A simple algorithm for computing the document array

Felipe A. Louza*

Department of Computing and Mathematics, University of São Paulo,
Ribeirão Preto, Brazil

Abstract

We present a simple algorithm for computing the document array given the string collection and its suffix array as input. Our algorithm runs in linear time using constant additional space for strings from constant alphabets.

Keywords: Text indexing, Document array, String collections

1. Introduction

The suffix array (SA) is a fundamental data structure in string processing that is commonly accompanied by the document array when indexing string collections (e.g. [3, 4, 5, 6, 7]). Given a collection of $d$ strings of total length $N$, the document array is an array of integers $DA[1, N]$ in the range $[1, d + 1]$ that gives which document each suffix in the suffix array belongs to.

It is well-known that $DA$ can be represented in a compact form by using a bitvector $bit[1, N]$ with support to rank operations, requiring $N + o(N)$ bits of space [8]. However, there are applications where $DA$ must be accessed sequentially (e.g. [9, 10, 11, 12, 13]), and having the array $DA[1, N]$ computed explicitly is important.

In this paper we show how to compute $DA$ given the string collection and its suffix array as input in linear time. Our algorithm reuses the space of $SA$ to store auxiliary arrays used to compute $DA$. We reconstruct $SA$ in its original space during $DA$ computation. The workspace of our algorithm, that is, the

*Corresponding author
Email address: louza@usp.br (Felipe A. Louza)
extra space used in addition to the input and output, is $O(\sigma \log N)$ bits, where $\sigma$ is the alphabet size. Therefore, for constant alphabets, the workspace of our algorithm is constant.

2. Background

Let $T$ be a string of length $n$, over an alphabet $\Sigma$ of size $\sigma$, such that $T[n] = \$\$ is an end-marker symbol that does not occur elsewhere in $T$ and precedes every symbol of $\Sigma$. $T[i, j]$ denotes the substring from $T[i]$ to $T[j]$ inclusive, for $1 \leq i \leq j \leq n$. A suffix of $T$ is a substring $T[i, n]$. We define $\text{rank}_c(T, i)$ as the number of occurrences of symbol $c$ in $T[1, i]$. The string $T$ is stored in $N \log \sigma$ bits of space.

The suffix array (SA) \cite{1} for $T$ is an array of integers in the interval $[1, n]$ that provides the lexicographical order of all suffixes of $T$. The inverted permutation of SA, denoted as ISA, is defined as $\text{ISA}[\text{SA}[i]] = i$. SA can be computed in $O(n)$ time using $O(\sigma \log n)$ bits of workspace \cite{14}. The arrays SA and ISA use $N \log N$ bits of space each one.

The Burrows-Wheeler transform (BWT) \cite{15} of $T$ is obtained by sorting all $n$ rotations of $T$ in a conceptual matrix $M$, and taking the last column $L$ as the BWT. It can also be defined through the relation

$$\text{BWT}[i] = T[\text{SA}[i]] - 1 \mod n. \tag{1}$$

The Last-to-First (LF) mapping states that the $k^{th}$ occurrence of a symbol $c$ in column $L$ of $M$ and the $k^{th}$ occurrence of $c$ in the first column $F$ correspond to the same symbol in $T$. Let $C[c]$ be the number of symbols $c' < c$ in $T[1, N]$. We define

$$\text{LF}(i, c) = C[c] + \text{rank}_c(\text{BWT}, i) \tag{2}.$$ 

We use shorthand $\text{LF}(i)$ for $\text{LF}(i, \text{BWT}[i])$. $\text{LF}(i)$ may be computed on-the-fly in $O(\log \sigma)$ time querying a wavelet tree \cite{16} for the rank queries on Equation 2. The wavelet tree requires additional $(N + o(N)) \log \sigma$ bits of space.
LF-mapping allows us to navigate $T$ right-to-left, given $T[k] = \text{BWT}[i]$, then $T[k-1] = \text{BWT}[(\text{LF}(i))]$. $T$ can be reconstructed backwards from $\text{BWT}$ starting with $T[n] = \text{BWT}[1] = \$$. and repeatedly applying $\text{LF}$ for $n$ steps.

### 2.1. String collections

Let $\mathcal{T} = T_1, T_2, \ldots, T_d$ be a collection of $d$ strings of lengths $n_1, n_2, \ldots, n_d$. The suffix array of $\mathcal{T}$ is the $\text{SA}$ built for the concatenation of all strings $T^{\text{cat}}[1, N] = T_1 T_2 \ldots T_d \#$, with total size $N = \sum_{i=1}^{d} (n_i) + 1$ and a new end-marker $\#$ < $. $\text{SA}$ can be computed in $O(N)$ time using $O(\sigma \lg N)$ bits of workspace \[1\], such that an end-marker $\$ from string $T_i$ is smaller than a $\$ from string $T_j$ iff $i < j$, which is equivalent to using $d$ different end-markers as separators, without increasing the alphabet size.

The $\text{BWT}$ may also be generalized for string collections. $\text{BWT}$ of $\mathcal{T}$ is obtained from $\text{SA}$ and $T^{\text{cat}}$ through the Equation \[1\] However, LF-mapping through Equation \[2\] does not work for symbols $\$$, since the $k^{th}$ symbol $\$ in column $L$ does not (necessarily) corresponds to the $k^{th}$ symbol $\$ in column $F$, in this case $\text{LF}(i, \$)$ is undefined \[7\].

$\text{LF}(i)$ can be pre-computed in an array $\text{LF}[1, n]$ through Equation \[8\] such that $\text{LF}$ still works for $\$-symbols. Given $\text{SA}$ and $\text{ISA}$, we have

$$\text{LF}[i] = \text{ISA}[(\text{SA}[i] - 1) \mod n]$$

The array $\text{LF}$ uses $N \lg N$ bits of space.

### 2.2. Document array

The document array $(\text{DA})$ is an array of integers in the interval $[1, d+1]$ that tells us which document $j \in \mathcal{T}$ each suffix in the $\text{SA}$ belongs to \[2\]. We define $\text{DA}[i] = j$ iff suffix $T^{\text{cat}}[\text{SA}[i], N]$ came from string $T_j \in \mathcal{T}$. $\text{DA}[1] = d + 1$ for the last suffix $T^{\text{cat}}[N, N] = \#$. $\text{DA}[1, N]$ uses $N \lg(d + 1)$ bits of space.

The array $\text{DA}[1, N]$ can also be represented using a wavelet tree \[16\], within the same $N \lg(d + 1)$ bits with additional functionalities \[3\]. $\text{DA}$ can still be compressed using grammars when the string collection is repetitive \[18\].
2.3. Related work

Given $T^{cat}$ and SA, the document array DA can be constructed in $O(N)$ time using $N \log N$ additional bits to store ISA, such that $DA[ISA[i]] = j$ for $i = \ell_{j-1}, \ldots, \ell_j$, with $\ell_0 = 1$ and $\ell_j = \sum_{k=1}^{j} n_k$, see [19, Alg. 5.29].

DA can also be computed in the same fashion as the text $T^{cat}$ is reconstructed from its BWT. Given $T^{cat}$ and SA, we can compute ISA[1, N] and then array LF[1, N] (Equation 3). DA is obtained in $O(N)$ time during the BWT inversion using $N \log N$ bits of workspace, see [19, Alg. 7.30]. In particular, in Section 3 we show an alternative algorithm that reuses the space of SA to compute LF without ISA. Our algorithm uses $O(\sigma \log N)$ bits of workspace and reconstructs SA during DA computation.

**Lightweight alternative.** DA can be computed using a compact data structure composed by a bitvector $bit[1, N]$ with rank support operation. $bit$ is built over $T^{cat}[1, N]$, such that

$$bit[i] = 1 \text{ iff } T^{cat}[i] = \$ \text{ and } bit[i] = 0, \text{ otherwise.}$$

(4)

$DA[i]$ can be obtained using $bit$ and SA as follows [19, Alg. 7.29]:

$$DA[i] = \text{rank}_1(bit, SA[i]) + 1,$$

(5)

$bit[1, N]$ can be pre-processed in $O(N)$ time so that rank queries are supported in $O(1)$ time using additional $o(N)$ bits [20]. This procedure computes DA in $O(N)$ time using $N + o(N)$ bits of workspace.

3. Computing DA

In this section we show how to compute DA from SA and $T^{cat}$ in $O(N)$ time using $O(\sigma \log N)$ bits of workspace, which is constant when $\sigma = O(1)$.

At a glance, we traverse $T^{cat}[1, N]$ from right-to-left applying the LF mapping $N$ times. We compute $DA[1], DA[LF^1(1)], DA[LF^2(1)], \ldots, DA[LF^{N-1}(1)]$. Starting with $doc = d+1$, each $DA[i]$ receives $doc$, and whenever $T^{cat}[SA[i] - 1] = \text{BWT}[i] = \$ \text{ doc is decremented by one.}$
Recall that we cannot traverse $T_{\text{cat}}[1, N]$ with the LF-mapping given on-the-fly by Equation 2. Alternatively, given the BWT of $T_{\text{cat}}$ and an auxiliary array $C[1, \sigma]$ initialized with $C[c]$ equal to the number of symbols $c'<c$ in $T_{\text{cat}}[1, N]$, we can pre-compute correct LF entries for every position with a corresponding BWT symbol $c \neq \$ as follows. For $i = 1, \ldots, N$, $\text{LF}[i] = C[\text{BWT}[i]]$, and $C[\text{BWT}[i]]$ is incremented by one. The resulting (incorrect) LF-positions, corresponding to $\text{BWT}[i] = \$$, will be in the interval $[2, d + 1]$. These values will be computed correctly by Algorithm 1 on-the-fly during the right-to-left $T_{\text{cat}}[1, N]$ traversal.

Algorithm 1. The algorithm starts with SA stored in $A[1, N]$. We use $N \log N$ bits to store $A[1, N]$, and $N \log(d + 1)$ bits to store $DA[1, N]$. First, we overwrite SA with the BWT in $A[1, N]$ (Lines 1-3). Then, we overwrite the BWT with the LF-array computed as described above (Lines 4-6). Recall that positions with $A[i] \in [2, d + 1]$ are not correct. In the sequel, $DA[1, N]$ is computed while SA is reconstructed in the space of $A[1, N]$ as follows. Initially, $pos = 1$ and $doc = d + 1$ (Lines 7-8). At each step $i = N, \ldots, 1$ (Lines 9-18), the value in $A[pos]$ (corresponding to $\text{LF}(pos)$) is stored in a temporary variable (Line 10) and replaced by $SA[pos] = i$ (Line 11), then $DA[pos] = doc$ (Line 12). Whenever $tmp \in [2, d + 1]$, $\text{BWT}(pos)$ is a $\$-symbol and we have to compute correctly its LF-mapping. In particular, when we reach the first $tmp \in [2, d + 1]$, we reach the BWT position corresponding to the $d$th $\$-symbol in $T_{\text{cat}}$ (the last one), because we traverse $T_{\text{cat}}[1, N]$ right-to-left, and its correct LF-mapping is $tmp = d + 1$. The next iteration we reach $tmp \in [2, d + 1]$ we are at the BWT position corresponding to the $(d - 1)^{th}$ $\$-symbol in $T_{\text{cat}}$, and $tmp = d$, and so on. Therefore, whenever $tmp \in [2, d + 1]$ we update $tmp$ with the correct LF-mapping value stored in $doc$ (Line 14), and $doc$ is decremented by one for the next iterations (Line 15). The next step will visit position $pos = tmp = \text{LF}(pos)$ (Line 17). At the end, $DA[1, N]$ is completely computed and $SA$ is reconstructed in the same space of $A[1, N]$. 

5
Algorithm 1: Computing DA from $T^\text{cat}$, SA[1, N] and C[1, $\sigma$].

1 for $i \leftarrow 1$ to $N$ do
2 \hspace{1em} A[i] $\leftarrow T^\text{cat}[A[i] - 1 \mod N]$ ; // $A = \text{BWT}$
3 end
4 for $i \leftarrow 1$ to $N$ do
5 \hspace{1em} A[i] $\leftarrow C[A[i]]++$; // $A = \text{LF}$
6 end
7 $pos \leftarrow 1$;
8 $doc \leftarrow d + 1$;
9 for $i \leftarrow N$ downto 1 do
10 \hspace{1em} tmp $\leftarrow A[pos]$; // $tmp = A[pos] = LF(pos)$
11 \hspace{1em} A[pos] $\leftarrow i$; // $A[pos] = SA[pos]$
12 \hspace{1em} DA[pos] $\leftarrow doc$; // $DA[pos] = DA[pos]$
13 \hspace{1em} if $tmp \leq d + 1$ then // $BWT(pos) == $ $
14 \hspace{2em} tmp \leftarrow doc$;
15 \hspace{2em} doc $\leftarrow doc - 1$;
16 \hspace{1em} end
17 \hspace{1em} pos $\leftarrow tmp$; // $pos = tmp = LF(pos)$
18 end

Theoretical costs. The number of steps is $N$ and only array $C[1, \sigma]$ was needed in addition to the input and output. Therefore, the algorithm runs in $O(N)$ time, using $O(\sigma \lg N)$ bits of workspace.

4. Experimental results

We compared our algorithm with the lightweight alternative described in Section 2.3. We evaluated two versions of this procedure, using compressed (BIT_SD) and plain bitvectors (BITPLAIN). We used C++ and SDSL library version 2.0. The algorithms receive as input the concatenated string ($T^\text{cat}$) and its suffix array (SA), which was computed using gSACA-K [17]. Our
algorithm was implemented in ANSI C. The source codes are available at
https://github.com/felipelouza/gsa-is/.

The experiments were conducted on a machine with Debian GNU/Linux 8 64
bits OS (kernel 3.16.0-4) with processor Intel Xeon E5-2630 v3 20M Cache 2.40-
GHz, 386 GB of RAM and a 13 TB SATA disk. We used real data collections
described in Table 1.

Table 1: Datasets. We used 32-bits integers to store $SA[1, N]$ when $N < 2^{31}$ (2GB), otherwise
we used 64-bits. The document array $DA[1, N]$ is stored using 32-bits integers, since $d$ is always
smaller than $2^{31}$. Each symbol of $T^{rev}$ uses 1 byte.

| Dataset  | $\sigma$ | $N/2^{30}$ | $d$  | $N/d$ | longest string |
|----------|----------|------------|------|-------|----------------|
| revision | 203      | 0.39       | 20,433 | 20,527 | 2,000,452 |
| influenza| 15       | 0.56       | 394,217 | 1,516  | 2,867    |
| reads    | 4        | 2.87       | 32,621,862 | 94    | 101      |
| pages    | 205      | 3.74       | 1,000  | 4,019,585 | 362,724,758 |
| wikipedia| 208      | 8.32       | 3,903,703 | 2,288  | 224,488  |
| proteins | 25       | 15.77      | 50,825,784 | 333   | 36,805   |

- **pages**: repetitive collection from a snapshot of Finnish-language Wikipedia.
  Each document is composed by one page and its revisions.

- **revision**: the same as **pages**, except that each revision is a separate document.

- **influenza**: repetitive collection of the genomes of influenza viruses.

- **wikipedia**: collection of pages from English-language of Wikipedia.

- **reads**: collection of DNA reads from Human Chromosome 14 (library 1).

- **proteins**: collection of protein sequences from Uniprot/TrEMBL 2015.

---

1. http://jltsiren.kapsi.fi/data/fiwiki.bz2
2. ftp://ftp.ncbi.nih.gov/genomes/INFLUENZA/influenza.fna.gz
3. http://algo2.iti.kit.edu/gog/projects/ALENEX15/collections/ENWIKIBIG/
4. http://gage.ccb.org/data/index.html
Table 2 shows the running time (in seconds) and workspace (in KB) of each algorithm. The workspace is the peak space used subtracted by the space used for the input, $T_{ext}[1, N]$ and $SA[1, N]$, and for the output, $DA[1, N]$.

**Results.** **BIT.PLAIN** was the fastest algorithm in all tests. **BIT.PLAIN** was 2.19 times faster than **BIT.SD**, and 5.73 times faster than Alg. 1, on the average. **BIT.SD** was still 3 times faster than Alg. 1, which shows that Alg. 1 is not competitive in practice. On the other hand, Alg. 1 was the only algorithm that kept the workspace constant, namely 1 KB for inputs smaller than $2^{31}$ (2 GB) and 2 KB otherwise, which correspond to the space used by the auxiliary array $C[1, \sigma]$ used to compute $LF$. The workspace of **BIT.PLAIN** and **BIT.SD** were much larger, **BIT.PLAIN** spent 0.16 × $N$ bytes, whereas **BIT.SD** spent 0.003 × $N$ bytes, on the average.

| Dataset   | Time (seconds) |             |             | Workspace (KB) |             |             |
|-----------|----------------|-------------|-------------|----------------|-------------|-------------|
|           |                | Alg. 1      | **BIT.PLAIN** | **BIT.SD**   | Alg. 1      | **BIT.PLAIN** | **BIT.SD** |
| revision  | 60.88          | 11.74       | 20.37       | 1              | 64,002      | 44          |
| influenza | 109.13         | 20.48       | 41.24       | 1              | 91,168      | 704         |
| reads     | 931.35         | **150.40**  | 549.65      | 2              | 470,389     | 38,980      |
| pages     | 762.91         | 141.99      | **141.25**  | 2              | 613,341     | 4           |
| wikipedia | 2,947.59       | **450.64**  | 1,054.08    | 2              | 1,363,147   | 7,096       |
| protein   | 7,007.87       | **1,211.13**| 2,899.63    | 2              | 2,583,532   | 69,423      |

Table 2: Running time and workspace.

**Acknowledgments.** We thank the anonymous reviewers for comments that improved the manuscript. We thank Giovanni Manzini, Travis Gagie and Nicola Prezza for helpful discussions.

**Funding.** F.A.L. was supported by the grants #2017/09105-0 and #2018/21509-2 from the São Paulo Research Foundation (FAPESP).
References

[1] U. Manber, E. W. Myers, Suffix arrays: A new method for on-line string searches, SIAM J. Comput. 22 (5) (1993) 935–948.

[2] S. Muthukrishnan, Efficient algorithms for document retrieval problems, in: Proc. ACM-SIAM Symposium on Discrete Algorithms (SODA), ACM/SIAM, 2002, pp. 657–666.

[3] N. Välimäki, V. Mäkinen, Space-efficient algorithms for document retrieval, in: Proc. Annual Symposium on Combinatorial Pattern Matching (CPM), 2007, pp. 205–215.

[4] D. Belazzougui, G. Navarro, D. Valenzuela, Improved compressed indexes for full-text document retrieval, J. Discrete Algorithms 18 (2013) 3–13.

[5] T. Kopelowitz, G. Kucherov, Y. Nekrich, T. A. Starikovskaya, Cross-document pattern matching, J. Discrete Algorithms 24 (2014) 40–47.

[6] T. Gagie, A. Hartikainen, K. Karhu, J. Kärkkäinen, G. Navarro, S. J. Puglisi, J. Sírén, Document retrieval on repetitive string collections, Inf. Retr. Journal 20 (3) (2017) 253–291.

[7] J. Sírén, E. Garrison, A. M. Novak, B. Paten, R. Durbin, Haplotype-aware graph indexes, in: Proc. WABI, 2018, pp. 4:1–4:13.

[8] K. Sadakane, Succinct data structures for flexible text retrieval systems, J. Discrete Algorithms 5 (1) (2007) 12–22.

[9] E. Ohlebusch, S. Gog, Efficient algorithms for the all-pairs suffix-prefix problem and the all-pairs substring-prefix problem, Information Processing Letters 110 (3) (2010) 123–128.

[10] M. Arnold, E. Ohlebusch, Linear Time Algorithms for Generalizations of the Longest Common Substring Problem, Algorithmica 60 (4) (2011) 806–818.
[11] W. H. Tustumi, S. Gog, G. P. Telles, F. A. Louza, An improved algorithm for the all-pairs suffix-prefix problem, J. Discret. Algorithms 37 (2016) 34–43.

[12] F. A. Louza, G. P. Telles, S. Gog, Z. Liang, Computing Burrows-Wheeler Similarity Distributions for String Collections, in: Proc. SPIRE, 2018, pp. 285–296.

[13] L. Egidi, F. A. Louza, G. Manzini, G. P. Telles, External memory BWT and LCP computation for sequence collections with applications, in: Proc. WABI, 2018, pp. 10:1–10:14.

[14] G. Nong, Practical linear-time O(1)-workspace suffix sorting for constant alphabets, ACM Trans. Inform. Syst. 31 (3) (2013) 1–15.

[15] M. Burrows, D. J. Wheeler, A block-sorting lossless data compression algorithm, Tech. rep., Digital SRC Research Report (1994).

[16] R. Grossi, A. Gupta, J. S. Vitter, High-order entropy-compressed text indexes, in: Proc. ACM-SIAM Symposium on Discrete Algorithms (SODA), ACM/SIAM, 2003, pp. 841–850.

[17] F. A. Louza, S. Gog, G. P. Telles, Inducing enhanced suffix arrays for string collections, Theor. Comput. Sci. 678 (2017) 22–39.

[18] G. Navarro, S. J. Puglisi, D. Valenzuela, Practical compressed document retrieval, in: Proc. SEA, 2011, pp. 193–205.

[19] E. Ohlebusch, Bioinformatics Algorithms: Sequence Analysis, Genome Rearrangements, and Phylogenetic Reconstruction, Oldenbusch Verlag, 2013.

[20] J. I. Munro, Tables, in: Proc. FSTTCS, Vol. 1180 of LNCS, Springer, 1996, pp. 37–42.

[21] S. Gog, T. Beller, A. Moffat, M. Petri, From theory to practice: Plug and play with succinct data structures, in: Proc. SEA, Vol. 8504 of LNCS, Springer, 2014, pp. 326–337.