Formalized Quantum Stochastic Processes and Hidden Quantum Models with Applications to Neuron Ion Channel Kinetics

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Abstract

A new class of formal latent-variable stochastic processes called hidden quantum models (HQM’s) is defined in order to clarify the theoretical foundations of ion channel signal processing. HQM’s are based on quantum stochastic processes which formalize time-dependent observation. They allow the calculation of autocovariance functions which are essential for frequency-domain signal processing. HQM’s based on a particular type of observation protocol called independent activated measurements are shown to be distributionally equivalent to hidden Markov models yet without an underlying physical Markov process.

Since the formal Markov processes are non-physical, the theory of activated measurement allows merging energy-based Eyring rate theories of ion channel behavior with the more common phenomenological Markov kinetic schemes to form energy-modulated quantum channels.
Using the simplest quantum channel model consistent with neuronal membrane voltage-clamp experiments, activation eigenenergies are calculated for the **Hodgkin-Huxley K⁺ and Na⁺ ion channels**. It is also shown that maximizing entropy under constrained activation energy yields noise spectral densities approximating $S(f) \sim 1/f^\alpha$, thus offering a biophysical explanation for the ubiquitous 1/f-type noise in neurological signals.

**Keywords and phrases:** Quantum measurement problem, stochastic processes, Hidden Markov models, neuronal noise, ion channels, Markov kinetics, maximum entropy, Hodgkin-Huxley, 1f noise

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1 **Introduction**

1.1 **Purpose**

This paper will present the theory of **formal quantum stochastic processes**, including **activated measurement processes**, as well as **hidden quantum models**.

We will show that the hidden quantum models to which activated measurement processes give rise are actually formal hidden Markov models with **no underlying associated physical Markov process**.

Applications to **neuronal ion channels** will be outlined which will be developed in more depth in subsequent publications:

**Natural Ion Channels**

The derivation of the the newly-defined **activation energy operators** $E_{K^+}$, $E_{Na^+}$ of the Hodgkin-Huxley channels from their well-known reaction rate matrices $A_{K^+}$, $A_{Na^+}$ using the models developed in §3.1

**Maximum Entropy and 1/f-type noise**

The proof that that maximum entropy equilibrium in a population of **energy-modulated ion channels** can give rise to $S(f) \sim 1/f^\alpha$ noise spectra by a generalized van der Ziel-McWhorter [1][3] formalism (§3.2).
1.2 Significance

Abstract mesoscale models of neurological tissue and signals are a key bridge between the hyper-detailed nanoscale structural and behavioral analyses of ion channels obtained by modern biochemical techniques [4] and the overly-simplified macroscale (i.e., neuron-level) synthetic neuron models pioneered by McCulloch and Pitts [5] in the 1940’s which are the basis for current neural network technology. Improved models in the mesoscale have significance for biological neuroscience and neurology, neurological signal processing and engineering, as well as synthetic neural networks for massive computation.

The concepts defined in this work go well beyond the standard Markov kinetics which have been used since the 1960’s [6] to account for ion channel permeability distributions. By basing the ion channel theory on a simplified but highly flexible and biophysically plausible quantum mechanical formalism the mesoscale model is brought closer to the underlying nanostructure while still retaining its mathematical tractibility.

1.2.1 Quantum Measurements and Signal Processing

A subtle limitation of standard formulations of quantum mechanics is that it has difficulty with the concept of sampling a physical system over time because of unsolved metaphysical problems concerning the role of the observer and wavefunction collapse. (Refer to the Appendix § 5.1 for a summary of quantum mechanics.) This is seen most clearly in the paradoxical “watched pot theorem” [7] which seems to imply that sampled quantum systems are always frozen into their initial states.

As a consequence, while standard quantum mechanics has no trouble defining what is meant by the state $\Psi (t)$ of a system at a single time $t$, the ambiguities resulting from the Measurement Problem (§ 5.1) makes it difficult to assign meaning to a state $\Psi (t_1, t_2)$ which is supposed to represent the system at two distinct times $t_1 < t_2$.

This causes enormous practical problems for signal processing in quantum systems because the standard autocovariance function

$$R_A (t_1, t_2) \overset{\text{def}}{=} \mathbb{E} [(A_{t_1} - \mu_A (t_1)) \cdot (A_{t_2} - \mu_A (t_2)) \mid \Psi (t_1, t_2)]$$

on which the entire theory of power spectra rests is thus ill-defined.

More generally, it is essential for ion channel signal processing to be able to calculate the higher moments $\mathbb{E} [A_{t_1} \cdots A_{t_n} \mid \Psi (\cdot)]$ of observables $A_1, \cdots A_n$ at distinct times $t_1 < \cdots < t_n$ for systems prepared in quantum state $\Psi$. 

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The calculation of $\mathbb{E}[A_1(t_1)A_2(t_2) \mid \Psi(\cdot)]$ requires knowledge of every measurement made by the observer during the closed interval $[t_1, t_2]$ because every such measurement caused the system to collapse into a measurement state at the moment it was made (cf. Def. 5.1.0.7). Different observer behaviors would lead to completely different average values of the product $A_1(t_1) \cdot A_2(t_2)$. So a concept of a state which intuitively corresponds to $\Psi(t_1, t_2)$ can only be defined in the context of the knowledge of the observer’s actions through time and must implicitly encode that knowledge into some form of measurement protocol.

This is in contrast to the calculation of $\mathbb{E}[A_1(t_1) \mid \Psi]$ at time $t_1$ for which the closed “interval” $[t_1, t_1]$ leaves room for, at most, a single measurement. For this reason, completely general observer-independent statements such as Born’s Axiom Eq. ?? are possible at a single time, but not over a non-trivial time interval.

These ambiguities are resolved in §2.2.1 by defining formal quantum stochastic processes (Def. 2.2.1.1) which satisfy the Hilbert space analog of the standard Kolmogoroff marginalization conditions (cf. §2.1.1). Examples of such processes and measurement protocols are presented in Ex. 2.2.1.1 and Def. 2.2.2.3.

This seems to be a satisfactory and plausible extension of standard quantum dynamics.

1.2.2 Ion Channels and Models

The ion channel equations Eq.10 and Eq.11, which are based on the formalism of quantum measurement processes, are the core of the research and provide a solvable mathematical translation between kinetic rate coefficients and the new concept of conformation energy operators (§3.1).

They provide a physical explanation based on thermal activation energy for these rate constants which, as in Hodgkin-Huxley’s original work, are usually regarded as unexplained phenomenological parameters. Apart from the conceptual clarity gained by reducing formerly unexplained coefficients to physical categories, the detailed solutions for the energy eigenvalues in terms of experimentally-determinable rates may provide insight into the nanoscale behavior of actual ion channels.

These energy solutions also allow the setting up of energy constraints for obtaining maximum entropy population distributions. This is the key idea for the derivation of the neural noise models of §1.2.3.

In the other direction, the energy operators may be designed to induce desired behavior in simulated ion channels and neural membranes for neu-
ropharmacological or other types of investigations. Energy operators may also be tuned by evolutionary algorithms to form synthetic neural membranes for large-scale network computation.

1.2.3 Neuronal Noise

Because of its power and ubiquity in all neural recordings, from the individual channel level up to cortical electroencephalograms (EEG), the study of neuronal noise continues to be a major field of research [8–10].

![Figure 1: EEG power spectrum showing visual response spikes embedded in $1/f^\alpha$ noise.](image)

Most of the noise created by the electrical activity in neural tissue can be characterized as $1/f$-type noise; that is, its power spectrum is of the general form $1/f^\alpha$ over a significant range of frequencies $f$ for some constant $\alpha$ (Fig. 1). Data taken from [11]). As discussed in detail in [12–15], since the 1970’s it has been thought that $1/f$-type neural noise must originate somewhere other than ion channels because hidden Markov models generate autocovariance functions of the wrong form [16].

A major consequence of this research is the demonstration that, contrary to the standard belief, a population of hidden quantum ion channels under appropriate energy constraints does yield $1/f$-type permeability noise when allowed to find a maximum entropy equilibrium. This is a result of the additional mathematical flexibility afforded by the quantum channel model.

Moreover, the noise exponent $\alpha(T)$ depends on the ambient temperature $T$ in a calculable manner which could afford critical laboratory tests of the theory.

1.2.4 The Biophysics of Ion Channels

Section 2.1.1 distinguishes between theoretical statistical models of ion channels given by Kolmogoroff’s Theorem [17] and the physical processes which gives rise to the observed behavior.
These distinctions are key to understanding the ion channel literature. It has been known since the 1960’s [?] that the physical stochastic processes represented by channel permiabilities or conductances have the formal structure of hidden Markov models. By Kolmogoroff’s Theorem, these conductance processes can certainly be simulated by underlying concrete Markov processes.

As a result, channel researchers have always assumed there are Markov processes controlling the successive transitions of the physical conformations of channels. But this is actually an unwarranted step since we cannot know if any particular simulated channel process is the actual physical process driving the distributional evolution because there is no general uniqueness corollary of Kolmogoroff’s Theorem.

In fact, the physically-based quantum activated measurement processes which are defined in §2.2.2 will be proven to give rise to formal hidden Markov models (i.e., HMM distribution functions) yet with no underlying physical Markov process.

Therefore, if all we know about a channel is its formal permiability or conductance distribution, there is no scientific way to choose between a hidden quantum model (HQM) and a hidden Markov model as the physical process. However, when the results of this research are considered which prove HQM’s can give rise to $1/f$-type noise (§3.2) while HMM’s cannot [15], the evidence will clearly favor the new HQM channel processes.

Thus, based on our research, the theoretical existence of simple concrete simulations may have completely distorted the historical understanding of ion channel behavior.

2 Classical and Quantum Stochastic Processes

2.1 Classical Stochastic Processes

Several observations first need to be made concerning general stochastic processes. In particular, we need to make conceptual distinctions between formal and concrete processes and also between simulated and physical processes.

2.1.1 Process Classification

By an (absolutely continuous, real-valued, Borel [18]) formal stochastic process $\{X_t\}_{T_1 \leq t \leq T_2}$ on the time interval $[T_1, T_2]$ we mean the specification
of a family
\[ \{p_{t_1, \ldots, t_n}(x_1, \ldots, x_n) \mid T_1 \leq t_1 < \cdots < t_n \leq T_2, 0 \leq n < \infty \} \]
of probability distributions on \( \mathbb{R} \) satisfying the **Kolmogoroff consistency** or **marginalization** conditions: \( p_{\emptyset}(\cdot) = 1 \) for \( n = 0 \) and
\[
p_{t_{11}, \ldots, \hat{t}_i, \ldots, t_n}(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \int_{-\infty}^{\infty} p_{t_1, \ldots, \hat{t}_i, \ldots, t_n}(x_1, \ldots, x_i, \ldots, x_n) \, dx_i \tag{1}\]
for every \( n > 0, T_1 \leq t_1 < \cdots < t_n \leq T_2, \) and \( 1 \leq i \leq n, \) where the caret \( \hat{\cdot} \) indicates an item missing from a list. Thus the **form** of the probability distribution functions through time are given.

On the other hand, by a **concrete** stochastic process we mean the specification of an underlying sample space \( \Omega \) with probability measure \( \text{Prob}(\cdot) \) together with a collection \( \{X_t(\omega) \mid \omega \in \Omega\}_{T_1 \leq t \leq T_2} \) of (Borel) random variables \( X_t \) on \( \Omega. \) A concrete process (with absolutely continuous probability) always generates a unique formal process given by
\[
p_{t_1, \ldots, t_n}(x_1, \ldots, x_n) \overset{\text{def}}{=} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \text{Prob}\{\omega \in \Omega \mid X_t_1(\omega) \leq x_1, \cdots, X_t_n(\omega) \leq x_n\}. \]

However, the converse (i.e., that every formal process comes from at least one concrete process) requires a strong theorem due to Kolmogoroff \[17\] which utilizes subtle topological properties of the real numbers. (There are, in fact, well-defined and consistent formal processes with non-real values which have no underlying sample space \((\Omega, \text{Prob}(\cdot))\) and so are not concrete in the above sense.)

Moreover, Kolmogoroff’s Theorem has no **uniqueness** guarantee and this brings us to the second distinction.

By a **simulated** process we mean an abstract or computer-based concrete process which yields the same formal distribution functions as the physical process. By a **physical** process we mean the actual mechanism in nature which generates random values whose frequency distributions we are studying.

Because of the lack of uniqueness in Kolmogoroff’s Theorem, even if the formal distributions are known exactly, there may be many distinct simulations none of which may actually represent the underlying physical process.
2.1.2 Latent Processes and Hidden Markov Models

A hidden stochastic model explains the dependencies in a manifest random process \( \{ X_t \}_{T_1 \leq t \leq T_2} \) in terms of an underlying, latent process \( \{ Y_t \}_{T_1 \leq t \leq T_2} \) with prior distribution functions

\[
\{ \pi_{t_1, \ldots, t_n}(y_1, \ldots, y_n) \}_{T_1 \leq t_1 < \cdots < t_n \leq T_2}
\]

and a conditional distribution function \( f(x \mid y) \) chosen so that the posterior distribution functions of \( \{ X_t \} \) are

\[
p_{t_1, \ldots, t_n}(x_1, \ldots, x_n) = \int \cdots \int f(x_1 \mid y_1) \cdots f(x_n \mid y_n) \pi_{t_1, \ldots, t_n}(y_1, \ldots, y_n) \, dy_1 \cdots dy_n.
\] (2)

Intuitively, one can think of the \( Y \) process as dealing numbered cards from a finite pack (thus showing dependent choices) while the \( X \) process consists of independent selections from numbered urns corresponding to the \( Y \) cards dealt.

When the latent process is a formal discrete-state, continuous-time process \( I \) with prior distributions

\[
\{ \pi_{t_1, \ldots, t_n}(i_1, \ldots, i_n) \}_{T_1 \leq t \leq T_2}
\]

and we are given conditional distributions \( f_i(x) \) then the posterior distribution functions may be written

\[
p_{t_1, \ldots, t_n}(x_1, \ldots, x_n) \overset{\text{def}}{=} \sum_{i_1, \ldots, i_n} f_{i_1}(x_1) \cdot \cdots \cdot f_{i_n}(x_n) \pi_{t_1, \ldots, t_n}(i_1, \ldots, i_n).
\] (3)

In the special case in which the latent process \( I \) is Markov, Eq.3 constitutes a formal (cf. §2.1) hidden Markov model (HMM).

By the discussion in §2.1, for every formal HMM there exists at least one concrete discrete-state Markov process \( \{ I_t \}_{T_1 \leq t \leq T_2} \) with distributions Eq.3. However, in §2.2.2 we will show that Eq.3 can also arise from systems with no latent physical Markov process.

2.2 Quantum Stochastic Processes and Hidden Quantum Models

In this section the research approach to stochastic processes will be outlined. A summary of the quantum mechanical background necessary for understanding the concepts and notation is presented in the Appendix §5.1.
See the Appendix §5.2 for a discussion of tensor products and the partial trace.

Note that for simplicity of presentation all Hilbert spaces are assumed to be separable. Thus, when $\mathbb{H}$ is non-trivial, $1 \leq \dim(\mathbb{H}) \leq \aleph_0$.

### 2.2.1 Quantum Stochastic Processes

**Definition 2.2.1.1.** Let $\mathbb{H}$ be the configuration space of a quantum system. A **formal quantum stochastic process** $\Psi(\cdot)$ on the time interval $[T_1, T_2]$ is a family

$$\{ \Psi(t_1, \ldots, t_n) \mid T_1 \leq t_1 < \cdots < t_n \leq T_2, 0 \leq n < \infty \}$$

of non-negative definite, self-adjoint, trace class operators $\mathbb{H}^{\otimes n} \equiv \mathbb{H} \otimes \cdots \otimes \mathbb{H}$ (cf. §5.2) satisfying the marginalization conditions: $\Psi(\emptyset) = 1$ for $n = 0$ and

$$\Psi(t_1, \ldots, \hat{t}_k, \ldots, t_n) = \text{tr}_k(\Psi(t_1, \ldots, t_k, \ldots, t_n)),$$

for all $t_1 < \cdots < t_k < \cdots < t_n$, $1 \leq k \leq n$, where $\text{tr}_k(\cdot)$ is the partial trace along the $k^{th}$ dimension defined in Def. 5.2.0.9 and, as before, the caret $\hat{}$ above a symbol indicates an item missing from a list.

**Remark 2.2.1.1.** The $n = 0$ condition $\Psi(\emptyset) = 1$ implies that every $\Psi(t_1, \ldots, t_n)$ is a state on $\mathbb{H}^{\otimes n}$; i.e., $\text{tr}(\Psi(t_1, \ldots, t_n)) = 1$.

**Remark 2.2.1.2.** Quantum marginalization is the quantum mechanical generalization of the Kolmogoroff consistency conditions Eq.1 for classical formal stochastic processes.

**Remark 2.2.1.3.** It is important to interpret Def. 2.2.1.1 correctly with respect to observers and measurements. As discussed in §1.2.1, the process $\Psi(\cdot)$ must be interpreted as excluding any further disturbances to the system. It implicitly “codes” every measurement or observation that made on the system during the time interval $[T_1, T_2]$. Any additional observation would change the process itself.

**Remark 2.2.1.4.** However, probability and expectation calculations such as Born’s Axiom Def. 5.1.0.6 are allowed because they do not constitute measurements or observations in the strict quantum mechanical sense. That there are valid quantum systems for which observables have average values but no actual values is an expression of the paradoxical nature of quantum logic.
Definition 2.2.1.2. Let $\Psi(\cdot)$ be a formal quantum stochastic process. Let $\mathcal{A}$ be a set of observables of the system. For every $A \in \mathcal{A}$, let $A$ denote the operator on $\mathbb{H}$ which corresponds to $A$. The set $\mathcal{A}$ is called correlatable with respect to $\Psi(\cdot)$ if, for every $t_1 < \cdots < t_n$ and every $A_1, \ldots, A_n \in \mathcal{A}$ the product $A_1(t_1) \cdots A_n(t_n)$ has a frequentist average value $E[A_1(t_1) \cdots A_n(t_n) | \Psi]$ in systems prepared in state $\Psi(t_1, \ldots, t_n)$ which satisfies the Generalized Born Axiom

$$E[A_1(t_1) \cdots A_n(t_n) | \Psi] = \text{tr}(A_1 \otimes \cdots \otimes A_n \cdot \Psi(t_1, \ldots, t_n)).$$

(5)

Definition 2.2.1.3. A concrete quantum stochastic process consists of a formal quantum stochastic process $\Psi(\cdot)$ together with a set $\mathcal{A}$ of observables which are correlatable with respect to $\Psi(\cdot)$.

Remark 2.2.1.5. It must be noted that Eq.5 is a genuine extension of the standard axioms of quantum mechanics. Cf. §4.1 for a discussion.

Example 2.2.1.1. Let $U(t) = e^{-\frac{\pi^2 \sqrt{-1}}{\hbar} \lambda t}$ be the Schrödinger representation of the additive group of $\mathbb{R}$, where $\lambda$ is the Hamiltonian of the system. Let $\psi_0 \in \mathbb{H}$ be an initial wavefunction. Define the wavefunction at time $t$ (the solution to Schrödinger’s Equation)

$$\psi(t) \overset{\text{def}}{=} U(t) \psi_0,$$

and the trace class operators

$$\Psi(t_1, \ldots, t_n) \overset{\text{def}}{=} \left(\psi(t_1) \psi(t_1) \psi(t_1) \cdots \psi(t_n) \psi(t_n) \psi(t_n) \right),$$

for $t_1 < \cdots < t_n$ and $n \geq 0$. Then $\Psi(\cdot)$ is a formal quantum stochastic process. We can take $\mathcal{A}$ to be all possible observables under the Schrödinger measurement protocol: never measure anything during the interval $[T_1, T_2]$. Thus the system has never been disturbed away from its dynamic evolution.

We will then have moments

$$E[A_1(t_1) \cdots A_n(t_n) | \Psi] = \text{tr}(A_1 \Psi(t_1)) \cdots \text{tr}(A_n \Psi(t_n))$$

$$= E[A_1(t_1) | \Psi] \cdots E[A_n(t_n) | \Psi],$$

so all observables are uncorrelated at distinct times.

2.2.2 Activated Measurement Processes

Recall (§5.1) a state $\Psi$ is a measurement state of an observable $A$ if $A \Psi = \Psi A$.  

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In this section we describe an important type of quantum operation called **activated measurement** which is meant to abstract the double procedure of absorption of energy while in a measurement state of an observable \( A \) followed by the measurement of \( A \) itself. This collapses the system into a new measurement state (§5.1) and may also return energy back to the environment.

This will yield a new type of quantum stochastic process which models Poisson-distributed measurement protocols.

**Remark 2.2.2.1.** A good image of activation is **shaking dice in a cup** before they are rolled while the subsequent measurement corresponds to turning the cup upside down on the table **without lifting it to see the result**. (If the cup were lifted, this would constitute an “observation” as was defined in §5.1 and the system would collapse further to the particular eigenstate of \( A \) corresponding the value observed.)

**Definition 2.2.2.1.** An **activator** for an observable \( A \) is linear operator \( Q \) such that for every measurement state \( \Psi \) of \( A \) the operator \( Q\Psi Q^H \) is a state; that is, \( \text{tr} \left( Q\Psi Q^H \right) = 1 \). (Note that the operation \( Q \left( \cdot \right) Q^H \) preserves non-negative definiteness.) We call \( Q\Psi Q^H \) an **activated state** of \( A \).

**Remark 2.2.2.2.** Unitary operators \( U \) are activators for every observable since
\[
\text{tr} \left( Q\Psi Q^H \right) = \text{tr} \left( \Psi Q^H Q \right) = \text{tr} \left( \Psi I \right) = 1.
\]
In particular, the dynamic operator \( U(t) = e^{-\frac{2\pi i}{\hbar} H t} \), with \( H \) the system’s Hamiltonian, is an activator for every observable.

**Remark 2.2.2.3.** As a result, if \( Q \) is an activator for \( A \) and \( U \) is unitary then \( UQ \) is also an activator for \( A \).

**Lemma 2.2.2.1.** Let \( A \) be an observable. If the linear operator \( Q \) is an activator for \( A \) then \( \|Q\psi\| = \|\psi\| \) for every eigenvector of \( A \). If \( A \) is compact (in particular if \( \mathbb{H} \) is finite), the converse holds.

**Proof.** Let \( Q \) be an activator and \( \psi \) be an eigenvector of \( A \). Then \( \Psi \overset{\text{def}}{=} \frac{1}{\|\psi\|^2} \psi \psi^H \) is a measurement state of \( A \). By hypothesis
\[
1 = \text{tr} \left( Q\Psi Q^H \right) = \frac{1}{\|\psi\|^2} \text{tr} \left( (Q\psi)(Q\psi)^H \right) = \frac{1}{\|\psi\|^2} \|Q\psi\|^2.
\]
Conversely let \( Q \) be compact and preserve the norms of eigenvectors. Since \( A \) is compact, it has a discrete spectrum \( \{\alpha_i \mid 1 \leq i \leq \text{dim} \left( \mathbb{H} \right) \} \) and every
eigenspace is finite-dimensional. Let $\psi_1, \ldots, \psi_{d_i}$ be an orthonormal basis for the $i^{th}$ eigenspace. Then the projection operator onto this eigenspace is

$$P_i = \sum_{i=1}^{d_i} \psi_i \psi_i^H$$

and every term $\psi_i \psi_i^H$ is a measurement state. From this we see that $\text{tr} (P_i) = d_i$ and

$$\text{tr} (Q P_i Q^H) = \sum_{i=1}^{d_i} \|Q \psi_i\|^2 = \sum_{i=1}^{d_i} 1 = d_i = \text{tr} (P_i)$$

since $\|Q \psi_i\| = \| \psi_i \| = 1$ by hypothesis. So $Q (\cdot) Q^H$ preserves the trace of every projection $P_i$. The general $A$-measurement state is the mixture $\Psi = \sum a_i P_i$, for which $a_i \geq 0$ and $\sum a_i \text{tr} (P_i) = 1$. Thus

$$\text{tr} (Q \Psi Q^H) = \sum_i a_i \text{tr} (Q P_i Q^H) = \sum_i a_i \text{tr} (P_i) = 1$$

and so $Q \Psi Q^H$ is a state. Therefore $Q$ is an activator for $A$. □

Remark 2.2.2.4. Thus $Q$ is an activator for a compact observable $A$ if and only every column of the matrix of $Q$ in the orthonormal basis for $A$ has norm 1. This is a useful criterion.

Activated states for $A$ are quantum states but generally not $A$-measurement states. In a measurement state we can regard $A$ as having some particular but unobserved value but activation causes quantum interference and so the possible $A$-eigenstates superpose. This motivates the following:

Definition 2.2.2.2. Let $Q$ be an activator for observable $A$ and let $\Psi$ be a measurement state of $A$. An activated measurement of $A$ consists of activating $\Psi$ by $Q$ followed by measuring (but not observing) $A$. This yields a new measurement state of $A$ denoted $\Psi' = M_{Q,A}[\Psi]$.

According to Cor. 5.1.0.1, the activated measurement operator may be written explicitly as

$$M_{Q,A}[\Psi] = \int_{\sigma(A)} P_\alpha Q \Psi Q^H P_\alpha \, d\alpha,$$  \hspace{1cm} (6)

where $\sigma (A)$ is the spectrum of $A$ and $P_\alpha$ is the projection operator onto the $\alpha$ eigenspace.
Definition 2.2.2.3. Let \( Q \) be an activator for observable \( A \), \( \Psi_0 \) be a measurement state of \( A \), and let \( \lambda > 0 \) be a constant. A family \( \{ \Psi(t_1, \ldots, t_n) \mid T_1 \leq t_1 < \cdots < t_n \leq T_2 \} \) for which \( \Psi(\emptyset) = 1 \) and which solves the infinite system
\[
\begin{align*}
\Psi(T_1) &= \Psi_0 \\
\frac{\partial}{\partial t_1} \Psi(t_1) &= \lambda \left( M_{Q,A} [\cdot] - I \right) \Psi(t_1), \quad T_1 < t_1 \leq T_2 \\
\Psi(t_1, \cdots, t_{n-1}, t_n) &= \Psi(t_1, \cdots, t_{n-1}) \otimes \Psi(t_{n-1}), \quad n \geq 2 \\
\frac{\partial}{\partial t_n} \Psi(t_1, \cdots, t_{n-1}, t_n) &= \lambda \Psi(t_1, \cdots, t_{n-1}) \otimes (M_{Q,A} [\cdot] - I) \Psi(t_{n-1}), \quad t_{n-1} < t_n \leq T_2
\end{align*}
\]
is called an activated measurement process with rate \( \lambda \).

Lemma 2.2.2.2. With the hypotheses of Def. 2.2.2.3, the activated measurement process \( \Psi(\cdot) \) exists, is unique, and is a formal quantum stochastic process.

Corollary 2.2.2.1. With the hypotheses of Def. 2.2.2.3, \( \Psi(t_1, \ldots, t_{n-1}, t_n) \) satisfies \( \cdots \) for \( t_n \geq t_{n-1} \) and where \( M_{Q,A}^{(k)} \) denotes \( k \) applications of the activated measurement operator.

From Cor. 2.2.2.1, it is clear that an activated measurement process is a Poisson mixture [17] of states. In the classical case, this would be a posterior distribution with a latent Poisson prior \( k \) representing the random number of activation/measurement cycles performed during the time interval \([t_{n-1}, t_n]\). In the quantum case, however, all the prior possibilities are superposed. Thus, in some sense, \( k = 0, 1, 2, \ldots \) cycles are performed together in every interval \([t_{n-1}, t_n]\).

2.2.3 Hidden Quantum Models

Definition 2.2.3.1. Let \( \mathfrak{A} \) be a measure space thought of as the possible observational values of a manifest process \( A \). An \( \mathbb{H} \)-conditional distribution is an integrable function \( f(\cdot) \) from \( \mathfrak{A} \) into the non-negative definite, compact, self-adjoint operators on \( \mathbb{H} \) such that \( \int f(\alpha) \, d\alpha = I \), where the convergence is in the operator norm.

Remark 2.2.3.1. When \( \mathbb{H} \) is separable and a fixed basis is chosen, the coefficients of each \( f(\alpha) \) form a non-negative definite, compact, and hermitian matrix \( [f_{ij}(\alpha)]_{1 \leq i,j \leq \dim(\mathbb{H})} \) of integrable complex-valued functions. Moreover, these functions satisfy \( \int f_{ij}(\alpha) \, d\alpha = \delta_{ij} \), the Kronecker delta.
Definition 2.2.3.2. Let $f(\cdot)$ be an $\mathbb{H}$–conditional distribution on the measure space $\mathfrak{A}$ and let $\Psi(\cdot)$ be a formal quantum stochastic process with configuration space $\mathbb{H}$. Then the formal hidden quantum model (HQM) generated by prior $\Psi(\cdot)$ and conditional $f(\cdot)$ is the formal stochastic process with posterior distribution functions

$$p_{t_1,\ldots,t_n}(\alpha_1,\ldots,\alpha_n) = \text{tr} (f(\alpha_1) \otimes \cdots \otimes f(\alpha_n) \cdot \Psi(t_1,\ldots,t_n)). \quad (7)$$

Remark 2.2.3.2. This definition is the quantum generalization of the classical latent model Eq.2.

Remark 2.2.3.3. Using the generalized Born relation Eq.5 this can be informally interpreted as stating there is some observable $A$ whose measured values are contained in the space $\mathfrak{A}$ and for which

$$\text{Prob}_{\Psi}[A(t_1) = \alpha_1,\ldots,A(t_n) = \alpha_n] = \mathbb{E}[X_{\alpha_1}(A(t_1)) \cdots X_{\alpha_n}(A(t_n)) | \Psi(\cdot)],$$

where $X_{\alpha}(\cdot)$ denotes the indicator functions on $\mathfrak{A}$: $X_{\alpha}(\beta) \overset{\text{def}}{=} \delta_{\alpha\beta}$.

Remark 2.2.3.4. An important feature of HQM’s is that, according to Def. 2.2.3.2 measurements of the manifest $A$ process are entirely classical: they do not disturb the hidden quantum stochastic process in any way. One can view them as macroscopic “readings” such as the global voltage through a patch of neuronal membrane [24]:

Definition 2.2.3.3. Let $f(\cdot)$ be an $\mathbb{H}$–conditional distribution on the measure space $\mathfrak{A}$. The distribution $f(\cdot)$ is called compatible if the operators $f(\alpha)$ and $f(\beta)$ commute for every $\alpha, \beta \in \mathfrak{A}$; that is, if $f(\alpha) \cdot f(\beta) = f(\beta) \cdot f(\alpha)$ for all $\alpha, \beta \in \mathfrak{A}$. A hidden quantum model is called compatible when it has a compatible conditional distribution.

For a compatible HQM, there is at least one orthonormal basis $\{\phi_i | 1 \leq i \leq \text{dim}(\mathbb{H})\}$ of wavefunctions for $\mathbb{H}$ in which the matrix of every $f(\alpha)$ diagonalizes. Letting $\{f_i(\alpha) | 1 \leq i \leq \text{dim}(\mathbb{H})\}$ be the diagonal coefficients in this basis and using Eq[7] and Eq[13] we will have posterior distribution

$$p_{t_1,\ldots,t_n}(\alpha_1,\ldots,\alpha_n) = \sum_{i_1,\ldots,i_n} f_{i_1}(\alpha_1) \cdots f_{i_n}(\alpha_n) \Psi_{i_1,\ldots,i_n}(t_1,\ldots,t_n), \quad (8)$$

where $\Psi_{i_1,\ldots,i_n}(t_1,\ldots,t_n)$ is the $(i_1,\ldots,i_n)^{th}$ main diagonal coefficient of the prior state $\Psi(t_1,\ldots,t_n)$ in the basis $\{\phi_i | 1 \leq i \leq \text{dim}(\mathbb{H})\}$.
Remark 2.2.3.5. Typically $\Psi(t_1, \cdots, t_n)$ does not itself diagonalize in this basis. But because $f(\alpha)$ does, it is only the diagonal coefficients of $\Psi(t_1, \cdots, t_n)$ which enter into the defining trace formula Eq.7.

Remark 2.2.3.6. It is clear that, except for notation, the posterior distribution Eq.8 of a compatible hidden quantum model has precisely the same form as a classical discrete-state, continuous-time formal hidden model Eq.3. Moreover, there are concrete hidden stochastic processes which simulate this posterior. However, there is a vast conceptual difference in terms of physical models (cf. §2.1.1).

The latent process $I$ of a classical discrete-state hidden model takes one and only one of the possible outcomes $i = 1, 2, \ldots$ at every instant $t$ of time. In the usual language, the latent process $I$ (Markov or not) is “in state $i$ at time $t$”.

But this is far from the case with a hidden quantum model:

For an HQM, the latent quantum process is almost never “in” one of the basic pure states $\phi_i^H$. At best, $\Psi(t_1, \cdots, t_n)$ may be coherent in the $\mathbf{f}(\cdot)$ basis so that its diagonal terms vanish but even that situation represents a mixture. In fact, since a basis in which the conditional $\mathbf{f}(\cdot)$ diagonalizes generally bears no direct relation to the physical process which gives rise to the prior distribution $\Psi(\cdot)$, we really need to regard the underlying state as in a simultaneous superposition of all its latent possibilities, just like Schrödinger’s famous cat.

Of course, if all we are studying is the posterior distribution $p_{t_1, \cdots, t_n}(\alpha_1, \ldots, \alpha_n)$ then these subtleties are irrelevant. However, as will be seen in §3.1 if we provisionally accept latent quantum explanations for well-known classical hidden models such as Hodgkin-Huxley ion channels, we may sometimes extract new levels of information (like activation energy matrices §10) not readily available from the posteriors alone. Moreover, as will be seen, hidden quantum layers allow canonical population models through the formalism of tensor products of configuration spaces (cf. §5.2) which suggest new explanations for phenomena such as $1/f$–type noise (§3.2).

Remark 2.2.3.7. Incompatible conditionals $\mathbf{f}(\cdot)$, which do not uniformly diagonalize, are tractable and useful HQM’s with posterior distributions which are an extension of the classical hidden model formula Eq.3. They will be studied in a future publication.
2.2.4 Hidden Quantum and Markov Models

In this section, hidden activated measurement models are defined and shown to generate to hidden Markov models as posterior distributions.

**Definition 2.2.4.1.** A hidden activated measurement model is a hidden quantum model (Def. 2.2.3.2) whose prior quantum stochastic process (Def. 2.2.1.1) is an activated measurement process (Def. 2.2.1.1). A hidden activated measurement model is called diagonalizable if the observable defining the prior quantum process is compact and the conditional distribution (Def. 2.2.3.1) is compatible (Def. 2.2.3.3).

We can then prove:

**Theorem 2.2.4.1.** Every diagonalizable hidden activated measurement model has posterior distribution which is a formal hidden Markov model (§2.1.2). Conversely, every formal hidden Markov model can be generated as the posterior of some diagonalizable hidden activated measurement model.

**Remark 2.2.4.1.** This theorem demonstrates there are formal hidden Markov models which derive from latent physical processes which are not Markov. There are no Markov states through which the activated measurement process is cycling, not even the eigenstates in the diagonal basis for the observable $\mathbf{A}$. The system is generally in mixed quantum states before activation and incoherent quantum states while activated, almost never in an eigenstate.

Furthermore, the converse shows that if the only distribution available concerning a physical process is a posterior formal hidden Markov model (as occurs with Hodgkin-Huxley type ion channels §3.1) then there is no method to distinguish the situation of an underlying Markov process from an underlying hidden quantum process. They are scientifically indistinguishable and the decision between them must be made on other grounds.

**Definition 2.2.4.2.** For any matrix or vector $\mathbf{A}$ then $\mathbf{A}^{[2]}$ denotes the real matrix of the same order whose entries are the squared moduli $|a|^2$ of the entries $a$ of $\mathbf{A}$.

**Remark 2.2.4.2.** It is very important to note that, even when the matrix or vector $\mathbf{A}$ derives from an operator of wavefunction, the squared moduli $\mathbf{A}^{[2]}$ is not an operator. It is basis-dependent and does not transform like an operator. It is best thought of as a simple array of real numbers which depends on an implicit basis.
Proof of Thm. 2.2.4.1. Let $A$ be the compact observable, $Q$ its activator, $Ψ_0$ the initial $A$–measurement state, and $f (·)$ the compatible conditional.

Let $ϕ_1, ϕ_2, \ldots$ be an orthonormal eigenbasis for $A$. We first characterize the finite iterates $M_{Q,A}^{(k)} [Ψ_0]$.

Every $M_{Q,A}^{(k)} [Ψ_0]$, for $k = 0, 1, \ldots$ is an $A$–measurement state by definition so can be expressed as a coherent mixture

$$M_{Q,A}^{(k)} [Ψ_0] = \sum_i a_i^{(k)} (ϕ_i ϕ_i^H),$$

where $a_i^{(k)} ≥ 0$ are real coefficients such that $\sum_i a_i^{(k)} = 1$. Activation then yields

$$Q M_{Q,A}^{(k)} [Ψ_0] Q^H = \sum_i a_i^{(k)} (ξ_i ξ_i^H),$$

where $ξ_i = Q ϕ_i$ is the $i$th column of the operator $Q$ in the $\{ϕ_j\}$ basis. So, by Cor. 5.1.0.1 and noting that the $i$th projection operator is the pure state $P_i = ϕ_i ϕ_i^H$, we have

$$M_{Q,A}^{(k+1)} [Ψ_0] = M_{Q,A} \left[ M_{Q,A}^{(k)} [Ψ_0] \right]$$

$$= \sum_i P_i M_{Q,A}^{(k)} [Ψ_0] Q^H P_i$$

$$= \sum_i \sum_j a_j^{(k)} (ϕ_j^H ξ_j) (ξ_j^H ϕ_i) (ϕ_i ϕ_i^H)$$

$$= \sum_i \sum_j |⟨ϕ_i, ξ_j⟩|^2 a_j^{(k)} (ϕ_i ϕ_i^H)$$

$$= \sum_i a_i^{(k+1)} (ϕ_i ϕ_i^H)$$

by definition of $a_i^{(k+1)}$.

Since $|⟨ϕ_i, ξ_j⟩|^2$ is the $(i, j)$–entry of $Q^2$ (Def. 2.2.4.2), we obtain the recursion $a^{(k+1)} = Q^2 \cdot a^{(k)}$ and therefore

$$a^{(k)} = (Q^2)^k \cdot a^{(0)},$$

(9)

where $a^{(k)}$ is the column vector with entries $a_i^{(k)}$.

Note that by Lem. 2.2.2.1 we have $∥ξ_j∥^2 = ∥Q ϕ_j∥^2 = 1$ and therefore $Q^2$ is a stochastic matrix [17]; that is, its entries are non-negative and its column sums are 1.
Therefore, from Eq. 9, the mixing coefficients $a^{(k)}$ in the $A$–basis are the probabilities of a formal, discrete-time Markov process with transition matrix $Q^{[2]}$. (But it is important to note that there is no physical Markov process which is transitioning between states $\phi_1, \phi_2, \ldots$; the quantum states are actually $M_{Q,A}^{(k)}[\Psi_0]$, each of which is a mixture of all $\phi_1, \phi_2, \ldots$)

Thus, by a standard result [17], the mixing coefficients

$$a(t) \overset{\text{def}}{=} e^{\lambda (M_{Q,A} \cdot \cdot \cdot - I)t} a(0)$$

corresponding to the state

$$\Psi(t) = \sum_i a_i(t) (\phi_i \phi_i^H)$$

are the probabilities of a formal, continuous-time, Poisson-Markov process with transition matrix $Q^{[2]}$ and Poisson rate $\lambda$. Therefore the full quantum stochastic process $\Psi(t_1, \ldots, t_n)$ given by Eq. ?? is the tensor product of such Poisson-Markov states. The

Since the conditional $f(\cdot)$ is compatible, there is an orthonormal basis $\psi_1, \psi_2, \ldots$ in which every $f(\alpha)$ diagonalizes. We now want to express the diagonal entries $\pi_{t_1,\ldots,t_n}(i_1,\ldots,i_n)$ of the matrix of $\Psi(t_1,\ldots,t_n)$ in the $\{\psi_i\}$ basis.

Let $U$ be the change-of-basis matrix so that $U$ is unitary and $\psi_i = \sum_j U_{ij} \phi_j$. ⋯

□

3 Ion Channels

3.1 Ion Channels and Models

Definition 3.1.0.3. Let $K$ be the kinetic rate matrix for the HMM of an ion channel. (For example, obtained from the Hodgkin-Huxley model).

Let $p_\infty$ be the equilibrium distribution of the Markov process. Let $\pi$ be a complex vector such that $\pi^{[2]} = p_\infty$. Let $D_\pi$ be an invertible matrix satisfying $D_\pi \cdot \pi = p_\infty$ and $1 \cdot D_\pi = \pi^H$ where $1 \overset{\text{def}}{=} [1, 1, \ldots, 1]$ (For example, we could use $D_\pi \overset{\text{def}}{=} \text{diag}(\pi^H)$ if all entries of $p_\infty$ are positive.)

Finally let

$$\lambda(T) = \frac{k_B T_0}{h} Q^{(T-T_0)/10}$$
be the Poisson transition rate as a function of absolute temperature $T$ where $\mu$ is a dimensionless Poisson efficiency constant. (Hodgkin-Huxley use $Q_{10} = 3$.)

Then the configuration energy operator $E$ is the unique nonnegative definite hermitian operator solving

$$K = \lambda(T) \cdot D_\pi \cdot e^{-E/k_BT} (I - \pi \pi^H) \cdot D_\pi^{-1}. \quad (10)$$

**Definition 3.1.0.4.** Let $U$ be the matrix in configuration coordinates which diagonalized the configuration energy operator $E$. Define $W \overset{\text{def}}{=} D_\pi^{-1} U |^2$ and $\tilde{E} \overset{\text{def}}{=} W^{-1} E W$.

The ion activator $Q(T)$ is a choice of an operator for an appropriate range of temperatures $T$ which satisfies

$$Q(T)^{[2]} = I + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \cdot 1 - e^{-\tilde{E}/k_BT}. \quad (11)$$

**Theorem 3.1.0.2.** Let $Q(T)$ be an ion activator for the channel and $T$ a valid temperature.

i. $Q(T)$ is the matrix in the $E$–basis of an activator for the the energy operator $E$.

ii. The hidden quantum model generated by the activator $Q(T)$ and the ion’s conductivity posteriors has precisely the same distribution functions as the hidden Markov model associated with the kinetic rate matrix $K$.

iii. As $T \to 0$, the channel becomes frozen in any initial state. As $T \to \infty$, the channel jumps to equilibrium instantly.

### 3.2 Neuronal Noise

**Lemma 3.2.0.1.** At equilibrium, an HMM with rate matrix $K$ is 2nd-order stationary with autocovariance function

$$R(t) = \mu_f e^{-K|t|} R_\infty \mu_f^T + \delta(t) \cdot \sigma_f^2 p_\infty,$$

where $R_\infty \overset{\text{def}}{=} \text{diag}(p_\infty) - p_\infty p_\infty^T$ and $\mu_f$, $\sigma_f^2$ are the row vectors of means and variances of the conditional distributions.
4 Conclusion

4.1 Quantum Foundations

As pointed out in Rem. 2.2.1.5, Eq. 5 is a genuine extension of the axioms of quantum mechanics. It cannot be derived from standard formulations since none of those assign a meaning to the concept of statistical moments $\mathbb{E}[A_1(t_1) \cdots A_n(t_n) | \Psi]$ at distinct times. Interpreting this concept is certainly challenging as we are not allowed to imagine performing actual measurements at times $t_1, \ldots, t_n$ to calculate the product $A_1(t_1) \cdots A_n(t_n)$ since every such measurements would cause wavefunction collapse.

We do not minimize these foundational issues. Nevertheless, conventional quantum formulations were created with physics in mind, which is satisfied if the statistics at a single time $t$ are well-defined, while signal processing simply cannot do without the concept of correlated behavior at distinct times. For this reason the notion of a correlatable set $\mathcal{A}$ of observables (Def. 2.2.1.2) was introduced: we regard the problem of identifying the set $\mathcal{A}$ of appropriate observables for a particular quantum process $\Psi(\cdot)$ as a key part of the formalization of any application area.

Similar remarks apply to the definition of hidden quantum models (Def. 2.2.3.2) which were meant to bridge the gulf separating underlying quantum-level processes from macroscopic classical instrument readings. Such a concept was needed, at the minimum, to explain the excellent fit of classical Markov kinetics to the ion channel noise measurements in which we were particularly interested.

5 Appendices

5.1 Precis of Non-Relativistic Quantum Mechanics (7, 20–22)

General quantum [7, 21, 22] and statistical mechanics [25, 26] as well as stochastic processes [17, 27] have a vast and accessible literature. The following is a minimalist quantum theory compatible with any detailed quantum model of ion channels.

Let $\mathbb{H}$ be a fixed separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ [23]. The space $\mathbb{H}$ is the configuration space of the quantum system. Vectors $\Psi \in \mathbb{H}$ are sometimes called wavefunctions or "kets" in Dirac's terminology [20].

In order to simplify the presentation of quantum-based reasoning, in this
Appendix we fix an orthonormal basis for $\mathbb{H}$ so that all vectors are (possibly infinite) columns of complex numbers, dual vectors (Dirac’s “bras” [20]) are rows, and continuous linear operators are (possibly infinite) square matrices.

In this way, operators act by matrix multiplication, the inner product $\langle \psi, \phi \rangle$ is just $\psi^H \phi$ where $(\cdot)^H$ denotes the hermitian transpose (i.e., the adjoint [?]), and the projection operator $P_\psi$ onto a unit vector $\psi$ is the rank-1 matrix $P_\psi = \psi \psi^H$.

We will regard bounded, normal operators as compact [23] (so that orthonormal bases of eigenvectors and countable spectral decomposition exist) and even non-degenerate (so eigenspaces are 1-dimensional) when convenient. We will often treat $\mathbb{H}$ as finite-dimensional to simplify formulas. These simplifications can all be replaced by appropriate functional analysis generalizations.

An operator $A$ is of trace class [19] if

$$\sum_i \langle \phi_i, (A^H A)^{1/2} \phi_i \rangle < \infty$$

for some orthonormal basis $\phi_1, \phi_2, \ldots$. Trace class operators are always compact [19], hence continuous.

The trace of a trace class operator $A$ is the (absolutely convergent) series

$$\text{tr}(A) \equiv \sum_i \langle \phi_i, A \phi_i \rangle.$$

This value is independent of the orthonormal basis chosen. With our fixed-basis simplification, the trace is just the sum of the diagonal entries of the matrix $A$.

Note that the set $T(\mathbb{H})$ of trace class operators is closed under convergence in the operator norm and also by left- and right-multiplication by bounded operators $B(\mathbb{H})$ [19]. Thus, for $A \in T(\mathbb{H})$ and $B \in T(\mathbb{H})$, we have both $AB, BA \in T(\mathbb{H})$ and $\text{tr}(AB) = \text{tr}(BA)$.

(In §5.2 the definition of trace is generalized. See Def. 5.2.0.9.)

A state of the system [7,22] is then a non-negative definite, self-adjoint operator $\Psi$ on $\mathbb{H}$ of trace class such that $\text{tr}(\Psi) = 1$.

The set $S(\mathbb{H})$ of all states is a convex subset of the set of bounded operators; that is, if $\Psi_1, \Psi_2, \cdots \in S(\mathbb{H})$ and $a_1, a_2, \cdots$ are non-negative real numbers such that $\sum_i a_i = 1$ then $\sum a_i \Psi_i \in S(\mathbb{H})$. The expression $\sum a_i \Psi_i$ is called a mixture of the states $\Psi_1, \Psi_2, \cdots$. 

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The boundary $\partial S(H)$ of $S(H)$ consists of the **pure** states: those which cannot be expressed as a non-trivial mixture of other states. The state $\Psi$ is pure if and only if it is a projection onto a 1-dimensional subspace; i.e., $\Psi = P_\psi = \psi \psi^H$ for some unit vector $\psi$ which is unique up to multiplication by a complex number of modulus 1.

If $\psi_1, \psi_2, \cdots$ is an orthonormal basis, then a state $\Psi$ is **coherent** with respect to this basis if it is a mixture of the projections $P_{\psi_1}, P_{\psi_2}, \cdots$ and **incoherent** otherwise. A state is coherent with respect to $\psi_1, \psi_2, \cdots$ if and only if it diagonalizes in that basis. If $a_1, a_2, \cdots$ are the mixture coefficients of the coherent state $\Psi$, then it is consistent to interpret this state by saying the physical system has probability $a_1$ of being in the state $P_{\psi_1}$, probability $a_2$ of being in the state $P_{\psi_2}$, etc. However it is critical to interpret these probabilities in a **frequentist** sense [28] because the absence of so-called “hidden variables” excludes the possibility of an underlying Kolmogoroff state space [29].

Real-valued observables $A$ of the system are associated in 1-1 manner with (not necessarily bounded) self-adjoint operators $A$ whose spectrum $\sigma(A)$ [23, 30] are precisely the possible measurements of $A$. Letting $\text{Prob}[\cdot | \Psi]$ denote relative frequencies for systems in prepared in state $\Psi$ then a central postulate of quantum theory is:

**Definition 5.1.0.5. Born’s Axiom (probability form)**

Let $\alpha \in \sigma(A)$ be a value of the observable $A$. Observations on systems prepared in state $\Psi$ will produce the result $A = \alpha$ with frequency

$$\text{Prob}[A = \alpha | \Psi] = \text{tr}(P_\alpha \cdot \Psi),$$

where $P_\alpha$ is the projection operator onto the $\alpha$–eigenspace of $A$.

For bounded operators and using an appropriate measure on the spectrum $\sigma(A)$ of $A$ [22, 23] we can (informally) write a spectral decomposition $A = \int \alpha \ P_\alpha \ d\alpha$. This yields an equivalent but more useful form of Def. 5.1.0.5 in terms of expected values:

**Definition 5.1.0.6. Born’s Axiom (expectation form)**

Let $A$ be a bounded observable. Systems prepared in state $\Psi$ will produce a frequency average value for $A$ given by

$$E[A | \Psi] = \int_{\sigma(A)} \alpha \cdot \text{Prob}[A = \alpha | \Psi] \ d\alpha = \int_{\sigma(A)} \alpha \cdot \text{tr}(P_\alpha \cdot \Psi) \ d\alpha = \text{tr} \int_{\sigma(A)} \alpha \ P_\alpha \ d\alpha \cdot \Psi$$

$$= \text{tr}(A \cdot \Psi).$$

(12)
Terminology concerning experiments, measurements, observations, and the like is not standardized but we will adopt the following conventions:

- **An experiment** specifies a measurable partition \( \sigma(A) = \bigcup_i E_i \), \( E_i \cap E_j = \emptyset \) for \( i \neq j \), of the measure space \( \sigma(A) \) of possible values of an observable \( A \). The experiment itself consists of determining one and only one index \( i \) such that \( A \in E_i \). It is conceptually important that no information other than this index be produced by the experiment. It is also important, however, that some value for \( A \) is obtained even though the experiment compresses the information about this value to a single index.

- An experiment is **reductive** (or of the **first kind** \([7]\)) if, after a determination \( A \in E_i \) is made, then repetitions of the experiment on the system will always produce the same result. The experiment reduces all states to the subset of \( \mathcal{S}(\mathcal{H}) \) for which \( A \) is certain to take values in \( E_i \).

- An experiment is **nonreductive** (or of the **second kind** \([7]\)) if it is not reductive. An example of a nonreductive experiment is the determination of a particle’s momentum by bombarding it with other particles.

- A **measurement** is an experiment in which the partition consists of the single set \( E = \sigma(A) \); i.e., some value for \( A \) is obtained but no information about it is available. Note that a measurement is trivially reductive. This is not a meaningless experiment: because of the Measurement Axiom Def. 5.1.0.7 discussed subsequently, the act of obtaining some value for \( A \) will change the state.

- An **observation** is an experiment using the singleton partition \( \sigma(A) = \bigcup_\alpha \{\alpha\} \). A reductive observation of a non-degenerate observable reduces (or **collapses**) any state \( \Psi \) to a wavefunction: \( \Psi \rightarrow \psi_\alpha \psi_\alpha^H, \|\psi_\alpha\| = 1 \).

- A state \( \Psi \) is a **measurement state** for the observable \( A \) if \( A\Psi = \Psi A \). In non-degenerate cases this is equivalent to: \( \Psi \) is coherent with respect to an orthonormal basis of eigenvectors of \( A \); that is, it is a mixture of pure eigenstates of \( A \). It is only in measurement states that we can regard \( A \) as having some definite but unknown value. Otherwise there is “quantum interference” between the possible values for \( A \) and we have to regard the observable as having **all** values simultaneously.

  The classic example of quantum interference is the two-slit photon experiment in which the wavefunction \( \psi(x, y) \) describing the probability of a photon hitting a point \((x, y)\) on the target screen is not a measurement state for the observable corresponding to the slit though which a photon passed. As a result the photons must be regarded as passing through both slits simultaneously even though they are particles.
Every reductive experiment on $A$ collapses the prior state to a measurement state. In fact, we have the following

**Definition 5.1.0.7. Measurement (or Observation or Projection) Axiom**

Let $\sigma(A) = \bigcup_i E_i$, $E_i \cap E_j = \emptyset$ for $i \neq j$ be the partition associated with an experiment on the observable $A$. Prepare an ensemble of systems in state $\Psi$. Then:

(i.) The determination $A \in E_i$ will occur with frequency

$$\text{Prob}[E_i | \Psi] \overset{\text{def}}{=} \int_{E_i} \text{tr}(P_\alpha \Psi) \ d\alpha,$$

where $P_\alpha$ is the projection onto the $\alpha$–eigenspace. (Another form of Born’s Axiom.)

(ii.) If the experiment is reductive, the state $\Psi$ will reduce to the $A$–measurement state

$$\Psi_{E_i} \overset{\text{def}}{=} \frac{1}{\text{Prob}[E_i | \Psi]} \int_{E_i} P_\alpha \Psi P_\alpha \ d\alpha.$$

**Corollary 5.1.0.1. Measurement and Observation**

(i.) If a measurement is made of the observable $A$ in state $\Psi$ (so that the value is unknown), the state will reduce to the $A$–measurement state

$$\Psi_A \overset{\text{def}}{=} \int_{\sigma(A)} P_\alpha \Psi P_\alpha \ d\alpha.$$

(ii.) If $A$ is non-degenerate and the experiment is reductive, the observation $A = \alpha$ made in state $\Psi$ will reduce the system to the pure $A$–measurement state

$$\Psi_\alpha = \frac{1}{\|\psi_{\alpha}\|^2} \psi_{\alpha} \psi_{\alpha}^H.$$

The latter example is referred to as **wavefunction collapse**. The Measurement Axiom and its consequences cause enormous philosophical difficulties for the foundation of quantum mechanics because of the apparent physical role of the observer: the final physical state of the system seems to depend upon what the observer “knows”. What observer? How does he know? How does the system “know” he knows? The famous **Schrödinger’s**
Cat paradox in which a cat inside a closed box is neither alive nor dead until we look at it is the classic example of the still unresolved issues of this Measurement Problem.

Dynamics is modeled as a unitary representation $U(t)$ of the additive group of $\mathbb{R}$. In the typical case, $U(t) = e^{-\frac{2\pi}{\hbar}Ht}$, where $H$ is the Hamiltonian of the system \[7,20–22\].

If no disturbance by an experiment is made on the system between its initial state $\Psi_0$ and time $t$, the system then will be in state $\Psi(t) = U(t) \cdot \Psi_0 \cdot U(t)^\dagger$, which is the integrated form of Schrödinger’s Equation.

The function $\Psi(\cdot)$ characterizes all stochastic properties of the system as long as there is no disturbance. Thus, in some sense, every observable $A$ defines a physical stochastic process \(\{A(t)\}_{-\infty < t < \infty}\) for which we have

$$\mu_A(t) \overset{\text{def}}{=} E[A(t) \mid \Psi] = \text{tr}(A \cdot \Psi(t)).$$

However, even if $\Psi_0$ is a measurement state for $A$, generally $U(t) \cdot \Psi_0 \cdot U(t)^\dagger$ will not be one (unless $A$ is a conserved quantity which means it commutes with every $U(t)$). So we have the paradoxical situation that the process $A(t)$ always has an average value but not always a particular value.

5.2 Tensor Products of Configuration Spaces and the Partial Trace

The tensor product $\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n$ of separable Hilbert spaces $\mathbb{H}_1, \mathbb{H}_2, \cdots, \mathbb{H}_n$ can be defined \[?\] as the complex Hilbert space with basis consisting of the formal expressions $\phi_{i_1}^1 \otimes \phi_{i_2}^2 \otimes \cdots \otimes \phi_{i_n}^n$, where $1 \leq i_1 \leq \dim(\mathbb{H}_1), \ldots, 1 \leq i_n \leq \dim(\mathbb{H}_n)$, and every $\{\phi_{i}^k\}_{1 \leq i \leq \dim(\mathbb{H}_k)}$ is a fixed basis for $\mathbb{H}_k$. Thus

$$\dim(\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n) = \dim(\mathbb{H}_1) \times \cdots \times \dim(\mathbb{H}_n).$$

The inner product on $\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n$ is defined on basis elements by

$$\langle \phi_{i_1}^1 \otimes \cdots \otimes \phi_{i_n}^n, \phi_{j_1}^1 \otimes \cdots \otimes \phi_{j_n}^n \rangle \overset{\text{def}}{=} \langle \phi_{i_1}^1, \phi_{j_1}^1 \rangle \cdots \langle \phi_{i_n}^n, \phi_{j_n}^n \rangle$$

and then extended to the whole space by linearity.

It follows that this definition is basis-independent and, if every $\{\phi_{i}^k\}_{1 \leq i \leq \dim(\mathbb{H}_k)}$ is orthonormal, then so is

$$\{\phi_{i_1}^1 \otimes \phi_{i_2}^2 \otimes \cdots \otimes \phi_{i_n}^n\}_{1 \leq i_1 \leq \dim(\mathbb{H}_1), \ldots, 1 \leq i_n \leq \dim(\mathbb{H}_n)}.$$
For a single Hilbert space $\mathbb{H}$ and for $n > 0$ then we define $\mathbb{H}^\otimes n \overset{\text{def}}{=} \bigotimes_{i=1}^n \mathbb{H}$. We can also consistently define $\mathbb{H}^\otimes 0 \overset{\text{def}}{=} \mathbb{C}$, the complex scalar field.

The tensor product $\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n$ satisfies an important universal mapping property [?]. Let $\mathbb{K}$ be a complex vector space and $f : \mathbb{H}_1 \times \cdots \times \mathbb{H}_n \to \mathbb{K}$ be a function which is linear in each variable separately. Then there is a unique linear operator $\otimes f : \mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n \to \mathbb{K}$ such that

$$(\otimes f)(\psi_1 \otimes \cdots \otimes \psi_n) = f(\psi_1, \cdots, \psi_n)$$

for all $\psi_1 \in \mathbb{H}_1, \ldots, \psi_n \in \mathbb{H}_n$.

In particular, linear operators $A_1, \ldots, A_n$ on $\mathbb{H}_1, \ldots, \mathbb{H}_n$ define a unique linear operator $A_1 \otimes \cdots \otimes A_n$ on $\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n$ satisfying

$$(A_1 \otimes \cdots \otimes A_n)(\psi_1 \otimes \cdots \otimes \psi_n) \overset{\text{def}}{=} (A_1 \psi_1) \otimes \cdots \otimes (A_n \psi_n),$$

for all $\psi_1 \in \mathbb{H}_1, \ldots, \psi_n \in \mathbb{H}_n$. If $A_1, \ldots, A_n$ are bounded then clearly $A_1 \otimes \cdots \otimes A_n$ is as well with $\|A_1 \otimes \cdots \otimes A_n\| = \|A_1\| \cdots \|A_n\|$.

Example 5.2.0.1. Suppose $\{\phi_k^i \mid 1 \leq i \leq d_k\}$ is a basis for $\mathbb{H}_k$ in which $A_k$ has coefficient matrix $[a_{ij}^k \mid 1 \leq i, j \leq d_k]$, for $k = 1, \ldots, n$. Then, in basis $\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_n} \mid 1 \leq i_1 \leq d_1, \ldots, 1 \leq i_n \leq d_n\}$, the operator $A_1 \otimes \cdots \otimes A_n$ has coefficient matrix

$$[a_{(i_1 \cdots i_n)(j_1 \cdots j_n)} \mid 1 \leq i_1, j_1 \leq d_1, \ldots, 1 \leq i_n, j_n \leq d_n]$$

where

$$a_{(i_1 \cdots i_n)(j_1 \cdots j_n)} = a_{i_1 j_1}^1 \cdots a_{i_n j_n}^n. \quad (13)$$

This follows from uniqueness by verifying the universal property on basis elements then extending to the whole space.

Definition 5.2.0.8. Let $A$ be a bounded linear operator on $\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n$. Let $1 \leq k \leq n$ and $\phi \in \mathbb{H}_k$ be fixed. By the Riesz Representation Theorem [?] there is a unique bounded linear operator $A \mid_k \phi$ on $\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_k \otimes \cdots \otimes \mathbb{H}_n$ such that for all $\psi_1, \chi_1 \in \mathbb{H}_1, \ldots, \psi_{k-1}, \chi_{k-1} \in \mathbb{H}_{k-1}, \psi_{k+1}, \chi_{k+1} \in \mathbb{H}_{k+1}, \ldots, \psi_n, \chi_n \in \mathbb{H}_n$ we have

$$\langle \psi_1 \otimes \cdots \otimes \psi_{k-1} \otimes \psi_{k+1} \otimes \cdots \otimes \psi_n, (A \mid_k \phi) (\chi_1 \otimes \cdots \otimes \chi_{k-1} \otimes \chi_{k+1} \otimes \cdots \otimes \chi_n) \rangle = \langle \psi_1 \otimes \cdots \otimes \psi_{k-1} \otimes \phi \otimes \psi_{k+1} \otimes \cdots \otimes \psi_n, A (\chi_1 \otimes \cdots \otimes \chi_{k-1} \otimes \phi \otimes \chi_{k+1} \otimes \cdots \otimes \chi_n) \rangle,$$
where the caret \( \hat{\cdot} \) above a symbol indicates an item missing from a list; that is, \( \phi \) is inserted into both \( k \)th places prior to the application of \( A \) and the inner product. We call \( A \mid_k \phi \) a restriction of \( A \) along the \( k \)th dimension.

**Definition 5.2.0.9.** Let \( A \) be a trace class operator \([?]\) on \( \mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n \) and let \( \phi_1, \phi_2, \cdots \) be an orthonormal basis for \( \mathbb{H}_k \) with \( 1 \leq k \leq n \). Then the partial trace of \( A \) along the \( k \)th dimension is

\[
\text{tr}_k (A) \overset{\text{def}}{=} \sum_{i=1}^{\dim(H_k)} (A \mid_k \phi_i),
\]

where the convergence is in the operator norm. Note that \( \text{tr}_k (A) \) is a bounded linear operator on \( \mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_k \otimes \cdots \otimes \mathbb{H}_n \).

**Remark 5.2.0.3.** Since \( A \) is a trace class operator it can be shown that \( \text{tr}_k (A) \) exists and is independent of the orthonormal basis chosen.

### 5.3 Symbols and Conventions

- "\( \overset{\text{def}}{=} \)" means "equal by definition" and "\( \overset{\text{def}}{\iff} \)" means "logically equivalent by definition."
- \( h \) is Planck’s constant with units Joule-sec.
- \( k_B \) is Boltzmann’s constant with units Joule/\( ^\circ \)K.
- \( \eta \) is a dimensionless Markov transition efficiency.
- Italic "\( T \)" denotes absolute temperature with units \( ^\circ \)K.
- Non-italic superscript \((\cdot)^T\) denotes the matrix transpose without complex conjugation.
- Unless otherwise indicated, the scalar field is the complex numbers \( \mathbb{C} \).
- \( \mathbb{N} \) are the non-negative integers.
- \( \mathbb{H} \) denotes a general Hilbert space and \( \mathbb{H}^\otimes n \overset{\text{def}}{=} \mathbb{H} \otimes \cdots \otimes \mathbb{H} \).
- \( \mathcal{B}(\mathbb{H}) \) are the bounded linear operators on \( \mathbb{H} \), \( \mathcal{T}(\mathbb{H}) \subseteq \mathcal{B}(\mathbb{H}) \) are the trace class operators, and \( \mathcal{S}(\mathbb{H}) \subseteq \mathcal{T}(\mathbb{H}) \) are the states.
- \((\cdot)^*\) is complex conjugation applied to scalars, vectors, or matrices, \(|z| = \sqrt{z \cdot z^*}\) is absolute value, and \( \text{sgn} (z) \overset{\text{def}}{=} z^*/|z|\) is the complex sign function.
- \((\cdot)^{\text{H}}\) is the hermitian transpose \(((\cdot)^*)^T\).
- If \( A \) is a matrix of any order with complex entries, then \( A^{[2]} \) denotes the real matrix of the same order whose entries are the squared moduli \( |a|^2 \) of the entries \( a \) of \( A \). (cf. Thm. 2.2.4.1 for the use of this operation.)
• $I_M$ is the $(M \times M)$ identity matrix or operator with the subscript omitted if it is clear from context.
• $\text{det}(\cdot)$ is the determinant.
• $\text{diag}(\vec{a})$ is the $(M \times M)$ diagonal matrix whose diagonal is the $M$-dimensional (row or column) vector $\vec{a}$.
• $\text{tr}(\cdot)$ is the trace operator on square matrices or trace class operators.
• Unless otherwise indicated, “vector” means “column vector.” Note this is the dual of the most common convention for Markov processes so certain expressions such as the Chapman-Kolmogoroff Equations (17) may appear transposed.
• $\delta(t)$ is the delta function.
• $E[\cdot]$ is the expected-value operator, $\text{Var}[\cdot]$ is variance, and $\text{Prob}(\cdot)$ will be used informally for “the probability of”.
• The caret $\hat{\cdot}$ above a symbol indicates a item missing from a list.
• The symbols “$i,j$” will always be used as integer indices and never to denote the imaginary unit $\sqrt{-1}$.
• When required all Hilbert spaces are assumed to be separable and all bounded normal operators will be assumed to be compact [23] so that an orthonormal basis of eigenvectors and a countable spectral decomposition exist.

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