Cramer’s rules for the solution to the two-sided restricted quaternion matrix equation.

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Abstract

Weighted singular value decomposition (WSVD) of a quaternion matrix and with its help determinantal representations of the quaternion weighted Moore-Penrose inverse have been derived recently by the author. In this paper, using these determinantal representations, explicit determinantal representation formulas for the solution of the restricted quaternion matrix equations, \( AXB = D \), and consequently, \( AX = D \) and \( XB = D \) are obtained within the framework of the theory of column-row determinants. We consider all possible cases depending on weighted matrices.

Keywords Weighted singular value decomposition, Weighted Moore-Penrose inverse, Quaternion matrix, Matrix equation, Cramer rule

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1 Introduction

Let \( \mathbb{R} \) and \( \mathbb{C} \) be the real and complex number fields, respectively. Throughout the paper, we denote the set of all \( m \times n \) matrices over the quaternion skew field

\[
\mathbb{H} = \{ a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \}
\]

by \( \mathbb{H}^{m \times n} \), and by \( \mathbb{H}_r^{m \times n} \) the set of all \( m \times n \) matrices over \( \mathbb{H} \) with a rank \( r \). Let \( M(n, \mathbb{H}) \) be the ring of \( n \times n \) quaternion matrices and \( I \) be the identity matrix with the appropriate size. For \( A \in \mathbb{H}^{n \times m} \), we denote by \( A^\ast \), rank \( A \) the conjugate transpose (Hermitian adjoint) matrix and the rank of \( A \). The matrix \( A = (a_{ij}) \in \mathbb{H}^{n \times n} \) is Hermitian if \( A^\ast = A \).

The definitions of the Moore-Penrose inverse [1] and the weighted Moore-Penrose inverse [2] can be extended to quaternion matrices as follows.

The Moore-Penrose inverse of \( A \in \mathbb{H}^{m \times n} \), denoted by \( A^\dagger \), is the unique matrix \( X \in \mathbb{H}^{n \times m} \) satisfying the following equations [1],

\[
AXA = A; \quad (1) \\
XAX = X; \quad (2) \\
(AX)^\ast = AX; \quad (3) \\
(XA)^\ast =XA. \quad (4)
\]

Let Hermitian positive definite matrices \( M \) and \( N \) of order \( m \) and \( n \), respectively, be given. For \( A \in \mathbb{H}^{m \times n} \), the weighted Moore-Penrose inverse of
A is the unique solution $X = A_{M,N}^\dagger$ of the matrix equations (1) and (2) and the following equations in $X$ [3]:

$$(3M) \ (MAX)^* = MAX; \ (4N) \ (NXA)^* = NXA.$$ 

In particular, when $M = I_m$ and $N = I_n$, the matrix $X$ satisfying the equations (1), (2), (3M), (4N) is the Moore-Penrose inverse $A^\dagger$.

A basic method for finding the Moore-Penrose inverse is based on the singular value decomposition (SVD). It is also available for quaternion matrices, (see, e.g. [4,5]). The weighted Moore-Penrose inverse $A_{M,N}^\dagger \in \mathbb{C}^{m \times n}$ (over complex or real fields) has the explicit expressing by the weighted singular value decomposition (WSVD) that at first has been obtained in [6] by Cholesky factorization. In [7], WSVD of real matrices with singular weights has been derived using weighted orthogonal matrices and weighted pseudoorthogonal matrices.

Recently, by the author, WSVD has been expanded to quaternion matrices.

**Theorem 1.1** [8] Let $A \in H^{m \times n}$, $M$ and $N$ be positive definite matrices of order $m$ and $n$, respectively. Denote $A^\# = N^{-1}A^*M$. There exist $U \in H^{m \times m}$, $V \in H^{n \times n}$ satisfying $U^*MU = I_m$ and $V^*N^{-1}V = I_n$ such that $A = UDV^*$, where $D = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$. Then the weighted Moore-Penrose inverse $A_{M,N}^\dagger$ can be represented

$$A_{M,N}^\dagger = N^{-1}V \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*M,$$ 

where $\Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$ and $\sigma_i^2$ is the nonzero eigenvalues of $A^\#A$ or $AA^\#$, which coincide.

By using WSVD, within the framework of the theory of column-row determinants, limit and determinantal representations of the quaternion weighted Moore-Penrose inverse has been derived ibidem as well.

But why determinantal representations of generalized inverses are so important? When we return to the usual inverse, its determinantal representation is the matrix with cofactors in entries that gives direct method of its finding and makes it applicable in Cramer’s rule for systems of linear equations. The same be wanted for generalized inverses. But there is not so unambiguous even for complex or real matrices. Therefore, there are various determinantal representations of generalized inverses because searches of their explicit more applicable expressions are continuing (see, e.g. [9–13]).

The understanding of the problem for determinantal representing of generalized inverses as well as solutions and generalized inverse solutions of quaternion matrix equations, only now begins to be decided due to the theory of column-row determinants introduced in [14,15].

Song at al. [16,17] have studied the weighted Moore-Penrose inverse over the quaternion skew field and obtained its determinantal representation within the framework of the theory of column-row determinants as well. But WSVD of quaternion matrices has not been considered and for obtaining a determinantal representation there was used auxiliary matrices which different from $A$,
and weights \( M \) and \( N \). Despite this in [17], Cramer’s rule of the quaternion restricted matrix equation \( AXB = D \) has been derived with the help obtained determinantal representations of the weighted Moore-Penrose inverse.

The main goals of the paper are obtaining Cramer’s rule for the quaternion restricted matrix equation \( AXB = D \), and consequently, \( AX = D \) and \( XB = D \) using the determinantal representations of the weighted Moore-Penrose inverse obtained by WSVD in [3]. We consider all possible cases with respect to weights of \( A \) and \( B \).

It need to note that currently the theory of column-row determinants of quaternion matrices is active developing. Within the framework of column-row determinants, determinantal representations of various kind of generalized inverses, (generalized inverses) solutions of quaternion matrix equations recently have been derived as by the author (see, e.g. [18–22]) so by other researchers (see, e.g. [23–26]).

In this chapter we shall adopt the following notation.

Let \( \alpha := \{ \alpha_1, \ldots, \alpha_k \} \subseteq \{1, \ldots, m\} \) and \( \beta := \{ \beta_1, \ldots, \beta_k \} \subseteq \{1, \ldots, n\} \) be subsets of the order \( 1 \leq k \leq \min\{m, n\} \). By \( A_{\alpha}^\beta \) denote the submatrix of \( A \) determined by the rows indexed by \( \alpha \), and the columns indexed by \( \beta \). Then, \( A_{\alpha}^\beta \) denotes a principal submatrix determined by the rows and columns indexed by \( \alpha \). If \( A \in M(n, \mathbb{H}) \) is Hermitian, then by \( |A_{\alpha}^\beta| \) denote the corresponding principal minor of \( \det A \), since \( A_{\alpha}^\beta \) is Hermitian as well. For \( 1 \leq k \leq n \), denote by \( L_{k,n} : = \{ \alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n \} \) the collection of strictly increasing sequences of \( k \) integers chosen from \( \{1, \ldots, n\} \). For fixed \( i \in \alpha \) and \( j \in \beta \), let

\[
I_{r,m}\{i\} := \{ \alpha : \alpha \in L_{r,m}, i \in \alpha \}, \quad J_{r,n}\{j\} := \{ \beta : \beta \in L_{r,n}, j \in \beta \}.
\]

The paper is organized as follows. We start with some basic concepts and results from the theories of row-column determinants and of quaternion matrices in Section 2. Cramer’s rules for the quaternionic restricted matrix equation \( AXB = D \), and consequently, \( AX = D \) and \( XB = D \) are derived in Section 3. All possible cases are considered in the three subsections of Section 3. In Section 4, we give numerical an example to illustrate the main results.

## 2 Preliminaries

### 2.1 Elements of the theory of column-row determinants and quaternion inverse matrices

For a quadratic matrix \( A = (a_{ij}) \in M(n, \mathbb{H}) \) can be define \( n \) row determinants and \( n \) column determinants as follows.

Suppose \( S_n \) is the symmetric group on the set \( I_n = \{1, \ldots, n\} \).

**Definition 2.1** [17] The \( i \)th row determinant of \( A = (a_{ij}) \in M(n, \mathbb{H}) \) is

\[
\det A_i = \sum_{\sigma \in S_n} (-1)^{i+\sigma(1)} a_{i\sigma(1)} a_{i+1\sigma(2)} \cdots a_{i+n-1\sigma(n)}
\]

where \( \sigma \) is a permutation of \( S_n \).
Proposition 2.3 If for 

\[ \text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{\sigma(i) - 1} \prod_{j=1}^{n} a_{i_j \cdot k_j \cdot i_j + 1} \ldots a_{i_j + i_j} i_j, \]

\[ \sigma = (i_{k_1} i_{k_1 + 1} \ldots i_{k_1 + l_1}) (i_{k_2} i_{k_2 + 1} \ldots i_{k_2 + l_2}) \ldots (i_{k_r} i_{k_r + 1} \ldots i_{k_r + l_r}), \]

with conditions \( i_{k_2} < i_{k_3} < \ldots < i_{k_r} \) and \( i_{k_s} < i_{k_{t+s}} \) for \( t = 2, \ldots, r \) and \( s = 1, \ldots, l_t. \)

Definition 2.2 [14] The \( j \)th column determinant of \( A = (a_{ij}) \in M(n, \mathbb{H}) \) is defined for all \( j = 1, \ldots, n \) by putting

\[ \text{cdet}_j A = \sum_{\tau \in S_n} (-1)^{\tau(i) - 1} \prod_{j=1}^{n} a_{j_{k_r+1} j_{k_r} + 1} \ldots a_{j_{k_r + i_j} j_{k_r} + 1} a_{j_{k_r + i_j} j_{k_r + i_j} + 1}, \]

\[ \tau = (j_{k_r+1} j_{k_r} + 1 j_{k_r}) (j_{k_r+1} j_{k_r} + 1 j_{k_r+1} j_{k_r}) (j_{k_r+1} j_{k_r} + 1 j_{k_r+1} j_{k_r}), \]

with conditions, \( j_{k_2} < j_{k_3} < \ldots < j_{k_r} \) and \( j_{k_t} < j_{k_{t+s}} \) for \( t = 2, \ldots, r \) and \( s = 1, \ldots, l_t. \)

Suppose \( A^{ij} \) denotes the submatrix of \( A \) obtained by deleting both the \( i \)th row and the \( j \)th column. Let \( a_{i,j} \) be the \( j \)th column and \( a_{i,j} \) be the \( i \)th row of \( A \). Suppose \( A_{i,j} (b) \) denotes the matrix obtained from \( A \) by replacing its \( j \)th column with the column \( b \), and \( A_{i,j} (b) \) denotes the matrix obtained from \( A \) by replacing its \( i \)th row with the row \( b \). We note some properties of column and row determinants of a quaternion matrix \( A = (a_{ij}) \), where \( i \in I_n, j \in J_n \) and \( I_n = J_n = \{1, \ldots, n\}. \)

Proposition 2.1 [14] If \( b \in \mathbb{H} \), then

\[ \text{rdet}_i A_{i,j} (b \cdot a_{i,j}) = b \cdot \text{rdet}_i A, \quad \text{cdet}_j A_{i,j} (a_{i,j} \cdot b) = \text{cdet}_j A \cdot b, \]

for all \( i = 1, \ldots, n. \)

Proposition 2.2 [14] If for \( A \in M(n, \mathbb{H}) \) there exists \( t \in I_n \) such that \( a_{i,j} = b_{j} + c_{j} \) for all \( j = 1, \ldots, n \), then

\[ \text{rdet}_i A = \text{rdet}_i A_{i,t} (b) + \text{rdet}_i A_{i,t} (c), \quad \text{cdet}_j A = \text{cdet}_j A_{i,t} (b) + \text{cdet}_j A_{i,t} (c), \]

where \( b = (b_1, \ldots, b_n), c = (c_1, \ldots, c_n) \) and for all \( i = 1, \ldots, n. \)

Proposition 2.3 [14] If for \( A \in M(n, \mathbb{H}) \) there exists \( t \in J_n \) such that \( a_{i,t} = b_{i} + c_{i} \) for all \( i = 1, \ldots, n \), then

\[ \text{rdet}_j A = \text{rdet}_j A_{i,t} (b) + \text{rdet}_j A_{i,t} (c), \quad \text{cdet}_j A = \text{cdet}_j A_{i,t} (b) + \text{cdet}_j A_{i,t} (c), \]

where \( b = (b_1, \ldots, b_n)^T, c = (c_1, \ldots, c_n)^T \) and for all \( j = 1, \ldots, n. \)
Remark 2.1 Let $r\det_i A = \sum_{j=1}^{n} a_{ij} \cdot R_{ij}$ and $c\det_j A = \sum_{i=1}^{n} L_{ij} \cdot a_{ij}$ for all $i, j = 1, \ldots, n$, where by $R_{ij}$ and $L_{ij}$ denote the right and left $(ij)$th cofactors of $A \in M(n, \mathbb{H})$, respectively. It means that $r\det_i A$ can be expand by right cofactors along the $i$th row and $c\det_j A$ can be expand by left cofactors along the $j$th column, respectively, for all $i, j = 1, \ldots, n$.

The main property of the usual determinant is that the determinant of a non-invertible matrix must be equal zero. But the row and column determinants don’t satisfy it, in general. Therefore, these matrix functions can be consider as some pre-determinants. The following theorem has a key value in the theory of the column and row determinants.

Theorem 2.1 [14] If $A = (a_{ij}) \in M(n, \mathbb{H})$ is Hermitian, then $r\det_i A = \cdots = r\det_n A = c\det_1 A = \cdots = c\det_n A \in \mathbb{R}$.

Due to Theorem 2.1 we can define the determinant of a Hermitian matrix $A \in M(n, \mathbb{H})$ by putting, $\det A := r\det_1 A = c\det_n A$, for all $i = 1, \ldots, n$. By using its row and column determinants, the determinant of a quaternion Hermitian matrix has properties similar to the usual determinant. These properties are completely explored in [14][15] and can be summarized in the following theorems.

Theorem 2.2 If the $i$th row of a Hermitian matrix $A \in M(n, \mathbb{H})$ is replaced with a left linear combination of its other rows, i.e. $a_i = c_1 a_i, + \ldots + c_k a_i,$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{i, i_l\} \subset I_n$, then

$$r\det_i A_i, (c_1 a_i, + \ldots + c_k a_i) = c\det_i A_i, (c_1 a_i, + \ldots + c_k a_i) = 0.$$

Theorem 2.3 If the $j$th column of a Hermitian matrix $A \in M(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $a_j = a_j, c_1 + \ldots + a_j c_k$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{i, i_l\} \subset J_n$, then

$$c\det_j A_j, (a_j, c_1 + \ldots + a_j c_k) = r\det_j A_j, (a_j, c_1 + \ldots + a_j c_k) = 0.$$

The following theorem about determinantal representation of an inverse matrix of Hermitian follows immediately from these properties.

Theorem 2.4 [15] If a Hermitian matrix $A \in M(n, \mathbb{H})$ is such that $\det A \neq 0$, then there exist a unique right inverse matrix $(RA)^{-1}$ and a unique left inverse matrix $(LA)^{-1}$, and $(RA)^{-1} = (LA)^{-1} =: A^{-1}$, which possess the following determinantal representations:

$$(RA)^{-1} = \frac{1}{\det A} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix},$$ (6)

$$(LA)^{-1} = \frac{1}{\det A} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix},$$ (7)
where \( \det A = \sum_{j=1}^{n} a_{ij} \cdot R_{ij} = \sum_{i=1}^{n} L_{ij} \cdot a_{ij} \),

\[
R_{ij} = \begin{cases} 
- \operatorname{rdet}_j A_{ii}^j (a_{i}) , & i \neq j; \\
\operatorname{rdet}_k A_{ii}^j , & i = j,
\end{cases}
L_{ij} = \begin{cases} 
- \operatorname{cdet}_j A_{jj}^i (a_{j}) , & i \neq j; \\
\operatorname{cdet}_k A_{jj}^i , & i = j.
\end{cases}
\]

The submatrix \( A_{ij}^{ii} (a_{i}) \) is obtained from \( A \) by replacing the \( j \)th column with the \( i \)th column and then deleting both the \( i \)th row and column, \( A_{ij}^{jj} (a_{j}) \) is obtained by replacing the \( i \)th row with the \( j \)th row, and then by deleting both the \( j \)th row and column, respectively. \( I_n = \{1, \ldots, n\} \), \( k = \min \{ I_n \setminus \{i\} \} \), for all \( i, j = 1, \ldots, n \).

**Theorem 2.5** If \( A \in M(n, \mathbb{H}) \), then \( \det AA^* = \det A^* A \).

**Definition 2.3** For \( A \in M(n, \mathbb{H}) \), the double determinant of \( A \) is defined by putting, \( \text{ddet} A := \det AA^* = \det A^* A \).

For arbitrary \( A \in M(n, \mathbb{H}) \), we have the following theorem on determinantal representations of its inverse.

**Theorem 2.6** The necessary and sufficient condition of invertibility of \( A \in M(n, \mathbb{H}) \) is \( \text{ddet} A \neq 0 \). Then there exists \( A^{-1} = (LA)^{-1} = (RA)^{-1} \), where

\[
(LA)^{-1} = (A^* A)^{-1} A^* = \frac{1}{\text{ddet} A} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\
L_{12} & L_{22} & \cdots & L_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}
\]

and

\[
(RA)^{-1} = A^* (AA^*)^{-1} = \frac{1}{\text{ddet} A^*} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\
R_{12} & R_{22} & \cdots & R_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}
\]

Moreover, the following criterion of invertibility of a quaternion matrix can be obtained.

**Theorem 2.7** If \( A \in M(n, \mathbb{H}) \), then the following statements are equivalent.

i) \( A \) is invertible, i.e. \( A \in GL(n, \mathbb{H}) \);

ii) rows of \( A \) are left-linearly independent;

iii) columns of \( A \) are right-linearly independent;

iv) \( \text{ddet} A \neq 0 \).
2.2 Some provisions of quaternion eigenvalues

Due to real-scalar multiplying on the right, quaternion column-vectors form a right vector $R$-space, and, by real-scalar multiplying on the left, quaternion row-vectors form a left vector $R$-space denoted by $\mathcal{H}_r$ and $\mathcal{H}_l$, respectively. It can be shown that $\mathcal{H}_r$ and $\mathcal{H}_l$ possess corresponding $\mathbb{H}$-valued inner products by putting $\langle x, y \rangle_r = y^*_1x_1 + \cdots + y^*_nx_n$ for $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathcal{H}_r$, and $\langle x, y \rangle_l = x_1y_1 + \cdots + x_ny_n$ for $x, y \in \mathcal{H}_l$ that satisfy the inner product relations, namely, conjugate symmetry, linearity, and positive-definiteness but with specialties

$$\langle x\alpha + y\beta, z \rangle = \langle x, z \rangle\alpha + \langle y, z \rangle\beta \quad \text{when } x, y, z \in \mathcal{H}_r$$

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \text{when } x, y, z \in \mathcal{H}_l,$$

for any $\alpha, \beta \in \mathbb{H}$. A set of vectors from $\mathcal{H}_r$ and $\mathcal{H}_l$ can be orthonormalized in particular by the GramSchmidt process with corresponding projection operators

$$\text{proj}_u(v) := \frac{(u, v)_r}{(u, u)_r}u,$$

$$\text{proj}_u(v) := \frac{(u, v)_l}{(u, u)_l}u$$

for $\mathcal{H}_r$ and $\mathcal{H}_l$, respectively. Due to the above, the following definition makes sense.

**Definition 2.4** Suppose $U \in M(n, \mathbb{H})$ and $U^*U =UU^* = I$, then the matrix $U$ is called unitary.

Clear, that columns of $U$ form a system of normalized vectors in $\mathcal{H}_r$, rows of $U^*$ is a system of normalized vectors in $\mathcal{H}_l$.

Due to the noncommutativity of quaternions, there are two types of eigenvalues. A quaternion $\lambda$ is said to be a left eigenvalue of $A \in M(n, \mathbb{H})$ if $A \cdot x = \lambda \cdot x$, and a right eigenvalue if $A \cdot x = x \cdot \lambda$ for some nonzero quaternion column-vector $x$. Then, the set $\{ \lambda \in \mathbb{H} | Ax = \lambda x, x \neq 0 \in \mathbb{H}^n \}$ is called the left spectrum of $A$, denoted by $\sigma_l(A)$. The right spectrum is similarly defined by putting, $\sigma_r(A) := \{ \lambda \in \mathbb{H} | Ax = x\lambda, x \neq 0 \in \mathbb{H}^n \}$.

The theory on the left eigenvalues of quaternion matrices has been investigated in particular in [29][31]. The theory on the right eigenvalues of quaternion matrices is more developed [32][37]. We consider this is a natural consequence of the fact that quaternion column vectors form a right vector space for which left eigenvalues seem to be "exotic" because of their multiplying from the left.

We present the some known results from the theory of right eigenvalues. It’s well known that if $\lambda$ is a nonreal eigenvalue of $A$, so is any element in the equivalence class containing $[\lambda]$, i.e. $[\lambda] = \{ x | x = u^{-1}\lambda u, u \in \mathbb{H}, \| u \| = 1 \}$.

**Theorem 2.8** [32] Any quaternion matrix $A \in M(n, \mathbb{H})$ has exactly $n$ eigenvalues which are complex numbers with nonnegative imaginary parts.
Those eigenvalues $h_1 + k_1i, \ldots, h_n + k_ni$, where $k_t \geq 0$ and $h_t, k_t \in \mathbb{R}$ for all $t = 1, \ldots, n$, are said to be the standard eigenvalues of $A$.

**Theorem 2.9** \[32\] Let $A \in M(n, \mathbb{H})$. Then there exists a unitary matrix $U$ such that $U^*AU$ is an upper triangular matrix with diagonal entries $h_1 + k_1i, \ldots, h_n + k_ni$ which are the standard eigenvalues of $A$.

**Corollary 2.1** \[36\] Let $A \in M(n, \mathbb{H})$ with the standard eigenvalues $h_1 + k_1i, \ldots, h_n + k_ni$. Then $\sigma_r = \{h_1 + k_1i\} \cup \cdots \cup \{h_n + k_ni\}$.

**Corollary 2.2** \[36\] Let $A \in M(n, \mathbb{H})$ be given. Then, $A$ is Hermitian if and only if there are a unitary matrix $U \in M(n, \mathbb{H})$ and a real diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $A = UDU^*$, where $\lambda_1, \ldots, \lambda_n$ are right eigenvalues of $A$.

The right and left eigenvalues are in general unrelated \[38\], but it is not for Hermitian matrices. Suppose $A \in M(n, \mathbb{H})$ is Hermitian and $\lambda \in \mathbb{R}$ is its right eigenvalue, then $A \cdot x = x \cdot \lambda = \lambda \cdot x$. This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, $\lambda \in \mathbb{R}$, the matrix $\lambda I - A$ is Hermitian.

**Definition 2.5** If $t \in \mathbb{R}$, then for a Hermitian matrix $A$ the polynomial $p_A(t) = \det(tI - A)$ is said to be the characteristic polynomial of $A$.

**Lemma 2.1** \[15\] If $A \in M(n, \mathbb{H})$ is Hermitian, then $p_A(t) = t^n - d_1t^{n-1} + d_2t^{n-2} + \cdots + (-1)^n d_n$, where $d_k$ is the sum of principle minors of $A$ of order $k$, $1 \leq k < n$, and $d_n = \det A$.

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues, which are its right eigenvalues as well.

### 2.3 Determinantal representations of the Moore-Penrose and weighted Moore-Penrose inverses over the quaternion skew field

Within the framework of the theory of column-row determinants, we have the following theorem on determinantal representations of the quaternion Moore-Penrose inverse.

**Theorem 2.10** \[5\] If $A \in \mathbb{H}^{m \times n}_r$, then the Moore-Penrose inverse $A^+ = (a^+_{ij}) \in \mathbb{H}^{n \times m}$ possess the following determinantal representations:

1. If $r < \min\{m, n\}$, then

$$a^+_{ij} = \frac{\sum_{\beta \in J_{r,n}} \text{cdet}_{i,\{\beta\}} ((A^*A)_{ij} (A^*_{\beta})_{\beta})}{\sum_{\beta \in J_{r,n}} \det(A^*A)_{\beta}}. \quad (8)$$
or

\[ a^\dagger_{ij} = \frac{\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j ((\mathbf{A}\mathbf{A}^*)_j, (\mathbf{a}^*_j)) \alpha}{\sum_{\alpha \in I_{r,m}} \det(\mathbf{A}\mathbf{A}^*) \alpha}. \]  

(9)

(ii) If \( r = n \), then

\[ a^\dagger_{ij} = \frac{\text{cdet}_i((\mathbf{A}^*\mathbf{A})_i)(\mathbf{a}^*_j)}{\det(\mathbf{A}^*\mathbf{A})}. \]

(10)

or (9) when \( n < m \).

(iii) If \( r = m \), then

\[ a^\dagger_{ij} = \frac{\text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j, (\mathbf{a}^*_j))}{\det(\mathbf{A}\mathbf{A}^*)}. \]

(11)

or (8) when \( m < n \).

Even though the eigenvalues of \( \mathbf{A}^\sharp \mathbf{A} \) and \( \mathbf{A}\mathbf{A}^\sharp \) are real and nonnegative, they are not Hermitian in general. Therefore, the following two cases are considered, when \( \mathbf{A}^\sharp \mathbf{A} \) and \( \mathbf{A}\mathbf{A}^\sharp \) both or one of them are Hermitian, and when they are non-Hermitian. Denote the \((ij)\)th entry of \( \mathbf{A}^\dagger_{M,N} \) by \( a^\dagger_{ij} \) for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

**Theorem 2.11** [8] Let \( \mathbf{A} \in \mathbb{H}^{m \times n} \). If \( \mathbf{A}^\sharp \mathbf{A} \) or \( \mathbf{A}\mathbf{A}^\sharp \) are Hermitian, then the weighted Moore-Penrose inverse \( \mathbf{A}^\dagger_{M,N} = \left( a^\dagger_{ij} \right) \in \mathbb{H}^{n \times m} \) possess the following determinantal representations, respectively,

(i) If \( r < \min\{m, n\} \), then

\[ a^\dagger_{ij} = \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i(\mathbf{A}^\sharp \mathbf{A})_i(\mathbf{a}^*_j) \beta \]

\[ \sum_{\beta \in J_{r,n}} |(\mathbf{A}^\sharp \mathbf{A})_\beta|, \]

(12)

or

\[ a^\dagger_{ij} = \sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_j, (\mathbf{a}^*_j)) \alpha \]

\[ \sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^*)_\alpha|, \]

(13)

(ii) If \( \text{rank} \mathbf{A} = n < m \), then

\[ a^\dagger_{ij} = \frac{\text{cdet}_i(\mathbf{A}^\sharp \mathbf{A})_i(\mathbf{a}^*_j)}{\det(\mathbf{A}^\sharp \mathbf{A})}, \]

(14)

or the determinantal representation (12) can be applicable as well.
(iii) If rank \( A = m < n \), then

\[
a_{ij}^\dagger = \frac{\text{rdet}_j(AA^\sharp)_{ij}}{\det(AA^\sharp)}.
\]

(15)

or the determinantal representation (13) can be applicable as well.

Denote \( M_{ij} = \left( m_{ij}^{(\dagger)} \right) \), \( N_{ij}^{-\dagger} = \left( n_{ij}^{(-\dagger)} \right) \), and \( \tilde{A} := M_{ij} A N_{ij}^{-\dagger} = (\tilde{a}_{ij}) \in \mathbb{H}^{m \times n} \), then \( N_{ij}^{-\dagger} A^* M_{ij} = \tilde{A}^* = (\tilde{a}_{ij}^*), \quad (M_{ij} A N_{ij}^{-\dagger})^\dagger = \tilde{A}^\dagger = (\tilde{a}_{ij}^\dagger) \).

**Theorem 2.12** \[8\] Let \( A \in \mathbb{H}^{m \times n} \).

(i) If \( A^\sharp A \) is non-Hermitian, then the weighted Moore-Penrose inverse \( A_{M,N}^\dagger = \left( a_{ij}^\dagger \right) \in \mathbb{H}^{n \times m} \) possess the determinantal representations

(a) if \( r < n \),

\[
a_{ij}^\dagger = \frac{\sum_k n_{ik}^{(-\dagger)} \sum_{\beta \in J_{r,n}} \text{cdet}_k \left( \left( \tilde{A}^* \tilde{A} \right)^{\dagger} \cdot \left( \hat{a}_{ij} \right)^{\beta} \right)}{\sum_{\beta \in J_{r,n}} \text{det}(\tilde{A}^* \tilde{A})^{\beta}},
\]

(16)

where \( \hat{a}_{ij} \) is the \( j \)th column of \( N_{ij}^{-\dagger} A^* M_{ij} \);

(b) if \( r = n \),

\[
a_{ij}^\dagger = \frac{\text{cdet}_i(A^* M A)_{ij} \cdot \left( \hat{a}_{ij} \right)}{\text{det}(A^* M A)},
\]

(17)

where \( \hat{a}_{ij} \) is the \( j \)th column of \( A^* M \) for all \( j = 1, \ldots, m \).

(ii) If \( A A^\sharp \) is non-Hermitian, then \( A_{M,N}^\dagger = \left( a_{ij}^\dagger \right) \) possess the determinantal representation

(a) if \( r < m \),

\[
a_{ij}^\dagger = \frac{\sum_l \sum_{\alpha \in I_{r,m}} \text{rdet}_l \left( \left( \tilde{A} A^\dagger \right)_{l} \cdot \left( \hat{a}_{i\alpha} \right)^{\alpha} \cdot m_{ij}^{(\dagger)} \right)}{\sum_{\alpha \in I_{r,m}} \text{det}(\tilde{A} A^\dagger)^{\alpha}},
\]

(18)

where \( \hat{a}_{i\alpha} \) is the \( \alpha \)th row of \( N_{ij}^{-1} A^* M_{ij} \);

(b) if \( r = m \),

\[
a_{ij}^\dagger = \frac{\text{rdet}_j(AN_{ij}^{-1} A^*)_{ij} \cdot \left( \hat{a}_{ij} \right)}{\text{det}(AN_{ij}^{-1} A^*)},
\]

(19)

where \( \hat{a}_{ij} \) is the \( \alpha \)th row of \( N_{ij}^{-1} A^* \) for all \( i = 1, \ldots, n \).
3 Cramer’s Rule for Two-sided Restricted Quaternionic Matrix Equation

Definition 3.1 For an arbitrary matrix over the quaternion skew field, \(A \in \mathbb{H}^{m \times n}\), we denote by

- \(\mathcal{R}_r(A) = \{y \in \mathbb{H}^{m \times 1} : y = Ax, x \in \mathbb{H}^{n \times 1}\}\), the column right space of \(A\),
- \(\mathcal{N}_r(A) = \{x \in \mathbb{H}^{n \times 1} : Ax = 0\}\), the right null space of \(A\),
- \(\mathcal{R}_l(A) = \{y \in \mathbb{H}^{1 \times n} : y = xA, x \in \mathbb{H}^{1 \times m}\}\), the row left space of \(A\),
- \(\mathcal{N}_l(A) = \{x \in \mathbb{H}^{1 \times m} : xA = 0\}\), the left null space of \(A\).

It is easy to see, if \(A \in \mathbb{H}^{n \times n}\), then \(\mathcal{R}_r \oplus \mathcal{N}_r = \mathbb{H}^{n \times 1}\), and \(\mathcal{R}_l \oplus \mathcal{N}_l = \mathbb{H}^{1 \times n}\).

Suppose that \(A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{p \times q}\). Denote

- \(\mathcal{R}_r(A, B) := \mathcal{N}_r(Y) = \{Y = AXB : X^{n \times p}\}\),
- \(\mathcal{N}_r(A, B) := \mathcal{R}_r(X) = \{X^{n \times p} : AXB = 0\}\),
- \(\mathcal{R}_l(A, A) = \mathcal{R}_l(Y) = \{Y = AXB : X^{n \times q}\}\),
- \(\mathcal{N}_l(A, B) := \mathcal{N}_l(X) = \{X^{n \times p} : AXB = 0\}\).

Lemma 3.1 [17] Suppose that \(A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{p \times q}, M, N, P,\) and \(Q\) are Hermitian positive definite matrices of order \(m, n, p,\) and \(q\), respectively. Denote \(A^\dagger = N^{-1}A^*M\) and \(B^\dagger = Q^{-1}B^*P\). If \(D \subset \mathcal{R}_r(\AA^\dagger, BB^\dagger)\) and \(D \subset \mathcal{R}_l(\AA^\dagger, BB^\dagger)\),

\[
\begin{align*}
AXB &= D, \\
\mathcal{R}_r(X) &\subset N^{-1}\mathcal{R}_r(A^*), \quad \mathcal{N}_r(X) \supset P^{-1}\mathcal{N}_r(B^*), \\
\mathcal{R}_l(X) &\subset \mathcal{R}_l(A^*M), \quad \mathcal{N}_l(X) \supset \mathcal{N}_l(B^*)Q 
\end{align*}
\]

then the unique solution of \((20)\) with the restrictions \((21)\) - \((22)\) is

\[
X = A^\dagger_{M,N}DB^\dagger_{P,Q}.
\]

In this chapter, we get determinantal representations of \((23)\) that are intrinsically analogs of the classical Cramer’s rule. We will consider several cases depending on whether the matrices \(A^\dagger\) and \(BB^\dagger\) are Hermitian or not.

3.1 The Case of Both Hermitian Matrices \(A^\dagger\) and \(BB^\dagger\).

Denote \(\bar{D} = A^\dagger DB^\dagger\).

Theorem 3.1 Let \(A^\dagger\) and \(BB^\dagger\) be Hermitian. Then the solution \((25)\) possess the following determinantal representations.
(i) If \( \text{rank } A = r_1 < n \) and \( \text{rank } B = r_2 < p \), then

\[
x_{ij} = \frac{\beta \in J_{r_1,n} \{i\}}{\sum_{\beta \in J_{r_1,n}} |(A^\dagger A)^\beta| \sum_{\alpha \in I_{r_2,p}} |(BB)^\alpha|}
\]

or

\[
x_{ij} = \frac{\alpha \in I_{r_2,p} \{j\}}{\sum_{\beta \in J_{r_1,n}} |(A^\dagger A)^\beta| \sum_{\alpha \in I_{r_2,p}} |(BB)^\alpha|}
\]

where

\[
d_j^B := \left( \sum_{\alpha \in I_{r_2,p} \{j\}} \text{rdet}_j \left( (BB)^\alpha \cdot (d_k^A) \right) \right) \in \mathbb{H}^{n \times 1}
\]

\[
d_i^A := \left( \sum_{\beta \in J_{r_1,n} \{i\}} \text{cdet}_i \left( (A^\dagger A)^\beta \cdot (d_l^B) \right) \right) \in \mathbb{H}^{1 \times p}
\]

are the column-vector and the row-vector, respectively. \( d_k^A \) and \( d_l^B \) are the \( k \)th row and the \( l \)th column of \( \tilde{D} \) for all \( k = 1, ..., n, l = 1, ..., p \).

(ii) If \( \text{rank } A = n \) and \( \text{rank } B = p \), then

\[
x_{ij} = \frac{\text{cdet}_i (A^\dagger A) \cdot (d_j^B)}{\text{det}(A^\dagger A) \cdot \text{det}(BB^\dagger)}
\]

or

\[
x_{ij} = \frac{\text{rdet}_j (BB^\dagger) \cdot (d_i^A)}{\text{det}(A^\dagger A) \cdot \text{det}(BB^\dagger)}
\]

where

\[
d_j^B := \left( (BB^\dagger) \cdot (d_k^A) \right) \in \mathbb{H}^{n \times 1}
\]

\[
d_i^A := \left( (A^\dagger A) \cdot (d_l^B) \right) \in \mathbb{H}^{1 \times p}
\]

(iii) If \( \text{rank } A = n \) and \( \text{rank } B = r_2 < p \), then

\[
x_{ij} = \frac{\text{cdet}_i \left( (A^\dagger A) \cdot (d_j^B) \right)}{\text{det}(A^\dagger A) \cdot \sum_{\alpha \in I_{r_2,p}} |(BB)^\alpha|}
\]

or

\[
x_{ij} = \frac{\sum_{\alpha \in I_{r_2,p} \{j\}} \text{rdet}_j \left( (BB)^\alpha \cdot (d_i^A) \right)}{\text{det}(A^\dagger A) \cdot \sum_{\alpha \in I_{r_2,p}} |(BB)^\alpha|}
\]

where \( d_j^B \) is (26) and \( d_i^A \) is (31).
(iv) If \( \text{rank } A = r_1 < n \) and \( \text{rank } B = p \), then

\[
x_{i,j} = \frac{\text{rdet}_j(BB^t)_{j,j} \cdot (d^A_i)}{\sum_{\beta \in J_{r_1,n}} (A^2A)^{\beta}_{\beta} \cdot \det(BB^t)},
\]

or

\[
x_{i,j} = \sum_{\beta \in J_{r_1,n}} \sum_{\alpha \in I_{r_2,p}} \frac{\text{cdet}_i ((A^2A)_{i,i} (d^B_{i,j}))^{\beta}_{\alpha}}{\sum_{\beta \in J_{r_1,n}} (A^2A)^{\beta}_{\beta} \cdot \det(BB^t)},
\]

where \( d^B_{i,j} \) is \((30)\) and \( d^A_i \) is \((27)\).

**Proof.** (i) If \( A \in \mathbb{H}^{m \times n}_1, B \in \mathbb{H}^{p \times q}_r \) and \( r_1 < n, r_2 < p \), then, by Theorem 2.11 the weighted Moore-Penrose inverses \( A^\dagger = \left(a^\dagger_{i,j}\right) \in \mathbb{H}^{n \times m} \) and \( B^\dagger = \left(b^\dagger_{i,j}\right) \in \mathbb{H}^{n \times p} \) possess the following determinantal representations, respectively,

\[
a^\dagger_{i,j} = \frac{\sum_{\beta \in J_{r_1,n}} \text{cdet}_i ((A^2A)_{i,i} (a^2_{i,j}))^{\beta}_{\beta}}{\sum_{\beta \in J_{r_1,n}} (A^2A)^{\beta}_{\beta} \cdot \det(BB^t)},
\]

\[
b^\dagger_{i,j} = \frac{\sum_{\alpha \in I_{r_2,p}} \text{rdet}_j ((BB^t)_{j,j} (b^2_{i,j}))^{\alpha}_{\alpha}}{\sum_{\beta \in J_{r_1,n}} (A^2A)^{\beta}_{\beta} \cdot \det(BB^t)}.
\]

By Lemma 3.1 \( X = A^\dagger_{M,N}DB^\dagger_{P,Q} \) and entries of \( X = (x_{i,j}) \) are

\[
x_{i,j} = \sum_{s=1}^{q} \left( \sum_{k=1}^{m} a^\dagger_{i,k} d_{k,s} \right) b^\dagger_{s,j},
\]

for all \( i = 1, \ldots, n, j = 1, \ldots, p \).

Denote by \( \hat{d}_{s} \) the \( s \)-th column of \( A^2D =: \hat{D} = \left(\hat{d}_{i,j}\right) \in \mathbb{H}^{n \times q} \) for all \( s = 1, \ldots, q \). It follows from \( \sum_{k} a^\dagger_{i,k} d_{k,s} = \hat{d}_{s} \) that

\[
\sum_{k=1}^{m} a^\dagger_{i,k} d_{k,s} = \sum_{k=1}^{m} \frac{\sum_{\beta \in J_{r_1,n}} \text{cdet}_i ((A^2A)_{i,i} (a^2_{i,k}))^{\beta}_{\beta} \cdot d_{k,s}}{\sum_{\beta \in J_{r_1,n}} (A^2A)^{\beta}_{\beta} \cdot \det(BB^t)} = \frac{\sum_{\beta \in J_{r_1,n}} (A^2A)^{\beta}_{\beta} \cdot \det(BB^t)}{\sum_{\beta \in J_{r_1,n}} (A^2A)^{\beta}_{\beta}}.
\]
Suppose $e_s$ and $e_t$ are the unit row-vector and the unit column-vector, respectively, such that all their components are 0, except the $s$th components, which are 1. Substituting (39) and (57) in (38), we obtain

$$x_{ij} = \frac{\sum_{s=1}^{q} \sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( (A^2A)_{\beta} (d_s) \right)_{\beta} \beta \sum_{\alpha \in I_{r_2,p}(j)} \text{rdet}_j \left( (BB^2)_{\alpha} (b^s) \right)_{\alpha}}{\sum_{\beta \in J_{1,n}} \left| (A^2A)_{\beta} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^2)_{\alpha} \right|}.$$  

Since

$$d_s = \sum_{l=1}^{n} e_l d_{ls}, \quad b^s = \sum_{t=1}^{p} b_{st} e_t, \quad \sum_{s=1}^{q} d_{ls} b^s = d_{lt},$$

then we have

$$x_{ij} = \frac{\sum_{t=1}^{p} \sum_{l=1}^{n} \sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( (A^2A)_{\beta} (d_s) \right)_{\beta} \beta \sum_{\alpha \in I_{r_2,p}(j)} \text{rdet}_j \left( (BB^2)_{\alpha} (b^s) \right)_{\alpha}}{\sum_{\beta \in J_{1,n}} \left| (A^2A)_{\beta} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^2)_{\alpha} \right|} = \frac{\sum_{t=1}^{p} \sum_{l=1}^{n} \sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( (A^2A)_{\beta} \tilde{d}_{lt} \right)_{\beta} \beta \sum_{\alpha \in I_{r_2,p}(j)} \text{rdet}_j \left( (BB^2)_{\alpha} \right)_{\alpha}}{\sum_{\beta \in J_{1,n}} \left| (A^2A)_{\beta} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^2)_{\alpha} \right|}.$$  

Denote by

$$d_t^A := \sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( (A^2A)_{\beta} (d_s) \right)_{\beta} \beta \sum_{\alpha \in I_{r_2,p}(j)} \text{rdet}_j \left( (BB^2)_{\alpha} (b^s) \right)_{\alpha}$$

the $t$th component of a row-vector $d^A_t = (d^A_{t1}, ..., d^A_{tp})$ for all $t = 1, ..., p$. Substituting it in (40), we have

$$x_{ij} = \frac{\sum_{t=1}^{p} d^A_t \sum_{\alpha \in I_{r_2,p}(j)} \text{rdet}_j \left( (BB^2)_{\alpha} (e_t) \right)_{\alpha}}{\sum_{\beta \in J_{1,n}} \left| (A^2A)_{\beta} \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^2)_{\alpha} \right|}.$$  

Since $\sum_{t=1}^{p} d^A_t e_t = d^A_l$, then it follows (25).

If we denote by

$$a_{ij}^B := \sum_{t=1}^{p} \tilde{d}_{lt} \sum_{\alpha \in I_{r_2,p}(j)} \text{rdet}_j \left( (BB^2)_{\alpha} (e_t) \right)_{\alpha} = \sum_{\alpha \in I_{r_2,p}(j)} \text{rdet}_j \left( (BB^2)_{\alpha} (\tilde{d}_l) \right)_{\alpha},$$

then

$$\sum_{t=1}^{p} d^A_t e_t = d^A_l,$$
the $l$th component of a column-vector $d^B_j = (d^B_{1j}, ..., d^B_{nj})^T$ for all $l = 1, ..., n$ and substitute it in (40), we obtain
\[ x_{ij} = \sum_{l=1}^{n} \sum_{\beta \in J_{r_1, n}} c_{\beta} \det_i \left( (A^\sharp A)_{i,i} (e_i, l) \right) \beta \ d^B_{lj} \sum_{\beta \in J_{r_1, n}} \left| (BB^\sharp)^\alpha \right|_{\alpha}. \]
Since $\sum_{l=1}^{n} e_i d^B_{lj} = d^B_{j}$, then it follows (24).

(ii) If rank $A = n$ and rank $B = p$, then by Theorem 2.11 the weighted Moore-Penrose inverses $A^\dagger_{M,N}$ and $B^\dagger_{P,Q}$ possess the following determinantal representations, respectively,
\[ a^\dagger_{ij} = \frac{\det_i (A^\sharp A)_{i,i} (a^\sharp_{ij}) (A^\sharp A)}{\det(A^\sharp A)} \quad (41) \]
\[ b^\dagger_{ij} = \frac{\det_j (BB^\sharp)_{j,j} (b^\sharp_{ij}) (BB^\sharp)}{\det(BB^\sharp)} \quad (42) \]
By their substituting in (43) and pondering ahead as in the previous case, we obtain (28) and (29).

(iii) If $A \in \mathbb{H}_m^{r_1}$, $B \in \mathbb{H}_p^{r_2}$ and $r_1 = n$, $r_2 < p$, then, for the weighted Moore-Penrose inverses $A^\dagger_{M,N}$ and $B^\dagger_{P,Q}$, the determinantal representations (11) and (36) are more applicable to use, respectively. By their substituting in (43) and pondering ahead as in the previous case, we finally obtain (32) and (33) as well.

(iv) In this case for $A^\dagger_{M,N}$ and $B^\dagger_{P,Q}$, we use the determinantal representations (11) and (37), respectively. \(\square\)

Corollary 3.1 Suppose that $A \in \mathbb{H}_m^{m \times n}$, $D \in \mathbb{H}_m^{m \times p}$, $M$, $N$ are Hermitian positive definite matrices of order $m$ and $n$, respectively, $A^\sharp A$ is Hermitian. Denote $D^\natural = A^\sharp D$. If $D \subset \mathcal{R}_r(AA^\sharp)$ and $D \subset \mathcal{R}_l(A^\sharp A)$,
\[ AX = D, \quad (43) \]
\[ \mathcal{R}_r(X) \subset \mathcal{N}^{-1} \mathcal{R}_r(A^\sharp), \quad \mathcal{R}_l(X) \subset \mathcal{R}_l(A^\sharp) \mathcal{M}, \quad (44) \]
then the unique solution of (43) with the restrictions (44) is
\[ X = A^\dagger_{M,N} D \]
which possess the following determinantal representations.

(i) If rank $A = r_1 < n$, then
\[ x_{ij} = \frac{\sum_{\beta \in J_{r_1, n}} \beta \ c_{\beta} \det_i \left( (A^\sharp A)_{i,i} (d^\natural_{ij}) \right) \beta}{\sum_{\beta \in J_{r_1, n}} \left| (A^\sharp A)_{i,i} \right|^\beta}, \]

(ii) If rank $A = n$ and rank $B = p$, then by Theorem 2.11 the weighted Moore-Penrose inverses $A^\dagger_{M,N}$ and $B^\dagger_{P,Q}$ possess the following determinantal representations, respectively,
where \( \hat{d}_{.,j} \) are the \( j \)th column of \( \hat{D} \) for all \( i = 1, ..., n, j = 1, ..., p \).

(ii) If \( \text{rank } A = n \), then

\[
x_{ij} = \frac{\text{cdet}_i(A^tA) \cdot (\hat{d}_{.,j})}{\det(A^tA)},
\]

**Proof.** The proof follows evidently from Theorem 3.1 when \( B \) be removed, and unit matrices insert instead \( P, Q \).

**Corollary 3.2** Suppose that \( B \in \mathbb{H}_{r_2}^{p \times q} \), \( D \in \mathbb{H}^{n \times q} \), \( P \), and \( Q \) are Hermitian positive definite matrices of order \( p \) and \( q \), respectively, \( BB^\dagger \) is Hermitian. Denote \( \hat{D} = DB^\dagger \). If \( D \subset R_r(B^\dagger B) \) and \( D \subset R_i(BB^\dagger) \),

\[
XB = D,
\]

\[
N_r(X) \supset P^{-1}N_r(B^\dagger), N_i(X) \supset N_i(B^\dagger)Q,
\]

then the unique solution of (45) with the restrictions (46) is

\[
X = DB^\dagger_{P,Q}
\]

which possess the following determinantal representations.

(i) If \( \text{rank } B = r_2 < p \), then

\[
x_{ij} = \frac{\sum_{\alpha \in I_{r_2,q}} \text{rdet}_j \left( (BB^\dagger)_{j, \cdot} (\hat{d}_{.,i}) \right)}{\sum_{\alpha \in I_{r_2,q}} |(BB^\dagger)_{\alpha}|},
\]

where \( \hat{d}_{.,i} \) are the \( i \)th row of \( \hat{D} \) for all \( i = 1, ..., n, j = 1, ..., p \).

(ii) If \( \text{rank } B = p \), then

\[
x_{ij} = \frac{\text{rdet}_j (BB^\dagger)_{j, \cdot} (\hat{d}_{.,i})}{\det(BB^\dagger)}.
\]

**Proof.** The proof follows evidently from Theorem 3.1 when \( A \) be removed and unit matrices insert instead \( M, N \).

### 3.2 The Case of Both Non-Hermitian Matrices \( A^tA \) and \( BB^\dagger \)

Denote \( \tilde{A} := M^\dagger AN^{-\frac{1}{2}} = (\tilde{a}_{ij}) \in \mathbb{H}_{m \times n}, \tilde{A}^\ast = N^{-\frac{1}{2}}A^\ast M^\dagger \) and \( \tilde{B} := P^\dagger BQ^{-\frac{1}{2}} = (\tilde{b}_{ij}) \in \mathbb{H}_{p \times q}, \tilde{B}^\ast = Q^{-\frac{1}{2}}B^\ast P^\dagger \).

**Theorem 3.2** Let \( A^tA \) and \( BB^\dagger \) be both non-Hermitian. Then the solution (45) possess the following determinantal representations.
(i) If \( \text{rank } A = r_1 < n \) and \( \text{rank } B = r_2 < p \), then
\[
x_{ij} = \frac{\sum_k n_{ik}^{(-\frac{1}{2})} \sum_{\beta \in J_{1\cdot n}(k)} \cdet_k \left( (\tilde{A}^* \tilde{A})^\beta_k (d^B_{ij})^\beta \right)}{\sum_{\beta \in J_{1\cdot n}} \left| (\tilde{A}^* \tilde{A})^\beta \right| \sum_{\alpha \in I_{2\cdot p}} \left| (BB^*)^\alpha \right|},
\]
(iii) If \( \text{rank } A = r_1 < n \) and \( \text{rank } B = r_2 < p \), then
\[
x_{ij} = \frac{\sum_{l} \sum_{\alpha \in I_{2\cdot p}(l)} \rdet_l \left( (\tilde{B}B^*)^\alpha \right) \cdet_k \left( (\tilde{A}^* \tilde{A})^\beta_k (d^A_{ij})^\beta \right)}{\sum_{\beta \in J_{1\cdot n}} \left| (\tilde{A}^* \tilde{A})^\beta \right| \sum_{\alpha \in I_{2\cdot p}} \left| (BB^*)^\alpha \right|},
\]
where
\[
d^B_{ij} = \left( \sum_{l} \sum_{\alpha \in I_{2\cdot p}(l)} \rdet_l \left( (\tilde{B}B^*)^\alpha \right) \left( d^A_{ij} \right) \right) \in \mathbb{H}^{n \times 1}
\]
\[
d^A_{ij} = \left( \sum_{k} n_{ik}^{(-\frac{1}{2})} \sum_{\beta \in J_{1\cdot n}(k)} \cdet_k \left( (\tilde{A}^* \tilde{A})^\beta_k (d^B_{ij})^\beta \right) \right) \in \mathbb{H}^{1 \times p}
\]
are the column-vector and the row-vector, respectively. \( \tilde{d}_l \) and \( \tilde{d}_f \) are the \( t \)th row and the \( f \)th column of \( \tilde{D} := N^{-\frac{1}{2}} A^* MDQ^{-1} B^* P^{\frac{1}{2}} = (\tilde{d}_{ij}) \in \mathbb{H}^{n \times p} \) for all \( t = 1, \ldots, n, f = 1, \ldots, p \).

(ii) If \( \text{rank } A = n \) and \( \text{rank } B = p \), then
\[
x_{ij} = \frac{\cdet_i \left( (A^* MA)^i \right) \left( d^B_{ij} \right)}{\det(A^* MA) \cdot \det(BQ^{-1} B^*)},
\]
(iii) If \( \text{rank } A = n \) and \( \text{rank } B = r_2 < p \), then
\[
x_{ij} = \frac{\cdet_i \left( (A^* MA)^i \right) \left( d^B_{ij} \right)}{\det(A^* MA) \cdot \sum_{\alpha \in I_{2\cdot p}} \left| (BB^*)^\alpha \right|},
\]
or
\[
x_{ij} = \frac{\sum l \sum_{\alpha \in I_{r_2,p}} \text{rdet}_l \left( \left( \tilde{B} \tilde{B}^* \right)_l \left( d_A^\alpha \right)_l \cdot m_{ij}^{(\beta)} \right)}{\det(A^*M) \cdot \sum_{\alpha \in I_{r_2,p}} \left| \left( \tilde{B} \tilde{B}^* \right)_\alpha \right|},
\]
(56)
where \( d_B^\beta_{ij} \) is (49) and \( d_A^\alpha \) is (54).

(iv) If \( \text{rank } A = r_1 < n \) and \( \text{rank } B = p \), then
\[
x_{ij} = \frac{\text{rdet}_j(BQ^{-1}B^*)_j \cdot (d_A^\alpha)}{\sum_{\beta \in J_{r_1,n}} \left( \tilde{A}^* \tilde{A} \right)^\beta_{\beta} \cdot \det(BQ^{-1}B^*)},
\]
(57)
or
\[
x_{ij} = \frac{\sum n \left(-\frac{1}{2}\right) \sum_{\beta \in J_{r_1,n}} \text{cdet}_k \left( \left( \tilde{A}^* \tilde{A} \right)_k \left( d_B^\beta \right)_\beta \right)}{\sum_{\beta \in J_{r_1,n}} \left( \tilde{A}^* \tilde{A} \right)^\beta_{\beta} \cdot \det(BQ^{-1}B^*)},
\]
(58)
where \( d_B^\beta \) is (50) and \( d_A^\alpha \) is (54).

Proof. (i) If \( A \in \mathbb{H}^{m \times n} \), \( B \in \mathbb{H}^{p \times q} \) are both non-Hermitian, and \( r_1 < n \), \( r_2 < p \), then, by Theorem 2.12, the weighted Moore-Penrose inverses \( A^\dagger = \left( a^\dagger_{ij} \right) \in \mathbb{H}^{n \times m} \) and \( B^\dagger = \left( b^\dagger_{ij} \right) \in \mathbb{H}^{q \times p} \) possess the following determinantal representations, respectively,
\[
a^\dagger_{ij} = \frac{\sum n \left(-\frac{1}{2}\right) \sum_{\beta \in J_{r_1,n}} \text{cdet}_k \left( \left( \tilde{A}^* \tilde{A} \right)_k \left( \tilde{a}^\dagger_{ij} \right) \right)^\beta_{\beta}}{\sum_{\beta \in J_{r_1,n}} \left( \tilde{A}^* \tilde{A} \right)^\beta_{\beta}},
\]
(59)
where \( \tilde{a}^\dagger_{ij} \) is the \( j \)th column of \( N^{-\frac{1}{2}} A^* M \);
\[
b^\dagger_{ij} = \frac{\sum_l \sum_{\alpha \in I_{r_2,p}} \text{rdet}_l \left( \left( \tilde{B} \tilde{B}^* \right)_l \left( \tilde{b}^\dagger_{ij} \right) \right)^\alpha_{\alpha} \cdot m_{ij}^{(\beta)}}{\sum_{\alpha \in I_{r_2,p}} \left| \left( \tilde{B} \tilde{B}^* \right)_\alpha \right|},
\]
(60)
where \( \tilde{b}^\dagger_{ij} \) is the \( i \)th row of \( Q^{-1}B^*P^\dagger \). By Lemma 3.1, \( X = A^\dagger_{M,N} DB^\dagger_{P,Q} \) and entries of \( X = (x_{ij}) \) are
\[
x_{ij} = \sum_s \left( \sum_{t=1}^m a^\dagger_{it} d_{ts} \right) b^\dagger_{sj},
\]
(61)
for all \( i = 1, ..., n, \ j = 1, ..., p \).
Denote by $\hat{d}_s$ the $s$th column of $N^{-\frac{1}{2}}A^*MD =: \hat{D} = (\hat{d}_{ij}) \in \mathbb{H}^{n \times q}$ for all $s = 1, \ldots, q$. It follows from $\sum_t \hat{a}_t d_{ts} = \hat{d}_s$ that

$$
\sum_{t=1}^m a^t_id_{ts} = \sum_{t=1}^m \sum_{k} n_{ik} \sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} (\hat{a}_t) \right)^{\beta} \cdot d_{ts} =
$$

$$
\sum_{s=1}^q n_{ik} \sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} (\hat{d}_s) \right)^{\beta} \cdot d_{ts} =
$$

$$
\sum_{s=1}^q \sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} (\hat{d}_s) \right)^{\beta} \cdot d_{ts} =
$$

$$
\sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} \right)^{\beta} \cdot d_{ts}.
$$

(62)

Suppose $e_s$ and $e_s$ are the unit row-vector and the unit column-vector, respectively, such that all their components are 0, except the $s$th components, which are 1. Substituting (62) and (60) in (61), we obtain

$$
x_{ij} = \sum_{s=1}^q n_{ik} \sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} (\hat{d}_s) \right)^{\beta} \cdot d_{ts} =
$$

$$
\sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} \right)^{\beta} \cdot d_{ts}.
$$

Since

$$
\hat{d}_s = \sum_{l=1}^n e_l \hat{d}_{ls}, \quad \hat{b}_s = \sum_{t=1}^p \hat{b}_{st} e_t, 
$$

then we have

$$
x_{ij} =
$$

$$
\sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} \right)^{\beta} \cdot d_{ts}.
$$

(63)

Denote by

$$
d_{it}^s := \sum_{k} n_{ik} \sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} (\hat{d}_s) \right)^{\beta} =
$$

$$
\sum_{f=1}^n n_{ik} \sum_{\beta \in J_{1,n}} \text{cdet}_k \left( \left( \hat{A}^* \hat{A} \right)_{\beta k} (\hat{e}_f) \right)^{\beta} d_{ft}.
$$
the $t$th component of the row-vector $d^A_t = (d^A_{t1}, ..., d^A_{tp})$ for all $t = 1, ..., p$.
Substituting it in (63), we have

$$x_{ij} = \frac{\sum_{t=1}^{p} d^A_t \sum_{l} \sum_{\alpha \in I_{2,p}(l)} r \det_l \left( (\tilde{B}B^*)_{l} \cdot (e_t) \right)_{\alpha} \cdot m_{ij}^{(\tilde{q})}}{\sum_{\beta \in J_{1,n}} (\tilde{A} \cdot \tilde{A})_{\beta} \sum_{\alpha \in I_{2,p}} (\tilde{B}B^*)_{\alpha}}.$$ 

Since $\sum_{t=1}^{p} d^A_t e_t = d^A$, then it follows (68).

If we denote by

$$\sum_{t=1}^{p} d^B_{ft} \sum_{l} \sum_{\alpha \in I_{2,p}(l)} r \det_l \left( (\tilde{B}B^*)_{l} \cdot (\tilde{d}_f) \right)_{\alpha} \cdot m_{ij}^{(\tilde{q})} =$$

$$\sum_{l} \sum_{\alpha \in I_{2,p}(l)} r \det_l \left( (\tilde{B}B^*)_{l} \cdot (\tilde{d}_f) \right)_{\alpha} \cdot m_{ij}^{(\tilde{q})} =: d^B_{fj}$$

the $f$th component of the column-vector $d^B_{f} = (d^B_{f1}, ..., d^B_{fn})^T$ for all $f = 1, ..., n$ and substitute it in (63), then

$$x_{ij} = \frac{\sum_{f=1}^{n} \sum_{k} n_{ik}^{(-\frac{q}{2})} \sum_{\beta \in J_{1,n}(k)} c \det_k \left( (A^*A)_{k} \cdot (e_{f}) \right)_{\beta} \cdot d^B_{jf}}{\sum_{\beta \in J_{1,n}} (A^*A)_{\beta} \sum_{\alpha \in I_{2,p}} (BB^*)_{\alpha}.}$$

Since $\sum_{f=1}^{n} e_{f} d^B_{jf} = d^B_{j}$, then it follows (70).

(ii) If rank $A = n$ and rank $B = p$, then by Theorem 2.12 the weighted Moore-Penrose inverses

$A^\dagger_{M,N} = (a^\dagger_{ij}) \in \mathbb{H}^{n \times m}$ and $B^\dagger_{P,Q} = (b^\dagger_{ij}) \in \mathbb{H}^{q \times p}$

possess the following determinantal representations, respectively,

$$a^\dagger_{ij} = \frac{\det_i(A^*MA)_i(\tilde{a}_{ij})}{\det(A^*MA)},$$

$$b^\dagger_{ij} = \frac{r \det_j(BQ^{-1}B^*)_j(\tilde{b}_{ij})}{\det(BQ^{-1}B^*)}.$$ 

where $\tilde{a}_{ij}$ is the $j$th column of $A^*M$ for all $j = 1, \ldots, m$, and $\tilde{b}_{ij}$ is the $i$th row of $Q^{-1}B^*$ for all $i = 1, \ldots, n$.

By their substituting in (65), we obtain

$$x_{ij} = \frac{\sum_{t=1}^{p} \sum_{f=1}^{n} \det_i(A^*MA)_i(e_{f}) \tilde{d}_f r \det_j(BQ^{-1}B^*)_j(e_{f})}{\det(A^*MA)\det(BQ^{-1}B^*)},$$

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where \( \tilde{d}_{ft} \) is the \((ft)\)th entry of \( \tilde{D} := A^*MDQ^{-1}B^* \) in this case. Denote by

\[
d^A_{it} := \text{cdet}_i(A^*MA)_{.,t} (\tilde{d}_t)
\]

the \(t\)th component of the row-vector \( d^A_t = (d^A_{i1}, ..., d^A_{ip}) \) for all \( t = 1, ..., p \). Substituting it in (63), it follows (51).

Similarly, we can obtain (52).

(iii) If \( A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{p \times q} \) and \( r_1 = n, r_2 < p \), then, for the weighted Moore-Penrose inverses \( A^*_{M,N} \) and \( B^*_{P,Q} \), the determinantal representations (64) and (59) are more applicable to use, respectively. By their substituting in (61) and pondering ahead as in the previous case, we finally obtain (55) and (56) as well.

(iv) In this case for \( A^*_{M,N} \) and \( B^*_{P,Q} \), we use the determinantal representations (59) and (65), respectively. □

**Corollary 3.3** Suppose that \( A \in \mathbb{H}^{m \times n}, D \in \mathbb{H}^{m \times p}, M, N \) are Hermitian positive definite matrices of order \( m \) and \( n \), respectively, and \( A^* \) is non-Hermitian. If \( D \subset \mathbb{R}_{r_1}(AA^*) \) and \( D \subset \mathbb{R}_{r_2}(A^*A) \), then the unique solution \( X = A^*_{M,N}D \) of the equation \( AX = D \) with the restrictions (44) possess the following determinantal representations.

(i) If \( \text{rank} A = r_1 < n \), then

\[
x_{ij} = \frac{\sum_k n_k^{(-\frac{1}{2})} \sum_{\beta \in J_{r_1 \times r_1}} \text{cdet}_k \left( \left( A^*A \right)_{.,\beta} \right)^\beta}{\sum_{\beta \in J_{r_1 \times n}} \left( A^*A \right)_{.,\beta}^\beta},
\]

where \( \tilde{d}_{j} \) are the \( j \)th column of \( \tilde{D} = N^{-\frac{1}{2}}A^*MD \) for all \( i = 1, ..., n \), \( j = 1, ..., p \).

(ii) If \( \text{rank} A = n \), then

\[
x_{ij} = \frac{\text{cdet}_i (A^*MA)_{.,i} (\tilde{d}_j)}{\det (A^*MA)},
\]

where \( \tilde{d}_{j} \) are the \( j \)th column of \( \tilde{D} = A^*MD \).

**Proof.** The proof follows evidently from Theorem 3.2 when \( B \) be removed and unit matrices insert instead \( P, Q \).

**Corollary 3.4** Suppose that \( B \in \mathbb{H}^{p \times q}, D \in \mathbb{H}^{n \times q}, P, Q \) are Hermitian positive definite matrices of order \( p \) and \( q \), respectively, and \( BB^* \) is non-Hermitian. If \( D \subset \mathbb{R}_r(B^*B) \) and \( D \subset \mathbb{R}_l(BB^*) \), then the unique solution \( X = DB^*_{P,Q} \) of the equation \( XB = D \) with the restrictions (46) possess the following determinantal representations.
(i) If rank $B = r_2 < p$, then

$$x_{ij} = \frac{\sum_{\alpha \in I_{r_2,q}} \text{rdet}_j \left( \left( \tilde{B}\tilde{B}^* \right)_j \left( \tilde{d}_{i.} \right) \right)_\alpha}{\sum_{\alpha \in I_{r_2,q}} \left( \left( \tilde{B}\tilde{B}^* \right)_\alpha \right)} ,$$

where $\tilde{d}_{i.}$ are the $i$th row of $\tilde{D} = DQ^{-1}B^*P$ for all $i = 1, ..., n, j = 1, ..., p$.

(ii) If rank $B = p$, then

$$x_{ij} = \frac{\text{rdet}_j \left( \left( BQ^{-1}B^* \right) \right)_j \left( \tilde{d}_{i.} \right)}{\det \left( \left( BQ^{-1}B^* \right) \right)} ,$$

where $\tilde{d}_{i.}$ are the $i$th row of $\tilde{D} = DQ^{-1}B^*$.

**Proof.** The proof follows evidently from Theorem 3.2 when $A$ be removed and unit matrices insert instead $M, N$.

### 3.3 Mixed Cases

In this subsection we consider mixed cases when only one from the pair $A^\sharp A$ and $BB^\sharp$ is non-Hermitian. We give this theorems without proofs, since their proofs are similar to the proof of Theorems 3.1 and 3.2.

**Theorem 3.3** Let $A^\sharp A$ be Hermitian and $BB^\sharp$ be non-Hermitian. Then the solution (23) possess the following determinantal representations.

(i) If rank $A = r_1 < n$ and rank $B = r_2 < p$, then

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,n} \setminus \{i\}} \text{cdet}_i \left( \left( A^\sharp A \right)_i \left( d_{.j} \right) \right)_{\beta}}{\sum_{\beta \in J_{r_1,n}} \left( \left( A^\sharp A \right)^{\beta} \right) \sum_{\alpha \in I_{r_2,p}} \left( \tilde{B}\tilde{B}^* \right)_{\alpha}^{\beta} } ,$$

or

$$x_{ij} = \sum_{l} \sum_{\alpha \in I_{r_2,p}} \text{rdet}_l \left( \left( \tilde{B}\tilde{B}^* \right)_l \left( d_{i.} \right) \right)_{\alpha}^{\beta} \cdot m_{ij}^{(4)} \sum_{\beta \in J_{r_1,n}} \left( \left( A^\sharp A \right)^{\beta} \right) \sum_{\alpha \in I_{r_2,p}} \left( \tilde{B}\tilde{B}^* \right)_{\alpha}^{\beta} ,$$

where

$$d_{i.}^{(4)} = \left( \sum_{l} \sum_{\alpha \in I_{r_2,p}} \text{rdet}_l \left( \left( \tilde{B}\tilde{B}^* \right)_l \left( d_{i.} \right) \right)_{\alpha}^{\beta} \cdot m_{ij}^{(4)} \right) \in \mathbb{H}^{n \times 1} \quad (67)$$
\[ d_i^A = \left( \sum_{\beta \in J_{r_1,n,\{i\}}} \text{cdet}_i \left( (A^2A)_i \left( \tilde{d}.f \right) \right) \right)^{\beta} \in H^{1 \times p} \quad (68) \]

are the column-vector and the row-vector, respectively. \( \tilde{d}_t \) and \( \tilde{d}.f \) are the \( t \)th row and the \( f \)th column of \( \tilde{D} = A^2DQ^{-1}B^*P^\perp \) for all \( t = 1, \ldots, n \), \( f = 1, \ldots, p \).

(ii) If rank \( A = n \) and rank \( B = p \), then

\[
x_{ij} = \frac{\text{cdet}_i (A^2A)_i (d^B_j)}{\det(A^2A) \cdot \det(BQ^{-1}B^*)},
\]

or

\[
x_{ij} = \frac{\text{rdet}_j (BQ^{-1}B^*)_j (d^A_i)}{\det(A^2A) \cdot \det(BQ^{-1}B^*)},
\]

where

\[
d^B_j := \left( \text{rdet}_j (BQ^{-1}B^*)_j \left( \tilde{d}_t \right) \right) \in H^{n \times 1}, \quad (69)
\]

\[
d^A_i := \left( \text{cdet}_i (A^2A)_i \left( \tilde{d}.f \right) \right) \in H^{1 \times p}, \quad (70)
\]

\( \tilde{d}_t \) and \( \tilde{d}.f \) are the \( t \)th row and \( f \)th column of \( \tilde{D} = A^2DQ^{-1}B^* \).

(iii) If rank \( A = n \) and rank \( B = r_2 < p \), then

\[
x_{ij} = \frac{\text{cdet}_i ((A^2A)_i) (d^B_j)}{\det(A^2A) \cdot \sum_{\alpha \in I_{r_2,p}} \left[ (\tilde{B}B^*)^\alpha \right]},
\]

or

\[
x_{ij} = \frac{\sum_{l} \sum_{\alpha \in I_{r_2,p}} \text{rdet}_l \left( (\tilde{B}B^*)^l \right) (d^A_i)^\alpha \cdot m_{ij}^{(l)}}{\det(A^2A) \cdot \sum_{\alpha \in I_{r_2,p}} \left[ (\tilde{B}B^*)^\alpha \right]},
\]

where \( d^B_j \) is (67) and \( d^A_i \) is (68).

(iv) If rank \( A = r_1 < n \) and rank \( B = p \), then

\[
x_{ij} = \frac{\text{rdet}_j (BQ^{-1}B^*)_j (d^A_i)}{\sum_{\beta \in J_{r_1,n,\{i\}}} \left| (A^2A)^\beta \right| \cdot \det(BQ^{-1}B^*)},
\]

or

\[
x_{ij} = \frac{\sum_{\beta \in J_{r_1,n,\{i\}}} \text{cdet}_i ((A^2A)_i) (d^B_j)^\beta}{\sum_{\beta \in J_{r_1,n,\{i\}}} \left| (A^2A)^\beta \right| \cdot \det(BQ^{-1}B^*)},
\]

where \( d^B_j \) is (69) and \( d^A_i \) is (70).
Theorem 3.4 Let $A^4A$ be non-Hermitian, and $BB^4$ be Hermitian. Denote $\tilde{D} := A^*DB^3$. Then the solution (23) possess the following determinantal representations.

(i) If $\text{rank } A = r_1 < n$ and $\text{rank } B = r_2 < p$, then

$$x_{ij} = \frac{\sum_k n_{ik}^{(-\frac{1}{2})} \sum_{\beta \in J_{r_1,n}} \text{cdet}_k \left( (\hat{A}^*\hat{A})_k (d^B_{ij}) \right)_{\beta}}{\sum_{\beta \in J_{r_1,n}} \left( (\hat{A}^*\hat{A})_{\beta} \right)_{\beta} \sum_{\alpha \in I_{r_2,p}} |(BB^3)_{\alpha}|},$$

or

$$x_{ij} = \frac{\sum_{\alpha \in I_{r_2,p}} \text{rdet}_j \left( (BB^3)_{j} (d^A) \right)_{\alpha}}{\sum_{\beta \in J_{r_1,n}} \left( (\hat{A}^*\hat{A})_{\beta} \right)_{\beta} \sum_{\alpha \in I_{r_2,p}} |(BB^3)_{\alpha}|},$$

where

$$d^B_{ij} = \left( \sum_{\alpha \in I_{r_2,p}} \text{rdet}_j \left( (BB^3)_{j} (d^A_t.) \right)_{\alpha} \right) \in \mathbb{H}^{n \times 1} \tag{71}$$

$$d^A_{ij} = \left( \sum_k n_{ik}^{(-\frac{1}{2})} \sum_{\beta \in J_{r_1,n}} \text{cdet}_k \left( (\hat{A}^*\hat{A})_k (d^A_f) \right)_{\beta} \right) \in \mathbb{H}^{1 \times p} \tag{72}$$

are the column-vector and the row-vector, respectively. $d_t$ and $d_f$ are the $t$th row and the $f$th column of $\tilde{D} := N^{-\frac{1}{2}}A^*MDB^3$ for all $t = 1, ..., n$, $f = 1, ..., p$.

(ii) If $\text{rank } A = n$ and $\text{rank } B = p$, then

$$x_{ij} = \frac{\text{cdet}_i(A^*MA)_i (d^B_{ij})}{\det(A^*MA) \cdot \det(BB^3)},$$

or

$$x_{ij} = \frac{\text{rdet}_j (BB^3)_j (d^A_{ij})}{\det(A^*MA) \cdot \det(BB^3)},$$

where

$$d^B_{ij} := \left( \text{rdet}_j (BB^3)_j (\hat{d}_t.) \right) \in \mathbb{H}^{n \times 1}, \tag{73}$$

$$d^A_{ij} := \left( \text{cdet}_i(A^*MA)_i (\hat{d}_f) \right) \in \mathbb{H}^{1 \times p}, \tag{74}$$

$\hat{d}_t, \hat{d}_f$ are the $t$th row and $f$th column of $\tilde{D} = A^*MDB^3$. 

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(iii) If \( \text{rank } A = n \) and \( \text{rank } B = r_2 < p \), then

\[
x_{ij} = \frac{\text{cdet}_i \left( (A^*MA)_i (d^B) \right)}{\text{det}(A^*MA) \cdot \sum_{\alpha \in I_{r_2,p}} |(BB^p)_\alpha|},
\]

or

\[
x_{ij} = \frac{\sum_{\alpha \in I_{r_2,p}} \text{rdet}_j \left( (BB^p)_j (d^A) \right)_\alpha}{\text{det}(A^*MA) \cdot \sum_{\alpha \in I_{r_2,p}} |(BB^p)_\alpha|},
\]

where \( d^B \) is (71) and \( d^A \) is (74).

(iv) If \( \text{rank } A = r_1 < n \) and \( \text{rank } B = p \), then

\[
x_{ij} = \frac{\text{rdet}_j \left( (BB^p)_j (d^A) \right)}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_\beta^\beta \right| \cdot \text{det}(BB^p)},
\]

or

\[
x_{ij} = \frac{\sum_{\beta \in J_{r_1,n}} \text{cdet}_i \left( (A^*A)_i (d^B) \right)_\beta}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_\beta^\beta \right| \cdot \text{det}(BQ^{-1}B^*)},
\]

where \( d^B \) is (73) and \( d^A \) is (72).

3.4 An example

Let us consider the restricted matrix equation

\[
AXB = D,
\]

\[
\mathcal{R}_r(X) \subset N^{-1} \mathcal{R}_r(A^*), \mathcal{N}_r(X) \supset P^{-1} \mathcal{N}_r(B^*),
\]

\[
\mathcal{R}_l(X) \subset \mathcal{R}_l(A^*)M, \mathcal{N}_l(X) \supset \mathcal{N}_l(B^*)Q
\]

where

\[
A = \begin{pmatrix} k & -j & j \\ 1 & 0 & k \end{pmatrix}, \quad N = \begin{pmatrix} 5 & 0 & -4j \\ 0 & 4 & 0 \\ 4j & 0 & 5 \end{pmatrix}, \quad M = \begin{pmatrix} 5 & 4k \\ -4k & 5 \end{pmatrix}, \quad D = \begin{pmatrix} i & -j & k \\ -k & 0 & j \end{pmatrix},
\]

\[
B = \begin{pmatrix} k & -j & j \\ 0 & 1 & i \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & i & 0 \\ -i & 2 & -j \\ 0 & j & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 2.5 & -1.5j \\ 1.5j & 2.5 \end{pmatrix}.
\]

Since

\[
A^* = \begin{pmatrix} -k & 1 & i \\ 0 & -j & -k \end{pmatrix}, \quad B^* = \begin{pmatrix} -k & 0 \\ j & 1 \\ -j & -i \end{pmatrix},
\]

\[
A^* = \begin{pmatrix} k & -j & j \\ 1 & 0 & k \end{pmatrix}, \quad B^* = \begin{pmatrix} 5 & 0 \\ -4k & 5 \\ j & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & -j & k \\ j & 0 & 5 \end{pmatrix}, \quad P = \begin{pmatrix} -k & 1 \\ 0 & -j & -i \end{pmatrix}.
\]
\[ \det(AA^*) = \det \begin{pmatrix} 3 & -i + k \\ i - k & 2 \end{pmatrix} = 4, \quad \det(BB^*) = \det \begin{pmatrix} 3 & -j + k \\ j - k & 2 \end{pmatrix} = 4, \]

then \( \text{rank} A = \text{rank} B = 2. \)

Due to Theorem 2.4, we can obtain the inverses

\[ Q^{-1} = \frac{1}{3} \begin{pmatrix} 5 & -4i & -3k \\ 4i & 5 & 3j \\ 3k & -3j & 3 \end{pmatrix}, \quad N^{-1} = \frac{1}{9} \begin{pmatrix} \frac{5}{4} & 0 & \frac{4i}{4} \\ 0 & \frac{1}{4} & 0 \\ -\frac{4i}{4} & 0 & \frac{5}{4} \end{pmatrix}. \]

It is easy to verify that the both matrices \( A^\dagger A = N^{-1}A^*MA \) and \( BB^\dagger = BQ^{-1}B^*P \) are not Hermitian. Hence, we shall find the solution of (75) by (57).

So,

\[ x_{ij} = \frac{\text{rdet}_j(BQ^{-1}B^*), (d^A_i)}{\sum_{\beta \in J_{2,3}} \left| \left( \tilde{A}^\dagger \tilde{A} \right)_\beta \right| \cdot \det(BQ^{-1}B^*)}, \quad i = 1, 2, 3, \quad j = 1, 2, \quad (76) \]

where \( d^A_i \) is \( (70) \), namely

\[ d^A_i = \left( \sum_k n_{ik} (-1)^k \sum_{\beta \in J_{2,3}(k)} \text{cdet}_k \left( \left( \tilde{A}^\dagger \tilde{A} \right)_\beta \left( \tilde{d}_j \right) \right) \right) \in \mathbb{H}^{1 \times 2}. \quad (77) \]

To obtain \( N^{-\frac{1}{2}} \), we firstly find the eigenvalues of \( N \) which are the roots of the characteristic polynomial

\[ p(\lambda) = \det \begin{pmatrix} \lambda - 5 & 0 & 4j \\ 0 & \lambda - 4 & 0 \\ -4j & 0 & \lambda - 5 \end{pmatrix} = \lambda^3 - 14\lambda^2 + 49\lambda - 36 \Rightarrow \begin{cases} \lambda_1 = 1, \\ \lambda_2 = 4, \\ \lambda_3 = 9. \end{cases} \]

By computing the associated eigenvectors and after their orthonormalization, we obtain the unitary matrix \( U \) whose columns are this eigenvectors.

\[ U = \begin{pmatrix} 0.5 - 0.5j & 0 & 0.5 + 0.5j \\ 0 & 1 & 0 \\ 0.5 + 0.5j & 0 & 0.5 - 0.5j \end{pmatrix}. \]

Finally, we have

\[ N^{-\frac{1}{2}} = U^*D U = \begin{pmatrix} 2 & 0 & -j \\ 0 & 2 & 0 \\ j & 0 & 2 \end{pmatrix}, \]

where \( D = \text{diag}(1, 2, 3) \) is the diagonal matrix with 1, 2, 3 on the principal diagonal. Then by Theorem 2.4

\[ N^{-\frac{1}{2}} = \begin{pmatrix} \frac{5}{3} & 0 & \frac{4}{3} \\ 0 & \frac{1}{2} & 0 \\ -\frac{4}{3} & 0 & \frac{5}{3} \end{pmatrix}. \]
Similarly, $\mathbf{M} = \begin{pmatrix} 2 & k \\ -k & 2 \end{pmatrix}$, $\mathbf{P} = \frac{1}{2} \begin{pmatrix} 3 & -j \\ j & 3 \end{pmatrix}$. Further, we find

$$
\bar{\mathbf{D}} := \mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M} \mathbf{D} \mathbf{Q}^{-1} \mathbf{B}^* \mathbf{P} \mathbf{N}^{-\frac{1}{2}} = 
\begin{pmatrix}
\frac{-7}{3} + \frac{125}{6} i + \frac{144}{3} j - \frac{187}{3} k & \frac{47}{2} - \frac{133}{3} i + \frac{61}{3} j - \frac{67}{3} k \\
\frac{35}{3} + \frac{238}{6} i - \frac{101}{3} j + \frac{199}{6} k & \frac{47}{3} + \frac{76}{6} i - 11 j - \frac{232}{6} k
\end{pmatrix},
$$

$$
\bar{\mathbf{A}} = \mathbf{M} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} = 
\begin{pmatrix}
\frac{52}{9} i - 3 j + \frac{2}{3} k & \frac{5}{4} i + 3 j - \frac{2}{3} k \\
-3 i - \frac{56}{9} j - 3 k & \frac{3}{2} + \frac{5}{3} i - \frac{5}{9} k
\end{pmatrix},
$$

$$
\bar{\mathbf{A}}^* \bar{\mathbf{A}} = 
\begin{pmatrix}
\frac{82}{9} - \frac{5}{4} i - 3 j + \frac{2}{3} k & \frac{3}{2} + \frac{5}{3} i - \frac{5}{9} k \\
\frac{3}{2} + \frac{5}{3} i - \frac{5}{9} k & \frac{3}{2} + \frac{5}{3} i - \frac{5}{9} k
\end{pmatrix},
$$

$$
\sum_{\beta \in J_{2,3}} \left| \left( \bar{\mathbf{A}}^* \bar{\mathbf{A}} \right)_{\beta} \right| = \det \left( \frac{82}{5} i - 3 j + \frac{2}{3} k \right) - \frac{5}{4} i + 3 j - \frac{2}{3} k + \det \left( -3 i - \frac{56}{9} j - 3 k \right) = \frac{5}{4} + \frac{5}{4} + 2 = 4.5,
$$

$$
\det \left( \mathbf{B} \mathbf{Q}^{-1} \mathbf{B}^* \right) = \det \left( \frac{10}{1 + 2 i + 4 j - k} \right) = 8.
$$

Now, we obtain components of the row-vectors (77), $\mathbf{d} = (d_{11}, d_{12})$.

$$
d_{11}^A = \frac{2}{3} \left( \text{cdet}_1 \left( \frac{-7}{3} + \frac{125}{6} i + \frac{144}{3} j - \frac{187}{3} k \right) - \frac{5}{6} i + 3 j - \frac{2}{3} k \right) + \text{cdet}_1 \left( \frac{-7}{3} + \frac{125}{6} i + \frac{144}{3} j - \frac{187}{3} k \right) + \frac{1}{3} j \left( \text{cdet}_2 \left( \frac{5}{3} i - \frac{56}{9} j - 3 k \right) \frac{35}{3} + \frac{238}{6} i - \frac{101}{3} j + \frac{199}{6} k \right) - \frac{5}{4} i + 3 j - \frac{2}{3} k + \frac{232}{6} k
$$

Similarly, we obtain $d_{12}^A = \frac{13}{12} - \frac{581}{24} i + \frac{39}{4} j + \frac{101}{8} k$. Moreover,

$$
d_{12}^A = \left( \frac{232}{3} + \frac{1763}{9} i + \frac{1093}{18} j + \frac{1795}{12} k, \frac{419}{6} + \frac{3035}{36} i - \frac{751}{9} j - \frac{1189}{9} k \right)
$$

$$
d_{13}^A = \left( \frac{116}{9} - \frac{239}{12} i + \frac{25}{36} j + \frac{299}{9} k, \frac{117}{24} + \frac{95}{6} i - \frac{99}{8} j + \frac{245}{12} k \right).
$$

Finally, we have
\[ x_{11} = \frac{\text{rdet}_1(BQ^{-1}B^*)_1}{\sum_{\beta \in J_{2,3}} \left| (\tilde{A} \cdot \tilde{A})_\beta \right| \cdot \text{det}(BQ^{-1}B^*)} = \]

\[ = \frac{1}{36} \text{rdet}_1 \left( \begin{array}{c}
- \frac{1}{12} - \frac{45}{32}i + \frac{15}{8}j - \frac{253}{8}k \\
- \frac{1}{12} + \frac{531}{24}i + \frac{39}{4}j + \frac{131}{8}k \\
\end{array} \right) = \]

\[ = \frac{1013}{864} + \frac{1}{144}i - \frac{359}{864}j + \frac{173}{144}k. \]

Similarly, we obtain

\[ x_{12} = \frac{19}{288} - \frac{2459}{864}i - \frac{257}{864}j + \frac{3247}{864}k, \]

and

\[ x_{21} = \frac{1162}{324} - \frac{8935}{1296}i - \frac{5983}{1296}j + \frac{1759}{432}k, \]

\[ x_{22} = \frac{1631}{432} + \frac{1285}{324}i - \frac{817}{324}j - \frac{10631}{432}k, \]

\[ x_{31} = \frac{127}{864} + \frac{83}{864}i - \frac{545}{864}j - \frac{329}{864}k, \]

\[ x_{32} = \frac{311}{1296} + \frac{367}{1296}i - \frac{949}{1296}j + \frac{77}{36}k. \]

Note that we used Maple with the package CLIFFORD in the calculations.

4 Conclusion

In this paper, previously obtained determinantal representations of the quaternion weighted Moore-Penrose inverse have been used to derive explicit determinantal representation formulas for the solution of the two-sided restricted quaternionic matrix equation, \( AXB = D \), within the framework of the theory of column-row determinants (also previously introduced by the author).

References

[1] R. A. Penrose, Generalized inverse for matrices, Proc. Camb. Philos. Soc. 51 (1955) 406–413.

[2] A. Ben-Israel, T.N.E. Grenville, Generalized Inverses: Theory and Applications. Springer-Verlag, Berlin, 2002.

[3] K.M. Prasad, R.B. Bapat, A note of the Khatri inverse, Sankhya: Indian J. Stat. 54 (1992) 291-295.

[4] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21-57.

[5] I.I. Kyrchei, Determinantal representation of the Moore-Penrose inverse matrix over the quaternion skew field, J. Math. Sci. 180(1) (2012) 23-33.
[6] C.F. Van Loan, Generalizing the singular value decomposition, *SIAM J. Numer. Anal.* 13 (1976) 76–83.

[7] E. F. Galba, Weighted singular decomposition and weighted pseudoinversion of matrices, *Ukr. Math. J.* 48(10) (1996) 1618-1622.

[8] I.I. Kyrchei, Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse, Appl. Math. Comput. 309 (2017) 1-16.

[9] P. Stanimirovic’, M. Stankovic’, Determinantal representation of weighted Moore-Penrose inverse, Mat. Vesnik 46 (1994) 41-50.

[10] X. Liu, Y. Yu, H. Wang, Determinantal representation of weighted generalized inverses, Appl. Math. Comput. 218(7) (2011) 3110-3121.

[11] X. Liu, G. Zhu, G. Zhou, Y. Yu, An analog of the adjugate matrix for the outer inverse $A^{(2)}_{T,S}$, Math. Problem in Eng. 2012, Article ID 591256 (2012) 14 pages.

[12] I.I. Kyrchei, Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules, Linear Multilinear Algebra 56 (4) (2008) 453-469.

[13] I.I. Kyrchei, Cramer’s rule for generalized inverse solutions, In: Advances in Linear Algebra Research, I.I. Kyrchei (Ed.), Nova Sci. Publ., New York, pp. 79-132, 2015.

[14] I.I. Kyrchei, Cramer’s rule for quaternion systems of linear equations. Fundamentalnaya i Prikladnaya Matematika 13(4) (2007) 67-94.

[15] I.I. Kyrchei, The theory of the column and row determinants in a quaternion linear algebra, In: Advances in Mathematics Research 15, A.R. Baswell (Ed.), Nova Sci. Publ., New York, pp. 301-359, 2012.

[16] G. Song, Q. Wang, H. Chang, Cramer rule for the unique solution of restricted matrix equations over the quaternion skew field, Comput Math. Appl. 61 (2011) 1576-1589.

[17] G.J. Song, Determinantal representation of the generalized inverses over the quaternion skew field with applications, Appl. Math. Comput. 219 (2012) 656–667.

[18] I.I. Kyrchei, Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations, Linear Algebra Appl. 438(1) (2013) 136–152.

[19] I.I. Kyrchei, Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations, Appl. Math. Comput. 238 (2014) 193–207.
[20] I.I. Kyrchei, Determinantal representations of the W-weighted Drazin inverse over the quaternion skew field, Appl. Math. Comput. 264 (2015) 453–465.

[21] I.I. Kyrchei, Explicit determinantal representation formulas of W-weighted Drazin inverse solutions of some matrix equations over the quaternion skew field, Math. Problem. in Eng. 2016 Article ID 8673809 (2016) 13 pages.

[22] I.I. Kyrchei, Determinantal representations of the Drazin and W-weighted Drazin inverses over the quaternion skew field with applications, In: Quaternions: Theory and Applications, S. Griffin (Ed.), New York: Nova Sci. Publ., pp.201-275, 2017.

[23] A. Kleyn, I. Kyrchei, Relation of row-column determinants with quaside-terminants of matrices over a quaternion algebra, In: Advances in Linear Algebra Research. I. Kyrchei (Ed.), Nova Sci. Publ., New York, pp.299-324, 2015.

[24] G.J. Song, C.Z. Dong, New results on condensed Cramers rule for the general solution to some restricted quaternion matrix equations, J. Appl. Math. Comput. 53 (2017) 321–341.

[25] G.J. Song, Bott-Duffin inverse over the quaternion skew field with applications, J. Appl. Math. Comput. 41 (2013) 377-392.

[26] G.J. Song, Characterization of the W-weighted Drazin inverse over the quaternion skew field with applications, Electron. J. Linear Algebra 26 (2013) 1–14.

[27] Y. Wei, H. Wu, The representation and approximation for the weighted Moore-Penrose inverse, Appl. Math. Comput. 121 (2001) 17-28.

[28] I.V. Sergienko, E.F. Galba, V.S. Deineka,Limiting representations of weighted pseudoinverse matrices with positive definite weights. Problem regularization, Cybernetics and Systems Analysis 39(6) (2003) 816-830.

[29] L. Huang, W. So, On left eigenvalues of a quaternionic matrix, Linear Algebra Appl. 323 (2001) 105-116.

[30] W. So, Quaternionic left eigenvalue problem, Southeast Asian Bulletin of Mathematics 29 (2005) 555-565.

[31] R. M. W. Wood, Quaternionic eigenvalues, Bull. Lond. Math. Soc. 17 (1985) 137-138.

[32] J.L. Brenner, Matrices of quaternions, Pac. J. Math. 1 (1951) 329-335.

[33] E. Macías-Virgós, M.J. Pereira-Sáez, A topological approach to left eigenvalues of quaternionic matrices, Linear Multilinear Algebra 62(2) (2014) 139–158.
[34] A. Baker, Right eigenvalues for quaternionic matrices: a topological approach, Linear Algebra Appl. 286 (1999) 303-309.

[35] T. Dray, C. A. Manogue, The octonionic eigenvalue problem, Advances in Applied Clifford Algebras 8(2) (1998) 341-364.

[36] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21-57.

[37] D.R. Farenick, B.A.F. Pidkowich, The spectral theorem in quaternions, Linear Algebra Appl. 371 (2003) 75-102.

[38] F. O. Farid, Q.W. Wang, F. Zhang, On the eigenvalues of quaternion matrices, Linear Multilinear Algebra 59(4) (2011) 451–473.

[39] R.A. Horn, C.R. Johnson, Matrix Analysis. Cambridge etc., Cambridge University Press 1985.