Ph.D. Thesis

Martingale Hardy spaces
and summability of the one dimensional
Vilenkin-Fourier series

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Abstract

The classical theory of Fourier series deals with decomposition of a function into sinusoidal waves. Unlike these continuous waves the Vilenkin (Walsh) functions are rectangular waves. Such waves have already been used frequently in the theory of signal transmission, multiplexing, filtering, image enhancement, codic theory, digital signal processing and pattern recognition. The development of the theory of Vilenkin-Fourier series has been strongly influenced by the classical theory of trigonometric series. Because of this it is inevitable to compare results of Vilenkin series to those on trigonometric series. There are many similarities between these theories, but there exist differences also. Much of these can be explained by modern abstract harmonic analysis, which studies orthonormal systems from the point of view of the structure of a topological group.

In this PhD thesis we discuss, develop and apply this fascinating theory connected to modern harmonic analysis. In particular we make new estimations of Vilenkin-Fourier coefficients and prove some new results concerning boundedness of maximal operators of partial sums. Moreover, we derive necessary and sufficient conditions for the modulus of continuity so that norm convergence of the partial sums is valid and develop new methods to prove Hardy type inequalities for the partial sums with respect to the Vilenkin systems. We also do the similar investigation for the Fejér means. Furthermore, we investigate some Nörlund means but only in the case when their coefficients are monotone. Some well-know examples of Nörlund means are Fejér means, Cesàro means and Nörlund logarithmic means. In addition, we consider Riesz logarithmic means, which are not example of Nörlund means. It is also proved that these results are the best possible in a special sense. As applications both some well-known and new results are pointed out.

This PhD is written as a monograph consisting of four Chapters: Preliminaries, Fourier coefficients and partial sums of Vilenkin-Fourier series on martingale Hardy spaces, Vilenkin-Fejér means on martingale Hardy spaces, Vilenkin-Nörlund means on martingale Hardy spaces. It is based on 15 papers with the candidate as author or coauthor, but also some new results are presented for the first time.

In Chapter 1 we first present some definitions and notations, which are crucial for our further investigations. After that we also define some summability methods and remind about some classical facts and results. We investigate some well-known results and prove new estimates for the kernels of these summability methods, which are very important to prove our main results. Moreover, we define martingale Hardy spaces and construct martingales, which help us to prove sharpness of our main results in the later chapters.

Chapter 2 is devoted to present and prove some new and known results about Vilenkin-Fourier coefficients and partial sums of martingales in Hardy spaces. First, we show that Fourier coefficients of martingales are not uniformly bounded when \( 0 < p < 1 \). By applying these results we prove some known Hardy and Paley type inequalities with a new method. After that we investigate partial sums with respect to Vilenkin systems and prove boundedness of maximal operators of partial sums. Moreover, we find necessary and sufficient...
conditions for the modulus of continuity for which norm convergence of partial sums hold and we present a new proof of a Hardy type inequality for it.

In Chapter 3 we investigate some analogous problems concerning the partial sums of Fejér means. First we consider some weighted maximal operators of Fejér means and prove some boundedness results for them. After that we apply these results to find necessary and sufficient conditions for the modulus of continuity for which norm convergence of Fejér means hold. Finally, we prove some new Hardy type inequalities for Fejér means, which is a main part of this PhD thesis. We also prove sharpness of all our main results in this Chapter.

In Chapter 4 we consider boundedness of maximal operators of Nörlund means. After that we prove some strong convergence theorems for these summability methods. Since Fejér means, Cesàro means are well-know examples of Nörlund means some well-known and new results are pointed out. We also investigate Riesz and Nörlund logarithmic means simultaneously at the end of this chapter.
Preface

This PhD thesis is written as a monograph based on the following publications:

[1] G. Tephnadze, The maximal operators of logarithmic means of one-dimensional Vilenkin-Fourier series, Acta Math. Acad. Paedagog. Nyházi. 27 (2011), no. 2, 245-256.

[2] G. Tephnadze, Fejér means of Vilenkin-Fourier series. Studia Sci. Math. Hungar. 49 (2012), no. 1, 79-90.

[3] G. Tephnadze, A note on the Fourier coefficients and partial sums of Vilenkin-Fourier series. Acta Math. Acad. Paedagog. Nyházi. 28 (2012), no. 2, 167-176.

[4] G. Tephnadze, On the maximal operators of Vilenkin-Fejér means. Turkish J. Math. 37 (2013), no. 2, 308-318.

[5] G. Tephnadze, On the maximal operators of Vilenkin-Fejér means on Hardy spaces. Math. Inequal. Appl. 16 (2013), no. 1, 301-312.

[6] G. Tephnadze, On the Vilenkin-Fourier coefficients, Georgian Math. J. 20 (2013), no. 1, 169-177.

[7] G. Tephnadze, On the partial sums of Vilenkin-Fourier series. J. Contemp. Math. Anal. 49 (2014), no. 1, 23-32.

[8] G. Tephnadze, A note on the norm convergence by Vilenkin-Fejér means. Georgian Math. J. 21 (2014), no. 4, 511–517.

[9] I. Blahota and G. Tephnadze, Strong convergence theorem for Vilenkin-Fejér means. Publ. Math. Debrecen 85 (2014), no. 1-2, 181-196.

[10] L. E. Persson and G. Tephnadze, A note on Vilenkin-Fejér means, on the martingale Hardy space \( H_p \), Bulletin of TICMI 18 (2014), no. 1, 55-64.

[11] L. E. Persson, G. Tephnadze and P. Wall, Maximal operators of Vilenkin-Nörlund means, J. Fourier Anal. Appl. 21 (2015), no. 1, 76-94.

[12] L. E. Persson, G. Tephnadze and P. Wall, Some new \((H_p, L_p)\) type inequalities of maximal operators of Vilenkin-Nörlund means with non-decreasing coefficients, J. Math. Inequal. (to appear).

[13] L. E. Persson and G. Tephnadze, A sharp boundedness result concerning some maximal operators of Vilenkin-Fejér means, Mediterr. J. Math. (to appear).

[14] I. Blahota, L. E. Persson and G. Tephnadze, On the Nörlund means of Vilenkin-Fourier series, Czechoslovak Math. J. (to appear).

[15] I. Blahota and G. Tephnadze, A note on maximal operators of Vilenkin-Nörlund means, Acta Math. Acad. Paedagog. Nyházi. (to appear).

Remark: Also some new results which can not be found in these papers appear in this PhD thesis for the first time.
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1 Preliminaries

1.1 Basic Notations

Denote by $\mathbb{N}_{+}$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_{+} \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the groups $Z_{m_i}$ with the product of the discrete topologies of $Z_{m_i}$.

The direct product $\mu$ of the measures

$$\mu_k(j) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on $G_{m_k}$ with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The elements of $G_m$ are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots) \quad (x_j \in Z_{m_j}) .$$

It is easy to give a base for the neighborhoods of $G_m$:

$$I_0(x) := G_m,$n \in \mathbb{N} .$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x, n \in \mathbb{N}) .$$

Let

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m \quad (n \in \mathbb{N}) .$$

If we define $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I}_n := G_m \setminus I_n$, then

$$\overline{I}_N = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1} = \left( \bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_{k,l}^N \right) \bigcup \left( \bigcup_{k=1}^{N-1} I_{k,N}^N \right) ,$$

(1.1)

where

$$I_{k,l}^N := \begin{cases} I_N(0, \ldots, 0, x_k \neq 0, \ldots, 0, x_l \neq 0, x_{l+1}, \ldots, x_{N-1}, \ldots), \\ \text{for } k < l < N , \\ I_N(0, \ldots, 0, x_k \neq 0, x_{k+1} = 0, \ldots, x_{N-1} = 0, x_N, \ldots), \\ \text{for } l = N . 
\end{cases}$$

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The norm (or quasi-norm when $0 < p < 1$) of the space $L_p(G_m)$ ($0 < p < \infty$) is defined by

$$
\|f\|_p := \left(\int_{G_m} |f|^p \, d\mu\right)^{1/p}.
$$

The space $\text{weak } - L_p(G_m)$ consists of all measurable functions $f$, for which

$$
\|f\|_{\text{weak } - L_p} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < +\infty.
$$

The norm of the space of continuous functions $C(G_m)$ is defined by

$$
\|f\|_C := \sup_{x \in G_m} |f(x)| < \infty.
$$

The best approximation of $f \in L_p(G_m)$ ($1 \leq p < \infty$) is defined as

$$
E_n(f, L_p) := \inf_{\psi \in P_n} \|f - \psi\|_p,
$$

where $P_n$ is set of all Vilenkin polynomials of order less than $n \in \mathbb{N}$.

The modulus of continuity of $f \in L_p(G_m)$ and $f \in C(G_m)$ are defined by

$$
\omega_p \left( \frac{1}{M_n}, f \right) := \sup_{h \in I_n} \|f(\cdot - h) - f(\cdot)\|_p
$$

and

$$
\omega_C \left( \frac{1}{M_n}, f \right) := \sup_{h \in I_n} \|f(\cdot - h) - f(\cdot)\|_C,
$$

respectively.

If we define the so-called generalized number system based on $m$ in the following way :

$$
M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),
$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$
n = \sum_{j=0}^{\infty} n_j M_j,
$$

where $n_j \in \mathbb{Z}_{m_j}$ ($j \in \mathbb{N}_+)$ and only a finite number of $n'_j$’s differ from zero.

Let

$$
\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\} \quad \text{and} \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},
$$

that is $M_{|n|} \leq n \leq M_{|n|+1}$. Set

$$
d(n) := |n| - \langle n \rangle, \quad \text{for all} \quad n \in \mathbb{N}.
For the natural number \( n = \sum_{j=1}^{\infty} n_j M_j \), we define functions \( v \) and \( v^* \) by (for details see Lukomskii [38])

\[
v(n) := \sum_{j=1}^{\infty} |\delta_{j+1} - \delta_j| + \delta_0, \quad v^*(n) := \sum_{j=1}^{\infty} \delta_j^*,
\]

where

\[
\delta_j = \text{sign}(n_j) = \text{sign}(\ominus n_j) \quad \text{and} \quad \delta_j^* = |\ominus n_j - 1| \delta_j
\]

and \( \ominus \) is the inverse operation for

\[
a_k \oplus b_k := (a_k + b_k) \mod m_k.
\]

Next, we introduce on \( G_m \) an orthonormal systems, which are called Vilenkin systems.

At first, we define the complex-valued function \( r_k(x) : G_m \to \mathbb{C} \), the generalized Rademacher functions, by

\[
r_k(x) := \exp \left( 2\pi i x_k/m_k \right), \quad (i^2 = -1, x \in G_m, \ k \in \mathbb{N})
\]

Now, define Vilenkin systems \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) as:

\[
\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).
\]

The Vilenkin systems are orthonormal and complete in \( L_2(G_m) \) (see e.g. Vilenkin [77]). Specifically, we call this system the Walsh-Paley system when \( m \equiv 2 \).

Next, we introduce some analogues of the usual definitions in Fourier-analysis. If \( f \in L_1(G_m) \) we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to Vilenkin systems in the usual manner:

\[
\hat{f}(n) := \int_{G_m} f(x) \psi_n(x) d\mu, \quad (n \in \mathbb{N}),
\]

\[
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+),
\]

\[
D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+),
\]

respectively.

The \( n \)-th Lebesgue constant is defined in the following way:

\[
L_n := \|D_n\|_1.
\]
1.2 Definition and Examples of Nörlund Means and Its Maximal Operators

Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of nonnegative numbers. The \( n \)-th Nörlund means for the Fourier series of \( f \) is defined by

\[
t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,
\]

where

\[
Q_n := n - 1 \sum_{k=0}^{n-1} q_k.
\]

We always assume that \( q_0 > 0 \) and

\[
\lim_{n \to \infty} Q_n = \infty.
\]

In this case it is well-known that the summability method generated by \( \{q_k : k \geq 0\} \) is regular if and only if

\[
\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0.
\]

Concerning this fact and related basic results, we refer to [40].

The next remark is due to Persson, Tephnadze and Wall [51]:

**Remark 1.1**

a) Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-increasing. Then the summability method generated by \( \{q_k : k \in \mathbb{N}\} \) is regular.

b) Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing. Then the summability method generated by \( \{q_k : k \in \mathbb{N}\} \) is not always regular.

**Proof:** Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-increasing. Then

\[
\frac{q_{n-1}}{Q_n} \leq \frac{q_{n-1}}{nq_{n-1}} = \frac{1}{n} \to 0, \text{ when } n \to \infty.
\]

According to [135] we conclude that in this case the summability method is regular.

Now, we prove part b) of theorem and construct Nörlund mean, with non-decreasing coefficients \( \{q_k : k \in \mathbb{N}\} \), which is not regular.

Let \( \{q_k = 2^k : k \in \mathbb{N}\} \). Then

\[
Q_n = \sum_{k=0}^{n-1} 2^k = 2^n - 1 \leq 2^n
\]
and
\[ \frac{q_{n-1}}{Q_n} = \frac{2^{n-1}}{2^{n-1}-1} \geq \frac{2^{n-1}}{2^n} = \frac{1}{2} \not\to 0, \text{ when } n \to \infty. \]

By using again (1.3) we obtain that when the sequence \( \{q_k : k \in \mathbb{N}\} \) is non-decreasing, then the summability method is not always regular.

The proof is complete. \( \square \)

Let \( t_n \) be Nörlund means with monotone and bounded sequence \( \{q_k : k \in \mathbb{N}\} \), such that
\[ q := \lim_{n \to \infty} q_n > c > 0. \]

Then, if the sequence \( \{q_k : k \in \mathbb{N}\} \) is non-decreasing, we get that
\[ nq_0 \leq Q_n \leq nq. \]

In the case when the sequence \( \{q_k : k \in \mathbb{N}\} \) is non-increasing, then
\[ nq \leq Q_n \leq nq_0. \] (1.4)

In both cases we can conclude that
\[ \frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \text{ when } n \to \infty. \] (1.5)

In the special case when \( \{q_k = 1 : k \in \mathbb{N}\} \), we get Fejér means
\[ \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f. \]

The \((C, \alpha)\)-means (Cesàro means) of the Vilenkin-Fourier series are defined by
\[ \sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f, \]
where
\[ A_0^\alpha := 0, \quad A_n^\alpha := \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, \ldots \]

It is well-known that (see e.g. Zygmund [88])
\[ A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \] (1.6)
\[ A_n^\alpha - A_{n-1}^\alpha = A_{n-1}^{\alpha-1}, \quad A_n^\alpha \sim n^\alpha. \] (1.7)
In the literature, there is the notion of Riesz means \( (R, \alpha) \text{-means} \) of the Fourier series. Let \( \beta^n \) denote the Nörlund mean, where

\[
\{ q_0 = 1, \ q_k = k^{\alpha-1} : k \in \mathbb{N}_+ \},
\]

that is

\[
\beta^n f := \frac{1}{Q_n} \sum_{k=1}^{n} (n - k)^{\alpha-1} S_k f, \quad 0 < \alpha < 1.
\]

It is obvious that

\[
\left| q_n - q_{n+1} \right| / n^{\alpha-2} = O(1), \quad \text{when} \ n \to \infty.
\] (1.8)

and

\[
q_0 / Q_n = O \left( \frac{1}{n^\alpha} \right), \quad \text{when} \ n \to \infty.
\] (1.9)

The \( n \)-th Nörlund logarithmic mean \( L_n \) and the Riesz logarithmic mean \( R_n \) are defined by

\[
L_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n - k}, \quad R_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{k},
\]

respectively, where

\[
l_n := \sum_{k=1}^{n-1} \frac{1}{k}.
\]

Up to now we have considered Nörlund mean in the case when the sequence \( \{ q_k : k \in \mathbb{N} \} \) is bounded but now we consider Nörlund summabilities with unbounded sequence \( \{ q_k : k \in \mathbb{N} \} \).

Let \( \alpha \in \mathbb{R}_+, \beta \in \mathbb{N}_+ \) and

\[
\log^{(\beta)} x := \underbrace{\log \ldots \log}_\beta x.
\]

If we define the sequence \( \{ q_k : k \in \mathbb{N} \} \) by

\[
\left\{ q_0 = 0 \text{ and } q_k = \log^{(\beta)} k^\alpha : k \in \mathbb{N}_+ \right\},
\]

then we get the class of Nörlund means with non-decreasing coefficients:

\[
\kappa_{n,\beta} f := \frac{1}{Q_n} \sum_{k=1}^{n} \log^{(\beta)} (n - k)^\alpha S_k f.
\]

First we note that \( \kappa_{n,\beta} \) are well-defined for every \( n \in \mathbb{N}_+ \), if we rewrite them as:

\[
\kappa_{n,\beta} f := \sum_{k=1}^{n} \frac{\log^{(\beta)} (n - k)^\alpha}{Q_n} S_k f.
\]
It is obvious that

\[ \frac{n}{2} \log^{(\beta)} \frac{n^\alpha}{2\alpha} \leq Q_n \leq n \log^{(\beta)} n^\alpha. \]

It follows that

\[ \frac{q_{n-1}}{Q_n} \leq \frac{c \log^{(\beta)} (n-1)^\alpha}{n \log^{(\beta)} n^\alpha} = O \left( \frac{1}{n} \right) \to 0, \text{ when } n \to \infty. \] (1.10)

For the function \( f \) we consider the following maximal operators:

\[
S^* f := \sup_{n \in \mathbb{N}} |S_n f|, \quad t^* f := \sup_{n \in \mathbb{N}} |t_n f|, \quad \sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|,
\]

\[
\sigma^\alpha, \ast f := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f|, \quad R^* f := \sup_{n \in \mathbb{N}} |R_n f|, \quad L^* f := \sup_{n \in \mathbb{N}} |L_n f|,
\]

\[
\kappa^\alpha, \beta, \ast f := \sup_{n \in \mathbb{N}} |\kappa_n^\alpha, \beta f|, \quad \beta^\alpha, \ast f := \sup_{n \in \mathbb{N}} |\beta_n^\alpha f|.
\]

We also consider the following weighted maximal operators:

\[
\widetilde{t}^*_{p, \alpha} f := \sup_{n \in \mathbb{N}} \frac{|t_n f|}{(n+1)^{1/p-1-\alpha}}, \quad (0 < p < 1/(1+\alpha), \ 0 < \alpha \leq 1),
\]

\[
\widetilde{t}_\alpha^* f := \sup_{n \in \mathbb{N}} \frac{|t_n f|}{n \log^{1+\alpha}(n+1)}, \quad (0 < \alpha \leq 1),
\]

\[
\widetilde{\sigma}^\alpha, \ast f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n^\alpha f|}{(n+1)^{1/p-1-\alpha}}, \quad (0 < p < 1/(1+\alpha), \ 0 < \alpha \leq 1),
\]

\[
\widetilde{\beta}^\alpha, \ast f := \sup_{n \in \mathbb{N}} \frac{|\beta_n^\alpha f|}{(n+1)^{1/p-1-\alpha}}, \quad (0 < p < 1/(1+\alpha), \ 0 < \alpha \leq 1),
\]

\[
\tilde{\sigma}^\alpha, \ast f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n^\alpha f|}{\log^{1+\alpha}(n+1)}, \quad \tilde{\beta}^\alpha, \ast f := \sup_{n \in \mathbb{N}} \frac{|\beta_n^\alpha f|}{\log^{1+\alpha}(n+1)}, \quad (0 < \alpha < 1),
\]

\[
\tilde{\sigma}_p := \sup_{n \in \mathbb{N}} \frac{|\sigma_n|}{(n+1)^{1/p-2}}, \quad \tilde{\kappa}^\alpha, \beta, \ast f := \sup_{n \in \mathbb{N}} \frac{|\kappa_n^\alpha, \beta f|}{(n+1)^{1/p-2}}, \quad (0 < p < 1/2),
\]

and

\[
\tilde{\sigma}^* f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\log^2(n+1)}, \quad \tilde{\kappa}^\alpha, \beta, \ast f := \sup_{n \in \mathbb{N}} \frac{|\kappa_n^\alpha, \beta f|}{\log^2(n+1)}.
\]
1.3 Dirichlet Kernels and Lebesgue Constants with Respect to Vilenkin Systems

It is easy to see that

\[ S_n f (x) = \int_{G_m} f (t) \sum_{k=0}^{n-1} \psi_k (x - t) d\mu (t) \]

\[ = \int_{G_m} f (t) D_n (x - t) d\mu (t) = (f * D_n) (x). \]

The next well-known identities with respect to Dirichlet kernels (see Lemmas 1.2 and 1.3, Corollaries 1.4 and 1.5) will be used many time in the proofs of our main results. The first equality can be found in Vilenkin [77] and the second identity in Gát and Goginava [19]:

**Lemma 1.2** Let \( n \in \mathbb{N}, 1 \leq s_n \leq m_n - 1 \). Then

\[ D_{j+M_n} = D_{M_n} + \psi_{M_n} D_j = D_{M_n} + r_n D_j, \quad j \leq (m_n - 1) M_n \]  

(1.11)

and

\[ D_{s_n M_n - j} = D_{s_n M_n} - \psi_{s_n M_n - 1} D_j, \quad j < M_n. \]  

(1.12)

**Lemma 1.3** Let \( n \in \mathbb{N} \) and \( 1 \leq s_n \leq m_n - 1 \). Then

\[ D_{s_n M_n} = D_{M_n} \sum_{k=0}^{s_n-1} \psi_{k M_n} = D_{M_n} \sum_{k=0}^{s_n-1} r_k^n \]  

(1.13)

and

\[ D_n = \psi_n \left( \sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j-n_j}^{m_{j-1}} r_{j}^k \right), \]  

(1.14)

for \( n = \sum_{i=0}^{\infty} n_i M_i \).

**Corollary 1.4** Let \( n \in \mathbb{N} \). Then

\[ D_{M_n+1} = \prod_{k=0}^{n} \left( \sum_{s=0}^{m_{k-1}} r_{k}^s \right). \]

**Corollary 1.5** Let \( n \in \mathbb{N} \). Then

\[ D_{M_n} (x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases} \]

We also need the following estimate (see Tephnadze [71]):
Lemma 1.6 Let \( x \in I_s \setminus I_{s+1}, s = 0, \ldots, N - 1. \) Then
\[
\int_{I_N} |D_n (x - t)| \, d\mu (t) \leq \frac{cM_s}{M_N},
\]
where \( c \) is an absolute constant.

**Proof:** By combining (1.14) in Lemma 1.3 and Corollary 1.5 we have that
\[
|D_n (x)| \leq \sum_{j=0}^{l} n_j D_{M_j} (x) = \sum_{j=0}^{l} n_j M_j \leq cM_l.
\]

Since \( t \in I_N \) and \( x \in I_s \setminus I_{s+1}, s = 0, \ldots, N - 1, \) we obtain that \( x - t \in I_s \setminus I_{s+1}. \) By using the estimate above we get that
\[
|D_n (x - t)| \leq cM_s
\]
and
\[
\int_{I_N} |D_n (x - t)| \, d\mu (t) \leq \frac{cM_s}{M_N}.
\]

The proof is complete.

To study the Dirichlet kernels we need an estimate of some sums of Rademacher functions. This Lemma can be found in Persson and Tephnadze [53].

Lemma 1.7 Let \( n \in \mathbb{N}, \) and \( x_n = 1. \) Then
\[
\left| \sum_{u=0}^{s_n-1} r_n^u (x) \right| \geq 1, \text{ for some } 1 \leq s_n \leq m_n - 1
\]
and
\[
\left| \sum_{u=1}^{s_n-1} r_n^u (x) \right| \geq 1, \text{ for some } 2 \leq s_n \leq m_n - 1.
\]

**Proof:** Let \( x_n = 1. \) Then we readily get that
\[
\left| \sum_{u=0}^{s_n-1} r_n^u (x) \right| = \left| \frac{r_n^{s_n} (x) - 1}{r_n (x) - 1} \right| = \frac{\sin (\pi s_n x_n / m_n)}{\sin (\pi x_n / m_n)} = \frac{\sin (\pi s_n / m_n)}{\sin (\pi / m_n)} \geq 1.
\]
Lemma 1.8

Let

\[ \sum_{u=1}^{s_n-1} r_n^u (x) = \left| r_n (x) \frac{r_n^{s_n-1} (x) - 1}{r_n (x) - 1} \right| = \frac{\sin \left( \pi (s_n - 1) / m_n \right)}{\sin \left( \pi / m_n \right)} \geq 1. \]

Analogously, we can prove that

\[ |D_n| = |D_{n-M[n]}| \geq M_{(n)}. \]

Proof: Let \( x \in I_{(n)} \setminus I_{(n)+1} \). Since

\[ n = n_{(n)} M_{(n)} + \sum_{j=(n)}^{\lfloor n/2 \rfloor - 1} n_j M_j + n_{[n]} M_{[n]} \]

and

\[ n - M_{[n]} = n_{(n)} M_{(n)} + \sum_{j=(n)}^{\lfloor n/2 \rfloor - 1} n_j M_j + \left( n_{[n]} - 1 \right) M_{[n]}, \]

if we apply Lemma 1.7, Corollary 1.5 and (1.14) in Lemma 1.3 we can conclude that

\[ \left| D_{n-M[n]} \right| \geq \left| D_{M_{(n)}} \right| \sum_{s=m_{(n)} - n_{(n)}}^{m_{(n)} - 1} r_{(n)}^s - \left| D_{M_{(n)}} \sum_{j=(n)+1}^{\lfloor n/2 \rfloor} D_{M_j} \sum_{s=m_j-n_j}^{m_{(n)} - 1} r_{(n)}^s \right| \]

\[ = D_{M_{(n)}} \left| \sum_{s=m_{(n)} - n_{(n)}}^{m_{(n)} - 1} r_{(n)}^s \right| = D_{M_{(n)}} r_{(n)}^{m_{(n)} - \alpha_{(n)}} \sum_{s=0}^{n_{(n)}-1} r_{(n)}^s \geq D_{M_{(n)}} \geq M_{(n)}. \]

Analogously we can show that

\[ |D_n| = D_{M_{(n)}} \sum_{s=0}^{n_{(n)}} r_{(n)}^s \geq M_{(n)} \]

and the proof is complete.

The next Lemma can be found in Tephnadze [71]:

Lemma 1.8 Let \( n \in \mathbb{N} \), \( |n| \neq \langle n \rangle \) and \( x \in I_{(n)} \setminus I_{(n)+1} \). Then

\[ |D_n| = |D_{n-M[n]}| \geq M_{(n)}. \]

The proof is complete. 

The next Lemma can be found in Lukomskii [38]:
Lemma 1.9 Let \( n = \sum_{i=1}^{\infty} n_i M_i \). Then
\[
\frac{1}{4\lambda} v(n) + \frac{1}{\lambda} v^*(n) + \frac{1}{2\lambda} \leq L_n \leq \frac{3}{2} v(n) + 4v^*(n) - 1,
\]
where \( \lambda := \sup_{n \in \mathbb{N}} m_n \).

The next result for the Walsh system can be found in the book [55] and for bounded Vilenkin systems in the book [2].

Corollary 1.10 Let \( q_{nk} = M_{2nk} + M_{2nk-2} + M_2 + M_0 \). Then
\[
\frac{n_k}{2\lambda} \leq \| D_{q_{nk}} \|_1 \leq \lambda n_k,
\]
where \( \lambda := \sup_{n \in \mathbb{N}} m_n \).

Proof: The proof readily follows by just using Theorem 1.9 and the following identity \( v(q_{nk}) = 2n_k \). Thus, we leave out the details.

1.4 Fejér Kernels with respect to Vilenkin systems

It is obvious that
\[
\sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} (D_k * f)(x)
\]
\[
= (f * K_n)(x) = \int_{G_m} f(t) K_n(x-t) \, d\mu(t).
\]

We frequently use the following well-known result, which was proved in Gát [17]:

Lemma 1.11 Let \( n > t, t, n \in \mathbb{N} \). Then
\[
K_{M_n}(x) = \begin{cases} 
M_{\frac{1-r_t(x)}{2}}, & x \in I_t \setminus I_{t+1}, x - x_t e_t \in I_n, \\
M_{\frac{1+r_t(x)}{2}}, & x \in I_n, \\
0, & \text{otherwise}.
\end{cases}
\]

The proof of the next lemma can easily be done by using last lemma (c.f. also the book [2] and Tephnadze [67, 68]):
Lemma 1.12 Let $n \in \mathbb{N}$ and $x \in I_N^{k,l}$, where $k < l$. Then

$$K_{M_n}(x) = 0, \text{ if } n > l.$$  \hspace{1cm} (1.15)

and

$$|K_{M_n}(x)| \leq cM_k.$$  \hspace{1cm} (1.16)

Moreover,

$$\int_{G_m} |K_{M_n}(x)| \, d\mu \leq c< \infty,$$  \hspace{1cm} (1.17)

where $c$ is an absolute constant.

We also need the following useful result:

Lemma 1.13 Let $t, s_n, n \in \mathbb{N}$, and $1 \leq s_n \leq m_n - 1$. Then

$$s_n M_n K_{s_n M_n} = \sum_{l=0}^{s_n-1} \left( \sum_{i=0}^{l-1} r_n^i \right) M_n D_{M_n} + \left( \sum_{l=0}^{s_n-1} r_n^l \right) M_n K_{M_n}$$  \hspace{1cm} (1.18)

and

$$|K_{s_n M_n}(x)| \geq \frac{M_n}{2\pi s_n}, \text{ for } x \in I_{n+1} \left(e_{n-1} + e_n\right).$$  \hspace{1cm} (1.19)

Moreover, if $x \in I_t \setminus I_{t+1}$, $x - x_t e_t \notin I_n$ and $n > t$, then

$$K_{s_n M_n}(x) = 0.$$

Remark 1.14 This result was proved by Blahota and Tephnadze [5], but here we will give a completely different and simpler proof.

Proof: We can write that

$$s_n M_n K_{s_n M_n} = \sum_{l=0}^{s_n-1} \sum_{k=lm_n}^{(l+1)m_n-1} D_k$$  \hspace{1cm} (1.20)

$$= \sum_{l=0}^{s_n-1} \sum_{k=lm_n}^{(l+1)m_n-1} D_k = \sum_{l=0}^{s_n-1} M_n D_{k+lm_n}.$$

Let $0 \leq k < M_n$. Then, by using (1.13) in Lemma 1.2 we have that

$$D_{k+lm_n} = \sum_{m=0}^{lM_n-1} \psi_m + \sum_{m=lM_n}^{lM_n+k-1} \psi_m.$$
\[ = D_{lM_n} + \sum_{m=0}^{k-1} \psi_{m+lM_n} = D_{lM_n} + r_n^l \sum_{m=0}^{k-1} \psi_m + r_n^l D_k. \]

According to (1.20) we readily get that

\[ s_n M_n K s_n M_n = \sum_{l=0}^{s_n-1} \sum_{k=0}^{M_n-1} D_{k+lM_n} \]

\[ = \sum_{l=0}^{s_n-1} M_n D_{M_n} + \left( \sum_{l=0}^{s_n-1} \sum_{k=0}^{M_n-1} D_k \right) \]

\[ = \sum_{l=0}^{s_n-1} M_n D_{M_n} + M_n K_{M_n}. \]

The first part of the Lemma is proved.

Now, let \( t, s_n, n \in \mathbb{N}, n > t, x \in I_t \setminus I_{t+1} \). If \( x - x_t e_t \notin I_n \), then by combining Corollary 1.5 and Lemma 1.11 we obtain that

\[ D_{M_n}(x) = K_{M_n}(x) = 0. \] (1.21)

Let \( x \in I_{n+1} \left( e_{n-1} + e_n \right) \). By Lemma 1.11 we have that

\[ |K_{M_n}(x)| = \frac{M_{n-1}}{|1 - r_{n-1}(x)|} = \frac{M_{n-1}}{2 \sin \pi/m_{n-1}}. \]

By combining Lemmas 1.7 and 1.11 and the first part of Lemma 1.13 we immediately get that

\[ |s_n M_n K_{s_n M_n}(x)| = \left| \sum_{l=0}^{s_n-1} r_n^l (x) M_n K_{M_n}(x) \right| \]

\[ \frac{M_n M_{n-1}}{2 \sin \pi/m_{n-1}} \geq \frac{M_n M_{n-1} m_{n-1}}{2 \pi} \geq \frac{M_n^2}{2 \pi}. \]

Now, let \( t, s_n, n \in \mathbb{N}, n > t, x \in I_t \setminus I_{t+1} \). If \( x - x_t e_t \notin I_n \), then, by using the first part of Lemma 1.13 with identities (1.21) we immediately get that

\[ K_{s_n M_n}(x) = 0. \]

The proof is complete. 

In the same paper Blahota and Tephnadze [5] also proved the following result:
Lemma 1.15 Let \( n = \sum_{i=1}^{r} s_{ni} M_{ni} \), where \( n_1 > n_2 > \cdots > n_r \geq 0 \) and \( 1 \leq s_{ni} < m_{ni} \) for all \( 1 \leq i \leq r \) as well as \( n^{(k)} = n - \sum_{i=1}^{k} s_{ni} M_{ni} \), where \( 0 < k \leq r \). Then

\[
nK_n = \sum_{k=1}^{r} \left( \prod_{j=1}^{r} r_{nj}^{s_{nj}} \right) s_{nk} M_{nk} K_{sn_k M_{nk}} + \sum_{k=1}^{r-1} \left( \prod_{j=1}^{r} r_{nj}^{s_{nj}} \right) n^{(k)} D_{sn_k M_{nk}}.
\]

Proof: Let \( k, n \in \mathbb{N} \), \( 0 \leq k < M_n \). If we use the identity (1.11) in Lemma 1.2 we readily get that

\[
nK_n = \sum_{k=1}^{n} D_k = \sum_{k=1}^{n} D_k + \sum_{k=sn_1 M_{n_1} + 1}^{n} D_k
\]

\[
= s_{n_1} M_{n_1} K_{sn_1 M_{n_1}} + \sum_{k=1}^{n^{(1)}} D_{k+sn_1 M_{n_1}}
\]

\[
= s_{n_1} M_{n_1} K_{sn_1 M_{n_1}} + \sum_{k=1}^{n^{(1)}} (D_{sn_1 M_{n_1}} + r_{sn_1 M_{n_1}})
\]

\[
= s_{n_1} M_{n_1} K_{sn_1 M_{n_1}} + n^{(1)} D_{sn_1 M_{n_1}} + r_{sn_1 M_{n_1}}
\]

If we calculate \( n^{(1)} K_{n^{(1)}} \) in similar way, we get that

\[
n^{(1)} K_{n^{(1)}} = s_{n_2} M_{n_2} K_{sn_2 M_{n_2}} + n^{(2)} D_{sn_2 M_{n_2}} + r_{sn_2 M_{n_2}}
\]

so

\[
nK_n = s_{n_1} M_{n_1} K_{sn_1 M_{n_1}} + r_{sn_1 M_{n_1}} + n^{(1)} D_{sn_1 M_{n_1}} + r_{sn_1 M_{n_1}}
\]

By using this method successively with \( n^{(2)} K_{n^{(2)}}, \ldots, n^{(r-1)} K_{n^{(r-1)}} \), we obtain that

\[
nK_n = \sum_{k=1}^{r} \left( \prod_{j=1}^{r} r_{nj}^{s_{nj}} \right) s_{nk} M_{nk} K_{sn_k M_{nk}}
\]

\[
+ \left( \prod_{j=1}^{r} r_{nj}^{s_{nj}} \right) n^{(r)} K_{n^{(r)}} + \sum_{k=1}^{r-1} \left( \prod_{j=1}^{r} r_{nj}^{s_{nj}} \right) n^{(k)} D_{sn_k M_{nk}}.
\]

Since \( n^{(r)} = 0 \) we conclude that the proof is complete.

Corollary 1.16 Let \( n \in \mathbb{N} \). Then

\[
n | K_n | \leq c \sum_{l=0}^{n} M_l | K_{M_l} | \leq c \sum_{l=0}^{n} M_l | K_{M_l} | \quad (1.22)
\]
and
\[ \sup_n \int_{G_m} |K_n| \, d\mu \leq c < \infty, \] (1.23)
where \( c \) is an absolute constant.

**Remark 1.17** Corollary [L.16] is known (see the book [2]), but it is also a simple consequence of Lemmas [L.12] and [L.15].

The next lemma can be found as a part of a more general result by Persson and Tephnadze [53]:

**Lemma 1.18** Let \( n \in \mathbb{N}, \langle n \rangle \neq |n| \) and \( x \in I_{\langle n \rangle + 1} (e_{\langle n \rangle} - 1 + e_{\langle n \rangle}) \). Then
\[ |nK_n| = \left| (n - M_{|n|}) K_{M_{|n|}} \right| \geq \frac{M_{\langle n \rangle}^2}{2\pi \lambda}, \]
where \( \lambda := \sup m_n \).

**Proof:** Let \( x \in I_{\langle n \rangle + 1}^{(n)-1,(n)} \). Since
\[ n = n_{\langle n \rangle} M_{\langle n \rangle} + \sum_{j=\langle n \rangle}^{|n|-1} n_j M_j + n_{|n|} M_{|n|} \]
and
\[ n - M_{|n|} = n_{\langle n \rangle} M_{\langle n \rangle} + \sum_{j=\langle n \rangle}^{|n|-1} n_j M_j + (n_{|n|} - 1) M_{|n|}, \]
if we combine (1.13), (1.19) and invoke Corollary 1.5, Lemmas 1.11 and 1.15 we obtain that
\[ n \left| K_n \right| = (n - M_{|n|}) \left| K_{n - M_{|n|}} \right| \]
\[ = \left| \prod_{j=1}^{\langle n \rangle - 1} \psi_{M_{n_j}}^j \right| s_{\langle n \rangle} M_{\langle n \rangle} K_{s_{\langle n \rangle} M_{\langle n \rangle}} \right| = \left| s_{\langle n \rangle} M_{\langle n \rangle} K_{s_{\langle n \rangle} M_{\langle n \rangle}} \right| \]
\[ = \left| \sum_{l=0}^{s_{\langle n \rangle}} \left( \sum_{i=0}^{l-1} r_{\langle n \rangle}^i \right) M_{\langle n \rangle} D_{M_{\langle n \rangle}} + \left( \sum_{l=0}^{s_{\langle n \rangle} - 1} r_{\langle n \rangle}^l \right) M_{\langle n \rangle} K_{M_{\langle n \rangle}} \right| \]
\[ = \left| \sum_{l=0}^{s_{\langle n \rangle} - 1} r_{\langle n \rangle}^l M_{\langle n \rangle} K_{M_{\langle n \rangle}} \right| \geq \left| M_{\langle n \rangle} K_{M_{\langle n \rangle}} \right| \geq \frac{M_{\langle n \rangle}^2}{2\pi \lambda}. \]

The proof is complete.

The next result is proved in Blahota, Gát and Goginava [9] and [10], (see also Tephnadze [70]):

\[ G. \text{Tephnadze} \]
Lemma 1.19 Let \( 2 < n \in \mathbb{N}_+ \), \( k \leq s < n \) and \( q_n = M_{2n} + M_{2n-2} + \ldots + M_2 + M_0 \). Then
\[
q_{n-1} |K_{q_{n-1}}(x)| \geq \frac{M_{2k}M_{2s}}{8},
\]
for
\[
x \in I_{2n}(0, \ldots, x_{2k} \neq 0, 0, \ldots, 0, x_{2s} \neq 0, x_{2s+1}, \ldots, x_{2n-1}),
\]
\( k = 0, 1, \ldots, n-3 \), \( s = k + 2, k + 3, \ldots, n-1 \).

The next lemma can be found in Blahota and Tephnadze [5] (see also Tephnadze [67] and [68]):

Lemma 1.20 Let \( x \in I_{N}^{k,l} \), \( k = 0, \ldots, N-2 \), \( l = k+1, \ldots, N-1 \). Then
\[
\int_{I_{N}} |K_{n}(x-t)| \, d\mu(t) \leq \frac{cM_{l}M_{k}}{nM_{N}}.
\]

Let \( x \in I_{N}^{k,N} \), \( k = 0, \ldots, N-1 \). Then
\[
\int_{I_{N}} |K_{n}(x-t)| \, d\mu(t) \leq \frac{cM_{k}}{M_{N}},
\]
where \( c \) is an absolute constant.

**Proof:** Let \( x \in I_{N}^{k,l} \), for \( 0 \leq k < l \leq N-1 \) and \( t \in I_{N} \). Since \( x - t \in I_{N}^{k,l} \) and \( n \geq M_{N} \), by combining Lemma 1.11 and (1.22) in Corollary 1.16 we obtain that
\[
n |K_{n}(x)|
\]
\[
\leq \frac{c}{nM_{N}} \sum_{i=0}^{l} M_{i} \int_{I_{N}} |K_{M_{i}}(x-t)| \, d\mu(t) \leq \frac{c}{nM_{N}} \sum_{i=0}^{l} M_{i}M_{k} \leq cM_{k}M_{l}
\]
and
\[
\int_{I_{N}} |K_{n}(x-t)| \, d\mu(t) \leq \frac{cM_{k}M_{l}}{nM_{N}}.
\]  \( \text{(1.24)} \)

Let \( x \in I_{N}^{k,N} \). Then by applying Lemma 1.11 and (1.22) in Corollary 1.16 we have that
\[
\int_{I_{N}} n |K_{n}(x-t)| \, d\mu(t) \leq \sum_{i=0}^{\lfloor n \rfloor} M_{i} \int_{I_{N}} |K_{M_{i}}(x-t)| \, d\mu(t).
\]  \( \text{(1.25)} \)

Let
\[
\begin{aligned}
x &= (0, \ldots, 0, x_{k} \neq 0, \ldots, x_{N-1} = 0, x_{N}, x_{N+1}, x_{q}, \ldots, x_{\lfloor n \rfloor-1}, \ldots) , \\
t &= (0, \ldots, 0, x_{N}, \ldots, x_{q-1}, t_{q} \neq x_{q}, t_{q+1}, \ldots, t_{\lfloor n \rfloor-1}, \ldots) , \quad q = N, \ldots, \lfloor n \rfloor - 1.
\end{aligned}
\]
By using Lemmas 1.11 and 1.12 in (1.25) it is easy to see that
\[
\int_{I_N} |K_n(x-t)| \, d\mu(t) \leq \frac{c}{n} \sum_{i=0}^{q-1} M_i \int_{I_N} M_k \, d\mu(t) \leq \frac{cM_kM_k}{nM_N} \leq \frac{cM_k}{M_N}.
\] (1.26)

Let
\[
\begin{align*}
x &= (0, \ldots, 0, x_m \neq 0, 0, \ldots, 0, x_N, x_{N+1}, x_q, \ldots, x_{|n|-1}, \ldots), \\
t &= (0, \ldots, x_N = 0, \ldots, x_{|n|-1}, \ldots).
\end{align*}
\]

If we apply again Lemmas 1.11 and 1.12 in (1.25) we obtain that
\[
\int_{I_N} |K_n(x-t)| \, d\mu(t) \leq \frac{c}{n} \sum_{i=0}^{n-1} M_i \int_{I_N} M_k \, d\mu(t) \leq \frac{cM_k}{M_N}.
\] (1.27)

By combining (1.24), (1.26) and (1.27) we can complete the proof. □

Also the next lemma is due to Tephnadze [67, 68], but it is also a simple consequence of Lemma 1.20.

**Lemma 1.21** Let \(x \in I_N^{k,l}, k = 0, \ldots, N - 1, l = k + 1, \ldots, N\). Then
\[
\int_{I_N} |K_n(x-t)| \, d\mu(t) \leq \frac{cM_lM_k}{M_N^2}, \text{ for } n \geq M_N,
\]
where \(c\) is an absolute constant.

**Proof:** Since \(n \geq M_N\) if we apply Lemma 1.20 we immediately get a proof of this estimate. □

1.5 **Nörlund Kernels with respect to Vilenkin systems**

A representation
\[
t_n f(x) = \int_G f(t) F_n(x-t) \, d\mu(t)
\]
plays a central role in the sequel, where
\[
F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k
\]
is the so-called Nörlund kernel.

In this section we study Nörlund kernels with respect to Vilenkin systems. The next results (Lemmas 1.22-1.27) are due to Persson, Tephnadze and Wall [51]:

G. Tephnadze
Lemma 1.22 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers, satisfying the condition
\[
\frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{when} \quad n \to \infty. \tag{1.28}
\]
Then
\[
|F_n| \leq \frac{c}{n} \left\{ \sum_{j=0}^{\lfloor n \rfloor} M_j \left| K_{M_j} \right| \right\},
\]
where \( c \) is an absolute constant.

**Proof:** First, we invoke Abel transformation to obtain the following identities
\[
Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^{n} q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n \tag{1.29}
\]
and
\[
F_n = \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n \right). \tag{1.30}
\]
Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing. Then, by using (1.28), we get that
\[
\frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) + q_0 \right) \leq \frac{q_{n-1}}{Q_n} \leq \frac{c}{n}.
\]

Under condition (1.28) if we apply (1.22) in Corollary 1.16 and use the equalities (1.29) and (1.30) we immediately get that
\[
|F_n| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) + q_0 \right) \right) \sum_{i=0}^{\lfloor n \rfloor} M_i \left| K_{M_i} \right|
\]
\[
= \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) + q_0 \right) \right) \sum_{i=0}^{\lfloor n \rfloor} M_i \left| K_{M_i} \right|
\]
\[
\leq \frac{q_{n-1}}{Q_n} \sum_{i=0}^{\lfloor n \rfloor} M_i \left| K_{M_i} \right| \leq \frac{c}{n} \sum_{i=0}^{\lfloor n \rfloor} M_i \left| K_{M_i} \right|.
\]

The proof is complete by just combining the estimates above.
Corollary 1.23 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then

\[
\sup_n \int_{G_m} |F_n| \, d\mu \leq c < \infty,
\]

where c is an absolute constant.

**Proof:** If we apply (1.23) in Corollary 1.16 and invoke the identities (1.29) and (1.30) we readily get the proof. So, we leave out the details.

Lemma 1.24 Let \( n \geq M_N \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then

\[
\left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j \right| \leq \frac{c}{M_N} \left( \sum_{j=0}^{\lfloor n \rfloor} M_j |K_{M_j}| \right),
\]

where c is an absolute constant.

**Proof:** Let \( M_N \leq j \leq n \). By using (1.22) in Corollary 1.16 we get that

\[
|K_j| \leq \frac{1}{j} \sum_{l=0}^{\lfloor j \rfloor} M_l |K_{M_l}| \leq \frac{1}{M_N} \sum_{l=0}^{\lfloor n \rfloor} M_l |K_{M_l}|.
\]

Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing. Then

\[
\sum_{j=M_N}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n \leq \sum_{j=0}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n
\]

\[
= \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n = Q_n.
\]

By using Abel transformation we can write that

\[
\left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j \right|
\]

\[
= \left| \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n \right) \right|
\]

\[
\left( \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n \right) \right) \frac{1}{M_N} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}|
\]

\[
\leq \frac{1}{M_N} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}|.
\]

The proof is complete. \( \blacksquare \)
Lemma 1.25 Let \( x \in I_N^{k,l} \), \( k = 0, \ldots, N-2, l = k + 1, \ldots, N-1 \) and \( \{ q_k : k \in \mathbb{N} \} \) be a sequence of non-decreasing numbers, satisfying condition (1.28). Then

\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{cM_lM_k}{nM_N}.
\]

Let \( x \in I_N^{k,N} \), \( k = 0, \ldots, N-1 \). Then

\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{cM_k}{M_N}.
\]

Here \( c \) is an absolute constant.

**Proof:** Let \( x \in I_N^{k,l} \), for \( 0 \leq k < l \leq N-1 \) and \( t \in I_N \). First, we observe that \( x - t \in I_N^{k,l} \). Next, we apply Lemma 1.22 and invoke (1.15) and (1.16) in Lemma 1.12 to obtain that

\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{c|n|}{n} \sum_{i=0}^{|n|} M_i \int_{I_N} |K_{M_i}(x-t)| \, d\mu(t)
\]

\[
\leq \frac{c}{n} \sum_{i=0}^{|n|} M_i \sum_{i=0}^l M_iM_k \, d\mu(t) \leq \frac{cM_kM_i}{nM_N}
\]

and the first estimate is proved.

Now, let \( x \in I_N^{k,N} \). Since \( x - t \in I_N^{k,N} \) for \( t \in I_N \), by combining Lemmas 1.11 and 1.22 with (1.15) and (1.16) in 1.12 we have that

\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{c}{n} \sum_{i=0}^{|n|-1} M_i \sum_{i=0}^l M_k \, d\mu(t) \leq \frac{cM_k}{M_N}.
\]

By combining (1.31) and (1.32) we complete the proof.

Lemma 1.26 Let \( n \geq M_N \), \( x \in I_N^{k,l} \), \( k = 0, \ldots, N-1, l = k + 1, \ldots, N \) and \( \{ q_k : k \in \mathbb{N} \} \) be a sequence of non-decreasing sequence, satisfying condition (1.28). Then

\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{cM_lM_k}{M_N^2},
\]

where \( c \) is an absolute constant.
Proof: Since $n \geq M_N$ if we apply Lemma 1.25 we immediately get the proof. □

Next, we state analogical estimate, but now without any restriction like (1.28):

**Lemma 1.27** Let $x \in I_N^{k,l}$, $k = 0, \ldots, N - 1$, $l = k + 1, \ldots, N$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing sequence. Then

$$
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j (x - t) \right| d\mu(t) \leq \frac{c M_l M_k}{M_N^2},
$$

where $c$ is an absolute constant.

Proof: Let $x \in I_N^{k,l}$, for $0 \leq k < l \leq N - 1$ and $t \in I_N$. Since $x - t \in I_N^{k,l}$ and $n \geq M_N$, if we combine Lemmas 1.11 and 1.24 and invoke (1.15) and (1.16) in Lemma 1.12 we readily obtain that

$$
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j (x - t) \right| d\mu(t) \leq c M_l M_k M_N^2. \tag{1.33}
$$

and the first estimate is proved.

Now, let $x \in I_N^{k,N}$. Since $x - t \in I_N^{k,N}$ for $t \in I_N$, if we apply again Lemmas 1.11, 1.22 and (1.15) and (1.16) in Lemma 1.12 we can conclude that

$$
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j (x - t) \right| d\mu(t) \leq c M_l M_k M_N^2 \tag{1.34}
$$

By combining (1.33) and (1.34) we complete the proof. □

The next results (Lemmas 1.28, 1.34) are due to Blahota, Persson and Tephnadze [7]:

\[ G.Tephnadze \]
Lemma 1.28 Let $s_n M_n < r \leq (s_n + 1) M_n$, where $1 \leq s_n \leq m_n - 1$. Then

$$Q_r F_r = Q_r D_{s_n M_n} - \psi_{s_n M_n-1} \sum_{l=1}^{s_n M_n-2} (q_{r-s_n M_n+l} - q_{r-s_n M_n+l+1}) l K_l$$

$$- \psi_{s_n M_n-1} (s_n M_n - 1) Q_{r-1} K_{s_n M_n-1} + \psi_{s_n M_n} Q_{r-s_n M_n} F_{r-s_n M_n}.$$ 

Remark 1.29 We note that Lemma 1.28 is true for every Nörlund mean, without any restriction on the generative sequence $\{q_k : k \in \mathbb{N}\}$.

Proof: Let $s_n M_n < r \leq (s_n + 1) M_n$, where $1 \leq s_n \leq m_n - 1$. It is easy to see that

$$Q_r F_r = \sum_{k=1}^{r} q_{r-k} D_k = \sum_{l=1}^{s_n M_n} q_{r-l} D_l + \sum_{l=s_n M_n+1}^{r} q_{r-l} D_l := I + II. \quad (1.35)$$

We apply (1.12) in Lemma 1.2 and invoke Abel transformation to obtain that

$$I = \sum_{l=0}^{s_n M_n-1} q_{r-s_n M_n+l} D_{s_n M_n-l} \quad (1.36)$$

$$= \sum_{l=1}^{s_n M_n-1} q_{r-s_n M_n+l} D_{s_n M_n-l} + q_{r-s_n M_n} D_{s_n M_n}$$

$$= D_{s_n M_n} \sum_{l=0}^{s_n M_n-1} q_{r-s_n M_n+l} - \psi_{s_n M_n-1} \sum_{l=1}^{s_n M_n-1} q_{r-s_n M_n+l} D_l$$

$$= (Q_r - Q_{r-s_n M_n}) D_{s_n M_n}$$

$$- \psi_{s_n M_n-1} \sum_{l=1}^{s_n M_n-2} (q_{r-s_n M_n+l} - q_{r-s_n M_n+l+1}) l K_l$$

$$- \psi_{s_n M_n-1} (s_n M_n - 1) Q_{r-1} K_{s_n M_n-1}.$$ 

By using (1.11) in Lemma 1.2 we can rewrite $II$ as

$$II = \sum_{l=1}^{r-s_n M_n} q_{r-s_n M_n-l} D_{l+s_n M_n} \quad (1.37)$$

$$= Q_{r-s_n M_n} D_{s_n M_n} + \psi_{s_n M_n} Q_{r-s_n M_n} F_{r-s_n M_n}.$$ 

The proof is complete by just combining (1.35), (1.37).
Lemma 1.30 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers satisfying the condition
\[
\frac{1}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{when} \quad n \to \infty. \tag{1.38}
\]
Then
\[
|F_n| \leq \frac{c_\alpha}{n} \left\{ \sum_{j=0}^{\lfloor n \rfloor} M_j |K_{M_j}| \right\},
\]
where \( c \) is an absolute constant.

**Proof:** Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-increasing satisfying condition \((1.38)\).

Then
\[
\frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right)
\]
\[
\leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} - (q_{n-j} - q_{n-j-1}) + q_0 \right)
\]
\[
\leq \frac{2q_0 - q_{n-1}}{Q_n} \leq \frac{2q_0}{Q_n} \leq \frac{c}{n}.
\]

If we apply \((1.22)\) in Corollary 1.16 and invoke equalities \((1.29)\) and \((1.30)\) we immediately get that
\[
|F_n| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \right)^{\lfloor n \rfloor} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}|
\]
\[
= \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} - (q_{n-j} - q_{n-j-1}) + q_0 \right) \right)^{\lfloor n \rfloor} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}|
\]
\[
\leq \frac{2q_0 - q_{n-1}}{Q_n} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}| \leq \frac{2q_0}{Q_n} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}|
\]
\[
\leq \frac{c}{n} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}|.
\]

The proof is complete by combining the estimates above.

Corollary 1.31 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers satisfying condition \((1.38)\). Then
\[
\sup_n \int_{G_m} |F_n| d\mu \leq c_\alpha < \infty,
\]
where \( c_\alpha \) is an absolute constant depending only on \( \alpha \).
Proof: By applying Lemma 1.30 we readily get the proof. So, we leave out the details.

Lemma 1.32 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers, \( 0 < \alpha < 1 \), and

\[
\frac{1}{Q_n} = O\left( \frac{1}{n^{\alpha}} \right), \text{ when } n \to \infty \tag{1.39}
\]

and

\[
q_n - q_{n+1} = O\left( \frac{1}{n^{2-\alpha}} \right), \text{ when } n \to \infty. \tag{1.40}
\]

Then

\[
|F_n| \leq \frac{c_\alpha}{n^\alpha} \left\{ \sum_{j=0}^{|n|} M_j^\alpha \left| K_{M_j} \right| \right\},
\]

where \( c_\alpha \) is an absolute constant depending only on \( \alpha \).

Proof: According to the fact that \( \{q_k : k \in \mathbb{N}\} \) is a sequence of non-negative and non-increasing numbers we have two cases:

1. \( \lim_{k \to \infty} q_k \geq c > 0 \),
2. \( \lim_{k \to \infty} q_k = 0 \),

In the first case we obtain that (1.4) and (1.5) are satisfied. Since the case

\[
\frac{n}{Q_n} = O(1), \text{ when } n \to \infty,
\]

have already been considered in Lemma 1.22 we can exclude it.

Hence, we may assume that

\[
q_n = o(1), \quad \text{when } n \to \infty. \tag{1.41}
\]

By combining (1.40) and (1.41) we immediately get that

\[
q_n = \sum_{l=n}^{\infty} (q_l - q_{l+1}) \leq \sum_{l=n}^{\infty} \frac{c}{l^{2-\alpha}} \leq \frac{c}{n^{1-\alpha}}
\]

and

\[
Q_n = \sum_{l=0}^{n-1} q_l \leq \sum_{l=1}^{n} \frac{c}{l^{1-\alpha}} \leq cn^\alpha.
\]

Let \( M_n < k \leq M_{n+1} \). It is easy to see that

\[
Q_k \left| D_{sM_n} \right| \leq c M_n^\alpha \left| D_{sM_n} \right|. \tag{1.42}
\]
and
\begin{equation}
(sM_n - 1)q_{k-1}|K_{sM_n-1}| \leq c\kappa^{\alpha-1}M_n|K_{sM_n-1}| \leq cM_n^\alpha|K_{sM_n-1}|.
\end{equation}

Let
\[ n = s_{n_1}M_{n_1} + s_{n_2}M_{n_2} + \cdots + s_{n_r}M_{n_r}, \quad n_1 > n_2 > \cdots > n_r, \]
and
\[ n^{(k)} = s_{n_{k+1}}M_{n_{k+1}} + \cdots + s_{n_r}M_{n_r}, \quad 1 \leq s_{n_l} \leq m_l - 1, \quad l = 1, \ldots, r. \]

By combining (1.42), (1.43) and Lemma 1.28 we have that
\[
|Q_nF_n| \leq c_\alpha \left( M_{n_1}^\alpha |D_{s_{n_1}M_{n_1}}| + \sum_{l=1}^{s_{n_1}M_{n_1}-1} (n^{(1)} + l)^{\alpha-2} |lK_l| + M_{n_1}^\alpha |K_{s_{n_1}M_{n_1}-1}| \right).
\]

By repeating this process \( r \) times we get that
\[
|Q_nF_n| \leq c_\alpha \sum_{k=1}^{r} \left( M_{n_k}^\alpha |D_{s_{n_k}M_{n_k}}| + \sum_{l=1}^{s_{n_k}M_{n_k}-1} (n^{(k)} + l)^{\alpha-2} |lK_l| + M_{n_k}^\alpha |K_{s_{n_k}M_{n_k}-1}| \right)
\]
\[ := I + II + III. \]

We combine Corollary 1.5 and Lemma 1.11 and invoke (1.13) in Lemma 1.3 to obtain that
\[
I \leq c_\alpha \sum_{k=1}^{|n|} M_k^\alpha |D_{s_kM_k}| \leq c_\alpha \sum_{k=1}^{|n|} M_k^\alpha |K_{M_k}|.
\]
and
\[
III \leq c_\alpha \sum_{k=1}^r M_{n_k}^\alpha |K_{s_{n_k}M_{n_k}} - M_{n_k}D_{s_{n_k}M_{n_k}}| \leq c_\alpha \sum_{k=1}^r M_{n_k}^\alpha |K_{s_{n_k}M_{n_k}}| + c_\alpha \sum_{k=1}^r M_{n_k}^\alpha |D_{s_{n_k}M_{n_k}}| \leq c_\alpha \sum_{k=1}^r M_k^\alpha |K_{M_k}|.
\]
Next, we can rewrite $II$ as

$$II = c_\alpha \sum_{k=1}^{r} s_{n+1}^{M_{n+1}} \left( \sum_{l=1}^{s_{n+1}^{M_{n+1}}} (n^{(k)} + l)^{\alpha-2} |lK_l| \right)$$

$$+ c_\alpha \sum_{k=1}^{r} s_{n+1}^{M_{n+1}} \left( \sum_{l=1}^{s_{n+1}^{M_{n+1}}} (n^{(k)} + l)^{\alpha-2} |lK_l| \right)$$

$$:= II_1 + II_2.$$

For $II_1$ we find that

$$II_1 \leq c_\alpha \sum_{k=1}^{r} s_{n+1}^{\alpha-2} M_{n+1}^{\alpha-2} \sum_{l=1}^{s_{n+1}^{M_{n+1}}} |lK_l|$$

$$\leq c_\alpha \sum_{k=1}^{n_1} M_{k}^{\alpha-2} \sum_{l=1}^{M_{k-1}} |lK_l|$$

$$= c_\alpha \sum_{k=1}^{n_1} M_{k}^{\alpha-2} \sum_{i=1}^{k} \sum_{l=M_{i-1}}^{M_{i-1}} |lK_l|$$

$$\leq c_\alpha \sum_{k=1}^{n_1} M_{k}^{\alpha-2} \sum_{i=1}^{k} M_{i} \sum_{j=0}^{M_{i}} |K_{M_j}|$$

$$\leq c_\alpha \sum_{k=0}^{n_1} M_{k}^{\alpha-1} \sum_{j=0}^{k} M_{j} |K_{M_j}|$$

$$= c_\alpha \sum_{j=0}^{n_1} M_{j} |K_{M_j}| \sum_{k=j}^{n_1} M_{k}^{\alpha-1}$$

$$\leq c_\alpha \sum_{j=0}^{n_1} M_{j}^{\alpha} |K_{M_j}| .$$

Moreover,

$$II_2 \leq c_\alpha \sum_{k=1}^{r} s_{n+1}^{M_{n+1}} \left( \sum_{l=M_{k+1}}^{s_{n+1}^{M_{n+1}}} l^{\alpha-2} |lK_l| \right)$$

$$\leq c_\alpha \sum_{k=1}^{r} s_{n+1}^{M_{n+1}} \left( \sum_{l=M_{k+1}}^{s_{n+1}^{M_{n+1}}} l^{\alpha-2} |lK_l| \right)$$

$$\leq c_\alpha \sum_{k=1}^{r} s_{n+1}^{M_{n+1}} \left( \sum_{l=M_{k+1}}^{s_{n+1}^{M_{n+1}}} l^{\alpha-2} |lK_l| \right) .$$
\[
\leq c_\alpha \sum_{k=1}^{r} \sum_{i=n_k+1}^{n_k} M_i^{\alpha-2} M_i \sum_{j=0}^{i} M_j |K_{M_j}|
\]
\[
= c_\alpha \sum_{k=1}^{r} \sum_{i=n_k+1}^{n_k} M_i^{\alpha-1} \sum_{j=0}^{i} M_j |K_{M_j}|
\]
\[
\leq c_\alpha \sum_{i=1}^{n_1} M_i^{\alpha-1} \sum_{j=0}^{i} M_j |K_{M_j}|
\]
\[
\leq c_\alpha \sum_{j=0}^{n_1} M_j^\alpha |K_{M_j}|.
\]

The proof is complete by just combining the estimates above. 

\[\square\]

**Corollary 1.33** Let \(0 < \alpha \leq 1\) and \(\{q_k : k \in \mathbb{N}\}\) be a sequence of non-increasing numbers satisfying the conditions (1.39) and (1.40). Then
\[
\sup_n \int_{G_m} |F_n| \, d\mu \leq c_\alpha < \infty,
\]
where \(c_\alpha\) is an absolute constant depending only on \(\alpha\).

**Proof**: If we use Lemma 1.32 we readily get the proof. Thus, we leave out the details. \[\square\]

**Lemma 1.34** Let \(0 < \alpha \leq 1\) and \(\{q_k : k \in \mathbb{N}\}\) be a sequence of non-increasing numbers satisfying the conditions (1.39) and (1.40). Then
\[
\int_{I_N} |F_m(x-t)| \, d\mu(t) \leq \frac{c_\alpha M_i^\alpha M_k}{m^\alpha M_N}, \text{ for } x \in I_N^{k,l},
\]
where \(k = 0, \ldots, N-2, l = k+2, \ldots, N-1\). Moreover,
\[
\int_{I_N} |F_m(x-t)| \, d\mu(t) \leq \frac{c_\alpha M_k}{M_N}, \text{ for } x \in I_N^{k,N}, \ k = 0, \ldots, N-1.
\]

Here \(c_\alpha\) is an absolute constant depending only on \(\alpha\).

**Proof**: Let \(x \in I_N^{k,l}\), where \(k = 0, \ldots, N-2, l = k+2, \ldots, N-1\). Since \(x-t \in I_N^{k,l}\), for \(t \in I_N\) if we apply Lemma 1.32 with (1.15) and (1.16) in Lemma 1.12 we can conclude that
\[
|F_m(x-t)| \leq \frac{c_\alpha}{m^\alpha} \sum_{i=0}^{l} M_i^\alpha |K_{M_i} (x-t)|
\]
\[
\leq \frac{c_\alpha}{m^\alpha} \sum_{i=0}^{l} M_i^\alpha M_k \leq \frac{c_\alpha M_i^\alpha M_k}{m^\alpha}.
\]

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Let \( x \in I_N^{k,l} \), for some \( 0 \leq k < l \leq N - 1 \). Since \( x - t \in I_N^{k,l} \), for \( t \in I_N \) and \( m \geq M_N \) from (1.44) we readily obtain that
\[
\int_{I_N} |F_m(x - t)| \, d\mu(t) \leq \frac{c_\alpha M_k^\alpha M_l}{m^\alpha M_N}. \tag{1.45}
\]

Let \( x \in I_N^{k,N} \), \( k = 0, \ldots, N - 1 \). Then, by applying Lemma 1.32 we have that
\[
\int_{I_N} |F_m(x - t)| \, d\mu(t) \leq \frac{c_\alpha}{m^\alpha} \sum_{i=0}^{\lfloor \frac{m}{\alpha} \rfloor} M_i^\alpha \int_{I_N} |K_{M_i}(x - t)| \, d\mu(t).
\]

Let \( x \in I_N^{k,N} \), \( k = 0, \ldots, N - 1 \), \( t \in I_N \) and \( x_q \neq t_q \), where \( N \leq q \leq \lfloor m \rfloor - 1 \). By using Lemma 1.11 and estimate (1.46) we get that
\[
\int_{I_N} |F_m(x - t)| \, d\mu(t) \leq \frac{c_\alpha}{m^\alpha} \sum_{i=0}^{\lfloor \frac{m}{\alpha} \rfloor} M_i^\alpha \int_{I_N} M_k \, d\mu(t) \leq \frac{c_\alpha M_k}{M_N}.
\]

Let \( x \in I_N^{k,N} \), \( k = 0, \ldots, N - 1 \), \( t \in I_N \) and \( x_N = t_N, \ldots, x_{\lfloor m \rfloor - 1} = t_{\lfloor m \rfloor - 1} \). By again applying Lemma 1.11 and estimate (1.46) we have that
\[
\int_{I_N} |F_m(x - t)| \, d\mu(t) \leq \frac{c_\alpha}{m^\alpha} \sum_{i=0}^{\lfloor \frac{m}{\alpha} \rfloor - 1} M_i^\alpha \int_{I_N} M_k \, d\mu(t) \leq \frac{c_\alpha M_k}{M_N}, \tag{1.48}
\]

The proof follows by combining (1.45), (1.47) and (1.48).

**Corollary 1.35** Let \( 0 < \alpha < 1 \), \( m \geq M_N \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers satisfying the conditions (1.39) and (1.40). Then there exists an absolute constant \( c_\alpha \), depending only on \( \alpha \), such that
\[
\int_{I_N} |F_m(x - t)| \, d\mu(t) \leq \frac{c_\alpha M_k^\alpha M_N}{M_{l+\alpha}^1}, \text{ for } x \in I_N^{k,l}
\]
where \( k = 0, \ldots, N - 1, l = k + 2, \ldots, N \).

**Proof:** The proof readily follows Lemma 1.34 if we use additional condition \( n \geq M_N \).

**Remark 1.36** For some sequences \( \{q_k : k \in \mathbb{N}\} \) of non-increasing numbers conditions (1.39) and (1.40) can be true or false independently.
1.6 INTRODUCTION TO THE THEORY OF MARTINGALE HARDY SPACES

The $\sigma$-algebra generated by the intervals

$$\{I_n(x) : x \in G_m \}$$

will be denoted by $\mathcal{F}_n(n \in \mathbb{N})$.

A sequence $f = (f^{(n)} : n \in \mathbb{N})$ of integrable functions $f^{(n)}$ is said to be a martingale with respect to the $\sigma$-algebras $\mathcal{F}_n(n \in \mathbb{N})$ if (for details see e.g. Weisz [83])

1) $f_n$ is $\mathcal{F}_n$ measurable for all $n \in \mathbb{N}$,
2) $S_{M_n}f_m = f_n$ for all $n \leq m$.

The martingale $f = (f^{(n)}, n \in \mathbb{N})$ is said to be $L_p$-bounded $(0 < p \leq \infty)$ if $f^{(n)} \in L_p$ and

$$\|f\|_p := \sup_{n \in \mathbb{N}} \|f_n\|_p < \infty.$$ 

If $f \in L_1(G_m)$, then it is easy to show that the sequence $F = (E_n f : n \in \mathbb{N})$ is a martingale. This type of martingales is called regular. If $1 \leq p \leq \infty$ and $f \in L_p(G_m)$ then $f = (f^{(n)}, n \in \mathbb{N})$ is $L_p$-bounded and

$$\lim_{n \to \infty} \|E_n f - f\|_p = 0$$

and consequently $\|F\|_p = \|f\|_p$ (see [46]). The converse of the latest statement holds also if $1 < p \leq \infty$ (see [46]): for an arbitrary $L_p$-bounded martingale $f = (f^{(n)}, n \in \mathbb{N})$ there exists a function $f \in L_p(G_m)$ for which $f^{(n)} = E_n f$. If $p = 1$, then there exists a function $f \in L_1(G_m)$ of the preceding type if and only if $f$ is uniformly integrable (see [46]), namely, if

$$\limsup_{y \to \infty} \frac{1}{\mathbb{N}} \int_{\{|f_n| > y\}} |f_n(x)| \, d\mu(x) = 0.$$ 

Thus the map $f \to f := (E_n f : n \in \mathbb{N})$ is isometric from $L_p$ onto the space of $L_p$-bounded martingales when $1 < p \leq \infty$. Consequently, these two spaces can be identified with each other. Similarly, the space $L_1(G_m)$ can be identified with the space of uniformly integrable martingales.

Analogously, the martingale $f = (f^{(n)}, n \in \mathbb{N})$ is said to be $weak-L_p$-bounded $(0 < p \leq \infty)$ if $f^{(n)} \in L_p$ and

$$\|f\|_{weak-L_p} := \sup_{n \in \mathbb{N}} \|f_n\|_{weak-L_p} < \infty.$$ 

The maximal function of a martingale $f$ is defined by

$$f^* := \sup_{n \in \mathbb{N}} |f^{(n)}|.$$
In the case $f \in L_1(G_m)$, the maximal functions are also given by

$$f^*(x) := \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \, d\mu(u) \right|.$$ 

For $0 < p < \infty$ the Hardy martingale spaces $H_p$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$ 

If $f = (f^{(n)} : n \in \mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \psi_i \, d\mu.$$

The next Lemma can be found in [84] (see also book [55]):

**Lemma 1.37** If $f \in L_1$, then the sequence $F := (S_{M_n} f : n \in \mathbb{N})$ is a martingale and

$$\|F\|_{H_p} \sim \left\| \sup_{n \in \mathbb{N}} |S_{M_n} f| \right\|_p.$$ 

Moreover, if $F := (S_{M_n} f : n \in \mathbb{N})$ is a regular martingale generated by $f \in L_1$, then

$$\hat{F}(k) = \int_{G_m} f(x) \psi_k(x) \, d\mu(x) = \hat{f}(k), \quad k \in \mathbb{N}. $$

A bounded measurable function $a$ is a p-atom if there exist an interval $I$ such that

$$\int_I a \, d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

Next, we note that the Hardy martingale spaces $H_p(G_m)$ for $0 < p \leq 1$ have atomic characterizations (see e.g. Weisz [83 84]):

**Lemma 1.38** A martingale $f = (f^{(n)} : n \in \mathbb{N})$ is in $H_p (0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k : k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N},$

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \quad \text{a.e.}, \quad (1.49)$$

where

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$
Moreover,
\[ \| f \|_{H^p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}, \]
where the infimum is taken over all decomposition of \( f = (f^{(n)} : n \in \mathbb{N}) \) of the form (1.49).

By using atomic characterization it can be easily proved that the following Lemmas hold (see e.g. Weisz [84]):

**Lemma 1.39** Suppose that an operator \( T \) is sub-linear and for some \( 0 < p \leq 1 \)
\[ \int_{I} |Ta|^p d\mu \leq c_p < \infty \]
for every \( p \)-atom \( a \), where \( I \) denotes the support of the atom. If \( T \) is bounded from \( L_\infty \) to \( L_\infty \), then
\[ \| Tf \|_p \leq c_p \| f \|_{H^p}. \]
Moreover, if \( p < 1 \), then we have the weak \((1,1)\) type estimate
\[ \lambda \mu \{ x \in G_m : |Tf(x)| > \lambda \} \leq \| f \|_1 \]
for all \( f \in L_1 \).

**Lemma 1.40** Suppose that an operator \( T \) is sub-linear and for some \( 0 < p \leq 1 \)
\[ \sup_{\lambda > 0} \lambda^p \mu \left\{ x \in I : |Tf| > \lambda \right\} \leq c_p < +\infty \]
for every \( p \)-atom \( a \), where \( I \) denote the support of the atom. If \( T \) is bounded from \( L_\infty \) to \( L_\infty \), then
\[ \| Tf \|_{\text{weak}-L_p} \leq c_p \| f \|_{H^p}. \]
Moreover, if \( p < 1 \), then
\[ \lambda \mu \{ x \in G_m : |Tf(x)| > \lambda \} \leq \| f \|_1 \]
for all \( f \in L_1 \).

The concept of modulus of continuity in \( H^p (p > 0) \) is defined by
\[ \omega_{H^p} \left( \frac{1}{M_n}, f \right) := \| f - S_{M_n} f \|_{H^p}. \]

We need to understand the meaning of the expression \( f - S_{M_n} f \) where \( f \) is a martingale and \( S_{M_n} f \) is function. So, we give an explanation in the following remark:
Remark 1.41 Let \( 0 < p \leq 1 \). Since

\[
S_{M_n} f = f^{(n)}, \text{ for } f = (f^{(n)} : n \in \mathbb{N}) \in H_p
\]

and

\[
(S_{M_k} f^{(n)} : k \in \mathbb{N}) = (S_{M_k} S_{M_n} f, k \in \mathbb{N})
\]

\[
= (S_{M_0} f, \ldots, S_{M_{n-1}} f, S_{M_n} f, S_{M_{n+1}} f, \ldots)
\]

\[
= (f^{(0)}, \ldots, f^{(n-1)}, f^{(n)}, f^{(n)}, \ldots)
\]

we obtain that

\[
f - S_{M_n} f = (f^{(k)} - S_{M_k} f : k \in \mathbb{N})
\]

is a martingale, for which

\[
(f - S_{M_n} f)^{(k)} = \begin{cases} 
0, & k = 0, \ldots, n, \\
(f^{(k)} - f^{(n)}), & k \geq n + 1,
\end{cases}
\]

(Watari [79] showed that there are strong connections between

\[
\omega_p \left( \frac{1}{M_n}, f \right), \quad E_{M_n} (L_p, f) \quad \text{and} \quad \|f - S_{M_n} f\|_p, \quad p \geq 1, \quad n \in \mathbb{N}.
\]

In particular,

\[
\frac{1}{2} \omega_p \left( \frac{1}{M_n}, f \right) \leq \|f - S_{M_n} f\|_p \leq \omega_p \left( \frac{1}{M_n}, f \right) \quad (1.51)
\]

and

\[
\frac{1}{2} \|f - S_{M_n} f\|_p \leq E_{M_n} (L_p, f) \leq \|f - S_{M_n} f\|_p.
\]

Remark 1.42 Since

\[
\|f\|_{H_p} \sim \|f\|_p,
\]

when \( p > 1 \), by applying (1.51), we obtain that

\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) \sim \omega_p \left( \frac{1}{M_n}, f \right).
\]
1.7 EXAMPLES OF $p$-ATOMS AND $H_p$ MARINGALES

The next two Examples can be found in the papers [9] and [10] by Blahota, Gát and Goginava (see also: Example 1.43)

Let $0 < p \leq 1$ and $\lambda = \sup_n m_n$. Then the function

$$a_k := \frac{M_{\alpha_k}^{1/p-1}}{\lambda} \left( D_{M_{\alpha_k}+1} - D_{M_{\alpha_k}} \right)$$

is a $p$-atom. Moreover,

$$\|a_k\|_{H_p} \leq 1.$$

Proof: Here we present the proof from Tephnadze [67] and [68]. Since

$$\text{supp}(a_k) = I_{\alpha_k}, \quad \int_{I_{\alpha_k}} a_k d\mu = 0$$

and

$$\|a_k\|_{\infty} \leq \frac{M_{\alpha_k}^{1/p-1}}{\lambda} M_{\alpha_k+1} \leq M_{\alpha_k}^{1/p} = (\text{supp } a_k)^{-1/p},$$

we conclude that $a_k$ is a $p$-atom.

Moreover, by using the orthonormality of Vilenkin functions, we find that

$$S_{M_n} \left( D_{M_{\alpha_k}+1} - D_{M_{\alpha_k}} \right) = \left\{ \begin{array}{ll} 0, & n = 0, \ldots, \alpha_k, \\ D_{M_{\alpha_k+1}} - D_{M_{\alpha_k}}, & n \geq \alpha_k + 1, \end{array} \right.$$ 

and

$$\sup_{n \in \mathbb{N}} \left| S_{M_n} \left( D_{M_{\alpha_k+1}} (x) - D_{M_{\alpha_k}} (x) \right) \right| = \left| D_{M_{\alpha_k+1}} (x) - D_{M_{\alpha_k}} (x) \right|, \quad \text{for all } x \in G_m.$$ 

If we invoke Lemma 1.37 we obtain that

$$\|a_k\|_{H_p} = \frac{M_{\alpha_k}^{1/p-1}}{\lambda} \left\| D_{M_{\alpha_k+1}} - D_{M_{\alpha_k}} \right\|_{H_p} = \frac{M_{\alpha_k}^{1/p-1}}{\lambda} \left\| D_{M_{\alpha_k+1}} - D_{M_{\alpha_k}} \right\|_p = \frac{M_{\alpha_k}}{\lambda} \left( \int_{I_{\alpha_k} \setminus I_{\alpha_k+1}} M_{\alpha_k}^p \, d\mu + \int_{I_{\alpha_k+1}} (M_{\alpha_k+1} - M_{\alpha_k})^p \, d\mu \right)^{1/p}$$

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\[ M^{1/p-1} \left( \frac{m_{\alpha_k} - 1}{M_{\alpha_k + 1}} M^{p}_{\alpha_k} + \frac{(m_{\alpha_k} - 1)^p}{M_{\alpha_k + 1}} M^{p}_{\alpha_k} \right)^{1/p} \leq \frac{M^{1/p-1}}{\lambda} \cdot M^{1-1/p} \leq 1. \]

The proof is complete. \[ \square \]

**Example 1.44** Let \( 0 < p \leq 1 \) and

\[ f_k = D_{M2^{n_k + 1}} - D_{M2^{n_k}}. \]

Then

\[ \hat{f}_k (i) = \begin{cases} 1, & i = M_{2^{n_k}}, \ldots, M_{2^{n_k} + 1} - 1, \\ 0, & \text{otherwise}, \end{cases} \]  

(1.52)

and

\[ S_{i} f_k = \begin{cases} D_{i} - D_{M2^{n_k}}, & i = M_{2^{n_k}} + 1, \ldots, M_{2^{n_k} + 1} - 1, \\ f_k, & i \geq M_{2^{n_k} + 1}, \\ 0, & \text{otherwise}. \end{cases} \]  

(1.53)

Moreover,

\[ \| f_k \|_{H^p} \leq \lambda M^{-1/p}_{2^{n_k}}, \]  

(1.54)

where \( \lambda = \sup_{n} m_n \).

**Proof:** The proof follows by using Example 1.43 in the case when \( \alpha_k = 2n_k \). We leave out the details. \[ \square \]

The next three examples of regular martingales will be used frequently and it can be found in Persson and Tephnadze \[52\]:

**Example 1.45** Let \( M_k \leq n < M_{k+1} \) and \( S_nf \) be the \( n \)-th partial sum with respect to Vilenkin systems, where \( f \in H_p \) for some \( 0 < p \leq 1 \). Then \( S_nf \in L_1 \) for every fixed \( n \in \mathbb{N} \) and

\[ \| S_n f \|_{H^p} \leq \left( \sup_{0 \leq l \leq k} |S_{M_l} f| \right)_{p} + \| S_n f \|_{p} \]

\[ \leq \left\| \hat{S}_{#} f \right\|_{p} + \| S_n f \|_{p}. \]

**Proof:** We consider the following martingale

\[ f_{#} = (S_{M_l} S_n, \ l \in \mathbb{N}) \]

\[ f_{#} := (S_{M_k} S_n f, \ k \geq 1) = (S_{M_0}, \ldots, S_{M_k} f, \ldots, S_n f, \ldots, S_n f) \].

It immediately follows that

\[ \| S_n f \|_{H^p} \leq \left\| \sup_{0 \leq l \leq k} |S_{M_l} f| \right\|_p + \| S_n f \|_p \leq \left\| \hat{S}_{#} f \right\|_p + \| S_n f \|_p. \]

The proof is complete. \[ \square \]
Example 1.46 Let $M_k \leq n < M_{k+1}$ and $\sigma_n f$ be $n$-th Fejér means with respect to Vilenkin systems, where $f \in H_p$ for some $0 < p \leq 1$. Then $\sigma_n f \in L_1$ for every fixed $n \in \mathbb{N}$ and

$$
\| \sigma_n f \|_{H_p} \leq \left\| \sup_{0 \leq l \leq k} |\sigma_{M_l} f| \right\|_p + \| \sigma_n f \|_p \leq \| \tilde{\sigma}_{\#} f \|_p + \| \sigma_n f \|_p.
$$

Proof: We consider the following martingale

$$
f_{\#} = (S_{M_k} \sigma_n f, k \geq 1)
$$

and if we follow analogous steps as in Example 1.45 we readily can complete the proof. Thus, we leave out the details.

The next five Examples of martingales will be used many times to prove sharpness of our main results (c.f. the papers [66], [69], [71], [72], [73] by Tephnadze).

Example 1.47 Let $0 < p \leq 1$, $\lambda = \sup_n m_n, \{\lambda_k : k \in \mathbb{N}\}$ be a sequence of real numbers $\mathbb{R}$, such that

$$
\sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p < \infty
$$

and $\{a_k : k \in \mathbb{N}\}$ be a sequence of $p$-atoms, defined by

$$
a_k := \frac{M_1^{1/p-1}}{\lambda} \left( D_{M_2a_k+1} - D_{M_2a_k} \right).
$$

Then $f = \{f^{(n)} : n \in \mathbb{N}\}$, where

$$
f^{(n)} := \sum_{\{k : 2^a_k < n\}} \lambda_k a_k
$$

is a martingale, $f \in H_p$ and

$$
\hat{f}(j) = \begin{cases} 
\frac{\lambda_k M_{1/p-1}^{1/p}}{\lambda}, & j \in \{M_{2a_k}, \ldots, M_{2a_k+1} - 1\}, k \in \mathbb{N}_+, \\
0, & j \notin \bigcup_{k=1}^{\infty} \{M_{2a_k}, \ldots, M_{2a_k+1} - 1\}.
\end{cases}
$$

Moreover,

$$
\omega_{H_p} \left( \frac{1}{M_n}, f \right) = O \left( \sum_{\{k : 2^a_k \geq n\}} |\lambda_k|^p \right)^{1/p}, \text{ when } n \to \infty,
$$

for all $0 < p \leq 1$. 

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Let \( M_{2\alpha_l-1+1} \leq j \leq M_{2\alpha_l}, l \in \mathbb{N}_+ \). Then
\[
S_j f = S_{M_{2\alpha_l-1+1}} = \sum_{\eta=0}^{l-1} \frac{\lambda_\eta M_{2\alpha_l}^{1/p-1}}{\lambda} \left( D_{M_{2\alpha_l+1}} - D_{M_{2\alpha_l}} \right).
\] (1.59)

Let \( M_{2\alpha_l} \leq j < M_{2\alpha_l+1}, l \in \mathbb{N}_+ \). Then
\[
S_j f = S_{M_{2\alpha_l}} + \frac{\lambda_1 M_{2\alpha_l}^{1/p-1} \psi_{2\alpha_l} D_{j-M_{2\alpha_l}}}{\lambda}
\]
\[
= \sum_{\eta=0}^{l-1} \frac{\lambda_\eta M_{2\alpha_l}^{1/p-1}}{\lambda} \left( D_{M_{2\alpha_l+1}} - D_{M_{2\alpha_l}} \right) + \frac{\lambda_1 M_{2\alpha_l}^{1/p-1} \psi_{M_{2\alpha_l}} D_{j-M_{2\alpha_l}}}{\lambda}.
\] (1.60)

**Proof:** Since
\[
S_{M_n} a_k = \begin{cases} a_k, & \alpha_k < n, \\ 0, & \alpha_k \geq n, \end{cases}
\]
by applying Lemma 1.38 and (1.55) we can conclude that \( f \in H_p \). Furthermore, in view of (1.50) in Remark 1.41 we immediately get also the estimate of \( H_p \) modulus of continuity of the martingale \( f \).

Let \( j \notin \bigcup_{k=1}^{\infty} \{M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1\} \). By using orthonormality of Vilenkin functions we find that
\[
\hat{a}_k(j) = \int_{G_m} M_{2\alpha_k}^{1/p-1} \left( D_{M_{2\alpha_k+1}} - D_{M_{2\alpha_k}} \right) \overline{\psi_i} d\mu = 0
\]
and
\[
\hat{f}(j) = \lim_{n \to \infty} \int_{G_m} \sum_{\{k: 2\alpha_k < n\}} \lambda_k a_k \overline{\psi_i} d\mu = 0
\] (1.61)

Let \( j \in \{M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1\} \), for some \( k \in \mathbb{N} \). Then, for every \( l \in \mathbb{N} \) we can conclude that
\[
\hat{a}_l(j) = \frac{M_{2\alpha_k}^{1/p-1}}{\lambda} \int_{G_m} \left( D_{M_{2\alpha_k+1}} - D_{M_{2\alpha_k}} \right) \overline{\psi_i} d\mu = \frac{M_{2\alpha_l}^{1/p-1}}{\lambda} \delta_{l,k},
\]
where \( \delta_{l,k} \) is Kronecker symbol. It follows that
\[
\hat{f}(j) = \lim_{n \to \infty} \sum_{\{l: 2\alpha_k < n\}} \lambda_l \int_{G_m} a_l \overline{\psi_i} d\mu
\] (1.62)
\[ = \lim_{n \to \infty} \sum_{\{l: 2\alpha_l < n\}} \frac{\lambda_l M_{2\alpha_l}^{1/p-1}}{\lambda} \delta_{l,k} = \frac{\lambda k M_{2\alpha_k}^{1/p-1}}{\lambda}. \]

Let \( M_{2\alpha_l-1} + 1 \leq j \leq M_{2\alpha_l} \). In the view of (1.59) we can conclude that
\[
S_j f = S_{M_{2\alpha_l-1}+1} f = \sum_{v=0}^{M_{2\alpha_l-1}+1-1} \hat{f}(v) \psi_v, \quad (1.63)
\]
\[
= \sum_{\eta=0}^{l-1} \sum_{v=M_{2\alpha\eta}}^{M_{2\alpha\eta}+1-1} \frac{\lambda \eta M_{2\alpha\eta}^{1/p-1}}{\lambda} \psi_v
\]
\[
= \sum_{\eta=0}^{l-1} \frac{\lambda \eta M_{2\alpha\eta}^{1/p-1}}{\lambda} \left( D_{M_{2\alpha\eta}+1} - D_{M_{2\alpha\eta}} \right).
\]

Let \( M_{2\alpha_l} \leq j < M_{2\alpha_{l+1}} \). We apply (1.60) and invoke (1.11) in Lemma 1.2 to obtain that
\[
S_j f = S_{M_{2\alpha_l}} f + \sum_{v=M_{2\alpha_l}}^{j-1} \hat{f}(v) \psi_v, \quad (1.64)
\]
\[
= \sum_{\eta=0}^{l-1} \frac{\lambda \eta M_{2\alpha\eta}^{1/p-1}}{\lambda} \left( D_j - D_{M_{2\alpha_l}} \right)
\]
\[
= \sum_{\eta=0}^{l-1} \frac{\lambda \eta M_{2\alpha\eta}^{1/p-1}}{\lambda} \left( D_j - D_{M_{2\alpha_l}} \right)
\]

If we take (1.63) obtained by \( j = M_{2\alpha_l} \) into account we can rewrite (1.64) as
\[
S_j f = \sum_{\eta=0}^{l-1} \frac{\lambda \eta M_{2\alpha\eta}^{1/p-1}}{\lambda} \left( D_{M_{2\alpha\eta}+1} - D_{M_{2\alpha\eta}} \right)
\]
\[
+ \frac{\lambda M_{2\alpha_l}^{1/p-1}}{\lambda} \left( D_j - D_{M_{2\alpha_l}} \right).
\]

By using (1.50) we get that
\[
(f - S_{M_n} f)^{(k)} = \begin{cases} 0, & k = 0, \ldots, n, \\ f^{(k)} - f^{(n)}, & k \geq n + 1, \end{cases} \quad (1.65)
\]
\[
= \begin{cases} 0, & k = 0, \ldots, n, \\ \sum_{l=n+1}^{k} \mu_l S_{M_l} a_l, & k \geq n + 1. \end{cases}
\]
According to Lemma 1.38 we conclude that (1.65) is an atomic decomposition of the martingale \( f - S_{M_n}f \in H_p \) and

\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) \sim \left( \sum_{k=n+1}^{\infty} |\mu_k|^p \right)^{1/p} < \infty.
\]

The proof is complete.

Sometimes, we will use the martingale defined in the next example.

**Example 1.48** Let \( 0 < p \leq 1, \lambda = \sup_{n \in \mathbb{N}} m_n, \{\lambda_k : k \in \mathbb{N}\} \) be a sequence of real numbers \( \mathbb{R} \), such that

\[
\sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p < \infty.
\]

and \( \{a_k : k \in \mathbb{N}\} \) be a sequence of \( p \)-atoms, defined by

\[
a_k := \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda} \left( D_{M_{|\alpha_k|}+1} - D_{M_{|\alpha_k|}} \right),
\]

where \( |\alpha_k| := \max \{j \in \mathbb{N} : (\alpha_k)_j \neq 0\} \) and \( (\alpha_k)_j \) denotes the \( j \)-th binary coefficient of \( \alpha_k \in \mathbb{N}_+ \). Then \( f = (f^{(n)} : n \in \mathbb{N}) \), where

\[
f^{(n)} := \sum_{\{k : |\alpha_k| < n\}} \lambda_k a_k.
\]

is a martingale, \( f \in H_p \) and

\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) = O \left( \sum_{\{k : |\alpha_k| \geq n\}} |\lambda_k|^p \right)^{1/p}, \text{ when } n \to \infty,
\]

for all \( 0 < p \leq 1 \). Moreover

\[
\hat{f}(j) = \begin{cases} 
\frac{\lambda_k M_{|\alpha_k|}^{1/p-1}}{\lambda}, & j \in \{M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1\}, k \in \mathbb{N}_+, \\
0, & j \notin \bigcup_{k=1}^{\infty} \{M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1\}.
\end{cases}
\]

Let \( M_{|\alpha_l|+1} \leq j \leq M_{|\alpha_l|}, l \in \mathbb{N}_+ \). Then

\[
S_j f = S_{M_{|\alpha_l|+1}} = \sum_{\eta=0}^{l-1} \frac{\lambda_{\eta} M_{|\alpha_l|}^{1/p-1}}{\lambda} \left( D_{M_{|\alpha_l|}+1} - D_{M_{|\alpha_l|}} \right).
\]
Let $M_{[\alpha l]} \leq j < M_{[\alpha l] + 1}, l \in \mathbb{N}_+$. Then

\[
S_j f = S_{M_{[\alpha l]}} + \frac{\lambda l M_{[\alpha l]}^{1/p-1} \psi_{M_{[\alpha l]}} D_{j-M_{[\alpha l]}}}{\lambda}
\]

(1.70)

\[
= \sum_{\eta=0}^{l-1} \frac{\lambda \eta M_{[\alpha \eta]}^{1/p-1}}{\lambda} \left( D_{M_{[\alpha \eta] + 1}} - D_{M_{[\alpha \eta]}} \right) + \frac{\lambda l M_{[\alpha l]}^{1/p-1} \psi_{M_{[\alpha l]}} D_{j-M_{[\alpha l]}}}{\lambda}.
\]

**Proof:** The proof is quite analogously to the proof of Example 1.47. Hence, we leave out the details.

For our further investigations it is convenient to fix sequence $\{\lambda_k : k \in \mathbb{N}\}$. In many cases such appropriate choice can be the following sequence

\[\left\{ \lambda/\alpha_k^{1/2} : k \in \mathbb{N} \right\}.
\]

**Example 1.49** Let $0 < p \leq q \leq 1$, $\lambda = \sup_{n \in \mathbb{N}} m_n$ and $\{\alpha_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that

\[
\sum_{k=0}^{\infty} \frac{1}{\alpha_k^{p/2}} < \infty,
\]

(1.71)

\[
\sum_{\eta=0}^{k-1} \frac{M_{2\alpha \eta}^{1/p}}{\alpha_{\eta}^{1/2}} < \frac{M_{2\alpha_k}^{1/p}}{\alpha_k^{1/2}},
\]

(1.72)

and

\[
\frac{32 \lambda M_{2\alpha_{k-1}}^{1/p}}{\alpha_k^{1/2}} \left\{ \begin{array}{ll}
M_{2\alpha_k}^{1/p} & \text{if } p = q \\
\frac{M_{2\alpha_k}^{1/p-1/q}}{\alpha_k^{3/2}} & \text{if } p < q
\end{array} \right.
\]

(1.73)

For the given sequence $\{\alpha_k : k \in \mathbb{N}\}$, which satisfy conditions (1.71)-(1.73) we define a martingale $f = (f^{(n)} : n \in \mathbb{N})$, where

\[f^{(n)} := \sum_{\{k : 2\alpha_k < n\}} \lambda_k a_k,
\]

and

\[
\lambda_k = \frac{\lambda}{\alpha_k^{1/2}} \text{ and } a_k := \frac{M_{2\alpha_k}^{1/p-1}}{\lambda} \left( D_{M_{2\alpha_k+1}} - D_{M_{2\alpha_k}} \right).
\]

Then $f \in H_p$ and

\[\hat{f}(j)\]

(1.74)

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For every \( l \in \mathbb{N} \) we have the following identities

\[
S_j f = \begin{cases} 
S_{M_{2\alpha_{l-1}+1}}, & M_{2\alpha_{l-1}+1} \leq j \leq M_{2\alpha_l}, \\
S_{M_{2\alpha_l}} + \frac{M_{2\alpha_l}^{1/p-1}}{\alpha_l^{1/2}} \psi_{M_{2\alpha_l}} \sum_{\eta=0}^{M_{2\alpha_l}+1} \left(D_{2\alpha_{\eta+1}} - D_{2\alpha_{\eta}}\right), & M_{2\alpha_l} \leq j < M_{2\alpha_{l+1}}.
\end{cases}
\]

(1.75)

Moreover,

\[
|S_j f| \leq \frac{2\lambda M_{2\alpha_k-1}^{1/p-1}}{\alpha_k^{1/2}}, \text{ for all } 0 \leq j \leq M_{2\alpha_k} \text{ and } 0 < p \leq q \leq 1
\]

(1.76)

and

\[
|S_{M_{2\alpha_k+1}} f| \geq \frac{M_{2\alpha_k}^{1/p-1}}{4\alpha_k^{1/2}}, \text{ for } 0 < p < q < 1.
\]

(1.77)

**Proof:** First, we note that such an increasing sequence \( \{\alpha_k : k \in \mathbb{N}\} \) which satisfies conditions (1.71)-(1.73), can be constructed.

By applying (1.71) we conclude that condition (1.55) is satisfied and we can invoke Lemma 1.71 to obtain that the martingale \( f = (f^{(n)} : n \in \mathbb{N}) \in H_p. \)

Since

\[
\frac{\lambda_k M_{2\alpha_k}^{1/p-1}}{\lambda} = \frac{M_{2\alpha_k}^{1/p-1}}{\alpha_k^{1/2}},
\]

according to (1.61) and (1.62), we find that Fourier coefficients are given by equality (1.74).

Let \( M_{2\alpha_{l-1}+1} + 1 \leq j \leq M_{2\alpha_l} \), where \( 1 \leq l \leq k. \) In view of (1.74) we can conclude that

\[
S_j f = S_{M_{2\alpha_{l-1}+1}} f = \sum_{v=0}^{M_{2\alpha_{l-1}+1}-1} \hat{f}(v) \psi_v
\]

(1.78)

\[
\sum_{\eta=0}^{l-1} \sum_{v=M_{2\alpha_{\eta}}}^{M_{2\alpha_{\eta+1}}-1} \frac{M_{2\alpha_{\eta}}^{1/p-1}}{\alpha_{\eta}^{1/2}} \psi_v
\]

\[
= \sum_{\eta=0}^{l-1} \frac{M_{2\alpha_{\eta}}^{1/p-1}}{\alpha_{\eta}^{1/2}} \left(D_{2\alpha_{\eta+1}} - D_{2\alpha_{\eta}}\right).
\]

Let \( M_{2\alpha_l} \leq j < M_{2\alpha_{l+1}} \), where \( 1 \leq l \leq k. \) We apply (1.74) again and invoke (1.11) in Lemma 1.74 to obtain that

\[
S_j f = S_{M_{2\alpha_l}} f + \sum_{v=M_{2\alpha_l}}^{j-1} \hat{f}(v) \psi_v
\]

(1.79)
By combining (1.78) and (1.79) we get the identities given by (1.75).

Moreover, if we take (1.78) obtained by letting $j = M_{2\alpha_l}$ we can rewrite (1.79) as

$$S_j f = \sum_{\eta=0}^{l-1} \frac{M_{2\alpha_{\eta}}^{1/p-1}}{\alpha_{\eta}^{1/2}} \left(D_{M_{2\alpha_{\eta+1}}} - D_{M_{2\alpha_{\eta}}}ight)$$

(1.80)

Let $M_{2\alpha_{l-1}} + 1 \leq j \leq M_{2\alpha_l}$, where $1 \leq l \leq k$. Then, according to (1.72) and (1.80), we get that

$$|S_j f| \leq \sum_{\eta=0}^{l-1} \frac{\lambda M_{2\alpha_{\eta}}^{1/p}}{\alpha_{\eta}^{1/2}}$$

(1.81)

$$= \sum_{\eta=0}^{l-2} \frac{\lambda M_{2\alpha_{\eta}}^{1/p}}{\alpha_{\eta}^{1/2}} + \frac{\lambda M_{2\alpha_{l-1}}^{1/p}}{\alpha_{l-1}^{1/2}} \leq 2 \lambda M_{2\alpha_{l-1}}^{1/p} \leq \frac{2\lambda M_{2\alpha_{k-1}}^{1/p}}{\alpha_{k-1}^{1/2}}.$$

Let $M_{2\alpha_l} \leq j < M_{2\alpha_{l+1}}$, where $1 \leq l \leq k - 1$. In view of (1.72) and (1.80) we can conclude that

$$|S_j f| \leq \sum_{\eta=0}^{l-1} \frac{\lambda M_{2\alpha_{\eta}}^{1/p}}{\alpha_{\eta}^{1/2}} + \frac{M_{2\alpha_{l}}^{1/p-1} j}{\alpha_{l}^{1/2}}$$

(1.82)

$$= \frac{\lambda M_{2\alpha_{l}}^{1/p}}{\alpha_{l}^{1/2}} + \frac{\lambda M_{2\alpha_{l}}^{1/p}}{\alpha_{l}^{1/2}} = \frac{\lambda M_{2\alpha_{l}}^{1/p}}{\alpha_{l}^{1/2}} \leq \frac{2\lambda M_{2\alpha_{k-1}}^{1/p}}{\alpha_{k-1}^{1/2}}.$$
\[ M_1^{1/p} - M_2^{1/p} - S_{M_2k - 1 + 1}f \geq M_1^{1/p} - M_{2\alpha k}^{1/p} - \frac{M_{2\alpha k}^{1/p - 1/2}}{\alpha_k} \geq M_2^{1/p} - \frac{M_{2\alpha k}^{1/p - 1/2}}{\alpha_k} \geq M_2^{1/p} - \frac{M_{2\alpha k}^{1/p - 1/2}}{16\alpha_k^{3/2}} \]

The proof is complete. \( \square \)

**Remark 1.50** It is obvious that if we cancel condition (1.73) in Example 1.49 we still get that \( f = (f^{(n)} : n \in \mathbb{N}) \in H_p \) and equality (1.72) and estimate (1.76) is valid. Moreover, (1.80)-(1.81) are still fulfilled.

In the next chapter we many times will use a martingale, with the following fixed sequence \( \{\lambda_k : k \in \mathbb{N}\} : \)

**Example 1.51** Let \( 0 < p \leq 1, \lambda = \sup_{n \in \mathbb{N}} m_n \) and \( \{\Phi_k : k \in \mathbb{N}\} \) be sequence of non-negative and nondecreasing numbers, such that

\[ \sum_{k=1}^{\infty} \frac{1}{\Phi_k^{p/4} M_{2\alpha k}} < \infty. \] (1.83)

For this given sequence \( \{\alpha_k : k \in \mathbb{N}\} \), which satisfies condition (1.83) we define a martingale \( f = (f^{(n)} : n \in \mathbb{N}) \), where

\[ f^{(n)} := \sum_{\{k: 2\alpha k < n\}} \lambda_k a_k \]

and

\[ \lambda_k = \frac{\lambda}{\Phi_k^{1/4} M_{2\alpha k}}, \quad a_k = \frac{M_{2\alpha k}^{1/p - 1/2}}{\lambda} \left( D_{M_{2\alpha k} + 1} - D_{M_{2\alpha k}} \right). \]

Then \( f = (f^{(n)} : n \in \mathbb{N}) \in H_p \). Moreover,

\[ \hat{f}(j) = \begin{cases} \frac{M_{2\alpha k}^{1/p - 1/2}}{\Phi_k M_{2\alpha k}}, & j \in \{M_{2\alpha k}, \ldots, M_{2\alpha k+1} - 1\}, \quad k \in \mathbb{N}_+, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{M_{2\alpha k}, \ldots, M_{2\alpha k+1} - 1\}. \end{cases} \] (1.84)
Proof: The proof is quite analogously to the proof of Example 1.47. Hence, we leave out the details.

Sometimes an appropriate choice of sequence will be the following

\[ \left\{ \lambda_k = \frac{\lambda}{M_k^{1/p}} : k \in \mathbb{N} \right\}. \]

**Example 1.52** Let \( 0 < p \leq 1 \), \( \lambda = \sup_{n \in \mathbb{N}} m_n \) and

\[ f^{(n)}(n) = \sum_{\{k : 2M_k < n\}} \lambda_k a_k, \]

where

\[ \lambda_k := \frac{\lambda}{M_k^{1/p}} \quad \text{and} \quad a_k := \frac{M_k^{1/p - 1}}{\lambda} \left( D_{M_{2M_k+1}} - D_{M_{2M_k}} \right). \]

Then \( f = (f^{(n)} : n \in \mathbb{N}) \in H_p \), for all \( 0 < p \leq 1 \). Moreover,

\[ \hat{f}(j) = \begin{cases} \frac{M_i^{1/p - 1}}{M_i^{1/p}}, & j \in \{M_{2M_i}, \ldots, M_{2M_i+1} - 1\}, i \in \mathbb{N}_+, \\ 0, & j \notin \bigcup_{i=0}^{\infty} \{M_{2M_i}, \ldots, M_{2M_i+1} - 1\}, \end{cases} \quad (1.85) \]

and

\[ \omega_{H_p} \left( \frac{1}{M_n}, f \right) = O \left( \frac{1}{n^{1/p}} \right), \quad \text{when} \quad n \to \infty. \quad (1.86) \]

Let \( M_{2M_i} \leq j < M_{2M_{i+1}}, l \in \mathbb{N}_+. \) Then

\[ S_j f = S_{M_{2M_i}} + \frac{M_{2M_i}^{1/p - 1} \psi_{M_{2M_i}} D_j - M_{2M_i}}{M_i^{1/p}}. \quad (1.87) \]

**Proof:** This example is a simple consequence of Example 1.47 if we take \( \alpha_k = M_k \) and \( \lambda_k = \lambda/M_k^{1/p} \). Since

\[ \sum_{k=0}^{\infty} |\lambda_k|^p \leq c \sum_{k=0}^{\infty} \frac{1}{M_k} \leq c \sum_{k=0}^{\infty} \frac{1}{2^k} < c < \infty, \]

we obtain that \( f = (f^{(n)} : n \in \mathbb{N}) \in H_p \), for all \( 0 < p \leq 1 \).

In view of (1.67) we can conclude that

\[ \omega_{H_p} \left( \frac{1}{M_n}, f \right) \sim \left( \sum_{\{k : 2M_k \geq n\}} \frac{1}{M_k} \right)^{1/p} \leq \left( \sum_{\{k : 2^{k+1} \geq n\}} \frac{1}{2^k} \right)^{1/p}. \]
\[
\leq \left( \sum_{\{k; \, k \geq \log_2 n \}} \frac{1}{2^k} \right)^{1/p} \leq \left( \frac{1}{2^\log_2 n - 1} \right)^{1/p} = O\left( \frac{1}{n^{1/p}} \right) \rightarrow \infty, \text{ when } n \rightarrow \infty.
\]

If we apply (1.60) obtained by taking \( \alpha_k = M_k \) we immediately get equality (1.87). So the proof is complete.
2 Fourier coefficients and partial sums of Vilenkin-Fourier series on Martingale Hardy spaces

2.1 Some classical results on the Vilenkin-Fourier coefficients and partial sums of Vilenkin-Fourier series

According to the Riemann-Lebesgue lemma (for details see e.g. the book [55]) we obtain that
\[ \hat{f}(k) \to 0, \quad \text{when} \quad k \to \infty, \]
for each \( f \in L_1 \).

Moreover,
\[ |\hat{f}(n)| \leq \frac{1}{2} \omega_1 \left( \frac{1}{M_n}, f \right), \]
where \( M_N \leq n \leq M_{N+1} \) and \( f \in L_1 \).

It is well-known (see e.g. the books [2] and [55]) that if \( f \in L_1 \) and the Vilenkin series
\[ T(x) = \sum_{j=0}^{\infty} c_j \psi_j(x) \]
converges to \( f \) in \( L_1 \)-norm, then
\[ c_j = \int_{G_m} \overline{\hat{f}(j)} \mu \]
i.e. in this case the approximation series must be a Vilenkin-Fourier series. An analogous result is true also if the Vilenkin series converges uniformly on \( G_m \) to an integrable function \( \hat{f} \).

Since \( H_1 \subset L_1 \) it yields that \( \hat{f}(k) \to 0 \) when \( k \to \infty \), for every \( f \in H_1 \). The classical inequality of Hardy type is well known in the trigonometric as well as in the Vilenkin-Fourier analysis. Namely,
\[ \sum_{k=1}^{\infty} \frac{1}{k} \left| \frac{\hat{f}(k)}{k} \right| \leq c \| f \|_{H_1}, \]
where the function \( f \) belongs to the Hardy space \( H_1 \) and \( c \) is an absolute constant. This was proved in the trigonometric case by Hardy and Littlewood [37] (see also the book [11]) and for the Walsh system it was proved in the book [55].

Weisz [81, 83] generalized this result for the bounded Vilenkin systems and proved that there is an absolute constant \( c_p \), depending only on \( p \), such that
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \left( \frac{\hat{f}(k)}{k} \right)^p \leq c_p \| f \|_{H_p}, \quad (f \in H_p, \ 0 < p \leq 2) \]
Paley [48] proved that the Walsh-Fourier coefficients of a function \( f \in L_p \) (\( 1 < p < 2 \)) satisfy the condition
\[
\sum_{k=1}^{\infty} \left| \hat{f}(2^k) \right|^2 < \infty.
\]

This result fails to hold for \( p = 1 \). However, it in fact holds for functions \( f \in L_1 \), such that the maximal function \( f^* \) belongs \( L_1 \), i.e. \( f \in H_1 \) (see e.g. the book [11]).

Let \( 0 < p \leq 1 \) and \( f \in H_p \). For Vilenkin systems the following inequality was proved by Simon and Weisz [62]:
\[
\left( \sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 \right)^{1/2} \leq c_p \| f \|_{H_p}.
\]

(2.2)

By using the Lebesgue constants we easily obtain that \( S_{n_k}f \) convergence to \( f \) in \( L_1 \)-norm, for every integrable function \( f \), if and only if \( \sup_k L_{n_k} \leq c < \infty \). There are various results when \( p > 1 \). It is also well-known that (see e.g. the book [55])
\[
\| S_n f \|_p \leq c_p \| f \|_p, \quad \text{when} \ p > 1,
\]
but it can be proved also a more stronger result (see e.g. the book [55]):
\[
\| S^* f \|_p \leq c_p \| f \|_p, \quad \text{when} \ f \in L_p, \ p > 1.
\]

Uniform and point-wise convergence and some approximation properties of partial sums in \( L_1 \) norms was investigated by Goginava [30] (see also [31]), Schneider [64] and Avdispahić and Memić [1]. Fine [12] derived sufficient conditions for the uniform convergence, which are in complete analogy with the Dini-Lipschitz conditions. Gulićev [36] estimated the rate of uniform convergence of a Walsh-Fourier series using Lebesgue constants and modulus of continuity. Uniform convergence of subsequences of partial sums was investigated also in Goginava and Tkebuchava [34]. This corresponding problem was considered for Vilenkin groups \( G_m \) by Fridli [13] and Gát [21].

It is known (for details see e.g. the books [55] and [81] [83]) that the subsequence \( S_{M_n} \) of the partial sums is bounded from the martingale Hardy space \( H_p \) to the Lebesgue space \( L_p \), for all \( p > 0 \). However, (see Tephnadze [72]) there exists a martingale \( f \in H_p \) \((0 < p < 1)\), such that
\[
\sup_{n \in \mathbb{N}} \| S_{M_{n+1}} f \|_{weak-L_p} = \infty.
\]

The reason of the divergence of \( S_{M_{n+1}} f \) is that when \( 0 < p < 1 \) the Fourier coefficients of \( f \in H_p \) are not uniformly bounded (see Tephnadze [71]). However, Simon [60] proved that there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[
\sum_{k=1}^{\infty} \| S_k f \|_p^{2-p} \leq c_p \| f \|_{H_p}^p, \quad (0 < p < 1)
\]
for all \( f \in H_p \).

It is also well-known that Vilenkin systems do not form bases in the space \( L_1 \). Moreover, there is a function in the Hardy space \( H_1 \), such that the partial sums of \( f \) are not bounded in the \( L_1 \)-norm. However, in Gát [18] the following strong convergence result was obtained:

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \left\| \frac{S_k f - f}{k} \right\|_1 = 0
\]

for all \( f \in H_1 \).

For the trigonometric analogue see Smith [63] and for the Walsh-Paley system see Simon [58].

Moreover, there exists an absolute constant \( c \), such that

\[
\frac{1}{\log n} \sum_{k=1}^{n} \left\| \frac{S_k f}{k} \right\|_1 \leq c \left\| f \right\|_{H_1} \quad (n = 2, 3, \ldots)
\]

and

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \left\| \frac{S_k f}{k} \right\|_1 = \left\| f \right\|_{H_1},
\]

for all \( f \in H_1 \).

2.2 Estimations of the Vilenkin-Fourier coefficients on martingale Hardy spaces

In this section we prove that the Fourier coefficients of \( f \in H_p, 0 < p < 1 \), are not uniformly bounded. Moreover, we prove a new estimate of the Vilenkin-Fourier coefficients (see Theorem 2.1). By applying this estimate we can obtain an independent proof of (2.1) and (2.2).

The next theorem can be found in Tephnadze [71]:

**Theorem 2.1** a) Let \( 0 < p < 1 \) and \( f \in H_p \). Then there exists an absolute constant \( c_p \), depending only on \( p \), such that

\[
\left| \hat{f}(n) \right| \leq c_p n^{1/p-1} \left\| f \right\|_{H_p}.
\]

b) Let \( 0 < p < 1 \) and \( \{\Phi_n : n \in \mathbb{N}\} \) be any non-decreasing sequence, satisfying the condition

\[
\lim_{n \to \infty} \frac{n^{1/p-1}}{\Phi_n} = \infty. \quad (2.3)
\]

Then there exists a martingale \( f \in H_p \), such that

\[
\lim_{n \to \infty} \frac{\left| \hat{f}(n) \right|}{\Phi_n} = \infty.
\]
2. Fourier coefficients and partial sums of Vilenkin-Fourier series on Martingale Hardy spaces

Proof: Since
\[
\left( \sum_{k=0}^{\infty} |\lambda_k| \right)^p \leq \sum_{k=0}^{\infty} |\lambda_k|^p, \quad (0 < p \leq 1)
\]
by applying Lemma 1.38 we obtain that the proof of part a) will be complete if we show that
\[
\frac{|\hat{a}(n)|}{(n+1)^{1/p-1}} \leq c < \infty, \quad \text{when } 0 < p < 1,
\]
for every \( p \)-atom \( a \), where \( I \) denotes the support of the atom.

Let \( a \) be an arbitrary \( p \)-atom with support \( I \) and \( \mu(I) = M_N^{-1} \). We may assume that \( I = I_N \). It is easy to see that \( \hat{a}(n) = 0 \) when \( n \leq M_N \). Therefore, we can suppose that \( n > M_N \).

According to the estimate \( \|a\|_{\infty} \leq M_N^{1/p} \) we can write that
\[
\frac{|\hat{a}(n)|}{(n+1)^{1/p-1}} \leq \frac{1}{(n+1)^{1/p-1}} \int_{I_N} \|a\|_{\infty} \, d\mu
\]
(2.4)
\[
\leq \frac{1}{(n+1)^{1/p-1}} \int_{I_N} M_N^{1/p} \, d\mu = \frac{M_N^{1/p} |I_N|}{(n+1)^{1/p-1}}
\]
\[
\leq \frac{M_N^{1/p-1}}{(n+1)^{1/p-1}} \leq c < \infty, \quad \text{when } 0 < p < 1.
\]

It follows that
\[
|\hat{f}(n)| \leq c_p n^{1/p-1} \|f\|_{H_p}
\]
and the proof of the part a) is complete.

Let \( 0 < p < 1 \) and \( \{\Phi_n : n \in \mathbb{N}\} \) be any non-decreasing, non-negative sequence, satisfying condition (2.3). Then, for every \( 0 < p < 1 \), there exists an increasing sequence \( \{\alpha_k : k \in \mathbb{N}\} \) of positive integers such that
\[
\lim_{k \to \infty} \frac{M_{2\alpha_k}^{(1/p-1)/2}}{\Phi_{2\alpha_k}^{1/2}} = \infty
\]
and
\[
\sum_{k=0}^{\infty} \frac{\Phi_{2\alpha_k}^{p/2}}{M_{2\alpha_k}^{(1-p)/2}} < \infty. \quad (2.5)
\]

Let \( f = (f(n), n \in \mathbb{N}) \) be from Example 1.47 where
\[
\lambda_k = \frac{\chi \Phi_{2\alpha_k}^{1/2}}{M_{2\alpha_k}^{(1/p-1)/2}}.
\]
In view of (2.5) we conclude that (1.55) is satisfied and by using Example 1.47 we obtain that \( f \in H_p \).

By now using (1.81) with \( \lambda_k \) defined by (2.6) we readily see that
\[
\hat{f}(j) = \begin{cases} 
M^{(1/p-1)/2} \Phi^{1/2}_{M2\alpha_k}, & j \in \{M2\alpha_k, \ldots, M2\alpha_{k+1} - 1\}, \ k \in \mathbb{N}, \\
0, & j \notin \bigcup_{k=1}^{\infty} \{M2\alpha_k, \ldots, M2\alpha_{k+1} - 1\}.
\end{cases}
\]

By combining (2.6) and (2.7) we find that
\[
\lim_{n \to \infty} \frac{\hat{f}(n)}{\Phi_n} \geq \lim_{k \to \infty} \frac{\hat{f}(M2\alpha_k)}{\Phi_{M2\alpha_k}} 
\geq \lim_{k \to \infty} \frac{M^{(1/p-1)/2} \Phi^{1/2}_{M2\alpha_k}}{\Phi_{M2\alpha_k}} 
\geq \lim_{k \to \infty} \frac{M^{(1/p-1)/2}}{\Phi^{1/2}_{M2\alpha_k}} = \infty.
\]

The proof is complete.

2.3 Hardy and Paley type inequalities on martingale Hardy spaces

By using Theorem 2.1 we prove Hardy type inequality (2.1) for \( 0 < p \leq 1 \). Moreover, we also show (see Tephnadze [69]) that the sequence \( \{1/k^{2-p} : k \geq 0\} \) in the inequality below can not be improved:

**Theorem 2.2**

a) Let \( 0 < p \leq 1 \) and \( f \in H_p \). Then there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[
\sum_{k=1}^{\infty} \left| \frac{\hat{f}(k)}{k^{2-p}} \right|^p \leq c_p \|f\|_{H_p}^p.
\]

b) Let \( \{\Phi_n : n \in \mathbb{N}\} \) be any non-decreasing sequence, satisfying the condition
\[
\lim_{n \to \infty} \Phi_n = +\infty.
\]

Then there exists a martingale \( f \in H_p \), such that
\[
\sum_{k=1}^{\infty} \left| \frac{\hat{f}(k)}{k^{2-p}} \Phi_k \right|^p \Phi_k = \infty, \ (0 < p \leq 2).
\]
Proof: By applying Lemma 1.38 we obtain that the proof of the part a) will be complete, if we show that
\[
\sum_{k=1}^{\infty} \frac{|\hat{a}(k)|^p}{k^{2-p}} < c < \infty.
\]
for every \( p \)-atom \( a \) \((0 < p \leq 1)\), where \( I \) denotes the support of the atom. Let \( a \) be an arbitrary \( p \)-atom with support \( I \) and \( \mu(I) = M_N^{-1} \). Analogously we may assume that \( I = \mathbb{I}_N \) and \( n > M_N \).

First, we consider the case \( 0 < p < 1 \). According to (2.4) in Theorem 2.1 we readily get that
\[
\sum_{k=M_N+1}^{\infty} \frac{|\hat{a}(k)|^p}{k^{2-p}} \leq \sum_{k=M_N+1}^{\infty} \frac{1}{k} \frac{|\hat{a}(k)|}{k^{1/p-1}} \leq \sum_{k=M_N+1}^{\infty} \frac{1}{k} M_N^{1-p} \leq M_N^{1-p} \sum_{k=M_N+1}^{\infty} \frac{1}{k^{2-p}} \leq c_p < \infty.
\]
Let \( p = 1 \). Hence, Schwatz and Bessel inequalities imply that
\[
\sum_{k=M_N+1}^{\infty} \frac{|\hat{a}(k)|}{k} \leq \left( \sum_{k=M_N+1}^{\infty} |\hat{a}(k)|^2 \right)^{1/2} \left( \sum_{k=M_N+1}^{\infty} \frac{1}{k^2} \right)^{1/2} \leq \frac{1}{M_N^{1/2}} \|a\|_2
\]
\[
= \frac{1}{M_N^{1/2}} \left( \int_{\mathbb{I}_N} |a|^2 \, d\mu \right)^{1/2}
\]
\[
= \frac{1}{M_N^{1/2}} \left( \int_{\mathbb{I}_N} M_N^2 \, d\mu \right)^{1/2}
\]
\[
= \frac{1}{M_N^{1/2}} M_N^{1/2} < c < \infty.
\]
The proof of part a) is complete.

Next we note that for every $0 < p < 1$ there exists an increasing sequence \( \{ \alpha_k \geq 2 : k \in \mathbb{N} \} \) of positive integers such that

\[
\lim_{k \to \infty} \Phi_{M_{2 \alpha_k}}^{1/2} = \infty
\]

and

\[
\sum_{k=1}^{\infty} \frac{1}{\Phi_{M_{2 \alpha_k}}^{p/4}} < \infty.
\] (2.9)

Let \( f = (f^{(n)}, n \in \mathbb{N}) \) be the martingale defined in Example 1.51. According to (2.9) we conclude that \( f \in H_p \).

Moreover, by using (1.84) from Example 1.51 and obvious estimates we find that

\[
\sum_{l=1}^{M_{2 \alpha_k+1} - 1} \left| \hat{f}(l) \right|^p \Phi_l \frac{M_{2 \alpha_k+1} - 1}{l^{2-p}} \\
= \sum_{n=1}^{k} \sum_{l=M_{2 \alpha_n}}^{M_{2 \alpha_k+1} - 1} \left| \hat{f}(l) \right|^p \Phi_l \frac{M_{2 \alpha_k+1} - 1}{l^{2-p}} \\
\geq \sum_{l=M_{2 \alpha_k}}^{M_{2 \alpha_k+1} - 1} \left| \hat{f}(l) \right|^p \Phi_l \frac{M_{2 \alpha_k+1} - 1}{l^{2-p}} \\
\geq c \Phi_{M_{2 \alpha_k}} \sum_{l=M_{2 \alpha_k}}^{M_{2 \alpha_k+1} - 1} \left| \hat{f}(l) \right|^p \Phi_l \frac{M_{2 \alpha_k+1} - 1}{l^{2-p}} \\
\geq c \Phi_{M_{2 \alpha_k}} \Phi_{M_{2 \alpha_k}}^{1/4} \frac{1}{\Phi_{M_{2 \alpha_k}}^{p/4}} \sum_{l=M_{2 \alpha_k}}^{M_{2 \alpha_k+1} - 1} \frac{1}{l^{2-p}} \\
\geq c \Phi_{M_{2 \alpha_k}}^{1/2} M_{2 \alpha_k}^{1-p} \frac{1}{M_{2 \alpha_k+1}^{2-p}} \\
\geq c \Phi_{M_{2 \alpha_k}}^{1/2} \to \infty, \quad \text{when} \quad k \to \infty.
\]

The proof is complete.

By using Theorem 2.1 we can also prove a Paley type inequality (2.2) for $0 < p \leq 1$.

Moreover, we show (see Tephnadze [69]) that the sequence \( \left\{ 1/M_k^{2-2/p} : k \in \mathbb{N} \right\} \) in the inequality below can not be improved:

\[ G.Tephnadze \]
Theorem 2.3  

a) Let $0 < p \leq 1$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$
\left( \sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 \right)^{1/2} \leq c_p \|f\|_{H_p}.
$$

b) Let $0 < p \leq 1$ and $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence satisfying the condition

$$
\lim_{n \to \infty} \Phi_n = +\infty.
$$

Then there exists a martingale $f \in H_p$, such that

$$
\sum_{k=1}^{\infty} \frac{\Phi_{M_k}}{M_k^{2/p-2}} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 = \infty.
$$

**Proof:** First, we consider case $0 < p < 1$. By applying Lemma 1.38 for the proof of the part a) it suffices to show that

$$
\sum_{k=N+1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{a}(jM_k) \right|^2 \leq c < \infty
$$

for every $p$-atom $a$ with support $I = I_N$.

According to (2.4) in Theorem 2.1 we readily get that

$$
\sum_{k=N+1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{a}(jM_k) \right|^2
\leq \sum_{k=N+1}^{\infty} m_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{a}(jM_k) \right|^2 M_k^{1/p-1}
\leq \sum_{k=N+1}^{\infty} m_k^{2-2/p} \sum_{j=1}^{m_k-1} \left( \frac{M_k^{1/p-1}}{(jM_k)^{1/p-1}} \right)^2
\leq M_N^{2/p-2} \sum_{k=N+1}^{\infty} m_k^2 \frac{1}{M_k^{2/p-2}} \leq c_p < \infty.
$$

Analogously to (2.8) we can prove this result also in the case when $p = 1$, so the proof of part a) is complete for every $0 < p \leq 1$.

On the other hand, to prove part b) we use the construction of martingale from Example 1.51 (see also part b) of Theorem 2.2). According to (1.84) in Example 1.51 we get that

$$
\sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2
\leq \frac{1}{M_{2\alpha_1}^{2/p-2}} \sum_{k=1}^{\infty} M_{2\alpha_1}^{2/p} \sum_{j=1}^{m_{2\alpha_1}-1} \left| \hat{f}(jM_{2\alpha_1}) \right|^2
$$

.
\[ \geq M_{2\alpha k}^{2-2/p} \Phi M_{2\alpha k} \sum_{j=1}^{m_{\alpha k}-1} |\widehat{f}(jM_{2\alpha k})|^2 \]

\[ \geq c M_{2\alpha k}^{2-2/p} \Phi M_{2\alpha k} \sum_{j=1}^{m_{\alpha k}-1} \frac{M_{2\alpha k}^{2/p-2}}{\Phi M_{2\alpha k}^{1/2}} \]

\[ \geq c \Phi_{M_{2\alpha k}}^{1/2} \rightarrow \infty, \quad \text{when} \quad k \rightarrow \infty. \]

The proof is complete. \[ \blacksquare \]

2.4 Maximal operators of partial sums of Vilenkin-Fourier series on martingale Hardy spaces

In this section we will consider weighted Maximal operators of partial sums of Vilenkin-Fourier series and prove \((H_p, L_p)\) and \((H_p, \text{weak} - L_p)\) type inequalities. We also show sharpness of these theorems in a special sense.

The next theorem can be found in Tephnadze [72].

**Theorem 2.4** a) Let \(0 < p < 1\). Then the maximal operator

\[ \widetilde{S}_p^* f := \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n + 1)^{1/p-1}} \]

is bounded from the Hardy space \(H_p\) to the Lebesgue space \(L_p\).

b) Let \(0 < p < 1\) and \(\{\Phi_n : n \in \mathbb{N}\} \) be any non-decreasing sequence satisfying the condition

\[ \lim_{n \to \infty} \frac{n^{1/p-1}}{\Phi_n} = +\infty. \]  \hspace{1cm} (2.10)

Then

\[ \sup_{k \in \mathbb{N}} \frac{\|S_{M_{2\alpha k}+1} f_k\|_{\text{weak} - L_p}}{\|f_k\|_{H_p}} = \infty. \]

**Proof:** Let \(0 < p < 1\). Since \(\widetilde{S}_p^*\) is bounded from \(L_\infty\) to \(L_\infty\) by Lemma [1.39] we obtain that the proof of part a) will be complete if we show that

\[ \int_I |\widetilde{S}_p^* a|^p d\mu \leq c < \infty \]

for every \(p\)-atom \(a\), where \(I\) denotes the support of the atom.
Let \( a \) be a \( \rho \)-atom with support \( I \) and \( \mu (I) = M_N \). We may assume that \( I = I_N \). It is easy to see that \( S_n(a) = 0 \) when \( n \leq M_N \). Therefore, we can suppose that \( n > M_N \).

Since \( \| a \|_\infty \leq M_N^{1/p} \), it yields that

\[
|S_n(a)| \leq \int_{I_N} |a(t)||D_n(x-t)| \, d\mu(t)
\]

\[
\leq \|a\|_\infty \int_{I_N} |D_n(x-t)| \, d\mu(t)
\]

\[
\leq M_N^{1/p} \int_{I_N} |D_n(x-t)| \, d\mu(t).
\]

Let \( 0 < p < 1 \) and \( x \in I_s \setminus I_{s+1} \). From Lemma 1.6 we get that

\[
\frac{|S_n a(x)|}{(n+1)^{1/p-1}} \leq c M_N^{1/p-1} M_s
\]

(2.11)

Since \( n > M_N \) we can conclude that

\[
\frac{|S_n a(x)|}{(n+1)^{1/p-1}} \leq c M_s, \text{ for } x \in I_s \setminus I_{s+1}, \ 0 \leq s \leq N - 1.
\]

(2.12)

The expression on the right-hand side of (2.12) does not depend on \( n \). Therefore,

\[
\left| \tilde{S}_p a(x) \right| \leq c M_s.
\]

(2.13)

By combining (1.1) and (2.13) we obtain that

\[
\int_{I_N} \left| \tilde{S}_p a(x) \right|^p \, d\mu(x)
\]

\[
= \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \tilde{S}_p a(x) \right|^p \, d\mu(x)
\]

\[
\leq c_p \sum_{s=0}^{N-1} \frac{M_p}{M_s} < c_p < \infty.
\]

The proof of part a) is complete.

Let \( 0 < p < 1 \). Under condition (2.10) there exists positive integers \( \{n_k : k \in \mathbb{N} \} \) such that

\[
\lim_{k \to \infty} \frac{(M_{2n_k} + 2)^{1/p-1}}{\Phi_{M_{2n_k}+2}} = \infty, \quad 0 < p < 1.
\]
To prove part b) we will apply the \( p \)-atoms from Example 1.44. By combining (1.52) and (1.53) we obtain that
\[
\frac{|S_{M2n_k+1} f_k|}{\Phi_{M2n_k+2}} = \frac{|D_{M2n_k+1} - D_{M2n_k}|}{\Phi_{M2n_k+2}} = \frac{|\psi_{M2n_k}|}{\Phi_{M2n_k+2}} = 1.
\]
Hence,
\[
\mu \left\{ x \in G_m : \frac{|S_{M2n_k+1} f_k(x)|}{\Phi_{M2n_k+2}} \geq \frac{1}{\Phi_{M2n_k+2}} \right\} = 1. \quad (2.14)
\]
By combining (1.54) and (2.14) we have that
\[
\frac{1}{\Phi_{M2n_k+2}} \left( \mu \left\{ x \in G_m : \frac{|S_{M2n_k+1} f_k|}{\Phi_{M2n_k+2}} \geq \frac{1}{\Phi_{M2n_k+2}} \right\} \right)^{1/p} \geq \frac{1}{\Phi_{M2n_k+2} M_{2n_k}^{1-1/p}} = \left( \frac{M_{2n_k} + 2}{\Phi_{M2n_k+2}} \right)^{1/p-1} \rightarrow \infty, \quad \text{when} \ k \rightarrow \infty.
\]
The proof is complete.

The next corollary is very important for our further investigation:

**Corollary 2.5** Let \( 0 < p < 1 \) and \( f \in H_p \). Then there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[
\|S_n f\|_p \leq c_p (n+1)^{1/p-1} \|f\|_{H_p}, \quad n \in \mathbb{N}_+.
\]

**Proof:** According to part a) of Theorem 2.4 we conclude that
\[
\left\| \frac{S_n f}{(n+1)^{1/p-1}} \right\|_p \leq \left\| \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}} \right\|_p \leq c_p \|f\|_{H_p}, \quad n \in \mathbb{N}_+.
\]
The proof is complete.
Corollary 2.6 Let $0 < p < 1$ and $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence, satisfying the condition (2.10). Then there exists a martingale $f \in H_p$, such that
\[
\sup_{n \in \mathbb{N}} \left\| \frac{S_n f}{\Phi_n} \right\|_{\text{weak-}L_p} = \infty.
\]

Corollary 2.7 Let $0 < p < 1$ and $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence, satisfying the condition (2.10). Then there exists a martingale $f \in H_p$, such that the following maximal operator
\[
\sup_{n \in \mathbb{N}} \frac{|S_n f|}{\Phi_n}
\]
is not bounded from the Hardy space $H_p$ to the space $\text{weak-}L_p$.

The next result can be found in Tephnadze [72].

Theorem 2.8 a) Let $f \in H_1$. Then the maximal operator
\[
\tilde{S}^* f := \sup_{n \in \mathbb{N}} \frac{|S_n f|}{\log (n + 1)}
\]
is bounded from the Hardy space $H_1$ to the space $L_1$.

b) Let $q_n = M_{2n} + M_{2n-2} + \ldots + M_0$ and $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence, satisfying the condition
\[
\lim_{n \to \infty} \frac{\log n}{\Phi_n} = +\infty.
\]

Then
\[
\sup_k \left\| \frac{S_{q_n k} f_k}{\Phi_{q_n k+1}} \right\|_1 = \infty.
\]

Proof: Since $\tilde{S}^*$ is bounded from $L_\infty$ to $L_\infty$, according Lemma [1.39] it is sufficies to show that
\[
\int_I \left| \tilde{S}^* a \right| d\mu \leq c < \infty
\]
for every $p$-atom $a$, where $I$ denotes the support of the atom. We may assume that $I = I_N$. Since $S_n (a) = 0$ when $n \leq M_N$, we can suppose that $n > M_N$.

Since $\|a\|_\infty \leq M_N$ we have that
\[
|S_n (a)| \leq M_N \int_{I_N} |D_n (x - t)| d\mu (t).
\]
Let \( x \in I_s \setminus I_{s+1} \). From Lemma 1.6 we get that
\[
\frac{|S_n a(x)|}{\log (n+1)} \leq \frac{M_s}{\log (n+1)}.
\]
Since \( n > M_N \) we can conclude that
\[
\frac{|S_n a(x)|}{\log (n+1)} \leq \frac{cM_s}{N}.
\tag{2.16}
\]
The expression on the right-hand side of (2.16) does not depend on \( n \). Thus,
\[
|\tilde{S}^s a(x)| \leq \frac{cM_s}{N}, \quad \text{for } x \in I_s \setminus I_{s+1}, \ 0 \leq s \leq N - 1.
\tag{2.17}
\]
By combining (1.1) and (2.17) we obtain that
\[
\int_{I_N} |\tilde{S}^s a(x)| \, d\mu(x)
= \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} |\tilde{S}^s a(x)| \, d\mu(x)
\leq c \frac{N}{N} \sum_{s=0}^{N-1} M_s \leq \frac{cN}{N} < c < \infty.
\]
The proof of part a) is complete.

Next, we note that under condition (2.15) there exists \( \{n_k : k \in \mathbb{N}\} \), such that
\[
\lim_{k \to \infty} \frac{\log q_{n_k+1}}{\Phi_{q_{n_k+1}}} = \infty.
\]
Let \( f_k \) be 1-atom from Example 1.44. By applying Corollary 1.10 with (1.54) we have that
\[
\left\| \frac{S_{q_{n_k} f_k(x)}}{\Phi_{q_{n_k+1}}} \right\|_{H_1}
\geq \frac{1}{\left\| f_k \right\|_{H_1}} \left( \left\| \frac{D_{q_{n_k}}}{\Phi_{q_{n_k+1}}} \right\|_1 - \left\| \frac{D_{M_{2q_k}}}{\Phi_{q_{n_k+1}}} \right\|_1 \right)
\geq \frac{c}{\Phi_{q_{n_k+1}}} (\log q_{n_k} - 1)
\geq \frac{c \log q_{n_k+1}}{\Phi_{q_{n_k+1}}} \to \infty, \text{ when } k \to \infty.
\]
Hence, also part b) is proved so the proof is complete.

The next corollary is very important for our further investigation:
Corollary 2.9 Let $f \in H_1$. Then there exists an absolute constant $c$, such that
\[ \|S_nf\|_1 \leq c \log (n+1) \|f\|_{H_1}, \quad n \in \mathbb{N}_+. \]

Proof: According to part a) of Theorem 2.8 we readily conclude that
\[ \left\| \frac{S_nf}{\log (n+1)} \right\|_1 \leq \frac{\sup_{n \in \mathbb{N}} |S_nf|}{\log (n+1)} \leq c \|f\|_{H_1}, \quad n \in \mathbb{N}_+. \]
The proof is complete.

We also point out two more consequences of Theorem 2.8:

Corollary 2.10 Let $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence, satisfying the condition (2.15). Then there exists a martingale $f \in H_1$ such that
\[ \sup_{n \in \mathbb{N}} \left\| \frac{S_nf}{\Phi_n} \right\|_1 = \infty. \]

Corollary 2.11 Let $0 < p < 1$ and $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence, satisfying the condition (2.15). Then there exists a martingale $f \in H_1$ such that the following maximal operator
\[ \sup_{n \in \mathbb{N}} \frac{|S_nf|}{\Phi_n} \]
is not bounded from the Hardy space $H_1$ to the Lebesgue space $L_1$.

Now, we formulate a theorem about boundedness of a restricted maximal operator of partial sums. This result is presented for the first time.

Theorem 2.12 a) Let $0 < p \leq 1$ and $\{\alpha_k : k \in \mathbb{N}\}$ be a subsequence of positive numbers, such that
\[ \sup_{k \in \mathbb{N}} \rho(\alpha_k) = \kappa < \infty. \quad (2.18) \]
Then the maximal operator
\[ \widetilde{S}^\star, \triangle f := \sup_{k \in \mathbb{N}} |S_{\alpha_k}f| \]
is bounded from the Hardy space $H_p$ to the space $L_p$.

b) Let $0 < p < 1$ and $\{\alpha_k : k \in \mathbb{N}\}$ be a subsequence of positive numbers satisfying the condition
\[ \sup_{k \in \mathbb{N}} \rho(\alpha_k) = \infty. \quad (2.19) \]
Then there exists a martingale $f \in H_p$ such that
\[ \sup_{k \in \mathbb{N}} \|S_{\alpha_k}f\|_{weak-L_p} = \infty, \quad (0 < p < 1). \]
Proof: By using (1.14) in Lemma 1.3 and Corollary 1.5 we easily conclude that if condition (2.18) holds, then
\[
\|D_{\alpha_k}\|_1 \leq c \sum_{j=\langle \alpha_k \rangle}^{|\alpha_k|} \|D_M\|_1 \\
\leq \sum_{j=\langle \alpha_k \rangle}^{|\alpha_k|} 1 = \rho(\alpha_k) + 1 \leq c < \infty.
\]

It follows that \(\tilde{S}^{*,\Delta}\) is bounded from \(L_\infty\) to \(L_\infty\). By Lemma 1.39 we obtain that the proof of part a) will be complete if we show that
\[
\int_{I_N} |\tilde{S}^{*,\Delta} a| \, d\mu \leq c < \infty
\]
for every \(p\)-atom \(a\) with support \(I = I_N\). Since \(S_{\alpha_k}(a) = 0\) when \(\alpha_k \leq M_N\), we can suppose that \(\alpha_k > M_N\).

Let \(t \in I_N\) and \(x \in I_s \setminus I_{s+1}, 1 \leq s \leq \langle \alpha_k \rangle - 1\). By using (1.14) in Lemma 1.3 we get that
\[
D_{\alpha_k}(x - t) = 0
\]
and
\[
|S_{\alpha_k}(a)| = 0. \tag{2.20}
\]

Let \(0 < p \leq 1, t \in I_N\) and \(x \in I_s \setminus I_{s+1}, \langle \alpha_k \rangle \leq s \leq N - 1\). By applying the fact that \(\|a\|_\infty \leq M_N\) and Lemma 1.6 we find that
\[
|S_{\alpha_k}(a)| \leq M_N^{1/p} \int_{I_N} |D_{\alpha_k}(x - t)| \, d\mu(t) \leq c_p M_N^{1/p-1} M_s. \tag{2.21}
\]
Set
\[
\rho := \min_{k \in \mathbb{N}} \langle \alpha_k \rangle.
\]
Then, in view of (2.20) and (2.21) we can conclude that
\[
|\tilde{S}^{*,\Delta} a(x)| = 0, \text{ for } x \in I_s \setminus I_{s+1}, 0 \leq s \leq \rho \tag{2.22}
\]
and
\[
|\tilde{S}^{*,\Delta} a(x)| \leq c_p M_N^{1/p-1} M_s, \text{ for } x \in I_s \setminus I_{s+1}, \rho \leq s \leq N - 1. \tag{2.23}
\]

By the definition of \(\rho\) there exists at least one index \(k_{i_0} \in \mathbb{N}\) such that \(\rho = \langle \alpha_{k_{i_0}} \rangle\). It follows that
\[
N - \rho = N - \langle \alpha_{k_{i_0}} \rangle \leq |\alpha_{k_{i_0}}| - \langle \alpha_{k_{i_0}} \rangle. \tag{2.24}
\]
\[ \leq \sup_{k \in \mathbb{N}} \rho (\alpha_k) = \zeta < c < \infty \]

and

\[ \frac{M_N^{1-p}}{M_\varrho^{1-p}} \leq \lambda^{(N-\varrho)(1-p)} \leq \lambda^{\zeta(1-p)} < c < \infty, \tag{2.25} \]

where \( \lambda = \sup_k m_k \).

Let \( p = 1 \). We combine (2.22)-(2.24) and invoke identity (1.1) to obtain that

\[ \int_{I_N} \left| \mathcal{S}_{s, \Delta} a(x) \right| \, d\mu(x) = \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \mathcal{S}_{s, \Delta} a(x) \right| \, d\mu(x) = \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \mathcal{S}_{s, \Delta} a(x) \right| \, d\mu(x) \leq c \sum_{s=0+1}^{N-1} \frac{M_s}{M_s} = c \sum_{s=\varrho+1}^{N-1} 1 \leq \zeta < c < \infty. \]

Let \( 0 < p < 1 \). According to (1.1) by using (2.22), (2.23) and (2.25) we obtain that

\[ \int_{I_N} \left| \mathcal{S}_{s, \Delta} a(x) \right|^p \, d\mu(x) = \sum_{s=\varrho}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \mathcal{S}_{s, \Delta} a(x) \right|^p \, d\mu(x) \leq c_p M_1^{1-p} \sum_{s=\varrho+1}^{N-1} \frac{M_s^p}{M_s} = c_p M_N^{1-p} \sum_{s=\varrho+1}^{N-1} \frac{1}{M_s^{1-p}} \leq \frac{c_p M_N^{1-p}}{M_\varrho^{1-p}} < c < \infty. \]

The proof of part a) is proved.

Under condition (2.19) we readily get that

\[ \sup_{k \in \mathbb{N}} \frac{M_{|n_k|}}{M_{(n_k)}} = \infty. \]
Moreover, there exists a sequence \( \{ \alpha_k : \alpha_k \in \mathbb{N} \} \subset \{ n_k : \alpha_k \in \mathbb{N} \} \) such that \( \alpha_0 \geq 3 \) and

\[
\lim_{k \to \infty} \frac{M_{\alpha_k}^{(1-p)/2}}{M_{\alpha_k}^{(1-p)/2}} = \infty
\]

and

\[
\sum_{k=0}^{\infty} \frac{M_{\alpha_k}^{(1-p)/2}}{M_{\alpha_k}^{(1-p)/2}} < c < \infty.
\] (2.26)

Let \( f = (f^{(n)} : n \in \mathbb{N}) \) be the martingale defined in Example 1.48, where

\[
\lambda_k = \frac{\lambda M_{\alpha_k}^{(1/p-1)/2}}{M_{\alpha_k}^{(1/p-1)/2}}.
\] (2.27)

Under condition (2.26) we can conclude that \( f \in H_p \).

By now using (1.68) with \( \lambda_k \) defined by (2.27) we readily see that

\[
\hat{f}(j) = \begin{cases} 
M_{\alpha_k}^{(1/p-1)/2} M_{\alpha_k}^{(1/p-1)/2}, & j \in \{ M_{\alpha_k}, \ldots, M_{\alpha_k+1} \}, \ k \in \mathbb{N}, \\
0, & j \notin \bigcup_{k=0}^{\infty} \{ M_{\alpha_k}, \ldots, M_{\alpha_k+1} \}.
\end{cases}
\]

Let \( M_{\alpha_k} < j < \alpha_k \). By using (1.70) obtained by \( l = k \) in the case when \( \lambda_k \) are given by expression (2.27), we immediately get that

\[
S_{\alpha_k} f = S_{M_{\alpha_k}} f + M_{\alpha_k}^{(1/p-1)/2} M_{\alpha_k}^{(1/p-1)/2} \psi_{M_{\alpha_k}} D_{\alpha_k-M_{\alpha_k}}
\]

\[
= I + II.
\]

According to part a) of Theorem 2.12 for \( I \) we have that

\[
\| I \|^p_{weak-L^p} \leq \left\| S_{M_{\alpha_k}} f \right\|^p_{weak-L^p} \leq c_p \| f \|^p_{H^p} < \infty.
\]

Under condition (2.19) we can conclude that

\[
\langle \alpha_k \rangle \neq |\alpha_k| \quad \text{and} \quad \langle \alpha_k - M_{\alpha_k} \rangle = \langle \alpha_k \rangle.
\]

Let \( x \in I_{\alpha_k} \setminus I_{\alpha_k+1} \). By using Lemma 1.8 we obtain that

\[
|D_{\alpha_k-M_{\alpha_k}}| \geq cM_{\alpha_k}
\]
and
\[ |II| = M^{(1/p-1)/2}M^{(1/p-1)/2} D_{\alpha_k - M_{\alpha_k}} \geq M^{(1/p+1)/2} M^{(1/p-1)/2}. \]

It follows that
\[ \|II\|_{weak-L_p}^p \geq c_p (M^{(1/p+1)/2} M^{(1/p-1)/2})^p \mu \left\{ x \in G_m : |II| \geq c_p M^{(1/p+1)/2} M^{(1/p-1)/2} \right\} \]
\[ \geq c_p M^{(1-p)/2} M^{(1+p)/2} \mu \left\{ I_{\langle \alpha_k \rangle - 1, \langle \alpha_k \rangle} \right\} \geq c_p \frac{M^{(1-p)/2}}{M^{(1-p)/2}}. \]

Hence, for large \( k \),
\[ \|S_{\alpha_k} f\|_{weak-L_p}^p \geq \|II\|_{weak-L_p}^p - \|I\|_{weak-L_p}^p \]
\[ \geq \frac{1}{2} \|II\|_{weak-L_p}^p \geq \frac{c_p M^{(1-p)/2}}{M^{(1-p)/2}} \rightarrow \infty, \text{ when } k \rightarrow \infty. \]

The proof is complete.

We also mention the following consequences:

**Corollary 2.13** Let \( p > 0 \) and \( f \in H_p \). Then the maximal operator
\[ \tilde{S}_n^* f := \sup_{n \in \mathbb{N}} |S_{M_n} f| \]
is bounded from the Hardy space \( H_p \) to the space \( L_p \).

**Corollary 2.14** Let \( p > 0 \) and \( f \in H_p \). Then the maximal operator
\[ \sup_{n \in \mathbb{N}_+} |S_{M_n + M_{n-1}} f| \]
is bounded from the Hardy space \( H_p \) to the space \( L_p \).

**Corollary 2.15** Let \( p > 0 \) and \( f \in H_p \). Then the maximal operator
\[ \sup_{n \in \mathbb{N}_+} |S_{M_n + 1} f| \]
is not bounded from the Hardy space \( H_p \) to the space \( L_p \).
2.5 Norm convergence of partial sums of Vilenkin-Fourier series on martingale Hardy spaces

By applying Corollaries 2.5 and 2.9 we find necessary and sufficient conditions for the modulus of continuity of martingale Hardy spaces $H_p$, for which the partial sums of Vilenkin-Fourier series convergence in $L_p$-norm. We also study sharpness of these results. All results in this section can be found in Tephnadze [72].

**Theorem 2.16** Let $0 < p < 1$ and $f \in H_p$. Then there exists an absolute constant $c_p$ depending only on $p$ such that

$$
\|S_n f\|_{H_p} \leq c_p n^{1/p - 1} \|f\|_{H_p}.
$$

**Remark 2.17** We note that the asymptotic behaviour of the sequence $\{n^{1/p - 1} : n \in \mathbb{N}\}$ in Theorem 2.16 can not be improved (c.f. part b) of Theorem 2.4).

**Proof**: According to Corollaries 2.5 and 2.13 if we invoke Example 1.45 we can conclude that

$$
\|S_n f\|_{H_p} \leq \|S^# f\|_p + \|S_n f\|_p
\leq \|f\|_{H_p} + c_p n^{1/p - 1} \|f\|_{H_p}
\leq c_p n^{1/p - 1} \|f\|_{H_p}.
$$

The proof is complete. ■

**Theorem 2.18** Let $f \in H_1$. Then there exists an absolute constant $c$ such that

$$
\|S_n f\|_{H_1} \leq c \log n \|f\|_{H_1}.
$$

**Remark 2.19** We note that the asymptotic behaviour of the sequence $\{\log n : n \in \mathbb{N}\}$ in Theorem 2.18 can not be improved (c.f. part b) of Theorem 2.3).

**Proof**: According to Corollaries 2.9 and 2.13 if we invoke Example 1.45 we can write that

$$
\|S_n f\|_{H_1} \leq \|S^# f\|_1 + \|S_n f\|_1
\leq \|f\|_{H_1} + c \log n \|f\|_{H_1}
\leq c \log n \|f\|_{H_1}.
$$

The proof is complete. ■

**Theorem 2.20** Let $p > 0$ and $f \in H_p$. Then there exists an absolute constant $c_p$ depending only on $p$ such that

$$
\|S_{M_n} f\|_{H_p} \leq c_p \|f\|_{H_p}.
$$
Proof: In the view of Corollary 2.13 and Example 1.45 we can conclude that

\[
\|S_n f\|_{H^p} \leq \left\| \sup_{0 \leq l \leq n} |S_{M_l} f| \right\|_p \leq \|f\|_{H^p}.
\]

The proof is complete.

Theorem 2.21 Let \(0 < p < 1\), \(f \in H_p\) and \(M_k < n \leq M_{k+1}\). Then there is an absolute constant \(c_p\) depending only on \(p\) such that

\[
\|S_n f - f\|_{H^p} \leq c_p n^{1/p - 1} \omega_{H^p} \left( \frac{1}{M_k}, f \right).
\]

Proof: Let \(0 < p < 1\) and \(M_k < n \leq M_{k+1}\). By using Corollary 2.5 we immediately get that

\[
\|S_n f - f\|_{H^p} \leq \|S_n (S_{M_k} f - f)\|_{H^p} + \|S_{M_k} f - f\|_{H^p}
\]
\[
\leq c_p (n^{1-p} + 1) \|S_{M_k} f - f\|_{H^p}
\]
\[
\leq c_p n^{1-p} \omega_{H^p} \left( \frac{1}{M_k}, f \right).
\]

The proof is complete.

Theorem 2.22 Let \(f \in H_1\) and \(M_k < n \leq M_{k+1}\). Then there is an absolute constant \(c\) such that

\[
\|S_n f - f\|_{H_1} \leq c \log n \omega_{H_1} \left( \frac{1}{M_k}, f \right).
\]

Proof: Let \(M_k < n \leq M_{k+1}\). Then, by using Corollary 2.9 we immediately get that

\[
\|S_n f - f\|_{H_1} \leq c \log n \omega_{H_1} \left( \frac{1}{M_k}, f \right).
\]

The proof is complete.
**Theorem 2.23**  

a) Let \( 0 < p < 1 \), \( f \in H_p \) and  
\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) = o \left( \frac{1}{M_n^{1/p-1}} \right), \text{ when } n \to \infty.
\]

Then  
\[
\| S_k f - f \|_{H_p} \to 0, \text{ when } k \to \infty.
\]

b) For every \( 0 < p < 1 \) there exists a martingale \( f \in H_p \), for which  
\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) = O \left( \frac{1}{M_n^{1/p-1}} \right), \text{ when } n \to \infty
\]
and  
\[
\| S_k f - f \|_{\text{weak-}L_p} \nrightarrow 0, \text{ when } k \to \infty.
\]

**Proof:** Let \( 0 < p < 1 \), \( f \in H_p \) and  
\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) = o \left( \frac{1}{M_n^{1/p-1}} \right), \text{ when } n \to \infty.
\]

By using Theorem 2.21 we immediately get that  
\[
\| S_n f - f \|_{H_p} \to 0, \text{ when } n \to \infty.
\]

The proof of part a) is complete.

Let \( f = \left( f^{(n)} : n \in \mathbb{N} \right) \) be the martingale defined in Example 1.47, where  
\[
\lambda_k = \frac{\lambda}{M_{2\alpha_k}} \quad \text{where } \lambda = \sup_{n \in \mathbb{N}} m_n,
\]

(2.29)

Since  
\[
\sum_{k=0}^{\infty} \frac{\lambda^p}{M_{2\alpha_k}^{1-p}} < c < \infty
\]
we conclude that \( f \in H_p \). Moreover, according to (1.58) we find that  
\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) = O \left( \sum_{k=n}^{\infty} \frac{1}{M_k^{1/p-1}} \right)
\]
\[
= O \left( \frac{1}{M_n^{1/p-1}} \right), \text{ when } n \to \infty.
\]

By now using (1.57) with \( \lambda_k \) defined by (2.29) we readily see that
\( \hat{f}(j) = \begin{cases} 
1, & j \in \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}, k \in \mathbb{N}_+, \\
0, & j \notin \bigcup_{k=1}^{\infty} \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}.
\end{cases} \) (2.30)

According to (2.30) and Corollary 1.5 we can establish that

\[
\limsup_{k \to \infty} \| S_{M_{2\alpha_k}+1} f - f \|_{\text{weak}-L^p} \\
\geq \limsup_{k \to \infty} \| S_{M_{2\alpha_k}} f + \hat{f}(M_{2\alpha_k}) \psi_{M_{2\alpha_k}} - f \|_{\text{weak}-L^p} \\
\geq \limsup_{k \to \infty} \left( \| \psi_{M_{2\alpha_k}} \|_{\text{weak}-L^p} - \| f - S_{M_{2\alpha_k}} f \|_{\text{weak}-L^p} \right) \\
\geq \limsup_{k \to \infty} (1 - o(1)) = 1.
\]

This completes the proof of the part b) and the proof is complete.

**Theorem 2.24**

a) Let \( f \in H_1 \) and

\( \omega_{H_1} \left( \frac{1}{M_n}, f \right) = o \left( \frac{1}{n} \right) \), when \( n \to \infty \).

Then

\( \| S_k f - f \|_{H_1} \to 0, \text{ when } k \to \infty. \)

b) There exists a martingale \( f \in H_1 \), for which

\( \omega_{H_1} \left( \frac{1}{M_{2M_n}}, f \right) = O \left( \frac{1}{M_n} \right) \), when \( n \to \infty \)

and

\( \| S_k f - f \|_1 \to 0, \text{ when } k \to \infty. \)

**Proof:** Let \( f \in H_1 \) and

\( \omega_{H_1} \left( \frac{1}{M_n}, f \right) = o \left( \frac{1}{n} \right) \), when \( n \to \infty \).

By using Corollary 2.9 we immediately get that

\( \| S_n f - f \|_{H_1} \to \infty, \text{ when } n \to \infty. \)

For the proof of part b) we use the martingale defined in Example 1.52 for \( p = 1 \). Then, \( f \in H_1 \) and by applying (1.86) we can conclude that

\( \omega_{H_1} \left( \frac{1}{M_n}, f \right) = O \left( \frac{1}{n} \right), \text{ when } n \to \infty. \)
By combining (1.85) and (1.87) with Corollaries 1.10 and 1.5 we find that
\[
\limsup_{k \to \infty} \| S_{M_k} f - f \|_1 = \limsup_{k \to \infty} \| \psi_{M_k} D_{M_k} f \|_1 + S_{M_k} f - f \|_1
\]
\[
- \limsup_{k \to \infty} \left( \frac{1}{M_{2k}} \| D_{M_k} f \|_1 + \| S_{M_k} f - f \|_1 \right)
\]
\[
\geq c - \limsup_{k \to \infty} \left( \sum_{i=k+1}^{\infty} \frac{1}{M_{2i}} - \frac{1}{M_{2k}} \right) = c > 0.
\]

The proof is complete.

2.6 Strong convergence of partial sums of Vilenkin-Fourier series on martingale Hardy spaces

In this section we prove a Hardy type inequality for partial sums with respect to Vilenkin systems. This result was first proved by Simon [60]. Here we use some new estimations and present a simpler proof, which is due to Tephnadze [72]. We also show sharpness of this result in the special sense (see Tephnadze [69]).

**Theorem 2.25** (Simon [60])

a) Let \( 0 < p < 1 \) and \( f \in H_p \). Then there is an absolute constant \( c_p \), depending only on \( p \), such that
\[
\sum_{k=1}^{\infty} \frac{\| S_k f \|_p^p}{k^{2-p}} \leq c_p \| f \|_{H_p}^p.
\]

b) Let \( 0 < p < 1 \) and \( \{ \Phi_n : n \in \mathbb{N} \} \) be any non-decreasing sequence satisfying the condition
\[
\lim_{n \to \infty} \Phi_n = +\infty.
\]

Then there exists a martingale \( f \in H_p \) such that
\[
\sum_{k=1}^{\infty} \frac{\| S_k f \|_{\text{weak-}L_p}^p}{k^{2-p} \Phi_k} = \infty.
\]

**Proof:** Let \( 0 < p < 1 \). By applying (1.1) with (2.11) in Theorem 2.4 we have that
\[
\sum_{k=M_N}^{\infty} \frac{\| S_k f \|_p^p}{k^{2-p}}
\]

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\[ \leq \sum_{k=M}^{\infty} \frac{1}{k} \int_{I_N} \left| S_k a(x) \right|^p \frac{d\mu(x)}{k^{1/p - 1}} \]
\[ = \sum_{k=M}^{\infty} \frac{1}{k} \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| S_k a(x) \right|^p \frac{d\mu(x)}{k^{1/p - 1}} \]
\[ \leq c \sum_{k=M}^{\infty} \frac{1}{k} \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| M_N^{1/p-1} M_s \right|^p \frac{d\mu(x)}{k^{1/p - 1}} \]
\[ \leq c_p M_N^{1-p} \sum_{k=M}^{\infty} \frac{1}{k^{2-p}} \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} M_s^p \frac{d\mu(x)}{k^{1/p - 1}} \]
\[ \leq c_p M_N^{1-p} \sum_{k=M}^{\infty} \frac{1}{k^{2-p}} \sum_{s=0}^{N-1} M_s^{p-1} d\mu \]
\[ + c_p M_N^{1-p} \sum_{k=M}^{\infty} \frac{1}{k^{2-p}} \leq c_p < \infty. \]

This completes the proof of the part a).

Next, we note that under condition (2.31) there exists an increasing sequence \( \{\alpha_k \geq 2 : k \in \mathbb{N}_+\} \) of positive integers such that
\[ \sum_{k=1}^{\infty} \frac{1}{\Phi_{M_{2\alpha_k}}^{1/4}} < \infty. \]  
(2.32)

Let \( f = (f^{(k)} : k \in \mathbb{N}) \) be the martingale defined in Example 1.51. By using (2.32) we conclude that the martingale \( f \in H_p \).

Let \( M_{\alpha_k} \leq j < M_{\alpha_k+1} \). By using (1.84) if we repeat the steps which leads to (1.70) obtained by \( l = k \) in the case when \( \lambda_k = 1/\Phi_{M_{2\alpha_k}}^{1/4} \) we obtain that
\[
S_j f = \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_k}^{1/p-1}}{\Phi_{M_{2\alpha_k}}^{1/4}} \left( D_{M_{2\alpha_k}+1} - D_{M_{2\alpha_k}} \right) \\
+ \frac{M_{2\alpha_k}^{1/p-1} \psi_{M_{2\alpha_k}} D_{j-M_{2\alpha_k}}}{\Phi_{M_{2\alpha_k}}^{1/4}} \\
:= I + II.
\]

We calculate each term separately. By using Corollary 1.5 with condition \( \alpha_n \geq 2 \) \((n \in \mathbb{N})\) for I we find that
\[ D_{M_{2\alpha_n}} = 0, \text{ for } x \in G_m \setminus I_1 \]
and

$$I = 0, \quad \text{for } x \in G_m \setminus I_1.$$  

Denote by $N_0$ the subset of positive integers $\mathbb{N}_+$, for which $\langle n \rangle = 0$. Then every $n \in N_0$, $M_k < n < M_{k+1}$ ($k > 1$) can be written as

$$n = n_0 M_0 + \sum_{j=1}^{k} n_j M_j,$$

where $n_0 \in \{1, ..., m_0 - 1\}$ and $n_j \in \{0, ..., m_j - 1\}$, ($j \in \mathbb{N}_+$).

Let $j \in N_0$ and $x \in G_m \setminus I_1 = I_0 \setminus I_1$. According to Lemma 1.8 we find that

$$|D_{\alpha_k M_k} - M_{\alpha_k} M_k| \geq c M_0 \geq c > 0$$

and

$$|II| = \frac{M_{2\alpha_k}^{1/p - 1}}{\Phi_{M_{2\alpha_k}}^{1/4}} |D_{j - M_{2\alpha_k}} (x)| = \frac{M_{2\alpha_k}^{1/p - 1}}{\Phi_{M_{2\alpha_k}}^{1/4}}.$$

It follows that

$$|S_j f (x)| = |II| = \frac{M_{2\alpha_k}^{1/p - 1}}{\Phi_{M_{2\alpha_k}}^{1/4}}, \quad \text{for } x \in G_m \setminus I_1,$$

and

$$\|S_j f\|_{\text{weak-}L_p} \geq \frac{M_{2\alpha_k}^{1/p - 1}}{2 \Phi_{M_{2\alpha_k}}^{1/4}} \mu \left( x \in G_m \setminus I_1 : |S_j f (x)| > \frac{M_{2\alpha_k}^{1/p - 1}}{2 \Phi_{M_{2\alpha_k}}^{1/4}} \right)^{1/p}$$

$$= \frac{M_{2\alpha_k}^{1/p - 1}}{2 \Phi_{M_{2\alpha_k}}^{1/4}} |G_m \setminus I_1| \geq \frac{c M_{2\alpha_k}^{1/p - 1}}{\Phi_{M_{2\alpha_k}}^{1/4}}.$$

Since

$$\sum_{n \in N_0 \setminus \{M_k, M_k \leq n \leq M_{k+1}\}} 1 \geq c M_k,$$

where $c$ is an absolute constant, by applying (2.33) we obtain that

$$\sum_{j=1}^{M_{2\alpha_k}^{1/p - 1}} \frac{\|S_j f\|_{\text{weak-}L_p}^p}{j^{2-p}} \Phi_j \geq \sum_{j=M_{2\alpha_k}}^{M_{2\alpha_k+1} - 1} \frac{\|S_j f\|_{\text{weak-}L_p}^p}{j^{2-p}} \Phi_j.$$
Fourier coefficients and partial sums of Vilenkin-Fourier series on Martingale Hardy spaces

\[ \geq \Phi_{M2^{\alpha k}} \sum_{\{j \in \mathbb{N}_0 : M2^{\alpha k} \leq j \leq M2^{\alpha k+1}\}} \frac{\|S_j f\|_{\text{weak} - L_p}^p}{j^{2-p}} \]

\[ \geq c \Phi_{M2^{\alpha k}} \frac{M^{1-p}_{2^{\alpha k}}}{\Phi_{M2^{\alpha k}}} \sum_{\{j \in \mathbb{N}_0 : M2^{\alpha k} \leq j \leq M2^{\alpha k+1}\}} \frac{1}{j^{2-p}} \]

\[ \geq c \Phi_{M2^{\alpha k}}^{3/4} M^{1-p}_{2^{\alpha k}} \sum_{\{j \in \mathbb{N}_0 : M2^{\alpha k} \leq j \leq M2^{\alpha k+1}\}} \frac{1}{M^{2-p}_{2^{\alpha k+1}}} \]

\[ \geq c \Phi_{M2^{\alpha k}}^{3/4} \sum_{\{j \in \mathbb{N}_0 : M2^{\alpha k} \leq j \leq M2^{\alpha k+1}\}} 1 \]

\[ \geq c \Phi_{M2^{\alpha k}}^{3/4} \to \infty, \text{ when } k \to \infty. \]

Hence, also the part b) is proved so the proof is complete.

2.7 An application concerning estimations of Vilenkin-Fourier coefficients

The following inequalities follow from our results:

**Corollary 2.26** Let \( 0 < p < 1 \) and \( f \in H_p \). Then there exists an absolute constant \( c_p \), depending only on \( p \), such that

\[ \left| \hat{f}(n) \right| \leq c_p n^{1/p - 1} \|f\|_{H_p}, \]

\[ \sum_{k=1}^{\infty} \frac{\left| \hat{f}(k) \right|^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (f \in H_p, \ 0 < p < 1) \]

and

\[ \left( \sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left| \hat{f}(jM_k) \right|^2 \right)^{1/2} \leq c_p \|f\|_{H_p}. \]

**Remark 2.27** The first inequality was first proved by Tephnadze [73], the second by Weisz [81, 83] and the third also by Weisz [62]. The proofs are different and simpler than the original ones.
Proof: Let $0 < p < 1$. Since
\[
|\hat{f}(n)| = |S_{n+1}f - S_n f| \leq |S_{n+1}f| + |S_n f| \tag{2.34}
\]
and
\[
\frac{|\hat{f}(n)|}{(n+1)^{1/p-1}} \leq \frac{|S_{n+1}f|}{(n+1)^{1/p-1}} + \frac{|S_n f|}{n^{1/p-1}}.
\]
By using part a) of Theorem 2.4 we can write that
\[
\frac{|\hat{f}(n)|}{(n+1)^{1/p-1}} \leq \left\| \frac{S_{n+1}f}{(n+1)^{1/p-1}} \right\|_p + \left\| \frac{S_n f}{n^{1/p-1}} \right\|_p
\]
\[
\leq c_p \|f\|_{H^p}.
\]
It follows that
\[
|\hat{f}(n)| \leq c_p n^{1/p-1} \|f\|_{H^p}
\]
and the first inequality is proved.

To prove the second inequality we use (2.34) again. We find that
\[
\frac{|\hat{f}(n)|}{(n+1)^{2/p-1}} \leq \frac{|S_{n+1}f|}{(n+1)^{2/p-1}} + \frac{|S_n f|}{n^{2/p-1}}.
\]
By combining (2.35) and Theorem 2.4 we get that
\[
\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^p}{n^{2/p}} \leq \sum_{n=1}^{\infty} \left( \left\| \frac{S_{n+1}f}{(n+1)^{2/p-1}} \right\|_p + \left\| \frac{S_n f}{n^{2/p-1}} \right\|_p \right)^p
\]
\[
\leq \sum_{n=1}^{\infty} \left( \left\| \frac{S_{n+1}f}{(n+1)^{2/p-1}} \right\|_p^p + \left\| \frac{S_n f}{n^{2/p-1}} \right\|_p^p \right)
\]
\[
\leq \sum_{n=1}^{\infty} \left( \left\| S_{n+1}f \right\|_p^p + \left\| S_n f \right\|_p^p \right).
\]

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\[ \leq 2 \sum_{n=1}^{\infty} \frac{\| S_n f \|_p^p}{n^{2-p}} \leq c_p \| f \|_{H_p}^p. \]

and also the second inequality is proved. The third inequality can be proved analogously so we leave the details.

The proof is complete. \hfill \blacksquare
3 Vilenkin-Fejér means on martingale Hardy spaces

3.1 Some classical results on Vilenkin-Fejér means

In the one-dimensional case Yano [78] proved that
\[ \|K_n\| \leq 2 \quad (n \in \mathbb{N}). \]

Consequently,
\[ \|\sigma_n f - f\|_p \to 0, \quad \text{when} \quad n \to \infty, \quad (f \in L_p, \quad 1 \leq p \leq \infty). \]

However (see [39, 55]) the rate of convergence can not be better than \( O(n^{-1}) \) \((n \to \infty)\) for non-constant functions. a.e, if \( f \in L_p, \quad 1 \leq p \leq \infty \) and
\[ \|\sigma_{M_n} f - f\|_p = o\left(\frac{1}{M_n}\right), \quad \text{when} \quad n \to \infty, \]
then \( f \) is a constant function.

Fridli [14] used dyadic modulus of continuity to characterize the set of functions in the space \( L_p \), whose Vilenkin-Fejér means converge at a given rate. It is also known that (see e.g books [2] and [55])
\[ \|\sigma_n f - f\|_p \leq c_p \omega_p\left(\frac{1}{M_n}, f\right) + c_p \sum_{s=0}^{N-1} \frac{M_s}{M_N} \omega_p\left(\frac{1}{M_s}, f\right), \quad (1 \leq p \leq \infty, \quad n \in \mathbb{N}). \]

By applying this estimate, we immediately obtain that if \( f \in \text{lip}(\alpha,p) \), i.e.,
\[ \omega_p\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{M_n^{\alpha}}\right), \quad n \to \infty, \]
then
\[ \|\sigma_n f - f\|_p = \begin{cases} O\left(\frac{1}{M_n}\right), & \text{if} \; \alpha > 1, \\ O\left(\frac{N}{M_N}\right), & \text{if} \; \alpha = 1, \\ O\left(\frac{1}{M_N^{\alpha}}\right), & \text{if} \; \alpha < 1. \end{cases} \]

Weisz [85] considered the norm convergence of Fejér means of Vilenkin-Fourier series and proved that
\[ \|\sigma_k f\|_p \leq c_p \|f\|_{H_p}, \quad p > 1/2 \quad \text{and} \quad f \in H_p. \quad (3.1) \]

This result implies that
\[ \frac{1}{n^{2p-1}} \sum_{k=1}^{n} \|\sigma_k f\|_p^p \leq c_p \|f\|_{H_p}^p, \quad (1/2 < p < \infty). \]
If (3.1) hold for $0 < p \leq 1/2$, then we would have that

$$
\frac{1}{\log^{[1/2+p]} \left( n \sum_{k=1}^{n} \frac{\| \sigma_k f \|^p}{k^{2-2p}} \right)^p} \leq c_p \| f \|^p_{H_p}, \quad (0 < p \leq 1/2). \tag{3.2}
$$

However, in Tephnadze [66] it was shown that the assumption $p > 1/2$ in (3.1) is essential. In particular, it was proved that there exists a martingale $f \in H_{1/2}$ such that

$$
\sup_{n \in \mathbb{N}} \| \sigma_n f \|_{1/2} = +\infty.
$$

For Vilenkin systems in [70] it was proved that (3.2) holds, though inequality (3.1) is not true for $0 < p \leq 1/2$.

In the one-dimensional case the weak type inequality

$$
\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \| f \|_1, \quad (f \in L_1, \ \lambda > 0)
$$

can be found in Zygmund [88] for the trigonometric series, in Schipp [54] for Walsh series and in Pál, Simon [49] for bounded Vilenkin series. Fuji [16] and Simon [57] verified that $\sigma^*$ is bounded from $H_L$ to $L_1$. Weisz [85] generalized this result and proved the boundedness of $\sigma^*$ from the martingale space $H_p$ to the Lebesgue space $L_p$ for $p > 1/2$. Simon [56] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava [27], (see also [9] and [10]). Weisz [86] proved that $\sigma^*$ is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$. In [67] and [68] (for Walsh system see [28]) it was proved that the maximal operator $\tilde{\sigma}_p^*$ with respect to Vilenkin systems defined by

$$
\tilde{\sigma}_p^* := \sup_{n \in \mathbb{N}} \frac{|\sigma_n|}{(n+1)^{1/2-2p} \log^{[1/2+p]} (n+1)},
$$

where $0 < p \leq 1/2$ and $[1/2 + p]$ denotes integer part of $1/2 + p$, is bounded from the Hardy space $H_p$ to the Lebesgue space $L_p$. Moreover, the order of deviant behavior of the $n$-th Fejér mean was given exactly. As a corollary we get that

$$
\| \sigma_n f \|_p \leq c_p (n+1)^{1/2-2p} \log^{[1/2+p]} (n+1) \| f \|_{H_p}.
$$

For Walsh-Kaczmarz system analogical theorems were proved in [33] and [69].

For the one-dimensional Vilenkin-Fourier series Weisz [85] proved that the maximal operator

$$
\sigma^# f = \sup_{n \in \mathbb{N}} |\sigma_{M_n} f|
$$

is bounded from the martingale Hardy space $H_p$ to the Lebesgue space $L_p$ for $p > 0$. For the Walsh-Fourier series Goginava [30] proved that the operator $|\sigma_{2n} f|$ is not bounded from the space $H_p$ to the space $H_p$, for $0 < p \leq 1$. 
3.2 DIVERGENCE OF VILENKIN-FEJÉR MEANS ON MARTINGALE HARDY SPACES

**Theorem 3.1** a) There exist a martingale \( f \in H_{1/2} \) such that

\[
\sup_{n \in \mathbb{N}} \| \sigma_n f \|_{1/2} = +\infty.
\]

b) Let \( 0 < p < 1/2 \). There exist a martingale \( f \in H_p \), such that

\[
\sup_{n \in \mathbb{N}} \| \sigma_n f \|_{\text{weak-}L_p} = +\infty.
\]

**Remark 3.2** This result for \( p = 1/2 \) can be found in Tephnadze [66]. By interpolation automatically follows the second part of this Theorem. Here, we make a more concrete proof of this fact by even constructing a martingale \( f \in H_p \), for which the Vilenkin-Fejér means are not bounded in the space \( \text{weak-}L_p \).

**Proof:** Let \( f = (f^{(n)} : n \in \mathbb{N}) \) be the martingale defined in Example [1.49] in the case when \( p = q = 1/2 \). We can write that

\[
\sigma_{q\alpha_k} f = \frac{1}{q\alpha_k} \sum_{j=1}^{M_{2\alpha_k}} S_j f + \frac{1}{q\alpha_k} \sum_{j=M_{2\alpha_k}+1}^{q\alpha_k} S_j f
\]

:= \( I + II \).

According to (1.76) in Example [1.49] we can conclude that

\[
|I| \leq \frac{1}{q\alpha_k} \sum_{j=1}^{M_{2\alpha_k}} |S_j f| \tag{3.3}
\]

\[
\leq \frac{2\lambda M_{2\alpha_{k-1}}^2}{\alpha_{k-1}^{1/2}} \frac{M_{2\alpha_k}}{q\alpha_k} \leq \frac{2\lambda M_{2\alpha_{k-1}}^2}{\alpha_{k-1}^{1/2}}.
\]

By applying (1.75) obtained by letting \( l = k \) we can rewrite \( II \) as

\[
II = \left( \frac{q\alpha_k - M_{2\alpha_k}}{q\alpha_k} \right) S_{M_{2\alpha_k}}
\]

\[
+ \frac{M_{2\alpha_k}}{\alpha_{k}^{1/2} q\alpha_k} \sum_{j=M_{2\alpha_k}+1}^{q\alpha_k} D_j - M_{2\alpha_k}
\]

:= \( II_1 + II_2 \).

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According to (1.76) for \( j = M_{2\alpha_k} \) and \( p = 1/2 \) we find that

\[
|II_1| \leq |S_{M_{2\alpha_k}}| \leq \frac{2\lambda M_{\alpha_{k-1}}^2}{\alpha_{k-1}^{1/2}}.
\]  

(3.4)

In view of (3.3) and (3.4) we invoke estimate (1.73) for \( p = q = 1/2 \) to conclude that

\[
|II_1| \leq \frac{2\lambda M_{2\alpha_{k-1}}^2}{\alpha_{k-1}^{1/2}} \leq \frac{M_{\alpha_k}^2}{16\alpha_k^{3/2}}.
\]

and

\[
|I| \leq \frac{2\lambda M_{2\alpha_{k-1}}^2}{\alpha_{k-1}^{1/2}} \leq \frac{M_{\alpha_k}^2}{16\alpha_k^{3/2}}.
\]

Let

\[
x \in I_{2^{2\alpha_k}}^{2(\eta+1)}, \quad \eta = \left[ \frac{\alpha_k}{2} \right] + 1, \ldots, \alpha_k - 3.
\]

By applying Lemma 1.19 we have that

\[
4q_{\alpha_k - 1} |K_{q_{\alpha_k - 1}}(x)| \geq M_{2\eta}M_{2(\eta+1)} \geq M_{2\eta}^2.
\]

Hence, for \( II_2 \) we readily get that

\[
|II_2| = \frac{M_{2\alpha_k}}{\alpha_k^{1/2} q_{\alpha_k} \psi_{M_{2\alpha_k}}} \sum_{j=1}^{q_{\alpha_k - 1}} |D_j| \geq \frac{q_{\alpha_k - 1}}{2\alpha_k^{1/2}} |K_{q_{\alpha_k - 1}}| \geq \frac{M_{2\eta}^2}{8\alpha_k^{1/2}}.
\]

Since \( M_{2\eta} \geq M_{\alpha_k} \) we obtain that

\[
|\sigma_{q_{\alpha_k}} f(x)| \geq |II_2| - (|I| + |II_1|) \geq \frac{1}{8\alpha_k^{1/2}} \left( M_{2\eta}^2 - \frac{M_{\alpha_k}^2}{2\alpha_k} \right).
\]  

(3.5)

From (3.5) it follows that

\[
|\sigma_{q_{\alpha_k}} f(x)| \geq \frac{eM_{2\eta}^2}{\alpha_k^{1/2}}, \quad x \in I_{2^{2\alpha_k}}^{2(\eta+1)},
\]
where

\[ \eta = \left\lfloor \frac{\alpha_k}{2} \right\rfloor + 1, \ldots, \alpha_k - 3, \]

Thus,

\[
\int_{G_m} \left| \sigma_{\alpha_k} f(x) \right|^{1/2} \, d\mu(x)
\]

\[ \geq \sum_{\eta=[\alpha_k/2]+1}^{\alpha_k-3} \sum_{x_{2\eta+3}=0}^{m_{2\eta+3}-1} \sum_{x_{2\alpha_k-1}=0}^{m_{2\alpha_k-1}-1} \int_{f_{2\alpha_k}^{2\eta+2(\eta+1)}} \left| \sigma_{\alpha_k} f(x) \right|^{1/2} \, d\mu(x)
\]

\[ \geq \frac{c}{\alpha_k^{1/4}} \sum_{\eta=[\alpha_k/2]+1}^{\alpha_k-3} \frac{m_{2\eta+3} \ldots m_{2\alpha_k-1}}{M_{2\alpha_k}} M_{2\eta}
\]

\[ \geq \frac{c}{\alpha_k^{1/4}} \sum_{\eta=[\alpha_k/2]+1}^{\alpha_k-3} \frac{M_{2\eta}}{M_{2\eta+2}}
\]

\[ \geq \frac{c}{\alpha_k^{1/4}} \sum_{\eta=[\alpha_k/2]+1}^{\alpha_k-3} 1
\]

\[ \geq \alpha_k^{3/4} \rightarrow \infty, \text{ when } k \rightarrow \infty. \]

The proof of part a) is complete so we turn to the proof of part b). Let \( 0 < p < 1/2 \). Let \( f = (f^{(n)} : n \in \mathbb{N}) \) be the martingale defined in Example 1.49 in the case when \( 0 < p < q = 1/2 \). We can write that

\[
\sigma_{M_{2\alpha_k}+1} f
\]

\[ = \frac{1}{M_{2\alpha_k} + 1} \sum_{j=0}^{M_{2\alpha_k}} S_j f + \frac{S_{M_{2\alpha_k}+1} f}{M_{2\alpha_k} + 1}
\]

\[ := III + IV. \]

We combine (1.76) and (1.77) and invoke (1.73) in the case when \( p < q = 1/2 \) to obtain the following estimates:

\[
|III| \leq \frac{M_{2\alpha_k}^{1/p}}{M_{2\alpha_k} + 1} \frac{2\lambda M_{2^{\alpha_k-1}}^{1/p}}{\alpha_k^{1/2} \alpha_k}
\]

\[ \leq \frac{2\lambda M_{2^{\alpha_k-1}}^{1/p}}{\alpha_k^{1/2}} \leq \frac{M_{1/p}^{1-p/2}}{16\alpha_k^{3/2}}
\]
and

$$|IV| \geq \frac{|S_{M_{2\alpha_k}+1}f|}{M_{2\alpha_k}+1} \geq \frac{M_{2\alpha_k}^{1/p-2}}{2\alpha_k}.$$ 

Let $$x \in G_m$$. We conclude that

$$|\sigma_{M_{2\alpha_k}+1}f(x)| \geq |IV| - |III|$$

$$\geq \frac{M_{2\alpha_k}^{1/p-2}}{2\alpha_k^{1/2}} - \frac{M_{2\alpha_k}^{1/p-2}}{16\alpha_k^{3/2}} \geq \frac{M_{2\alpha_k}^{1/p-2}}{4\alpha_k^{1/2}}.$$ 

It follows that

$$\frac{M_{2\alpha_k}^{1/p-2}}{4\alpha_k^{1/2}} \left( \mu \left\{ x \in G_m : \left| \sigma_{M_{2\alpha_k}+1}f(x) \right| \geq \frac{M_{2\alpha_k}^{1/p-2}}{4\alpha_k^{1/2}} \right\} \right)^{1/p} \geq \frac{M_{2\alpha_k}^{1/p-2}}{4\alpha_k^{1/2}} \to \infty, \text{ when } k \to \infty.$$ 

Hence, also part b) is proved so the proof is complete. 

3.3 Maximal Operators of Vilenkin-Fejér Means on Martingale Hardy Spaces

In this subsection we consider weighted Maximal operators of Fejér means of Vilenkin-Fourier series and prove \((H_p, L_p)\) and \((H_p, \text{weak} - L_p)\) type inequalities. In all cases we also show sharpness in a special sense.

First we note the following consequence of Theorem 3.1:

**Corollary 3.3**

a) There exist a martingale \(f \in H_{1/2}\) such that

$$\|\sigma^*f\|_{1/2} = +\infty.$$ 

b) Let \(0 < p < 1/2\). There exist a martingale \(f \in H_p\) such that

$$\|\sigma^*f\|_{\text{weak} - L_p} = +\infty.$$ 

The next theorem can be found in Tephnadze [68], but here we will give a simpler proof of part b).

**Theorem 3.4**

a) Let \(0 < p < 1/2\). Then the maximal operator

$$\bar{\sigma}^*_p f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{(n+1)^{1/p-2}}$$

is bounded from the Hardy martingale space $H_p$ to the Lebesgue space $L_p$.

b) Let $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence satisfying the condition

$$\lim_{n \to \infty} \frac{(n + 1)^{1/p - 2}}{\Phi_n} = +\infty.$$  \hfill (3.6)

Then

$$\sup_{k \in \mathbb{N}} \frac{\|\sigma_{M_{2n_k + 1}} f_k\|_{weal-L_p}}{\|f_k\|_{H_p}} = \infty.$$

**Proof:** First place we note that $\sigma_n$ is bounded from $L_\infty$ to $L_\infty$ (see (1.23) in Corollary 1.16). Hence, by Lemma 1.39 the proof of Theorem 3.4 will be complete if we show that

$$\int_{I_N} \left( \sup_{n \in \mathbb{N}} \frac{|\sigma_n a|}{(n + 1)^{1/p - 2}} \right)^p d\mu \leq c < \infty$$

for every p-atom $a$, where $I$ denotes the support of the atom.

Let $a$ be an arbitrary p-atom with support $I$ and $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $\sigma_n(a) = 0$ when $n \leq M_N$. Therefore we can suppose that $n > M_N$.

Since $\|a\|_\infty \leq M_N^{1/p}$ it follows that

$$\frac{|\sigma_n(a)|}{(n + 1)^{1/p - 2}} \leq \frac{1}{(n + 1)^{1/p - 2}} \int_{I_N} |a(t)| |K_n(x - t)| d\mu(t)$$

$$\leq \frac{\|a\|_\infty}{(n + 1)^{1/p - 2}} \int_{I_N} |K_n(x - t)| d\mu(t)$$

$$\leq c M_N^{1/p} \int_{I_N} |K_n(x - t)| d\mu(t).$$

Let $x \in I^k_N$, $0 \leq k < l \leq N$. From Corollary 1.21 we can deduce that

$$\frac{|\sigma_n(a)|}{(n + 1)^{1/p - 2}} \leq \frac{c_p M_N^{1/p} M_l M_k}{M_N^{1/p - 2} M_N^2} = c_p M_l M_k.$$  \hfill (3.7)

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The expression on the right-hand side of (3.7) does not depend on $n$. Thus,
\[
\left| \tilde{\sigma}_p^* a (x) \right| \leq c_p M_l M_k, \quad \text{for } x \in I_N^{k,l}, \ 0 \leq k < l \leq N. \quad (3.8)
\]

By using (3.8) with identity (1.1) we obtain that
\[
\int_{I_N} \left| \tilde{\sigma}_p^* a (x) \right|^p \, d\mu (x) \\
= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j \in \{l+1, \ldots, N-1\}} \int_{I_N^{k,l}} \left| \tilde{\sigma}_p^* a (x) \right|^p \, d\mu (x) \\
+ \sum_{k=0}^{N-1} \int_{I_N^K} \left| \tilde{\sigma}_p^* a (x) \right|^p \, d\mu (x) \\
\leq c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_l^{p+1} \ldots M_{N-1}^{p-1}}{M_N^{p}} M_l^p M_k^p \\
+ c_p \sum_{k=0}^{N-1} \frac{1}{M_N} M_N^p M_k^p \\
\leq c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_l^p M_k^p}{M_l^{1-2p}} + c_p \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^{k-1}} \leq c_p \leq c_p < \infty.
\]

We estimate each term separately.
\[
I = c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l^{1-2p} M_k^p} \\
\leq c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l^{1-2p}} \\
\leq c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{2^{l(1-2p)}} \\
\leq c_p \sum_{k=0}^{N-2} \frac{1}{2^{k(1-2p)}} < c_p < \infty.
\]

It is obvious that
\[
II \leq \frac{c_p}{M_N^{1-2p}} \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^p} \quad (3.10)
\]
≤ \frac{c_p}{M^{1-2p}_N} < c < \infty.

The proof of the part a) is complete by combining the estimates above.

Let 0 < p < 1/2. Under condition (3.6) there exists an increasing sequence of positive integers \( \{\lambda_k : k \in \mathbb{N}\} \) such that

\[
\lim_{k \to \infty} \frac{\lambda_k^{1/p-2}}{\Phi_{\lambda_k}} = \infty.
\]

It is evident that for every \( \lambda_k \) there exists a positive integers \( m_k^* \) such that \( q_{m_k^*} \leq \lambda_k < 2q_{m_k^*} \). Since \( \{\Phi_n : n \in \mathbb{N}\} \) is a non-decreasing function we have that

\[
\lim_{k \to \infty} \frac{M^{1/p-2}_{2m_k}}{\Phi_{M_{2m_k}+1}} \geq \frac{1}{2k} \lim_{k \to \infty} \frac{(M_{2m_k} + 1)^{1/p-2}}{\Phi_{M_{2m_k}+1}} \geq c \lim_{k \to \infty} \frac{\lambda_k^{1/p-2}}{\Phi_{\lambda_k}} = \infty.
\]

Let \( \{n_k : k \in \mathbb{N}\} \subset \{m_k^* : k \in \mathbb{N}\} \) be a sequence of positive numbers such that

\[
\lim_{k \to \infty} \frac{M^{1/p-2}_{2n_k}}{\Phi_{M_{2n_k}+1}} = \infty
\]

and \( f_k \) be the atom defined in Example 1.44.

By combining (1.52) and (1.53) in Example 1.44 we find that

\[
\frac{\left| \sigma_{M_{2n_k}+1} f_k \right|}{\Phi_{M_{2n_k}+1}} = \frac{1}{\Phi_{M_{2n_k}+1} (M_{2n_k} + 1)} \left| \sum_{j=0}^{M_{2n_k}+1} S_j f_k \right| = \frac{1}{\Phi_{M_{2n_k}+1} (M_{2n_k} + 1)} \left| S_{M_{2n_k}+1} f_k \right| = \frac{1}{\Phi_{M_{2n_k}+1} (M_{2n_k} + 1)} \left| D_{M_{2n_k}+1} - D_{M_{2n_k}} \right|
\]
\[
\frac{1}{\Phi_{M_{2n_k} + 1}(M_{2n_k} + 1)} |\psi_{M_{2n_k}}| \geq \frac{1}{\Phi_{M_{2n_k} + 1}(M_{2n_k} + 1)} c \frac{M_{2n_k} \Phi_{M_{2n_k} + 1}}{M_{2n_k} \Phi_{M_{2n_k} + 1}}.
\]

Hence,
\[
\mu \left\{ x \in G_m : \frac{|\sigma_{M_{2n_k} + 1} f_k(x)|}{\Phi_{M_{2n_k} + 1}} \geq \frac{c}{M_{2n_k} \Phi_{M_{2n_k} + 1}} \right\} \geq \mu(G_m) = 1.
\]

Therefore, by using (1.54) in Example 1.44 we get that
\[
\frac{M_{2n_k} c}{\Phi_{M_{2n_k} + 1}} \left( \mu \left\{ x \in G_m : \frac{|\sigma_{M_{2n_k} + 1} f_k(x)|}{\Phi_{M_{2n_k} + 1}} \geq \frac{c}{M_{2n_k} \Phi_{M_{2n_k} + 1}} \right\} \right)^{1/p} \geq \|f_k\|_{H_p}.
\]

Thus, also part b) is proved so the proof is complete. \hfill \blacksquare

We also point out the following consequence of Theorem 3.4, which we need later on.

**Corollary 3.5** a) Let \(0 < p < 1/2\) and \(f \in H_p\). Then there exists an absolute constant \(c_p\), depending only on \(p\), such that
\[
\|\sigma_n f\|_p \leq c_p (n + 1)^{1/p-2} \|f\|_{H_p}, \quad n \in \mathbb{N}_+.
\]

**Proof:** According to part a) of Theorem 3.4 we conclude that
\[
\left\| \frac{\sigma_n f}{(n + 1)^{1/p-2}} \right\|_p \leq \left\| \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{(n + 1)^{1/p-2}} \right\|_p \leq c \|f\|_{H_p}, \quad n \in \mathbb{N}_+.
\]

The proof is complete. \hfill \blacksquare

We also mention the following corollaries:
Corollary 3.6 Let \( \{ \Phi_n : n \in \mathbb{N} \} \) be any non-decreasing sequence satisfying the condition (3.6). Then there exists a martingale \( f \in H_p \) such that
\[
\sup_{n \in \mathbb{N}} \left\| \frac{\sigma_n f}{\Phi_n} \right\|_{weak-L_p} = \infty.
\]

Corollary 3.7 Let \( \{ \Phi_n : n \in \mathbb{N} \} \) be any non-decreasing sequence satisfying the condition (3.6). Then the following maximal operator
\[
\sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\Phi_n}
\]
is not bounded from the Hardy space \( H_p \) to the space \( weak - L_p \).

The next theorem can be found in Tephnadze [67].

Theorem 3.8 a) The maximal operator
\[
\tilde{\sigma} f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\log^2 (n + 1)}
\]
is bounded from the Hardy space \( H_{1/2} \) to the Lebesgue space \( L_{1/2} \).

b) Let \( \{ \Phi_n : n \in \mathbb{N} \} \) be any non-decreasing sequence satisfying the condition
\[
\lim_{n \to \infty} \frac{\log^2 (n + 1)}{\Phi_n} = +\infty.
\]
(3.11)

Then
\[
\sup_{k \in \mathbb{N}} \left\| \frac{\sigma_{q_n k} f_k}{\Phi_{q_n k}} \right\|_{1/2}^{1/2} = \infty.
\]

**Proof:** First we note that \( \sigma_n \) is bounded from \( L_{\infty} \) to \( L_{\infty} \) (see (1.23) in Corollary 1.16). Hence, according to Lemma 1.39 to prove part a) it suffices to show that
\[
\int_{I_N} \left( \sup_{n \in \mathbb{N}} \frac{|\sigma_n a|}{\log^2 (n + 1)} \right)^{1/2} d\mu \leq c < \infty
\]
for every 1/2-atom \( a \), where \( I \) denotes the support of the atom.

Let \( a \) be an arbitrary 1/2-atom with support \( I \) and \( \mu (I) = M_N^{-1} \). We may assume that \( I = I_N \). It is easy to see that \( \sigma_n (a) = 0 \) when \( n \leq M_N \). Therefore we can suppose that \( n > M_N \).
Since \(\|a\|_{\infty} \leq M_N^2\) we obtain that
\[
\frac{|\sigma_n(a)|}{\log^2(n+1)} \leq \frac{1}{\log^2(n+1)} \int_{I_N} |a(t)||K_n(x-t)| \, d\mu(t)
\]
\[
\leq \frac{\|a\|_{\infty}}{\log^2(n+1)} \int_{I_N} |K_n(x-t)| \, d\mu(t)
\]
\[
\leq \frac{M_N^2}{\log^2(n+1)} \int_{I_N} |K_n(x-t)| \, d\mu(t).
\]

Let \(x \in I_{N}^{k,l}, 0 \leq k < l \leq N\). Then, from Corollary 1.21 it follows that
\[
\frac{|\sigma_n(a)|}{\log^2(n+1)} \leq cM_N^2 \frac{M_lM_k}{N^2} = \frac{cM_lM_k}{N^2}.
\]

The expression on the right-hand side of (3.12) does not depend on \(n\). Therefore,
\[
|\tilde{\sigma}^*a(x)| \leq \frac{cM_lM_k}{N^2}, \quad \text{for} \quad x \in I_N^{k,l}, 0 \leq k < l \leq N.
\]

(3.13)

By applying (3.13) with identity (1.1) we obtain that
\[
\int_{I_N} |\tilde{\sigma}^*a(x)|^{1/2} \, d\mu(x)
\]
\[
= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{m_j=0}^{m_j-1} \int_{I_N^{k,l}} |\tilde{\sigma}^*a(x)|^{1/2} \, d\mu(x)
\]
\[
+ \sum_{k=0}^{N-1} \int_{I_N^{k,k}} |\tilde{\sigma}^*a(x)|^{1/2} \, d\mu(x)
\]
\[
\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \ldots m_{N-1} M_l^{1/2} M_k^{1/2}}{M_N} \quad M_N^{1/2} \quad M_k^{1/2} \quad N
\]
\[
+ c \sum_{k=0}^{N-1} \frac{1}{M_N} \quad M_N^{1/2} \quad M_k^{1/2} \quad N := I + II.
\]

We estimate each term separately.
\[
I \leq \frac{c}{N} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_l^{1/2} M_k^{1/2}}{M_l^{1/2}} \leq \frac{c}{N} \sum_{k=0}^{N-2} 1 \leq c < \infty.
\]
Moreover,
\[ II \leq \frac{1}{M_{1/2}^N} \sum_{k=0}^{N-1} M_k^{1/2} \leq \frac{c}{N} < c < \infty. \]

The proof of part a) is complete.

Let \( \{ \lambda_k : k \in \mathbb{N} \} \) be an increasing sequence of positive integers such that
\[
\lim_{k \to \infty} \frac{\log^2 \left( \frac{\lambda_k + 1}{\Phi_k} \right)}{\Phi_k} = \infty.
\]

It is evident that for every \( \lambda_k \) there exists a positive integers \( m'_k \) such that \( q_{m'_k} \leq \lambda_k < q_{m'_k + 1} < cq_{m'_k} \). Since \( \Phi_n \) is a non-decreasing function we have that
\[
\lim_{k \to \infty} \frac{\left( m'_k \right)^2}{\Phi_{q_{m'_k}}} \geq c \lim_{k \to \infty} \frac{\log^2 \left( \lambda_k + 1 \right)}{\Phi_k} = \infty.
\]

Let \( \{ n_k : k \in \mathbb{N} \} \subset \{ m'_k : k \in \mathbb{N} \} \) be a subsequence of positive numbers \( \mathbb{N}_+ \) such that
\[
\lim_{k \to \infty} \frac{n_k^2}{\Phi_{q_{n_k}}} = \infty
\]

and let \( f_k \) be the atom defined in Example 1.44. We combine (1.52) and (1.53) in Example 1.44 and invoke (1.11) in Lemma 1.2 to obtain that
\[
\left| \frac{\sigma_{q_{n_k}} f_k}{\Phi_{q_{n_k}}} \right|
\]
\[
= \frac{1}{\Phi_{q_{n_k}} q_{n_k}} \sum_{j=1}^{q_{n_k}} |S_j f_k|
\]
\[
= \frac{1}{\Phi_{q_{n_k}} q_{n_k}} \sum_{j=M_{2n_k} + 1}^{q_{n_k}} |S_j f_k|
\]
\[
= \frac{1}{\Phi_{q_{n_k}} q_{n_k}} \sum_{j=M_{2n_k} + 1}^{q_{n_k}} (D_j - D_{M_{2n_k}})
\]
\[
= \frac{1}{\Phi_{q_{n_k}} q_{n_k}} \sum_{j=1}^{q_{n_k} - 1} (D_{j + M_{2n_k}} - D_{M_{2n_k}})
\]
\[
= \frac{1}{\Phi_{q_{n_k}} q_{n_k}} \sum_{j=1}^{q_{n_k} - 1} D_j
\]
\[
= \frac{1}{\Phi_{q_{nk}}} q_{nk-1} K_{q_{nk}-1}.
\]

Let \( x \in I_{2n^2}^{2l} \). Then, in view of Lemma 1.19, we can conclude that
\[
\left| \frac{\sigma_{q_{nk}} f_k(x)}{\Phi_{q_{nk}}} \right| \geq c M_{2s} M_{2l} \frac{M_{2n_k}^{1/2} M_{2l}^{1/2}}{\Phi_{q_{nk}}}.
\]

Hence,
\[
\int_{G_m} \left| \frac{\sigma_{q_{nk}} f_k}{\Phi_{q_{nk}}} \right|^{1/2} d\mu
\geq \sum_{s=0}^{n_k-3} \sum_{l=s+1}^{n_k-1} \sum_{m_{2l+1}=0}^{m_{2n_k}-1} \ldots \sum_{x_{2n_k-1}=0}^{m_{2n_k}-1} \int_{I_{2n_k}^{2l}} \left| \frac{\sigma_{q_{nk}} f_k}{\Phi_{q_{nk}}} \right|^{1/2} d\mu
\geq c \sum_{s=0}^{n_k-3} \sum_{l=s+1}^{n_k-1} \sum_{m_{2l+1}=0}^{m_{2n_k}-1} \frac{M_{2s}^{1/2} M_{2l}^{1/2}}{M_{2n_k}^{1/2} \Phi_{q_{nk}}}^{1/2}
\geq c n_k \frac{M_{2n_k}^{1/2} \Phi_{q_{nk}}^{1/2}}{M_{2n_k}^{1/2} \Phi_{q_{nk}}^{1/2}}.
\]

From (1.54) in Example 1.44 we have that
\[
\left( \int_{G_m} \left| \frac{\sigma_{q_{nk}} f_k}{\Phi_{q_{nk}}} \right|^{1/2} d\mu \right)^2 \geq \frac{c n_k^2}{M_{2n_k}^{1/2} \Phi_{q_{nk}}} M_{2n_k}^{1/2}
\geq \frac{c n_k^2}{M_{2n_k}^{1/2} \Phi_{q_{nk}}} M_{2n_k}^{1/2}
\geq \frac{c n_k^2}{\Phi_{q_{nk}}} \to \infty, \quad \text{when} \quad k \to \infty.
\]

Thus, also part b) is proved so the proof is complete.

We also point out the following consequence of Theorem 3.8 which we need later on.

**Corollary 3.9** Let \( f \in H_{1/2} \). Then there exists an absolute constant \( c \) such that
\[
\| \sigma_n f \|_{1/2} \leq c \log^2 (n + 1) \| f \|_{H_{1/2}}, \quad n \in \mathbb{N}_+.
\]
Proof: According to part a) of Theorem 3.8 we readily conclude that
\[
\left\| \frac{\sigma_n f}{\log^2 (n+1)} \right\|_{1/2} \leq \left\| \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\log^2 (n+1)} \right\|_{1/2} \leq c \|f\|_{H_{1/2}}, \quad n \in \mathbb{N}_+.
\]
The proof is complete.

We also mention the following corollaries:

Corollary 3.10 Let \( \{\Phi_n : n \in \mathbb{N}\} \) be any non-decreasing sequence, satisfying the condition (3.11). Then there exists a martingale \( f \in H_{1/2} \), such that
\[
\sup_{n \in \mathbb{N}} \left\| \frac{\sigma_n f}{\Phi_n} \right\|_{1/2} = \infty.
\]

Corollary 3.11 Let \( \{\Phi_n : n \in \mathbb{N}\} \) be any non-decreasing sequence, satisfying the condition (3.11). Then the following maximal operator
\[
\sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\Phi_n}
\]
is not bounded from the Hardy space \( H_{1/2} \) to the Lebesgue space \( L_{1/2} \).

The next results was proved by Tephnadze [53].

Theorem 3.12 a) Let \( 0 < p \leq 1/2 \) and \( \{n_k : k \in \mathbb{N}\} \) be a subsequence of positive numbers such that
\[
\sup_{k \in \mathbb{N}} \rho(n_k) = \kappa < c < \infty.
\]
Then the maximal operator
\[
\tilde{\sigma}^{*\Delta} f := \sup_{k \in \mathbb{N}} |\sigma_{n_k} f|
\]
is bounded from the Hardy space \( H_p \) to the Lebesgue space \( L_p \).

b) Let \( 0 < p < 1/2 \) and \( \{n_k : k \in \mathbb{N}\} \) be a subsequence of positive numbers satisfying the condition
\[
\sup_{k \in \mathbb{N}} \rho(n_k) = \infty.
\]
Then there exists an martingale \( f \in H_p \) such that
\[
\sup_{k \in \mathbb{N}} \|\sigma_{n_k} f\|_{\text{weak}-L_p} = \infty, \quad (0 < p < 1/2).
\]
**Proof:** Since $\tilde{\sigma}^{*, \Delta}$ is bounded from $L_\infty$ to $L_\infty$, by using Lemma 1.39, we obtain that the proof of part a) is complete if we show that
\[
\int_{I_N} |\tilde{\sigma}^{*, \Delta} a (x)| < c < \infty,
\]
for every $p$-atom $a$ with support $I_N$ and $\mu(I_N) = M_N^{-1}$. Analogously to Theorem 3.8 we may assume that $n_k > M_N$.

Since $\|a\|_\infty \leq M_N^{1/p}$ we find that
\[
|\sigma_{n_k} (a)| \leq \int_{I_N} |a (t)||K_{n_k} (x - t)| d\mu (t)
\]
\[
\leq \|a\|_\infty \int_{I_N} |K_{n_k} (x - t)| d\mu (t)
\]
\[
\leq M_N^{1/p} \int_{I_N} |K_{n_k} (x - t)| d\mu (t).
\] (3.16)

Let $x \in I_N^{i,j}$ and $i < j < \langle n_k \rangle$. Then $x - t \in I_N^{i,j}$ for $t \in I_N$ and, according to Lemma 1.11 we obtain that
\[
|K_{M_l} (x - t)| = 0, \text{ for all } \langle n_k \rangle \leq l \leq |n_k|.
\]

By applying (3.16) and (1.22) in Lemma 1.16 we get that
\[
|\sigma_{n_k} a (x)| \leq M_N^{1/p} \sum_{l = \langle n_k \rangle}^{n_k} \int_{I_N} |K_{M_l} (x - t)| d\mu (t) = 0,
\]
for $x \in I_N^{i,j}$, $0 \leq i < j < \langle n_k \rangle \leq l \leq |n_k|$.

Let $x \in I_N^{i,j}$, where $\langle n_k \rangle \leq j \leq N$. Then, in view of Corollary 1.21 we have that
\[
\int_{I_N} |K_{n_k} (x - t)| d\mu (t) \leq \frac{cM_iM_j}{M_N^2}.
\]

By using again (3.16) we obtain that
\[
|\sigma_{n_k} a (x)| \leq c_p M_N^{1/p - 2} M_i M_j.
\]

Set
\[
\varrho := \min_{k \in N} \langle \alpha_k \rangle.
\]

Then
\[
|\tilde{\sigma}^{*, \Delta} a (x)| = 0, \text{ for } x \in I_N^{i,j}, \ 0 \leq i < j \leq \varrho \tag{3.17}
\]
and
\[ |\tilde{\sigma}^{*,\Delta} a(x)| \leq c_p M_N^{1/p-2} M_i M_j, \text{ for } x \in I_N^{i,j}, \quad i < j \leq N - 1. \tag{3.18} \]

Analogously to (2.24) we can conclude that
\[ N - \varrho \leq N \tag{3.19} \]

and
\[ \frac{M_N^{1-2p}}{M_{\tilde{\varrho}}^{1-2p}} \leq \lambda^{(N-\varrho)(1-2p)} \leq \lambda^{\varrho(1-2p)} < c < \infty, \tag{3.20} \]

where \( \lambda = \sup_k m_k. \)

Let \( p = 1/2. \) By combining (3.17)-(3.19) with (1.1) we get that
\[
\int_{I_N} |\tilde{\sigma}^{*,\Delta} a|^{1/2} d\mu = \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \int_{I_N^{i,j}} |\tilde{\sigma}^{*,\Delta} a|^{1/2} d\mu + \sum_{i=0}^{N-1} \int_{I_N^{i,N}} |\tilde{\sigma}^{*,\Delta} a|^{1/2} d\mu \\
\leq \sum_{i=0}^{\varrho-1} \sum_{j=i+1}^{N-1} \int_{I_N^{i,j}} |\tilde{\sigma}^{*,\Delta} a|^{1/2} d\mu + \sum_{i=0}^{N-1} \int_{I_N^{i,N}} |\tilde{\sigma}^{*,\Delta} a|^{1/2} d\mu \\
+ \sum_{i=0}^{\varrho} \sum_{j=i+1}^{N-1} \int_{I_N^{i,j}} |\tilde{\sigma}^{*,\Delta} a|^{1/2} d\mu + \sum_{i=0}^{N-1} \int_{I_N^{i,N}} |\tilde{\sigma}^{*,\Delta} a|^{1/2} d\mu \\
\leq c \sum_{i=0}^{\varrho} M_i \frac{1}{M_{\varrho+1}} \sum_{j=\varrho+1}^{N-1} \frac{M_j^{1/2}}{M_j} \\
+ \sum_{i=0}^{N-2} M_i^{1/2} \sum_{j=i+1}^{N-1} \frac{1}{M_j^{1/2}} + c_p \sum_{i=0}^{N-1} \frac{M_i^{1/2}}{M_N^{1/2}} \\
\leq c M_{\varrho}^{1/2} \frac{1}{M_{\varrho}^{1/2}} + c \sum_{i=0}^{N-2} M_i^{1/2} \frac{1}{M_i^{1/2}} + c \\
\leq N - \varrho + c \leq c < \infty.
\]

Let \( 0 < p < 1/2. \) By combining (3.17), (3.18) and (3.20) with (1.1) we have that
\[
\int_{I_N} |\tilde{\sigma}^{*,\Delta} a|^{p} d\mu
\]
\begin{align*}
&= \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \int_{I_{i,j}^N} |\bar{\sigma}_i a|^p d\mu \\
&\quad + \sum_{i=0}^{N-1} \int_{I_{i}^N} |\bar{\sigma}_i a|^p d\mu \\
&\leq \sum_{i=0}^{\varrho - 1} \sum_{j=i+1}^{N-1} \int_{I_{i,j}^N} |\bar{\sigma}_i a|^p d\mu \\
&\quad + \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \int_{I_{i,j}^N} |\bar{\sigma}_i a|^p d\mu \\
&\quad + \sum_{i=0}^{N-1} \int_{I_{i}^N} |\bar{\sigma}_i a|^p d\mu \\
&\leq c_p M_N^{1-2p} \sum_{i=0}^{\varrho} M_i^p \sum_{j=\varrho+1}^{N-1} \frac{1}{M_j^{1-p}} \\
&\quad + M_N^{1-2p} \sum_{i=0}^{N-2} M_i^p \sum_{j=i+1}^{N-1} \frac{1}{M_j^{1-p}} \\
&\quad + c_p \sum_{i=0}^{N-1} \frac{M_i^p}{M_N^p} \\
&\leq \frac{c_p M_N^{1-2p}}{M_{\varrho}^{1-2p}} + c_p \leq c_p \lambda \left( |n_k - \langle n_k \rangle| (1-2p) \right) + c_p < \infty.
\end{align*}

The proof of part a) is complete.

Let \( \{n_k : k \in \mathbb{N}\} \) be a sequence of positive numbers satisfying condition (3.15). Then

\begin{equation}
\sup_{k \in \mathbb{N}} \frac{M_{|n_k|}}{M_{n_k}} = \infty. \tag{3.21}
\end{equation}

Under condition (3.21) there exists a sequence \( \{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\} \) such that

\begin{equation}
\sum_{k=0}^{\infty} \frac{M_{|\alpha_k|}^{(1-2p)/2}}{M_{|\alpha_k|}^{(1-2p)/2}} < c < \infty. \tag{3.22}
\end{equation}

Let \( f = (f^{(n)} : n \in \mathbb{N}) \) be the martingales defined in Example 1.48 where

\begin{equation}
\lambda_k = \frac{\lambda M_{|\alpha_k|}^{(1/p-2)/2}}{M_{|\alpha_k|}^{(1/p-2)/2}}, \quad \lambda = \sup_{n \in \mathbb{N}} m_n. \tag{3.22}
\end{equation}

By applying (1.38) we can conclude that \( f \in H_p \).

By now using (1.68) with \( \lambda_k \) defined by (3.22) we obtain that

\begin{equation}
\hat{f}(j) \tag{3.23}
\end{equation}
\[
\begin{aligned}
M_{\alpha_k}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2}, \quad &j \in \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\}, \ k \in \mathbb{N}_+,
0, \quad &j \notin \bigcup_{k=0}^{\infty} \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\}.
\end{aligned}
\]

Therefore,
\[
\sigma_{\alpha_k} f = \frac{1}{\alpha_k} \sum_{j=1}^{M_{\alpha_k}} S_j f + \frac{1}{\alpha_k} \sum_{j=M_{\alpha_k}+1}^{\alpha_k} S_j f
:= I + II.
\]

Let \(M_{\alpha_k} < j \leq \alpha_k\). Then, according to (3.23) we have that
\[
S_j f = S_{M_{\alpha_k}} f + M_{\alpha_k}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2} D_{j-M_{\alpha_k}} f.
\]

By applying (3.24) we can rewrite \(II\) as
\[
II = \frac{\alpha_k - M_{\alpha_k}}{\alpha_k} S_{M_{\alpha_k}} f + \frac{M_{\alpha_k}^{1/2p} M_{\langle \alpha_k \rangle}^{(1/p-2)/2}}{\alpha_k} \sum_{j=M_{\alpha_k}+1}^{\alpha_k} D_{j-M_{\alpha_k}} f
:= II_1 + II_2.
\]

In view of Corollary 2.13 for \(II_1\) we find that
\[
\|II_1\|_{\text{weak}-L_p}^p
\leq \left(\frac{\alpha_k - M_{\alpha_k}}{\alpha_k}\right)^p \left\|S_{M_{\alpha_k}} f\right\|_{\text{weak}-L_p}^p
\leq c_p \|f\|_{H_p}^p < \infty.
\]

By using part a) of Theorem 3.12 for \(I\) we obtain that
\[
\|I\|_{\text{weak}-L_p}^p = \left(\frac{M_{\alpha_k}}{\alpha_k}\right)^p \left\|\sigma_{M_{\alpha_k}} f\right\|_{\text{weak}-L_p}^p
\leq c \|f\|_{H_p}^p \leq c_p < \infty.
\]

Under condition (3.15) we can conclude that
\[
\langle \alpha_k \rangle \neq |\alpha_k| \quad \text{and} \quad \langle \alpha_k - M_{\alpha_k} \rangle = \langle \alpha_k \rangle.
\]
Let $x \in I^{(\alpha_k)-1, (\alpha_k)}$. According to Lemma 1.18 we get that

$$|II_2| = \left| \sum_{j=1}^{M^{1/2p}M^{(1/p-2)/2}(\alpha_k - M_{[\alpha_k]})} D_j \right|$$

$$= \left| \frac{M^{1/2p}M^{(1/p-2)/2}}{\alpha_k} \left( \alpha_k - M_{[\alpha_k]} \right) \right| K_{\alpha_k - M_{[\alpha_k]}}$$

$$\geq cM^{1/2p-1}M^{(1/p-2)/2}(\alpha_k - M_{[\alpha_k]}) \left| K_{\alpha_k - M_{[\alpha_k]}} \right|$$

$$\geq cM^{1/2p-1}M^{(1/p+2)/2}.$$

It follows that

$$\|II_2\|_{p \text{-weak-} L_p}^p \geq c_p \left( M^{(1/p-2)/2}M^{(1/p+2)/2} \right) \mu \left\{ x \in G_m : |II_2| \geq c_p M^{(1/p-2)/2}M^{(1/p+2)/2} \right\} - c_p$$

$$\geq c_p M^{1/2-p}M^{1/2+2p} \mu \left\{ I^{(\alpha_k)-1, (\alpha_k)} \right\}$$

$$\geq \frac{c_p M^{1/2-p}}{M^{1/2-p}} \rightarrow \infty, \text{ when } k \to \infty.$$

By now combining the estimates above we can conclude that

$$\|\sigma_{\alpha_k} f\|_{p \text{-weak-} L_p}^p$$

$$\geq \|II_2\|_{p \text{-weak-} L_p}^p - \|II_1\|_{p \text{-weak-} L_p}^p - \|I\|_{p \text{-weak-} L_p}^p$$

$$\geq \frac{1}{2} \|II_2\|_{p \text{-weak-} L_p}^p$$

$$\geq \frac{c_p M^{1/2-p}}{M^{1/2-p}} \rightarrow \infty, \text{ when } k \to \infty.$$

The proof is complete.

From the part a) of Theorem 3.12 follows immediately the following well-known results of Weisz [85]:

**Corollary 3.13** Let $p > 0$. Then the maximal operator

$$\sigma^# := \sup_{n \in \mathbb{N}} |\sigma_{M_n} f|$$

is bounded from the Hardy space $H_p$ to the Lebesgue space $L_p$. 
Proof: It is obvious that
\[ |M_n| = [M_n] = n \]
and
\[ \sup_{n \in \mathbb{N}} \rho(M_n) = 0 < c < \infty. \]

It follows that condition (3.14) is satisfied and the proof is complete by just applying part a) of Theorem 3.12.

Our next results reads:

Theorem 3.14 Let \( 0 < p \leq 1 \). Then the operator \(|\sigma_{M_n} f|\) is not bounded from the martingale Hardy space \( H_p \) to the martingale Hardy space \( H_p \).

Remark 3.15 This result for the Walsh system can be found in Goginava [30] and for bounded Vilenkin systems in the paper of Persson and Tephnadze [52].

Proof: Let \( f_k \) be the martingale from Example 1.44. By combining (1.11) in Lemma 1.2 and (1.52), (1.53) in Example 1.44 we find that

\[
\sigma_{M_{2n_k+1}} f_k
= \frac{1}{M_{2n_k+1}} \sum_{j=1}^{M_{2n_k+1}} S_j f_k
= \frac{1}{M_{2n_k+1}} \sum_{j=M_{2n_k}+1}^{M_{2n_k+1}} S_j f_k
= \frac{1}{M_{2n_k+1}} \sum_{j=M_{2n_k}+1}^{M_{2n_k+1}} (D_j - D_{M_{2n_k}})
= \frac{1}{M_{2n_k+1}} \sum_{j=1}^{M_{2n_k}} (D_{j+M_{2n_k}} - D_{M_{2n_k}})
= \frac{\psi_{M_{2n_k}}}{M_{2n_k+1}} \sum_{j=1}^{M_{2n_k}} D_j = \frac{\psi_{M_{2n_k}}}{m_{2n_k}} K_{M_{2n_k}}.
\]

It is evident that
\[
S_{M_N} \left( \left| \sigma_{M_{2n_k+1}} f_k (x) \right| \right)
= \int_{G_m} \left| \sigma_{M_{2n_k}} f_k (t) \right| D_{M_N} (x - t) d\mu (t)
\]

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Let 

\[ \psi_j(t) = 1, \quad \text{for} \quad t \in I_{2n_k}, \quad j = 0, 1, \ldots, M_{2n_k} - 1 \]

we obtain that

\[ S_{MN} \left( \left| \sigma_{M_{2n_k+1}} f_k(x) \right| \right) \]

\[ \geq c M_{2n_k} \frac{1}{M_{2n_k}} D_{MN}(x), \quad N = 0, 1, \ldots, M_N - 1 \]

and

\[ \sup_{1 \leq N < M_{2n_k}} S_{MN} \left( \left| \sigma_{M_{2n_k+1}} f_k(x) \right| \right) \]

\[ \geq \sup_{1 \leq N < M_{2n_k}} D_{MN}(x). \]

Let \( x \in I_s/I_{s+1} \), for some \( s = 0, 1, \ldots, 2n_k - 1 \). Then, from Corollary 1.5 it follows that

\[ \sup_{1 \leq N < M_{2n_k}} S_{MN} \left( \left| \sigma_{M_{2n_k+1}} f_k(x) \right| \right) \geq c M_s. \]

According to Lemma 1.37 for every \( 0 < p \leq 1 \) it yields that

\[ \left\| \sigma_{M_{2n_k+1}} f_k \right\|_{H_p}^p = \left\| \sup_{1 \leq N < M_{2n_k}} S_{MN} \left( \left| \sigma_{M_{2n_k+1}} f_k(x) \right| \right) \right\|_{H_p}^p \]

\[ \geq \int_{G_m} \left( \sup_{1 \leq N < M_{2n_k}} S_{MN} \left( \left| \sigma_{M_{2n_k+1}} f_k(x) \right| \right) \right)^p d\mu(x) \]

\[ \geq \sum_{s=1}^{2n_k-1} \int_{I_s \setminus I_{s+1}} \left( \sup_{1 \leq N < M_{2n_k}} S_{MN} \left( \left| \sigma_{M_{2n_k+1}} f_k(x) \right| \right) \right)^p d\mu(x) \]

Let \( 0 < p < 1 \). Then

\[ \left\| \sigma_{M_{2n_k+1}} f_k \right\|_{H_p}^p \geq c \sum_{s=1}^{2n_k-1} \frac{M_s^p}{M_s} = c_p > 0. \]
Let $p = 1$. We find that
\[
\left\| \sigma_{M_{2n_k+1}} f_k \right\|_{H^1} \geq c \sum_{s=1}^{2n_k-1} \frac{M_s}{M_{2n_k}} \geq c n_k.
\] (3.26)

By combining (1.54) and (3.25) with (3.26) in Example 1.44 we can conclude that
\[
\left\| \sigma_{M_{2n_k+1}} f_k \right\|_{H^p} \geq c_{\phi} n_k \quad \text{for} \quad k \to \infty,
\] when $0 < p < 1$

and
\[
\left\| \sigma_{M_{2n_k+1}} f_k \right\|_{H^1} \geq c n_k \quad \text{for} \quad k \to \infty.
\]

The proof is complete.

**Corollary 3.16** Let $p > 0$. Then the maximal operator
\[
\sigma^\# := \sup_{n \in \mathbb{N}} \left| \sigma_{M_n} f \right|
\]
is bounded from the Hardy space $H^p$ to the Lebesgue space $L^p$, but it is not bounded from the Hardy space $H^p$ to the Hardy space $H^p$.

**Proof:** Since
\[
\left| \sigma_{M_n} f \right| \leq \sigma^\# f
\] (3.27)
this result follows immediately from Corollary 3.13 and Theorem 3.14 so the proof is complete.

**Corollary 3.17** Let $p > 0$. Then the operator $\left| \sigma_{M_n} f \right|$ is bounded from the Hardy space $H^p$ to the space Lebesgue $L^p$, but is not bounded from the Hardy space $H^p$ to the Hardy space $H^p$.

**Proof:** By using (3.27) this result immediately follows Corollary 3.13 and Theorem 3.14 so the proof is complete.
3.4 Norm Convergence of Vilenkin-Fejér Means on Martingale Hardy Spaces

In this section we find necessary and sufficient conditions for the modulus of continuity of martingale Hardy spaces $H_p$ for which the partial sums of Vilenkin-Fourier series convergence in $L_p$-norm. We also study sharpness of these results.

Theorems 3.18–3.22 can be found in Persson and Tephnadze [52] and Theorems 3.23–3.24 are proved in Tephnadze [71].

**Theorem 3.18** Let $0 < p < 1/2$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$\|\sigma_n f\|_{H_p} \leq c_p n^{1/p-2} \|f\|_{H_p}, \quad (0 < p < 1).$$

**Remark 3.19** We note that the asymptotic behaviour of the sequence $\{n^{1/p-2} : n \in \mathbb{N}\}$ in Theorem 3.18 can not be improved (c.f. part b) of Theorem 3.8).

**Proof:** According to Theorem 3.18 and Corollary 3.9 if we invoke Example 1.46 we can conclude that

$$\|\sigma_n f\|_{H_p} \leq \|\sigma^# f\|_p + \|\sigma_n f\|_p \leq \|f\|_{H_p} + c_p n^{1/p-2} \|f\|_{H_p} \leq c_p n^{1/p-2} \|f\|_{H_p}.$$ \hfill \(\square\)

**Theorem 3.20** Let $f \in H_{1/2}$. Then there exists an absolute constant $c$ such that

$$\|\sigma_n f\|_{H_{1/2}} \leq c \log^2 n \|f\|_{H_{1/2}}.$$ 

**Remark 3.21** We note that the asymptotic behaviour of the sequence of sequence $\{\log^2 n : n \in \mathbb{N}\}$ in Theorem 3.18 can not be improved (c.f. part b) of Theorem 3.8).

**Proof:** According to Theorem 3.18 and Corollary 3.9 if we invoke Example 1.46 we can write that

$$\|\sigma_n f\|_{H_{1/2}} \leq \|\sigma^# f\|_{1/2} + \|\sigma_n f\|_{1/2} \leq \|f\|_{H_{1/2}} + c \log^2 n \|f\|_{H_{1/2}} \leq c \log^2 n \|f\|_{H_{1/2}}.$$ \hfill \(\square\)
Theorem 3.22 Let $0 < p < 1$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$\| \sigma_{M_n} f \|_{H_p} \leq c_p \| f \|_{H_p}, \quad (0 < p < 1).$$

**Proof:** In the view of Corollary 3.16 and Example 1.46 we can conclude that

$$\| \sigma_{M_n} f \|_{H_p} \leq \left( \sup_{0 \leq l \leq n} | \sigma_{M_l} f | \right) \leq \| f \|_{H_p}.$$

The proof is complete. \qed

Theorem 3.23 a) Let $0 < p < 1/2$, $f \in H_p$ and

$$\omega_p \left( \frac{1}{M_n}, f \right) = o \left( \frac{1}{M_n^{1/p - 2}} \right) \text{ when } n \to \infty. \quad (3.29)$$

Then

$$\| \sigma_n f - f \|_{H_p} \to 0, \text{ when } n \to \infty.$$

b) Let $0 < p < 1/2$. Then there exists a martingale $f \in H_p$ for which

$$\omega \left( \frac{1}{M_n}, f \right)_{H_p} = O \left( \frac{1}{M_n^{1/p - 2}} \right) \text{ when } n \to \infty$$

and

$$\| \sigma_n f - f \|_{\text{weak-}L_p} \to 0, \text{ when } n \to \infty.$$

**Proof:** Let $f \in H_p$, $0 < p < 1/2$, and $M_N < n \leq M_{N+1}$. By a routine calculation we immediately get that

$$\sigma_n S_{MN} f - S_{MN} f$$

\hspace{1em} (3.30)

$$= \frac{1}{n} \sum_{k=0}^{M_N} S_k S_{MN} f + \frac{1}{n} \sum_{k=M_{N+1}}^{n} S_k S_{MN} f - S_{MN} f$$

$$= \frac{1}{n} \sum_{k=0}^{M_N} S_k f + \frac{1}{n} \sum_{k=M_{N+1}}^{n} S_{MN} f - S_{MN} f$$

$$= \frac{1}{n} \sum_{k=0}^{M_N} S_k f + \frac{n - M_N}{n} S_{MN} f - S_{MN} f$$

$$= \frac{M_N}{n} \sigma_{MN} f - \frac{M_N}{n} S_{MN} f$$

$$= \frac{M_N}{n} (S_{MN} \sigma_{MN} f - S_{MN} f).$$

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\[
M_n S_{MN} (\sigma_{MN} f - f).
\]

Hence,

\[
\|\sigma_n f - f\|_{H_p}^p \leq \|\sigma_n f - \sigma_n S_{MN} f\|_{H_p}^p
\]

\[
+ \|\sigma_n S_{MN} f - S_{MN} f\|_{H_p}^p + \|S_{MN} f - f\|_{H_p}^p
\]

\[
= \|\sigma_n (S_{MN} f - f)\|_{H_p}^p + \|S_{MN} f - f\|_{H_p}^p + \|\sigma_n S_{MN} f - S_{MN} f\|_{H_p}^p
\]

\[
\leq c_p (n^{1-2p} + 1) \omega_{H_p}^p \left( \frac{1}{M_n}, f \right) + \|S_{MN} (\sigma_{MN} f - f)\|_{H_p}^p.
\]

Let \( p > 0 \) and \( f \in H_p \). According to Corollary 2.13 and Theorem 3.22 we get that

\[
\|S_{M_k} f - f\|_{H_p} \to 0, \quad \text{when } k \to \infty, \quad (p > 0)
\]

and

\[
\|\sigma_{M_k} f - f\|_{H_p} \to 0, \quad \text{when } k \to \infty, \quad (p > 0).
\]

Hence,

\[
\|\sigma_n S_{MN} f - S_{MN} f\|_{H_p}^p
\]

\[
= \frac{M_n^p}{n^p} \|S_{MN} (\sigma_{MN} f - f)\|_{H_p}^p
\]

\[
\leq \|S_{MN} (\sigma_{MN} f - f)\|_{H_p}^p
\]

\[
\leq \|\sigma_{MN} f - f\|_{H_p}^p \to 0, \quad \text{when } k \to \infty.
\]

It follows that under condition (3.29) we have that

\[
\|\sigma_n f - f\|_{H_p} \to 0, \quad \text{when } n \to \infty.
\]

This completes the proof of part a) so we turn to the part b).

First, we consider case \( 0 < p < 1/2 \). Let \( f = (f^n : n \in \mathbb{N}) \) be the martingale defined in Lemma 1.47 where

\[
\lambda_k = \frac{\lambda}{M_{2\alpha_k}^{1/p-2}}.
\]

By combining (1.55) and (1.58) in Lemma 1.47 we conclude that the martingale \( f \in H_p \) and

\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) \leq \left( \sum_{\{k, 2\alpha_k \geq n\}} \frac{1}{M_{2\alpha_k}^{1-2p}} \right)^{1/p}
\]

\[
= O \left( \frac{1}{M_n^{1/p-2}} \right), \quad \text{when } n \to \infty.
\]
A simple calculation gives that
\[\hat{f}(j) = \begin{cases} 
M_{2\alpha_i}, & j \in \{M_{2\alpha_i}, \ldots, M_{2\alpha_i+1} - 1\}, i \in \mathbb{N}_+, \\
0, & j \notin \bigcup_{i=0}^{\infty} \{M_{2\alpha_i}, \ldots, M_{2\alpha_i+1} - 1\}.
\end{cases} \tag{3.33}\]

By using (3.33) we obtain that
\[
\|\sigma_{M_{2\alpha_k}+1}f - f\|_{\text{weak}-L_p} = \left\| \frac{M_{2\alpha_k}\sigma_{M_{2\alpha_k}}f}{M_{2\alpha_k}+1} + \frac{S_{M_{2\alpha_k}}f}{M_{2\alpha_k}+1} + \frac{M_{2\alpha_k}\psi_{M_{2\alpha_k}+1}f}{M_{2\alpha_k}+1} - \frac{M_{2\alpha_k}f}{M_{2\alpha_k}+1} \right\|_{\text{weak}-L_p}
\geq \frac{M_{2\alpha_k}}{M_{2\alpha_k}+1} \| \psi_{M_{2\alpha_k}}\|_{\text{weak}-L_p}
- \frac{M_{2\alpha_k}}{M_{2\alpha_k}+1} \| \sigma_{M_{2\alpha_k}}f - f \|_{\text{weak}-L_p}
- \frac{1}{M_{2\alpha_k}+1} \| S_{M_{2\alpha_k}}f - f \|_{\text{weak}-L_p}
\geq \frac{M_{2\alpha_k}}{M_{2\alpha_k}+1} - o(1), \text{ when } k \to \infty.
\]

Therefore,
\[
\limsup_{k \to \infty} \| \sigma_{M_{2\alpha_k}+1}f - f \|_{\text{weak}-L_p}
\geq \limsup_{k \to \infty} \frac{M_{2\alpha_k}}{M_{2\alpha_k}+1} \geq c > 0.
\]

Hence also part b) is proved so the proof is complete.

**Theorem 3.24** a) Let \(f \in H_{1/2}\) and
\[
\omega_{H_{1/2}} \left( \frac{1}{M_n}, f \right) = o \left( \frac{1}{n^2} \right), \text{ when } n \to \infty. \tag{3.34}\]

Then
\[
\| \sigma_n f - f \|_{H_{1/2}} \to 0, \text{ when } n \to \infty.
\]

b) There exists a martingale \(f \in H_{1/2}\) for which
\[
\omega_{H_{1/2}} \left( \frac{1}{M_n}, f \right) = O \left( \frac{1}{n^2} \right), \text{ when } n \to \infty
\]
and
\[
\| \sigma_n f - f \|_{1/2} \not\to 0, \text{ when } n \to \infty.
\]

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Proof: Let \( f \in H_{1/2} \) and \( M_N < n \leq M_{N+1} \). By using identity (3.30) we find that
\[
\| \sigma_n f - f \|_{H_{1/2}}^{1/2} \leq \| \sigma_n f - \sigma_n S_M f \|_{H_{1/2}}^{1/2} 
\]
\[
+ \| \sigma_n S_M f - S_M f \|_{H_{1/2}}^{1/2} + \| S_M f - f \|_{H_{1/2}}^{1/2} 
\]
\[
= \| \sigma_n (S_M f - f) \|_{H_{1/2}}^{1/2} + \| S_M f - f \|_{H_{1/2}}^{1/2} + \| \sigma_n S_M f - S_M f \|_{H_{1/2}}^{1/2} 
\]
\[
\leq c ( \log^2 n + 1 ) \omega_{H_{1/2}}^{1/2} \left( \frac{1}{M_N}, f \right) + \| \sigma_n S_M f - S_M f \|_{H_{1/2}}^{1/2} 
\]

Let \( p > 0 \) and \( f \in H_p \). By combining (3.31), (3.32) and equality (3.30) we have that
\[
\| \sigma_n S_M f - S_M f \|_{H_{1/2}}^{1/2} \quad \text{(3.35)}
\]
\[
= \frac{M_N}{n} \| S_M (\sigma_M f - f) \|_{H_{1/2}}^{1/2} \to 0 \quad \text{when} \quad k \to \infty.
\]

It follows that under condition (3.34) we obtain that
\[
\| \sigma_n f - f \|_{H_{1/2}} \to 0, \quad \text{when} \quad n \to \infty.
\]

and the proof of part a) is complete.

Let \( f = (f^{(n)} : n \in \mathbb{N}) \) be a martingale from the Lemma 1.52 where \( p = 1/2 \). Then \( f \in H_{1/2} \) and
\[
\omega_{H_{1/2}} \left( \frac{1}{M_n}, f \right) = O \left( \frac{1}{n^2} \right), \quad \text{when} \quad n \to \infty.
\]

Hence,
\[
\sigma_{qM_k} f - f = M_{2M_k} q_{M_k}^2 f + \frac{1}{q_{M_k}} \sum_{j=M_{2M_k+1}}^{M_{M_k}} S_j f - M_{2M_k} f q_{M_k}^2 - q_{M_k-1} f q_{M_k}.
\]

Let \( M_{2M_k} < j \leq q_{M_k} \). By combining (1.85) and (1.87) we have that
\[
\frac{1}{q_{M_k}} \sum_{j=M_{2M_k+1}}^{M_{M_k}} S_j f
\]
\[
= \frac{q_{M_k-1} S_{2M_k} f}{q_{M_k}} + \frac{q_{M_k-1}}{q_{M_k}^2} \sum_{j=1}^{q_{M_k-1}} D_j
\].
\[
= \frac{q_{M_k-1}S_{M_2M_k}f}{q_{M_k}} + \frac{M_{2M_k}\psi_{M_2M_k}q_{M_k-1}K_{q_{M_k-1}}}{q_{M_k}M_{M_k}^2}.
\]

By using (3.36) we get that
\[
\|\sigma_{q_{M_k}}f - f\|_{1/2}^2
\geq \frac{c}{M_k}\|q_{M_k-1}K_{q_{M_k-1}}\|_{1/2}^2
\]
\[
- \left(\frac{M_{2M_k}}{q_{M_k}}\right)^{1/2}\|\sigma_{M_2M_k} f - f\|_{1/2}^2
\]
\[
- \left(\frac{q_{M_k-1}}{q_{M_k}}\right)^{1/2}\|S_{M_2M_k} f - f\|_{1/2}^2.
\]

Let
\[
x \in I_{2M_k}^{2s,2\eta}, \quad s = \eta + 2, \eta + 3, M_k - 2.
\]

By applying Lemma 1.19 we find that
\[
q_{M_k-1}\left|K_{q_{M_k-1}}(x)\right| \geq \frac{M_{2\eta}M_{2s}}{4}.
\]

Hence,
\[
\int_{G_m} |q_{M_k-1}K_{q_{M_k-1}}|^{1/2} \, d\mu
\]
\[
\geq c \sum_{\eta=1}^{M_k-4} \sum_{s=\eta+2}^{M_k-2} \sum_{x_{2s+1}=0}^{m_{2s+1}-1} \cdots \sum_{x_{2\eta_k-1}=0}^{m_{2\eta_k-1}-1} \int_{I_{2M_k}^{2s,2\eta}} |q_{M_k-1}K_{q_{M_k-1}}|^{1/2} \, d\mu
\]
\[
\geq c \sum_{\eta=1}^{M_k-4} \sum_{s=\eta+2}^{M_k-2} \frac{1}{2M_{2s}M_{2\eta}^{1/2}M_{2s}^{1/2}} \geq cM_k.
\]

By combining (3.37) and (3.38) we obtain that
\[
\limsup_{k \to \infty} \|\sigma_{q_{M_k}}f - f\|_{1/2} \geq c > 0.
\]

Thus also the part b) is proved so the proof is complete.
3.5 Strong convergence of Vilenkin-Fejér means on martingale Hardy spaces

The first result in this section is due to Blahota and Tephnadze [5]:

**Theorem 3.25** a) Let $0 < p < 1/2$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$\sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p.$$ 

b) Let $0 < p < 1/2$ and $\{\Phi_k : k \in \mathbb{N}\}$ be any non-decreasing sequence satisfying the conditions $\Phi_n \uparrow \infty$ and

$$\lim_{k \to \infty} \frac{k^{2-2p}}{\Phi_k} = \infty. \quad (3.39)$$

Then there exists a martingale $f \in H_p$ such that

$$\sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_{\text{weak}-L_p}^p}{\Phi_k} = \infty.$$ 

**Proof:** According to Lemma 1.38 it suffices to show that

$$\sum_{m=1}^{\infty} \frac{\|\sigma_m a\|_p^p}{m^{2-2p}} \leq c < \infty$$

for every $p$-atom $a$ with support $I$, $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$ and $n > M_N$.

Let $x \in I_N$. Since $\sigma_n$ is bounded from $L_\infty$ to $L_\infty$ (see (1.23) in Corollary 1.16) and $\|a\|_{\infty} \leq M_N^{1/p}$ we obtain that

$$\int_{I_N} |\sigma_m a|^p \, d\mu \leq c \frac{\|a\|_{\infty}^p}{M_N} \leq c < \infty, \quad 0 < p < 1/2.$$ 

Hence,

$$\sum_{m=1}^{\infty} \frac{\int_{I_N} |\sigma_m a|^p \, d\mu}{m^{2-2p}} \leq \sum_{m=1}^{\infty} \frac{1}{m^{2-2p}} \leq c < \infty.$$ 

It is easy to see that

$$|\sigma_m a(x)|$$

$$\leq \int_{I_N} |a(t)||K_m(x-t)| \, d\mu(t).$$
$$\leq \|a(x)\|_{\infty} \int_{I_N} |K_m(x-t)| \, d\mu(t)$$

$$\leq M_N^{1/p} \int_{I_N} |K_m(x-t)| \, d\mu(t).$$

Let $x \in I_N^{k,l}$, $0 \leq k < l \leq N$. Then, from Corollary 1.21 it follows that

$$|\sigma_m a(x)| \leq c_p M_l M_k M_N^{1/p-2}.$$  (3.40)

By combining (1.1) and (3.40) if we invoke also estimates (3.9) and (3.10) we obtain that

$$\int_{I_N} |\sigma_m a(x)|^p \, d\mu(x)$$  (3.41)

$$= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{m_{j-1}}^{m_{j-1}} \int_{I_N} |\sigma_m a(x)|^p \, d\mu(x)$$

$$+ c_p \sum_{k=0}^{N-1} \int_{I_N} |\sigma_m a(x)|^p \, d\mu(x)$$

$$\leq c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_{l+1} \cdots M_{N-1}}{M_N} M_l^p M_k^p M_N^{1-2p}$$

$$+ c_p \sum_{k=0}^{N-1} \frac{M^p}{M_N} M_N^{1-p}$$

$$\leq c_p M_N^{1-2p} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_l^p M_k^p}{M_l} + c_p \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^p}$$

$$\leq c_p M_N^{1-2p}.$$

Let $0 < p < 1/2$. By using (3.41) we get that

$$\sum_{m=M_N+1}^{\infty} \frac{\int_{I_N} |\sigma_m a(x)|^p \, d\mu(x)}{m^{2-2p}}$$

$$\leq \sum_{m=M_N+1}^{\infty} \frac{c M_N^{1-2p}}{m^{2-p}} < c < \infty.$$  

and the proof of part a) is complete.

Under condition (3.39) there exists an increasing numbers $\{\alpha_k : k \in \mathbb{N}\}$, such that

$$\lim_{k \to \infty} \frac{c M_{\alpha_k}^{2-2p}}{\Phi M_{\alpha_k}^{\alpha_k+1}} = \infty.$$  

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There exists a sequence \( \{ \alpha_k : k \in \mathbb{N} \} \subset \{ n_k : k \in \mathbb{N} \} \) such that \(|\alpha_k| > 2\), for \( k \in \mathbb{N} \)

and

\[
\sum_{\eta=0}^{\infty} \frac{\Phi_{M_{|\alpha_k|+1}}^{1/2}}{M_{|\alpha_k|}^{1-p}} = m_{|\alpha_k|}^{1-p} \sum_{\eta=0}^{\infty} \frac{\Phi_{M_{|\alpha_k|+1}}^{1/2}}{M_{|\alpha_k|}^{1-p}} < c < \infty.
\]

Let \( f \) be the martingale defined in Example 1.48 in the case when

\[
\lambda_k = \frac{\lambda \Phi_{M_{|\alpha_k|+1}}^{1/2}}{M_{|\alpha_k|}^{1/p-1}}.
\]

By applying (3.42) we can conclude that \( f \in H_p \).

By now using (3.43) with \( \lambda_k \) defined by (1.68) we readily get that

\[
\hat{f}(j) = \begin{cases} 
\Phi_{M_{|\alpha_k|+1}}^{1/2}, & j \in \{ M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1 \}, \; k \in \mathbb{N}, \\
0, & j \notin \bigcup_{k=0}^{\infty} \{ M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1 \}.
\end{cases}
\]

Hence,

\[
\sigma_{\alpha_k} f = \frac{1}{\alpha_k} \sum_{j=1}^{M_{|\alpha_k|}} S_j f + \frac{1}{\alpha_k} \sum_{j=M_{|\alpha_k|}+1}^{\alpha_k} S_j f
\]

\[
= III + IV.
\]

Let \( M_{|n_k|} < j < \alpha_k \). According to (3.44) and (1.69), we can write that

\[
S_j f
\]

\[
= \sum_{\eta=0}^{k-1} \Phi_{M_{|\alpha_k|+1}}^{1/2} \left( D_{M_{|\alpha_k|}+1} - D_{M_{|\alpha_k|}} \right)
\]

\[
+ \Phi_{M_{|\alpha_k|+1}}^{1/2} \psi_{M_{|\alpha_k|}} D_j - M_{|\alpha_k|}.
\]

By using (3.46) in IV we obtain that

\[
IV = \frac{\alpha_k - M_{|n_k|}}{\alpha_k} \sum_{\eta=0}^{k-1} \Phi_{M_{|\alpha_k|+1}}^{1/2} \left( D_{M_{|\alpha_k|}+1} - D_{M_{|\alpha_k|}} \right).
\]
\[
\frac{\Phi^{1/2p}_{M|\alpha_k|+1}}{\alpha_k} \sum_{j=M|\alpha_k|+1}^{\alpha_k} D_{j-M|\alpha_k|} := IV_1 + IV_2.
\]

We calculate each term separately. First we define a set of positive numbers \( n \in \mathbb{N}_1 \), for which \( \langle n \rangle = 1 \), that is,

\[
\mathbb{N}_1 := \left\{ n \in \mathbb{N} : n = n_1 M_1 + \sum_{i=2}^{\lceil n \rceil} n_i M_i \right\},
\]

where \( n_1 \in \{1, \ldots, m_1 - 1\} \) and \( n_i \in \{0, \ldots, m_i - 1\} \), for \( i \geq 2 \).

Let \( \alpha_k \in \mathbb{N}_1 \) and \( x \in I^0_2 \). Since \( \alpha_k - M|\alpha_k| \in \mathbb{N}_2 \) and \( |\alpha_k| \neq \langle \alpha_k \rangle \) for every \( |\alpha_k| > 2 \), by applying Lemma 1.18 we find that

\[
|IV_2| = \frac{c_p \Phi^{1/2p}_{M|\alpha_k|+1}}{\alpha_k} \left| \sum_{j=1}^{\alpha_k-M|\alpha_k|} D_j(x) \right|
\]

\[
= \frac{c_p \Phi^{1/2p}_{M|\alpha_k|+1}}{\alpha_k} \left| (\alpha_k - M|\alpha_k|) K_{\alpha_k-M|\alpha_k|}(x) \right|
\]

\[
\geq \frac{c_p \Phi^{1/2p}_{M|\alpha_k|+1}}{\alpha_k}.
\]

Let \( x \in I^0_2, n \geq 2 \) and \( 1 \leq s_n \leq m_n - 1 \). By combining Corollary 1.5, Lemma 1.11 and (1.18) in Lemma 1.13 we have that

\[
K_{s_n, M_n}(x) = D_{M_n}(x) = 0, \text{ for } n \geq 2.
\]

Since \( |\alpha_k| > 2, k \in \mathbb{N}, \) we obtain that

\[
IV_1 = 0, \text{ for } x \in I^0_2.
\]

Moreover, if we invoke (1.69) and (1.70) with (3.43) we get that

\[
S_j f = \begin{cases} 
\Phi^{1/2p}_{M|\alpha_s|+1}(M|\alpha_s|+1) \psi_{M|\alpha_s|} D_{j-M|\alpha_s|}, & M|\alpha_s| < j \leq M|\alpha_s|+1, \ s \in \mathbb{N}_+ \\
0, & M|\alpha_s|+1 < j \leq M|\alpha_{s+1}|, \ s \in \mathbb{N}_+
\end{cases}
\]

and

\[
III = \frac{1}{n} \sum_{n=0}^{k-1} \Phi^{1/2p}_{M|\alpha_n|+1} (M|\alpha_n|+1) \psi_{M|\alpha_n|} \sum_{v=M|\alpha_n|+1}^{M|\alpha_n|+1} D_{v-M|\alpha_n|} \tag{3.49}
\]
Let $0 < p < 1/2$, $n \in \mathbb{N}_2$ and $M_{|\alpha|} < n < M_{|\alpha|+1}$. By combining (3.45)-(3.49) we find that

$$\| \sigma_n f \|_{\text{weak} - L^p} \geq \frac{c_p \Phi^{1/2}_M |\alpha_k|+1}{\alpha_k^p} \mu \left\{ x \in I_2^{0,1} : |IV_2| \geq \frac{c_p \Phi^{1/2}_M |\alpha_k|+1}{\alpha_k} \right\} \geq \frac{c_p \Phi^{1/2}_M |\alpha_k|+1}{M^p |\alpha_k|+1}. \]

Since

$$\sum_{\{n \in \mathbb{N}_1 : M_k \leq n \leq M_{k+1}\}} 1 \geq cM_k,$$

we obtain that

$$\sum_{n=1}^{\infty} \frac{\| \sigma_n f \|_{\text{weak} - L^p}^p}{\Phi_n} \geq \sum_{\{n \in \mathbb{N}_1 : M_{|\alpha|} < n < M_{|\alpha|+1}\}} \frac{\| \sigma_n f \|_{\text{weak} - L^p}^p}{\Phi_n} \geq \frac{c_p}{\Phi^{1/2}_M |\alpha_k|+1} \sum_{\{N_1 : M_{|\alpha|} < n < M_{|\alpha|+1}\}} \frac{1}{M^p |\alpha_k|+1} \geq \frac{c_p M^{1-p}}{\Phi^{1/2}_M |\alpha_k|+1} \to \infty, \text{ when } k \to \infty.$$

Therefore also part b) is proved so the proof is complete.

**Corollary 3.26** Let $0 < p < 1/2$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$\sum_{k=1}^{\infty} \| \sigma_k f \|_{H_p}^p \leq c_p \| f \|_{H_p}^p,$$
**Proof:** By combining (3.28) and Theorem 3.25 we have that
\[
\sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_{L_p}^p}{k^{2-2p}} \leq \sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_{L_p}^p + \|\sigma f\|_{L_p}^p}{k^{2-2p}}
\]
\[
\leq c \sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_{L_p}^p}{k^{2-2p}} + c \sum_{k=1}^{\infty} \frac{\|f\|_{L_p}^p}{k^{2-2p}}
\]
\[
\leq c \|f\|_{L_p}^p,
\]
and the proof is complete. \[\blacksquare\]

**Corollary 3.27** Let \(0 < p < 1/2\) and \(f \in H_p\). Then there exists an absolute constant \(c_p\), depending only on \(p\), such that
\[
\frac{1}{n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{L_p}^p}{k^{1-2p}} \leq c_p \|f\|_{L_p}^p,
\]
\[
\frac{1}{n} \sum_{k=1}^{n} \frac{\|\sigma_k f - f\|_{L_p}^p}{k^{1-2p}} = 0,
\]
and
\[
\frac{1}{n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{L_p}^p}{k^{1-2p}} = \|f\|_{L_p}^p.
\]

In Blahota and Tephnadze [5] also the endpoint case \(p = 1/2\) was considered and the following result was proved:

**Theorem 3.28** Let \(f \in H_{1/2}\). Then there exists an absolute constant \(c\) such that
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{L_{1/2}}^{1/2}}{k^{1/2}} \leq c \|f\|_{L_{1/2}}^{1/2}.
\]

**Proof:** In view of Lemma 1.38 it suffices to show that
\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|\sigma_m a\|_{L_{1/2}}^{1/2}}{m^{1/2}} \leq c < \infty
\]
for every $p$-atom $a$ with support $I$, $\mu (I) = M_N^{-1}$. Analogously to the previous Theorems we may assume that $I = I_N$ and $n > M_N$.

Let $x \in I_N$. Since $\sigma_n$ is bounded from $L_\infty$ to $L_\infty$ (see (1.23) in Corollary 1.16) and $\|a\|_\infty \leq M_N^2$ we obtain that

$$\int_{I_N} |\sigma_m a|^{1/2} d\mu \leq \frac{\|a\|_\infty^{1/2}}{M_N} \leq c < \infty.$$ 

Hence,

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\int_{I_N} |\sigma_m a|^{1/2} d\mu}{m} \leq \frac{c}{\log n} \sum_{m=1}^{n} \frac{1}{m} \leq c < \infty.$$ 

It is easy to see that

$$|\sigma_m a (x)| \leq \int_{I_N} |a (t)| |K_m (x - t)| d\mu (t)$$

$$\leq \|a (x)\|_\infty \int_{I_N} |K_m (x - t)| d\mu (t)$$

$$\leq M_N^2 \int_{I_N} |K_m (x - t)| d\mu (t).$$

Let $x \in I_N^{k,l}$, $0 \leq k < l < N$. Then, from Lemma 1.20 it follows that

$$|\sigma_m a (x)| \leq \frac{c M_l M_k M_N}{m}.$$  \hspace{1cm} (3.50)

Let $x \in I_N^{k,N}$, $0 \leq k < N$. Then, according to Lemma 1.20 we have that

$$|\sigma_m a (x)| \leq c M_N^2 \frac{M_k}{M_N} \leq c M_k M_N.$$  \hspace{1cm} (3.51)

By combining (1.1) with (3.50) and (3.51) we find that

$$\int_{I_N} |\sigma_m a (x)|^{1/2} d\mu (x)$$

$$= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0}^{m_{j-1}} \int_{I_N^{k,l}} |\sigma_m a (x)|^{1/2} d\mu (x).$$
\[ \sum_{k=0}^{N-1} \int_{I_N} |\sigma_{m,a}(x)|^{1/2} d\mu(x) \]
\[ + \sum_{k=0}^{N-1} \int_{I_N} |\sigma_{m,a}(x)|^{1/2} d\mu(x) \leq c \sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \cdots m_{N-1} M_{M}^{1/2} M_{k}^{1/2} M_{N}^{1/2}}{M_{N} m^{1/2}} \]
\[ + c \sum_{k=0}^{N-1} \frac{M_{k}^{1/2} M_{N}^{1/2}}{M_{N}^{1/2}} \]
\[ \leq cM_{N}^{1/2} \sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} \frac{M_{l}^{1/2} M_{k}^{1/2}}{m^{1/2} M_{l}} \]
\[ + c \sum_{k=0}^{N-1} \frac{M_{k}^{1/2}}{M_{N}^{1/2}} \leq \frac{cM_{N}^{1/2} N}{m^{1/2}} + c. \]

It follows that
\[ \frac{1}{\log n} \sum_{m=M_{N}+1}^{n} \left( \frac{cN^{1/2} m^{-3/2}}{m} + \frac{c}{m} \right) < c < \infty. \]

The proof is complete.

**Corollary 3.29** Let \( f \in H_{1/2} \). Then
\[ \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_{k,f}\|_{H_{1/2}}^{1/2}}{k} \leq \|f\|_{H_{1/2}}^{1/2}, \]
\[ \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_{k,f} - f\|_{H_{1/2}}^{1/2}}{k} = 0 \]
and
\[ \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_{k,f}\|_{H_{1/2}}^{1/2}}{k} = \|f\|_{H_{1/2}}^{1/2}. \]

**Proof:** According to Theorem 3.28 we have that
\[ \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_{k,f}\|_{H_{1/2}}^{1/2}}{k} \]
\[ \leq \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_1^{1/2} + \|\sigma^# f\|_1^{1/2}}{k} \]

\[ \leq \frac{c}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_1^{1/2} + \|f\|_1^{1/2}}{k} \]

\[ \leq \frac{c}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_1^{1/2}}{k} + \frac{c}{\log n} \sum_{k=1}^{n} \frac{\|f\|_1^{1/2}}{k} \]

\[ \leq c \|f\|_{H^{1/2}}^{1/2} . \]

The first inequality is proved. Analogously we can prove the second and third statement. We leave out the details.
4 VILENKIN-NÖRLUND MEANS ON MARTINGALE HARDY SPACES

4.1 SOME CLASSICAL RESULTS ON VILENKIN-NÖRLUND MEANS

It is well-known in the literature that the so-called Nörlund means are generalizations of the Fejér, Cesàro and logarithmic means. The Nörlund summation is a general summability method. Therefore it is of prior interest to study the behavior of operators related to Nörlund means of Fourier series with respect to orthonormal systems.

In [69] it was proved that there exists a martingale \( f \in H_p \), (0 < p ≤ 1), such that the maximal operator of Nörlund logarithmic means \( L^* \) is not bounded in the Lebesgue space \( L_p \). Riesz logarithmic means with respect to the trigonometric system was studied by a lot of authors. We mention, for instance, the paper by Szasz [65] and Yabuta [76]. These means with respect to the Walsh and Vilenkin systems were investigated by Simon [56] and Gát [18]. Blahota and Gát [3] considered norm summability of Nörlund logarithmic means and showed that Riesz logarithmic means \( R_n \) have better approximation properties on some unbounded Vilenkin groups than the Fejér means. Moreover, in [74] it was proved that the maximal operator of Riesz’s means is bounded from the Hardy space \( H_p \) to the Lebesgue space \( L_p \) for \( p > 1/2 \) but not when 0 < p ≤ 1/2.

Móricz and Siddiqi [41] investigate the approximation properties of some special Nörlund means of Walsh-Fourier series of \( L_p \) functions in norm. The case when \( \{q_k = 1/k : k \in \mathbb{N}\} \) was excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [19] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space \( L_1 \). In particular, they gave a negative answer to the question of Móricz and Siddiqi [41]. Gát and Goginava [20] proved that for each measurable function satisfying

\[
\phi(u) = o\left(u \log^{1/2} u\right), \quad \text{when } u \to \infty,
\]

there exists an integrable function \( f \) such that

\[
\int_{G_m} \phi(|f(x)|) \, d\mu(x) < \infty
\]

and that there exists a set with positive measure such that the Walsh-logarithmic means of the function diverges on this set. Fridli, Manchanda and Siddiqi [15] improved and extended results of Móricz and Siddiqi [41] to dyadic homogeneous Banach spaces and Martingale Hardy spaces.

In [26] Goginava investigated the behavior of Cesàro means of Walsh-Fourier series in detail. In the two-dimensional case approximation properties of Nörlund and Cesàro means were considered by Nagy (see [42], [43] and [45]). The maximal operator \( \sigma_{\alpha,*}^\alpha \) (0 < \( \alpha < 1 \)) of the \((C, \alpha)\) means of Vilenkin systems was investigated by Weisz [80]. In this paper Weisz proved that \( \sigma_{\alpha,*}^\alpha \) is bounded from the martingale space \( H_p \) to the Lebesgue space \( L_p \) for
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p > 1/(1 + α). Goginava [24] gave a counterexample which shows that boundedness does not hold for 0 < p ≤ 1/(1 + α). Weisz and Simon [61] showed that the maximal operator σ^{α,*} is bounded from the Hardy space H_{1/(1+α)} to the space weak − L_{1/(1+α)}.

4.2 MAXIMAL OPERATORS OF N~ ORLUND MEANS ON MARTINGALE HARDY SPACES

In this section we first state our main result concerning the maximal operator of the N~ orlund summation method (see (1.2)). We also show that this result is in a sense sharp. The proof can be found in Persson, Tephnadze and Wall [50].

Theorem 4.1 a) The maximal operator t^* of the summability method (1.2) with non-decreasing sequence \{q_k : k ∈ N\} is bounded from the Hardy space H_{1/2} to the space weak − L_{1/2}.

The statement in a) is sharp in the following sense:

b) Let 0 < p < 1/2 and \{q_k : k ∈ N\} be a non-decreasing sequence satisfying the condition

$$\frac{q_0}{Q_n} ≥ \frac{c}{n}, \quad (c > 0).$$

(4.1)

Then there exists a martingale f ∈ H_p, such that

$$\sup_{n ∈ N} \|t_n f\|_{weak−L_p} = ∞.$$

Proof: Let the sequence \{q_k : k ∈ N\} be non-decreasing. By combining (1.29) and (1.30) and using Abel transformation we get that

$$|t_n f| ≤ \left| \frac{1}{Q_n} \sum_{j=1}^{n} q_{n−j} S_j f \right|$$

$$≤ \frac{1}{Q_n} \left( \sum_{j=1}^{n−1} \left| q_{n−j} − q_{n−j−1} \right| j |σ_j f| + q_0 n |σ_n f| \right)$$

$$≤ \frac{c}{Q_n} \left( \sum_{j=1}^{n−1} (q_{n−j} − q_{n−j−1}) j + q_0 n \right) σ^* f ≤ σ^* f$$

so that

$$t^* f ≤ σ^* f.$$

(4.2)

In view of (4.2) we can conclude that the maximal operators t^* is bounded from the Hardy space H_{1/2} to the space weak − L_{1/2}. The proof of part a) is complete.

Let f := (f^{(n)}, n ∈ N) be the martingale defined in Example 1.49 in the case when 0 < p < q = 1/2. We can write that

$$t_{M_{2a_k+1}} f = \frac{1}{Q_{M_{2a_k+1}}} \sum_{j=0}^{M_{2a_k}} q_{M_{2a_k}+1−j} S_j f + \frac{q_0}{Q_{M_{2a_k+1}}} S_{M_{2a_k}+1} f.$$
According to (1.76) in Example 1.49 we can conclude that
\[
|I| \leq \frac{1}{Q_{M_{2\alpha k}+1}} \sum_{j=0}^{M_{2\alpha k}} q_j \left| S_{M_{2\alpha k}+1-j} f \right|
\] (4.3)
\[
\leq \frac{2\lambda M_{2\alpha k}^{1/p}}{\alpha_{k-1}^{1/2}} \frac{1}{Q_{M_{2\alpha k}+1}} \sum_{j=0}^{M_{2\alpha k}} q_{M_{2\alpha k}+1-j}
\leq \frac{2\lambda M_{2\alpha k}^{1/p}}{\alpha_{k-1}^{1/2}} \frac{M_{2\alpha k}^{1/p-2}}{16\alpha_k^{3/2}}.
\]

If we now apply (1.77) for \( II \) we find that
\[
|II| = \frac{q_0}{Q_{M_{2\alpha k}+1}} \left| S_{M_{2\alpha k}+1} f \right| \geq \frac{q_0}{Q_{M_{2\alpha k}+1}} \frac{M_{2\alpha k}^{1/p-1}}{4\alpha_k^{1/2}}.
\] (4.4)

Without lost the generality we may assume that \( c = 1 \) in (4.1). By combining (4.3) and (4.4) we obtain that
\[
\left| t_{M_{2\alpha k}+1} f \right| \geq |II| - |I|
\geq \frac{q_0}{Q_{M_{2\alpha k}+1}} - \frac{4\lambda M_{2\alpha k}^{1/p-2}}{\alpha_k^{3/2}}
\geq \frac{M_{2\alpha k}^{1/p-2}}{4\alpha_k^{1/2}} - \frac{4\lambda M_{2\alpha k}^{1/p-2}}{8\alpha_k^{1/2}}.
\]

On the other hand,
\[
\mu \left\{ x \in G_m : \left| t_{M_{2\alpha k}+1} f (x) \right| \geq \frac{M_{2\alpha k}^{1/p-2}}{8\alpha_k^{1/2}} \right\} = \mu (G_m) = 1.
\] (4.5)

Let \( 0 < p < 1/2 \). Then
\[
\frac{M_{2\alpha k}^{1/p-2}}{8\alpha_k^{1/2}} \cdot \mu \left\{ x \in G_m : \left| t_{M_{2\alpha k}+1} f (x) \right| \geq \frac{M_{2\alpha k}^{1/p-2}}{8\alpha_k^{1/2}} \right\}
\] (4.6)
\[
= \frac{M_{2\alpha k}^{1/p-2}}{8\alpha_k^{1/2}} \rightarrow \infty, \text{ when } k \rightarrow \infty.
\]

The proof is complete.
Theorem 4.2  a) Let $0 < p < 1/2$ and the sequence $\{q_k : k \in \mathbb{N}\}$ be non-decreasing. Then the maximal operator

$$\tilde{t}_{p,1}^* f := \sup_{n \in \mathbb{N}} \frac{|t_n f|}{(n+1)^{1/p-2}}$$

is bounded from the Hardy martingale space $H_p$ to the Lebesgue space $L_p$.

Remark 4.3 Since the Fejér means are examples of Nörlund means with non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ we immediately obtain from part b) of Theorem 3.4 that the asymptotic behaviour of the sequence of weights

$$\left\{\frac{1}{(k+1)^{1/p-2}} : k \in \mathbb{N}\right\}$$

in Nörlund means in Theorem 4.2 can not be improved.

Proof: The idea of proof is similar to that of part a) of Theorem 3.4, but in more general situation so we give the details.

First we note that $t_n$ is bounded from $L_\infty$ to $L_\infty$ (see Corollary 1.23). Let $a$ be an arbitrary $p$-atom, with support $I$ and $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $S_n(a) = t_n(a) = 0$, when $n \leq M_N$. Therefore, we can suppose that $n > M_N$.

Hence,

$$t_n(a) = \frac{1}{Q^n} \sum_{k=1}^{n} q_{n-k} S_k(a)$$

$$= \frac{1}{Q^n} \sum_{k=M_N}^{n} q_{n-k} S_k(a)$$

$$= \frac{1}{Q^n} \sum_{k=M_N}^{n} q_{n-k} \int_{I_N} a(x) D_k(x-t) \, d\mu(t).$$

Since $\|a\|_\infty \leq M_N^{1/p}$ it follows that

$$\frac{|t_n(a)|}{(n+1)^{1/p-2}} \leq \frac{\|a\|_\infty}{(n+1)^{1/p-2}} \int_{I_N} \left| \frac{1}{Q^n} \sum_{k=M_N}^{n} q_{n-k} D_k(x-t) \right| \, d\mu(t)$$

$$\leq \frac{M_N^{1/p}}{(n+1)^{1/p-2}} \int_{I_N} \left| \frac{1}{Q^n} \sum_{k=M_N}^{n} q_{n-k} D_k(x-t) \right| \, d\mu(t).$$

Let $x \in I_N^{k,l}$, $0 \leq k < l \leq N$. From Lemma 1.27 we can deduce that

$$\frac{|t_n(a)|}{(n+1)^{1/p-2}} \leq \frac{c_p M_N^{1/p} M_l M_k}{M_N^{1/p-2} M_N^2} = c_p M_l M_k.$$

(4.8)
The expression on the right-hand side of (4.8) does not depend on \( n \). Therefore,

\[
|\tilde{t}_{p,1} a(x)| \leq c_p M_1 M_k, \quad \text{for} \quad x \in I_{N}^{k,l}, \quad 0 \leq k < l \leq N. \tag{4.9}
\]

By combining (1.1) with (4.9) we obtain that

\[
\int_{I_N} |\tilde{t}_{p,1} a(x)|^p \, d\mu(x) \\
= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0}^{m_{l,j}-1} \int_{I_N^{k,l}} |\tilde{t}_{p,1} a(x)|^p \, d\mu(x) \\
+ \sum_{k=0}^{N-1} \int_{I_N^{k,N}} |\tilde{t}_{p,1} a(x)|^p \, d\mu(x) \\
\leq c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \cdots m_{N-1}}{M_N} M_p^p M_k^p + c_p \sum_{k=0}^{N-1} \frac{1}{M_N} M_p^p M_k^p \\
\leq c_p \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_p^p M_k^p}{M_l} + c_p \sum_{k=0}^{N-1} \frac{M_p^p}{M_N^{1-p}} < c < \infty.
\]

The proof is complete by using this estimate and Lemma 1.39.

**Theorem 4.4** Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing. Then the maximal operator

\[
\tilde{t}^*_{1,f} := \sup_{n \in \mathbb{N}} \frac{|t_n f|}{\log^2 (n + 1)}
\]

is bounded from the Hardy space \( H_{1/2} \) to the Lebesgue space \( L_{1/2} \).

**Remark 4.5** Since the Fejér means are examples of Nörlund means with non-decreasing sequence \( \{q_k : k \in \mathbb{N}\} \) we immediately obtain from part b) of Theorem 3.8 that the asymptotic behaviour of the sequence of weights

\[
\left\{ 1/ \log^2 (n + 1) : n \in \mathbb{N} \right\}
\]

in Nörlund means in Theorem 4.4 can not be improved.

**Proof:** The idea of proof is similar as that of part a) of Theorem 3.8 but since this case is more general we give the details.

Analogously to Theorem 4.2 we may assume that \( n > M_N \) and \( a \) be a \( p \)-atom with support \( I = I_N \). Since \( \|a\|_\infty \leq M_N^2 \) if we apply (4.7) we obtain that

\[
\frac{|t_n (a)|}{\log^2 (n + 1)}
\]

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Let \( x \in I_{N}^{k,l}, 0 \leq k < l \leq N \). Then, from Lemma 1.27 it follows that
\[
|t_n(a)| \leq \frac{cM_lM_k}{\log^2(n+1)} M_N^{1/2} M_k^{1/2} N
\]
and
\[
|\tilde{t}_1^* a(x)| \leq \frac{cM_lM_k}{N^2}. \tag{4.10}
\]

By combining (1.1) with (4.10) we find that
\[
\int_{I_N^{k,l}} |\tilde{t}_1^* a(x)|^{1/2} d\mu(x)
\]
\[
= \sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} |\tilde{t}_1^* a(x)|^{1/2} d\mu(x)
\]
\[
= \sum_{k=0}^{N-1} \int_{I_N^{k,l}} |\tilde{t}_1^* a(x)|^{1/2} d\mu(x)
\]
\[
\leq c \sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} m_{l+1} \cdots m_{N-1} M_l^{1/2} M_k^{1/2} N
\]
\[
+ c \sum_{k=0}^{N-1} M_l^{1/2} M_k^{1/2} N
\]
\[
\leq c \sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} M_k^{1/2} N
\]
\[
+ c \sum_{k=0}^{N-1} M_l^{1/2} N
\]
\[
\leq c \sum_{k=0}^{N-1} M_k^{1/2} N
\]
\[
\leq c < \infty.
\]

The proof is complete.

The next results deal with a Nörlund means with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \).
We begin by stating a divergence result for all such summability methods when \( 0 < p < 1/2 \),
which can be found in Persson, Tephnade and Wall [50].
**Theorem 4.6** Let $0 < p < 1/2$. Then, for all Nörlund means with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \) there exists a martingale \( f \in H_p \) such that

\[
\sup_{n \in \mathbb{N}} \|t_n f\|_{\text{weak-}L_p} = \infty.
\]

**Proof:** For the proof we use the martingale defined in Example 1.49 (see also the part b) of Theorem 4.1).

It is obvious that for every non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \) it automatically holds that

\[
\frac{q_0}{Q_{M2\alpha_k + 1}} \geq \frac{1}{M2\alpha_k + 1}.
\]

Since

\[
t_{M2\alpha_k + 1} f = \frac{1}{Q_{M2\alpha_k + 1}} \sum_{j=0}^{M2\alpha_k} q_{M2\alpha_k + 1-j} S_j f + \frac{q_0}{Q_{M2\alpha_k + 1}} S_{M2\alpha_k + 1} f := I + II.
\]

by combining (4.3) and (4.4) we see that

\[
|t_{M2\alpha_k + 1} f| \geq |II| - |I| \geq \frac{M^{1/p - 2}_{2\alpha_k}}{8\alpha_k^{1/2}}.
\]

Analogously to (4.5) and (4.6) we get that

\[
\sup_{k \in \mathbb{N}} \left\| t_{M2\alpha_k + 1} f \right\|_{\text{weak-}L_p} = \infty.
\]

The proof is complete.

**Corollary 4.7** Let $0 < p < 1/2$ and \( t_n \) be Nörlund means with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \). Then the maximal operator \( t^* \) is not bounded from the martingale Hardy space \( H_p \) to the space \( \text{weak} - L_p \), that is there exists a martingale \( f \in H_p \), such that

\[
\sup_{n \in \mathbb{N}} \|t^* f\|_{\text{weak-}L_p} = \infty.
\]

Next, we present necessary condition for the Nörlund means with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \), when $1/2 \leq p < 1$, which can be found in Persson, Tephnade and Wall [50].
Theorem 4.8  a) Let $0 < p < 1/(1+\alpha)$, $0 < \alpha \leq 1$, and $\{q_k : k \in \mathbb{N}\}$ be a non-increasing sequence satisfying the condition

\[
\lim_{n \to \infty} \frac{q^n}{n^{\alpha}} = c > 0, \ 0 < \alpha \leq 1.
\]

Then there exists a martingale $f \in H_p$ such that

\[
\sup_{n \in \mathbb{N}} \|t_n f\|_{\text{weak}-L_p} = \infty.
\]

b) Let $\{q_k : k \in \mathbb{N}\}$ be a non-increasing sequence satisfying the condition

\[
\lim_{n \to \infty} \frac{n^{\alpha}}{q^n} = \infty, \ (0 < \alpha \leq 1).
\]

Then there exists a martingale $f \in H_{1/(1+\alpha)}$, such that

\[
\sup_{n \in \mathbb{N}} \|t_n f\|_{\text{weak}-L_{1/(1+\alpha)}} = \infty.
\]

Proof: Under condition (4.11) there exists an increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$ of positive integers such that

\[
\frac{M_{2\alpha_k}^\alpha}{Q_{M_{2\alpha_k}+1}} \geq c, \ k \in \mathbb{N}
\]

and the estimates (1.71)-(1.73) are satisfied. To prove part a) we use the martingale defined in Example 1.49 in the case when $0 < p < q = 1/(1+\alpha)$.

Since

\[
t_{M_{2\alpha_k}+1} f = \frac{1}{Q_{M_{2\alpha_k}+1}} \sum_{j=0}^{M_{2\alpha_k}+1-j} q_{M_{2\alpha_k}+1-j} S_j f + \frac{1}{Q_{M_{2\alpha_k}+1}} S_{M_{2\alpha_k}+1} f
\]

:= I + II,

by combining (4.3) and (4.4), we get that

\[
|t_{M_{2\alpha_k}+1} f| \geq |II| - |I|
\]

\[
= \frac{M_{2\alpha_k}^{1/p-1}}{4\alpha_k^{1/2}} \frac{Q_{M_{2\alpha_k}+1}}{Q_{M_{2\alpha_k}+1}^{1/p}} - \frac{2\lambda M_{\alpha_k-1}^{1/p}}{\alpha_k^{1/p}}.
\]

Without lost the generality we may assume that $c = 1$ in (4.11). By using (1.73) we find that

\[
|t_{M_{2\alpha_k}+1} f|
\]
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and

\[
\frac{M_{2\alpha_k}^{1/p-\alpha}}{8\alpha_k} \cdot \mu \left\{ x \in \mathcal{G}_m : \left| t_{M_{2\alpha_k}+1} f \right| \geq \frac{M_{2\alpha_k}^{1/p-\alpha}}{8\alpha_k} \right\} = \frac{M_{2\alpha_k}^{1/p-\alpha}}{8\alpha_k} \rightarrow \infty, \quad \text{when } k \rightarrow \infty.
\]

Which means that (4.12) holds and part (a) is proved.

Under condition (4.13) there exists an increasing sequence \( \{\alpha_k : k \in \mathbb{N}\} \) which satisfies the conditions

\[
\sum_{k=0}^{\infty} Q_{M_{2\alpha_k}+1}^{1/2(1+\alpha)} \leq c < \infty, \tag{4.15}
\]

\[
\sum_{\eta=0}^{k-1} Q_{M_{2\alpha_\eta}+1} M_{2\alpha_\eta}^{\alpha/2+1} \leq Q_{M_{2\alpha_k}+1} M_{2\alpha_k}^{\alpha/2+1} \tag{4.16}
\]

and

\[
32\lambda Q_{M_{2\alpha_{k-1}}+1} M_{2\alpha_{k-1}}^{\alpha/2+1} < \frac{M_{2\alpha_k}^{\alpha/2}}{Q_{M_{2\alpha_k}+1}^{1/2}}, \tag{4.17}
\]

where \( \lambda = \sup_n m_n \).

Let the martingale defined in the Example 1.47, where \( \lambda_k \) is defined by

\[
\lambda_k = \frac{\lambda Q_{M_{2\alpha_k}+1}^{1/2}}{M_{2\alpha_k}^{\alpha/2}}
\]

for which the sequence \( \{\alpha_k : k \in \mathbb{N}\} \) satisfies conditions (4.15)-(4.17) and \( \alpha_k \) are given by (1.56) for \( p = 1/(1+\alpha) \). If we apply (4.15) analogously we can conclude that \( f \in H_{1/(1+\alpha)} \). Hence,

\[
\hat{f}(j) = \begin{cases} 
Q_{M_{2\alpha_k}+1}^{1/2} M_{2\alpha_k}^{\alpha/2}, & j \in \{M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1\} \quad k \in \mathbb{N}, \\
0, & j \notin \bigcup_{k=1}^{\infty} \{M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1\},
\end{cases}
\]

and

\[
t_{M_{2\alpha_k}+1} f
\]

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By combining (4.20) and (4.21) we find that

\[
M \leq \sum_{j=0}^{M_{2\alpha k}+1} q_{M_{2\alpha k}+1-j} S_j f + \frac{1}{Q_{M_{2\alpha k}+1}} S_{M_{2\alpha k}+1} f := III + IV.
\]

Let \( M_{2\alpha s} < j \leq M_{2\alpha s+1} \), where \( s = 0, \ldots, k-1 \). From (4.18) we have that

\[
|S_j f| \leq \sum_{\eta=0}^{s-1} Q_{M_{2\alpha q}+1}^{1/2} M_{2\alpha q}^{\alpha/2} \left( D_{M_{2\alpha q}+1} - D_{M_{2\alpha q}} \right)
\]

\[
+ Q_{M_{2\alpha q}+1}^{1/2} M_{2\alpha q}^{\alpha/2} \left( D_j - D_{M_{2\alpha s}} \right) \leq 4\lambda Q_{M_{2\alpha s}+1}^{1/2} M_{2\alpha q}^{\alpha/2+1}
\]

Let \( M_{\alpha s+1}+1 \leq j \leq M_{\alpha s} \), where \( s = 1, \ldots, k \). Then

\[
|S_j f| = \sum_{\eta=0}^{s-1} Q_{M_{2\alpha q}+1}^{1/2} M_{2\alpha q}^{\alpha/2} \left( D_{M_{2\alpha q}+1} - D_{M_{2\alpha q}} \right)
\]

\[
\leq \lambda \sum_{\eta=0}^{s-1} Q_{M_{2\alpha q}+1} M_{2\alpha q}^{\alpha/2+1} \leq 2\lambda Q_{M_{2\alpha s-1}+1} M_{2\alpha s-1}^{\alpha/2+1}
\]

\[
\leq 2\lambda Q_{M_{2\alpha k-1}+1} M_{2\alpha k-1}^{\alpha/2+1}
\]

and

\[
|III| \leq \frac{1}{Q_{M_{2\alpha k}+1}} \sum_{j=0}^{M_{2\alpha k}} q_{M_{2\alpha k}+1-j} |S_j f|
\]

\[
\leq 2\lambda Q_{M_{2\alpha k-1}+1} M_{2\alpha k-1}^{\alpha/2+1} \frac{1}{Q_{M_{2\alpha k}+1}} \sum_{j=0}^{M_{2\alpha k}} q_{M_{2\alpha k}+1-j}
\]

\[
\leq 2\lambda Q_{M_{2\alpha k-1}+1} M_{2\alpha k-1}^{\alpha/2+1}.
\]

If we apply (4.18) and (4.19) we get that

\[
|IV| \geq Q_{M_{2\alpha k}+1}^{1/2} M_{2\alpha k}^{\alpha/2} \frac{q_0}{Q_{M_{2\alpha k}+1}} \left| \frac{D_{M_{2\alpha k}+1} - D_{M_{2\alpha k}}}{Q_{M_{2\alpha k}+1}} \right|
\]

\[
- \frac{1}{Q_{M_{2\alpha k}+1}} \left| S_{M_{2\alpha k}} f \right|
\]

\[
\geq \frac{q_0 M_{2\alpha k}^{\alpha/2}}{Q_{M_{2\alpha k}+1}^{1/2}} - 2\lambda Q_{M_{2\alpha k-1}+1} M_{2\alpha k-1}^{\alpha/2+1} \geq \frac{M_{2\alpha k}^{\alpha/2}}{4K_{M_{2\alpha k}+1}}.
\]

By combining (4.20) and (4.21) we find that

\[
|t_{M_{2\alpha k}+1} f| \geq |IV| - |III|
\]
\[
\geq \frac{M_{2\alpha k}^{\alpha/2}}{4Q_{M_{2\alpha k+1}}^{1/2}} - 2\lambda Q_{M_{2\alpha k-1}} M_{2\alpha k-1}^{\alpha/2+1} \geq \frac{M_{2\alpha k}^{\alpha/2}}{8Q_{M_{2\alpha k+1}}^{1/2}}.
\]

Hence, it yields that

\[
M_{2\alpha k}^{\alpha/2} \left( 1 + \frac{\alpha}{2} \right) Q_{M_{2\alpha k+1}}^{1/2} M_{2\alpha k}^{\alpha/2+1} \geq \frac{M_{2\alpha k}^{\alpha/2}}{8Q_{M_{2\alpha k+1}}^{1/2}}.
\]

\[
\mu \left( \{ x \in G_m : |t_{M_{2\alpha k+1}} f (x) | \geq \frac{M_{2\alpha k}^{\alpha/2}}{8Q_{M_{2\alpha k+1}}^{1/2}} \} \right) \to \infty, \ 	ext{ when } k \to \infty.
\]

Which means that \[4.14\] holds and the proof is complete.

**Corollary 4.9** Let \(0 < p < 1/(1 + \alpha)\), \(0 < \alpha \leq 1\) and \(\{q_k : k \in \mathbb{N}\}\) be a non-increasing sequence satisfying the condition \(4.11\). Then there exists a martingale \(f \in H_p\) such that

\[\|t^* f\|_{\text{weak-}L_p} = \infty.\]

**Corollary 4.10** Let \(\{q_k : k \in \mathbb{N}\}\) be a non-increasing sequence satisfying the condition \(4.13\). Then there exists an martingale \(f \in H_{1/(1+\alpha)}\) such that

\[\|t^* f\|_{\text{weak-}L_{1/(1+\alpha)}} = \infty.\]

Our next result reads:

**Theorem 4.11** a) The maximal operator \(t^*\) of the Nörlund summability method with non-increasing sequence \(\{q_k : k \in \mathbb{N}\}\), satisfying the condition \(1.39\) and \(1.40\) is bounded from the Hardy space \(H_{1/(1+\alpha)}\) to the space \(\text{weak-}L_{1/(1+\alpha)}\), for \(0 < \alpha \leq 1\).

b) Let \(0 < \alpha \leq 1\) and \(\{q_k : k \in \mathbb{N}\}\) be a non-increasing sequence satisfying the conditions

\[
\lim_{n \to \infty} \frac{n^\alpha}{Q_n} \geq c_\alpha > 0
\]

and

\[
|q_n - q_{n+1}| \geq c_\alpha n^{\alpha} - 2, \ n \in \mathbb{N}.
\]

Then there exists a martingale \(f \in H_{1/(1+\alpha)}\) such that

\[
\sup_{n \in \mathbb{N}} \|t_n f\|_{1/(1+\alpha)} = \infty.
\]

**Remark 4.12** Part a) of this result can be found in [50], but below we give a different proof. Part b) of the theorem is new.

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Proof: By Lemma 4.3.40, the proof of part a) is complete if we show that
\[ t \mu \left\{ x \in I_N : t f \geq t^{1+\alpha} \right\} \leq c < \infty, \quad t \geq 0, \]
for every $1/(1+\alpha)$-atom $a$. We may assume that $a$ is an arbitrary $1/(1+\alpha)$-atom with support $I$, $\mu(I) = M_N^{-1}$ and $I = I_N$. It is easy to see that $t_m(a) = 0$, when $m \leq M_N$. Therefore we can suppose that $m > M_N$.

Let $x \in I_N$. Since $t_m$ is bounded from $L_\infty$ to $L_\infty$ (the boundedness follows from Corollary 1.33) and $\|a\|_\infty \leq c M_N^{1/(1+\alpha)}$ we obtain that
\[
|t_m a(x)| \leq \int_{I_N} |a(t)||F_m(x-t)| \, d\mu(t)
\leq \|a\|_\infty \int_{I_N} |F_m(x-t)| \, d\mu(t)
\leq M_N^{1+\alpha} \int_{I_N} |F_m(x-t)| \, d\mu(t).
\]

Let $x \in I_N^k$, $0 \leq k < l \leq N$ and $m > M_N$. From Corollary 1.35 we get that
\[ |t_m a(x)| \leq c \alpha M_k M_l^\alpha. \quad (4.24) \]
The expression on the right-hand side of (4.24) does not depend on $m$. Hence,
\[ |t^\ast a(x)| \leq c \alpha M_k M_l^\alpha \quad (4.25) \]

Let $n \geq N$. According to (4.25) we conclude that
\[ \mu \left\{ x \in I_N : t^\ast f \geq c \alpha M_n^{1+\alpha} \right\} = 0. \]

Thus, we can suppose that $0 < n < N$. Let $\lambda = \sup_n m_n$ and $[x]$ denotes the integer part of $x$. It is obvious that for fixed $\lambda$ there exists a positive number $\theta$ so that $\lambda^{1/\theta} \leq 2$. Then, for every $k < n$, it yields that
\[
M_k M_n^\alpha \left[ (n-k)/\theta \right] \leq M_k M_n^\alpha \lambda^{(n-k-1)/\theta}
\leq M_k M_n^\alpha \left( \lambda^{1/\theta} \right)^{n-k-1} \leq M_k M_n^\alpha 2^{n-k-1}
\leq M_k M_n^\alpha m_k m_{k+1} \ldots m_{n-1} \leq M_n^{1+\alpha}.
\]

It is obvious that if $n + [(n - k - 1)/\lambda] > N$, for some $k < l < N$ we readily get that
\[ c \alpha M_k M_l^\alpha \leq c \alpha M_k M_N^\alpha \leq c \alpha M_k M_n^\alpha \left[ (n-k-1)/\lambda \right] \leq c \alpha M_n^{1+\alpha}. \]

It follows that for such $k < l < N$ we have the following estimate
\[ |t_m a(x)| \leq c \alpha M_n^{1+\alpha}, \quad \text{for } x \in I_N^{k,l}. \]
and
\[
\mu \left\{ x \in I_N^{k,l} : t^* f \geq c_\alpha M_n^{1+\alpha} \right\} = 0.
\]

Therefore, we may assume that \( n + \left\lfloor (n-k-1)/\lambda \right\rfloor \leq N. \)

By combining (1.1) and (4.25) we obtain that
\[
\mu \left\{ x \in I_N^{k,l} : t^* f \geq c_\alpha M_n^{1+\alpha} \right\} 
\leq \sum_{k=n}^{N-1} \sum_{l=k+1}^{N} \sum_{j=0}^{m_{j-1}} |T_{N}^{k,l}| 
+ \sum_{k=0}^{n} \sum_{l=n+\left\lfloor (n-k-1)/\lambda \right\rfloor}^{N-1} \sum_{j=0}^{m_{j-1}} |T_{N}^{k,l}| 
\leq \sum_{k=n+1}^{N} \sum_{l=k+1}^{N} \frac{1}{M_l} + \sum_{k=0}^{n} \sum_{l=n+\left\lfloor (n-k-1)/\theta \right\rfloor}^{N} \frac{1}{M_l} \leq \frac{c}{M_n}.
\]

Hence,
\[
\sup_{n \in \mathbb{N}} M_n \mu \left\{ x \in I_N^{k,l} : t^* f \geq M_n^{1+\alpha} \right\} \leq c < \infty.
\]

Part a) is proved.

Under condition (4.22) there exists an increasing sequence \( \{ \alpha_k : k \in \mathbb{N} \} \) of positive integers such that
\[
\frac{M_{2\alpha_k+1}^{\alpha}}{Q_{M_{2\alpha_k+1}}} > c_\alpha > 0, \quad k \in \mathbb{N},
\]
and estimates (1.71)-(1.73) are satisfied. To prove part b) of Theorem 4.8 we use the martingale defined in Example 1.49 in the case when \( p = q = 1/(1+\alpha) \).

In particular, as we proved there, the martingale belongs to the space \( H_{1/(1+\alpha)} \).

Moreover,
\[
\hat{f}(j) = \begin{cases} 
\frac{M_{2\alpha_k+1}^{\alpha}}{M_k}, & j \in \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}, \; k \in \mathbb{N}, \\
0, & j \notin \bigcup_{k=1}^{\infty} \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}.
\end{cases}
\]

We can write that
\[
t_{M_{2\alpha_k}+M_{2\alpha}} f = \frac{1}{Q_{M_{2\alpha_k}+M_{2\alpha}}} \sum_{j=0}^{M_{2\alpha_k}} q_{M_{2\alpha_k}+M_{2\alpha}-j} S_j f 
+ \frac{1}{Q_{M_{2\alpha_k}+M_{2\alpha}}} \sum_{j=M_{2\alpha_k}+1}^{M_{2\alpha_k}+M_{2\alpha}} q_{M_{2\alpha_k}+M_{2\alpha}-j} S_j f := I + II.
\]
According to (1.76) we can conclude that

\[ |I| \leq \frac{1}{Q M_{2\alpha_k} + M_{2s}} \sum_{j=0}^{M_{2\alpha_k}} q^{M_{2\alpha_k} + M_{2s} - j} |S_j f| \] (4.27)

\[ \leq \frac{2\lambda M_{2\alpha_k}^{\alpha+1}}{\alpha_{k-1}^{1/2} Q M_{2\alpha_k} + M_{2s}}\sum_{j=0}^{M_{2\alpha_k}} q^{M_{2\alpha_k} + M_{2s} - j} \]

\[ \leq \frac{2\lambda M_{2\alpha_k}^{\alpha+1}}{\alpha_{k-1}^{1/2}} \leq \frac{M_{\alpha_k}^\alpha}{16 \alpha_k^{3/2}}. \]

Let \( x \in I_s/I_{s+1} \) and \( M_{2\alpha_k} + 1 \leq j \leq M_{2\alpha_k} + M_{2s} \). In view of the second inequality of (1.75) in the case when \( l = k \) and \( p = 1/(1 + \alpha) \) we can write that

\[ S_j f = S_{M_{2\alpha_k}} f + \frac{M_{\alpha_k}^\alpha \psi_{M_{2\alpha_k}} D_{j-M_{2\alpha_k}}}{\alpha_k^{1/2}}. \]

Hence, it yields that

\[ II = \frac{1}{Q M_{2\alpha_k} + M_{2s}} \sum_{j=M_{2\alpha_k}+1}^{M_{2\alpha_k} + M_{2s}} q^{M_{2\alpha_k} + M_{2s} - j} \frac{M_{\alpha_k}^\alpha \psi_{M_{2\alpha_k}} D_{j-M_{2\alpha_k}}}{\alpha_k^{1/2}} \]

\[ + \frac{1}{Q M_{2\alpha_k} + M_{2s}} \sum_{j=M_{2\alpha_k}+1}^{M_{2\alpha_k} + M_{2s}} q^{M_{2\alpha_k} + M_{2s} - j} S_{M_{2\alpha_k}} f \]

\[ := II_1 + II_2. \]

By using again (1.76) we get that

\[ |II_2| \leq \frac{2\lambda M_{2\alpha_k}^{\alpha+1}}{\alpha_{k-1}^{1/2} Q M_{2\alpha_k} + M_{2s}}\sum_{j=M_{2\alpha_k}+1}^{M_{2\alpha_k} + M_{2s}} q^{M_{2\alpha_k} + M_{2s} - j} \]

\[ \leq \frac{2\lambda M_{2\alpha_k}^{\alpha+1}}{\alpha_{k-1}^{1/2}} \leq \frac{M_{\alpha_k}^{\alpha+1}}{16 \alpha_k^{3/2}}. \]

Let \( x \in I_s/I_{s+1} \), for \( \lfloor \alpha_k/2 \rfloor < s \leq \alpha_k \). Then, according to (4.26) we find that

\[ |II_1| = \frac{1}{Q M_{2\alpha_k} + M_{2s}} \left| \frac{\psi_{M_{2\alpha_k}} M_{2\alpha_k}^\alpha}{\alpha_k^{1/2}} \sum_{j=1}^{M_{2s}} q^{M_{2s} - j} D_j \right| \]

\[ = \frac{M_{2\alpha_k}^\alpha}{\alpha_k^{1/2} Q M_{2\alpha_k} + M_{2s}} \left| \sum_{j=1}^{M_{2s}} q^{M_{2s} - j} \right|. \]
We invoke Abel transformation and apply (4.23) to get that

\[
\left| \sum_{j=1}^{M_2s} q_{M_2s-j} \right| 
\]

\[
\geq \sum_{j=1}^{M_2s} (q_{M_2s-j} - q_{M_2s-j-1}) \frac{j(j+1)}{2} 
\]

\[
\geq c_\alpha M_{2s}^2 \sum_{j=[M_2s/2]}^{M_2s} |q_{M_2s-j} - q_{M_2s-j-1}| j^2 
\]

\[
\geq c_\alpha M_{2s}^2 \sum_{j=[M_2s/2]}^{M_2s} |q_{M_2s-j} - q_{M_2s-j-1}| 
\]

\[
\geq c_\alpha M_{2s}^2 \sum_{j=1}^{[M_2s/2]} |q_j - q_{j+1}| 
\]

\[
\geq c_\alpha M_{2s}^2 \sum_{j=1}^{[M_2s/2]} \frac{1}{j_\alpha} \geq c_\alpha M_{2s}^{\alpha-1} M_{2s}^2 \geq c_\alpha M_{2s}^{\alpha+1}. 
\]

Hence,

\[
|II_1| \geq \frac{c_\alpha}{\alpha_k^{1/2}} \left| \sum_{j=1}^{M_2s} q_{M_2s-j} \right| \geq \frac{c_\alpha}{\alpha_k^{1/2}}. 
\]

By now using the estimates above we obtain that

\[
\int_{G_m} |t_{2\alpha_k} M_{2s} f|^{1/(1+\alpha)} d\mu \geq |II_1| - |II_2| - |I| \quad (4.28) 
\]

\[
\geq \frac{c_\alpha}{\alpha_k^{1/2}} M_{2s}^{1+\alpha} - \frac{4\lambda}{\alpha_k^{3/2}} \geq \frac{c_\alpha}{\alpha_k^{1/2}} M_{2s}^{1+\alpha}. 
\]

By combining (4.27) and (4.28) we find that

\[
\int_{G_m} |t^* f|^{1/(1+\alpha)} d\mu
\]
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\[
≥ \sum_{s=[\alpha_k/2]}^{\alpha_k-1} \int_{I_s/I_{s+1}} \left| t_{M2^{\alpha_k}+M2s}f \right|^{1/(1+\alpha)} \, d\mu
\]

\[
≥ c_\alpha \sum_{s=[\alpha_k/2]}^{\alpha_k-1} \frac{M_{2s}}{M_{2s}^{1/2(1+\alpha)}} \geq c_\alpha \sum_{s=[\alpha_k/2]}^{\alpha_k-1} \frac{1}{\alpha_k^{1/2(1+\alpha)}}
\]

\[
≥ \frac{c_\alpha}{\alpha_k^{1/2(1+\alpha)}} \sum_{s=[\alpha_k/2]}^{\alpha_k-1} 1
\]

\[
≥ \frac{c_\alpha \alpha_k}{\alpha_k^{1/2(1+\alpha)}} \geq c_\alpha \alpha_k^{1/2} \to \infty, \text{ when } k \to \infty.
\]

The proof is complete. \[\blacksquare\]

Our next result reads:

**Theorem 4.13** Let \( f \in H_p \), where \( 0 < p < 1/ (1 + \alpha) \) for some \( 0 < \alpha \leq 1 \), and \{\( q_k : k \in \mathbb{N} \)\} be a sequence of non-increasing numbers satisfying conditions (1.39) and (1.40). Then the maximal operator \( \sim^* t_{p,\alpha} := \frac{|t_nf|}{(n+1)^{1/p-1-\alpha}} \) is bounded from the martingale Hardy space \( H_p \) to the Lebesgue space \( L_p \).

b) Let \{\( \Phi_n : n \in \mathbb{N}_+ \)\} be any non-decreasing sequence, satisfying the condition

\[
\lim_{n \to \infty} (n+1)^{1/p-1-\alpha} / \Phi_n = +\infty.
\]

Then there exists Nörlund means with non-increasing sequence \{\( q_k : k \in \mathbb{N} \)\} satisfying the conditions (4.22) and (4.23) such that

\[
\sup_{k \in \mathbb{N}} \left\| \frac{t_{M2^{\alpha_k}+M2k}}{\Phi_{M2^{\alpha_k}+1}} \right\|_{w\text{-}1-L_p} = \infty.
\]

**Remark 4.14** Part a) can be found in the paper Blahota and Tepnadze [6], while part b) has not been stated before for such a general case.

**Proof:** Since the Nörlund means \( t_n \) are bounded from \( L_\infty \) to \( L_\infty \) (the boundedness follows from Corollary 1.33), according to Lemma 1.39 it actually suffices to show that

\[
\int_{T_N} \left| t_{p,\alpha} \right|^p \, d\mu < c.
\]
for some constant $c$ and every $p$-atom $a$. We may assume that $a$ is an arbitrary $p$-atom with support $I$, $\mu(I) = M_N^{-1}$ and $I = I_N$. It is easy to see that $S_n(a) = t_n(a) = 0$ when $n \leq M_N$. Therefore we can suppose that $n > M_N$.

Let $x \in I_N$. Since $\|a\|_\infty \leq M_N^{1/p}$ we obtain that

$$|t_n a(x)| \leq \int_{I_N} |a(t)||F_n(x-t)| d\mu(t)$$

$$\leq \|a\|_\infty \int_{I_N} |F_n(x-t)| d\mu(t) \leq M_N^{1/p} \int_{I_N} |F_n(x-t)| d\mu(t).$$

Let $x \in I_{N}^{k,l}$, $0 \leq k < l \leq N$. Then, from Lemma 1.34 we get that

$$|t_n a(x)| \leq c_{\alpha,p} M_N^{1/p-1} M_k^\alpha M_l^{k}. \quad (4.30)$$

Let $x \in I_{N}^{k,N}$, $0 \leq k < N$. Then, according to Lemma 1.34 we have that

$$|t_n a(x)| \leq c_{\alpha,p} M_N^{1/p-1} M_k. \quad (4.31)$$

Let $x \in I_{N}^{k,l}$, $0 \leq k < l \leq N$. Since $n > M_N$ we can conclude that

$$\left|\frac{t_n a(x)}{n^{1/p-1-\alpha}}\right| \leq c_{\alpha,p} M_k^\alpha. \quad (4.32)$$

The expression on the right-hand side of (4.32) does not depend on $n$. Hence, we can conclude that

$$\left|\frac{t_p a(x)}{n^{1/p-1-\alpha}}\right| \leq c_{\alpha,p} M_l^\alpha M_k. \quad (4.33)$$

for $x \in I_{N}^{k,l}$, $0 \leq k < l \leq N$.

By combining (1.1) and (4.33) we obtain that

$$\int_{I_N} \left|\frac{t_p a(x)}{n^{1/p-1-\alpha}}\right|^p d\mu$$

$$= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{m_l=0}^{m_{l+1}-1} \int_{I_N^{k,l}} \left|\frac{t_p a(x)}{n^{1/p-1-\alpha}}\right|^p d\mu$$

$$+ \sum_{k=0}^{N-1} \int_{I_N^{k,N}} \left|\frac{t_p a(x)}{n^{1/p-1-\alpha}}\right|^p d\mu$$

$$\leq c_{\alpha,p} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l} M_l^{\alpha p} M_k^p + c_{\alpha,p} \sum_{k=0}^{N-1} \frac{1}{M_N} M_N^{1-p} M_k^p.$$
\[
\leq c_{\alpha,p} \sum_{k=0}^{N-2} M_k^p \sum_{l=k+1}^{N-1} M_l^{1-\alpha p} + c_{\alpha,p} \sum_{k=0}^{N-1} M_k^p \leq c_{\alpha,p} < \infty.
\]

The proof of part a) is complete.

Let \( 0 < p < 1/(1 + \alpha) \). Under condition (4.29) there exists positive integers \( n_k \) such that
\[
\lim_{k \to \infty} \frac{(M_{2n_k} + 1)^{1/p - 1 - \alpha}}{\Phi_{M_{2n_k} + 1}} = \infty, \quad 0 < p < 1/(1 + \alpha).
\]

To prove part b) we apply the \( p \)-atoms defined in Example 1.44. Under conditions (1.39) and (1.40), if we invoke (1.52) and (1.53) we find that
\[
\left| \frac{t_{M_{2n_k} + 1} f_k}{\Phi_{M_{2n_k} + 1}} \right| = \frac{|S_{M_{2n_k} + 1}|}{Q_{M_{2n_k} + 1} \Phi_{M_{2n_k} + 1}} = \frac{t_{M_{2n_k} + 1}}{Q_{M_{2n_k} + 1} \Phi_{M_{2n_k} + 1}} = \frac{c_{\alpha}}{M_{2n_k}^\alpha \Phi_{M_{2n_k} + 1}}.
\]

Hence,
\[
\mu \left\{ x \in G_m : \frac{|t_{M_{2n_k} + 1} f_k(x)|}{\Phi_{M_{2n_k} + 1}} \geq \frac{c_{\alpha}}{M_{2n_k}^\alpha \Phi_{M_{2n_k} + 1}} \right\} = 1. \quad (4.34)
\]

By combining (1.54) and (4.34) we have that
\[
\frac{c_{\alpha}}{M_{2n_k}^\alpha \Phi_{M_{2n_k} + 1}} \left( \mu \left\{ x \in G_m : \frac{|S_{M_{2n_k} + 1} f_k(x)|}{\Phi_{M_{2n_k} + 1}} \geq \frac{c_{\alpha}}{M_{2n_k}^\alpha \Phi_{M_{2n_k} + 1}} \right\} \right)^{1/p} \leq \|f_k\|_{\ell_p}
\]
\[
\geq c_{\alpha} M_{2n_k}^{1/p - 1 - \alpha} \geq c_{\alpha} \left( M_{2n_k} + 1 \right)^{1/p - 1 - \alpha} \to \infty, \quad \text{when} \quad k \to \infty.
\]

The proof is complete.

**Corollary 4.15** Let \( 0 < p < 1/(1 + \alpha) \) and \( f \in H_p \). Then there exists an absolute constant \( c_{p,\alpha} \), depending only on \( p \) and \( \alpha \), such that
\[
\|t_n f\|_p \leq c_{p,\alpha} (n + 1)^{1/p - 1 - \alpha} \|f\|_{H_p}, \quad n \in \mathbb{N}_+.
\]

**Proof:** According to part a) of Theorem 4.13 we conclude that
\[
\left\| \frac{t_n f}{(n + 1)^{1/p - 1 - \alpha}} \right\|_p \leq \sup_{n \in \mathbb{N}} \left\| \frac{|t_n f|}{(n + 1)^{1/p - 1 - \alpha}} \right\|_p.
\]
The proof is complete.

\[ \leq c_{p, \alpha} \| f \|_{H_p}, \quad n \in \mathbb{N}_+. \]

Corollary 4.16 Let \( \{ \Phi_n : n \in \mathbb{N} \} \) be any non-decreasing sequence satisfying the condition \( (4.29) \). Then there exists a martingale \( f \in H_p \) such that

\[ \sup_{n \in \mathbb{N}} \left\| \frac{t_n f}{\Phi_n} \right\|_{\text{weak-}L_p} = \infty. \]

Corollary 4.17 Let \( \{ \Phi_n : n \in \mathbb{N} \} \) be any non-decreasing sequence satisfying the condition \( (4.29) \). Then the maximal operator

\[ \sup_{n \in \mathbb{N}} \frac{|t_n f|}{\Phi_n} \]

is not bounded from the Hardy space \( H_p \) to the space \( \text{weak-}L_p \).

We now formulate our final result in this section.

Theorem 4.18 Let \( f \in H_{1/(1+\alpha)} \), where \( 0 < \alpha \leq 1 \) and \( \{ q_k : k \in \mathbb{N} \} \) be a sequence of non-increasing numbers satisfying the conditions \( (1.39) \) and \( (1.40) \). Then there exists an absolute constant \( c_\alpha \) depending only on \( \alpha \) such that the maximal operator

\[ \tilde{t}_\alpha^* := \frac{|t_n f|}{\log^{1+\alpha}(n+1)} \]

is bounded from the martingale Hardy space \( H_{1/(1+\alpha)} \) to the Lebesgue space \( L_{1/(1+\alpha)} \).

b) Let \( \{ \Phi_n : n \in \mathbb{N}_+ \} \) be any non-decreasing sequence satisfying the condition

\[ \lim_{n \to \infty} \frac{\log^{1+\alpha}(n+1)}{\Phi_n} = +\infty. \quad (4.35) \]

Then there exists Nörlund means with non-increasing sequence \( \{ q_k : k \in \mathbb{N} \} \) satisfying the conditions \( (4.22) \) and \( (4.23) \) such that

\[ \sup_{k \in \mathbb{N}} \left\| \frac{\sup_n \left| t_n f_k / \Phi_n \right|}{\| f \|_{H_{1/(1+\alpha)}}} \right\|_{1/(1+\alpha)} = \infty. \]

Remark 4.19 Part a) of this result can be found in the paper Bhahota and Tephnadze [7]. Part b) has not been stated before for such a general case.
**Proof:** According to Lemma [1.39] the proof of part a) will be complete if we show that

\[
\int_{I_N} \left| {t_\alpha a}^* \right|^{1/(1+\alpha)} \, d\mu < \infty
\]

for every \(1/(1+\alpha)\)-atom \(a\). We may assume that \(a\) is an arbitrary \(1/(1+\alpha)\)-atom with support \(I, \mu(I) = M_N^{-1}\) and \(I = I_N\). It is easy to see that \(t_m(a) = 0\), when \(m \leq M_N\). Therefore we can suppose that \(m > M_N\).

Let \(x \in I_N\). Since \(t_m\) is bounded from \(L_\infty\) to \(L_\infty\) (the boundedness follows from Corollary [1.33]) and \(\|a\|_\infty \leq M_N^{1/(1+\alpha)}\) we obtain that

\[
\left| t_m a(x) \right| \leq \int_{I_N} \left| a(t) \right| \left| F_m(x-t) \right| d\mu(t)
\]

\[
\leq \left\| a(x) \right\|_\infty \int_{I_N} \left| F_m(x-t) \right| d\mu(t)
\]

\[
\leq M_N^{1+\alpha} \int_{I_N} \left| F_m(x-t) \right| d\mu(t).
\]

Let \(x \in I_{k,l}^N, 0 \leq k < l < N\). From Lemma [1.34] we get that

\[
\left| t_m a(x) \right| \leq \frac{c_\alpha M_k M_l^\alpha M_N^\alpha}{m^\alpha}.
\]

(4.36)

Let \(x \in I_{k,N}^N, 0 \leq k < N\). In the view of Lemma [1.34] we have that

\[
\left| t_m a(x) \right| \leq c_\alpha M_k M_N^\alpha.
\]

(4.37)

Let \(x \in I_{k,l}^N, 0 \leq k < l \leq N\). Since \(n > M_N\) we can conclude that

\[
\left| \frac{t_m a(x)}{\log^{1+\alpha} n} \right| \leq \frac{c_\alpha M_k M_l^\alpha}{N^{1+\alpha}}.
\]

(4.38)

The expression on the right-hand side of (4.38) does not depend on \(n\). Hence,

\[
\left| \tilde{t}_\alpha^* a(x) \right| \leq \frac{c_\alpha M_k M_N^\alpha}{N^{1+\alpha}},
\]

(4.39)

for \(x \in I_{k,l}^N, 0 \leq k < l \leq N\).

According to (1.1) and (4.39) we obtain that

\[
\int_{I_N} \left| \tilde{t}_\alpha^* a \right|^{1/(1+\alpha)} \, d\mu
\]

\[
= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0}^{m_j-1} \sum_{j \in \{l+1, \ldots, N-1\}} \int_{I_N} \left| \tilde{t}_\alpha^* a \right|^{1/(1+\alpha)} \, d\mu
\]
The proof of part a) is complete.

Under condition (4.35) there exists a positive integers \( m'_k \) such that
\[
M_{2m'_k + 1} \leq \lambda_k < 2M_{2m'_k + 1}.
\]
Since \( \Phi_n \) is a non-decreasing function we have that
\[
\lim_{k \to \infty} \frac{(m'_k)^{1+\alpha}}{\Phi_{M_{2m'_k + 1}}} = \infty.
\]

Let \( \{ n_k : k \in \mathbb{N}_+ \} \subset \{ m'_k : k \in \mathbb{N}_+ \} \) be a sequence of positive numbers such that
\[
\lim_{k \to \infty} \frac{r_{n_k}^{1+\alpha}}{\Phi_{M_{2n_k + 1}}} = \infty.
\]

To prove part b) we use the \( 1/(1+\alpha) \)-atoms defined in Example 1.44. If we apply (1.11) in Lemma 1.2 with (1.52) and (1.53) and invoke Abel transformation we get that
\[
\left| \frac{t_{M_{2n_k} + M_2, f_k}}{\Phi_{M_{2n_k} + M_2}} \right| = \frac{1}{\Phi_{M_{2n_k} + M_2}} \sum_{j=M_{2n_k} + 1}^{M_{2n_k} + M_2} q_{M_{2n_k} + M_2 - j} \left( D_j - D_{M_{2n_k}} \right)
\]
\[
= \frac{1}{\Phi_{M_{2n_k} + M_2}} \sum_{j=1}^{M_{2n_k}} q_{M_{2n_k} - j} \left( D_j + M_{2n_k} - D_{M_{2n_k}} \right)
\]
\[
= \frac{1}{\Phi_{M_{2n_k} + M_2}} \sum_{j=1}^{M_{2n_k}} q_{M_{2n_k} - j} D_j
\]
\[
= \frac{1}{\Phi_{M_{2n_k} + M_2}} \sum_{j=1}^{M_{2n_k}} q_{M_{2n_k} - j} D_j
\]
\[
G.Tephnadze,
Let \( x \in I_{2s}/I_{2s+1}, \ s = [n_k/2], \ldots, n_k. \) If we again use abel transformation, then under the conditions (4.22) and (4.23) we find that

\[
\left| \frac{t_{M_{2n_k}+M_{2s}} f_k}{\Phi_{M_{2n_k}+M_{2s}}} \right| \geq \frac{c}{\Phi_{M_{2n_k}+M_{2s}}} M_{2n_k}^\alpha \sum_{j=1}^{M_{2s}} \left( q_{M_{2s}-j} - q_{M_{2s}-j-1} \right) j^2
\]

\[
\geq \frac{c}{\Phi_{M_{2n_k}+M_{2s}}} \sum_{j=[M_{2s}]/2}^{M_{2s}} \left| q_{M_{2s}-j} - q_{M_{2s}-j-1} \right| j^2
\]

\[
\geq \frac{c M_{2s}^2}{\Phi_{M_{2n_k}+M_{2s}}} \sum_{j=1}^{[M_{2s}]/2} \left| q_j - q_{j+1} \right|
\]

\[
\geq \frac{c M_{2s}^2}{\Phi_{M_{2n_k}+M_{2s}}} \sum_{j=1}^{[M_{2s}]/2} j^{\alpha-2} \geq \frac{c M_{2s}^2 M_{2s}^{\alpha-1}}{\Phi_{M_{2n_k}+M_{2s}}} \Phi_{M_{2n_k}+M_{2s}}^\alpha
\]

\[
\geq \frac{c M_{2s}^\alpha + 1}{\Phi_{M_{2n_k}+M_{2s}}} \Phi_{M_{2n_k}+M_{2s}}^\alpha
\]

Hence,

\[
\int_{G_m} \left( \sup_n \left| t_n f_k \right| \Phi_n \right)^{1/(1+\alpha)} d\mu
\]

\[
\geq \sum_{s=[n_k/2]}^{n_k-1} \int_{I_{2s}/I_{2s+1}} \left| \frac{t_{M_{2n_k}+M_{2s}} f_k}{\Phi_{M_{2n_k}+M_{2s}}} \right|^{1/(1+\alpha)} d\mu
\]

\[
\geq \frac{c}{\Phi_{M_{2n_k+1}}^{1/(1+\alpha)}} \sum_{s=[n_k/2]}^{n_k-1} M_{2s}^{\alpha/(1+\alpha)} d\mu
\]

\[
\geq \frac{c}{\Phi_{M_{2n_k+1}}^{1/(1+\alpha)}} \sum_{s=[n_k/2]}^{n_k-1} \frac{M_{2s}^{\alpha+1}}{M_{2n_k}^{\alpha/(1+\alpha)}} \frac{1}{M_{2s}^{\alpha+1}}
\]

\[
\geq \frac{c n_k}{\Phi_{M_{2n_k+1}}^{1/(1+\alpha)}} \Phi_{M_{2n_k+1}}^{1/(1+\alpha)}
\]

Therefore, by also using (1.54) for \( p = 1/(1+\alpha) \) we have that

\[
\left( \int_{G_m} \left( \sup_n \frac{t_n f_k}{\Phi_n} \right)^{1/(1+\alpha)} d\mu \right)^{1+\alpha}
\]

\[
\| f_k \|_{H_{1/(1+\alpha)}}
\]
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\[ \geq \frac{cn_k^{1+\alpha}}{M_{2n_k}^\alpha \Phi_{M_{2n_k}+1}} M_{2n_k}^\alpha \geq \frac{cn_k^{1+\alpha}}{\Phi_{M_{2n_k}+1}} \to \infty, \text{ when } k \to \infty. \]

The proof is complete. \[ \blacksquare \]

**Corollary 4.20** Let \( \{ \Phi_n : n \in \mathbb{N}_+ \} \) be any non-decreasing sequence satisfying the condition (4.35). Then the following maximal operator

\[ \sup_n \left| \frac{t_n f}{\Phi_n} \right| \]

is not bounded from the Hardy space \( H_{1/(1+\alpha)} \) to the space \( L_{1/(1+\alpha)} \).

### 4.3 Strong Convergence of Nörlund Means on Martingale Hardy Spaces

The first result in this section is due to Persson, Tephnadze and Wall [51].

**Theorem 4.21** Let \( 0 < p < 1/2 \), \( f \in H_p \) and the sequence \( \{ q_k : k \in \mathbb{N} \} \) be non-decreasing. Then there exists an absolute constant \( c_p \) depending only on \( p \) such that

\[ \sum_{k=1}^{\infty} \| t_k f \|_p^p \leq c_p \| f \|_{H_p}^p. \]

**Remark 4.22** Since the Fejér means are examples of Nörlund means with non-decreasing sequence \( \{ q_k : k \in \mathbb{N} \} \) we immediately obtain from part b) of Theorem 3.25 that the asymptotic behaviour of the sequence of weights

\[ \{ 1/k^{2-2p} : k \in \mathbb{N} \} \]

in Nörlund means in Theorem 4.27 can not be improved.

**Proof:** The proof is similar to that of part a) of Theorem 3.25 but since this case is more general we give the details.

By Lemma 1.38 the proof is complete if we show that

\[ \sum_{m=1}^{\infty} \| t_m a \|_p^p \leq c_p \] \hspace{1cm} (4.40)

for every \( p \)-atom \( a \) with support \( I, \mu(I) = M_N^{-1} \). We may assume that \( I = I_N \). It is easy to see that \( S_\alpha(a) = t_n(a) = 0 \), when \( n \leq M_N \). Therefore, we can suppose that \( n > M_N \).
Let \( x \in I_N \). Since Nörlund means \( t_n \) with non-decreasing sequence \( \{q_k : k \in \mathbb{N}\} \) are bounded from \( L_\infty \) to \( L_\infty \) (the boundedness follows from Corollary 1.23) and \( \|a\|_\infty \leq M_{N}^{1/p} \), we obtain that

\[
\int_{I_N} |t_m a|^p \, d\mu \\
\leq \frac{\|a\|_\infty^p}{M_N} \leq 1, \quad 0 < p \leq 1/2.
\]

Hence,

\[
\sum_{m=1}^{\infty} \frac{\int_{I_N} |t_m a|^p \, d\mu}{m^{2-2p}} \leq c < \infty.
\] (4.41)

It is easy to see that

\[
|t_n a(x)| = \left| \int_{I_N} a(t) \frac{1}{Q_{m-k=M_N}} \sum_{k=M_N}^{n} q_{n-k} D_k(x-t) \, d\mu(t) \right| \\
\leq \int_{I_N} |a(t)| \left| \frac{1}{Q_{m-k=M_N}} \sum_{k=M_N}^{n} q_{n-k} D_k(x-t) \right| \, d\mu(t) \\
\leq \|a\|_\infty \int_{I_N} \left| \frac{1}{Q_{m-k=M_N}} \sum_{k=M_N}^{n} q_{n-k} D_k(x-t) \right| \, d\mu(t) \\
\leq M_N^{1/p} \int_{I_N} \left| \frac{1}{Q_{m-k=M_N}} \sum_{k=M_N}^{n} q_{n-k} D_k(x-t) \right| \, d\mu(t).
\]

Let \( x \in I_N^{k,l} \), \( 0 \leq k < l \leq N \). Then, in the view of Lemma 1.27, we get that

\[
|t_m a(x)| \leq c_p M_{l} M_{k} M_N^{1/p-2}, \quad \text{for} \ 0 < p < 1/2.
\] (4.42)

According to (1.1) with (4.42) we find that

\[
\int_{I_N} |t_m a|^p \, d\mu
\]

(4.43)

\[
= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{m_j-1}^{m_{j-1}} \int_{I_N^{k,l}} |t_m a|^p \, d\mu
\]
Moreover, according to (4.43), we get that
\[
\sum_{m=M_N+1}^{\infty} \frac{\int_{I_N} |t_m a|^p \, d\mu}{m^{2-2p}} \leq c_p \sum_{m=M_N+1}^{\infty} \frac{M_N^{1-2p}}{m^{2-2p}} < c < \infty, \quad (0 < p < 1/2).
\]

Now, by combining this estimate with (4.41) we obtain (4.40) so the proof is complete. □

Also the next theorem is proved in Persson, Tephnadze and Wall [51].

**Theorem 4.23** Let \( f \in H_{1/2} \) and the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing satisfying condition (1.28). Then there exists an absolute constant \( c \), such that
\[
\frac{1}{\log n} \sum_{k=1}^{n} \left\| t_k f \right\|_{H_{1/2}}^{1/2} \leq c \left\| f \right\|_{H_{1/2}}^{1/2}.
\]

**Proof:** The proof is similar to that of part a) of Theorem 3.28 but since this case is more general we present the details.

According to Lemma 1.38 it suffices to show that
\[
\frac{1}{\log n} \sum_{m=1}^{n} \left\| t_m a \right\|_{1/2}^{1/2} \leq c
\]
for every \( p \)-atom \( a \) with support \( I, \mu(I) = M_N^{-1} \). We may assume that \( I = I_N \). It is easy to see that \( S_n(a) = t_n(a) = 0 \) when \( n \leq M_N \). Therefore we can suppose that \( n > M_N \).
Let \( x \in I_N \). Since \( t_n \) is bounded from \( L_\infty \) to \( L_\infty \) (the boundedness follows from Corollary 1.23) and \( \|a\|_\infty \leq M_N^2 \) we obtain that
\[
\int_{I_N} |t_m a|^{1/2} \, d\mu \leq \frac{\|a\|_\infty^{1/2}}{M_N} \leq c < \infty.
\]
Hence,
\[
\frac{1}{\log n} \sum_{m=1}^{n} \int_{I_N} |t_m a|^{1/2} \, d\mu \leq \frac{\|a\|_\infty^{1/2}}{M_N} \leq c < \infty, \quad n \in \mathbb{N}.
\]
It is easy to see that
\[
|t_m a(x)| \leq \int_{I_N} |a(t)| |F_m(x-t)| \, d\mu(t)
\]
\[
\leq \|a\|_\infty \int_{I_N} |F_m(x-t)| \, d\mu(t)
\]
\[
\leq M_N^2 \int_{I_N} |F_m(x-t)| \, d\mu(t).
\]
Let \( x \in I_N^{k,l} \), \( 0 \leq k < l < N \). Then, in the view of Lemma 1.25 we get that
\[
|t_m a(x)| \leq \frac{cM_l M_k M_N}{m}.
\]
Let \( x \in I_N^{k,N} \). Then, according to Lemma 1.25 we find that
\[
|t_m a(x)| \leq cM_k M_N.
\]
By combining (4.45) and (4.46) with (1.1) we can conclude that
\[
\int_{I_N} |t_m a(x)|^{1/2} \, d\mu(x)
\]
\[
\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \cdots m_{N-1}}{M_N} \frac{M_{l/2}^1 M_{k/l/2}^1 M_{N/l}^1}{m^{1/2}}
\]
\[
+c \sum_{k=0}^{N-1} \frac{1}{M_N} M_{k/l/2}^1 M_{N/l}^1
\]
\[
\leq cM_N^{1/2} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_{l/2}^1 M_{k/l/2}^1}{m^{1/2} M_{l}^1} + c \sum_{k=0}^{N-1} \frac{M_{k/l/2}^1}{M_{N/l}^1}.
\]
It follows that
\[
\frac{1}{\log n} \sum_{m=M_N+1}^{n} \frac{\int_{I_N} |t_m a(x)|^{1/2} d\mu(x)}{m} \leq \frac{c M_N^{1/2} N}{m^{1/2}} + c.
\]

The proof is complete by just combining (4.44) and (4.47).

Next, we investigate Nörlund means with non-increasing sequence \(\{q_k : k \in \mathbb{N}\}\). At first we consider the case \(0 < p < 1/(1 + \alpha)\) where \(0 < \alpha < 1\). For details see the paper of Blahota and Tephnadze [6].

**Theorem 4.24** Let \(f \in H_p\), where \(0 < p < 1/(1 + \alpha)\), \(0 < \alpha \leq 1\) and \(\{q_k : k \in \mathbb{N}\}\), be a sequence of non-increasing numbers satisfying the conditions \((1.39)\) and \((1.40)\). Then there exists an absolute constant \(c_{\alpha,p}\), depending only on \(\alpha\) and \(p\) such that

\[
\sum_{k=1}^{\infty} \|t_k f\|_{H_p}^p \leq c_{\alpha,p} \|f\|^p_{H_p}.
\]

**Proof:** By Lemma [1.38] the it suffices to show that

\[
\sum_{m=1}^{\infty} \frac{\|t_m a\|_{H_p}^p}{m^{2-(1+\alpha)p}} \leq c_{\alpha,p} < \infty
\]

for every \(p\)-atom \(a\). Analogously to the proofs of the previous theorems we may assume that \(a\) be an arbitrary \(p\)-atom with support \(I\), \(\mu(I) = M_N^{-1}\) and \(I = I_N\) and \(m > M_N\).

Let \(x \in I_N\). Since \(t_m\) is bounded from \(L_\infty\) to \(L_\infty\) (the boundedness follows from Corollary [1.33]) and \(\|a\|_{\infty} \leq M_N^{1/p}\) we obtain that

\[
\int_{I_N} |t_m a|^p d\mu \leq \|a\|_{\infty}^p M_N^{-1} \leq 1.
\]

Hence

\[
\sum_{m=M_N}^{\infty} \frac{\int_{I_N} |t_m a|^{1/(1+\alpha)} d\mu}{m^{2-(1+\alpha)p}} \leq \sum_{m=1}^{\infty} \frac{1}{m^{2-(1+\alpha)p}} \leq c_{\alpha,p} < \infty.
\]
According to (1.1) and (4.30)-(4.31) we can conclude that

\[
\sum_{m=M_N+1}^{\infty} \frac{\int_{I_N} |t_m a|^p d\mu}{m^{2-(1+\alpha)p}}
\]

\[
= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j \in \{l+1, \ldots, N-1\}}^{m_j-1} \frac{\int_{I_{k,l}} |t_m a|^p d\mu}{m^{2-(1+\alpha)p}}
\]

\[
+ \sum_{m=M_N+1}^{N} \sum_{k=0}^{n} \frac{\int_{I_{k,N}} |t_m a|^p d\mu}{m^{2-(1+\alpha)p}}
\]

\[
\leq c_{\alpha,p} \sum_{m=M_N+1}^{\infty} \left( \frac{M_N^{1-p} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_{k,l}^{po} M_k^p}{M_l^{1-po}}}{m^{2-p} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l^{1-po}}} + \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^{1-p}} \sum_{k=0}^{N-1} \frac{M_k^p}{M_N} \right).
\]

Since

\[
\sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{M_{k,l}^{po} M_k^p}{M_l^{1-po}} \leq \sum_{k=0}^{N-2} M_k^p \sum_{l=k+1}^{N-1} \frac{1}{M_l^{1-po}}
\]

\[
\leq \sum_{k=0}^{N-2} \frac{1}{M_k^{1-po}} \frac{M_k^p}{M_N^{1-p}} \leq \sum_{k=0}^{N-2} \frac{1}{M_k^{1-(p+1)}} < c < \infty
\]

and

\[
\sum_{k=0}^{N-1} \frac{M_k^p}{M_N} \leq M_N^{p-1} < \infty
\]

we obtain that

\[
\sum_{m=M_N+1}^{\infty} \frac{\int_{I_N} |t_m a|^p d\mu}{m^{2-(1+\alpha)p}}
\]

\[
< c_{\alpha,p} \sum_{m=M_N+1}^{\infty} \frac{1}{m^{2-p}} + c_{\alpha,p} \sum_{m=M_N+1}^{\infty} \frac{1}{m^{2-(1+\alpha)p}} \leq c_{\alpha,p} < \infty.
\]

The proof is complete.

The final result in this section is due to Blahota, Persson and Tephnadze [7].

**Theorem 4.25** Let \( f \in H_{1/(1+\alpha)} \) where \( 0 < \alpha \leq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers satisfying the conditions (1.39) and (1.40). Then there exists an absolute constant \( c_\alpha \) depending only on \( \alpha \) such that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|t_{mf}\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}}{m} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}}^{1/1+\alpha}.
\]
Proof: By Lemma 1.38 the proof is complete if we show that

$$ \frac{1}{\log n} \sum_{m=1}^{n} \frac{\|t_m a\|^{1/(1+\alpha)}}{m} \leq c_\alpha < \infty $$

for every $1/(1+\alpha)$-atom $a$. Analogously to the proofs of previous results we may assume that $a$ is an arbitrary $1/(1+\alpha)$-atom with support $I$, $\mu(I) = M_N^{-1}$ and $I = I_N$ and $m > M_N$.

Let $x \in I_N$. Since $t_m$ is bounded from $L_\infty$ to $L_\infty$ (the boundedness follows from Corollary 1.33) and $\|a\|_\infty \leq M_N^{1+\alpha}$ we obtain that

$$ \int_{I_N} |t_m a(x)|^{1/(1+\alpha)} \, d\mu $$

$$ \leq \|a\|^{1/(1+\alpha)}_\infty / M_N \leq 1. $$

Hence

$$ \frac{1}{\log n} \sum_{m=M_N}^{n} \frac{\int_{I_N} |t_m a(x)|^{1/(1+\alpha)} \, d\mu}{m} $$

$$ \leq \frac{1}{\log n} \sum_{m=1}^{n} \frac{1}{m} \leq c_\alpha < \infty. $$

According to (1.1) together with (4.36)-(4.37) we can conclude that

$$ \frac{1}{\log n} \sum_{m=M_N+1}^{n} \frac{\int_{I_N} |t_m a(x)|^{1/(1+\alpha)} \, d\mu}{m} $$

$$ = \frac{1}{\log n} \sum_{m=M_N+1}^{n} \frac{\sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} \sum_{j=1}^{N} \int_{I_N} |t_m a(x)|^{1/(1+\alpha)} \, d\mu}{m} $$

$$ + \frac{1}{\log n} \sum_{m=M_N+1}^{n} \frac{\sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} \sum_{j=1}^{N} \int_{I_N} |t_m a(x)|^{1/(1+\alpha)} \, d\mu}{m} $$

$$ \leq \frac{c_\alpha}{\log n} \sum_{m=M_N+1}^{n} \frac{M_N^{\alpha/(1+\alpha)} M_N^{1/(1+\alpha)} M_N^{\alpha/(1+\alpha)} M_N^{1/(1+\alpha)} m_{l+1} \cdots m_{N-1}}{M_N} $$

Since

$$ \sum_{k=0}^{N-1} \sum_{l=k+1}^{N-1} \sum_{j=1}^{N} M_l^{\alpha/(1+\alpha)} M_k^{1/(1+\alpha)} m_{l+1} \cdots m_{N-1} / M_N $$
\[
\sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} M^{\alpha/(1+\alpha)}_l M^{1/(1+\alpha)}_k \frac{1}{M_l} \leq \sum_{k=0}^{N-2} M^{1/(1+\alpha)}_k \frac{1}{M^{1/(1+\alpha)}_k} \leq \sum_{k=0}^{N-2} 1 \leq N
\]

and

\[
\sum_{k=0}^{N-1} M^{1/(1+\alpha)}_k M^{\alpha/(1+\alpha)}_N \leq M^{1/(1+\alpha)}_N M^{\alpha/(1+\alpha)}_N \leq M_N
\]

we obtain that

\[
\frac{1}{\log n} \sum_{m=M_N+1}^{n} \frac{\int_{I_{MN}} |t_m a|^{1/(1+\alpha)} \, d\mu}{m} \leq \frac{c_\alpha}{\log n} \left( \sum_{m=M_N+1}^{n} \frac{N M^{\alpha/(1+\alpha)}_N + \frac{1}{m}}{m^{\alpha/(1+\alpha)+1}} \right) < c_\alpha < \infty.
\]

The proof is complete.

### 4.4 Maximal Operators of Riesz and Nörlund Logarithmic Means on Martingale Hardy Spaces

In our previous sections we investigated Nörlund means with non-increasing sequences \(\{q_k : k \in \mathbb{N}\}\), but the case when \(q_k = 1/k\) was excluded, since this sequence does not satisfy the condition (1.39) for any \(0 < \alpha \leq 1\). On the other hand, Riesz logarithmic means are not examples of Nörlund means. In this subsection we fill up this gap simultaneously for both cases.

Both theorems in this section are due to Tephnadze [75].

**Theorem 4.26** a) The maximal operator of Riesz logarithmic means \(R^*\) is bounded from the Hardy space \(H_{1/2}\) to the space \(\text{weak} - L_{1/2}\).

b) Let \(0 < p \leq 1/2\). Then there exists a martingale \(f \in H_p\) such that

\[
\|R^* f\|_p = +\infty.
\]
Proof: By using Abel transformation we obtain that

$$R_n f = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\sigma_j f}{j+1} + \frac{\sigma_n f}{l_n}.$$  

Consequently,

$$R^* f \leq c \sigma^* f.$$  \hspace{1cm} (4.48)

Since $\sigma^*$ is bounded from the martingale Hardy space $H_{1/2}$ to the space weak $- L_{1/2}$ by using (4.48) we can conclude that

$$\|R^* f\|_{\text{weak} - L_{1/2}} \leq c \|f\|_{H_{1/2}}$$

and the proof of part a) is complete.

Let $f = (f^{(n)} : n \in \mathbb{N})$ be martingale defined in Example 1.49.

Set $q_n^s = M_{2n} + M_{2s} - 1, n > s$. Then we can write that

$$R_{q_n^s} f = \frac{1}{l_{q_n^s}} \sum_{j} S_j f$$

$$= \frac{1}{l_{q_n^s}} \sum_{j=1}^{M_{2\alpha_k}} S_j f + \frac{1}{l_{q_n^s}} \sum_{j=M_{2\alpha_k}+1}^{q_n^s} S_j f := I + II.$$  

According to (1.76) we have that

$$|I| \leq \frac{1}{l_{q_n^s}} \sum_{j=1}^{M_{2\alpha_k}-1} |S_j f(x)|$$

$$\leq \frac{1}{\alpha_k} \frac{2\lambda M_{2\alpha_k-1}^{1/p}}{\alpha_k^{1/2}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{1}{j}$$

$$\leq \frac{2\lambda M_{2\alpha_k-1}^{1/p}}{\alpha_k^{1/2}} \leq \frac{2\lambda M_{\alpha_k}^{1/p}}{\alpha_k^{3/2}}.$$  \hspace{1cm} (4.49)

Let $M_{2\alpha_k} \leq j \leq q_n^s$. According to the second inequality of (1.75) in the case when $l = k$, we deduce that

$$S_j f = S_{M_{2\alpha_k}} f + \frac{M_{2\alpha_k}^{1/p-1} \psi_{M_{2\alpha_k}} D_j - M_{2\alpha_k}}{\alpha_k^{1/2}}.$$  \hspace{1cm} (4.49)
Hence, we can rewrite $II$ as

$$II = \frac{1}{l_{q_{\alpha k}}^{q_{\alpha k}}} \sum_{j=M_{2\alpha k}}^{q_{\alpha k}} S_{M_{2\alpha k}} f \frac{j}{j}$$

$$+ \frac{1}{l_{q_{\alpha k}}^{q_{\alpha k}}} M^{1/p-1}_{2\alpha k} \psi_{M_{2\alpha k}}^{1/2} \sum_{j=M_{2\alpha k}}^{q_{\alpha k}} D_{j-M_{2\alpha k}}$$

$$:= II_1 + II_2.$$  

In view of (1.76) we find that

$$|II_1| \leq \frac{1}{l_{q_{\alpha k}}^{q_{\alpha k}}} \sum_{j=M_{2\alpha k}}^{q_{\alpha k}} \frac{1}{j} \left| S_{M_{2\alpha k}} f \right|$$

$$\leq \left| S_{M_{2\alpha k}} f \right| \leq \frac{2\lambda M_{1/p}^{1/p-1}_{2\alpha k}}{\alpha_{k-1}}.$$  

Let $0 < p \leq 1/2$, $x \in I_{2s} \setminus I_{2s+1}$ for $s = \lfloor 2\alpha_k/3 \rfloor, \ldots, \alpha_k$. Since

$$\sum_{j=0}^{M_{2\alpha k}-1} D_j(x) \geq \sum_{j=0}^{M_{2\alpha k}-1} \frac{j}{j+M_{2\alpha k}} \geq \sum_{j=0}^{M_{2\alpha k}-1} \frac{j}{2M_{2\alpha k}} \geq \frac{c M_{2s}^2}{M_{2\alpha k}}$$

we obtain that

$$|II_2| = \frac{1}{l_{q_{\alpha k}}^{q_{\alpha k}}} M^{1/p-1}_{2\alpha k} \psi_{M_{2\alpha k}} \sum_{j=0}^{M_{2\alpha k}-1} \left| D_j \left( x \right) \right|$$

$$\geq \frac{c}{\alpha_{k}} M_{2\alpha k}^{1/p-1} M_{2s}^2 \geq \frac{c M_{2\alpha k}^{1/p-2}}{\alpha_{k}^{3/2}}.$$  

By combining (1.71)-(1.73) with (4.50)-(4.52) we get that

$$\left| R_{q_{\alpha k}} f \right| = |II_2 - I - II_1|$$

$$\geq |II_2| - |I| - |II_1| \geq |II_2| - \frac{2\lambda M_{1/p}^{1/p}}{\alpha_{k}^{3/2}}$$

$$\geq \frac{c M_{2\alpha k}^{1/p-2}}{\alpha_{k}^{3/2}} - \frac{c M_{2\alpha k}^{1/p}}{\alpha_{k}^{3/2}}$$
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\[
\geq \frac{c M_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}}.
\]

Hence,

\[
\int_{G_m} |R^* f(x)|^p d\mu(x) 
\geq c \sum_{s=\lfloor 2\alpha_k/3 \rfloor}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} |R_{q_{\alpha_k}^s} f(x)|^p d\mu(x) 
\geq c \sum_{s=\lfloor 2\alpha_k/3 \rfloor}^{\alpha_k} \int c M_{2\alpha_k}^{1-2p} M_{2s}^{2p} \alpha_k^{3p/2} d\mu(x) 
\geq c \sum_{s=\lfloor 2\alpha_k/3 \rfloor}^{\alpha_k} M_{2\alpha_k}^{1-2p} M_{2s}^{2p-1} \alpha_k^{3p/2} 
\geq \begin{cases} 
c_{p^{2\alpha_k(1-2p)/3}}, & \text{when } 0 < p < 1/2, \\
c_{p^{1/4} \alpha_k}, & \text{when } p = 1/2, \rightarrow \infty, \text{when } k \rightarrow \infty.
\end{cases}
\]

The proof is complete. ■

**Theorem 4.27** Let $0 < p \leq 1$. Then there exists a martingale $f \in H_p$ such that

\[
\|L^* f\|_p = +\infty.
\]

**Proof:** We write that

\[
L_{q_{\alpha_k}^s} f = \frac{1}{l_{q_{\alpha_k}} \sum_{j=1}^{q_{\alpha_k}} S_j f} \sum_{j=1}^{q_{\alpha_k}} S_j f 
= \frac{1}{l_{q_{\alpha_k}} \sum_{j=1}^{M_{2\alpha_k}-1} S_j f} + \frac{1}{q_{\alpha_k} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}} S_j f} := III + IV.
\]

In the view of (1.76) for $III$ we get the following estimate:

\[
|III| \leq \frac{1}{\alpha_k} \sum_{j=0}^{M_{2\alpha_k}-1} \frac{1}{q_{\alpha_k} - j} \frac{M_{2\alpha_k-1}^{1/p}}{\alpha_k^{1/2}} \leq \frac{M_{2\alpha_k-1}^{1/p}}{\alpha_k^{1/2}} \leq \frac{M_{2\alpha_k-1}^{1/p}}{\alpha_k^{3/2}}.
\]

Moreover, according to the second inequality of (1.75) in the case when $l = k$, (see also (4.49)) we can rewrite $IV$ as

\[
IV = \frac{1}{l_{q_{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}} S_{M_{2\alpha_k}} f} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}} S_{M_{2\alpha_k}} f
\]
By now applying (1.76) again we have that
\[
|IV_1| \leq \frac{2\lambda M_{\alpha_k-1}^{1/p}}{\alpha_k^{1/2}} \leq \frac{2\lambda M_{\alpha_k}^{1/p}}{\alpha_k^{3/2}}. \tag{4.56}
\]

Let \(x \in I_{2s} \setminus I_{2s+1}, s = [2\alpha_k/3], \ldots, \alpha_k\). Since
\[
\sum_{j=0}^{M_{2s}-1} \frac{D_j}{M_{2s} - j} = \sum_{j=0}^{M_{2s}-1} \frac{j}{M_{2s} - j} = \sum_{j=0}^{M_{2s}-1} \frac{M_{2s}}{M_{2s} - j} - \sum_{j=0}^{M_{2s}-1} \frac{M_{2s} - j}{M_{2s} - j}
\]
we obtain that
\[
|IV_2| = \frac{1}{l_{q_{\alpha_k}}^{1/p-1}} M_{\alpha_k}^{1/p} \left| \sum_{j=0}^{M_{2s}-1} \frac{D_j}{M_{2s} - j} \right| \geq \frac{c_p M_{\alpha_k}^{1/p-1}}{\alpha_k^{3/2}} M_{2s} - \frac{2\lambda M_{\alpha_k}}{\alpha_k^{3/2}} s M_{2s}, \quad x \in I_{2s}/I_{2s+1}. \tag{4.57}
\]

By combining (1.73) with (4.53)-(4.57) for \(x \in I_{2s} \setminus I_{2s+1}, s = [2\alpha_k/3], \ldots, \alpha_k\) and \(0 < p \leq 1\) we get that
\[
\left| L_{q_{\alpha_k}}^{s} f(x) \right| = |III + IV_1 + IV_2| \geq |IV_2| - |III| - |IV_1|
\]
\[
\geq \frac{c_p M_{\alpha_k}^{1/p-1}}{\alpha_k^{3/2}} M_{2s} - \frac{2\lambda M_{\alpha_k}}{\alpha_k^{3/2}} s M_{2s}.
\]

Consequently,
\[
\int_{G_m} \left| L^s f(x) \right|^p d\mu(x) \geq \sum_{s=[2\alpha_k/3]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left| L^s f(x) \right|^p d\mu(x)
\]
\[
\geq \sum_{s=[2\alpha_k/3]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left| L_{q_{\alpha_k}}^{s} f(x) \right|^p d\mu(x)
\]
\[
\geq c_p \sum_{s=[2\alpha_k/3]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} M_{2\alpha_k}^{1-p} \frac{M_{2s}^p}{\alpha_k^{p/2}} d\mu \geq c_p \sum_{s=[2\alpha_k/3]}^{\alpha_k} \frac{M_{2\alpha_k}^{1-p}}{\alpha_k^{p/2}} M_{2s}^{p-1}
\]
\[
\geq \left\{ \begin{array}{ll}
\frac{c_p 2^{\alpha_k(1-p)/3}}{\alpha_k^{p/2}}, & \text{when } 0 < p < 1, \\
\frac{c_p 1}{\alpha_k^{1/2}}, & \text{when } p = 1,
\end{array} \right.
\]
\[
\rightarrow \infty, \text{ when } k \to \infty.
\]

The proof is complete.

\[\blacksquare\]
4.5 Applications

First we consider Nörlund means \( t_n \) with monotone and bounded sequence \( \{ q_k : k \in \mathbb{N} \} \).

The results in our previous sections in particular imply the following results:

**Theorem 4.28**

a) Let \( f \in H_{1/2} \) and \( t_n \) be Nörlund means with monotone and bounded sequence \( \{ q_k : k \in \mathbb{N} \} \). Then there exists an absolute constant \( c \) such that

\[
\| t^* f \|_{\text{weak-}L_{1/2}} \leq c \| f \|_{H_{1/2}}.
\]

b) There exists a martingale \( f \in H_{1/2} \), such that

\[
\| t^* f \|_{1/2} = \infty.
\]

**Theorem 4.29**

a) Let \( p > 1/2 \), \( f \in H_p \) and \( t_n \) be Nörlund means with monotone and bounded sequence \( \{ q_k : k \in \mathbb{N} \} \). Then there exists an absolute constant \( c_p \) depending only on \( p \) such that

\[
\left\| \sim t_{1,p} f \right\|_p \leq c_p \| f \|_{H_p}.
\]

b) Let \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be a non-decreasing function satisfying the condition (3.6). Then the following maximal operator

\[
\sup_{n \in \mathbb{N}} \frac{|t_n f|}{\Phi_n}
\]

is not bounded from the Hardy space \( H_p \) to the space weak \( - L_p \).

**Theorem 4.30**

a) Let \( f \in H_{1/2} \) and \( t_n \) be Nörlund means with monotone and bounded sequence \( \{ q_k : k \in \mathbb{N} \} \). Then there exists an absolute constant \( c \) such that

\[
\left\| \sim t_{1} f \right\|_{1/2} \leq c \| f \|_{H_{1/2}}.
\]

b) Let \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be a non-decreasing function satisfying the condition (3.7). Then the following maximal operator

\[
\sup_{n \in \mathbb{N}} \frac{|B_n f|}{\Phi_n}
\]

is not bounded from the Hardy space \( H_{1/2} \) to the Lebesgue space \( L_{1/2} \).

**Theorem 4.31**

Let \( 0 < p < 1/2 \), \( f \in H_p \) and \( t_n \) be Nörlund means with monotone and bounded sequence \( \{ q_k : k \in \mathbb{N} \} \). Then there exists an absolute constant \( c_p \) depending only on \( p \) such that

\[
\sum_{m=1}^{\infty} \frac{\| t_m f \|^p}{m^{2-2p}} \leq c_p \| f \|^p_{H_p}.
\]
Theorem 4.32 Let \( f \in H_{1/2} \) and \( t_n \) be Nörlund means with monotone and bounded sequence \( \{q_k : k \in \mathbb{N}\} \). Then there exists an absolute constant \( c \) such that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|t_m f\|^{1/2}}{m} \leq c \|f\|^{1/2}_{H_{1/2}}, \quad n = 2, 3, \ldots
\]

Next we remark that \( \sigma_n \) are Nörlund means with monotone and bounded sequence \( \{q_k : k \in \mathbb{N}\} \) and \( \kappa_{\alpha,\beta}^\ast \) are concrete examples of Nörlund means with non-decreasing and unbounded sequence \( \{q_k : k \in \mathbb{N}\} \) but they have similar boundedness properties. The results in our previous sections in particular imply the following results:

Theorem 4.33 a) Let \( f \in H_{1/2} \). Then there exists absolute constant \( c \) such that

\[
\|\sigma^\ast f\|_{\text{weak} - L_{1/2}} \leq c \|f\|_{H_{1/2}}
\]

and

\[
\|\kappa_{\alpha,\beta}^\ast f\|_{\text{weak} - L_{1/2}} \leq c \|f\|_{H_{1/2}}.
\]

b) There exists a martingale \( f \in H_{1/2} \) such that

\[
\|\sigma^\ast f\|_{1/2} = \infty
\]

and

\[
\|\kappa_{\alpha,\beta}^\ast f\|_{1/2} = \infty.
\]

Theorem 4.34 a) Let \( 0 < p < 1/2 \) and \( f \in H_p \). Then there exists an absolute constant \( c_p \) depending only on \( p \) such that

\[
\|\tilde{\sigma}_p^\ast f\|_p \leq c_p \|f\|_{H_p}
\]

and

\[
\|\tilde{\kappa}_{\alpha,\beta}^\ast f\|_p \leq c_p \|f\|_{H_p}.
\]

b) Let \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be a non-decreasing function satisfying the condition (3.6). Then the following maximal operators

\[
\sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\Phi_n} \quad \text{and} \quad \sup_{n \in \mathbb{N}} \frac{|\kappa_{\alpha,\beta}^\ast|}{\Phi_n}
\]

are not bounded from the Hardy space \( H_p \) to the space weak \( \text{L}_p \).

Theorem 4.35 a) Let \( f \in H_{1/2} \). Then there exists an absolute constant \( c \) such that

\[
\|\sigma^\ast f\|_{1/2} \leq c \|f\|_{H_{1/2}}
\]
and

\[ \left\| \kappa^{\alpha,\beta} f \right\|_{1/2}^{1/2} \leq c \| f \|_{H_{1/2}}. \]

(b) Let \( \varphi : \mathbb{N}_+ \rightarrow [1, \infty) \) be a nondecreasing function satisfying the condition (3.11). Then the following maximal operators

\[ \sup_{n \in \mathbb{N}} \frac{\sigma_n f}{\Phi_n} \quad \text{and} \quad \sup_{n \in \mathbb{N}} \frac{\kappa_n^{\alpha,\beta}}{\Phi_n} \]

are not bounded from the Hardy space \( H_{1/2} \) to the Lebesgue space \( L_{1/2} \).

**Theorem 4.36** Let \( 0 < p < 1/2 \) and \( f \in H_p \). Then there exists an absolute constant \( c_p \) depending only on \( p \) such that

\[ \sum_{m=1}^{\infty} \left\| \sigma_m f \right\|^p_{m^{2-2p}} \leq c_p \| f \|^p_{H_p} \]

and

\[ \sum_{m=1}^{\infty} \left\| \kappa_m^{\alpha,\beta} f \right\|^p_{m^{2-2p}} \leq c_p \| f \|^p_{H_p}. \]

**Theorem 4.37** Let \( f \in H_{1/2} \). Then there exists an absolute constant \( c \) such that

\[ \frac{1}{\log n} \sum_{m=1}^{n} \left\| \sigma_m f \right\|^{1/2}_{m^{1/2}} \leq c \| f \|^{1/2}_{H_{1/2}}, \quad n = 2, 3, \ldots, \]

and

\[ \frac{1}{\log n} \sum_{m=1}^{n} \left\| \kappa_m^{\alpha,\beta} f \right\|^{1/2}_{m^{1/2}} \leq c \| f \|^{1/2}_{H_{1/2}}, \quad n = 2, 3, \ldots. \]

Furthermore, we note that it is obvious that Cesàro \((C, \alpha)\) and Riesz \((R, \alpha)\) means are examples of Nörlund means with non-increasing sequence satisfying the conditions (1.39) and (1.40). It follows that all results concerning such summability methods are true also for the Cesàro \((C, \alpha)\) and Riesz \((R, \alpha)\) means. Hence, the results in our previous sections in particular imply the following results:

**Theorem 4.38** a) Let \( 0 < p < 1/2 \). Then there exists a martingale \( f \in H_p \) such that

\[ \sup_{n \in \mathbb{N}} \left\| \sigma_n^\alpha f \right\|_{\text{weak}-L_p} = \infty \]

and

\[ \sup_{n \in \mathbb{N}} \left\| \beta_n^\alpha f \right\|_{\text{weak}-L_p} = \infty. \]
Theorem 4.39  a) Let $0 < \alpha \leq 1$ and $f \in H_{1/(1+\alpha)}$. Then there exists absolute constant $c_\alpha$ depending only on $\alpha$ such that
\[ \|\sigma_{\alpha,*} f\|_{\text{weak}-L_{1/(1+\alpha)}} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}} \]
and
\[ \|\beta_{\alpha,*} f\|_{\text{weak}-L_{1/(1+\alpha)}} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}} . \]

b) Let $0 < \alpha \leq 1$. Then there exists a martingale $f \in H_{1/(1+\alpha)}$ such that
\[ \|\sigma_{\alpha,*} f\|_{1/(1+\alpha)} = \infty \]
and
\[ \|\beta_{\alpha,*} f\|_{1/(1+\alpha)} = \infty . \]

Theorem 4.40  a) Let $0 < \alpha \leq 1$, $0 < p < 1/(1+\alpha)$ and $f \in H_p$. Then there exists an absolute constant $c_{\alpha,p}$ depending only on $\alpha$ and $p$ such that
\[ \left\| \tilde{\sigma}_{\alpha,*} f \right\|_p \leq c_{\alpha,p} \|f\|_{H_p} \]
and
\[ \left\| \tilde{\beta}_{\alpha,*} f \right\|_p \leq c_{\alpha,p} \|f\|_{H_p} . \]

b) Let $0 < \alpha \leq 1$ and $\varphi : \mathbb{N}_+ \to [1, \infty)$ be a non-decreasing function satisfying the condition (4.29). Then the following maximal operators
\[ \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\Phi_n} \text{ and } \sup_{n \in \mathbb{N}} \frac{|\beta_n f|}{\Phi_n} \]
are not bounded from the Hardy space $H_p$ to the Lebesgue space $L_{1/(1+\alpha)}$.

Theorem 4.41  a) Let $0 < \alpha \leq 1$ and $f \in H_{1/(1+\alpha)}$. Then there exists an absolute constant $c_\alpha$ depending only on $\alpha$ such that
\[ \left\| \tilde{\sigma}_{\alpha,*} f \right\|_{1/(1+\alpha)} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}} \]
and
\[ \left\| \tilde{\beta}_{\alpha,*} f \right\|_{1/(1+\alpha)} \leq c_\alpha \|f\|_{H_{1/(1+\alpha)}} . \]

b) Let $0 < \alpha \leq 1$ and $\varphi : \mathbb{N}_+ \to [1, \infty)$ be a nondecreasing function satisfying the condition (4.35). Then the following maximal operators
\[ \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\Phi_n} \text{ and } \sup_{n \in \mathbb{N}} \frac{|\beta_n f|}{\Phi_n} \]
are not bounded from the Hardy space $H_{1/(1+\alpha)}$ to the Lebesgue space $L_{1/(1+\alpha)}$. 
Theorem 4.42 Let $0 < \alpha \leq 1$, $0 < p < 1/(1 + \alpha)$ and $f \in H_p$. Then there exists an absolute constant $c_{\alpha,p}$ depending only on $\alpha$ and $p$ such that

$$\sum_{m=1}^{\infty} \frac{\|\sigma_m^{\alpha} a\|_p^p}{m^{(1+\alpha)(1-p)}} \leq c_{\alpha,p} \|f\|_{H_p}^p$$

and

$$\sum_{m=1}^{\infty} \frac{\|\beta_m^{\alpha} a\|_p^p}{m^{(1+\alpha)(1-p)}} \leq c_{\alpha,p} \|f\|_{H_p}^p.$$ 

Theorem 4.43 Let $0 < \alpha \leq 1$ and $f \in H_{1/(1+\alpha)}$. Then there exists an absolute constant $c_{\alpha}$ depending only on $\alpha$ such that

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|\sigma_m^{\alpha} f\|_{1/(1+\alpha)}^{1/(1+\alpha)}}{m} \leq c_{\alpha} \|f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}$$

and

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|\beta_m^{\alpha} f\|_{1/(1+\alpha)}^{1/(1+\alpha)}}{m} \leq c_{\alpha} \|f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}.$$ 

Finally, we present some applications concerning almost everywhere convergence of some summability methods. To study almost everywhere convergence of some summability methods is one of the most difficult topics in Fourier analysis. They involve techniques from function theory and Hardy spaces.

In most applications the a.e. convergence of $\{T_n : n \in \mathbb{N}\}$ can be established for $f$ in some dense class of $L_1(G_m)$. In particular, the following result play an important role for studying this type of questions (see e.g. the books [35], [55] and [88]).

Lemma 4.44 Let $f \in L_1$ and $T_n : L_1 \to L_1$ be some sub-linear operators and

$$T^* := \sup_{n \in \mathbb{N}} |T_n|.$$ 

If

$$T_n f \to f \ a.e. \ \text{for every } f \in S$$

where the set $S$ is dense in the space $L_1$ and the maximal operator $T^*$ is bounded from the space $L_1$ to the space weak $L_1$, that is

$$\sup_{\lambda > 0} \lambda \mu \{x \in G_m : |T^* f(x)| > \lambda\} \leq \|f\|_1,$$

then

$$T_n f \to f, \ a.e. \ \text{for every } f \in L_1(G_m).$$
Remark 4.45  Since the Vilenkin function $\psi_m$ is constant on $I_n(x)$ for every $x \in G_m$ and $0 \leq m < M_n$, it is clear that each Vilenkin function is a complex-valued step function, that is, it is a finite linear combination of the characteristic functions

$$\chi(E) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

On the other hand, notice that, by Corollary [L5] (Paley lemma), it yields that

$$\chi(I_n(t))(x) = \frac{1}{M_n} \sum_{j=0}^{M_n-1} \psi_j(x-t), \quad x \in I_n(t),$$

for each $x, t \in G_m$ and $n \in \mathbb{N}$. Thus each step function is a Vilenkin polynomial. Consequently, we obtain that the collection of step functions coincides with a collection of Vilenkin polynomials $\mathcal{P}$. Since the Lebesgue measure is regular it follows from Lusin theorem that given $f \in L_1$ there exist Vilenkin polynomials $P_1, P_2, \ldots$, such that $P_n \to f$ a.e. when $n \to \infty$. This means that the Vilenkin polynomials are dense in the space $L_1$.

By using Lemma 4.44, remarks 4.45 and the results in our previous sections we in particular obtain the following a.e. convergence results.

Theorem 4.46 Let $f \in L_1$ and $t_n$ be the regular Nörlund means with non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then

$$t_n f \to f \quad \text{a.e., when } n \to \infty.$$  

Proof: Since

$$S_n P = P, \quad \text{for every } P \in \mathcal{P},$$

we obtain that

$$t_n P \to P \quad \text{a.e., when } n \to \infty,$$

for every regular Nörlund mean with non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$, where $P \in \mathcal{P}$ is dense in the space $L_1$ (see Remark 4.45).

On the other hand, by combining Lemma 1.40 and Theorem 4.1 we obtain that the maximal operator of Nörlund means, with non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying conditions (1.38) is bounded from the space $L_1$ to the space weak $- L_1$, that is,

$$\sup_{\lambda > 0} \lambda \mu \{x \in G_m : |t^* f(x)| > \lambda\} \leq \|f\|_1.$$  

According to Lemma 4.44 we obtain that under condition (1.38) we have a.e. convergence

$$t_n f \to f \quad \text{a.e., when } n \to \infty.$$  

The proof is complete.
Corollary 4.47 Let \( f \in L_1 \). Then
\[
\sigma_n f \to f \quad a.e., \quad \text{when} \ n \to \infty,
\]
and
\[
\kappa_n^{\alpha, \beta} f \to f \quad a.e., \quad \text{when} \ n \to \infty.
\]

Proof: Since \( \sigma_n \) and \( \kappa_n \) are regular Nörlund means with non-decreasing sequence \( \{q_k : k \in \mathbb{N}\} \) (see (1.5) and (1.10)) the proof is complete by just using Theorem 4.46.

Theorem 4.48 Let \( f \in L_1 \) and \( t_n \) be Nörlund means with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \) satisfying conditions (1.39) and (1.40). Then
\[
t_n f \to f \quad a.e., \quad \text{when} \ n \to \infty.
\]

Proof: By using part b) of Theorem 1.1 we get that every Nörlund mean, with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \) is regular. Since
\[S_n P = P, \quad \text{for every} \ P \in \mathcal{P}\]
we obtain that \( t_n P \to P, \ \text{when} \ n \to \infty, \ \text{where} \ P \in \mathcal{P} \) is dense in the space \( L_1 \) (see Remark 4.45).

On the other hand, by combining Lemma 1.40 and part a) of Theorem 4.11 we obtain that the maximal operator of Nörlund means with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \) satisfying conditions (1.39) and (1.40) is bounded from the space \( L_1 \) to the space weak \( -L_1 \), that is
\[
\sup_{\lambda > 0} \lambda \mu \{x \in G_m : |t^* f(x)| > \lambda\} \leq \|f\|_1.
\]

According to Lemma 4.44 we obtain that under conditions (1.39) and (1.40) we have a.e. convergence
\[
t_n f \to f \quad a.e., \quad \text{when} \ n \to \infty.
\]

The proof is complete.

Corollary 4.49 Let \( f \in L_1 \). Then
\[
\sigma_n^{\alpha} f \to f \quad a.e., \quad \text{when} \ n \to \infty, \ \text{when} \ 0 < \alpha < 1.
\]
and
\[
\beta_n^{\alpha} f \to f \quad a.e., \quad \text{when} \ n \to \infty, \ \text{when} \ 0 < \alpha < 1.
\]

Proof: By combining (1.6)-(1.7) with (1.8)-(1.9) we obtain that the Nörlund means \( \sigma_n^{\alpha} \) and \( \beta_n^{\alpha} \), with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \) satisfy conditions (1.39) and (1.40).

Hence, the proof is complete by just using Theorem 4.48.
Corollary 4.50 Let $f \in L_1$ and $t_n$ be Nörlund means with monotone and bounded sequence \( \{q_k : k \in \mathbb{N}\} \). Then
\[
t_n f \rightarrow f \quad \text{a.e., when} \quad n \rightarrow \infty.
\]

Proof: The proof is analogously to the proofs of Theorems 4.46 and 4.48 so we leave out the details. \hfill \blacksquare

Theorem 4.51 Let $f \in L_1$. Then
\[
R_n f \rightarrow f, \quad \text{a.e. when} \quad n \rightarrow \infty.
\]

Proof: The proof is analogously to the proofs of Theorems 4.46 and 4.48 so we leave out the details. \hfill \blacksquare


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