On multidimensional fixed-point theorems and their applications

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Abstract

The purpose of this paper is to present some multidimensional fixed-point theorems and their applications. For this, we provide a multidimensional fixed point theorem and then using this theorem we prove the existence and uniqueness of a solution of a nonlinear Hammerstein integral equation. Moreover, we provide an example to illustrate the hypotheses and the abstract result of this paper.

1 Introduction and Preliminaries

Many problems which arise in mathematical physics, engineering, biology, economics and etc., lead to mathematical models described by nonlinear integral equations. For instance, the Hammerstein integral equations appear in nonlinear physical phenomena such as electro-magnetic fluid dynamics, reformulation of boundary value problems with a nonlinear boundary condition (see [8]). A Hammerstein integral equation is introduced as follows

\[ x(t) = \int_{a}^{b} G(t,s)H(s,x(s))ds + p(t). \]

The aim of this paper is to investigate this integral equation under a certain conditions of \( G \) and \( H \). For this, we use the methods of multidimensional fixed point theorems. The concept of multidimensional fixed point i.e., \( \Upsilon \)-fixed point was introduced by Roldán et al. [12, 13]. This notion covers the concepts of coupled, tripled, quadruple fixed point. We refer the reader to the references [6, 9, 10, 11, 14] in which were introduced the concept of coupled, tripled, quadruple fixed points and obtained related theorems. The uniqueness and existence theorems of multidimensional fixed point and their applications to nonlinear integral equations, matrix equations and the system of matrix equations have been developed in [2-5], [7]. In this paper, by using multidimensional fixed point theorems, we prove the existence and uniqueness of solution of a nonlinear Hammerstein integral equation under a certain conditions of \( G \) and \( H \). Moreover, we provide an example to illustrate the hypotheses and the abstract result of this paper. Let us introduce some necessary concepts and tools which help us to formulate our theorems. Denote by \((X,d,\preceq)\) a partially ordered metric space.

Definition 1.1. An ordered metric space \((X,d,\preceq)\) is called regular if it satisfies the following:

- if \( \{x_m\} \) is a nondecreasing sequence and \( \{x_m\} \xrightarrow{d} x \), then \( x_m \preceq x \) for all \( m \);

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- if \( \{y_m\} \) is a nonincreasing sequence and \( \{y_m\} \overset{d}{\to} y \) then \( y_m \geq y \) for all \( m \).

Taking a natural number \( k \geq 2 \) we consider the set \( \Lambda_k = \{1, 2, \ldots, k\} \). Let \( \{A, B\} \) be a partition of \( \Lambda_k \) that is \( A \cup B = \Lambda_k \) and \( A \cap B = \emptyset \). Using this partition and partially ordered metric space \((X, d, \preceq)\) we define a \( k \)-dimensional partially ordered metric space \((X^k, d_k, \preceq_k)\) as follows:

- the \( k \)-cartesian power of a set \( X \)
  \[
  X^k = \underbrace{X \times X \times \cdots \times X}_k = \{(x = (x_1, x_2, \ldots, x_k)) : |x_i| \in X \text{ for all } i \in \Lambda_k\};
  \]

- the maximum metric \( d_k : X^k \times X^k \to [0, +\infty) \), given by
  \[
  d_k(x, y) = \max_{1 \leq i \leq k} \{d(x_i, y_i)\},
  \]
  where \( x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in X^k \);

- the partial order w.r.t \( \{A, B\} \) that is, for any \( x = (x_1, x_2, \ldots, x_k) \) and \( y = (y_1, y_2, \ldots, y_k) \in X^k \) we have
  \[
  x \preceq_k y \iff \left\{ \begin{array}{l}
  x_i \leq y_i, \text{ if } i \in A, \\
  x_i \geq y_i, \text{ if } i \in B.
  \end{array} \right.
  \]

It is easy to see that if \((X, d)\) is a complete metric space, then \((X^k, d_k)\) is a complete metric space.

**Definition 1.2.** We say that a mapping \( F : X^k \to X \) has the mixed monotone property w.r.t partition \( \{A, B\} \), if \( F \) is monotone nondecreasing in arguments of \( A \) and monotone nonincreasing in arguments of \( B \).

We define the following set of mappings:

\[
\Omega_{A, B} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(A) \subseteq A, \sigma(B) \subseteq B\},
\]
\[
\Omega'_{A, B} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}.
\]

Let \( \Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_k) \) be \( k \)-tuple of mappings of \( \sigma_i : \Lambda_k \to \Lambda_k \) such that \( \sigma_i \in \Omega_{A, B} \) if \( i \in A \) and \( \sigma_i \in \Omega'_{A, B} \) if \( i \in B \). In the sequel we consider only such kind of \( k \)-tuple of mappings.

**Definition 1.3.** A point \( x = (x_1, x_2, \ldots, x_k) \in X^k \) is called \( \Upsilon \)-fixed point of a mapping \( F : X^k \to X \) if

\[
F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(k)}) = x_i
\]

for all \( i \in \Lambda_k \).

## 2 A Multidimensional fixed point theorem

In this section we provide a multidimensional fixed point theorem which will be used in the next section. We need the following definition.

**Definition 2.1.** A function \( \psi : [0, +\infty) \to [0, +\infty) \) is called altering distance function, if \( \psi \) is continuous, monotonically increasing and \( \psi(\{0\}) = \{0\} \).
The following theorem has been obtained by Akhadkulov et. al in [2].

**Theorem 2.2.** Let \((X, d, \preceq)\) be a complete partially ordered metric space. Let \(\Upsilon : \Lambda_k \to \Lambda_k\) be a \(k\)-tuple mapping \(\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_k)\) such that \(\sigma_i \in \Omega_{\Lambda, \mathcal{B}}\) if \(i \in \mathcal{A}\) and \(\sigma_i \in \Omega'_{\Lambda, \mathcal{B}}\) if \(i \in \mathcal{B}\). Let \(F : X^k \to X\) be a mapping which obeys the following conditions:

(i) there exists an altering distance function \(\psi\), an upper semi-continuous function \(\theta : [0, +\infty) \to [0, +\infty)\) and a lower semi-continuous function \(\varphi : [0, +\infty) \to [0, +\infty)\) such that for all \(x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k)\), \(x, y\) with \(x \preceq_k y\) we have

\[
\psi(d(F(x), F(y))) \leq \theta(d_k(x, y)) - \varphi(d_k(x, y))
\]

where \(\theta(0) = \varphi(0) = 0\) and \(\psi(x) - \theta(x) + \varphi(x) > 0\) for all \(x > 0\);

(ii) there exists \(x^0 = (x^0_1, x^0_2, \ldots, x^0_k)\) such that \(x_i^0 \preceq_i \Upsilon(x)\) for all \(i \in \Lambda_k\);

(iii) \(F\) has the mixed monotone property w.r.t \(\{\mathcal{A}, \mathcal{B}\}\);

(iv) (a) \(F\) is continuous or

(b) \((X, d, \preceq)\) is regular.

Then \(F\) has a \(\Upsilon\)-fixed point. Moreover

(v) if for any \(x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k)\) there exists a point \(z = (z_1, z_2, \ldots, z_k)\) such that \(x \preceq_k z\) and \(y \preceq_k z\), then \(F\) has a unique \(\Upsilon\)-fixed point \(x^* = (x^*_1, x^*_2, \ldots, x^*_k)\).

### 3 An application of Theorem 2.2

In this section, we apply Theorem 2.2 to a nonlinear Hammerstein integral equation to show the existence and uniqueness of solution. Let \(T > 1\) be a real number. Consider the following nonlinear Hammerstein integral equation on \(C([1, T])\):

\[
x(t) = \int_1^T G(t, s) \left[ \sum_{i=1}^{2m} f_i(s, x(s)) \right] ds + p(t), \quad t \in [1, T].
\]

In order to show the existence of a solution of equation (3.1) we assume:

(a) \(f_i : [1, T] \times \mathbb{R} \to \mathbb{R}, \quad 1 \leq i \leq 2m\) are continuous;

(b) \(p : [1, T] \to \mathbb{R}\) is continuous;

(c) \(G : [1, T] \times [1, T] \to [0, \infty)\) is continuous;

(d) there exist positive constants \(\eta_1, \eta_2, \ldots, \eta_{2m}\) such that

\[
\max_{1 \leq i \leq 2m} \eta_i \leq \left(\frac{2m}{\int_{0}^{T} G(t, s) ds} \right)^{-1}
\]

for all \(1 \leq i \leq 2m\) and

\[
0 \leq f_{2i-1}(s, y) - f_{2i-1}(s, x) \leq \eta_{2i-1} \log \left(1 + y - x\right),
\]

\[
-\eta_{2i} \log \left(1 + y - x\right) \leq f_{2i}(s, y) - f_{2i}(s, x) \leq 0
\]

for all \(x, y \in \mathbb{R}, \quad y \geq x\) and \(1 \leq i \leq m\).
(e) there exist continuous functions $y_1^0, y_2^0, \ldots, y_{2m}^0 : [1, T] \rightarrow \mathbb{R}$ such that $y_{2r-1}^0(t) \leq H_{2r-1}(t), 1 \leq r \leq m$ and $y_{2r}^0(t) \geq H_{2r}(t), 1 \leq r \leq m$ for all $t \in [0, T]$ where

$$H_1(t) = \int_1^T G(t, s) \left[ \sum_{i=1}^{2m} f_i(s, y_i^0(s)) \right] ds + p(t)$$

and

$$H_r(t) = \int_1^T G(t, s) \left[ \sum_{i=1}^{2m-r+1} f_i(s, y_i^0(s)) + \sum_{\ell=0}^{r-2} f_{2m-\ell}(s, y_{r-1-\ell}^0(s)) \right] ds + p(t),$$

for $2 \leq r \leq 2m$.

Note that the equation (3.1) has been studied in [1], under the similar assumptions. The main difference is the contraction condition i.e., the assumption (e). We have the following.

**Theorem 3.1.** Under assumptions (a)-(e), equation (3.1) has a unique solution in $C[1, T]$.

**Proof.** The proof of this theorem is similar to the proof of the main theorem of [1]. Therefore we give only the sketch of the proof. First, we define necessary notions as follow. Let

$$X = \{1, 3, 5, \ldots, 2m-1\} \text{ and } B = \{2, 4, 6, \ldots, 2m\}.$$ 

We choose $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_{2m})$ as follows:

$$\Upsilon = \begin{pmatrix}
\sigma_1(1) & \sigma_1(2) & \cdots & \sigma_1(2m) \\
\sigma_2(1) & \sigma_2(2) & \cdots & \sigma_2(2m) \\
\sigma_3(1) & \sigma_3(2) & \cdots & \sigma_3(2m) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{2m}(1) & \sigma_{2m}(2) & \cdots & \sigma_{2m}(2m)
\end{pmatrix} = \begin{pmatrix}
1 & 2 & \cdots & 2m - 2 & 2m - 1 & 2m \\
2 & 3 & \cdots & 2m - 1 & 2m & 1 \\
3 & 4 & \cdots & 2m & 1 & 2 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
2m & 1 & \cdots & 2m - 3 & 2m - 2 & 2m - 1
\end{pmatrix}$$

Next we consider the operator $\mathbb{A} : X^{2m} \rightarrow X$,

$$\mathbb{A}(x) = \mathbb{A}(x_1, x_2, \ldots, x_{2m}) = \int_1^T G(t, s) \left[ \sum_{i=1}^{2m} f_i(s, x_i(s)) \right] ds + p(t),$$

where $t \in [1, T]$ and $x = (x_1, x_2, \ldots, x_{2m}) \in X^{2m}$. Further, we show $\mathbb{A}$ satisfies all conditions of Theorem 2.2. Let $x = (x_1, x_2, \ldots, x_{2m}), z = (z_1, z_2, \ldots, z_{2m}) \in X^{2m}$. We define a metric in $X^{2m}$ as follows:

$$d_{2m}(x, z) = \max_{i \in \Lambda_{2m}} \{ d(x_i, z_i) \} = \max_{i \in \Lambda_{2m}} \{ \max_{1 \leq t \leq T} | x_i(t) - z_i(t) | \}.$$
Step 1. We claim that the operator $A$ satisfies the first condition of Theorem 2.2 with

$$\psi(x) = x, \quad \theta(x) = \log(1 + x) \quad \text{and} \quad \varphi(x) = 0.$$ 

Indeed, from assumption (d) it follows that

$$A(z_1, z_2, \ldots, z_{2m})(t) - A(x_1, x_2, \ldots, x_{2m})(t) =$$

$$\int_1^T G(t, s) \left[ \sum_{i=1}^{2m} f_i(s, z_i(s)) - f_i(s, x_i(s)) \right] ds \leq$$

$$2m \left( \max_{1 \leq i \leq 2m} \eta_i \right) \left( \max_{1 \leq i \leq 2m} \int_1^T G(t, s) ds \right) \cdot \log \left( 1 + d_{2m}(x, z) \right)$$

for any $x = (x_1, x_2, \ldots, x_{2m}), z = (z_1, z_2, \ldots, z_{2m}) \in X^{2m}$ with $x \preceq z$. Hence

$$d(A(x), A(z)) \leq \log \left( 1 + d_{2m}(x, z) \right)$$

that is

$$\psi(d(A(x), A(z))) \leq \theta(d_{2m}(x, z)) - \varphi(d_{2m}(x, z)).$$

One can easily see $\psi(x) - \theta(x) + \varphi(x) = x - \log(1 + x) > 0$ for all $x > 0$.

Step 2. There exists $y = (y_1^0, y_2^0, \ldots, y_{2m}^0) \in X^{2m}$ such that the operator $A$ satisfies the second condition of Theorem 2.2. The proof of this claim follows from assumption (e).

Step 3. The operator $A$ has mixed monotone property w.r.t $\{A, B\}$. The proof of this claim follows from assumptions (c) and (d).

Step 4. We claim that $A : X^{2m} \to X$ is continuous. Indeed, for any $x, z \in X^{2m}$ verifying $d_{2m}(x, z) \leq \delta$ we have

$$\left| A(x) - A(z) \right| \leq \int_0^T G(t, s) \sum_{i=1}^{2m} \left| f_i(s, x_i(s)) - f_i(s, z_i(s)) \right| ds$$

$$\leq \gamma d_{2m}(x, z),$$

due to the assumption (d), where $\gamma = 2m\eta \max_{1 \leq t \leq T} \int_1^T G(t, s) ds$. Hence $A$ is continuous. We have shown that the operator $A$ satisfies the conditions $(i) - (iv)$ of Theorem 2.2. It implies that $A$ has a $Y$-fixed point $x^* = (x_1^*, x_2^*, \ldots, x_{2m}^*)$. That is

$$A(x_1^*, x_2^*, x_3^*, \ldots, x_{2m}^*) = x_1^*,$$

$$A(x_2^*, x_3^*, \ldots, x_{2m}^*, x_1^*) = x_2^*,$$

$$\vdots$$

$$A(x_{2m}^*, x_1^*, \ldots, x_{2m-1}^*) = x_{2m}^*.$$ 

It is obvious, for any $x = (x_1, x_2, \ldots, x_{2m}), y = (y_1, y_2, \ldots, y_{2m}) \in X^{2m}$ there exists a $q = (q_1, q_2, \ldots, q_{2m}) \in X^{2m}$ such that $x \preceq q$ and $y \preceq q$. Indeed, consider the functions $q_i : [1, T] \to \mathbb{R}$

$$q_i(s) = \max\{x_i(s), y_i(s)\}, \quad s \in [1, T].$$
Since \( x_i(s) \) and \( y_i(s) \) are continuous on \([1, T]\), the functions \( q_i(s) \) are continuous on \([1, T]\) and \( x_i(s) \leq q_i(s), y_i(s) \leq q_i(s) \) for all \( 1 \leq i \leq 2m \). Therefore \( A \) has a unique \( \Upsilon \)-fixed point \( x^* = (x^*_1, x^*_2, \ldots, x^*_{2m}) \). Next we show

\[
x_1^* = x_2^* = \ldots = x_{2m}^*.
\]

If \( x^* = (x^*_1, x^*_2, \ldots, x^*_{2m}) \) is the \( \Upsilon \)-fixed point of \( A \), then \( y^* = (y^*_1, y^*_2, \ldots, y^*_{2m}) \) is also a \( \Upsilon \)-fixed point of \( A \), where \( y^*_i = x^*_{i+1} \) \( 1 \leq i \leq 2m - 1 \) and \( y^*_{2m} = x^*_1 \). However, \( A \) has the unique \( \Upsilon \)-fixed point. Therefore \( x^* = y^* \) hence

\[
x_1^* = x_2^* = \ldots = x_{2m}^*.
\]

Finally, we have shown that there exists a continuous function \( x^*(t) \) such that

\[
x^*(t) = A(x^*, x^*, \ldots, x^*)(t) = \int_1^T G(t, s) \left[ \sum_{i=1}^{2m} f_i(s, x^*(s)) ds \right] + p(t).
\]

This proves Theorem 3.1.

\[\square\]

4 Illustrative example

In this section, we provide a representative example to illustrate how Theorem 3.1 can be applied in solving nonlinear Hammerstein integral equation. Let \( T > 1 \). Consider the following class of nonlinear Hammerstein integral equations.

\[
x(t) = \frac{1}{2\ln T} \int_1^T \frac{1}{t s} \ln \left( \frac{s + x(s)}{sx(s)} \right) ds + \alpha t - \frac{1}{2} \ln \frac{1 + \alpha}{\alpha \sqrt{T}} \cdot \frac{1}{t}, \quad \text{where} \quad \alpha > 1.
\]

**Theorem 4.1.** For every \( \alpha > 1 \), the equation (4.1) has a unique solution in \( C([1, T]) \).

**Proof.** Denote

\[
f_1(s, t) = \ln(s + t), \quad f_2(s, t) = -(\ln s + \ln t),
\]

\[
G(t, s) = \frac{1}{2\ln T} \cdot \frac{1}{t s}, \quad p(t) = \alpha t - \frac{1}{2} \ln \frac{1 + \alpha}{\alpha \sqrt{T}} \cdot \frac{1}{t},
\]

where \( s \in [1, T] \) and \( t \geq 1 \). It is easy to see that the equation (4.1) can be presented as the equation (3.1) by using these notations. Our next goal is to show that the equation (4.1) satisfies assumptions (a) – (e). One can easily see that the functions \( f_1, f_2, G \) and \( p \) are continuous. We show that the assumption (d) is satisfied. A simple calculation shows that

\[
2 \max_{1 \leq t \leq T} \int_1^T G(t, s) = 2 \max_{1 \leq t \leq T} \frac{1}{2\ln T} \int_1^T \frac{ds}{ts} = 1.
\]

Let \( y \geq x \geq 1 \). One can see that

\[
0 \leq f_1(s, y) - f_1(s, x) = \ln(s + y) - \ln(s + x) = \ln \left( 1 + \frac{y - x}{s + x} \right) \leq \ln(1 + y - x);
\]

\[
- \ln(1 + y - x) \leq - \ln \left( 1 + \frac{y - x}{x} \right) = - \left( \ln y - \ln x \right) = f_2(s, y) - f_2(s, x) \leq 0.
\]
We next show that the assumption (e) is fulfilled with \( y_1^0(t) = \alpha t / 2 \) and \( y_2^0(t) = 3\alpha t / 2 \). It is easily seen that

\[
H_1(t) = \frac{1}{2\ln T} \int_1^T \frac{1}{ts} \ln \left( \frac{2 + \alpha}{3\alpha s} \right) ds + \alpha t - \frac{1}{2} \ln \frac{1 + \alpha}{\alpha \sqrt{T}} \cdot \frac{1}{t},
\]

\[
H_2(t) = \frac{1}{2\ln T} \int_1^T \frac{1}{ts} \ln \left( \frac{2 + 3\alpha}{\alpha s} \right) ds + \alpha t - \frac{1}{2} \ln \frac{1 + \alpha}{\alpha \sqrt{T}} \cdot \frac{1}{t}.
\]

Evaluating the integrals in (4.2) the functions \( H_1 \) and \( H_2 \) can be simplified as follow.

\[
H_1(t) = \alpha t + \frac{1}{2t} \cdot \ln \left( \frac{2 + \alpha}{3(1 + \alpha)} \right),
\]

\[
H_2(t) = \alpha t + \frac{1}{2t} \cdot \ln \left( \frac{2 + 3\alpha}{1 + \alpha} \right).
\]

(4.3)

The task is now to show

\[
y_1^0(t) \leq H_1(t) \quad \text{and} \quad H_2(t) \leq y_2^0(t).
\]

(4.4)

For this, we first show that

\[
\frac{2 + 3\alpha}{1 + \alpha} \leq e^\alpha \quad \text{for} \quad \alpha > 1.
\]

(4.5)

Let

\[
k(\alpha) := e^\alpha - \frac{2 + 3\alpha}{1 + \alpha}.
\]

One can check that

\[
k'(\alpha) = e^\alpha - \frac{1}{(1 + \alpha)^2} > 0
\]

since \( \alpha \) is positive. It implies that \( k(\alpha) \) is increasing and, in consequence, \( k(\alpha) \geq k(1) = e - 2.5 > 0 \). From inequality (4.5) it follows that

\[
\ln \left( \frac{2 + 3\alpha}{1 + \alpha} \right) \leq \alpha t^2
\]

(4.6)

since \( t \geq 1 \). Dividing by \( 2t \) and adding \( \alpha t \) to the both side of (4.6) yields

\[
H_2(t) = \alpha t + \frac{1}{2t} \cdot \ln \left( \frac{2 + 3\alpha}{1 + \alpha} \right) \leq \frac{3\alpha t}{2} = y_2^0(t).
\]

(4.7)

Since \( \alpha > 1 \), it follows that

\[
\ln \left( \frac{3 + 3\alpha}{2 + \alpha} \right) \leq \ln \left( \frac{2 + 3\alpha}{1 + \alpha} \right).
\]

As a consequence of the last inequality and the inequality (4.6) we obtain

\[
\ln \left( \frac{3 + 3\alpha}{2 + \alpha} \right) \leq \alpha t^2.
\]

Similarly as above, dividing by \( -2t \) and adding \( \alpha t \) to the both side of the last inequality we obtain

\[
H_1(t) = \alpha t + \frac{1}{2t} \cdot \ln \left( \frac{2 + \alpha}{3 + 3\alpha} \right) \geq \frac{\alpha t}{2} = y_1^0(t).
\]

(4.8)

Combing inequalities (4.7) and (4.8) we can conclude that the assumption (e) is fulfilled with \( y_1^0(t) = \alpha t / 2 \) and \( y_2^0(t) = 3\alpha t / 2 \). This finishes the proof of Theorem 4.1.

Remark 4.2. The aim of Theorem 4.1 is to show a strategy of applying abstract assumptions (a)-(e) of Theorem 3.1 in some concrete examples. The reader can check that the solution of the equation (4.1) is \( x(t) = \alpha t \).
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