Implicit and fully discrete approximation of the supercooled Stefan problem in the presence of blow-ups

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Abstract

We consider two approximation schemes of the one-dimensional supercooled Stefan problem and prove their convergence, even in the presence of finite time blow-ups. All proofs are based on a probabilistic reformulation recently considered in the literature. The first scheme is a version of the time-stepping scheme studied in V. Kaushansky, C. Reisinger, M. Shkolnikov, and Z. Q. Song, arXiv:2010.05281, 2020, but here the flux over the free boundary and its velocity are coupled implicitly. Moreover, we extend the analysis to more general driving processes than Brownian motion. The second scheme is a Donsker-type approximation, also interpretable as an implicit finite difference scheme, for which global convergence is shown under minor technical conditions. With stronger assumptions, which apply in cases without blow-ups, we obtain additionally a convergence rate arbitrarily close to 1/2. Our numerical results suggest that this rate also holds for less regular solutions, in contrast to explicit schemes, and allow a sharper resolution of the discontinuous free boundary in the blow-up regime.

1 Introduction

The classical PDE formulation of the one-dimensional one-phase Stefan problem is

\begin{align}
\partial_t u &= \frac{1}{2} \partial_{xx} u, \quad x > \Lambda_t, \quad t \geq 0, \\
u(t, \Lambda_t) &= 0, \quad t \geq 0, \\
\dot{\Lambda}_t &= \frac{\alpha}{2} \partial_x u(t, \Lambda_t), \quad t \geq 0, \\
u(0, x) &= f(x), \quad x \geq 0 \quad \text{and} \quad \Lambda_0 = 0.
\end{align}

\text{(1)}
In physics terminology, \(-f\) is the initial temperature in a liquid, where we will consider \(f \geq 0\) corresponding to the supercooled regime; \(-u(t, \cdot)\) is the temperature in the liquid phase at time \(t\); and \(\Lambda_t\) is the location of the liquid-solid boundary at time \(t\). The above relationship between the flux \(\partial_x u\) at the free boundary \(\Lambda_t\) and the growth rate \(\Lambda_t\) of the solid phase is known as Stefan condition. It expresses the energy conservation at the interface, in integrated form,

\[
\Lambda_t = \alpha \left( 1 - \int_{\Lambda_t}^{\infty} u(t, x) \, dx \right).
\]

Variants of the Stefan problem, originally proposed in [38], have been studied in great detail in the applied mathematics literature. In particular, it has been established in a string of works starting in the 1970s (see, e.g., [37, 22, 16, 17, 12, 23, 28, 31, 11, 24, 41]) that the supercooled Stefan problem may exhibit finite time blow-ups, whereby continuous solutions \(t \mapsto \Lambda_t\) cease to exist.

As closed-form solutions are rarely available for this type of free boundary problems, one typically has to resort to numerical methods. We refer to [4, 33] for a survey of numerical schemes. These works treat classical cases with different boundary conditions, where sign requirements (related to well-posedness without blow-ups) on the initial or boundary data – usually that the temperature of the material in the liquid phase is positive – are imposed. In the supercooled regime, where such conditions are not satisfied, regularization techniques to prevent the formation of blow-ups in finite time can be applied. For instance, in [27] the effect of kinetic undercooling as a regularizing mechanism is analysed and it is shown how the (unregularized) supercooled Stefan problem can be recovered asymptotically. These asymptotics are also compared to numerical solutions of the unregularized problem. The authors there follow (for both the regularized and the unregularized case) the so-called method of lines considered in [15]. More precisely, the continuous time PDE systems are discretized and solved at successive times as a sequence of free boundary problems for ordinary differential equations. They can in turn be tackled via the so-called method of invariant embedding described in [32], which gives rise to explicitly solvable Riccati equations. Note that in the numerical studies of the unregularized case the blow-up behaviour cannot be fully reproduced due to truncation errors. Further alternative (less physical) regularization approaches are discussed in [19], albeit without a thorough numerical analysis. In [2], the ill-posed Stefan problem for melting a superheated solid, which is mathematically identical to the supercooled Stefan problem, is analysed. There, similarly as in [27], the method of lines is applied and existing numerical results from the literature in the blow-up regime are reproduced. In [1], blow-up points to one-phase Stefan problems, however with Dirichlet boundary conditions, are treated and studied numerically. Let us finally mention the recent paper [31], which analyses the Fisher–Stefan model, a generalization of the well-known Fisher–KPP model, in the context of biological invasion, where the speed of the moving boundary is related to the flux of population there. By rescaling this problem, it can be compared to the supercooled Stefan problem. In this context, [31] provides new links between these models and a numerical scheme for the Fisher–Stefan model.

Despite this plethora of articles on ill-posed Stefan problems, to our knowledge, with the exception of the particle simulation scheme of [26], no provably convergent algorithm is known for the blow-up scenario. We contribute to this literature by proposing a class of numerical schemes for which we can prove global convergence,
more specifically, that the discrete approximation of the free boundary converges to the true free boundary at all continuity points.

The following simple finite difference scheme is a canonical example of the schemes we will consider for (1), where \( u^k_i \) is an approximation to \( u(kh, i\sqrt{h}) \) for \( i, k > 0 \) and some fixed mesh widths \( h > 0 \) in time and \( \sqrt{h} \) in space, respectively:

\[
\begin{align*}
\frac{u^k_i - u^{k-1}_i}{h} &= \frac{1}{2} \frac{u^{k-1}_{i+1} - 2u^{k-1}_i + u^{k-1}_{i-1}}{h^2}, \quad k > 0, \; i > i_k, \\
i_k &= \left\lfloor \frac{\alpha}{\Delta x} \left( 1 - \sum_{i=k+1}^{\infty} u^i_i \right) \right\rfloor, \quad k > 0, \\
\frac{u^0_i}{\sqrt{h}} &= \int_{i\sqrt{h}}^{(i+1)\sqrt{h}} f(x) \, dx, \quad i \geq 0, \quad \text{and} \quad u^k_i = 0, \quad k > 0, \; i \leq i_k.
\end{align*}
\]

We will analyse and implement a scheme of type (2) in Section 4 with a slightly perturbed initial condition, as motivated later. Note that (2) is reminiscent of the forward Euler approximation of the heat equation, but is nonetheless an implicit nonlinear equation through the dependence of the discrete free boundary \( i_k \sqrt{h} \) on the solution \( (u^k)_i \) at time \( kh \).

Our analysis is based on the following probabilistic reformulation of the problem:

\[
\begin{align*}
X_t &= X_{0-} + B_t - \Lambda_t, \\
\tau &= \inf\{t \geq 0 : X_t \leq 0\}, \\
\Lambda_t &= \alpha \mathbb{P}(\tau \leq t),
\end{align*}
\]

where \( X_{0-} \) is a real-valued non-negative random variable, \( \alpha > 0 \), and \( B \) is a standard Brownian motion independent of \( X_{0-} \). We will also study extensions where \( B \) is replaced by a more general continuous-time process.

The link between (3) and (1) is found by noticing that over sufficiently small times the forward density \( p(t, \cdot) \), \( t \in [0, T] \) of the absorbed process \( X_t \mathbf{1}_{\{\tau > t\}} \), \( t \in [0, T] \) on \((0, \infty)\) satisfies the initial-boundary value problem

\[
\begin{align*}
\dot{p}(t, x) &= \frac{1}{2} \partial_{xx}p + \dot{\Lambda}_t \partial_x p, \quad x \geq 0, \quad t \in [0, T], \\
p(0, x) &= f(x), \quad x \geq 0 \quad \text{and} \quad p(t, 0) = 0, \quad t \in [0, T], \\
\Lambda_t &= \alpha \left( 1 - \int_0^\infty p(t, x) \, dx \right), \quad t \in [0, T],
\end{align*}
\]

as long as \( X_{0-} \) has a sufficiently regular density \( f \) supported on \([0, \infty)\). Applying the transformation \( u(t, x) := p(t, x - \Lambda_t), \; x \geq \Lambda_t \) then leads to (1) (see [10] for a rigorous proof).

The problem (3) and its variants have recently been used to model systemic risk of interconnected financial systems. There, \( \Lambda_t \) models the proportion of banks defaulted at time \( t \), and \( \alpha \) the strength of interbank borrowing (see e.g. [20, 34, 21, 7] for details and further references). Equations with similar mean-field effects through thresholding derive from integrate-and-fire models of neurons as considered in [9].

Besides the applicability of (3) to systemic risk modeling and neuroscience, which is justified by certain propagation of chaos results (see [9, 8]), the probabilistic formulation (3) has also several mathematical advantages. Most notably, it allows for a rigorous definition of global solutions to (1) even in the presence of blow-ups.
For this it is however necessary to first establish appropriate solution concepts for the probabilistic version of (3). Indeed, due to the singular interaction, (3) does not have unique solutions in general. Therefore, two physically and economically meaningful notions of solutions have been developed: so-called physical solutions and minimal solutions. A rigorous characterisation of unique physical solutions was given under certain technical assumptions on the initial conditions in [10]. The concept of minimal solutions, which are by definition unique, was in detail treated in [8]. In particular, it was shown there that the minimal solution is also a physical one. It is still an open question if uniqueness of physical solutions holds in general, but it is clear that the minimal solution coincides with the minimal physical one.

Among the numerical schemes introduced recently for the approximation of (3) are those using time-stepping (see [26]), iterations with heat kernels (see [30]) and particle systems (see [25]), where in the latter the numerical schemes are based on the corresponding propagation of chaos result.

Motivated by promising numerical experiments based on implicit instead of explicit schemes, we here first prove a global convergence result for an implicit version of the Euler time-stepping scheme, as considered in [26]. Furthermore, we extend this result to more general drivers than Brownian motion and allow for processes that satisfy the so-called crossing property (see [8]), which holds e.g. for fractional Brownian motion with arbitrary Hurst parameter in \( (0,1) \) and some non-degenerate continuous semi-martingales (see Example 2.1).

To further reduce the numerical complexity we then consider a scheme that – in contrast to the time-stepping algorithm – does not require a (time-consuming) Monte Carlo simulation of the corresponding particle system. The main result in the second part of the paper is that convergence in the Brownian case still holds after a Donsker approximation of the driving Brownian motion, i.e. the increments over single time-steps are approximated by random variables matching the first two moments of the Brownian increments. Under more restrictive conditions, in particular ruling out blow-ups, a rate of \( 1/2 \) (modulo a log factor) is proven.

The computational significance of this result is that by use of suitable discrete random variables, the discrete measure is supported on a recombining binomial tree and can be computed deterministically by a recursion over time-steps without simulation. The resulting discrete equations, of which (2) is a special case, are then interpretable as approximation schemes (finite difference schemes) for (1). We refer to these as fully discrete schemes, in contrast to the schemes based on time-stepping approximations alone, which require additional (particle) approximations.

We observe the following advantage in computational complexity. If the step size in time is \( h \), the spacing of the tree nodes is \( h^{1/2} \). On a given time horizon and finite spatial domain (by truncation of the real line), the total number of nodes, and hence computational complexity, is therefore \( O(h^{-3/2}) \). Although we only prove an order of convergence in the case of no blow-ups, we find empirically that in all regimes studied (jumps, no jumps, different regularities) the error is of order \( h^{1/2} \); see Section 4 for details. Hence, to achieve an error of \( \epsilon \), the computational complexity is \( O(\epsilon^{-3}) \). In contrast to that, for the Monte Carlo time-stepping scheme of [23, 26], the simulation error using \( n \) particles is \( O(n^{-1/2}) \), at cost \( O(n) \), and the time stepping error in the most regular case is \( h^{1/2} \). Setting \( n \) proportional to \( [h^{-1}] \), the computational complexity is then \( O(\epsilon^{-4}) \) to achieve an error of \( \epsilon \). Moreover, for low regularity, the convergence of the (explicit) time-stepping scheme is found to be arbitrarily slow in [26], and hence the computational complexity for
prescribed accuracy arbitrarily high. For the Donsker scheme, we do not observe such a phenomenon, but rather similar convergence rates in regular and irregular regimes. At the same time, the computational time is also considerably lower.

Throughout the paper, we denote by $\bar{\nu}$ the measure on $\bar{\mathbb{R}}$ endowed with the Skorokhod $M_1$-topology. For properties of the $M_1$-topology we refer to [40] and [8, Appendix A]. For $x \in D([-1, \infty))$ we denote by $\text{Disc}(x)$ the set of discontinuity points of $x$. We also define the path functional $\lambda_t(x) := \mathbb{1}_{x \in \mathbb{R}}(\inf_{0 \leq s \leq t} x_s)$, such that $\lambda_t(x) = 0$ if $x_\tau > 0$ on $[0, t]$ and $\lambda_t(x) = 1$ otherwise. The space of continuous functions on $[0, \infty)$ is denoted by $C([0, \infty))$ and endowed with the topology of compact convergence, i.e., $w_n \to w$ in $C([0, \infty))$ if and only if $w_n|_K \to w|_K$ uniformly for every compact $K \subseteq [0, \infty)$. We define the space of cumulative distribution functions on $[0, \infty]$ by

$$M := \{ \ell: \mathbb{R} \to [0, 1] \mid \ell \text{ càdlàg and increasing}, \ell_{0-} = 0, \ell_{\infty} = 1 \},$$

where $\mathbb{R}$ is the two-point compactification of $\mathbb{R}$. We endow $M$ with the topology induced by the Lévy metric, i.e., $\ell^n \to \ell$ in $M$ if and only if $\ell^n_t \to \ell_t$ for every $t \in [0, \infty) \setminus \text{Disc}(\ell)$. This topology turns $M$ into a compact Polish space. Furthermore, we define $E := C([0, \infty)) \times M$ and endow it with the product topology. For $\alpha > 0$, we define $\iota_{\alpha}: E \to D([-1, \infty))$ for $w \in C([0, \infty))$ and $\ell \in M$ as

$$\iota_{\alpha}(w, \ell)_t := \begin{cases} w_0 & t \in [-1, 0) \\ w_t - \alpha \ell_t & t \in [0, \infty) \end{cases}.$$

Throughout the paper, we denote by $x$ a generic element of $D([-1, \infty))$, by $w$ a generic element of $C([0, \infty))$ and by $\ell$ a generic element of $M$. If $S$ is a Polish space, we denote the space of probability measures on $S$ by $\mathcal{P}(S)$ and endow it with the topology of weak convergence, i.e., we say that $\mu_n \to \mu$ in $\mathcal{P}(S)$ if $\int_S F(x) d\mu_n(x) \to \int_S F(x) d\mu(x)$ for all $F \in C_b(S; \mathbb{R})$, where $C_b(S; \mathbb{R})$ denotes the space of continuous bounded functions from $S$ to $\mathbb{R}$. If $\mu \in \mathcal{P}(S)$ and $F: S \to \mathbb{R}$, we denote the integral of $F$ with respect to $\mu$ also with brackets, i.e., we write $\int_S F(x) \ d\mu(x) = \langle \mu, F \rangle$. Furthermore, if $\nu$ is the pushforward of the measure $\mu$ with respect to the map $T$, we denote this by $T(\mu) = \nu$. In particular, with a slight abuse of notation, if $\xi$ denotes a measure on $\bar{\mathbb{E}}$, $\iota_{\alpha}(\xi)$ denotes its pushforward on $D([-1, \infty))$.  

1.1 Notation

Throughout the paper, $D([-1, \infty))$ denotes the space of càdlàg functions on $[-1, \infty)$ endowed with the Skorokhod $M_1$-topology. For properties of the $M_1$-topology we refer to [40] and [8, Appendix A].
2 Convergence of the time-stepping scheme

In this section we prove convergence of an implicit version of the time-stepping scheme considered in [26] for more general driving processes than Brownian motion. Specifically, consider the following McKean–Vlasov problem

\[
\begin{align*}
X_t &= X_{0-} + Z_t - \Lambda_t, \\
\tau &= \inf\{t \geq 0 : X_t \leq 0\}, \\
\Lambda_t &= \alpha P(\tau \leq t),
\end{align*}
\]

where \(X_{0-} \in \mathbb{R}, \alpha > 0\), and \(Z\) is a continuous stochastic process with \(Z_0 = 0\) independent of \(X_{0-}\) satisfying the following crossing property,

\[
P\left(\tau < \infty, \inf_{0 \leq s \leq h} (Z_{t+s} - Z_t) = 0, \ h > 0,\right) = 0,
\]

for every stopping time \(\tau\) with respect to the natural filtration of \(X_{0-} + Z\). Note that the crossing property we use here is slightly more general than the one defined in [9, 8, 35] since we require (8) to hold for any finite stopping time, not just for the hitting time of zero. Note also that (8) holds for every stopping time if and only if it holds for every bounded stopping time.

Example 2.1. a) Let \(Z := M + Y\), where \(M\) is a continuous local martingale and \(Y\) is 1/2-Hölder continuous on compacts. Suppose additionally that for every \(K \in \mathbb{N}\) there is a strictly positive random variable \(\epsilon_K\) such that \(\langle M \rangle_t - \langle M \rangle_s \geq \epsilon_K(t-s)\) for \(0 \leq s \leq t \leq K\), where \(\langle M \rangle\) is the quadratic variation of \(M\). Then, the proof of Theorem 3.5 in [3] shows that \(Z\) satisfies the crossing property (8). In particular, we may choose \(Z\) to be Brownian motion.

b) The process \(Z := B^H\), where \(B^H\) is fractional Brownian motion with Hurst parameter \(H \in (0, 1)\) satisfies the crossing property (8) by [36, Theorem 1.1].

We start by establishing existence (and uniqueness) of minimal solutions which are defined as follows. We call a solution \((X, \tau, \Lambda)\) to the McKean–Vlasov problem (7) minimal, if for every solution \((X, \tau, \Lambda)\) to (7) we have

\[
\Lambda_t \leq \Lambda_t, \quad t \geq 0.
\]

We introduce the fixed-point operator \(\Gamma\), defined for \(\ell \in M\),

\[
\begin{align*}
X_t[\ell] &= X_{0-} + Z_t - \alpha \ell_t, \\
\tau[\ell] &= \inf\{t \geq 0 : X_t[\ell] \leq 0\}, \\
\Gamma[\ell]_t &= P(\tau[\ell] \leq t).
\end{align*}
\]

As the next proposition shows, \(\Gamma\) is continuous on \(M\).

**Proposition 2.2.** The operator \(\Gamma\): \(M \to M\) is continuous.

**Proof.** Suppose that \(\ell^n \to \ell\) in \(M\). Define

\[
\xi^n := \text{law}((X_{0-} + Z, \ell^n)), \quad \xi := \text{law}((X_{0-} + Z, \ell)).
\]

Then, \(\xi^n \to \xi\) in \(\mathcal{P}(\bar{E})\). Set \(\mu_n := \tau_n(\xi^n), \mu := \tau_n(\xi)\) and note that \(\langle \mu_n, \lambda_t \rangle = \Gamma[\ell^n]_t, \langle \mu, \lambda_t \rangle = \Gamma[\ell]_t\), where \(\lambda\) was defined in Section 1.1.
By Theorem 4.2 in [8] and the continuous mapping theorem, $\mu_n \to \mu$ in $\mathcal{P}(D([-1, \infty)))$.

For $x \in D([-1, \infty))$ set $\tau_0(x) := \inf\{t \geq 0 : x_t \leq 0\}$ and define $\tau_\ell(w) := \inf\{t \geq 0 : w_t - \alpha \ell t \leq 0\}$. Fixing $h > 0$, we compute

$$\mu\{\{x \in D([-1, \infty)) : \tau_0(x) < \infty, \inf_{0 \leq s \leq h}(x_{\tau_0 + s} - x_{\tau_0}) = 0\}\}$$

(11)

$$= \xi\{\{(w, \ell) \in E : \tau_0(w - \alpha \ell) < \infty, \inf_{0 \leq s \leq h}[(w_{\tau_0 + s} - w_{\tau_0}) - \alpha(\ell_{\tau_0 + s} - \ell_{\tau_0})] = 0\}\}$$

$$\leq \xi\{\{(w, \ell) \in E : \tau_\ell < \infty, \inf_{0 \leq s \leq h}(w_{\tau_\ell + s} - w_{\tau_\ell}) = 0\}\}$$

$$= \mathbb{P}\left(\tau_\ell < \infty, \inf_{0 \leq s \leq h}(Z_{\tau_\ell + s} - Z_{\tau_\ell}) = 0\right) = 0,$$

where the inequality is due to the fact that $\ell \in M$ is increasing. Noting that the analogue of Lemma 5.5 in [8] holds with the assumption that (11) vanishes for every $h > 0$, it follows that

$$\lim_{n \to \infty} \Gamma[\ell^n] = \lim_{n \to \infty} \langle \mu_n, \lambda \rangle = \langle \mu, \lambda \rangle = \Gamma[\ell].$$

\[\square\]

**Theorem 2.3.** There is a (unique) minimal solution to (7), and it is given by

$$\Lambda = \alpha \lim_{k \to \infty} \Gamma^{(k)}[0],$$

(12)

where the convergence is understood as $\Gamma^{(k)}[0] \to \alpha^{-1} \Lambda$ in $M$.

**Proof.** This is analogous to the proof of Proposition 2.3 in [8], making use of Proposition 2.2 above. \[\square\]

For $\Delta > 0$, define a time-discretized version $Z^\Delta$ by $Z^\Delta_t := Z^\frac{\alpha}{\Delta} \left|_{\Delta} \right.$ for $t \geq 0$. In analogy to the situation in continuous time, we define a fixed-point operator and use it to construct the minimal solution in this time-discretized version of the McKean–Vlasov problem and prove continuity of the operator in a suitable sense.

**Lemma 2.4.** For $\ell \in M$, set

$$\begin{cases}
X^\Delta_\ell[t] = X_{0-} + Z^\Delta_\ell - \alpha \ell \left|_{\Delta} \right., \\
\tau^\Delta_\ell[t] = \inf\{t \geq 0 : X^\Delta_\ell[t] \leq 0\}, \\
\Lambda_\ell[t] = \mathbb{P}\left(\tau^\ell[t] \leq t\right).
\end{cases}$$

(13)

Suppose that $\ell^n_{\left|_{\Delta} \right.} \to \ell_{\left|_{\Delta} \right.}$ in $M$ and that $\ell_0^{\alpha} = \ell_0$. Assume in addition that either law($X_{0-}$) is atomless or that law($Z_\ell$) is atomless for every $t > 0$. Then

$$\lim_{n \to \infty} \Lambda_\ell^n = \Lambda_\ell \text{ in } M.$$

**Proof.** Note that the condition $\ell^n_{\left|_{\Delta} \right.} \to \ell_{\left|_{\Delta} \right.}$ in $M$ implies $\ell^n_k \to \ell_k$ for $k \in \mathbb{N}$. 7
The assumption $\ell_0^n = \ell_0$ implies $\mathbb{P} (\tau^\Delta[\ell^n] = 0, \tau^\Delta[\ell] > t) = 0$. For any $t \geq 0$, 
\[
\Gamma_\Delta[\ell^n] - \Gamma_\Delta[\ell] \leq \mathbb{P} (\tau^\Delta[\ell^n] \leq t, \tau^\Delta[\ell] > t) 
= \sum_{k = 1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P} (\tau^\Delta[\ell^n] = k\Delta, \tau^\Delta[\ell] > t) 
\leq \sum_{k = 1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P} (X^\Delta_0[\ell^n] \leq 0, X^\Delta_0[\ell] > 0) 
= \sum_{k = 1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P} (X_{0-} + Z_{k\Delta} - \alpha\ell^n_{k\Delta} \leq 0, X_{0-} + Z_{k\Delta} - \alpha\ell_{k\Delta} > 0) 
= \sum_{k = 1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{P} (X_{0-} + Z_{k\Delta} \in (\alpha\ell_{k\Delta}, \alpha\ell^n_{k\Delta})). 
\]

Now if $\text{law}(Z_t)$ is atomless for every $t > 0$, using the independence of $Z$ and $X_{0-}$ we may rewrite this as
\[
\sum_{k = 1}^{\lfloor \frac{t}{\Delta} \rfloor} \mathbb{E} \left[ \mathbb{P} (x + Z_{k\Delta} \in (\alpha\ell_{k\Delta}, \alpha\ell^n_{k\Delta})) | x = x_{0-} \right],
\]
and the dominated convergence theorem yields that $\limsup_{n \to \infty} \Gamma_\Delta[\ell^n] \leq \Gamma_\Delta[\ell]$. The same argument works if we assume that $\text{law}(X_{0-})$ is atomless, as we see by noticing that $\mathbb{P} (X_{0-} + Z_{k\Delta} \in (\alpha\ell_{k\Delta}, \alpha\ell^n_{k\Delta})) = \mathbb{E} \left[ \mathbb{P} (X_{0-} + z \in (\alpha\ell_{k\Delta}, \alpha\ell^n_{k\Delta})) | z = Z_{k\Delta} \right]$. Interchanging the roles of $\ell^n$ and $\ell$ in the estimates then yields the claim. \hfill \Box

**Definition 2.5.** We say that $\Lambda^\Delta$ solves the discretized McKean–Vlasov problem if $\alpha \Gamma_\Delta[\alpha^{-1} \Lambda^\Delta] = \Lambda^\Delta$. If $\Lambda^\Delta$ is such that $\Lambda^\Delta \leq \Lambda^\Delta$ for every $\Lambda^\Delta$ that solves the discretized McKean–Vlasov problem, we call $\Lambda^\Delta$ minimal. We shall refer to the scheme as implicit time-stepping scheme.

**Remark 2.6.** With Definition 2.5, $\Lambda^\Delta$ is a solution if and only if
\[
\Lambda^\Delta_{k\Delta} = \alpha \mathbb{P} \left( \min_{0 \leq i \leq k} \{ X_{0-} + Z_i \Delta - \Lambda^\Delta_{i\Delta} \} \leq 0 \right), \tag{14}
\]
for every $k \in \mathbb{N}$. Since $\Lambda^\Delta_{k\Delta}$ appears on both sides of (14), this is an implicit equation for $\Lambda^\Delta$ (the solution of which is not unique in general), therefore we call our time-stepping scheme implicit. If we define an alternative notion of solution through (14) by taking the minimum on the right-hand side over $\{0, \ldots, k-1\}$ instead of $\{0, \ldots, k\}$, this gives a recursion for $\Lambda^\Delta$, and we obtain the time-stepping scheme of [26], which we refer to as explicit in the following.

**Proposition 2.7.** There is a (unique) minimal solution of the discretized McKean–Vlasov problem and it is given by
\[
\Lambda^\Delta = \alpha \lim_{k \to \infty} \Gamma^{(k)}_\Delta[0]. \tag{15}
\]

**Proof.** $\Gamma_\Delta$ is monotone in the sense that if $\ell^1 \leq \ell^2$ then $\Gamma_\Delta[\ell^1] \leq \Gamma_\Delta[\ell^2]$. Therefore,
\[
0 \leq \Gamma_\Delta[0] \leq \Gamma^{(2)}_\Delta[0] \leq \ldots
\]
This allows us to define \( \hat{\ell}^\Delta := \lim_{k \to \infty} \Gamma^{(k)}_\Delta[0] \). Then \( \hat{\ell}^\Delta \) lies in \( M \) and by construction \( \lim_{k \to \infty} \Gamma^{(k)}_\Delta[0] = \tilde{\ell}^\Delta \) in \( M \). Note that \( \Gamma^{(k)}_\Delta[0] \) is a step function with jumps at times \( \Delta \mathbb{N} \), and so the same holds for \( \hat{\ell}^\Delta \). This implies that \( \Gamma^{(k)}_\Delta[0] \mid_{\Delta} = \Gamma^{(k)}[0]_t \) and \( \hat{\ell}^\Delta \mid_{\Delta} = \ell^\Delta_t \) for \( t \geq 0 \). As \( \Gamma^{(k)}[0]_t = \Gamma[0]_t \) for \( 0 \leq s < \Delta \), it follows that \( \hat{\ell}^\Delta_0 = \Gamma^{(k)}_\Delta[0]_0 \) for every \( k \in \mathbb{N} \), and therefore we may apply Lemma 2.4 to find

\[
\Gamma[\hat{\ell}^\Delta] = \Gamma[\lim_{k \to \infty} \Gamma^{(k)}_\Delta[0]] = \lim_{k \to \infty} \Gamma[\Gamma^{(k)}_\Delta[0]] = \hat{\ell}^\Delta,
\]

so \( \hat{\Lambda} := \alpha \hat{\ell}^\Delta \) is a solution of the discretized McKean–Vlasov problem. If \( \Lambda^\Delta \) is another solution, then since \( 0 \leq \alpha^{-1} \Lambda^\Delta \), and \( \hat{\ell}^\Delta \) is monotone we have \( \alpha \Gamma[0] \leq \alpha \Gamma[\alpha^{-1} \Lambda^\Delta] = \Lambda^\Delta \). Proceeding inductively we obtain \( \alpha \Gamma[0] \leq \Lambda^\Delta \) and therefore \( \Lambda^\Delta \leq \Lambda^\Delta \) for every continuity point of \( \Lambda^\Delta \). By right-continuity, it follows that \( \Lambda^\Delta \) is minimal.

**Remark 2.8.** In contrast to the explicit scheme in [26], the time-stepping scheme we propose to solve here requires us to solve the implicit condition that \( \Lambda^\Delta \) is the minimal solution of [14] through the iteration [15]. The analogous results we prove here hold as well for the explicit version of the scheme, but we expect that the explicit version smoothes out the jumps while the implicit scheme should not, and therefore the implicit scheme should lead to a more accurate approximation of jump sizes and jump times. We observe this empirically in Section 4.

We are now ready to state the main result of this section.

**Theorem 2.9.** Choose a sequence \( \Delta_n > 0 \) such that the resulting discretizations are nested, i.e., \( \Delta_n \mathbb{N} \subseteq \Delta_{n+1} \mathbb{N} \) for all \( n \in \mathbb{N} \), and such that \( \lim_{n \to \infty} \Delta_n = 0 \). Assume in addition that either law(\( X_{0-} \)) is atomless or that law(\( Z_t \)) is atomless for every \( t > 0 \). Then \( \lim_{n \to \infty} \Lambda^{\Delta_n} = \Lambda \) in \( M \).

We collect some properties of the space \( M \) in the next lemma.

**Lemma 2.10.** Let \( \ell^n, \ell \in M \). The following statements are equivalent.

(a) \( \ell^n \to \ell \) in \( M \).

(b) Let \( t \) be a continuity point of \( \ell \). Then, for every \( \epsilon > 0 \), there is a \( \delta > 0 \) with

\[
\lim_{n \to \infty} \sup_{s \in [t-\delta, t+\delta]} |\ell^n_s - \ell_s| \leq \epsilon.
\]

(c) There is a co-countable set \( D \subseteq [0, \infty) \) such that \( \ell^n_t \to \ell_t \) for \( t \in D \).

**Proof.** [(a) \implies (b)]: Fix \( \epsilon > 0 \). As \( \ell \) is continuous at \( t \) and \( \ell \) has at most countably many discontinuity points, there is \( \delta > 0 \) such that both \( t + \delta \) and \( t - \delta \) are continuity points and \( |\ell_s - \ell_r| < \frac{\epsilon}{2} \) for \( s, r \in [t - \delta, t + \delta] \). Choose \( n_0 \in \mathbb{N} \) large enough such that \( |\ell^n_{t+\delta} - \ell_{t+\delta}| < \frac{\epsilon}{2} \) and \( |\ell^n_{t-\delta} - \ell_{t-\delta}| < \frac{\epsilon}{2} \) for all \( n \geq n_0 \). Then, for \( s \in [t - \delta, t + \delta] \) we have

\[
\ell^n_s - \ell_s \leq \ell^n_{t+\delta} - \ell_s < \ell_{t+\delta} - \ell_s + \frac{\epsilon}{2} < \epsilon,
\]

\[
\ell^n_s - \ell_s \geq \ell^n_{t-\delta} - \ell_s > \ell_{t-\delta} - \ell_s - \frac{\epsilon}{2} > -\epsilon,
\]

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To distinguish the two schemes we use here of $\ell$ and hence $\lim \ell$ we obtain that $x_\xi$ we then have $J$ then have $\lim$. Hence $\ell^n \to \ell$ in $M$. The reverse implication is clear.

**Proposition 2.11.** Let $\Delta_n \to 0$ and suppose that $\ell^n \to \ell$ in $M$. Then,

$$
\lim_{n \to \infty} \Gamma_{\Delta_n}[\ell^n] = \Gamma[\ell].
$$

**Proof.** Step 1: We first show that $(X_{0-} + Z^{\Delta_n}, \ell^n_{[\Delta_n]}{\mid}_{\Delta_n}) \to (X_{0-} + Z, \ell)$ in $E$. Fix $T > 0$. For $\omega \in \Omega$ and $\epsilon > 0$, by uniform continuity there almost surely is $\delta > 0$ such that $|Z_t(\omega) - Z_s(\omega)| < \epsilon$ whenever $|s - t| < \delta$ and $s, t \in [0, T]$. For $\Delta_n < \delta$, we then have $|Z_t(\omega) - Z^{\Delta_n}_t(\omega)| < \epsilon$, $t \in [0, T]$, and hence $\lim_{n \to \infty} \|Z - Z^{\Delta_n}\|_{C([0,T])} = 0$ almost surely. Let $t$ be a continuity point of $\ell$. Write $g^n := \ell^n_{[\Delta_n]}{\mid}_{\Delta_n}$. Fix $\epsilon > 0$, and choose $\delta > 0$ as in Lemma 2.10(b). We then have

$$
\limsup_{n \to \infty} |g^n - \ell_t| \leq \limsup_{n \to \infty} \sup_{s \in [t-\delta, t+\delta]} |\ell^n_s - \ell_s| + \limsup_{n \to \infty} |\ell_{\frac{t}{\Delta_n}} - \ell_n| \leq \epsilon.
$$

As $\epsilon > 0$ was arbitrary, we see that $g^n \to \ell$ in $M$.

Step 2: Setting $\xi_n := \text{law}((X_{0-} + Z^{\Delta_n}, g^n))$ and $\xi := \text{law}((X_{0-} + Z, \ell))$, by Step 1 we have $\xi_n \to \xi$ in $\mathcal{P}(E)$. By the same reasoning as in the proof of Proposition 2.2 we obtain that $\langle \iota_\alpha(\xi_n), \lambda \rangle \to \langle \iota_\alpha(\xi), \lambda \rangle$ in $M$ with $\lambda$ being defined in Section 1.1. Since $\Gamma_{\Delta_n}[\ell^n]_t = \langle \iota_\alpha(\xi_n), \lambda_t \rangle$ and $\Gamma[\ell]_t = \langle \iota_\alpha(\xi), \lambda_t \rangle$, this yields the claim.

We are now prepared to prove Theorem 2.9.

**Proof of Theorem 2.9** A straightforward induction using Proposition 2.11 shows that $\lim_{n \to \infty} \Gamma_{\Delta_n}[\ell^n]_t = \Gamma[\ell]_t$. Since $\Delta_n \subseteq \Delta_{n+1} \subseteq \Delta_{n} \subseteq \Delta_{n+1} \subseteq \Delta_{n+1}$ for any $\ell \in M$. Set $J := \text{Disc}(\Delta) \cup \{\text{Disc}(\Gamma_{\Delta_n}[\ell^n]_t) \mid k \in \mathbb{N}\} \cup \bigcup_{n \geq 0} \Delta_n \mathbb{N}$. Then $J$ is countable, and for $t \notin J$ we have by Proposition 2.17

$$
\Delta_t = \alpha \lim_{k \to \infty} \Gamma_{\Delta_n}[\ell^n]_t = \alpha \lim_{k \to \infty} \Gamma_{\Delta_n}[\ell^n]_t = \alpha \lim_{n \to \infty} \Gamma_{\Delta_n}[\ell^n]_t = \lim_{n \to \infty} \Delta_{\Delta_n}^n,
$$

where we may exchange the order of the limits because $\Gamma_{\Delta_n}[\ell^n]_t$ is increasing both in $n$ and in $k$. Lemma 2.10(c) yields the claim.

### 3 A Donsker-type approximation

In this section, we consider the numerical approximation of a special case of the McKean–Vlasov problem [7] with $Z = B$, with $B$ a Brownian motion independent of $X_{0-}$. The idea is to replace the Brownian motion by its Donsker approximation, i.e., for $h > 0$ we consider

\[^1\text{To distinguish the two schemes we use here } h \text{ rather than } \Delta \text{ for the time step.}\]
\[
\begin{aligned}
\begin{cases}
X_t^h = X_0^h + B_t^h - \Lambda_t^h, \\
\tau_t^h = \inf\{t \geq 0 : X_t^h \leq 0\}, \\
\Lambda_t^h = \alpha \mathbb{P} \left( \tau_t^h \leq t \right),
\end{cases}
\end{aligned}
\]

(18)

where

\[
B_t^h = \sqrt{h} \sum_{k=0}^{\lfloor t/h \rfloor} Y_k,
\]

and \((Y_i)_{i \in \mathbb{N}}\) is an i.i.d. sequence of random variables satisfying \(\mathbb{E}[Y_1] = 0\) and \(\mathbb{E}[Y_1^2] = 1\). We repeat the same reasoning as previously, obtaining the minimal solution of (18) through a fixed-point iteration. We call a solution \(\Lambda^h\) of (18) minimal, if \(\Lambda^h \leq \Lambda^h\) for every \(\Lambda^h\) that solves (7). The minimal solution will then give rise to the implicit Donsker approximation scheme presented in Section 4.

Lemma 3.1. Assume that \(X_{0-}^h\) is atomless. Define the operator \(\Gamma^h\) via

\[
\begin{aligned}
X_t^h[\ell] &= X_0^h + B_t^h - \alpha \ell_{\lfloor t/h \rfloor}^h, \\
\tau_t^h[\ell] &= \inf\{t \geq 0 : X_t^h[\ell] \leq 0\}, \\
\Gamma^h[\ell]_t &= \mathbb{P} \left( \tau_t^h[\ell] \leq t \right).
\end{aligned}
\]

Then it holds that

\[
\Lambda^h = \alpha \lim_{k \to \infty} \Gamma^h(0).
\]

(20)

Proof. By the same arguments given in Lemma 2.4, if \(\ell_n^h \to \ell_{\lfloor t/h \rfloor}^h\) in \(M\), then \(\lim_{n \to \infty} \Gamma^h[\ell_n^h] = \lim_{n \to \infty} \Gamma^h[\ell]\) in \(M\). Reasoning as in the proof of Theorem 2.9, using Step 1 of Proposition 2.11, we obtain (20).

Remark 3.2. If we were to pose problem (18) on a finite time horizon \([0, T]\), the analogous result to Lemma (3.1) would hold by the same reasoning. In particular, since the solution operator for the finite time horizon is the restriction to \([0, T]\) of \(\Gamma^h\), the representation (20) shows that the minimal solution for the problem on \([0, T]\) is simply the restriction of the global minimal solution.

From here on, we will denote by \(\Lambda^h\) the minimal solution in the Donsker approximation with initial condition

\[
X_{0-}^h = \left( \left\lfloor \frac{X_0 - \sqrt{h}}{\sqrt{h}} \right\rfloor + \log(h)^2 \right) \sqrt{h}.
\]

(21)

The reasons for this choice are twofold: first, to make the theoretical results applicable to the numerical scheme of Section 4, we replace \(X_{0-}\) by a space-discretized version of itself, given by \(\left\lfloor \frac{X_0 - \sqrt{h}}{\sqrt{h}} \right\rfloor \sqrt{h}\); second, for technical reasons, we require a perturbation term of order \(O(\sqrt{h} \log(h)^2)\) in the initial condition.

The goal of this section is to show that, with the above choice of initial condition (21), \(\Lambda^h \to \Lambda\) in \(M\) as \(h \to 0\). We require some additional assumptions.

Assumptions 3.3. We assume that

(i) \(X_{0-}\) has a bounded density, which we denote by \(V_{0-}\).

(ii) The moment generating function of \(Y_1\) exists in a neighbourhood of 0, i.e., \(\mathbb{E}[\exp(uY_1)] < \infty\) for some \(\delta > 0\) and \(|u| < \delta\).
3.1 Convergence results

On an abstract level, the proof of convergence has two main parts: (i) proving that limit points of solutions to the Donsker problem are solutions to the McKean–Vlasov problem; (ii) identifying limit points of minimal solutions of the Donsker problem as the minimal solution of the McKean–Vlasov problem. Typically, part (ii) is the more challenging one. Fortunately, a similar approach as in [8] works.

The central idea of this approach is to find a sequence of optimization problems that allows us to relate the minimal solution of the (perturbed) Donsker problem to the minimal solution of the McKean–Vlasov equation, showing that the latter dominates the former asymptotically as $h \to 0$.

The following lemma constitutes the main technical ingredient of this proof. As a rough intuition, setting $\ell = \Lambda$, the lemma allows us to quantify how far away the minimal solution of the McKean–Vlasov problem is from being a solution to the Donsker problem when $h$ is small.

Lemma 3.4. Suppose that Assumptions 3.3 are satisfied. Then, there is a constant $C > 0$ depending on $T$ and $\|V_0\|_\infty$ such that for all sufficiently small $h > 0$ it holds

$$\sup_{t \in \mathbb{R}} \sup_{\ell \in [0, T]} |\Gamma_h[\ell]_t - \Gamma[\ell]_t| \leq C \sqrt{h} \log(T/h),$$

where $\Gamma_h$ is defined as in Lemma 3.1 with $X_0 = X_0 - h\tilde{X}$. The proof of the above lemma requires some preparations. The convergence rate in (22) is essentially the convergence rate of the Donsker approximation with respect to the Prokhorov metric, which we transfer to the estimate above through a Lipschitz mapping theorem.

Definition 3.5. Let $(S, d)$ be a Polish space. For $\mu, \nu \in \mathcal{P}(S)$ define the Prokhorov distance between $\mu$ and $\nu$ as

$$\rho^S_P(\mu, \nu) = \inf\{\epsilon > 0 : \mu(B) \leq \nu(B') + \epsilon \text{ for all Borel sets } B\},$$

where $B' = \{x \in S : d_S(x, B) < \epsilon\}$. If $X, Y$ are random variables on $S$ we will write $\rho^S_P(X, Y)$ in place of $\rho_P(\text{law}(X), \text{law}(Y))$.

It is well-known that the Prokhorov metric metrizes weak convergence (in the probabilistic sense) and $(\mathcal{P}(S), \rho^S_P)$ is a Polish space. By [40, Theorem 3.4.2], Lipschitz maps preserve the Prokhorov metric in the sense that $\rho^S_P(g(X), g(Y)) \leq (1 + K)\rho^S_P(X, Y)$ for any $K$-Lipschitz map $g$. We will need a slight generalization of this result, which allows $g$ to depend on an additional random variable that is independent of the Lipschitz component.

Theorem 3.6. Let $(S, d), (S', d')$, $(R, d_R)$ be Polish spaces and suppose that $g : S' \times S \to R$ is $K$-Lipschitz in the second variable, i.e.,

$$d_R(g(x, y), g(x, z)) \leq K d_S(y, z), \quad x \in S', \ y, z \in S.$$  

Suppose furthermore that the random variables $X, Y, Z$ satisfy $X \perp Y$, $X \perp Z$. Then,

$$\rho^R_P(g(X, Y), g(X, Z)) \leq (1 + K)\rho^S_P(Y, Z).$$

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Proof. We may assume wlog that $K \geq 1$. By the Strassen-Dudley representation theorem [13 Corollary 11.6.4], there is a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ with random variables $Y, \tilde{Z}$ satisfying $\mathbb{L}(Y) = \mathbb{L}(\tilde{Y}), \mathbb{L}(\tilde{Z}) = \mathbb{L}(Z)$ such that $\rho^\mathcal{K}_p(Y, Z) = \rho_K(Y, \tilde{Z})$, where $\rho_K$ is the Ky-Fan metric, i.e.,

$$\rho_K(Y, \tilde{Z}) = \inf \{ \epsilon > 0 : \tilde{\mathbb{P}}(d(\tilde{Y}, \tilde{Z}) > \epsilon) \leq \epsilon \}.$$ 

Now set $\Omega = \Omega \times \tilde{\Omega}$ and $\mathbb{P} = \mathbb{P} \otimes \tilde{\mathbb{P}}$, then $\mathbb{L}((X, \tilde{Y})) = \mathbb{L}(X, Y)$ and $\mathbb{L}((X, \tilde{Z})) = \mathbb{L}((X, Z))$ by construction of $(\Omega, \mathbb{P})$ and the independence assumption. In particular, we have $\rho_p^\mathcal{K}(g(X, Y), g(X, Z)) = \rho_p^\mathcal{K}(g(X, \tilde{Y}), g(X, \tilde{Z}))$. Noting the general fact that $\rho_p \leq \rho_K$, we consider $\epsilon > 0$ such that $\tilde{\mathbb{P}}(d(\tilde{Y}, \tilde{Z}) > \epsilon/K) \leq \epsilon/K$. Then,

$$\mathbb{P}(d_H(g(X, \tilde{Y}), g(X, \tilde{Z})) > \epsilon) \leq \mathbb{P}(d(\tilde{Y}, \tilde{Z}) > \epsilon/K) \leq \epsilon/K \leq \epsilon.$$

It follows that

$$\rho_K(g(X, \tilde{Y}), g(X, \tilde{Z})) = \inf \{ \epsilon > 0 : \tilde{\mathbb{P}}(d(\tilde{Y}, \tilde{Z}) > \epsilon/K) \leq \epsilon/K \} = K \rho_K(\tilde{Y}, \tilde{Z}).$$

We have obtained that

$$\rho_p^\mathcal{K}(g(X, Y), g(X, Z)) \leq \rho_K(g(X, \tilde{Y}), g(X, \tilde{Z})) \leq K \rho_K(\tilde{Y}, \tilde{Z}) = K \rho_p^\mathcal{K}(Y, Z).$$

\[\square\]

Example 3.7. We illustrate with an example that Theorem 3.6 does not hold in general without the independence assumption. We consider $S = S' = R = \mathbb{R}$, we set $\mathbb{L}((X, Y)) = \frac{1}{2} \delta_{(0,0)} + \frac{1}{2} \delta_{(1,1)}$, and $\mathbb{L}((X, Z)) = \frac{1}{2} \delta_{(0,1)} + \frac{1}{2} \delta_{(1,0)}$. Then, $\mathbb{L}(X) = \mathbb{L}(Y) = \mathbb{L}(Z) = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1$, so in particular $\rho_p^\mathcal{K}(Y, Z) = 0$. However, taking $g(x, y) := |x - y|$, we see that $\mathbb{L}(g(X, Y)) = \delta_0$ and $\mathbb{L}(g(X, Z)) = \delta_1$, so $\rho_p^\mathcal{K}(g(X, Y), g(X, Z)) = 1$.

Corollary 3.8. Suppose $g : S' \times S \to \mathbb{R}$ is $K$-Lipschitz in the second variable. Suppose that $X \perp Y, X \perp Z$ and that the cdf of $g(X, Y)$ is Lipschitz with constant $L$. Then, for all $x \in \mathbb{R}$ we have

$$\| \mathbb{P}(g(X, Y) \leq x) - \mathbb{P}(g(X, Z) \leq x) \| \leq (1 + L)(1 \vee K) \rho_p^\mathcal{K}(Y, Z). \quad (26)$$

Proof. Denote the cdf of $g(X, Y)$ by $F$ and that of $g(X, Z)$ by $G$. It holds that

$$\sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq (1 + L) \inf \{ \epsilon > 0 : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \ \forall x \in \mathbb{R} \},$$

we have

$$\leq (1 + L) \rho_p^\mathcal{K}(g(X, Y), g(X, Z)).$$

The claim now follows from Theorem 3.6. \[\square\]

We state two technical lemmata for future reference.

Lemma 3.9. Let $x^1, x^2$ be c\`adl\`ag paths. Then,

$$| \inf_{0 \leq s \leq t} x^1_s - \inf_{0 \leq s \leq t} x^2_s | \leq \sup_{0 \leq s \leq t} | x^1_s - x^2_s |$$

Proof. Let $\epsilon > 0$ and choose $r \in [0, t]$ such that $x^2_r \leq \inf_{0 \leq s \leq t} x^2_s + \epsilon$. Then,

$$\inf_{0 \leq s \leq t} x^1_s - \inf_{0 \leq s \leq t} x^2_s \leq x^1_r - x^2_r + \epsilon \leq \sup_{0 \leq s \leq t} | x^1_s - x^2_s | + \epsilon.$$

Reversing the roles and noting that $\epsilon > 0$ was arbitrary proves the claim. \[\square\]
Lemma 3.10. Let $X_{0-}$ admit a bounded density, denoted by $V_{0-}$ and let $Y^1, Y^2$ be càdlàg and such that $X_{0-}$ is independent of $(Y^1, Y^2)$. Then, it holds that

$$\|P\left(\inf_{0 \leq s \leq t} X_{0-} + Y^1_s \leq 0\right) - P\left(\inf_{0 \leq s \leq t} X_{0-} + Y^2_s \leq 0\right)\| \leq \|V_{0-}\|_{\infty} \mathbb{E}\left[\sup_{0 \leq s \leq t} |Y^1_s - Y^2_s|\right].$$

Proof. Set $I := [-\inf_{s \leq t} Y^1_s, -\inf_{s \leq t} Y^2_s] \cup [-\inf_{s \leq t} Y^1_s, -\inf_{s \leq t} Y^2_s]$ and note that by Lemma 3.9 the length of $I$ is bounded by $\sup_{0 \leq s \leq t} |Y^1_s - Y^2_s|$. We argue as in the proof of Lemma 2.1 in [29], using the assumption on $X_{0-}$ and the general inequality $P(A) - P(B) \leq P(A \setminus B)$,

$$\begin{align*}
\left|P\left(\inf_{0 \leq s \leq t} X_{0-} + Y^1_s \leq 0\right) - P\left(\inf_{0 \leq s \leq t} X_{0-} + Y^2_s \leq 0\right)\right| &\leq P(X_{0-} \in I) \\
&= \mathbb{E}\left[\int_0^\infty 1_I(t) V_{0-}(t) \, dt\right] \\
&\leq \|V_{0-}\|_{\infty} \mathbb{E}\left[\sup_{0 \leq s \leq t} |Y^1_s - Y^2_s|\right].
\end{align*}$$

\hfill \square

Proof of Lemma 3.10. Step 1: We transfer the convergence rate on $[0, 1]$ to $[0, T]$ for $T > 0$. Set

$$S := \{x \in D[0, 1] \colon \text{Disc}(x) \subseteq h\mathbb{Q}\}, \quad R := \left\{x \in D[0, T] \colon \text{Disc}(x) \subseteq \frac{h}{T} \mathbb{Q}\right\}. \quad (27)$$

In words, $S$ is the set of càdlàg paths on $[0, 1]$ that can only jump at times that are rational multiples of $h$. Note that if we endow the sets $S$ and $R$ with the uniform metric, they are separable and hence Polish spaces. By Theorem 1.16 in [6] we have $\rho^R(B^h, W) \leq C_{\sqrt{T}} \log(1/h')$ for sufficiently small $h' > 0$. Define $f : S \to R$ by $f(x) = \sqrt{T}x/T$, then $f$ is $\sqrt{T}$-Lipschitz. Theorem 3.6 and the self-similarity of the Donsker approximation and Brownian motion give

$$\rho^R(B^h, W) = \rho^R(\sqrt{T}B^{h/T}, \sqrt{T}W/T) = \rho^R(f(B^{h/T}), f(W)) \leq C \frac{(1 + \sqrt{T})}{\sqrt{T}} \sqrt{h} \log(T/h),$$

for $h > 0$ sufficiently small, which concludes Step 1.

Step 2: Fix $\ell \in M$ and define the map $g_{\ell} : \mathbb{R} \times R \to \mathbb{R}$ by $g_{\ell}(x_0, x) := \inf_{0 \leq s \leq t} x_0 + x_s - \alpha \ell$. Note that $g(x_0, \cdot)$ is 1-Lipschitz for every $x_0 \in \mathbb{R}$ by Lemma 3.9. Denote the cdf of $g(X_{0-}, B)$ by $F$. We check that $F$ is Lipschitz continuous: for $x, y \in \mathbb{R}$, setting $Y^1 := B - \alpha \ell - x$ and $Y^2 := B - \alpha \ell - y$ in Lemma 3.10, we obtain

$$|F(x) - F(y)| \leq \|V_{0-}\|_{\infty} |x - y|. \quad (28)$$

By Corollary 3.8 we therefore obtain for $t \in [0, T]$ and $h > 0$ small enough

$$|P(g_{\ell}(X_{0-}, B^h) \leq 0) - P(g_{\ell}(X_{0-}, B) \leq 0)| \leq C(1 + \|V_{0-}\|_{\infty}) \frac{(1 + \sqrt{T})}{\sqrt{T}} \sqrt{h} \log(T/h).$$

\footnote{Note that in [6], the result is stated to hold for all $h > 0$, but the proof reveals that it only holds for sufficiently small $h > 0$.}
Since with \( X_{0^-}^h := \left\lfloor \frac{X_{0^-}}{\sqrt{h}} \right\rfloor \sqrt{h} \) it holds that \( |X_{0^-} - X_{0^-}^h| \leq \sqrt{h} \) almost surely, setting \( Y^1 := -\sqrt{h} + B^h - \alpha \ell \) and \( Y^2 := B^h - \alpha \ell \) in Lemma 3.10 we find
\[
P(g_t(X_{0^-}^h, B^h) \leq 0) - P(g_t(X_{0^-}, B^h) \leq 0)
\leq \|V_0\|_\infty \sqrt{h},
\]
and arguing analogously for the downwards estimate we find
\[
\|P(g_t(X_{0^-}^h, B^h) \leq 0) - P(g_t(X_{0^-}, B^h) \leq 0)\| \leq \|V_0\|_\infty \sqrt{h}.
\] (29)
Noting that \( \Gamma_t[\ell_t] = P(g_t(X_{0^-}^h, B^h) \leq 0) \) and \( \Gamma_t[\ell] = P(g_t(X_{0^-}, B) \leq 0) \), taking the supremum over \( t \in [0, T] \) and absorbing some terms into the constant \( C \), the claim follows.

\[\]
Lemma 3.11. Let \( \tilde{\Lambda}^h \) be the minimal solution to (18) with initial condition given by (21) and suppose that Assumptions 3.3 are satisfied. Then, for any \( t > 0 \),
\[
\limsup_{h \rightarrow 0} \tilde{\Lambda}^h \leq \Lambda_t.
\] (30)

Proof. Define the sequence of optimal values
\[
V_t^h = \inf_{\ell \in M} c_h(\ell), \quad c_h(\ell) = \alpha \ell + \frac{\alpha}{\sqrt{h} \log(h)^2} \sup_{|s| \leq |t|} |\Gamma_h[\ell]| - \ell_s|,
\] (31)
where \( \Gamma_h \) is defined as in Lemma 3.1.

Step 1: We show that \( V_t^h \) is asymptotically dominated by \( \Lambda \) as \( h \rightarrow 0 \). By Lemma 3.4 for small enough \( h > 0 \) we have
\[
V_t^h \leq c_h(\alpha^{-1} \tilde{\Lambda}) = \Lambda_t + \frac{\alpha}{\sqrt{h} \log(h)^2} \sup_{|s| \leq |t|} |\Gamma_h[\Lambda] - \Lambda_s| \leq \Lambda_t + \alpha C \frac{\log(t/h)}{\log(h)^2}\] (32)
and therefore \( \limsup_{h \rightarrow 0} V_t^h \leq \Lambda_t \).

Step 2: We show that \( V_t^h \) asymptotically dominates \( \tilde{\Lambda}^h \). Choose a sequence \( (L^h)_{h \geq 0} \) with \( L^h \in M \) such that \( c_h(L^h) - V_t^h \rightarrow 0 \) as \( h \rightarrow 0 \). Suppose that \( \sup_{s \in [0, t]} |\Gamma_h[L^h_s] - L^h_s| > \alpha^{-1} h^{1/2} \log(h)^2 \) for some \( h \), then we have \( c_h(L^h) > c_h(\alpha^{-1} \tilde{\Lambda}^h) \), where \( \tilde{\Lambda}^h \) is a solution to (18) with \( X_{0^-}^h := \left\lfloor \frac{X_{0^-}}{\sqrt{h}} \right\rfloor \sqrt{h} \). We may therefore assume wlog that \( \sup_{s \in [0, t]} |\Gamma_h[L^h_s] - L^h_s| \leq \alpha^{-1} h^{1/2} \log(h)^2 \) for all \( h > 0 \). For the sake of brevity, we write \( d_h := \alpha^{-1} h^{1/2} \log(h)^2 \) in the following. We then have
\[
L_s^h :\Gamma_h[L^h_s] - d_h \quad s \in [0, t].
\] (33)
For \( \ell \in M \) define the operator
\[
\tilde{\Gamma}_h[\ell] := P\left( \inf_{0 \leq s \leq t} \left( \left\lfloor \frac{X_{0^-}}{\sqrt{h}} \right\rfloor + |\log(h)^2| \right) \sqrt{h} + B_s^h - \alpha \ell_s \right) \leq 0\). \] (34)
Inequality (33) and the monotonicity of \( \Gamma_h \) imply
\[
L_s^h :\Gamma_h[-d_h] - d_h = \tilde{\Gamma}_h[0] - d_h, \quad s \in [0, t].
\] (35)
Substituting this again into inequality (33) yields
\[ L_s^h \geq \Gamma_h \tilde{\Gamma}_h[0] - d_h, \quad s \in [0, t]. \]

Proceeding inductively, we obtain \( L_s^h \geq \tilde{\Gamma}_h^{(k)}[0] - d_h \) for every \( s \in [0, t] \) and \( k \in \mathbb{N} \), and taking the limit as \( k \to \infty \), by Lemma 3.10 we obtain \( \alpha L_s^h \geq \Delta_s^h - \alpha d_h \) for \( s \in [0, t] \). Since \( d_h \to 0 \) as \( h \to 0 \), combined with the definition of \( c_h \) and Step 1, this yields
\[ \limsup_{h \to 0} \Delta_s^h \leq \limsup_{h \to 0} \alpha L_t^h \leq \limsup_{h \to 0} c_h(L_t^h) = \limsup_{h \to 0} V_t^h \leq \Lambda_t. \]

\[ \square \]

**Theorem 3.12.** Suppose that Assumptions 3.3 are satisfied. Then, the sequence of (perturbed) minimal solutions to the Donsker problem converges to the minimal solution to the McKean–Vlasov problem, i.e., it holds that
\[ \lim_{h \to 0} \Delta_s^h = \Lambda_t, \quad (36) \]
for every \( t \geq 0 \) that is a continuity point of \( \Lambda \).

**Proof.** Choose a sequence \( (h_n)_{n \geq 1} \) such that \( h_n > 0 \) and \( \lim_{n \to \infty} h_n = 0 \). By compactness of \( M \), after passing to a subsequence if necessary, we may assume that \( \Delta_n \to \Lambda \) for some \( \Lambda \) with \( \alpha^{-1} \Lambda \in M \). If \( x \) is a càdlàg path, we introduce the notation \( \Gamma_\alpha[x] := \alpha \Gamma[\alpha^{-1}x] \), such that \( \Lambda \) solves (37) iff \( \Gamma_\alpha[\Lambda] = \Lambda \). We use the notation \( \Gamma_\alpha^n \) in the analogous way. We show that \( \Lambda \) solves the McKean-Vlasov problem. By Theorem 2.2 \( \Gamma \) is continuous, so for \( t \notin \text{Disc}(\Lambda) \cap \text{Disc}(\Gamma[\Lambda]) \) we have
\[ |\Gamma_\alpha[\Lambda]_t - \Lambda_t| = \lim_{n \to \infty} |\Gamma_\alpha^n[\Delta_n^h]_t - \Delta_n^h| \leq \limsup_{n \to \infty} \left[ |\Gamma_\alpha[\Delta_n^h]_t - \Gamma_\alpha^n[\Delta_n^h]_t| + |\Gamma_\alpha^n[\Delta_n^h]_t - \tilde{\Gamma}_h^n[\Delta_n^h]_t| \right]. \]

where \( \tilde{\Gamma}_h^n[x] \) is defined as \( \alpha \tilde{\Gamma}_h[n, \alpha^{-1}x] \) with \( \tilde{\Gamma}_h \) defined in (34) and we used that \( \Delta_n^h = \Gamma_\alpha^n[\Delta_n^h] \) since \( \Delta_n^h \) is a solution to the perturbed Donsker problem. Due to Lemma 3.4 the first term above vanishes as \( n \to \infty \). For the second term, setting \( Y^1 := \sqrt{h_n} \left( 1 + [\log(h_n)^2] \right) + B^h - \Delta_n^h \) and \( Y^2 := -\sqrt{h_n} + B^h - \Delta_n^h \), we have
\[ \alpha \mathbb{P} \left( X_{0^-} + \inf_{s \leq t} Y_s^1 \leq 0 \right) \leq \Gamma_\alpha^n[\Delta_n^h]_t \leq \Gamma_\alpha^n[\Delta_n^h]_t \leq \alpha \mathbb{P} \left( X_{0^-} + \inf_{s \leq t} Y_s^2 \leq 0 \right), \]
so by Lemma 3.10 it follows that
\[ |\Gamma_\alpha^n[\Delta_n^h]_t - \Gamma_\alpha^n[\Delta_n^h]_t| \leq \alpha \|V_0\|_\infty \sqrt{h_n} \left( 2 + [\log(h_n)^2] \right). \]

Therefore, \( \Gamma_\alpha[\Lambda]_t = \Lambda_t \). By right-continuity of \( \Lambda \) and \( \Gamma_\alpha[\Lambda] \), we find \( \Lambda = \Gamma_\alpha[\Lambda] \). By definition, this implies \( \Lambda \geq \Lambda \). On the other hand, for \( t \notin \text{Disc}(\Lambda) \), Lemma 3.11 yields
\[ \Lambda_t = \limsup_{n \to \infty} \Delta_n^h \leq \Lambda_t. \]

By right-continuity, we find \( \Lambda \leq \Lambda \) and hence \( \Lambda = \Lambda \).
In the case $\alpha \| V_0 - \|_\infty < 1$, we also get a rate of convergence as shown in the next theorem. By [29, Theorem 2.2 and the comment below], this corresponds to the weak feedback regime, where uniqueness and in particular continuity of solutions holds true.

**Theorem 3.13.** Suppose that Assumptions 3.3 are satisfied and that additionally $\alpha \| V_0 - \|_\infty < 1$. Then, there is a constant $C$ depending on $T, \alpha$ and $\| V_0 - \|_\infty$ such that

$$\sup_{t \in [0,T]} |\Lambda_t - \Lambda_h^t| \leq C \log(h)^2 \sqrt{h},$$

for $h > 0$ sufficiently small.

**Proof.** Applying Lemma 3.10 with $Y^1 = B - \alpha t^1$ and $Y^2 = B - \alpha t^2$ shows

$$\sup_{t \in [0,T]} |\Gamma^1[t] - \Gamma^2[t]| \leq \alpha \| V_0 - \|_\infty \sup_{t \in [0,T]} |\ell^1_t - \ell^2_t|.$$  \hspace{1cm} (37)

Recalling the notation of the proof of Theorem 3.12 we have $\Lambda_h = \tilde{\Lambda}_h$, since $\Lambda_h$ is a solution to the perturbed Donsker problem and $\Gamma^\alpha[A] = \Lambda$. Moreover,

$$\sup_{t \in [0,T]} |\Lambda_t - \Lambda_h^t| = \sup_{t \in [0,T]} |\Gamma^\alpha[A]_t - \tilde{\Gamma}_h[A]_t|$$

$$\leq \sup_{t \in [0,T]} |\Gamma^\alpha[A]_t - \Gamma^\alpha_h[A]_t| + \sup_{t \in [0,T]} |\Gamma^\alpha_h[A]_t - \tilde{\Gamma}_h[A]_t|$$

$$+ \sup_{t \in [0,T]} |\tilde{\Gamma}_h[A]_t - \Gamma^\alpha_h[A]_t|$$

$$\leq \alpha \| V_0 - \|_\infty \sup_{t \in [0,T]} |\Lambda_t - \Lambda_h^t| + \alpha C \sqrt{h} \log(T/h)$$

$$+ \alpha \| V_0 - \|_\infty \sqrt{h}(2 + \log(h)^2),$$

where the last estimate follows from (37), Lemma 3.4 and the third term is estimated as in the proof of Theorem 3.12. Rearranging terms and absorbing some into the constant $C$, the claim follows. \hfill \Box

**Remark 3.14.** Note that the choice $\sqrt{h} \log(h)^2$ for the perturbation of the initial condition was somewhat arbitrary, since the proofs of the previous theorems work for any function $\delta(h)$ that converges to zero slowly enough such that

$$\lim_{h \to 0} \frac{\sqrt{h} \log(T/h)}{\delta(h)} = 0$$

for every $T > 0$. In particular, we could consider $\sqrt{h} \epsilon \log(h)^2$ for any $\epsilon > 0$, which suggests that the perturbation can be ignored in practice, and we will indeed ignore it in the numerical tests in the next section. It seems natural to conjecture that the result should also hold with $\epsilon = 0$, and this would indeed follow from Conjecture 6.10 in [8], however we do not have a proof at this point.

4 Numerical tests

In this section, we first describe our implementation of the Donsker scheme from Section 3 and then present results from numerical tests. We also compare them with the implicit and explicit time-stepping scheme.
4.1 A tree-type scheme for the Donsker approximation

We consider (18) and specify \( Y_k \) to be Rademacher random variables, i.e. \( \mathbb{P}(Y_k = -1) = \mathbb{P}(Y_k = 1) = 1/2 \). We introduce a time mesh \( t_k = kh, k \geq 0 \) an integer, and

\[
 u^k_i = \mathbb{P}(X^h_{t_k} = i \sqrt{h} + \Lambda^h_{t_k}, r^h > t_k),
\]

for \( i \in \mathbb{Z} \), using the notation defined in Section 3 from which it follows

\[
 \Lambda^h_{t_k} = \alpha \left( 1 - \sum_{i=\lceil t_k \rceil + 1}^{\infty} u^i_k \right), \quad i_k = \left\lfloor \frac{\Lambda^h_{t_k}}{\sqrt{h}} \right\rfloor.
\]

We have that

\[
 u^k_i = \begin{cases}
 \frac{1}{2} u^{k-1}_{i-1} + \frac{1}{2} u^{k-1}_{i+1}, & i > i_k + 1, \\
 \frac{1}{2} u^{k-1}_{i+1}, & i = i_k + 1, \\
 0, & i < i_k + 1,
\end{cases}
\]

for \( k > 0 \). This is an implicit scheme as \( u^k \) and \( \Lambda^h \) are implicitly coupled. The recurrence relation (38) has the resemblance of a binomial tree, shifted by the interaction term, and can be rearranged into the finite difference scheme (2) for recurrence relation (38). It can also be interpreted as a special type of semi-Lagrangian scheme (see [5, 14]) for the forward equation (4).

4.2 Iterative solution

Note that (38) is a non-linear equation through the dependence of \( i_k \) on \( u^k \) via \( \Lambda^h_{t_k} \). Lemma 3.1 suggests a fixed-point iteration to solve simultaneously for \( \Lambda^h_{t_k} \) and the vector \( u^k \) for each \( k \). We assume that we know the cdf of \( X_{0-} \) exactly, and therefore we can calculate,

\[
 \mathbb{P}(X^h_{0-} = i \sqrt{h}) = \mathbb{P}(X_{0-} \in [i \sqrt{h}, (i + 1) \sqrt{h})),
\]

for all \( i \in \mathbb{Z} \). To calculate \( u^0_i \) for \( i \in \mathbb{Z} \), we need to determine \( \Lambda^0_{t_k} \) first, which we obtain as in Lemma 3.1 through the iteration initialized with \( \lambda^0 = 0 \) and

\[
 \lambda^{n+1} = \alpha \sum_{j=0}^{t^n} \mathbb{P}(X^h_{0-} = j \sqrt{h}), \quad t^n = \left\lfloor \frac{\lambda^n}{\sqrt{h}} \right\rfloor.
\]

The iteration terminates when \( t^{n+1} = t^n \), which happens after at most \( \lfloor \alpha / \sqrt{h} \rfloor \) iterations, since \( \lambda^n < \lambda^{n+1} \) and hence \( t^n < t^{n+1} \) as well. This yields \( \Lambda^h_{t_k} \). We then calculate \( u^0_i \) via

\[
 u^0_i = 0 \text{ for } i \leq \left\lfloor \frac{\Lambda^h_{t_k}}{\sqrt{h}} \right\rfloor \quad \text{and} \quad u^0_i = \mathbb{P}(X^h_{0-} = i \sqrt{h}) \text{ for } i > \left\lfloor \frac{\Lambda^h_{t_k}}{\sqrt{h}} \right\rfloor.
\]

The vectors \( u^k = (u^k_i) \) are then calculated recursively through (38), using a local in time version of the iteration in Lemma 3.1 that iterates only over the scalar loss at each time point. Set \( \lambda^0 = \Lambda^0_{t_k-1} \), \( t^0 = \left\lfloor \lambda^0 / \sqrt{h} \right\rfloor \), and for \( n \geq 0 \),

\[
 u^{k,n+1}_i = \begin{cases}
 \frac{1}{2} u^{k-1}_{i-1} + \frac{1}{2} u^{k-1}_{i+1}, & i > t^n + 1, \\
 \frac{1}{2} u^{k-1}_{i+1}, & i = t^n + 1, \\
 0, & i < t^n + 1,
\end{cases}
\]

\[
 \lambda^{n+1} = \alpha \left( 1 - \sum_{i=t^n+1}^{\infty} u^{k,n+1}_i \right), \quad t^{n+1} = \left\lfloor \frac{\lambda^{n+1}}{\sqrt{h}} \right\rfloor.
\]
Expressing this in terms of $\Gamma_h$, if we define

$$
\hat{\Lambda}^{(0)} := \begin{cases}
\Delta_t^h, & t \in [0, t_{k-1}], \\
\Delta_{t_{k-1}}^h, & t \in [t_{k-1}, t_k]
\end{cases}
$$

and set $\hat{\Lambda}^{(n+1)} = \Gamma_h[\hat{\Lambda}^{(n)}]$, then $\Lambda^{n+1}$ above is equal to $\Gamma_h[\hat{\Lambda}^{(n)}]_{t_k}$. We convince ourselves that this computes the minimal solution: As in the case of the initial condition, the iteration is increasing in $n$ for the losses, $\theta^{n+1} \geq \lambda^n$ and $\lambda^{n+1} \geq \lambda^n$, and the iteration terminates in at most $\lceil \alpha/\sqrt{h} \rceil$ iterations (because $\ell^n$ can only take values in $\{t_{k-1}, \ldots, \lfloor \alpha/\sqrt{h} \rfloor\}$). If $n_0$ is the smallest $n$ such that $\theta^{n_0} = \lambda^n$, then $\hat{\Lambda}^{(n_0)}$ solves (15) on $[0, t_k]$, and therefore $\Lambda^{(n_0)} \leq \Lambda^h$ for $t \in [0, t_k]$ by Remark 4.2. On the other hand, since $\Lambda^h$ is increasing, it holds that $\hat{\Lambda}^{(n_0)} \leq \Lambda^h$, and by the monotonicity of $\Gamma_h$ and a straightforward induction it follows that $\Lambda^{(n)} \leq \Lambda^h$ for $n \in \mathbb{N}$ and hence $\hat{\Lambda}^{(n_0)} = \Lambda^h$ for $t \in [0, t_k]$.

4.3 Explicit and implicit particle scheme

Here, we briefly discuss the implementation of the time-stepping scheme specified by (13) and Definition 2.5.

The particle method in [26], which we refer to here as explicit particle scheme, is given by, for a fixed number $n \geq 1$ of particles,

$$
X_{\Delta k \Delta}^{\Delta n,(m)} = X_{0-} + Z_{\Delta k \Delta}^{\Delta n,(m)} - \Lambda_{\Delta k \Delta}^{\Delta n},
$$

$$
\Lambda_{\Delta k \Delta}^{\Delta n} = \frac{\alpha}{n} \sum_{m=1}^{n} \mathbb{1}_{\left\{ \min_{0 \leq i < k} X_{\Delta k \Delta}^{\Delta n,(m)} \right\}},
$$

where we interpret the minimum over an empty set as $\infty$. Similar to Section 4.2 for the fully discrete scheme, $\Lambda^{\Delta}$ is implicitly defined and can be found iteratively:

$$
\Lambda_{\Delta k \Delta}^{\Delta n,(0)} = \frac{\alpha}{n} \sum_{m=1}^{n} \mathbb{1}_{\left\{ \min_{0 \leq i < k} X_{\Delta k \Delta}^{\Delta n,(m)} \right\}},
$$

$$
\Lambda_{\Delta k \Delta}^{\Delta n,(j)} = \frac{\alpha}{n} \sum_{m=1}^{n} \mathbb{1}_{\left\{ \min_{0 \leq i < k} X_{\Delta k \Delta}^{\Delta n,(m)} \right\}},
$$

where we interpret the minimum over an empty set as $\infty$. Similar to Section 4.2 for the fully discrete scheme, $\Lambda^{\Delta}$ is implicitly defined and can be found iteratively:

$$
\Lambda_{\Delta k \Delta}^{\Delta n,(0)} = \frac{\alpha}{n} \sum_{m=1}^{n} \mathbb{1}_{\left\{ \min_{0 \leq i < k} X_{\Delta k \Delta}^{\Delta n,(m)} \right\}},
$$

$$
\Lambda_{\Delta k \Delta}^{\Delta n,(j)} = \frac{\alpha}{n} \sum_{m=1}^{n} \mathbb{1}_{\left\{ \min_{0 \leq i < k} X_{\Delta k \Delta}^{\Delta n,(m)} \right\}},
$$

It is clear that $\Lambda_{\Delta k \Delta}^{\Delta n,(j)}$ is increasing in $j$, and that it terminates in finitely many iterations in a fixed-point, which has to be the minimal solution.

4.4 Numerical results

In this section, we analyse computational aspects and, especially, the numerical accuracy of the scheme.

As a first example, we consider a $\Gamma(k, \theta)$ distribution for $X_{0-}$ with $k = 2$ and $\theta = 1/3$. Note that the initial density $V_{0-}$ is globally Lipschitz in this case. For
simplicity, we here set \( u_0^i = V_{0,i} \). Furthermore, we fix the time interval \([0, T] = [0, 0.02]\).

We first examine iteration (39) for the Donsker scheme. For two different values of \( \alpha \) and an increasing number of time points \( N \) such that \( h = 1/N \), Table 1 gives the number of iterations before termination, first averaged over all time steps, and then the maximum number of iterations for any time step.

| time points \( N \) | 100    | 200    | 400    | 800    | 1600   | 3200   |
|---------------------|--------|--------|--------|--------|--------|--------|
| \( \alpha = 0.5 \)  | 1.0200 | 1.0150 | 1.0125 | 1.0088 | 1.0063 | 1.0047 |
| av. iter.           | 2      | 2      | 2      | 2      | 2      | 2      |
| max. iter.          | 1.2800 | 1.1800 | 1.1175 | 1.0775 | 1.0513 | 1.0341 |
| \( \alpha = 1.5 \)  | 16     | 18     | 20     | 20     | 22     | 23     |
| av. iter.           | 1.2800 | 1.1800 | 1.1175 | 1.0775 | 1.0513 | 1.0341 |
| max. iter.          | 16     | 18     | 20     | 20     | 22     | 23     |

Table 1: Average and maximum number of iterations over all \( N \) time points for \( \alpha = 0.5 \) (no jump) and \( \alpha = 1.5 \) (jump).

In the regular case without a jump (\( \alpha = 0.5 \)), there are never more than 2 iterations needed, while the average number is close to 1. This is explained by the fact that \( i_k \neq i_{k-1} \) only if \( \Lambda_k^h - \Lambda_{k-1}^h > \alpha \sqrt{h} \), so in the regular regime, where \( \Lambda_k^h \) is differentiable, a change of \( i_k \) will only happen every \( O(1/\sqrt{h}) \) time steps, and usually by only 1. Put differently, because of the monotonicity of \( \Lambda_k^h \) in \( k \) and that of the iteration, the total number of iterations summed up over all time steps is bounded by \( \alpha/\sqrt{h} \). So the average number of iterations is \( 1 + O(\sqrt{h}) \).

In the presence of a jump (\( \alpha = 1.5 \)), a larger number of iterations is needed at the time of the jump, but this number only grows mildly under mesh refinement, and on average the number of iterations is still close to 1.

In either case, therefore, the computational cost of the iteration amounts to less than 10% of the overall cost for reasonably fine time meshes.

Figure 1a shows the numerical free boundary \( p_t, \Lambda_k^h/\alpha \), labeled ‘implicit’, and scaled by \( 1/\alpha \) to show how much mass was absorbed at the boundary. The solution is compared to an approximation computed with a simplified scheme where the iteration is stopped after the first iteration. The latter can be considered an explicit treatment where the interaction term computed at each time point is used to compute the density at the following point, and is hence labeled ‘explicit’, in contrast to scheme (38) where \( p_k^h \) and \( \Lambda_k^h \) are implicitly coupled. This reveals that the explicit scheme smoothes out the jump and takes a significant time for the losses to ‘catch up’. In contrast, the implicit scheme (38) reproduces a sharp jump, i.e., for sufficiently small \( h \) the increment \( \Delta_{ik+1}^h - \Delta_{ik}^h \) is small for all but one \( k \), while this largest increase does not go to 0 as \( h \) diminishes, but converges to the true jump size.

Let us now turn to the convergence of \( L^h = \Lambda^h/\alpha \) as the step size \( h \) goes to 0 in the Donsker scheme. Figure 1b shows the error estimator \( 2(L_T^h - L_{2T}^h) \) for decreasing \( h \), for \( \alpha = 0.5 \) and \( \alpha = 1.5 \), and otherwise the same parameters as earlier. In both cases the order of convergence appears to be 0.5, irrespective of the jump that occurs for \( \alpha = 1.5 \).

\[\text{We thank Andreas Søjmark for a discussion on the implicit treatment of jumps.}\]

\[\text{Assuming the error to be } L_T - L_T^h \approx c h^{1/2}, \text{ we find more precisely } L_T^h - L_{2T}^h \approx (\sqrt{2} - 1)(L_T - L_T^h).\]
Figure 1: Convergence of $L^h = \frac{\Delta}{\alpha}$ in the Donsker scheme.

(a) Jump for $\alpha = 1.5$, explicit vs. implicit.  
(b) Implicit Donsker, $\alpha = 0.5$ and $\alpha = 1.5$.

Figure 2: Convergence of $L^\Delta = \frac{\Delta}{\alpha}$ in the time-stepping schemes for $\alpha = 1.5$.

(a) Jump for $\alpha = 1.5$, explicit vs. implicit.  
(b) Mesh convergence, explicit vs. implicit.

We now analyse the same example in the jump regime ($\alpha = 1.5$), for the implicit and explicit Euler-type time-stepping scheme (i.e. without Donsker approximation), whose precise difference is explained in Remark 2.6. In the first two experiments, we fix a seed to generate $n = 100\,000$ sample paths for the particle method detailed in Section 4.3, and vary the number $N$ of time points. The relatively small number of particles is chosen to keep the computational time similar to the Donsker scheme.

In Figure 2a we observe the same phenomenon as in Figure 1a for the Donsker scheme when comparing the resolution of the jump between explicit and implicit schemes. Indeed, the explicit scheme again smoothes out the jump and takes more time to converge, in a way already seen in [26] (see in particular Figure 3 there). More precisely, the implicit scheme with $N = 100$ behaves again similarly to the explicit scheme with $N = 400$ (compare Figure 1a). Concerning the convergence of $L^\Delta = \frac{\Delta}{\alpha}$ (for $\Delta = 1/N$), both the implicit and explicit time-stepping scheme show more irregular behaviour than the implicit Donsker approximation as illustrated in Figure 2b. This is likely to be a consequence of the Monte Carlo error which is quite high due to the relatively small sample size. Figure 3 quantifies this further by showing the error estimator $4|\bar{t}_{\alpha}^{2\Delta} - t_{\alpha}^{\Delta}|$ for
the jump times \( t^\Delta \), where \( t^\Delta \) denotes the jump time for mesh size \( \Delta \), identified by
\[
t^\Delta = \Delta \arg\max_{0 < k \leq N} \{ \Delta^\Lambda_k - \Delta^\Lambda_{k-1} \}.
\]
Also shown is the error estimator \( 4|J^{2\Delta} - J^\Delta| \) for the jump size, where \( J^\Delta = L^\Lambda - L^\Lambda_{\Delta-\Delta} \). Here, we choose between \( N = 2 \) and approximately 2000 timesteps, and 2000\( N \) samples, to reduce the Monte Carlo error together with the time stepping error.

The jump times appear to converge with order 1/2 for both the explicit and implicit scheme, where the implicit scheme is slightly more accurate by a constant factor. \(^6\) This is consistent with the earlier observation in Figure 2a. The simple estimate \( J^\Delta \) of the jump size does not converge to the true jump size of around 0.78 for the explicit scheme, but fluctuates around 0.27. This is due to the fact that we only consider changes over a single time step. In contrast, the jump size in the implicit scheme appears to converge with order 1/2, albeit with relatively high variance.

One key advantage of the Donsker scheme is the avoidance of Monte Carlo sampling, which explains its outperformance over the time-stepping algorithm in terms of computational complexity. We shall therefore focus in the following examples solely on the implicit Donsker approximation, for which we investigate two further cases of intermediate regularity, inspired by \([26]\).

First, we consider the jump regime with \( \alpha = 1.5 \) and vary \( k \) in the \( \Gamma(k, 1/3) \) initial distribution of \( X_0 \) to \( k = 3/2 \) and 5/4, such that the density is only Hölder 1/2 and 1/4, respectively. As seen from Figure 4a, the empirical convergence order is still 0.5 in all cases, even though for small \( N \) the lower regularity of the initial density is noticeable.

Second, we consider the initial density
\[
V_{0-}(x) = \begin{cases} 
\frac{1}{\alpha} - cx^a, & 0 \leq x \leq A, \\
0, & x > A,
\end{cases}
\]
for \( \alpha > 0 \) and \( a > 0 \), which we vary in the tests, and where \( A > 0 \) is determined by \( \int_0^A V_{0-}(x) \, dx = 1 \) for given \( c > 0 \), the latter being sufficiently small. Moreover,

5The factor 4 accounts for extrapolation of the timestepping error and additional Monte Carlo error.
6The missing data points in the implicit case are explained by 0 change of the jump time for coarse time steps and resulting undefined values in the log-log plot.
Different $\Gamma(k, 1/3)$ initial values, alpha = 1.5.

Initial densities as in Figure 4.

Figure 4: Mesh convergence, implicit Donsker.

we let $T = 10^{-4}$ be small enough to precede a possible discontinuity. Here, the convergence order of the explicit time-stepping scheme in [26] is $1/(2(a + 1))$ (see Theorem 1.5 and Table 1 there). Remarkably, the asymptotic order of our scheme appears to be 0.5 irrespective of $a$, see Figure 4b.

References

[1] T. Aiki and H. Imai. Blow-up points to one phase Stefan problems with Dirichlet boundary conditions. In Modelling and Optimization of Distributed Parameter Systems Applications to Engineering, pages 83–89. Springer, 1996.

[2] J. M. Back, S.W. McCue, and T.J. Moroney. Numerical study of two ill-posed one phase Stefan problems. ANZIAM J., 52:C430–C446, 2010.

[3] C. Bender. Simple arbitrage. Ann. Appl. Probab., 22(5):2067 – 2085, 2012.

[4] J. Caldwell and Y.Y. Kwan. Numerical methods for one-dimensional Stefan problems. Comm. Numer. Meth. Eng., 20(7):535–545, 2004.

[5] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes. ESAIM: Math. Mod. Numer. Anal., 29(1):97–122, 1995.

[6] M. Csörgő and L. Horváth. Weighted Approximations in Probability and Statistics. Wiley Series in Probability and Statistics. Wiley, 1993.

[7] C. Cuchiero, C. Reisinger, and S. Rigger. Optimal bailout strategies resulting from the drift controlled supercooled Stefan problem. arXiv preprint arXiv:2111.01783, 2021.

[8] C. Cuchiero, S. Rigger, and S. Svaluto-Ferro. Propagation of minimality in the supercooled Stefan problem. Ann. Appl. Probab., in press.

[9] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré. Particle systems with a singular mean-field self-excitation. Application to neuronal networks. Stoch. Proc. Appl., 125(6):2451–2492, 2015.

[10] F. Delarue, S. Nadtochiy, and M. Shkolnikov. Global solutions to the supercooled Stefan problem with blow-ups: regularity and uniqueness. arXiv preprint arXiv:1902.05174, 2019.
[11] J.N. Dewynne, S.D. Howison, J.R. Ockendon, and W.Q. Xie. Asymptotic behavior of solutions to the Stefan problem with a kinetic condition at the free boundary. *J. Austral. Math. Soc. Ser. B*, 31(1):81–96, 1989.

[12] E. DiBenedetto and A. Friedman. The ill-posed Hele–Shaw model and the Stefan problem for supercooled water. *Trans. Amer. Math. Soc.*, 282(1):183–204, 1984.

[13] R. M. Dudley. *Real Analysis and Probability: 0*. Chapman and Hall/CRC, 2018.

[14] M. Falcone and R. Ferretti. *Semi-Lagrangian approximation schemes for linear and Hamilton–Jacobi equations*. SIAM, 2013.

[15] A. Fasano, G.H. Meyer, and M. Primicerio. On a problem in the polymer industry: theoretical and numerical investigation of swelling. *SIAM J. Math. Anal.*, 17(4):945–960, 1986.

[16] A. Fasano and M. Primicerio. New results on some classical parabolic free-boundary problems. *Q. Appl. Math.*, 38(4):439–460, 1980/81.

[17] A. Fasano and M. Primicerio. A critical case for the solvability of Stefan-like problems. *Math. Methods Appl. Sci.*, 5(1):84–96, 1983.

[18] A. Fasano, M. Primicerio, S.D. Howison, and J.R. Ockendon. On the singularities of one-dimensional Stefan problems with supercooling. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 215–226. Birkhäuser, Basel, 1989.

[19] A. Fasano, M. Primicerio, S.D. Howison, and J.R. Ockendon. Some remarks on the regularization of supercooled one-phase Stefan problems in one dimension. *Quarterly of applied mathematics*, 48(1):153–168, 1990.

[20] B. Hambly, S. Ledger, and A. Søjmark. A McKean–Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.*, 29(4):2338–2373, 08 2019.

[21] B. Hambly and A. Søjmark. An SPDE model for systemic risk with endogenous contagion. *Financ. Stoch.*, 23(3):535–594, 2019.

[22] M.A. Herrero and J.L. Velázquez. Singularity formation in the one-dimensional supercooled Stefan problem. *Eur. J. Appl. Math.*, 7(2):119–150, 1996.

[23] S.D. Howison, J.R. Ockendon, and A.A. Lacey. Singularity development in moving-boundary problems. *Quart. J. Mech. Appl. Math.*, 38(3):343–360, 1985.

[24] S.D. Howison and W.Q. Xie. Kinetic undercooling regularization of supercooled Stefan problems. In *Mathematical models for phase change problems (Óbidos, 1988)*, volume 88 of *Internat. Ser. Numer. Math.*, pages 227–237. Birkhäuser, Basel, 1989.

[25] V. Kaushansky and C. Reisinger. Simulation of a simple particle systems interacting through hitting times. *Disc. Cont. Dyn. Sys. B*, (10), 2019.

[26] V. Kaushansky, C. Reisinger, M. Shkolnikov, and Z. Q. Song. Convergence of a time-stepping scheme to the free boundary in the supercooled Stefan problem. *Ann. Appl. Probab.*, in press.

[27] J.R. King and J.D. Evans. Regularization by kinetic undercooling of blow-up in the ill-posed Stefan problem. *SIAM J. Appl. Math.*, 65(5):1677–1707, 2005.
A.A. Lacey and J.R. Ockendon. Ill-posed free boundary problems. *Control Cybernet.*, 14(1-3):275–296 (1986), 1985.

S. Ledger and A. Søjmark. Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime. *B. Lond. Math. Soc.*, 52(3):448–463, 2020.

A. Lipton, V. Kaushansky, and C. Reisinger. Semi-analytical solution of a McKean–Vlasov equation with feedback through hitting a boundary. *Eur. J. Appl. Math.*, pages 1–34, 2019.

S.W. McCue, M. El-Hachem, and M.J. Simpson. Traveling waves, blow-up, and extinction in the Fisher–Stefan model. *Stud. Appl. Math.*, 148(2):964–986, 2022.

G.H. Meyer. One-dimensional parabolic free boundary problems. *SIAM Review*, 19(1):17–34, 1977.

S.L. Mitchell and M. Vynnycky. Finite-difference methods with increased accuracy and correct initialization for one-dimensional Stefan problems. *Appl. Math. Comput.*, 215(4):1609–1621, 2009.

S. Nadtochiy and M. Shkolnikov. Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.*, 29(1):89–129, 02 2019.

S. Nadtochiy and M. Shkolnikov. Mean field systems on networks, with singular interaction through hitting times. *Ann. Probab.*, 48(3):1520–1556, 05 2020.

R. Peyre. Fractional Brownian motion satisfies two-way crossing. *Bernoulli*, 23(4B):3571 – 3597, 2017.

B. Sherman. A general one-phase Stefan problem. *Q. Appl. Math.*, 28(3):377–382, 1970.

J. Stefan. Über einige Probleme der Theorie der Wärmeleitung. *Sitzungber.*, *Wien, Akad. Mat. Natur.*, 98:473–484, 1889.

A. Visintin. Stefan problem with a kinetic condition at the free boundary. *Ann. Mat. Pura Appl. (4)*, 146:97–122, 1987.

W. Whitt. *Stochastic-process limits: an introduction to stochastic-process limits and their application to queues*. Springer Science & Business Media, 2002.

W.Q. Xie. The Stefan problem with a kinetic condition at the free boundary. *SIAM J. Math. Anal.*, 21(2):362–373, 1990.