ON THE LARGE N LIMIT OF 3D
AND 4D HERMITIAN MATRIX MODELS

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Abstract

The large $N$ limit of the hermitian matrix model in three and four Euclidean space-time dimensions is studied with the help of the approximate Renormalization Group recursion formula. The planar graphs contributing to wave function, mass and coupling constant renormalization are identified and summed in this approximation. In four dimensions the model fails to have an interacting continuum limit, but in three dimensions there is a non trivial fixed point for the approximate RG relations. The critical exponents of the three dimensional model at this fixed point are $\nu = 0.665069$ and $\eta = 0.19882$. The existence (or non existence) of the fixed point and the critical exponents display a fairly high degree of universality since they do not seem to depend on the specific (non universal) assumptions made in the approximation.
1) Introduction.

Since ’t Hooft’s original proposal [1], the large $N$ limit has been regarded as one of the most promising attempts to understand strongly coupled gauge theories. The observation that a field theory simplifies in the presence of a large symmetry group is, in fact, more general and has led, among other things, to an understanding of many linear and non linear $\sigma$ models in dimension\(^1\) $2 \leq d < 4$ [2], to the exact solution of matrix models in dimension $d \leq 1$ [3] and to the solution of $d = 2$ QCD [4].

The great simplification arising in the study of $\sigma$ models is the vector-like nature of the fields, i.e. the fact that the number of degrees of freedom grows like $N$. This allows one to reduce the sum of all the leading ”cactus” diagrams to the sum of a finite number of effective graphs, providing in this way a tractable perturbative expansion. The leading term in the expansion already captures the relevant features of the various models, including spontaneous symmetry breaking (or the lack of it in $d = 2$), confinement, dynamical generation of relevant operators, etc...

The situation is more complicated in the case of matrix models, of which QCD is the prime example. There, the number of degrees of freedom grows like $N^2$ and one is faced with the arduous task of summing the leading ”planar” diagrams. This problem has been solved in [5] for the single matrix model ($d = 0$), and for the quantum mechanical anharmonic oscillator ($d = 1$), with its remarkable equivalence to a free Fermi gas. These low dimensional models have been shown to describe $d = 2$ quantum gravity and non critical string theory [3].

Alas, progress has been much slower in the study of higher dimensional matrix models. The factorization properties of the expectation values of observables [6] has led to the conjecture [7] that the leading order in $1/N$ should be dominated by a single field configuration, the ”master field”. Very recently the precise meaning of the master field has been clarified using non commutative probability theory [8]. This recent development is very exciting but is also somewhat worrisome: the connection between the master field and the knowledge of the connected Green functions is so explicit that it might signify that finding the master field is just as hard as exactly solving the large $N$ theory without it! It seems likely that one would have to develop approximations to this construction, treating the large $N$ theory as an ”exact” theory in its own right.

In this paper, we develop an approximation to $d = 3$ and $d = 4$ hermitian matrix models with quartic potential using one of the oldest tools in the non perturbative formulation of quantum field theory: the approximate renormalization group (RG) recursion formula [9]. We make some

\(^1\) Throughout this paper, the letter $d$ refers to the dimension of (Euclidean) space-time.
modification to this technique to make it more suitable for the problem at hand: on the one
hand, we relax the restriction of no wave function renormalization [10] to allow for a non trivial
anomalous dimension $\eta$ and, on the other hand, we restrict our attention to the renormalization
of the quartic term in the interaction. The first change is crucial because, to leading order in $1/N$,
there are already an infinite number of diagrams contributing to $\eta$; neglecting them would ruin the
structure of the RG flow. The second assumption is not crucial, but it simplifies the analysis by
allowing one to perform explicitly the integrals in the recursion formula using the results from the
single $d = 0$ matrix model and to reduce the single recursion step from an integral equation to an
algebraic one.

The main results of this investigation are as follows. In $d = 3$ the model has a non trivial fixed
point in addition to the Gaussian one. The large $N$ critical exponents at this point are computed
to be $\eta = 0.19882$ and $\nu = 0.66517$. In $d = 4$ the model fails to have a non Gaussian large $N$ limit;
its continuum limit is a free theory. This last result may very well be due to the roughness of the
approximation and should not discourage us to study the $d = 4$ theory further.

The paper is organized as follows. In section two, we present the model and qualitatively de-
scribe its renormalization in the large $N$ limit. In section three, we give a sketch of a perturbative
calculation to third order in the coupling constant. This section is not needed for the further calcu-
lations which are intrinsically non-perturbative and it is only given for comparison. In section four,
we derive the approximate recursion formula. All the assumptions that go into its derivation are
spelled out and explicit expressions for wave function, mass and coupling constant renormalization
are obtained in the large $N$ limit. In section five, we study the RG equation derived, establish the
existence of a non Gaussian fixed point for $d = 3$ and compute the critical exponents. We also
comment on the non existence of a non Gaussian fixed point in $d = 4$. All the results for the $d = 0$
hermitian model [5] of direct relevance for this paper are collected in the appendix.

2) The hermitian matrix model and its RG.

The model we study in this paper is the $d = 3, 4$ hermitian matrix model with quartic
interaction. The field variable is a $N \times N$ hermitian matrix $\Phi(x) = \Phi(x)$. In the spirit of Wilson’s
RG, we assume the presence of an effective cut-off $\Lambda_0$ in momentum space. Choosing the mass
unit to be $\Lambda_0$ itself, allows us to set $\Lambda_0 = 1$ and represent all quantities in dimensionless units. In
this way we write the Fourier transform of the field as

$$
\Phi(x) = \int_{0 \leq |k| \leq 1} \frac{d^d k}{(2\pi)^d} e^{ix \cdot k} \Phi(k).
$$

(1)
The Euclidean action for $\Phi$, reads

$$S[\Phi] = \frac{1}{2} \int d^d x \ tr \left( (\nabla \Phi)^2 + r_0 \Phi^2 \right) + \frac{u_0}{N} \int d^d x \ tr \Phi^4$$

$$= \frac{1}{2} \int_{0 \leq |k| \leq 1} \frac{d^d k}{(2\pi)^d} (k^2 + r_0) \ tr \left( \Phi(k)\Phi(-k) \right)$$

$$+ \frac{u_0}{N} \int_{0 \leq |k_1,2,3| \leq 1} \frac{d^d k_1 d^d k_2 d^d k_3}{(2\pi)^{3d}} \ tr \left( \Phi(k_1)\Phi(k_2)\Phi(k_3)\Phi(-k_1 - k_2 - k_3) \right). \tag{2}$$

It is well known that, in the $1/N$ expansion, the leading term for the free energy is of order $O(N^2)$. Also, the correct normalization for singlet operators, in order for them to have a finite vacuum expectation value, is as follows: a factor of $1/\sqrt{N}$ for every power of the field and a factor of $1/N$ for every trace; e.g.:

$$\frac{1}{N^{1+k/2}} \ tr \Phi^k; \quad \frac{1}{N^{2+(k+l)/2}} \ tr \Phi^k \ tr \Phi^l \ \cdots \tag{3}$$

Action (2) is, of course, just a truncation of the most general action in theory space consistent with the symmetries of the problem (throughout the paper we deal with a parity preserving theory). A generic term in the most general action would contain products of traces of powers of the field multiplied by (short ranged) functions of their momenta and by the appropriate powers of $1/N$:

$$\approx \frac{1}{N^p} \int \frac{d^d k_1}{(2\pi)^d} \cdots u(k_1 \cdots) \ tr \left( \Phi(k_1)\Phi(k_2)\cdots \right) \ tr \left( \Phi(k_i)\Phi(k_{i+1})\cdots \right) \cdots. \tag{4}$$

That (2) is a consistent truncation can only be checked at the end. However, it should at least seem reasonable because it includes the three most relevant operators at the Gaussian point. Actually, there is a fourth such operator: $(v_0/N^2) \int d^d x \left( tr \Phi^2 \right)^2$. One might be tempted simply to dismiss such an operator as trivial because it can be removed by introducing an auxiliary singlet field $\lambda$ to the action $-(N^2/v_0) \int d^d x \left( \lambda - (v_0/N^2) \ tr \left( \Phi^2 \right)^2 \right)$, or because it only affects the correlation functions of the theory through the insertion of vacuum diagrams via a contact interaction. However, this is not quite enough; if such a term was generated by the RG it would affect mass renormalization and its effect would have to be taken into account.

At the cost of being pedantic, we begin by showing that such a term in not generated by applying the RG transformation on (2) and therefore it may be excluded a priori. This boils down to showing that actions of the form (2) are mapped into themselves by the RG, modulo terms of higher Gaussian dimension. There might be some confusion on this point since it is well known that the usual exponential relations between the generating function of the connected and disconnected Green functions, as well as the Legendre transformation between the generating function of the
connected Green function and the one particle irreducible vertices, are not valid in the large $N$ limit. Why then should the Wilsonian action re-exponentiate on itself without generating terms non linear in the trace? The solution to this puzzle is that there are leading Green functions in $1/N$ that are not planar (fig. 1) and, while they do not appear in the generating function, they do contribute to the Wilsonian action.

**Fig 1.** The usual decomposition of the four point function into its connected components. The last contribution is not "planar" but should be included when computing the renormalized action.

Let us address this and other related issues by carefully carrying on the first few steps in the renormalization programme. We must integrate out the higher momenta from the action and obtain a new action that, when rescaled, will yield information about the RG flow. Let us then begin by splitting the field variable into slow and fast components $Φ = \bar{Φ} + φ$ with momenta smaller and larger than $1/2$ respectively:

$$\bar{Φ}(x) = \int_{0 \leq |k| \leq 1/2} \frac{d^d k}{(2\pi)^d} e^{ix \cdot k} \bar{Φ}(k) \quad \text{and} \quad φ(x) = \int_{1/2 < |k| \leq 1} \frac{d^d k}{(2\pi)^d} e^{ix \cdot k} φ(k).$$

Action (2) decomposes into

$$S[Φ] = S[\bar{Φ}] + σ[\bar{Φ}, φ] + S[φ],$$

where $S[\bar{Φ}]$ and $S[φ]$ are the same as (2) with the appropriate restriction on the momenta and

$$σ[\bar{Φ}, φ] = \frac{u_0}{N} \int d^d x \text{ tr} \left( 4φ\bar{Φ}^2 + 4φ^2\bar{Φ}^2 + 2(φ\bar{Φ})^2 + 4φ^3\bar{Φ} \right).$$

(Of course, the quadratic term factorizes by momentum conservation). Let us denote, for any local functional $F[\bar{Φ}, φ]$ of the fields,

$$\langle F \rangle = \lim_{N \to \infty} \frac{\int Dφ e^{-S[φ]} F}{\int Dφ e^{-S[φ]}}.$$
The denominator in (8) is such that $< 1 >= 1$. It removes all vacuum diagrams just as in the ordinary case. The relation between Green functions with and without vacuum diagrams is the familiar one because every leading vacuum diagram is planar and vice versa.

Expanding $\exp(-\sigma)$ to forth order in $\bar{\Phi}$ and keeping only those terms that are allowed by parity one obtains

$$
\left\langle e^{-\sigma} \right\rangle = 1 - \frac{u_0}{N} \int d^d x \left(4 \text{ tr } \phi^2 \bar{\Phi}^2(x) + 2 \text{ tr } (\phi \bar{\Phi})^2(x) \right)
+ \frac{1}{2} \frac{u_0^2}{N^2} \int d^d x d^d y \left(32 \text{ tr } \phi \bar{\Phi}^3(x) \text{ tr } \phi \bar{\Phi}^3(y) + 16 \text{ tr } \phi \bar{\Phi}(x) \text{ tr } \phi \bar{\Phi}(y) \right)
+ 16 \text{ tr } \phi \bar{\Phi}^2(x) \text{ tr } \phi \bar{\Phi}^2(y) + 16 \text{ tr } \phi \bar{\Phi}^2(x) \text{ tr } (\phi \bar{\Phi})^2(y) + 4 \text{ tr } (\phi \bar{\Phi})^2(x) \text{ tr } (\phi \bar{\Phi})^2(y)
- \frac{1}{6} \frac{u_0^3}{N^3} \int d^d x d^d y d^d z \left(192 \text{ tr } \phi \bar{\Phi}^2(x) \text{ tr } \phi \bar{\Phi}^3(y) \text{ tr } \phi \bar{\Phi}(z) \right)
+ 96 \text{ tr } (\phi \bar{\Phi})^2(x) \text{ tr } \phi \bar{\Phi}(y) \text{ tr } \phi \bar{\Phi}(z)
+ \frac{1}{24} \frac{u_0^4}{N^4} \int d^d x d^d y d^d z d^d w \left(256 \text{ tr } \phi \bar{\Phi}^3(x) \text{ tr } \phi \bar{\Phi}^3(y) \text{ tr } \phi \bar{\Phi}(z) \text{ tr } \phi \bar{\Phi}(w) \right).$

(9)

(This is not a perturbative expansion; each term contains diagrams with an arbitrary number of $\phi^4$ vertices.)

The first thing to notice is that none of the four terms containing the operator $\text{ tr } (\phi \bar{\Phi})^2$ in (9) contributes to the expression. Terms like those in fig. 2.a yield a contribution proportional to $(1/N)(\text{ tr } \bar{\Phi})^2$. This term would be leading ($O(N^2)$) if $\text{ tr } \bar{\Phi}$ was allowed to pick up a vacuum expectation value, but it is subleading in a parity invariant theory:

$$
\left\langle \frac{1}{N} (\text{ tr } \bar{\Phi})^2 \right\rangle = N^2 \times \left\langle \frac{1}{N^3} (\text{ tr } \bar{\Phi})^2 \right\rangle = N^2 \times \left( \frac{1}{N^{3/2}} \text{ tr } \bar{\Phi} \right)^2 + O(N^{-2})
= N^2 \times \left(0 + O(N^{-2}) \right) = O(1).

(10)

Terms like those in fig. 2.b ”would be” leading but they vanish identically because of momentum conservation. In fact, all diagrams containing a sub-diagram with one $\Phi$ leg and one $\phi$ leg (fig. 3) vanish identically by momentum conservation. This implies that the term $\text{ tr } \phi \bar{\Phi}^3(x) \text{ tr } \phi \bar{\Phi}^3(y)$ in (9) (fig. 2.c) also vanishes identically. Finally, notice that a term like the one in fig. 2.d is planar but it is subleading because the fields $\bar{\Phi}$ are not connected at the outside of the diagram.
Fig 2.) Various contributions to the renormalized action in t’ Hooft’s double line notation. All external lines represent the slow field $\bar{\Phi}$ and all the internal lines the fast field $\phi$. 2.a) is subleading if the trace does not pick up a vacuum expectation value. 2.b) and 2.c) are identically zero by momentum conservation. 2.d) is planar but subleading.

Fig 3.) Same notation as in fig 2. All diagrams containing this sub-diagram vanish.
Considering all the remaining terms and performing the decomposition into connected components one obtains:

\[
\langle e^{-\sigma} \rangle = 1 - \frac{4u_0}{N} \int d^d x \left\langle \text{tr} \, \phi^2 \Phi^2(x) \right\rangle_{\text{conn.}} + \frac{8u_0^2}{N^2} \int d^d x d^d y \left\langle \text{tr} \, \phi^3 \Phi(x) \text{tr} \, \phi^3 \Phi(y) \right\rangle_{\text{conn.}} \\
+ \frac{8u_0^2}{N^2} \left( \int d^d x \left\langle \text{tr} \, \phi^2 \Phi^2(x) \right\rangle_{\text{conn.}} \right)^2 + \frac{8u_0^2}{N^2} \int d^d x d^d y \left\langle \text{tr} \, \phi^2 \Phi^2(x) \text{tr} \, \phi^2 \Phi^2(y) \right\rangle_{\text{conn.}} \\
- \frac{32u_0^3}{N^3} \int d^d x \left\langle \text{tr} \, \phi^2 \Phi^2(x) \right\rangle_{\text{conn.}} \int d^d y d^d z \left\langle \text{tr} \, \phi^3 \Phi(y) \text{tr} \, \phi^3 \Phi(z) \right\rangle_{\text{conn.}} \\
- \frac{32u_0^3}{N^3} \int d^d x d^d y d^d z \left\langle \text{tr} \, \phi^2 \Phi^2(x) \text{tr} \, \phi^3 \Phi(y) \text{tr} \, \phi^3 \Phi(z) \right\rangle_{\text{conn.}} + \frac{32u_0^4}{N^4} \left( \int d^d x d^d y \left\langle \text{tr} \, \phi^3 \Phi(x) \text{tr} \, \phi^3 \Phi(y) \right\rangle_{\text{conn.}} \right)^2 \\
+ \frac{32u_0^4}{3N^4} \int d^d x d^d y d^d z d^d w \left\langle \text{tr} \, \phi^3 \Phi(x) \text{tr} \, \phi^3 \Phi(y) \text{tr} \, \phi^3 \Phi(z) \text{tr} \, \phi^3 \Phi(w) \right\rangle_{\text{conn.}} \, .
\]

(11)

The only relevant point is that the next to last quantity \((\int d^d x d^d y \left\langle \text{tr} \, \phi^3 \Phi(x) \text{tr} \, \phi^3 \Phi(y) \right\rangle_{\text{conn.}})^2\) is multiplied by the factor \(32 = 3 \times 256/24\), in other words we had to use all three disconnected Green functions in fig. 1 since we were taking the vacuum expectation value of four singlets (\(\text{tr} \, \phi^3 \Phi\)). This makes it possible to re-exponentiate (11) without generating terms non linear in the trace. If we denote by \(\tilde{S}[\Phi]\) the exponential:

\[
e^{-\tilde{S}[\Phi]} = \left\langle e^{-\sigma[\Phi,\phi]} \right\rangle,\]

(12)

then

\[
\tilde{S}[\Phi] = \frac{4u_0}{N} \int d^d x \left\langle \text{tr} \, \phi^2 \Phi^2(x) \right\rangle_{\text{conn.}} - \frac{8u_0^2}{N^2} \int d^d x d^d y \left\langle \text{tr} \, \phi^3 \Phi(x) \text{tr} \, \phi^3 \Phi(y) \right\rangle_{\text{conn.}} \\
- \frac{8u_0^3}{N^3} \int d^d x d^d y \left\langle \text{tr} \, \phi^2 \Phi^2(x) \text{tr} \, \phi^2 \Phi^2(y) \right\rangle_{\text{conn.}} \\
+ \frac{32u_0^3}{N^3} \int d^d x d^d y d^d z \left\langle \text{tr} \, \phi^2 \Phi^2(x) \text{tr} \, \phi^3 \Phi(y) \text{tr} \, \phi^3 \Phi(z) \right\rangle_{\text{conn.}} \\
- \frac{32u_0^4}{3N^4} \int d^d x d^d y d^d z d^d w \left\langle \text{tr} \, \phi^3 \Phi(x) \text{tr} \, \phi^3 \Phi(y) \text{tr} \, \phi^3 \Phi(z) \text{tr} \, \phi^3 \Phi(w) \right\rangle_{\text{conn.}} \, .
\]

(13)

So far, our discussion has been completely general and no approximation has been made, except for the truncation to operators containing no more than four fields \(\Phi\). Of course, the
identification and evaluation of the diagrams contributing to (13) is still a formidable problem and we must make further approximations; these will be made in section four. Before that, in the next section, we present the sketch of a perturbative calculation as a warm up.

3) A perturbative calculation.

A perturbative expansion to \(O(u_0^3)\) around the Gaussian point and to leading order in \(1/N\) results in the diagrammatic expansion for \(\tilde{S}[\Phi]\) given in fig. 4.

Fig 4.) Perturbative expansion of the renormalized action in single line notation. The external lines represent the field \(\bar{\Phi}\) and the internal ones the field \(\phi\).
All diagrams that are forbidden by momentum conservation have been omitted. The symmetry factors are all powers of 2 and this is typical of planar expansions. One can compare these factors with those in [5], keeping in mind that there is an overall factor 2 of difference in the two-point function and an overall factor 4 in the four-point function. This is so because, here, the external legs are attached to the slow field $\Phi$ acting as a background field. One must also remember the explicit factor of $1/N$ in front of the four-point functions.

Just as an example, the first diagram in fig. 4 corresponds to the expression

$$4u_0 \int d^d x \text{tr} (\tilde{\Phi}^2(x)) G(x,x);$$

whereas, say, the tenth diagram corresponds to

$$-\frac{8u_0^2}{N} \int d^d x d^d y \text{tr} (\tilde{\Phi}^2(x) \tilde{\Phi}^2(y)) G(x,y)^2.$$  

In (14) and (15), the quantity $G(x,y)$ is given by the tree level propagator for $\phi$:

$$G(x,y) = \left\langle N^{-2} \text{tr} \phi(x) \phi(y) \right\rangle_0 = \int_{0 \leq |k| \leq 1} \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-y)} G(k)$$

$$G(k) = \begin{cases} 0 & \text{for } |k| \leq \frac{1}{2} \\ \frac{1}{k^2 + r_0} & \text{for } \frac{1}{2} < |k| \leq 1 \end{cases}$$

The symbol $\left\langle \cdots \right\rangle_0$ is the Gaussian average and one can also write, for each component of the matrix,

$$\left\langle \phi^\alpha_\beta(x) \phi^\gamma_\delta(y) \right\rangle_0 = \delta^\alpha_\delta \delta^\gamma_\beta G(x,y) + O\left(\frac{1}{N}\right).$$

One could then perform the integrals and obtain a perturbative expression for $\tilde{S}$, but this will not be done explicitly because the main purpose of this section was to present the relevant diagrams with their relative weights for later comparison. Let us therefore go on to the next section and present those approximations that allow one to obtain non-perturbative results.

### 4) The approximate RG formula in the large N limit.

We must now make some different approximations in order to compute (13) without using perturbation theory. The problem is, of course, that in the expression for $\tilde{S}$ we want to keep the contribution of integrals of arbitrary higher number of loops and we do not know how to do it exactly. The first thing we do is to set all incoming momenta (the momenta of the slow field $\Phi$) to
zero, in order to obtain the "ultra local" contributions to $\tilde{S}$. This approximation is equivalent to
the truncation we have chosen for (2) and it is not an independent assumption, but it should only be applied to
those terms that are already ultra local in (2), namely, mass term and interaction. If we applied it to the kinetic
term we would rule out the possibility of anomalous dimension for the field. This would be too strong since, in the
large $N$ limit, there is an infinite number of planar diagrams contributing to wave function renormalization. Let us therefore start with the mass and the interaction terms, leaving the analysis of the kinetic term for later.

In the spirit of the approximate recursion formula, we now make two further assumptions:
1. Replace every propagator (regardless of its momentum dependence) by $1/(1 + r_0)$
2. Replace every loop integral by a constant $c_d$

The actual value of the constant $c_d$ is not important. To be specific, we will assume

$$c_d = \int_{1/2 < |k| \leq 1} \frac{d^d k}{(2\pi)^d} \approx \begin{cases} 
0.014776 & \text{for } d = 3 \\
0.002968 & \text{for } d = 4
\end{cases}.$$  \hspace{1cm} (18)

Now we must identify those diagrams that contribute to $\tilde{S}$ and estimate the relative powers of $u_0$, $1/(1 + r_0)$ and $c_d$ in each of them. Since all external momenta are set to zero, momentum conservation implies that not only diagrams containing sub-diagrams like the one in fig. 3, but all diagrams that can be disconnected by cutting one internal $\phi$ line vanish, no matter how many external $\bar{\Phi}$ lines are attached to it. Therefore, only one particle irreducible diagrams contribute, as one would naively have expected. 2) The great simplification in considering only planar diagrams is that, with the assumptions above, the sum of these diagrams can be explicitly evaluated by using the single matrix model (c.f.r. appendix). None of the further assumptions usually made in the approximate recursion formula, (such as forbidding those diagrams with an odd number of internal lines at some vertex,) need to be made. The situation is summarized in fig. 5. The diagram in fig. 5.a is exactly forbidden; the diagram in fig 5.b is forbidden under the assumption of zero incoming momentum; the diagram in fig. 5.c, that would be forbidden by the ordinary approximation, is in fact allowed, as it should.

2) This observation is not directly relevant to the calculation of the corrections to $r_0$ and $u_0$ because, due to the particular nature of quartic interactions, all one particle reducible diagrams with two or four external legs have a sub-diagram of the kind in fig. 3; but it should be kept in mind if one wants to compute diagrams with more than four external legs.
Some planar diagrams in single line notation. 5.a) has already been shown to be identically zero. 5.b) vanishes in the approximations used in this section. 5.c) does not vanish and it is accounted for in our formulas.

To estimate the relative powers of $u_0$, $1/(1+r_0)$ and $c_d$, we make use of the following topological formulas: for a particular diagram with $V$ vertices, $E > 0$ external $\Phi$ legs, $P$ propagators and $L$ momentum loops, the following two relations apply

$$4V = E + 2P \quad \text{and} \quad V - P + L = 1. \quad \text{(19)}$$

The reader should not be confused by the second relation in (19). In (19), $L$ refers to the number of loop momentum integrals, not color loops. Among all diagrams with $L$ momentum loops, the planar limit picks out those with the maximum number $L_{\text{col}}^{\text{max}}$ of color loops. For $E > 0$ these two numbers coincide, whereas for vacuum diagrams $L_{\text{col}}^{\text{max}} = L + 1$.

For a given number $E$ of external legs, we can solve for $P$ and $L$ as function of the number of vertices $V$. We can therefore write the mass term ($E = 2$) $\tilde{S}_{r_0}$ and the interaction term ($E = 4$) $\tilde{S}_{u_0}$ in $\tilde{S}$ as follows

$$\tilde{S}_{r_0}[\Phi] = (1 + r_0)f_{r_0}\left(\frac{c_d u_0}{(1 + r_0)^2}\right) \int d^d x \, \text{tr} \, \Phi^2(x),$$

$$\tilde{S}_{u_0}[\Phi] = \frac{u_0}{N} f_{u_0}\left(\frac{c_d u_0}{(1 + r_0)^2}\right) \int d^d x \, \text{tr} \, \Phi^4(x), \quad \text{(20)}$$

where $f_{r_0}(g)$ and $f_{u_0}(g)$ are given in terms of the single matrix model proper vertices (c.f.r. appendix):

$$f_{r_0}(g) = \frac{\Gamma_2(g) - 1}{2} \quad \text{and} \quad f_{u_0}(g) = \frac{\Gamma_4(g) - 4g}{4g}. \quad \text{(21)}$$
After all that has been said, the proof follows easily. First, we match the powers of $g$ in the single matrix model with the powers of $u_0$ in the full quantum theory by dividing by the overall factors of 2 and 4 already encountered in perturbation theory.\textsuperscript{3) \textsuperscript{3)}} Second, we subtract the tree level terms, already present in $S[\Phi]$ (6) and not to be included in $\tilde{S}[\Phi]$ (13). Finally, we match the powers of $c_d$ and $1/(1 + r_0)$ with the help of the topological formulas (19).

We must now face the problem of wave function renormalization by relaxing the condition of zero external momenta. Let $q << 1/2$ be the external momentum of the fields $\Phi$. First we isolate those diagrams that contribute to the renormalization: they are those with two external legs connected at two different points in space-time. Terms where the two external legs are connected to the same point do not have any explicit dependence on the external momenta and they should not be included. The decomposition of the one particle irreducible two point function into these two kind of terms is given in fig. 6.

\begin{diagram}
\begin{tikzpicture}
\begin{scope}
\draw[very thick] (-1,0) -- (1,0);
\draw[very thick] (-0.5,0.5) -- (0.5,-0.5);
\draw[very thick] (-0.5,-0.5) -- (0.5,0.5);
\end{scope}
\node at (-1.5,0) {$\times$};
\node at (1.5,0) {$+$};
\node at (3,0) {2};
\end{tikzpicture}
\end{diagram}

\textbf{Fig. 6.} A useful decomposition of the vertex $\Gamma_2$. Solid blobs represent connected Green functions and crossed blobs represent one particle irreducible vertices. The last term contributes to wave function renormalization.

Fig. 6 can be checked explicitly for the single matrix model (c.f.r. appendix):

$$\Gamma_2(g) = 1 + 8gG_2(g) - 4g \Gamma_4(g) G_2^3(g).$$

(22)

Only the last term in (22) contributes to wave function renormalization. To find such contribution we must take the derivative with respect to $q^2$ of each contributing diagram and then set $q^2 = 0$.\textsuperscript{3) \textsuperscript{3)}}

\textsuperscript{3) The explicit factor of $g$ at the denominator in the definition of $f_{u_0}$ is there simply because we have factored out one power of $u_0$ in the definition of $\tilde{S}_{u_0}$, it does not represent a singularity at the origin!}
Again, this is hard to do exactly because there may be more than one $q$ dependent propagator in a diagram. We overcome this difficulty by making the further approximation, (justified by dimensional analysis and also by acting with the derivative on a single propagator):

$$\left. \frac{d}{dq^2} \right|_{0} \to -\frac{1}{1 + r_0}.$$  \quad (23)

This allows us to estimate the planar contributions without any effort. It is an approximation in the same spirit as those made before. It should correctly capture the qualitative behavior and the order of magnitude of the correction. By using it, we obtain the last piece of $\tilde{S}$:

$$\tilde{S}_{q^2}[\bar{\Phi}] = f_{q^2}\left(\frac{c_d u_0}{(1 + r_0)^2}\right) \int d^d x \text{tr} \left(\nabla \bar{\Phi}(x)\right)^2, \quad (24)$$

where (remembering the $1/2$ normalization factor as before),

$$f_{q^2}(g) = 2g \Gamma_4(g) G_2^3(g). \quad (25)$$

Action (13) is therefore specified in terms of three known functions $f_{r_0}, f_{u_0}$ and $f_{q^2}$ of

$$g(u_0, r_0) = \frac{c_d u_0}{(1 + r_0)^2}. \quad (26)$$

These three functions can be written as rational functions of a single function of $g$ (c.f.r. appendix)

$$a^2(g) = \frac{\sqrt{1 + 48g} - 1}{24g} \quad (27)$$

as follows

$$f_{r_0}(g) = \frac{(a^2 - 1)(a^2 - 3)}{2a^2(4 - a^2)} \approx 4g - 40g^2 + 832g^3 + \cdots$$

$$f_{u_0}(g) = \frac{27(5 - 2a^2)}{(4 - a^2)^3} - 1 \approx -8g + 224g^2 - 7296g^3 + \cdots \quad (28)$$

$$f_{q^2}(g) = \frac{(a^2 - 1)^2(5 - 2a^2)}{18a^2(4 - a^2)} \approx 8g^2 - 256g^3 + \cdots.$$

A plot of these three functions is given in fig. 7. The range in which they are real is $-1/48 \leq g < \infty$; they are analytic at $g = 0$ and their expansion given in (28) matches the perturbative expansion in fig. 4 if one makes the the same approximations as we have made here.
Fig 7.) A plot of the three functions described in the text. \( f_{r_0} \) is the mass renormalization, \( f_{u_0} \) the coupling constant renormalization and \( f_{q^2} \) the wave function renormalization. \( f_{u_0} \) approaches a finite limit as \( g \rightarrow \infty \) whereas \( f_{r_0} \) and \( f_{q^2} \) diverge as \( \sqrt{g} \).

Let us summarize what has been done so far. We started with the action \( S[Φ] \) (2), and after integrating out the fast modes \( φ \) we obtained a new action for the slow fields \( \bar{Φ} \):

\[
S'[\bar{Φ}] = S[\bar{Φ}] + S[Φ] = \frac{1}{2} \int d^d x \left[ 1 + 2 f_{q^2} \left( \frac{c_d u_0}{(1 + r_0)^2} \right) \right] \text{tr} \left( \nabla Φ(x)^2 \right) \\
+ \frac{1}{2} \int d^d x \left[ r_0 + 2(1 + r_0)f_{r_0} \left( \frac{c_d u_0}{(1 + r_0)^2} \right) \right] \text{tr} \bar{Φ}(x)^2 + \frac{u_0}{N} \int d^d x \left[ 1 + f_{u_0} \left( \frac{c_d u_0}{(1 + r_0)^2} \right) \right] \text{tr} \bar{Φ}(x)^4.
\]  

(29)

To obtain the RG equations we have to rescale co-ordinates \( x \rightarrow 2x, \ (k \rightarrow k/2) \) and redefine a new field

\[
\Phi(x) = 2^{(d-2)/2} \sqrt{1 + 2 f_{q^2} \left( \frac{c_d u_0}{(1 + r_0)^2} \right)} \bar{Φ}(2x)
\]

(30)

in terms of which the action becomes again of the form (2), but with a new mass term and a new coupling constant

\[
S[Φ] = \frac{1}{2} \int d^d x \text{tr} \left( (\nabla Φ)^2 + r_1 Φ^2 \right) + \frac{u_1}{N} \int d^d x \text{tr} Φ^4.
\]

(31)
This is just the first step of the RG transformation. The same relation between \( r_0, u_0 \) and \( r_1, u_1 \) holds for a generic step between \( r_l, u_l \) and \( r_{l+1}, u_{l+1} \). We have therefore obtained the RG equations for the hermitian matrix model in this approximation:

\[
\begin{align*}
    r_{l+1} &= 4r_l + 2(1 + r_l)f_{r_0}(g) + 2 \left(1 + 2f_q^2(g)\right) \frac{1 + f_{r_0}(g)}{(1 + 2f_q^2(g))^2}, \\
    u_{l+1} &= 2^{4-d}u_l + 2 \left(1 + f_{u_0}(g)\right) \frac{1 + f_{u_0}(g)}{(1 + 2f_q^2(g))^2},
\end{align*}
\]  

(32)

where \( g = g(u_l, r_l) \) is given in (26). In the next and final section we investigate these equations to determine the dynamics of the theory.

5) Study of the RG. Fixed point and critical exponents.

The iteration of eqs. (32) determines the RG flow in the two dimensional space spanned by two co-ordinates \( r \) and \( u \). At a fixed point \( r^*, u^* \), any further iteration leaves the co-ordinates invariant, i.e., \( r^* = r_l = r_{l+1} \) and \( u^* = u_l = u_{l+1} \). Let us also define, for convenience, the quantity

\[
g^* = g(u^*, r^*) = \frac{c_d u^*}{(1 + r^*)^2}.
\]  

(33)

At the fixed point, eqs. (32) then become

\[
\begin{align*}
    r^* &= 4r^* + 2(1 + r^*)f_{r_0}(g^*) + 2 \left(1 + 2f_q^2(g^*)\right) \frac{1 + f_{r_0}(g^*)}{(1 + 2f_q^2(g^*))^2}, \\
    u^* &= 2^{4-d}u^* + 2 \left(1 + f_{u_0}(g^*)\right) \frac{1 + f_{u_0}(g^*)}{(1 + 2f_q^2(g^*))^2},
\end{align*}
\]  

(34)

By eliminating the solution \( u^* = 0 \) (Gaussian point) from the second eq. in (34), we obtain an algebraic equation in terms of the variable \( g^* \) alone:

\[
2^{d-4}(1 + 2f_q^2(g^*))^2 = 1 + f_{u_0}(g^*).
\]  

(35)

For \( d = 4 \), one can see right away that the only solution is \( g^* = 0 \). In fact, for \( g > 0 \) (physical region), it is always \( f_q^2(g) > 0 \) and \( f_{u_0}(g) < 0 \) (fig. 7), thus preventing (35) from having non trivial solutions\(^4\). Within our approximation, this lack of a non trivial fixed point in four dimensions is rather robust since it only depends on the relative sign of \( f_q^2 \) and \( f_{u_0} \). However, it should not be interpreted as a "no go" result because more subtle approximations might reveal a richer structure.

Things are much better in \( d = 3 \). We can immediately see that eq. (35) must have a non trivial solution in the physical region by considering the asymptotic behaviour of the two sides of

\(^4\) One can also check that there are no solutions in the interval \([-1/48, 0]\)
the equation as \( g \to 0 \) and \( g \to \infty \). As \( g \to 0 \), the l.h.s. \( \to 1/2 \) and the r.h.s. \( \to 1 \), whereas as \( g \to \infty \), the r.h.s. approaches a finite value and the l.h.s. blows up. The two functions must then cross somewhere. Numerically, the only two solution in the whole interval \([-1/48, +\infty]\) are the Gaussian point and \( g^* \approx 0.408655 \). By plugging in this last value to the first of (34) one obtains \( r^* = -0.642902 \) and, recalling the definitions of \( g^* \) (33) and of \( c_d \) (18), one also obtains \( u^* = 3.52675 \).

If one studies the evolution of this fixed point as \( d \) goes from 3 to 4, one finds that it approaches the Gaussian one. In this sense, our fixed point is "Wilson-Fisher-like". On the other hand, our result does not require using the \( \epsilon \) expansion, and it is a truly large \( N \) result because it hinges on the sum of planar graphs and the consequent asymptotic behavior of the functions \( f_{r_0}, f_{u_0} \) and \( f_{q^2} \) for large \( g \). As a final comment, notice that, if we neglect wave function renormalization by setting \( f_{q^2} \equiv 0 \), the fixed point goes away, yet another indication of the different nature of the calculation.

To compute the large \( N \) critical exponent \( \nu \) we linearize the RG equations (32) near the fixed point. Numerically:

\[
\begin{align*}
    r_{l+1} - r^* &= 1.97107 (r_l - r^*) + 0.303152 (u_l - u^*) \\
    u_{l+1} - u^* &= 6.11699 (r_l - r^*) + 0.690315 (u_l - u^*).
\end{align*}
\]

The largest eigenvalue \( \lambda_{\text{max.}} \) of the matrix

\[
    M = \begin{pmatrix} 1.97107 & 0.303152 \\ 6.11699 & 0.690315 \end{pmatrix}
\]

is \( \lambda_{\text{max.}} = 2.8355 \), yielding a critical exponent

\[
    \nu = \frac{\log 2}{\log \lambda_{\text{max.}}} = 0.665069.
\]

As a "check" of universality, one can compute the dependence of \( M \) on the non universal quantity \( c_d \) and notice that, while the off diagonal entries of \( M \) depend on \( c_d \), the characteristic polynomial (i.e., trace and determinant) does not; and therefore neither does \( \nu \).

To compute the anomalous dimension \( \eta \) of the matrix field we simply express the wave function renormalization as a power of 2 and take the logarithm:

\[
    \eta = \log \frac{1 + 2 f_{q^2}(g^*)}{2} = 0.19882.
\]

The exponent \( \eta \) is also independent on \( c_d \).

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The numerical calculations have been performed with the program Mathematica© running on a SPARK station.
Appendix: Some useful facts about the single hermitian matrix model.

In this appendix we collect, without proof, some basic facts about the single matrix model that are used throughout the paper. All details can be found in [5].

Let $\phi$ be an $N \times N$ hermitian matrix. The action of the $d = 0$ hermitian matrix model we are interested in is

$$S^{d=0}[\phi] = \text{tr} \left( \frac{1}{2} \phi^2 + \frac{g}{N} \phi^4 \right).$$

(40)

For every function $F(\phi)$ we define

$$\left\langle F(\phi) \right\rangle_{d=0} = \lim_{N \to \infty} \int \frac{d^N \phi}{Z} e^{-S^{d=0}[\phi]} F(\phi) \left\langle \phi \right\rangle_{d=0}.$$  

(41)

The Green functions are defined as

$$G_{2p}(g) = \left\langle \frac{1}{N^{1+p}} \text{tr} \phi^{2p} \right\rangle_{d=0} + O(\frac{1}{N^2}),$$

(42)

where the factors of $1/N$ are chosen so that the leading (planar) contribution is finite.

The relation between the two and four point Green functions $G_2$ and $G_4$, the connected ones $C_2$ and $C_4$ and the vertices $\Gamma_2$ and $\Gamma_4$ is

$$C_2 = G_2$$

$$C_4 = G_4 - 2G_2^2$$

$$\Gamma_2 = C_2^{-1} = G_2^{-1}$$

$$\Gamma_4 = -C_4 C_2^{-4} = (2G_2^2 - G_4)G_2^{-4}$$

(43)

The $g$ dependence of all planar Green functions and vertex functions can be written in terms of rational functions of $a^2(g)$ where

$$12ga^4 + a^2 - 1 = 0 \quad \text{i.e.,} \quad a^2(g) = \frac{\sqrt{1 + 48g} - 1}{24g}.$$  

(44)

In particular, for the Green functions of interest:

$$G_2 = \frac{1}{3} a^2(4 - a^2) \approx 1 - 8g + 144g^2 - 3456g^3 + \cdots$$

$$G_4 = a^4(3 - a^2) \approx 2 - 36g + 864g^2 - 24192g^3 + \cdots$$

$$C_2 = \frac{1}{3} a^2(4 - a^2) \approx 1 - 8g + 144g^2 - 3456g^3 + \cdots$$

$$C_4 = -\frac{a^4}{9} (1 - a^2)(5 - 2a^2) \approx -4g + 160g^2 - 5760g^3 + \cdots$$

$$\Gamma_2 = \frac{3}{a^2(4 - a^2)} \approx 1 + 8g - 80g^2 + 1664g^3 + \cdots$$

$$\Gamma_4 = \frac{9(1 - a^2)(5 - 2a^2)}{a^4(4 - a^2)^4} \approx 4g - 32g^2 + 896g^3 + \cdots$$

(45)
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5) The literature on the large N limit and on the renormalization group is immense. I have been forced to leave out many relevant papers and to refer to recent reviews whenever possible.