Bosonization of Cooper pairs and novel Bose-liquid superconductivity and superfluidity in high-$T_c$ cuprates and other systems

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Abstract

The universal criteria for bosonization of Cooper pairs and novel Bose-liquid superconductivity and superfluidity in pseudogap matters (high-$T_c$ cuprates and other systems with low Fermi energies) are formulated by using the uncertainty principle and the composite-boson mean field theory. We have established that the often discussed $s$- or $d$-wave superconductivity occurring in the fermionic limit of Cooper pairs can exist in conventional superconductors (e.g., ordinary metals and heavily overdoped cuprates) with large Fermi energies but the Fermi-liquid (BCS-type) superconductivity is not characteristic of underdoped to overdoped cuprates with low Fermi energies. The unusual superconducting order parameter in high-$T_c$ cuprates and other pseudogap matters cannot be determined as the BCS-like ($s$- or $d$-wave) gap. We argue that many experimental data (including tunneling and angle-resolved photoemission data) are not accurate to identify the true superconducting order parameter in high-$T_c$ cuprates. We show that the unconventional superconductivity/superfluidity occurring in the bosonic limit of Cooper pairs would exist in low Fermi energy systems where the bosonic Cooper pairs are formed at a pseudogap temperature $T^*$ above the superconducting/superfluid transition temperature $T_c$ and then part of such Cooper pairs condenses into a Bose superfluid at $T_c$. Diamagnetism of bosonic Cooper pairs exists in high-$T_c$ cuprates below $T^*$. Upon lowering the temperature, the pair condensation of attracting bosons occurs first at $T_c$ and then their single particle condensation sets in at $T^*$ lower than $T_c$ (in three dimensions (3D)) or at $T = 0$ (in two dimensions (2D)). The coherent single particle and pair condensates of bosons exist as two distinct superfluid phases and arise from an effective attraction between bosons in some domains of momentum space. By solving the mean field equations for high-$T_c$ cuprates, the novel superconducting states (i.e., a vortex-like state existing below the temperature $T_v = T_c^{2D}$ lower than $T^*$ but higher than $T_c = T_c^{3D}$ as well as two superconducting phases below $T_c$) and their properties characterized by the boson superfluid stiffness are self-consistently determined and compared with the key experimental findings. The mechanisms responsible for the novel Bose-liquid superconductivity and superfluidity could be common to a wider class of exotic superconductors/superfluids, including quantum liquids, atomic Fermi gases and low-density nuclear matter.

1. Introduction

The conventional superconductivity in simple metals with large Fermi energies $E_F >> 1$ eV and small phonon energies (Debye energies) $\hbar \omega_D$ is well described in terms of the Bardeen-Cooper-Schrieffer (BCS) condensation of weakly-bound (large) Cooper pairs [1]. However, in more complex systems which are of significant current interest in condensed matter physics and beyond, our understanding of the phenomena of superconductivity and superfluidity is still far from satisfactory. The BCS picture, in which electrons form Cooper pairs as a result of conventional electron-phonon interactions, is now believed to account well for the great majority of metallic superconductors. But there is a growing number of exotic systems, including the unconventional high-$T_c$ cuprate superconductors and other related systems (e.g., heavy-fermion and organic compounds, liquid $^3$He, ultracold atomic Fermi gases and low-density nuclear matter), in which superconductivity/superfluidity appears anomalous and where the origin of this phenomenon remains controversial.

After the discovery of high-$T_c$ superconductivity in doped copper oxides (cuprates), various mechanisms have been proposed for unconventional superconductivity, especially in the cuprates. Many of the proposed
mechanisms for unconventional cuprate superconductivity are based on the BCS-like ($s$- or $d$-wave) pairing correlations and on the usual Bose-Einstein condensation (BEC) of an ideal Bose-gas of tightly-bound Cooper pairs and other bosonic quasiparticles (e.g., bipolarons and holons) \cite{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}. In these two limiting cases the appearance of superconductivity in the BCS-like and BEC regimes is assumed possible. However, a very controversial question in the theory of superconductivity is the relation of both the BCS-like pairing of carriers and the usual BEC of small Cooper pairs to the unconventional superconductivity. The BCS-type Cooper pairing may have a certain relation to the unconventional superconductivity, so it is necessary to study the formation of Cooper pairs with the determination of their genuine nature, which is very important in the establishment of the following key scenarios of superconducting phase transitions. In some versions of the theory of high-$T_c$ cuprate superconductivity \cite{2, 3, 9}, the crossover from BCS-like pairing regime to real-space pairing or BEC regime \cite{14, 15, 16} is considered. Renewed interest in this crossover arose with the study of the anomalous behavior of high-$T_c$ materials. Unlike conventional weak-coupling BCS superconductors, the high-$T_c$ cuprate compounds falling between the BCS and BEC limits exhibit the new physics and are characterized by small Cooper pairs, which may have the bosonic nature. For these reasons, the BCS-Eliashberg theory, which is a very good approximation for ordinary metals, turned out to be inadequate for the description of unconventional superconductivity in high-$T_c$ cuprates where the Fermi energy $\epsilon_F$ becomes comparable with the energy $\hbar \omega_o$ of the optical phonons and the Eliashberg theory based on the adiabatic approximation $\epsilon_F/\hbar \omega_o >> 1$ breaks down. Further, some authors have attempted to describe the superconductivity in high-$T_c$ cuprates in terms of the BCS-BEC crossover. According to the Landau criterion \cite{17}, BEC of an ideal Bose gas of small real-space pairs and Cooper pairs is irrelevant to the superconductivity (superfluidity) phenomenon. Therefore, the superfluid transition in liquid $^4$He should not be considered as the usual BEC of Bose particles. Because the liquid $^4$He is strongly interacting Bose system and not an ideal Bose gas which undergoes a BEC. Evans and Imry emphasized \cite{18} that the superfluid phase in $^4$He is best identified with a nonvanishing coherence parameter of attracting bosons rather than with the presence of BEC in ideal or repulsive Bose gases where condensation can exist without coherence.

For a long time, the basic questions concerning the true origins of the unusual superconducting/superfluid states in high-$T_c$ cuprates and other exotic systems remain open. In these systems, unconventional interactions may take place between pairs of quasiparticles, leading to new and unidentified states of matter. Actually, the doped cuprates exhibit pseudogap phenomena \cite{10} and diamagnetism \cite{19, 20} above the superconducting transition temperature $T_c$ and a $\lambda$-like superconducting transition at $T_c$ \cite{21} just like the $\lambda$ transition in liquid $^4$He. The pseudogap formation, diamagnetism, vortex-like excitation, high-$T_c$ superconductivity and quantum criticality occur in the cuprates between their insulating state at low doping and their normal metallic state at high doping \cite{10, 12, 13}. Attempts to understand the new physics of high-$T_c$ superconductors led to the assumptions of the importance of superconducting fluctuations \cite{22, 23}. It was argued that the fluctuations of the BCS-like order parameter could be responsible for the retaining of Cooper pairs and superconductivity on short length scales (i.e., in small islands) at temperatures higher than $T_c$. Further, the so-called $d$-wave superconductivity just like the superconducting fluctuation is widely discussed in various contexts and does not fundamentally differ from the BCS superconductivity. But the validity of these scenarios for superconductivity in high-$T_c$ cuprates is not justified. Because both the superconducting fluctuation model and the $d$-wave superconductivity model fails to account for the $\lambda$-like transition and other unusual superconducting properties of the cuprates. Remarkably, the cuprate compounds in the intermediate doping regime exhibit unexplained exotic properties inherent in unconventional superconductors (e.g., heavy-fermion systems \cite{17, 24}), quantum liquids ($^3$He and $^4$He) \cite{17, 18, 25, 26, 27} and low-density nuclear matter \cite{28} but at high doping levels they are similar to ordinary metals \cite{10, 29} and high-density nuclear matter. Apparently, the essential physics of underdoped to overdoped cuprates, heavy-fermion and organic superconductors, superfluid $^4$He, superfluid atomic Fermi gases and superfluid low-density nuclear matter may be described by a two-stage Fermi-Bose-liquid model \cite{30, 31} and controlled by the formation of bosonic Cooper pairs and by the attractive interactions between these composite bosons. This idea opens the way to consider that in more complex pseudogap matters, the Bose-liquid superconductivity/superfluidity might occur rather than the BCS-type Fermi-liquid superconductivity/superfluidity. Another important fact is that the superfluidity in ultracold atomic Fermi gases with an extremely high transition temperature with respect to the Fermi temperature $T_F \approx 5T_c$ defies also a BCS-like description \cite{32}. The above arguments, together with the experimental evidences for vortex-like
state above $T_c$, two distinct superconducting/superfluid phases below $T_c$, a $\lambda$-like phase transition at $T_c$ and a first-order phase transition somewhat below $T_c$ in high-$T_c$ cuprates and other unconventional superconductors and superfluid $^3$He seemed to make the BCS-like scenario as hopeless to explain fully the unconventional superconductivity (superfluidity) and stimulated the search for radically new mechanisms. At present, the greatest part of the available experimental data is not accurate enough to identify the superconducting order parameter in high-$T_c$ cuprates and the different interpretations of the experimental results for the pseudogap, vortex-like excitations and diamagnetism persisting in the normal state of underdoped to overdoped cuprates are often misleading. In particular, a prolonged dispute about the s-wave or the d-wave superconducting gap in high-$T_c$ cuprates, which is determined by using the single particle angle-resolved photoemission spectroscopy (ARPES) and tunneling spectroscopy, is already deadlocked. Thus, the properties of the pseudogap and superconducting phases of these intricate materials are the central issues in the search for the mechanism of high-$T_c$ cuprate superconductivity. In order to obtain the novel types of superconductivity and superfluidity in low Fermi energy systems, there are two problems to be solved. First, the crossover from fermionic limit of large Cooper pairs to bosonic limit of small Cooper pairs is not well understood yet. Second, the BEC in ideal Bose gases of small Cooper pairs exists without coherence. The above interrelated two problems can be solved by considering the real possibility of the bosonization of Cooper pairs and the superfluid condensation of the attractive Bose gases of Cooper pairs.

The present paper is devoted to discussion of these two important questions. First we consider the possibility of formation of bosonic Cooper pairs and formulate the criterion for bosonization of Cooper pairs in high-$T_c$ cuprates and other pseudogap matters by using the uncertainty principle. Then we consider the condensation of the attractive Bose gases of Cooper pairs into a superfluid Bose-liquid and the superfluidity of Bose-liquid with coherent single particle and pair condensates, which arise from an effective attraction between bosonic Cooper pairs in some domains of momentum space, within the composite-boson mean field theory. By solving the mean field equations for attractive Bose systems and closely examining the possible superconducting/superfluid states arising in high-$T_c$ cuprates and other classes of exotic matters, we find that bosonic Cooper pairs and novel types of superconductivity and superfluidity may indeed exist in such systems. We then describe in detail the novel superconducting/superfluid properties of these systems and their experimental manifestations. We discuss the capabilities of the existing experimental techniques for identifying the true superconducting order parameter in high-$T_c$ cuprates. Further, we describe the entire doping-temperature phase diagram of high-$T_c$ cuprates from Mott insulator to the heavily overdoped regime and the existence regions of the distinct superconducting states below $T_c$ and the possible pseudogap, diamagnetic and vortex-like states above $T_c$. The origins of vortices in high-$T_c$ cuprates above $T_c$ and in thin $^3$He superfluid film on porous substrate are explained naturally as the destruction of the bulk superconductivity (superfluidity) and the remnant quasi-two-dimensional (2D) superconductivity (superfluidity) above $T_c$. We show that the superfluid Bose-liquid model provides a fairly good quantitative description of unconventional superconductivity (superfluidity) observed in high-$T_c$ cuprates, heavy-fermion and organic compounds, quantum liquids ($^3$He and $^4$He) and ultracold atomic Fermi gases. Finally, the basic principles of novel superconductivity (superfluidity) in these systems described by a two-stage Fermi-Bose-liquid model are formulated.

2. Criterion for bosonization of Cooper pairs

There is now much experimental evidence that polaronic carriers are present in doped cuprates \[33,35\] and they have effective masses $m_e \approx (2 - 3)m_e$ \[29,33\] (where $m_e$ is the free electron mass) and binding energies $E_p \approx (0.06 - 0.12)$ eV \[34\]. In lightly doped cuprates, polarons tend to form real-space pairs, which are localized bipolarons. In conventional metals, fermionic Cooper pairs and superconductivity appear simultaneously at $T_c$. The situation, however, is different in high-$T_c$ cuprates in which the electron-phonon interactions are unconventional and the Fermi energy $\varepsilon_F$ of polarons is comparable with the energy $\hbar \omega_0$ of the high-frequency optical phonons. These superconductors are characterized by low Fermi energies $\varepsilon_F \approx (0.1 - 0.3)$ eV \[36\] and high-energy optical phonons $\hbar \omega_0 \approx (0.04 - 0.08)$ eV \[33,36\]. Therefore, the Cooper pairing of polarons may occur in the normal state of high-$T_c$ cuprates at a characteristic temperature $T^\ast$ \[31,37\]. In these materials the attractive interaction mechanism (e.g., due to exchange of static and dynamic phonons) between the carriers operating in the energy range $\{-E_p + \hbar \omega_0\}, \{E_p + \hbar \omega_0\}$ is more effective than in the simple BCS picture-in-the narrow energy range $\{-\hbar \omega_D, \hbar \omega_D\}$. For such strong pairing interactions, it is predicted that the pseudogap phase has a BCS-like dis-
persion given by $E(k) = \sqrt{\epsilon_F^2(k) + \Delta^2_F}$ (where $\epsilon_F(k)$ is the energy of fermionic quasiparticles measured from the Fermi energy $\epsilon_F$, $k$ is the quasiparticle momentum), but the BCS-like gap $\Delta_F$ is no longer the superconducting order parameter and opens on the Fermi surface at $T^* > T_c$ [31]. Various experiments showed that a BCS-like excitation gap indeed persists as a pseudogap well above the measured critical temperature for superconductivity/superfluidity in high-$T_c$ cuprates and atomic Fermi gases [32]. In particular, the formation of such a pseudogap at the precursor Cooper pairing of polarons above the measured critical temperature for superconductivity/superfluidity in high-$T_c$ cuprates and atomic Fermi gases [32]. In particular, the formation of such a pseudogap at the precursor Cooper pairing of polarons with antiparallel spins is manifested in the diamagnetic pseudogap at the precursor Cooper pairing of polarons Fermi gases [32]. In particular, the formation of such a pseudogap at the precursor Cooper pairing of polarons with antiparallel spins is manifested in the diamagnetic pseudogap at the precursor Cooper pairing of polarons Fermi gases [32].

As the binding between fermions increases, Fermi gas of weakly-bound Cooper pairs evolves into Bose gas of tightly-bound Cooper pairs, as pointed out by Leggett [15]. This is the most interesting crossover regime, since a Fermi system passes from a BCS-like Fermi-liquid limit to a normal Bose gas limit with decreasing $\epsilon_F$. Thus, it is a challenging problem to find the criterion for bosonization of Cooper pairs in such Fermi systems. If the size of the Cooper pairs $a_c(T)$ is much larger than the average distance $R_c$ between them, the bosonization of such Cooper pairs cannot be realized due to their strong overlapping, as argued by Bardeen and Schrieffer [33] [39]. However, the composite (bosonic) nature of Cooper pairs becomes apparent when $a_c \sim R_c$. At $R_c \gtrsim a_c$, the fermions cannot move from one Cooper pair to another one and the non-overlapping Cooper pairs behave like bosons. The criterion for bosonization of polaronic Cooper pairs can be determined from the uncertainty relation [40]

$$\Delta x \cdot \Delta E \cong \frac{(h\Delta k)^2}{2m_p} \frac{1}{2\Delta k}$$

(1)

where $\Delta x$ and $\Delta E$ are the uncertainties in the coordinate and energy of attracting polaronic carriers, $\Delta k$ is the uncertainty in the wave vector of polarons. The expression $(h\Delta k)^2/2m_p$ represents the uncertainty in the kinetic energy of polarons, which is of order $\epsilon_F$, whereas $\Delta k$ would be of the order of $1/R_c$. Taking into account that $\Delta x$ is of order $a_c$ and $\Delta E$ would be of the order of the characteristic energy $\epsilon_A$ of the attractive interaction between polarons, Eq. (1) can be written as

$$\frac{R_c}{a_c} \cong \frac{\epsilon_A}{\epsilon_F} \gtrsim 1$$

(2)

This ratio is universal criterion for the bosonization of Cooper pairs in low Fermi energy systems, in particular, in high-$T_c$ cuprates (for which $\epsilon_A$ is replaced by $E_p + \hbar \omega_0$) and other exotic superconductors, liquid $^3$He and ultracold atomic Fermi gases. The criterion in Eq. (2) is well satisfied at $\epsilon_A \gtrsim 0.5 \epsilon_F$, where $\epsilon_F \approx 0.016 - 0.025$ eV (for UPt$_3$ [41]), $\epsilon_F \approx 0.1 - 0.3$ eV (for organic compounds [36]), $\epsilon_F \approx 4.4 \times 10^{-4}$ eV (for liquid $^3$He [42]) and $\epsilon_F \approx 10^{-10}$ eV (for ultracold atomic Fermi gases [43]). For the mass density of nuclear matter $\rho_M \approx 10^{11}$ g/cm$^3$ [44], we find $\epsilon_F \approx 0.38$ MeV. Then the deuteron-like bosonic Cooper pairs in low-density nuclear matter are formed at $\epsilon_A > 0.19$ MeV. We can now formulate the following key postulates:

1. the BCS-type superconductivity and superfluidity would occur in the fermionic limit of Cooper pairs and could exist in high-density nuclear matter and conventional superconductors (e.g., ordinary metals and heavily overdoped cuprates) with large Fermi energies, where the pseudogap is absent and the superconducting state is characterized by the BCS-like order parameter and the onset temperature of Cooper pairing $T^*$ coincides with $T_c$;

2. the high-$T_c$ cuprates exhibiting pseudogap behaviors at $\epsilon_F <\! < 1$ eV and other pseudogap matters (e.g., heavy-fermion and organic compounds, liquid $^3$He, atomic Fermi gases and low-density nuclear matter) could be in the bosonic limit of Cooper pairs and the novel (non-BCS-type) superconductivity and superfluidity would occur in such systems, where the pseudogap coexists with the unusual superconducting order parameter below $T_c$ [31]. In pseudogap matters, a two-stage superconducting/superfluid transition process involves the formation of bosonic Cooper pairs at $T^* > T_c$ and the subsequent superfluid Bose condensation of these bosons at $T_c$.

3. Composite-boson mean field theory of superfluidity and its experimental confirmation

The repulsive interaction is less realistic for the problem of Bose superfluids (including $^4$He). In unconventional superconductors/superfluids with $\epsilon_F \lesssim 2 \epsilon_A$, Cooper pairs behave as composite bosons and would undergo a BEC in the noninteracting particle approximation without superfluidity at $T = T_{BEC} >> T_c$. Here we show that the superconductivity/superfluidity in these systems is driven by the condensation of the attractive Bose gases of Cooper pairs with low densities. Such composite bosons repel one another at small distances between them and their net interaction is attractive at large distances. The Hamiltonian of a Bose gas inter-
acting via a pair potential $V_B(k - k')$ has the form

$$H_B = \sum_k [\tilde{c}_k^\dagger \tilde{c}_k + \frac{1}{2} \Delta_B(k)(c^\dagger_{k'} c_k + c_{k'} c_{-k})],$$

where $\tilde{c}_k = \epsilon(k) - \mu_B + V_B(0) \rho_B + \chi_B(0)$, $\epsilon(k) = \hbar^2 k^2 / 2m_B$, $\mu_B$ is the chemical potential, $\Delta_B(k) = (1/\Omega) \sum_{k'} V_B(k - k') < c_{-k'} c_{k'} >$ is the coherence parameter, $\chi_B(0) = (1/\Omega) \sum_k n_0(k)$, $\epsilon(k) = c^\dagger_k c_k$ is the particle number operator, $\rho_B = (1/\Omega) \sum_k n_0(k)$, $m_B = 2m_p$ is the mass of bosonic Cooper pairs, $c^\dagger_k (c_k)$ is the creation (annihilation) operator of bosons with the wave vector $k$, $\Omega$ is the volume of the system.

The Hamiltonian in Eq. (3) is diagonalized by the Bogoliubov transformations of Bose operators and the quasiparticle spectrum has the form $E_B(k) = \sqrt{\tilde{\epsilon}_B^2(k) - \Delta_B^2(k)}$, which is gapless for $k = 0$ and $k' = 0$ provided $\tilde{\mu}_B = -\mu_B + V_B(0) \rho_B + \chi_B(0) = |\Delta_B(0)|$. If $E_B(k = 0)$, the $k = 0$ and $k' = 0$ terms in the summation of the equations for $\Delta_B(k)$, $\Delta_B(k')$ and $\mu_B$ are considered separately according to the procedure proposed in Ref. 11.

Further, in order to simplify the solutions of the equations for $\Delta_B(k)$, $\chi_B(k)$ and $\rho_B$, the interboson interaction potential may be chosen in a simple separable form

$$V_B(k - k') = \begin{cases} V_{BB} - V_B & \text{for } \epsilon(k) < \epsilon(k'), \\
V_B & \text{for } \epsilon_A < \epsilon(k) < \epsilon(k'), \\
0 & \text{for } \epsilon(k) < \epsilon_A, \end{cases}$$

where $\epsilon_A$ and $V_{BB}$ are the cutoff parameters for attractive $V_B$ and repulsive $V_{BB}$ parts of $V_B(k - k')$, respectively. Then the three-dimensional (3D) equations for determining the coherence (e.g., superconducting order) parameter $\Delta_{SC} = \Delta_2$ and the condensation temperature $T_c$ of attracting bosons can be written as:

$$\frac{2}{D_B V_B} = \int_0^{\epsilon_A} \frac{\text{coth} \left[ \sqrt{\epsilon^2 + \mu_B^2} - \Delta_2^2 \right]}{\sqrt{\epsilon^2 + \mu_B^2} - \Delta_2^2} \text{d} \epsilon,$$

$$\frac{2 \rho_B}{D_B} = \int_0^{\epsilon_A} \frac{\text{coth} \left[ \sqrt{\epsilon^2 + \mu_B^2} - \Delta_2^2 \right]}{\sqrt{\epsilon^2 + \mu_B^2} - \Delta_2^2} \times \frac{\sqrt{\epsilon^2 + 2 \mu_B \gamma}}{2k_BT} \text{d} \epsilon,$$

where $D_B = m_B^2 / \sqrt{2\pi^4 \hbar^3}$, $V_B = V_{BB} - V_{BB}[1 + V_{BB} T]^{-1}$, $\rho_B = \rho_B^0$, $\epsilon_A = \epsilon_A^0$, $\mu_B = \mu_B^0$. Solutions of Eqs. (5) and (6) allow us to examine closely the possible superconducting/superfluid states arising in attractive 3D Bose systems. Below $T_c$, the excitation spectrum $E_B(k)$ has a gap $\Delta_2 = \sqrt{\mu_B - \Delta_2^2}$ and satisfies the Landau criterion for superfluidity. For $\gamma_B = D_B V_B \sqrt{\epsilon_A^0}$ less than a threshold value $\gamma_{Bc}$, however, $E_B(k)$ becomes gapless at $T \leq T_c << T_c^*$ (for $\gamma_B < \gamma_{Bc}$) or at $T \leq T_c^* < T_c$ (for $\gamma_B > \gamma_{Bc}$). For $T = 0$ and $\epsilon_B^0 / k_B T_{REC} = 10 - 50$, the energy gap $\Delta_2$ vanishes at the critical values of the interboson coupling constant $\gamma_B = \gamma_{Bc}^* = 1.4 - 2.0$. When the interboson coupling is weak ($\gamma_B << 1$), the coherence parameter $\Delta_B$ is proportional to the density of condensed bosons $\rho_B$ (i.e., $\Delta_B \propto \rho_B V_B$) 46, 47. For $\gamma_B < \gamma_{Bc}$ and $\Delta_2 = 0$, Eqs. (5) and (6) become

$$\frac{2}{D_B V_B} = \frac{\rho_B}{D_B} + \int_0^{\epsilon_A} \frac{\text{coth} \left[ \sqrt{\epsilon^2 + \mu_B^2} / 2k_BT \right]}{\sqrt{\epsilon^2 + \mu_B^2}} \text{d} \epsilon,$$

$$\frac{2 \rho_B}{D_B} = \frac{\rho_B}{D_B} + \int_0^{\epsilon_A} \frac{\text{coth} \left[ \sqrt{\epsilon^2 + 2 \mu_B \gamma} / 2k_BT \right]}{\sqrt{\epsilon^2 + 2 \mu_B \gamma}} \times \frac{\sqrt{\epsilon^2 + 2 \mu_B \gamma}}{2k_BT} \text{d} \epsilon,$$

where $\rho_B^0$ is the density of condensed bosons with $k = 0$ and $\epsilon = 0$.

Equations (5) and (6) similar to those of the BCS theory for fermions have collective solutions for the attractive interboson interaction $V_B$. The superconducting/superfluid state is characterized by the coherence (macroscopic order) parameter $\Delta_B$ which vanishes at $T = T_c$, that marks the vanishing of a macroscopic superfluid condensate of attracting bosons. For $T \leq T_c^*$, the gapless and linear (at small $k$), phonon-like spectrum $E_B(k)$ in the superfluid state is similar to the excitation spectrum in superfluid $^4$He and satisfies also the criterion for superfluidity, i.e., the critical velocity of quasiparticles $v_c = \hbar / (\partial E_B(k) / \partial k)_{\text{min}} > 0$ satisfies the condition for the existence of superfluidity. By solving the mean field equations (5) and (6) for $\Delta_2 > 0$ and $\Delta_2 = 0$, we find that the condensation of boson pairs at $T > T_c^*$ will correspond to a smaller value of both the chemical potential $\mu_B$ and the order parameter $\Delta_B < \mu_B$, while the single particle condensation of bosons at $T \leq T_c^*$ will correspond to a much larger value of the chemical potential $\mu_B = \Delta_2$. In this case, the pair condensation of attracting bosons occurs first at $T_c^*$ and their single particle condensation takes place at $T_{c1}^* < T_c$ (for $\gamma_B < 1$) or at $T_{c1}^* < T_c^*$ (for $\gamma_B > 1$). First the $\lambda$-like second-order phase transition in high-$T$ cuprates occurs at $T_c$. Then, a new first-order phase transition from pair condensation state to single particle condensation one occurs at...
\[ T \leq T^*_c \] and the true superconducting order parameter \( \Delta_{SC}(T) \) shows a pronounced kink-like behavior near \( T^*_c \). We see that two distinct superconducting states of a 3D attractive Bose gas of Cooper pairs in high-\( T_c \) cuprates are characterized by the integer \( \hbar/2e \) (at \( T \leq T^*_c \) and \( \tilde{\mu}_B(T) = \Delta_B(T) \)) and half-integer \( \hbar/4e \) (at \( T > T^*_c \) and \( \tilde{\mu}_B(T) > \Delta_B(T) \)) magnetic flux quantizations. The first-order phase transition was actually observed in high-\( T_c \) cuprates [48]–[59] and in superfluid \( ^3\text{He} \) (where the transition between the \( A \) and \( B \) phases occurs at \( T^*_c = T_{AB} \)) [26]–[50]. A similar phase transition was also observed in heavy-fermion systems [17, 41, 51]. Some experiments [26, 50] indicate that the superconducting order parameter \( \Delta_{SC}(T) \) in the cuprates has a kink-like feature near the characteristic temperature \( T_c' \) (\( \lesssim 0.6T_c \)). The half-integer circulation quantum \( \hbar/2m_e \) (where \( m_e \) is the mass of \( ^4\text{He} \) atoms) observed in superfluid \( ^3\text{He} \) [53] and the formation of \( \alpha \)-clusters in exotic nuclear matter [54] are equally well explained by the pair condensation of \( ^4\text{He} \) atoms and deuteron-like Cooper pairs. The half-quantum vortices in the superfluid \( ^3\text{He}-\text{A} \) were discussed in Ref. [55]. The microscopic origins of the half-quantum vortices \( \hbar/4e \) and \( \hbar/4m_3 \) in high-\( T_c \) cuprates and superfluid \( ^3\text{He}-\text{A} \) are associated with the excitations of pair condensates of bosonic Cooper pairs rather than other effects. The superfluidity of \( ^4\text{He}-\text{B} \) is caused by the single particle condensation of attracting bosonic Cooper pairs. Thus, the single particle and pair condensates of bosonic Cooper pairs are different superconducting/superfluid phases in high-\( T_c \) cuprates and other pseudogap matters. The occurrence of novel superconductivity/superfluidity in these systems is characterized by a non-zero coherence parameter \( \Delta_B \) which defines the bond strength of all condensed bosons - boson superfluid stiffness. Therefore, excitations of a superfluid Bose condensate of Cooper pairs in high-\( T_c \) cuprates are really many-particle ones and cannot be measured by single-particle spectroscopies, as noted also in Ref. [56]. In these systems the gapless superconductivity/superfluidity occurs due to the vanishing of the gap \( \Delta_B \) in \( E_B(k) \) at \( T \leq T^*_c \) and is not associated with the point or line nodes of the BCS-like gap assumed in some \( p \)- and \( d \)-wave pairing models. The frictionless flow of Bose condensate would be possible under the condition \( \Delta_B > 0 \). While the BCS-like fermionic excitation gap \( \Delta_F \) characterizing the bond strength of Cooper pairs may exist as the pseudogap and its formation is not accompanied by the superconducting transition [57]–[58]. The other key superconducting/superfluid properties of high-\( T_c \) cuprates and related systems will be discussed below.

4. Experimental manifestations of novel superconducting and superfluid properties

We now discuss the novel superconducting properties and show that the kink-like features of \( \Delta_{SC}(T) \) are responsible for the kink-like behaviors of the critical current \( J_c(T) \) and the critical magnetic fields \( (H_{11}(T) \) and \( H_{22}(T) \)) near \( T^*_c \), as observed in various high-\( T_c \) cuprates [59]–[60]–[61]. The critical current density can be written as

\[
J_c(T) = 2e\rho_s(T)v_c(T),
\]

where \( \rho_s(T) = \rho_B - \rho_n \) is the density of the superfluid part of condensed bosons, \( \rho_n = -(1/3m_B)\int(d\mu_B/dE_B)p^2[4\pi^2p^2/(2\pi\hbar)^3]dp \) is the density of the normal part of a 3D Bose-liquid, \( p = \sqrt{2m_B\mu} \), \( n_B = \exp(E_B/k_BT) - 1 \)^{-1}, \( v_c(T) = \sqrt{\mu_B(T) + \Delta_c(T)/m_B} \) is the critical velocity of superfluid carriers. The kink-like behavior of \( J_c(T) \) in YBa\(_2\)Cu\(_3\)O\(_y\) (YBCO) film is shown in Fig. [6] The lower critical magnetic field \( H_{11} \) is determined from the relation

\[
H_{11}(T) = \frac{\ln \chi(T)}{\sqrt{\chi(T)}} H_c(T),
\]

where \( \chi(T) = \lambda_\perp(T)/\xi_c(T) \) is the Ginzburg-Landau parameter, \( \lambda_\perp(T) = [m_Bc^2/16\pi^2\rho_s(T)]^{1/2} \) is the London penetration depth, \( \xi_c(T) = \hbar/\sqrt{2m_B\Delta_B(T)} \) is the coherence length of bosonic superconductors, \( H_c(T) = 4\pi R I_c(1/T)/c \) is the thermodynamic critical magnetic field, \( R \) is the radius of a superconducting wire, \( c \) is the velocity of light. The kink-like behaviors of \( H_{11}(T) \)
(in YBCO) and upper critical magnetic field $H_{c2}(T)$ (in Bi$_2$Sr$_2$CuO$_6$ (Bi-2201) with $T_c \lesssim 15$ K near $T_c^*$ are shown in Fig. 2. A peak in the specific heat of high-$T_c$ cuprates was also observed at $T_c^*$ below which $H_{c2}(T)$ suddenly increased. Further, an abrupt jump-like increase of the critical velocity $v_c(T) = \sqrt{\Delta g(T)/m_B}$ three times and such a change of the superfluid density at $T_c \approx (0.6 - 0.7)T_c$ were observed in superfluid $^3$He [55].

Clearly, the sharp increasing of $\Delta g(T)$ at the vanishing of the gap $\Delta$ in $E_B(k)$ near $T_c$ leads to the jump-like increasing of $v_c(T)$ and superfluid density $\rho_s(T)$ at $T \approx T_c$.

![Image](70x272 to 280x418)

Figure 2: Temperature dependences of the critical magnetic fields $H_{c1}(T)$ and $H_{c2}(T)$ measured in superconducting cuprates. (a) Solid line is the fit of the experimental data (solid) for $H_{c1}(T)$ in YBCO [56] using the parameters $\mu_B = 1.7 \times 10^{19}$ cm$^{-3}$, $m_0 = 4.3m_e$, $R = 0.01cm$ and $\varepsilon_{BA} = 0.18$ eV. (b) Solid line is the fit of equation $H_{c2}(T) = \sqrt{\Delta H(T)H_{c1}(T)}$ to the experimental data (solid) for $H_{c2}(T)$ in Bi-2201 [61] using the parameters $\mu_B = 0.1 \times 10^{19}$ cm$^{-3}$, $m_0 = 5m_e$, $R = 0.5 \times 10^{-3}$ cm and $\varepsilon_{BA} = 0.13$ eV. Dashed line is by the Werthamer-Helfand-Hohenberg theory (see Ref. [61]). Insets show the kink-like behaviors of $\Delta g_{c}(T)$ near $T_c$.

The specific heat of a 3D superfluid Bose-liquid $C_s(T)$, diverges as $C_s(T) \sim (T - T_c)^{-0.5}$ near $T_c$ (where $\Delta g(T) \ll \mu_B(T) \ll k_BT_c$) [46, 47] and will exhibit a $\lambda$-like anomaly at $T_c$, as observed in high-$T_c$ cuprates [21, 27], organic superconductors [17] and superfluid $^4$He (see Fig. 1.9a in Ref. [42]). Such a behavior of $C_s(T)$ is similar to that of superfluid $^4$He. Note that, as $T$ approaches $T_c$ from below, the temperature dependences of $\mu_B$ and $\Delta g$ are defined as

$$\mu_B(T) \approx \mu_B(T_c) \left[1 + a(T_c - T)^{0.5}\right]$$

(11)

and

$$\Delta g(T) \approx 2\mu_B(T_c) \sqrt{\mu}(T_c - T)^{0.25},$$

(12)

where $a = 2(\xi_0\gamma_B T_c)^{-0.5}(\varepsilon_{BA}/k_BT_c)^{0.25}$ and $c_0 = \frac{\rho}{12}/3.912$. Therefore, the temperature derivatives of $\mu_B$ and $\Delta g$ entering the expression for $C_s(T)$ give rise to a pronounced $\lambda$-like divergence. By introducing the quantity of superfluid matter $\nu_B = N_B/N_A$ (where $N_B$ is the number of attracting bosonic Cooper pairs and $N_A$ is the Avogadro number, which is equal to the number of CuO$_2$ formula unit per unit volume) and the molar fraction of the superfluid bosonic carriers defined by $f_s = \nu_B/\nu$ (where $\nu = N/N_A$ is the amount of doped matter), we now write the molar specific heat of the superfluid Bose-gas in high-$T_c$ cuprates as

$$C_s(T) = f_s \frac{C_s(T)}{\nu_B} = f_s \frac{\mu_B k_BT_c^2}{2\mu_B(T_c)} \int_0^{\epsilon_{BA}} \frac{d\epsilon}{\sqrt{\epsilon^{3/2} \varepsilon_{BA}}} \times$$

$$\left\{ \frac{E_\nu^2(\epsilon)}{2(\nu - \mu_B(T_c)) \nu_B} \right\}$$

(13)

Here we have accounted for that $\Omega/\nu_B = N_B\nu_B/\nu_B = \nu_B N_A$ and $\nu_B = 1/\rho_B$. In doped cuprates the carriers are distributed between the polaronic band and the impurity band (with Fermi energy $\varepsilon_{FI}$) and the normal-state specific heat $C_n(T)$ above $T_c$ is calculated by considering three contributions from the excited components of Cooper pairs, the ideal Bose-gas of Cooper pairs and the unpaired carriers bound to impurities [63]. The fraction $f_p$ of carriers residing in the polaronic band and the other fraction $f_I$ of carriers residing in the impurity band are taken into account in comparing the specific heat $C_s(T)$ with the experiment. The total electronic specific heat $C_e(T) = C_s(T) + C_n(T)$ below $T_c$ is calculated and compared with the experimental data for $C_s(T)$ in cuprates (Fig. 3). The calculated results for $(\lambda_1(0)/\lambda_2(T_c))^2$ are also compared with the experimental data (Fig. 4).

Analytical solutions of Eqs. (5) and (6) near $T_c$ [45, 46] allow to estimate the bulk superconducting/superfluid transition temperature as

$$T_c \approx T_c^{3D} \approx T_{BEC} \left[1 + c_0\gamma_B \sqrt{\frac{k_BT_{BEC}}{\varepsilon_{BA}}} \right].$$

(14)
parameters $m$ compared with experimental values for $C$ (solid line) and $\lambda$ (black circles). According to [63], $C_c(T)$ is calculated by using the parameters $\varepsilon_f = 0.12$ eV, $\varepsilon_{FF} = 0.012$ eV, $f_p = 0.3$, $f_t = 0.7$, while superconducting contribution $C_{sc}(T)$ to $C(T)$ is calculated by using the parameters $\rho_B = 1.6 \times 10^{10}$ cm$^{-3}$, $m_B = 2.5 m_0$, $\tilde{\rho}_B(T_c) = 1.6$ meV and $f_t = 0.03$. The inset shows the calculated temperature dependence of $C_c(T)/T$ (solid line) compared with experimental $C_c(T)/T$ data for LSCO [27] (black circles). According to [63], $C_c(T)/T$ is calculated by using the parameters $\varepsilon_f = 0.1$ eV, $\varepsilon_{FF} = 0.06$ eV, $f_p = 0.4$, $f_t = 0.6$, while $C_{sc}(T)/T$ is calculated by using the parameters $\rho_B = 1.4 \times 10^{10}$ cm$^{-3}$, $m_B = 2.7 m_0$, $\tilde{\rho}_B(T_c) = 0.5$ meV and $f_t = 0.012$.

Figure 3: Temperature dependence of the specific heat of $\text{HgBa}_2\text{Cu}_2\text{O}_8$ measured near $T_c$ and above $T_c$ [23]. Solid line is the calculated curve for comparing with experimental points (black circles).}

Figure 4: Temperature dependence of $(\lambda_f(T)/\lambda_c(T))^2$ (solid line) is calculated by using the parameters $\rho_B = 1.29 \times 10^{10}$ cm$^{-3}$, $m_B = 5 m_0$ and $\varepsilon_{BA} = 0.08$ eV and compared with experimental data (o) for YBCO film [65].

$$T_{\text{REC}} = 3.31 \hbar^2 \rho_B^{2/3} / k_B m_B, \gamma_B < < 1.$$ By solving now the mean field equations for attractive 2D Bose systems, we see that the pair condensation of bosons will take place in the temperature range $0 < T < T_c$, while their single particle condensation occurs only at $T = 0$. The condensation temperature of attracting 2D bosons for arbitrary $\gamma_B = D_B \tilde{V}_B$ is given by [47]

$$T_c^{2D} = \frac{T_0}{\ln \left[ 1 - \exp \left( -\frac{2\pi}{2\gamma_B} \right) \right]}$$

which is in agreement with the result of Ref. [66] at $\gamma_B < < 1$, where $T_0 = 2\pi \hbar^2 \rho_B / k_B m_B$ and $\tilde{V}_B$ depends now on $I_B = D_B \ln(\varepsilon_{BR}/\varepsilon_{BA})$ and $D_B = m_B/2\pi\hbar$. When the effective (renormalized) mass of interacting bosons depends on $\rho_B$, the mass of bosons $m_B$ in the expressions for $T_{\text{REC}}$ and $T_0$ should be replaced by $m_B^* = m_B[1 - \rho_B V_B(0)/\varepsilon_{BR}]^{-1}$ [47]. Thus, both the $T_c$ and the $T_{\text{REC}}^{2D}$ is mainly controlled by $\rho_B$ and $\gamma_B$.

5. Vortex-like excitations above $T_c$ and gapless superconductivity and superfluidity below $T_c$

It would also be interesting to discuss the conditions for (i) the diamagnetism in the pseudogap state and vortex formation observed above the bulk superconducting transition temperature $T_c$ in high-$T_c$ cuprates, (ii) the vortex formation observed above the bulk superfluid transition temperature $T_J$ in liquid $^3$He, and (iii) the gapless superconductivity and superfluidity observed in high-$T_c$ cuprates and other systems somewhat below $T_c$ or far below $T_c$. We first discuss the origins of the gapless excitations and other effects observed in the superconducting/superfluid state of high-$T_c$ cuprates and related systems and then the vortex-like Nernst effect and diamagnetism observed above $T_c$ in high-$T_c$ cuprates and the vortex-like state observed above the $\lambda$-transition temperature $T_{\lambda}$ in liquid $^3$He.

So far, most researchers confuse the $s$-, $p$- and $d$-wave pairing states of fermionic quasiparticles in high-$T_c$ cuprates and other exotic systems with the usual superconducting/superfluid states. Therefore, in many cases the origins of the gapless superconductivity and superfluidity and gapless excitations, which are manifested in the power law temperature dependences of the superconducting/superfluid properties of high-$T_c$ cuprates and other systems, are rashly attributed to the nodes of $p$- and $d$-wave BCS-like gaps. From the above considerations, it follows that the unconventional superconductor/superfluid exhibiting a pseudogap behavior above $T_c$ at $\varepsilon_F < < 1$ eV is not in the fermionic limit of Cooper pairs but in the bosonic limit...
we find
By taking \( \gamma \)
T in the temperature range superconductivity persists at quasi-2D grain boundaries 2D grain boundaries than in the bulk and the residual 2D Bose systems. It follows that the superconducting vortex-like Nernst signals observed in high-T cuprates [67] and their nonexistence above \( T_c^* \) up to \( T_c \) in high-\( T_c \) cuprates.

We now return to the issue of the vortex-like excitations above \( T_c \). In 3D high-\( T_c \) cuprates with \( m_p \approx 2.1 m_e, m_B \approx 2 m_p, m_B^* \approx 1.05 m_B \) and \( \rho_B \approx 4.2 \times 10^{13} \text{cm}^{-2} \), we find \( T_{\text{BEC}} \approx 80 \text{ K} \). For quasi-2D grain boundaries in these systems, we use the values of \( m_p \approx 3m_e, m_B \approx 2m_p, m_B^* \approx 1.1m_B \) and \( \rho_B \approx 2.5 \times 10^{13} \text{cm}^{-2} \). We then obtain \( T_0 \approx 210 \text{ K} \). Further, we find \( T_{\text{BEC}}^{3D} \approx 1.135 T_{\text{BEC}} \approx 91 \text{ K} \) for \( \gamma_B = 0.3 \) and \( \epsilon_{\text{BA}}/k_B T_{\text{BEC}} = 10 \). By taking \( \gamma_B = 0.3 \) for quasi-2D grain boundaries, we find \( T_{\text{BEC}}^{2D} \approx 0.68 T_0 \approx 143 \text{ K} \). Within the superfluid Bose-liquid model in the mean-field approximation, thus the highest \( T_c \) is expected to arise in quasi-2D Bose systems. It follows that the superconducting transition temperature in the cuprates is higher at quasi-2D grain boundaries than in the bulk and the residual superconductivity persists at quasi-2D grain boundaries in the temperature range \( T_c < T < T_c^*(= T_c^{2D}) \), i.e., the stability of high-\( T_c \) superconductivity in cuprates is greater in quasi-2D than in 3D systems. Therefore, the vortex-like Nernst signals observed in high-\( T_c \) cuprates [10, 68, 69] are caused by the destruction of the bulk superconductivity in the 3D-to-2D crossover region and are associated with the existence of superconductivity at quasi-2D grain boundaries rather than with other effects. There is some confusion in the literature concerning the origins of the vortex-like and diamagnetic states, which have been found in unconventional cuprate superconductors above \( T_c \) [19, 20, 68, 69]. We argue that the vortex-like Nernst signals are not associated with the diamagnetic signal persisting above \( T_c \), since the vortex-like state should persist up to superconducting transition temperature \( T_{\text{BEC}}^{2D} = T_c \) at quasi-2D grain boundaries, while the diamagnetism above \( T_c \) is associated with the formation of bosonic Cooper pairs (with zero spin) and would persist up to pseudogap temperature \( T^* \geq T_c \). Another grain boundary effect is that the gap \( \Delta_g \) in the excitation spectrum of a 2D superfluid Bose condensate at \( T \neq 0 \) is larger than that in the excitation spectrum of a 3D superfluid Bose condensate at \( T > T_c \). Hence, the half-integer \( h/4e \) magnetic flux quantization is better manifested in the 3D-to-2D crossover region than in the bulk, as indeed observed at quasi-2D grain boundaries and in thin films of high-\( T_c \) cuprates [6], where the half-quantum vortices are associated with the excitations of pair condensate of bosonic Cooper pairs. Similarly, the new vortex topology in thin \( ^4 \text{He} \) superfluid film on porous media might be intermediate between the bulk superfluid liquid and flat superfluid film configuration, as discussed in Ref. [70]. This vortex-like state existing at temperatures \( T_1 < T < T_{\text{BEC}}^{2D} \) can be also interpreted as a result of the crossover from 3D to 2D nature of the superfluid state and formation of 3D vortices at the destruction of the bulk superfluidity in the 3D-to-2D crossover region (i.e., in thin \( ^4 \text{He} \) film on porous substrate).

6. The full phase diagram of the normal and superconducting states of high-\( T_c \) cuprates

The undoped cuprate compounds are antiferromagnetic (AF) insulators. Because the strong electron correlations (i.e., the strong Coulomb interactions between two holes on the same copper sites) drive these systems into the AF Mott insulating state. However, the strong Coulomb interactions of the lattice scale disappear in doped cuprates [13]. The distinctive feature of the doped cuprates is the polarizability of their crystal lattice in the presence of charge carriers introduced by doping. The self-trapping and pairing of doping carriers are more favorable in such polar materials than in non-polar solids. In the lightly doped cuprates, the strong and unconventional electron-phonon interactions are responsible for the existence of localized carriers and (bi)polaronic insulating state. Actually, a small level of doping (e.g., \( x \approx n/a = 0.02 \approx 0.03 \) [33, 61], where \( n \) is the density of doping carriers, \( n_a = 1/V_a \) is the density of the lattice atoms, \( V_a \) is the volume per CuO unit in the cuprates) results in the disappearance of AF order, the system undergoes a transition from the AF insulator to the (bi)polaronic insulator. Upon further doping, the cuprate compounds are converted into a pseudogap metal (above \( T_c \)) or a non-BCS high-\( T_c \) superconductor (below \( T_c \)).

The above results show that the high-\( T_c \) cuprates are characterized by low density of condensing (attracting) bosons \( \rho_B << n \). Here the true superconducting transition temperature \( T_c \) (the onset temperature of the \( \lambda \)-like second order phase transition) is determined by postulating that superconductivity in these systems originates from the superfluid condensation of a fraction of the normal-state Cooper pairs and is associated with a microscopic separation between superfluid and normal
bosonic carriers. Such a microscopic phase separation will likely occur just like the phase separation into the regions of a Bose solid (high-density limit) and a dilute Bose gas (low-density limit) described in Ref. [71]. The values of $T_c$ in non-BCS cuprate superconductors are actually determined by low densities of bosons and only a part of preformed Cooper pairs is involved in the superfluid Bose condensation. In 3D systems, the density of condensing (attracting) bosons is related to $n$ as $\rho_B = f_B n << n$, where $f_B$ is the fraction of superfluid bosons. According to Eqs. (14) and (15), $T_c$ first increases nearly as $T_c \sim (f_B n_x x)^{2/3}$ (in the 3D case) and $T_c \sim (f_B^2 n_x x)$ (in the 2D case), then reaches the maximum at optimal doping and exhibits the saturating or decreasing tendency with increase of $T_c/\rho_B^2$.

Thus, both curves $T_c$ and $T_v$ in high-$x$ing or decreasing tendency with increase of $x$ and $m_B^*$.

This general advantage of quasi-2D versus 3D systems is that superconductivity can be observed in a wider region of the phase diagram in the former than in the latter. The normal state of high-$T_c$ cuprates exhibits a pseudogap behavior [32]. Further, the onset temperature of the first-order phase transition $T_p$ separates two distinct superconducting phases of 3D high-$T_c$ cuprates, which arise at pair and single particle condensations of attracting bosonic Cooper pairs. The entire phase diagram of Bi$_2$Sr$_2$CaCu$_2$O$_8$ (Bi-2212) from Mott insulator to the heavily overdoped regime is shown in Fig. 5 where the characteristic temperatures $T_p$, $T_s$, and $T_v$ describe three distinct superconducting regimes, whereas two unusual metallic states exist below the crossover temperatures $T_p$ and $T^*$. The vortex-like state exists in the temperature range $T_c < T < T_v$, while the diamagnetic state persists up to the BCS-like pseudogap formation temperature $T^*$. The characteristic temperatures $T_p$, $T_s$, and $T_v$ describe three distinct superconducting regimes, whereas two unusual metallic states exist below the crossover temperatures $T_p$ and $T^*$. The vortex-like state exists in the temperature range $T_c < T < T_v$, while the diamagnetic state persists up to the BCS-like pseudogap formation temperature $T^*$.

7. Discussion and conclusions

In this paper, we have studied the bosonization of Cooper pairs and novel superconductivity/superfluidity in high-$T_c$ cuprates and other pseudogap matters, which are characterized by low Fermi energies $\epsilon_F << 1$eV. We have shown that these phenomena would occur in doped cuprate compounds, heavy-fermion and organic systems, liquid $^3$He and atomic Fermi gases under certain conditions. The universal and correct criterion for bosonization of Cooper pairs in such systems is formulated by using the uncertainty principle. We found that the condition $\epsilon_F < 2\epsilon_A$ is favorable for the formation of bosonic Cooper pairs and novel superconducting/superfluid states in low Fermi energy systems. By the use of this criterion, we might be able to realize an unconventional superconductor/superfluid not in the BCS-type but in the type of bosonic Cooper pairs. In this case the superconductivity/superfluidity is not simply caused by Cooper pairing on a Fermi surface and the formation of a BCS-like gap $\Delta_F$ does not necessarily lead the system to a superfluid state. Because the underlying mechanism of superconductivity/superfluidity in a Fermi system depends on the fermionic or bosonic nature of Cooper pairs. When the Fermi energy $\epsilon_F$ becomes comparable with the characteristic energy $\epsilon_A$ of the effective attraction between fermions, superconducting and superfluid matters are in the bosonic regime $\epsilon_F \lesssim 2\epsilon_A$.

The BCS-type superconductivity/superfluidity would occur in the fermionic limit of Cooper pairs (see also Ref. [39]) and can exist in conventional Fermi systems, in particular, in ordinary metals ($\epsilon_F/\hbar\omega_D \sim 10^2$) and heavily overdoped cuprates (at $E_F = 0$ and $\epsilon_F >> \epsilon_A = \hbar\omega_D$), where, unlike in unconventional cuprate superconductors, the superconducting state is characterized by the BCS-like order parameter and the onset temperature of Cooper pairing $T^*$ coincides with $T_c$. However, high-$T_c$ cuprates and other unconventional superconductors/superfluids could be in the bosonic limit of Cooper pairs and their Fermi energy is so small [36].
that the size of the Cooper pair $a_c$ becomes small and less than the spatial separation between two Cooper pairs. Hence, the BCS-Eliashberg-like theory of superfluid Fermi-liquid cannot be used to elucidate the mechanisms of superconductivity in many systems at $\varepsilon_F < 2\varepsilon_A$. According to the two-stage Fermi-Bose-liquid model [31, 57], the BCS-like pairing theory of fermions, applied to high-$T_c$ cuprates and other related systems, can describe the formation of Cooper pairs and pseudogap at $T^* > T_c$, but it fails to account for their novel superconducting/superfluid states and properties. Here we have demonstrated that novel types of superconductivity/superfluidity occurring in the bosonic limit of Cooper pairs exist in high-$T_c$ cuprates and other low Fermi energy systems. In high-$T_c$ cuprates the bosonic Cooper pairs (with zero spin) and diamagnetic state are already formed at a temperature $T^*$ well above $T_c$, but high-$T_c$ superconductivity is only established when the part of such composite bosons condenses into a Bose superfluid at $T_c$. In these and other related systems, the formation of a BCS-like pairing state of fermions is a necessary, but not a sufficient, condition for the appearance of superconductivity/superfluidity.

Actually, the discussed BCS-like pairing of fermions at $T^* > T_c$ may be considered as a first step toward a more complete treatment of Bose-type superconductivity/superfluidity in such systems. Therefore, two main criteria for the occurrence of unconventional superconductivity/superfluidity in Fermi systems described by a two-stage Fermi-Bose-liquid model are following: (i) the BCS-like order parameter $\Delta_F$ should appear first at $T^* > T_c$, and (ii) the BCS-like order parameter (or energy gap) $\Delta_F$ and the new coherence parameter $\Delta_B$ (defining the boson superfluid stiffness) should coexist below $T_c$. The latter criterion is a necessary and sufficient condition of the novel superconductivity/superfluidity. The above results clearly demonstrate that superconductivity/superfluidity of bosonic Cooper pairs just like superfluidity of $^{3}$He atoms is well described by the mean field theory of attracting bosons and the true superconducting/superfluid phase is identified with the coherence parameter $\Delta_B$ appearing below $T_c$. As the temperature is decreased, the pair condensation of attracting bosons occurs first at $T^*_c$. Further decrease of the temperature leads to their single particle condensation at $T^*_c$ somewhat below $T_c$ (in three dimensions) or at $T = 0$ (in two dimensions). The gapless superconductivity/superfluidity occurs below $T^*_c$ due to the vanishing of the gap $\Delta_c$ in $\varepsilon_F(k)$ at $T < T^*_c$. The coherent single particle and pair condensates of bosonic Cooper pairs exist as the two different superfluid phases and arise from an effective attraction between these composite bosons in some domains of momentum space. According to the superfluid Bose-liquid model, the cuprate high-$T_c$ superconductivity is more robust in quasi-two-dimensions than in three dimensions, i.e., $T_c$ is higher in quasi-2D than in 3D systems. We see therefore that, three different superconducting phases exist in high-$T_c$ cuprates where the coherent pair condensate of bosonic Cooper pairs persists up to the temperature $T_c = T^{2D} > T^{3D}$ at quasi-2D grain boundaries as the superfluid phase and the coherent pair and single particle condensates of such composite bosons in 3D systems exist as the two distinct superfluid phases below $T_c = T^{3D}$. It follows that the persistence of the vortex-like excitations in high-$T_c$ cuprates above $T_c$ is caused by the destruction of the bulk superconductivity. The existence of such vortices is expected below the temperature $T_c$ lower than $T^*$ but higher than $T_c$. One of the important conclusions is that diamagnetism in the pseudogap state and vortex formation above $T_c$ in high-$T_c$ cuprates are unrelated phenomena.

Clearly, the condensate and excitations of a Bose-liquid are unlike those of a BCS-like Fermi liquid. Therefore, not all the experimental methods are able to identify the true superconducting order parameter in high-$T_c$ cuprates and other pseudogap matters. For example, the single-particle tunneling spectroscopy and ARPES provide information about the excitations gaps at the Fermi surface but fail to identify the true superconducting order parameter appearing below $T_c$ in non-BCS superconductors. Actually, a prolonged discussion of the origin of unconventional superconductivity in the cuprates on the basis of tunneling and ARPES data has nothing to do with the true mechanism of high-$T_c$ cuprate superconductivity. We note here that the superconducting/superfluid order parameter in high-$T_c$ cuprates and other pseudogap matters should not be identified as a BCS-like gap and the gapless superconductivity/superfluidity in these systems should also not be attributed to the point and line nodes of the BCS-like ($p$- and $d$-wave) gaps.

There is now experimental evidence that the BCS-like fermionic excitation gap $\Delta_F$ exists as a pseudogap in high-$T_c$ cuprates [10, 12] and other related superconductors [72, 73, 74] and atomic Fermi gases [32, 75, 76]. In non-BCS superconductors, the superconducting/superfluid order parameter $\Delta_B$ appearing below $T_c$ and the BCS-like gap $\Delta_F$ opening on the Fermi surfaces above $T_c$ have different origins. Unconventional high-$T_c$ superconductivity in cuprates is controlled by the coherence parameter (superfluid stiffness) $\Delta_B \sim \rho_B$ rather than BCS-like pairing gap $\Delta_F \gg \Delta_B$ and appears under the coexistence of two order param-
eters $\Delta_F$ and $\Delta_{SC}(\equiv \Delta_B)$. The BCS-like pseudogap is therefore a necessary ingredient for high-$T_c$ superconductivity in the cuprates. Some selected experimental techniques can provide information about the new superconducting order parameter. The above presented results show that the thermodynamic methods and the methods of critical current and magnetic field measurements are sensitive to the identification of $\Delta_{SC}(T)$ in unconventional superconductors.

Thus, the criterion for bosonization of Cooper pairs $\varepsilon_F \lesssim 2 \varepsilon_A$ allows us to find the real applicability boundary (which up to now remains unknown) between BCS-type and Bose-type regimes of superconductivity/superfluidity. This criterion and other necessary and sufficient criteria formulated here should be satisfied for the occurrence of the unconventional superconductivity/superfluidity. The above theoretical predictions and their experimental confirmations speak strongly in favor of the existence of novel superconducting/superfluid states, which arise in high-$T_c$ cuprates and other pseudogap matters at single particle and pair condensations of attracting bosonic Cooper pairs. The critical behavior of a superfluid Bose liquid of Cooper pairs near $T_c$ is similar to that of liquid $^4$He near the $\lambda$-transition. Within the mean field theory of a superfluid Bose-liquid, it is possible to describe the following unexplained features of unconventional superconductors and superfluids: (i) the key features of the phase diagrams of high-$T_c$ cuprates (e.g., vortex-like state existing at temperatures $T_c < T < T^{2D}_c$ and two distinct superconducting phases below $T_c$), (ii) the two distinct superconducting phases in heavy-fermion systems below $T_c$, (iii) the superfluid $A$ and $B$ phases in $^3$He, (iv) the superfluid phase in $^4$He below $T_A$ and the vortex-like state existing at temperatures $T_A < T < T^{2D}_c$ in the crossover regime between the bulk superfluid liquid and thin $^4$He superfluid film, (v) the unconventional superfluidity in ultracold atomic Fermi gases, (vi) the $\alpha$-clustering in nuclei (i.e. $\alpha$-particle structure of nuclei) and high stability of magic and twice magic nuclei, which is associated with the single particle and pair condensations of attracting bosonic Cooper pairs of nucleons both in proton and in neutron subsystems. We have shown that this theory provides a consistent picture of the distinctive superconducting properties (e.g., $\lambda$-like second-order phase transition at $T^*_c$, first-order phase transition and kink-like temperature dependences of superconducting parameters near $T^*_c$) of high-$T_c$ cuprates and other related materials.

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