instance, a cluster in the form of a triangle with an SS diagonal as the hypotenuse is sufficient for the Ising model on the conventional SS lattice.\textsuperscript{2} Here we consider the same model but with additional interaction $J_3$ along all the diagonals of the squares without SS bonds [Fig. 1(d)]. To construct the ground-state structures for this model one small cluster is insufficient; one should use two different clusters: a square with an SS diagonal and a square without it [Fig. 2(a)]. However, there is no crucial difference in comparison with the one-cluster approach. The main idea is the same and consists in introducing “free” coefficients which account for the fact that the energy contribution of sites and some bonds can be distributed between clusters in various ways. (As will be clear later, “free” coefficients make it possible to equalize and minimize the energies of two or more cluster configurations at once). Each edge belongs to two squares (one of each type) and each site belongs to four squares (two of each type). The energy contribution of an edge (site) can be distributed in various ways between the squares sharing this edge (site). The way of distribution is determined by the “free” coefficients $\alpha$, $\beta$, and $\gamma$ in the expressions for the energy contributions of different squares [see Fig. 2(b)]

$$
e = J_3 \sigma_1 \sigma_3 + \alpha J_1 (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_4 \sigma_1)$$

$$- \gamma h [\sigma_1 \sigma_3 + (1 - \beta)(\sigma_2 + \sigma_4)],$$

$$\tilde{e} = J_3 (\sigma_1 \sigma_3 + \sigma_2 \sigma_4) + (1 - \alpha) J_1 (\sigma_1 \sigma_2 + \sigma_2 \sigma_3$$

$$+ \sigma_3 \sigma_4 + \sigma_4 \sigma_1) - \frac{1 - \gamma}{2} h (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4).$$

Here $e$ and $\tilde{e}$ are the energy contributions of the squares with SS bond and without SS bond, respectively; $J_1$, $J_2$, and $J_3$ are the parameters of interaction along the edges, SS diagonals, and ordinary diagonals (only for the squares without SS diagonals), respectively; $h$ is the applied magnetic field. The numbers $\sigma_1$, $\sigma_2$, $\sigma_3$, and $\sigma_4$ ($\sigma_i = \pm 1$) define the spin configuration of a square. There are nine configurations of a square with an SS di-
TABLE I: Basic rays and basic sets of configurations for the Ising model on the extended Shastry-Sutherland lattice.

| $r_1$ | Basic ray \((h, J_1, J_2, J_3)\) | Basic set of configurations $R_\alpha$ | Full-dimensional structures | “Free” coefficients |
|-------|----------------------------------|---------------------------------------|-----------------------------|--------------------|
| $r_1$ | \((0, 0, -1)\)                   | \(1, \bar{1}, 3, 9, 10, \bar{10}\)   | Arbitrary                   |
| $r_2$ | \((0, -1, 0)\)                   | \(1, \bar{1}, 3, 7, 8\)              | Arbitrary                   |
| $r_3$ | \((0, -1, 0)\)                   | \(1, \bar{1}, 5, 8\)                | $\alpha = 0$              |
| $r_4$ | \((0, -1, 2, 0)\)                | \(1, \bar{1}, 5, 9\)                | $\alpha = 1$              |
| $r_5$ | \((0, 1, 2, 0)\)                 | \(3, 4, 4, 5, 9\)                  | $\alpha = 1$              |
| $r_6$ | \((2, 0, 0, 1)\)                 | \(1, 5, 6, 7, 8\)                  | $\gamma = 0$              |
| $r_7$ | \((4, 1, 0, 0)\)                 | \(1, 3, 4, 6, 7, 10\)              | $\alpha = 1, \beta = \frac{1}{2}, \gamma = 1$ |
| $r_8$ | \((1, 0, 1, 0)\)                 | \(1, 4, 5, 6, 9, 10\)              | $\beta = 1, \gamma = 1$ |
| $r_9$ | \((2, 1, 0, 1)\)                 | \(3, 4, 5, 6, 7, 8\)              | $\alpha = 0, \gamma = 0$ |
| $r_{10}$ | \((2, 1, 2, -1)\)          | \(3, 4, 9, 10\)                 | $\alpha = 1, \beta = \frac{1}{2}, \gamma = 1$ |
| $r_6$ | \((-2, 0, 0, 1)\)                | \(1, 5, 6, 7, 8\)                  | $\gamma = 0$              |
| $r_7$ | \((-4, 1, 0, 0)\)                | \(1, 3, 4, 6, 7, \bar{10}\)        | $\alpha = 1, \beta = \frac{1}{2}, \gamma = 1$ |
| $r_8$ | \((-1, 0, 1, 0)\)                | \(1, 4, 5, 6, 9, \bar{10}\)        | $\beta = 1, \gamma = 1$ |
| $r_7$ | \((-2, 1, 0, 1)\)                | \(3, 4, 5, 6, 7, 8\)              | $\alpha = 0, \gamma = 0$ |
| $r_{10}$ | \((-2, 1, 2, -1)\)             | \(3, 4, 9, \bar{10}\)            | $\alpha = 1, \beta = \frac{1}{2}, \gamma = 1$ |

$^a$Configurations marked by asterisk enter the structures in blocks shown in Fig. 4.

$^b$Condition symmetric to the condition for $r_{10}$.

agonal and six configurations of a square without SS diagonal: open and solid circles denote spin down (−1) and spin up (+1), respectively; the squares of different types are separated by the symbol ||.

As in our previous studies, here we use the method of basic rays (vectors) and basic sets of cluster configurations. In brief it is as follows.

Consider the parameter space \((h, J_1, J_2, J_3)\) of the model. Each ground-state structure corresponds to a region in this space. The region can be presented as a solution of a set of uniform linear inequalities; therefore, it is a polyhedral cone. Let us recall that a polyhedral cone is a conical hull—that is, all the linear combinations with nonnegative coefficients—of a set of vectors. It is completely determined by its edges or vectors along them. Among the polyhedral cones corresponding to all possible ground-state structures of the model, the most important are those with the dimensionality equal to the dimensionality of the parameter space. We refer to these regions and corresponding structures as full-dimensional.

If a structure is a ground-state one at two points of the parameter space, then it is a ground-state structure in the entire line segment connecting these points. This is a convexity property; the region corresponding to a structure is always convex. The convexity property makes it possible to determine the ground-state structures in any point of the parameter space if all edges of the full-dimensional regions (basic rays) and all ground-state structures in these edges are known.

First we determine the cluster configurations which generate ground-state structures and then the structures themselves. By definition, a set of cluster configurations generates a structure if all the cluster configurations in the structure belong to the set. A set of cluster configurations generates all the structures at the point \((h, J_1, J_2, J_3)\) of the parameter space if the following three conditions are satisfied: (1) values of the “free” coefficients $\alpha$, $\beta$, and $\gamma$ in Eqs. (1) can be chosen in such a manner that all the configurations of the set have the same energy, (2) this energy is lower than the energies of all the remaining configurations, and (3) at least one structure can be generated by the configurations of the set. It should be noticed that this group of conditions is sufficient but not necessary; only the third condition is necessary.

The basic rays and the corresponding sets of cluster configurations (basic sets) of the model under consideration are listed in Table I. Configurations of clusters of different types are separated by the symbol ||. If configurations of a type of cluster are not indicated in a set, then all the configurations of this type belong to the set (i.e., the configuration of this type of cluster can be arbitrary). The last column of Table I contains the values of the “free” coefficients for which the configurations belonging to the basic set have the same energy which is lower than the energies of the remaining configurations (in the corresponding ray, of course).

Having determined the full-dimensional regions and structures on the basis of Table I, we can show that the set of basic rays is complete. This will be done later. It should be noticed that we present here only the final results, the correctness of which can be easily verified. In reality, first we found the full-dimensional ground-state
structures of the model (all these but one are determined in Ref. 9) and only then we calculated the basic rays.

In all the basic rays, except for \( r_{10} \) and \( r_{10}^* \), all the structures generated by the configurations of the corresponding basic set are ground-state structures in these rays. As to the rays \( r_{10} \) and \( r_{10}^* \), an additional condition should be satisfied. For the ray \( r_{10} \) this condition can be formulated as follows: the configurations marked by the asterisk enter the ground-state structures in the blocks shown in Fig. 4 and these blocks do not overlap either with themselves or with other squares. (For the symmetric ray \( r_{10}^* \) the symmetric condition should be satisfied). Let us prove this.

Using Eq. (1), one can calculate the energies of all the configurations of the clusters at the point \( h = 2J_1 \), \( J_2 = 2J_1 \), and \( J_3 = -J_1 \) (the ray \( r_{10} \)) for the following values of the “free” coefficients: \( \alpha = 1 \), \( \beta = 1/2 \), and \( \gamma = 1 \).

\[
\begin{align*}
\bullet \bullet \bullet \bullet (\alpha = 1, \beta = 1/2, \gamma = 1) & : \bullet \bullet \bullet \bullet (2J_1) \quad \bullet \bullet \bullet \bullet (4J_1) \quad (2J_2) \quad (12J_1) \quad (0) \quad (2J_1) \quad (4J_1).
\end{align*}
\]

The zero energy level is shifted here by \( 2J_1 > 0 \). The energy value indicated after each group of configurations is the same for all the members of the group. We refer to the squares with negative, zero, and positive energies as “negative,” “zero,” and “positive” squares, respectively.

Let us prove that, in the ray \( r_{10} \), the energy of an arbitrary structure cannot be negative. Near each negative square \( \bullet \bullet \bullet \bullet \) in the region shown in Fig. 3(a), at least one square among the three squares of the region is positive. Let us group all the squares in a structure in the following way: each zero square forms a group by itself; each negative square enters the group of the positive square situated near it in the region shown in Fig. 3(a). If a negative square enters two or three groups, then its energy is distributed into equal parts between these groups. It is easy to see that, for such a grouping rule, positive squares \( \bullet \bullet \bullet \bullet \) and \( \bullet \bullet \bullet \bullet \) cannot form a group with any negative square. Squares \( \bullet \bullet \bullet \bullet \) and \( \bullet \bullet \bullet \bullet \) can form groups with one negative square only, but the energy of the groups is positive. Square \( \bullet \bullet \bullet \bullet \) can form a zero-energy group with two negative squares. In this case, however, other groups with positive energies are formed inevitably [Fig. 3(b)].

The square \( \bullet \bullet \bullet \bullet \) can form a single zero-energy group with a negative square in the way shown in Fig. 3(b). The only square that can form a group with negative energy \( (-J_1) \) is the square \( \bullet \bullet \bullet \bullet \) [Fig. 3(d)]. However, then there appears one of the two positive squares, \( \bullet \bullet \bullet \bullet \) or \( \bullet \bullet \bullet \bullet \). The first one forms a group with the energy \( 3J_1 \) which cannot be compensated by the negative energy of the group formed by the square \( \bullet \bullet \bullet \bullet \). The second square \( \bullet \bullet \bullet \bullet \) can form a zero-energy group, but in this case the situation is repeated and so the negative-energy group of the first square \( \bullet \bullet \bullet \bullet \) generates an infinite half-stripe in which all the remaining groups have zero energies. Hence, the number of negative-energy groups in the structure [Fig. 3(d)] can be infinitesimal only.

To summarize, the ground-state structures in the ray \( r_{10} \) can contain, along with zero squares, the positive squares \( \bullet \bullet \bullet \bullet \) and \( \bullet \bullet \bullet \bullet \) but only in combi-

III. FULL-DIMENSIONAL GROUND-STATE STRUCTURES AND REGIONS

Table I represents a complete solution of the ground-state problem under consideration. Using this Table and the convexity property, it is easy to determine the full-dimensional ground-state structures and regions. All one has to do is to find the subsets of the basic sets of configurations with the following properties: (1) each of them is a subset of at least four basic sets and the linear hull of the corresponding basic vectors is full-(four-)dimensional; (2) the cluster configurations of such a subset generate at least one structure.

The full-dimensional regions and the corresponding subsets of configurations which generate all the ground-state structures in these regions are listed in Table II. The first column gives the numbers of full-dimensional regions. Taking into account the symmetry of the model...
with respect to the field inversion with simultaneous flip of all spins, we indicate only the regions (structures) with zero and positive magnetization. We denote a region (structure) with negative magnetization by the same number as the symmetric region (structure) but with a bar over the number. The third column of Table II gives energies per site of structures and, in the square brackets, the fractional contents of cluster configurations in the structures (see Ref. [13]). If the fractional contents of configurations in a structure can vary (region 7), then there is a degeneracy marked by the word “Disorder.” (In region 6, there is degeneracy as well, but the fractional contents of configurations do not vary). The fourth, fifth, sixth, and seventh columns indicate, respectively: magnetization of the structure, basic rays that define the corresponding region, the number of three-dimensional faces of the region, and the conditions for the existence of the region in the \((h, J_2)\)-plane.

The full-dimensional ground-state structures of the model under consideration are depicted in Figs. 5 and 6. They are constructed with the cluster configurations listed in Table II. All the full-dimensional structures, except for structures 6 and 7, are fully determinate. In the regions 6 an 7, an infinite number of the ground-state structures occur: phases 6 and 7 are disordered (Fig. 6). It is easy to see that the disorder of phase 7 is one-dimensional: it is ordered in one direction and disordered in another.

It is more difficult to determine the character (i.e., the dimensionality) of disorder in phase 6. In structures of phase 6 (and even at the boundary between phases 4 and 6), the structural element shown in Fig. 7 generates an infinite half-stripe. On the basis of this fact one can prove that the disorder of phase 6 is one-dimensional. The number of squares \(\varnothing\) and \(\varnothing\) is infinitesimal in comparison with the number of remaining squares. Although their energies are equal to the energies of other squares which generate structure 6, these squares represent a kind of zero-energy defects. Hence, structure 6 can be considered as a simple mixture of structures 6a and 6b depicted in Fig. 5.

It is interesting that the full-dimensional structure 9 is chiral. This is the best seen on the lattice shown in Fig. 1(c). Two different structures 9 are possible: a structure with the SS bonds twisted counterclockwise around “open” and “solid” squares and its chiral twin with the SS bonds twisted clockwise (Fig. 8). Hence, the interaction \(J_3\) lifts the degeneracy of the Ising dimer phase (opposite spins on the SS diagonals\(\varnothing\) and thus two ordered phases are produced: one of these is chiral. It should be noticed
FIG. 5: Full-dimensional ground-state structures of the Ising model on the extended SS lattice (for $h \geq 0$). Phases 1, 3, 4, and 5 are, respectively, the fully polarized phase, the Néel phase, the $1/3$-plateau phase (or the UUD phase), and the collinear phase. Phases 8 and 9 are chessboard phases. Phases 10, 6a, and 6b are $1/2$-plateau phases. A simple mixture of structures 6a and 6b is the ground-state structure for region 6. Unit cells are indicated.

FIG. 6: Full-dimensional ground-state structures of the Ising model on the extended SS lattice. Disordered phases 6 and 7. In structure 6 the squares-“defects” are shown.

that this structure is not chiral on the Archimedean lattice [Fig. 1(a)] with equal interactions $J_1$ and $J_2$. Chiral structures are ubiquitous in nature, however, their emergence as a result of spontaneous symmetry breaking remains unclear.\[15\]

IV. GROUND-STATE STRUCTURES IN THE 3-FACES OF THE FULL-DIMENSIONAL REGIONS

Let us consider now the ground-state structures in the three-dimensional faces (3-faces) of the full-dimensional regions. A 3-face is defined by a subset of the set of basic vectors for a full-dimensional region. An algorithm for the determination of 3-faces of a full-dimensional region is
FIG. 7: A structural element in phases 4 and 6 (as well as in their boundary) and an infinite half-stripe generated by this element.

FIG. 8: Chirality of structure 9. The SS bonds twist counterclockwise around “open” and “solid” squares (left structure) or clockwise (right structure) but in the same manner for all the squares.

described in Ref. [13]. It is a simple mathematical problem and we do not reproduce the description of this algorithm here. Having found the 3-faces of all the full-dimensional regions, one can easily prove the completeness of the set of basic vectors. It is sufficient to prove that each 3-face is a 3-face for two full-dimensional regions and, hence, the full-dimensional regions fill the whole parameter space without gaps and overlaps.

Knowing the basic vectors for a 3-face, it is easy to determine the set of cluster configurations which generates all the ground-state structures in this 3-face. This set is the intersection of the basic sets of configurations for the basic vectors of the 3-face. Then one can describe the corresponding ground-state structures.

Basic vectors for the 3-faces of full-dimensional regions as well as corresponding sets of cluster configurations are presented in Table III. In this Table, a 3-face is denoted by the numbers of full-dimensional regions (phases) which share this 3-face. If a configuration of a set is not compatible with other configurations of the set (i.e., the configuration cannot enter any ground-state structure in this 3-face), then it is separated by the symbol —. The character of transition between two regions (phases) depends on the ground-state structures at their boundary (i.e., at their common 3-face). If the set of configurations for a 3-face generates only the structures which are the ground-state ones for the full-dimensional regions sharing this 3-face and nothing else, then there is a first-order phase transition between these regions (notation “Jump” in Table III). This kind of phase transitions exists between phases 1 and 1̅, 1 and 3, 1 and 9, 3 and 8, as well as 4 and 10 (and also between pairs of symmetric phases). If for any value of magnetization from the interval between the values of magnetization for two neighboring full-dimensional phases, at least one ground-state structure can be constructed, then the transition between these phases is continuous (notation “Cont.” in Table III). As it is clear from Table III, the most part of transitions just possess this property. The situation between phases 4 and 9 is more complicated: there are both discontinuous and continuous phase transitions between them.

An interesting and important question concerns the disorder and entropy at the boundaries of full-dimensional regions, in particular, in the 3-faces. Let us consider, for instance, a 3-face between the regions 1 and 10. A typical example of a structure in this 3-face is shown in Fig. 9(a): the structure consists of diagonal spin-down chains (along $J_3$ bonds) separated by diagonal spin-up chains whose number can be arbitrary odd but not less than three. This is a simple mixture of structures 1 and 10. It is clear that order exists along these chains but there is disorder in the perpendicular direction. Such a disorder can be called one-dimensional. It does not lead to macroscopic degeneracy: the entropy per site tends to zero if dimensions of the lattice tend to infinity. A similar simple mixtures of full-dimensional structures exist also at boundaries of phases 1 and 5, 1 and 8, 3 and 4, 3 and 8, as well as 3 and 10. At the boundary between phases 9 and 10, the ground-state structures are not a simple mixture of full-dimensional structures 9 and 10 but rather a kind of their hybrid, although the disorder is one-dimensional there. They look like structure 10 with additional spin-down diagonal chains normal to the similar chains of structure 10 [Fig. 9(b)].

Another kind of disorder occurs, for instance, at the boundary between phases 1 and 6. The ground-state structures at this boundary can be obtained from a mixture of structures 6a and 6b (which is a ground state in region 6) by flipping a part of spins down [Fig. 9(c)]. It is clear that this disorder is two-dimensional since infinite number of local changes can be made in the structures within the bounds of the ground state. Every fifth spin in the structure depicted in Fig. 9(c) is “free,” that is, can be directed downward as well as upward. Hence, the entropy per site is equal to $\frac{1}{5}\ln 2$ or maybe more, since Fig. 9(b) does not exhaust all structures.

In Table III, the dimensionality of disorder is indicated in the fourth column. To find it is sometimes not so easy as in the cases considered above. For instance, we could not determine the dimensionality of disorder at the boundary between phases 4 and 5 (see Fig. 10). The disorder in this 3-face is most likely one-dimensional but it still should be proved. Let us analyze the disorder at some other 3-faces where such analysis is nontrivial.

One can see from Fig. 11 that at the boundary between phases 3 and 5 as well as 5 and 8 the disorder exists in two perpendicular directions independently. However, at the first boundary the disorder is two-dimensional (and the entropy per site is nonzero) and at the second one the disorder is one-dimensional. The structures at the boundary between phases 5 and 8 are interesting in the
TABLE III: Basic rays and ground-state configurations for the 3-faces of the full-dimensional regions for the Ising model on the extended SS lattice ($h \geq 0$). The dimensionality of disorder is indicated for continuous transitions.

| Regions | Basic rays of the 3-face | Ground-state configurations for the 3-face | Transition between phases | Conditions for existence in the plane ($h, J_2$) |
|---------|-------------------------|-------------------------------------------|--------------------------|-----------------------------------------------|
| 1, 1    | $r_1, r_2, r_3, r_4$    | ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ | Jump                     | $J_1 < 0, J_2 > \min[-2J_1, -2(J_1 + J_3)], J_3 < |J_1|$ |
| 1, 3    | $r_1, r_2, r_3, r_7$    | Jump                                      |                          | $J_1 > 0, J_2 < 0, J_3 < 0$                   |
| 1, 5    | $r_3, r_4, r_8, r_6$    | Cont. (1)                                |                          | $J_1 < 0, J_2 > \max[0, -2(J_1 + J_3)], J_3 > 0$ |
| 1, 6    | $r_6, r_7, r_8$         | Cont. (2)                                |                          | $J_1 > 0, J_2 > 0, J_3 > 0$                   |
| 1, 7    | $r_2, r_6, r_7$         | Cont. (2)                                |                          | $J_1 > 0, J_2 > 0, J_3 > 0$                   |
| 1, 8    | $r_2, r_3, r_6$         | Cont. (1)                                |                          | $0 < -J_1 < J_3, J_2 < 0$                     |
| 1, 9    | $r_1, r_4, r_8$         | Jump                                     |                          | $J_1 < 0, J_2 > -2J_1, J_3 < 0$               |
| 1, 10   | $r_1, r_7, r_8$         | Cont. (1)                                |                          | $J_1 > 0, J_2 > 0, J_3 < 0$                   |
| 3, 4    | $r_5, r_9, r_7, r_{10}$ | Cont. (1)                                |                          | $\max(0, -2J_3) < J_2 < \min[2J_1, 2(J_1 - J_3)], J_1 > 0, J_3 < J_1$ |
| 3, 5    | $r_5, r_9, r_{10}$      | Cont. (2)                                |                          | $J_2 = 2(J_1 - J_3), 0 < J_3 < J_1$           |
| 3, 7    | $r_2, r_7, r_9$         | Cont. (2)                                |                          | $J_2 < 0, 0 < J_3 < J_1$                      |
| 3, 8    | $r_2, r_9, r_{10}$      | Cont. (1)                                |                          | $J_2 < 0, J_3 = J_1$                          |
| 3, 9    | $r_1, r_{10}, r_7, r_{10}$ | Jump                              |                          | $J_1 > 0, J_2 > 2J_1, J_3 < 0$               |
| 3, 10   | $r_1, r_7, r_{10}$      | Cont. (1)                                |                          | $J_1 > 0, 0 < J_2 < \min(2J_1, -2J_3), J_3 < 0$ |
| 4, 5    | $r_5, r_8, r_9$         | Cont. (?)                               |                          | $J_2 > 2(J_1 - J_3), 0 < J_3 < J_1$           |
| 4, 6    | $r_7, r_8, r_9$         | Cont. (1)                                |                          | $J_2 > 0, 0 < J_3 < J_1$                      |
| 4, 9    | $r_5, r_8, r_{10}$      | Jump + Cont. (1)                        |                          | $J_2 > 2J_1, -J_1 < J_3 < 0$                 |
| 4, 10   | $r_7, r_8, r_{10}$      | Jump                                     |                          | $J_2 > -2J_3, -J_1 < J_3 < 0$                |
| 5, 6    | $r_6, r_8, r_9$         | Cont. (2)                                |                          | $0 < J_1 < J_3, J_2 > 0$                      |
| 5, 8    | $r_6, r_9, r_6, r_6, r_6$ | Cont. (1)                       |                          | $J_2 = 0, J_3 > |J_1|$                        |
| 5, 9    | $r_1, r_8, r_5, r_8$    | Cont. (2)                                |                          | $J_2 > 2|J_1|, J_3 = 0$                       |
| 6, 7    | $r_6, r_7, r_9$         | Cont. (2)                                |                          | $J_1 > 0, J_2 = 0, J_3 > 0$                   |
| 7, 8    | $r_2, r_6, r_9$         | Cont. (2)                                |                          | $0 < J_1 < J_3, J_2 < 0$                      |
| 9, 10   | $r_1, r_8, r_{10}$      | Cont. (1)                                |                          | $J_3 < -J_1 < 0, J_2 > 2J_1$                 |

\[^a^]Configurations marked by asterisk enter the structures in blocks shown in Fig. 4.

FIG. 9: Examples of the ground-state structures at the boundaries between phases (a) 1 and 10 as well as (b) 9 and 10. The disorder at these boundaries is one-dimensional. (c) An example of ground-state structures at the boundary between phases 1 and 6. Each gray circle can be either open or solid, hence, the disorder is two-dimensional at this boundary. If all the gray circles are open, then we obtain a mixture of structures 6a and 6b, which is a ground-state structure in region 6.
An interesting set of structures also occurs at the boundary between phases 4 and 9. The disorder at this boundary is one-dimensional, since the structure is completely determined by a zigzag-stripe example of which is depicted in Fig. 12. The structure closest to structure 9 in this set is an ordered structure with $m/m_s = 1/5$ [Fig. 13(a)]. Hence, there is a jump between structure 9 and this structure and then a continuous transition to phase 4, that is, at the boundary between phases 4 and 9 there occur both a jump and a continuous transition. In addition to the structures of the type shown in Fig. 12, there is one structure of another type at the boundary between phases 4 and 9. This is a structure with $m/m_s = 1/4$ shown in Fig. 13(b). All these structures (except for structure 4) are chiral (see Fig. 8 and its explanation in the text).

Another set of structures, where it is difficult to determine the dimensionality of disorder, occurs at the boundary between phases 4 and 6. An example of a structure at this boundary is shown in Fig. 14. It seems that the disorder is two-dimensional, however, a more profound analysis shows that the disorder is nevertheless one-dimensional since, as in phase 6, the structural element shown in Fig. 7 generates an infinite half-stripe.

At the boundary between phases 3 and 4 a collection of stripe structures exists. Most probably, some of these give rise to fractional magnetization plateaus in TmB$_4$. We do not describe these structures here, since this is done in Ref. 3. The interaction $J_3$ does not lift the degeneracy at this boundary.

V. GROUND-STATE STRUCTURES AT THE 2-FACES OF FULL-DIMENSIONAL REGIONS

Although the ground states in the 2-faces of full-dimensional regions are not so important as the ground states in the 3-faces, let us consider them for completeness. We denote a 2-face by two basic vectors (which generate it) in curly brackets. In Table III, vectors of each 3-face are ordered in the way that each pair of neighboring vectors (the first and the last are neighbors as well) generates a 2-face. The set of ground-state configurations for a 2-face is an intersection of the basic sets of configurations for these vectors. The 2-faces of full-dimensional regions for $h \geq 0$ and the ground-state configurations for them are presented in Table IV. The third, fourth, fifth, and
FIG. 12: An example of ground-state structures at the boundary between phases 4 and 9. The rectangles formed by the ferromagnetic chains together with “open” and “solid” squares are shown as well as one zigzag-stripe that generates the whole structure. The structure is chiral: the SS bonds twist clockwise around “open” squares and counterclockwise around “solid” ones (and vice versa for the chiral twin).

Sixth columns in Table IV indicate, respectively, the dimensionality of disorder in a 2-face, the full-dimensional regions sharing this 2-face, the coordinates of the point that corresponds to this 2-face in the plane \((h, J_2)\), and the conditions for the existence of such a point in this plane (if the coordinates are not indicated in parenthesis, then the 2-face does not intersect the plane \((h, J_2)\) or lies completely in it for some values of \(J_1\) and \(J_3\)). It is easy to determine the dimensionality of disorder in the most part of 2-faces, taking into account that the disorder in a 2-face cannot be lower than the disorder in a 3-face bounded by this 2-face. This task is difficult, however, for some 2-faces. Let us consider such 2-faces. In the 2-face \(\{r_3, r_4\}\), the disorder is two-dimensional. This is clear from Fig. 15. It is more difficult to show the disorder in the 2-face \(\{r_5, r_{10}\}\) is two-dimensional too. This is illustrated in Fig. 16, where the zigzag-strips going from left to right can both descend and ascend a little. In the 2-face \(\{r_8, r_{10}\}\) the disorder is two-dimensional. This is clear from Fig. 17, where each pair of gray circles should contain one open and one solid circle.

Finally, let us note that the disorder is two-dimensional in all the basic rays.

VI. GROUND-STATE PHASE DIAGRAMS AND FRACTIONAL MAGNETIZATION PLATEAUS

To make our results more usable, we present all the types of ground-state phase diagrams in the plane \((h, J_2)\). There are three types of diagrams if \(J_1 < 0\) (Fig. 18) and four types if \(J_1 > 0\) (Fig. 19). In these diagrams, the lines between neighboring regions correspond to the 3-faces and the points where three or more regions converge correspond to the 2-faces. If there is a first-order phase transition between the neighboring phases, then the line separating the corresponding regions is solid red; if the transition is continuous, then the line is solid black. The dash-dotted green line between regions 4 and 9 corresponds to a jump along with a continuous transition. Re-
TABLE IV: Basic rays and ground-state configurations for the 2-faces of the full-dimensional regions for the Ising model on the extended SS lattice ($h \geq 0$).

| 2-face for the 2-face | Ground-state configurations | Disorder | Full-dimensional structures | Coordinates in the plane ($h, J_2$) | Conditions for existence in the plane ($h, J_2$) |
|-----------------------|-----------------------------|----------|-----------------------------|-----------------------------------|-----------------------------------------------|
| {r₁, r₂}             | ![Image](image1.png)         | 0        | 1, ăr, 3                   | $h = 0$, $J_1 = 0$                | $J_2 < 0$, $J_3 < 0$                         |
| {r₁, r₃}             | ![Image](image2.png)         | 0        | 1, ăr, 9                   | $(0, -2J_1)$                      | $J_1 < 0$, $J_3 < 0$                         |
| {r₁, r₅}             | ![Image](image3.png)         | 1        | 1, 3, 10                   | $(4J_1, 0)$                       | $J_2 > 0$, $J_3 < 0$                         |
| {r₁, r₆}             | ![Image](image4.png)         | 1        | 1, 9, 10                   | $h = J_2$, $J_1 = 0$              | $J_2 > 0$, $J_3 < 0$                         |
| {r₁, r₇}             | ![Image](image5.png)         | 1        | 3, 9, 10                   | $(2J_1, 2J_2)$                    | $J_3 < -J_1$                                 |
| {r₂, r₃}             | ![Image](image6.png)         | 1        | 1, ăr, 8                   | $h = 0$, $J_3 = -J_1$             | $J_1 < 0$, $J_2 < 0$                         |
| {r₂, r₅}             | ![Image](image7.png)         | 2        | 1, 7, 8                    | $h = 2J_1$, $J_1 = 0$             | $J_2 < 0$, $J_3 > 0$                         |
| {r₂, r₆}             | ![Image](image8.png)         | 2        | 1, 3, 7                    | $h = 4J_1$, $J_3 = 0$             | $J_1 > 0$, $J_2 < 0$                         |
| {r₂, r₇}             | ![Image](image9.png)         | 2        | 3, 7, 8                    | $h = 2J_1$, $J_3 = J_1$           | $J_3 > 0$, $J_2 < 0$                         |
| {r₃, r₄}             | ![Image](image10.png)        | 2        | 1, ăr, 5                   | $(0, -2J_1 - 2J_3)$               | $0 < J_3 < -J_1$                              |
| {r₃, r₆}             | ![Image](image11.png)        | 2        | 1, 5, 8                    | $(2J_1 + 2J_3, 0)$                | $0 < J_1 < J_3$                               |
| {r₄, r₅}             | ![Image](image12.png)        | 2        | 1, 5, 9                    | $h = J_2 + 2J_1$, $J_3 = 0$      | $-J_2 < 2J_1 < 0$, $J_3 > 0$                 |
| {r₅, r₆}             | ![Image](image13.png)        | 2        | 4, 5, 9                    | $h = J_2 - 2J_1$, $J_3 = 0$      | $0 < 2J_1 < J_2$, $J_3 > 0$                  |
| {r₅, r₇}             | ![Image](image14.png)        | 2        | 3, 4, 5                    | $(2J_1, 2J_1 - 2J_3)$             | $0 < J_3 < J_1$                               |
| {r₅, r₈}             | ![Image](image15.png)        | 2        | 3, 4, 9                    | $(-2J_1, 2J_1)$                   | $-J_1 < J_3$                                 |
| {r₆, r₇}             | ![Image](image16.png)        | 2        | 1, 6, 7                    | $(4J_1 + 2J_3, 0)$                | $J_1 > 0$, $J_3 > 0$                         |
| {r₆, r₈}             | ![Image](image17.png)        | 2        | 1, 5, 6                    | $h = J_2 + 2J_3$, $J_1 = 0$      | $J_2 > 0$, $J_3 > 0$                         |
| {r₆, r₉}             | ![Image](image18.png)        | 2        | 5, 6, 7, 8                 | $(2J_3, 0)$                      | $0 < J_1 < J_3$                               |
| {r₇, r₈}             | ![Image](image19.png)        | 2        | 1, 4, 6, 10                | $h = J_2 + 4J_1$, $J_3 = 0$      | $J_1 > 0$, $J_2 > 0$                         |
| {r₇, r₉}             | ![Image](image20.png)        | 2        | 3, 4, 6, 7                 | $(4J_1 - 2J_3, 0)$                | $0 < J_3 < J_1$                               |
| {r₈, r₁₀}            | ![Image](image21.png)        | 1        | 3, 4, 10                   | $(4J_1 + 2J_3, -2J_3)$            | $-J_1 < J_3 < 0$                              |
| {r₉, r₁₀}            | ![Image](image22.png)        | 2        | 4, 5, 6                    | $h = J_2 + 2J_1$, $J_1 = J_3$    | $J_1 > 0$, $J_2 > 0$                         |
| {rᵢ, rᵢ}            | ![Image](image23.png)        | 2        | 3, 5, 8                    | $J_1 = J_3$, $J_2 = 0$           | $J_1 > 0$                                    |

*Configurations marked by asterisk enter the structures in blocks shown in Fig. 4.

FIG. 15: An example of a ground-state structure at the boundary between phases 1, 1, and 5 (2-face {r₃, r₄}), which shows that the disorder at this boundary is two-dimensional.

FIG. 16: Example of a ground-state structure at the boundary between phases 3, 4, and 9 (2-face {r₅, r₁₀}). The rectangles formed by the ferromagnetic chains together with “open” and “solid” squares are shown.
regions 6, 7, and 10, which give rise to an 1/2-plateau, are colored and region 4, which gives rise to a 1/3-plateau, is shaded.

As one can see from Fig. 19 and Table II, the width of the 1/2-plateau generated by phase 10 is equal to 2$J_2$ and $4J_1$ for sequences of phases $3 \rightarrow 10 \rightarrow 1$ and $9 \rightarrow 10 \rightarrow 1$; for all the rest of sequences that give rise to this plateau its width is equal to $|4J_3|$. The widths of zero plateaus and fractional ones for various sequences of phases as well as conditions for their existence are presented in Table V.

Which sequence of phases corresponds to the magnetization process in ErB$_4$? A single 1/2-plateau has been observed in this compound, therefore, only five initial sequences in Table V are possible. However, since interactions $J_1$ and $J_2$ are antiferromagnetic and approximately equal and interaction $J_3$ is ferromagnetic and relatively large only the sequence $3 \rightarrow 10 \rightarrow 1$ remains. This contradicts the statement of Ref. [5] about the magnetic structure of ErB$_4$ for zero field. According to this reference, there should be structure AF1 rather than the Néel phase (AF3). The structure AF1, in the ground-state phase diagram ($J_1 = J_2$, $h = 0$) presented in this reference, should correspond to our structure 5, however, some other structure is depicted there; maybe this is a simple inadvertence.

On the basis of the ground-state structures that we have found previously the authors of Ref. [18] have constructed a ground-state phase diagram for $J_1 = J_2 > 0$ and for arbitrary $J_3$ and then obtained numerically a spin supersolid phase in the quantum model with strong Ising anisotropy. They argue that this ground state exists in ErB$_4$ which, without magnetic field, has magnetic structure 5. However, structure 5 is possible only under the condition $J_3 > J_1 > 0$ which looks unrealistic. It seems that experimental results in favor of this structure in ErB$_4$ without magnetic field[19] and the experimental results yielding that interaction $J_3$ is ferromagnetic and large are contradictory.

The structure with $m/m_s = 1/2$, presented in Ref. [4] for TmB$_4$ is erroneous as well. It is a mixture of structures 1 and 5. In the model under consideration, this structure exists at the boundary between phases 1 and 5 where $J_1 < 0$. The authors of Refs. [4] and [5] state that, in TmB$_4$, interaction $J_1 \approx J_2$ is antiferromagnetic, therefore, it is unlikely that this structure could occur in this compound.

It is also unlikely that, in TmB$_4$, the structure $(3,7)_4$ with $m/m_s = 1/9$ presented in Ref. [4] (Fig. 20) could exist. In comparison with the structure $(3,4)_4$ that we proposed in Ref. [4] all the antiferromagnetic chains in this structure are shifted. In the model under consideration, the structure $(3,7)_4$ exists at the boundary between phases 3 and 7, that is, for positive values of $J_1$ and $J_3$ but negative values of $J_2$.

In Ref. [5] the following experimental values of interaction parameters for TmB$_4$ are presented: $J_1 = J_2 = 0.85$ K and $J_3 = 0.3$ K. These correspond to the sequence of phases $3 \rightarrow 4 \rightarrow 6 \rightarrow 1$, however, then the 1/3-plateau is approximately twice as wide as the 1/2-plateau. [If $J_3$ were ferromagnetic ($J_3 = -0.3$ K), then the 1/2-plateau would be twice as wide]. The 1/3-plateau has not been observed in TmB$_4$. Maybe the longer-range interactions, giving rise to the fractional plateaus 1/6, 1/7..., remove at the same time the 1/3-plateau.

In Ref. [17] to explain the appearance of fractional magnetization plateaus in TmB$_4$, a spin-electron model has been considered. For the 1/2-plateau the structure 6b (or symmetric one) has been obtained. The SS diagonals are not shown in the figures of Ref. [17] therefore we do not know whether the structures obtained there are of the same type as in Ref. [4] or as in Ref. [5] (Fig. 20). It is easy to show that the interaction $J_4$ (the next-nearest neighbors along edges) lifts the degeneracy in phase 6 and stabilizes one of the structures 6a or 6b. The ferromagnetic (antiferromagnetic) interaction $J_4$ stabilizes the structure 6a[16-18] (6b).

It is interesting that in the case where all the three interactions are antiferromagnetic ($J_1 > 0$, $J_2 > 0$, and $J_3 > 0$) the full-dimensional phases for $h \geq 0$ (1, 3, 4, 5, and 6) can be considered as one-dimensional ordering of ferro- and antiferromagnetic chains. A question arises: In order to study the effect of longer-range interactions on ground states, is it possible to consider a one-dimensional model with effective interactions between chains instead of the two-dimensional one? 220.
FIG. 18: Ground-state phase diagrams of the Ising model on the extended SS lattice for $J_1 < 0$. Black and red lines correspond to continuous phase transitions and jumps, respectively. (See also Table and Figs. 5 and 6).

FIG. 19: Ground-state phase diagrams of the Ising model on the extended SS lattice for $J_1 > 0$. Black and red lines correspond to continuous phase transitions and jumps, respectively; a green dash-dotted line denotes a jump together with a continuous phase transition. The regions which give rise to an $1/2$-plateau are colored. The region of $1/3$-plateau phase is shaded. (See also Table and Figs. 5 and 6).

VII. SUMMARY

To conclude, we have determined a complete solution of the ground-state problem for an Ising model on the extended SS lattice with an additional interaction $J_3$ along the diagonals of "empty" squares. We have used the method of basic vectors and basic sets of cluster configurations that was proposed in our previous works. Here, however, we generalize the method and consider configurations of two clusters at once. We have constructed the ground-state phase diagrams and studied the ground-state structures in the full-dimensional regions as well as at their three- and two-dimensional boundaries. This
TABLE V: The widths of the fractional plateaus for various ways of transitions from the zero-field phase to the ferromagnetic one.

| Sequence of phases | Width of zero plateau | Width of 1/3-plateau | Width of 1/2-plateau | Conditions for existence |
|--------------------|-----------------------|----------------------|----------------------|-------------------------|
| 3 – 10 – 1         | 4J1 – J2              | 0                    | 2J2                  | J1 > 0, J3 < 0, 0 < J2 < min[2J1, –2J3] |
| 3 – 7 – 1          | 4J1 – 2J3             | 0                    | 4J3                  | 0 < J3 < J1             |
| 5 – 6 – 1          | J2 + 2J3              | 0                    | 4J3                  | 0 < J1 < J3             |
| 8 – 7 – 1          | 2J3                   | 0                    | 4J3                  | 0 < J1 < J3             |
| 9 – 10 – 1         | J2                    | 0                    | 4J1                  | J3 < –J1 < 0, J2 > 2J1  |
| 3 – 4 – 10 – 1     | 4J1 – 2J2 – 2J3       | 3J2 + 6J3            | –4J3                 | –J1 < J3 < 0, –2J3 < J3 < 2J1 |
| 9 – 4 – 10 – 1     | –2J1 + J2 – 2J3       | 6(J1 + J3)           | –4J3                 | –J1 < J3 < 0, J2 > 2J1  |
| 3 – 4 – 6 – 1      | 4J1 – 2J2 – 2J3       | 3J2                  | 4J3                  | 0 < J3 < J1, 0 < J2 < 2(J1 – J3) |
| 5 – 4 – 6 – 1      | –2J1 + J2 + 4J3       | 6(J1 – J3)           | 4J3                  | 0 < J3 < J1, J2 > 2(J1 – J3) |

FIG. 20: Structure (3,4), for the 1/9-plateau in TmB4, proposed in Ref. 4, and structure (3,7), proposed for this plateau in Ref. 4. These two structures differ by the positions of the antiferromagnetic chains. Unit cells are indicated.

FIG. 21: Structure (3,4), for the 1/9-plateau in TmB4, proposed in Ref. 4, and structure (3,7), proposed for this plateau in Ref. 4. These two structures differ by the positions of the antiferromagnetic chains. Unit cells are indicated.

made it possible to establish that the additional interaction J3 gives rise to an 1/2-plateau. This plateau can correspond to three different phases (depending on the interaction parameters), two of which are partially disordered. In addition to the 1/2-plateau, another fractional plateau is possible—with the magnetization 1/3. As we have shown earlier, this is a single fractional plateau in the Ising model on the conventional SS lattice. For some relations between the interaction parameters, the 1/2-plateau can be a single fractional plateau in the Ising model on the extended SS lattice as well. Hence, it might be reasonable to believe that we have explained the origin of a single fractional plateau—with m/mn = 1/2—in ErB4. As to TmB4, where no 1/3-plateau but an 1/2-plateau and a sequence of other fractional plateaus were observed, the theoretical explanation of the magnetization curve in this compound requires to study the effect of longer-range interactions. However, the knowledge of the ground-states at the boundaries of the full-dimensional regions of the model under consideration makes it possible to draw some conclusions about the effect of such interactions.

The advantages of the analysis of the ground states at the boundaries of full-dimensional regions become obvious when the results of this paper are compared to the results of the previous one. Having constructed the ground-state phase diagram for the Ising model on the conventional SS lattice and having analyzed which of the ground states at boundaries of full-dimensional regions become full-dimensional if a small additional interaction J3 is switched on, we have obtained all the full-dimensional ground-state structures of the Ising model on the extended SS lattice except for structure 8 (since it occurs under the condition J3 > |J1|).

The results obtained here and in Ref. 3 can be useful for the numerical studies of the origin of the fractional magnetization plateaus in SS magnets (with Ising anisotropy) for nonzero temperatures.

In the present paper, a development of analytical methods for the determination of ground states of lattice-gas models and equivalent spin models is presented. These methods can be used to investigate the structure of substitutional alloys, that is a very important problem (a recent initiative of US President Barack Obama gives evidence of this). In our future paper we will develop the method and show how to study the effect of longer-range interactions.

VIII. ACKNOWLEDGMENTS

The author is grateful to T. Verkholyak and I. Stasyuk for useful discussions and suggestions and to O. Kocherga for correction of the text.

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