On the Notion of Quantum Copulas

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Abstract

Working with multivariate probability distributions Sklar introduced the notion of copula in 1959, which turned out to be a key concept to understand the structure of distributions of composite systems. Roughly speaking Sklar proved that a joint distribution can be represented with its marginals and a copula. The main goal of this paper is to present a quantum analogue of the notion of copula. Our main theorem states that for any state of a composite quantum system there exists a unique copula such that the state and the copula are connected to each other by invertible matrices, moreover, they are both separable or both entangled. So considering copulas instead of states is a separability preserving transformation and efficiently decreases the dimension of the state space of composite systems. The method how we prove these results draws attention to the fact that theorem for states can be achieved by considering the states as quantum channels and using theorems for channels and different kind of positive maps.

1 Introduction

A famous problem was proposed by Fréchet in 1951 about multivariate probability distributions [5]. Sklar managed to obtain an outstanding result in Fréchet’s problem by introducing the notion of copula in 1959 [19], which is known as Sklar’s theorem. Roughly speaking Sklar proved that a joint distribution can be represented with its marginals and a copula. It means that the copula has all the dependence information of random variables [14].

First let us recall the Sklar’s theorem in the classical two dimensional setting. The theorem states that every multivariate distribution function \( H : \mathbb{R}^2 \to [0, 1] \) of a random vector \((X_1, X_2)\), that is

\[
H(x_1, x_2) = \Pr (X_1 \leq x_1, X_2 \leq x_2),
\]

(1)

can be expressed in terms of its marginals \( F_i(x_i) = \Pr (X_i \leq x_i) \) and a copula function \( C : [0, 1]^2 \to [0, 1] \), such that

\[
H(x_1, x_2) = C(F_1(x_1), F_2(x_2))
\]

(2)

holds, moreover, the copula function is unique on the set \( \text{Ran} F_1 \times \text{Ran} F_2 \). So in the continuous case the copula function is unique.

In noncommutative probability theory the above mentioned notions and constructions are difficult to work with, but the idea behind Sklar’s theorem can be adopted. In quantum mechanical setting,
an $n$-level quantum system can be described on an $n$ dimensional Hilbert space and the state space $\mathcal{D}_n$ of this quantum system can be identified by the set of self-adjoint, positive semi definite $n \times n$ matrices with trace one. Quantum mechanical axioms postulate [15] that the Hilbert space of a composite system is the tensor product of the Hilbert spaces associated with the components. If a composite quantum system consists of two subsystems with state spaces $\mathcal{D}_m$ and $\mathcal{D}_n$ then the state space of the composite system is $\mathcal{D}_{mn}$. Transferring the idea of Sklar’s theorem to quantum setting one finds that the role of classical distribution function (Eq. 1) is played by an element $\rho \in \mathcal{D}_{mn}$ and the marginals can be identified by states generated by partial trace, namely by $\text{Tr}_1 \rho \in \mathcal{D}_m$ and $\text{Tr}_2 \rho \in \mathcal{D}_n$. Equation (2) suggests to interpret the quantum copula as a state $C_\rho \in \mathcal{D}_{mn}$ with uniform marginals. It means that $\text{Tr}_1 C_\rho$ and $\text{Tr}_2 C_\rho$ should be the most mixed states, which can be written as $\text{Tr}_1 C_\rho = \frac{1}{m} I_m$ and $\text{Tr}_2 C_\rho = \frac{1}{n} I_n$, where $I_m$ denotes the $n \times n$ identity matrix. We will call these states to precopulas and denote their set by $\mathcal{C}_{mn}$.

We use the concept of Hilbert’s pseudometric and the corresponding Birkhoff–Hopf theorem for strictly positive quantum channels to construct a copula from a given state. It turns out that for any state $\rho \in \mathcal{D}_{mn}$ there are unique matrices $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$ such that $\rho' = (A^* \otimes B^*) \rho (A \otimes B)$. Our main theorem states that for any state $\rho \in \mathcal{D}_{mn}$ there exists a unique copula $\tilde{\chi} \in \mathcal{C}_{mn}$ such that $\rho$ and any representative $\chi$ of $\tilde{\chi}$ are connected, moreover, $\chi$ is separable if and only if $\rho$ is separable. So considering copulas instead of states is a separability preserving transformation, moreover the set of copulas are considerably lower dimensional than the set of states and thus for bipartite systems, we can give a simplified but exact picture of the separable-entangled structure of the state space. For instance, in the theory of quantum steering, which is a type of quantum correlation intermediate between entanglement and Bell nonlocality, the copula of a two-qubit state is proved to be useful because its “steering” properties are the same as for the state itself [3 12 15].

2 Preliminary lemmas and notations

We use $\mathcal{M}_n$ to denote $n \times n$ matrices, $\mathcal{M}_{n,sa}$ to denote the self-adjoint elements of $\mathcal{M}_n$, $\mathcal{M}_n^+$ the cone of self-adjoint positive semi-definite ones [15 15 16].

The state space of an $n$-level quantum system arises as the intersection of $\mathcal{M}_n^+$ and the hyperplane of trace one matrices, that is

$$\mathcal{D}_n = \{ \rho \in \mathcal{M}_n^+ \mid \text{Tr}(\rho) = 1 \}.$$  

Invertible elements in $\mathcal{M}_n^+$ are denoted by $\mathcal{M}_n^{++}$ and $\mathcal{D}_n$ stands for the set of invertible density matrices. If we would like to emphasize the underlying field, then we write $\mathcal{M}_n(\mathbb{K})$, $\mathcal{M}_{n,sa}(\mathbb{K})$, $\mathcal{M}_n^+(\mathbb{K})$, $\mathcal{M}_n^{++}(\mathbb{K})$, etc., where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The notation $\text{Eig}(A)$ stands for the set of eigenvalues of an element $A \in \mathcal{M}_n$.

Assume that a composite quantum system consists of two subsystems with state spaces $\mathcal{D}_m$ and $\mathcal{D}_n$ respectively. According to the axioms of quantum mechanics, the state space of the composite system is $\mathcal{D}_{mn}$ [13 16]. Product states in $\mathcal{D}_{mn}$ are of the form $\rho^1 \otimes \rho^2$, where $\rho^1 \in \mathcal{D}_n$ and $\rho^2 \in \mathcal{D}_m$.

**Definition 1.** A state $\rho \in \mathcal{D}_{mn}$ is separable (or classically correlated), if it can be written as a
convex combination of product states i.e.

$$\rho = \sum_{i=1}^{r} p_i \rho_1^i \otimes \rho_2^i,$$

where $\rho_1^i \in \mathcal{D}_n$, $\rho_2^i \in \mathcal{D}_m$ and $(p_i)_{1 \leq i \leq r}$ is a probability vector.

Non-separable states are called entangled. We use the notations $\mathcal{D}_{\text{sep}}^{mn}$ and $\mathcal{D}_{\text{ent}}^{mn}$ for the set of separable and entangled quantum states, respectively. Roughly speaking, separability means that the subsystems are in the state $\rho_1^i$ and $\rho_2^i$ with probability $p_i$. The information theoretical aspect of entanglement observed by Schrödinger himself: "Best possible knowledge of the whole does not include the best possible knowledge of its parts." \[16\].

Definition 2. A linear map $\Phi: \mathcal{M}_n \rightarrow \mathcal{M}_m$ is said to be

i. positive if $\Phi(\mathcal{M}_n^+) \subseteq \mathcal{M}_m^+$,

ii. strictly positive or positivity improving if $\Phi(\mathcal{M}_n^+ \setminus \{0\}) \subseteq \mathcal{M}_m^+$,

iii. completely positive if the map $I_{\mathcal{M}_k} \otimes \Phi: \mathcal{M}_k \otimes \mathcal{M}_n \rightarrow \mathcal{M}_k \otimes \mathcal{M}_m$ is positive for all $k \in \mathbb{N}$, where $I_{\mathcal{M}_k}$ is the $\mathcal{M}_k \rightarrow \mathcal{M}_k$ identity map.

Let us introduce the matrix units $(E_{ij})_{i,j=1,...,n}$, where $E_{ij}$ denotes the matrix whose entries are all 0 except in the $ij$-th cell, where it is 1. One can associate to every linear map $\Phi: \mathcal{M}_n \rightarrow \mathcal{M}_m$ an element of $\mathcal{M}_n \otimes \mathcal{M}_m \cong \mathcal{M}_{mn}$ in the following way

$$\rho_\Phi = \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}).$$

According to Choi’s theorem on completely positive maps, the map $\Phi \mapsto \rho_\Phi$ establishes a bijection between completely positive maps and positive semi-definite matrices. This correspondence is called Choi–Jamiolkowski isomorphism and its inverse is denoted by $\rho \mapsto \Phi_\rho$.

Theorem 1 (Choi–Jamiolkowski \[11\]). A linear map $\Phi: \mathcal{M}_n \rightarrow \mathcal{M}_m$ is completely positive if and only if the matrix $\rho_\Phi$ is positive semi-definite.

Let us denote the identity operator within $\mathcal{M}_m$ by $I_m$. Consider the map $\Psi_{nm}: \mathcal{M}_n \rightarrow \mathcal{M}_m$ defined by $\Psi_{nm}(X) = \text{Tr}(X) I_m$, which is obviously strictly positive. If $\Phi: \mathcal{M}_n \rightarrow \mathcal{M}_m$ is an arbitrary positive map then for every $\varepsilon > 0$ the map $\Phi + \varepsilon \Psi_{nm}$ is strictly positive. This observation leads to the following lemma.

Lemma 1. The set $\{ \Phi_\rho \mid \rho \in \mathcal{D}_{mn} \}$ consists of strictly positive maps.

Proof. If $\rho \in \mathcal{D}_{mn}$, then exists $\varepsilon > 0$ such that $\rho - \varepsilon I_{mn}$ is still positive semi-definite. By Choi–Kraus’ theorem, $\Phi_{\rho - \varepsilon I_{mn}}$ is completely positive, so positive. Using the previous observation we get that

$$\Phi_\rho = \Phi_{\rho - \varepsilon I_{mn}} + \varepsilon \Phi_{I_{mn}} = \Phi_{\rho - \varepsilon I_{mn}} + \varepsilon \Psi_{nm}$$

is strictly positive. \[\square\]

The Hilbert–Schmidt scalar product of matrices $X, Y \in \mathcal{M}_n$ is defined by

$$\langle X, Y \rangle = \text{Tr}(X^* Y).$$

For a linear map $\Phi: \mathcal{M}_n \rightarrow \mathcal{M}_m$ its adjoint $\Phi^*: \mathcal{M}_m \rightarrow \mathcal{M}_n$ with respect to the Hilbert–Schmidt scalar product is determined by the equation

$$\langle \Phi(X), Y \rangle = \langle X, \Phi^*(Y) \rangle \quad \forall X \in \mathcal{M}_n \forall Y \in \mathcal{M}_m.$$
Lemma 2. If $\Phi : M_n \rightarrow M_m$ is a strictly positive linear map, then its Hilbert–Schmidt adjoint $\Phi^* : M_m \rightarrow M_n$ is also strictly positive.

Proof. Let $\Phi : M_n \rightarrow M_m$ be a strictly positive linear map and $B \in M_m^+ \setminus \{0\}$ be an arbitrary matrix. Contrary to the lemma, assume that the matrix $\Phi^*(B)$ has $\lambda \leq 0$ as eigenvalue with unit length eigenvector $v$. Consider the orthogonal projection $P = v \langle v, \cdot \rangle$ onto the one-dimensional subspace generated by $v$. Note that because of strict positivity of $\Phi$, we have $\Phi(P) > 0$. Then we get the following contradiction

$$0 \geq \lambda = \text{Tr} (\lambda v \langle v, \cdot \rangle) = \text{Tr} (\Phi^*(B)v \langle v, \cdot \rangle) = \text{Tr} (\Phi^*(B)P) = \langle \Phi(P), B \rangle = \text{Tr}(\Phi(P)B) > 0.$$ 

To a matrix $A \in M_k$ one can associate the left and right multiplication operators $L_A, R_A : M_k \rightarrow M_k$ that act like $X \mapsto L_A(X) = AX$ 
$X \mapsto R_A(X) =XA$.

Lemma 3. Let $A \in M_n$, $B \in M_m$ and $\Phi : M_n \rightarrow M_m$ be a linear map. Then we have the following identities.

$$\rho_{L_B \circ \Phi} = (I_n \otimes B)\rho_{\Phi} \quad \rho_{R_B \circ \Phi} = \rho_{\Phi}(I_n \otimes B)$$
$$\rho_{\Phi \circ L_A} = (A^T \otimes I_m)\rho_{\Phi} \quad \rho_{\Phi \circ R_A} = \rho_{\Phi}(A^T \otimes I_m)$$

Proof. These identities can be verified by direct calculations.

$$\rho_{L_B \circ \Phi} = \sum_{i,j=1}^{n} E_{ij} \otimes B\Phi(E_{ij}) = (I_n \otimes B) \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}) = (I_n \otimes B)\rho_{\Phi}$$
$$\rho_{R_B \circ \Phi} = \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij})B = \left( \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}) \right) (I_n \otimes B) = \rho_{\Phi}(I_n \otimes B)$$

For $A = E_{kl}$, we have

$$\rho_{\Phi \circ L_A} = \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{kl}E_{ij}) = \sum_{j=1}^{n} E_{ij} \otimes \Phi(E_{kj}) = \sum_{i,j=1}^{n} E_{ik}E_{ij} \otimes \Phi(E_{ij}) = (A^T \otimes I_m)\rho_{\Phi}$$

and similarly

$$\rho_{\Phi \circ R_A} = \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}E_{kl}) = \sum_{i=1}^{n} E_{ik} \otimes \Phi(E_{il}) = \sum_{i,j=1}^{n} E_{ik}E_{lj} \otimes \Phi(E_{ij}) = \rho_{\Phi}(A^T \otimes I_m).$$

The general case follows from this by the linearity of maps $A \mapsto L_A$ and $A \mapsto R_A$. 

For the set of positive semi-definite matrices we have

i. $M_n^+ + M_n^+ \subseteq M_n^+$,

ii. $M_n^+ \cap -M_n^+ = \{0\}$,
iii. $\lambda \mathcal{M}_n^+ \subseteq \mathcal{M}_n^+$ holds for all $\lambda \geq 0$,

which means $\mathcal{M}_n^+$ is a cone in the real vector space of self-adjoint matrices i.e. $\mathcal{M}_n^+ \subseteq \mathcal{M}_{n,sa}$. The set $\mathcal{M}_n^+$ is a closed set with nonempty interior. Now, we introduce the Hilbert metric $[7]$ on this cone. In the following we present the definitions and theorems to our settings. The general form of these results can be found for example in [6, 11].

**Definition 3.** Birkhoff’s version of Hilbert’s pseudometric is defined on $\mathcal{M}_n^+ \setminus \{0\}$ by

$$d_H(A, B) = \log \left( \frac{M(A, B)}{m(A, B)} \right),$$

where

$$M(A, B) = \inf \{ \lambda \in \mathbb{R} \mid \lambda B - A \in \mathcal{M}_n^+ \},$$

$$m(A, B) = \sup \{ \lambda \in \mathbb{R} \mid A - \lambda B \in \mathcal{M}_n^+ \}.$$ 

The pseudometric $d_H$ is symmetric, satisfies the triangle inequality, for every $A, B \in \mathcal{M}_n^+ \setminus \{0\}$ we have $d_H(A, B) = 0$ if and only if there exists $c \in \mathbb{R}^+$ such that $A = cB$ and $d_H$ remains invariant under scaling by positive constant i.e. $d_H(A, B) = d_H(cA, B) = d_H(A, cB)$ for every $c \in \mathbb{R}^+$. So $d_H$ is a projective metric, it measures the distance of rays and not elements.

Notice that for $A, B \in \mathcal{M}_n^{++}$ positive-definite and invertible matrices, one has

$$\text{Eig} \left( B^{-1/2} AB^{-1/2} \right) = \text{Eig} \left( AB^{-1} \right),$$

where $\text{Eig}(X)$ denotes the set of eigenvalues of $X$. Therefore on the space of positive semi-definite matrices, the Birkhoff–Hilbert metric has the following simpler form

$$d_H(A, B) = \log \left( \frac{\max(\text{Eig} \left( AB^{-1} \right))}{\min(\text{Eig} \left( AB^{-1} \right))} \right).$$

Using this form, we can see that matrix inversion is an isometric map on $\mathcal{M}_n^{++}$ with respect to $d_H$. Similarly, for general positive matrices $A, B \in \mathcal{M}_n^+ \setminus \{0\}$ one has

$$d_H(A, B) = \begin{cases} 
\log \left( \frac{\max(\text{Eig} \left( AB^{-1} \right))}{\min(\text{Eig} \left( AB^{-1} \right) \setminus \{0\})} \right) & \text{if } \text{supp}A = \text{supp}B, \\
\infty & \text{if } \text{supp}A \neq \text{supp}B,
\end{cases}$$

where $\text{supp}X$ denotes the orthogonal subspace to $\text{Ker}X$ and the inverse is denoting the pseudo inverse (inverse on the support). From this form we can see that the pseudo inversion is an isometric map on $\mathcal{M}_n^+ \setminus \{0\}$ with respect to $d_H$. Moreover, the space $\mathcal{M}_n^+ \setminus \{0\}$ is complete with respect to $d_H$. We will intensively use these observations in the proof of Theorem 4.

**Definition 4.** Given a strictly positive linear mapping $\Phi : \mathcal{M}_{n,sa} \to \mathcal{M}_{m,sa}$, the projective diameter of $\Phi$ is

$$\Delta(\Phi) = \sup \{ d_H(\Phi(A), \Phi(B)) \mid A, B \in \mathcal{M}_n^{++} \}.$$ 

Moreover, the Birkhoff contraction ratio of $\Phi$ is given by

$$\delta(\Phi) = \inf \{ \lambda \in \mathbb{R}^+ \mid d_H(\Phi(A), \Phi(B)) \leq \lambda d_H(A, B) \text{ for all } A, B \in \mathcal{M}_n^{++} \}.$$ 

5
Birkhoff-Hopf theorem on linear maps between cones provides upper bound for the Birkhoff contraction ratio in terms of the projective diameter. This classical result was proved first by Birkhoff [1, 2] and similar theorems were discovered by Hopf [8, 9] who was apparently unaware of Birkhoff’s work. It is valid in much more general settings, but the following restricted version will be enough for our purposes.

**Theorem 2 (Birkhoff-Hopf).** If \( \Phi : M_{n,sa} \rightarrow M_{m,sa} \) is a strictly positive linear map, then
\[
\delta(\Phi) = \tanh\left(\frac{1}{4}\Delta(\Phi)\right),
\]
where \( \tanh(\infty) = 1 \).

Finally we cite Sinkhorn’s theorem, since our main theorem is a similar result in a similar setting.

**Theorem 3 (Sinkhorn [17]).** If \( A \) is an \( n \times n \) matrix with strictly positive elements, then there exist diagonal matrices \( D_1 \) and \( D_2 \) with strictly positive diagonal elements such that \( D_1 AD_2 \) is doubly stochastic. The matrices \( D_1 \) and \( D_2 \) are unique modulo multiplying the first matrix by a positive number and dividing the second one by the same number.

3 From states to copulas

We define an equivalence relation among bipartite quantum states such that each equivalence class consists only of either separable or entangled states.

**Definition 5.** We say that \( \rho \in D_{mn} \) and \( \rho' \in D_{mn} \) are connected if there exist invertible matrices \( A \in M_n \) and \( B \in M_m \) such that
\[
\rho' = (A^* \otimes B^*)\rho(A \otimes B).
\]

Connectivity of states \( \rho \) and \( \rho' \) is denoted by \( \rho \leftrightarrow \rho' \).

Easy to check the following observation about the connectivity of states.

**Lemma 4.** Connected states are mutually separable or entangled.

Now let us define the first candidates to copulas, namely those composite states which have uniform marginals.

**Definition 6.** A state \( \rho \in D_{nm} \) is called as a precopula if it has uniform marginals. The set of precopulas is denoted by \( C'_{mn} \), that is
\[
C'_{mn} = \left\{ \rho \in D_{mn} \bigg| \Tr_1(\rho) = \frac{1}{m} I_m, \ Tr_2(\rho) = \frac{1}{n} I_n \right\}.
\]

For states \( \rho, \rho' \in C'_{mn} \) we write \( \rho \sim \rho' \) if and only if there exist unitaries \( U \in M_n \) and \( V \in M_m \) such that
\[
\rho' = (U^* \otimes V^*)\rho(U \otimes V).
\]

Since, \( \rho \) is an equivalence relation on \( C'_{mn} \) we can take the quotient space
\[
C_{mn} = C'_{mn} / \sim.
\]

Elements of the quotient space \( C_{mn} \) are called copulas.
To prove our main result we use the fact that the marginals of a state can be obtained as applying the quantum channel and its adjoint defined by the state to the identity matrix.

**Lemma 5.** A density matrix \( \rho \in D_{nm} \) has marginals \( \rho^{(1)} \in D_n \) and \( \rho^{(2)} \in D_m \) if and only if
\[
\Phi_\rho(I_n) = \rho^{(2)} \quad \Phi^*_\rho(I_m) = \rho^{(1)}.
\]

**Proof.** The second marginal can be expressed as
\[
\rho^{(2)} = \text{Tr}_1 \rho = \sum_{i,j=1}^{n} \text{Tr}(E_{ij}) \Phi_\rho(E_{ij}) = \sum_{i=1}^{n} \Phi_\rho(E_{ii}) = \Phi_\rho(I_n).
\]
For the first marginal, we have
\[
\rho^{(1)} = \text{Tr}_2 \rho = \sum_{i,j=1}^{n} E_{ij} \text{Tr}(\Phi_\rho(E_{ij})) = \sum_{i,j=1}^{n} E_{ij} \text{Tr}(\Phi^*_\rho(I_m) E_{ij}),
\]
where the last expression is nothing else but the matrix \( \Phi^*_\rho(I_m) \) in the basis \( (E_{ij})_{i,j=1,...,n} \), which verifies the equality \( \rho^{(1)} = \Phi^*_\rho(I_m) \).

According to the previous lemma the density matrix \( \rho \in D_{nm} \) is a precopula if and only if \( \Phi_\rho(I_n) = \frac{1}{m} I_m \) and \( \Phi^*_\rho(I_m) = \frac{1}{n} I_n \). The next theorem is about a kind of fixed point property of strictly positive maps. We have modified and adopted to our settings the theorem which appeared first in [6].

**Theorem 4.** For any strictly positive linear map \( \Phi : M_n \to M_m \) there exist unique matrices \( \varphi_0 \in M^{++}_n \) and \( \varphi_1 \in M^{++}_m \) up to a positive multiplicative constant such that
\[
\Phi(\varphi_0^{-1}) = \frac{1}{m} \varphi_1^{-1}, \quad \Phi^*(\varphi_1) = \frac{1}{n} \varphi_0.
\]

**Proof.** We mimic the proof of Theorem 6 in [6]. Let us define the matrix inversion as function \( i : M^{++}_n \to M^{++}_n, i(\rho) = \rho^{-1} \) for every \( \rho \in M^{++}_n \). Consider the map \( T : M^{++}_n \to M^{++}_n \) defined as
\[
T = i \circ \Phi^* \circ i \circ \Phi.
\]
We show that \( T \) is a contraction with respect to the Hilbert metric. As it was mentioned the matrix inversion is an isometry. If \( \Phi : M^{++}_n \to M^{++}_m \) is a contraction then by Lemma 2 the same holds for \( \Phi^* : M^{++}_m \to M^{++}_n \), so in this case \( T \) is a composition of contractions and isometries.

To prove that \( \Phi \) is a contraction we estimate its projective diameter. Since the Hilbert metric is invariant under scaling by positive scalars we can restrict our attention to \( D_n \) which leads to the following estimation.
\[
\Delta(\Phi) = \sup \left\{ d_H(\Phi(A), \Phi(B)) | A, B \in M^{++}_n \right\} = \sup \left\{ d_H(\Phi(\rho), \Phi(\rho')) | \rho, \rho' \in D_n \right\}
\]
The map \( \Phi : M_n \to M_m \) is strictly positive thus it sends all the states in \( D_n \) to \( M^{++}_m \), so the map \( \Phi(\rho) \to d_H(\Phi(\rho), \Phi(\rho')) \) is continuous on the compact set \( D_n \times D_n \). This implies that \( \Delta(\Phi) < \infty \) and by Theorem 2, we have \( \delta(\Phi) = \tanh \left( \frac{1}{4} \Delta(\Phi) \right) < 1 \), which means that \( \Phi \) is a contraction.

By Banach fixed-point theorem we can conclude that there exists a unique fixed ray given by the state \( \varphi \in D_n \) such that \( T(\varphi) = \lambda \varphi \) for some \( \lambda > 0 \).
Let us define \( \varphi_1 = \frac{1}{m}(i \circ \Phi)(\varphi) \) and \( \varphi_0 = n\Phi^*(\varphi_1) \). In this case
\[
\varphi_0^{-1} = i(\varphi_0) = \frac{m}{n}(i \circ \Phi^* \circ i \circ \Phi)(\varphi) = \frac{m}{n}T(\varphi) = \frac{\lambda m}{n} \varphi,
\]
therefore
\[
\Phi(\varphi_0^{-1}) = \frac{\lambda m}{n} \Phi(\varphi) = \frac{\lambda}{n} \varphi_1^{-1}.
\]
Now we can determine the value of the parameter \( \lambda \) from the following equation.
\[
n = \text{Tr}(I_n) = \text{Tr}(\varphi_0^{-1} \varphi_0) = \text{Tr}(\varphi_0^{-1} n \Phi^*(\varphi_1)) = n \langle \varphi_0^{-1}, \Phi^*(\varphi_1) \rangle
\]
\[
= n \langle \Phi(\varphi_0^{-1}), \varphi_1 \rangle = n \text{Tr} \Phi(\varphi_0^{-1}) \varphi_1 = n \text{Tr} \frac{\lambda}{n} \varphi_1^{-1} \varphi_1 = \lambda m
\]
Since \( \lambda = \frac{m}{n} \), the equations \( \Phi(\varphi_0^{-1}) = \frac{1}{m} \varphi_1^{-1} \) and \( \Phi^*(\varphi_1) = \frac{1}{n} \varphi_0 \) are fulfilled. □

The next theorem formulates the main result of this paper. It establishes a separability-preserving transformation in composite systems between states and copulas. We prove that each state in \( D_{mn} \) is connected to a well-defined copula. This leads to the conclusion that copulas describe the dependence between the subsystems. We use the language of completely positive maps and the Choi-Jamiołkowski isomorphism to prove this.

**Theorem 5.** For any element \( \rho \in D_{mn} \), there exists a unique copula \( \tilde{\chi} \in C_{mn} \) such that for any representative \( \chi \) of \( \tilde{\chi} \) one has \( \rho \leftrightarrow \chi \).

**Proof.** Let \( \rho \in D_{mn} \) be arbitrary. By Lemma 1 \( \Phi_\rho : M_n \to M_m \) is strictly positive. Theorem 4 guarantees that there exist unique matrices \( \varphi_0 \in M_n^{++} \) and \( \varphi_1 \in M_m^{++} \) up to a positive multiplicative constant such that
\[
\Phi_\rho(\varphi_0^{-1}) = \frac{1}{m} \varphi_1^{-1}
\]
\[
\Phi_\rho^*(\varphi_1) = \frac{1}{n} \varphi_0.
\]
Taking any factorization \( \varphi_0 = \psi_0^* \psi_0 \), \( \varphi_1 = \psi_1^* \psi_1 \), for the map
\[
\Theta : M_n \to M_m \quad X \mapsto \psi_1 \Phi_\rho \left( \psi_0^{-1} X (\psi_0^{-1})^* \right) \psi_1^*
\]
we have
\[
\Theta(I_n) = \frac{1}{m} I_m
\]
\[
\Theta^*(I_m) = \frac{1}{n} I_n.
\]
Using Lemma 2 we conclude that the state \( \rho_\Theta \in D_{mn} \) has uniform marginals, so \( \rho_\Theta \in C_{mn}' \). According to Lemma 3
\[
\rho_\Theta = \left( (\psi_0^{-1})^T \otimes \psi_1 \right) \rho \left( (\psi_0^{-1})^T \otimes \psi_1 \right)^*.
\]
On the other hand, factorization of \( \varphi_0 \) and \( \varphi_1 \) is unique up to unitary equivalence. If \( \varphi_0 = \psi_0^* \psi_0 \), \( \varphi_1 = \psi_1^* \psi_1 \), then there exists unitaries \( U_0 \in M_n \) and \( U_1 \in M_m \) such that \( \psi_0' = U_0 \psi_0 \) and \( \psi_1' = U_1 \psi_1 \). The composite state corresponding to the transform defined by \( \varphi_0' \) and \( \varphi_1' \) is
\[
\left( (\psi_0'^{-1})^T \otimes \psi_1' \right) \rho \left( (\psi_0'^{-1})^T \otimes \psi_1' \right)^* = \left( (U_0^{-1})^T \otimes U_1 \right) \rho_\Theta \left( (U_0^{-1})^T \otimes U_1 \right)^*
\]
and therefore represents the same copula as \( \rho_\Theta \). □
The proof of the previous theorem shows that for any state $\rho \in \mathcal{D}_{nm}$ there are unique matrices $A$ and $B$ up to multiplication by unitaries such that $(A \otimes B)\rho (A \otimes B)^*$ is a precopula. Similar phenomena can be discovered in Sinkhorn’s theorem (Theorem 3), where for a matrix $D$ with strictly positive elements there exist diagonal matrices $A$ and $B$ up to a positive multiplicative constant (i.e., one can take the matrices $cA$ and $\frac{1}{c}B$ for $c \in \mathbb{R}^+)$ such that the matrix $ADB$ is doubly stochastic.

4 Conclusions

We introduced the notion of copula as an equivalence class of those states which have uniform marginals and we proved that for every state there exists a unique copula connected to the state. The construction of the copula of a given state uses Banach fixed-point theorem, so we do not have explicit formula for the corresponding copula, but numerically can be computed with arbitrary precision. We have found very fast convergence for randomly generated qubit-qubit states.

The following are examples for connecting open question to copulas. How one can reconstruct a state from its copula and its marginals? How one can parametrize the space of copulas? Which quantities are hereditary from states to copulas, namely for given states $D_1, D_2$ and the corresponding copulas $\chi_1, \chi_2$ for which quantity $H$ holds the equality $H(D_1, D_2) = H(\chi_1, \chi_2)$. Is it true for some kind of entropy or distance measure?

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