Network formed by movements of random walkers on a Bethe lattice

Nobutoshi Ikeda
Tohoku Seikatsu Bunka Junior College,
1-18-2 Niji-no-Oka, Izumi-ku, Sendai 981-8585, Japan
E-mail: ikeda.nobu@nifty.com

Abstract. We investigate a stochastic model of network formation where short-cut edges are assumed to be created between vertices in traces of random walkers. The network initially starts from a tree-like structure (Bethe lattice) with a finite number of shells, and develops into a complex network with many circuits generated by the movement of random walkers. We show that the resulting network has a power-law in the degree distribution with an exponent smaller than 2, and demonstrate the robustness against attacks on hubs in the networks. While scale-free networks without a degree correlation are usually vulnerable to attacks on its hubs, the robustness of the network connectivity in this model comes from a self-similar structure of the network. It is interesting that a simple stochastic process like random walks can cause various structures widely seen in real networks on tree-like graphs which play an important role in the graph theory.

1. Introduction
One of the essential problems in studies on random networks is the presence of common structures in large-scaled real networks such as the Internet, social networks, and biological networks [1], given that most real networks are not constructed by design, but by random events in the development of the networks. For example, the power law in degree distribution, which is usually called scale-free property, is a representative character found in many real networks. Preferential attachment of new vertices in the growing process of networks is a simple but probable rule which reproduces the power law in degree distribution [2]. Since the recognition of the importance of preferential attachment, many models based on the growth property of networks have been proposed. However, there have only been a few models which consider short-cut creations stimulated by local events depending on the local structure of networks.

In this paper, we will focus on short-cut creations stimulated by movements of random walkers. We have studied this process in previous works in which we showed that the structure of the initial graph plays an essential role in the development of networks [3]. This result can be explained by the fact that the initial structure of a network affects the diffusion property of the random walker that stimulates the new creation of edges. Here, we take a Bethe lattice as the initial graph. Bethe lattices are not only used in theoretical approaches to statistical mechanics on graphs [4, 5], but also in information science including search problems. Here we will show that the Bethe lattice can be developed into a complex network by the addition of many edges which are stimulated by the repeated movements of many random walkers starting from the root vertex.
Tolerance of networks to random or intentional attacks is an important property of networks, because connections between constituents of a system are usually necessary to maintain the function of the network [6, 7]. It is known that scale-free networks without degree correlation are usually vulnerable to attacks on its hubs. However, the many circuits generated by the movement of random walkers are expected to provide a robust structure against attacks on hubs in the network. We will demonstrate to what extent the movement of random walkers can provide a robust structure to tree-like structures.

The model based on the movement of random walkers is described in the following sections. In section 3, we show numerical results for degree distribution and discuss the time evolution of networks. In Section 4, we show the numerical results for the fraction of the maximum component after the deletion of hubs in the resulting network. Section 5 summarizes the results.

2. Model

The network model investigated in this paper is constructed by the following process (Fig. 1):

(i) A Bethe lattice with the coordination number 3 is prepared as an initial graph, and a random walker starts from the root vertex in the graph.

(ii) At each time step, the walker moves randomly to one of the vertices that is directly connected to the vertex where the walker began the move.

(iii) At each time step, if the vertex where the walker lands and the vertex where the walker landed two steps previously are not yet joined, a short-cut edge is created between them.

(iv) Steps (ii) and (iii) are iterated until the walker reaches a vertex in the Lth shell from the root vertex where L is a constant integer. Once the walker reaches the Lth shell, the walker is removed, leaving short-cut edges that the walker has created. A new walker is born on the root vertex and begins to move (goes to step (ii)).

Note that the walker is assumed to be able to pass through not only the edges in the original lattice but also the edges created by the movement of the walker. We will examine the subgraph $g(t)$ constructed by vertices that the walkers have visited until time step $t$, since degrees of other vertices remain the same as the initial one.

In this model, it is obvious that the network changes into a complete graph by the saturation of number of edges, because the number of vertices included in 21 shells, $N_L = 3 \times (2^{21} - 1)$.

3. Degree distribution

This section presents the numerical results for degree distributions for cases when the condition $L = 21$ is employed. Note that the number of vertices $V$ in subgraph $g(t)$ being considered is by far smaller than the number of vertices included in 21 shells, $N_L = 3 \times (2^{21} - 1)$.

Fig. 2 shows the power-law behavior in the degree distributions $V(k)$ of subgraph $g(t)$. The value of the power-law exponent $\gamma$ which is defined by $V(k) \sim k^{-\gamma}$ is about 2, but seems to change with time. Fig. 3 presents the dependence of numerically obtained power-law exponents on the number of vertices in $g(t)$, $V$. The result shows that $\gamma$ takes a value smaller than 2 in most time intervals and changes slowly to a smaller value as $V$ increases.

The time dependence of the power-law exponent can be understood by the time dependence of $V$ by using a relation derived in our early works [3, 8], $\gamma = 3 - 1/\alpha$, where $\alpha$ is defined by the relation $V \sim t^\alpha$. Our numerical calculation shows that $V$ can be described by $V \sim t^\alpha$, given that $\alpha$ is not a constant but a function of time with a very slow rate of decrease. According to the relation $\gamma = 3 - 1/\alpha$, the power-law exponent must change from 2 to 1, corresponding to decrease in $\alpha$ from 1 to 0.5. A situation when the value of $\alpha$ is smaller than 0.5 implies that
Figure 1. Initially, a random walker who can move to one of the nearest neighboring vertices (11, 12, 13) per one time step starts from the root vertex. At time $t = 2$, if the walker reaches vertex 22 via vertex 11, a new edge is created between the root vertex and vertex 22 (dashed line). In the next time step $t = 3$, the walker may move to vertex 33, or vertex 34, or vertex 11, or the root vertex (At this stage, the walker can move on the dashed line). Corresponding to the actual movement, a short-cut will be created between 11 and 33 or between 11 and 34, or a short-cut will not be created.

A nearly complete graph begins to form, because, if so, $2dE/dt < dV^2/dt \to 0 (t \to \infty)$, where $E$ is the number of edges in $g(t)$. Note that this premonitory sign of formation of a nearly complete subgraph is observed far earlier before $V$ reaches to $N_L$. From that standpoint, the time evolution of the subgraph is similar to cases when one-dimensional lattice is employed as an initial lattice, in which a nearly complete subgraph begins to form immediately [3, 9]. However, there is a clear difference between both cases in the time order needed to form a nearly complete subgraph.

The above discussion infers that the rate of changes in $\gamma$ from 2 to 1 becomes small as the coordination number of the initial Bethe lattice becomes large, because it becomes easier for the walker to diffuse in the Bethe lattice as the coordination number of Bethe lattice increases.

4. Robustness against attacks

In order to examine the robustness of the subgraph $g(t)$ against intentional attacks, we have calculated the fraction of the largest connected components of the graph after the deletion of vertices in the descending order of rank of degree. In Fig. 4, a numerical result for $g(t)$ of 12,000 vertices is compared with the case where a graph is generated by preferential attachments in the growing network. As shown in the figure, the critical point where the subgraph is broken down is larger than that for the graph generated by preferential attachments.

Intuitively, the robustness is associated with the self-similar structure of the network, for vertices in a Bethe lattice are equivalent with each other. From the model assumptions, it is expected that the degree of a vertex is approximately proportional to the frequency of visits of walkers to the vertex. Therefore, the topological structure around the vertex with the second-largest degree is expected to be similar to that around the vertex with the largest degree. However, this similarity is not exact because vertices being considered are only in a finite
Figure 2. Number of vertices with degree $k$, $V(k)$, in subgraph $g(t)$ at different time steps. One is the case when $V = \sum_k V(k) = 2,000$ (circles), and the other is the case when $V = 12,000$ (squares). The slopes are estimated to be $-2.2$ and $-1.83$, respectively.

subgraph $g(t)$ of the Bethe lattice.

Figure 3. Numerical results for the dependence of power-law exponent $\gamma$ on the number of vertices $V$ in subgraph $g(t)$.

Figure 4. Fraction of the largest component $S$ versus fraction of deleted vertices $p$ (solid line). The number of vertices is initially 12,000. SF represents (dotted line) a scale-free network with mean degree 4.
5. Summary
We have demonstrated that the addition of edges based on the movement of random walkers provides tree-like graph complex properties such as the power-law in degree distribution, highly clustered structure, and robustness against intentional attacks. The value of power-law exponent is initially about 2, and modulated to a smaller value by the further addition of edges. Although the complex structure can be kept only in a transient state until the number of edges becomes a saturated state in terms of the number of edges, the relaxation time is very large and can be controlled by some parameters including coordination number and the number of shells being considered in the initial Bethe lattice. The robust structure of the resulting network may be associated with the self-similar structure of the network. It should be noted that only a simple random phenomenon is required for obtaining the complex and robust structure of graphs. Such knowledge about controlling network properties in random phenomena should be useful in further studies on the emergence of functions of networks.

References
[1] Newman M E J 2003 SIAM Rev. 45 167
[2] Barabási A-L and Albert R 1999 Science 286 509
[3] Ikeda N 2010 Physica A 389 3336
[4] Boccaletti S, Latora V, Morenod Y, Chavezf M, and Hwanga D-U 2006 Physics Reports 424 175
[5] Dorogovtsev S N, Goltsev A V and Mendes J F F 2008 Rev. Mod. Phys. 80 1275
[6] Cohen R, Erez K, ben-Avraham D, and Havlin S 2001 Phys. Rev. Lett. 86 3682
[7] Goltsev A V, Dorogovtsev S N and Mendes J F F 2008 Phys. Rev. E 78 051105
[8] Ikeda N 2009 Mod. Phys. Lett. B 23 2073
[9] Ikeda N 2008 J. Phy. A : Math. Theor. 41 235005