One-dimensionality of the minimizers in the large volume limit for a diffuse interface attractive/repulsive model in general dimension

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Abstract
In this paper we consider the diffuse interface generalized antiferromagnetic model with local/nonlocal attractive/repulsive terms in competition studied in [9]. The parameters of the model are denoted by \( \tau \) and \( \varepsilon \): the parameter \( \tau \) represents the relative strength of the local term with respect to the nonlocal one, while the parameter \( \varepsilon \) describes the transition scale in the Modica–Mortola type term. Restricting to a periodic box of size \( L \), with \( L \) multiple of the period of the minimal one-dimensional minimizers, in [9] the authors prove that in any dimension \( d \geq 1 \) and for small but positive \( \tau \) and \( \varepsilon \) (eventually depending on \( L \)), the minimizers are non-constant one-dimensional periodic functions. In this paper we prove that periodicity and one-dimensionality of minimizers occurs also in the zero temperature analogue of the thermodynamic limit, namely as \( L \to +\infty \).

Mathematics Subject Classification 49N99 · 74G05

1 Introduction
In this paper we consider the following short-range attractive long-range repulsive (SALR) energy functional. For \( L, J, \varepsilon > 0, d \geq 1, p \geq d + 2, u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d; [0, 1]) \) and \( [0, L)^d \)-periodic, define

\[
\tilde{F}_{J,L,\varepsilon}(u) := \frac{J}{L^d} \left[ 3\varepsilon \int_{[0,L)^d} \|\nabla u(x)\|_1^2 \, dx + \frac{3}{\varepsilon} \int_{[0,L)^d} W(u(x)) \, dx \right] - \frac{1}{L^d} \int_{\mathbb{R}^d} \int_{[0,L)^d} |u(x + \xi) - u(x)|^2 K(\xi) \, dx \, d\xi,
\]

(1.1)
where, for \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d, \|y\|_1 = \sum_{i=1}^d |y_i|, \) \( W(t) = t^2(1-t)^2 \) and \( K_p(\zeta) = \frac{1}{(\|\zeta\|_1 + 1)^p}. \)

In a suitable range of \( J \), the short range attractive term of Ginzburg–Landau type favoring uniform phases competes with the nonlocal repulsive term favoring oscillations and, as a result, periodic patterns are expected to emerge. The fact that regular periodic structures emerge solely from the minimization of symmetric short-range attractive and long-range repulsive interactions in competition is known in the field as energy-driven pattern formation conjecture and to show it rigorously proves to be a very difficult task, in most cases still open. The two main difficulties in this kind of problems are the phenomenon of symmetry breaking, namely the fact that the functionals retain more symmetries (in this case symmetry w.r.t. coordinate permutations) than their ground states, and the nonlocality of the interactions, which make any perturbation argument a delicate task.

For the functional (1.1) there exists a critical constant

\[
J_c = \int |\zeta_1| K(\zeta) \, d\zeta
\]

such that, whenever \( J > J_c \), minimizers are trivial, namely \( u \equiv 0 \) or \( u \equiv 1 \). Below this constant, whenever \( \varepsilon \) and \( \tau = J_c - J \) are sufficiently small, one-dimensional periodic ground states are expected to emerge, giving rise to symmetry breaking and pattern formation. Since the period of the ground states for the functional (1.1) will converge to infinity as \( \tau \to 0 \), in order to study their structure it is convenient to rescale the functional in order to have that the width of the optimal period for one-dimensional functions and their energy to 0, in order to study their structure it is convenient to rescale the functional in order to have that the width of the optimal period for one-dimensional functions and their energy are of order \( O(1) \). The rescaling is done considering the fact that, among one-dimensional characteristic functions \( u = \chi_E \) for the sharp interface limit of (1.1) as \( \varepsilon \to 0 \), minimizers are periodic stripes of width of order \( \tau^{-1/\beta} \), where \( \beta = p - d - 1 \). In particular, setting

\[
x = \tau^{-1/\beta} \tilde{x}, \quad \zeta = \tau^{-1/\beta} \tilde{\zeta}, \quad L = \tau^{-1/\beta} \tilde{L},
\]

\[\tilde{u}(\tilde{x}) = u(x), \quad \tilde{F}_{J,L,\varepsilon}(u) = \tau^{1+1/\beta} F_{\tau,L,\varepsilon}(\tilde{u})\]

and finally dropping the tildas, one has that the rescaled functional has the form

\[
F_{\tau,L,\varepsilon}(u) = \frac{1}{L^d} \left[ \mathcal{M}_{\alpha,\varepsilon,\tau}(u, [0, L]^d) \left( \int_{\mathbb{R}^d} K(\zeta)|\zeta_1| \, d\zeta - 1 \right) - \int_{\mathbb{R}^d} \int_{[0,L]^d} |u(x) - u(x + \zeta)|^2 K_{\tau,p}(\zeta) \, dx \, d\zeta \right],
\]

where for \( \alpha > 0 \)

\[
\mathcal{M}_\alpha(u, [0, L]^d) = 3 \alpha \int_{[0,L]^d} \|
abla u(x)\|^2 \, dx + \frac{3}{\alpha} \int_{[0,L]^d} W(u(x)) \, dx,
\]

\( \alpha_{\varepsilon,\tau} = \varepsilon \tau^{1/\beta} \) and

\[
K_{\tau,p}(\zeta) = \frac{1}{(\|\zeta\|_1 + \tau^{1/\beta})^p}.
\]

For fixed \( \tau > 0 \) and \( \varepsilon > 0 \), consider first for all \( L > 0 \) the minimal value obtained by \( F_{\tau,L,\varepsilon} \) on \([0, L]^d\)-periodic one-dimensional functions (denoted by \( U^p_{\text{per}} \)) and then the minimal among these values as \( L \) varies in \((0, +\infty)\). We will denote this value by \( C^*_\tau,\varepsilon \), namely

\[
C^*_\tau,\varepsilon := \inf_{L>0} \inf_{u \in U^p_{\text{per}}} F_{\tau,L,\varepsilon}(u).
\]
By the reflection positivity technique, in [16] it is shown that such value is attained by periodic one-dimensional functions with possibly infinite and not unique periods.

In [9] we proved in collaboration with A. Kerschbaum that, for $\tau$ and $\varepsilon$ sufficiently small, there exist periodic functions of finite period $2h$ for which the energy value $C_{\tau,\varepsilon}^*$ is attained and, up to translations, the following property holds

$$g((2k+1)h+t) = 1 - g((2k+1)h-t) \quad \text{for all } k \in \mathbb{N} \cup \{0\}, \ t \in [0, h]. \quad (1.6)$$

We denote any of such finite optimal periods $2h$ (which may not be unique) as $2h_{\tau,\varepsilon}^*$.

The main result obtained in [9] is the following

**Theorem 1.1** [9, Theorem 1.1] Let $2h_{\tau,\varepsilon}^*$ be an admissible optimal period and let $L = 2kh_{\tau,\varepsilon}^*$, $k \in \mathbb{N}$. Then there exist $\tau_L > 0$, $\varepsilon_L > 0$ such that, for any $0 < \tau \leq \tau_L$ and $0 < \varepsilon \leq \varepsilon_L$ the minimizers of (1.2) are one-dimensional periodic functions of period $2h_{\tau,\varepsilon}^*$ satisfying (1.6).

In this paper, we prove that one-dimensionality and periodicity of minimizers of (1.2) holds also in the so-called large volume limit, namely the zero temperature analogue of the thermodynamic limit. More precisely, we show that the range of parameters $\tau$, $\varepsilon$ in which one-dimensionality and periodicity of minimizers is observed can be fixed independently on how large $L$ is. The following holds.

**Theorem 1.2** Let $d \geq 1$, $p \geq d + 2$ and $h_{\tau,\varepsilon}^*$ be an optimal period for fixed $\tau$, $\varepsilon > 0$. Then there exists $\bar{\tau}$ and $\bar{\varepsilon}$, such that for every $\tau < \bar{\tau}$ and $\varepsilon < \bar{\varepsilon}$, one has that for every $k \in \mathbb{N}$ and $L = 2kh_{\tau,\varepsilon}^*$, the minimizers of $F_{\tau,L,\varepsilon}$ are optimal one-dimensional functions of width $h_{\tau,\varepsilon}^*$.

We notice that the existence of minimizers for the functional $F_{\tau,L,\varepsilon}$ for any $L$, $\tau$ and $\varepsilon$ can be shown applying the direct method in the calculus of variations. Indeed, it is not difficult to see that a control of the energy $F_{\tau,L,\varepsilon}$ implies a control on the $W^{1,2}$-norm (being the term involving the kernel a volume term controlled by a constant) and that the functional is lower semicontinuous w.r.t. weak convergence in the set of $[0, L]^d$-periodic functions in $W^{1,2}_{\text{loc}}(\mathbb{R}^d; [0, 1])$.

In the next section we insert our results in their scientific context and explain the main difficulties and the general strategy of the proof.

### 1.1 Scientific context and main approach

For the sharp interface limit of $F_{\tau,L,\varepsilon}$ as $\varepsilon \to 0$, namely

$$F_{\tau,L}(E) := \frac{1}{L^d} \left[ \text{Per}_1(E; [0, L]^d) \left( \int_{\mathbb{R}^d} K_{\tau,p}(\xi) |\xi_1| \, d\xi - 1 \right) - \int_{\mathbb{R}^d} \int_{[0,L]^d} |\chi_E(x) - \chi_E(x + \xi)| K_{\tau,p}(\xi) \, dx \, d\xi \right], \quad (1.7)$$

where $E \subset \mathbb{R}^d$, $d \geq 2$, one-dimensionality and periodicity of minimizers in the thermodynamic limit has been first proved in [17] in the discrete setting (for exponents $p > 2d$) and in [11,15] in the continuous one (for exponents $p \geq d + 2$). In [19] symmetry breaking has been first proved in the continuous setting for exponents $p > 2d$. In [21] the results of [11] have been recently extended to a small range of exponents below $p = d + 2$. In [14] the authors prove that minimizers are periodic stripes even under the imposition of an arbitrary nontrivial volume constraint, provided $\tau$ is sufficiently small.

For the most physically relevant exponents such as $p = d + 1$ (thin magnetic films), $p = d$ (3D micromagnetics) and $p = d - 2$ (diblock copolymers), a rigorous proof of
pattern formation in dimension $d \geq 2$ is a challenging open problem. The main difficulty is to prove that symmetry breaking occurs, namely that minimizers of (1.7) have less symmetries than the functional itself, on symmetric domains. For the discrete analogue of (1.7) and $p > 2d$ this issue is resolved in [17]. For the above continuous setting it was resolved in [19], [11] and [21]. In this cases symmetry under coordinate permutations is broken by the one-dimensionality of the ground states. It is expected that the same symmetry breaking holds for the functional analogous to (1.7) where the 1-norm is substituted by the Euclidean norm, namely full rotational symmetry breaking is expected to occur as well.

The difficulties in lowering the exponents of the kernel are due to the increased nonlocality of the interactions. Another family of kernels which is physically relevant and widely used in the literature is the Yukawa or screened Coulomb kernel (see e.g. [1,2,6,18,20,28]). For this type of kernels one-dimensionality and periodicity of minimizers in the thermodynamic limit has been proved in [12] (see also [13]). The diffuse interface counterpart of (1.7), namely (1.2), is expected to describe more closely the physical picture due to the presence of continuous phase transitions (see e.g. the famous model for block copolymers introduced by Ohta and Kawasaki in [25]). For the diffuse interface problem, even less results are available in the literature on the structure of minimizers. This is due to the fact that in this setting the geometry of possible phase transitions is much richer. In particular, other phenomena such as small amplitude oscillations and slow transitions may occur, adding mathematical difficulty to the problem of showing exact one-dimensionality and requiring new estimates. In dimension $d = 1$, in [27] the authors show that close to the local minima of the corresponding sharp interface problem there are local minima of the approximating diffuse one. In [16], the authors show that the constant $C_{\tau,\varepsilon}^*$ in the one-dimensional problem is attained on periodic functions of possibly infinite period. The only result in dimension $d \geq 2$ concerning one-dimensionality and periodicity of minimizers in the regime $J < J_c$, $J \sim J_c$, is the one given in [9]. In [9] one-dimensionality and periodicity of minimizers for $d \geq 2$ was proved (see Theorem 1.1), for $L$ multiple of an optimal admissible period $h_{\tau,\varepsilon}^*$ and a range of positive $\tau, \varepsilon$ depending on $L$.

The aim of this paper is to show that a range of parameters for which pattern formation is observed can be chosen independently of $L$, no matter how large $L$ is (see Theorem 1.2). This is important since it allows to remove the $[0, L]^d$-periodic boundary conditions imposed in order to deal with finite energies (the natural physical setting is the whole $\mathbb{R}^d$) and show how the boundedness of the energy allows indeed to control deviations from one-dimensional profiles on arbitrarily large domains. The main difficulty in passing from Theorem 1.1 to Theorem 1.2 lies in the fact that rigidity estimates for sets of finite energy in the asymptotic limit as $\tau$ and $\varepsilon$ tend to 0 degenerate as $L \to +\infty$. Roughly speaking, a bound on the $\Gamma$-limit energy $\mathcal{F}_{\tau,\varepsilon}$ as $\varepsilon, \tau \to 0$ on a cube of edge $L$ gives a geometric control on the deviations from one-dimensional profiles which is weaker and weaker as $L \to +\infty$.

In order to overcome this we adopt, as in [11] and in [12], a multiscale approach, whose main steps are the following:

- a localization through averaging of the functional (1.2) on small cubes of size $l < L$;
- a decomposition of the localized functional into geometric terms penalizing deviations from being one-dimensional in different ways;
- a partition of $[0, L]^d$ into sets $A \cup A_1 \cup \cdots \cup A_d$ where $z \in A_i$ if on the cube $Q_l(z)$ the function $u$ is $L^1$-close to stripes with boundaries orthogonal to $e_i$;
- rigidity, stability and one-dimensional optimization arguments on slices in order to show that whenever $\tau = \tau(l)$ and $\varepsilon = \varepsilon(l)$ are sufficiently small, $[0, L]^d = A_i$ for some $i \in \{1, \ldots, d\}$.
• when $[0, L)^d = A_i$, hence the symmetry is broken, conclude with a stability argument that $u = u(x_i)$.

However, not only the above-mentioned rigidity, stability and one-dimensional optimization arguments are necessarily different from the ones of [11] due to the fact that the space of competitors is now given by functions instead of sets, but also the decomposition of the functional into localized terms is considerably different. In particular, a clever reformulation of the functional (1.2) together with a disintegration argument are introduced in order to identify the contributions of $y$ on the subspace orthogonal to $\partial_k y$ for $k = 1$.

In the following, let $N = \{1, 2, \ldots \}$. Let $(e_1, \ldots, e_d)$ be the canonical basis in $\mathbb{R}^d$ and for $y \in \mathbb{R}^d$ let $y_i = \langle y, e_i \rangle$ and $y_i^\perp := y - y_i e_i$, where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. For $y \in \mathbb{R}^d$, we denote by $\|y\|_1 = \sum_{i=1}^d |y_i|$ its 1-norm and we define $\|y\|_\infty = \max_i |y_i|$. With a slight abuse of notation, we will sometimes identify $y_i^\perp \in [0, L)^d$ with its projection on the subspace orthogonal to $e_i$ or as an element of $\mathbb{R}^{d-1}$.

For $z \in [0, L)^d$ and $r > 0$, we also define

$$Q_r(z) = \{x \in \mathbb{R}^d : \|x - z\|_\infty \leq r\} \quad \text{and} \quad Q_r^\perp(x_i^\perp) = \{z_i^\perp : \|x_i^\perp - z_i^\perp\|_\infty \leq r\}.$$

For every $i \in \{1, \ldots, d\}$ and for all $x_i^\perp \in [0, L)^{d-1}$, we define the slices of $u$ in direction $e_i$ as

$$u_{x_i^\perp} : \mathbb{R} \to [0, 1], \quad u_{x_i^\perp}(s) := u(se_i + x_i^\perp).$$

Notice that whenever $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d; \mathbb{R})$ then $u_{x_i^\perp} \in W^{1,2}_{\text{loc}}(\mathbb{R}; \mathbb{R})$ for almost every $x_i^\perp$. We denote by $\partial_i$ the partial derivatives of a function with respect to $e_i$, $i \in \{1, \ldots, d\}$.

Given a measurable set $A \subset \mathbb{R}^k$ with $k \in \{1, \ldots, d\}$, we denote by $|A|$ its $k$-dimensional Lebesgue measure (or if $A$ is contained in some $k$-dimensional plane of $\mathbb{R}^d$, its Hausdorff $k$-dimensional measure), being always clear from the context which will be the dimension $k$.
Moreover, let $\chi_A : \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \in \mathbb{R}^d \setminus A.
\end{cases}
$$

A set $E \subset \mathbb{R}^d$ is of (locally) finite perimeter if the distributional derivative of $\chi_E$ is a (locally) finite measure. We denote by $\partial E$ the reduced boundary of $E$ and by $v^E$ the exterior normal to $E$.

For some $\nu$, $\partial E$ is of the form $V$ Lebesgue null sets, is of the form $E$ dimensional subspace orthogonal to $E$. Notice that the constant 3 in (1.1) is chosen in such a way that

$$
\int_{\partial E \cap \{0\}} \nu \cdot \nu^E \, d\mathcal{H}^{d-1}(x)=1.
$$

So that the constant in front of the 1-perimeter in (2.1) is equal to 1.

Theorem 2.1 As $\varepsilon \to 0$, the functionals $\mathcal{F}_{\tau, L, \varepsilon}$ $\Gamma$-converge in $BV_{\text{loc}}(\mathbb{R}^d; [0, 1])$ to the functional

$$
\mathcal{F}_{\tau, L}(u) := \begin{cases} 
\frac{1}{L^d} \left[ \frac{\text{Per}_1(E; [0, L]^d)}{L^d} \left( \int_{\mathbb{R}^d} K_{\tau, p}(\xi) |\xi_1| \, d\xi - 1 \right) \right] \\
- \int_{\mathbb{R}^d} \int_{[0, L]^d} |\chi_E(x) - \chi_E(x + \xi)| K_{\tau, p}(\xi) \, dx \, d\xi \\
+ \infty
\end{cases}
$$

if $u = \chi_E$ (2.1)

otherwise.

Notice that the constant 3 in (1.1) is chosen in such a way that

$$
6 \int_0^1 t(1-t) \, dt = 1,
$$

so that the constant in front of the 1-perimeter in (2.1) is equal to 1.

The kernel $K_{\tau, p}$ is, as shown in [11], reflection positive, namely it satisfies the following property: the function

$$
\hat{K}_{\tau, p}(t) := \int_{\mathbb{R}^{d-1}} K_{\tau, p}(t, \xi_2, \ldots, \xi_d) \, d\xi_2 \cdots d\xi_d.
$$

is the Laplace transform of a nonnegative function.

In the rest of the paper we will fix $p \geq d + 2$ and thus write for simplicity of notation $K_{\tau}$ instead of $K_{\tau, p}$ and $\hat{K}_{\tau}$ instead of $\hat{K}_{\tau, p}$.

Regarding the limit functional (2.1), we recall the following result, obtained in [11] for $p \geq d + 2$ and extended to a range of exponent below $d + 2$ in [21] (Fig. 1).

Theorem 2.2 Let $d \geq 1$, $L > 0$. Then, there exists $0 < \tilde{\alpha} < 1$ and $\tilde{\tau}_L > 0$ such that, for all $p = d + 2 - \alpha$ with $\alpha \leq \tilde{\alpha}$ and for all $0 < \tau \leq \tilde{\tau}_L$, the minimizers of the functional $\mathcal{F}_{\tau, L}$ in (2.1) are periodic unions of stripes.

In the above result, a periodic union of stripes of width $h$ is by definition a set which, up to Lebesgue null sets, is of the form $V_i \perp \hat{E} e_i$ for some $i \in \{1, \ldots, d\}$, where $V_i \perp$ is the $(d - 1)$-dimensional subspace orthogonal to $e_i$ and $\hat{E} \subset \mathbb{R}$ with $\hat{E} = \bigcup_{k=0}^{N} (2kh + v, (2k + 1)h + v)$ for some $v \in \mathbb{R}$ and some $N \in \mathbb{N}$ (Fig. 2).
One-dimensionality of the minimizers...

Fig. 1 Minimizers of $\mathcal{F}_{\tau,L}$ under the conditions of Theorem 2.2, namely periodic stripes of width and distance $h_{\tau,L} > 0$ with the property that $L = 2kh_{\tau,L}$ for some $k \in \mathbb{N}$

Fig. 2 Minimizers of the diffuse interface functional $\mathcal{F}_{\tau,L,\varepsilon}$ in Theorems 1.1 and 1.2

Let us now recall the decomposition of the functional $\mathcal{F}_{\tau,L,\varepsilon}$ obtained in [9, Section 3]. In [9, Proposition 3.1] one gets the following lower bound for the functional $\mathcal{F}_{\tau,L,\varepsilon}$:

$$\mathcal{F}_{\tau,L,\varepsilon}(u) \geq \frac{1}{L^d} \sum_{i=1}^{d} \left\{ \int_{[0,L]^d} \left[ -\mathcal{M}_{\alpha,\varepsilon,\tau}^{\parallel}(u, x_{i}^{\perp}, [0, L]) + \mathcal{G}_{\alpha,\varepsilon,\tau}^{\parallel}(u, x_{i}^{\perp}, [0, L]) \right] dx_{i}^{\perp} + I_{\tau,L}^{i}(u) \right\} + \frac{1}{L^d} W_{\tau,L,\varepsilon}(u),$$

(2.2)

where

$$\mathcal{M}_{\alpha,\varepsilon,\tau}^{\parallel}(u, x_{i}^{\perp}, [s, t]) := 3\alpha_{\varepsilon,\tau} \int_{[s,t]\cap[\nabla u(x_{i}+x_{i}^{\perp})\neq 0]} |\partial_{i} u_{x_{i}^{\perp}}(\rho)\|\nabla u(\rho e_{i} + x_{i}^{\perp})\|_{1} d\rho$$

$$+ \frac{3}{\alpha_{\varepsilon,\tau}} \int_{[s,t]\cap[\nabla u(x_{i}+x_{i}^{\perp})\neq 0]} W(u_{x_{i}^{\perp}}(\rho)) \frac{|\partial_{i} u_{x_{i}^{\perp}}(\rho)|}{\|\nabla u(\rho e_{i} + x_{i}^{\perp})\|_{1}} d\rho,$$

(2.3)
the elementary inequality

\[ \mathcal{T}^i_{\alpha_\tau}(a, x_i^+, [0, L]) := \mathcal{M}^i_{\alpha_\tau}(u, x_i^+, [0, L]) \int_{\mathbb{R}} |\xi_i| K_\tau(\xi_i) \, d\xi_i \]

\[ - \int_{\mathbb{R}} \int_0^L |u_{x_i^+}(x_i) - u_{x_i^+}(x_i + \xi_i e_i)|^2 K_\tau(\xi_i) \, dx_i \, d\xi_i, \]

(2.4)

and where

\[ W_{\tau,\mu}(u) = \frac{3(C_\tau - 1)}{\alpha_\tau} \int_{[\mathcal{V} u = 0] \cap [0, L]^d} W(\mu(x)) \, dx, \quad C_\tau = \int_{\mathbb{R}} |\xi_i| K_\tau(\xi_i) \, d\xi_i. \]  

(2.5)

In particular, since showing that the minimizers for the r.h.s. of (2.2) are one-dimensional implies that the minimizers for \( \mathcal{F}^i_{\tau,\mu} \) are one-dimensional, this allows us to reduce to prove one-dimensionality of the minimizers for the lower bound functional (i.e., the r.h.s. of (2.2)).

Given the numerous slicing arguments, it is also convenient to define the slicing of \( \mathcal{T}^i_{\tau,\mu} \) as follows

\[ \mathcal{T}^i_{\tau,\mu}(a, x_i^+, [0, L]) := \frac{1}{d} \int_{[\mathcal{V} u = 0] \cap [0, L]^d} \left[ (x(x + \xi_i e_i) - u(x)) - (x(x + \xi_i) - u(x + \xi_i^+)) \right]^2 K_\tau(\xi_i) \, dx \, d\xi_i. \]  

(2.7)

and where \( x = x_i e_i + x_i^+ \).

We also recall the estimate contained in Lemma 4.3 in [9], namely that for all \( \rho \in \mathbb{R} \) the following holds:

\[ |\rho| \mathcal{M}^i_{\alpha_\tau}(u, x_i^+, [0, L]) = \int_0^L \mathcal{M}^i_{\alpha_\tau}(u, x_i^+, [s, s + \rho]) \, dx_i \]

\[ \geq \int_0^L |\omega(u_{x_i^+}(s + \rho)) - \omega(u_{x_i^+}(s))| \, ds, \]  

(2.8)

where \( \omega: [0, 1] \to [0, 1] \) is defined by

\[ \omega(t) = \int_0^t 6\sqrt{W(s)} \, ds = 3t^2 - 2t^3. \]  

(2.9)

In the first equality of (2.8) one uses \([0, L]^d\)-periodicity of \( u \) and in the second inequality the elementary inequality \( a^2 + b^2 \geq 2ab \).

As observed in [9, Remark 4.1], the function \( \omega \) satisfies the following inequality: for \( a, b \in [0, 1] \) with \( a = b + t, \, t > 0 \)

\[ \frac{\omega(a) - \omega(b)}{|a - b|^2} = \frac{6b(1 - b - t)}{t} + 3 - 2t \geq 3 - 2t \geq 1. \]  

(2.10)

and equality in the last inequality holds if and only if \( a = 1 \) and \( b = 0 \).

In order to measure the \( L^1 \) distance of the functions \( u \) to stripes having boundaries orthogonal to a certain direction we will need the following definition.

**Definition 2.3** For every \( \eta \) we denote by \( \mathcal{A}^i_\eta \) the family of all sets \( F \) such that

(i) they are union of stripes with boundaries orthogonal to \( e_i \);
(ii) their connected components of the boundary are distant at least $\eta$.

We denote by

$$D^j_\eta(u, Q) := \inf \left\{ \frac{1}{|Q|} \int_Q |u - \chi_F| : F \in \mathcal{A}^j_\eta \right\}$$

and

$$D_\eta(u, Q) = \inf_i D^i_\eta(u, Q). \quad (2.11)$$

Also in the cases of functions, such a distance enjoys some useful properties, that we list below.

**Remark 2.4** (i) The map $z \mapsto D^j_\eta(u, Q_j(z))$ is Lipschitz, with Lipschitz constant $C_d/l$, where $C_d$ is a constant depending only on the dimension $d$.

(ii) For every $v > 0$ there exists $\sigma_0 = \sigma_0(v)$ such that for every $\sigma \leq \sigma_0$ whenever $D^j_\eta(u, Q_j(z)) \leq \sigma$ and $D^i_\eta(u, Q_i(z)) \leq \sigma$ with $i \neq j$ for some $\eta > 0$, it holds

$$\min \left\{ \|u\|_{L^1(Q_j(z))}, \|1 - u\|_{L^1(Q_i(z))} \right\} \leq vl^d. \quad (2.12)$$

Finally, we recall from [9] (Lemma 5.6) the following

**Lemma 2.5** Let $u$ be such that $D^j_\eta(u, Q_j(z)) \leq \tilde{\sigma}$ for some $j \neq i$. Then, for any $\alpha > 0$ and $|s_0 - t_0| \leq \alpha$, if $\tilde{\sigma}$ is sufficiently small

$$\int_{|\xi_j^-| < \alpha} \int_{s_0 - \alpha}^{t_0 - x_j + \alpha} \left\{ \frac{1}{4} - u(x_j^+ + \xi_j^+ + x_je_j) - u(x_j^+ + \xi_j^+ + x_je_j) \right\}^2 \right. \left. d\xi_j \ dx_j \right\} \geq \frac{1}{8} \alpha^{d+1}. \quad (2.13)$$

### 3 Decomposition of the functional

The aim of this section is to determine a localization on scale $l < L$ and a decomposition of the functional $\mathcal{F}_{T, L, \varepsilon}$ into geometric quantities which will be able to control deviations from one-dimensional profiles. The final decomposition is contained in Lemma 3.1. As observed in Sect. 1.1, in order to deal with the diffuse character of the short-range attractive term the decomposition differs significantly from the one used for the sharp interface model. The $[0, L]^d$ periodicity of $u$ and a disintegration argument are here essential. Moreover, at the end of this section, once the localized functionals are defined we will state the Local Rigidity Proposition 3.2.

In order to introduce the decomposition of Lemma 3.1, we need some preliminary definitions.

For $a \in [0, L)$, $b \in \mathbb{R}$ with $a < b$ define

$$\Omega(a, b) = \left\{ (s, \rho) \in [0, L] \times (0, +\infty) : [s, s + \rho] \supset [a, b] \right\}$$

$$\cup \left\{ (s, \rho) \in [0, L] \times (-\infty, 0) : [s + \rho, s] \supset [a, b] \right\}. \quad (3.1)$$

If $b < a$, let

$$\Omega(a, b) = \left\{ (s, \rho) \in [0, L] \times (0, +\infty) : [s, s + \rho] \supset [b, a] \right\}$$

$$\cup \left\{ (s, \rho) \in [0, L] \times (-\infty, 0) : [s + \rho, s] \supset [b, a] \right\}. \quad (3.2)$$

and for all $\rho \in \mathbb{R}$ let

$$G(\rho) = |\rho| \min(|\rho|, L). \quad (3.3)$$
Notice that, for any \( s \in [0, L) \), \( \rho \in \mathbb{R} \)

\[
\int_s^{s+L} \int_{\mathbb{R}} \chi_{[(a,b);(s,\rho)\in\Omega(a,b)]}(a, b) \, db \, da = G(\rho).
\] (3.4)

In the following, whenever not clear from the context whether \( \rho > 0 \) or \( \rho < 0 \), we will use for simplicity the following compact notation for the interval with boundary points \( s, s+\rho \):

\[
J(s, s + \rho) = \begin{cases} 
\{ t \in \mathbb{R} : s \leq t \leq s + \rho \} & \text{whenever } \rho > 0 \\
\{ t \in \mathbb{R} : s + \rho \leq t \leq s \} & \text{whenever } \rho < 0.
\end{cases}
\] (3.5)

Then, for any \( i \in \{1, \ldots, d\} \), \( x_i^\perp \in [0, L)^d-1 \) and for any interval \( I \subset [0, L) \) define

\[
R_{i,\tau,\varepsilon}(u, x_i^\perp, I) = -\nabla_{x_i,\tau} (u, x_i^\perp) + \int_{[s,\rho)\in\Omega(a,b)} G^{-1}(\rho) \left( \nabla_{x_i,\tau} (u, x_i^\perp, J(s, s + \rho)) - (u_{x_i^\perp}(s) - u_{x_i^\perp}(s + \rho))^2 \right) K_\tau(\rho) \, ds \, d\rho \, db \, da,
\] (3.6)

\[
V_{i,\tau}(u, x_i^\perp, I) = \frac{1}{2d} \int_{[s,\rho)\in\Omega(a,b)} G^{-1}(\rho) f_u(x_i^\perp, s, y_i^\perp, s + \rho) K_\tau(\rho) (y_i^\perp - x_i^\perp)^2 \, dy_i^\perp \, ds \, d\rho \, db \, da
\] (3.7)

with

\[
f_u(x_i^\perp, s, y_i^\perp, s + \rho) = [(u(x_i^\perp + (s + \rho)e_i) - u(x_i^\perp + s e_i)) - (u(x_i^\perp + y_i^\perp + (s + \rho)e_i) - u(x_i^\perp + y_i^\perp + se_i))]^2,
\] (3.8)

and

\[
W_{i,\tau}(u, x) = \frac{1}{2d} \int_{\mathbb{R}^d} f_u(x_i^\perp, x_i, y_i^\perp, y_i) K_\tau(y - x) \, dy.
\] (3.9)

We can now state the main result of this section

**Lemma 3.1** Let \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d, [0, 1]) \) be \([0, L)^d\)-periodic. One has that

\[
\mathcal{F}_{\tau,L,\varepsilon}(u) \geq \frac{1}{L^d} \sum_{i=1}^d \int_{[0,L)^d} \tilde{F}_{i,\tau,\varepsilon}(u, Q_i(z)) \, dz,
\] (3.10)

where

\[
\tilde{F}_{i,\tau,\varepsilon}(u, Q_i(z)) = \frac{1}{L^d} \int_{Q_i^\perp(z_i^\perp)} R_{i,\tau,\varepsilon}(u, x_i^\perp, (z_i - 1/2, z_i + 1/2)) \, dx_i^\perp
\]

\[
+ \frac{1}{L^d} \int_{Q_i^\perp(z_i^\perp)} V_{i,\tau}(u, x_i^\perp, (z_i - 1/2, z_i + 1/2)) \, dx_i^\perp
\]

\[
+ \frac{1}{L^d} \int_{Q_i(z)} W_{i,\tau}(u, x) \, dx
\]

\[
+ \frac{3(C_{\tau} - 1)}{d} \int_{|\nabla u = 0| \cap Q_i(z)} \frac{W(u(x))}{\alpha_{\varepsilon,\tau}} \, dx.
\] (3.11)
**Proof** Step 1

Recall the lower bound (2.2) obtained in [9, Proposition 3.1], namely

\[
\mathcal{F}_{\tau,L,\varepsilon}(u) \geq \frac{1}{L^d} \sum_{i=1}^d \left[ \int_{[0,L)^{d-1}} \left[ -\mathcal{M}_{\alpha_{\varepsilon,\tau}}(u, x_i^+, [0, L]) + \mathcal{G}_{\alpha_{\varepsilon,\tau}}^i(u, x_i^+, [0, L]) \right] dx_i^+ + \mathcal{T}^i_{\tau,L}(u) \right] dx_i^+ + \frac{1}{L^d} \mathcal{W}_{\tau,L,\varepsilon}(u),
\]

(3.12)

where \(\mathcal{M}_{\alpha_{\varepsilon,\tau}}^i, \mathcal{G}_{\alpha_{\varepsilon,\tau}}^i, \mathcal{T}^i_{\tau,L}, \mathcal{W}_{\tau,L,\varepsilon}\) are defined in (2.3)–(2.6).

Using the periodicity of \(u\) w.r.t. \([0, L)^d\), as recalled in (2.8) one has that

\[
|\rho|\mathcal{M}_{\alpha_{\varepsilon,\tau}}(u, x_i^+, [0, L]) = \int_0^L \mathcal{M}_{\alpha_{\varepsilon,\tau}}(u, x_i^+, J(s, s + \rho)) \, ds,
\]

(3.13)

where by convention we recall that \(J(s, s + \rho)\) is defined as in (3.5).

Therefore

\[
\mathcal{G}_{\alpha_{\varepsilon,\tau}}^i(u, x_i^+, [0, L]) = \int_0^L \int_{\mathbb{R}^{d-1}} \left[ \mathcal{M}_{\alpha_{\varepsilon,\tau}}(u, x_i^+, J(s, s + \rho)) - (u_{x_i^+}(s) - u_{x_i^+}(s + \rho))^2 \right] \mathcal{K}(\rho) \, d\rho \, ds.
\]

(3.14)

On the other hand, recall that

\[
\mathcal{T}^i_{\tau,L}(u) = \int_{[0,L)^{d-1}} \mathcal{T}^i_{\tau}(u, x_i^+, [0, L]) \, dx_i^+,
\]

(3.15)

where

\[
\mathcal{T}^i_{\tau}(u, x_i^+, [0, L]) := \frac{1}{d} \int_0^L \int_{\mathbb{R}^{d-1}} f_u(x_i^+, s, y_i^+, s + \rho) K_{\tau}(\rho e_i + (y_i^+ - x_i^+)) \, dy_i^+ \, d\rho \, ds
\]

(3.16)

and \(f_u\) as in (3.8).

Hence, decomposing half of the term \(\mathcal{T}^i_{\tau,L}\) as in (3.15)–(3.16) and leaving half of it as it is, it follows immediately that

\[
\mathcal{T}^i_{\tau,L}(u) = \int_{[0,L)^{d-1}} \left[ \frac{1}{2d} \int_0^L \int_{\mathbb{R}^{d-1}} f_u(x_i^+, s, y_i^+, s + \rho) K_{\tau}(\rho e_i + (y_i^+ - x_i^+)) \, dy_i^+ \, d\rho \, ds \right] dx_i^+ + \frac{1}{2d} \int_{[0,L)^d} \mathcal{W}_{\tau,L}(u, x) \, dx.
\]

(3.17)

Hence, by (3.12), (3.14) and (3.17) one has that

\[
\mathcal{F}_{\tau,L,\varepsilon}(u) \geq \frac{1}{L^d} \sum_{i=1}^d \int_{[0,L)^{d-1}} \left[ -\mathcal{M}_{\alpha_{\varepsilon,\tau}}(u, x_i^+, [0, L]) \right.
\]

\[
\left. + \int_0^L \int_{\mathbb{R}^{d-1}} \left[ \mathcal{M}_{\alpha_{\varepsilon,\tau}}(u, x_i^+, J(s, s + \rho)) - (u_{x_i^+}(s) - u_{x_i^+}(s + \rho))^2 \right] \mathcal{K}(\rho) \, d\rho \, ds \right)
\]

\[
+ \frac{1}{2d} \int_0^L \int_{\mathbb{R}^{d-1}} f_u(x_i^+, s, y_i^+, s + \rho) K_{\tau}(\rho e_i + (y_i^+ - x_i^+)) \, dy_i^+ \, d\rho \, ds \right] dx_i^+ + \frac{1}{2d} \int_{[0,L)^d} \mathcal{W}_{\tau,L}(u, x) \, dx
\]

\[
+ \frac{1}{L^d} \mathcal{W}_{\tau,L,\varepsilon}(u).
\]

(3.18)
Step 2 By (3.3), (3.4) and Fubini Theorem, for any function \( A(s, \rho) \) with \( A(\cdot, \rho) \) \( L \)-periodic one has that

\[
\int_0^L \int \frac{\partial}{\partial s} A(s, \rho) \, ds \, d\rho = \int_0^L \int \frac{\partial}{\partial s} A(s, \rho) G^{-1}(\rho)G(\rho) \, d\rho \, ds
\]

where

\[
\int \frac{\partial}{\partial s} A(s, \rho) G^{-1}(\rho)G(\rho) \, d\rho \, ds = \int \frac{\partial}{\partial s} A(s, \rho) G^{-1}(\rho) \int_{\Theta(a,b)} x_{(a,b)}(s) \, ds \, d\rho \, ds
\]

and

\[
\int \frac{\partial}{\partial s} A(s, \rho) G^{-1}(\rho) \int_{\Theta(a,b)} x_{(a,b)}(s) \, ds \, d\rho \, ds = \int \frac{\partial}{\partial s} A(s, \rho) G^{-1}(\rho) \int_{\Theta(a,b)} x_{(a,b)}(s) \, ds \, d\rho \, ds
\]

Applying (3.19) to the second and the third term of the r.h.s. of (3.18) and recalling the definitions of \( R_{i, \tau, \varepsilon} \) and \( V_{i, \tau} \) given in (3.6) and (3.7), one obtains

\[
F_{\tau, L, \varepsilon}(u) \geq \frac{1}{L^d} \sum_{i=1}^d \left\{ \int_{[0,L]^d} R_{i, \tau, \varepsilon}(u, x_i^+, [0, L]) + V_{i, \tau}(u, x_i^+, [0, L]) \right\} dx_i^+
\]

\[
+ \frac{1}{2dL^d} \int_{[0,L]^d} W_{i, \tau}(u, x) \, dx
\]

\[
+ \frac{1}{L^d} W_{\tau, L, \varepsilon}(u).
\]

Step 3 By periodicity of \( u \) w.r.t. \([0, L]^d\), as in [11], one has that

\[
F_{\tau, L, \varepsilon}(u) \geq \frac{1}{L^d} \int_{[0,L]^d} \tilde{F}_{\tau, \varepsilon}(u, Q_l(z)) \, dz
\]

with

\[
\tilde{F}_{\tau, \varepsilon}(u, Q_l(z)) := \sum_{i=1}^d \tilde{F}_{i, \tau, \varepsilon}(u, Q_l(z))
\]

and

\[
\tilde{F}_{i, \tau, \varepsilon}(u, Q_l(z)) = \frac{1}{L^d} \int_{Q_l(z)} \left\{ R_{i, \tau, \varepsilon}(u, x_i^+, [0, L]) + V_{i, \tau}(u, x_i^+, Q_l(z)) \right\} dx_i^+
\]

\[
+ \frac{3(\tau - 1)}{2dL^d} \int_{Q_l(z)} W_{i, \tau}(u, x) \, dx
\]

where

\[
C_\tau = \int_{[0, L]^d} |\xi_i| \, d\xi_i.
\]

From Theorems 2.1, 2.2 and Definition 2.3 one has the following

\textbf{Proposition 3.2} (Local Rigidity) For every \( M > 1, \), \( \sigma > 0 \), there exist \( \hat{\tau}, \hat{\eta}, \hat{\varepsilon} > 0 \) such that whenever \( \tau < \hat{\tau}, \varepsilon < \hat{\varepsilon} \) and \( \tilde{F}_{\tau, \varepsilon}(u, Q_l(z)) < M \) for some \( z \in [0, L]^d \) and \( u \in W^{1,2}_{\text{loc}}([0, 1]) \) \([0, L]^d\)-periodic, with \( L > 1 \), then it holds \( D_{\eta}(u, Q_l(z)) \leq \sigma \) for every
\[ \eta < \tilde{\eta}. \text{Moreover } \tilde{\eta} \text{ can be chosen independent on } \sigma. \text{Notice that } \tilde{\epsilon}, \hat{\epsilon} \text{ and } \tilde{\eta} \text{ are independent of } L. \]

4 Preliminary lemmas

This section contains all the preliminary lemmas which are needed to prove Theorem 1.2. Such lemmas provide a series of estimates which show an increase of the localized energy when \( u \) deviates in different ways from being one-dimensional. This section, together with Sect. 3, contains the main novelties w.r.t. the study of the corresponding sharp interface problem, while Sect. 5 contains the main underlying strategy and shows how the lemmas of this section enter in the proof of Theorem 1.2.

Let us start recalling from [9] the following facts:

Lemma 4.1 Let \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d; [0, 1]) \) be \([0, L)^d\)-periodic. For all \( i \in \{1, \ldots, d\} \), \( x_\perp \in [0, L)^{d-1}, s \in [0, L), \rho \in \mathbb{R} \)

\[
\overline{\mathcal{M}}_{\alpha\varepsilon, \tau}(u, x_\perp^i, J(s, s + \rho)) - (u_{x_\perp^i}(s) - u_{x_\perp^i}(s + \rho))^2 \geq 0 \quad (4.1)
\]

and if \(|u_{x_\perp^i}(s) - u_{x_\perp^i}(s + \rho)| \leq 1 - \delta \)

\[
\frac{1}{1 + 2\delta} \overline{\mathcal{M}}_{\alpha\varepsilon, \tau}(u, x_\perp^i, J(s, s + \rho)) - (u_{x_\perp^i}(s) - u_{x_\perp^i}(s + \rho))^2 \geq 0. \quad (4.2)
\]

The proof, follows immediately from the fact that \( \overline{\mathcal{M}}_{\alpha\varepsilon, \tau}(u, x_\perp^i, J(s, s + \rho)) = |\omega(u_{x_\perp^i}(s)) - \omega(u_{x_\perp^i}(s + \rho))| \), where \( \omega(t) = 3t^2 - 2t^3 \) is the optimal energy function for the Modica–Mortola term and on the fact that \(|\omega(a) - \omega(b)| \geq (3 - 2|a - b|)|a - b| \) (see (2.8) and (2.9)).

In the following lemma, penalization of functions \( u \) whose Modica–Mortola term is large on a small interval is shown. In particular, since the minimal energy is negative whenever \( \tau \) and \( \varepsilon \) are sufficiently small and since \( V_{i, \tau} \) and \( W_{i, \tau} \) in (3.11) are nonnegative, whenever we show that under some conditions also the third term contributing to the energy, i.e. \( R_{i, \tau, \varepsilon} \), is positive, then the corresponding configuration is surely not optimal.

Lemma 4.2 Let \( \Upsilon > 1 \). Then there exists \( C > 0 \), \( \eta_0 = \eta_0(\Upsilon) \) and \( \tau_0 > 0 \) with \( \tau_0^{1/\beta} \leq \eta_0 \) such that for every \( \tau \leq \tau_0 \), \( k \geq 1 \) and for all \( x_\perp^i \in [0, L)^d, \tilde{a} \in [0, L) \) and \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d; [0, 1]) \) \([0, L)^d\)-periodic such that \( \overline{\mathcal{M}}^i_{\alpha\varepsilon, \tau}(u, x_\perp^i, [\tilde{a} - \eta_0, \tilde{a} + \eta_0]) \geq k\Upsilon \), then \( R_{i, \tau, \varepsilon}(u, x_\perp^i, (\tilde{a} - \eta_0/2, \tilde{a} + \eta_0/2)) \geq k\Upsilon > 0. \)

Proof First of all, observe that for any \( \eta_0 \) such that \( \overline{\mathcal{M}}^i_{\alpha\varepsilon, \tau}(u, x_\perp^i, [\tilde{a} - \eta_0, \tilde{a} + \eta_0]) \geq k\Upsilon \) and for all \( s, s + \rho \in \Omega(\tilde{a} - \eta_0, \tilde{a} + \eta_0) \)

\[
\overline{\mathcal{M}}^i_{\alpha\varepsilon, \tau}(u, x_\perp^i, J(s, s + \rho)) - (u_{x_\perp^i}(s) - u_{x_\perp^i}(s + \rho))^2 \geq \frac{\Upsilon - 1}{\Upsilon} \overline{\mathcal{M}}^i_{\alpha\varepsilon, \tau}(u, x_\perp^i, J(s, s + \rho))
\]

\[
+ \left( \frac{1}{\Upsilon} \overline{\mathcal{M}}^i_{\alpha\varepsilon, \tau}(u, x_\perp^i, J(s + \rho) - u_{x_\perp^i}(s + \rho) - u_{x_\perp^i}(s))^2 \right) \geq \frac{\Upsilon - 1}{\Upsilon} \overline{\mathcal{M}}^i_{\alpha\varepsilon, \tau}(u, x_\perp^i, J(s, s + \rho)) \quad (4.3)
\]
where in the last inequality have used that \(\overline{M}_{i}^{\delta}(u, x_1^+, J(s, s + \rho)) \geq \overline{M}_{i}^{\delta}(u, x_1^+, [\tilde{\alpha} - \eta_0, \tilde{\alpha} + \eta_0]) \geq \Upsilon \) and \(|u_{x_1}^{-}(s) - u_{x_1}^{-}(s + \rho)| \leq 1\). Recalling (4.1) and then using Fubini Theorem, one has that

\[
R_{i, \tau, \varepsilon}(u, x_1^+, (\tilde{\alpha} - \eta_0/2, \tilde{\alpha} + \eta_0/2)) = -\overline{M}_{i}^{\delta}(u, x_1^+, [\tilde{\alpha} - \eta_0/2, \tilde{\alpha} + \eta_0/2])
\]

\[
+ \frac{1}{4} \int_{(s, \rho) \in \Omega(\tilde{\alpha} - \eta_0, \tilde{\alpha} + \eta_0), \{\rho\} \leq 4\eta_0} \left( \overline{M}_{i}^{\delta}(u, x_1^+, J(s, s + \rho)) - (u_{x_1}^{-}(s) - u_{x_1}^{-}(s + \rho))^2 \right) \tilde{K}(\rho) \, d\rho \, ds \, db \, da
\]

\[
\geq k \left( -\overline{\chi} + \frac{\tilde{C}(\Upsilon - 1)}{\eta_0^{1/2}} \right)
\]

Now observe that by direct computation, whenever \((s, \rho) \in \Omega(\tilde{\alpha} - \eta_0, \tilde{\alpha} + \eta_0)\),

\[
\int_{\tilde{\alpha} - \eta_0/2}^{\tilde{\alpha} + \eta_0/2} \int_{\mathbb{R}} \chi((a, b); (s, \rho) \in \Omega(a, b))(a, b) \, da \, db = |\rho| \eta_0.
\]

hence (recalling the definition of \(G(\rho)\) in (3.3) and the fact that \(|\rho| \leq 4\eta_0\) with \(4\eta_0 \ll L\))

\[
G^{-1}(\rho) \int_{\tilde{\alpha} - \eta_0/2}^{\tilde{\alpha} + \eta_0/2} \int_{\mathbb{R}} \chi((a, b); (s, \rho) \in \Omega(a, b))(a, b) \, da \, db = \eta_0 |\rho| \geq \frac{1}{4}.
\]

Inserting (4.5) in (4.4), one obtains

\[
R_{i, \tau, \varepsilon}(u, x_1^+, (\tilde{\alpha} - \eta_0/2, \tilde{\alpha} + \eta_0/2)) \geq -\overline{M}_{i}^{\delta}(u, x_1^+, [\tilde{\alpha} - \eta_0/2, \tilde{\alpha} + \eta_0/2])
\]

\[
+ \frac{1}{4} \int_{(s, \rho) \in \Omega(\tilde{\alpha} - \eta_0, \tilde{\alpha} + \eta_0), \{\rho\} \leq 4\eta_0} \left( \overline{M}_{i}^{\delta}(u, x_1^+, J(s, s + \rho)) - (u_{x_1}^{-}(s) - u_{x_1}^{-}(s + \rho))^2 \right) \tilde{K}(\rho) \, d\rho \, ds \, db \, da
\]

Now since \((s, \rho) \in \Omega(\tilde{\alpha} - \eta_0, \tilde{\alpha} + \eta_0)\), we can use (4.3) and get

\[
R_{i, \tau, \varepsilon}(u, x_1^+, (\tilde{\alpha} - \eta_0/2, \tilde{\alpha} + \eta_0/2)) \geq -\overline{M}_{i}^{\delta}(u, x_1^+, [\tilde{\alpha} - \eta_0/2, \tilde{\alpha} + \eta_0/2])
\]

\[
+ \frac{1}{4} \overline{M}_{i}^{\delta}(u, x_1^+, [\tilde{\alpha} - \eta_0, \tilde{\alpha} + \eta_0]) \int_{(s, \rho) \in \Omega(\tilde{\alpha} - \eta_0, \tilde{\alpha} + \eta_0), \{\rho\} \leq 4\eta_0} \left( \frac{1}{|\rho| + \tau^{1/\beta}} \right) \, d\rho \, ds \, db
\]

\[
\geq k \left( -\overline{\chi} + \frac{\tilde{C}(\Upsilon - 1)}{\eta_0^{1/2}} \right)
\]

where the last two inequalities hold provided \(\tau^{1/\beta} \leq \eta_0\) and \(\eta_0\) is sufficiently small depending on \(\Upsilon\).

As a consequence, the following holds.
Corollary 4.3 Let $\Upsilon > 1$ and $\eta_0, \tau_0$ as in Lemma 4.2. Then, for all $u \in W^{1,2}_{\text{loc}}(\mathbb{R}; [0, 1])$ such that $R_{i, \tau, \epsilon}(u, x_i^\perp, I) < 0$ and for all $\tau \leq \tau_0$, it holds

$$\overline{\mathcal{N}}_{\alpha \epsilon, \tau}(u, x_i^\perp, I) \leq 2\Upsilon \max \left\{ \frac{|I|}{\eta_0}, 1 \right\}. \quad (4.7)$$

Proof Let $I = (a, b)$ and let $a = t_0 < t_1 < \cdots < t_N = b$ be a partition of $I$ into intervals $[t_j, t_{j+1})$ such that the following holds:

$$t_{j+1} = \inf \{ t \in I, t \geq t_j + \eta_0/2 : \overline{\mathcal{N}}_{\alpha \epsilon, \tau}(u, x_i^\perp, [t_j, t)) \geq \Upsilon \}.$$ 

W.l.o.g., assume that $|I| \geq \eta_0$. Otherwise by Lemma 4.2 one has that (4.7) holds. Let $s \geq 1$ and let us define the following sets:

$$A_s(\eta_0) := \{ j \in \{0, N - 1\} : |t_{j+1} - t_j| \leq \eta_0 \text{ and } \overline{\mathcal{N}}_{\alpha \epsilon, \tau}(u, x_i^\perp, [t_j, t_{j+1})) = s\Upsilon \}$$

$$A_0(\eta_0) := \{ j \in \{0, N - 2\} : |t_{j+1} - t_j| \geq \eta_0 \}.$$ 

By definition, if $j \in A_0$ then $\overline{\mathcal{N}}_{\alpha \epsilon, \tau}(u, x_i^\perp, [t_j, t_{j+1})) = \Upsilon$. One has that

$$\overline{\mathcal{N}}_{\alpha \epsilon, \tau}(u, x_i^\perp, I) \leq \sum_{j \in A_0(\eta_0)} \Upsilon + \sum_{s \geq 1} \sum_{j \in A_s(\eta_0)} s\Upsilon = \Upsilon \# A_0(\eta_0) + \sum_{s \geq 1} s\Upsilon \# A_s(\eta_0). \quad (4.8)$$

Assume now that (4.7) does not hold. Hence, since $\# A_0(\eta_0) \leq \frac{|I|}{\eta_0}$ and using (4.8), one has that

$$\Upsilon \# A_0(\eta_0) \leq \frac{\Upsilon |I|}{\eta_0} \leq \frac{1}{2} \Upsilon \# A_0(\eta_0) + \frac{1}{2} \sum_{s \geq 1} s\Upsilon \# A_s(\eta_0). \quad (4.9)$$

On the other hand, by Lemma 4.2 and (4.9)

$$R_{i, \tau, \epsilon}(u, x_i^\perp, I) = \sum_{j \in A_0(\eta_0)} R_{i, \tau, \epsilon}(u, x_i^\perp, [t_j, t_{j+1})) + \sum_{s \geq 1} \sum_{j \in A_s(\eta_0)} R_{i, \tau, \epsilon}(u, x_i^\perp, [t_j, t_{j+1}))$$

$$\geq -\Upsilon \# A_0(\eta_0) + \Upsilon \sum_{s \geq 1} s\# A_s(\eta_0)$$

$$\geq 0,$$

thus reaching a contradiction. \hfill \Box

The following lemma shows that non optimal configurations are also those for which oscillations of amplitude close to 1 happen at a scale larger than $\tau^{1/\beta}$.

Lemma 4.4 Let $\delta_0 \geq \tau^{1/\beta}, \delta > 0$. Then, there exists $\tau_1 > 0$ such that for all $\tau \leq \tau_1$, for any $x_i^\perp \in [0, L]^{d-1}, \tilde{a} \in [0, L)$ and $c > 0$, if for any $s, s + \rho \in [\tilde{a} - c - \delta_0/2, \tilde{a} + c + \delta_0/2]$ and $|\rho| \leq \delta_0$ and $u \in W^{1,2}_{\text{loc}}(\mathbb{R}; [0, 1])$ $[0, L)^{d}$-periodic one has that

$$|u_{x_i^\perp}(s) - u_{x_i^\perp}(s + \rho)| \leq 1 - \delta,$$

then $R_{i, \tau, \epsilon}(u, x_i^\perp, (\tilde{a} - c, \tilde{a} + c)) > 0$.\hfill \openbullet

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Proof Recalling (4.1) and (4.2) and then using Fubini Theorem, one has that
\[ R_{i,\tau,\varepsilon}(u, x_i^\perp, (\bar{a} - c, \bar{a} + c)) \geq -\mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, [\bar{a} - c, \bar{a} + c]) \]
\[ + \int_{\bar{a} - c}^{\bar{a} + c} \int_{\mathbb{R}} \int_{\{s, \rho\} \in \Omega(a, b) \cap \{s, \rho\} \in \mathbb{R}^d} G^{-1}(\rho) \hat{K}_\tau(\rho) \]
\[ - \frac{2\delta}{1 + 2\delta} \mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, \mathcal{J}(s, s + \rho)) \int_{\mathbb{R}} \int_{\{s, \rho\} \in \Omega(a, b) \cap \{s, \rho\} \in \mathbb{R}^d} G^{-1}(\rho) \hat{K}_\tau(\rho) \]
\[ \geq -\mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, [\bar{a} - c, \bar{a} + c]) + \int_{\{s, \rho\} \in \Omega(a, b) \cap \{s, \rho\} \in \mathbb{R}^d} G^{-1}(\rho) \hat{K}_\tau(\rho) \]
\[ \cdot \frac{2\delta}{1 + 2\delta} \mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, \mathcal{J}(s, s + \rho)) \int_{\mathbb{R}} \int_{\{s, \rho\} \in \Omega(a, b) \cap \{s, \rho\} \in \mathbb{R}^d} G^{-1}(\rho) \hat{K}_\tau(\rho). \]

Then observe that
\[ \int_{\bar{a} - c}^{\bar{a} + c} \int_{\mathbb{R}} \chi_{\{s, \rho\} \in \Omega(a, b) \cap \{s, \rho\} \in \mathbb{R}^d}(a, b) \int_{\mathbb{R}^2} \frac{G^{-1}(\rho) \hat{K}_\tau(\rho)}{\int_{\mathbb{R}^2} G^{-1}(\rho) \hat{K}_\tau(\rho)} \]
and thus, recalling (3.3) and assuming w.l.o.g. that \( \delta_0 \ll L \), one has that
\[ G^{-1}(\rho) \int_{\mathbb{R}^2} \frac{G^{-1}(\rho) \hat{K}_\tau(\rho)}{\int_{\mathbb{R}^2} G^{-1}(\rho) \hat{K}_\tau(\rho)} \geq \frac{|J(s, s + \rho) \cap [\bar{a} - c, \bar{a} + c]|}{|\rho|}. \]

In particular, when \( s, \rho \in [\bar{a} - c - \frac{1}{4} \tau^{1/\beta}, \bar{a} + c + \frac{1}{4} \tau^{1/\beta}] \) and \( \frac{1}{4} \tau^{1/\beta} \leq |\rho| \leq \delta_0 \), one has that
\[ G^{-1}(\rho) \int_{\mathbb{R}^2} \frac{G^{-1}(\rho) \hat{K}_\tau(\rho)}{\int_{\mathbb{R}^2} G^{-1}(\rho) \hat{K}_\tau(\rho)} \geq \frac{1}{2}. \]

Restricting further the domain of integration in (4.10) to the pairs \( s, \rho \in [\bar{a} - c - \frac{1}{4} \tau^{1/\beta}, \bar{a} + c + \frac{1}{4} \tau^{1/\beta}] \) and inserting (4.11), we obtain
\[ R_{i,\tau,\varepsilon}(u, x_i^\perp, (\bar{a} - c, \bar{a} + c)) \geq -\mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, [\bar{a} - c, \bar{a} + c]) \]
\[ + \frac{\delta}{1 + 2\delta} \int_{\{s, \rho\} \in \Omega(a, b) \cap \{s, \rho\} \in \mathbb{R}^d} \mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, \mathcal{J}(s, s + \rho)) \hat{K}_\tau(\rho) \int_{\mathbb{R}^2} \frac{\hat{K}_\tau(\rho)}{\int_{\mathbb{R}^2} G^{-1}(\rho) \hat{K}_\tau(\rho)}. \]

Then observe that, whenever \( \frac{1}{2} \tau^{1/\beta} \leq |\rho| \leq \tau^{1/\beta} \)
\[ \int_{[\bar{a} - c - \frac{1}{4} \tau^{1/\beta}, \bar{a} + c + \frac{1}{4} \tau^{1/\beta}]} \mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, \mathcal{J}(s, s + \rho)) \int_{\mathbb{R}^2} \frac{\hat{K}_\tau(\rho)}{\int_{\mathbb{R}^2} G^{-1}(\rho) \hat{K}_\tau(\rho)}. \]
Hence, inserting (4.13) in (4.12) one gets
\[ R_{i,\tau,\varepsilon}(u, x_i^\perp, (\bar{a} - c, \bar{a} + c)) \geq \mathcal{M}_{i,\tau,\varepsilon}(u, x_i^\perp, [\bar{a} - c, \bar{a} + c]) \]
\[ \left( -1 + \frac{\delta}{1 + 2\delta} \int_{\frac{1}{2} \tau^{1/\beta} \leq |\rho| \leq \tau^{1/\beta}} \frac{|\rho|}{4(|\rho| + \tau^{1/\beta})^3} d\rho \right) \]
which is positive provided \( \tau \) is sufficiently small depending on \( \delta \). \( \square \)

**Corollary 4.5** Let \( 1 < \gamma \leq \frac{17}{16}, \delta > 0 \) and let \( \eta_0 \) as in Lemma 4.2, \( \delta_0 \ll \eta_0 \) as in Lemma 4.4 and \( \tau_2 \leq \min\{\tau_0, \tau_1\} \), then, for all \( \tau \leq \tau_2 \), for all \( x_i^\perp \in [0, L)^{d-1}, \bar{a} \in [0, L) \), whenever \( R_{i,\tau,\varepsilon}(u, x_i^\perp, (\bar{a} - \eta_0/2, \bar{a} + \eta_0/2)) < 0 \) for some \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d, [0, 1]) \)
\( [0, L)^{d-1} \)-periodic then...
1. \( \mathcal{M}_{\alpha, \tau}^i (u, x_i^\top, [\bar{a} - \eta_0, \bar{a} + \eta_0]) \leq \Upsilon \)
2. There exist \( s_0 < t_0 \in [\bar{a} - \eta_0/2 - \delta_0/2, \bar{a} + \eta_0/2 + \delta_0/2] \) with \( |t_0 - s_0| \leq \delta_0 \) such that 
   \[ |u_{x_i^\top}^+(s_0) - u_{x_i^\top}^+(t_0)| \geq (1 - \delta). \]

In particular, 
\[ R_{i, \tau, \varepsilon} (u, x_i^\top, (\bar{a} - \eta_0/2, \bar{a} + \eta_0/2)) \geq -\Upsilon \] (4.14)
and
\[ \forall t \in [t_0, \bar{a} + \eta_0], \quad |u_{x_i^\top}^+(t) - u_{x_i^\top}^+(t_0)| \leq \frac{1}{4} + \sqrt{2\delta} \] (4.15)
\[ \forall s \in [\bar{a} - \eta_0, s_0], \quad |u_{x_i^\top}^+(s) - u_{x_i^\top}^+(s_0)| \leq \frac{1}{4} + \sqrt{2\delta}. \] (4.16)

**Proof** The only estimates which are not restatements of Lemma 4.2 and Lemma 4.4 are (4.15) and (4.16).

To this aim, notice that by (4.2),
\[ \mathcal{M}_{\alpha, \tau}^i (u, x_i^\top, [t_0, \bar{a} + \eta_0]) + \mathcal{M}_{\alpha, \tau}^i (u, x_i^\top, [\bar{a} - \eta_0, s_0]) = \mathcal{M}_{\alpha, \tau}^i (u, x_i^\top, [\bar{a} - \eta_0, \bar{a} + \eta_0]) \]
\[ - \mathcal{M}_{\alpha, \tau}^i (u, x_i^\top, [s_0, t_0]) \]
\[ \leq \frac{17}{16} - (u_{x_i^\top}(s_0) - u_{x_i^\top}(t_0))^2 \]
\[ \leq \frac{1}{16} + 2\delta. \]

Thus, by (4.1),
\[ |u_{x_i^\top}^+(s) - u_{x_i^\top}^+(s_0)| \leq \sqrt{\mathcal{M}_{\alpha, \tau}^i (u, x_i^\top, [s, s_0])} \leq \frac{1}{4} + \sqrt{2\delta} \quad \forall s \in [\bar{a} - \eta_0, s_0], \] (4.17)
\[ |u_{x_i^\top}^+(t) - u_{x_i^\top}^+(t_0)| \leq \sqrt{\mathcal{M}_{\alpha, \tau}^i (u, x_i^\top, [t_0, t])} \leq \frac{1}{4} + \sqrt{2\delta} \quad \forall t \in [t_0, \bar{a} + \eta_0]. \] (4.18)

In particular, the upper bound \( \Upsilon \leq \frac{17}{16} \) enters only in the proof of (4.15) and (4.16).

The following lemma contains a localized version of the one-dimensional minimization of the slices of the functional. In comparison with the sharp interface problem, where a simpler periodic extension argument is used thanks to the concentrated character of the perimeter on the boundary points and on the simpler geometry of characteristic functions, one has to single out suitable points where to perform a periodic extension of the given profile \( u \) and additionally to take care of the growth of the Modica–Mortola term of the periodic extension.

At this purpose we will use Corollary 4.3.

**Lemma 4.6** There exists \( C_0 > 0 \) with the following property. Let \( u \in W_{\text{loc}}^{1,2} (\mathbb{R}^d; [0, 1]) \) \([0, L]^d\)-periodic, \( x_i^\top \in [0, L)^{d-1} \) and \( I \subset [0, L) \) be an open interval. Then, for all \( \tau \leq \tau_2 \) with \( \tau_2 \) as in Corollary 4.5 it holds
\[ R_{i, \tau, \varepsilon} (u, x_i^\top, I) \geq C_{\tau, \varepsilon}^* |I| - C_0, \] (4.19)
where \( C_{\tau, \varepsilon}^* \) was defined in (1.5).

**Proof** Let \( I = [x_1, x_2] \). Let \( 1 < \Upsilon \leq \frac{17}{16} \) and let \( \eta_0 \) be as in Lemma 4.2 and Corollary 4.5. W.l.o.g., we can assume that
\[ R_{i, \tau, \varepsilon} (u, x_i^\top, [x_1, x]) < 0 \quad \text{and} \quad R_{i, \tau, \varepsilon} (u, x_i^\top, (x, x_2)) < 0 \quad \text{for all} \quad x \in (x_1, x_2). \] (4.20)
Indeed, if the contrary holds, setting \( \bar{I} = I \setminus [x_1, x) \) or \( \bar{I} = I \setminus (x, x_2] \) whenever the result of the lemma holds for \( \bar{I} \) then it holds also for \( I \), since it holds
\[
R_{t, \tau, \epsilon}(u, x_j^\perp, \bar{I}) \geq R_{t, \tau, \epsilon}(u, x_j^\perp, \bar{I}) \geq C_{t, \tau, \epsilon} |\bar{I}| - C_0 \geq C_{t, \tau, \epsilon} |I| - C_0,
\]
where in the last inequality we used the fact that \( C_{t, \tau, \epsilon} \) is a negative constant. W.l.o.g., we can also assume that \( |I| \geq 5 \eta_0 \). Indeed, if the contrary holds, since \( \tau \leq \tau_2 \) one can use (4.14) and then (4.19) holds with \( C_0 = -\gamma \).

Let us then set \( I = [\bar{x}, \bar{y}] \) with \( |\bar{y} - \bar{x}| \geq 5 \eta_0 \). By the assumptions on \( I \) and Corollary 4.5, whenever \( \tau \leq \tau_2 \) then there exist \( s_0 < t_0 \in [\bar{x} - \delta_0/2, \bar{x} + \eta_0 + \delta_0/2] \) with \( |t_0 - s_0| \leq \delta_0 \) such that \(|u_{x_j^\perp}(s_0) - u_{x_j^\perp}(t_0)| \geq (1 - \delta)\) and \( \epsilon_1 < t_1 \in [\bar{y} - \eta_0 - \delta_0/2, \bar{y} + \delta_0/2] \) with \( |t_1 - s_1| \leq \delta_0 \) such that \(|u_{x_j^\perp}(s_1) - u_{x_j^\perp}(t_1)| \geq (1 - \delta)\). In particular, there exist \( \bar{s} \in [\bar{x} - \delta_0/2, \bar{x} + \eta_0 + \delta_0/2] \) and \( \bar{t} \in [\bar{y} - \eta_0 - \delta_0/2, \bar{y} + \delta_0/2] \) such that \( u_{x_j^\perp}(\bar{s}) = u_{x_j^\perp}(\bar{t}) = \frac{1}{2} \).

Then we define a \([\bar{t} - \bar{s}]\)-periodic function \( \bar{u} \) on \( \mathbb{R} \) by the usual symmetric reflection of \((u_{x_j^\perp})_{|t, \bar{t}}\) (i.e. as in (1.6) with \( h = |\bar{t} - \bar{s}| \)) and set \( \bar{u}(x) = \bar{u}(x_j^\perp) \) for all \( x \in \mathbb{R}^d \). By optimality of the energy density \( C_{t, \tau, \epsilon}^* \) on one-dimensional periodic functions one has that
\[
R_{t, \tau, \epsilon}(\bar{u}, x_j^\perp, [\bar{s}, \bar{t}]) \geq C_{t, \tau, \epsilon}^* |\bar{t} - \bar{s}| \geq C_{t, \tau, \epsilon}^* |I| + \delta_0 C_{t, \tau, \epsilon}^*.
\]

Thus, since by the assumptions on \( I \) one has that
\[
R_{t, \tau, \epsilon}(u, x_j^\perp, I) - R_{t, \tau, \epsilon}(u, x_j^\perp, [\bar{s}, \bar{t}]) \geq -2 \gamma,
\]
we are left to prove that
\[
R_{t, \tau, \epsilon}(u, x_j^\perp, [\bar{s}, \bar{t}]) - R_{t, \tau, \epsilon}(\bar{u}, x_j^\perp, [\bar{s}, \bar{t}]) \geq -\tilde{C}. \tag{4.21}
\]

Indeed, one can further reduce (4.21) to prove that
\[
R_{t, \tau, \epsilon}(u, x_j^\perp, [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2]) - R_{t, \tau, \epsilon}(\bar{u}, x_j^\perp, [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2]) \geq -\tilde{C}, \tag{4.22}
\]
since by construction \([\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2] \subset [\bar{s}, \bar{t}] \) but \([\bar{s}, \bar{t}] \setminus [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2] \subset [\bar{x} - \eta_0/2, \bar{x} + \eta_0 + \delta_0] \cup [\bar{y} - \eta_0 - \delta_0, \bar{y} + \eta_0/2] \), intervals on which (since \( \delta_0 \ll \eta_0/2 \)) by assumption \( R_{t, \tau, \epsilon} \geq \gamma \). Notice also that
\[
\inf \{|s - t| : s \in [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2], t \in \mathbb{R} \setminus [\bar{s}, \bar{t}] \} \geq \eta_0/2. \tag{4.23}
\]

Since \( u_{x_j^\perp} = \bar{u}_{x_j^\perp} \) on \([\bar{s}, \bar{t}] \supset [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2] \), one has that
\[
R_{t, \tau, \epsilon}(u, x_j^\perp, [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2]) - R_{t, \tau, \epsilon}(\bar{u}, x_j^\perp, [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2])
\]
\[
\geq \int_{\bar{x} + 3 \eta_0/2}^{\bar{y} - 3 \eta_0/2} \int_{\mathbb{R}} \int_{(s, \rho) \in \Omega(a, b) \cap \{(|s - \bar{t}|) \cup (s + \rho \notin [\bar{s}, \bar{t}])\}} G^{-1}(\rho)
\]
\[
\cdot \left( \overline{\lambda}_{a, \tau, \epsilon}(u, x_j^\perp, j(s, s + \rho)) - \overline{\lambda}_{a, \tau, \epsilon}(\bar{u}, x_j^\perp, j(s, s + \rho)) \right) \hat{K}(\rho) \, ds \, d\rho \, db \, da
\]
\[
+ \int_{\bar{x} + 3 \eta_0/2}^{\bar{y} - 3 \eta_0/2} \int_{\mathbb{R}} \int_{(s, \rho) \in \Omega(a, b) \cap \{(|s - \bar{t}|) \cup (s + \rho \notin [\bar{s}, \bar{t}])\}} G^{-1}(\rho) \cdot \left( |\bar{u}_{x_j^\perp}(s) - \bar{u}_{x_j^\perp}(s + \rho)|^2 \right) \hat{K}(\rho) \, ds \, d\rho \, db \, da. \tag{4.24}
\]

Let us first estimate the second term in the r.h.s. of (4.24). Observe that
\[
\int_{\bar{x} + 3 \eta_0/2}^{\bar{y} - 3 \eta_0/2} \int_{\mathbb{R}} \chi_{\Omega(a, b) \cap \{s \in [\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2] \cap J(s, s + \rho)\}}(a, b) \, db \, da = |\rho||[\bar{x} + 3 \eta_0/2, \bar{y} - 3 \eta_0/2] \cap J(s, s + \rho)|. \tag{4.25}
\]

\( \square \) Springer
Moreover, since either s or s + ρ in the above integrals do not belong to [\bar{s}, \bar{t}] and at the same time [s, s + ρ] has to contain a point in [\bar{x} + 3\eta_0/2, \bar{y} - 3\eta_0/2], then by (4.23) |\rho| ≥ \eta_0/2.

Then, using that |u|, |\hat{u}| ≤ 1 and Fubini Theorem one obtains

\[
\int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{\bar{y}} G^{-1}(\rho) \cdot \left( (\hat{u}_{\bar{x}}^+(s) - \hat{u}_{\bar{x}}^+(s + \rho))^2 - (u_{\bar{x}}^+(s) - u_{\bar{x}}^+(s + \rho))^2 \right) \hat{K}_\tau(\rho) \, ds \, d\rho \, db \, da \\
≥ - \int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{\bar{y}} G^{-1}(\rho) \cdot \left( (\hat{u}_{\bar{x}}^+(s) - \hat{u}_{\bar{x}}^+(s + \rho))^2 - (u_{\bar{x}}^+(s) - u_{\bar{x}}^+(s + \rho))^2 \right) \hat{K}_\tau(\rho) \, ds \, d\rho \\
≥ -2 \int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{-\infty}^{\bar{y}} \hat{K}_\tau(s - t) \, ds \, dt \\
≥ -\hat{C}(\eta_0)
\] (4.26)

with \(\hat{C}(\eta_0) \sim \frac{1}{\eta_0^2}\).

Let us now deal with the first term in the r.h.s. of (4.24). By positivity of the Modica–Mortola term, Fubini Theorem and (4.25), one has that

\[
\int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{\bar{y}} G^{-1}(\rho) \cdot \left( \mathcal{M}_{\bar{x}, \tau}(\hat{u}, x_\tau^+, J(s, s + \rho)) - \mathcal{M}_{\bar{x}, \tau}(\hat{u}, x_\tau^+, J(s, s + \rho)) \right) \hat{K}_\tau(\rho) \, ds \, d\rho \, db \, da \\
≥ - \int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{\bar{y}} G^{-1}(\rho) \mathcal{M}_{\bar{x}, \tau}(\hat{u}, x_\tau^+, J(s, s + \rho)) \hat{K}_\tau(\rho) \, ds \, d\rho \\
≥ - \int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{\bar{y}} G^{-1}(\rho) \mathcal{M}_{\bar{x}, \tau}(\hat{u}, x_\tau^+, J(s, s + \rho)) \hat{K}_\tau(\rho) \, ds \, d\rho.
\] (4.27)

Now notice that, by (4.20) and since \(\hat{u}\) is obtained by periodic reflection of \(u_{[\bar{s}, \bar{t}]}\) on each interval \([s, s + \rho]\) as above \(R_{t, \varepsilon, \tau}(\hat{u}, x_\tau^+, [s, s + \rho]) < 0\). Therefore, Corollary 4.3 holds implying that

\[
\mathcal{M}_{\bar{x}, \tau}(\hat{u}, x_\tau^+, J(s, s + \rho)) \leq \frac{2\gamma |\rho|}{\eta_0}.
\] (4.28)

Substituting (4.28) into (4.27) one gets

\[
- \int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{\bar{y}} G^{-1}(\rho) \mathcal{M}_{\bar{x}, \tau}(\hat{u}, x_\tau^+, J(s, s + \rho)) \hat{K}_\tau(\rho) \, ds \, d\rho \\
≥ - \int_{\bar{x}+3\eta_0/2}^{\bar{x}+3\eta_0/2} \int_{\bar{y}} |s - t| \hat{K}_\tau(s - t) \, ds \, dt \\
≥ -\hat{C}(\eta_0),
\] (4.29)

with \(\hat{C}(\eta_0) \sim \frac{1}{\eta_0^2}\), thus concluding the proof of (4.22), hence of the lemma. □

In the next lemma we give a lower bound of the energy on cubes where \(u\) is either close to 0 or close to 1 in \(L^1\).

**Lemma 4.7** There exists a constant \(C_1 > 0\) such that the following holds. Let \(u \in W^{1,2}_{{\text{loc}}}([0, 1]; [0, L]^d)\) periodic be such that

\[
\min\{\|u - 1\|_{L^1(\Omega_1(\eta))}, \|u\|_{L^1(\Omega_1(\eta))}\} \leq v_1^d,
\] (4.30)
for some $v > 0$. Let $1 < \Upsilon \leq \frac{17}{16}$. Then, provided $\tau$ is sufficiently small,

$$\tilde{F}_{i,\tau,e}(u, Q_i(z)) \geq -C_1 \frac{\Upsilon v d}{\eta_0}, \quad (4.31)$$

where $\eta_0 = \eta_0(\Upsilon)$ is as in Corollary 4.5.

**Proof** By assumption, we assume w.l.o.g. that $\|u - 1\|_{L^1(Q_i(z))} \leq v l^d$. In particular, by Chebyshev inequality

$$\left| \left| \left\{ t \leq \frac{3}{8} \right\} \cap Q_i(z) \right| \right| \leq \frac{8}{5} v l^d. \quad (4.32)$$

Let $z_i - l/2 = t_0 < t_1 < \cdots < t_N = z_i + l/2$ be a partition of $Q_i(z)$ into intervals $[t_k, t_{k+1})$ of size $\eta_0$ (with eventually $|t_N - t_{N-1}| \leq \eta_0$). Then,

$$\tilde{F}_{i,\tau,e}(u, Q_i(z)) \geq \frac{1}{l^d} \int_{Q_i^+(z_i)} \mathcal{R}_{i,\tau,e}(u, x_i^+, Q_i^+(z_i)) \, dx_i^+$$

$$\geq \frac{1}{l^d} \int_{Q_i^+(z_i)} \sum_{k \in \mathcal{K}(Q_i^+(z_i))} \mathcal{R}_{i,\tau,e}(u, x_i^+, [t_k, t_{k+1})) \, dx_i^+, \quad (4.33)$$

$$\mathcal{K}(Q_i^+(z_i)) = \{ k \in \{1, \ldots, N \} : R_{i,\tau,e}(u, x_i^+, [t_k, t_{k+1})) < 0 \}.$$ 

In particular, by Corollary 4.5, given $1 < \Upsilon \leq \frac{17}{16}$ and $\delta \ll 1$, there exist $\eta_0$ and $\tau_2$ such that for any $\tau \leq \tau_2$

$$R_{i,\tau,e}(u, x_i^+, ([t_k, t_{k+1}))) \geq -\Upsilon \quad (4.34)$$

and there exist $\bar{s} < \bar{t} \in [t_k - \delta_0/2, t_{k+1} + \delta_0/2]$ with $|\bar{t} - \bar{s}| \leq \delta_0 \ll \eta_0$ such that $|u_{x_i^+}(\bar{s}) - u_{x_i^+}(\bar{t})| \geq (1 - \delta)$. Moreover,

$$\forall t \in [\bar{t}, t_{k+1} + \eta_0/2), \quad |u_{x_i^+}(t) - u_{x_i^+}(\bar{t})| \leq \frac{1}{4} + \sqrt{2\delta.} \quad (4.35)$$

$$\forall s \in [t_k - \eta_0/2, \bar{s}], \quad |u_{x_i^+}(s) - u_{x_i^+}(\bar{s})| \leq \frac{1}{4} + \sqrt{2\delta}. \quad (4.36)$$

Hence, when $\delta$ is sufficiently small, for any $k \in \mathcal{K}(Q_i^+(z_i))$ there exist an interval of size at least $\eta_0/4$ on which $u \leq \frac{3}{8}$. By construction, there exist at least $\frac{1}{\Upsilon} \# \mathcal{K}(Q_i^+(z_i))$ of such intervals which are disjoint. Then, inserting this information together with (4.34) and (4.32) in (4.33) one obtains that

$$\tilde{F}_{i,\tau,e}(u, Q_i(z)) \geq \frac{1}{l^d} \int_{Q_i^+(z_i)} \sum_{k \in \mathcal{K}(Q_i^+(z_i))} \mathcal{R}_{i,\tau,e}(u, x_i^+, [t_k, t_{k+1})) \, dx_i^+$$

$$\geq -\frac{1}{l^d} \int_{Q_i^+(z_i)} \Upsilon \# \mathcal{K}(Q_i^+(z_i)) \, dx_i^+$$

$$\geq -\frac{1}{l^d} \int_{Q_i^+(z_i)} \Upsilon \left[ \left| \left\{ u_{x_i^+} \leq \frac{3}{8} \right\} \cap Q_i(z) \right| \right] \frac{d}{\eta_0/4}$$

$$\geq -C_1 \frac{\Upsilon v}{\eta_0}. \quad (4.37)$$

Summing over $i \in \{1, \ldots, d\}$ one obtains (4.31). \qed
The following lemma roughly shows that, whenever the function $u$ on a subset of a slice in direction $e_j$ is close to a stripe with boundaries orthogonal to $e_j$ for some $j \neq i$, then the contribution to the energy $\bar{F}_{i,\tau,\varepsilon}$ is positive. It is the counterpart of the local stability Lemma 7.8 in [11].

**Lemma 4.8** Let $\eta_0$, $\tau_2$ be as in Corollary 4.5. Then, there exists $0 < \tau_3 \leq \tau_2$, $\tilde{\sigma} > 0$ (independent of $l$) such that for every $\tau \leq \tau_3$ and $\sigma \leq \tilde{\sigma}$ the following holds: let $u \in W^{1,2}_\text{loc}(\mathbb{R}^d; [0,1]) [0, L)^d$-periodic, $x_i^\perp \in Q_i^+(z)$ and $z$ s.t.

$$D^j_i(\eta, Q_i(z)) \leq \sigma$$

for some $j \neq i$. \hfill (4.38)

Then,

$$R_{i,\tau,\varepsilon}(u, x_i^\perp, (z_i - l/2 + \eta_0, z_i + l/2 - \eta_0)) + V_{i,\tau}(u, x_i^\perp, (z_i - l/2 + \eta_0, z_i + l/2 - \eta_0)) \geq 0$$ \hfill (4.39)

and equality holds if and only if $u = u(x_i^\perp)$.

**Proof** Let $z_i - l/2 + \eta_0 = t_0 < t_1 < \cdots < t_N = z_i + l/2 - \eta_0$ be a partition of $(z_i - l/2 + \eta_0, z_i + l/2 - \eta_0)$ into intervals of length $\eta_0$ (with possibly $|t_N - t_{N-1}| \leq \eta_0$). Let

$$\mathcal{K}(Q_i^+(z)) = \{ \varepsilon \in [1, \ldots, N] : R_{i,\tau,\varepsilon}(u, x_i^\perp, [t_k, t_{k+1}]) < 0 \}.$$

By Corollary 4.5, for every $\varepsilon \in \mathcal{K}(Q_i^+(z))$ one has the following:

- $\overline{M}_{i,\tau,\varepsilon}(u, x_i^\perp, [t_k - \eta_0/2, t_{k+1} + \eta_0/2)) \leq \Upsilon$
- $R_{i,\tau,\varepsilon}(u, x_i^\perp, [t_k, t_{k+1}]) \geq -\Upsilon$
- $\exists \tilde{s} < \tilde{t} \in [t_k - \delta_0, t_{k+1} + \delta_0]$ with $|\tilde{s} - \tilde{t}| \leq \delta_0$ and $|u_{x_i^\perp}(\tilde{s}) - u_{x_i^\perp}(\tilde{t})| > 1 - \delta$.

In particular, assuming w.l.o.g. that $u_{x_i^\perp}(\tilde{s}) > u_{x_i^\perp}(\tilde{t})$, since $\delta_0 \ll \eta_0/8$ and $\overline{M}_{i,\tau,\varepsilon}(u, x_i^\perp, [t_k - \eta_0/2, t_{k+1} + \eta_0/2)) \leq \Upsilon$, by (4.17) and (4.18) one has that, taking $\delta$ sufficiently small,

$$u_{x_i^\perp} \geq \frac{5}{8} \quad \text{on}[t_k - \eta_0/2, \tilde{s}],$$

$$u_{x_i^\perp} \leq \frac{3}{8} \quad \text{on}[\tilde{t}, t_{k+1} + \eta_0/2].$$ \hfill (4.40)

Hence, using the positivity of $f_{\eta}$, Fubini Theorem and the fact that

$$\int_{t_{k} - \eta_0/2}^{t_{k+1} + \eta_0/2} \int_{\mathbb{R}} \chi((a,b) \in \Omega(a,b))|\Pi(a,b)| \Pi d\Pi = |\rho||[t_k - \eta_0/2, t_{k+1} + \eta_0/2] \cap J(s, s + \rho)|,$$

one has that

$$V_{i,\tau}(u, x_i^\perp, [t_k - \eta_0/2, t_{k+1} + \eta_0/2)) \geq \frac{1}{2d} \int_{t_k - \eta_0/2}^{t_{k+1} + \eta_0/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \Pi \Omega(a,b) \cap \Omega(\tilde{s}, \tilde{t}) \int_{\mathbb{R}^{d-1}} G^{-1}(\rho) \cdot f_u(x_i^\perp + \zeta_i^\perp, s + \rho) K_\tau(\rho_{e_i} + \zeta_i^\perp) \Pi d\Pi \Pi \Pi d\Pi d\rho$$

$$\geq \frac{1}{2d} \int_{\mathbb{R}^{d-1}} f_u(x_i^\perp + \zeta_i^\perp, s + \rho) K_\tau(\rho_e + \zeta_i^\perp) \Pi d\Pi \rho d\Pi$$

$$\geq \frac{1}{16d} \int_{(s, \rho) \in \mathcal{M}_i^\perp} \int_{(s, \rho) \in \mathcal{M}_i^\perp} f_u(x_i^\perp, s, \zeta_i^\perp + \rho, s + \rho) K_\tau(\rho e_i + \zeta_i^\perp) \Pi d\Pi \rho d\Pi.$$
Now observe that, for \((s, \rho) \in \Omega(\hat{s}, \hat{r})\) with \(\frac{\eta_0}{8} \leq |\rho| \leq \frac{\eta_0}{4}\), due to (4.40) and (4.41) one has that
\[
f_u(x_i^\perp, s, \xi_i^\perp, s + \rho) \geq \left( \frac{1}{4} - \left[ u(x_i^\perp + \xi_i^\perp + s e_i) - u(x_i^\perp + \xi_i^\perp + (s + \rho) e_i) \right] \right)^2.
\]
Hence,
\[
\frac{1}{16d} \int_{(s,\rho)(\Omega(\hat{s},\hat{r}),|\rho| \leq \frac{\eta_0}{4})} \int_{\mathbb{R}^d + 1} f_u(x_i^\perp, s, \xi_i^\perp, s + \rho) K_\tau (\rho e_i + \xi_i^\perp) \, d\xi_i^\perp \, ds \, d\rho
\]
\[
\geq \frac{1}{16d} \int_{\tilde{s} - \alpha}^{\tilde{s}} \int_{\tilde{r} - \alpha}^{\tilde{r} + \alpha} \int_{|\xi_i^\perp| < \alpha} \left( \frac{1}{4} - \left[ u(x_i^\perp + \xi_i^\perp + s e_i) - u(x_i^\perp + \xi_i^\perp + (s + \rho) e_i) \right] \right)^2 \frac{d\xi_i^\perp \, d\rho \, ds}{(3\alpha + \delta_0 + \tau^{1/\beta})^p}
\]
where \(\alpha < \eta_0/4\), and in the last inequality we used that \(|\rho| \leq 2\alpha + \delta_0\).

By Lemma 2.5, if \(\delta_0 \leq \alpha\) (as in this case, since \(\delta_0 \ll \eta_0/8\)) and \(\sigma\) is sufficiently small,
\[
\frac{1}{16d} \int_{\tilde{s} - \alpha}^{\tilde{s}} \int_{\tilde{r} - \alpha}^{\tilde{r} + \alpha} \int_{|\xi_i^\perp| < \alpha} \left( \frac{1}{4} - \left[ u(x_i^\perp + \xi_i^\perp + s e_i) - u(x_i^\perp + \xi_i^\perp + (s + \rho) e_i) \right] \right)^2 \frac{d\xi_i^\perp \, d\rho \, ds > \alpha^{d+1}}{8 \cdot 16d(3\alpha + \delta_0 + \tau^{1/\beta})^p}.
\]

Then, assuming that \(0 < \tau \leq \tau_3\) is such that \(\alpha \geq \tau_3^{1/\beta}\) one has that \(1/(3\alpha + \delta_0 + \tau^{1/\beta})^p \geq 1/(5\alpha)^p\) and thus since \(p \geq d + 2\) one has that
\[
V_{i,\tau}(u, x_i^\perp, [t_k - \eta_0/2, t_{k+1} + \eta_0/2]) \geq \frac{1}{8 \cdot 16d \cdot 5^p \alpha}.
\]

To conclude, we observe that
\[
R_{i,\tau,\varepsilon}(u, x_i^\perp, V_{i,\tau}(u, x_i^\perp, [t_k - \eta_0/2, t_{k+1} + \eta_0/2])) \geq -\gamma + \frac{1}{8 \cdot 16d \cdot 5^p \alpha}
\]
and the r.h.s. of the above inequality is strictly positive provided \(\alpha\) is chosen such that
\[
-\gamma + \frac{1}{8 \cdot 16d \cdot 5^p \alpha} > 0.
\]
Thus, (4.39) holds. Moreover, since whenever \(R_{i,\tau,\varepsilon} < 0\) then by the above \(R_{i,\tau,\varepsilon} + V_{i,\tau} > 0\), one has that equality in (4.39) holds if and only if \(R_{i,\tau,\varepsilon} = 0\) and \(V_{i,\tau} = 0\), which implies \(u = u(x_i^\perp)\).

In the following lemma we estimate from below the contributions on intervals of slices in directions \(e_i\) to the energy \(\tilde{F}_{i,\tau,\varepsilon}\). In order to obtain such estimates, we will use the various lemmas and corollaries proved in this section.

**Lemma 4.9** Let \(\tilde{\sigma}, \tau_3 > 0\) as in Lemma 4.8. Let \(\sigma \leq \tilde{\sigma}, \tau \leq \tau_3\) and \(l > C_0/(-C_{\tau,\varepsilon}^\ast)\), where \(C_0\) is the constant appearing in Lemma 4.6. Let \(\zeta \in [0, \ell)\) and let \(\tilde{\eta} > 0\).

The following statements hold: there exists \(M_0\) constant independent of \(l\) (but depending on the dimension) such that for all \(u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d; [0, 1]) [0, \ell)^d\) and \(-\tilde{\eta} > 0\),

\[
\text{Lemma 4.9} \quad \text{Let} \quad \tilde{\sigma}, \tau_3 > 0 \quad \text{as in Lemma 4.8. Let} \quad \sigma \leq \tilde{\sigma}, \tau \leq \tau_3 \quad \text{and} \quad l > C_0/(-C_{\tau,\varepsilon}^\ast), \quad \text{where} \quad C_0 \quad \text{is the constant appearing in Lemma 4.6. Let} \quad \zeta \in [0, \ell)\text{ and let} \quad \tilde{\eta} > 0.
\]

The following statements hold: there exists \(M_0\) constant independent of \(l\) (but depending on the dimension) such that for all \(u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d; [0, 1]) [0, \ell)^d\) and \(-\tilde{\eta} > 0\),

\[
\text{Lemma 4.9} \quad \text{Let} \quad \tilde{\sigma}, \tau_3 > 0 \quad \text{as in Lemma 4.8. Let} \quad \sigma \leq \tilde{\sigma}, \tau \leq \tau_3 \quad \text{and} \quad l > C_0/(-C_{\tau,\varepsilon}^\ast), \quad \text{where} \quad C_0 \quad \text{is the constant appearing in Lemma 4.6. Let} \quad \zeta \in [0, \ell)\text{ and let} \quad \tilde{\eta} > 0.
\]
(i) Let \( J \subset \mathbb{R} \) an interval such that for every \( s \in J \) one has that \( D^j_{\eta}(u, Q_l(z_i^+ + se_i)) \leq \sigma \) with \( j \neq i \). Then
\[
\int_{J} \tilde{F}_{i,\tau,\varepsilon}(u, Q_l(z_i^+ + se_i)) \, ds \geq -\frac{M_0}{l}. \tag{4.43}
\]

Moreover, if \( J = [0, L] \), then
\[
\int_{J} \tilde{F}_{i,\tau,\varepsilon}(u, Q_l(z_i^+ + se_i)) \, ds \geq 0 \tag{4.44}
\]
and equality holds only if \( u = u(x_i^+) \).

(ii) Let \( J = (a, b) \subset \mathbb{R} \). If for \( s = a \) and \( s = b \) it holds \( D^j_{\eta}(u, Q_l(z_i^+ + se_i)) \leq \sigma \) with \( j \neq i \), then
\[
\int_{J} \tilde{F}_{i,\tau,\varepsilon}(u, Q_l(z_i^+ + se_i)) \, ds \geq |J|C^*_\tau,\varepsilon - \frac{M_0}{l}, \tag{4.45}
\]
otherwise
\[
\int_{J} \tilde{F}_{i,\tau,\varepsilon}(u, Q_l(z_i^+ + se_i)) \, ds \geq |J|C^*_\tau,\varepsilon - M_0l. \tag{4.46}
\]

Moreover, if \( J = [0, L] \), then
\[
\int_{J} \tilde{F}_{i,\tau,\varepsilon}(u, Q_l(z_i^+ + se_i)) \, ds \geq |J|C^*_\tau,\varepsilon. \tag{4.47}
\]

**Proof** Given Lemmas 4.2–4.8, the proof proceeds in a way similar to that followed in Lemma 7.9 of [11].

Let us first prove (i). For simplicity of notation, we may assume without loss of generality that \( a = 0 \) and \( b = l' \), namely \( J = [0, l'] \). For any \( x_i, y_i \in [0, L] \) we also set
\[
\begin{align*}
R_{i,\tau,\varepsilon}(u, x_i^+, (x_i, y_i)) &= \int_{x_i}^{y_i} r_{i,\tau,\varepsilon}(u, x_i^+, a) \, da, \\
V_{i,\tau}(u, x_i^+, (x_i, y_i)) &= \int_{x_i}^{y_i} v_{i,\tau}(u, x_i^+, a) \, da.
\end{align*}
\]

From the definition of \( \tilde{F}_{i,\tau,\varepsilon} \) given in (3.21) and since \( W_{i,\tau} \geq 0 \), we have that
\[
\int_{J} \tilde{F}_{i,\tau,\varepsilon}(u, Q_l(z_i^+ + se_i)) \, ds \geq \frac{1}{l^d} \int_{J} \int_{Q_l^+(z_i^+)} \left\{ R_{i,\tau,\varepsilon}(u, x_i^+, Q_l'(s)) + V_{i,\tau}(u, x_i^+, Q_l'(s)) \right\} \, dx_i^+ \, ds
\]
\[
= \frac{1}{l^d} \int_{J} \int_{Q_l^+(z_i^+)} \int_{a-l/2}^{a+l/2} \left\{ r_{i,\tau,\varepsilon}(u, x_i^+, a) + v_{i,\tau}(u, x_i^+, a) \right\} \, da \, dx_i^+ \, ds
\]
\[
= \frac{1}{l^{d-1}} \int_{J} \int_{Q_l^+(z_i^+)} \int_{l-j/2}^{l+j/2} \left\{ |a-l/2, a+l/2| \cap [0, l'] \right\} \left\{ r_{i,\tau,\varepsilon}(u, x_i^+, a) + v_{i,\tau}(u, x_i^+, a) \right\} \, da,
\]
where in order to obtain the last line we used Fubini Theorem.

Let us now estimate the last term in (4.48).

One has that
\[
\begin{align*}
&\frac{1}{l^{d-1}} \int_{J} \int_{Q_l^+(z_i^+)} \int_{l-j/2}^{l+j/2} \left\{ |a-l/2, a+l/2| \cap [0, l'] \right\} \left\{ r_{i,\tau,\varepsilon}(u, x_i^+, a) + v_{i,\tau}(u, x_i^+, a) \right\} \, da
\end{align*}
\]
\[
= \frac{1}{l^{d-1}} \int_{J} \int_{Q_l^+(z_i^+)} \int_{|a-l/2, a+l/2| \cap [0, l'] \cap [a-l/2, a+l/2]} \left\{ r_{i,\tau,\varepsilon}(u, x_i^+, a) + v_{i,\tau}(u, x_i^+, a) \right\} \, da
\]
\[
+ \frac{1}{l^{d-1}} \int_{Q_i^j(\zeta_j^i)} \int_{L-1/2}^{L+1/2} \int_{-\eta_0}^{\eta_0} \frac{\|a - l/2, a + l/2 \cap [0, l']\|}{l} \left\{ \hat{r}_{i, \tau, \psi}(u, x_i^\perp, a) + v_{i, \tau}(u, x_i^\perp, a) \right\} da
\geq -2\Upsilon \frac{\eta_0}{l} + \frac{1}{l^{d-1}} \int_{Q_i^j(\zeta_j^i)} \int_{L-1/2}^{L+1/2} \int_{-\eta_0}^{\eta_0} \frac{\|a - l/2, a + l/2 \cap [0, l']\|}{l} \left\{ r_{i, \tau, \psi}(u, x_i^\perp, a) + v_{i, \tau}(u, x_i^\perp, a) \right\} da.
\]

where in the last inequality we used (4.14). As in Lemma 4.8, one can now partition \([-l/2 + \eta_0, l' + l/2 - \eta_0]\) in intervals of size \(\eta_0\) and notice that on the intervals \([t_k, t_k+1)\) of this partition such that \(R_{i, \tau, \psi}(u, x_i^\perp, [t_k, t_k+1)) = \int_{t_k}^{t_{k+1}} r_{i, \tau, \psi}(u, x_i^\perp, a) da < 0\), then

\[
\int_{t_k}^{t_{k+1}} \frac{\|a - l/2, a + l/2 \cap [0, l']\|}{l} \hat{r}_{i, \tau, \psi}(u, x_i^\perp, a) da \geq -\Upsilon_{\alpha, \tau} (u, x_i^\perp, [t_k, t_k+1)) \geq -\Upsilon.
\]

On the other hand,

\[
\int_{t_k}^{t_{k+1}} \frac{\|a - l/2, a + l/2 \cap [0, l']\|}{l} v_{i, \tau}(u, x_i^\perp, a) da \geq \frac{\eta_0}{l} \int_{t_k}^{t_{k+1}} v_{i, \tau}(u, x_i^\perp, a) da
\geq \frac{\eta_0}{l} V_{i, \tau}(u, x_i^\perp, [t_k, t_k+1]),
\]

and provided \(\bar{\sigma}\) and \(\tau_3\) are sufficiently small, since for every \(s \in [0, l']\) one has that \(D^f_\sigma(u, Q_l(\zeta_j^i + se)) \leq \sigma\) with \(j \neq i\), as in Lemma 4.8 one has that

\[-\Upsilon + \frac{\eta_0}{l} V_{i, \tau}(u, x_i^\perp, [t_k, t_k+1]) > 0,
\]

thus proving (4.43) with \(M_0 = \frac{2\Upsilon \eta_0}{l}\).

Let us now prove (4.44). In this case, by periodicity of \(u\) w.r.t. \([0, L)^d\) and Fubini Theorem,

\[
\int_0^L \int_{Q_i^j(\zeta_j^i)} \left\{ R_{i, \tau, \psi}(u, x_i^\perp, Q_l^j(s)) + V_{i, \tau}(u, x_i^\perp, Q_l^j(s)) \right\} dx_i^\perp ds
= \int_{Q_i^j(\zeta_j^i)} \int_0^L \left\{ R_{i, \tau, \psi}(u, x_i^\perp, [0, L]) + V_{i, \tau}(u, x_i^\perp, [0, L]) \right\} dx_i^\perp.
\]

Hence (4.44) holds since we can apply directly Lemma 4.8 without having points close to the boundary.

Let us now prove (ii). W.l.o.g. let us assume that \(J = (0, l')\).

As in (4.48) one has that

\[
\int J \hat{r}_{i, \tau, \psi}(u, Q_l(\zeta_j^i + se)) ds \geq \frac{1}{l^{d-1}} \int_{Q_i^j(\zeta_j^i)} \int_0^L \left\{ R_{i, \tau, \psi}(u, x_i^\perp, Q_l^j(s)) + V_{i, \tau}(u, x_i^\perp, Q_l^j(s)) \right\} dx_i^\perp ds
= \frac{1}{l^{d-1}} \int_{Q_i^j(\zeta_j^i)} \int_{L-1/2}^{L+1/2} \int_{-\eta_0}^{\eta_0} \frac{\|a - l/2, a + l/2 \cap [0, l']\|}{l} \left\{ r_{i, \tau, \psi}(u, x_i^\perp, a) + v_{i, \tau}(u, x_i^\perp, a) \right\} da
= \frac{1}{l^{d-1}} \int_{Q_i^j(\zeta_j^i)} \int_{L-1/2}^{L+1/2} \int_{-\eta_0}^{\eta_0} \frac{\|a - l/2, a + l/2 \cap [0, l']\|}{l} \left\{ r_{i, \tau, \psi}(u, x_i^\perp, a) + v_{i, \tau}(u, x_i^\perp, a) \right\} da
+ \frac{1}{l^{d-1}} \int_{Q_i^j(\zeta_j^i)} \int_{L-1/2}^{L+1/2} \left\{ r_{i, \tau, \psi}(u, x_i^\perp, a) + v_{i, \tau}(u, x_i^\perp, a) \right\} da.
\]
As in the proof of (4.43), if for \( s = 0 \) and \( s = l' \) it holds \( D^l \hat{h}(u, Q_l(z^i + s e_i)) \leq \sigma \), then one has that
\[
\frac{1}{l^{d-1}} \int_{Q^l(z^i)} \int_{[-l/2,l/2] \cup [l'/2,l'+l/2]} \frac{[a - l/2, a + l/2] \cap [0,l']}{l} \{ r_{i,\tau,\epsilon}(u, x^i, a) + v_{i,\tau}(u, x^i, a) \} da \\
\geq -4 \frac{\tau}{l} n_0.
\] (4.49)

On the other hand, by Lemma 4.6 and the assumption \( l > C_0/(-C_\tau^* \epsilon) \)
\[
\frac{1}{l^{d-1}} \int_{Q^l(z^i)} \int_{l/2}^{l'-l/2} \{ r_{i,\tau,\epsilon}(u, x^i, a) + v_{i,\tau}(u, x^i, a) \} da \\
\geq \frac{1}{l^{d-1}} \int_{Q^l(z^i)} R_{i,\tau,\epsilon}(u, x^i, (l/2, l' - l/2)) dx^i \\
\geq C_{\tau,\epsilon}^* |J| - lC_{\tau,\epsilon}^* - C_0 \\
\geq C_{\tau,\epsilon}^* |J|.
\] (4.50)

Thus, (4.45) follows combining (4.49) and (4.50).

If instead either for \( s = 0 \) or for \( s = l' \) it holds \( D^l \hat{h}(u, Q_l(z^i + s e_i)) > \sigma \), then we partition the intervals \([ -l/2, l/2 ] \cup [ l'/2, l' + l/2 ]\) into intervals of size \( \eta_0 \), on which by (4.14) one has that \( R_{i,\tau,\epsilon} \geq -\gamma \). In this way we get
\[
\frac{1}{l^{d-1}} \int_{Q^l(z^i)} \int_{[-l/2,l/2] \cup [l'/2,l'+l/2]} \frac{[a - l/2, a + l/2] \cap [0,l']}{l} \{ r_{i,\tau,\epsilon}(u, x^i, a) + v_{i,\tau}(u, x^i, a) \} da \\
\geq -2 \gamma \frac{l}{\eta_0},
\] (4.51)

being \( l/\eta_0 \) an upper bound for the number of disjoint intervals of length \( \eta_0 \) inside an interval of length \( l \).

Thus, (4.46) follows from (4.50) and (4.51).

The proof of (4.47) proceeds using the \( L \)-periodicity of the contributions as done for (4.44). 

\[ \square \]

5 Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2, bringing together the lemmas of the previous section in order to show the optimality of one-dimensional periodic functions in a range of \( \tau, \epsilon \) independent of \( L \). This part of the proof follows closely the strategy adopted in Section 7 of [11].

The sets defined in the proof and the main estimates will depend on a set of parameters \( l, \gamma, \eta_0, \sigma, \rho, M, \eta, \tau, \) and \( \epsilon \). If suitably chosen, they lead to the proof Theorem 1.2.

Let us first specify how the parameters are chosen, and their dependence on each other. The reason for such choices will be clarified during the proof.

Let \( 0 < -C^* < -C_{\tau,\epsilon}^* \) for all sufficiently small \( \epsilon \) and \( \tau \). Such a \( C^* \) exists by the \( \Gamma \)-convergence of the functionals \( F_{\tau,\epsilon} \) to a functional which is finite only on stripes as \( \tau, \epsilon \to 0 \). For the sharp interface problem (i.e., the \( \Gamma \)-limit as \( \epsilon \to 0 \)) one can also explicitly compute such constants and see this directly (see [21]).

We fix a family of parameters as follows:

- Let \( 1 < \gamma \leq \frac{17}{16} \) and let \( \eta_0 = \eta_0(\gamma) \) as in Lemma 4.2.
Then we fix \( l > 0 \) s.t.
\[
\begin{align*}
\frac{dC(d, \Upsilon, \eta_0)}{-C^*}, \frac{C_0}{-C^*}
\end{align*}
\]
(5.1)
where \( C(d, \Upsilon, \eta_0) \) is a constant (depending on \( d, \Upsilon \) and \( \eta_0 \)) that appears in (5.11), and \( C_0 \) is the constant which appears in the statement of Lemma 4.6.

- Choose \( \nu = \frac{1}{l} \).
- Let \( \sigma = \min\{\sigma_0, \tilde{\sigma}\} \) with \( \sigma_0 = \sigma_0(\nu) \) as in (2.12) and \( \tilde{\sigma} \) as in Lemma 4.8.
- Thanks to Remark 2.4 (i), we then fix \( \rho \sim \sigma l \).

\[
\begin{align*}
\forall z, z' \text{ s.t. } D_\eta(E, Q_l(z)) \geq \sigma, \quad |z - z'|_\infty \leq \rho \quad \Rightarrow \quad D_\eta(E, Q_l(z')) \geq \sigma / 2.
\end{align*}
\]
(5.3)
- Then we fix \( M \) such that
\[
\begin{align*}
\frac{M \rho}{2d} > M_0 l,
\end{align*}
\]
(5.4)
where \( M_0 \) is the constant appearing in Lemma 4.9.
- Finally, let \( \tau > 0, \varepsilon > 0 \) satisfy the following
\[
\begin{align*}
\tau \leq \tau_3 \quad \text{as in Lemma 4.8}
\end{align*}
\]
(5.5)
\[
\tau \text{ is such that Lemma 4.7 holds}
\]
(5.6)
\[
\tau \leq \hat{\tau}, \varepsilon \leq \hat{\varepsilon} \quad \text{as in Proposition 3.2}.
\]
(5.7)
- We fix also \( \hat{\eta} = \hat{\eta}(M, l) \) as in Proposition 3.2.

Given such parameters, let us prove Theorem 1.2 for any \( L > l \) of the form \( L = 2kh_{\tau, \varepsilon}^* \), with \( k \in \mathbb{N} \).

Let \( u \) be a minimizer of \( F_{\tau, \varepsilon} \). Since \( u \) is \([0, L)^d\)-periodic, we can consider \( u \) as defined on \( \mathbb{T}_L^d \), where \( \mathbb{T}_L^d \) is the \( d \)-dimensional torus of size \( L \). Thus the problem is naturally defined on the torus. Hence with a slight abuse of notation, we will denote by \([0, L)^d\) the cube of size \( L \) with the usual identification of the boundary.

**Decomposition of \([0, L)^d\):**

We define
\[
\tilde{A}_0 := \left\{ z \in [0, L)^d : D_\eta(u, Q_l(z)) \geq \sigma \right\}.
\]
Hence, by Lemma 3.2, for every \( z \in \tilde{A}_0 \) one has that \( \tilde{F}_{\tau, \varepsilon}(u, Q_l(z)) > M \).

Let us denote by \( \tilde{A}_{-1} \) the set of points
\[
\tilde{A}_{-1} := \left\{ z \in [0, L)^d : \exists i, j \text{ with } i \neq j \text{ s.t. } D_\eta(u, Q_l(z)) \leq \sigma, D_\eta(u, Q_l(z)) \leq \sigma \right\}.
\]
One can easily see that \( \tilde{A}_0 \) and \( \tilde{A}_{-1} \) are closed.

By the choice of \( \rho \) made in (5.2), (5.3) holds, namely for every \( z \in \tilde{A}_0 \) and \( |z - z'|_\infty \leq \rho \) one has that
\[
D_\eta(u, Q_l(z')) > \sigma / 2.
\]
Moreover, since \( \sigma \) satisfies (2.12) with \( \nu = \frac{1}{l} \), when \( z \in \tilde{A}_{-1} \), then one has that
\[
\min\left\{ \|u - 1\|_{L^1(Q_l(z))}, \|u\|_{L^1(Q_l(z))} \right\} \leq \tau d.
\]
Thus, using Lemma 4.7 with \( \nu = 1/l \), one has that
\[
\tilde{F}_{\tau, \varepsilon}(u, Q_l(z)) \geq -C_1 \frac{\Upsilon d}{\eta_0 l}.
\]
Moreover, let now \( z' \) such that \( |z - z'|_\infty \leq 1 \) with \( z \in \tilde A_{-1} \). It is not difficult to see that if \( \|u - 1\|_{L^1(Q(z))} \leq l^{d-1} \) then \( \|u - 1\|_{L^1(Q(z'))} \leq \tilde C d^{d-1} \). Thus from Lemma 4.7, one has that
\[
\tilde F_{\tau, \varepsilon}(u, Q_l(z')) \geq -\frac{\tilde C d^{d-1}}{\eta_0 l},
\]
where \( \tilde C d = d C_1 \tilde C_d \).

The above observations motivate the following definitions
\[
A_0 := \left\{ z' \in [0, L)^d : \exists z \in \tilde A_0 \text{ with } |z - z'|_\infty \leq \rho \right\},
\]
(5.9)
\[
A_{-1} := \left\{ z' \in [0, L)^d : \exists z \in \tilde A_{-1} \text{ with } |z - z'|_\infty \leq 1 \right\},
\]
(5.10)

By the choice of the parameters and the observations above, for every \( z \in A_0 \) one has that \( \tilde F_{\tau, \varepsilon}(u, Q_l(z)) > M \) and for every \( z \in A_{-1} \), \( \tilde F_{\tau, \varepsilon}(u, Q_l(z)) \geq -\left( \tilde C d^{d-1}/l \eta_0 \right) \).

For simplicity of notation let us denote by \( A := A_0 \cup A_{-1} \).

The set \( [0, L)^d \setminus A \) has the following property: for every \( z \in [0, L)^d \setminus A \), there exists \( i \in \{1, \ldots, d\} \) such that \( D^i_{\eta}(u, Q_l(z)) \leq \sigma \) and for every \( k \neq i \) one has that \( D^k_{\eta}(u, Q_l(z)) > \sigma \).

Given that \( A \) is closed, we consider the connected components \( C_1, \ldots, C_n \) of \( [0, L)^d \setminus A \). The sets \( C_i \) are path-wise connected.

Let us now show the following claim: given a connected component \( C_j \) one has that there exists \( i \) such that \( D^i_{\eta}(u, Q_l(z)) \leq \sigma \) for every \( z \in C_j \) and for every \( k \neq i \) one has that \( D^k_{\eta}(u, Q_l(z)) > \sigma \). Indeed, suppose that there exists \( z', z' \in C_j \) such that \( D^i_{\eta}(u, Q_l(z)) \leq \sigma \) and \( D^k_{\eta}(u, Q_l(z')) \leq \sigma \) with \( i \neq k \) and take a continuous path \( \gamma : [0, 1] \to C_j \) such that \( \gamma(0) = z \) and \( \gamma(1) = z' \). From our hypothesis, we have that \( \{ s : D^i_{\eta}(u, Q_l(\gamma(s))) \leq \sigma \} \neq \emptyset \) and there exists \( \tilde s \in \partial \{ s : D^i_{\eta}(u, Q_l(\gamma(s))) \leq \sigma \} \cap \partial \{ s : D^j_{\eta}(u, Q_l(\gamma(s))) \leq \sigma \} \) for some \( j \neq i \). Let \( \tilde z = \gamma(\tilde s) \). Thus there are points arbitrary close to \( \tilde z \in C_j \) such that \( D^i_{\eta}(u, Q_l(\tilde z)) \leq \sigma \) and \( D^j_{\eta}(u, Q_l(\tilde z)) \leq \sigma \). From the continuity of the maps \( z \mapsto D^i_{\eta}(u, Q_l(z)) \), \( z \mapsto D^j_{\eta}(u, Q_l(z)) \), we have that \( \tilde z \in A_{-1} \), which contradicts our assumption. We will say that \( C_j \) is oriented in direction \( e_i \) if there is a point \( z \in C_j \) such that \( D^i_{\eta}(u, Q_l(z)) \leq \sigma \). Because of the above being oriented along direction \( e_i \) is well-defined.

We will denote by \( A_j \) the union of the connected components \( C_j \) such that \( C_j \) is oriented along the direction \( e_i \).

Let us now summarize the important properties that will be used in the following

(i) \( \) The sets \( A = A_{-1} \cup A_0, A_1, A_2, \ldots, A_d \) form a partition of \( [0, L)^d \).

(ii) \( \) The sets \( A_{-1}, A_0 \) are closed and \( A_i, i > 0 \), are open.

(iii) \( \) For every \( z \in A_i \), we have that \( D^i_{\eta}(u, Q_l(z)) \leq \sigma \).

(iv) \( \) There exists \( \rho \) (independent of \( L, \tau, \varepsilon \)) such that if \( z \in A_0 \), then \( \exists z' \) s.t. \( Q_\rho(z') \subset A_0 \) and \( z \in Q_\rho(z') \). If \( z \in A_{-1} \) then \( \exists z' \) s.t. \( Q_1(z') \subset A_{-1} \) and \( z \in Q_1(z') \).

(v) \( \) For every \( z \in A_j \) and \( z' \in A_j \) one has that there exists a point \( \tilde z \) in the segment connecting \( z \) to \( z' \) lying in \( A_0 \cup A_{-1} \).

The proof of Theorem 1.2 can thus be reduced to the following

**Proposition 5.1** Let \( B = \bigcup_{i>0} A_i \). For every \( i \) and for our choices of \( \tau, \varepsilon \), it holds
\[
\frac{1}{L^d} \int_B \tilde F_{\tau, \varepsilon}(u, Q_l(z)) \, dz + \frac{1}{d L^d} \int_A \tilde F_{\tau, \varepsilon}(u, Q_l(z)) \, dz \geq \frac{C^*_{\tau, \varepsilon}|A_i|}{L^d} - C(d, \eta_0) \frac{|A|}{L^d}
\]
(5.11)
for some constant \( C(d, \eta, \eta_0) \) depending on the dimension \( d \), on \( \eta \) and \( \eta_0 \).
Indeed, assuming \((5.11)\), we can sum over \(i\) and obtain that, since \(l\) satisfies \((5.1)\)

\[
\mathcal{F}_{\tau, L, \varepsilon}(u) \geq \frac{1}{L^d} \int_{[0, L)^d} \tilde{F}_{i, \tau, \varepsilon}(u, Q_i(z)) \, dz \geq \frac{C_{\tau, \varepsilon}}{L^d} \sum_{i=1}^{d} |A_i| - \frac{dC(d, \Upsilon, \eta_0)|A|}{L^d} \\
\geq C_{\tau, \varepsilon}^* - \frac{C_{\tau, \varepsilon}^* |A|}{L^d} - \frac{dC(d, \Upsilon, \eta_0)}{L^d} |A| \geq C_{\tau, \varepsilon}^* ,
\]

where in the above \(C_{\tau, \varepsilon}^* \) is the energy density of optimal stripes of stripes \(h_{\tau, \varepsilon}^*\) and we have used that \(C_{\tau, \varepsilon}^* < 0\) and that \(|A| + \sum_{i=1}^{d} |A_i| = \|0, L\|^d = L^d\).

Notice that, in the inequality above, equality holds only if \(|A| = 0\) and therefore by \((v)\) only if there is just one \(A_i, i > 0\) with \(|A_i| > 0\). Therefore, it has been proved that there exists \(i > 0\) with \(A_i = [0, L)^d\). Finally, let us consider

\[
\frac{1}{L^d} \int_{[0, L)^d} \tilde{F}_{i, \tau, \varepsilon}(u, Q_i(z)) \, dz = \frac{1}{L^d} \int_{[0, L)^d} \tilde{F}_{i, \tau, \varepsilon}(u, Q_i(z)) \, dz \\
+ \frac{1}{L^d} \sum_{j \neq i} \int_{[0, L)^d} \tilde{F}_{j, \tau, \varepsilon}(u, Q_i(z)) \, dz
\]

\((5.12)\)

\[
\frac{1}{L^d} \int_{B_{t_i^+}^\perp} \tilde{F}_{i, \tau, \varepsilon}(u, Q_i(t_i^+ + s e_i)) \, ds + \frac{1}{dL^d} \int_{A_i^\perp} \tilde{F}_{i, \tau, \varepsilon}(u, Q_i(t_i^+ + s e_i)) \, ds \geq \frac{C_{\tau, \varepsilon}^* |A_i|}{L^d} \\
- C(d, \Upsilon, \eta_0) \frac{|A_i^\perp|}{L^d}
\]

\((5.14)\)

Indeed by integrating \((5.14)\) over \(t_i^+\) we obtain \((5.11)\).

Notice also that \(B_{t_i^+}\) is a finite union of intervals. Indeed, being a union of intervals follows from \((ii)\) and finiteness follows from condition \((v)\) on the decomposition. Indeed, for every point that does not belong to \(B_{t_i^+}\) because of \((iv)\) there is a neighborhood of fixed positive size that is not included in \(B_{t_i^+}\). Let \(\{I_1, \ldots, I_n\}\) such that \(\bigcup_{j=1}^{n} I_j = B_{t_i^+}\) with \(I_j \cap I_k = \emptyset\) whenever \(j \neq k\). We can further assume that \(I_i \leq I_{i+1}\), namely that for every \(s \in I_i\) and \(s' \in I_{i+1}\) it holds \(s \leq s'\). By construction there exists \(J_j \subset A_i^\perp\) such that \(I_j \leq J_j \leq I_{j+1}\).

Whenever \(J_j \cap A_{0, t_i^+} \neq \emptyset\), we have that \(|J_j| > \rho\) and whenever \(J_j \cap A_{-1, t_i^+} \neq \emptyset\) then \(|J_j| > 1\).
Thus we have that
\[
\frac{1}{L^d} \int_{B_{t_1^+}^i} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds + \frac{1}{dL^d} \int_{A_{t_1^+}^i} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \\
\geq \sum_{j=1}^n \frac{1}{L^d} \int_{I_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds + \frac{1}{dL^d} \sum_{j=1}^n \int_{J_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \\
\geq \frac{1}{L^d} \sum_{j=1}^n \left( \int_{I_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds + \frac{1}{2d} \int_{J_{j-1} \cup J_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \right),
\]
where in order to obtain the third line from the second line, we have used periodicity and \( J_0 := J_n \).

Let first \( I_j \subset A_{i, t_1^+} \). By construction, we have that \( \partial I_j \subset A_{i, t_1^+} \).

If \( \partial I_j \subset A_{-1, t_1^+} \), by using our choice of parameters we can apply (4.45) in Lemma 4.9 and obtain
\[
\frac{1}{L^d} \int_{I_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \geq \frac{1}{L^d} \left( |I_j| C^\ast_{\tau, \epsilon} - M_0 \right).
\]

If \( \partial I_j \cap A_{0, t_1^+} \neq \emptyset \), by using our choice of parameters, we can apply (4.46) in Lemma 4.9, and obtain
\[
\frac{1}{L^d} \int_{I_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \geq \frac{1}{L^d} \left( |I_j| C^\ast_{\tau, \epsilon} - M_0 \right).
\]

On the other hand, if \( \partial I_j \cap A_{0, t_1^+} \neq \emptyset \), we have that either \( J_j \cap A_{0, t_1^+} \neq \emptyset \) or \( J_{j-1} \cap A_{0, t_1^+} \neq \emptyset \). Thus
\[
\frac{1}{2dL^d} \int_{J_{j-1}} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \\
\geq \frac{M \rho}{2dL^d} - \frac{|J_{j-1} \cap A_{-1, t_1^+}| \hat{C}_d \Upsilon}{2d \eta_0 L^d} - \frac{|J_j \cap A_{-1, t_1^+}| \hat{C}_d \Upsilon}{2d \eta_0 L^d},
\]
where \( \hat{C}_d \) is the constant in (5.8).

Since \( M \) satisfies (5.4), in both cases \( \partial I_j \subset A_{-1, t_1^+} \) or \( \partial I_j \cap A_{0, t_1^+} \neq \emptyset \), we have that
\[
\frac{1}{L^d} \int_{I_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \\
+ \frac{1}{2dL^d} \int_{J_{j-1}} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \\
\geq \frac{C^\ast_{\tau, \epsilon} |I_j|}{L^d} - \frac{|J_{j-1} \cap A_{-1, t_1^+}| \hat{C}_d \Upsilon}{2d \eta_0 L^d} - \frac{|J_j \cap A_{-1, t_1^+}| \hat{C}_d \Upsilon}{2d \eta_0 L^d}.
\]

If \( I_j \subset A_{k, t_1^+} \) with \( k \neq i \) from (4.43) in Lemma 4.9 it holds
\[
\frac{1}{L^d} \int_{I_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \geq - \frac{M_0}{L^d}.
\]

In general for every \( J_j \) we have that
\[
\frac{1}{dL^d} \int_{J_j} \tilde{F}_{\tau, \epsilon}(u, Q_l(t_1^+ + se_i)) \, ds \geq \frac{|J_j \cap A_{0, t_1^+}| M}{dL^d} - \frac{\hat{C}_d \Upsilon}{d \eta_0 L^d} |J_j \cap A_{-1, t_1^+}|.
\]
For $I_j \subset A_{k,i}^+$ such that $(J_j \cup J_{j-1}) \cap A_{0,i}^+ \neq \emptyset$ with $k \neq i$, we have that

$$
\frac{1}{L^d} \int_{I_j} \tilde{F}_{i,j}(u, Q_i(t_j^+ + se_i)) \, ds
$$

$$
+ \frac{1}{2dL^d} \int_{J_{j-1}} \tilde{F}_{\tau,\varepsilon}(u, Q_i(t_j^+ + se_i)) \, ds
$$

$$
+ \frac{1}{2dL^d} \int_{J_j} \tilde{F}_{\tau,\varepsilon}(u, Q_i(t_j^+ + se_i)) \, ds
$$

$$
\geq - \frac{M_0}{L^d} + \frac{M_0}{2dL^d} - \frac{|J_{j-1} \cap A_{-1,1}^+| |\hat{C}_d\varepsilon|}{d\eta_l L^d} - \frac{|J_j \cap A_{-1,1}^+| |\hat{C}_d\varepsilon|}{d\eta_l L^d}.
$$

where the last inequality is true due to (5.4).

For $I_j \subset A_{k,i}^+$ such that $(J_j \cup J_{j-1}) \subset A_{-1,1}^+$ with $k \neq i$, we have that

$$
\frac{1}{L^d} \int_{I_j} \tilde{F}_{i,j}(u, Q_i(t_j^+ + se_i)) \, ds
$$

$$
+ \frac{1}{2dL^d} \int_{J_{j-1}} \tilde{F}_{\tau,\varepsilon}(u, Q_i(t_j^+ + se_i)) \, ds
$$

$$
+ \frac{1}{2dL^d} \int_{J_j} \tilde{F}_{\tau,\varepsilon}(u, Q_i(t_j^+ + se_i)) \, ds
$$

$$
\geq - \frac{M_0}{L^d} + \frac{M_0}{2dL^d} - \frac{|J_{j-1} \cap A_{-1,1}^+| |\hat{C}_d\varepsilon|}{d\eta_l L^d} - \frac{|J_j \cap A_{-1,1}^+| |\hat{C}_d\varepsilon|}{d\eta_l L^d}.
$$

where in the last inequality we have used that $|J_j \cap A_{-1,1}^+| \geq 1$, $|J_{j-1} \cap A_{-1,1}^+| \geq 1$.

Summing over $j$, and taking $C(d, \varepsilon, \eta_0) = \max \left( M_0, \frac{\hat{C}_d\varepsilon}{d\eta_l} \right)$, one obtains (5.14) as desired.

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