We derive the noncommutative torus compactification of M(atrix) theory directly from the string theory by imposing mixed boundary conditions on the membranes. The relation of various dualities in string theory and M(atrix) theory compactification on the noncommutative torus are studied.

1 Introduction

Compacitification of the M(atrix) model on tori $T^d$ [1] have lead to interesting consequences, including super Yang Mills theories on d+1 dimensional dual tori for lower dimensions and non trivial theories for higher dimensions. It was observed recently [2] that compactification of the M(atrix) model on a two dimensional "noncommutative" torus $T^\theta$, leads to a non local gauge theory defined on a "noncommutative" torus. Here the torus $T^\theta$ is "noncommutative" in the sense of A. Connes' noncommutative geometry [3], defined by the noncommutative $C^*$ algebra generated by the elements $U_1, U_2$ with,

$$U_1 U_2 = e^{2i\pi \theta} U_2 U_1.$$

(1)

$\theta$ being the noncommutativity parameter of the torus $T^\theta$. It was then observed [2,4] that this compactification on $T^\theta$ is equivalent to an M theory three form $C$, with nonzero value for one of the indices in the minus direction $C_{-12}$.

Genuine noncommutativity of space-time in the string theory and D-branes was first observed in [5], where coordinates of D-brane embedding become noncommutative. In M(atrix) model, however, the large scale coordinates of the center of mass of the system are still commuting variables, and remain so for the ordinary toroidal compactification.

It is therefore important to understand fully the connection among the various appearances of noncommutativity in string theory, M-theory, and M(atrix) model. In this paper we will make one such connection, relating the Connes’ noncommutative geometry’s resurgence in M(atrix) compactification [2] to a phenomenon of noncommutativity of space due to mixed boundary conditions in string theory [6,7]. In [7], it was shown that in the presence of a Kalb-Ramond antisymmetric field $F$, strings satisfying mixed boundary conditions
at $\sigma = 0, \pi$, have noncommuting center of mass coordinates. It was then observed in [7] that a moving membrane in M-theory, results in the above string configuration. The similarity these two kind of noncommutativities cries for an explanation. In this paper we attempt to provide one.

2 Mixed Boundary Conditions

Following [6,7] we consider the action

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma [\eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu g^{ab} + \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu] + \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau A_i \partial_\tau \zeta^i,$$

(2)

Variation of $X^\mu$ gives the boundary conditions for $\sigma = 0, \pi$, with

$$F = F_{12} = B_{12} - A_{[1,2]}.$$  

(3)

We next impose the canonical commutation relations on $X^1, X^2$ and their conjugate momenta i.e. $P^1, P^2$:

$$\{X^\mu(\sigma, \tau), P^\mu(\sigma', \tau)\} = i\eta^{\mu\nu} \delta(\sigma - \sigma'),$$

leading to the nontrivial result

$$\{X^1(\sigma, \tau), X^2(\sigma', \tau)\} = 2\pi i F \theta(\sigma - \sigma'),$$

(4)

for the space coordinates of the membrane. Considering the mode expansions for $X^1$ and $X^2$ consistent with our boundary conditions, results in the noncommutativity of the center of mass coordinates

$$x^1 = \int X^1(\sigma, \tau) \, d\sigma, \quad x^2 = \int X^2(\sigma, \tau) \, d\sigma.$$

$$[x^1, x^2] = -2\pi i F.$$  

(5)

This noncommutativity of space coordinates of the mixed membrane, which formally is a consequence of the appearance of momenta $p_i, i = 1, 2$, in the expression for $x^i$, innocent as it looks, reflects the zero brane distribution inside the D-membrane [6,7] and therefore is closely related to the noncommutativity of the D-brane dynamics [5].

3 Mixed Brane Wrapping

In order to recover the noncommutativity of the torus of the M(atrix) model compactifications, in the presence of Kalb-Ramond field, we need to consider
and understand wrapping of our mixed branes over a torus. Thereby we will be able to compute the mass spectrum of the wrapped mixed brane and find its symmetries, thus reproducing the spectrum and the symmetries of the M(atrix) model noncommutative torus compactification.

To do so, it is convenient to view the mixed brane as a T-dual of a D-string obliquely wound on a torus. Thus we first take a D-string which winds around a torus with cycles of radii \( R_1 \) and \( R_2 \) making an angle \( \alpha \) with each other:

\[
\tau = \frac{R_2}{R_1} e^{i \alpha} = \tau_1 + i \tau_2, \quad \rho = i R_1 R_2 \sin \alpha = i \rho_2. \quad (6)
\]

The D-string is located at angle \( \phi \) with the \( R_1 \) direction such that it winds \( n_1 \) times around \( R_1 \) and \( n_2 \) times around \( R_2 \). Hence

\[
\cot \phi = \frac{n_1}{n_2 \Im \tau} + \cot \alpha. \quad (7)
\]

Imposing the boundary conditions for the open strings attached to the oblique D-string, their mode expansions are

\[
X^i = x^i + p^i \tau + L^i \sigma + \text{Oscil.}, \quad i = 1, 2. \quad (8)
\]

where \( p^i \) and \( L^i \), in an usual complex notation, are:

\[
p = r \frac{n_1 + n_2 \tau}{|n_1 + n_2 \tau|^2} \sqrt{\frac{\tau_2}{\rho_2}} ; r \in \mathbb{Z}. \quad (9)
\]

\[
L = q \frac{\rho(n_1 + n_2 \tau)}{|n_1 + n_2 \tau|^2} \sqrt{\frac{\tau_2}{\rho_2}} ; q \in \mathbb{Z}. \quad (10)
\]

As we see, \( p \) is parallel to the D-string and \( L \) is normal to it. It is interesting to note that the length of \( L \) is an integer multiple of a minimum length. This is the length of a string stretched between two consecutive cycles of the wound D-string.

Mass of such an open string is found to be

\[
M^2 = |p + L|^2 + \mathcal{N} = \frac{\tau_2}{|n_1 + n_2 \tau|^2} \frac{|r + q \rho|^2}{\rho_2} + \mathcal{N}. \quad (11)
\]

The spectrum is manifestly invariant under the two \( SL(2, \mathbb{Z}) \)'s of the torus acting on \( \rho \) and \( \tau \) respectively.

Applying a T-duality: \( R_1 \rightarrow \frac{1}{R_1} \) or equivalently \( \tau \leftrightarrow \rho \), we obtain the mass spectrum of the open strings compactified on a noncommutative torus. Moreover, we observe that:

\[
F^{-1} = \frac{n_1}{n_2 \rho_2} + \cot \alpha. \quad (12)
\]
The advantage of T-duality apart from providing the spectrum on non-commutative torus, is to give its dependence on the integers $n_1$ and $n_2$ which now can be interpreted as the number of times the membrane is wrapped.

4 M(atrix) Theory on $T^\theta$

We have observed that membranes in string theory, with mixed boundary condition, exhibit noncommuting space coordinates; the amount of noncommutativity determined by the Kalb-Ramond field on the membrane.

In the strong coupling limit this field is a pull back of the three form $C$ of the M-theory with non-zero component $C_{012}$ [7]. Therefore the M(atrix) description of this configuration must involve non-vanishing $C_{-12}$ as M(atrix) model is purported to be M-theory in the infinite momentum frame [8]. It remains to compactify [9] the M(atrix) theory of [8] on the two directions $X^1$ and $X^2$ i.e, to find operators $U_1$ and $U_2$ with the property [9]

$$U_iX_iU_i^{-1} = X_i + R_i.$$  \hspace{1cm} (13)

But now, the coordinates must also satisfy:

$$[X^1, X^2] = -2\pi i F.$$ \hspace{1cm} (14)

It is the standard noncommutative torus as discussed in [2,3], with the $C^*$ algebra defining the geometry, generated by one pair of elements $U_1$ and $U_2$ as in eq.(1), with

$$\theta = \rho_2 \cot \alpha.$$ \hspace{1cm} (15)

It is straightforward to see that the mass spectrum for the mixed membranes is thus translated to the M(atrix) model context and reproduces the spectrum of [2]. The $SL(2, Z)$ symmetries discussed in section 3 are therefore also the symmetries of the M(atrix) theory. Specifically, after T-duality, our $SL(2, Z)$ acting on the $\rho$ modulus becomes the non-classical $SL(2, Z)_N$ of [2].

References

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