Verification Studies for the Noh Problem using Non-ideal Equations of State and Finite Strength Shocks

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The Noh verification test problem is extended beyond the commonly studied ideal gamma-law gas to more realistic equations of state (EOSs): including the stiff gas, the Noble-Abel gas, and the Carnahan-Starling EOS for hard-sphere fluids. Self-similarity methods are used to solve the Euler compressible flow equations, which in combination with the Rankine-Hugoniot jump conditions, provide a tractable general solution. This solution can be applied to fluids with EOSs that meet criterion, such as being a convex function and having a corresponding bulk modulus. For the planar case, the solution can be applied to shocks of arbitrary strength, but for cylindrical and spherical geometries it is required that the analysis be restricted to strong shocks. The exact solutions are used to perform a variety of quantitative code verification studies of the Los Alamos National Laboratory Lagrangian hydrocode FLAG.

1 Introduction

The Noh problem [1,2,3] is a well-studied and widely used verification test problem in the field of computational hydrodynamics of compressible fluids. Typically, it is applied to an ideal polytropic gas [2,4] and serves as a tool for testing the accuracy of sophisticated hydrocodes. Axford first obtained analytic solutions to the Noh problem for the non-ideal stiff gas EOS and for a specialized form of the Mie-Grüneisen EOS [5]. Recently the topic was revisited and generalized by Ramsey et al. [3], whose formal solutions we employ here.

In this work we review the general solution of the Noh problem, which is most rigorously derived using Lie group methods [3], and then apply the general solution to three more realistic, non-ideal EOSs: the stiffened gas [6,7,8,9,10,11,12], the Noble-Abel gas [13,14,15,16,17,18,19,20,21,22,23,24], and the Carnahan-Starling EOS for a model fluid composed of hard spheres [25,26,27]. The exact solutions for these non-ideal EOSs are then compared to the numerical results obtained using the Los Alamos National Laboratory (LANL) compressible-flow code FLAG [28].

The Noh problem consists of a strong shock forming as a compressible gas moves at a constant velocity towards a rigid wall (in planar geometry) or as the gas is compressed with constant radial velocity towards a central axis (cylindrical) or a point (spherical) in the curvilinear cases. (See Figs. 1 and 2) Alternatively, and equivalently, in planar geometry the problem can be visualized as shock wave driven by a piston moving at a constant velocity traveling into a quiescent ideal gas [2]. The problem is defined by specifying the geometry; the EOS of the fluid; and the initial velocity, density, and specific internal energy of the fluid. It is typically applied to an ideal gas initialized at zero pressure and energy (a convenient, but rather unrealistic, initial condition corresponding to a temperature of absolute zero), giving rise to an infinitely strong shock since the speed of sound at the initial conditions is zero. In this paper, we explore extensions of the Noh problem beyond these
idealized conditions.

The purpose of this investigation is threefold: to demonstrate the feasibility of extending hydrodynamic verification studies to more physically realistic materials and initial conditions; to use the results to assess a particular hydrocode’s ability to accurately capture shock formation in the Noh problem; and to elucidate the spatial convergence order of the hydrocode under non-ideal gas conditions. In previous studies, acceptable accuracy has been found to be first order in spatial convergence tests for hydrocodes including any discontinuities [29,30].

We begin by reviewing the general solution of the Noh problem derived using the conservation equations, self-similarity methods, and Rankine-Hugoniot jump conditions. Following this, we describe the EOSs and the motivation for choosing these particular forms for the verification studies. Specific analytic solutions for each EOS are then compared with numerical results obtained using the Lagrangian hydrocode FLAG, so that the spatial convergence order can be calculated. Outcomes for the relative error between numerical and exact results and the observed spatial convergence order are presented for all three non-ideal EOSs, with a particular focus on planar geometry, where finite strength shocks can also be considered. Results for infinitely strong shocks in cylindrical and spherical geometries using the Noble-Abel EOS are also presented. We conclude with a summary of our findings and ideas for future work.

2 Theory

An analytic solution to the Noh problem can be constructed from piecewise isentropic solutions of the inviscid one-dimensional Euler compressible flow equations in combination with the Noh initial and boundary conditions and the Rankine-Hugoniot jump conditions [11] applied across the shock interface. In the absence of an external field, the one-dimensional Euler equations for mass, momentum and internal energy are [7,31]

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \left( \frac{\partial u}{\partial r} + \frac{mu}{r} \right) = 0 \quad (1)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \quad (2)
\]

\[
\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + K \left( \frac{\partial u}{\partial r} + \frac{mu}{r} \right) = 0 \quad (3)
\]

where \( t \) denotes time and \( r \) represents distance from the rigid wall, center axis, or center point depending on whether the geometry is planar, cylindrical, or spherical (distinguished by \( m = 0, 1 \) or 2, respectively). The quantities \( \rho, u, p, \) and \( K \) represent density, velocity, pressure, and adiabatic bulk modulus, respectively. The density and pressure state variables are related to the specific internal energy through an EOS of the form \( e(\rho, p) \) or \( p(\rho, e) \). The EOS enters into the analysis via the adiabatic bulk modulus, \( K \) [5], defined as

\[
K(p, \rho) \equiv \rho \left( \frac{\partial p}{\partial \rho} \right)_e = \rho \left( \frac{\partial p}{\partial \rho} \right)_e + \frac{p}{\rho} \left( \frac{\partial p}{\partial e} \right)_\rho \quad (4)
\]

where \( S \) denotes entropy. The second identity conveniently expresses the bulk modulus as a function of \( e \) and \( \rho \), common Lagrangian independent variables, making it straightforward to derive \( K \) from an EOS written in the form \( p(e, \rho) \).

Lie group methods [31,32] leverage the symmetry properties of the one-dimensional Euler equations to construct a similarity variable of the form

\[
\xi = \frac{ar}{t} \quad (5)
\]

where \( a \) is a constant parameter with units of inverse velocity. Substitution of Eq. (5) into Eqs. (1)-(3) enables the reduction of the PDEs in question to ODEs in terms of \( \xi \),

\[
-\xi \frac{dp}{d\xi} + au \frac{dp}{d\xi} + \rho \left( au \frac{d\rho}{d\xi} + \frac{mau}{\xi} \right) = 0 \quad (6)
\]

\[
-\xi \frac{d\rho}{d\xi} + au \frac{d\rho}{d\xi} + \frac{a dp}{p \xi} = 0 \quad (7)
\]

\[
-\xi \frac{dp}{d\xi} + au \frac{dp}{d\xi} + K \left( au \frac{d\rho}{d\xi} + \frac{mau}{\xi} \right) = 0. \quad (8)
\]

The Noh solution is piecewise solution, with the discontinuity in the flow interpreted as a shock wave. As a result, we apply the Rankine-Hugoniot jump conditions [11] at the position of the shock, \( r_s(t) \), to ensure conservation of mass, momentum, and energy across the discontinuity

\[
(u_2 - D)p_2 = (u_1 - D)p_1 \quad (9)
\]

\[
p_2 + (u_2 - D)p_2u_2 = p_1 + (u_2 - D)p_1u_1 \quad (10)
\]

\[
e_2 + \frac{p_2}{p_2} + \frac{1}{2}(u_2 - D)^2 = e_1 + \frac{p_1}{p_1} + \frac{1}{2}(u_1 - D)^2 \quad (11)
\]

where the subscripts 1 and 2 represent the unshocked and shocked regions, respectively, and where \( D = dr_s/dt \) is the shock velocity. For Eqs. (9)-(11) to be invariant under the same group transformations as the PDEs (i.e., for Eqs. (9), (11) to be expressible solely in terms of the PDEs, \( \xi \), \( D = \text{const.} \) and thus,

\[
r_s(t) = \frac{t}{a} = D \cdot t. \quad (12)
\]
2.1 General Noh Solution

The Rankine-Hugoniot jump conditions and the conditions unique to the Noh problem nicely simplify the ODE form of the one-dimensional compressible Euler equations, Eqs. (6)-(8), to a general solution. The following procedure may be used to solve the Noh problem:

1. By applying the constant velocity condition in the unshocked region, Eqs. (6)-(8) can be reduced and integrated to get the general solution for first the unshocked density, then the unshocked pressure and lastly, the unshocked energy.

2. The condition of zero velocity can be applied to Eqs. (6) and (8) within the shocked region. From this we notice that it must be the case that both pressure and density in the shocked region are constant.

3. The density in the shocked region can be calculated by taking the conservation of mass equation from the jump conditions, Eq. (9), and substituting $u_2 = 0$, $u_1 = u_0$, $D = 1/a$ and $P_1$.

4. The pressure in the shocked region is found by taking the conservation of momentum equation from the jump conditions, Eq. (10), substituting for $u_2$, $u_1$, $P_1$ and $P_2$ and then solving for pressure.

5. The energy in the shocked region can also be calculated by taking the conservation of energy equation from the jump conditions, Eq. (11), substituting for $u_2$, $u_1$, $P_1$ and $P_2$ and then solving for energy.

In this fashion, the general solution for the unshocked region is found to be

$$\rho_1 = \rho_0 \left(1 - \frac{u_0 t}{r}\right)$$

$$p_1 = p_0 = \text{constant}$$

$$u_1 = u_0 < 0.$$ (13) (14) (15)

The general solution for the shocked region is

$$\rho_2 = \rho_0 \left(1 - au_0\right)^{m+1}$$

$$p_2 = p_0 - \frac{\rho_0 u_0}{a} \left(1 - au_0\right)$$

$$e_2 = e_1 + \frac{1}{2} u_0^2 - \frac{au_0 p_0}{\rho_2}$$

$$u_2 = 0.$$ (16) (17) (18) (19)

Lastly, we can apply the condition $u_2 = 0$ to the conservation of momentum Rankine-Hugoniot jump condition, Eq. (10), and solve for the shock speed to get

$$D = \frac{p_1 - p_2}{\rho_2 u_1}.$$ (20)

Given Eq. (12), the shock location, $r_s(t)$ is determined to be

$$r_s(t) = D \cdot t = \left(\frac{p_1 - p_2}{\rho_2 u_1}\right) t.$$ (21)

The last element required is the EOS in the form $e_2(\rho_2, p_2)$ and $e_1(\rho_1, p_1)$ to create a system of equations with a unique closed-form solution.

2.1.1 Constraints on the Initial Conditions for the Curvilinear cases

The constant velocity Noh condition imposes a severe constraint on the initial pressure in cylindrical and spherical geometries. Since $u_1 = u_0$ and $p_1 = p_0$ are both constant, the energy Euler equation, Eq. (8), reduces to

$$K \left(\frac{mau}{\xi}\right) = 0.$$ (22)
Since $u$ is a negative constant in the unshocked region and $a$ must be a non-zero constant to have a nontrivial solution then either $m = 0$ or $K = 0$. In the planar case ($m = 0$), this analysis shows that the general solutions can be applied to an EOS that is a convex function and has an corresponding bulk modulus form or with any initial pressure since $K$ need not equal zero. However, if implementing the curvilinear cases ($m \neq 0$), $K$ must be equal to zero. Since $K = \rho c_s^2$, where $c_s$ is the is the sound speed in the fluid, this implies that $c_s = 0$ and that the shock will be infinitely strong.

The requirement of a zero initial sound speed in cylindrical and spherical geometries, in turn, constrains the initial pressure. Substituting $K = 0$ into Eq. (4) leads to

$$\left( \frac{\partial e}{\partial p} \right)_p = \frac{p}{\rho^2}$$

(23)

which can be integrated to give

$$e(\rho, p) + \frac{p}{\rho} = f(p)$$

(24)

where $f(p)$ is an arbitrary function. This is satisfied by one of these constraints, recalling this is applied to the unshocked region,

$$\rho_1 = \text{constant}$$

(25)

or

$$e_1 + \frac{p_1}{\rho_1} = \text{constant.}$$

(26)

Note in the general Noh solution, the energy, density and pressure are constant in all cases with exception of the unshocked density, $\rho_1$, in Eq. (17) when $m = 1$ or 2. Thus, Eq. (25) cannot be true so Eq. (26) must be applied to meet the conditions of conservation and the Noh problem conditions. Substituting the density solution from the unshocked region, Eq. (13), and $p_1 = p_0$, into Eq. (26) provides

$$e_1 + \frac{p_0}{\rho_1} \left( 1 - \frac{u_0 f}{r} \right)^{-m} = \text{constant}$$

(27)

implying that $p_0 = 0$ and $e_1 = \text{constant}$ when $m \neq 0$. Thus, the curvilinear cases can only be studied provided that $p_0 = 0$ [3].

The primary focus of this paper is the planar geometry case when $m = 0$, where the solution is piecewise constant. An example for the cylindrical and spherical geometries will be studied with the Noble-Abel EOS for infinitely strong shocks.

### 2.2 Equations of State

In its original incarnation [1], and nearly every application since [2], the Noh problem used the ideal polytropic gas EOS [4,10] initialized at zero pressure and specific internal energy, to model the thermodynamic behavior of the fluid,

$$p(\rho, e) = \rho e(\gamma - 1)$$

(28)

where $\gamma$ is the polytropic constant. Typically in Noh studies $\gamma$ is set equal to 5/3, which corresponds to a monatomic ideal gas; however, it can be viewed simply as an adjustable parameter and empirically set to other values depending on the fluid and the thermodynamic path being modeled [4].

While Eq. (28) admits a particularly simple solution to the Noh problem, its behavior (as well as the conventional initial conditions) is far removed from most materials of practical interest. Thus, there is significant value, from a verification and validation perspective, in extending the domain of applicability of the Noh problem to more realistic EOSs and initial conditions. We consider here three more complex EOSs: the stiff gas, useful for metals and liquids; the Noble-Abel, useful for explosive gases; and the Carnahan-Starling equation for a hypothetical atomatic model fluid composed of rigid, noninterpenetrating, hard spheres.

The stiff gas EOS [3,5,6,7,8,9,10], may be viewed as either a modification of the ideal polytropic gas EOS or a simplification (Taylor expansion in density) of the Mie-Grüneisen EOS when compressions are modest. In its most general form, it may be written

$$p(\rho, e) = p_0 + \left[ c_s^2 - (\gamma - 1) \frac{p_0}{\rho_0} \right] (\rho - \rho_0) + p(\gamma - 1)(e - e_0)$$

(29)

where $p_0$, $\rho_0$, and $e_0$ denote the pressure, density, and specific internal energy at some convenient reference condition (e.g., the initial conditions of the problem of interest) and $\gamma$ is an adjustable constant such that $\Gamma = \gamma - 1$ plays the role of an effective Grüneisen parameter [11].

If we choose our reference state such that $p_0$ and $e_0$ are zero, then Eq. (29) simplifies to the more common form

$$p = \rho e(\gamma - 1) + c_s^2 (\rho - \rho_0).$$

(30)

This form of the stiff gas equation has been found to be qualitatively accurate for a variety of metals. Though
similar in appearance to the ideal gas, Eq. (28), it enables
time, which can be empirically
to the effect of interatomic attractions
the number of internal degrees of freedom of a
to model shock compression of salt water up to 8
The significance of Eq. (29) and its simplifications, Eq. (30) and Eq. (33), is that the EOS remains simple enough algebraically to permit useful analytical manipulations, such as an exact solution to the Noh problem [3][5], but is realistic enough to provide qualitatively accurate predictions for shock compression of liquids and metals, thus enabling code verification studies on physically realistic materials. In the remainder of this work, we focus on the Eq. (30) form of the stiff gas EOS, where the density, $\rho_0$, is at a convenient reference density [7].

The second EOS we consider is the Noble-Abel equation [13][14][15][16][17] (also sometimes referred to as the Clausius I [18][19] or Hirn EOS [20][21][22]), which takes the form

$$ p(\rho, e) = \rho e(\gamma - 1)/1 - b\rho $$

(34)

where the adjustable parameter $b$, referred to as the co-volume (units of volume per mass), is intended to account, approximately, for the finite size of the molecules (compare with Eq. (28) [19][21]). The original idea for this type of correction to the ideal gas traces its roots to Bernouilli [23][24], and reemerges in Hirn’s, Clausius’s, and van der Waals’s subsequent EOS developments [33][20][21][23] in the nineteenth century. For dilute gases, $b$ may be deduced from the second virial coefficient, and it can take on negative as well as positive values if the temperature is low enough that intermolecular attractive forces decrease the pressure below the ideal gas value. In hot or strongly compressed fluids, where it is generally more accurate, it takes on positive values (subject to the constraint $pb < 1$) and functions to increase the pressure above the ideal gas value.

Eq. (34) has found extensive applicability in ballistics modeling [13][14][15], where it provides a simple and reasonably accurate EOS for propellant gases at the high densities and temperatures experienced in guns. It has also been used to model the release of hydrogen into air from high pressure tanks [16], and in modeling the transfer and discharge of various other compressed gases [17].

The third EOS we consider is the Carnahan-Starling equation [25][26][27]. In contrast to the polytropic ideal gas, stiff gas, and Noble-Abel EOSs, which can be empirically fitted to a variety of fluids, the Carnahan-Starling EOS was developed for a very specific model of a prototypical atomistic fluid, namely one composed of noninterpenetrating, hard spheres. The hard-sphere fluid has long been of interest in the field of statistical thermodynamics, in part because the simple form of the interatomic potential greatly simplifies calculations of properties such as virial coefficients and radial distribution functions; in part because the specific internal energy is exactly that of a monatomic ideal gas; and in part, because x-ray scattering experiments have shown that the local structure of dense real fluids, including liquid metals, is dominated by the harshly repulsive component of the interatomic potential, which can be well approximated by a hard sphere of an appropriately chosen diameter. Viewed this way, the hard-sphere fluid becomes a simple reference system about which the effects of interatomic attractions present in real fluids can be added perturbatively [26][24].

Carnahan and Starling [25] observed an approximate, serendipitous recursion relationship between the second through the sixth virial coefficients of the hard-sphere fluid which, when postulated to apply equally well to the higher virial coefficients, allowed the infinite series to be summed exactly. The resulting EOS takes the form

$$ p(\rho, e) = \rho e(\gamma - 1)Z(\eta) $$

(35)

where $Z(\eta)$ is the compressibility factor

$$ Z(\eta) = \frac{1 + \eta + \eta^2 - \eta^3}{(1 - \eta)^3} $$

(36)

and $\eta = b\rho$ is the packing fraction of the spheres. Here again, $b$ denotes the co-volume, now interpreted as the volume per mass occupied by the rigid spheres.

In extensive comparisons with Monte Carlo and molecular dynamic simulations of hard-sphere fluids, Eq. (35) has been found to be remarkably accurate over nearly the entire fluid range, from the dilute, ideal gas limit, $\eta \to 0$, up to packing fractions, $\eta > 0.4$ [25][25][27].
2.3 Ideal Gas Equation

The equation for the ideal gas equation is Eq. (28) and can be rearranged in terms of energy as

\[ e = \frac{p}{\rho(\gamma - 1)}. \]  (37)

Ramsey et al. [3] discuss how the general solution to the Noh problem (Eqs. (13)–(21)) can be combined with the EOS to calculate the shock speed \( D = 1/a \). If we consider only the planar \((m = 0)\) case, then the shock location is

\[ r_s(t) = \left[ \frac{A_i + (3 - \gamma)\sqrt{p_0u_0}}{4\sqrt{p_0}} \right] t \]  (38)

where

\[ A_i = \sqrt{16\gamma p_0 + (\gamma + 1)^2p_0u_0^2}. \]  (39)

The density for planar ideal gas solution for the unshocked region is \( \rho_1 = \rho_0 \) and the density for the planar ideal gas solution for the shocked region is

\[ \rho_2 = \rho_0 \left[ 1 - \frac{4\sqrt{p_0u_0}}{A_i + (3 - \gamma)\sqrt{p_0u_0}} \right]. \]  (40)

As shown in Section 2.1.1 for \( m = 1 \) or 2 solutions exist only in the infinitely strong shock case \((K_0 = 0, p_0 = 0)\) [3]. The shock location is

\[ r_s(t) = Dt = -\frac{(\gamma - 1)u_0t}{2}. \]  (41)

The density in the unshocked region is

\[ \rho_1 = \rho_0 \left(1 - \frac{u_0t}{r}\right)^m \]  (42)

and the density in the shocked region is

\[ \rho_2 = \rho_0 \left(\frac{\gamma + 1}{\gamma - 1}\right)^m. \]  (43)

2.4 Stiff Gas Equation

The stiff gas EOS, Eq. (30), may be written in terms of energy as

\[ e = \frac{p - c_s^2(\rho - \rho_0)}{\rho(\gamma - 1)} \]  (44)

Combining the EOS with the general solution, the shock speed is

\[ r_s(t) = \left[ A_s + (3 - \gamma)\sqrt{p_0u_0} \right] \frac{t}{4\sqrt{p_0}} \]  (45)

where

\[ A_s = \sqrt{16(c^2p_0 + \gamma p_0) + (\gamma + 1)^2p_0u_0^2}. \]  (46)

The density in the unshocked region is \( \rho_1 = \rho_0 \) and the density in the shocked region is

\[ \rho_2 = \rho_0 \left[ 1 - \frac{4\sqrt{p_0u_0}}{A_s + (3 - \gamma)\sqrt{p_0u_0}} \right]. \]  (47)

2.5 Noble-Abel Gas Equation

The Noble-Abel EOS is Eq. (34) may be written in terms of energy as

\[ e = \frac{p(1 - bp)}{\rho(\gamma - 1)}. \]  (48)

Using the general Noh solution and the EOS, the shock speed is

\[ r_s(t) = \left[ \frac{2\rho_0u_0^2(2bp_0 + \gamma - 1) + 4\gamma p_0}{A_{NA} + \rho_0u_0(4bp_0 + \gamma - 3)} \right] t \]  (49)

where

\[ A_{NA} = \sqrt{\rho_0 \left[(\gamma + 1)^2\rho_0u_0^2 - 16\gamma p_0(bp_0 - 1)\right]}. \]  (50)

The density for the planar solution for the unshocked region is \( \rho_1 = \rho_0 \) and the density for the planar solution for the shocked region is

\[ \rho_2 = \rho_0 \left[-u_0A_{NA} + 4\gamma p_0 + (\gamma + 1)p_0u_0^2\right] \frac{2\rho_0u_0^2(2bp_0 + \gamma - 1) + 4\gamma p_0}{2\rho_0u_0^2(2bp_0 + \gamma - 1) + 4\gamma p_0}. \]  (51)

As with the ideal gas solution, for \( m = 1 \) or 2 Noble-Abel gas solutions exist only in the infinitely strong shock case \((K_0 = 0, p_0 = 0)\) [3]. The shock speed is

\[ r_s(t) = \frac{(\gamma - 1)u_0t}{2(bp_2 - 1)}. \]  (52)

Then the density for the Noble-Abel gas in the unshocked
region is
\[ \rho_1 = \rho_0 \left( 1 - \frac{u_0 t}{r} \right)^m \] (53)

and the density for the Noble-Abel gas in the shocked region is
\[ \rho_2 = \rho_0 \left( \frac{\gamma + 1 - 2bp_2}{\gamma - 1} \right)^{m+1} \] (54)

which must be solved using an implicit solver.

2.6 Carnahan-Starling Gas Equation
The Carnahan-Starling gas EOS, Eq. (55), may be rearranged in terms of energy as
\[ e = \frac{p}{\rho(\gamma - 1)Z} \] (55)

Combining this with Eqs. (13)-(21) the planar Carnahan-Starling fluid solution for the unshocked region is
\[ \rho_1 = \rho_0 \] (56)
\[ p_1 = p_0 \] (57)
\[ e_1 = \frac{p_0}{\rho_0(\gamma - 1)Z(p_0)} \] (58)
\[ u_1 = u_0. \] (59)

Given \( u_2 = 0 \), the variables, \( a, \rho_2, p_2 \) and \( e_2 \) can be solved implicitly from the system,
\[ \rho_2 = \rho_0 \left( 1 - au_0 \right) \] (60)
\[ p_2 = p_0 - \frac{\rho_0 u_0}{a} (1 - au_0) \] (61)
\[ e_2 = e_1 + \frac{1}{2} u_0^2 - \frac{au_0 p_0}{\rho_2} \] (62)
\[ e_2 = \frac{p_2}{\rho_2(\gamma - 1)Z(p_2)}. \] (63)

3 Verification Studies Results
Verification studies compare numerical implementations with analytically derived solutions. These studies are done in order to evaluate the accuracy and performance of numerical techniques and algorithms, in this case, for a hydrocode based on Euler conservation laws. The Los Alamos Lagrangian hydrocode FLAG uses a compressible flow algorithm featuring artificial viscosity and ancillary grid stability methods. The solver implements explicit, geometry-compatible, finite volume spatial Lagrangian staggered-grid hydrodynamics. This method uses zones that are assigned by drawing lines between nodes. The densities, pressures, and specific internal energies live at zone centers and the velocities live at the nodes [28], [35].

Artificial viscosity is added to the hyperbolic continuity equation and then the momentum equation is altered to make up for the added term, providing a smoother equation that is easier to solve numerically as the model now produces a stable solution on the Lagrangian grids. In compression, the quadratic viscosity coefficient is 2.0 and the linear viscosity coefficient is 0.3. The linear viscosity coefficient in expansion is 0.3 [36, 37]. These constructs are used to capture the zero-thickness of the jump discontinuity on a grid of finite-thickness and are mostly responsible for expectation that the spatial convergence is first order [29, 30]. The other factor responsible for the first-order behavior is the excessive wall heating produced by the shock reflecting at the origin. This is caused by the pressure initially being computed and then energy/temperature added to the system, via the EOS, to compensate for a shortage in density [38]. The convergence order will remain first order despite refining the spatial mesh size tremendously because of the presence of the artificial viscosity and the excessive wall heating.

FLAG results for the pressure, density, and specific internal energy as a function of time, position, and mesh resolution are compared with the exact solutions for the ideal and the non-ideal gases (Figs. 4, 6, 8, 10, 12, 14, 16, 18, 20, and 22). All results use the same initial density \( \rho_0 = 1 \text{g/cm}^3 \), the same adiabatic coefficient \( \gamma = 5/3 \) and are measured at time \( t = 0.6 \mu s \). First-order spatial convergence with respect to mesh size is observed with \( \Delta x = 0.01, 0.005, 0.0025, \) and 0.00125 cm. The verification tool Exactpack [39, 40] analyzes the results by calculating the L1-norm
\[ \| y^E - y^F \|_1 \approx \frac{1}{N} \sum_{i=1}^{N} w_i | y^E_i - y^F_i | \] (64)

where \( y^E \) is the exact solution and \( y^F \) is the FLAG data. The weights \( w_i \) are determined by the dimensionality of the problem. For the 1D Cartesian problem, they are the zone lengths. The L1-norm with respect to the mesh size is fitted on a log-log scale to a line in the form
\[ \ln \left( \| y^E - y^F \|_1 \right) = p \ln (\Delta x) + c. \] (65)

The following figure is an example of the output from Exactpack of the convergence order study.
Fig. 3: Convergence study for the ideal gas case when $p_0 = 0 \text{Mbar}$

The slope, $p$, is determined from Eq. (65) and is known as the convergence order fitting this equation,

$$
\|y^E - y^S\|_1 = e^c (\Delta t)^p .
$$

The convergence orders for a variety of parameters are presented in Tables 1-4.

In the density plots (shown in the left columns), the points are the FLAG results at indicated mesh resolutions and the line represents the exact solution. In the right adjacent column is the correspond relative error between the FLAG results and the exact solution. Results are shown for the planar ideal gas in Figs. 4 and 5, the planar stiff gas in Figs. 6-9, the Noble-Abel gas in Figs. 10-17, and the Carnahan-Starling liquid in Figs. 18-22. A sample of a closer look is given for the relative error at the origin and at the discontinuity for the Carnahan-Starling equation as seen in Fig. 22.
Fig. 4: Density plots of planar ideal gas, $\gamma = 5/3$, $p_0 = 1.0\,\text{g/cm}^3$.

Fig. 5: Relative error plots of planar ideal gas, $\gamma = 5/3$, $p_0 = 1.0\,\text{g/cm}^3$. 
Fig. 6: Density plots of planar stiff gas, $\gamma = 5/3$, $\rho_0 = 1.0\,\text{g/cm}^3$, $p_0 = 0.0\,\text{Mbar}$

Fig. 7: Relative error plots of planar stiff gas, $\gamma = 5/3$, $\rho_0 = 1.0\,\text{g/cm}^3$, $p_0 = 0.0\,\text{Mbar}$
Fig. 8: Density plots of planar stiff gas, $\gamma = 5/3$, $p_0 = 1.0 g/cm^3$, $c_s = 0.6 cm/\mu s$

Fig. 9: Relative error plots of planar stiff gas, $\gamma = 5/3$, $p_0 = 1.0 g/cm^3$, $c_s = 0.6 cm/\mu s$
Fig. 10: Density plots of planar Noble-Abel gas, $\gamma = 5/3$, $p_0 = 0.00\text{Mbar}$.

(a) $p_0 = 0.00\text{Mbar}$

(b) $p_0 = 0.01\text{Mbar}$

(c) $p_0 = 0.10\text{Mbar}$

(d) $p_0 = 1.00\text{Mbar}$

Fig. 11: Relative error plots of planar Noble-Abel gas, $\gamma = 5/3$, $p_0 = 1.00\text{Mbar}$, $\rho_0 = 0.00\text{g/cm}^3$, $b = 0.1\text{cm}^3/\text{g}$.
Fig. 12: Density plots of planar Noble-Abel gas, $\gamma = 5/3$, $\rho_0 = 1.0\text{g/cm}^3$, $p_0 = 0.0\text{Mbar}$

Fig. 13: Relative error plots of planar Noble-Abel gas, $\gamma = 5/3$, $\rho_0 = 1.0\text{g/cm}^3$, $p_0 = 0.0\text{Mbar}$
Fig. 14: Density plots of cylindrical Noble-Abel gas, $\gamma = 5/3$, $\rho_0 = 1.0\text{g/cm}^3$, $p_0 = 0.0\text{Mbar}$

Fig. 15: Relative error plots of cylindrical Noble-Abel gas, $\gamma = 5/3$, $\rho_0 = 1.0\text{g/cm}^3$, $p_0 = 0.0\text{Mbar}$
Fig. 16: Density plots of spherical Noble-Abel gas, $\gamma = 5/3$, $\rho_0 = 1.0\, \text{g/cm}^3$, $p_0 = 0.0\, \text{Mbar}$

Fig. 17: Relative error plots of spherical Noble-Abel gas, $\gamma = 5/3$, $\rho_0 = 1.0\, \text{g/cm}^3$, $p_0 = 0.0\, \text{Mbar}$
Fig. 18: Density plots of planar Carnahan-Starling model, $\gamma = 5/3$, $\rho_0 = 1.0\text{g/cm}^3$, $p_0 = 0.0\text{Mbar}$

Fig. 19: Relative error plots of planar Carnahan-Starling model, $\gamma = 5/3$, $\rho_0 = 1.0\text{g/cm}^3$, $p_0 = 0.0\text{Mbar}$
Fig. 20: Density plots of planar Carnahan-Starling model, $\gamma = 5/3$, $\rho_0 = 1.0 \text{g/cm}^3$, $b = 0.1 \text{cm}^3/\text{g}$

Fig. 21: Relative error plots of planar Carnahan-Starling model, $\gamma = 5/3$, $\rho_0 = 1.0 \text{g/cm}^3$, $b = 0.1 \text{cm}^3/\text{g}$
Fig. 22: Density for the Planar Carnahan-Starling Gas where \( b = 0.1 \text{ cm}^3/\text{g} \) and \( p = 0.1 \text{ Mbar} \). The relative error of the density due to excessive wall heating and the relative error of the density at the shock.

Table 1: Table of convergence orders for planar stiff gas. The units for \( c_s \) are cm/µs and for \( p_0 \) the units are Mbar.

| \( p_0 \) | 0.00 | 0.01 | 0.10 | 1.00 |
|---|---|---|---|---|
| 0.0 | 0.998 | 0.971 | 1.017 | 0.997 |
| 0.1 | 0.993 | 1.062 | 0.989 | 0.974 |
| 0.5 | 0.998 | 0.978 | 1.001 | 0.988 |
| 0.6 | 1.018 | 0.988 | 1.017 | 0.987 |
| 1.0 | 1.011 | 0.975 | 1.008 | 1.039 |
| 1.5 | 0.973 | 1.087 | 0.975 | 0.989 |
| 2.0 | 0.987 | 1.019 | 1.019 | 0.985 |

Table 2: Table of convergence orders for Planar Noble-Abel gas. The units for \( b \) are cm³/g.

| \( p_0 \) | 0.00 | 0.01 | 0.10 | 1.00 |
|---|---|---|---|---|
| -0.20 | 1.031 | 1.083 | 1.028 | 0.955 |
| -0.10 | 1.078 | 1.064 | 0.960 | 1.007 |
| 0.00 | 0.998 | 0.987 | 1.017 | 0.997 |
| 0.01 | 0.979 | 0.938 | 0.994 | 1.005 |
| 0.10 | 1.030 | 0.988 | 1.036 | 0.956 |
| 0.20 | 1.000 | 1.027 | 1.006 | 1.001 |
| 0.40 | 0.924 | 0.966 | 0.964 | 0.976 |

Table 3: Table of convergence orders for Noble-Abel gas as strong shocks.

| \( p_0 \) | 0.00 | 0.01 | 0.10 | 1.00 |
|---|---|---|---|---|
| 0.000 | 0.998 | 0.970 | 0.900 |
| 0.001 | 1.003 | 0.960 | 0.925 |
| 0.010 | 0.979 | 0.985 | 1.045 |
| 0.100 | 1.030 | 0.858 | 0.771 |
| 0.200 | 1.000 | 0.865 | 0.907 |
| 0.400 | 0.924 | 1.253 | 0.923 |

Table 4: Table of convergence orders for Planar Carnahan-Starling gas.

| \( p_0 \) | 0.00 | 0.01 | 0.10 | 1.00 |
|---|---|---|---|---|
| -0.10 | 0.954 | 1.139 | 0.963 | 0.994 |
| 0.00 | 0.998 | 0.987 | 1.017 | 0.997 |
| 0.01 | 1.061 | 1.008 | 1.070 | 1.015 |
| 0.05 | 0.903 | 0.984 | 1.044 | 0.999 |
| 0.10 | 0.978 | 1.049 | 1.050 | 0.963 |
| 0.20 | 0.961 | 1.000 | 0.913 | 0.993 |
| 0.30 | 1.166 | 1.040 | 1.045 | 0.973 |
| 0.40 | 0.900 | 0.919 | 0.884 | 0.976 |
In every case, the code behaves consistently with the exact solution with the exception of the common Lagrangian inaccuracy at wall and the shock front. There are trends for each EOS that are apparent by changing the parameters. One simple trend that is exhibited over all the EOSs is the effect of the increase in initial pressure on the shock speed and density. In all cases, as the unshocked pressure increases, the shocked density decreases and the shock speed increases. This is due to the fact that increasing the initial pressure increases the sound speed at all densities, which then leads the shock to move faster. Also, as the co-volume, \( b \), increases in the Carnahan-Starling gas (Fig. 18) and Noble-Abel gas (Fig. 14) or as the reference sound speed, \( c_s \), increases in the material of the stiff gas (Fig. 6), the density in the shocked region decreases and the shock speed increases. When the reference sound speed of the stiff gas increases, the shock moves faster for the same reason as when the initial pressure is increased. By increasing the co-volume in the Noble-Abel gas and the Carnahan-Starling gas, the gas becomes less compressible, which means it cannot become as dense as the ideal gas resulting in the shock front to be further from the wall at \( t = 0.66 \mu \text{s} \). When these EOSs have a negative co-volume, the molecules attract each other resulting in higher shocked densities and lower shock speeds than ideal gas.

More interesting than the general trends are the effects of the change in parameters on the areas of the hydrocode where the largest errors arise: the origin and the discontinuity. From the relative error plots shown in Figs. 7, 9, 11, 13, 15, 17, 19, and 21, the computed error in the density at \( r = 0 \) decreases as both the initial pressure increases and the co-volume, \( b \), or sound speed, \( c_s \), increases. Moreover, as those parameters increase, the spatial extent of the error at the shock discontinuity becomes shorter and wider. At first it seems that this could be attributed to using a model that is less like the ideal gas, however, the negative-co-volume studies in the Noble-Abel results negate this assumption. It appears that the drop in relative error at the discontinuity is connected to the shock’s position relative to the origin. The closer the shock is to the origin, the more wall heating affects the location of the discontinuity in the results. Therefore, the results suggest the further the discontinuity is from the wall, the more accurately FLAG performs in determining the jump discontinuity location.

Also, note that for the infinitely strong shock cases, seen in the Noble-Abel gas of Figs. 14, 17 the wall heating increases with increasing curvilinearity. In these cylindrical and spherical implementations, the change in the compression term, \( b \), more dramatically effects the wall heating than it did in the planar case. This is a consequence of the flow converging on an axis or a single point.

4 Conclusion

Our results indicate how the use of non-ideal EOS in verification studies can provide a more physically realistic, but still a computationally tractable tool for verifying and validating complex hydrocodes. Proper verification studies that feature shocks typically exhibit first-order spatial convergence [29, 30]. Our studies demonstrate that the LANL hydrocode FLAG models the Noh solution with first-order accuracy when compared to the exact solutions. The Noh problem is such a fundamental problem that these results suggest that non-ideal EOSs and non-zero initial pressures can be applied to other shock problems with expected accuracy in the hydrocode. Additionally, the decrease in the amount of error as the shock speed increases is significant. We observe that the higher the initial pressure and/or the larger the initial bulk modulus, the smaller the magnitude of the error both at the wall and the shock front.

The results from this verification study show that the FLAG hydrocode is sufficient for representing the exact solution. Further analysis of the current studies could be done by examining pointwise verification metrics such as the shock location as a function of the non-ideal parameters. Future work would be to try additional EOSs, change the adiabatic constant, \( \gamma \), look at shorter time scales, and vary the initial density. In addition to modifying the parameters for the studies, it would be interesting to compare to a verification study utilizing an Eulerian hydrocode such as the LANL code, RAGE [31]. Eulerian code was shown to give better results than the Lagrangian code as discussed by Rider [38]. In our results, the lowest convergence orders were seen in the curvilinear cases so this should be the focus of the comparison study. Finally, the exact solution can be derived by applying Lie group methods to other problems such as the Noh-2 problem [1], the Sedov problem [42], the Guderley problem [43], and the Sod Problem [44]. Verification studies would be needed to show that the FLAG hydrocode performed as expected for these problems since they are considered more complex than the Noh problem.

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