HARMONIC COHOMOLOGY
OF SYMPLECTIC FIBER BUNDLES

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Abstract. We show that every de Rham cohomology class on the total space of a symplectic fiber bundle with closed Lefschetz fibers admits a Poisson harmonic representative in the sense of Brylinski. The proof is based on a new characterization of closed Lefschetz manifolds.

1. Introduction and main result

Suppose $P$ is a Poisson manifold [10] with Poisson tensor $\pi$. Let $d$ denote the de Rham differential on $\Omega(P)$ and write $\iota_\pi$ for the contraction with the Poisson tensor. Recall that Koszul’s [5] codifferential $\delta := [\iota_\pi, d] = \iota_\pi d - di_\pi$ satisfies $\delta^2 = 0$ and $[d, \delta] = d\delta + \delta d = 0$. Differential forms $\alpha \in \Omega(P)$ with $d\alpha = 0 = \delta\alpha$ are called (Poisson) harmonic. Brylinski [2] asked for conditions on a Poisson manifold which imply that every de Rham cohomology class admits a harmonic representative.

In the symplectic case, this question has been settled by Mathieu. Recall that a symplectic manifold $(M, \omega)$ of dimension $2n$ is called Lefschetz if, for all $k$,

$$\omega^k \wedge H^{n-k}(M; \mathbb{R}) = H^{n+k}(M; \mathbb{R}).$$

According to Mathieu [3] (see [11] for an alternative proof) a symplectic manifold is Lefschetz if it satisfies the Brylinski conjecture; i.e. every de Rham cohomology class of $M$ admits a harmonic representative.

In this paper we study the Brylinski problem for smooth symplectic fiber bundles [7]. Recall that the total space of a symplectic fiber bundle $P \to B$ is canonically equipped with the structure of a Poisson manifold obtained from the symplectic form on each fiber. Locally, the Poisson structure on $P$ is product-like; that is, every point in $B$ admits an open neighborhood $U$ such that there exists a fiber preserving Poisson diffeomorphism $P|_U \cong M \times U$. Here $M$ denotes the typical symplectic fiber, equipped with the corresponding Poisson structure, and $U$ is considered as a trivial Poisson manifold. This renders the symplectic foliation of $P$ particularly nice, for its leaves coincide with the connected components of the fibers of the bundle $P \to B$.

The aim of this paper is to establish the following result, providing a class of Poisson manifolds which satisfy the Brylinski conjecture.

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Theorem 1. Let $M$ be a closed symplectic Lefschetz manifold, and suppose $P \rightarrow B$ is a smooth symplectic fiber bundle with typical symplectic fiber $M$. Then every de Rham cohomology class of $P$ admits a Poisson harmonic representative. Moreover, the analogous statement for compactly supported cohomology holds true.

This result, as well as a characterization of closed Lefschetz manifolds similar to Theorem 2 below, has been established in the first author’s diploma thesis, employing slightly different methods than those of the present work; see [3].

The proof presented in Section 3 below is based on a handlebody decomposition $\emptyset = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ of $B$. Given a cohomology class of $P$, we will inductively produce representatives which are harmonic on $P|_{B_k}$, for increasing $k$. The crucial problem, of course, is to extend harmonic forms across the handle, from $P|_{B_k}$ to $P|_{B_{k+1}}$. This issue is addressed in Theorem 2; see also Lemma 6.

2. Extension of harmonic forms

Let $M$ be a closed symplectic manifold and consider the trivial symplectic fiber bundle $P := M \times \mathbb{R}^p \times D^q$ where $D^q$ denotes the $q$-dimensional closed unit ball.

In other words, the Poisson structure on $P$ is the product structure obtained from the symplectic form on $M$ and the trivial Poisson structure on $\mathbb{R}^p \times D^q$. Note that the boundary $\partial P = M \times \mathbb{R}^p \times \partial D^q$ is a Poisson submanifold. It turns out that the Lefschetz property of $M$ is equivalent to harmonic extendability of forms, from $\partial P$ to $P$.

To formulate this precisely, we need to introduce some notation that will be used throughout the rest of the paper. For every Poisson manifold $P$ we let $Z(P) := \{\alpha \in \Omega(P) \mid d\alpha = 0\}$ and $Z_0(P) := \{\alpha \in \Omega(P) \mid d\alpha = 0 = \delta \alpha\}$ denote the spaces of closed and harmonic differential forms, respectively. Moreover, we write $H_0(P) := \ker(\delta) \cap \ker(\delta) / \text{img}(d) \cap \ker(\delta)$ for the space of de Rham cohomology classes which admit a harmonic representative, $H_0(P) \subseteq H(P)$. If $\iota : S \hookrightarrow P$ is a Poisson submanifold, then the relative complex $\Omega(P,S) := \{\alpha \in \Omega(P) \mid \iota^* \alpha = 0\}$ is invariant under $\delta$, and we define the relative harmonic cohomology $H_0(P,S) \subseteq H(P,S)$ in an analogous manner. Finally, if $Q$ is a Poisson manifold and $B$ is a smooth manifold, we let $\Omega_{vc}(Q \times B)$ denote the space of forms with vertically compact support (with respect to the projection $Q \times B \rightarrow Q$), and define the harmonic cohomology with vertically compact supports $H_{vc,0}(Q \times B) \subseteq H_{vc}(Q \times B)$ in the obvious way.

Here is the main result that will be established in this section.

Theorem 2. Let $M$ be a closed symplectic manifold, suppose $p,q \in \mathbb{N}_0$, and consider the Poisson manifold $P := M \times \mathbb{R}^p \times D^q$. Then the following are equivalent:

(i) $M$ is Lefschetz; i.e. $H_0(M) = H(M)$ according to [6].

(ii) $H_0(P,\partial P) = H(P,\partial P)$.

(iii) $H_{vc,0}(P \setminus \partial P) = H_{vc}(P \setminus \partial P)$ with respect to the projection along $D^q \setminus \partial D^q$.

(iv) If $\alpha \in Z(P)$ is harmonic on a neighborhood of $\partial P$, then there exists $\beta \in \Omega(P)$, supported on $P \setminus \partial P$, so that $\alpha + d\beta$ is harmonic on $P$.

(v) If $\alpha \in Z(P)$ and $\iota^* \alpha = 0$, then there exists $\beta \in \Omega(P)$ with $\iota^* \beta = 0$, so that $\alpha + d\beta$ is harmonic on $P$. Here $\iota : \partial P \hookrightarrow P$ denotes the canonical inclusion.

An essential ingredient for the proof of Theorem 2 is the following $d\delta$-lemma.

Lemma 1 ($d\delta$-Lemma, [4 8]). A closed symplectic manifold is Lefschetz if and only if $\ker(\delta) \cap \text{img}(d) = \text{img}(d\delta)$. 

Lemma 2. Suppose $G$ is a connected compact Lie group acting on a Poisson manifold $P$ via Poisson diffeomorphisms, and let $r : \Omega(P \times I) \to \Omega(P \times I)^G$, $r(\alpha) := \int_G g^* \alpha \, dg$, denote the standard projection onto the space of $G$-invariant forms, $I := [0,1]$. Then there exists an operator $A : \Omega(P \times I) \to \Omega(P \times I)$, commuting with $\iota, i_\pi$ and $\delta$, so that $A(\alpha) = \alpha$ in a neighborhood of $P \times \{1\}$ and $A(\alpha) = r(\alpha)$ in a neighborhood of $P \times \{0\}$, for all $\alpha \in \Omega(P \times I)$.

Proof. Choose finitely many smoothly embedded closed balls $D_i \subseteq G$ such that $\bigcup_i D_i = G$. Let $\lambda_i$ denote a partition of unity on $G$ so that $\text{supp}(\lambda_i) \subseteq D_i$. Choose smooth contractions $h_i : D_i \times I \to G$ so that $h_i(g, t) = g$ for $t \leq 1/3$ and $h_i(g, t) = e$ for $t \geq 2/3$, $g \in D_i$. Here $e$ denotes the neutral element of $G$. Using the maps $\phi_{i,g} : P \times I \to P \times I$, $\phi_{i,g}(x,t) := (h_i(g, t) \cdot x, t)$, $g \in D_i$, we define the operator $A : \Omega(P \times I) \to \Omega(P \times I)$ by

$$A(\alpha) := \sum_i \int_{D_i} \lambda_i(g) \phi_{i,g}^* \alpha \, dg,$$

where integration is with respect to the invariant Haar measure of $G$. It is straightforward to verify that $A$ has the desired properties; the relations $[A, \iota_\pi] = 0 = [A, \delta]$ follow from the fact that each $\phi_{i,g}$ is a Poisson map. 

The following application of Lemma 2 will be used in the proof of Theorem 2.

Lemma 3. Let $M$ be a symplectic manifold and consider the Poisson manifold $P := M \times \mathbb{R}^p \times A^q$ where $A^q := \{\xi \in \mathbb{R}^q \mid \frac{1}{2} \leq |\xi| \leq 1\}$ denotes the $q$-dimensional annulus. Moreover, suppose $\alpha \in \Omega(P)$ is harmonic on a neighborhood of $\partial_+P := M \times \mathbb{R}^p \times \partial D^q$. Then there exist $\beta \in \Omega(P)$, supported on $P \setminus \partial_+P$, and $\beta_1, \beta_2 \in Z_0(M)$, so that $\tilde{\alpha} := \alpha + d\beta$ is harmonic on $P$ and $\tilde{\alpha} = \sigma^* \beta_1 + \sigma^* \beta_2 + \rho^* \theta$ in a neighborhood of $\partial_-P := M \times \mathbb{R}^p \times \frac{1}{2} \partial D^q$. Here $\sigma : P \to M$ and $\rho : P \to \partial D^q$ denote the canonical projections, and $\theta$ denotes the standard volume form on $\partial D^q$.

Proof. W.l.o.g., we may assume $\alpha \in Z_0(P)$ and $\alpha = \tau^* \gamma$ in a neighborhood of $\partial_-P$ where $\gamma \in Z_0(M \times \partial D^q)$ and $\tau : (\sigma, \rho) : P \to M \times \partial D^q$ denotes the canonical projection. Applying the operator $A$ from Lemma 2 to $\alpha$, we obtain $\tilde{\alpha} \in Z_0(P)$ so that $\tilde{\alpha} = \alpha$ in a neighborhood of $\partial_+P$ and $\tilde{\alpha} = \tau^* \tilde{\gamma}$ in a neighborhood of $\partial_-P$, where $\tilde{\gamma} \in Z_0(M \times \partial D^q)$ is $SO(q)$-invariant. We conclude that $\tilde{\gamma}$ is of the form $\tilde{\gamma} = \beta_1 + \beta_2 \wedge \theta$ with $\beta_1, \beta_2 \in Z_0(M)$, whence $\tilde{\alpha} = \sigma^* \beta_1 + \sigma^* \beta_2 + \rho^* \theta$ in a neighborhood of $\partial_-P$. Clearly, there exists $\beta \in \Omega(P)$, supported on $P \setminus \partial_+P$, such that $\tilde{\alpha} - \alpha = d\beta$.

Lemma 4. Let $P$ be a Poisson manifold, and suppose $B$ is an oriented smooth manifold with boundary. Then integration along the fibers $\int_B : \Omega_{\partial B}(P \times B) \to \Omega(P)$ commutes with $i_\pi$ and $\delta$.

Proof. The relation $i_\pi \int_B \alpha = \int_B i_\pi \alpha$ is obvious. Combining this with Stokes’ theorem, that is $[\int_B, \iota^*] = \int_{\partial B} \iota^*$, we obtain $[\int_B, \delta] = [\int_B, i_\pi, [\iota^*]] = [\int_B, i_\pi, [\iota^*]] = [\iota^*, [\iota^*]] = 0$.

Here $\iota : P \times \partial B \to P \times B$ denotes the canonical inclusion.

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1To be specific, in the case $q = 1$ we assume $\theta(-1) = -1/2$ and $\theta(1) = 1/2$, so that $\int_{\partial D^q} \theta = 1$ with respect to orientation on $\partial D^q$ induced from the standard orientation of $D^q$.  

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Lemma 5. Suppose $Q$ is a Poisson manifold, and consider the Poisson manifold $P := Q \times D^q$. Then the Thom (Künneth) isomorphism restricts to an isomorphism of harmonic cohomology; i.e. $H^*_0(Q) = H^*_c(\partial P \setminus \partial P) = H^*_0(P \setminus \partial P)$.

Proof. Choose $\eta \in \Omega^0(D^q)$, supported on $D^q \setminus \partial D^q$, such that $\int_{D^q} \eta = 1$. Clearly, the chain map $\Omega(Q) \to \Omega_c(\partial P \setminus \partial P) \subseteq \Omega(P, \partial P)$, $\alpha \mapsto \alpha \wedge \eta$, commutes with $\delta$. This map induces the Thom isomorphism which therefore preserves harmonicity. Its inverse is induced by integration along the fibers $\int_{D^q} : \Omega(P, \partial P) \to \Omega(Q)$, and this commutes with $\delta$ too; see Lemma 4.

Now the table is served, and we proceed to the

Proof of Theorem 2. Set $Q := M \times \mathbb{R}^p$ and note that the isomorphism $H(Q) = H(M)$ induced by the canonical projection restricts to an isomorphism of harmonic cohomology $H_0(Q) = H_0(M)$. The equivalence of the first three statements thus follows from Lemma 5. Let us continue by showing that (iii) implies (iv). Assume $\alpha \in Z(P)$ is harmonic on a neighborhood of $\partial P$. Let $p : P \setminus (M \times \mathbb{R}^p \times \{0\}) \to \partial D^q$ and $\sigma : P \to M$ denote the canonical projections. In view of Lemma 3 we may w.l.o.g. assume $\alpha = \sigma^* \beta_1 + \sigma^* \beta_2 \wedge \rho^* \theta$ in a neighborhood of $\partial P$ where $\beta_1, \beta_2 \in Z_0(M)$ and $\theta$ denotes the standard volume form on $\partial D^q$. Using Stokes’ theorem for integration along the fiber of $M \times D^q \to M$, we obtain

$$\beta_2 = \int_{\partial D^q} j^* \alpha = -d \int_{D^q} j^* \alpha \in \text{img}(d) \cap \ker(\delta),$$

where $j : M \times D^q \to M \times \{0\} \times D^q \subseteq P$ denotes the canonical inclusion. By the $d\delta$-lemma, Lemma 1, we thus have $\beta_2 = d\delta \gamma$ for some differential form $\gamma$ on $M$. Let $\lambda$ be a smooth function on $P$, identically 1 in a neighborhood of $\partial P$, identically 0 near $M \times \mathbb{R}^p \times \{0\}$, and constant in the $M$-direction. Then $\tilde{\alpha} := \sigma^* \beta_1 + d(\delta \sigma^* \gamma \wedge \lambda \rho^* \theta)$ is harmonic on $P$, and $\alpha - \tilde{\alpha} = 0$ in a neighborhood of $\partial P$. Hence, using (ii), we find $\beta \in \Omega(P)$, supported on $P \setminus \partial P$, so that $\alpha - \tilde{\alpha} + d \beta$ is harmonic on $P$. Thus, $\beta$ has the desired property. Let us next show that (iv) implies (v). Suppose $\alpha \in Z(P)$ and $\delta \alpha = 0$. Clearly, there exists $\beta_1 \in \Omega(P)$, with $\iota^* \beta_1 = 0$, so that $\tilde{\alpha} := \alpha + d \beta_1$ satisfies $\iota^* \tilde{\alpha} = \tilde{\alpha}$ near $\partial P$, where $r : P \setminus (M \times \mathbb{R}^p \times \{0\}) \to \partial D^q$ denotes the canonical radial retraction. Particularly, $\tilde{\alpha}$ is harmonic on a neighborhood of $\partial P$. According to (iv) there exists $\beta_2 \in \Omega(P)$, supported on $P \setminus \partial P$, so that $\tilde{\alpha} + d \beta_2$ is harmonic on $P$. The form $\beta := \beta_1 + \beta_2$ thus has the desired property. Obviously, (v) implies (ii).

3. Proof of Theorem 1.

Choose a proper Morse function $f$ on $B$, bounded from below, so that the preimage of each critical value consists of a single critical point. We label the critical values in increasing order $c_0 < c_1 < \cdots$, and choose regular values $r_k$ so that $c_{k-1} < r_k < c_k$. By construction, the sublevel sets $B_k := \{ f(x) \leq r_k \}$ provide an increasing filtration of $B$ by compact submanifolds with boundary, $\emptyset = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$. The statement in Theorem 1 is an immediate consequence of the following:

Lemma 6. Suppose $\alpha \in Z(P)$ is a closed form which is harmonic on a neighborhood of $P|_{B_k}$. Then there exists $\beta \in \Omega(P)$, supported on $P|_{B_{k+2} \setminus B_k}$, such that $\alpha + d \beta$ is harmonic on a neighborhood of $P|_{B_{k+1}}$. 


Proof. Let \( q \) denote the Morse index of the unique critical point in \( B_{k+1} \setminus B_k \), and set \( p := \dim B - q \). Recall [9] that there exists an embedding \( j : \mathbb{R}^p \times \partial D^q \to B_{k+1} \setminus B_k \) so that \( j(\mathbb{R}^p \times \partial D^q) = j(\mathbb{R}^p \times D^q) \cap \partial B_k \). Moreover, there exists a vector field \( X \) on \( B \), supported on \( B_{k+2} \setminus B_k \), so that its flow \( \varphi_t \) maps \( B_{k+1} \) into any given neighborhood of \( \partial B_k \cup j(\{0\} \times D^q) \), for sufficiently large \( t \).

Trivializing the symplectic bundle \( P \) over the image of \( j \), we obtain an isomorphism of Poisson manifolds \( j^* P \cong M \times \mathbb{R}^p \times D^q \). Using Theorem 2(iv), we may thus assume that there exists an open neighborhood \( U \) of \( \partial B_k \cup j(\{0\} \times D^q) \) so that \( \alpha \) is harmonic on \( P \mid U \). Let \( \tilde{X} \) denote the horizontal lift of \( X \) with respect to a symplectic connection [7] on \( P \), and denote its flow at time \( t \) by \( \tilde{\varphi}_t \). Clearly, each \( \tilde{\varphi}_t \) is a Poisson map. Moreover, there exists \( t_0 \) so that \( \tilde{\varphi}_{t_0} \) maps \( P \mid B_{k+1} \) into \( P \mid U \). Thus, \( \tilde{\varphi}_{t_0}^* \alpha \) is harmonic on \( P \mid B_{k+1} \). Furthermore, \( \tilde{\varphi}_{t_0}^* \alpha - \alpha = d\beta \), where \( \beta := \int_0^{t_0} \tilde{\varphi}_t^* i_X \alpha \, dt \) is supported on \( P \mid B_{k+2} \setminus B_k \). □

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