EXPONENTIALS OF BOUNDED NORMAL OPERATORS

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Abstract. The present paper is mainly concerned with equations involving exponentials of bounded normal operators. Conditions implying commutativity of those normal operators are given. This is carried out without the known $2\pi i$-congruence-free hypothesis. It is also a continuation of a recent work by the corresponding author.

1. Introduction

First, we assume the reader is familiar with notions and results on Bounded Operator Theory. Some important references are [3] and [15]. We suppose that all operators are linear and defined on a complex Hilbert space, designated by $H$. The set of all these operators is denoted by $B(H)$ which is a Banach algebra.

Let us just say a few words about notations. It is known that any linear bounded operator $T$ may be expressed as $A + iB$ where $A$ and $B$ are self-adjoint. In fact, it is known that $A = \frac{T + T^*}{2}$ and $B = \frac{T - T^*}{2i}$. Then we call $A$ the real part of $T$ and we write $\text{Re} T$; $B$ the imaginary part of $T$ and we write $\text{Im} T$. It is also well-known that $T$ is normal if and only if $AB = BA$.

The following standard result will be useful.

Lemma 1. Let $T$ be a self-adjoint operator such that $e^T = I$. Then $T = 0$.

We include a proof for the reader’s convenience.

Proof. Assume $e^T = I$ and let $x \in \sigma(T)$. Then $e^x = 1$. Since $T$ is self-adjoint, $x$ is real and hence $e^x = 1$ implies $x = 0$ only. Since $\sigma(T)$ is never empty, $\sigma(T) = \{0\}$. Again, since $T$ is self-adjoint, by the "spectral radius theorem", we have

$$\|T\| = r(T) = 0 \implies T = 0.$$ 

We will also be using the celebrated Fuglede theorem, which we recall for the reader’s convenience (for more versions, see [6, 8, 10]). For a proof, see e.g. [3] or [15].

Theorem 1 (Fuglede). Let $A, N \in B(H)$. Assume that $N$ is normal. Then

$$AN = NA \implies AN^* = N^* A$$

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The exponential of an operator appears in many areas of science, and in mathematics in particular. For example, it appears when solving a problem of the type $X' = AX$ where $A$ is an operator. It is also met when dealing with semi-groups, the Stone theorem, the Lie product formula, the Trotter product formula, the Feynman-Kac formula, (bounded) wave operators, etc... See [5, 7, 12, 13, 14].

Commutativity of operators and its characterization is one of the most important topics in Operator Theory. Thus when the commutativity of exponentials implies that of operators becomes an interesting problem. In this paper we are mainly concerned with problems of this sort. Many authors have worked previously on similar questions (see [11, 16, 17, 18, 20]). However, they all used what is known as the $2\pi i$-congruence-free hypothesis (and similar hypotheses). G. Bourgeois (2) dropped that hypothesis but he only worked in low dimensions. Very recently, M. H. Mortad (see [9]) gave a different approach to this problem for normal operators, by bypassing the $2\pi i$-congruence-free hypothesis. He used the well-known cartesian decomposition of normal operators as $A + iB$ where $A$ and $B$ are commuting self-adjoint operators, so that the following result may be applied (whose proof may be found in e.g. [20])

**Theorem 2.** Let $A$ and $B$ be two self-adjoint operators defined on a Hilbert space. Then

$$e^A e^B = e^B e^A \iff AB = BA.$$  

Using a result on similarities (due to S. K. Berberian in [1]), the following two results were then obtained (both appeared in [9])

**Proposition 1.** Let $N$ be a normal operator with cartesian decomposition $A + iB$. Let $S$ be a self-adjoint operator. If $\sigma(B) \subset (0, \pi)$, then

$$e^S e^N = e^N e^S \iff SN = NS.$$ 

**Remark.** Going back to the proof of Proposition 1, we see that we may take $(-\frac{\pi}{2}, \frac{\pi}{2})$ in lieu of $(0, \pi)$ without any problem. Hence the same results hold with this new interval. Thus any self-adjoint operator (remember that its imaginary part then must vanish) obeys the given condition on the spectrum.

**Theorem 3.** Let $N$ and $M$ be two normal operators with cartesian decompositions $A + iB$ and $C + iD$ respectively. If $\sigma(B), \sigma(D) \subset (0, \pi)$, then

$$e^M e^N = e^N e^M \iff MN = NM.$$ 

In this paper, we investigate further this question and accordingly, we obtain more results. We have tried to keep the proofs as simple as possible, so that we would allow a broader audience to read the paper with ease. Another virtue of these proofs, is that in some cases, they may even be applied to prove known results which use the $2\pi i$-congruence-free hypothesis, for example. Let us now give a sample of already known results on the topic of the present paper.

**Theorem 4** (Hille, [4]). Let $A$ and $B$ be both in $B(H)$ such that $e^A = e^B$. If $\sigma(A)$ is incongruent (mod $2\pi i$), then $A$ and $B$ commute.

**Theorem 5** (Schmoeger, [18]). Let $A$ and $B$ be both in $B(H)$. Then

1. If $A + B$ is normal, $\sigma(A + B)$ is generalized $2\pi i$-congruence-free and

$$e^A e^B = e^B e^A = e^{A+B},$$

then $AB = BA$. 

(2) If $A$ is normal, $\sigma(A)$ is generalized $2\pi i$-congruence-free and
\[ e^A = e^B, \]
then $AB = BA$.

**Theorem 6** (Schmoeger, [19]). Let $A$ and $B$ be both in $B(H)$ such that $e^A = e^B$. Assume that $A$ is normal.

1. If $r(A) < \pi$, then $AB = BA$ (where $r(A)$ is the spectral radius of $A$).
2. If $\sigma(A)$ satisfies
\[ \sigma(A) \subseteq \{ z \in \mathbb{C} : |Im z| \leq \pi \} \]
and
\[ \sigma(A) \cap \sigma(A + 2\pi i) \subseteq \{ i\pi \}, \]
then $A^2B = BA^2$. If $i\pi \notin \sigma_p(A)$ or $-i\pi \notin \sigma_p(A)$, then $AB = BA$.

Throughout this paper, the reader will see that with simple hypotheses, we shall get the same conclusions as above.

2. An Example

The following example (s) will recalled on in the next section, mainly as a counterexample.

**Example 1.** Let
\[ A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}. \]
Then $A$ is clearly normal. By computing integers powers of $A$ we may easily check that
\[ e^A = \begin{pmatrix} \cos \pi & \sin \pi \\ -\sin \pi & \cos \pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I. \]
Next, we have
\[ \text{Im}A = \frac{A - A^*}{2i} = \begin{pmatrix} 0 & -i\pi \\ i\pi & 0 \end{pmatrix} \]
and hence $\sigma(\text{Im}A) = \{ \pi, -\pi \}$. This signifies that $\sigma(\text{Im}A)$ cannot be inside an open interval of length equals to $\pi$ (a hypothesis which will play an important role in our proofs).

Now let
\[ B = \begin{pmatrix} \pi & -2\pi \\ \pi & -\pi \end{pmatrix}. \]
We may also show that $e^B = -I$. Finally, it is easily verifiable that $A$ and $B$ do not commute.

3. Main Results

We start by giving a very important result of the paper. We have

**Theorem 7.** Let $A$ be in $B(H)$. Let $N \in B(H)$ be normal such that $\sigma(\text{Im}N) \subset (0, \pi)$. Then
\[ Ae^N = e^N A \iff AN = NA. \]
Proof. Of course, we are only concerned with proving the implication \( \Rightarrow \). The normality of \( N \) implies that of \( e^N \), and so by the Fuglede theorem

\[
Ae^N = e^N A \iff Ae^{N^*} = e^{N^*} A
\]

or

\[
A^* e^N = e^N A^*.
\]

Hence

\[
(A + A^*)e^N = e^N (A + A^*) \quad \text{or} \quad (\text{Re} A)e^N = e^N (\text{Re} A)
\]

so that

\[
e^{\text{Re} A} e^N = e^N e^{\text{Re} A}.
\]

But, \( \text{Re} A \) is self-adjoint, so Proposition 1 applies and then gives

\[
(\text{Re} A)N = N(\text{Re} A).
\]

Similarly, we find that

\[
(\text{Im} A)e^N = e^N (\text{Im} A)
\]

and as \( \text{Im} A \) is self-adjoint, similar arguments to those applied before yield

\[
(\text{Im} A)N = N(\text{Im} A).
\]

Therefore, \( AN = NA \), establishing the result. \( \Box \)

Remark. The hypothesis \( \sigma(\text{Im} N) \subset (0, \pi) \) cannot merely be dropped. Take \( N \) to be the operator \( A \) in Example 1 and take \( A \) to be any operator which does not commute with \( N \). Then

\[
Ae^N = e^N A = -A \quad \text{but} \quad AN \neq NA.
\]

Next we give the first consequence of the previous result. We have

**Theorem 8.** Let \( A \) and \( B \) be both in \( B(H) \). Assume that \( A + B \) is normal such that \( \sigma(\text{Im}(A + B)) \subset (0, \pi) \). If

\[
e^A e^B = e^B e^A = e^{A+B},
\]

then \( AB = BA \).

Proof. We have

\[
e^{A+B} e^A = e^B e^A e^A = e^A e^B e^A = e^A e^{A+B}.
\]

Since \( A + B \) is normal and \( \sigma(\text{Im}(A + B)) \subset (0, \pi) \), Theorem 7 gives

\[
(A + B)e^A = e^A (A + B) \quad \text{or} \quad B e^A = e^{A+B}
\]

for \( A \) commutes with \( e^A \). Now, right multiplying both sides of the previous equation by \( e^B \) leads to

\[
B e^A e^B = e^A B e^B = e^A e^B B
\]

or

\[
B e^{A+B} = e^{A+B} B.
\]

Applying again Theorem 7, we see that

\[
B(A + B) = (A + B)B \quad \text{or} \quad AB = BA.
\]

The proof is thus complete. \( \Box \)

**Corollary 1.** Let \( A \in B(H) \). Then

\[
e^A e^{A^*} = e^{A^*} e^A = e^{A^*+A} \quad \iff \quad A \text{ is normal.}
\]
Proof. We need only prove the implication "\(\Rightarrow\)". It is plain that \(A + A^*\) is self-adjoint. Hence the remark below Proposition 1 combined with Theorem 8 give us

\[ AA^* = A^* A. \]

□

We have yet another consequence of Theorem 7 (cf. Theorem 6).

**Corollary 2.** Let \(A\) be normal such that \(\sigma(\text{Im}A) \subset (0, \pi)\). Let \(B \in B(H)\). Then

\[ e^A = e^B \implies A^2B = BA^2. \]

**Proof.** We obviously have

\[ e^B(e^B)B = (Be^B)e^B. \]

So since \(e^A = e^B\), we have

\[ e^A(e^A)B = (Be^A)e^A. \]

By Theorem 7 we obtain

\[ Ae^A B = Be^A A. \]

Hence

\[ e^A(AB) = (BA)e^A. \]

Applying Theorem 7 once more yields

\[ A(AB) = (BA)A \text{ or } A^2B = BA^2, \]

completing the proof. □

**Corollary 3.** Let \(A\) be normal such that \(\sigma(\text{Im}A) \subset (0, \pi)\). Let \(B \in B(H)\). Then

\[ e^A = e^B \iff AB = BA. \]

**Remark.** By Example 1, \(e^A = e^B = -I\), but \(AB \neq BA\), showing again the importance of the assumption \(\sigma(\text{Im}A) \subset (0, \pi)\).

**Proof.** We obviously have

\[ Be^B = e^B B \]

so that

\[ Be^A = e^A B. \]

Theorem 7 does the remaining job, i.e. it gives the commutativity of \(A\) and \(B\). □

**Corollary 4.** Let \(A\) and \(B\) be two self-adjoint operators. Then

\[ e^A = e^B \iff A = B. \]

**Proof.** Observe first that we are only concerned with establishing the implication "\(\Rightarrow\)". By the remark below Proposition 1 we may get

\[ e^A = e^B \implies e^A e^B = e^B e^A \implies AB = BA. \]

Hence

\[ I = e^A e^{-A} = e^A e^{-B} = e^{A-B} \]

since \(A\) and \(B\) commute. But \(A - B\) is obviously self-adjoint, so Lemma 1 gives \(A = B\). □
Remark. The previous corollary is actually a consequence of Theorem 2. Authors who did not argue as we did usually considered it as a consequence of their results. But, with the proof given here, we clearly see that we only need Theorem 2 and Lemma 1.

Remark. Of course, the previous corollary also generalizes Lemma 1.

**Corollary 5.** Let $A$ be normal. Then we have

$$A \text{ is self-adjoint } \iff e^{iA} \text{ is unitary.}$$

**Proof.** The implication "$\Rightarrow"$ is well-known. Let us prove the reverse implication. By the normality of $A$, we have

$$e^{iA - iA^*} = e^{iA}e^{-iA^*} = e^{iA}(e^{iA^*})^* = I.$$  

Since $iA - iA^*$ is self-adjoint, Lemma 1 gives $A = A^*$, which completes the proof. $\Box$

We now come to a result that appeared in [11] and [18]. It reads: If $A$ is self-adjoint, $\sigma(A) \subseteq [-\pi, \pi]$ and $e^{iA} = e^{B}$, then $B^* = -B$ if $B$ is normal. Here is an improvement of this result.

**Proposition 2.** If $A$ is self-adjoint and $e^{iA} = e^{B}$, then $B^* = -B$ whenever $B$ is normal.

**Proof.** It is clear that $e^{iA}$ is unitary. We also have

$$e^{B^*} = e^{-iA} \text{ and } e^{-B} = e^{-iA}.$$  

Thus

$$e^{-B} = e^{B^*} \text{ so that } e^{B^* + B} = I$$

because $B$ is normal. However, $B + B^*$ is always self-adjoint, whence $B^* = -B$ by Lemma 1. $\Box$

4. Conclusion

The results of this paper as well as those of the paper [9] should be easily generalized to unital Banach algebras.

Theorem 7 is very important here. The very simple and interesting proof of Corollary 2 or Corollary 3 could not have been achieved if we had not Theorem 7 in hand. Also, as mentioned in the introduction, most of the proofs, for instance that of Corollary 1, may be adopted to prove the results that use the $2\pi i$-congruence-free hypothesis and similar hypotheses.

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