Bracket notation for the ‘coefficient of’ operator

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When \( G(z) \) is a power series in \( z \), many authors now write \( \left[ z^n \right] G(z) \) for the coefficient of \( z^n \) in \( G(z) \), using a notation introduced by Goulden and Jackson in [5, p. 1]. More controversial, however, is the proposal of the same authors [5, p. 160] to let \( \left[ z^n/n! \right] G(z) \) denote the coefficient of \( z^n/n! \), i.e., \( n! \) times the coefficient of \( z^n \). An alternative generalization of \( \left[ z^n \right] G(z) \), in which we define \( \left[ F(z) \right] G(z) \) to be a linear function of both \( F \) and \( G \), seems to be more useful because it facilitates algebraic manipulations. The purpose of this paper is to explore some of the properties of such a definition. The remarks are dedicated to Tony Hoare because of his lifelong interest in the improvement of notations that facilitate manipulation.

Informal introduction. In this paper \( \left[ z^2 + 2z^3 \right] G(z) \) will stand for the coefficient of \( z^2 \) plus twice the coefficient of \( z^3 \) in \( G(z) \), when \( G(z) \) is a function of \( z \) for which such coefficients are well defined. More generally, if \( F(z) = f_0 + f_1z + f_2z^2 + \cdots \) and \( G(z) = g_0 + g_1z + g_2z^2 + \cdots \), we will let

\[
\left[ F(z) \right] G(z) = f_0g_0 + f_1g_1 + f_2g_2 + \cdots
\]

be the “dot product” of the vectors \( (f_0, f_1, f_2, \ldots) \) and \( (g_0, g_1, g_2, \ldots) \), assuming that the infinite sum exists. Still more generally, if \( F(z) = \cdots + f_{-2}z^{-2} + f_{-1}z^{-1} + f_0 + f_1z + f_2z^2 + \cdots \) and \( G(z) = \cdots + g_{-2}z^{-2} + g_{-1}z^{-1} + g_0 + g_1z + g_2z^2 + \cdots \) are doubly infinite series, we will write

\[
\left[ F(z) \right] G(z) = \cdots + f_{-2}g_{-2} + f_{-1}g_{-1} + f_0g_0 + f_1g_1 + f_2g_2 + \cdots ,
\]

again assuming convergence. (It is convenient to write ‘\( z^{-1} \)’ for \( 1/z \), as in [8].) The right side of (1) is symmetric in \( F \) and \( G \), so we have a commutative law:

\[
\left[ F(z) \right] G(z) = \left[ G(z) \right] F(z).
\]

There also is symmetry between positive and negative powers:

\[
\left[ F(z) \right] G(z) = \left[ F(z^{-1}) \right] G(z^{-1}).
\]

In particular, we will write \( \left[ 1 \right] G(z) \) for the constant term \( g_0 \) of a given doubly infinite power series \( G(z) = \sum_n g_n z^n \). Notice that \( \left[ z^n \right] G(z) = \left[ 1 \right] z^{-n} G(z) \) and in fact

\[
\left[ F(z) \right] G(z) = \left[ 1 \right] F(z^{-1}) G(z),
\]
when the product of series is defined in the usual way:

\[ \sum_n h_n z^n = \left( \sum_n f_n z^n \right) \left( \sum_n g_n z^n \right) \quad \iff \quad h_n = \sum_{j+k=n} f_j g_k . \]  

(5)

Relation (4) gives us a useful rule for moving factors in and out of brackets:

\[ [F(z)] G(z) H(z) = [F(z) G(z^-)] H(z) . \]  

(6)

Both sides reduce to \([1] F(z^-) G(z) H(z)\), so they must be equal. This rule is most often applied in a simple form such as

\[ [z^n] z^3 H(z) = [z^{n-3}] H(z) , \]

but it is helpful to remember the general principle (6). Similarly,

\[ [F(z) G(z)] H(z) = [F(z)] G(z^-) H(z) . \]  

(7)

A paradox. So far the extended bracket notation seems straightforward and innocuous, but if we start to play with it in an undisciplined fashion we can easily get into trouble. For example, one of the first uses we might wish to make of relation (1) is

\[ [z^n] \frac{1}{1-z} G(z) = g_n + g_{n+1} + g_{n+2} + \cdots , \]  

(8)

because \(z^n/(1-z) = z^n + z^{n+1} + z^{n+2} + \cdots\). This, unfortunately, turns out to be dangerous, if not outright fallacious.

The danger is sometimes muted and we might be lucky. For example, if we try combining (8) with (7) in the case \(G(z) = 1/(1-z)\) and \(H(z) = (1-z)^2 = 1-2z+z^2\), we get

\[ \left[ \frac{z^n}{1-z} \right] (1-z)^2 = [z^n] \frac{(1-z)^2}{1-z} = [z^n] (z^2 - z) . \]  

(9)

Sure enough, the sum \(h_n + h_{n+1} + h_{n+2} + \cdots\) is nonzero in this case only when \(n = 2\) and \(n = 1\), and (9) gives the correct answer. So far so good.

But (7) and (8) lead to a contradiction when we apply them to the trivial case \(F(z) = H(z) = 1\) and \(G(z) = 1/(1-z)\):

\[ 1 = \left[ \frac{1}{1-z} \right] 1 = [1] \frac{1}{1-z} = [1] \frac{-z}{1-z} = 0 . \]  

(10)

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What went wrong?

Formal analysis. To understand the root of the paradox (10), and to learn when (6) and (7) are indeed valid rules of transformation, we need to know the basic properties of double power series \( \sum g_n z^n \). The general theory can be found in Henrici [6, §4.4]; we will merely sketch it here.

If \( G(z) \) is analytic in an annulus \( \alpha < |z| < \beta \), it has a unique double series representation \( G(z) = \sum g_n z^n \). Conversely, any double power series that converges in an annulus defines an analytic function there. The proof is based on the contour integral formula

\[
G(z) = \frac{1}{2\pi i} \oint_{|t|=\beta'} \frac{G(t) \, dt}{(t-z)} - \frac{1}{2\pi i} \oint_{|t|=\alpha'} \frac{G(t) \, dt}{(t-z)},
\]

where \( \alpha' \) is between \( |z| \) and \( \alpha \) while \( \beta' \) is between \( |z| \) and \( \beta \). The quantity \( 1/(t-z) \) can be expanded as \( t^{-1}(1+z/t+z^2/t^2+\cdots) \) when \( |t| > |z| \) and as \( -z^{-1}(1+t/z+t^2/z^2+\cdots) \) when \( |t| < |z| \).

If \( F(z) \) and \( G(z) \) are both analytic for \( \alpha < |z| < \beta \), their product \( H(z) \) is an analytic function whose coefficients are given by (5). Moreover, the infinite sum over all \( j \) and \( k \) with \( j+k = n \) in (5) is absolutely convergent: The terms are \( O((\alpha'/\beta')^k) \) as \( k \to +\infty \) and \( O((\beta'/\alpha')^k) \) as \( k \to -\infty \).

The coefficients of \( G(z) \) in its double power series depend on \( \alpha \) and \( \beta \). For example, suppose \( G(z) = 1/(2-z) \); we have

\[
\frac{1}{2-z} = \begin{cases} 
\frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots, & \text{when } |z| < 2; \\
-z^2 \cdots - 4z^{-3} - \cdots, & \text{when } |z| > 2.
\end{cases}
\]

Thus if \( F(z) = 1/(2-z) + 1/(2-z^{-1}) \), there are three expansions

\[
F(z) = \begin{cases} 
\frac{1}{2} + \left(\frac{1}{4} - 1\right)z + \left(\frac{1}{8} - 2\right)z^2 + \left(\frac{1}{16} - 4\right)z^3 + \cdots, & |z| < \frac{1}{2}; \\
\cdots + \frac{1}{8}z^{-2} + \frac{1}{4}z^{-1} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots, & \frac{1}{2} < |z| < 2; \\
\cdots + \left(\frac{1}{16} - 4\right)z^{-3} + \left(\frac{1}{8} - 2\right)z^{-2} + \left(\frac{1}{4} - 1\right)z^{-1} + \frac{1}{2}, & |z| > 2.
\end{cases}
\]

Here’s another example, this time involving a function that has an essential singularity instead of a pole:

\[
e^{z/(1-z)} = \begin{cases} 
1 + z + \frac{3}{2}z^2 + \frac{13}{6}z^3 + \frac{73}{24}z^4 + \cdots, & |z| < 1; \\
e^{-e^{-z} - \frac{e^{-z}}{2}z^2 - \frac{e^{-z}}{6}z^3 - \frac{e^{-z}}{24}z^4 + \cdots, & |z| > 1.
\end{cases}
\]

The coefficients when \( |z| < 1 \) are \( P_n/n! \), where \( P_n \) is the number of “sets of lists” of order \( n \) [9].
Explaining the paradox. The dependency of coefficients on $\alpha$ and $\beta$ makes our notation $[F(z)]G(z)$ ambiguous; that is why we ran into trouble in the paradoxical “equation” (10). We can legitimately use bracket notation only when the context specifies a family of “safe” functions—functions with well defined coefficients.

The basic definition of $[F(z)]G(z)$ in (4) should be used only if the product $F(z^-)G(z)$ is safe. Operation (6), which moves a factor $G(z)$ into the bracket, should be used only if $F(z^-)G(z)H(z)$ is safe. Operation (7), which removes a factor $G(z)$ from the bracket, should be used only if $F(z^-)G(z^-)H(z)$ is safe.

The root of our problem in (10) begins in (8), where we used the expansion

$$F(z) = \frac{z^n}{1-z} = z^n + z^{n+1} + z^{n+2} + \cdots;$$

in other words, $F(z^-) = z^{-n}/(1-z^-) = z^{-n} + z^{-n-1} + z^{-n-2} + \cdots$. The latter expansion is valid only when $|z| > 1$, so the bracket notation of (8) refers to coefficients in the region $1 < |z| < \infty$. In the last step of (10), however, we said that $[1](-z/(1-z)) = 0$, using coefficients from the region $|z| < 1$. The correct result for $|z| > 1$ is

$$[1] \frac{-z}{1-z} = [1] \frac{1}{1-z^-} = [1] (\cdots + z^{-2} + z^- + 1) = 1.$$

Bracket notation is most often used when $|z|$ is small, so we should actually forget the “rightward sum” appearing in equation (8); it hardly ever yields the formula we want. The “leftward sum” rule

$$\left[ \frac{z^n}{1-z^-} \right] G(z) = \cdots + g_{n-2} + g_{n-1} + g_n$$

should be used instead, because $z^n/(1-z^-) = \cdots + z^{n-2} + z^{n-1} + z^n$ is valid for $|z^-| < 1$. When the bracket notation $[F(z)]G(z)$ is being used in the annulus $(\alpha, \beta)$, the functions $F(z^-)$ and $G(z)$ should be analytic in $(\alpha, \beta)$. Note that $f(z^-)$ is analytic in $(\alpha, \beta)$ if and only if $f(z)$ is analytic in $(\beta^-, \alpha^-)$.

Formal series. Manipulations of generating functions are often done on formal power series, when the coefficients are arbitrary and convergence is disregarded. However, formal power series are not allowed to be infinite in both directions; a formal series $G(z) = \sum_n g_n z^n$ is generally required to be a “formal Laurent series”—a series in which $g_n = 0$ for all sufficiently negative values of $n$. We shall call such series $L$-series for short. Similarly, we shall say that a reverse formal Laurent series, in which $g_n = 0$ for all sufficiently positive values of $n$, is an $R$-series. A power series is both an $L$-series and an $R$-series if and only if it is a polynomial in $z$ and $z^-$. Henrici [6, §1.2–1.8] shows that the normal operations on power series—addition, subtraction, multiplication, division by nonzero, differentiation, composition—can all be done
rigorously on \(L\)-series without regard to convergence. Thus \(L\)-series are “safe” functions: We can define bracket notation \([F(z)]G(z)\) by rule (4) whenever \(F(z)\) is an \(R\)-series and \(G(z)\) is an \(L\)-series. Convergence is not then an issue. This definition provides the default meaning of bracket notation, whenever no other context is specified. The transformations in (7) and (8) are valid when the functions inside brackets are \(R\)-series and the functions outside brackets are \(L\)-series. Equations (2) and (3) should not be used unless \(F\) and \(G\) are both \(L\)-series and \(R\)-series.

In such cases paradoxes do not rear their ugly heads. The ill-fated equation (8) may fail, but equation (15) is always true.

**Additional properties.** The bracket notation satisfies several identities in addition to (2), (3), (6), and (7), hence we can often transform formulas in which it appears. In the first place, the operation is linear in both operands:

\[
[a F(z) + b G(z)] H(z) = a[F(z)] H(z) + b[G(z)] H(z); \quad (16)
\]

\[
[F(z)](a G(z) + b H(z)) = a[F(z)] G(z) + b[F(z)] H(z). \quad (17)
\]

In the second place, there is a general multiplication law

\[
[F_1(z) F_2(z)] G_1(z) G_2(z) = \sum_k ([F_1(z)z^k] G_1(z))[F_2(z)z^{-k}] G_2(z)) \quad (18)
\]

If \(F_1(z) = F_2(z) = 1\), this equation is simply the special case \(n = 0\) of (5), and for general \(F_1\) and \(F_2\) it follows from the special case because we can replace \(G_1(z)\) and \(G_2(z)\) by \(F_1(z)G_1(z)\) and \(F_2(z)G_2(z)\) using (7).

We also have

\[
[F(z^m)] G(z^m) = [F(z)] G(z) \quad (19)
\]

for any nonzero integer \(m\); this equation, which includes (3) as the special case \(m = -1\), follows immediately from (4) because \(1[H(z)] = [1]H(z^m)\). Equation (19) suggests that we generalize bracket notation to functions that are sums over nonintegral powers, in which case \(m\) would not need to be an integer. Then we could write (19) as

\[
[F(z)] G(z^m) = [F(z^{1/m})] G(z), \quad m \neq 0. \quad (19')
\]

Such generalizations, extending perhaps to integrals as well as to sums, may prove to be quite interesting, but they will not be pursued further here.

If \(a\) is any nonzero constant, we have \([1]H(az) = [1]H(z)\). This rule implies that \([1]F(z)G(az) = [1]F(az)G(z)\), and (4) yields

\[
[F(z)] G(az) = [F(az)] G(z). \quad (20)
\]
The special case where $F(z)$ is simply $z^m$ is, of course, already familiar:

$$[z^m] G(az) = [(az)^m] G(z) = a^m [z^m] G(z).$$

Bracket notation also interacts with differentiation in interesting ways. We have, for instance,

$$[z^-] G'(z) = 0 \quad (21)$$

for any function $G(z) = \sum_{n=-\infty}^{\infty} g_n z^n$. More significantly,

$$[F(z)] z G'(z) = [z F'(z)] G(z). \quad (22)$$

Equation (21) is essentially the special case $F(z) = 1$ of (22), but we can also derive (22) from (21): Let $H(z) = F(z^-)G(z)$; then $0 = [1] z H'(z) = [1] z \left(F(z^-)G'(z) - z^{-2} F'(z)G(z)\right)$, hence $[1] F(z^-) z G'(z) = [1] z^{-2} F'(z^-)G(z)$, which is (22).

Let $\vartheta$ be the operator $z \frac{d}{dz}$. Then (22) implies by induction on $m$ that

$$[F(z)] \vartheta^m G(z) = [\vartheta^m F(z)] G(z)$$

for all integers $m \geq 0$, and we have

$$[F(z)] P(\vartheta) G(z) = [P(\vartheta) F(z)] G(z) \quad (23)$$

for any polynomial $P$. If $F(z) = \sum_n f_n z^n$ and $G(z) = \sum_n g_n z^n$, both sides of (23) evaluate to $\sum_n P(n) f_n g_n$.

Additional variables. When $G(w, z)$ is a bivariate generating function we also wish to write $[w^m z^n]$ for the coefficient of $w^m z^n$ in $G$. In general we can define

$$[F(w, z)] G(w, z) = [1] F(w^-, z^-) G(w, z), \quad (24)$$

extending (4).

Variables must be clearly distinguished from constants. If $w$ and $z$ are both variables, we have for instance $[z] wz = 0$, while if $w$ is constant we have $[z] wz = w$. If the set of variables is not clear from the context, we can specify it by writing its elements as subscripts on the brackets. For example,

$$[F(w) G(z)]_{w,z} H(w, z) = [G(z)]_z ([F(w)]_w H(w, z)) \quad (25)$$

because the former is $[1]_{w,z} F(w^-)G(z^-)H(w, z)$ while the latter is

$$[1]_z (G(z^-) [1]_w (F(w^-)H(w, z))) = [1]_z [1]_w G(z^-) F(w^-) H(w, z)$$
and \([1]_{w,z} = [1]_w[1]_z\).

After we have evaluated the parenthesis on the right side of (25), the ambiguity disappears, because \(w\) is no longer present. For example, if \(m \geq 0\) we have
\[
[w^m z^n] \frac{1}{1 - wF(z)} = [z^n] \left( [w^m]_w \frac{1}{1 - w(z)} \right) = [z^n] F(z)^m, \tag{26}
\]
where brackets without subscripts assume that both \(w\) and \(z\) are variables. Similarly
\[
[w^m z^n] e^{wF(z)} = [z^n] \frac{F(z)^m}{m!}, \tag{27}
\]
\[
[w^m z^n] G(wF(z)) H(z) = [z^n] F(z)^m H(z) [w^m] G(w). \tag{28}
\]

Suppose \(w\) and \(z\) are variables. Then laws (19) and (20) extend to
\[
[F(w,z)] G(aw,z) = [F(aw,z)] G(w,z), \quad a \neq 0; \tag{29}
\]
\[
[F(w^m,z)] G(w^m,z) = [F(w,z)] G(w,z), \quad \text{integer } m \neq 0; \tag{30}
\]
\[
[F(w,w^m z)] G(w,w^m z) = [F(w,z)] G(w,z); \tag{31}
\]
and we have indeed the general rule
\[
[F(a^{-w^k z^l}, b^{-w^m z^n})] G(aw^k z^l, bw^m z^n) = [F(w,z)] G(w,z) \tag{32}
\]
when \(a \neq 0, b \neq 0, \) and \(\left\lfloor \frac{k \ l}{m \ n} \right\rfloor \neq 0, \) i.e., \(kn \neq lm.\) A similar formula applies with respect to any number of variables.

The following example from the theory of random graphs \([3, (10.10) \text{ and } (10.14)]\) illustrates how these rules are typically applied. Suppose we want to evaluate the coefficient of \([w^m z^n]\) in the expression \(e^{U(wz)/w+V(wz)}\), where \(U\) and \(V\) are known functions with \(U(0) = 0\). The two-variable problem is reduced to a one-variable problem as follows:
\[
[w^m z^n] e^{U(wz)/w+V(wz)} = [(w^{-})^{n-m}(wz)^n] e^{U(wz)w^{-}+V(wz)}
\]
\[
= [w^{n-m} z^n] e^{U(z)w+V(z)}
\]
\[
= \frac{1}{(n-m)!} [z^n] U(z)^{n-m} e^{V(z)}, \tag{33}
\]
by (32) with \(F(w,z) = w^{n-m} z^n, G(w,z) = e^{U(z)w+V(z)}, a = b = 1, k = -1, l = 0, m = n = 1.\) The final step uses (28) with \(F(z) = U(z), G(w) = e^w, \) and \(H(z) = e^{V(z)}.\)

As before, we need to check that the functions are safe before we can guarantee that such manipulations are legitimate. For formal power series, the functions inside brackets
should be \(R\)-series and the functions outside should be \(L\)-series. This condition holds in each step of (33) because \(U(0) = 0\).

**Additional identities.** The bracket notation also obeys more complex laws that deserve further study. For example, Gessel and Stanton [4, Eq. (3)] have shown among other things that

\[
[F(w, z)] \frac{G(w, z)}{1 - wz} = [F(w(1 + z^-), z(1 + w^-))] G \left( \frac{w}{1 + z}, \frac{z}{1 + w} \right). \tag{34}
\]

If we set \(F(w, z) = w^k z^l\) and \(G(w, z) = (1 + w)^m(1 + z)^n/(1 - wz)^{m+n}\), Gessel and Stanton observe that we obtain Saalschütz’s identity after some remarkable cancellation:

\[
\sum_r \binom{m}{k - r} \binom{n}{l - r} \binom{m + n + r}{r} = [w^k(1 + z^-)^k z^l(1 + w^-)^l] (1 + w)^m(1 + z)^n
= [w^k z^l] (1 + w)^{m+l}(1 + z)^{n+k}
= \binom{m + l}{k} \binom{n + k}{l}. \tag{35}
\]

And if we set \(F(w, z) = w^{l+n}z^{m+n}\), \(G(w, z) = (w - z)^{l+m}/(1 - wz)^{l+m}\), the left side of (34) reduces to

\[
[w^{l+n}z^{m+n}] \frac{(w - z)^{l+m}}{(1 - wz)^{l+m+1}} = (-1)^m \binom{l + m + n}{l} \frac{(l + m + n)!}{l! m! n!}; \tag{36}
\]

the right side is

\[
[w^{l+n}(1 + z^-)^{l+n} z^{m+n}(1 + w^-)^{m+n}] (w - z)^{l+m}
= [w^{l+n}z^{m+n}] (w - z)^{l+m}(1 + w)^{m+n}(1 + z)^{l+n}
= \sum_k (-1)^{k+m} \binom{l + m}{k + m} \binom{m + n}{k + n} \binom{n + l}{k + l}. \tag{37}
\]

The fact that (36) = (37) is Dixon’s identity [7, exercise 1.2.6–62].

Equation (34) can be generalized to \(n\) variables, and we can replace the ‘1’ on the right by any nonzero constant \(a\):

\[
[F(z_1, \ldots, z_n)] \frac{G(z_1, \ldots, z_n)}{1 - z_1 \ldots z_n}
= [F(z_1(a + z_2^-), \ldots, z_n(a + z_1^-))] G \left( \frac{z_1}{a + z_2}, \ldots, \frac{z_n}{a + z_1} \right). \tag{38}
\]
It suffices to prove this when $F(z_1, \ldots, z_n) = 1$ and $G(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}$, in which case both sides are 0 unless $m_1 = \cdots = m_n \leq 0$, when both sides are 1. Equation (38) holds in particular when $n = 1$:

$$[F(z)] \frac{G(z)}{1 - z} = \left[ F(1 + az) \right] G \left( \frac{z}{a + z} \right), \quad a \neq 0. \quad (39)$$

Returning to the case of a single variable, we should also state the general rule for composition of series:

$$G(F(z)) = \sum_n F(z)^n \left[ z^n \right] G(z). \quad (40)$$

Special conditions are needed to ensure that this infinite sum is well defined.

**Lagrange’s inversion formula.** Let $F(z) = f_1 z + f_2 z^2 + f_4 z^3 + \cdots$, with $f_1 \neq 0$, and let $G(z)$ be the inverse function so that

$$F(G(z)) = G(F(z)) = z. \quad (41)$$

Lagrange’s celebrated formula for the coefficients of $G$ can be expressed in bracket notation in several ways; for example, we have

$$n\left[ z^n \right] G(z)^m = m\left[ z^{-m} \right] F(z)^{-n}, \quad (42)$$

for all integers $m$ and $n$.

One way to derive (42), following Paule [10], is to note first that (40) implies

$$z^m = G(F(z))^m = \sum_k F(z)^k \left[ z^k \right] G(z)^m. \quad (43)$$

Differentiating with the $\vartheta$ operator and dividing by $F(z)^n$ yields

$$\frac{m z^n}{F(z)^n} = \sum_k k F(z)^{k-1-n} \vartheta F(z) \left[ z^k \right] G(z)^m. \quad (44)$$

Now we will study the constant terms of (44). If $k \neq n$,

$$[1] \frac{F(z)^{k-1-n} \vartheta F(z)}{k - n} = 0, \quad (45)$$

by (22). And if $k = n$,

$$[1] \frac{\vartheta F(z)}{F(z)} = \left[ \frac{f_1 + 2 f_2 z + 3 f_3 z^2 + \cdots}{f_1 + f_2 z + f_3 z^2 + \cdots} \right] = 1, \quad (46)$$
because \( f_1 \neq 0 \). Therefore the constant terms of (44) are

\[ [1] \frac{mz^m}{F(z)^n} = n [z^n] G(z)^m; \]

this is Lagrange’s formula (42).

**Conclusions.** Many years of experience have confirmed the great importance of generating functions in the analysis of algorithms, and we can reasonably expect that some fluency in manipulating the “coefficient-of” operator will therefore be rewarding.

If, for example, we are faced with the task of simplifying a formula such as

\[ \sum_k \binom{m}{k} [z^{n-k}] F(z)^k, \]

a rudimentary acquaintance with the properties of brackets will tell us that it can be written as \( \sum_k \binom{m}{k} [z^n] z^k F(z)^k \) and then summed to yield

\[ [z^n](1 + z F(z))^m. \]

We have seen several examples above in which formulas that are far less obvious can be derived rapidly by bracket manipulation, when we use quantities more general than monomials inside the brackets.

In most applications we use bracket notation in connection with formal Laurent series, in which case it is important to remember that our identities for \([F(z)] G(z)\) require \( G(z) \) to have only finitely many negative powers of \( z \) while \( F(z) \) must have only finitely many positive powers. If we write, for example,

\[ \left[ \frac{z^n}{z - 1} \right] G(z), \]

we should think of the quantity in brackets as an infinite series

\[ z^{n-1} + z^{n-2} + z^{n-3} + \ldots \]

that descends to arbitrarily negative powers of \( z \); the bracket notation then denotes the sum \( g_{n-1} + g_{n-2} + g_{n-3} + \ldots \), which will be finite. We have seen that other interpretations of bracket notation are possible for functions analytic in an annulus; but great care must be taken to avoid paradoxes in such cases, hence the extra effort might not be worthwhile.

Bracket notation, like all notations, is “dispensable,” in the sense that we can prove the same theorems without it as with it. But the use of a good notation can shorten proofs and help us see patterns that would otherwise be difficult to perceive.
Let us close with one more example, illustrating that the notation (47) helps to simplify some of the formulas in [2]. The coupon collector’s problem asks for the expected number of trials needed to obtain \( n \) distinct coupons from a set \( C \) of \( m \) given coupons, where each trial independently produces coupon \( c \) with probability \( p(c) \). Theorem 2 of [2] says, when rewritten in the notation discussed above, that this expected number is

\[
\int_0^\infty \left[ \frac{z^n}{z-1} \right] \prod_{c \in C} (1 + z(e^{p(c)t} - 1)) e^{-t} \, dt.
\] (48)

We can evaluate (48) by expanding the integrand as follows:

\[
\left[ \frac{z^n}{z-1} \right] \prod_{c \in C} (1 + z(e^{p(c)t} - 1)) = \sum_{B \subseteq C} \prod_{c \in B} (e^{p(c)t} - 1)
\]

\[
= \sum_{A \subseteq B \subseteq C} (-1)^{|B|-|A|} e^{p(A)t}
\]

\[
= \sum_{A \subseteq B \subseteq C} e^{p(A)t} \sum_{|A| \leq k < n} (-1)^k e^{p(A)(|C| - |A|)}
\]

\[
= \sum_{A \subseteq C} e^{p(A)t} (-1)^{n-1-|A|} e^{p(A)(|C| - n)}
\]

where \( p(A) \) denotes \( \sum_{a \in A} p(a) \). The integral (48) therefore is

\[
\sum_{A \subseteq C} (-1)^{n-1-|A|} e^{p(A)(|C| - n)} \left( \frac{|C| - |A| - 1}{|C| - n} \right) / (1 - p(A)).
\] (49)

(This is Corollary 3 of [2], which was stated without proof.)

Related work. Steven Roman’s book on umbral calculus [11] develops extensive properties of his notation \( \langle G(t) \mid F(x) \rangle \), which equals \( \sum_{n \geq 0} f_n g_n \) when \( F(x) = \sum_{n \geq 0} f_n x^n \) and \( G(t) = \sum_{n \geq 0} g_n t^n/n! \); the function \( F(x) \) in these formulas must be a polynomial. Thus, if \( D \) is the operator \( d/dx \), Roman’s \( \langle G(t) \mid F(x) \rangle \) is the constant term of the polynomial \( G(D)F(x) \). Chapter 6 of [11] considers generalizations in which \( \langle G(t) \mid F(x) \rangle \) is defined to be \( \sum_{n \geq 0} f_n g_n \) when \( G(t) = \sum_{n \geq 0} g_n t^n/c_n \) and \( c_n \) is an arbitrary sequence of constants; the case \( c_n = 1 \) corresponds to the special case of bracket notation \( [F(z)]G(z) \) when \( F \) and \( G \) involve no negative powers of \( z \). Roman traces the theory back to a paper by Morgan Ward [12].
G. P. Egorychev’s book [1] includes a great many examples that demonstrate the value of coefficient extraction in the midst of formulas.

**Open problems.** One reason formal power series are usually restricted to $L$-series is that certain doubly infinite power series are divisions of zero. For example, $\sum_{n=-\infty}^{\infty} z^n$ is a divisor of zero because multiplication by $1 - z$ annihilates it. (This series causes no problem in the theory of non-formal power series because it does not converge for any value of $z$.) All double series having the form $\sum_n n^m \alpha^n z^n$ for $\alpha \neq 0$ and integer $m \geq 0$ can also be shown to be divisors of zero. Question: Do there exist divisors of zero besides finite linear combinations of the double series just mentioned? Conjecture: There is no nonzero double series $F(z)$ such that $e^z F(z) = 0$. (A counterexample would necessarily be divergent.)

It may be possible and interesting to extend the theory of formal Laurent series to arbitrary functions of the form $F(z) \sum_n g_n z^n$, where $g_n$ is zero for all sufficiently negative $n$ and where $F(z)$ is analytic for $0 < |z| < \infty$.

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