C-spaces, generalized geometry and double field theory

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Abstract

We construct a class of C-spaces associated with closed 3-forms. We show that they depend only on the class of 3-form in $H^3(M, \mathbb{Z})$ and that induce a generalized geometry structure on the spacetime. We also explain their relation to gerbes. C-spaces are constructed after introducing additional coordinates at the open sets and at their double overlaps of a spacetime generalizing the standard construction of Kaluza-Klein spaces for 2-forms. C-spaces are not manifolds and satisfy the topological geometrization condition. Double manifolds arise as local subspaces of C-spaces that cannot be globally extended. This indicates that for the global definition of double field theories additional coordinates are needed. We explore several other aspect of C-spaces like their topology and relation to Whitehead towers, and also describe the construction of C-spaces for closed k-forms.
1 Introduction

Double field theory (DFT) has been introduced to provide a geometric interpretation of the T-duality symmetries and to describe string theory in a T-duality covariant way, see [1, 2, 3] for early works and [4]-[10] for more recent developments. More general proposals include $E_{11}$ [12, 13] and exceptional field theories, see eg [14]-[24], and reviews [25, 26, 27] and references with. For the construction of DFTs, the spacetime $M$ is enhanced with additional coordinates, leading to double spaces $D_M$ which have dimension twice that of spacetime. So far the construction of local actions relies on two ingredients. First, the use of infinitesimal transformations to prove invariance, and second the application of the strong section condition. These infinitesimal transformations combine the spacetime diffeomorphisms and the gauge transformations of the $B$-field that act on a generalized metric. This generalized metric is constructed from both the spacetime metric and the $B$-field. This is interpreted as a geometrization of $B$-field. The strong section condition in effect restricts the fields and their infinitesimal transformations to dependent on either the spacetime or dual coordinates. More recently several suggestions have been made to integrate the infinitesimal transformations of double field theory leading to the construction of finite transformations of the double spaces and those of the associated fields [28, 27, 29, 30]. Another suggestion is to employ a non-trivial split metric on the extended spaces [31]. Similar results also hold for the exceptional field theories, however see also [32, 33].

The global definition of double field theories remains an open problem. Using the solution of the strong section condition for the spacetime presented in [28, 27], it has been shown in [34] that the patching of double spaces constructed is consistent if and only if the 3-form field strength is exact. In section 4, we shall strengthen this statement. The C-spaces that we propose below resolve this global patching problem.

To identify the spaces which can implement the geometrization of the $B$-field in the context of DFT, it has also been argued in [34] that one necessary ingredient is the topological geometrization condition. This can be stated as follows: Given a manifold $M$, eg spacetime, and a closed k-form $\omega^k$, a space $C_M$ satisfies the topological geometrization condition, if and only if there is a projection $\pi : C_M \rightarrow M$ such that $\pi^* \omega^k$ represents the trivial class in $H^k(C_M)$.

Given $M$ and $\omega^k$, this definition does not uniquely specify $C_M$. There are several constructions of C-spaces via K-theory and homotopy theory. The latter applies for any manifold and for any form of any degree. The standard examples of C-spaces are circle bundles over $M$ which satisfy the topological geometrization property for closed 2-forms, and implement the geometrization of the Maxwell fields in the context of Kaluza-Klein theory.

In this paper, a construction of C-spaces, $C_M^{[\omega^3]}$, is proposed for every closed 3-form,
\( \omega^3 \), on a manifold \( M \) provided that \( [\omega^3] \in H^3(M, \mathbb{Z}) \) which is suitable for applications in DFT. The construction involves the introduction of new coordinates associated with the gauge transformations of the transition functions of \( \omega^3 \) at double overlaps with respect to both the Čech and de Rham differentials. This leads to an additional

- (local) 1-form coordinate \( y^1 \) for every open set of spacetime, as for double spaces,
- and a new angular coordinate \( \theta \) at every double intersection of two open sets.

Exploring the consistency of the patching conditions given in (2.6) at triple and 4-fold overlaps, it leads to the requirement that \( \omega^3 \) must represent a class in \( H^3(M, \mathbb{Z}) \) as expected from the Dirac quantization condition. In addition, it is demonstrated that \( C_M^{[\omega^3]} \) depends on \( [\omega^3] \), ie it is independent from the choice of a representative of the cohomology class \( [\omega^3] \). From construction is apparent that \( C_M^{[\omega^3]} \) are not manifolds, in particular they do not have a well-defined dimension. Nevertheless they can be described in some detail using the transition functions and the additional coordinates. For example, one can show that \( C_M^{[\omega^3]} \) satisfy the topological geometrization condition.

This construction of C-spaces for closed 3-forms is related to gerbes. In particular, we explain how from \( C_M^{[\omega^3]} \) one can construct the gerbe transition functions that arise in the approach of [35]. However the construction of \( C_M^{[\omega^3]} \) involves the open sets and their double overlaps, as well as the triple and 4-fold overlaps, in an essential way, and the emphasis is on the object itself rather than its transition functions on \( M \). This is more close in spirit to the definition of gerbes in terms of sheafs [36] but without the complications of category theory. Furthermore, the construction of \( C_M^{[\omega^3]} \) leads to the emergence of generalized geometry on \( M \) as described by Hitchin and Gualtieri [37, 38]. In particular we shall show that \( C_M^{[\omega^3]} \) induces a bundle over \( M \) which is the extension of \( T^*M \) with \( T^*M \). As result one can define a generalized metric and carry out generalized differential geometry calculus on \( M \).

To get some insight into the topological structure of \( C_M^{[\omega^3]} \), we consider the nerve of the good cover of \( M \) which provides a chain complex description of \( M \). We find that every 2-simplex in the nerve of \( M \) together with the new angular coordinates give rise to a \( \mathbb{C}P^2 \) in \( C_M^{[\omega^3]} \). We use this to raise the question whether this construction of \( C_M^{[\omega^3]} \) is related to Whitehead towers. Furthermore, we construct, \( C_T^{[\omega^3]} \), which is the C-space of 3-torus with a 3-form flux. We demonstrate that \( C_T^{[\omega^3]} \) resloves the patching problems of the double space construction of [27] for this model.

To elucidate the relation between C-spaces and double spaces, we revisit the global properties of the double spaces. We show that the mere use of the strong section condition, ie without invoking any information about the transformation of the generalized fields, together with the requirement of the general covariance of the spacetime imply that the double space must be diffeomorphic to \( T^*M \). Such a space cannot satisfy the topological geometrization property. Moreover if the transition functions of the B-field are related in a linear way to those of the dual coordinates, then the 3-form flux is exact.

However, we shall demonstrate that \( y^1 \) and the corresponding coordinates for double spaces transform differently.
On the other hand, the C-spaces, $C_M^{[\omega^3]}$, include the double spaces as local subspaces. In particular, the double spaces arise as subspaces of $C_M^{[\omega^3]}$ by taking the new angular coordinates $\theta$ to vanish. This can be consistently done only at appropriate open sets and not globally over the whole spacetime $M$. Therefore double spaces can only provide a local description DFTs, ie on a patch of $M$. For the global definition of DFTs over $M$ additional coordinates are required.

The construction of C-spaces, $C_M^{[\omega^k]}$, can be generalized to every $k$-form, $\omega^k$, which represents a class in $H^k(M, \mathbb{Z})$. This proceeds in a similar way to that of $C_M^{[\omega^3]}$. However, the construction of $C_M^{[\omega^k]}$ requires the presence of additional coordinates which are introduced at the multiple intersections of open sets of $M$. The properties of $C_M^{[\omega^k]}$ are also similar, ie $C_M^{[\omega^k]}$ satisfy the topological geometrization condition and depend on the class of $\omega^k$ in $H^k(M, \mathbb{Z})$. Their construction also has a Kaluza-Klein interpretation. The extended space associated with a $k$-form, which is the generalization of a double space for $k > 3$, can be seen as local subspaces of $C_M^{[\omega^k]}$. This again indicates that more coordinates are need for the global description of exceptional field theories.

There is a construction of C-spaces in the context of homotopy theory using Whitehead towers. Here we revisit the theory and point out that the Whitehead towers construction for 2-forms coincides, up to homotopy, with the standard circle bundle construction of Kaluza-Klein spaces. Then we review some of the properties of Whitehead towers construction for closed 3-forms and ask the question how these are related to $C_M^{[\omega^3]}$ spaces.

This paper has been organized as follows. In section 2, we describe the construction of $C_M^{[\omega^3]}$. In section 3, we explain the relation of $C_M^{[\omega^3]}$ to generalized geometry and gerbes. In section 4, we investigate some of the topological properties of $C_M^{[\omega^3]}$ and investigate the 3-torus with 3-form flux C-space. In section 5, we explore the application of C-spaces to DFT. In section 6, we construct C-spaces for closed k-forms. In section 7, we explore the relation between C-spaces and Whitehead towers, and in section 8, we give our conclusions.

## 2 C-spaces for closed 3-forms

### 2.1 C-spaces for closed 2-forms

Before, we proceed to give the patching conditions of C-spaces associated with closed 3-forms, let us briefly review the standard Kaluza-Klein space, $C_M^{[\omega^2]}$, for 2-forms. Let $M$ be a manifold and \{${U_\alpha}$\}$_{\alpha \in I}$ be a good cover$^5$ of $M$, for the precise definition see eg$^3$.

Moreover suppose that $\omega^2$ represents a class in $H^2(M, \mathbb{R})$. Then within the Čech-de Rham theory applying the Poincaré lemma on the open sets $U_\alpha$ as well as their $U_{\alpha\beta}$ and $U_{\alpha\beta\gamma}$ intersections$^6$,

$$\omega^2 = dA^1_\alpha, \quad -A^1_\alpha + A^1_\beta = da^0_{\alpha\beta}, \quad -a^0_{\alpha\beta} - a^0_{\beta\gamma} - a^0_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma}. \quad (2.1)$$

$^5$Good covers exist for compact and non-compact manifolds and are essential in Čech- de Rham theory.

$^6$We use the notation $U_{\alpha_0...\alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ for the k-fold intersections or overlaps of open sets.
and 4-fold intersections, one finds that
\[ M \text{ mod } 2\pi \mathbb{Z} = 0 \] (2.2)

which is consistent at triple overlaps \( U_{\alpha\beta\gamma} \) if and only if \( n_{\alpha\beta\gamma} \in \mathbb{Z} \) and so \( \frac{1}{2\pi}[\omega^2] \in H^2(M, \mathbb{Z}) \).

Taking the exterior derivative of patching condition, one finds that \( d\tau - A^1 = d\tau - A^1 \) and so \( d\tau - A^1 \) is globally defined on the total space \( C_M^3 \). Thus \( \pi^*\omega^2 = -d(d\tau - A^1) \) is an exact form on \( C_M^3 \), and so \( C_M^3 \) satisfies the topological geometrization condition. Of course \( C_M^3 \) is a circle bundle on \( M \) with first Chern class given by \( \frac{1}{2\pi}[\omega^2] \).

### 2.2 Patching C-spaces for closed 3-forms

To begin the construction of \( C_M^3 \) spaces, suppose \( M \) be a manifold and \( \omega^3 \) be a closed 3-form on \( M \). For the applications in DFT, \( M \) is the spacetime and \( \omega^3 \) is the NS-NS 3-form field strength. In addition let \( \{ U_\alpha \}_{\alpha \in I} \) be a good cover of \( M \) as for 2-forms in the previous section. Applying the Poincaré lemma on the open sets \( U_\alpha \) as well double, triple and 4-fold intersections, one finds that

\[
\omega^3 = dB^3_\alpha, \quad -B^2_\alpha + B^2_\beta = da^0_{\alpha\beta}, \quad -a^1_{\alpha\beta} - a^1_{\gamma\alpha} = da^0_{\alpha\beta\gamma}, \\
-a^0_{\beta\gamma\delta} + a^0_{\alpha\gamma\delta} - a^0_{\alpha\beta\delta} + a^0_{\alpha\beta\gamma} = 2\pi n_{\alpha\beta\gamma\delta}
\] (2.3)

respectively, where \( n_{\alpha\beta\gamma\delta} \) are constants and the combinatorics of the open set labels follow from the definition of the Čech differential, see [6.1]. \( B_\alpha \) are the 2-form gauge potentials of \( \omega^3 \) on each \( U_\alpha \), and \( \{ a^0_{\alpha\beta}, a^0_{\alpha\beta\gamma} \} \) are the patching or transition “functions” of \( \omega^3 \) at double and triple overlaps. Moreover if \( \frac{1}{2\pi}\omega^3 \) represents a class in \( H^3(M, \mathbb{Z}) \), then \( n_{\alpha\beta\gamma\delta} \in \mathbb{Z} \) on all 4-fold overlaps, \( U_{\alpha\beta\gamma\delta} \). All the patching data are skew-symmetric under the exchange of open set labels, i.e. \( a^1_{\alpha\beta} = -a^1_{\beta\alpha} \) and similarly for the rest.

The gauge potentials \( B_\alpha \) and the transition functions \( \{ a^0_{\alpha\beta}, a^0_{\alpha\beta\gamma} \} \) are not uniquely defined. In fact, the gauge potentials are defined up to the gauge transformations

\[ B'_\alpha = B_\alpha + d\alpha^1_\alpha \] (2.4)

and similarly the transition functions are defined up to gauge transformations as

\[
a^1_{\alpha\beta} = a^1_{\alpha\beta} - \xi^1_{\alpha\beta} + \zeta^1_{\alpha\beta} + d\xi^0_{\alpha\beta}, \\
a^0_{\alpha\beta\gamma} = a^0_{\alpha\beta\gamma} - \xi^0_{\alpha\beta\gamma} - \zeta^0_{\alpha\beta\gamma} - \zeta^0_{\gamma\alpha}.
\] (2.5)

These gauge transformations are the only ones compatible with the closure of \( \omega^3 \).

The construction of \( C_M^3 \) proceeds with the introduction of new coordinates \( y_\alpha \) and \( \theta_{\alpha\beta} \) associated with the open sets \( U_\alpha \) and the double overlaps \( U_{\alpha\beta} \), respectively. These are new coordinates in addition those of the spacetime. They should be thought in the same way as the Kaluza-Klein coordinate \( \tau \) we have introduced for the description of \( C_M^3 \) in the previous section. Though \( y_\alpha \) is assigned the degree of a 1-form. In addition, one imposes the gluing transformations

\[-y^1_\alpha + y^1_\beta + d\theta_{\alpha\beta} = a^1_{\alpha\beta}, \]

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\[
(\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} + a^0_{\alpha\beta\gamma}) = 0 \mod 2\pi \mathbb{Z},
\] (2.6)
on \ U_{\alpha\beta} \text{ and } U_{\alpha\beta\gamma}.

Using the second condition in (2.3), one finds that consistency of the first condition on triple overlaps yields
\[
d(\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} + a^0_{\alpha\beta\gamma}) = 0.
\] (2.7)
This is implied from the second condition in (2.6). Next investigating the consistency of the second condition of (2.6) are 4-fold overlaps and after using the last condition in (2.3), one finds that
\[
n_{\alpha\beta\gamma\delta} = 0 \mod \mathbb{Z}.
\] (2.8)
This is satisfied provided that \(\frac{1}{2}\omega\) represents a class in \(H^3(M, \mathbb{Z})\).

One of the questions that arises in imposing (2.6) is how one is supposed to think about these new coordinates and their gluing transformations. The coordinates should be thought in the same way as in the usual construction of a circle bundle over a manifold utilizing the patching conditions of a manifold together with those of a closed 2-form. For the gluing transformations, this particularly applies to the second patching condition which involves triple overlaps and three coordinates rather than double overlaps and two coordinates which is the usual patching conditions for manifolds. To give some insight into this question, one can view the usual patching of manifolds as follows. Given two charts, ie open sets and coordinates adapted to each one of the sets, the patching condition at the double intersection relates the coordinates of first chart to the coordinates of the second chart, and vice versa. Now if we introduce additional coordinates \(\theta_{\alpha\beta}\) at each double overlap, the second patching condition in (2.6) specifies how the three coordinates, each one associated with one of the three double overlaps that contribute to the triple overlap, are related.

The C-spaces \(C_{[\omega^3]}^M\) constructed with the above patching conditions are not manifolds. To see this, first observe that by construction there is a projection \(\pi : C_{[\omega^3]}^M \to M\). The inverse image \(\pi^{-1}(x)\) of \(x \in M\) has different dimension. If \(x \in U_{\alpha}\) and \(x \notin U_{\alpha_0...\alpha_k}\), \(\pi^{-1}(x) = \mathbb{R}^n\). While if \(x \in U_{\alpha\beta}\) and \(x \notin U_{\alpha\beta\gamma}\), then \(\pi^{-1}(x) = \mathbb{R}^n \times S^1\). Finally if \(x \in U_{\alpha\beta\gamma}\), then \(\pi^{-1}(x) = \mathbb{R}^n \times T^2\). As a consequence \(C_{[\omega^3]}^M\) does not have a well-defined dimension.

### 2.3 Dependence on \(\omega^3\)

Here we shall investigate whether or not \(C_{[\omega^3]}^M\) depends on the representative \(\omega^3\) of the class \(\frac{1}{2\pi}[\omega^3] \in H^3(M, \mathbb{Z})\). Suppose that \(\omega^3\) is another representative of \([\omega^3]\), ie \([\omega^3] = [\omega^3]\). Then there is a globally defined 2-form \(\omega^3\) such that \(\omega^3 = \omega^3 + du^2\). Thus \(b^2_\alpha = b^2_\alpha + u^2_\alpha\). Since \(u^2_\alpha = u^2_\beta\) at double overlaps the dependence on \(u\) drops out and so \(a^0_{\alpha\beta}\) does not dependent on the choice of representative of \([\omega^3]\). As a consequence the transition functions of \(C_{[\omega^3]}^M\) do not depend on the representative of \([\omega^3]\).

There is additional gauge redundancy in the definition of \(b_\alpha\) and that of the transition transition functions given in (2.4) and (2.5), respectively. This is eliminated we perform the compensating transformations
\[
y'^1_\alpha = y^1_\alpha + \zeta^1_\alpha, \quad \theta'^{\alpha\beta} = \theta^{\alpha\beta} + \zeta^0_{\alpha\beta},
\] (2.9)
on the new coordinates. As a result $C_{M}^{[\omega]}$ does not depend on the choices made including that of the representative of $[\omega]$.

In addition to these, one should also investigate a more subtle choice in the construction of $C_{M}^{[\omega]}$ that of the good cover $\{U_{a}\}_{a \in I}$. Although for every choice of a good cover on $M$, one can construct a $C_{M}^{[\omega]}$, it is not apparent that there is a large enough class of good covers which give the “same” $C_{M}^{[\omega]}$. This will require a better understanding of the class of objects that contain $C_{M}^{[\omega]}$ so their notion of equivalence can be established. We shall not duel on this question. The expectation is that for every choice of a good cover, or at least for a large class of good covers, all $C_{M}^{[\omega]}$ are at least homotopic.

### 2.4 Topological geometrization condition

It has been argued in [34] that any spaces which geometrizes a k-form flux has to be a C-space, i.e. there must be a projection from the C-space on the spacetime and that the pull back of the k-form flux must represent the trivial class in the C-space.

Here we shall demonstrate that $C_{M}^{[\omega]}$ is a C-space. As we have mentioned, there is a projection $\pi$ from $C_{M}^{[\omega]}$ onto $M$. Next taking the differential of the first patching condition in (2.6), one finds that

$$-dy_{\alpha}^{1} + dy_{\beta}^{1} = da_{\alpha\beta}^{1}.$$  \hspace{2cm} (2.10)

Using the second condition in (2.3), this can be rewritten as

$$dy_{\alpha}^{1} - B_{\alpha}^{2} = dy_{\beta}^{1} - B_{\beta}^{2}.$$  \hspace{2cm} (2.11)

Therefore $dy^{1} - B^{2}$ is globally defined on $C_{M}^{[\omega]}$. As $\pi^{*}\omega^{3} = -d(dy^{1} - B^{2})$, $\pi^{*}\omega^{3}$ is exact on $C_{M}^{[\omega]}$. Therefore $C_{M}^{[\omega]}$ satisfies the topological geometrization property.

### 3 Relation to gerbes and generalized geometry

#### 3.1 Gerbes

In the definition of [35], a gerbe is the object which represents a class in $H^{3}(M, \mathbb{Z})$ in the same way that a circle bundle represents a class in $H^{2}(M, \mathbb{Z})$. It is expected that given a manifold $M$ and a class in $H^{3}(M, \mathbb{Z})$, in a certain sense, the topology of the gerbe is specified uniquely. Then the gerbes are investigated via their transition functions. To relate the transition functions of a gerbe as defined in [35] to the transition functions we use here, we note that

$$g_{\alpha_\beta\gamma} = e^{u_{0}^{\alpha_\beta\gamma}}.$$  \hspace{2cm} (3.1)

Then the second condition in (2.3) reads as

$$g_{\beta_\gamma_\delta}^{-1}g_{\alpha_\gamma_\delta}g_{\alpha_\beta_\delta}^{-1}g_{\alpha_\beta_\gamma} = 1.$$  \hspace{2cm} (3.2)
which can be recognized as the patching condition for gerbes on 4-fold overlaps.

One of the aims of this article is the construction of C-spaces that can apply to double field theory. So the emphasis is not only on the transition functions but rather in the description of a particular object that represents a class in $H^3(M, \mathbb{Z})$. It is not expected that each class in $H^3(M, \mathbb{Z})$ is represented with a unique such object unless additional requirements are put in place. In fact, this is not the case even for the elements of $H^2(M, \mathbb{Z})$. In particular these can be represented with complex line bundles $L$ as well. Furthermore $L$ and the direct sum $L \oplus I$, where $I$ is the trivial $I$ line bundle, represent the same class in $H^2(M, \mathbb{Z})$. However they have different geometric properties which can be essential in certain applications. Furthermore, in the construction of $C^{[\omega^3]}_M$ the introduction of $y^1$ and $\theta$ coordinates at the open sets and double overlaps, and how they patch according to (2.3), have been essential for the applications considered here. Presumably there are other spaces with different geometric properties from $C^{[\omega^3]}_M$ that represent the same class in $H^3(M, \mathbb{Z})$.

3.2 Generalized geometry

The C-space $C^{[\omega^3]}_M$ induces naturally a generalized geometry structure on $M$. This can be seen from the transition functions of $dy^1$,

$$-dy^1_\alpha + dy^1_\beta = da^1_{\alpha\beta},$$

(3.3)

ie $dy^1_\alpha$ patches as a 1-form on $M$ accompanied with a shift with the transition function of $B_\alpha$. This assertion regarding the degree of $dy^1_\alpha$ requires some explanation. We have assigned the degree of a 1-form on $y^1$ from the start. So $dy^1$ has degree 2. However $y^1$ is an independent coordinate. So, from the perspective of $M$, $dy^1$ transforms as an 1-form. For the rest of the section, we shall neglect the grading of $y$. Therefore $dy$ span $T^*M$. In fact, the patching condition (3.3) defines an extension of the tangent bundle with respect to the cotangent bundle as

$$0 \rightarrow T^*M \rightarrow E \rightarrow TM \rightarrow 0.$$  

(3.4)

This is the first step required to define a generalized structure on $M$ as it is described in [37, 38].

For example, one can define a generalized metric $G$ from a metric $g$ on $M$ and the $B$ form as follows. We have seen that $dy - B$ is globally defined on $M$. Using this, we can write

$$G = g_{ij}dx^idx^j + g^{ij}(dy_i - B_{ik}dx^k)(dy_j - B_{jl}dx^l),$$

(3.5)

where $dy_i = \frac{\partial}{\partial y^i} dy$. This is the expected form of a generalized metric written in the basis $(dx^i, dy_i)$ of vectors and 1-forms.
4 Some topological aspects of $C^\omega_M$ and an example

4.1 Topological aspects

One way to get an insight into the topological structure of the C-space is to investigate $C^\omega_M$ in a chain complex approximation of the spacetime. Given a good cover $\{U_\alpha\}_{\alpha \in I}$ on $M$, one can associate a chain complex with $M$ the nerve $N$ of the cover, see eg \[39\]. $N$ is constructed as follows. One introduces a vertex for each open set $U_\alpha$ of the cover. Two vertices are joined by a side if and only if the corresponding open sets intersect $U_{\alpha\beta} \neq \emptyset$. The faces of three sides are filled if and only if the corresponding three open sets have a common intersection $U_{\alpha\beta\gamma} \neq \emptyset$, and so on. The cohomology of this chain complex is exactly the same as the de Rham cohomology or singular cohomology depending on the coefficients.

Let us now focus how the information from the additional coordinates of $C^\omega_M$ can be stored on the nerve $N$. This particularly applies to the angular coordinates $\theta_{\alpha\beta}$ as the $y^1_\alpha$ coordinates are contractible. It is apparent from the construction of $C^\omega_M$ that the vertices of $N$ do not alter as there are no angular coordinates associated to open sets. However a circle is associated to every point of every side of $N$ as these represent the intersection of two open sets. Furthermore at every point on a face of $N$ one should associate a 2-torus. This is because of the second patching condition in (2.6) as the three angular coordinates associated to each side are restricted to two.

Therefore one can describe this construction at a face of $N$ as follows. The 2-tori of the face degenerate to circles at each of the three sides, and in turn, the circles at the sides and the tori of the face degenerate to a point at they approach the vertices. Such a structure is reminiscent\footnote{This construction can be adapted to construct the universal bundle classifying spaces for any group, see eg \[40\].} to that of $\mathbb{C}P^2$. To see this consider the algebraic equation of $S^5$,

$$w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3 = 1.$$  \hspace{1cm} (4.1)

Setting $t_1 = w_1\bar{w}_1$, $t_2 = w_2\bar{w}_2$ and $t_3 = w_3\bar{w}_3$, this can be seen as the defining equation of a 2-simplex. The three phases of the complex numbers $w_1$, $w_2$ and $w_3$ associate a circle at every vertex, a 2-torus at every point of a side, and a 3-torus at every point of the face. As $\mathbb{C}P^2$ is the base space of the fibration, $S^1 \rightarrow S^5 \rightarrow \mathbb{C}P^2$, where $S^1$ acts from the right on the triplet $(w_1, w_2, w_3)$, a circle in removed from every point of the simplex leading to the picture describe above for $N$. As a result the topology of $C^\omega_M$ is different from that of spacetime $M$. As we shall see $C^\omega_M$ or rather $C^\omega_\infty$, appears also in the homotopy approach to C-spaces using Whitehead towers.

4.2 The C-space of 3-torus with a 3-form flux

The construction of $C^\omega_M$ described in section 2 is general and applies to every manifold with a good cover equipped with a closed 3-form which represents a class in $H^3(M, \mathbb{Z})$. 

As good covers exist on manifolds, one can construct $C_{M}^{[\nu]}$ for all smooth solutions of supergravity theories including the NS5-brane solution. Here we construct the C-space of a 3-torus with a 3-form flux. This example was initially investigated from the perspective of double spaces in [27]. Later it was explored from the patching point of view in [34] where it was found that the construction depends on the choice of the atlas on $T^3$. Another feature of the construction was that a quantization condition was imposed at the triple overlaps rather than the 4-fold overlaps which are involved in the Dirac quantization condition of 3-forms field strengths.

We shall follow the notation of [34] where all the data regarding the patching conditions of the 3-form flux can be found. The patching conditions of the C-space are

$$-y_{\alpha_1}^1 + y_{\alpha_2}^1 + d\theta_{\alpha_3} = a_{\alpha_1 \alpha_2}^1,$$

$$\left(\theta_{\alpha_1 \alpha_2} + \theta_{\alpha_2 \alpha_3} + \theta_{\alpha_3 \alpha_1} + a_{\alpha_1 \alpha_2 \alpha_3}^0\right) = 0 \mod 2\pi \mathbb{Z},$$

(4.2)

where we have set $\alpha_1 = i_1 j_1 k_1$ and so on. In the atlas we have chosen on $T^3$, the components of $a_{\alpha_1 \alpha_2}$ and $a_{\alpha_1 \alpha_2 \alpha_3}$ are linear in the coordinates of $T^3$. However the above patching conditions do not depend on this choice. This particularly applies to the second condition in (4.2) as the consistency required for it deals to $n_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \in \mathbb{Z}$ on 4-fold overlaps. Since $n_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}$ are constant for any choice of an atlas, the quantization condition is atlas independent. This should be contrasted with the double field theory computation which arises after taking all the angular coordinates to zero. As a result of (2.7), consistency in this case requires that the components of $da_{\alpha_1 \alpha_2 \alpha_3}$ are constant and should vanish up to some period. As $da_{\alpha_1 \alpha_2 \alpha_3}$ is a local 1-form, the constancy of its components is an atlas dependent statement [34].

5 DFT on double manifolds

5.1 Revisiting the patching of double manifolds

In the formulation of double field theory so far, one introduces a new set of coordinates $z$ in addition to those of the spacetime $x$ and imposes on all fields and their transformations the strong section condition which reads

$$\frac{\partial}{\partial x^i} A \frac{\partial}{\partial z_i} B + \frac{\partial}{\partial x^i} B \frac{\partial}{\partial z_i} A = 0 \ , \ \frac{\partial}{\partial x^i} \frac{\partial}{\partial z_i} A = 0.$$  

(5.1)

Setting for $A$ and $B$ the infinitesimal local transformations $\delta x^i$ and $\delta z_i$ of $x^i$ and $z_i$, respectively, and assuming that $\delta x^i$ must be arbitrary functions of $x$, which is required in

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5 The dilaton singularity does not affect the construction.

8 Strictly speaking one should introduce a third open set on $S^1$, $U_3 = (-\pi/4, \pi/4)$, so that the cover is a good cover. As the transition functions between $U_1$ and $U_3$, and $U_2$ and $U_3$ are the identity, there is no change in the computations on $S^1$ and the effects of $U_3$ have already been taken into account via the choice of $n_x$.

10 Usually the dual coordinates $z$ are denoted with $y$. Here we denote them with $z$ to distinguish them from those of the C-space as they have different transformation properties.
order to account for all reparameterizations the spacetime\(^{11}\), one concludes that the most general solutions to the above conditions are

\[
\delta x^i = \xi^i(x), \quad \delta y_i = \kappa_i(x).
\]

(5.2)

In particular, the second equation in (5.1) implies that \(\delta x^i\) can depend only on \(x\). Then the first equation for \(A = \delta x^i\) and \(B = \delta z_i\) implies that \(\delta z_i\) can dependent only on \(x\) as well. These infinitesimal transformations can be integrated to give

\[
x^\prime_i = x^i(x^j), \quad z_i' = z_i - \kappa_i(x).
\]

(5.3)

Moreover in \([28, 27]\), \(u_i\) is related linearly to the gauge transformations of the \(b\) field. To investigate the global properties of double field theories, these transformations are interpreted as patching conditions,

\[
x^\alpha_i = x^\alpha_i(x^\beta), \quad y_\alpha = y_\beta - \kappa_\alpha\beta,
\]

(5.4)

where we have introduced a good cover \(\{U_\alpha\}_{\alpha \in I}\) on the spacetime \(M\).

The strong section condition has another solution where \(z\) and \(x\) exchange places, this is the solution for the dual space. It also has many more solutions\(^{12}\) provided that one weakens the requirement that \(\delta x^i\) must be an arbitrary function of \(x\) and does not allow for general reparametrization of \(M\) but this breaks general covariance.

So in order to allow for reparametrization of spacetime, one is forced to patch the theory with transformations of the type (5.4). If this is the case, then

\[
\kappa_\alpha\beta + \kappa_\beta\gamma + \kappa_\gamma\alpha = 0.
\]

(5.5)

Using the results of \([34]\), one concludes that this is possible if and only if the double space is diffeomorphic to \(D_M = T^*M\).

This result is independent from the form of finite transformations on the fields and other geometric considerations. It is a consequence of the application of the strong section condition. \textit{Thus if one uses the strong section condition to describe the double theory and allows for general reparametrizations of the spacetime coordinates, then one is led to the conclusion that the double space is \(T^*M\).}

This has immediate consequences. \(T^*M\) is contractible to \(M\), so \(\pi^*\omega^3\) is not trivial in \(T^*M\). Thus this space does not satisfy the topological geometrization condition. Furthermore, if the transition of functions of \(\omega^3\) at double overlaps are related via a linear transformations to \(\kappa\), then \(\omega^3\) is exact [34].

### 5.2 Relation of double spaces to C-spaces

Now let us compare the results of the previous section with those we have obtained for the \(C_M^{(\omega^3)}\) spaces in section 2. In particular, let us compare the second patching condition

\(^{11}\)It is required for example for the construction of a maximal atlas on the spacetime.

\(^{12}\)One can easily construct many power series solutions.
of (5.4) with the first patching condition in (2.6). It is clear (2.6) reduces to (5.4) only
when the new coordinate \( \theta_{\alpha\beta} \) is chosen\(^1\) as
\[
\theta_{\alpha\beta} = 0 ,
\]
y\(^1\) = z and \( \kappa_{\alpha\beta} = a_{\alpha\beta}^1 \). This choice cannot be made everywhere on \( M \) consistent with the data. Thus the double spaces are local subspaces of \( C^3_M \).

Although the geometric aspects of DFT on \( C^3_M \) have not been developed, it is clear
from the topological considerations presented that for the global definition of DFT additional coordinates are required. The mere introduction of \( z \) coordinates in the context
of double spaces is not sufficient to geometrize the topological changes of \( \omega^3 \), and to give
a global definition of double spaces. The examination of the example of [27] from the
patching point of view in [34] and in section 4.2 supports this assertion. However, it is
not apparent how the additional coordinates \( \theta \) can be inserted in the description of DFTs.

6 C-spaces for closed k-forms

6.1 The construction of \( C^k_M \)

The construction of C-spaces for \( \omega^k \) closed forms, \( C^k_M \), can be done in a way similar to
that for \( C^3_M \). To simplify the discussion it is convenient to introduce the Čech differential \( \delta \). As before we choose a good cover \( \{ U_\alpha \}_{\alpha \in I} \) on \( M \) and define
\[
\delta \lambda^m_{\alpha_0...\alpha_p} = \sum_{i=0}^{p} (-1)^i \lambda^m_{\alpha_0...\alpha_i...\alpha_{i+1}...\alpha_p} ,
\]
where \( \lambda^m \) is a m-form defined at p-overlaps and restricted upon applying \( \delta \) to \( (p+1) \)-
overlaps, and \( \hat{\alpha}_i \) means that the label \( \alpha_i \) is omitted. As before all these forms defined
at the various overlaps are skew-symmetric under the exchange of the labels of the open
sets. For example,
\[
\delta \lambda^m_{\alpha_0...\alpha_1} = -\lambda^m_{\hat{\alpha}_0...\alpha_1} + \lambda^m_{\alpha_0...\hat{\alpha}_1} ,
\]
on \( U_{\alpha_0}\alpha_1 \). Observe that \( \delta^2 = 0 \) and \( d\delta = \delta d \).

Applying the Poincaré lemma, the Čech-de Rham expansion of a k-form at multiple
overlaps is
\[
\omega^k_\alpha = dA^{k-1}_\alpha , \quad \delta A^{k-1}_{\alpha_0...\alpha_1} = d\alpha^{k-2}_{\alpha_0...\alpha_1} , \ldots , \delta a^{k-\ell}_{\alpha_0...\alpha_\ell} = da^{k-\ell-1}_{\alpha_0...\alpha_\ell} , \ldots , \delta a^0_{\alpha_0...\alpha_k} = 2\pi n_{\alpha_0...\alpha_k} \quad (6.3)
\]
where \( n_{\alpha_0...\alpha_k} \) are constants. Again \( \frac{1}{2\pi} \omega^k \) represents a class in \( H^k(M, \mathbb{Z}) \), if \( n_{\alpha_0...\alpha_k} \in \mathbb{Z} \).
The transition functions of the \( \omega^k \) are not unique. Rather they are specified up to the
gauge transformations
\[
a^{k-\ell}_{\alpha_0...\alpha_{\ell+1}} = a^{k-\ell}_{\alpha_0...\alpha_{\ell}} + d\zeta^{k-\ell-1}_{\alpha_0...\alpha_{\ell+1}} + \delta \zeta^{k-\ell}_{\alpha_0...\alpha_{\ell+1}} .
\]
\(^1\)If \( \theta_{\alpha\beta} \) was not identified mod2\( \pi \mathbb{Z} \), it would have been sufficient to choose it as a function of \( U_{\alpha\beta} \).
To construct $C_M^{[\omega^k]}$, introduce coordinates $y_{a_0\ldots a_\ell}^{k-\ell}$ and impose the patching conditions
\begin{align}
\delta y_{a_0\ldots a_{\ell+1}}^{k-\ell} + dy_{a_0\ldots a_{\ell+1}}^{k-\ell-1} &= a_{a_0\ldots a_{\ell+1}}^{k-\ell}, \quad \ell = 2, \ldots, k-1, \\
(\delta y_{a_0\ldots a_k}^0 - a_{a_0\ldots a_k}^0) &= 0 \mod 2\pi ,
\end{align}
(6.5)
where now $y^0$ denote the new angular coordinates. After acting with $\delta$, it is clear from the last patching condition that consistency requires that $n_{a_0\ldots a_{k+1}} \in \mathbb{Z}$ and $\frac{1}{2\pi} \omega^k$ represents a class in $H^k(M, \mathbb{Z})$. This is the Dirac quantization condition. Note that the construction begins with the introduction of a new coordinate which locally is a $(k-2)$-form coordinates. Again this implies that the exceptional spaces are diffeomorphic to $\Lambda^{k-2} \mathbb{C}$ for the $(k-2)$-form coordinates. Consequently, one can write a generalized metric in a way similar to that of DFTs.

Furthermore, many more coordinate are needed in analogy with DFTs.

For the (k-2)-form coordinates, which extends the generalized geometry considerations beyond the co-tangent bundle and has applications in exceptional field theories. As $dy^k - A^k$ is globally defined on $C_M^{[\omega^k]}$, one can write a generalized metric in a way similar to that of $C_M^{[\omega^k]}$ presented in section 3.2. $C_M^{[\omega^k]}$ provides also a model for a k-gerbe.

In the context of exceptional field theories, the strong section condition, under similar assumptions to the DFT case, will lead to a patching condition
\begin{align}
- z_{a}^{k-2} + z_{\beta}^{k-2} &= \kappa_{a\beta}^{k-2}.
\end{align}
(6.8)
for the $(k-2)$-form coordinates. Again this implies that the exceptional spaces are diffeomorphic to $\Lambda^{k-2}(M)$. Such a space cannot satisfy the topological geometrization condition. Furthermore if $\frac{1}{2\pi} \omega^k$ is related to the transition functions of $\omega^k$ at double overlaps with a linear map, then $\omega^k$ represents the trivial class in cohomology. The exceptional spaces are local subspaces of $C_M^{[\omega^k]}$ where all coordinates of the latter apart from $x$ and $y^{k-2}$ are set to zero. These topological considerations lead to the conclusion that for the global definition of exceptional field theories many more coordinates are needed in analogy with DFTs.

6.2 Applications

Most of the properties and applications we have explored for $C_M^{[\omega^k]}$ can be extended to $C_M^{[\omega^k]}$. Selectively, $C_M^{[\omega^k]}$ induces an extension $E$ of $TM$ with respect to the bundle of $(k-2)$-forms $\Lambda^{k-2}(M)$ as
\begin{align}
0 \to \Lambda^{k-2}(M) \to E \to TM \to 0
\end{align}
(6.7)
which extends the generalized geometry considerations beyond the co-tangent bundle and has applications in exceptional field theories. As $dy^k - A^k$ is globally defined on $C_M^{[\omega^k]}$, one can write a generalized metric in a way similar to that of $C_M^{[\omega^k]}$ presented in section 3.2. $C_M^{[\omega^k]}$ provides also a model for a k-gerbe.

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7 Whitehead Towers and $C_M^{[\omega^k]}$

As it has been mentioned in [34], that there is a construction of C-spaces in homotopy theory realized by the Whitehead towers. These are sequences of fibrations such that

$$M \xleftarrow{p_1} X_1 \xrightarrow{p_2} X_2 \xleftarrow{p_3} X_3 \xrightarrow{p_4} \ldots \quad (7.1)$$

where the fibre associated with the $p_n$ projection is the Eilenberg-MacLane space $K(n, \pi_{n-1})$, $\pi_\ell = \pi_\ell(M)$ are the homotopy groups of $M$, and $X_n$ is $n$-connected, ie $\pi_\ell(X_n) = 0$ for $\ell \leq n$ and also $\pi_\ell(X_n) = \pi_\ell(M)$ for $\ell > n$. The Eilenberg-MacLane space $K(m, A)$ has the property that $\pi_\ell(K(m, A)) = 0$ unless $\ell = m$ in which case $\pi_m(K(m, A)) = A$ for any abelian group $A$.

Assuming that $M$ is connected, the description of $X_1$ begins with the construction of an auxiliary space $Y_1$ which is derived from $M$ after adding cells to kill all the higher homotopy groups than $\pi_1$. $M$ is included in $Y_1$. Then a point $z$ is chosen in $Y_1$, and $X_1$ is defined as all paths that begin at $z$ and end in $M$ as $M \subset Y_1$. Then $p_1$ is defined as the end point projection of the paths. It turns out that the fibre of this fibration is homotopic to the loop space $\Omega(Y_1)$ which is the fibre over $z$. As by construction $Y_1 = K(\pi_1, 1)$, one concludes from the homotopy exact sequence of path fibrations that $\Omega(Y_1) = K(\pi_1, 0)$. As the only non-vanishing homotopy group is $\pi_0(K(\pi_1, 0)) = \pi_1$, from the homotopy exact sequence of the fibration $M \xleftarrow{p_1} X_1$ one finds that $X_1$ homotopic to the universal cover of $M$, ie $X_1$ is simple connected, and $\pi_\ell(X_1) = \pi_\ell(M)$ for all $\ell > 1$. This construction can be repeated for $X_1$ to yield $X_2$ and so on.

Next assume that $M$ is simply connected so that we can go straight to the fibration $M \xrightarrow{p_2} X_2$. The fibre in this case is $K(\pi_2, 1)$ as $M$ is simply connected $\pi_2 = H_2(M, \mathbb{Z}) = H^2(M, \mathbb{Z})$. Since $\pi_1(X_2) = \pi_2(X_2) = 0$, $H^2(X_2, \mathbb{Z}) = 0$ and so $X_2$ realizes the topological geometrization property for $M$ and for all closed 2-forms on $M$. Furthermore, the construction is homotopic to the usual Kaluza-Klein reduction. This is because for $\pi_2 = \bigoplus_m \mathbb{Z}$, the fibre $K(\bigoplus_m \mathbb{Z}, 1)$ can be chosen up to a homotopy as $T^m$. Though there is a difference between $X_2$ and $C_M^{[\omega^2]}$ as the former by construction topologically geometrizes all closed 2-forms while the latter topologically geometrizes only $\omega^2$. Of course one can repeat the process to construct the C-spaces for all closed 2-forms in which case it will be homotopic to $X_2$.

Next let us go one step up the Whitehead tower. Assume that $M$ is 2-connected. In such case, $X_3$ is 3-connected and realizes the topological geometrization property for $M$ and for all closed 3-forms on $M$. Furthermore, for $\pi_3 = \mathbb{Z}$, $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. This can be easily seen form the homotopy sequence of the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ as $n \to \infty$. As $\mathbb{C}P^\infty$ is also identified as $BU(1)$, the universal classifying space of $S^1$ bundles, $X_3$ is a fibration over $M$ fibres the space of $S^1$ bundles reminiscent of the gerbes according to [36]. It is also reminiscent of the emergence of $\mathbb{C}P^2$ in the exploration of the topological structure of $C_M^{[\omega^3]}$. These raise the question how $C_M^{[\omega^3]}$ is related to $X_3$, and whether the former can become a model for the latter. Clearly, the same question can be raised for the rest of the cases.
8 Concluding remarks

We have proposed a C-space, $C_M^{[\omega^3]}$, for any closed 3-form $\omega^k$ on a manifold $M$ which represents a class $\frac{1}{3!} [\omega^3] \in H^3(M, \mathbb{Z})$. These have been constructed by introducing appropriate new coordinates and after imposing suitable transition functions which are related to the transition functions of $M$ and the patching data of $\omega^3$ as arise in the Čech-de Rham theory. $C_M^{[\omega^3]}$ are not manifolds. It is confirmed that $C_M^{[\omega^3]}$ satisfy the topological geometrization condition and induce a generalized geometry structure on the spacetime. The double spaces of DFTs are included as local subspaces in $C_M^{[\omega^3]}$. An interpretation of this is that for the global definition of DFTs additional coordinates are required. We argue that these new coordinates are necessary on topological grounds and this should not depend of the details of geometry. However how these can enter in the existing local description of DFTs remains an open problem.

We have also generalized the construction of C-spaces for any closed k-form on $M$, and we have established that $C_M^{[\omega^k]}$ have similar properties to those of $C_M^{[\omega^3]}$. It is expected that these spaces are required for the global definition of exceptional field theories.

The construction of $C_M^{[\omega^3]}$ can be done starting from any spacetime with a good cover and a closed 3-form. As a result such spaces can be found for all relevant supergravity backgrounds including those of the NS5-branes. Here we have explored in detail the 3-torus with a 3-form flux model of [27]. We demonstrate how several puzzles associated with the construction of double spaces for this model [34] are resolved via the use of C-spaces.

Another method to topologically geometrize $k$-forms in the context of homotopy theory is that of Whitehead towers. It was emphasized that for simply connected manifolds, the Whitehead construction coincides with the construction of $C_M^{[\omega^3]}$ which in turn is the usual Kaluza-Klein space of circle fibrations. This raises the question whether $C_M^{[\omega^3]}$ can be also related to the Whitehead construction and in particular whether the former provide a model for the latter. Such a relation will elucidate the topological structure of C-spaces.

Although C-spaces resolve the global patching problem of double spaces, the additional coordinates which are non-linear, are still too special to allow for a full covariance under all required symmetries, diffeomorphisms and dualities, without any further assumptions on the structure of spacetime. Nevertheless, they may prove to be useful way to proceed. In addition, the understanding how to incorporate the additional coordinates in DFT may lead to some new insights into the structure of these theories.

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