Research Article

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Sharp estimates on the first Dirichlet eigenvalue of nonlinear elliptic operators via maximum principle

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Abstract: In this paper, we study optimal lower and upper bounds for functionals involving the first Dirichlet eigenvalue \( \lambda_F(p, \Omega) \) of the anisotropic \( p \)-Laplacian, \( 1 < p < +\infty \). Our aim is to enhance, by means of the \( P \)-function method, how it is possible to get several sharp estimates for \( \lambda_F(p, \Omega) \) in terms of several geometric quantities associated to the domain. The \( P \)-function method is based on a maximum principle for a suitable function involving the eigenfunction and its gradient.

Keywords: Dirichlet eigenvalues, anisotropic operators, optimal estimates

MSC 2010: 35P30, 49Q10

1 Introduction

Given a bounded domain \( \Omega \subset \mathbb{R}^N \) and \( p \in ]1, +\infty[ \), let us consider the first Dirichlet eigenvalue of the anisotropic \( p \)-Laplacian, that is,

\[
\lambda_F(p, \Omega) = \min_{\psi \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} F(\nabla \psi)^p \, dx}{\int_{\Omega} |\psi|^p \, dx},
\]

where \( F: \mathbb{R}^N \rightarrow [0, +\infty[ \), \( N \geq 2 \), is a convex, even, \( 1 \)-homogeneous \( C^3, \beta(\mathbb{R}^N \setminus \{0\}) \)-function such that \( [F^p]_\xi \) is positive definite in \( \mathbb{R}^N \setminus \{0\} \), \( 1 < p < +\infty \). We are interested in the study of optimal lower and upper bounds for functionals involving \( \lambda_F(p, \Omega) \). In this order of ideas, our aim is to enhance how these estimates may be obtained as a consequence of a maximum principle for a function which involves an eigenfunction and its gradient, namely, the so-called \( P \)-function, introduced by L.E. Payne in the case of the classical Euclidean Laplace operator. We refer the reader to the book by Sperb [29] and the references contained therein for a survey on the \( P \)-function method in the Laplacian case and its applications. More precisely, if \( u \) is a positive eigenfunction associated to \( \lambda_F(p, \Omega) \), we introduce the function

\[
P := (p - 1)F^p(\nabla u) + \lambda_F(p, \Omega)(u^p - M^p),
\]

where \( M \) is the maximum value of \( u \). We show that the function \( P \) verifies a maximum principle in \( \overline{\Omega} \) in order to get a pointwise estimate for the gradient in terms of \( u \). This is the starting point to prove several useful
bounds, involving quantities which depend on the domain $\Omega$. As a matter of fact, the use of the $P$-function method in the anisotropic setting has been studied in the recent paper [14]. Here, the authors consider the $p$-anisotropic torsional rigidity $T_{F}^{p^{-1}}(p, \Omega) = \frac{1}{\Pi_{F}(\Omega)^{p}}$ and show optimal bounds for two functionals involving $T_{F}^{p^{-1}}(p, \Omega)$ and some geometric quantities related to the domain. In this spirit, we aim to analyze the case of the eigenvalue problem. Given a convex, bounded domain $\Omega \subset \mathbb{R}^{N}$, our main results can be summarized as follows: We prove the anisotropic version of the Hersch inequality for $\lambda_{F}(p, \Omega)$, namely, that

$$\lambda_{F}(p, \Omega) \geq \left( \frac{\pi_{p}}{2} \right)^{p} \frac{1}{R_{F}(\Omega)^{p}}$$

(1.3)

where $R_{F}(\Omega)$ is the anisotropic inradius defined in Section 2, and

$$\pi_{p} := \frac{(p-1)^{\frac{p}{2}}}{2} \int_{0}^{1} \frac{dt}{1 - t^{\frac{p}{p-1}}} = 2\pi \left( p - 1 \right)^{\frac{p}{2}} \frac{1}{p \sin \frac{\pi}{p}}.$$ 

(1.4)

Regarding the Euclidean setting, for $p = 2$, inequality (1.3) has been proved by Hersch [18], improved in [27] and generalized for any $p$ in [19] (see also [26]). In the general anisotropic case or $p = 2$, it has been studied in [6, 33]. Another consequence of the maximum principle for $P$ that we obtain is the inequality

$$\left( \frac{p-1}{p} \right)^{p^{-1}} \left( \frac{\pi_{p}}{2} \right)^{p} \leq \lambda_{F}(p, \Omega) M_{v_{\Omega}}^{p^{-1}},$$

(1.5)

where $v_{\Omega}$ is the positive maximizer of (1.2) such that

$$T_{F}(p, \Omega) = \int_{\Omega} v_{\Omega} \, dx,$$

and $M_{v_{\Omega}}$ is the maximum of $v_{\Omega}$. Inequality (1.5), in the Euclidean case ($p = 2$), has been first proved in [24] and then studied also, for instance, in [17, 29, 31].

The last main result we show is the following: Let $u$ be a first eigenfunction relative to $\lambda_{F}(p, \Omega)$, and consider the so-called anisotropic “efficiency ratio”

$$E_{F}(p, \Omega) := \frac{\|u\|_{p^{-1}}}{|\Omega|^{\frac{1}{p-1}} \|u\|_{\infty}}.$$

Then we prove that

$$E_{F}(p, \Omega) \leq \frac{1}{(p-1)^{\frac{p}{2}}} \left( \frac{2}{\pi_{p}} \right)^{p},$$

(1.6)

where $\pi_{p}$ is defined in (1.4). In the Euclidean case and $p = 2$, this inequality is due to Payne and Stakgold, who proved it in [25].

Finally, we show the optimality in (1.3) and (1.5), while the optimality of (1.6) in the class of convex sets is still an open problem.

As a matter of fact, the convexity assumption in (1.3), (1.5) and (1.6) can be weakened, since they are also valid in the case of smooth domains with anisotropic nonnegative mean curvature (see Section 2 for the definition).

In the present paper, we also emphasize the relation of $\lambda_{F}(p, \Omega)$ to the so-called anisotropic Cheeger constant $h_{F}(\Omega)$ (see Section 3 for the definition). Indeed, in the class of convex sets, we prove the validity of a Cheeger-type inequality for $\lambda_{F}$, as well as a reverse Cheeger inequality.

The paper is organized as follows: In Section 2, we fix the notation and recall some basic facts regarding the eigenvalue problem for the anisotropic $p$-Laplacian and the torsional rigidity $T_{F}(p, \Omega)$. Section 3 is
devoted to the study of $h_\rho(\Omega)$. More precisely, we recall the definition and the main properties, and we prove optimal lower and upper bounds for $h_\rho(\Omega)$ in terms of the anisotropic inradius $R_\rho(\Omega)$ of a convex set $\Omega$. In Section 4, we prove that the $\phi$-function in (1.1) verifies a maximum principle. Finally, in Section 5, we prove the quoted results (1.3), (1.5), (1.6) and a reverse Cheeger inequality investigating also the optimality issue.

\section{Notation and preliminaries}

Throughout the paper, we will consider a convex, even, 1-homogeneous function

$$\xi \in \mathbb{R}^N \mapsto F(\xi) \in [0, +\infty[,$$

that is, a convex function such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R}, \ \xi \in \mathbb{R}^N,$$

and such that

$$a|\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^N,$$

for some constant $a > 0$. The hypotheses on $F$ imply that there exists $b \geq a$ such that

$$F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^N.$$ 

Moreover, throughout the paper, we will assume that $F \in C^{\beta}(\mathbb{R}^N \setminus \{0\})$, and

$$[F^p]\xi(\xi)$$

is positive definite in $\mathbb{R}^N \setminus \{0\}$

with $1 < p < +\infty$.

The hypothesis (2.3) on $F$ ensures that the operator

$$\Omega_p u := \text{div}\left(\frac{1}{p} \nabla \xi |F^p|(\nabla u)\right)$$

is elliptic, hence there exists a positive constant $\gamma$ such that

$$\frac{1}{p} \sum_{i=1}^n \nabla_\xi^2[F^p](\eta)\xi_i\xi_j \geq \gamma|\eta|^{p-2}|\eta|^2$$

for any $\eta \in \mathbb{R}^n \setminus \{0\}$ and for any $\xi \in \mathbb{R}^n$. The polar function $F^\circ: \mathbb{R}^N \to [0 , +\infty[\ 0 , +\infty[\ 0$ of $F$ is defined as

$$F^\circ(\nu) = \sup_{\xi \neq 0} \frac{\langle \xi, \nu \rangle}{F(\xi)}.$$ 

It is easy to verify that also $F^\circ$ is a convex function which satisfies properties (2.1) and (2.2). Furthermore,

$$F(\nu) = \sup_{\xi \neq 0} \frac{\langle \xi, \nu \rangle}{F^\circ(\xi)}.$$ 

From the above property, it holds that

$$|\langle \xi, \eta \rangle| \leq F(\xi)F^\circ(\eta) \quad \text{for all } \xi, \eta \in \mathbb{R}^N.$$ 

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^N : F^\circ(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin. We put $\kappa_N = |\mathcal{W}|$, where $|\mathcal{W}|$ denotes the Lebesgue measure of $\mathcal{W}$. More generally, we denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is, the Wulff shape centered at $x_0$ with measure $\kappa_N r^N$ and $\mathcal{W}_r(0) = \mathcal{W}_r$.

The following properties of $F$ and $F^\circ$ hold true:

$$\langle F(\xi), \xi \rangle = F(\xi), \quad \langle F^\circ(\xi), \xi \rangle = F^\circ(\xi) \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\};$$

$$F(F^\circ(\xi)) = F^\circ(F(\xi)) = 1 \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\},$$

$$F^\circ(\xi)F(\xi) = F(\xi)F^\circ(\xi) = \xi \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$
2.1 Anisotropic mean curvature

Let \( \Omega \) be a \( C^2 \)-bounded domain, let \( n_E(x) \) be the unit outer normal at \( x \in \partial \Omega \), and let \( u \in C^2(\overline{\Omega}) \) such that \( \Omega_t = \{ u > t \}, \partial \Omega_t = \{ u = t \} \) and \( \nabla u \neq 0 \) on \( \partial \Omega_t \). The anisotropic outer normal \( n_F \) to \( \partial \Omega_t \) is given by

\[ n_F(x) = F_{x}(n_E(x)) = F_{x}(-\nabla u), \quad x \in \partial \Omega. \]

It holds

\[ F^0(n_F) = 1. \]

The anisotropic mean curvature of \( \partial \Omega_t \) is defined as

\[ \mathcal{H}_F(x) = \text{div}(n_F(x)) = \text{div}[\nabla_F(-\nabla u(x))], \quad x \in \partial \Omega_t. \]

It holds that

\[ \frac{\partial u}{\partial n_F} = \nabla u \cdot n_F = \nabla u \cdot F_x(-\nabla u) = -F(\nabla u). \tag{2.5} \]

In [14], it has been proved that, for a smooth function \( u \) on its level sets \( \{ u = t \} \), it holds

\[ \Omega_p u = F^{p-2}(\nabla u)\left( \frac{\partial u}{\partial n_F} \mathcal{H}_F + (p - 1)\frac{\partial^2 u}{\partial n^2_F} \right). \tag{2.6} \]

Finally, we recall the definition of the anisotropic distance from the boundary and the anisotropic inradius. Let us consider a bounded domain \( \Omega \), that is a connected, open set of \( \mathbb{R}^N \) with nonempty boundary. The anisotropic distance of \( x \in \Omega \) to the boundary of \( \partial \Omega \) is the function

\[ d_F(x) = \inf_{y \in \partial \Omega} F^0(x - y), \quad x \in \Omega. \]

We stress that when \( F = | \cdot | \), then \( d_F = d_E \), the Euclidean distance function from the boundary. It is not difficult to prove that \( d_F \in W^{1,\infty}_0(\Omega) \), and using the property of \( F \), we have

\[ F(\nabla d_F(x)) = 1 \quad \text{a.e. in } \Omega. \tag{2.7} \]

Moreover, we recall that \( \Omega \) is convex, and the anisotropic distance function is concave. The quantity

\[ R_F(\Omega) = \sup \{ d_F(x), x \in \Omega \} \tag{2.8} \]

is called the anisotropic inradius of \( \Omega \). For further properties of the anisotropic distance function, we refer the reader to [10].

2.2 The first Dirichlet eigenvalue for \( \Omega_p \)

Let \( \Omega \) be a bounded, open set in \( \mathbb{R}^N \), \( N \geq 2, 1 < p < +\infty \), and consider the eigenvalue problem

\[ \begin{cases} -\Omega_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \tag{2.9} \]

The smallest eigenvalue, denoted by \( \lambda_F(p, \Omega) \), has the following well-known variational characterization:

\[ \lambda_F(p, \Omega) = \min_{\varphi \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} F^p(\nabla \varphi) \, dx}{\int_{\Omega} |\varphi|^p \, dx}. \tag{2.10} \]

The following two results which enclose the main properties of \( \lambda_F(p, \Omega) \) hold true. We refer the reader, for example, to [4, 15].

**Theorem 2.1.** If \( \Omega \) is a bounded, open set in \( \mathbb{R}^N \), \( N \geq 2 \), there exists a function \( u_1 \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega}) \) which achieves the minimum in (2.10) and satisfies the problem (2.9) with \( \lambda = \lambda_F(p, \Omega) \). Moreover, if \( \Omega \) is connected, then \( \lambda_F(p, \Omega) \) is simple, that is, the corresponding eigenfunctions are unique up to a multiplicative constant, and the first eigenfunctions have constant sign in \( \Omega \).
In the following proposition, the scaling and monotonicity properties of $\lambda_F(p, \Omega)$ are recalled.

**Proposition 2.2.** Let $\Omega$ be a bounded, open set in $\mathbb{R}^N$, $N \geq 2$. Then the following properties hold:

1. For $t > 0$, it holds $\lambda_F(p, t\Omega) = t^p \lambda_F(p, \Omega)$.
2. If $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, then $\lambda_F(p, \Omega_1) \leq \lambda_F(p, \Omega_2)$.
3. For all $1 < p < s < +\infty$, we have $p[\lambda_F(p, \Omega)]^{\frac{1}{p}} < s[\lambda_F(s, \Omega)]^{\frac{1}{s}}$.

### 2.3 Anisotropic $p$-torsional rigidity

In this subsection, we summarize some properties of the anisotropic $p$-torsional rigidity. We refer the reader to [12] for further details.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, and let $1 < p < +\infty$. Throughout the paper, we will denote by $q$ the Hölder conjugate of $p$,

$$q := \frac{p}{p - 1}.$$

Let us consider the torsion problem for the anisotropic $p$-Laplacian

$$\begin{cases}
-\Omega_p \nu := -\text{div}(F^{p-1}(\nabla \nu)F(\nabla \nu)) &= 1 \quad \text{in } \Omega, \\
\nu &= 0 \quad \text{on } \partial \Omega.
\end{cases} \quad (2.11)$$

By classical result, there exists a unique solution of (2.11), that we will always denote by $\nu_{\Omega}$, which is positive in $\Omega$. Moreover, by (2.3) and letting $F \in C^1(\mathbb{R}^n \setminus \{0\})$, then $\nu_{\Omega} \in C^{1,q}(\Omega) \cap C^3(\{\nabla \nu_{\Omega} \neq 0\})$ (see [21, 30]).

The anisotropic $p$-torsional rigidity of $\Omega$ is

$$T_F(p, \Omega) = \int_{\Omega} F(\nabla \nu_{\Omega})^p \, dx = \int_{\Omega} \nu_{\Omega} \, dx.$$

The following variational characterization for $T_F(p, \Omega)$ holds:

$$T_F(p, \Omega)^{p-1} = \max_{\psi \in W_0^{1,p}(\Omega) \setminus \{0\}} \left( \int_{\Omega} |\psi| \, dx \right)^p \int_{\Omega} F(\nabla \psi)^p \, dx,$$  \hspace{1cm} (2.12)

and the solution $\nu_{\Omega}$ of (2.11) realizes the maximum in (2.12).

By the maximum principle, $M_{\nu_{\Omega}} \leq M_{\nu_0}$ holds, where $M_{\nu_0}$ is the maximum of the torsion function in $\Omega$. Finally, we recall the following estimates for $M_{\nu_0}$ contained in [14].

**Theorem 2.3.** Let $\Omega$ be a bounded, convex open set in $\mathbb{R}^N$, and $R_F$ the anisotropic inradius defined in (2.8). Then

$$\frac{R_F^q(\Omega)}{qN^{q-1}} \leq M_{\nu_0} \leq \frac{R_F^q(\Omega)}{q}. \quad (2.13)$$

### 3 Anisotropic Cheeger constant

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The total variation of a function $u \in BV(\Omega)$ with respect to $F$ is (see [3])

$$\int_{\Omega} |\nabla u|_F = \sup \left\{ \int_{\Omega} u \, \text{div} \, \sigma \, dx : \sigma \in C_0^1(\Omega; \mathbb{R}^N), F^o(\sigma) \leq 1 \right\}.$$

This yields the following definition of anisotropic perimeter of $K \subset \mathbb{R}^N$ in $\Omega$:

$$P_F(K) = \int_{\mathbb{R}^N} |\nabla \chi_K|_F = \sup \left\{ \int_K \text{div} \, \sigma \, dx : \sigma \in C_0^1(\mathbb{R}^N; \mathbb{R}^N), F^o(\sigma) \leq 1 \right\}. \quad (3.1)$$
It holds that
\[
P_F(K) = \int_{\partial^* K} F(n_E) \, d\mathcal{H}^{N-1},
\]
where \(\mathcal{H}^{N-1}\) is the \((N-1)\)-dimensional Hausdorff measure in \(\mathbb{R}^N\), \(\partial^* K\) is the reduced boundary of \(F\) and \(n_E\) is the Euclidean unit outer normal to \(K\) (see [3]).

An isoperimetric inequality for the anisotropic perimeter holds, namely, \(\mathcal{W}_R\) is the Wulff shape such that \(|\mathcal{W}_R| = |K|\), then
\[
P_F(K) \geq P_F(\mathcal{W}_R) = N\kappa_n |K|^{1 - \frac{1}{N}},
\]
and the equality holds if and only if \(\Omega\) is a Wulff shape (see for example [2, 5, 16]). The following lemma will play a key role in order to investigate on optimality issue of the quoted results.

**Lemma 3.1.** Let \(\Omega_{a,k} = [a, a + k] \times [-k, k]^N\) be an \(N\)-rectangle in \(\mathbb{R}^N\), and suppose that \(R_F(\Omega_{a,k}) = a F^0(e_1)\). Then
\[
\lim_{k \to +\infty} \frac{P_F(\Omega_{a,k})}{|\Omega_{a,k}|} = \frac{1}{a F^0(e_1)}.
\]

**Proof.** First observe that (see [14])
\[
F^0(e_1) F(e_1) = 1.
\]
By definition of anisotropic perimeter, we get
\[
\frac{P_F(\Omega_{a,k})}{|\Omega_{a,k}|} = \frac{2(2k)^{N-1} F(e_1) + O(k^{N-2})}{2^N k^{N-1} a},
\]
hence, using (3.4) and passing to the limit, we get (3.3). The anisotropic Cheeger constant associated to an open, bounded set \(\Omega \subseteq \mathbb{R}^N\) is defined as
\[
h_F(\Omega) = \inf_{K \subseteq \Omega} \frac{P_F(K)}{|K|}.
\]
We recall that for a given bounded, open set in \(\mathbb{R}^N\), the Cheeger inequality states that
\[
\lambda_F(p, \Omega) \geq \left( \frac{h_F(\Omega)}{p} \right)^p.
\]
This inequality, well known in the Euclidean case after the paper by Cheeger ([8]) for \(p = 2\), has been proved in [20] in the anisotropic case. We refer the reader to [22] and the references contained therein for a survey on the properties of the Cheeger constant in the Euclidean case.

It is known (see [20] and the references therein) that if \(\Omega\) is a Lipschitz bounded domain, there exists a Cheeger set, that is, a set \(K_\Omega\) for which
\[
h_F(\Omega) = \frac{P_F(K_\Omega)}{|K_\Omega|}.
\]
When \(\Omega = \mathcal{W}_R\), we immediately get \(K_{\mathcal{W}_R} = \mathcal{W}_R\) and
\[
h_F(\mathcal{W}_R) = \frac{N}{R}.
\]
We observe that usually the Cheeger set \(K_\Omega\) is not unique, nevertheless, \(\Omega\) is convex (see, for instance, [1, 7, 20]).

**Theorem 3.2.** If \(\Omega\) is a bounded, convex domain, there exists a unique convex Cheeger set.

The next results give an upper bound for the Cheeger constant in terms of the anisotropic inradius of \(\Omega\).

**Proposition 3.3.** If \(\Omega\) is a bounded, open set in \(\mathbb{R}^N\), then
\[
h_F(\Omega) \leq \frac{N}{R_F(\Omega)}.
\]
Moreover, the equality holds if \(\Omega\) is a Wulff shape.
Proof. By definition, the constant $h_F(\Omega)$ is monotonically decreasing with respect to the set inclusion. Then, by (3.6) and the definition of anisotropic inradius, we get inequality (3.7).\hfill\Box

Regarding a lower bound for the anisotropic Cheeger constant in terms of the inradius of $\Omega$, we have the following:

**Proposition 3.4.** If $\Omega$ is a bounded, open, convex set in $\mathbb{R}^N$, then

$$\frac{1}{R_F(\Omega)} \leq h_F(\Omega).$$

(3.8)

Moreover, the inequality is optimal for a suitable sequence of $N$-rectangular domains.

Proof. Using (2.7), (2.1) and the coarea formula, we have, for a bounded, convex set $K \subseteq \Omega$, that

$$|K| = \int_{\Omega} F(\nabla d_F) \, dx = \int_0^{R_F(K)} \int_{d_F = t} F(n_E) \, d\sigma = \int_0^{R_F(K)} P_F(d_F \leq t) \, dt \leq P_F(K) R_F(K).$$

Hence, since $K$ is convex, $R_F(K) \leq R_F(\Omega)$, then

$$\frac{P_F(K)}{|K|} \geq \frac{1}{R_F(\Omega)}.$$

Passing to the infimum on $K$, we get (3.8). The optimality follows immediately from (3.3).\hfill\Box

More generally, for convex sets, the following result holds (see also, for instance, [13], where the case $N = 2$ is given with a different proof):

**Proposition 3.5.** If $\Omega \subset \mathbb{R}^N$ is a bounded, open, convex set, then

$$\frac{P_F(\Omega)}{|\Omega|} \leq \frac{N}{R_F(\Omega)}.$$

For the Wulff shape, the equality holds.

Proof. Let $x_0 \in \Omega$ be such that $R_F(\Omega) = d_F(x_0)$. By the concavity of $d_F$, we have

$$d_F(x_0) - d_F(x) \leq -\nabla d_F(x) \cdot (x - x_0) = n_E(x) \cdot (x - x_0)|\nabla d_F(x)|.$$

Hence, for $x \in \partial \Omega$, it holds that $R_F(\Omega) \leq n_E(x) \cdot (x - x_0)|\nabla d_F(x)|$. By the divergence theorem and observing also that $F(n_E) = \frac{1}{|\nabla n_E|^2}$, we have

$$|\Omega| = \int_{\Omega} \text{div}(x - x_0) \, dx = \frac{1}{N} \int_{\partial \Omega} (x - x_0) \cdot n_E(x) \, d\sigma \geq \frac{R_F(\Omega)}{N} \int_{\partial \Omega} \frac{1}{|\nabla d_F(x)|} \, d\sigma = \frac{R_F(\Omega) P_F(\Omega)}{N},$$

and this completes the proof.\hfill\Box

**Remark 3.6.** We observe that the inequality in the inequality of Proposition 3.5 holds, in general, also for other kinds of convex sets. For example, if $N = 2$ and $F = \mathcal{E}$, the equality holds for circles with two symmetrical caps (see, for instance, [28]).

An immediate consequence of the anisotropic isoperimetric inequality is the following:

**Theorem 3.7.** Let $\Omega$ be a bounded, open set. Then

$$h_F(\Omega) \geq h_F(W_R),$$

(3.9)

where $W_R$ is the Wulff shape such that $|\Omega| = |W_R|$, and the equality holds if and only if $\Omega$ is a Wulff shape.

Proof. Let $K \subseteq \Omega$. Then, if $|W_R| = |K|$, by (3.2), we have

$$\frac{P_F(K)}{|K|} \geq \frac{P_F(W_R)}{|W_R|} \geq \frac{P_F(W_R)}{|W_R|} = h(W_R).$$

Passing to the infimum on $K$, we get the result.\hfill\Box
Remark 3.8. Let \( \Omega \) be an open, bounded set of \( \mathbb{R}^N \). Then inequality (3.7) implies
\[
h_F(\Omega) - h_F(W_R) \leq N \left( \frac{1}{R_f(\Omega)} - \frac{1}{R} \right). \tag{3.10}
\]

When \( \Omega \) is convex, (3.10) can be read as a stability result for (3.9).

In [12], the following upper bound for \( \lambda_F(p, \Omega) \) is proved in terms of volume and anisotropic perimeter of \( \Omega \) for convex domains.

Theorem 3.9. Let \( \Omega \subset \mathbb{R}^N \) be a bounded, convex, open set. Then
\[
\lambda_F(p, \Omega) \leq \left( \frac{\pi_p}{2} \right)^p \left( \frac{P_F(\Omega)}{|\Omega|} \right)^p, \tag{3.11}
\]
where \( \pi_p \) is defined in (1.4) and \( P_F(\Omega) \) is the anisotropic perimeter of \( \Omega \) defined in (3.1).

The following reverse anisotropic Cheeger inequality holds (see [23] for the Euclidean case with \( N = p = 2 \)).

Proposition 3.10. Let \( \Omega \subset \mathbb{R}^N, N \geq 2 \), be a bounded, open, convex set. Then
\[
\lambda_F(p, \Omega) \leq \left( \frac{\pi_p}{2} \right)^p h_F(\Omega)^p, \tag{3.12}
\]
where \( \pi_p \) is defined in (1.4).

Proof. Let \( K_\Omega \subseteq \Omega \) be the convex Cheeger set of \( \Omega \). Since \( \lambda_F(\cdot, \cdot) \) is monotonically decreasing by set inclusion, then by (3.11), we have
\[
\lambda_F(p, \Omega) \leq \lambda_F(p, K_\Omega) \leq \left( \frac{\pi_p}{2} \right)^p \left( \frac{P_F(K_\Omega)}{|K_\Omega|} \right)^p = \left( \frac{\pi_p}{2} \right)^p h_F(\Omega)^p.
\]
The equality sign holds in the limiting case when \( \Omega \) approaches a slab. This will be shown in Theorem 5.8.

4 The \( \mathcal{P} \)-function

In order to give some sharp lower bound for \( \lambda_F(p, \Omega) \), we will use the so-called \( \mathcal{P} \)-function method. Let us consider the general problem
\[
\begin{cases}
-\nabla_p w = f(w) & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{cases} \tag{4.1}
\]
where \( f \) is a nonnegative \( C^1(0, +\infty) \cap C^0([0, +\infty[) \)-function, and define
\[
\mathcal{P}(x) := \frac{p-1}{p} F^p(\nabla w(x)) - \int_{w(x)}^{\max w} f(s) \, ds.
\]
The following result is proved in [9].

Proposition 4.1. Let \( \Omega \) be a domain in \( \mathbb{R}^N, N \geq 2 \), and let \( w \in W^{1,p}_0(\Omega) \) be a solution of (4.1). Set
\[
d_{ij} := \frac{1}{F(\nabla u_\Omega)} \partial_{\xi_i} \left[ \frac{F^p}{p} (\nabla u_\Omega) \right].
\]
Then it holds that
\[
(d_{ij} \mathcal{P}_i) - b_k \mathcal{P}_k \geq 0 \quad \text{in } \{ \nabla u_\Omega \neq 0 \},
\]
where
\[
b_k = \frac{p-2}{F^3(\nabla w)} F_{\xi_i}(\nabla w) \mathcal{P}_{\xi_i} F_{\xi_i}(\nabla w) + \frac{2p-3}{F^2(\nabla w)} \left( \frac{F_{\xi_i}(\nabla w) \mathcal{P}_{\xi_i}}{p-1} - f(w) F_{\xi_i}(\nabla w) \right).
\]
As a consequence of the previous result, we get the following maximum principle for $\mathcal{P}$.

**Theorem 4.2.** Let $\Omega$ be a bounded $C^{2,\alpha}$-domain in $\mathbb{R}^N$, $N \geq 2$, with nonnegative anisotropic mean curvature $\mathcal{H}_F \geq 0$ on $\partial \Omega$, and let $w > 0$ be a solution to the problem (4.1). Then

$$
\mathcal{P}(x) = \frac{p - 1}{p} F^p(\mathcal{V}w(x)) - \max_{w(x)} \int_{w(x)}^w f(s) \, ds \leq 0 \quad \text{in } \overline{\Omega},
$$

that is, the function $\mathcal{P}$ achieves its maximum at the points $x_M \in \Omega$ such that $w(x_M) = \max_{\partial \Omega} w$.

**Proof.** Let us denote by $\mathcal{C}$ the set of the critical points of $w$, that is, $\mathcal{C} = \{ x \in \overline{\Omega} : \nabla w(x) = 0 \}$. Being $\partial \Omega \in C^{2,\alpha}$, by the Hopf lemma (see, for example, [11]), $\mathcal{C} \cap \partial \Omega = \emptyset$.

Applying Proposition 4.1, the function $\mathcal{P}$ verifies a maximum principle in the open set $\Omega \setminus \mathcal{C}$. Then we have

$$
\max_{\Omega \setminus \mathcal{C}} \mathcal{P} = \max_{\partial (\Omega \setminus \mathcal{C})} \mathcal{P}.
$$

Hence, one of the following three cases occurs:

1. The maximum point of $\mathcal{P}$ is on $\partial \Omega$.
2. The maximum point of $\mathcal{P}$ is on $\mathcal{C}$.
3. The function $\mathcal{P}$ is constant in $\overline{\Omega}$.

In order to prove the theorem, we have to show that statement (1) cannot happen. Let us compute the derivative of $\mathcal{P}$ in the direction of the anisotropic normal $n_F$ in the sense of (2.5). Hence, on $\partial \Omega$, we get

$$
\frac{\partial \mathcal{P}}{\partial n_F} = \frac{p - 1}{p} \frac{\partial}{\partial n_F} \left( -\frac{\partial w}{\partial n_F} \right)^p + f(w) \frac{\partial w}{\partial n_F} = -(p - 1) \left( \frac{\partial w}{\partial n_F} \right)^{p-1} \frac{\partial^2 w}{\partial n_F^2} + f(w) \frac{\partial w}{\partial n_F}.
$$

where last identity follows by (2.6). On the other hand, if a maximum point $\mathcal{X}$ of $\mathcal{P}$ is on $\partial \Omega$, by the Hopf lemma, either $\mathcal{P}$ is constant in $\Omega$, or $\frac{\partial \mathcal{P}}{\partial n_F}(\mathcal{X}) > 0$. Hence, since $\mathcal{H}_F \geq 0$, we have a contradiction. \qed

**Remark 4.3.** Let $\Omega$ be a bounded, open, convex set, and let us consider $u$ a positive eigenfunction relative to the first eigenvalue $\lambda_F(p, \Omega)$ of problem (2.9). Then, with $M$ denoting $\max_{\partial \Omega} u$, inequality (4.2) becomes

$$
(p - 1) F^p(\mathcal{V}u) \leq \lambda_F(p, \Omega) (M^p - u^p) \quad \text{in } \overline{\Omega}.
$$

Integrating over $\Omega$ in both sides of (4.3) and recalling that $u$ satisfies problem (2.9), we get

$$
\int_{\Omega} u^p \leq \frac{M^p |\Omega|}{p}.
$$

By the definition of $\pi_p$ in (1.4), we have

$$
\frac{\pi_p}{2} = \int_0^{(p-1)\frac{\pi}{2}} \left( 1 - \frac{t^p}{p - 1} \right)^{-\frac{1}{p}} dt = \int_0^{M(p-1)\frac{\pi}{2}} \left( M^p - \frac{t^p}{p - 1} \right)^{-\frac{1}{p}} dt,
$$

where $u$ is a first positive eigenfunction of $-\mathcal{V}_p$. Let us consider the function

$$
\Phi(s) = \left( \frac{\pi_p}{2} \right)^{\frac{1}{p^*}} - \left( \int_{s(p-1)\frac{\pi}{2}}^{M(p-1)\frac{\pi}{2}} \frac{dt}{(M^p - \frac{t^p}{p - 1})^{\frac{1}{p}}} \right)^{\frac{1}{p^*}}, \quad s \in [0, M].
$$

**Proposition 4.4.** Let $\Omega$ be a bounded $C^2$-domain in $\mathbb{R}^N$, $N \geq 2$, with nonnegative anisotropic mean curvature $\mathcal{H}_F \geq 0$ on $\partial \Omega$. Then the following inequalities hold:

$$
\Phi(u(x)) \leq \frac{p}{p - 1} \lambda_F(p, \Omega)^{\frac{1}{p^*}} \mathcal{V}_\Omega(x),
$$

$$
\Phi'(u) F(\mathcal{V}u) \leq \frac{p}{p - 1} \lambda_F(p, \Omega)^{\frac{1}{p^*}} F(\mathcal{V}_\Omega) \quad \text{on } \partial \Omega,
$$

where $\mathcal{V}_\Omega$ is the stress function of $\Omega$. 

Proof. In order to prove (4.4), we will show that

$$-Q_p[\psi] \leq -Q_p\left[ \frac{p}{p-1} \lambda_F(p, \Omega)^{\frac{p-1}{p}} \nu_{\Omega} \right] = \lambda_F(p, \Omega)\left( \frac{p}{p-1} \right)^{p-1}.$$  \hfill (4.6)

By the comparison principle, being $\Phi(u) = \nu_{\Omega} = 0$ on $\partial \Omega$, then (4.4) holds.

With $\varphi(u)$ denoting

$$-\int_{u(p^{-1})}^{M(p^{-1})} \left( M^p - \frac{t^p}{p-1} \right)^{-\frac{1}{p}} dt,$$

we have

$$\Phi'(u) = q(p-1)^{\frac{1}{p}} \varphi(u)^{q-1}(M^p - u^p)^{\frac{1}{p}},$$

and

\[
\Phi''(u) = -q(q-1)(p-1)^{\frac{1}{p}} \varphi(u)^{q-2}(M^p - u^p)^{-\frac{1}{p}} + q(p-1)^{\frac{1}{p}} \varphi(u)^{q-1}(M^p - u^p)^{-\frac{1}{p} - 1} u^{p-1} \\
= q(p-1)^{\frac{1}{p}} \varphi(u)^{q-1}(M^p - u^p)^{-\frac{1}{p}} \left[ \frac{u^{p-1}}{M^p - u^p} - \frac{(q-1)(p-1)^{\frac{1}{p}} \varphi(u)^{-1}}{(M^p - u^p)^{\frac{1}{p}}} \right] \\
= \Phi'(u) \left[ \frac{u^{p-1}}{M^p - u^p} - \frac{(q-1)(p-1)^{\frac{1}{p}} \varphi(u)^{-1}}{(M^p - u^p)^{\frac{1}{p}}} \right] = \Phi'(u)\Psi(u),
\]

where we denoted the last square bracket with $\Psi(u)$. Then

$$Q_p \Phi(u) = \text{div}(\Psi) = q(p-1)F(\nabla u)^{p-1}F_{\xi}(\nabla u)$$

$$= (\Phi')^{p-1}Q_p u + (p-1)(\Phi')^{p-2} \Phi''(u) F(\nabla u)^p$$

$$= (\Phi')^{p-1}[-\lambda_F(p, \Omega) u^{p-1} + (p-1)F(\nabla u)^p \Psi(u)].$$

To prove the claim, we need to show that (4.6) holds, that is,

$$(\Phi')^{p-1}[-\lambda_F(p, \Omega) u^{p-1} + (p-1)F(\nabla u)^p \Psi(u)] + q^{p-1} \lambda_F(p, \Omega) \geq 0.$$  

Substituting, we get

$$-\lambda_F(p, \Omega) u^{p-1} + (p-1)F(\nabla u)^p \left[ \frac{u^{p-1}}{M^p - u^p} - \frac{(q-1)(p-1)^{\frac{1}{p}} \varphi(u)^{-1}}{(M^p - u^p)^{\frac{1}{p}}} \right] + \frac{\lambda_F(p, \Omega) [M^p - u^p]^{\frac{p-1}{p}}}{(p-1)^{\frac{1}{p}} \varphi(u)}$$

$$= [(p-1)^{-\frac{1}{p}} \varphi(u)^{-1} [M^p - u^p]^{1-\frac{1}{p}} - u^{p-1}] \left[ \lambda_F(p, \Omega) - \frac{(p-1)F(\nabla u)^p}{M^p - u^p} \right].$$

The function in the last square brackets is nonnegative by (4.2). To conclude, we show that the function

$$B(u) := [M^p - u^p]^{1-\frac{1}{p}} - (p-1)^{1-\frac{1}{p}} u^{p-1} \varphi(u)$$

is nonnegative. This is true, since $B(M) = 0$ and $B' \leq 0$. This concludes the proof of (4.4). Finally, by computing the derivative of $\Phi$ with respect to the anisotropic normal $n_F$ on $\partial \Omega = \{ u = 0 \}$, we have

$$\frac{\partial \Phi}{\partial n_F} = \nabla \Phi \cdot F_{\xi}(-\nabla u) = -\Phi'(u) F(\nabla u)$$

on $\partial \Omega$.

Recalling (4.4), by the Hopf lemma, we get

$$\frac{\partial \Phi}{\partial n_F} \geq \frac{p}{p-1} \lambda_F(p, \Omega)^{\frac{1}{p}} \frac{\partial \nu_{\Omega}}{\partial n_F}$$

on $\partial \Omega$.

Then

$$\Phi'(u) F(\nabla u) \leq \frac{p}{p-1} \lambda_F(p, \Omega)^{\frac{1}{p}} F(\nabla \nu_{\Omega})$$

on $\partial \Omega$,

which is (4.5), and this concludes the proof of the theorem. \hfill $\square$
5 Applications

Now we prove several inequalities involving \( \lambda_F(p, \Omega) \), \( R_F(\Omega) \), \( h_F(\Omega) \), \( M_{\nu_0} \), \( E_F(p, \Omega) \). The main estimates that we prove using the \( \Phi \)-function method (Theorem 5.1, Proposition 5.2, Theorem 5.3 and Theorem 5.6) are stated for \( C^2 \)-bounded domains in \( \mathbb{R}^N \) with nonnegative anisotropic mean curvature. Actually, for bounded, convex sets, the \( C^2 \)-regularity is not needed. This can be proved approximating \( \Omega \) in the Hausdorff distance by an increasing sequence of strictly convex smooth domains contained in \( \Omega \). A similar argument has been used, for example, in [14].

5.1 Anisotropic Hersch inequality

**Theorem 5.1.** Let \( \Omega \) be a bounded \( C^2 \)-domain in \( \mathbb{R}^N \), \( N \geq 2 \), with nonnegative anisotropic mean curvature \( \mathcal{H}_F \geq 0 \) on \( \partial \Omega \). Then the following anisotropic Hersch inequality holds:

\[
\lambda_F(p, \Omega) \geq \left( \frac{\pi_p}{2} \right)^{\frac{p}{p-1}} \frac{1}{R_F(\Omega)^{\frac{p}{p-1}}},
\]

where \( R_F(\Omega) \) is the anisotropic inradius defined in (2.8).

**Proof.** Let \( u \) be a positive eigenfunction relative to \( \lambda_F(p, \Omega) \), and let \( v \) be a direction of \( \mathbb{R}^N \). Let \( M = \max_{\Omega} u \). Then by Theorem 4.2 with \( f(w) = Aw^{p-1} \) and property (2.4), we have

\[
\frac{\partial u}{\partial v} = (\nabla u, v) \leq F(\nabla u) F^0(v) \leq \left( \frac{\lambda_F(p, \Omega)}{p - 1} \right)^{\frac{1}{p-1}} (M^p - u^p)^{\frac{1}{p-1}} F^0(v).
\]

Let us denote by \( x_M \) the point of \( \Omega \) such that \( M = u(x_M) \), by \( \overline{x} \in \partial \Omega \) the point such that \( F^0(x_M - \overline{x}) = d_F(x_M) \) and by \( v \) the direction of the straight line joining the points \( x_M \) and \( \overline{x} \). Then by (5.2), since \( F^0(\overline{x} - x_M) \leq R_F(\Omega) \), we get

\[
\int_0^{M^{(\Omega)}} \frac{1}{(M^p(\Omega) - u^p)^{\frac{1}{p-1}}} du \leq \left( \frac{\lambda_F(p, \Omega)}{p - 1} \right)^{\frac{1}{p-1}} F^0(\overline{x} - x_M) \leq \left( \frac{\lambda_F(p, \Omega)}{p - 1} \right)^{\frac{1}{p-1}} R_F(\Omega).
\]

By definition of (1.4) and a change of variables, we get

\[
\int_0^{M^{(\Omega)}} \frac{1}{(M^p(\Omega) - u^p)^{\frac{1}{p-1}}} du = \frac{1}{(p - 1)^{\frac{1}{p-1}}} \frac{\pi_p}{2}.
\]

Finally, joining (5.3) and (5.4), we get inequality (5.1). \( \square \)

The equality sign in (5.1) holds in the limiting case when \( \Omega \) approaches a slab. This will be shown in Theorem 5.8.

From the Hersch inequality (5.1) and the bound (3.7), the following immediately holds:

**Proposition 5.2.** Let \( \Omega \) be a bounded \( C^2 \)-domain in \( \mathbb{R}^N \), \( N \geq 2 \), with nonnegative anisotropic mean curvature \( \mathcal{H}_F \geq 0 \) on \( \partial \Omega \). Then

\[
\lambda_F(p, \Omega) \geq \left( \frac{\pi_p}{2N} \right)^{\frac{p}{p-1}} h_F^p(\Omega).
\]

Hence, for \( p \geq \frac{2N}{\pi_p} \), inequality (5.5) gives a better constant than (3.5).

5.2 An upper bound for the efficiency ratio

As a consequence of the Theorem 4.4, we obtain the following inequality:
Theorem 5.3. Let $\Omega$ be a bounded $C^2$-domain in $\mathbb{R}^N$, $N \geq 2$, with nonnegative anisotropic mean curvature $\mathcal{K}_F \geq 0$ on $\partial \Omega$. Then
\[
\left( \frac{p-1}{p} \right)^{p-1} \left( \frac{\pi_p}{2} \right)^p \leq \lambda_F(p, \Omega) M_{\nu_\Omega}^{p-1},
\]
where $M_{\nu_\Omega} = \max \nu_{\Omega}$.

Proof. The proof is a direct consequence of Theorem 4.4 and of the definition (1.4) of $\pi_p$. Indeed, by (4.4) and the explicit expression of $\Phi$, evaluating both sides at the maximizer $x_m$ of $u$, we obtain
\[
\left( \frac{p-1}{p} \right)^{p-1} \left( \frac{\pi_p}{2} \right)^p \leq \nu_{\Omega}^{p-1}(x_m) \lambda_F(p, \Omega) \leq \lambda_F(p, \Omega) M_{\nu_\Omega}^{p-1},
\]
which is the desired inequality (5.6).

The equality sign in (5.6) holds in the limiting case when $\Omega$ approaches a slab. This will be shown in Theorem 5.8.

Remark 5.4. We observe that the functional involved in Theorem 5.3 is related to other functionals studied in literature. Indeed, it holds that
\[
\lambda_F(p, \Omega) \left( \frac{T_F(p, \Omega)}{|\Omega|} \right)^{p-1} \leq \lambda_F(p, \Omega) M_{\nu_\Omega}^{p-1} \leq \left( \frac{|\Omega|M_{\nu_\Omega}}{T_F(p, \Omega)} \right)^{p-1}.
\]
The functional on the left-hand side of (5.7) has been studied, for example, in [32] for $p = 2$ in the Euclidean case. The functional on the right-hand side of (5.7) has been investigated, for instance, in [17] for $p = 2$ in the Euclidean case and in [14] for any $p$ in the anisotropic setting.

Remark 5.5. Using the upper bound in (2.13) and (5.6) we directly get the anisotropic Hersch inequality for $\lambda_F(p, \Omega)$:
\[
\lambda_F(p, \Omega) \geq \left( \frac{\pi_p}{2} \right)^p \frac{1}{R_F(\Omega)^p}.
\]
Let $u$ be the first eigenfunction relative to $\lambda_F(p, \Omega)$, and let us define the anisotropic efficiency ratio as
\[
E_F(p, \Omega) := \frac{|\nabla u|_{L^p}^{p-1}}{|\Omega|^{1/p} \|u\|_{L^\infty}}.
\]
We stress that, by Remark 4.3 and the Hölder inequality, for open, bounded, convex sets, we obtain the following upper bound for (5.8):
\[
E_F^p(p, \Omega) \leq \frac{1}{p}.
\]

Actually, as a consequence of Theorem 4.4, we get the following upper bound for $E_F(p, \Omega)$ which, in the Euclidean case, is due to Payne and Stakgold [25]:

Theorem 5.6. Let $\Omega$ be a bounded $C^2$-domain in $\mathbb{R}^N$, $N \geq 2$, with nonnegative anisotropic mean curvature $\mathcal{K}_F \geq 0$ on $\partial \Omega$. Then
\[
E_F(p, \Omega) \leq (p-1)^{-\frac{1}{p}} \left( \frac{2}{\pi_p} \right)^{\frac{2}{p^*}}.
\]

Proof. Passing to the power $p-1$ in both sides of (4.5), integrating on $\partial \Omega$ and using the equations, by the divergence theorem, we have
\[
(p-1)^{\frac{1}{2}} \int_\Omega u^{p-1} dx \leq M^{p-1}|\Omega|.
\]
That gives the following upper bound for the efficiency ratio $E_F$:
\[
E_F(p, \Omega) = \frac{|\nabla u|_{L^p}^{p-1}}{|\Omega|^{1/p} \|u\|_{L^\infty}} \leq \frac{1}{(p-1)^{\frac{1}{2}}} \left( \frac{2}{\pi_p} \right)^{\frac{2}{p^*}}.
\]

Remark 5.7. We observe that the bound in (5.10) improves the one given in (5.9).
Finally, we are in the position to give the following optimality result.

**Theorem 5.8.** The equality sign in (3.8), (3.11), (3.12), (5.1), (5.6) and in the upper bound of (2.13) holds in the limiting case when \( \Omega \) approaches a suitable infinite slab.

**Proof.** Let \( \Omega \) be a bounded, open, convex set of \( \mathbb{R}^N \). Then by (5.1), (3.12) and the definition of \( h_F \), we get
\[
\left( \frac{\pi_p}{2} \right)^p \leq R_F(\Omega)^p \lambda_F(p, \Omega) \leq \left( \frac{\pi_p}{2} \right)^p (h_F(\Omega)R_F(\Omega))^p \leq \left( \frac{\pi_p}{2} \right)^p \left( \frac{P_F(\Omega)R_F(\Omega)}{\left| \Omega \right|} \right)^p,
\]
and by (5.6), (3.11) and the upper bound in (2.13), we get
\[
\left( \frac{p - 1}{p} \right)^{-1} \left( \frac{\pi_p}{2} \right)^p \leq \lambda_F(p, \Omega)M_{v_0}^{p-1} \leq \left( \frac{p - 1}{p} \right)^{-1} \left( \frac{\pi_p}{2} \right)^p \left( \frac{P_F(\Omega)R_F(\Omega)}{\left| \Omega \right|} \right)^p.
\]
Choosing \( \Omega = \Omega_{a,k} \) in (5.11) and (5.12) as in Lemma 3.1 and passing to the limit, we get the required optimality. \( \Box \)

**Remark 5.9.** For a general planar, open, convex set, in [31], in the Euclidean case, the author proves the following result:
\[
\lambda(\Omega)M_{v_0} \leq \left( \frac{\pi^2}{8} \right) \left( 1 + 7 \cdot 3 \frac{2}{p} \right) \left( \frac{W(\Omega)}{d(\Omega)} \right)^{\frac{1}{2}},
\]
where \( \lambda(\Omega) \) is the first Dirichlet eigenvalue of \( -\Delta \), \( d(\Omega) \) denotes the Euclidean diameter and \( W(\Omega) \) the width. Then, for a planar, open, convex set and \( p = 2 \), in the Euclidean case, the equality in (5.6) holds for the sets such that \( \frac{W(\Omega)}{d(\Omega)} \to 0 \).

**Remark 5.10.** The slab is not optimal for \( E_F(p, \Omega) \). Indeed, if, for example, \( N = p = 2 \) and \( F = E = (\sum x_i^2) \frac{1}{2}, \) for any rectangle \( R \), it holds that \( E_F(2, R) = \left( \frac{2}{p} \right)^2 \).

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