Scaling of space and timelike response of confined relativistic particles

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Abstract

The response of a relativistic particle bound in a linear confining well is calculated as a function of the momentum and energy transfer, \( \mathbf{q}, \nu \). At large values of \(|\mathbf{q}|\) the response exhibits scaling in the variable \( \tilde{y} = \nu - |\mathbf{q}| \), which is proportional to the Nachtmann variable, \( \xi \). The approach to scaling is studied at smaller values of \(|\mathbf{q}|\). Scaling occurs at \( \nu \sim |\mathbf{q}| \) at relatively small \(|\mathbf{q}|\), and its validity extends over the entire \( \xi \) range as \(|\mathbf{q}|\) increases; this behavior is observed in electron-proton scattering. About 10% of the response at large \(|\mathbf{q}|\) is in the timelike region where \( \nu > |\mathbf{q}| \), and it is necessary to include it to fulfill the particle number sum rule. The Gross-Llewellyn Smith and Gottfried sum rules are discussed in the context of these results.

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Deep inelastic scattering (DIS) of leptons by hadrons is generally discussed in the framework of the naive parton model and the QCD-improved parton model using the operator product expansion.\[1\] This approach has been very successful in determining the evolution of the structure functions as a function of the square of the four-momentum transferred to the hadron.\[2\] In the leading order of the model the hadron is approximated by a collection of noninteracting quarks and gluons. The struck quark is assumed to be on the mass-shell both before and after its interaction with the electron. Based on this assumption, the leading order response is predicted to be in the spacelike region for which the energy transfer $\nu$ is less than the magnitude of momentum transfer, $|q|$, as a consequence of the inequality,

$$\nu = \sqrt{|k + q|^2 + m_q^2} - \sqrt{|k|^2 + m_q^2} \leq |q|. \quad (1)$$

Here $k$ and $m_q$ are the momentum and mass of the struck quark, respectively. The predicted response is discontinuous at the boundary $|q| = \nu$ between space and timelike regions.

The conventional variables of the parton model, $Q^2 = |q|^2 - \nu^2$ and the Bjorken $x =
\[ Q^2/2M\nu, \] used to describe the DIS structure functions of a hadron of mass \( M \), are confined to the spacelike region of the \(|q|\)-\( \nu \) plane for positive values of \( Q^2 \) accessible in lepton scattering experiments, as shown in Fig. 1. Therefore we study the response, \( R(q, \nu) \) as a function of \( \nu \) and \(|q|\) in the rest frame of the system, as is common practice in the many-body theory (MBT). Lines of constant \(|q|\) in Fig. 1 cross the photon line (\( \nu = |q| \)) and go into the timelike region. The observed \((Q^2 > 0)\) DIS response is limited to a narrow region in the \(|q|\)-\( \nu \) plane illustrated in Fig. 1. It is bounded by the elastic limit, \( \nu_{el} = \sqrt{q^2 + M^2} - M \) on one side, and by the photon line on the other. In the limit of large \(|q|\) the width of the observed response at fixed \(|q|\) is \( M \). Lines of constant \( Q^2 \) intersect the elastic limit curve at \( x = 1 \) and approach the photon line at small \( x \).

For a hypothetical scalar probe, the response is given by:

\[
R(q, \nu) = \sum_I \langle I | \sum_j e^{i q \cdot r_j} | 0 \rangle^2 \delta(E_I - E_0 - \nu) \tag{2}
\]

where \( \sum_j \) is over all the particles and the \( \sum_I \) over all energy eigenstates. It is viewed as the distribution of the strength of the state \( \sum_j e^{i q \cdot r_j} | 0 \rangle \) over the energy eigenstates of the system having momentum \( q \). It is not necessarily zero in the timelike, \( \nu > |q| \) region.

The natural scaling variable in the MBT approach to DIS is \( \tilde{y} = \nu - |q| \). At large \(|q|\) the response is expected to depend only on \( \tilde{y} \), and not on \( q \) and \( \nu \) independently. This variable is equivalent to the Nachtmann variable \( \xi \) since

\[
\xi = \frac{1}{M} (|q| - \nu) = -\frac{1}{M} \tilde{y}. \tag{3}
\]

In the limit of large \( Q^2 \) the \( \xi = x \), thus \( \tilde{y} \) scaling includes Bjorken scaling. However, both \( \tilde{y} \) and \( \xi \) span both spacelike and timelike regions at fixed \(|q|\) unlike \( x \) at fixed \( Q^2 \).

The particle number sum rule in MBT is obtained by integrating the response at large \(|q|\) over all \( \nu > 0 \):

\[
\int_0^\infty R(q, \nu) d\nu = \sum_I \langle 0 | \sum_i e^{-i q \cdot r_i} | I \rangle \langle I | \sum_j e^{i q \cdot r_j} | 0 \rangle = \sum_{i,j} \langle 0 | e^{i q \cdot (r_j - r_i)} | 0 \rangle. \tag{4}
\]

When \( q \) is large only the \( i = j \) terms in the above sum contribute, and therefore the integral gives the number of particles in the system. In contrast the sums of the response in the parton model are obtained by integrating the response over \( \xi > 0 \) at fixed \( Q^2 \). These sums
FIG. 2: $E_{n,\ell}^2$ in GeV$^2$ plotted against angular momentum $\ell$ for the first ten values of the radial quantum number $n$. The lines are linear fits to the calculated values.

will fulfill the particle number sum rule only if the response in the timelike region is zero. As mentioned earlier, the response of a collection of noninteracting particles lies in the spacelike region. Interaction effects, however, can shift a part of the strength to the timelike region. Evidence for shifts caused by interactions is discussed in Ref. [3].

We have studied the exact response of a simple “toy” model which contains the basic features of relativity and confinement to obtain further insights on the possible response in the timelike region and it’s effects on the sum rules. In this model we assume that the response of the hadron is due to a single light valence quark confined within the hadron by its interaction with an infinitely massive color charge. We model this interaction by a linear flux-tube potential, and use the single particle Hamiltonian,

$$H = \sqrt{|p|^2 + m_q^2 + \sqrt{\sigma} r}$$

containing the relativistic kinetic energy operator. In the limit $m_q = 0$ used here, the $H$ can be cast in the form:

$$H = \sigma^{1/4} \left( \sqrt{|p'|^2 + r'} \right),$$

where $p' = p/\sigma^{1/4}$, and $r' = \sigma^{1/4} r$ are dimensionless. The response $R(|q|, \nu)$ of the model then depends only on the dimensionless variables $|q'| = |q|/\sigma^{1/4}$ and $\nu' = \nu/\sigma^{1/4}$. The main
FIG. 3: The response for $|q| = 10$ GeV versus the variable, $\tilde{y} = \nu - |q|$, for various $\Gamma_0$.

conclusions of this work are independent of the assumed value of $\sigma$; however, we show results in familiar units using the typical value $\sqrt{\sigma} = 1$ GeV/fm.

The model may be viewed as that of a meson with a heavy antiquark or that of a baryon with a heavy diquark. It is obviously too simple to address the observed response of hadrons. For example, it omits the sea quarks and radiative gluon effects contained in the DGLAP equations [1, 2] to describe scaling violations. Nevertheless its exact solutions are interesting and useful to study scaling, the approach to scaling, and the contribution of the timelike region to sum rules. A similar model has been considered by Isgur et al. [3].

The eigenstates of this Hamiltonian have the usual quantum numbers $n, \ell, \text{ and } m$. They are expanded in the basis of $j_\ell(k_\alpha r)Y^m_\ell(\hat{r})$. For each value of $\ell \leq 100$ we consider 200 ($\alpha = 1, \ldots, 200$) basis states each having $j_\ell(k_\alpha R) = 0$, for $R = 10$ fm, and 300 basis states for $R = 15$ fm. The Hamiltonian is diagonalized within a sphere of radius $R$. The ground and relevant excited states have rather small radii, and are not influenced by this boundary condition. For example, the energies of the lowest 100 eigenstates of each $\ell$ and the calculated responses at $|q| \leq 10$ GeV are essentially the same for $R = 10$ and 15 fm. Figure 2 shows a plot of the square of the energy eigenvalue for a given $n$ versus $\ell$. The calculated energies lie on linear Regge trajectories having slopes within 10% of the classical estimate $4\sqrt{\sigma}$. 
FIG. 4: The response for values of $|q| \geq 3$ GeV versus the scaling variable, $\tilde{y} = \nu - |q|$. 

The response is calculated including all states with $n \leq 100$ and $\ell \leq 100$ in Eq. (2). It is a sequence of $\delta$ functions at $\nu = E_{n,\ell} - E_0$. We verify that the full strength of the integrated response, 

$$\int_0^{\infty} R(q, \nu) d\nu = 1,$$

is obtained in this truncated basis for all values of the momentum transfer considered in this work with $< 0.02 \%$ error. In order to obtain a smooth response we assume decay widths for all the excited states dependent on the excitation energy $\nu$:

$$\Gamma(\nu \geq E_t) = \Gamma_0 \left(1 - e^{-\left(\nu - E_t\right)/E_s}\right).$$

Here $E_t$ is the threshold excitation energy for meson emission below which the width is zero, and $E_s$ parameterizes the approach to a constant width $\Gamma_0$ at $\nu > E_t$. We use $E_t = E_s = 100$ MeV and various values of $\Gamma_0$ for illustration. Only the response at small values of $|q|$ is sensitive to $E_t$ and $E_s$. The response including decay widths is given by:

$$R(q, \nu) = \sum_l |\langle I| e^{i q \cdot r} |0 \rangle|^2 \left(\frac{\Gamma(\nu)}{2\pi}\right) \frac{1}{(E_I - E_0 - \nu)^2 + \frac{1}{4} \Gamma^2(\nu)}.$$

The dependence of the response at $|q| = 10$ GeV on $\Gamma_0$ is shown in Fig.3. Since this response has $\nu \gg E_t$ and $E_s$, it depends only on $\Gamma_0$. 


FIG. 5: The approach to scaling of the response for values of $|q| \leq 2$ GeV and $|q| = 10$ GeV versus the scaling variable, $\tilde{y} = \nu - |q|$.

Figure 4 shows the response calculated for values of $|q| \geq 3$ GeV as a function of $\tilde{y}$ for $\Gamma_0 = 100$ MeV. The scaling behavior is clearly exhibited; at large $|q|$ the $R(|q|, \nu)$ becomes a function $f(\tilde{y})$ alone. This scaling is equivalent to $\xi$ scaling via (Eq. 3).

In Fig. 5 we show the response for $\Gamma_0 = 100$ MeV, at various values of $|q| \leq 2$ GeV compared with that for $|q| = 10$ GeV, to study the approach to scaling. At small $|q|$ the scattering is dominated by resonances, and the first inelastic peak is due to the lowest excited state with $n = 1$ and $\ell = 1$, 335 MeV above the ground state. In our toy model, the elastic scattering occurs at $\nu = 0$ or $\tilde{y} = -|q|$, since our hadron is heavy. This elastic scattering contribution is omitted from Fig. 5. It has a strength of $\{0.85, 0.53, 0.26, 0.10, 0.010, 0.00086\}$ for $|q| = \{0.25, 0.5, 0.75, 1.0, 1.5, 2.0\}$ GeV. The energy dependence of $\Gamma$ [Eq. (8)] implies that the inelastic response is zero for $\nu < E_t$ or equivalently $\tilde{y} < E_t - |q|$.

For $\tilde{y} \sim 0$, i.e. for small $\xi$, the response approximately scales at relatively small values of $|q|$, comparable to $\sigma^{1/4}$. As $|q|$ increases, the range over which scaling occurs is extended to more negative values of $\tilde{y}$, i.e. to larger values of $\xi$. The contribution of each resonance shifts to lower $\tilde{y}$ and decreases in magnitude following the $R(|q| \rightarrow \infty, \tilde{y})$. This behavior is seen in the experimental data on the proton and deuteron and interpreted as evidence for quark-hadron duality. Thus the toy model seems to describe some of the observed properties.
of the DIS response of nucleons. It exhibits \( \bar{y} \) or equivalently \( \xi \) scaling at large \( |q| \) as observed [3], and an approach to \( \xi \) scaling similar to that seen in recent experiments.

The \( R(|q|, \nu) \), and therefore the \( f(\bar{y}) \) extend into the timelike \( (\bar{y} > 0) \) region. The sum-rule given by Eq.(7), counts the number of particles in the target. It is necessary to integrate over the timelike region to fulfill this sum rule. In the limit \( \Gamma_0 = 0 \) about 9.6% of the sum is in that region independent of \( \sqrt{\sigma} \). It increases to 13.7% for \( \Gamma_0 = 100 \text{ MeV} \) and \( \sqrt{\sigma} = 1 \text{ GeV/fm} \). The response expressed as \( R(Q^2, \xi) \) also scales at large \( Q^2 \) where \( |q| \) is necessarily large. It becomes a function of \( \xi \) alone. However, the integral:

\[
\int_0^{\infty} R(Q^2 \to \infty, \xi) d\xi = \int_0^{\infty} R(|q| \to \infty, \nu) d\nu \leq 0.904, \tag{10}
\]

because the contribution of the timelike region is omitted. The maximum value, 0.904 of the integral is obtained for \( \Gamma_0 = 0 \). Here we have defined \( \xi = |q| - \nu \) without the conventional \( 1/M \) scale [Eq.(3)].

The baryon number, or Gross-Llewellyn Smith (GLS) \( S_{GLS} \) sum, is obtained by integrating the neutrino scattering responses over the range \( 0 < x < 1 \). In absence of any timelike response, in the limit \( Q^2 \to \infty \), \( S_{GLS} = 3 \), the number of quarks minus the number antiquarks in a nucleon. The GLS sum has gluon radiative corrections that have been calculated at next-to next-to leading order (i.e. up to \( O(\alpha_3^3) \)) using pQCD [9]. The experimental values of \( S_{GLS} \) obtained from revised CCFR \( xF_3 \) are \( 2.55 \pm 0.06(\text{stat}) \pm 0.10(\text{syst}) \) and \( 2.80 \pm 0.13(\text{stat}) \pm 0.18(\text{syst}) \) at \( Q^2 = 3.16 \) and \( 12.59 \text{ GeV}^2 \), respectively. They have been used to determine \( \alpha_s(Q^2) \) assuming that the difference between 3 and the above experimental values is entirely due to gluon radiative effects.

Recent measurements of the Drell-Yan cross section ratios, \( \sigma(p + d)/\sigma(p + p) \) from E866 at Fermilab [11, 12] determine the ratio of \( \bar{d}(x)/\bar{u}(x) \) and, in turn, the integral,

\[
\int_0^1 dx [\bar{u}(x) - \bar{d}(x)] = -0.118 \pm 0.012 \tag{11}
\]

to a high level of precision. The valence isospin, or Gottfried, sum

\[
S_G = \int_0^1 \frac{dx}{x} [F_2^p(x) - F_2^n(x)] = \frac{1}{3} + \frac{2}{3} \int_0^1 dx \left[ \bar{u}(x) - \bar{d}(x) \right] \tag{12}
\]

has been measured by the NM Collaboration [13]. The analysis of their complete data set gives \( S_G = 0.216 \pm 0.027 \) [14, 13] essentially independent of \( Q^2 \) on the interval \( 0.5 < Q^2 < 10 \).
GeV$^2$. Using these measurements we find a contribution due to valence quarks of,

\[ S_G - \frac{2}{3} \int_0^1 dx \left[ \bar{u}(x) - \bar{d}(x) \right] = 0.294 \pm 0.030, \quad (13) \]

about 10% below the expected result of $\frac{1}{3}$, with comparable error.

The present work suggests that these sums over the spacelike region could be smaller than the theoretical expectation due to some of the response being shifted into the timelike region. Such a shift is due to the bound nature of the quarks in the hadron and is therefore a nonperturbative effect.

The prevalent models of valence quark structure functions [1] predict a large valence quark response in the spacelike region adjacent to the photon line which diverges as $\sim x^{-0.5}$. Even though the $x < 0.1$ region is dominated by sea quarks, contributions from this region make up a large part of the experimental sums; about 50% of $S_{GLS}$ [14] and about 30% of $S_G$ [17]. Response having $\xi < 0.1$ is within $< 0.1 M \sim 94$ MeV of the photon line. It is possible that there is some response beyond the photon line, moved into the timelike region due to nonperturbative effects associated with binding.

It is known from $e^+e^-$ annihilation experiments that the vacuum has timelike response, and the nucleon presumably couples to it. In the toy model we have ignored this problem in order to focus on the possibility that the nucleon can have timelike response due to the bound quarks in the nucleon being off the mass-shell.

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