STABILITY OF DYNAMIC GROWTH OF TWO ANTI-SYMMETRIC CRACKS USING PDS-FEM

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This paper studies the stability of dynamic crack growth in a homogeneous body, carrying out a numerical experiment of a plate with two anti-symmetric cracks. PDS-FEM proposed by the authors is extended to dynamic state and used in the numerical experiment. It is shown that while a common process is not found for the crack growth, there are two dominant patterns for the final crack configuration. The first pattern is anti-symmetric, indicating the stability of the homogeneous body solution, and the second pattern is not anti-symmetric, suggesting that the solution becomes unstable. It is also shown that higher loading rate tends to shift the crack configuration to the second pattern, losing the stability of the solution.

Key Words: stability/instability, dynamic crack growth, particle discretization scheme, finite element method, Monte-Carlo simulation

1. INTRODUCTION

The stability of dynamic crack growth has been a challenging problem in solid continuum mechanics; for instance, see a list of references related to numerical computation. Effects of linear/non-linear material properties or boundary conditions as well as initial configuration on the crack growth have been examined. In this paper, we seek to provide a new viewpoint, the material heterogeneity effect, emphasizing that mathematically, the stability is a nature of a solution and examined by adding certain perturbation to a problem.

Based on the above idea, our target is a weakly heterogeneous body, which is made by adding distribution of small material heterogeneity to an ideally homogeneous body. Thus, it is regarded that a crack growth solution is unstable, if it is changed non-negligibly from a solution of the ideally homogeneous body. A numerical experiment which uses a set of such weakly heterogeneous bodies is carried out for the stability analysis of dynamic crack growth; these bodies correspond to experiment samples used for actual failure tests.

In order to carry out a numerical experiment, we extend PDS-FEM (Particle-Discretization-Scheme FEM) to dynamic state. PDS-FEM is originally formulated for Lagrangian at quasi-static state, and hence the extension to dynamic state is straightforward. Special attentions, however, have to be paid to time integration since cracking releases strain energy. High robustness is required for the time integration, and we adopt Hamiltonian formulation so that most robust algorithm which is proposed in the field of computational quantum mechanics is employed for the time integration.

Based on experimental experiences, it is natural to expect that the crack growth stability depends on the loading rate. That is, crack growth leads to shattering at higher loading, while crack growth becomes smoother at slower loading. The
dependence of the crack growth on the loading rate is investigated in the numerical experiment.

The content of the present paper is as follows: First, in Section 2, we discuss modeling of weak heterogeneity. It is explained that modeling is made by selecting candidates of possible crack extensions. The extension of PDS-FEM to dynamic state is explained in Section 3. Discretized Hamiltonian is derived so that a robust algorithm is applied to the time integration. In Section 4, a numerical experiment is carried out for a plate in which a pair of anti-symmetric cracks are located. The stability of the dynamic crack growth is analyzed based on the results which use a wide range of loading rates. Concluding remarks are pointed out in Section 5.

It should be noted that this paper is regarded as the extension of the authors’ study of evaluating the variability in cracking by using PDS-FEM\(^{(4,5,6)}\). The extension is aimed at analyzing dynamic state; another problem of dynamic state is studied\(^{(6)}\). The variability is due to the distribution of local heterogeneities, which is modeled by using a set of PDS-FEM models with different meshing. While the accuracy of PDS-FEM with linear tetrahedron elements is basically the same as ordinary FEM with linear tetrahedron elements, the accuracy of evaluating the cracking variability is evaluated by checking the convergence of the statistical characteristics of cracking. In this paper, the statistical characteristics are computed for the location of the crack path, and it is confirmed that the mean and standard deviation of the location converge as around 200 models with different meshing are simulated; the simulation could be regarded as a Monte Carlo simulation which uses different models.

2. MODELLING OF WEAKLY HETEROGENEITY FOR CRACKING

For brittle materials, it is usually observed that a crack propagates in an unpredictable manner, when subjected to dynamic loading. For instance, kinking and branching are induced during the process of crack growth, or shattering due to multiple cracking is observed at higher loading rates. This unpredictable pattern of the crack path is in contrast with the smooth path of fatigue cracks which are subjected to quasi-static loading.

We find disadvantages in applying fracture mechanics to the stability analysis of dynamic crack growth. This is mainly because fracture mechanics is aimed at clarifying a condition for a pre-existing crack to start to grow; this condition is sufficient to prevent cracking due to fatigue, thermal stress or stress corrosion. Less attention is paid to clarify the mechanism of how a crack grows subjected to dynamic loading.

Another disadvantage of applying fracture mechanics to the stability analysis is the use of stress intensity factor or a related quantity. Estimating the stress intensity factor requires some amount of numerical computation. No data are found for data related to spatial distribution of fracture toughness that is a critical value of the stress intensity factor; it is inherently assumed that fracture toughness is uniformly distributed even though the value may change for each specimen or sample.

The key task of this paper is a numerical experiment that uses a set of weakly heterogeneous bodies. We\(^{(6)}\) take a simple treatment of weak heterogeneity, as

material properties are uniform except for a parameter of fracture, and cracking is allowed only on some of predetermined weak plane segments.

It is natural to measure the weakness of the plane segment in terms of material strength. Thus, instead of a fracture mechanics based criterion, we adopt a material strength criterion, i.e., if the average stress over a plane segment exceeds the strength, a crack is initiated on the segment or extends itself to the segment. For simplicity, we assume only tensile failure, ignoring shear failure. As will be explained later, PDS-FEM uses a boundary facet as a set of such pre-determined weak plane segments for possible crack extensions.

The stress averaged over a plane segment can be interpreted as the surface integration of stress that is required for a crack to grow all over the segment; while the stress near the crack tip is singular, the surface integration on the segment is bounded. The critical value at which cracking take place on the plane segment varies depending on the area of the segment as well as the segment edge at which the crack tip reaches. It is surely possible to apply a fracture mechanics based criterion in a numerical experiment of PDS-FEM, although the material strength criterion is used in this paper.

As studied by the authors\(^{(4,5)}\), the mesh size is regarded as a parameter which represents the degree of material heterogeneity; the size becomes smaller as the distribution of material parameters is closer to being uniform. For the stability analysis of dynamic crack growth, however, we may not need to identify the degree of material het-
erogeneity or the mesh size, since a smaller mesh size surely better represents the weak heterogeneity. Thus, in the numerical experiment presented in Section 4, finer meshing is used near the tip of the original crack, to allow a wider choice of crack extension, while meshing becomes coarser farther from the crack tip.

3. EXTENSION OF PDS-FEM TO DYNAMIC STATE

On the viewpoint of the numerical computation, it is not easy to analyze the crack growth, except for determining geometry of the crack extension, since cracking releases strain energy stored in a region which surrounds the crack extension. A robust algorithm that is able to compute such an abrupt change in strain energy is required. The algorithm is also required to guarantee symplecticity, i.e., the total energy is conserved during the crack growth process.

A robust algorithm of time integration has been studied in the field of computational quantum mechanics. To implement such an algorithm, we formulate the dynamic extension of PDS-FEM using Hamiltonian. It is certainly true that the use of Hamiltonian is rare in continuum mechanics as well as numerically less efficient in the sense that the degree-of-freedom is doubled. However, for accurate time integration, we take this uncommon formulation; the numerical efficiency is not much worse than a time integration scheme which uses velocity and acceleration such as the Newmark method.

We start from the following Lagrangian of a linearly and isotropically elastic solid, denoted by $B$, with elasticity $c$ and density $\rho$:

$$
\mathcal{L}[\mathbf{u}, \dot{\mathbf{u}}, \mathbf{c}, \sigma] = \int_B \left[ \frac{1}{2} \epsilon : \epsilon - \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \sigma : (\nabla \mathbf{u} - \epsilon) \right] \, dv,
$$

(1)

where $\mathbf{u}$, $\epsilon$ and $\sigma$ are displacement, strain and stress, respectively; $\cdot$ and $:$ stand for the first and second contraction; and $\nabla(\cdot)$ and $(\cdot)$ are the spatial and temporal derivative of $(\cdot)$. It is readily derived that $\delta(\int \mathcal{L} \, dt) = 0$ with respect to $\epsilon$ and $\sigma$ leads to $\sigma = \mathbf{c} : \epsilon$ and $\epsilon = \text{sym}(\nabla \mathbf{u})$ with sym being the symmetric part of the second-order tensor, and that $\delta(\int \mathcal{L} \, dt) = 0$ with respect to $\mathbf{u}$ leads to the wave equation, $\rho \ddot{\mathbf{u}} = \nabla \cdot (\mathbf{c} : (\nabla \mathbf{u}))$.

(1) Discretization of Lagrangian by means of PDS-FEM

PDS-FEM uses dual domain decomposition of $B$, the Voronoi and Delaunay tessellations denoted by $\{\Phi^\alpha\}$ and $\{\Psi^\beta\}$, and employs characteristic functions of $\Phi^\alpha$ and $\Psi^\beta$, denoted by $\phi^\alpha$ and $\psi^\beta$, as basis functions of discretizing function and its spatial derivatives, respectively. Functions used in $\mathcal{L}$ are discretized as

$$
\mathbf{u}(x, t) = \sum_\alpha \mathbf{u}^\alpha(t) \phi^\alpha(x),
$$

$$
\epsilon(x, t) = \sum_\beta \epsilon^\beta(t) \psi^\beta(x),
$$

$$
\sigma(x, t) = \sum_\beta \sigma^\beta(t) \psi^\beta(x).
$$

(2)

Substituting Eqs. (2) into Eq. (1) and computing $\delta(\int \mathcal{L} \, dt) = 0$ with respect to $\epsilon^\beta$ and $\sigma^\beta$, we obtain $\sigma^\beta = \mathbf{c} : \epsilon^\beta$ and $\epsilon^\beta = \text{sym}(\sum_\alpha \mathbf{b}^{\beta\alpha} \otimes \mathbf{u}^\alpha)$, where the component of $\mathbf{b}^{\beta\alpha}$ is

$$
b_i^{\beta\alpha} = \int \partial \phi^\alpha / \partial x_i \psi^\beta \, dv. \tag{3}
$$

Computing $\delta(\int \mathcal{L} \, dt) = 0$ with respect to $\mathbf{u}$, we arrive at

$$
M^\alpha \ddot{\mathbf{u}}^\alpha(t) + \sum_\alpha \mathbf{K}^{\alpha\alpha'} \cdot \mathbf{u}^\alpha(t) = 0, \tag{4}
$$

where $M^\alpha$ is the mass of $\Phi^\alpha$ and the component of $\mathbf{K}^{\alpha\alpha'}$ is

$$
K_{ik}^{\alpha\alpha'} = \sum_\beta c_{ijkl} b_j^{\beta\alpha} l_i^{\beta\alpha'}.
$$

(5)

Note that $\mathbf{K}^{\alpha\alpha'}$ is an element stiffness matrix of PDS-FEM. This $\mathbf{K}^{\alpha\alpha'}$ coincides with an element stiffness matrix of FEM with linear tetrahedron elements.

It should be noted that Eq. (4) automatically leads to a lumped mass matrix. No approximation is needed to derive the lumped mass matrix, unlike ordinary FEM. This is the advantage of PDS-FEM, since, as shown in Eq. (2), displacement is discretized as a set of rigid body displacement, or a continuum is regarded as an assembly of rigid body particles; see similar but different treatment of particle-like discretization.

A crack or discontinuity in displacement is readily expressed in terms of discontinuous basis functions of $\{\phi^\alpha\}$ provided that it exists only on the interface between the neighboring Voronoi tessellations. It is straightforward to re-compute $\mathbf{b}^{\beta\alpha}$ of Eq. (3) and $\mathbf{K}^{\alpha\alpha'}$ of Eq. (5) when a crack passes through the interface between $\Phi^\alpha$ and $\Phi^\beta$, denoted by $S^{\alpha\alpha'}$. Indeed, due to cracking, new boundaries that correspond to $S^{\alpha\alpha'}$ are created, and they are excluded from the integration over $B$. This leads to the neglecting $\nabla \phi^\alpha$ on $S^{\alpha\alpha'}$, and the value of $\mathbf{b}^{\alpha\beta}$ and $\mathbf{K}^{\alpha\alpha'}$ is changed accordingly.
(2) Discretized Hamiltonian for PDS-FEM

The governing equation for \( u^a \), Eq. (4), is regarded as a discretized Lagrangian equation of the following discretized Lagrangian:

\[
\mathcal{L} = \sum \frac{1}{2} u^a K^{\alpha\alpha} \cdot u^a' - \sum \frac{1}{2} M^a \dot{u}^a \cdot \dot{u}^a',
\]

which is a function of \( u^a \) and \( \dot{u}^a \). Note that the second term gives kinetic energy since PDS-FEM automatically derives a lumped mass matrix.

Once \( \mathcal{L} \) is given, it is straightforward to transform it to the discretized Hamiltonian, denoted by \( \mathcal{H} \), i.e.,

\[
\mathcal{H} = \sum p^a \cdot q^a - \mathcal{L},
\]

where \( p^a = \frac{\partial \mathcal{L}}{\partial \dot{q}^a} \) and \( q^a = u^a \) are the momentum and displacement of \( \Phi^a \). As expected, \( \mathcal{H} \) is explicitly expressed in terms of \( K^{\alpha\alpha} \) and \( M^a \) as

\[
\mathcal{H} = \sum \frac{1}{2} q^a \cdot K^{\alpha\alpha} \cdot q^a' + \frac{1}{2 M^a} p^a \cdot p^a',
\]

and the Hamiltonian equations are

\[
\frac{d}{dt} \left[ \begin{array}{c} p^a \\ q^a \end{array} \right] = \left[ -\frac{1}{M^a} \left( \sum K^{\alpha\alpha} \cdot q^a' \right) \right].
\]

This is the governing equation for \((p^a, q^a)\).

We take advantage of the bilateral symplectic algorithm\(^{(11)}\) as a robust algorithm of the time integration of Eq. (9). The main advantage of this algorithm is that in order to achieve the accuracy of the order of \( \Delta t^N \) with \( \Delta t \) and \( N \) being time increment and an integer, it needs \( 2N \) times iteration for the interval of \( 2\Delta t \). For simplicity, omitting superscript \( a \) and using superscript for the iteration number in the interval of \( 2\Delta t \), this algorithm is formulated as follows:

\[
q^n = q^{n-1} + b_n p^{n-1} \Delta t,
\]

\[
p^n = p^{n-1} - a_n K^n \cdot p^{n-1} \Delta t,
\]

for \( n = 1 \) to \( N \),

\[
p^n = p^{n-1} - b_{n-N} K^{n-1} \cdot p^{n-1} \Delta t,
\]

\[
q^n = q^{n-1} + a_{n-N} p^{n-1} \Delta t
\]

for \( n = N + 1 \) to \( 2N \), with \([q^0, p^0]\) and \([q^{2N}, p^{2N}]\) being \([q, p]\) at \( t \) and \( t + 2\Delta t \). It is the set of constants, \((a_n, b_n)\), that guarantee the accuracy of the order \( \Delta t^N \). In the present paper, we use \( N = 2 \), for which \((a_1, b_1) = (\frac{1}{2}, 0)\) and \((a_2, b_2) = (1, 1)\); see Chen\(^{(12)}\) for the detailed examination of the time integration accuracy.

4. NUMERICAL EXPERIMENT

(1) Problem setting

Sato et al.\(^{(13)}\) solved a 2D plane strain problem of anti-symmetric cracks which grow in an ideally homogeneous body, assuming quasi-static state. Following his numerical study, we study a thin plate of \( 5 \times 24.5 \times 140 \text{ mm} \), which includes two anti-symmetric parallel cracks of length 0.6 mm; see Fig. 1. It is assumed that the material is linearly elastic and that a fracture criterion is a material strength one; see Section 2. The material properties are similar to those of epoxy; see Table 1.

![Fig. 1 Plate model with initial cracks.](image)

The displacement boundary condition is posed; the bottom end of the model is fixed, and the top end is pulled up in longitude direction. The final displacement is set as \( U = 0.236 \text{ mm} \). The loading rate is the velocity of the top end, denoted by \( \dot{U} \). In view of the P-wave velocity of the assumed material being 2288 m/s, we use 0.1% of the P-wave velocity as a reference, i.e., \( \dot{V_r} = 0.252 \text{ m/s} \), and examine four loading rates, namely, \( \dot{U}/\dot{V_r} = 1, 3, 9, 27 \). Following the PDS-FEM discretization, we model the crack tip as a notch of the height 0.6 mm; the vertical surface of the notch is discretized by using 2 elements. The average mesh size is 1.0 mm at the top and bottom surfaces of the notch. Due to this discretization, we set the time increment as \( \Delta t = 269, 89.7, 29.9, 2.03 \times 10^{-9} \text{ s} \) for \( \dot{U}/\dot{V_r} = 1, 3, 9, 27 \), respectively. It should be emphasized that the crack tip velocity is assumed to be infinite in the present analysis or a whole facet is fully broken during one increment of loading; less attention is paid to the wave propagation analysis in which a target frequency must be specified for a given time.
increment and an element dimension.

(2) Crack growth process

A Monte-Carlo simulation is used for this numerical experiment of the stability analysis. For a plate model of the identical configuration, different Voronoi and Delaunay tessellations are applied so that models with distinct set of crack plane candidates are generated. Random generation of the tessellation is applied, with a model which has ill-shaped configuration for Delaunay tetrahedrons being excluded. Special cares\(^{(12)}\) are taken not to generate models of similar tessellations; see Chen\(^{(12)}\) in detail.

As a typical example of crack growth process, \textbf{Fig. 2} shows snapshots of growing crack for \(\dot{U}/V_r = 1\). As is seen, the anti-symmetry is lost during the crack growth process, which is in contrast with the quasi-static state simulation made by Sato \textit{et al.}\(^{(13)}\). The crack growth process changes when the loading rate is different; see \textbf{Fig. 3} for \(\dot{U}/V_r = 27\).

For other models, the anti-symmetry is lost during the crack growth process as well, and the process changes depending on the loading rate. It is hard to find common crack growth process: each model has its own process, such as the right crack grows first to some extent and then the left one grows, or both the cracks grow simultaneously until only one grows. The process is literally \textit{chaotic}, and strongly depends on the initial setting of crack extension candidates that are given by the Voronoi boundaries.

(3) Crack path configuration

For simplicity, the crack path configuration at the final loading step is used to quantitatively analyze the stability of the crack path solution. The
crack configuration is measured at the location of the crack that is averaged in the plate thickness direction. The average and standard deviation of the location are plotted in Fig. 4 and Fig. 5 for the case of $\dot{U}/V_r = 1$ and 27, respectively; the horizontal axis is the number of models and the vertical axis is the difference of the vertical coordinate of the crack path at the designated horizontal coordinate from its converged value. As is seen, the average and standard deviation are almost converged when the model number exceeds 100.

The probability density function (PDF) of crack configuration is obtained by using the results of 200 models. A grid of $24,000 \times 20,000$ is used, and the probability of cracking at the $(i,j)$ grid is computed as

$$P_{ij} = \frac{N_{ij}}{N}$$  \hspace{1cm} (12)$$

where $N_{ij}$ is the number of models in which cracking takes place at the $(i,j)$ grid and $N$ is the total number of models.

It should be noted that PDF is anti-symmetric; the probability that the left crack forms a certain configuration is identical with the probability the right crack forms the anti-symmetric image of the configuration. The anti-symmetry of the configuration, however, is rarely observed for each model. To emphasize the loss of anti-symmetry, we rotate the model by 180 degree so that only the left crack grows mainly. The true PDF is easily obtained by taking the anti-symmetric part of PDF computed in this manner.

In Fig. 6, presented are PDF’s of the final crack configuration computed in the above mentioned manner for the four cases of $\dot{U}/V_r = 1, 3, 9, 27$. Unlike the crack growth process, we
can observe some patterns. For the lowest loading rate \( \dot{U}/V_r = 1 \), regions where probability density of crack passing is concentrated form an anti-symmetric pattern, which seems similar to the anti-symmetric crack configuration computed by Sato et al.\textsuperscript{13).} For the highest loading rate \( \dot{U}/V_r = 27 \), regions of high probability density appear a branch from the anti-symmetric pattern. As the loading rate increases, this branch has higher values of the probability density.

There exist two dominant patterns for PDF of the crack configuration, even though there are no common growth processes through which the cracks form the final configuration shown in Fig. 6. These two patterns are schematically presented in Fig. 7. Pattern 1 (or the anti-symmetric PDF) shows that the solution of the ideally homogeneous body is stable in the sense that the configuration does not change significantly even in the presence of weak heterogeneity. As the loading rate increases, Pattern 1 becomes less dominant, which implies the stability loss of the solution of the ideally homogeneous body. However, the concentration of PDF becomes close to Pattern 2, which suggests the presence of an attractor to which the crack configuration solution tends to be close to.

It is of interest to quantitatively examine the shift from Pattern 1 to Pattern 2 due to the increase in the loading rate. In Table 2, the ra-
Table 2  Ratio of probability for two crack configuration patterns.

|        | Pattern 1 [%] | Pattern 2 [%] |
|--------|---------------|---------------|
| U      | 100           | 0             |
| 3      | 68            | 32            |
| 9      | 8             | 92            |
| 27     | 0             | 100           |

Fig. 7  Two dominant patterns of crack configuration.

The ratio of Pattern 1 and Pattern 2 is summarized for the four loading rates. As is seen, Pattern 1 is smoothly shifted to Pattern 2 as the loading rate increases. This numerical experiment does not find a critical loading rate which sharply separates Patterns 1 and 2.

5. CONCLUDING REMARKS

The numerical experiment shows that while the crack growth process is not stable, the crack configuration is stable under lower loading rate. The configuration loses the stability as the loading rate increases. This implies the increase in the crack configuration variability when a plate is cracked due to dynamic loading. However, there exists a domain pattern for PDS, and there is a certain dominant crack configuration even though it is not anti-symmetric.

What the present paper clarifies is the stability of the solution that is numerically computed by PDS-FEM. More realistic problem setting should be made to study cases of an actual material. We are now conducting the comparison of the numerical experiment with actual experimental data, in order to verify the present conclusion on the nature of the stability of the dynamic crack growth.

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