MINIMAL PRIME AGES, WORDS AND PERMUTATION GRAPHS

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Abstract. This paper is a contribution to the study of hereditary classes of finite graphs. We classify these classes according to the number of prime structures they contain. We consider such classes that are minimal prime: classes that contain infinitely many primes but every proper hereditary subclass contains only finitely many primes. We give a complete description of such classes. In fact, each one of these classes is a well-quasi-ordered age and there are uncountably many of them. Eleven of these ages are almost multichainable: they remain well-quasi-ordered when labels in a well-quasi-ordering are added, hence have finitely many bounds. Five ages among them are exhaustible. Among the remaining ones, only countably many remain well-quasi-ordered when one label is added, and these have finitely many bounds (except for the age of the infinite path and its complement). The others have infinitely many bounds.

Except for six examples, members of these ages we characterize are permutation graphs. In fact, every age which is not among the eleven ones is the age of a graph associated to a uniformly recurrent word on the integers.

A description of minimal prime classes of posets and bichains is also provided.

Our results support the conjecture that if a hereditary class of finite graphs does not remain well-quasi-ordered by adding labels in a well-quasi ordered set to these graphs, then it is not well-quasi-ordered if we add just two constants to each of these graphs.

Our description of minimal prime classes uses a description of minimal prime graphs [61] and previous work by Sobrani [65, 66] and the authors [45, 51] on properties of uniformly recurrent words and the associated graphs. The completeness of our description is based on classification results of Chudnovsky, Kim, Oum and Seymour [18] and Malliaris and Terry [42].

1. Introduction and presentation of the results

This paper is a contribution to the study of hereditary classes of finite graphs. We classify these classes according to their proper subclasses. With this idea, our simplest classes are those who contain finitely many proper subclasses, hence these classes are finite. At the next level, there are the classes who contain infinitely many proper subclasses, but every proper subclass contains only finitely many. That is such classes are infinite but proper subclasses are finite. It is a simple exercise based on Ramsey’s theorem that there are only two such classes: the class of finite cliques and the class of their complements. Pursuing this idea further, we would like to attach a rank to each class, preferably an ordinal. If we do this, it turns out that a class has a rank if and only if the set of its proper subclasses ordered by set

Date: June 6, 2022.
2000 Mathematics Subject Classification. 05C30, 06F99, 05A05, 03C13.
Key words and phrases. ordered set; relational structure; indecomposability; primality; graph; permutation; permutation graph; age; hereditary class; well-quasi-order, uniformly recurrent sequences.
*Corresponding author. Supported by Canadian Defence Academy Research Program, NSERC and LABELX MILYON (ANR-10-LABX-0070) of Université de Lyon within the program "Investissements d’Avenir (ANR-11-IDEX-0007)" operated by the French National Research Agency (ANR).
inclusion is well founded. This latter condition amounts to the class being well-quasi-ordered (this follows from Higman’s characterization of well-quasi-orders [26]). This puts forward the importance of well-quasi-ordered hereditary classes.

A basic construction of well-quasi-ordered hereditary classes of finite graphs and more generally of finite structures goes as follows: chose a finite hereditary class of finite binary structures and take its closure under lexicographical sums over elements of the class. The fact that this latter class is well-quasi-ordered is a consequence of a theorem of Higman [26]. An important property of such a class is that it contains only finitely many prime structures (see Definition 2). A concrete example of such a class is the class of finite cographs (prime structures in this class have cardinality at most two). A natural question then arises: under what conditions a class that contains infinitely many primes is well-quasi-ordered?

Among hereditary classes which contain infinitely many prime members, we show that there are minimal ones with respect to set inclusion (Theorem 13). Furthermore, we show that the minimal ones are well-quasi-ordered ages (Theorem 14). We obtain some general results that we are able to refine in some special cases like graphs, ordered sets, and bichains. We give a complete description of minimal prime ages of graphs (Theorem 14), of posets, and bichains (Corollary 46). It turns out that there are $2^{\aleph_0}$ such ages (Corollary 41). Eleven of these ages are almost multichainable; they remain well-quasi-ordered when labels in a well-quasi-ordering are added, five being exhaustible. Among the remaining ones, countably many remain well-quasi-ordered when one label is added and these have finitely many bounds (except for the age of the infinite path and its complement). The others have infinitely many bounds (Theorem 17).

Except for six examples, members of these ages we characterize are permutation graphs. In fact, every age which is not among the eleven ones is the age of a graph associated to a uniformly recurrent word on the integers (this is a consequence of Theorems 36, 37 and 39). This result supports the conjecture that if a hereditary class of finite graphs does not remain well-quasi-ordered by adding labels in a well-quasi-ordered set to these graphs, then it is not well-quasi-ordered if we add just two constants to each of these graphs.

Our description of minimal prime classes uses a description of minimal prime graphs [61] and previous work by Sobrani [65, 66] and the authors [45, 51] on properties of uniformly recurrent words and the associated graphs. The completeness of our description is based on classification results of Chudnovsky, Kim, Oum and Seymour [18] and Malliaris and Terry [42].

2. Organisation of the paper

In section 3 we present some prerequisites on graphs, posets and words. In section 4 we consider binary relational structures with a finite signature, we give the definition of a minimal prime hereditary class of binary structures and prove their existence. Section 4 contains also the proof of Theorem 14 (see subsection 4.2) and a proof of Theorem 18 (see subsection 4.2.1). In Section 5 we start with the classification results of Chudnovsky, Kim, Oum and Seymour [18] and Malliaris and Terry [42]. Then, we present our main results on minimal prime ages. In Section 5.5 we look at the number of bounds of our minimal prime ages. In section 6 we provide a proof of Theorem 46 and a characterization of order types of realizers of transitive orientations of 0-1 graphs. In section 7 we characterize the modules of a 0-1 graph. We prove among other things, that if $G_\mu$ is not prime, then $\mu$ contains large
factors of 0’s or 1’s. Section 3 is devoted to the study of the relation between embeddings of 0-1 words and their corresponding graphs. Results obtained in this section will be used in the proof of Theorem 37. In section 9 we give a proof of Theorem 37. Theorem 39 is proved in section 10. In section 11 we investigate bounds of 0-1 graphs and give a proof of Theorems 47.

3. Prerequisites

3.1. Graphs, posets and relations. This paper is mostly about graphs and posets. Sometimes, we will need to consider binary relational structures, that is ordered pairs $R := (V, (\rho_i)_{i \in I})$ where each $\rho_i$ is a binary relation or a unary relation on $V$. The set $V$, sometimes denoted by $V(R)$, is the domain or base of $R$. The sequence $s := (n_i)_{i \in I}$ of arity $n_i$ of $\rho_i$ is the signature of $R$ (this terminology is justified since we may identify a unary relation on $V$, that is a subset $U$ of $V$, with the binary relation made of pairs $(u, u)$ such that $u \in U$). We denote by $\Omega_s$ the collection of finite structures of signature $s$. In the sequel we will suppose the signature finite, i.e. $I$ finite. For example, we will consider bichains, i.e., relational structures $R := (V, (\leq', \leq''))$ made of a set $V$ and two linear orders $\leq'$ and $\leq''$ on $V$.

The framework of our study is the theory of relations as developed by Fraïssé and subsequent investigators. At the core is the notion of embeddability, a quasi-order between relational structures. We recall that a relational structure $R$ is embeddable in a relational structure $R'$, and we set $R \preceq R'$, if $R$ is isomorphic to an induced substructure of $R'$. Several important notions in the study of these structures, like hereditary classes, ages, bounds, derive from this quasi-order. For example, a class $\mathcal{C}$ of relational structures, of signature $s$, is hereditary if it contains every relational structure that embeds into a member of $\mathcal{C}$. The age of a relational structure $R$ is the class $\text{Age}(R)$ of all finite relational structures, considered up to isomorphy, which embed into $R$. This is an ideal of $\Omega_s$ that is a nonempty, hereditary and up-directed class $\mathcal{C}$ (any pair of members of $\mathcal{C}$ are embeddable in some element of $\mathcal{C}$). A characterization of ages was given by Fraïssé (see chapter 10 of [23]). Namely, a class $\mathcal{C}$ of finite relational structures is the age of some relational structure if and only if $\mathcal{C}$ is an ideal of $\Omega_s$. We recall that a bound of a hereditary class $\mathcal{C}$ of finite relational structures (e.g. graphs, ordered sets) is any relational structure $R \not\in \mathcal{C}$ such that every proper induced substructure of $R$ belongs to $\mathcal{C}$. For a wealth of information on these notions see [23].

3.1.1. Graphs. Unless otherwise stated, the graphs we consider are undirected, simple and have no loops. That is, a graph is a pair $G := (V, E)$, where $E$ is a subset of $[V]^2$, the set of 2-element subsets of $V$. Elements of $V$ are the vertices of $G$ and elements of $E$ its edges. The complement of $G$ is the graph $\overline{G}$ whose vertex set is $V$ and edge set $\overline{E} := [V]^2 \setminus E$. If $A$ is a subset of $V$, the pair $G|_A := (A, E \cap [A]^2)$ is the graph induced by $G$ on $A$. A path is a graph $P$ such that there exists a one-to-one map $f$ from the set $V(P)$ of its vertices into an interval $I$ of the chain $\mathbb{Z}$ of integers in such a way that $(u, v)$ belongs to $E(P)$, the set of edges of $P$, if and only if $|f(u) - f(v)| = 1$ for every $u, v \in V(P)$. If $I = \{1, \ldots, n\}$, then we denote that path by $P_n$; its length is $n - 1$ (so, if $n = 2$, $P_2$ is made of a single edge, whereas if $n = 1$, $P_1$ is a single vertex.

3.1.2. Posets. Throughout, $P := (V, \preceq)$ denotes an ordered set (poset), that is a set $V$ equipped with a binary relation $\preceq$ on $V$ which is reflexive, antisymmetric and transitive. We say that two elements $x, y \in V$ are comparable if $x \preceq y$ or $y \preceq x$, otherwise, we say they
are incomparable. The dual of $P$ denoted $P^*$ is the order defined on $V$ as follows: if $x, y \in V$, then $x \leq y$ in $P^*$ if and only if $y \leq x$ in $P$.

According to Szpilrajn [67], every order $\preceq$ on a set $V$ has a linear extension, that is a linear (or total) order $\preceq$ on the $V$ such that $x \preceq y$ whenever $x \leq y$, for all $x, y \in V$. Let $P := (V, \preceq)$ be a poset. A realizer of $P$ is a family $\mathcal{L}$ of linear extensions of the order of $P$ whose intersection is the order of $P$. Observe that the set of all linear extensions of $P$ is a realizer of $P$. The dimension of $P$, denoted $\dim(P)$, is the least cardinal $d$ for which there exists a realizer of cardinality $d$ [19]. It follows from the Compactness Theorem of First Order Logic that an order is intersection of at most $n$ linear orders ($n \in \mathbb{N}$) if and only if every finite restriction of the order has this property. Hence, the class of posets with dimension at most $n$ is determined by a set of finite obstructions, each obstruction is a poset $Q$ of dimension $n + 1$ such that the deletion of any vertex of $Q$ leaves a poset of dimension $n$; such a poset is said critical. For $n \geq 2$ there are infinitely many critical posets of dimension $n + 1$. For $n = 2$, critical posets of dimension three (and hence finite comparability graphs of critical posets of dimension three) were characterized by Kelly [30]. Beyond, the task is considered as hopeless.

### 3.1.3. Comparability and incomparability graphs

The comparability graph, respectively the incomparability graph, of a poset $P := (V, \preceq)$ is the graph, denoted by $\text{Comp}(P)$, respectively $\text{Inc}(P)$, with vertex set $V$ and edges the pairs $\{u, v\}$ of comparable distinct vertices (that is, either $u < v$ or $v < u$) respectively incomparable vertices. A graph $G := (V, E)$ is a comparability graph if the edge set is the set of comparabilities of some order on $V$. From the Compactness Theorem of First Order Logic, it follows that a graph is a comparability graph if and only if every finite induced subgraph is a comparability graph. Hence, the class of comparability graphs is determined by a set of finite obstructions. The complete list of minimal obstructions was determined by Gallai [28] (see [10] for an English translation). The list can also be found in [69] Figures 4(a) and 4(b).

### 3.1.4. Permutation graphs

A graph $G := (V, E)$ is a permutation graph if there is a linear order $\preceq$ on $V$ and a permutation $\sigma$ of $V$ such that the edges of $G$ are the pairs $\{x, y\} \in [V]^2$ which are reversed by $\sigma$.

Denoting by $\preceq_\sigma$ the set of oriented pairs $(x, y)$ such that $\sigma(x) \leq \sigma(y)$, the graph is the comparability graph of the poset whose order is the intersection of $\preceq$ and the dual of $\preceq_\sigma$. Hence, a permutation graph is the comparability graph of an order intersection of two linear orders, that is the comparability graph of an order of dimension at most two [19]. The converse holds if the graph is finite. As it is well known, a finite graph $G$ is a permutation graph if and only if $G$ and $\overline{G}$ are comparability graphs [19]; in particular, a finite graph is a permutation graph if and only if its complement is a permutation graph.

The comparability graph of an infinite order which is intersection of two linear orders is not necessarily a permutation graph. A one way infinite path is a permutation graph, but the complement of this infinite path is not a permutation graph. There are examples of infinite posets which are intersection of two linear orders and whose comparability and incomparability graphs are not permutation graphs. For an example see Figure 8. However, via the Compactness Theorem of First Order Logic, an infinite graph is the comparability graph of a poset intersection of two linear orders if an only if each finite induced subgraph
is a permutation graph (sometimes these graphs are called permutation graphs, while there is no possible permutation involved). For more about permutation graphs, see [32], [70].

3.1.5. Initial segment, ideal. An initial segment of a poset $P := (V, \leq)$ is any subset $I$ of $V$ such that $x \in V$, $y \in I$ and $x \leq y$ imply $x \in I$. An ideal is any nonempty initial segment $J$ of $P$ which is up-directed (that is $x, y \in J$ implies $x, y \leq z$ for some $z \in J$). If $X$ is a subset of $V$, the set $\downarrow X := \{y \in V : y \leq x \text{ for some } x \in X\}$ is the least initial segment containing $X$, we say that it is generated by $X$. If $X$ is a singleton, say $X = \{x\}$, we denote by $\downarrow x$, instead of $\downarrow X$, this initial segment and say that it is principal. We denote by $I(P)$, resp. $\text{Id}(P)$, the set of initial segments, respectively ideals, of $P$, ordered by set inclusion.

3.2. Well-quasi-order. We present the notion of well-quasi-order and introduce the notion of better-quasi-order; we refer to [39]. A poset is well-founded if every nonempty subset has some minimal element. Such a poset has a level decomposition $(P_\alpha)_{\alpha \in \text{Ch}(P)}$ indexed by ordinal numbers. Level $P_\alpha$ is the set of minimal elements of $P \setminus \bigcup\{P_\beta : \beta < \alpha\}$ and $h(P)$, the height of $P$, is the least ordinal $\alpha$ such that $P_\alpha = \emptyset$. The poset is level-finite if each level $P_\alpha$ is finite. A quasi-ordered-set (quoset) $Q$ is well-quasi-ordered (w.q.o.), if every infinite sequence of elements of $Q$ contains an infinite increasing subsequence. If $Q$ is an ordered set, this amounts to say that every nonempty subset of $Q$ contains infinitely many minimal elements (this number being non zero). Equivalently, $Q$ is w.q.o. if and only if it contains no infinite descending chain and no infinite antichain.

3.2.1. Better-quasi-order. Proofs that some classes of countable structures are w.q.o. under embeddability may require a strengthening of that notion, e.g; the notion of better-quasi-order (b.q.o) (see Subsection 3.1.4). We just recall that b.q.o.’s are w.q.o’s. As for w.q.o.’s, finite sets and well-ordered sets are b.q.o.’s, finite unions, finite products, subsets and images of b.q.o.s by order preserving maps are b.q.o.’s. (see [23] for more). Nash-Williams 1965 [44] p.700, asserted that “one is inclined to conjecture that most w.q.o. sets which arise in a reasonably ‘natural’ manner are likely to be b.q.o.” It is not known if the answer is positive for hereditary classes of finite graphs. The first classes to consider are probably those which are minimal prime. Due to the description of these classes, the answer is positive.

3.2.2. Labelled classes. Among classes of structures which are w.q.o. under the embeddability quasi-order some remain w.q.o. when the structures are labelled by the elements of a quasi-order. Precisely, let $C$ be a class of relational structures, e.g., graphs, posets, etc., and $Q$ be a quasi-ordered set or a poset. If $R \in C$, a labelling of $R$ by $Q$ is any map $f$ from the domain of $R$ into $Q$. Let $C \cdot Q$ denotes the collection of $(R, f)$ where $R \in C$ and $f : R \rightarrow Q$ is a labelling. This class is quasi-ordered by $(R, f) \leq (R', f')$ if there exists an embedding $h : R \rightarrow R'$ such that $f(x) \leq (f' \circ h)(x)$ for all $x \in R$. We say that $C$ is very well-quasi-ordered (vw.q.o. for short) if for every finite $Q$, the class $C \cdot Q$ is w.q.o. The class $C$ is hereditary w.q.o. if $C \cdot Q$ is w.q.o. for every w.q.o. $Q$. The class $C$ is n-w.q.o. if for every $n$-element poset $Q$, the poset $C \cdot Q$ is w.q.o. The class $C$ is $n^-$-w.q.o. if the class $C_{n^-}$ of $(R, a_1, \ldots, a_n)$ where $R \in C$ and $a_1, \ldots, a_n \in R$ is w.q.o. We do not know if these four notions are different. In the case of posets covered by two chains (that is of width at most two) we proved that they are identical [60].
We will use the notion of hereditary well-quasi-ordering in Theorems [11] and [19] and the notion of 1'-well-quasi-ordering in Lemma [21]. We recall the following result (Proposition 2.2 of [50]).

**Theorem 1.** Provided that the signature $s$ is bounded, the cardinality of bounds of every hereditary and hereditary w.q.o. subclass of $\Omega_s$ is bounded.

### 3.2.3. Jónsson posets

**Definition 1.** A poset $P$ is a Jónsson poset if it is infinite and every proper initial segment has a strictly smaller cardinality than $P$.

Jónsson posets were introduced by Oman and Kearnes [28]. Countable Jónsson posets were studied and described in [51, 49, 5]. We recall (see Proposition 3.1 [5]):

**Theorem 2.** Let $P$ be a countable poset. The following propositions are equivalent.

(i) $P$ is Jónsson;
(ii) $P$ is well-quasi-ordered and each ideal distinct from $P$ is finite;
(iii) $P$ is level-finite, has height $\omega$, and for each $n < \omega$, there is $m < \omega$ such that each element of height at most $n$ is below every element of height at least $m$.

**Lemma 3.** Every infinite well-founded poset $P$ which is level finite contains an initial segment which is Jónsson.

**Proof.** We apply Zorn’s Lemma to the set $J$ of infinite initial segments of $P$ included in the first $\omega$-levels. For that, we prove that $J$ is closed under intersections of nonempty chains. Indeed, let $C$ be a nonempty chain (with respect to set inclusion) of members of $J$. Set $J := \cap C$. Let $n < \omega$, let $P_n$ be the $n$-th level of $P$ and $C_n := \{C \cap P_n : C \in C\}$. The members of $C_n$ are finite, nonempty and linearly ordered by set inclusion. Hence, $J_n := \cap C_n$ is nonempty. Since $J = \cup\{J_n : n \in \mathbb{N}\}, J \in J$.

Jónsson posets are behind the study of minimal prime hereditary classes (See Theorem 13 in Section [4]).

### 3.3. Words

Let $\Sigma$ be a finite set. A $\Sigma$-sequence is any map $u$ from an interval $I$ of the set $\mathbb{Z}$ of integers in $\Sigma$. The set $I$ is the domain of $u$. Two $\Sigma$-sequences $u$ and $u'$ are isomorphic if there is a translation $t$ on $\mathbb{Z}$ mapping the domain $I$ of $u$ onto the domain $I'$ of $u'$ so that $u(i) = u'(t(i))$ for all $i \in I$. If the domain of a $\Sigma$-sequence $u$ is $\{0, \ldots, n - 1\}$, $\mathbb{N}$, $\mathbb{N}^* := \{0, -1, \ldots, -n \ldots\}$ or $\mathbb{Z}$, the sequence is a word. Words appear as representatives of equivalence classes of sequences. Except if their domain is $\mathbb{Z}$, the representatives are unique. The elements of $\Sigma$ are called letters and $\Sigma$ is the alphabet. When the alphabet is $\{0, 1\}$, we use the terminology 0-1 sequences or 0-1 word. If $u$ is a 0-1 sequence with domain $I$ and if $I'$ is a subset of $I$, the restriction of $u$ to $I'$ is denoted by $u_{|I'}$. If $I$ is finite, the sequence $u$ is finite and the length of $u$, denoted $|u|$, is the number of elements of $I$. We denote by $\square$ the empty sequence. If $u$ is a finite word and $v$ is a word, finite or infinite with domain $\mathbb{N}$, the concatenation of $u$ and $v$ is the word $uv$ obtained by writing $v$ after $u$. If $v$ has domain $\mathbb{N}^*$, the word $vu$ is defined similarly. A word $v$ is a factor of $u$ if $u = u_1vu_2$. This defines an order on the collection $\Sigma^*$ of finite words, the factor ordering.
3.3.1. **Hereditary classes of words.** A subset \( \mathcal{C} \) of \( \Sigma^* \) is **hereditary** if it contains every factor of every member of \( \mathcal{C} \). In other words, this is an initial segment of \( \Sigma^* \) ordered with the factor ordering. The **age** of a word \( u \) is the set \( \text{Fac}(u) \) of all its finite factors endowed with the factor ordering. This is a hereditary subset of \( \Sigma^* \). In fact, if the alphabet is at most countable, a set \( \mathcal{C} \) of finite words is the age of a word \( u \) if and only if \( \mathcal{C} \) is an **ideal** for the factor ordering. Note that the domain of \( u \) is not necessarily \( \mathbb{N} \).

A nonempty subset \( \mathcal{C} \) of \( \Sigma^* \) is **inexhaustible** if it is not reduced to the empty word and if for every \( v \in \mathcal{C} \) there is some \( w \) such that \( vwv \in \mathcal{C} \).

**Lemma 4.** A hereditary class \( \mathcal{C} \) of finite words is inexhaustible if and only if \( \mathcal{C} \) is an union of inexhaustible ages.

**Proof.** \( \Rightarrow \)

**Claim 1.** If \( \mathcal{D} \) is an inexhaustible subset of \( \Sigma^* \) then \( \downarrow \mathcal{D} \) is inexhaustible.

**Proof of Claim 1.** Let \( u \in \mathcal{D} \). There exists \( u' \in \mathcal{D} \) such that \( u \) is a factor of \( u' \). We write \( u' := u'_1 uu'_2 \). Since \( \mathcal{D} \) is inexhaustible there exists \( w' \) such that \( u'w'u' \in \mathcal{D} \). Let \( w := u'_2 w'u'_1 \). The word \( uwu \) is a factor of \( u'w'u' \) hence is in \( \downarrow \mathcal{D} \). \( \square \)

**Claim 2.** If \( \mathcal{D} \) is an inexhaustible subset of \( \Sigma^* \), then for all \( u \in \mathcal{D} \) there is a sequence \( u_0, \ldots, u_n \ldots \) satisfying \( u_0 = u \), \( u_{n+1} := u_n v_n u_n \), \( u_n \in \mathcal{D} \) and \( u_{n+1} \in \mathcal{D} \). By construction, the set \( \mathcal{E} := \{u_n : n \in \mathbb{N}\} \) is an inexhaustible set.

**Claim 3.** If \( \mathcal{C} \) is inexhaustible and hereditary, then for all \( u \in \mathcal{C} \) there exists a inexhaustible age \( \mathcal{A} \) such that \( u \in \mathcal{A} \subseteq \mathcal{C} \).

**Proof of Claim 3.** We apply Claim 2 with \( \mathcal{D} := \mathcal{C} \). The set \( \mathcal{E} \) defined in Claim 2 is inexhaustible. The set \( \downarrow \mathcal{E} \) is the age of the word \( u_{\infty} \) having the words \( u_n \) as prefixes for \( n \geq 0 \).

\( \Leftarrow \) Obvious. \( \square \)

A word \( u \) is **recurrent** if every finite factor occurs infinitely often. This amounts to the fact that \( \text{Fac}(u) \) is inexhaustible. In fact:

**Theorem 5.** Let \( \mu \) be a nonempty 0-1 sequence on an interval of \( \mathbb{Z} \). The following are equivalent.

(i) \( \text{Fac}(\mu) \) is inexhaustible;
(ii) \( \mu \) is recurrent;
(iii) There exists a word \( \nu \) on \( \mathbb{Z} \) so that \( \text{Fac}(\mu) = \text{Fac}(\nu) = \text{Fac}(\nu_{\mathbb{N}}) = \text{Fac}(\nu_{\mathbb{N}^*}) \).

This is a variation of Proposition 2 in section II-2.3 page 40 of [51]. For results along this lines, see [7].

A word \( u \) is **uniformly recurrent** if for every \( n \in \mathbb{N} \) there exists \( m \in \mathbb{N} \) such that each factor \( u(p) \ldots u(p+n) \) of length \( n \) occurs as a factor of every factor of length \( m \).

The fact that a word is uniformly recurrent can be expressed in terms of properties of its age ordered by the factor ordering.

**Theorem 6.** Let \( u \) be a word with domain \( \mathbb{N} \) over a finite alphabet. The following properties are equivalent:

(i) \( u \) is uniformly recurrent;
(ii) Fac(u) is inexhaustible and well-quasi-ordered;
(iii) Fac(u) is a countable Jónsson poset.

The only nontrivial implication is (ii) ⇒ (iii) (see Lemma II-2.5 of Ages belordonnés in [51] p. 47). See Theorem 5 in [59].

Let u and v be two words. The word v is a prefix of u if u = vu'. This is a suffix of v if v = u''v. These relations define two orders on the collection Σ* of finite words, the prefix and the suffix orders. The prefix order, as well as the suffix order are (ordered) trees (for each u ∈ Σ*, the set of elements below is a chain). A first consequence is that every ideal is a chain. A more significant consequence, is that if C is an initial segment of Σ* for one of these orders, C is a w.q.o. if and only if it is an union of finitely many chains (indeed, if C is a w.q.o. then as every w.q.o. this is a finite union of ideals (see [39]). From this we deduce:

**Theorem 7.** Let u be a word with domain N over a finite alphabet. Then Fac(u) is well-quasi-ordered for the prefix order, respectively, the suffix order, if and only if u is ultimately periodic, respectively, periodic.

*Proof.* Suppose that u is w.q.o. for the prefix order. Then Fac(u) is a finite union of ideals. For each integer n ∈ N, the set pref_n(u) := {u|m| : n ≤ m} is an ideal. This ideal been included in a finite union of initial segments is included in one of them, and in fact equal. Thus there are only finitely many sets of the form pref_n(u). If for n < n', pref_n(u) = pref_n'(u) then the sequences pref_n(u) and pref_n'(u) give the same word. It follows that u is ultimately periodic. If u is w.q.o. for the suffix order, we observe first that u is recurrent. Then we apply Theorem 5 there is a word w on Z such that Fac(w) = Fac(u). With the same argument as above, the set for n ∈ Z of suff_n(w) := {w|m| : n ≤ m} is finite. This ensures that w is periodic. It follows that u is periodic too. The converses of these implications are obvious.

3.3.2. Bounds of hereditary classes of words. Let C be a hereditary class of finite words. A **bound** of C is any finite word v ∉ C such that every proper factor of v belongs to C. Equivalently, if v := v_0...v_{n-1}, then v is a bound of C if and only if v ∉ C and the words v_0...v_{n-2} and v_1...v_{n-2} belong to C.

Let u be a word and p be a nonnegative integer. The word u is **periodic** and p is a period if u(i) = u(i + p) whenever i and i + p belong to the domain of u.

The following result is Proposition 3 in section II-2.6 page 54 of [51]. This result was never published. For completeness we include its proof here.

**Theorem 8.** Let µ be an infinite periodic word of period p > 0. Then the bounds of Fac(µ) have length at most p.

*Proof.* The proof is based on the following remark due to Roland Assous. Namely, a word v is periodic, with period p > 0, if and only if every two factors w' and w'' of v, both having length p, contain the same letters and each of these letters occur the same number of times in w' and w''.

We now prove the theorem. Let v be a finite word of length at least p + 1 so that each factor of length p is a factor of the word µ. It follows from the above remark that the word v is periodic with period p, hence v is of the form w...ww' = (wn)w' where w' is a prefix of w. Since µ is periodic of period p and w is a factor of µ we infer that w(n + 1) is a factor of µ, hence v is a factor of µ. Thus v is not a bound of Fac(µ).
The following is a consequence of Proposition 6 in II-2.6, page 60 of [51]. This result was never published. For completeness we include a proof (in fact two) here.

**Theorem 9.** Let \( \mu \) be a uniformly recurrent and non periodic word. Then \( \text{Fac}(\mu) \) has infinitely many bounds.

**Proof.** We give two proofs.

1) Let \( \mu \) be a uniformly recurrent and non periodic word. Suppose for a contradiction that \( \text{Fac}(\mu) \) has finitely many bounds and let \( m \) be the maximum length of bounds of \( \text{Fac}(\mu) \). Let \( v \in \text{Fac}(\mu) \) of length at least \( m \). Since \( \mu \) is uniformly recurrent, \( \text{Fac}(\mu) \) is inexhaustible (Theorem 6) it contains a word of the form \( vwv \). Since the bounds of \( \text{Fac}(\mu) \) have lengths at most \( m \) we infer that \( \text{Fac}(\mu) \) contains all periodic words of the form \((vw)\ldots(vw)\) and hence contains the set of factors of the infinite periodic word \( \mu' := (vw)vw\ldots \). Since \( \text{Fac}(\mu) \) is Jónsson (Theorem 6) and \( \text{Fac}(\mu') \) is infinite, \( \text{Fac}(\mu) = \text{Fac}(\mu') \). It follows that \( \mu \) is periodic. A contradiction.

2) Our second proof is based on the following properties of regular languages. 

**Claim.** Let \( C \) be a hereditary class of finite words (ordered by the factor relation). If \( C \) has a finite number of bounds, then \( C \) is a regular language. And if \( C \) is an infinite regular language it contains \( \text{Fac}(\mu) \) where \( \mu \) is an infinite periodic word.

**Proof of the Claim.** The first implication is immediate. Indeed, for every finite word \( v \), the set \( \uparrow v := \{ w : v \text{ is a factor of } w \} \) is regular language (in fact \( \uparrow v = \Sigma^*v\Sigma^* \), where \( \Sigma \) is the alphabet). It follows from Kleene’s Theorem that the complement of \( \uparrow v \), that is \( C \setminus \uparrow v \) is a regular language. If \( C \) has a finite number of bounds, then 

\[
C = \bigcup \{ \Sigma^* \setminus \uparrow v : v \text{ bound of } C \}
\]

is a finite intersection of regular languages and is therefore regular.

For the second implication we use the Pumping Lemma for regular languages [6]. Since \( C \) is a regular language, there are finite words \( u, v_1, v_2 \) such that for all \( n \in \mathbb{N} \), \( v_1u^nv_2 \in C \). Let \( \mu := uu \ldots \). Then \( \text{Fac}(\mu) \subseteq C \).

The existence of infinitely many bounds to a non periodic and uniformly recurrent word follows from the Claim. Indeed, let \( \mu \) be a non periodic uniformly recurrent word. Then \( \text{Fac}(\mu) \) cannot be regular. Otherwise, it follows from the second part of the Claim that \( \text{Fac}(\mu) \) contains the set of factor of an infinite periodic word \( w \). Since \( \mu \) is uniformly recurrent \( \text{Fac}(\mu) = \text{Fac}(w) \) and therefore \( \mu \) is periodic, contradicting our assumption. The required conclusion now follows from the first part of the Claim.

4. **Minimal prime hereditary classes**

In this section we present the definition and properties of minimal prime hereditary classes of finite binary structures. We introduce the notion of minimal prime structure and we conclude with the notion of almost chainability. Results for graphs, given in the subsequent sections, are more precise. Most of the results presented here were included in Chapter 5 of the thesis of the first author [45].

We start with the notion of a module.
Definition 2. Let $R := (V, (\rho_i)_{i \in I})$ be a binary relational structure. A module of $R$ is any subset $A$ of $V$ such that

$$(x \rho_i a \iff x \rho_i a') \text{ and } (a \rho_i x \iff a' \rho_i x) \text{ for all } a, a' \in A \text{ and } x \notin A \text{ and } i \in I.$$ 

The empty set, the singletons in $V$ and the whole set $V$ are modules and are called trivial. (sometimes in the literature, modules are called interval, autonomous or partitive sets). If $R$ has no nontrivial module, it is called prime or indecomposable.

For example, if $R := (V, \leq)$ is a chain, its modules are the ordinary intervals of the chain. If $R := (V, (\leq, \leq'))$ is a bichain then $A$ is a module of $R$ if and only if $A$ is an interval of $(V, \leq)$ and $(V, \leq')$.

The notion of module goes back to Fraïssé [22] and Gallai [25], see also [21]. A fundamental decomposition result of a binary structure into modules was obtained by Gallai [25] for finite binary relations (see [21] for further extensions). We recall the compactness result of Ille [27].

Theorem 10. A binary structure $R$ is prime if and only if every finite subset $F$ of its domain extends to a finite set $F'$ such that $R|_{F'}$ is prime.

We consider the class $\text{Prim}_s := \text{Prim}(\Omega_s)$ of finite binary structures of signature $s$ which are prime. We set $\text{Prim}(C) := \text{Prim}_s \cap C$ for every $C \subseteq \Omega_s$.

We say that a subclass $\mathcal{D}$ of $\text{Prim}_s$ is hereditary if it contains every member of $\text{Prim}_s$ which can be embedded into some member of $\mathcal{D}$.

4.1. Hereditary classes containing finitely many prime structures. The following result (see Proposition 5.2 of [47]) improves a result of [1] for hereditary classes of finite permutations.

Theorem 11. Let $\mathcal{C}$ be a hereditary class of finite binary structures containing only finitely many prime structures. Then $\mathcal{C}$ is hereditarily well-quasi-ordered. In particular, $\mathcal{C}$ has finitely many bounds.

The following result, due independently to Delhommé [20] and McKay [43] extends Thomassé’s result on the well-quasi-order character of the class of countable series-parallel posets [68], which extends the famous Laver’s theorem [35] on the well-quasi-order character of the class of countable chains.

Theorem 12. Let $\mathcal{C}$ be a hereditary class of $\Omega_s$. If $\text{Prim}(\mathcal{C})$ is finite, then the collection $\mathcal{C}^{\leq \omega}$ of countable $R$ such that Age($R$) $\subseteq \mathcal{C}$ is well-quasi-ordered by embeddability.

In fact, Delhommé and McKay obtain a stronger conclusion of Theorem 12. If Prim($\mathcal{C}$) is finite, and $Q$ is a better-quasi-order then, the class of members of $\mathcal{C}^{\leq \omega}$ labelled by $Q$ is better-quasi-ordered (this implication is false if b.q.o. is replaced by w.q.o). In particular, if $Q$ is finite, this class is w.q.o. This case follows from Theorem 12 above. Indeed, we may view structures labelled by $Q$ as binary structures. In this new class, say $\mathcal{D}$, modules are unchanged, hence there are only finitely many primes and thus the class $\mathcal{D}^{\leq \omega}$ is w.q.o. We will use this observation in the proof of Theorem 25 below.

4.2. Hereditary classes containing infinitely many prime structures. In this subsection, we report some results included in [45]. We consider hereditary classes containing infinitely many prime structures. We show that each such a class contains one which is minimal with respect to inclusion.
Definition 3. A hereditary class $C$ of $\Omega_\mu$ is minimal prime if it contains infinitely many prime structures, while every proper hereditary subclass contains only finitely many prime structures.

This notion appears in the thesis of the first author \cite{45} (see Theorem 5.12, p. 92, and Theorem 5.15, p. 94 of \cite{45}).

Due to their definition, minimal prime ages ordered by set inclusion form an antichain with respect to set inclusion.

We have immediately (cf. Théorème 5.14 p.93 of \cite{45}).

Theorem 13. A hereditary class $C$ of $\Omega_s$ is minimal prime if and only if $\text{Prim}(C)$ is a Jónsson poset which is cofinal in $C$.

Proof. Let $C$ be a minimal prime class. By definition, $\text{Prim}(C)$ is infinite. Let $I$ be a proper hereditary subclass of $\text{Prim}(C)$. The initial segment $\downarrow I$ in $\Omega_s$ is a proper subclass of $C$. Hence $I$ is finite. Thus $\text{Prim}(C)$ is Jónsson. Let $C' := \downarrow \text{Prim}(C)$. If $C' \neq C$ then since $C$ is minimal prime, $\text{Prim}(C') = \text{Prim}(C)$ is finite, which is impossible. This proves that the forward implication holds.

Conversely, suppose that $\text{Prim}(C)$ is a Jónsson poset which is cofinal in $C$. Then $C$ is infinite. If $C$ is not minimal prime there is a proper hereditary subclass $C'$ of $C$ such that $\text{Prim}(C')$ is infinite. Since $\text{Prim}(C)$ is Jónsson, $\text{Prim}(C') = \text{Prim}(C)$. Since $C' = \downarrow \text{Prim}(C')$ and $\text{Prim}(C)$ is cofinal in $C$, this yields $C' = C$. A contradiction. $\square$

We have:

Theorem 14. Every hereditary class of finite binary structures (with a given finite signature), which contains infinitely many prime structures contains a minimal prime hereditary subclass.

For the proof of Theorem 14 we will need the following lemma which is a special case of Theorem 4.6 of \cite{5}.

Lemma 15. $\text{Prim}_s$ is level finite.

Proof. Suppose for a contradiction that there exists an integer $n \geq 0$ such that the level $\text{Prim}(n)$ of $\text{Prim}_s$ is infinite and choose $n$ smallest with this property. Define

$$C := \{R \in \Omega_s : R < S \text{ for some } S \in \text{Prim}(n)\}.$$ 

Then $C$ is a hereditary class of $\Omega_s$ containing only finitely many prime structures. It follows from Theorem 14 that $C$ is hereditary well-quasi-ordered and hence has finitely many bounds. This is not possible since the elements of $\text{Prim}(n)$ are bounds of $C$. $\square$

The proof of Theorem 14 goes as follows. Let $C$ be a hereditary class of $\Omega_s$ such that $J := \text{Prim}(C)$ is infinite. Since $\text{Prim}_s$ is level finite, Lemma 14 ensures that $J$ contains an initial segment $D$ which is Jónsson. According to Theorem 13 $\downarrow D$ is minimal prime. This completes the proof. $\square$

4.2.1. Another proof of Lemma 15. We prove the finiteness of the levels of $\text{Prim}_s$ via the properties of critical primality. A binary structure $R$ is critically prime if it is prime and $R_{\setminus V(R)\setminus \{x\}}$ is not prime for every $x \in V(R)$. Note that $|V(R)|$ has at least four elements. This notion of critical primality was introduced by Schmerl and Trotter \cite{63}. Among results given in their paper, we have the following theorem (this is Theorem 5.9, page 204):
Theorem 16. Let $R$ be a prime binary structure of order $n \geq 7$. Then there are distinct $c, d \in V(R)$ such that $R_{\{V(R)\setminus \{c,d\}}$ is prime.

In their paper, Schmerl and Trotter give examples of critically prime structures within the class of graphs, posets, tournaments, oriented graphs and binary relational structures. The set of critical prime structures within each of these classes is a finite union of chains.

Decompose $\text{Prim}_s$ into levels; in level $i$, with $i \leq 2$, are the structures of order zero, one or two.

For structures $R$ in $\text{Prim}_s$ of order at least 2, we have the following relationship between the height $h(R)$ in $\text{Prim}_s$ and its order, $|V(R)|$ (which is the height of $R$ in $\Omega_s$).

\[ (1) \quad h(R) \leq |V(R)| \leq 2(h(R) - 1). \]

The first inequality is obvious. For the second, we use induction on $n := h(R) \geq 2$. The basis step $n = 2$ is trivially true. Suppose $n > 2$. Let $S$ be prime such that $S$ embeds in $R$ with $h(S) = n - 1$. From the induction hypothesis, $|V(S)| \leq 2(h(S) - 1) = 2(n - 2)$. According to Theorem 13, $|V(R)| - 2 \leq |V(S)|$. Hence $|V(R)| - 2 \leq 2(n - 2)$. Therefore $|V(R)| \leq 2(n - 1)$.

Lemma 15 follows from the second inequality in (1) since there are only finitely many structures of a given order.

With Theorem 10 and 14, one gets:

Corollary 17. The age of any infinite prime structure contains a minimal prime age.

With Lemma 3 and Theorem 11 we get:

Theorem 18. Every minimal prime hereditary class is the age of some prime structure; furthermore this age is well-quasi-ordered.

Proof. Let $C$ be a minimal prime hereditary class. We first prove that it is the age of a prime structure. It follows from Theorem 13 that $C = \downarrow D$ where $D$ is Jónsson. Since $D$ is Jónsson, it is up-directed. Thus $C$ is an age. Since $D$ is up-directed and countable, it contains a cofinal sequence $R_0 \leq R_1 \leq \ldots < R_n \leq \ldots$. We may define the limit $R$ of these $R_n$. Since the $R_n$’s are prime, $R$ is prime and $\text{Age}(R) = C$.

Next we prove that $C$ is w.q.o. Since $D$ is Jónsson, it is w.q.o. To prove that $C$ is w.q.o., let $R \in C$ and consider $C \setminus (\uparrow \{R\})$. In order to prove that $C$ is w.q.o. it is enough to prove that $C \setminus (\uparrow \{R\})$ is w.q.o. by embeddability. Indeed, an antichain that contains $R$ must be in $C \setminus (\uparrow \{R\})$. Now to prove that $C \setminus (\uparrow \{R\})$ is w.q.o. we note that since $C \setminus (\uparrow \{R\})$ is a proper hereditary class in $C$, it contains only finitely many primes. It follows from Theorem 11 that $C \setminus (\uparrow \{R\})$ is w.q.o. .

As mentioned in subsection 3.2, it is not known if a hereditary class of finite graphs which is w.q.o. is b.q.o.

Problem 1. Is every minimal prime hereditary class of finite binary structures b.q.o.?

It is known that Jónsson posets are b.q.o. [51] [17] but the argument in the proof of Theorem 18 does not give the b.q.o. character of the class $C$. In the case of graphs, minimal prime hereditary classes divide into two types. Those which are almost multichainable and those which are the ages of some special graphs. The b.q.o. character of these classes can be obtained by an extension of Higman’s theorem to b.q.o. (see Remark 5). We give below an improvement of Theorem 18 based on properties of the kernel of a relational structure.
4.3. Inexhaustibility, kernel and minimality. The kernel of a relational structure $R$ with domain $V$ is the set

$$\ker(R) : \{x \in V : \text{Age}(R|_{V \setminus \{x\}}) \neq \text{Age}(R)\}.$$ 

The kernel is an invariant of the age in the sense that if $R$ and $R'$ have the same age then there is an isomorphism $f$ from $\ker(R)$ onto $\ker(R')$ such that (a) every restriction of $f$ to every finite subset $F$ of $\ker(R)$ extends to every finite superset $F'$ of $F$ to an embedding of $R|_{F'}$ in $R'$ and (b) the same property holds for $f^{-1}$. An age $\mathcal{A}$ is inexhaustible, or has the disjoint embedding property, if two arbitrary members of the age can be embedded into a third member in such a way that their domains are disjoint. As it is easy to see, the kernel of a relational structure $R$ is empty if and only if $\text{Age}(R)$ is inexhaustible. We say that an age $\mathcal{C}$ which is not inexhaustible is exhaustible. It is almost inexhaustible if the kernel of some $R$ with $\text{Age}(R) = \mathcal{C}$ is finite.

The notion of inexhaustibility was introduced by Fraïssé in the sixties. The notion of kernel was introduced in [51] and studied in several papers [52], [54], and [53] (see Lemme IV-3.1 p. 37), first for structures with finite signature. The general case was considered in [55].

We prove:

**Theorem 19.** If $\mathcal{C}$ is a minimal prime class of binary structures, then $\mathcal{C}$ is almost inexhaustible.

In order to prove Theorem 19 we recall two facts below. The first one is in [52] see III.1.3, p. 323.

**Lemma 20.** An element $a \in V(R)$ belongs to $\ker(R)$ if and only if there is some finite subset $A$ of $V(R)$ containing all the images of $a$ by the local automorphisms defined on $A$.

We extract the second fact from [51] Corollaire p.6 in “Caractérisation combinatoire et topologique des âges les plus simples”. For reader’s convenience, we give a proof.

**Lemma 21.** Let $R := (V, (\rho_i)_{i \in I})$ be a relational structure made of finitely many binary relations. If $\text{Age}(R)_{1-} := \{(S, a) : S \in \text{Age}(R), a \in V(S)\}$ is well-quasi-ordered, then $\ker(R)$ is finite.

**Proof.** Suppose that $\ker(R)$ is infinite. We built a sequence $(R|_{A_n}, a_n)$ of elements of $\text{Age}(R)_{1-}$ such that no two members of the sequence have a common extension belonging to $\text{Age}(R)_{1-}$. In particular these members form an infinite antichain of $\text{Age}(R)_{1-}$. We pick $a_0 \in \ker(R)$ and $A_0$ given by Lemma 20. Suppose $(A_n, a_n)$ defined for $n < m$, pick $a_m \in V \setminus \bigcup_{n \leq m} A_n$, select $A$ given by Lemma 20 and set $A_m := A \cup \bigcup_{n < m} A_n$. \qed

Let $\mathcal{C}$ be a class of finite binary structures $S := (F, (\rho_i)_{i \in I})$ with a finite signature $s$. Denote by $\mathcal{C}^{1+}$ the class of $S := (F, (\rho_i)_{i \in I})$ such that there is some $a \in F$ such that $S|_{F \setminus \{a\}} \in \mathcal{C}$.

The following lemma is Proposition 5.32 p. 105 of [19] and Theorem 4.5 page 20 of [10]. A similar fact, but non explicit, appears in the proof of Theorem 4.24 p.267 of [38]. For reader’s convenience, we give a proof.

**Lemma 22.** Let $\mathcal{C}$ be a hereditary class of binary structures. If the members of $\mathcal{C}$ are not necessarily finite and if these members when labelled by any better-quasi-order form a well-quasi-order, then $\mathcal{C}^{1+}$ has the same property. If $\mathcal{C}$ is made of finite structures and is hereditarily well-quasi-ordered, then $\mathcal{C}^{1+}$ is hereditarily well-quasi-ordered.
Proof. Let $I$ be such that each $S \in \mathcal{C}$ is of the form $S := (F, (\rho_i)_{i \in I})$. Let $W$ be a w.q.o. By hypothesis, the set $(2 \times 2 \times 2)^I$ is finite, hence with the equality ordering it is wqo. The direct product $W'$ of $W$ with $(2 \times 2 \times 2)^I$ is w.q.o. We code members of $\mathcal{C}^+$ labelled by $W$ by members of $\mathcal{C}$ labelled by $W'$. Indeed, for each $S := (F, (\rho_i)_{i \in I}) \in \mathcal{C}^+$ we select $a \in F$ such that $S_{F \setminus \{a\}} \in \mathcal{C}$ and we label $S_{F \setminus \{a\}}$ by the map $g_a$ defined for $x \in F \setminus \{a\}$ by $g_a(x) := (\rho_i(a, x), \rho_i(x, a), \rho_i(a, a))_{i \in I}$. Now if $f$ is a labelling of $F$ in $W$, we associate the labelling $f'$ of $F \setminus \{a\}$ by setting $f' := (f_{F \setminus \{a\}}, g_a)$. By construction, if $S, S' \in \mathcal{C}$, an embedding $h$ from the labelled structure $S_{F \setminus \{a\}}$ in the labelled structure $S'_{F \setminus \{a'\}}$ will extend to an embedding of the labelled structure $S$ in the labelled structure $S'$ with $a$ mapped to $a'$. The conclusion follows.

We deduce:

**Corollary 23.** Let $R := (V, (\rho_i)_{i \in I})$ be a relational structure made of finitely many binary relations and let $a \in V$. If $\text{Age}(R_{\setminus \{a\}})$ is hereditarily well-quasi-ordered, then $\text{Age}(R)$ is hereditarily well-quasi-ordered.

**Proof.** If $a \notin \ker(R)$, there is nothing to prove. If $a \in \ker(R)$, we set $\mathcal{C} := \text{Age}(R_{\setminus \{a\}})$. We observe that $\text{Age}(R) \subseteq \mathcal{C}^+$ and we apply Lemma 22.

**Proof of Theorem 19.** Let $\mathcal{C}$ be a minimal prime class and $R$ such that $\text{Age}(R) = \mathcal{C}$. Suppose that $\ker(R)$ is nonempty. Let $a \in \ker(R)$. Then $\text{Age}(R_{\setminus \{a\}}) \neq \text{Age}(R) = \mathcal{C}$. Since $\mathcal{C}$ is minimal prime, $\text{Age}(R_{\setminus \{a\}})$ contains only finitely many primes. Theorem 11 asserts that $\text{Age}(R_{\setminus \{a\}})$ is hereditarily wqo. Corollary 23 asserts that $\text{Age}(R)$ is hereditarily w.q.o. Lemma 24 asserts that $\ker(R)$ is finite. With that the proof is complete.

Since each hereditary well-quasi-ordered class has finitely many bounds (Theorem 11) we have only countably many exhaustible minimal prime classes.

**Corollary 24.** There are at most countably many minimal prime classes $\mathcal{C}$ such that $\mathcal{C}$ is exhaustible.

**Problem 2.** (1) Is it true that $|\ker(\mathcal{R})| \leq 2$ if $\text{Age}(R)$ minimal prime?

(2) Is the number of exhaustible minimal prime ages finite?

As we will see, the answers are positive if one considers minimal prime classes of graphs. In this case, there are only five examples with a nonempty kernel.

4.4. Links with an other notion of minimality.

**Definition 4.** A binary relational structure $R$ is minimal prime if $R$ is prime and $R$ embeds in every induced indecomposable substructure with the same cardinality.

Several examples of graphs and posets are given in [61].

**Problem 3.** Is it true that the age of a minimal prime binary structure is necessarily minimal prime?

Even in the case of graphs we do not know the answer. The converse is false in the sense that there are minimal prime ages of graphs such that no graph with that age is minimal prime.

We prove:
Theorem 25. If $C$ is minimal prime and exhaustible, then every binary prime structure $R$ with $\text{Age}(R) = C$ embeds a minimal prime structure.

The proof relies on Theorem 12 and Lemma 22.

We prove first the following.

Lemma 26. If $C$ is minimal prime and exhaustible then $C^\omega$ is well-quasi-ordered.

Proof. Let $R$ with $\text{Age}(R) = C$. Pick $a \in \ker(R)$. Let $D := \text{Age}(R|_{V(R) \setminus \{a\}})$. This age contains only finitely many primes. From Theorem 12, $D^\omega$ is well-quasi-ordered. Furthermore, members of $D^\omega$ when labelled by any finite set form a well-quasi-ordered set. According to Lemma 22, $(D^\omega)^+1$ has the same property. Next, $C^\omega \subseteq (D^\omega)^+1$. Indeed, every member of $C^\omega$ has a copy $R'$ in a countable extension $R''$ of $R$ having the same age as $R$ hence $\text{Age}(R'|_{V(R') \setminus \{a\}}) \subseteq C$. Hence, $C^\omega$ is well-quasi-ordered.

Next,

Lemma 27. Let $C$ be a hereditary class of $\Omega$. If $C^\omega$ is well founded then every prime member of $C^\omega$, if any, embeds a minimal one.

Proof of Theorem 25. Let $R$ be a prime structure with $\text{Age}(R) = C$. According to Lemma 26, $C^\omega$ is well-quasi-ordered. According to Lemma 27, $R$ embeds a minimal prime member. □

4.5. Primality and almost multichainability. A relational structure $R$ is almost multichainable if its domain $V$ is the disjoint union of a finite set $F$ and a set $L \times K$ where $K$ is a finite set, for which there is a linear order $\leq$ on $L$, satisfying the following condition:

- For every local isomorphism $h$ of the chain $C := (L, \leq)$ the map $(h, 1_K)$ extended by the identity on $F$ is a local isomorphism of $R$ (the map $(h, 1_K)$ is defined by $(h, 1_K)(x, y) := (h(x), y)$).

The notion of almost multichainability was introduced in [51] (see [58] for further references and discussions). The special case $|K| = 1$ is the notion of almost chainability introduced by Fraïssé. The use of this notion in relation with the notion of primality is illustrated in several papers, notably [9], [57].

We recall 1. of Theorem 4.19 p.265 of [58].

Proposition 28. The age of an almost multichainable structure is hereditarily well-quasi-ordered.

The proof of Proposition 28 given in [58] consists to interpret members of the age by words over a finite alphabet and apply Higman’s Theorem on words. In fact, the extension of Higman’s Theorem to b.q.o. tells us that the age of an almost multichainable structure is hereditarily b.q.o.

With Theorem [1] we have:

Theorem 29. If the signature is bounded, the cardinality of bounds of the age of an almost multichainable structure is bounded.

Proposition 28 extends a little bit.

Proposition 30. If $C$ is the age of a almost multichainable structure, then the collection $C^\omega$ of countable structures whose ages are included in $C$ is b.q.o. and in fact hereditary b.q.o.
For a proof, see [60]. Applying Lemma 27 and Proposition 30, we have:

**Theorem 31.** If $R$ is almost multichainable, then $\text{Age}(R)$ is hereditarily well-quasi-ordered. Hence, it has finitely many bounds. Every prime $R'$ with the same age (if any) contains a minimal prime structure.

**Problem 4.** If a minimal prime age is $2^−$-well-quasi-ordered, this is the age of an almost multichainable binary relational structure.

The answer is positive for graphs. Indeed, the minimal prime ages which are not ages of multichainable graphs are ages of some special graphs, the $G_n$'s, their ages are not $2^−$-w.q.o. Some are $1^−$-w.q.o. (when $\mu$ is periodic). For more, see Remark 6.

5. Minimal prime ages of graphs

Our description of minimal prime ages of graphs is based on several results. First a previous description of unavoidable prime graphs in large finite prime graphs of Chudnovsky, Kim, Oum and Seymour [18], see also Malliaris and Terry [42]. Next a study of graphs associated to 0-1 sequences.

5.1. Unavoidable prime graphs. We introduce some finite prime graphs. Fix an integer $n \geq 1$.

- The bipartite half-graph of height $n$ $H_n$, is a graph with $2n$ vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $a_i$ is adjacent to $b_j$ if and only if $i \leq j$ and such that $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are independent sets.

- The half split graph of height $n$ $H_n^\prime$, is a graph with $2n$ vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $a_i$ is adjacent to $b_j$ if and only if $i \leq j$ and such that $\{a_1, \ldots, a_n\}$ is an independent set and $\{b_1, \ldots, b_n\}$ is a clique (a graph is a split graph if its vertices can be partitioned into a clique and an independent set).

- Let $H_{n,1}$ be the graph obtained from $H_n^\prime$ by adding a new vertex adjacent to $a_1, \ldots, a_n$ (and no others). Let $H_n^*$ be the graph obtained from $H_n^\prime$ by adding a new vertex adjacent to $a_1$ (and no others).

- The thin spider with $n$ legs is a graph with $2n$ vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $\{a_1, \ldots, a_n\}$ is an independent set and $\{b_1, \ldots, b_n\}$ is a clique, and $a_i$ is adjacent to $b_j$ if and only if $i = j$. The thick spider with $n$ legs is the complement of the thin spider with $n$ legs. In particular, it is a graph with $2n$ vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $\{a_1, \ldots, a_n\}$ is an independent set $\{b_1, \ldots, b_n\}$ is a clique, and $a_i$ is adjacent to $b_j$ if and only if $i \neq j$. A spider is a thin spider or a thick spider. In Item (4) of Theorem 33 we consider the extension of this notion to infinite sets.

- A sequence of distinct vertices $v_0, \ldots, v_m$ in a graph $G$ is called a chain from a set $I \subseteq V(G)$ to $v_m$ if $m \geq 2$ is an integer, $v_0, v_1 \in I, v_2, \ldots, v_m \notin I$, and for all $i > 0, v_{i-1}$ is either the unique neighbor or the unique non-neighbor of $v_i$ in $\{v_0, \ldots, v_{i-1}\}$. The length of a chain $v_0, \ldots, v_m$ is $m$.

The following is due to Chudnovsky et al [18]:

**Theorem 32** (Theorem 1.2 of [18]). For every integer $n \geq 3$ there is $N$ such that every prime graph with at least $N$ vertices contains one of the following graphs or their complements as an induced subgraph.
Figure 1. Unavoidable prime finite graphs (this is Figure 1 from [18])

1. The 1-subdivision of \( K_{1,n} \) (denoted by \( K^1_{1,n} \)).
2. The line graph of \( K_{2,n} \).
3. The thin spider with \( n \) legs.
4. The bipartite half-graph of height \( n \).
5. The graph \( H^1_{n,1} \).
6. The graph \( H^*_n \).
7. A prime graph induced by a chain of length \( n \).

Malliaris and Terry prove in [42] an infinitary version of Theorem 32 for infinite graphs, then use it to prove Theorem 33. Their result is the following.

**Theorem 33** (Theorem 6.8 of [42]). An infinite prime graph \( G \) contains one of the following.

1. Copies of \( H_n, \overline{H}_n, H^*_n, \overline{H}^*_n, H^1_{n,1}, \overline{H}^1_{n,1} \) for arbitrarily large finite \( n \),
2. Prime graphs induced by arbitrarily long finite chains,
3. \( K^1_{1,\omega} \) or its complement,
4. The line graph of \( K_{2,\omega} \) or its complement,
5. A spider with \( \omega \) many legs.

The graphs mentioned in the last three items and some infinite versions of the graphs in Item 1 were considered in [61]. In addition, the following characterization of unavoidable infinite prime graphs without infinite clique (or infinite independent set) was given.
**Theorem 34** (Theorem 2 of [61]). An infinite prime graph which does not contain an infinite clique embeds one of the following:

1. The bipartite half-graph of height $\omega$.
2. The infinite one way path.
3. The 1-subdivision of $K_{1,\omega}$.
4. The complement of the line graph of $K_{2,\omega}$.

The graphs mentioned in Theorem 34 are depicted in Figure 2.

5.2. Eleven almost multichainable graphs and their ages. Let $\mathcal{M}$ be the graphs $G_0, G_1, G_3, G_4, G_5$ and $G_6$ depicted in Figures 2 and 3. Let $\mathcal{M}$ be the list of these graphs and their complements. Let $\mathcal{L}$ be the set of the ages of these graphs and of their complements. It should be noted that the graphs $G_5, G_6$ have the same age, hence $\mathcal{L}$ has eleven members.
Theorem 35. Members of $M$ and their complements are almost multichainable and minimal prime. Members of $L$ are distinct and minimal prime, five of them are exhaustible.

Proof. An inspection of the six members of $M$ shows that $G_0, G_1$ and $G_4$ are multichainable with an empty kernel, the three others are almost multichainable with a one-element kernel, in the case of $G_3$ and $G_5$, and a two-element kernel in the case of $G_6$. This gives three exhaustible ages; with the ages of their complements added (and since $G_5$ and $G_5$ have the same age) this gives five exhaustible ages. The fact that these graphs are minimal prime is given in [61]. The second part of the theorem, notably the fact that the ages are distinct and minimal prime is detailed in Chapter 6 page 109 of the first author’s thesis [45]. □

The only prime graphs occurring in Theorem 32 and 33 and not in Theorem 35 are chains. Chains can be represented by words on the alphabet $\{0,1\}$. They will give rise to uncountably many minimal prime ages. We study these graphs and their ages in the next subsection.

5.3. Graphs associated to 0-1 sequences.

Definition 5. To a word $\mu$ we associate the graph $G_\mu$ whose vertex set $V(G_\mu)$ is $\{-1,0,\ldots,n-1\}$ if the domain of $\mu$ is $\{0,\ldots,n-1\}$, $\{-1\}$ or $\mathbb{N}$ if the domain of $\mu$ is $\mathbb{N}$, and $\mathbb{N}^*$ or $\mathbb{Z}$ if the domain of $\mu$ is $\mathbb{N}^*$ or $\mathbb{Z}$. For two vertices $i,j$ with $i < j$ we let $\{i,j\}$ be an edge of $G_\mu$ if and only if

$$\begin{align*}
\mu_j &= 1 \text{ and } j = i + 1, \text{ or } \\
\mu_j &= 0 \text{ and } j \neq i + 1.
\end{align*}$$

For instance, if $\mu$ is the word defined on $\mathbb{N}$ by setting $\mu_i = 1$ for all $i \in \mathbb{N}$, then $G_\mu$ is the infinite one way path on $\{-1\} \cup \mathbb{N}$. Note that if $\mu'$ is the word defined on $\mathbb{N}$ by setting $\mu_i' = 1$ for all $i \in \mathbb{N} \setminus \{1\}$ and $\mu_1' = 0$, then $G_{\mu'}$ is also the infinite one way path. In particular, the graphs $G_\mu$ and $G_{\mu'}$ have the same age but $\mu$ and $\mu'$ do not have the same sets of finite factors.

![Figure 4. 0-1 words of length two and their corresponding graphs.](image1)

![Figure 5. Two distinct 0-1 sequences with isomorphic corresponding graphs.](image2)

Remark 1. If $I$ is an interval of $\mathbb{N}$ and $\mu := (\mu_i)_{i \in I}$ is a 0-1 sequence, then $\overline{G_\mu} = G_{\overline{\mu}}$, where $\overline{\mu} := (\overline{\mu_i})_{i \in I}$ is the 0-1 sequence defined by $\overline{\mu}(i) := \mu(i)+1$ and $+ \text{ is the addition modulo } 2.$
Remark 2. Given a 0-1 graph defined on $\mathbb{N} \cup \{-1\}$ or on $\mathbb{N}^*$ there does not exist necessarily a 0-1 graph on $\mathbb{Z}$ with the same age.

Indeed, (a) Let $\mu := 100111 \ldots$ be an infinite word on $\mathbb{N}$ (the corresponding graph is depicted in (a) of Figure 6). There does not exist a word $\mu'$ on $\mathbb{N}^*$ or $\mathbb{Z}$ such that $\text{Age}(G_{\mu'}) = \text{Age}(G_{\mu})$. (b) Let $\nu := \ldots 11100$ be an infinite word on $\mathbb{N}^*$ (the corresponding graph is depicted in (b) of Figure 6). There does not exist a word $\nu'$ on $\mathbb{N}$ or $\mathbb{Z}$ such that $\text{Age}(G_{\nu'}) = \text{Age}(G_{\nu})$.

Proof of (a): Every vertex of the graph $G_{\mu}$ has finite degree. Suppose for a contradiction that there exists a word $\mu'$ on $\mathbb{N}^*$ or $\mathbb{Z}$ such that $\text{Age}(G_{\mu'}) = \text{Age}(G_{\mu})$. Then there exists $i \in \mathbb{Z}$ such that $\mu'(i) = 0$ because otherwise $G_{\mu'}$ would be a path and hence $\text{Age}(G_{\mu'}) \neq \text{Age}(G_{\mu})$. But then the vertex $i$ of $G_{\mu'}$ would have infinite degree which is impossible since every vertex of the graph $G_{\mu}$ has finite degree.

Proof of (b): The graph $G_{\nu}$ has two vertices of infinite degree. Suppose for a contradiction that there exists a word $\nu'$ on $\mathbb{N}$ or $\mathbb{Z}$ such that $\text{Age}(G_{\nu}) = \text{Age}(G_{\nu'})$. Then $\nu'$ must take the value 0 on an infinite subset of $I$ of $\mathbb{Z}$ because otherwise every vertex of $G_{\nu'}$ would have finite degree which is impossible since $\text{Age}(G_{\nu}) = \text{Age}(G_{\nu'})$. Let $I' \subseteq I$ be an infinite set of nonconsecutive integers. Then $G_{\nu'}$ induces an infinite clique on $I'$. This is not possible since the only cliques of $G_{\nu}$ have cardinality 3.

Remark 3. Given a word $\nu$ we associate the graph $G_{\nu}$ whose vertex set $V(G_{\nu})$ is $\{-n+1, \ldots , 0, 1\}$ if the domain of $\nu$ is $\{-n+1, \ldots , 0\}$, $\mathbb{N}$ or $\mathbb{Z}$ if the domain of $\nu$ is $\mathbb{N}$ or $\mathbb{Z}$ respectively, and $\mathbb{N}^* \cup \{1\}$ if the domain of $\nu$ is $\mathbb{N}^*$. For two vertices $i, j$ with $i < j$ we let \{i, j\} be an edge of $G_{\nu}$ if and only if
\[
\nu_i = 1 \text{ and } j = i + 1, \text{ or } \\
\nu_i = 0 \text{ and } j \neq i + 1.
\]

If $\nu$ is of domain $\{0, \ldots , n - 1\}$, $\mathbb{N}$, $\mathbb{N}^*$ or $\mathbb{Z}$ define $\nu^*$ to be the sequence of domain is $\{-n+1, \ldots , 0\}$, $\mathbb{N}^*$, $\mathbb{N}$ or $\mathbb{Z}$ respectively by setting $\nu^*(i) := \nu(-i)$. Then $G_{\nu^*}$ and $G_{\nu}$ are isomorphic.

Remark 4. For every word $\mu$ the graph $G_{\mu}$ is the union of at most two infinite cliques and at most two infinite independent sets.

To see that, let $\mu$ be a 0-1 sequence on an infinite interval $J$ of $\mathbb{Z}$.
If \( \mu \) takes the value 0 or the value 1 finitely many times, then there exists a finite interval \( K \) of \( J \) such that \( G_{\mu \mid J \setminus K} \) has at most two connected components and either each connected component is an infinite path or the complement of an infinite path.

If \( \mu \) takes the values 0 and 1 infinitely many times, let \( J_0 := \{ j \in J : \mu(j) = 0 \} \) and \( J_1 := \{ j \in J : \mu(j) = 1 \} \). For \( i \in \{0,1\} \) let \( C_i := \{ \min(J_0) + i + 2k : k \in \mathbb{N} \} \). Note that it is possible for \( C_0 \) or \( C_1 \) to be empty, for an example consider the periodic sequence \( \mu := 011011 \ldots \). Then \( \{C_0, C_1\} \) is a partition of \( J_0 \) and \( G_\mu \) induces a clique on \( C_0 \) and on \( C_1 \). Similarly, for \( i \in \{0,1\} \) let \( I_i := \{ \min(J_1) + i + 2k : k \in \mathbb{N} \} \). Note that it is possible for \( I_0 \) or \( I_1 \) to be empty, for an example consider the periodic sequence \( \mu := 100100 \ldots \). Then \( \{I_0, I_1\} \) is a partition of \( J_1 \) and \( G_\mu \) induces an independent set on \( I_0 \) and on \( I_1 \).

Here is our first result.

**Theorem 36.** For every 0-1 word \( \mu \) the age \( \text{Age}(G_\mu) \) consists of permutation graphs.

The proof of Theorem 36 is given in Section 6. It follows from the Compactness Theorem of First Order Logic and Lemma 48. It was brought to us by Brignall [12] that chains are the same objects as pin sequences (see [13] Subsection 2.6. p.41).

The next result is about the number of hereditary classes of finite permutation graphs. It is easy to prove and well known that there are \( 2^{\aleph_0} \) such classes. This is due to the existence of infinite antichains among finite permutation graphs.

In general, it is not true that two words with different sets of finite factors give different ages. But, we prove:

**Theorem 37.** Let \( \mu \) and \( \mu' \) be two words. If \( \mu \) is recurrent and \( \text{Age}(G_\mu) \subseteq \text{Age}(G_{\mu'}) \), then \( \text{Fac}(\mu) \subseteq \text{Fac}(\mu') \).

Using this result and the fact that there are \( 2^{\aleph_0} \) 0-1 recurrent words with distinct sets of factors, we obtain the following.

**Corollary 38.** There are \( 2^{\aleph_0} \) ages of permutation graphs.

The ages we obtain in Theorem 38 are not necessarily well-quasi-ordered. To obtain well-quasi-ordered ages, we consider graphs associated to uniformly recurrent sequences.

**Theorem 39.** Let \( \mu \) be a 0-1 sequence on an infinite interval of \( \mathbb{Z} \). The following propositions are equivalent.

(i) \( \mu \) is uniformly recurrent.
(ii) \( \mu \) is recurrent and \( \text{Age}(G_\mu) \) is minimal prime.

The proofs of Theorem 37 and 39 are given in Section 9 and 10.

As it is well known, there are \( 2^{\aleph_0} \) uniformly recurrent words with distinct sets of factors (e.g. Sturmian words with different slopes, see Chapter 6 of [62]). With Theorem 37 we get:

**Corollary 40.** There are \( 2^{\aleph_0} \) ages of permutation graphs which are minimal prime.

Theorem 38 asserts that minimal prime ages are well-quasi-ordered. Since minimal prime ages are incomparable when ordered by set-inclusion, it follows from Corollary 40 that the set of well-quasi-ordered ages of permutation graphs, when ordered by set inclusion, has an uncountable antichain. On the other hand, observe that the chains are countable.
Problem 5. Does every uncountable set of ages of permutation graphs, when ordered by set inclusion, contain an uncountable antichain of ages?

Remark 5. When \( \mu \) is uniformly recurrent, the age \( C \) of \( G_\mu \) is w.q.o. since it is minimal prime. In fact, it is b.q.o. Indeed, since Fac(\( \mu \)) is Jónsson, it is b.q.o. (see [51, 17]). From the extension of Higman’s Theorem to b.q.o, the set \((\text{Fac}(\mu))^*\) of finite sequences of members of Fac(\( \mu \)), once equipped with the Higman’s ordering of finite sequences, is b.q.o. If \( s := (u_0, \ldots, u_k) \in (\text{Fac}(\mu))^* \), we may represent it by a sequence \( u'_0, \ldots, u'_k \) of factors of \( \mu \) in such a way that \( u'_i \) is before \( u'_{i+1} \) and not contiguous to it. The graph induced by \( G_\mu \) on this union of factors does not depend of the representation. Denote it by \( G(s) \). Observe that the map which associate \( G(s) \) to each \( s \) is order preserving. It follows that its range is b.q.o. Once observed that this range is \( C \), the result follows.

Remark 6. If \( \mu \) is periodic, the collection \( C^{\omega} \), of countable \( G \) such that \( \text{Age}(G) \subseteq C \) is 1\(-\)w.q.o. (and, in fact, 1\(-\)b.q.o. But if \( \mu \) is uniformly recurrent and not periodic, \( C^{\omega} \) is not w.q.o. (indeed, the sequence of \( G_n := G_{\mu_{[n, \omega[}}, n \in \mathbb{N} \), is strictly decreasing). This simple fact is a reason for using uniformly recurrent sequences in the theory of relations.

If \( \mu \) is an infinite word, then \( \text{Age}(G_\mu) \) is not 2\(-\)w.q.o. However,

Theorem 41. If \( \mu \) is an infinite word, \( \text{Age}(G_\mu) \) is 1\(-\)w.q.o. if and only if \( \mu \) is periodic.

Proof. Suppose that \( \text{Age}(G_\mu) \) is 1\(-\)w.q.o. We claim that Fac(\( u \)) is w.q.o. for the suffix order. According to Theorem [4], this implies that \( \mu \) is periodic. The proof of our claim is based on the following observation. Let \( w := w_0 \ldots w_n \) and \( w' := w'_0 \ldots w'_n \) be two finite words. Then \( w \) is a suffix of \( w' \) if and only if the labelled graph \( (G_w, n) \) embeds in the labelled graph \( (G_{w'}, n') \). Indeed, if \( n \) is mapped to \( n' \), then since \( n - 1 \) is the unique neighbour or nonneighbour of \( n \) in \( G_w \) we infer that \( n - 1 \) is mapped to the unique neighbour or nonneighbour of \( n' \) in \( G_w' \). Hence, the labelled graphs obtained by deleting \( n \) and \( n' \) and labelling them \( n - 1 \) and \( n' - 1 \) embed in each other. Conversely, if \( \mu \) is periodic, then according to Theorem [4] Fac(\( \mu \)) is w.q.o. for the prefix and the suffix order. We prove first that the collection of \( (G_w, a_w) \), where \( w \in \text{Fac}(\mu) \) and \( a_w \) is a constant, is w.q.o. (decompose each \( G_w \) into an initial part and a final part containing only the label \( a_w \). An infinite sequence of such labelled graphs yields two infinite sequences; extract an increasing sequence from the first and then an increasing sequence from the corresponding sequence. This yields an increasing sequence). From that fact, the proof that \( \text{Age}(G_\mu) \) is w.q.o. is as in Remark [5].

Permutation graphs come from posets and from bichains. Let us recall that a bichain is relational structure \( R := (V, (\le', \le'')) \) made of a set \( V \) and two linear orders \( \le' \) and \( \le'' \) on \( V \). If \( V \) is finite and has \( n \) elements, there is a unique permutation \( \sigma \) of \( \{1, \ldots, n\} \) for which \( R \) is isomorphic to the bichain \( C_\sigma := ((\{1, \ldots, n\}, \le, \le_\sigma) \) where \( \le \) is the natural order on \( n := \{1, \ldots, n\} \) and \( \le_\sigma \) is the linear order defined by \( i \le_\sigma j \) if \( \sigma(i) \le \sigma(j) \).

If we represent bichains by permutations, embeddings between bichains is equivalent to the pattern containment between the corresponding permutations, see Cameron [16].

To a bichain \( R := (V, (\le', \le'')) \), we may associate the intersection order \( o(R) := (V, \le' \cap \le'') \) and to \( o(R) \) its comparability graph.

The following is Theorem 67 from [60].
Theorem 42. (1) Let $P := (V, \leq)$ be a poset. Then $\text{Age}(\text{Inc}(P))$ is minimal prime if and only if $\text{Age}(\text{Comp}(P))$ is minimal prime. Furthermore, $\text{Age}(P)$ is minimal prime if and only if $\text{Age}(\text{Inc}(P))$ is minimal prime and $\downarrow \text{Prim}(\text{Age}(P)) = \text{Age}(P)$.

(2) Let $B := (V, (\leq_1, \leq_2))$ be a bichain and $o(B) := (V, \leq \cap \leq_2)$. Then $\text{Age}(B)$ is minimal prime if and only if $\text{Age}(o(B))$ is minimal prime and $\downarrow \text{Prim}(\text{Age}(B)) = \text{Age}(B)$.

With (2) of Theorem 42 and Corollary 40 we have:

Theorem 43. There are $2^{20}$ ages of bichains and permutation orders which are minimal prime.

5.4. A complete characterization of minimal prime ages of graphs.

Theorem 44. A hereditary class $\mathcal{C}$ of finite graphs is minimal prime if and only if $\mathcal{C} = \text{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$, or $\mathcal{C} \in \mathcal{L}$.

Proof. $\Rightarrow$. Follows from Theorems 35 and 39 and Chapter 6 page 109 of the first author’s thesis [45].

$\Rightarrow$. Follows essentially from Theorem 32. Let $\mathcal{C}$ be a minimal prime age. Then $\mathcal{C}$ contains infinitely many prime graphs of one of the types given in Theorem 32. If for an example, $\mathcal{C}$ contains infinitely many chains, that is graphs of the form $G_\mu$ for $\mu$ finite, then, since it is minimal prime, we claim that this is the age of some $G_\mu$ with $\mu$ uniformly recurrent. Indeed, let $\mathcal{A}$ be an age containing 0-1 graphs $G_w$ for arbitrarily long finite words $w$. We prove that $\mathcal{A}$ contains the age of a graph $G_\mu$ where $\mu$ is a uniformly recurrent word. Indeed, let $W$ be the set of finite words $w$ such that $G_w \in \mathcal{A}$. Clearly, $W$ is an infinite hereditary set of finite words. It follows from Lemma 3 that $W$ contains an initial segment $U$ which is Jónsson. It follows from the equivalence $(i) \iff (iii)$ of Theorem 3 that $U = \text{Fac}(\mu)$ where $\mu$ is a uniformly recurrent word. We now prove that $\text{Age}(G_\mu) \subseteq \mathcal{A}$. Let $H \in \text{Age}(G_\mu)$. There exists then $w \in \text{Fac}(\mu)$ such that $H$ is an induced subgraph of $G_w$. But $w \in \text{Fac}(\mu) \subseteq W$. Thus $G_w \in \mathcal{A}$ as required. For the other cases, use the structure of the infinite graphs described in Figures 2 and 3.

Theorem 45. (1) A minimal prime hereditary class $\mathcal{C}$ of finite graphs is hereditary well-quasi-ordered if and only if $\mathcal{C} \in \mathcal{L}$.

(2) A minimal prime hereditary class $\mathcal{C}$ of finite graphs remains well-quasi-ordered when just one label is added if and only if $\mathcal{C} = \text{Age}(G_\mu)$ for some periodic 0-1 word on $\mathbb{N}$, or $\mathcal{C} \in \mathcal{L}$.

The corresponding characterization of minimal prime ages of posets and bichains will follow from Theorem 42 and a careful examination of our list of graphs to decide which graphs are comparability graphs.

Corollary 46. (1) A hereditary class $\mathcal{C}$ of finite comparability graphs is minimal prime if and only if $\mathcal{C} = \text{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$, or $\mathcal{C} \in \{\text{Age}(G_0), \text{Age}(G_1), \text{Age}(G_3), \text{Age}(G_5), \text{Age}(G_6), \text{Age}(\overline{G}_0)\}$.

(2) A hereditary class $\mathcal{C}$ of finite permutation graphs is minimal prime if and only if $\mathcal{C} = \text{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$, or $\mathcal{C} \in \{\text{Age}(G_1), \text{Age}(\overline{G}_1), \text{Age}(G_5), \text{Age}(G_6), \text{Age}(\overline{G}_0)\}$.

We end this section with the following conjecture.

Conjecture 1. Every infinite prime graph embeds one of the graphs depicted in Figures 2 and 3, or a graph $G_\mu$ for some 0-1 sequence $\mu$ on an interval of $\mathbb{Z}$.  

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5.5. **Bounds of minimal prime hereditary classes.** We recall that a **bound** of a hereditary class \( C \) of finite structures (e.g. graphs, ordered sets) is any structure \( R \not\in C \) such that every proper induced substructure of \( R \) belongs to \( C \).

As we have seen in Theorem 44 minimal prime ages of graphs belong either to \( \mathcal{L} \), in which case they have finitely many bounds since they are ages of multichainable graphs (Theorem 31), or they are of the form \( \text{Age}(G_\mu) \) with \( \mu \) uniformly recurrent.

If \( \mu \) is a 0-1 periodic word, \( \text{Age}(G_\mu) \) may have infinitely many bounds. This is the case if \( \mu \) is constant. For the remaining cases within uniformly recurrent sequences, we have the following.

**Theorem 47.** Let \( \mu \) be a 0-1 uniformly recurrent word.

1. If \( \mu \) is non periodic, then \( \text{Age}(G_\mu) \) has infinitely many bounds;
2. If \( \mu \) is periodic and non constant, then \( \text{Age}(G_\mu) \) has finitely many bounds.

In [14], Brignall et al provided an example of a hereditary class of permutation graphs which are w.q.o., have finitely many bounds, but are not labelled w.q.o. solving negatively a conjecture of Korpelainen et al [33].

As stated in (2) of Theorem 47, ages of 0-1 graphs corresponding to periodic and non constant words provide infinitely many examples of such classes. Note these classes are \( 1^- \)-w.q.o.

6. **A proof of Theorem 36 and a characterization of order types of realizers of transitive orientations of 0-1 graphs**

Let \( P : (V, \leq) \) be a poset. An element \( x \in V \) is extremal if it is maximal or minimal.

**Lemma 48.** Let \( w := w_0 \ldots w_{n-1} \) be a finite word with \( n \geq 2 \) and \( w' := w_0 \ldots w_{n-2} \). Then every realizer \((L_{w'},M_{w'})\) of a transitive orientation of \( G_{w'} \) on \( \{-1,0,\ldots,n-2\} \) (if any) such that \( n-2 \) is extremal in \( L_{w'} \) or in \( M_{w'} \) extends to a realizer \((L_w,M_w)\) of a transitive orientation of \( G_w \) on \( \{-1,0,\ldots,n-1\} \) such that \( n-1 \) is extremal in \( L_w \) or in \( M_w \).

**Proof.** Let \((L_{w'},M_{w'})\) be a realizer of a transitive orientation \( P_{w'} \) of \( G_{w'} \) on \( \{-1,0,\ldots,n-2\} \) such that \( n-2 \) is extremal in \( L_{w'} \) or in \( M_{w'} \). We may assume without loss of generality that \( n-2 \) is maximal in \( L_{w'} \) or in \( M_{w'} \). Otherwise, consider \( P_{w'}^* \) and the pair \((L_{w'}^*,M_{w'}^*)\). Note that \( P_{w'}^* \) is a transitive orientation of \( G_{w'} \), the pair \((L_{w'}^*,M_{w'}^*)\) is a realizer of \( P_{w'}^* \), and \( n-2 \) is maximal in \( L_{w'}^* \) or in \( M_{w'}^* \) (this is because \( n-2 \) is minimal in \( L_{w'} \) or in \( M_{w'} \)). We then extend \((L_{w'}^*,M_{w'}^*)\) to a realizer of \( P_{w'}^* \) with the desired property. The dual of this realizer is a realizer of \( P_w \) with the required property. We may also suppose that \( n-2 \) is maximal in...
\(L_{w'}\), because otherwise, we interchange the roles of \(L_{w'}\) and \(M_{w'}\).

- If \(w_{n-1} = 1\), then \(\{n - 2, n - 1\}\) is the unique edge of \(G_w\) containing \(n - 1\). Clearly \(P_w := P_{w'} \cup \{(n - 1, n - 2)\}\) is a transitive orientation of \(G_w\). Let \(L_w\) be the linear order obtained from \(L_{w'}\) so that \(n - 1\) appears immediately before \(\max(L_{w'}) = n - 2\) and larger than all other elements and let \(M_w\) be the linear order obtained from \(M_{w'}\) by letting \(n - 1\) smaller than all elements of \(M_{w'}\). Clearly, \((L_w, M_w)\) is a realizer of \(P_w\) and by construction \(n - 1\) is minimal in \(P_w\) and in \(M_w\).

- Else if \(w_{n-1} = 0\), then \(\{n - 2, n - 1\}\) is the unique non edge of \(G_w\) containing \(n - 1\). Since \(n - 2\) is maximal in \(P_{w'}\) we infer that \(P_w := P_{w'} \cup \{(x, n - 1) : x \in \{-1, 0, \ldots, n - 3\}\}\) is a transitive orientation of \(G_w\) in which \(n - 1\) and \(n - 2\) are incomparable. Let \(L_w\) be the linear order obtained from \(L_{w'}\) so that \(n - 1\) appears immediately before \(\max(L_{w'}) = n - 2\) and larger than all other elements and let \(M_w\) be the linear order obtained from \(M_{w'}\) by letting \(n - 2\) larger than all elements of \(M_{w'}\). Clearly, \((L_w, M_w)\) is a realizer of \(P_w\) (indeed, \(n - 2\) and \(n - 1\) are incomparable in \(L_w \cap M_w\) and for all \(x \in \{-1, 0, \ldots, n - 3\}\), \(x < n - 1\) in \(L_w \cap M_w\) proving that \(\{L_w, M_w\}\) is a realizer of \(P_w\)). By construction \(n - 1\) is maximal in \(P_w\) and \(M_w\). The proof of the lemma is now complete. \(\square\)

**Lemma 49.** Let \(\mu\) be a 0-1 sequence defined on an interval \(I\) of \(\mathbb{Z}\). Then \(G_{\mu}\) is a comparability graph and an incomparability graph. In particular, if \(I\) is finite, then \(G_{\mu}\) is a permutation graph.

**Proof.** We consider two cases.

(a) \(I\) is finite or \(I\) is a final segment of \(\mathbb{Z}\) bounded below. Write \(I := \{i_0, \ldots, i_n, \ldots\}\) and define for every \(n\) a realizer \((L_n, M_n)\) of a transitive orientation of the restriction of \(G_{\mu}\) to \(\{i_0 - 1, i_0, \ldots, i_n\}\). For that, use Lemma 48 and induction on \(n\). Note that for \(n = 0\), the restriction of \(G_{\mu}\) to \(\{i_0 - 1, i_0\}\) is either a 2-element independent set or a 2-element clique, and these are permutation graphs. Then \((L_{\mu}, M_{\mu})\) where \(L_{\mu} := \bigcup_{n \in I} L_n\) and \(M_{\mu} := \bigcup_{n \in I} M_n\) is a realizer of a transitive orientation of \(G_{\mu}\). Hence \(G_{\mu}\) a comparability and an incomparability graph, and a permutation graph if \(I\) is finite.

(b) \(I\) is an initial segment of \(\mathbb{Z}\). In this case, if \(F\) is any finite subset of \(I\), let \(J\) be a finite interval of \(I\) containing \(F\). Let \(w\) be the restriction of \(\mu\) to \(J \setminus \{\min(J)\}\). Then the graph induced by \(G_{\mu}\) on \(J\) is \(G_w\). It follows from (a) that \(G_w\) is a permutation graph, hence \(G_{\mu|F}\) is permutation graph. It follows from the Compactness Theorem of First Order Logic that \(G_{\mu}\) is a comparability and an incomparability graph. \(\square\)

Theorem 30 readily follows from Lemma 49.

The remainder of this section is devoted to characterizing the order types of linear extensions in a realizer of a transitive orientation of the graph \(G_{\mu}\) in the case \(\mu\) is a 0-1 word on \(\mathbb{N}\).

**Lemma 50.** Let \(w := w_0 \ldots w_{n-1}\) be a finite word with \(n \geq 3\). If \((L_w, M_w)\) is a realizer of a transitive orientation of \(G_w\) on \(\{-1, 0, \ldots, n - 1\}\) constructed step by step by means of Lemma 48, then for all \(0 \leq k \leq n - 3\) the set \(\{k + 2, \ldots, n - 1\}\) does not meet the intervals of \(L_w\) and \(M_w\) generated by \(\{-1, 0, \ldots, k\}\).

**Proof.** Let \(k \in \{0, \ldots, n - 3\}\) and \(j \in \{k + 2, \ldots, n - 1\}\).

**Case 1:** \(j > k + 2\).

Suppose \(w_j = 0\). Then \(j\) is adjacent to all vertices of \(\{-1, 0, \ldots, k + 1\}\). It follows from
the algorithm described in Lemma \[48\] that in a transitive orientation of \(G_w\) the vertex \(j\) is larger than all elements of \(-1,0,\ldots,k+1\) or the vertex \(j\) is smaller than all elements of \(-1,0,\ldots,k+1\). Hence, if \(\{L_w,M_w\}\) is a realizer of a transitive orientation of \(G_w\), then \(j\), in both \(L_w\) and \(M_w\), is either above all elements of \(-1,0,\ldots,k+1\) or is below all elements of \(-1,0,\ldots,k+1\). Hence, \(j \notin I\). We now consider the case \(w_j = 1\). Then \(j\) is not adjacent to any vertex of \(-1,0,\ldots,k+1\). Hence, if \(\{L_w,M_w\}\) is a realizer of a transitive orientation of \(G_w\), then \(j\) is either above all elements of \(-1,0,\ldots,k+1\) in \(L_w\) and below all elements of \(-1,0,\ldots,k+1\) in \(M_w\), or \(j\) is below all elements of \(-1,0,\ldots,k+1\) in \(L_w\) and above all elements of \(-1,0,\ldots,k+1\) in \(M_w\). Hence, \(j \notin I\).

**Case 2: \(j = k + 2\).**

We may assume without loss of generality that \(k + 1\) is maximal in the restriction of a transitive orientation \(P\) of \(G_w\) and \(L_w\) to \(-1,0,\ldots,k+1\) (otherwise consider the dual of \(P\) which is a transitive orientation of the restriction of \(G_w\) to \(-1,0,\ldots,k+1\)). It follows from the algorithm described in Lemma \[48\] that \(k + 2 \notin I\) as required.

As it is customary, we denote by \(\omega\) the order type of \(\mathbb{N}\), by \(\omega^*\) the order type of its dual and by \(\omega^* + \omega\) the order type of \(\mathbb{Z}\).

The proof of the following Lemma is easy and is left to the reader.

**Lemma 51.** (1) The intersection of two linear orders of order type \(\omega\) is a w.q.o.

(2) The intersection of two linear orders of order types \(\omega\) and \(\omega^*\) has no infinite chains.

(3) The intersection of two linear orders of order types \(\omega\) and \(\omega^* + \omega\) is well founded.

**Corollary 52.** Let \(\mu\) be a word on \(\mathbb{N}\). If \((L,M)\) is a realizer of a transitive orientation of \(G_\mu\), then the order types of \(L\) and \(M\) embed into \(\omega^* + \omega\). Furthermore, if \(\mu\) has finitely many 0’s or 1’s, then the order types of \(L\) and \(M\) embed into \(\omega\) or \(\omega^*\), else at least one of \(L\) and \(M\) have order type \(\omega^* + \omega\).

**Proof.** Let \((L,M)\) be a realizer of a transitive orientation of \(G_\mu\). According to Lemma \[50\] for every \(k \in \mathbb{N}\), the least interval of \(L\) containing \(-1,0,\ldots,k\) is included in \(-1,0,\ldots,k+1\). Hence \(L\) is a countable increasing union of finite intervals, proving that \(L\) embeds in \(\mathbb{Z}\).

If \(\mu\) has finitely many 0’s or 1’s, then there exists a final interval \(I\) of \(\mathbb{N}\) such that the restriction of \(G_\mu\) to \(I\) is an infinite one way path or the complement of an infinite one way path. It can be easily seen that the order types in a realizer of transitive orientations of an infinite one way path or its complement are \(\{\omega,\omega\}\) or \(\{\omega,\omega^*\}\) or \(\{\omega^*,\omega^*\}\). Since \(\mathbb{N} \setminus I\) is an initial segment of \(\mathbb{N}\) we have that the order types of a linear extension in a realizer \(P_\mu\) are \(\{\omega,\omega\}\) or \(\{\omega,\omega^*\}\) or \(\{\omega^*,\omega^*\}\). Next we suppose that \(\mu\) has infinitely many 0’s and 1’s. There exists then two infinite subsets of nonconsecutive integers \(J\) and \(K\) so that \(\mu\) is constant on \(J\) and \(K\), and \(\mu\) takes the value 1 on \(J\) and takes the value 0 on \(K\). Then \(P_\mu\) has an infinite antichain, induced by the set \(J\), and an infinite chain, induced by the set \(K\). The order types of a linear extension in a realizer of \(P_\mu\) cannot be \(\{\omega,\omega\}\) or \(\{\omega^*,\omega^*\}\) because otherwise \(P_\mu\) or its dual is w.q.o and hence has no infinite antichains. The order types of a linear extension in a realizer of \(P_\mu\) cannot be \(\{\omega,\omega^*\}\) either because otherwise all chains of \(P_\mu\) would be finite.

We now provide examples of \(P_\mu\) that have realizers of type \((\omega,\omega^* + \omega)\) and \((\omega^* + \omega,\omega^* + \omega)\).

**Example 1.** Let \(\mu := 001100110011\ldots\). The order types of a linear extension in a realizer of \(P_\mu\) are \(\omega\) and \(\omega^* + \omega\). Indeed, an embedding of \(P_\mu\) into \(\mathbb{N} \times \mathbb{Z}\) is depicted in Figure \[8\].
follows easily that $P_\mu$ has a realizer of type $(\omega, \omega^* + \omega)$. Since $P_\mu$ is prime it has a unique realizer up to a transposition.

**Example 2.** Let $\mu := 011011011 \ldots$. The order types of a linear extension in a realizer of $P_\mu$ is $\mathbb{Z}$. Indeed, an embedding of $P_\mu$ into $\mathbb{Z} \times \mathbb{Z}$ is depicted in Figure 9. It follows easily that $P_\mu$ has a realizer of type $(\omega^* + \omega, \omega^* + \omega)$. Since $P_\mu$ is prime it has a unique realizer up to a transposition.

We should mention that in the first example $G_\nu$ nor its complement are permutation graphs, while in the second example both are.

### 7. Modules in $G_\mu$

The aim of this section is to characterize the modules of a 0-1 graph. We prove among other things, that if $G_\mu$ is not prime, then $\mu$ contains large factors of 0’s or 1’s. Results of this section will be used in Section 8 to derive properties of embeddings between 0-1 graphs.

We recall that if $G := (X, E)$ is a graph, then a subset $M$ of $X$ is called a *module* in $G$ if for every $x \notin M$, either $x$ is adjacent to all vertices of $M$ or $x$ is not adjacent to any vertex of $M$.

The following lemma will be useful.

**Lemma 53.** A graph and its complement have the same set of modules. In particular, $G_\mu$ and $G_{\overline{\mu}}$ have the same modules.
Figure 9. An embedding into $\mathbb{Z} \times \mathbb{Z}$ of a transitive orientation of the graph corresponding to the periodic 0-1 sequence $\mu := 011011011 \ldots$.

Lemma 53 and Remark 1 of subsection 5.3 combined together will allow us to simplify proofs. Indeed, if we are arguing on the value of $\mu$ on a particular integer $i$ we may only consider the case $\mu(i) = 0$ (or $\mu(i) = 1$).

We recall some properties of modules in a graph. The proof of the following lemma is easy and is left to the reader (see [24]).

**Lemma 54.** Let $G = (V, E)$ be a graph. The following propositions are true.

1. The intersection of a nonempty set of modules is a module (possibly empty).
2. The union of two modules with nonempty intersection is a module.
3. For two modules $M$ and $N$, if $M \setminus N \neq \emptyset$, then $N \setminus M$ is a module.

Let $G := (X, E)$ be a graph and $\{x, y, z\} \subseteq X$. We say that $z$ separates $x$ and $y$ if $\{z, x\}$ is an edge and $\{z, y\}$ is not and edge, or vice versa. For instance,

- if $I$ is an interval of $\mathbb{N}$, $\mu$ is a 0-1 sequence on $I$ and $i \in I$, then $i$ separates $i - 1$ and $j$ for all $j < i - 1$ in $G_\mu$. (Indeed, $\{j, i\}$ is an edge if and only if $\{i - 1, i\}$ is not an edge).

**Lemma 55.** Let $G$ be a graph and $\{x, y, z\} \subseteq V(G)$. If $z$ separates $x$ and $y$ and if $x$ and $y$ belong to a module in $G$, then $z$ belongs to that module.

**Lemma 56.** Let $\mu$ be a 0-1 sequence on an interval $I := \{i_1, \ldots, i_n, \ldots\}$ of $\mathbb{N}$ and let $i_0 := i_1 - 1$. Let $J \subseteq \{i_0\} \cup I$ be a nonempty subset. Let $J^-$ be the maximal initial segment of $J$ which is
an interval of \( \{i_0\} \cup I \). If \( J \) is not an interval of \( \{i_0\} \cup I \), then \( J^- \) is a module of \( G_{\mu_1J} \). In particular, if \( J^- \) is not a singleton, then \( G_{\mu_1J} \) is not prime.

Proof. If \( J \) is not an interval of \( \{i_0\} \cup I \), \( J \setminus J^- \) is nonempty. Furthermore, no element of \( J \setminus J^- \) separates two elements of \( J^- \) (indeed, the element \( i_k := \max(J^-) + 1 \) does not belong to \( J^- \) and is the only element of \( I \) that separates two elements of \( J^- \)). Therefore \( J^- \) is a module of \( G_{\mu_1J} \). If \( J^- \) is not a singleton, then since it is distinct from \( J \), it is a nontrivial module of \( G_{\mu_1J} \) and therefore \( G_{\mu_1J} \) is not prime.

\[ \square \]

Corollary 57. Let \( \mu \) be a 0-1 word on \( \mathbb{N} \) and let \( F \subseteq \{-1\} \cup \mathbb{N} \) be such that \( G_{\mu_1F} \) is prime. Then \( F \setminus \{\min(F)\} \) is an interval of \( \mathbb{N} \).

Proof. We apply Lemma 56 with \( J := F \setminus \{\min(F)\} \). It follows that if \( J \) is not an interval of \( I \), then \( J^- \) is a module of \( G_{\mu_1J} \). Since no element of \( J \setminus J^- \) separates two elements of \( \{\min(F)\} \cup J^- \) we infer that \( \{\min(F)\} \cup J^- \) is a module of \( G_{\mu_1F} \) which is prime. Hence, if \( J^- \) is not empty \( \{\min(F)\} \cup J^- \) is a nontrivial module of \( G_{\mu_1F} \) which is impossible. This proves that \( F \setminus \{\min(F)\} \) is an interval of \( \mathbb{N} \) as required.

\[ \square \]

Lemma 58. Let \( I := \{i_1, \ldots, i_n, \ldots\} \) be an interval of \( \mathbb{N} \) of cardinality at least 3, \( \mu \) be a 0-1 sequence on \( I \) and \( i_0 = i_1 - 1 \). For \( k \geq 2 \), \( \{i_0, \ldots, i_{k-1}\} \) is a module of \( G_\mu \setminus \{i_j\} \) if and only if \( k = j \).

Proof. We only need to prove the forward implication. Suppose that \( \{i_0, \ldots, i_{k-1}\} \) is a module of \( G_\mu \setminus \{i_j\} \). Then \( i_j \notin \{i_0, \ldots, i_{k-1}\} \) and hence \( k \leq j \). Since \( i_k \) is the only vertex that separates \( i_{k-1} \) and \( i_{k-2} \) (recall that \( k \geq 2 \)) we infer that \( j = k \).

\[ \square \]

Corollary 59. Let \( I := \{i_1, \ldots, i_n\} \) be an interval of \( \mathbb{N} \) of cardinality at least 3, \( \mu \) be a 0-1 sequence on \( I \) and \( i_0 = i_1 - 1 \). We suppose \( G_\mu \) is prime and let \( x \in \{i_0\} \cup I \). If \( G_\mu \setminus \{x\} \) is prime, then \( x \in \{i_0, i_1, i_n\} \).

In the next lemma we state some properties of modules of \( G_\mu \) when \( \mu \) is a word on \( \mathbb{N} \). It follows that a nontrivial module of \( G_\mu \) with at least three elements is necessarily the whole domain of \( G_\mu \) minus a singleton.

Lemma 60. Let \( I := \{i_1, \ldots, i_n, \ldots\} \) be an interval of \( \mathbb{N} \), \( i_0 = i_1 - 1 \) and let \( \mu \) be a 0-1 sequence on \( I \). Let \( M \) be a nontrivial module of \( G_\mu \).

(1) \( \text{Let } i_j, i_k \in M \text{ with } j < k. \)

(a) If \( i_{k+1} \in I \), then \( i_{k+1} \in M \).
(b) \( \{m \in I : i_k \leq m\} \subseteq M \).
(c) exactly one of \( i_0 \) and \( i_1 \) is in \( M \).

(2) The largest final segment \( F \) of \( I \) included in \( M \) is nonempty.

(3) Assume \( F \) has at least two elements. Then

(a) \( \mu \) is constant on \( F \setminus \{\min(F)\} \) and \( \mu(\min(F)) \neq \mu(\min(F) + 1) \).
(b) \( \{m \in I : m \leq \min(F) - 1\} \subseteq M \).
(c) \( F = I \) or \( F = I \setminus \{i_1\} \).

Proof. (1) Let \( i_j, i_k \in M \) with \( j < k \).

(a) Suppose \( i_{k+1} \in I \). Since \( i_j, i_k \in M \) and \( i_{k+1} \) separates \( i_j \) and \( i_k \) we infer that \( i_{k+1} \in M \).

This proves Item (1)(a).
(b) Since $M$ contains at least two distinct elements, Item (1)(b) now follows by repeatedly applying Item (1)(a).

(c) Suppose for a contradiction that $\{i_0, i_1\} \subseteq M$. It follows from Item (1)(b) that $I \subseteq M$. This is impossible since $M$ is nontrivial. This proves that at least one of $i_0$ or $i_1$ is not in $M$. Suppose that $M \cap \{i_0, i_1\} = \emptyset$. Let $k$ be the smallest positive integer such that $i_k \in M$ (note that $k \geq 2$). Since $M$ is nontrivial there exists some $j > k$ such that $i_j \in M$. Since $i_{k-1} \notin M$, $i_{k-1}$ cannot separate $i_k$ and $i_j$. Since $i_{k-1}$ and $i_k$ are consecutive, $\mu(i_k) \neq \mu(i_j)$. Hence, $i_0$ separates the two elements $i_k$ and $i_j$ of $M$. Since $i_0 \notin M$ and $M$ is a module, we obtain a contradiction. This proves Item (1)(c).

(2) Let $F$ be the largest final segment of $I$ included in $M$. We prove that $F$ is nonempty. Indeed, since $M$ is nontrivial, it has at least two elements $i_j, i_k$ with $j < k$. From Item (1)(b) it follows that either $i_{k+1} \in I$ and hence the final segment $\{m \in I : i_k \leq m\}$ is nonempty and is a subset of $M$. Or, $\max(\{I\}) = i_k$ and $\{i_k\}$ is a final segment of $I$ and belongs to $M$.

(3) (a) Let $l, m \in F \setminus \{\min(F)\}$. Since $\min(F)-1 \notin M$ and $M$ is a module, we infer that the vertex $\min(F)-1$ of $G_\mu$ is either adjacent to both $l$ and $m$ or not adjacent to both $l$ and $m$. Thus $\mu(l) = \mu(m)$ and $\mu$ is constant on $F \setminus \{\min(F)\}$ as required. Since $\min(F)$ and $\min(F)+1$ are elements of $M$ and $\min(F)-1$ and $\min(F)$ are consecutive in $I$ we must have $\mu(\min(F)) = \mu(\min(F)+1) + 1$, that is $\mu(\min(F)) = \mu(\min(F)+1)$, proving Item (3)(a).

(b) It follows from (3)(a) that every element $m \in I \cup \{i_0\}$ such that $m < \min(F)-1$ is adjacent to $\min(F)$ but not adjacent to $\min(F)+1$ or vice versa. Since $\min(F)$ and $\min(F)+1$ are elements of $M$ and $M$ is a module we infer that $m \in M$. Hence, $\{m \in I : m \leq \min(F)-2\} \subseteq M$ proving item (3)(b).

(c) It follows from (1)(c) and (3)(b) that $\min(F) \in \{i_1, i_2\}$. If $\min(F) = i_1$, then $F = I$. Else if $\min(F) = i_2$, then $F = I \setminus \{i_1\}$. This completes the proof of (3)(c) and of the lemma.

The proof of the lemma is now complete.

Let $I$ be an interval of $\mathbb{N}$ and $G_\mu$ be the graph on $\{\min(I)-1\} \cup I$ associated to a sequence $\mu$ defined on $I$. In the following proposition we characterize the modules of $G_\mu$. We prove that if $\mu \notin \{011, 100, 001, 110\}$, then $G_\mu$ has at most one nontrivial module. Also, if $I$ is finite, then a nontrivial module of $G_\mu$ has necessarily cardinality 2 or $|I|$.

**Proposition 61.** Let $I := \{i_1, \ldots, i_n, \ldots\}$ be an interval of $\mathbb{N}$, $i_0 := i_1 - 1$ and let $\mu$ be a 0-1 sequence on $I$.

1. If $\mu \notin \{011, 100, 001, 110\}$, then $G_\mu$ has at most one nontrivial module.
2. If $M$ is a nontrivial module of $G_\mu$, then either $M = I$ or $M = \{i_0\} \cup I \setminus \{i_1\}$, or $I$ is finite, $I = \{i_1, \ldots, i_n\}$ and $M = \{i_0, i_n\}$ or $M = \{i_1, i_n\}$.

**Proof.** We recall that a graph on at most two vertices is prime. Hence, if $G_\mu$ has a nontrivial module, then $|I| \geq 2$.

**Claim:** If $|I| = 2$, then $G_\mu$ has exactly one nontrivial module.

**Proof of Claim:** We only consider the case $\mu(i_1) = 0$ and deduce the other case by considering $\overline{\mu}$. By inspection, if $\mu(i_2) = 0$, then $\{i_0, i_2\}$ is the only nontrivial module of $G_\mu$. If $\mu(i_2) = 1$, then $\{i_1, i_2\}$ is the only nontrivial module of $G_\mu$. □
We use the characterisation of modules of $G_\mu$ found in Item (2). We consider all possible pairs of such modules.

Case 1. $\{i_0, i_n\}$ and $\{i_1, i_n\}$ are both modules of $G_\mu$.
Since $\{i_0, i_n\} \cap \{i_1, i_n\} \neq \emptyset$ and the union of two modules with nonempty intersection is a module (see Item (2) of Lemma 53), we infer that $A := \{i_0, i_n\} \cup \{i_1, i_n\} = \{i_0, i_1, i_n\}$ is a module of $G_\mu$. It follows from Item (2) that $A$ is trivial. Since $A$ has 3 elements we infer that $A = \{i_0\} \cup I$. This implies that $|I| = 2$ and hence $n = 2$. We derive a contradiction from the Claim.

Case 2. $I$ and $\{i_0\} \cup I \setminus \{i_1\}$ are both modules of $G_\mu$.
Since the intersection of two modules is a module we infer that $A := I \cap (\{i_0\} \cup I \setminus \{i_1\}) = I \setminus \{i_1\}$ is a module of $G_\mu$. It follows from Item (2) that $A$ is trivial. Hence, $A = \emptyset$ or $A$ is a singleton or $A = \{i_0\} \cup I$. This last case is not possible. The case $A = \emptyset$ is also not possible because otherwise $I = \{i_1\}$, which contradicts $|I| \geq 2$. We are left with the case $A$ is a singleton, that is $I$ has two elements. We derive a contradiction from the Claim.

Case 3. $\{i_0, i_n\}$ and $I$ are both modules of $G_\mu$.
We apply Item (3) of Lemma 53 with $M := \{i_0, i_n\}$ and $N := I$. Since $M \setminus N \neq \emptyset$, then $A := N \setminus M = I \setminus \{i_n\}$ is a module of $G_\mu$. It follows from Item (2) that $A$ is trivial. Hence, $A = \emptyset$, $A$ is a singleton or $A = \{i_0\} \cup I$. This last case is not possible. The case $A = \emptyset$ is also not possible because otherwise $I = \{i_n\}$, which contradicts $|I| \geq 2$. We are left with the case $I$ is a singleton, that is $I$ has two elements. We derive a contradiction from the Claim in the case $|I| = 2$.

Case 4. $\{i_1, i_n\}$ and $\{i_0\} \cup I \setminus \{i_1\}$ are both modules of $G_\mu$.
We apply Item (3) of Lemma 53 with $M := \{i_1, i_n\}$ and $N := \{i_0\} \cup I \setminus \{i_1\}$. Since $M \setminus N \neq \emptyset$, then $A := N \setminus M = \{i_0\} \cup I \setminus \{i_1, i_n\}$ is a module of $G_\mu$. It follows from Item (2) that $A$ is trivial. Hence, $A = \emptyset$, $A$ is a singleton or $A = \{i_0\} \cup I$. This last case and the case $A = \emptyset$ are not possible. We are left with the case $A$ is a singleton. Since $i_0 \notin I \setminus \{i_1, i_n\}$ we infer that $I \setminus \{i_1, i_n\} = \emptyset$ and hence $I$ has at most two elements. We derive a contradiction from $|I| \geq 2$ in the case $I$ is a singleton, and from the Claim in the case $|I| = 2$.

Case 5. $\{i_1, i_n\}$ and $I$ are both modules of $G_\mu$.
Since $I$ is a module it follows from (3)(c) of Lemma 60 that $\mu$ is constant on $I \setminus \{i_1\}$ and $\mu(i_1) \neq \mu(i_n)$. Then $n \leq 3$ because otherwise $i_2$ separates $i_1$ and $i_n$ contradicting our assumption that $\{i_1, i_n\}$ is a module in $G_\mu$. It follows that $\mu = 100$ or $\mu = 011$.

Case 6. $\{i_0, i_n\}$ and $\{i_0\} \cup I \setminus \{i_1\}$ are both modules of $G_\mu$.
Since $\{i_0\} \cup I \setminus \{i_1\}$ is a module it follows from (3)(c) of Lemma 60 that $\mu$ is constant on $I \setminus \{i_1, i_2\}$ and $\mu(i_2) \neq \mu(i_n)$. Then $n \leq 3$ because otherwise $i_{n-1}$ separates $i_0$ and $i_n$ contradicting our assumption that $\{i_0, i_n\}$ is a module in $G_\mu$. It follows that $\mu = 110$ or $\mu = 001$.

(2) Let $M$ be nontrivial module of $G_\mu$. Suppose first that $M$ has cardinality at least 3. Let $F$ be the largest final segment of $I$ included in $M$. From (2) of Lemma 60 $F$ is nonempty. Since $M$ has at least 3 elements, it follows from Item (1)(b) and (1)(c) of Lemma 60 that $F$ has at least two elements. It follows from (3)(c) of Lemma 60 that $F = I$ or $F = I \setminus \{i_1\}$. Since $F \subseteq M \subseteq I \cup \{i_0\}$, if $F = I$, then $M = I$. Else, it follows from
(1)(c) of Lemma 60 that \(i_0 \in M\). Hence, \(M = \{i_0\} \cup I \setminus \{i_1\}\).

We now consider the case \(M\) has exactly two elements. It follows from Item (1)(b) of Lemma 60 that \(i_n \in M\). It follows from Item (1)(c) of Lemma 60 that exactly one of \(i_0\) and \(i_1\) is in \(M\). Hence, \(M = \{i_0, i_n\}\) or \(M = \{i_1, i_n\}\).

The proof of the proposition is now complete. \(\square\)

Several corollaries will now follow.

**Corollary 62.** Let \(\mu\) be a 0-1 word on \(\mathbb{N}\). The graph \(G_{\mu}\) is prime if and only if \(\mu \notin \{011111\ldots, 100000\ldots, 0011111\ldots, 1100000\ldots\}\).

**Proof.** We prove the following equivalence: the graph \(G_{\mu}\) is not prime if and only if \(\mu \notin \{011111\ldots, 100000\ldots, 0011111\ldots, 1100000\ldots\}\).

\(\Rightarrow\) Let \(M\) be a nontrivial module of \(G_{\mu}\). Since \(I\) is infinite it follows from (1) of Proposition 61 that \(M = \mathbb{N}\) or \(M = \{-1\} \cup \mathbb{N} \setminus \{0\}\). It follows from (3) (a) of Lemma 60 that \(\mu \notin \{011111\ldots, 100000\ldots, 0011111\ldots, 1100000\ldots\}\) as required.

\(\Leftarrow\) Easy. \(\square\)

Since all of the 0-1 sequences in \(\{011111\ldots, 100000\ldots, 0011111\ldots, 1100000\ldots\}\) are not recurrent we get this.

**Corollary 63.** Let \(\mu\) be a recurrent 0-1 word on \(\mathbb{N}\). Then the graph \(G_{\mu}\) is prime.

We have a similar conclusion to the corollary if we consider words on \(\mathbb{N}^*\) or \(\mathbb{Z}\) but not necessarily recurrent.

**Lemma 64.** Let \(\mu\) be a 0-1 word on \(\mathbb{N}^*\) or on \(\mathbb{Z}\). Then the graph \(G_{\mu}\) is prime.

**Proof.** As in (2) of Lemma 60 if \(M\) is a module of \(G_{\mu}\), then the largest final segment \(F\) of \(\mathbb{N}^*\) or of \(\mathbb{Z}\) included in \(M\) is nonempty. Suppose for a contradiction that \(F \neq \mathbb{N}^*\) and \(F \neq \mathbb{Z}\) and let \(n := \min(F)\). Then \(n - 1 \notin M\) because otherwise \(F \cup \{n - 1\}\) is a final segment included in \(M\) and \(F \in F \cup \{n - 1\}\) contradicting the maximality of \(F\). Since \(M\) is a module and \(n - 1 \notin M\) we infer that \(n - 1\) must be either adjacent to both \(n\) and \(n + 1\) or nonadjacent to both \(n\) and \(n + 1\). Since \(n - 1\) and \(n\) are consecutive in \(\mathbb{Z}\) we have \(\mu(n) \neq \mu(n + 1)\). But then every \(k < n - 1\) separates \(n\) and \(n + 1\). It follows from our assumption that \(M\) is a module that \(\{k : k < n - 1\} \notin M\). Hence, \(M = \mathbb{N}^* \setminus \{n - 1\}\) or \(M = \mathbb{Z}^* \setminus \{n - 1\}\). We get a contradiction since \(n - 3 \in M\) and \(n - 1\) separates \(n - 2\) and \(n - 3\). \(\square\)

In the next proposition we show that \(G_{\mu}\) not being prime forces the sequence \(\mu\) to have a large factor of 0’s or of 1’s.

**Proposition 65.** Let \(I := \{i_1, \ldots, i_n\}\) be a finite interval of \(\mathbb{N}\), \(i_0 = i_1 - 1\), and let \(\mu\) be a 0-1 sequence on \(I\). Suppose \(G_{\mu}\) is not prime and let \(M\) be a nontrivial module of \(G_{\mu}\).

**Case 1.** \(M\) has cardinality 2. Then either \(M = \{i_0, i_1\}\) and either \((n = 2\) and \((\mu = 00\) or \(\mu = 11\))\), or \(n > 2\) and \((\mu = 100\ldots010\) or \(\mu = 011\ldots101\)), or \(M = \{i_1, i_n\}\), and either \((n = 3\) and \((\mu = 100\) or \(\mu = 011\))\), or \(n > 3\) and \((\mu = 1100\ldots010\) or \(\mu = 0011\ldots101\)).

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Case 2. $M$ has cardinality $n$. Then either $M = I$ and $(\mu = 100\ldots0$ or $\mu = 011\ldots1)$, or

$$M = \{i_0\} \cup I \setminus \{i_1\} \text{ and } (\mu = 0011\ldots1 \text{ or } \mu = 1100\ldots0).$$

In particular, $M$ induces a path or the complement of a path in $G_\mu$.

Proof. Since $G_\mu$ and $G_\overline{\mu}$ have the same modules (this follows from Lemma 43) we may assume without loss of generality that $\mu(i_n) = 0$.

Case 1. Suppose $M$ has exactly two elements. It follows from Item (2) of Proposition 61 that either $M = \{i_0, i_n\}$ or $M = \{i_1, i_n\}$. Suppose $M = \{i_0, i_n\}$. It follows from our assumption $\mu(i_n) = 0$ that $i_n$ is not adjacent to $i_{n-1}$ and $i_n$ is adjacent to $i_k$ for all $k < n - 1$. Since $\{i_0, i_n\}$ is a module we infer that $i_0$ cannot be adjacent to $i_{n-1}$ and $i_0$ is adjacent to $i_k$ for all $1 \leq k < n - 1$. It follows that $\mu(i_{n-1}) = 1$ if $i_{n-1} \neq i_1$, and $\mu(i_{n-1}) = 0$ if $i_{n-1} = i_1$, that is if $n = 2$. Furthermore, $\mu(i_k) = 0$ for all $1 < k < n - 1$ and $\mu(i_1) = 1$. Thus $\mu = 00$ if $n = 2$ and $\mu = 100 \ldots 010$ if $n > 2$.

Suppose $M = \{i_1, i_n\}$. It follows from our assumption $\mu(i_n) = 0$ that $i_n$ is not adjacent to $i_{n-1}$ and $i_n$ is adjacent to $i_k$ for all $k < n - 1$. Hence, $i_1$ cannot be adjacent to $i_{n-1}$ and $i_1$ is adjacent to $i_0$ and to $i_k$ for all $1 < k < n - 1$. It follows that $(\mu(i_{n-1}) = 0$ if $n = 3$ and $(\mu(i_{n-1}) = 1$ if $n > 3$) and $\mu(i_k) = 0$ for all $1 < k < n - 1$ and $\mu(i_1) = 1$. Then $\mu = 100$ if $n = 3$ and $\mu = 1100 \ldots 010$ otherwise.

Case 2. Suppose $M$ has exactly $n$ elements. It follows from Item (2) of Proposition 61 that $M = I$ or $M = \{i_0\} \cup I \setminus \{i_1\}$.

Suppose $M = I$. It follows from (2) (a) of Lemma 50 that $\mu$ is constant on $I \setminus \{i_1\}$ and $\mu(i_1) \neq \mu(i_2)$. It follows from our assumption $\mu(n) = 0$ that $\mu(i_1) = 1$ and $\mu(i_k) = 0$ for all $2 \leq k \leq n$, in which case $\mu$ induces the complement of a path on $M$.

Suppose $M = \{i_0\} \cup I \setminus \{i_1\}$. It follows from (2) (a) of Lemma 50 that $\mu$ is constant on $I \setminus \{i_2\}$ and $\mu(i_2) \neq \mu(i_3)$. It follows from our assumption $\mu(n) = 0$ that $\mu(i_2) = 1$, then $\mu(i_k) = 0$ for all $3 \leq k \leq n$ and $\mu(i_1) = 1$, in which case $\mu$ induces the complement of a path on $I$.

For a set $X$ of finite words let $l_i(X)$ be the supremum, over all words $\mu$ in $X$, of the length of factors of $i$’s in $\mu$. Let $l(X) := \max\{l_0(X), l_1(X)\}$. For a 0-1 sequence $\mu$ we let $l(\mu) := l(\text{Fac}(\mu))$. Note that $l(\mu) = l(\overline{\mu})$. We should mention that if $\mu$ uniformly recurrent and non constant, then $l(\mu)$ is finite.

Corollary 66. Let $X$ be an infinite set of finite words such that $l(X)$ is finite. Then for every $w \in X$ such that $|w| > l(X) + 4$ the graph $G_w$ is prime.

Proof. Let $w \in X$ be such that $|w| > l(X) + 4$ and suppose for a contradiction that $G_w$ is not prime. It follows from Proposition 53 that $w$ has a factor of 0’s or of 1’s of length at least $|w| - 4$. Hence, $|w| - 4 \leq l(X)$. This contradicts our assumption $|w| > l(X) + 4$.

Corollary 67. If $X$ is an infinite initial segment of $\{0, 1\}^*$, then the set $X'$ of $u \in X$ such that $G_u$ is prime is infinite.
Proof. If \( X \) contains factors of 0’s or factors of 1’s of arbitrary large length, then the corresponding graphs are clearly prime. Otherwise, \( l(X) \) is finite and the conclusion follows from Corollary 66.

\[ \square \]

**Corollary 68.** Let \( I := \{i_1, \ldots, i_n\} \) be a finite interval of \( \mathbb{N} \), \( i_0 := i_1 - 1 \), and let \( \mu \) be a 0-1 sequence on \( I \). Suppose \( G_\mu \) is prime but at least one of \( G_\mu \backslash \{i_0\} \) and \( G_\mu \backslash \{i_n\} \) and \( G_\mu \backslash \{i_1\} \) is not prime. Then \( \mu \) has \( 0^{n-6} \) or \( 1^{n-6} \) as a factor.

**Proof.** (1) The graph \( G_\mu \backslash \{i_0\} \) is isomorphic to the graph \( G_\mu' \) where \( V(G_\mu') := \{i_1, \ldots, i_n\} \) and \( \mu' := \mu_{\{i_1, \ldots, i_n\}} \). If \( G_\mu \backslash \{i_0\} \) is not prime, then the graph \( G_\mu' \) is not prime and we can apply Proposition 65 to this graph with \( n' := n - 1 \) and deduce that \( \mu' \) has \( 0^{n'-4} \) or \( 1^{n'-4} \) as a factor. Hence, \( \mu \) has \( 0^{n-5} \) or \( 1^{n-5} \) as a factor.

The case \( G_\mu \backslash \{i_n\} \) not prime can be treated similarly. Apply Proposition 65 to the graph \( G_\mu' \) where \( V(G_\mu') := \{i_0, \ldots, i_{n-1}\} \) and \( \mu' := \mu_{\{i_1, \ldots, i_{n-1}\}} \) and \( n' := n - 1 \).

(2) Suppose \( G_\mu \backslash \{i_1\} \) is not prime and let \( M \) be a nontrivial module. If \( M = \{i_2, \ldots, i_n\} \), then \( i_0 \) must be either adjacent to all elements of \( M \) or adjacent to non. Thus \( \mu \) is constant on \( M \), that is \( \mu_{\{i_2, \ldots, i_n\}} = 0^{n-1} \) or \( \mu_{\{i_2, \ldots, i_n\}} = 1^{n-1} \). If \( M \neq \{i_2, \ldots, i_n\} \), then \( M \) is a nontrivial module of \( G_\mu \) where \( V(G_\mu') := \{i_2, \ldots, i_n\} \) and \( \mu' := \mu_{\{i_2, \ldots, i_n\}} \). We then apply Proposition 65 to \( G_\mu' \) with \( n' = n - 2 \) and deduce that \( \mu' \) has \( 0^{n-6} \) or \( 1^{n-6} \) as a factor. For the remainder of the proof we may assume that \( M \) meets \( \{i_2, \ldots, i_n\} \) in a singleton and since \( M \) is nontrivial \( i_0 \in M \). Let \( k \neq 0 \) be such that \( i_k \in M \). Since \( i_{k+1} \) separates \( i_k \) from \( i_0 \) we infer that \( k + 1 > n \). This shows that \( k = n \), that is \( M = \{i_0, i_n\} \). Suppose \( \mu(i_n) = 1 \). Then no vertex in \( \{i_2, \ldots, i_{n-2}\} \) is adjacent to \( i_n \). Since \( M \) is a module no vertex in \( \{i_2, \ldots, i_{n-2}\} \) is adjacent to \( i_0 \) and therefore \( \mu \) is constant on \( \{i_2, \ldots, i_{n-2}\} \) and takes the value 1. Thus \( \mu \) has \( 1^{n-3} \) as factor. If \( \mu(i_n) = 0 \), then we obtain that \( \mu \) has \( 0^{n-3} \) as factor.

\[ \square \]

8. Embeddings between 0-1 graphs

In this section we study the relation between embeddings of words and embeddings of the corresponding 0-1 graphs, see for example Proposition 69. Results obtained in this section will be used in the proof of Theorem 69.

**Lemma 69.** Let \( \mu \) be a 0-1 sequence on an interval of \( I \) of \( \mathbb{N} \). Let \( \{i_0, i_1, i_2, i_3\} \subseteq I \) be such that \( i_0 < i_1 < i_2 < i_3 \). If \( G_{\mu\restriction\{i_0,i_1,i_2,i_3\}} \) is isomorphic to a \( P_4 \), then \( \{i_1,i_2,i_3\} \) is an interval of \( \mathbb{N} \) and \( \mu \) can be any 0-1 word of length 3.

**Proof.** Since a \( P_4 \) is prime it follows from Corollary 57 that \( \{i_1, i_2, i_3\} \) is an interval of \( \mathbb{N} \). Since \( P_4 \) is isomorphic to its complement follows that if \( G_{\mu\restriction\{i_0,i_1,i_2,i_3\}} \) is isomorphic to \( P_4 \), then so is \( G_{\mu\restriction\{i_0,i_1,i_2,i_3\}} \). So we may assume without loss of generality that \( \mu(i_3) = 1 \). If \( \mu(i_2) = 0 \), then \( \{i_1, i_2\} \) is not an edge of \( G_{\mu\restriction\{i_0,i_1,i_2,i_3\}} \) and \( \{i_0, i_2\} \) is an edge of \( G_{\mu\restriction\{i_0,i_1,i_2,i_3\}} \). Hence, \( \mu(i_1) = 1 \) if \( i_1 \) is a successor of \( i_0 \) and \( \mu(i_1) = 0 \) otherwise. If \( \mu(i_2) = 1 \), then \( \{i_1, i_2\} \) is an edge of \( G_{\mu\restriction\{i_0,i_1,i_2,i_3\}} \) and \( \{i_0, i_2\} \) is not an edge of \( G_{\mu\restriction\{i_0,i_1,i_2,i_3\}} \). Hence, \( \mu(i_1) = 1 \) if \( i_1 \) is a successor of \( i_0 \) and \( \mu(i_1) = 0 \) otherwise.

\[ \square \]

We denote by \( 1^k \) the constant word of length \( k \) whose all letters are 1, that is \( 1^k := 11 \ldots 1 \). Similarly we define \( 0^k \).
Lemma 70. Let $\mu$ a 0-1 sequence on an interval $I$ of $\mathbb{N}$. Let $\{i_0, i_1, \ldots, i_{k-1}\} \subseteq I$ be such that $k \geq 5$ and $i_0 < i_1 < \cdots < i_{k-1}$. If $G_{\mu^1}(i_0, \ldots, i_{k-1})$ is isomorphic to $P_k$, then $\{i_1, \ldots, i_{k-1}\}$ is an interval of $\mathbb{N}$ and $\mu_{\{i_1, \ldots, i_{k-1}\}} = 1^k$ and $\mu_{\{i_1i_2\}}$ can be any 0-1 word of length 2.

Proof. Suppose $G_{\mu^1}(i_0, \ldots, i_{k-1})$ is isomorphic to $P_k$. Since $P_k$ is prime for $k \geq 4$ it follows from Corollary 57 that $\{i_1, \ldots, i_{k-1}\}$ is an interval of $\mathbb{N}$. Then $\mu(i_{k-1}) = 1$ because otherwise $i_{k-1}$ would be a vertex of degree at least 3 in $P_k$ and this is impossible. Similarly, we have $\mu_{i_{k-2}} = 1$. Since $i_{k-1}$ is a vertex of degree 1 in $G_{\mu^1}(i_1, \ldots, i_k)$, which is isomorphic to $P_k$, we infer that $G_{\mu^1}(i_0, \ldots, i_{k-2})$ is isomorphic to $P_{k-1}$. The required conclusion follows from Lemma 69 and an induction on $k \geq 5$.\hfill $\Box$

Lemma 71. Let $\mu$ be a 0-1 sequence on an interval $J$ of $\mathbb{N}$. Let $I := \{i_0, i_1, \ldots, i_n\}$ be a finite interval of $\mathbb{N}$ with $n \geq 2$ and let $w$ be a 0-1 sequence on $I \setminus \{i_0\}$. Suppose $G_w$ embeds into $G_{\mu}$ and let $f$ be such an embedding. If $f(i_n) = \max(f(I))$, then $f(\{i_2, \ldots, i_n\})$ is an interval of $J$ and $f$ is strictly increasing on $\{i_2, \ldots, i_n\}$ and $\mu_{f(\{i_3, \ldots, i_n\})} = w_3 \ldots w_n$.

Proof. Let $w = w_1 \ldots w_n$. We notice at once that we can assume without loss of generality that $w_n = 1$. Indeed, if $w_n = 0$, then we consider $\overline{w}$ and $\overline{f}$ and recall that $G_{\overline{w}}$ is the complement of $G_w$. Furthermore, two graphs embed in each other if and only if their corresponding complements embed in each other. Let $f$ be an embedding of $G_w$ into $G_{\mu}$ such that $f(i_n) = \max(f(I))$. If $n = 2$, there is nothing to prove. Next we suppose $n \geq 3$. It follows from our assumption $w_n = 1$ that $i_n$ has degree 1 in $G_w$ and $i_{n-1}$ is its unique neighbour. Since $f$ is an embedding we infer that $f(i_n)$ has degree 1 in $f(G_w)$. It follows from this and $n \geq 3$ and $f(i_n) = \max(f(I))$ that $\mu(f(i_n)) = 1$. Hence, $f(i_n) - 1$ is the unique neighbour of $f(i_n)$ in $G_{\mu}$ satisfying $f(i_n) - 1 < f(i_n)$, and therefore in $f(G_w)$. Since $f$ is an embedding we must have $f(i_{n-1}) = f(i_n) - 1$. The proof of the lemma follows by induction on $n \geq 3$.\hfill $\Box$

It should be noted that the lemma is best possible. Indeed, $f(i_1) < f(i_0)$ is possible in general.

Lemma 72. Let $I := \{i_0, i_1, \ldots, i_n\}$ be a finite interval of $\mathbb{N}$ with $n \geq 7$ and let $w$ be a 0-1 sequence on $I \setminus \{i_0\}$ so that $G_{\mu^1}$ is prime. Let $\mu$ be a 0-1 sequence on an interval $J$ of $\mathbb{N}$. Suppose $G_w$ embeds into $G_{\mu}$ and let $f$ be such an embedding. Let $f(I) := \{j_0, j_1, \ldots, j_n\}$ so that $j_0 < j_1 < \ldots < j_n$. If $f(i_n) \in \{j_0, j_1\}$, then $w$ and $\mu$ have $0^{n-7}$ or $1^{n-7}$ as a factor.

Proof. Let $w := w_1 \ldots w_n$. As in the proof of Lemma 71 we may assume without loss of generality that $w_n = 1$. Then $i_n$ has degree 1 in $G_w$ and $i_{n-1}$ is its unique neighbour. Let $f$ be an embedding of $G_w$ into $G_{\mu}$ and suppose $f(i_n) \in \{j_0, j_1\}$. Since $f$ is an embedding we infer that $f(i_n)$ has degree 1 in $f(G_w)$. Furthermore, since $G_w$ is prime, $f(G_w)$ is prime too and therefore $\{j_1, \ldots, j_n\}$ is an interval of $\mathbb{N}$ (Corollary 57).

Case 1. $f(i_n) = j_0$.

Let $k \in \mathbb{N}$ be such that $j_k := f(i_{n-1})$. It follows from Corollary 58 that we may assume $k \notin \{1, n\}$. Since $f$ is an embedding and $j_0 = f(i_n)$ it follows that $j_k$ is the unique neighbour of $j_0$ in $f(G_w)$. It follows from this and $k \notin \{1, n\}$ that $\mu(j_k) = 0$ and $\mu$ is constant on $\{j_2, \ldots, j_n\} \setminus \{j_k\}$ and takes the value 1. In particular, $j_k$ has at least $j_0$ and $j_{k+1}$ as neighbours.

If $w_{n-1} = 1$, then $i_{n-1}$ has degree 2 in $G_w$ and since $f$ is an embedding $j_k = f(i_{n-1})$
has degree 2 in \( f(G_w) \). It follows from \( k \not\in \{1, n\} \) and \( \mu(j_k) = 0 \) that \( k = 2 \). In particular, \( \mu_1(i_3, \ldots, i_n) = 1^{n-2} \) and \( f(G_w) \) embeds \( P_{n-1} \). Since \( f \) is an embedding we infer that \( G_w \) embeds \( P_{n-1} \). It follows from Lemma 70 that \( w \) has \( 1^{n-4} \) as a factor.

**Case 2.** \( f(i_n) = j_1 \).

Let \( k \in \mathbb{N} \) be such that \( j_k = f(i_{n-1}) \). It follows from Corollary 68 that we may assume \( k \not\in \{1, n\} \). Since \( f \) is an embedding and \( j_1 = f(i_n) \) it follows that \( j_k \) is the unique neighbour of \( j_1 \) in \( f(G_w) \). It follows from this and \( k \not\in \{1, n\} \) that:

(a) \( k = 2 \) and \( \mu \) is constant on \( \{j_2, \ldots, j_n\} \) and takes the value 1, or

(b) \( k > 2 \) and \( \mu(i_2) = \mu(j_k) = 0 \) and \( \mu \) is constant on \( \{j_3, \ldots, j_n\} \setminus \{i_k\} \) and takes the value 1.

If \( w_{n-1} = 1 \), then \( i_{n-1} \) has degree 2 in \( G_w \) and since \( f \) is an embedding \( j_k = f(i_{n-1}) \) has degree 2 in \( f(G_w) \). Then only case (a) holds. Indeed, if not \( j_k \) would be adjacent to \( j_{k+1}, j_1 \) and \( j_0 \) and hence has degree 3 which is impossible. Thus \( \mu_1(j_2, \ldots, j_n) = 1^{n-1} \). In particular, \( f(G_w) \) has an induced \( P_n \) and since \( f \) is an embedding we infer that \( G_w \) has an induced \( P_n \). It follows from Lemma 70 that \( w \) has \( 1^{n-3} \) as a factor.

**Else if** \( w_{n-1} = 0 \), then \( i_{n-1} \) has degree \( n - 1 \) in \( G_w \). Since \( f \) is an embedding we infer that \( j_k = f(i_{n-1}) \) has degree \( n - 1 \) in \( f(G_w) \). Then only case (b) holds. Indeed, if not \( j_k \) would be adjacent only to \( j_1, j_2 \) and hence has degree 2 which is impossible. This forces \( k = n - 1 \). It follows that \( \mu_1(j_3, \ldots, j_{n-1}) = 1^{n-5} \). In particular, \( f(G_w) \) has an induced \( P_{n-4} \) and since \( f \) is an embedding we infer that \( G_w \) has an induced \( P_{n-4} \). It follows from Lemma 70 that \( w \) has \( 1^{n-7} \) as a factor.

\[ \square \]

**Proposition 73.** Let \( \mu \) be a recurrent word on \( \mathbb{N} \) such that \( l(\mu) \) is finite. Let \( w := w_0 \ldots w_{n-1} \) be a finite word such that \( n > l(\mu) + 7 \). If \( G_w \) embeds into \( G_\mu \) and \( f \) is such an embedding, then \( f(-1), f(0) < f(1) < f(2) < \ldots < f(n-1) \) and either \( \{f(-1), f(1), f(2), \ldots, f(n-1)\} \) or \( \{f(0), f(1), f(2), \ldots, f(n-1)\} \) is an interval of \( \mathbb{N} \) and \( w_2 \ldots w_n \) is a factor of \( \mu \).

**Proof.** It follows from Corollary 66 that \( G_w \) is prime. It follows from our assumption \( n > l(\mu) + 7 \) and Corollary 68 that \( G_w \setminus \{-1\} \) and \( G_w \setminus \{0\} \) and \( G_w \setminus \{n-1\} \) are also prime. Since \( f \) is an embedding it follows that in \( f(G_w) \) removal of one of the vertices \( f(-1) \) or \( f(0) \) or \( f(n-1) \) leaves a prime graph. Since \( G_w \) is prime and \( f \) is embedding it follows that \( f(G_w) \) is also prime. It follows from Corollary 57 that \( I := f(V(G_w)) \setminus \{\min(f(V(G_w)))\} \) is an interval of \( \mathbb{N} \). It follows from Corollary 59 that \( f(n-1) \in \{\min(f(V(G_w))), \min(I), \max(I)\} \). It follows from Lemma 72 that \( f(n-1) = \max(f(V(G_w))) \). The required conclusion follows then from Lemma 71. \[ \square \]

**Corollary 74.** Let \( \mu \) be a recurrent word on \( \mathbb{N} \) such that \( l(\mu) < 4 \). Let \( u, v \) be finite words such that \( |v| \geq 3 \) and \( n > l(\mu u) + 4 \) and \( G_{vu} \) is prime. If \( G_{vu} \) embeds into \( G_\mu \), then \( u \in \text{Fac}(\mu) \).

**Proof.** Follows from Proposition 73 applied to \( w := vu \). \[ \square \]

**Lemma 75.** If \( \mu \) is recurrent word and \( u \in \text{Fac}(\mu) \), then there exists \( v \in \{0, 1\}^* \) such that \( |v| \geq 4 \) and \( vu \in \text{Fac}(\mu) \) and \( G_{vu} \) is prime.
Proof. We consider several cases.

Case 1. $\mu$ has $1^4$ as a factor.

We can write $\mu = \alpha 1^4 \mu'$ where $\alpha$ is a finite word and $\mu'$ is an infinite 0-1 sequence. Since $\mu$ is recurrent $\text{Fac}(\mu) = \text{Fac}(1^4 \mu') = \text{Fac}(\mu')$. Hence, we may assume without loss of generality that $\alpha$ is the empty word. Let $u \in \text{Fac}(\mu')$. There exists then $\beta \in \text{Fac}(\mu')$ such that $1^4 \beta u \in \text{Fac}(\mu')$. It follows from Proposition 65 that $G_{1^4 \beta u}$ is prime. Choose $v := 1^4 \beta$.

Case 2. $\mu$ has $0^4$ as a factor.

We apply Case 1 to $\overline{\mu}$ and $\overline{u}$.

Case 3. $l(\mu) < 4$. Let $u \in \text{Fac}(\mu)$. Since $\mu$ is recurrent there exists $v \in \text{Fac}(\mu)$ such that $vu \in \text{Fac}(\mu)$ and $|v| \geq 4$ and $|vu| > l(\mu) + 4$. It follows from Corollary 66 that $G_{vu}$ is prime.

\endproof

Lemma 76. Let $\mu$ be a word on an interval $I$ of $\mathbb{N}$ and let $w := w_1 \ldots w_n$ be any finite word. If $G_{1^4 w}$ embeds into $G_{\mu}$, then $1w$ is a factor of $\mu$.

Proof. We notice that when it must be the case from Proposition 65 that $G_{1^4 w}$ is prime. Let $f$ be an embedding of $G_{1^4 w}$ into $G_{\mu}$. Then the image of $G_{1^4 w}$ under $f$ is prime. We write $f(V(G_{1^4 w})) = \{i_0, i_1, i_2, i_3, i_4, j_1, \ldots, j_n\}$ so that $i_0 < \ldots < i_4 < j_1 < \ldots < j_n$. It follows from Corollary 57 that $\{i_1, i_2, i_3, i_4, j_1, j_2, \ldots, j_n\}$ is an interval of $\mathbb{N}$.

We use induction on the length $n \geq 1$ of $w$ to prove the following statement: $\mu|_{\{i_4, j_1, \ldots, j_n\}} = 1w_1 \ldots w_n$ and if $w \neq 1^n$, then for all $i \in \{1, \ldots, n\}$, $f$ maps the vertex of $G_{1^4 w}$ corresponding to $w_i$ to the vertex $v_i$.

For the basis case suppose $w \in \{0, 1\}$. If $w = 1$, then $G_{1^4 w} = G_{15}$ is a path on six vertices. Since $f$ is an embedding we infer that $f(G_{1^4 w})$ is a path on six vertices. It follows from Lemma 70 that $\mu(u_4) = \mu(v_1) = 1$ and hence $1w = 11$ is a factor of $\mu$ as required. Now suppose $w = 0$ and note that $G_{1^4 w}$ has exactly one vertex of degree four. We prove that $\mu(j_1) = 0$. Suppose for a contradiction that $\mu(j_1) = 1$. Then $\mu(i_4) = 0$ because otherwise $f(G_{1^4 w})$ won’t have a vertex of degree four and since $f$ is an embedding neither will $G_{1^4 w}$ which is impossible. But then in $f(G_{1^4 w})$ the vertex $i_4$ which has degree four is adjacent to the vertex $j_1$ which has degree one and hence in $G_{1^4 w}$ the vertex of degree four is adjacent to a vertex of degree one and this is not possible. A contradiction. Hence, our supposition that $\mu(v_1) = 1$ is false, that is $\mu(j_1) = 0$ as required. Now since $f(G_{1^4 w}) \setminus \{j_1\}$ is a path on five vertices it follows from Lemma 70 that $\mu(i_4) = 1$ and hence $1w = 10$ is a factor of $\mu$ as required.

Next we consider the inductive case. We first note that if $w = 1^n$, then $G_{1^4 w}$ is a path on $n + 5$ vertices. We apply Lemma 70 with $k = n + 5$ and deduce that $\mu|_{\{i_3, i_4, j_1, \ldots, j_n\}} = 1^{n+2}$ and hence $1w$ is a factor of $\mu$. We now assume that $w \neq 1^n$. Suppose that $w_1 \ldots w_{n-1} = 1^{n-1}$. Then $G_{1^4 w_1 \ldots w_{n-1}}$ is a path on $n + 4$ vertices. It follows from Lemma 70 that $\mu|_{\{i_3, i_4, j_1, \ldots, j_{n-1}\}} = 1^{n+1}$. From our assumption that $w \neq 1^n$ we deduce that $w_n = 0$. Hence, $G_{1^4 w}$ has a unique vertex of degree $n + 4$ and this vertex is associated to $w_n$. Since $f$ is an embedding and $\mu|_{\{i_3, i_4, j_1, \ldots, j_{n-1}\}} = 1^{n+1}$ it follows that the image under $f$ of the vertex associated to $w_n$ must be $j_n$ and $j_n$ has degree $n + 4$. This shows that $\mu(j_n) = 0$ and hence $1w$ is a factor of $\mu$.

Next we suppose that $w_1 \ldots w_{n-1} = 1^{n-1}$. By the induction hypothesis $\mu|_{\{i_3, j_1, \ldots, j_{n-1}\}} = 1w_1 \ldots w_{n-1}$ and for all $i \in \{1, \ldots, n - 1\}$, $f$ maps the vertex of $G_{1^4 w}$ corresponding to $w_i$ to
the vertex $j_i$. We note that $j_{n-1}$ is the unique neighbour or the unique non neighbour of $j_n$ in $f(G_{1^+w})$. Since $f$ is an embedding it follows that $j_n$ is the image under $f$ of the vertex of $G_{1^+w}$ corresponding to $w_n$ and $\mu(j_n) = w_n$. This completes the proof of the lemma. □

**Corollary 77.** Let $\mu$ be a word on an interval $I$ of $\mathbb{N}$ and let $w := w_1 \ldots w_n$ be any finite word. If $G_{0^+w}$ embeds into $G_{\mu}$, then $0w$ is a factor of $\mu$.

**Proof.** We apply Lemma 76 to $\bar{\mu}$ and recall that $G_{1^+\bar{\mu}}$ embeds into $G\bar{\mu}$ if and only if the complement of $G_{1^+\bar{\mu}}$, which is $G_{0^+w}$, embeds into the complement of $G\bar{\mu}$, which is $G_{\mu}$.

**Corollary 78.** Let $\mu$ be a word on $\mathbb{N}$.

1. If $w$ is a bound of $\mu$, then $G_{1^+w}$ and $G_{0^+w}$ do not embed into $G_{\mu}$.
2. If $(w_i)_{i \in I}$, $I \subseteq \mathbb{N}$, is an antichain (with respect to the factor ordering) of finite words such that no $w_i$ starts with 1, then $(G_{1^+w_i})_{i \in I}$ is an antichain of (permutation) graphs.

**Proof.** (1) The fact that $G_{1^+w}$ does not embed into $G_{\mu}$ follows from Lemma 76. The fact that $G_{0^+w}$ does not embed into $G_{\mu}$ follows from Corollary 77.

(2) Suppose for a contradiction that there exists $i \neq j$ be such that $G_{1^+w_i}$ embeds into $G_{1^+w_j}$. It follows from Lemma 76 that $1w_i$ is a factor of $1^4w_j$. Since $w_i$ does not start with 1 we infer that $w_i$ is a factor of $w_j$. This is impossible since by assumption the sequence $(w_i)_{i \in I}$ is an antichain of words.

□

9. A Proof of Theorem 37

We prove the following strengthening of Theorem 37. For that we introduce first the following notation: if $X$ is a set of finite 0-1 words we set $G_X := \{G_w : w \in X\}$ and

$$\downarrow G_X := \{H : H \text{ embeds into some } G_w \in G_X\}.$$  

**Theorem 79.** Let $\mu$ be a recurrent word and $X$ be an initial segment of $\{0,1\}^*$ for the factor ordering. If $\text{Age}(G_{\mu}) \subseteq \downarrow G_X$ then $\text{Fac}(\mu) \subseteq X$.

**Proof.** Let $u \in \text{Fac}(\mu)$. We prove that $u \in X$. According to Lemma 75 since $\mu$ is recurrent and $u \in \text{Fac}(\mu)$ there is some $v \in \{0,1\}^*$ with $|v| \geq 3$ such that $vu \in \text{Fac}(\mu)$ and $G_{vu}$ is prime. Since $G_{vu} \in \text{Age}(G_{\mu}) \subseteq \downarrow G_X$, $G_{vu}$ embeds in $G_w$ for some $w \in X$. If $l(\mu) < 4$ it follows from Corollary 74 that $u$ is a factor of $w$. If $l(\mu) \geq 4$ then there is $u' \in \text{Fac}(\mu)$ such that $u$ is a factor of $u'$ and either $0^4u'$ or $1^4u'$ is a factor of $\mu$. It follows from Lemma 76 and Corollary 77 applied to either $v = 0^4$ or $v = 1^4$ that $u'$ is a factor of $w$, and so is $u$. Hence, $u \in X$. □

Theorem 37 now follows by observing that $\text{Age}(G_{\mu'}) = \downarrow G_{\text{Fac}(\mu')}$ and then applying Theorem 79 to $X := \text{Fac}(\mu')$.

10. A Proof of Theorem 39

**Proof.** (i) ⇒ (ii). Let $\mu$ be uniformly recurrent. Then trivially, $\mu$ is recurrent. Since for two infinite sequence $\tau$ and $\tau'$, the equality $\text{Fac}(\tau) = \text{Fac}(\tau')$ implies $\text{Age}(G_{\tau}) = \text{Age}(G_{\tau'})$, it follows from Theorem 5 that we may assume that $\mu$ is a word on $\mathbb{N}$. It follows from Corollary 63 that $G_{\mu}$ is prime. Hence, from Theorem 10 it follows that the set of prime graphs in $\text{Age}(G_{\mu})$ is cofinal in $\text{Age}(G_{\mu})$ hence infinite. Now let $\mathcal{C}$ be a proper age of $\text{Age}(G_{\mu})$. We prove that $\mathcal{C}$ contains only a finite number of prime graphs. If $\mathcal{C}$ contains
restrictions on intervals of \( \mathbb{N} \) of arbitrarily large length, then according to Corollary 66 \( \mathcal{C} \) contains finite prime graphs of arbitrarily large length and therefore \( \mathcal{C} = \text{Age}(G'_n) \). Else, \( \mathcal{C} \) contains only restrictions to factors of \( \mu \) of bounded length. It follows from Corollary 57 that every prime member of \( \mathcal{C} \) of cardinality \( m \) induces an interval of \( \mathbb{N} \) of cardinality \( m - 1 \). Therefore prime members of \( \mathcal{C} \) have bounded cardinality. That is, there are only finitely many prime members of \( \mathcal{C} \).

(ii) \( \Rightarrow \) (i). First \( \text{Fac}(\mu) \) is infinite since \( \mu \) is recurrent. Next, let \( X \) be an infinite initial segment of \( \text{Fac}(\mu) \). We claim that \( X = \text{Fac}(\mu) \). Corollary 67 asserts that the set \( X' := \{ u \in X : G_u \text{ is prime} \} \) is infinite. Since \( \downarrow G_X \) contains infinitely many prime and \( \text{Age}(G'_\mu) \) is minimal prime, \( \downarrow G_X = \text{Age}(G'_\mu) \). Since \( \text{Age}(G'_\mu) \subseteq \downarrow G_X \), Theorem 79 asserts that \( \text{Fac}(u) \subseteq X \). This proves our claim.\( \square \)

11. Bounds of 0-1 graphs: a proof of Theorem 47.

Let \( \mu \) be a 0-1 sequence. Then every bound of \( \text{Age}(G'_\mu) \) is one of the following types:

1. Finite graphs that are not comparability graphs and that are minimal with this property.
2. Finite comparability graphs of critical posets of dimension three (see subsection 3.1.2).
3. Finite comparability graphs of posets of dimension two, that is finite permutation graphs.

For example if \( \mu = 11111..., \) then the bounds of \( \text{Age}(G'_\mu) \) listed according to their type are:

(a) Odd cycles of length at least 5. These are of type (1).
(b) Even cycles of length at least 6. These are of type (2).
(c) The complete bipartite graph \( K_{1,3} \) and the complete graph \( K_3 \). These are of type (3).

Let \( \mu \) be a 0-1 sequence. If \( \mu \) contains factors of 1’s of arbitrarily length, then it follows from Lemma 70 that \( G'_\mu \) embeds \( P_k \) for infinitely many \( k \)'s, hence \( \text{Age}(G'_\mu) \) contains the age of an infinite path. The cycles \( C_k \) are bounds of the infinite path and form an infinite antichain. Since cycles of length at least five are not permutation graphs, these cycles are bounds of \( \text{Age}(G'_\mu) \). Now suppose that neither \( P_k \) nor \( \bar{P}_k \) embed in \( G'_\mu \). In particular, \( \mu \) has infinitely many 1’s and 0’s, that is \( G'_\mu \) has an infinite independent set and an infinite clique.

If we put an upper bound on the length of paths and of complement of paths in members of the lists of Gallai [25] and Kelly [30], there are only finitely many such members, hence \( \text{Age}(G'_\mu) \) has only finitely many bounds of type (1) and finitely many bounds of type (2). Hence,

Theorem 80. If the age of \( G'_\mu \) does not contain the age of the infinite path nor of its complement, then it has only finitely many bounds which are not permutation graphs.

It is tempting to think that candidates for bounds of \( \text{Age}(G'_\mu) \) of type (3) are graphs of the form \( G_w \) where \( w \) is a bound of \( \text{Fac}(\mu) \). This is false.

Lemma 81. Let \( \mu \) be a recurrent 0-1 sequence on \( \mathbb{N} \) and \( w := w_1 \ldots w_n \) be a finite word. If \( w_2 \ldots w_n \) is a factor of \( \mu \), then \( G_w \) embeds into \( G'_\mu \).

Proof. Suppose \( w_2 \ldots w_n \) is a factor of \( \mu \). Let \( \{j_2, \ldots, j_n\} \subseteq \mathbb{N} \) be such that \( \mu(j_k) = w_k \) for all \( 2 \leq k \leq n \). Since \( \mu \) is recurrent we may assume that there are at least three elements of \( \mathbb{N} \cup \{-1\} \) before \( j_2 \). Let \( j_1 := j_2 - 1 \) and \( w'_1 := \mu(j_1) \). If \( w'_1 = w_1 \), then \( w \) is a factor of \( \mu \) and hence \( G_w \) embeds into \( G'_\mu \). Else if \( w'_1 \neq w_1 \), then we set \( j_0 := j_1 - 2 \). It follows that \( G_w \) is isomorphic to \( G_{\mu'} \).

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Corollary 82. Let \( \mu \) be a recurrent 0-1 sequence on \( \mathbb{N} \) and \( w \) be a finite word. If \( w \) is a bound of \( \text{Fac}(\mu) \), then \( G_w \) embeds into \( G_\mu \).

11.1. Proof of (1) of Theorem 47. We show first how to construct a bound of \( \text{Age}(G_\mu) \) using a bound of \( \mu \).

Lemma 83. Let \( \mu \) be a recurrent 0-1 sequence on \( \mathbb{N} \) with \( l(\mu) \) finite. Let \( w = w_1 \ldots w_n \) be a finite word such that \( n > l(\mu) + 7 \).

(1) If \( w = w_1 \ldots w_n \) is a bound of \( \mu \) and \( w_0 \in \{0,1\} \) is such that \( w_0 \ldots w_{n-1} \) is a factor of \( \mu \) and \( w' := w_0 w_1 \ldots w_n \), then \( G_{w'} \) is a bound of \( \text{Age}(G_\mu) \).

(2) If \( G_w \) is a bound of \( \text{Age}(G_\mu) \), then \( w_2 \ldots w_n \) is a bound of \( \text{Fac}(\mu) \).

Proof. (1) We need to prove that \( G_{w'} \) does not embed into \( G_\mu \) and that deleting any vertex from \( G_{w'} \) yields a graph that embeds into \( G_\mu \). We notice at once that it follows from our assumption \( n > l(\mu) + 7 \) that \( G_{w'} \) is prime. We first prove that \( G_{w'} \) does not embed into \( G_\mu \). Suppose not and let \( f \) be an embedding of \( G_{w'} \) into \( G_\mu \). Then \( f(i_n) = \max(f(V(G_{w'}))) \) because otherwise it follows from Corollary 68 that \( w' \) has \( 0^{n-6} \) or \( 1^{n-6} \) as a factor. Since \( w_0 \ldots w_{n-1} \) is a factor of \( \mu \) we infer that \( n-7 < l(\mu) \) contradicting our assumption that \( n > l(\mu) + 7 \). This proves that \( f(i_n) = \max(f(V(G_{w'}))) \). It follows then for Lemma 72 that \( w \) is a factor of \( \mu \) contradicting our assumption that \( w \) is a bound of \( \text{Fac}(\mu) \). This proves that \( G_{w'} \) does not embed into \( G_\mu \).

Next we prove that deleting any vertex from \( G_{w'} \) yields a graph that embeds into \( G_\mu \). Set \( V(G_{w'}) = \{-1,0,\ldots,n\} \). First we consider the graph \( G_{w'} \setminus \{-1\} \) and observe that it is isomorphic to \( G_w \). It follows from Corollary 82 that \( G_w \) embeds into \( G_\mu \). We now consider the graph \( G_{w'} \setminus \{n\} \) and observe that it is isomorphic to \( G_{w_0 \ldots w_{n-1}} \). Since \( w_0 \ldots w_{n-1} \) is a factor of \( \mu \) we infer that \( G_{w_0 \ldots w_{n-1}} \) is an induced subgraph of \( G_\mu \). Let \( k \notin \{-1,n\} \) and consider the graph \( G_{w'} \setminus \{k\} \). Then \( G_{w'} \setminus \{-1,\ldots,k-1\} \) is the graph \( G_{w_0 \ldots w_{k-1}} \) and \( G_{w'} \setminus \{k+1,\ldots,n\} \) is the graph \( G_{w_{k+2} \ldots w_n} \). Since \( w_0 \ldots w_{k-1} \) and \( w_{k+2} \ldots w_n \) are factors of \( \mu \) the graphs \( G_{w_0 \ldots w_{k-1}} \) and \( G_{w_{k+2} \ldots w_n} \) are induced subgraphs of \( G_\mu \), and hence, so is \( G_{w'} \setminus \{k\} \). This completes the proof of (1).

(2) Suppose \( G_w \) is a bound of \( \text{Age}(G_\mu) \). Then \( w \) cannot be a factor of \( \mu \) and it follows from Lemma 81 that \( w \) is not a bound of \( \text{Fac}(\mu) \). Hence, \( w \) has a factor which is a bound of \( \text{Fac}(\mu) \).

We prove that \( w_2 \ldots w_n \) is a bound of \( \text{Fac}(\mu) \), that is \( w_2 \ldots w_n \) is a not a factor of \( \mu \) and both words \( w_3 \ldots w_n \) and \( w_2 \ldots w_{n-1} \) are factors of \( \mu \). The fact that \( w_2 \ldots w_n \) is a factor of \( \mu \) follows from Lemma 81 and the fact that \( G_w \) does not embed in \( G_\mu \). Next we prove that \( w_3 \ldots w_n \) and \( w_2 \ldots w_{n-1} \) are factors of \( \mu \). It follows from our assumption \( n > l(\mu) + 7 \) and Corollary 68 that \( G_w \) is prime. Next we set \( V(G_w) := \{i_0,i_1,\ldots,i_n\} \) so that \( w \) is a word on \( \{i_0,\ldots,i_n\} \). It follows from Corollary 68 that \( G_w \setminus \{i_0\} \) and \( G_w \setminus \{i_n\} \) are prime. It follows from our assumption that \( G_w \) is a bound of \( \text{Age}(G_\mu) \) that \( G_w \setminus \{i_0\} \) and \( G_w \setminus \{i_n\} \) embed in \( G_\mu \). It follows from Lemma 72 and our assumption \( n > l(\mu) + 7 \) that if \( f \) and \( g \) are such embeddings then \( f(i_n) = \max(f(V(G_w \setminus \{i_0\})) \) and \( g(i_{n-1}) = \max(g(V(G_w \setminus \{i_n\})) \). Lemma 72 yields that \( \mu f(i_3,\ldots,i_n)) = w_3 \ldots w_n \) and \( \mu g(i_2,\ldots,i_{n-1}) = w_2 \ldots w_{n-1} \). This proves that \( w_3 \ldots w_n \) and \( w_2 \ldots w_{n-1} \) are factors of \( \mu \) as required.

The proof of (1) of Theorem 47 follows from Theorem 9 and (1) of Lemma 83.
11.2. Proof of (2) of Theorem 47. We notice at once that if $\mu$ is a period and $u$ is a period, then $\mathfrak{P}$ is periodic and $\mathfrak{P}$ is a period.

**Lemma 84.** Let $\mu$ be a 0-1 word on $\mathbb{N}$, let $I := \{i_0, i_1, \ldots, i_{n-1}\} \subseteq \mathbb{N} \cup \{-1\}$ so that $i_0 < i_1 < \cdots < i_{n-1}$ and $H := G_{\mu | I}$ be an induced subgraph of $G_{\mu}$. Let $j < k < n - 1$. If $i_j$ is adjacent to all vertices in $\{i_k, \ldots, i_{n-1}\}$, then $\mu$ is constant on $\{i_{k+1}, \ldots, i_{n-1}\}$ and takes the value 0. In particular, if $l(\mu)$ is finite and $\{i_{k+1}, \ldots, i_{n-1}\}$ is an interval of $\mathbb{N}$, then $n - l(\mu) - 1 \leq k$.

**Proof.** Straightforward.

**Lemma 85.** Let $\mu$ be a 0-1 word on $\mathbb{N}$ such $l(\mu)$ is finite. Let $J := \{j_0, j_1, \ldots, j_k\} \subseteq \mathbb{N}$ be such that $j_0 < j_1 < \cdots < j_k$ and $\{j_1, \ldots, j_k\}$ is an interval of $\mathbb{N}$ and $k > l(\mu) + 5$. Then $G := G_\mu \upharpoonright J$ is prime.

**Proof.** Suppose for a contradiction that $G$ is not prime. Let $M$ be a nontrivial module of $G$. Then $M \cap \{j_1, \ldots, j_k\}$ is a module of $G \setminus \{j_0\}$. It follows from our assumption that $k > l(\mu) + 5$ and Corollary 85 that $G \setminus \{j_0\}$ is prime. Hence, $M \cap \{j_1, \ldots, j_k\}$ is either empty, reduced to a singleton or is equal to $\{j_1, \ldots, j_k\}$. Since $M$ is nontrivial we infer that $M = \{j_1, \ldots, j_k\}$ or $M \cap \{j_1, \ldots, j_k\}$ is a singleton. If $M = \{j_1, \ldots, j_k\}$, then $j_0$ must be either adjacent to all elements of $M$ or adjacent to none. Thus $\mu$ is constant on $M$, that is $k \leq l(\mu) < k - 5$. A contradiction. Else if $M \cap \{j_1, \ldots, j_k\}$ is a singleton, then $M = \{j_0, j_m\}$ for some $1 \leq m \leq k$. Necessarily $m = k$, because otherwise $j_{m+1}$ separates $j_m$ from $j_0$. That is $M = \{j_0, j_k\}$. Suppose $\mu(j_k) = 1$. Then no vertex in $\{j_2, \ldots, j_{k-2}\}$ is adjacent to $j_k$. Since $M$ is a module, no vertex in $\{j_2, \ldots, j_{k-2}\}$ is adjacent to $j_0$ and therefore $\mu$ is constant on $\{j_2, \ldots, j_{k-2}\}$ and takes the value 1. Thus $\mu$ has $1^{k-3}$ as factor. If $\mu(j_k) = 0$, then we obtain that $\mu$ has $0^{k-3}$ as factor. Therefore, $k - 3 \leq l(\mu)$ and from our assumption $k > l(\mu) + 5$ we get $k - 3 < k - 5$ which is impossible.

**Corollary 86.** Let $\mu$ be a 0-1 word on $\mathbb{N}$ such $l(\mu)$ is finite. Let $J := \{j_0, j_1, \ldots, j_k\} \subseteq \mathbb{N}$ be such that $j_0 < j_1 < \cdots < j_k$ and $\{j_1, \ldots, j_k\}$ is an interval of $\mathbb{N}$ and $k > l(\mu) + 6$. Let $G := G_\mu \upharpoonright J$ and $x \in J$. Then $G \setminus \{x\}$ is prime if and only if $x \in \{j_0, j_1, j_k\}$.

**Lemma 87.** Let $\mu$ be a 0-1 word on $\mathbb{N}$ such that $l(\mu)$ is finite. Let $\{i_0, i_1, \ldots, i_{n-1}\} \subseteq \mathbb{N}$ be such that $i_0 < i_1 < \cdots < i_{n-1}$ and $\{i_1, \ldots, i_{n-1}\}$ is an interval of $\mathbb{N}$ and $n > l(\mu) + 8$. Let $x \notin \mathbb{N}$ and $H$ be the graph whose vertex set is $\{i_0, i_1, \ldots, i_{n-1}\} \cup \{x\}$ and edge set $E := E(G_{\mu \upharpoonright \{i_0, i_1, \ldots, i_{n-1}\}}) \cup \{(i_1, x), (i_2, x), \ldots, (i_{n-1}, x)\}$. Then $H$ does not embed into $G_\mu$.

**Proof.** Suppose for a contradiction that $H$ embeds into $G_\mu$ and let $f$ be such an embedding. Then $f$ induces an embedding of $H \setminus \{x\}$ into $G_\mu$. It follows from Lemma 57 that $H \setminus \{x\}$ is prime. According to Corollary 57 the image of $H \setminus \{x\}$ under $f$ decomposes into a point $y$ and an interval $J$ to its right. It follows from Corollary 86 that $f(\{i_0, i_1, i_{n-1}\}) = \{y, \min(J), \max(J)\}$. It follows from Lemma 72 that $f(i_{n-1}) = \max(J)$. Hence, $f(\{i_0, i_1\}) = \{y, \min(J)\}$. Now, we argue on the possible position of $f(x)$. Suppose that $f(x)$ is to the left of $f(i_{n-1})$. Since $\{f(i_2), \ldots, f(i_{n-1})\}$ is an interval of $\mathbb{N}$ we infer that $f(x)$ is to the left of $f(i_2)$. Since $f(x)$ is adjacent to all vertices in $\{f(i_1), f(i_2), \ldots, f(i_{n-1})\}$ it follows from Lemma 84 that $\mu$ is constant on $\{f(i_2), \ldots, f(i_{n-1})\}$. Hence, $n - 2 \leq l(\mu)$. From our assumption that $n > l(\mu) + 8$ we get $n < n - 8$ which is impossible. Now suppose that $f(x)$ is to the right of $f(i_{n-1})$. Since $f(x)$ is adjacent to all vertices in $\{f(1), \ldots, f(n - 1)\}$ it is adjacent to $f(i_{n-2})$ and $f(i_{n-1})$. It follows that $\mu(f(x)) = 0$. Thus $f(x)$ is adjacent to $f(i_0)$,
hence \( x \) is adjacent to \( i_0 \) in \( H \). A contradiction. This proves that our supposition \( H \) embeds into \( G_\mu \) is false.

A vertex \( x \) of a graph \( G \) is \(-1\)-extremal if either \( x \) is not adjacent to at most one vertex of \( V(G) \setminus \{x\} \) or if \( x \) is adjacent to at most one vertex of \( V(G) \setminus \{x\} \). Note that if \( x \) is \(-1\)-extremal in \( G \), then \( x \) is also \(-1\)-extremal in \( \overline{G} \).

**Lemma 88.** Let \( \mathcal{C} \) be a hereditary class of finite graphs which is \( 1^-\)-well-quasi-ordered. Then \( \mathcal{C} \) has only finitely many bounds having a \(-1\)-extremal vertex.

*Proof.* Since \( \mathcal{C} \) is w.q.o. there are only finitely many bounds of \( \mathcal{C} \) having a vertex adjacent to all other vertices. Let \( (G_n)_{n \in \mathbb{N}} \) be a sequence of bounds of \( \mathcal{C} \) such that each \( G_n \) has a \(-1\)-extremal vertex \( x_n \). We may suppose that there is a unique vertex \( y_n \) distinct from \( x_n \) and not adjacent to \( x_n \). Let \( H_n := G_n[V(G_n) \setminus \{x_n\}] \). Since \( \mathcal{C} \) is \( 1^-\)-well-quasi-ordered from the sequence \( (H_n, y_n) \) we can extract an increasing subsequence. Clearly, if \( (H_n, y_n) \) embeds into \( (H_m, y_m) \), then \( G_n \) embeds into \( G_m \). This contradicts the fact that \( \{G_n : n \in \mathbb{N}\} \) forms an antichain.

**Corollary 89.** Let \( \mu \) be a periodic 0-1 sequence on \( \mathbb{N} \). Then there are only finitely many bounds of \( \text{Age}(G_\mu) \) having a \(-1\)-extremal vertex.

*Proof.* Follows from Lemma 88 and the fact that \( \text{Age}(G_\mu) \) is \( 1^-\)-well-quasi-ordered.

**Lemma 90.** Let \( \mu \) be a periodic 0-1 sequence on \( \mathbb{N} \). Then the number of non prime bounds of \( \text{Age}(G_\mu) \) is finite.

*Proof.* Let \( (G_n)_{n \in \mathbb{N}} \) be a sequence of bounds of \( \text{Age}(G_\mu) \). Suppose \( G_n \) is not prime. Let \( M_n \) be a nontrivial module of \( G_n \) and \( x_n \) any vertex of \( M_n \). Let \( H_n := G_n[V(G_n) \setminus M_n] \). Since \( M_n \) is nontrivial and \( G_n \) is a bound we infer that \( H_n \) and \( M_n \) are elements of \( \text{Age}(G_\mu) \). Since \( \text{Age}(G_\mu) \) is w.q.o there exists an infinite subset \( I \) of \( \mathbb{N} \) so that the sequence \( (G_n)_{n \in I} \) is increasing with respect to embeddability. Since \( \text{Age}(G_\mu) \) is \( 1^-\)-well-quasi-ordered we infer that we can extract from the sequence \( (H_n, x_n)_{n \in I} \) an increasing subsequence. Then note that if \( (H_n, x_n) \) embeds into \( (H_m, x_m) \) and \( M_n \) embeds into \( M_m \), then \( G_n \) embeds into \( G_m \).

We now prove (2) of Theorem 47. Let \( \mu \) be a non constant and periodic 0-1 sequence on \( \mathbb{N} \) and let \( H \) be a bound of \( G_\mu \). It follows from Lemma 40 that we may assume that \( H \) is prime. Since the examples of critically prime graphs of Schmerl and Trotter [63] split into two totally ordered sets with respect to embeddability we may assume that \( H \) is not critically prime. There exists then \( x \in V(H) \) such that \( H \setminus \{x\} \) is prime. Since \( H \) is a bound of \( G_\mu \) we infer that \( H \setminus \{x\} \) embeds into \( G_\mu \). Let \( f_x \) be such an embedding. We write \( f_x(V(H \setminus \{x\})) := \{i_0, i_1, \ldots, i_n\} \) so that \( i_0 < i_1 < \ldots < i_n \). Since \( H \setminus \{x\} \) is prime it follows from Corollary 57 that \( \{i_1, \ldots, i_n\} \) is an interval of \( \mathbb{N} \). Since \( \mu \) is periodic \( l(\mu) \) is finite. For \( n > l(\mu) + 5 \), \( G_\mu \uparrow \{i_1 < \ldots < i_n\} \) is prime, hence \( H \setminus \{x, f_x^{-1}(i_0)\} \) is prime. We may assume that \( \mu(i_0) = 0 \) (if not consider \( G_{\mu} = G_{\overline{\mu}} \) and \( \overline{H} \) and note that \( \overline{\mu} \) is also periodic). By Lemma 88 we may assume that \( H \) has no \(-1\)-extremal vertices. It follows that \( \{x, f_x^{-1}(i_0)\} \) is not an edge of \( H \) (otherwise \( f_x^{-1}(i_0) \) is \(-1\)-extremal in \( H \)). We now consider the graph \( H \setminus \{f_x^{-1}(i_0)\} \). Let \( g_{i_0} \) be an embedding of \( H \setminus \{f_x^{-1}(i_0)\} \) into \( G_\mu \). For \( 1 \leq k \leq n \), we define \( i'_k := g_{i_0}(f_x^{-1}(i_k)) \).

Suppose \( n > l(\mu) + 7 \). It follows then from Proposition 63 that every embedding of \( H \setminus \{f_x^{-1}(i_0)\} \) in \( G_\mu \) maps \( \{i'_2, \ldots, i'_n\} \) into an interval and in that order. Hence, such an
embedding agree with $f_x$ and $g_{i_n}$. From our assumption that $\mu(i_n) = 0$ and $\{x, f_x^{-1}(i_n)\}$ is not an edge of $H$ we deduce that $g_{i_n}(x)$ is to the right of $i'_n$. Indeed, if $g_{i_n}(x)$ is to the left of $i'_n$, then since $\mu(i'_n) = \mu(i_n) = 0$ we infer that $g_{i_n}(x)$ is on the left of $i'_3$. But then $\{g_{i_n}(x), i'_n\}$ is an edge, therefore $\{x, f_x^{-1}(i_0)\}$ is an edge of $H$ hence $x$ is $\sim$-1-extremal, which is not possible. Thus, $g_{i_n}(x)$ is to the right of $i'_n$. It follows then that $g_{i_n}(x)$ is either adjacent to all vertices in $\{i'_1, \ldots, i'_{n-1}\}$ or adjacent to none. This last case is not possible, otherwise $x$ would be $\sim$-1-extremal. So we are left with the case that $g_{i_n}(x)$ is adjacent to all vertices in $\{i'_1, \ldots, i'_{n-1}\}$. Since $x$ is not $\sim$-1-extremal, $x$ is adjacent to all vertices of $H \setminus \{f_x^{-1}(i_0), f_x^{-1}(i_n)\}$ and not adjacent to either $f^{-1}(i_0)$ or $f^{-1}(i_n)$. It follows from Lemma 87 that $H \setminus \{i_n\}$ does not embed into $G_\mu$. This contradicts our assumption that $H$ is a bound of Age($G_\mu$).

12. CONCLUSION

This work on hereditary classes of finite graphs containing relatively few primes put a light on hereditary classes which are well-quasi-ordered and also on those made of permutation graphs. The result of [18] was crucial in proving that our list of hereditary classes of graphs which are minimal prime was complete. Kim [31] obtained for tournaments a result similar to Chudnovski and al. [18]. It remains to see if similar results to ours can be obtained in the case of tournaments; and also, if they shed light on the case of binary relations and binary relational structures and allow to solve the problems mentioned in the text about minimal prime hereditary classes. Among question which interest us are first the rank of minimal prime classes of permutation graphs; in this respect note that it is unknown if there are hereditary well-quasi-ordered classes of graphs with arbitrary countable rank (see [55]).

Next, the question to know whether or not well-quasi-ordered hereditary classes of finite graphs are better-quasi-ordered.

A consequence of our study is the existence of an uncountable antichain of well-quasi-ordered ages of permutation graphs. The existence of uncountably many well-quasi-ordered ages of binary structures was obtained in 1978 [51]. This was obtained by means of a coding via uniformly recurrent sequences. The same existence for graphs, permutation graphs or posets, is a non trivial fact which requires work. The same coding than the one we use in this paper was used first in 1992 [63] and in 2002 [66]. In Chapter 5 of [45] the first author proved with a simpler coding the existence of uncountably many hereditary classes of oriented graphs which are minimal prime. We conclude by mentioning the existence of uncountably many well-quasi-ordered ages of permutation graphs with distinct enumeration functions (alias profile) due to Brignall and Vatter [11].

ACKNOWLEDGEMENTS

The authors would like to sincerely thank Robert Brignall for bringing to their attention several informations, notably on pin sequences and labelled classes of permutations.

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