ON THE FREENESS AND PROJECTIVENESS OF BREUIL-KISIN MODULE

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Abstract. In the work we have considered Breuil-Kisin module over the ring of witt vectors $W(\kappa)$ over the residue field $\kappa$ of characteristic $p$ and a finite flat $\mathbb{Z}_p$-algebra $R$. Then considered Breuil-Kisin modules $M$ over the ring $W(\kappa)$ and taking the action of $R$ on $W(\kappa)$, we get again a Breuil-Kisin module $M$ over the ring $R \otimes_{\mathbb{Z}_p} W(\kappa)$. We have studied freeness and projectiveness of this module.

1. Introduction

Consider the $p$-adic field $\mathbb{Q}_p$ with residue field $\mathbb{F}_p$ and its finite extension $K$ with ring of integers $\mathcal{O}_K$ and residue field $\kappa$. Let $W(\kappa)$ be the ring of witt vectors over the residue field $\kappa$. Let $\pi \in \mathcal{O}_K$ and $E(u)$ be the minimal (Eisenstein) polynomial of $\pi$ over $W(\kappa)$. Define $\mathfrak{D} := W(\kappa)[[u]]$. Let $\varphi_{\mathfrak{D}} : W(\kappa)[[u]] \to W(\kappa)[[u]]$ be the endomorphism which extends the Frobenious on $W(\kappa)$ such that $\varphi_{\mathfrak{D}}(u) = u^p$.

**Definition 1.1.** [2] A Breuil-Kisin module of height 1 is a finite generated $W(\kappa)[[u]]$-module $M$ equipped with $\varphi_{\mathfrak{D}}$-semi-linear endomorphism $\varphi_M : M \to M$ whose linearisation $1 \otimes \varphi_M : \mathfrak{D} \otimes_{\varphi_{\mathfrak{D}}} M \to M$ has cokernel killed by $E(u)$.

**Definition 1.2.** The category of finite free $\mathfrak{D} = W(k)[[u]]$-module $M$, with an injective semi-linear map

$$\varphi_M : M \to M$$

such that the cokernel of the linearisation

$$\varphi^*(M) := \mathfrak{D} \otimes_{\varphi_{\mathfrak{D}}} M \xrightarrow{1 \otimes \varphi_M} M$$

is killed by $E(u)$, where $\varphi_{\mathfrak{D}}$ is an extension of Frobenius on $W$, denoted $BT^\varphi_{\mathfrak{D}}$.

**Definition 1.3.** [3] Let $p$ be a prime number, and $h$ an integer $\geq 0$, then a $p$-divisible group $G$ over complete Noetherian local ring $R$ of height $h$ is an inductive system

$$G = (G_v, i_v), \ v \geq 0,$$
where

1. $G_v$ is a finite group scheme over $R$ of order $p^{vh}$,
2. for each $v \geq 0$,

$$0 \rightarrow G_v \overset{i_v}{\rightarrow} G_{v+1} \overset{p^v}{\rightarrow} G_{v+1}$$

is exact sequence (i.e., $G_v$ can be identified via $i_v$ with the kernel of multiplication by $p^v$ in $G_{v+1}$).

**Definition 1.4.** [6] Let $G$ be a $p$-divisible group over an integral domain $O$ with field of fraction $K$ having characteristic 0. The

$$\cdots \overset{i_3}{\rightarrow} G_3 \overset{i_2}{\rightarrow} G_2 \overset{i_1}{\rightarrow} G_1$$

be the inverse system associated to the $p$-divisible group $G$. Then the Tate module of $G$ is

$$T_p(G) := \lim_{\leftarrow n} G_n(K^{alg}),$$

where we take the limit over the $i_n$ maps.

The category of $p$-divisible groups (Barsotti-Tate group) over $O_K$ is denoted as $BT(O_K)$. Now it is well known that the category of $p$-divisible groups $BT(O_K)$ is equivalent to the category of Breuil-Kisin modules $BT_{/O}$, which was conjectured by Breuil [1] and proved by Kisin [4].

The Breuil-Kisin modules are simple modules defined over the ring of witt vectors. Take a finite flat $\mathbb{Z}_p$-algebra $R$ and consider $p$-divisible group $G$ over $O_K$ endowed with an $R$-action. This can be checked that the newly constructed $M$ is a $(R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]]$-module. The natural question:

Is the $(R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]]$-module free and projective?

The freeness of the $(R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]]$-module does not follow for general $R$. For example, consider the ring $R = \mathbb{Z}_p[x]/x^2$. This acts as 0 on any Breuil-Kisin module. So the Breuil-Kisin module would not be free or even projective over $R = \mathbb{Z}_p[x]/(x^2)$. Another obstruction is that $Spec(R)$ might not be connected; then $M$ could be projective, but have different ranks on different components of spectra of $R$, $Spec(R)$, hence not free. Further the ring $(R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]]$ is not necessarily local, for example $W(\kappa) \otimes_{\mathbb{Z}_p} W(\kappa)$ decomposes as a direct product of $[\kappa : \mathbb{F}_p]$ copies of $W(\kappa)$. So we can have only special cases and criterions in order to get affirmative answer of the above question which we have proved in the next section.
2. Results

Theorem 2.1. If \( R = \mathcal{O}_K \) is the ring of finite extension of \( \mathbb{Q}_p \) (i.e., \( R \) is regular or gorestein), then \( M \) is \( (R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]] \)-free module.

Proof. Let \( R = \mathcal{O}_K \) be the ring of integers of finite extension of \( \mathbb{Q}_p \). Then \( R \otimes_{\mathbb{Z}_p} W(\kappa) \) is a direct product of rings of integers of finite extensions of \( \mathbb{Q}_p \), i.e.,
\[
R \otimes_{\mathbb{Z}_p} W(\kappa) = \bigotimes_{i=1}^{\infty} \mathcal{O}_{L_i},
\]
where \( L_i \) are finite extensions of \( \mathbb{Q}_p \) and \( \mathcal{O}_{L_i} \) are the rings of integers of \( L_i \). Thus it is sufficient to show that if \( L \) is a finite extension of \( \mathbb{Q}_p \) then an \( \mathcal{O}_L[[u]] \)-module \( M \) which is finite free \( \mathbb{Z}_p[[u]] \)-module is also a free \( \mathcal{O}_L[[u]] \)-module. When \( M = 0 \), the statement is trivial. Let \( M \) be non-zero. Since \( \mathcal{O}_L[[u]] \) is regular local ring, all finitely generated modules have finite projective dimension. By Auslander-Buchsbaum formula, we have
\[
\text{Projdim}(M) + \text{depth}(M) = \text{depth}(\mathcal{O}_L[[u]]) = 2.
\]
\( M \) being a finite free \( \mathbb{Z}_p[[u]] \)-module, clearly \( \{u, p\} \) is a regular sequence for \( M \) as \( \mathcal{O}_L[[u]] \)-module. So the depth of \( M \) as \( \mathcal{O}_K[[u]] \)-module is equal to 2 i.e., \( \text{depth}(M) = 2 \). So from (2.1), we get \( \text{Projdim}(M) = 0 \). This means that \( M \) is \( \mathcal{O}_L[[u]] \)-free module and hence \( M \) is \( (R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]] \)-free module. This completes the proof. \( \square \)

Corollary 2.1.1. If \( R = \mathcal{O}_K \) is the ring of finite extension of \( \mathbb{Q}_p \) (i.e., \( R \) is regular or gorestein), and if there is a finite flat morphism \( R \to R' \), then \( M \) is finite free \( (R' \otimes_{\mathbb{Z}_p} W(\kappa))[[u]] \)-module.

Proof. It is sufficient to show that if \( R \to R \) is a finite flat morphism between regular local rings and if \( M \) is an \( R' \)-module that is finite free over \( R \), then \( M \) is finite free module over \( R' \). But this follows using the same argument as in Theorem (2.1). \( \square \)

If we emphasize on projectiveness rather than freeness, we get a positive answer.

Theorem 2.2. For an arbitrary \( \mathbb{Z}_p \)-algebra \( R \), if \( M/uM \) is finitely generated projective \( R \)-module such that \( M \) is \( u \)-adically complete, separated and \( u \)-torsion free, then \( M \) is finitely generated and projective as \( (R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]] \)-module.

Proof. As \( M \) is a Breuil-Kisin module over \( W(\kappa) \), \( M \) is projective module over \( W(\kappa)[[u]] \). Now for the given \( \mathbb{Z}_p \)-algebra \( R \), the \( (R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]] \)-module \( M \) will be finitely generated and projective if and only if \( M/uM \) is finite projective as an \( R \)-module. Further, it is projective module over \( (R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]] \) if and only if it is projective over \( R[[u]] \). But \( R[[u]] \)-module
$M$ is finite projective if it is $u$-torsion free, $u$-adically complete, separated, and $M/uM$ is a finite projective $R$-module. For, it is sufficient to show that $M/u^iM$, $i \in \mathbb{N}$ is a projective $R[u]/u^i$-module. Since $M$ is $u$-torsion free, the multiplication map $M \to M$ by $u$ is injective. In other word, there exists a short exact sequence of $R$-modules

$$0 \to M/u^iM \xrightarrow{u^i} M/u^{i+j}M \to M/u^jM \to 0.$$  

From this sequence, we see $M/u^iM$ is $R[u]/u^i$-module and using induction on $i$, $M/u^iM$ is $R$-module of finite representation. But given that $M/uM$ is finite generated. So the proof follows.

\[ \square \]

**Theorem 2.3.** Let $R$ be a ring with the property that every finitely generated and $u$-torsion free $R$-module is projective then $M$ is projective as $(R \otimes_{\mathbb{Z}_p} W(\kappa))[u]$-module.

**Proof.** Since $M/uM$ is projective over $W(\kappa)$, it is $u$-torsion free, and therefore projective over $R$. Hence $M$ is projective over $(R \otimes_{\mathbb{Z}_p} W(\kappa))[u]$ by Theorem (2.2). \[ \square \]

Finally we answer the question of freeness of $M$ over $(R \otimes_{\mathbb{Z}_p} W(\kappa))[u]$. In this case we need to permute the action of Frobenius.

**Theorem 2.4.** If $R$ contains all embeddings $\sigma$ of $W(\kappa)$, then $M$ is free $R \otimes_{\mathbb{Z}_p} W(\kappa)[u]$-module.

**Proof.** Given the $p$-divisible group (ignoring the $R$ action), we have $M$ is free as $W(\kappa)[u]$-module. Since $R$ contains all embeddings of $W(\kappa)$, we get

$$R \otimes_{\mathbb{Z}_p} W(\kappa) \cong \prod_{\sigma} R,$$

where $\sigma$ runs over the embeddings of $W(\kappa)$ into $R$. By using same reason, we get

$$M \cong \bigoplus_{\sigma} M_{\sigma}.$$  

Since $M$ is projective, each summand $M_{\sigma}$ is projective and hence free $W(\kappa)[u]$-module. Equivalently, $M_{\sigma}$ is a projective and hence free $R[[u]]$-module by Auslander-Buchsbaum theorem. Finally, the Frobenius permutes the $M_{\sigma}$ transitively and so each $M_{\sigma}$ has the same rank, which implies that $M$ is actually free as $(R \otimes_{\mathbb{Z}_p} W(\kappa))[u]$-module. \[ \square \]

**Theorem 2.5.** If $R$ be the ring with the property that

$$R \otimes_{\mathbb{Z}_p} W(\kappa) \cong \bigoplus R',$$
where $\oplus R'$ denotes direct sum of finitely many copies of ring of integers $R'$ in the compositum of the fraction fields $\text{Frac}(R)$ and $\text{Frac}(W(\kappa))$, then the $(R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]]$-module $M$ is free.

**Proof.** In the proof we use the property that the Frobenius $\varphi$ permutes the components of the spectra ring $\text{Spec}(R \otimes_{\mathbb{Z}_p} W(\kappa))$. Now we have the following property of $R$,

$$R \otimes_{\mathbb{Z}_p} W(\kappa) \cong \oplus R', \ R' \in \text{Frac}(R) \cdot \text{Frac}(W(\kappa)).$$

Any finite projective $R'[[u]]$-module is free, and so the module $M$ is free over each components of $(R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]]$. So $M$ is free module of some rank over each components of $R \otimes_{\mathbb{Z}_p} W(\kappa)$. Now Frobenius $\varphi$ permutes cyclically over the components of $\text{Spec}(R \otimes_{\mathbb{Z}_p} W(\kappa))$ and since the pullback $\varphi^*$ is injective, all the ranks must be equal. Hence $M$ is a free $(R \otimes_{\mathbb{Z}_p} W(\kappa))$-module in this case. \hfill $\square$

Now we digress our attention to the $p$-adic Tate module $T_p(G)$ of the $p$-divisible group $G$, which will be a motivation for further work in this direction. We can recover the from Breuil-Kisin module easily: Taking the $\phi$-fixed points $M \otimes_{\mathcal{O}} W(C^\phi)$, where $C^\phi$ is the fraction field of $\mathcal{O}_{C^\phi}$, which in turn is the inverse limit of $\mathcal{O}_{C/p} \leftarrow \mathcal{O}_{C/p} \leftarrow \mathcal{O}_{C/p} \leftarrow \cdots$, where $C$ is the $p$-adic completion of algebraic closure $\bar{K}$ of finite extension of the $p$-adic field $\mathbb{Q}_p$. From the work of [4], we have an inverse process. That is, we start with a $p$-adic Tate module $T_p(G)$ of the $p$-divisible group $G$ and get Breuil-Kisin module. Hence the further question remains to check whether the $(R \otimes_{\mathbb{Z}_p} W(\kappa))$-module $M$ is free when the $p$-adic Tate module $T_p(G)$ of the $p$-divisible group $G$ is free.

### 3. Conclusion

The paper deals with Breuil-Kisin module over the ring of witt vectors $W(\kappa)$ over the residue field $\kappa$ of characteristic $p$ and a finite flat $\mathbb{Z}_p$-algebra $R$. Then considered Breuil-Kisin modules $M$ over the ring $W(\kappa)$ and taking the action of $R$ on $W(\kappa)$, we get again a Breuil-Kisin module $M$ over the ring $R \otimes_{\mathbb{Z}_p} W(\kappa)$. We have studied freeness and projectiveness of this module under various impositions. The question remains open, how does the $p$-adic Tate module $T_p(G)$ of the $p$-divisible group $G$ influence the freeness of the $(R \otimes_{\mathbb{Z}_p} W(\kappa))[[u]]$-module $M$.

**Acknowledgement:** The authors are grateful to David Savitt, Brian Conrad and Mark Kisin for their helpful comments and suggestions. The second author is grateful to The Council Of Scientific and Industrial Research (CSIR), Government of India, for the award of JRF (Junior Research Fellowship).
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