Cellular Resolutions of Monomial Modules

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Abstract: We construct a canonical free resolution for arbitrary monomial modules and lattice ideals. This includes monomial ideals and defining ideals of toric varieties, and it generalizes our joint results with Irena Peeva for generic ideals [BPS], [PS].

Introduction

Given a field $k$, we consider the Laurent polynomial ring $T = k[x_1^\pm, \ldots, x_n^\pm]$ as a module over the polynomial ring $S = k[x_1, \ldots, x_n]$. The module structure comes from the natural inclusion of semigroup algebras $S = k[N^n] \subset k[Z^n] = T$. A monomial module is an $S$-submodule of $T$ which is generated by monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}, a \in Z^n$. Every monomial module $M$ has a unique minimal generating set. Of special interest are the two cases when this generating set is finite or when it forms a group under multiplication. In the first case $M$ is isomorphic to a monomial ideal in $S$. In the second case $M$ coincides with the lattice module

$$M_L := S \{ x^a \mid a \in L \} = k \{ x^b \mid b \in \mathbb{N}^n + L \} \subset T,$$

for some sublattice $L \subset \mathbb{Z}^n$ whose intersection with $\mathbb{N}^n$ is the origin $0 = (0, \ldots, 0)$.

We shall derive free resolutions of $M$ from regular cell complexes whose vertices are the generators of $M$ and whose faces are labeled by the least common multiples of their vertices. The basic theory of such cellular resolutions is developed in Section 1.

Our main result is the construction of the hull resolution in Section 2. We rescale the exponents of the monomials in $M$, so that their convex hull in $\mathbb{R}^n$ is a polyhedron $P$ whose bounded faces support a free resolution of $M$. This resolution is new and interesting even for monomial ideals. It need not be minimal, but, unlike minimal resolutions, it respects symmetry and is free from arbitrary choices.

In Section 3 we relate the lattice module $M_L$ to the $\mathbb{Z}^n/L$-graded lattice ideal

$$I_L = \langle x^a - x^b \mid a - b \in L \rangle \subset S.$$

This class of ideals includes ideals defining toric varieties. We express the cyclic $S$-module $S/I_L$ as the quotient of the infinitely generated $S$-module $M_L$ by the action of $L$. In fact, we like to think of $M_L$ as the “universal cover” of $I_L$. Many questions about $I_L$ can thus be reduced to questions about $M_L$. In particular, we obtain the hull resolution of a lattice ideal $I_L$ by taking the hull resolution of $M_L$ modulo $L$.

This paper is inspired by the work of Barany, Howe and Scarf [BHS] who introduced the polyhedron $P$ in the context of integer programming. The hull resolution generalizes results in [BPS] for generic monomial ideals and in [PS] for generic lattice ideals. In these generic cases the hull resolution is minimal. We remark that deriving algebraic syzygies from polytopes is a natural operation also in other contexts such as associative algebras [AB] and algebraic K-theory [KS].
1 Cellular resolutions

Fix a subset \( \{ a_j \mid j \in I \} \subset \mathbb{Z}^n \), for an index set \( I \) which need not be finite, and let \( M \) be the monomial module generated by the monomials \( m_j = x^{a_j}, j \in I \). Let \( X \) be a regular cell complex having \( I \) as its set of vertices, and equipped with a choice of an incidence function \( \varepsilon(F,F') \) on pairs of faces. We recall from [BH, Section 6.2] that \( \varepsilon \) takes values in \( \{0,1,-1\} \), that \( \varepsilon(F,F') = 0 \) unless \( F' \) is a facet of \( F \), that \( \varepsilon(\{j\},\emptyset) = 1 \) for all vertices \( j \in I \), and that for any codimension 2 face \( F' \) of \( F \),

\[
\varepsilon(F,F_1)\varepsilon(F_1,F') + \varepsilon(F,F_2)\varepsilon(F_2,F') = 0
\]

where \( F_1, F_2 \) are the two facets of \( F \) containing \( F' \). The prototype of a regular cell complex is the set of faces of a convex polytope. The incidence function \( \varepsilon \) defines a differential \( \partial \) which can be used to compute the homology of \( X \). Define the augmented oriented chain complex \( \tilde{C}(X;k) = \bigoplus_{F \in X} kF \), with differential

\[
\partial F = \sum_{F' \in X} \varepsilon(F,F') F'.
\]

The reduced cellular homology group \( \tilde{H}_i(X;k) \) is the \( i \)th homology of \( \tilde{C}(X;k) \), where faces of \( X \) are indexed by their dimension. The oriented chain complex \( C(X;k) = \bigoplus_{F \in X, F \neq \emptyset} kF \) is obtained from \( \tilde{C}(X;k) \) by dropping the contribution of the empty face. It computes the ordinary homology groups \( H_i(X;k) \) of \( X \).

The cell complex \( X \) inherits a \( \mathbb{Z}^n \)-grading from the generators of \( M \) as follows. Let \( F \) be a nonempty face of \( X \). We identify \( F \) with its set of vertices, a finite subset of \( I \). Set \( m_F := \text{lc}(\{ m_j \mid j \in F \}) \). The exponent vector of the monomial \( m_F \) is the join \( a_F := \bigvee \{ a_j \mid j \in F \} \) in \( \mathbb{Z}^n \). We call \( a_F \) the degree of the face \( F \).

Homogenizing the differential \( \partial \) of \( C(X;k) \) yields a \( \mathbb{Z}^n \)-graded chain complex of \( S \)-modules. Let \( SF \) be the free \( S \)-module with one generator \( F \) in degree \( a_F \). The cellular complex \( F_X \) is the \( \mathbb{Z}^n \)-graded \( S \)-module \( \bigoplus_{F \in X, F \neq \emptyset} SF \) with differential

\[
\partial F = \sum_{F' \in X, F' \neq \emptyset} \varepsilon(F,F') \frac{m_F}{m_{F'}} F'.
\]

The homological degree of each face \( F \) of \( X \) is its dimension.

For each degree \( b \in \mathbb{Z}^n \), let \( X_{\leq b} \) be the subcomplex of \( X \) on the vertices of degree \( \leq b \), and let \( X_{< b} \) be the subcomplex of \( X_{\leq b} \) obtained by deleting the faces of degree \( b \). For example, if there is a unique vertex \( j \) of degree \( \leq b \), and \( a_j = b \), then \( X_{\leq b} = \{ j \}, X_{< b} = \{ \emptyset \} \). A full subcomplex on no vertices is the acyclic complex \( \{ \} \), so if there are no vertices of degree \( \leq b \), then \( X_{\leq b} = X_{< b} = \{ \} \).

The following proposition generalizes [BPS, Lemma 2.1] to cell complexes:

**Proposition 1.1** The complex \( F_X \) is a free resolution of \( M \) if and only if \( X_{< b} \) is acyclic over \( k \) for all degrees \( b \). In this case we call \( F_X \) a cellular resolution of \( M \).
Proof. The complex $F_X$ is $\mathbb{Z}^n$-graded. The degree $b$ part of $F_X$ is precisely the oriented chain complex $C(X \subseteq b; k)$. Hence $F_X$ is a free resolution of $M$ if and only if $H_0(X \subseteq b; k) \cong k$ for $x^b \in M$, and otherwise $H_i(X \subseteq b; k) = 0$ for all $i$ and all $b$. This condition is equivalent to $\tilde{H}_i(X \subseteq b; k) = 0$ for all $i$ (since $x^b \in M$ if and only if $\emptyset \in X \subseteq b$) and thus to $X \subseteq b$ being acyclic.

Remark 1.2 Fix $b \in \mathbb{Z}^n$. The set of generators of $M$ of degree $\preceq b$ is finite. It generates a monomial module $M_{\preceq b}$ isomorphic to an ideal (up to a degree shift).

Corollary 1.3 The cellular complex $F_X$ is a resolution of $M$ if and only if the cellular complex $F_{X_{\preceq b}}$ is a resolution of the monomial ideal $M_{\preceq b}$ for all $b \in \mathbb{Z}^n$.

Proof. This follows from Proposition 1.1 and the identity $(X_{\preceq b})_{\preceq c} = X_{\preceq b \wedge c}$.

Remark 1.4 A cellular resolution $F_X$ is a minimal resolution if and only if any two comparable faces $F' \subset F$ of the complex $X$ have distinct degrees $a_F \neq a_{F'}$.

The simplest example of a cellular resolution is the Taylor resolution for monomial ideals [Tay]. The Taylor resolution is easily generalized to arbitrary monomial modules $M$ as follows. Let $\{m_j \mid j \in I\}$ be the minimal generating set of $M$. Define $\Delta$ to be the simplicial complex consisting of all finite subsets of $I$, equipped with the standard incidence function $\varepsilon(F, F') = (-1)^j$ if $F \setminus F'$ consists of the $j$th element of $F$. The Taylor complex of $M$ is the cellular complex $F_\Delta$.

Proposition 1.5 The Taylor complex $F_\Delta$ is a resolution of $M$.

Proof. By Proposition 1.1 we need to show that each subcomplex $\Delta_{\preceq b}$ of $\Delta$ is acyclic. $\Delta_{\preceq b}$ is the full simplex on the set of vertices $\{j \in I \mid a_j \preceq b\}$. This set is finite by Remark 1.2. Hence $\Delta_{\preceq b}$ is a finite simplex, which is acyclic.

The Taylor resolution $F_\Delta$ is typically far from minimal. If $M$ is infinitely generated, then $\Delta$ has faces of arbitrary dimension and $F_\Delta$ has infinite length. Following [BPS, §2] we note that every simplicial complex $X \subset \Delta$ defines a submodule $F_X \subset F_\Delta$ which is closed under the differential $\partial$. We call $F_X$ the restricted Taylor complex supported on $X$. $F_X$ is a resolution of $M$ if and only if the hypothesis of Proposition 1.1 holds, with cellular homology specializing to simplicial homology.

Example 1.6 Consider the monomial ideal $M = \langle a^2b, ac, b^2, bc^2 \rangle$ in $S = k[a, b, c]$. Figure 1 shows a truncated “staircase diagram” of unit cubes representing the monomials in $S \setminus M$, and shows two simplicial complexes $X$ and $Y$ on the generators of $M$. Both are two triangles sharing an edge. Each vertex, edge or triangle is labeled by its degree. The notation $210$, for example, represents the degree $(2, 1, 0)$ of $a^2b$. 


By Proposition 1.1, the complex $X$ supports the minimal free resolution $F_X = 0 \to S^2 \to S^{a^2b} \to S^{bc^2} \to M \to 0$. The complex $Y$ fails the criterion of Proposition 1.1, and hence $F_Y$ is not exact: if $b = (1, 2, 1)$ then $Y \preceq b$ consists of the two vertices $ac$ and $b^2$, and is not acyclic.

We next present four examples which are not restricted Taylor complexes.

**Example 1.7** Let $M$ be a Gorenstein ideal of height 3 generated by $m$ monomials. It is shown in [BH1, §6] that the minimal free resolution of $M$ is the cellular resolution $F_X : 0 \to S \to S \to S \to S \to 0$ supported on a convex $m$-gon.

**Example 1.8** A monomial ideal $M$ is co-generic if its no variable occurs to the same power in two distinct irreducible components $\langle x_{r_1}^{i_1}, x_{r_2}^{i_2}, \ldots, x_{r_s}^{i_s} \rangle$ of $M$. It is shown in [Stu2] that the minimal resolution of a co-generic monomial ideal is a cellular resolution $F_X$ where $X$ is the complex of bounded faces of a simple polyhedron.

**Example 1.9** Let $u_1, \ldots, u_n$ be distinct integers and $M$ the module generated by the $n!$ monomials $x_{\pi(1)}^{u_1}x_{\pi(2)}^{u_2}\cdots x_{\pi(n)}^{u_n}$ where $\pi$ runs over all permutations of $\{1, 2, \ldots, n\}$. Let $X$ be the complex of all faces of the permutohedron [Zie, Example 0.10], which is the convex hull of the $n!$ vectors $(\pi(1), \ldots, \pi(n))$ in $\mathbb{R}^n$. It is known [BLSWZ, Exercise 2.9] that the $i$-faces $F$ of $X$ are indexed by chains

$$\emptyset = A_0 \subset A_1 \subset \ldots \subset A_{n-i} \subset A_{n-i} = \{u_1, u_2, \ldots, u_n\}.$$  

We assign the following monomial degree to the $i$-face $F$ indexed by this chain:

$$x^a_F = \prod_{j=1}^{n-i} \prod_{r \in A_j \setminus A_{j-1}} x_r^{\max\{A_j \setminus A_{j-1}\}}.$$  

It can be checked (using our results in §2) that the conditions in Proposition 1.1 and Remark 1.4 are satisfied. Hence $F_X$ is the minimal free resolution of $M$.  

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Figure 1
Example 1.10  Let $S = k[a, b, c, d, e, f]$. Following [BH, page 228] we consider the Stanley-Reisner ideal of the minimal triangulation of the real projective plane $\mathbb{RP}^2$,

$$M = \langle abc, abf, ace, ade, bcd, bde, bef, cdf, cef \rangle.$$ 

The dual in $\mathbb{RP}^2$ of this triangulation is a cell complex $X$ consisting of six pentagons. The ten vertices of $X$ are labeled by the generators of $M$. We illustrate $X \simeq \mathbb{RP}^2$ as the disk shown on the left in Figure 2; antipodal points on the boundary are to be identified. The small pictures on the right will be discussed in Example 2.14.

![Figure 2](image)

If $\text{char } k \neq 2$ then $X$ is acyclic over $k$ and the cellular complex $F_X$ coincides with the minimal free resolution $0 \to S^6 \to S^{15} \to S^{10} \to M$. If $\text{char } k = 2$ then $X$ is not acyclic over $k$, and the cellular complex $F_X$ is not a resolution of $M$.  

Returning to the general theory, we next present a formula for the Betti number $\beta_{i, b} = \dim \text{Tor}_i(M, k)_b$ which is the number of minimal $i$th syzygies in degree $b$. The degree $b \in \mathbb{Z}^n$ is called a Betti degree of $M$ if $\beta_{i, b} \neq 0$ for some $i$.

**Theorem 1.11**  If $F_X$ is a cellular resolution of a monomial module $M$ then

$$\beta_{i, b} = \dim H_i(X_{\leq b}, X_{< b}; k) = \dim \tilde{H}_{i-1}(X_{< b}; k),$$

where $H_*$ denotes relative homology and $\tilde{H}_*$ denotes reduced homology.

**Proof.** We compute $\text{Tor}_i(M, k)_b$ as the $i$th homology of the complex of vector spaces $(F_X \otimes_S k)_b$. This complex equals the chain complex $\tilde{C}(X_{\leq b}, X_{< b}; k)$ which computes the relative homology with coefficients in $k$ of the pair $(X_{\leq b}, X_{< b})$. Thus

$$\text{Tor}_i(M, k)_b = H_i(X_{\leq b}, X_{< b}; k).$$

Since $X_{\leq b}$ is acyclic, the long exact sequence of homology groups looks like

$$0 = \tilde{H}_i(X_{\leq b}; k) \to H_i(X_{\leq b}, X_{< b}; k) \to \tilde{H}_{i-1}(X_{< b}; k) \to \tilde{H}_{i-1}(X_{\leq b}; k) = 0.$$ 

We conclude that the two vector spaces in the middle are isomorphic.
A subset $Q \subset \mathbb{Z}^n$ is an order ideal if $b \in Q$ and $c \in \mathbb{N}^n$ implies $b - c \in Q$. For a $\mathbb{Z}^n$-graded cell complex $X$ and an order ideal $Q$ we define the order ideal complex $X_Q = \{ F \in X \mid a_F \in Q \}$. Note that $X_{<b}$ and $X_{\leq b}$ are special cases of this.

**Corollary 1.12** If $F_X$ is a cellular resolution of $M$ and $Q \subset \mathbb{Z}^n$ an order ideal which contains the Betti degrees of $M$, then $F_{X_Q}$ is also a cellular resolution of $M$.

**Proof.** By Corollary 1.3 and the identity $(X_Q)_{\leq b} = (X_{\leq b})_Q$, it suffices to prove this for the case where $M$ is a monomial ideal and $X$ is finite. We proceed by induction on the number of faces in $X \setminus X_Q$. If $X_Q = X$ there is nothing to prove. Otherwise pick $c \in \mathbb{Z}^n \setminus Q$ such that $X_{\leq c} = X$ and $X_{<c} \neq X$. Since $c$ is not a Betti degree, Theorem 1.11 implies that the complex $X_{<c}$ is acyclic. For any $b \in \mathbb{Z}^n$, the complex $(X_{<c})_{\leq b}$ equals either $X_{<c}$ or $X_{\leq b \wedge c}$ and is hence acyclic. At this point we replace $X$ by the proper subcomplex $X_{<c}$, and we are done by induction. \[\Box\]

By Proposition 1.5 and Theorem 1.11, the Betti numbers $\beta_{i,b}$ of $M$ are given by the reduced homology of $\Delta_{<b}$. Let us compare that formula for $\beta_{i,b}$ with the following formula which is due independently to Hochster [Ho] and Rosenknop [Ros].

**Corollary 1.13** The Betti numbers of $M$ satisfy $\beta_{i,b} = \dim \tilde{H}_i(K_b;k)$ where $K_b$ is the simplicial complex $\{ \sigma \subseteq \{1, \ldots, n\} \mid M$ has a generator of degree $\preceq b - \sigma \}$. Here each face $\sigma$ of $K_b$ is identified with its characteristic vector in $\{0,1\}^n$.

**Proof.** For $i \in \{1, \ldots, n\}$ consider the subcomplex of $\Delta_{<b}$ consisting of all faces $F$ with degree $a_F \preceq b - \{i\}$. This subcomplex is a full simplex. Clearly, these $n$ simplices cover $\Delta_{<b}$. The nerve of this cover by contractible subsets is the simplicial complex $K_b$. Therefore, $K_b$ has the same reduced homology as $\Delta_{<b}$. \[\Box\]

## 2 The hull resolution

Let $M$ be a monomial module in $T = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. In this section we apply convexity methods to construct a canonical cellular resolution of $M$. For $a \in \mathbb{Z}^n$ and $t \in \mathbb{R}$ we abbreviate $t^a = (t^{a_1}, \ldots, t^{a_n})$. Fix any real number $t$ larger than $(n+1)! = 2 \cdot 3 \cdot \cdots \cdot (n+1)$. We define $P_t$ be the convex hull of the point set

$$\{ t^a \mid a \text{ is the exponent of a monomial } x^a \in M \} \subset \mathbb{R}^n.$$ 

The set $P_t$ is a closed, unbounded $n$-dimensional convex polyhedron.

**Proposition 2.1** The vertices of the polyhedron $P_t$ are precisely the points $t^a = (t^{a_1}, \ldots, t^{a_n})$ for which the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ is a minimal generator of $M$.

**Proof.** Suppose $x^a \in M$ is not a minimal generator of $M$. Then $M$ contains both $x^{a+e_i} = x^a x_i$ and $x^{a-e_i} = x^a / x_i$ for some $i$. The line segment $\text{conv}\{t^{a-e_i}, t^{a+e_i}\}$ lies in $P_t$ and contains $t^a$ in its relative interior. Therefore $t^a$ is not a vertex of $P_t$. 

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Next, suppose $x^a \in M$ is a minimal generator of $M$. Let $v = t^{-a}$, so $v \cdot t^a = n$. For any other exponent $b$ of a monomial in $M$, we have $b_i \geq a_i + 1$ for some $i$, so
\[
v \cdot t^b = \sum_{j=1}^{n} t^{b_j-a_j} \geq t^{b_i-a_i} \geq t > (n+1)! > n.
\]
Thus, the inner normal vector $v$ supports $t^a$ as a vertex of $P_t$.  

**Corollary 2.2**  $P_t = \mathbb{R}_+^n + \text{conv} \left\{ t^a \mid x^a \text{ is a minimal generator } x^a \text{ of } M \right\}$.

Our first goal is to establish the following combinatorial result.

**Theorem 2.3**  The face poset of the polyhedron $P_t$ is independent of $t$ for $t > (n+1)!$. The same holds for the subposet of all bounded faces of $P_t$.

**Proof.**  The face poset of $P_t$ can be computed as follows. Let $C_t \subseteq \mathbb{R}^{n+1}$ be the cone spanned by the vectors $(t^a, 1)$ for all minimal generators $x^a$ of $M$ together with the unit vectors $(e_i, 0)$ for $i = 1, \ldots, n$. The faces of $P_t$ are in order-preserving bijection with the faces of $C_t$ which do not lie in the hyperplane “at infinity” $x_{n+1} = 0$. A face of $P_t$ is bounded if and only if the corresponding face of $C_t$ contains none of the vectors $(e_i, 0)$. It suffices to prove that the face poset of $C_t$ is independent of $t$.

For any $(n+1)$-tuple of generators of $C_t$ consider the sign of their determinant
\[
\text{sign } \det \begin{bmatrix} e_{i_0} \cdots e_{i_r} & t^{a_{j_1}} & \cdots & t^{a_{j_{n-r}}} \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix} \in \{-1, 0, +1\}. \tag{1}
\]

The list of these signs forms the (possibly infinite) oriented matroid associated with the cone $C_t$. It is known (see e.g. [BLWSZ]) that the face poset of $C_t$ is determined by its oriented matroid. It therefore suffices to show that the sign of the determinant in (1) is independent of $t$ for $t > (n+1)!$. This follows from the next lemma.  

**Lemma 2.4**  Let $a_{ij}$ be integers for $1 \leq i, j \leq r$. Then the Laurent polynomial $f(t) = \det((t^{a_{ij}})_{1 \leq i, j \leq r})$ either vanishes identically or has no real roots for $t > r!$.

**Proof.**  Suppose that $f$ is not zero and write $f(t) = c_\alpha t^\alpha + \sum_\beta c_\beta t^\beta$, where the first term has the highest degree in $t$. For $t > r!$ we have the chain of inequalities
\[
|\sum_\beta c_\beta \cdot t^\beta| \leq \sum_\beta |c_\beta| \cdot t^\beta \leq (\sum_\beta |c_\beta|) \cdot t^{\alpha-1} < r! \cdot t^{\alpha-1} < t^\alpha \leq |c_\alpha \cdot t^\alpha|.
\]
Therefore $f(t)$ is nonzero, and $\text{sign}(f(t)) = \text{sign}(c_\alpha)$.  

In the proof of Theorem 2.3 we are using Lemma 2.4 for $r = n + 1$. Lev Borisov and Sorin Popescu constructed examples of matrices which show that the exponential lower bound for $t$ is necessary in Lemma 2.4, and also in Theorem 2.3.

We are now ready to define the hull resolution and state our main result. The hull complex of a monomial module $M$, denoted $\text{hull}(M)$, is the complex of bounded faces of the polyhedron $P_t$ for large $t$. Theorem 2.3 ensures that $\text{hull}(M)$ is well-defined and depends only on $M$. The vertices of $\text{hull}(M)$ are labeled by the generators of $M$, by Proposition 2.1, and hence the complex $\text{hull}(M)$ is $\mathbb{Z}^n$-graded. Let $F_{\text{hull}(M)}$ be the complex of free $S$-modules derived from $\text{hull}(M)$ as in Section 1.
Theorem 2.5  The cellular complex $F_{\text{hull}(M)}$ is a free resolution of $M$.

Proof. Let $X = (\text{hull}(M))_{\le b}$ for some degree $b$; by Proposition 1.1 we need to show that $X$ is acyclic. This is immediate if $X$ is empty or a single vertex. Otherwise choose $t > (n + 1)!$ and let $v = t^{-b}$. If $t^a$ is a vertex of $X$ then $a < b$, so
\[ v \cdot t^a = t^{-b} \cdot t^a < t^{-b} \cdot t^b = n, \]
while for any other $x^c \in M$ we have $c_i \ge b_i + 1$ for some $i$, so
\[ v \cdot t^a = t^{-b} \cdot t^c \ge t^{c_i-b_i} \ge t > n. \]
Thus, the hyperplane $H$ defined by $v \cdot x = n$ separates the vertices of $X$ from the remaining vertices of $P_t$. Make a projective transformation which moves $H$ to infinity. This expresses $X$ as the complex of bounded faces of a convex polyhedron, a complex which is known to be contractible, e.g. [BLSWZ, Exercise 4.27 (a)].

We call $F_{\text{hull}(M)}$ the hull resolution of $M$. Let us see that the hull resolution generalizes the Scarf complex introduced in [BPS]. This is the simplicial complex
\[ \Delta_M = \{ F \subseteq I \mid m_F \ne m_G \text{ for all } G \subseteq I \text{ other than } F \}. \]
The Scarf complex $\Delta_M$ defines a subcomplex $F_{\Delta_M}$ of the Taylor resolution $F_\Delta$.

Proposition 2.6  For any monomial module $M$, the Scarf complex $\Delta_M$ is a subcomplex of the hull complex $\text{hull}(M)$.

Proof. Let $F = \{ x^{a_1}, \ldots, x^{a_p} \}$ be a face of $\Delta_M$ with $m_F = \text{lcm}(F) = x^a$. Consider any injective map $\sigma : \{1,\ldots,p\} \to \{1,\ldots,n\}$ such that $a_{i,\sigma(i)} = u_i$ for all $i$. Compute the inverse of the $p \times n$-matrix $(t^{a_i-\sigma(i)})$, and let $v^{\sigma}(t)'$ be the sum of the column vectors of that inverse matrix. By augmenting the $p$-vector $v^{\sigma}(t)'$ with additional zero coordinates, we obtain an $n$-vector $v^{\sigma}(t)$ with the following properties:
(i) $v^{\sigma}(t) \cdot t^{a_1} = v^{\sigma}(t) \cdot t^{a_2} = \cdots = v^{\sigma}(t) \cdot t^{a_p} = 1$;
(ii) $v^j(t) = 0$, for all $j \notin \text{image}(\sigma)$;
(iii) $v_j(t) = t^{-u_j} + \text{lower order terms in } t$, for all $j \in \text{image}(\sigma)$.

By taking a convex combination of the vectors $v^{\sigma}(t)$ for all possible injective maps $\sigma$ as above, we obtain a vector $v(t)$ with the following properties:
(iv) $v(t) \cdot t^{a_1} = v(t) \cdot t^{a_2} = \cdots = v(t) \cdot t^{a_p} = 1$;
(v) $v_j(t) = c_j \cdot t^{-u_j} + \text{lower order terms in } t$ with $c_j > 0$, for all $j \in \{1,\ldots,n\}$.

For any $x^b \in M$ which is not in $F$ there exists an index $\ell$ such that $b_\ell \ge u_\ell + 1$. This implies $v(t) \cdot t^b \ge c_\ell \cdot t^{b_\ell - u_\ell} + \text{lower order terms in } t$, and therefore $v(t) \cdot t^b > 1$ for $t \gg 0$. We conclude that $F$ defines a face of $P_t$ with inner normal vector $v(t)$.

A binomial first syzygy of $M$ is called generic if it has full support, i.e., if no variable $x_i$ appears with the same exponent in the corresponding pair of monomial generators. We call $M$ generic if it has a basis of generic binomial first syzygies. This is a translation-invariant generalization of the definition of genericity in [BPS].
Lemma 2.7  If $M$ is generic, then for any pair of generators $m_i$, $m_j$ either the corresponding binomial first syzygy is generic, or there exists a third generator $m$ which strictly divides the least common multiple of $m_i$ and $m_j$ in all coordinates.

Proof. Suppose that the syzygy formed by $m_i$ and $m_j$ is not generic, and induct on the length of a chain of generic syzygies needed to express it. If the chain has length two, then the middle monomial $m$ divides lcm$(m_i, m_j)$. Moreover, because the two syzygies involving $m$ are generic, this division is strict in each variable. If the chain is longer, then divide it into two steps. Either each step represents a generic syzygy, and we use the above argument, or by induction there exists an $m_j$ strictly dividing the degree of one of these syzygies in all coordinates, and we are again done. ■

Lemma 2.8  Let $M$ be a monomial module and $F$ a face of hull$(M)$. For every monomial $m \in M$ there exists a variable $x_j$ such that deg$_{x_j}(m) \geq$ deg$_{x_j}(m_F)$.

Proof. Suppose that $m = x^u$ strictly divides $m_F$ in each coordinate. Let $t^{a_1}, \ldots, t^{a_p}$ be the vertices of $F$ and consider their barycenter $v(t) = \frac{1}{p} \cdot (t^{a_1} + \cdots + t^{a_p}) \in F$. The $j$th coordinate of $v(t)$ is a polynomial in $t$ of degree equal to deg$_{x_j}(m_F)$. The $j$th coordinate of $t^u$ is a monomial of strictly lower degree. Hence $t^u < v(t)$ coordinatewise for $t \gg 0$. Let $w$ be a nonzero linear functional which is nonnegative on $\mathbb{R}^n_+$ and whose minimum over $P_t$ is attained at the face $F$. Then $w \cdot v(t) = w \cdot a_1 = \cdots = w \cdot a_p$, but our discussion implies $w \cdot t^u < w \cdot v(t)$, a contradiction. ■

Theorem 2.9  If $M$ is a generic monomial module then hull$(M)$ coincides with the Scarf complex $\Delta_M$ of $M$, and the hull resolution $F_{\text{hull}(M)} = F_{\Delta_M}$ is minimal.

Proof. Let $F$ be any face of hull$(M)$ and $x^{a_1}, \ldots, x^{a_p}$ the generators of $M$ corresponding to the vertices of $F$. Suppose that $F$ is not a face of $\Delta_M$. Then either

(i) lcm$(x^{a_1}, \ldots, x^{a_{i-1}}, x^{a_{i+1}}, \ldots, x^{a_p}) = m_F$ for some $i \in \{1, \ldots, p\}$, or

(ii) there exists another generator $x^u$ of $M$ which divides $m_F$ and such that $t^u \notin F$.

Consider first case (i). By Lemma 2.8 applied to $m = x^{a_i}$ there exists $x_j$ such that deg$_{x_j}(x^{a_i}) =$ deg$_{x_j}(m_F)$, and hence deg$_{x_j}(x^{a_i}) =$ deg$_{x_j}(x^{a_k})$ for some $k \neq i$. The first syzygy between $x^{a_i}$ and $x^{a_k}$ is not generic, and, by Lemma 2.7, there exists a generator $m$ of $M$ which strictly divides lcm$(x^{a_i}, x^{a_k})$ in all coordinates. Since lcm$(x^{a_i}, x^{a_k})$ divides $m_F$, we get a contradiction to Lemma 2.8.

Consider now case (ii). For any variable $x_j$ there exists $i \in \{1, \ldots, p\}$ such that deg$_{x_j}(x^{a_i}) =$ deg$_{x_j}(m_F) \geq$ deg$_{x_j}(x^u)$. If the inequality is an equality $=$, then the first syzygy between $x^u$ and $x^{a_i}$ is not generic, and Lemma 2.7 gives a new monomial generator $m$ which strictly divides $m_F$ in all coordinates, a contradiction to Lemma 2.8. Therefore $\geq$ is a strict inequality $>$ for all variables $x_j$. This means that $x^u$ strictly divides $m_F$ in all coordinates, again a contradiction to Lemma 2.8.

Hence both (i) and (ii) lead to a contradiction, and we conclude that every face of hull$(M)$ is a face of $\Delta_M$. This implies hull$(M) = \Delta_M$ by Proposition 2.6. The resolution $F_{\Delta_M}$ is minimal because no two faces in $\Delta_M$ have the same degree. ■
In this paper we are mainly interested in nongeneric monomial modules for which the hull complex is typically not simplicial. Nevertheless the possible combinatorial types of facets seem to be rather limited. Experimental evidence suggests:

**Conjecture 2.10** Every face of $\text{hull}(M)$ is affinely isomorphic to a subpolytope of the $(n - 1)$-dimensional permutohedron and hence has at most $n!$ vertices.

By Example 1.9 it is easy to see that any subpolytope of the $(n - 1)$-dimensional permutohedron can be realized as the hull complex of suitable monomial ideal.

The following example, found in discussions with Lev Borisov, shows that the hull complex of a monomial module need not be locally finite:

**Example 2.11** Let $n = 3$ and $M$ the monomial module generated by $x_1^{-1}x_2$ and \{ $x_1^i x_3^{-i} \mid i \in \mathbb{Z}$ \}. Then every triangle of the form \{ $x_1^{-1}x_2$, $x_1^i x_3^{-i}$, $x_2 x_3^{-i-1}$ \} is a facet of $\text{hull}(M)$. In particular, the vertex $x_1^{-1}x_2$ of $\text{hull}(M)$ has infinite valence.

For a generic monomial module $M$ we have the following important identity

$$\text{hull}(\text{hull}(M) \leq b) = \text{hull}(M) \leq b.$$ 

See equation (5.1) in [BPS]. This identity can fail if $M$ is not generic:

**Example 2.12** Consider the monomial ideal $M = \langle a^2 b, ac, b^2, bc^2 \rangle$ studied in Example 1.6 and let $b = (2, 1, 2)$. Then $\text{hull}(M \leq b)$ is a triangle, while $\text{hull}(M) \leq b$ consists of two edges. The vertex $b^2$ of $\text{hull}(M)$ “eclipses” the facet of $\text{hull}(M \leq b)$.

The hull complex $\text{hull}(M)$ is particularly easy to compute if $M$ is a squarefree monomial ideal. In this case we have $P_t = P_1$ for all $t$. Moreover, if all square-free generators of $M$ have the same total degree, then the faces of their convex hull are precisely the bounded faces of $P_t$. Theorem 2.5 implies the following corollary.

**Corollary 2.13** Let $a_1, \ldots, a_p$ be 0-1-vectors having the same coordinate sum. Then their boundary complex, consisting of all faces of the convex polytope $P = \text{conv}\{a_1, \ldots, a_p\}$, defines a cellular resolution of the ideal $M = \langle x^{a_1}, \ldots, x^{a_p} \rangle$.

**Example 2.14** Corollary 2.13 applies to the Stanley-Reisner ideal of the real projective plane in Example 1.10. Here $P$ is a 5-dimensional polytope with 22 facets, corresponding to the 22 cycles on the 2-complex $X$ of length $\leq 6$. Representatives of these three cycle types, and supporting hyperplanes of the corresponding facets of $P$, are shown on the right in Figure 2. This example illustrates how the hull resolution encodes combinatorial information without making arbitrary choices.
3 Lattice ideals

Let $L \subseteq \mathbb{Z}^n$ be a lattice. In this section we study (cellular) resolutions of the lattice module $M_L$ and of the lattice ideal $I_L$. Let $S[L]$ be the group algebra of $L$ over $S$. We realize $S[L]$ as the subalgebra of $k[x_1, \ldots, x_n, z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ spanned by all monomials $x^a z^b$ where $a \in \mathbb{N}^n$ and $b \in L$. Note that $S = S[L]/(z^a - 1 \mid a \in L)$.

**Lemma 3.1** The lattice module $M_L$ is an $S[L]$-module, and $M_L \otimes_{S[L]} S = S/I_L$.

**Proof.** The $k$-linear map $\phi : S[L] \to M_L$, $x^a z^b \mapsto x^a z^b$ defines the structure of an $S[L]$-module on $M_L$. Its kernel $\ker(\phi)$ is the ideal in $S[L]$ generated by all binomials $x^u - x^v z^{u-v}$ where $u, v \in \mathbb{N}^n$ and $u - v \in L$. Clearly, we obtain $I_L$ from $\ker(\phi)$ by setting all $z$-variables to 1, and hence $(S[L]/\ker(\phi)) \otimes_{S[L]} S = S/I_L$. □

We define a $\mathbb{Z}^n$-grading on $S[L]$ via $\deg(x^a z^b) = a + b$. Let $\mathcal{A}$ be the category of $\mathbb{Z}^n$-graded $S[L]$-modules, where the morphisms are $\mathbb{Z}^n$-graded $S[L]$-module homomorphisms of degree 0. The polynomial ring $S = k[x_1, \ldots, x_n]$ is graded by the quotient group $\mathbb{Z}^n/L$ via $\deg(x^a) = a + L$. Let $\mathcal{B}$ be the category of $\mathbb{Z}^n/L$-graded $S$-modules, where the morphisms are $\mathbb{Z}^n/L$-graded $S$-module homomorphisms of degree 0. Clearly, $M_L$ is an object in $\mathcal{A}$, and $M_L \otimes_{S[L]} S = S/I_L$ is an object in $\mathcal{B}$.

**Theorem 3.2** The categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

**Proof.** Define a functor $\pi : \mathcal{A} \to \mathcal{B}$ by the rule $\pi(M) := M \otimes_{S[L]} S$. This functor weakens the $\mathbb{Z}^n$-grading of objects in $\mathcal{A}$ to a $\mathbb{Z}^n/L$-grading. The properties of $\pi$ cannot be deduced from the tensor product alone, which is poorly behaved when applied to arbitrary $S[L]$-modules; e.g., $S$ is not a flat $S[L]$-module. Further, the categories $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic; we are only claiming that they are equivalent.

We apply condition iii) of [Mac, §IV.4, Theorem 1]: It is enough to prove that $\pi$ is full and faithful, and that each object $N \in \mathcal{B}$ is isomorphic to $\pi(M)$ for some object $M \in \mathcal{A}$. To prove that $\pi$ is full and faithful, we show that for any two modules $M, M' \in \mathcal{A}$ it induces an identification $\text{Hom}_\mathcal{A}(M, M') = \text{Hom}_\mathcal{B}(\pi(M), \pi(M'))$.

Because each module $M \in \mathcal{A}$ is $\mathbb{Z}^n$-graded, the lattice $L \subseteq S[L]$ acts on $M$ as a group of automorphisms, i.e. the multiplication maps $z^b : M_a \to M_{a+b}$ are isomorphisms of $k$-vector spaces for each $b \in L$, compatible with multiplication by each $x_i$. For each $a \in \mathbb{Z}^n/L$, the functor $\pi$ identifies the spaces $M_a$ for $a \in \alpha$ as the single space $\pi(M)_\alpha$. A morphism $f : M \to M'$ in $\mathcal{A}$ is a collection of $k$-linear maps $f_a : M_a \to M'_a$, compatible with the action by $L$ and with multiplication by each $x_i$. A morphism $g : \pi(M) \to \pi(M')$ in $\mathcal{B}$ is a collection of $k$-linear maps $g_\alpha : \pi(M)_\alpha \to \pi(M')_\alpha$, compatible with multiplication by each $x_i$. For each $a \in \mathbb{Z}^n/L$, the functor $\pi$ identifies the maps $f_a$ for $a \in \alpha$ as the single map $\pi(f)_\alpha$.

It is clear from this discussion that $\pi$ takes distinct morphisms to distinct morphisms. Given a morphism $g \in \text{Hom}_\mathcal{B}(\pi(M), \pi(M'))$, define a morphism $f \in \text{Hom}_\mathcal{A}(M, M')$ by the rule $f_a = g_a$ for $a \in \alpha$. We have $\pi(f) = g$, establishing the desired identification of Hom-sets. Hence $\pi$ is full and faithful.
Finally, let \( N = \bigoplus_{\alpha \in \mathbb{Z}^n/L} N_\alpha \) be any object in \( \mathcal{B} \). We define an object \( M = \bigoplus_{a \in \mathbb{Z}^n} M_a \) in \( \mathcal{A} \) by setting \( M_a := N_\alpha \) for each \( a \in \alpha \), by lifting each multiplication map \( x_i : N_\alpha \to N_{a, e_i} \) to maps \( x_i : M_a \to M_{a+e_i} \) for \( a \in \alpha \), and by letting \( z_b \) act on \( M \) as the identity map from \( M_a \) to \( M_{a+b} \) for \( b \in L \). The module \( M \) satisfies \( \pi(M) = N \), showing that \( \pi \) is an equivalence of categories. ■

Theorem 3.2 allows us to resolve the lattice module \( M_L \in \mathcal{A} \) in order to resolve the quotient ring \( \pi(M_L) = S/I_L \in \mathcal{B} \), and conversely.

**Corollary 3.3** A \( \mathbb{Z}^n \)-graded complex of free \( S[L] \)-modules,

\[
C : \quad \ldots \xrightarrow{f_2} S[L]^{\beta_1} \xrightarrow{f_1} S[L]^{\beta_0} \xrightarrow{f_0} S[L] \to M_L \to 0,
\]

is a (minimal) free resolution of \( M_L \) if and only if its image

\[
\pi(C) : \quad \ldots \xrightarrow{\pi(f_2)} S^{\beta_1} \xrightarrow{\pi(f_1)} S^{\beta_0} \xrightarrow{\pi(f_0)} S \to S/I_L \to 0,
\]

is a (minimal) \( \mathbb{Z}^n/L \)-graded resolution of \( S/I_L \) by free \( S \)-modules.

**Proof.** This follows immediately from Lemma 3.1 and Theorem 3.2. ■

Since \( S[L] \) is a free \( S \)-module, every resolution \( C \) as in the previous corollary gives rise to a resolution of \( M_L \) as a \( \mathbb{Z}^n \)-graded \( S \)-module. We demonstrate in an example how resolutions of \( M_L \) over \( S \) are derived from resolutions of \( S/I_L \) over \( S \).

**Example 3.4** Let \( S = k[x_1, x_2, x_3] \) and \( L = \ker \{ 1 \ 1 \ 1 \} \subset \mathbb{Z}^3 \). Then \( \mathbb{Z}^3/L \simeq \mathbb{Z}, I_L = \langle x_1 - x_2, x_2 - x_3 \rangle \), and \( M_L \) is the module generated by all monomials of the form \( x_1^i x_2^j x_3^{k-1} \). The ring \( S/I_L \) is resolved by the Koszul complex

\[
0 \to S(-2) \xrightarrow{\begin{bmatrix} x_2 - x_3 \\ x_2 - x_1 \end{bmatrix}} S(-1)^2 \xrightarrow{\begin{bmatrix} x_1 - x_2 & x_2 - x_3 \end{bmatrix}} S \to S/I_L.
\]

This is a \( \mathbb{Z}^3/L \)-graded complex of free \( S \)-modules. An inverse image under \( \pi \) equals

\[
0 \to S[L](-1, 1, 0) \xrightarrow{\begin{bmatrix} x_2 - x_3 z_1 z_2^{-1} \\ x_2 - x_1 z_2 z_3^{-1} \end{bmatrix}} S[L](-1, 0, 0) \oplus S[L](-0, 1, 0) \xrightarrow{\begin{bmatrix} x_1 - x_2 z_1 z_3^{-1} & x_2 - x_3 z_2 z_3^{-1} \end{bmatrix}} S[L] \to M_L.
\]

Writing each term as a direct sum of free \( S \)-modules, for instance, \( S[L](-1, 1, 0) = \bigoplus_{i+j+k=2} S(-i, j, k) \), we get a \( \mathbb{Z}^3 \)-graded minimal free resolution of \( M_L \) over \( S \):

\[
0 \to \bigoplus_{i+j+k=2} S(-i, j, k) \to \bigoplus_{i+j+k=1} S(-i, j, k)^2 \to \bigoplus_{i+j+k=0} S(-i, j, k) \to M_L.
\]

Our goal is to define and study cellular resolutions of the lattice ideal \( I_L \). Let \( X \) be a \( \mathbb{Z}^n \)-graded cell complex whose vertices are the generators of \( M_L \). Each cell \( F \in X \) is identified with its set of vertices, regarded as a subset of \( L \). The cell complex \( X \) is called *equivariant* if \( F \in X \) and \( b \in L \) implies that \( F + b \in X \), and if the incidence function satisfies \( \varepsilon(F, F') = \varepsilon(F + b, F' + b) \) for all \( b \in L \).
Lemma 3.5  If $X$ is an equivariant $\mathbb{Z}^n$-graded cell complex on $M_L$ then the cellular complex $F_X$ has the structure of a $\mathbb{Z}^n$-graded complex of free $S[L]$-modules.

Proof. The group $L$ acts on the faces of $X$. Let $X/L$ denote the set of orbits. For each orbit $F \in X/L$ we select a distinguished representative $F \in F$, and we write $\text{Rep}(X/L)$ for the set of representatives. The following map is an isomorphism of $\mathbb{Z}^n$-graded $S$-modules, which defines the structure of a free $S[L]$-module on $F_X$:

\[
\bigoplus_{F \in \text{Rep}(X/L)} S[L] \cdot e_F \cong \bigoplus_{F \in X} S \cdot e_F = F_X, \quad z^b \cdot e_F \mapsto e_{F+b}.
\]

The differential $\partial$ on $F_X$ is compatible with the $S[L]$-action on $F_X$ because the incidence function is $L$-invariant. For each $F \in \text{Rep}(X/L)$ and $b \in L$ we have

\[
\partial(z^b \cdot e_F) = \partial(e_{F+b}) = \sum_{F' \in X, F' \neq \emptyset} \varepsilon(F+b, F'+b) \frac{m_{F'+b}}{m_{F+b}} e_{F'+b} \varepsilon(F, F') \frac{m_F}{m_{F'}} z^b \cdot e_{F'} = z^b \cdot \partial(e_F).
\]

Clearly, the differential $\partial$ is homogeneous of degree 0, which proves the claim. ■

Corollary 3.6  If $X$ is an equivariant $\mathbb{Z}^n$-graded cell complex on $M_L$ then the cellular complex $F_X$ is exact over $S$ if and only if it is exact over $S[L]$.

Proof. The $\mathbb{Z}^n$-graded components of $F_X$ are complexes of $k$-vector spaces which are independent of our interpretation of $F_X$ as an $S$-module or $S[L]$-module. ■

If $X$ is an equivariant $\mathbb{Z}^n$-graded cell complex on $M_L$ such that $F_X$ is exact, then we call $F_X$ an equivariant cellular resolution of $M_L$.

Corollary 3.7  If $F_X$ is an equivariant cellular (minimal) resolution of $M_L$ then $\pi(F_X)$ is a (minimal) resolution of $S/I_L$ by $\mathbb{Z}^n/L$-graded free $S$-modules. ■

We call $\pi(F_X)$ a cellular resolution of the lattice ideal $I_L$. Let $Q$ be an order ideal in the quotient poset $\mathbb{N}^n/L$. Then $Q+L$ is an order ideal in $\mathbb{N}^n + L$, and the restriction $F_{X_{Q+L}}$ is a complex of $\mathbb{Z}^n$-graded free $S[L]$-modules. We set $\pi(F_X)_Q := \pi(F_{X_{Q+L}})$. This is a complex of $\mathbb{Z}^n/L$-graded free $S$-modules. Corollary 1.12 implies

Proposition 3.8  If $\pi(F_X)$ is a cellular resolution of $I_L$ and $Q$ is an order ideal in $\mathbb{N}^n/L$ which contains all Betti degrees then $\pi(F_X)_Q$ is a cellular resolution of $I_L$.

In what follows we shall study two particular cellular resolutions of $I_L$.

Theorem 3.9  The Taylor complex $\Delta$ on $M_L$ and the hull complex $\text{hull}(M_L)$ are equivariant. They define cellular resolutions $\pi(F_\Delta)$ and $\pi(F_{\text{hull}(M_L)})$ of $I_L$. 

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Proof. The Taylor complex $\Delta$ consists of all finite subsets of generators of $M_L$. It has an obvious $L$-action. The hull complex also has an $L$-action: if $F = \text{conv}(\{t^{a_1}, \ldots, t^{a_n}\})$ is a face of hull$(M_L)$ then $z^b \cdot F = \text{conv}(\{t^{a_1+b}, \ldots, t^{a_n+b}\})$ is also a face of hull$(M_L)$ for all $b \in L$. In both cases the incidence function $\varepsilon$ is defined uniquely by the ordering of the elements in $L$. To ensure that $\varepsilon$ is $L$-invariant, we fix an ordering which is $L$-invariant; for instance, order the elements of $L$ by the value of an $\mathbb{R}$-linear functional whose coordinates are $\mathbb{Q}$-linearly independent.

Both $\pi(F_\Delta)$ and $\pi(F_{\text{hull}(M_L)})$ are cellular resolutions of $I_L$ by Corollary 3.7.

The Taylor resolution $\pi(F_\Delta)$ of $I_L$ has the following explicit description. For $\alpha \in \mathbb{N}^n/L$ let $\text{fiber}(\alpha)$ denote the (finite) set of all monomials $x^b$ with $b \in \alpha$. Thus $S_\alpha = k \cdot \text{fiber}(\alpha)$. Let $E_i(\alpha)$ be the collection of all $i$-element subsets $I$ of fiber$(\alpha)$ whose greatest common divisor $\gcd(I)$ equals 1. For $I \in E_i(\alpha)$ set $\text{deg}(I) := \alpha$.

**Proposition 3.10** The Taylor resolution $\pi(F_\Delta)$ of a lattice ideal $I_L$ is isomorphic to the $\mathbb{Z}^n/L$-graded free $S$-module $\bigoplus_{\alpha \in \mathbb{N}^n/L} S \cdot E_i(\alpha)$ with the differential

$$
\partial(I) = \sum_{m \in I} \text{sign}(m, I) \cdot \gcd(I\setminus\{m\}) \cdot [I\setminus\{m\}].
$$

In this formula, $[I\setminus\{m\}]$ denotes the element of $E_i-1(\alpha - \text{deg}(\gcd(I\setminus\{m\})))$ which is obtained from $I\setminus\{m\}$ by removing the common factor $\gcd(I\setminus\{m\})$.

**Proof.** For $b \in \mathbb{Z}^n$ let $F_i(b)$ denote the collection of $i$-element subsets of generators of $M_L$ whose least common multiple equals $b$. For $J \in F_i(b)$ we have $\text{lcm}(J) = x^b$. The Taylor resolution $F_\Delta$ of $M_L$ equals $\bigoplus_{b \in \mathbb{Z}^n/L} S \cdot F_i(b)$ with differential

$$
\partial(J) = \sum_{m \in J} \text{sign}(m, J) \cdot \frac{\text{lcm}(J)}{\text{lcm}(J\setminus\{m\})} \cdot J\setminus\{m\}.
$$

There is a natural bijection between $F_i(b)$ and $E_i(b + L)$, namely, $J \mapsto \{x^b/x^c \mid x^c \in J\} = I$. Under this bijection we have $\frac{x^b}{\text{lcm}(J\setminus\{m\})} = \gcd(I\setminus\{m\})$.

The functor $\pi$ identifies each $F_i(b)$ with $E_i(b + L)$ and it takes (3.2) to (3.1).

**Corollary 3.11** Let $Q$ be an order ideal in $\mathbb{N}^n/L$ which contains all Betti degrees. Then $\pi(F_\Delta)_Q = \bigoplus_{\alpha \in Q} S E_i(\alpha)$ with differential (3.1) is a cellular resolution of $I_L$.

**Proof.** This follows from Proposition 3.8, Theorem 3.9 and Proposition 3.10.

**Example 3.12** (Generic lattice ideals) The lattice module $M_L$ is generic (in the sense of §2) if and only if the ideal $I_L$ is generated by binomials with full support. Suppose that this holds. It was shown in [PS] that the Betti degrees of $I_L$ form an order ideal $Q$ in $\mathbb{N}^n/L$. Theorem 2.9 and Proposition 3.8 imply that the resolution $\pi(F_\Delta)_Q$ is minimal and coincides with the hull resolution $\pi(F_{\text{hull}(M_L)})$. 

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The remainder of this section is devoted to the hull resolution of $I_L$. We next show that the hull complex $\text{hull}(M_L)$ is locally finite. This fact is nontrivial, in view of Example 2.11. It will imply that the hull resolution has finite rank over $S$.

Write each vector $a \in L \subseteq \mathbb{Z}^n$ as difference $a = a^+ - a^-$ of two nonnegative vectors with disjoint support. A nonzero vector $a \in L$ is called primitive if there is no vector $b \in L \setminus \{a, 0\}$ such that $b^+ \leq a^+$ and $b^- \leq a^-$. The set of primitive vectors is known to be finite [St, Theorem 4.7]. The set of binomials $x^a^+ - x^a^-$ were $a$ runs over all primitive vectors in $L$ is called the Graver basis of the ideal $I_L$. The Graver basis contains the universal Gröbner basis of $I_L$ [St, Lemma 4.6].

**Lemma 3.13** If $\{0, a\}$ is an edge of $\text{hull}(M_L)$ then $a$ is a primitive vector in $L$.

**Proof.** Suppose that $a = (a_1, \ldots, a_n)$ is a vector in $L$ which is not primitive, and choose $b = (b_1, \ldots, b_n) \in L \setminus \{a, 0\}$ such that $b^+ \leq a^+$ and $b^- \leq a^-$. This implies $t^{b_i} + t^{a_i} - b_i \leq 1 + t^{a_i}$ for $t \gg 0$ and $i \in \{1, \ldots, n\}$. In other words, the vector $t^b + t^a - b$ is componentwise smaller or equal to the vector $t^0 + t^a$. We conclude that the midpoint of the segment $\text{conv}\{t^0, t^a\}$ lies in $\text{conv}\{t^b, t^a - b\} + \mathbb{R}_+^n$, and hence $\text{conv}\{t^0, t^a\}$ is not an edge of the polyhedron $P_t = \text{conv}\{t^c : c \in L\} + \mathbb{R}_+^n$. ■

**Theorem 3.14** The hull resolution $\pi(F_{\text{hull}(M_L)})$ is finite as an $S$-module.

**Proof.** By Lemma 3.13 the vertex $0$ of $\text{hull}(M_L)$ lies in only finitely many edges. It follows that $0$ lies in only finitely many faces of $\text{hull}(M_L)$. The lattice $L$ acts transitively on the vertices of $\text{hull}(M_L)$, and hence every face of $\text{hull}(M_L)$ is $L$-equivalent to a face containing $0$. The faces containing $0$ generate $F_{\text{hull}(M_L)}$ as an $S[L]$-module, and hence they generate $\pi(F_{\text{hull}(M_L)})$ as an $S$-module. ■

A minimal free resolution of a lattice ideal $I_L$ generally does not respect symmetries, but the hull resolution does. The following example illustrates this point.

**Example 3.15** *(The hypersimplicial complex as a hull resolution)*

The lattice $L = \ker_{\mathbb{Z}}(1 \ 1 \ \cdots \ \ 1)$ in $\mathbb{Z}^n$ defines the toric ideal

$I_L = \langle x_i - x_j : 1 \leq i < j \leq n \rangle$.

The minimal free resolution of $I_L$ is the Koszul complex on $n - 1$ of the generators $x_i - x_j$. Such a minimal resolution does not respect the action of the symmetric group $S_n$ on $I_L$. The hull resolution is the Eagon-Northcott complex of the matrix

$\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n
\end{bmatrix}$. This resolution is not minimal but it retains the $S_n$-symmetry of $I_L$.

It coincides with the hypersimplicial complex studied by Gel’fand and MacPherson in [GM, §2.1.3]. The basis vectors of the hypersimplicial complex are denoted $\Delta^I_i$ where $I$ is a subset of $\{1, 2, \ldots, n\}$ with $|I| \geq 2$ and $\ell$ is an integer with $1 \leq \ell \leq |I| - 1$. We have $\Delta^I_i(x_j) \mapsto x_i - x_j$ and the higher differentials act as

$\Delta^I_{\ell} \mapsto \sum_{i \in I} \text{sign}(i, I) \cdot x_i \cdot \Delta^I_{\ell-1} \Delta^I_{\ell-1} - \sum_{i \in I} \text{sign}(i, I) \cdot \Delta^I_{\ell-1} \Delta^I_{\ell-1}$,

where the first sum is zero if $\ell = 1$ and the second sum is zero if $\ell = |I| - 1$. ■
Remark 3.16  Our study suggests a curious duality of toric varieties, under which the coordinate ring of the primal variety is resolved by a discrete subgroup of the dual variety. More precisely, the hull resolution of $I_L$ is gotten by taking the convex hull in $\mathbb{R}^n$ of the points $t^a$ for $a \in L$. The Zariski closure of these points (as $t$ varies) is itself an affine toric variety, namely, it is the variety defined by the lattice ideal $I_{L^\perp}$ where $L^\perp$ is the lattice dual to $L$ under the standard inner product on $\mathbb{Z}^n$.

For instance, in Example 3.15 the primal toric variety is the line $(t, t, \ldots, t)$ and the dual toric variety is the hypersurface $x_1 x_2 \cdots x_n = 1$. That hypersurface forms a group under coordinatewise multiplication, and we are taking the convex hull of a discrete subgroup to resolve the coordinate ring of the line $(t, t, \ldots, t)$.

Example 3.17  (The rational normal quartic curve in $P^4$)

Let $L = \ker_{\mathbb{Z}} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$. The minimal free resolution of the lattice ideal $I_L$ looks like $0 \to S^3 \to S^8 \to S^6 \to I_L$. The primal toric variety in the sense of Remark 3.16 is a curve in $P^4$ and the dual toric variety is the embedding of the 3-torus into affine 5-space given by the equations

$$x_2 x_3^2 x_4^2 x_5^2 = x_1^2 x_2^2 x_3^2 x_4 = 1.$$  

Here the hull complex $\text{hull}(M_L)$ is simplicial, and the hull resolution of $I_L$ has the format $0 \to S^4 \to S^{16} \to S^{20} \to S^9 \to I_L$. The nine classes of edges in $\text{hull}(M_L)$ are the seven quadratic binomials in $I_L$ and the two cubic binomials $x_3 x_4^2 - x_1 x_5^2$, $x_2^2 x_3 - x_1^2 x_5$.

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