(T,L)-type rotational surface in three dimensional Lorentz-Minkowski space

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Abstract. In this work, we study on the third Laplace-Beltrami operator of timelike rotational surface of (T,L)-type in three dimensional Lorentz-Minkowski space.

1. Introduction

Rotational surfaces have complicated geometric structures in Lorentz-Minkowski 3-space \( \mathbb{L}^3 \). \( \mathbb{L}^3 \) has distinguished axes of rotation, namely, spacialike, timelike and lightlike axes.

A helicoidal surface and a rotational surface have an isometric relation by Bour’s theorem is shown by Ikawa [3]. Güler classified the spacialike (and timelike) helicoidal (and rotational) surfaces with lightlike profile curve of spacialike, timelike and lightlike axes [2] in \( \mathbb{L}^3 \). He also investigate the properties of the \((S,L)\)-type rotational surfaces in [1].

In this paper, we give a study on the timelike rotational surface with lightlike profile curve of \((T,L)\)-type in Lorentz-Minkowski 3-space. In section 2, we recall some basic notions of the Lorentzian geometry. We give the definition of the timelike rotational surface with lightlike profile in section 3. In section 4, we study on the third Laplace-Beltrami operator of timelike rotational surface of \((T,L)\)-type in three dimensional Lorentz-Minkowski space.

2. Preliminaries

In this section, we will obtain a lightlike profile curve in the Lorentz-Minkowski 3-space. In the rest of this paper we shall identify a vector \((a, b, c)\) with its transpose \( (a, b, c)^t \).

The Lorentz-Minkowski 3-space \( \mathbb{L}^3 \) is the Euclidean space \( \mathbb{E}^3 \) provided with the inner product
\[
\langle \vec{p}, \vec{q} \rangle = p_1q_1 + p_2q_2 - p_3q_3,
\]
where \( \vec{p}, \vec{q} \in \mathbb{L}^3 \). We say that a Lorentzian vector \( \vec{p} \) is spacialike (resp. lightlike and timelike) if \( \vec{p} = 0 \) or \( \langle \vec{p}, \vec{p} \rangle > 0 \) (resp. \( \vec{p} \neq 0; \langle \vec{p}, \vec{p} \rangle = 0 \) and \( \langle \vec{p}, \vec{p} \rangle < 0 \)).
The norm of the vector is defined by \( \|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p}\rangle} \). Lorentzian vector product \( \mathbf{p} \times \mathbf{q} \) of \( \mathbf{p} \) and \( \mathbf{q} \) is defined as follows

\[
\mathbf{p} \times \mathbf{q} = (p_2q_3 - q_2p_3)e_1 + (q_1p_3 - p_1q_3)e_2 + (q_1p_2 - p_1q_2)e_3.
\]

Now we define a non-degenerate rotational surface in \( \mathbb{L}^3 \). For an open interval \( I \subset \mathbb{R} \), let \( \gamma : I \to \Pi \) be a curve in a plane \( \Pi \) in \( \mathbb{L}^3 \), and let \( \ell \) be a straight line in \( \Pi \) which does not intersect the curve \( \gamma \). A rotational surface in \( \mathbb{L}^3 \) is defined as a non-degenerate surface rotating a curve \( \gamma \) around a line \( \ell \) (these are called the profile curve and the axis, respectively). If the axis \( \ell \) is timelike in \( \mathbb{L}^3 \), then we may suppose that \( \ell \) is the line spanned by the vector (0,0,1). The semi–orthogonal matrix given as follow is the subgroup of the Lorentzian group that fixes the above vectors as invariant

\[
T(v) = \begin{pmatrix}
\cos(v) & -\sin(v) & 0 \\
\sin(v) & \cos(v) & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad v \in \mathbb{R}.
\]

The matrix \( T \) can be found by solving the following equations simultaneously \( T\ell = \ell, T^\epsilon T = \varepsilon \), where \( \varepsilon = \text{diag}(1,1,-1) \), \( \det T = 1 \).

A surface in \( \mathbb{L}^3 \) is timelike surface if the \( \det I = EG - F^2 < 0 \), where \( E, F, G \) are the coefficients of the first fundamental form. Parametrization of the profile curve \( \gamma \) is given by

\[
\gamma(u) = (\zeta(u), \mu(u), \eta(u)),
\]

where \( \zeta(u), \mu(u) \) and \( \eta(u) \) are differentiable functions for all \( u \in \mathbb{R} \setminus \{0\} \). If \( \gamma(u) \) lightlike curve, \( < \gamma', \gamma' > = 0 \), and \( \eta = \int \sqrt{\zeta'^2 + \mu'^2}du \). A rotational surface in \( \mathbb{L}^3 \) with the timelike axis which is spanned by the vector (0,0,1) is

\[
R(u,v) = T(v)\gamma(u).
\]

A lightlike profile curve in \( \mathbb{L}^3 \) is given by

\[
(2.1) \quad \gamma(u) = (u^2, u, \int \sqrt{4u^2 + 1}du),
\]

where \( \int \sqrt{4u^2 + 1}du = 2^{-1}u\sqrt{4u^2 + 1} + 2^{-2}\sinh^{-1}(2u) + c \quad (c = \text{const.}) \). We use the lightlike profile curve \( \gamma \) in \((2.1)\) in the next sections.

### 3. Timelike Rotational Surface of (T,L)-type

We give a rotational surface by types of axis and profile curve, and we write it as (axis’s type, profile curve’s type)—type; for example, (\( T, L \) — type) mean that the surface has a timelike axis and a lightlike profile curve. We give rotational surfaces with lightlike profile curve that are used to obtain the main theorem in this paper. If the profile curve \( \gamma \) is a lightlike curve, then the rotational surface is a timelike surface with timelike axis and it has (\( T, L \) — type).

When the axis \( \ell \) is timelike, there is a Lorentz transformation by which the axis \( \ell \) is transformed to the (0,0,1) axis of \( \mathbb{L}^3 \). If the profile curve is \( \gamma(u) = (u^2, u, \eta(u)) \), then timelike rotational surface can be written as

\[
(3.1) \quad R(u, v) = \begin{pmatrix}
u^2\cos(v) - u\sin(v) \\
u^2\sin(v) + u\cos(v) \\
\eta(u)
\end{pmatrix}.
\]
Proposition 1. A timelike rotational surface with lightlike profile curve of \((T,L)\)-type (see Fig. 1) is as follows

\[
R(u,v) = \begin{pmatrix}
  u^2 \cos(v) - u \sin(v) \\
  u^2 \sin(v) + u \cos(v) \\
  \frac{1}{2}u \sqrt{4u^2 + 1} + \frac{1}{4} \sinh^{-1}(2u)
\end{pmatrix}
\]

in the Lorentz-Minkowski 3-space, where \(u, v \in \mathbb{R} \setminus \{0\} \).

Proposition 2. If an \((T,L)\)-type a timelike rotational surface with lightlike profile curve is as above, then its Gauss map is

\[
e = \frac{1}{\sqrt{\det I}} \begin{pmatrix}
  -u(\sin(v) + u \cos(v)) \eta' \\
  -u(\cos(v) + u \sin(v)) \eta' \\
  -2u^3 + u
\end{pmatrix}
\]

in the Lorentz-Minkowski 3-space, where \(\det I = -u^4\), \(\eta = \eta(u) = \frac{u \sqrt{4u^2 + 1}}{2} + \frac{\sinh^{-1}(2u)}{4}\), \(u, v \in \mathbb{R} \setminus \{0\}\).

Figure 1 Timelike rotational surface with lightlike profile curve of \((T,L)\)-type

4. The Third Laplace-Beltrami

In this section, we study on the third Laplace-Beltrami operator of the timelike rotational surface with lightlike profile curve of \((T,L)\)-type. We assume that the axis \(\ell = (0,0,1)\) is a timelike vector, profile curve \(\gamma(u) = (u^2, u, \eta(u))\) is a lightlike curve, and \(\eta(u) = 2^{-1}u \sqrt{4u^2 + 1} + 2^{-2} \sinh^{-1}(2u), u \in \mathbb{R} \setminus \{0\}\).

Now, let \(x = x(u^1, u^2)\) be a surface of three dimensional Lorentz-Minkowski space defined in domain \(D\). The same for the functions \(\phi, \psi\). Let \(n = n(u^1, u^2)\) be the normal vector of the surface. We write

\[
g_{ij} = \langle x_i, x_j \rangle, \quad b_{ij} = \langle x_{ij}, n \rangle, \quad e_{ij} = \langle n_i, n_j \rangle.
\]

The equations of Weingarten are

\[
x_i = b_{ij} e^j_r n_r
\]

\[
= -g_{ij} b^j_r n_r,
\]
\[ n_i = -e_{ij}b^{jr}x_r, \]

\[ = -b_{ij}g^{jr}x_r, \]

where \( x_i = \frac{\partial x}{\partial u} \). Then the first parameter Beltrami is defined

\[ \text{grad}^{III}(\phi, \psi) = e_{ik}\psi_k. \]

Using following expressions

\[ \text{grad}^{III}(\phi) = \text{grad}^{III}(\phi, \phi) = e_{ik}\phi_k, \]

\[ \text{grad}^{III}\phi = \text{grad}^{III}(\phi, n) = e_{ik}\phi_n, \]

the second parameter Beltrami is defined

\[ \Delta^{III}\phi = -e_{ik}\text{grad}^{III}\phi_i. \]

Using the last relation we get the expression the third Laplace-Beltrami operator of the function \( \phi \). So, we have the third fundamental form (see [4] for details) as follow

\[ \Delta^{III} = -\frac{\sqrt{|\det I|}}{\det II} \left[ \frac{\partial}{\partial u} \left( \frac{Z\phi_u - Y\phi_v}{\sqrt{|\det I|} \det II} \right) - \frac{\partial}{\partial v} \left( \frac{Y\phi_u - X\phi_v}{\sqrt{|\det I|} \det II} \right) \right], \]

where the coefficients of the first (resp., second, and third) fundamental form of the function \( \phi \) is \( E,F,G \) (resp., \( L,M,N \), and \( X,Y,Z \)), \( \det I = EG - F^2 \), \( \det II = LN - M^2 \), \( X = EM^2 - 2FLM + GL^2 \), \( Y = EMN - FLN + GLM - FM^2 \), \( Z = GM^2 - 2FMN + EN^2 \).

Next, we find the third Laplace-Beltrami operator of the timelike rotational surface.

**Theorem 1.** The third Laplace-Beltrami operator of the timelike rotational surface with lightlike profile curve of \((T,L)\)-type (in (3.1)) is

\[ \Delta^{III}R = (\Delta^{III}R_1, \Delta^{III}R_2, \Delta^{III}R_3) \]

in the Lorentz-Minkowski 3-space, such that

\[ \Delta^{III}R_1 = \frac{u^4}{\Phi} \left[ \left( 224u^9 + 760u^7 + 468u^5 + 4u^3 + 6u \right) \sin(v) \right. \]

\[ + \left( 64u^{10} + 80u^8 + 100u^6 - 94u^4 + 188u^2 \right) \cos(v) \]

\[ + \left( -96u^9 - 696u^7 - 220u^5 + 148u^3 + 120u \right) \sin(3v) \]

\[ + \left( 160u^8 - 572u^6 - 710u^4 + 160u^2 - 12 \right) \cos(3v) \]

\[ + \left( 344u^7 - 188u^5 - 556u^3 + 48u \right) \sin(5v) \]

\[ + \left( 16u^8 - 452u^6 - 214u^4 - 432u^2 + 12 \right) \cos(5v) \]

\[ + \left( -56u^7 + 48u^5 - 4u^3 - 75u \right) \sin(7v) \]

\[ + \left( -44u^6 + 64u^4 + 90u^2 + 6 \right) \cos(7v) \]

\[ + \left( 12u^5 - 8u^3 - 9u \right) \sin(9v) \]

\[ + \left( -16u^6 + 18u^4 - 6u^2 - 6 \right) \cos(9v) \],
\[ \Delta^{III} R_2 = -\frac{2u^4}{\Phi} \left[ (64u^{10} - 272u^8 - 1540u^6 - 182u^4 - 264u^2) \sin(v) \\
+ (160u^9 - 8u^7 - 876u^5 - 332u^3 - 6u) \cos(v) \\
+ (-128u^8 + 692u^6 - 318u^4 + 692u^2 + 12) \sin(3v) \\
+ (96u^9 - 232u^7 + 1148u^5 - 164u^3 + 120u) \cos(3v) \\
+ (16u^8 - 564u^6 - 338u^4 + 60u^2 + 12) \sin(5v) \\
+ (120u^7 - 572u^5 + 196u^3 - 48u) \cos(5v) + (148u^6 - 148u^4 \\
- 46u^2 - 6) \sin(7v) + (56u^7 + 168u^5 - 28u^3 - 75u) \cos(7v) \\
+ (-16u^6 + 18u^4 - 6u^2 - 6) \sin(9v) + (12u^5 + 8u^3 \\
+ 9u) \cos(9v), \right] \\
\]

\[ \Delta^{III} R_3 = -\frac{u^4}{\Theta} \left\{ [2^{-4}u(32u^6 + 83u^4 + 65u^2 + 36)] \sin(2v) \\
- 2^{-4}u^2(32u^6 + 100u^4 + 51u^2 + 22) \cos(2v) \\
- 2^{-4}u(24u^6 + 78u^4 + 43u^2 + 12) \sin(4v) \\
+ 2^{-4}(38u^6 - 32u^4 - 25u^2 + 8u^8 - 9) \cos(4v) \\
+ 2^{-4}(4u^6 + 27u^4 + 5u^2 - 12) \sin(6v) \\
- 2^{-4}u^2(16u^4 - 27u^2 - 22) \cos(6v) \\
- 2^{-5}u(12u^4 + 5u^2 - 12) \sin(8v) \\
+ 2^{-6}(16u^6 - 24u^4 - 21u^2 + 9) \cos(8v) \\
+ 2^{-6}(80u^8 + 256u^6 + 224u^4 + 121u^2 + 27) |\eta|^2 \\
+ 2^{-4}u^2(32u^6 - 24u^4 + 109u^2 - 87) \sin(2v) \\
+ 2^{-2}u^3(4u^4 - 3u^2 + 24) \cos(2v) \\
+ 2^{-4}(104u^6 - 58u^4 + 93u^2 - 6) \sin(4v) \\
- 2^{-3}u(16u^6 - 50u^4 + 14u^2 - 9) \cos(4v) \\
- 2^{-4}u^2(20u^4 - 69u^2 + 19) \sin(6v) \\
- 2^{-2}u^3(19u^2 - 8) \cos(6v) \\
- 2^{-5}(20u^4 - 31u^2 - 6) \sin(8v) \\
+ 2^{-5}u(16u^4 - 32u^2 + 15) \cos(8v) \\
+ 2^{-5}u(64u^6 + 112u^4 - 8u^2 - 51) |\eta|^2 \right\} \]

in Lorentz-Minkowski 3-space, where

\[ \Phi = |\eta|^2 [u \sin(4v) - (2u^3 + 4u) \sin(2v) + (-4u^2 + 3) \cos^2(2v) \\
+ 2u^2 \cos(2v) - 2u^4 - u^2 - 3] \cdot [64u^7 + 192u^5 + 176u^3 \\
+ 144u) \sin(2v) + (64u^8 - 144u^6 - 56u^2) \cos(2v) \\
- (64u^5 + 112u^3 + 24u) \sin(4v) + (48u^6 - 88u^2 - 36) \cos(4v) \\
+ (32u^5 - 56u^3 - 48u) \sin(6v) + (-16u^4 + 56u^2) \cos(6v) \\
+ (-16u^3 + 12u) \sin(8v) + (16u^4 - 28u^2 + 9) \cos(8v) \\
+ 32u^8 + 112u^6 + 216u^4 + 116u^2 + 27], \]
\( \Theta = \eta^2[3 \cdot 2^{-5} u (32 u^{10} + 136 u^8 + 312 u^6 - 424 u^4 - 249 u^2 - 90) \sin(2v) \\
+ 2^{-5} u^2 (32 u^8 + 152 u^6 + 256 u^4 + 132 u^2 + 33) \cos(2v) \\
- 3 \cdot 2^{-6} u (192 u^8 + 672 u^6 + 896 u^4 + 444 u^2 + 45) \sin(4v) \\
+ 3 \cdot 2^{-4} (8u^{10} + 8u^8 - 12u^6 - 81u^4 - 60) \cos(4v) \\
+ 2^{-6} u (176 u^8 + 528 u^6 + 300 u^4 - 203 u^2 - 270) \sin(6v) \\
- 2^{-6} u^2 (48 u^6 - 272 u^4 - 648 u^2 - 297) \cos(6v) \\
- 3 \cdot 2^{-5} u (28 u^6 + 20 u^4 - 46 u^2 - 9) \sin(8v) \\
+ 3 \cdot 2^{-7} (32 u^8 - 120 u^6 - 204 u^4 + 66 u^2 + 27) \cos(8v) \\
+ 3 \cdot 2^{-6} u (16 u^6 + 20 u^4 - 59 u^2 + 18) \sin(10v) \\
+ 9 \cdot 2^{-6} u^2 (16 u^2 + 11) \cos(10v) - 2^{-6} u (48 u^4 - 76 u^2 + 27) \sin(12v) \\
+ 2^{-8} (64 u^6 - 192 u^4 + 144 u^2 - 27) \cos(12v) \\
+ 2^{-7} (128 u^{12} + 768 u^{10} + 2496 u^8 + 3576 u^6 + 2652 u^4 \\
+ 1170 u^2 + 135)], \\
\eta(u) = u \sqrt{\frac{u^2 + 1}{2}} + \frac{\sinh^{-1}(2u)}{4} \text{ and } u, v \in \mathbb{R} \setminus \{0\}. \\
\eta'(u) = u \sqrt{\frac{u^2 + 1}{2}} + \frac{\sinh^{-1}(2u)}{4} \text{ and } u, v \in \mathbb{R} \setminus \{0\}. \\
\text{Since } \det I = -u^4 < 0, R(u,v) \text{ is a timelike surface. Therefore, the coefficients of the third fundamental form are} \\
X = 2u \eta^2[4u^2 \sin(2v) + 4u \cos(2v) + 2 \sin(4v) \\
- u \cos(4v) + 2u^3 + u], \\
Y = 2^{-1} \eta^2[(-12u^3 + 8u) \sin(2v) + (8u^4 - 4u^2) \cos(2v) \\
(4u^3 - 2u) \sin(4v) + (6u^2 + 3) \cos(4v) - 8u^4 - 3], \\
\text{and} \\
Z = 2^{-1} \eta^2[(-8u^3 + 8u) \sin 2v + (16u^4 + 4u^2) \cos 2v \\
+ (4u^3 + 6u) \sin 4v + (-4u^4 + u^2 + 3) \cos 4v \\
- 10u^4 - 3u^2 - 3]. \\
\text{Hence, we get} \\
\Delta^{I\!I\!I} R = - \frac{1}{u^2 \det I} \left( \frac{\partial \Psi}{\partial u} - \frac{\partial \Omega}{\partial v} \right),
where the matrices $\Psi$ and $\Omega$ are
\[
\Psi := \frac{(GM^2 - 2FNM)R_u - (-FLN + GLM - FM^2) R_v}{u^2 \det II},
\]
\[
\Omega := \frac{(-FLN + GLM - FM^2) R_u - (-2FLM + GL^2) R_v}{u^2 \det II},
\]
and
\[
\det II = \eta' \frac{u}{u^2} \left[ (-2u^3 - 4u) \sin(2v) + 2u^2 \cos(2v) + u \sin(4v) \right.
\]
\[
+ \left. (-2u^2 + 3/2) \cos(4v) - 2u^4 - 3u^2 - 3/2 \right].
\]

After some calculations we get the components of the third Laplace-Beltrami of the (T-L)-type timelike rotational surface $\Delta^{III} R_1, \Delta^{III} R_2, \Delta^{III} R_3$.

**Corollary 4** The mean curvature and the Gaussian curvature of the timelike rotational surface with lightlike profile curve of (T,L)—type in (3.1) is, respectively, as follow
\[
H = -\eta' \frac{u}{u^4} \left( u^3 \sin(2v) + 2u^2 \cos(2v) + u^4 \right),
\]
and
\[
K = \eta' \frac{u}{u^6} \left[ - (2u^3 + 4u) \sin(2v) + 2u^2 \cos(2v) 
\right.
\]
\[
+ u \sin(4v) + (-2u^2 + 3/2) \cos(4v)
\]
\[
-2u^4 - 3u^2 - 3/2 \right].
\]

When the rotational surface is minimal, i.e.
\[
-\frac{\sqrt{4u^2 + 1}}{u^4} \left( u^3 \sin(2v) + 2u^2 \cos(2v) + u^4 \right) = 0,
\]
then the $u$ solutions are
\[
u_{1,2} = \mp 2^{-1} i, \quad u_{3,4} = \mp \sqrt{2 \cos(2v) + 2^{-2} \sin(2v) - 2^{-1} \sin(2v)},
\]
where $u \neq 0$.

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