Self-dual two-forms and divergence-free vector fields

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Abstract

Beginning with the self-dual two-forms approach to the Einstein equations, we show how, by choosing basis spinors which are proportional to solutions of the Dirac equation, we may rewrite the vacuum Einstein equations in terms of a set of divergence-free vector fields, which obey a particular set of chiral equations. Upon imposing the Jacobi identity upon these vector fields, we reproduce a previous formulation of the Einstein equations linked with a generalisation of the Yang-Mills equations for a constant connection on flat space. This formulation suggests the investigation of some new aspects of the self-dual two-forms approach. In the case of real Riemannian metrics, these vector fields have a natural interpretation in terms of the torsion of the natural almost-complex-structure on the projective spin-bundle.

1. Introduction

If one adopts a null tetrad approach to Lorentzian geometry in four dimensions, one quickly finds that, due to the structure of the Lorentz group, the spin connection and curvature quantities that arise break up naturally into self-dual and anti-self-dual quantities, related to one another by complex conjugation [1]. Given that the complexified space of 2-forms also has a direct sum decomposition into self-dual and anti-self-dual 2-forms, and that the self-dual part of the spin connection defines a natural connection on the space of self-dual 2-forms, it is perhaps not too surprising that one can recast the equations of Lorentzian geometry solely in the language of self-dual 2-forms and the self-dual spin-connection [2, 3], along with the reality condition that self-dual objects are the complex conjugate of their anti-self-dual counterparts.

One can adopt a similar approach in the more general context of complex-Riemannian geometry, with the usual description in terms of the bundle of orthonormal frames and the natural SO(4, C) connection being replaced with the bundle of self-dual 2-forms, and its sl(2, C) connection. Real slices of Lorentzian, real-Riemannian or ultra-hyperbolic signature then arise when suitable reality conditions are satisfied.

The reformulation of four-dimensional geometry in such terms becomes of special interest when we consider metrics which satisfy the Einstein condition

\[ r = \frac{s}{4} g, \]

where \( r \) and \( s \) denote the Ricci tensor and scalar curvature of the metric \( g \). This condition is equivalent to the condition that the self-dual spin-connection have self-dual curvature. Although this observation has been the basis of extensive work on the Lorentzian Einstein equations, it also seems to be of interest in the context of compact Riemannian Einstein manifolds. In this case the self-dual spin-connection is an self-dual SU(2) Yang-Mills field with second chern class \( c_2 = -\frac{1}{2}(\chi + \frac{3}{2} r) \). A large amount of information has accumulated concerning the moduli spaces of such fields on generic four-manifolds [4], at least for small values of \( c_2 \), although it is not clear at this point whether this information is generally useful in the study of existence and uniqueness of Einstein metrics[5].

\[ ^1 \text{I am grateful to Andrew Chamblin for discussions on this point.} \]
Another area in which this alternative description of the Einstein condition is useful is in the
context of Ricci-flat metrics with anti-self-dual Weyl tensor in which case the self-dual connection
is flat. If we specialise to a simply connected, real Riemannian manifold \( M \), a Ricci-flat anti-
self-dual metric automatically defines a hyper-kähler structure. The three Kähler forms associated
with this hyper-kähler structure can then be identified with the self-dual 2-forms of our formalism.
Alternatively [5], the integrability condition for the three complex-structures can be recast as the
self-dual Yang-Mills equations for a constant connection on flat space, with the connection taking
values in \( \text{Lsdiff} M \), the algebra of divergence-free vector fields on \( M \). Building upon this result, it
was shown that one could reformulate the full vacuum Einstein equations in terms of such a set of
divergence-free vector fields [5]. Unlike the self-dual 2-forms approach to the Einstein equations
mentioned above, this reformulation is symmetrical between the self-dual and anti-self-dual parts
of the gravitational field and, although the final equations bear some resemblance to the full Yang-
Mills equation for a constant connection on flat space, they also included interaction terms between
self-dual and anti-self-dual parts of the gravitational field.

Our aim here is to relate the self-dual 2-forms approach to the Einstein equations with the
divergence-free vector field approach. We begin by reviewing the description of four-dimensional
Lorentzian geometry in terms of two component spinors, null tetrads and self-dual 2-forms. We then
show that by choosing appropriate spinor bases and carrying out a related conformal transformation,
we can describe any metric in terms of a set of divergence-free vector fields, which obey a set of
equations which are explicitly chiral in nature. Reversing the argument, we see that the vector field
approach suggests the investigation of a particular set of divergence-free vector fields which occur
naturally in the self-dual 2-forms approach. These divergence-free vector fields essentially carry the
information of the self-dual spin connection and, for Riemannian metrics, can be interpreted in terms
of the torsion of an almost-complex-structure on the tangent space of our manifold. This almost-
complex-structure is the projection to \( TM \) of the horizontal part of the natural almost-complex-
structure on the projective spin bundle which arises when we consider metrics with anti-self-dual
Weyl tensor. The Einstein condition tells us that the Riemann tensor evaluated on any anti-self-dual
null bivector in the complexification of the tangent space must commute with the almost-complex-
structure.

Most of the reality conditions used in this paper are those necessary for the description of
metrics of Lorentzian signature. Since, however, Section 5 is concerned with Riemannian metrics,
we have included an appendix devoted to a general discussion of the reality conditions for metrics of
Riemannian and ultra-hyperbolic signature.

2. Self-dual two-forms

We assume we are on a smooth, oriented, real four-manifold \( M \) with a pseudo-Riemannian
metric, \( g \), of Lorentzian signature \((+ − − −)\). The manifold \((M, g)\) comes naturally equipped with
the bundle of exterior p-forms, \( \Lambda^p \). The Hodge map is the unique vector bundle isomorphism
\( * : \Lambda^k \rightarrow \Lambda^{4-k} \),
defined by
\[
\alpha \wedge * \beta = g(\alpha, \beta) \nu, \quad \alpha, \beta \in \Lambda^k,
\] (2.1)
where \( g(\alpha, \beta) \) is the product on \( \Lambda^k \) induced by the metric, and \( \nu \) is the volume form defining the
orientation \[7, 8\]. In the particular case of four dimensional Lorentzian spacetimes, the Hodge map
acts as an endomorphism of \( \Lambda^2 \) with \( *^2 = -1 \), so the space \( \Lambda^2 \) has a natural complex-structure \[8, 9\].
Therefore the complexified bundle of 2-forms \( \Lambda^2 = \Lambda^2 \otimes \mathbb{C} \) decomposes as \( \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_- \) where
\( \Lambda^2_{\pm} \) are the bundles of self-dual and anti-self-dual 2-forms:
\[
\Lambda^2_{\pm} = \{ \lambda \in \Lambda^2_c : * \lambda = \pm i \lambda \}.
\] (2.2)
Due to the use of the complexification of \( \Lambda^2 \) in the decomposition, complex conjugation defines
isomorphisms $\Lambda^2_c \cong \Lambda^{2\mp}_c$. Note that the spaces $\Lambda^2_c$ are orthogonal with respect to the product $g$ when it is extended, by linearity, to $\Lambda^2_c$.

This decomposition of $\Lambda^2_c$ has a straightforward interpretation in terms of Lie algebras. Since our metric has Lorentzian signature, the orthonormal frame bundle has structure group $SO(1,3)$. As an $SO(1,3)$ module, $\Lambda^2$ is isomorphic to the Lie algebra $so(1,3)$ and, although the algebra $so(1,3)$ is simple, its complexification $so(1,3) \otimes \mathbb{C}$ decomposes as $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$. As in the Riemannian case \cite{4, 8}, we now use this isomorphism at the group level, where $Spin(1,3) \otimes \mathbb{C} \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, to introduce the 2-dimensional complex vector bundles $\mathbb{V}^\pm$ of self-dual and anti-self-dual spinors. Since these are $SL(2,\mathbb{C})$ bundles, there naturally come equipped with symplectic forms $\epsilon_{\pm}$, which can be used to define isomorphisms $\mathbb{V}^\pm \cong (\mathbb{V}^\pm)^*$ between $\mathbb{V}^\pm$ and their duals $(\mathbb{V}^\pm)^*$. As in the decomposition of $\Lambda^2_c$ complex conjugation defines isomorphisms $\mathbb{V}^\pm \cong \overline{\mathbb{V}^\mp}$, denoted by $\pi \mapsto \overline{\pi}$.

Given the basic spin-bundles $\mathbb{V}^\pm$, general spin-bundles are constructed by taking appropriate symmetric products, with a field $\phi \in \Gamma(S^m\mathbb{V}^- \otimes S^n\mathbb{V}^+)$ transforming under the irreducible representation $\left(\begin{array}{l} m \\ n \end{array}\right)$ of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Extending complex conjugation to the higher bundles, the bundles $S^m\mathbb{V}^- \otimes S^n\mathbb{V}^+$ inherit real structures for each positive integer $m$. In particular, from the vector representation of $SO(1,3)$, we deduce that the complexified tangent bundle of a Lorentzian four-manifold is isomorphic to the product of spin-spaces

$$T_c M \equiv TM \otimes \mathbb{C} \cong \mathbb{V}^- \otimes \mathbb{V}^+,$$

with the real tangent bundle $TM$ corresponding to products of spinors invariant under the real structure on $\mathbb{V}^- \otimes \mathbb{V}^+$. Using this isomorphism, we can translate a given complexified tensor field into a section of a given spinor bundle, and then reduce it to irreducible spinor parts by use of the $SL(2,\mathbb{C})$ invariant forms $\epsilon_{\pm}$, and the decomposition of the tensor product

$$S^m\mathbb{V}^\pm \otimes S^n\mathbb{V}^\pm \cong \bigoplus_{k=0}^{\min(m,n)} S^{m+n-k}\mathbb{V}^\pm.$$

For example, the tensor product of $\Lambda^1_c$ with itself decomposes as

$$\otimes^2 \Lambda^1_c \cong \Lambda^2_c \oplus \Lambda^2_c \oplus S^2_0 \Lambda^1_c \oplus \mathbb{C} g,$$

where $S^2_0 \Lambda^1_c$ denotes the trace-free symmetric tensors (the trace being defined with the metric $g$). We can then identify the complexified metric in the form

$$g \cong \epsilon_+ \otimes \epsilon_-.$$  \hspace{1cm} (2.3)

Any complex null vector $v \in (T_c M)_x$ can be written

$$v \cong \pi \otimes \lambda, \quad \text{where} \quad \pi \in (\mathbb{V}^-)_x, \quad \lambda \in (\mathbb{V}^+)_x,$$

and a real null vector $v \in (TM)_x$ may be expressed as

$$v \cong \pi \otimes \overline{\pi}, \quad \text{where} \quad \pi \in (\mathbb{V}^-)_x, \quad \overline{\pi} \in (\mathbb{V}^+)_x.$$

In particular, if we introduce a local basis $\{\epsilon_A : A = 0, 1\}$ for $\mathbb{V}^-$, and the dual basis $\{\epsilon^A\}$ for $(\mathbb{V}^-)^*$, we can, without loss of generality, assume that the bases are orthonormal, in the sense that the components of $\epsilon_-$ with respect to $\{\epsilon_A\}$ are

$$\epsilon_{AB} = \epsilon_-(\epsilon_A, \epsilon_B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \hspace{1cm} (2.4)$$

The dual bundle $(\mathbb{V}^\pm)^*$ also inherits a symplectic structure $\epsilon^*_\pm$ which, relative to the basis $\{\epsilon^A\}$, has components

$$\epsilon^{AB} = \epsilon^*_+(\epsilon^A, \epsilon^B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \hspace{1cm} (2.5)$$

3
We can raise and lower spinor indices using the components of the symplectic forms \( \epsilon_{\pm} \) according to the standard conventions

\[
\lambda^A = \epsilon^{AB} \lambda_B, \quad \lambda_A = \epsilon_{BA}.
\]

Similar orthonormal bases, denoted \( \{ \epsilon_{A'} : A' = 0', 1' \} \) and \( \{ \epsilon^{A'} \} \) can be introduced for for \( V^+ \), and \( (V^+)^* \). We can, without loss of generality, choose the bases \( \epsilon_A \) and \( \epsilon_{A'} \) to be related by complex conjugation with

\[
\overline{\epsilon_A} = \epsilon_{A'}, \quad \overline{\epsilon_{A'}} = \epsilon_A.
\]

(2.6)

If we denote these local bases for \( V^\pm \) by \( \epsilon_{A} \equiv (o, \iota) \) and \( \epsilon_{A'} \equiv (o', \iota') \), then we can define the null basis for \( T_cM \approx o \otimes o \) \( e_1 \approx o \otimes o' \), \( e_2 \approx \iota \otimes \iota' \), \( e_3 \approx o \otimes \iota' \), \( e_4 \approx \iota \otimes o' \).

(2.7)

The components of the metric (2.3) with respect to this basis is given by the matrix

\[
\eta_{ab} \equiv g(e_a, e_b) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]

Therefore, in terms of the dual basis \( \{ \epsilon^a \} \) for \( \Lambda^1_c \), the metric may be expressed as

\[
g = \epsilon^1 \otimes \epsilon^2 + \epsilon^2 \otimes \epsilon^1 - \epsilon^3 \otimes \epsilon^4 - \epsilon^4 \otimes \epsilon^3.
\]

(2.8)

Similarly, the inverse metric \( g^\sharp \in \Gamma(S^2(T_cM)) \) takes the form

\[
g^\sharp = e_1 \otimes e_2 + e_2 \otimes e_1 - e_3 \otimes e_4 - e_4 \otimes e_3.
\]

(2.9)

Given the complex conjugation laws (2.6) for the spinor bases, we deduce that the vector fields \( \{ e_i \} \) obey the reality relations

\[
\overline{e_1} = e_1, \quad \overline{e_2} = e_2, \quad \overline{e_3} = e_4, \quad \overline{e_4} = e_3,
\]

and the real tangent space takes the form

\[
TM = \text{Span}_\mathbb{R} \left( e_1, e_2, e_3 + e_4, i (e_3 - e_4) \right).
\]

The complex metric (2.8) then indeed restricts to a real metric of Lorentzian signature on the real tangent space.

If we consider the Riemann curvature tensor of the metric \( g \), then, using the metric, this may be viewed as self-adjoint map \( R : \Lambda^2 \rightarrow \Lambda^2 \) given by

\[
R(e^a \wedge e^b) = \frac{1}{2} R^{cd}_{\quad ab} e^c \wedge e^d,
\]

where \( \{ e^a \} \) is a local orthonormal basis for \( \Lambda^1_c \). In terms of the decomposition \( \Lambda^2_c = \Lambda^2_{c^+} \oplus \Lambda^2_{c^-} \), \( R \) can be put in block form

\[
\begin{pmatrix}
+W & \Phi \\
\Phi^* & -W
\end{pmatrix},
\]

(2.10)

relative to orthonormal bases \( \{ \pm \lambda^i \} \) for \( \Lambda^2_{c^\pm} \). Using the isomorphism between tensors and spinors introduced above, \( +W \) and \( -W \) correspond to the \( S^4V^+ \) and \( S^4V^- \) parts of the Riemann tensor,
which can be identified with the self-dual and anti-self-dual parts of the Weyl tensor respectively. $\Phi$ corresponds to the $S^2V^- \otimes S^2V^+$ part of the curvature, which can be identified with the trace-free part of the Ricci tensor:

$$\Phi = r - \frac{s}{4}g,$$

where $s = \text{tr} r$ denotes the scalar curvature. In Lorentzian signature, we may choose the bases $\{\pm \lambda^i\}$ to be complex conjugate to one another, in which case $^+W$ and $^-W$ are complex conjugates, $\Phi$ viewed as a $3 \times 3$ matrix is Hermitian, and the scalar curvature $s$ is real.

In precisely four dimensions, there is an alternative description of conformal geometry. Suppose we introduce a set of three linearly-independent complex 2-forms $\{\Sigma^i : i = 1, 2, 3\}$ on a real four-manifold $M$. These 2-forms will be self-dual with respect to a unique conformal class of metrics on $M$. If these 2-forms are orthogonal to their complex conjugates, with $\Sigma^i \wedge \Sigma^j = 0, \quad i, j = 1, 2, 3,$

then the conformal structure is of Lorentzian signature. These complex 2-forms may be combined into a single $sl(2, \mathbb{C})$ valued two-form:

$$\Sigma = \frac{i}{2} \sum \tau_i,$$

where $\tau_i$ are the Pauli matrices. In the same way that a frame for $TM$ defines an isomorphism between $(TM)_x$ and $\mathbb{R}^4$, the form $\Sigma$ defines an isomorphism between $(\Lambda^2_x, \mathbb{C})$ and $\mathbb{C}^3$. Given $\Sigma$, we may define the unique $sl(2, \mathbb{C})$-valued connection $\gamma$ on the vector bundle $\Lambda^2_x$ by the condition

$$d\Sigma + [\gamma, \Sigma] = 0,$$

and the associated curvature

$$R = d\gamma + \frac{1}{2} [\gamma, \gamma].$$

Given that the 2-forms $\Sigma^i$ define an isomorphism $(\Lambda^2_x, \mathbb{C}) \cong \mathbb{C}^3$, we can, by means of a GL(3, $\mathbb{C}$) transformation, choose the $\Sigma^i$ to obey the orthonormality condition

$$\Sigma^i \wedge \Sigma^j = i \delta^{ij} \nu,$$

with $\nu$ a real volume element on $M$. Similarly, the complex conjugate basis obey the relation

$$\Sigma^i \wedge \Sigma^j = -i \delta^{ij} \nu,$$

along with the condition (2.11).

In $sl(2, \mathbb{C})$ language, since we can identify the adjoint representation space of $sl(2, \mathbb{C})$ with $S^2V^+$, we can represent the form $\Sigma \in \Gamma(\Lambda^2_x \otimes sl(2, \mathbb{C}))$ by its components $\Sigma_{A'}B'$ with respect to the bases $\{\epsilon_A\}$ and $\{\epsilon^{A'}\}$ for $V^+$ and $(V^+)^*$ introduced above, with the condition:

$$\Sigma_{A'}B' = 0 \quad \Rightarrow \quad \Sigma^{A'B'} = \Sigma^{B'A'}.$$

The orthonormality condition (2.14) then becomes

$$\Sigma_{A'B'} \wedge \Sigma^{C'D'} = i \epsilon_{(A'}^{C'} \epsilon_B)^{D'} \nu.$$  

This means [2, 3] that there exists a basis, $\{\epsilon^a \cong \epsilon^{A'A'}\}$, of $\Lambda^1$ unique up to an $sl(2, \mathbb{C})$ rotation of the basis $\{\epsilon_A\}$, with the property that

$$\Sigma_{A'B'} = \frac{1}{2} \epsilon_{AB} \epsilon^{AA'} \wedge \epsilon^{BB'}.$$
In terms of this basis, the metric may be written
\[ g = \epsilon_{AB} \epsilon_{A'B'} \epsilon^{AA'} \otimes \epsilon^{BB'}. \] (2.16)
We can also construct a basis for the space \( \Lambda^2_c \) of anti-self-dual 2-forms given by
\[ \Sigma^{A'B'} = \frac{1}{2} \epsilon_{A'B'} \epsilon^{AA'} \wee \epsilon^{BB'}. \]
which are orthogonal to the self-dual 2-forms \( \Sigma_{A'B'} \) in the sense that
\[ \Sigma^{AB} \wee \Sigma^{A'B'} = 0. \] (2.17)
When we identify \( \Sigma \) in this way with the metric and tetrad, the connection \( \gamma \) of equation (2.12) becomes the self-dual part of the standard spin-connection, \( \Gamma_{ab} \), defined by
\[ d\epsilon^a + \Gamma_{ab} \wee \epsilon^b = 0, \quad \Gamma_{ab} = \Gamma_{c\,a} \epsilon^c. \] (2.18)
The curvature \( R \) in (2.13) can then be identified with the self-dual part of the Riemann curvature. In the notation of equation (2.10) we therefore have
\[ R_{A'B'} = +W_{A'B'C'D'} \Sigma^{C'D'} + \Phi_{ABA'B'} \Sigma^{AB} + \frac{s}{12} \Sigma_{A'B'}. \] (2.19)
The condition that the metric \( g \) be Einstein is that \( \Phi = 0 \), which, from (2.19), we see is equivalent to the condition that the self-dual spin-connection has self-dual curvature [2, 7, 9]:
\[ ^*R = R. \] (2.20)
The Einstein equations are fully characterised by equations (2.12), (2.15) and (2.20). In particular, this interpretation of the Einstein condition only involves the frame and connection for \( \Lambda^2_c \) and there is no dependence (at least explicitly) upon the properties of the anti-self-dual part of the gravitational field.

We now introduce a dual basis \( \{e_a\} \) for \( T_c M \), where
\[ <\epsilon^a, e_b> = \delta^a_b. \]
When acting as a differential operator, we will denote \( e_a \) by \( \nabla_a \). We define the commutator coefficients, \( C_{ab}^c \), of the vector fields \( e_a \) by the relation
\[ [e_a, e_b] = C_{ab}^c e_c, \]
so that
\[ C_{ab}^c = <[e_a, e_b], \epsilon^c>. \]
If we decompose the spin-connection \( \Gamma^a_{\,b} \) and the commutator coefficients \( C_{ab}^c \) into spinor terms according to the formulae
\[ \Gamma_{a\,b}^c \simeq \epsilon_{b'} \gamma_{A'A'B'}^c + \epsilon_B \gamma_{AA'B'}^c, \] (2.21)
\[ C_{ab}^c \simeq \epsilon_{A'B'} C_{AB}^c \gamma_{C'}^c + \epsilon_A C_{A'B'} C_{C'}^c, \] (2.22)
and use the standard relationship between the spin-connection and the commutator coefficients for a pseudo-orthonormal basis
\[ \Gamma_{abc} = \frac{1}{2} [C_{acb} - C_{abc} - C_{cba}], \] (2.23)
then we find that

\[ C_{A'B'C'} = \epsilon_{(A'C'} \gamma_{B')}^D C - \gamma_{(A'B')}^C C', \tag{2.24} \]

\[ \gamma_{AA'B'C'} = -\frac{1}{2} \left[ C_{(A'B'C')A} + C_{B'C'A}^A + C_{ABD}^{(B'C)}_{B'C'}A \right]. \tag{2.25} \]

3. Divergence-free vector fields

Up to now, our discussion has been of completely general tetrads and metrics. It is known, however, that if we consider Ricci-flat metrics with anti-self-dual Weyl tensor, it is often advantageous to partially fix the internal frame \( e_i \) for \( TM' \), and perform a related conformal transformation \([5]\). More precisely \([10]\), if we have a conformal structure which is self-dual, with \(+W = 0\), then it is possible to find a representative metric within the conformal class where the inverse metric takes the form

\[ \hat{g}^\dagger = \eta^{ab} \hat{e}_a \otimes \hat{e}_b, \]

for a fixed constant internal metric \( \eta \), and where the basis vectors \( \hat{e}_a \) obey the relation:

\[ [\hat{e}_a, \hat{e}_b] = -\frac{1}{2} \epsilon_{abcd} [\hat{e}_c, \hat{e}_d], \]

where

\[ \epsilon_{abcd} = \begin{cases} 1 & \text{if } abcd \text{ an even permutation of } 1234 \\ -1 & \text{if } abcd \text{ an odd permutation of } 1234 \\ 0 & \text{otherwise} \end{cases} \]

and indices are raised and lowered with the object \( \eta \) and its inverse. If we have a physical metric which is anti-self-dual and Ricci-flat, we may also take the vectors \( \hat{e}_a \) to be divergence-free with respect to some volume element \( \hat{\omega} \), with

\[ \mathcal{L}_{\hat{e}_a} \hat{\omega} = 0. \]

Defining the function \( f \) by

\[ \hat{\omega}(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4) = f^2, \]

then we obtain the physical, Ricci-flat inverse metric by the conformal transformation

\[ g^\dagger = f^2 \hat{g}^\dagger. \]

Given that such a conformal transformation is useful in the study of anti-self-dual Ricci-flat metrics, it is natural to ask whether one can develop a similar approach to the full Einstein equations, without the anti-self-duality constraint \([1]\). Since this divergence-free condition is not preserved under a general internal rotation of our vector basis, however, it is necessary to study what, if any, restrictions such a choice of gauge places on the geometry. In particular, to make the transformation to divergence-free vector fields, it is necessary to choose our vector basis so that the inverse metric is written

\[ g^\dagger = \eta^{ab} e_a \otimes e_b, \tag{3.1} \]

where the vectors \( \{e_a\} \) satisfy the condition

\[ C_{ab}^b = -\nabla_a (\log f), \tag{3.2} \]

for some function \( f \). Our aim in this section is to show that we can always achieve this condition by an internal rotation of basis vectors, for any inverse metric \( g^\dagger \).

Beginning with any real Lorentzian metric, we can, without loss of generality, complexify the tangent space and define a local basis \( \{e_a\} \) for \( T_c M \) where the inverse metric takes the form \([2,3]\). With the standard spin-connection defined by equation \([2.18]\), and the components of the
spin-connection and the commutator coefficients related by equation (2.23), we see that the gauge condition (3.2) may be rewritten
\[ \Gamma^b_{\ a} = \nabla_a (\log f). \]
This in turn implies that
\[ \text{div} \left( f^{-1} e_a \right) = 0, \] (3.3)
where, for an arbitrary vector field \( v \in \Gamma(T_c M) \), we have defined
\[ \text{div} v = \delta v^\flat, \]
where \( \delta : \Lambda^{p+1} \rightarrow \Lambda^p \) is codifferentiation [8].

As in the previous section, we can use the isomorphism \( T_c M \cong \mathbb{V}^- \otimes \mathbb{V}^+ \) to introduce bases \( \{ \epsilon_A \}, \{ \epsilon_A' \} \) for \( \mathbb{V}^- \) and \( \mathbb{V}^+ \) in terms of which we can write the basis \( e_a \) as
\[ e_a \sim = \epsilon_A \otimes \epsilon_A'. \]

Defining spinor fields
\[ \alpha_A = f^{-1/2} \epsilon_A, \quad \alpha_{A'} = f^{-1/2} \epsilon_{A'}, \] (3.4)
then, from (3.3) we deduce that we require
\[ \epsilon_+ (D^- \alpha_A, \alpha_{A'}) + \epsilon_- (D^+ \alpha_{A'}, \alpha_A) = 0, \] (3.5)
where
\[ D^\pm : \Gamma(\mathbb{V}^\pm) \rightarrow \Gamma(\mathbb{V}^\mp) \]
are the standard Dirac operators, defined by pulling back the connection on the frame bundle to an \( \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \) on the full spin-bundle, then projecting onto the separate \( \mathfrak{sl}(2, \mathbb{C}) \) factors to give connections on \( \mathbb{V}^\pm \). Assuming for the moment that we are given solutions \( \alpha_A, \alpha_{A'} \) of equation (3.5), we can straightforwardly reconstruct new spinor bases with the required gauge properties. Explicitly, we define
\[ \chi = \epsilon_- (\alpha_0, \alpha_1), \quad \chi' = \epsilon_+ (\alpha_{0'}, \alpha_{1'}), \]
then the spinor fields
\[ \tilde{\epsilon}_A = \chi^{-1/2} \alpha_A, \quad \tilde{\epsilon}_{A'} = \chi'^{-1/2} \alpha_{A'}, \]
constitute normalised spinor bases. The vector fields
\[ c_a \cong \tilde{\epsilon}_A \otimes \tilde{\epsilon}_{A'} \]
form a normalised basis for \( T_c M \) which satisfy the condition (3.2) with the function \( f \) given by
\[ f = (\chi \chi')^{-1/2}. \]

It only remains to show that equation (3.5) does actually admit solutions. In Lorentzian and ultra-hyperbolic signatures, we simply note that (3.5) is automatically satisfied if we choose \( \alpha_I \) and \( \alpha_{I'} \) to satisfy the Weyl equation
\[ D^- \alpha_A = 0, \quad D^+ \alpha_{A'} = 0. \] (3.6)
On a general space of Lorentzian or ultra-hyperbolic signature, equations (3.6) will each have two linearly-independent solutions, and so we have a well defined new basis. In the Riemannian case, there is a vanishing theorem for solutions of the Weyl equation on compact manifolds with non-negative non-vanishing scalar curvature. However in this case we note that if we combine \( \alpha_A \) and \( \alpha_{A'} \) into a pair of Dirac spinors \( \psi_1 = \alpha_0 \oplus \alpha_{0'}, \psi_2 = \alpha_1 \oplus \alpha_{1'} \), then (3.5) is automatically satisfied if
ψ_i are eigenspinors of the Dirac operator. The existence of independent solutions then follows from the general theory of elliptic operators.

A brief note on the reality conditions for the spinor bases in each real signature is perhaps in order. If we are working with a Lorentzian metrics then, as explained in the previous section, we can assume the original spinor bases \{ε_A\}, \{ε_A'\} are related by complex conjugation, with \( \bar{ε}_A = ε_A' \). Similarly, we can take the solutions \( α_A, α_A' \) to be complex conjugates, in which case the fields \( χ \) and \( χ' \) are complex conjugates.

As explained in the Appendix, in the case of ultra-hyperbolic signature, we can take the original spinor bases \{ε_A\}, \{ε_A'\}, and the spinor fields \( α_A \) and \( α_A' \) to be real and independent. In this case, therefore, \( χ \) and \( χ' \) are automatically real.

In the Riemannian case, the quaternion maps \( j_{±} \) on \( V_{±} \) can be assumed to act on spinor bases as
\[
j_{−}ε_A = δ_{AB}ε^B, \quad j_{+}ε_A' = −δ_{A'B'}ε^{B'} .
\]
Similarly, due to the SU(2) nature of the connections on \( V_{±} \), we may assume that the spinor fields \( α_A \) and \( α_A' \) obey
\[
j_{−}α_A = δ_{AB}α^B, \quad j_{+}α_A' = −δ_{A'B'}α^{B'} .
\]
Due to the reality of the symplectic forms \( ε_{±} \), the functions \( χ \) and \( χ' \) are automatically real.

An important point to note, since we are about to consider conformal transformations, is that the function \( f \) is real in each signature.

### 4. Conformal transformations

Given a physical metric \( g \), and a basis \( e_i \) for \( T_c M \), which obey the gauge condition (3.2). We now define a new set of vector fields
\[
\hat{e}_a = f e_a , \tag{4.1}
\]
and metric \( \hat{g} \) satisfying
\[
\hat{g} = f^{-2} g . \tag{4.2}
\]
In terms of the spinor decomposition of the metric, we take the symplectic forms to transform as
\[
\hat{ε}_{±} = f^{-1} ε_{±} .
\]
In order to preserve the normalisation of the spin-bases and to conform with (4.1), we take the bundles \( V_{±} \) to have conformal weight \(-\frac{1}{2}\) with the spin-bases transforming as
\[
ε_A = f^{-1/2} ε_A , \quad ε_A' = f^{-1/2} ε_A' , \quad ε^A = f^{1/2} ε^A , \quad ε^{A'} = f^{1/2} ε^{A'} .
\]

Using the symplectic forms \( ε_{±} \), we can view the curvature spinors \( -W, Φ \) as sections of \( S^4(V^-)^* \) and \( S^2(V^-)^* ⊗ S^2(V^+)^* \) respectively. Defining
\[
\hat{Υ} = \hat{d}(\log f) ∈ \Gamma(Λ^1) , \tag{4.3}
\]
we now recall that under the conformal transformation defined in equation (4.2) the curvature spinors transform as follows [1, 8]: the curvature spinors transform as follows [4, 5]:
\[
^+W = \hat{^+W} , \tag{4.4}
Φ = \hat{Φ} − 2 \left( \nabla \hat{Υ} − \hat{Υ} ⊗ \hat{Υ} \right) − \frac{1}{2} \left( \hat{δ} \hat{Υ} + |\hat{Υ}|^2 \right) \hat{g} , \tag{4.5}
f^2 s = \hat{s} + 6 \hat{δ} \hat{Υ} − 6 |\hat{Υ}|^2 . \tag{4.6}
\]
In order to simplify the analysis slightly, it is helpful to introduce a pair of arbitrary spinor fields \( \pi \in \Gamma(V^-) \), \( \lambda \in \Gamma(V^+) \). We assume that these fields behave under conformal transformations as

\[
\hat{\pi} = \pi, \quad \hat{\lambda} = \lambda.
\]

We then consider the scalar fields

\[
\Psi \equiv W(\lambda, \lambda, \lambda), \quad \hat{\Psi} \equiv \hat{W}(\hat{\lambda}, \hat{\lambda}, \hat{\lambda}),
\]

\[
\Phi = \Phi(\lambda, \lambda, \pi, \pi), \quad \hat{\Phi} = \hat{\Phi}(\hat{\lambda}, \hat{\lambda}, \hat{\pi}, \hat{\pi}).
\]

Since the spinor fields \( W \) and \( \Phi \) are totally symmetric in the relevant spinor indices, we can recover all of the information of \( W \) and \( \Phi \) from the quantities \( \Psi \) and \( \hat{\Phi} \) as the spinor fields \( (\lambda, \pi) \) vary over \( \Gamma(V^\pm) \).

We now wish to rewrite the curvature components of the physical metric \( \hat{g} \) in terms of the spin-connection of the unphysical metric \( g \). To do this, we need to expand \( W, \hat{\Phi} \) and \( \hat{s} \) in terms of the spin-connection \( \gamma_{A'B'} \).

From equations (2.13), (2.17), (2.19) and (2.15), we find that

\[
\hat{\Psi} = \hat{\lambda}^A \hat{\lambda}^{B'} \hat{\lambda}^{C'} \hat{\lambda}^{D'} \left[ \hat{\nabla}_{AA'} \hat{\gamma}^{A'}_{B'C'D'} - \hat{C}_{A'B'}^{E'E'} \hat{\gamma}_{E'E'}^{C'D'} + \hat{\gamma}_{AA'B'C'} \hat{\gamma}^{A'}_{D'D'} \right],
\]

\[
\hat{\Phi} = 2 \hat{\lambda}^A \hat{\lambda}^{B'} \hat{\pi}^{A} \hat{\pi}^{B} \left[ \hat{\nabla}_{AC} \hat{\gamma}^{C'}_{B'C'} - \hat{C}_{AB}^{CC'} \hat{\gamma}_{CC'}^{B'B'} + \hat{\gamma}_{AC} \hat{\gamma}^{C'}_{B'B'} \right],
\]

\[
\hat{s} = -4 \left[ \hat{\nabla}_{AA'} \hat{\gamma}^{A'}_{B'B'} - \hat{C}_{A'B'}^{CC'} \hat{\gamma}_{CC'}^{A'B'} + \hat{\gamma}_{AA'B'C'} \hat{\gamma}^{A'}_{B'B'} \right].
\]

Our goal is now to use these equations to rewrite \( \Psi, \Phi \) and \( s \) in terms of the commutators of the vector fields \( \hat{e}_a \). We now define new fields

\[
\hat{\chi}_{AA'} = \hat{C}_{AB}^{B'} A', \quad \hat{\chi}_{A'A} = \hat{C}_{A'B'}^{B'} A.
\]

In terms of these fields we have

\[
\hat{\Gamma}_a \equiv \hat{C}_{ab}^b = -\hat{\chi}_a - \hat{\chi}_a.
\]

Using the fact that

\[
\hat{\gamma}_{AA'B'C'} = -\hat{C}^{A'}_{A(B'C')} A + \frac{1}{2} \hat{\Gamma}_a (B'\epsilon_{C'}) A, \]

we find that

\[
f^2 \Psi = \hat{\lambda}^A \hat{\lambda}^B \hat{\lambda}^C \hat{\lambda}^D \left[ \hat{\nabla}_A \hat{\hat{\gamma}}^{C'}_{B'C'} D' A + 2 \hat{\hat{C}}_{A'B'}^{E'E'} \hat{\hat{C}}_{C'D'E'E'} \right]
\]

\[
f^2 \Phi = 2 \hat{\lambda}^A \hat{\lambda}^B \hat{\pi}^A \hat{\pi}^B \left[ \hat{\nabla}^{C'}_A \hat{\hat{\gamma}}^{C'}_{B'C'} + \hat{\hat{C}}_{AB}^{CC'} \hat{\hat{\gamma}}^{C'}_{C'B'C'} - \hat{\epsilon}^{D'E'} A \hat{\hat{C}}_{(A'D')} (A' \hat{\hat{C}}_{(B'E')} B) \right]
\]

\[
f^2 \hat{s} = 2 \left[ \hat{\nabla}_a \hat{\chi}^a - \hat{\hat{C}}_{AB'C'}^{A} \hat{\hat{C}}^{A'} \hat{A} \hat{\hat{C}}_{C'B'} \right]
\]

We also find that, by construction, the commutator functions \( \hat{C}_{ab}^c \) obey the condition

\[
\hat{C}_{ab}^c = 2 \hat{\nabla}_a \log f.
\]

This condition is equivalent to the existence of \( \hat{\omega} \in \Gamma(\Lambda^4) \) with the properties that

\[
\mathcal{L}_{\hat{e}_a} \hat{\omega} = 0
\]

\[
(4.10)
\]
\[ \omega(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4) = f^2 \]  

(4.11)

where \( \mathcal{L} \) denotes Lie derivative. In other words, the vector fields \( \{\hat{e}_a\} \) are divergence-free with respect to the volume element \( \omega \).

**Remarks**

The content of the vacuum Einstein equations is the vanishing of the expression for \( \Phi \) and \( s \) as given in equations (4.8) and (4.9), and the divergence-free condition (4.10). There are a few properties of these equations that are of interest.

1. If \( \hat{C}_{AB}^{\mu \nu} = 0 \), we automatically have that the physical metric is Ricci-flat with anti-self-dual Weyl tensor \( \hat{\eta} \). More generally, if \( \hat{C}_{(A'B'C')C} = 0 \) then the Weyl tensor of both the physical and unphysical metric is anti-self-dual. Conversely, the vanishing of the self-dual Weyl tensor is equivalent to the condition that the totally symmetrised part of the spin-connection \( \hat{\gamma}_{(A'B'C')C} \) be pure gauge, which in turn implies that \( \hat{C}_{(A'B'C')C} \) be pure gauge.

2. Only the expression for the trace-free Ricci tensor, equation (4.11), has any explicit dependence on the unprimed coefficients \( \hat{C}_{AB}^{CC'} \). It is geometrically reasonable that the Ricci tensor should depend on both sides of the commutator: If \( \Phi = 0 \), the self-dual spin-connection \( \gamma \) is a self-dual SL(2, \( \mathbb{C} \)) Yang-Mills field. As such, this connection will be integrable on anti-self-dual null planes. However, in a space with algebraically general anti-self-dual Weyl tensor, there are no integrable anti-self-dual null planes (in the above terminology, \( \hat{C}_{AB}^{CC'} \neq 0 \)). Only if the anti-self-dual Weyl tensor vanishes can we fix a local frame for \( T_s M \) with \( \hat{C}_{AB}^{CC'} \) vanishing within the present formalism. We then have our full quota of anti-self-dual null planes, and we are left with a problem that involves only the primed coefficients \( \hat{C}_{A'B'}^{CC'} \).

3. Equations (4.7), (4.8), (4.9) and (4.10) are accompanied by the Jacobi identity for the vector fields \( \{\hat{e}_a\} \), which we write in the form

\[ \epsilon^{abcd} [\hat{e}_b, [\hat{e}_c, \hat{e}_d]] = 0 \]  

(4.12)

We now note that we may rewrite (4.8) and (4.9) in the form

\[ f^2 \Phi = 2 \chi^A \chi^{B'} \chi^{C'} \chi^{D'} \left[ \frac{1}{2} < \hat{e}^A, [\hat{e}^{A'}, \hat{e}^{B'}, \hat{e}^{C'}, \hat{e}^{D'}] > , \xi_{BB'} > + \hat{\chi}_{AB} \hat{\chi}_{A'B} + \hat{C}_{AB}^{CC'} \hat{C}_{A'B'}^{CC'} \right], \]

\[ f^2 s = - < [\hat{e}^{AA'}, [\hat{e}_{BA'}, \hat{e}^{BB'}] > , \xi_{AB'} > + 2 \hat{\chi}_C \hat{\chi}^C. \]

Using (4.12), and translating all spinor indices back into tetrad indices, we find that the Einstein tensor of the physical metric \( g \) obeys the relation

\[ f^2 \left( r(v, v) - \frac{g}{2} g(v, v) \right) = v^a v^b [\hat{\eta}^{cd} < [\hat{e}_c, [\hat{e}_d, \hat{e}_a]] , \xi_b > + 2 \hat{\chi}_a \hat{\chi}_b - \hat{\eta}_{ab} \hat{\chi}_c \hat{\chi}^c - 2 \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}^c ), \]

where \( v \) is an arbitrary vector field, and

\[ + \hat{C}_{ab}^{c} = \frac{1}{2} \left[ \hat{C}_{ab}^{c} + i \epsilon_{abc} \hat{C}_{de}^{c} \right], \quad - \hat{C}_{ab}^{c} = \frac{1}{2} \left[ \hat{C}_{ab}^{c} - i \epsilon_{abc} \hat{C}_{de}^{c} \right], \]

and

\[ \hat{\chi}_a = - + \hat{C}_{ab}^{b}, \quad \hat{\chi}_a = - - \hat{C}_{ab}^{b}. \]

The Ricci-flatness of the physical metric is then summarised in the equation

\[ \hat{\eta}^{cd} < [\hat{e}_c, [\hat{e}_d, \hat{e}_a]] , \xi_b > = - 2 \hat{\chi}_{(a} \hat{\chi}_{b)} + \hat{\eta}_{ab} \hat{\chi}_c \hat{\chi}^c + 2 + \hat{C}_{(a}^{\ c} \hat{C}_{(\ b)}^{\ c} - \hat{C}_{a}^{\ b} \hat{C}_{a}^{\ c} \]

(4.13)
along with the divergence-free condition (4.11). The conformal factor \( f \) which defines the transformation back to the physical metric is deduced from equation (4.11).

Equation (4.13) is a form of the Einstein equations discussed in [6], where it was noted that the object on the left-hand-side of the equation can be interpreted as a generalisation of the Yang-Mills operator for a constant connection on flat space time, \( \mathbb{M} \), taking values in the algebra of divergence-free vector fields on the auxiliary four-manifold \( \mathbb{M} \). The term on the right-hand-side of (4.13) is a source term, which is purely an interaction between the self-dual and anti-self-dual parts of the gravitational field. In the current context, this equation is interesting because it is symmetrical between the self-dual and anti-self-dual parts of the gravitational field, even though the formalism we started does not explicitly have such a symmetry. Whether a chiral or non-chiral approach to the Einstein equations is more generally useful probably remains to be seen, especially since most chiral formalisms are very sensitive to the dimension of the space-time we choose to work with.

5. Complex structures and torsion

We have shown in the previous section how, starting from the self-dual 2-form approach to the Einstein equations, we can partially fix spin-frames in a way that leads to a chiral description of the Einstein equations in terms of a frame of divergence-free vector fields.

Conversely, a solution of the vacuum Einstein equations in the latter approach would correspond to a set of vector fields \( \hat{e}_i \) and a function \( f \) which were solutions of equations (4.13), (4.10) and (4.11). One can then return to the physical metric, and directly reconstruct the elements of the self-dual 2-form approach, by simply reversing the conformal transformation given in equations (4.1) and (4.2). Although there is therefore a direct way of returning to the self-dual variables, the connection with the divergence-free vector field approach does seem to suggest investigation of some other aspects of the self-dual 2-forms approach.

In this section, we will concentrate on real Riemannian spaces, since this seems to be the context where the geometrical interpretation is most clear. We begin by restating the equations of the self-dual two-form formalism in this signature. We have a set of 2-forms, \( \Sigma_{A'B'} \), and an \( \text{su}(2) \) connection \( \gamma \) which obey the equations

\[
\Sigma_{A'B'} \Sigma^{C'D'} = \epsilon^{(A'C'} \epsilon_{B')D'} \nu, \tag{5.1}
\]

\[
d \Sigma + [\gamma, \Sigma] = 0, \tag{5.2}
\]

\[
R_{A'B'} = +W_{A'B'C'D'} \Sigma^{C'D'} + \Phi_{ABA'B'} \Sigma^{AB} + \frac{s}{4} \Sigma_{A'B'}. \tag{5.3}
\]

For the moment, we concentrate on finding the natural analogues of the divergence-free equation (4.10) and the equation for the conformal factor (4.11) in terms of the 2-forms \( \Sigma_{A'B'} \).

We begin by can defining the dual, \( \tilde{\nu} \in \Gamma(\wedge^4 TM) \), of the volume form \( \nu \) by the condition

\[
\tilde{\nu}(\nu) = 1.
\]

The self-duality of the 2-forms \( \Sigma_{A'B'} \) is expressed by the relation

\[
\tilde{\nu}(\Sigma_{A'B'}) = i\Sigma_{A'B'}.
\]

We also find that

\[
\tilde{\nu}(\epsilon^{AA'} \wedge \Sigma^{B'C'}) = \epsilon^{A'(B' \epsilon^{C'})A}.
\]

We now define the set of vector fields

\[
\mathbf{v}^{A'B'} = \tilde{\nu}(\ast d \Sigma^{A'B'}) \tag{5.4}
\]

These vector fields are automatically divergence-free with respect to \( \nu \) in the sense that

\[
\mathcal{L}_{\mathbf{v}} \Sigma_{A'B'} \nu = 0.
\]
From the definition of $v^{A'B'}$ along with (2.25), it follows that
\[
v_{A'B'} = -C_{A'B'} e_c + \Gamma_{A(A'} e_{B')}^A.
\]
Rewriting the right-hand-side of this relation in terms of the unphysical vector fields $\{\hat{e}_i\}$, we find that
\[
v_{A'B'} = -f^{-2} \hat{C}_{A'B'}^c \hat{e}_c = -\frac{1}{2} f^{-2} \hat{e}^{AB} [\hat{e}_{AA'}, \hat{e}_{BB'}].
\] (5.5)
Thus, the divergence-free vector fields that naturally occur in the self-dual 2-form approach are, up to a factor, simply the self-dual part of the commutator of the conformally transformed vector fields.

By reversing the conformal transformation (1.2), we deduce that
\[\nu = f^2 \hat{\omega}.
\]
Therefore, given the function $f$, the 4-form $\hat{\omega}$ and the self-dual part of the commutator $[\hat{e}_i, \hat{e}_j]$, we may directly reconstruct the vectors $v_{A'B'}$ and the volume form $\nu$. This in turn, via equation (5.4), gives us $d\Sigma_{A'B'}$. From this, along with the fact that the $\Sigma_{A'B'}$ are self-dual with respect to the volume form $\nu$, means that locally we can reconstruct the $\Sigma_{A'B'}$ up to ambiguity
\[
\Sigma_{A'B'} \mapsto \Sigma_{A'B'} + d\chi_{A'B'},
\]
where
\[^{*}d\chi_{A'B'} = d\chi_{A'B'}.
\]
In the particular case when $\hat{C}_{A'B'} = 0$, we see from equations (4.7), (4.8) and (4.9) that the physical metric is Ricci-flat and has anti-self-dual Weyl tensor (3). In this case, $v_{A'B'} = 0$ and so $d\Sigma_{A'B'} = 0$. These equations, along with the orthogonality condition (2.15) completely determine, on a simply connected region, metrics with anti-self-dual Riemann tensor (1).

The form of $v_{A'B'}$ in equation (1.3) suggests that it is the obstruction to the integrability of self-dual null planes in the complexified tangent space. Since all such planes are integrable if and only if the Weyl tensor is anti-self-dual, this suggests that the vector fields $v_{A'B'}$ should be related to the self-dual part of the Weyl tensor.

Suppose we consider simply connected, Riemannian spaces with a Ricci-flat, anti-self-dual metric, then we may interpret the objects above in terms of hyper-kähler geometry. As such, we have a metric $g$ which is Hermitian with respect to three integrable complex-structures $(I, J, K)$ on $TM$ which obey the quaternion algebra:
\[
I^2 = J^2 = K^2 = -1, \quad IJ = K, \quad JK = I, \quad KI = J,
\] (5.6)
which are covariantly constant:
\[\nabla I = \nabla J = \nabla K = 0.
\] (5.8)
If the complex-structures are integrable, then $(I, J, K)$ define a hyper-complex structure, and the conformal structure of the metric $g$ is anti-self-dual. The integrability condition for the covariant constancy of the complex-structures then implies that the self-dual spin-connection is flat, and therefore that the metric $g$ is Ricci-flat.

More concretely, it is always possible to choose a null basis for $T_cM$ where the inverse metric is as in equation (2.3), and where $(I, J, K)$ may be represented in matrix form as
\[
\begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & 0 & 0 & i
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & i & 0 \\
0 & 0 & 0 & i \\
i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{pmatrix}.
\] (5.9)
In terms of the null basis we have used, the integrability conditions for these structures implies that there exist functions $a, b, c, d$ with

$$[e_1, e_4] = ae_1 + be_4, \quad [e_2, e_3] = ce_2 + de_3, \quad [e_1, e_2] + [e_3, e_4] = -de_1 + be_2 + ae_3 - ce_4. \tag{5.10}$$

If these conditions are satisfied, then the Weyl tensor is anti-self-dual. Demanding that the complex-structures be covariantly constant requires satisfaction of the integrability condition that the structures commute with the Riemann curvature

$$[R(X, Y), I] = [R(X, Y), J] = [R(X, Y), K] = 0, \quad \forall X, Y \in \Gamma(T_{\varepsilon}M).$$

Since, in four dimensions, $(I, J, K)$ define (via the metric) a three-dimensional subspace of $\Lambda^2$, which we can define to be $\Lambda^2+$, this tells us that the self-dual part of the Riemann curvature of the metric $g$ must vanish. As such, the metric $g$ is Ricci-flat and anti-self-dual.

Given the three almost-complex-structures $(I, J, K)$, it is useful to notice that if $(a, b, c)$ are the components of a unit vector in $\mathbb{R}^3$, then the combination $aI + bJ + cK$ also defines an almost-complex-structure on $TM$, so a hyper-kähler manifold actually admits an $S^2$ worth of almost-complex-structures $[12]$. We view this $S^2$ as a complex projective line $\mathbb{P}_1$, which is itself constructed from two copies $U, U'$ of the complex plane with coordinates $\zeta, \zeta'$, related by $\zeta = (\zeta')^{-1}$ on $U \cap U'$. We then combine the three almost-complex-structures into the single object

$$J_\zeta = \frac{1}{1 + \zeta \bar{\zeta}} \left( (1 - \zeta \bar{\zeta}) I + (\zeta + \bar{\zeta}) J + i(\zeta - \bar{\zeta}) K \right).$$

If $v$ is a vector with $Iv = iv$, then we define $w = v + \zeta K v$ which automatically has the property that $J_\zeta w = iw [12]$. As such, if we assume $(I, J, K)$ are represented as in equation (5.9), then a local basis for the $(1, 0)$ space of $J_\zeta$ is given by

$$e_1 + i\zeta e_3, \quad e_4 + i\zeta e_2,$$

and in a similar fashion we find that a basis for the $(0, 1)$ space is given by

$$e_2 + i\bar{\zeta} e_4, \quad e_3 + i\bar{\zeta} e_1.$$

If we ask that $J_\zeta$ is integrable, $\forall \zeta \in \mathbb{C}$, we recover equations (5.10). More precisely, if we define the Nijenhuis torsion tensor $N$ by

$$4N(X, Y) = [X, Y] + J[IX, Y] + J[X, JY] - [JX, JY],$$

for $X, Y \in T_{\Phi}M$, then on the $(1, 0)$ space of $J_\zeta$ defined above we find that

$$(1 + \zeta \bar{\zeta}) N(e_1 + i\zeta e_3, e_4 + i\zeta e_2) = \left[ C_{14}^3 + i\zeta \left( C_{12}^3 + C_{34}^3 \right) + \zeta^2 C_{23}^3 \right.$$$$
- i\zeta \left( C_{13}^1 + i\zeta \left( C_{12}^1 + C_{34}^1 \right) + \zeta^2 C_{23}^2 \right) \left( e_3 + i\bar{\zeta} e_1 \right)$$

$$\left( C_{14}^2 + i\zeta \left( C_{12}^2 + C_{34}^2 \right) + \zeta^2 C_{23}^2 \right.$$$$- i\zeta \left( C_{13}^4 + i\zeta \left( C_{12}^4 + C_{34}^4 \right) + \zeta^2 C_{23}^4 \right) \left( e_2 + i\bar{\zeta} e_4 \right).$$

The parts of $C_{ab}^c$ which occur in this expression are the totally symmetrised parts of the symmetrised coefficients $C_{(A'B'C')C}$, which in turn correspond to the conformally invariant part of the vector fields $v_{A'B'}$. As such, when the conformal structure is not anti-self-dual, the torsion of the almost-complex-structure $J_\zeta$ is directly related to the divergence-free vector fields $v_{A'B'}$.

Treating the parameter $\zeta$ as a constant, the tensor $J_\zeta$ is covariantly constant when we have a hyper-kähler structure. In the more general case we wish to consider, the obstruction to the covariant constancy of $J_\zeta$ is given by

$$([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) J_\zeta = [R(X, Y), J_\zeta]. \tag{5.11}$$
If the metric is Einstein, with $\Phi = 0$, the right-hand-side of this equation vanishes if take the bivector $X \wedge Y$ to be anti-self-dual. As such, the condition that the metric be Einstein becomes the condition that the Riemann tensor, viewed as a 2-form with values in $\text{End}(T_x M)$, commutes with the almost-complex-structure $J_{\zeta}$ when acting on any anti-self-dual bi-vector. If the Weyl tensor is self-dual, then such an anti-self-dual bi-vector would define an integrable anti-self-dual null plane $\Sigma$, with equation (5.11) holding for any $X, Y \in T_x \Sigma$. Since such planes would be integrable, we can interpret the vanishing of the left-hand-side of equation (5.11) in this case as the integrability condition for the tensor $J_{\zeta}$ to be covariantly constant on the surface $\Sigma$. In the general case when $\tilde{W} \neq 0$, however, equation (5.11) has no such interpretation as an integrability condition.

Perhaps the most compact way of viewing is in quaternionic notation. Generally, we translate objects with values in the vector bundle associated with the vector representation of $SO(4)$ into quaternion valued objects, and those taking values in the vector bundle corresponding to the adjoint representation of $SU(2)$ into objects with values in the imaginary quaternions. For example, given a standard orthonormal tetrad $\epsilon_i$ for the Riemannian metric $g = \sum_{i=0}^{3} \delta_{ij} \epsilon_i \otimes \epsilon_j$, we combine the 1-forms $\epsilon_i$ into the quaternion-valued 1-form

$$\epsilon = \epsilon_0 + i \epsilon_1 + j \epsilon_2 + k \epsilon_3,$$

with quaternionic conjugate

$$\overline{\epsilon} = \epsilon_0 - i \epsilon_1 - j \epsilon_2 - k \epsilon_3.$$

In terms of these forms we may write the metric as

$$g = \frac{1}{2} [\epsilon \otimes \overline{\epsilon} + \overline{\epsilon} \otimes \epsilon],$$

which is invariant under the $SU(2) \times SU(2)/\mathbb{Z}_2$ action corresponding to the left or right multiplication of $\epsilon$ by unit modulus quaternions. The form $\epsilon \wedge \overline{\epsilon}$ gives us an imaginary-quaternion-valued self-dual 2-form

$$\epsilon \wedge \overline{\epsilon} = -2i [\epsilon_0 \wedge \epsilon_1 + \epsilon_2 \wedge \epsilon_3] - 2j [\epsilon_0 \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_1] - 2k [\epsilon_0 \wedge \epsilon_3 + \epsilon_1 \wedge \epsilon_2],$$

whilst $\overline{\epsilon} \wedge \epsilon$ furnishes us with imaginary-quaternion-valued anti-self-dual 2-form,

$$\overline{\epsilon} \wedge \epsilon = 2i [\epsilon_0 \wedge \epsilon_1 - \epsilon_2 \wedge \epsilon_3] + 2j [\epsilon_0 \wedge \epsilon_2 - \epsilon_3 \wedge \epsilon_1] + 2k [\epsilon_0 \wedge \epsilon_3 - \epsilon_1 \wedge \epsilon_2].$$

The complex structures $(I, J, K)$ we considered earlier are modelled upon left multiplication by $(i, j, k)$ of the form $\epsilon$. The self-dual forms appearing in equation (5.12) are the corresponding Kähler forms when the metric $g$ is hyper-kähler.

We may now encode the information of the self-dual spin-connection into the imaginary-quaternion-valued 1-form

$$\Gamma = \frac{i}{2} [\Gamma_{01} + \Gamma_{23}] + \frac{j}{2} [\Gamma_{02} + \Gamma_{31}] + \frac{k}{2} [\Gamma_{03} + \Gamma_{12}],$$

which has the property that

$$d (\epsilon \wedge \overline{\epsilon}) = 2 \Gamma \wedge \epsilon \wedge \overline{\epsilon}.$$

The curvature of this connection is the imaginary-quaternion-valued 2-form

$$R = d \Gamma + \frac{1}{2} [\Gamma, \Gamma] = +W\lambda^+ + \Phi\lambda^- + \frac{s}{12} \epsilon \wedge \overline{\epsilon},$$

where we have adopted the notation of equation (2.19), and $\pm \lambda$ are orthonormal bases for $\Lambda^2 \pm$.
The volume form $\nu$ is deduced from the relation

$$(\epsilon \wedge \tau) \wedge (\epsilon \wedge \tau) = -24\nu.$$  

Given this volume form, we may define the imaginary-quaternion-valued vector field

$$v = \tilde{\nu}(d(\epsilon \wedge \tau)),$$

with the property that

$$\mathcal{L}_v \nu = 0.$$  

It follows from equation (5.13) that the vector field $v$ carries a large amount of the information of the self-dual spin-connection $\gamma$. If the metric is Einstein, the two-form $R$ is self-dual and therefore vanishes when evaluated on any anti-self-dual bivector. Therefore, the action of left multiplication by $(i,j,k)$ of the form $\epsilon$ will be covariantly constant on anti-self-dual bivectors in the sense that the left hand side of equation (5.11) will vanish. Unfortunately, it does not seem possible to find any straightforward interpretation of this condition directly in terms of the vector fields $v$.

Similarly, whether there exists a concise geometrical interpretation of the Einstein equations in terms of the almost-complex-structure $J_\tau$ and its torsion which could serve as a useful alternative to the usual self-dual two-forms approach remains to be seen. More correctly, we should look on the object $J_\tau$ as the projection to $TM$ of the horizontal part of the natural almost-complex-structure on the projective spin-bundle $PV^+$ [7]. This almost-complex-structure on $PV^+$ is integrable if and only if the Weyl tensor on $M$ is anti-self-dual. If we allow ourselves to consider the full spin-bundle $V$, there do seem to be differential forms which capture some of the content of the Einstein equations [3, 13]. Since approaches, however, are generally non-chiral, requiring information about both the natural almost-complex-structure $J^+$ on $V^+$, and the corresponding structure $J^-$ on $V^-$. As such, it would appear that there may some redundancy in these descriptions. The underlying geometry of these constructions also seems rather unclear.

6. Conclusion

The main objective of this paper is to give a more concrete derivation of the results of [6] where the Einstein equations were reformulated in terms of a set of divergence-free vector fields on an auxiliary four-manifold, in a way which seemed to make some connection with the Yang-Mills equations. We have demonstrated how these equations naturally arise, after a conformal transformation, if we choose spinor bases which satisfied the Dirac equation. Since our starting point was the self-dual two-form formalism the equations we arrive at are actually a chiral version of the $LsdiffM$ equations, which reduce to the form given in [3] with the help of the Jacobi identity.

With the divergence-free vector approach in mind, we then investigated some other aspects of the self-dual two-forms approach. In particular, there naturally arises a particular set of divergence-free vector fields related to the self-dual two-forms. In the Riemannian sector, these vector fields may be interpreted in terms of the torsion of projection to the tangent space of the natural almost-complex-structure on the projective spin-bundle of the space. The natural statement of the Einstein condition in this context is simply that the almost-complex-structure commutes with the Riemann curvature evaluated on an anti-self-dual bivector. Unless the anti-self-dual Weyl tensor vanishes, however, this condition on the almost-complex-structure cannot be interpreted as an integrability condition. This is a similar dilemma to the one we face when we interpret the Einstein condition in terms of the self-dual spin-connection being an self-dual SU(2) Yang-Mills field. In this case, the self-duality condition on the Yang-Mills connection may be interpreted as the integrability condition for the existence of particular holomorphic vector bundles over the projective spin-bundle of our space, but only if the space itself is self-dual [7]. Alternatively, the self-dual Yang-Mills equations are only integrable on an self-dual background, the obstruction to their integrability being precisely the anti-self-dual part of the Weyl tensor.

It should be noted that all considerations have been local in nature. In the use of spinor techniques, we are implicitly assuming the existence of a spin structure. This does not seem to
be a significant problem, however, since locally any manifold is spin and also a large proportion of the
spinor analysis we use carries through with only a projective spin structure, to which there is
no topological obstruction. A more important global problem is that we are implicitly assuming
our metric may be described in terms of the non-vanishing linearly-independent set of vector fields
\{e_i\}. The existence of such a set of vector fields is known to place restrictions on the cohomology of
the underlying manifold \[14\]. In our case, we require four non-vanishing linearly-independent vector
fields on a four-dimensional manifold, which implies that the tangent bundle of the manifold is trivial
(implicitly implying that the manifold admits a spin-structure anyway). However, it is possible to
develop a gauge invariant version of the formalism used here, where full SO(1, 3) invariance is restored
and the internal bases are no longer constrained by the divergence-free condition \[15\]. Although such
a formalism could be used globally, it is essentially equivalent to the usual compacted spin-coefficient
formalism \[1\]. It does, however, lead to a straight-forward derivation of the main points of the
Light-Cone-Cut description of conformally Einstein spaces \[16\].

Acknowledgments

The author is grateful to thank Riccardo Capovilla for several illuminating discussions concern-
ing the self-dual two-form approach to General Relativity, and to Andrew Chamblin for discussions
on global aspects of four-dimensional Einstein manifolds. He would also like to thank the University
of Newcastle for the Sir Wilfred Hall fellowship, under which this work was carried out.

Appendix: Reality Conditions

Although the reality conditions for real manifolds with metrics of Lorentzian signature were
discussed in Section \[3\] for the sake of generality we here discuss the reality conditions for metrics of
other real signatures in four dimensions. As such, we begin with a complex four-manifold \(M\) with a
holomorphic metric \(g\). For each \(x \in M\), a frame for \(T_x M\) defines an isomorphism
\(T_x M \cong \mathbb{C}^4 \cong \mathbb{C}(2)\), with a vector \(V \in T_x M\) being mapped to a \(2 \times 2\) matrix via
\[
\tilde{V} = \frac{1}{\sqrt{2}} \begin{pmatrix}
V^0 + V^3 & V^1 - iV^2 \\
V^1 + iV^2 & V^0 - V^3
\end{pmatrix} \in \mathbb{C}(2).
\] (A.1)

We then have
\[
\det \tilde{V} = \frac{1}{2} g(V, V). \tag{A.2}
\]

The map \(\tilde{V} \mapsto L\tilde{V}R\), where \(L, R \in \text{SL}(2, \mathbb{C})\) gives an explicit description of the isomorphism
\(\text{SO}(4, \mathbb{C}) \cong \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})/\mathbb{Z}_2\). Passing to the double cover, with \(\text{Spin}(4, \mathbb{C}) \cong \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})\), we introduce the two-dimensional complex vector bundles of spinors, \(V^\pm\), associated with
the separate \(\text{SL}(2, \mathbb{C})\) factors. Being \(\text{SL}(2, \mathbb{C})\) bundles, these bundles inherit symplectic forms, denoted \(\epsilon^\pm\), which define isomorphisms \(V^\pm \cong (V^\pm)^*\). The form of the metric given in equation \(\text{(A.3)}\) then agrees with that of equation \(\text{(2.3)}\). The map \(\text{(A.4)}\) gives the isomorphism between \(T_x M\) and \(V^+ \times V^-\). The 2-form \(\Sigma\) introduced in Section \(\text{[3]}\) defines an isomorphism between \((\Lambda^2)^*\), and the
adjoint representation space of \(\text{SL}(2, \mathbb{C})\), so we can choose a basis for \(V^+\) where \(\Sigma\) is represented by
a trace-free \(2 \times 2\) matrix of complex 2-forms.

We wish to find the real structures satisfied by \(\Sigma\) and the bundles \(V^\pm\) which characterise real
pseudo-Riemannian metric of signature \((0, 4)\) and \((2, 2)\) which we will refer to as Riemannian and
ultra-hyperbolic respectively.
In the Riemannian case, Spin(4) ∼= SU(2) × SU(2), so the bundles \( \mathbb{V}^\pm \), in addition to the symplectic forms mentioned above, inherit Hermitian forms denoted \( <, >_\pm \), which define isomorphisms \( \mathbb{V}^\pm \cong (\mathbb{V}^\pm)^* \). Alternatively, since SU(2) ∼= Sp(1), the bundles have a quaternionic structure, with anti-linear isomorphisms \( j_\pm : \mathbb{V}^\pm \to \mathbb{V}^\pm \), such that \( j_\pm^2 = -1 \). We can then identify the Hermitian structures in the form

\[
< u, v >_- = \epsilon_-(u, jv), \quad u, v \in \mathbb{V}^-, \\
< u, v >_+ = \epsilon_+(ju, v), \quad u, v \in \mathbb{V}^+,
\]

with

\[
\epsilon_\pm(du, dv) = \epsilon_\pm(u, v)
\]

in both cases. Similarly, the bundles \( S^m\mathbb{V}^- \otimes S^n\mathbb{V}^+ \) inherit a quaternionic structure when \( m + n \) is an odd integer, and a real structure when \( m + n \) is even.

In the case of an ultra-hyperbolic metric, Spin(2, 2) ∼= SL(2, \mathbb{R}) × SL(2, \mathbb{R}) and the bundles \( \mathbb{V}^\pm \) inherit real structures i.e. anti-linear maps \( \sigma_\pm : \mathbb{V}^\pm \to \mathbb{V}^\pm \) with \( \sigma_\pm^2 = 1 \). Therefore, \( \mathbb{V}^\pm \) arise naturally as the complexification of the bundles of real spinors, \( \mathbb{V}_r \), which are invariant under the maps \( \sigma_\pm \). The higher spinor bundles \( S^m\mathbb{V}^- \otimes S^n\mathbb{V}^+ \) inherit real structures, which single out real spinors to be those which lie in \( S^m\mathbb{V}^-_r \otimes S^n\mathbb{V}^+_r \).

If we wish, we can choose local bases for \( \mathbb{V}^\pm \) adapted to the particular real structures present in each signature. We may always choose bases \( \epsilon_A = (o, i) \) for \( \mathbb{V}_r^+, \) \( (o', l') \) for \( \mathbb{V}_r^-, \), which are orthonormal with respect to the symplectic forms \( \epsilon_\pm \) in the sense of equation (2.4). In the ultra-hyperbolic case, we may choose these basis spinors to lie in the real spin-bundles \( \mathbb{V}_r^\pm \), so identifying the real structures \( \sigma_\pm \) with complex conjugation, we have

\[
\text{Ultra – hyperbolic :} \quad \tau = o, \quad \tau = i, \quad \tau = o', \quad \tau = l'.
\]

In this basis, a spinor is real if its components are real. In particular the matrix-valued 2-form \( \Sigma \) becomes a real trace-free matrix, the components of the Weyl spinor \( -\mathbb{W} \) and the Ricci spinor \( \Phi \) are real. Finally, the vector fields \( \{e_i\} \) of equation (2.7) are real, and span the real tangent space.

In the Riemannian case, taking components of spinors in \( \mathbb{V}_r^\pm \) with respect to the bases defines isomorphisms \( \mathbb{V}_r^\pm \cong \mathbb{C}^2 \cong \mathbb{H} \). Given the form of the map (A.3), and the action of \( \text{SL}(2, \mathbb{C}) \times \text{SLTC} \) as \( \psi \mapsto L\psi R \), we represent elements of \( \mathbb{V}_r^\pm \) as row vectors, transforming as \( \mathbb{V}_r^+ \in \pi \mapsto \pi R \) and elements of \( \mathbb{V}_r^- \) as column vectors, transforming as \( \mathbb{V}_r^- \in \psi \mapsto L\psi \). Given the components \( (z_1, z_2) \) of an element of \( \mathbb{V}_r^+ \), we map this to \( w = z_1 + z_2j \in \mathbb{H} \). The action of \( R \) now corresponds to right multiplication by a unit modulus quaternion. The Sp(1) property means that this action is compatible with left multiplication of \( w \) by the unit quaternion \( j \), which acts as \( (z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2}) \). This is the model for the quaternionic structure \( j_- \) on \( \mathbb{V}_r^+ \). Similarly, the action of the quaternionic structure \( j_- \) on \( \mathbb{V}_r^- \) corresponds to right multiplication by \( -j \), which is compatible with the action \( L \) on \( \mathbb{V}_r^- \), which corresponds to left multiplication by a unit quaternion \( l \).

In terms of the bases \( (o, i) \), and \( (o', l') \) we may therefore explicitly take \( j_\pm \) to act as

\[
\begin{align*}
j_-(o) &= i, \\
j_-(-i) &= -o, \\
j_+(o') &= -l', \\
j_+(l') &= o',
\end{align*}
\]

or, more compactly

\[
\text{Riemannian :} \quad j_-\epsilon_A = \delta_{AB}\epsilon_B, \quad j_+\epsilon_{A'} = -\delta_{A'B'}\epsilon^{B'}.
\]

Identifying the real structures that occur on higher bundles may be identified with complex conjugation, with respect to these spinor bases, the object \( \Sigma \) becomes a trace-free skew-Hermitian matrix of two-forms. The self-dual Weyl spinor \( +\mathbb{W} \), as a real element of \( S^4\mathbb{V}^+ \), has components which satisfy

\[
+\mathbb{W}_{A'B'C'D'} = \delta_{A'E'}\delta_{B'F'}\delta_{C'G'}\delta_{D'H'} + \mathbb{W}_{E'F'G'H'}.
\]

Therefore, if the Weyl spinor is non-vanishing, the roots of the quartic polynomial \( +\mathbb{W}(\pi, \pi, \pi, \pi) \), where \( \pi \in \mathbb{V}^+ \), occur in pairs of the form \( \zeta = (\lambda, -1/\lambda) \), where \( \zeta = \pi_0/\pi_1 \), is an affine coordinate
Therefore, in terms of the standard classification of Lorentzian Weyl tensors\cite{1}, the self-dual Weyl tensor of a Riemannian metric is either algebraically general, type–D or flat\cite{17}. Finally, the complex vector fields \(\{e_i\}\) of equation (2.7) obey the reality conditions:

\[
\begin{align*}
\overline{e_1} &= -e_2, & \overline{e_2} &= -e_1, & \overline{e_3} &= e_4, & \overline{e_4} &= e_3,
\end{align*}
\]

and the real tangent space takes the form

\[
\text{Riemannian : } T_xM = \text{Span}_{\mathbb{R}} (i(e_1+e_2), e_1-e_2, i(e_3-e_4), e_3+e_4).
\]  

(A.3)

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