Topo-Geometric Model MZ: Feded Objects

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Summary: The Topo-geometric approach MZ requires notions of both vector geometry and affine geometry, and also uses topology concepts to transform the elementary notions of presenting a linear program problem. These different forms or presentation models of problems related to the Topo-geometric approach MZ are initially presented in this article and will be followed by elementary results demonstrated by the use of the supported objects of the supported model. Thus, this article is the precursor of the determination of redundant constraints by the Topo-geometric model MZ. The energized objects will thus constitute the language of the proofs of the subsequent propositions.

Keywords: Linear programming, mathematical writing of the model, writing of the Topo-geometric model MZ, algorithm.

1 INTRODUCTION
This paper constitutes the basis of the Topo-geometric model MZ in search of redundant constraints in a problem of linear programming. This article follows the work on the Topo-Geometric MZ model published in MADA-ETI, ISSN 2220-0673, Vol.2, 2016, www.madarevues.gov.mg which develops the basic concept of Topo-geometric models, and presents the powered objects in search of the redundant constraints of a linear programming model. All the mathematical object classes will first be presented from the simplest to the most complicated, to model the real concept allowing the decision making. Together with the presentation of each class, the existing operations and relationships will be analyzed; the representation of technical constraints by affine half-spaces is the most important part of this work.

2 THE MEAN OBJECTIVES
2.1 The scalars
A scalar is any real number. Symbolically, a scalar will be represented with a Latin or Greek or lowercase letter. Examples: $\alpha, \beta, \gamma, \ldots, x, y, z, \ldots$.

2.2 Indexes
It is a positive or null integer that will be attached to an object of a model, as a unique identifier. Symbolically the indexes will be represented by $i, j, k, \ldots$.

To represent the possible values of an index, the following notions will be used:

\[ i = 1, 2, \ldots, n \]

An object can have two indexes as identifiers. This is the case with matrices, for example.

It is represented by

\[ a_{ij} \]

which designates the matrix:

\[
\begin{bmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{m1} & \cdots & a_{mn}
\end{bmatrix}
\]

To fully exploit the indexes, the symbol $\sum$ (summation) will also be used.

So, to represent

\[ c_1 x_1 + c_2 x_2 + \ldots + c_n x_n, \]

it will be simply written:

\[ \sum_{j=1}^{n} c_j x_j \]
3 THE ELEMENTARY OBJECTS

3.1 Decision spaces
Take again the problems LPP in its known form which consists of finding the unknown-real \( x_1, x_2, \ldots, x_n \), and maximize:

\[
Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n
\]  

(3.01)

And which satisfy the following conditions:

\[
\begin{align*}
& \quad a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
& \quad a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\
& \quad a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m 
\end{align*}
\]  

(3.02)

\[
\begin{align*}
& \quad x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0
\end{align*}
\]  

(3.03)

where the coefficients:

\[
\begin{align*}
& \quad (c_j) j = 1, \ldots, n \\
& \quad (a_{ij}) i = 1, \ldots, n j = 1, \ldots, n \\
& \quad (b_j) i = 1, n
\end{align*}
\]

Each line of the conditions (3.02) is called a technical constraint.

Conditions (3.03) are called non-negativity constraints.

Definition 3.01: decision-action-point-vectors

We call decision-action any n-tuple

\[
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{pmatrix}
\]

Whose \( X_1, X_2, \ldots, X_n \) are elements of \( \mathbb{R}^n \)

It will be noted later:

\[
\begin{pmatrix}
X_j \\
\vdots \\
X_n
\end{pmatrix}
\]  

(3.04)

Mathematically, we see that it is an element of \( \mathbb{R}^n \) relative to a given base. It is also a point in the affine space \( \mathbb{R}^n \) relative to a given landmark.

An action decision will be represented by an affine point. Which brings us to the decision-action definition?

Notation: an action decision is noted using a capital Greek letter: \( A, B, X, Y \)

Definition 3.02: decision marker - action - decision space

We call decision-action benchmark, the positive orthonormal geometric benchmark, noted:

\[
\begin{pmatrix}
O \\
U_1 \\
\vdots \\
U_n
\end{pmatrix}
\]

related to the aforementioned LPP problem. In this reference, the geometric origin \( O \) has a particular meaning: it represents the decision to do nothing, that is, if

\[
O = \begin{pmatrix}
x_{10} \\
x_{20} \\
\vdots \\
x_{n0}
\end{pmatrix}, \quad x_{10} = x_{12} = x_{30} = \ldots = x_{nm} = 0
\]

To each decision, variable \( x_j \) corresponds the unit vector \( U_j \).

The affine space \( \mathbb{R}^n \), equipped with the reference \( (O, U_1, U_2, \ldots, U_n) \) is called the decision space relating to the LPP problem to be solved. Despite this striking resemblance to Euclidean affine geometry, a limiting aspect will be presented from the start.

3.2 Postulate

In the topo-geometric theory MZ, the orthonormal affine frame of the decision space action of the problem LPP to be solved is unique.

This assumption means that:

- The concept of basic change does not exist.
- The concept of change of origin does not exist.

However, following a presolved analysis, it may be possible that \( n \) changes, which completely modifies the LPP problem.

4 OPERATIONS ON ACTION-DECISIONS

Given an LPP problem, according to definition 3.01, a decision is represented by a point in the decision-action space and / or by a vector of the vector space \( \mathbb{R}^n \).

This dual nature (both affine and vectorial) of decision-actions is very important.

It allows to express:

- The creation of other stock-decisions based on existing stock-decisions.
- The search for other decision-actions according to a given direction.

4.1 Amplifier of an action decision

Let the decision-action space relate to the reference.

Let \( X_0 \) be an action decision and \( \alpha \) a given positive real number.

We call amplifier of \( X_0 \) using \( \alpha \), the generation of an action decision \( X \) such that

\[ X = \alpha X_0 \]

Mathematically, it is therefore the multiplier of vector \( X_0 \) by the scalar \( \alpha \).

4.2 Addition of two decision-actions

Let \( X_0 \) and \( X_1 \) be two decision-actions in a LPP problem.

The addition of \( X_0 \) and \( X_1 \) is called the creation of a new action decision \( X \) such that

\[ X = X_0 + X_1 \]

Note 4.01

The combination of 2.3.1 and 2.3.2 allows us to model the conic combination concept of two or more action-decisions.

Let the decision-actions \( X_1, X_2, \ldots, X_k \).

The conic combination of these decisions is the new decision \( X \) defined by:

\[ X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_k X_k \]

\( \alpha_1, \alpha_2, \ldots, \alpha_k \) are positive real numbers or zero.
We call resource vector \( V \) the vector derived from \( X \) as decision-action. Let \( X \) be an action decision, satisfying the constraint (2.03)

\[
X = (x_j) \quad \text{with} \quad x_j \geq 0
\]

but

\[
(x_j) = \sum_{j=1}^{n} x_j U_j \quad \text{with} \quad x_j \geq 0
\]

4.3 Search axe defined by decision-action \( X_o \) and direction \( U \)

Let \( X_0 \) and \( U \) be action-decisions.

We call search axis starting from \( X_0 \) and direction \( U \) the set of decision actions, noted \( \mathbb{R} (X_0, U) \) defined by

\[
\mathbb{R} (X_0, U) = \{ X \in \mathbb{R}^n / X = X_0 + aU, \quad a \in \mathbb{R} \text{ and } a \geq 0 \}
\]

Mathematically, it is the affine ray derived from \( X_0 \) and the vector \( U \).

Conceptually, the research idea is justified by the fact that in practice, any solver is based on a search algorithm starting from an initial point.

Thus, from a given decision-action \( X_0 \), and a given search direction \( U \), one can search the points in the universe of decisions that will satisfy the constraints (3.02) and (3.03). This research idea is not confined to a single research direction. It is also possible to adopt simultaneously two search directions \( V_1 \) and \( V_2 \). Hence the concept of “research plane”. Starting from \( X_0 \) and direction \( V_1 \) and \( V_2 \).

4.4 Plan of research starting from a decision-action \( X_0 \) and directions \( V_1 \) and \( V_2 \)

Let \( X_0 \) be a decision-action point and \( V_1 \) and \( V_2 \) two decision-action vectors of it.

We call search plane starting from \( X_0 \) and directions \( V_1 \) and \( V_2 \), the set noted \( \mathbb{P} (X_0, V_1, V_2) \) defined as follows :

\[
\mathbb{P} (X_0, V_1, V_2) = \{ X \in \mathbb{R}^n / X = X_0 + a_1 V_1 + a_2 V_2, \text{ where } a_1 \in \mathbb{R} \text{ and } a_2 \in \mathbb{R} \}
\]

By generalizing, we have :

4.5 Research of \( X_0 \) direction and \( V_1, V_2, \ldots, V_k \) vectors decisions

Let \( X_0 \) be a decision-action point associated with a LPP.

Let \( V_1, V_2, \ldots, V_k \), \( k \) decision-action vectors of the same problem LPP.

We call vertex cone \( X_0 \) and directions \( V_1, V_2, \ldots, V_k \), the action decision set denoted:

\[
\mathbb{C} (X_0, V_1, V_2, \ldots, V_k) \quad \text{defined as follows:}
\]

\[
\mathbb{C} (X_0, V_1, V_2, \ldots, V_k)
\]

It should be noted at once that research axes and research plans are only specific cases of research axes.

5 THE TECHNICAL CONSTRAINTS

In practice, any action-decision always has an impact. In the field of linear programming, two categories of impacts can be distinguished:

- Impact-performance.
- Resource impacts.

In this study, we only deal with the impact-resources.

Definition 5.01 resource impact

Let \( X \) be an action decision point of a given LPP problem. Let \( V \) be a null vector of \( \mathbb{R}^n \). We call resource impact according to \( V \), the scalar product of \( X \) and \( V \) in \( \mathbb{R}^n \) noted:

\[
\exists (X, V) = <X, V>
\]

Where \(<X, V>\) denotes the scalar product \( U \) with \( V \).

The vector \( V \) is called unit consumption vector \( V \) of the given resource.

Definition 5.02 constraint resources

In our theory, we assume that a constraint that is logical or material may be associated with a resource that is always limited in the associated LPP problem.

Let \( A = (a_i)_{j=1...n} \) be a vector \( V \) of consumption of a given resource.

Let \( b \) be the limit value of the resource, the zone of respect of the consumption of the resource limited by \( b \) is the set noted \( ZR (A, b) \) defined by :

\[
ZR (A, b) = \{ X \in \mathbb{R}^n / - A \cdot X \leq b \}
\]

In affine geometry, this set is none other than the negative half-space delimited by the noted affine hyperplane. \( HP (A, b) \) defined by:

\[
HP (A, b) = \{ X \in \mathbb{R}^n / - A \cdot X = b \}
\]

NB: The vector \( A \) is not an action-decision it is linked to a given resource and allows to calculate the impact of an action \( X \) decision on this resource.

6 The constraints of non-negativity

The non-negativity constraints translate the fact that the elementary decisions \( x_i \), where \( i = 1... n \) must be non-negatives, is:

\[
x_i \geq 0 \quad \forall \ i = 1, 2... n
\]

Let us note immediately that the relations:

\[
x_i \geq 0 \quad \forall \ i = 1, 2... n
\]

can be written:

\[
x_i \leq 0 \quad \forall \ i = 1, 2... n
\]

Expression equivalent to:

\[
- U, X \geq 0
\]

Therefore, a non-negativity constraint can also be considered as a resource constraint delimited by \( ZR = (-U, 0) \) called non-negativity zone ZNN.

However the combination of all non-negativity constraints has a particular geometric meaning that we call “ non-negativity cone ”.

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6.1 Non-negativity cone

We call non-negativity cone noted \( C_o^+ \) the cone of search from point-decision \( O \) and direction of all decision vectors:

\[
U_i, \; i = 1, 2, \ldots, n
\]

Geometrically we have:

\[
C_o^+ = \left\{ X \in \mathbb{R}^n \mid X = \sum_{i=1}^{n} a_i U_i \; \text{where} \; a_i \geq 0 \right\} \quad (6.01)
\]

\[
= \text{Cone}(O, U_1, U_2, \ldots, U_n)
\]

Note 6.01:

It is obvious that \( C_o^+ = \bigcap_{i=1}^{n} ZNN_i \)

6.2 Non-negativity face

It is to be recalled that the geometric topo reference of the LPP problem is immutable \((O, U_0, U_2, \ldots, U_n)\).

This means that we cannot change the place of a vector \( U_i \) of this reference, for all \( i \), for all \( i = 1, 2, \ldots, n \).

We call the non-negativity face, denoted by \( FNN_i \), the decision point set defined as follows:

\[
FNN_i = \left\{ P(O, U_i, U_n) \; \text{for} \; i = 1 \right\} \cup \left\{ P(O, U_{i-1}, U_i) \; \text{for} \; i = 2, 3, \ldots, n \right\}
\]

6.3 Non-negativity axes

For all \( i = 1, 2, \ldots, n \), we call non-negativity axis, \( ANN_i \), the set of decision point defined as follows:

\[
ANN_i = \mathbb{R} (O, U_i) \; i = 1, 2, \ldots, n \quad (6.02)
\]

Note that \( ANN_i \) is none other than the search axis starting from \( O \) and with the direction \( U_i \).

Recall the concept of redundant constraints.

7 REDUNDANT CONSTRAINTS

A redundant constraint is a constraint which can be deleted from a system of linear constraints without changing the feasible region or acceptable solution area.

If we look at the next system of \( m \) and \( n \) constraints of linear inequality, no negative \((m \geq n)\), we can adopt the matrix writing:

\[
AX \leq B, \; \; X \geq 0, \quad (7.01)
\]

But

\[
A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m, \; X \in \mathbb{R}^n,
\]

And

\[
0 \in \mathbb{R}^m.
\]

Let

\[
AT_k \leq b_i
\]

be the \( k \)th constraint of system \((7.01)\) and let

\[
S = \{ X \in \mathbb{R}^n \mid \; AT_k X \leq b_i, \; X \geq 0 \}
\]

The acceptable solution area associated with the system \((7.01)\).

Let

\[
S_k = \{ X \in \mathbb{R}^n \mid \; AT_k X \leq b, \; X \geq 0, \; i \neq k \}
\]

The acceptable solution area associated to the constraint:

\[
A_i X \leq b, \; i = 1, 2, \ldots, m, \; i \neq k
\]

of the system.

The \( k \)th constraint:

\[
A_k X \leq b_k \quad (1 \leq k \leq m)
\]

is redundant for the system \((7.01)\) if and only if

\[
S = S_k.
\]

Definition 7.01

Redundant constraints can be classified as weak and strong redundant constraints. 1.3.2

Low redundancy constraints

The constraint \( AT_i X \leq b_i \) is weakly redundant if she is redundant and \( AT_i X = b_i \) for all \( X \in S \).

8 RELATIONSHIP BETWEEN MZ OBJECTS

The relationships described in this section are the basic ones. More complex relationships will be discussed in the next chapter.

Moreover, the combinations between point-decisions and point-vectors have already been seen. Therefore, only the following points will be dealt with:

- The relation between a decision-point and a constraint-technical zone.

- The relation between a research axis and a zone of technical constraints.

- The relation between a decision-point and a technical constraints area.

8.1 Relationship between a decision-point and a technical constraint zone

Since a point-decision is represented with a point, and a zone of technical constraints is represented by a half-space, a set of decision-point, the essential same basic relation between one of these two object classes, is the ensemble relation of belonging.

Let \( X_0 \) be a decision point and \( ZCT(A, b) \) a constraint zone.

The relationship \( X_0 \in ZCT(A, b) \) means \( \langle A, X_0 \rangle \preceq b \)

Since in the LPP problem model, the constraints are indexed, i.e. numbered from 1 to \( n \), the corresponding technical constraints areas will also be noted:

\[
ZCT_i, \; i = 1, 2, \ldots, n
\]

This makes it possible to represent the system of techniques as follows:

Let \( X_0 \) be a decision-point satisfying all the technical constraints of the problem.

We have:

\[
< A_i, X_0 > \leq b_i \quad \text{for} \; i = 1, 2, \ldots, n
\]

This can be written:

\[
X_0 \in ZCT_i \quad \text{For every} \; i = 1, 2, \ldots, n
\]

From where,

\[
X_0 \in \bigcap_{i=1}^{m} ZCT_i \quad (8.01)
\]
Relationship that is a basis for redundancy and infeasibility analysis. Moreover, when \( X_0 \) does not belong to a ZCT, we also say that \( X_0 \) is exterior to ZCT, as the outer term, and its inner opposite has a topological connotation, the topological definitions of these terms will be recalled.

### 8.1.1 Projection of a point on a hyperplane

Let ZCT be the area of technical constraints delimited by the HP hyperplane \((A_i, b_i)\) and let \( X_0 \) be any decision-point. We call projection of \( X_0 \) on the hyperplane HP \((A_i, b_i)\) the point noted \( X' \) such that:

\[
X_0 \text{ belongs to HP (} A_i, b_i) \text{ and is collinear to the vector } A_i.
\]

\( X_0 \) is the intersection of HP \((A_i, b_i)\) with the straight line passing through \( X_0 \) and whose direction vector is \( A_i \).

**Proof:**

\[
\bullet
\]

Mathematically this right is defined by:

\[
X \in \mathbb{R}^n \quad \text{with} \quad X = X_0 + \alpha A_i, \quad \alpha \in \mathbb{R}
\]

The intersection is written, \( X_0 = X_0 + \alpha A_i \),

\[
< A_i, X_0 > = b_i
\]

\[
< A_i, X_0 + \alpha A_i > = b_i
\]

\[
< A_i, X_0 > + \alpha < A_i, A_i > = b_i
\]

\[
\Rightarrow \alpha < A_i, A_i > = b_i - < A_i, X_0 >
\]

\[
\Rightarrow \alpha = \frac{b_i - < A_i, X_0 >}{< A_i, A_i >}
\]

Finally, we have:

\[
X_0 = X_0 + \left[ \frac{b_i - < A_i, X_0 >}{< A_i, A_i >} \right] A_i
\]

(8.02)

\( \bullet \)

Moreover, the distance between \( X_0 \) and \( X'_0 \) denoted \( d(X_0, X'_0) \) is equal to:

\[
d(X_0, X'_0) = \frac{|b_i - < A_i, X_0 >|}{< A_i, A_i >}
\]

(8.03)

This distance is also called the distance from the point \( X_0 \) to the hyperplane HP \((A_i, b_i)\).

### 8.1.2 Orientation of the vector \( A_i \) with respect to ZCT

**Proposition 8.01**

\( A_i \) is always oriented from inside to outside of ZCT.

**Proof:**

\( \bullet \)

Let \( X_0 \) a ZCT point that does not belong to the hyperplane HP \((A_i, b_i)\):

\[
X_0 \in \text{ZCT}, \quad \text{and} \quad X_0 \notin \text{HP}(A_i, b_i)
\]

Is \( X_0' \) the projection of \( X_0 \) on \( \text{HP}(A_i, b_i) \):

We have:

\[
X_0' = X_0 + \left[ \frac{b_i - < A_i, X_0 >}{< A_i, A_i >} \right] A_i
\]

The product of \( X_0'X_0 \) with \( A_i \) is:

\[
< X_0'X_0, A_i > = \left( \frac{b_i - < A_i, X_0 >}{< A_i, A_i >} \right) < A_i, A_i >
\]

\[
< X_0'X_0, A_i > = b_i - < A_i, X_0 >
\]

but \( X_0 \in \text{ZCT} \) which means:

\[
< A_i, X_0 > \leq b_i \quad \text{and} \quad X_0 \notin \text{HP}(A_i, b_i)
\]

So \( < A_i, X_0 > \leq b_i \).

Therefore we have:

\[
< A_i, X_0 > < b_i
\]

From where:

\[
b_i - < A_i, X_0 > > 0
\]

Finally we get the following relation:

\[
< X_0'X_0, A_i > > 0
\]

8.1.3 Direction of a decision vector with respect to ZCT

**8.1.3.1 External orientation**

We say that \( V \) has an external orientation with respect to ZCT, if \( < V, A_i > \) is strictly negative.

**8.1.3.2 Parallel orientation**

We say that \( V \) has a parallel orientation with respect to ZCT, if \( < V, A_i > \) is equal to 0.

**8.1.3.3 Internal orientation**

We say that \( V \) has an internal orientation with respect to ZCT, if \( < V, A_i > \) is strictly positive.

**Note 8.01:**

The qualification “parallel” refers to the fact that \( V \) is parallel to HP \((A_i; b_i)\), that is to say at the ZCT, boundary.

8.1.4 Characterization of an Inner Point of ZCT

**Proposition 8.02**

\( X_0 \) is an inner point of ZCT if and only if:

\[
< A_i, X_0 > < b_i
\]

**Proof:**

\( \bullet \)

a) Let \( X_0 \) be a point inside ZCT.

According to 2.7.1.3.1, there exists a strictly positive reality \( r \) such that the ball whose center is \( X_0 \) and with a radius \( r \) is entirely contained in ZCT.
Logically, this means that if \( X_0' \) designates the projection of \( X_0 \) on \( HP(A, b) \), then the distance between \( X_0 \) and \( X_0' \) is equal to:

\[
\frac{b - <A, \ X_0>}{\sqrt{<A, \ A>}}
\]

And

\[
r \leq \frac{b - <A, \ X_0>}{\sqrt{<A, \ A>}} \quad \text{but} \ r \text{ is strictly positive,}
\]

\[
\frac{b - <A, \ X_0>}{\sqrt{<A, \ A>}} \text{ is strictly positive from where}
\]

\[
\frac{b - <A, \ X_0>}{\sqrt{<A, \ A>}} \text{ is alsostrictly positive}.
\]

b) Reciprocally, \( X_0 \) is a point of \( ZCT_i \) such that:

\[
< A, \ X_0 > < b_i
\]

Let us show that this is an inside point of \( ZCT_i \).

Let \( X_0' \) be the projection of \( X_0 \) on the hyperplane. We saw that:

The distance between \( X_0 \) and \( X_0' \) is equal to:

\[
\frac{b - <A, \ X_0>}{\sqrt{<A, \ A>}}
\]

Let

\[
r = \frac{1}{2} \frac{b - <A, \ X_0>}{\sqrt{<A, \ A>}}
\]

We have \( r > 0 \);

and it is obvious that the ball whose center is \( X_0 \) and radius \( r \) is entirely included in \( ZCT_i \), which means that \( X_0 \) is an internal point of \( ZCT_i \).

8.1.4.1 S border point

A point \( X_0 \in \mathbb{R}^n \) is said to be a border point of \( S \) if:

\( X_0 \) is an element of \( S \);

Whatever the positive real number \( r \) is, the open ball of center \( X_0 \) and radius \( r \) contains both elements of \( S \) other than \( X_0 \) and elements of \( \overline{S} \) (the complement of \( S \) in Euclidean affine space \( \mathbb{R}^n \)).

8.1.5 S border

Let \( S \) be a non-empty set in the Euclidean affine space with the orthonormal coordinate system

\( (O, U_1, U_2, ..., U_n) \). We call the boundary of \( S \) the set noted \( Fr(S) \) containing all the boundary points of \( S \).

Proposition 8.03

We characterize the border points of the \( ZCT_i \) by:

\[
Fr(ZCT_i) = HP(A_i; b_i)
\]

Proof:

\[
\star
\]

Let's first show the first inclusion:

\[
Fr(ZCT_i) \subset HP(A_i; b_i)
\]

Let \( X_0 \) be an element of \( Fr(ZCT_i) \), that means that \( X_0 \) is also an element of \( ZCT_i \), so,

\[
<A_i, \ X_0 > \leq b_i
\]

If \( X_0 \not\in HP(A_i; b_i) \), this means that \( <A_i, \ X_0 > \neq b_i \)

From where,

\[
<A_i, \ X_0 > < b_i
\]

meaning that \( X_0 \) is an inner point of \( ZCT_i \), which contradicts the fact that \( X_0 \) is a border point of \( ZCT_i \).

Reciprocal:

\[
HP(A_i; b_i) \subset Fr(ZCT_i)
\]

Let \( X_0 \) be a point of \( HP(A_i; b_i) \) which means that:

\[
<X_i, \ X_0 > = b_i
\]

Let \( r \) be any strictly positive real number and \( X_1 \) the point defined by:

\[
X_1 = X_0 + \frac{r}{2} \frac{A_i}{\sqrt{<A_i, A_i>}}
\]

Note that the distance between \( X_0 \) and \( X_1 \) is equal to \( \frac{r}{2} \), which is strictly less than \( r \).

Moreover, according to Proposition 8.01, \( A_i \) is always oriented towards the outside of \( ZCT_i \), so \( X_1 \) do not belong to \( ZCT_i \).

So \( X_1 \) belongs to the open ball of center \( X_0 \) and radius \( r \). Finally, let us show that the open ball of center \( X_0 \) and radius \( r \) contains points of \( ZCT_i \) other than \( X_0 \).

Let \( X_2 \) the point defined by:

\[
X_2 = X_0 - \frac{r}{2} \frac{A_i}{\sqrt{<A_i, A_i>}}
\]

We obtain:

\[
<A_i, \ X_2 > = < A_i, \ X_0 > - \frac{r}{2} \frac{1}{\sqrt{<A_i, A_i>}} < A_i, A_i >
\]

\[
< A_i, \ X_2 > = < A_i, \ X_0 > - \frac{r}{2}
\]

But

\[
<A_i, \ X_0 > = b_i \quad \Rightarrow \quad < A_i, \ X_2 > = b_i - \frac{r}{2}
\]

As

\[
r > 0 \quad \Rightarrow \quad \frac{r}{2} > 0
\]

From where

\[
b_i - \frac{r}{2} < b_i
\]

We can write:

\[
<A_i, \ X_2 > < b_i
\]
Which means that $\tilde{X}_2$ belongs to ZCT.

In summary, every strictly positive real number in the open ball of center $X_0$ and radius $r$ contains both elements of ZCT, other than $X_0$ and elements of ZCT.

This means that $X_0$ is a frontier point of ZCT.

Finally, in combination with the two way, we have:

$$Fr(ZCT) \subseteq HP(A_i, b_i)$$

and

$$Fr(ZCT) \supseteq HP(A_i, b_i)$$

So we have

$$Fr(ZCT) = HP(A_i, b_i)$$

**Proposition 8.04**

The zone of technical constraints ZCT is topologically closed.

**Proof:**

Let $X_0$ be a point not belonging to ZCT (where ZCT denotes the complement of ZCT, in the affine space). As $X_0 \notin ZCT_i$

So

$$\langle A_i, X_0 \rangle > b_i$$

Let $X_0'$ be the projection of $X_0$ on $HP(A_i, b_i)$. We showed that:

The distance between $X_0'$ and $X_0$ is equal to:

$$\frac{|b_i - \langle A_i, X_0 \rangle|}{\sqrt{\langle A_i, A_i \rangle}}.$$ 

But

$$\langle A_i, X_0 \rangle > b_i \Rightarrow b_i - \langle A_i, X_0 \rangle < 0$$

so

$$|b_i - \langle A_i, X_0 \rangle| = -(b_i - \langle A_i, X_0 \rangle) = \langle A_i, X_0 \rangle - b_i$$

The distance between $X_0$ and the border $HP(A_i, b_i)$ is equal to:

$$\frac{\langle A_i, X_0 \rangle - b_i}{\sqrt{\langle A_i, A_i \rangle}}.$$

Let:

$$r = \frac{1}{2} \frac{\langle A_i, X_0 \rangle - b_i}{\sqrt{\langle A_i, A_i \rangle}}$$

It is easy to show that the open ball of center $X_0$ and radius $r$ is entirely contained in $ZCT$.

Therefore $ZCT$ is open.

So ZCT is closed.

In summary of Proposition 8.03 and Proposition 8.04, each zone of ZCT technical constraints is closed and their boundary is none other than the hyperplane $HP(A_i, b_i)$.

### 8.2 Relationship between a search axis $R(X_0, V)$ and a technical constraint zone ZCT

Consider an area of ZCT, technical constraints. Let also be the radius $R(X_0, V)$ coming from $X_0$ and direction vector $V$, representing a search axis. This relationship is based on the existence and uniqueness of the solution of the equation:

$$\alpha \langle A_i, V \rangle = b_i - \langle A_i, X_0 \rangle, \quad \alpha \text{ is the unknown.}$$

**Proof:**

Since the two objects are sets of decision points, the main combination that can be imagined between them is the intersection. In addition, since the area of technical constraint is closed and the hyperplane HP $(A_i, b_i)$ is closed, we will be much more interested in the intersection of the radius $R(X_0, V)$ with this boundary HP $(A_i, b_i)$. Indeed, if $X_0$ is outside of ZCT, then such an intersection gives us the point of entry from the outside to the inside of ZCT, along the radius. On the other hand, if $X_0$ is inside ZCT, it gives the exit point of ZCT.

Let $I$ denote this point of intersection, as $I$ belongs to $R(X_0, V)$

We have $I = X_0 + \alpha V$ avec $\alpha > 0$

Since $I$ also belongs to HP $(A_i, b_i)$, we can write:

$$\langle A_i, I \rangle = b_i$$

By combining these two relationships, we have:

$$\langle A_i, X_0 + \alpha V \rangle = b_i$$

By developing, we get

$$\langle A_i, X_0 \rangle + \alpha \langle A_i, V \rangle = b_i$$

which is an equation where $\alpha$ is the unknown.

This equation gives:

$$\alpha \langle A_i, V \rangle = b_i - \langle A_i, X_0 \rangle \quad (8.04)$$

The existence and uniqueness of $\alpha$ depend on the values of $b_i - (A_i, X_0)$ and $(A_i, V)$, hence $X_0$ and $V$.

#### 8.2.1 Where $X_0$ is outside ZCT; and where $V$ is not facing inwards from ZCT

The equation:

$$\alpha \langle A_i, V \rangle = b_i - \langle A_i, X_0 \rangle$$

has no solution.

**Proof:**

This case is mathematically translated by:

$$\{\langle A_i, X_0 \rangle \leq b_i \text{ and } \langle A_i, V \rangle \geq 0\}$$

leads that, $b_i - \langle A_i, X_0 \rangle < 0$

Also as

$$\alpha \geq 0 \text{ et } \langle A_i, V \rangle = b_i - \langle A_i, X_0 \rangle < 0$$
which is impossible.
In this case I do not exist.

8.2.2 Where $X_0$ is outside ZCT, and where $V$ is inward ZCT.
The equation:
$$\alpha \langle A, V \rangle = b_i - \langle A, X_0 \rangle$$
has a unique solution.

$$\alpha = \frac{b_i - \langle A, X_0 \rangle}{\langle A, V \rangle} > 0$$

Proof:

We have:
$$b_i - \langle A, X_0 \rangle < 0 \text{ and } \langle A, V \rangle > 0$$
which gives
$$\alpha = \frac{b_i - \langle A, X_0 \rangle}{\langle A, V \rangle} > 0$$
which is in addition a unique value.

In this case we say that we have a single point of entry starting from $X_0$, and moving along the axis of $R (X_0, V)$.

8.2.3 Where $X_0$ is on ZCT, and where $V$ is outside ZCT.
The equation:
$$\alpha \langle A, V \rangle = b_i - \langle A, X_0 \rangle$$
admits a null solution.

Proof:

This case results in:
$$b_i - \langle A, X_0 \rangle = 0 \text{ and } \langle A, V \rangle > 0$$
The only solution available is:
$$\alpha = 0$$
Meaning that, $I = X_0$.

8.2.4 Where $X_0$ is on the ZCT, border and where $V$ is parallel oriented to HP $(A_i, b_i)$.
The equation:
$$\alpha \langle A, V \rangle = b_i - \langle A, X_0 \rangle$$
has an infinity of solutions.

Proof:

Mathematically we translate this case by:
$$b_i - \langle A, X_0 \rangle = 0 \text{ and } \langle A, V \rangle = 0$$
This gives us infinity of solutions. In fact, $R(X_0, V)$ is included in HP $(A_i ; b_i)$, and the intersection is none other than $R (X_0, V)$.

8.2.5 Where $X_0$ is on the ZCT, border and where $V$ is inward ZCT.
The equation:
$$\alpha \langle A, V \rangle = b_i - \langle A, X_0 \rangle$$
has a unique solution.

Proof:

In this case:
$$b_i - \langle A, X_0 \rangle = 0 \text{ et } \langle A, V \rangle < 0$$
which leads to:
$$\alpha = 0$$
That is to say $I = X_0$, unique solution.

8.2.6 Where $X_0$ is inside ZCT, and where $V$ is outside ZCT.
The equation:
$$\alpha \langle A, V \rangle = b_i - \langle A, X_0 \rangle$$
has a unique solution.

$$\alpha = \frac{b_i - \langle A, X_0 \rangle}{\langle A, V \rangle}$$

Proof:

This case results in:
$$b_i - \langle A, X_0 \rangle = 0 \text{ et } \langle A, V \rangle < 0$$
which leads to:
$$\alpha = 0$$
Hence the unique solution:
$$\alpha = \frac{b_i - \langle A, X_0 \rangle}{\langle A, V \rangle}$$

8.2.7 Case where $X_0$ is inside ZCT, and where $V$ is parallel oriented to ZCT.
The equation:
$$\alpha \langle A, V \rangle = b_i - \langle A, X_0 \rangle$$
has no solution.

Proof:

As in the previous case, we have:
$$b_i - \langle A, X_0 \rangle = 0 \text{ but } \langle A, V \rangle = 0$$
Which leads to an impossibility, meaning that $R (X_0, V)$ will never intercept the border of ZCT.

8.2.8 Where $X_0$ is inside ZCT, and where $V$ is inward ZCT.
The equation:
$$\alpha \langle A, V \rangle = b_i - \langle A, X_0 \rangle$$
has no solution.
Proof:

As recently, $b_i - \langle A_i, X_0 \rangle > 0$ and $\langle A_i, V \rangle > 0$

In this case, the equation

$\alpha \langle A_i, V \rangle = b_i - \langle A_i, X_0 \rangle$

has no positive solution, meaning that $R(X_0, V)$ will never intercept the border of $ZCT_i$.

8.2.9 Algorithm for determining the intersection of a search axis $R(X_0, V)$ with the technical constraint zone $ZCT_i$.

The previous eight cases can be summarized in the following algorithm:

- Case where there is no intersection.

  No intersection:

  - Case with $\langle A, X_i \rangle > 0$ and $\langle A, V \rangle < 0$

    We have a single point of intersection

    $\left\{ X_0 + \frac{b_i - \langle A_i, X_0 \rangle}{\langle A_i, V \rangle} V \right\}$

    - If $\langle A, X_0 \rangle = b_i$ and $\langle A, V \rangle < 0$.

      There is a unique solution:

      $\{X_0\}$

      - If $\langle A, X_0 \rangle = b_i$ and $\langle A, V \rangle > 0$.

        There is a unique solution:

        $\{X_0 + \frac{b_i - \langle A_i, X_0 \rangle}{A_i, V} V\}$

        - If $\langle A, X_0 \rangle (b_i)$ and $\langle A, V \rangle \geq 0$

          No solution: $\emptyset$

8.3 Relationship between a search-plane $C(X_0, V)$ and the technical constraint zones $ZCT_i$

As for section 8.2, these two objects are sets of decision points, so this section will describe their intersection.

Let $I$ be such a point. It is thus of the form:

$I = X_0 + \alpha_1 V_1 + \alpha_2 V_2$ where $\alpha_1$ and $\alpha_2$ are positives.

More like $I$ belongs to the border of $ZCT_i$, we can write:

$\langle A_i, I \rangle = b_i$

Which leads to:

$\langle A_i, X_0 + \alpha_1 V_1 + \alpha_2 V_2 \rangle = b_i$

From where

$\langle A_i, X_0 \rangle + \alpha_1 \langle A_i, V_1 \rangle + \alpha_2 \langle A_i, V_2 \rangle = b_i$

Let

$\alpha_1 \langle A_i, V_1 \rangle + \alpha_2 \langle A_i, V_2 \rangle = b_i - \langle A_i, X_0 \rangle$

which is a linear equation with two unknowns variables, $\alpha_1$ et $\alpha_2$ positive.

As for 2.6.2, the existence and uniqueness of the solutions to this problem depend on the three objects $X_0, V_1$ et $V_2$.

- For $X_0$, there are three possibilities: to be outside of $ZCT_i$, be on the border of $ZCT_i$, or be inside of $ZCT_i$.

- For $V_1$, there are three possibilities: be outward facing from $ZCT_i$, or be oriented parallel to $ZCT_i$, or be oriented towards the inside of $ZCT_i$.

- For $V_2$, the possibilities are the same as for $V_1$.

All in all, we have $3 \times 3 \times 3$, that is to say, 27 possible cases to be made.

8.4 Search along an axis $R(X_0, V)$ in the ZNN non-negativity zone

The ZNN non-negativity zone has been defined as

$ZNN = \bigcup_{i=1}^{n} ZNN_i$

where

$ZNN_i = ZR(-U_i, \emptyset)$ (8.05)

Conceptually, this means that we assimilate $ZNN_i$ to a resource constraint zone, and if a $X_0$ point belongs to $ZNN_i$, this means that $X_0$ satisfies the constraint associated with $ZNN_i$.

It is therefore a stage situation.

The concept of research implies that there is a situation or state of deposition, and that from this situation, one move to another situation or state. And we have already seen that this research, when it is linear, can be modeled by the concept of axis of research starting from a given point and moving according to a given vector $R(X_0, V)$.

In this section, the starting point of the search is the point $X_0$, which is supposed to be located in the ZNN non-negativity zone. Since the new decisions found must have remained in ZNN, it is logical to study the conditions under which this search leads us out of ZNN. And as in the case of ZCT, technical constraint zones, we will need to define the boundary concept in ZNN.

For $ZNN_i$, there is no problem.

Indeed,

$Fr(ZNN_i) = HP(-U_i, 0)$

given that $ZNN_i$ is only half affine delimited by the hyperplane $HP(-U_i, O)$.

And as $HP(-U_i, 0)$ is included in $ZNN_i$, we know that $ZNN_i$ is a closed area.

8.4.1 Frontier of the ZNN; non-negativity zone

Proposition 8.05

$Fr(ZNN) = \bigcup_{i=1}^{n} \left( HP(-U_i, 0) \cap C_0 \right)$
Proof:

♣

It suffices to show that for every j, \( HP(-U_j,0) \cap C^+_0 \) is a function of ZNN.

Step 1:
Let us show that every X point of \( HP(-U_j,0) \cap C^+_0 \) is a border point of ZNN. Let \( r \) be any positive real number. This is to show that the open ball of center X and radius \( r \) denoted \( B(X,r) \), contains both an element of ZNN other than X and an element of ZNN other than X is:

\[
X = (x_i)_{i=1,...,n}
\]

The fact that \( X \in HP(-U_j,0) \cap C_0^+ \) means that \( x_i \geq 0, \forall i = 1...n \) and that \( x_j \geq 0 \)

Let us take the point \( X' = (x'_i)_{i=1,...,n} \) defined as follows:

\[
x'_i = \begin{cases} 
  x_i & \text{for all } i \neq j \\
  \frac{1}{2} & \text{if } i = j
\end{cases}
\]

Similarly let us take the point:

\[
x''_i = \begin{cases} 
  x_i & \text{for all } i \neq j \\
  \frac{1}{2} & \text{if } i = j
\end{cases}
\]

It is obvious that:

\( X' \in \text{ZNN} \) and has \( X'' \in \text{ZNN} \)

Moreover, it is also obvious that:

\( X' \in B(X,r) \) and \( X'' \in B(X,r) \)

This shows that the open ball \( B(x,r) \) contains both the point \( X \) which is the element outside ZNN and the point \( X'' \) which is the element in ZNN, and it is obvious that \( X' \) is different from \( X \) and that \( X'' \) is different from \( X \).

Therefore \( X \) is a border point of ZNN.

2nd step:
Let us show that \( HP(-U_j,0) \cap C^+_0 \) and \( \text{d}_j \) the distance between \( X \) and \( X_j \)

As

\[
X' \neq X \quad \forall j = 1,...,n
\]

Then \( \text{d}_j \neq 0 \quad \forall j = 1,...,n \)
Let

\[
r = \frac{1}{2} \min(d_j) \quad \forall j = 1,...,n
\]

It is obvious that \( r > 0 \) and that the open ball \( B(X,r) \) is included in ZNN.

There fore,

\[
\bigcup_{j=1}^{n} HP(-U_j,0) \cap C_0^+ = Fr(\text{ZNN})
\]

Note 8.02

\( HP(-U_j,0) \cap C_0^+ \) is the cone

\[
C(O,U_1,U_2,...,U_{j-1},U_{j+1},...,U_n)
\]

We will call it " conic wall of non-negativity j " Noted \( \text{MCNN}_j \).

Proof :

♣

\[
\text{MCNN}_j = HP(-U_j,0) \cap C_0^+ \cap \text{ZNN}
\]

et \( \text{ZNN} = \bigcup_{j=1}^{n} \text{MCNN}_j \)

In other words, the ZNN border is none other than the conic wall meeting of non-negativity.

Note further that

\[
\text{MCNN}_j \subset HP(U_j,0)
\]

So

\[
\text{MCNN}_j \subset Fr(\text{ZNN})_j
\]

8.4.2 Exit point of ZNN following R (X_0, V)

Proposition 8.06

We assume that \( X_0 \) is in ZNN. Since ZNN is closed, the exit point is the point of contact or intersection between \( R(X_0,V) \) and \( Fr(\text{ZNN})_j \).

Proof

♣

Let \( F_j \) be the point of \( R(X_0,V) \) et \( Fr(\text{ZNN})_j \) which is none other than \( HP(-U_j,0) \).

Given that \( C_0^+ = \text{ZNN} \), it means that

\[
X \neq \bigcup_{j=1}^{n} HP(U_j,0) \cap C_0^+
\]

It means that:

\[
X \neq \bigcup_{j=1}^{n} HP(U_j,0) \cap C_0^+
\]

\[
I_j = \frac{1}{2}(X + \alpha_j V) \quad \text{with } \alpha_j \geq 0
\]
Similarly

\[ I \in \text{HP}\left(-U_j, 0\right) , \]

So we check the equation:

\[ -U_j, I > = 0 \quad \text{ou} \quad -U_j, I >= 0 \]

\[ < U_j, X_0, \alpha_j V > = 0 \]

From where,

\[ < U_j, X_0 > + \alpha_j < U_j, V > = 0 \]

Finally, we obtain the equation:

\[ \alpha_j < U_j, V > = - < U_j, X_0 > \]

with the condition \( \alpha_j \geq 0 \).

Based on the results of section 5.2, we can say that the research axis \( R\left(X_0, V\right) \) leads us outside of ZNNj only if \( V \) is oriented outside ZNNj, that is, if

\[ < -U_j, v > > 0 , \]

in which case,

\[ \alpha_j = - < U_j, X_0 > \]

\[ < U_j, V > \]  

(8.06)

Let \( X_0 = (X_0^i)_{i=1,...,n} \)

\( V = (v_j)_{j=1,...,n} \) and \( < U_j, X_0 >= x_0^i \)

\[ < U_j, V >= v_j \]

The expression of \( \alpha_j \) becomes:

\[ \alpha_j = \frac{-X_0^i}{v_j} \]  

(8.07)

Which is good non-negative because \( x_0^i \geq 0 \) and \( v_j < 0 \).

Hence the proposal:

**Proposition 8.07**

The research axis is in the non-negativity zone ZNN only if IN (V) is non-empty. In this case, the exit point I is given by:

\[ I = X_0 + \alpha V . \]

Or

\[ \alpha = \min \left\{ \frac{-x_0^i}{v_j} \right\} \]  

(8.08)

\[ j \in \text{IN} (V) \]

Proof:

Given that \( R\left(X_0, V\right) \) is a totally ordered set, the exit point of I such that verify:

\[ I = \min \left( I_j \right) \]

\[ j \in \text{IN} (V) \] where \( I_j = X_0 + \alpha V \)  

(8.09)

The Proof is immediate. The output point I is none other than the first smallest \( I_j \).

**9 CONCLUSION**

Through this article, we presented all the art mathematical object classes required for topo-geometric modeling MZ and the mean objectives associated with Topo-geometric definitions, as well as the conventional operations and properties of constraints LPP defining hyperplanes, are mentioned.

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