On completeness of orbits
of Killing vector fields

P.T. Chruściel*
Institute for Theoretical Physics
University of California
Santa Barbara, California 93106–4030

March 24, 2022

Abstract

A Theorem is proved which reduces the problem of completeness
of orbits of Killing vector fields in maximal globally hyperbolic, say
vacuum, space–times to some properties of the orbits near the Cauchy
surface. In particular it is shown that all Killing orbits are complete
in maximal developments of asymptotically flat Cauchy data, or of
Cauchy data prescribed on a compact manifold.

1 Introduction

In any physical theory a privileged role is played by solutions of the field
equation which exhibit special symmetries. In general relativity there exist
several ways for a solution to be symmetric: there might exist

1. a Killing vector field $X$ on the space–time $(M,g)$, or there might exist

2. an action of a group $G$ on $M$ by isometries, and finally there might
perhaps exist

*On leave from the Institute of Mathematics of the Polish Academy of Sciences. Sup-
ported in part by a KBN grant # 2 1047 9101, an NSF grant # PHY 89–04035 and the
Alexander von Humboldt Foundation. e-mail: piotr@sbitp.ucsb.edu
3. a Cauchy surface $\Sigma \subset M$ and a group $G$ which acts on $\Sigma$ while preserving the Cauchy data.

It is natural to enquire what are the relationships between those notions. Clearly 2 implies 1, but 1 does not need to imply 2 (remove e.g. points from a space–time on which an action of $G$ exists). With a little work one can show [11, 8, 6] that 3 implies 1, and actually it is true [6] that 3 implies 2, when $M$ is suitably chosen. The purpose of this paper is to address the question, do there exist natural conditions on $(M, g)$ under which 1 implies 2? Recall [3] that given a Cauchy data set $(\Sigma, \gamma, K)$, where $\Sigma$ is a three–dimensional manifold, $\gamma$ is a Riemannian metric on $\Sigma$, and $K$ is a symmetric two–tensor on $\Sigma$, there exists a unique up to isometry vacuum space–time $(M, \gamma)$, which is called the maximal globally hyperbolic vacuum development of $(\Sigma, \gamma, K)$, with an embedding $i : \Sigma \to M$ such that $i^*g = \gamma$, and such that $K$ corresponds to the extrinsic curvature of $i(\Sigma)$ in $M$. $(M, \gamma)$ is inextendible in the class of globally hyperbolic space–times with a vacuum metric. This class of space–times is highly satisfactory to work with, as they can be characterized by their Cauchy data induced on some Cauchy surface. Let us also recall that in globally hyperbolic vacuum space–times $(M, g)$, the question of existence of a Killing vector field $X$ on $M$ can be reduced to that of existence of appropriate Cauchy data for $X$ on a Cauchy surface $\Sigma$ (cf. e.g. [6]). In this paper we show the following:

**Theorem 1.1** Let $(M, g)$ be a smooth, vacuum, maximal globally hyperbolic space–time with Killing vector field $X$ and Cauchy surface $\Sigma$. The following conditions are equivalent:

1. There exists $\epsilon > 0$ such that for all $p \in \Sigma$ the orbits $\phi_s(p)$ of $X$ through $p$ are defined for all $s \in [-\epsilon, \epsilon]$.

2. The orbits of $X$ are complete in $M$.

It should be said that though this result seems to be new, it is a relatively straightforward consequence of the results in [3].

The following example$^1$ shows that some conditions on the behaviour of the orbits on the Cauchy surface are necessary in general: Let $\Sigma$ be a

---

$^1$I am grateful to R.Wald and B.Schmidt for discussions concerning this point.
connected component of the unit spacelike hyperboloid in Minkowski space–
time $R^{1+3}$, let $(M, g)$ be the domain of dependence of $\Sigma$ in $R^{1+3}$ with the
obvious flat metric, let $X$ be the Killing vector $\partial/\partial t$. $M$ is maximal globally
hyperbolic (cf. e.g. Proposition 2.4 below), $\Sigma$ is a complete Riemannian
manifold, the Lorentzian length of $X$ is uniformly bounded on $\Sigma$, nevertheless
no orbits of $X$ are complete in $M$.

As a Corollary of Theorem 1.1 one obtains, nevertheless, the following
(cf. Section 3 for precise definitions):

**Corollary 1.2** Let $(M, g)$ be a smooth, vacuum, maximal globally hyperbolic
space–time with Killing vector $X$ and with an achronal spacelike hypersurface
$\Sigma$. Suppose that either

1. $\Sigma$ is compact, or
2. $(\Sigma, \gamma, K)$ is asymptotically flat, or
3. $(\Sigma, \gamma, K)$ are Cauchy data for an asymptotically flat exterior region in
   a (non–degenerate) black–hole space–time.

Then the orbits of $X$ are complete in $D(\Sigma)$.

The difference between cases 2 and 3 above is, roughly speaking, the
following: In point 2 above $\Sigma$ is a complete Riemannian submanifold of $M$
without boundary. On the other hand, in point 3 above $\Sigma$ is a complete
Riemannian submanifold of $M$ with a compact boundary $\partial \Sigma$, and the Killing
vector is assumed to be tangent to $\partial \Sigma$; cf. the beginning of Section 3 for a
longer discussion of the relevant notions. [It should also be pointed out, that
we necessarily have $M = D(\Sigma)$ in point 1 above by 4. In point 3 above,
however, $M = D(\Sigma)$ cannot hold, cf. Definition D2, Section 4.

We have stated Theorem 1.1 and Corollary 1.2 in the vacuum, but they
clearly hold for any kind of well posed hyperbolic system of equations for the
metric $g$ coupled with some matter fields. All that is needed is a local existence
theorem for the coupled system, together with uniqueness of solutions in
domains of dependence. In particular, Theorem 1.1 will still be true e.g. for
metrics satisfying the Einstein – Yang–Mills – Higgs equations. Corollary
1.2 will still hold for the Einstein – Yang–Mills – Higgs equations, provided
both the gravitational field and the matter fields satisfy appropriate fall–off
conditions in the asymptotically flat case. We plan to discuss this elsewhere.
The use and applicability of Theorem 1.1 and Corollary 1.2 is rather wide: all non–purely–local results about space–times with Killing vectors assume completeness of their orbits. Let us in particular mention the theory of uniqueness of black–holes. Clearly it is essential to also classify those black–holes in which the orbits of the Killing field are not complete, and which are thus not covered by the existing theory. Consider then a stationary black hole space–time \((M, g)\) in which an asymptotically flat Cauchy surface exists but in which the Killing orbits are *not* complete: Corollary 1.2 shows that \((M, g)\) can be enlarged to obtain a space–time with complete Killing orbits. As another application, let us also mention the recent work of Wald and this author \([7]\), where the question of existence of maximal hypersurface in asymptotically stationary space–times is considered. Corollary 1.2 shows that the hypothesis of completeness of the orbits of the Killing field made in \([7]\) can be removed, when the space–time under consideration is vacuum (or satisfies some well behaved field equations) and e.g. maximal.

**Acknowledgements.** Most of the work on this paper was done when the author was visiting the Max Planck Institut für Astrophysik in Garching; he is grateful to Jürgen Ehlers and to the members of the Garching relativity group for hospitality. Useful discussions with Berndt Schmidt and Robert Wald are acknowledged.

## 2 Proof of Theorem 1.1

In this Section we shall prove Theorem 1.1. Let us start with a somewhat weaker result:

**Theorem 2.1** Let \((M, g)\) be a smooth, vacuum, maximal globally hyperbolic space-time with Cauchy surface \(\Sigma\) and with a Killing vector field \(X\), with \(g, X \in C^\infty\). Then the orbits of \(X\) in \(M\) are complete if and only if

(i) there exists \(\epsilon > 0\) such that for all \(p \in \Sigma\) the orbits \(\phi_s(p)\) of \(X\) are defined for all \(s \in [-\epsilon, \epsilon]\), and

(ii) for \(s \in [-\epsilon, \epsilon]\) the sets \(\phi_s(\Sigma)\) are achronal.

**Proof:** Let us start by showing necessity: Point (i) is obvious, consider point (ii). As the orbits of \(X\) are complete, the flow of \(X\) (defined as the solution of the equations \(\frac{d\phi_s(p)}{ds} = X \circ \phi_s(p)\), with initial value \(\phi_0(p) = p\)) is defined for all \(p \in M\) and all \(s \in \mathbb{R}\). Suppose there exists \(s_1 \in M\) and a
timelike path $\Gamma : [0, 1] \to M$ with $\Gamma(0) \in \phi_s(\Sigma)$ and $\Gamma(1) \in \phi_s(\Sigma)$. Then $\phi_s(\Gamma)$ would be a timelike path with $\phi_s(\Gamma(0)) \in \Sigma, \phi_s(\Gamma(1)) \in \Sigma$, which is not possible as $\Sigma$ is achronal. Hence (i) and (ii) are necessary.

To show sufficiency, we shall need the following proposition:

**Proposition 2.2** Let $(M_a, g_a), a = 1, 2,$ be vacuum globally hyperbolic space-times with Cauchy surfaces $\Sigma_a$, and suppose that $(M_2, g_2)$ is maximal. Let $O \subset M_1$ be a (connected) neighbourhood of $\Sigma_1$ and suppose there exists a one-to-one isometry $\Psi_O : O \to M_2$, such that $\Psi_O(\Sigma_1)$ is achronal. Then there exists a one-to-one isometry

$$\Psi : M_1 \to M_2,$$

such that $\Psi|_O = \Psi_O$.

**Remarks:**

1. When $\Psi_O(\Sigma_1) = \Sigma_2$, this result can be essentially found in [3]. The proof below is a rather straightforward generalization of the arguments of [3], cf. also [4, 9]. Although we assume smoothness of the metric throughout this paper for the sake of simplicity, we have taken some care to write the proof below in a way which generalizes with no essential difficulties to the case where low Sobolev–type differentiability of the metric is assumed.

2. The condition that $\psi_O(\Sigma_1)$ is achronal is necessary, which can be seen as follows: Let $M_1 = \mathbb{R}^2$ with the standard flat metric, set $\Sigma_1 = \{t = 0\}$. Let $\sim_a$ be the equivalence relation defined as $(t, x) \sim_a (t + a, x + 1)$, where $a$ is a number satisfying $|a| < 1, a \neq 0$. Define $M_2 = M_1/\sim_a$ with the naturally induced metric, $O = (-a/3, a/3) \times \mathbb{R}$, $\psi_O = i_{M_1}|_O$, where $i_{M_1}$ is the natural projection: $i_{M_1}(p) = [p]_{\sim_a}$. $M_2$ is causal geodesically complete; the function $t - ax : M_1 \to \mathbb{R}$ defines, by passing to the quotient, a time function on $M_2$ the level sets of which are Cauchy surfaces. It follows that $M_2$ is maximal globally hyperbolic. Clearly $\psi_O(\Sigma_1)$ is not achronal, and there is no one-to-one isometry from $M_1$ to $M_2$.

**Proof:** Consider the collection $\mathcal{X}$ of all pairs $(U, \Psi_U)$, where $U \subset M_1$ is a globally hyperbolic neighbourhood of $\Sigma_1$ (with $\Sigma_1$ - Cauchy surface for
(\mathcal{U}, g_1|_{\mathcal{U}}), and \Psi_{\mathcal{U}} : \mathcal{U} \to M_2 is an isometric diffeomorphism between \mathcal{U} and \Psi_{\mathcal{U}}(\mathcal{U}) \subset M_2 satisfying \Psi_{\mathcal{U}}|_{\Sigma_1} = \Psi_{\mathcal{O}}|_{\Sigma_1}. \mathcal{X} can be ordered by inclusion: (\mathcal{U}, \mathcal{U}') \leq (\mathcal{V}, \mathcal{V}') if \mathcal{U} \subset \mathcal{V} and if \Psi_{\mathcal{V}}|_{\mathcal{U}} = \Psi_{\mathcal{U}}. Let (\mathcal{U}_\alpha, \Psi_{\mathcal{U}_\alpha})_{\alpha \in \Omega} be a chain in \mathcal{X}, set \mathcal{W} = \cup_{\alpha \in \Omega} \mathcal{U}_\alpha, define \Psi_{\mathcal{W}} : \mathcal{W} \to M_2 by \Psi_{\mathcal{W}}|_{\mathcal{U}_\alpha} = \Psi_{\mathcal{U}_\alpha}; clearly (\mathcal{W}, \Psi_{\mathcal{W}}) is a majorant for (\mathcal{U}_\alpha, \Psi_{\mathcal{U}_\alpha})_{\alpha \in \Omega}. From the set theory axioms (cf. e.g. [10][Appendix]) it is easily seen that \mathcal{X} forms a set, we can thus apply Zorn’s Lemma [10] to conclude that there exist maximal elements (\tilde{M}, \Psi) in \mathcal{X}. Let then (\tilde{M}, \Psi) be any maximal element, by definition (\tilde{M}, g_1|_{\tilde{M}}) is thus globally hyperbolic with Cauchy surface \Sigma_1, and \Psi is a one-to-one isometry from \tilde{M} into M_2 such that \Psi|_{\Sigma_1} = \Psi_{\mathcal{O}}|_{\Sigma_1}. By e.g. Lemma 2.1.1 of [6] we have

\Psi|_{\tilde{M} \cap \mathcal{O}} = \Psi_{\mathcal{O}}|_{\tilde{M} \cap \mathcal{O}}. \quad (2.2)

We have the following:

**Lemma 2.3** Under the hypotheses of Proposition 2.2, suppose that (\mathcal{O}, \Psi_{\mathcal{O}}) is maximal. Then the manifold

\[ M' = (M_1 \sqcup M_2)/\Psi_{\mathcal{O}} \]

is Hausdorff.

**Remark:** Recall that \sqcup denotes the disjoint union, while (M_1 \sqcup M_2)/\Psi is the quotient manifold (M_1 \sqcup M_2)/\sim, where p_1 \in M_1 is equivalent to p_2 \in M_2 if p_2 = \Psi(p_1).

**Proof:** Let p, q \in M' be such that there exist no open neighbourhoods separating p and q; clearly this is possible only if (interchanging p with q if necessary) we have p \in \partial \mathcal{O} and q \in \partial \Psi_{\mathcal{O}}(\mathcal{O}). Consider the set \mathcal{H} of “non-Hausdorff points” \tilde{p}' in M' such that \tilde{p}' = i_{M_1}(p) for some p \in M_1, where i_{M_1} is the embedding of M_1 into M'; \mathcal{H} is closed and we have \mathcal{H} \subset \partial \mathcal{O}.

Suppose that \mathcal{H} \neq \emptyset, changing time orientation if necessary we may assume that \mathcal{H} \cap I^+(\Sigma_1) \neq \emptyset; let \tilde{p}' \in \mathcal{H} \cap I^+(\Sigma_1). We wish to show that there necessarily exists p \in \mathcal{H} such that

\[ J^-(p) \cap \mathcal{H} \cap I^+(\Sigma_1) = \{p\}. \quad (2.3) \]

If (2.3) holds with p = \tilde{p}' we are done, otherwise consider the (non-empty) set \mathcal{Y} of causal paths \Gamma : [0, 1] \to I^+(\Sigma) such that \Gamma(0) \in \mathcal{H}, \Gamma(1) = \tilde{p}'. \mathcal{Y} is directed by inclusion: \Gamma_1 < \Gamma_2 if \Gamma_1([0, 1]) \subset \Gamma_2([0, 1]). Let \{\Gamma_\alpha\}_{\alpha \in \Omega}
be a chain in \( \mathcal{Y} \), set \( \Gamma = \bigcup_{\alpha \in \Omega} \Gamma_\alpha([0,1]) \), consider the sequence \( p_\alpha = \Gamma_\alpha(0) \). Clearly \( \Gamma \subset J^+(\Sigma_1) = I^+(\Sigma_1) \cup \Sigma_1 \), and global hyperbolicity implies that \( \Gamma \) must be extendible, thus \( \Gamma_\alpha(0) \) accumulates at some \( p_* \in I^+(\Sigma_1) \cup \Sigma_1 \). As \( \mathcal{O} \) is an open neighbourhood of \( \Sigma_1 \) the case \( p_* \in \Sigma_1 \) is not possible, hence \( p_* \in I^+(\Sigma_1) \) and consequently \( \Gamma \in \mathcal{Y} \). It follows that every chain in \( \mathcal{Y} \) has a majorant, and by Zorn’s Lemma \( \mathcal{Y} \) has maximal elements. Let then \( \bar{\Gamma} \) be any maximal element of \( \mathcal{Y} \), setting \( p = \bar{\Gamma}(0) \) the equality (2.3) must hold.

We now claim that (2.3) also implies
\[
J^-(p) \cap \partial \mathcal{O} \cap I^+(\Sigma_1) = \{p\}. \tag{2.4}
\]
Suppose, on the contrary, that there exists \( q \in (J^-(p) \cap \partial \mathcal{O} \cap I^+(\Sigma_1)) \setminus \{p\} \); let \( q_i \in \mathcal{O} \) be a sequence such that \( q_i \to q \). We can choose \( q_i \) so that \( q_{i+1} \in I^+(q_i) \). Global hyperbolicity of \( \mathcal{O} \) implies that for \( i > i_o \), for some \( i_o \), there exist timelike paths \( \Gamma_i : [0,1] \to \mathcal{O}, \Gamma_i([0,1]) \subset \mathcal{O} \), \( \Gamma_i(0) = q_i \), \( \Gamma_i(1) = p \). Let \( \tilde{p} \in M_2 \) be a non–Hausdorff partner of \( p \) such that the curves \( \Psi_\mathcal{O}(\Gamma_i) \) have \( \tilde{p} \) as an accumulation point. We have \( \Psi_\mathcal{O}(\Gamma_i) \subset J^+(\Psi_\mathcal{O}(q_{i_o})) \cap J^-(\tilde{p}) \) which is compact by global hyperbolicity of \( M_2 \), hence there exists a subsequence \( \Psi_\mathcal{O}(q_{i_o}) \) converging to some \( q \in M_2 \). This implies that \( q \) and \( \tilde{q} \) constitute a “non-Hausdorff pair” in \( M' \), contradicting (2.3), and thus (2.4) must be true.

Let \( p_1 \in M_1, p_2 \in M_2, i_{M_1}(p_1) = i_{M_2}(p_2) \), be any non-Hausdorff pair in \( M' \) such that (2.3) holds with \( p = p_1 \). Let \( x^\mu \) be harmonic coordinates defined in a neighbourhood \( \mathcal{O}_2 \) of \( p_2 \). [Such coordinates can be e.g. constructed as follows: Let \( t_0 \) be any time function defined in some neighbourhood \( \mathcal{O}_2 \) of \( p_2 \), such that \( t_0(p_2) = 0 \), set \( \mathcal{I}_\tau = \{ p \in \mathcal{O}_2 : t_0(p) = \tau \} \). Passing to a subset of \( \mathcal{O}_2 \) if necessary there exists a global coordinate system \( x^\mu_0 \) defined on \( \mathcal{I}_0 \); again passing to a subset of \( \mathcal{O}_2 \) if necessary we may assume that \( \mathcal{O}_2 \) is globally hyperbolic with Cauchy surface \( \mathcal{I}_0 \). Let \( x^\mu \in C^\infty(\mathcal{O}_2) \) be the (unique) solutions of the problem
\[
\square_{g_2} x^\mu = 0,
\]
\[
\left. x^0 \right|_{\mathcal{I}_0} = 0, \quad \left. \frac{\partial x^0}{\partial t} \right|_{\mathcal{I}_0} = 1, \quad \left. x^i \right|_{\mathcal{I}_0} = x^i_0, \quad \left. \frac{\partial x^i}{\partial t} \right|_{\mathcal{I}_0} = 0, \tag{2.5}
\]
where \( \square_\gamma \) is the d’Alembert operator of a metric \( \gamma \). Passing once more to a globally hyperbolic subset of \( \mathcal{O}_2 \) if necessary, the functions \( x^\mu \) form a coordinate system on \( \mathcal{O}_2 \).] We can choose \( \epsilon > 0 \) such that
1. \( \text{int} D^+(\mathcal{I}_-) \subset \mathcal{O}_2 \),
2. \( p \in \text{int} D^+(\mathcal{I}_-) \),
3. \( \mathcal{I}_- \subset \Psi_{\mathcal{O}}(\mathcal{O}) \).

Define
\[
\hat{\mathcal{I}} = \Psi_{\mathcal{O}}^{-1}(\mathcal{I}_-).
\]

Let \( y^\mu \in C^\infty(D(\hat{\mathcal{I}})) \) be the (unique) solutions of the problem
\[
\Box_{g_1} y^\mu = 0, \\
y^\mu|_{\hat{\mathcal{I}}} = x^\mu \circ \Psi_{\mathcal{O}}|_{\hat{\mathcal{I}}}, \\
\left. \frac{\partial y^\mu}{\partial n} \right|_{\hat{\mathcal{I}}} = \left. \frac{\partial (x^\mu \circ \Psi_{\mathcal{O}})}{\partial n} \right|_{\hat{\mathcal{I}}},
\]
where \( \frac{\partial}{\partial n} \) is the derivative in the direction normal to \( \hat{\mathcal{I}} \). By isometry invariance of the wave equation we have
\[
y^\mu|_{D(\hat{\mathcal{I}}) \cap \mathcal{O}} = x^\mu \circ \Psi_{\mathcal{O}}|_{D(\hat{\mathcal{I}}) \cap \mathcal{O}}.
\]

Set
\[
\mathcal{U} = \mathcal{O} \cup \text{int} D^+(\hat{\mathcal{I}}),
\]
and for \( p \in \mathcal{U} \) define
\[
\Psi_{\mathcal{U}}(p) = \begin{cases} 
\Psi_{\mathcal{O}}(p), & p \in \mathcal{O}, \\
q : \text{where } q \text{ is such that } x^\mu(q) = y^\mu(p), & p \in \text{int} D^+(\hat{\mathcal{I}}). 
\end{cases}
\]

From (2.6) it follows that \( \Psi_{\mathcal{U}} \) is a smooth map from \( \mathcal{U} \) to \( M_2 \). Clearly \( \mathcal{U} \) is a globally hyperbolic neighbourhood of \( \Sigma_1 \), and \( \Sigma_1 \) is a Cauchy surface for \( \mathcal{U} \). Note that \( \mathcal{O} \) is a proper subset of \( \mathcal{U} \), as \( p_1 \in \text{int} D^+(\hat{\mathcal{I}}) \) but \( p_1 \notin \mathcal{O} \). It follows from uniqueness of solutions of Einstein equations in harmonic coordinates that \( \Psi_{\mathcal{U}} \) is an isometry. To prove that \( \Psi_{\mathcal{U}} \) is one-to-one, consider \( p, q \in \mathcal{U} \) such that \( \Psi_{\mathcal{U}}(p) = \Psi_{\mathcal{U}}(q) \). Changing time orientation if necessary we may suppose that \( p \in I^+(\Sigma_1) \). By hypothesis we have \( I^+(\Psi_{\mathcal{O}}(\Sigma_1)) \cap I^-(\Psi_{\mathcal{O}}(\Sigma_1)) = \emptyset \), hence \( q \in I^+(\Sigma_1) \). Let \( [0,1] \ni s \to \Gamma(s) \) be a timelike path from \( \Sigma_1 \) to \( q \), let \( \Gamma_1(s) \) be a connected component of \( \Psi_{\mathcal{U}}^{-1}(\Psi_{\mathcal{U}}(\Gamma)) \) which contains \( \{p\} \). Consider the set \( \Omega = \{ s \in [0,1] : \Gamma(s) = \Gamma_1(s) \} \). Since \( \Psi_{\mathcal{U}}|_{\mathcal{O}} = \Psi_{\mathcal{O}} \) which is one-to-one, \( \Omega \) is non-empty. By continuity of \( \Gamma_1 \) and \( \Gamma \), \( \Omega \) is closed. Since
is locally one-to-one (being a local diffeomorphism), \( \Omega \) is open. It follows that \( \Omega = [0,1] \), hence \( p = q \), and \( \Psi_U \) is one-to-one as claimed.

We have thus shown, that \( (O, \Psi_O) \leq (U, \Psi_U) \) and \( (O, \Psi_O) \neq (U, \Psi_U) \) which contradicts maximality of \( (O, \Psi_O) \). It follows that \( M' \) is Hausdorff, as we desired to show.

Returning to the proof of Proposition 2.2, let \( \tilde{M}, \Psi \) be maximal. If \( \tilde{M} = M_1 \) we are done, suppose then that \( \tilde{M} \neq M_1 \). Consider the manifold

\[
M' = (M_1 \sqcup M_2) / \Psi.
\]

By Lemma 2.3, \( M' \) is Hausdorff. We claim that \( M' \) is globally hyperbolic with Cauchy surface \( \Sigma' = i_{M_2}(\Sigma_2) \approx \Sigma_2 \), where \( i_{M_a} \) denotes the canonical embedding of \( M_a \) in \( M' \). Indeed, let \( \Gamma' \subset M' \) be an inextendible causal curve in \( M' \), set \( \Gamma_1 = i_{M_1}^{-1}(\Gamma' \cap i_{M_1}(M_1)) \), \( \Gamma_2 = i_{M_2}^{-1}(\Gamma' \cap i_{M_2}(M_2)) \). Clearly \( \Gamma_1 \cup \Gamma_2 \neq \emptyset \), so that either \( \Gamma_1 \neq \emptyset \), or \( \Gamma_2 \neq \emptyset \), or both. Let the index \( a \) be such that \( \Gamma_a \neq \emptyset \). If \( \hat{\Gamma}_a \) were an extension of \( \Gamma_a \) in \( M_a \), then \( i_{M_a}(\hat{\Gamma}_a) \) would be an extension of \( \Gamma' \) in \( M' \), which contradicts maximality of \( \Gamma' \), thus \( \Gamma_a \) is inextendible. Suppose that \( \Gamma_1 \neq \emptyset \); as \( \Gamma_1 \) is inextendible in \( M_1 \) we must have \( \Gamma_1 \cap \Sigma_1 = \{p_1\} \) for some \( p_1 \in \Sigma_1 \). We thus have \( \Psi(p_1) \in \Gamma_2 \), so that it always holds that \( \Gamma_2 \neq \emptyset \). By global hyperbolicity of \( M_2 \) and inextendibility of \( \Gamma_2 \) it follows that \( \Gamma_2 \cap \Sigma_2 = \{p_2\} \) for some \( p_2 \in \Sigma_2 \), hence \( \Gamma' \cap i_{M_2}(\Sigma_2) = \{i_{M_2}(p_2)\} \). This shows that \( i_{M_2}(\Sigma_2) \) is a Cauchy surface for \( M' \), thus \( M' \) is globally hyperbolic. As \( \tilde{M} \neq M_1 \) we have \( M' \neq M_2 \) which contradicts maximality of \( M_2 \). It follows that we must have \( \tilde{M} = M_2 \), and Proposition 2.2 follows.

Returning to the proof of Theorem 2.1, choose \( s \in [-\epsilon/2, \epsilon/2] \). There exists a globally hyperbolic neighborhood \( O_s \) of \( \Sigma \) such that the map \( \phi_s(p) \) is defined for all \( p \in O_s \):

\[
O_s \ni p \to \phi_s(p) \in M.
\]

\( \phi_s(\Sigma) \) is achronal by hypothesis, and Proposition 2.2 shows that there exists a map \( \hat{\phi}_s : M \to M \) such that \( \hat{\phi}_s|_{O_s} = \phi_s \). For \( s \in \mathbb{R} \) let \( k \) be the integer part of \( 2s/\epsilon \), define \( \hat{\phi}_s : M \to M \) by

\[
\hat{\phi}_s = \hat{\phi}_{s-k\epsilon/2} \circ \hat{\phi}_{k\epsilon/2} \circ \ldots \circ \hat{\phi}_{\epsilon/2}.
\]

9
It is elementary to show that $\hat{\phi}_s$ satisfies
\[
\frac{d\hat{\phi}_s}{ds} = X \circ \hat{\phi}_s,
\]
and Theorem 2.1 follows. \qed

Let us point out the following useful result:

**Proposition 2.4** Let $(M, g)$ be a maximal globally hyperbolic vacuum spacetime. Suppose that $\tilde{\Sigma} \subset M$ is an achronal spacelike submanifold, and let $(\tilde{\gamma}, \tilde{K})$ be the Cauchy data induced by $g$ on $\tilde{\Sigma}$. Then $(D(\tilde{\Sigma}), g|_{D(\tilde{\Sigma})})$ is isometrically diffeomorphic to the maximal globally hyperbolic vacuum development $(\tilde{M}, \tilde{\gamma})$ of $(\tilde{\Sigma}, \tilde{\gamma}, \tilde{K})$.

**Proof:** By maximality of $(\tilde{M}, \tilde{\gamma})$, there exists a map $\Psi : D(\tilde{\Sigma}) \to \tilde{M}$ which is a smooth isometric diffeomorphism between $D(\tilde{\Sigma})$ and $\Psi(D(\tilde{\Sigma}))$. By standard local uniqueness results for vacuum Einstein equations there exists a globally hyperbolic neighborhood $\tilde{\mathcal{O}}$ of $\tilde{\mathcal{I}}(\tilde{\Sigma})$ in $\tilde{M}$, where $\tilde{\mathcal{I}}$ is the embedding of $\tilde{\Sigma}$ in $\tilde{M}$, and a map $\Phi : \tilde{\mathcal{O}} \to D(\tilde{\Sigma}) \subset M$ which is an isometric diffeomorphism between $\tilde{\mathcal{O}}$ and $\Phi(\tilde{\mathcal{O}})$. By Proposition 2.2, $\Phi(\tilde{\mathcal{O}})$ can be extended to a map $\Phi : \tilde{M} \to M$ which is an isometric diffeomorphism between $\tilde{M}$ and $\Phi(\tilde{M})$. Clearly we must have $\Phi(\tilde{M}) \subset D(\tilde{\Sigma})$, so that one obtains $\Psi \circ \Phi = id_{\tilde{M}}$, $\Phi \circ \Psi = id_{D(\tilde{\Sigma})}$, and the result follows. \qed

To prove Theorem 1.1, i.e., to remove the hypothesis (ii) of Theorem 2.1, more work is needed. Let $t_\pm(p) \in \mathbb{R} \cup \{\pm \infty\}$, $t_-(p) < 0 < t_+(p)$ be defined by the requirement that $(t_-(p), t_+(p))$ is the largest connected interval containing 0 such that the solution $\phi_s(p)$ of the equation $\frac{d\phi_s(p)}{ds} = X \circ \phi_s(p)$ with initial condition $\phi_0(p) = p$ is defined for all $s \in (t_-(p), t_+(p))$. From continuous dependence of solutions of ODE’s upon parameters it follows that for every $\delta > 0$ there exists a neighborhood $\mathcal{O}_{p, \delta}$ of $p$ such that for all $q \in \mathcal{O}_{p, \delta}$ we have $t_+(q) \geq t_+(p) - \delta$ and $t_-(q) \leq t_-(p) + \delta$. In other words, $t_+$ is a lower semi-continuous function and $t_-$ is an upper semi-continuous function.

We have the following:

**Lemma 2.5** Let $p \in I^+(\Sigma)$, $q \in J^-(p) \cap I^+(\Sigma)$, suppose that $t_+(p) \geq \tau_0$. If $t_+(q) < \tau_0$, then there exists $s \in [0, t_+(q))$ such that $\phi_s(q) \in \Sigma$.

**Proof:** Let $\gamma(s)$ be any future directed causal curve with $\gamma(0) = q$, $\gamma(1) = p$. Suppose that $t_+(p) \geq \tau_0 > t_+(q)$, let $(s, 1]$ be the largest interval
such that $t_+(\gamma(s)) > t_+(q)$ for all $s \in (s_-, 1]$. By lower semi-continuity of $t_+$ we have $(s_-, 1] \neq \emptyset$. Consider the one-parameter family of causal paths

$$[0, t_+(q)] \times (s_-, 1] \ni (\tau, s) \mapsto \tilde{\gamma}_\tau(s) = \phi_\tau(\gamma(s)).$$

Suppose that for all $s \in [0, t_+(q))$ we have $\phi_s(q) \notin \Sigma$. Global hyperbolicity of $M$ implies that for all $s \in [0, t_+(p))$ we have $\phi_s(q) \in I^+(\Sigma)$, consequently for any $r \in I^+(\phi_s(q))$ it also holds that $r \in I^+(\Sigma)$; hence $\tilde{\gamma}_\tau(s) \in J^+(\Sigma)$ for all $\tau, s \in [0, t_+(q)] \times (s_-, 1]$. As $\Sigma$ is a Cauchy surface, for each $\tau$ the curve $\tilde{\gamma}_\tau$ must be past-extendible. Let thus $\gamma_\tau(s)$ be any past extension of $\tilde{\gamma}_\tau$, for $\tau \in [0, t_+(q)]$ define $\psi_\tau = \tilde{\gamma}_\tau(s_-)$. It is elementary to show that $\psi_\tau = \phi_\tau(\gamma(s_-))$, so that $t_+(\gamma(s_-)) > t_+(q)$. This, however, contradicts the definition of $s_-$, and the result follows.

**Proof of Theorem 1.1**: Suppose there exists $s_0 \in [-\epsilon, \epsilon]$ such that $\phi_{s_0}(\Sigma)$ is not achronal. Let $\Gamma : [0, 1] \to M$ be a timelike curve such that $\Gamma(0), \Gamma(1) \in \phi_{s_0}(\Sigma)$. Changing $X$ to $-X$ if necessary we may assume $s_0 < 0$; changing time orientation if necessary we may suppose that $\Gamma(1) \in J^+(\Sigma)$. We have $t_+|_\Sigma \geq \epsilon$, hence $t_+|_{\phi_{s_0}(\Sigma)} = (t_+ + |s_0|)|_{\Sigma} \geq \epsilon$.

Let $q \in I^- (\Gamma(1)) \cap J^+(\Sigma)$. By Lemma 2.3 either $t_+(q) \geq \epsilon$, or there exists $s \in [0, t_+(q))$ such that $\phi_s(q) \in \Sigma$. In that last case we have $t_+(q) - s = t_+(\phi_s(q)) \geq \epsilon$, hence $t_+(q) \geq \epsilon$, and in either case we obtain $t_+(q) \geq \epsilon$. It follows that

$$t_+|_{\Gamma \cap J^+(\Sigma)} \geq \epsilon. \quad (2.8)$$

If $\Gamma(0) \in J^+(\Sigma)$ we thus obtain

$$t_+|_{\Gamma} \geq \epsilon. \quad (2.9)$$

Consider the case $\Gamma(0) \in J^-(\Sigma)$. We have $t_+(\Gamma(0)) \geq \epsilon$ and by an argument similar to the one above (using the time-dual version of Lemma 2.3) we obtain

$$t_+|_{\Gamma \cap J^-(\Sigma)} \geq \epsilon,$$

and by global hyperbolicity we can again conclude that $(2.9)$ holds. Eq. $(2.9)$ shows that $\phi_{-s_0}(\Gamma)$ is a timelike curve satisfying $\phi_{-s_0}(\Gamma(0)), \phi_{-s_0}(\Gamma(1)) \in \Sigma$. This, however, contradicts achronality of $\Sigma$. We therefore conclude that for all $s \in [-\epsilon, \epsilon]$ the hypersurfaces $\phi_s(\Sigma)$ are achronal. Theorem 1.1 follows now from Theorem 2.1. \qed
3 Proof of Corollary [1.2]

Before passing to the proof of Corollary [1.2], it seems appropriate to present some definitions.

**Definition 3.1** We shall say that an initial data set \((\Sigma, \gamma, K)\) for vacuum Einstein equations is asymptotically flat if \((\Sigma, \gamma)\) is a complete Riemannian manifold (without boundary), with \(\Sigma\) of the form

\[
\Sigma = \Sigma_{\text{int}} \bigcup_{i=1}^{I} \Sigma_i,
\]

for some \(I < \infty\). Here we assume that \(\Sigma_{\text{int}}\) is compact, and each of the ends \(\Sigma_i\) is diffeomorphic to \(\mathbb{R}^3 \setminus B(R_i)\) for some \(R_i > 0\), with \(B(R_i)\) — coordinate ball of radius \(R_i\). In each of the ends \(\Sigma_i\) the metric is assumed to satisfy the hypotheses of the boost theorem, Theorem 6.1 of [5].

The hypotheses of Theorem 6.1 of [5] will hold if e.g. there exists \(\alpha > 0\) such that in each of the ends \(\Sigma_i\) we have

\[
0 \leq k \leq 4 \quad |\partial_i \cdots \partial_k (\gamma_{ij} - \delta_{ij})| \leq Cr^{-\alpha-k},
\]

\[
0 \leq k \leq 3 \quad |\partial_i \cdots \partial_k K_{ij}| \leq Cr^{-\alpha-k-1},
\]

for some constant \(C\).

To motivate the next definition, consider a space–time with some number of asymptotically flat ends, and with a black hole region. In such a case there might be a Killing vector field defined in, say, the domain of outer communication of the asymptotically flat ends. It could, however, occur, that there is no Killing vector field defined on the whole space–time — a famous example of such a space–time has been considered by Brill [1], yielding a space–time in which no asymptotically flat maximal surfaces exist. Alternatively, there might be a Killing vector field defined everywhere, however, there might be some non-asymptotically flat ends in \(M\). [As an example, consider a space–like surface in the Schwarzschild–Kruskal–Szekeres space–time in which one end is asymptotically flat, and the second is “asymptotically hyperboloidal”.

\[\text{[2]}\] The differentiability threshold of Theorem 6.1 of [5] can actually be weakened to \(s \geq 3\). Similarly, the differentiability threshold in Theorem 6.2 of [5] can be weakened to \(s \geq 4\), and probably also to \(s \geq 3\).
In such cases one would still like to claim that the orbits of $X$ are complete at least in the exterior region. We shall see that this is indeed the case, under some conditions which we spell out below:

**Definition 3.2** Consider a stably causal Lorentzian manifold $(M, g)$ with an achronal spacelike surface $\hat{\Sigma}$. Let $\Sigma \subset \hat{\Sigma}$ be a connected submanifold of $\hat{\Sigma}$ with smooth compact boundary $\partial \Sigma$, and let $(\gamma, K)$ be the Cauchy data induced by $g$ on $\Sigma$. Suppose finally that there exists a Killing vector field $X$ defined on $D(\Sigma)$. We shall say that $(\Sigma, \gamma, K)$ are Cauchy data for an asymptotically flat exterior region in a (non-degenerate) black-hole space-time if the following hold:

1. The closure $\bar{\Sigma} \equiv \Sigma \cup \partial \Sigma$ of $\Sigma$ is of the form (3.1), with $\Sigma_{\text{int}}$ and $\Sigma_i$ satisfying the requirements of Definition 3.1.

2. [From eq. (3.3) below it follows that $X$ can be extended by continuity to $D(\Sigma)$.] We shall require that $X$ be tangent to $\partial \Sigma$.

An example of the behaviour described in Definition 3.2 can be observed in the Schwarzschild–Kruskal–Szekeres space-time $M$, when $\hat{\Sigma}$ is taken as a standard $t = 0$ surface, $\Sigma$ is the part of $\hat{\Sigma}$ which lies in one asymptotic end of $M$, and $\partial \Sigma$ is the set of points where the usual Killing vector $X$ (which coincides with $\partial/\partial t$ in the asymptotic regions) vanishes. Such $\partial \Sigma$’s are usually called “the bifurcation surface of a bifurcate Killing horizon”. An example in which $X$ does not vanish on $\partial \Sigma$ is given by the Kerr space-time, when $X$ is taken to coincide with $\partial/\partial t$ in the asymptotic region, and $\partial \Sigma$ is the intersection of the black hole and of the white hole with respect to the asymptotic end under consideration.

The notion of non-degeneracy referred to in definition 3.2 above is related to the non-vanishing of the surface gravity of the horizon: Indeed, it follows from [12] that in situations of interest the behaviour described in Definition 3.2 can only occur if the surface gravity of the horizon is constant on the horizon, and does not vanish.

With the above definitions in mind, we can now prove Corollary 1.2:

**Proof of Corollary 1.2:** Suppose first that $\Sigma$ is compact. We have

$$ t_+|_{\Sigma} \geq \epsilon $$

13
for some $\epsilon > 0$, because a lower semi-continuous function attains its infimum on a compact set (cf. e.g. [13]), and the result follows from Theorem 1.1. [Here we could also use Theorem 2.1: the hypersurfaces $\phi_s(\Sigma), s \in [-\epsilon, \epsilon]$, are compact and spacelike and hence achronal by [2].]

Consider next the case $(\Sigma, \gamma, K) -$ asymptotically flat. Let $(M, g)$ be the maximal globally hyperbolic development of $(\Sigma, \gamma, K)$. A straightforward extension of the boost-theorem [5] using domain of dependence arguments shows that $M$ contains a subset of the form

$$M_1 = ([\delta_0, \delta] \times \Sigma_{\text{int}}) \bigcup_{i=1}^{I} \Omega_i,$$

with some $\delta > 0$, where each of the $\Omega_i$’s is a boost-type domain:

$$\Omega_i = \{(t, \vec{x}) \in \mathbb{R}^4 : |\vec{x}| \geq R_i, |t| \leq \delta + \theta(r - R_i)\},$$

with some $\theta > 0$. Let $X$ be a Killing vector field on $M$. As is well known, $X$ satisfies the equations

$$\nabla_\mu \nabla_\nu X_\alpha = R_{\lambda \mu \nu \alpha} X^\lambda. \quad (3.3)$$

Under the hypotheses of Theorem 6.1 of [3], a simple analysis of (3.3) shows that in each $\Omega_i$ there exists $\alpha > 0$ and a constant (perhaps vanishing) matrix $\Lambda^\mu_\nu = \Lambda^\mu_\nu(i)$ such that

$$0 \leq j \leq 2 \quad \partial_{i_1} \ldots \partial_{i_j} [X^\mu - \Lambda^\mu_\nu x^\nu] = O(r^{1-\alpha-j}) \quad (3.4)$$

From equations (3.2) and (3.4) one easily shows that there exists $\epsilon > 0$ such that for all $p \in \Sigma$ the orbit $\phi_s(p)$ of $X$ through $p$ remains in $M_1$ for $|s| \leq \epsilon$. This shows that in the asymptotically flat case the hypotheses of Theorem 1.1 are satisfied as well, and the second part of Corollary 1.2 follows.

Consider finally point 3 of Corollary 1.2. Let $\hat{X}$ be any vector field (not necessarily Killing) defined in a neighbourhood $O$ of $\partial \Sigma$ such that $\hat{X}|_{\partial \Sigma \cap D(\Sigma)} = X$.

[Because $D(\Sigma)$ is not a smooth manifold, a little work is needed to show that an extension $\hat{X}$ of $X$ exists. A possible construction goes as follows: Define $\psi^\mu = X^\mu|_{\Sigma}, \chi^\mu = n^a \nabla_a X^\mu|_{\Sigma}$, where $n^a$ is the field of unit normals to $\Sigma$. Because $\partial \Sigma$ is smooth in $\hat{\Sigma}$, there exists smooth extensions $\hat{\psi}^\mu$ and $\hat{\chi}^\mu$ of $\chi^\mu$ and of $\psi^\mu$ from $\Sigma$ to $\hat{\Sigma}$. On $D(\Sigma)$ let $\hat{X}$ be the unique solution of the problem

$$\begin{align*}
\square \hat{X}^\mu &= -R_{\alpha \mu}^\alpha \hat{X}^\alpha \\
\hat{X}^\mu|_{\Sigma} &= \psi^\mu, \\
\hat{n}^a \nabla_a \hat{X}^\mu|_{\Sigma} &= \chi^\mu
\end{align*} \quad (3.5)$$

]
where \( \hat{n}^\alpha \) is the field of unit normals to \( \hat{\Sigma} \) and \( R^\mu_{\alpha\nu} \) is the Ricci tensor of \( g \).

We have \( \hat{X}|_{D(\Sigma)} = X \) by uniqueness of solutions of (3.5).

Returning to the main argument, without loss of generality we may assume that the neighbourhood \( \mathcal{O} \) of \( \partial \Sigma \) is covered by normal geodesic coordinates based on \( \partial \Sigma \):

\[
\mathcal{O} = \{(q, t, x) : q \in \partial \Sigma, (t, x) \in B(\epsilon) \subset \mathbb{R}^2\},
\]

for some \( \epsilon > 0 \), where \( B(\epsilon) \) is a coordinate ball of radius \( \epsilon \). We have \( \partial \Sigma \cap \mathcal{O} = \{(q, t, x) : t = x = 0\} \), and we can also assume that \( \overline{\mathcal{O}} \) is a compact subset of \( M \). For \( p \in \mathcal{O} \) and \( s \in ([t_-(p), t_+(p)]) \) let \( \phi_s(p) \) be the orbit of \( \hat{X} \) through \( p \). There exists \( \epsilon > 0 \) such that \( t_+(p)|_{\mathcal{O}} \geq \epsilon \), \( t_-(p)|_{\mathcal{O}} \leq -\epsilon \). Consider \( p \in \mathcal{O} \cap D(\Sigma) \), thus \( p = (q, t, x) \), with \( q \in \partial \Sigma, (t, x) \in B(\epsilon) \); changing \( x \) to \(-x\) if necessary we also have \(|t| < x\). By construction of the coordinates \((q, t, x)\) the straight lines \( q = q_0, t = \alpha s, x = \beta s, \alpha, \beta \in \mathbb{R} \), are affinely parametrized geodesics. Now for \(|s| \leq \epsilon \) \( \hat{\phi}_s : \mathcal{O} \cap D(\Sigma) \to M \) are isometries, hence in \( \mathcal{O} \cap D(\Sigma) \) the maps \( \hat{\phi}_s \) carry geodesics into geodesics and preserve affine parametrization. It follows that the \( \hat{\phi}_s \)'s must be of the form

\[
\mathcal{O} \cap D(\Sigma) \ni (q, x^\mu) \to \phi_s(q, x^\mu) = \hat{\phi}_s(q, x^\mu) = (\psi_s(q), \Lambda(s, q)^\mu_\nu x^\nu),
\]

for some map \( \psi_s : \partial \Sigma \to \partial \Sigma \), where we have set \( x^\mu = (t, x) \), and where \( \Lambda(s, q) \) is a Lorentz boost. Consequently, we can find \( 0 < \delta \leq \epsilon \) and a conditionally compact neighbourhood \( \mathcal{U} \) of \( \partial \Sigma \), \( \mathcal{U} \subset \mathcal{O} \), such that for all \( p \in \mathcal{U} \cap \Sigma \) and for \( s \in [-\delta, \delta] \) we shall have \( \phi_s(p) \in D(\Sigma) \). The result follows now from the arguments of the proof of parts 1 and 2 of this Corollary. \( \square \)

References

[1] Brill D., On spacetimes without maximal surfaces. In Proc. of the Third Marcel Grossman Meeting (H. Ning, ed.), North Holland, 1982.

[2] Budic R., Isenberg J., Lindblom L., Yasskin P., On the Determination of Cauchy Surfaces from Intrinsic Properties. Comm. Math. Phys. 61 (1978) 87-95.

[3] Choquet-Bruhat Y., Geroch R., Global Aspects of the Cauchy Problem. Comm. Math. Phys. 14 (1969) 329-335.

---

3The argument that follows is essentially due to R. Wald.
[4] Choquet-Bruhat Y., York J., *The Cauchy Problem*. In *General Relativity and Gravitation* (ed. A. Held), Plenum, 1980.

[5] Christodoulou D., O’Murchadha N., *The boost problem in general relativity*. Commun. Math. Phys. 80 (1981) 271–300.

[6] Chruściel P.T., *On Uniqueness in the Large of Solutions of Einstein’s Equations (“Strong Cosmic Censorship”).* Proceedings of the CMA, Vol. 27, Australian National University, Canberra 1991.

[7] Chruściel P.T., Wald R., *Maximal Hypersurfaces in Asymptotically Stationary Space–Times*. MPA preprint 708, Garching 1992.

[8] Fischer A.E., Marsden J.E., Moncrief V., *The structure of the space of solutions of Einstein’s equations. I. One Killing field*. Ann. Inst. Henri Poincaré 33 (1980) 147–194.

[9] Hawking S., Ellis G., *The Large Structure of Spacetime*. Cambridge U. Press, 1973.

[10] Kelley J.G., *General Topology*. Van Nostrand, 1967.

[11] Moncrief V., *Spacetime symmetries and linearization stability of the Einstein equations*. Jour. Math. Phys. 16 (1975) 493–498.

[12] Rácz I., Wald R.M., *Extensions of Spacetimes with Killing Horizons*. Class. Quantum Grav. 9 (1992) 2643–2656.

[13] Struwe M., *Variational Methods*. Springer, 1990.