THE UNIFORM MORDELL–LANG CONJECTURE

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Abstract. The Mordell–Lang conjecture for abelian varieties states that the intersection of an algebraic subvariety $X$ with a subgroup of finite rank is contained in a finite union of cosets contained in $X$. In this article, we prove a uniform version of this conjecture, meaning that the number of cosets necessary does not depend on the ambient abelian variety. To achieve this, we prove a general gap principle on algebraic points that extends the gap principle for curves embedded into their Jacobians, previously obtained by Dimitrov–Gao–Habegger and Kühne. Our new gap principle also implies the full uniform Bogomolov conjecture in abelian varieties.

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1. Introduction

Throughout this article, $F$ is an algebraically closed field of characteristic $0$; particularly relevant cases are $F = \mathbb{Q}$ and $F = \mathbb{C}$. Furthermore, we let $A$ be an abelian variety defined over $F$ and consider an ample line bundle $L$ on $A$. The translates of abelian subvarieties by closed points in $A$ are called cosets.

The main result of our article is the following uniform version of the well-known Mordell–Lang conjecture (see [Lan62, p. 138]).

**Theorem 1.1** (Uniform Mordell–Lang Conjecture). For all integers $g,d \geq 0$, there exists a constant $c(g,d) > 0$ with the following property. Let $X \subseteq A$ be an irreducible closed subvariety and $\Gamma \subseteq A(F)$ a subgroup of finite rank. Then the intersection $X(F) \cap \Gamma$ is covered by at most

$$c(\dim A, \deg_L X)^{1+\text{rk} \Gamma}$$

cosets contained in $X$.

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For readers’ convenience, let us restate this theorem in a more explicit fashion. For every polarized abelian variety of dimension $g$, every irreducible closed subvariety $X \subseteq A$ of degree $d$ (with respect to the given polarization) and every subgroup $\Gamma$ of finite rank $\rho$, it claims the existence of cosets

$$x_i + B_i \subseteq X, \quad 1 \leq i \leq N \leq c(g, d)^{1+\rho},$$

such that

$$X(F) \cap \Gamma = \bigcup_{i=1}^{N} (x_i + B_i)(F) \cap \Gamma.$$  

The original Mordell–Lang conjecture asserts that finitely many cosets

$$x_i + B_i \subseteq X, \quad 1 \leq i \leq N,$$

are sufficient to cover the intersection $X(F) \cap \Gamma$ as in (1.1), without suggesting any further quantitative control on their number. This original version, which combines both the Manin–Mumford conjecture [Ray83] and the Mordell conjecture [Fal83], was established by Faltings [Fal91,Fal94], following work of Hindry [Hin88] and Vojta [Voj91]. An explicit upper bound for the number of cosets was obtained by Rémond [Rém00a], which additionally depends on the ambient abelian variety $A$ via its Faltings height $h_{\text{Fal}}(A)$.

The novelty here is the complete removal of this very dependence on the ambient abelian variety $A$, confirming a folklore expectation that can be found for example in [DP07, Conj. 1.8]. It is known under the name of uniform Mordell–Lang conjecture because the number $N$ in (1.1) must depend on $g$, $d$ and $\rho$. For curves embedded in their Jacobian, it dates back to a question of Mazur [Maz86, top of p. 234], which has been answered affirmatively by work of Dimitrov, the first-named author, Habegger and the third-named author [DGH21, Küh21]. Readers may profit from the survey of the first-named author [Gao21] for an overview of these previous works and their implications for rational points on algebraic curves. Let us remark that before these works, the only uniform results of Mordell-Lang type were obtained by David and Philippon [DP07, Théorème 1.13] for subvarieties of self-products of an elliptic curve. It should be also noted that they give a completely explicit constant in this special case. In this regard, it is interesting to ask whether the present arguments can yield explicit upper bounds on the number of cosets similar to those of David and Philippon [DP07] or if substantial new ideas are necessary.

Our proof of Theorem 1.1 relies on the ideas established in the series of work [DGH19, DGH21, DGH22, Küh21] building upon Vojta’s approach [Voj91] to the Mordell conjecture. However, several new difficulties arise in the higher-dimensional case considered here; see §1.3 for a short discussion. A comprehensive outline of all other sections can be found at the end of the introduction.

1.1. The Ueno locus and first reductions. The Ueno locus of $X$ is the union of positive dimensional cosets contained in $X$. A result of Kawamata [Kaw81, Thm. 4] states that it is Zariski closed. Write $X^\circ$ for its complement in $X$; then $X^\circ$ is a Zariski open set. Note that the Ueno locus of a smooth, proper curve of genus $g \geq 2$ embedded into its Jacobian is empty.
A recursive argument in §7.4 reduces Theorem 1.1 to the following weaker statement.

Theorem 1.1’. For all integers \(g, d \geq 0\), there exists a constant \(c(g, d) > 0\) with the following property. Let \(X \subseteq A\) be an irreducible closed subvariety and \(\Gamma \subseteq A(F)\) a subgroup of finite rank. Then

\[
\#X^\circ(F) \cap \Gamma \leq c(\dim A, \deg_X L)^{1+\text{rk}\Gamma}.
\]

Besides reducing to Theorem 1.1’, we also use a specialization argument of Masser [Mas89] to assume \(F = \mathbb{Q}\) in our main arguments. This is essential for most of our arithmetically flavored techniques revolving around the theory of heights. Although Theorem 1.1 could itself be understood as a purely geometric assertion (e.g. for \(F = \mathbb{C}\)), we heavily rely on arithmetic tools throughout our proof.

1.2. A generalized gap principle. Our approach is modeled on Vojta’s proof of the Mordell conjecture [Voj91] and its later refinements, notably the quantitative ones obtained by Rémond [Rémo00a, Rémo00b]. His method leads to a dichotomy between algebraic points of large and small Néron–Tate height, which we call large and small points for simplicity.

A uniform count of large points can be done by using the work of Rémond [Rémo00a, Rémo00b], which provides explicit, generalized versions of Mumford’s and Vojta’s inequalities. Compared to the case of curves in their Jacobians dealt with in [DGH19, DGH21], some extra work is actually needed to establish Mumford’s inequality. In particular, we need to invoke the induction hypothesis twice to handle large points. We give a detailed account in Appendix A without claiming originality.

Our main contribution is a uniform count of small points, and we achieve this by establishing another kind of gap principle for algebraic points. In the case of curves in their Jacobians, preceding work [DGH21, Küh21] has been subsumed under such a New Gap Principle [Gao21, Thm. 4.1]. In this article, we take the same perspective and generalize it as follows. We say that a subvariety \(X \subseteq A\) generates \(A\) if the smallest abelian subvariety containing \(X - X\) is \(A\). Furthermore, we let \(\hat{h}_{L \otimes [-1]^* L}\) denote the Néron–Tate height on \(A(\mathbb{Q})\) associated with the symmetric ample line bundle \(L \otimes [-1]^* L\).

Theorem 1.2 (New Gap Principle). There exist positive constants \(c_1 = c_1(\dim A, \deg_L X)\) and \(c_2 = c_2(\dim A, \deg_L X)\) with the following property: For any irreducible closed subvariety \(X \subseteq A\) that generates \(A\), the set

\[
\left\{ P \in X^\circ(\mathbb{Q}) : \hat{h}_{L \otimes [-1]^* L}(P) \leq c_1 \max\{1, h_{\text{Fal}}(A)\} \right\},
\]

is contained in some Zariski closed \(X' \subseteq X\) with \(\deg_L(X') < c_2\).

Let us remark that this theorem has been predicted by [Gao21, Conj.10.5’]. The two examples constructed at the end of [Gao21, §10.2] show that, in contrast to the case of curves embedded into their Jacobians, one can neither get rid of the assumption that \(X\) generates \(A\) nor assert that (1.3) is a finite set of uniformly bounded cardinality.

We remark that Theorem 1.2 implies a uniform version of the Bogomolov conjecture, generalizing [Küh21, Thm.3] in the case of curves embedded in their Jacobians. Since this result is of independent interest, we state it here explicitly as well.
Theorem 1.3 (Uniform Bogomolov Conjecture). There exist positive constants \( c_3 = c_3(\dim A, \deg_L X) \) and \( c_4 = c_4(\dim A, \deg_L X) \) with the following property: For each irreducible subvariety \( X \) of \( A \), we have

\[
\# \left\{ P \in X^\circ(\overline{\mathbb{Q}}) : \hat{h}_{L(\mathbb{Q})}(P) \leq c_3 \right\} < c_4.
\]

We emphasize that as in the case of curves, to obtain Theorem 1.1 it is necessary to use the New Gap Principle for (1.3) instead of (1.4) because the dichotomy of large and small points is in comparison to \( \max \{ 1, h_{\text{Fal}}(A) \} \). Note that unlike the case of curves embedded into their Jacobians, this theorem is not formally a special case of Theorem 1.2 although a deduction from Theorem 1.2 is easy and given in 

An analogue of Theorem 1.3 for curves in algebraic tori was proven by Bombieri–Zannier [BZ95] (compare also [DP99, AD06]). For the case of abelian varieties considered here, Theorem 1.3 was known in some selected cases [DP07, DKY20, Kühl21]. Let us note that all these results except for [Kühl21] have rather explicit constants, in contrast to our general Theorem 1.3 here.

After the first version of the current article appeared as a preprint, Yuan [Yua21] gave another proof of Theorem 1.2 in the case of curves embedded into their Jacobians previously considered in [DGH21, Kühl21]. His approach has the advantage to work in the function field case (in positive characteristic) as well. It relies on previous work of Zhang, Cinkir, and de Jong [Zha93, Zha10, Cini11, dJ18] for lower bounds on the self-intersection numbers of the admissible canonical bundles of curves over global fields. Little is known in this direction for subvarieties of higher dimension and it seems very hard to generalize Yuan’s proof to the setting considered here. He also uses the new theory of Yuan–Zhang [YZ21] on adelic line bundles over quasi-projective varieties.

1.3. Ideas of the proof. Non-degenerate subvarieties of abelian schemes, a notion introduced by Habegger [Hab13] and extensively studied by the first-named author [Gao20], have played a central role in previous work [DGH19, DGH21, DGH22, Kühl21] and continue to do so in the current article. They derive their importance from the fact that they are the natural setting for both

1. the height inequality of [DGH21] Thm. 1.6 and B.1], which allows a comparison of the Néron–Tate height and the height on the base variety, as well as

2. the equidistribution theorem [Kühl21] Thm. 1].

More recently, Yuan and Zhang have reproven both these results using their general theory of adelic line bundles over quasi-projective varieties [YZ21].

A starting problem in the current paper is hence the construction of an appropriate non-degenerate subvariety. In the case of curves embedded in their Jacobians, by the quasi-finiteness of the Torelli map, we could restrict ourselves to consider subvarieties of an abelian scheme \( A \to S \) of maximal variation, that is, the moduli map from \( S \) to the moduli space of abelian varieties is generically finite. In the current paper, we need a space parametrizing all subvarieties of a fixed degree in abelian varieties of a fixed dimension and polarization. While there is a natural candidate, the Hilbert scheme, the moduli map from it to the moduli space of abelian varieties has positive dimensional fibers. This makes the construction of the relevant non-degenerate subvarieties significantly harder than the case of curves.
We resolve this problem by showing that the moduli map on the total space, when restricted to the universal family over an open subset of the Hilbert scheme, becomes still generically finite after taking a high enough fibered power (Lemma 3.3), as inspired by the second-named author’s work [Ge21 §3]. Then [Gao20, Thm. 10.1] gives us the desired non-degeneracy (Proposition 3.4). We expect the idea of this construction to be applicable in other settings, for example to study uniformity problems for semiabelian varieties, which would extend in particular the uniform results of Bombieri–Zannier [BZ95] on algebraic tori, and maybe even to study related problems in some families of dynamical systems. The need for non-degeneracy make it necessary to shift back and forth to fibered products. In the course of this, we have to control the exceptional sets appearing. For this purpose, we provide a technical key Lemma 4.3.

To prove uniform bounds as in Theorem 1.1′, it is important to work with only finitely many families at all times. While this is automatic in the setting for curves embedded into their Jacobians (by an induction on the dimension of the subvariety), we have to carefully handle the invariants involved, in particular the polarization type. Even if one is only interested in the case of principal polarization, our inductive proof generally invokes abelian variety of any polarization degree. We use various techniques to overcome these problems, including a classical result of Mumford and the Poincaré biextension on the universal abelian variety.

1.4. Outline of the article. In §2 we review basic facts on abelian varieties and polarization types. In particular, we give a bound Lemma 2.5 on the degree of the abelian subvariety generated by $X$, using an argument suggested to us by Marc Hindry. This allows us to avoid any dependence on the polarization degree $\deg_L(A)$ itself in the final results.

In §3 we construct the families of non-degenerate subvarieties used in our main arguments. We also deduce the key result to establish their non-degeneracy here (Proposition 3.4) from the results of [Gao20].

In §4 we apply the height inequality [DGH21, Thm.1.6 and B.1] to these non-degenerate subvarieties. The main result of this section is Proposition 4.1 which roughly proves an analogue of Theorem 1.2, albeit invoking more invariants, with $c_{1}'$ replaced by the set

$$\left\{ x \in X^\circ(\overline{\mathbb{Q}}) : \hat{h}_{L^\circ[-1]^*L}(x) \leq c_1' \max\{1, h_{\text{Fal}}(A)\} - c_3' \right\}$$

for certain uniform $c_1', c_3' > 0$. Note that this set is exactly (1.3) if $h_{\text{Fal}}(A) \geq \max\{1, 2c_3'/c_1'\}$ up to modifying the constant, and becomes trivially empty if $h_{\text{Fal}}(A) < c_3'/c_1'$. For counting purposes, a key technical Lemma 4.3 allows us to use the non-degenerate fibered products of the (in general degenerate) Hilbert schemes constructed in §3.

The main result of §5 is Proposition 5.1, which again has the same form as Theorem 1.2 but without $h_{\text{Fal}}(A)$; here, the set (1.3) replaced by the set

$$\left\{ x \in X^\circ(\overline{\mathbb{Q}}) : \hat{h}_{L^\circ[-1]^*L}(x) \leq c_3'' \right\}$$

for a certain uniform $c_3'' > 0$. Its proof invokes the equidistribution theorem [Kühl21, Thm. 1] for the same non-degenerate subvarieties as in §4. The proof follows the classical strategy of Ullmo [Ull98] and Zhang [Zha98] with substantial new technical difficulties. Lemma 4.3 is again used.
In §6, we combine Propositions 4.1 and 5.1 into the desired gap principle stated in Theorem 1.2.

In §7, we show how to deduce the uniform Mordell–Lang conjecture by combining the gap principle with a result of Rémond. In addition, we use a specialization argument of Masser to reduce to the case $F = \mathbb{Q}$ (Lemma 7.3). We also include an argument to deduce Theorem 1.1 from Theorem 1.1′.

In §8, we show how to deduce the uniform Bogomolov conjecture from the new gap principle (Theorem 1.2).

In Appendix A, we give a more detailed account of Rémond’s result that is one of the two essential ingredients for the proof in §7.

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2. Preliminaries on abelian varieties

In this section, we include some basic results on abelian varieties. Throughout the section, let $A$ be an abelian variety defined over $\overline{\mathbb{Q}}$ of dimension $g$.

2.1. Polarizations. Let $\text{Pic}(A)$ be the Picard group of $A$ and $\text{Pic}^0(A)$ be the connected component of the identity. For $L \in \text{Pic}(A)$, denote its Chern class by $c_1(L) \in H^2(A(\mathbb{C}), \mathbb{Z})$. Let $A^\vee$ be the dual abelian variety of $A$. Then $\text{Pic}^0(A) = A^\vee(\overline{\mathbb{Q}})$.

For each $a \in A(\overline{\mathbb{Q}})$, let $t_a : A \to A$ be the translation-by-$a$ map. Given $L \in \text{Pic}(A)$, it induces a group homomorphism between dual abelian varieties $\phi_L : A \to A^\vee$ by sending $a \in A(\overline{\mathbb{Q}})$ to $t_a^* L \otimes L^{-1} \in \text{Pic}^0(A)$.

When $L$ is ample, the homomorphism $\phi_L$ is moreover an isogeny, i.e. a surjective group homomorphism with finite kernel, in which case we say $\phi_L$ is a polarization of $A$. The polarization is called a principal polarization if $\phi_L$ is an isomorphism.

Convention. We will use the following convention. By a polarized abelian variety, we mean a pair $(A, L)$ where $A$ is an abelian variety and $L$ is an ample line bundle. We say that $(A, L)$ is defined over $\overline{\mathbb{Q}}$ if both $A$ and $L$ are defined over $\overline{\mathbb{Q}}$.

The following Lemma is [Deb05, Theorem 6.10], which says that the polarization is uniquely determined by the Chern class of the line bundle.

**Lemma 2.1.** Let $L$ and $L'$ be two ample line bundles on $A$. Then $L$ and $L'$ define the same polarization if and only if $c_1(L) = c_1(L')$. 
Next we define the polarization type. Let \((A, L)\) be a polarized abelian variety.

Write \(A(\mathbb{C}) = \mathbb{C}^g/\Lambda\) for some lattice \(\Lambda \subseteq \mathbb{C}^g\) of rank \(2g\). There is a canonical isomorphism between \(H^2(A(\mathbb{C}), \mathbb{Z})\) and \(\text{Alt}^2(\Lambda, \mathbb{Z})\), the group of \(\mathbb{Z}\)-bilinear alternating forms \(\Lambda \times \Lambda \rightarrow \mathbb{Z}\). Thus \(c_1(L)\) defines a \(\mathbb{Z}\)-bilinear alternating form

\[
E: \Lambda \times \Lambda \rightarrow \mathbb{Z}.
\]

As \(L\) is ample, \(c_1(L)\) is positive definite, and hence \(E\) is non-degenerate. So there exists a basis \((\gamma_1, \ldots, \gamma_{2g})\) of \(\Lambda\) under which the matrix of \(E\) is

\[
\begin{bmatrix}
0 & D \\
-D & 0
\end{bmatrix}
\]

where \(D = \text{diag}(d_1, \ldots, d_g)\) with \(d_1| \cdots |d_g\) positive integers; see [Deb05, Proposition 6.1]. Moreover, the matrix \(D\) is uniquely determined. We say \(D\) is the polarization type of \((A, L)\). Define the Pfaffian by

\[
\text{Pf}(L) := \det(D).
\]

**Lemma 2.2.** Let \(L\) be an ample line bundle on \(A\). Then

(i) \(\dim H^0(A, L) = \text{Pf}(L)\).

(ii) \(\deg_L(A) = g! \cdot \text{Pf}(L)\).

(iii) let \(f: A' \rightarrow A\) be an isogeny, then \(\dim H^0(A', f^*L) = \deg(f) \dim H^0(A, L)\).

(iv) there exist an abelian variety \(A_0\), an ample line bundle \(L_0\) on \(A_0\) defining a principal polarization, and an isogeny \(u_0: A \rightarrow A_0\) such that \(L \cong u_0^* L_0\); moreover, \(\deg(u_0) = \deg_L(A)/g!\).

**Proof.** For (i), (ii) and (iii), it suffices to prove the assertions over \(\mathbb{C}\). Then (i) is [BL04, Corollary 3.2.8], (ii) is just the Riemann–Roch theorem [BL04, Section 3.6] and (iii) is [Deb05, Corollary 6 to Proposition 6.12]. (iv) is [Mum74, pp. 216, Corollary 1 and its proof].

We often work with symmetric ample line bundles for Néron-Tate heights. For this purpose, we need the following lemma.

**Lemma 2.3.** Let \(L\) and \(L'\) be two symmetric ample line bundles on \(A\). If \(c_1(L') = c_1(L)\), then \(\hat{h}_L = \hat{h}_{L'}\).

**Proof.** We have \(c_1(L' \otimes L^{\otimes -1}) = c_1(L') - c_1(L) = 0\). So \(L' \otimes L^{\otimes -1} \in \text{Pic}^0(A) = \text{A}^\vee(\overline{\mathbb{Q}})\). The morphism of abelian varieties \(\phi_L: A \rightarrow A^\vee, a \mapsto t^*_a L \otimes L^{\otimes -1}\), is an isogeny because \(L\) is ample. So there exists \(a \in A(\overline{\mathbb{Q}})\) such that \(\phi_L(a) = L' \otimes L^{\otimes -1}\). Thus \(L' \cong t^*_a L\). Therefore for each \(x \in A(\overline{\mathbb{Q}})\), we have

\[
\hat{h}_{L'}(x) = \hat{h}_{t^*_a L}(x) = \hat{h}_L(a + x).
\]

Take \(x \in A(\overline{\mathbb{Q}})_{\text{tor}}\), then \(\hat{h}_{L'}(x) = \hat{h}_{L}(x) = 0\) for the symmetric ample line bundles \(L\) and \(L'\). So (2.3) yields \(0 = \hat{h}_L(a)\). But then \(a \in A(\overline{\mathbb{Q}})_{\text{tor}}\) since \(L\) is symmetric ample, and hence (2.3) yields \(\hat{h}_{L'}(x) = \hat{h}_L(x)\) for all \(x \in A(\overline{\mathbb{Q}})\).
2.2. Degree estimates. The degree of a closed subvariety $X$ of $A$ with respect to an ample line bundle $L$ is defined as follows. If $X$ is irreducible, set $\deg_L X := c_1(L)^{\dim X} \cap [X]$ where $[X]$ is the cycle of $A$ given by $X$. For general $X$, set $\deg_L X := \sum_i \deg_L X_i$ where $X = \bigcup X_i$ is the decomposition into irreducible components. We use the notation from \cite{Ful98} Chapters 1 and 2] freely in the following.

**Lemma 2.4.** Assume that $L$ is very ample. Let $Y$ and $Y'$ be irreducible subvarieties of $A$. Then

$$\deg_L (Y + Y') \leq 4^{\dim Y + \dim Y'} \deg_L Y \cdot \deg_L Y'. \quad (2.4)$$

**Proof.** Write $p_i \colon A \times A \to A, (x, y) \mapsto (x, y)$, for the natural projection to the $i$-th factor. Then $L^{p2} := p_1^* L \otimes p_2^* L$ is an ample line bundle on $A \times A$ and we have $c_1(L^{p2}) = p_1^* c_1(L) + p_2^* c_1(L)$. For readability, write $d$ and $d'$ for the dimension of $Y$ and $Y'$, respectively. By \cite{Ful98} Prop. 2.5 and Rmk. 2.5.3, it follows that

$$\deg_{L^{p2}} (Y \times Y') = c_1(L^{p2})^{d + d'} \cap [Y \times Y'] \quad (2.5)$$

$$= \sum_{i=0}^{d + d'} \binom{d + d'}{i} (c_1(p_1^* L)^i \cap c_1(p_2^* L)^{d + d' - i} \cap [Y \times Y'])$$

For reasons of dimension, the only non-vanishing term in this sum is for $i = d$. Therefore,

$$\deg_{L^{p2}} (Y \times Y') = (d + d') \deg_L Y \cdot \deg_L Y'. \quad (2.6)$$

Consider the isogeny

$$\alpha \colon A \times A \to A \times A, \quad (x, y) \mapsto (x + y, x - y),$$

of degree $2^q$. We recall that $c_1(\alpha^* L^{p2}) = 2c_1(L^{p2})$ by \cite{HS00} Prop. A.7.3.3]. By (2.6), it follows that

$$\deg_{\alpha^* L^{p2}} (Y \times Y') = 2^{d + d'} \binom{d + d'}{d} \deg_L Y \cdot \deg_L Y'.$$

We are ready to prove (2.4). Indeed, using $p_1(\alpha(Y \times Y')) = Y + Y'$ we obtain

$$\deg_L (Y + Y') \leq \deg_{p_1^* L \otimes p_2^* L} (\alpha(Y \times Y')) \leq \deg_{\alpha^* L \otimes \alpha^* L} (Y \times Y') = \deg_{\alpha^* L^{p2}} (Y \times Y');$$

the first inequality follows for example from \cite{Dil22} Lem. 2.4] and the second one from the projection formula \cite{Ful98} Prop. 2.5(c)]. We conclude the proof by combining the last two inequalities and using $(d + d') \leq 2^{d + d'}$.

**Lemma 2.5.** Let $X$ be an irreducible subvariety of $A$ and let $A'$ denote the abelian subvariety generated by $X - X$. Then $\deg_L A' \leq g \deg_L (X)^{2^q}.$

**Proof.** Replacing $L$ by $L^{p3}$, we may and do assume that $L$ is very ample. We write $r = \dim X$

The ascending chain

$$(X - X) \subseteq (X - X) + (X - X) \subseteq (X - X) + (X - X) + (X - X) \subseteq \cdots$$
of closed irreducible subvarieties becomes stationary as soon as two consecutive elements are equal. By dimension reasons, we infer that

\[ A' = (X - X) + (X - X) + \cdots + (X - X). \]

To conclude the proof, we claim the inequality

\[ \deg_L((X - X) + \cdots + (X - X)) \leq 8^{2k} \deg_L(X)^{2g} \]

for every \( k \geq 1 \). For \( k = 1 \), we apply (2.4) to \( Y = X \) and \( Y' = -X \) and get

\[ \deg_L(X - X) \leq 8^r \deg_L(X)^2. \]

Similarly, we get

\[ \deg_L((X - X) + \cdots + (X - X)) \leq 8^{kr} \cdot \deg_L((X - X) + \cdots + (X - X)) \cdot \deg_L(X - X) \]

for every integer \( k \geq 2 \). Combining these inequalities, we obtain

\[ \deg_L(A') \leq 8^{(g+2)r} \deg_L(X - X)^g < 8^{(g+1)r} \deg_L(X)^{2g}, \]

whence the assertion of the lemma. \( \square \)

3. HILBERT SCHEMES AND NON-DEGENERACY

Let \( r \geq 1 \) and \( d \geq 1 \) be two integers. In this section, we work over \( \overline{\mathbb{Q}} \), i.e. all objects and morphisms are defined over \( \overline{\mathbb{Q}} \) unless said otherwise.

The goal of this section is to introduce the restricted Hilbert schemes (3.6) and prove a non-degeneracy result, Proposition 3.4. This is a main new ingredient to prove the Uniform Mordell–Lang Conjecture for higher dimensional subvarieties of abelian varieties (compared to the case of curves).

As we will work with Hilbert schemes, we make the following convention. All schemes are assumed to be separated and of finite type over the base. By a variety defined over \( \overline{\mathbb{Q}} \), we mean a reduced scheme over \( \overline{\mathbb{Q}} \). Hence an integral scheme (i.e. a reduced irreducible scheme) is the same as an irreducible variety.

For general knowledge of Hilbert schemes and Hilbert polynomials, we refer to [Gro62], [ACG11, Chap.9] or [Kol99, §1.1].

3.1. Abelian variety. Let \( A \) be an abelian variety and let \( L \) be a very ample line bundle. All degrees below will be with respect to \( L \).

There are finitely many possibilities for the Hilbert polynomials of irreducible subvarieties of \( A \) of dimension \( r \) and degree \( d \); see [Gro62] Thm.2.1(b) and Lem.2.4][1]. The key point to prove this finiteness result is to compare the Hilbert scheme and the Chow variety. Let \( \Xi \) be this finite set of polynomials.

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[1] In the notation of [Gro62] Thm.2.1(b) and Lem.2.4, \( X = A, \mathcal{S} = \text{Spec} \overline{\mathbb{Q}}, \mathcal{O}_X(1) = L, K = \overline{\mathbb{Q}}, Y \) the irreducible subvariety in question, \( E \) the set of the classes of all the structural sheaves of such \( Y \)'s, and \( \mathcal{G} \) the structural sheaf of a such \( Y \).
Let $\mathcal{H}_{r,d}(A) := \bigcup_{P \in \Xi} \mathcal{H}_P(A)$, where $\mathcal{H}_P(A)$ is the Hilbert scheme which parametrizes subschemes of $A$ with Hilbert polynomial $P$. As $\Xi$ is a finite set, $\mathcal{H}_{r,d}(A)$ is of finite type over $\overline{\mathbb{Q}}$ and is projective.

There exists a universal family $\mathcal{X}_{r,d}(A) \to \mathcal{H}_{r,d}(A)$ endowed with a natural closed $\mathcal{H}_{r,d}(A)$-immersion

$$\mathcal{X}_{r,d}(A) \xrightarrow{\pi_A} A \times \mathcal{H}_{r,d}(A)$$

where $\pi_A$ is the projection to the second factor. Over each point $s \in \mathcal{H}_{r,d}(A)(\overline{\mathbb{Q}})$, the fiber $\mathcal{X}_{r,d}(A)_s$ is precisely the subscheme of $A$ parametrized by $s$, and the horizontal immersion is the natural closed immersion $\mathcal{X}_{r,d}(A)_s$ in $A$.

By definition of Hilbert schemes, the morphism $\pi_A|_{\mathcal{X}_{r,d}(A)}$ is flat and proper. Define

$$\mathcal{H}_{r,d}(A)^o := \{ s \in \mathcal{H}_{r,d}(A)(\overline{\mathbb{Q}}) : \mathcal{X}_{r,d}(A)_s \text{ is an integral subscheme of } A \}$$

endowed with the reduced induced subscheme structure; hence it is a subvariety of $\mathcal{H}_{r,d}(A)$ which is quasi-projective over $\overline{\mathbb{Q}}$. Then $\mathcal{H}_{r,d}(A)^o$ parametrizes all irreducible subvarieties $X$ of $A$ with $\dim X = r$ and $\deg_L X = d$ (by our choice of $\Xi$ above). By [Gro67, Thm.12.2.4.(viii)], $\mathcal{H}_{r,d}(A)^o$ is Zariski open in $\mathcal{H}_{r,d}(A)$. For each irreducible component $V$ of $\mathcal{H}_{r,d}(A)$, the intersection $V \cap \mathcal{H}_{r,d}(A)^o$ is either Zariski open dense in $V$ or is empty.

**Lemma 3.1.** Let $V$ be a (not necessarily irreducible) subvariety of $\mathcal{H}_{r,d}(A)^o$. Then there exist $m_0$ points $P_1, \ldots, P_{m_0} \in A(\overline{\mathbb{Q}})$ such that the Zariski closed subset of $V$ defined by

$$\{ [X] \in V(\overline{\mathbb{Q}}) : P_1 \in X(\overline{\mathbb{Q}}), \ldots, P_{m_0} \in X(\overline{\mathbb{Q}}) \}$$

has dimension 0, i.e. this set is a non-empty finite set.

**Proof.** Fix $[X_0] \in V(\overline{\mathbb{Q}})$. We claim for each $k \in \{0, \ldots, \dim V\}$ the following statement.

**Claim** There are finitely many points $P_{k,1}, \ldots, P_{k, n_k}$ in $X_0(\overline{\mathbb{Q}})$ such that $V_k := \{ [X] \in V(\overline{\mathbb{Q}}) : P_{k,1} \in X(\overline{\mathbb{Q}}), \ldots, P_{k, n_k} \in X(\overline{\mathbb{Q}}) \}$ has dimension $\leq \dim V - k$.

Notice that this claim immediately yields the lemma by taking $k = \dim V$; the set $V_{\dim V}$ thus obtained is non-empty since it contains $[X_0]$.

We prove this claim by induction on $k$. The base step $k = 0$ trivially holds true because we can take any finite set of points in $X_0(\overline{\mathbb{Q}})$.

Assume the claim holds true for $0, \ldots, k - 1$. We have thus obtained point $P_{k-1,1}, \ldots, P_{k-1, n_{k-1}} \in X_0(\overline{\mathbb{Q}})$ such that $V_{k-1} := \{ [X] \in V(\overline{\mathbb{Q}}) : P_{k-1,1} \in X(\overline{\mathbb{Q}}) \text{ for all } i \in 1, \ldots, n_{k-1} \}$ has dimension $\leq \dim V - k + 1$.

For each irreducible component $W$ of $V_{k-1}$ with $\dim W > 0$, there exists some $[X_W] \in W(\overline{\mathbb{Q}})$ such that $X_W \neq X_0$ as (irreducible) subvarieties of $A$. Take $P_W \in (X_0 \setminus X_W)(\overline{\mathbb{Q}})$. Then $\{ [X] \in W(\overline{\mathbb{Q}}) : P_W \in X(\overline{\mathbb{Q}}) \}$ has dimension $\leq \dim W - 1 \leq \dim V - k + 1 - 1 = \dim V - k$.

Thus it suffices to take $\{P_{k,1}, \ldots, P_{k, n_k}\} := \{P_{k-1,1}, \ldots, P_{k-1, n_{k-1}}\} \cup \bigcup_W \{P_W\}$ with $W$ running over all positive dimensional irreducible components of $V_{k-1}$. □
For each \( m \geq 1 \), set \( \mathcal{X}_{r,d}(A)^{[m]} := \mathcal{X}_{r,d}(A) \times_{H_{r,d}(A)} \ldots \times_{H_{r,d}(A)} \mathcal{X}_{r,d}(A) \). Then (3.1) induces

\[
\begin{array}{ccc}
\mathcal{X}_{r,d}(A)^{[m]} \times_{H_{r,d}(A)} H_{r,d}(A)^{\circ} & \xrightarrow{\sim} & A^m \times H_{r,d}(A)^{\circ} \\
\pi_{A}^{[m]} & \xrightarrow{\sim} & A^m \\
\mathcal{X}_{r,d}(A)^{\circ} & \xrightarrow{\sim} & H_{r,d}(A)^{\circ}
\end{array}
\]

where \( \pi_{A}^{[m]} \) is the natural projection.

**Lemma 3.2.** Let \( V \) be a (not necessarily irreducible) subvariety of \( H_{r,d}(A)^{\circ} \). Then there exists \( m_0 \geq 1 \) with the following property. For each \( m \geq m_0 \), there exists \( P \in A^m(\overline{\mathbb{Q}}) \) such that \( (\pi_{A}^{[m]}|_{\mathcal{X}_{r,d}(A)^{[m]} \times_{H_{r,d}(A)} V})^{-1}(P) \) has dimension 0, i.e. is a non-empty finite set.

**Proof.** Let \( P_1, \ldots, P_{m_0} \in A(\overline{\mathbb{Q}}) \) be from Lemma 3.1. For each \( m \geq m_0 \), set \( P_k = P_{m_0} \) for each \( k \geq m_0 \). Set \( P := (P_1, \ldots, P_m) \in A^m(\overline{\mathbb{Q}}) \). To prove the lemma, it suffices to prove that \( (\pi_{A}^{[m]}|_{\mathcal{X}_{r,d}(A)^{[m]} \times_{H_{r,d}(A)} V})^{-1}(P) \) is a non-empty finite set.

It is clear that \( \pi_{A}^{[m]} \vert_{\{P\} \times V} \) is an isomorphism. Therefore \( \pi_{A}^{[m]} \) induces an isomorphism

\[
\left( \{P\} \times V \right) \cap \mathcal{X}_{r,d}(A)^{[m]} \cong \pi_{A}^{[m]} \left( \left( \{P\} \times V \right) \cap \mathcal{X}_{r,d}(A)^{[m]} \right).
\]

Notice that the left hand side of this isomorphism is precisely \( (\pi_{A}^{[m]}|_{\mathcal{X}_{r,d}(A)^{[m]} \times_{H_{r,d}(A)} V})^{-1}(P) \). So it suffices to prove that the right hand side of this isomorphism is a non-empty finite set. A direct computation shows that the right hand side is

\[
\{[X] \in V(\overline{\mathbb{Q}}) : P_i \in X(\overline{\mathbb{Q}}), \ldots, P_{m_0} \in X(\overline{\mathbb{Q}})\},
\]

which is non-empty finite by Lemma 3.1. Hence we are done. \( \square \)

### 3.2. Family version.

Let \( \pi_{\text{univ}} : \mathfrak{A}_g \to A_g \) be the universal abelian variety over the fine moduli space of principally polarized abelian varieties of dimension \( g \) with level-4-structure. For each \( b \in A_g(\overline{\mathbb{Q}}) \), the abelian variety parametrized by \( b \) is \( (\mathfrak{A}_g)_b = (\pi_{\text{univ}})^{-1}(b) \).

Fix a symmetric relatively ample line bundle \( \mathfrak{L}_g \) on \( \mathfrak{A}_g/A_g \) satisfying the following property: for each principally polarized abelian variety \( (A, L) \) parametrized by \( b \in A_g(\overline{\mathbb{Q}}) \), we have \( c_1(\mathfrak{L}_g|_{(\mathfrak{A}_g)_b}) = 2c_1(L) \); see [MiK94, Prop.6.10] for the existence of \( \mathfrak{L}_g \).

Then \( \mathfrak{L}_g^{\otimes 4} \) is relatively very ample on \( \mathfrak{A}_g/A_g \).

There are finitely many possibilities for the Hilbert polynomials of irreducible subvarieties of \( (\mathfrak{A}_g)_b \) (for all \( b \in A_g(\overline{\mathbb{Q}}) \)) of dimension \( r \) and degree \( d \) with respect to \( \mathfrak{L}_g^{\otimes 4}|_{(\mathfrak{A}_g)_b} \); see [Gro62, Thm.2.1(b) and Lem.2.4] \[2\] Let \( \Xi \) be this finite set of polynomials.

Consider the Hilbert scheme

\[
H := \bigcup_{P \in \Xi} H_P(\mathfrak{A}_g/A_g) \xrightarrow{\text{m}} A_g
\]

\[2\] In the notation of [Gr62, Thm.2.1(b) and Lem.2.4], \( X = \mathfrak{A}_g, S = A_g, O_X(1) = \mathfrak{L}_g^{\otimes 4}, K = \overline{\mathbb{Q}}, Y \) the irreducible subvariety in question, \( E \) the set of the classes of all the structural sheaves of such \( Y \)'s, and \( \mathfrak{S} \) the structural sheaf of a such \( Y \).
with $H_P(\mathfrak{X}/A_g)$ the $A_g$-scheme representing the functor \{schemes over $A_g$\} $\rightarrow$ \{sets\}, $T \mapsto$ \{subschemas $W$ of $\mathfrak{X}_g \times_{A_g} T$ which are proper flat over $T$ and have Hilbert polynomial $P$\}. It is known that each $H_P(\mathfrak{X}/A_g)$ is a projective $A_g$-scheme.

Then $H$ is an $A_g$-scheme with $\iota_H$ the structural morphism. The morphism $\iota_H$ is of finite type since $\Xi$ is a finite set.

There is a universal family endowed with a closed $H$-immersion

$$\mathcal{S} := \mathcal{S}_{r,d}(\mathfrak{X}_g/A_g) \hookrightarrow \mathfrak{X}_g \times_{A_g} H =: \mathcal{A}_H \quad \xrightarrow{\pi} \quad H$$

with $\pi|_{\mathcal{S}}$ flat and proper.

To ease notation, for each $H$-scheme $\mathcal{G} \rightarrow H$ and for each morphism $S \rightarrow H$, we will denote by $\mathcal{G}_S := \mathcal{G} \times_H S$. In particular, this applies to $\mathcal{S}$ and $\mathcal{A}_H$; if $S$ is a variety, then $\mathcal{A}_S \rightarrow S$ is an abelian scheme, and $\mathcal{S}_S$ is a subvariety of $\mathcal{A}_S$ which dominates $S$.

As for (3.2), we define

$$H^o := H_{r,d}(\mathfrak{X}_g/A_g) := \{s \in H(\overline{\mathbb{Q}}) : \mathcal{S}_s \text{ is an integral subscheme of } \mathcal{A}_s\}$$

defined with the reduced induced subscheme structure; it is thus a quasi-projective variety defined over $\overline{\mathbb{Q}}$. We will call $H^o$ the restricted Hilbert scheme.

By our choice of $\Xi$ above (3.3), $H^o(\overline{\mathbb{Q}})$ parametrizes all pairs $(X, (A, L))$ consisting of a principally polarized abelian variety defined over $\overline{\mathbb{Q}}$ and an irreducible subvariety $X$ of $A$ defined over $\overline{\mathbb{Q}}$ with $\dim X = r \geq 1$ and $\deg_{A_g[1]} X = 4$. By [Gro67, Thm.12.2.4.(viii)] applied to $\pi|_{\mathcal{S}}$, $H^o$ is Zariski open in $H$. For each irreducible component $S$ of $H$, the intersection $H^o \cap H^o$ is either Zariski open dense in $S$ or is empty. Moreover for the structural morphism $\iota_H : H \rightarrow A_g$, we have that $\iota_H|_{\mathcal{S}}(b)$ is precisely the $H_{r,d}(A^o_b)$ defined in (3.2).

Write $\iota_{H^o} := \iota_H|_{H^o}$.

**Convention.** From now on, we will forget the scheme $H$ and only work with the variety $H^o$ defined over $\overline{\mathbb{Q}}$; notice that $H^o$ is in general not irreducible.

For each $m \geq 1$, set $\mathcal{S}_{H^o}^{[m]}$ (resp. $\mathcal{A}_{H^o}^{[m]}$) to be the $m$-fibered power of $\mathcal{S}_{H^o}$ over $H^o$ (resp. of $\mathcal{A}_{H^o}$ over $H^o$). Then we have a commutative diagram

$$\mathcal{S}_{H^o}^{[m]} \rightarrow \mathcal{A}_{H^o}^{[m]} \rightarrow \mathfrak{X}_g^{[m]} \quad \xrightarrow{\pi^{[m]}} \quad \mathcal{A}_g^{[m]} \quad \xrightarrow{\text{univ.}^{[m]}} \quad H^o \quad \xrightarrow{\iota_{H^o}} \quad A_g$$

Over each $b \in A_g(\overline{\mathbb{Q}})$, this diagram restricts to (3.3) (with $A = (\mathfrak{X}_g)_b$) completed by $(\mathfrak{X}_g)_b^{[m]} \rightarrow \{b\}$ and $H_{r,d}(\mathfrak{X}_g)_b^{[m]} \rightarrow \{b\}$.

**Lemma 3.3.** Let $S$ be an irreducible subvariety of $H^o$. Then there exists $m_0 \geq 1$ with the following property. For each $m \geq m_0$, $\iota^{[m]}|_{\mathcal{S}_{H^o}^{[m]}}$ is generically finite.

[3] Here $(A, L)$ gives rise to a point $b \in A_g(\overline{\mathbb{Q}})$, and we identify $A$ with $(\mathfrak{X}_g)_b$. 

Proof. Take \( b \in \varrho_{H^0}(S)(\overline{Q}) \). Use the notation from (3.3) with \( A = A_b \). For any \( P \in (\pi_{\text{univ},[m]})^{-1}(b)(\overline{Q}) = A_b^m(\overline{Q}) \), we have

\[
\left( \varrho_{H^0}([m])^{-1}(P) \right) = \left( \varrho_{A_b}([m])^{-1}(P) \right),
\]

where \( V = (\varrho_{H^0}([s])^{-1}(b) \subseteq (\varrho_{H^0}([s]))^{-1}(b) = \text{H}_{r,d}(A_b) \). Thus there exists \( P \) such that \( \left( \varrho_{H^0}([m])^{-1}(P) \right) \) has dimension 0 by Lemma 3.2.

Thus it remains to prove that \( \mathcal{X}_{S}^r[m] \) is irreducible. To do this, by [Gro67, Prop.9.7.8] it suffices to prove that the geometric generic fiber \( \mathcal{X}_{\eta} \) is irreducible. This is true because \( \mathcal{X}_{s} \) is irreducible for all \( s \in H^0(\overline{Q}) \) (by definition of \( H^0 \)) and [Gro67, Prop.9.7.8].

3.3. Non-degeneracy. We keep the notation from the previous subsection.

Later on, we wish to apply the height inequality of Dimitrov–Gao–Habegger [DGH21, Thm.1.6 and Thm.B.1] and Kühne’s equidistribution result [Küh21, Thm.1] to appropriate abelian schemes and non-degenerate subvarieties to achieve the desired uniform bound. Thus it is fundamental to construct some non-degenerate subvarieties to apply these tools.

For our purpose and inspired by the second-named author’s [Ge21, Prop.3.4], we prove the following proposition. In fact, it is precisely [Ge21, Prop.3.4] adapted to our context of restricted Hilbert schemes, and the main idea of the proof is to apply the first-named author’s [Gao20, Thm.10.1] to the situation considered in Lemma 3.3.

Proposition 3.4. Let \( S \subseteq H^0 \) be an irreducible subvariety. Consider \( \mathcal{X}_{S} \subseteq A_S \to S \).

Use \( \eta \) to denote the geometric generic point of \( S \). Assume that the following hypotheses hold true on the geometric generic fiber \( A_{\eta} \) of \( A_S/S \):

(i) \( \mathcal{X}_{\eta} \) is an irreducible subvariety of \( A_{\eta} \);
(ii) \( \mathcal{X}_{S} \) generates \( A_{s} \) for each \( s \in S(\mathbb{C}) \);
(iii) the subvariety \( \mathcal{X}_{\eta} \) has finite stabilizer.

Then there exists \( m_0 \geq 1 \) such that \( \mathcal{X}_{S}^r[m] \) is a non-degenerate subvariety of \( A_{S}^r[m] \) for each \( m \geq m_0 \).

Non-degenerate subvarieties defined over \( \mathbb{C} \) are defined as in [Gao21, Defn.6.1] (we need both descriptions later on). A subvariety defined over \( \overline{Q} \) is said to be non-degenerate if its base change to \( \mathbb{C} \) is non-degenerate.

Proof. We wish to invoke [Gao20, Thm.10.1.(i)] with \( t = 0 \). In fact, as the conventions of [Gao20] are somewhat different from standard terminologies, we will apply the formulation of [Gao21, Thm.6.5.(i)].

All hypotheses of [Gao21, Thm.6.5] are satisfied: indeed, hypothesis (a) clearly holds true because we fixed \( r \geq 1 \) at the beginning of this section, hypothesis (b) holds true by (ii), and hypothesis (c) holds true by (iii). Thus we can apply [Gao21, Thm.6.5.(i)] to the abelian scheme \( A_S \to S \) and the subvariety \( \mathcal{X}_{S} \). Therefore \( \mathcal{X}_{S}^r[m] \) is non-degenerate if \( m \geq \dim S \) and \( \varrho_{H^0}(S)^d \) is generically finite. By Lemma 3.3 there exists \( m_0 \geq 1 \) such that \( \varrho_{H^0}(S)^d[m] \) is generically finite for each \( m \geq m_0 \). So the conclusion holds true with \( m_0 \) replaced by \( \max\{m_0, \dim S\} \).

The following lemma is useful.
Lemma 3.5. Let $S \subseteq H^0$ be a (not necessarily irreducible) subvariety. Define the subset

\[(3.9) \quad S_{gen} := \{ s \in S(\overline{Q}) :\text{ the stabilizer of } \mathcal{X}_s \text{ in } A_s \text{ is finite, and } \mathcal{X}_s \text{ generates } A_s \} .\]

Endow each irreducible component $S'$ of $\overline{S}_{gen}$ (the Zariski closure of $S_{gen}$ in $S$) with the reduced induced subscheme structure. Then $\mathcal{X}_{S'} \subseteq A_{S'} \to S'$ satisfies the hypotheses (i)–(iii) of Proposition 3.4.

Proof. Let $\bar{q}$ be the geometric generic point of $S'$.

For (i): assume $\mathcal{X}_{\bar{q}}$ has $n$ irreducible components. Then $\mathcal{X}_{\bar{q}}$ has $n$ irreducible components for $s' \in U(\overline{Q})$ with $U$ a Zariski open dense subset of $S'$; see [Gro67, Prop.9.7.8]. By definition of $H^0$, $\mathcal{X}_{s'}$ is irreducible for all $s' \in H^0(\overline{Q})$. Hence $n = 1$. So $\mathcal{X}_{\bar{q}}$ has 1 irreducible component, and hence (i) is established.

(ii) clearly holds true.

For (iii): let $C$ be the neutral component of $\text{Stab}_{A_0}(\mathcal{X}_{\bar{q}})$, then there exists a quasi-finite dominant morphism $\rho : S'' \to S'$ such that $C$ extends to an abelian subscheme $C$ of $A_{S''} = A_{S'} \times_{S'} S'' \to S''$. Now that $\rho(S'')$ contains a Zariski open dense subset of $S'$, there exists a point $s' \in \rho(S'')(\overline{Q}) \cap S_{gen}$ as in the previous paragraph. Take $s'' \in S''(\overline{Q})$ over $s' \in S'(\overline{Q})$, then $A_{s''}$ can be identified with $A_{s'}$ under the natural projection $A_{S''} = A_{S'} \times_{S'} S'' \to A_{S''}$. Notice that $\mathcal{X}_{s''}$ is identified with $\mathcal{X}_{s'}$ under the same projection. Thus $C_{s''} \subseteq \text{Stab}_{A_{s''}}(\mathcal{X}_{s''}) = \text{Stab}_{A_{s'}}(\mathcal{X}_{s'})$. But $\text{Stab}_{A_{s'}}(\mathcal{X}_{s'})$ is finite by definition of $S_{gen}$ (3.9). So $C_{s''}$ is the origin of $A_{s''}$. So $C$ is the zero section of $A_{S''} \to S''$, and hence $C = C_{\bar{q}}$ is the origin of $A_{\bar{q}}$. Thus we have established (iii). □

4. APPLYING THE HEIGHT INEQUALITY

Proposition 4.1. Let $g$, $l$, $r \leq g$ and $d$ be positive integers. There exist constants $c_1' = c_1'(g, l, r, d) > 0$, $c_2' = c_2'(g, l, r, d) > 0$ and $c_3' = c_3'(g, l, r, d) > 0$ satisfying the following property. For

- each polarized abelian variety $(A, L)$ of dimension $g$ defined over $\overline{Q}$ with $\deg_L A = l$,
- and each irreducible subvariety $X$ of $A$, defined over $\overline{Q}$, with $\dim X = r$ and $\deg_L X = d$, such that $X$ generates $A$,

the set

\[(4.1) \quad \Sigma := \{ x \in X^0(\overline{Q}) : h_{L \otimes L_-}(x) \leq c_1' \max\{1, h_{Fal}(A)\} - c_3' \},\]

where $L_- := [-1] \ast L$, satisfies the following property: $\Sigma \subseteq X'(\overline{Q})$ for some Zariski closed $X' \subseteq X$ with $\deg_L(X') < c_2'$.

The key to prove this proposition is to put all $(A, L)$ and $X$ into finitely many families. Briefly speaking, up to some reductions it suffices to prove the proposition for $(A, L)$ principally polarized; then one works with the universal abelian variety over the fine moduli space and consider the restricted Hilbert scheme from (3.6). The core of this section is the result in family Proposition 4.2.
4.1. A result in family. Let $\mathfrak{A}_g \to \mathfrak{A}_g$ be the universal abelian variety over the fine moduli space of principally polarized abelian varieties with level-4-structure. There exists a symmetric relatively ample line bundle $\mathfrak{L}_g$ on $\mathfrak{A}_g/\mathfrak{A}_g$ satisfying the following property: for each principally polarized abelian variety $(A, L)$ parametrized by $b \in \mathfrak{A}_g(\overline{\mathbb{Q}})$, we have $c_1(\mathfrak{L}_g|_{\mathfrak{A}_g}) = 2c_1(L)$; see [MFK94, Prop.6.10]. In particular if we identify $A = (\mathfrak{A}_g)_b$, then $c_1(\mathfrak{L}_g|_A) = c_1(L \otimes L_{-})$ for $L_{-} := [-1]^*L$.

Use the notation in (3.2). For each $1 \leq r \leq g$ and each $d \geq 1$, let $H^0 := H^0_{r,d}(\mathfrak{A}_g/\mathfrak{A}_g)$ be the restricted Hilbert scheme from (3.6); it is a variety defined over $\overline{\mathbb{Q}}$ whose $\overline{\mathbb{Q}}$-points parametrizes all pairs $(X, (A, L))$ of a principally polarized abelian variety $(A, L)$ and an irreducible subvariety $X$ of $A$ with $\dim X = r$ and $\deg \Sigma_{\mathfrak{A}_g}^r|_A X = d$.

For $A_{H^0} := \mathfrak{A}_g \times_{\mathfrak{A}_g} H^0$, we have a commutative diagram (3.7) with $m = 1$

$$
\begin{array}{ccc}
\mathfrak{A}_{H^0} & \xrightarrow{\pi} & \mathfrak{A}_g \\
\downarrow & & \downarrow \\
H^0 & \xrightarrow{\iota_{H^0}} & \mathfrak{A}_g
\end{array}
$$

where $\pi|_{\mathfrak{A}_{H^0}} : \mathfrak{A}_{H^0} \to H^0$ is the universal family. Every object in the diagram (4.2) is a variety defined over $\overline{\mathbb{Q}}$, and every morphism is defined over $\overline{\mathbb{Q}}$.

Set $L := \iota^*\mathfrak{L}_g^\otimes 4$. Then $L$ is relatively very ample on $A_{H^0}$.

Fix $\mathcal{M}$ an ample line bundle on a compactification $\overline{\mathfrak{A}_g}$ of $\mathfrak{A}_g$ defined over $\overline{\mathbb{Q}}$. The morphism $\iota_{H^0}$ extends to a morphism $\iota_{H^0} : H^0 \to \overline{\mathfrak{A}_g}$ for some compactification $\overline{H^0}$ of $H^0$, i.e. $\overline{H^0}$ is a projective variety defined over $\overline{\mathbb{Q}}$ which contains $H^0$ as a Zariski open dense subset. Set $\mathcal{M} := (\iota_{H^0})^*\mathcal{M}$.

**Proposition 4.2.** Let $S \subseteq H^0$ be a (not necessarily irreducible) subvariety defined over $\overline{\mathbb{Q}}$. Let $S_{gen} \subseteq S(\overline{\mathbb{Q}})$ be the subset defined by (3.9).

There exist constants $c'_1 = c'_1(r, d, S) > 0$, $c'_2 = c'_2(r, d, S) > 0$ and $c'_3 = c'_3(r, d, S) > 0$ such that the following property holds true. For each $s \in S_{gen}$, the set

$$
\Sigma_s := \left\{ x \in S_{s}^0 : \hat{h}_{L}(x) \leq c'_1 \max\{1, h_{H^0,M}(s)\} - c'_3 \right\}
$$

satisfies the following property: $\Sigma_s \subseteq X'(\overline{\mathbb{Q}})$ for some proper Zariski closed $X' \subseteq S_s$ with $\deg L_s(X') < c'_2$.

This proposition is the generalization of Dimitrov–Gao–Habegger’s result for the universal curve [DGH21, Prop.7.1]. The current formulation is closer to the one presented in the first-named author’s survey [Gao21, Prop.7.2]. The proof of Proposition 4.2 is in line with these results; the main new ingredients are Proposition 3.3 (to replace [Gao20, Thm.1.2] for the non-degeneracy of a new family) and Lemma 4.2 (which generalizes [DGH21, Lem.6.3] to treat subvarieties other than curves).

**Proof.** It suffices to prove the proposition for $S$ irreducible.

From now on, assume $S$ is irreducible. We prove this proposition by induction on $\dim S$. The proof of the base step $\dim S = 0$ is in fact contained in the induction.

Endow each irreducible component $S'$ of $S_{gen}$ with the reduced induced subscheme structure.
Lemma 3.3 allows us to invoke Proposition 3.4 for $\mathcal{X}_S \subseteq \mathcal{A}_S \to S'$. Let $m_0(S') > 0$ be from Proposition 3.4, and let $m = \max_{S'}\{m_0(S')\}$ with $S'$ running over all irreducible components of $S_{\text{gen}}$. Then $\mathcal{X}^m_{S'}$ is a non-degenerate subvariety of $\mathcal{A}^m_{S'}$. Notice that $m = m(S) > 0$ depends only on $S$.

Let $\mathcal{X}^m_{S'}$ be the Zariski open subset of $\mathcal{X}^m_{S'}$ as in [Gao21, Thm.7.1]; it is defined over $\mathbb{Q}$ since $\mathcal{X}^m_{S'}$ is, and is dense in $\mathcal{X}^m_{S'}$ since $\mathcal{X}^m_{S'}$ is non-degenerate.

Consider the abelian scheme $\pi^m: \mathcal{A}^m_{\mathcal{H}} \to \mathcal{H}^m$.

Consider $S' \setminus \pi^m(\mathcal{X}^m_{S'})$ endowed with the reduced induced subscheme structure; it has dimension $\leq \dim S' - 1 \leq \dim S - 1$. Let $S_1, \ldots, S_k$ be the irreducible components of $\bigcup_{S'} S' \setminus \pi^m(\mathcal{X}^m_{S'})$ with $S'$ running over all irreducible components of $S_{\text{gen}}$. Then the set $\{S_1, \ldots, S_k\}$ is uniquely determined by $S$ and $m$. But $m = m(S) > 0$ depends only on $S$. So the set $\{S_1, \ldots, S_k\}$ is uniquely determined by $S$.

Let $s \in S_{\text{gen}}(\mathbb{Q})$. Then either $s \in \bigcup_{i=1}^{k} S_i(\mathbb{Q})$ or $s \in \pi^m(\mathcal{X}^m_{S'})(\mathbb{Q})$ for some irreducible component $S'$ of $S_{\text{gen}}$.

**Case (i)** $s \in \bigcup_{i=1}^{k} S_i(\mathbb{Q})$.

For each $i \in \{1, \ldots, k\}$, we can apply the induction hypothesis to $S_i$ since $\dim S_i \leq \dim S - 1$. So we obtain constants $c'_{i,1} = c'_{i,1}(r, d, S_i) > 0$, $c'_{i,2} = c'_{i,2}(r, d, S_i) > 0$, and $c'_{i,3} = c'_{i,3}(r, d, S_i) > 0$ such that for each $s \in S_i(\mathbb{Q})$, the set

\[
\Sigma_{i,s} := \left\{ x \in \mathcal{X}^m_{S'_i}(\mathbb{Q}) : \hat{h}_L(x) \leq c'_{i,1} \max\{1, h_{\mathcal{F}_{\mathcal{A}}}^m(s)\} - c'_{i,3} \right\}
\]

is contained in $X'(\mathbb{Q})$ for some proper Zariski closed $X' \subseteq \mathcal{X}_s$ with $\deg_{\mathcal{L}_s}(X') < c'_{i,2}$.

Let $c'_{\text{deg},1} = \min_{1 \leq i \leq k}\{c'_{i,1}\} > 0$, $c(r, d, S_{\text{deg},2}) = \max_{1 \leq i \leq k}\{c'_{i,2}\} > 0$ and $c'_{\text{deg},3} = \max_{1 \leq i \leq k}\{c'_{i,3}\} > 0$. We have seen above that the set $\{S_1, \ldots, S_k\}$ is uniquely determined by $S$. So the constants $c'_{\text{deg},1}$, $c'_{\text{deg},2}$ and $c'_{\text{deg},3}$ thus obtained depend only on $g$, $d$, $r$ and $S$. Moreover, for each $s \in \bigcup_{i=1}^{k} S_i(\mathbb{Q})$, the set

\[
\Sigma_{\text{deg},s} := \left\{ x \in \mathcal{X}^m_{S_s}(\mathbb{Q}) : \hat{h}_L(x) \leq c'_{\text{deg},1} \max\{1, h_{\mathcal{F}_{\mathcal{A}}}^m(s)\} - c'_{\text{deg},3} \right\}
\]

must be contained in $\Sigma_{i,s}$ for some $i$. So $\Sigma_{\text{deg},s}$ is contained in $X'(\mathbb{Q})$ for some proper Zariski closed $X' \subseteq \mathcal{X}_s$ with $\deg_{\mathcal{L}_s}(X') \leq c'_{\text{deg},2}$. This concludes for Case (i).

**Case (ii)** $s \in \pi^m(\mathcal{X}^m_{S'})(\mathbb{Q})$ for some irreducible component $S'$ of $S_{\text{gen}}$. In this case, $(\mathcal{X}^m_{S'})(s) \neq \emptyset$. So $\mathcal{X}^m_{S'} = (\mathcal{X}^m_{S'})(s) \neq (\mathcal{X}^m_{S'} \setminus \pi^m(\mathcal{X}^m_{S'}))(s)$.

By the height inequality of Dimitrov–Gao–Habegger [DGH21, Thm.1.6] (here, we take the version of [Gao21, Thm.7.1]), there exist constants $c = c(S') > 0$ and $c' = c'(S')$ such that

\[
\hat{h}_L(x_1) + \cdots + \hat{h}_L(x_m) \geq c h_{\mathcal{F}_{\mathcal{A}}}^m(s) - c'
\]

for all $(x_1, \ldots, x_m) \in (\mathcal{X}^m_{S'})_{\mathcal{H}}(\mathbb{Q})$.

Set $c'_{S',1} = c/m$ and $c'_{S',3} = (c + c')/m$, with $c$ and $c'$ from (4.6). Then $c'_{S',1}$ and $c'_{S',3}$ depend only on $S'$ and $m$. Consider

\[
\Sigma_{S',s} := \left\{ x \in \mathcal{X}^m_{S'}(\mathbb{Q}) : \hat{h}_L(x) \leq c'_{S',1} \max\{1, h_{\mathcal{F}_{\mathcal{A}}}^m(s)\} - c'_{S',3} \right\} \subseteq \mathcal{X}_{S'}(\mathbb{Q})
\]
We claim that \( \Sigma_{S',s}^m \cap \mathcal{Z}_{S'}^m(s) = \emptyset \). Indeed assume not, then there exists \((x_1, \ldots, x_m) \in \Sigma_{S',s}^m \cap \mathcal{Z}_{S'}^m(s) \). This contradicts the height inequality (4.3) above. So \( \Sigma_{S',s}^m \subseteq (\mathcal{Z}_{S'}^m(s) \setminus \mathcal{Z}_{S'}^m(s)) \).

Recall the assumption \( \mathcal{Z}_{S'}^m \neq \mathcal{Z}_{S'}^m(s) \) for this case. Thus we are allowed to apply Lemma 4.3 to \( X = \mathcal{Z}_{s}, L = \mathcal{L}_s|_X, Z = (\mathcal{Z}_{S'}^m(s) \setminus \mathcal{Z}_{S'}^m(s)) \) and \( \Sigma = \Sigma_{S',s} \). So there exists a Zariski closed subset \( X' \) of \( \mathcal{Z}_{s} \) such that

(i) \( X' \subseteq \mathcal{Z}_{s} \);

(ii) \( \deg_{\mathcal{L}_s}(X') < c(m, r, d, \deg_{\mathcal{L}_{S, s}^m}(\mathcal{Z}_{S'}^m(s) \setminus \mathcal{Z}_{S'}^m(s))) \);

(iii) \( \Sigma_{S',s} \subseteq X'(\overline{\mathbb{Q}}) \).

But \( \deg_{\mathcal{L}_{S, s}^m}(\mathcal{Z}_{S'}^m(s) \setminus \mathcal{Z}_{S'}^m(s)) \) is bounded above solely in terms of \( S' \) and \( m \). Hence property (ii) above can be simplified to be \( \deg_{\mathcal{L}_s}(X') < c_{S',2} \), with \( c_{S',2} \) depending only on \( m, r, d \) and \( S' \). In summary, the set \( \Sigma_{S',s} \) defined in (4.7) is contained in \( X'(\overline{\mathbb{Q}}) \) for some proper Zariski closed \( X' \subseteq \mathcal{Z}_{s} \) with \( \deg_{\mathcal{L}_s}(X') < c_{S',2} \).

Let \( c'_{S',1} = \min_{s} \{ c'_{S',1} \} > 0, c'_{S',2} = \max_{s} \{ c'_{S',2} \} > 0 \) and \( c'_{S',3} = \max_{s} \{ c'_{S',3} \} > 0 \), where \( S' \) runs over all irreducible components of \( S_{\text{gen}} \); see (3.3) for the definition of \( S_{\text{gen}} \). The Zariski closed subset \( \overline{S_{\text{gen}}} \) of \( S \) is uniquely determined by \( S \), and hence the set \( \{ S' \} \) is uniquely determined by \( S \). Moreover \( m = m(S) \) does not depend on \( S \). So the constants \( c'_{S',1}, c'_{S',2} \) and \( c'_{S',3} \) thus obtained depend only on \( r, d \) and \( S \). The discussion above yields the following assertion: for each \( s \in \bigcup_{S'} \pi_{\overline{m}}(\mathcal{Z}_{S'}^m(s)) \), the set

\[
\Sigma_{s}^* := \left\{ x \in \mathcal{Z}_{S'}^m(s) : h_{\mathcal{L}_s}(x) \leq c'_{S',1} \max\{1, h_{\mathcal{L}_{S, s}^m}(s)\} - c'_{S',3} \right\}
\]

must be contained in \( \Sigma_{S',s} \) for some \( S' \); so \( \Sigma_{s}^* \) is contained in \( X'(\overline{\mathbb{Q}}) \) for some proper Zariski closed \( X' \subseteq \mathcal{Z}_{s} \) with \( \deg_{\mathcal{L}_s}(X') < c'_{S',3} \). This concludes for Case (ii).

Now let \( s \in S_{\text{gen}} \) be an arbitrary point. Then we are either in Case (i) or Case (ii). So the proposition follows by letting \( c' = \min \{ c'_{S',1}, c'_{S',2} \} \) if \( c'_{S',1}, c'_{S',2} > 0 \) and \( c'_{S',3} = \max \{ c'_{S',3} \} > 0 \); all the constants involved depend only on \( g, d, S \) (see (4.3) and (4.8)).

The following lemma replaces [DGH1, Lem.6.3] when we work in higher dimensional cases. Let \( k \) be an algebraically closed field and all varieties are assumed to be defined over \( k \). Let \( M \geq 1 \) be an integer.

**Lemma 4.3.** Let \( X \) be an irreducible projective variety with a very ample line bundle \( L \). Let \( Z \subseteq X^M \) be a proper closed subvariety. There exists

\[
c = c(M, \dim X, \deg_{\mathcal{L}_X} X, \deg_{\mathcal{L}_{BM}}(Z)) > 0
\]
such that for any subset \( \Sigma \subseteq X(k) \) satisfying \( \Sigma^M \subseteq Z(k) \), there exists a proper closed subvariety \( X' \) of \( X \) with \( \Sigma \subseteq X'(k) \) and \( \deg_{\mathcal{L}_X}(X') < c \).

**Proof.** We prove this lemma by induction on \( M \). The base step \( M = 1 \) is trivial by taking \( X' := Z \).

Assume the lemma is proved for \( 1, \ldots, M - 1 \geq 1 \). Let \( q : X^M \to X \) be the projection to the first factor. Let \( Y \) be any irreducible component of \( Z \). Let \( Z' \) be the union of all \( q \)’s with \( q(Y) = X \), and \( Z'' \) be the union of the other components.
Set $\Sigma' := q(\Sigma^M \cap Z'(k))$. As $\Sigma^M \subseteq Z(k)$, we have $\Sigma^M \subseteq Z(k) \cup Z''(k)$. Then clearly $(\Sigma \setminus \Sigma'') \times \Sigma^{M-1} \subseteq Z'(k)$.

By the Fiber Dimension Theorem, for a generic point $P \in X(k)$, the fiber $q|_Z^{-1}(P)$ has dimension $\leq \dim Z' - \dim X < \dim X^M - \dim X = \dim X^{M-1}$. Let

$$W := \{ P \in X(k) : \{ P \} \times X^{M-1} \subseteq Z' \}.$$

Then the upper semicontinuity theorem says that $W$ is Zariski closed in $X$, which must furthermore be proper.

\[ \text{Case } \Sigma \setminus \Sigma'' \nsubseteq W(k) \] Take $P \in \Sigma \setminus \Sigma''$ which is not in $W(k)$. Then $\{ P \} \times X^{M-1} \nsubseteq Z'$. Let $Z_1 := \{ P \} \times X^{M-1} \cap Z'$. Since $\{ P \} \times \Sigma^{M-1} \subseteq Z'(k)$, we can apply the induction hypothesis to $M - 1$, $\Sigma \subseteq X(k)$ and $Z_1$, by identifying $\{ P \} \times X^{M-1}$ with $X^{M-1}$. So there exists a proper Zariski closed subset $X' \subseteq X$ such that $\Sigma \subseteq X'(k)$ and

$$\deg_L(X') < c(M - 1, \dim X, \deg_L X, \deg_{L^{M-1}}(Z_1)).$$

But $\deg_{L^{M-1}}(Z_1) = \deg_{L^{M-1}}(Z' \cap \{ P \} \times X^{M-1})$ and

$$\deg_{L^{M-1}}(Z' \cap \{ P \} \times X^{M-1}) \leq \deg_{L^{M-1}}(Z') \deg_{L^{M-1}}(X^{M-1})$$

by Bézout’s Theorem. Using (2.6) inductively by taking $Y_1 := X$ and $Y_2 := X^i$ for $i = 1, 2, \ldots, M - 2$, we have

$$\deg_{L^{M-1}}(X^{M-1}) = \frac{((M - 1) \dim X)!}{((\dim X)!)^{M-1}} (\deg_L X)^{M-1}$$

Moreover $\deg_{L^{M-1}}(Z') \leq \deg_{L^{M-1}}(Z)$. Overall, we get

$$\deg_{L^{M-1}}(Z_1) \leq \frac{((M - 1) \dim X)!}{((\dim X)!)^{M-1}} \deg_{L^{M-1}}(Z) \deg_L X^{M-1}.$$ 

Hence the right hand side of (4.9) is only related to $M, \dim X, \deg_L X, \deg_{L^{M-1}}(Z)$.

\[ \text{Case } \Sigma \setminus \Sigma'' \subseteq W(k) \] In this case, $(\Sigma \setminus \Sigma'') \times X^{M-1} \subseteq Z'$. Namely, for any $x_0 \in X^{M-1}(k)$, we have $(\Sigma \setminus \Sigma'') \times \{ x_0 \} \subseteq Z'(k)$. Since $Z'$ is proper subset of $X^M$, there exists some $x_0 \in X^{M-1}(k)$ such that $(X \times \{ x_0 \}) \cap Z' \neq X \times \{ x_0 \}$. Thus $(X \times \{ x_0 \}) \cap Z'' = X'' \times \{ x_0 \}$ for some proper closed subset $X'' \subseteq X$. Now we have $\Sigma \setminus \Sigma'' \subseteq Z''(k)$, and

$$\deg_L X'' \leq \deg_{L^{M-1}}(X \times \{ x_0 \}) \deg_{L^{M-1}}(Z) = \deg_L X \deg_{L^{M-1}}(Z)$$

by Bézout’s Theorem.

For $\Sigma'' \subseteq q(Z'')$, note that $q(Z'')$ is a proper closed subset of $X$ by definition. Moreover we have

$$\deg_L q(Z'') \leq \deg_{L^{M-1}}(Z'') \leq \deg_{L^{M-1}}(Z).$$

Thus for this case, it suffices to take $X' := X'' \cup q(Z'')$. We are done. \qed

4.2. Proposition 4.4. for principally polarized abelian varieties.

**Proposition 4.4.** Let $g$, $r$ and $d$ be positive integers. There exist constants $c_1 = c_1(g, r, d) > 0$, $c_2 = c_2(g, r, d) > 0$ and $c_3 = c_3(g, r, d) > 0$ satisfying the following property. For

- each principally polarized abelian variety $(A, L)$ defined over $Q$ of dimension $g$,
- and each irreducible subvariety $X$ of $A$, defined over $Q$, with $\dim X = r$ and $\deg_L X = d$ such that $X$ generates $A$,
the set
\[ \Sigma := \left\{ x \in X^\circ(\overline{\mathbb{Q}}) : \hat{h}_{L \otimes L}(x) \leq c'_1 \max\{1, h_{\text{Fal}}(A)\} - c'_3 \right\}, \]
where \( L_{-} := [-1]^*L \), satisfies the following property: \( \Sigma \subseteq X' \) for some proper Zariski closed \( X' \subsetneq X \) with \( \deg_{L}(X') \prec c'_2 \).

**Proof.** Let \( \mathfrak{A}_{g} \to \mathfrak{A}_{g} \) be the universal abelian variety over the moduli space of principally polarized abelian varieties of dimension \( g \) with level-4-structure.

Each principally polarized abelian variety \((A, L)\) defined over \( \overline{\mathbb{Q}} \) gives rise to a point \( b \in \mathfrak{A}_{g}(\overline{\mathbb{Q}}) \). Thus \( A \cong (\mathfrak{A}_{g})_{b} \). In the rest of the proof, we will identify \( A \) with \( (\mathfrak{A}_{g})_{b} \).

Recall the symmetric relatively ample line bundle \( \mathfrak{L}_{g} \) on \( \mathfrak{A}_{g}/\mathfrak{A}_{g} \) at the beginning of \[ \text{4.1} \] in particular \( c_{1}(\mathfrak{L}_{g}|_{A}) = c_{1}(\mathfrak{L}_{g}|_{(\mathfrak{A}_{g})_{b}}) = 2c_{1}(L) = c_{1}(L \otimes L_{-}) \).

Let \( X \) be an irreducible subvariety of \( A \) which generates \( A \). Assume \( d = \deg_{L}(X) \) and \( r = \dim X \). We may assume \( r \geq 1 \) because otherwise the result is trivial. We have
\[ \deg_{\mathfrak{A}_{g}(\overline{\mathbb{Q}})} X = (4c_{1}(\mathfrak{L}_{g}|_{A}))^{r} \cdot [X] = (8c_{1}(L))^{r} \cdot [X] = 8^{r} \deg_{L}(X) = 8^{r} d. \]

Consider the restricted Hilbert scheme \( \mathbb{H}^\circ := \mathbb{H}_{r,8d}(\mathfrak{A}_{g}/\mathfrak{A}_{g}) \) from \[ \text{3.6} \] and the commutative diagram\[ \begin{array}{ccc} \mathcal{X}_{\mathbb{H}^\circ} & \xrightarrow{\iota} & \mathbb{H}^\circ \xrightarrow{\pi} \mathfrak{A}_{g} \\ \downarrow \pi & & \downarrow \iota_{\text{univ}} \\ \mathbb{H}^\circ & \xrightarrow{\iota} & \mathfrak{A}_{g} \end{array} \]
where \( \pi|_{\mathcal{X}_{\mathbb{H}^\circ}} : \mathcal{X}_{\mathbb{H}^\circ} \to \mathbb{H}^\circ \) is the universal family. All objects in this diagram are varieties defined over \( \overline{\mathbb{Q}} \). Then the pair \((X, (A, L))\) is parametrized by a point \( s \in \mathbb{H}^\circ(\overline{\mathbb{Q}}) \). Thus \( X = \mathcal{X}_{s} \) and \( A = A_{s} = (\mathfrak{A}_{g})_{b} \) and \( \iota_{\mathbb{H}^\circ}(s) = b \).

For the line bundle \( \mathcal{M} = \iota_{\mathbb{H}^\circ}^{*}\mathfrak{M} \) on \( \mathbb{H}^\circ \) defined above Proposition\[ \text{4.2} \] we have \( h_{\mathbb{H}^\circ,\mathcal{M}}(s) = h_{\mathfrak{A}_{g},\mathfrak{M}}(\iota_{\mathbb{H}^\circ}(s)) = h_{\mathfrak{A}_{g},\mathfrak{M}}(b) \). By fundamental work of Faltings\[ \text{MB85, Thm.1.1} \], \( h_{\text{Fal}}(A) \) is bounded from above in terms of \( h_{\mathfrak{A}_{g},\mathfrak{M}}(b) = h_{\mathbb{H}^\circ,\mathcal{M}}(s) \) and \( g \) only. More precisely,
\[ \max\{1, h_{\text{Fal}}(A)\} \leq c(g)h_{\mathbb{H}^\circ,\mathcal{M}}(s) + c'(g)\log(h_{\mathbb{H}^\circ,\mathcal{M}}(s) + 2) \leq (c(g) + c'(g))h_{\mathbb{H}^\circ,\mathcal{M}}(s) + 2c'(g) \]
for \( c(g) > 0 \) and \( c'(g) > 0 \).

For the line bundle \( \mathcal{L} = \iota^{*}\mathfrak{L}_{g}^{\otimes 4} \) on \( \mathfrak{A}_{g} \) defined above Proposition\[ \text{4.2} \] we have \( \mathcal{L}|_{s} = \mathfrak{L}_{g}^{\otimes 4}|_{s} = \mathfrak{L}_{g}^{\otimes 4}|_{(\mathfrak{A}_{g})_{b}} \), and hence \( \mathcal{L}|_{A} = \mathfrak{L}_{g}^{\otimes 4}|_{A} \) for the identification \( A = A_{s} = (\mathfrak{A}_{g})_{b} \) fixed above. Thus \( c_{1}(\mathcal{L}|_{A}) = 4c_{1}(L \otimes L_{-}) \). Moreover, both \( \mathcal{L}|_{A} \) and \( (L \otimes L_{-})^{\otimes 4} \) are symmetric and ample on \( A \), so Lemma\[ \text{2.3} \] implies
\[ \hat{h}_{\mathcal{L}|_{A}}(x) = 4\hat{h}_{L \otimes L_{-}}(x) \] for each \( x \in A(\overline{\mathbb{Q}}) \).

Let \( \mathbb{H}^\circ_{\text{gen}} \) be the subset of \( \mathbb{H}^\circ(\overline{\mathbb{Q}}) \) as defined by \[ \text{3.9} \]. If the stabilizer of \( X \) in \( A \) (denoted by \( \text{Stab}_{A}(X) \)) has positive dimension, then \( X^\circ = \emptyset \) by definition of Ueno locus and hence the proposition trivially holds true. Thus we may and do assume that \( \text{Stab}_{A}(X) \) is finite. Then the point \( s \in \mathbb{H}^\circ(\overline{\mathbb{Q}}) \) which parametrizes \((X, (A, L))\) is in \( \mathbb{H}^\circ_{\text{gen}} \), As above, we identify \( X = \mathcal{X}_{s} \) and \( A = A_{s} \).

Apply Proposition\[ \text{4.2} \] to \( S = \mathbb{H}^\circ \). Then we obtain constants \( c'_1, c'_2 \) and \( c'_3 \) depending only on \( r, 8^{r}d \) and \( \mathbb{H}^\circ \) (and hence only on \( g, d \) and \( r \)). The set \( \Sigma_{s} := \)
\( x \in \mathcal{X}_s^c(\overline{\mathbb{Q}}) = X^c(\overline{\mathbb{Q}}) : \hat{h}_{c|A}(x) \leq c_1 \max\{1, h_{Fal}(s)\} - c_3 \) is contained in \( X' \), for some proper Zariski closed \( X' \subseteq X \) with \( \deg_{c|A}(X') < c_2 \).

Notice that \( \deg_L(X') \leq \deg_{c|A}(X') \) since \( 8c_1(L) = c_1(L|A) \), and hence \( \deg_L(X') < c_2 \).

By (4.11) and (4.12), we have

\[
\left\{ x \in X^c(\overline{\mathbb{Q}}) : \hat{h}_{L \otimes L_-}(x) \leq \frac{c_1}{4(c(g) + c'(g))} \max\{1, h_{Fal}(A)\} - \frac{c_2 + 2c_1}{4} \right\} \subseteq \Sigma_s
\]

Hence we are done with \( c_1' \) replaced by \( c_1/4(c(g) + c'(g)) \) and \( c_3' \) replaced by \( (c_3' + 2c_1')/4 \). \( \square \)

### 4.3. Proof of Proposition 4.1

Let \((A, L)\) be a polarized abelian variety with \( \deg_L A = l \) and let \( X \) be an irreducible subvariety with \( \deg_L X = d \) such that \( X \) is not contained in any proper subgroup of \( A \).

By Lemma 2.2 (iv), there exist a principally polarized abelian variety \((A_0, L_0)\) defined over \( \overline{\mathbb{Q}} \) and an isogeny \( u_0 : A \to A_0 \) such that \( L = u_0^* L_0 \); moreover, \( \deg(u_0) = l/g! \). A basic property of Faltings heights [Ray85, Prop.1.4.1] is

\[
|h_{Fal}(A) - h_{Fal}(A_0)| \leq \frac{1}{2} \log \deg u_0 = \frac{1}{2} \log(l/g!).
\]

It is well known that there exists an isogeny \( u : A_0 \to A \) such that \( u_0 \circ u = [\deg u_0] \) on \( A_0 \). So \( u^* L = (u_0 \circ u)^* L_0 = [l/g!]^* L_0 \), and \( \deg u = (\deg u_0)^{2g-1} = (l/g!)^{2g-1} \).

Let \( X_0 \) be an irreducible component of \( u^{-1}(X) \). Then \( u(X_0) = X \). Moreover, \( X_0 \) is not contained in any proper subgroup of \( A_0 \).

By the projection formula for intersection numbers, we have

\[
(4.14) \quad d' := \deg_{u^* L} X_0 = c_1(u^* L)^{g - \dim X} \cdot [X_0] = \deg(u|X_0) c_1(L)^{g - \dim X} \cdot [X] \leq \deg(u) \deg_{L} X = d(l/g!)^{2g-1}.
\]

By the definition of Ueno locus, we have \( X^o = u(X_0)^o \subseteq u(X_0) \).

Let \( c_1'(g, r, d') > 0, c_2'(g, r, d') > 0 \) and \( c_3'(g, r, d') > 0 \) be the constants from Proposition 4.4 with \( d' \) replaced by \( d' \). Set \( c_1' := \min_{1 \leq d' \leq l/g!} \left\{ c_1'(g, r, d')\right\} > 0, c_2' := \max_{1 \leq d' \leq l/g!} \left\{ c_1'(g, r, d')\right\} > 0 \) and \( c_3' := \max_{1 \leq d' \leq l/g!} \left\{ c_1'(g, r, d')\right\} > 0 \). Then \( c_1', c_2' \) and \( c_3' \) depend only on \( g, r \) and \( d \). Proposition 4.4 yields the following assertion: the set \( \Sigma : = \left\{ x \in X_0^c(\overline{\mathbb{Q}}) : \hat{h}_{L_0 \otimes (L_0)_-}(x) \leq c_1 \max\{1, h_{Fal}(A_0)\} - c_3' \right\} \) is contained in \( X_0^c(\overline{\mathbb{Q}}) \) for some Zariski closed irreducible \( X_0 \subseteq X_0 \) with \( \deg_{L_0}(X') \leq c_2' \).

Let \( \Sigma = \left\{ x \in X^o(\overline{\mathbb{Q}}) : (l/g!)^2 \hat{h}_{L \otimes L_-}(x) \leq c_1 \max\{1, h_{Fal}(A)\} - c_3 - (1/2) \log(l/g!) \right\} \).

As \( X^c \subseteq u(X_0^o) \) and \( u^*(L \otimes L_-) \cong (L_0 \otimes (L_0)_-) \otimes (l/g!)^2 \) (recall that \( u^* L = [l/g!]^* L_0 \)), (4.13) yields \( \Sigma \subseteq u(\Sigma_0) \).

Let \( X' := u(X_0^o) \). Then \( X' \) is proper Zariski closed in \( X, \Sigma \subseteq X'(\overline{\mathbb{Q}}) \), and

\[
\deg_{L} X' = c_1(L)^{g - \dim X'} \cdot [X'] 
\leq c_1(u^* L)^{g - \dim X'} \cdot [X_0'] 
= \deg_{u^* L} X'_0 \leq c_2'.
\]

Replace \( c_1 \) by \((l/g!)^2 c_1'\) and \( c_3 \) by \((l/g!)^2 (c_3' + (1/2) \log(l/g!))\). Then Proposition 4.4 holds true.
5. Applying equidistribution

**Proposition 5.1.** Let $g$, $l$, $r$ and $d$ be positive integers. There exist constants $c'_2 = c'_2(g,l,r,d) > 0$ and $c''_3 = c''_3(g,l,r,d) > 0$ satisfying the following property. For

- each polarized abelian variety $(A, L)$ of dimension $g$ defined over $\overline{\mathbb{Q}}$ with $\deg L A = l$,
- and each irreducible subvariety $X$ of $A$, defined over $\overline{\mathbb{Q}}$, with $\dim X = r$ and $\deg L X = d$, such that $X$ generates $A$

the set

$$
\Sigma := \left\{ x \in X'_{\mathbb{Q}} : \hat{h}_{L \otimes L} (x) \leq c''_3 \right\},
$$

where $L_- = [-1]^* L$, is contained in $X' \subseteq X$ with $\deg L (X') < c'_2$.

The basic idea to prove this proposition is similar to the proof of Proposition 4.1, i.e. to put all $(A, L)$ and $X$ into (finitely many) families over the components of the restricted Hilbert scheme from (3.6). The core of this section is the result in family Proposition 5.2.

### 5.1. A result in family

We retain the notation from §4.1. In particular $\pi^{univ} : \mathcal{A}_g \to \mathcal{A}_g$ is the universal abelian variety over the finite moduli space of principally polarized abelian varieties with level-$4$-structure, and $\mathcal{L}_g$ is a symmetric relatively ample line bundle $\mathcal{L}_g$ on $\mathcal{A}_g/\mathcal{A}_g$ satisfying the following property: for each principally polarized abelian variety $(A, L)$ parametrized by $b \in \mathcal{A}_g(\overline{\mathbb{Q}})$, we have $c_1(\mathcal{L}_g|_{\mathcal{A}_g(b)}) = 2c_1(L)$.

For each $1 \leq r \leq g$ and each $d \geq 1$, let $H^r \subseteq H^r, d(\mathcal{A}_g/\mathcal{A}_g)$ be the restricted Hilbert scheme from (3.6): it is a variety defined over $\overline{\mathbb{Q}}$ whose $\overline{\mathbb{Q}}$-points parametrizes all pairs $(X, (A, L))$ of a principally polarized abelian variety $(A, L)$ and an irreducible subvariety $X$ of $A$ with $\dim X = r$ and $\deg_{\mathcal{L}_g \otimes L} X = d$.

We have a commutative diagram ((3.7) with $m = 1$)

$$
\begin{array}{ccc}
\mathcal{X}_H \hookrightarrow & \mathcal{A}_H & \xrightarrow{\pi} \mathcal{A}_g \\
\downarrow & \downarrow \pi \downarrow & \downarrow \pi^{univ} \\
H^r & \xrightarrow{\mathcal{L}_g} & \mathcal{A}_g
\end{array}
$$

where $\pi|_{\mathcal{X}_H} : \mathcal{X}_H \to H^r$ is the universal family. Every object in the diagram (4.2) is a variety defined over $\overline{\mathbb{Q}}$, and every morphism is defined over $\overline{\mathbb{Q}}$. Set $\mathcal{L}_r := \iota^* \mathcal{L}_g^\otimes 4$.

**Proposition 5.2.** Let $S \subseteq H^r$ be a (not necessarily irreducible) subvariety defined over $\overline{\mathbb{Q}}$. Let $S_{gen} \subseteq S(\overline{\mathbb{Q}})$ be the subset defined by (3.9).

There exist constants $c'_2 = c'_2(r,d,S) > 0$ and $c''_3 = c''_3(r,d,S) > 0$ such that the following property holds true. For each $s \in S_{gen}$, the set

$$
\Sigma_s := \left\{ x \in \mathcal{X}_s^0 (\overline{\mathbb{Q}}) : \hat{h}_L (x) \leq c''_3 \right\}
$$

satisfies the following property: $\Sigma_s \subseteq X'(\overline{\mathbb{Q}})$ for some proper Zariski closed $X' \subseteq \mathcal{X}_s$ with $\deg_{\mathcal{L}_s} (X') < c'_2$. 

This proposition is the generalization of the third-named author’s [Küh21, Prop.21] on families of curves, with which the proof of Proposition 12.2 is in line; the main new ingredients are the non-degeneracy construction Proposition 3.4, replacing [Gao20, Thm.1.2’], and Lemma 4.3 which generalizes [DGH21, Lem.6.3] to treat higher-dimensional subvarieties instead of curves.

For readers’ convenience, we divide the proof into 4 steps as in the first-named author’s survey [Gao21, Prop.8.3]. Briefly speaking, the goal of the first 3 steps is to run a family version of the classical approach of Ullmo–Zhang (for the Bogomolov Conjecture) to obtain a generic lower bound on heights closely related to our height of interest \( \hat{h}_L \). In Step 4, we then deduce Proposition 5.2 for this height bound. Step 4 is similar to the proof of Proposition 4.2, where Lemma 4.3 is applied to get the desired \( X' \); the main difference of this step and the proof of Proposition 4.2 is that the height inequality of Dimitrov–Gao–Habegger [DGH21, Thm.1.6 and B.1] is replaced by the height bound achieved from the previous steps.

This family version of the Ullmo–Zhang approach differs from the classical one mainly in two aspects. First and naturally, we use a version of equidistribution that applies to families of abelian varieties; for our purpose, the results of the third-named author [Küh21, Thm.1] or the more general results of Yuan–Zhang [YZ21, Thm.6.7] are sufficient replacements of the classical equidistribution result of Szpiro–Ullmo–Zhang [SUZ97]. Second and in contrast to [SUZ97], a new condition, namely non-degeneracy as defined in [DGH21, Defn.B.4], has to be verified for the subvarieties under consideration. For this purpose we apply [Gao20, Thm.10.1]. Notice that in the case of a single abelian variety, every subvariety is non-degenerate so this is a genuinely new aspect of the family case.

**Proof of Proposition 5.2** Decomposing \( S \) into its irreducible components, it suffices to prove the proposition for \( S \) irreducible.

From now on, we assume that \( S \) is irreducible. We prove the proposition by induction on \( \dim S \). The proof of the base step \( \dim S = 0 \) is in fact contained in the induction. We also point out that the induction hypothesis will be applied only in the last step (Step 4).

**Step 1** Construct non-degenerate subvarieties.

Endow each irreducible component \( S' \) of \( S_{\text{gen}} \) with the reduced induced subscheme structure. Lemma 3.5 allows us to invoke Proposition 3.4 for \( \mathcal{A}_{S'} \subseteq A_{S'} \rightarrow S' \).

Let \( m_0(S') > 0 \) be from Proposition 3.4 and let \( m = \max_{S'} \{ m_0(S') \} \) with \( S' \) running over all irreducible components of \( S_{\text{gen}} \). Then \( \mathcal{A}_{S'}^{[m]} \) is a non-degenerate subvariety of \( A_{S'}^{[m]} \) by Proposition 3.4. Notice that \( m = m(S) > 0 \) depends only on \( S \).

By definition of non-degenerate subvariety [Gao21, Defn.6.1], for the Betti form \( \omega_m \) of \( A_{S'}^{[m]} \), there exists a point \( x \in (\mathcal{A}_{S'}^{[m]} )_{\text{sm}}(\mathbb{C}) \) such that

\[
(\omega_m |_{\mathcal{A}_{S'}^{[m]}(\mathbb{C})})^\wedge \dim \mathcal{A}_{S'}^{[m]}(\mathbb{C}) x \neq 0.
\]

By generic smoothness, there exists a Zariski open dense subset \( U \) of \( (S')_{\text{sm}} \) such that \( \pi^{[m]} |_{\mathcal{A}_{S'}^{[m],\text{sm}}(\pi^{[m]} )^{-1}(U)} \) is a smooth morphism. We have some freedom to choose \( x \), and we may and do assume \( x \) lies above \( U \).
For each $M \geq 1$, recall the proper $S'$-morphism

\[(5.5) \quad \mathcal{D}_M^{[m]} : (\mathcal{A}^{[m]}_{S'})_{[M+1]} \to (\mathcal{A}^{[m]}_{S'})_{[M]} \]

fiberwise defined by $(\bar{a}_0, a_1, \ldots, a_M) \mapsto (a_1 - a_0, \ldots, a_M - a_0)$, with each $a_i \in \mathcal{A}^{[m]}_{S'}(\overline{\mathbb{Q}})$.

Let $\eta$ be the geometric generic point of $S'$. Invoke \cite[proof of Lem.3.1]{Zha98} to $\mathcal{D}_M^{[m]} \subseteq \mathcal{A}^{[m]}_{S'}$. Then there exists an $M_0(S') > 0$ with the following property: For each $M \geq M_0(S')$, a generic fiber of $\mathcal{D}_M^{[m]}$ is a $\text{Stab}_{\mathcal{A}^{[m]}_{S'}} \mathcal{D}_M^{[m]}$-orbit; here $\text{Stab}_{\mathcal{A}^{[m]}_{S'}}(\mathcal{D}_M^{[m]})$ is viewed as a subgroup of $(\mathcal{A}^{[m]}_{S'})_{[M+1]}$ via the diagonal embedding. Notice that $\text{Stab}_{\mathcal{A}^{[m]}_{S'}}(\mathcal{D}_M^{[m]})$ is finite by definition of $S_{\text{gen}}$.

Take $M = \max_S \{M_0(S')\} > 0$ with $S'$ running over all irreducible components of $S_{\text{gen}}$. Then $M = M(S') > 0$ depends only on $S$.

We have an $S'$-morphism $(\text{id, } \mathcal{D}_M^{[m]}): \mathcal{A}^{[m]}_{S'} \times_{S'} (\mathcal{A}^{[m]}_{S'})_{[M+1]} \to \mathcal{A}^{[m]}_{S'} \times_{S'} (\mathcal{A}^{[m]}_{S'})_{[M]}$. Consider its restriction to $\mathcal{D}^{[m]}_{S'} \times_{S'} (\mathcal{D}^{[m]}_{S'})_{[M+1]} = \mathcal{D}^{[m]}_{S'}$.

In $\mathcal{A}^{[m]}_{S'}$, we have a non-degenerate subvariety $\mathcal{D}^{[m]}_{S'}$. Moreover, $\mathcal{D}(\mathcal{D}^{[m]}_{S'})$ is non-degenerate in $\mathcal{A}^{[m]}_{S'} \times_{S'} (\mathcal{A}^{[m]}_{S'})_{[M]}$. Indeed, $\mathcal{D}(\mathcal{D}^{[m]}_{S'}) = \mathcal{D}(\mathcal{D}^{[m]}_{S'}) \times_{S'} (\mathcal{D}^{[m]}_{S'})_{[M+1]} = \mathcal{D}^{[m]}_{S'} \times_{S'} (\mathcal{D}^{[m]}_{S'})_{[M+1]} \subseteq (\mathcal{A}^{[m]}_{S'})_{[M]} - \mathcal{A}^{[m]}_{S'} \times_{S'} (\mathcal{A}^{[m]}_{S'})_{[M]}$, and hence is non-degenerate because $\mathcal{D}^{[m]}_{S'}$ is non-degenerate; see \cite[Lem.6.2]{Gao21}.

Now we have obtained the two desired non-degenerate subvarieties $\mathcal{D}^{[m]}_{S'}$ in $\mathcal{A}^{[m]}_{S'}$ and $\mathcal{D}(\mathcal{D}^{[m]}_{S'})$ in $\mathcal{A}^{[m]}_{S'}$. Moreover using the notations below (5.5), a generic fiber of $\mathcal{D}|_{\mathcal{D}^{[m]}_{S'}}$ is a $G_{\eta} := \{0\}$ of $\text{Stab}_{\mathcal{A}^{[m]}_{S'}}(\mathcal{D})$-orbit. Moreover, $G_{\eta}$ is a finite group and hence extends to a finite étale $S'$-group subscheme $G$ of $\mathcal{A}^{[m]}_{S'} \to S'$.

**Step 2** Choose suitable functions $f_{S',1}$, $f_{S',2}$ and constant $\epsilon_{S'} > 0$ for later applications of equidistribution.

Write $\omega_{m(M+2)}$ for the Betti form on $\mathcal{A}^{[m+1]}_{S'}$ and $\omega_{m(M+1)}$ for the Betti form on $\mathcal{A}^{[m+1]}_{S'}$. Let $\mu_{S',1}$ be the measure on $\mathcal{D}^{[m+2]}_{S'}(\mathbb{C})$ as in \cite[Thm.1]{Kuh21}; it equals $k_{S',1} \cdot \omega_{m(M+2)}^{\text{dim} \mathcal{D}^{[m+2]}_{S'}}$ with a constant $k_{S',1} > 0$.

Let $\mu_{S',2}$ be the measure on $\mathcal{D}(\mathcal{D}^{[m+2]}_{S'})(\mathbb{C})$ as in \cite[Thm.1]{Kuh21}; it equals $k_{S',2} \cdot \omega_{m+1}^{\text{dim} \mathcal{D}(\mathcal{D}^{[m+2]}_{S'})}$ with a constant $k_{S',2} > 0$.

We start by proving $\mu_{S',1} \neq \mathcal{D}^{*} \mu_{S',2}$.

For the point $x \in (\mathcal{A}^{[m]}_{S'})_{[m]}(\mathbb{C})$ from (5.4), denote by $\Delta_{x}$ the point $(x, \ldots, x)$ in $(\mathcal{D}^{[m]}_{S'})_{[m+1]}(\mathbb{C}) = \mathcal{D}^{[m]}_{S'}(\mathbb{C})$. Then $(x, \Delta_{x}) \in (\mathcal{D}^{[m]}_{S'} \times_{S'} (\mathcal{D}^{[m]}_{S'})_{[M+1]})(\mathbb{C})$, which is furthermore a smooth point which is our choice of $x$ below (5.4). We have $(\mu_{S',1}(x, \Delta_{x}) = 0$; see \cite[Lem.6.3]{Gao21}.
On the other hand, \( \mathcal{D}_{M_0}^{[m]}(\Delta_x) \) is the origin of fiber of \((A^{[m]}_s)^{[M]} \to S'\) in question (which we call \((A^{[m]}_s)^{[M]}\)), so \( \mathcal{D}_{M_0}^{[m]} \big|_{\mathcal{D}^{-1}(\mathcal{D}(\Delta_x))} \) contains the diagonal of \( \mathcal{D}_{s_{\sigma}} \subseteq A^{[m]}_s \) in \((A^{[m]}_s)^{[M]}\) (which for the moment we denote by \( \Delta_{\mathcal{D}_{s_{\sigma}}} \)). Therefore for the morphism \( \mathcal{D} = (\text{id}, \mathcal{D}_{M_0}^{[m]})) \) from (5.6), \( \mathcal{D}^{-1}(\mathcal{D}(\Delta_x)) \) contains \((\Delta_x, \Delta_{\mathcal{D}_{s_{\sigma}}} \Delta) \). Thus \( \dim \mathcal{D}^{-1}(\mathcal{D}(\Delta_x)) > 0 \), so the linear map

\[
d\mathcal{D} : T(\Delta_x) \mathcal{D}^{[M+2]}_s \to T_{\mathcal{D}(\Delta_x)} \mathcal{D}(\mathcal{D}^{[M+2]}_s)
\]

has non-trivial kernel. This implies \( \mathcal{D}^{[s]}(\Delta_x, \Delta_{\mathcal{D}_{s_{\sigma}}} \Delta) = 0 \).

Thus we get \( \mu_{S',1} \neq \mathcal{D}^{[s]} \mu_{S',2} \) by looking at their evaluations at \((\Delta_x, \Delta_x)\).

Thus there exist a constant \( \epsilon_{S'} > 0 \) and a function \( f_{S',1} \in C^0_c(\mathcal{D}(\mathcal{D}^{[M+2]}_s)^{[an]}) \) such that

\[
(5.7) \quad \left| \int_{\mathcal{D}^{[M+2]}_s} f_{S',1} \mu_{S',1} - \int_{\mathcal{D}^{[M+2]}_s} f_{S',1} \mathcal{D}^{[s]} \mu_{S',2} \right| > \epsilon_{S'}.
\]

We finish this step by showing that we can choose such an \( f_{S',1} \) with the following property: there exists a unique \( f_{S',2} \in C^0_c(\mathcal{D}(\mathcal{D}^{[M+2]}_s)^{[an]}) \) such that \( f_{S',1} = f_{S',2} \circ \mathcal{D} \).

Indeed, let \( G \) be the finite \( S'\)-group from the end of Step 1. Then our \( \mathcal{D} \) from (5.6) satisfies the following property: there exists a Zariski open dense subset \( V \) of \( \mathcal{D}(\mathcal{D}^{[M+2]}_s) \subseteq A^{[m]}_s \) such that \( \mathcal{D}|_{\mathcal{D}^{-1}(V)} : \mathcal{D}^{-1}(V) \to V \) is finite étale and that each of its fibers \( \mathcal{D}^{-1}(y) \) is a \( G \)-orbit for each \( y \in V(\mathbb{C}) \); here \( y \mapsto s \in S' \) means the fiber of \( G \to S' \) over \( s \). Moreover, we can assume that \( V \) is sufficiently small such that each point \( \sigma \in G_{\overline{\eta}} \) extends to a section \( \sigma_U : U \to G_{\overline{\eta}} \subset A^{[m]}_U \) where \( U \subset S' \) is the image of \( V \). We may and do assume that \( f_{S',1} \) is supported in \( \mathcal{D}^{-1}(V) \). For each \( \sigma \in G_{\overline{\eta}} \), by abuse of notation we also use \( \sigma_U \) to denote the translation of \( A^{[m]}_U \) defined by it. As \( \sigma \) is the \((M+1)\)-fold self-product of a translation of \( A^{[m]}_s \), we have \( \mathcal{D} \sigma = \mathcal{D} \) and \( f_{S',1} \sigma_U := \sum_{\sigma \in G_{\overline{\eta}}} f_{S',1} \circ \sigma \sigma_U \) equals \( f_{S',2} \circ \mathcal{D} \) for an \( f_{S',2} \in C^0_c(\mathcal{D}(V)) \). Notice that \( \sigma^*_U \mu_{S',1} = \mu_{S',1} \) over \( U \) for each \( \sigma \in G_{\overline{\eta}} \) because \( \mu_{S',1} \) is obtained from the Betti form. Thus (5.7) holds true if we replace \( f_{S',1} \) by \( f_{S',1}^* \) and so it suffices to replace \( f_{S',1} \) by \( f_{S',1}^* \).

**Step 3** Prove some height lower bound on \( \mathcal{D}^{[M+2]}_s \) or \( \mathcal{D}(\mathcal{D}^{[M+2]}_s) \).

We apply the third-named author’s equidistribution result [Küh21, Thm.1] twice. In fact, to desired lower bound it is more convenient to apply its corollary under the formulation of [Gao21, Cor.8.2].

Apply [Gao21, Cor.8.2] to \( \mathcal{D}^{[M+2]}_s \), \( f_{S',1} \) and \( \epsilon_{S'} \). We thus obtain a constant \( \delta_{\epsilon_{S',1}} > 0 \) and a Zariski closed proper subset \( Z_{S',1} := Z_{f_{S',1}} \in \mathcal{D}^{[M+2]}_s \). Apply [Gao21, Cor.8.2] to \( \mathcal{D}(\mathcal{D}^{[M+2]}_s) \), \( f_{S',2} \) and \( \epsilon_{S'} \). We thus obtain a constant \( \delta_{\epsilon_{S',2}} > 0 \) and a Zariski closed proper subset \( Z_{S',2} := Z_{f_{S',2}} \in \mathcal{D}(\mathcal{D}^{[M+2]}_s) \).

Let \( \delta_{S'} := \min\{\delta_{\epsilon_{S',1}}, \delta_{\epsilon_{S',2}}\} > 0 \), and let \( Z_{S'} = Z_{S',1} \cup \mathcal{D}^{-1}(Z_{S',2}) \cup Z_{S',3} \), where \( Z_{S',3} \) is the largest Zariski closed subset of \( \mathcal{D}^{[M+2]}_s \) on which \( \mathcal{D} \) is not finite. Then \( Z_{S'} \) is Zariski closed in \( \mathcal{D}^{[M+2]}_s \), and is proper because \( \mathcal{D} \) is generically finite. If a point
\(x \in (X^{[m(M+2)]} \setminus Z_{S'})_\mathbb{Q}\) is such that \(\hat{h}_{L^{\mathbb{Q}(M+2)}(x)} < \delta_{S'}\) and \(\hat{h}_{L^{\mathbb{Q}(M+1)}}(\mathcal{D}(x)) < \delta_{S'}\), then case (ii) of [Gao21, Cor.8.2] holds true for both \(x, f_{S',1}, \mu_{S',1}\) and \(\mathcal{D}(x), f_{S',2}, \mu_{S',2}\). Thus

\[
\left| \int_{\mathcal{D}^{[m(M+2)]} \setminus \mathcal{D}} f_{S',1}\mu_{S',1} - \frac{1}{\#O(x)} \sum_{y \in O(x)} f_{S',1}(y) \right| < \epsilon_{S'} \quad \text{and} \quad \left| \int_{\mathcal{D}^{[m(M+1)]} \setminus \mathcal{D}} f_{S',2}\mu_{S',2} - \frac{1}{\#O(\mathcal{D}(x))} \sum_{y \in O(\mathcal{D}(x))} f_{S',2}(y) \right| < \epsilon_{S'}
\]

where \(O(\cdot)\) is the Galois orbit. But \(\frac{1}{\#O(x)} \sum_{y \in O(x)} f_{S',1}(y) = \frac{1}{\#O(\mathcal{D}(x))} \sum_{y \in O(\mathcal{D}(x))} f_{S',2}(y)\) because \(f_{S',1} = f_{S',2} \circ \mathcal{D}\). So we have

\[
\left| \int_{\mathcal{D}^{[m(M+2)]} \setminus \mathcal{D}} f_{S',1}\mu_{S',1} - \int_{\mathcal{D}^{[m(M+2)]} \setminus \mathcal{D}} f_{S',2}\mu_{S',2} \right| \leq 2\epsilon_{S'}.
\]

This contradicts \((5.7)\) because \(f_{S',1} = f_{S',2} \circ \mathcal{D}\).

Hence for each point \(x \in (X^{[m(M+2)]} \setminus Z_{S'})_\mathbb{Q}\), we are in one of the following alternatives.

(i) Either \(\hat{h}_{L^{\mathbb{Q}(M+2)}(x)} \geq \delta_{S'}\),

(ii) or \(\hat{h}_{L^{\mathbb{Q}(M+1)}}(\mathcal{D}(x)) \geq \delta_{S'}\).

Before moving on, let us take a closer look at the constants obtained in the first three steps. First of all, the two integers \(m = m(S) > 0\) and \(M = M(S) > 0\) were taken in Step 1 and depend only on \(S\). We obtain in Step 3 a proper closed subvariety \(Z_{S'}\) of \(X^{[m(M+2)]} \setminus \mathcal{D}\) and a constant \(\delta_{S'}\) for each irreducible component \(S'\) of \(S^\text{gen}\); à priori they depend on the function \(f_{S',1}\) and the constant \(\epsilon_{S'} > 0\) chosen in Step 2. However, if \(S', m\) and \(M\) are fixed, then the measures \(\mu_{S',1}\) and \(\mu_{S',2}\) in Step 2 are fixed, and so one can fix a choice of \(f_{S',1}\) and \(\epsilon_{S'} > 0\). Therefore, we can and do view \(Z_{S'}\) and \(\delta_{S'} > 0\) to depend only on \(S', m\) and \(M\).

**Step 4.** Conclude with a similar argument for Proposition 4.2.

Again, we are divided into two cases.

Consider the abelian scheme \(\pi^{[m(M+2)]} : A^{[m(M+2)]} \to H^c\).

Consider \(S' \setminus \pi^{[m(M+2)]}(X^{[m(M+2)]} \setminus Z_{S'})\) endowed with the reduced induced subscheme structure; it has dimension \(\leq \dim S' - 1 \leq \dim S - 1\). Let \(S_1, \ldots, S_k\) be the irreducible components of \(\bigcup_{S'} S' \setminus \pi^{[m(M+2)]}(X^{[m(M+2)]} \setminus Z_{S'})\) with \(S'\) running over all irreducible components of \(S^\text{gen}\). Then the set \(\{S_1, \ldots, S_k\}\) is uniquely determined by \(S\) and \(m\) and \(M\). But \(m = m(S) > 0\) and \(M = M(S) > 0\) depend only on \(S\). So the set \(\{S_1, \ldots, S_k\}\) is uniquely determined by \(S\).

**Case (i)\( s \in \bigcup_{i=1}^k S_i(\mathbb{Q})\).**

For each \(i \in \{1, \ldots, k\}\), we can apply the induction hypothesis to \(S_i\) since \(\dim S_i \leq \dim S - 1\). So we obtain constants \(c_{i,2}^n = c_i(r, d, S_i) > 0\) and \(c_{i,3}^n = c_i^n(r, d, S_i) > 0\) such that for each \(s \in S_i(\mathbb{Q})\), the set

\[
(5.8) \quad \Sigma_{i,s} := \left\{ x \in X^c_s(\mathbb{Q}) : \hat{h}_{L^c}(x) \leq c_{i,3}^n \right\}
\]

is contained in \(X' (\mathbb{Q})\) for some proper Zariski closed \(X' \subseteq X_s\) with \(\deg_{L^c}(X') < c_{i,2}^n\).

Let \(c_{\text{deg},2}^n = \max_{1 \leq i \leq k} \{c_{i,2}^n\} > 0\) and \(c_{\text{deg},3}^n = \min_{1 \leq i \leq k} \{c_{i,3}^n\} > 0\). We have seen above that the set \(\{S_1, \ldots, S_k\}\) is uniquely determined by \(S\). So \(c_{\text{deg},2}^n\) and \(c_{\text{deg},3}^n\) thus obtained
depend only on \(r, d\) and \(S\). Moreover, for each \(s \in \bigcup_{i=1}^{k} S_i(\overline{Q})\), the set
\[
\Sigma_{\text{deg},s} := \left\{ x \in \mathcal{R}^\circ_s(\overline{Q}) : \hat{h}_L(x) \leq c'_{\text{deg},3} \right\}
\]
must be contained in \(\Sigma_{i,s}\) for some \(i\); so \(\Sigma_{\text{deg},s}\) is contained in \(X'(\overline{Q})\) for some proper Zariski closed \(X' \subsetneq \mathcal{R}_s\) with \(\deg_{L_s}(X') < c'_{\text{deg},2}\). This concludes for Case (i).

**Case (ii)** \(s \in \pi^{[m(M+2)]}(\mathcal{R}_{S'}^{[m(M+2)]} \setminus Z_{S'}(\overline{Q}))\) for some irreducible component \(S'\) of \(S_{\text{gen}}\). In this case, \(\mathcal{R}_{S'}^{[m(M+2)]} = (\mathcal{R}_{S'}^{[m(M+2)]})_s \neq (Z_{S'})_s\).

Set \(c''_{S',3} = \delta_{S'}/4m(M+2)\). Then \(c''_{S',3}\) depend only on \(S', m\) and \(M\). Consider
\[
\Sigma_{S',s} = \left\{ x \in \mathcal{R}^\circ_s(\overline{Q}) : \hat{h}_L(x) \leq c''_{S',3} \right\} \subseteq \mathcal{R}_s(\overline{Q})\)

We claim that \(\Sigma_{S',s}^{m(M+2)} \subseteq (Z_{S'})_s(\overline{Q})\). Indeed assume not, then there exists \(x := (x_1, \ldots, x_{m(M+2)}) \in (\Sigma_{S',s}^{m(M+2)} \setminus (Z_{S'})_s(\overline{Q}))\). Then \(\hat{h}_{L_s}^{m(M+2)}(x) = \sum_{i=1}^{m(M+2)} \hat{h}_L(x_i) \leq m(M+2)c''_{S',3} < \delta_{S'}\). On the other hand, each component of \(\mathcal{R}(x)\) is of the form \(x_k\) or of the form \(x_i - x_j\) for some \(i\) and \(j\), and \(\hat{h}_L(x_j - x_i) \leq 2\hat{h}_L(x_j) + 2\hat{h}_L(x_i) \leq 4c''_{S',3}\). So \(\hat{h}_{L_s}^{m(M+1)}(\mathcal{R}(x)) \leq m(M+1)4c''_{S',3} < \delta_{S'}\). Thus we have reached a contradiction to the conclusion of Step 3. This establishes the claim.

Recall the assumption \(\mathcal{R}_s^{m(M+2)} \neq (Z_{S'})_s\) for this case. Thus we are allowed to apply Lemma 5.9 to \(X = \mathcal{R}_s, L = L_s|_{\mathcal{R}_s}, Z = (Z_{S'})_s\) and \(\Sigma = \Sigma_{S',s}\). So there exists a Zariski closed subset \(X'\) of \(\mathcal{R}_s\) such that
(i) \(X' \subsetneq \mathcal{R}_s\);
(ii) \(\deg_{L_s}(X') \leq c(m(M+2), r, d, \deg_{L^\circ_s}(Z_{S'})_s)\);
(iii) \(\Sigma_{S',s} \subseteq X'(\overline{Q})\).

But \(\deg_{L^\circ_s}(Z_{S'})_s\) depends only on \(S', m\) and \(M\). Hence property (ii) above can be simplified to be \(\deg_{L_s}(X') \leq c'_{S',2}\), with \(c'_{S',2}\) depending only on \(r, d, S', m\) and \(M\). In summary, the set \(\Sigma_{S',s}\) defined in (5.10) is contained in \(X'(\overline{Q})\) for some proper Zariski closed \(X' \subsetneq \mathcal{R}_s\) with \(\deg_{L_s}(X') \leq c'_{S',2}\).

Let \(c''_2 = \max_{S'}\{c''_{S',2}\} > 0\) and \(c''_3 = \min_{S'}\{c''_{S',3}\} > 0\), where \(S'\) runs over all irreducible components of \(S_{\text{gen}}\) defined by (5.3). The subset \(S_{\text{gen}}\) of \(S\) is uniquely determined by \(S\), and hence the set \(\{S'\}\) is uniquely determined by \(S\). Moreover \(m = m(S) > 0\) and \(M = M(S) > 0\) depend only on \(S\). So the constants \(c''_2\) and \(c''_3\) thus obtained depend only on \(r, d\) and \(S\). The discussion above yields the following assertion: for each \(s \in \bigcup_{S'} \pi^m(\mathcal{R}^{[m(M+2)]}_{S'} \setminus Z_{S'}(\overline{Q}))\), the set
\[
\Sigma^*_s := \left\{ x \in \mathcal{R}^\circ_s(\overline{Q}) : \hat{h}_L(x) \leq c''_3 \right\}
\]
must be contained in \(\Sigma_{S',s}\) for some \(S'\); so \(\Sigma^*_s\) is contained in \(X'(\overline{Q})\) for some proper Zariski closed \(X' \subsetneq \mathcal{R}_s\) with \(\deg_{L_s}(X') < c''_2\). This concludes for Case (ii).

Now let \(s \in S_{\text{gen}}\) be an arbitrary point. Then we are either in Case (i) or Case (ii). So the proposition holds true by letting \(c''_2 = \max\{c''_{\text{deg},2}, c''_3\} > 0\) and \(c''_3 = \min\{c''_{\text{deg},3}, c''_3\} > 0\); all the constants involved depend only on \(r, d\) and \(S\) (see (5.10) and (5.11)).
5.2. Proof of Proposition 5.1. The deduction of Proposition 5.1 from Proposition 5.2 follows from an almost verbalized copy of the proof of Proposition 4.4 and the argument in §4.3 except that the arguments needed here are simpler because we no longer need to deal with the Faltings height \( h_{\text{Fal}}(A) \). Instead of repeating the proof, we hereby give a brief explanation.

Let \((A, L)\) be a polarized abelian varieties defined over \( \overline{\mathbb{Q}} \) with \( \deg_L A = l \). Let \( X \) be an irreducible subvariety of \( A \), defined over \( \overline{\mathbb{Q}} \), with \( \dim X = r \) and \( \deg_L X = d \) such that \( X \) generates \( A \). We may and do assume \( r \geq 1 \) because otherwise the proposition is trivial.

We start with the case where \((A, L)\) is principally polarized; notice that \( l = g! \).

Let \( \mathfrak{A}_g \rightarrow A_g \) be the universal abelian variety and let \( \mathfrak{L}_g \) be the symmetric relatively ample line bundle on \( \mathfrak{A}_g/A_g \), both as at the beginning of §5.4.

Now \((A, L)\) gives rise to a point \( b \in A_g(\overline{\mathbb{Q}}) \) such that \((\mathfrak{A}_g)_b \cong A \) and \( c_1(\mathfrak{L}_g|_A) = 2c_1(L) = c_1(L \otimes L_-) \) for \( L_- = [-1]^*L \). We have \( \deg_{\mathfrak{L}_g} X = 8^d d \) by (4.10).

Consider the restricted Hilbert scheme \( H^r := H^r_{\text{Fal}}(\mathfrak{A}_g/A_g) \) from (3.6) and retain the commutative diagram (5.2)

\[
\begin{array}{ccc}
\mathfrak{A}_g \xrightarrow{i} A_g & \xrightarrow{i} \mathfrak{A}_g \\
\downarrow \pi & & \downarrow \pi^\text{univ} \\
H^r \xrightarrow{i_{H^r}} H^r_{\text{Fal}} & & \end{array}
\]

where \( \pi|_{\mathfrak{A}_g} : \mathfrak{A}_g \rightarrow H^r \) is the universal family. All objects in this diagram are varieties defined over \( \overline{\mathbb{Q}} \). Then the pair \((X, (A, L))\) is parametrized by a point \( s \in H^r(\overline{\mathbb{Q}}) \). Thus \( X = \mathfrak{A}_s \) and \( A = A_s = (\mathfrak{A}_g)_b \), and \( i_{H^r}(s) = b \).

For the line bundle \( L = i^* \mathfrak{L}_g \), we have seen that \( \hat{h}_{L|A} = 4 \hat{h}_{L \otimes L_-} \) as height functions on \( A(\overline{\mathbb{Q}}) \) in (4.12).

Let \( H^r_{\text{gen}} \) be the subset of \( H^r(\overline{\mathbb{Q}}) \) as defined by (3.9). If the stabilize of \( X \) in \( A \) (denoted by \( \text{Stab}(X) \)) has positive dimension, then \( X^0 = \emptyset \) by definition of Ueno locus and hence the proposition trivially holds true. Thus we may and do assume that \( \text{Stab}(X) \) is finite. Then the point \( s \in H^r(\overline{\mathbb{Q}}) \) which parametrizes \((X, (A, L))\) is in \( H^r_{\text{gen}} \).

Apply Proposition 5.2 to \( S = H^r \). Then we obtain constants \( c_2'' \) and \( c_3'' \) depending only on \( r, 8^d d \) and \( H^r \) (and hence only on \( g, d \) and \( r \)). The set \( \Sigma := \{ x \in \mathfrak{A}_s(\overline{\mathbb{Q}}) : \hat{h}_{L|A}(x) \leq c_3'' \} \) contains in \( X' \), for some proper Zariski closed \( X' \subseteq X \) with \( \deg_{\mathfrak{L}_A}(X') \leq c_2'' \). Thus Proposition 5.1 for this case holds true because \( \deg_L X' < \deg_{\mathfrak{L}_A}(X') \) (since \( c_1(\mathfrak{L}_A) = 4c_1(\mathfrak{L}_g|_A) = 8c_1(L) \)).

Now let us turn to arbitrary \((A, L)\). By Lemma 2.2(iv) and the inverse isogeny, there exist a principally polarized abelian variety \((A_0, L_0)\) defined over \( \overline{\mathbb{Q}} \) and an isogeny \( u : A_0 \rightarrow A \) such that \( u^*L = [l/g!]^*L_0 \); see below (1.13). In particular, \( u^*(L \otimes L_-) \cong (L_0 \otimes (L_0)_-) \otimes ([l/g]!)^2 \).

Let \( X_0 \) be an irreducible component of \( u^{-1}(X) \). Then \( \deg_{\text{Fal}} X_0 \leq d([l/g]!)^{2g-1} \) by (1.14), and \( X_0 \) is not contained in any proper subgroup of \( A_0 \). Thus we can apply the
conclusion for the principally polarized case to $X_0 \subseteq A_0$ and get two constants $c''_2$ and $c''_3$ depending only on $g$, $r$, $l$ and $d$.

By the definition of Ueno locus, we have $X^\circ = u(X_0)^\circ \subseteq u(X_0^\circ)$. So we have $\Sigma \subseteq u(\Sigma_0)$ for $\Sigma := \left\{ x \in X^\circ(\overline{Q}) : \hat{h}_{L\otimes L_-}(x) \leq c_3' \right\}$ and $\Sigma_0 = \left\{ x_0 \in X_0^\circ(\overline{Q}) : \hat{h}_{L_0\otimes(L_0)_-}(x_0) \leq (l/g!)^2 c_3'' \right\}$. Now the conclusion follows from the principally polarized case with $c''_3$ replaced by $(l/g!)^2 c''_3$.

6. Proof of the Gap Principle (Theorem 1.2)

In this section we combine Proposition 4.1 and Proposition 5.1 to finish the proof of the generalized New Gap Principle (Theorem 1.2) with the same argument for curves [Gao21, Prop.9.2]; this argument is eventually related to [DGH22, Prop.2.3].

Proposition 6.1. Let $g$, $l$, $r$, and $d$ be positive integers. There exist constants $c_1 = c_1(g, l, r, d) > 0$ and $c_2 = c_2(g, l, r, d) > 0$ satisfying the following property. For

- each polarized abelian variety $(A, L)$ of dimension $g$ defined over $\overline{Q}$ with $\deg_L A = l$,
- and each irreducible subvariety $X$ of $A$ defined over $\overline{Q}$ with $\dim X = r$ and $\deg_L X = d$, such that $X$ generates $A$,

the set

$$P \in X^\circ(\overline{Q}) : \hat{h}_{L\otimes L_-}(P) \leq c_1 \max\{1, h_{Fal}(A)\}$$

where $L_- := [-1]^*L$, is contained in a proper Zariski closed $X' \subseteq X$ with $\deg_L X' < c_2$.

Proof. Let $(A, L)$ and $X$ be as in the proposition.

By Proposition 4.1 there exist constants $c'_1 = c'_1(g, l, r, d) > 0$, $c'_2 = c'_2(g, l, r, d) > 0$ and $c'_3 = c'_3(g, l, r, d) > 0$ such that

$$P \in X^\circ(\overline{Q}) : \hat{h}_{L\otimes L_-}(P) \leq c'_1 \max\{1, h_{Fal}(A)\} - c'_3$$

is contained in a proper Zariski closed $X' \subseteq X$ with $\deg_L X' < c'_2$.

By Proposition 5.1 there exist constants $c''_2 = c''_2(g, l, r, d) > 0$ and $c''_3 = c''_3(g, l, r, d) > 0$ such that

$$P \in X^\circ(\overline{Q}) : \hat{h}_{L\otimes L_-}(P) \leq c''_3$$

is contained in a proper Zariski closed $X' \subseteq X$ with $\deg_L X' < c''_2$.

Now set

$$c_1 := \min\left\{ \frac{c''_3}{\max\{1, 2c'_3/c'_1\}}, \frac{c'_1}{2} \right\} \quad \text{and} \quad c_2 := \max\{c'_2, c''_2\}.$$ 

We will prove that these are the desired constants.

To prove this, it suffices to prove the following claim.

Claim: If $P \in X^\circ(\overline{Q})$ satisfies $\hat{h}_{L\otimes L_-}(P) \leq c_1 \max\{1, h_{Fal}(A)\}$, then $P$ is in either the set (6.2) or the set (6.3).

\footnote{Again, we first of all get constants $c''_{0,2}(g, r, d') > 0$ and $c''_{0,3}(g, r, d') > 0$ for each $1 \leq d' \leq d$, and then set $c''_{0,2} := \max_{1 \leq d' \leq d} \{c''_{0,2}(g, r, d')\}$ and $c''_{0,3} := \min_{1 \leq d' \leq d} \{c''_{0,3}(g, r, d')\}$.}
Let us prove this claim. Suppose \( P \in X^\circ(\overline{\mathbb{Q}}) \) is not in (6.2) or (6.3), i.e., \( \hat{h}_{L \otimes L^-}(P) > c'_1 \max\{1, h_{\text{Fal}}(A)\} - c'_3 \max\{1, 2h_{\text{Fal}}(A)\} \). We wish to prove \( \hat{h}_{L \otimes L^-}(P) > c_1 \max\{1, h_{\text{Fal}}(A)\} \).

We split up to two cases on whether \( \max\{1, h_{\text{Fal}}(A)\} \leq \max\{1, 2c'_3/c'_1\} \).

In the first case, i.e., \( \max\{1, h_{\text{Fal}}(A)\} \leq \max\{1, 2c'_3/c'_1\} \), we have

\[
\hat{h}_{L \otimes L^-}(P) > c'_3 \max\{1, h_{\text{Fal}}(A)\} = \frac{c'_3}{\max\{1, 2c'_3/c'_1\}} \max\{1, h_{\text{Fal}}(A)\} \geq c_1 \max\{1, h_{\text{Fal}}(A)\}.
\]

In the second case, i.e., \( \max\{1, h_{\text{Fal}}(A)\} > \max\{1, 2c'_3/c'_1\} \), we have \( c'_3 > (c'_1/2) \max\{1, h_{\text{Fal}}(A)\} \) and hence

\[
\hat{h}_{L \otimes L^-}(P) > \frac{c'_1}{2} \max\{1, h_{\text{Fal}}(A)\} \geq c_1 \max\{1, h_{\text{Fal}}(A)\}.
\]

Hence we are done. \( \square \)

**Proof of Theorem 7.1.** Let \( A \) be an abelian variety of dimension \( g \), let \( L \) be an ample line bundle, and let \( X \) be an irreducible subvariety which generates \( A \). Assume all these objects are defined over \( \overline{\mathbb{Q}} \). Then \( (A, L) \) is a polarized abelian variety.

Write \( d = \deg_L X, r = \dim X, l = \deg_L A \). Then \( l \leq c(g, d) \) by Lemma 2.5.

Since \( X \) generates \( A \), we have that \( r \geq 1 \). Thus we can apply Proposition 6.1 to \( (A, L) \) and \( X \). Then we obtain constants \( c_1 = c_1(g, l, r, d) > 0 \) and \( c_2 = c_2(g, l, r, d) > 0 \) such that the set

\[
\Sigma := \{ P \in X^\circ(\overline{\mathbb{Q}}) : \hat{h}_{L \otimes L^-}(P) \leq c_1 \max\{1, h_{\text{Fal}}(A)\} \}
\]

is contained in a proper Zariski closed \( X' \subseteq X \) with \( \deg_L X' < c_2 \).

Now we can conclude by replacing \( c_1 \) by \( \min_{1 \leq r \leq g, 1 \leq l \leq c(g, d)} \{ c_1(g, l, r, d) \} > 0 \) and replacing \( c_2 \) by \( \max_{1 \leq r \leq g, 1 \leq l \leq c(g, d)} \{ c_2(g, l, r, d) \} > 0 \). \( \square \)

**7. Proof of the Uniform Mordell–Lang Conjecture (Theorem 1.1)**

**7.1. A theorem of Rémond.** In this subsection, we work over \( \overline{\mathbb{Q}} \).

We start by recalling the following result, which is a consequence of Rémond’s generalized Vojta’s Inequality [Rémond00b, Thm.1.2] for points in \( X^\circ(\overline{\mathbb{Q}}) \), the generalized Mumford’s Inequality [Rémond00a, Thm.3.2] for points in \( X^\circ(\overline{\mathbb{Q}}) \cap \Gamma \), and the technique to remove the height of the subvariety [Rémond00a §3.b)].

Let \( A \) be an abelian variety of dimension \( g \), and \( L \) be a symmetric ample line bundle on \( A \).

Let \( X \) be an irreducible subvariety of \( A \), and \( \Gamma \) be a finite rank subgroup of \( A(\overline{\mathbb{Q}}) \). We say that the assumption (Hyp pack) holds true for \( (A, L) \), \( X \) and \( \Gamma \), if there exists a constant \( c_0 = c_0(g, \deg_L X) > 0 \) satisfying the following property: for each \( P_0 \in X(\overline{\mathbb{Q}}) \),

\[
\left\{ P - P_0 \in (X^\circ(\overline{\mathbb{Q}}) - P_0) \cap \Gamma : \hat{h}_L(P - P_0) \leq c_0^{-1} \max\{1, h_{\text{Fal}}(A)\} \right\} \leq c_0^{rk\Gamma + 1}.
\]

**Theorem 7.1** (Rémond). Assume that (Hyp pack) holds true for all \( (A, L), X, \Gamma \) (as above) such that \( X \) generates \( A \).

Then for each polarized abelian variety \( (A, L) \) with \( L \) symmetric, each irreducible subvariety \( X \) of \( A \) and each finite rank subgroup \( \Gamma \) of \( A(\overline{\mathbb{Q}}) \), we have

\[
\#X^\circ(\overline{\mathbb{Q}}) \cap \Gamma \leq c(g, \deg_L X, \deg_L A)^{rk\Gamma + 1}.
\]
A more detailed proof of this theorem can be found in Appendix A. We hereby give a brief explanation by taking David–Philippon’s formulation of Rémond’s result.

Proof. We start with a dévissage. More precisely, we reduce to the case where

\[(\text{Hyp}) : \quad X \text{ generates } A.\]

Indeed, let \(A'\) be the abelian subvariety of \(A\) generated by \(X - X\). Then \(X \subseteq A' + Q\) for some \(Q \in A(\overline{\mathbb{Q}})\). The subgroup \(\Gamma'\) of \(A(\overline{\mathbb{Q}})\) generated by \(\Gamma\) and \(Q\) has rank \(\leq \text{rk} \Gamma + 1\). We have \((X - Q)^{\circ} = X^{\circ} - Q\) by definition of the Ueno locus, \((X^{\circ}(\overline{\mathbb{Q}}) - Q) \cap \Gamma' = X^{\circ}(\overline{\mathbb{Q}}) \cap \Gamma'\) and \(\deg_L(X - Q) = \deg_L X\). By Lemma 2.3, \((\text{Hyp})\) yields \(\deg_L A' \leq c'(g, \deg_L X)\). Therefore if \((\text{Hyp})\) holds true for \(X - Q \subseteq A', L|_{A'}\) and \(\Gamma' \cap A'(\overline{\mathbb{Q}})\), then \(\#(X^{\circ}(\overline{\mathbb{Q}}) \cap \Gamma \leq c(g, \deg_L X)^{\text{rk} \Gamma + 1} \leq c(g, \deg_L X)^{\text{rk} \Gamma + 2}\). So we can conclude by replacing \(c\) with \(c^2\). Thus we are reduced to the case where \((\text{Hyp})\) holds true.

We take the formulation of David–Philippon [DP07, Thm.6.8] of Rémond’s result [Rém00a].

Let \(c_{\text{NT}}\) and \(h_1\) be as from [DP07, Thm.6.8]. It is known that \(c_{\text{NT}}, h_1 \leq c' \max\{1, h_{\text{Fal}}(A)\}\) for some \(c' = c'(g, \deg_L X) > 0\); see [DP07, equation (6.41)].

Let \(\eta \geq 1\) be a real number. Then

\[(\text{Hyp})\) allows us to apply \((7.1)\) to \(X - P_0\) for each \(P_0 \in X(\overline{\mathbb{Q}})\). Set \(R = (\eta c' \max\{1, h_{\text{Fal}}(A)\})^{1/2}\) and \(r = (c_0^{-1} \max\{1, h_{\text{Fal}}(A)\})^{1/2}\) with \(c_0\) from \((7.1)\).

Consider the real vector space \(\Gamma \otimes \mathbb{R}\) endowed with the Euclidean norm \(|\cdot| = h_L^{1/2}\).

By an elementary ball packing argument, any subset of \(\Gamma \otimes \mathbb{R}\) contained in a closed ball of radius \(R\) centered at \(0\) is covered by at most \((1 + 2R/r)^{\text{rk} \Gamma}\) closed balls of radius \(r\) centered at the elements \(P - P_0\) of the given subset \((7.1)\); see [Rém00a, Lem.6.1]. Thus the number of balls in the covering is at most \((1 + 2\sqrt{\eta c' c_0})^{\text{rk} \Gamma}\). But each closed ball of radius \(r\) centered at some \(P - P_0\) in \((7.1)\) contains at most \(c\) elements by \((7.1)\). So

\[(7.4)\) \# \(\left\{ P \in X^{\circ}(\overline{\mathbb{Q}}) \cap \Gamma : h_L(P) \leq \eta c' \max\{1, h_{\text{Fal}}(A)\} \right\} \leq c_0(1 + 2\sqrt{\eta c' c_0})^{\text{rk} \Gamma}.\]

So \((7.3)\) and \((7.4)\) yield, for each real number \(\eta \geq 1,\)

\[(7.5)\) \# \(\left\{ P \in X^{\circ}(\overline{\mathbb{Q}}) \cap \Gamma : h_L(P) \leq \eta \max\{1, c_{\text{NT}}, h_1\} \right\} \leq c_0(1 + 2\sqrt{\eta c' c_0})^{\text{rk} \Gamma}.\]

Thus [DP07, Thm.6.8] implies

\[(7.6)\) \# \(X^{\circ}(\overline{\mathbb{Q}}) \cap \Gamma \leq (c'' r^{\text{rk} \Gamma + 1} \cdot c_0(1 + 2\sqrt{c'' c_0})^{\text{rk} \Gamma}\)

for some \(c'' = c''(g, \deg_L X) > 0\). Therefore \((7.2)\) holds true by letting \(c = (c'' c_0(1 + 2\sqrt{c'' c_0}))^2\).

7.2. Proof of Theorem 1.1 over \(\overline{\mathbb{Q}}\). Now we are ready to prove Theorem 1.1 over \(\overline{\mathbb{Q}}\)

In view of Rémond’s result (Theorem 7.1) cited above, the most important ingredient is the following proposition.

**Proposition 7.2.** Let \(A\) be an abelian variety of dimension \(g\) and \(L\) be an ample line bundle on \(A\). Then with \(L\) replace by \(L \otimes L_-\), \((\text{Hyp} \text{ pack})\) holds true for each irreducible subvariety \(X\) of \(A\) which generates \(A\) and each finite rank subgroup \(\Gamma\) of \(A(\overline{\mathbb{Q}})\).
Proof. Write $d = \deg_L X$.
We prove this result by induction on $r := \dim X$.
The base step is $r = 0$, in which case trivially holds true.
For an arbitrary $r \geq 1$. Assume the theorem is proved for $0, 1, \ldots, r - 1$.

We wish to prove (7.7) with $L$ replaced by $L \otimes L$. Let $P_0 \in X(\bar{\mathbb{Q}})$. Then $\deg_L (X - P_0) = \deg_L X = d$. Moreover, $(X - P_0)^{(\mathbb{Q})} = X^0(\mathbb{Q}) - P_0$ by definition of the Ueno locus. Notice that $X - P_0$ still generates $A$ because $(X - P_0) - (X - P_0) = X - X$.

Apply Theorem 1.2 to $X - P_0$. Then we have constants $c_1 = c_1(g, d) > 0$ and $c_2 = c_2(g, d) > 0$ such that for $\left\{ P - P_0 \in X^0(\mathbb{Q}) - P_0 : \hat{h}_{L \otimes L}(P - P_0) \leq c_1 \max\left\{ 1, h_{\text{Fal}}(A) \right\} \right\}$ is contained in a proper Zariski closed $X' \subseteq X - P_0$ with $\deg_L X' < c_2$. In particular, the number of irreducible components of $X'$ is $< c_2$.

Let $X^\dagger$ be an irreducible component of $X'$. Then $\dim X^\dagger \leq \dim X - 1 \leq r - 1$. Let $A^\dagger$ be the abelian subvariety of $A$ generated by $X^\dagger - X^\dagger$. Then $X^\dagger \subseteq A^\dagger + Q$ for some $Q \in A(\mathbb{Q})$. Now $\dim(X^\dagger - Q) = \dim X^\dagger \leq r - 1$.

Let $\Gamma^\dagger$ be the subgroup of $A(\mathbb{Q})$ generated by $\Gamma$ and $Q$. Then $\text{rk}\Gamma^\dagger \leq \text{rk}\Gamma + 1$. Apply the induction hypothesis to $A^\dagger$, $L|A^\dagger$, the irreducible subvariety $X^\dagger - Q$ and $\Gamma^\dagger \cap A^\dagger(\mathbb{Q})$. Then we have

\begin{equation}
\#(X^\dagger - Q)^{(\mathbb{Q})} \cap \Gamma^\dagger \leq (c^\dagger)^{\text{rk}\Gamma^\dagger + 1} \leq (c^\dagger)^{\text{rk}\Gamma + 2}
\end{equation}

for some $c^\dagger = c^\dagger(\dim A^\dagger, \deg_L (X^\dagger - Q)) > 0$. But $\dim A^\dagger \leq \dim A = g$ and $\deg_L (X^\dagger - Q) = \deg_L X^\dagger \leq \deg_L X' < c_2 = c_2(g, d)$.

But $(X^\dagger - Q)^{(\mathbb{Q})} = (X^\dagger)^{(\mathbb{Q})} - Q$ by definition of the Ueno locus, and $Q \in \Gamma^\dagger$. So (7.7) yields

\begin{equation}
\#(X^\dagger)^{(\mathbb{Q})} \cap \Gamma^\dagger \leq (c^\dagger)^{\text{rk}\Gamma + 2}
\end{equation}

Now that $(X^\dagger)^{(\mathbb{Q})} \cap \Gamma \subseteq \bigcup_{X^\dagger}(X^\dagger)^{(\mathbb{Q})} \cap \Gamma^\dagger$ with $X^\dagger$ running over all irreducible components of $X'$ and $\Gamma^\dagger$ constructed accordingly, (7.8) implies

\begin{equation}
\#(X^\dagger)^{(\mathbb{Q})} \cap \Gamma \leq c_2 \max\{c^\dagger, c_3, \}^{\text{rk}\Gamma + 2} \leq c_3^{\text{rk}\Gamma + 1}
\end{equation}

where $c_3 = (\max X^\dagger\{c_2, c^\dagger\}) > 0$ depends only on $g$ and $d$.

But $\left\{ P - P_0 \in X^0(\mathbb{Q}) - P_0 : \hat{h}_{L \otimes L}(P - P_0) \leq c_1 \max\left\{ 1, h_{\text{Fal}}(A) \right\} \right\} \subseteq X'(\mathbb{Q})$ by construction of $X'$. Moreover, $(X')^\circ \supseteq X^0 \cap X'$ by definition of the Ueno locus. So (7.9) yields

\begin{equation}
\#(X')^\circ(\mathbb{Q}) \cap \Gamma \leq c_2 \max\{c^\dagger, c_3, \}^{\text{rk}\Gamma + 2} \leq c_3^{\text{rk}\Gamma + 1}
\end{equation}

with $c_2 = \max\{c_1, c_2, c^\dagger\} > 0$ which depends only on $g$ and $d$. Hence we are done. \hfill \Box

Proof of Theorem 1.4 with $F = \mathbb{Q}$. Let $A$ be an abelian variety of dimension $g$ and $L$ be an ample line bundle on $A$. Let $X$ be a closed irreducible subvariety of $A$. Assume that all varieties are defined over $\mathbb{Q}$. Set $l = \deg_L A$ and $d = \deg_L X$. Let $\Gamma$ be a subgroup of $A(\mathbb{Q})$ of finite rank.

Notice that $\deg_{L \otimes L}(X) = 2^{\dim X} \deg_L X \leq 2^gd$ and $\deg_{L \otimes L} A = 2^g \deg_L A = 2^gl$.

Let $X^\circ$ be the complement of the Ueno locus of $X$.

As before, we start by reducing to the case where $X$ generates $A$. Indeed, let $A'$ be the abelian subvariety of $A$ generated by $X - X$. Then $X \subseteq A' + Q$ for some $Q \in A(\mathbb{Q})$. 

\[ \text{THE UNIFORM MORDELL–LANG CONJECTURE} \]
The subgroup $\Gamma'$ of $A(\bar{\mathbb{Q}})$ generated by $\Gamma$ and $Q$ has rank $\leq \text{rk}\Gamma + 1$. We have $(X - Q)^\circ = X^\circ - Q$ by definition of the Ueno locus, $(X^\circ(\bar{\mathbb{Q}}) - Q) \cap \Gamma \subseteq (X^\circ(\bar{\mathbb{Q}}) - Q) \cap \Gamma' = X^\circ(\bar{\mathbb{Q}}) \cap \Gamma'$ and $\deg_L(X - Q) = \deg_L X$. If (1.2) holds true for $X - Q \subseteq A' \setminus \{1\}$ and $\Gamma' \cap A'(\mathbb{Q})$, then $\#X^\circ(\bar{\mathbb{Q}}) \cap \Gamma \leq c(g, d)^{\text{rk}\Gamma + 1} \leq c(g, d)^{\text{rk}\Gamma + 2}$. So we can conclude by replacing $c$ with $c^2$. Thus we are reduced to the case where $X$ generates $A$. In particular, we have $l \leq c'(g, d)$ by Lemma 2.5.

By Proposition 7.2 and Theorem 7.1 we have $\#X^\circ(\bar{\mathbb{Q}}) \cap \Gamma \leq c(g, l, d)^{\text{rk}\Gamma + 1}$. Thus (1.2) holds true because $l \leq c'(g, d)$. Hence we are done. □

7.3. From $\bar{\mathbb{Q}}$ to arbitrary $F$ of characteristic 0. The following lemma of specialization allows us to pass from $\bar{\mathbb{Q}}$ to $F$.

Lemma 7.3. Assume Theorem 7.1 holds true for $F = \bar{\mathbb{Q}}$. Then Theorem 7.1 holds true, under the extra assumptions that $\Gamma$ is finitely generated, for arbitrary $F$ of characteristic 0 with $F = \mathbb{F}$.

Proof. Assume Theorem 7.1 holds true for $F = \bar{\mathbb{Q}}$. Then we obtain a function $c : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that Theorem 7.1 holds true with this function $c$ viewed as a constant depending only on $g$ and the degree of the subvariety in question.

Now let $F$ be an arbitrary algebraic closed field of characteristic 0. Let $A$, $L$, $X$ and $\Gamma$ be as in Theorem 7.1 with $\Gamma$ finitely generated. Write $\rho = \text{rk}\Gamma$. Let $\gamma_1, \ldots, \gamma_r \in A(F)$ be generators of $\Gamma$ with $\gamma_{r+1}, \ldots, \gamma_r$ torsion.

There exists a field $K$, finitely generated over $\bar{\mathbb{Q}}$, such that $A$, all elements of $\text{End}(A)$, $X$, and $\gamma_1, \ldots, \gamma_r$ are defined over $K$. Then $K$ is the function field of some regular, irreducible quasi-projective variety $V$ defined over $\bar{\mathbb{Q}}$.

Up to replacing $V$ by a Zariski open dense subset, we have

- $A$ extends to an abelian scheme $\mathcal{A} \rightarrow V$ of relative dimension $g$,
- $L$ extends to a relatively ample line bundle $\mathcal{L}$ on $\mathcal{A}/V$,
- each element of $\text{End}(A)$ extends to an element of $\text{End}(\mathcal{A}/V)$ (this can be achieved since $\text{End}(A)$ is a finitely generated group); in particular, each abelian subvariety $B$ of $A$ extends to an abelian subscheme $\mathcal{B}$ of $\mathcal{A} \rightarrow V$,
- $X$ extends to a flat family $\mathcal{X} \rightarrow V$ (i.e., $X$ is the generic fiber of $\mathcal{X} \rightarrow V$),
- $\gamma_1, \ldots, \gamma_r$ extend to sections of $\mathcal{A} \rightarrow V$; we retain the symbols $\gamma_1, \ldots, \gamma_r$ for these sections. Use $\Gamma_V$ to denote the sub-$\mathcal{V}$-group of $\mathcal{A}$ generated by $\gamma_1, \ldots, \gamma_r$.

Moreover, up to replacing $V$ by a Zariski open dense subset, we may choose $\mathcal{X}$ over $V$ to be a “Good model” as explained in [Maz00, pp.219–220] (which uses [Hin88, App.1, Lemma A]) such that the Ueno locus of $\mathcal{X}_v$ is the specialization of the Ueno locus of $X$ for each $v \in V(\bar{\mathbb{Q}})$. Hence $\mathcal{X}_v$ is the specialization of $X^\circ$.

As $\mathcal{X} \rightarrow V$ is flat, we have $\deg_L X = \deg_{\mathcal{L}_v} \mathcal{X}_v$ for each $v \in V(\bar{\mathbb{Q}})$.

We define the specialization of $\Gamma$ at $v$, which we denote with $\Gamma_v$, to be the subgroup of $\mathcal{A}_v(\bar{\mathbb{Q}})$ generated by $\gamma_1(v), \ldots, \gamma_r(v)$. There exists then a specialization homomorphism $\Gamma \rightarrow \Gamma_v$ for each $v \in V(\bar{\mathbb{Q}})$. Note that $\text{rk}\Gamma_v \leq \rho$.

The extension of elements of $\text{End}(A)$ to elements of $\text{End}(\mathcal{A}/V)$ yields a specialization $\text{End}(A) \rightarrow \text{End}(\mathcal{A}_v)$ for each $v \in V(\bar{\mathbb{Q}})$. Denote this map by $\alpha \mapsto \alpha_v$.

---

[5] We can do this because $\text{End}(A)$ is a finitely generated group.
Set \( \Theta := \{ v \in V(\overline{\mathbb{Q}}) : \Gamma \rightarrow \Gamma_v \text{ is injective and } \text{End}(A) \cong \text{End}(A_v) \} \).

Masser [Mas89, Main Theorem and Scholium 1] and [Mas96, Main Theorem] guarantee that \( \Theta \) is Zariski dense in \( V \).

Let \( v \in \Theta \). Then \( \#X^0(F) \cap \Gamma \leq \#X^0_v(\overline{\mathbb{Q}}) \cap \Gamma_v \). By the result over \( \overline{\mathbb{Q}} \), we have \( \#X^0_v(\overline{\mathbb{Q}}) \cap \Gamma_v \leq c(g, \deg_{L_v} X_v)^{1+\kappa_{L_v}^r} \leq c(g, \deg_{L} X)^{1+\rho} \). Hence we are done. \( \square \)

Now we are ready to finish the proof of Theorem 1.1.

**Proof of Theorem 1.1** for arbitrary \( F \). Let \( F \) be an arbitrary algebraic closed field of characteristic 0. Let \( A, L \) and \( X \) be as in Theorem 1.1. Let \( \Gamma \) be a subgroup of \( A(F) \) of finite rank \( \rho \).

By the definition of a finite rank group, there exists a finitely generated subgroup \( \Gamma_0 \) of \( A(F) \) with rank \( \rho \) such that
\[
\Gamma \subseteq \{ x \in A(F) : [N]x \in \Gamma_0 \text{ for some } N \in \mathbb{N} \}.
\]

Moreover, we may choose such a \( \Gamma_0 \) satisfying that \( \Gamma_0 = \text{End}(A) \cdot \Gamma_0 \).

For each \( n \in \mathbb{N} \), define
\[
\frac{1}{n} \Gamma_0 := \{ x \in A(F) : [n]x \in \Gamma_0 \}.
\]

Then \( \frac{1}{n} \Gamma_0 \) is again a finitely generated subgroup of \( A(F) \) of rank \( \rho \), and is invariant under \( \text{End}(A) \).

We have proved Theorem 1.1 over \( \overline{\mathbb{Q}} \) in \( \S 7.2 \). Thus Theorem 1.1 holds true for \( \frac{1}{n} \Gamma_0 \) and our \( F \) by the specialization result above (Lemma 7.3). So there exists a constant \( c = c(g, \deg_{L} X) > 0 \) such that
\[
\#X^0(F) \cap \frac{1}{n} \Gamma_0 \leq c^{1+\rho} \tag{7.11}
\]

Note that \( \{ \frac{1}{n} \Gamma_0 \}_{n \in \mathbb{N}} \) is a filtered system and \( \Gamma \subseteq \bigcup_{n \in \mathbb{N}} \frac{1}{n} \Gamma_0 \). But the bound (7.11) is independent of \( n \). So
\[
\#X^0(F) \cap \Gamma \leq c^{1+\rho}.
\]

This is precisely Theorem 1.1. Hence we are done. \( \square \)

**7.4. From Theorem 1.1 to Theorem 1.1.** Now that we have proved Theorem 1.1, we can conclude for Theorem 1.1 with the following lemma.

**Lemma 7.4.** Assume Theorem 1.1 holds true for all \( (A, L), X, \) and \( \Gamma \). Then Theorem 1.1 also holds true.

**Proof.** Using the same argument as in the proof of Theorem 1.1 with \( F = \overline{\mathbb{Q}} \) at the end of \( \S 7.2 \), we may and do assume that \( X \) generates \( A \).

We start with the following finer description of the Ueno locus of \( X \). Let \( \Sigma(X; A) \) be the set of abelian subvarieties \( B \subseteq A \) with \( \dim B > 0 \) satisfying: \( x + B \subseteq X \) for some \( x \in A(F) \), and \( B \) is maximal for this property. Then for each \( B \in \Sigma(X; A) \), there exists a closed subvariety \( X_B \) of \( X \) such that the Ueno locus of \( X \) is \( \bigcup_{B \in \Sigma(X; A)} (X_B + B) \).

The union above can be expressed in a quantitative way. First, by Bogomolov [Bog81, Thm. 1], each \( B \in \Sigma(X; A) \) satisfies \( \deg L B \leq c_3 \) for some constant \( c_3 = c_3(\dim A, \deg_{L} X) > 0 \), and hence \( \#\Sigma(X; A) \leq c_4 = c_4(\dim A, \deg_{L} X) \) by [Rém00a, Prop. 4.1]; here we used...
Lemma 2.5. Next, $X_B$ can be constructed as follows. Let $B^\perp$ be a complement of $B$, i.e. $B \cap B^\perp$ is finite and $B + B^\perp = A$. It is possible to choose such a $B^\perp$ with $\deg_L B^\perp \leq c_\delta^2(g, \deg_L A, \deg_L B)$; see [MW93]. Then we can choose $X_B := \bigcap_{b \in B(F)} (X - b) \cap B^\perp$. Notice that by dimension reasons, this intersection must be a finite intersection of at most $\dim X \leq \dim A$ members. So $\deg_L X_B \leq c_\delta(\dim A, \deg_L X)$ by Bézout’s Theorem and Lemma 2.5. In particular, $X_B$ has at most $c_\delta$ irreducible components $X_{B,1}, \ldots, X_{B,m_B}$.

As the $B_i$’s in (1.1) satisfies $x_i + B_i \subseteq X$ and $\dim B_i > 0$, we may and do assume $B_i \in \Sigma(X; A)$ by definition of the Ueno locus. Now (1.1) becomes

$$X(F) \cap \Gamma = \bigcup_{B \in \Sigma(X; A)} \bigcup_{j=1}^{n_B} (x_{B,j} + B)(F) \cap \Gamma \bigcap X^\circ(F) \cap \Gamma.$$  

Moreover, each $x_{B,j}$ can be chosen to be in $X_{B}^\circ(F) \cap \Gamma$, where $X_{B}^\circ = \bigcup_{k=1}^{n_B} X_{B,k}^\circ$. See [Rémi99, Lem. 4.6]; notice that $p|X_B$ is finite for the quotient $p: A \to A/B$. In particular, $n_B \leq \# X_{B}^\circ(F) \cap \Gamma$.

Let us bound $n_B$ for each $B \in \Sigma(X; A)$. Applying Theorem 1.1 to each irreducible component $X_{B,k}$ of $X_B$, we get $\# X_{B,k}^\circ(F) \cap \Gamma \leq c_\epsilon(1 + \rk \Gamma)$ for some $c = c(\dim A, \deg_L X_{B,k}) > 0$. But we have seen that $X_B$ has at most $c_\delta$ components and that $\deg_L X_{B,k} \leq \deg_L X_B \leq c_\delta$. So $\# X_{B,k}^\circ(F) \cap \Gamma$, and hence $n_B$, is at most $c_\delta(\dim A, \deg L X)_{1+\rk \Gamma}$.

From the bounds on $\Sigma(X; A)$ and $n_B$ above, we get from (7.12) that $N \leq c_4_{\epsilon}^{1+\rk \Gamma} + \# X^\circ(F) \cap \Gamma$. Hence Theorem 1.1 holds true by applying Theorem 1.1 again to $X^\circ(F) \cap \Gamma$.

8. Proof of the uniform Bogomolov conjecture (Theorem 1.3)

Proof of Theorem 1.3. Let $A$ be an abelian variety of dimension $g$, let $L$ be an ample line bundle, and let $X$ be an irreducible subvariety. Assume all these objects are defined over $\mathbb{Q}$.

Write $d = \deg_L X$ and $r = \dim X$. Let $c = c(g)$ be the constant from Lemma 2.5. Let $c_2' = c_2'(g, \deg L, r, d) > 0$ and $c_3'' = c_3''(g, \deg L, r, d) > 0$ be from Proposition 5.1. Moreover, set $c_3 := \min_{1 \leq r \leq g} \{c_2'(g, \deg L, r, d)\} > 0$ and $c_2 := \max_{1 \leq r \leq g} \{c_2'(g, \deg L, r, d)\} > 0$; both $c_3$ and $c_2$ depend only on $g$ and $d$.

We prove the theorem by induction on $r$. The base step $r = 0$ trivially holds true.

For arbitrary $r$, assume the theorem is proved for $0, \ldots, r - 1$.

Let $A'$ be the abelian subvariety of $A$ generated by $X - X$, then $\deg_L A' \leq c d^g$ by Lemma 2.5. We can apply Proposition 5.1 to $X$ and $(A', L|_{A'})$ to conclude that the set

$$\Sigma := \left\{ P \in X^\circ(\mathbb{Q}) : \hat{h}_{L_{\leq L}}(P) \leq c_3 \right\},$$

where $L_{\leq L} = [-1]^* L$, is contained in $X^\circ(\mathbb{Q})$, for some proper Zariski closed $X' \subseteq X$ with $\deg_L(X') < c_2$. Each irreducible component of $X'$ has dimension $\leq r - 1$, and $X'$ has at most $c_2$ irreducible components. Hence the conclusion follows by applying the induction hypothesis to each irreducible component of $X'$ and appropriately adjusting $c_3$ and $c_2$. \qed
Appendix A. Rémond’s theorem revisited

The goal of this appendix is to give a more detailed proof of Rémond’s theorem, which we cited as Theorem [7.1] to make the current paper more complete. We will explain how Rémond’s generalized Vojta’s Inequality, generalized Mumford’s Inequality, and the technique to remove the height of the subvariety together imply the desired Theorem 7.1. The proof follows closely the arguments presented in [Rémond 00a].

We work over \( \overline{\mathbb{Q}} \).

Let \( A \) be an abelian variety and let \( L \) be a symmetric ample line bundle on \( A \). To ease notation, we may and do assume \( L \) is very ample and gives a projectively normal closed immersion into some projective space, by replacing \( L \) by \( L^{\otimes 4} \).

Let us restate Theorem 7.1.

Let \( X \) be an irreducible subvariety of \( A \), and \( \Gamma \) be a finite rank subgroup of \( A(\overline{\mathbb{Q}}) \). We say that the assumption (Hyp pack) holds true for \((A, L), X \) and \( \Gamma \), if there exists a constant \( c_0 = c_0(g, \deg_L X) > 0 \) satisfying the following property: for each \( P_0 \in X(\overline{\mathbb{Q}}) \),

\[
\left\{ P \in (X^0(\overline{\mathbb{Q}}) - P_0) \cap \Gamma : \hat{h}_L(P - P_0) \leq c_0^{-1} \max\{1, h_{\text{Fal}}(A)\} \right\} \leq c_0^{\text{rk} \Gamma + 1}.
\]

\[ \text{Theorem 7.1. Assume that (Hyp pack) holds true for all } (A, L), X, \Gamma \text{ (as above) such that } X \text{ generates } A. \]

Then for each polarized abelian variety \((A, L)\) with \( L \) symmetric and very ample, each irreducible subvariety \( X \) of \( A \) and each finite rank subgroup \( \Gamma \) of \( A(\overline{\mathbb{Q}}) \), we have

\[
\#X^0(\overline{\mathbb{Q}}) \cap \Gamma \leq c(g, \deg_L X, \deg_L A)^{\text{rk} \Gamma + 1}. \tag{7.2}
\]

Moreover, we may and do assume that

(Property c): As a function, \( c \) is increasing in all three invariables.

In what follows, we will introduce many constants \( c_4, c_5, \ldots \). All these constants are assumed to depend only on \( g, \deg_L X, \) and \( \deg_L A \) unless stated otherwise.

A.1. Preliminary setup. We have \( H^0(A, L) = \deg_L A/g! \) by Lemma 2.2. Thus \( A \) can be embedded into the projective space \( \mathbb{P}^{\deg_L A/g! - 1} \) using global sections of \( H^0(A, L) \). Thus the integer \( n \) in [Rémond 00a, Rémond 00b] can be taken to be \( \deg_L A/g! - 1 \).

The closed immersion \( A \subseteq \mathbb{P}^{\deg_L A/g! - 1} \) defines a height function \( h : A(\overline{\mathbb{Q}}) \to \mathbb{R} \). The Tate Limit Process then gives rise to a height function

\[
\hat{h}_L : A(\overline{\mathbb{Q}}) \to [0, \infty), \quad P \mapsto \lim_{N \to \infty} \frac{h([N]^2 P)}{N^4}.
\]

For \( P, Q \in A(\overline{\mathbb{Q}}) \) we set \( \langle P, Q \rangle = (\hat{h}_L(P + Q) - \hat{h}_L(P) - \hat{h}_L(Q))/2 \) and often abbreviate \( |P| = \hat{h}_L(P)^{1/2} \). The notation \( |P| \) is justified by the fact that it induces a norm after tensoring with the reals.

It follows from Tate’s construction that there exists a constant \( c_{NT} \geq 0 \), which depends on \( A \), such that \( |\hat{h}_L(P) - h(P)| \leq c_{NT} \) for all \( P \in A(\overline{\mathbb{Q}}) \).

Let \( h_1 \) denote the Weil height of the polynomials defining the addition and the substraction on \( A \).

It is known that

\[
c_{NT}, h_1 \leq c'(g, \deg_L A) \max\{1, h_{\text{Fal}}(A)\}. \tag{A.1}
\]
See [DP07, equation (6.41)]. Alternatively this can be deduced from [DGH21] (8.4) and (8.7).

Finally for any irreducible subvariety \( X \) of the projective space \( \mathbb{P}^{\deg_L A/g!-1} \), one can define the height \( h(X) \); see [BGS94].

### A.2. Generalized Voja’s Inequality

**Theorem A.1** ([Rém00b, Thm.1.1]). There exist constants \( c_4 = c_4(g, \deg_L X, \deg_A) > 0 \) and \( c_5 = c_5(g, \deg_L X, \deg_A) > 0 \) with the following property. If \( P_0, \ldots, P_{\dim X} \in X^\circ(\overline{\mathbb{Q}}) \) satisfy

\[
\langle P_i, P_{i+1} \rangle \geq \left( 1 - \frac{1}{c_4} \right) |P_i||P_{i+1}| \quad \text{and} \quad |P_{i+1}| \geq c_4|P_i|,
\]

then

\[
|P_0|^2 \leq c_5 \max\{1, h(X), h_{\text{Fal}}(A)\}.
\]

**Proof.** This follows immediately from [Rém00b, Thm.1.1] (with \( n = \deg_L A/g! - 1 \)) and (A.1) and \( \dim X \leq g \). \( \square \)

### A.3. Generalized Mumford’s Inequality

Let \( D_{\dim X} : X^{\dim X+1} \to \overline{A}^{\dim X} \) be the morphism defined by \((x_0, x_1, \ldots, x_{\dim X}) \mapsto (x_1 - x_0, \ldots, x_{\dim X} - x_0)\).

**Proposition A.2** ([Rém00a, Prop.3.4]). There exist constants \( c_4 = c_4(g, \deg_L X, \deg_A) > 0 \) and \( c_5 = c_5(g, \deg_L X, \deg_A) > 0 \) with the following property. Let \( P_0 \in X(\overline{\mathbb{Q}}) \). Suppose \( P_1, \ldots, P_{\dim X} \in X(\overline{\mathbb{Q}}) \) with \( (P_0, P_1, \ldots, P_{\dim X}) \) isolated in the fiber of \( D_{\dim X} : X^{\dim X+1} \to \overline{A}^{\dim X} \). If

\[
\langle P_0, P_i \rangle \geq \left( 1 - \frac{1}{c_4} \right) |P_0||P_i| \quad \text{and} \quad |P_0| - |P_i| \leq \frac{1}{c_4}|P_0|
\]

then

\[
|P_0|^2 \leq c_5 \max\{1, h(X), h_{\text{Fal}}(A)\}.
\]

**Proof.** This follows immediately from [Rém00a, Prop.3.4] (with \( n = \deg_L A/g! - 1 \)) and (A.1) and \( \dim X \leq g \). \( \square \)

**Proposition A.3** ([Rém00a, Prop.3.3]). Let \( \Xi \subseteq X^\circ(\overline{\mathbb{Q}}) \). We are in one of the following alternatives.

(i) Either for any \( x \in X(\overline{\mathbb{Q}}) \), there exist pairwise distinct \( x_1, \ldots, x_{\dim X} \in \Xi \) such that \((x, x_1, \ldots, x_{\dim X})\) is isolated in the fiber of \( D_{\dim X} : X^{\dim X+1} \to \overline{A}^{\dim X} \);

(ii) or \( \Xi \) is contained in a proper Zariski closed subset \( X' \subsetneq X \) with \( \deg X' < (\deg_L X)^{2^{\dim X}} \).

**Proof.** Denote by 0 the origin of the abelian variety \( A \). We may and do assume that the stabilizer of \( X \) in \( A \), denoted by \( \text{Stab}(X) \) has dimension 0; otherwise \( X^\circ = \emptyset \) and the proposition trivially holds true.

The points in the fiber of \( D_{\dim X} \) in question can be written as \((x+a, x_1+a, \ldots, x_{\dim X}+a)\) with \( a \) running over the \( \overline{\mathbb{Q}} \)-points of \((X-x) \cap (X-x_1) \cap \cdots \cap (X-x_{\dim X})\). Thus \((x, x_1, \ldots, x_{\dim X})\) is isolated in the fiber of the image of \( X^{\dim X+1} \to \overline{A}^{\dim X} \) if and only if

\[
\text{dim}_0(X-x) \cap (X-x_1) \cap \cdots (X-x_{\dim X}) = 0.
\]
Assume we are not in case (i). Then there exists \(i_0 \leq \dim X - 1\) satisfying the following property. There are pairwise distinct points \(x_1, \ldots, x_{i_0}\) such that for \(W := (X - x) \cap (X - x_2) \cap \cdots \cap (X - x_{i_0})\), we have

\[(\text{A.3}) \quad \dim_0 W = \dim_0 W \cap (X - y) = \dim X - i_0 \quad \text{for all } y \in \Xi.\]

Let \(C_1, \ldots, C_s\) be the irreducible components of \(W\) passing through 0 with \(\dim C_j = \dim X - i_0 \geq 1\). Then \(s \leq \sum_{j=1}^s \deg C_j \leq (\deg X)^{i_0+1} \leq (\deg X)^{\dim X}\) by Bézout’s Theorem. Moreover,

\[
\dim_0 W \cap (X - y) = \dim_0 W = \dim X - i_0 \iff C_j \subseteq X - y \quad \text{for some } j \in \{1, \ldots, s\}
\]

\[
\iff y \in \bigcap_{c \in C_j(\overline{\mathbb{Q}})} (X - c) \quad \text{for some } j \in \{1, \ldots, s\}.
\]

So (A.3) is equivalent to \(\Xi \subseteq \bigcup_{j=1}^s \bigcap_{c \in C_j(\overline{\mathbb{Q}})} (X - c)\). Each \(\bigcap_{c \in C_j(\overline{\mathbb{Q}})} (X - c)\) is a finite intersection of at most \(\dim X\) members because of dimension reasons. So \(\deg \bigcap_{c \in C_j(\overline{\mathbb{Q}})} (X - c) \leq (\deg X)^{\dim X}\) by Bézout’s Theorem. Moreover, each irreducible component of \(\bigcap_{c \in C_j(\overline{\mathbb{Q}})} (X - c)\) has dimension < \(\dim X\) because \(\dim \text{Stab}(X) = 0\). Hence we are in case (ii) by setting \(X' = \bigcup_{j=1}^s \bigcap_{c \in C_j(\overline{\mathbb{Q}})} (X - c)\). So we are done. \(\square\)

### A.4. Removing \(h(X)\).

**Lemma A.4** ([Rémi00a, Lem.3.1]). Assume \(S \subseteq X(\overline{\mathbb{Q}})\) a finite set. Assume that each equidimensional subvariety \(Y \supseteq S\) of \(X\) of dimension \(\dim X - 1\) satisfies \(\deg_L Y > \deg_L A(\deg_L X)^2/g!\). Then

\[
h(X) \leq (\deg_L A/g! + 1)^{\dim X + 1} \deg_L X \left( \max_{x \in S} h(x) + 3 \log(\deg_L A/g!) \right).
\]

**Proof.** This is precisely [Rémi00a, Lem.3.1] with \(n = \deg_L A/g! - 1\). \(\square\)

### A.5. Proof of Theorem 7.1

Let \(X\) be an irreducible subvariety of \(A\), and let \(\Gamma\) be a subgroup of \(A(\overline{\mathbb{Q}})\) of finite rank.

We start by reducing to the case where

\[(\text{Hyp}) : \quad X \text{ generates } A.\]

Indeed, let \(A'\) be the abelian subvariety of \(A\) generated by \(X\). Then \(X \subseteq A' + Q\) for some \(Q \in A(\overline{\mathbb{Q}})\). The subgroup \(\Gamma'\) of \(A(\overline{\mathbb{Q}})\) generated by \(\Gamma\) and \(Q\) has rank \(\leq \rk \Gamma + 1\). We have \((X - Q)^\circ = X^\circ - Q\) by definition of the Ueno locus, \((X^\circ(\overline{\mathbb{Q}}) - Q) \cap \Gamma \subseteq (X^\circ(\overline{\mathbb{Q}}) - Q)\). Therefore if (7.2) holds true for \(X - Q \subseteq A', L|A'\), and \(\Gamma' \cap A(\overline{\mathbb{Q}})\), then \(\#X^\circ(\overline{\mathbb{Q}}) \cap \Gamma \leq c(g, \deg_L X)^{\rk + 2}\). So we can conclude by replacing \(c\) with \(c^2\). Thus we are reduced to the case where (Hyp) holds true.

Now, assume (Hyp). We prove (7.2) by induction on

\[(\text{A.4}) \quad r := \dim X.\]

The base step is \(r = 0\), in which case trivially holds true. For an arbitrary \(r \geq 1\). Assume (7.2) holds true for \(0, 1, \ldots, r - 1\).
Observe that both Theorem A.1 and Proposition A.2 hold with $c_4$ and $c_5$ replaced by some larger value. We let $c_4$ (resp. $c_5$) denote the maximum of both constants $c_4$ (resp. $c_5$) from these two theorems. Both constants depend only on $g$, $\deg_L X$ and $\deg_L A$.

**Step 1** Handle large points by both inequalities of Rémond. The goal of this step is to prove the following bound: there exists a constant $c_6 = c_6(g, \deg_L X, \deg_L A) > 0$ such that

$$
(A.5) \quad \# \left\{ P \in X^o(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}_L(P) \geq c_3 \max\{1, h(X), h_{Fal}(A)\} \right\} \leq c_6^{k\Gamma+1}.
$$

The proof follows a standard classical argument involving the inequalities of Vojta and Mumford. Consider the rK$\Gamma$-dimensional real vector space $\Gamma \otimes \mathbb{R}$ endowed with the Euclidean norm $| \cdot | = \hat{h}_L^{1/2}$. We may and do assume $rK\Gamma \geq 1$. By elementary geometry, the vector space can be covered by at most $[(1 + (8c_4)^{1/2})^{rK\Gamma}]$ cones on which $\langle P, Q \rangle \geq (1 - 1/c_4) |P||Q|$ holds.

Let $P_0, P_1, P_2, \ldots, P_N \in X^o(\overline{\mathbb{Q}}) \cap \Gamma$ be pairwise distinct points in one such cone such that

$$
(A.6) \quad c_3 \max\{1, h(X), h_{Fal}(A)\} < |P_0|^2 \leq |P_1|^2 \leq |P_2|^2 \leq \cdots \leq |P_N|^2.
$$

Notice that

$$
(A.7) \quad \langle P_i, P_j \rangle \geq \left(1 - \frac{1}{c_4}\right) |P_i||P_j| \quad \text{for all } i, j \in \{0, \ldots, N\}.
$$

Set $N' := (\deg_L X)^{2r}c(g, (\deg_L X)^{2r}, \deg_L A)^{rK\Gamma+1} + 1$, with $c$ the constant from (7.2).

Consider the subset $\Xi_j = \{P_{j+1}, \ldots, P_{j+N'}\}$ with $j \in \{0, \ldots, N - N'\}$; it has $N'$ pairwise distinct elements. We claim that $\Xi$ cannot be contained in a proper Zariski closed subset $X' \subset X$ with $\deg_L X' \leq (\deg_L X)^{2\dim X}$. Indeed if such an $X'$ exists, then by definition of the Ueno locus we have $(X')^o \supseteq X^o \cap X'$. So $\Xi \subseteq (X')^o(\overline{\mathbb{Q}}) \cap \Gamma$. As $\dim X' < \dim X = r$, we can apply the induction hypothesis (7.2) to each irreducible component of $X'$. As $X' \subset X$ has $\leq (\deg_L X)^{2r}$ irreducible components and each component has degree $\leq (\deg_L X)^{2r}$, we then get $\#\Xi \leq (\deg_L X)^{2r}c(g, (\deg_L X)^{2r}, \deg_L A)^{rK\Gamma+1}$. This contradicts our choice of $N'$ because $c$ is increasing in all the three variables (Property c).

By Proposition A.3 applied to $\Xi_j$ and $P_j$, we then get pairwise distinct points $P_{i_1}, \ldots, P_{i_r} \in \Xi_j$ such that $(P_j, P_{i_1}, \ldots, P_{i_r})$ is isolated in the fiber of $D_r : X^{r+1} \rightarrow A^r$. But the hypotheses of Proposition A.2 cannot hold true by (A.7) and (A.7). So there exists $k \in \{i_1, \ldots, i_r\}$ such that $|P_k| - |P_j| > \frac{1}{c_4}|P_j|$. As $k \leq j + N'$, we then have

$$
|P_{j+N'}| > \left(1 + \frac{1}{c_4}\right) |P_j|.
$$

This holds true for each $j \in \{0, \ldots, N - N'\}$. So

$$
(A.8) \quad |P_{j+N'}|^k > \left(1 + \frac{1}{c_4}\right)^k |P_j|,
$$

for all $j \geq 0$ and $k \geq 1$.

Next we choose an integer $M \geq 0$ such that $(1 + 1/c_4)^M \geq c_4$. We may and do assume that $M$ depends only on $g$, $\deg_L X$ and $\deg_L A$ (since $c_4$ does). Then by (A.8), $|P_{(k+1)MN'}| > (1 + 1/c_4)^M |P_{kMN'}| \geq c_4 |P_{kMN'}|$ for each $k \geq 0$. 

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We claim \( N < rMN' \leq gMN' \). Indeed, assume \( N \geq rMN' \). Then we have \( r + 1 \) pairwise distinct points \( P_0, P_{MN'}, P_{2MN'}, \ldots, P_{rMN'} \). The hypotheses of Theorem \[A.3\] for these points cannot hold true by \( (A.6) \) and \( (A.7) \). Thus there exists \( k \) such that \( |P_{(k+1)MN'}| < c_4|P_{kMN'}| \). But this contradicts the conclusion of last paragraph. So we much have \( N < rMN' \leq gMN' \).

Recall that we have covered \( \Gamma \otimes \mathbb{R} \) by at most \( [(1 + (8c_4)^{1/2})^{rk\Gamma}] \) cones and each cone contains \( < gMN' \) points \( P \in X^\circ(\overline{Q}) \cap \Gamma \) with \( h_L(P) = |P|^2 \geq c_5 \max\{1, h(X), h_{Fal}(A)\} \). Thus

\[
\# \left\{ P \in X^\circ(\overline{Q}) \cap \Gamma : \hat{h}_L(P) \geq c_5 \max\{1, h(X), h_{Fal}(A)\} \right\} \leq (1 + (8c_4)^{1/2})^{rk\Gamma} gMN'.
\]

All constants on the right hand side depend only on \( g, \deg_L X \) and \( \deg_L A \). So \( (A.5) \) holds true by choosing \( c_6 \) appropriately.

**Step 2** Remove the dependence on \( h(X) \). More precisely, set

\[
(A.9) \quad c_7 := \deg_L A(\deg_L X)^2/g! \cdot c(g, \deg_L A(\deg_L X)^2/g!, \deg_L A) \quad \text{and} \quad N'' := c_7^{rk\Gamma+1} + 1.
\]

The goal of this step is to prove: There exist positive constants \( c_8, c_9, c_{10}, \) depending only on \( g, \deg_L X, \) and \( \deg_L A \) with the following property. If \( P_0, \ldots, P_{N''} \) are pairwise distinct points in \( X^\circ(\overline{Q}) \cap \Gamma \), then

\[
(A.10) \quad \# \left\{ P \in X^\circ(\overline{Q}) \cap \Gamma : \hat{h}_L(P - P_0) \geq c_3 \max_{1 \leq i \leq N''} \hat{h}_L(P_i - P_0) + c_9 \max\{1, h_{Fal}(A)\} \right\} \leq c_7^{rk\Gamma+1}.
\]

The proof follows closely [Rémond, Prop.3.6]. We wish to apply Lemma \[A.4\] to \( X - P_0 \) and the set \( S = \{P_i - P_0 : 0 \leq i \leq N''\} \). Let us verify the hypothesis. Let \( Y \subseteq X - P_0 \) with \( Y \) equidimensional of dimension \( \dim X - 1 = r - 1 \) and \( S \subseteq Y(\overline{Q}) \), and set \( Y' := Y + P_0 \). Then \( P_i \in (Y')(\overline{Q}) \cap \Gamma \). Each irreducible component of \( Y' \) has dimension \( \leq r - 1 \). Thus we can apply induction hypothesis \( (7.2) \) to each irreducible component of \( Y' \). So \( N'' \leq \sum_{Y''} c(g, \deg_L Y'', \deg_L A)^{rk\Gamma+1} \), with \( Y'' \) running over all irreducible components of \( Y' \). Since \( \deg_L Y = \deg_L Y' = \sum_{Y''} \deg_L Y'' \) and \( c \) is increasing in all the three variables (Property \( c \)), this bound implies

\[
N'' \leq \deg_L Y' \cdot c(g, \deg_L Y, \deg_L A)^{rk\Gamma+1}.
\]

We claim \( \deg_L Y > \deg_L A(\deg_L X)^2/g! \). Indeed, assume \( \deg_L Y \leq \deg_L A(\deg_L X)^2/g! \). Then \( N'' \leq \deg_L A(\deg_L X)^2/g! \cdot c(g, \deg_L A(\deg_L X)^2/g!, \deg_L A)^{rk\Gamma+1} \) because \( c \) is increasing in all the three variables (Property \( c \)). This contradicts the definition of \( N'' \) from \( (A.9) \).

Thus the assumption of Lemma \[A.4\] is satisfied. So

\[
h(X) \leq c_{11}(g, \deg_L X, \deg_L A) \left( \max_{1 \leq i \leq N''} \hat{h}_L(P_i - P_0) + \max\{1, h_{Fal}(A)\} \right).
\]

Here we also used \( (A.1) \). Thus by \( (A.5) \), we can find the desired constants \( c_8, c_9, c_{10} \) such that \( (A.10) \) holds true.

**Step 3** Prove the following alternative.

(i) Either \( \# X^\circ(\overline{Q}) \cap \Gamma \leq N''(8c_8 + 1)^{rk\Gamma} + c_7^{rk\Gamma+1} \),
(ii) or there exists \( Q \in X^\circ(Q) \cap \Gamma \) such that \(#\{P \in X^\circ(Q) \cap \Gamma : \hat{h}_L(P - Q) \geq 2c_9 \max\{1, h_{\text{Fal}}(A)\}\}\leq c_{10}^{k\Gamma+1}.

The proof follows closely [Rém00a Prop.3.7]. Assume we are not in case (i), i.e. \(#X^\circ(Q) \cap \Gamma > N''(8c_8 + 1)^{k\Gamma} + c_{10}^{k\Gamma+1}\). Let \(c_{12}\) be the smallest real number such that there exists \( Q \in X^\circ(Q) \cap \Gamma \) with

\[
\#\{P \in X^\circ(Q) \cap \Gamma : |P - Q| \geq c_{12}\} \leq c_{10}^{k\Gamma+1}.
\]

Consider the set \( \Xi := \{P \in X^\circ(Q) \cap \Gamma : |P - Q| \leq c_{12}\} \) in the \(\Gamma\)-dimensional Euclidean space \((\Gamma \otimes \mathbb{R}, |\cdot|)\). Then \#\(\Xi\) \#\(X^\circ(Q) \cap \Gamma - \#\{P \in X^\circ(Q) \cap \Gamma : |P - Q| \geq c_{12}\} > N''(4c_8 + 1)^{k\Gamma}\). By an elementary ball packing argument, \(\Xi\) (being a subset of \(\Gamma \otimes \mathbb{R}\) contained in a closed ball of radius \(c_{12}\) centered at \(Q\)) is covered by at most \((8c_8 + 1)^{k\Gamma}\) balls of radius \(c_{12}/4c_8\) centered at points in \(X^\circ(Q) \cap \Gamma\); see [Rém00a Lem.6.1]. By the Pigeonhole Principle, one of the balls contains \(\geq N'' + 1\) points in \(X^\circ(Q) \cap \Gamma\), say \(P_0, P_1, \ldots, P_{N''}\). We have \(|P_i - P_0| \leq c_{12}/2c_8\) for each \(i\). Thus (A.10) yields

\[
\#\left\{P \in X^\circ(Q) \cap \Gamma : \hat{h}_L(P - P_0) \geq c_{12}^2/4 + c_9 \max\{1, h_{\text{Fal}}(A)\}\right\} \leq c_{10}^{k\Gamma+1}.
\]

Therefore by the minimality of \(c_{12}\), we have \(c_{12}^2 \leq c_{12}^2/4 + c_9 \max\{1, h_{\text{Fal}}(A)\}\), and hence \(c_{12}^2 \leq 2c_9 \max\{1, h_{\text{Fal}}(A)\}\). So we are in case (ii) by (A.11).

**Step 4** Conclude by the standard packing argument.

Recall our assumption (Hyp) that \(X\) generates \(A\). The assumption of Theorem 7.1 says that (Hyp pack) holds true for \(X\) and \(\Gamma\), i.e. we have (7.1).

Assume we are in case (i) from Step 3. Recall the definition of \(N'' = c_7^{k\Gamma+1} + 1\) from (A.9). Then \#\(X^\circ(Q) \cap \Gamma \leq c^{k\Gamma+1}\) for \(c := 2\max\{(c_7 + 1)(8c_8 + 1), c_{10}\}\). Hence we can conclude for this case.

Assume we are in case (ii) from Step 3. Set \(R = (2c_9 \max\{1, h_{\text{Fal}}(A)\})^{1/2}\) and \(R_0 = (c_{10}^{-1} \max\{1, h_{\text{Fal}}(A)\})^{1/2}\). By an elementary ball packing argument, any subset of \(\Gamma \otimes \mathbb{R}\) contained in a closed ball of radius \(R\) centered at \(Q\) is covered by at most \((1 + 2R/R_0)^{k\Gamma}\) closed balls of radius \(R_0\) centered at the elements \(P - P_0\) with \(P\) from the given subset (7.1); see [Rém00a Lem.6.1]. Thus the number of balls in the covering is at most \((1 + 2\sqrt{2c_9c_0})^{k\Gamma}\). But each closed ball of radius \(r\) centered at some \(P - P_0\) in (7.1) contains at most \(c\) elements by (7.1). So

\[
\#\left\{P \in X^\circ(Q) \cap \Gamma : \hat{h}_L(P - Q) \leq 2c_9 \max\{1, h_{\text{Fal}}(A)\}\right\} \leq c_0(1 + 2\sqrt{2c_9c_0})^{k\Gamma}.
\]

Thus we have \#\(X^\circ(Q) \cap \Gamma \leq c_0(1 + 2\sqrt{2c_9c_0})^{k\Gamma} + c_{10}^{k\Gamma+1}\). So \#\(X^\circ(Q) \cap \Gamma \leq c^{k\Gamma+1}\) for \(c := 2\max\{c_0, 1 + 2\sqrt{2c_9c_0}, c_{10}\}\). Hence we can conclude are this case.

Therefore, it suffices to take \(c = 2\max\{(c_7 + 1)(8c_8 + 1), c_0, 1 + 2\sqrt{2c_9c_0}, c_{10}\}\), which is a constant depending only on \(g, \deg_L X\) and \(\deg_L A\). We are done.

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