GENERIC FREE SUBGROUPS AND STATISTICAL HYPERBOLICITY

SUZHEN HAN AND WEN-YUAN YANG

Abstract. This paper studies the generic behavior of $k$-tuple elements for $k \geq 2$ in a proper group action with contracting elements, with applications towards relatively hyperbolic groups, CAT(0) groups and mapping class groups. For a class of statistically convex-cocompact action, we show that an exponential generic set of $k$ elements for any fixed $k \geq 2$ generates a quasi-isometrically embedded free subgroup of rank $k$. For $k = 2$, we study the sprawl property of group actions and establish that the class of statistically convex-cocompact actions is statistically hyperbolic in a sense of M. Duchin, S. Lelièvre, and C. Mooney. For any proper action with a contracting element, if it satisfies a condition introduced by Dal’bo-Otal-Peigné and has purely exponential growth, we obtain the same results on generic free subgroups and statistical hyperbolicity.

1. Introduction

1.1. Motivation and background. Suppose that a group $G$ admits a proper and isometric action on a proper geodesic metric space $(Y, d)$. The group $G$ is assumed to be non-elementary: it is not virtually cyclic. An element $g \in G$ is called contracting if for some basepoint $o \in Y$, an orbit $\{h^n \cdot o : n \in \mathbb{Z}\}$ is a contracting subset, and the map $n \in \mathbb{Z} \mapsto h^n o \in G$ is a quasi-isometric embedding. Here a subset $X$ is called contracting if any metric ball disjoint with $X$ has a uniformly bounded projection to $X$ (see [34, 8]). It is clear that this definition does not depend on the choice of the basepoint.

The prototype of a contracting element is a hyperbolic isometry on Gromov-hyperbolic spaces, but more interesting examples are furnished by the following:

- hyperbolic elements in relatively hyperbolic groups or groups with nontrivial Floyd boundary (see [24, 25]);
- rank-1 elements in CAT(0) groups (see [4, 8]);
- certain infinite order elements in certain small cancellation groups (see [2]);
- pseudo-Anosov elements in mapping class groups of closed oriented surfaces with genus greater than two acting on Teichmüller space (see [34]).

In [49], the second-named author proved that, for a class of statistically convex-cocompact actions defined below, the set $X$ of contracting elements is exponentially generic in the ball model:

$$\frac{|X \cap B_n|}{|B_n|} \to 1$$

exponentially fast, where $B_n := \{g \in G : d(o, go) \leq n\}$.

Along this line, the goal of this paper is to continue the study of generic properties for $k$-tuples of elements in $G$ for a fixed $k \geq 2$. To that end, we introduce a few more notations. We fix a basepoint $o \in Y$ and denote $|g| = d(o, go)$ for easy notation. Denote $G^{(k)} = \{(u_1, \ldots, u_k) :$
\( u_i \in G \) and \( B^{(k)}_n = \{(u_1, \ldots, u_k) \in G^{(k)} : |u_i| \leq n\}. \) When \( k \) is understood, we write \( \overrightarrow{u} \) for \((u_1, \ldots, u_k)\), and \( |\overrightarrow{u}| \) for \( \max\{|u_i| : 1 \leq i \leq k\}. \)

The asymptotic density of a subset \( X \subseteq G^{(k)} \) in ball model is defined as

\[
\mu(X) = \lim_{n \to \infty} \frac{|X \cap B^{(k)}_n|}{|B^{(k)}_n|}
\]

if the limit exists. If the convergence happens exponentially fast, we denote \( \mu(X)^{exp} = \lambda \in [0,1] \).

We shall be interested in the extreme cases \( \mu(X)^{exp} = 1 \) (resp. \( \mu(X) = 1 \)) which are called exponentially generic (resp. generic). By definition, the complement of an (exponentially) generic set is called (exponentially) negligible.

The generic property of \( k \)-tuple of elements has been studied using random walks in various class of groups with negative curvature. Let \( \mu \) be a probability measure with finite support on the group \( G \) so that the support generates \( G \) as a semi-group. A \( \mu \)-random walk is a product of a sequence of independent identical \( \mu \)-distributed random variables on \( G \). In our setting, Sisto [44] proved that the \( n \)-th step of a simple random rank lands on a contracting element with asymptotic probability one. In mapping class groups, this was obtained by Maher for pseudo-Anosov elements, and the most general result is, as far as we know, due to Maher and Tiozzo [33] for any non-elementary action on a hyperbolic space where random elements are being loxodromic. When \( k \geq 2 \), Gilman, Miasnikov, and Osin [27] proved in hyperbolic groups that two simple random walks on the Cayley graph stay at a ping-pong position in \( n \)-steps with asymptotic probability one so that they generate an undistorted free group of rank 2. The same result holds in non-virtually solvable linear groups [1] and in mapping class groups [41, 46, 32] for two independent \( \mu \)-random walks. In fact, most of these works are stated in a general class of groups with hyperbolic embedded subgroups called b Dahmani, Guirardel and Osin [13] and equivalently, the class of acylindrical hyperbolic groups in the sense of Osin [36]. It is worth pointing out that a proper action with a contracting element is acylindrical hyperbolic by a result of Sisto [44]. However, our first goal is to address the analogue of generic free subgroups using counting measure as above instead of probability measure from random walks.

In fact, studying the generic properties of \( k \)-tuple elements in a counting measure is not a new idea. In [17], M. Duchin, S. Lelièvre, and C. Mooney initiated a study of sprawl property of pair of points in the space. The notion of statistical hyperbolicity is then introduced to capture negative curvature in a statistical sense. Roughly speaking, the intuitive meaning could be explained as follows: consider the annular set

\[
A(n, \Delta) = \{g \in G : ||g| - n| \leq \Delta\}
\]

for \( \Delta > 0 \). On average, a random pair of points \( x, y \) on an annular set \( A(n, \Delta) \) of the group has the distance \( d(xo, yo) \) nearly equal to \( 2n \). We formulize this concept using both annuli and balls.

**Definition 1.1.** Let \( G \) admit a proper action on a geodesic metric space \((Y, d)\). Define

\[
E_B(G) = \lim_{n \to +\infty} \frac{1}{|B_n|^2} \sum_{x, y \in B_n} \frac{d(x, y)}{n},
\]

and for a constant \( \Delta > 0 \),

\[
E_A(G, \Delta) = \lim_{n \to +\infty} \frac{1}{|A(n, \Delta)|^2} \sum_{x, y \in A(n, \Delta)} \frac{d(x, y)}{n}.
\]
if the limit exists. The action is called statistically hyperbolic in annuli (resp. in balls) if 
\[ E_A(G, \Delta) = 2 \] for any sufficiently large \( \Delta > 0 \) (resp. \( E_B(G) = 2 \)).

Remark. In \cite{17} this definition was introduced using annular model with \( \Delta = 0 \) in the Cayley graph of groups. Here we consider also the quantity \( E_B(G) \) without involving the extra parameter \( \Delta \). In our results, we obtain \( E_A(G, \Delta) = E_B(G) = 2 \) along the same line of proofs.

The non-examples include elementary groups, \( \mathbb{Z}^d \) for \( d \geq 2 \), and the integer Heisenberg group for any finite generating set among the others (cf. \[17\]). In the opposite, the exact value of \( E_B(G) = 2 \) indeed happens for many groups with certain negative curvature from a point of view of coarse geometry. For instance, non-elementary relatively hyperbolic groups are statistical hyperbolic for any finite generating set (cf. \cite{17, 35}). Moreover, the statistical hyperbolicity is preserved under certain direct product of a relatively hyperbolic group and a group. And the lamplighter groups \( \mathbb{Z}_m \wr \mathbb{Z} \) where \( m \geq 2 \) are statistical hyperbolic for certain generating sets \cite{17}.

The notion of statistical hyperbolicity could be considered for any metric space with a measure as in \cite{17}, rather than our definition using a counting measure. In this direction, it was proved in the same paper that for any \( m, p \geq 2 \), the Diestel-Leader graph \( DL(m, p) \) is statistically hyperbolic. The statistical hyperbolicity for Teichmüller space with various measures was proved by Dowdall, Duchin and Masur in \cite{15}.

The second goal of the paper is to generalize these results in a very general class of proper actions using counting measures from orbits in Definition 1.1. In what follows, we shall describe our results in detail.

1.2. Main results. In order to expose our results, we first give a quick overview of the various classes of actions under consideration in this study. First of all, we consider the class of statistically convex-cocompact actions introduced in \cite{49} which generalizes a convex-cocompact action in a statistical sense. Making this idea precise requires a notion of growth rate of a subset \( X \) in \( G \):

\[ \delta_X = \limsup_{n \to \infty} \frac{\ln |X \cap B_n|}{n}. \]

It is clear that the value \( \delta_X \) does not depend on the choice of the basepoint. By abuse of language, a geodesic between two sets \( A \) and \( B \) is a geodesic between \( a \in A \) and \( b \in B \).

Given constants \( 0 \leq M_1 \leq M_2 \), let \( \mathcal{O}_{M_1, M_2} \) be the set of element \( g \in G \) such that there exists some geodesic \( \gamma \) between \( B(o, M_2) \) and \( B(go, M_2) \) with the property that the interior of \( \gamma \) lie outside \( N_{M_1}(Go) \).

Definition 1.2 (SCC Action). If there exist positive constants \( M_1, M_2 > 0 \) such that \( \delta_{\mathcal{O}_{M_1, M_2}} < \delta_G < \infty \), then the proper action of \( G \) on \( Y \) is called statistically convex-cocompact (SCC).

The idea to define the set \( \mathcal{O}_{M_1, M_2} \) is to look at the action of the fundamental group of a finite volume Hadamard manifold on its universal cover. It is then easy to see that for appropriate constants \( M_1, M_2 > 0 \), the set \( \mathcal{O}_{M_1, M_2} \) coincides with the union of cusp subgroups up to a finite Hausdorff distance. The assumption in SCC actions was called a parabolic gap condition by Dal’bo, Otal and Peigné in \cite{14}. One of motivations of this study is to push forward the analogy between the concave set \( \mathcal{O}_{M_1, M_2} \) and the (union of) parabolic cusp regions. This allows us to draw conclusions for the SCC actions through the analogy with the geometrically finite actions, which have been well studied in last twenty years.

Moreover, our study suggests considering a class of proper actions satisfying a more general condition introduced at the same paper \cite{14}. The condition, reformulated below, is proved to be
equivalent to the finiteness of Bowen-Margulis-Sullivan (BMS) measure on the geodesic flow of the unit tangent bundle of a geometrically finite Hadamard manifold in [14], and later for any Hadamard manifold by Pit and Shapira [39, Theorem 2].

**Definition 1.3 (DOP condition).** The group action of $G$ on $Y$ satisfies the Dal’bo-Otal-Peigné (DOP) condition if there exist two positive constants $M_1, M_2 > 0$ such that

$$\sum_{g \in O_{M_1, M_2}} |g| \exp(-\delta_G |g|) < \infty$$

**Remark.** We remark that, in the setting of negatively curved manifolds, the DOP condition is called positive recurrent by Pit and Shapira in [39], whereas the notion of SCC actions is called strongly positive recurrent by Shapira and Tapie in [43]. We thank Rémi Coulon to bring these references to our attention.

The concept of the geodesic flow is non-applicable in a general geodesic metric space with negative curvature such as contracting property. However, the definition of the DOP condition could be always made, and so could be understood as substitute of finite BMS measures in a general metric space. One of Roblin’s results [42, Théorème 4.1] stated in the setting of a geometrically finite Hadamard manifold characterized the finiteness of BMS measures by a purely exponentially growth (PEG) of the action:

$$|B_n| \asymp \exp(\delta_G n).$$

Hence, the class of proper actions with purely exponential growth should be viewed as equivalents of DOP conditions. We expect this relation persists in a very general setting, and remark that it is indeed true for the class of geometrically finite action on a $\delta$-hyperbolic space in [48] (weaker than the setting of Roblin).

Our first main result establishes that generic $k$-tuple elements are the free basis of a free group with quasi-isometrically embedded property for the above two class of actions.

**Theorem 1.4.** Assume that a non-elementary group $G$ acts properly on a geodesic metric space $(Y, d)$ with a contracting element. If $G$ satisfies the DOP condition and has purely exponentially growth. Then for any $k \geq 2$, the set of all tuples $(u_1, \cdots, u_k) \in G^{(k)}$ so that $u_1, \cdots, u_k$ generate a free subgroup of rank $k$ in $G$ is generic in $G^{(k)}$. Moreover, these free subgroups are quasi-isometrically embedded with contracting images.

When the action is SCC, the above assumptions hold, and moreover, we can obtain an exponential convergence rate for the above conclusion.

**Theorem 1.5.** Assume that a non-elementary group $G$ admit a SCC action on a geodesic metric space $(Y, d)$ with a contracting element. Then for any $k \geq 2$, the set of all $(u_1, \cdots, u_k) \in G^{(k)}$ for which $u_1, \cdots, u_k$ generate a free subgroup of rank $k$ in $G$ is exponentially generic in $G^{(k)}$. Moreover, these free subgroups are quasi-isometrically embedded with contracting images.

A group generated by a finite set acts cocompactly on its Cayley graph, so our results apply for this particular case. A finitely generated subgroup $H$ is called undistorted if the inclusion $H \subset G$ is quasi-isometrically embedded with respect to word metrics.

**Corollary 1.6.** Let $G$ be a non-elementary group with a finite generating set $S$. If $G$ has a contracting element, then the set of all $(u_1, \cdots, u_k) \in G^{(k)}$ for which $u_1, \cdots, u_k$ generate an undistorted free subgroup of rank $k$ in $G$ is exponentially generic in $G^{(k)}$. 
To illustrate consequences of previous results, we remark that the following examples of groups with contracting elements with respect to the Cayley graph:

1. any relatively hyperbolic group $G$ acts on a Cayley graph $\mathcal{G}(G, S)$ with respect to a finite generating set $S$. See [25].

2. any group $G$ with non-trivial Floyd boundary acts on a Cayley graph $\mathcal{G}(G, S)$ with respect to a finite generating set $S$. [25].

3. the right-angled Artin (Coxeter) groups with respect to the standard generating set, if they are not virtually direct product. [5, 7, 12].

4. the Gr"(1/6)-labeled graphical small cancellation group $G$ with finite components labeled by a finite set $S$ acts on the Cayley graph $\mathcal{G}(G, S)$. See [2].

Thus, by Corollary 1.6, the list of these examples all have the generic free basis property. We remark that this result is even new in the class of relatively hyperbolic groups.

We next explain an application of Theorem 1.5 about surface group extensions. Let $\text{Mod}(\Sigma_g)$ be the mapping class group of a closed oriented surface $\Sigma_g$ of genus $g \geq 2$. Combining the results of Minsky [34] and Eskin-Mirzakhani-Rafi [19] we know that the action of $\text{Mod}(\Sigma_g)$ on Teichmüller space $T(\Sigma_g)$ is a SCC action with a contracting element. By Theorem 1.5, we obtain the exponential genericity of $k$-tuple elements $(u_1, u_2, \cdots, u_k)$ being free basis in the counting measure from Teichmüller metric. Denote $\Gamma := \langle u_1, u_2, \cdots, u_k \rangle$. Marking a point $p \in \Sigma_g$, the Bireman exact sequence in [9] gives an extension $E_\Gamma$ in $\text{Mod}(\Sigma_g, p)$ of the surface group $\pi_1(\Sigma_g, p)$ by $\Gamma$ as follows

$$1 \to \pi_1\Sigma_g \to E_\Gamma \to \Gamma \to 1.$$  

We refer the reader to the reference [20] for related facts about $\text{Mod}(\Sigma_g)$ and $T(\Sigma_g)$.

In [21], Farb and Mosher studied when the extension is a hyperbolic group and showed that, when $\Gamma$ is a Schottky group, this is equivalent to the quasi-convexity of $\Gamma$-orbits in $T(\Sigma_g)$.

In Theorem 1.5, the quasi-isometrically embedded image of the free group $\Gamma$ are contracting and thus quasi-convex in the sense of Farb and Mosher. Thus, by [21, Theorem 1.1], the free group $\Gamma$ is convex-cocompact in their sense, so the following result holds.

**Theorem 1.7.** The set of $k$-tuples of elements $(u_1, u_2, \cdots, u_k)$ in $\text{Mod}(\Sigma_g)$ with the hyperbolic extension in $\text{Mod}(\Sigma_g, p)$ is exponentially generic.

Our second main result obtains the statistical hyperbolicity for the exact class of actions as in Theorem 1.4 and in particular for statistically convex-cocompact actions.

**Theorem 1.8.** Let a non-elementary group $G$ act properly on $(Y, d)$ with a contracting element satisfying DOP condition and purely exponentially growth. Then $G$ is statistically hyperbolic in balls and annuli. In particular, if the action is SCC, then $G$ is statistically hyperbolic in balls and annuli.

**Remark.** Motivated by the distinction between SCC action and a general proper action, one may wonder whether there is a significant convergence rate of $E_A(G, \Delta)$ or $E_B(G)$ under SCC actions. This is, however, not true even in free groups: a simple computation as Example 4.2 shows that the convergence rate is of order $\frac{1}{n}$. Hence, we have no assertion on the convergence speed.

Except the class of SCC actions, the action of discrete groups on CAT(-1) spaces provides a source of examples with DOP condition and purely exponential growth. For example, combining
obtain that the finiteness of the Bowen-Margulis-Sullivan measure on the geodesic flow is equivalent to either have purely exponential growth or satisfy the DOP condition. Hence, we obtain the following corollary.

**Theorem 1.9.** Suppose that the Bowen-Margulis-Sullivan measure on the unit tangent bundle of a Hadamard manifold is finite. Then the fundamental group action on the universal covering is statistically hyperbolic in balls and annuli. Moreover, the generic pair of elements generate a free group of rank 2 with uniform quasi-isometric embedding.

If a hyperbolic $n$-manifold for $n \geq 2$ is geometrically finite, then the BMS measure is always finite \[45\]. We thus have the following corollary in Kleinian groups, which seems to be not recorded in literatures. Note that examples of non-geometrically finite Kleinian groups with finite BMS measures are constructed for $n \geq 4$ by Peigné in \[38\].

**Corollary 1.10.** Geometrically finite Kleinian groups are statistically hyperbolic and have generic free basis property.

For the action of mapping class groups on Teichmüller space, we then have the following corollary, which could be thought of as a discrete analogue of the result in \[14\].

**Corollary 1.11.** The action of mapping class groups on Teichmüller space is statistically hyperbolic with respect to the Teichmüller metric.

Of course, the action of a group on the Cayley graph is SCC, so if there exists a contracting element, then it is statistically hyperbolic. This allows us to give new examples of groups with statistically hyperbolic property in the original sense \[17\].

**Corollary 1.12.** The following classes of groups are statistically hyperbolic with respect to word metrics.

1. A $Gr^\ell(\frac{1}{6})$-labeled graphical small cancellation group $G$ with finite components labeled by a finite set $S$ acts on the Cayley graph $\mathcal{F}(G, S)$ with respect to the finite generating set $S$.
2. Right-angled Artin (Coxeter) groups are statistically hyperbolic with respect to the standard generating set, if they are not virtually direct product.

We point out that it is not clear to us whether the above two classes of groups are statistically hyperbolic for every generating set. Note that they include non-relatively hyperbolic examples of groups (cf. \[6, 30\]). Hence, it would be interesting to know to which extent the statistical hyperbolicity for every generating set characterizes the class of relatively hyperbolic groups.

**The structure of this paper** Section 2 discusses the notions and relevant facts of contracting elements, SCC actions and the DOP condition. The main technical contribution is given in Section 3 and provides useful characteristics of several negligible sets. In Section 4 a generic set of elements is then singled out to complete the proofs of Main Theorems.

2. **Preliminaries**

In this section, we will introduce some preliminaries. First we fix some notations and conventions.
2.1. Notations and Conventions. Let \((Y, d)\) be a proper geodesic metric space. The \(r\) neighborhood of a subset \(X \subseteq Y\) is denoted by \(N_r(X)\). We denote \(\|X\|\) by the diameter of a subset \(X \subseteq Y\) and \(d_{\text{Haus}}(X_1, X_2)\) by the Hausdorff distance of two subsets \(X_1, X_2 \subseteq Y\). Given a point \(y \in Y\), and a subset \(X \subseteq Y\), let \(\Pi_X(y)\) be the set of point \(x \in X\) such that \(d(y, x) = d(y, X)\). The projection of a subset \(A \subseteq Y\) to \(X\) is then \(\Pi_X(A) := \cup_{a \in A} \Pi_X(a)\).

The path \(\gamma\) in \(Y\) under consideration is always assumed to be rectifiable with arc-length parametrization \([0, |\gamma|] \to \gamma\), where \(|\gamma|\) denotes the length of \(\gamma\). Denote by \(\gamma_-\), \(\gamma_+\) the initial and terminal points of \(\gamma\) respectively. For any two parameters \(a < b \in [0, |\gamma|]\), we denote by \((\gamma(a), \gamma(b))\gamma := \gamma([a, b])\) and \((\gamma(a), \gamma(b))\gamma := \gamma((a, b))\) the closed (resp. open) subpath of \(\gamma\) between \(a\) and \(b\). For any \(x, y \in Y\), we denote by \([x, y]\) a choice of geodesic in \(Y\) from \(x\) to \(y\).

Given a property (P), a point \(z \in \gamma\) is called the entry point satisfying (P) if \(\|\gamma_-, z\|\) is minimal among the points \(z \in \gamma\) with the property (P). A point \(w\) on \(\gamma\) is called the exit point satisfying (P) if \(\|w, \gamma_+\|\) is minimal among the points \(w \in \gamma\) with the property (P).

A path \(\gamma\) is called a \((\lambda, c)\)-quasi-geodesic for \(\lambda \geq 1\), \(c \geq 0\) if the following holds
\[
|\beta| \leq \lambda \cdot d(\beta_-, \beta_+) + c
\]
for any rectifiable subpath \(\beta\) of \(\alpha\).

Let \(\beta, \gamma\) be two paths in \(Y\). Denote by \(\beta \cdot \gamma\) (or simply \(\beta \gamma\)) the concatenated path provided that \(\beta_- = \gamma_+\).

Let \(f, g\) be real-valued functions with domain understood in the context. Then \(f \prec_a \prec g\) means that there is a constant \(a > 0\) depending on parameters \(c_i\) such that \(f < ag\). The symbols \(\succ_{c_i}\) and \(\sim_{c_i}\) are defined analogously. For simplicity, we shall omit \(c_i\) if they are universal constants.

We say a sequence \(\{a_n\} \subseteq \mathbb{R}\) of numbers converges to a number \(\lambda \in \mathbb{R}\) exponentially fast, denoted by \(a_n \xrightarrow{\text{exp}} \lambda\), if
\[
|\lambda - a_n| \leq c\theta^n
\]
for some constant \(\theta \in (0, 1)\) and a positive constant \(c > 0\).

**Remark.**
(1) It is clear that the (exponential) genericity is preserved by taking any finite intersection and finite union. This fact shall be often used implicitly.

(2) If \(X\) is exponentially negligible, then \(\delta_X < \delta_G\), by which we call \(X\) growth tight in \([47]\).

Note that if \(G\) has purely exponentially growth, then a growth tight set is exponentially negligible. In this paper, the group actions under consideration always have purely exponentially growth, so we do not distinguish these two notions.

2.2. Contracting Property. We fix a preferred class of quasi-geodesics \(\mathcal{L}\), which contains at least all geodesics in \(Y\).

**Definition 2.1** (Contracting subset). A subset \(X \subseteq Y\) is called \(\kappa\)-contracting with respect to \(\mathcal{L}\) if for any quasi-geodesic \(\gamma \in \mathcal{L}\) with \(d(\gamma, X) \geq \kappa\), we have \(\Pi_X(\gamma) \leq \kappa\). A collection of \(\kappa\)-contracting subsets is referred to as a \(\kappa\)-contracting system (with respect to \(\mathcal{L}\)).

We first note the following examples in various contexts.

**Examples 2.2.**
(1) Quasi-geodesics and quasi-convex subsets are contracting with respect to the set of all quasi-geodesics in hyperbolic spaces.

(2) Fully quasi-convex subgroups (and in particular, maximal parabolic subgroups) are contracting with respect to the set of all quasi-geodesics in relatively hyperbolic groups (see \([24]\, Proposition 8.2.4]\)).
The subgroup generated by a hyperbolic element is contracting with respect to the set of all quasi-geodesics in groups with non-trivial Floyd boundary (see [47, Section 7]).

(4) Contracting segments in CAT(0)-spaces in the sense of Bestvina and Fujiwara are contracting here with respect to the set of geodesics (see [8, Corollary 3.4]).

(5) The axis of any pseudo-Anosov element is contracting relative to geodesics in Teichmüller spaces by Minsky [34].

Convention 2.3. In view of the above examples, the preferred collection \( L \) in the sequel will always be the set of all geodesics in \( Y \).

The notion of a contracting subset is equivalent to the following one considered by Minsky [34]. The proof given in [8, Corollary 3.4] for CAT(0) spaces is valid in the general case. In this paper, we will always work with the above definition of the contracting property.

Lemma 2.4. A subset \( X \) is contracting in \( Y \) if and only if any open ball \( B \) missing \( X \) has a uniformly bounded projection to \( X \).

We collect some properties of contracting sets that will be used later on. The proof is straightforward and is left to the interested reader.

Lemma 2.5. Let \( X \) be a contracting set.

1. **(Quasi-convexity)** \( X \) is \( \sigma \)-quasi-convex for a function \( \sigma : \mathbb{R}_+ \to \mathbb{R}_+ \): given \( r \geq 0 \), any geodesic with endpoints in \( N_r(X) \) lies in the neighborhood \( N_{\sigma(r)}(X) \).

2. **(Finite neighborhood)** Let \( Z \) be a set with finite Hausdorff distance to \( X \). Then \( Z \) is contracting.

3. There exists a constant \( C > 0 \) such that for any geodesic segment \( \gamma \),

\[
\| \Pi_X(\{\gamma^-, \gamma^+\}) \| = \| \Pi_X(\gamma) \| < C.
\]

In most situations, we are interested in a contracting system \( X \) with a \( \nu \)-bounded intersection for a function \( \nu : \mathbb{R}_+ \to \mathbb{R}_+ \geq 0 \) if the following holds

\[
\forall X \neq X' \in X, \| N_r(X) \cap N_r(X') \| \leq \nu(r)
\]

for any \( r \geq 0 \). This property is, in fact, equivalent to a bounded projection property of \( X \): there exists a constant \( B > 0 \) such that

\[
\Pi_X(X') \leq B
\]

for \( X \neq X' \in X \). See [47] for further discussions.

An infinite subgroup \( H < G \) is called **contracting** if for some (hence any by [49, Proposition 2.4.2]) \( o \in Y \), the subset \( Ho \) is contracting in \( Y \).

An element \( h \in G \) is called **contracting** if the subset \( \langle h \rangle o \) is contracting, and the orbital map \( n \in \mathbb{Z} \mapsto h^n o \in Y \) is a quasi-isometric embedding. The set of contracting elements is preserved under conjugacy.

Let \( H \) be a contracting subgroup. We define a group \( E(H) \) as follows:

\[
E(H) := \{ g \in G : \exists r > 0, gHo \leq N_r(Ho), Ho \leq N_r(gHo) \}.
\]

For a contracting element \( h \), we have the following result about \( E(h) := E(\langle h \rangle) \) (see [49, Lemma 2.11]).

Lemma 2.6. Assume that \( G \) acts properly on \( (Y, d) \). For a contracting element \( h \), the following statements hold:

1. \( E(h) = \{ g \in G : \exists n > 0, gh^n g^{-1} = h^n \text{ or } gh^n g^{-1} = h^{-n} \} \).
Lemma 2.7. For any $C > 0$, let $\gamma$ be a geodesic with interior dose not meet $N_C(A)$. Then
\[ d_{Haus}(\Pi_{N_C(A)}(\gamma), \Pi_A(\gamma)) \leq C. \]
In particular, if $C$ is a contracting constant of $A$, then we have $\Pi_{N_C(A)}(\gamma) \leq 3C$.

Proof. For any $x \not\in N_C(A)$, it is sufficient to prove
\[ d_{Haus}(\Pi_{N_C(A)}(x), \Pi_A(x)) \leq C. \]
For any $y \in \Pi_A(x)$, take some $[x, y]$. Let $z$ be the point of $[x, y]$ such that $d(y, z) = C$. Now for each $z' \in N_C(A)$, there exists $y' \in A$ such that $d(y', z') \leq C$. Since
\[ d(x, z') + d(z', y') \geq d(x, y') \geq d(x, y) = d(x, z) + d(z, y), \]
we have $d(x, z') \geq d(x, z)$, which means that $z \in \Pi_{N_C(A)}(x)$. Thus
\[ \Pi_A(x) \subseteq \Pi_{N_C(A)}(x). \]
Let $z \in \Pi_{N_C(A)}(x)$. Since $z \in N_C(A)$, there exists $y \in A$ so that $d(y, z) \leq C$. Take some $y' \in \Pi_A(x)$, then there exists $z' \in \Pi_{N_C(A)}(x)$ so that $d(x, y') = d(x, z') + C$ by the above discussion. Then
\[ d(x, y) \leq d(x, z) + d(y, z) \leq d(x, z) + C = d(x, z') + C = d(x, y'). \]
So $y \in \Pi_A(x)$, $\Pi_{N_C(A)}(x) \subseteq N_C(A)$. \hfill \Box

Lemma 2.8. Let $C > 0$ be the contraction constant of $A$ and $\alpha, \beta$ be two geodesics with the same initial endpoint. If $x$ is the entry point of $\alpha$ into $N_C(A)$ and $\beta \cap B(x, 4C) = \emptyset$, then $\beta \cap N_C(A) = \emptyset$.

Proof. If $\beta \cap N_C(A) \neq \emptyset$, then let $y \in \beta$ be the entry point of $\beta$ in $N_C(A) \neq \emptyset$. We have
\[ d(x, y) \leq C + \|\Pi_A([\alpha_-, x, \alpha])\| + \|\Pi_A([\beta_-, y, \beta])\| + C \leq 4C \]
which proves the lemma. \hfill \Box

Since $gN_C(A) = N_C(gA)$ for every $g \in G$, the following lemma is a consequence of Lemma 2.7 and Lemma 2.6.

Lemma 2.9. For any $C \geq 0$, the collection $\mathcal{X} = \{gN_C(A) : g \in G\}$ is a contracting system with bounded projection.

2.3. Admissible Path. Let $\mathcal{X}$ be a contracting system with a bounded intersection property. The following notion of an admissible path will be used to obtain a quasi-geodesic path.

Definition 2.10 (Admissible Path). Given $D, \tau \geq 0$ and a function $R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a path $\gamma$ is called $(D, \tau)$-admissible in $Y$, if $\gamma$ is a concatenation of geodesic subpaths $p_0, q_1, p_1 \cdots q_n p_n$ ($n \in \mathbb{N}$), $p_0, p_n$ could be trivial), the endpoints of $p_i$ are in some $X_i \in \mathcal{X}$ for each $i$, and satisfies the following called Long Local and Bounded Projection properties:

(LL1) Each $p_i$ has length bigger than $D$, except that $(p_i)_- = \gamma_-$ or $(p_i)_+ = \gamma_+$;

(BP) For each $X_i$, we have $\max\{\|\Pi_{X_i}(q_i)\|, \|\Pi_{X_i}(q_{i+1})\|\} \leq \tau$, where $q_0 := \gamma_-$ and $q_{n+1} := \gamma_+$ by convention.
(LL2) Either \( X_i \neq X_{i+1} \) have \( R \)-bounded intersection or \( q_{i+1} \) has length bigger than \( D \).

We need the following result from [47, Corollary 3.2].

**Proposition 2.11.** Let \( \kappa \) be the contraction constant of \( X \). For any \( \tau > 0 \), there are constants \( D_0 = D_0(\kappa, \tau) > 0, \Lambda = \Lambda(\kappa, \tau) > 0 \) such that given \( D > D_0 \) any \( (D, \tau) \)-admissible path is \( (\Lambda, 0) \)-quasi-geodesic.

We refer the reader to [47, 49] for further discussions about admissible path.

2.4. SCC actions and barrier-free elements. We recall the notion of a barrier-free element from [49].

**Definition 2.12.** Fix constants \( \nu, M > 0 \).

1. Given \( \nu > 0 \) and \( g \in G \), we say that a geodesic \( \gamma \) contains an \( (\nu, g) \)-barrier if there exists a element \( z \in G \) so that

\[
\max\{d(z \cdot o, \gamma), d(z \cdot go, \gamma)\} \leq \nu. 
\]

If no such \( z \in G \) exists so that (2) holds, then \( \gamma \) is called \( (\nu, g) \)-barrier-free.

2. An element \( f \in G \) is \( (\nu, M, g) \)-barrier-free if there exists an \( (\nu, g) \)-barrier-free geodesic between \( B(o, M) \) and \( B(go, M) \).

We have chosen two parameters \( M_1, M_2 \) so that the definition of a statistically convex-cocompact action [12] is flexible and easy to verify. It is enough to take \( M_1 = M_2 = M \) in our use. Henceforth, we set \( \mathcal{O}_M := \mathcal{O}_{M, M} \) for easy of notation. When the SCC action contains a contracting element, the definition is independent of the basepoint (see [49]).

Given \( \nu, M > 0 \) and any \( g \in G \), let \( V_{\nu, M, g} \) be the collection of all \( (\nu, M, g) \)-barrier-free elements of \( G \). The following results will be key in next sections.

**Proposition 2.13.** [49] If \( G \) admit a SCC action on a proper geodesic space \((Y, d)\) with a contracting element, then

1. \( G \) has purely exponentially growth.
2. Let \( M_0 \) be the constant in the definition of SCC action, then for any \( M > M_0 \), there exists \( \nu = \nu(M) > 0 \) such that \( V_{\nu, M, g} \) is exponentially negligible for any \( g \in G \).

It is easy to see from the proof of [49, Corollary 4.5] that the following conclusion holds in a general proper action.

**Proposition 2.14.** Suppose that a group \( G \) acts properly on a proper geodesic space \((Y, d)\) with a contracting element, then for any \( M > 0 \), there exists \( \nu = \nu(M) > 0 \) so that

\[
\sum_{n=1}^{+\infty} |V_{\nu, M, g} \cap A(n, \Delta)| \exp(-n\delta_G) < +\infty
\]

for any \( g \in G \).

2.5. The DOP condition. This subsection collects several useful consequences of the Dal’bo-Otal-Peigné condition. For any \( 0 \leq n_1 \leq n_2 \), we consider the following annulus-like set

\[
A([n_1, n_2], \Delta) := \{g \in G : n_1 - \Delta \leq d(o, go) \leq n_2 + \Delta\}.
\]

Usually, we consider the \( (\rho, \Delta) \)-annulus \( A([\rho n, n], \Delta) \) for \( \rho \in [0, 1] \). For simplicity, we write \( A([\rho n, n]) \) if \( \Delta = 0 \), and assume that \( \rho n \) are integers.
Observe that
\begin{equation}
\sum_{g \in O_{M_1, M_2}} |g| \exp(-\delta_G |g|) \lesssim \Delta \sum_{n=1}^{+\infty} n |O_{M_1, M_2} \cap A(n, \Delta)| \exp(-n \delta_G),
\end{equation}
for any $\Delta > 0$. Indeed, this follows from the fact that any $g \in O_{M_1, M_2}$ is contained in a uniform number of annular sets $A(n, \Delta)$ where $n \geq 1$. Consequently,
\begin{equation}
\sum_{g \in O_{M_1, M_2}} \exp(-\delta_G |g|) < \infty.
\end{equation}

Thus, if $G$ admit a SCC action on $Y$, then the action satisfies the DOP condition. We remark that the formula (3) turns out to be true for any proper action of $G$ on $(Y, d)$ with a contracting element: the methods in [49] can be invoked to prove (3). This generality is not used here and so the details are left to interested reader.

For any $\Delta > 0$, let
\[ O_M(n, \Delta) := O_M \cap A(n, \Delta) \cup \{1\}, \ V_{\nu, h}(n, \Delta) := V_{\nu, h} \cap A(n, \Delta). \]

The following elementary lemma will be needed in the next section.

**Lemma 2.15.** Assume that the proper group action satisfies the DOP condition. For any $1 > \varepsilon > 0$ and any $\Delta > 0$, we have
\begin{align*}
(1) & \quad \lim_{n \to \infty} \sum_{\varepsilon n \leq l \leq n} n |O_M(l, \Delta)| \exp(-l \delta_G) = 0. \\
(2) & \quad \lim_{n \to \infty} \sum_{\varepsilon n \leq l \leq n} \sum_{l_1 + l_2 + l_3 = l} (l_1 + 1) |O_M(l_1, \Delta)| \cdot |V_{\nu, h}(l_2, \Delta)| \cdot |O_M(l_3, \Delta)| \cdot \exp(-n \delta_G) = 0.
\end{align*}

When the action is SCC, the convergence is exponentially fast.

**Proof.** By definition of the DOP condition, we obtain
\[ \sum_{n=0}^{+\infty} |O_M(n, \Delta)| \exp(-n \delta_G) < \infty. \]
from the formulae (3) and (4). By the Cauchy criterion of series, we know
\[ \lim_{n \to \infty} \sum_{\varepsilon n \leq l \leq n} l |O_M(l, \Delta)| \exp(-l \delta_G) = 0 \]
where the convergence is exponential fast when the action is SCC. The first statement (1) thus follows from the following
\[ \sum_{\varepsilon n \leq l \leq n} \varepsilon n |O_M(l, \Delta)| \exp(-l \delta_G) \leq \sum_{\varepsilon n \leq l \leq n} l |O_M(l, \Delta)| \exp(-l \delta_G). \]

By Proposition 2.14, we have $\sum_{n=1}^{+\infty} |V_{\nu, h}(n, \Delta)| \exp(-n \delta_G) < \infty$, where the partial sum converges exponentially fast when the action is SCC. The second statement then follows from the convergence of the Cauchy product of three convergent series. The proof is finished. \qed
At last, we introduce a slightly general notion of negligibility using $(\rho, \Delta)$-annulus. Fix a number $\rho \in (0, 1]$ and $\Delta > 0$. We say that a set $K \subset G$ is negligible in the $(\rho, \Delta)$-annulus if the following holds

\begin{equation}
|K \cap A([\rho n, n], \Delta)| \to 0.
\end{equation}

If the convergence is exponentially fast, it is exponentially negligible.

The following lemma clarifies its role in proving the genericity in the next sections. It follows immediately from the purely exponential growth.

**Lemma 2.16.** Assume that the proper group action has purely exponential growth. For any $0 < \rho < 1$, we have $|A([\rho n, n])| \asymp \rho \exp(\delta_G n)$ and

\begin{equation*}
\frac{|A([\rho n, n]) \times A([\rho n, n])|}{|B_n|} \exp 1, \frac{|A([\rho n, n])|}{|B_n|} \exp 1.
\end{equation*}

Hence, in order to prove that a set $K$ is (exponentially) negligible in $G$, we can assume that $K \subset A([\rho n, n])$ to simplify the discussion for a certain choice of $\rho \in (0, 1)$. That is to say, we only need to prove that $K$ is (exponentially) negligible in $(\rho, \Delta)$-annulus. And, it turns out that the proof of (5) for $\rho = 1$ is much more simple than that for $\rho \in (0, 1)$. Therefore, we shall consider the big annulus instead of the usual one in next sections.

The same consideration applies in the case of $G^{(2)}$ where $K$ is assumed to be in $A([\rho n, n]) \times A([\rho n, n])$. 

3. **Negligible subsets**

Throughout this section, let $G$ admit a proper action on a proper geodesic metric space $(Y, d)$ with a contracting element. If the group action satisfies the DOP condition, then we take $\nu, M > 0$ to satisfy the definition of DOP condition and Proposition 2.14. When the action is SCC, the constants $\nu, M > 0$ are given by Proposition 2.13. We denote $O_M = O_{M,M}$, $V_{\nu,h} = V_{\nu,M,h}$ for simplicity.

The goal of this section is to provide some negligible sets under the above assumptions. Moreover, these are exponentially negligible when the group action is SCC. We suggest that the reader only reads the definition of these sets first and then read the proof of the theorems in next section, finally return to the proof that these sets are negligible.

In all results obtained in what follows, we assume in the DOP case and have in the SCC case by Proposition 2.13 that $G$ has purely exponentially growth:

\begin{equation*}
|B_n| \asymp \exp(\delta_G n) \asymp \Delta |A(n, \Delta)|
\end{equation*}

for any $\Delta \gg 0$. We fix such a constant $\Delta$. This estimate will be used implicitly several times.

3.1. **Elements with definite barrier-free proportion.** This subsection defines three negligible subsets of elements with definite proportion with(out) certain properties.

For any $\varepsilon \in (0, 1)$, let $U(\varepsilon)$ be the set of elements $u \in G$ such that some geodesic $\alpha = [o, uo]$ contains a subsegment $\alpha^\varepsilon$ of length $\varepsilon |u| \text{ outside } N_M(Go)$. That is to say,

\begin{equation}
U(\varepsilon) = \{u \in G : \exists \alpha = [o, uo], \alpha^\varepsilon \subset \alpha \text{ s.t. } |\alpha^\varepsilon| \geq \varepsilon |\alpha|, \alpha^\varepsilon \cap N_M(Go) = \emptyset\}
\end{equation}

**Lemma 3.1.** If the action has PEG and satisfies the DOP condition, then for any $\varepsilon \in (0, 1)$ and $1 \geq \rho > \varepsilon$, we have $U(\varepsilon)$ is negligible in $(\rho, \Delta)$-annulus.

Moreover, if the action is SCC, then $U(\varepsilon)$ is exponentially negligible.
Proof. Assume first that the group action satisfies the PEG and DOP condition.

Fix any \( 1 > \rho > \varepsilon \). By Lemma 2.16 we only need to show that \( \frac{|U(\varepsilon) \cap A([\rho n, n])|}{|A([\rho n, n])|} \to 0 \) as \( n \to \infty \). Consider any \( g \in U(\varepsilon) \cap A([\rho n, n]) \) and denote \( |g| = k \), then \( \rho n \leq k \leq n \). By definition of \( U(\varepsilon) \), there exists a geodesic \( \alpha = [o, go] \) such that

(7) \( \alpha \) contains a subsegment of length \( \varepsilon k \) which lies outside \( N_M(Go) \).

Among those, we consider the first maximal open segment \( (x, y)_\alpha \) of \( \alpha \) which lies outside \( N_M(Go) \) and whose length is bigger than \( \varepsilon k \in [\varepsilon \rho n, n] \).

According to the length and the position of \( (x, y)_\alpha \), we subdivide \( U(\varepsilon) \cap A([\rho n, n]) \) into a sequence of subsets as follows.

For \( 0 \leq i \leq (1 - \varepsilon)n, \varepsilon \rho n \leq l \leq n \), define \( U_i^l \) to be the set of element \( g \in U(\varepsilon) \cap A([\rho n, n]) \) such that the segment \( (x, y)_\alpha \subseteq \alpha \) defined as above satisfies \( d(o, x) = i \) and \( d(x, y) = l \). Then we have the following decomposition,

\[
U(\varepsilon) \cap A([\rho n, n]) = \bigcup_{\varepsilon \rho n \leq l \leq n} \bigcup_{0 \leq i \leq (1 - \varepsilon)n} U_i^l.
\]

For any \( g \in U_i^l \), there exists a geodesic \( \alpha = [o, go] \) such that \( \alpha_{(i, i+l)} \) lies outside \( N_M(Go) \) and \( \max\{d(\alpha(i), wo), d(\alpha(i + l), vo)\} \leq M \) for some \( u, v \in G \). Now we can write \( g = u(u^{-1}v)(v^{-1}g) \), where

\[
u \in A(i, M), u^{-1}v \in O_M(l, 2M), v^{-1}g \in A([\rho n - l - i, n - l - i], M) \subseteq B_{n-l-i+M}.
\]

Set \( \Delta = 2M \). We assumed that \( G \) has purely exponential growth, so

\[
|A(n, \Delta)| \asymp_{\Delta} \exp(\delta_G n) \asymp |B_n|.
\]

We thus obtain

\[
|U(\varepsilon) \cap A([\rho n, n])| \leq \sum_{\varepsilon \rho n \leq l \leq n} \sum_{0 \leq i \leq (1 - \varepsilon)n} |U_i^l| \leq \sum_{\varepsilon \rho n \leq l \leq n} |A(i, \Delta)| \cdot |O(l, \Delta)| \cdot |B_{n-l-i+\Delta}| \sim_{\Delta} \sum_{\varepsilon \rho n \leq l \leq n} \exp(i\delta_G) \cdot |O_M(l, \Delta)| \cdot \exp((n - l - i)\delta_G) \sim_{\Delta} \sum_{\varepsilon \rho n \leq l \leq n} n|O_M(l, \Delta)| \exp((n - l)\delta_G).
\]

Therefore, the negligibility of \( U(\varepsilon) \) follows from Lemma 2.16.
If the group action is SCC, then there exists $0 < \delta < \delta_G$ such that $|O_M(l, \Delta)| \asymp \exp(l\delta_G)$. The above computation goes without changes, and so we get

$$|U(\varepsilon) \cap A([\rho n, n])| \asymp \delta \sum_{\varepsilon \rho n \leq i \leq n} n|O_M(l, \Delta)| \exp((n - l)\delta_G)$$

$$\asymp \delta \sum_{\varepsilon \rho n \leq i \leq n} n \exp(l\delta_G) \cdot \exp((n - l)\delta_G)$$

$$\asymp \delta n^2 \exp(-\delta_G \varepsilon \rho n) \exp(n\delta_G).$$

Hence, in this case, $U(\varepsilon)$ is exponentially negligible. \hfill \Box

Let $h \in G$ be a contracting element with the axis $Ax(h) = E(h) \cdot o$, where $E(h)$ is the maximal elementary subgroup given in Lemma 2.6.

Given $\varepsilon \in (0, 1)$ and $C > 0$, consider the following set of elements $g \in G$ such that an $\varepsilon$-percentage of $[o, go]$ is contained in some translate of $Ax(h)$.

(8) $W(\varepsilon, h, C) = \{g \in G : \exists \alpha = [o, go], \alpha \subset \alpha \cap NC(fAx(h))$ s.t. $|\alpha| \geq \varepsilon|\alpha|$ for some $f \in G\}.

**Lemma 3.2.** Assume that the action has PEG. For any $0 < \varepsilon < \rho \leq 1$ and $C > 0$, we have $W(\varepsilon, h, C)$ is exponentially negligible in $(\rho, \Delta)$-annuli in $G$.

**Proof.** Since $h \in G$ is contracting, and by definition, $i \mapsto h^io$ is a quasi-isometric embedding, we have $|(h) \cap B_n| \asymp n$. By Lemma 2.6, we have $|E(h) \cdot \langle h \rangle| < \infty$, so the following holds

$$|E(h) \cap B_n| \asymp n.$$

As before, we want to show $\lim_{n \to \infty} \frac{|W(\varepsilon, h, C) \cap A([\rho n, n])|}{|A([\rho n, n])|} = 0$. Let $g \in W(\varepsilon) \cap A([\rho n, n])$, so $\rho n \leq j \leq n$, where $j := |u|$. By definition of $W(\varepsilon, h, C)$, there exists $\alpha = [o, go], i \in [0, (1 - \varepsilon)]j$ and $f \in G, k \in E(h)$ such that

$$d(\alpha(i), f o) \leq C, \ d(\alpha(i + \varepsilon j), f ko) \leq C.$$

Thus, we have $f \in A(i, C)$ and $d(o, ko) \leq \varepsilon j + 2C \leq \varepsilon n + 2C$, which yields that $k \in E(h) \cap B_{\varepsilon n + 2C}$. Consequently, we can write $g = fk((fk)^{-1}g)$ where $(fk)^{-1}g \in B_{n - \varepsilon \rho n + C}$. This gives the following

$$W(\varepsilon, h, C) \cap A([\rho n, n]) \subseteq \bigcup_{i=0}^{(1 - \varepsilon)n} A(i, C) \cdot (E(h) \cap B_{\varepsilon n + 2C}) \cdot B_{n - \varepsilon \rho n + C}.$$

Since $G$ has purely exponentially growth, we have the following estimate:

$$|W(\varepsilon, h, C) \cap A([\rho n, n])| \leq \sum_{i=0}^{(1 - \varepsilon)n} |A(i, C)| \cdot |E(h) \cap B_{\varepsilon n + 2C}| \cdot |B_{n - \varepsilon \rho n + C}|$$

$$\asymp n \cdot n \cdot \exp((1 - \varepsilon \rho)\delta_G)$$

which clearly concludes the proof of the result. \hfill \Box

We now introduce the third negligible sets of elements which have a fixed percentage being barrier-free. To be precise, we need a bit more notation. Let $\alpha$ be a geodesic and $\varepsilon_1 \leq \varepsilon_2 \in [0, 1]$. We denote by $\alpha_{[\varepsilon_1, \varepsilon_2]}$ the subsegment $\alpha([\varepsilon_1 n, \varepsilon_2 n])$ of $\alpha$, where $n = |\alpha|$.

Given $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $h \in G$, we define

(9) $V(\varepsilon_1, \varepsilon_2, h) = \{g \in G : \exists \alpha = [o, go], s.t. \alpha_{[\varepsilon_1, \varepsilon_2]} \text{ is } (\nu, h)\text{-barrier-free}\}$. 
Lemma 3.3. Fix $\rho \in (0, 1]$, and choose any $\varepsilon_1 < \varepsilon_2 \in (0, \rho)$ so that $\varepsilon_2 \rho \in (\varepsilon_1, \varepsilon_2)$. Let $h$ be any element. If our group action satisfies the DOP condition and PEG, then $V(\varepsilon_1, \varepsilon_2, h)$ is negligible in $(\rho, \Delta)$-annuli in $G$.

Moreover, if the action is SCC, then $V(\varepsilon_1, \varepsilon_2, h)$ is exponentially negligible in $G$.

Proof. By Lemma 2.10 it suffices to prove that \[
\frac{|V(\varepsilon_1, \varepsilon_2, h) \cap A([\rho m, n])|}{|A([\rho m, n])|} \rightarrow 0 \text{ as } n \rightarrow \infty.
\] Let $g \in V(\varepsilon_1, \varepsilon_2, h) \cap A([\rho m, n])$ and denote $|g| = k$, so $\rho m \leq k \leq n$. By definition of $V(\varepsilon_1, \varepsilon_2, h)$, there exists a geodesic $\alpha = [o, ga]$, so that $\alpha([\varepsilon_1 k, \varepsilon_2 k])$ is $(\nu, h)$-barrier-free. Set $x = \alpha(\varepsilon_1 n), y = \alpha(\varepsilon_2 \rho m)$. By the choice of $\varepsilon_2 \rho \in (\varepsilon_1, \varepsilon_2)$, we see that $[x, y]_\alpha = \alpha([\varepsilon_1 n, \varepsilon_2 \rho m])$ is a subsegment of $\alpha([\varepsilon_1 k, \varepsilon_2 k])$, and thus is $(\nu, h)$-barrier-free.

We now subdivide our discussion into three cases, the first two of which could be viewed degenerate cases of the third one. However, we treat them separately in order to illustrate the latter one.

Case 1. Assume that $x, y \in N_M(\mathcal{G}o)$ so there exists $u, v \in G$ such that
\[
d(x, uo) \leq M, d(y, vo) \leq M.
\]
Thus, $[x, y]_\alpha$ is a $(\nu, h)$-barrier-free geodesic between $B(uo, M)$ and $B(vo, M)$. So $u^{-1}v \in \mathcal{V}_{\nu, h}$.

Denote $\varepsilon = \varepsilon_2 \rho - \varepsilon_1 > 0$. Since $d(x, y) = \varepsilon n$ and $|d(uo, vo) - d(x, y)| \leq 2M$, we have
\[
u^{-1}v \in \mathcal{V}_{\nu, h}(\varepsilon n, 2M).
\]
Clearly we have,
\[
u \in A(\varepsilon_1 n, M), \quad v^{-1}g \in A(k - \varepsilon_2 \rho m, M).
\]
Therefore, setting $\Delta = 2M$, we obtain that $g = u(u^{-1}v)(v^{-1}g)$ lies the following set
\[
A(\varepsilon_1 n, \Delta) \cdot \mathcal{V}_{\nu, h}(\varepsilon n, \Delta) \cdot A(k - \varepsilon_2 \rho m, \Delta).
\]

Case 2. Assume that one of $\{x, y\}$ lies outside $N_M(\mathcal{G}o)$. Let’s assume first that $x \in N_M(\mathcal{G}o), y \notin N_M(\mathcal{G}o)$, so there exists $u \in G$ such that $d(x, uo) \leq M$. Consider the maximal open segment $(y_1, y_2)$ of $\alpha$ which contains $y$ but lies outside $N_M(\mathcal{G}o)$. Hence, there exists $v_1, v_2 \in G$ such that $d(y_i, vi, o) \leq M$ for $i = 1, 2$. By definition, we have $v_1^{-1}v_2 \in \mathcal{O}_M$.

Set $s = d(o, y_1) \in [\varepsilon_1 n, \varepsilon_2 \rho m], t = d(o, y_2) \in [\varepsilon_2 \rho m, k]$, where $n \geq k \geq \rho m$. Thus, $d(y_1, y_2) = t - s$, and $|d(v_1, o, v_2 o) - (t - s)| \leq 2M \leq \Delta$. This means that
\[
v_1^{-1}v_2 \in \mathcal{O}_M(t - s, \Delta).
\]
Similarly as above, we have that
\[
u^{-1}v_1 \in \mathcal{V}_{\nu, h}(s - \varepsilon_1 n, \Delta), \quad v_2^{-1}g \in A(k - t, \Delta).
\]

Consequently, the element $g = u(u^{-1}v_1)(v_1^{-1}v_2)(v_2^{-1}g)$ lies the following set
\[
A(\varepsilon_1 n, \Delta) \cdot \mathcal{V}_{\nu, h}(s - \varepsilon_1 n, \Delta) \cdot \mathcal{O}_M(t - s, \Delta) \cdot A(k - t, \Delta)
\]
where $s \in [\varepsilon_1 n, \varepsilon_2 \rho m]$ and $t \in [\varepsilon_2 \rho m, k]$.

Similarly, when $x \notin N_M(\mathcal{G}o)$ and $y \in N_M(\mathcal{G}o)$, we obtain
\[
g \in A(i, \Delta) \cdot \mathcal{O}_M(j - i, \Delta) \cdot \mathcal{V}_{\nu, h}(\varepsilon_2 \rho m - j, \Delta) \cdot A(k - \varepsilon_2 \rho m, \Delta),
\]
where $i \in [0, \varepsilon_1 n], j \in [\varepsilon_1 n, \varepsilon_2 \rho m]$.

Case 3. We now consider the general case that $x, y \notin N_M(\mathcal{G}o)$. Recall that $\varepsilon = \varepsilon_2 \rho - \varepsilon_1$. By Lemma 3.1, the set $U(\varepsilon)$ is negligible. Without loss of generality, we can assume that $g \notin U(\varepsilon)$. This implies that $[x, y]_\alpha \cap N_M(\mathcal{G}o) \neq \emptyset$. Indeed, if not, then the geodesic segment $[x, y]_\alpha$ lies...
outside $N_M(\Go)$. Since $[x, y]_\alpha$ is a subsegment of $\alpha = [o, uo]$ of length $(\varepsilon_2 \rho - \varepsilon_1)n$ outside $N_M(\Go)$, we obtain $g \in U(\varepsilon_2 \rho - \varepsilon_1)$, that is a contradiction.

Hence, consider the maximal open segments $(x_1, x_2)_\alpha, (y_1, y_2)_\alpha$ of $\alpha$ outside $N_M(\Go)$ which contain $x, y$ respectively. Since $[x, y]_\alpha \cap N_M(\Go) \neq \emptyset$, these two intervals are disjoint.

Denote $i = d(o, x_1), j = d(o, x_2)$ and $s = d(o, y_1), t = d(o, y_2)$. Then $i \in [0, \varepsilon_1 n], j < s \in [\varepsilon_1 n, \varepsilon_2 pn], t \in [\varepsilon_2 pm, k]$, where $k \in [pn, n]$. By the same reasoning as in the previous two cases, we have

$$g \in A(i, \Delta) \cdot O_M(j - i, \Delta) \cdot V_{\nu, h}(s - j, \Delta) \cdot O_M(t - s, \Delta) \cdot A(k - t, \Delta)$$

for each $g \in V(\varepsilon_1, \varepsilon_2) \cap A([\rho n, n])$ with $|g| = k$.

Note that $\varepsilon_2 \rho \in (\varepsilon_1, \varepsilon_2)$ and $k \in [\rho n, n]$. We look at the index set

$$\Lambda = \{(i, j, s, t) \in \mathbb{N}^4 : 0 \leq i \leq \varepsilon_1 n \leq j \leq s \leq \varepsilon_2 pn \leq t \leq n\},$$

over which, we define

$$V_{(i, j), (s, t)} := A(i, \Delta) \cdot O_M(j - i, \Delta) \cdot V_{\nu, h}(s - j, \Delta) \cdot O_M(t - s, \Delta) \cdot B_{n - t + \Delta}.$$

Combining (10), (11) and (12), we have the following decomposition

$$V(\varepsilon_1, \varepsilon_2) \cap A([\rho n, n]) \subseteq \bigcup_{(i, j, s, t) \in \Lambda} V_{(i, j), (s, t)},$$

up to a negligible set $U(\varepsilon)$.

To conclude the proof, it remains to show that the right-hand set in (13) is negligible. For that purpose, we consider a triple of lengths $(l_1, l_2, l_3)$ with $l_1 + l_2 + l_3 = t \in [\varepsilon n, n]$. We observe that there are at most $(l_1 + 1)$ indexes $(i, j, s, t) \in \Lambda$ satisfying $j - i = l_1, s - j = l_2, t - s = l_3$. In fact, we can choose some $i \in [0, \varepsilon_1 n]$ first, and once $i$ is fixed, then $j, s, t$ are all determined by the triple $(l_1, l_2, l_3)$. However, the choice of $i$ can only change from $\varepsilon_1 n - l_1$ to $\varepsilon_1 n$, so we have at most $l_1 + 1$ many $(i, j, s, t) \in \Lambda$ falling in the same triple $(l_1, l_2, l_3)$.

For each $V_{(i, j), (s, t)}$ with $j - i = l_1, s - j = l_2, t - s = l_3$, we have the following estimate:

$$|V_{(i, j), (s, t)}| \leq |A(i, \Delta)| \cdot |O_M(j - i, \Delta)| \cdot |V_{\nu, h}(s - j, \Delta)| \cdot |O_M(t - s, \Delta)| \cdot |B_{n - t + \Delta}|$$

$$\times \exp(n \delta_G) \cdot |O_M(l_1, \Delta)| \cdot |V_{\nu, h}(l_2, \Delta)| \cdot |O_M(l_3, \Delta)| \cdot \exp((n - l_1 - l_2 - l_3) \delta_G),$$

where we used $|B_{n - t + \Delta}| \approx \exp((n - t) \delta_G)$ since the action has purely exponential growth.

Since the indexes $(i, j, s, t) \in \Lambda$ can be grouped according to the triple $(l_1, l_2, l_3)$, we obtain

$$\sum_{(i, j, s, t) \in \Lambda} |V_{(i, j), (s, t)}| \exp(n \delta_G)$$

$$\leq \sum_{\varepsilon n \leq l \leq n} \left(\sum_{l_1 + l_2 + l_3 = t} |O_M(l_1, \Delta)| \cdot |V_{\nu, h}(l_2, \Delta)| \cdot |O_M(l_3, \Delta)| \exp((-l_1 - l_2 - l_3) \delta_G)\right).$$

This tends 0 as $n \to \infty$ by Lemma 2.15(2). We conclude that $V(\varepsilon_1, \varepsilon_2, h) \cap A([\rho n, n])$ is negligible. When the action is SCC, the above inequality tends to 0 exponentially fast. The proof of the result is complete. \qed
3.2. Negligible pairs of elements. The goal of Theorem 1.7 is to show a random pair \((u_1, u_2) \in G^2\) generates a free group of rank 2. We now define two negligible sets of 2-tuples \((u_1, u_2) \in G^2\), whose properties shall fail to be a free basis.

For any \(u \in G\), let \(\alpha = [o, uo]\) be any geodesic with length parametrization \(\alpha(t)\). Define \(\overline{\alpha} = [o, u^{-1}o]\) to be the geodesic with parametrization \(\overline{\alpha}(t) := u^{-1}\alpha(|u| - t)\).

Given \(0 < \varepsilon_1 < \varepsilon_2 < 1\) and \(C > 0\), let \(Z(\varepsilon_1, \varepsilon_2)\) be the set of \(u \in G\) such that for some \(\alpha = [o, uo]\), one of the following holds:

1. \(\overline{\alpha}\) intersect the \(C\) neighborhood of the subsegment \(\alpha_{[\varepsilon_1, \varepsilon_2]}\) of \(\alpha\)
2. \(\alpha\) intersect the \(C\) neighborhood of the subsegment \(\overline{\alpha}_{[\varepsilon_1, \varepsilon_2]}\) of \(\overline{\alpha}\)

In other words,

\[
Z(\varepsilon_1, \varepsilon_2, C) = \{ u \in G : \exists \alpha = [o, uo], s.t. \overline{\alpha} \cap N_C(\alpha_{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset \text{ or } \alpha \cap N_C(\overline{\alpha}_{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset \}.
\]

Lemma 3.4. Let \(0 < \varepsilon_1 < \varepsilon_2 \leq 1 - \varepsilon_1 < \rho < 1\) and \(C > 0\). If our group action satisfies the DOP condition and purely exponential growth, then \(Z(\varepsilon_1, \varepsilon_2)\) is negligible in \((\rho, \Delta)\)-annuli in \(G\). Moreover, if the action is SIC, then \(Z(\varepsilon_1, \varepsilon_2, C)\) is exponentially negligible in \(G\).

Proof. For any \(u \in Z(\varepsilon_1, \varepsilon_2) \cap A([pm, n]) - U(\varepsilon_1, \varepsilon_2)\), there exists \(\alpha = [o, uo]\) satisfying the condition in the definition of \(Z(\varepsilon_1, \varepsilon_2)\). Denote \(j = |u|\) and then \(pm \leq j \leq n\).

Without loss of generality, assume that \(\overline{\alpha} \cap N_C(\alpha_{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset\). By definition, there exists \(\varepsilon_1 \leq i \leq \varepsilon_2\), so that \(\overline{\alpha} \cap N_C(\alpha(i)) \neq \emptyset\). Thus, there exists \(s \in [i - C, i + C]\) such that

\[
d(\overline{\alpha}(s), \alpha(i)) \leq C.
\]

Set \(x = \alpha(i), y = \overline{\alpha}(s) = u^{-1}\alpha(j - s)\). Thus, \(d(x, y) \leq C\).

We follow a similar analysis as in the proof of Lemma 3.3.

Case 1. Assume that \(x, y \in N_M(\alpha)\), so there exist \(v, w \in G\) such that

\[
d(x, vo) \leq M, \quad d(y, wo) \leq M.
\]

This implies \(d(vo, wo) \leq d(x, y) + 2M \leq 2M + C\), so \(v^{-1}w \in B_{2M+C}\). Since \(\overline{\alpha}(0) = o\) and \(s \in [i - C, i + C]\), we have \(d(o, y) = s\) and then \(d(o, wo) - s \leq M + C\). Thus,

\[
v \in A(i, M), w \in A(i, M + C).
\]

We can now write \(u = v(v^{-1}uw)(w^{-1}v)\), where

\[
\begin{align*}
d(uwo, vo) & \leq d(o, uy) - d(o, x) + 2M \\
& \leq d(o, \alpha(j - s)) - d(o, \alpha(i)) + 2M \\
& \leq j - s - i + 2M \leq j - 2i + C.
\end{align*}
\]

which implies \(v^{-1}uw \in A(j - 2i, 2M + C)\).

Noting that \(v \in A(i, M)\), the set of elements \(u\) in this case belongs to the following set

\[
A(i, M) \cdot A(j - 2i, 2M + C) \cdot B_{2M+C} \cdot A(i, M).
\]

Case 2. Assume that one of \(\{x, y\}\) lies outside \(N_M(\alpha)\). For definiteness, assume that \(x \in N_M(\alpha), y \notin N_M(\alpha)\); the other case is symmetric. Then there exists \(w \in G\) such that \(d(x, wo) \leq M\). Consider the maximal open segment \((y_1, y_2)\) of \(\overline{\alpha}\) which contains \(y\) but lies outside \(N_M(\alpha)\). Hence, there exists \(w \in G\) such that \(d(y_1, wo) \leq M\).

Since \(u \notin U(\varepsilon_1, \varepsilon_2)\) is assumed and then \(u^{-1} \notin U(\varepsilon_1, \varepsilon_2)\) by definition, we obtain that \(d(y_1, y_2) < \varepsilon_1//8\). Thus we have \(d(y_1, y) \leq d(y_1, y_2) < \varepsilon_1//8\). This yields

\[
d(vo, wo) \leq d(vo, x) + d(x, y) + d(y, y_1) + d(y_1, wo) \leq \varepsilon_1//8 + 2M + C.
\]
Hence, we can also write $u = v(v^{-1}uw)(w^{-1}v)v^{-1}$, where

$$v \in A(i, M), v^{-1}w \in B_{\frac{\Delta}{2}j + 2M + C}, v^{-1}uw \in B_{j - 2i + \frac{\Delta}{2}j + 2M + C}.$$ 

**Case 3.** Assume $x, y \notin N_M(G_0)$. Consider the maximal open segment $(x_1, x_2)_\alpha$ (resp. $(y_1, y_2)_\alpha$) of $\alpha$ (resp. $\beta$) which contains $x$ (resp. $y$) but lies outside $N_M(G_0)$. Then there exist $v, w \in G$ such that $d(x_1, vo) \leq M, d(y_1, wo) \leq M$. Similar argument as above we have the following conclusion: we can write

$$u = v(v^{-1}uw)(w^{-1}v)v^{-1},$$

where

$$v \in B_{i + M + C}, v^{-1}w \in B_{\frac{\Delta}{2}j + 2M + C}, v^{-1}uw \in B_{j - 2i + \frac{\Delta}{2}j + 2M + C}.$$ 

Set $\Delta = 2M + C$. Summarizing the above three cases, we have

$$|Z(\varepsilon_1, \varepsilon_2) \cap A([\rho n, n]) \setminus U(\varepsilon_1 \frac{1}{8})|$$

$$\leq 2 \sum_{j = \rho n}^{\rho n + \varepsilon_1} |B_{i + \Delta} | \cdot | B_{j - 2i + \frac{\Delta}{2}j + \Delta} | \cdot | B_{\frac{\Delta}{2}j + \Delta} |$$

$$\times (1 - \rho)n \cdot (\varepsilon_2 - \varepsilon_1)n \cdot \exp((1 - \frac{\varepsilon_1}{2})n)$$

where the last line used

$$|B_{i + \Delta} \cdot | B_{j - 2i + \frac{\Delta}{2}j + \Delta} \cdot | B_{\frac{\Delta}{2}j + \Delta} | \asymp \exp(\delta_G(j + \frac{\varepsilon_1}{2}j - i))$$

which follows from the purely exponentially growth.

This shows that $Z(\varepsilon_1, \varepsilon_2, C) \cap A([\rho n, n]) \setminus U(\varepsilon_1 \frac{1}{8})$ is negligible. By Lemma 3.1 $U(\varepsilon_1 \frac{1}{8})$ is negligible. Thus the conclusion follows.

Fix $0 < \varepsilon_1 < \varepsilon_2 < \rho < 1$ and $C > 0$. Let $T(\varepsilon_1, \varepsilon_2, C)$ be the set of $(u_1, u_2) \in G \times G$ with the following property:

there exist two geodesics $\alpha := [o, u_1o], \beta := [o, u_2o]$ such that neither of them disjoint the $C$ neighborhood of the $[\varepsilon_1, \varepsilon_2]$-interval of the other. In other words,

(15)

$$T(\varepsilon_1, \varepsilon_2, C) = \left\{ (u_1, u_2) \in G \times G : \exists \alpha := [o, u_1o], \beta := [o, u_2o], s.t. \alpha \cap N_C(\beta_{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset \right\}.$$ 

**Lemma 3.5.** For any $0 < \varepsilon_1 < \varepsilon_2 \leq 1 - \varepsilon_1 < \rho < 1$ and $C > 0$, if our group action satisfies the DOP condition and PEG condition, then $T(\varepsilon_1, \varepsilon_2, C)$ is negligible in $(\rho, \Delta)$-annuli in $G \times G$.

Moreover, if the action is SCC, then $T(\varepsilon_1, \varepsilon_2, C)$ is exponentially negligible in $G \times G$.

**Proof.** Since the union of two (exponentially) negligible sets is (exponentially) negligible, without loss of generality, we can assume that for all $(u_1, u_2) \in T(\varepsilon_1, \varepsilon_2, C)$, we have

$$\beta \cap N_C(\alpha_{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset.$$ 

Choose $1 - \varepsilon_1 < \rho < 1$. By Lemma 2.10 we can assume further that $(u_1, u_2)$ belongs to $T(\varepsilon_1, \varepsilon_2) \cap \left( A([\rho n, n]) \times A([\rho n, n]) \right)$.

Denote $n_1 = |u_1|$. By definition of $T(\varepsilon_1, \varepsilon_2, C)$, there exists $i \in [\varepsilon_1 n_1, \varepsilon_2 n_1]$ so that $\beta \cap N_C(\alpha(i)) \neq \emptyset$. Denote $x = \alpha(i)$ and $\Delta = C + 2M$. We proceed by a similar argument as before.
we have that $U \in v M$, $d H$. Hence, we have the upper bound on pairs $(u, v)$ follows so these pairs $(u, v)$ such that $\rho n \leq n$. Then $(u_1, u_2)$ can be written as $(v(v^{-1}u_1), v(v^{-1}u_2))$, where $v^{-1}u_1 \in A(n_1 - i, M), v^{-1}u_2 \in A(n_2 - i, C + M)$. Note that $n_1 \in [\rho m, n]$. In this case, we bound by above the number of elements $(u_1, u_2)$ as follows

\[
\leq \sum_{\rho n \leq n_1 \leq n} |A(i, \Delta)| \cdot |A(n_1 - i, \Delta)| \cdot |A([\rho m - n - i], \Delta)| \\
\sim n \exp((2 - \varepsilon_1)\delta_G n) = o(\exp(2\delta_G n)),
\]

so these pairs $(u_1, u_2) \in T(\varepsilon_1, \varepsilon_2)$ are exponentially negligible.

**Case 2.** Otherwise, consider the maximal open segment $(x_1, x_2)_{a, i}$ of $\alpha^i$, which contains $x$ but lies outside $N_M(Go)$. Denote $j := d(o, x_1), l := d(o, x_2)$. Thus $0 \leq j \leq i$ and $i < l \leq n_1$.

**Subcase 2.1** $l - j \geq \frac{\varepsilon_1}{2}n_1$, then $u_1 \in U(\frac{\varepsilon_1}{2})$. Since $U(\frac{\varepsilon_1}{2})$ is negligible in $G$ by Lemma 3.1, we have that $U(\frac{\varepsilon_1}{2}) \times G$ is negligible as well in $G \times G$.

**Subcase 2.2** $l - j < \frac{\varepsilon_1}{2}n_1$. As before, there exist $v_1, v_2 \in G$ such that $d(x_1, v_1o) \leq M, d(x_2, v_2o) \leq M$. Thus, $v_1 \in A(j, M)$.

Then $(u_1, u_2)$ can be written as $(v_1(v_1^{-1}v_2)(v_2^{-1}u_1), v_1(v_1^{-1}v_2)(v_2^{-1}u_2))$, where $v_1^{-1}v_2 \in A(l - j, 2M), v_2^{-1}u_1 \in A(n_1 - l, M), v_2^{-1}u_2 \in A((n_2 - i) + (l - i), C + M)$

We consider the index set

\[\Lambda = \{(n_1, i, j, l) \in \mathbb{Z}^4 : \rho m \leq n_1 \leq n, \varepsilon_1n_1 \leq i \leq \varepsilon_2n_1, 0 \leq j \leq l \leq j + \frac{\varepsilon_1}{2}n_1\} \}

Hence, we have the upper bound on pairs $(u_1, u_2)$ of the second case as follows

\[
\leq \sum_{(n_1, i, j, l) \in \Lambda} |A(j, \Delta)| \cdot |A(l - j, \Delta)| \cdot |A(n_1 - l, \Delta)| \cdot |A([\rho m + l - 2i, n + l - 2i], \Delta)| \\
\sim \sum_{(n_1, i, j, l) \in \Lambda} \exp((n + n_1 + l - 2i)\delta_G) \\
\sim n^4 \exp((2 - \frac{\varepsilon_1}{2})n\delta_G) = o(\exp(2\delta_G n)).
\]

Therefore, in this case, we have proved the negligibility of $T(\varepsilon_1, \varepsilon_2)$. The proof is complete. $\square$

4. The proof of the Theorems

This section is devoted to the proof of the theorems of this paper.

4.1. Generically free subgroups. Let $\Lambda > 0$. Denote by $\mathcal{F}^{(k)}$ by the set of $k$-tuples

\[\{u_1, \ldots, u_k\} \in G^{(k)}\]

such that

1. $\langle u_1, u_2, \ldots, u_k \rangle$ is a free group of rank $k$ consisting of contracting elements except the identity,

2. the map $h \in \langle u_1, u_2, \ldots, u_k \rangle \to ho \in Y$ is a $(\Lambda, 0)$-quasi-isometrically embedded map.
Let $f(u_1, \ldots, u_k)$ be the free group generated by the $k$-tuple $\{u_1, \ldots, u_k\}$. In order to prove that $F^{(k)}$ is generic in $G^{(k)}$, the idea is to construct a generic subset $E \subseteq G^{(k)}$, such that for any $\overrightarrow{u} = (u_1, \ldots, u_k) \in E$ and any nontrivial freely reduced word $W \in f(u_1, \ldots, u_k)$, we can construct an admissible path from $o$ to $W_o$ that satisfies the conditions of Proposition 2.11 and thus the path is a quasi-geodesic by the same proposition. This then concludes the proof of Theorem 1.4.

To be clear, we fix some notations and constants at the beginning (the reader is encouraged to read the proof first and return here until the constant appears).

**Setup**

1. We denote by $C > 0$ the contraction constant for the contracting system $\{gAx(h) : g \in G\}$. Assume that $C$ satisfies Lemma 2.5 as well.
2. Let $X = \{gN_C(A) : g \in G\}$ be the contracting system of Lemma 2.9 with contraction constant $\kappa$. We denote $X = N_C(A)$.
3. If our group action satisfies DOP condition, then constants $\nu, M > 0$ are given by definition of DOP condition and Proposition 2.14. Moreover if the group action is SCC, then $\nu, M > 0$ are given by Proposition 2.18.
4. Let $D = D(\kappa, 9C) > 16C$ be the constant of admissible paths given by Proposition 2.11.
5. Take $m > 0$ so that $|h^m| > D + 2\nu$. This can be done since $h$ is a contracting element, then we have $n \in \mathbb{Z} \mapsto h^n \in G$ is a quasi-isometric embedding map of $\mathbb{Z} \to G$.

We refer the reader to the definitions of the set $V(2\varepsilon, 1-2\varepsilon, h^m)$ in (9), the set $W(\varepsilon, C)$ in (8), the set $Z(\varepsilon, 1-\varepsilon, C)$ in (14) and the set $T(\varepsilon, 1-\varepsilon, C)$ in (15).

**Lemma 4.1.** Fix $1 > \rho > \frac{8}{7}$ and $\varepsilon \in (1-\rho, \frac{1}{7})$. The subset $E$ of all $\overrightarrow{u} = (u_1, \ldots, u_k) \in G^{(k)}$ satisfying the following conditions is generic.

1. $|u_i| \geq \rho|\overrightarrow{u}|$ for $1 \leq i \leq k$.
2. $u_i^{\pm 1} \not\in V(2\varepsilon, 1-2\varepsilon, h^m) \cup W(\varepsilon, C) \cup Z(\varepsilon, 1-\varepsilon, C)$ for $1 \leq i \leq k$.
3. $(u_i^{\pm 1}, u_j^{\pm 1}) \not\in T(\varepsilon, 1-\varepsilon, C)$ for $i \neq j \in \{1, 2, \ldots, k\}$.

When the action is SCC, the set $E$ is exponentially generic.

**Proof.** It suffices to show that the set of $\overrightarrow{u} \in G^{(k)}$ in each statement as above is generic. It is clear that our choice of $\rho, \varepsilon$ satisfy all the condition of the lemmas in Section 3. Hence the assertion (1) is given by Lemma 2.10. The assertion (2) is a consequence of Lemmas 3.3, 3.2 and 3.4 together. And the assertion (3) follows from Lemma 3.5.

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4**. For notational simplicity, we give the proof for $k = 2$. Let $E$ be the subset of $G \times G$ provided by Lemma 4.1. It suffices to show that $E$ in contained in $F^{(2)}$.

Fix a choice $(u_1, u_2) \in E$. We choose a geodesic $\alpha = [o, u_1o]$, and denote by $\overrightarrow{\alpha} := [o, u_1^{-1}o]$ a geodesic from $o$ to $u_1^{-1}o$. Similarly, we define $\beta = [o, u_2o]$ and its reverse $\overrightarrow{\beta} := [o, u_2^{-1}o]$. Denote $n_1 = |u_1|$ and $n_2 = |u_2|$.

Let $W$ be a non-trivial freely reduced word in $f(u_1, u_2)$. We shall prove that the evaluation of the word $W$ in $G$ gives a non-trivial contracting element. For this purpose, we can assume without loss of generality that $W$ is cyclically reduced so that the bi-infinite word $W^\infty = \cdots W \cdot W \cdot W \cdots$ is reduced, written explicitly as

$$W^\infty = \cdots x_{-1}x_0x_1x_2 \cdots x_j \cdots$$
where each \( x_j \in \{ u_1, u_2, u_1^{-1}, u_2^{-1} \} \).

Associated with the bi-infinite word \( W^\infty \), we construct a bi-infinite path \( \gamma \) as a concatenation of geodesic segments \( \gamma_j \) for \( j \in \mathbb{Z} \) as follows:

\[
\gamma = \ldots \gamma^{-1} \cdot \gamma^0 \cdot \gamma^1 \cdot \gamma^2 \cdot \ldots \cdot \gamma^j \ldots,
\]

where \( \gamma^0 = [o, x_0o] \) and for \( j \geq 1 \), \( \gamma^j \) is the \( x_1 \ldots x_{j-1} \)-translate of \( \alpha \) or \( \beta \) depending on \( x_j \).

We now describe a procedure to convert the path \( \gamma \) to be an admissible path by truncating certain subpaths.

Since \( u_1, u_2 \notin V(2\varepsilon, 1 - 2\varepsilon, h^m) \) defined in (9), we have that each \( \gamma^j \) contains a \((\nu, h^m)\)-barrier in \( \gamma^j_{[2\varepsilon,(1-2\varepsilon)]} \). Then there exists an element \( g_j \in G \) such that

\[
\max\{d(g_j o, \gamma^j_{[2\varepsilon,(1-2\varepsilon)]}), \ d(g_j h^m o, \gamma^j_{[2\varepsilon,(1-2\varepsilon)]})\} \leq \nu \leq C.
\]

We denote \( X = N_C(A) \). Let \( v_j, w_j \) be the entry and exit point of \( \gamma^j \) into \( N_C(g_j A) = g_j X \) respectively. We must have

\[
d(v_j, w_j) \geq d(o, h^m o) - 2\nu.
\]

Since \( u_1, u_2 \notin W(\varepsilon, C) \) defined in (3), we have \( d(v_j, w_j) \leq \varepsilon \min\{ |u_1|, |u_2| \} \leq \varepsilon n \), and so \( v_j, w_j \in \gamma^j_{[\varepsilon,(1-\varepsilon)]} \) follows from the definition of \( W(\varepsilon, C) \). In other words, the subsegment \( [v_j, w_j]_{\gamma^j} \) of \( \gamma^j \) is contained in \( g_j X \) by Lemma 2.5.

We now truncate the subpath \( [v_{j-1}, \gamma^j]\cdot[\gamma_-, w_j] \) from \( \gamma \) and replace it with a geodesic. The resulting path is given as follows

\[
\beta = \ldots [v_{-1}, w_{-1}]_{\gamma^{-j}}\cdot[w_{-1}, v_0]\cdot[w_0, w_{|\gamma^0|}]\cdot[w_{|\gamma^0|}, v_1]\cdot[v_1, w_{|\gamma^1|}]\cdot[w_{|\gamma^1|}, v_2]\cdot\ldots\cdot[w_{j-1}, v_j]\cdot[v_j, w_j]_{\gamma^j}\ldots,
\]

where \( [w_{j-1}, v_j] \) is a choice of geodesics between \( w_{j-1} \) and \( v_j \) for any \( j \in \mathbb{Z} \).

By Lemma 2.9 \( X = \{ gN_C(A) : g \in G \} \) is a contracting system with bounded projection. In the following claim, we shall consider the admissible path associated with \( \{ g_j X : j \in \mathbb{Z} \} \).

**Claim** \( \beta \) is an \((D, \tau)\)-admissible path.

**Proof of Claim.** First of all, we have \( v_j, w_j \in g_j X \) and \( d(v_j, w_j) \geq |h^m| - 2\nu \geq D \), thus the condition (LL1) is satisfied.

Recall that \( W^\infty \) is a freely reduced word over \( \{ u_1, u_2, u_1^{-1}, u_2^{-1} \} \), so the pair of any two adjacent letters \( (x_j, x_{j+1}) \) does not belong to \( Z(\varepsilon, 1 - \varepsilon, 4C) \) and \( T(\varepsilon, 1 - \varepsilon, 4C) \). Since \( v_j, w_j \in \gamma^j_{[\varepsilon,(1-\varepsilon)]} \), we derive from Lemma 3.1 that

\[
\forall j \in \mathbb{Z}, \ \gamma^{j-1} \cap N_{4C}([v_j, w_j]_{\gamma^j}) = \emptyset, \ \gamma^{j+1} \cap N_{4C}([v_j, w_j]_{\gamma^j}) = \emptyset.
\]

For simplicity, we write \( X_j := g_j X_j \). Using Lemma 2.8 we have

(16) \[
\forall j \in \mathbb{Z}, \ \gamma^{j-1} \cap X_j = \emptyset, \ \gamma^{j+1} \cap X_j = \emptyset.
\]

Thus, by Lemma 2.7 we obtain \( \|\Pi_{X_j}([w_j, \gamma^j_-])\| \leq 3C, \|\Pi_{X_j}(\gamma^{j+1})\| \leq 3C \).

For any \( j \in \mathbb{Z} \), we have

\[
\|\Pi_{X_j}([w_j, v_{j+1}])\| \leq \|\Pi_{X_j}([w_j, v_{j+1}])\| + C
\]

\[
\leq \|\Pi_{X_j}([w_j, \gamma^j_-])\| + \|\Pi_{X_j}(\gamma^{j+1})\| + C
\]

\[
\leq \|\Pi_{X_j}([w_j, \gamma^j_-])\| + \|\Pi_{X_j}(\gamma^{j+1})\| + 3C
\]

\[
\leq 6C + 3C \leq 9C.
\]
A similar estimate as above shows
\[ \|\Pi_{X_j}([w_{j-1}, v_j])\| \leq 9C. \]
Thus the condition (BP) is satisfied.

From (16), we have \( X_j \neq X_{j+1} \) for all \( j \in \mathbb{Z} \). Then all conditions in the definition of admissible paths are verified. Thus, \( \beta \) is a \((D, 9C)\)-admissible path.

By Proposition 2.11, we know that \( \beta \) is a \((\Lambda, 0)\)-quasi-geodesic and it is contracting. Thus, every non-trivial freely reduced word gives a non-trivial contracting element so \( \langle u_1, u_2 \rangle \) is a free group of rank 2.

This implies that \( \langle u_1, u_2 \rangle \) generates a free group of rank 2 consisting of contracting elements such that the orbital map is \((\Lambda, 0)\)-quasi-isometrically embedded. This concludes the proof of Theorem 1.5. \( \square \)

**Proof of Theorem 1.6.** If a non-elementary group \( G \) admit a proper SCC action on \((Y, d)\) with a contracting element, then the corresponding set defined in Lemma 4.1 is exponentially generic since these sets provided in Section 4 are exponentially negligible. Therefore, \( F^{(k)} \) is exponentially generic in \( G^{(k)} \). \( \square \)

### 4.2. Statistical hyperbolicity.

**Proof of Theorem 1.8 for Annuli Case.** Choose any \( 0 < \varepsilon < 1 \). Let
\[ E = U(\varepsilon^2, C) \cup V(2\varepsilon, 3\varepsilon, h^m) \cap W(\varepsilon, C). \]
Then By Lemma 3.1, 3.3 and 3.2 together, we have \( \lim_{n \to +\infty} \frac{|K \cap A(n, \Delta)|}{|A(n, \Delta)|} = 0 \) for some \( \Delta > 0 \).

Now we fix such a \( \Delta \).

For any \( x \in A(n, \Delta) \setminus E \), we fix a geodesic \( \alpha = [o, xo] \), and consider the following set
\[ K_x = \{ z \in A(n, \Delta) : \exists \beta = [o, zo], s.t. \beta \cap N_{4C}(a[\varepsilon,4\varepsilon]) \neq \emptyset \}. \]

We shall show that \( K_x \) is negligible:

\[ \lim_{n \to \infty} \frac{|K_x|}{|A(n, \Delta)|} = 0 \]
for each \( x \). Set \( n_1 = |x| \), then \( n - \Delta \leq n_1 \leq n + \Delta \). We carry out the same analysis as in the proof of Lemma 3.5 to bound \( |K_x| \): given any element \( z \in K_x \) of length \( n_2 \), if \( \alpha(i) \) intersect \( N_{4C}(Go) \) for some \( \varepsilon n_1 \leq i \leq 4\varepsilon n_1 \), then we can write \( (x, z) \) as \( (v_1(v_1^{-1}x), v_1(v_1^{-1}z)) \) for some \( v_1 \in A(i, M) \), such that \( v_1^{-1}z \in A(n_2 - i, 2\Delta) \); otherwise we can write \( (x, z) \) as \( (v_2(v_2^{-1}x), v_2(v_2^{-1}z)) \) for some \( i \leq l \leq i + \frac{\varepsilon}{2} n_1 \) and some \( v_2 \in A(l, M) \), such that \( v_2^{-1}z \in A((n_2 - i) + (l - i), 2\Delta) \) (by our choice of \( x \notin U(\varepsilon^2) \), subcase 2.1 of Lemma 3.5 can not happen).

If we introduce the index sets
\[ \Lambda_1 = \{(n_2, i) \in \mathbb{Z}^2 : n - \Delta \leq n_2 \leq n + \Delta, \varepsilon n_1 \leq i \leq 4\varepsilon n_1 \}, \]
\[ \Lambda_2 = \{(n_2, i, l) \in \mathbb{Z}^3 : \rho n \leq n_2 \leq n, \varepsilon n_1 \leq i \leq 4\varepsilon n_1, i \leq l \leq i + \frac{\varepsilon}{2} n_1 \}, \]
then we have
\[ |K_x| \leq \sum_{(n_2, i) \in \Lambda_1} |A(n_2 - i, 2\Delta)| + \sum_{(n_2, i, l) \in \Lambda_2} |A(n_2 + l - 2i, 2\Delta)| \]
\[ < \exp((1 - \varepsilon)n\delta_G) + n^3 \exp((1 - \varepsilon^2)n\delta_G), \]

where \( \delta_G \) is a constant depending on \( G \). Therefore, \( K_x \) is negligible for each \( x \), and hence \( K \) is negligible.
which implies (17) from $|A(n, \Delta)| \asymp \exp(\delta_G n)$.

The next step is to bound the distance between $x_0$ with the orbit point $y_0$ outside $K_\Delta$. **Claim** For any $y \in A(n, \Delta) \setminus K_\Delta$, we have $d(x, y) \geq 2(n - 4\varepsilon n - 4\varepsilon \Delta - \Delta - 4C)$.

**Proof of Claim.** Since $x \notin V(2\varepsilon, 3\varepsilon, h^m)$ in (9), $\alpha$ contains a $(\nu, h^m)$-barrier in $\alpha_{[2\varepsilon, 3\varepsilon]}$, so there exists an element $g \in G$ such that

\[
\max \{d(g_0, \alpha_{[2\varepsilon, 3\varepsilon]}), d(gh^{m_0}, \alpha_{[2\varepsilon, 3\varepsilon]})\} \leq \nu \leq C.
\]

We denote $X = N_C(A)$. Let $v, w$ be the entry and exit point of $\alpha$ into $gX$ respectively, so that

\[
d(v, w) \geq d(o, h^m o) - 2\nu > D.
\]

Since $x \notin W(\varepsilon, C)$ in (9), this implies that $v, w \in \alpha_{[\varepsilon, 4\varepsilon]}$. Thus, $d(o, w) \geq \varepsilon n_1$ and $d(w, xo) \geq (1 - 4\varepsilon)n_1$.

For any $y \in A(n, \Delta) \setminus K_\Delta$, we know from the definition of $K_\Delta$ that for any geodesic $\beta = [o, yo]$, $\beta \cap N_{4C}(\alpha_{[\varepsilon, 4\varepsilon]}) = \emptyset$. Thus, we have $\beta \cap gX = \emptyset$ by Lemma 2.8.

If we choose $d(o, h^m o) - 2\nu > D \geq 16C$ as in the setup, then for any $\gamma = [x_0, yo]$, we have $\gamma \cap gX \neq \emptyset$. Indeed, if $\gamma \cap gX = \emptyset$, we will then obtain a contradiction:

\[
d(v, w) \leq \|\Pi_X([v, 1])\| + \|\Pi_X([1, y])\| + \|\Pi_X([y, x])\| + \|\Pi_X([x, w])\|
\]

\[
\leq \|\Pi_X([1, v])\| + \|\Pi_X(\beta)\| + \|\Pi_X(\gamma)\| + \|\Pi_X([w, x])\| + 4C
\]

\[
\leq 12C + 4C < D
\]

from a projection argument.

Let $u$ be the entry point of $\gamma$ in $gX$. Then $d(u, w) \leq 4C$ by the contracting property of $X$. Hence,

\[
d(xo, u) \geq d(xo, w) - d(u, w) \geq n_1 - 4\varepsilon n_1 - 4C.
\]

Since $y \in A(n, \Delta)$, we have

\[
n - \Delta \leq d(o, yo) \leq d(o, w) + d(w, u) + d(u, yo) \leq 4\varepsilon n_1 + 4C + d(u, yo)
\]

which yields

\[
d(u, yo) \geq n - 4\varepsilon n_1 - \Delta - C.
\]

We finally obtain

\[
d(xo, yo) = d(xo, u) + d(u, yo) \geq 2(n - 4\varepsilon n - 4\varepsilon \Delta - \Delta - 4C)
\]

concluding the proof of the claim. \qed

Let us return to the proof of the theorem. By the above claim, we have

\[
\sum_{x, y \in A(n, \Delta)} d(xo, yo) \geq 2(n - 4\varepsilon n - 4\varepsilon \Delta - \Delta - 4C) \cdot (|A(n, \Delta)| - |E|) \cdot (|A(n, \Delta)| - |K_\Delta|).
\]

Notice $\lim_{n \to \infty} \frac{|K_\Delta|}{|A(n, \Delta)|} = 0$ and $\lim_{n \to \infty} \frac{|E|}{|A(n, \Delta)|} = 0$. Then we obtain

\[
\liminf_{n \to \infty} \frac{1}{|A(n, \Delta)|^2} \sum_{x, y \in A(n, \Delta)} \frac{d(xo, yo)}{n} \geq 2(1 - 4\varepsilon).
\]

Since $\varepsilon$ is arbitrary, we have $E_A(G, \Delta) = 2$. \qed
Proof of Theorem 1.8 for Ball Case. The proof is almost identical to that in annuli case. We only point out the difference in the proof.

Choose any $\frac{1}{2} < \rho < 1$ and any $0 < \varepsilon < \frac{\rho}{8}$. Let $E = U(\frac{\varepsilon}{2}) \cup V(2\varepsilon, 3\varepsilon) \cup W(\varepsilon)$. Then by Lemma 5.1, 5.3 and 5.2 together, we have $\lim_{n \to +\infty} \frac{|E\cap B_n|}{|B_n|} = 0$.

For any $x \in A([\rho n, n]) \setminus E$, set $n_1 = |x|$, then $\rho n \leq n_1 \leq n$. We fix a geodesic $\alpha = [o, xo]$ and consider

$$K_x = \{ z \in A([\rho n, n]) : \exists \beta = [o, zo], s.t. \beta \cap N_{4C}(\alpha_{[\rho n, n]}) \neq \emptyset \}.$$

By the same argument as in annuli case, we have $\lim_{n \to +\infty} \frac{|K_x|}{|A([\rho n, n])|} = 0$

Claim For any $y \in A([\rho n, n]) \setminus K_x$, we have $d(x, y) \geq 2\rho n - 8\varepsilon n - 8C$.

Proof of Claim. The proof is the same as that in the annuli case, except that we now use the big annulus $A([\rho n, n])$. Note that

$$d(xo, u) \geq d(xo, w) - d(u, w) \geq n_1 - 4\varepsilon n_1 - 4C,$$

where $n_1 \in [\rho n, n]$. Since $y \in A([\rho n, n])$, we have

$$\rho n \leq d(o, yo) \leq d(o, w) + d(u, w) + d(u, yo) \leq 4\varepsilon n_1 + C + d(u, yo),$$

from which we have $d(u, yo) \geq \rho n - 4\varepsilon n_1 - 4C$. So $d(xo, yo) = d(xo, u) + d(u, yo) \geq 2\rho n - 8\varepsilon n - 8C$. \hfill \Box

The same computation as above in annuli case gives

$$\liminf_{n \to \infty} \frac{1}{|B_n|^2} \sum_{x, y \in B_n} \frac{d(x, y)}{n} \geq 2\rho - 8\varepsilon.$$

Since $\varepsilon$ can be made arbitrary small and $\rho$ can be arbitrary close to 1, then we obtain $E_B(G) = 2$. \hfill \Box

Examples 4.2. We carry out a concrete example to explain the convergence speed of $E_A(G, \Delta) = 2$ of a statistical hyperbolic group is at most of order $O(n^{-1})$. Consider the free group $\mathbb{F}(a, b)$ and its Cayley graph with respect to the free generators $\{a, b\}$. It is easy to calculate

$$\frac{1}{|A(n, 0)|^2} \sum_{x, y \in A(n, 0)} d(x, y) = \frac{3}{4} \cdot 2n + \frac{1}{4} \cdot \frac{2}{3} \cdot (2n - 2) + \frac{1}{3} \cdot \frac{2}{3} \cdot (2n - 4) + \cdots + 0.$$

Thus we obtain

$$\frac{1}{|A(n, 0)|^2} \sum_{x, y \in A(n, 0)} |d(x, y) - 2| = \frac{n - 1}{2n} \cdot \frac{3}{4} \cdot \left(\frac{1}{3} - \frac{1}{3^2}\right) - \frac{1}{2} \cdot \frac{1}{2n} + \frac{3}{4n} \cdot \left(\frac{1}{3} - \frac{1}{3^2}\right) = O\left(\frac{1}{n}\right).$$

References

1. R. Aoun Random subgroups of linear groups are free. Duke Math. J. 160 (2011), no. 1, 117C173.
2. N. Arzhantseva, C. Cashen, D. Gruber, and D. Hume, Contracting geodesics in infinitely presented graphical small cancellation groups, arXiv:1602.03767.
3. G. Arzhantseva, C. Cashen, and J. Tao, Growth tight actions, Pacific Journal of Mathematics 278 (2015), 1-49.
4. W. Ballmann, Lectures on spaces of nonpositive curvature, Birkäuser, Basel, 1995.
5. J. Behrstock and R. Charney, Divergence and quasimorphisms of right-angled artin groups, Mathematische Annalen 352 (2012), 339-356.
6. J. Behrstock, C. Drutu, and L. Mosher, Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity, Math. Annalen 344 (2009), no. 3, 543-595.
[7] J. Behrstock, M. Hagen, and A. Sisto, \textit{Thickness, relative hyperbolicity, and randomness in Coxeter groups}, To appear in Algebr. Geom. Topol.

[8] M. Bestvina and K. Fujiwara, \textit{A characterization of higher rank symmetric spaces via bounded cohomology}, Geometric and Functional Analysis 19 (2009), no. 1, 11-40.

[9] Joan S. Birman, \textit{Braids, links, and mapping class groups}, Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, NJ, 1974.

[10] B. Bowditch, \textit{Relatively hyperbolic groups}, Int. J. Algebra Comput. (2012), no. 22, p1250016.

[11] M. Bridson and A. Haefliger, \textit{Metric spaces of non-positive curvature}, vol. 319, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1999.

[12] R. Charney and H. Sultan, \textit{Contracting boundaries of CAT(0) spaces}, J Topology 8 (2015), no. 1, 93–117.

[13] F. Dahmani, V. Guirardel, and D. Osin, \textit{Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces}, arXiv:1111.7048, to appear in Memoirs AMS, 2011.

[14] F. Dal’bo, P. Otal, and M. Peigné, \textit{Séries de Poincaré des groupes géométriquement finis}, Israel Journal of Math. 118 (2000), no. 3, 109-124.

[15] S. Dowdall, M. Duchin, and H. Masur, \textit{Statistical hyperbolicity in Teichmüller space}, Geometric and Functional Analysis 24 (2014), no.3, 748-795.

[16] C. Drutu and M. Sapir, \textit{Tree-graded spaces and asymptotic cones of groups}, Topology 44 (2005), no. 5, 959-1058, With an appendix by D. Osin and M. Sapir.

[17] M. Duchin, S. Lellèvre, and C. Mooney, \textit{statistical hyperbolicity in groups}, Algebr. Geom. Topol. 12(2012), no. 1, 1-18.

[18] M. Duchin and C. Mooney, \textit{Fine asymptotic geometry of the Heisenberg group}, Indiana University Math Journal 63 (2014), no.3, 885-916.

[19] A. Eskin, M. Mirzakhani, and K. Rafi, \textit{Counting geodesics in a stratum}, to appear Inventiones Mathematicae, Arxiv: 1206.5574, 2012.

[20] B. Farb and D. Margalit, \textit{A primer on mapping class groups}, Princeton Mathematical Series, vol. 49, Princeton University Press, 2012.

[21] B. Farb and L. Mosher, \textit{Convex cocompact subgroups of mapping class groups}, Geom. Topol. 6 (2002), 91–152.

[22] W. Floyd, \textit{Group completions and limit sets of Kleinian groups}, Inventiones Math. 57 (1980), 205C218.

[23] V. Gerasimov, \textit{Expensive convergence groups are relatively hyperbolic}, Geom. Funct. Anal. 19 (2009), 137C169.

[24] V. Gerasimov and L. Potyagailo, \textit{Quasiconvexity in the relatively hyperbolic groups}, Journal für die reine und angewandte Mathematik (Crelle Journal) (2015).

[25] V. Gerasimov and L. Potyagailo, \textit{Quasi-isometries and Floyd boundaries of relatively hyperbolic groups}, J. Eur. Math. Soc. 15 (2013), 2115-2137.

[26] E. Ghys, P. de la Harpe, eds., \textit{Sur les Groupes Hyperboliques d’après Mikhael Gromov}, Progress in Math., 83, Birkhäuser, 1990.

[27] R. Gilman, A. Miasnikov, D. Osin, \textit{Exponentially generic subsets of groups}, Illinois J. Math. 54 (1) (2010) 371–388.

[28] M. Gromov, \textit{Hyperbolic groups}, Essays in group theory (S Gersten, editor), vol. 1, pp. 75C263, Springer New York, 1987.

[29] M. Gromov, \textit{Random walk in random groups}, GAFA 13 (2003), no. 1, 73C146.

[30] D. Gruber and A. Sisto, \textit{Infinitely presented graphical small cancellation groups are acylindrically hyperbolic}, to appear in Ann. Inst. Fourier, arXiv:1408.4488, 2014.

[31] A. Karlsson, \textit{Free subgroups of groups with non-trivial Floyd boundary}, Comm. Algebra. 31 (2003), 5361C5376.

[32] J. Maher and A. Sisto \textit{Random subgroups of acylindrically hyperbolic groups and hyperbolic embeddings}, to appear in International Mathematics Research Notices

[33] J. Maher and G. Tiozzo, \textit{Random walks on weakly hyperbolic groups}, to appear in Journal für die reine und angewandte Mathematik.

[34] Y. Minsky, \textit{Quasi-projections in Teichmüller space}, J. Reine Angew. Math. 473 (1996), 121-136.

[35] J. Osborne, W.Y. Yang, \textit{Statistical hyperbolicity of relatively hyperbolic groups}, Algebr. Geom. Topol., 16 (2016) 2143-2158.

[36] D. Osin, \textit{Acylicdrically hyperbolic groups}, Trans. Amer. Math. Soc. 368 (2016), no. 2, 851–888.
[37] D. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, vol. 179, Mem. Amer. Math. Soc., 2006.

[38] M. Peigné, *On the Patterson-Sullivan measure of some discrete groups of isometries*, Israel Journal of Math. 133 (2002), 77C88.

[39] V. Pit and B. Schapira, *Finiteness of Gibbs measures on non-compact manifolds with pinched negative curvature*, to appear in Annales Institut Fourier, available at https://arxiv.org/abs/1610.03255 (2018).

[40] I. Rivin, *Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms*, Duke Math. J. 142 (2008), no. 2, 353–379.

[41] I. Rivin, *Zariski density and genericity*, Int. Math. Res. Not. IMRN 19 (2010), 3649-3657.

[42] T. Rohlin, *Ergodicité et équidistribution en courbure négative*, no. 95, Mémoires de la SMF, 2003.

[43] B. Schapira and S. Tapie, *Regularity of entropy, geodesic currents and entropy at infinity*, arXiv:1802.04991

[44] A. Sisto, *Contracting elements and random walks*, arXiv:1112.2666, accepted in Crelle’s Journa, 2011.

[45] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Etudes Sci. Publ. Math. No. 50 (1979), p. 171–202.

[46] G. Tiozzo and S. Taylor, *Random extensions of free groups and surface groups are hyperbolic*, Int. Math. Res. Notices (2016), no. 1, 294–310.

[47] W.Y. Yang, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc., 157(2014), 297-319.

[48] W.Y. Yang, *Purely exponential growth of cusp-uniform actions*, Preprint, Arxiv: 1602.07897, to appear in Ergodic theory and dynamical systems, 2016.

[49] W.Y. Yang, *Statistically convex-cocompact actions of groups with contracting elements*, Preprint 2016, To appear in International Mathematics Research Notices.

Beijing International Center for Mathematical Research (BICMR), Beijing University, No. 5 Yiheyuan Road, Haidian District, Beijing, China

E-mail address: suzhenhan@pku.edu.cn
E-mail address: yabziz@gmail.com