Quasinormal modes, quasiperiodic oscillations and shadow of rotating regular black holes in non-minimally coupled Einstein-Yang-Mills theory

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In this paper we obtain an effective metric describing a regular and rotating magnetic black hole (BH) solution with a Yang-Mills electromagnetic source in Einstein-Yang-Mills (EYM) theory using the Newman–Janis algorithm via the non-complexification radial coordinate procedure. We then study the BH shadow and the quasinormal modes (QNMs) for massless scalar and electromagnetic fields and the quasiperiodic oscillations (QPOs). To this end, we also study the embedding diagram for the rotating EYM BH. The energy conditions, shadow curvature radius, topology and the dynamical evolution of scalar and electromagnetic perturbations using the time domain integration method are investigated. We show that the shadow radius decreases by increasing the magnetic charge, while the real part of QNMs of scalar and electromagnetic fields increases by increasing the magnetic charge. This result is consistent with the inverse relation between the shadow radius and the real part of QNMs. In addition, we have studied observational constraints on the EYM parameter $\lambda$ via frequency analysis of QPOs and the EHT data of shadow cast by the M87 central black hole. We also find that the decaying rate of the EYM BH is slower than that of the neutral and ends up with a tail. We argue that the rotating EYM black hole can be distinguished from the Kerr-Newman black hole with a magnetic charge based on the difference between the angular diameters of their shadows.

I. INTRODUCTION

It is generally believed that most of the giant elliptical and spiral galaxies contain supermassive black holes (SMBHs) at their galactic centers. For instance, the masses of SMBHs at the centers of Milky Way spiral galaxy and M87 elliptical galaxy are four million and six billion solar masses, respectively. Besides having huge masses, these SMBHs also possess spins (or angular momenta). Depending on the spacetime geometry, a BH can capture light received from nearby stars or accretion disks into bound orbits. A large collection of light orbits constitutes a “photon sphere” around the BH. If the orbit of light is unstable, then photons (quanta of electromagnetic field) can either fall into the BH or escape to infinity (or a distant observer at a finite distance). The Event Horizon Telescope (EHT) collaboration has detected the first shadow images of the SMBH at the center of M87 galaxy [1,2]. With this image, it is observed that the diameter of the center BH shadow was approximately 52 micro-arc-second with a deviation of less than 10% from circularity which leads to a measurement of the central mass of 6.5 billion solar mass. Importantly, these precise observations could provide a potential window to explore, distinguish or constrain physically viable BH solutions that exhibit small deviations from the Kerr metric. The distortion in the size and magnification of the shadow images provides information about the BH properties (such as its mass and spin) and the nearby geometry (the Schwarzschild, Kerr or modified Kerr spacetime). Moreover, the shadow image is a manifestation of strong gravitational lensings which can be used to distinguish various forms of BH spacetimes and naked singularities. Some of such studies on BH shadows in various gravitational theories were given in [3–14].

In the literature, numerous static and spherically symmetric BH solutions have been derived in the modified gravity theories (MGTs). However, the task of deriving the exact rotating black hole solutions analytically by solving the coupled field equations in any MGT has remained daunting due to the complexity of the
non-linear partial differential equations of the underlying theory. For instance, under reasonable assumptions of stationary, axial symmetry and asymptotic flatness, the governing equations in f(R) gravity are highly non-linear having the fourth order derivatives, while in the general Horndeski theories the field equations are second order. Still one is able to generate the metrics of stationary and axisymmetric BHs using the Newman–Janis algorithm (NJA) [15] and its modifications by starting with any seed static and spherically symmetric spacetime [16]. Among the modifications to NJA there is the noncomplexification procedure of the radial coordinate [16]. This method has been extensively used in the literature for obtaining rotating BH solutions [17–41]. From the astrophysical and astronomical perspectives, almost all known candidates of BHs are rotating. The signature of rotation of a BH would be determined by the distortion of its shadow images or deviation from the spherical symmetry. The solution obtained by NJA method is acceptable only if the resulting solution is free from geometrical pathologies, and satisfies the energy conditions, causality and regularity everywhere except at some spacetime singularities, while allowing the existence of a spatial hypersurface where a timelike Killing vector becomes null.

A rigorous proof about the existence of an infinite number of BH solutions to the Einstein-Yang-Mills (EYM) equations with the gauge group SU(2) for any event horizon was provided in [42]. In the literature, slowly rotating non-abelian BHs, numerical rotating BHs in the minimally coupled EYM theory as well as nonstatic spherically symmetric EYM BHs were previously derived [43–46], in addition to the static, spherically symmetric constant curvature BHs [47]. Recently, new BH solutions have been also derived by adding Lorentz group symmetry in the minimally coupled EYM theory [48] and loop quantum corrections [49]. In this paper, we focus on the non-minimally coupled EYM theory where the curvature couples with the SU(2) gauge fields non-trivially [50, 51]. Our aim is to test the non-minimally coupled EYM theory via constructing rotating BHs, and then systematically investigate the consistency of the theory with the current and forthcoming observations, including the observations of M87 BH shadow. Furthermore, we would like to relate the shadow size with the quasi-normal modes (QNMs) of the BHs. Here gravitational waves will be treated as massless particles propagating along null geodesics and slowly leaking to infinity.

Among numerous astrophysical events, the quasiperiodic oscillations (QPOs) are very common phenomena in the X-ray power density spectra of stellar-mass BHs. The frequency of QPOs can be related to the matter orbiting in the vicinity of the innermost stable circular orbit (ISCO) of the BH. The appearance of two peaks at 300 Hz and 450 Hz in the X-ray power density spectra of Galactic microquasars, representing possible occurrence of a lower QPO and of an upper QPO in a ratio of 3 to 2, has stimulated a lot of theoretical works to explain the value of the 3/2-ratio. Some theoretical models, including the parametric resonance, forced resonance and Keplerian resonance, have been proposed. Therefore, the study of QPOs not only help us understand the physical processes in BH mechanics, but also provides a powerful approach to explore the nature of the BH spacetime in the strong field regime.

The structure of our paper is laid out as follows: In section II we review the non-minimally coupled EYM theory and the static BH solution. Henceforth, we apply the NJA modified by the noncomplexification procedure of the radial coordinate to generate the rotating counterpart of the static solution. In sections III and IV we study the embedding diagram and energy conditions, respectively. In sections V and VI we study the geometrical and astronomical features of the BH shadows and constraints on the free parameters. In Section VII we investigate the curvature radius and its relation with the topology of the shadow. Section VIII is devoted to the investigation of QNMs of the static BH and their relationship with the radius of the shadow, as well as the dynamical evolution of scalar and electromagnetic perturbations. Section IX is devoted to QPOs and their resonances. First, we derive the generic expressions for the radial and vertical QPOs, and then apply them to the rotating solution. In particular, we show how to obtain good and complete curve fits to the data of three microquasars. Finally, in Section X we discuss our main results and provide some concluding remarks. There are also two appendices, in which we provide the exact results and provide some concluding remarks. There are also two appendices, in which we provide the exact expressions of the Einstein tensor and of some physical quantities pertaining to section IX respectively.

II. ROTATING REGULAR EINSTEIN-YANG-MILLS BH

Let us start by writing down the action of the non-minimally coupled EYM theory in four-dimensional spacetimes is given by [50, 51]

\[
S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[ R + \frac{1}{2} F_{\mu\nu}^2 F^{\mu\nu} + \frac{1}{2} R^{\alpha\beta\mu\nu} F_{\alpha\beta}^a F_{\mu\nu}^a \right],
\]

(1)

in which \( g \) is the determinant of the metric tensor and \( R \) is the Ricci scalar. Furthermore, the Greek indices run from 0 to 3, while the Latin indices run from 1 to 3. On the other hand, the Yang-Mills (YM) tensor \( F_{\mu\nu}^a \) is connected to the YM potential \( A_\mu^a \) by the following relation

\[
F_{\mu\nu}^a = \nabla_\mu A_\nu^a - \nabla_\nu A_\mu^a + f_{(a)}^{(b)(c)} A_\mu^b A_\nu^c.
\]

(2)

In the last equation \( \nabla_\mu \) represents the covariant derivative and \( f_{(a)}^{(b)(c)} \) denote the real structure constants of the 3-parameters YM gauge group SU(2). The tensor
\( R^{\alpha \beta \mu \nu} \) is given by [51]

\[
R^{\alpha \beta \mu \nu} = \frac{\xi_1}{2} \left( g^{\alpha \mu} g^{\beta \nu} - g^{\alpha \nu} g^{\beta \mu} \right) + \frac{\xi_2}{2} \left( R^{\alpha \mu} g^{\beta \nu} - R^{\alpha \nu} g^{\beta \mu} + R^{\beta \nu} g^{\alpha \mu} - R^{\beta \mu} g^{\alpha \nu} \right) + \xi_3 R g^{\alpha \beta \mu \nu},
\]

in which \( R^{\alpha \beta} \) and \( R^{\alpha \beta \mu \nu} \) are the Ricci and Riemann tensors respectively. In addition, \( \xi_i (i = 1, 2, 3) \) are the non-minimally coupled parameters between the YM field and the gravitational field. With the assumptions that the gauge field is characterized by the Wu-Yang ansatz and \( \xi_1 = -\xi, \xi_2 = 4\xi, \xi_3 = -6\xi \) along with \( \xi > 0 \), a regular, static and spherically symmetric BH was found [50,52]

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{g(r)} + h(r)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

with \( f(r) = g(r), h(r) = r^2 \) and

\[
g(r) = 1 + \left( \frac{r^4}{r^4 + 2\lambda} \right) \left( -\frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi \epsilon_0 c^4 r^2} \right).
\]

Note that \( \lambda = \frac{1}{2} Q^2 \), when \( M \) is the BH mass and \( Q \) is the magnetic charge. When \( \lambda = 0 \) and \( Q = 0 \), the above metric reduces to the Schwarzschild BH. Furthermore, the total effective energy-momentum tensor consists of the pure Yang-Mills field and the effect of the coupling between the gravity and the Yang-Mills field [50,51]. From the Einsteins field equation the energy density, the radial and tangential pressures are derived as follows

\[
\rho = -p_r = \frac{Q^2 (r^4 - 6\lambda) + 16Mr\lambda}{(r^4 + 2\lambda)^2},
\]

\[
p_\theta = p_\phi = \frac{Q^2 (r^4 - 24\lambda r^4 + 12\lambda^2) - 8M\lambda r (6\lambda - 5r^4)}{(r^4 + 2\lambda)^3}.
\]

Now, we apply the modified NJ algorithm recently proposed in [16] to the static metric [41]. The essence of the procedure is to drop the complexification of the \( r \) coordinate normally done in the NJ algorithm [15], as there does not exist a unique way to carry out it [16]. Dropping the complexification of \( r \) implies dropping the complexification of the metric functions \( f(r), g(r) \) and \( h(r) \). Taking this advantage, Azreg-Aïnou replaced them by \( F \equiv F(r, a, \theta), G \equiv G(r, a, \theta) \) and \( H \equiv H(r, a, \theta) \), respectively,

\[
f(r) \rightarrow F(r, a, \theta), \quad g(r) \rightarrow G(r, a, \theta), \quad h(r) \rightarrow H(r, a, \theta).
\]

This combined algorithm should be called NJAAA algorithm or just NJAAA. Then, the remaining steps, as described in detail in [16], lead to the explicit expressions for \( F/H \) and \( G/H \),

\[
\frac{F}{H} = \frac{gh + a^2 \cos^2 \theta}{(k(r) + a^2 \cos^2 \theta)^2}, \quad \frac{G}{H} = gh + a^2 \cos^2 \theta,
\]

where \( k(r) = \sqrt{g/f} \), for which the rotating metric takes the form

\[
ds^2 = -\frac{(g(r)h(r) + a^2 \cos^2 \theta)H}{(k(r) + a^2 \cos^2 \theta)^2} dt^2 + \frac{H dr^2}{g(r)h(r) + a^2}
- 2a \sin^2 \theta \left( k(r) - g(r)h(r) \right) (H d\theta + H d\phi)^2 + H \sin^2 \theta \left( \frac{2k(r) - g(r)h(r) + a^2 \cos^2 \theta}{(k(r) + a^2 \cos^2 \theta)^2} \right) + 1 d\phi^2.
\]

Now since \( f(r) = g(r) \), and \( h(r) = r^2 \), one finds \( k(r) = h(r) = r^2 \). Furthermore, the function \( H(r, \theta, a) \) is still arbitrary and can be chosen so that the cross-term \( G_{\theta \theta} \) of the Einstein tensor vanishes, i.e. \( G_{\theta \theta} = 0 \), which yields the differential equation

\[
(h(r) + a^2 y^2)^2 (3H_r y_g - 2HH_{ryg}) = 3a^2 h_r H^2,
\]

where \( y \equiv \cos \theta \). It can be shown that the solution of the above equation takes the form [16],

\[
H = \Sigma \equiv r^2 + a^2 \cos^2 \theta.
\]

Thus, summarizing all the above, the metric of the rotating BH finally reads

\[
ds^2 = -c^2 \left( 1 - \frac{2Y(r)r}{\Sigma} \right) dt^2 - 2ac \sin^2 \theta \frac{2Y(r)r}{\Sigma} dtd\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{[(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \sin^2 \theta]}{\Sigma} d\phi^2,
\]

where

\[
Y(r) = \frac{r(1 - g(r))}{2}, \quad \Delta(r) = g(r)h(r) + a^2
- r^2 - \frac{\rho}{(r^4 + 2\lambda)} \left( \frac{2GM}{c^2 r} - \frac{GQ^2}{4\pi \epsilon_0 c^4 r^2} \right) + a^2.
\]

The above metric is an effective metric describing a regular and rotating magnetic black hole (BH) solution with a Yang-Mills electromagnetic source in the non-minimal Einstein-Yang-Mills theory. Metric (12) reduces to the Kerr-Newman BH with a magnetic charge instead of the electric charge if \( \lambda = 0 \). Thus, by continuity it is certainly an exact solution to the field equations (Eq. (7) of [50]) at least for small \( \lambda \). As we shall see in the subsequent sections, it is also free from geometrical pathologies and satisfies the energy conditions outside the outer horizon. In addition, it is free of spacetime singularity, too, as its curvature and Kretschmann scalar invariants
are all regular for $\lambda > 0$,

$$R = \frac{8\lambda r^2[Q^2(5r^4 - 6\lambda) + \mathcal{M}(-6r^5 + 20\lambda r)]}{(r^2 + a^2 \cos^2 \theta)(r^4 + 2\lambda)^3},$$

(15)

$$R^\mu_\nu R_{\mu\nu} = \frac{P(r, \cos^2 \theta, \mathcal{M}, Q^2, a^2, \lambda)}{(r^2 + a^2 \cos^2 \theta)^6(r^4 + 2\lambda)^6},$$

(16)

where $P$ is a polynomial of its arguments and finite, $\mathcal{M} = GM/c^2$ and $Q^2 = GQ^2/(4\pi\epsilon_0 c^4)$.

### A. Effective energy-momentum tensor

In the NJAAA the rotating solution is sourced by an anisotropic fluid the effective energy-momentum tensor $T^\mu_\nu$, which is diagonal in the orthonormal basis $(e_t, e_r, e_\theta, e_\phi)$ defined by\^[17, 18] 

$$
\begin{align*}
\rho & = (h + a^2, 0, 0, 0), \\
\Theta & = \sqrt{\Lambda/(1, 0, 0, 0)}, \\
\phi & = (0, 0, 1, 0), \\
\psi & = (a \sin^2 \theta, 0, 0, 1) / \sqrt{\Sigma \sin \theta}.
\end{align*}
\tag{17}
$$

Being given by the expression $T^\mu_\nu = \rho e^\mu_i e^\nu_j + p_r e^\mu_i e^\nu_r + p_\phi e^\mu_\theta e^\nu_\phi$, where $\rho$ is the energy density and $(p_r, p_\theta, p_\phi)$ are the components of the pressure, it can be shown that the effective energy-momentum tensor has the following components (examples of detailed calculations can be found in\^[17, 18, 27, 28])

$$
\rho = -p_r = \frac{r^4[Q^2(r^4 - 6\lambda) + 16Mr\lambda]}{(r^2 + a^2 \cos^2 \theta)^2(r^4 + 2\lambda)^2},
$$

$$
p_\theta = p_\phi = \frac{r^2}{(r^2 + a^2 \cos^2 \theta)^2(r^4 + 2\lambda)^3} \left[8Mr\lambda[2(5r^4 - 6\lambda) + (3r^4 - 10\lambda)a^2 \cos^2 \theta] + Q^2[r^8(24\lambda^2 + 12\lambda) - 4\lambda(5a^4 - 6\lambda)a^2 \cos^2 \theta]\right].
$$

(18)

We can therefore say that rotating effective metric\^[12] is a solution to the field equation derived from the action given by Eq. (1) if all matter terms are replaced by the effective energy-momentum as defined in Eq. (17) and in Eq. (18). To have more elucidating expressions we seek their Taylor expansions about $\lambda = 0$ in order to compare them with their counterparts of the Kerr–Newman black hole

$$
\rho = -p_r = \frac{Q^2}{(r^2 + a^2 \cos^2 \theta)^2} - \frac{2(5Q^2 - 8Mr)}{r^4(r^2 + a^2 \cos^2 \theta)^2} \lambda + \cdots ,
$$

$$
p_\theta = p_\phi = \frac{Q^2}{(r^2 + a^2 \cos^2 \theta)^2} + \frac{1}{r^6(r^2 + a^2 \cos^2 \theta)^2} \left[8Mr(5r^2 + 3a^2 \cos^2 \theta) - 10Q^2(3r^2 + 2a^2 \cos^2 \theta)\right] \lambda + \cdots ,
$$

(19)

where the first term in each expression corresponds to the Kerr-Newman component of the energy-momentum tensor. We see clearly that the corrections added to the Kerr–Newman counterparts can be neglected recalling that most observers are at large spatial distances from the sources. These corrections, proportional to $\lambda$, behave as the inverse of $r^7$ in the limit of large $r$ while the leading Kerr–Newman terms behave as the inverse of $r^4$.

We have discussed some relevant observable quantities and there remain some other observable quantities, mainly the usual electromagnetic fields and their Yang–Mills extensions. In the literature there are ansätze\^[44] for the general expressions of electromagnetic and Yang–Mills fields but no exact analytical solutions were found, see, for example,\^[45, 53, 54] and references therein. The determination of the electromagnetic and Yang–Mills fields of the rotating black hole is more involved than the determination of the metric itself. This necessitates the resolution of coupled nonlinear differential equations and to the best of our knowledge only numerical solutions are available in the literature (see\^[45, 53, 54] and references therein). However, for our current purpose, such solutions are not needed.

### B. Shape of the ergoregion

After obtaining the rotating BH solution\^[12], now let us turn to investigate its shape of the ergoregion. Usually, one plots the shape of the ergoregion in the $xz$-plane. The corresponding horizons of our BH can be found by solving the following equation $\Delta = 0$,

$$
\Delta = g(r)h(r) + a^2 = 0.
$$

(20)

Meantime, the so-called static limit or ergo-surface, inner and outer, is obtained via $g_{tt} = 0$, i.e.,

$$
r^2g(r) + a^2 \cos^2 \theta = 0.
$$

(21)

From Fig. 1 we observe that in general for a given $Q$ and $\lambda$, one gets two horizons if $a < a_c$. However, when $a = a_c$ (the blue line) the two horizons coincide, which means that we have an extremal BH with degenerate horizons. It is interesting to note that going beyond this critical value, $a > a_c$, one can see that event horizons no longer exist and the solution represents a compact object without horizons and singularities at the center. Moreover, by varying the angular momentum parameter $a$ while having constant value of magnetic charge, say, $Q = 0.3$ and a constant value of the parameter $\lambda$, say, $\lambda = 0.1$ one can see the effect of the magnetic charge on the surface horizon and ergoregion given in Fig. 2. For a given domain of parameters we find that at some for the angular momentum $a = 0.817$, the horizon disappears. In Fig. 3 we depict the effect of the magnetic charge on the black hole horizons and ergoregions by varying the parameter $\lambda$, with having constant values
From Eqs. (22) and (25), we find that the strong energy condition (SEC) is not satisfied. In other words, the matter supporting this configuration is exotic, although we note that the cosmological constant does not satisfy the SEC either.

III. EMBEDDING DIAGRAM

In this section, we investigate the geometry of the BH spacetime, by embedding it into a higher-dimensional Euclidean space. To this purpose, let us consider the equatorial plane $\theta = \pi/2$ at a fixed moment $t = \text{const}$, for which the metric can be written as

$$ds^2 = \frac{dr^2}{1 - \frac{b(r)}{r}} + R^2 d\phi^2,$$

where

$$b(r) = \frac{r^5}{(r^4 + 2\lambda)} \left( \frac{2M}{r} - \frac{Q^2}{r^2} \right) - \frac{a^2}{r},$$

$$R(r) = \left[ r^2 + a^2 + \frac{a^2 r^4}{(r^4 + 2\lambda)} \left( \frac{2M}{r} - \frac{Q^2}{r^2} \right) \right]^{1/2}.$$

Let us embed this reduced BH metric into three-dimensional Euclidean space in the cylindrical coordinates,

$$ds^2 = dz^2 + dR^2 + R^2 d\phi^2 = \left[ \left( \frac{dR}{dr} \right)^2 + \left( \frac{dz}{dr} \right)^2 \right] dr^2 + R^2 d\phi^2.$$

From Eqs. (22) and (25), we find that

$$\frac{dz}{dr} = \pm \sqrt{\frac{r}{r - b(r)} - \left( \frac{dR}{dr} \right)^2},$$

where $b(r)$ is given by Eq. (23). Note that the integration of the last expression cannot be accomplished analytically. Invoking numerical techniques allows us to illustrate the embedding diagrams given in Fig. 4. It is seen that by varying the parameter $\lambda$ the black hole geometry is significantly changed.

IV. ENERGY CONDITIONS

In this section we are going to explore the energy conditions for the rotating EYM BH. For this purpose we use the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ along with the effective energy-momentum tensor represented by a properly chosen tetrad of vectors given by $T^a_{\mu\nu} = e^a_\mu e^\nu_b T^{\mu\nu}$, where $T^{ab} = (\rho, p_r, p_\theta, p_\phi)$. In terms of the orthogonal basis, the non-vanishing components of the energy momentum tensor are given as follows [17],

$$\rho = \frac{1}{8\pi} \epsilon^\mu_\nu \epsilon^\nu_\mu G_{\mu\nu}, \quad p_r = \frac{1}{8\pi} \epsilon^\mu_\nu \epsilon^\nu_\mu G_{r\mu},$$

$$p_\theta = \frac{1}{8\pi} \epsilon^\mu_\nu \epsilon^\nu_\mu G_{\phi\nu}.$$

The Einstein tensor $G_{\mu\nu}$ is given in Appendix A. Using the orthogonal bases given by Eq. (16) the corresponding physical quantities defined in Eq. (27) now read,

$$\rho = -p_r - \frac{2\Sigma r^2 Y'(r)}{8\pi \Sigma^2},$$

$$p_\phi = p_\theta = p_r - \frac{\Sigma r Y'(r) + 2Y(r)}{8\pi \Sigma}.$$

In Fig. 5 by varying the parameter $\lambda$ we plot out the quantities $\rho$ and $\rho + p_r + 2p$, where $p = p_\theta = p_\phi$, for a given values of $(Q, a, M)$, from which we find that the strong energy condition (SEC) is not satisfied. In other words, the matter supporting this configuration is exotic, although we note that the cosmological constant does not satisfy the SEC either.
FIG. 2 Surface horizon (blue color) and ergoregion (red color) of the BH for different values of \( a \). For \( Q = 0.3 \), and \( a = 0.817 \) we find that the horizon disappears. On the other hand, the magnetic charge has strong effects on the ergoregion surface.

V. SHADOW OF THE ROTATING BHS

In order to find the contour of a BH shadow, we need to separate the null geodesic equations in the general rotating spacetime (12), by using the Hamilton-Jacobi equation given by

\[
\frac{\partial S}{\partial \sigma} = -\frac{1}{2} \eta^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},
\]

where \( \sigma \) is the affine parameter, \( S \) is the Jacobi action. In order to find a separable solution we express the action in terms of the known constants of the motion as follows

\[
S = \frac{1}{2} \mu^2 \sigma - Et + J\phi + S_r(r) + S_\theta(\theta),
\]

where \( \mu \) is the mass of the test particle, \( E = -p_t \) the conserved energy, and \( J = p_\phi \) the conserved angular momentum (with respect to the symmetry axis). For a photon, we have \( \mu = 0 \). From these equations it is straightforward to obtain the following equations of motion (see...
FIG. 3  Surface horizon (blue color) and ergoregion (red color) of the BH for different values of \( \lambda \) and fixed \( a \) and \( Q \). For small values of \( \lambda \) the black hole has two horizons but if we increase \( \lambda \) an extremal black hole is obtained and the two horizons coincide. When \( \lambda = 0.6225 \) we find that the horizon disappears.

for instance [16],

\[
\begin{align*}
\Sigma \frac{dt}{d\sigma} &= \frac{r^2 + a^2}{\Delta} [E(r^2 + a^2) - af] - a(aE \sin^2 \theta - f), \\
\Sigma \frac{d\phi}{d\sigma} &= \frac{a}{\Delta} [E(r^2 + a^2) - af] - \left( aE - \frac{f}{\sin^2 \theta} \right), \\
\Sigma \frac{dr}{d\sigma} &= \pm \sqrt{\Re(r)}, \\
\Sigma \frac{d\theta}{d\sigma} &= \pm \sqrt{\Theta(\theta)},
\end{align*}
\]

(31)

where

\[
\begin{align*}
\Re(r) &= [X(r)E - af]^2 - \Delta(r) \left[ K + (J - aE)^2 \right], \\
\Theta(\theta) &= K + a^2 E^2 \cos^2 \theta - J^2 \cot^2 \theta.
\end{align*}
\]

(32)

(33)

with \( X(r) = (r^2 + a^2) \). The function \( \Delta(r) \) is defined by Eq. (14), while \( K \) is the Carter separation constant. If we define the following two quantities \( \xi = J/E \) and \( \eta = K/E^2 \), and make use the fact the the unstable circular photon orbits in the general rotating spacetime must satisfy \( \Re(r)(r_{ph}) = 0, \Re(r)'(r_{ph}) = 0 \) and \( \Re(r)'' \geq 0 \), we
FIG. 4  The BH spacetime embedded in a three-dimensional Euclidean space. Left panel: We choose \( a = 0.3 \) and \( \lambda = 0.1 \). Right panel: We choose \( a = 0.3 \) and \( \lambda = 0.5 \). In both plots we have used \( M = 1 \) along with \( Q = 0.3 \).

FIG. 5  Top left panel: Plot of \( \rho \) with \( \lambda = 0.1 \) and \( a = 0.3 \). Top right panel: Plot of \( \rho + 2p + 2p \) with \( \lambda = 0.1 \) and \( a = 0.3 \). Bottom left panel: Plot of \( \rho \) with \( \lambda = 0.5 \) and \( a = 0.3 \). Bottom Right panel: Plot of \( \rho + 2p + 2p \) with \( \lambda = 0.5 \) and \( a = 0.3 \). Note that we have used \( x = \cos \theta \) and set \( M = 1 \) along with \( Q = 0.3 \) in all the plots.

obtain (see, for example \[41\])

\[
\begin{align*}
\left[ X(r_{ph}) - a\xi \right]^2 - \Delta(r_{ph}) \left[ \eta + (\xi - a)^2 \right] &= 0, \\
2X'(r_{ph}) \left[ X(r_{ph}) - a\xi \right] - \Delta'(r_{ph}) \left[ \eta + (\xi - a)^2 \right] &= 0,
\end{align*}
\tag{34}
\]

\[
\begin{align*}
\left[ X(r_{ph}) - a\xi \right]^2 - \Delta(r_{ph}) \left[ \eta + (\xi - a)^2 \right] &= 0, \\
2X'(r_{ph}) \left[ X(r_{ph}) - a\xi \right] - \Delta'(r_{ph}) \left[ \eta + (\xi - a)^2 \right] &= 0,
\end{align*}
\tag{35}
\]

where \( r = r_{ph} \) is the radius of the unstable photon orbit. Furthermore, if we eliminate \( \eta \) from the last two equations and then solve for \( \xi \), we find that \[41\]

\[
\xi = \frac{X_{ph} \Delta_{ph}' - 2 \Delta_{ph} X_{ph}'}{a \Delta_{ph}^2},
\tag{36}
\]

where we note that the subscript “ph” indicates that the quantities are evaluated at \( r = r_{ph} \). Equations \(36\) and \(37\) give the general expressions for the critical impact parameters \( \xi \) and \( \eta \) of the unstable photon orbits, which describe the contour of the shadow.

The unstable photon orbits form the boundary of the shadow. The apparent shape of the shadow is obtained by using the celestial coordinates \( \alpha \) and \( \beta \), which lie in the celestial plane perpendicular to the line joining the
FIG. 6 Variation in the shape of the shadow of the rotating BH described by the metric (12) for different values of $a$ and $Q$.

observer and the center of the spacetime geometry. The coordinates $\alpha$ and $\beta$ are defined by

$$\alpha = \lim_{r_0 \to \infty} \left( -r_0^2 \sin \Theta_0 \frac{d\phi}{dr} \right)_{(r_0, \Theta_0)}$$  \hspace{1cm} (38)

$$\beta = \lim_{r_0 \to \infty} \left( r_0^2 \frac{d\theta}{dr} \right)_{(r_0, \Theta_0)} \hspace{1cm} (39)$$

where $(r_0, \Theta_0)$ are the position coordinates of the observer. After taking the limit, we obtain

$$\alpha = -\frac{\kappa}{\sin \Theta_0}$$  \hspace{1cm} (40)

$$\beta = \pm \sqrt{\eta + a^2 \cos^2 \Theta_0 - \zeta^2 \cot^2 \Theta_0}$$  \hspace{1cm} (41)

The shadow is constructed by using the unstable photon orbit radius $r_{ph}$ as a parameter and then plotting out $\alpha$ and $\beta$ using Eqs. (36), (37), (40) and (41). In Fig. (6) we show the effect of the magnetic charge by varying $Q$ and a given values of $(a, M, \lambda)$. It is observed that the black hole shadow radius decreases with the increase of $Q$. On the other hand, in Fig. (7) we show the effect of the magnetic charge by varying $\lambda$ and a given values of $(a, M, Q)$. It is observed that the black hole shadow radius monotonically decreases with the increase of $\lambda$. 

FIG. 7 Variation in the shape of the shadow of the rotating BH described by the metric (12) for different values of \(a\) and \(\lambda\).

Although the effect is very small compared to Fig. (6). Thus, for any \(Q > 0\) and \(\lambda > 0\), we see that the shadow radius is smaller compared to the Kerr–Newman black hole with a magnetic charge. As we see from Fig. 7, the effect of magnetic charge on the shadow radius is very small when we increase \(\lambda\) and, as a result, the EYM black hole is hard to be distinguishable from the Kerr-Newman black hole based on their shadows. The small effect of \(\lambda\) can be understood from the fact that if we consider a Taylor expansion of \(g(r)\) around \(\lambda\), we obtain

\[
g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{2(2Mr - Q^2)\lambda}{r^6} + \ldots \tag{42}
\]

[working in natural units]

thus the leading correction term behave as the inverse of \(r^6\). Despite the fact that the effect of \(\lambda\) is small, we are going to elaborate more on the possibility of distinguishing a rotating Kerr-Newman black hole with magnetic charge from a EYM black hole based on the physical observable such as the shadow radius \(R_s\) and the angular diameter \(\theta_s = 2R_sM/D\), where \(M\) is the black hole mass and \(D\) is the distance between the black hole and the ob-
The size of the shadow is given in terms of the observer’s viewing angle \( \theta_0 \). In the present work, we are going to use an expression for the typical shadow radius of rotating black holes obtained by Jusufi [55]

\[
R_s = \frac{\sqrt{2}}{2} \left( \sqrt{r_0^2 + \left( \frac{r_0}{g'(r)} \right)_0^2} + \sqrt{r_0^2 - \left( \frac{r_0}{g'(r)} \right)_0^2} \right),
\]

provided the black hole shadow is viewed from the equatorial plane. In addition, the radius of circular null geodesics \( r_0^2 \) for the prograde/retrograde orbit must be chosen such that both are outside of the horizon and can be obtained by solving the equation [55]

\[
r_0^2 - \frac{2r_0}{g'(r)} g(r_0) + 2a \sqrt{\frac{2r_0}{g'(r)} g(r_0)} = 0.
\]

In particular, we are going to use the M87 black hole with \( M = 6.5 \times 10^9 M_\odot \) and \( D = 16.8 \) Mpc. The angular diameter can be further expressed as \( \theta_0 = 2 \times 9.87098 \times 10^{-6} R_s (M/M_\odot) (1 \text{ kpc}/D) \) \( \mu \text{as} \). In Table I we show the numerical values obtained for the typical shadow radius of a rotating EYM black hole by varying the parameter \( \lambda \). From these numerical results we can see that, as \( \lambda \) increases, the shadow radius and the corresponding angular diameter decrease while the numerical values for the angular diameter are in the range \( 42 \pm 3 \mu \text{as} \) reported in [1][2]. In other words, as \( \lambda \) increases, it is more easy to distinguish the EYM black hole from the Kerr-Newman black hole since the difference between their angular diameters given in terms of \( \Delta \theta_0 \) increases.

Now we would like to study the observables of the shadow, which is useful for us to fit the observed data and determine the values of the black hole parameters.

Let us first introduce several characteristic points, the right point \((a_r, 0)\), left point \((a_l, 0)\), top point \((a_t, \beta_t)\) and bottom point \((a_b, -\beta_b)\) of the shape. According to the symmetry of the shadow, one easily gets \( a_t = a_b \) and \( \beta_t = -\beta_b \). Following Ref. [56], we can construct the size and distortion of the shadow. The size of the shadow is described by the reference circle passing the top, bottom and right points of the shadow. The reference circle cuts the \( \alpha \) axis at \((\tilde{a}_t, 0)\).

The radius of the reference circle can be calculated with these characteristic points

\[
R_s = \frac{(a_t - a_r)^2 + \beta_t^2}{2(a_t - a_r)}.
\]

In the following, we will focus on two distortions \( \delta_s \) and \( k_s \), which can be defined as

\[
\delta_s = \frac{a_t - \tilde{a}_l}{R_s},
\]

\[
k_s = \frac{\beta_t - \beta_b}{a_t - a_l}.
\]

For the nonrotating black hole, we can get \( \delta_s = 0 \) and \( k_s = 1 \), which means the shadow shape is a standard circle. However when the black hole spin is nonzero, both these distortions deviates from these values.

In order to show how these two distortions vary with \( \lambda \), we plot them in Fig. [8] for \( Q = 0.4 \) and \( M = 1 \). From the figures, we can find that for low spin, the influence of \( \lambda \) on distortions \( \delta_s \) and \( k_s \) is very tiny. \( \delta_s \) and \( k_s \) almost keep 0 and 1, respectively. These indicates the shadow shapes are very close to standard circle. For \( \theta_0 = \frac{\pi}{2} \), both \( \delta_s \) and \( k_s \) increases with \( \lambda \) and approach to their maximal values for the extremal black holes. For example when \( a = 0.8 \), the distortion \( \delta_s \) takes 14% and \( k_s \) takes 1.12, indicating the shadows have a big deformation from a standard circle. For \( \theta_0 = \frac{\pi}{7} \), we find that the distortion \( k_s \) still increases with \( \lambda \) for different black hole spin. However \( \delta_s \) decreases, which is resulted by the decrease of the shadow size. On the other sides, comparing with these figures, we can easily obtained the result that both the distortions get smaller with the decrease if \( \theta_0 \). So decreasing with \( \theta_0 \), the shadows get less deformation.

In summary, if the observer locates near the equatorial plane, both the observables \( \delta_s \) and \( k_s \) of the shadows increase with \( \lambda \). If the observer is far off the equatorial plane, \( \delta_s \) decreases while \( k_s \) increase with \( \lambda \). This provides us a possible way to test the magnetic black hole in EYM theory by making use the shadows.

### VI. OBSERVATIONAL CONSTRAINTS

We can apply the numerical results of shadow size to the black hole of M87. The first M87 Event Horizon Telescope (EHT) results published the image of shadow of black hole with a ring diameter of \( 42 \pm 3 \mu \text{as} \) [1][2]. Adopting this measurement value and distance \( D = 16.8 \pm 0.8 \text{Mpc} \), we performed the Monte-Carlo simulations for the parameters space \((M, Q, \lambda)\). The constraints on the parameter \( \lambda \) and mass of M87 are shown in Fig. [9]. In 95% confidence level, the parameter \( \lambda \) is constrained as \( \lambda = 0.53 \pm 0.03 \) where we have applied the prior \( \lambda > 0 \). The mass of M87 is estimated
FIG. 8 Observables $\delta_s$ and $k_s$ for $Q=0.4$ and $M=1$. The spin is set as $a=0.1, 0.3, 0.5$, and $0.8$ from bottom to top. In the top left and top right plots, we fixed $\theta = \pi/2$, whereas in the bottom left and bottom right figures, we used $\theta = \pi/6$.

as $M = (7.52 \pm 1.85) \times 10^9 M_\odot$ which covers the range of value derived by EHT $M = (6.5 \pm 0.7) \times 10^9 M_\odot$ in Schwarzschild black hole. From above results, we found that there is a large parameter range to fit the EHT shadow size, so it is necessary to compare the above constraints with those obtained from other astrophysical observations.

VII. CURVATURE RADIUS AND TOPOLOGY OF SHADOW

It is believed that the curvature radius has an important application in testing the BH shadow. Here we aim to study the curvature radius and then discuss its topology following [57, 58].

Since the curvature is parameterized by the length parameter, we first show the perimeter of the shadow. For a given shadow, its perimeter can be calculated with the following formula

$$l_s = 2 \int \sqrt{(\partial_r \alpha)^2 + (\partial_r \beta)^2} dr,$$

(48)

where $\alpha$ and $\beta$ are the celestial coordinates describing the shadow [57], as mentioned above. The factor 2 comes from the $Z_2$ symmetry of the shadow. For $a = 0.5, Q = 0.4$ and $M = 1$, the perimeter $l_s$ slightly decreases as $\lambda$ increases (from 31.4936 to 30.9322 as $\lambda$ varies from 0.01 to its maximum value 0.3752, corresponding to an extremal BH).

Since $\alpha$ and $\beta$ are parameterized by $r$, we can adopt these forms to calculate the local curvature of the shadow $R$ and $l$ in terms of $r$ as done in Refs. [57, 58]. Finally, we plot $R$ versus $l$ in Fig. 10. In this plot, the first point on the perimeter of the shadow has the largest curvature value $R$, corresponding, by convention, to $l = 0$ (where $l$ is the shadow segment length), and the last point corresponds to $l = l_s/2$, allowing to drop the $Z_2$ symmetry in the plot. We find that the curvature $R$ first decreases with the length parameter $l$, and then increases. This result is consistent with that of the Kerr BH shadows [57]. Along each curve, there are one maximum and one minimum. In particular, the maximum increases while the minimum decreases with increasing the parameter $\lambda$.

When the shapes of the shadow are obtained from astronomical observations, we can use the curvature radius to fit the results and then obtain the values of the BH parameters. In [58], we discuss several different ways to determine the BH spin and the inclination angle of the observer for a Kerr BH. These provide possible applications on testing the nature of a BH through
As we know, topology plays an important role in physics. In our investigation, topology can be used to describe differences between BHs and horizonless solutions \[57\]. To reveal the particular topological properties of the shadow we introduce the topological covariant quantity \(\delta\) \[57\],

\[
\delta = \frac{1}{2\pi} \left( \int \frac{dl}{R} + \sum_i \theta_i \right). \tag{49}
\]

Since in the current case, the light ring is always unstable, the second term vanishes. Taking \(a=0.5\), \(Q=0.4\) and \(M = 1\) as an example, we numerically calculate \(\delta\) in terms of \(\lambda\). The result is displayed in Fig. 11. From this figure we see that, for small \(\lambda\), \(\delta\) assumes the value 1. While for \(\lambda > 0.3752\), \(\delta\) decreases indicating a topological change corresponding to the transition from rotating BH \(\rightarrow\) rotating horizonless solution. Note that for \(a = 0.5\), \(Q = 0.4\), \(M = 1\), \(\lambda < 0.3752\), the rotating solution is a BH with more than one horizon; for \(\lambda = 0.3752\) it is an extremal BH; and for \(\lambda > 0.3752\) it is a horizonless solution. This indicates a possible topological phase transition from a BH to a horizonless solution, a kind of gravitational vacuum condensate stars or gravastars without both an event horizon and a singularity at the center \[59\]. Gravastars are compact objects and may arise due to the Bose-Einstein condensation in gravitational systems resulting with an interior structure filled with vacuum energy and with an exterior effective Schwarzschild geometry if \(a = 0\). So, the variation of \(\lambda\) indicates a change of the rotating solution from a BH to a compact object. Therefore, we conclude that the deviation from 1 of \(\delta\) is a topological phase transition. The behavior of \(\delta\) can act as a topological quantity to reflect the topological information of the spacetime structure.

VIII. CONNECTION BETWEEN THE SHADOW RADIUS AND QNMS

It is well known that QNMs in the eikonal regime are related to the angular velocity of the last circular null geodesic, while the imaginary part was related to the
### A. QNMs of a scalar field

Before we consider the problem of QNMs, let us point out that in this section we are going to simplify the problem by setting the rotation of the black hole to zero, i.e., $a = 0$. For the metric (4), we introduce the tortoise coordinate,

$$dr_* = \frac{dr}{f(r)},$$

in order to study perturbations of a massless scalar field, described by the equation

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \Phi) = 0. \quad (54)$$

Separation of variables of the function $\Phi$ in terms of the spherical harmonics yields

$$\Phi(t, r, \theta, \phi) = \frac{1}{r} e^{-i\omega t} Y_l(r, \theta) \Psi(r), \quad (55)$$

with $l = 0, 1, 2, \ldots$ being the multipole numbers. Then, one can show that the perturbations are governed by a Schrödinger wave-like equation

$$\frac{d^2 \Psi}{dr_*^2} + (\omega^2 - V_S(r)) \Psi = 0, \quad (56)$$

where the function $\Psi$ satisfies the following boundary conditions

$$\Psi(r_*) = C_{\pm} \exp (\pm i \omega r_*), \quad r \to \pm \infty, \quad (57)$$

where $\omega$ can be written in terms of the real and imaginary parts, i.e., $\omega = \omega_R + i \omega_I$, where the imaginary part is proportional to the decay rate of a given mode. The effective potential $V_S(r)$ of the perturbations for the scalar field is given by,

$$V_S(r) = \left[ 1 + \left( \frac{r^4}{r^4 + 2\lambda} \right) \left( -\frac{2M}{r} + \frac{Q^2}{r^2} \right) \right] \times \left[ \frac{l(l+1)}{r^2} + 2M^2 - 2Q^2 r^2 - 12M \lambda r + 4Q^2 \lambda \right]. \quad (58)$$

To solve Eqs. (56) and (58) with the boundary conditions (57), we use the WKB approximation to compute the quasi-normal frequencies. The WKB method is widely used for numerical computations of QNMs and is based on the analogy with the problem of wave scattering near the peak of a potential barrier in quantum mechanics, where $\omega$ plays a role of energy [68, 69]. In this work we are going to use the sixth order WKB approximation for calculating QNMs developed by Konopty [70].

In Table IV and Table V, we present the results for the scalar perturbations by varying the magnetic charge $Q$ and the parameter $\lambda$, respectively. Note that we have not presented the calculations of QNMs for the fundamental mode $l = n = 0$ in Table IV and Table V. This is simply related to the fact that the WKB method is applicable only when $l > n$ and does not give a satisfactory precision for this fundamental mode.
In this section we proceed to study the effect of the magnetic charge on the propagation of the electromagnetic field. To do so, we recall the wave equations for a test electromagnetic field,

\[ \frac{1}{\sqrt{-g}} \partial_{\nu} \left( \sqrt{-g} g^{\alpha\mu} \partial_{\nu} (A_{\nu,\sigma} - A_{\lambda,\nu}) \right) = 0. \] (59)
The real and imaginary parts of the quasinormal frequencies of the scalar field with different values of $Q$. In all these cases we have set $\lambda = 0.1$.

| spin 0 | $l = 1, n = 0$ | $l = 2, n = 0$ | $l = 2, n = 1$ |
|-------|----------------|----------------|----------------|
| $\lambda$ | $\omega (WKB)$ | $\omega (WKB)$ | $\omega (WKB)$ |
| 0.1 | 0.2942-0.0968 i | 0.4857-0.0958 i | 0.4662-0.2924 i |
| 0.2 | 0.2958-0.0969 i | 0.4882-0.0959 i | 0.4689-0.2927 i |
| 0.3 | 0.2984-0.0971 i | 0.4925-0.0961 i | 0.4734-0.2932 i |
| 0.4 | 0.3023-0.0972 i | 0.4989-0.0963 i | 0.4801-0.2937 i |
| 0.5 | 0.3076-0.0974 i | 0.5076-0.0965 i | 0.4892-0.2941 i |
| 0.6 | 0.3149-0.0977 i | 0.5193-0.0966 i | 0.5015-0.2940 i |
| 0.7 | 0.3242-0.0969 i | 0.5350-0.0963 i | 0.5177-0.2927 i |
| 0.8 | 0.3369-0.0954 i | 0.5562-0.0950 i | 0.5392-0.2883 i |

The real and imaginary parts of the quasinormal frequencies of the scalar field with different values of $\lambda$. In all these cases we have set $Q = 0.3$.

| spin 1 | $l = 1, n = 0$ | $l = 2, n = 0$ | $l = 2, n = 1$ |
|-------|----------------|----------------|----------------|
| $\lambda$ | $\omega (WKB)$ | $\omega (WKB)$ | $\omega (WKB)$ |
| 0.1 | 0.2501-0.0917 i | 0.4598-0.0940 i | 0.4392-0.2875 i |
| 0.2 | 0.2515-0.0918 i | 0.4623-0.0942 i | 0.4418-0.2878 i |
| 0.3 | 0.2541-0.0920 i | 0.4665-0.0943 i | 0.4464-0.2883 i |
| 0.4 | 0.2579-0.0923 i | 0.4728-0.0946 i | 0.4530-0.2888 i |
| 0.5 | 0.2631-0.0926 i | 0.4815-0.0948 i | 0.4621-0.2893 i |
| 0.6 | 0.2702-0.0928 i | 0.4931-0.0949 i | 0.4744-0.2893 i |
| 0.7 | 0.2798-0.0925 i | 0.5088-0.0947 i | 0.4907-0.2880 i |
| 0.8 | 0.2928-0.0911 i | 0.5301-0.0934 i | 0.5126-0.2835 i |

The real and imaginary parts of the quasinormal frequencies of the electromagnetic field with different values of $Q$. In all these cases we have set $\lambda = 0.1$.

| spin 1 | $l = 1, n = 0$ | $l = 2, n = 0$ | $l = 2, n = 1$ |
|-------|----------------|----------------|----------------|
| $\lambda$ | $\omega (WKB)$ | $\omega (WKB)$ | $\omega (WKB)$ |
| 0.0 | 0.2526-0.0932 i | 0.4650-0.0955 i | 0.4443-0.2921 i |
| 0.1 | 0.2546-0.0920 i | 0.4665-0.0944 i | 0.4464-0.2883 i |
| 0.2 | 0.2558-0.0908 i | 0.4681-0.0931 i | 0.4482-0.2841 i |
| 0.3 | 0.2570-0.0894 i | 0.4697-0.0918 i | 0.4499-0.2796 i |
| 0.4 | 0.2589-0.0878 i | 0.4713-0.0903 i | 0.4512-0.2747 i |
| 0.5 | 0.2594-0.0861 i | 0.4729-0.0886 i | 0.4521-0.2694 i |
| 0.6 | 0.2602-0.0842 i | 0.4745-0.0869 i | 0.4522-0.2639 i |
| 0.7 | 0.2606-0.0823 i | 0.4760-0.0849 i | 0.4516-0.2583 i |

The four-potential $A_\mu$ can be expanded in terms of the 4-dimensional vector spherical harmonics as,

$$A_\mu (t, r, \theta, \phi) = \sum_{l, m} \left[ \begin{array}{l} 0 \\ 0 \\ -\frac{n(t,r)}{\sin(\theta)} \partial_\theta Y_{l m}(\theta, \phi) \\ f(t,r) \partial_\phi Y_{l m}(\theta, \phi) \end{array} \right] + \left[ \begin{array}{l} h(t,r) \partial_\phi Y_{l m}(\theta, \phi) \\ k(t,r) \partial_\phi Y_{l m}(\theta, \phi) \end{array} \right], \quad (60)$$

in which $Y_{l m}(\theta, \phi)$ denotes the spherical harmonics. Without going to details, we find the following second-order differential equation for the radial part

$$\frac{d^2 \Psi (r_s)}{dr_s^2} + \left[ \frac{\omega^2 - V_E (r_s)}{r_s^2} \right] \Psi (r_s) = 0, \quad (61)$$

with the effective potential

$$V_E (r) = \left[ 1 + \left( \frac{r^4}{r^4 + 2\lambda} \right) \left( \frac{-2M}{r} + \frac{Q^2}{r^2} \right) \right] \frac{l(l+1)}{r^2}. \quad (62)$$

In Table VI and Table VII we show the results for the electromagnetic perturbations by varying the magnetic charge $Q$ and the parameter $\lambda$, respectively. From Fig. (12) we see that the effective potentials for both fields are indeed affected by the magnetic charge $Q$. From Fig. (13) we see that by increasing the magnetic charge $Q$, while having a constant $\lambda$, the real part of QNMs describing scalar and electromagnetic fields increase. A similar result is obtained when we increase the parameter $\lambda$ while having a fixed value of $Q$, namely the real part of QNMs increase monotonically, as can be seen in Fig. (14). Although in this case the effect of the magnetic charge on the real part of QNMs is smaller compared to the first case. From Tables IV-VII, it can also be seen that, in general, the absolute values of the imaginary part of QNMs decreases with the increase of the magnetic charge $Q$ and $\lambda$, respectively. This means that the field perturbations in the spacetime of EYM black hole having $Q > 0$ or $\lambda > 0$ oscillate more rapidly compared to the vacuum Schwarzschild BH, however in terms of damping, the field perturbations decay more slowly compared to the Schwarzschild BH. In addition to that, we see that for the scalar field perturbations the values of the real part of QNMs in absolute values are higher than those for the electromagnetic field perturbations (see Tables IV-VII). Thus the scalar field perturbations will oscillate more rapidly compared to the electromagnetic field perturbations, in the same time the scalar field ones damp more rapidly than electromagnetic field ones. Once we compute the real part of QNMs and find that $\omega_R$ increases with $Q$ with a constant $\lambda$, we can make use of the inverse relation between $\omega_R$ and the shadow radius $R_S$

$$R_S (Q) = \lim_{l \to 1+} \left. \frac{l + \frac{1}{2}}{\omega_R (Q)} \right|_{\lambda = \text{const}}, \quad (63)$$
which decreases with increasing $Q$ as can be seen from Fig. 15 (left panel). This fact is verified in Fig. 6 where we have shown that the shadow radius decreases by increasing $Q$. Similarly, having the real part of QNMs with a varying $\lambda$ and a constant $Q$, we can use

$$R_5(\lambda) = \lim_{l \to >1} \frac{l + \frac{1}{2}}{\omega_5(\lambda)} |Q = \text{const.},$$

and show that the shadow radius monotonically decreases with increasing $\lambda$, as can be seen from Fig. 14 (right panel). This is consistent with Fig. 7 where we have shown that the shadow radius decreases by increasing $\lambda$. Finally, we can compare the numerical results for the real part of QNMs obtained from the shadow radius presented in Tables II-III, with the ones obtained via the WKB method presented in Tables IV-VII. We observe that even for the fundamental modes with small $l$, the accuracy between the two methods works well for the case of the scalar field perturbations. Increasing $l$, the accuracy between the two methods increases.

C. The time domain integration method

Let us now explore the the dynamical evolution of the scalar and electromagnetic perturbations using the time domain integration method. Toward this goal, first we simplify our computations by employing the following relation

$$2M = f(r_h, \lambda, Q) = r_h + \frac{Q^2}{r_h} + \frac{2 \lambda}{r_h^2}. \tag{65}$$

This means that we fix the mass of the BH and analyze when the line $y = 2M$ meets the curve $y = f(r_h, \lambda, Q)$ at the point of minimum. We call this mass as the critical mass, $M = M_c$, and the horizon is denoted by $r = r_H$. In other words, the Cauchy and event horizons coincide. With that information in mind, we can rewrite Eq. (5) as follows

$$g(r) = \frac{\left(r - r_h\right)^2}{r^4 + 2\lambda} \left[ r^2 + \frac{2 \lambda}{r_h^2} \left( \frac{2 r}{r_h} + 1 \right) \right]. \tag{66}$$

Introducing the tortoise coordinate $r_* = \int dr/g(r)$, we find that it is possible to write the wave equations (56) and (61) as follows

$$\frac{\partial^2 \Phi}{\partial r_*^2} - \frac{\partial^2 \Phi}{\partial t^2} = V_{S/E}(r)\Phi, \tag{67}$$

where $V_{S/E}(r)$ represents the effective potential for the scalar and electromagnetic field, respectively. One can determine the oscillation shape of the QNMs, by utilizing the finite difference method to study the dynamical evolution of the field perturbations in the time domain and examine the stability of the EYM BH. To do so, we first re-write the wave equation in terms of the variables $u$ and $v$,

$$\frac{\partial^2 \Phi}{\partial u \partial v} + \frac{1}{4} V(r)\Phi = 0, \tag{68}$$

where $u = t - r_*$ and $v = t + r_*$, respectively. To solve this two-dimensional wave equation we use a numerical method known as the finite difference method based on the following equation [71]

$$\Phi(u + \delta u, v + \delta v) = \Phi(u, v) + \Phi(u + \delta u, v) - \Phi(u, v) - \delta u \delta v \Theta \left[ \frac{n \Phi(u + \delta u, v) + \Phi(u, v + \delta v)}{8} \right] + O(e^4), \tag{69}$$

where

$$\Theta = V \left( \frac{2 v - 2 u + \delta v - \delta u}{4} \right). \tag{70}$$

Next, we suppose the initial perturbation as a gaussian pulse, centered on $v_c$ given by

$$\Phi(u = u_0, v) = e^{-\frac{(v - v_c)^2}{2 \sigma^2}}. \tag{71}$$

With this initial condition, from Eqs. (69) and (70), we find numerically the function $\Phi$, and plot it out in Figs. (16) and (17) for different values of $l$. From these figures we can see that the decaying rates of the scalar and electromagnetic perturbations in the EYM BH spacetime are slower than that of the Schwarzschild BH, and end up in a tail. This conclusion is also supported by our numerical results obtained above by the WKB approximations.

IX. QUASIPERIODIC OSCILLATIONS (QPOS)

For the numerical calculations to be carried out in this section, we need the physical values of some physical constants including the solar mass $M_{\odot} = 1.9888 \times 10^{30}$, the gravitational constant $G = 6.673 \times 10^{-11}$, and the speed of light in vacuum $c = 299792458$, all given in SI units. These same constants will be written explicitly in some subsequent formulas of this section.

In the power spectra of Fig. 3 of Ref. [72], we clearly see two peaks at 300 Hz and 450 Hz, representing, respectively, the possible occurrence of the lower $v_L = 300$ Hz quasi-periodic oscillation (QPO), and of the upper $v_U = 450$ Hz QPO from the Galactic microquasar GRO J1655-40. Similar peaks have been obtained for the microquasars XTE J1550-564 and GRS 1915+105 obeying the remarkable relation, $v_U/v_L = 3/2$ [73]. Some of the physical quantities of these three microquasars and their uncertainties are as follows [72, 74]:

$$\text{GRO J1655-40: } \frac{M}{M_{\odot}} = 6.30 \pm 0.27, \frac{a}{r_g} = 0.70 \pm 0.05 \nu_U = 450 \pm 3 \text{ Hz, } \nu_L = 300 \pm 5 \text{ Hz}. \tag{72}$$
FIG. 15  Left panel: The plot of the shadow radius as a function of the magnetic charge $Q$ obtained directly from the real part of QNMs given by Eq. (58). We have set $\lambda = 0.1$. Right panel: The plot of the shadow radius as a function of the parameter $Q$ obtained from the real part of QNMs given by Eq. (59). We have set $Q = 0.3$.

FIG. 16  The time domain profile for the scalar perturbations. Left panel: The red curve is the time domain profile for the EYM BH and the blue curve for the Schwarzschild BH with $l = 0$. Right panel: The red curve is the time domain profile for the EYM BH and the blue curve for the Schwarzschild BH with $l = 1$. We have set $\lambda = 0.1$ and $r_h = 1$ in both plots.

The time domain profile for the electromagnetic perturbations. Left panel: The red curve is the time domain profile for the EYM BH and the blue curve for the Schwarzschild BH with $l = 1$. Right panel: The red curve is the time domain profile for the EYM BH and the blue curve for the Schwarzschild BH with $l = 2$. We have set $\lambda = 0.1$ and $r_h = 1$ in both plots.

XTE J1550-564: $\frac{M}{M_\odot} = 9.1 \pm 0.6, \ \frac{a}{r_g} = 0.405 \pm 0.115 \ \nu_U = 276 \pm 3 \text{ Hz}, \ \nu_L = 184 \pm 5 \text{ Hz}, \ (73)$

GRS 1915+105: $\frac{M}{M_\odot} = 14.0 \pm 4.4, \ \frac{a}{r_g} = 0.99 \pm 0.01 \ \nu_U = 168 \pm 3 \text{ Hz}, \ \nu_L = 113 \pm 5 \text{ Hz}, \ (74)$
where \( r_g \equiv GM/c^2 \).

These twin values of the QPOs are most certainly due to the phenomenon of resonance which occurs in the vicinity of the ISCO, where the accreting particles perform radial and vertical oscillations around almost circular orbits. These two oscillations couple generally non-linearly to yield resonances in the power spectra [75,76].

So, in the first part of this section we will be concerned with stable circular orbits in the symmetry plane and their perturbations, since these orbits are mostly borrowed by in-falling matter in accretion processes.

From now on we consider stable circular orbits in the \( \theta = \pi/2 \) plane. First of all, we need to set up the equations governing an unperturbed circular motion. Once this is done, we will derive the equations that describe a perturbed circular motion around a stable unperturbed circular motion. In a third step we will separate out the set of equations governing the perturbed circular motion.

The unperturbed circular motion is a geodesic motion obeying the equation,

\[
\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0, \tag{75}
\]

where \( u^\mu = dx^\mu/d\tau \) is the four-velocity. Here the connection \( \Gamma^\mu_{\alpha\beta} \) is related to the unperturbed metric (12). For a circular motion in the equatorial plane (\( \theta = \pi/2 \)), \( u^\mu = (u^t, 0, 0, u^\phi) = u^t (1, 0, 0, \omega) \), where \( \omega = d\phi/dt \) is the angular velocity of the test particle. The only equation describing such a motion is the \( r \) component of (75) and the normalization condition \( g_{\mu\nu} u^\mu u^\nu = -c^2 \), which...
take, respectively, the following forms,
\[ \partial_t g_{tt}(u^t)^2 + 2\partial_t g_{t\phi} u^t u^\phi + \partial_t g_{\phi\phi}(u^\phi)^2 = 0, \]
\[ g_{tt}(u^t)^2 + 2g_{t\phi} u^t u^\phi + g_{\phi\phi}(u^\phi)^2 = -c^2, \]
where the metric and its derivatives are evaluated at \( \theta = \pi/2 \). From them, we obtain
\[ \omega = -\partial_t g_{t\phi} \pm \sqrt{(\partial_t g_{t\phi})^2 - \partial_t g_{tt} \partial_t g_{\phi\phi}}, \]
\[ u^t = \frac{c}{\sqrt{- (g_{tt} + 2\partial_t g_{t\phi} \omega + g_{\phi\phi} \omega^2)}}, \]
\[ u^\phi = \omega u^t, \]
where the upper sign corresponds to prograde circular orbits and the lower sign corresponds to retrograde orbits.

If the motion is perturbed, the actual position is now denoted by \( X^\mu = x^\mu + \eta^\mu \) and the 4-velocity by \( U^\mu = u^\mu + \eta^\mu \). We substitute these into the expressions of \( (78) \) and \( (82) \) and finally arrive at \( (77) \) and \( (78) \) by
\[ \ddot{\eta}^\mu + 2\Gamma^\mu_{\alpha\beta} u^\alpha \eta^\beta + \partial_t \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \eta^\gamma = 0, \]
where the background connection \( \Gamma^\mu_{\alpha\beta} \) and its derivatives are all evaluated at \( \theta = \pi/2 \). As shown in \( (78) \), Eqs. \( (80) \) decouple and take the form of oscillating radial (in the \( \theta = \pi/2 \) plane) and vertical (perpendicular to the \( \theta = \pi/2 \) plane) motions obeying the following harmonic equations:
\[ \ddot{\eta}^\mu + \Omega_{\alpha}^\mu \eta^\alpha = 0, \quad \ddot{\eta}^\theta + \Omega_{\phi}^\theta \eta^\phi = 0. \]

The locally measured frequencies \( (\Omega_r, \Omega_\theta) \) are related to the spatially-remote observer’s frequencies \( (v_r, v_\theta) \) by
\[ v_r = \frac{1}{2\pi} \frac{1}{u^t} \Omega_r, \quad v_\theta = \frac{1}{2\pi} \frac{1}{u^t} \Omega_\theta, \]
where \( u^t \) is given in \( (78) \) and \( (78) \)
\[ \Omega_{\alpha}^\mu = (\partial_{\theta} g_{ij}) u^i u^j, \quad (i, j = t, \phi), \]
\[ \Omega_{\phi}^\theta = (\partial_{t} g_{ijk} - 4\Gamma^t_{ik} \Gamma^t_{jr}) u^i u^j u^k, \quad (i, j, k = t, \phi). \]
In these expressions the summations extend over \( (t, \phi) \). It is understood that all the functions appearing in \( (78) \), \( (82) \) and \( (84) \) are evaluated at \( \theta = \pi/2 \). In terms of
\[ x \equiv \frac{r}{r_g}, \quad a_0 \equiv \frac{a}{r_g}, \quad \lambda_0 \equiv \frac{\lambda}{r_g}, \quad r_g \equiv \frac{GM}{c^2}, \]
the expressions \( (v_r, v_\theta) \) measured in Hz take the form
\[ v_r = \frac{c^3}{2\pi GM \sqrt{N_n / D_n}}, \quad v_\theta = \frac{c^3}{2\pi GM \sqrt{N_\theta / D_\theta}}, \]
where \( (N_n, N_r, D_r, N_\theta, D_\theta) \) are given in Appendix B.

As we mentioned earlier, the twin values of the QPOs observed in the microquasars are most certainly due to the phenomenon of resonance resulting from the coupling of the vertical and radial oscillatory motions \( [75, 76] \). The most common models for resonances are parametric resonance, forced resonance and Keplerian resonance. It is the general belief that the resonance observed in the three microquasars \( (22, 73) \) and \( (74) \) is of the nature of the parametric resonance and is given by
\[ v_U(v_r, v_\theta), \quad \Omega = v_r, \]
with
\[ \frac{v_\theta}{v_r} = \frac{n}{2}, \quad n \in \mathbb{N}^+. \]

In most of the applications of the parametric resonance one considers the case \( n = 1 \) \( (79, 82) \), where in this case \( v_r \) is the natural frequency of the system and \( v_\theta \) is the parametric excitation \( (T_\theta = 2T_r, \text{the corresponding periods}) \) that is, the vertical oscillations supply energy to the radial oscillations causing resonance \( (82) \). However, since \( v_\theta > v_r \) in the vicinity of ISCO, where accretion occurs and QPO resonance effects take place, the lower possible value of \( n \) is 3 and in this case \( v_r \) becomes the parametric excitation that supplies energy to the vertical oscillations.

Thus, the observed ratio \( v_U / v_L = 3/2 \) is theoretically justified by making the assumptions \( (87) \) and \( (88) \) with \( n = 3 \). Numerically we have to show that the plot of \( v_\theta \) \( (v_r) \) versus \( M/M_\odot \) crosses the upper (lower) mass band error, given in \( (22), (73) \) and \( (74) \) as \( a_0 \) assumes values in its defined band error and \( \lambda_0 > 0 \) runs within some interval to be defined later as the interval of its constrained values.

Curves that fit the upper and lower oscillation frequencies of the uncharged test particles to the observed frequencies (in Hz) of the microquasars GRO J1655-40, XTE J1550-564 and GRS 1915+105 at the 3/2 resonance radius are presented in Fig. [17]. In these plots each microquasar is treated as a rotating EYM BH \( (12) \) with \( \lambda_0 = 0 \). The black curves represent \( v_U = v_\theta \) versus \( M/M_\odot \), the blue curves represent \( v_L = v_r \) versus \( M/M_\odot \) with \( v_U / v_L = 3/2 \), and the green curves represent the mass limits as given in \( (22), (73) \) and \( (74) \). For comparison we start with the upper left panel where the microquasar GRO J1655-40 is treated as a Kerr BH \( (\lambda_0 = 0) \) with \( M/M_\odot = 6.3 \) and \( a/r_g = 0.70 \). We see that the black (blue) curve does not cross the upper (lower) mass error band. In the other remaining three panels, where each microquasar is treated as a rotating EYM BH \( (12) \), we see how the black (blue) curve crosses the upper (lower)
mass error band for each of the microquasars. The curve fittings allow us to fix the following limits for $\lambda_0$:

$$0 < \lambda_0 \lesssim 4.3.$$  \hspace{1cm} (89)

It is worth noting that the ratio $\nu_U/\nu_L = 3/2$ may sometimes admit two $x$-roots. In our plot we have chosen the root that is closer to $\chi_{\text{ISCO}}$ where the events of accretion and QPOs occur.

For completeness and comparison, two other plots (not shown in this paper) similar to those in the upper left panel of Fig. [17] have been sketched for the microquasars XTE J1550-564 and GRS 1915+105 treating them as the Kerr BH ($\lambda_0 = 0$). For the microquasar XTE J1550-564 the plots show no intersections of the curves ($\nu_U/\nu_L$) with the mass error bands, and for the microquasar GRS 1915+105 intersections exist but these are certainly due to the large mass band error for this microquasar [74].

Other curves that fit the data of the three microquasars [72], [73] and [74] have been given either via the immersion of a Schwarzschild BH into a test magnetic field [83] or via the consideration of generalized theories of gravity [84].

\section{X. CONCLUSION}

In this paper we have obtained a rotating regular magnetic BH solution of the EYM theory, by applying the NJAAA to a spherical symmetric solution. We have then investigated the ergosurface and the BH shadow. We have found that the magnetic charge $Q$ causes deformations to both of the size and shape of the BH shadow. For a given value of the angular momentum $a$ and the inclination angle $\theta_0$, the presence of the magnetic charge $Q$ shrinks the shadow and enhances its deformation with respect to the shadow of the Kerr spacetime. In other words, the shadow radius decreases due to the presence of the magnetic charge $Q$. Among other things, we have constructed the embedding diagram for the rotating EYM BH and examined the energy conditions. In particular, it has been found that the strong energy condition in general is not satisfied. The particular topological property of the shadow has been revealed upon studying the behavior of the topological quantity $\delta$ as a function of $\lambda$. At some critical value $\lambda_c$ we have found that there is a possible topological phase transition. In this transition the rotating EYM BH first becomes extremal and then turns to a horizonless compact object without spacetime singularities at the center.

In addition, we have studied the connection between the real part of QNMs in the eikonal limit and the shadow radius. First, using the WKB approximation to the sixth order we have shown that the quasi-normal frequencies in the spacetime of the EYM BH deviate from those of the Schwarzschild BH, that is, $\omega_R$ increases with increasing $Q$. We have shown that the same result is obtained if we fix the magnetic charge $Q$ and increase the parameter $\lambda$, although the effect is very small. This suggest that the shadow radius $R_S$ decreases due to the inverse relation given by Eq.(58) and Eq. (59), respectively. We have verified this result by means of the geodesic approach with the shadow images given in Figs. 6-7. Despite the fact the effect of $\lambda$ is small, we have used the M87 black hole parameters and shown that the rotating EYM black hole can be distinguished from the Kerr-Newman black hole with a magnetic charge. The difference between the angular diameters of their shadows is given by the interval $\Delta \theta_\delta \in (0.11 - 0.61) \mu$as with $\lambda \in (0.1 - 0.5)$. In addition, we studied observational constraints on the EYM parameter $\lambda$ via frequency analysis of QPOs and the EHT data of shadow cast by the M87 central black hole. It is interesting to note that EHT data offers more tighter constraints on the parameter $\lambda$ as compared to QPO’s associated with microquasars.

We have also examined the dynamical evolution of the scalar and electromagnetic perturbations using the time domain integration. We have shown that the decaying rates of the scalar and electromagnetic perturbations in the rotating EYM BH are slower than that of the Schwarzschild BH, and end up in a tail.

Finally, we have considered the QPOs and their resonances generated by a test particle undergoing a circular motion in the symmetric plane of the rotating EYM BH. We have employed the usually put-forward assumptions: $\nu_U = \nu_\theta$, $\nu_L = \nu_r$ with $\nu_U/\nu_L = 3/2$. With these assumptions, we have explored in details the effects of the parameter $\lambda_0$ on the frequencies of QPOs. For the uncharged rotating EYM BHs, we have shown that the value of $\lambda_0$ lies in an interval bounded below by 0 and above by 4.3. This has allowed us to obtain good and complete curve fittings for the three microquasars GRO J1655-40, XTE J1550-564 and GRS 1915+105, all treated as rotating neutral EYM BHs.

\section*{APPENDIX A: EINSTEIN FIELD EQUATIONS}

The non-vanishing components of the Einstein tensor are given by [16].
\[ G_{tt} = \frac{2Y'(a^4 \cos^4 \theta - a^4 \cos^2 \theta + a^2 r^2 + r^4 - 2Y r^3)}{\Sigma^3} - \frac{a^2 r \sin^2 \theta Y''}{\Sigma^3}, \]
\[ G_{rr} = -\frac{2Y' r^2}{\Delta \Sigma}, \]
\[ G_{\theta \theta} = -\frac{Y'' a^2 r^2 \cos^2 \theta + 2Y' a^2 \cos^2 \theta + Y'' r^3}{\Sigma}, \]
\[ G_{t \phi} = \frac{a \sin^2 \theta}{\Sigma^3} \left[ r(a^2 + r^2) \Sigma Y'' \right. \]
\[ \left. + 2Y' \left( (a^2 + r^2)a^2 \cos^2 \theta - a^2 r^2 - r^3(r - 2Y) \right) \right], \]
\[ G_{\phi \phi} = -\frac{\sin^2 \theta}{\Sigma^3} \left[ r(a^2 + r^2)^2 \Sigma Y'' \right. \]
\[ \left. + 2a^2 Y' \left( \cos^2 \theta \left( a^4 + 3a^2 r^2 + 2r^4 - 2Y r^3 \right) \right. \]
\[ \left. - a^2 r^2 - r^4 + 2Y r^3 \right) \right]. \] (90)

**APPENDIX B: QPOS’ EXPRESSIONS**

In terms of \( x, a_0, \) and \( \lambda_0, \) the quantities \( N, N_r, D_r, N_\theta, \) and \( D_\theta \) appearing in (86) are given by
\[ N = \frac{1}{[(x^4 + 2\lambda_0)^2 - a_0^2(x^5 - 6x\lambda_0)]^2} \left[ 2^{3/2}a_0^3 \sqrt{x^4 - 6\lambda_0} \left( x^8 - 4x^4\lambda_0 - 12\lambda_0^2 \right) \\
+ 2x^{7/2}a_0 \sqrt{x^4 - 6\lambda_0} \left( 3x^8 + 4x^4\lambda_0 - 4\lambda_0^2 \right) + \left( x^4 + 2\lambda_0 \right)^2 \left( -3x^7 + x^8 + 2x^3\lambda_0 + 4x^4\lambda_0 + 4\lambda_0^2 \right) \\
+ a_0^4 \left( -3x^{12} - 3x^{13} + 20x^8\lambda_0 + 6x^9\lambda_0 - 12x^4\lambda_0^2 + 60x^5\lambda_0^2 + 72x\lambda_0^3 \right) \right], \] (91)

\[ N_r = \left( x^5 + 2x\lambda_0 \right)^2 \left( -6x^7 + x^8 - 12x^3\lambda_0 + 36x^4\lambda_0 - 60\lambda_0^2 \right) + a_0^4 \left( -3x^{13} + 70x^9\lambda_0 - 324x^5\lambda_0^2 + 72x\lambda_0^3 \right) \\
+ 2x^{5/2}a_0 \sqrt{x^4 - 6\lambda_0} \left( 6x^{11} + 3x^{12} + 24x^7\lambda_0 - 46x^8\lambda_0 + 24x^3\lambda_0^2 - 92x^4\lambda_0^2 + 24\lambda_0^3 \right) \\
+ a_0^4 \left( -6x^{14} - 15x^{15} - 3x^{16} + 24x^{10}\lambda_0 + 190x^{11}\lambda_0 + 40x^{12}\lambda_0 + 72x^6\lambda_0^2 - 464x^7\lambda_0^2 + 184x^8\lambda_0^2 - 792x^3\lambda_0^3 \right) \\
+ 160x^4\lambda_0^3 - 48\lambda_0^4 \right) + 2a_0^4 \left( x^{23/2} \sqrt{x^4 - 6\lambda_0} - 3x^{25/2} \sqrt{x^4 - 6\lambda_0} - 48x^{15/2} \sqrt{x^4 - 6\lambda_0} \\
- 46x^{17/2} \sqrt{x^4 - 6\lambda_0} + 144x^{27/2} \sqrt{x^4 - 6\lambda_0} - 92x^{9/2} \sqrt{x^4 - 6\lambda_0} + 24\lambda_0^3 \sqrt{x^4 - 6\lambda_0} \right), \] (92)

\[ D_r = x(x^4 + 2\lambda_0) \left[ 2^{3/2}a_0^3 \sqrt{x^4 - 6\lambda_0} (x^8 - 4x^4\lambda_0 - 12\lambda_0^2) + 2x^{7/2}a_0 \sqrt{x^4 - 6\lambda_0} (3x^8 + 4x^4\lambda_0 - 4\lambda_0^2) \\
+ \left( x^4 + 2\lambda_0 \right)^2 \left( -3x^7 + x^8 + 2x^3\lambda_0 + 4x^4\lambda_0 + 4\lambda_0^2 \right) + a_0^4 \left( -3x^{12} - 3x^{13} + 20x^8\lambda_0 + 6x^9\lambda_0 - 12x^4\lambda_0^2 \\
+ 60x^5\lambda_0^2 + 72x\lambda_0^3 \right) \right], \] (93)

\[ N_0 = (x^4 - 6\lambda_0) (x^5 + 2x\lambda_0)^2 - 2x^{5/2}a_0 \sqrt{x^4 - 6\lambda_0} (3x^8 + 4x^4\lambda_0 - 4\lambda_0^2) + a_0^4 \left( 3x^9 - 20x^5\lambda_0 + 12x\lambda_0^2 \right) \\
+ a_0^4 \left( 9x^{11} + 3x^{12} - 44x^7\lambda_0 + 10x^5\lambda_0 - 60x^3\lambda_0^2 + 4x^4\lambda_0^2 - 8\lambda_0^3 \right) + a_0^4 \left( -4x^{15/2} \sqrt{x^4 - 6\lambda_0} \\
- 6x^{17/2} \sqrt{x^4 - 6\lambda_0} + 24x^{27/2} \sqrt{x^4 - 6\lambda_0} - 8x^{9/2} \sqrt{x^4 - 6\lambda_0} + 24\lambda_0^3 \sqrt{x^4 - 6\lambda_0} \right), \] (94)

\[ D_\theta = x \left[ 2^{3/2}a_0^3 \sqrt{x^4 - 6\lambda_0} (x^8 - 4x^4\lambda_0 - 12\lambda_0^2) + 2x^{7/2}a_0 \sqrt{x^4 - 6\lambda_0} (3x^8 + 4x^4\lambda_0 - 4\lambda_0^2) \\
+ \left( x^4 + 2\lambda_0 \right)^2 \left( -3x^7 + x^8 + 2x^3\lambda_0 + 4x^4\lambda_0 + 4\lambda_0^2 \right) + a_0^4 \left( -3x^{12} - 3x^{13} + 20x^8\lambda_0 + 6x^9\lambda_0 - 12x^4\lambda_0^2 \\
+ 60x^5\lambda_0^2 + 72x\lambda_0^3 \right) \right]. \] (96)

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