SPIN $q$–WHITTAKER POLYNOMIALS
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Abstract. We introduce and study a one-parameter generalization of the $q$–Whittaker symmetric functions. This is a family of multivariate symmetric polynomials, whose construction may be viewed as an application of the procedure of fusion from integrable lattice models to a vertex model interpretation of a one-parameter generalization of Hall–Littlewood polynomials from [Bor17, BP16a, BP16b]. We prove branching and Pieri rules, standard and dual (skew) Cauchy summation identities, and an integral representation for the new polynomials.

Contents
1. Introduction 1
2. Higher spin vertex model 6
3. Spin Hall–Littlewood functions 9
4. Fusion 15
5. Spin $q$–Whittaker polynomials 20
6. Combinatorial formulae 30
7. Cauchy identities and Pieri rules 34
8. Integral representation of spin $q$–Whittaker polynomials 38
References 40

1. Introduction

1.1. Background. The $q$–deformed class one $\mathfrak{gl}_n$ Whittaker functions, or $q$–Whittaker functions for short, were defined in [GLO10] as a special class of joint eigenfunctions of the $q$–deformed Toda chain Hamiltonians [Eti99, Rui90] with the support in the positive Weyl chamber. In the limit $q \to 1$ they reduce to classical $\mathfrak{gl}_n$ Whittaker functions [GLO12], while for general $q$ they are themselves the limiting case of a more general family of symmetric functions, the Macdonald polynomials [Mac95]. In the last decade they have come to play a prominent role in integrable probability via the $q$–Whittaker processes [BC14, BP14]. These are a rather general class of stochastic processes, which not only degenerate at $q \to 1$ to the Whittaker processes of O’Connell [O'C12] used to describe random directed polymers but, in a different direction, to the continuous time $q$–deformed totally asymmetric simple exclusion process ($q$–TASEP). For a survey of these properties, and their connection with KPZ (Kardar–Parisi–Zhang) universality, we refer the reader to [BC14, Chapters 3–5], [BP14].

A somewhat perpendicular approach to KPZ-type observables has recently been proposed in [BP16a, BP16b], in the setting of a higher spin version of the six-vertex model from statistical mechanics [Bax07]. The chief object of study in these works has been the height function of the vertex model, a random (non-negative, integer-valued) variable which lives
on the vertices of the lattice. Once again, the theory of symmetric functions turns out to be indispensable to the calculations: a family of symmetric, rational functions and their associated orthogonality relations play a pivotal role in writing exact integral expressions for the $q$-moments of the height function. These rational functions were originally introduced in [Bor17] as a one-parameter deformation of the Hall–Littlewood functions [Mac95] (which are yet another family encompassed by the more general Macdonald polynomials). In this work we refer to the former rational functions as spin Hall–Littlewood functions, in reference to the spin parameter $s$ which they carry.

Even more recently, some direct equivalences between expectations of observables in Macdonald processes (and their degenerations) and those in higher spin vertex models have been remarked [Bor16], see also [OP16]. While these equivalences (once guessed) can be readily proved by comparing integral formulae, their origin remains fairly mysterious. A more conceptual version of the equivalence [Bor16], albeit in the limit to Hall–Littlewood processes, has now been exposed [BBW16]. A natural angle towards better understanding these equivalences is to search for a family of symmetric functions which unifies all of the classes listed above, and to study in detail the properties of such a family. This remains beyond the scope of the present work, although we mention that promising steps in this direction have been made in [GdGW17], where a mutual generalization of Macdonald polynomials and spin Hall–Littlewood functions was explicitly constructed.

In fact, rather than unifying the known families, in this paper we will add one more family to the list: these we term spin $q$–Whittaker polynomials. The name is chosen to reflect the fact that they are a one-parameter, $s$-deformation of $q$–Whittaker polynomials; they degenerate to the latter at $s = 0$. This means that the spin $q$–Whittaker polynomials have a similar footing in the general theory as do the spin Hall–Littlewood functions, but now on the $t = 0$ side of the Macdonald “coin” (whereas the spin Hall–Littlewood functions lie on the $q = 0$ side). We refer the reader to Figure 1 for a survey of the symmetric function landscape.

This analogy between the spin Hall–Littlewood functions and spin $q$–Whittaker polynomials is not accidental: one can construct the latter directly from the former, using the

![Figure 1. A table of symmetric functions, indicating their parameter-dependence.](image-url)
technique of fusion in integrable lattice models \cite{KRSS1981, KR87}. To see how this can transpire, consider the right hand side of the Cauchy identity for Hall–Littlewood polynomials (here we use a parameter \(q\), rather than the conventional \(t\)),

\[
\Pi(u_1, u_2, \ldots, u_m; v_1, v_2, \ldots, v_n) := \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1 - q u_i v_j}{1 - u_i v_j},
\]

and specialize one set of these variables to a geometric progression with ratio \(q\), letting \((u_1, u_2, \ldots, u_m) = (u, qu, \ldots, q^{m-1}u)\). We obtain

\[
\Pi(u, qu, \ldots, q^{m-1}u; v_1, v_2, \ldots, v_n) = \prod_{j=1}^{n} \frac{1 - q^m uv_j}{1 - uv_j}.
\]

After analytic continuation in \(q^m\) effected by sending \(q^m u \mapsto -x\), and setting \(u = 0\), the product turns into \(\prod_{j=1}^{n} (1 + xv_j)\). This we recognize as the right hand side of the dual Cauchy identity between a Hall–Littlewood polynomial and a \(q\)–Whittaker polynomial (in one variable). On the other hand, such geometric specializations are known to be a key ingredient in fusion. It is therefore reasonable to suggest that \(q\)–Whittaker polynomials might be recovered by applying the fusion technique to the lattice model construction of Hall–Littlewood polynomials. This indeed turns out to be the case; surprisingly, we were unable to find a precise statement of this fact in the literature.

In light of this observation, one can play the same game taking the spin Hall–Littlewood functions (and their lattice model construction, developed in \cite{Bor17, BP16a, BP16b}) as the starting point: doing so ultimately leads us to our definition of the spin \(q\)–Whittaker polynomials.

As is common in the theory of symmetric functions \cite{Mac95}, we define skew spin \(q\)–Whittaker polynomials \(F_{\lambda/\nu}(x_1, \ldots, x_m)\) parametrized by pairs of partitions \(\lambda, \nu\), as well as non-skew ones \(F_\lambda(x_1, \ldots, x_m)\) parametrized by single partitions \(\lambda\); the latter correspond to taking \(\nu = \emptyset\) in the former. Let us list a few of their properties that we prove below.

- \(F_{\lambda/\nu}(x_1, \ldots, x_m)\) is a symmetric, \textit{inhomogeneous} polynomial in \((x_1, \ldots, x_m)\).
- For any pair of partitions \(\lambda, \nu\) the following stability relation holds:

\[
F_{\lambda/\nu}(x_1, \ldots, x_{m-1}, -s) = F_{\lambda/\nu}(x_1, \ldots, x_{m-1}).
\]

At \(s = 0\) this recovers the usual stability property of \(q\)–Whittaker polynomials.

- The spin \(q\)–Whittaker polynomials satisfy a standard branching rule

\[
F_{\lambda/\mu}(x_1, \ldots, x_{m+n}) = \sum_{\nu} F_{\nu/\mu}(x_1, \ldots, x_m) F_{\lambda/\nu}(x_{m+1}, \ldots, x_{m+n}).
\]

- The one-variable skew spin \(q\)–Whittaker polynomial is given by

\[
F_{\lambda/\nu}(x) = \begin{cases} 
\prod_{i \geq 1} (x; q)_{\lambda_i - \nu_i} ((-s/x; q)_{\nu_i - \lambda_i + 1} (q; q)_{\lambda_i - \lambda_{i+1}} & \lambda \triangleright \nu, \\
0, & \text{otherwise}.
\end{cases}
\]

Together with the branching rule, this yields a simple “interlacing” construction of the spin \(q\)–Whittaker polynomials (or equivalently, one in terms of Gelfand–Tsetlin patterns).
• Define the dual spin $q$–Whittaker polynomials by

$$F_{\lambda/\nu}^*(x_1, \ldots, x_m) = \prod_{i,j \geq 1} \frac{(s^2; q)_{\lambda_i - \lambda_{i+1}} (q; q)_{\nu_i - \nu_{i+1}}}{(q; q)_{\lambda_i - \lambda_{i+1}} (s^2; q)_{\nu_i - \nu_{i+1}}} \cdot F_{\lambda/\nu}(x_1, \ldots, x_m).$$

Then the following skew Cauchy-type summation identities hold: For arbitrary partitions $\mu$ and $\nu$, one has

$$\sum_{\lambda} F_{\lambda/\mu}(x_1, \ldots, x_m) F_{\lambda/\nu}^*(y_1, \ldots, y_n) = \prod_{i=1}^m \prod_{j=1}^n \frac{(-sx_i; q)_{\infty}(-sy_j; q)_{\infty}}{(s^2; q)_{\infty}(x_iy_j; q)_{\infty}} \sum_{\kappa} F_{\nu/\kappa}(x_1, \ldots, x_m) F_{\mu/\kappa}^*(y_1, \ldots, y_n).$$

For $\mu = \nu = \emptyset$, this turns into a Cauchy-type identity

$$\sum_{\lambda} F_{\lambda}(x_1, \ldots, x_m) F_{\lambda}(y_1, \ldots, y_n) = \prod_{i=1}^m \prod_{j=1}^n \frac{(-sx_i; q)_{\infty}(-sy_j; q)_{\infty}}{(s^2; q)_{\infty}(x_iy_j; q)_{\infty}},$$

which is a multivariate generalization of the $q$–Gauss summation theorem.

There are also dual (skew) Cauchy identities that combine spin $q$–Whittaker polynomials and stable spin Hall–Littlewood functions, see Section 7.3 below.

• The following integral representation holds:

$$F_{\lambda}(x_1, \ldots, x_m) = \oint_C \frac{du_1}{2\pi i u_1} \cdots \oint_C \frac{du_L}{2\pi i u_L} \prod_{1 \leq i < j \leq L} \frac{u_i - u_j}{u_i - qu_j} \prod_{i=1}^L \frac{1 - su_i}{u_i - s} \left( \prod_{j=1}^m (1 + u_ix_j)^{\lambda_i} \right),$$

where $L = \lambda_1$ denotes the largest part of $\lambda$ and the contour $C$ is as in Figure 6 below.

The vertex model interpretation of the spin Hall–Littlewood functions makes it natural to add countably many inhomogeneity parameters to their definition; most of their properties remain intact [BP16a]. One can do the same for the spin $q$–Whittaker polynomials; the branching rule remains the same, and in the formula for the one-variable specialization above one needs to add index $i$ to the spin parameter $s$. This preserves the (skew) dual Cauchy identities (that need to involve stable spin Hall–Littlewood functions with the same inhomogeneities), but appears to destroy the ordinary (skew) Cauchy identities. There is also an integral representation that generalizes the one above. We decided to leave the inhomogeneous case out of this work in order not to cloud the arguments with more involved notation.

1.2. Layout of paper. In Section 2 we recall the basic features of the integrable higher spin vertex model studied in [Bor17, BP16a, BP16b], and then in Section 3 we use it to define the spin Hall–Littlewood rational functions as lattice model partition functions. In Section 4 we gather a number of results on the fusion procedure as applied to the vertex model of Section 2 leading to the construction of integrable Boltzmann weights in equation (35) that form the foundation of the remainder of the paper. We remark that, at $s = 0$, these weights degenerate into those used by Korff in Sections 3 and 6 of [Kor13]. This is expected, since
contains an integrable lattice construction of the $q$–Whittaker polynomials which is exactly the $s = 0$ specialization of our results here.

In Section 5 we apply the fusion procedure to the spin Hall–Littlewood functions, and use the resulting partition functions to define spin $q$–Whittaker polynomials. Along the way, we also define a stable version of the spin Hall–Littlewood functions, which are later paired with the spin $q$–Whittaker polynomials in dual Cauchy summation identities. We formulate the spin $q$–Whittaker polynomials algebraically using certain monodromy matrix operators in the model (35), and derive key commutation relations between these operators. In Section 6 we use the lattice construction of the spin $q$–Whittaker polynomials to derive their branching rules, and show that they reduce to (ordinary) $q$–Whittaker polynomials by specializing $s = 0$. Section 7 contains a list of various Cauchy, dual Cauchy and Pieri identities which are all proven by means of the commutation relations derived in Section 5. We conclude, in Section 8, with a multiple integral expression for the the spin $q$–Whittaker polynomials.

1.3. Notation. Throughout the paper we use a number of partition-related terminologies, which we summarize below. These are all standard in the combinatorics literature, with one exception: we include parts of size zero in our partitions, rather than the usual practice of truncating a partition after its last positive part.

A partition $\lambda$ is a finite non-increasing sequence of non-negative integers $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0$, where $\lambda_i$ is called a part of $\lambda$ for all $1 \leq i \leq \ell$. The empty partition $\varnothing$ is the trivial sequence consisting of no parts. The length of a partition $\lambda$ is the number of parts which comprise it, and denoted by $\ell(\lambda)$. We let $\text{Part}_\ell$ denote the set $\{\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0\}$ of all partitions of length $\ell$. Similarly, $\text{Part}_\ell^+$ will denote the set $\{\lambda_1 \geq \cdots \geq \lambda_m \geq 1\}$ of all partitions with purely positive parts, whose length is bounded by $\ell$. It is sometimes convenient to write a partition in terms of its part-multiplicities $m_i(\lambda) = \#\{j : \lambda_j = i\}$, i.e. by writing $\lambda = 0^{m_0}1^{m_1}2^{m_2}\ldots$ where $m_\ell \equiv m_i(\lambda)$. The conjugate of a partition $\lambda$, denoted $\lambda'$, is the partition with parts $\lambda'_i = \#\{j : \lambda_j \geq i\}$.

For two positive partitions $\lambda, \mu$ we write $\lambda \supset \mu$ if $\ell(\lambda) \geq \ell(\mu)$ and the inequality $\lambda_i \geq \mu_i$ holds for all $1 \leq i \leq \ell(\mu)$. Similarly, we write $\lambda \succ \mu$ and say that $\lambda$ interlaces $\mu$ if $0 \leq \ell(\lambda) - \ell(\mu) \leq 1$ and $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all $1 \leq i \leq \ell(\mu)$ (where the final inequality $\mu_{\ell(\mu)} \geq \lambda_{\ell(\mu)+1}$ is omitted in the case $\ell(\lambda) = \ell(\mu)$).

We make frequent use of $q$–Pochhammer symbols, which are defined as follows:

\[
(a;q)_m = \begin{cases} 
\prod_{1 \leq i \leq m} (1 - a q^{i-1}), & m > 0, \\
1, & m = 0, \\
\prod_{1 \leq i \leq -m} (1 - a q^{-i})^{-1}, & m < 0.
\end{cases}
\]

We will also tacitly assume that $|q| < 1$ so that, in particular, $(a;q)_\infty$ makes sense. Most of our equations depend on another parameter $s$; we will also assume that $|s| < 1$, since this prevents $s$ from taking values which would lead to divergences because of vanishing denominators.

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2. Higher spin vertex model

2.1. Vertex weights and Yang–Baxter equation. Following Section 2 of [BP16b], we define a vertex model consisting of SW $\rightarrow$ NE oriented paths on a square grid. Horizontal edges of the grid can be occupied by at most one lattice path, but no restriction is placed on the number of paths which traverse a vertical edge. Every intersection of horizontal and vertical gridlines constitutes a vertex, and each vertex is assigned a Boltzmann weight that depends on the local configuration of lattice paths about that intersection. Assuming conservation of lattice paths through a vertex, four types of vertex are possible. We indicate these vertices and their explicit weights below:

\[
\begin{align*}
\text{(1)} & \quad w_u(g, 0; g, 0) = \frac{1 - sq^g u}{1 - su} \\
\text{(2)} & \quad w_u(g + 1, 0; g, 1) = \frac{(1 - s^2 q^g) u}{1 - su} \\
\text{(3)} & \quad w_u(g, 1; g + 1, 0) = \frac{1 - q^{g+1}}{1 - su} \\
\text{(4)} & \quad w_u(g, 1; g, 1) = \frac{u - sq^g}{1 - su}
\end{align*}
\]

where $g$ is any non-negative integer (representing the number of paths which sit at a vertical edge), $s$ and $q$ are fixed global parameters of the model, and $u$ is a local variable called the spectral parameter. We denote the Boltzmann weight of a vertex in two equivalent ways, and interchange between the two according to convenience:

\[
(2) \quad w_u \left( \begin{array}{c} j \\ i \\ k \\ \ell \end{array} \right) \equiv w_u(i, j; k, \ell), \quad i, k \in \mathbb{Z}_{\geq 0}, \quad 0 \leq j, \ell \leq 1.
\]

It is conventional to relax the constraint of lattice path conservation through each vertex, which can be done by extending the Boltzmann weights (2) to all values of $i, j, k, \ell$ and assuming that $w_u(i, j; k, \ell) = 0$ for all $i + j \neq k + \ell$. The common factor $1 - su$ in the denominator of each vertex (1) is to ensure that the empty vertex has weight 1, i.e. $w_u(0, 0; 0, 0) = 1$.

Define an $n$-vertex by concatenating $n$ vertices vertically, with summation assumed over all internal vertical edges:

\[
(3) \quad w_{\{u_1, \ldots, u_n\}} \left( \begin{array}{c} k \\ j_n \\ \vdots \\ j_1 \\ i \\ \ell_n \\ \vdots \\ \ell_1 \end{array} \right) \equiv w_{\{u_1, \ldots, u_n\}}(i, \{j_1, \ldots, j_n\}; k, \{\ell_1, \ldots, \ell_n\})
\]

It is easily seen that the weight of an $n$-vertex (3) is always factorized into weights of individual vertices (1): knowing three of the edge states surrounding a vertex determines the fourth by lattice path conservation, and this constrains each of the internal edges in (3) to assume a unique value (or it causes the whole weight to vanish if $i + j_1 + \cdots + j_n \neq k + \ell_1 + \cdots + \ell_n$, when conservation is impossible).
Definition 2.1. Let \( W = \text{Span}\{|j\rangle\}_{0 \leq j \leq 1} \cong \mathbb{C}^2 \) be a two-dimensional vector space, and for all \( 1 \leq i \leq n \) let \( W_i \) denote a copy of \( W \). The \( n \)-vertex operator \( \mathcal{W}_{\{u_1,...,u_n\}}(i; k) \) acts linearly on \( W_1 \otimes \cdots \otimes W_n \) as follows:

\[
\mathcal{W}_{\{u_1,...,u_n\}}(i; k) : |\ell_1\rangle_1 \otimes \cdots \otimes |\ell_n\rangle_n \mapsto \sum_{0 \leq j_1,...,j_n \leq 1} w_{\{u_1,...,u_n\}}(i, \{j_1,\ldots,j_n\}, k, \{\ell_1,\ldots,\ell_n\}) |j_1\rangle_1 \otimes \cdots \otimes |j_n\rangle_n.
\]

This in fact defines an infinite family of operators, since \( i \) and \( k \) can be any non-negative integers.

Proposition 2.2. Let \( \mathcal{W}_{\{u_1,u_2\}}(i; k) \in \text{End}(W_1 \otimes W_2) \) be a 2-vertex operator as defined above, with \( i, k \in \mathbb{Z}_{\geq 0} \). The Yang–Baxter equation holds:

\[
\mathcal{P} \circ \mathcal{R}(u_2/u_1) \circ \mathcal{W}_{\{u_1,u_2\}}(i; k) = \mathcal{W}_{\{u_1,u_2\}}(i; k) \circ \mathcal{P} \circ \mathcal{R}(u_2/u_1),
\]

where the \( \mathcal{R} \)-matrix is given by

\[
\mathcal{R}(u) = \begin{pmatrix}
1 - qu & 0 & 0 & 0 \\
0 & q(1-u) & 1-q & 0 \\
0 & (1-q)u & 1-u & 0 \\
0 & 0 & 0 & 1 - qu
\end{pmatrix} \in \text{End}(W_1 \otimes W_2),
\]

and \( \mathcal{P} \in \text{End}(W_1 \otimes W_2) \) is the permutation operator, with action \( \mathcal{P} : |a\rangle_1 \otimes |b\rangle_2 \mapsto |b\rangle_1 \otimes |a\rangle_2 \) for all vectors \( |a\rangle, |b\rangle \in W \).

Proof. By direct computation, using the vertex weights (1) and the explicit realization of the permutation operator,

\[
\mathcal{P} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in \text{End}(W_1 \otimes W_2).
\]

See also Section 2 of [Bor17] for the derivation of (1) from the \( U_q(\widehat{sl}_2) \) Yang–Baxter equation in the tensor product \( W_1 \otimes W_2 \otimes V_I \), where \( V_I \) denotes a highest weight representation with weight \( I \).

\[\square\]

2.2. Dual vertex weights. We will at times adopt an alternative convention for the vertex weights (1). The change in convention is brought about inverting the spectral parameter \( u \) in (1), then multiplying all vertices by \( (u - s)/(1 - su) \). Since the Yang–Baxter equation (1) is preserved under this transformation (up to inversion of \( u_1 \) and \( u_2 \)), this does not damage the integrability of the model, although it will be essential to ensure that certain infinite partition functions which we study are well-defined. The alternative weights are as shown
Graphically, we distinguish such vertices from their counterparts by using a coloured background. If we complement the states that live on horizontal edges (i.e. draw a path if the edge is unoccupied, delete a path if the edge is occupied), we find that the vertices are converted to

\[ w_u^*(g, 1; g, 1) \quad w_u^*(g + 1, 1; g, 0) \quad w_u^*(g, 0; g + 1, 1) \quad w_u^*(g, 0; g, 0) \]

where we have also reversed the orientation of all paths, so that they propagate NW → SE. In this way we define a set of dual vertex weights \( w_u^*(i, j; k, \ell) \), whose non-zero values are indicated in (7). There is a strong similarity between the vertices and those of the starting vertex model. Indeed, by reflecting the vertices about their central horizontal axis, we almost recover those of (1) – the only difference is that the weights of the two middle vertices have changed slightly. It turns out that this discrepancy can be cured by a simple gauge transformation of the weights. More precisely, find that

\[ w_u(i, j; k, \ell) = \frac{(q; q)_k}{(s^2; q)_k} w_u^*(k, j; i, \ell) \frac{(s^2; q)_i}{(q; q)_i}, \quad i, k \in \mathbb{Z}_{\geq 0}, \quad 0 \leq j, \ell \leq 1. \]

2.3. **Partition states.** Let \( V = \text{Span}\{ |m_i\rangle \}_{m_i \geq 0} \) be an infinite-dimensional vector space, and for all \( i \geq 0 \) let \( V_i \) denote a copy of \( V \). Further, let \( V = \otimes_{i \geq 0} V_i \), the global vector space obtained by tensoring the local spaces \( V_i \) for all \( i \geq 0 \). It can be viewed as the span of pure tensors \( \otimes_{i \geq 0} |m_i\rangle_i \) with \( m_i \in \mathbb{Z}_{\geq 0} \), all but finitely many of which are 0.

Consider a partition \( \lambda = 0^{m_0} 1^{m_1} 2^{m_2} \ldots \) expressed in terms of its of part multiplicities \( m_i(\lambda) \). The part multiplicities can be obtained as the difference between adjacent parts in the conjugate partition \( \lambda' \), as follows:

\[ m_i(\lambda) = \lambda'_i - \lambda'_{i+1}, \]

since \( \lambda'_i = \sum_{j \geq i} m_j(\lambda) \) for all \( i \geq 0 \). To every partition \( \lambda \) we associate a unique state in \( V \):

\[ |\lambda\rangle = \bigotimes_{i \geq 0} |m_i\rangle_i = \bigotimes_{i \geq 0} |\lambda'_i - \lambda'_{i+1}\rangle_i \in V. \]
Similarly, one can define dual partition states \( \langle \lambda | = \bigotimes_{i \geq 0} (m_i)\rangle \in \mathcal{V}^* \), with the orthonormal action \( \langle \lambda | \mu \rangle = \delta_{\lambda, \mu} \) for all partitions \( \lambda, \mu \).

In the coming sections, we will study the vertex model (1) on a square lattice with infinitely many columns, labelled from left to right by non-negative integers. In that situation, we shall identify the vector space \( V \) with the \( i \)th column of the lattice, with \( |m\rangle \), encoding a vertical edge in that column which is occupied by \( m \) paths.

In Section 3, we shall define, for all (positive) partitions \( \lambda = 1^{m_1} 2^{m_2} \ldots \), the natural objects are partitions \( \lambda \) with strictly positive parts (i.e. the traditional notion of a partition), since the number of zero parts in \( \lambda \) becomes irrelevant. This leads us to define, for all (positive) partitions \( \lambda = 1^{m_1} 2^{m_2} \ldots \), the state

\[
\langle \lambda | = \bigotimes_{i \geq 1} |m_i\rangle \in \mathcal{V}^* ,
\]

and dual state \( \langle \lambda | = \bigotimes_{i \geq 1} \langle m_i | \in \mathcal{V}^* \).

3. Spin Hall–Littlewood functions

3.1. Setup of the lattice for \( F_\lambda (u_1, \ldots, u_\ell) \). Following [BP16b], we study the vertex model (1) on the quadrant \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \). Horizontal lines are oriented from left to right and numbered from bottom to top, while vertical lines are oriented from bottom to top and numbered from left to right. The \( i \)th horizontal line is assigned spectral parameter \( u_i \), where \( i \in \mathbb{Z}_{\geq 1} \). The boundary conditions are fixed as follows:

1. There is an incoming lattice path at the external left edge of every horizontal line.
2. The external bottom edge of every vertical line is unoccupied.

**Definition 3.1.** Let \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0) \in \text{Part}_\ell \) be a partition. The spin Hall–Littlewood function \( F_\lambda (u_1, \ldots, u_\ell) \) is defined as the partition function of \( \mathbb{Z}_{\geq 0} \times \{1, \ldots, \ell\} \) in the model (1), whose left and bottom edge boundary conditions are given by 1 and 2 as above, and whose \( i \)th external top edge is occupied by exactly \( m_i(\lambda) \) paths for all \( i \geq 0 \). See Figure 2, left panel.

**Remark 3.2.** The lattice used in the definition of \( F_\lambda (u_1, \ldots, u_\ell) \) has infinitely many vertical lines, but it remains a meaningful definition. Indeed, for a given \( \lambda \), only the vertical lines numbered 0 to \( \lambda_1 \) will contribute non-trivially to the partition function. It is obvious that all vertical lines beyond this will be unoccupied, giving rise to infinitely many copies of the vertex \( 0 \quad 0 \quad 0 \) , which has weight 1. A similar remark applies to all infinite partition functions studied in this section.

By allowing more general boundary conditions at the base of the lattice, we can extend the definition to skew Young diagrams:

**Definition 3.3.** Let \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_{\ell+n} \geq 0) \in \text{Part}_{\ell+n} \) and \( \mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0) \in \text{Part}_n \) be two partitions. The skew spin Hall–Littlewood function \( F_{\lambda/\mu} (u_1, \ldots, u_\ell) \) is defined as the partition function of \( \mathbb{Z}_{\geq 0} \times \{1, \ldots, \ell\} \) in the model (1), whose left edge boundary satisfies 1 as above, whose \( i \)th external top edge is occupied by exactly \( m_i(\lambda) \) paths, and whose \( i \)th external bottom edge is occupied by exactly \( m_i(\mu) \) paths for all \( i \geq 0 \). The
Figure 2. Left panel: boundary conditions used to calculate $F_{\lambda}(u_1, \ldots, u_4)$ for $\lambda = (4, 3, 3, 1)$. Right panel: boundary conditions used to calculate $F_{\lambda/\mu}(u_1, \ldots, u_4)$ for $\lambda = (5, 4, 2, 2, 1, 0)$, $\mu = (4, 1, 1)$.

Figure 3. Left panel: boundary conditions used to calculate $G_{\lambda}(v_1, \ldots, v_5)$ for $\lambda = (5, 4, 1, 0)$. Right panel: boundary conditions used to calculate $G_{\lambda/\mu}(v_1, \ldots, v_5)$ for $\lambda = (4, 3, 3, 1)$, $\mu = (3, 2, 0, 0)$.

(ordinary) spin Hall–Littlewood function $F_\lambda$ is recovered in the special case $\mu = \emptyset$. See Figure 2 right panel.

3.2. Setup of the lattice for $G_\lambda(v_1, \ldots, v_n)$. Here we use an alternative set of boundary conditions:

1. The external left edge of every horizontal line is unoccupied.
2. The external bottom edge of each vertical line is unoccupied, with the exception of the 0th, which is occupied by $\ell$ incoming paths, for some $\ell \geq 0$.

Definition 3.4. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0) \in \text{Part}_\ell$ be a partition. The dual spin Hall–Littlewood function $G_\lambda(v_1, \ldots, v_n)$ is defined as the partition function of $\mathbb{Z}_{\geq 0} \times \{1, \ldots, n\}$ in the model (1), whose left and bottom edge boundary conditions are given by 1 and 2 as above, and whose $i$th external top edge is occupied by exactly $m_i(\lambda)$ paths for all $i \geq 0$. Notice that we impose no relation between the values of $\ell$ and $n$. See Figure 3 left panel.

Once again, this definition can be extended to skew Young diagrams by allowing for more general boundary conditions at the base of the lattice:

Definition 3.5. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0) \in \text{Part}_\ell$ and $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0) \in \text{Part}_\ell$ be two partitions of equal length. The dual skew spin Hall–Littlewood function $G_{\lambda/\mu}(v_1, \ldots, v_n)$ is defined as the partition function of $\mathbb{Z}_{\geq 0} \times \{1, \ldots, n\}$ in the model (1), whose left edge boundary conditions are given by 1 as above, whose $i$th external top edge is
occupied by exactly $m_i(\lambda)$ paths, and whose $i^{th}$ external bottom edge is occupied by exactly $m_i(\mu)$ paths for all $i \geq 0$. The (ordinary) dual spin Hall–Littlewood function $G_\lambda$ is recovered in the special case $\mu = 0^\ell$. See Figure 3, right panel.

### 3.3. The function $G^*_\lambda/\mu(v_1, \ldots, v_n)$

There is a slight modification of the dual polynomials in Section 3.2 that turns out to be important for correctly stating the Cauchy identity between $F_\lambda$ and $G_\lambda$ [Bor17, BP16b]. Let $\lambda$ and $\mu$ be two partitions in the set $\text{Part}^\ell$. We define

\begin{equation}
G^*_\lambda/\mu(v_1, \ldots, v_n) := \frac{c_\lambda(q, s)}{c_\mu(q, s)} G_\lambda/\mu(v_1, \ldots, v_n), \quad c_\lambda(q, s) := \frac{(q; q)_\ell}{(s^2; q)_\ell} \prod_{i \geq 0} \frac{(s^2; q)_{m_i(\lambda)}}{(q; q)_{m_i(\lambda)}}.
\end{equation}

In the special case $\mu = 0^\ell$, one has $m_0(\mu) = \ell$ and $m_i(\mu) = 0$ for all $i \geq 1$, so that $c_\mu(q, s) = 1$. Then (11) reduces to

\[G^*_\lambda(v_1, \ldots, v_n) = c_\lambda(q, s) G_\lambda(v_1, \ldots, v_n).\]

**Proposition 3.6.** The function $G^*_\lambda/\mu(v_1, \ldots, v_n)$ is equal to the partition function of $\mathbb{Z}_{\geq 0} \times \{1, \ldots, n\}$ in the model (7), whose external left edges are all unoccupied, whose $i^{th}$ external top edge is occupied by exactly $m_i(\mu)$ paths, and whose $i^{th}$ external bottom edge is occupied by exactly $m_i(\lambda)$ paths for all $i \geq 0$ (see Figure 4).

**Proof.** This is a straightforward consequence of the relation (8). Starting from the partition function as described in the Proposition, we apply the conjugation (8) to every vertex in the lattice. This conjugation can be effected by multiplying the whole partition function by

\[\prod_{i \geq 0} \frac{(s^2; q)^{m_i(\mu)}}{(q; q)^{m_i(\mu)}} \frac{(q; q)_{m_i(\lambda)}}{(s^2; q)_{m_i(\lambda)}} ,\]

and at the end of this transformation we obtain the partition function as described in Definition 3.5. We conclude that the partition function that we started with is equal to

\[
\prod_{i \geq 0} \frac{(q; q)^{m_i(\mu)}}{(s^2; q)^{m_i(\mu)}} \frac{(s^2; q)^{m_i(\lambda)}}{(q; q)^{m_i(\lambda)}} G_\lambda/\mu(v_1, \ldots, v_n) = \frac{c_\lambda(q, s)}{c_\mu(q, s)} G^*_\lambda(v_1, \ldots, v_n),
\]

completing the proof. □
3.4. Row operators. So far we defined the spin Hall–Littlewood functions as partition functions in the vertex model \([1]\), but it is also possible to formulate them algebraically. For that, we now introduce finite row operators, whose action is specified in terms of the partition function of a single row of vertices:

\[
w_u \left( \begin{array}{cccc}
  & & & \\
  & j & & \\
  & & \ddots & \\
  & & & \ell
\end{array} \right) = w_u \left( \{i_0, \ldots, i_L\}, j; \{k_0, \ldots, k_L\}, \ell \right),
\]

where \(L\) is any non-negative integer. Introduce the monodromy matrix

\[
T(u) = \begin{pmatrix} T_u(0;0) & T_u(0;1) \\ T_u(1;0) & T_u(1;1) \end{pmatrix} \equiv \begin{pmatrix} A_L(u) & B_L(u) \\ C_L(u) & D_L(u) \end{pmatrix}
\]

whose entries \(T_u(j;\ell)\) are operators acting linearly on \(V_0 \otimes \cdots \otimes V_L\) as follows:

\[
T_u(j;\ell) : |k_0\rangle_0 \otimes \cdots \otimes |k_L\rangle_L \mapsto \sum_{i_0, \ldots, i_L \geq 0} w_u \left( \{i_0, \ldots, i_L\}, j; \{k_0, \ldots, k_L\}, \ell \right) |i_0\rangle_0 \otimes \cdots \otimes |i_L\rangle_L.
\]

It is immediate, from \(L + 1\) applications of the Yang–Baxter equation \([1]\), that the monodromy matrix satisfies the intertwining relation

\[
\mathcal{P} \circ \mathcal{R}(u/v) \circ \left( T(v) \otimes T(u) \right) = \left( T(u) \otimes T(v) \right) \circ \mathcal{P} \circ \mathcal{R}(u/v),
\]

as an identity in \(\text{End}(W_1 \otimes W_2 \otimes V_1 \otimes \cdots \otimes V_L)\). This encodes sixteen bilinear relations among the entries of the monodromy matrix, which are collectively known as the Yang–Baxter algebra. See, for example, Chapter VII of \([KBI93]\) for a complete list of the relations. We will primarily be interested in the following ones:

\[
[ A_L(u), A_L(v) ] = [ B_L(u), B_L(v) ] = [ C_L(u), C_L(v) ] = [ D_L(u), D_L(v) ] = 0,
\]

(13)

\[
(1 - qu/v)D_L(v)C_L(u) = (1 - u/v)C_L(u)D_L(v) + (1 - q)D_L(u)C_L(v).
\]

(14)

It is convenient to define a “starred” version of the \(B\)- and \(D\)-operators:

\[
B^*_L(u) := \left( \frac{u - s}{1 - su} \right)^{L+1} B_L(u^{-1}), \quad D^*_L(u) := \left( \frac{u - s}{1 - su} \right)^{L+1} D_L(u^{-1}).
\]

(15)

This has the effect of modifying these operators, so that they are constructed in precisely the same way as above, but now using the alternative vertices \([6]\).

3.5. Infinite volume limit. It is possible to take the length of the row operators to infinity, by sending \(L \to \infty\) in \([12]\). However some care is needed in doing so, since the behaviour of \(A_L(u), B_L(u), C_L(u), D_L(u)\) in the limit depends on how they are normalized. One finds that

\[
A(u) := \lim_{L \to \infty} A_L(u), \quad C(u) := \lim_{L \to \infty} C_L(u)
\]

We point out that our conventions regarding the operators \([12]\) are different to those of \([BP16a]\). When viewed as rows in partition functions, the operators \([12]\) act from top to bottom, whereas those in \([BP16b]\) act from bottom to top. Also, the labelling of the operators in \([BP16b]\) is obtained from our labelling under \(B \leftrightarrow C\).
make sense as written; for the remaining two operators it is necessary to use the normalization \([15]\), leading to

\[
B^*(u) := \lim_{L \to \infty} B_L^*(u), \quad D^*(u) := \lim_{L \to \infty} D_L^*(u).
\]

Now let us examine what happens to the commutation relation \((14)\) in the limit \(L \to \infty\). We firstly invert \(v\), then multiply the whole relation by \((v - s)^{L+1}/(1 - sv)^{L+1}\):

\[
(1 - quv)D_L^*(v)C_L(u) = (1 - uv)C_L(u)D_L^*(v) + (1 - q) \left( \frac{(u - s)(v - s)}{(1 - su)(1 - sv)} \right)^{L+1} D_L^*(u)C_L(v).
\]

Assuming that \(u, v, s\) are chosen such that \(|(u - s)(v - s)| < |(1 - su)(1 - sv)|\), the second term of the above equation vanishes when \(L \to \infty\), and we obtain

\[
(1 - quv)D^*(v)C(u) = (1 - uv)C(u)D^*(v).
\]

3.6. **Algebraic formulation of spin Hall–Littlewood functions.** Using the definition of the row operators from the previous section, the following result is immediate:

**Proposition 3.7.** Fix two partitions \(\lambda \in \text{Part}_{\ell+n}\) and \(\mu \in \text{Part}_n\). Then

\[
F_{\lambda/\mu}(u_1, \ldots, u_\ell) = \langle \mu | C(u_1) \ldots C(u_\ell) | \lambda \rangle,
\]

where we have used the notation \([9]\) for partition states, namely, \(|\lambda\rangle = \otimes_{i \geq 0} |m_i(\lambda)\rangle_i \in \mathbb{V}\) and \(\langle \mu | = \otimes_{i \geq 0} \langle m_i(\mu) |_i \in \mathbb{V}^*\). Similarly, fix two partitions \(\lambda, \mu \in \text{Part}_\ell\). Then

\[
G_{\lambda/\mu}(v_1, \ldots, v_n) = \langle \mu | A(v_1) \ldots A(v_n) | \lambda \rangle,
\]

\[
G^*_{\lambda/\mu}(v_1, \ldots, v_n) = \langle \lambda | D^*(v_n) \ldots D^*(v_1) | \mu \rangle.
\]

**Proof.** These expectation values can be directly compared with the partition functions written down in Sections 3.1–3.3. Note that each one is clearly symmetric in its rapidity variables, by virtue of the commutation relations \([13]\). \(\square\)

3.7. **Proving the skew Cauchy identity.** Equipped with the algebraic expressions of Proposition 3.7, it is possible to establish many interesting properties of the symmetric functions \(F_{\lambda/\mu}\) and \(G_{\lambda/\mu}\). Here we shall recall the proof of one such property, the **skew Cauchy identity.** We refer the reader to [Bor17, BP16b] for a more detailed exposition of other properties.

**Theorem 3.8.** Fix two partitions \(\mu \in \text{Part}_n\) and \(\nu \in \text{Part}_{\ell+n}\), for some \(\ell, n \geq 1\). For any complex numbers \(u_1, \ldots, u_\ell; v_1, \ldots, v_m\) such that \(|(u_i - s)(v_j - s)| < |(1 - su_i)(1 - sv_j)|\) for all \(i, j\), the following summation identity holds:

\[
\sum_{\lambda} F_{\lambda/\mu}(u_1, \ldots, u_\ell) G^*_{\lambda/\nu}(v_1, \ldots, v_m) = \left( \prod_{i=1}^{\ell} \prod_{j=1}^{m} \frac{1 - qu_iv_j}{1 - u_iv_j} \right) \sum_{\kappa} F_{\nu/\kappa}(u_1, \ldots, u_\ell) G^*_{\mu/\kappa}(v_1, \ldots, v_m),
\]

where the left hand side is summed over partitions \(\lambda \in \text{Part}_{\ell+n}\), and the right hand side over partitions \(\kappa \in \text{Part}_n\) (observe that the right hand sum is finite, since the summand vanishes if \(\kappa \not\subset \nu\) or if \(\kappa \not\subset \mu\).
Corollary 3.9. Assuming \( |(u_i - s)(v_j - s)| < |(1 - su_i)(1 - sv_j)| \) for all \( i, j \), the following identity holds:

\[
\sum_{\lambda} F_{\lambda}(u_1, \ldots, u_\ell) G^*_{\lambda}(v_1, \ldots, v_m) = \frac{(q; q)_\ell}{\prod_{i=1}^{\ell} (1 - su_i)} \prod_{i=1}^{\ell} \prod_{j=1}^{m} \left( \frac{1 - qu_i v_j}{1 - u_i v_j} \right),
\]

where the sum is taken over all partitions \( \lambda \in \text{Part}_\ell \).

**Proof.** We make the choice \( \mu = \emptyset \) and \( \nu = 0^\ell \) in (17). This converts the left hand side into that of (20), while the sum on the right hand side now consists of a single term, corresponding to \( \kappa = \emptyset \). One can easily show that

\[
F_{0^\ell}(u_1, \ldots, u_\ell) = \frac{(q; q)_\ell}{\prod_{i=1}^{\ell} (1 - su_i)}, \quad G^*_{\emptyset}(v_1, \ldots, v_m) = 1.
\]

This accounts for the new factors appearing on the right hand side of (20), and completes the proof. \( \square \)

\footnote{It is clear from its definition as a partition function that \( F_{0^\ell}(u_1, \ldots, u_\ell) \) is equal to the \( \ell \)-vertex \( u_{\{u_1, \ldots, u_\ell\}}(0; \{1, \ldots, 1\}; \ell, \{0, \ldots, 0\}) \), whose weight is precisely \( (q; q)_\ell \prod_{i=1}^{\ell} (1 - su_i)^{-1} \).}
4. Fusion

4.1. Specializing variables to geometric progressions. Specializing spectral parameters to geometric progressions plays a crucial role in the fusion procedure. Since the spin $q$–Whittaker polynomials are ultimately to be obtained under such specializations, we develop some rather general notation here which will prove useful in our calculations.

Let $\{J_0, \ldots, J_n\}$ be some arbitrary set of positive integers and further, define their partial sums $J_i := \sum_{k=0}^i J_k$. We will often write $J_n \equiv J$ for the sum of all $J_i$. Then for any set of variables $\{u_1, \ldots, u_J\}$, we define its $\{J_0, \ldots, J_n; z_0, \ldots, z_n\}$-specialization to be

\[
\Big\{ u(J_{i-1}+1), u(J_{i-1}+2), \ldots, u(J_{i-1}+J_i) \Big\} = \{ z_i, qz_i, \ldots, q^{J_i-1}z_i \}
\]

for all $0 \leq i \leq n$, where by agreement $J_{-1} = 0$. We use the notation $\rho_{\{J_0, \ldots, J_n; z_0, \ldots, z_n\}}(\cdot)$ to indicate that a $\{J_0, \ldots, J_n; z_0, \ldots, z_n\}$-specialization has been taken.

For the purposes of fusion itself, we will only require the $n = 0$ case of the above. In that case we have a single positive integer $J_0 \equiv J$, and a $\{J; u\}$-specialization of $\{u_1, \ldots, u_J\}$ is simply given by

\[
\{u_1, u_2, \ldots, u_J\} = \{u, qu, \ldots, q^{J-1}u\}.
\]

4.2. Fusion. We recall the basics of the fusion procedure in integrable lattice models [KRS81, KR87], mainly following the conventions of Section 5 of [BP16b] (see also [CPT16], for a more probabilistic interpretation of fusion). The central object is the $J$-vertex, as defined in (3), whose spectral parameters $\{u_1, u_2, \ldots, u_J\}$ have been $\{J; u\}$-specialized; i.e. specialized as in (22). One then defines a “fused” vertex as follows:

\[
w_u^{(J)}(j \quad k \quad i) = \sum_{0 \leq a_1, \ldots, a_J \leq 1; |a| = j} \frac{q^{\sum_{m=1}^{J} (m-1)a_m}}{Z_J(j)} \times w_{\{u, qu, \ldots, q^{j-1}u\}}(a_j \quad b_j \quad \vdots \quad a_1 \quad b_1),
\]

obtained by fixing the bottom and top of the $J$-vertex to the states $i$ and $k$, respectively, while summing the left and right edges over all possible ways of assigning $j$ and $\ell$ arrows to $J$ sites. Here we have defined the normalization

\[
Z_J(j) = \sum_{0 \leq c_1, \ldots, c_J \leq 1; |c| = j} q^{\sum_{m=1}^{J} (m-1)c_m} = q^{j(j-1)/2} \left( \frac{q; q}_J(q; q)^J J^{-j}. \right)
\]

Graphically, we represent fused vertices by the intersection of a thick horizontal and vertical line. This is supposed to indicate that up to $J$ lattice paths can now occupy a horizontal edge, while the occupation numbers along vertical edges continue to be unbounded. As before we interchange freely between the graphical version of vertices, and a purely algebraic
ways of distributing the summation over the internal horizontal edges. These internal edges are summed over all proved in exactly the same way. We take the right hand side of (25) and explicitly perform a right hand side of (25) is thus

\[ \frac{q^{\sum_{m=1}^{J}(m-1)a_m}}{Z_J(J)} \times w_{\{u,qu,\ldots,q^{J-1}u\}} \]

\[
\begin{bmatrix}
  k_0 \\
  a_1 \\
  \vdots \\
  c_1 \\
  i_0 \\
\end{bmatrix} \\
\begin{bmatrix}
  k_1 \\
  c_1 \\
  \vdots \\
  b_1 \\
  i_1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_J \\
  c_J \\
  \vdots \\
  b_J \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  k_0 \ldots k_L \\
  a_1 \ldots a_J \ldots b_1 \ldots b_J \\
  i_0 \ldots i_L \\
\end{bmatrix}
\]

\[ w_{\{u,qu,\ldots,q^{J-1}u\}} \]

Proving. We restrict our attention to the case \( L = 1 \), since the result for larger values of \( L \) is proved in exactly the same way. We take the right hand side of (25) and explicitly perform the summation over the internal horizontal edges. These internal edges are summed over all ways of distributing \( n \) arrows, where (by conservation) \( n = i_0 + j - k_0 = k_1 + \ell - i_1 \). The right hand side of (25) is thus

\[
\sum_{a_1,\ldots,a_J:|a|=j}^{b_1,\ldots,b_J:|b|=-\ell} \sum_{c_1,\ldots,c_J:|c|=n}^{q^{\sum_{m=1}^{J}(m-1)a_m}} \times w_{\{u,qu,\ldots,q^{J-1}u\}} \]

\[
\begin{bmatrix}
  k_0 \\
  a_1 \\
  \vdots \\
  c_1 \\
  i_0 \\
\end{bmatrix} \\
\begin{bmatrix}
  a_J \\
  c_J \\
  \vdots \\
  b_J \\
\end{bmatrix}
\]

or using algebraic notation,

\[
\sum_{a_1,\ldots,a_J:|a|=j}^{b_1,\ldots,b_J:|b|=-\ell} q^{\sum_{m=1}^{J}(m-1)a_m} \times w_{\{u,qu,\ldots,q^{J-1}u\}} \]

\[
\begin{bmatrix}
  k_0 \\
  a_1 \\
  \vdots \\
  c_1 \\
  i_0 \\
\end{bmatrix} \\
\begin{bmatrix}
  a_J \\
  c_J \\
  \vdots \\
  b_J \\
\end{bmatrix}
\]

\[ w_{\{u,qu,\ldots,q^{J-1}u\}} \]

In order to progress further, we note the following property on 2-vertices, which can be easily checked for all \( i, k \in \mathbb{Z}_{\geq 0} \) [BP16b]:

\[
\sum_{0\leq a_1,a_2\leq 1} q^{a_2} w_{\{u,qu\}}(i,\{a_1,a_2\};k,\{0,1\}) = \sum_{0\leq a_1,a_2\leq 1} q^{a_2+1} w_{\{u,qu\}}(i,\{a_1,a_2\};k,\{1,0\}).
\]
Because the spectral parameters in (26) form geometric sequences with ratio $q$, we are able to apply the relation (28) repeatedly to the first $J$-vertex in (26), treating all indices apart from $a_1, \ldots, a_J$ as fixed. We find that

\begin{equation}
\sum_{a_1, \ldots, a_J; |a|=j}^\infty \frac{q^{j(m-1)a_m}}{Z_j(J)} w_{\{u, qu, \ldots, q^{J-1}u\}} \left( i_0, \{a_1, \ldots, a_J\}; k_0, \{c_1, \ldots, c_J\} \right)
\end{equation}

\begin{equation}
= \sum_{a_1, \ldots, a_J; |a|=j}^\infty \frac{q^{j(m-1)a_m}}{Z_j(J)} w_{\{u, qu, \ldots, q^{J-1}u\}} \left( i_0, \{a_1, \ldots, a_J\}; k_0, \{1, \ldots, 1, 0, \ldots, 0\} \right)
\end{equation}

\begin{equation}
= \frac{\sum_{m=1}^{J} (m-1)c_m}{Z_n(J)} \sum_{a_1, \ldots, a_J; |a|=j}^\infty \frac{q^{j(m-1)a_m}}{Z_j(J)} w_{\{u, qu, \ldots, q^{J-1}u\}} \left( i_0, \{a_1, \ldots, a_J\}; k_0, \{d_1, \ldots, d_J\} \right),
\end{equation}

where the latter expression is by definition equal to $w_u^{(j)}(i_0, j; k_0, n) / Z_n(J)$. Using this result in (27), the sums over $c_1, \ldots, c_J$ and $b_1, \ldots, b_J$ can then be taken. The result of the calculation is $\sum_n w_u^{(j)}(i_0, j; k_0, n) w_u^{(j)}(i_1, n; k_1, \ell)$, which is just the left hand side of (27).

4.3. Recursion relation. It is easy to see that the fused vertex (23) obeys the following recursion relation, obtained by summing over all possible configurations of the lowest vertex in (23):

\[
w_u^{(j)}(i, j; k, \ell) = \frac{q^j Z_j(J - 1)}{Z_j(J)} \sum_{n=0}^1 w_u(i - n, i; n) w_u^{(j-1)}(i - n, j; k, \ell - n) + \frac{q^{j-1} Z_{j-1}(J - 1)}{Z_j(J)} \sum_{n=0}^1 w_u(i - n + 1, i; n) w_u^{(j-1)}(i - n + 1, j - 1; k, \ell - n).
\]

Simplifying and converting to algebraic notation, this reads

\begin{equation}
w_u^{(j)}(i, j; k, \ell) = \frac{q^j - q^j}{1 - q^j} \sum_{n=0}^1 w_u(i, 0; i - n, n) w_u^{(j-1)}(i - n, j; k, \ell - n) + \frac{1 - q^j}{1 - q^j} \sum_{n=0}^1 w_u(i, 1; i - n + 1, n) w_u^{(j-1)}(i - n + 1, j - 1; k, \ell - n).
\end{equation}

The recursion relation (30) can be solved to yield an explicit formula for the fused vertex weights. The solution is, however, rather complicated and involves $q$-hypergeometric functions [Man14, Bor17, CP16]:

\begin{equation}
w_u^{(j)}(i, j; k, \ell) = \sum_{i+j=k+\ell} \frac{(-1)^{\ell-i} q^{i(j+2j-1)/2} s^{j-k} u^i (u/s; q)_{\ell-i} (s^2; q)_i}{(su; q)_{k+\ell} (q^{j+j-1}; q)_{j-\ell} (q; q)_i (s^2; q)_k} 4H_3 \left( \begin{array}{c} q^{-k}, q^{-\ell}, q^{j+1} s, q^{j+1} u \\ q, q \end{array} | q, q \right),
\end{equation}
where
\[ \tilde{\phi}_r(i, j; k, \ell) = \sum_{k=0}^{n} z^k (q^{-n}; q)_k \prod_{i=1}^{r} (a_i; q)_k (b_i q^k; q)_{n-k}. \]

A higher-rank version of the Boltzmann weights \[31\] has recently been considered in \[BM16\].

One can write a fused version of the Yang–Baxter equation \[4\], which now features an \( R \)-matrix acting in a tensor product of spin-\( J_1/2 \) and spin-\( J_2/2 \) representations \[Man14\] (whereas the \( R \)-matrix \[5\] acts in a tensor product of spin-1/2 representations). The entries of the \( R \)-matrix have essentially the same functional form as \[31\], however we will bypass writing this Yang–Baxter equation in its full generality, quoting instead a much simpler special case of it below.

4.4. Simplified vertex weights after setting \( u = s \). The vertex weights \[31\] greatly simplify at \( u = s \), as is explained in \[Bor17, BP16b, BM16\]. Quoting Proposition 6.10 from \[BP16b\], we have
\[ w_s^{(J)}(i, j; k, \ell) = (1_{i+j=k+\ell})(1_{i\geq \ell})(-sq^{j+\ell})(q^{-J}; q)_\ell (s^2 q^J; q)_{i-\ell} (q; q)_k, \]
for \( i, k \in \mathbb{Z}_{\geq 0} \), \( 0 \leq j, \ell \leq J \). Let us also define an \( R \)-matrix with components given by
\[ R^{(J_1 J_2)}(i, j; k, \ell) = (1_{i+j=k+\ell})(1_{i\geq \ell})(q^{-I_1}; q)_\ell (q^{-I_2}; q)_{i-\ell} (q; q)_k, \]
where \( 0 \leq i, k \leq I \) and \( 0 \leq j, \ell \leq J \).

**Theorem 4.2.** Let \( J_1, J_2 \) be two positive integers, and fix two triples \( (i_1, i_2, i_3), (j_1, j_2, j_3) \) such that \( 0 \leq i_1, j_1 \leq J_1 \), \( 0 \leq i_2, j_2 \leq J_2 \) and \( i_3, j_3 \in \mathbb{Z}_{\geq 0} \). The Yang–Baxter equation holds:
\[ \sum_{k_1=0}^{J_1} \sum_{k_2=0}^{J_2} \sum_{k_3=0}^{\infty} R^{(J_1 J_2)}(i_2, i_1; k_2, k_1) w_s^{(J_1)}(i_3, k_1; k_3, j_1) w_s^{(J_2)}(k_3, k_2; j_3, j_2) = \sum_{k_1=0}^{J_1} \sum_{k_2=0}^{J_2} \sum_{k_3=0}^{\infty} w_s^{(J_2)}(i_3, i_2; k_2, k_3) w_s^{(J_1)}(k_3, i_1; j_3, j_1) R^{(J_2 J_1)}(k_2, k_1; j_2, j_1). \]

**Proof.** We will not give a detailed proof of \[31\]. It can be derived straightforwardly by writing a \( (J_1 + J_2) \)-vertex version of the Yang–Baxter equation \[4\], then taking its \( \{J_1, J_2; s, s\} \) specialization to effect the desired fusion. \( \square \)

4.5. Analytic continuation. We see that \( q^{I_1} \) and \( q^{I_2} \) play the role of spectral parameters in the relation \[34\], even though the \( R \)-matrix \[33\] does not depend purely on the ratio of these parameters. This suggests that one should analytically continue \( q^{I_1} \) and \( q^{I_2} \) to arbitrary complex values, when \[34\] continues to hold (after removing the bounds on \( i_1, j_1 \) and \( i_2, j_2 \)). This will be a key idea in our construction of the spin \( q \)-Whittaker polynomials.

Let us perform the substitution \( q^I \mapsto -x/s \) in \[32\], where \( x \in \mathbb{C} \). We call the resulting weight \( W_x(i, j; k, \ell) \); it is given by
\[ W_x(i, j; k, \ell) = (1_{i+j=k+\ell})(1_{i\geq \ell}) x^\ell (-sq^j; q)_\ell (-sx; q)_{i-\ell} (q; q)_k. \]
for all \( i, j, k, \ell \in \mathbb{Z}_{\geq 0} \). Similarly, under \( q' \mapsto -x/s \) and \( q' \mapsto -y/s \), the \( \mathcal{R} \)-matrix (33) becomes

\[
\mathcal{R}_{x,y}(i, j; k, \ell) = (1_{i+j=k+\ell})(1_{i+j=\ell})(x/y)^{\ell} \left( \frac{-s/x}{q}\frac{x/y}{q_{i-\ell}} \frac{(q; q)_{i}}{(q; q)_{i+\ell}} \right),
\]

for all \( i, j, k, \ell \in \mathbb{Z}_{\geq 0} \).

**Corollary 4.3.** Let \((i_1, i_2, i_3)\) and \((j_1, j_2, j_3)\) be two triples of non-negative integers. The following identity holds for arbitrary \( x, y \in \mathbb{C} \):

\[
\sum_{k_1, k_2, k_3 = 0}^{\infty} \mathcal{R}_{x,y}(i_2, i_1; k_2, k_1)W_y(i_3, k_1; k_3, j_1)W_x(k_3, k_2; j_3, j_2) = \sum_{k_1, k_2, k_3 = 0}^{\infty} W_x(i_3, i_2; k_2, k_1)W_y(k_3, i_1; j_3, k_1)\mathcal{R}_{x,y}(k_2, k_1; j_2, j_1).
\]

**Proof.** After fixing finite triples of non-negative integers \((i_1, i_2, i_3)\) and \((j_1, j_2, j_3)\), equation (34) holds for infinitely many values of \( J_1 \) with fixed \( J_2 \), and infinitely many values of \( J_2 \) with fixed \( J_1 \). Furthermore, both sides of (34) are rational functions in \( q^{J_1} \) and \( q^{J_2} \); the functions must then be equal for all \( q^{J_1} \in \mathbb{C} \) and \( q^{J_2} \in \mathbb{C} \). \( \square \)

**Remark 4.4.** The \( s = 0 \) case of the Boltzmann weights (35), and of the Yang–Baxter equation (36), first appeared in Section 3 of [Kor13].

4.6. **Dual vertex weights.** Up to this point we have examined the fusion procedure as applied to \( J \)-vertices in the vertex model (11). Clearly, we could also adapt this approach to \( J \)-vertices in the dual model (7); in fact the steps of Sections 4.1, 4.5 can be repeated virtually without modification, because of equation (8), which relates the vertices (1) and (7) up to a conjugation applied to their vertical edges. We obtain, for generic spectral parameter \( u \), a dual set of weights

\[
w^d_u(J) \left( \begin{array}{c} j \hline i \end{array} \right)^{\ell} \equiv w^d_u(i, j; k, \ell)
\]

which are determined by the relation

\[
w^d_u(J)(i, j; k, \ell) = \frac{(q; q)_k}{(s^2; q)_k} w^s_u(J)(k, i; j, \ell) \frac{(s^2; q)_i}{(q; q)_i}, \quad i, k \in \mathbb{Z}_{\geq 0}, \quad 0 \leq j, \ell \leq J,
\]

where \( w^d_u(J)(i, j; k, \ell) \) is given by (31). It then follows from (32) that, at \( u = s \), one has

\[
w^d_u(J)(i, j; k, \ell) = (1_{j+k=i+\ell})(1_{k+\ell}) \frac{-s q^J \ell}{(q; q)_{i}} \frac{(s^2 q^J; q)_{k-\ell}}{(q; q)_{k}}.
\]

Finally, after analytic continuation in \( q^J \) and substituting \( q^J \mapsto -x/s \), we arrive at the vertex model

\[
W^d_x \left( \begin{array}{c} j \hline i \end{array} \right)^{\ell} \equiv W^d_x(i, j; k, \ell) = (1_{j+k=i+\ell})(1_{k+\ell}) x^{\ell} \frac{-s x q^J \ell}{(q; q)_{i}} \frac{(-s x q^J; q)_{k-\ell}}{(q; q)_{k}}.
\]
related to (35) via the transformation

\[ W_x(i, j; k, \ell) = \frac{(q; q)_k}{(s^2; q)_k} W_x^*(k, j; i, \ell) \frac{(s^2; q)_i}{(q; q)_i}, \quad i, j, k, \ell \in \mathbb{Z}_{\geq 0}. \]

5. Spin \(q\)-Whittaker polynomials

5.1. Geometric specialization of \(F_\lambda(u_1, \ldots, u_\mathcal{J})\). Fix a set of positive integers \(\{J_0, \ldots, J_n\}\) and let \(\mathcal{J} = \sum_{i=0}^{n} J_i\) denote their sum. Let \(F_\lambda(u_1, \ldots, u_\mathcal{J})\) be a spin Hall–Littlewood function as defined in Section 3.1.

Following Section 4.1, we take a \(\{J_0, \ldots, J_n; z_0, \ldots, z_n\}\)-specialization of \(F_\lambda(u_1, \ldots, u_\mathcal{J})\), as dictated by (21). When applying this specialization to \(F_\lambda(u_1, \ldots, u_\mathcal{J})\), we group its horizontal spectral parameters into \(n + 1\) successive geometric progressions with ratio \(q\), meaning that we are in the position to fuse the horizontal lines \((\mathcal{J}_i - 1 + 1, \mathcal{J}_i - 1 + J_i)\) for all \(0 \leq i \leq n\). The fusion is particularly simple: we note that the left and right boundary conditions of the lattice correspond with trivial summations over \(a_1^{(i)}, \ldots, a_{J_i}^{(i)}\) such that \(\sum_{k=1}^{J_i} a_k^{(i)} = J_i\) and \(b_1^{(i)}, \ldots, b_{J_i}^{(i)}\) such that \(\sum_{k=1}^{J_i} b_k^{(i)} = 0\), for all \(0 \leq i \leq n\), where \(a\)’s and \(b\)’s are as in Theorem 4.1. We can therefore apply Theorem 4.1 to obtain

\[ \rho_{\{J_0, \ldots, J_n; z_0, \ldots, z_n\}}(F_\lambda) = \]

\[ \begin{pmatrix}
  \vdots \\
  w_{z_n}^{(J_n)}(J_n) \\
  \vdots \\
  w_{z_1}^{(J_1)}(J_1) \\
  w_{z_0}^{(J_0)}(J_0)
\end{pmatrix}
\]

where every vertex within the lattice is of the form (31), and where we have indicated the value of the spin \(J_i\) and the spectral parameter \(z_i\) of each fused row by writing \(w_{z_i}^{(J_i)}(\cdot)\) around it. Note that the partition state \(\lambda\) along the top of the lattice is chosen such that \(\sum_{i \geq 0} m_i(\lambda) = \mathcal{J}\), otherwise this partition function vanishes trivially.

5.2. The limit \(J_0 \to \infty\). In the next step of the calculation we set \(z_0 = s\), which converts all vertices in the \(0\)th row of the lattice to the simplified form (32), and take \(J_0\) to infinity. In order to obtain a non-zero result, we also shift the value of \(m_0\) at the top of the \(0\)th column, by assuming that \(m_0 \geq J_0\).

Consider the vertex at the intersection of the \(0\)th row and column, as indicated by the arrow in (40). In taking \(J_0 \to \infty\), its incoming data becomes \(i = 0, j = \infty\), which by the form of (32) constrains the outgoing data to \(k = \infty, \ell = 0\). The resulting Boltzmann weight is \((q; q)_\infty/(s^2; q)_\infty\). The remainder of the \(0\)th row then only gives rise to empty vertices,
which have weight 1. The result of the calculation is thus

\[ \rho_{\{\infty,J_1,\ldots,J_n,s,z_1,\ldots,z_n\}}(F_{\{\infty,m_1,m_2,\ldots\}}) = \frac{(q;q)_{\infty}}{(s^2;q)_{\infty}} \times \]

\[ w_{z_n}^{(J_n)} \left( \begin{array}{c} J_n \\ \vdots \\ J_1 \\ 0 \\ \vdots \\ 0 \end{array} \right) \]

where we have removed the 0th row of the lattice entirely.

5.3. Stable spin Hall–Littlewood functions. Before proceeding further in our study of equation (41), we mention a particular case of it that will later be important: namely, the case \( J_1 = \cdots = J_n = 1 \). In this case the \( n \) rows of the lattice are not fused at all, and we return to the model of Section 5 but now with infinitely many lattice paths entering and leaving the 0th column of the lattice. The vertices in the 0th column then have Boltzmann weights of the form

\[ w_u \left( \begin{array}{cccc} j & \infty & \ell & \infty \end{array} \right) = \frac{u^\ell}{1 - su}, \quad 0 \leq j, \ell \leq 1, \]

and we remark that they are independent of the value of \( j \), the left edge state. We use this partition function to define stable spin Hall–Littlewood functions:

\[ \tilde{F}_\lambda(u_1, \ldots, u_n) := \langle \emptyset | \tilde{C}(u_1) \cdots \tilde{C}(u_n) | \lambda \rangle, \quad \tilde{C}(u) := (1 - su) \langle \infty | 0 C(u) | \infty \rangle_0, \]

where \( \lambda = 1^{m_1}2^{m_2} \cdots \in \text{Part}^+_n \) is a positive partition, and where we have employed the notation (10), i.e. \( \langle \emptyset | = \langle 0 \rangle_1 \otimes \langle 0 \rangle_2 \otimes \cdots \) and \( | \lambda \rangle = | m_1 \rangle_1 \otimes | m_2 \rangle_2 \otimes \cdots \). The multiplicative factor \( (1 - su) \) appearing in the definition of \( \tilde{C}(u) \) is to cancel the same factor appearing in the denominator of (42). The operators \( \tilde{C}(u) \) live in \( \text{End}(\tilde{V}) \) and are given explicitly by

\[ \tilde{C}_L(u) : | k_1 \rangle_1 \otimes \cdots \otimes | k_L \rangle_L \mapsto \sum_{0 \leq j \leq 1} \sum_{i_1, \ldots, i_L \geq 0} u^j w_u \left( \{ i_1, \ldots, i_L \}, j; \{ k_1, \ldots, k_L \}, 0 \right) | i_1 \rangle_1 \otimes \cdots \otimes | i_L \rangle_L, \]

with \( \tilde{C}(u) = \lim_{L \to \infty} \tilde{C}_L(u) \).

For any positive partition \( \lambda \in \text{Part}^+_{n-1} \), the functions (43) satisfy the stability equation

\[ \tilde{F}_\lambda(u_1, \ldots, u_{n-1}, 0) = \tilde{F}_\lambda(u_1, \ldots, u_{n-1}), \]

which can be easily verified using the lattice interpretation of \( \tilde{F}_\lambda \) (namely, (41) with all \( J_i = 1 \)). Indeed, analysing the top row of the partition function (41), we find that the only configuration which does not have a common factor of \( u_n \) (and accordingly, does not vanish
when $u_n = 0$) is the following one:

![Diagram](image)

Using the Boltzmann weights \((1)\) with $u = 0$, we see that the weight of this configuration is 1. The property \((44)\) is now immediate.

The functions \((43)\) appeared in essentially the same form in \([GdGW17]\). There an explicit formula for $\tilde{F}_\lambda$, as a sum over the symmetric group, was obtained:

\begin{equation}
\tilde{F}_\lambda(u_1, \ldots, u_n) = \frac{(1 - q)^n}{(q; q)_n - \ell(\lambda)} \times \sum_{\sigma \in S_n} \sigma \left\{ \prod_{1 \leq i < j \leq n} \left( \frac{u_i - qu_j}{u_i - u_j} \right) \prod_{i=1}^{\ell(\lambda)} \left( \frac{u_i}{u_i - s} \right) \prod_{i=1}^{n} \left( \frac{u_i - s}{1 - su_i} \right) \right\}.
\end{equation}

This equation is parallel to similar formulae for $F_\lambda$ and $G_\lambda$ obtained in \([Bor17]\, Section 5\) and \([BP16b]\, Section 4\). In fact, one can easily derive \((45)\) by starting from the symmetrization formula for $G_\lambda$ in \([Bor17]\) and taking the limit in which the number of lattice paths on the 0th column becomes infinite. Doing so leads to the partition function as shown in \((41)\), but in which the left edge states are $J_1 = \cdots = J_n = 0$ instead of $J_1 = \cdots = J_n = 1$. This discrepancy between the left edge states is actually irrelevant, in view of the fact that the vertices \((42)\) do not depend on the value of $j$.

One can also define a dual version of the stable spin Hall–Littlewood functions:

$$\tilde{F}_\lambda(v_1, \ldots, v_n) := \langle \lambda | \tilde{B}^* (v_n) \cdots \tilde{B}^* (v_1) | \emptyset \rangle, \quad \tilde{B}^* (v) := (1 - sv) \langle 0 | D^*(v) | \infty \rangle_0,$$

for all $\lambda \in \text{Part}_n^+$. This time, the operators $\tilde{B}^* (v) \in \text{End}(\tilde{V})$ have the explicit action

$$\tilde{B}^*_L(v) : |k_1\rangle_1 \otimes \cdots \otimes |k_L\rangle_L \mapsto \sum_{0 \leq j < 1} \sum_{i_1, \ldots, i_L \geq 0} v^j w_v^j \left( \{ i_1, \ldots, i_L \}; \{ k_1, \ldots, k_L \}, 0 \right) |i_1\rangle_1 \otimes \cdots \otimes |i_L\rangle_L,$$

with $\tilde{B}^* (v) = \lim_{L \to \infty} \tilde{B}^*_L(v)$. By virtue of the relation \((8)\), it is then evident that

\begin{equation}
\tilde{F}_\lambda^* (v_1, \ldots, v_n) = \bar{c}_\lambda(q; s) \tilde{F}_\lambda (v_1, \ldots, v_n), \quad \bar{c}_\lambda(q, s) := \prod_{i \geq 1} \frac{(s^2; q)_{m_i(\lambda)}}{(q; q)_{m_i(\lambda)}}.
\end{equation}

**Remark 5.1.** It is easily verified from \((15)\) and \((46)\) that $\tilde{F}_\lambda$ degenerates to the dual Hall–Littlewood polynomial $Q_\lambda$ at $s = 0$, while $\tilde{F}_\lambda^*$ becomes equal to the $P_\lambda$ Hall–Littlewood polynomial.

### 5.4. Lattice construction of $F_\lambda(x_1, \ldots, x_n)$.

Returning to \((11)\), we set $z_1 = \cdots = z_n = s$, which converts all vertex weights to the form \((32)\). The weights in the 0th column have a particularly simple expression. Indeed, taking the $i \to \infty$ limit of \((32)\), we find that

\begin{equation}
w_s^{(j)}(\infty; j; k, \ell) = 1_{k=\infty} (-sq_j^{-1})^\ell (q^{-j}; q)_{\ell} (s^2q^j; q)_{\infty},
\end{equation}

with $s = \cdots = s_n = s$. By virtue of the relation \((8)\), it is then evident that
since the value of \( \ell \) must remain finite (otherwise the factor \((-sq^\ell)^{ \ell} \) vanishes). Because of the indicator function \( 1_{k=\infty} \), if infinitely many paths enter a vertex vertically from below, infinitely many paths will also exit vertically from above; this ensures that every vertex in the 0\(^{th}\) column has a Boltzmann weight of the form (47). It is then clear that the 0\(^{th}\) column of the lattice always produces the factor \( \prod_{i=1}^{n} (s^2 q^{i}; q)_{\infty} / (s^2 q)_{\infty} \) (irrespective of the path configuration along the right edges of the vertices in this column).

The weights (47) are independent of the value of \( j \), meaning that the partition function (41) does not depend on its left-edge states \( \{ J_1, \ldots, J_n \} \). We may therefore reassign the left-edge states to any values; the most natural choice is \( \{ 0, \ldots, 0 \} \). Furthermore, we can effectively remove the restriction that internal horizontal edge states in the \( i \)\(^{th}\) row be bounded by \( J_i \), since the Boltzmann weights (32) vanish whenever \( \ell > J_i \).

In light of these observations, we see that \( \rho(\infty, J_1, \ldots, J_n; s, s, \ldots, s) (F_{(\infty, m_1, m_2, \ldots)} \ldots (F_{(\infty, m_1, m_2, \ldots)} \ldots \) depends on \( J_1, \ldots, J_n \) only via \( q^{J_1}, \ldots, q^{J_n} \). After dividing by the common factor \( \prod_{i=1}^{n} (s^2 q^{i}; q)_{\infty} / (s^2 q)_{\infty} \), it is easily shown to be polynomial in each \( q^{J_i} \), where the boundedness of degree follows from the fact that \( m_i = 0 \) for all \( i > N \), for sufficiently large \( N \). Since we know this polynomial at infinitely many values \( q^{J_i} \in \{ q, q^2, \ldots, \} \), we can analytically continue in \( q^{J_i} \), writing \( q^{J_i} = -x_i / s \), for all \( 1 \leq i \leq n \). This takes us to the partition function

\[
(48) \quad \prod_{i=1}^{n} \frac{(s^2 q)_{\infty}}{(-s x_i q)_{\infty}}
\]

where each vertex is now of the form (35), and we have dropped the factor \( (q; q)_{\infty} / (s^2 q)_{\infty} \) from (41), which turns out to be an unnecessary artefact of the calculation.

**Definition 5.2.** Fix a positive partition \( \lambda \) and let \( \lambda' = 1^{m_1(\lambda')} 2^{m_2(\lambda')} \ldots \) be its conjugate. The spin \( q \)-Whittaker polynomial \( F_{\lambda}(x_1, \ldots, x_n) \) is defined to be equal to the partition function shown in (48) with vertex weights (35), where \( m_i \equiv m_i(\lambda') \) for all \( i \geq 1 \).

**Remark 5.3.** A couple of comments are in order regarding the partition function (48). First, it is a meaningful way to define \( F_{\lambda} \), even though it contains infinitely many columns. This can be argued, once again, using the fact that only the columns numbered 0 to \( \lambda'_1 \) will contribute non-trivially to the partition function; the remaining columns will only feature vertices of the form \( 0 \begin{array}{c} 0 \end{array} 0 \), with weight 1.

Second, it might not be clear at this stage why we have chosen the \( m_i \) in (48) to correspond with the part-multiplicities of the conjugate partition \( \lambda' \), rather than \( \lambda \) itself. Our justification for doing so is the \( F_{\lambda} \), as defined, reduce to the \( q \)-Whittaker polynomials at \( s = 0 \); see Section 6.3.

**Definition 5.4.** Fix positive partitions \( \lambda, \mu \) such that \( \lambda \supset \mu \), and let \( \lambda' = 1^{m_1(\lambda')} 2^{m_2(\lambda')} \ldots \) and \( \mu' = 1^{m_1(\mu')} 2^{m_2(\mu')} \ldots \) be their conjugates. The skew spin \( q \)-Whittaker polynomial \( F_{\lambda/\mu}(x_1, \ldots, x_n) \) is defined as the partition function shown in Figure 5 (left panel) with
vertex weights \((35)\), divided by \(\prod_{i=1}^{n} (-sx_i; q)_{\infty}/(s^2; q)_{\infty}\). It reduces to the (ordinary) spin \(q\)-Whittaker polynomial for \(\mu = \emptyset\).

5.5. Lattice construction of \(G_\lambda(x_1, \ldots, x_n)\). In Sections 5.1, 5.2 and 5.4 we gave a systematic approach that allows one to start from a spin Hall–Littlewood function \(F_\lambda\) and transform it into a spin \(q\)-Whittaker polynomial (up to adjustments of the normalization). One can now repeat this procedure, applying it to the dual spin Hall–Littlewood function \(G_\lambda\).

This time we begin by fixing a set of positive integers \(\{K_1, \ldots, K_n\}\), with \(K = \sum_{i=1}^{n} K_i\) denoting their sum, and let \(G_\lambda(v_1, \ldots, v_K)\) be the dual spin Hall–Littlewood function as defined in Section 3.2. We then take a \(\{K_1, \ldots, K_n; s, \ldots, s\}\)-specialization of \(G_\lambda(v_1, \ldots, v_K)\), and send the number of lattice paths entering and exiting the 0th column to infinity (this can always be achieved, since one can send both \(\ell\) and \(m_0(\lambda)\) in Definition 3.4 to infinity, while keeping \(\ell - m_0(\lambda)\) finite).

After appropriate analytic continuation in \(q^{K_1}, \ldots, q^{K_n}\), the result of the calculation is direct passage to the partition function \((48)\), which we already obtained by applying the fusion procedure to \(F_\lambda\). We conclude that no new function is obtained in this case; accordingly, we make the identification \(G_{\lambda/\mu}(x_1, \ldots, x_n) \equiv F_{\lambda/\mu}(x_1, \ldots, x_n)\) for all partitions \(\lambda, \mu\), and will not mention \(G_{\lambda/\mu}(x_1, \ldots, x_n)\) again in the sequel.

5.6. The polynomial \(F_{\lambda/\mu}^*(x_1, \ldots, x_n)\). Motivated by equation \((46)\), we make the following definition:

\[
F_{\lambda/\mu}^*(x_1, \ldots, x_n) := \frac{\tilde{c}_{\lambda'}(q, s)}{\tilde{c}_{\mu'}(q, s)} F_{\lambda/\mu}(x_1, \ldots, x_n), \quad \tilde{c}_{\lambda'}(q, s) = \prod_{i \geq 1} \frac{(s^2; q)_{m_i(\lambda')}}{(q; q)_{m_i(\lambda')}} \prod_{i \geq 1} \frac{(s^2; q)_{\lambda_i-\lambda_{i+1}}}{(q; q)_{\lambda_i-\lambda_{i+1}}}.
\]

We refer to these as dual (skew) spin \(q\)-Whittaker polynomials.

**Proposition 5.5.** Fix positive partitions \(\lambda, \mu\) such that \(\lambda \supset \mu\), and let \(\lambda' = 1^{m_1(\lambda')}2^{m_2(\lambda')}\ldots\) and \(\mu' = 1^{m_1(\mu')}2^{m_2(\mu')}\ldots\) be their conjugates. The dual skew spin \(q\)-Whittaker polynomial \(F_{\lambda/\mu}^*(x_1, \ldots, x_n)\) is equal to the partition function shown in Figure 5 (right panel) with vertex weights \((38)\), divided by \(\prod_{i=1}^{n} (-sx_i; q)_{\infty}/(s^2; q)_{\infty}\).
**Proof.** This is proved using the relation (39), and following essentially the same steps as in the proof of Proposition 3.6.

5.7. **Fused row operators.** In analogy with Section 3.4, we now define row operators which can be used to express the spin $q$–Whittaker polynomials in a purely algebraic way. By horizontally concatenating the vertices (35) or (38), and summing over the states on all internal lattice edges, we obtain fused row vertices:

\[
W_x \left( \begin{array}{ccc}
  & k_1 & \ldots & k_L \\
  j & \ldots & \ldots & \ldots & i_1 \ldots i_L \\
\end{array} \right) = W_x(\{i_1, \ldots, i_L\}, j; \{k_1, \ldots, k_L\}, \ell)
\]

\[
W_x^* \left( \begin{array}{ccc}
  & k_1 & \ldots & k_L \\
  j & \ldots & \ldots & \ldots & i_1 \ldots i_L \\
\end{array} \right) = W_x^*(\{i_1, \ldots, i_L\}, j; \{k_1, \ldots, k_L\}, \ell)
\]

where all indices $\{i_1, \ldots, i_L\}, j, \{k_1, \ldots, k_L\}, \ell$ take values in $\mathbb{Z}_{\geq 0}$. From this, introduce two (infinite dimensional) monodromy matrices

\[
T(x) = \begin{pmatrix}
  T_x(0; 0) & T_x(0; 1) & \cdots \\
  T_x(1; 0) & T_x(1; 1) & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix}, \quad T^*(x) = \begin{pmatrix}
  T_x^*(0; 0) & T_x^*(0; 1) & \cdots \\
  T_x^*(1; 0) & T_x^*(1; 1) & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix},
\]

whose entries act linearly on $V_1 \otimes \cdots \otimes V_L$ as follows:

\[
T_x(j; \ell) : |k_1\rangle_1 \otimes \cdots \otimes |k_L\rangle_L \mapsto \sum_{i_1, \ldots, i_L \geq 0} W_x(\{i_1, \ldots, i_L\}, j; \{k_1, \ldots, k_L\}, \ell)|i_1\rangle_1 \otimes \cdots \otimes |i_L\rangle_L,
\]

\[
T_x^*(j; \ell) : |k_1\rangle_1 \otimes \cdots \otimes |k_L\rangle_L \mapsto \sum_{i_1, \ldots, i_L \geq 0} W_x^*(\{i_1, \ldots, i_L\}, j; \{k_1, \ldots, k_L\}, \ell)|i_1\rangle_1 \otimes \cdots \otimes |i_L\rangle_L.
\]

Similarly to the algebraic construction of the stable spin Hall–Littlewood functions, it turns out that certain linear combinations of the monodromy matrix entries $T_x(j; \ell)$ and $T_x^*(j; \ell)$ are the most useful for constructing the spin $q$–Whittaker polynomials. We define two such linear combinations:

\[
C(x) := \lim_{L \to \infty} \left( \sum_{i=0}^{\infty} x^i \frac{(-s/x;q)_i}{(q;q)_i} T_x(i; 0) \right), \quad B^*(x) := \lim_{L \to \infty} \left( \sum_{i=0}^{\infty} x^i \frac{(-s/x;q)_i}{(q;q)_i} T_x^*(i; 0) \right),
\]

passing also to the semi-infinite lattice, so that both $C(x), B^*(x) \in \text{End}(\bar{V})$. The coefficients in the above sums originate from the right hand side of (47) with $q^f = -s/x$; see the proof of Proposition 5.6 below.

---

3The fused row operators that we study have previously appeared, in the $s = 0$ case, in [Kor13, Kor16] and [DP15]; these works also exposed a direct correspondence with Baxter’s $Q$-operator.
5.8. **Algebraic formulation of spin $q$–Whittaker polynomials.** Putting together the definitions of the previous subsections, we are now ready to express the spin $q$–Whittaker polynomials as expectation values of the operators (49).

**Proposition 5.6.** Let $n$ be a positive integer and fix two positive partitions $\lambda, \mu$ such that $\lambda \supset \mu$. Letting $\lambda', \mu'$ be the corresponding conjugate partitions, we have
\begin{align}
F_{\lambda/\mu}(x_1, \ldots, x_n) &= \langle \mu'|C(x_1) \cdots C(x_n)|\lambda' \rangle, \\
F^*_{\lambda/\mu}(y_1, \ldots, y_n) &= \langle \lambda|B^*(y_1) \cdots B^*(y_n)|\mu' \rangle,
\end{align}
where $|\lambda'\rangle = \otimes_{i \geq 1} |m_i(\lambda')\rangle_i$, and similarly for the remaining state vectors.

**Proof.** For the proof of (50), we start from the lattice expression for $F_{\lambda/\mu}(x_1, \ldots, x_n)$, shown on the left panel of Figure 5. Counting from bottom to top, the $i$th vertex in the 0th column is of the form $0 \underbrace{\thickspace \cdot \cdot \cdot \cdot \cdot}_i \infty$, with Boltzmann weight
\[
W_{x_i} \left( \underbrace{0 \infty \cdots \infty}_{i} \right) = x_i^{\ell_i} \frac{(-s/x_i; q)_{\infty}}{(q; q)^{\ell_i}} \frac{(s^2; q)_{\infty}}{(-s x_i; q)_{\infty}},
\]
where $\ell_i \geq 0$ takes any non-negative integer value. Deleting the factor $\prod_{i=1}^n (-s x_i; q)_{\infty}/(s^2; q)_{\infty}$ which is common to the 0th column of the lattice (as per the definition of $F_{\lambda/\mu}(x_1, \ldots, x_n)$), and summing over all $\ell_1, \ldots, \ell_n \geq 0$, the $i$th row of the lattice then manifestly corresponds with the operator $C(x_i)$ (cf. equation (49), the definition of $C(x)$). This proves the formula (50).

The proof of (51) is completely analogous; there, one starts from the lattice expression for $F^*_{\lambda/\mu}(y_1, \ldots, y_n)$, shown on the right panel of Figure 5.

5.9. **Exchange relations.** In this section we prove some important properties of the operators (49). The first is their self-commutativity, Proposition 5.7. The relations (52) allow us to show that the spin $q$–Whittaker polynomials are symmetric in their variables, which is not obvious from their lattice definition. The remaining properties are the exchange relations (55), (56) and (57), in Proposition 5.9. These relations play a key role in deriving Cauchy-type summation identities involving the spin $q$–Whittaker polynomials.

**Proposition 5.7.** The operators (49) commute amongst themselves:
\begin{equation}
[C(x), C(y)] = [B^*(x), B^*(y)] = 0,
\end{equation}
for arbitrary complex parameters $x$ and $y$.

**Proof.** A particularly simple case of the Yang–Baxter equation (35) is recovered by choosing $i_1 = i_2 = j_1 = j_2 = 0$:
\[
\sum_{k_1, k_2, k_3 = 0}^\infty R_{x,y}(0, 0; k_2, k_1) W_y(i_3, k_1; k_3, 0) W_x(k_3, k_2; j_3, 0) = \\
\sum_{k_1, k_2, k_3 = 0}^\infty W_x(i_3, 0; k_3, k_2) W_y(k_3, 0; j_3, k_1) R_{x,y}(k_2, k_1; 0, 0).
\]
The summation over \( k_1 \) and \( k_2 \) becomes trivial, since \( R_{x,y}(i, j; k, \ell) = 0 \) unless \( i + j = k + \ell \). This constrains \( k_1 \) and \( k_2 \) to be zero, and we read (after dropping \( R_{x,y}(0, 0; 0, 0) = 1 \) from both sides of the equation)

\[
(53) \quad \sum_{k_3} W_y(i_3, 0; k_3, 0)W_x(k_3, 0; j_3, 0) = \sum_{k_3} W_x(i_3, 0; k_3, 0)W_y(k_3, 0; j_3, 0),
\]

which is true by inspection, since either side of this equation vanishes unless \( i_3 = j_3 = k_3 \). Moreover, using \( L + 1 \) iterations of \( (53) \) and applying the same logic as above, we obtain the following non-trivial commutation relation between row operators, generalizing \( (53) \):

\[
(54) \quad W_x\left( \begin{array}{cccc} m_0 & m_1 & \cdots & m_L \\ 0 & n_1 & \cdots & n_L \\ \end{array} \right) = W_y\left( \begin{array}{cccc} m_0 & m_1 & \cdots & m_L \\ 0 & n_1 & \cdots & n_L \\ \end{array} \right),
\]

where \( m_0, \ldots, m_L \) and \( n_0, \ldots, n_L \) are arbitrary non-negative integers. Sending \( m_0, n_0, L \to \infty \) and deleting the factor \((-sx;q)_\infty(-sy;q)_\infty/(s^2;q)_\infty^2\) which becomes common to both sides of \( (54) \) (cf. the proof of Proposition 5.6), we obtain precisely \( C(y)C(x) = C(x)C(y) \), as required.

The second relation, \([B^*(x), B^*(y)] = 0\), can be deduced from the first using the transformation property \( (39) \). \( \square \)

**Corollary 5.8.** The skew spin \( q \)-Whittaker polynomials are symmetric in their variables, by virtue of the commutativity \( (52) \) of the operators in \( (50) \) and \( (51) \).

**Proposition 5.9.** The following exchange relations hold:

\[
(55) \quad C(x)B^*(y) = \frac{(-sx;q)_\infty(-sy;q)_\infty}{(s^2;q)_\infty(xy;q)_\infty} B^*(y)C(x), \quad \text{for } |x|, |y| < 1,
\]

\[
(56) \quad \tilde{C}(u)B^*(x) = \left( \frac{1 + ux}{1 - su} \right) B^*(x)\tilde{C}(u),
\]

\[
(57) \quad C(x)\tilde{B}^*(u) = \left( \frac{1 + ux}{1 - su} \right) \tilde{B}^*(u)C(x),
\]

\[
(58) \quad \tilde{C}(u)\tilde{B}^*(v) = \left( \frac{1 - quv}{1 - uv} \right) \tilde{B}^*(v)\tilde{C}(u), \quad \text{for } |(u-s)(v-s)| < |(1-s)(1-sv)|.
\]

**Remark 5.10.** The \( s = 0 \) case of the relations \( (56) \) and \( (57) \) was previously obtained by Duval and Pasquier in \[DP15\] Section 6.

**Proof.** We begin by noting the relation

\[
\prod_{i=1}^l C(u_i) \prod_{j=1}^J D^*(v_j) = \prod_{i=1}^l \prod_{j=1}^J \frac{1 - qu_i v_j}{1 - u_i v_j} \prod_{j=1}^J D^*(v_j) \prod_{i=1}^l C(u_i)
\]
which holds in $\text{End}(\mathbb{V})$, provided $|(u_i - s)(v_j - s)| < |(1 - su_i)(1 - sv_j)|$ for all $1 \leq i \leq I$ and $1 \leq j \leq J$. The graphical version of this relation is as follows:

where we have assumed the arbitrary boundary conditions $|m_0\rangle_0 \otimes |m_1\rangle_1 \cdots \in \mathbb{V}$ and $\langle n_0|_0 \otimes \langle n_1|_1 \cdots \in \mathbb{V}^*$ at the top and base of the lattice, respectively. Each relation (55)–(58) can be deduced by specializing (59) in a different way, as we now show.

**Proof of (55).** Starting from (59), we perform an $\{I; s\}$-specialization of the variables $(u_1, \ldots, u_I)$ and a $\{J; s\}$-specialization of the variables $(v_1, \ldots, v_J)$. These specializations instigate fusion in the lattices on the left and right hand sides of (59), and we obtain the relation

$$
\rho_{\{I; s\}}^u \rho_{\{J; s\}}^v \left( \prod_{i=1}^I \prod_{j=1}^J \frac{1 - q u_i v_j}{1 - u_i v_j} \right) \times \begin{pmatrix}
w_s(I) \\ w_s(J)
\end{pmatrix}
= \begin{pmatrix}
w_s(I) \\ w_s(J)
\end{pmatrix},
$$

where the vertices that appear in these partition functions are either of the form (32) or (37). The multiplicative factor on the right hand side of (60) may be easily calculated; due to telescopic cancellations, one has

$$
\rho_{\{I; s\}}^u \rho_{\{J; s\}}^v \left( \prod_{i=1}^I \prod_{j=1}^J \frac{1 - q u_i v_j}{1 - u_i v_j} \right) = \frac{(s^2 q^I; q)_I}{(s^2; q)_I} = \frac{(s^2 q^I; q)_\infty (s^2 q^I; q)_\infty}{(s^2; q)_\infty (s^2 q^I + j; q)_\infty}.
$$

We now exploit the freedom to choose the states at the top and bottom of the lattices in (60), sending $m_0, n_0 \to \infty$ simultaneously. Summing explicitly over the possible contributions of

---

4In making these specializations, one should ensure that $|s^2 (1 - q^{-1}) (1 - q^{i-1})| < |(1 - s^2 q^{-1})(1 - s^2 q^{i-1})|$ for all $1 \leq i \leq I$, $1 \leq j \leq J$, since this is necessary for (59) to remain valid.
the 0\textsuperscript{th} column on either side of (60), and deleting some trivial common factors from the equation, we obtain

\begin{equation}
(61) \sum_{i=0}^{I} \sum_{j=0}^{J} (-sq^{I})^{i} (-sq^{J})^{j} (q^{-I}; q)_{i} (q^{-J}; q)_{j} \frac{w_{s}^{(J)}}{w_{s}^{(I)}} \left( \begin{array}{c}
m_{1} m_{2} \\
n_{1} n_{2} \end{array} \right) \left( \begin{array}{c}0 \\
0 \end{array} \right)
\end{equation}

\begin{equation}
= \frac{(s^{2}q^{I}; q)_{\infty} (s^{2}q^{J}; q)_{\infty}}{(s^{2}; q)_{\infty} (s^{2}q^{I+J}; q)_{\infty}} \times
\end{equation}

\begin{equation}
\sum_{i=0}^{I} \sum_{j=0}^{J} (-sq^{I})^{i} (-sq^{J})^{j} (q^{-I}; q)_{i} (q^{-J}; q)_{j} \frac{w_{s}^{(J)}}{w_{s}^{(I)}} \left( \begin{array}{c}
m_{1} m_{2} \\
n_{1} n_{2} \end{array} \right) \left( \begin{array}{c}0 \\
0 \end{array} \right),
\end{equation}

where $|m_{1}| \otimes |m_{2}| \otimes \cdots \in \tilde{V}$ and $\langle n_{1} \rangle \otimes \langle n_{2} \rangle \otimes \cdots \in \tilde{V}^{*}$ are two arbitrary states.

We relax the constraint that at most $I$ (resp. $J$) paths pass through each grey (coloured) horizontal edge, and replace both summations $\sum_{i=0}^{I} \left( \sum_{j=0}^{J} \right)$ in (61) by $\sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} \right)$. These modifications of (61) are legal, since they only add vanishing terms to the equation: if the summation indices satisfy $i > I$ or $j > J$ on either side of (61), the resulting term must vanish because of the factors $(q^{-I}; q)_{i}$ and $(q^{-J}; q)_{j}$, and a similar comment applies to the internal horizontal edge states.

Equation (61) then becomes an equality between two functions in $q^{I}$ and $q^{J}$, which holds for all $I \geq 1$ and $J \geq 1$. The final step is analytic continuation\footnote{This is possible because both sides of (61) are clearly jointly analytic in $q^{I}$ and $q^{J}$, given that $q$, $sq^{I}$ and $sq^{J}$ are all in the unit disc by assumption.} in $q^{I}$ and $q^{J}$. This replaces the operators appearing in the equation by those in (49), and converts the factor on the right hand side to $(-sx; q)_{\infty} (-sy; q)_{\infty} (s^{2}; q)_{\infty} (xy; q)_{\infty}$, completing the proof of (55).

\textbf{Proof of (56) and (57).} Let us focus firstly on the proof of (56). Returning to (59), we consider the special case $I = 1$ (while keeping $J$ generic), and take a $\{J; s\}$-specialization of the variables $(v_{1}, \ldots, v_{J})$. This results in the equation

\begin{equation}
\sum_{i=0}^{I} \sum_{j=0}^{J} (-sq^{I})^{i} (-sq^{J})^{j} (q^{-I}; q)_{i} (q^{-J}; q)_{j} \frac{w_{s}^{(J)}}{w_{s}^{(I)}} \left( \begin{array}{c}
m_{1} m_{2} \\
n_{1} n_{2} \end{array} \right) \left( \begin{array}{c}0 \\
0 \end{array} \right)
\end{equation}

\begin{equation}
= \left( \frac{1 - sq^{I} u}{1 - su} \right) \times
\end{equation}

\begin{equation}
\sum_{i=0}^{I} \sum_{j=0}^{J} (-sq^{I})^{i} (-sq^{J})^{j} (q^{-I}; q)_{i} (q^{-J}; q)_{j} \frac{w_{s}^{(J)}}{w_{s}^{(I)}} \left( \begin{array}{c}
m_{1} m_{2} \\
n_{1} n_{2} \end{array} \right) \left( \begin{array}{c}0 \\
0 \end{array} \right),
\end{equation}

in which the coloured rows consist of fused vertices (37), while the grey rows are unfused and consist of the vertices (11) of the original model.

We then repeat the procedure used in the proof of (53), sending $m_{0}, n_{0} \rightarrow \infty$ and summing explicitly over the possible contributions of the 0\textsuperscript{th} column on either side of (62).
case, we find that

\[
\sum_{i=0}^{1} \sum_{j=0}^{J} u^i (-sq^j)^j \frac{(q^{-j}; q)_j}{(q; q)_j} \frac{w^{s(J)}_s}{w_u} \bigg|_{m_1 \ m_2 \ \cdots} = (1 - sq^J) u_1 \ \cdots \ w^*_{s(J)} \bigg|_{m_1 \ m_2 \ \cdots} = \frac{1 - sq^J u}{1 - su} \times \sum_{i=0}^{1} \sum_{j=0}^{J} u^i (-sq^j)^j \frac{(q^{-j}; q)_j}{(q; q)_j} \frac{w^{s(J)}_s}{w_u} \bigg|_{m_1 \ m_2 \ \cdots}.
\]

After analytically continuing in \( q^j \) (letting \( q^j \mapsto -x/s \)) the factor on the right hand side of the commutation relation is converted to \( (1 + ux)/(1 - su) \), and (56) is immediate.

The proof of (57) is very similar, so we shall not present it in detail. For this proof, one considers the special case \( J = 1 \) of (59) (leaving \( I \) generic) and takes an \( \{I; s\} \)-specialization of \((u_1, \ldots, u_I)\). Sending \( m_0, n_0 \to \infty \) and analytically continuing in \( q^J \) then produces (57).

**Proof of (58).** This relation is the simplest of all: it corresponds to the special case \( I = J = 1 \) of (59). After taking \( m_0, n_0 \to \infty \) and expanding over all possible contributions from the 0th lattice column, we obtain

\[
\sum_{i=0}^{1} \sum_{j=0}^{1} u^i v^j \bigg|_{m_1 \ m_2 \ \cdots} = \frac{1 - quv}{1 - uv} \times \sum_{i=0}^{1} \sum_{j=0}^{1} u^i v^j \bigg|_{m_1 \ m_2 \ \cdots}
\]

which establishes the claim (58).

\[\square\]

6. **Combinatorial formulae**

In this section we examine some of the combinatorial properties of the spin \( q \)-Whittaker polynomials, arising from their definition as partition functions. In Sections 6.1 and 6.2 they are shown to satisfy a simple branching rule, with factorized coefficients when branching off a single variable. Furthermore, the one-variable skew \( q \)-Whittaker polynomials have the so-called interlacing property: they vanish unless their two participating partitions interlace.
In Section 6.3 we study the one-variable skew spin $q$–Whittaker polynomials at $s = 0$, and find agreement with the standard $q$–Whittaker polynomials.

6.1. Branching rules.

**Proposition 6.1.** Let $m, n$ be two positive integers and fix two positive partitions $\lambda, \mu$ such that $\lambda \supset \mu$. The skew spin $q$–Whittaker polynomials satisfy the branching rules

\begin{equation}
F_{\lambda/\mu}(x_1, \ldots, x_{m+n}) = \sum_{\nu} F_{\nu/\mu}(x_1, \ldots, x_m) F_{\lambda/\nu}(x_{m+1}, \ldots, x_{m+n}),
\end{equation}

\begin{equation}
F^*_\lambda/\mu(x_1, \ldots, x_{m+n}) = \sum_{\nu} F^*_\nu/\mu(x_1, \ldots, x_m) F^*_\lambda/\nu(x_{m+1}, \ldots, x_{m+n}),
\end{equation}

where both summations are taken over all partitions $\nu$ such that $\lambda \supset \nu \supset \mu$.

**Proof.** This is an easy consequence of the algebraic expressions for the spin $q$–Whittaker polynomials. We start from (50) in the case of $m+n$ variables $(x_1, \ldots, x_{m+n})$, inserting the identity $\sum_{\nu} |\nu|' \langle \nu |' \rangle$ after the $m^{th}$ $\mathbb{C}$-operator. We find that

\begin{equation*}
F_{\lambda/\mu}(x_1, \ldots, x_{m+n}) = \sum_{\nu} \langle \mu | \mathcal{C}(x_1) \cdots \mathcal{C}(x_m) | \nu' \rangle \langle \nu | \mathcal{C}(x_{m+1}) \cdots \mathcal{C}(x_{m+n}) | \lambda' \rangle,
\end{equation*}

and (63) follows immediately by reapplying (50).

Note that the second branching rule (64) follows trivially from the first, by multiplying through by $\bar{c}_{\lambda/\mu}(q, s)/\bar{c}_{\mu'}(q, s)$. \hfill \Box

6.2. One-variable skew spin $q$–Whittaker polynomials.

**Theorem 6.2.** The one-variable skew spin $q$–Whittaker polynomials are given explicitly by

\begin{equation}
F^*_{\mu/\nu}(x) = \begin{cases} 
\sum_{i=1}^{x|\mu|-|\nu|} \prod_{i \geq 1} \frac{(-s/x; q)_{\mu_i-\nu_i} (-sx; q)_{\nu_i-\mu_{i+1}} (q; q)_{\mu_i-\mu_{i+1}}}{(q; q)_{\mu_i-\nu_i} (q; q)_{\nu_i-\mu_{i+1}} (s^2; q)_{\mu_i-\mu_{i+1}}}, & \mu \succ \nu, \\
0, & \text{otherwise},
\end{cases}
\end{equation}

\begin{equation}
F^*_{\mu/\nu}(x) = \begin{cases} 
\sum_{i=1}^{x|\mu|-|\nu|} \prod_{i \geq 1} \frac{(-s/x; q)_{\mu_i-\nu_i} (-sx; q)_{\nu_i-\mu_{i+1}} (q; q)_{\mu_i-\mu_{i+1}}}{(q; q)_{\mu_i-\nu_i} (q; q)_{\nu_i-\mu_{i+1}} (s^2; q)_{\nu_i-\mu_{i+1}}}, & \mu \succ \nu, \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

**Proof.** We begin by writing down $F_{\mu/\nu}(x)$ as a sum of single-row partition functions:

\begin{equation}
F_{\mu/\nu}(x) = \langle \nu' | \mathcal{C}(x) | \mu' \rangle = \sum_{j=0}^{\infty} x^j \frac{(-s/x; q)_j}{(q; q)_j} \times W_x \left( \begin{array}{c} m'_1 \\ m'_i \\ m'_M \end{array} \right),
\end{equation}

where we use the abbreviations $m'_i = m_i(\mu') = \mu_i - \mu_{i+1}$ and $n'_i = m_i(\nu') = \nu_i - \nu_{i+1}$, and $M = \max\{\ell(\mu), \ell(\nu)\}$ (all vertices beyond the $M^{th}$ column will have weight equal to 1, so we may suppress them). We now read off the Boltzmann weights one by one. We start from
the rightmost vertex in the product. Since \( m'_M = \mu_M \) and \( n'_M = \nu_M \) by the very definition of \( M \), the weight of this vertex is given by

\[
W_x \left( \begin{array}{c}
\mu_M \\
\nu_M \\
\hline
j \\
\hline
0
\end{array} \right) = (1_{j = \mu_M - \nu_M}) \frac{(-sx; q)_\nu_M (q; q)_\mu_M}{(q; q)_\nu_M (s^2; q)_\mu_M},
\]

constraining the number of paths passing through the left edge to \( \mu_M - \nu_M \) (and in particular, vanishing if \( \nu_M > \mu_M \)). Now observe that the vertex in the \( i \)th column has a Boltzmann weight of the form

\[
W_x \left( \begin{array}{c}
\mu_i - \mu_{i+1} \\
\nu_i - \nu_{i+1} \\
\hline
j \\
\hline
\nu_i - \nu_{i+1}
\end{array} \right) = (1_{j = \mu_i - \nu_i}) (1_{\nu_i \geq \mu_{i+1}}) x^{\mu_i - \nu_i - 1} \frac{(-s/x; q)_{\mu_i - \nu_i} (q; q)_{\nu_i - \mu_{i+1}} (s^2; q)_{\nu_i - \mu_{i+1}}}{(q; q)_{\nu_i - \mu_{i+1}} (q; q)_{\nu_i - \mu_{i+1}} (s^2; q)_{\nu_i - \mu_{i+1}}}.
\]

Indeed, this clearly holds for \( i = M \), and since \( j \) is constrained to the value \( \mu_i - \nu_i \), we conclude inductively that it holds for all \( 1 \leq i \leq M \). The indicator functions present in (68) ensure that \( \mu_i \geq \nu_i \geq \mu_{i+1} \) for all \( 1 \leq i \leq M \); the total contribution of the 1st to \( M \)th vertices is therefore

\[
F_{\mu/\nu}(x) = \prod_{i=1}^{M} \left( x^{\mu_i - \nu_i} \frac{(-s/x; q)_{\mu_i - \nu_i} (-sx; q)_{\nu_i - \mu_{i+1}} (q; q)_{\mu_i - \mu_{i+1}}}{(q; q)_{\nu_i - \mu_{i+1}} (s^2; q)_{\nu_i - \mu_{i+1}}} \right), \quad \mu \succ \nu,
\]

provided that \( \mu \succ \nu \), vanishing otherwise. Combining this with the prefactor in (67), and noting that the summation over \( j \) is constrained to the value \( j = \mu_1 - \nu_1 \), we conclude that

\[
F_{\mu/\nu}(x) = \prod_{i=1}^{M} \left( x^{\mu_i - \nu_i} \frac{(-s/x; q)_{\mu_i - \nu_i} (-sx; q)_{\nu_i - \mu_{i+1}} (q; q)_{\mu_i - \mu_{i+1}}}{(q; q)_{\nu_i - \mu_{i+1}} (s^2; q)_{\nu_i - \mu_{i+1}}} \right), \quad \mu \succ \nu,
\]

completing the proof of (65).

The second formula (66) follows immediately from (65) by multiplying it by

\[
\frac{\tilde{e}_{\mu'}(q, s)}{\tilde{e}_{\nu'}(q, s)} = \prod_{i \geq 1} \left( \frac{(s^2; q)_{\mu_i - \mu_{i+1}}}{(q; q)_{\mu_i - \mu_{i+1}}} \right) \left( \frac{(q; q)_{\nu_i - \mu_{i+1}}}{(s^2; q)_{\nu_i - \mu_{i+1}}} \right).
\]

\[\Box\]

**Corollary 6.3.** Let \( \mu \) be a positive partition of length \( \ell \). Then \( F_{\lambda/\mu}(x_1, \ldots, x_m) \) vanishes if the length of \( \lambda \) exceeds \( \ell + m \).

**Proof.** By \( m \) iterations of the branching rule, we have

\[
F_{\lambda/\mu}(x_1, \ldots, x_m) = \sum_{\mu(0) \prec \mu(1) \prec \cdots \prec \mu(m)} \prod_{i=1}^{m} F_{\mu(i)/\mu(i-1)}(x_i),
\]

(70)
where we define \( \nu^{(0)} = \mu \) and \( \nu^{(m)} = \lambda \), and sum over the remaining \( m - 1 \) partitions. Because each partition \( \nu^{(i)} \) interlaces \( \nu^{(i-1)} \), its length can be at most one greater than its predecessor. It follows that \( \ell(\nu^{(m)}) \equiv \ell(\lambda) \) is maximally \( \ell(\nu^{(0)}) + m \equiv \ell(\mu) + m \).  

**Corollary 6.4.** The skew spin \( q \)-Whittaker polynomials satisfy the stability relation

\[
F_{\lambda/\mu}(x_1, \ldots, x_{m-1}, -s) = F_{\lambda/\mu}(x_1, \ldots, x_{m-1}),
\]

for all partitions \( \lambda, \mu \).

**Proof.** Isolating the dependence on \( x_m \) in (70) and setting \( x_m = -s \), we have

\[
F_{\lambda/\mu}(x_1, \ldots, x_{m-1}, -s) = \sum_{\nu^{(0)} \prec \nu^{(1)} \prec \cdots \prec \nu^{(m)}} F_{\lambda/\mu^{(m)}}(-s) \prod_{i=1}^{m-1} F_{\nu^{(i)}/\nu^{(i-1)}}(x_i),
\]

where we have defined \( \nu^{(0)} = \mu \) and \( \nu^{(m)} = \lambda \), as before. Examining equation (65) for the one-variable skew spin \( q \)-Whittaker polynomials, we note that \( F_{\lambda/\nu}(x) \) contains the factor \( \prod_{i \geq 1} (-s/x; q)_{\lambda_i - \nu_i} \), which vanishes when \( x = -s \) if \( \lambda_i > \nu_i \) for any \( i \). Furthermore, it is clear that \( F_{\lambda/\nu}(-s) = 1 \) when \( \lambda = \nu \). We conclude that \( F_{\lambda/\nu}(-s) = 1_{\lambda=\nu} \), and therefore

\[
F_{\lambda/\mu}(x_1, \ldots, x_{m-1}, -s) = \sum_{\nu^{(0)} \prec \nu^{(1)} \prec \cdots \prec \nu^{(m-1)}} \prod_{i=1}^{m-1} F_{\nu^{(i)}/\nu^{(i-1)}}(x_i),
\]

where the restriction \( \nu^{(m-1)} = \lambda \) is now assumed. The final expression then matches (70) in the case of \( m - 1 \) variables. □

6.3. **Reduction to \( q \)-Whittaker polynomials.** At \( s = 0 \), the spin \( q \)-Whittaker polynomials reduce to ordinary \( q \)-Whittaker polynomials. This can be easily deduced from the explicit form (65), (66) of the one-variable skew spin \( q \)-Whittaker polynomials:

\[
F_{\mu/\nu}(x) \bigg|_{s=0} = \begin{cases} x^{|\mu| - |\nu|} \prod_{i \geq 1} \frac{(q; q)_{\mu_i - \nu_i + 1}}{(q; q)_{\mu_i - \nu_i} (q; q)_{\nu_i - \mu_i + 1}}, & \mu > \nu, \\
0, & \text{otherwise},
\end{cases}
\]

\[
F^{*}_{\mu/\nu}(x) \bigg|_{s=0} = \begin{cases} x^{|\mu| - |\nu|} \prod_{i \geq 1} \frac{(q; q)_{\nu_i - \mu_i + 1}}{(q; q)_{\mu_i - \nu_i} (q; q)_{\nu_i - \mu_i + 1}}, & \mu > \nu, \\
0, & \text{otherwise},
\end{cases}
\]

which matches precisely with the one-variable skew Macdonald polynomials \( P_{\mu/\nu}(x; q, t) \) and \( Q_{\mu/\nu}(x; q, t) \) at \( t = 0 \) (see Example 2 (b) in Section 6, Chapter VI of [Mac95]), and set \( t = 0 \). The partition function (50) thus reduces precisely to Korff’s lattice model construction of the \( q \)-Whittaker polynomials [Kor13], at \( s = 0 \).
In this section we derive a series of identities for the spin $q$–Whittaker polynomials. The first of these is a skew Cauchy identity, which is proved using the exchange relation (55) for the fused row operators (in this way, the proof directly mirrors that of (17), the skew Cauchy identity for the spin Hall–Littlewood functions). It reduces to a non-skew Cauchy identity for trivial skew Young diagrams (with an empty bottom partition), and that identity, in turn, can be considered as a multi parameter generalization of the $q$–Gauss summation theorem.

The second is a skew dual Cauchy identity, involving both a spin $q$–Whittaker polynomial and a stable spin Hall–Littlewood function, proved using the exchange relation (56). The appearance of a spin $q$–Whittaker polynomial and a spin Hall–Littlewood function in the same summation identity is suggestive of the existence of an involution which maps between the two families, much as $q$–Whittaker and Hall–Littlewood polynomials are related under the Macdonald involution [Mac95]. It would be very interesting to find such an involution, since it would provide some hope for the unification of spin $q$–Whittaker polynomials and spin Hall–Littlewood functions as specializations of a single “spin Macdonald” function.

Finally, we conclude with Pieri rules for the spin $q$–Whittaker polynomials. These are derived as simple corollaries of the skew Cauchy and dual skew Cauchy identities.

### 7.1. Cauchy identity for spin $q$–Whittaker polynomials.

**Theorem 7.1.** Fix two positive integers $m$ and $n$, and let $\mu$ and $\nu$ be two partitions. The spin $q$–Whittaker polynomials satisfy the following summation identity (assuming all parameters are in the unit disc):

\[
\sum_{\lambda} \mathcal{F}_{\lambda/\mu}(x_1, \ldots, x_m) \mathcal{F}_{\lambda/\nu}^*(y_1, \ldots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \frac{(-sx_i; q)_{\infty} (-sy_j; q)_{\infty}}{(s^2; q)_{\infty} (x_i y_j; q)_{\infty}} \right) \sum_{\kappa} \mathcal{F}_{\nu/\kappa}(x_1, \ldots, x_m) \mathcal{F}_{\mu/\kappa}^*(y_1, \ldots, y_n),
\]

(71)

with the left hand sum taken over all partitions $\lambda$ such that $\lambda' \supset \mu'$ and $\lambda' \supset \nu'$, and the right hand sum taken over all partitions $\kappa$ such that $\kappa' \subset \mu'$ and $\kappa' \subset \nu'$.

**Proof.** This is essentially a repetition of the steps used to prove Theorem 3.8. One starts by writing down the expectation value

\[
\mathcal{E}_{\mu,\nu}(x_1, \ldots, x_m; y_1, \ldots, y_n) := \langle \mu' | (C(x_1) \ldots C(x_m)) B^*(y_n) \ldots B^*(y_1) | \nu' \rangle,
\]

which, by virtue of (50)–(51), clearly expands as

\[
\mathcal{E}_{\mu,\nu}(x_1, \ldots, x_m; y_1, \ldots, y_n) = \sum_{\lambda} \mathcal{F}_{\lambda/\mu}(x_1, \ldots, x_m) \mathcal{F}_{\lambda/\nu}^*(y_1, \ldots, y_n).
\]

(72)

On the other hand, by $mn$ iterations of the exchange relation (55), one has

\[
\mathcal{E}_{\mu,\nu}(x_1, \ldots, x_m; y_1, \ldots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \frac{(-sx_i; q)_{\infty} (-sy_j; q)_{\infty}}{(s^2; q)_{\infty} (x_i y_j; q)_{\infty}} \right) \langle \mu' | B^*(y_n) \ldots B^*(y_1) C(x_1) \ldots C(x_m) | \nu' \rangle
\]

(73)
we recover the well-known

\[ q^7.2. \]

\( q \) satisfy the following Cauchy identity (assuming all parameters are in the unit disc):

corresponds with \( \kappa \) trivializes the sum on the right hand side of (71): the only term remaining in this sum simply by multiplying (77) by \( \tilde{\eta} \).

This is immediate from equation (71), by choosing \( \mu = \nu = \emptyset \). Such a choice trivializes the sum on the right hand side of (71): the only term remaining in this sum corresponds with \( \kappa = \emptyset \). Since \( F_{\emptyset} = F_{\emptyset}^* = 1 \), the result follows.

Corollary 7.2. For any two positive integers \( m \) and \( n \), the spin \( q \)-Whittaker polynomials satisfy the following Cauchy identity (assuming all parameters are in the unit disc):

\[
\sum_{\lambda} F_{\lambda}(x_1, \ldots, x_m) F^*_{\lambda}(y_1, \ldots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{(-sx_i; q)_{\infty} (-sy_j; q)_{\infty}}{(s^2; q)_{\infty} (x_i y_j; q)_{\infty}}.
\]

Note that this reproduces the Cauchy identity for ordinary \( q \)-Whittaker polynomials by setting \( s = 0 \).

Proof. This is immediate from equation (71), by choosing \( \mu = \nu = \emptyset \). Such a choice trivializes the sum on the right hand side of (71): the only term remaining in this sum corresponds with \( \kappa = \emptyset \). Since \( F_{\emptyset} = F_{\emptyset}^* = 1 \), the result follows.

7.2. \( q \)-Gauss summation identity as special case. Taking the \( m = n = 1 \) case of (74), we recover the well-known \( q \)-Gauss summation identity

\[
\sum_{i=0}^{\infty} \frac{(-s/x; q)_i (-s/y; q)_i}{(s^2; q)_i (q; q)_i} (xy)^i = \frac{(-sx; q)_{\infty} (-sy; q)_{\infty}}{(s^2; q)_{\infty} (xy; q)_{\infty}}.
\]

To check that the left hand side of (75) is indeed recovered from (74), we note that at \( m = n = 1 \) the left hand side of (74) is summed over all partitions \( \lambda \) of at most one part. Using the explicit form of the one-variable spin \( q \)-Whittaker polynomials (65) and (66), the left hand side of (74) becomes

\[ 1 + \sum_{i=1}^{\infty} F_i(x) F^*_i(y) = 1 + \sum_{i=1}^{\infty} \left( x_i \frac{(-s/x; q)_i}{(s^2; q)_i} \right) \left( y_i \frac{(-s/y; q)_i}{(q; q)_i} \right), \]

and the formula (75) follows immediately.

7.3. Dual Cauchy identity.

Theorem 7.3. Fix two positive integers \( m \) and \( n \), and let \( \mu \) and \( \nu \) be two partitions. The stable spin Hall–Littlewood functions and spin \( q \)-Whittaker polynomials satisfy the following summation identity\(^6\):\(^6\)

\[
\sum_{\lambda} \tilde{F}_{\lambda/\mu}(u_1, \ldots, u_m) F^*_{\lambda/\nu}(x_1, \ldots, x_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1 + u_i x_j}{1 - su_i} \sum_{\kappa} F_{\nu/\kappa}(u_1, \ldots, u_m) F^*_{\mu/\kappa'}(x_1, \ldots, x_n).
\]

\(^6\)We could also write this identity as

\[
\sum_{\lambda} \tilde{F}_{\lambda/\mu}(u_1, \ldots, u_m) F^*_{\lambda/\nu}(x_1, \ldots, x_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1 + u_i x_j}{1 - su_i} \sum_{\kappa} F^*_{\lambda/\kappa}(u_1, \ldots, u_m) F_{\mu/\kappa'}(x_1, \ldots, x_n),
\]

simply by multiplying (77) by \( \tilde{c}_{\nu}(q,s)/\tilde{c}_{\mu}(q,s) \), and redistributing factors within the summations. The algebraic origin of this alternative identity is, of course, the commutation relation (74).
We take the specialization functions and spin Corollary 7.4. For any two positive integers stable spin Hall–Littlewood functions and skew spin \( q \) which, using (43), (51) and (56), can be expanded in two different ways in terms of skew starting point is the expectation value Theorem 7.5. Cauchy identity for stable spin Hall–Littlewood functions. □

Corollary 7.6. For any two positive integers \( m \) and \( n \), the stable spin Hall–Littlewood functions and spin \( q \)–Whittaker polynomials satisfy the following dual Cauchy identity:

(78)

\[
\sum_{\lambda} \tilde{F}_{\lambda}(u_{1}, \ldots, u_{m}) \tilde{F}_{\lambda'}^{*}(x_{1}, \ldots, x_{n}) = \sum_{\lambda} \tilde{F}_{\lambda}(u_{1}, \ldots, u_{m}) \tilde{F}_{\lambda'}^{*}(x_{1}, \ldots, x_{n}) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \frac{1 + u_{i}x_{j}}{1 - su_{i}} \right).
\]

Again, we note that this reproduces the correct dual Cauchy identity between Hall–Littlewood \((78)\) and \( q \)–Whittaker polynomials by setting \( s = 0 \).

Proof. This follows from equation (77) by choosing \( \mu = \nu = \emptyset \). □

7.4. **Cauchy identity for stable spin Hall–Littlewood functions.**

**Theorem 7.5.** Fix two positive integers \( m \) and \( n \), and let \( \mu \) and \( \nu \) be two partitions. The stable spin Hall–Littlewood functions satisfy the summation identity

(79)

\[
\sum_{\lambda} \tilde{F}_{\lambda/\mu}(u_{1}, \ldots, u_{m}) \tilde{F}_{\lambda/\nu}^{*}(v_{1}, \ldots, v_{n}) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \frac{1 - qu_{i}v_{j}}{1 - u_{i}v_{j}} \right) \sum_{\kappa} \tilde{F}_{\nu/\kappa}(u_{1}, \ldots, u_{m}) \tilde{F}_{\mu/\kappa}^{*}(v_{1}, \ldots, v_{n}),
\]

assuming that \(|(u_{i} - s)(v_{j} - s)| < |(1 - su_{i})(1 - sv_{j})|\) for all \( i, j \).

Proof. This can be established in the same way as in the proof of the Cauchy identity (77) for (ordinary) spin Hall–Littlewood functions, but now using the commutation relation (58). □

Corollary 7.6. For any two positive integers \( m \) and \( n \), the stable spin Hall–Littlewood functions satisfy the Cauchy identity

\[
\sum_{\lambda} \tilde{F}_{\lambda}(u_{1}, \ldots, u_{m}) \tilde{F}_{\lambda}^{*}(v_{1}, \ldots, v_{n}) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \frac{1 - qu_{i}v_{j}}{1 - u_{i}v_{j}} \right),
\]

assuming that \(|(u_{i} - s)(v_{j} - s)| < |(1 - su_{i})(1 - sv_{j})|\) for all \( i, j \).

Proof. This is the \( \mu = \nu = \emptyset \) case of (79). □

7.5. **Pieri rules.** In symmetric function theory, the **Pieri rules** constitute the simplest types of product formulae, *i.e.* they are rules for multiplying a certain symmetric function by (typically) a more elementary one. Here we list two such formulae, which arise as special cases of the Cauchy and dual Cauchy identities for skew polynomials.

**First Pieri rule for \( \tilde{F}_{\lambda} \).** We take the specialization \( n = 1 \), \( \mu = \emptyset \) of the identity (77). This trivializes the summation on the right hand side of (77), forcing \( \kappa = \emptyset \). We then read

\[
\sum_{\lambda} \tilde{F}_{\lambda}(x_{1}, \ldots, x_{m}) \tilde{F}_{\lambda/\nu}^{*}(y) = \prod_{i=1}^{m} \left( \frac{(-sx_{i}; q)_{\infty}(-sy_{i}; q)_{\infty}}{(s^{2}; q)_{\infty}(x_{i}y_{i}; q)_{\infty}} \right) \tilde{F}_{\nu}(x_{1}, \ldots, x_{m}).
\]
We take the summation on its right hand side to the single term $\kappa$ polynomial $P$. You young diagram $\nu$ the multiplicative factor appearing on the right hand side in terms of spin $q$–Whittaker polynomials. We are easily able to do that, using the Cauchy identity (71) itself at $n = 1$, and the explicit expression (66) for the one-variable dual spin $q$–Whittaker polynomial. We obtain the equation

$$\sum_{\lambda} F_{\lambda}(x_1, \ldots, x_m) F_{\lambda/\mu}(y) = \left(1 + \sum_{i=1}^{\infty} y^i \frac{(s/y; q)_i}{(q; q)_i} F_i(x_1, \ldots, x_m) \right) F_{\nu}(x_1, \ldots, x_m).$$

This identity is the natural $s$-generalization of the “horizontal” Pieri rule for $q$–Whittaker polynomials: indeed, at $s = 0$ the $y$ variable becomes a generating parameter that can be dropped from both sides of the equation, and we obtain

$$\sum_{\lambda > \nu: |\lambda| - |\nu| = i} P_{\lambda}(x_1, \ldots, x_m) \varphi'_{\lambda/\nu}(q) = \frac{1}{(q; q)_i} P_{\lambda}(x_1, \ldots, x_m) P_{\nu}(x_1, \ldots, x_m),$$

where $\varphi'_{\lambda/\nu}(q) : = \prod_{j>1} \frac{(q; q)_{\nu_j - \nu_{j+1}}}{(q; q)_{\lambda_j - \nu_j + 1}}$,

expressing the product of a one-row $q$–Whittaker polynomial $P$ and a general $q$–Whittaker polynomial $P_{\nu}$ as a sum over the ways of adding weight $i$ horizontal strips to the starting Young diagram $\nu$.

Second Pieri rule for $F_{\lambda}$. We take the $m = 1, \nu = \emptyset$ specialization of (76). This reduces the summation on its right hand side to the single term $\kappa = \emptyset$, and we find that

$$\sum_{\lambda} F_{\lambda}(x_1, \ldots, x_n) F_{\lambda/\mu'}^*(u) = \prod_{j=1}^{n} \left(1 + \frac{ux_j}{1-su}\right) F_{\mu}(x_1, \ldots, x_n),$$

where we have also conjugated all partitions appearing in the identity. Expanding the multiplicative factor on the right hand side using the dual Cauchy identity (78), we then have

$$\sum_{\lambda} F_{\lambda}(x_1, \ldots, x_n) \tilde{F}_{\lambda/\mu'}^*(u) =$$

$$\left(1 + \frac{u(1-s^2)}{1-su} \sum_{i=1}^{n} \frac{u-s}{1-su} i^{i-1} F_{1^i}(x_1, \ldots, x_n) \right) F_{\mu}(x_1, \ldots, x_n).$$

This identity, in turn, plays the role of an $s$-generalization of the “vertical” Pieri rule for $q$–Whittaker polynomials. At $s = 0$, it becomes

$$\sum_{\lambda' > \mu': |\lambda'| - |\mu'| = i} P_{\lambda}(x_1, \ldots, x_n) \psi_{\lambda'/\mu'}(q) = P_{1^i}(x_1, \ldots, x_n) P_{\mu}(x_1, \ldots, x_n),$$

where $\psi_{\lambda'/\mu'}(q) : = \prod_{j>1: m_j(\mu) = m_j(\lambda) + 1} (1 - q^{m_j(\mu)})$,

expressing the product of a one-column $q$–Whittaker polynomial $P$ and a general $q$–Whittaker polynomial $P_{\mu}$ as a sum over the ways of adding weight $i$ vertical strips to the starting Young diagram $\mu$. 

37
8. Integral representation of spin $q$–Whittaker polynomials

We conclude with an elegant integral formula for the spin $q$–Whittaker polynomials. The derivation of this formula is based on a known integral expression for the spin Hall–Littlewood function $G_\lambda$, found in [Bor17]. Indeed, since the spin $q$–Whittaker polynomials are obtained in a systematic way by the fusion/analytic continuation procedure of Section 5, we need only apply these steps to the existing integral formula for $G_\lambda$ and observe what we obtain at the end of the calculation.

8.1. Multiple integral formula for $G_\lambda(v_1, \ldots, v_N)$. Let $n \geq k$ be two positive integers, and begin by fixing a partition $\lambda \in \text{Part}_n$, whose first $n-k$ parts are positive and whose final $k$ parts are equal to zero; i.e. we have $\lambda_i \geq 1$ for all $1 \leq i \leq n-k$, and $\lambda_{n-k+1} = \cdots = \lambda_n = 0$. Choose another integer $N$ such that $N \geq n-k$. Quoting equation (7.8) from [Bor17], the following integral formula for spin Hall–Littlewood functions holds:

$$G_\lambda(v_1, \ldots, v_N) = (s^2; q)_n \times$$

$$\oint_C \frac{du_1}{2\pi i} \cdots \oint_C \frac{du_n}{2\pi i} \prod_{1 \leq i < j \leq n} \left( \frac{u_i - u_j}{u_i - qu_j} \right) \prod_{i=1}^{n} \left( \frac{1}{1 - su_i} \right) \frac{1}{(u_i - s)} \prod_{j=1}^{N} \frac{1 - qu_i v_j}{1 - u_i v_j},$$

where all integrations take place along the same positively-oriented contour $C$. This contour is chosen such that 1. The points $s^{-1}$ and $\{v_1^{-1}, \ldots, v_N^{-1}\}$ lie outside $C$; 2. The points $\{s, sq, \ldots, sq^{n-1}\}$ lie inside $C$; 3. The image of $C$ under multiplication by $q$, denoted $qC$, lies completely inside $C$. An example of a suitable contour is shown in Figure 6.

Following Section 7 of [Bor17], we integrate over the $u_n, \ldots, u_{n-k+1}$ contours explicitly, starting with $u_n$ and working backwards sequentially. This is a straightforward calculation, in view of the assumption $\lambda_{n-k+1} = \cdots = \lambda_n = 0$. For each $j = n, \ldots, n-k+1$, one readily sees that the $u_j$ contour only surrounds a simple pole at $u_j = sq^{n-j}$, whose residue can be taken immediately. After performing these integrations, the formula then reads

$$(80) \ G_\lambda(v_1, \ldots, v_N) = \frac{(s^2; q)_n}{(s^2; q)_k} \prod_{j=1}^{N} \left( \frac{1 - sq^k v_j}{1 - sv_j} \right) \times \oint_C \frac{du_1}{2\pi i} \cdots \oint_C \frac{du_{n-k}}{2\pi i}$$
\[ \prod_{1 \leq i < j \leq n-k} \left( \frac{u_i - u_j}{u_i - q u_j} \right) \prod_{i=1}^{n-k} \left( \frac{1}{1 - su_i} \left( \frac{1 - su_i}{u_i - s} \right) \lambda_i \prod_{j=1}^{N} \frac{1 - qu_i v_j}{1 - u_i v_j} \right), \]

cf. equation (7.9) in [Bor17].

8.2. Fusion combined with analytic continuation. We now apply the following steps to the integral (\textbf{S0}): 1. We send \( n, k \to \infty \) while keeping \( n - k \) fixed and finite. For simplicity, we write \( n - k = \ell \); 2. We take a \( \{ K_1, \ldots, K_m; s, \ldots, s\} \)-specialization of the variables \((v_1, \ldots, v_N)\), where it is assumed that \( N = K_1 + \cdots + K_m \); 3. The resulting expression depends on \( K_1, \ldots, K_m \) only via \( q^{K_1}, \ldots, q^{K_m} \), allowing us to analytically continue in these variables, letting \( q^{K_i} \to -x_i/s \) for all \( 1 \leq i \leq m \); 4. We normalize by dividing by \( \prod_{i=1}^{m}(-sx_i ; q)_\infty/(s^2 ; q)_\infty \), which must be a common factor of the final expression.

As we explained in Section 5.5, performing these steps to the spin Hall–Littlewood function \( G_\lambda(v_1, \ldots, v_N) \) transforms it exactly into the spin \( q \)-Whittaker polynomial \( F_{\lambda'}(x_1, \ldots, x_m) \), where \( \lambda' \) is the conjugate partition of \((\lambda_1, \ldots, \lambda_\ell)\). The result of these calculations will be, therefore, a multiple integral formula for \( F_{\lambda'}(x_1, \ldots, x_m) \).

Applying the first step, we see immediately that

\[ G_\lambda(v_1, \ldots, v_N) \big|_{\text{Step 1}} = \prod_{j=1}^{N} \left( \frac{1}{1 - s v_j} \right) \times \]

\[ \oint_{C} \frac{d u_1}{2 \pi i u_1} \cdots \oint_{C} \frac{d u_\ell}{2 \pi i u_\ell} \prod_{1 \leq i < j \leq \ell} \left( \frac{u_i - u_j}{u_i - q u_j} \right) \prod_{i=1}^{\ell} \left( \frac{1}{1 - su_i} \left( \frac{1 - su_i}{u_i - s} \right) \lambda_i \prod_{j=1}^{N} \frac{1 - qu_i v_j}{1 - u_i v_j} \right), \]

where the contour of integration \( C \) is as before: all points \( sq^i, i \in \mathbb{Z}_{\geq 0} \), and the point 0, are contained within it. The geometric specialization\(^7\) of the second step yields

\[ G_\lambda(v_1, \ldots, v_N) \big|_{\text{Steps 1,2}} = \prod_{i=1}^{m} \left( \frac{s^2 q^{K_i} ; q \infty}{(s^2 ; q)_\infty} \right) \times \]

\[ \oint_{C} \frac{d u_1}{2 \pi i u_1} \cdots \oint_{C} \frac{d u_\ell}{2 \pi i u_\ell} \prod_{1 \leq i < j \leq \ell} \left( \frac{u_i - u_j}{u_i - q u_j} \right) \prod_{i=1}^{\ell} \left( \frac{1}{1 - su_i} \left( \frac{1 - su_i}{u_i - s} \right) \lambda_i \prod_{j=1}^{m} \frac{1 - su_i q^{K_j}}{1 - su_i} \right), \]

and finally, the analytic continuation of the third step gives

\[ G_\lambda(v_1, \ldots, v_N) \big|_{\text{Steps 1,2,3}} = \prod_{i=1}^{m} \left( \frac{-sx_i ; q \infty}{(s^2 ; q)_\infty} \right) \times \]

\[ \oint_{C} \frac{d u_1}{2 \pi i u_1} \cdots \oint_{C} \frac{d u_\ell}{2 \pi i u_\ell} \prod_{1 \leq i < j \leq \ell} \left( \frac{u_i - u_j}{u_i - q u_j} \right) \prod_{i=1}^{\ell} \left( \frac{1}{1 - su_i} \left( \frac{1 - su_i}{u_i - s} \right) \lambda_i \prod_{j=1}^{m} \frac{1 + u_i x_j}{1 - su_i} \right). \]

After the normalization required by the fourth step (noting that the correct overall factor does indeed emerge from the calculation) we arrive at our desired formula, which we now quote as a theorem:

\(^7\)Note that setting each \( v \) variable to \( sq^i \) for some \( i \in \mathbb{Z}_{\geq 0} \) is consistent with the assumption that the points \( \{ v_j^{-1} \} \) lie outside of \( C \).
Theorem 8.1. Let $m$ be a positive integer, and $\lambda, \lambda'$ denote a partition and its conjugate, chosen such that $\ell(\lambda) = \lambda'_1 \leq m$. The spin $q$–Whittaker polynomial $F_{\lambda}(x_1, \ldots, x_m)$ is given by the integral expression

$$F_{\lambda}(x_1, \ldots, x_m) = \oint \frac{du_1}{2\pi i u_1} \cdots \oint \frac{du_L}{2\pi i u_L} \prod_{1 \leq i < j \leq L} \left( \frac{u_i - u_j}{u_i - qu_j} \right) \prod_{i=1}^{L} \left( \frac{1 - su_i}{u_i - s} \right)^{\lambda'_i} \left( \prod_{j=1}^{m} (1 + u_i x_j) \right) \left( \prod_{k=1}^{m+1} (1 - su_k) \right),$$

where $L = \lambda_1$ denotes the largest part of $\lambda$ and the contour $C$ is as specified above; see Figure 6. An alternative way to arrive at this formula would be to use the orthogonality of $F_{\lambda}$’s (cf. [Bor17, BP16a, BP16b]) and the dual Cauchy identity (78), above.

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