Tidal radii of main sequence stars - I. Physical tidal radius, semi-analytic model and their implications

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ABSTRACT

A star is tidally disrupted by a supermassive black hole when their separation is shorter than the so-called “tidal radius". This quantity is often estimated on an order-of-magnitude basis without reference to the star’s internal structure. Using MESA models for main sequence stars and fully general relativistic dynamics, we find the physical tidal radius for complete disruption $R_t$ for a $10^6 M_\odot$ black hole (BH). We find that across a factor $\sim 20$ in stellar mass $M_\star$, i.e., $0.15 M_\odot \lesssim M_\star \lesssim 3 M_\odot$, $R_t \simeq 27 r_g$, where $r_g$ is the BH’s gravitational radius. When comparing the physical tidal radius with the commonly used order of magnitude estimate $r_t$, we find that $R_t \simeq 1.05 - 1.45 r_t$ for stars with $0.15 M_\odot \leq M_\star \leq 0.5 M_\odot$, but between $0.5 M_\odot$ and $1 M_\odot$, $R_t$ drops to $\simeq 0.45 r_t$, and it remains at this value up to at least $10 M_\odot$. The near-constancy of $R_t$ implies a weaker dependence of the full disruption rate on stellar mass than when predicted with $r_t$. The characteristic width of the energy distribution of the debris $\Delta E$ ranges from $\simeq 1.2 \Delta E$ for low-mass stars to $\simeq 0.35 \Delta E$ for higher-mass stars, where $\Delta E = GM_{BH} R_\star / R_t^2$. We present analytic fits for the $M_\star$ dependence of $R_t$ and $\Delta E$; these fits lead to analytic expressions for the time of peak mass fallback rate and the maximal mass fallback rate. Our results also bear on the fraction of events leading to fast or slow circularization, as well as on the character of the tidal event occurring when the remnant of a partial disruption returns to the black hole. Using a semi-analytic model, we show that $R_t$ is primarily determined by the star’s central density rather than its mean density. For high-mass stars, the full tidal disruption rate is roughly $1/4$ the partial disruption rate, while this ratio is close to unity for low-mass stars.

Keywords: black hole physics − gravitation − hydrodynamics − galaxies:nuclei − stars: stellar dynamics

1. INTRODUCTION

Supermassive black holes (SMBHs) reside in the nuclei of virtually every nearby massive galaxy (Kormendy & Ho 2013). The orbits of stars around the central BH are stochastically perturbed by weak gravitational encounters with other stars. Occasionally these perturbations place stars on orbits taking them so close to the BH that they are tidally disrupted, losing part or all of their mass in a tidal disruption event (TDE). Roughly half the mass torn off the star swings far out from the stellar orbit’s pericenter and then returns. The energy it releases as it falls deeper into the black hole potential generates a luminous flare.

Many examples of tidal disruption events have now been seen. Since the detection of the first TDE candidates (Komossa & Bade 1999) in the ROSAT all-sky survey (Truemper 1982), greatly improved searches have been conducted, including X-ray surveys such as the XMM-Newton slew survey (Saxton et al. 2008) and UV/optical surveys, e.g., the GALEX Deep Imaging Survey (Gezari et al. 2006), Pan-STARRS (Chambers et al. 2016), PTF (Law et al. 2009) and ASAS-SN (Holoien et al. 2016). From these, dozens of transients
have been identified as TDE candidates (Komossa 2015 and van Velzen 2018). In the near future, this number is likely to grow rapidly with detections by ongoing surveys like the Zwicky Transient Facility (ZTF) (Graham et al. 2019) and upcoming surveys, e.g., the eROSITA All-Sky Survey (Merloni et al. 2012) and the Large Synoptic Survey Telescope (LSST) (LSST Science Collaboration et al. 2009).

The “tidal radius” is the distance close enough to the BH that a star suffers a TDE. More precisely, this term is usually reserved for the largest stellar pericenter such that a star is completely disrupted. Because the orbital speed at the tidal radius for an event involving a main-sequence star and a supermassive black hole is much larger than stellar orbital speeds in the surrounding galaxy, the cross section for a tidal disruption scales linearly with the tidal radius; it is therefore a crucial parameter for determining the event rate. The tidal radius also affects the properties of TDE outcomes because the orbital energies of bound and unbound stellar debris are characterized by the energy spread within the star when the star reaches the tidal sphere of the BH (Rees 1988).

Despite the many ways in which the tidal radius is determined, it is often determined to only order-of-magnitude precision. In this traditional method, a comparison between the star’s self-gravity and tidal gravity at the stellar surface leads to (Hills 1988)

\[ r_t = \left( \frac{M_{BH}}{M_*} \right)^{1/3} R_* \]

\[ \simeq 47 \left( \frac{M_{BH}}{10^6 M_\odot} \right)^{-2/3} \left( \frac{M_*}{1 M_\odot} \right)^{-1/3} \left( \frac{R_*}{1 R_\odot} \right) r_g, \]

where \( M_* \) and \( R_* \) are the stellar mass and radius, respectively. \( M_{BH} \) is the mass of the BH and \( r_g \) is the gravitational radii of the BH, \( r_g = GM_{BH}/c^2 \). Phinney (1989) suggested an improved estimate for \( r_t \) with an extra prefactor \((k/f)^{1/6}\) that takes into account the internal structure of the star. Here, \( k \) is the star’s apsidal motion constant and \( f \) is its binding energy in units of \( GM_*^2 / R_* \). For fully convective (radiative) stars, \((k/f)^{1/6} = 0.82 \) (0.52). However, this extra factor has not been properly tested and is not widely used in TDE studies. This conventional tidal distance \( r_t \) undoubtedly sets a characteristic length scale for interesting tidally-driven behavior, but it is highly unlikely to be the tidal radius.

A number of previous studies have attempted to study tidal disruptions more quantitatively, either analytically (e.g., Kosovichev & Novikov 1992; Diener et al. 1995) or numerically (e.g., Guillochon & Ramirez-Ruiz 2013; Mainetti et al. 2017; Goicovic et al. 2019; Law-Smith et al. 2019; Golightly et al. 2019). One drawback shared by all these studies is that they treat the problem in Newtonian gravity, even though the relevant pericenter distances \( r_p \) are typically only a few tens of \( r_g \). Another is that many of these studies (all but the last three cited) model the stars as polytropes. These most recent studies by Goicovic et al. (2019), Law-Smith et al. (2019) and Golightly et al. (2019) adopt a more realistic stellar model, one generated by the stellar evolution code MESA. However, they consider a limited selection of stellar masses (only \( 1 M_\odot \) in Goicovic et al. 2019, \( 1 M_\odot \) and \( 3 M_\odot \) at several ages in Law-Smith et al. 2019, \( 0.3 M_\odot, 1 M_\odot \) and \( 3 M_\odot \) at three different ages in Golightly et al. 2019). Goicovic et al. (2019) found a physical tidal radius slightly less than \( r_t/2 \); Law-Smith et al. (2019) focused primarily on stellar age-dependence, rather than distinguishing full and partial disruptions; Golightly et al. (2019) focused on the fallback rate’s time-dependence for a single value of \( r_p/r_t \). These examples immediately raise the question of how large the correction is for stars of other masses, with other internal density profiles.

The primary goal of this series of studies is to accurately identify the physical tidal radius, which we denote by \( R_t \), for realistic main-sequence (MS) stars over a wide range of masses (\( 0.15 M_\odot \leq M_* \leq 10 M_\odot \)). In so doing, we are also able to determine quantitatively the stellar-mass dependence of numerous other TDE properties. For each of eight masses in this mass range, we conducted a series of simulations of stars on parabolic orbits (\( 1 - e \simeq 10^{-8} \) for eccentricity \( e \)) approaching non-spinning \( 10^6 M_\odot \) black holes with pericenters differing from one simulation to the next by \( 0.05-0.2 r_t \). This procedure allowed us to closely bracket \( R_t \) for each mass, as well as to study the properties of both partial disruptions (\( r_p > R_t \)) and more deeply-penetrating total disruptions (\( r_p < R_t \)). These simulations (using HARM3D: Noble et al. 2009), treated the hydrodynamics in terms of full general relativity, including fully relativistic tidal stresses. Their initial conditions were taken from the stellar evolution code MESA and correspond to an age halfway through the main-sequence lifetime of stars born with solar abundances. Self-gravity was calculated by a (Newtonian) Poisson solver in a frame co-moving with the star’s center-of-mass and whose metric is exactly Minkowski at the center-of-mass.

In this paper, the first in a series of four papers, we consider a canonical case: a non-spinning \( 10^6 M_\odot \) SMBH. Here we give an overview of this series’ principal findings: the physical tidal radius as a function of stellar mass (Section 2), the energy scale of stellar debris (Section 3), a semi-analytic model that predicts both \( R_t \)
and the remnant mass as a function of $r_p$ for partial disruptions (Section 4). The new functional dependences on stellar mass we have uncovered (for which we provide analytic fits) have numerous implications and lead to revised estimates of the peak fallback rate, the peak fallback time, TDE rates, and the relative frequency of full and partial disruptions, as well as new predictions about the properties of unbound ejecta and stellar remnants. These are summarized in section 5. We conclude with a summary of our findings in Section 6.

The other three papers in this series provide details. In Ryu et al. (2019a) (Paper 2), we present detailed descriptions of our simulation setup and the results for full disruptions; Ryu et al. (2019b) (Paper 3) reports our results relevant to partial disruptions; and Ryu et al. (2019c) (Paper 4) shows how these quantitative results depend on black hole mass due to the changing magnitude of relativistic corrections to the tidal stress.

2. PHYSICAL TIDAL DISTANCE $R_T$

We have determined the physical tidal radius for MS stars disrupted by a non-spinning $10^6 M_\odot$ BH as a function of mass for eight different masses, presenting it in units of both $r_g$ and $r_t$ in Table 1 and Figure 1. Hereafter all masses are measured in solar mass and stellar radii in solar radius.

The contrast in $R_t$ across the range of mass is much smaller than predicted by the traditional order-of-magnitude estimate $r_t$. Moreover, unlike $r_t$, $R_t$ is not monotonic with $M_\ast$: it is almost constant from $M_\ast \simeq 0.15$ to 3. The average value of $R_t$ over this mass range is

$$R_t(0.15 \leq M_\ast \leq 3) \simeq 26.9 \, r_g; \quad (2)$$

(see Figure 1). The largest and smallest values of $R_t/r_g$ are 22 (for $M_\ast = 0.15$) and 33 (for $M_\ast = 3$), so that the maximum departure is only 20% either up or down.

On the other hand, the ratio $\Psi \equiv R_t/r_t$ has a sharp transition near $0.4 \lesssim M_\ast \lesssim 1$. For low-mass stars ($M_\ast \leq 0.5$), which are predominantly convective, $\Psi \simeq 1 - 1.45$. For higher mass stars ($M_\ast \geq 1$), which are predominantly radiative, $\Psi \simeq 0.45$ (see right panel of Figure 1).

The behavior of $\Psi$ for all stars is well-described by an analytic formula (shown by the dotted curve in the right panel of Figure 1),

$$\Psi(M_\ast) = \frac{1.47 + \exp[(M_\ast - 0.669)/0.137]}{1 + 2.34 \exp[(M_\ast - 0.669)/0.137]} \quad (3)$$

The large coefficient of $M_\ast$ in the exponentials conveys how sharp the transition is from low-mass to high-mass stars.

At both mass extremes, $\Psi$ becomes nearly constant. At the high-mass end, $\Psi$ asymptotically approaches $\simeq 0.43$. In the low-mass limit, $\Psi$ increases only slowly toward lower mass. However, it does so at a higher value ($\simeq 1.4$) than found by previous work studying $\gamma = 5/3$ polytropes ($\simeq 1.1$: Guillochon & Ramirez-Ruiz 2013; Mainetti et al. 2017), even though this should be a very good approximation for these nearly isentropic stars. In Paper 4 we argue that this contrast is due to our use of fully relativistic tidal stresses because the offset de-

![Table 1. The physical tidal radii $R_t$ for MS stars encountering a $10^6 M_\odot$ non-spinning black hole. $r_t'$ and $r_t''$ are $(k/f)^{1/6}$-corrected $r_t$, which is $(k/f)^{1/6}(M_{BH}/M_t)^{1/3} R_t$, but with different choices of $R_t$. Here, $(k/f)^{1/6}$ refers to a correction factor suggested by Phinney (1989). For these estimates, we take $(k/f)^{1/6} = 0.82$ for $M_t < 1$ and 0.52 for $M_t \geq 1$. For $r_t'$, $R_t = M_t^{1-\xi}$ where $\xi = 0.2$ for $M_t < 1$ and 0.4 for $M_t \geq 1$ (Kippenhahn & Weigert 1994). For $r_t''$, we use the stellar radius of our MESA stellar models. In the last column $\Delta E/\Delta t$ is the ratio of the actual characteristic debris energy width (containing 90% of the total mass) to the order-of-magnitude estimate.

| $M_\ast$ | $r_t/r_g$ | $r_t'/r_g$ | $r_t''/r_g$ | $R_t/r_g$ | $\Psi = R_t/r_t$ | $\Xi = \Delta E/\Delta t$ |
|---------|---------|-----------|-----------|-----------|----------------|----------------|
| 0.15    | 0.17    | 15.2      | 12.5      | 22.1 ± 0.8 | 1.45 ± 0.05   | 0.67           |
| 0.30    | 0.30    | 21.2      | 17.4      | 26.5 ± 1.1 | 1.25 ± 0.05   | 0.75           |
| 0.40    | 0.37    | 23.9      | 19.6      | 30.0 ± 1.4 | 1.25 ± 0.05   | 0.75           |
| 0.50    | 0.46    | 27.4      | 22.5      | 28.9 ± 1.4 | 1.05 ± 0.05   | 0.85           |
| 0.70    | 0.69    | 36.4      | 29.8      | 24.6 ± 1.4 | 0.675 ± 0.025 | 1.09           |
| 1.0     | 1.0     | 47.5      | 24.7      | 22.5 ± 1.2 | 0.475 ± 0.025 | 1.47           |
| 3.0     | 2.4     | 79.8      | 41.5      | 33.9 ± 2.0 | 0.425 ± 0.025 | 1.83           |
| 10      | 5.6     | 123       | 45.3      | 64.7      | 52.1 ± 3.1    | 1.79           |

1 For explanatory convenience, we categorize stars into “low-mass” ($M_\ast \leq 0.5$) and “high-mass” ($M_\ast \geq 1$) based on the properties of TDE outcomes. Consequently, these mass ranges may be different from those typically used in stellar evolution studies.
creases for smaller $M_{\text{BH}}$ and increases for larger $M_{\text{BH}}$ as would be expected for a relativistic effect.

In addition to presenting our results on the physical tidal radius as a function of stellar mass, Table 1 also shows the three other extant means for estimating $R_t$ for any given $M_*$: $r_1$ and two implementations of the $(k/f)^{1/6}$ correction factor suggested by Phinney (1989). The version we call $r'_1$ applies the correction factor to the broken power-law parameterization (Kippenhahn & Weigert 1994) of the mass-radius relation; the version we call $r''_1$ uses the mass-radius relation found from MESA for stars halfway through their main-sequence lifetimes. As illustrated by the left panel of Figure 1, $r_1$ can be substantially discrepant from the actual value of $R_t$, too small for low-mass stars, too large for high-mass stars. Comparing $r'_1$ and $r''_1$ to our results for $R_t$, it is clear that the $(k/f)^{1/6}$ factor captures the qualitative fact of the abrupt decline in $\Psi$ near $M_* \simeq 1$, but, $r''_1$, its presumably more accurate form, underestimates $\Psi$ for low-mass stars by $\simeq 50\%$ and overestimates it for high-mass stars by $\simeq 20\%$. Curiously, the broken power-law fit to the mass-radius relation diminishes the error at both ends of the mass spectrum.

The nominal tidal radius $r_t$ in units of $r_g$ scales $\propto M_{\text{BH}}^{-2/3}$. In Newtonian gravity, this relation captures the only physical length scale, $R_*$, and we might therefore expect that $\Psi$ is independent of $M_{\text{BH}}$. However, relativistic gravity introduces a new length scale, $r_g$. As we have already discussed for the case of $M_{\text{BH}} = 10^6$, the comparison of $R_t$ found from Newtonian tides applied to polytropes (17 $r_g$ for $M_* = 0.15$) to $R_t$ found from relativistic tides applied to convective stars (22 $r_g$ for $M_* = 0.15$) suggests that relativistic gravity alters $\Psi$ by $10-30\%$ when $R_t \simeq 20 r_g$. One might then expect that Newtonian scaling would be a fairly good approximation for smaller black hole masses. On the other hand, for larger black hole masses, relativistic effects should become more important, potentially introducing a significant implicit dependence of $\Psi$ on black hole mass. This effect is explored in detail in Paper 4.

3. ENERGY DISTRIBUTION OF THE TIDAL DEBRIS

In the conventional picture of TDEs (Rees 1988), the characteristic magnitude $\Delta E$ of the tidal debris energy is determined by the star’s spin-up as it passes inside the tidal radius, and its distribution within the range from $-\Delta E$ to $+\Delta E$ is assumed to be flat. In the traditional formalism,

$$\Delta \epsilon = \frac{G M_{\text{BH}} R_*}{r_t^2} = \left( \frac{R_*}{r_g} \right) \left( \frac{r_t}{r_g} \right)^2 c^2$$

is chosen as the unit of debris energy and it is assumed that $\Delta E = \Delta \epsilon$. However, in the previous section, we have numerically determined the physical tidal radius $R_t$. It is then natural to introduce a corresponding characteristic energy scale,

$$\Delta E = \frac{\Delta \epsilon}{\Psi^2} = \frac{G M_{\text{BH}} R_*}{R_t^2}.$$  (5)

As expected, we find the energy distribution $dM/dE$ to be quite closely symmetrical around $E = 0$. To measure $\Delta E$ from our data, we define it as the energy range containing 90% of the total mass (the cumulative mass

![Figure 1. The physical tidal radius $R_t(M_*)$. (Left) $R_t/r_g$. The mean value and the extreme range of $R_t/r_g$ for $0.15 \leq M_* \leq 3$ are marked as a horizontal line and a shaded region. (Right) $\Psi \equiv R_t/r_t$. The numerical values (filled points) and the analytic fit given in Equation 3 (red dotted curve).](image-url)
distribution $M(> E)$ for unbound ejecta is shown in Figure 2) $\Delta E$ can then be measured in either of the two energy units, $\Delta \epsilon$ or $\Delta \mathcal{E}$. For fully-disrupted low-mass stars, $\Delta E/\Delta \epsilon \simeq 0.7$–1, but it increases to $\simeq 1.5$–2 for high-mass stars. In terms of $\Delta \mathcal{E}$, the spread of debris energy is $\simeq 1$–1.4 for low-mass stars and $\simeq 0.35$ for high-mass stars (Figure 3). Thus, neither analytic estimate for the energy scale, Equation 4 or 5, is correct to better than a factor of two, with the actual energy generally lying between the two of them.

Fitting our numerical results we provide functional relationships between $\Delta E$ and $M_\star$ in terms of either $\Delta \epsilon$ or $\Delta \mathcal{E}$ (See Figure 3). For the former, we find

$$\Xi \equiv \frac{\Delta E(M_\star)}{\Delta \epsilon} = \frac{0.620 + \exp \left(\left[ M_\star - 0.674 \right]/0.212 \right)}{1 + 0.553 \exp \left(\left[ M_\star - 0.674 \right]/0.212 \right)}.$$  \hspace{1cm} (6)

(Both expressions are illustrated in Figure 3.)

In the low-mass case, the energy distribution $dM/dE$ drops extremely sharply for $|E| > \Delta E$, as has previously been found from simulations using polytropic stars (Cheng & Evans 2013; Guillochon & Ramirez-Ruiz 2013; Yalinewich et al. 2019). However, in the high-mass case $dM/dE \propto e^{-k|E|/\Delta \epsilon}$ with $k \simeq 2.5$–3 for $|E| > \Delta E$ (Paper 2).

In strong partial disruptions, $\Delta E$ for a given mass star is very similar to its value for a full disruption, but it decreases modestly for weaker events. The greater contrast with complete disruptions is that $dM/dE$ is significantly depressed for energies near 0, with the width of the depression expanding for weaker disruptions (Paper 3). This latter effect is the primary cause for the contrast in the cumulative distributions shown in the two panels of Figure 2.

4. SEMI-ANALYTIC MODEL FOR PHYSICAL TIDAL RADIUS AND REMNANT MASS

We have shown numerically that the traditional order-of-magnitude model for tidal radii needs to be corrected with order-unity coefficients in order to match quantitatively the behavior of realistic main sequence stars. Here we show how a natural generalization of the original tidal radius argument, augmented by a single free parameter, can be used both to deepen our understanding of the order-unity coefficients and to predict how
much mass is lost in a partial disruption. A qualitative version of this argument was made by Li et al. (2002), but was never applied to actual stellar structures.

Suppose that the mass outside radius $R$ is tidally stripped when the self-gravity at that location is less than a factor $\zeta$ times the tidal force applied at that radius. In other words,

$$\frac{GM(R)}{R^2} < \frac{\zeta GM_{BH} R}{r_p^3},$$  \hspace{1cm} (8)

where $M(R)$ is the enclosed mass inside a spherical radius $R$ and $r_p$ is the pericenter distance of an orbit. Replacing $r_p$ with $\psi r_1$ and using the definition of $r_1$ (Equation 1), we rewrite Equation 8 (with equality) as

$$\frac{\zeta}{3} \psi \left[ \frac{M(R)}{M_*} \right] \left[ \frac{R^2}{r_1^3} \right] = 1.$$  \hspace{1cm} (9)

Defining $\rho_* = M_* (4\pi R_*^2/3)^{-1}$ and $\bar{\rho}(R) = M(R)/(4\pi R^2/3)^{-1}$ we finally have

$$\psi = \frac{r_p}{r_1} = \left[ \frac{\rho_*}{\bar{\rho}(R)} \right]^{1/3}.$$

Thus, for a given pericenter distance $r_p$ and density profile (e.g., MESA density profile), we can, by solving Equation 10, immediately determine the radius $R$ beyond which the mass of the star is lost due to tidal forces. The enclosed mass $M(R)$ at the radius $R$ corresponds to the remnant mass.

In searching for $R_t$, we ran simulations for numerous partial disruptions with varying $r_p$ and studied the properties of the partially disrupted stars including the remnant mass. We will discuss our results in detail in Paper 3, but here we merely use the results. Using the remnant mass from the partial disruption simulations, we estimate $[\bar{\rho}_*/\bar{\rho}(R)]^{1/3}$ at each $r_p/r_1$, as shown in Figure 4. More explicitly, using the MESA enclosed mass profile $M(R)$ for the star, we found $R$ such that the enclosed mass equals the remnant mass at each $r_p/r_1$, then estimated $[\bar{\rho}_*/\bar{\rho}(R)]^{1/3}$ at that $R$ as $(M_*/R_*^2)/(M(R)/R^2)$. As expected from Equation 10, $[\bar{\rho}_*/\bar{\rho}(R)]^{1/3}$ and $r_p/r_1$ are linearly correlated; the slope of this correlation $\zeta \approx 9.8$ for all remnants. This result confirms the validity of Equation 10. It also means that the remnant mass produced when a star passes a black hole with a given pericenter outside $R_t$ can be easily determined by use of MESA models for the original structure of the star.

The limit of $R \rightarrow 0$ corresponds to a complete disruption. In that case, $\bar{\rho} = \rho_c = \lim_{R \rightarrow 0} \rho(R)$. In other words, $\Psi$ can be determined solely from the ratio between the star’s central density $\rho_c$ and its mean density $\bar{\rho}_*$ with the correction factor $\zeta$, i.e.,

$$\Psi \approx \left[ \frac{\bar{\rho}_*}{\rho_c} \right]^{1/3}.$$  \hspace{1cm} (11)

It follows that $R_t$ is determined solely from the central density $\rho_c$:

$$R_t = \Psi r_1 \approx \zeta^{1/3} \left( \frac{\bar{\rho}_*}{\rho_c} \right)^{1/3} r_1,$$

$$\approx \left[ \frac{M_{BH}}{\rho_c} \right]^{1/3},$$  \hspace{1cm} (12)

where $\zeta = [3\zeta/(4\pi)]^{1/3} \approx 1.32$.

The opposite limit, the pericenter distance outside which no mass is lost, is also instructive. To explore it, we begin by dividing Equation 10 by $\Psi$ using Equation 11. This yields

$$\frac{r_p}{R_t} = \left[ \frac{\rho_c}{\bar{\rho}(R)} \right]^{1/3}.$$  \hspace{1cm} (13)

The maximum pericenter for losing any mass is then

$$R_t = \left( \frac{\rho_c}{\bar{\rho}_*} \right)^{1/3} R_t.$$  \hspace{1cm} (14)

However, we already know that $R_t$ is related to $\rho_c$ through equation 12. This fact brings us to an expression for the same result in terms of $r_1$:

$$\hat{R}_t = \zeta^{1/3} r_1 \approx 2.1 r_1.$$  \hspace{1cm} (15)
independent of $M_*$. This result is supported by a more detailed comparison with our simulations (Paper 3).

As a final use of this simple model, we find a link between our empirical fit for remnant mass (Paper 3),

$$\frac{M_{\text{rem}}}{M_*} = 1.0 - \left( \frac{r_p}{R_\star} \right)^{-3}, \quad (16)$$

and that predicted by the semi-analytic model, which gives us a direct relationship between three dimensionless spatial scales, e.g., $r_p/R_\star(= \Psi)$, $r_p/R_\star(= \psi)$ and $R/R_\star$. Inserting $M_{\text{rem}}/M_\star$ above into Equation 9 and rearranging the terms, we have:

$$\frac{R}{R_\star} = \left( \frac{\psi^3 - \Psi^3}{\zeta} \right)^{1/3}. \quad (17)$$

This relation behaves correctly in simple limits: at $\psi = \Psi, R = 0$, and at $\psi = \Psi, R/R_\star \approx 1$ with no more than 5% errors. Thus, by combining our purely empirical fit for the mass-loss–pericenter relation in a partial disruption with the result of our physically-motivated semi-analytic model, we find a simple expression for the radius within the star such that the mass outside it is lost in a partial disruption.

5. IMPLICATIONS

Our analytic fits to $\Psi(M_\star)$ and $\Xi(M_\star)$ enable us to translate the simple conventional formulae linking stellar mass and black hole mass to properties more closely linked to observations into more accurate expressions. The quantitative contrast with the older formalism can have significant implications for observations; we consider a few here.

5.1. Orbital properties of stellar debris

The energy spread directly determines the mass return rate and the time of peak mass return for the bound debris as well as the energy and velocity of the unbound debris.

5.1.1. Peak mass return: time and rate

The mass fallback rate of stellar debris on ballistic orbits is (Rees 1988; Phinney 1989),

$$\dot{M}_{\text{fb}} = \frac{dM}{dE} \left| \frac{dE}{dt} \right| = \frac{2\pi GM_{\text{BH}}}{3} \frac{dM}{dE} t^{-5/3}, \quad (18)$$

For a flat energy distribution $dM/dE$, the peak fallback rate emerges when the most tightly bound stellar debris returns to the BH. Although our energy distributions for low-mass stars exhibit a steep drop at $|E| \geq \Delta E$, the same distributions for high-mass stars show interestingly-wide tails.

As a result, the shapes of the mass fallback curves are different for low-mass and high-mass stars. Fallback rates from low-mass disruptions show sharp peaks close to the time at which 5% of the total mass (10% of the bound mass) has returned. On the other hand, the fallback rate curves for high-mass stars show relatively flat peaks, beginning at the time when 5% of the total mass has returned. If we define $t_{\text{peak}}$ as the time at which the most bound 10% returns, it is exactly the time at which debris with $\Delta E$ returns, $P_{\Delta E} = \frac{3}{\sqrt{2}} GM_{\text{BH}} \Delta E^{-3/2}$.

Using the fitting formula for $\Xi(M_\star)$ (Equation 6), $t_{\text{peak}}$ can be written as

$$t_{\text{peak}} \approx P_\Delta = \frac{\pi}{\sqrt{2}} \frac{GM_{\text{BH}}}{\Delta E^{3/2}},$$

$$\approx 0.11 \text{ yr} \Xi^{-3/2} M_\star^{-1} R_\star^{3/2} \left( \frac{M_{\text{BH}}}{10^6} \right)^{1/2}. \quad (19)$$

We can rewrite Equation 18 as:

$$\dot{M}_{\text{fb}} = \left( \frac{M_\star}{3P_\Delta} \right) \left( \frac{dM}{dE} \Delta E \right) \left( \frac{t}{P_\Delta} \right)^{-5/3}. \quad (20)$$

The peak fallback rate $\dot{M}_{\text{peak}}$ at $t = t_{\text{peak}}$ is

$$\dot{M}_{\text{peak}} \approx f \frac{M_\star}{P_{\text{peak}}},$$

$$\approx 1.49 M_\odot \text{ yr}^{-1} \left( \frac{f}{0.5} \right) \Xi^{3/2} M_\star^2 R_\star^{-3/2} \left( \frac{M_{\text{BH}}}{10^6} \right)^{-1/2}, \quad (21)$$

where the correction factor $f$ accounts for the different shape of the energy distribution near the tails. We take $f = 1$ for $M_\star \leq 0.5$ and 0.5 for $M_\star > 0.5$.

To show the full dependence of these quantities on $M_\star$, it is useful to make a power-law fit to $R_\star(M_\star)$. From $0.15 \leq M_\star \leq 3$, the MESA data are very well fit by $R_\star \propto M_\star^{0.68}$. Using this relation, we can write,

$$t_{\text{peak}} \propto \Xi^{-3/2} M_\star^{0.32} M_{\text{BH}}^{1/2}, \quad (22)$$

$$\dot{M}_{\text{peak}} \propto \Xi^{3/2} M_\star^{0.68} M_{\text{BH}}^{-1/2}. \quad (23)$$

As shown in Figure 5, $\Psi, \Delta E/\Delta \xi$ and $\Xi^{-3/2}$ have very similar functional behavior. All three have nearly-constant values between 1 and 2 for $M_\star < 0.5$, drop by a factor of a few from $M_\star = 0.5$ to $M_\star = 1$, and then have nearly-constant values between 0.3 and 0.5 for $M_\star > 1$. Because $t_{\text{peak}}$ follows $\Xi^{-3/2}$ multiplied by a small power of $M_\star$, its $M_\star$ dependence is tilted upward only gently relative to that of the correction factor. On the other hand, $\dot{M}_{\text{peak}}$ is proportional to the inverse of the correction factor multiplied by $M_\star^{0.68}$. Both factors in $\dot{M}_{\text{peak}}$ rise with increasing $M_\star$, giving it a nearly linear overall dependence on $M_\star$. 
Relativistic effects do not alter the dependence on $M_*$, but they change both $R_t$ and $\Delta E$. Through the latter, they alter the dependence of $t_{\text{peak}}$ and $M_{\text{peak}}$ on $M_{\text{BH}}$ (see Paper 4). The corresponding correction term, which we denote by $\Xi_2$, can be incorporated as a multiplicative term like $\Xi_2 t_{\text{peak}} \propto \Xi_2^{-1/2} M_{\text{BH}}^{1/2}$ and $M_{\text{peak}} \propto \Xi_2^{-3/2} M_{\text{BH}}^{-1/2}$. $\Xi_2^{-3/2}$ increases monotonically with $M_{\text{BH}}$, rising from 1 to 1.8 over the span $10^6 \leq M_{\text{BH}} \leq 10^7$. Thus, for $M_{\text{BH}} = 10^7$, $t_{\text{peak}}$ is longer and $M_{\text{peak}}$ is smaller by a factor of 2 than an extrapolation from $M_{\text{BH}} = 10^6$ assuming the Newtonian scaling relation. We will discuss these effects in more detail in Paper 4.

5.1.2. Unbound debris energy and speed at infinity

The energy of the most highly-bound matter determines the time of peak mass-return; the energy of the most highly-unbound matter determines the fastest speed of the ejecta that never return to the black hole, as well as the total amount of energy available for deposition in surrounding gas. From Equations 5 and 7, we find that the total energy is $\sim 10^{49} - 10^{51}$ erg, and the greatest speed at infinity for the bulk of the ejecta mass is $\sim 6 \times 10^4 \Xi^{1/2} M_*^{1/3} R_*^{-1/2} (M_{\text{BH}}/10^6)^{1/6} \text{ km s}^{-1}$. Over the wide range of masses in which $R_* \sim M_*^{0.88}$, this speed is very weakly dependent on both $M_*$ and $M_{\text{BH}}$. Because the energy scale is comparable to that of a supernova remnant, one might expect that when the unbound debris shocks against whatever gas surrounds the black hole, there would be radio emission. Such emission has, in fact, been seen in several cases, with the most useful data coming from ASASSN-14li: Alexander et al. 2016; van Velzen et al. 2016. Using the equipartition formalism for synchrotron self-absorbed spectra (Barniol Duran et al. 2013), Krolik et al. (2016) found that the linear scale of the emission region in ASASSN-14li expanded at a constant speed $\simeq (1.45 - 2) \times 10^4 \text{ km s}^{-1}$ (see also Alexander et al. 2016) for a comparable estimate). Because these speeds are similar to those expected for the fastest-moving unbound ejecta, Krolik et al. (2016) suggested that the unbound ejecta are, indeed, responsible.

Our results, if anything, strengthen that conclusion for two reasons. First, the characteristic energy spread we find is larger than the conventional estimate for all $M_* > 0.7$, and larger by a factor $\simeq 1.8$ for $M_* \gtrsim 3$. Second, we also find that for $M_* > 0.7$, the energy distribution has a significant tail extending beyond $\Delta E$ (see Figure 2). Consequently, if this event involved a higher-mass star, the encounter would not be subject to the constraint of a highly-penetrating event suggested by Yalinewich et al. (2019).

5.2. TDE rate

The physical tidal radius directly affects the TDE rate. As mentioned earlier, the cross section for an encounter with a pericenter less than $R_t$ is linearly proportional to $R_t$, so the rate of full tidal disruptions for a given mass is likewise linearly proportional to $R_t$.

5.2.1. TDE rate for different stellar mass

We have shown that $R_t$ is much more weakly dependent on $M_*$ than $r_t$ is. This fact alters the relative rates of full tidal disruptions predicted for different stellar masses. The fraction of total disruptions involving stars with $M_* \lesssim 0.5$ ($R_t > r_t$) should be rather larger than would be predicted on the basis of the order-of-magnitude criterion $r_t$ while the fraction of total disruptions involving high-mass stars is smaller.

5.2.2. Relative frequency between full and partial disruption

The ratio $R_{4L}/(R_t - R_4)$ is the ratio between the cross sections for full and partial disruptions. For low-mass stars it is $\simeq 1$, but for high-mass stars it is $\simeq 0.25$. In other words, for low-mass stars there should be roughly as many partial disruptions (including rather weak ones) as total disruptions, but for high-mass stars, the partial disruption rate should be $\simeq 4 \times$ the total disruption rate. Because $R_t \simeq 2.1 r_t$ for all mass stars, the left panel of Figure 1 shows that although the rate of full disruptions per star varies rather little as a function of mass, the rate of partial disruptions per star increases significantly for higher-mass stars.

5.3. Fraction of events with rapid or slow circularization

![Figure 5.](image-url)
We can also think about the fraction of total disruptions with short pericenter distances at which relativistic effects can potentially play a major role, e.g., significant relativistic apsidal precession by, say, \( \gtrsim \pi/3 \) at \( r_p \lesssim 10 r_g \). Such a large apsidal precession angle can move the inter-stream point of intersection from near the debris orbit apocenters (\( \sim 100 R_t \)) to close to \( \sim R_t \). As a result, when \( r_p \gtrsim 10 r_g \), circularization and accretion inflow can be substantially slower than for smaller pericenters (Shiokawa et al. 2015). Because the cross section for an encounter with \( < R_t \) is linear in \( r_p \), we find that when stars with \( 0.3 < M_\star < 3 \) pass close to a non-spinning \( 10^6 M_\odot \) SMBH, 40% of full disruptions circularize rapidly. This fraction drops quickly for smaller mass black holes, but increases toward unity as \( M_{BH} \) approaches \( 10^7 \).

5.4. Multiple TDEs of a MS star

Due to slightly asymmetric mass-loss, the specific orbital energy of remnants with respect to the black hole becomes very slightly non-zero (\( \simeq 10^{-3} \Delta E \)). We refer to those with positive specific energy as “unbound” population and those with negative energy as “bound” population.

When the remnant is unbound, its speed at infinity is generally \( \simeq 100 - 300 \text{ km s}^{-1} \) (see Paper 3). At this speed, it can escape the influence radius \( r_{\text{in}} \) of the central BH, but reach only to distances a few to ten times farther. The orbits of bound remnants are very nearly parabolic (\( 1 - e \simeq 10^{-5} \)) and have semimajor axes \( a \simeq 0.03 - 0.5 \text{ pc} < r_{\text{in}} \). Since their angular momenta are much smaller than those for a circular orbit, there is a possibility that they can approach the BH sufficiently close for another tidal disruption event when they return back to the galactic center.

The pericenter distance is determined primarily by their specific angular momentum when returning to the black hole. However, the angular momenta of bound and unbound remnants are subject to stellar encounters. In general, for unbound populations and less tightly bound remnants (e.g., \( a \simeq 0.5 \text{ pc} \)), the specific angular momentum, and therefore the star’s next pericenter distance, is usually increased by a factor of a few (see Paper 3). However, more tightly bound stars generally suffer little angular momentum change and therefore return to the black hole with the same pericenter as during the initial encounter.

Whether the former pericenter is close enough for a full disruption upon return depends on the thermal and rotational state of the remnant. Most remnants have higher entropy and faster rotation than main sequence stars of the same mass (See Paper 3). The global thermal time scale of tightly bound remnants is relatively long compared to their traveling times. So tightly bound stars would return back to the BH without significant changes in their internal structures. These stars are then more likely to suffer a strong tidal event than less tightly bound stars, suggesting that multiple TDEs of a MS star are a plausible scenario. This also implies that some of the observed TDE candidates are not necessarily from ordinary MS stars, but possibly from stars which had been partially disrupted once or more. Disruptions of such strongly disturbed stars could differ in interesting ways from those of main sequence stars.

6. SUMMARY

In this paper we have presented the principal results from our suite of relativistic tidal disruption simulations using realistic main sequence stellar structures for stars of many different masses. Subsequent papers in this series will fill in the details, both of our methods and of our results.

However, several broad themes can be seen clearly from this summary of our calculations. The first is that the order-of-magnitude estimate \( r_t \) for the maximum pericenter yielding a full disruption can easily be wrong by factors of order unity. It is too small for low-mass stars and too large for high-mass stars. The polytropic approximation is appropriate for fully-convective stars, but even for them relativistic corrections are noticeable for black hole masses of \( 10^6 \) and will be greater for larger masses. Because the sense of the correction to \( r_t \) runs opposite to the dependence of stellar radius on stellar mass, the net result of these combined effects is a physical tidal radius that is roughly constant over the range \( M_\star \simeq 0.1 - 3 \), a range spanning the overwhelming majority of all stars.

A second is that the commonly-used estimate for the spread in energy of the tidal debris also requires quantitative adjustment for stars that are not fully-convective, i.e., \( M_\star > 0.5 \). The actual energy spread can be nearly twice as great as previously thought. Because the characteristic orbital period of the debris scales as the -3/2 power of the most-bound material, the mass-return timescale can be shorter than thought by a factor as much as \( \approx 3 \) and the maximum rate of mass-return correspondingly higher.

For both the physical tidal radius and the energy spread of the debris, we have developed easy-to-use analytic expressions closely fitting our simulation results that describe our corrections to the order-of-magnitude expressions in common use. With these, it is easy to calculate significantly more accurate estimates for the time of peak mass-return and the maximal rate of mass-
return, both quantities of strong interest to observations.

The tidal radius and the characteristic energy spread of debris together determine a great deal about the outcome of these events. The former controls estimates of rates—we now see, for example, that the probability per unit time that a star is completely disrupted has only a weak dependence on stellar mass, contrary to the conclusion associated with use of $r_t$ for the physical tidal radius. The spread in energy determines the maximum speed at infinity for the bulk of the unbound ejecta, controlling how rapidly the ejecta rush out through surrounding gas, shocking it and possibly engendering radio emission.

Lastly, the approach to complete disruption, i.e., how the magnitude of mass loss increases with diminishing pericenter, can be used for further insight into the process of tidal disruption. A simple semi-analytic model applied to our results reveals that, perhaps not surprisingly in hindsight, the primary quantity used in the traditional order-of-magnitude estimate, the mean density of the star, is closely related to the largest pericenter at which some matter is torn off the star, but it is the central density that determines the radius within which a star is fully disrupted.

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