On the Padovan \( p \)-circulant numbers

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Abstract: In this paper, we define Padovan \( p \)-circulant numbers by using circulant matrices which are obtained from the characteristic polynomials of the Padovan \( p \)-numbers. Then, we derive the permanental and the determinantal representations of the Padovan \( p \)-circulant numbers by using certain matrices which are obtained from the generating matrix of Padovan \( p \)-circulant sequence. Also, we obtain the combinatorial representation, the exponential representation and the sums of the Padovan \( p \)-circulant numbers by the aid of the generating function and the generating matrix of the Padovan \( p \)-circulant sequence.

Keywords: Padovan \( p \)-circulant Sequence, Matrix, Representation.

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1 Introduction

It is well-known that Padovan sequence is defined by the following equation:

\[ P(n) = P(n - 2) + P(n - 3) \]

for \( n \geq 3 \), where \( P(0) = P(1) = P(2) = 1 \).

Deveci and Karaduman defined \([8]\) the Padovan \( p \)-numbers as shown:
\[ p_{n+p+2} = p_{n+p} + 2p_{n+p} \]

for any given \( p = 2, 3, 4, \ldots \) and \( n \geq 1 \) with initial conditions \( p_{n+p} = 2p_{n+p} = \cdots = p_{n+p} = 0, \ p_{n+p+1} = 1 \) and \( p_{n+p+2} = 0 \).

Suppose that the \((n+k)\)th term of a sequence is defined recursively by a linear combination of the preceding \( k \) terms:

\[ a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1} \]

where \( c_0, c_1, \ldots, c_{k-1} \) are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix \( A \) be defined by

\[ A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
c_0 & c_1 & c_2 & \cdots & c_{k-3} & c_{k-2} & c_{k-1}
\end{bmatrix}, \]

then

\[ A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix} \]

for \( n \geq 0 \).

Many of the numbers obtained by using homogeneous linear recurrence relations and their miscellaneous properties have been studied by some authors; see for example [1, 3, 5–7, 9–11, 13, 18–20]. In this paper, we define Padovan \( p \)-circulant numbers and then we obtain their some properties such as the generating matrix, the Binet formula, the combinatorial representation, the generating function, exponential representation.

## 2 The Padovan \( p \)-circulant numbers

We define the Padovan \( p \)-circulant numbers for \( n \geq 1 \) as follows:

\[ x_{n+p+3} = x_{n+p+2} - x_{n+p} - x_n \] (1)

with initial constants \( x_1 = \cdots = x_{p+2} = 0 \) and \( x_{p+3} = 1 \), where \( p \geq 2 \).

It is important to note that equation (1) is a \((p+3)\)-th order homogeneous linear recurrence relation example of the arbitrary-order equation (1.1) in [16].

By equation (1), we can write the following companion matrix:
Lemma 2.1. The characteristic equation of the Padovan $p$-circulant sequence $x^{p+3} - x^{p+2} + x^p + 1 = 0$ does not have multiple roots.

Proof. There is a similar proof in [8]. Let $f(x) = x^{p+3} - x^{p+2} + x^p + 1$ and suppose that $z$ is a multiple root of $f(x)$. Then, since $z$ is a multiple root, $z$ is a root of $f'(x)$, that is, $f'(z) = (p + 3) z^{p+2} - (p + 2) z^{p+1} + p z^{p-1} = z^{p-1} ((p + 3) z^3 - (p + 2) z^2 + p) = 0$. Since $f(0) \neq 0$, we consider the equations $(p + 3) z^3 - (p + 2) z^2 + p = 0$. Thus we obtain

$$z_1 = \frac{(\sqrt[3]{-p-2})}{3(p+3) \left( -25p^3 - 150p^2 + 3\sqrt{3} \left( 23p^6 + 276p^5 + 1230p^4 + 2380p^3 + 1563p^2 - 288p - 219p + 16 \right) \right)^{\frac{1}{3}}} \left( -25p^3 - 150p^2 + 3\sqrt{3} \left( 23p^6 + 276p^5 + 1230p^4 + 2380p^3 + 1563p^2 - 288p - 219p + 16 \right) \right)^{\frac{1}{3}} - \frac{p}{3(p+3)}$$

$$z_2 = \frac{3 \sqrt[3]{(p+3)} (1 + \sqrt{3})(-p-2)^2}{3 \sqrt[3]{(p+3)} (1 + \sqrt{3})(-p-2)^2}$$

and

$$C_p = [c_{ij}]_{(p+3) \times (p+3)} = \begin{bmatrix} 1 & 0 & -1 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ \end{bmatrix}$$

The matrix $C_p$ is called the Padovan $p$-circulant matrix. It is easy to see that

$$(C_p)^\alpha = \begin{bmatrix} x_{p+3} \\ x_{p+2} \\ \vdots \\ x_1 \end{bmatrix} = \begin{bmatrix} x_{\alpha+p+3} \\ x_{\alpha+p+2} \\ \vdots \\ x_{\alpha+1} \end{bmatrix}$$

for $\alpha \geq 0$. Also, by an inductive argument, we may write

$$(C_p)^\alpha = \begin{bmatrix} x_{\alpha+p+3} & x_{\alpha+p+4} - x_{\alpha+p+3} & x_{\alpha+p+5} - x_{\alpha+p+4} & -x_{\alpha+3} & -x_{\alpha+4} & \cdots & -x_{\alpha+p+2} \\ x_{\alpha+p+2} & x_{\alpha+p+3} - x_{\alpha+p+2} & x_{\alpha+p+4} - x_{\alpha+p+3} & -x_{\alpha+2} & -x_{\alpha+3} & \cdots & -x_{\alpha+p+1} \\ x_{\alpha+p+1} & x_{\alpha+p+2} - x_{\alpha+p+1} & x_{\alpha+p+3} - x_{\alpha+p+2} & -x_{\alpha+1} & -x_{\alpha+2} & \cdots & -x_{\alpha+p} \\ x_{\alpha+p} & x_{\alpha+p+1} - x_{\alpha+p} & x_{\alpha+p+2} - x_{\alpha+p+1} & -x_{\alpha} & -x_{\alpha+1} & \cdots & -x_{\alpha+p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{\alpha+2} & x_{\alpha+3} - x_{\alpha+2} & x_{\alpha+4} - x_{\alpha+3} & x_{\alpha-3} & -x_{\alpha-2} & \cdots & -x_{\alpha+1} \\ x_{\alpha+1} & x_{\alpha+2} - x_{\alpha+1} & x_{\alpha+3} - x_{\alpha+2} & x_{\alpha-2} & -x_{\alpha-1} & \cdots & -x_{\alpha} \end{bmatrix}$$

for $\alpha \geq p$. We easily derive that $\det (C_p)^\alpha = (-1)^{p+\alpha}$. Now we concentrate on the Binet formula for the Padovan $p$-circulant numbers by the aid of the determinantal representation.
Let \( G = \begin{pmatrix} 1 \end{pmatrix} \). Since the eigenvalues of the matrix \( G \) are \( p \)-th roots of unity, we get
\[
\det G = \det \begin{pmatrix} 1 \end{pmatrix} = 1.
\]

By Lemma 2.1, if \( \det V \neq 0 \), the matrix \( V \) is invertible. Then we obtain \( V^p = V^p G_p \). Since \( \det V^p \neq 0 \), the matrix \( V^p \) is invertible. Then we obtain \( (V^p)^{-1} C_p V^p = G_p \). Thus, the matrix \( C_p \) is similar to \( G_p \). So we get \( (C_p)^\alpha V^p = V^p (G_p)^\alpha \) for \( \alpha \geq p \) and \( \alpha \geq 2 \). Then we can write the following linear system of equations:
\[
\begin{cases}
\sum_{i=1}^{p} c_{i,1}^\alpha (x_1)^{p+2} + c_{i,2}^\alpha (x_1)^{p+1} + \cdots + c_{i,p+3}^\alpha = (x_1)^{\alpha+3} - i \\
\sum_{i=1}^{p} c_{i,1}^\alpha (x_2)^{p+2} + c_{i,2}^\alpha (x_2)^{p+1} + \cdots + c_{i,p+3}^\alpha = (x_2)^{\alpha+3} - i \\
\sum_{i=1}^{p} c_{i,1}^\alpha (x_{p+3})^{p+2} + c_{i,2}^\alpha (x_{p+3})^{p+1} + \cdots + c_{i,p+3}^\alpha = (x_{p+3})^{\alpha+3} - i
\end{cases}
\]
for \( \alpha \geq p \) and \( \alpha \geq 2 \). Therefore, for each \( i, j = 1, 2, \ldots, p + 3 \), we obtain \( c_{i,j}^\alpha \) as follows,
\[
c_{i,j}^\alpha = \frac{\det V_{i,j}^p}{\det V^p}.
\]

So we have the following useful results.

**Corollary 2.3.** Let \( x_n \) be the \( n \)-th the Padovan \( p \)-circulant number for \( p \geq 2 \). Then
\[
x_n = -\frac{\det V_{n,n}^p}{\det V^p}
\]
for \( 4 \leq n \leq p + 3 \).

Now we consider the permanental representations of the Padovan \( p \)-circulant numbers.
Definition 2.1. Let $M = [m_{i,j}]$ be a $u \times v$ real matrix and let $r^1, r^2, \ldots, r^u$ and $c^1, c^2, \ldots, c^v$ be respectively, the row and column vectors of $M$. If $r^\alpha$ contains exactly two non-zero entries, then $M$ is contractible on row $\alpha$. Similarly, $M$ is contractible on column $\beta$ provided $c^\beta$ contains exactly two non-zero entries.

Let $x_1, x_2, \ldots, x_u$ be row vectors of the matrix $M$ and let $M$ be contractible in the $\alpha$-th column with $m_{i,\alpha} \neq 0, m_{j,\alpha} \neq 0$ and $i \neq j$. Then the $(u-1) \times (v-1)$ matrix $M_{ij,\alpha}$ obtained from $M$ by replacing the $i$-th row with $m_{i,\alpha}x_j + m_{j,\alpha}x_i$ and deleting the $j$-th row and the $\alpha$-th column is called the contraction in the $\alpha$-th column relative to the $i$-th row and the $j$-th row.

In [2], Brualdi and Gibson obtained that $\text{per}(M) = \text{per}(N)$ if $M$ is a real matrix of order $u > 1$ and $N$ is a contraction of $M$.

Let $k \geq p + 3$ be a positive integer and suppose that $G(k, p) = [g^{k,p}_{i,j}]$ is the $k \times k$ super-diagonal matrix, defined by

$$g^{k,p}_{i,j} = \begin{cases} 1, & \text{if } i = t \text{ and } j = t \text{ for } 1 \leq t \leq k - 1, \\ -1, & \text{if } i = t \text{ and } j = t + 2 \text{ for } 1 \leq t \leq k - 2 \text{ and } i = t \text{ and } j = t + p + 2 \text{ for } 1 \leq t \leq k - p - 2, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$G(k, p) = \begin{bmatrix} 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Theorem 2.4. For $k \geq p + 3$ and $p \geq 2$,

$$\text{per}(G(k, p)) = x_{k+p+3}.$$  

Proof. We will use the induction method on $k$. Suppose that the equation holds $k \geq p + 3$, then we show that the equation holds for $k + 1$. If we expand the $\text{per}(G(k, p))$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\text{per}(G(k + 1, p)) = \text{per}(G(k, p)) - \text{per}(G(k - 2, p)) - \text{per}(G(k - p - 2, p)).$$
Since $\text{per}G(k+p) = x_{k+p+3}$, $\text{per}G(k-2,p) = x_{k+p+1}$ and $\text{per}G(k-p-2,p) = x_{k+1}$, we easily obtain that $\text{per}G(k+1,p) = x_{k+p+4}$.

Let $k \geq p + 3$ and let $Y(k,p) = \begin{bmatrix} y_{k,p} \\ i,j \end{bmatrix}$ be the $k \times k$ matrix, defined by

$$y_{k,p}^{i,j} = \begin{cases} 1 & \text{if } i = t \text{ and } j = t \text{ for } 1 \leq t \leq k - p - 2 \\
 & \text{and} \\
1 & \text{if } i = t + 1 \text{ and } j = t \text{ for } 1 \leq t \leq k - 1, \\
 & \text{and} \\
-1 & \text{if } i = t \text{ and } j = t + 2 \text{ for } 1 \leq t \leq k - p - 2 \\
 & \text{and} \\
0 & \text{otherwise.} \\
\end{cases}$$

Now we define the $k \times k$ matrix $L(k,p) = \begin{bmatrix} l_{k,p} \\ i,j \end{bmatrix}$ as follows:

$$(k - p - 3) \text{ th down}$$

$$L(k,p) = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & Y(k-1,p) \\ 0 \\ 0 \end{bmatrix}$$

where $k > p + 3$.

Then we can give other permanental representations than the above.

**Theorem 2.5.**

(i). For $k \geq p + 3$,

$$\text{per}Y(k,p) = -x_k.$$

(ii). For $k > p + 3$,

$$\text{per}L(k,p) = -\sum_{i=1}^{k-1} x_i.$$

**Proof.** (i). Suppose that the equation holds for $k \geq p + 3$, then we show that the equation holds for $k + 1$. If we expand the $\text{per}Y(k,p)$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\text{per}Y(k+1,p) = \text{per}Y(k,p) - \text{per}Y(k-2,p) - \text{per}Y(k-p-2,p) = -x_k + x_{k-2} + x_{k-p-2} = -x_{k+1}.$$

The conclusion is obtained.

(ii). If we extend the $\text{per}L(k,p)$ with respect to the first row, we write

$$\text{per}L(k,p) = \text{per}L(k-1,p) + \text{per}Y(k-1,p).$$

By the results of Theorem 2.4 and Theorem 2.5. (i) and the inductive argument, the proof is easily seen. 

\[\square\]
A matrix $M$ is called convertible if there is an $n \times n (1, -1)$-matrix $K$ such that $\text{per} M = \det (M \circ K)$, where $M \circ K$ denotes the Hadamard product of $M$ and $K$.

Let $k > p + 3$ and let $R$ be the $k \times k$ matrix, defined by

$$R = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & -1 & 1 & 1 \\
1 & \cdots & 1 & 1 & -1 & 1
\end{bmatrix}.$$  

It is easy to see that $\text{per} G (k, p) = \det (G (k, p) \circ R)$, $\text{per} Y (k, p) = \det (Y (k, p) \circ R)$ and

$$\text{per} L (k, p) = \det (L (k, p) \circ R).$$

Then we have the following useful results.

**Corollary 2.6.** For $k > p + 3$,

$$\det (G (k, p) \circ R) = x_{k+p+3},$$

$$\det (Y (k, p) \circ R) = -x_k$$

and

$$\det (L (k, p) \circ R) = -\sum_{i=1}^{k-1} x_i.$$  

Let $K (k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K (k_1, k_2, \ldots, k_v) = \begin{bmatrix}
k_1 & k_2 & k_3 & \cdots & k_v \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}.$$  

See [14, 15] for more information about the companion matrix.

**Theorem 2.7.** (Chen and Louck [4]). The $(i, j)$ entry $k_{i,j}^{(u)} (k_1, k_2, \ldots, k_v)$ in the matrix $K^u (k_1, k_2, \ldots, k_v)$ is given by the following formula:

$$k_{i,j}^{(u)} (k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \left( \frac{t_1 + \cdots + t_v}{t_1, \ldots, t_v} \right) k_1^{t_1} \cdots k_v^{t_v} \quad (3)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = u - i + j$,

$$\left( \frac{t_1 + \cdots + t_v}{t_1, \ldots, t_v} \right) = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$$

is a multinomial coefficient, and the coefficients in (3) are defined to be 1 if $u = i - j$.

Then we have the following Corollary for the Padovan $p$-circulant numbers.
Corollary 2.8. Let $x_\alpha$ be the $\alpha$th Padovan $p$-circulant number. Then

$$x_\alpha = - \sum_{(t_1,t_2,\ldots,t_p+3)} \frac{t_4 + t_5 + \cdots + t_{p+3}}{t_1 + t_2 + \cdots + t_{p+3}} \times \left( \frac{t_1 + \cdots + t_{p+3}}{t_1, \ldots, t_{p+3}} \right) (-1)^{t_3 + t_{p+3}}$$

$$= - \sum_{(t_1,t_2,\ldots,t_p+3)} \frac{t_5 + t_6 + \cdots + t_{p+3}}{t_1 + t_2 + \cdots + t_{p+3}} \times \left( \frac{t_1 + \cdots + t_{p+3}}{t_1, \ldots, t_{p+3}} \right) (-1)^{t_3 + t_{p+3}}$$

$$= \ldots$$

$$= - \sum_{(t_1,t_2,\ldots,t_p+3)} \frac{t_{p+3}}{t_1 + t_2 + \cdots + t_{p+3}} \times \left( \frac{t_1 + \cdots + t_{p+3}}{t_1, \ldots, t_{p+3}} \right) (-1)^{t_3 + t_{p+3}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+3)t_{p+3} = \alpha$.

Proof. In Theorem 2.7, if we choose $v = p + 3$ and $i = j$ such that $4 \leq i, j \leq p + 3$, then the proof is immediately seen from (2). \qed

The generating function of the Padovan $p$-circulant numbers is given by

$$g^p(y) = \frac{y^{p+2}}{1 - y + y^3 + y^{p+3}},$$

where $p \geq 2$.

Note that the generating function $g^p(y)$ is, in effect, a generalization of the main result in Section 2 of [17].

Now we give an exponential representation for the Padovan $p$-circulant numbers with the following Theorems.

Theorem 2.9. The Padovan $p$-circulant numbers have the following exponential representation:

$$g^p(y) = y^{p+2} \exp \left( \sum_{i=1}^{\infty} \frac{(y)^i}{i} (1 - y^2 + y^{p+2})^i \right).$$

Proof. Since

$$\ln g^p(y) = \ln \frac{y^{p+2}}{1 - y + y^3 + y^{p+3}}$$

$$= \ln y^{p+2} - \ln (1 - y + y^3 + y^{p+3})$$

and

$$- \ln (1 - y + y^3 + y^{p+3}) = - \left[ -y(1 - y^2 - y^{p+2}) - \frac{1}{2} y^2 (1 - y^2 - y^{p+2})^2 - \cdots \right.$$

$$\left. - \frac{1}{n} y^n (1 - y^2 - y^{p+2})^n - \cdots \right],$$

it is clear that

$$\ln \frac{g^p(y)}{y^{p+2}} = \sum_{i=1}^{\infty} \frac{(y)^i}{i} (1 - y^2 + y^{p+2})^i.$$

Thus we have the conclusion. \qed
Now we give the sums of the Padovan $p$-circulant numbers.
Let $S_\alpha = \sum_{n=1}^\alpha x_n$ and suppose that $M_p$ is the $(p + 4) \times (p + 4)$ matrix such that

$$M_p = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & C_p \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{bmatrix}.$$

Then it can be shown by induction that

$$(M_p)^\alpha = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
S_{\alpha+p+2} & S_{\alpha+p+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
S_\alpha & 0 & \cdots & 0
\end{bmatrix} (C_p)^\alpha.$$

3 Conclusion

We have given Padovan $p$-circulant numbers. These sequences had defined by using circulant matrices which had obtained from the characteristic polynomials of the Padovan $p$-numbers. Also, we have given relationships between Padovan $p$-circulant numbers and the generating matrices of these sequences. Then we have obtained some properties of the Padovan $p$-circulant numbers such as the Binet formula, permanental, determinantal, combinatorial, exponential representations and we have derived a formula for the sums of the Padovan $p$-circulant numbers.

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