SPLIT INVOLUTION COUPLED TO

ACTUAL GAUGE SYMMETRY

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ABSTRACT

The split involution quantization scheme, proposed previously for pure second-class constraints only, is extended to cover the case of the presence of irreducible first-class constraints. The explicit Sp(2)-symmetry property of the formalism is retained to hold. The constraint algebra generating equations are formulated and the Unitarizing Hamiltonian is constructed. Physical operators and states are defined in the sense of the new equivalence criterion that is a natural counterpart to the Dirac's weak equality concept as applied to the first-class quantities.

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1 Introduction

In previous paper [1] of the present authors the split involution formalism has been proposed for canonical quantization of dynamical systems with pure second–class constraints.

The formalism implies no extra variables to be introduced with the purpose of converting original constraints into effective ones of the first-class.

On the other hand, the total set of original second–class constraints is supposed to be polarized by splitting into two interchangeable subsets, $T_a^\mu$, $a = 1, 2$, to satisfy the so-called "split involution" relations

$$(\bar{\hbar})^{-1}[T_a^\mu, T_b^\nu] = U_{\mu\nu}^{\rho\sigma}T_{\rho}^b,$$ (1.1)

symmetrized in their superscripts $a, b$.

Besides, the Hamiltonian $H$ is supposed to satisfy the relations (1.2)

$$(\bar{\hbar})^{-1}[H, T_a^\mu] = V_{\mu}^{\nu}T_a^\nu.$$ (1.2)

One generates the "gauge" algebra, initiated by the relations (1.1), (1.2) by solving the equations

$$[Q_a, Q_b] = 0, \quad [Q_a, H] = 0,$$ (1.3)

for the Fermions $Q^a$ and Boson $H$ in the form of a series expansion in ghost powers

$$Q^a = C^\mu T_\mu^a + \ldots, \quad H = H + \ldots.$$ (1.4)

Then one constructs the complete Unitarizing Hamiltonian of the theory in the following $Sp(2)$–symmetric form

$$H_{\text{complete}} = H + \epsilon_{ab}(\bar{\hbar})^{-2}[Q_b, [Q^a, B]]$$ (1.5)

where $B$ is a "gauge–fixing" Bosonic operator. Being the physical quantities defined in an appropriate way, they do not depend on a particular choice of a "gauge" operator $B$. This independence is quite a nontrivial feature of the split involution scheme, because pure second–class constraints do not generate an actual gauge symmetry.

The algebra generating equations (1.3) as well as the Hamiltonian (1.5) possess the $Sp(2)$–covariant form which is characteristic to the formalism developed in Refs. [2, 3] to quantize gauge–invariant theories in a ghost–antighost symmetric fashion. However, the number of

\[1\] It goes without saying that arbitrary second–class constraints (whose Fermionic component number is divisible by 4) and Hamiltonian can be transformed locally to the polarized basis subjected to eqs. (1.1), (1.2). What is not so evident that there exists a valuable set of relativistic dynamical systems such that the Dirac’s hamiltonization procedure, being applied directly to the original relativistic Lagrangian, just produces the polarized constraint basis.
ghosts (and antighosts) introduced in the formalism [2, 3] is twice as compared with the corresponding number in the split involution theory. Moreover, the ghost numbers of the generating operators \((Q^1, Q^2)\) are \((+1, +1)\) in the split involution scheme, while in the ghost–antighost symmetric theory these numbers are \((+1, -1)\).

In the present paper we generalize the split involution formalism by including original first–class constraints into it. When doing this we retain the explicit \(Sp(2)\)-symmetry property of the method to hold.

We assign ghost canonical pairs to constraints of both the classes and require the ghost number operators \(G'\) and \(G''\) of the first and second class, respectively, to be conserved separately. In accordance with this requirement, a pair of the ghost number values, denoted by \(gh'\) and \(gh''\), is assigned to each admitted operator of the theory.

Then we formulate the extended version of the gauge algebra generating equations. We require the generating operator of the first–class constraint algebra to be nilpotent modulo contributions similar to the gauge–fixing term in r.h.s. of (1.5). Thereby we define the equivalence criterion that is a natural counterpart to the Dirac’s weak equality concept as applied to the first–class quantities. The conservation property of the first–class generating operator is also formulated in the sense of the new equivalence criterion proposed.

The constraint algebra generating equations are shown to possess the group of automorphisms that enables one to make the first (resp. second)–class constraints be a set of momenta (resp. a set of canonical pairs). The maximal group of automorphisms is given by semidirect product of three groups that are: ghost–dependent canonical transformations, \(c\)-numerical symplectomorphisms, and exact shifts initiated by the new equivalence criterion.

In terms of the constraint algebra generating operators we construct the complete Unitarizing Hamiltonian of the theory. We modify the definition (1.5) by adding the genuine gauge-fixing term required by the presence of original first–class constraints.

Finally, we formulate the definitions of physical operators and physical states in the sense of the new equivalence criterion.

**Notations and Conventions.** As usual, \(\varepsilon(A)\) represents the Grassmann parity of the quantity \(A\).

If \(n = n_+ + n_-\) is the total number of some superobjects, then \(n_+(n_-)\) indicates the number of Bosons (Fermions) among them.

The standard supercommutator of the operators \(A, B\) is defined by the formula

\[
[A, B] \equiv AB - BA(-1)^{\varepsilon(A)\varepsilon(B)}.
\]  
(1.6)

By \(\varepsilon^{ab}\) we denote the constant \(Sp(2)\)-invariant tensor

\[
\varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]  
(1.7)

while its inverse is denoted as \(\varepsilon_{ab}\).
\[ \varepsilon^{ab}\varepsilon_{bc} = \delta^a_c. \]

We also use the standard notations for symmetrization

\[ A^{(ab)} \equiv A^{ab} + A^{ba}, \]  

and antisymmetrization

\[ A^{[ab]} \equiv A^{ab} - A^{ba}. \]

Greek indices of first(second)–class constraints are taken from the first(second), half of the Greek alphabet, \( \alpha, \ldots, \lambda(\mu, \ldots, \omega) \). The same convention holds for related quantities.

By \( \text{gh}'(A) \) (\( \text{gh}''(A) \)) we denote the first(second)–class ghost number of quantity \( A \).

The other notation is clear from the context.

## 2 Constraint Algebra

Let

\[ (q^i, p_i), \quad i = 1, \ldots, n = n_+ + n_- , \]  

\[ \varepsilon(q^i) = \varepsilon(p_i) \equiv \varepsilon_i, \quad \text{gh}'(q^i) = -\text{gh}'(p_i) = 0, \quad \text{gh}''(q^i) = -\text{gh}''(p_i) = 0, \]  

\[ (q^i)^\dagger = q^i, \quad (p_i)^\dagger = p_i(-1)^{\varepsilon_i}, \]  

be a set of the original phase variable operators whose equal–time nonzero super–commutation relations are

\[ (\hbar^2)^{-1}[q^i, p_j] = \delta^i_j. \]

Further let us suppose the Hamiltonian,

\[ H = H(p, q), \quad \varepsilon(H) = 0, \]  

and the constraint operators,

\[ T_\alpha = T_\alpha(p, q), \quad \varepsilon(T_\alpha) \equiv \bar{\varepsilon}_\alpha, \]  

\[ \alpha = 1, \ldots, m' = m'_+ + m'_-, \]
\[ T_a^\mu = T_\mu^a(p, q), \quad \varepsilon(T_\mu^a) \equiv \varepsilon_\mu, \quad (2.8) \]

\[ a = 1, 2; \quad \mu = 1, \ldots, m'' = m''_+ + m''_-, \quad (2.9) \]

\[ m''_+ = 2k, \quad m_\pm \equiv m'_\pm + m''_\pm < n_\pm, \quad (2.10) \]

To satisfy the following involution relations

\[ (\iota \hbar)^{-1}[T_\mu^a, T_\nu^b] = U_\mu^a \delta_\nu^b, \quad (2.11) \]

\[ (\iota \hbar)^{-1}[T_\mu^a, T_\alpha^a] = \tilde{U}_\mu^a T_\beta^a + U_\mu^a T_\nu^a, \quad (2.12) \]

\[ (\iota \hbar)^{-1}[T_\alpha^a, T_\beta] = \tilde{U}_\alpha^a T_\gamma^a + \frac{1}{2} \varepsilon_{ab} W_\alpha^\mu (T_\nu^\delta \delta_\rho^b - T_\mu^b \delta_\delta^\rho T_\nu^\delta \rho_\mu (1)^{\varepsilon_\mu \varepsilon_\nu} - \iota \hbar U_\nu^b) T_\rho^a, \quad (2.13) \]

\[ (\iota \hbar)^{-1}[H, T_\mu^a] = V_\nu^a T_\rho^a, \quad (2.14) \]

\[ (\iota \hbar)^{-1}[H, T_\alpha^a] = \tilde{V}_\alpha^a T_\beta^a + \frac{1}{2} \varepsilon_{ab} W_\alpha^\mu (T_\nu^\delta \delta_\rho^b - T_\mu^b \delta_\delta^\rho T_\nu^\delta \rho_\mu (1)^{\varepsilon_\mu \varepsilon_\nu} - \iota \hbar U_\nu^b) T_\rho^a, \quad (2.15) \]

where the structure coefficient operators are some functions of the original phase variables (2.1), and the following antisymmetry properties are supposed to hold:

\[ U_\mu^a = -U_\nu^a (1)^{\varepsilon_\mu \varepsilon_\nu}, \quad \tilde{U}_\mu^a = -\tilde{U}_\nu^a (1)^{\varepsilon_\mu \varepsilon_\nu}, \quad (2.16) \]

\[ W_\alpha^\mu = -W_\alpha^\nu (1)^{\varepsilon_\alpha \varepsilon_\nu} = -W_\nu^\mu (1)^{\varepsilon_\alpha \varepsilon_\nu}, \quad (2.17) \]

\[ W_\mu^a = -W_\nu^a (1)^{\varepsilon_\mu \varepsilon_\nu} = -W_\nu^a (1)^{\varepsilon_\mu \varepsilon_\nu}. \quad (2.18) \]

Let us also require the supercommutators

\[ \Delta_{\mu \nu}^{ab} \equiv (\iota \hbar)^{-1}[T_\mu^a, T_\nu^b], \quad (2.19) \]

enumerated by collective indices \((a, \mu), (b, \nu)\), to form an invertible operator–valued matrix:

\[ \Delta \quad \Rightarrow \quad \exists \Delta^{-1} \quad (2.20) \]

This condition implies the constraints (2.8) to be of the second–class.

In their own turn the involution relations (2.12),(2.13) imply the constraints (2.6) to be of the first class. Let us require for these constraints to commute with the operators (2.1)
to give an operator–valued supermatrix whose invertible Bose–Bose and Fermi–Fermi blocks
are of the maximal possible sizes \( m'_+ \times m'_+ \) and \( m'_- \times m'_- \), respectively, which requirement
is an operator version to the irreducibility condition.

As for the second-class constraints, they are irreducible due to the condition (2.20).

The irreducibility property determines the quantum rules of ”dividing by constraints”,
i.e. characteristic form of the most general operator solution to the basic set of homogeneous linear equations

\[
Z^\mu T^a_\mu + \tilde{Z}^{a\alpha} T_\alpha = 0, \quad (2.21)
\]

\[
Z^{(a\mu} T^{b)}_\mu + \tilde{Z}^{a\alpha} T_\alpha = 0, \quad \tilde{Z}^{[a\beta] \alpha} = 0, \quad (2.22)
\]

\[
Z^{a\mu} T^c_\mu + \text{cycle}(a, b, c) = 0, \quad Z^{[a\beta] \mu} = 0, \quad (2.23)
\]

\[
Z^{\mu\nu} \frac{1}{2} \varepsilon_{ab} (T^b_\rho \delta^\rho_\nu - T^b_\nu \delta^\rho_\rho) (-1)^{\varepsilon_a \varepsilon_b} - i \hbar U^{b\alpha}_{\nu \mu} T^a_\mu + \tilde{Z}^{\alpha} T_\alpha = 0, \quad Z^{\mu\nu} = -Z^{\nu\mu} (-1)^{\varepsilon_a \varepsilon_b}, \quad (2.24)
\]

which are obtained by applying the Jacobi identity to all the involution relations (2.11) –
(2.15). In the Appendix these equations will be considered in more details.

It would be just desirable to avoid imposing further restrictions on the constraint algebra
(2.11) – (2.15). Unfortunately, we are unable to prevent such restrictions for the present.
Therefore we have to impose the following extra condition on the structure coefficients \( \tilde{U}^{a\gamma}_{\mu\alpha} \)
entering the cross–sector relation (2.12) that involves constraints of the both classes:

\[
(i\hbar)^{-1} [T^a_\mu, \tilde{U}^{b\beta}_{\nu\alpha}] - (i\hbar)^{-1} [T^{a\mu}_\nu, \tilde{U}^{b\beta}_{\nu\gamma}] (-1)^{\varepsilon_a \varepsilon_b} - \tilde{U}^{a\gamma}_{\mu\alpha} \tilde{U}^{b\beta}_{\nu\gamma} = (T^b_\rho \delta^\rho_\nu \delta^\gamma_\mu - i \hbar \tilde{U}^{b\beta}_{\rho\gamma} \delta^\mu_\rho \delta^\gamma_\nu) (-1)^{\varepsilon_a \varepsilon_b}, \quad (2.25)
\]

where the new structure coefficient operators \( \tilde{U}^{a\gamma}_{\mu\alpha} \) are supposed to possess the antisymmetry property

\[
\tilde{U}^{a\gamma\rho}_{\mu\alpha} = -\tilde{U}^{a\gamma\rho}_{\mu\alpha} (-1)^{\varepsilon_a \varepsilon_b + \varepsilon_a \varepsilon_b + \varepsilon_a \varepsilon_b}. \quad (2.26)
\]

Let us consider the status of the restriction (2.25). By applying the Jacoby identity to the
constraint algebra (2.11) – (2.15) and then making use of the above mentioned quantum
“rules of dividing by constraints”, one can show the operators \( \tilde{U}^{a\beta}_{\mu\alpha} \) to satisfy the relation
that differs from the one (2.25) by the extra contribution

\[
\tilde{U}^{a\gamma\lambda}_{\mu\alpha} (T^\lambda_\nu \delta^\beta_\gamma - T^\gamma_\nu \delta^\beta_\nu) (-1)^{\varepsilon_a \varepsilon_b} - i \hbar \tilde{U}^{a\beta}_{\gamma\lambda} \quad (2.27)
\]
to r.h.s. Thus, in fact, the condition (2.25) is equivalent to the requirement for the contribution (2.27) to vanish.

On the other hand, one can consider the cross–sector relation (2.12) to be the covariant constancy property of the constraints, being the structure coefficients $\tilde{U}_{a\beta}^{\mu\alpha}, U_{a\rho}^{\mu\nu}$ treated to serve as the connection components. From this viewpoint, l.h.s. of (2.25) is nothing else but the corresponding curvature components. The condition (2.25), being treated classically, requires for the curvature to vanish on the second–class constraint surface, while the algebra (2.11) – (2.13) itself implies a weaker condition to be satisfied that the curvature components should vanish on the surface of all the constraints.

Now let us comment in brief the most characteristic features of the involution relations (2.11) – (2.15).

First of all we observe that the split involution relations (2.11), (2.14) retain their original form [1] specific to the pure second–class constraint case. Further, the cross–sector constraint supercommutators are actually restricted in two respects: the operators $\tilde{U}_{a\beta}^{\mu\alpha}$ are subordinated to the relations (2.25), and the operators $U_{a}^{\nu\mu}$ do not possess their own $Sp(2)$–indices.

Finally, let us turn to the first–class constraint involution relations (2.13), (2.15). Being these relations treated classically, second–class constraints are allowed to contribute only quadratically, which assertion is a consequence of the Jacoby identity. Such quadratic contributions are just represented by the second and third terms in r.h.s. of (2.13), (2.15), and these terms possess the specific structure characterized by the antisymmetry property of the coefficients $\varepsilon_{ab}W^{\mu\nu}$ in their indices $a, b$ and $\mu, \nu$. However, at $\hbar \neq 0$ second–class constraints appear to be allowed quantum–mechanically to contribute to (2.13), (2.15) linearly with the effective coefficients $-\frac{1}{2}i\hbar\varepsilon_{ab}W^{\mu\nu}U_{b\rho}^{\mu\nu}$. These linear quantum contributions, represented by the fourth terms in r.h.s. of (2.13), (2.15), are necessary in order to provide the operator compatibility of the formal constraint algebra.

Given the initial operators (2.5), (2.6), (2.8), the involution relations (2.11) – (2.15) serve to determine the lowest structure coefficient operators $U_{\mu\nu}^{a\rho}, \tilde{U}_{\mu\alpha}^{a\beta}, U_{a}^{\nu\mu}, \tilde{U}_{a\beta}^{\mu\alpha}, W_{a}^{\mu\nu}, V_{a}^{\nu\mu}, \tilde{V}_{a}^{\beta}, W_{a}^{\mu\nu}$ (2.28) up to a natural arbitrariness.

By making use of the Jacoby identity together with the irreducibility property of the constraints, one can derive the necessary compatibility conditions to the involution relations (2.11) – (2.15). These new conditions, including the one (2.25), contain new structure coefficient operators to be determined at this level. On the other hand, these relations reduce to an admissible extent the arbitrariness in the preceding–level structure coefficient operators. Continuing this procedure, one generates, step by step, an infinite gauge algebra initiated by the operators (2.5), (2.6), (2.8).

In the next Section we formulate the generating equations that give automatically an
3 Constraint algebra generating equations

As a next step let us introduce the ghost phase variable operators. We assign a ghost canonical pair to each first–class constraint operator:

\[
T_\alpha \rightarrow (C'^\alpha, \bar{P}'_\alpha), \quad \alpha = 1, \ldots, m',
\]

\[
\varepsilon(C'^\alpha) = \varepsilon(\bar{P}'_\alpha) = \tilde{\varepsilon}_\alpha + 1,
\]

\[
gh'(C'^\alpha) = -gh'(\bar{P}'_\alpha) = 1, \quad gh''(C'^\alpha) = -gh''(\bar{P}'_\alpha) = 0.
\]

\[
(C'^\alpha)\dagger = C'^\alpha, \quad (\bar{P}'_\alpha)\dagger = -\bar{P}'_\alpha(-1)^{\tilde{\varepsilon}_\alpha}.
\]

In the same way we assign a ghost canonical pair to each \((a = 1, 2)\)–pair of the second–class constraint operators \((2.8)\),

\[
T_\mu \rightarrow (C''^\mu, \bar{P}''_\mu), \quad \mu = 1, \ldots, m'',
\]

\[
\varepsilon(C''^\mu) = \varepsilon(\bar{P}''_\mu) = \varepsilon_\mu + 1,
\]

\[
gh'(C''^\mu) = -gh'(\bar{P}''_\mu) = 0, \quad gh''(C''^\mu) = -gh''(\bar{P}''_\mu) = 1
\]

\[
(C''^\mu)\dagger = C''^\mu, \quad (\bar{P}''_\mu)\dagger = -\bar{P}''_\mu(-1)^{\varepsilon_\mu}.
\]

The equal-time nonzero supercommutators of the ghost operators introduced are

\[
(i\hbar)^{-1}[C'^\alpha, \bar{P}'_\beta] = \delta'^\alpha_\beta, \quad (i\hbar)^{-1}[C''^\mu, \bar{P}''_\nu] = \delta''^\mu_\nu.
\]

Further, introduce the generating operators

\[
\Omega''(q, p, C', \bar{P}', C'', \bar{P}''), \quad \varepsilon(\Omega) = 1,
\]

\[
gh'(\Omega) = 0, \quad gh''(\Omega) = 1,
\]

\[
\Omega(q, p, C', \bar{P}', C'', \bar{P}''), \quad \varepsilon(\Omega) = 1,
\]
\[ gh'(\Omega) = 1, \quad gh''(\Omega) = 0, \quad (3.13) \]

\[ K(q, p, C', \bar{P}', C'', \bar{P}'') = 0, \quad (\varepsilon(K) = 0), \quad (3.14) \]

\[ gh'(K) = 2, \quad gh''(K) = -2, \quad (3.15) \]

\[ \mathcal{H}(q, p, C', \bar{P}', C'', \bar{P}'') = 0, \quad (\varepsilon(\mathcal{H}) = 0), \quad (3.16) \]

\[ gh'(\mathcal{H}) = 0, \quad gh''(\mathcal{H}) = 0, \quad (3.17) \]

\[ \Lambda(q, p, C', \bar{P}', C'', \bar{P}'') = 1, \quad (3.18) \]

\[ gh'(\Lambda) = 1, \quad gh''(\Lambda) = -2, \quad (3.19) \]

and subordinate them to the following generating equations:

\[ [\Omega^a, \Omega^b] = 0, \quad (\Omega^a)^\dagger = \Omega^a, \quad (3.20) \]

\[ [\Omega^a, \Omega] = 0, \quad (\Omega)^\dagger = \Omega, \quad (3.21) \]

\[ [\Omega, \Omega] = \varepsilon_{ab}(i\hbar)^{-1}[\Omega^b, [\Omega^a, K]], \quad (K)^\dagger = K, \quad (3.22) \]

\[ [\Omega^a, \mathcal{H}] = 0, \quad (\mathcal{H})^\dagger = \mathcal{H}, \quad (3.23) \]

\[ [\Omega, \mathcal{H}] = \varepsilon_{ab}(i\hbar)^{-1}[\Omega^b, [\Omega^a, \Lambda]], \quad (\Lambda)^\dagger = \Lambda. \quad (3.24) \]

Let us seek for a solution to these equations in the form of \(C\bar{P}\)-ordered series expansion in ghost powers:

\[ \Omega^a = C'^{\mu}T_\mu^a + \frac{1}{2}(-1)^{\varepsilon_\nu}C'^{\mu
u}C'^{\mu\rho}U_{\mu\nu}^{\alpha\beta}\bar{P}_\rho(-1)^{\varepsilon_\rho} + (-1)^{\varepsilon_\alpha}C'^{\alpha}C'^{\mu}U_{\mu\alpha}^{\alpha\beta}\bar{P}_\beta(-1)^{\varepsilon_\beta} + \ldots, \quad (3.25) \]

\[ \Omega = C'^{\alpha}T_\alpha + \frac{1}{2}(-1)^{\varepsilon_\beta}C'^{\alpha\beta}C'^{\alpha\gamma}U_{\alpha\beta\gamma}\bar{P}_\gamma(-1)^{\varepsilon_\gamma} + (-1)^{\varepsilon_\alpha}C'^{\alpha\mu}U_{\mu\alpha\nu}\bar{P}_\nu(-1)^{\varepsilon_\nu} + \ldots, \quad (3.26) \]
\[ K = \frac{1}{2}(-1)^{\tilde{\epsilon}_\beta} C^\alpha_\beta C^\alpha_\mu W^\mu_\nu \tilde{\mathcal{P}}^\nu_\mu (\mathcal{P})^{\tilde{\epsilon}_\nu} + \ldots, \]  
\[ (3.27) \]

\[ \mathcal{H} = H - C^\mu_\nu V_\mu_\nu (\mathcal{P})^{\tilde{\epsilon}_\nu} - C^\alpha_\nu \tilde{V}^\alpha_\beta \tilde{\mathcal{P}}^\nu_\beta (\mathcal{P})^{\tilde{\epsilon}_\beta} + \ldots, \]  
\[ (3.28) \]

\[ \Lambda = \frac{1}{2} C^\mu_\nu W_\alpha_\mu \tilde{\mathcal{P}}^\nu_\nu (\mathcal{P})^{\tilde{\epsilon}_\nu} + \ldots. \]  
\[ (3.29) \]

Of course, we have chosen the \( C \tilde{\mathcal{P}} \)–ordering only for the sake of convenience of the general analysis. Depending on a particular representation of constraints some other choice of ghost ordering may appear to be more relevant, such as the Weyl– or Wick–ordering in field–theory case.

By inserting the expansions (3.25) – (3.29) into the left generating equations in (3.20) – (3.24), one obtains to the second order in ghosts just the constraint involution relations (2.11) – (2.15), whereas to higher orders in ghosts we obtain all the higher structure relations of the gauge algebra initiated by the given operators (2.5), (2.6), (2.8). On the other hand, the right equations in (3.20) – (3.24) determine the properties of the constraints and higher structure coefficients with respect to the Hermitian conjugation. Thus the equations (3.20) – (3.24) describe the gauge algebra generating mechanism comprehensively.

The following Existence Theorem holds for the proposed generating equations (3.20) – (3.24): if the constraint involution relations (2.11) – (2.5) are satisfied together with the conditions (2.20), (2.25) and the ones requiring for the first-class constraints \( T_\alpha \) to be irreducible in the above formulated sense, then there also exist all the higher structure coefficients in the expansions (3.25) – (3.29) and, thus, there exists a formal solution of the algebra generating equations. Besides, it can be shown that all the Hermiticity properties in (3.20) – (3.24) can also be satisfied by the solution obtained.

The algebra generating equations (3.20) – (3.24) admit the following group of automorphisms:

\[ A = A_1 \cdot A_2 \cdot A_3 \]  
\[ (3.30) \]

where \( A_1 \) is the standard unitary group

\[ \Omega^a \rightarrow U^{-1} \Omega^a U, \]  
\[ (3.31) \]

\(^2\text{In particular, the relation (2.25) is generated by the left equation (3.20) to the } C'(C'')^2 \mathcal{P}^- \text{–order, whereas the corresponding contribution to } \Omega^a \text{ is of the form (see also eq. (A.10) of the Appendix)}\]

\[ \frac{1}{2}(-1)^{\epsilon_\mu + \epsilon_\nu + \epsilon_\alpha} C^\alpha C^\mu C^\nu \tilde{\mathcal{P}}^\alpha_\mu \tilde{\mathcal{P}}^\nu_\nu (\mathcal{P})^{\tilde{\epsilon}_\nu}. \]
\[ \Omega \rightarrow U^{-1} \Omega U, \quad K \rightarrow U^{-1} K U, \quad (3.32) \]

\[ \mathcal{H} \rightarrow U^{-1} \mathcal{H} U, \quad \Lambda \rightarrow U^{-1} \Lambda U, \quad (3.33) \]

\[ A_2 = GL(2, R) \text{ is the group of } c\text{-numerical nondegenerate linear transformations} \]

\[ \Omega^a \rightarrow S_\theta^b \Omega^b, \quad \Omega \rightarrow \Omega, \quad \mathcal{H} \rightarrow \mathcal{H}, \quad (3.34) \]

\[ K \rightarrow \lambda^{-1} K, \quad \Lambda \rightarrow \lambda^{-1} \Lambda, \quad \lambda \equiv \det(S_\theta^b), \quad (3.35) \]

\[ A_3 \text{ is the group of exact shifts} \]

\[ \Omega^a \rightarrow \Omega^a, \quad (3.36) \]

\[ \Omega \rightarrow \Omega + \varepsilon_{ab} (\hbar)^{-2} [\Omega^b, [\Omega^a, \Xi]], \quad (3.37) \]

\[ K \rightarrow K + 2(\hbar)^{-1} [\Omega, \Xi] + \varepsilon_{ab} (\hbar)^{-3} [[\Omega^b, \Xi], [\Omega^a, \Xi]] + (\hbar)^{-1} [\Omega^a, X_a], \quad (3.38) \]

\[ \mathcal{H} \rightarrow \mathcal{H} + (\hbar)^{-1} [\Omega, \Psi] + \varepsilon_{ab} (\hbar)^{-1} [\Omega^b, [\Omega^a, \Phi]] + (\hbar)^{-1} [\Omega^a, \Phi] = 0, \quad (3.39) \]

\[ \Lambda \rightarrow \Lambda + (\hbar)^{-1} [\Xi, \mathcal{H}] + (\hbar)^{-1} [\Omega, \Phi] + \frac{1}{2} (\hbar)^{-1} [K, \Psi] + (\hbar)^{-2} [\Xi, [\Omega, \Psi]] + \varepsilon_{ab} (\hbar)^{-3} [[\Xi, \Omega^b], [\Omega^a, \Phi]] + (\hbar)^{-1} [\Omega^a, Y_a]. \quad (3.40) \]

Under the premises of the Existence Theorem the group of automorphisms (3.30) is the maximal possible one and, thus, describes the natural arbitrariness of a solution to the algebra generating equations (3.20) – (3.24) comprehensively.

The exact shift transformations (3.37) – (3.40) enable one to make the new operators \( \bar{K} \) and \( \bar{\Lambda} \) vanish. Then one can apply the ghost–dependent canonical transformations (3.31) – (3.33) to make the generating operators take the Abelian form

\[ \Omega^a_{\text{abelian}} = C^{a\mu} t_\mu, \quad \Omega_{\text{abelian}} = C^{a\alpha} t_\alpha, \quad (3.41) \]

\[ [t^{a}_{\mu}, t^{b}_{\nu}] = 0, \quad [t^{a}_{\mu}, t^{a}_{\alpha}] = 0, \quad [t_{\alpha}, t_{\beta}] = 0, \quad (3.42) \]

\[ [\mathcal{H}_{\text{abelian}}, t^{a}_{\mu}] = 0, \quad [\mathcal{H}_{\text{abelian}}, t^{a}_{\alpha}] = 0. \quad (3.43) \]
4 Unitarizing Hamiltonian

Introduce now the following new canonical variable operators which are the antighosts:

\[(\mathcal{P}^\alpha, \bar{C}_\alpha), \quad \alpha = 1, \ldots, m'\]  \hspace{1cm} (4.1)

\[\varepsilon(\mathcal{P}^\alpha) = \varepsilon(\bar{C}_\alpha) = \bar{\varepsilon}_\alpha + 1,\]  \hspace{1cm} (4.2)

\[gh'(\mathcal{P}^\alpha) = -gh'(\bar{C}_\alpha) = 1, \quad gh''(\mathcal{P}^\alpha) = -gh''(\bar{C}_\alpha) = 0,\]  \hspace{1cm} (4.3)

\[(\mathcal{P}^{\alpha\mu}, \bar{C}^{\mu}_\alpha), \quad \mu = 1, \ldots, m''\]  \hspace{1cm} (4.4)

\[\varepsilon(\mathcal{P}^{\alpha\mu}) = \varepsilon(\bar{C}^{\mu}_\alpha) = \varepsilon_\mu + 1,\]  \hspace{1cm} (4.5)

\[gh'(\mathcal{P}^{\alpha\mu}) = -gh'(\bar{C}^{\mu}_\alpha) = 0, \quad gh''(\mathcal{P}^{\alpha\mu}) = -gh''(\bar{C}^{\mu}_\alpha) = 1,\]  \hspace{1cm} (4.6)

\[gh'(\mathcal{P}^{\alpha\mu}) = -gh'(\bar{C}^{\mu}_\alpha) = 0, \quad gh''(\mathcal{P}^{\alpha\mu}) = -gh''(\bar{C}^{\mu}_\alpha) = 1,\]  \hspace{1cm} (4.7)

\[\varepsilon(\mathcal{P}^{\alpha\mu}) = \varepsilon(\bar{C}^{\mu}_\alpha) = \varepsilon_\mu + 1,\]  \hspace{1cm} (4.8)

and dynamically-active Lagrange multipliers:

\[(\lambda_\alpha, \pi_\alpha), \quad \alpha = 1, \ldots, m'\]  \hspace{1cm} (4.9)

\[\varepsilon(\lambda_\alpha) = \varepsilon(\pi_\alpha) = \bar{\varepsilon}_\alpha,\]  \hspace{1cm} (4.10)

\[gh'(\lambda_\alpha) = -gh'(\pi_\alpha) = 0, \quad gh''(\lambda_\alpha) = -gh''(\pi_\alpha) = 0,\]  \hspace{1cm} (4.11)

\[\varepsilon(\lambda_\alpha) = \varepsilon(\pi_\alpha) = \varepsilon_\alpha,\]  \hspace{1cm} (4.12)

\[(\lambda_\alpha)^\dagger = \lambda_\alpha(-1)^{\bar{\varepsilon}_\alpha}, \quad (\pi_\alpha)^\dagger = \pi_\alpha,\]  \hspace{1cm} (4.13)

\[\varepsilon(\lambda_\mu), \quad a = 1, 2, \quad \mu = 1, \ldots, m'',\]  \hspace{1cm} (4.14)

\[gh'(\lambda_\mu) = gh''(\lambda_\mu) = 0.\]  \hspace{1cm} (4.15)
\((\lambda^a_\mu)^\dagger = \lambda^a_\mu\). 

The equal–time nonzero supercommutators of the new operators introduced are

\[ (i\hbar)^{-1}[\mathcal{P}^{\alpha \mu}, \bar{\mathcal{C}}^\beta_\nu] = \delta^\alpha_\beta, \quad (i\hbar)^{-1}[\mathcal{P}^{\mu \nu}, \bar{\mathcal{C}}^\mu_\nu] = \delta^\mu_\nu, \] 

\[ (i\hbar)^{-1}[\lambda^a_\alpha, \pi_\beta] = \delta^a_\beta, \quad (i\hbar)^{-1}[\lambda^a_\mu, \lambda^b_\nu] = \varepsilon^{ab} d_{\mu \nu}, \]

where a constant matrix \(d_{\mu \nu}\) is supposed to be invertible and possesses the following symmetry properties

\[ d_{\nu \mu} = d_{\mu \nu} (-1)^{\varepsilon_{\mu \nu}}, \quad d_{\nu \mu}^* = d_{\mu \nu}. \]

Let us extend the generating operators (3.10), (3.12) by including the phase variable operators (4.1), (4.5), (4.9), (4.14) via the formulae

\[ Q = \Omega + \mathcal{P}^{\alpha \mu} \pi_\alpha, \]

\[ Q^a = \Omega^a + \mathcal{P}^{\mu \nu} \lambda^a_\mu, \quad a = 1, 2, \]

so that

\[ \varepsilon(Q) = 1, \quad gh'(Q) = 1, \quad gh''(Q) = 0, \]

\[ \varepsilon(Q^a) = 1, \quad gh'(Q^a) = 0, \quad gh''(Q^a) = 1. \]

The extended operators \(Q, Q^a\) satisfy the same equations (3.20) – (3.24) as their minimal–sector counterparts \(\Omega, \Omega^a\) do.

The complete Unitarizing Hamiltonian of the theory reads

\[ H_{\text{complete}} = \mathcal{H} + (i\hbar)^{-1}[Q, F] + \varepsilon_{ab}(i\hbar)^{-2}[Q^b, [Q^a, B]], \quad [Q^a, F] = 0, \]

where

\[ \varepsilon(F) = 1, \quad gh'(F) = -1, \quad gh''(F) = 0, \]

\[ \varepsilon(B) = 0, \quad gh'(B) = 0, \quad gh''(B) = -2. \]

\[ (F)^\dagger = -F, \quad (B)^\dagger = -B. \]
The gauge-fixing operators $F$ and $B$ may depend on the total set of phase variables of the extended phase space. In the simplest case these gauge operators can be chosen in the form

$$F = \lambda^\alpha \tilde{P}_\alpha + (\chi^\alpha + C^\mu_\nu V^\nu_\mu \tilde{P}_\mu (1)^{\varepsilon_v + \varepsilon_\alpha}) \tilde{C}_\alpha + \ldots, \quad (i\hbar)^{-1}[T_\mu, \chi^\alpha] = V^\nu_\alpha T^\alpha_\nu,$$

$$B = \tilde{P}^\mu_\nu \tilde{C}^\nu_\mu d^\nu, \quad d^\mu_\nu d^\nu\rho = \delta^\rho_\mu.$$  

Further, let us introduce the ghost number operators

$$G' = \frac{1}{2} (C^\mu_\alpha \tilde{P}_\alpha (1)^{\varepsilon_v} - \tilde{P}_\alpha C^\mu_\alpha) + \frac{1}{2} (P^\alpha_\mu \tilde{C}_\alpha (1)^{\varepsilon_v} - \tilde{C}_\alpha P^\mu_\alpha),$$

$$G'' = \frac{1}{2} (C^\mu_\nu \tilde{P}_\mu (1)^{\varepsilon_v} - \tilde{P}_\mu C^\mu_\nu) + \frac{1}{2} (P^\mu_\nu \tilde{C}_\mu (1)^{\varepsilon_v} - \tilde{C}_\mu P^\mu_\nu).$$

Then we have

$$(i\hbar)^{-1}[G', A] = g h'(A) A, \quad G' \Phi = gh'(|\Phi\rangle |\Phi\rangle),$$

$$(i\hbar)^{-1}[G'', A] = g h''(A) A, \quad G'' \Phi = gh''(|\Phi\rangle |\Phi\rangle).$$

The total ghost number operator is naturally defined as

$$G = G' + G''.$$  

As a next step, let us define the physical operators and physical states. An operator $O$ is called the physical one iff

$$gh'(O) = gh''(O) = 0,$$

$$[Q^a, O] = 0, \quad [Q, O] = \varepsilon_{ab} (i\hbar)^{-1}[Q^b, [Q^a, E]].$$

Of course, the Hamiltonian (4.24) is a physical operator just in the sense of this definition. A state $|\Phi\rangle$ is called the physical one iff

$$gh'(|\Phi\rangle) = gh''(|\Phi\rangle) = 0,$$

$$Q^a |\Phi\rangle = 0, \quad Q |\Phi\rangle = \varepsilon_{ab} (i\hbar)^{-1} Q^b Q^a |E\rangle.$$  

The physical matrix elements $\langle \Phi | O | \Phi_1 \rangle$ depend neither on the arbitrariness (3.30) in determining the generating operators $\Omega^a, \Omega, K, H, \Lambda$, nor on the arbitrariness of r.h.s. of eqs. (4.36), (4.38).
Let $\Gamma$ be the total set of phase variable operators of the extended phase space, and let $\Gamma(t)$ satisfies the Heisenberg equations governed by the Unitarizing Hamiltonian (4.24). Then the physical matrix elements $\langle \Phi|O(\Gamma(t))|\Phi_1 \rangle$ do not depend on a particular choice of gauge–fixing operators $F$ and $B$.

5 Further Generalization and Geometric Interpretation

It has been implied in the above considerations that the second–class constraints themselves retain their algebraic properties to be the same as they are in the pure second–class case. In particular, no first–class constraints enter the split involution relations (1.1), (1.2) actually.

In this section we intend to generalize the set of constraint algebra generating equations in order to make it possible for the first–class constraints contribute explicitly to the modified split involution relations.

The main idea can be explained as follows. Let the original second–class constraints $T^a_\mu$ are allowed to contain the first–class admixture. Let us suppose that the corresponding admixture to the generating operators $\Omega^a$ is representable in the form

$$(\imath\hbar)^{-1}[A^a,\Omega],$$

where new ghost–dependent operators $A^a$ are introduced,

$$\varepsilon(A^a) = 0, \quad \text{gh}'(A^a) = -1, \quad \text{gh}''(A^a) = 1, \quad (A^a)\dagger = A^a,$$

and $\Omega$ is the first–class generating operator to be determined selfconsistently.

It is quite natural to require for the pure second–class generating operators

$$\Omega^a - (\imath\hbar)^{-1}[A^a,\Omega]$$

(5.3)

to satisfy the equations similar to the above–given ones (3.20), (3.21):

$$[\Omega^a - (\imath\hbar)^{-1}[A^a,\Omega],\Omega^b - (\imath\hbar)^{-1}[A^b,\Omega]] = 0,$$

(5.4)

$$[\Omega^a - (\imath\hbar)^{-1}[A^a,\Omega],\Omega] = 0,$$

(5.5)

Besides, we have to subordinate the first–class generating operator $\Omega$ to the equation similar the one (3.22):

$$[\Omega,\Omega] = \varepsilon_{ab}(\imath\hbar)^{-1}[\Omega^b - (\imath\hbar)^{-1}[A^b,\Omega],[\Omega^a - (\imath\hbar)^{-1}[A^a,\Omega],K]].$$

(5.6)

In the same way we formulate the equations similar to the above–given ones (3.23), (3.24):
\[ [\Omega^b - (i\hbar)^{-1}[A^b, \Omega], H] = 0, \tag{5.7} \]

\[ [\Omega, H] = \varepsilon_{ab} (i\hbar)^{-1}[\Omega^b, [\Omega^a, \Omega], [\Omega^a, \Omega, \Lambda]]. \tag{5.8} \]

The generating operators \( \Omega^a, \Omega, K, H, \Lambda \) are searched in the form of the corresponding series expansions (3.25) – (3.29), whereas the new operators \( A^a \) are expanded in ghost powers as

\[
A^a = C'''^a B^a_{\beta} B^a_{\gamma} (\frac{1}{i}) \varepsilon_{\beta \gamma} + \frac{1}{2} (-1)^{\varepsilon_{\delta \beta}} C'''^a B^a_{\rho} B^a_{\gamma} (\frac{1}{i}) \varepsilon_{\rho \gamma} + \ldots. \tag{5.9} \]

Here we refrain from considering in details the explicit form of a constraint algebra generated by eqs. (5.4) – (5.8) to the lowest order in ghosts. The only comment to be given here concerns the modified cross–sector relations. Instead of (2.12) we have:

\[
(i\hbar)^{-1}[T^a_\alpha, T^a_\beta] = \tilde{T}^a_\alpha T^a_\beta + U^a_\mu T^a_\nu + \frac{1}{2} (\delta^a_\alpha \tilde{T}^a_\beta - \delta^a_\beta \tilde{T}^a_\alpha) (\frac{1}{i}) \varepsilon_{\alpha \beta} + \frac{1}{i} \hbar \tilde{T}^a_\alpha T^a_\beta (\frac{1}{i}) \varepsilon_{\alpha \beta} - \tilde{U}^a_\beta T^a_\alpha. \tag{5.10} \]

Being treated at the classical level, these relations determine the quantities \( \tilde{X}^a_\alpha \) to serve as coefficients of a linear dependence between the cross–sector supercommutators \( \{T^a_\alpha, T^a_\beta\} \) and the pure first–class–sector ones \( \{T^a_\alpha, T^a_\beta\} \).

The following Existence Theorem apparently holds for the generating equations (5.4) – (5.8): if these equations are satisfied to the lowest order in ghosts and, besides, the equations (5.4) themselves are satisfied to the \( C'(C'')^2 \)–order, then there exists a formal solution for the generating operators \( \Omega^a, \Omega, K, H, \Lambda \) to all orders in ghosts.

It is an interesting circumstance that l.h.s. of eqs. (5.4) – (5.8) possess the structure of a natural first–class counterpart of the well–known Dirac’s bracket, being eq.(5.5) represented in the equivalent form

\[
(i\hbar)^{-1}[A^a, \frac{1}{i}[\Omega, \Omega]] = [\Omega^a, \Omega]. \tag{5.11} \]

to determine the "Lagrange multiplier" operators \( A^a \) to an admissible extent. It is just the form that generalizes in the most natural way the lowest–order relations (5.10).

Now, let us consider an interesting geometric extension to the set of eqs. (5.4) – (5.8). First of all, introduce a pair of real Bosonic ghost parameters \( \xi_a \),

\[
\varepsilon(\xi_a) = 0, \quad \text{gh}'(\xi_a) = 1, \quad \text{gh}''(\xi_a) = -1, \quad \xi_a^* = \xi_a. \tag{5.12} \]

Next, let us define the \( \xi \)–dependent generating operators
\[\bar{\Omega}, \bar{K}, \bar{H}, \bar{\Lambda}, \bar{A}^a\] (5.13)

to satisfy the equations

\[\left[\bar{\Omega}, \bar{\Omega}\right] = \varepsilon_{ab}(ih)^{-1}\left[\nabla^b\bar{\Omega}, \left[\nabla^a\bar{\Omega}, \bar{K}\right]\right],\] (5.14)

\[\nabla^a\nabla^b\bar{\Omega} = 0, \quad \nabla^a\bar{K} = 0,\] (5.15)

\[\left[\bar{\Omega}, \bar{H}\right] = \varepsilon_{ab}(ih)^{-1}\left[\nabla^b\bar{\Omega}, \left[\nabla^a\bar{\Omega}, \bar{\Lambda}\right]\right],\] (5.16)

\[\nabla^a\bar{H} = 0, \quad \nabla^a\bar{\Lambda} = 0,\] (5.17)

where

\[\nabla^a \equiv \partial^a - (ih)^{-1} \text{ad} \bar{A}^a, \quad \partial^a \equiv \frac{\partial}{\partial \xi^a}\] (5.18)

stand for the covariant \(\xi\)-derivative components, so that for arbitrary \(E(\xi)\) we have

\[\nabla^a E = \partial^a E - (ih)^{-1}\left[\bar{A}^a, E\right].\] (5.19)

We suppose the connection \(\bar{A}^a\) to be flat:

\[\partial^{[a} \bar{A}^{b]} - (ih)^{-1}\left[\bar{A}^{a}, \bar{A}^{b}\right] = 0.\] (5.20)

It follows immediately from eqs. (5.14) – (5.17), (5.20) that

\[\nabla^a[\bar{\Omega}, \bar{\Omega}] \equiv 2[\nabla^a\bar{\Omega}, \bar{\Omega}] = 0,\] (5.21)

\[\left[\nabla^a\bar{\Omega}, \nabla^b\bar{\Omega}\right] = 0,\] (5.22)

\[\left[\nabla^a\bar{\Omega}, \bar{H}\right] = 0.\] (5.23)

If one denotes:

\[\bar{\Omega}|_{\xi=0} \equiv \Omega, \quad \partial^a\bar{\Omega}|_{\xi=0} \equiv \Omega^a,\] (5.24)

\[\bar{A}^a|_{\xi=0} \equiv A^a, \quad \bar{K}|_{\xi=0} \equiv K,\] (5.25)

\[\bar{H}|_{\xi=0} \equiv H, \quad \bar{\Lambda}|_{\xi=0} \equiv \Lambda,\] (5.26)
the equations (5.14), (5.16), (5.21) – (5.23), being taken at \( \xi = 0 \), just reproduce the set of eqs. (5.4) – (5.8).

Due to the flatness condition (5.20) there exists a \( \xi \)–dependent canonical transformation that results for the connection components \( \bar{A}^a \) in their vanishing. Thus one returns naturally to the case considered in previous Sections.

Further, let us extend the operator \( \bar{\Omega} \) via the formula

\[
\bar{Q} \equiv \bar{\Omega} + \mathcal{P}^\alpha \pi_\alpha + \mathcal{P}^{\mu \nu} \lambda_\mu^a \xi_a. \tag{5.27}
\]

Then the \( \xi \)–dependent Unitarizing Hamiltonian reads

\[
\bar{H}_{\text{complete}} \equiv \bar{\mathcal{H}} + (i\hbar)^{-1}[\bar{Q}, \bar{F}] + \varepsilon_{ab}(i\hbar)^{-2} [\nabla^b \bar{Q}, [\nabla^a \bar{Q}, \bar{B}]], \tag{5.28}
\]

where \( \xi \)–dependent gauge–fixing operators \( \bar{F}, \bar{B} \) should satisfy the conditions

\[
\nabla^a F = 0 \quad [F, \nabla^a \bar{Q}] = 0, \quad \nabla^a B = 0. \tag{5.29}
\]

Finally, let us define the \( \xi \)–dependent physical operators and states. An operator \( \bar{\mathcal{O}} \) is called the physical one iff:

\[
\frac{\partial}{\partial \xi} \bar{\mathcal{O}} = \frac{\partial^2}{\partial \xi^2} \bar{\mathcal{O}} = 0, \tag{5.30}
\]

\[
[\bar{Q}, \bar{\mathcal{O}}] = \varepsilon_{ab}(i\hbar)^{-1} [\nabla^b \bar{Q}, [\nabla^a \bar{Q}, \bar{B}]], \tag{5.31}
\]

\[
\nabla^a \bar{\mathcal{O}} = 0, \quad \nabla^a E = 0. \tag{5.32}
\]

A state \( |\bar{\Phi}\rangle \) is called the physical one iff

\[
\frac{\partial}{\partial \xi} |\bar{\Phi}\rangle = \frac{\partial^2}{\partial \xi^2} |\bar{\Phi}\rangle = 0, \tag{5.33}
\]

\[
\bar{Q}|\bar{\Phi}\rangle = \varepsilon_{ab}(i\hbar)^{-1} (\nabla^b \bar{Q})(\nabla^a \bar{Q})|\bar{E}\rangle, \tag{5.34}
\]

\[
\nabla^a |\bar{\Phi}\rangle = 0, \quad \nabla^a |\bar{E}\rangle = 0, \tag{5.35}
\]

where the covariant derivative operators \( \nabla^a \) are applied to arbitrary state \( |\ldots\rangle \) via the formula: \( \nabla^a |\ldots\rangle \equiv (\partial^a - (i\hbar)^{-1} A^a)|\ldots\rangle \).

By construction, the physical matrix elements are \( \xi \)–independent:

\[
\langle \bar{\Phi}|\bar{\mathcal{O}}|\bar{\Phi}_1\rangle = \langle \Phi|\mathcal{O}|\Phi_1\rangle \tag{5.36}
\]
where unbared operators and states in r.h.s. coincide with the corresponding bared ones taken at $\xi = 0$. (Of course, one can choose another fixed point $\xi_0$ instead of $\xi = 0$.)

The physical matrix elements (5.36) are also independent of a particular choice of operators $F$, $B$, $E$ and states $|E\rangle$ entering eqs. (5.28) – (5.35) taken at $\xi = 0$.

6 Conclusion

So, we have extended the split involution formalism to cover the case of the presence of irreducible first–class constraints. Thereby the miraculous supersymmetry yielded by the split involution relations is coupled to the actual gauge symmetry initiated by the original first–class constraints.

The most characteristic feature of the formalism proposed is the appearance of the new equivalence criterion explicitly–quadratic in second–class constraints that is a natural counterpart to the Dirac’s weak equality concept as applied to the first–class quantities.

It is quite evident from this viewpoint that all the double–supercommutator contributions in (3.22), (3.24), (3.39), (4.24), (4.36) as well as the quadratic operator in r.h.s. of (4.38) are of the same origin.

All the main results are extendable in a straightforward way to cover the case of finite–stage reducibility of the first and second-class constraints included.

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7 Appendix. Quantum Rules of Dividing by Constraints

In this Appendix we represent the general solution to the equation (2.21) – (2.24).

First of all, let us introduce the following remarkable operators:

$$W_n \equiv \sum_{m=0}^{n-1} \Omega_m(C, \bar{P})|_{C \rightarrow -i\hbar \frac{\partial}{\partial \bar{P}}(-1)^c(\bar{P})}, \quad (A.1)$$

$$W_n^a \equiv \sum_{m=0}^{n-1} \Omega_m^a(C, \bar{P})|_{C \rightarrow -i\hbar \frac{\partial}{\partial \bar{P}}(-1)^c(\bar{P})}, \quad (A.2)$$

where $C \equiv (C', C''')$, $\bar{P} \equiv (\bar{P}', \bar{P}'')$ is a condensed notation for ghost operators, and

$$\Omega_m \sim (C)^{m+1}(\bar{P})^m, \quad \Omega_m^a \sim (C)^{m+1}(\bar{P})^m \quad (A.3)$$

are the corresponding homogeneous monomials entering the ghost power series expansions to the generating operators $\Omega$, $\Omega^a$. 

\[ \Omega = \sum_{m=0}^{\infty} \Omega_m, \quad \Omega^a = \sum_{m=0}^{\infty} \Omega^a_m. \quad (A.4) \]

In particular we have

\[ \Omega_0 = C^{\alpha \alpha} T_\alpha, \quad (A.5) \]

\[ \Omega_1 = \frac{1}{2} (-1)^{\bar{\epsilon} a} C^{\mu \beta} C^{\alpha a}_{\mu \beta} \bar{U}^\gamma_{\alpha \beta} \bar{P}^\gamma_\gamma (-1)^{\bar{\epsilon} \gamma} + (-1)^{\bar{\epsilon} h} C^{\mu \nu} \bar{U}^\nu_{\mu a} \bar{P}^\nu_\nu (-1)^{\bar{\epsilon} \nu}, \quad (A.6) \]

\[ \Omega_2 = \frac{1}{12} (-1)^{\bar{\epsilon} a + \bar{\epsilon} a \bar{\epsilon} a} C^{\mu \gamma} C^{\nu a} \bar{U}^\gamma_{\alpha \beta} \bar{P}^\gamma_\alpha \bar{P}^\gamma_\delta (-1)^{\bar{\epsilon} \lambda} + \frac{1}{2} \sum_{\nu} (-1)^{\bar{\epsilon} a + \bar{\epsilon} a \bar{\epsilon} a} C^{\mu \nu} \bar{U}^\nu_{\mu a} \bar{P}^\nu_\nu (-1)^{\bar{\epsilon} \nu} + \frac{1}{4} (-1)^{\epsilon \nu + \bar{\epsilon} a \nu} C^{\mu \nu} \bar{U}^{\nu a}_{\mu a} \bar{P}^\nu_\nu (-1)^{\epsilon \nu}, \quad (A.7) \]

\[ \Omega_2^a = \frac{1}{12} (-1)^{\bar{\epsilon} a + \bar{\epsilon} a \bar{\epsilon} a} C^{\mu \rho} C^{\nu a} \bar{U}^{a \beta}_{\nu a} \bar{P}^\nu_\rho \bar{P}^\nu_\rho (-1)^{\epsilon \rho} + \frac{1}{2} (-1)^{\epsilon \rho + \bar{\epsilon} a \epsilon \rho} C^{\mu \nu} \bar{U}^{\nu a}_{\mu a} \bar{P}^\nu_\rho \bar{P}^\nu_\rho (-1)^{\epsilon \rho} + \frac{1}{4} (-1)^{\bar{\epsilon} a + \bar{\epsilon} a \bar{\epsilon} a} C^{\mu \nu} \bar{U}^{\nu a}_{\mu a} \bar{P}^\nu_\rho \bar{P}^\nu_\rho (-1)^{\epsilon \rho}. \quad (A.9) \]

As applied from the right to arbitrary \( C \bar{P} \)-ordered polinomial of the highest power \( n \) in ghost momenta \( \bar{P} \), the operators (A.1), (A.2) possess the important formal properties

\[ W_n W^a_{n-1} + W^a_n W_{n-1} = 0, \quad W^{\{a W^{\mu b} \}}_n = 0, \quad (A.11) \]

\[ W_n W_{n-1} = \frac{1}{2} \Delta_n \big|_{C \rightarrow -i \hbar \frac{\partial}{\partial \bar{P}}} (-1)^{\epsilon (\bar{P})}, \quad (A.12) \]

where

\[ \Delta \equiv \epsilon_{ab} (i \hbar)^{-1} [\Omega^b, [\Omega^a, K]] = \sum_{n=2}^{\infty} \Delta_n \big|_{C \rightarrow -i \hbar \frac{\partial}{\partial \bar{P}}} (-1)^{\epsilon (\bar{P})}, \quad \Delta_n \sim (C)^n (\bar{P})^{n-2}, \quad (A.13) \]

\[ \Delta_2 = \bar{W}_2 W^a_2 W^a_1 \epsilon_{ab}, \quad \bar{W}_2 = (i \hbar)^{-1} K_2 \big|_{C \rightarrow -i \hbar \frac{\partial}{\partial \bar{P}}} (-1)^{\epsilon (\bar{P})}, \quad (A.14) \]

and \( K_m \) is the \( (\bar{P})^{m} \)-order in the expansion (3.27).

Now, let us consider the equation (2.21) to represent it in the form
\[ Z_1 W_1^a + \tilde{Z}_1^a W_1 = 0, \quad (A.15) \]

where

\[ Z_1 \equiv Z_\mu \bar{P}_\mu (-1)^\varepsilon, \quad \tilde{Z}_1^a \equiv \tilde{Z}_a^{\alpha} \bar{P}_\alpha (-1)^\varepsilon. \quad (A.16) \]

It can be shown that the general solution for \( Z_1, \tilde{Z}_1^a \) is

\[ Z_1 = (E_3 W_3^b + \tilde{E}_2 W_2^a) \varepsilon_{ab} + \tilde{E}_2 W_2, \quad (A.17) \]

\[ \tilde{Z}_1^a = \tilde{E}_2 W_2^a + \tilde{E}_2 W_2, \quad (A.18) \]

where

\[ E_3 \sim (\bar{P}^\rho)^3, \quad \tilde{E}_2 \sim \bar{P}^\rho \bar{P}^\rho, \quad \tilde{E}_2 \sim (\bar{P}^\rho)^2 \quad (A.19) \]

are arbitrary operators.

By eliminating ghost operators from the representations (A.17), (A.18), one decodes the general solution for \( Z^\mu, \tilde{Z}^{\alpha\alpha} \) in the form

\[ Z^\mu = \tilde{E}^{\alpha\nu} \Pi^\mu_{\nu\alpha} + (E^\tau\rho \Pi^\tau_{\rho\sigma} + i\hbar \tilde{E}_{a}^{\beta\alpha} W_\alpha^\xi_{\sigma\nu}) \Pi^\mu_{\nu\xi\varepsilon_{ba}}, \quad (A.20) \]

\[ \tilde{Z}^{\alpha\alpha} = \tilde{E}^{\beta\mu} \tilde{\Pi}^{\alpha\alpha}_{\mu\beta} + \tilde{E}_{a}^{\alpha\gamma} \tilde{\Pi}^{\alpha}_{a\nu\gamma}, \quad (A.21) \]

where

\[ \Pi^\nu_{\mu\alpha} \equiv -T_{a}^{\delta\mu} (-1)^\varepsilon_{a}^{\varepsilon_{\nu}} \varepsilon - i\hbar U_{\mu\alpha}^\nu, \quad (A.22) \]

\[ \Pi^\sigma_{\mu\nu\rho} \equiv [(\delta^\sigma_{\mu} \tilde{\Pi}^{\tau\alpha}_{\nu\rho} - \delta^\tau_{\mu} \tilde{\Pi}^{\sigma\alpha}_{\nu\rho} (-1)^\varepsilon_{\sigma\tau}) (-1)^\varepsilon_{\mu\nu\rho} + \text{cycle}(\rho, \mu, \nu)] + (i\hbar)^2 U_{\mu\nu\rho}^{a\sigma\tau}, \quad (A.23) \]

\[ \bar{\Pi}^{\rho\nu}_{\mu\alpha} \equiv \frac{1}{2} (T_{a}^{\rho\nu} - T_{\nu}^{\rho\mu} (-1)^\varepsilon_{\rho\nu}) - i\hbar U_{\mu\rho}^{\alpha\nu}, \quad (A.24) \]

\[ \Pi^{\rho\nu}_{\mu\nu} \equiv T_{\mu}^{\rho\mu} - T_{\nu}^{\rho\nu} (-1)^\varepsilon_{\rho\nu} - i\hbar U_{\mu\nu}^{\alpha\rho}, \quad (A.25) \]

\[ \tilde{\Pi}^{\beta\alpha}_{\mu\alpha} \equiv T_{\mu}^{\beta\alpha} - i\hbar U_{\mu\alpha}^{\beta\alpha}, \quad (A.26) \]

\[ \tilde{\Pi}^{\gamma\mu}_{\alpha\beta} \equiv T_{\gamma}^{\gamma\mu} - T_{\beta}^{\gamma\alpha} (-1)^\varepsilon_{\alpha\beta} - i\hbar U_{\alpha\beta}^{\gamma}, \quad (A.27) \]
Next, let us consider the equation (2.22) to represent it in the form

\[ Z_1^{(a} W_1^{b)} + \tilde{Z}_1^{ab} W_1 = 0, \quad \tilde{Z}_1^{[ab]} = 0, \]  

(A.28)

where

\[ Z_1^{a} \equiv Z^{a\mu} \tilde{\mathcal{P}}''_{\mu} (-1)^{\varepsilon_\mu}, \quad \tilde{Z}_1^{ab} \equiv \tilde{Z}^{aba} \tilde{\mathcal{P}}'' \varepsilon^{a}. \]  

(A.29)

The general solution for \( Z_1^{a}, \tilde{Z}_1^{ab} \) is given by the formulae

\[ Z_1^{a} = E_2 W_2^{a} + \tilde{E}_2^{a} W_2 + \frac{1}{2} \tilde{E}_2^{ac} W_2^{b} \varepsilon_{bc}, \]  

(A.30)

\[ \tilde{Z}_1^{ab} = \tilde{E}_2^{(a} W_2^{b)} + \tilde{E}_2^{ab} W_2, \quad \tilde{Z}_1^{[ab]} = 0, \]  

(A.31)

where

\[ E_2 \sim (\tilde{\mathcal{P}}'')^2, \quad \tilde{E}_2^{a} \sim \tilde{\mathcal{P}}'' \tilde{\mathcal{P}}', \quad \tilde{E}_2^{ab} \sim (\tilde{\mathcal{P}}')^2 \]  

(A.32)

are arbitrary operators.

By decoding the representations (A.30), (A.31) one obtains the general solution for \( Z^{a\mu}, \tilde{Z}^{aba} : \)

\[ Z^{a\mu} = (E^{\rho \mu} \delta_{c}^{a} + \frac{1}{2} i h \tilde{E}^{ab\beta\alpha} W_{ab\alpha\beta} \varepsilon_{cb} ) \Pi_{\nu}^{\mu c} + \tilde{E}^{a\alpha\nu} \Pi_{\nu}^{\mu \alpha}, \]  

(A.33)

\[ \tilde{Z}^{aba} = \tilde{E}^{(a\beta\mu} \Pi_{\mu}^{ab} ) + \tilde{E}^{ab\gamma\beta} \Pi_{\beta}^{\alpha}. \]  

(A.34)

Further, let us represent the equation (2.23) in the form

\[ Z_1^{ab} W_1^c + \text{cycle}(a, b, c) = 0, \quad Z_1^{[ab]} = 0, \]  

(A.35)

where

\[ Z_1^{ab} \equiv Z^{ab\mu} \tilde{\mathcal{P}}'' (-1)^{\varepsilon_\mu}. \]  

(A.36)

The general solution is given by the formula

\[ Z_1^{ab} = E_2^{(a} W_2^{b)} \]  

(A.37)

where

\[ E_2^{a} \sim (\tilde{\mathcal{P}}'')^2 \]  

(A.38)

are arbitrary operators.
It follows from (A.37) that the general solution for $Z^{ab\mu}$ is of the form

$$Z^{ab\mu} = E^{(a\rho\nu} \Pi^{\mu b)}_{\rho\nu}. \quad (A.39)$$

Finally, let us turn to the equation (2.24) as represented in the form

$$Z_2 W_2^b W_1^a \varepsilon_{ab} + \tilde{Z}_1 W_1 = 0, \quad (A.40)$$

where

$$Z_2 \equiv -\frac{1}{2} Z^{\mu \nu} \bar{\mathcal{P}}^\mu_{\rho} \bar{\mathcal{P}}^\nu_{\rho} (-1)^{\varepsilon_\nu}, \quad \tilde{Z}_1 \equiv \tilde{Z}^\alpha \bar{\mathcal{P}}'_\alpha (-1)^{\tilde{\varepsilon}_a} \quad (A.41)$$

The general solution is given by the formulae

$$Z_2 = -\bar{E}_3 W_3 + E_3^b W_3^a \varepsilon_{ab} - \frac{1}{2} \tilde{E}_2 \bar{W}_2, \quad (A.42)$$

$$Z_1 = \tilde{E}_2 W_2 + \tilde{E}_3 W_3^b W_2^a \varepsilon_{ab}, \quad (A.43)$$

where

$$E_3^a \sim (\bar{\mathcal{P}}')^3, \quad \bar{E}_3 \sim (\bar{\mathcal{P}}')^2 \mathcal{P}', \quad \tilde{E}_2 \sim (\mathcal{P}')^2 \quad (A.44)$$

are arbitrary operators.

By decoding the representations (A.42), (A.43), one obtains the general solution for $Z^{\mu \nu}$, $\tilde{Z}^\alpha$ in the form

$$Z^{\mu \nu} = \tilde{E}^{\alpha \sigma \rho} \Pi^{\mu \nu}_{\rho \sigma \alpha} + E^{\beta \sigma \rho} \Pi^{\mu \nu \alpha}_{\rho \sigma \beta} \varepsilon_{ab} - \i \hbar \tilde{E}^{\beta \alpha} W^{\mu \nu}_{\alpha \beta}, \quad (A.45)$$

$$\tilde{Z}^\alpha = \tilde{E}^{\gamma \beta} \Pi^{\alpha}_{\beta \gamma} - \bar{E}^{\gamma \nu \mu} \bar{\Pi}^{\beta \nu \gamma} \bar{\Pi}^{\alpha \mu \beta} \varepsilon_{ab}, \quad (A.46)$$

where

$$\Pi^{\rho \sigma}_{\mu \nu \alpha} \equiv (\delta^\rho_{\mu} \bar{\Pi}^{\sigma}_{\nu \alpha} (-1)^{\varepsilon_\mu \varepsilon_\nu} - \delta^\sigma_{\nu} \bar{\Pi}^{\rho}_{\pi \alpha} (-1)^{\tilde{\varepsilon}_\alpha (\varepsilon_\mu + \varepsilon_\nu)}) -$$

$$-(\delta^\rho_{\mu} \bar{\Pi}^{\sigma}_{\nu \alpha} (-1)^{\varepsilon_\nu \varepsilon_\mu} - \delta^\sigma_{\nu} \bar{\Pi}^{\rho}_{\mu \alpha} (-1)^{\tilde{\varepsilon}_\alpha (\varepsilon_\mu + \varepsilon_\nu)}) (-1)^{\varepsilon_\mu \varepsilon_\nu} + (\i \hbar)^2 U_{\mu \nu \alpha}^{\rho \sigma}, \quad (A.47)$$

$$\bar{\Pi}^{\rho}_{\mu \alpha} \equiv -\frac{1}{2} T_{\alpha} \delta^\rho_{\mu} (-1)^{\varepsilon_\alpha \varepsilon_\mu} - \i \hbar U_{\mu \alpha}^{\rho}, \quad (A.48)$$

$$\bar{\Pi}^{\beta \rho}_{\mu \alpha} \equiv (\bar{\Pi}^{\rho \sigma}_{\mu \alpha} \delta^\beta_{\sigma} + \bar{\Pi}^{\beta \sigma}_{\rho \alpha} \delta^\rho_{\sigma} (-1)^{\varepsilon_\rho (\varepsilon_\alpha + \varepsilon_\beta)}) -$$

$$-\bar{\Pi}^{\beta \rho}_{\nu \alpha} \delta^\rho_{\mu} (-1)^{\varepsilon_\nu (\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\nu)} (-1)^{\varepsilon_\mu \tilde{\varepsilon}_\alpha} + (\i \hbar)^2 U_{\mu \alpha}^{\beta \rho}, \quad (A.49)$$

$$\bar{\Pi}^{\beta \rho}_{\mu \alpha} \equiv \frac{1}{2} T_{\alpha} \delta^\beta_{\mu} - \i \hbar U_{\mu \alpha}^{\beta \rho}, \quad (A.50)$$
References

[1] I.A.Batalin, S.L.Lyakhovich, I.V.Tyutin, Mod.Phys. Lett. A7 (1992) 1931.

[2] I.A.Batalin, P.M.Lavrov, I.V.Tyutin, J.Math. Phys. 31 (1990) 6

[3] I.A.Batalin, P.M.Lavrov, I.V.Tyutin, J.Math. Phys. 31 (1990) 2708