Near Maximum-Likelihood Performance of Some New Cyclic Codes Constructed in the Finite-Field Transform Domain

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Abstract—It is shown that some well-known and some new cyclic codes with orthogonal parity-check equations can be constructed in the finite-field transform domain. It is also shown that, for some binary linear cyclic codes, the performance of the iterative decoder can be improved by substituting some of the dual code codewords in the parity-check matrix with other dual code codewords formed from linear combinations. This technique can bring the performance of a code closer to its maximum-likelihood performance, which can be derived from the erroneous decoded codeword whose euclidean distance with the respect to the received block is smaller than that of the correct codeword. For (63, 37), (93, 47) and (105, 53) cyclic codes, the maximum-likelihood performance is realised with this technique.

I. INTRODUCTION

Low-density parity-check (LDPC) codes [1],[2] form a class of \((n, k)\) linear block codes, where \(n\) is the codeword length and \(k\) is the information length, that can approach near capacity performance. The good performance of LDPC codes is attributed to the code representation and the use of an iterative decoder. It is an essential condition that the code representation does not contain more than one parity-check equation checking on the same two or more bit positions, i.e. no cycles of length 4. The avoidance of these short cycles is important to allow convergence of the iterative decoder [3]. The best performance gains to date have been obtained with long LDPC codes, i.e. several thousand bits in length.

There are many applications where short LDPC codes can be potentially useful. Applications such as watermarking, thin data storage, command/control data reporting and packet communications require blocks of data ranging from 32 to 512 bits to be either robustly protected or reliably transmitted. We concentrate on cyclic LDPC codes of similar lengths in this paper. The particular class of cyclic codes we consider are difference set cyclic (DSC) codes [4] and one-step majority logic decodable (OSMLD) codes [5] which have orthogonal parity-check equations on each bit position, thus there are no cycles of length 4. For short block lengths, these cyclic codes\(^1\) have been shown to outperform the ad-hoc computer design (random) counterpart of the same code-rate and block length [6]. For an \((n, k)\) LDPC code, the ad-hoc computer design code has \(n - k\) parity-check equations but the cyclic code has \(n\) parity-check equations that can be used by the iterative decoder. Consequently, cyclic codes exhibit better convergence than the random LDPC codes when iteratively decoded.

In this paper, we present a modified Belief-Propagation (BP) iterative decoder that can perform near maximum-likelihood (ML) performance for binary transmission over the additive-white-Gaussian-noise (AWGN) channel. It is also shown that, for certain cyclic codes, the modified iterative decoder can achieve ML performance.

The organisation of this paper is as follows. Section II gives a brief review on cyclic code construction method in the finite-field transform domain. Section III introduces the idea of a more-likely codeword and its relationship to ML and iterative decoders. We present modification to the iterative decoder in section IV and some simulation results of the modified decoder are presented in section V. Section VI contains the conclusions.

II. FINITE-FIELD TRANSFORM DOMAIN CONSTRUCTION OF BINARY CYCLIC CODES

There are relatively few OSMLD and DSC codes. As shown in [7], we have extended these codes by a construction method that works in the finite-field transform domain, which is also known as the Mattson-Solomon domain. We briefly review the construction method in this section.

Let \(n\) be a positive odd integer and \(\text{GF}(2^n)\) be the splitting field for \(1 + x^n\) over \(\text{GF}(2)\). We assume that \(\alpha\) is the generator for \(\text{GF}(2^n)\) and \(T_\alpha(x)\) is the polynomial with coefficients in \(\text{GF}(2^n)\) and degree \(\leq n - 1\). Let us denote \(\mathcal{F} = \{f_1(z), f_2(z), \ldots, f_t(z)\}\), where \(f_i(z) \in T_1(z)\) is an irreducible polynomial, such that \(\prod_{i \leq 1 \leq t} f_i(z) = 1 + z^n\). For each \(f_i(z)\), there is a corresponding primitive idempotent\(^2\),

\[^{1}\]We will refer the OSMLD and DSC codes as cyclic codes from this point onwards.

\[^{2}\]A binary polynomial, \(e(x)\), is an idempotent if the property of \(e(x) = e(x)^2 = e(x^2)\) mod \(1 + x^n\) is satisfied.
\[
\theta(z) = \frac{z(1 + z^n)f_i(z)}{f_i(z)} + \delta(1 + z^n)
\]

where \( f_i(z) = \frac{\partial}{\partial z} f_i(z) \), \( f_i(z) \in T_1(z) \) and the integer \( \delta \) is:

\[
\delta = \begin{cases} 
1 & \text{if } \deg(f_i(z)) \text{ is odd}, \\
0 & \text{otherwise}.
\end{cases}
\]

where \( \deg(a(x)) \) represents the degree of the polynomial \( a(x) \).

Let \( a(x) \in T_n(u) \), the finite-field transform or Mattson-Solomon (MS) polynomial of \( a(x) \) is:

\[
A(z) = MS(a(x)) = \sum_{j=0}^{n-1} a(\alpha^{-j})z^j
\]

\[
a(x) = MS^{-1}(A(z)) = \frac{1}{n} \sum_{i=0}^{n-1} A(\alpha^i)x^i
\]

where \( A(z) \in T_n(u) \).

Let \( \mathcal{I} \subseteq \{1, 2, \ldots, t\} \), we define \( f(z) = \prod_{i \in \mathcal{I}} f_i(z) \) and \( \theta(z) = \sum_{i \in \mathcal{I}} \theta_i(z) \), where \( f(z), \theta(z) \in T_1(z) \). Let us define a binary polynomial \( u(x) = MS(\theta(z)) \). Since the MS polynomial of a binary polynomial is an idempotent and vice-versa [8], \( u(x) \) is an idempotent with coefficients in GF(2). If we write \( u(x) = u_0 + u_1x + \ldots + u_{n-1}x^{n-1} \) then, from equation 3

\[
u_i = \frac{1}{n} \theta(\alpha^i), \quad \forall i \in \{0, 1, \ldots, n-1\}.
\]

The idempotent \( u(x) \) can be used to describe an \( (n, k) \) binary cyclic code which has a parity-check polynomial, \( h(x) \), of degree \( k \) and a generator polynomial, \( g(x) \), of degree \( n-k \).

The polynomial \( h(x) \) is a divisor of the idempotent \( u(x) \), i.e. \( (u(x), 1 + x^n) = h(x) \) and \( u(x) = m(x)h(x) \) where \( m(x) \) contains the repeated factors and/or non-factors of \( 1 + x^n \).

From the theories above, we can summarise that:

1) The weight of \( u(x) \) is equal to the number of \( n \)th roots of unity which are roots of \( f(z) \). Note that for \( 0 \leq i \leq n-1 \), \( \theta(\alpha^i) = 1 \) if and only if \( f(\alpha^i) = 0 \) and from equation 4, \( u_i = 1 \) if and only if \( \theta(\alpha^i) = 1 \).

In the other words, \( u_i = 1 \) precisely when \( f(\alpha^i) = 0 \), giving \( wt(u(x))^3 = deg(f(z)) \). Clearly, \( wt(u(x)) = \sum_{i \in \mathcal{I}} \deg(f_i(z)) \).

2) Since \( \theta(z) = MS(u(x)) \), the number of zeros of \( u(x) \) which are roots of unity is clearly \( n - wt(\theta(z)) \).

In general, \( wt(u(x)) \) is much lower than \( wt(h(x)) \) and as such, we can derive a low-density parity-check matrix from \( u(x) \) and apply iterative decoding on it. The parity-check matrix of the resulting code consists of the \( n \) cyclic shifts of \( x^n u(x^{-1}) \). Since the idempotent \( u(x) \) is orthogonal on each bit position, the resulting LDPC code has no cycles of length 4 in the bipartite graph and the true minimum-distance, \( d_{min} \), of the code is simply \( wt(1 + u(x)) \), see [9, Theorem 10.1] for the proof.

Table I shows some examples of cyclic codes derived from this technique. From Table I, it is clear that our technique can also be used to construct the well-known OSMLD and DSC codes.

| \( (n, k) \) | \( u(x) \) | \( d_{min} \) |
|-------|---------|--------|
| (21, 11) | \( 1 + x^2 + x^7 + x^8 + x^{11} \) | 6 |
| (63, 37) | \( 1 + x^4 + x^3 + x^7 + x^{15} + x^{20} + x^{31} + x^{41} \) | 9 |
| (73, 45) | \( 1 + x + x^3 + x^7 + x^{15} + x^{31} + x^{36} + x^{54} + x^{63} \) | 10 |
| (93, 47) | \( 1 + x^3 + x^9 + x^{21} + x^{28} + x^{45} + x^{59} \) | 8 |
| (105, 53) | \( 1 + x^7 + x^8 + x^{21} + x^{23} + x^{49} + x^{53} \) | 8 |
| (255, 175) | \( 1 + x^3 + x^7 + x^{15} + x^{26} + x^{31} + x^{53} + x^{98} + x^{107} + x^{127} + x^{140} + x^{176} + x^{197} + x^{215} \) | 17 |
| (341, 205) | \( 1 + x^3 + x^7 + x^{15} + x^{31} + x^{54} + x^{63} + x^{98} + x^{109} + x^{127} + x^{170} + x^{197} + x^{219} + x^{255} \) | 16 |
| (511, 199) | \( 1 + x + x^3 + x^7 + x^{15} + x^{31} + x^{63} + x^{82} + x^{100} + x^{127} + x^{152} + x^{165} + x^{201} + x^{255} + x^{296} + x^{305} + x^{331} + x^{403} \) | 19 |
| (511, 259) | \( 1 + x^{31} + x^{42} + x^{93} + x^{115} + x^{217} + x^{240} + x^{261} + x^{360} + x^{420} + x^{450} + x^{465} \) | 13 |

### III. More Likely Codewords in Relation to ML and Iterative Decoders

Realising an optimum decoder for any coded system is NP-complete [10]. For general \( (n, k) \) binary linear codes, the optimum decoding complexity is proportional to \( \min\{2^k, 2^{n-k}\} \). Due to this complexity, the optimum decoder can only be realised for very short or very high-rate or very low-rate codes. An ML decoder is the optimum decoder in terms of minimising the frame-error-rate (FER). An ML decoder will output a codeword that has the closest euclidean distance [5] to the received block.

The iterative decoder is a suboptimal decoder approximating ML performance. In decoding LDPC codes, the BP iterative decoder can produce a codeword that is not identical to
the transmitted codeword. This is illustrated by the two-dimensional representation of the ML decision criterion shown in Fig. 1. Points R and A represent the received block and transmitted/correct codeword respectively. The point B represents a codeword whose euclidean distance with the respect to R is smaller than that of A. If the iterative decoder outputs codeword B then a decoding error is produced, but an ML decoder will also make an error. We classify codeword B as a more likely (mrl) codeword [11], [12]. By counting the number of mrl codewords produced in a simulation, we can derive an mrl-FER curve. A similar technique has been used by Dorsch [13], but the metric was based on the hamming distance rather than the euclidean distance, i.e. hard-decisions rather than soft-decisions.

The significance of the mrl codewords is that an ML decoder either outputs correct codewords or mrl codewords. The percentage of mrl codewords output from the iterative decoder gives us a performance indication of how close the iterative decoder is from the ML decoder for the same code. The mrl-FER provides the lower-bound on the ML-performance of a code in comparison to the Maximum-Likelihood-Asymptote (MLA) which provides the upper-bound.

IV. IMPROVED BELIEF-PROPAGATION DECODER

For the (63,37) cyclic code, it has been noticed that the standard BP decoder produces many codewords that are neither correct nor mrl. The number of incorrect codewords is much larger than the number of mrl codewords output. Based on these findings and the fact that every codeword will satisfy all $2^{n-k}$ parity-check equations, we should be able to improve the performance of the BP decoder by extending the number of parity-check equations in the parity-check matrix, denoted as $H$. However, this is likely to be true if the extended parity-check matrix have low-density and does not contain many short cycles.

For any linear codes, additional parity-check equations can be formed from the linear combinations of the equations in $H$. These additional parity-check equations form the high weight codewords of the dual code and appending them to $H$ will introduce many short cycles.

The proposed modified BP decoder does not extend the number of parity-check equations in $H$. Instead, we generate a set of parity-check equations, denoted as $H'$, by taking the linear combinations of those equations in $H$. A subset of $H'$ is substituted into $H$ resulting in a modified parity-check matrix, labelled as $H$. The overall procedures is described in Algorithm 1 [14]. Note that the selection of the parity-check equations may be made on a random basis or may correspond to a predetermined sequence.

Algorithm 1 Modified Belief-Propagation Iterative Decoder

Input:
- $r$ ⇐ received vector
- $H$ ⇐ original parity-check matrix of the code
- $H'$ ⇐ a set of parity-check equations not in $H$
- $T$ ⇐ number of trials
- $\psi$ ⇐ number of selections

Output: a codeword with the minimum euclidean distance

1: Perform BP decoding and $d_0$ ⇐ decoded output.
2: $d_E(d_0, r)$ ⇐ euclidean distance between $d_0$ and $r$.
3: $d_r' \leftarrow d_0$ and $d_E^{\text{min}} \leftarrow d_E(d_0, r)$
4: for $\tau = 1$ to $T$, do
5: for $i = 1$ to maximum number of iterations, do
6: Pick $\psi$ parity-check equations from $H'$.
7: Substitute them into $H$ to generate $H_r$.
8: Based on $H_r$, perform the check nodes (horizontal) and bit nodes (vertical) processing as in standard BP algorithm,
9: $d_r$ ⇐ denote the decoded output,
10: $d_E(d_r, r)$ ⇐ euclidean distance between $d_r$ and $r$.
11: if $d_r^2 = 0$ then
12: Stop the algorithm
13: end if
14: end for
15: if $(d_E(d_r, r) < d_E^{\text{min}})$ and $(d_r^2 = 0)$ then
16: $d_r' \leftarrow d_r$ and $d_E^{\text{min}} \leftarrow d_E(d_r, r)$
17: end if
18: end for
19: Output $d_r'$.

V. SIMULATION RESULTS

In this section, we present simulation results of the modified BP decoder for some cyclic codes designed using the approach discussed in section II. The selection of the parity-check equations is made on a random basis. It is assumed that the simulation system employs BPSK modulation mapping the symbols 0 and 1 to -1 and +1 respectively.

Fig. 2 shows the FER performance of the (63,37) cyclic code. It is shown that the modified BP decoder, provided enough substitutions and trials are used, can achieve the ML performance as indicated by the mrl-FER and the FER of the modified BP decoder that produce the same curve. Compared to the standard BP decoder, at a FER of $10^{-3}$ a gain of approximately 0.9 dB is obtained by using the modified BP decoder. In addition, it can be seen that, at a FER of $10^{-3}$, the performance of the code is within 0.4 dB of the sphere-packing-bound [15],[16] for a code of length 63 and code-rate of 0.587 after allowing for the coding loss attributable to binary transmission. Table II shows how close the performance of the modified BP decoder is to the ML decoder. With the standard BP decoder more than 50% mrl codewords are found in the low signal-to-noise ratio (SNR) region, but in the
moderate SNR region we can only find a few mrl codewords. With just single substitution and 50 trials, the modified BP decoder is able to increase the percentage of mrl codewords found to be higher than 50%. ML performance is achieved with 8 substitutions and 300 trials.

Fig. 3, 4 and 5 show the FER performance of the (93, 47), (105, 53) and (341, 205) cyclic codes respectively. Both of the (93, 47) and (105, 53) codes achieve ML performance with the modified BP decoder. For the (341, 205) code, the modified BP decoder produces a gain of approximately 0.4dB with the respect to the standard BP decoder. Due to the code length, there are very few mrl codewords observed indicating that a better decoder is required. Table III summarises, at the FER of $10^{-3}$, the amount of gain obtained with the modified decoder with the respect to the standard BP decoder and the distance from the sphere packing bound after allowing binary transmission loss.

VI. CONCLUSIONS

Construction of cyclic LDPC codes using idempotents and MS polynomials can produce a large number of cyclic codes that are free from cycle of length 4. Some of the codes are already known such as the DSC codes, but others are new. An important feature of this approach is the ability to increase the $d_{\text{min}}$ of the codes by taking into account additional irreducible factors of $1 + z^n$ and so steadily decrease the sparseness of the parity-check matrix. As an example, consider that we want to design a cyclic code of length 63. If we let $f(z) = 1 + z + z^6$, we obtain a (63, 31) cyclic code with $d_{\text{min}}$ of 7. Now, if the irreducible polynomial $1 + z + z^2$ is also taken into account, the resulting cyclic code is the (63, 37) code which has $d_{\text{min}}$ of 9. The row or column weight of the parity-check matrices for the former and latter codes are 6 and 8 respectively.

By substituting the parity-check equations in the parity-check matrix with other codewords of the dual code derived from their linear combinations, the performance of the BP decoder can be improved. For the (63, 37), (93, 47) and (105, 53) cyclic codes, the modified BP decoder has been shown to achieve ML performance.

Although the substitution method introduces cycles of length 4, these cycles do not pose a lasting negative effect on the iterative decoder. By substituting at every iteration, the effect of these short cycles is broken and simulation results have shown that this can improve the decoding performance.

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TABLE II
PERCENTAGE OF MRL CODEWORDS AGAINST $E_b/N_o$(dB) OF THE (63, 37)
CYCLIC CODE

| $E_b/N_o$ | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | %   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Standard BP decoder             |     |     |     |     |     |     |     |
| $%$   | 73  | 41  | 27  | 23  | 16  | 9  |     |

Substitutions: 1, Trials: 50

| $E_b/N_o$ | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | %   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| $%$   | 90  | 94  | 90  | 82  | 74  | 61  |     |

Substitutions: 8, Trials: 300

| $E_b/N_o$ | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | %   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| $%$   | 100 | 100 | 100 | 100 | 100 | 100 |     |

TABLE III
PERFORMANCE GAIN WITH THE RESPECT TO BP DECODER AND DISTANCE FROM SPHERE-PACKING-BOUND† AT THE FER OF $10^{-3}$

| Codes | Gain with the respect to BP decoder | Distance from sphere-packing-bound† |
|-------|------------------------------------|-------------------------------------|
| (63, 37) | 0.9 dB                            | 0.4 dB                                |
| (93, 47) | 1.1 dB                            | 0.8 dB                                |
| (105, 53) | 2.0 dB                            | 0.9 dB                                |
| (341, 205) | 0.4 dB                            | 0.7 dB                                |

† Sphere-packing-bound offset by binary transmission loss.

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