The Bell and probability inequalities are not violated when non-commutation is applied according to quantum principles

Louis Sica$^{1,2}$

$^1$Institute for Quantum Studies, Chapman University, Orange, CA & Burtonsville, MD, 20866, USA
$^2$Inspire Institute, Inc., Alexandria, VA 22303, USA
Email: lousica@jhu.edu

The Bell inequalities in three and four correlations may be re-derived in a general form showing that the corresponding number of data sets of ±1's identically satisfies them regardless of whether they are randomly or deterministically generated. When the data sets become infinite in size in the random case, and assuming convergence of the correlation estimates, the inequalities become constraints on the correlation functions of the mutually cross-correlated data sets. Replacing the correlations in the inequalities by the corresponding physical model-based probabilities that produce them results in inequalities in probabilities. Under the special assumption of a wide-sense-stationary (WSS) random process, the Wigner inequality in probabilities results in the three variables case. This inequality is violated by probabilities that produce Bell states, since these states are inconsistent with the assumption of a WSS process. They are also inconsistent with quantum non-commutation as occurs in the case of more than one spin measurement on each of two particles. When all the correlations or probabilities are computed according to quantum principles, however, the corresponding version of the Bell inequality is satisfied.

I. INTRODUCTION

The Bell inequality in probability averaged correlations may be shown to follow from an algebraic relation that must be identically satisfied by finite data sets [1], while the Wigner inequality [2] in probabilities requires consideration of the physical properties of a system before an appropriate probability description may be formulated. The purpose of the present paper is to discuss misunderstandings that have led to correlational and probability inequalities violations, and the relation between correlational and probability inequalities.

In the laboratory, one can only measure photon counts having values of +1 or -1 according to the labeling of the detectors used in Bell experiments (See Figure). From these counts, resulting correlations are computed. It has been shown previously [1] that such correlations satisfy the Bell inequalities under the general condition that the appropriate number of data sets simply exists, regardless of other characteristics, such as whether or not they are random.

The specification of the physical process to be characterized has an important effect on inequalities in probabilities, such as the inequality of Wigner. Below, the Wigner inequality will be obtained from the Bell form under the assumption of
sufficient symmetry, but without the specific assumption that the Bell cosine correlation represents the correlations of all variable pairs, which is mathematically impossible. The probability inequality may take other forms, however, in situations without such symmetry.

That the correlation and probability forms of Bell inequalities are logically connected has been shown previously by Sica [3]. However, considering the general importance of Bell inequalities, and the continuing controversies in their interpretation [4], it is appropriate to review the Bell theorem and the quantum probabilities related to it.

Various forms of inequality in three and four variables arose originally from Bell’s concept of a stochastic process representation of quantum entanglement [5]. For the stochastic process that Bell chose, one value of a random variable may correspond to an indefinite number of instrumental readouts. Bell represented the readouts with a function \( A(a, \lambda) \), where \( a \) is an instrument setting and \( \lambda \) is a random variable. Thus, \( A(a, \lambda) \) is a random function for which various relevant readouts occur at different values of instrument setting \( a \) for each value of parameter \( \lambda \). There is no implication that accessing a readout at \( a \) affects accessing a readout at \( a' \), or vice versa, for a given realization of the random process. The physical situation might be exemplified by a macroscopic object with a physical property depending on time or space coordinates but having random values resulting from random initial conditions. (It is somewhat analogous to a quantum process producing states characterized by a set of commuting observables [6].)

Without seeming to realize its implications, Bell assumed a particular kind of random process, termed wide-sense-stationary (WSS) [7]. He used the correlation functional form computed from quantum mechanics for commuting measurements on a pair of entangled spins (that suggests WSS) for correlations involving a third non-commuting measurement as well. The WSS process Bell assumed is defined as one in which the correlation of readouts at any two instrument settings \( a_i \) and \( a_j \) is given by a function of the form \( f(a_i - a_j) \) depending on the difference of coordinates. Even for classical processes, such as those in optics, for example [8], this functional form is frequently an approximation, and it may evolve to a more complex function for different values of the coordinates.

II. BELL INEQUALITY FOR DATA SETS

Data sets are defined herein by the property of being able to be written down. They may be obtained from experimental observations, from predictions of experimental observations (also known as counterfactuals), or from a combination of the two. The use of predicted results together with those of measurements has led to considerable confusion in the case of the Bell theorem and inequalities, since
they involve more than the one measurement on each particle that can be obtained in a Bell experiment without encountering quantum mechanical noncommutation. If the number of measurements is extended beyond the two that commute, the noncommutation involves changes in the form of probabilities invoked, and in the resulting data correlations, as will be shown below.

The original Bell inequality holds far more generally then Bell realized, since it must be identically satisfied by the appropriate number of data sets. The inequality as derived by Bell [5], however, resulted after average correlations were first computed for a stochastic process that Bell later assumed to be WSS. The problem that arises is not that the Bell inequality does not hold for the specific assumption of a WSS process (it does), but that the Bell-state correlation function assumed cannot hold for such a process, and leads to contradictions when more than two measurements are carried out.

The basic mathematical fact is that it does not matter whether data are measured or predicted, or even whether they are random or deterministic. *The Bell inequality must be identically satisfied as a fact of algebra that may consequently be applied to random processes.*

Assume three data sets exist and are labeled with instrument settings \(a, b,\) and \(c\). The data set items are denoted by \(a_i, b_i,\) and \(c_i\) with \(N\) members in each set. The numerical values of all items of data are \(\pm 1\). One may form the equation

\[
a_i b_i - a_i c_i = a_i (b_i - c_i)
\]

and sum this equation over the \(N\) data triplets assumed to exist. After dividing by \(N\), one obtains

\[
\frac{\sum_i^N a_i b_i}{N} - \frac{\sum_i^N a_i c_i}{N} = \frac{\sum_i^N a_i b_i (1 - b_i c_i)}{N}.
\]

Taking absolute values of both sides,

\[
\left| \frac{\sum_i^N a_i b_i}{N} - \frac{\sum_i^N a_i c_i}{N} \right| = \left| \frac{\sum_i^N a_i b_i (1 - b_i c_i)}{N} \right| \leq \frac{\sum_i^N |1 - b_i c_i|}{N},
\]

or

\[
\left| \frac{\sum_i^N a_i b_i}{N} - \frac{\sum_i^N a_i c_i}{N} \right| \leq 1 - \frac{\sum_i^N b_i c_i}{N}.
\]

Equation (2.4) is the Bell inequality in its most general form. The author unexpectedly discovered this result some time ago [1] by asking the following question: If one performs a laboratory experiment for which the number of data
items $N$ per set is finite, to what extent do random fluctuations of the correlation estimates result in violation of the inequality (2.4)? Astonishingly, the answer as shown above, is that the inequality is always satisfied identically. (No assumption has been made other than that the data exist.) The data do not necessarily have to be random, with the result that the correlation estimates in (2.4) do not then represent probabilistically computable functions.

Suppose, however, that the data derive from a random process, and that the correlation estimates in (2.4) converge to probabilistically computable correlations as $N \to \infty$. The resulting correlations must then satisfy an inequality of the same form as (2.4):

$$\left| \langle ab \rangle - \langle ac \rangle \right| \leq 1 - \langle bc \rangle .$$

(2.5a)

This is the inequality derived by Bell [5] except that Bell replaced the variable $c$ on the right-hand-side by a variable $a' = -c$. In that case, the inequality may be written

$$\langle ab \rangle - \langle ac \rangle - \langle a'b \rangle \leq 1 .$$

(2.5b)

(Below, $b' = c$ will be used as appropriate to the argument to be made at that time.)

It is critically important to understand that (2.4) is a result that holds generally for any three arbitrary data sets, while the Bell relation (2.5a,b) does not hold for arbitrary correlations. This is because one can make up any data sets whatsoever, and they will satisfy (2.4), but made up correlations not derived from three data sets will not necessarily satisfy (2.5a,b). It follows that if (2.5a,b) is violated, no data sets exist that result in the proposed correlations.

It should be noted that the assumptions used in deriving (2.5a) from (2.4) can also be used to derive a four variable Bell inequality. Assuming that there exist four data sets with members $a_i, a'_i, b_i, b'_i$, of length $N$, with each item equal to $\pm 1$, then for each group of four data items from the four respective data sets, one has (by inspection)

$$-2 \leq a_i (b_i + b'_i) + a'_i (b'_i - b_i) \leq 2 .$$

(2.6)

Summing over $N$ in (2.6), and dividing by $N$ leads to

$$-2 \leq \frac{\sum_i a_i (b_i + b'_i)}{N} + \frac{\sum_i a'_i (b'_i - b_i)}{N} \leq 2 .$$

(2.7a)

Again assuming convergence (the law-of-large-numbers) to limits as $N \to \infty$, a common form of Bell inequality used by experimentalists results:

$$C(ab) + C(ab') + C(a'b) - C(a'b') \leq 2 .$$

(2.7b)

The difficulty of applying the three variable inequality (2.4) to an entangled state in which more than two measurements are non-commutative, is amplified in the case of a four variable inequality.
III. HOW CORRELATIONS FROM EXPERIMENTAL DATA VIOLATE
BELL INEQUALITIES

The frequent statement that correlations obtained from experimental data violate Bell inequalities involves a misconception of the conditions under which Bell inequalities hold. Relations (2.5a,b) as follow from (2.4), and that of (2.7b) from (2.7a), result from the cross-correlations of three or four data sets, respectively. However, if correlations are computed from individual variable pairs, each of which was acquired in an independent experimental realization i.e., run, the correlations may be different depending on the experimental requirements of the random process considered. Note that in the case of separate runs for each correlation, there would be six data sets instead of the three used in (2.4) and eight instead of four in relation (2.7a).

To understand the non-innocuous effect of this, consider an analogous situation that would be realized if one replaced the correlation $\bar{ab}$ for two jointly Gaussian random variables both measured together on each individual member of a population, with the product $\bar{a}\bar{b}$, of averages of $a$ and $b$ measured separately over the same population. The product of the averages does not in general equal the average of the product, i.e., the correlation, unless the joint probability is a product of probabilities, one for each variable. (Note that it may occur that $\bar{a}=\bar{b}=0$ while $\bar{ab}\neq0$.)

Similarly, three correlations, each individually computed from three separate experimental runs using six data sets, are not in general equal to the correlations obtained from data acquisition in which the three items of data are simultaneously acquired and correlated at each realization of a random process. The effect of three items of data present at once for variables equal to $\pm1$ may be seen in the fact that $(a_i b_i)(a_i c_i) = (b_i c_i)$. The product of the third variable pair is already determined by the product of the first two, if they are all present at once in a realization. This is not the case when pairs from different runs are used since the value of $a$ will vary randomly in different runs. Note that this fact holds for any pair of variables used in (2.4-2.5) since each variable occurs in two correlations, e.g., $(b_i c_i)(b_i a_i) = (c_i a_i)$, etc.

Hess has pointed out [9] that similar facts, and inequalities related to that of Bell, have been known to mathematicians since Boole. Pure mathematics determines a third correlation when data for two out of the three are specified. It is also critical to note that nothing in the proof of (2.4, 2.5) indicates that the correlations need have the same functional form. They may all be different or all the same, but their mutual functional forms will be constrained.

There is at least one stochastic process for which all the correlations may be measured individually as well as jointly, and that is the case of a WSS process mentioned above. That was the process that Bell assumed, extrapolating from the
form of correlation function, \(-\cos(\theta_1 - \theta_2)\), that characterizes spin measurements on two entangled particles at detector angles \(\theta_1\) and \(\theta_2\), and that was later found to characterize optical photons in an entangled state (with a factor of 2 multiplying the angular difference). However, if this functional form is assumed to represent all three correlations, the Bell inequality is violated. Therefore, contrary to popular belief, no three sets of \(\pm 1\)'s under cross-correlation can produce the correlation functions in question, as a fact of mathematics. Processes resulting in triple data sets can produce only two correlations of the triplet in this form.

*These facts do not disagree with predictions of quantum mechanics.* Quantum mechanics (QM) allows one measurement to be carried out on each of the particles of an entangled pair. A third measurement is non-commutative with one of the other two, and is therefore conditional on it [6,10]. This leads to a third correlation of a different form satisfying the Bell inequality. The fact of such non-commutation is fundamental and widely treated in quantum mechanics texts. Due to this fact also, the experimental arrangement or calculation procedure necessary to obtain values for a third variable and associated correlations is different from that used to obtain data for either of the first two correlations.

One may now consider relation (2.7a,b) for a finite value of \(N\) in the special case of a WSS process and four data sets. The WSS properties have been assumed to represent entanglement by experimentalists and theoreticians alike, after Bell’s mistaken assumption of their applicability. One may write (2.7a) in the form

\[
-2 \leq C(ab) + \delta_a(ab) + C(ab') + \delta_a(ab') + C(a'b) + \delta_b(a'b) - C(a'b') - \delta_b(a'b') \leq 2, \tag{2.9}
\]

where as before, the \(C(xy)\) functions are assumed to represent the limiting forms for the correlation estimates as \(N \to \infty\). Since the inequality cannot be violated for data sets that are jointly present and cross-correlated, the \(\delta_{xy}\)'s represent random differences between the infinite and finite averages for the four correlations. The \(\delta_{xy}\)'s must sum to zero in this case where the four variables’ values are all obtained in each realization of the experiment.

By contrast if the data are taken in four independent runs using the same instrument settings, inequality (2.9) for the same WSS process becomes

\[
-2? \leq C(a_{1}\hat{b}_{1}) + \delta_{1}(a_{1}\hat{b}_{1}) + C(a_{2}\hat{b}_{2}) + \delta_{2}(a_{2}\hat{b}_{2}) + C(a_{3}\hat{b}_{3}) + \delta_{3}(a_{3}\hat{b}_{3}) - C(a_{4}\hat{b}_{4}) - \delta_{4}(a_{4}\hat{b}_{4}) \leq 2?, \tag{2.10}
\]

where the subscripts 1...4 indicate the experimental run number used to compute the correlation, and the question marks indicates possible violation of the \(\pm 2\) limits of the previous inequality. The Bell inequality form of (2.10) may now be violated since eight data sets and not four are used, and the \(\delta_{xy}\)'s no longer need to sum to
zero, even though the limiting correlations $C(xy)$ are the same for the WSS process assumed.

Thus, relation (2.10) may be written more simply as

\[-2? \leq C(ab) + \delta_n(a_1 b_1) + C(ab') + \delta_n(a_2 b_2) + C(a'b) + \delta_n(a_3 b_3) - C(a'b') - \delta_n(a_4 b_4) \leq 2?, \quad (2.11)\]

where subscripts on the variables in the $\delta_n$'s still indicate the experimental run, but they have been removed for the probability averaged correlations, since these are the same whether measured in independent runs or in one run with four measurement outcomes read at each realization in the WSS case. The errors in each of the four runs for $N$ data pairs exist and are finite. However, assuming that the law of large numbers holds, the $\delta_n$'s become small as $N$ becomes large. Thus, the inequality (2.11) would be expected to be violated, but by smaller and smaller values as $N$ becomes larger.

Bell made the assumption [5] that the random process applicable to a triplet of polarization or spin measurements is WSS, and this is also widely assumed in the four variable case of (2.7b) by those interpreting experimental data in a way that violates the Bell inequality (2.7b). When the mathematical facts leading to (2.5) or (2.7b) are considered, however, it becomes clear that the measurement results in QM experiments do not correspond to a WSS process. If they did, violation of the corresponding Bell inequality would necessarily be small, i.e., of the order of four standard deviations rather than 102 standard deviations as reported in [11].

However, for a process that cannot be WSS because some measurements are non-commutative, the correlation functional forms at different detector settings vary as well as the procedures necessary to obtain them. Given that one of the correlations necessary for Bell inequality application is conditional on data from previous correlations, it is not surprising that a different correlation results from data obtained in a separate run than from data previously obtained. When correlations are explicitly computed from data, or predicted from quantum mechanical principles for three or four data sets, it is found that the mutual correlations are not all of the Bell form, and that the relevant Bell inequality is now satisfied. Since correction of the errors of neglecting non-commutation and the assumption of a WSS process eliminates violation of the Bell inequalities (that are a fact of algebra), the basis for claims deduced from inequality violation is eliminated.

It should be noted in passing that a major example of quantum non-commutation, the Pauli spin matrices, originated in a classical representation of three dimensional rotations by two dimensional matrices [12]. This, as well as examples such as non-commutation of classical light polarization measurements,
indicate that non-commutation is a fact that cannot be neglected in either classical
or quantum physics.

IV. CORRELATION MEASUREMENTS THAT SATISFY QUANTUM
MECHANICS AND THE BELL INEQUALITY

1. A third measurement may be taken in sequence after one of the first two [13].
The path indicated by the final detector triggered indicates the previous
measurement outcome (by retrodiction). Three data sets are obtained and the Bell
inequality is satisfied.
2. A second way of obtaining three data sets is to measure \( C(ab) \) and \( C(ab') \) in two
runs, thus obtaining four data sets [14]. From processing these as indicated in [14]
one may obtain effectively three data sets and compute:

\[
C(bb') = C(bb'|a=1)P(a=1) + C(bb'|a=-1)P(a=-1). \quad (4.1)
\]

3. An equivalent result may be obtained from quantum mechanical predictions of
the data using the quantum probabilities resulting from entanglement.

V. INTERACTION BETWEEN DETECTORS

If measurements are made on two particles, one of the measurements occurs
first, except in circumstances of infinite time precision. Assuming that \( A \) is set so as
to be measured before \( B \) or \( C \) by an infinitesimal amount, assumed pickup from
detector \( A \) to \( B \) or \( A \) to \( C \), still leads to three data sets using the procedures of Sec.
IV. Then the three variable inequality would hold even for the corrupted data. Observed
correlational functional forms could then be compared with quantum mechanical predictions.

VI. THE WIGNER FORM OF THE BELL INEQUALITY FOR A WSS
PROCESS

It is first assumed that relation (2.5) is applied to a WSS process using
probabilities with the same symmetry as those obtained from QM for two entangled
spins in a Bell state. (Note that the detailed functional froms are not used, only the
symmetries.) Thus for \( C(a,b) \)

\[
P_+ (a,b) = P_+ (a,b), \quad P_- (a,b) = P_- (a,b), \quad (6.1)
\]

with normalization condition

\[
2P_+(a,b) + 2P_-(a,b) = 1, \quad (6.2)
\]

so that correlation \( C(a,b) \) is given by
This becomes

\[ C(a,b) = 4P_{+a}(a,b) - 1 \]  \hspace{1cm} (6.3b)

after making use of the normalization condition (6.2).

Since the variable pairs in (2.5b) are on opposite sides of a Bell apparatus, one may use (6.3b) for correlations of other variable pairs by appropriately changing the variable names. Using (6.3b) with these variable name-changes, and inserting them into (2.5b), an inequality in probabilities follows:

\[ P_{+a}(a,b) \leq P_{+a}(a,c) + P_{+a}(a',b) \]  \hspace{1cm} (6.4)

where \( a \) and \( a' \) are on one side of the Bell apparatus and \( b \) and \( c \) are on the other. (However, \( a' = -c \) is necessary to maintain the connection of (6.4) to the original Bell inequality.)

**VII. QUANTUM CORRELATIONS SATISFY THE INEQUALITY**

Inequality (6.4) is violated by quantum probabilities corresponding to Bell correlations based on entanglement. This occurs because the correlation on the right-hand-side of (2.5a) is constrained by the left-hand correlations \( C(ab) \) and \( C(ac) \) whose existence requires data that determines the right-hand side. The correlation \( C(bc) \) thereby determined cannot have the same form as the previous correlations if the latter have the Bell cosine form. Changing variable \( c \) to \( b' \) with these two variables both on the right or \( B \)–side of a Bell apparatus, (2.5a) becomes

\[ \left| \langle ab \rangle - \langle ab' \rangle \right| \leq 1 - \langle bb' \rangle . \]  \hspace{1cm} (6.4a)

Correlation \( C(bb') \) must then be computed using the QM conditional probabilities \( P(b|a) \) and \( P(b'|a) \) based on the joint probabilities from which \( C(ab) \) and \( C(ab') \) are computed.

Using the established probabilities for \( P_{+a} \) and \( P_{-a} \), the correlation \( C(bb') \) may be computed [14]:

\[ \langle bb' \rangle = \sum_{bb'} b'b' P(b|a=1)P(b'|a=1)P(a=1) + \sum_{bb'} b'b' P(b|a=-1)P(b'|a=-1)P(a=-1) \]
\[ = \cos(\theta_b - \theta_a) \cos(\theta_{b'} - \theta_{a'}) \]  \hspace{1cm} (6.4b)

Using the contracted notation \( \theta(ba) = \theta_b - \theta_a \), and the Bell cosine correlations, inequality (6.4a) becomes
\[-\cos \theta(a,b) + \cos \theta(a,b') \leq 1 - \cos \theta(b,a) \cos \theta(b',a) = 1 - \cos \theta(a,b) \cos \theta(a,b') \]  \hfill (6.4c)

\[-2 \sin \frac{\theta(a,b) + \theta(a,b')}{2} \sin \frac{\theta(a,b') - \theta(a,b)}{2} \leq \sin^2 \frac{\theta(a,b) + \theta(a,b')}{2} + \sin^2 \frac{\theta(a,b) - \theta(a,b')}{2}, \] \hfill (6.4d)

after use of appropriate trigonometric identities. One may replace the difference of correlations on the left-hand-side of (6.4c) by expressions in probabilities \(P_a\), but the same result occurs in (6.4d). Since

\[ (a^2 + b^2) - 2|a||b|(|a| - |b|)^2 \leq 0, \]
\[ (a^2 + b^2) \geq 2|a||b|, \] \hfill (6.5d)

the Bell inequality in (6.4d) is satisfied.

Thus, when probabilities resulting from QM are used, (6.4a) is satisfied as demanded by basic mathematics. Deductions of non-locality or non-reality, if based on Bell inequality violation, no longer follow.

**VIII. CONCLUSION**

The Bell inequality was originally developed as a theorem in probability theory and applied to quantum mechanical correlations resulting from entanglement, under the unstated assumption of a wide-sense-stationary (WSS) stochastic process. However, when expressed in a general form immediately applicable to laboratory data, the inequality is satisfied by any data sets of \( \pm 1 \)'s after assuming merely that they exist. This is the case for both three and four variable data sets and corresponding inequalities. If probabilities with symmetries appropriate to entangled spins are assumed to generate the correlations of a WSS process, and substituted in the three variable Bell inequality, the Wigner inequality in probabilities results. However, the constraint of this inequality is not consistent with QM probabilities for a sequence of three measurements on two particles that involves both commutation and non-commutation.

When it is shown that the Bell inequality follows from basic facts of algebra independently of randomness, the logical deductions that follow from inequality violation are dramatically impacted. For a random process, the Bell inequality disallows the Bell cosine correlation as the functional form for all three or four correlations, as well as the probabilities corresponding to these correlations. This agrees with QM principles according to which certain sets of observations commute, but it necessitates probabilities conditional on prior measurement outcomes for those that do not. The requirements of QM are thus consistent with the basic constraints of algebra that produce the Bell inequality. **When these requirements are met, the Bell inequalities are satisfied.**
An essential step necessary to understanding the errors in the Bell theorem may be missed if it is believed that all classical measurements commute. Neither three spatial rotations, nor sequences of polarization measurements on classical waves commute, which is consistent with the non-commutation of quantum mechanical spin and polarization measurements, respectively. These facts as well as the result that the Bell inequality follows purely from algebra and the existence of data sets, have been central to unraveling the logical errors that lead to claims that QM data violate a Bell inequality.

The Bell theorem is ordinarily interpreted to imply that one cannot construct a local hidden variables account of quantum correlations. If the logic of the theorem is flawed, however, it does not follow thereby that the converse is true. The author believes that the definition and construction of a proper hidden variables theory that accounts for entanglement-based correlations is a complex task involving open issues. A discussion of this is beyond the scope of the present work.

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Figure. Schematic of Bell experiment in which a source sends two particles to two detectors having angular settings $\theta_a$ and $\theta_b$ and/or counterfactual settings $\theta_a'$ and $\theta_b'$. While one measurement operation on the A-side, e.g. at setting $\theta_a$, commutes with one on the B-side at $\theta_b$, any additional measurements at either $\theta_a'$ or $\theta_b'$ are non-commutative with prior measurements at $\theta_a$ and $\theta_b$, respectively.

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