A spectral characterization of isomorphisms on $C^*$-algebras

R. Brits, F. Schulz, and C. Touré

Abstract. Following a result of Hatori et al. (J Math Anal Appl 326:281–296, 2007), we give here a spectral characterization of an isomorphism from a $C^*$-algebra onto a Banach algebra. We then use this result to show that a $C^*$-algebra $A$ is isomorphic to a Banach algebra $B$ if and only if there exists a surjective function $\phi : A \to B$ satisfying (i) $\sigma(\phi(x)\phi(y)\phi(z)) = \sigma(xyz)$ for all $x, y, z \in A$ (where $\sigma$ denotes the spectrum), and (ii) $\phi$ is continuous at $1$. In particular, if (in addition to (i) and (ii)) $\phi(1) = 1$, then $\phi$ is an isomorphism. An example shows that (i) cannot be relaxed to products of two elements, as is the case with commutative Banach algebras. The results presented here also elaborate on a paper of Brešar and Špenko (J Math Anal Appl 393:144–150, 2012), and a paper of Bourhim et al. (Arch Math 107:609–621, 2016).

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1. Introduction. In general, $A$ will be a unital and complex Banach algebra, with the unit denoted by $1$. The invertible group of $A$ will be denoted by $G(A)$. If $x \in A$, then the spectrum of $x$ (relative to $A$) is the (necessarily non-empty and compact) set $\sigma(x, A) := \{\lambda \in \mathbb{C} : \lambda 1 - x \notin G(A)\}$, and the spectral radius of $x \in A$ is defined as $\rho(x, A) := \sup\{|\lambda| : \lambda \in \sigma(x, A)\}$. For $x \in A$, $C_{\{x\}}$ denotes the bicommutant of the set $\{x\}$. Recall that if $y \in C_{\{x\}}$, then $\sigma(y, A) = \sigma(y, C_{\{x\}})$, in which case we shall simply write $\sigma(y)$ for the spectrum of $y$, and $\rho(y)$ for the spectral radius of $y$; the same convention will be used whenever the algebra under consideration is clear from the context. We shall further write $\text{Rad}(A)$ for the radical of $A$, and $Z(A)$ for the centre of $A$, which is, by definition, equal to $C_{\{1\}}$.

The literature on the topic of spectral preservers is quite extensive, but closest to the current paper are perhaps [4–8] and, more recently, [2]. We should point out that the results and arguments presented in the current paper have
no reliance on the existence of an essential socle as is often the case with papers on this topic; the proofs of the results here rely fundamentally on a connection between the following three theorems, as well as the well-known Lie-Trotter Formula ([1, p. 67]) for exponentials:

**Theorem 1.1** ([3, Theorem 2.6]). Let $A$ be a unital and semisimple Banach algebra, and let $a, b \in A$. Then $a = b$ if and only if $\sigma(ax) = \sigma(bx)$ for all $x \in A$ satisfying $\rho(x - 1) < 1$. In particular, $a = b$ if and only if $\sigma(ae^y) = \sigma(be^y)$ for all $y \in A$.

**Theorem 1.2** ([5, Corollary 3.3]). Let $A$ be a unital, semisimple, and commutative Banach algebra, and let $B$ be a unital and commutative Banach algebra. Suppose that $\phi$ is a map from $A$ onto $B$ such that the equations

1. $\phi(1) = 1$,
2. $\sigma(\phi(x)\phi(y)) = \sigma(xy)$ for all $x, y \in A$

hold. Then $B$ is semisimple and $\phi$ is an isomorphism.

**Theorem 1.3** ([2, Theorem 6.4]). If $A$ and $B$ are unital semisimple Banach algebras, and $\phi$ is a surjection from $A$ onto $B$ satisfying $\sigma(\phi(x)\phi(y)\phi(z)) = \sigma(xyz)$ for all $x, y, z \in A$, then $\phi(1)$ is a central invertible element of $B$ for which $\phi(1)^3 = 1$ and $\phi(1)^2\phi$ is a multiplicative map from $A$ to $B$.

The current paper is in fact motivated by Theorem 1.2, which is the main result of [5], the results in [4, Section 4.1], as well as [2, Theorem 6.4]. It is not hard to see that Theorem 1.2 fails in even finite-dimensional non-commutative cases. For example, take $A = B = M_n(\mathbb{C}) \oplus M_k(\mathbb{C})$ where $m, k \geq 2$, and define $\phi : A \to B$ by $\phi((a, b)) = (a^t, xba^{-1})$ where $a^t$ denotes the transpose of the matrix $a$ and $x$ is any fixed invertible element of $M_k(\mathbb{C})$. Then $\phi(1) = 1$, $\sigma(\phi(x)\phi(y)) = \sigma(xy)$ for all $x, y \in A$, $\phi$ is surjective (and $\phi$ is injective, linear and continuous). But clearly $\phi$ is not an automorphism or an anti-automorphism. Concerning the preceding example, it is worthwhile to mention that the complete description of multiplicatively spectrum preservers on $M_n(\mathbb{C})$ is well-known and, up to a multiple factor of $\pm 1$, is either an automorphism or an anti-automorphism.

In pursuance of the main results in Section 2, we need the following:

**Lemma 1.4.** Let $A$ be a semisimple Banach algebra, and let $\phi$ be a function from $A$ onto a Banach algebra $B$ satisfying

$$\sigma\left(\prod_{i=1}^{m} \phi(x_i)\right) = \sigma\left(\prod_{i=1}^{m} x_i\right)$$

for all $x_i \in A$, $1 \leq m \leq 3$. Then

1. $\phi$ is bijective,
2. $B$ is semisimple,
3. $\phi$ is multiplicative, i.e. $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$,
4. $\phi(1) = 1$ and $\phi(\lambda x) = \lambda \phi(x)$ for all $x \in A$, $\lambda \in \mathbb{C}$,
5. if $x, y \in A$, then $xy = yx$ $\iff$ $\phi(x)\phi(y) = \phi(y)\phi(x)$,
6. if $x \in A$, then $y \in C_{\{x\}}$ $\iff$ $\phi(y) \in C_{\{\phi(x)\}}$. 

Define for each $x \in A$, $C_{\{x\}}/\text{Rad} (C_{\{x\}})$ is isomorphic to $C_{\{\phi(x)\}}/\text{Rad} (C_{\{\phi(x)\}})$.

**Proof.** (i) Suppose $\phi(x) = \phi(y)$. Then, using (1), we have

$$\sigma (\phi(x)\phi(z)) = \sigma (\phi(y)\phi(z)) \Rightarrow \sigma (xz) = \sigma (yz) \text{ for all } z \in A.$$

Since $A$ is semisimple, Theorem 1.1 gives $x = y$.

(ii) This follows from [2, Lemma 3.2].

(iii) Fix $x, y \in A$. Then

$$\sigma (\phi(x)\phi(y)\phi(z)) = \sigma (xyz) = \sigma (\phi(xy)\phi(z)) \text{ for all } z \in A.$$ Since $\phi$ is surjective and $B$ is semisimple, Theorem 1.1 gives $\phi(xy) = \phi(x)\phi(y)$.

(iv) The first part follows from

$$\sigma (\phi(1)\phi(z)) = \sigma (z) = \sigma (1\phi(z)) \text{ for all } z \in A,$$

and the second part from

$$\sigma (\phi(\lambda x)\phi(z)) = \sigma (\lambda xz) = \sigma (\lambda \phi(x)\phi(z)) \text{ for all } z \in A.$$

(v) If $xy = yx$, then, since $\phi$ is multiplicative,

$$\phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x).$$

Since $\phi$ is also bijective, the reverse implication holds.

(vi) Let $y \in C_{\{x\}}$, and suppose $\phi(a) \in B$ commutes with $\phi(x)$. By (v), $a$ commutes with $x$, and therefore with $y$. Again by (v), it follows that $\phi(y)$ commutes with $\phi(a)$ which proves the forward implication. The reverse implication holds similarly.

(vii) Define

$$\tilde{\phi} : C_{\{x\}}/\text{Rad} (C_{\{x\}}) \rightarrow C_{\{\phi(x)\}}/\text{Rad} (C_{\{\phi(x)\}})$$

by

$$\tilde{\phi} (w + \text{Rad} (C_{\{x\}})) = \phi(w) + \text{Rad} (C_{\{\phi(x)\}}).$$

If $w, u \in C_{\{x\}}$ and $r \in \text{Rad} (C_{\{x\}})$ with $w = u + r$, then, for each $y \in C_{\{x\}},$

$$wy = uy + ry \Rightarrow \sigma (wy) = \sigma (uy + ry)$$

$$\Rightarrow \sigma (wy) = \sigma (uy)$$

$$\Rightarrow \sigma (\phi(w)\phi(y)) = \sigma (\phi(u)\phi(y)).$$

It follows from Theorem 1.1 that $\phi(w) - \phi(u) \in \text{Rad} (C_{\{\phi(x)\}})$ and hence that $\tilde{\phi}$ is well-defined. Since

$$\tilde{\phi} (1 + \text{Rad} (C_{\{x\}})) = 1 + \text{Rad} (C_{\{\phi(x)\}}),$$

and

$$\sigma (\tilde{\phi}(a)\tilde{\phi}(b)) = \sigma (ab) \text{ for all } a, b \in C_{\{x\}}/\text{Rad} (C_{\{x\}}),$$

Theorem 1.2 implies that $\tilde{\phi}$ is an isomorphism. \qed
Define now an auxiliary map $\psi x$ since then, by the Fuglede–Putnam–Rosenblum theorem [1, Theorem 6.2.5], $\phi$ satisfies the following properties:

(i) $\sigma (\prod_{i=1}^m \phi (x_i)) = \sigma (\prod_{i=1}^m x_i)$ for all $x_i \in A$, $1 \leq m \leq 3$.

(ii) $\phi$ is continuous at 1.

Proof. If $\phi$ is an isomorphism from $A$ onto $B$, then (i) follows trivially, and (ii) follows from Johnson’s continuity theorem [1, Corollary 5.5.3]. We prove the reverse implication: Let $a \in G(A)$. If $x_n \to a$, then $x_n a^{-1} \to 1$. By continuity of $\phi$ at 1, together with Lemma 1.4(iii), it follows that

$$\lim_n \phi (x_n) \phi (a^{-1}) = \lim \phi (x_n a^{-1}) = \phi (1) = 1.$$ 

Hence $\lim_n \phi (x_n) = \phi (a)$, and so $\phi$ is continuous on $G(A)$. If $x \in A$ is normal, then, by the Fuglede–Putnam–Rosenblum theorem [1, Theorem 6.2.5], $C_{(x)}$ is a commutative $C^*$-subalgebra of $A$, and hence semisimple. From Lemma 1.4(vi), it follows that the map

$$\hat{\phi} : C_{(x)} \to C_{(\phi (x))}$$

defined by $\hat{\phi}(y) = \phi (y)$ for $y \in C_{(x)}$ is surjective, and it satisfies (1). So, by Lemmas 1.2 and 1.4, we have that $\hat{\phi}$ is an isomorphism; from this, we may conclude that

$$\phi (e^x) = e^{\phi (x)}$$

if $x$ is normal. (2)

If $x, y \in S$, then, using the Lie-Trotter Formula together with the continuity of $\phi$ on $G(A)$,

$$e^{\phi (x+y)} = \phi \left( e^{x+y} \right) = \phi \left( \lim_n \left( e^{\frac{x}{n}} e^{\frac{y}{n}} \right)^n \right) = \lim_n \phi \left( \left( e^{\frac{x}{n}} e^{\frac{y}{n}} \right)^n \right) = \lim_n \left( e^{\frac{\phi (x)}{n}} e^{\frac{\phi (y)}{n}} \right)^n = e^{\phi (x) + \phi (y)}.$$ 

Replacing $x, y$ in the above equation by respectively $tx, ty$ where $t \in \mathbb{R}$, it follows that $e^{t \phi (x+y)} = e^{t (\phi (x) + \phi (y))}$ from which differentiation with respect to $t$ gives $\phi (x + y) = \phi (x) + \phi (y)$. Thus, the restriction of $\phi$ to $S$ is a linear map. Define now an auxiliary map $\psi_\phi : A \to B$ by

$$\psi_\phi (x) := \phi \left( \text{Re} \; x \right) + i \phi \left( \text{Im} \; x \right).$$ (3)

Since the restriction of $\phi$ to the real Banach space $S$ is linear, and $1 \in S$, continuity of $\phi$ at 1 implies continuity of the restriction of $\phi$ to $S$. Therefore, since $x \mapsto \text{Re} \; x$ and $x \mapsto \text{Im} \; x$ are continuous on $A$, $\psi_\phi$ is continuous on $A$. Using the fact that $\phi$ is additive on $S$, together with Lemma 1.4(iv), it is a
simple matter to show that \( \psi_\phi \) is linear on \( A \). Let \( x = a + ib \) where \( a = \text{Re} \ x \) and \( b = \text{Im} \ x \). Then, since \( a, b \in \mathcal{S} \), and since \( \phi \) is multiplicative,

\[
\psi_\phi(x)^2 = (\phi(a) + i\phi(b))^2 = \phi(a^2 - b^2) + i [\phi(ab) + \phi(ba)],
\]

and

\[
\psi_\phi(x^2) = \phi(a^2 - b^2) + i\phi(ab + ba).
\]

But, since \( \phi \) is additive on \( \mathcal{S} \) and multiplicative,

\[
\phi((a + b)^2) = \phi(a + b)^2 = (\phi(a) + \phi(b))^2
\]

from which expansion and comparison imply that \( \phi(ab) + \phi(ba) = \phi(ab + ba) \). Thus \( \psi_\phi(x)^2 = \psi_\phi(x^2) \) holds for each \( x \in A \). By induction, it then follows that \( \psi_\phi(x)^{2^n} = \psi_\phi(x^{2^n}) \) holds for each \( m \in \mathbb{N} \). From this, we obtain

\[
\psi_\phi(e^x) = \psi_\phi\left( \lim_n (1 + x/2^n)^{2^n} \right) = \lim_n \psi_\phi\left( (1 + x/2^n)^{2^n} \right)
\]

\[
= \lim_n \psi_\phi (1 + x/2^n)^{2^n} = \lim_n (1 + \psi_\phi(x)/2^n)^{2^n}
\]

\[
= e^{\psi_\phi(x)}.
\]

But we also have

\[
\phi(e^x) = \phi\left( e^{\text{Re} \ x + i \text{Im} \ x} \right) = \phi\left( \lim_n \left( e^{\frac{\text{Re} \ x}{n} + i \frac{\text{Im} \ x}{n}} \right)^n \right)
\]

\[
= \lim_n \phi\left( \left( e^{\frac{\text{Re} \ x}{n} + i \frac{\text{Im} \ x}{n}} \right)^n \right) = \lim_n \left( \phi\left( e^{\frac{\text{Re} \ x}{n}} \right) \phi\left( e^{i \frac{\text{Im} \ x}{n}} \right) \right)^n
\]

\[
= \lim_n \left( e^{\phi(\text{Re} \ x) + i \phi(\text{Im} \ x)} \right)^n = e^{\phi(\text{Re} \ x) + i \phi(\text{Im} \ x)} = e^{\psi_\phi(x)}.
\]

Therefore

\[
\phi(e^x) = e^{\psi_\phi(x)} = \psi_\phi(e^x) \text{ for all } x \in A. \quad (4)
\]

Now, if \( x \in A \) is normal, then, by combining (2) and (4), we obtain

\[
e^{\lambda \phi(x)} = \phi(e^{\lambda x}) = e^{\lambda \psi_\phi(x)}, \lambda \in \mathbb{C},
\]

from which it follows that

\[
\phi(x) = \psi_\phi(x) \text{ if } x \text{ is normal.} \quad (5)
\]

Let \( x, y \in A \) be arbitrary. If \( t > 0 \) is sufficiently small (which we keep fixed), then each of the sets \( \sigma(1 + tx), \sigma(1 + ty), \text{ and } \sigma((1 + tx)(1 + ty)) \) does not contain zero, and does not separate 0 from infinity. So it follows from [1, Theorem 3.3.6] that there exist \( a, b, c \in A \) such that

\[
1 + tx = e^a, \quad 1 + ty = e^b, \quad \text{and } (1 + tx)(1 + ty) = e^c.
\]

So, by (4), it follows, on the one hand, that

\[
\phi((1 + tx)(1 + ty)) = \psi_\phi(1 + t(x + y) + t^2xy)
\]

\[
= 1 + t(\psi_\phi(x) + \psi_\phi(y)) + t^2\psi_\phi(xy),
\]
and on the other hand, using the fact that \( \phi \) is multiplicative together with (4), that
\[
\phi ((1 + tx)(1 + ty)) = \phi(1 + tx)\phi(1 + ty)
= \psi_\phi(1 + tx)\psi_\phi(1 + ty)
= (1 + t\psi_\phi(x))(1 + t\psi_\phi(y))
= 1 + t(\psi_\phi(x) + \psi_\phi(y)) + t^2\psi_\phi(x)\psi_\phi(y).
\]
Comparison of the two expressions yields \( \psi_\phi(xy) = \psi_\phi(x)\psi_\phi(y) \), and so \( \psi_\phi \) is linear and multiplicative. We proceed to show that (i) and (ii) hold, it follows from [2, Lemma 3.6] that
\[
\text{it suffices to prove the reverse implication. So, assuming that (i) and (ii) hold,}
\]
\[
\text{we have that (5) that}
\]
\[
\phi(a) = \phi(h)\phi(u) = \psi_\phi(h)\psi_\phi(u) = \psi_\phi(hu) = \psi_\phi(a),
\]
and so \( \psi_\phi \) agrees with \( \phi \) on \( G(A) \). Notice further, since \( \phi \) is a spectrum preserving multiplicative bijection from \( A \) onto \( B \), we have that \( \phi(G(A)) = G(B) \).

Let \( b \in B \) be arbitrary, and fix \( \lambda \in \mathbb{C} \) such that \( |\lambda| > \rho(b) \). If we write \( b = \lambda 1 + (b - \lambda 1) \) and observe that both terms in the decomposition belong to \( G(A) \), then it follows that \( b - \lambda 1 = \phi(a) \) for some \( a \in G(A) \) and \( \lambda 1 = \phi(\lambda 1) \). Hence, from (6), we have that
\[
b = \phi(\lambda 1) + \phi(a) = \psi_\phi(\lambda 1) + \psi_\phi(a) = \psi_\phi(\lambda 1 + a)
\]
which shows that \( \psi_\phi \) is surjective. So, since \( \psi_\phi \) is now an isomorphism from \( A \) onto \( B \), to obtain the required result it suffices to show that \( \psi_\phi \) agrees with \( \phi \) everywhere on \( A \): Fix \( x \in A \) arbitrary. Then, for each \( y \in A \),
\[
\sigma(\phi(x)\phi(y)) = \sigma(xy) = \sigma(\psi_\phi(x)\psi_\phi(y)).
\]
In particular, (7) and (4) imply that for each \( y \in A \),
\[
\sigma(\phi(x)\psi_\phi(e^y)) = \sigma(\phi(x)\phi(e^y)) = \sigma(\psi_\phi(x)\psi_\phi(e^y)).
\]
Since the exponentials in \( B \) are precisely the images of the exponentials in \( A \) under \( \psi_\phi \), it follows from Theorem 1.1 that \( \phi(x) = \psi_\phi(x) \) and hence \( \phi = \psi_\phi \).

**Corollary 2.2.** Let \( A \) be a \( C^* \)-algebra, and \( B \) be a Banach algebra. Then \( A \) is isomorphic to \( B \) if and only if there exists a surjective map \( \phi : A \to B \) such that
\begin{enumerate}[(i)]
  \item \( \sigma(\phi(x)\phi(y)\phi(z)) = \sigma(xy)z \) for all \( x, y, z \in A \).
  \item \( \phi \) is continuous at \( 1 \).
\end{enumerate}

**Proof.** It suffices to prove the reverse implication. So, assuming that (i) and (ii) hold, it follows from [2, Lemma 3.6] that \( B \) is semisimple. If we define \( \psi : A \to B \) by
\[
\psi(x) := \phi(1)^2\phi(x),
\]
then Theorem 1.3 gives that \( \psi \) is multiplicative. It is then clear that \( \psi \) satisfies condition (i) in Theorem 2.1. Obviously, continuity of \( \psi \) at \( 1 \) follows from continuity of \( \phi \) at \( 1 \), and surjectivity of \( \psi \) from surjectivity of \( \phi \) together with
invertibility of \( \phi(1) \). So \( \psi \) satisfies the hypothesis, and hence the conclusion, of Theorem 2.1.

In Corollary 2.2, one cannot expect \( \phi \) to be an isomorphism. For example, let \( A = B = M_n(\mathbb{C}) \) and define \( \phi : A \to B \) by \( \phi(a) = (-1/2 + i\sqrt{3}/2)a \). Then \( \sigma(\phi(x)\phi(y)\phi(z)) = \sigma(xyz) \) for all \( x, y, z \in A \), \( \phi \) is surjective, and \( \phi \) is continuous. But \( \phi \) is not an automorphism. We should note here that this exemplifies one of two possible scenarios in the case where \( A = B = M_n(\mathbb{C}) \); the complete description of maps on \( M_n(\mathbb{C}) \) preserving the spectrum of the triple product of matrices is well-known and, up to a multiplication by a cube root of unity, is either an automorphism or an anti-automorphism.

As a simple consequence of Corollary 2.2, we can now improve Theorem 2.1. For example, we only require the spectral assumption to hold for \( m = 2, 3 \):

**Corollary 2.3.** Let \( A \) be a \( C^* \)-algebra, and \( B \) be a Banach algebra. Then a surjective map \( \phi : A \to B \) is an isomorphism from \( A \) onto \( B \) if and only if \( \phi \) satisfies the following properties:

(i) \( \sigma(\prod_{i=1}^{m} \phi(x_i)) = \sigma(\prod_{i=1}^{m} x_i) \) for all \( x_i \in A \), \( 2 \leq m \leq 3 \).

(ii) \( \phi \) is continuous at \( 1 \).

**Proof.** With \( m = 3 \) we have, as in the proof of Corollary 2.2, that \( \psi : A \to B \) defined by \( \psi(x) := \phi(1)^2 \phi(x) \), \( x \in A \), is an isomorphism. With \( m = 2 \) the spectral mapping theorem says that each \( \lambda \in \sigma(\phi(1)) \) is a cube root, as well as a square root of 1 from which it follows that \( \sigma(\phi(1)) = \{1\} \). Since \( \phi(1) \in Z(B) \) and \( B \) is semisimple, \( \phi(1) = 1 \) and hence \( \phi \) is an isomorphism.

Alternatively, one may formulate the following equivalent statement:

**Corollary 2.4.** Let \( A \) be a \( C^* \)-algebra, and \( B \) be a Banach algebra. Then a surjective map \( \phi : A \to B \) is an isomorphism from \( A \) onto \( B \) if and only if \( \phi \) satisfies the following properties:

(i) \( \sigma(\phi(x)\phi(y)\phi(z)) = \sigma(xyz) \) for all \( x, y, z \in A \).

(ii) \( \phi(1) = 1 \).

(iii) \( \phi \) is continuous at \( 1 \).

In Theorem 1.2, removing the requirement \( \phi(1) = 1 \) from the hypothesis would still give the result that \( B \) is semisimple, and that \( A \) and \( B \) are isomorphic, but with the isomorphism given by the map \( \psi(x) := \phi(1)\phi(x) \); this can be shown using essentially the same arguments as in the proof of Corollary 2.2 above. In [4], Brešar and Špenko consider the condition \( \sigma(\phi(x)\phi(y)\phi(z)) = \sigma(xyz) \) for all \( x, y, z \in A \) where \( \phi \) is map from a Banach algebra \( A \) onto a semisimple Banach algebra \( B \). One difficulty that arises here is that the spectral assumption on \( \phi \) does not seem to imply the semisimplicity of \( A \) (in contrast to the case where \( A \) is the assumed semisimple Banach algebra, with semisimplicity of \( B \) following via the spectral assumption). By application of Theorem 1.1, one can nevertheless improve [4, Corollary 4.1] to hold for all \( x, y \) rather than for all invertible \( x, y \), and one can therefore omit the linearity assumption in [4, Corollary 4.2] with the same conclusion. As a final remark, the example in the paragraph preceding Lemma 1.4 shows that the spectral...
assumption (i) in Theorem 2.1 and Corollary 2.2 cannot be relaxed to products of two elements which leaves open the question of whether the continuity assumption (ii) on $\phi$ is perhaps superfluous?

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**References**

[1] Aupetit, B.: A Primer on Spectral Theory. Universitext. Springer, Berlin (1991)

[2] Bourhim, A., Mashreghi, J., Stepanyan, A.: Maps between Banach algebras preserving the spectrum. Arch. Math. **107**, 609–621 (2016)

[3] Braatvedt, G., Brits, R.: Uniqueness and spectral variation in Banach algebras. Quaest. Math. **36**, 155–165 (2013)

[4] Brešar, M., Špenko, Š.: Determining elements in Banach algebras through spectral properties. J. Math. Anal. Appl. **393**, 144–150 (2012)

[5] Hatori, O., Miura, T., Tagaki, H.: Unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative. J. Math. Anal. Appl. **326**, 281–296 (2007)

[6] Molnár, L.: Some characterizations of the automorphisms of $B(H)$ and $C(X)$. Proc. Am. Math. Soc. **130**, 111–120 (2002)

[7] Rao, N.V., Roy, A.K.: Multiplicatively spectrum-preserving maps of function algebras. Proc. Am. Math. Soc. **133**, 1135–1142 (2005)

[8] Rao, N.V., Roy, A.K.: Multiplicatively spectrum-preserving maps of function algebras II. Proc. Edinb. Math. Soc. **48**, 219–229 (2005)

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