REMARKS ON 1-MOTIVIC SHEAVES

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Abstract. We generalize the construction of the category of 1-motives with torsion $^tM_1$ in [4] as well as the construction of the category of 1-motivic sheaves $Shv_1$ in [3] to perfect fields (without inverting the exponential characteristic). We extend a result in [3] showing that $^tM_1$ and $Shv_1$ have equivalent bounded derived categories. Over a field of characteristic zero, the previous constructions work also for Laumon 1-motives, i.e., allowing additive factors and formal $k$-groups.

1. Introduction

Let $k$ be a field of characteristic 0. In [3] the authors introduce the category of 1-motivic sheaves $Shv_1$ and show that $D^b(Shv_1)$ and the bounded derived category of 1-motives with torsion $D^b( ^tM_1 )$ are both equivalent to the thick subcategory of Voevodsky’s triangulated category of motives $DM^{eff}_{gm}(k)$ generated by motives of smooth curves. When $k$ is a perfect field of positive characteristic $p$, the classical definition of $^tM_1$ in [4] doesn’t work well, for example it does not provide an abelian category. One possibility is then to invert $p$ as done in [3] getting a $\mathbb{Z}[1/p]$-linear abelian category of 1-motives with torsion. Also the definition of 1-motivic sheaves can be extended to the characteristic $p$ context paying attention to invert $p$-multiplications and a comparison result for bounded derived categories that generalizes the one above still holds (cf. [3], 3.9). As the authors explain, if one is interested in comparison results with Voevodsky’s category $DM$, one can not avoid to invert the exponential characteristic.

In the following sections, we show that an integral definition of $^tM_1$ is possible over any perfect field, i.e., without inverting the exponential characteristic, if we allow finite connected $k$-group schemes in the component of degree $-1$ of 1-motives. We get then an abelian category $^tM_1^a$ that contains the category of Deligne’s 1-motives as a full exact subcategory. Also the definition of 1-motivic sheaves works “integrally” over any perfect field passing to the fppf topology. Both constructions are equivalent to the ones in [3] over a field of characteristic 0. Furthermore we show that $D^b(Shv_1^{fppf})$ and $D^b( ^tM_1^a )$ are equivalent, i.e., we get an integral version of [3], 3.9.2, without passing through Voevodsky’s category $DM$.

When working over a field $k$ of characteristic 0, it is natural to ask if the previous definitions and proofs extend to Laumon 1-motives, i.e., to those 1-motives with additive factors in degree 0 and formal $k$-groups in degree $-1$ (cf. [11]). The abelian category of (generalized) 1-motives with torsion $^tM_1^a$ that extends
the category of Laumon’s 1-motives has already been introduced in [2]. The aim there, was to produce a new one-dimensional “sharp de Rham cohomology” of algebraic varieties following the construction of classical one-dimensional de Rham cohomology in [8]. In this paper we define the category of generalized 1-motivic sheaves $\text{Shv}_{1}^{a}$; this category contains $\text{Shv}_{1}^{\text{fppf}}$ as an abelian subcategory but also new objects such as the quotient of a smooth commutative $k$-group $G$ by its formal completion at the origin $\hat{G}$ (cf. [3, 2.3]). We show then that $D^{b}(t\mathcal{M}^{\text{et}}_{1})$ is equivalent to $D^{b}(\text{Shv}_{1}^{s})$. We hope that 1-motivic sheaves can help to develop a theory of generalized 1-motives over fields of positive characteristic.

2. The derived category of 1-motives

Notations: Let $k$ be any perfect field. We say that a $k$-group scheme is discrete if it is finitely generated locally constant for the étale topology (cf. [3], 1.1.1). Let $\mathcal{C}E$ be the category of commutative $k$-group schemes extension of a discrete group scheme by a finite commutative connected group scheme. (Recall that such connected group schemes are flat and that the extension is automatically split.) Denote by $\mathcal{M}_{1}$ the category of Deligne 1-motives.

2.1. The category of 1-motives with torsion.

2.1.1. Definition. An effective 1-motive with torsion is a complex $M = [u: L \to G]$ of $k$-group schemes where: $L$ is an object in $\mathcal{C}E$ and $G$ is semi-abelian. An effective morphism $M \to M'$ is a map of complexes $(f,g)$, with $f: L \to L'$, $g: G \to G'$ morphisms of group schemes. $M$ is said to be étale if $L$ is étale.

Denote by $t\mathcal{M}^{\text{eff,fl}}_{1}$ the category of effective 1-motives and by $t\mathcal{M}^{\text{eff,ét}}_{1}$ the full subcategory of étale ones. The category $\mathcal{M}_{1}$ of Deligne’s 1-motives is the full subcategory of $t\mathcal{M}^{\text{eff,ét}}_{1}$ consisting of those $M$ with $L$ torsion-free.

2.1.2. Definition. Let $\Sigma$ be the class of quasi-isomorphisms of effective 1-motives with torsion, i.e., the class of effective maps $(f,g): M \to M'$ where $g$ is an isogeny, $f$ is faithfully flat and $\text{Ker}(f) = \text{Ker}(g)$ is a finite group scheme. Define then the category of 1-motives with torsion $t\mathcal{M}^{\text{fl}}_{1}$ as the localization of $t\mathcal{M}^{\text{eff,ét}}_{1}$ at $\Sigma$. Similarly we get the category of étale 1-motives with torsion $t\mathcal{M}^{\text{ét}}_{1}$.

The category $t\mathcal{M}^{\text{ét}}_{1}$ was firstly introduced in [4] (for $k$ of characteristic 0 and denoted as $\mathcal{M}_{1}$) and then in [3] (over any perfect field and denoted as $t\mathcal{M}_{1}$). It was proved to be equivalent to the category of Mixed Hodge Structures of level $\leq 1$ for $k = \mathbb{C}$ (cf. [4], 1.5). For $k$ of characteristic 0, one has that

$$t\mathcal{M}_{1} = t\mathcal{M}^{\text{ét}}_{1} = t\mathcal{M}^{\text{fl}}_{1}$$

is an abelian category (cf. [4], [3]).

Over a field of positive characteristic $p$

$$t\mathcal{M}_{1} = t\mathcal{M}^{\text{ét}}_{1}$$

is a full subcategory of our category $t\mathcal{M}^{\text{fl}}_{1}$ that becomes abelian up to inverting $p$-multiplications (cf. [3], C.5.3).
We will show below that $t\mathcal{M}_1^{\text{fl}}$ is indeed an abelian category.

Before proving this fact, observe that in the characteristic zero case, starting with an effective 1-motive $M = [L \to G]$ and an isogeny $g: G' \to G$, by pull-back one always gets a q.i. $(f, g) : [L' \to G'] \to M$. Over fields of positive characteristic this is not always the case if $L$ is forced to be a discrete group because there are isogenies with connected kernel. Hence one can generalize the construction of the category of 1-motives with torsion in [4] either inverting $p$-multiplications as done in [3] or accepting to work with non étale finite group schemes as we do.

To show that $t\mathcal{M}_1^{\text{fl}}$ is an abelian category, we will follow the analogous proof for $t\mathcal{M}_1[1/p]$ in [3], Appendix C.

2.1.3. Lemma. Morphisms in $\Sigma$ are simplifiable on the left and on the right.

Proof. (cf. [3], C.2.3.) Let $(f, g) \in \Sigma$. As $f : L \to L'$, $g : G \to G'$ are epimorphisms, they are simplifiable on the right. Suppose given a $(f', g')$ with $ff' = 0$, $gg' = 0$. As $g$ is an isogeny, say of degree $n$, the $n$ multiplication on $G$ factors through $g$ and $ng' = 0$. Hence $g' = 0$; furthermore $f' = 0$ because $f'$, $g'$ factor both through $\text{Ker}(g) = \text{Ker}(f)$.

2.1.4. Lemma. $\Sigma$ admits the calculus of right fractions.

Proof. (cf. [3], C.2.4.) Let $(f', g') : M'' \to M'$ be a q.i. and $(f, g) : M \to M'$ be an effective map. Define $\tilde{G}$ as the reduced subgroup of the identity component of $G \times_{G'} G''$. It is a semi-abelian group scheme isogenous to $G$. Define then $\tilde{M} := [L \to \tilde{G}]$ via pull-back. This result, together with the previous Lemma, says that $\Sigma$ is a left multiplicative system.

Also [3], C.2.6 in loc.cit. works the same. Moreover:

2.1.5. Proposition. The categories $t\mathcal{M}_1^{\text{eff, fl}}, \mathcal{M}_1$ have all finite limits and colimits. The canonical functor $t\mathcal{M}_1^{\text{eff, fl}} \to t\mathcal{M}_1^{\text{fl}}$ is left exact and faithful.

Proof. (cf. [3], C.1.3.) For the definition of the kernel of an effective morphism $\varphi = (f, g)$ take $\text{Ker}(\varphi) = [\text{Ker}^0(f) \to \text{Ker}^0(g)]$ where $\text{Ker}^0(g)$ is the reduced subgroup of the identity component of the kernel (as group schemes) of $g$ and $\text{Ker}^0(f)$ is the pull-back of $\text{Ker}^0(g)$ along $\text{Ker}(f) \to \text{Ker}(g)$. The cokernel of $\varphi$ is the cokernel as group schemes in each degree. For the last statement, see [3], C.5.1.

2.1.6. Definition. An effective morphism $(f, g) : M \to M'$ is strict if $g$ has smooth connected kernel, i.e., the kernel of $g$ is still semi-abelian.

2.1.7. Proposition. Any effective morphism $\varphi : M \to M'$ factors as $\sigma \tilde{\varphi} = \varphi$ with $\sigma$ a quasi-isomorphism and $\tilde{\varphi}$ strict.

Proof. The proof in [3], 1.3, works the same.
2.1.8. Example. Let $n$ be a positive integer and consider the $n$-multiplication
$n: \mathbb{G}_m \to \mathbb{G}_m$. It factors as
\[ \mathbb{G}_m \to [\mu_n \to \mathbb{G}_m] \to \mathbb{G}_m \]
where the first map is a strict morphism and the second one is a quasi-isomorphism. Observe that the 1-motive in the middle is not a 1-motive with torsion in the sense of [4] if $n$ is not invertible in $k$.

We can generalize results [3], C.5.3 and [4], 1.3 getting:

2.1.9. Theorem. i) $\mathcal{M}_1^{fl}$ is an abelian category.

ii) Given a short exact sequence of 1-motives in $\mathcal{M}_1^{fl}$
\[ 0 \to M' \to M \to M'' \to 0 \]
this can be represented (up to isomorphisms) by a sequence of effective 1-motives that is exact as sequence of complexes.

iii) The natural functor $\mathcal{M}_1 \to \mathcal{M}_1^{fl}$ is fully faithful and makes $\mathcal{M}_1$ an exact sub-category of $\mathcal{M}_1^{fl}$.

Proof. For i) one observes that results C.4.2, C.4.4, C.5.2 in [3] still hold and hence the proofs in [3], C.5.3, [4], 1.3, work the same. Also ii) follows immediately as corollary to i). For iii) one follows the proof in [3], C.7.1. □

We will see that all informations needed to understand the bounded derived category of $\mathcal{M}_1^{fl}$ are all encoded in $\mathcal{M}_1^{fl}$ and indeed in the following subcategory:

2.1.10. Definition. Denote by $\mathcal{M}_1^*$ the full subcategory of $\mathcal{M}_1$ whose objects are $[u: L \to G]$ with $\ker u = 0$.

2.1.11. Remark. Observe that there are no non-trivial q.i. in $\mathcal{M}_1^*$. Moreover, given two quasi-isomorphic 1-motives with torsion $M_i = [u_i: L_i \to G_i]$, $i = 1, 2$, $\ker(u_1)$ is trivial if and only if $\ker(u_2)$ is trivial. In particular, $M = [u: L \to G]$ is quasi-isomorphic to a 1-motive in $\mathcal{M}_1^*$ if and only if $\ker(u) = 0$.

2.1.12. Lemma. $\mathcal{M}_1^*$ is a full subcategory of $\mathcal{M}_1^{fl}$ closed by kernels, closed by extensions and generating. Moreover, given a monomorphism $M \to M'$ in $\mathcal{M}_1^{fl}$ with $M'$ in $\mathcal{M}_1^*$ then also $M$ is in $\mathcal{M}_1^*$.

Proof. $\mathcal{M}_1^*$ is a full subcategory of $\mathcal{M}_1$ and the latter is a full subcategory of $\mathcal{M}_1^{fl}$. Given a morphism $\varphi: M \to M'$ in $\mathcal{M}_1^*$, this factors through a strict morphism $\tilde{\varphi}: M \to \tilde{M}$ with $\sigma: \tilde{M} \to M'$ a q.i. (cf. 2.1.7). As the kernel of $\tilde{\varphi}$ is the complex of kernels, it is an object of $\mathcal{M}_1^*$. Similarly one proves the last assertion.

Given a short exact sequence $M^* = 0 \to M' \to M \to M'' \to 0$ in $\mathcal{M}_1^{fl}$ with $M', M''$ in $\mathcal{M}_1^*$, by Theorem 2.1.9 ii) and Remark 2.1.11 we get $\ker(u) = 0$. Moreover $\mathcal{M}_1$ is closed by extensions and $M$ is (up to quasi-isomorphisms) a Deligne 1-motive $\tilde{M}$. Hence $M$ is q.i. to an object of $\mathcal{M}_1^*$.

To see that $\mathcal{M}_1^*$ is generating, we have to see that for any 1-motive with torsion $M = [u: L \to G]$ there exists an epimorphism $\varphi: M' \to M$ with $M'$ in $\mathcal{M}_1^*$. Observe that given a group scheme $L$ in $\mathcal{CE}$ there always exists an abelian
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variety $B$ and a monomorphism $\tilde{a}: L \to B$; define then $\tilde{M} := [\tilde{u}: L \to B \times G]$, $\tilde{u} = (a, u)$, and $\tilde{\varphi} = (id_L, p_G): \tilde{M} \to M$ with $p_G$ the usual projection map. It is clear that $\ker(\tilde{u}) = 0$ and $\tilde{\varphi}$ is a strict epimorphism and hence it remains an epimorphism in $^1M^h_1$ (cf. [3], C. 5.2). Define now, $M' = [L_{fr} \to B \times G/\tilde{u}(L_{tor})]$ with $L_{fr}, L_{tor}$ respectively the free and torsion subgroup of $L$. The 1-motive $M'$ is q.i. to $\tilde{M}$ and lies in $M^h_1$. □

2.2. The category of 1-motivic sheaves.

2.2.1. Definition. A sheaf $F$ for the fpf topology over $\text{Spec}(k)$ is 1-motivic if there exists a morphism of sheaves $b: G \to F$ with $G$ a semi-abelian scheme over $k$ and $\ker b, \coker b$ in $CE$. The map $b$ is said to be normalized if $\ker b$ is étale torsion-free.

In particular, we have an exact sequence

$$0 \to L \to G \xrightarrow{b} F \to E \to 0$$

with $L$ and $E$ in $CE$. Denote by $\text{Shv}^{\text{fpf}}_1$ the category of 1-motivic sheaves. For $k$ of characteristic 0, the category $\text{Shv}^{\text{fpf}}_1$ is equivalent to the category $\text{Shv}_1$ defined in [3] (cf. [3], 3.3.2). We will explain in Section A.1 the relation between $\text{Shv}_1$ in [3] and our $\text{Shv}^{\text{fpf}}_1$ over general perfect fields.

Denote by $\text{Shv}^{\text{fpf}}_0$ the full subcategory of $\text{Shv}^{\text{fpf}}_1$ consisting of those $F$ with $G = 0$, i.e., it is equivalent to $CE$. Proposition [3], 3.2.3, generalizes immediately.

2.2.3. Proposition. a) In Definition 2.2.1 we may choose $b$ normalized.

b) Given two 1-motivic sheaves $F, F'$, normalized morphisms $b: G \to F$, $b': G' \to F'$ and a morphism of sheaves $\varphi: F' \to F$ there exists a unique homomorphism of group schemes $\varphi_G: G' \to G$ above $\varphi$.

c) Given a 1-motivic sheaf $F$, a morphism $b: G \to F$ as above with $b$ normalized is uniquely (up to isomorphisms) determined by $F$.

d) $\text{Shv}^{\text{fpf}}_1$ and $\text{Shv}^{\text{fpf}}_0$ are exact abelian sub-categories of the category of fpf sheaves over $\text{Spec}(k)$.

2.2.4. Definition. Denote by $\text{Shv}^*_1$ the full subcategory of $\text{Shv}^{\text{fpf}}_1$ consisting of those $F$ such that there exists a $b: G \to F$ with $b$ epimorphism i.e., $E = 0$.

Observe that $\text{Hom}(F, L) = 0$ for $F$ in $\text{Shv}^*_1$ and $L$ in $CE$.

2.2.5. Lemma. For a 1-motivic sheaf $F$ there exist unique (up to isomorphisms) $F^*$ in $\text{Shv}^*_1$, $E$ in $\text{Shv}_0$ and an exact sequence

$$0 \to F^* \to F \to E \to 0.$$

Proof. Take simply $E = \text{Coker} b$ for any $b: G \to F$ as in 2.2.1. □

The "dual" of Lemma 2.1.12 holds:

For the torsion subgroup, it is well-known that any finite flat commutative group can be embedded in an abelian variety. For a discrete group $L$, one may work on a finite separable extension of $k$ on which $L$ becomes constant and then descend by restriction of scalars.
2.2.6. **Lemma.** Shv\(^*_1\) is a full subcategory of Shv\(_1\), closed by cokernels, closed by extensions and cogenerating. Moreover given an epimorphism \(\mathcal{F} \to \mathcal{F}'\) in Shv\(_1\) with \(\mathcal{F}\) in Shv\(^*_1\) then also \(\mathcal{F}'\) is an object of Shv\(^*_1\).

**Proof.** The only non-trivial fact is that Shv\(^*_1\) is cogenerating, i.e., that for any 1-motivic sheaf \(\mathcal{F}\) there exists a \(\mathcal{F}'\) in Shv\(^*_1\) and a monomorphism \(\varphi: \mathcal{F} \to \mathcal{F}'\).

By Proposition A.2.2, \(\mathcal{F}\) is \(\text{Coker}(F_1 \to F_0)\) with \(F_1\) in \(\mathcal{C}E\) and \(F_0\) extension of a group scheme in \(\mathcal{C}E\) by a semi-abelian group scheme. It is sufficient to prove that \(F_0\) embeds in a semi-abelian group \(G'\) and then take \(\mathcal{F}' = \text{Coker}(F_1 \to G')\).

Moreover, we can treat separately the case \(E = \text{Coker} b\) étale and connected.

Suppose \(E\) étale. Then the extension \(F_0\) of \(E\) by \(G\) splits over a suitable finite separable extension \(k'\) of \(k\). Let \(f: \text{Spec}(k') \to \text{Spec}(k)\) so that \(f^*F_0 = F_{0,k'}\) is isomorphic to \(G_{k'} \times E_{k'}\). We may assume that \(E\) is constant over \(k'\).

Embed \(E_{k'}\) into an abelian variety \(A_{k'}\) over \(k'\). Then we have a monomorphism \(E \to f_*A_{k'}\), where \(f_*A_{k'}\) is still an abelian variety, the Weil restriction of \(A_{k'}\).

Moreover, \(G = f_*f'^*G\) is a monomorphism. Hence we have a monomorphism \(F_0 \to f_*f^*F_0 = (f_*f'^*G) \times f_*A_{k'}\) where the latter is semi-abelian.

Suppose \(E\) finite connected of order \(n\). Let \(nF_0\) denotes the kernel of the \(n\)-multiplication on \(F_0\), and similar notation for \(nG\). Then \(nF_0\) is extension of \(E\) by \(nG\), hence finite and \(F_0/nF_0 \cong G\). Let then \(f: nF_0 \to B\) be an embedding into an abelian variety \(B\) and let \(f': F_0 \to C\) be the push-out of \(F_0\) along \(f\). As \(C\) is extension of \(G\) by \(B\) it is a semi-abelian group scheme and we are done. \(\square\)

2.3. **Equivalence on bounded derived categories.** Consider the following picture:

\[
\begin{array}{cccc}
\mathcal{M}_1 & \xrightarrow{d} & \mathcal{M}_1^\text{fl} & \xrightarrow{A} \text{Shv}^\text{fppf}_1 \\
\uparrow{\iota} & & \downarrow{\alpha} \\
\mathcal{M}_1^* & & & \\
\end{array}
\]

where \(A\) is the functor that maps an effective 1-motive with torsion \([u: L \to G]\) (with \(u\) in general neither a monomorphism nor normalized) to the 1-motivic sheaf \(\text{Coker} u\) that lies in \(\text{Shv}^*_1\). As \(A\) sends quasi-isomorphisms to isomorphisms, it extends to \(\mathcal{M}_1^\text{fl}\). The functor \(d\) is the usual embedding. We denote by \(\iota\) the inclusion functor \(\mathcal{M}_1^* \to \mathcal{M}_1\), by \(\alpha\) the obvious composition.

2.3.2. **Lemma.** The functor \(\alpha\) provides an equivalence between \(\mathcal{M}_1^*\) and the full subcategory \(\text{Shv}^*_1\) of \(\text{Shv}^\text{fppf}_1\).

**Proof.** It is immediate to prove that \(\alpha\) is fully faithful applying [2.2.3]. Furthermore, given a \(\mathcal{F}\) in \(\text{Shv}^*_1\) and a normalized \(b: G \to \mathcal{F}\), the Deligne 1-motive \([u: \text{Ker} b \to G]\) satisfies \(\text{Ker} u = 0\) and \(\text{Coker} u = \mathcal{F}\). \(\square\)

In order to see that \(A\) provides an equivalence of bounded derived categories, we check the following facts:
2.3.3. **Lemma.** Denote by $N^b(Shv^*_1)$ the full subcategory of $K^b(Shv^*_1)$ consisting of complexes that are acyclic as complexes of 1-motivic sheaves. The natural functor

$$K^b(Shv^*_1)/N^b(Shv^*_1) \to D^b(Shv^f_1)$$

is an equivalence of categories.

**Proof.** It follows from [10], 13.2.2 and Lemma 2.2.6.

2.3.4. **Lemma.** Denote by $N^b(M^*_1)$ the full subcategory of $K^b(M^*_1)$ consisting of complexes that are acyclic as complexes of 1-motives with torsion. The natural functor

$$K^b(M^*_1)/N(M^*_1) \to D^b(tM^f_1)$$

is an equivalence of categories.

**Proof.** By Lemma 2.1.12 the "dual" conditions required in [10], 13.2.2 ii) are satisfied. One checks that the "dual" statement of [10], 13.2.1 holds the same and hence one can apply [10], 10.2.7 ii).

Similar, with $M_1$ in place of $M^*_1$:  

2.3.5. **Lemma.** Denote by $N^b(M^*_1)$ the full subcategory of $K^b(M^*_1)$ consisting of complexes that are acyclic as complexes of 1-motives with torsion. The natural functor

$$D^b(M^*_1) := K^b(M^*_1)/N(M^*_1) \to D^b(tM^f_1)$$

is an equivalence of categories.

It remains to check that the functor $a$ preserves the exact structures.

2.3.6. **Lemma.** Let $M^*$ be a complex in $K^b(M^*_1)$. Then $M^* \in N(M^*_1)$ if and only if $a(M^*) \in N(Shv^*_1)$. In particular $a$ induces an equivalence of categories

$$K^b(tM^*_1)/N(tM^*_1) \to K^b(Shv^*_1)/N(Shv^*_1).$$

**Proof.** For the only if part it is sufficient to check the case of an acyclic complex of objects of $tM^f_1$, $M^* = 0 \to M^0 \to M^1 \to M^2 \to 0$ with $\text{Ker}(u_i) = 0$ (cf. Lemma 2.1.12/proof); indeed any short exact sequence is represented up to q.i. by an acyclic complex of effective 1-motives (cf. Thm. 2.1.9 ii) and Remark 2.1.11). The result follows from the usual ker-coker sequence. For the if part, suppose to have a complex $M^*$ of objects in $tM^f_1$ such that all $\text{Ker}(u_i) = 0$ and $\partial^i$: $Coker d^i \to \text{Ker} d^{i+1}$ are monomorphisms such that $a(\partial^i)$ become isomorphisms. We know that $\text{Ker} d^{i+1}$ is in $M^*_1$, because kernel of a morphism in $M^*_1$; hence also $\text{Coker} d^i$ is in $M^*_1$ and $a(\partial^i) = 0$ implies that $\partial^i$ is a q.i. in $M^*_1$, hence an isomorphism.

All the Lemmas above provide immediately the main result of this section that generalizes [3], 1.6.1, 3.9.2:

2.3.7. **Theorem.** We have canonical equivalences of categories

$$D^b(M^*_1) \cong D^b(tM^f_1) \cong D^b(Shv^f_1)$$

where $D^b(M^*_1)$ was defined in Lemma 2.3.6.
2.3.8. **Remark.** For $X$ a smooth projective $k$-variety the sheaf $\text{Pic}_{X/k}$ is clearly $1$-motivic being representable by a group scheme of finite type whose reduced identity component is an abelian variety. In [3], 3.4.1 the authors prove that the relative Picard functor is $1$-motivic for the étale topology as soon as $X$ is smooth over $k$, i.e. with the notations in (A.1.1) $\pi_*\text{Pic}_{X/k}$ is a sheaf in $\text{Shv}_1$. Unfortunately the proof in [3], 3.4.1 does not work in the fppf context and it is not clear at the present if a similar result holds.

3. **The derived category of Laumon’s $1$-motives.**

**Notations:** Let in this section $k$ be a field of characteristic $0$. Denote by $\mathcal{M}_1^a$ the category of Laumon 1-motives (cf. [11], [2]). The category $\mathcal{M}_1$ of Deligne 1-motives over $k$ is a full subcategory of $\mathcal{M}_1^a$. Denote by $\text{Ab}/k$ the category of fppf sheaves on the category of affine schemes over $k$, by $\text{For}/k$ the category of formal $k$-groups and by $\text{Alg}/k$ the category of (non necessarily connected) algebraic $k$-groups. Both $\text{For}/k$ and the category of algebraic $k$-groups may be viewed as thick subcategories of $\text{Ab}/k$. Given a formal $k$-group $L$, $L^0$ denotes its maximal connected subgroup and $L^\text{ét}: = L/L^0$.

In the following we generalize to Laumon 1-motives constructions and results of Section 2.

3.1. **The category of generalized $1$-motives with torsion.**

3.1.1. **Definition.** A (generalized) effective $1$-motive with torsion is a complex $[L \to G]$ of sheaves in $\text{Ab}/k$ where: $L$ is a formal $k$-group and $G$ is a connected algebraic $k$-group. An effective morphism is a map of complexes.

Denote by $\mathcal{M}_1^{\text{eff},a}$ the category of (generalized) effective 1-motives. Recall that an effective morphism $(f, g): M \to M'$ is a quasi-isomorphism if $g$ is an isogeny and $\text{Ker}(f) = \text{Ker}(g)$ is a finite group scheme.

3.1.2. **Definition.** The category of (generalized) $1$-motives with torsion $\mathcal{M}_1^a$ is the localization of $\mathcal{M}_1^{\text{eff},a}$ with respect to the multiplicative class of quasi-isomorphisms.

The category $\mathcal{M}_1^a$ was introduced in [2] showing that it is abelian and, when $k = \mathbb{C}$, equivalent to the category of Formal Hodge Structures of level $\leq 1$; see also [1] for the “torsion free” case. This result generalizes equivalences in [4] and [8] for Deligne’s 1-motives.

3.1.3. **Proposition.** The canonical functor $d: \mathcal{M}_1^a \to \mathcal{M}_1^{\text{eff},a}$ makes $\mathcal{M}_1^a$ an exact subcategory of $\mathcal{M}_1^{\text{eff},a}$ in the sense of Quillen.

Denote by $\mathcal{M}_1^{a*}$ the full subcategory of $\mathcal{M}_1^a$ whose objects are those $[u: L \to G]$ with $\text{Ker} u = 0$. Observe that there are no non-trivial quasi-isomorphisms in $\mathcal{M}_1^a$.

3.1.4. **Lemma.** $\mathcal{M}_1^{a*}$ (resp. $\mathcal{M}_1^a$) is a full subcategory of $\mathcal{M}_1^{\text{eff},a}$ closed by kernels, closed by extensions and generating. Moreover, given a monomorphism $M \to M'$ in $\mathcal{M}_1^a$ with $M'$ in $\mathcal{M}_1^{a*}$ (resp. $\mathcal{M}_1^a$) then also $M$ is in $\mathcal{M}_1^{a*}$ (resp. $\mathcal{M}_1^a$).

**Proof.** Almost word by word as the proof of [2.1.12].
3.2. Generalized 1-motivic sheaves.

3.2.1. Definition. A sheaf \( \mathcal{F} \) in \( \text{Ab}/k \) is 1-motivic if there exists a morphism of sheaves \( b: G \to \mathcal{F} \) with \( G \) a smooth connected algebraic \( k \)-group, \( \text{Ker}(b), \text{Coker}(b) \) formal \( k \)-groups. The morphism \( b \) is said to be normalized if \( \text{Ker}(b) \) is torsion free.

Hence a generalized 1-motivic sheaf \( \mathcal{F} \) fits in a sequence

\[
0 \to L \to F \to G \to \mathcal{F} \to E \to 0
\]

with \( L, E \) formal \( k \)-groups and \( G \) a smooth connected algebraic \( k \)-group.

Denote by \( \text{Shv}_1^a \) the category of (generalized) 1-motivic sheaves and by \( \text{Shv}_0^a \) the subcategory equivalent to \( \text{For}/k \), consisting of those \( \mathcal{F} \) with \( G = 0 \).

3.2.3. Examples. The category \( \text{Shv}_1^\text{fppf} \) is equivalent to the full subcategory of \( \text{Shv}_1^a \) consisting of those \( \mathcal{F} \) with \( L, E \) discrete. One has to be careful because the first category consists of sheaves for the fppf topology on the category of schemes over \( k \) while the second is a subcategory of \( \text{Ab}/k \), i.e., sheaves for the fppf topology on the category of affine schemes over \( k \). However, let \( \pi: (\text{Sch}/k)_{\text{fppf}} \to (\text{Aff}/k)_{\text{fppf}} \) be the canonical morphism of sites. By \[12\], III, 3.1 \( \pi_* \) is exact and \( \mathcal{F} \cong \pi_* \pi^* \mathcal{F} \). Moreover, proceeding as in \[A.1\] one finds that \( \pi^* X = X \) for commutative group schemes and that \( \pi^* \) is exact on 1-motivic sheaves viewed as cokernels of morphisms of commutative group schemes. (cf. \[A.2\], Appendix \[B\].)

- Let \( M = [L \to G] \) be a Deligne \( k \)-1-motive. It is shown in \[5\] that for \( M^2 = [u: L \to G^2] \) a universal \( \mathbb{G}_a \)-extension of \( M \) the sheaf \( \text{Coker}(u) \) is the sheaf of \( \mathbb{G}_1 \)-extensions of \( M \) by \( \mathbb{G}_m \). As \( G^2 \) is a connected algebraic \( k \)-group but, in general not a semi-abelian group scheme, and \( L \) is a discrete group, \( \text{Coker}(u) \) is an example of generalized 1-motivic sheaf that is not 1-motivic in the sense of \[2.2.1\].

- Let \( A, A' \) be dual abelian varieties over \( k \) and \( \hat{A}' \) the completion of \( A' \) at the origin. The Laumon 1-motive \( \hat{A}' \to A' \), for \( i \) the canonical embedding, is the Cartier dual of \( [0 \to A^2] \) where \( A^2 \) is the universal \( \mathbb{G}_a \)-extension of \( A \) (cf. \[11\]). The sheaf of \( \mathbb{G}_m \)-extensions of \( A^2 \) is isomorphic to \( \text{Coker}(i) \) and is a generalized 1-motivic sheaf.

- More generally, given a smooth connected algebraic \( k \)-group \( G \), let \( \hat{G} \) be its formal completion at the origin. Then \( [\iota: \hat{G} \to G] \) is a generalized 1-motive and \( \text{Coker}(\iota) \) is a 1-motivic sheaf.

Proposition \[2.2.3\] generalizes easily.

3.2.4. Proposition. a) In Definition \[3.2.1\] we may choose \( b \) normalized.

b) Given two 1-motivic sheaves \( \mathcal{F}, \mathcal{F}' \), normalized morphisms \( b: G \to \mathcal{F}, b': G' \to \mathcal{F}' \) and a map \( \varphi: \mathcal{F}' \to \mathcal{F} \) in \( \text{Ab}/k \), there exists a unique homomorphism of group schemes \( \varphi_0: G' \to G \) above \( \varphi \). In particular \( \text{Shv}_1 \) is a full subcategory of \( \text{Ab}/k \).

c) Given a 1-motivic sheaf \( \mathcal{F} \), a morphism \( b: G \to \mathcal{F} \) as above with \( b \) normalized is uniquely (up to isomorphisms) determined by \( \mathcal{F} \).

d) \( \text{Shv}_1^a \) and \( \text{Shv}_0^a \) are exact abelian subcategories of \( \text{Ab}/k \).
Proof. a) as in [3]. b) as in [3] replacing Lemma 3.1.5 in loc. cit. by [2], Lemma A. 4.5. For the uniqueness of $\varphi_G$ one uses [2], Lemma A. 4.4. c) follows from b). Point d) is proved with the same construction as in [3]. □

Denote by $\text{Shv}^{a*}_1$ the full subcategory of $\text{Shv}^a_1$ consisting of those objects with $\text{Coker}(b) = 0$.

3.2.5. Lemma. $\text{Shv}^{a*}_1$ is a full subcategory of $\text{Shv}^a_1$, closed by cokernels, closed by extensions and cogenerating. Moreover given an epimorphism $F \rightarrow F'$ in $\text{Shv}^a_1$ with $F$ in $\text{Shv}^{a*}_1$ then also $F'$ is an object of $\text{Shv}^{a*}_1$.

Proof. (cf. 2.2.6). Again the problem is to see that $\text{Shv}^{a*}_1$ is cogenerating, i.e., that for any 1-motivic sheaf $F$ there exists a $F'$ in $\text{Shv}^{a*}_1$ and a monomorphism $\varphi: F \rightarrow F'$. Consider the extension (3.2.2) and let $F^{a*} = G/L$. We can treat separately the connected and étale cases.

If $E$ is formal connected, from Propositions C.0.10 and C.0.12 it follows that $F = F^{a*} \times E$. Let $E \rightarrow \mathbb{G}^a_n$ be an embedding; then the induced map $F \rightarrow F^{a*} \times \mathbb{G}^a_n$ is a monomorphism.

If $E$ and $L$ are both discrete, $F$ comes from a sheaf in $\text{Shv}^{fppf}_1$ and the result follows from 2.2.6.

If $L$ is formal connected and $E$ is étale, by C.0.11 it holds $F = H/L$ for $H$ an extension of $E$ by $G$. Proceeding as in the proof of 2.2.6 (étale case) we can embed $H$ into a connected algebraic group over a finite field extension of $k$. Hence by restriction of scalars, we get a monomorphism $H \rightarrow H'$ with $H'$ connected algebraic $k$-group and $F'$ is the cokernel of $L \rightarrow H'$.

As in the classical case any generalized 1-motivic sheaf $F$ can be viewed as extension of a formal $k$-group $E$ by a 1-motivic sheaf $F^{a*}$ in $\text{Shv}^{a*}_1$.

3.3. Equivalence on bounded derived categories. We have the following picture that generalizes 2.3.1

\[
\begin{array}{ccc}
\mathcal{M}^{a*}_1 & \xrightarrow{d} & \mathcal{M}^a_1 & \xrightarrow{A} & \text{Shv}^a_1 \\
& \downarrow & & \searrow & \\
\mathcal{M}^{a*}_1 & & & & \\
\end{array}
\]

with $A$ that maps $[u: L \rightarrow G]$ to $\text{Coker} u$. Again we can show that:

3.3.2. Lemma. The functor $a$ provides an equivalence between $\mathcal{M}^{a*}_1$ and the full subcategory $\text{Shv}^{a*}_1$ of $\text{Shv}^a_1$.

Thanks to 3.2.5 and 3.1.4 results in 2.3.3 2.3.4 2.3.6 can be generalized and hence one gets:

3.3.3. Theorem. Let denote by $N(\mathcal{M}^a_1)$ the subcategory of $K^b(\mathcal{M}^a_1)$ consisting of complexes that are acyclic as complexes in $K^b(\mathcal{M}^a_1)$ and let $D^b(\mathcal{M}^a_1)$ be the localization $K^b(\mathcal{M}^a_1)/N(\mathcal{M}^a_1)$. We have the following equivalences of categories $D^b(\mathcal{M}^a_1) \cong D^b(\mathcal{M}^a_1) \cong D^b(\text{Shv}^a_1)$.
4. 1-MOTIVES WITH COTORSION AND CARTIER DUALITY

Cartier duality on the category of 1-motives does not extend to an anti-equivalence on the category 1-motives with torsion: it is necessary to introduce a new category, the category of 1-motives with cotorsion as done in [3], 1.8, in order to get a reasonable duality result. We show in this section that duality results in loc.cit. extend both to the category $tM_1^{\text{eff}, \text{fl}}$ and to $tM_1^{\text{mult}}$.

4.1. $k$ perfect. Let notations be as in Section 2.

4.1.1. Definition. Let $tM_1^{\text{eff}, \text{fl}}$ be the category of complexes $M = [u: L \to G]$ where $L$ is a torsion free, discrete group scheme over $k$ and $G$ is a commutative group scheme extension of an abelian scheme $A$ by a commutative $k$-group $Q$ that is product of a group scheme of multiplicative type and a finite group scheme $N$. $M$ is said to be multiplicative if $Q$ is of multiplicative type. The category of 1-motives with cotorsion $tM_1^{\text{fl}}$ is the localization of $tM_1^{\text{eff}, \text{fl}}$ at the class of quasi-isomorphisms. Similarly for the subcategory $tM_1^{\text{mult}}$ of 1-motives with cotorsion of multiplicative type.

Observe that we may suppose that $N$ in the above definition contains no group schemes of multiplicative type. Furthermore, a quasi-isomorphism $(f, g): M \to M'$ is such that $f, g$ have trivial kernel and $\text{Coker}(f) = \text{Coker}(g)$ is finite étale. The category of 1-motives with cotorsion in [3], denoted there by $tM_1$, is with our notations, the full subcategory $tM_1^{\text{mult}}$ of $tM_1^{\text{fl}}$.

4.1.2. Lemma. Cartier duality on $M_1$ extends to a contravariant additive functor $(\ )^*: tM_1^{\text{eff}, \text{fl}} \to tM_1^{\text{eff}, \text{fl}}$ which sends quasi-isomorphisms to quasi-isomorphisms.

Proof. (cf. [3], 1.8.3.) The effective 1-motive with cotorsion $M^* = [u': L' \to G']$ associated to the 1-motive with torsion $M = [u: L \to G]$ is defined as follows: let $M_A = [L \to A]$ be the 1-motive obtained via the composition of $u$ with the projection $G \to A$. Then

- $L'$ is the Cartier dual of the maximal torus in $G$;
- $G'$ is the commutative $k$-group that represents the fppf sheaf over $k$

$$\text{Ext}(M_A, \mathbb{G}_m): S \hookrightarrow \text{Ext}_S(M_A, \mathbb{G}_m)$$

- $u': \text{Hom}(T, \mathbb{G}_m) \to \text{Ext}(M_A, \mathbb{G}_m)$ is the push-out morphism for the sequence

$$0 \to T \to M \to M_A \to 0.$$ (4.1.3)

Observe that the sequence $0 \to A \to M_A \to L[1] \to 0$ provides a sequence of sheaves

$$0 \to \text{Hom}(L, \mathbb{G}_m) \to \text{Ext}(M_A, \mathbb{G}_m) \to \text{Ext}(A, \mathbb{G}_m) = A' \to 0$$

where the exactness on the right is due to [12], III, 4.17 and the local triviality of $\mathbb{G}_m$-torsors. As $L$ is product of its discrete part and a finite connected part, $G'$ satisfies the condition in Definition 4.1.1.
For the assertion on quasi-isomorphisms one can use the same proof as in [3]. □

4.1.4. Proposition. The functor $(\cdot)^*$ in Lemma 4.1.2 induces an anti-equivalence of abelian categories

$$(\cdot)^*: \mathcal{M}_1^\mu \to \mathcal{M}_1^\mu.$$  

Moreover, Cartier duality is an exact functor on $\mathcal{M}_1$ and hence it induces a triangulated self-duality on $\mathcal{D}^b(\mathcal{M}_1)$.

Proof. (cf. [3], 1.8.4.) Observe that that the proof in loc.cit. part a), works also without inverting the the exponential characteristic, i.e., it works for $\mathcal{M}_1 = \mathcal{M}_1^{\text{et}}$ and $\mathcal{M}_1 = \mathcal{M}_1^{\text{mult}}$. More generally it works for our categories $\mathcal{M}_1^\mu$ and $\mathcal{M}_1^\mu$. Also the other assertions can be proved with the same arguments. □

Hence we can generalize [3], 1.8.6:

4.1.5. Theorem. The natural functor $\mathcal{M}_1 \to \mathcal{M}_1^\mu$ is fully faithful and induces an equivalence of categories

$$\mathcal{D}^b(\mathcal{M}_1) \cong \mathcal{D}^b(\mathcal{M}_1^\mu).$$

Cartier duality exchanges $\mathcal{M}_1^\mu$ and $\mathcal{M}_1^\mu$ inside $\mathcal{D}^b(\mathcal{M}_1)$.

4.2. $k$ of characteristic 0. Let notations be as in Section 3. We can introduce also generalized 1-motives with cotorsion.

4.2.1. Definition. Let $\mathcal{M}_1^{\text{eff},a}$ be the category of complexes $M = [u: L \to G]$ where $L$ is a formal $k$-group with torsion free étale part and $G$ is a commutative group scheme extension of an abelian scheme $A$ by an affine $k$-group $Q$ product of a vector group by a group of multiplicative type. The category of (generalized) 1-motives with cotorsion $\mathcal{M}_1^a$ is the localization of $\mathcal{M}_1^{\text{eff},a}$ at the class of quasi-isomorphisms.

Repeating the arguments of the previous subsection, we can prove that

4.2.2. Proposition. i) Cartier duality on $\mathcal{M}_1^a$ induces an anti-equivalence of abelian categories

$$(\cdot)^*: \mathcal{M}_1^a \to \mathcal{M}_1^a,$$

that is exact. Moreover, Cartier duality is an exact functor on $\mathcal{M}_1^a$ and hence it induces a triangulated self-duality on $\mathcal{D}^b(\mathcal{M}_1^a)$.

ii) The natural functor $\mathcal{M}_1^a \to \mathcal{M}_1^a$ is fully faithful and induces an equivalence of categories

$$\mathcal{D}^b(\mathcal{M}_1^a) \cong \mathcal{D}^b(\mathcal{M}_1^a).$$

Cartier duality exchanges $\mathcal{M}_1^a$ and $\mathcal{M}_1^a$ inside $\mathcal{D}^b(\mathcal{M}_1^a)$.

One should be cautious with the construction of the Cartier dual of a generalized 1-motive with (co-)torsion. However, as $\text{Ext}(L, \mathbb{G}_m) = 0$ for $L$ a formal $k$-group ([2], A.4.6, [12], III, 4.17) one can proceed as in the classical case.


APPENDIX A. More results on 1-motivic sheaves.

We collect in this section results used in the proof of the equivalence of bounded derived categories in Section 2. Moreover we describe the relation between our definition of 1-motivic sheaves and the one in [3]. We provide also an alternative definition of 1-motivic sheaf as cokernel of a morphism of group schemes $F_1 \rightarrow F_0$ where $F_1$ is an object in $\mathcal{CE}$, i.e., product of a discrete group by a finite connected group scheme and $F_0$ is a commutative group scheme over $k$ extension of an object of $\mathcal{CE}$ by a semi-abelian group scheme. Following this idea, one can construct a category of presentations $S_1$ equivalent to $\text{Shv}^{\text{fppf}}$.

Notations: Let notations be as in Section 2 and denote by $p$ the exponential characteristic of $k$. Denote furthermore by $\text{Sm}/k$ the category of smooth separated $k$-schemes.

A.1. Comparison of topologies. Let $\text{Shv}_1'$ be the full subcategory of $\text{Shv}_1^{\text{fppf}}$ whose objects are those $\mathcal{F}$ as in (2.2.2) with $L, E$ discrete. It coincides with $\text{Shv}_1^{\text{fppf}}$ in characteristic 0.

Let $\text{Shv}_1^{\text{et}}$ be the category of étale sheaves on $(\text{Sm}/k)$ that fit in a sequence as (2.2.2) with $G$ semi-abelian, $L, E$ discrete. The definition of the category of 1-motivic sheaves $\text{Shv}_1$ in [3], is with our notations

$$\text{Shv}_1 := \text{Shv}_1^{\text{et}}[1/p].$$

Denote by $\text{Shv}_1^{\text{et}, *}$ the full subcategory of $\text{Shv}_1^{\text{et}}$ consisting of sheaves with $E = 0$. Let $\pi: (\text{Sch}/k)_{\text{fppf}} \rightarrow (\text{Sm}/k)_{\text{ét}}$ be the usual morphism of sites. We have the following picture

(A.1.1) \[
\begin{array}{c}
\text{Shv}_1' & \longrightarrow & \text{Shv}_1^{\text{fppf}} & \longrightarrow & \text{Sh}((\text{Sch}/k)_{\text{fppf}}) \\
\pi^* & & & & & \pi^*
\end{array}
\]

\[
\begin{array}{c}
\text{Shv}_1^{\text{et}, *} & \longrightarrow & \text{Shv}_1^{\text{et}} & \longrightarrow & \text{Sh}((\text{Sm}/k)_{\text{ét}}) \\
\pi^* & & & & \pi^*
\end{array}
\]

where $\pi^*$ restricted to $\text{Shv}_1^{\text{et}}$ or $\text{Shv}_1^{\text{et}, *}$ are exact equivalences of categories with quasi-inverse $\pi_*$. We spend some words on these facts (see also [3], 3.3.2) First of all observe that for $X$ a smooth $k$-scheme it holds $\pi^* X = X$ (proof as in [12], p. 69). Moreover, $\pi_*\pi^* X = X$. Indeed, let $U$ be a smooth $k$-scheme. Then $\Gamma(U, \pi_*\pi^* X) = \Gamma(U, \pi^* X)$ by definition of $\pi_*$; we have just seen that the last group equals $\Gamma(U, X)$ and this does not change when working with the étale or the flat topology. Observe now that if we prove that $R^i\pi_*\pi^* G = R^i\pi_* G = 0$ (and similar for $L$) we get that also $R^i\pi_*\pi^* (G/L) = R^i\pi_* (G/L) = 0$; hence $\pi_*$ and $\pi_*\pi^*$ are exact on 1-motivic sheaves and $\pi_*\pi^* \mathcal{F} \cong \mathcal{F}$ for any 1-motivic sheaf $\mathcal{F}$.

Consider then a $k$-group scheme $F$; it holds $R^i \pi_* F = 0$ if and only if for all $X$ smooth over $k$, $H^i(X_{\text{ét}}, F) = H^i(X_{\text{fppf}}, F)$. But this follows from [9], 11.7.

It remains to check that $\pi^*$ is exact on 1-motivic sheaves. Now the functor $\pi^*$ is right exact and then $\pi^* L = L, \pi^* G = G$ imply that it sends $0 \rightarrow L \rightarrow G \rightarrow \mathcal{F}^* \rightarrow 0$ into an exact sequence. Moreover it is exact on $\text{Shv}^{\text{et}, *}$.

Indeed given a
short exact sequence of 1-motivic sheaves
(A.1.2) \[ 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \]
in \( \text{Shv}^{\text{ét},*} \), let \( \mathcal{F}_i = \text{Coker}(v_i; L_i \to G_i) \) with \( v_i \) monomorphisms, \( L_i \) discrete group schemes, \( G_2, G_3 \) semi-abelian schemes and \( G_1 \) eventually extension of a discrete group by a semi-abelian group scheme, so that the exact sequence of \( \mathcal{F}_i \) induces exact sequences both on discrete groups \( L_i \) and on smooth \( k \)-groups \( G_i \).

As \( \pi^* \) preserves the exactness of these sequences of smooth group schemes as well as the exactness of the sequences \( L_i \to G_i \to \mathcal{F}_i \), it preserves the exactness of (A.1.2). To conclude, observe that by [3, 3.7.5], any 1-motivic sheaf in \( \text{Shv}^\text{ét}_1 \) is cokernel of a monomorphism \( F_0 \to F_1 \) where \( F_0 \) is a discrete group scheme containing \( L \) and \( F_0 \) is a smooth commutative group scheme with semi-abelian identity component \( G \). Hence, we have a diagram

(A.1.3) \[
\begin{array}{ccccccccc}
0 & \to & L & \to & G & \to & \mathcal{F}^* & \to & 0 \\
0 & \to & F_1 & \to & F_0 & \to & \mathcal{F} & \to & 0 \\
0 & \to & F_1/L & \to & F_0/G & \to & E & \to & 0
\end{array}
\]

where \( \pi^* \) is exact on the horizontal sequences and on the first two vertical sequences. Hence \( \pi^*\mathcal{F}^* \to \pi^*\mathcal{F} \) is a monomorphism. Using the fact that \( \pi^* \) is exact on \( \text{Shv}^{\text{ét},*}_1 \) and on discrete group schemes it is immediate to see that it is exact on \( \text{Shv}^{\text{ét}}_1 \). In particular \( \pi^*\pi_*\mathcal{F} \cong \mathcal{F} \).

A.1.4. Proposition. i) The category \( \text{Shv}^{\text{ét}}_1 \) is equivalent to the full subcategory \( \text{Shv}'_1 \) of \( \text{Shv}^{\text{fpf}}_1 \). The category \( \text{Shv}'_1 \) is cogenerating and for any epimorphism \( \mathcal{F}' \to \mathcal{F} \) with \( \mathcal{F}' \in \text{Shv}'_1 \) also \( \mathcal{F} \) is in \( \text{Shv}'_1 \). In particular, denote by \( N^b(\text{Shv}'_1) \) the bounded complexes of objects in \( \text{Shv}'_1 \) that are acyclic as complexes of 1-motivic sheaves. The natural functor

\[ K^b(\text{Shv}'_1)/N^b(\text{Shv}'_1) \to D^b(\text{Shv}^{\text{fpf}}_1) \]

is an equivalence of categories.

ii) If we endow \( \text{Shv}^{\text{ét}}_1 \) with the exact structure inherited from \( \text{Shv}^{\text{fpf}}_1 \), we have an equivalence of categories.

\[ K^b(\text{Shv}^{\text{ét}}_1)/N^b(\text{Shv}^{\text{ét}}_1) \to D^b(\text{Shv}^{\text{fpf}}_1) \cong D^b(M_1). \]

iii) \( D^b(\text{Shv}^{\text{fpf}}_1[1/p]) \) is equivalent to the thick subcategory \( d_{\leq 1}\text{DM}^{\text{eff}}_{gm,\text{ét}} \) of Voevodsky’s triangulated category of motives generated by motives of smooth curves.

Proof. For the first statement apply the same arguments as in 2.3.3. The second is then immediate. For the comparison result with Voevodsky’s category, observe first that the inclusion functor \( \text{Shv}'_1 \to \text{Shv}^{\text{fpf}}_1 \) provides an equivalence of categories \( \text{Shv}'_1[1/p] \to \text{Shv}^{\text{fpf}}_1[1/p] \) because any sheaf in \( \text{Shv}^{\text{fpf}}_1 \) coming from a finite
group scheme of order a power of $p$ becomes isomorphic to 0 in $\text{Shv}^{\text{fppf}}_1[1/p]$. Hence we can forget the connected component of $E$ in (2.2.2). Moreover, we have that $\text{Shv}^1_1[1/p]$ and $\text{Shv}^\text{et}_1[1/p]$ are equivalent and hence one concludes by \cite{3}, 3.9.2. □

A.2. Alternative definitions.

A.2.1. Lemma. Let $N$ be a finite connected commutative group scheme over $k$ and $L$ discrete. Then $\text{Ext}^2(N, L) = 0$

Proof. Let $n$ be the order of $N$. The homomorphism $\text{Ext}^1(N, L/nL) \to \text{Ext}^2(N, L)$ is epi and the first group is trivial over a perfect field. □

The following proposition provides an alternative definition of 1-motivic sheaves.

A.2.2. Proposition. An fppf sheaf $F$ on $k$ is 1-motivic if and only if

\begin{equation}
F = \text{Coker}(F_1 \xrightarrow{u} F_0) \tag{A.2.3}
\end{equation}

where $F_1$ is a discrete group scheme, $F_0$ is an extension of an object in $\mathcal{CE}$ by a semi-abelian group scheme $G$ and $u$ is a monomorphism. Denote by $L$ the pull-back of $F_1$ to $G$; we have then a diagram

\begin{equation}
\begin{array}{ccccccccc}
0 & \rightarrow & L & \rightarrow & G & \rightarrow & \mathcal{F}^* & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F_1 & \xrightarrow{u} & F_0 & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & F_1/L & \rightarrow & F_0/G & \rightarrow & E & \rightarrow & 0 \\
\end{array} \tag{A.2.4}
\end{equation}

For a 1-motivic sheaf $\mathcal{F}$ one can always find a presentation $F_1 \rightarrow F_0$ as above with $F_1$ étale torsion free. If $k$ is algebraically closed, there exists a diagram as above with $F_1/L = 0$.

Proof. The if part follows from diagram \[A.2.3\]. For the converse, suppose to have a normalized $b: G \rightarrow \mathcal{F}$ and consider the sequence

\[ \eta: 0 \rightarrow L \rightarrow G \rightarrow \mathcal{F} \rightarrow E \rightarrow 0. \]

If $\eta$ is isomorphic to the trivial one, $\eta$ is isomorphic to the push-out along $G \rightarrow G/L = \mathcal{F}^*$ of an extension $F_0$ of $E$ by $G$ and we get a diagram as above with $F_1/L = 0$. By Lemma \[A.2.1\] this is the case if $E$ is finite connected. Hence it remains to check only the case $E = E^\text{ ét}$ étale. Let $\mathcal{F}^\text{ ét}$ be the pull-back of $\mathcal{F}$ along $E^\text{ ét} \rightarrow E$. Now, Step 1 in the proof of \cite{3}, 3.7.5 shows that a presentation as above exists for $\mathcal{F}^\text{ ét}$. □

A.2.5. Remark. The Lemma above works the same for the categories $\text{Shv}^\text{ ét}_1$ and $\text{Shv}^\text{ ét}_1[1/p]$, hence an étale sheaf $\mathcal{F}$ on $(\text{Sm}/k)$ is 1-motivic in the sense of \cite{3} if and only if $\mathcal{F}$ fits in a diagram as above with $F_0$ a smooth commutative $k$-group and $F_1$ discrete. Indeed in \cite{3}, 3.7.5 it is showed that one can always find
such presentation with $F_0$ split extension of its component group by its identity component.

Moreover, in \[3\], 3.8.1 $\text{Shv}_1 = \text{Shv}^{\text{et}}_1[1/p]$ was showed to be equivalent to a category of presentation $S_1$ and this fact was used to extend the functor

$$\rho: \text{AbS} \rightarrow \text{HI}_{\text{et}}[1/p]$$

(notations as in loc.cit.) to a full embedding

$$\rho: \text{Shv}_1 \rightarrow \text{HI}_{\text{et}}[1/p].$$

Indeed, given a 1-motivic sheaf $\mathcal{F}$ (in $\text{Shv}_1^{\text{et}}$) one always has a presentation $F_1 \rightarrow F_0$ and one defines $\rho(\mathcal{F}) = \text{Coker}(\rho(F_1) \rightarrow \rho(F_0))$. As a sheaf, it is $\mathcal{F}$. Then \[3\] 3.8.2, says that the transfer structure does not depend on the presentation.

A.3. The category of presentations. Proposition \[A.2.2\] suggests how to construct a category of presentations for 1-motivic sheaves that plays the role of $S_1$ in \[3\].

A.3.1. Definition. Let $S_1^{\text{eff}}$ be the category of complexes of $k$-group schemes $F = [F_1 \rightarrow F_0]$ where i) $F_1$ is discrete, ii) $F_0$ is extension of an object in $\text{CE}$ by a semi-abelian group scheme $G$, iii) $u$ is a monomorphism. We call it the category of presentations. A presentation $u$ is normalized if $F_1$ is torsion free.

Observe the following facts:

A.3.2. Remark. • We have an embedding $\mathcal{M}_1^* \rightarrow S_1^{\text{eff}}$ that admits a right inverse on the subcategory of normalized presentations.

• We have seen in \[A.2.2\] that any 1-motivic sheaf $\mathcal{F}$ is $H_0(F_1)$ for a suitable presentation $F_1$ as above. The definition of $S_1^{\text{eff}}$ differs from the in \[3\], 3.7.1 but we will show that it plays the same role.

• For any presentation $F$ there exists a q.i. $F \rightarrow \bar{F}$ with $\bar{F}$ normalized: simply divide each term by $F_1_{\text{tor}}$.

A.3.3. Lemma. Given two presentations $F, F'$ of $\mathcal{F}$, we always find a third presentation $\tilde{F}$ and q.i. $F \rightarrow \tilde{F}, F' \rightarrow \tilde{F}$. More generally, given two presentations $F, F'$ and a morphism $\varphi: \mathcal{F} \rightarrow \mathcal{F}'$ we always find a third presentation $\tilde{F}$, a morphism $\psi: \mathcal{F} \rightarrow \tilde{F}$, and a q.i. $F' \rightarrow \tilde{F}$.

Proof. We may suppose both presentations normalized. Define $\tilde{F}_0$ as the push-out of $F_0$ along $G \cong G' \rightarrow F'_0$. Its identity component is $G$ and its component group is $F_0/G \times F'_0/G'$. It comes out with a canonical map $\tilde{F}_0 \rightarrow \mathcal{F}$. Take as $\tilde{F}_1$ its kernel. It is extension of $F_0/G$ by $F'_1$, hence discrete.

For the last statement construct a presentation $\tilde{F} = [\tilde{F}_1 \rightarrow \tilde{F}_0]$ of $\mathcal{F}$ as pull-back of $F'$ along $\varphi$. By the previous case, we know that there exist a third presentation $F''$ q.i. to $F$ and $\tilde{F}$. Let $\alpha: F'_1 \rightarrow F''_1$ be the induced map. Define $[\tilde{F}_1 = \hat{F}_1 \rightarrow \tilde{F}_0]$ as the push-out along $\alpha$ of $F'_0$. It is quasi-isomorphic to $F''_1$. Let $\psi$ be the composition of the canonical map $F''_1 \rightarrow \tilde{F}$ with the q.i. $F \rightarrow F''_1$. $\Box$

A.3.4. Lemma. Quasi-isomorphisms are simplifiable on the left and on the right. The set $\Sigma$ of quasi-isomorphisms admits a calculus of right and left fractions.
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Proof. Suppose given a q.i. \( f: F \to F'' \) and a morphism \( g: F'' \to F' \). If \( fg = 0 \), also the map on cokernels (i.e., on 1-motivic sheaves) is zero, hence \( F'' \to F' \cong F \) is trivial and \( g \) is homotopic to zero. Similarly for the right.

For the pull-back condition on the calculus of right fractions cf. [3] 3.7.4. The push-out condition for the calculus of left fractions is granted by Lemma A.3.3. □

A.3.5. **Definition.** Let \( \Sigma \) be the collection of quasi-isomorphisms of \( S_1^{\text{eff}} \) and \( S_1 \) the localization of the homotopy category of \( S_1^{\text{eff}} \) with respect to the image of \( \Sigma \).

Following the construction in [3], the functor \( F \to H_0(F) \) induces a functor \( h_0: S_1 \to \text{Shv}_{fppf}^1 \).

A.3.6. **Proposition** (cf. [3], 3.7.5). The functor \( h_0 \) is an equivalence of categories. In particular, \( S_1 \) is abelian.

Proof. \( h_0 \) is essentially surjective by Proposition A.2.2. To see that \( h_0 \) is faithful we may reduce to the case of an effective map \( f: F' \to F \) (cf. Lemma A.3.4). But \( h_0(f) = 0 \) implies that \( f \) is homotopic to 0. By Lemma A.3.3 \( h_0 \) is full. □

A.3.7. **Remark.** As in Remark A.2.5 we may restrict to work with the étale topology and discrete groups instead of \( CE \). Adapting in the obvious manner the definition of \( S_1 \) we get a category of presentations \( S_1^{\text{ét}} \) equivalent to \( \text{Shv}_1^{\text{ét}} \) and then \( S_1^{\text{ét}}[1/p] \) is equivalent to the category of presentations \( S_1 \) in [3].

APPENDIX B. PRESENTATIONS OF GENERALIZED 1-MOTIVIC SHEAVES.

Let notations be as in Section 3. A modified version of Proposition A.2.2 holds also for generalized 1-motivic sheaves.

B.0.8. **Lemma.** A sheaf \( F \) in \( \text{Ab}/k \) is (generalized) 1-motivic if and only if

\[
\mathcal{F} = \text{Coker}(F_1 \overset{u}{\to} F_0)
\]

where \( F_1 \) is a formal \( k \)-group, \( F_0 \) is extension of a formal \( k \)-group by a smooth connected algebraic \( k \)-group \( G \) and \( u \) is a monomorphism. We have then an diagram as (A.2.4) with \( L,E \) formal \( k \)-groups. Moreover, we can always find a \( F \) as above with \( F_1 \) torsion free. If \( E \) is connected, one may suppose \( F_1/L = 0 \) in (A.2.4); in the general case this holds after a finite separable field extension of \( k \).

Proof. We have to prove the “only if” assertion. For \( E \) connected the results follows immediately from C.0.10 C.0.12. In the general case, by Corollary C.0.17 \( \mathcal{F} = \mathcal{F}' \times E^0 \) with \( \mathcal{F}' \) extension of \( E^{\text{ét}} \) by \( \mathcal{F}^* \). Hence we have to check the case \( E = E^{\text{ét}} \). Let now \( \mathcal{F}^* := \text{Coker}(L^{\text{ét}} \to G) \). It comes out with an epimorphism \( \mathcal{F}^* \to \mathcal{F}^* \) whose kernel is \( L^0 \). The induced sequence

\[
0 \to L^0 \to \mathcal{F}^* \to \mathcal{F} \to E^{\text{ét}} \to 0
\]

is trivial by C.0.11 Hence \( \mathcal{F} \) is the push-out along the epimorphism \( \mathcal{F}^* \to \mathcal{F}^* \) of a 1-motivic sheaf \( \mathcal{F} \) that is extension of \( E^{\text{ét}} \) by \( \mathcal{F}^* \). By Proposition A.2.2 and
the first example in 3.2.3 \( \tilde{\mathcal{F}} = \text{Coker}(\tilde{F}_1 \to \tilde{F}_0) \) with \( \tilde{F}_0 \) a smooth \( k \)-group with identity component \( G \) and \( \tilde{F}_1 \) a discrete group. Define now \( F_0 := \tilde{F}_0 \) and \( F_1 \) as the kernel of the composition \( F_0 \to \tilde{F} \to \mathcal{F} \). It is a formal \( k \)-group extension of \( L^0 \) by \( \tilde{F}_1 \) and we are done. \( \square \)

### Appendix C. Some results on extensions

Throughout this section \( k \) will be a field of characteristic zero. We will work with generalized 1-motivic sheaves as in Section 3.2. In the following, we prove some results concerning extensions of formal \( k \)-groups and \( k \)-group schemes in the context of 1-motivic sheaves. We will denote by \( \mathbb{G}_{a,n} \) the scheme \( \text{Spec}(k[x]/(x^{n+1})) \).

**Lemma.** Let \( \mathcal{F}^* \) be a (generalized) 1-motivic sheaf quotient of a semi-abelian scheme \( G \) by a connected formal \( k \)-group \( L \). Then \( \text{Ext}^1(\hat{\mathbb{G}}_a, \mathcal{F}^*) = 0 \).

**Proof.** As the morphism \( L \to G \) factors through \( \hat{G} \), we may assume that \( L = \hat{G} \simeq \mathbb{G}_a \). We need the following results:

**Claim 1:** \( H^1(\mathbb{G}_{a,n}, \hat{\mathbb{G}}_a) = H^1(\mathbb{G}_{a,n}, \hat{G}) = 0 \). By the usual spectral sequence and \( \hat{\mathbb{G}}_a = \text{Hom}(\mathbb{G}_a, \mathbb{G}_m) \) the group \( H^1(\mathbb{G}_{a,n}, \hat{\mathbb{G}}_a) \) injects into \( \text{Ext}^1_{\mathbb{G}_{a,n}}(\mathbb{G}_a, \mathbb{G}_m) = 0 \), hence is trivial.

**Claim 2:** \( H^2(\mathbb{G}_{a,n}, \hat{\mathbb{G}}_a) = 0 \). First observe that \( \text{Ext}^2_{\mathbb{G}_{a,n}}(\mathbb{G}_a, \mathbb{G}_m) \) is torsion and hence trivial by [7], §8. To be honest in loc.cit. is required that the base is regular. However the result holds also in our special case because the proof is based on the torsion of groups \( H^q(\mathbb{G}_a, \mathbb{G}_m) \); now, by [12], III, 3.9 we may work with the étale topology and there is an equivalence for the étale sites of \( \mathbb{G}_a \) over \( \text{Spec}(k) = \mathbb{G}_a,0 \) and over \( \mathbb{G}_{a,n} \). Again the cited above spectral sequence provides an exact sequence

\[
H^0(\mathbb{G}_{a,n}, \text{Ext}^1(\mathbb{G}_a, \mathbb{G}_m)) \to H^2(\mathbb{G}_{a,n}, \hat{\mathbb{G}}_a) \to \text{Ext}^2_{\mathbb{G}_{a,n}}(\mathbb{G}_a, \mathbb{G}_m).
\]

This proves the claim because \( \text{Ext}^1(\mathbb{G}_a, \mathbb{G}_m) = 0 \) and \( \text{Ext}^2_{\mathbb{G}_{a,n}}(\mathbb{G}_a, \mathbb{G}_m) = 0 \).

**Claim 3:** \( \mathcal{F}^*(\mathbb{G}_{a,n}) = \mathcal{F}^*(\mathbb{G}_a, \mathbb{G}_m) = G(k) \) does not depend on \( n \). By the usual lifting property of smooth schemes \( G(\mathbb{G}_{a,n}) \to G(k) \) is surjective with kernel \( \hat{G}(\mathbb{G}_{a,n}) \). Hence Claim 1 implies \( \mathcal{F}^*(\mathbb{G}_{a,n}) = G(k) \).

**Claim 4:** \( H^i(\mathbb{G}_{a,n}, \mathcal{F}^*) = H^i(k, G) \) is torsion and does not depend on \( n \). By Claim 1 and Claim 2, \( H^i(\mathbb{G}_{a,n}, \hat{G}) = 0 \), \( i = 1, 2 \). Hence \( H^i(\mathbb{G}_{a,n}, G) = H^i(\mathbb{G}_{a,n}, \mathcal{F}^*) \). Moreover \( H^1(\mathbb{G}_{a,n}, G) = H^1(\mathbb{G}_{a,n}, \mathcal{F}^*) \) is torsion and does not depend on \( n \) (because the finite étale coverings of \( \mathbb{G}_{a,n} \) and \( k \) correspond bijectively and then apply cf. [12], III, 3.9).

Now, let \( \mathcal{F} \) be an extension of \( \hat{\mathbb{G}}_a \) by \( \mathcal{F}^* \). We shall prove that \( \mathcal{F} \) is trivial. Observe that \( \mathcal{F} \) provides a \( G \)-torsor \( \mathcal{F}_G \) over \( \mathbb{G}_{a,n} \) (compatible with pull-backs \( \mathbb{G}_{a,n-1} \to \mathbb{G}_{a,n} \)) whose class is torsion by Claim 4. Up to an isomorphism on \( \mathbb{G}_a \), hence on \( \mathcal{F} \), we may assume that \( \mathcal{F}_G \) is trivial. Hence there is a family of compatible sections \( s_n \in \mathcal{F}(\mathbb{G}_{a,n}) \) with \( s_n(0) = 0 \) that provide a “factor set”
γ: \hat{G}_a \times \hat{G}_a \to \mathcal{F}^*. By Claim 3, γ corresponds to a section in \mathcal{F}^*(k) and by γ(x,0) = 0, it follows that γ is trivial.

C.0.11. Lemma. Let E be a discrete group and L a connected formal k-group; then Ext^p(E,L) = 0 for all p.

Proof. By the usual Ext spectral sequence one gets that H^q(E,\hat{G}_a) = 0 for all q because \hat{G}_a = \text{Hom}(G_a,G_m) and Ext(G_a,G_m) = 0. Hence applying spectral sequences in [7] one gets that Ext^p(E,\hat{G}_a) = 0 for all p.

C.0.12. Lemma. Let \mathcal{F} be a (generalized) 1-motivic sheaf as in (3.2.2) with E connected formal k-group and L discrete. Then \mathcal{F} = \mathcal{F}^* \times E.

Proof. We may assume E = \hat{G}_a. Observe that by [12], III, 3.9

(C.0.13) H^i(G_a,n,G) = H^i(k,G), \quad H^i(G_a,n,L) = H^i(k,L), \quad i = 1, 2

can be calculated for the étale topology and are torsion. Hence

(C.0.14) H^1(G_a,n,\mathcal{F}^*) = H^1(k,\mathcal{F}^*) is torsion;

moreover the coboundary map

(C.0.15) \hat{G}_a(G_a,n) \to H^1(G_a,n,\mathcal{F}^*) is trivial

because the m-multiplication is an isomorphism on \hat{G}_a. To prove that \mathcal{F} is a split extension, observe that canonical maps

(C.0.16) \mathcal{F}(G_a,n) \to \mathcal{F}(G_a,n-1)

are surjective. Indeed this holds for \mathcal{F}^* because of the smoothness of L,G and [C.0.13]. Furthermore, it holds for E. Hence it holds in general because of [C.0.15].

By [C.0.16] we can now construct a compatible sequence of sections s_n \in \mathcal{F}(G_a,n) which induce the zero section on \mathcal{F}(k) and lifting the canonical section in \hat{G}_a(\hat{G}_a,n). Hence there exists a “system of factors”

γ: \hat{G}_a \times \hat{G}_a \to \mathcal{F}^*.

Let γ_n be the restriction of γ to G^2_a,n. By [C.0.14] \gamma_n \circ \gamma_n lifts to G for a suitable integer m that does not depend on n. Hence \gamma_n \circ \gamma_n lifts to G. However, as \mathcal{F} \cong [m]*\mathcal{F} we may assume that γ itself lifts to a γ': \hat{G}_a^2 \to G. As Mor(\hat{G}_a,L) = 0 (as functor of sets), γ' is still a system of factors. It factors through G and hence it is trivial.

C.0.17. Corollary. Any 1-motivic sheaf \mathcal{F} is product of a formal connected k-group E^0 by a 1-motivic sheaf \mathcal{F}' such that Coker b' is discrete.

Proof. It follows immediately from C.0.10 C.0.12

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