How Ominous is the Future Global Warming Premonition?

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Abstract

Global warming, the phenomenon of increasing global average temperature in the recent decades, is receiving wide attention due to its very significant adverse effects on climate. Whether global warming will continue even in the future, is a question that is most important to investigate. In this regard, the so-called general circulation models (GCMs) have attempted to project the future climate, and nearly all of them exhibit alarming rates of global temperature rise in the future.

Although global warming in the current time frame is undeniable, it is important to assess the validity of the future predictions of the GCMs. In this article, we attempt such a study using our recently-developed Bayesian multiple testing paradigm for model selection in inverse regression problems. An important premise in the context of our GCM forecast assessment hinges on the question that how probable the current global warming phenomenon is if the future predictions by the GCMs are correct. This is the inverse regression aspect since the future depends upon the current, but here we wish to learn about the present pretending it to be unknown, assuming that the future is known. Our multiple testing framework coherently compares the combination of inverse aspect with the forward, to yield the best model. The model we assume for the global temperature time series is based on Gaussian process emulation of the black box scenario, realistically treating the dynamic evolution of the time series as unknown.

We apply our ideas to datasets available from the Intergovernmental Panel on Climate Change (IPCC) website. The best GCM models selected by our method under different assumptions on future climate change scenarios do not convincingly support the present global warming pattern when only the future predictions are considered known.

We also consider all the GCM models under given assumptions on any particular future climate change scenario as an ensemble, which we model as multivariate time series, based on multidimensional Gaussian processes. Our results in the multivariate cases emphatically demonstrate that if the future GCM predictions are believed to be true, then the current global warming phenomenon must be highly unlikely. In other words, the GCM predictions available at the IPCC website do not seem to adequately represent the future climate change.

What is more, using our Gaussian process idea, we forecast the future temperature time series given the current one. Interestingly, our results do not support drastic future global warming predicted by almost all the GCM models. Indeed, we show that except the predictions of the best GCM model in the “Commitment” scenario, none other fall in the high density regions of our Bayesian forecasted time series.

Keywords: Bayesian multiple testing for model selection; General circulation models; Global warming; Inverse regression; Multivariate Gaussian process; Parallel computing.

1 Introduction

The gradual warming of the earth’s average surface temperature, known as global warming, is perhaps the gravest concern for environmental scientists. Overwhelming evidence from multiple
and independent sources of data have led the U.S. Global Change Research Program, the National Academy of Sciences, and the Intergovernmental Panel on Climate Change (IPCC) to independently conclude that global warming, particularly, in the recent decades, is undeniable. As per the records (see IPCC (2018), for example), compared to the pre-industrial baseline 1850 – 1900, the 2009 – 2015 time period was warmer by about 0.87°C, and that each decade is getting warmer by about 0.2°C. Such an alarming rate of increase is unprecedented, and even the prehistorical rates of global warming, such as the Paleocene-Eocene Thermal Maximum, fail to match the current rate of global warming (see, for example, Masson-Delmotte et al. (2013)). However, see Idso et al. (2013b) and the references therein who argue, providing details on past temperature records, that this global warming is not unprecedented.

Such global warming is considered responsible for increasing droughts, heat waves, increase in extreme wet or dry events within the monsoon period in India and East Asia, increase in frequencies of hurricanes and typhoons, increase in global sea level as a result of melting glaciers, expansion of deserts and much more. According to the IPCC, “human influence on climate has been the dominant cause of observed warming since the mid-20th century”, and this conclusion has been upheld by all scientific bodies. In fact, human activities are estimated to have caused approximately 1.0°C of global warming above pre-industrial levels. Scientific investigations reveal that (see Olivier and Peters (2019)) the emission of greenhouse gases, with over 90% of the impact from carbon dioxide and methane, has been a major contributing factor to global warming by human activities such as fossil fuel burning, agricultural emissions and deforestation. But also see Idso et al. (2013b) who write “The empirical observations cited above reveal a relationship opposite of what is expected if carbon dioxide and methane were the powerful greenhouse gases the IPCC claims them to be. Clearly, if there is anything at all that is unusual, unnatural, or unprecedented about Earth’s current surface air temperature, it is that it is so cold.” and de Lange and Carter (2013) who mention in their key findings section “There appears to be nothing unusual about the extremes of wetness and dryness experienced during the twentieth century, or about recent changes in ocean circulation, sea level, or heat content, that would require atmospheric carbon dioxide forcing to be invoked as a causative factor. Natural variability in the frequency or intensity of precipitation extremes and sea-level change occurs largely on decadal and multidecadal time scales, and this variability cannot be discounted as a major cause of recent changes where they have occurred.”

The IPCC has warned that if the warming increases by 1.5°C compared to the pre-industrial era 1850 – 1900, then human and natural systems would be at grave risk. The concerning news is that under the current conditions global warming is projected to surpass 2.8°C by the year 2100 (see Climate Action Tracker (2019)).

The climate projections are performed by the general circulation models (GCMs) that attempt to model the major climate system components, namely, atmosphere, land surface, ocean and sea ice, and the interactions among them. Expressing great confidence in such models, the IPCC has claimed that (see Lupo et al. (2013)) “development of climate models has resulted in more realism in the representation of many quantities and aspects of the climate system,” adding, “it is extremely likely that human activities have caused more than half of the observed increase in global average surface temperature since the 1950s”. However, Lupo et al. (2013) write “Confidence in a model is further based on the careful evaluation of its performance, in which model output is compared against actual observations. A large portion of this chapter, therefore, is devoted to the evaluation of climate models against real-world climate and other biospheric data. That evaluation, summarized in the findings of numerous peer-reviewed scientific papers described in the different subsections of this chapter, reveals the IPCC is overestimating the ability of current state-of-the-art GCMs to accurately simulate both past and future climate. The IPCC’s stated confidence in the models, as presented at the beginning of this chapter, is likely exaggerated. The many and varied model deficiencies discussed in this chapter indicate much work remains to be done before model simulations can
be treated with the level of confidence ascribed to them by the IPCC.” This was written seven years ago, and by now we expect the GCMs to have reduced their deficiencies and to yield more reliable climate projections.

The current GCM predictions by different GCMs available from the IPCC website http://www.ipcc-data.org/sim/gcm_global/index.html, under the assumptions of several future climate scenarios associated with greenhouse gas emissions, pertaining to the Special Report on Emissions Scenarios (SRES), a report by the IPCC published in 2000. According to the IPCC Fourth Assessment Report (AR4), published in 2007, there are three SRES, namely, A1B, A2 and B1. Brief descriptions of the assumptions, obtained from the IPCC website, are reproduced below for the reader’s convenience.

The key assumption for A1B is a future world of very rapid economic growth, low population growth and rapid introduction of new and more efficient technology. Major underlying themes are economic and cultural convergence and capacity building, with a substantial reduction in regional differences in per capita income. In this world, people pursue personal wealth rather than environmental quality.

SRES A2 corresponds to a very heterogeneous world. The underlying theme is that of strengthening regional cultural identities, with an emphasis on family values and local traditions, high population growth, and less concern for rapid economic development.

In SRES B1, a convergent world with the same global population as in the A1B is assumed but with rapid changes in economic structures toward a service and information economy, with reductions in materials intensity, and the introduction of clean and resource-efficient technologies.

Commitment is a non-SRES idealised scenario in which the atmospheric burdens of long-lived greenhouse gases are held fixed at AD2000 levels.

The scenarios A1B, A2, B1 and Commitment consist of 21, 17, 21 and 16 GCMs, respectively, each yielding a simulated global mean temperature time series in the duration 1900 – 2099. The HadCRUT4 observed near surface average global temperature dataset during the years 1850 – 2020 is also available from the IPCC website; see https://www.metoffice.gov.uk/hadobs/hadcrut4/data/current/download.html. But since the year 2020 is still ongoing, we find reasons to doubt the reliability of the last few data points, and as such, we shall consider the dataset ranging from 1850 – 2016. This dataset pertains to temperature anomalies in degree celsius relative to the years 1961 – 1990. Now, the most widely quoted value for the global average temperature for the 1961 – 1990 period is 14°C, which has been developed by Jones et al. (1999). Hence, we convert the HadCRUT4 temperature anomalies data to (approximate) actual temperatures by adding 14°C to the anomalies. We also convert the GCM-simulated actual temperatures, originally available in Kelvin, to degree celsius.

Figure 1.1 presents the diagrams of the HadCRUT4 dataset (thick, black line) and the GCM predictions. Observe that the GCM based global temperatures seem to significantly underestimate the observed global temperatures during the years 1900 – 2016. Moreover, their rates of increase seem to be much faster than that of the observed dataset. Hence, the sharp increase of most of the GCM based future temperatures till the end of this century, is perhaps not unquestionable. Observe that the future predictions of the Commitment models are more stable compared to the others.

Perhaps the most important ingredient in any statistical learning is quantification of uncertainty. The GCM results displayed in Figure 1.1 are devoid of any uncertainty quantification; at least we are unable to find any in the IPCC website. In the observed HadCRUT4 data context, an ensemble of 100 time series are available, which has been recommended by climatologists to quantify uncertainty in the observations to some extent. It seems that ensembles can be obtained even for GCM models, provided they are run with different initial conditions. But the models are deterministically dynamic, and non-probabilistic, so that rigorous statistical ways of uncertainty quantification need not apply. It is thus not clear how believable the future global
Figure 1.1: Visualization of the HadCRUT4 data (thick, black line) and the GCM based time series. The temperature is in °C.
warming forecasts presented in Figure 1.1 are. In fact, as detailed in Lupo et al. (2013), the leading scientific experts have placed no faith in the GCMs. For instance, Freeman Dyson has written (see Dyson (2007)), “I have studied the climate models and I know what they can do. The models solve the equations of fluid dynamics, and they do a very good job of describing the fluid motions of the atmosphere and the oceans. They do a very poor job of describing the clouds, the dust, the chemistry, and the biology of fields and farms and forests. They do not begin to describe the real world that we live in”. Green et al. (2009) tested whether the warming-trend forecasts used by the IPCC are more accurate than the standard benchmark forecast that there will be no change, using the historical HadCRUT3 observed dataset, which exhibited clear global warming till the present years. To their surprise, they found that the errors from the IPCC warming trend forecasts were nearly eight times greater than the errors from the no-change forecasts. Consequently, Green et al. (2009) recommend that the best policy is to do nothing about global warming.

The evaluation method of Green et al. (2009) was not based upon model based statistical or probabilistic methods and thus calls for more sophisticated analyses. In this article, we evaluate the global warming forecasts shown in Figure 1.1 in a rigorous footing using our recently-developed Bayesian methods. An important question in this regard is if the observed HadCRUT4 time series is plausible, given the GCM forecasts. This gives rise to an inverse regression problem in the following sense. The future temperature depends upon the present, our goal is to learn about the present, pretending it to be unknown, while the future is assumed to be known. Given each climate scenario, we then select the best GCM using our Bayesian multiple testing paradigm for model selection in inverse regression problems (Chatterjee and Bhattacharya (2020)). The multiple testing procedure, it must be mentioned, not only considers the inverse aspect; it combines the inverse aspect with the forward in a coherent Bayesian compound decision theoretic sense, to compare the models under consideration. Once the best models are selected, we then show that even for such best GCMs, the Bayesian posterior time series for the current years (1850—2016) do not convincingly support the observed HadCRUT4 data, given the future forecasts for the years 2017—2099.

It is important to discern that the actual model for climate dynamics must be infeasibly complex and in fact unknown. Even the GCMs, which are complex computer models, are nothing but black boxes to us. The purpose of this discussion is to make it clear that standard time series models are inappropriate for climate dynamics. As such, we consider modeling the logarithm of the global temperature at any year as a function of that at the previous year, plus some random error, where the function is assumed to be unknown and modeled appropriately by Gaussian process. The key idea has parallels with Bhattacharya (2007) and Ghosh et al. (2014).

Apart from the Bayesian model selection framework, we also treat the different GCM time series in any given climate scenario as an ensemble, and extend our univariate climate dynamics modeling to the multivariate situation, with multidimensional Gaussian processes replacing the previous one-dimensional Gaussian processes. The posterior distribution of the mean of the logarithm of the time series during 1850—2016, averaged over the dimensions (ensembles) in the corresponding climate scenario, is of interest in these cases. Our results in the multidimensional context very emphatically bear out that the HadCRUT4 data with its global warming trend must be highly implausible if the GCM forecasts are believed to be true.

Furthermore, given the observed HadCRUT4 data and our Gaussian process emulation model, we also provide Bayesian forecasts for the years 2017—2099, which show no evidence of drastic future global warming. Interestingly, as can be anticipated from panel (d) of Figure 1.1, only the forecasted time series by the best GCM model in the Commitment scenario fall in the high density regions of our Bayesian forecasted time series.

The general reader is likely to anticipate from the above discussions that computations associated with a study of such a proportion must be infeasibly complex. We assure this is
not so. We wrote all our codes in the C language as efficiently as possible, parallelizing them using the Message Passing Interface (MPI) protocol whenever relevant, for example, in the case of the Bayesian multiple testing procedure. In such a case, we implemented the Gaussian process models associated with the large number of GCM forecasts in the parallel computing architecture (VMWare) available at our institution. Very efficient and time-saving computations are the results of our parallel processing. Details will be presented in due course.

The rest of our article is structured as follows. In Section 2 we introduce our Gaussian process emulation model for climate dynamics, and discuss relevant prior choices in Section 3. The methods for Bayesian posterior inference regarding the current temperature time series given the future GCM simulations, and regarding future forecasts given the current temperature time series, are detailed in Section 4. In Section 5 we introduce our Bayesian multiple testing procedure in the context of best GCM selection in different climate scenarios, and provide details on our method of implementation in Section 6. The results of our best GCM selections and their detailed analyses are provided in Section 7. In Section 8, we model the ensemble of GCM-based future temperature time series in each climatic scenario as nonparametric multidimensional time series, driven by multidimensional Gaussian processes, and present the relevant theory and methods. The results and detailed analyses of our Bayesian multivariate Gaussian process emulation model for climate dynamics are presented in Section 9. In Section 10 we forecast the future global temperature with our Bayesian Gaussian process approach, conditional on the HadCRUT4 data, and compare our results with the GCM forecasts as well as with the analysis of Green et al. (2009). Finally, in Section 11, we summarize our contributions, along with relevant discussions.

2 Gaussian process based emulation process for nonparametric climate dynamics

Let \( \{ x_t : t = 0, 1, 2, \ldots \} \) denote a time series, here the logarithm of the global temperature time series. For time \( t \geq 1 \), we model \( x_t \) as

\[
x_t = f_t(x_{t-1}) + \epsilon_t, \tag{2.1}
\]

where \( \epsilon_t \sim N(0, \sigma^2_\epsilon) \) independently, for \( t \geq 1 \). In this article, we assume that \( x_0 \) is known. Crucially, we assume that \( f_t \) is an unknown function dependent on time \( t \). For any real \( z \), we write \( f_t(z) = f(t, z) \), where \( f(\cdot) \) is considered an unknown function on \( \mathbb{R}^+ \times \mathbb{R} \), which we shall model as a Gaussian process. Here \( \mathbb{R}^+ = [0, \infty) \) and \( \mathbb{R} = (-\infty, \infty) \).

Using the notation \( x_{t,u}^* = (t, x_u) \) following Ghosh et al. (2014), we re-write (2.1) as

\[
x_t = f(x_{t,t-1}^*) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2_\epsilon) \text{ independently.} \tag{2.2}
\]

Next, we provide details of the Gaussian process based model for \( f \).

2.1 Modeling the unknown time-varying functions using Gaussian process

We model \( f(\cdot) \) as a Gaussian processes with mean functions \( \mu_f(\cdot) = h(\cdot)'\beta_f \) and with \( h(x^*) = (1, x^*)' \) for any \( x^* \), and covariance function of the form \( \sigma_f^2 c_f(\cdot, \cdot) \). Here \( \sigma_f^2 \) is the process variance and \( c_f \) is the correlations function. Typically, for any \( z_1, z_2, c_f(z_1, z_2) = \exp\{-(z_1 - z_2)'R_f(z_1 - z_2)\} \), where \( R_f \) is a \( 2 \times 2 \)-dimensional diagonal matrix consisting of respective smoothness parameters \( \{r_{1,f}, r_{2,f}\} \). These choices of the correlation functions imply that the functions, modeled by the process realizations, are infinitely smooth.

Our model is thus associated with the parameter set \( \theta = (\theta_f, \sigma^2_\epsilon) \), where \( \theta_f = (\beta_f, \sigma^2_f, r_{1,f}, r_{2,f}) \). The choice of the priors on the parameters will be discussed subsequently, but we shall assume that all the components of \( \theta \) are a priori independent. Henceforth, abusing notation, we
shall denote densities and distributions using the notation \([\cdot]\) and conditional densities and distributions by \([\cdot|\cdot]\).

2.2 Hierarchical structure induced by our Gaussian process approach

Thus, for any \(T > 1\), our modeling strategy can be described in the following hierarchical form, with GP standing for Gaussian process:

\[
[x_t|f, \theta_f, x_{t-1}] \sim N\left(f(x_{t,t-1}), \sigma^2_f\right); \quad t = 1, \ldots, T; \tag{2.3}
\]

\[
[f(\cdot)|\theta_f] \sim GP\left(h(\cdot)|\beta_f, \sigma^2_f c_f(\cdot, \cdot)\right); \tag{2.4}
\]

\[
[\beta_f, \sigma^2_f, R_f, \sigma^2_f] = [\beta_f][\sigma^2_f][r_{1f}][r_{2f}][\sigma^2_f], \tag{2.5}
\]

where the components of \(\beta_f\) will also be considered independent \(a \ priori\). Forms of the prior distributions in (2.5) are provided in Section 3.

2.3 Joint distribution of \(\{x_t: t = 1, \ldots, T\}\)

Note that \([x_1 | x_0] \sim N(h(x_0)|\beta_f, \sigma^2_f + \sigma^2_e)\), but \([x_2 | x_1, x_0] = [f(2, x_1) + \epsilon_2 | x_1, x_0] = [f(2, f(1, x_0) + \epsilon_1) + \epsilon_2 | f(1, x_0) + \epsilon_1, x_0]\). Hence, the conditional distribution of \([x_t|x_{t-1}, x_0]\), for \(t \geq 2\), need not be straightforward to get hold of. In this regard, we adopt the procedure introduced by Bhattacharya (2007) which has also been successfully exploited in the nonparametric state-space modeling approach of Ghosh et al. (2014), to deal with this problem. The key idea is to conceptually simulate the entire function \(f\) modeled by the Gaussian process, and use the simulated process as a look-up table to obtain the conditional distributions of \(\{x_t: t \geq 2\}\).

2.3.1 The key concept

For simplicity of illustration, let \(x_t = f(x^*_{t,t-1})\). Now consider a table with the first column \(z \in \mathbb{R}^+ \times \mathbb{R}\) and the second column \(f(z)\). Existence of this table hinges on the implicit assumption that the entire process \(f(\cdot)\) is available. Given this table, conditional on \(x^*_{t,t-1}\) (equivalently, conditional on \(x_{t-1}\)), \(x_t = f(x^*_{t,t-1})\) can be obtained by looking-up the input \(x^*_{t,t-1}\) from the first column of the table and getting hold of the corresponding output value \(f(x^*_{t,t-1})\), located in the second column of the table. Thus, we refer to such a hypothetical table as a “look-up table”. In practice, we can construct a look-up table by simulating a realization of the Gaussian process \(f\) on a fine enough grid of inputs. Given this look-up table realization, simulation from the conditional distribution of \(f(x^*_{t,t-1})\), fixing \(x^*_{t,t-1}\) as known, will approximate \(x_t\) as accurately as we desire by making the grid as fine as required, thanks to the well-known interpolation property of Gaussian processes. Formalization of this key concept leads to the following detailed steps.

2.3.2 Auxiliary variables for emulating the look-up table

Note that given \(x_0\) we can simulate \(x_1 = f(x^*_1, 0) \sim N(h(x^*_1, 0)|\beta_f, \sigma^2_f)\), the marginal distribution of the Gaussian process prior. To simulate the rest of the dynamic sequence, we first need to generate the rest of the process \(\{f(x^*) : x^* \neq x^*_{1,0}\}\) for the look-up table approach.

In practice, it is not possible to have a simulation of this entire set \(\{f(x^*) : x^* \neq x^*_{1,0}\}\). We only have available a set of grid points \(G_n = \{z_1, \ldots, z_n\}\) obtained, perhaps, by Latin hypercube sampling (see, for example, Santner et al. (2003)) and a corresponding simulation of \(f\), given by \(D^*_n = \{f(z_1), \ldots, f(z_n)\}\), the latter having a joint multivariate normal distribution with mean

\[
E[D^*_n | \theta_f] = H D^*_f \beta_f \tag{2.6}
\]
and covariance matrix
\[ V[D_n^* | \theta_f] = \sigma_f^2 A_{f,D_n^*}, \] (2.7)
where \(H_{D_n^*}^T = [h(z_1), \ldots, h(z_n)]\) and \(A_{f,D_n^*}\) is a correlation matrix with the \((i,j)\)-th element \(c_{f}(z_i, z_j)\).

Given \((x_0, f(x_{1,0}^*))\), we simulate \(D_n^*\) from \([D_n^* | \theta_f, f(x_{1,0}^*), x_0]\). Note that the conditional \([D_n^* | f(x_{1,0}^*), x_{1,0}^*]\) has an n-variate normal distribution with mean vector
\[ E[D_n^* | \theta_f, f(x_{1,0}^*), x_0] = \mu_{g,D_n^*} = H_{D_n^*} \beta_f + s_{f,D_n^*}(x_{1,0}^*) (f(x_{1,0}^*) - h(x_{1,0}^*)' \beta_f) \] (2.8)
and covariance matrix
\[ V[D_n^* | \theta_f, f(x_{1,0}^*), x_0] = \sigma_f^2 \Sigma_{f,D_n^*}, \] (2.9)
where \(s_{f,D_n^*}(\cdot) = (c_f(\cdot, z_1), \ldots, c_f(\cdot, z_n))'\) and
\[ \Sigma_{f,D_n^*} = A_{f,D_n^*} - s_{f,D_n^*}(x_{1,0}^*) s_{f,D_n^*}(x_{1,0}^*)'. \] (2.10)

### 2.3.3 Distribution of \(x_t\) given \(D_{n}^*\)

Let us now deal with the conditional distribution \([x_t = f(x_{t-1}^*) | D_{n}^*, x_{t-1}, x_{t-2}, \ldots, x_1]\). Since the look-up table idea supports conditional independence, that is, given a simulation of the entire random function \(f\), \(x_t\) depends only upon \(x_{t-1}\) via \(x_t = f(x_{t-1}^*)\), it is sufficient to obtain the conditional distribution of \([f(x_{t-1}^*) | D_{n}^*, x_{t-1}]\); see Bhattacharya (2007) and Ghosh et al. (2014) for detailed arguments. This distribution is of course normal with mean
\[ \mu_t = h(x_{t-1}^*)' \beta_f + s_{g,D_n^*}(x_{t-1}^*) A_{f,D_n^*}^{-1} (D_n^* - H_{D_n^*} \beta_f) \] (2.11)
and variance
\[ \sigma_t^2 = \sigma_f^2 \left\{ 1 - s_{f,D_n^*}(x_{t-1}^*)' A_{f,D_n^*}^{-1} s_{f,D_n^*}(x_{t-1}^*) \right\}. \] (2.12)

For mathematical theory on the accuracy of the Markov approximation of the distributions of \(x_t\) given \(D_{n}^*\), see Ghosh et al. (2014).

### 2.3.4 Summary of the look-up table procedure

The look-up table idea involves the following steps, given that \(x_0\) is known:

1. Draw \(x_1 = f(x_{1,0}^*) \sim N(h(x_{1,0}^*)' \beta_f, \sigma_f^2)\).
2. Given \(x_0\) and \(x_1 = f(x_{1,0}^*)\), draw \(D_n^* \sim [D_n^* | \theta_f, f(x_{1,0}^*), x_0]\).
3. For \(t = 2, 3, \ldots\), draw \(x_t \sim [x_t = f(x_{t-1}^*) | \theta_f, D_n^*, x_{t-1}]\).

### 2.3.5 Joint distribution of \([x_1, \ldots, x_T, D_n^*]\)

So far we have discussed the situations where \(\epsilon_t = 0\), but our actual model (2.2) consists of non-zero \(\epsilon_t\) which are normally distributed with mean zero and variance \(\sigma_{\epsilon}^2\). In such case, once \(G_n\) and \(D_n^*\) are available, we write down the joint distribution of \([x_1, \ldots, x_T, D_n^*]\) conditional on the other parameters as
\[
[x_1, \ldots, x_T, D_n^* | \theta_f, \sigma_{\epsilon}^2] = [x_1 = f(x_{1,0}^*) + \epsilon_1 | x_0, \sigma_{\epsilon}^2] [D_n^* | \theta_f] \times \prod_{t=1}^{T-1} [x_{t+1} = f(x_{t+1,t}^*) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma_{\epsilon}^2]. \] (2.13)
In (2.13), \( x_t = f(x_{t,0}) + \epsilon_t \mid x_{t,0}, \sigma^2_\epsilon \sim N(h(x_{t,0})/\beta_f, \sigma^2_f + \sigma^2_\epsilon) \) and the distribution of \( \mathbf{D}_n \) is multivariate normal with mean and variance given by (2.6) and (2.7). The conditional distribution \( x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} \mid \mathbf{D}_n, x_t, \theta_f, \sigma^2_\epsilon \) is normal with mean
\[
\mu_x = h(x_{t+1})^{\prime} \beta_f + s_{f,D_n}(x_{t+1,t})^{\prime} \mathbf{A}_{f,D_n}^{-1} (\mathbf{D}_n - \mathbf{H}_{D_n} \beta_f)
\] (2.14)
and variance
\[
\sigma^2_x = \sigma^2_\epsilon + \sigma^2_f \left( 1 - s_{f,D_n}(x_{t+1,t})^{\prime} \mathbf{A}_{f,D_n}^{-1} s_{f,D_n}(x_{t+1,t}) \right).
\] (2.15)
Observe that in this case even if \( x_{t+1,t} \in \mathbf{G}_n \), due to the presence of the additive error term \( \epsilon_{t+1} \), the conditional variance of \( x_{t+1} \) is non-zero, equalling \( \sigma^2_\epsilon = \sigma^2_x \), the error variance.

### 2.3.6 Non-Markovian dependence structure of \( \{x_1, \ldots, x_T\} \)

Note that although conditionally on \( \mathbf{D}_n \) the variables \( x_t \) have a Markovian structure, if \( \mathbf{D}_n \) is integrated out from (2.13), then the marginalized distribution of \( \{x_1, \ldots, x_T\} \) is non-Markovian. In fact, the marginalized conditional distribution of \( x_{t+1} \) depends upon \( \{x_k : k < t+1\} \); (see also Bhattacharya (2007) and Ghosh et al. (2014)). An important issue discussed in this context by Bhattacharya (2007) and Ghosh et al. (2014) is that this strong marginalized dependence structure is the root of all numerical instabilities associated with the model implementation. Essentially, by sample path continuity of the underlying Gaussian process, \( x_0, x_1, \ldots, x_t \) will be often close to each other with high probability, particularly if \( \sigma^2_\epsilon \) and \( \sigma^2_f \) are small. This would render the relevant correlation matrix almost singular, which would be difficult to invert. Since such inversions are required for every \( t \in \{2, \ldots, T\} \) and at every iteration of any Monte Carlo simulation method, progress would be almost impossible when \( T \) is relatively large, with increasing computational cost for each \( t \) further aggravating the situation.

In contrast, if \( \mathbf{D}_n \) is retained, it is required to deal with \( [x_{t+1} | \mathbf{D}_n, x_t, \theta_f, \sigma^2_\epsilon] \), which requires computation of \( \mathbf{A}_{f,D_n}^{-1} \) only once, for all \( t \geq 2 \). This can be done even before beginning the simulation procedure. Moreover, invertibility of \( \mathbf{A}_{f,D_n}^{-1} \) can be ensured by the user, since the \((i,j)\)-th element of \( \mathbf{A}_{f,D_n}^{-1} \) is of the form \( c(f(z_i, z_j)) \), where \( z_1, \ldots, z_n \) are fixed constants, which can be judiciously chosen by the user to guarantee invertibility. Thus, retaining \( \mathbf{D}_n \) solves both the issues of numerical instability and computational burden inherent in the marginalized distribution of \( \{x_1, \ldots, x_T\} \). It is hence no wonder that retaining \( \mathbf{D}_n \) in the model is the only sensible decision.

### 3 Prior distributions for \( \theta_f \) and \( \sigma^2_\epsilon \)

We assume the following forms of the prior distributions:
\[
[\beta_f] \sim N_3 (\beta_{f,0}, \Sigma_{\beta_{f,0}}); \tag{3.1}
\]
\[
[\sigma^2_f] \propto (\sigma^2_f)^{-\left(\frac{n^2+2}{2}\right)} \exp \left(-\frac{\gamma_f}{2\sigma^2_f}\right); \quad \alpha_f, \gamma_f > 0; \tag{3.2}
\]
\[
[\sigma^2_\epsilon] \propto (\sigma^2_\epsilon)^{-\left(\frac{n^2+2}{2}\right)} \exp \left(-\frac{\gamma_\epsilon}{2\sigma^2_\epsilon}\right); \quad \alpha_\epsilon, \gamma_\epsilon > 0; \tag{3.3}
\]
\[
[\log(\tau_{i,f})] \sim N \left(\mu_{\tau_{i,f}}, \sigma^2_{\tau_{i,f}}\right); \quad \text{for } i = 1, 2. \tag{3.4}
\]
All the prior parameters are assumed to be known. Now we discuss our approach to selecting the prior parameters for our application our Bayesian model in simulation studies and real data application in the univariate situations.

As per (3.1), we set the prior of \( \beta_f \) to be trivariate normal with the identity matrix as the variance, that is, we set \( \Sigma_{\beta_{f,0}} = \mathbf{I}_3 \), where \( \mathbf{I}_3 \) is the 3-dimensional identity matrix. This choice
turned out to be appropriate as larger variances in the diagonal caused the posterior time series to explode with increasing time. For the mean \( \beta_{f,0} \), except the first component associated with the intercept, we set the rest of the components to zero. We set the first component of \( \beta_{f,0} \) to be the mean of the underlying logarithm of the time series data to be modeled, after thinning by 5 observations. This ensures that the intercept corresponds to the overall mean of the log time series.

For the choice of the parameters of the priors of \( \sigma^2_f \) and \( \sigma^2_c \) we first note that the mean is of the form \( \gamma/(\alpha - 2) \) and the variance is of the form \( 2\gamma^2/((\alpha - 2)^2(\alpha - 4)) \). Thus, if we set \( \gamma/(\alpha - 2) = a \), then the variance becomes \( 2a^2/(\alpha - 4) \). Here we set \( a = \hat{\sigma}^2/2 \) for both \( \sigma^2_f \) and \( \sigma^2_c \), where \( \hat{\sigma}^2 \) is the variance of the underlying log time series obtained after thinning by 5 observations. Again, this strategy is to ensure that the expected variability matches the data variability. For each of these priors we set \( \alpha = 4.01 \), so that the variance is of the form \( 200a^2 \).

In order to choose the parameters of the log-normal priors of the smoothness parameters \( r_{1f} \) and \( r_{2f} \), we set the mean of the log-normal prior with parameters \( \mu \) and \( \sigma^2 \), given by \( \exp(\mu + \sigma^2/2) \), to 1. This yields \( \mu = -\sigma^2/2 \). Since the variance of this log-normal prior is given by \( \exp(\sigma^2) - 1 \), the relation \( \mu = -\sigma^2/2 \) implies that the variance is \( \exp(\sigma^2) - 1 = \exp(-2\mu) - 1 \). We set \( \sigma^2 = 1 \), so that \( \mu = -0.5 \). This implies that the mean is 1 and the variance is approximately 2, for the priors of each smoothness parameter \( r_{i,f}; i = 1, 2 \). This prevents the smoothness parameters from being too large or too small. Indeed, if the smoothness parameters are too large then \( c_f(z_1,z_2) \approx 0 \) for \( z_1 \neq z_2 \), so that the correlation matrix is rendered almost the identity matrix. On the other hand, if the smoothness parameters are close to zero, then \( c_f(z_1,z_2) \approx 1 \) for \( z_1, z_2 \), making the correlation matrix almost singular. Both these undesirable situations are ruled out by our prior choice.

4 Posterior distributions of current and future time series in our dynamic Gaussian process approach

4.1 Posterior of current given the future

Let us assume that for any given GCM, the logarithms of the future mean global temperatures \( \{x_t: t = T_0 + 1, \ldots, T\} \) are given, where \( 1 \leq T_0 \leq T - 1 \). In our case, the times \( \{0, \ldots, T_0\} \) correspond to the current years (1850, \ldots, 2016) and the times \( \{T_0 + 1, \ldots, T\} \) correspond to the future years (2017, \ldots, 2099). Then assuming that \( x_0 \) is known, we can obtain the posterior distribution of the logarithms of the current mean global temperatures \( \{x_t: t = 1, \ldots, T_0\} \) as follows:

\[
[x_1, \ldots, x_{T_0}|x_{T_0+1}, \ldots, x_T] = \int [x_1, \ldots, x_{T_0}|D_n^*, x_{T_0+1}, \ldots, x_T, \theta_f, \sigma^2_c]d[D_n^*, \theta_f, \sigma^2_c|x_{T_0+1}, \ldots, x_T] \\
\approx \int [x_1, \ldots, x_{T_0}|D_n^*, \theta_f, \sigma^2_c]d[D_n^*, \theta_f, \sigma^2_c|x_{T_0+1}, \ldots, x_T]. \tag{4.1}
\]

The second approximate equality follows from the first equality since given \( D_n^* \), \( \{x_1, \ldots, x_{T_0}\} \) are conditionally approximately independent of \( \{x_{T_0+1}, \ldots, x_T\} \), “approximate” because \( x_{T_0} \) and \( x_{T_0+1} \) are not independent, even when \( D_n^* \) is conditioned upon. This approximate conditional independence ensures \( [x_1, \ldots, x_{T_0}|D_n^*, x_{T_0+1}, \ldots, x_T, \theta_f, \sigma^2_c] \approx [x_1, \ldots, x_{T_0}|D_n^*, \theta_f, \sigma^2_c] \). In our practical applications, however, we shall replace this approximate equality with equality. For well-chosen fine enough grid \( G_n \) this is not at all a serious issue.

Hence, if we can have simulations from the posterior \( [D_n^*, \theta_f, \sigma^2_c|x_{T_0+1}, \ldots, x_T] \), then we can
Thus, after obtaining MCMC samples from
\[ [x_1, \ldots, x_{T_0}] | D_n^*, \theta_f, \sigma^2_e ] = \prod_{t=0}^{T_0-1} [x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma^2_e], \]
where \([x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma^2_e]\) is normally distributed with mean and variance given by (2.14) and (2.15), respectively, for \(t = 0, 1, \ldots, T_0 - 1\).

To obtain samples from the posterior
\[
[D_n^*, \theta_f, \sigma^2_e | x_{T_0+1}, \ldots, x_T] \\
\propto [D_n^* | \theta_f][\theta_f][\sigma^2_e][x_{T_0+1}, \ldots, x_T][D_n^*, \theta_f, \sigma^2_e] \\
= [D_n^*, \theta_f][\theta_f][\sigma^2_e] \prod_{t=T_0}^{T-1} [x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma^2_e],
\]
we resort to Markov Chain Monte Carlo (MCMC) where we sample \(\beta_f\) and \(D_n^*\) from their respective full conditional distributions and the remaining parameters \(\{r_{1f}, r_{2f}, \sigma^2_f, \sigma^2_e\}\) using Transformation based Markov Chain Monte Carlo (TMCMC) introduced by Dutta and Bhattacharya (2014). In particular, we use the additive transformation, with judicious choice of the tuning constants.

### 4.2 Posterior of future given the current

Now, given \(\{x_1, \ldots, x_{T_0}\}\), which may be interpreted as the current observed log global mean temperatures, we can obtain the posterior distribution of the future log global mean temperatures \(\{x_{T_0+1}, \ldots, x_T\}\) in a similar manner. That is,
\[
[x_{T_0+1}, \ldots, x_T | x_1, \ldots, x_{T_0}] \\
= \int [x_{T_0+1}, \ldots, x_T | D_n^*, x_1, \ldots, x_{T_0}, \theta_f, \sigma^2_e] d[D_n^*, \theta_f, \sigma^2_e | x_1, \ldots, x_{T_0}] \\
= \int [x_{T_0+1}, \ldots, x_T | D_n^*, x_{T_0}, \theta_f, \sigma^2_e] d[D_n^*, \theta_f, \sigma^2_e | x_1, \ldots, x_{T_0}] \\
= [D_n^*, \theta_f, \sigma^2_e | x_1, \ldots, x_{T_0}, \theta_f, \sigma^2_e] \prod_{t=T_0-1}^{T-1} [x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma^2_e].
\]

Thus, after obtaining MCMC samples from
\[
[D_n^*, \theta_f, \sigma^2_e | x_1, \ldots, x_{T_0}] \\
\propto [D_n^* | \theta_f][\theta_f][\sigma^2_e][x_1, \ldots, x_{T_0}][D_n^*, \theta_f, \sigma^2_e] \\
= [D_n^*, \theta_f][\theta_f][\sigma^2_e] \prod_{t=T_0}^{T-1} [x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma^2_e],
\]
using the same techniques as for \(D_n^*, \theta_f, \sigma^2_e | x_{T_0+1}, \ldots, x_T\), we simulate from
\[
[x_{T_0+1}, \ldots, x_T | D_n^*, x_{T_0}, \theta_f, \sigma^2_e] = \prod_{t=T_0}^{T-1} [x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma^2_e],
\]
where \([x_{t+1} = f(x_{t+1,t}) + \epsilon_{t+1} | D_n^*, x_t, \theta_f, \sigma^2_e]\) is normally distributed with mean and variance given by (2.14) and (2.15), respectively, for \(t = T_0, 1, \ldots, T - 1\). This yields simulations from (4.2).
A Bayesian multiple testing framework for GCM selection in any given climate scenario

Given any climate scenario, let us consider GCMs \( \mathcal{M}_k; k = 1, \ldots, K \), from among which the best model needs to be selected. For our purpose, we adopt and extend the novel Bayesian multiple testing procedure for model selection introduced by Chatterjee and Bhattacharya (2020) that respects the inverse regression perspective of the models, in coherence with the forward aspect. It is important to mention that in statistics, model selection pertains to choosing the best model from among a set of models that attempt to fit a single dataset. However, in our present GCM case, there are \( K \) datasets generated by \( K \) GCMs in a given climate scenario. Our strategy will be to combine the \( K \) datasets into a single dataset by taking averages over the \( K \) GCMs for each time point, and then to invoke our Gaussian process based dynamics for the averaged time series, where the hyperparameters of the model are fixed using the mean and variance of the original GCM-specific simulated time series. This yields \( K \) different Gaussian process based models for the averaged time series, inheriting the main characteristics of the GCM-specific time series. The design our Bayesian multiple testing procedure ensures that the Gaussian process models will be compared with respect to their abilities to fit the averaged simulated future global temperature data in the forward sense, as well as their abilities to capture the HadCRUT4 data given the averaged GCM-simulated future global temperature data, in the inverse sense. Details follow.

Let us denote the logarithms of the observed current global mean temperatures (the HadCRUT4 data) by \( \{ x_t^{(0)} : t = 1, \ldots, T_0 \} \). For GCM \( \mathcal{M}_k \), let \( \{ x_t^{(k)} : t = 0, 1, \ldots \} \) denote the logarithms of its simulated global mean temperature time series, for \( k = 1, \ldots, K \). For \( t = 0, 1, \ldots \), let \( \bar{x}_t = K^{-1} \sum_{k=1}^{K} x_t^{(k)} \), and let this averaged time series \( \{ \bar{x}_t : t = 0, 1, \ldots \} \) be also modeled by the Gaussian process emulation procedure given by (2.3), (2.4) and (2.5), with parameters denoted by \( \theta^{(k)} = (\theta_f^{(k)}, \sigma^{(k)}_f)^2 \). The rationale behind this modeling strategy is simple: if the functional forms \( f(\cdot) \) associated with the individual time series \( \{ x_t^{(k)} : t = 0, 1, \ldots \} \) are unknown, then the functional form driving the dynamics of their average must also be unknown, which is again best modeled by a Gaussian process. In this regard, let \( [\bar{x}_{T_0+1}, \ldots, \bar{x}_T|\theta^{(k)}, \mathcal{M}_k] \) denote the density of the logarithms of the future global mean temperatures, averaged over all the models in the climate scenario under Gaussian process emulation model \( \mathcal{M}_k \), with its associated parameters \( \theta^{(k)} \).

We combine the competing models in the following mixture form:

\[
[\bar{x}_{T_0+1}, \ldots, \bar{x}_T|\theta] = \sum_{k=1}^{K} p_k[\bar{x}_{T_0+1}, \ldots, \bar{x}_T|\theta^{(k)}, \mathcal{M}_k],
\]

where \( \theta = (\theta^{(1)}, \ldots, \theta^{(K)}) \), \( 0 \leq p_k \leq 1, \) for \( k = 1, \ldots, K \) and \( \sum_{k=1}^{K} p_k = 1 \). Letting \( \zeta \) denote the allocation variable (model index), with \( P(\zeta = k) = p_k \), note that \( [\bar{x}_{T_0+1}, \ldots, \bar{x}_T|x_1^{(0)}, \ldots, x_{T_0}^{(0)}, \theta, \zeta = k] = [\bar{x}_{T_0+1}, \ldots, \bar{x}_T|x_1^{(0)}, \ldots, x_{T_0}^{(0)}, \theta^{(k)}, \mathcal{M}_k] \). We consider the Dirichlet prior for \( (p_1, \ldots, p_K) \) with parameters \( (\alpha_1, \ldots, \alpha_K) \), where \( \alpha_k > 0 \), for \( k = 1, \ldots, K \). In our problem, we shall set \( \alpha_k = 1 \), for all \( k = 1, \ldots, K \), for all the climate scenarios. Thus, the prior is uniform over the simplex, indicating no preference for any specific GCM \textit{a priori}. The priors for the parameters \( \theta^{(k)} \) remain the same as described in Section 3. Since for different \( k \) the prior depends upon the mean and variance of the underlying entire \( k \)-th GCM-simulated time series, the priors are all very distinct from one another. In fact, the distinctions among the priors induce distinguishing among the competing Bayesian models, since otherwise all of them have the same dynamic structure driven by Gaussian processes, started at the same known initial value \( x_0 \).

We let \( \{ \bar{x}_t : t = 1, \ldots, T_0 \} \) stand for the random quantities corresponding to \( \{ x_t^{(0)} : t = 1, \ldots, T_0 \} \),
whose posterior distribution will be of interest to us. In particular, it is of interest in evaluating how well this posterior captures the observed current log global mean temperatures, which we shall formalize in our multiple testing procedure. Towards this goal, for any $T_0$, dimensional vector $v_{T_0} = (v_1, \ldots, v_{T_0})$, and for some $c > 0$, let us define the following discrepancy measures in the spirit of Chatterjee and Bhattacharya (2020):

$$S_1^{(k)}(v_{T_0}) = \frac{1}{T_0} \sum_{t=1}^{T_0} \frac{|v_t - M(\bar{x}_t|x_{T_0+1}, \ldots, \tilde{x}_T, M_k)|}{\sqrt{Var(\bar{x}_t|x_{T_0+1}, \ldots, \tilde{x}_T, M_k) + c}} \quad (5.2)$$

where $M(\bar{x}_t|x_{T_0+1}, \ldots, \tilde{x}_T, M_k)$ stands for the posterior mode of $[\bar{x}_t|x_{T_0+1}, \ldots, \tilde{x}_T, M_k]$. Similarly, let

$$S_1^{(k)}(v_{T_0}) = \frac{1}{T_0} \sum_{t=1}^{T_0} \frac{(v_t - M(\bar{x}_t|x_{T_0+1}, \ldots, \tilde{x}_T, M_k))^2}{Var(\bar{x}_t|x_{T_0+1}, \ldots, \tilde{x}_T, M_k) + c} \quad (5.3)$$

In our examples, we set $c = 0.01$. Various other measures of discrepancy can be defined (see Bhattacharya (2013) for a discussion on such discrepancy measures; see also Mukhopadhyay and Bhattacharya (2013)), but for brevity we focus on these two measures in this paper.

Importantly, using discrepancy measures Bhattacharya (2013) introduced a novel Bayesian decision-theoretic methodology for Bayesian model assessment in inverse regression problems, which we shall adopt to assess goodness-of-fit of the best GCMs with respect to fitting the HadCRUT4 data, conditioned on the future GCM projections and our Bayesian dynamic Gaussian process emulation strategy.

With $\bar{x}_{T_0} = (\bar{x}_1, \ldots, \bar{x}_{T_0})$ and $x_{T_0}^{(0)} = (x_1^{(0)}, \ldots, x_{T_0}^{(0)})$, for a given discrepancy measure $S^{(k)}$, let $[\bar{\ell}_k, \bar{u}_k]$ denote the $(1 - \alpha)\%$ credible interval for the posterior distribution of $S^{(k)}(\bar{x}_{T_0})$ for any desired $\alpha \in (0, 1)$; in our application, we set $\alpha = 0.05$. Following the recommendation of Chatterjee and Bhattacharya (2020) for practical purposes (see (8.1) and (8.2) of Section 8 of their article) we now define the appropriate multiple hypotheses that we shall test for our Bayesian model selection purpose. For $k = 1, \ldots, K$,

$$H_{0k} : \zeta = k, S^{(k)}(\bar{x}_{T_0}) - S^{(k)}(x_{T_0}^{(0)}) \in [\bar{\ell}_k, \bar{u}_k] \quad (5.4)$$

versus

$$H_{1k} : \{\zeta \neq k\} \bigcup \{\zeta = k, S^{(k)}(\bar{x}_{T_0}) - S^{(k)}(x_{T_0}^{(0)}) \in [\bar{\ell}_k, \bar{u}_k]^{c}\} \quad (5.5)$$

where, for any set $A$, $A^c$ stands for its complement.

The hypotheses are so designed that the best model is chosen on the basis of both forward and inverse perspectives. To elucidate, note that to select the best model we first need to choose a model $[\bar{x}_{T_0+1}, \ldots, \tilde{x}_T|\theta^{(k)}, M_k]$ indexed by $\zeta = k$ which has high marginal posterior probability. This reflects the forward perspective of the model selection problem. Indeed, the posterior probability of $\{\zeta = k\}$ is proportional to its corresponding marginal density $[\bar{x}_{T_0+1}, \ldots, \tilde{x}_T|M_k] = \int [\bar{x}_{T_0+1}, \ldots, \tilde{x}_T|\theta^{(k)}, M_k] d(\theta^{(k)})$ (see (6.3) for details). This marginal density has interpretation in the forward sense only since it is not associated with the posterior distribution $[\bar{x}_1, \ldots, \bar{x}_{T_0}]|\bar{x}_{T_0+1}, \ldots, \tilde{x}_T, M_k]$, the latter to be interpreted as the inverse aspect of the problem.

The inverse sense in our multiple testing formalization is made explicit in the following way. In addition to selecting $\zeta = k$ with high marginal posterior probability, we demand that for such model

$$S^{(k)}(\bar{x}_{T_0}) - S^{(k)}(x_{T_0}^{(0)}) \in [\bar{\ell}_k, \bar{u}_k] \quad (5.6)$$
is also satisfied. Roughly, this condition demands that for \( \mathcal{M}_k \) to qualify as a good inverse regression model, the observed discrepancy measure \( S^{(k)}(\hat{x}^{(0)}_{T_0}) \) must be included in the desired credible intervals of the reference discrepancy measure \( S^{(k)}(\bar{x}_{T_0}) \). This reflects the inverse perspective since the reference discrepancy measure explicitly deals with the posterior \([\bar{x}_1, \ldots, \bar{x}_{T_0}| \bar{x}_{T_0}+1, \ldots, \bar{x}_T, \mathcal{M}_k] \) associated with the inverse regression problem. The key idea of the Bayesian goodness-of-fit test devised by Bhattacharya (2013) is based on the same principle.

Note that our Bayesian multiple hypotheses formulation (5.4) and (5.5) does not involve cross-validation, even though Chatterjee and Bhattacharya (2020) formulated the general Bayesian multiple testing framework for model and variable selection in problems involving covariates using inverse leave-one-out cross-validation with respect to posteriors associated with the covariates (see also Bhattacharya (2013)). Indeed, as must be evident from the very beginning, our current global climate change problem is not the traditional model selection problem. However, our Bayesian multiple testing procedure is based on similar principles introduced in Chatterjee and Bhattacharya (2020).

### 5.1 The Bayesian multiple testing procedure

Let

\[
\begin{align*}
  d_k &= \begin{cases} 
    1 & \text{if the } k\text{-th null hypothesis is rejected;} \\
    0 & \text{otherwise;} 
  \end{cases} \\
  r_k &= \begin{cases} 
    1 & \text{if } H_{1k} \text{ is true;} \\
    0 & \text{if } H_{0k} \text{ is true.}
  \end{cases}
\end{align*}
\]

Following Chatterjee and Bhattacharya (2020) (see also Müller et al. (2004)) and (Guindani et al. (2009)), let us define the true positives as

\[
TP = \sum_{k=1}^{K} d_k r_k,
\]

the posterior expectation of which is to be maximized subject to controlling the posterior expectation of the error term

\[
E = \sum_{k=1}^{K} d_k (1 - r_k).
\]

From the above notions it is clear that the optimal decision configuration can be obtained by minimizing the function

\[
\xi(d) = - \sum_{k=1}^{K} d_k E(r_k|\bar{x}_{T_0+1}, \ldots, \bar{x}_T) + \lambda \sum_{k=1}^{K} d_k E[(1 - r_k)|\bar{x}_{T_0+1}, \ldots, \bar{x}_T]
\]

\[
= -(1 + \lambda) \sum_{k=1}^{K} d_k \left( v_k - \frac{\lambda}{1 + \lambda} \right),
\]

with respect to all possible decision configurations of the form \( d = \{d_1, \ldots, d_K\} \), where \( \lambda > 0 \), and

\[
v_k = E(r_k|\bar{x}_{T_0+1}, \ldots, \bar{x}_T) = [H_{1k}|\bar{x}_{T_0+1}, \ldots, \bar{x}_T].
\]
is the posterior probability of the $k$-th alternative hypothesis. Letting $\beta = \lambda/(1 + \lambda)$ denote the penalizing constant, one can equivalently maximize

$$f_\beta(d) = \sum_{k=1}^{K} d_k (v_k - \beta)$$  \hspace{1cm} (5.9)$$

with respect to $d$ and obtain the optimal decision configuration. In this case, the optimal decision configuration $\hat{d} = \{\hat{d}_1, \ldots, \hat{d}_K\}$ is given by the following: for $k = 1, \ldots, K$,

$$\hat{d}_k = \begin{cases} 1 & \text{if } v_k > \beta; \\ 0 & \text{otherwise}. \end{cases}$$  \hspace{1cm} (5.10)$$

In our model selection setup, the least value of the penalty $\beta \in (0, 1)$ for which the decision configuration $\hat{d}_k = 0$ and $\hat{d}_k = 1$ for all $k \in \{1, \ldots, K\}\{\tilde{k}\}$ is obtained, for some $\tilde{k} \in \{1, \ldots, K\}$, yields the best model $M_{\tilde{k}}$. This is because in such a case, $v_\tilde{k} \leq \beta$, even though $\beta$ is reasonably small, suggesting that $H_{\tilde{o}k}$ has significant posterior probability. Since $v_k > \beta$ for all $k \in \{1, \ldots, K\}\{\tilde{k}\}$, the posterior probabilities of $H_{\tilde{o}k}$ for $k \in \{1, \ldots, K\}\{\tilde{k}\}$ are less substantial compared to that of $H_{\tilde{o}k}$. This indicates that $M_{\tilde{k}}$ is the best model among $M_k; k = 1, \ldots, K$. This key intuition is rigorously formalized in our Bayesian multiple testing procedure.

5.2 Error measures for our Bayesian multiple testing procedure

To discuss appropriate measures of error for our Bayesian multiple testing procedure, first let us define $\delta(d|x_{T_0+1}, \ldots, \bar{x}_T)$ to be the probability of choosing $d$ as the optimal decision configuration given data $x_{T_0+1}, \ldots, \bar{x}_T$ when a given multiple testing method is employed. Also, let $D$ be the set of all $K$-dimensional binary vectors, standing for all possible decision configurations.

As suitable posterior measures of Type-I and Type-II errors, Sarkar et al. (2008) defined posterior false discovery rate and false non-discovery rate, respectively, which we denote as conditional false discovery rate (cFDR) and conditional false non-discovery rate (cFNR). The measures, in our current setup, are given as the following:

$$cFDR = E \left[ \sum_{d \in D} \frac{\sum_{k=1}^{K} d_k (1 - r_k)}{\sum_{k=1}^{K} d_k \lor 1} \delta(d|x_{T_0+1}, \ldots, \bar{x}_T) \bigg| \bar{x}_{T_0+1}, \ldots, \bar{x}_T \right]$$

$$= \sum_{d \in D} \frac{\sum_{k=1}^{K} d_k (1 - v_k)}{\sum_{k=1}^{K} d_k \lor 1} \delta(d|x_{T_0+1}, \ldots, \bar{x}_T);$$

$$cFNR = E \left[ \sum_{d \in D} \frac{\sum_{k=1}^{K} (1 - d_k) r_k}{\sum_{k=1}^{K} (1 - d_k) \lor 1} \delta(d|x_{T_0+1}, \ldots, \bar{x}_T) \bigg| \bar{x}_{T_0+1}, \ldots, \bar{x}_T \right]$$

$$= \sum_{d \in D} \frac{\sum_{k=1}^{K} (1 - d_k) v_k}{\sum_{k=1}^{K} (1 - d_k) \lor 1} \delta(d|x_{T_0+1}, \ldots, \bar{x}_T).$$

Note that since in our multiple testing method the decision rule is non-randomized, $\delta(d|x_{T_0+1}, \ldots, \bar{x}_T)$ is either 1 or 0 depending on data $\{x_{T_0+1}, \ldots, \bar{x}_T\}$.

For our Bayesian purpose, following Chatterjee and Bhattacharya (2020), we shall consider the Bayesian measures $cFDR$ and $cFNR$ as Bayesian multiple testing error rates. These measures are also recommended by Chandra and Bhattacharya (2019) and Chandra and Bhattacharya (2020) since they are conditioned on the observed data and hence qualify as bona fide Bayesian measures.
6 Implementation of the Bayesian multiple testing procedure

6.1 Parallel computation of \([\bar{x}_1, \ldots, \bar{x}_{T_0}, \bar{x}_{T_0+1}, \ldots, \bar{x}_T, \mathcal{M}_k]\) for different GCMs and climate scenarios

Note that for conducting the Bayesian multiple hypotheses tests, we need to obtain samples from the posteriors \([\bar{x}_1, \ldots, \bar{x}_{T_0} | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, \mathcal{M}_k]\), for all \(k = 1, \ldots, K\), for any given climate scenario. These are required to evaluate the posterior probabilities of (5.6), associated with the inverse perspective.

The method of obtaining posterior samples from the above distributions is the same as described in Section 4.1, with the priors discussed in Section 3, but we need to select the grid \(G_n\) appropriately for creating the Gaussian process based look-up table. Note that the input grid \(G_n\) is a two-dimensional grid, the first component being the time component and the second being the real line. In our case, we re-label the times 1850–2099 as 0–249 and further divide the re-labeled times by 250 to have them lie in \([0, 1)\). We then divide up the interval \([0, 1)\) into \(n = 50\) equal sub-intervals and randomly simulate a value from each sub-interval. For the second component of \(G_n\), gridding the interval \([0, 5)\) instead of a large interval turned out to be more than adequate for our problem, particularly because we consider the logarithms of the time series rather than the actual time series. We divide up the interval \([0, 5)\) into \(n = 50\) equal sub-intervals and randomly simulate a value from each sub-interval. Thus, we construct \(G_n\) using component-wise Latin hypercube sampling with \(n = 50\).

For each model \([\bar{x}_1, \ldots, \bar{x}_{T_0} | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, \mathcal{M}_k]\), \(k = 1, \ldots, K\), we obtain 60,000 samples of \(\{\bar{x}_1, \ldots, \bar{x}_{T_0}\}\) following the method described in Section 4.1, discarding the first 10,000 as burn-in. Now recall that the climate scenarios A1B, A2, B1 and Commitment consist of 21, 17, 21 and 16 GCMs, respectively. That is, in all, there are 75 posteriors of the form \([\bar{x}_1, \ldots, \bar{x}_{T_0} | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, \mathcal{M}_k]\), and from each of them 60,000 realizations are to be simulated. This is an infeasible task if the models are implemented separately. However, we implement our code, written in C in accordance with the MPI protocol, in a parallel architecture associated with a VMWare consisting of 100 cores, running at 2.80 GHz speed, and having 1 TB memory. Specifically, we parallelize our computation by splitting 75 model implementations into 75 separate cores of our VMWare. The entire exercise takes less than an hour in our parallel implementation.

6.2 Obtaining the posterior model probabilities using Gibbs sampling

Recall that our multiple testing approach also requires computation of the posterior model probabilities \(\zeta = k | \bar{x}_{T_0+1}, \ldots, \bar{x}_T\). We propose (see also Chatterjee and Bhattacharya (2020)) Gibbs sampling for simulation-based computations of these probabilities, by sampling from the full conditionals \([\zeta | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, p_1, \ldots, p_K]\) and \([p_1, \ldots, p_K | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, \zeta]\) successively.

Note that given \(\zeta\), the posterior distribution of \((p_1, \ldots, p_K)\) is again a Dirichlet distribution with parameters \((\alpha_1 + I(\zeta = 1), \ldots, \alpha_K + I(\zeta = K))\). In other words, since \(\alpha_k = 1\) for \(k = 1, \ldots, K\), we have

\[
[p_1, \ldots, p_K | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, \zeta] = \text{Dirichlet}(1 + I(\zeta = 1), \ldots, 1 + I(\zeta = K)). \tag{6.1}
\]
Given \((p_1, \ldots, p_K)\), the posterior distribution of \(\zeta\) is given by

\[
[\zeta = k | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, p_1, \ldots, p_K] = \frac{p_k[\bar{x}_{T_0+1}, \ldots, \bar{x}_T | \mathcal{M}_k]}{\sum_{i=1}^K p_i[\bar{x}_{T_0+1}, \ldots, \bar{x}_T | \mathcal{M}_i]}; \quad k = 1, \ldots, K, \tag{6.2}
\]

where for any \(k\), letting \(D_n^{(k)}\) denote the look-up table associated with model \(\mathcal{M}_k\),

\[
[\bar{x}_{T_0+1}, \ldots, \bar{x}_T | \mathcal{M}_k] = \int [\bar{x}_{T_0+1}, \ldots, \bar{x}_T | D_n^{(k)}, \theta^{(k)}, \mathcal{M}_k]d[\theta^{(k)}]d[D_n^{(k)}]
= \int \prod_{t=T_0}^{T-1} [\bar{x}_{t+1} = f(t + 1, \bar{x}_t) + \epsilon_{t+1} \mid D_n^{(k)}, \bar{x}_t, \theta^{(k)}, \mathcal{M}_k]d[\theta^{(k)}]d[D_n^{(k)}]
\approx \frac{1}{N} \sum_{i=1}^N \prod_{t=T_0}^{T-1} [\bar{x}_{t+1} = f(t + 1, \bar{x}_t) + \epsilon_{t+1} \mid D_n^{(k)}, \bar{x}_t, \theta^{(k)}, \mathcal{M}_k], \tag{6.3}
\]

where \(\left\{ D_n^{(k)}, \theta^{(k)} : i = 1, \ldots, N \right\}\), for sufficiently large \(N\), is a set of simulations from the prior distributions of \(\theta^{(k)}\) and the distribution of the look-up table \(D_n^{(k)}\).

In practice, rather than simulating from the priors, we simulate \(\left\{ D_n^{(k)}, \theta^{(k)} : i = 1, \ldots, N \right\}\) from the posterior distributions of \(\theta^{(k)}\) and \(D_n^{(k)}\). The reason for this is the following. Simulating from the priors would lead to many realizations that are not well-supported by the data \(\{\bar{x}_{T_0+1}, \ldots, \bar{x}_T\}\), and these realizations would render the density \([\bar{x}_{T_0+1}, \ldots, \bar{x}_T | D_n^{(k)}, \theta^{(k)}, \mathcal{M}_k]\) extremely small, thus significantly reducing the effective simulation size. This issue is clearly much alleviated if the simulations correspond to the posterior distributions \([D_n^{(k)}, \theta^{(k)}] | \bar{x}_{T_0+1}, \ldots, \bar{x}_T, \mathcal{M}_k]\), since such realizations are well-supported by the data that has been conditioned upon. This strategy also led to numerically stable estimates of the marginal densities in all our cases.

Using the full conditional distributions (6.1) and (6.2), along with the aforementioned posterior-based computation of (6.3), we obtain 100,000 realizations from the posterior distribution of \((\zeta, p_1, \ldots, p_K)\) using Gibbs sampling, after discarding the first 10,000 iterations as burn-in.

### 6.3 Obtaining the posterior probabilities of the alternative hypotheses \(H_{1k}\)

Note that for \(k = 1, \ldots, K\), the posterior probability of \(H_{1k}\) is given by

\[
v_k = 1 - [\zeta = k, S^{(k)}(\bar{x}_{T_0}) - S^{(k)}(\bar{x}_{T_0}^{(0)}) \in [\bar{t}_k, \bar{u}_k]| \bar{x}_{T_0+1}, \ldots, \bar{x}_T]
= 1 - [\zeta = k | \bar{x}_{T_0+1}, \ldots, \bar{x}_T] \left[ S^{(k)}(\bar{x}_{T_0}) - S^{(k)}(\bar{x}_{T_0}^{(0)}) \in [\bar{t}_k, \bar{u}_k] \right] = k, \bar{x}_{T_0+1}, \ldots, \bar{x}_T. \tag{6.4}
\]

Hence, once we obtain realizations from the posteriors of \(S^{(k)}(\bar{x}_{T_0})\) for \(k = 1, \ldots, K\), and \((\zeta, p_1, \ldots, p_K)\), evaluation of \(v_k; k = 1, \ldots, K\), follows simply by Monte Carlo averaging associated with the two factors of (6.4).

### 7 GCM selection results

We implemented our Bayesian multiple testing procedure with both the discrepancy measures \(S_1^{(k)}\) and \(S_2^{(k)}\) given by (5.2) and (5.3), respectively. We denote the corresponding cFDRs by cFDR1 and cFDR2 and the corresponding cFNRs by cFNR1 and cFNR2, respectively. Figures 7.1 and 7.2 depict these Bayesian error measures as functions of the penalty \(\beta\), for all the four
climate scenarios A1B, A2, B1 and Commitment, with respect to both the discrepancy measures $S_1^{(k)}$ (red line) and $S_2^{(k)}$ (green line).

The discussions toward the ends of Section 5.1 and 5.2 point out that the first jump occurring in either of the graphs of cFDR or cFNR as functions of $\beta$, corresponds to the best model. In this regard, Figures 7.1 and 7.2 show that values of the penalty $\beta$ close to one are required to obtain the first jumps of cFDR and cFNR for both the discrepancy measures $S_1^{(k)}$ and $S_2^{(k)}$, for all the four climate scenarios. Thus, none of the selected models seem to be satisfactory. Also, all the jumps occur close to each other in all the cases, indicating that the best models are not significantly good compared to the other competing models.

In all the cases, $S_2^{(k)}$ performs relatively better than $S_1^{(k)}$ in the sense that the value of $\beta$ required for $S_2^{(k)}$ is somewhat less than the $S_1^{(k)}$ counterpart for selecting the best model. Among all the four climate scenarios, the Commitment scenario turns out to be the best since here the best model is selected for a value of $\beta$ that is lesser than those of the other scenarios.

In the case of A1B, $S_1^{(k)}$ and $S_2^{(k)}$ yielded two different best models, csiro_mk3.0 and

\[\text{Figure 7.1:} \quad \text{cFDR and cFNR for GCM selection in the climate scenarios A1B and A2 using Bayesian multiple testing.}\]
Figure 7.2: cFDR and cFNR for GCM selection in the climate scenarios B1 and Commitment using Bayesian multiple testing.
inmcm3, respectively. In the remaining climate scenarios, both the discrepancy measures $S_1^{(k)}$ and $S_2^{(k)}$ yielded the same best models. The best GCMs selected for the scenarios A2, B1 and Commitment, are ukmo_hadgem1, gfdl_cm2_0 and cnrm_cm3, respectively.

Figure 7.3 displays the posterior distribution of the time series $[\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{T_0}, \bar{x}_{T_0+1}, \ldots, \bar{x}_T]$ (note that $\bar{x}_0 = x_0$, since $x_0$ is assumed to be known) corresponding to the aforementioned best GCMs selected by our Bayesian multiple testing procedure, as colour plots. The progressively higher densities are represented by progressively intense colours. The thick black line is the HadCRUT4 data, which is the current global temperature (CGT) and the dashed line is the model based global temperature (MBGT), the simulated global temperatures by the underlying GCM. The other starred line stands for the average model based global temperature (AMBGT), which is the average over all the GCM based simulated time series in the respective climate scenario. All the time series are in degree celsius and in the log scale. Recall that the HadCRUT4 data is associated with the years 1850 – 2016 and the GCMs are associated with 1900 – 2099, which is why the time scales for the HadCRUT4 data and the GCM based simulated data are different.

Observe that except for B1 and Commitment most part of the observed HadCRUT4 data is not included in the high density regions of the corresponding posterior time series associated with the best GCMs. In fact, except the case of Commitment, all other posteriors strongly support lower temperatures than HadCRUT4. This is not surprising since Figure 1.1 show that the GCM-simulated time series significantly underestimate the HadCRUT4 data during the relevant time period, and so must the averaged GCM time series, and this is broadly consistent with the observations on Figure 7.3. Also observe that MBGT and AMBGT lie closer to the high density regions compared to CGT, which is again not unexpected as Figure 1.1 indicates.
Figure 7.3: The posteriors corresponding to the HadCRUT4 data or the current global temperature (CGT) conditional on GCM-based average time series are shown as colour plots with progressively higher densities depicted by progressively intense colours. Also shown are the HadCRUT4 data (CGT), GCM based time series (MBGT) and the average of GCM based time series (AMBGT). The temperature is in °C and in the log-scale.
Table 7.1: Goodness-of-fit check for the best GCMs with respect to averaged time series. Here 95% BCI stands for 95% Bayesian credible intervals.

| Model               | $S_1^{(k)}(\mathbf{x}_{T_0})$ | 95% BCI of $S_1^{(k)}(\bar{\mathbf{x}}_{T_0})$ | $S_2^{(k)}(\mathbf{x}_{T_0})$ | 95% BCI of $S_2^{(k)}(\bar{\mathbf{x}}_{T_0})$ |
|---------------------|-------------------------------|-----------------------------------------------|------------------------------|-----------------------------------------------|
| A1B (csiro_mk3_0)   | 0.126                         | [0.104,0.281]                                 | 0.024                        | [0.015,0.424]                                 |
| A1B (inmcm3_0)      | 0.001                         | [5 × 10^{-4},0.023]                           | 126 × 10^{-6}               | [7.324 × 10^{-6},0.048]                      |
| A2 (ukmo_hadgem1)   | 0.006                         | [0.003,0.048]                                 | 0.001                        | [9.047 × 10^{-5},0.082]                      |
| B1 (gfdl_cm2_0)     | 0.142                         | [0.107,1.048]                                 | 0.028                        | [0.017,2.420]                                |
| Commit (cnrm_cm3)   | 0.039                         | [0.119,1.599]                                 | 0.002                        | [0.021,3.848]                                |

Table 7.1, summarizing the goodness-of-fit of the posteriors to the HadCRUT4 data with respect to the discrepancy measures $S_1^{(k)}$ and $S_2^{(k)}$, tell a somewhat different story. The best GCM in the Commitment scenario seems to overfit the HadCRUT4 data in the sense that the observed discrepancies are too small to be included the 95% credible intervals of the reference discrepancy measures. Given the large variability of the time series as shown in panel (e) of Figure 7.3, which can also be gauged by the less colour intensities compared to the other panels, this result is not unexpected in retrospect. On the other hand, in the other cases, the observed discrepancies are included in the respective 95% credible intervals. Although again this seems surprising at the first glance, this is due the fact that the posterior time series relatively closer to the year 2017, where the GCM time series begins in our posterior formulation, well-captures the HadCRUT4 data, with relatively small posterior variability. Hence, even though the posteriors fail to perform well for the years closer to 1850, the overall goodness-of-fit still can not be declared as poor.

Figure 7.4 shows the posterior distributions of $[x_0, x_1, \ldots, x_{T_0+1}, x_T]$ associated with the individual time series for the best GCM models, rather than the averaged time series as shown in Figure 7.3. The overall story, however, did not seem to be very different compared to that told by Figure 7.3. Table 7.2, evaluating goodness-of-fit for these posteriors using the discrepancy measures, also provide similar inference as Table 7.1, where $\mathbf{x}_{T_0} = (x_1, \ldots, x_{T_0})$. 
Figure 7.4: The posteriors corresponding to the HadCRUT4 data or the current global temperature (CGT) conditional on individual best GCM time series are shown as colour plots with progressively higher densities depicted by progressively intense colours. Also shown are the HadCRUT4 data (CGT), GCM based time series (MBGT) and the average of GCM based time series (AMBGT). The temperature is in °C and in the log-scale.
8 GCM simulations as ensembles: extension of our Gaussian process emulation approach to the multivariate situation

So far, the inference with our one-dimensional Gaussian process approach demonstrated that although even the best GCM models are not as adequate as desired, it is not very easy to discard them since Tables 7.1 and 7.2 demonstrate quantitatively that in general their overall performances in fitting the observed current global temperatures are not particularly poor. However, Figures 7.3 and 7.4 show that a large part of the current global temperature data, beginning from 1851, fails to lie in the high density regions of the relevant posterior, which is clearly very disconcerting. Even though the Commitment scenario includes almost the entire beginning from 1851, fails to lie in the high density regions of the relevant posterior, which is clearly very disconcerting. Even though the Commitment scenario includes almost the entire current temperature data in its high posterior density region, the posterior variability turns out to be too high to render the fit satisfactory.

For further investigation we consider all the K GCM-based time series in any climate scenario as an ensemble of time series, and consider modeling them as multivariate (K-dimensional) time series, extending our one-dimensional Gaussian process emulation theory to multidimensional Gaussian process emulation. In this regard, for \( t = 0, 1, 2, \ldots \), let \( x_t = \left( x_t^{(1)}, \ldots, x_t^{(K)} \right)' \) be \( K \)-component vectors, corresponding to \( K \) different GCM based log time series \( x_t^{(k)}; k = 1, \ldots, K \). With \( \bar{x}_t = K^{-1} \sum_{k=1}^{K} x_t^{(k)} \), we shall be interested in the posterior \( [\bar{x}_1, \ldots, \bar{x}_T | x_{T_0+1}, \ldots, x_T] \), for predicting the logarithm of the observed current temperature data. Note that \( [\bar{x}_1, \ldots, \bar{x}_T | x_{T_0+1}, \ldots, x_T] \) is induced by \( [x_1, \ldots, x_{T_0} | x_{T_0+1}, \ldots, x_T] \) as the former is obtained from the latter by simply taking the averages of the components of \( x_t \), for each \( t = 1, \ldots, T_0 \). It is thus sufficient to build the multivariate Gaussian process emulation theory with respect to the \( K \)-dimensional vectors \( x_t \).

Our multivariate dynamic model is of the form

\[
x_t = f(x_{t-1}) + \epsilon_t, \quad \epsilon_t \sim N_K(0, \Sigma),
\]

where \( x_0 = x_0 \mathbf{1}_K \) is assumed known. Here \( \mathbf{1}_K \) is a \( K \)-dimensional vector with all components 1.

In the above, \( f(\cdot) = (f_1(\cdot), \ldots, f_K(\cdot))' \) is a function with \( K \) components. We assume that \( f(\cdot) \) is a \( K \)-variate Gaussian process with mean \( E[f(\cdot)] = \mathbf{B}_f^0 \mathbf{h}(\cdot) \) and covariance function \( \text{cov}(f(z_1), f(z_2)) = c_f(z_1, z_2) \Sigma_f \), for any \( (K + 1) \)-dimensional inputs \( z_1, z_2 \). Here \( \mathbf{h}(\cdot) = (h_1(\cdot), \ldots, h_m(\cdot))' \) and \( \mathbf{B}_f = (\beta_{1,f}, \ldots, \beta_{K,f}) \), where, for \( j = 1, \ldots, K \), \( \beta_{j,f} \) are \( m \)-dimensional column vectors. Note that \( h_1(\cdot) \equiv 1 \) corresponds to the intercept and \( h_2(\cdot), \ldots, h_m(\cdot) \) correspond to the components of \( (K + 1) \)-dimensional inputs \( z \). Hence, it is clear that \( m = K + 2 \). Also, \( c_f(z_1, z_2) = \exp \{ -(z_1 - z_2)' \mathbf{R}_f(z_1 - z_2) \} \), where \( \mathbf{R}_f \) is a diagonal matrix consisting of \( (K + 1) \) smoothness parameters, denoted by \( \{ r_{1,f}, \ldots, r_{(K+1),f} \} \).

### Table 7.2: Goodness-of-fit check for the best GCMs with respect to individual time series. Here 95% BCI stands for 95% Bayesian credible intervals.

| Model       | \( S_1^{(k)}(x_{T_0}^{(0)}) \) | 95% BCI of \( S_1^{(k)}(x_{T_0}) \) | \( S_2^{(k)}(x_{T_0}^{(0)}) \) | 95% BCI of \( S_2^{(k)}(x_{T_0}) \) |
|-------------|---------------------------------|-----------------------------------|-----------------------------|-----------------------------------|
| A1B (csiro_mkmk3.0) | 0.127                           | [0.099, 0.255]                    | 0.028                       | [0.014, 0.419]                    |
| A1B (inmcm3.0)      | 0.105                           | [0.098, 0.163]                    | 0.016                       | [0.013, 0.179]                    |
| A2 (ukmo_hadgem1)   | 0.088                           | [0.077, 0.152]                    | 0.015                       | [0.009, 0.186]                    |
| B1 (gfdl_cm2.0)     | 0.072                           | [0.062, 0.202]                    | 0.010                       | [0.005, 0.348]                    |
| Commit (crnr_cm3)   | 0.043                           | [0.179, 1.496]                    | 0.003                       | [0.049, 3.504]                    |
8.1 Distributions of $f(x^*_{1,0})$ and $D^*_n$

Conditional on $x_0$, $f(x^*_{1,0})$ is $K$-variate normal with mean $B_f' h(x^*_{1,0})$ and covariance matrix $\Sigma_f$. 

Now, $D_{z,v,K} = (f'(z_1), f'(z_2), \ldots, f'(z_n))'$ has an $nK$-variate normal distribution with mean

$$E[D_{z,v,K} \mid B_f, \Sigma_f, R_f] = \left( \begin{array}{c} B_f' h(z_1) \\ B_f' h(z_2) \\ \vdots \\ B_f' h(z_n) \end{array} \right) = \mu_{D_{z,v,K}} \text{ (say)} \quad (8.2)$$

and covariance matrix

$$V[D_{z,v,K} \mid B_f, \Sigma_f, R_f] = A_{f,D^*_n} \otimes \Sigma_f = \Sigma_{D_{z,v,K}} \text{ (say)}, \quad (8.3)$$

where "\otimes" denotes Kronecker product. Hence, the distribution of the $n \times K$-dimensional matrix $D^*_n = (f(z_1), f(z_2), \ldots, f(z_n))'$ is matrix normal:

$$[D^*_n \mid B_f, \Sigma_f, R_f] \sim \mathcal{N}_{n,K}(H_{D^*_n} B_f, A_{f,D^*_n}, \Sigma_f). \quad (8.4)$$

Conditionally on $(x_0, f(x^*_{1,0}))$, it follows that $D^*_n$ is $n \times K$-dimensional matrix-normal:

$$[D^*_n \mid f(x^*_{1,0}), x_0, B_f, \Sigma_f, R_f, \Sigma_{\epsilon}] \sim \mathcal{N}_{n,K}(\mu_{f,D^*_n}, \Sigma_{f,D^*_n}, \Sigma_f) \quad (8.5)$$

In $(8.5)$ $\mu_{f,D^*_n}$ is the mean matrix, given by

$$\mu_{f,D^*_n} = H_{D^*_n} B_f + s_{f,D^*_n}(x^*_{1,0})(f(x^*_{1,0})' - h(x^*_{1,0})'B_f), \quad (8.6)$$

and

$$\Sigma_{f,D^*_n} = A_{f,D^*_n} - s_{f,D^*_n}(x^*_{1,0})s_{f,D^*_n}(x^*_{1,0})'. \quad (8.7)$$

Here we slightly abuse notation to denote both univariate and multivariate versions of the mean matrix and the right covariance matrix by $\mu_{f,D^*_n}$ and $\Sigma_{f,D^*_n}$, respectively.

8.2 Joint distribution of $\{x_1, \ldots, x_T, D^*_n\}$

Note that

$$[x_1 \mid f(x_0), x_0, B_f, \Sigma_f] \sim \mathcal{N}_K(f(x^*_{1,0}), \Sigma_{\epsilon}), \quad (8.8)$$

and for $t = 1, \ldots, T$, the conditional distribution $[x_{t+1} = f(x^*_{t+1,t}) + \epsilon_{t+1} \mid D^*_n, x_t, B_f, \Sigma_f, R_f, \Sigma_{\epsilon}]$ is $K$-variate normal with mean

$$\mu_{x_t} = B_f' h(x^*_{t+1,t}) + (D^*_n - H_{D^*_n} B_f)' A_{f,D^*_n}^{-1} s_{f,D^*_n}(x^*_{t+1,t}) \quad (8.9)$$

and variance

$$\Sigma_{x_t} = \left[1 - s_{f,D^*_n}(x^*_{t+1,t}) A_{f,D^*_n}^{-1} s_{f,D^*_n}(x^*_{t+1,t})' \right] \Sigma_f + \Sigma_{\epsilon}. \quad (8.10)$$

Since $x_0$ is assumed to be known and the distribution of $D^*_n$ is given by $(8.4)$, the joint distribution is obtained by taking products of the individual distributions.
8.3 Prior distributions

We assume the following forms of the prior distributions:

\[ \begin{align*}
[B_f | \Sigma_f] & \sim \mathcal{N}_{m,K} \left( B_{f,0}, \Sigma_{B_f,0}, \psi \Sigma_f \right); \\
[\Sigma_f] & \propto |\Sigma_f|^{-\nu_f + K + 1 + \frac{1}{2}} \exp \left( -\frac{1}{2} \text{tr} \left( \Sigma_f^{-1} \Sigma_f \right) \right), \text{ with } \nu_f > K - 1; \\
[\Sigma_\epsilon] & \propto |\Sigma_\epsilon|^{-\nu_\epsilon + K + 1 + \frac{1}{2}} \exp \left( -\frac{1}{2} \text{tr} \left( \Sigma_\epsilon^{-1} \Sigma_\epsilon \right) \right), \text{ with } \nu_\epsilon > K - 1; \text{ and } \\
[\log(r_{i,f})] & \overset{iid}{\sim} \mathcal{N} \left( \mu_{R_f}, \sigma^2_{R_f} \right).
\end{align*} \]

For the prior of \( B_f \) we set \( \psi = 1 \), and except the first column of \( B_{f,0} \), we set all other columns of \( B_{f,0} \) to be null vectors. We set the first column of \( B_{f,0} \) to be the vector of means of the \( K \) GCM based time series thinned by 5 observations. Recall that these means are also used for the corresponding prior in the one-dimensional situation for model selection.

In the priors for \( \Sigma_f \) and \( \Sigma_\epsilon \), we set \( \nu_f = K \) and \( \nu_\epsilon = K \). For \( \Sigma_{B_f,0} \) and \( \Sigma_{\epsilon,0} \), we first let \( \hat{\Sigma} \) to be the empirical covariance matrix for the \( K \) GCM-based time series, thinned by 5 observations. Then we set \( \Sigma_{B_f,0} = \Sigma_{\epsilon,0} = \Sigma_f/2 \). Again, this choice is analogous to the previous one-dimensional setup.

For the log-normal priors of the smoothness parameters we set \( \mu_{R_f} = -0.5 \) and \( \sigma^2_{R_f} = 1 \). The choices imply as in the one-dimensional situation that the prior mean and the prior variance of each of the smoothness parameters are, respectively, 1 and 2 (approximately).

Thus, these prior choices are in keeping with the one-dimensional situation and have similar rationale as before.

8.4 Choice of the input grid \( \mathbf{G}_n \)

To set up the \((K + 1)\)-dimensional grid \( \mathbf{G}_n \) for the model-fitting purpose, we considered \([-5,5]^K\) to be a grid space for the \( K \)-dimensional variable \( \mathbf{z} \). We divide \([-5,5]\) into 50 equal sub-intervals and choose a point randomly from each of the 50 sub-intervals, in each dimension, yielding \( n = 50 \) \( K \)-dimensional points corresponding to \( \mathbf{z} \). For the first component of the grid, corresponding to the time component, we follow the strategy in the one-dimensional Gaussian process situation. That is, we first re-label the times \( 1850 - 2099 \) as \( 0 - 249 \) and further divide the re-labeled times by 250 to have them lie in \([0,1)\). Then, after dividing the interval \([0,1]\) into 50 equal sub-intervals we randomly simulate a value from each sub-interval, to complete construction of the input grid \( \mathbf{G}_n \). This grid choice turned out to be adequate for our purpose.

The rest of the multivariate Gaussian process emulation theory remains analogous to the corresponding univariate case, but the full conditionals of \( \mathbf{B}_f \) and \( \mathbf{D}^*_n \) are no longer available in standard form for simulating in the MCMC context, which is not analogous to the univariate context discussed in Section 4; see the supplement of Ghosh et al. (2014) for details. We use additive TMCMC to update the unknowns in the multidimensional situation. For updating the positive definite matrices \( \Sigma_f \) and \( \Sigma_\epsilon \), we represent the matrices in the Cholesky decomposition forms \( \mathbf{C}\mathbf{C}' \), where \( \mathbf{C} \) is a lower triangular matrix, and use additive TMCMC to update the non-zero elements in a single block. We implement our codes, written in C, in our VMWare. The implementations associated with A1B, A2, B1 and Commitment took about 30 hours 37 minutes, 18 hours 52 minutes, 29 hours 12 minutes and 16 hours 59 minutes, respectively.
9 Results for the multivariate climate dynamics

For the four climate scenarios, the posterior distributions of $[\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_T | x_{T_0+1}, \ldots, x_T]$ (where $\bar{x}_0 = x_0$, since $x_0$ is assumed to be known) are shown in Figure 9.1. Now, compared to the one-dimensional situations, severe under-estimation of the HadCRUT4 data by all the four climate scenarios is corroborated by this multivariate framework. And, Table 9.1 revealing severe underfits for all the four climate scenarios, confirms that even the discrepancy measures could not act as saviours this time.
Table 9.1: Goodness-of-fit check for ensembles of GCM time series with respect to \([\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{T_0}], x_{T_0+1}, \ldots, x_T]\). Here 95% BCI stands for 95% Bayesian credible intervals.

| Model | \(S_1^{(k)}(x_{T_0}^{(0)})\) | 95% BCI of \(S_1^{(k)}(x_{T_0})\) | \(S_2^{(k)}(x_{T_0}^{(0)})\) | 95% BCI of \(S_2^{(k)}(x_{T_0})\) |
|-------|------------------|-----------------|------------------|-----------------|
| A1B   | 3.580            | [0.690,0.872]   | 13.158           | [0.763,1.182]   |
| A2    | 3.807            | [0.689,0.871]   | 14.909           | [0.759,1.179]   |
| B1    | 3.872            | [0.688,0.870]   | 15.434           | [0.758,1.177]   |
| Commit| 3.711            | [0.690,0.870]   | 14.229           | [0.760,1.176]   |

Since \([\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{T_0}], x_{T_0+1}, \ldots, x_T]\) severely under-estimates the HadCRUT4 data, we now investigate how well the posterior \([x_0^{(max)}, x_1^{(max)}, \ldots, x_{T_0}^{(max)}], x_{T_0+1}, \ldots, x_T]\) can capture the observed current temperature data, where for \(t = 0, 1, 2, \ldots, x^{(max)}_t\) is the maximum of the components of \(x_t\). Figure 9.2 displays the relevant posterior time series as colour plots, along with the HadCRUT4 data (CGT) and the maximum of model based global temperature (MMGT) associated with the GCM simulations, in the log scales. Observe that CGT and MMGT are included in the supports, but it is doubtful how good the fits are, since the posterior variances are high and moreover for A2 and Commitment CGT and MMGT fall in low density regions. Table 9.2 shows that the fits are indeed not encouraging. Observe that A1B overfits with respect to both \(S_1^{(k)}\) and \(S_2^{(k)}\). With respect to \(S_1^{(k)}\), A2 slightly underfits, while the fit is adequate with respect to \(S_2^{(k)}\). Since \(S_2^{(k)}\) is generally a better performer than \(S_1^{(k)}\), one can considered to fit of A2 to be adequate. B1 seriously overfits with respect to both the discrepancy measures, while Commitment seriously underfits with respect to both \(S_1^{(k)}\) and \(S_2^{(k)}\).
Figure 9.2: The posteriors $[x_{0}^{(\text{max})}, x_{1}^{(\text{max})}, \ldots, x_{T_{0}}^{(\text{max})}] | \mathbf{x}_{T_{0}+1}, \ldots, \mathbf{x}_{T}$ are shown as colour plots with progressively higher densities depicted by progressively intense colours, along with the HadCRUT4 data (CGT) and the maximum of model based global temperature (MMBGT). The temperature is in °C and in the log-scale.
Table 9.2: Goodness-of-fit check for ensembles of GCM time series with respect to 
\[ x_0^{(\text{max})}, x_1^{(\text{max})}, \ldots, x_{T_0}^{(\text{max})}, x_{T_0+1}, \ldots, x_T \]. Here 95% BCI stands for 95% Bayesian credible intervals.

| Model | \( S_1^{(k)}(x_{T_0}^{(0)}) \) | 95% BCI of \( S_1^{(k)}(x_{T_0}^{(0)}) \) | \( S_2^{(k)}(x_{T_0}^{(0)}) \) | 95% BCI of \( S_2^{(k)}(x_{T_0}^{(0)}) \) |
|-------|-----------------|-------------------------------|-----------------|-------------------------------|
| A1B   | 0.216           | [0.693,0.891]                | 0.061           | [0.787,1.313]                |
| A2    | 0.893           | [0.692,0.888]                | 0.816           | [0.786,1.318]                |
| B1    | 0.303           | [0.690,0.891]                | 0.104           | [0.785,1.332]                |
| Commit| 1.256           | [0.671,0.879]                | 1.617           | [0.755,1.376]                |

10 Future climate forecast with our Bayesian Gaussian process dynamics model

Our detailed analyses of the GCM forecasts so far failed to justify their credibilities. This failure, however, seems to hold a great deal of positivity since the rapid future global warming foreboding that might eventually threaten life on earth, need not become the reality. However, it is not clear yet then what kind of climate change we can expect in the future. We attempt to answer this question, again with our Bayesian Gaussian process emulation theory, now forecasting the log global average temperature in the years 2017 − 2099 given the log HadCRUT4 dataset for the years 1850 − 2016, using the theory and strategies proposed in Section 4.2. Here we let the prior distributions remain the same as detailed in Section 3, except that the first component of \( \beta_{f,0} \) and \( \sigma^2 \) are now based upon thinning the log HadCRUT4 data by 5 observations. The input grid \( G_n \) remains the same as in the one-dimensional setup detailed in Section 6.1.

Our future climate prediction results are presented in Figure 10.1, along with the posterior modes associated with the Gaussian process forecasted global temperature (GPFGT), the best GCM-specific model based forecasted global temperature (MBFGT) and average model based forecasted global temperature (AMBFGT). In stark contrast with MBFGT and AMBFGT which show steep increase in the temperature in panels (a)-(d), the high posterior density regions of our Bayesian forecasts do not support increasing future global temperature. Only in the case of Commitment (panel (e)) MBFGT and AMBFGT tend to fall within the high posterior density regions of our Bayesian forecasts.

According to Green et al. (2009): “The benchmark forecast is that the global mean temperature for each year for the rest of this century will be within 0.5°C of the 2008 figure.” Thus, according to their prediction, the future global temperature should lie in the interval [13.895, 14.895]°C. This interval is included even within all the 50% credible intervals of our year-wise Bayesian posterior forecast distributions for 2017−2099. Thus, our results are broadly in agreement with the forecast of Green et al. (2009), and clearly do not support drastic global warming as projected by the GCMs.

11 Summary and discussion

As stated in Lupo et al. (2013) (see also the references therein), “When physicists, biologists, and other scientists who are unaware of the rules of forecasting attempt to make climate predictions, their forecasts are at risk of being no more reliable than those made by non-experts, even when they are communicated through complex computer models”. The GCMs are indeed complex computer models built by physicists, biologists, and other scientists. The future global warming forecasts yielded by such models have great bearing on the current world and particularly on the IPCC policymakers. But as discussed by Lupo et al. (2013) in great detail, major scientists of the world do not find much reason to pin faith on the global warming foreboding, and most of
Figure 10.1: The posteriors \([x_{T_0+1}, \ldots, x_T | x_1, \ldots, x_{T_0}]\) for future climate prediction are shown as colour plots, along with the posterior modes of the Gaussian process forecasted global temperature (GPFGT), best GCM-specific model based forecasted global temperature (MBFGT) and average model based forecasted global temperature (AMBFGT). The temperature is in °C and in the log-scale.
them, based on their experiments and experiences, are strongly critical of the abilities of GCM to adequately model so complex a system as world climate.

However, we are unaware of any significant and rigorous statistical research that evaluates the GCM-based global warming projections. Such a task, which is of global importance, must be seriously undertaken and no wonder statistics is the only subject that can promise to make justice to such an issue where quantification of uncertainties (in the predictions by the GCMs) plays the most important role. It is also very well-established that the Bayesian statistical paradigm is the most well-equipped to coherently deal with uncertainty quantifications.

As such, we undertake the task, in the Bayesian framework, of evaluating the global warming forecasts, with observed current temperature data and GCM-simulated data obtained from the IPCC website. We first consider dynamically but nonparametrically modeling temperature using Gaussian process emulation procedure, borrowing ideas from Bhattacharya (2007) and Ghosh et al. (2014). Such a modeling strategy seems to quite appropriate for nonparametrically addressing the uncertainties in dynamic climate change, in the absence of any known model framework for either the GCMs or the observed current temperature.

With such a Gaussian process based model we then attempt to address the question of how to select the best GCM within any climate scenario using the principle of our recently-developed Bayesian multiple testing procedure (Chatterjee and Bhattacharya (2020)) that provides rigorous assessment of the projected future dynamics (forward sense) of the GCMs as well as their abilities to predict the observed HadCRUT4 data (inverse regression sense), and yields the best model by comparing these combined abilities in a theoretically sound manner. The procedure, along with discrepancy measure based theory of Bayesian model adequacy test proposed in Bhattacharya (2013) provides the additional evaluation if the best models adequately satisfy the goodness-of-fit test with respect to fitting the HadCRUT4 data, given the future projections.

Such evaluations are previously contemplated upon in the climate context by other researchers: for example, Lupo et al. (2013), quoting Reifen and Toumi (2009), write “Expounding on this principle, Reifen and Toumi (2009) note, “with the ever increasing number of models, the question arises of how to make a best estimate prediction of future temperature change.” That is to say, which model should one use? With respect to this question, they note, “one key assumption, on which the principle of performance-based selection rests, is that a model which performs better in one time period will continue to perform better in the future.” In other words, if a model predicts past climate fairly well, it should predict future climate fairly well. The principle sounds reasonable enough, but does it hold true?”

To our knowledge, there does not exist any other sound statistical analysis in this respect. We believe that ours is a significant contribution from this perspective, where the nonparametric Gaussian process dynamics substantially adds to the overall novelty. Employment of efficient C-coding and parallel computing architectures ensure quite cheap computation, in spite of evaluations of a large number of GCMs using intricate Bayesian nonparametric models and methods.

Our results on model selection and evaluation fail to provide any evidence in favour of global warming, challenging all our evaluated GCMs along the way. To further strengthen our results regarding these, we model the GCM forecasts in any given climate scenario as ensembles, which we model by extending our one-dimensional climate dynamics to multidimensional climate dynamics, driven by multidimensional Gaussian processes. Attempts to predict the observed HadCRUT4 data given the future ensembles of GCM forecasts, resulted in conspicuously poor fits. This strongly reinforces that future global warming, as projected by the GCMs, need not turn out to be the reality.

Finally, based on our one-dimensional Gaussian process based climate dynamics, we provide Bayesian climate forecasts into the future, conditioned on the HadCRUT4 data. Our results quite persuasively demonstrate that the future does not hold the drastic global warming doom
for the world, in stark contrast with the GCM warnings. An important finding in this respect is that the best model of the Commitment scenario at least falls closer to our high posterior density regions of our future predictions as well as those of current HadCRUT4 prediction given the future simulations, compared to the other GCMs. Since the greenhouse gases are held fixed in the Commitment case, it is probably not unreasonable to question the role of greenhouse gases as important drivers of climate change. Our result in this regard seems to be broadly supported by some scientific experiments conducted by Hansen et al. (1998). Quoting these authors and criticizing IPCC’s faith in the state-of-the-art climate projections, Idso et al. (2013a) write “Hansen et al. (1998) examined the forcings of well-mixed greenhouse gases (CO2, CH4, N2O, and CFCs), tropospheric ozone, stratospheric ozone, tropospheric aerosols, forced cloud changes, vegetation and other planetary surface alterations, solar variability, and volcanic aerosols. That examination revealed so many uncertainties in the forcings that the researchers concluded, “the forcings that drive long-term climate change are not known with an accuracy sufficient to define future climate change.” Nevertheless, the IPCC has expressed confidence in projections of future climate, saying the temperature sensitivity of Earth’s climate system in response to a doubling of atmospheric CO2 concentrations “is likely to be in the range 2°C to 4.5°C with a best estimate of about 3°C, and is very unlikely to be less than 1.5°C [italics in the original]” (IPCC (2007)).”

In summary, our novel Bayesian models and methodologies provide a peek into the future world, which does not seem to be as gloomy as portrayed by the non-Bayesian scientists and the IPCC policymakers.

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