Hyperbolic Quadrature Method of Moments for the one-dimensional kinetic equation

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5. Conclusion, Perspectives
Considered kinetic model

Kinetic model on \( f(t, x, v) \)

\[
\frac{\partial f}{\partial t} + \nabla \cdot (v f) = S(f)
\]

physical transport

Source terms
Considered kinetic model

**Kinetic model on** $f(t, x, v)$

\[
\partial_t f + \partial_x \cdot (v f) = S(f)
\]

- **physical transport**
- **Source terms**

**Gas dynamics**

for example BGK source term: $S(f) = -\frac{f - f_{eq}}{Kn}$

Transition regime: $0.01 < Kn < 10$

Far from the Maxwellian equilibrium
Considered kinetic model

Kinetic model on $f(t, x, v)$

$$\frac{\partial f}{\partial t} + \nabla \cdot (v f) = S(f)$$

**physical transport**

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Population of inertial particles in a gas

for example drag source term: $S(f) = -\partial_v \cdot \left( \frac{v_g(t, x) - v}{St} f \right)$

Particle trajectory crossing for large enough particles (and $St$): $f$ is no more a Dirac delta function
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**Kinetic model on** $f(t, x, v)$

$$\frac{\partial}{\partial t} f + \nabla \cdot (v f) = S(f)$$

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Particle trajectory crossing for large enough particles (and $St$): $f$ is no more a Dirac delta function

- The kinetic model is too costly to solve with direct methods of Monte-Carlo type
- Moments $\int_{\mathbb{R}} v^k f(t, x, v) dv$ of order $k$ smaller than 1 or 2 are not enough to represent the distribution.
Moment method

Principle of the method

Write equations on a finite set of moments $\mathbf{m}_N = (m_0, m_1, \ldots, m_N)^t$:

$$\partial_t m_k + \partial_x m_{k+1} = S_k, \quad k = 0, 1, \ldots, N$$

(1)

Closure: express $m_{N+1}$ (and eventually the source terms $S_k$) as a function of $\mathbf{m}_N$.

Issues:
- $(m_0, m_1, \ldots, m_N, m_{N+1})^t$ is realizable
- The system (1) is globally hyperbolic
- Capture equilibrium state

Strategy
- Solve the Hamburger truncated moment problem:
  find a positive measure $\mu$ such that $\mathbf{m}_N = \int_{\mathbb{R}} (1, v, \ldots, v^N)^t d\mu(v)$.

  and set $m_{N+1} = \int_{\mathbb{R}} v^{N+1} d\mu(v)$
- Give directly $m_{N+1}$
Moment method

Principle of the method

Write equations on a finite set of moments $\mathbf{m}_N = (m_0, m_1, \ldots, m_N)^t$:

$$\partial_t m_k + \partial_x m_{k+1} = S_k, \quad k = 0, 1, \ldots, N$$  \hspace{1cm} (1)

Closure: express $m_{N+1}$ (and eventually the source terms $S_k$) as a function of $\mathbf{m}_N$.

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Examples of closure in the literature

- Grad closure [Grad, 1949] → hyperbolic only around the moments of the maxwellian distribution
- Entropy maximization [Levermore, 1996, Müller and Ruggeri, 1998] → high computational cost - not valid on the entire realizability domain
- Quadrature method of moment [McGraw, 1997, Fox, 2008] → weakly hyperbolic
Moment method

Moment space

**Definition**

The \( n^{th} \)-moment space \( \mathcal{M}_n \) is defined by

\[
\mathcal{M}_n = \left\{ m \in \mathbb{R}^{n+1} \mid \exists \mu \in \mathcal{M}_+(\mathbb{R}), \quad m = \int_{\mathbb{R}} (1, v, \ldots, v^n)^t d\mu(v) \right\}
\]

If \( m \) belongs to \( \mathcal{M}_n \), then it is said to be **realizable**.
If \( m \) belongs to the interior \( \text{Int} \mathcal{M}_n \) of \( \mathcal{M}_n \), it is said to be **strictly realizable**.

Characterized by the non-negativity of the **Hankel determinants**: \( n \geq 0 \)

\[
H_{2n} = \begin{vmatrix}
    m_0 & \cdots & m_n \\
    \vdots & \ddots & \vdots \\
    m_n & \cdots & m_{2n}
\end{vmatrix},
\]

**Theorem**

\( m_N = (m_0, m_1, \ldots, m_N)^t \) strictly realizable \( \iff \) \( H_{2k} > 0, \quad k \in \{0, 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor \} \)

\( m_N \in \partial \mathcal{M}_N \cap \mathcal{M}_n \Rightarrow H_0 > 0, \ldots, H_{2k-2} > 0, H_{2k} = 0, \ldots, H_{2\left\lfloor \frac{N}{2} \right\rfloor} = 0, k \leq \left\lfloor \frac{N}{2} \right\rfloor \).

*In the latter case, the only corresponding measure is a sum of \( k \) weighted Dirac delta functions.*

[Shohat and Tamarkin, 1943, Gautschi, 2004, Schm"udgen, 2017]
### Definition

**The $n^{th}$-moment space $\mathcal{M}_n$ is defined by**

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$$

If $m$ belongs to $\mathcal{M}_n$, then it is said to be **realizable**. If $m$ belongs to the interior $\text{Int} \mathcal{M}_n$ of $\mathcal{M}_n$, it is said to be **strictly realizable**.

Characterized by the non-negativity of the **Hankel determinants**: $n \geq 0$

$$
H_{2n} = \begin{vmatrix}
    m_0 & \ldots & m_n \\
    \vdots & \ddots & \vdots \\
    m_n & \ldots & m_{2n}
\end{vmatrix},
$$

First constraints for the strict realizability:

$$
m_0 > 0 \quad m_2 > \frac{m_1^2}{m_0} \\
m_4 > \frac{m_0 m_3^2 - 2m_1 m_2 m_3 + m_3^2}{m_2 m_0 - m_1^2} \quad \ldots
$$
### Hyperbolicity

#### Characteristic polynomial

**System on moments**

Equations on $m_N = (m_0, m_1, \ldots, m_N)^t$:

$$\partial_t m_N + \partial_x F(m_N) = \bar{S}$$

**Characteristic Polynomial**

Jacobian matrix

$$\frac{DF(m_N)}{Dm_N} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 \\
\frac{\partial m_{N+1}}{\partial m_0} & \frac{\partial m_{N+1}}{\partial m_1} & \frac{\partial m_{N+1}}{\partial m_2} & \cdots & \frac{\partial m_{N+1}}{\partial m_N}
\end{pmatrix}.$$ 

Characteristic polynomial

$$\bar{P}_{N+1}(X) = X^{N+1} - \sum_{i=0}^{N} \frac{\partial m_{N+1}}{\partial m_i} X^i$$
Moments - Central moments - Standardized moments

**moments:**

\[ m_k = \int_{\mathbb{R}} v^k f(v) \, dv \]

**central moments:** with \( \rho = m_0, \ u = \frac{m_1}{m_0} \) and \( f^c(c) = \frac{1}{\rho} f(c + \rho) \)

\[ C_k = \frac{1}{\rho} \int_{\mathbb{R}} (v - u)^k f(v) \, dv = \int_{\mathbb{R}} c^k f^c(c) \, dc \]

so that \( C_0 = 1 \) and \( C_1 = 0 \).

**standardized moments:** with \( \sigma = \sqrt{C_2}, \ f^s(s) = \frac{\sigma}{\rho} f(u + \sigma s) \)

\[ S_k = \frac{1}{m_0} \int_{\mathbb{R}} \left( \frac{v - u}{\sqrt{C_2}} \right)^k f(v) \, dv = \int_{\mathbb{R}} s^k f^s(s) \, ds \]

so that \( S_0 = 1, \ S_1 = 0 \) and \( S_2 = 1 \).

**link:**

\[ C_k = \sum_{i=0}^{k} \binom{k}{i} \left( -\frac{m_1}{m_0} \right)^{k-i} m_i, \quad m_k = \rho \left( \sum_{i=2}^{k} \binom{k}{i} u^{k-i} C_i + u^k \right), \quad S_k = \frac{C_k}{(C_2)^{k/2}}. \]
Hyperbolicity

Property of the characteristic polynomial

$m_N = (m_0, m_1, \ldots, m_N)^t$ be a realizable moment vector such that $m_0 > 0$ and $C_2 > 0$.

Linear functional $\langle \cdot \rangle_{m_N}$ on the space $\mathbb{R}[X]_N$

$$\langle X^k \rangle_{m_N} = m_k, \text{ for } k \in \{0, 1, \ldots, N\}.$$ 

Linear functional $\langle \cdot \rangle_{S_N}$ associated with the standardized moments $S_N = (S_0, \ldots, S_N)^t$:

$$\langle X^k \rangle := \langle X^k \rangle_{S_N} = S_k, \text{ for } k \in \{0, 1, \ldots, N\}.$$ 

Property of the scaled characteristic polynomial

Let us assume that the function $S_{N+1}$ does not depend on $(m_0, u, C_2)$, i.e., $S_{N+1}(S_3, \ldots, S_N)$. Then, the following polynomial

$$P_{N+1}(X) := \bar{P}_{N+1} \left( u + C_2^{1/2} X \right) C_2^{-(N+1)/2}$$

only depends on $(S_3, \ldots, S_N)$, and

$$\langle P_{N+1} \rangle = 0, \quad \langle P_{N+1}' \rangle = 0, \quad \langle XP_{N+1}' \rangle = 0.$$
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QMOM: Principles of the method

From a strictly realizable moment vector $\mathbf{m}_{2n-1}$

**Reconstruction**

reconstruct the discrete measure $\mu = \sum_{i=1}^{n} w_i \delta u_i$

in such a way that

$$\sum_{i=1}^{n} w_i u_i^k = m_k \quad k = 0, 1, \ldots, 2n - 1$$

**Closure**

$$m_{2n} = \int_{\mathbb{R}} v^{2n} d\mu = \sum_{i=1}^{n} w_i u_i^{2n}$$

It is the minimal value for this moment
**QMOM: Principles of the method**

From a strictly realizable moment vector $m_{2n-1}$

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Closure

$$m_{2n} = \int_{\mathbb{R}} v^{2n} d\mu = \sum_{i=1}^{n} w_i u_i^{2n}$$

It is the minimal value for this moment

From the standardized moments $S_{2n-1}$, with $\rho = m_0$, $u = m_1/m_0$, $\sigma = \sqrt{C_2}$

Reconstruction

reconstruct the discret measure

$$\mu = \sum_{i=1}^{n} \rho \omega_i \delta u + \sigma c_i$$

in such a way that

$$\sum_{i=1}^{n} \omega_i c_i^k = S_k \quad k = 0, 1, \ldots, 2n - 1$$

Closure

$$S_{2n} = \sum_{i=1}^{n} \omega_i c_i^{2n}$$
QMOM: Principles of the method

From a strictly realizable moment vector $m_{2n-1}$

Reconstruction

reconstruct the discrete measure $\mu = \sum_{i=1}^{n} w_i \delta u_i$

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$$\sum_{i=1}^{n} w_i u_i^k = m_k \quad k = 0, 1, \ldots, 2n - 1$$

Closure

$$m_{2n} = \int_{\mathbb{R}} v^{2n} d\mu = \sum_{i=1}^{n} w_i u_i^{2n}$$

It is the minimal value for this moment

Remarks

- The reconstruction $\mu$ is the only one possible for the moment vector $m_{2n}$.
- $m_{2n}$ is at the boundary of the moment space: $H_{2n} = 0$
PRINCIPLE OF THE METHOD

QMOM: Computation of the weights and abscissas

\( m_{2n-1} \): strictly realizable moment vector

**Orthogonal polynomials**

Family \((Q_k)_{k=0,...,n}\) of monic orthogonal polynomials for the scalar product \((p, q) \mapsto \langle pq \rangle\) of \(\mathbb{R}_n[X]\).

\[
Q_{k+1}(X) = (X - a_k)Q_k(X) - b_k Q_{k-1}(X)
\]

with \(Q_{-1} = 0\) and \(Q_0 = 1\).

The recurrence coefficients \(a_k\) and \(b_k\) can be found from the standardized moments using the Chebyshev algorithm [Chebyshev, 1859, Wheeler, 1974, Gautschi, 2004]

\[
a_k = \frac{\langle XQ_k^2 \rangle}{\langle Q_k^2 \rangle}, \quad b_k = \frac{\langle Q_k^2 \rangle}{\langle Q_{k-1}^2 \rangle} = \frac{H_{2k} H_{2k-4}}{H_{2k-2}^2}.
\]

example

\[
a_0 = 0, \quad a_1 = S_3, \quad a_2 = \frac{S_5 - S_3(2 + S_3^2 + 2H_4)}{H_4} \]

\[
b_0 = 1, \quad b_1 = 1, \quad b_2 = H_4, \quad b_3 = H_6/H_4^2
\]
Principle of the method

**QMOM: Computation of the weights and abscissas**

$m_{2n-1}$: strictly realizable moment vector

**Orthogonal polynomials**

Family $(Q_k)_{k=0,\ldots,n}$ of monic orthogonal polynomials for the scalar product $(p, q) \mapsto \langle pq \rangle$ of $\mathbb{R}_n[X]$.

$$Q_{k+1}(X) = (X - a_k)Q_k(X) - b_k Q_{k-1}(X)$$

with $Q_{-1} = 0$ and $Q_0 = 1$.

The recurrence coefficients $a_k$ and $b_k$ can be found from the standardized moments using the Chebyshev algorithm [Chebyshev, 1859, Wheeler, 1974, Gautschi, 2004]

The abscissas $c_i$ are the zeros of $Q_n$ and also the eigenvalues of the Jacobi matrix

$$J_n = \begin{pmatrix} a_0 & \sqrt{b_1} & & \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & \\ & \ddots & \ddots & \ddots \\ & & \sqrt{b_{n-2}} & a_{n-2} & \sqrt{b_{n-1}} \\ & & & \sqrt{b_{n-1}} & a_{n-1} \end{pmatrix}$$
Hyperbolicity of the QMOM method

Theorem

The QMOM closure $b_n = 0$ induces the following characteristic polynomial $P_{2n} = Q_n^2$ and the system is only weakly hyperbolic.

proof [Chalons et al., 2012, Huang et al., 2020]
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First version of the HyQMOM closure [Fox et al., 2018]

Extension of QMOM, adding one moment and one abscissa for the reconstruction [Fox et al., 2018]

three-node HyQMOM

reconstruction with an additional fixed abscissa \( \mu = w_0 \delta u + \sum_{i=1}^{2} w_i \delta u_i \) in such a way that

\[
    w_0 u^k + \sum_{i=1}^{n} w_i u_i^k = m_k \quad k = 0, 1, \ldots, 4
\]

Closure

in term of the standardized moments: \( S_5 = S_3 (2S_4 - S_3^2) \)

Theorem (Hyperbolicity)

Assuming that the vector \( m_4 \) is strictly realizable, then system with the three-node HyQMOM closure is hyperbolic.

Problem

- The generalization to a larger number of moment is not easy
- The eigenvalues of the problem do not tend to the ones of QMOM when \( H_4 \to 0 \)
New HyQMOM closure

[Idea:]

- Instead of looking at a reconstruction or at a closure on $S_{2n+1}$, one looks at $a_n$.
- Have a reduced characteristic polynomial on the form

$$P_{2n+1} = Q_n [(X - \alpha_n)Q_n - \beta_n Q_{n-1}]$$

such that $\beta_n$ tends to zero when $H_{2n} \to 0$.

[Theorem]

For all $n = 1, 2, \ldots$; let the monic polynomial $P_{2n+1}$ be given by

$$P_{2n+1} = Q_n [(X - \alpha_n)Q_n - \beta_n Q_{n-1}] \quad \alpha_n, \beta_n \in \mathbb{R}$$

Then, the following statements are equivalent:

(i) $\langle P_{2n+1} \rangle = 0$, $\langle P'_{2n+1} \rangle = 0$ and $\langle XP'_{2n+1} \rangle = 0$.

(ii) $\alpha_n = a_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k$ and $\beta_n = \frac{2n + 1}{n} b_n$. 

[Fox and Laurent, 2021]
New HyQMOM closure

[Fox and Laurent, 2021]

Theorem

For all $n = 1, 2, \ldots, 9$; the scaled characteristic polynomial can be written as

$$P_{2n+1} = Q_n [(X - \alpha_n)Q_n - \beta_n Q_{n-1}]$$

if and only if the closure on $S_{2n+1}$, defined through the coefficient $a_n$, and the coefficients $\alpha_n$ and $\beta_n$ are related to the recurrence coefficients $a_k$ and $b_k$ by

$$a_n = \alpha_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k, \quad \beta_n = \frac{2n+1}{n} b_n.$$ 

Proof using formal computation with matlab symbolic:

1. set the closure $a_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k$
2. compute the Standardized moments $S_{2n+1}$ with the reverse Chebyshev algorithm
3. compute the coefficients $c_k$ of $P_{2n+1}$
4. compute the polynomials $Q_k$, $k = 0, 1, \ldots, n$
5. compute $P_{2n+1} - Q_n [(X - a_n)Q_n - \frac{2n+1}{n} b_n Q_{n-1}]$
New HyQMOM closure

New HyQMOM closure [Fox and Laurent, 2021]

**Theorem**

For all \( n = 1, 2, \ldots, 9 \); the scaled characteristic polynomial can be written as

\[
P_{2n+1} = Q_n \left[ (X - \alpha_n)Q_n - \beta_n Q_{n-1} \right]
\]

if and only if the closure on \( S_{2n+1} \), defined through the coefficient \( a_n \), and the coefficients \( \alpha_n \) and \( \beta_n \) are related to the recurrence coefficients \( a_k \) and \( b_k \) by

\[
a_n = \alpha_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k, \quad \beta_n = \frac{2n+1}{n} b_n.
\]

**Examples**

- \( n = 1 \): \( S_3 = 0 \) (as for the Maxwellian reconstruction)
- \( n = 2 \): \( S_5 = \frac{1}{2} S_3(5S_4 - 3S_3^2 - 1) \) (different from the previous version: \( S_5 = S_3(2S_4 - S_3^2) \))
Hyperbolicity - Eigenvalues

Theorem

When $\beta_n > 0$, the $n + 1$ roots of $R_{n+1} = (X - \alpha_n)Q_n - \beta_n Q_{n-1}$ are real-valued and bound and separate the $n$ roots of $Q_n$.

comes from Christoffel–Darboux formula.

The roots of $P_{2n+1}$ are then the eigenvalues of the two following Jacobi matrices:

$$
\begin{pmatrix}
\sqrt{b_1} & a_1 & \sqrt{b_2} \\
a_0 & \sqrt{b_1} & \sqrt{b_2} \\
. & . & . \\
\sqrt{b_{n-2}} & a_{n-2} & \sqrt{b_{n-1}} \\
\sqrt{b_{n-1}} & a_{n-1}
\end{pmatrix},
\begin{pmatrix}
\sqrt{b_1} & \sqrt{b_2} & . & . & . \\
a_0 & \sqrt{b_1} & a_1 & \sqrt{b_2} & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
\sqrt{b_{n-1}} & a_{n-1} & \sqrt{\beta_n} & \alpha_n
\end{pmatrix}
$$
Theorem

When $\beta_n > 0$, the $n + 1$ roots of $R_{n+1} = (X - \alpha_n)Q_n - \beta_n Q_{n-1}$ are real-valued and bound and separate the $n$ roots of $Q_n$.

comes from Christoffel–Darboux formula.

Example of the evolution of the eigenvalues with $H_{2n}$

$S_3 = -1$

$(S_3, S_4, S_5) = (-1, 5, -8)$
**Theorem**

*When* $\beta_n > 0$, *the* $n+1$ *roots of* $R_{n+1} = (X - \alpha_n)Q_n - \beta_n Q_{n-1}$ *are real-valued and bound and separate the* $n$ *roots of* $Q_n$.

This comes from Christoffel–Darboux formula.

**Example of the evolution of the eigenvalues with* $H_{2n}$**

- For $n = 2$:
  
  $S_3 = -1$

- For $n = 3$:
  
  $(S_3, S_4, S_5) = (-1, 5, -8)$

The moment system with the HyQMOM closure is then hyperbolic, whatever the strictly realizable moment.
Properties - Practical computations

Practical computations

Closure, directly from the moments $m_{2n}$

1. compute the $(\bar{a}_k)_{k=0}^{n-1}$ and $(\bar{b}_k)_{k=0}^{n}$ from $m_{2n}$ with the Chebyshev algorithm
2. set the closure $\bar{a}_n = \frac{1}{n} \sum_{k=0}^{n-1} \bar{a}_k$
3. compute $m_{2n+1}$ using the reverse Chebyshev algorithm
Practical computations

Closure, directly from the moments $m_{2n}$

1. compute the $(\bar{a}_k)^{n-1}_{k=0}$ and $(\bar{b}_k)^n_{k=0}$ from $m_{2n}$ with the Chebyshev algorithm

2. set the closure $\bar{a}_n = \frac{1}{n} \sum_{k=0}^{n-1} \bar{a}_k$

3. compute $m_{2n+1}$ using the reverse Chebyshev algorithm

Eigenvalues of the system

eigenvalues of the two following Jacobi matrices:

\[
\begin{pmatrix}
\bar{a}_0 & \sqrt{\bar{b}_1} \\
\sqrt{\bar{b}_1} & \bar{a}_1 \\
& \ddots \\
& & \bar{a}_{n-2} & \sqrt{\bar{b}_{n-1}} \\
& & & \sqrt{\bar{b}_{n-1}} & \bar{a}_{n-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\bar{a}_0 & \sqrt{\bar{b}_1} \\
\sqrt{\bar{b}_1} & \bar{a}_1 \\
& \ddots \\
& & \bar{a}_{n-2} & \sqrt{\bar{b}_{n-1}} \\
& & & \sqrt{\bar{b}_{n-1}} & \bar{a}_{n-1}
\end{pmatrix}
\]
Properties - Practical computations

Practical computations

Closure, directly from the moments $m_{2n}$

1. compute the $(\tilde{a}_k)_{k=0}^{n-1}$ and $(\tilde{b}_k)_n^{n}$ from $m_{2n}$ with the Chebyshev algorithm
2. set the closure $\tilde{a}_n = \frac{1}{n} \sum_{k=0}^{n-1} \tilde{a}_k$
3. compute $m_{2n+1}$ using the reverse Chebyshev algorithm

Eigenvalues of the system

The eigenvalues of the two following Jacobi matrices:

$$
\begin{pmatrix}
\tilde{a}_0 & \sqrt{\tilde{b}_1} \\
\sqrt{\tilde{b}_1} & \tilde{a}_1 & \sqrt{\tilde{b}_2} \\
& \ddots & \ddots & \ddots \\
& \sqrt{\tilde{b}_{n-2}} & \tilde{a}_{n-2} & \sqrt{\tilde{b}_{n-1}} \\
& & \sqrt{\tilde{b}_{n-1}} & \tilde{a}_{n-1}
\end{pmatrix},
\begin{pmatrix}
\tilde{a}_0 & \sqrt{\tilde{b}_1} \\
\sqrt{\tilde{b}_1} & \tilde{a}_1 & \sqrt{\tilde{b}_2} \\
& \ddots & \ddots & \ddots \\
& \sqrt{\tilde{b}_{n-1}} & \tilde{a}_{n-1} & \sqrt{2n+1 \tilde{b}_n} \\
& & \sqrt{2n+1 \tilde{b}_n} & \tilde{a}_n
\end{pmatrix}
$$

Reconstruction

A reconstruction as a sum of weighted Dirac delta function corresponds to the closure. The abscissas and weights can be easily computed from the $(\tilde{a}_k, \tilde{b}_k)_{k=0,\ldots,n}$. 
## Outline

1. **Introduction**
   - Context
   - Moment method
   - Hyperbolicity

2. **QMOM**
   - Principle of the method
   - Hyperbolicity

3. **HyQMOM**
   - First version of HyQMOM
   - New HyQMOM closure
   - Properties - Practical computations

4. **Results**
   - Configuration
   - Results

5. **Conclusion, Perspectives**
The 1D Riemann problem

**Configuration**

**The 1D Riemann problem**

**Problem at the kinetic level**

Two homogeneous sprays, with Gaussian distribution and infinite Stokes, crossing.

Problem at the kinetic level

\[
\begin{align*}
\partial_t f + \partial_x (vf) &= 0, \\
f(v; 0, x) &= \mathcal{M}_\sigma(v - \bar{u}(x))
\end{align*}
\]

with \( \sigma = 1/3 \)

\[
\bar{u}(x) = \begin{cases} 
1 & \text{if } x < 0, \\
-1 & \text{otherwise.}
\end{cases}
\]

Analytical solution \( f(t, x, v) = \mathcal{M}_\sigma(v - \bar{u}(x - vt)) = \begin{cases} 
\mathcal{M}_\sigma(v - 1) & \text{if } v > x/t, \\
\mathcal{M}_\sigma(v + 1) & \text{otherwise.}
\end{cases} \)

\( t = 0: \)

![Graphs showing the analytical solution at t=0 and the interaction at x=0.](image)
Introduction

QMOM

HyQMOM

Results

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The 1D Riemann problem

Problem at the kinetic level

Two homogeneous sprays, with Gaussian distribution and infinite Stokes, crossing.

\[ \partial_t f + \partial_x (v f) = 0, \]

\[ f(v; 0, x) = M_\sigma (v - \bar{u}(x)) \]

with \( \sigma = 1/3 \)

\[ \bar{u}(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{otherwise}. \end{cases} \]

Analytical solution

\[ f(t, x, v) = M_\sigma (v - \bar{u}(x - vt)) = \begin{cases} M_\sigma (v - 1) & \text{if } v > x/t, \\ M_\sigma (v + 1) & \text{otherwise}. \end{cases} \]

Moment problem

\[ \partial_t m_k + \partial_x m_{k+1} = 0, \quad k = 0, \ldots, 2n \]

with the initial condition for the standardized moments

\[ \rho(0, x) = 1, \quad u(0, x) = \bar{u}(x), \quad C_2(0, x) = \sigma, \]

\[ \begin{cases} S_{2k-1} = 0, \\ S_{2k} = (2k - 1)S_{2k-2}, \end{cases} \quad k = 2, \ldots, n \]

Numerical scheme: HLL [Harten et al., 1983]
The 1D Riemann problem - Results

moments - cases n=2,3,4

Good behavior on this hard test case.
The 1D Riemann problem - Results

standardized moments - cases n=2, 3, 4

Good behavior on this hard test case.
The 1D Riemann problem - Results

first moments - case n=10

Close to the analytical solution.
The 1D Riemann problem - Results

first standardized moments - case n=10

Close to the analytical solution.
The moment method seems to converge to the solution of the kinetic equation when the number of moments increases.
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5. Conclusion, Perspectives
Conclusion and Perspectives

Conclusion

- Closure inducing a global hyperbolicity
- Include the Maxwellian distribution
- Good behavior at the boundary of the moment space
- Efficient algorithm to compute the closure and the eigenvalues

THANK YOU FOR YOUR ATTENTION
Conclusion and Perspectives

Conclusion

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Perspectives

- 2D-3D version of the HyQMOM closure
Conclusion and Perspectives

Conclusion

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Perspectives

- 2D-3D version of the HyQMOM closure

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Three terms recurrence relation for a sequence \((Q_k)_{k \geq 0}\) of orthogonal polynomials relative to \(\langle \cdot, \cdot \rangle\):

\[
Q_{k+1}(x) = (x - a_k) Q_k(x) - b_k Q_{k-1}(x).
\]

Chebyshev algorithm \([\text{Chebyshev, 1859, Wheeler, 1974, Gautschi, 2004}]\)

\[
Z_{k,p} = \langle Q_k x^p \rangle
\]

\[
\begin{align*}
Z_{-1,p} &= 0, \quad Z_{0,p} = m_p \\
Z_{k+1,p} &= Z_{k,p+1} - a_k Z_{k,p} - b_k Z_{k-1,p}.
\end{align*}
\]

\[
\begin{align*}
b_0 &= m_0, \quad a_0 = \frac{m_1}{m_0}, \quad \forall k > 0 \quad b_k &= \frac{Z_{k,k}}{Z_{k-1,k-1}}, \quad a_k = \frac{Z_{k,k+1}}{Z_{k,k}} - \frac{Z_{k-1,k}}{Z_{k-1,k-1}},
\end{align*}
\]