Dual Circumference and Collinear Sets

Vida Dujmović¹ · Pat Morin²

Received: 23 May 2019 / Revised: 28 August 2020 / Accepted: 29 January 2022 / Published online: 12 September 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
We show that, if an $n$-vertex triangulation $G$ of maximum degree $\Delta$ has a dual that contains a cycle of length $\ell$, then $G$ has a non-crossing straight-line drawing in which some set, called a collinear set, of $\Omega(\ell/\Delta^4)$ vertices lie on a line. Using the current lower bounds on the length of longest cycles in cubic 3-connected graphs, this implies that every $n$-vertex planar graph of maximum degree $\Delta$ has a collinear set of size $\Omega(n^{0.8}/\Delta^4)$.

Keywords Planar graphs · Collinear sets · Circumference · Untangling

Mathematics Subject Classification 05C35 · 05C10

1 Introduction

Throughout this paper, all graphs are simple and finite and have at least four vertices. For a planar graph $G$, we say that a set $S \subseteq V(G)$ is a collinear set if $G$ has a non-crossing straight-line drawing in which the vertices of $S$ are all collinear. A plane graph is a planar graph $G$ along with a particular non-crossing drawing of $G$. The dual $G^*$ of a plane graph $G$ is the graph whose vertex set $V(G^*)$ is the set of faces in $G$ and in which $fg \in E(G^*)$ if and only if the faces $f$ and $g$ of $G$ have at least one
edge in common. The circumference, \( c(G) \), of a graph \( G \) is the length of the longest cycle in \( G \). In Sect. 2, we prove the following theorem:

**Theorem 1.1** Let \( G \) be a triangulation of maximum degree \( \Delta \) whose dual \( G^* \) has circumference \( \ell \). Then \( G \) has a collinear set of size \( \Omega(\ell/\Delta^4) \).

The dual of a triangulation is a 3-connected cubic planar graph. The study of the circumference of 3-connected cubic planar graphs has a long and rich history going back to at least 1884 when Tait [27] conjectured that every such graph is Hamiltonian. In 1946, Tait’s conjecture was disproved by Tutte who gave a non-Hamiltonian 46-vertex example [28]. Repeatedly replacing vertices of Tutte’s graph with copies of Tutte’s graph, resulting in a family of graphs. Let \( G \) be a triangulation of maximum degree \( \Delta \). Theorem 1.1 improves to \( \Delta = 8 \).

**Corollary 1.2** Every \( n \)-vertex planar graph of maximum degree \( \Delta \) contains a collinear set of size \( \Omega(n^{0.8}/\Delta^4) \).

It is known that every planar graph \( G \) has a collinear set of size \( \Omega(\sqrt{n}) \) [9, 14]. Corollary 1.2 therefore improves on this bound for bounded-degree planar graphs and, indeed for the family of \( n \)-vertex planar graphs of maximum degree \( \Delta \in O(n^\delta) \), with \( \delta < 0.075 \). For example, the triangulations dual to Grünbaum and Walther’s construction have maximum degree \( \Delta \in O(\log n) \). As discussed below, this implies that there exists \( n \)-vertex triangulations of maximum degree \( O(\log n) \) whose largest collinear set has size \( O(n^{0.9859}) \). Corollary 1.2 implies that every \( n \)-vertex planar graph of maximum degree \( O(\log n) \) has a collinear set of size \( \Omega(n^{0.8}) \).

Recently, Dujmović et al. [15] have shown that every collinear set is free. That is, for any planar graph \( G \), any collinear set \( S \subseteq V(G) \), and any set \( X \subseteq \mathbb{R}^2 \) with \( |X| = |S| \), there exists a non-crossing straight-line drawing of \( G \) in which the vertices of \( S \) are drawn on the points of \( X \). Because of this, collinear sets have immediate applications in graph drawing and related areas. For applications of Corollary 1.2, including untangling [9, 11, 12, 14, 17, 20, 23, 25, 29], column planarity [3, 12, 14, 16], universal point subsets [1, 12–14], and partial simultaneous geometric drawings [2, 4,
7, 14, 16] the reader is referred to Dujmović [14] and Dujmović et al. [15, Sect. 1.1]. Corollary 1.2 gives improved bounds for all of these problems for planar graphs of maximum $\Delta = o(n^{0.075})$.

For example, it is known that every $n$-vertex planar geometric graph can be untangled while keeping some set of $\Omega(n^{0.25})$ vertices fixed [9] and that there are $n$-vertex planar geometric graphs that cannot be untangled while keeping any set of $\Omega(n^{0.4948})$ vertices fixed [10]. Although asymptotically tight bounds are known for paths [11], trees [17], outerplanar graphs [17], planar graphs of treewidth two [25], and planar graphs of treewidth three [12], progress on the general case has been stuck for 10 years due to the fact that the exponent 0.25 comes from two applications of Dilworth’s Theorem. Thus, some substantially new idea appears to be needed. By relating collinear/free sets to dual circumference, the current paper presents an effective new idea. Indeed, Corollary 1.2 implies that every bounded-degree $n$-vertex planar geometric graph can be untangled while keeping $\Omega(n^{0.4})$ vertices fixed. Even for bounded-degree planar graphs, $\Omega(n^{0.25})$ was the best previously-known lower bound. Our work opens two avenues for further progress:

- Lower bounds on the circumference of 3-connected cubic graphs are an active area of research. At the time of writing, the $\Omega(n^{0.8})$ lower bound of Liu et al. [22] is less than a year old. Any further progress on these lower bounds will translate immediately to an improved bound in Corollary 1.2 and all its applications.
- It is possible that the dependence on $\Delta$ can be removed from Theorem 1.2 and Corollary 1.2, thus making these results applicable to all planar graphs, regardless of maximum degree.

2 Proof of Theorem 1.2

Let $G$ be a plane graph. We treat the vertices of $G$ as points, the edges of $G$ as closed curves, and the faces of $G$ as closed sets (so that a face contains all the edges on its boundary and an edge contains both its endpoints). Whenever we consider subgraphs of $G$ we treat them as having the same embedding as $G$. Similarly, if we consider a graph $G'$ that is homeomorphic to $G$ then we assume that the edges of $G'$—each of which represents a path in $G$ whose internal vertices all have degree 2—inherit their embedding from the paths they represent in $G$.

Finally, if we consider the dual $G^*$ of $G$ then we treat it as a plane graph in which each vertex $f$ is represented as a point in the interior of the face $f$ of $G$ that it represents. The edges of $G^*$ are embedded so that an edge $fg$ is contained in the union of the two faces $f$ and $g$ of $G$, it intersects the interior of exactly one edge of $G$ that is common to $f$ and $g$, and this intersection consists of a single point.

A proper good curve $C$ for a plane graph $G$ is a Jordan curve with the following properties:

proper: for any edge $xy$ of $G$, $C$ either contains $xy$, intersects $xy$ in a single point (possibly an endpoint), or is disjoint from $xy$; and

\[ \text{proper: for any edge } xy \text{ of } G, \text{ } C \text{ either contains } xy, \text{ intersects } xy \text{ in a single point (possibly an endpoint), or is disjoint from } xy; \text{ and} \]
good: $C$ contains at least one point in the interior of some face of $G$.

Da Lozzo et al. [12] show that proper good curves define collinear sets:

**Theorem 2.1** In a plane graph $G$, a set $S \subseteq V(G)$ is a collinear set if and only if there is a proper good curve for $G$ that contains $S$.

For a triangulation $G$, let $v(G)$ denote the size of a largest collinear set in $G$. We will show that, for any triangulation $G$ of maximum degree $\Delta$ whose dual is $G^*$, $v(G) = \Theta(c(G^*)/\Delta^4)$ by relating proper good curves in $G$ to cycles in $G^*$.

As shown by Ravsky and Verbitsky [24, 25], the inequality $v(G) \leq c(G^*)$ is easy: If $G$ is a triangulation that has a proper good curve $C$ containing $k$ vertices, then a slight deformation of $C$ produces a proper good curve that contains no vertices. This curve intersects a cyclic sequence of faces $f_0, \ldots, f_{k'-1}$ of $G$ with $k' \geq k$. In this sequence, $f_i$ and $f_{(i+1) \mod k'}$ share an edge, for every $i \in \{0, \ldots, k'-1\}$, so this sequence is a closed walk in the dual $G^*$ of $G$. The properness of the original curve and the fact that each face of $G$ is a triangle ensures that $f_i \neq f_j$ for any $i \neq j$, so this sequence is a cycle in $G^*$ of length $k' \geq k$. Therefore, $c(G^*) \geq v(G)$. From the result of Grünbaum and Walther described above, this implies that there are $n$-vertex triangulations $G$ such that $v(G) = O(n^{0.9859})$.

The other direction, lower-bounding $v(G)$ in terms $c(G^*)$ is more difficult. Not every cycle $C$ of length $\ell$ in $G^*$ can be easily transformed into a proper good curve containing a similar number of vertices. In the next section, we describe three parameters $\tau$, $\rho$, and $\kappa$ of a cycle $C$ in $G^*$ and show that $C$ can always be transformed into a proper good curve containing $\Omega(\kappa)$ vertices of $G$.

### 2.1 Faces that are Touched, Pinched, and Caressed

Throughout the remainder of this paper, $G$ is a triangulation whose dual is $G^*$ and $C$ is a cycle in $G^*$. Refer to Fig. 1 for the following definitions. We say that a face $f$ of $G^*$

- is **touched** by $C$ if $f \cap C \neq \emptyset$;
- is **pinched** by $C$ if $f \cap C$ is a cycle or has more than one connected component; and
- is **caressed** by $C$ if it is touched but not pinched by $C$.

Since $C$ is almost always the cycle of interest, we will usually say that a face $f$ of $G^*$ is touched, pinched, or caressed, without specifically mentioning $C$. We will frequently use the values $\tau$, $\rho$, and $\kappa$ to denote the number of faces of $G^*$ in some region that are $\tau$ touched, $\rho$ pinched, or $\kappa$ caressed. Observe that, since every face that is touched is either pinched or caressed, we have the identity $\tau = \rho + \kappa$.

**Lemma 2.2** If $C$ caresses $\kappa$ faces of $G^*$ then $G$ has a proper good curve that contains at least $\kappa/4$ vertices so, by Theorem 2.1, $v(G) \geq \kappa/4$.

**Proof** Let $F$ be the set of faces in $G^*$ that are caressed by $C$. Each element $u \in F$ corresponds to a vertex of $G$ so we will treat $F$ as a set of vertices in $G$. Consider the subgraph $G[F]$ of $G$ induced by $F$. The graph $G[F]$ is planar and has $\kappa$ vertices.
Therefore, by the 4-Colour Theorem [26], $G[F]$ contains an independent set $F' \subseteq F$ of size at least $\kappa/4$.

We claim that there is a proper good curve for $G$ that contains all the vertices in $F'$. To see this, first observe that the cycle $C$ in $G^*$ already defines a proper good curve (that does not contain any vertices of $G$) that we also call $C$. We perform local modifications on $C$ so that it contains all the vertices in $F'$.

For any vertex $u \in F'$, let $w_0, \ldots, w_{d-1}$ denote the neighbours of $u$ in cyclic order. The curve $C$ intersects some contiguous subsequence $uw_i, \ldots, uw_j$ of the edges adjacent to $u$. Since $u$ is caressed, this sequence does not contain all edges incident to $u$. Therefore, the curve $C$ crosses the edge $w_{i-1}w_i$, then crosses $uw_i, \ldots, uw_j$, and then crosses the edge $w_jw_{j+1}$. We modify $C$ by removing the portion between the first and last of these crossings and replacing it with a curve that contains $u$ and is contained in the two faces $w_{i-1}uw_i$ and $w_jw_{j+1}$. (See Fig. 2.)

After performing this local modification for each $u \in F'$ we have a curve $C'$ that contains every vertex $u \in F'$. All that remains is to verify that $C'$ is good and proper for $G$. That $C'$ is good for $G$ is obvious. That $C'$ is proper for $G$ follows from the following two observations: (i) $C'$ does not contain any two adjacent vertices (since $F'$ is an independent set); and (ii) if $C'$ contains a vertex $u$, then it does not intersect the interior of any edge incident to $u$.

Lemma 2.2 reduces our problem to finding a cycle in $G^*$ that caresses many faces. It is tempting to hope that any sufficiently long cycle in $G^*$ caresses many faces, but this is not true; Fig. 3 shows that even a Hamiltonian cycle $C$ in $G^*$ may caress only four faces, two inside $C$ and two outside of $C$. In this example, there is an obvious sequence of faces $f_0, \ldots, f_k$, all contained in the interior of $C$ where $f_i$ shares an edge with $f_{i+1}$ for each $i \in \{0, \ldots, k-1\}$. The only caressed faces in the interior of $C$ are the endpoints $f_0$ and $f_k$ of this sequence.

Our strategy is to define a tree structure, $T_0$ on groups of faces contained in the interior of $C$ and a similar structure, $T_1$ on groups of faces in the exterior of $C$. We
will then show that every leaf of $T_0$ or $T_1$ contains a face caressed by $C$. In Fig. 3, the tree $T_0$ is the path $f_0, \ldots, f_k$ and, indeed, the leaves $f_0$ and $f_k$ of this tree are caressed by $C$. After a non-trivial analysis of the trees $T_0$ and $T_1$, we will eventually show that, if $C$ does not caress many faces, then $T_0$ and $T_1$ have many nodes, but few leaves. Therefore $T_0$ and $T_1$ have many degree-2 nodes. This abundance of degree-2 nodes makes it possible to perform a surgery on $C$ that increases the number of caressed faces. Performing this surgery repeatedly will then produce a curve $C$ that caresses many faces.

A path $P = v_1, \ldots, v_r$ in $G^*$ is a chord path (for $C$) if $v_1, v_r \in V(C)$ and $v_2, \ldots, v_{r-1} \notin V(C)$. Note that this definition implies that the interior vertices $v_2, \ldots, v_{r-1}$ of $P$ are either all contained in the interior of $C$ or all contained in the exterior of $C$.

**Lemma 2.3** Let $P$ be a chord path for $C$ and let $L$ and $R$ be the two faces of the graph formed by $P \cup C$ that each contain $P$ in their boundary. Then $R$ contains at least one face of $G^*$ that is caressed by $C$.

**Proof** The proof is by induction on the number, $t$, of faces of $G^*$ contained in $R$. If $t = 1$, then $R$ is a face of $G^*$ and it is caressed by $C$.

If $t > 1$, then consider the face $f$ of $G^*$ that is contained in $R$ and has the first edge of $P$ on its boundary. Refer to Fig. 4. Since $t > 1$, $X = R \setminus f$ is non-empty. The set $X$ may have several connected components $X_1, \ldots, X_k$, but each $X_i$ has a boundary that contains a chord path $P_i$ for $C$. We can therefore apply induction on $P_1$ (or any $P_i$) using $R = X_1$ in the inductive hypothesis. \(\square\)
2.2 Auxiliary Graphs and Trees: $H, \tilde{H}, T_0, \text{ and } T_1$

Refer to Fig. 5. Consider the auxiliary graph $H$ with vertex set $V(H) \subseteq V(G^*)$ and whose edge set consist of the edges of $C$ plus those edges of $G^*$ that belong to any face pinched by $C$. Let $v_0, \ldots, v_{t-1}$ be the clockwise cyclic sequence of vertices on some face $f$ of $G^*$ that is pinched by $C$. We identify three kinds of vertices that are special with respect to $f$ (see Fig. 6).

- A vertex $v_i$ is special of type A if $v_{i-1}v_i$ is an edge of $C$ and $v_iv_{i+1}$ is not an edge of $C$.
- A vertex $v_i$ is special of type B if $v_{i-1}v_i$ is not an edge of $C$ and $v_iv_{i+1}$ is an edge of $C$.
- A vertex $v_i$ is special of type Y if $v_i$ is not incident to any edge of $C$ and $v_i$ has degree 3 in $H$.

We say that a chord path $v_i, \ldots, v_j$ is a keeper with respect to $f$ if $v_i$ is special of type A, $v_j$ is special of type B, and none of $v_{i+1}, \ldots, v_{j-1}$ are special. We let $\tilde{H}$ denote the subgraph of $H$ containing all the edges of $C$ and all the edges of all paths that are keepers with respect to some pinched face $f$ of $G^*$.

It is worth emphasizing at this point that, by definition, every keeper is entirely contained in the boundary of at least one face $f$ of $G^*$. This property will be useful shortly.

Let $\tilde{H}'$ denote the graph that is homeomorphic to $\tilde{H}$ but does not contain any degree-2 vertices. That is, $\tilde{H}'$ is the minor of $\tilde{H}$ obtained by repeatedly contracting an edge incident a degree-2 vertex. The graph $\tilde{H}'$ naturally inherits an embedding from the embedding of $\tilde{H}$. This embedding partitions the edges of $\tilde{H}'$ into three sets:

- the set $B$ of edges that are contained in (the embedding of) $C$;
- the set $E_0$ of edges whose interiors are contained in the interior of (the embedding of) $C$; and
- the set $E_1$ of edges whose interiors are contained in the exterior of (the embedding of) $C$.

Observe that, for each $i \in \{0, 1\}$, the graph $H_i$ whose edges are exactly those in $B \cup E_i$ is outerplanar, since all vertices of $H_i$ are on a single face, whose boundary is $C$. Let $H_i^*$ be dual of $H_i$ and let $T_i$ be the subgraph of $H_i$ whose edges are all those dual to the edges of $E_i$. From the outerplanarity of $H_i$, it follows that $T_i$ is a tree.

Each vertex of $T_i$ corresponds to a face of $\tilde{H}$. From this point onwards, we will refer to the vertices of $T_i$ as nodes to highlight this fact, so that a node $u$ of $T_i$ is...
Fig. 5 | a the cycle $C$ in $G^*$ with faces classified as pinched or caressed; b the auxiliary graph $H$; c the auxiliary graph $\tilde{H}$ with keeper paths highlighted; d the trees $T_0$ and $T_1$

synonymous with the subset of $\mathbb{R}^2$ contained in the corresponding face of $\tilde{H}$. In the following, when we say that a node $u$ of $T_i$ contains a face $f$ of $G^*$ we mean that $f$ is one of the faces of $G^*$ whose union makes up $u$. The degree, $\delta_u$ of any node $u$ in $T_i$ is exactly equal to the number of keeper paths on the boundary of $u$.

The following lemma allows us to direct our effort towards proving that one of $T_0$ or $T_1$ has many leaves.

**Lemma 2.4** Each leaf $u$ of $T_i$ contains at least one face of $G^*$ that is caressed by $C$.

**Proof** The edge of $T_i$ incident to $u$ corresponds to a chord path $P$. The graph $P \cup C$ has two faces with $P$ on its boundary, one of which is $u$. The lemma now follows immediately from Lemma 2.3, with $R = u$. \(\square\)

We will make use of the following well-known property of 3-connected plane graphs.

**Lemma 2.5** If $G$ has $n \geq 4$ vertices then any two faces of $G^*$ share at most one edge.

**Proof** Suppose that two faces $f$ and $g$ share two edges $e_1$ and $e_2$. Then $e_1$ and $e_2$ form an edge cutset of $G^*$. If $G^*$ contains at least four vertices, then two of the endpoints of $e_1$ and $e_2$ form a vertex cutset of $G^*$ of size 2, contradicting the fact that $G^*$ is 3-connected. That $G^*$ contains at least four vertices follows from Euler’s formula, which gives the number of vertices in $G^*$ as $2n - 4 \geq 4$ for all $n \geq 4$. \(\square\)
Note that, as should be evident from Fig. 6, the number of faces in \( \tilde{H} \) is not lower bounded by any function of the number of faces in \( H \) and therefore the number of nodes in \( T_0 \) and \( T_1 \) is not lower bounded by any function of \( \ell \). Indeed, a single face of \( \tilde{H} \) may contain arbitrarily many faces of \( G^* \) that are touched by \( C \). The following important lemma shows that, when this happens, the corresponding node in \( T_0 \) or \( T_1 \) either has high degree or contains many faces of \( G^* \) that are caressed by \( C \). The latter case is obviously good for our purposes. The former case is also good because a vertex of degree \( \delta \) in any tree creates \( \delta - 2 \) leaves and, by Lemma 2.4, each leaf contains at least one caressed face.

For a node \( u \) of \( T_i \), we let \( \tau_u, \rho_u, \kappa_u, \) and \( \delta_u \) denote the number of touched face of \( T^* \) in \( u \), pinched faces of \( G^* \) in \( u \), the number of caressed faces of \( G^* \) in \( u \), and the degree of \( u \) in \( T_i \), respectively.

**Lemma 2.6** For any node \( u \) of \( T_i \), \( \rho_u \leq 2(\kappa_u + \delta_u) \).

Before proving Lemma 2.6, we point out that the leading constant 2 is tight. Figure 7 shows an example in which all \( \rho_u = 2k + 1 \) pinched faces of \( G^* \) are contained in a single (pink) node \( u \) of \( T_0 \) that contains \( \kappa_u = 0 \) caressed faces and has degree \( \delta_u = k + 2 \).

**Proof of Lemma 2.6** The proof is a discharging argument. We assign each pinched face in \( u \) a single unit of charge, so that the total charge is \( \rho_u \). We then describe a discharging
procedure that preserves the total charge and such that, after executing this procedure, the following conditions are satisfied:

(Post1) Each pinched face has no charge.
(Post2) Each caressed face has charge at most 2.
(Post3) Each keeper path has charge at most 2.

Since there is a bijection between keeper paths in \( u \) and edges of \( T_i \) incident to \( u \), this proves the result.

The discharging procedure is made up of two routines, an initialization procedure and a recursive procedure. The recursive procedure takes inputs \((L, R, P, c)\), where \( P \) is a chord path, \( L, R \subseteq u \), \( L \cap R = P \), \( L \) contains at least one face of \( G^\ast \), and \( 0 \leq c \leq 2 \) is a charge that we think of as resting on \( P \). The input \((L, R, P, c)\) must satisfy the following conditions:

(Pre1) Each face of \( G^\ast \) in \( L \) that shares an edge with \( P \) is pinched.
(Pre2) If \( c > 1 \) then \( P \) is contained in the boundary of a single face of \( G^\ast \) that is contained in \( L \).

The procedure guarantees that, after its completion, the charge of \( c \) that was resting on \( P \) has been moved into \( R \), any other charges in \( L \) are undisturbed, and the faces contained in \( R \) satisfy (Post1)–(Post3) but does not modify charges on faces and keeper paths in \( R \).

Before defining the recursive procedure itself, we will show how it is used by the initialization procedure. This initialization procedure takes an arbitrary pinched face \( f \) contained in \( u \). Since \( f \) is pinched, it has \( r \geq 2 \) chord paths \( P_1, \ldots, P_r \) on its boundary. For each \( i \in \{1, \ldots, r\} \), let \( L_i^- \) be the component of \( u \setminus P_i \) that contains \( f \), let \( L_i = L_i^- \cup P_i \), and let \( R_i = u \setminus L_i^- \). This initialization procedure guarantees that, after it runs, all the faces and chord paths in \( u \setminus R_1 \) satisfy (Post1)–(Post3) but does not modify charges on faces and keeper paths in \( R_1 \).

The initialization procedure works as follows: Since \( f \) is pinched it has a charge of 1 so we move the charge from \( f \) onto \( P_2 \) and apply the recursive procedure to \((L_2, R_2, P_2, 1)\). Since \( f \) is pinched and is the only face of \( L_2 \) that shares an edge with \( P_2 \), this satisfies (Pre1). Since the final argument \( c = 1 \), (Pre2) is vacuously satisfied. Once the recursive procedure is complete, conditions (Post1)–(Post3) are satisfied for all faces in \( f \cup R_2 \).

Next, we apply the recursive procedure on \((L_i, R_i, P_i, 0)\) for each \( i \in \{3, \ldots, r\} \). Since \( f \) is a pinched face, this satisfies (Pre1) and since the final argument \( c = 0 \) this satisfies (Pre2). Once the recursive procedure is complete conditions (Post1)–(Post3) are satisfied for all faces in \( f \cup R_i \) and does affect any charges in \( R_1 \).
Since every face and keeper path contained in $u$ is contained in $R_i$ for at most one $i$, the initialization procedure produces a distribution of charges that satisfies (Post1)--(Post3) for $u \setminus R_1$, as required.

Next we describe the recursive discharging procedure that takes $(L, R, P, c)$ satisfying (Pre1) and (Pre2) and moves charges in $R$, and the charge $c$ resting on $P$, so that they satisfy (Post1)--(Post3). There are several cases to consider (see Fig. 8):

1. $R$ contains no face of $G^*$ that is pinched by $C$. If $R$ contains no face of $G^*$ at all, then $R = P$ is a keeper path, in which case we leave a charge of $c$ on it and we are done. Otherwise $R$ contains at least one face of $G^*$ and Lemma 2.3 ensures that $R$ contains at least one caressed face $f$. We move the charge from $P$ onto $f$ and we are done.

2. $R$ contains a face $f$ of $G^*$ that is pinched by $C$ and that shares at least one edge with $P$. We consider three subcases, each illustrated in Fig. 8:

2.a $f$ contains neither endpoint of $P$. In this case, $R \setminus f$ has two distinct components, $R_1^-$ and $R_2^-$ each containing a distinct endpoint of $P$. For each $i \in \{1, 2\}$,
let $P_i$ be the chord path that separates $R_i^-$ from $u \setminus R_i^-$. Since $f$ is pinched, $f$ contains $r \geq 3$ chord paths $P_1, \ldots, P_r$. Indeed, if $P_1$ and $P_2$ were the only chord paths on $f$, then $f$ would be caressed. For each $i \in \{1, \ldots, r\}$, let $L_i^- = u \setminus P_i$, let $L_i = L_i^- \cup P_i$, and let $R_i = u \setminus L_i^-$. We split the charge $c$ on $P$ evenly between $P_1$ and $P_2$ and apply the recursive procedure on $(L_1, R_1, P_1, c/2)$ for each $i \in \{1, 2\}$. Next, we move the charge on $f$ to $P_3$ and apply the recursive procedure on $(L_3, R_3, P_3, 1)$. Finally, we apply the recursive procedure on $(L_i, R_i, P_i, 0)$ for each $i \in \{4, \ldots, r\}$. The recursive call $(L_1, R_1, P_1, c/2)$ satisfies (Pre1) because the path $P_1$ used in this recursive call is contained in the boundary of $f$ and $P$. In particular each face of $G^*$ contained in $u \setminus R_1$ that is incident to $P_1$ is either in $L$ and incident to $P$ or is the face $f$. The latter faces are pinched by (Pre1) and $f$ is pinched by definition. The recursive call on $(L_1, R_1, P_1, c/2)$ also satisfies (Pre2) since $c \leq 2$, so $c/2 \leq 1$. The same argument shows that the recursive call on $(L_2, R_2, P_2, c/2)$ satisfies (Pre1) and (Pre2). For each $i \in \{3, \ldots, r\}$, the recursive call on $(L_i, R_i, P_i, 1)$ satisfies (Pre1) because $P_i$ is contained in $f$ and $f$ is pinched and satisfies (Pre2) because the final argument is 1 for $i = 3$ and 0 for $i \in \{4, \ldots, r\}$.

2.b $f$ contains exactly one endpoint of $P$. In this case, $R \setminus f$ has one connected component $R_1^-$ that contains an endpoint of $P$. Since $f$ is pinched, $f$ has $r \geq 2$ chord paths $P_1, \ldots, P_r$ on its boundary, where $P_1$ separates $R_1^-$ from $u \setminus R_1$. Define $L_1, \ldots, L_r, R_1, \ldots, R_r$, and $P_2, \ldots, P_r$ as in the previous case. Because $f$ is pinched, it has one unit of charge on it, that we move onto $P_1$ before calling the recursive procedure on $(L_1, R_1, P_1, 1)$. This satisfies (Pre1) for the same reasons described in the previous case and satisfies (Pre2) because the final argument is 1. The path $P$ has a charge $c \leq 2$ which we move onto $P_2$ and call the recursive procedure on $(L_2, R_2, P_2, c)$. This recursive call satisfies (Pre1) because $f$ is pinched and it satisfies (Pre2) because $P_2$ is entirely contained in the boundary of $f$. Finally, for each $i \in \{3, \ldots, r\}$, we call the recursive procedure on $(L_i, R_i, P_i, 0)$. Clearly each of these calls also satisfies (Pre1) and (Pre2).

2.c $f$ contains both endpoints of $P$. We claim that, in this case, $P$ must be on the boundary of more than one face in $L$, otherwise $P$ would be a keeper path. To see this, observe that the face $f$ contains both the first edge $e_1$ and last edge $e_2$ of $P$. If $e_1 = e_2$ because $P$ is a single edge, then it is certainly a keeper, which is not possible since $P$ is in the interior of $u$. Otherwise, by Lemma 2.5, $e_1$ and $e_2$ are on the boundary of two different faces in $L$. Therefore, by (Pre2) $P$ has $c \leq 1$ units of charge assigned to it. Now, since $f$ is pinched, it has $r \geq 1$ chord paths $P_1, \ldots, P_r$, other than $P$ on its boundary. Define $L_1, \ldots, L_r$ and $R_1, \ldots, R_r$ as in the previous two cases. Now, $P$ has a charge $c \leq 1$ and, since it is pinched, $f$ has a charge of 1. We move these $c + 1$ units of charge from $P$ and $f$ onto $P_1$ and call the recursive procedure on $(L_1, R_1, P_1, c + 1)$. This satisfies (Pre1) since $f$ is pinched and satisfies (Pre2) since $P_1$ is entirely contained in the boundary of $f$. For each $i \in \{2, \ldots, r\}$ we then call the recursive procedure on $(L_i, R_i, P_i, 0)$. Clearly each of these calls satisfies (Pre1) and (Pre2).
3. $R$ contains at least one pinched face of $G^*$, but no pinched face in $R$ shares an edge with $P$. We claim that there is a single face, $g$ of $H$, contained in $R$, that contains all of $P$ on its boundary. Indeed, edges of $G^*$ not in $C$ are in $H$ only if they are on the boundary of some pinched face of $G^*$. Since no pinched face of $G^*$ in $R$ shares an edge with $P$, none of the edges incident to internal vertices of $P$ and contained in $R$ are part of $H$. Therefore, $P$ is on the boundary of a single face of $H$ that is contained in $R$. Let $f$ be the face of $G^*$ that is contained in $R$ and that contains the first edge of $P$. The face $f$ is touched by $C$ but not pinched, so it must be caressed. We move the $c$ units of charge from $P$ onto $f$. Since $g$ is a face of $H$, each edge of $g$ is either in $C$ or on the boundary of some face of $G^*$ that is pinched by $C$. Since $R$ contains at least one pinched face of $G^*$, there is at least one edge of $G$ that is not in $P$ or in $C$. Therefore, $R$ contains $k \geq 1$ pinched faces $f_1, \ldots, f_k$ of $G^*$, such that each $f_i$ shares part of a chord path $P_i$ with $g$. Consider one such $f_i$ and observe that $u \setminus f_i$ has $r_i \geq 2$ chord paths $P_{i,1}, \ldots, P_{i,r_i}$ on its boundary and use the convention that $P_{i,1} = P_i$. Define $L_{i,1}, \ldots, L_{i,r_i}$ and $R_{i,1}, \ldots, R_{i,r_i}$ in a manner analogous to $L_1, \ldots, L_r$ and $R_1, \ldots, R_r$ in the previous cases. On each such face $f_i$, we run the initialization procedure and this reorganizes the charges in $u \setminus R_{i,1}$ so that they satisfy (Post1)–(Post3) and does not modify charges in $L \cup g$. Doing this for each $i \in \{1, \ldots, k\}$ completes the description of the discharging procedure.

To complete the proof, first observe that if $u$ contains no pinched faces then the result is trivially true. Otherwise $u$ contains a pinched face $f$ such that one of the components $R_1$ of $u \setminus f$ contains no pinched faces. (The existence of such an $f$ is established by choosing $f$ so that the minimum number of faces in any component of $u \setminus f$ is minimum over all pinched faces $f$ in $u$.) Since $R_1$ contains no pinched faces, it contains no charges, so it already satisfies (Post1)–(Post3). Running the initialization procedure on $f$ will then redistribute charges so that they satisfy (Post1)–(Post3) for all faces and keeper paths in $u$.

\[\square\]

### 2.3 Bad Nodes

We say that a node of $T_i$ is **bad** if it has degree 2 and contains no face of $G^*$ that is caressed by $C$. We now move from studying individual nodes of $T_0$ and $T_1$ to studying global quantities associated with $T_0$ and $T_1$. From this point on, for each $i \in \{0, 1\}$,

- $\tau_i$, $\rho_i$, and $\kappa_i$ refer the total numbers of faces contained in nodes of $T_i$ that are touched, pinched, and caressed by $C$, respectively;
- $n_i$ refers to the number of nodes of $T_i$;
- $\delta_i = 2(n_i - 1)$ is the total degree of all nodes in $T_i$; and
- $b_i$ is the number of bad nodes in $T_i$.

**Lemma 2.7** If $\kappa_i \leq \tau_i/6$ then $n_i \geq \tau_i/8$.

**Proof** From Lemma 2.6 we know $\rho_i \leq 2(\kappa_i + \delta_i)$, so

$$\tau_i = \kappa_i + \rho_i \leq 3\kappa_i + 2\delta_i = 3\kappa_i + 4(n_i - 1) \leq \tau_i/2 + 4n_i,$$

\[\square\]
and reorganizing the left- and right-hand sides gives the desired result. □

**Lemma 2.8** For any $0 < \epsilon < 1$, if $b_i \leq (1 - \epsilon)n_i$, then $\kappa_i \geq \epsilon \tau_i / 24$.

**Proof** Partition the nodes of $T_i$ into the following sets:

- the set $B$ of bad nodes;
- the set $N_1$ of leaves;
- the set $N_{\geq 3}$ of nodes having degree at least 3;
- the set $N_2$ of nodes having degree 2 that are not bad.

Then

$$b_i = n_i - |N_1| - |N_{\geq 3}| - |N_2|$$

$$> n_i - 2|N_1| - |N_2| \quad \text{(since } |N_1| > |N_{\geq 3}|)$$

$$\geq n_i - 2\kappa_i - |N_2| \quad \text{(since, by Lemma 2.4, } \kappa_i \geq |N_1|)$$

$$\geq n_i - 3\kappa_i. \quad \text{(since each node in } N_2 \text{ contains a caressed face)}$$

Thus, we have

$$n_i - 3\kappa_i \leq b_i \leq (1 - \epsilon)n_i$$

and rewriting gives

$$\kappa_i \geq \frac{\epsilon n_i}{3}. \quad (1)$$

If $\kappa_i \geq \tau_i / 6$, then the proof is complete since $\tau_i / 6 > \tau_i / 24$. On the other hand, if $\kappa_i \leq \tau_i / 6$ then, by Lemma 2.7, $n_i \geq \tau_i / 8$. Combining this with (1) gives

$$\kappa_i \geq \frac{\epsilon n_i}{3} \geq \frac{\epsilon \tau_i}{24}. \quad \Box$$

**2.4 Interactions Between Bad Nodes**

We have now reached a point in which we know that the vast majority of nodes in $T_0$ and $T_1$ are bad nodes, otherwise Lemma 2.8 implies that a constant fraction of the faces touched by $C$ are caressed by $C$. At this point, we are ready to study interactions between bad nodes of $T_0$ and bad nodes of $T_1$.

**Lemma 2.9** If $u$ is a bad node then $u$ is a face of $G^*$.

**Proof** First observe that, since $u$ is bad, it has degree 2, so $C \cap u$ has exactly two connected components $C_1$ and $C_2$. Thus $u$’s boundary consists of $C_1$, $C_2$, and two chord paths $P_1$ and $P_2$. We first argue that there is a single face $g$ of $G^*$ that contains $C_1 \cup C_2$. If not, then $G^*$ must contain a path $P$ whose interior is in $u$ and has both endpoints on the boundary of $u$. There are a few cases to consider:

1. $P$ has both endpoints on $C_i$ for some $i \in \{1, 2\}$. In this case, $P$ is a chord path and, by Lemma 2.3, $u$ contains a face that is caressed by $C$, contradicting the assumption that $u$ is a bad node.
2. \( P \) has one endpoint on \( C_i \) and one endpoint on \( P_j \) for some \( i, j \in \{1, 2\} \). In this case, \( P \cup P_j \) contains a chord path with both endpoints on \( C_i \), again contradicting the assumption that \( u \) is a bad node.

3. \( P \) has one endpoint on \( P_1 \) and one endpoint on \( P_2 \). In this case, \( P \cup P_1 \cup P_2 \) contains a chord path with both endpoints on \( C_1 \), again contradicting the assumption that \( u \) is a bad node.

4. \( P \) has one endpoint on \( C_1 \) and one endpoint on \( C_2 \). The path \( P \) is not a keeper, otherwise it would have split \( u \) into two nodes. Therefore, it must be the case that \( P \) contains an internal vertex. Let \( S_1 \) be the set of internal vertices of \( P \) and let \( S_2 \) be the set of vertices on the boundary of \( u \), not including the endpoints of \( P \). Since \( G^* \) is 3-connected, there is a path from \( S_1 \) to \( S_2 \) that does not contain either endpoint of \( P \). The shortest such path, \( P' \), does not contain any edges of \( P \). Again, using portions of \( P, P_1, P_2, \) and \( P' \) we can construct a chord path, contained in \( u \), with both endpoints on \( C_1 \) or both endpoints on \( C_2 \), contradicting the assumption that \( u \) is a bad node.

This establishes that \( C_1 \cup C_2 \) is contained in the boundary of a single face \( g \) of \( G^* \). The boundary of \( g \) contains two disjoint paths \( P'_1 \) and \( P'_2 \) joining \( C_1 \) and \( C_2 \). We claim that \( P'_i \) is a keeper path, for each \( i \in \{1, 2\} \). Indeed, each internal \( x \) vertex of \( P'_i \) is either a vertex of \( P_j \) or is on the boundary of three faces: \( g \) and two faces that are not touched by \( C \). In either case, \( x \) is not special of type \( Y \). Therefore \( P'_i \) has endpoints that are special of type \( A \) and type \( B \) with respect to the pinched face \( g \) and has no internal
vertices that are special of type Y, so $P'_i$ is a keeper. Therefore $\{P'_1, P'_2\} = \{P_1, P_2\}$ since, otherwise $f$ would not be a face of $\tilde{H}$. Therefore $g = f$ so $f$ is a face of $G^*$. □

The following lemma shows that a bad node $u$ in $T_0$ and a bad node $w$ in $T_1$ share at most one edge of $C$.

**Lemma 2.10** Any two bad nodes $u$ of $T_i$ and $w$ of $T_j$ have at most one edge in common.

**Proof** By Lemma 2.9, $u$ and $w$ are each faces of $G^*$. Therefore, by Lemma 2.5, $u$ and $w$ share at most one edge. □

### 2.5 Really Bad Nodes

At this point we will start making use of the assumption that the triangulation $G$ has maximum degree $\Delta$, which is equivalent to the assumption that each face of $G^*$ has at most $\Delta$ edges on its boundary.

**Observation 2.11** If $G$ has maximum degree $\Delta$ and $C$ has length $\ell$, then the number of faces $\tau$ of $G^*$ touched by $C$ is at least $2\ell/\Delta$. At least $\ell/\Delta$ of these faces are in the interior of $C$ and at least $\ell/\Delta$ of these faces are in the exterior of $C$.

**Proof** Orient the edges of $C$ counterclockwise so that, for each edge $e$ of $C$, the face of $G^*$ to the left of $e$ is in $C$’s interior and the face of $G^*$ to the right of $e$ is in $C$’s exterior. Each face of $G^*$ has at most $\Delta$ edges. Therefore, the number of faces to the right of edges in $C$ is at least $\ell/\Delta$. The same is true for the number of faces of $G^*$ to the left of edges in $C$. □

For each node $u$ of $T_i$, we define $N(u)$ as the set of nodes in $T_0$ and $T_1$ (excluding $u$) that share an edge of $G^*$ with $u$. Note that $N(u)$ contains the neighbours of $u$ in $T_i$ as well as nodes of $T_{1-i}$ with which $u$ shares an edge of $C$. We say that a node $u$ is really bad if $u$ and all nodes in $N(u)$ are bad.

**Lemma 2.12** For each $i \in \{0, 1\}$ and each $0 < \alpha < 1/24$, if $G$ has maximum degree $\Delta$, $C$ has length $\ell$, and the number $\kappa$, of faces of $G^*$ caressed by $C$ is at most $\alpha \ell/\Delta$, then the number $b_i$ of really bad nodes in $T_i$ is at least $n_i - \alpha (120\Delta + 72)n_i$.

**Proof** Without loss of generality, let $i = 0$. From Observation 2.11, we know that $\tau_0 \geq \ell/\Delta$. Therefore, $\kappa_0 \leq \kappa \leq \alpha \ell/\Delta \leq \alpha \tau_0 \leq \tau_0/6$ so, by Lemma 2.7, $n_0 \geq \tau_0/8$. By Lemma 2.8, if $b_0 < (1 - 24\alpha)n_0$, then

$$\kappa > \kappa_0 \geq \alpha \tau_0 \geq \frac{\alpha \ell}{\Delta}.$$  

This violates our assumption that $\kappa \leq \alpha \ell/\Delta$. Therefore, we may assume that $b_0 \geq (1 - 24\alpha)n_0$.

We now want to study how many of the bad nodes in $T_0$ are really bad. Let $A$ be the set of nodes in $T_0$ that are not bad ;and partition $A$ into $A_1$ (leaves), $A_2$ (degree-2 nodes), and $A_{\geq 3}$ (nodes of degree at least 3). We make use of the following inequality:

$$|A_1| = 2 + \sum_{w \in A_{\geq 3}} (\delta_w - 2) \geq \sum_{w \in A_{\geq 3}} (\delta_w - 2) \geq \sum_{w \in A_{\geq 3}} \frac{\delta_w}{3},$$  (2)
which is true because \( x - 2 \geq x/3 \) for all \( x \geq 3 \). Now each node \( w \) in \( A \) can prevent at most \( \delta_w \) bad nodes of \( T_0 \) from being really bad. We count this as follows:

\[
\sum_{w \in A} \delta_w = \sum_{w \in A_1} \delta_w + \sum_{w \in A_2} \delta_w + \sum_{w \in A_{\geq 3}} \delta_w \leq |A_1| + 2|A_2| + 3|A_1|.
\]

Now, \( A_1 \) contains leaves of \( T_0 \) and, by Lemma 2.4, each leaf of \( T_0 \) contains a caressed face. Therefore \( |A_1| \leq \kappa \). Next, \( A_2 \) contains degree-2 nodes of \( T_0 \) that are not bad. If a node has degree-2 and contains no caressed face, then it is bad. Therefore each node in \( A_2 \) contains a caressed face. Therefore \( |A_2| \leq \kappa \), so \( |A_1| + 2|A_2| + 3|A_1| \leq 6\kappa \).

Picking up where we left off:

\[
\sum_{w \in A} \delta_w \leq |A_1| + 2|A_2| + 3|A_1| \leq 6\kappa \leq \frac{6\alpha \ell}{\Delta} \leq 48\alpha n_0,
\]

where the last inequality uses the fact that \( n_0 \geq \tau_0/8 \geq \ell/(8\Delta) \). That is, the set \( A \) of non-bad nodes in \( T_0 \) prevents at most \( 48\alpha n_0 \) bad nodes in \( T_0 \) from being really bad. Next we account for nodes in \( T_1 \) that prevent bad nodes in \( T_0 \) from being really bad.

Let \( A' \) be the set of nodes in \( T_1 \) that are not bad. For two nodes \( u \) in \( T_0 \) and \( w \) in \( T_1, w \in N(u) \) if and only if \( w \) and \( u \) share an edge of \( C \). The number of edges of \( C \) incident to a node \( w \) is at most \( \tau_w/\Delta \). Therefore, we can upper bound the number of bad nodes in \( T_0 \) that are prevented from being really bad by some node in \( T_1 \) as

\[
\sum_{w \in A'} \Delta \tau_w = \sum_{w \in A'} \Delta (\rho_w + \kappa_w)
\]

\[
\leq \sum_{w \in A'} (3\Delta \kappa_w + 2\Delta \delta_w)
\]

\[
\leq 3\Delta \kappa + \sum_{w \in A'} 2\Delta \delta_w
\]

\[
< 3\Delta \kappa + 12\Delta \kappa
\]

\[
= 15\Delta \kappa \leq 15\alpha \ell
\]

\[
\leq 120\alpha \Delta n_0.
\]

Therefore, the number of bad nodes in \( T_0 \) is \( b_0 \) and the number of these that are really bad is at least

\[
b_0 - \alpha (120\Delta + 48)n_0 \geq n_0 - \alpha (120\Delta + 72)n_0.
\]

We say that a node \( u \) is really really bad if all the nodes in \( N(u) \) are really bad. (Note that this implies that \( u \) is bad.) The following lemma extends Lemma 2.12 to really really bad nodes:

**Lemma 2.13** For each \( i \in \{0, 1\} \) and each \( 0 < \alpha < 1/24 \), if \( G \) has maximum degree \( \Delta \), \( C \) has length \( \ell \), and the number \( \kappa \), of faces of \( G^* \) caressed by \( C \) is at most \( \alpha \ell/\Delta \), then
The number $b_i$ of really really bad nodes in $T_i$ is at least $n_i - \alpha(\Delta + 1)(120\Delta + 72)n_i = n_i - O(\alpha\Delta^2)$.

**Proof** A node $u$ is a *fringe node* if it is really bad but not really really bad. A node $u$ is a *critical node* if it is bad but not really bad. Observe that every fringe node $u$ is in $N(w)$ for some critical node $w$. To bound the number of fringe nodes, it therefore suffices to bound $\sum_w |N(w)| \leq \sum_w \Delta$ where the sum is over all critical nodes and the inequality is due to Lemma 2.9, so $|N(w)| \leq \Delta$ for any bad node $w$.

By Lemma 2.12, the number of nodes that are not really bad, and hence the number of critical nodes, is at most $\alpha(120\Delta + 72)n_i$. Therefore, the number of fringe nodes is at most $\alpha\Delta(120\Delta + 72)n_i$. Any node that is not really really bad is either a fringe node or is not really bad. Therefore, the number of nodes that are really really bad is at least

$$n_i - \alpha(\Delta + 1)(120\Delta + 72)n_i.$$  

The following observation, illustrated in Fig. 10, follows from the fact that all the nodes it considers are bad and that $\bar{H}$ is a cubic graph, so each vertex of $\bar{H}$ is on the boundary of three faces. The second part of the figure shows an example in which $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_s\}$ are not disjoint. (In this example $b_1 = a_3$.)

**Observation 2.14** Let $x_1, \ldots, x_t, t \geq 1$, be a path in $T_0$ consisting entirely of really bad nodes. Then $C \cap \bigcup_{i=1}^{t} x_i$ consists of two paths $C_a$ and $C_b$ each having at least one edge and the subgraph of $T_1$ induced by $\bigcup_{i=1}^{t} N(x_i)$ is contained in two (not necessarily disjoint) paths $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$, $r, s \geq 1$, where $a_i$ contains an edge of $C_a$ for each $i \in \{1, \ldots, r\}$ and $b_i$ contains an edge of $C_b$ for each $i \in \{1, \ldots, s\}$. 
2.6 Tree/Cycle Surgery

We summarize the situation so far. By Lemma 2.2, finding a large collinear set is equivalent to finding a cycle in $G^*$ that caresses many faces. By existing results on the circumference of cubic triconnected graphs, $G^*$ has a cycle $C_0$ of length $\ell = \Omega(n^\alpha)$ for some $\alpha > 0.8$. Thus we assume that $G^*$ has a cycle $C_0$ of length $\ell$ and we want to show the existence of a cycle $C$ that caresses $\Omega(\ell/\Delta^4)$ faces.

Because each face of $G^*$ has at most $\Delta$ edges, $C_0$ touches $\Omega(\ell/\Delta)$ faces (Observation 2.11). To complete the proof of Theorem 1.2 we must deal with the situation where $C_0$ caresses $o(\ell/\Delta^4)$ faces and therefore each of $T_0$ and $T_1$ has $o(\ell/\Delta^4)$ leaves (Lemma 2.4), $\Omega(\ell/\Delta)$ nodes (Lemma 2.7), and the fraction of really really bad nodes in $T_0$ and $T_1$ is $1 - o(1/\Delta)$ (Lemma 2.13).

Figure 11 illustrates an extreme example of this situation. To handle cases like these, the only option is to perform surgery on the cycle $C$ to increase the number of caressed faces. We achieve this by performing a surgery that increases the number of leaves in $T_1$. This surgery is quite delicate and requires a particular node $u$ for which we have a good enough understanding of the faces of $\tilde{H}$ surrounding $u$ so that we can make a local modification of $C$ around $N(u)$ that is guaranteed to strictly increase the number of caressed faces.

**Proof of Theorem 1.2** By Lemma 2.2, it suffices to prove the existence of a cycle $C$ in $G^*$ that caresses $\Omega(\ell/\Delta^4)$ faces. We begin by applying Lemma 2.13 with $\alpha = \epsilon/\Delta^3$. For sufficiently small, but constant, $\epsilon$, Lemma 2.13 implies that $\kappa = \Omega(\ell/\Delta^4)$ or the number of nodes in $T_0$ that are not really really bad is at most $O(\epsilon n_0/\Delta)$. In the former case, $C$ caresses $\Omega(\ell/\Delta^4)$ faces of $G^*$ and we are done.

In the latter case, consider the forest obtained by removing all nodes of $T_0$ that are not really really bad. This forest has $(1 - O(\epsilon/\Delta))n_0$ nodes. We claim that it also has $O(\epsilon n_0/\Delta)$ components. To see why this is so, let $L$ be the set of leaves in $T_0$ and let $S$ be the set of non-leaf nodes in $T_0$ that are not really really bad. Observe that it is sufficient to upper bound the number, $k$, of components in $T_0 - S$.

Since removing a degree $d$ vertex from a graph increases the number of components by at most $d - 1$, we have $k \leq \sum_{u \in S} (\deg_{T_0}(u) - 1)$. Since $(S, L)$ is a partition of the nodes of $T_0$ that are not really really bad, we have $|L| \leq |S| + |L| = O(\epsilon n_0/\Delta)$. Recall that a standard fact about trees is that the number of leaves in a tree $G$ is exactly
2 + \sum_u (\deg_T(u) - 2), where the sum runs over all non-leaf nodes \( u \) of \( G \). Therefore,

\[
|L| \geq \sum_{u \in S} (\deg_{T_0}(u) - 2) = \sum_{u \in S} (\deg_{T_0}(u) - 1) - |S| = k - |S|.
\]

Therefore \( k \leq |S| + |L| = O(\epsilon n_0/\Delta) \), as claimed.

Thus the forest induced by all really really bad nodes of \( T_i \) has at most \( O(\epsilon n_0/\Delta) \) components, each of which is a path. At least one of these paths contains \( \Omega(\Delta/\epsilon) \) nodes. In particular, for a sufficiently small constant \( \epsilon \), one of these components, \( X \), has at least \( 5\Delta \) nodes.

Consider some node \( u \) in \( X \), and let \( C_a \) and \( C_b \) be the two components of \( u \cap C \). By Observation 2.14, the subgraph of \( T_1 \) induced by \( N(u) \) consists of two paths \( a_1, \ldots, a_r \) and \( b_1, \ldots, b_s \) of really bad nodes where each \( a_1, \ldots, a_r \) contains an edge of \( C_a \) and each of \( b_1, \ldots, b_s \) contains an edge of \( C_b \).

It follows from Lemma 2.10 that among any sequence of \( \Delta \) consecutive nodes in \( X \), at least one node has \( r \geq 2 \) and therefore \( |N(u)| \geq 5 \). Let \( u \) be any such node that is not among the first \( 2\Delta \) or last \( 2\Delta \) nodes of \( X \). Such a \( u \) always exists because \( X \) contains at least \( 5\Delta \) nodes. Let \( x_0 = u \). We now define notations for some of the nodes in the vicinity of \( u \) (refer to Fig. 12):

- there is a path \( x_2\Delta, \ldots, x_1, x_0, y_1, \ldots, y_2\Delta \) in \( T_0 \) consisting entirely of really really bad nodes;
- some really bad node \( a_1 \) of \( T_1 \) shares an edge with each of \( x_0, \ldots, x_i \) for some \( i \in \{1, \ldots, \Delta - 4\} \);
- some really bad node \( a_2 \) of \( T_1 \) shares an edge with \( a_1 \) and edge with \( x_0 \);
- some really bad node \( a_0 \neq a_2 \) of \( T_1 \) shares an edge with \( a_1 \) and with each of \( x_i, \ldots, x_{i+j} \) for some \( j \in \{0, \ldots, \Delta - 4\} \).
The surgery we perform focuses on the nodes \(u\) and \(a_1\). Consider the two components of \(C \cap a_1\). At least one of these components, \(p\), shares an edge with \(u\). By Lemma 2.10, the other component, \(q\), does not share an edge with \(u\). Imagine removing \(u\) from \(T_0\), thereby separating \(T_0\) into a component \(T_x\) containing \(x_1\) and a component \(T_y\) containing \(y_1\). Equivalently, one can think of removing the edges of \(u\) from \(C\) separating \(C\) into two paths \(C_x\) and \(C_y\) on the boundary of \(T_x\) and \(T_y\), respectively. Since \(q\) does not share an edge with \(u\), \(q \subseteq C_x\) or \(q \subseteq C_y\). We treat these cases separately:

1. \(q \subset C_x\). We transform this into case 2, by redefining \(u, x_1, y_1,\) and \(a_1\) as follows:
   By Lemma 2.5, \(a_1 \setminus C\) contains exactly two edges of \(G^*\) and exactly one of these edges, \(e\), is not incident to \(u\). Instead, \(e\) is incident to \(x_i\). We set \(u' = x_i, x_i' = x_{i-1}, y_i' = x_{i+1}\), and \(a_1' = a_1\). Observe that \(a_1'\) connects the two components of \(T_0 - u'\) and shares edges with \(u'\) and \(x_i'\). This is exactly the situation considered in case 2, next.

2. \(q \subset C_y\). At this point it is helpful to think of \(T_0, T_1,\) and \(C\) as a partition of \(\mathbb{R}^2\), where nodes of \(T_0\) are coloured red, nodes of \(T_1\) are coloured blue and \(C\) is the (purple) boundary between red and blue. To describe our modifications of \(C\), we imagine changing the colours of nodes. The effect that such a recolouring has on \(C\) is immediately obvious: It produces a 1-dimensional set \(C'\) that contains every (purple) edge contained in the red-blue boundary. The set \(C'\) is a collection of vertices and edges of \(G^*\). Therefore, if \(C'\) is a simple cycle, then \(C'\) defines a new pair of trees \(T_0'\) and \(T_1'\). Refer to the right two thirds of Fig. 13 for a simple (and misleading) example of what follows. For a full example, refer to Fig. 14.

The surgery we perform recolours \(x_0, x_1, \ldots, x_{i-1}\) blue and recolours \(a_1\) red. Observe that, because \(q \subset C_y\) and \(p\) contain an edge of \(x_i\), this implies that the red subset of \(\mathbb{R}^2\) is connected. Similarly, one can verify that \(T_1 - \{a_1\}\) contains two components, one containing \(a_2\) and one containing \(a_0\) and \(b_1\). The blue subset of \(\mathbb{R}^2\) is connected because it contains a path from \(a_2\) through \(u\) to \(b_1\). Therefore the red and blue subsets of \(\mathbb{R}^2\) are each connected and their common boundary \(C'\) is a simple cycle consisting of edges of \(G^*\). The new trees \(T_0'\) and \(T_1'\) are therefore well defined. We now make two claims that will complete our proof.

**Claim 2.15** For each \(i \in \{0, 1\}\), and each node \(w\) of \(T_i\) that is not bad, \(C \cap w = C' \cap w\). (Equivalently, for every face \(f\) of \(G^*\) that is not a bad node of \(T_0\) or \(T_1\), \(C \cap f = C' \cap f\).)

**Claim 2.16** The face \(a_0\) is caressed by \(C'\).

These two claims complete the proof because, together, they imply that \(C'\) caresses at least one more face of \(G^*\) than \(C\). Indeed, by definition, \(C\) did not caress any faces belonging to bad nodes. Therefore, the first claim implies that the faces of \(G^*\) caressed by \(C'\) are a superset of those caressed by \(C\). The face \(a_1\) is a bad node of \(T_i\) so it is not caressed by \(C\) but the second claim states that it is caressed by \(C'\). Therefore \(C'\) caresses at least one more face than \(C\). This surgery recolours at most \(\Delta - 2 \leq \Delta\) nodes of \(T_0\) and \(T_1\), so the difference in length between \(C\) and \(C'\) is at most \(\Delta^2\). If we start with a cycle \(C\) of length \(\ell\), then we can perform this surgery at least \(\ell/(4\Delta^2)\) times before the length of \(C\) decreases to less than \(\ell' = \ell/2\). If at some point during this process, we are no longer able to perform this operation, it

\(\mathfrak{S}\) Springer
is because \( C \) caresses \( \Omega(\ell'/\Delta^4) = \Omega(\ell'/\Delta^4) \) faces of \( G^* \) and we are done. If the process runs to completion, then by its end, the number of faces caressed by \( C \) is at least \( \ell/(4\Delta^2) \in \Omega(\ell/\Delta^2) \subset \Omega(\ell/\Delta^4) \) and we are also done. Thus, all that remains is to prove Claim 2.15 and Claim 2.16. To prove Claim 2.15, we observe that \( C \) and \( C' \) differ only on the boundaries of nodes that are recoloured. Thus, it is sufficient to show that all nodes in \( R = \bigcup \{ N(v) : v \in \{x_0, \ldots, x_{i-1}, a_1\} \} \) are bad. But this is immediate since \( x_0, \ldots, x_{i-1} \) are really really bad and \( a_1 \in N(x_0) \), so \( a_1 \) is really bad. Since every node in \( R \) share an edge with at least one of \( \{x_0, \ldots, x_{i-1}, a_1\} \), every node in \( R \) is therefore bad, as required. To prove Claim 2.16 we consider the boundary of the face \( a_0 \) of \( G^* \) after the recolouring operation. This boundary consists of, in cyclic order:

- An edge \( p_0p_1 \) shared between \( a_0 \) and \( a_1 \). This edge is in \( C' \) since \( a_0 \) is in \( T'_1 \) and \( a_1 \) is in \( T'_0 \). This edge has one endpoint, \( p_0 \), on the boundary of \( x_i \) (\( p_0 \) is also an endpoint of \( p \)).
- A path \( p_1, \ldots, p_\mu \) whose edges are shared with \( x_i, \ldots, x_{i+j} \). The nodes \( x_i, \ldots, x_{i+j} \) are in \( T_0 \) and are distinct from \( x_0, \ldots, x_{i-1} \), so these nodes are in \( T'_0 \). Therefore, \( p_1, \ldots, p_\mu \) is also contained in \( C' \).
- An edge \( p_\mu p_{\mu+1} \) shared between \( a_0 \) and another node \( a_{\mu-1} \neq a_1 \) of \( T_1 \). The faces of \( a_{\mu-1} \) are in \( T'_1 \) because \( a_1 \) is the only face that moves from \( T_1 \) to \( T'_0 \). (\( a_1 \) is the only face whose colour goes from blue to red.) The edge \( p_\mu p_{\mu+1} \) is therefore not contained in \( C' \).
A path $p_{\mu+1}, \ldots, p_v$ with $p_v = p_0$ that is contained in $C$. Let $C'_x$ be the path obtained by removing all edges on the boundary of $x_1, \ldots, x_{i-1}$ from $C_x$. Thus, the boundary of $C$ is partitioned into four paths: $C_y$, a path $P_1$ that contains $p$; $C'_x$, and a path $P_2$ that does not contain $p$. Without loss of generality, assume that these four paths occur in the order $C_y$, $P_1$, $C'_x$, $P_2$ when traversing $C$ clockwise. The path $p_{\mu+1}, \ldots, p_v$ ends at $p_v = p_0$, which is contained in $C_y$. This path must therefore either begin in $P_2$ or be entirely contained in $C_y$ since, otherwise it would contain an edge of $x_i$, contradicting Lemma 2.5. The edges of $P_2$ are not in $C'$. Therefore $p_{\mu+1}, \ldots, p_v$ begins with a (possibly empty) sequence of edges $p_{\mu+1}, \ldots, p_{\mu+k}$ not contained in $C'$ followed by a non-empty sequence $p_{\mu+k}, \ldots, p_v$ of edges that are contained in $C'$. Therefore the intersection $C' \cap a_0$ is a path $p_{\mu+k}, \ldots, p_v, p_1, \ldots, p_\mu$ so $a_0$ is caressed by $C'$. □

3 Discussion

It remains an open problem to eliminate the dependence of our results on the maximum degree, $\Delta$, of $G$. The next significant step is to resolve the following conjecture:
Conjecture 3.1 If $G$ is a triangulation whose dual $G^*$ has a cycle of length $\ell$, then $G^*$ has a cycle that caresses $\Omega(\ell)$ faces. (Therefore, by Lemma 2.2 and Theorem 2.1, $G$ has a collinear set of size $\Omega(\ell)$.)

Acknowledgements Much of this research took place during the Sixth Workshop on Order and Geometry held in Cia˛żeni, Poland, September 19–22, 2018. The authors are grateful to the organizers, Stefan Felsner and Piotr Micek, and to the other participants for providing a stimulating research environment.

References

1. Angelini, P., Binucci, C., Evans, W., Hurtado, F., Liotta, G., Mchedlidze, T., Meijer, H., Okamoto, Y.: Universal point subsets for planar graphs. In: 23rd International Symposium on Algorithms and Computation (Taipei 2012). Lecture Notes in Computer Science, vol. 7676, pp. 423–432. Springer, Heidelberg (2012)
2. Angelini, P., Evans, W., Frati, F., Gudmundsson, J.: SEFE without mapping via large induced outerplane graphs in plane graphs. J. Graph Theory 82(1), 45–64 (2016)
3. Barba, L., Evans, W., Hoffmann, M., Kusters, V., Saumell, M., Speckmann, B.: Column planarity and partially-simultaneous geometric embedding. J. Graph Algorithms Appl. 21(6), 983–1002 (2017)
4. Barba, L., Hoffmann, M., Kusters, V.: Column planarity and partial simultaneous geometric embedding for outerplanar graphs. In: 31th European Workshop on Computational Geometry (Ljubljana 2015). Book of Abstracts, pp. 53–56 (2015). http://eurocg15.ﬁ.uni-lj.si/pub/eurocg15-book-of-abstracts.pdf
5. Barnette, D.: Trees in polyhedral graphs. Can. J. Math. 18, 731–736 (1966)
6. Bilinski, M., Jackson, B., Ma, J., Yu, X.: Circumference of 3-connected claw-free graphs and large Eulerian subgraphs of 3-edge connected graphs. J. Comb. Theory Ser. B 101(4), 214–236 (2011)
7. Bläsius, T., Kobourov, S.G., Rutter, I.: Simultaneous embedding of planar graphs. In: Handbook of Graph Drawing and Visualization, pp. 349–381. CRC Press, Boca Raton (2013)
8. Bondy, J.A., Simonovits, M.: Longest cycles in 3-connected 3-regular graphs. Can. J. Math. 32(4), 987–992 (1980)
9. Bose, P., Dujmović, V., Hurtado, F., Langerman, S., Morin, P., Wood, D.R.: A polynomial bound for untangling geometric planar graphs. Discrete Comput. Geom. 42(4), 570–585 (2009)
10. Cano, J., Tóth, C.D., Urrutia, J.: Upper bound constructions for untangling planar geometric graphs. SIAM J. Discrete Math. 28(4), 1935–1943 (2014)
11. Cibulka, J.: Untangling polygons and graphs. Discrete Comput. Geom. 43(2), 402–411 (2010)
12. Da Lozzo, G., Dujmović, V., Frati, F., Mchedlidze, T., Roselli, V.: Drawing planar graphs with many collinear vertices. J. Comput. Geom. 9(1), 94–130 (2018)
13. Di Giacomo, E., Liotta, G., Mchedlidze, T.: How many vertex locations can be arbitrarily chosen when drawing planar graphs? (2012). arXiv:1212.0804
14. Dujmović, V.: The utility of untangling. J. Graph Algorithms Appl. 21(1), 121–134 (2017)
15. Dujmović, V., Frati, F., Gonçalves, D., Morin, P., Rote, G.: Every collinear set is free. In: 30th Annual ACM-SIAM Symposium on Discrete Algorithms (San Diego 2019), pp. 1521–1538. SIAM, Philadelphia (2019)
16. Evans, W., Kusters, V., Saumell, M., Speckmann, B.: Column planarity and partial simultaneous geometric embedding. In: 22nd International Symposium on Graph Drawing (Würzburg 2014). Lecture Notes in Computer Science, vol. 8781, pp. 259–271. Springer, Heidelberg (2014)
17. Goaoc, X., Kratochvíl, J., Okamoto, Y., Shin, C.-S., Spillner, A., Wolff, A.: Untangling a planar graph. Discrete Comput. Geom. 42(4), 542–569 (2009)
18. Grünbaum, B., Walther, H.: Shortness exponents of families of graphs. J. Comb. Theory Ser. A 14(3), 364–385 (1973)
19. Jackson, B.: Longest cycles in 3-connected cubic graphs. J. Comb. Theory Ser. B 41(1), 17–26 (1986)
20. Kang, M., Pikhurko, O., Ravsky, A., Schacht, M., Verbitsky, O.: Untangling planar graphs from a specified vertex position—hard cases. Discrete Appl. Math. 159(8), 789–799 (2011)
21. Kant, G., Bodlaender, H.L.: Triangulating planar graphs while minimizing the maximum degree. Inf. Comput. 135(1), 1–14 (1997)
22. Liu, Q., Yu, X., Zhang, Z.: Circumference of 3-connected cubic graphs. J. Comb. Theory Ser. B 128, 134–159 (2018)
23. Pach, J., Tardos, G.: Untangling a polygon. Discrete Comput. Geom. 28(4), 585–592 (2002)
24. Ravsky, A., Verbitsky, O.: On collinear sets in straight line drawings (2008). arXiv:0806.0253
25. Ravsky, A., Verbitsky, O.: On collinear sets in straight-line drawings. In: 37th International Workshop on Graph-Theoretic Concepts in Computer Science (Teplá 2011). Lecture Notes in Computer Science, vol. 6986, pp. 295–306. Springer, Heidelberg (2011)
26. Robertson, N., Sanders, D., Seymour, P., Thomas, R.: The four-colour theorem. J. Comb. Theory Ser. B 70(1), 2–44 (1997)
27. Tait, P.G.: Remarks on the colouring of maps. Proc. R. Soc. Edinb. Sect. A 10, 729 (1880)
28. Tutte, W.T.: On Hamiltonian circuits. J. Lond. Math. Soc. 21, 98–101 (1946)
29. Watanabe, M.: Open problem. In: 5th Czech–Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications (Prague 1998)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.