Piecewise dominant sequences and the cocenter of the cyclotomic quiver Hecke algebras

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Abstract
In this paper we study the cocenter of the cyclotomic quiver Hecke algebra $\mathcal{R}_\Lambda^\alpha$ associated to an arbitrary symmetrizable Cartan matrix $A = (a_{ij})_{i,j \in I}$, $\Lambda \in P^+$ and $\alpha \in Q^+_n$. We introduce a notion called “piecewise dominant sequence” and use it to construct some explicit homogeneous elements which span the cocenter of $\mathcal{R}_\Lambda^\alpha$. Our first main result shows that the minimal (resp., maximal) degree component of the cocenter of $\mathcal{R}_\Lambda^\alpha$ is spanned by the image of some KLR idempotent $e(\nu)$ (resp., some monomials $Z(\nu)e(\nu)$ on KLR $x_k$ and $e(\nu)$ generators), where each $\nu \in I^\alpha$ is piecewise dominant. As an application, we show that any weight space $L(\Lambda)_\Lambda - \alpha$ of the irreducible highest weight module $L(\Lambda)$ over $g(A)$ is nonzero (equivalently, $\mathcal{R}_\Lambda^\alpha \neq 0$) if and only if there exists a piecewise dominant sequence $\nu \in I^\alpha$. Finally, we show that the Indecomposability Conjecture on $\mathcal{R}_\Lambda^\alpha(K)$ holds if it holds when $K$ is replaced by a field of characteristic $0$. In particular, this implies $\mathcal{R}_\Lambda^\alpha(K)$ is indecomposable when $K$ is a field of arbitrary characteristic and $g$ is symmetric and of finite type.

Keywords  Cyclotomic quiver Hecke algebras · Categorification

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Contents
1 Introduction ............................................... 2
2 Preliminary ................................................ 5
3 Relations and $K$-linear generators of the cocenter ................................ 9
  3.1 $K$-linear generators ......................................... 9
  3.2 Positivity of the degree of the cocenter ............................... 12

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1 Introduction

The quiver Hecke algebras (also known as the KLR algebras) \( R_\alpha \) are some remarkable infinite families of \( \mathbb{Z} \)-graded algebras introduced by Khovanov–Lauda [12, 13], and independently by Rouquier [17, 18] around 2008. These algebras depend on a symmetrizable Cartan matrix \( A = (a_{ij})_{i,j \in I}, \alpha \in Q_n^+ \) and some polynomials \( \{Q_{ij}(u, v)i, j \in I\} \). These algebras play important roles in the categorification of the negative part \( U_q(g)^- \) of the quantum group \( U_q(g) \), where \( q = q(A) \). When the ground field \( K \) has characteristic 0, \( A \) is symmetric and the polynomial \( \{Q_{ij}(u, v)i, j \in I\} \) are chosen as \([18, \S 3.2.4]\), Rouquier [18], and independently Varagnolo–Vasserot [20] have proved that the categorification sends the indecomposable projective modules over the quiver Hecke algebra \( R_\alpha \) to the canonical bases of \( U_q(g)^- \).

For each dominant integral weight \( \Lambda \in P^+ \), Khovanov–Lauda, and Rouquier have also introduced a graded quotient \( R^\Lambda_\alpha \) of \( R_\alpha \), called the cyclotomic quiver Hecke algebra (also known as the cyclotomic KLR algebras). Khovanov and Lauda have conjectured that the category of finite dimensional projective modules over these \( R^\Lambda_\alpha \) should give a categorification of the integrable highest weight module \( L(\Lambda) \) over the quantum group \( U_q(g) \). Khovanov–Lauda’s Cyclotomic Categorification Conjecture was later proved by Kang and Kashiwara [10]. The cyclotomic quiver Hecke algebra \( R^\Lambda_\alpha \) behaves in many aspects similar to the cyclotomic Hecke algebra of type \( G(\ell, 1, n) \) (also known as the Ariki–Koike algebras). In fact, in the case of type \( A_\infty \) or affine type \( A_1^{(1)}(e-1) \), assuming the ground field \( K \) contains a primitive \( e \)-th root of unity, and \( \{Q_{ij}(u, v)i, j \in I\} \) are chosen as \([18, \S 3.2.4]\), Brundan and Kleshchev have proved in [3] that each \( R^\Lambda_\alpha \) is isomorphic to the block algebra of the cyclotomic Hecke algebra of type \( G(\ell, 1, n) \) corresponding to \( \alpha \), where \( \ell \) is the level of \( \Lambda \). Mathas and the first author of this paper have constructed a \( \mathbb{Z} \)-graded cellular basis for the cyclotomic quiver Hecke algebra \( R^\Lambda_\alpha \), and used this basis to construct a homogeneous symmetrizing form on \( R^\Lambda_\alpha \) (see [5, Remark 4.7], [7]). More recently, Mathas and Tubbenhauer [15, 16] have constructed \( \mathbb{Z} \)-graded cellular bases for the cyclotomic quiver Hecke algebra in types \( C_{\mathbb{Z}, 0}, C_{\mathbb{Z}, 1}, B_{\mathbb{Z}, 0}, A_{2\mathbb{Z}} \) and \( D_{e+1}^{(2)} \).

For the cyclotomic quiver Hecke algebras \( R^\Lambda_\alpha \) of general type, we have given in [8] a closed formula for the graded dimension of \( R^\Lambda_\alpha \). The formula depends only on the root system associated to \( A \) and the dominant weight \( \Lambda \) but not on the chosen ground field \( K \), which immediately implies that the cyclotomic quiver Hecke algebra \( R^\Lambda_\alpha (O) \) is free over \( O \) for any commutative ground ring \( O \). These graded dimension formulæ are also generalized to the cyclotomic quiver Hecke superalgebras in [9]. The \( i \)-restriction functor \( E_i \) and the \( i \)-induction functor \( F_i \) play key roles in Kang and Kashiwara’s proof of Khovanov–Lauda’s Cyclotomic Categorification Conjecture. Rouquier has noticed [17] that the biadjointness of \( E_i \) and \( F_i \) induces a natural homogeneous Frobenius form on \( R^\Lambda_\alpha \). Shan, Varagnolo and Vasserot [19] proved that this Frobenius form is a homogeneous symmetrizing form on \( R^\Lambda_\alpha \) of degree \(-d_{\Lambda, \alpha} \). The homogeneous symmetrizing form on \( R^\Lambda_\alpha \) implies that there is a \( \mathbb{Z} \)-graded linear isomorphism: \( \text{Tr}(R^\Lambda_\alpha) : R^\Lambda_\alpha / [R^\Lambda_\alpha, R^\Lambda_\alpha] \cong (Z(R^\Lambda_\alpha))^\Lambda (d_{\Lambda, \alpha}) \). Thus the study of the degree \( k \) component of the center \( Z(R^\Lambda_\alpha) \) can be transformed into the study of the
degree $d_{\Lambda,\alpha} - k$ component of the cocenter $\text{Tr}(R^\Lambda_\alpha)$. There are now two major unsolved open problems on $R^\Lambda_\alpha$ as follows:

**Center Conjecture 1.1** The center of $R^\Lambda_\alpha$ consists of symmetric elements in its KLR $x_k$ and $e(v)$ generators, where $k = 1, \ldots, n$, $v \in I^\alpha$.

**Indecomposability Conjecture 1.2** The algebra $R^\Lambda_\alpha$ is indecomposable.

Conjecture 1.2 is a slight generalization of [19, Conjecture 3.33] in that $K$ can be of positive characteristic, and it is actually a consequence of the Center Conjecture 1.1. There are a number of special cases where the above two conjectures were verified. For example, assume $\{Q_{ij}(u, v)|i, j \in I\}$ are given as [19, (11)], in the affine type $A$ case (i.e., $g$ is affine type $A_{\infty}$ or affine type $A^{(1)}_{\infty}$), Conjecture 1.2 was proved by Brundan and Kleshchev in [3]; if $\text{char } K = 0$, $g$ is symmetric and of finite type, then Conjecture 1.2 holds by [19, Remark 3.41]. For arbitrary $\{Q_{ij}(u, v)|i, j \in I\}$, assume $\alpha = \sum_{j=1}^n \alpha_{ij}$ with $\alpha_{i1}, \ldots, \alpha_{in}$ pairwise distinct, Conjecture 1.2 also holds by [9, Theorem 1.9]. Moreover, in this case the first author and Huang Lin also verified in [6] that Conjecture 1.1 holds via a “cocenter approach”.

The “cocenter approach” reduces Conjecture 1.1 to constructing a basis or even some explicit linear generators of the cocenter $\text{Tr}(R^\Lambda_\alpha)$. The current work is motivated by the study of the above two conjectures and the “cocenter approach” proposed in [6]. More precisely, we construct some explicit homogeneous elements which span the degree 0 and $d_{\Lambda,\alpha}$ components of the cocenter $\text{Tr}(R^\Lambda_\alpha)$. To state our main result, we need a new notion called “piecewise dominant sequences”. Let $v = (v_1, \ldots, v_n) \in I^\alpha$ (i.e., $v \in I^\alpha$ with $\alpha = \sum_{j=1}^n \alpha_{v,j}$). We can always write $v$ in the following form:

$$v = (v_1, \ldots, v_n) = (v^1, v^1, \ldots, v^1, \ldots, v^m, v^m, \ldots, v^m),$$

such that

$$v^j \neq v^{j+1}, \quad \forall 1 \leq j < m,$$

where $m, b_1, \ldots, b_m \in \mathbb{Z}_{\geq 1}$ with $\sum_{i=1}^m b_i = n$. We call $v = (v_1, \ldots, v_n)$ a **piecewise dominant sequence** with respect to $\Lambda$, if for any $1 \leq i \leq m$,

$$\ell_i(v) := \langle h_{\nu^i}, \Lambda - \sum_{j=1}^{b_1+\cdots+b_{i-1}} \alpha_{v,j} \rangle \geq b_i,$$

where $\Lambda$ and the pairing $\langle -, - \rangle$ are introduced in Sect. 2.

The following theorem is the first main result of our paper.

**Theorem 1.3** Suppose $\text{char } K = 0$. Then we have

1. $(\text{Tr}(R^\Lambda_\alpha))_{d_{\Lambda,\alpha}} = K - \text{Span} \left\{ Z(v) + [R^\Lambda_\alpha, R^\Lambda_\alpha] \mid v \text{ is piecewise dominant with respect to } \Lambda \right\}$;

2. $(\text{Tr}(R^\Lambda_\alpha))_0 = K - \text{Span} \left\{ e(v) + [R^\Lambda_\alpha, R^\Lambda_\alpha] \mid v \text{ is piecewise dominant with respect to } \Lambda \right\}$;

3. $\text{Tr}(R^\Lambda_\alpha) = K - \text{Span} \left\{ x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} e(v) + [R^\Lambda_\alpha, R^\Lambda_\alpha] \mid d_1, d_2, \ldots, d_n \in \mathbb{N}, v \in I^\alpha \text{ is piecewise dominant with respect to } \Lambda, \right. \left. 0 \leq d_j \leq \ell_i(v) - 1, \forall c_i-1 + 1 \leq j \leq c_i; \forall 1 \leq i \leq m. \right\}$
where \( c_i := \sum_{j=1}^{i} b_j \), and we refer the readers to Definition 4.10 for the definition of \( Z(v) \).

As an application, we recover [19, Theorem 3.31(a)] in an elementary way, which says 0 and \( d_{\Lambda, \alpha} \) are the minimal degree and maximal degree of the cocenter respectively when \( \text{char} \ K = 0 \). Remarkably, these generators are all monomials in the KLR’s \( x_k \) and \( e(v) \) generators.

In another application, we use Theorem 1.3 to give an easy criterion for the weight space \( L(\Lambda)_{\Lambda-\alpha} \) of the irreducible highest weight module \( L(\Lambda) \) over \( g(\Lambda) \) is nonzero (equivalently, \( \mathcal{R}^\Lambda_{\alpha} \neq 0 \)), which is the second main result of this paper.

**Theorem 1.4** Let \( K \) be a field of arbitrary characteristic, \( \Lambda \in P^+ \) and \( \alpha \in Q^+_n \). The following statements are equivalent:

1. \( \mathcal{R}^\Lambda_{\alpha}(K) \neq 0 \);
2. \( L(\Lambda)_{\Lambda-\alpha} \neq 0 \);
3. There is a piecewise dominant sequence \( v \in I^\alpha \) with respect to \( \Lambda \).

Actually, in the proof of Theorem 1.4, we construct some explicit elements \( S(v) \) for each piecewise dominant sequence \( v \), whose image is nonzero in the degree \( d_{\Lambda, \alpha} \) component of the cocenter \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \). Hence they are already non-zero in \( \mathcal{R}^\Lambda_{\alpha} \). Using \( S(v) \), we give a stronger form of Conjecture 1.2 in Conjecture 4.30. We also give a second proof of Theorem 1.4 in Sect. 4.3.

The following theorem is the third main result of this paper, which reduces the Indecomposability Conjecture 1.2 on \( \mathcal{R}^\Lambda_{\alpha}(K) \) to the case when \( K \) has characteristic 0.

**Theorem 1.5** Let \( K \) be a field with characteristic \( \text{char} \ K = p > 0 \), \( \Lambda \in P^+ \) and \( \alpha \in Q^+_n \). Suppose that each \( Q_{ij}(u, v) \) is defined over \( \mathbb{Z} \). If Conjecture 1.2 holds for the cyclotomic quiver Hecke algebra \( \mathcal{R}^\Lambda_{\alpha}(\mathbb{Q}) \), then Conjecture 1.2 holds for the cyclotomic quiver Hecke algebra \( \mathcal{R}^\Lambda_{\alpha}(K) \) too.

In particular, Theorem 1.5 implies that Conjecture 1.2 holds over arbitrary ground field when \( \{Q_{ij}(u, v)\mid i, j \in I\} \) are given as [18, §3.2.4], \( g \) is symmetric and of finite type, see Corollary 4.32.

The paper is organised as follows. In Sect. 2 we recall some basic definitions and properties of the quiver Hecke algebra \( \mathcal{R}_{\alpha} \) and its cyclotomic quotient \( \mathcal{R}^\Lambda_{\alpha} \). In Sect. 3 we investigate some relations inside the cocenter \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \). Theorem 3.7 gives a subset of \( K \)-linear generator for \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \), where its proof makes essentially use of Kang-Kashiwara’s categorification result Lemma 2.7. Lemma 3.19 and Corollary 3.20 give some useful relations inside the cocenter \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \). As an easy application of Theorem 3.7, Proposition 3.14 proves the positivity of the degrees of the cocenter \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \) over arbitrary ground field \( K \). The proof of our first main result Theorem 1.3 is given in Sect. 4.1, where we introduce a new notion called “piecewise dominant sequence” in Definition 4.4, and use this object to construct some explicit homogeneous elements which can span the maximal degree component as well as the minimal degree component of the cocenter of \( \mathcal{R}^\Lambda_{\alpha} \) when \( \text{char} \ K = 0 \). Moreover, a refined subset of \( K \)-linear homogeneous generators for the whole cocenter \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \) is also given when \( \text{char} \ K = 0 \). As an application, Corollary 4.21 recovers a result of [19, Theorem 3.31(a)] on the degree range of the cocenter \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \). Our second main result Theorem 1.4 in this paper is contained in Theorems 4.28 and 4.29, whose proof are given in Sect. 4.2. The two theorems give some criteria for which \( \mathcal{R}^\Lambda_{\alpha} \neq 0 \) (equivalently, \( L(\Lambda)_{\Lambda-\alpha} \neq 0 \)), where \( L(\Lambda) \) is the irreducible highest weight module over \( g \) of highest weight \( \Lambda \). More
Piewise dominant sequences and the cocenter

precisely, we show in Theorem 4.28 that $\mathfrak{R}^A_\alpha \neq 0$ if and only if there exists a piecewise dominant sequence $v \in I^\alpha$, and construct in Theorem 4.29 an explicit (nonzero) monomial weight vector in $L(\Lambda)_\Lambda$. Furthermore, we show in Lemma 4.35 that one can associate each piecewise dominant sequence a crystal path in the crystal graph $B(\Lambda)$ of $L(\Lambda)$, and show in Lemma 4.38 each vertex in the crystal graph $B(\Lambda)$ can be connected with the highest weight vector $v_\Lambda$ via a crystal path associated to some piecewise dominant sequence. Our third main result of this paper (Theorem 1.5) reduces the Indecomposability Conjecture 1.2 on $\mathfrak{S}^A_\alpha(K)$ to the case when $K$ is a field of characteristic 0. In particular, this implies $\mathfrak{S}^A_\alpha(K)$ is indecomposable when $K$ is a field of arbitrary characteristic, $\{Q_{ij}(u, v) | i, j \in I\}$ are given as $[18, \S 3.2.4]$ and $g$ is symmetric and of finite type.

2 Preliminary

In this section we shall give some preliminary definitions and results on the quiver Hecke algebras and their cyclotomic quotients. Throughout, unless otherwise stated, we shall assume that $K$ is a field of arbitrary characteristic.

Let $I$ be an index set. An integral square matrix $A = (a_{ij})_{i, j \in I}$ is called a symmetric generalized Cartan matrix if it satisfies

\begin{enumerate}
  \item $a_{ii} = 2, \forall i \in I$;
  \item $a_{ij} \leq 0$ ($i \neq j$);
  \item $a_{ij} = 0 \iff a_{ji} = 0$ ($i, j \in I$);
  \item there is a diagonal matrix $D = \text{diag}(d_i \in \mathbb{Z}_{>0} | i \in I)$ such that $DA$ is symmetric.
\end{enumerate}

A Cartan datum $(A, P, \Pi, P^+, \Pi^\vee)$ consists of

\begin{enumerate}
  \item a symmetrizable generalized Cartan matrix $A$;
  \item a free abelian group $P$ of finite rank, called the weight lattice;
  \item $\Pi = \{\alpha_i \in P | i \in I\}$, called the set of simple roots;
  \item $P^\vee := \text{Hom}(P, \mathbb{Z})$, called the dual weight lattice and $\langle -, - \rangle : P^\vee \times P \to \mathbb{Z}$, the natural pairing;
  \item $\Pi^\vee = \{h_i | i \in I\} \subset P^\vee$, called the set of simple coroots;
\end{enumerate}

satisfying the following properties:

\begin{enumerate}
  \item $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
  \item $\Pi$ is linearly independent,
  \item $\forall i \in I, \exists \Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I$.
\end{enumerate}

Those $\Lambda_i$ are called the fundamental weights. We set

$$P^+ := \{\Lambda \in P | \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\},$$

which is called the set of dominant integral weights. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the root lattice. Set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\alpha = \sum_{i \in I} k_i\alpha_i \in Q^+$, we define the height of $\alpha$ to be $|\alpha| = \sum_{i \in I} k_i$. For each $n \in \mathbb{N}$, we set

$$Q^+_n := \{\alpha \in Q^+ | |\alpha| = n\}.$$

Let $g = g(A)$ be the corresponding Kac–Moody Lie algebra associated to $A$ with Cartan subalgebra $\mathfrak{h} := Q \otimes_{\mathbb{Z}} P^\vee$. Since $A$ is symmetrizable, there is a symmetric bilinear form $(,)$
on \(\mathfrak{h}^*\) satisfying

\[
(\alpha_i, \alpha_j) = d_i a_{ij} \quad (i, j \in I) \quad \text{and} \quad \langle h_i, \Lambda \rangle = \frac{2(\alpha_i, \Lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any} \ \Lambda \in \mathfrak{h}^* \text{ and } i \in I.
\]

Let \(u, v\) be two commuting indeterminates over \(K\). We fix a matrix \((Q_{i,j})_{i,j \in I}\) in \(K[u, v]\) such that

\[
Q_{i,j}(u, v) = Q_{j,i}(v,u), \quad Q_{i,i}(u, v) = 0, \quad Q_{i,j}(u, v) = \sum_{k,q \geq 0} c_{i,j,k,q} u^k v^q, \quad \text{if} \ i \neq j.
\]

where \(c_{i,j, -a_{ij}, 0} \in K^\times\), and \(c_{i,j, k,q} \neq 0\) only if \(2(\alpha_i, \alpha_j) = -(\alpha_i, \alpha_i)k - (\alpha_j, \alpha_j)q\).

Let \(S_n = \langle s_1, \ldots, s_{n-1} \rangle\) be the symmetric group on \(\{1, 2, \ldots, n\}\), where \(s_i = (i, i+1)\) is the transposition on \(i, i+1\). Then \(S_n\) acts naturally on \(I^n\) by:

\[
wv := (v_{w^{-1}(1)}, \ldots, v_{w^{-1}(n)}),
\]

where \(v = (v_1, \ldots, v_n) \in I^n\). The orbits of this action is identified with element of \(Q_n^+\). Then \(I^\alpha := \{v = (v_1, \ldots, v_n) \in I^n \mid \sum_{j=1}^n \alpha_{v_j} = \alpha \}\) is the orbit corresponding to \(\alpha \in Q_n^+\).

**Lemma 2.1** [12, 13, 18] Let \(\alpha \in Q_n^+\). The elements in the following set form a \(K\)-basis of \(\mathcal{R}_\alpha\):

\[
\left\{ x_1^{c_1} \cdots x_n^{c_n} \psi_{u,v} e(v) \mid v \in I^\alpha, \ w \in S_n, \ c_1, \ldots, c_n \in \mathbb{N} \right\}.
\]

**Definition 2.2** Let \(\alpha \in Q_n^+\). The quiver Hecke algebra \(\mathcal{R}_\alpha\) associated with a Cartan datum \((\Lambda, P, \Pi, P^\vee, \Pi^\vee)\), \((Q_{i,j})_{i,j \in I}\) and \(\alpha \in Q_n^+\) is the associative algebra over \(K\) generated by \(e(v) \ (v \in I^n), \ x_k \ (1 \leq k \leq n), \ \tau_l \ (1 \leq l \leq n-1)\) satisfying the following defining relations:

\[
e(v)e(v') = \delta_{v,v'} e(v), \quad \sum_{v \in I^n} e(v) = 1,
\]

\[
x_k x_l = x_l x_k, \quad x_k e(v) = e(v)x_k,
\]

\[
\tau_l e(v) = e(s_l(v))\tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if} \ |k - l| > 1,
\]

\[
\tau_k^2 e(v) = Q_{v_k, v_{k+1}}(x_k, x_{k+1}) e(v),
\]

\[
(x_k x_l - x_l x_k) e(v) = \begin{cases} -e(v) & \text{if} \ l = k, v_k = v_{k+1}, \\ e(v) & \text{if} \ l = k+1, v_k = v_{k+1}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
(\tau_k \tau_l - \tau_l \tau_k) e(v) = \begin{cases} Q_{v_k, v_{k+1}}(x_k, x_{k+1}) - Q_{v_{k+2}, v_{k+1}}(x_{k+2}, x_{k+1}) e(v) & \text{if} \ v_k = v_{k+2}, \\ 0 & \text{otherwise}. \end{cases}
\]

In particular, \(\mathcal{R}_0 \cong K\), and \(\mathcal{R}_{\alpha_i}\) is isomorphic to \(K[x_1]\). For any \(\alpha \in Q_n^+\) and \(i \in I\), we set

\[
e(\alpha, i) = \sum_{v = (v_1, \ldots, v_n) \in I^\alpha} e(v_1, \ldots, v_n, i) \in \mathcal{R}_{\alpha + \alpha_i}.
\]
The algebra $\mathcal{R}_\alpha$ is $\mathbb{Z}$-graded whose grading is given by
\[ \deg e(v) = 0, \quad \deg x_k e(v) = (\alpha_{i_k}, \alpha_{v_k}), \quad \deg \tau_k e(v) = -(\alpha_{v_k}, \alpha_{v_{k+1}}). \]

Let $\Lambda \in P^+$ be a dominant integral weight. We now recall the cyclotomic quiver Hecke algebra $\mathcal{R}_\alpha^\Lambda$. For $1 \leq k \leq n$, we define
\[ a^\Lambda_\alpha(x_k) = \sum_{\nu \in I^\alpha} x_k^{(h_{\nu_k}, \Lambda)} e(\nu) \in \mathcal{R}_\alpha. \]

**Definition 2.3** Set $I_{\Lambda, \alpha} = \mathcal{R}_\alpha a^\Lambda_\alpha(x_1) \mathcal{R}_\alpha$. The cyclotomic quiver Hecke algebra $\mathcal{R}_\alpha^\Lambda$ is defined to be the quotient algebra:
\[ \mathcal{R}_\alpha^\Lambda = \mathcal{R}_\alpha / I_{\Lambda, \alpha}. \]

Sometimes we shall write $\mathcal{R}_\alpha^\Lambda(K)$ instead of $\mathcal{R}_\alpha^\Lambda$ in order to emphasize the ground field $K$.

In general, if $O$ is a commutative ring and $Q_{ij}(u, v) \in O[u, v]$ for any $i, j \in I$, then we can define the cyclotomic quiver Hecke algebra $\mathcal{R}_\alpha^\Lambda(O)$ over $O$.

For any $K$-algebra $A$, we define the center $Z(A) := \{ a \in A | ax = xa, \forall x \in A \}$, and define the cocenter $\text{Tr}(A)$ of $A$ to be the $K$-linear space $\text{Tr}(A) := A/[A, A]$, where $[A, A]$ is the $K$-subspace of $A$ generated by all commutators of the form $xy - yx$ for $x, y \in A$. Note that $\text{Tr}(A)$ is the 0-th Hochschild homology $\text{HH}_0(A)$ of $A$, while $Z(A)$ is the 0-th Hochschild cohomology $\text{HH}^0(A)$ of $A$.

Let $B$ be a $K$-algebra with an algebra homomorphism $i : B \to A$. Then $A$ naturally becomes a $(B, B)$-bimodule. For any $f \in \{ a \in A | ab = ba, \forall b \in B \}$, we define (following [19, (1)]
\[ \mu_f : A \otimes_B A \to A, \quad \sum_{(a)} a_1 \otimes a_2 \to \sum_{(a)} a_1 f a_2. \]

**Definition 2.4** Let $\Lambda \in P^+, \alpha \in Q_n^+$ and $\nu = (v_1, \ldots, v_n) \in I^\alpha$. We define
\[ d_{\Lambda, \alpha} := 2(\Lambda, \alpha) - (\alpha, \alpha). \]

For any $i \in I$ and $\beta \in Q^+$, we set
\[ \lambda_{i, \beta} = \langle h_i, \Lambda - \beta \rangle. \]

For any $\nu \in I^\alpha$ and $0 \leq k \leq n - 1$, we set
\[ \lambda_{k, \nu} = \left( h_{v_{k+1}}, \Lambda - \sum_{j=1}^{k} \alpha_{v_j} \right). \]

If $(v_1, \ldots, v_k) \in I^\beta$, then we have $\lambda_{v_{k+1}, \beta} = \lambda_{k, \nu}$.

Let $\alpha \in Q_n^+$ and $\nu \in \mathcal{R}_\alpha$. By convention, for any $m > n$, we shall often abbreviate the element
\[ z \sum_{i_{n+1}, \ldots, i_m \in I} e(\alpha, i_{n+1}, \ldots, i_m) \in \mathcal{R}_m := \bigoplus_{\alpha \in Q_m^+} \mathcal{R}_\alpha \]
as $z$. The same convention is also adopted for elements in $\mathcal{R}_\alpha^\Lambda$. The following result of Kang-Kashiwara will be used in the proof of the main results in this paper.

**Lemma 2.7** ([10, Theorem 5.2], [19, (6),(7)]) Let $\alpha \in Q_n^+, i \in I$ and $z \in e(\alpha, i) \mathcal{R}_\alpha^{\Lambda_{\alpha+i}} e(\alpha, i)$. 

\[ \tag*{\copyright Springer} \]
(1) If \( \lambda_{i, \alpha} \geq 0 \), then there are unique elements \( \pi(z) \in \mathcal{R}_\alpha^\Lambda e(\alpha - \alpha_i, i) \otimes \mathcal{R}_\alpha^\Lambda e(\alpha - \alpha_i, i) \mathcal{R}_\alpha^\Lambda \) and \( p_k(z) \in \mathcal{R}_\alpha^\Lambda \) such that,

\[
\lambda_{i, \alpha}^{-1} = \mu_{\tau_n}(\pi(z)) + \sum_{k=0}^{\lambda_{i, \alpha}} p_k(z)\lambda_{n+1}^k e(\alpha, i); \tag{2.8}
\]

(2) If \( \lambda_{i, \alpha} \leq 0 \), then there is a unique element \( \tilde{z} \in \mathcal{R}_\alpha^\Lambda e(\alpha - \alpha_i, i) \otimes \mathcal{R}_\alpha^\Lambda e(\alpha - \alpha_i, i) \mathcal{R}_\alpha^\Lambda \) such that

\[
z = \mu_{\tau_n}(\tilde{z}), \quad \mu_{x_2}(\tilde{z})e(\alpha, i) = 0, \quad \forall 0 \leq k \leq -\lambda_{i, \alpha} - 1. \tag{2.9}
\]

Let \( z \in e(\alpha, i) \mathcal{R}_\alpha^\Lambda e(\alpha, i), \) where \( \alpha \in Q_n^+, i \in I \). The following trivial observation will be used in the proof of Theorem 3.7 in Sect. 3.

If \( \lambda_{i, \alpha} \geq 0 \) then \( \pi(z) \neq 0 \) only if \( \alpha \geq \alpha_i \); if \( \lambda_{i, \alpha} < 0 \) then \( \alpha \geq \alpha_i \). \tag{2.10}

Let \( \alpha \in Q_n^+, \nu \in I^\alpha \) and \( i \in I \). Following [19, Theorem 3.8], we define

\[
\hat{e}_{i, \Lambda - \alpha} : e(\alpha, i) \mathcal{R}_\alpha^\Lambda e(\alpha, i) \rightarrow \mathcal{R}_\alpha^\Lambda
\]

\[
z \mapsto \begin{cases} p_{\lambda_{i, \alpha}}(z), & \text{if } \lambda_{i, \alpha} > 0; \\ \mu_{x_n}(\tilde{z}), & \text{if } \lambda_{i, \alpha} \leq 0. \end{cases}
\]

Let \( r_v \in K^\times \) be defined as [19, (62)]. For any \( v, v' \in I^\alpha, z \in \mathcal{R}_\alpha^\Lambda \), we define

\[
t_{\Lambda, \alpha}(e(v)ze(v')) := \begin{cases} 0, & \text{if } v \neq v'; \\ r_v\hat{e}_{1, \nu_1}\hat{e}_{2, \nu_2} \cdots \hat{e}_{n, \nu_n}(e(v)ze(v')), & \text{if } v = v', \end{cases}
\]

where \( \hat{e}_{k, v_k} \) is a map

\[
\hat{e}_{k, v_k} : e(v_1, \ldots, v_k) \mathcal{R}_\alpha^\Lambda \sum_{j=k+1}^n a_{v_j} e(v_1, \ldots, v_k) \rightarrow e(v_1, \ldots, v_{k-1}) \mathcal{R}_\alpha^\Lambda \sum_{j=k+1}^n a_{v_j} e(v_1, \ldots, v_k-1),
\]

which is the restriction of \( \hat{e}'_{v_k, \Lambda - \sum_{j=k+1}^n a_{v_j}} \). Note that the map \( \hat{e}_{v_k} \) was denote by \( \hat{e}_{v_k} \) in [19, A.3]. We extend \( t_{\Lambda, \alpha} \) linearly to a \( K \)-linear map \( t_{\Lambda, \alpha} : \mathcal{R}_\alpha^\Lambda \rightarrow K \).

**Lemma 2.11** [19, Proposition 3.10] The map \( t_{\Lambda, \alpha} : \mathcal{R}_\alpha^\Lambda \rightarrow K \) is a homogeneous symmetrizing form on \( \mathcal{R}_\alpha^\Lambda \) of degree \(-d_{\Lambda, \alpha}\).

In [8], we have obtained some closed formulae for the graded dimension of the cyclotomic quiver Hecke algebra \( \mathcal{R}_\alpha^\Lambda \) of arbitrary type.

**Lemma 2.12** [8, Theorem 1.1] Let \( \alpha \in Q_n^+ \) and \( \nu = (v_1, \ldots, v_n), v' = (v'_1, \ldots, v'_n) \in I^\alpha \). Then

\[
\dim_q e(v) \mathcal{R}_\alpha^\Lambda e(v') = \sum_{w \in S(v, v')} \prod_{t=1}^n \left( \left[ N^\Lambda(w, v, t) \right]_{v_t} q_{v_t}^{N^\Lambda(1, v, t)-1} \right).
\]

\footnote{We remark that there is a typo in [19, (6),(7)] where the element “\( e(\alpha, i) \)” was missing in the second term of the right-hand side of (2.8) and the term \( \mu_{x_n}(\tilde{z}) \) in (2.9).}
where $\mathcal{S}(v, v') := \{w \in \mathcal{S}_n | wv = v'\}$, $q$ is an indeterminate, $q_{v_i} := q^{d_{v_i}}$, $[m]_{v_i}$ is the quantum integer [8, (2.1)], $N^\Lambda(w, v, t)$ is defined as follows:

$$N^\Lambda(w, v, t) := \left(h_{v_i}, \Lambda - \sum_{j \in J_{w}^{\leq t}} \alpha_{v_j}\right), \quad J_{w}^{\leq t} := \{1 \leq j < t | w(j) < w(t)\}.$$

The above lemma shows that the dimension of $R^\Lambda_a(K)$ depends only on the root system associated to $A$ and the dominant weight $\Lambda$, but not on the chosen ground field $K$ and the polynomials $Q_{ij}(u, v)$. This implies that if each $Q_{ij}(u, v)$ is defined over $\mathbb{Z}$ then $R^\Lambda_a(\mathbb{Z})$ is free over $\mathbb{Z}$, and hence $O \otimes_{\mathbb{Z}} R^\Lambda_a(\mathbb{Z}) \cong R^\Lambda_a(O)$ for any commutative ground ring $O$. Thus we recover the following result of Ariki, Park and Speyer.

**Corollary 2.13** [1, Proposition 2.4] Suppose that each $Q_{ij}(u, v)$ is defined over $\mathbb{Z}$. For any commutative ground ring $O$, the cyclotomic quiver Hecke algebra $R^\Lambda_a(O)$ is a free $O$-module of finite rank.

For any $A_1, \ldots, A_p \in R^\Lambda_a$, we define the ordered product:

$$\prod_{1 \leq i \leq m} A_i := A_1 A_2 \cdots A_p.$$

### 3 Relations and $K$-linear generators of the cocenter

In this section we shall investigate some relations inside the cocenter $\text{Tr}(R^\Lambda_a)$ with a purpose of looking for some normal forms for the $K$-linear generators of the cocenter $\text{Tr}(R^\Lambda_a)$. We shall also analyze the range of the degrees of elements in $\text{Tr}(R^\Lambda_a)$. The main result of this section is Theorem 3.7.

#### 3.1 $K$-linear generators

Let $\alpha \in Q^+_n$. In this subsection, we will give a set of $K$-linear generators for the cocenter of the cyclotomic quiver Hecke algebra $R^\Lambda_a$.

Recall that a composition of $n$ is a sequence of non-negative integers $a = (a_1, a_2, \ldots, a_k)$ such that $\sum_{i=1}^k a_i = n$. If $a = (a_1, a_2, \ldots, a_k)$ is a composition of $n$ then we write $a \models n$.

Let $v = (v_1, \ldots, v_n) \in I^\alpha$. We define

$$C(v) := \left\{ b = (b_1, \ldots, b_m) \models n \left| \begin{array}{l}
 m, b_1, \ldots, b_m \geq 1, \text{ and } v_j = v_{j+1}, \text{ for any } 1 \leq i \leq m \text{ and any } \sum_{k=1}^{i-1} b_k + 1 \leq j < \sum_{k=1}^i b_k \end{array} \right. \right\}.$$

For any $b = (b_1, \ldots, b_m) \in C(v)$, we define $c := (c_0, c_1, \ldots, c_{m-1}, c_m)$, where

$$c_0 := 0, \quad c_j := b_1 + b_2 + \cdots + b_j, \quad j = 1, 2, \ldots, m. \quad (3.1)$$

For any $b = (b_1, \ldots, b_m) \in C(v)$, we can decompose $v$ as follows:

$$v = (v^1, v^2, \ldots, v^m), \quad (3.2)$$

where $m, b_1, \ldots, b_m \in \mathbb{Z}_{\geq 1}$ with $\sum_{i=1}^m b_i = n$, $v^1, \ldots, v^m \in I$. Note that it could happen that $v^j = v^{j+1}$ for some $1 \leq j < m$. 

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Recall the definition of $\lambda_{k,v}$ given in (2.6). We define

$$C^\Lambda (v) := \left\{ b = (b_1, \ldots, b_m) \in C(v) \mid \lambda_{ci,v} > 0, \forall 0 \leq i < m \right\}. \quad (3.3)$$

**Definition 3.4** Let $v \in I^\alpha$. We denote by $R^\Lambda_{v,1}$ the $K$-subspace of $R^\Lambda_\alpha$ spanned by the elements of the following form:

$$\prod_{0 \leq i < m} \left( x_{ci+1}^{n_{ci+1}} \tau_{ci+1} \tau_{ci+2} \cdots \tau_{ci+1-1} \right) e(v)$$

where $\{ c_j \mid 0 \leq j \leq m \}$ is defined using $b$ as in (3.1).

$$\left\{ b \in C^\Lambda (v), 0 \leq n_{ci+1} \leq \lambda_{ci,v}, \forall 0 \leq i < m, \right\}.$$  

**Remark 3.6** We remark that although we use the ordered product in the definition of (3.5), it is easy to see that any two terms $\left( x_{ci+1}^{n_{ci+1}} \tau_{ci+1} \tau_{ci+2} \cdots \tau_{ci+1-1} \right) e(v)$, $\left( x_{cj+1}^{n_{cj+1}} \tau_{cj+1} \tau_{cj+2} \cdots \tau_{cj+1-1} \right) e(v)$ in the product commute with each other.

For any $K$-linear spaces $M$, $N$, let $\omega : M \otimes N \to N \otimes M$ be the map uniquely determined by $x \otimes y \mapsto y \otimes x, \forall x, y \in M$. For any $K$-algebra $A$ and subspaces $M, N \subseteq A$, we define $\text{Mult} : M \otimes N \to A$ to be the multiplication map. For any subset $A$ of $R^\Lambda_\alpha$, we use $\overline{A}$ to denote the natural image of $A$ in the cocenter $R^\Lambda / [R^\Lambda, R^\Lambda]$ of $R^\Lambda_\alpha$.

**Theorem 3.7** For any $\alpha \in Q^+_n$, we have

$$\text{Tr}(R^\Lambda_\alpha) = R^\Lambda_\alpha / [R^\Lambda_\alpha, R^\Lambda_\alpha] = \sum_{v \in I^\alpha} \overline{R^\Lambda_{v,1}}.$$  

**Proof** We use induction on $n$. The case $n = 1$ is trivial. Now we suppose our Theorem holds for $n - 1 \geq 1$. Let $z \in e(v)R^\Lambda e(\mu)$. If $v \neq \mu$, then in the cocenter $\text{Tr}(R^\Lambda_\alpha)$ we have

$$ze(\mu) = e(\mu)z = \overline{0}.$$  

Hence we only need to consider the case when $v = \mu$.

Recall the definitions of $\pi(z), p_k(z), \omega$ in (2.8) and (2.9) of Lemma 2.7. For any $\alpha \in Q^+_n$ and $i \in I$ with $\alpha > \alpha_i$, we fix a $K$-linear injection

$$\iota_{\alpha,i} : R^\Lambda_\alpha e(\alpha - \alpha_i, i) \otimes_{R^\Lambda_{\alpha - \alpha_i}} e(\alpha - \alpha_i, i)R^\Lambda_\alpha \to R^\Lambda_\alpha e(\alpha - \alpha_i, i) \otimes_K e(\alpha - \alpha_i, i)R^\Lambda_\alpha,$$  

(3.8)

such that $\pi_{\alpha,i} \circ \iota_{\alpha,i} = \text{id}$, where $\pi_{\alpha,i}$ denotes the canonical surjection $R^\Lambda_\alpha e(\alpha - \alpha_i, i) \otimes_K e(\alpha - \alpha_i, i)R^\Lambda_\alpha$.

Let $R^\Lambda_n := \bigoplus_{\alpha \in Q^+_n} R^\Lambda_\alpha$. For each $i \in I$, we define a $K$-linear map

$$\text{pr}_{n,i} : \bigoplus_{\mu \in I^n} e(\mu)R^\Lambda_n e(\mu) \to \bigoplus_{v \in I^{n-1}} e(v)R^\Lambda_{n-1} e(v)$$

as follows:

$$\text{pr}_{n,i}(z) := \begin{cases} \mu \circ \omega \circ \iota_{\alpha - \alpha_i, i} \circ \pi(z), & \text{if } z \in e(\mu)R^\Lambda_\alpha e(\mu), \mu_n = i, \lambda_i, \alpha - \alpha_i > 0; \\ \text{Mult} \circ \omega \circ \iota_{\alpha - \alpha_i, i} \circ \overline{\omega(z)}, & \text{if } z \in e(\mu)R^\Lambda_\alpha e(\mu), \mu_n = i, \lambda_i, \alpha - \alpha_i \leq 0; \\ 0, & \text{otherwise}. \end{cases} \quad (3.9)$$
For any $0 \leq t \leq n - 1$, we define
\[ \text{pr}_{[t,n],i} := \text{pr}_{t+1,i} \circ \cdots \circ \text{pr}_{n,i}, \]
and we set $\text{pr}_{[n,n],i}$ to be the identity map on $\bigoplus_{\mu \in I^n} e(\mu)\mathbb{R}_n^L e(\mu)$. We set $\tau_0 := 0$ and $e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1}) := 0$ whenever $\alpha \not\geq (n - t + 1)\alpha_i$. We claim that for any $z \in e(\mu)\mathbb{R}_{\mu,1}^L e(\mu)$ with $\mu_n = i$, and any $0 \leq t < n$,
\[ z \in K \text{pr}_{[t,n],i}(e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1}) + \sum_{\rho \in I^a} \mathbb{R}_{\rho,1}^L + [\mathbb{R}_a^L, \mathbb{R}_a^L]. \]
(3.10)

Once this is proved, then the theorem follows by taking $t = 0$. So it suffices to prove the above claim.

We use downward induction on $t$. The case $t = n$ is clear from definition.

Suppose that our claim holds for $1 \leq t \leq n$. We now consider the $t - 1$ case. If $\alpha \not\geq (n - t + 1)\alpha_i$, then by induction hypothesis, $z \in \sum_{\rho \in I^a} \mathbb{R}_{\rho,1}^L + [\mathbb{R}_a^L, \mathbb{R}_a^L]$ and we are done. Henceforth we assume $\alpha \geq (n - t + 1)\alpha_i$.

Suppose $\lambda_i, \alpha - (n-t+1)\alpha_i \geq 0$. By Lemma 2.7, we have a decomposition
\[ \text{pr}_{[t,n],i}(e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1})) e(\alpha - (n - t + 1)\alpha_i, i) \]
\[ + \sum_{k=0}^{\lambda_i, \alpha - (n-t+1)\alpha_i} p_k \text{pr}_{[t,n],i}(e(\alpha - (n - t + 1)\alpha_i, i), e(\alpha - (n - t + 1)\alpha_i, i)). \]

Note that for each $0 \leq k \leq \lambda_i, \alpha - (n-t+1)\alpha_i - 1$,
\[ p_k \text{pr}_{[t,n],i}(e(\alpha - (n - t + 1)\alpha_i, i)) \in \mathbb{R}_{\alpha - (n-t+1)\alpha_i}^L. \]

The elements $x_t, \tau_t, \ldots, \tau_{n-1}$ centralize the image of $\mathbb{R}_{\alpha - (n-t+1)\alpha_i}^L$ in
\[ e(\alpha - (n - t + 1)\alpha_i, i) \mathbb{R}_{\alpha - (n-t+1)\alpha_i}^L e(\alpha - (n - t + 1)\alpha_i, i). \]

Thus we can apply induction hypothesis to $p_k \text{pr}_{[t,n],i}(e(\alpha - (n - t + 1)\alpha_i, i))$ to deduce that
\[ p_k \text{pr}_{[t,n],i}(e(\alpha - (n - t + 1)\alpha_i, i)) \tau_t \cdots \tau_{n-1} e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1}) \in \sum_{\rho \in I^a} \mathbb{R}_{\rho,1}^L + [\mathbb{R}_a^L, \mathbb{R}_a^L]. \]

It remains to consider the first term in the right-hand side of the above decomposition of $\text{pr}_{[t,n],i}(e(\alpha - (n - t + 1)\alpha_i, i))$ (3.10) we have $e(\alpha - (n - t + 1)\alpha_i, i)$ and we are done by induction hypothesis. Hence we can assume $\alpha \geq (n - t + 2)\alpha_i$.

Recall the definition of $\lambda_{\alpha, i}$ given in (3.8). We write
\[ \lambda_{\alpha - (n-t+1)\alpha_i, i} e(\mu(\text{pr}_{[t,n],i}(e(\alpha - (n - t + 2)\alpha_i, i))) \cdot K e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1})) \]
\[ = \sum_{l=1}^{r'_1} z'_{l1} e(\alpha - (n - t + 2)\alpha_i, i^{n-t+2}) \tau_{l-1} z'_{l2} \tau_t \cdots \tau_{n-1} e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1}) \]
\[ \notin K [\mathbb{R}_a^L, \mathbb{R}_a^L]. \]

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This proves our claim in this case.

Let \( \lambda_{i, \alpha - (n - t + 1)\alpha_i} < 0 \). Then \( \alpha \geq (n - t + 2)\alpha_i \) by (2.10). We can write

\[
\mu_{\lambda_{(n - t + 1)\alpha_i}} \circ \text{pr}_{[t, n], i}(z) = \sum_{l=1}^{r_1'} \hat{z}_{l1} \otimes \hat{z}_{l2},
\]

where

\[
\hat{z}_{l1} \in \mathcal{R}^\Lambda_{\alpha - (n - t + 1)\alpha_i} e(\alpha - (n - t + 2)\alpha_i, i),
\]

\[
\hat{z}_{l2} \in e(\alpha - (n - t + 2)\alpha_i, i) \mathcal{R}^\Lambda_{\alpha - (n - t + 1)\alpha_i}.
\]

Using the same argument in the proof of the last paragraph (with \( z'_{l1}, z'_{l2} \) replaced with \( \hat{z}_{l1}, \hat{z}_{l2} \)) and using the definition of \( \text{pr}_{[t-1, n], i} \), we can get that

\[
\text{pr}_{[t, n], i}(z) \tau_t \cdots \tau_{n-1} e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1})
\]

\[
= \mu_{\tau_t-1} \left( \text{pr}_{[t, n], i}(z) \tau_t \cdots \tau_{n-1} e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1}) \right)
\]

\[
= \text{pr}_{[t-1, n], i}(z) \tau_{t-1} \cdots \tau_{n-1} e(\alpha - (n - t + 1)\alpha_i, i^{n-t+1}) \pmod{[\mathcal{R}^\Lambda_{\alpha}, \mathcal{R}^\Lambda_{\alpha}]},
\]

Hence in this case our claim follows from induction hypothesis. This completes the proof of the whole theorem.

\[\square\]

### 3.2 Positivity of the degree of the cocenter

Let \( \alpha \in Q^+ \). The purpose of this subsection is to give an application of Theorem 3.7. We shall show that any element in the cocenter \( \text{Tr}(\mathcal{R}^\Lambda_{\alpha}) \) of \( \mathcal{R}^\Lambda_{\alpha} \) has degree \( \geq 0 \). Equivalently, this means any element in the center \( Z(\mathcal{R}^\Lambda_{\alpha}) \) of \( \mathcal{R}^\Lambda_{\alpha} \) has degree \( \leq d_{\Lambda, \alpha} \).
Let \( \ell, n \in \mathbb{N} \). Consider the cyclotomic nilHecke algebra \( \text{NH}^\ell_n \) of type \( A \) which is defined over \( K \). Applying [5, Theorem 3.7], we see that the center \( Z(\text{NH}^\ell_n) \) of the cyclotomic nilHecke algebra \( \text{NH}^\ell_n \) has a basis \( \{ z_{\mu} | \mu \in \mathcal{P}_0 \} \), where

\[
\mathcal{P}_0 = \left\{ (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \left| \sum_{j=1}^{\ell} |\lambda^{(j)}| = n, \lambda^{(j)} \in \{0, 1\}, 1 \leq j \leq \ell \right. \right\},
\]

see also [5, Definition 2.5]. The degree of each basis element \( z_{\mu} \) is explicitly known by [5, Definition 3.3]. In particular, we know that

\[
\left( Z(\text{NH}^\ell_n) \right)_j \neq 0 \text{ only if } 0 \leq j \leq 2\ell n - 2n^2,
\]

and \( \dim(\text{ZH}^\ell_n) = \dim(\text{ZH}^\ell_n)_{2\ell n - 2n^2} = 1 \). As a result, we can deduce that

\[
\text{Tr}(\text{NH}^\ell_n)_j \neq 0 \text{ only if } 0 \leq j \leq 2\ell n - 2n^2,
\]

and

\[
\dim \text{Tr}(\text{NH}^\ell_n)_0 = \dim \text{Tr}(\text{NH}^\ell_n)_{2\ell n - 2n^2} = 1.
\]

By [10, Corollary 4.4], \( \mathcal{R}_A^\Lambda \) is a finite dimensional \( K \)-linear space. Let \( v \in I^\alpha \). For each \( 1 \leq k \leq n \), we have that \( \deg x_k e(v) > 0 \). It follows that \( x_k e(v) \) is nilpotent in \( \mathcal{R}_A^\Lambda \).

Now let \( b = (b_1, \ldots, b_m) \in C(v) \). We define \( e = (e_0, c_1, \ldots, c_m) \) as in (3.1). For each \( 1 \leq j \leq m \), we use \( r_j \) to denote the nilpotent index of \( x_{c_j-1+1} e(v) \). That says,

\[
r_j := \min \left\{ l \geq 1 \left| (x_{c_j-1+1} e(v))^l = 0 \right. \right\}.
\]

We define

\[
\text{NH}_{b_1}^{r_1}, \ldots, r_m := \text{NH}^{r_1}_{\{1,2,\ldots,c_1\}} \otimes \text{NH}^{r_2}_{\{c_1+1,c_1+2,\ldots,c_2\}} \otimes \cdots \otimes \text{NH}^{r_m}_{\{c_m-1,\ldots,n\}}.
\]

where for each \( 1 \leq j \leq m \), \( \text{NH}^{r_j}_{\{c_j-1+1,c_j-1+2,\ldots,c_j\}} \) denote the cyclotomic nilHecke algebra with standard generators

\[
x_{c_j-1+1}, x_{c_j-1+2}, \ldots, x_{c_j}, \tau_{c_j-1+1}, \tau_{c_j-1+2}, \ldots, \tau_{c_j-1}.
\]

which is isomorphic to \( \text{NH}^{r_j}_{b_j} \). Clearly, the following correspondence

\[
x_k \mapsto x_k e(v), \quad \forall 1 \leq k \leq n,
\]

\[
\tau_{c_i-1+l} \mapsto \tau_l e(v), \quad \forall 1 \leq l \leq b_j, 1 \leq i \leq m,
\]

can be extended uniquely to a \( K \)-algebra homomorphism \( \pi_{b,v} : \text{NH}_{b_1}^{r_1}, \ldots, r_m \rightarrow \mathcal{R}_A^\Lambda \).

By construction, it is clear that \( \pi_b \) induces a homogeneous \( K \)-linear map:

\[
\pi_{b,v} : \text{Tr}(\text{NH}_{b_1}^{r_1}, \ldots, r_m) \rightarrow \text{Tr}(\mathcal{R}_A^\Lambda).
\]

As an easy application of Theorem 3.7, we recover one half of [19, Theorem 3.31(a)] (see Corollary 4.21 for another half of [19, Theorem 3.31(a)]). Their original proof used the categorical representation and an action of the loop algebra, while our proof is more direct and elementary.
Proposition 3.14 [19, Theorem 3.31(a)] The cocenter $\text{Tr}(\mathcal{R}_a^\Lambda)$ of $\mathcal{R}_a^\Lambda$ is always positively graded. In other words, for any $j \in \mathbb{Z}$,

$$(\text{Tr}(\mathcal{R}_a^\Lambda))_j \neq 0$$

only if $j \geq 0$.

Equivalently, $Z(\mathcal{R}_a^\Lambda)_j \neq 0$ only if $j \leq d_{\Lambda,\alpha}$.

Proof We consider all the $v \in I^a$ and all the decomposition of $v$ as in (3.2). Applying Theorem 3.7, we can deduce that the following homogeneous $K$-linear map

$$\pi := \sum_{v, b} \pi_{b, v} : \bigoplus_{v, b} \text{Tr}(\mathcal{R}_b^\Lambda) \rightarrow \text{Tr}(\mathcal{R}_a^\Lambda)$$

is surjective. Since $\text{Tr}(\mathcal{R}_b^\Lambda) \supseteq \text{Tr}(\mathcal{R}_b^{\alpha_1}) \otimes \cdots \otimes \text{Tr}(\mathcal{R}_b^{\alpha_d})$ positively graded, the same must be also true for $\text{Tr}(\mathcal{R}_a^\Lambda)$. This proves the proposition. \qed

Corollary 3.15 Let $\alpha \in Q_n^+$. Suppose that $\mathcal{R}_a^\Lambda \neq 0$. Then $d_{\Lambda,\alpha} \geq 0$.

Proof By [19, Proposition 3.10], there exists $0 \neq h \in \mathcal{R}_a^\Lambda$ of degree $d_{\Lambda,\alpha}$ such that $t_{\Lambda,\alpha}(h) \neq 0$. Hence $h \notin [\mathcal{R}_a^\Lambda, \mathcal{R}_a^\Lambda]$, or equivalently, $0 \neq h \in \text{Tr}(\mathcal{R}_a^\Lambda)$. Applying Proposition 3.14, we can deduce that $d_{\Lambda,\alpha} \geq 0$. \qed

Prof. Wei Hu has asked whether $d_{\Lambda,\alpha} \geq 0$ is sufficient to ensure that $\mathcal{R}_a^\Lambda \neq 0$. The following example shows that this is not the case.

Example 3.16 Let $t := \mathbb{Z}/3\mathbb{Z}$, $\mathfrak{g} := \hat{s}t_3$, $\Lambda = 4\Lambda_0$, $\alpha = \alpha_0 + 2\alpha_1$. Then we have $d_{\Lambda,\alpha} = 2 > 0$. But $\mathcal{R}_a^\Lambda = 0$, because otherwise it should be a block of the cyclotomic Hecke algebra of type $G(4, 1, 3)$. But the latter case can not happen by [7, Lemma 4.1] because there is no standard $t = (t^{(1)}, t^{(2)}, t^{(3)}, t^{(4)})$ whose residues multiset is equal to $\{0, 1, 1\}$.

In Theorem 4.28 of Sect. 4 we shall give a necessary and sufficient condition for which $\mathcal{R}_a^\Lambda \neq 0$.

3.3 Relations inside the cocenter

The purpose of this subsection is to present some relations inside the cocenter of $\mathcal{R}_a^\Lambda$. In fact, we shall deduced it from some relations inside the cocenter of the (non-cyclotomic) nil-Hecke algebra $\mathcal{N}_n$ [5, Definition 1.4] of type $A$.

By definition, the (non-cyclotomic) nil-Hecke algebra $\mathcal{N}_n$ of type $A$ is generated by $\tau_1, \ldots, \tau_{n-1}, x_1, \ldots, x_n$ with some defining relations. We refer the readers to [5, Definition 1.4] for the details. Note that the generators $\tau_r, x_k$ are denoted by $\psi_r, y_k$ in [5, Definition 1.4].

Lemma 3.17 Let $y := x_1^k \tau_1 \tau_2 \cdots \tau_{n-1} \in \mathcal{N}_n$.

(1) If $n = 2$, then we have

$$(k + 1)y + \sum_{k_1, k_2 \in \mathbb{N}, k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} + \sum_{k_1, k_2, k_1' \in \mathbb{N}, 1 \leq k_1' \leq k - 1, k_1' + k_2 = k - 1 - k_1} x_1^{k_1 + k_1'} x_2^{k_2} \in [\mathcal{N}_n, \mathcal{N}_n];$$
(2) If \( n > 2 \), then we have

\[
(k + 1)y + kx_1^{k-1}\tau_1\tau_2\cdots\tau_{n-2} + \sum_{k_1, k_2, k'_1, k'_2 \in \mathbb{N}, \atop 0 \leq k_1 \leq k-2, \atop k'_1 + k'_2 = k-1-k_1} x_1^{k_1+k'_1}x_2^{k_2}\tau_2\cdots\tau_{n-2}
\]

\[
+ \sum_{k_1, k_2, k'_1, k'_2 \in \mathbb{N}, \atop 1 \leq k_1 \leq k-1, \atop k'_1 + k'_2 = k-1-k_1} x_1^{k_1+k'_1}x_2^{k_2}\tau_2\cdots\tau_{n-1} \in \left[ \mathbb{N}_n, \mathbb{N}_n \right].
\]

In particular, in both cases if \( k = 0 \) then \( y \in \left[ \mathbb{N}_n, \mathbb{N}_n \right] \).

**Proof** (1) By assumption, \( n = 2 \). We can do the following calculation in \( \text{Tr}(\mathbb{N}_n) \):

\[
y \equiv x_1^k\tau_1(x_2\tau_1 - \tau_1x_1) \equiv x_1^k\tau_1x_2\tau_1 \equiv \tau_1x_1^k\tau_1x_2
\]

\[
\equiv - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1}x_2^{k_2}\tau_1x_2
\]

\[
\equiv - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1}x_2^{k_2}(1 + x_1\tau_1)
\]

\[
\equiv - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1}x_2^{k_2} - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1+1}x_2^{k_2}\tau_1
\]

\[
\equiv - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1}x_2^{k_2} - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1+1}\left( \sum_{k'_1, k'_2 \in \mathbb{N}, k'_1+k'_2=k-1} x_1^{k'_1}x_2^{k'_2} + \tau_1x_1^{k_2} \right)
\]

\[
\equiv - \sum_{k_1, k_2, k'_1, k'_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1}x_2^{k_2} - \sum_{k_1, k_2, k'_1, k'_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1+k'_1+1}x_2^{k'_2} - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1+1}\tau_1x_1^{k_2}
\]

\[
\equiv - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1}x_2^{k_2} - \sum_{k_1, k_2, k'_1, k'_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1+k'_1+1}x_2^{k'_2} - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_2}\tau_1x_1^{k_1+1}
\]

\[
\equiv - \sum_{k_1, k_2 \in \mathbb{N}, k_1+k_2=k-1} x_1^{k_1}x_2^{k_2} - \sum_{k_1, k_2, k'_1, k'_2 \in \mathbb{N}, 0 \leq k_1 \leq k-1, k'_1+k'_2=k-2} x_1^{k_1+k'_1+1}x_2^{k'_2} - \sum_{k_1, k_2 \in \mathbb{N}, 0 \leq k_1 \leq k-1, k'_1+k'_2=k-2} x_1^{k_1}x_2^{k_2+1}\tau_1x_1\mod \left[ \mathbb{N}_n, \mathbb{N}_n \right].
\]

This proves (1). In particular, this implies that \( \tau_1 \in \left[ \mathbb{N}_n, \mathbb{N}_n \right] \).
(2) By assumption, $n > 2$. We can do the following calculation in $\text{Tr}(\text{NH}_n)$:

$$y \equiv x_1^k \tau_1 \tau_2 \cdots \tau_{n-1} \left( x_n \tau_{n-1} - \tau_{n-1} x_n \right)$$

$$\equiv x_1^k \tau_1 \tau_2 \cdots \tau_{n-1} x_n \tau_{n-1}$$

$$\equiv \tau_{n-1} x_1^k \tau_1 \tau_2 \cdots \tau_{n-1} x_n$$

$$\equiv x_1^k \tau_1 \tau_2 \cdots \tau_{n-3} \tau_{n-1} x_n \tau_{n-2} \tau_{n-1} x_n$$

$$\equiv x_1^k \tau_1 \tau_2 \cdots \tau_{n-3} \tau_{n-2} \tau_{n-1} x_n \tau_{n-2} x_n$$

$$\equiv \tau_{n-2} x_1^k \tau_1 \tau_2 \cdots \tau_{n-3} \tau_{n-2} \tau_{n-1} x_n$$

as $\tau_{n-2}$ commutes with $x_n$.

$$\equiv x_1^k \tau_1 \tau_2 \cdots \tau_{n-3} \tau_{n-2} \tau_{n-1} x_n$$

(by braid relation)

$$\equiv x_1^k \tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1} x_n \tau_{n-3}$$

We repeat the same argument in the second last two steps by moving $\tau_{n-3}$ to the leftmost side and then applying braid relation to get a $\tau_{n-4}$ on the rightmost side, and then do the same to $\tau_{n-4}$, $\ldots$, $\tau_2$, $\tau_1$. Eventually, we shall get that

$$y \equiv \tau_1 x_1^k \tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1} x_n \quad (\text{mod } [\text{NH}_n, \text{NH}_n])$$

Hence, if $k = 0$, then we can deduce

$$y \equiv \tau_1 \tau_1 \tau_1 + \cdots \tau_{n-2} \tau_{n-1} x_n \equiv 0 \quad (\text{mod } [\text{NH}_n, \text{NH}_n])$$

Now suppose $k > 0$. We can deduce

$$y \equiv (\tau_1 x_1^k) \tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-1} x_n$$

$$\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}, k_1 + k_2 = k-1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-1} x_n$$

$$\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}, k_1 + k_2 = k-1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2} (x_{n-1} \tau_{n-1} + 1)$$

$$\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}, k_1 + k_2 = k-1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2}$$

$$- \left( \sum_{k_1, k_2 \in \mathbb{N}, k_1 + k_2 = k-1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2} \tau_{n-3} (\tau_{n-2} x_{n-1}) \tau_{n-1}$$

$$\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}, k_1 + k_2 = k-1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2}$$
\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2} x_{n-2} \tau_{n-2} \tau_{n-1}
\]

\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-3} \tau_{n-1} \pmod{[N_H, N_H]}
\]

Now, for the last term above, consider the linear map: \(\text{Tr}(N_{H2}) \rightarrow \text{Tr}(N_H)\) which is induced by the natural algebra homomorphism \(N_{H2} \rightarrow N_H\) given by \(x_1 \mapsto x_{n-1}, x_2 \mapsto x_n\) and \(\tau_1 \mapsto \tau_{n-1}\). Since \(-\left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-3} \tau_{n-1} \) commutes with \(\tau_{n-1}\), it follows from the \(k = 0\) case in the part (1) of the lemma that the last term must vanish in \(\text{Tr}(N_H)\). Therefore,

\[
y \equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2}
\]

\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-3} \tau_{n-2} \tau_{n-1}
\]

\[
\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2}
\]

\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_1+1 \cdots \tau_{n-4} \left( \tau_{n-3} x_{n-2} \right) \tau_{n-2} \tau_{n-1}
\]

\[
\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2}
\]

\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots x_{n-3} \tau_{n-3} \tau_{n-2} \tau_{n-1}
\]

\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}}^{k_1 + k_2 = k - 1} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-4} \tau_{n-2} \tau_{n-1} \pmod{[N_H, N_H]}
\]
Again, for the same reason as before, the $k = 0$ case in the part (2) of the lemma implies that the last term above must vanish in $\text{Tr}(NH_n)$. We repeat the same argument above and eventually we shall get that

\[
y \equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2} - \left( \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1+1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-1} \pmod{[NH_n, NH_n]}.
\]

Finally, we have

\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-2}
\]

\[
\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1} \left( \tau_1 x_1^{k_2} + \sum_{k_1', k_2' \in \mathbb{N}} x_1^{k_1'} x_2^{k_2'} \right) \right) \tau_2 \cdots \tau_{n-2}
\]

\[
\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1+k_2} \tau_1 + \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1+k_1'} x_2^{k_2'} \right) \tau_2 \cdots \tau_{n-2}
\]

\[
\equiv -k x_1^{k-1} \tau_1 \tau_2 \cdots \tau_{n-2} - \left( \sum_{k_1, k_1', k_2' \in \mathbb{N}} x_1^{k_1+k_1'} x_2^{k_2'} \right) \tau_2 \cdots \tau_{n-2} \pmod{[NH_n, NH_n]},
\]

where we have used the fact that $x_1^{k_2}$ commutes with $\tau_2 \cdots \tau_{n-2}$, and moved $x_1^{k_2}$ from the right end to the left end in the second equality.

Similarly, we have

\[
- \left( \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1+1} x_2^{k_2} \right) \tau_1 \tau_2 \cdots \tau_{n-1}
\]

\[
\equiv - \left( \sum_{k_1, k_2 \in \mathbb{N}} x_1^{k_1+1} \left( \tau_1 x_1^{k_2} + \sum_{k_1, k_1', k_2' \in \mathbb{N}} x_1^{k_1'} x_2^{k_2'} \right) \right) \tau_2 \cdots \tau_{n-1}
\]
Let $1 \leq m, m' \leq n$. We define $A_m$ to be the $K$-subalgebra of $R^A_\alpha$ generated by

$$\tau_w, x_j, \; w, j \in \mathcal{S}_{1,2,\ldots,m}, \; 1 \leq j \leq m.$$ 

We define $B_{m'-1}$ to be the $K$-subalgebra of $R^A_\alpha$ generated by

$$\tau_w, x_j, \; w \in \mathcal{S}_{m',m'+1,\ldots,n}, \; m' \leq j \leq n.$$ 

By convention, we set $A_0 = Ke(\alpha) = B_n$. We call $A_m$ the first $m$-th part of $R^A_\alpha$, while call $B_{m'-1}$ the last $(n - (m' - 1))$-th part of $R^A_\alpha$.

In particular,

$$A_m e(v) = K - \text{Span} \left\{ \tau_w x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} e(v) \mid w \in \mathcal{S}_{1,2,\ldots,m}, \; k_1, \ldots, k_m \in \mathbb{N} \right\},$$

$$B_{m'-1} e(v) = K - \text{Span} \left\{ \tau_w x_m^{k_m} x_{m+1}^{k_{m+1}} \cdots x_n^{k_n} e(v) \mid w \in \mathcal{S}_{m',m'+1,\ldots,n}, \; k_m, k_{m'+1}, \ldots, k_n \in \mathbb{N} \right\}.$$ 

**Lemma 3.19** Suppose that $y = y_1 x_{c_1+1}^{k_1} \tau_{c_1+1} \tau_{c_1+2} \cdots \tau_{c_1+1-1} y_2 e(v)$, where $k \in \mathbb{N}, y_1 \in A_{c_1} e(v), \; y_2 \in B_{c_1+1} e(v), \; 0 \leq t \leq m - 1$.

1. If $b_{t+1} = 2$, then we have

$$(k+1)y + y_1 \left( \sum_{k_1, k_2 \in \mathbb{N}, \atop k_1 + k_2 = k-1} x_{c_{t+1}}^{k_1} x_{c_{t+2}}^{k_2} + \sum_{k_1, k_1', k_2 \in \mathbb{N}, \atop 1 \leq k_1 \leq k-1, \atop k_1' + k_2 = k-1-k_1} x_{c_{t+1}}^{k_1+k_1'} x_{c_{t+2}}^{k_2} \right) y_2 e(v) \in [R^A_\alpha, R^A_\alpha];$$

2. If $b_{t+1} > 2$, then we have
\[(k + 1)y + y_1 + \left( k x_{c_t+1}^{k-1} \tau_{c_t+1} \tau_{c_t+2} \cdots \tau_{c_{t+1}} - 2 + \sum_{k_1, k_1' \in \mathbb{N}, \ 0 \leq k_1 \leq k-2, \ k_1' + k_1 = k-1} \sum_{k_2, k_2' \in \mathbb{N}, \ 1 \leq k_2 \leq k-1, \ k_2' + k_2 = k-1 - k_1} x_{c_{t+1}}^{k_1 + k_1'} x_{c_t+2}^{k_2} \tau_{c_t+2} \cdots \tau_{c_{t+1}} - 2 \right) y_2 e(v) \in [\mathcal{R}_\alpha^\Lambda, \mathcal{R}_\alpha^\Lambda].\]

In particular, in both cases if \( k = 0 \) then \( y \in [\mathcal{R}_\alpha^\Lambda, \mathcal{R}_\alpha^\Lambda] \).

**Proof** Consider the algebraic maps:

\[\text{NH}_{b_{t+1}} \to \mathcal{R}_\alpha \to \mathcal{R}_\alpha^\Lambda,\]

where the first map is defined by \( x_a \mapsto x_{a+c_t} e(v) \) and \( t_b \mapsto \tau_{b+c_t} e(v) \) for \( 1 \leq a \leq b_{t+1}, 1 \leq b \leq b_{t+1} - 1 \) and the second map is the natural surjection. This map induces a map between cocenters:

\[\text{Tr}(\text{NH}_{b_{t+1}}) \to \text{Tr}(\mathcal{R}_\alpha^\Lambda).\]

Now note that both \( y_1, y_2 \) commute with the image of \( \text{Tr}(\text{NH}_{c_{t+1}}) \), the Lemma follows from Lemma 3.17. \( \square \)

**Corollary 3.20** Suppose \( \text{char } K = 0 \). Let \( y = y_1 x_{c_{t+1}}^k \tau_{c_{t+1}} \tau_{c_{t+2}} \cdots \tau_{c_{t+1} - 1} y_2 e(v), \) where \( y_1 \in A_{c_t} e(v), \ y_2 \in B_{c_{t+1}} e(v) \).

1. If \( k < b_{t+1} - 1 \), then \( y \in [\mathcal{R}_\alpha^\Lambda, \mathcal{R}_\alpha^\Lambda] \).
2. If \( k = b_{t+1} - 1 \), then \( y \in K y_1 y_2 + [\mathcal{R}_\alpha^\Lambda, \mathcal{R}_\alpha^\Lambda] \).
3. For arbitrary \( k \geq 1 \), we have

\[y \in \sum_{c_{t+1} \leq i \leq c_{t+1} - 1 \atop 0 \leq k_i \leq k-1} K x_{c_{t+1}}^{k_1} \cdots x_{c_{t+1} - 1} y_1 y_2 + [\mathcal{R}_\alpha^\Lambda, \mathcal{R}_\alpha^\Lambda].\]

**Proof** Since \( \text{char } K = 0 \), it follows that \((k + 1) \cdot 1_K \) is invertible in \( K \). If \( k = 0 \), then the first part of the corollary follows from Lemma 3.19. In general, Lemma 3.19 gives an algorithm to rewrite the element \( y \) as a linear combination of some elements of the form

\[y' x_{c_{t+1}}^{k'-1} \tau_{c_{t+1}} \cdots \tau_{c_{t+1} - 2} y_2 e(v), \ y'' x_{c_{t+1}}^{k''} \tau_{c_{t+1}} + 2 \cdots \tau_{c_{t+1} + b} y_2 e(v), \quad (3.21)\]

such that \( k' \leq k - 2 \) and \( b' = b_{t+1} - 1, y_1' \in A_{c_t} e(v), y_2'' \in A_{c_{t+1}} e(v), y_2', y_2'' \in B_{c_{t+1}} e(v) \).

So the first part of the corollary follows from an induction on \( k \). For the second part, we induction on \( k \geq 1 \). If \( k = 1 \), then the first part of Lemma 3.19 implies

\[2y + y_1 y_2 \in [\mathcal{R}_\alpha^\Lambda, \mathcal{R}_\alpha^\Lambda].\]
from which the result follows. For $k > 1$, using the second part of Lemma 3.19 and the first part of the corollary, we have

\[(k + 1)y + y_1 \left( kx_{c_t+1}^{k-1} \tau_{c_t+1} \tau_{c_t+2} \cdots \tau_{c_t+2} + x_{c_t+2}^{k-2} \tau_{c_t+2} \cdots \tau_{c_t+1} \right) y_2 e(v) \in \mathcal{L}_x \mathcal{L}_\alpha \mathcal{L}^\Lambda_{a_\alpha} \mathcal{L}^\Lambda_{a_\alpha}.

Note that $k - 1 = (b_{t+1} - 1) - 1$ and $k - 2 = (b_{t+1} - 2) - 1$. We are now at the position to apply induction hypothesis to the last two terms to complete the second part of the corollary.

Finally, once again as Lemma 3.19 gives an algorithm to rewrite the element $y$ as a linear combination of some elements of the form (3.21), Part (3) of the lemma follows from a recursive application of Lemma 3.19. \[\Box\]

4 Piecewise dominant sequence and maximal degree and minimal degree components of the cocenter

The purpose of this section is to give the three main results of this paper. We shall introduce a new notion called “piecewise dominant sequence” and use it together with the main result in previous section to construct $K$-linear generators of both the maximal degree component and the minimal degree component of the cocenter $\text{Tr}(\mathcal{L}_x \mathcal{L}^\Lambda_{a_\alpha})$. In particular, we shall derive a new and simple criterion for which $\mathcal{L}_x \mathcal{L}^\Lambda_{a_\alpha} \neq 0$.

Let $v \in \mathcal{L}^\alpha$. There is a unique decomposition of $v$ as follows:

\[v = (v_1, \ldots, v_n) = (v_1^{1}, v_1^{1}, \ldots, v_1^{m}, v_2^{1}, v_2^{1}, \ldots, v_2^{m}, \ldots, v_n^{1}, v_n^{1}, \ldots, v_n^{m}),\]

which satisfies that

\[v^j \neq v^{j+1}, \quad \forall \, 1 \leq j < m,\]

where $m, b_1, \ldots, b_m \in \mathbb{Z}_{\geq 1}$ with $\sum_{j=1}^{m} b_j = n$. For each $1 \leq j \leq m$, we define

\[\ell_j(v) := \left( h_{v_j}, \Lambda - \sum_{t=1}^{c_j-1} \alpha_{v_t} \right),\]

where $c_0 := 0, c_j := \sum_{t=1}^{j} b_t, \forall \, 1 \leq j \leq m$. When $v$ is clear from the context, we shall write $\ell_j$ instead of $\ell_j(v)$ for simplicity.

4.1 Piecewise dominant sequence

Definition 4.4 Let $\Lambda \in P^+$ and $\alpha \in Q^+_n$. We call $v = (v_1, \ldots, v_n) \in \mathcal{L}^\alpha$ a piecewise dominant sequence with respect to $\Lambda$, if for the unique decomposition (4.1) of $v$ and any $1 \leq j \leq m$,

\[\ell_j = \ell_j(v) \geq b_j.\]

Example 4.6 Consider the cyclotomic nilHecke algebra $NH^\ell_n$ of type $A$. Let $v = (0, 0, \ldots, 0)$.

Then an easy computation shows that $v$ is piecewise dominant if and only if $\ell \geq n$, i.e., $NH^\ell_n \neq 0$. \[\text{Springer}\]
Lemma 4.7 Let \( v = (v_1, \ldots, v_n) \in I^\alpha \) and fix the unique decomposition (4.1) of \( v \). Then \( v \) is a piecewise dominant sequence with respect to \( \Lambda \) if and only if for each \( 1 \leq i \leq m \), there is an integer \( c_{i-1} + 1 \leq k_i' \leq c_i \) such that

\[
\left\langle h_{v_i}, \Lambda - \sum_{j=1}^{k_i'-1} \alpha_{v_j} \right\rangle \geq c_i - k_i' + 1. \tag{4.8}
\]

In this case, we denote the maximal value of each \( k_i' \) by \( k_i \), which can be taken as:

\[
k_i = \begin{cases} 
c_i, & \text{if } \ell_i - 2b_i \geq 0; \\
\ell_i + 2c_{i-1} - c_i + 1, & \text{if } \ell_i - 2b_i \leq -1.
\end{cases} \tag{4.9}
\]

Proof Suppose that \( v \) is a piecewise dominant sequence with respect to \( \Lambda \) with a unique decomposition as in (4.1). Then for each \( 1 \leq i \leq m \), we can simply take \( k_i' := c_{i-1} + 1 \). This proves one direction.

Conversely, suppose that for the unique decomposition of \( v \) as in (4.1) and for each \( 1 \leq i \leq m \), there is an integer \( c_{i-1} + 1 \leq k_i' \leq c_i \) such that (4.8) holds. Then we take \( k_i \) to be the maximal value of \( k_i' \) such that (4.8) holds. Let \( 1 \leq i \leq m \). Recall that

\[
\ell_i = \ell_i(v) = \left\langle h_{v_i}, \Lambda - \sum_{j=1}^{i-1} b_j \alpha_{v_j} \right\rangle.
\]

If \( \ell_i - 2b_i \geq 0 \), then we can take \( k_i = c_i \) such that (4.8) holds. If \( \ell_i - 2b_i \leq -1 \), then it is easy to see that the following inequalities:

\[
\begin{align*}
\ell_i - 2(k_i - c_{i-1}) &\geq c_i - 1 - k_i \\ 
\ell_i - 2(k_i + 1 - c_{i-1}) &< c_i - 1 - (k_i + 1) \\ 
c_{i-1} + 1 &\leq k_i \leq c_i,
\end{align*}
\]

has a unique solution \( k_i := \ell_i + 2c_{i-1} - c_i + 1 \). Combining this with the third inequality we can deduce that \( \ell_i \geq b_i \), we prove that \( v \) is a piecewise dominant sequence with respect to \( \Lambda \). \( \square \)

Definition 4.10 Let \( v \in I^\alpha \) be a piecewise dominant sequence with the unique decomposition as in (4.1). We define

\[
Z(v) = Z(v)_1 Z(v)_2 \cdots Z(v)_m,
\]

where for each \( 1 \leq i \leq m \),

\[
Z(v)_i := \begin{cases} 
x_{c_{i-1}+1}^{\ell_i-3} x_{c_{i-1}+2}^{c_{i-1}+2} \cdots x_{c_i}^{c_i-2b_i+1} e(v), & \text{if } \ell_i \geq 2b_i; \\
x_{c_{i-1}+1}^{\ell_i-1} x_{c_{i-1}+2}^{\ell_i-3} x_{c_{i-1}+3}^{c_{i-1}+2} \cdots x_{c_i}^{c_i-2b_i+1} e(v), & \text{if } b_i < \ell_i \leq 2b_i - 1; \\
x_{c_{i-1}+1}^{\ell_i-1} x_{c_{i-1}+2}^{\ell_i-3} x_{c_{i-1}+3}^{c_{i-1}+2} \cdots x_{c_i}^{c_i-2b_i+1} e(v), & \text{if } \ell_i = b_i.
\end{cases}
\]

Lemma 4.11 Suppose \( v \in I^\alpha \) is a piecewise dominant sequence with respect to \( \Lambda \), then \( \deg(Z(v)) = d_{\Lambda, \alpha} \).

Proof Let \( v \) be a piecewise dominant sequence with respect to \( \Lambda \) with a decomposition as in (4.1) which satisfies (4.2). There are two cases:

Case 1. \( \ell_i - 2b_i \geq 0 \). In this case, by definition,

\[
Z(v)_i = x_{c_{i-1}+1}^{\ell_i-1} x_{c_{i-1}+2}^{\ell_i-3} x_{c_{i-1}+3}^{c_{i-1}+2} \cdots x_{c_i}^{c_i-2b_i+1} e(v).
\]
A direct computation shows $\deg(Z(v)_i) = (\alpha, \nu)(\ell_i - b_i)$. 

Case 2. $\ell_i = b_i$. In this case, $\deg(Z(v)_i) = \deg(e(v)) = 0$.

Case 3. $b_i < \ell_i - 2b_i \leq -1$. In this case, by definition,

$$
\deg(Z(v)_i) = (\alpha, \nu)(\ell_i - b_i)\nu.
$$

In both cases we have

$$
d_{\Lambda, \nu} = 2(\alpha, \Lambda) - (\alpha, \nu) =
\sum_{i=1}^{m} (\alpha, \nu)b_i\langle h_i, \Lambda \rangle - 2 \sum_{i=1}^{m} \sum_{j<i} b_j b_j (\alpha, \nu) - \sum_{i=1}^{m} b_i^2 (\alpha, \nu)
\sum_{i=1}^{m} (\alpha, \nu)(\ell_i - b_i)\nu = \sum_{i=1}^{m} \deg(Z(v)_i) = \deg(Z(v)).
$$

This proves the lemma.

Lemma 4.12 Suppose $\text{char } K = 0$. Let $v \in I^\nu$ and $z \in R_{\nu, 1}^\Lambda$. If $v$ is not piecewise dominant with respect to $\Lambda$, then $z \in [(R_{\nu}^\Lambda, R_{\Lambda}^\Lambda)]$.

Proof Suppose $z \notin [(R_{\nu}^\Lambda, R_{\Lambda}^\Lambda)]$. Then applying Theorem 3.7, we see that modulo $[(R_{\nu}^\Lambda, R_{\Lambda}^\Lambda)]$, $\bar{z}$ is a non-zero linear combination of some elements of the form

$$
\left(\left\langle\sum_{i=1}^{L_1} \tau_i \cdots \tau_{i-1}\right\rangle\right) \left(\left\langle\sum_{i=1}^{L_{c_i+1}} \tau_{i+1} \cdots \tau_{c_i} \cdots \tau_{c_i+2} \cdots \tau_{c_i-1}\right\rangle\right) \cdots \left(\left\langle\sum_{i=1}^{L_{c_{m-1}+1}} \tau_{i+1} \cdots \tau_{c_{m-1}+1} \cdots \tau_{n-1}\right\rangle\right) e(v),
$$

where $b' = (b'_1, \ldots, b'_{m}) \in C^\Lambda(v), c'_i = \sum_{j=1}^{i} b'_j, \forall 1 \leq i \leq m', v \in I^\nu$ is decomposed as in (3.2), and

$$
0 \leq L_j \leq \left\langle h_{v_j}, \Lambda - \sum_{k=1}^{j-1} \alpha_{v_k}\right\rangle - 1, \quad \forall j \in \{1, c'_1 + 1, \ldots, c'_{m-1} + 1\}.
$$

(4.13)

Now Corollary 3.20 tells us the above element is non-zero in the cocenter only if the following holds:

$$
L_1 \geq c'_{1} - 1, \quad L_{c'_1+1} \geq c'_{2} - 1, \ldots, L_{c'_{m'-1}+1} \geq n - 1 - c'_{m'-1}.
$$

(4.14)

We fix one such $v$. Note that the decomposition of $v$ as in (3.2) is not necessarily the unique decomposition in (4.1). That says, it is possible $v^j = v^{j+1}$ for some $1 \leq j < m'$. However, we can incorporate some neighbouring parts in the decomposition of $v$ in (3.2) and turn it into the unique decomposition in (4.1). That says, we can find $0 = t_0 < t_1 < t_2 = \cdots < t_m = m'$ and $b_1, \ldots, b_m \in \mathbb{Z}^{\geq 1}$, such that

1. $b_i = \sum_{j=t_i-1}^{t_i} b'_j, \forall 1 \leq i \leq m$; and
2. $v_{c_i} \neq v_{c_i+1}, \forall 1 \leq i \leq m - 1$; and
3. $v_s = v_{s+1}, \forall c_i - 1 + 1 \leq s < c_i, 1 \leq i < m$,

where $c_0 := 0, c_j := \sum_{s=1}^{j} b_s, \forall 1 \leq j \leq m$. Now from (4.14) we can get that

$$
L_1 \geq b'_i - 1, \quad L_{b_1+1} \geq b'_{t_2} - 1, \ldots, L_{n-b_m+1} \geq b'_{t_m} - 1.
$$

(4.15)

Combining this with (4.13) and applying Lemma 4.7 we can deduce (by taking $k'_i := c_s - b'_s + 1$) that $v$ is piecewise dominant, a contradiction. 

\qed
We are now in a position to give the proof of our first main result Theorem 1.3 in this paper.

\[ \prod_{0 \leq i < m'} \left( x_{c_{i}+1}^{n_{c_{i}+1}} \tau_{i+1} \tau_{i+2} \cdots \tau_{m+1} \right) e(v), \] (4.16)

where \( b' = (b'_1, \ldots, b'_{m'}) \in C^A(v), 0 \leq n_{c_{i}+1} < \lambda_{c_{i},v}, c_{i} := \sum_{j=1}^{i} b'_j, \forall 0 \leq i < m', \) and \( v \) is piecewise dominant.

We fix a piecewise dominant sequence \( v \). Let \( b' \in C^A(v) \) and let \( \{ c'_j \mid 0 \leq j \leq m' \} \) be defined as above. As in Lemma 4.12, we can incorporate some neighbouring parts in the decomposition of \( v \) in (3.2) and turn it into the unique decomposition in (4.1). That says, we can find \( 0 = t_0 < t_1 < t_2 < \cdots < t_m = m' \) and \( b_1, \ldots, b_m \in \mathbb{Z}^+ \), such that

1. \( b_i = \sum_{j=t_{i-1}+1}^{t_i} b'_j, \forall 1 \leq i \leq m; \) and
2. \( v_r \neq v_{r+1}, \forall 1 \leq r < m; \) and
3. \( v_s = v_{s+1}, \forall c_{r-1} + 1 \leq s < c_r, 1 \leq r < m. \)

where \( c_0 := 0, c_j := \sum_{s=1}^{j} b_s, \forall 1 \leq j \leq m. \) In particular, \( (b'_1, b'_2, \ldots, b'_m) \) is a composition of \( b_i \) for each \( 1 \leq i \leq m. \) Now Part (3) of the theorem follows from Corollary 3.20 and the fact that

\[ n_{c_{j}+1} < \lambda_{c_{j},v} \leq \ell_{i}(v), \forall t_{i-1} \leq j < t_i. \]

It remains to prove Part (1).

Using Corollary 3.20(1), we can narrow our selection for the monomial (4.16) by requiring:

\[ b'_{j+1} - 1 \leq n_{c_{j}+1} < \lambda_{c_{j},v}, \forall t_{i-1} \leq j < t_i. \] (4.17)

as otherwise the whole term already lives inside \( \mathcal{R}_A^A, \mathcal{R}_A^A \). Taking \( j = t_i - 1 \) in (4.17), we deduce that

\[ c_i - (c_i - b'_t + 1) + 1 = b'_t \leq \lambda_{c_i-b'_t,v} = \ell_{i}(v) - 2b_t + 2b'_t = \left( h_{\nu'}, \Lambda - \sum_{j=1}^{c_i-b'_t} \alpha_{\nu_j} \right). \]

Hence we take \( k'_t = c_i - b'_t + 1 \) satisfying (4.8). Now recall the definition of \( k_i \) in Lemma 4.7. We get that

\[ c_i - b'_t + 1 \leq k_i, \]

hence

\[ b'_t \geq \sum_{s=1}^{i} b_s - k_i + 1 = c_i - k_i + 1. \] (4.18)

Note that any decomposition of \( v \) in (3.2) is a refinement of the unique decomposition in (4.1). For each \( b > 0 \), we set \( C^+(b) := \{(\rho_1, \ldots, \rho_s) \mid b|\rho_j > 0, \forall 1 \leq j \leq s\}. \) In the next step, we shall inductively construct for any \( 1 \leq i \leq m, \) a composition \( (b'_{t_i-1+1}, b'_{t_i-1+2}, \ldots, b'_{t_i}) \in C^+(b_i) \) satisfying (4.18), and a unique choice of power indices.
\(|n_{e',j+1}|t_i-1 \leq j < t_i| satisfying (4.17), such that inside Tr(\(\mathfrak{B}_\alpha^\Lambda\)), the degree of the following element (which is a consecutive sub-product extracted from (4.16)):

\[
\prod_{t_i-1 \leq j < t_i} \left( x_{e'_j+1}^{n_{e',j+1}} \tau_{c'_j+1} \tau_{c'_j+2} \cdots \tau_{c'_j+1} \right) e(v),
\]

(4.19)
is strictly bigger than the degree of the corresponding elements for other compositions or other choices of power indices, where \(c'_j = c_{i-1} + \sum_{s=1}^{j-t_i-1} b'_{t_i-1+s}, \forall t_i-1 \leq j < t_i\).

If \(\ell_i(v) \geq 2b_i\), then we consider the composition \(b' := (1, 1, \ldots, 1) \in C^+(b_i)\) and set the power indices \(n_{e',j+1} = \lambda c'_{j,v} - 1, \forall t_i-1 \leq j < t_i\). In this case, the element (4.19) coincides with \(Z(v)_i\). The condition \(n_{e',j+1} < \lambda c'_{j,v}\) implies that for all the other \((b'_{t_i-1+1}, b'_{t_i-1+2}, \ldots, b'_{t_i}) \in C^+(b_i)\), the degree of the corresponding element (4.19) is strictly less than \(\text{deg } Z(v)_i\).

Now assume \(\ell_i(v) \leq 2b_i - 1\). We consider a special composition

\[
(1, 1, \ldots, 1, c_i - k_i + 1) \in C^+(b_i)
\]

by setting \(t_i := t_i-1 + k_i - c_i-1, b'_i = c_i - k_i + 1\) and

\[
b'_{t_i-1+1} = b'_{t_i-1+2} = \cdots = b'_{t_i-1+k_i-1} = 1,
\]

\[
n_{e',j+1} = \lambda c'_{j,v} - 1 = \ell_i - 2(j-t_i-1) - 1, \quad \forall t_i-1 \leq j < t_i-1+k_i-c_i-1.
\]

Then the corresponding monomial (4.19) coincides with the following element

\[
Z'(v)_i := x_{c_i-1}^{\ell_i-1} x_{c_i-1}^{\ell_i-3} \cdots x_{c_i}^{\ell_i-2(k_i-c_i-1)+1} \tau_{k_i} \cdots \tau_{c_i-2} \tau_{c_i-1} e(v).
\]

In this case, for all the other \((b'_{t_i-1+1}, b'_{t_i-1+2}, \ldots, b'_{t_i}) \in C^+(b_i)\) satisfying (4.18), the degree of the corresponding monomial (4.19) is strictly lesser than \(\text{deg } Z'(v)_i\). This is because \(b'_i - 1\) is the number of consecutive KLR \(\tau_i\) generators on the rightmost part of the monomial (4.19). If \(b'_i > \sum_{s=1}^{t_i} b_i - k_i + 1\), then the corresponding monomial (4.19) would have less numbers of power of KLR \(x_k\) generators and more products of KLR \(\tau_i\) generators than \(Z'(v)_i\).

We define

\[
Z'(v) = Z'(v)_1 Z'(v)_2 \cdots Z'(v)_m,
\]

where for each \(1 \leq i \leq m\),

\[
Z'(v)_i := \begin{cases} 
Z(v)_i = x_{c_i-1}^{\ell_i-1} x_{c_i-1}^{\ell_i-3} \cdots x_{c_i}^{\ell_i-2(b_i-k_i)+1} e(v), & \text{if } \ell_i \geq 2b_i; \\
 x_{c_i-1}^{\ell_i-1} x_{c_i-1}^{\ell_i-3} \cdots x_{c_i}^{\ell_i-2(k_i-c_i-1)+1} \tau_{k_i} \cdots \tau_{c_i-2} \tau_{c_i-1} e(v), & \text{if } \ell_i \leq 2b_i - 1,
\end{cases}
\]

where \(k_i\) is as defined in (4.9). Now applying Corollary 3.20(2), we see that \(Z'(v)\) is some scalar multiple of \(Z(v)\) whose degree is exactly \(d_{\Lambda,\alpha}\) by Lemma 4.11. Combining these with the discussion in the last two paragraphs, we can deduce that the image of those \(Z(v)\) can form a spanning set for the degree \(d_{\Lambda,\alpha}\) component of the cocenter \(\text{Tr}(\mathfrak{B}_\alpha^\Lambda)\). This completes the proof of Part (3) of the theorem.

\(\Box\)

**Remark 4.20** We remark that if \(\text{char } K > 0\) then Part 3) of the above theorem may not hold. For example, by Lemma 3.19, inside \(\text{NH}_3^2\) we have

\[
x_1 \tau_1 \equiv x_1 \tau_1 (x_2 \tau_1 - \tau_1 x_1) \equiv x_1 \tau_1 x_2 \tau_1 \equiv (\tau_1 x_1 \tau_1) x_2 \\
\equiv -\tau_1 x_2 \equiv -x_1 \tau_1 - 1 \pmod{[\text{NH}_3^2, \text{NH}_3^2]}.
\]
which implies that \( 2x_1 \tau_1 \equiv 1 \pmod{[\text{NH}_2^3, \text{NH}_2^3]} \). Using [5, Corollary 5.10], we know that 

\[
t_{3A_0, 2a_0}(\tau_1x_1(x_1x_2)) = 1,
\]

which implies that \( x_1 \tau_1 \notin [\text{NH}_2^3, \text{NH}_2^3] \), hence (by (3.13)) the degree 0 component of the cocenter of \( \text{NH}_2^3 \) is spanned by \( x_1 \tau_1 + [\text{NH}_2^3, \text{NH}_2^3] \). However, if \( \text{char} \, K = 2 \) then \( 1 \in [\text{NH}_2^3, \text{NH}_2^3] \). Hence Part 3) of the above theorem does not hold for \( \text{NH}_2^3 \) in this case.

Another direct application of Theorem 3.7 and Lemma 4.12 is the following corollary, which recovers [19, Theorem 3.31(a)] in an elementary way.

**Corollary 4.21** [19, Theorem 3.31(a)] Suppose \( \text{char} \, K = 0 \). Then we have \( (\text{Tr}(\mathcal{R}_\alpha^\Lambda))_j \neq 0 \) only if \( j \leq 0 \).

**Proof** Suppose \( (\text{Tr}(\mathcal{R}_\alpha^\Lambda))_j \neq 0 \). Then \( j \geq 0 \) by Proposition 3.14. Now \( j \leq d_{\Lambda, \alpha} \) follows from the proof of Theorem 1.3. \( \square \)

**Remark 4.22** It is tempting to speculate that \( \{e(v) \mid v \text{ is piecewise dominant}\} \) is a basis of \( (\text{Tr}(\mathcal{R}_\alpha^\Lambda))_0 \). Unfortunately, this is not true. For example, in the type \( A_2 \) case we choose \( \Lambda = \Lambda_1 + \Lambda_2 \). We can write down all the piecewise dominant sequences with respect to \( \Lambda \) as follows (where \( \alpha \)'s are given below):

\[
\begin{align*}
\alpha &= 0 : \emptyset; \quad \alpha_1 : (1); \quad \alpha_2 : (2); \quad \alpha_1 + \alpha_2 : (1, 2), (2, 1); \quad \alpha_1 + 2\alpha_2 : (1, 2, 2); \\
2\alpha_1 + \alpha_2 : (2, 1, 1); \quad 2\alpha_1 + 2\alpha_2 : (2, 1, 2), (1, 2, 2, 1).
\end{align*}
\]

However, by the end of proof in [19, Theorem 3.31(c)], we have

\[
\sum_{\alpha \in Q^+} \dim (\text{Tr}(\mathcal{R}_\alpha^\Lambda))_0 = \sum_{\alpha \in Q^+} \dim V(\Lambda)_{\Lambda - \alpha} = \dim V(\Lambda) = 8 < 9.
\]

### 4.2 A criterion for \( \mathcal{R}_\alpha^\Lambda \neq 0 \)

In this subsection, we will give a criterion for which \( \mathcal{R}_\alpha^\Lambda \neq 0 \) via the existence of piecewise dominant sequences. Throughout this subsection, unless otherwise stated, \( K \) is a field of arbitrary characteristic.

**Definition 4.23** Let \( b := (b_1, \ldots, b_m) \) be a composition of \( n \). We define

\[
\mathcal{S}_b = \mathcal{S}_{(1, 2, \ldots, b_1)} \times \mathcal{S}_{(b_1 + 1, \ldots, b_1 + b_2)} \times \cdots \times \mathcal{S}_{(n - b_m + 1, \ldots, n)}
\]

which is the standard Young subgroup of \( \mathcal{S}_n \) corresponding to \( b := (b_1, \ldots, b_m) \).

For each composition \( b = (b_1, \ldots, b_m) \) of \( n \), we denote by \( w_{b_0} \) the unique longest element in \( \mathcal{S}_b \). In other words,

\[
w_{b_0} = w_{b_1, 0}^{(1)} \times w_{b_2, 0}^{(2)} \times \cdots \times w_{b_m, 0}^{(m)}
\]

where for each \( 1 \leq i \leq m \), \( w_{b_i, 0}^{(i)} \) is the unique longest element in the Young subgroup

\[
\mathcal{S}_{(b_1 + \cdots + b_{i-1} + 1, b_1 + \cdots + b_{i-1} + 2, \ldots, b_1 + \cdots + b_i)}.
\]

For each \( 1 \leq i \leq m \), we fix a reduced expression of \( w_{b_i, 0}^{(i)} \) and use it to define \( \tau_{b, i} := \tau_{w_{b_i, 0}^{(i)}} \).
Definition 4.24 Let \( v \) be a piecewise dominant sequence with respect to \( \Lambda \), with a decomposition (4.1) satisfying (4.2). Let \( \{c_i\}_{0 \leq i \leq m} \) and \( \{\ell_i\}_{1 \leq i \leq m} \) be defined as in (3.1) and (4.5) respectively. We define

\[
S(v) := \prod_{1 \leq i \leq m} \left( \tau_{b, i} x_{c_{i-1}+1}^{\ell_{i}-1} x_{c_{i-1}+2}^{\ell_{i}-2} \cdots x_{c_i}^{\ell_i-b_i} \right) e(v) \in \mathcal{R}_\alpha^\Lambda.
\]

By Definition 4.4 of piecewise dominant sequence, each power index of \( x_{c_{i-1}+j} \) in the product of the above big bracket is non-negative, hence the element \( S(v) \) is well-defined.

Recall the map \( \hat{\varepsilon}_{k, v_k} \) introduced in Sect. 2 after Lemma 2.7.

Lemma 4.25 Let \( v = (v_1, \ldots, v_n) \in I^\alpha \) be a piecewise dominant sequence with respect to \( \Lambda \), with a decomposition (4.1) satisfying (4.2). Write \( v' = (v_1, \ldots, v_{n-1}) \). Then \( v' \) is also piecewise dominant with respect to \( \Lambda \) and

\[
\hat{\varepsilon}_{R,v^m}(S(v)) = S(v').
\]

Proof The first statement follows from the definition of piecewise dominant sequence. It remains to prove the second statement. Assume \( b_m = 1 \). Then by Definition 4.4 we have \( \ell_m \geq 1 \) and Lemma 2.7,

\[
S(v) = S(v') x_n^{\ell_m-1} = p_{\ell_m-1}(S(v)) x_n^{\ell_m-1}.
\]

Hence, \( \hat{\varepsilon}_{R,v^m}(S(v)) = S(v') \).

Now assume \( b_m > 1 \). We set \( b'_m := (b_1, \ldots, b_m-1) \). Since \( \hat{\varepsilon}_{R,v^m} \) is \( \mathcal{R}_{n-1}^\Lambda \)-linear, we have

\[
\hat{\varepsilon}_{R,v^m}(S(v)) = \prod_{1 \leq i \leq m-1} \left( \tau_{b, i} x_{c_{i-1}+1}^{\ell_{i}-1} x_{c_{i-1}+2}^{\ell_{i}-2} \cdots x_{c_i}^{\ell_i-b_i} \right) e(v')
\]

\[
\times \hat{\varepsilon}_{R,v^m} \left( \tau_{b, m} x_{c_{m-1}+1}^{\ell_m-1} x_{c_{m-1}+2}^{\ell_m-2} \cdots x_n^{\ell_m-b_m} e(v) \right).
\]

It remains to show that

\[
\hat{\varepsilon}_{R,v^m} \left( \tau_{b, m} x_{c_{m-1}+1}^{\ell_m-1} x_{c_{m-1}+2}^{\ell_m-2} \cdots x_n^{\ell_m-b_m} e(v) \right) = \tau_{b', m} x_{c_{m-1}+1}^{\ell_m-1} x_{c_{m-1}+2}^{\ell_m-2} \cdots x_n^{\ell_m-b_m+1} e(v').
\]

By a similar calculation as in the first paragraph of the proof of [5, Lemma 5.6], we obtain

\[
\tau_{b, m} x_{c_{m-1}+1}^{\ell_m-1} x_{c_{m-1}+2}^{\ell_m-2} \cdots x_n^{\ell_m-b_m} e(v) = \mu_{\ell_{m-1}} \tau_{c_{m-1}+1}^{\ell_{m-2}} \cdots \tau_{n-2}^{\ell_{m-b_m}} e(v') \otimes \tau_{b', m} x_{c_{m-1}+1}^{\ell_m-1} x_{c_{m-1}+2}^{\ell_m-2} \cdots x_n^{\ell_m-b_m+1} e(v)
\]

\[
+ \sum_{a_1 + a_2 = \ell_{m-b_m}} \tau_{b', m} x_{c_{m-1}+1}^{a_1} \tau_{c_{m-1}+2}^{a_2} \cdots \tau_{c_{m-1}+j}^{\ell_{m-b_m}} e(v)
\]

\[
= \mu_{\ell_{m-1}} \left( \tau_{c_{m-1}+1}^{\ell_{m-2}} \cdots \tau_{n-2}^{\ell_{m-b_m}} e(v') \otimes \tau_{b', m} x_{c_{m-1}+1}^{\ell_m-1} x_{c_{m-1}+2}^{\ell_m-2} \cdots x_n^{\ell_m-b_m+1} e(v) \right)
\]

\[
+ \sum_{a_1 + a_2 = \ell_{m-b_m}} \tau_{b', m} x_{c_{m-1}+1}^{a_1} \tau_{c_{m-1}+2}^{a_2} \cdots \tau_{c_{m-1}+j}^{\ell_{m-b_m}} e(v),
\]

where in the second equality we used the fact that \( \tau_{b', m} \tau_j = 0 \) for any \( c_{m-1} + 1 \leq j \leq n-2 \), and \( x_{c_{m-1}+1}^{a_1} \tau_{c_{m-1}+2} \cdots \tau_{c_{m-1}+j} \tau_{c_{m-1}+j+1} \in \sum_{j=c_{m-1}+1}^{n-2} \tau_j \mathcal{R}_{\alpha}^\Lambda \) whenever \( a_1 < b_m - 2 \). Now there are two possibilities:
Case 1. $\ell_m - 2b_m > -2$, which corresponds to the case $\lambda_{\nu_n, \alpha - \alpha_{cm}} > 0$ in the notation of Lemma 2.7. In that case, the map $\hat{\epsilon}_{n, v^m}$ picks out the coefficient of $x_n^{\ell_m - 2b_m + 1}$ (i.e., set $a_2 = \ell_m - 2b_m + 1$). We get that

$$\hat{\epsilon}_{n, v^m} (\tau_{b, m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v))$$

$$= \tau_{b', m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v'),$$

where the second equality follows again from the same argument used in the last sentence of the previous paragraph.

Case 2. $\ell_m - 2b_m \leq -2$, which corresponds to the case $\lambda_{\nu_n, \alpha - \alpha_{cm}} \leq 0$ in the notation of Lemma 2.7. Note that

$$\ell_m - b_m - 1 - (b_m - 2) = \ell_m - 2b_m + 1 < 0.$$

By the same argument used in the last sentence of the paragraph above Case 1, we can deduce that

$$\sum_{a_1 + a_2 = \ell_m - b_m - 1 \atop a_1 \geq b_m - 2, a_2 \geq 0} \tau_{b', m} x_{c_{m-1}+1}^{a_1} x_{c_{m-1}+2}^{\ell_{m-1}} x_{c_{m-1}+3}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} x_{a_2} e(v) = 0.$$

For any $0 \leq k \leq -\ell_m + 2b_m - 3 = -(-\ell_m - 2b_m + 2) - 1$, it follows from the same argument as in the last paragraph of the proof of [5, Lemma 5.7],

$$\mu_{\nu_{n-1}}^{k} \left( \tau_{c_{m-1}+1} \cdots \tau_{c_{n-2}} x_{c_{n-1}}^{\ell_{m-b_m}} e(v') \otimes \tau_{b', m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v') \right)$$

$$= \tau_{b', m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v'),$$

Hence by Lemma 2.7,

$$\tilde{z} = \tau_{c_{m-1}+1} \cdots \tau_{c_{n-2}} x_{c_{n-1}}^{\ell_{m-b_m}} e(v') \otimes \tau_{b', m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v'),$$

where

$$\tilde{z} := \mu_{\nu_{n-1}} \left( \tau_{c_{m-1}+1} \cdots \tau_{c_{n-2}} x_{c_{n-1}}^{\ell_{m-b_m}} e(v') \otimes \tau_{b', m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v') \right).$$

By the definition of $\hat{\epsilon}_{n, v^m}$, we have that

$$\hat{\epsilon}_{n, v^m} (\tau_{b, m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v))$$

$$= \mu_{\nu_{n-1}}^{-(-\ell_m - 2b_m + 2)} \left( \tau_{c_{m-1}+1} \cdots \tau_{c_{n-2}} x_{c_{n-1}}^{\ell_{m-b_m}} e(v') \otimes \tau_{b', m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v') \right)$$

$$= \tau_{b', m} x_{c_{m-1}+1}^{\ell_{m-1}} x_{c_{m-1}+2}^{\ell_{m-2}} \cdots x_{c_m}^{\ell_{m-b_m+1}} e(v').$$

This completes the proof of the lemma. 

Recall the symmetrizing form $t_{\Lambda, \alpha} : \mathcal{R}_{\alpha}^{\Lambda} \to K$ introduced in Lemma 2.11.

**Corollary 4.26** Let $v \in \mathcal{L}^\alpha$ be a piecewise dominant sequence with respect to $\Lambda$. Then

$$t_{\Lambda, \alpha}(S(v)) \in K^\times.$$

In particular, $0 \neq S(v) \notin [\mathcal{R}_{\alpha}^{\Lambda}, \mathcal{R}_{\alpha}^{\Lambda}]$ and $0 \neq e(v) \in \mathcal{R}_{\alpha}^{\Lambda}$. 

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Proof This follows from the definition of $t_{\Lambda, \alpha}$ and Lemma 4.25.

Remark 4.27 If $v \in I^{\alpha}$ satisfies the stronger assumption that $v^i \neq v^j$ for any $1 \leq i \neq j \leq m$, then $v$ coincides with $\tilde{v}$ in the notation of [8, (5.1)]. In this special case, the number $\ell_i$ is the same as $N_i^\Lambda(\tilde{v})$ in the notation of [8, Definition 5.2], and the second part of [8, Theorem 5.4] can be reformulated as:

$$e(v) \neq 0 \text{ in } \mathcal{R}_\alpha^\Lambda \text{ if and only if } v \text{ is piecewise dominant with respect to } \Lambda.$$ 

Our Corollary 4.26 says that for those $v$ not satisfying the above stronger assumption, the “if part” of the above statement still holds. But the “only if part” of the above statement may be false. For example, let’s consider the type $A_2$ case again and choose $\Lambda = \Lambda_1 + \Lambda_2$, then [8, Theorem 5.34] implies $e(2, 1, 2) \neq 0$. But it’s easy to check by definition that $(2, 1, 2)$ is not piecewise dominant with respect to $\Lambda$.

Our second main result in this paper is contained in the following two theorems.

Theorem 4.28 Let $K$ be a field of arbitrary characteristic, $\Lambda \in P^*$ and $\alpha \in Q^+$. The following statements are equivalent:

1. $\mathcal{R}_\alpha^\Lambda(K) \neq 0$;
2. There is a piecewise dominant sequence $v \in I^\alpha$ with respect to $\Lambda$.

Proof Let $\mathcal{R}_\alpha^\Lambda(Q)$ be the cyclotomic quiver Hecke algebra defined by the same Cartan datum as $\mathcal{R}_\alpha^\Lambda(K)$ but using certain polynomials $\{Q^i_j(u, v)|i, j \in I\}$ defined over $Q$. By Lemma 2.12, we see that $\mathcal{R}_\alpha^\Lambda(K) \neq 0$ if and only if $\mathcal{R}_\alpha^\Lambda(Q) \neq 0$.

Suppose now $\mathcal{R}_\alpha^\Lambda(K) \neq 0$. Then $\mathcal{R}_\alpha^\Lambda(Q) \neq 0$. In particular,

$$\dim \text{Tr}(\mathcal{R}_\alpha^\Lambda(Q))_{d_{\Lambda, \alpha}} = \dim Z(\mathcal{R}_\alpha^\Lambda(Q))_0 \neq 0.$$ 

Applying Theorem 1.3, we can deduce that there is a piecewise dominant sequence $v \in I^\alpha$ with respect to $\Lambda$.

Conversely, suppose there is a piecewise dominant sequence $v \in I^\alpha$ with respect to $\Lambda$. By Corollary 4.26, we see that $S(v)$ is a non-zero element in $\mathcal{R}_\alpha^\Lambda(K)$ which implies $\mathcal{R}_\alpha^\Lambda(K) \neq 0$.

As an application, we obtain the following criterion for which $\Lambda - \alpha$ is a weight of the irreducible highest weight $g$-module $L(\Lambda)$.

Theorem 4.29 Suppose $L(\Lambda)$ is the irreducible highest weight $g$-module with highest weight $\Lambda$. Then $\Lambda - \alpha$ is a weight of $L(\Lambda)$ if and only if there is a piecewise dominant sequence $v \in I^\alpha$ with respect to $\Lambda$. In that case, if $v \in I^\alpha$ is a piecewise dominant sequence with respect to $\Lambda$, then $f_{v_n}f_{v_{n-1}} \cdots f_{v_1}v_\Lambda \neq 0$ is a nonzero weight vector in $L(\Lambda)_{\Lambda-\alpha}$, where $f_i$ denotes the Chevalley generator of the enveloping algebra $U(g)$ for each $i \in I$.

Proof The first statement follows from the equality $\dim L(\Lambda)_{\Lambda-\alpha} = \# \text{Irr}(\mathcal{R}_\alpha^\Lambda)[10, \text{Theorem 6.2}]$ and Theorem 4.28.

Let $\alpha \in Q^+$. For each $i \in I$, let

$$F_i^\Lambda : \text{Mod}(\mathcal{R}_\alpha^\Lambda) \rightarrow \text{Mod}(\mathcal{R}_{\alpha+\alpha_i}^\Lambda),$$

$$M \mapsto \mathcal{R}_{\alpha+\alpha_i}^\Lambda e(\alpha, i) \otimes_{\mathcal{R}_{\alpha+\alpha_i}^\Lambda} M.$$
be the induction functor introduced in [10]. Let \([F_i^\Lambda]: K(\text{Proj} \mathcal{R}_a^\Lambda) \to K(\text{Proj} \mathcal{R}_{a+\alpha}^\Lambda)\) be the induced map on the Grothendieck group of finite dimensional projective modules. Set \(F_i := q_i^{(\ell, \Lambda, \alpha)}[F_i^\Lambda]\). Then by [10, Theorem 6.2], we have
\[
[F_{\alpha}^\Lambda e(\nu)] = F_{v_0} F_{v_{n-1}} \cdots F_{v_1} [I\Lambda] = f_{v_0} f_{v_{n-1}} \cdots f_{v_1} v_\Lambda.
\]
Finally, by Corollary 4.26, if \(\nu \in I^\alpha\) is a piecewise dominant sequence with respect to \(\Lambda\), then \([F_{\alpha}^\Lambda e(\nu)] \neq 0\). Hence, \(f_{v_0} f_{v_{n-1}} \cdots f_{v_1} v_\Lambda \neq 0\), and it is a nonzero weight vector in \(L(\Lambda)_{\Lambda-\alpha}\).}

\[
\sum_{i \in I^\alpha} e(i) \text{ is a block idempotent of the corresponding cyclotomic Hecke algebra by [14] and [2], and hence a block idempotent of } \mathcal{R}_a^\Lambda.
\]

(3) By [6] or [9, Theorem 1.9], we also know that Conjectures 1.2 and 4.30 hold whenever \(\alpha = \sum_{j=1}^n \alpha_{i_j}\) with \(\alpha_{i_1}, \ldots, \alpha_{i_n}\) pairwise distinct.

For any prime number \(p > 0\), we use \(\hat{\mathbb{Z}}(p)\) to denote the localization of \(\mathbb{Z}\) at its maximal ideal \((p)\), and \(\hat{\mathbb{Z}}(p)\) to denote the completion of \(\mathbb{Z}(p)\) at its unique maximal ideal \(p\mathbb{Z}(p)\). Let \(\hat{\mathbb{Q}}(p)\) be the fraction field of \(\hat{\mathbb{Z}}(p)\). We are now in a position to give the proof of the third main result Theorem 1.5 in this paper.

**Proof of Theorem 1.5** Applying [19, Proposition 2.1(d)], we can assume without loss of generality that \(K = \mathbb{F}_p\), the finite field with \(p\) elements. Since \(\mathbb{Q} \hookrightarrow \hat{\mathbb{Q}}(p)\), Conjecture 4.30 holds for \(\mathcal{R}_a^\Lambda(\mathbb{Q})\) implies that it also holds for \(\mathcal{R}_a^\Lambda(\hat{\mathbb{Q}}(p))\). For any \(\mathcal{O} \in \{\hat{\mathbb{Z}}(p), K, \hat{\mathbb{Q}}(p)\}\), we set
\[
eq (\mathcal{O}) := \sum_{i \in I^\alpha} e(i) \mathcal{O}.
\]

(1) Conjecture 4.30 is a stronger form of the Indecomposability Conjecture 1.2. By [19, Remark 3.41], we know that when \(\text{char} K = 0\), \(\mathfrak{g}\) is symmetric and of finite type, and \(\{Q_{ij}((u, v))| i, j \in I\}\) are given as [19, (11)], Conjecture 4.30 holds.

(2) Let \(\mathfrak{g}\) be of type \(A_\infty\) or affine type \(A_{\ell-1}^{(1)}\) with \(e > 1\) and \((e, p) = 1\), where \(p := \text{char} K\) and \(\{Q_{ij}((u, v))| i, j \in I\}\) are given as [18, §3.2.4]. Then after a finite extension of the ground field \(K\), there is a Brundan-Kleshchev isomorphism between the cyclotomic quiver Hecke algebra \(\mathcal{R}_a^\Lambda\) and the block algebra of the cyclotomic Hecke algebra of type \(G(\ell, 1, n)\) [3] which corresponds to \(\alpha\). In this case, the Indecomposability Conjecture 1.2 holds because \(e(\alpha) := \sum_{i \in I^\alpha} e(i)\) is a block idempotent of the corresponding cyclotomic Hecke algebra by [14] and [2], and hence a block idempotent of \(\mathcal{R}_a^\Lambda\).

(3) By [6] or [9, Theorem 1.9], we also know that Conjectures 1.2 and 4.30 hold whenever \(\alpha = \sum_{j=1}^n \alpha_{i_j}\) with \(\alpha_{i_1}, \ldots, \alpha_{i_n}\) pairwise distinct.
which is a central idempotents in \( \mathcal{R}_a^\Lambda (O) \). In particular, \( e(\alpha)\hat{z}_{p(i)} \) is a lift of \( e(\alpha)_K \) in \( \mathcal{R}_a^\Lambda (K) \). Now Conjecture 4.30 holds for \( \mathcal{R}_a^\Lambda (\hat{Q}(p)) \) means \( e(\alpha)\hat{Q}(p) \) is a central primitive idempotent in \( \mathcal{R}_a^\Lambda (\hat{Q}(p)) \).

Note that

\[
\mathcal{R}_a^\Lambda (K) \cong K \otimes_{\mathbb{Z}(p)} \mathcal{R}_a^\Lambda (\hat{Z}(p)) \cong \mathcal{R}_a^\Lambda (\hat{Z}(p))/m\mathcal{R}_a^\Lambda (\hat{Z}(p)),
\]

where \( m \) is the unique maximal ideal of \( \hat{Z}(p) \). By Corollary 2.13, \( \mathcal{R}_a^\Lambda (\hat{Z}(p)) \) is free over \( \hat{Z}(p) \).

Suppose that \( e(\alpha)_K = e(1) \oplus \cdots \oplus e(k) \) is a decomposition of \( e(\alpha)_K \) into a direct sum of pairwise orthogonal (nonzero) central primitive idempotents in \( \mathcal{R}_a^\Lambda (K) \). Applying [4, Proposition 5.22, Theorem 6.7, §6, Exercise 8], for each \( 1 \leq i \leq k \), we can get a lift \( e(i) \) in \( \mathcal{R}_a^\Lambda (\hat{Z}(p)) \) of \( e(i) \), such that \( \{e(i)|1 \leq i \leq k\} \) is a set of pairwise orthogonal central idempotents in \( \mathcal{R}_a^\Lambda (\hat{Z}(p)) \) and \( \sum_{j=1}^{k} e(i) = e(\alpha)\hat{z}_{p(i)} \). Now we have

\[
e(\alpha)\hat{Q}(p) = (1\hat{Q}(p) \otimes \hat{z}_{p(i)} e(1)) \oplus \cdots \oplus (1\hat{Q}(p) \otimes \hat{z}_{p(i)} e(k)),
\]

is a decomposition of \( e(\alpha)\hat{Q}(p) \) into a direct sum of pairwise orthogonal central elements in \( \mathcal{R}_a^\Lambda (\hat{Q}(p)) \). Moreover,

\[
(1\hat{Q}(p) \otimes \hat{z}_{p(i)} e(j))^2 = (1\hat{Q}(p) \otimes \hat{z}_{p(i)} e(j)), \quad \forall 1 \leq j \leq k.
\]

By the discussion in the last paragraph, \( \mathcal{R}_a^\Lambda (\hat{Z}(p)) \) is a torsion-free \( \hat{Z}(p) \)-module. It follows that the canonical map \( \mathcal{R}_a^\Lambda (\hat{Z}(p)) \to \mathcal{R}_a^\Lambda (\hat{Q}(p)) \) is injective. Thus each \( 1\hat{Q}(p) \otimes \hat{z}_{p(i)} e(j) \) must be nonzero, hence is a (nonzero) central idempotent in \( \mathcal{R}_a^\Lambda (\hat{Q}(p)) \). We get a contradiction to the fact that \( e(\alpha)\hat{Q}(p) \) is a central primitive idempotent in \( \mathcal{R}_a^\Lambda (\hat{Q}(p)) \). This completes the proof of Theorem 1.5.

We end this subsection with the following corollary.

**Corollary 4.32** Let \( K \) be a field of arbitrary characteristic. Suppose that the polynomials \( \{Q_{ij}(u, v)|i, j \in I\} \) are given as [18, §3.2.4], \( g \) is either symmetric and of finite type, or \( g \) is of type \( A_{\infty} \) or affine type \( A_{(1)}^{(1)} \) with \( e > 1 \). Then Conjecture 1.2 holds.

**Proof** By assumption, each \( Q_{ij}(u, v) \) is defined over \( \mathbb{Z} \). If char \( K = 0 \), the theorem holds by [19, Remark 3.41], [3], [14] and [2] (see Remark 4.31). Now applying Theorem 1.5, we can deduce that the theorem still holds if char \( K > 0 \).

**Remark 4.33** We remark that in the second part of the above corollary we do not need any assumption that \( (e, p) = 1 \) when \( e > 1 \), where \( p \) is the characteristic of the ground field \( K \), while the approach of using Brundan-Kleshchev’s isomorphism (cf. Remark 4.31 2)) does need the extra assumption \( (e, p) = 1 \) because otherwise one can not find a primitive \( e \)th root of unity in any field extension of the ground field \( K \).

### 4.3 Relations with crystal basis

Let \( B = B(\Lambda) \) be the crystal base of the integral highest weight \( U_q(g) \)-module \( L(\Lambda) \). For each \( i \in I \), let \( \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\} \) be the corresponding Kashiwara operators, let \( \varepsilon_i, \varphi_i : B \to \mathbb{Z} \). 

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Lemma 4.34 [11, §4.2, (4.3)] For each $i \in I$ and $b \in B$,
$$\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \text{wt}(b) \rangle, \quad \varphi_i(b), \varepsilon_i(b) \geq 0.$$  

Lemma 4.35 Suppose $L(\Lambda)$ is the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\Lambda$.
\begin{equation}
\nu = (v^1, v^1, \ldots, v^m, v^m, \ldots, v^m) \in I^\alpha, \tag{4.36}
\end{equation}

is a piecewise dominant sequence with respect to $\Lambda$, such that $v^j \neq v^{j+1}, \forall 1 \leq j < m$, and $m, b_1, \ldots, b_m \in \mathbb{Z}_+^\perp$ with $\sum_{i=1}^m b_i = n$. Then there is a path in the crystal graph of $B$ of the following form:
\begin{equation}
\begin{array}{cccc}
v_A & \rightarrow & \cdots & \rightarrow \vdots & \rightarrow \cdots & \rightarrow v^n \rightarrow \cdots \rightarrow v_m \rightarrow b.
\end{array}
\end{equation}

We call (4.37) the crystal path associated to the piecewise dominant sequence $\nu = (v_1, \ldots, v_n)$ and define $b_\nu$ to be the endpoint $b$ of the path.

**Proof** Using Lemma 4.34, we can calculate
$$\varphi_{v^1}(b_\nu) = \langle h_{v^1}, \text{wt}(b_\nu) \rangle + \varepsilon_{v^1}(b_\nu) \geq \langle h_{v^1}, \text{wt}(b_\nu) \rangle \geq b_1,$$
where the last inequality follows from the definition of piecewise dominant sequence. It follows that $b_1^{(1)} := \tilde{f}_{v^1}^{b_1} v_A \in B$. Now we have $\text{wt}(b_1^{(1)}) = \text{wt}(v_A) - b_1 \alpha_{v^1} = \Lambda - b_1 \alpha_{v^1}$. It follows that
$$\varphi_{v^2}(b_1^{(1)}) = \langle h_{v^2}, \Lambda - b_1 \alpha_{v^1} \rangle + \varepsilon_{v^2}(b_1^{(1)}) \geq \langle h_{v^2}, \Lambda - b_1 \alpha_{v^1} \rangle \geq b_2,$$
by Lemma 4.34. Hence $b_2^{(2)} := \tilde{f}_{v^2}^{b_2} \tilde{f}_{v^1}^{b_1} v_A \in B$. In general, suppose that $b_t^{(i)} = \tilde{f}_{v^i}^{b_i} \cdots \tilde{f}_{v^1}^{b_1} v_A \in B$ is already defined, where $1 \leq t < m$. Then we have $\text{wt}(b_t^{(i)}) = \Lambda - \sum_{j=1}^t b_j \alpha_{v^j}$. It follows that
$$\varphi_{v^{t+1}}(b_t^{(i)}) = \left( h_{v^{t+1}}, \Lambda - \sum_{j=1}^t b_j \alpha_{v^j} \right) + \varepsilon_{v^{t+1}}(b_t^{(i)}) \geq \left( h_{v^{t+1}}, \Lambda - \sum_{j=1}^t b_j \alpha_{v^j} \right) \geq b_{t+1},$$
by Lemma 4.34 again. Hence $b_{t+1}^{(t+1)} := \tilde{f}_{v^{t+1}}^{b_{t+1}} \cdots \tilde{f}_{v^1}^{b_1} v_A \in B$. By an induction on $t$, we get a path in the crystal graph of $B$ of the form (4.37). \qed

Lemma 4.38 Suppose $L(\Lambda)$ is the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\Lambda$, $b \in B = B(\Lambda)$ with $\text{wt}(b) = \Lambda - \alpha, \alpha \in Q_+^n$. Then for any $i \in I$ satisfying $\varepsilon_i(b) > 0$, there is a path in the crystal graph of $B$ of the following form:
\begin{equation}
\begin{array}{cccc}
v_A & \rightarrow & \cdots & \rightarrow \vdots & \rightarrow \cdots & \rightarrow v_m \rightarrow \cdots \rightarrow v_m \rightarrow b, \tag{4.39}
\end{array}
\end{equation}

such that $v^m = i$ and it is the crystal path associated to the piecewise dominant sequence $\nu \in I^\alpha$ (see (4.36)) with respect to $\Lambda$. 

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Proof Suppose \( \Lambda - \alpha \) is a weight of \( L(\Lambda) \). We use induction on \( |\alpha| \). For any \( b \in \mathcal{B} \) with \( \text{wt}(b) = \Lambda - \alpha \), we can find \( i \in I \) such that \( b_0 := \varepsilon_i(b) > 0 \). Set \( b' := e_i^{b_0}b \). Then \( b' \in \mathcal{B} \) and \( \text{wt}(b') = \text{wt}(b) + b_0\alpha_i \). As a result,

\[
(h_i, \text{wt}(b')) = (h_i, \text{wt}(b)) + 2b_0 = \varphi_i(b) - b_0 + 2b_0 = \varphi_i(b) + b_0 \geq b_0,
\]

where in the second equality we have used Lemma 4.34. By induction hypothesis, there is a path in the crystal graph of \( \mathcal{B} \):

\[
v^{1} \to \cdots \to v^{1} \to \cdots \to v^{m-1} \to \cdots \to v^{m-1} \to b',
\]

where \( v^j \neq v^{j+1} \), \( 1 \leq j < m-1 \), and \( m-1, b_1, \ldots, b_{m-1} \in \mathbb{Z}^{\geq 1} \) with \( \sum_{i=1}^{m-1} b_i = n - b_0 \), such that

\[
v = (v^1, v^1, \ldots, v^1, \cdots, v^{m-1}, v^{m-1}, \cdots, v^{m-1}) \in \mathcal{P}^{\alpha-b_0\alpha_i},
\]

is a piecewise dominant sequence with respect to \( \Lambda \). By construction, \( \varepsilon_i(b') = 0 \). It follows that \( v^{m-1} \neq i \). Concatenating this path with the path \( b' \to \cdots \to b \) we prove the statement. \( \square \)

**Remark 4.40** We remark that the Theorem 4.29 can be deduced from Lemma 4.35 and Lemma 4.38, which also gives a second proof of Theorem 4.28. Although Lemma 4.38 says that for each \( b \in \mathcal{B} \), we can always find a crystal path associated to a piecewise dominant sequence, the crystal path of this kind may not be unique.

For any two piecewise dominant sequences \( \mu, v \in I^\alpha \), we define \( \mu \sim v \) if and only if \( b_\mu = b_v \). In particular, this defines an equivalence relation \( \sim \) on the set \( \mathcal{P} \mathcal{D}_\alpha \) of piecewise dominant sequences in \( I^\alpha \) with respect to \( \Lambda \). Let \( \mathcal{P} \mathcal{D}_\alpha / \sim \) be a set of representatives of all the equivalence classes in \( \mathcal{P} \mathcal{D}_\alpha \). We end this paper with the following conjecture.

**Conjecture 4.41** Let \( \alpha \in \mathbb{Q}_n^+ \). Suppose

\[
\mathcal{P} \mathcal{D}_\alpha / \sim = \{\mu^{(i)}|1 \leq i \leq m\}.
\]

Then the set of elements

\[
\left\{ e(\mu^{(i)}) + [\mathcal{R}_\alpha^\Lambda, \mathcal{R}_\alpha^\Lambda] \mid 1 \leq i \leq m \right\}
\]

forms a \( K \)-basis of the degree 0 component \( \text{Tr}(\mathcal{R}_\alpha^\Lambda)_0 \) of the cocenter of \( \mathcal{R}_\alpha^\Lambda \), and the following set of elements

\[
\left\{ 1_\mathbb{Q} \otimes_{\mathbb{Z}} [\mathcal{R}_\alpha^\Lambda e(\mu^{(i)})] \mid 1 \leq i \leq m \right\}
\]

forms a \( \mathbb{Q} \)-basis of \( \mathbb{Q} \otimes_{\mathbb{Z}} K(\text{Proj} \mathcal{R}_\alpha^\Lambda) \).

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References

1. Ariki, S., Park, E., Speyer, L.: Specht modules for quiver Hecke algebras of type C. Publ. Res. Inst. Math. Sci. 55(3), 565–626 (2019)
2. Brundan, J.: Centers of degenerate cyclotomic Hecke algebras and parabolic category $O$. Represent. Theory 12, 236–259 (2008)
3. Brundan, J., Kleshchev, A.: Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras. Invent. Math. 178, 451–484 (2009)
4. Curtis, C.W., Reiner, I.: Methods of Representation Theory, With Applications to Finite Groups and Orders, vol. I. Wiley Interscience, New York (1981)
5. Hu, J., Liang, X.F.: On the structure of cyclotomic nilHecke algebras. Pac. J. Math. 296(1), 105–139 (2018)
6. Hu, J., Lin, H.: On the center conjecture for the cyclotomic KLR algebras. arXiv:2204.11659 (2022) (preprint)
7. Hu, J., Mathas, A.: Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A. Adv. Math. 225(2), 598–642 (2010)
8. Hu, J., Shi, L.: Graded dimensions and monomial bases for the cyclotomic quiver Hecke algebras. arXiv:2108.05508 (2021) (preprint)
9. Hu, J., Shi, L.: Graded dimensions and monomial bases for the cyclotomic quiver Hecke superalgebras. arXiv:2111.03296 (2021) (preprint)
10. Kang, S.J., Kashiwara, M.: Categorification of highest weight modules via Khovanov–Lauda–Rouquier algebras. Invent. Math. 190, 699–742 (2012)
11. Kashiwara, M.: On crystal bases. In: Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, pp. 157–197. Amer. Math. Soc., Providence (1995)
12. Khovanov, M., Lauda, A.D.: A diagrammatic approach to categorification of quantum groups, I. Represent. Theory 13, 309–347 (2009)
13. Khovanov, M., Lauda, A.D.: A diagrammatic approach to categorification of quantum groups, II. Trans. Am. Math. Soc. 363, 2685–2700 (2011)
14. Lyle, S., Mathas, A.: Blocks of cyclotomic Hecke algebras. Adv. Math. 216, 854–878 (2007)
15. Mathas, A., Tubbenhauer, D.: Subdivision and cellularity for weighted KLR algebras. arXiv:2111.12949 (2021) (preprint)
16. Mathas, A., Tubbenhauer, D.: Cellularity for weighted KLRW algebras of types $B$, $A^{(2)}_2$, $D^{(2)}_2$. arXiv:2201.01998 (2022) (preprint)
17. Rouquier, R.: 2-Kac–Moody algebras. arXiv:0812.5023v1 (2008) (preprint)
18. Rouquier, R.: Quiver Hecke algebras and 2-Lie algebras. Algebr. Colloq. 19, 359–410 (2012)
19. Shan, P., Varagnolo, M., Vasserot, E.: On the center of quiver-Hecke algebras. Duke Math. J. 166(6), 1005–1101 (2017)
20. Varagnolo, M., Vasserot, E.: Canonical bases and KLR algebras. J. Reine Angew. Math. 659, 67–100 (2011)

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