Bouncing cosmology in modified Gauss-Bonnet gravity

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We explore bounce cosmology in $F(G)$ gravity with the Gauss-Bonnet invariant $G$. We reconstruct $F(G)$ gravity theory to realize the bouncing behavior in the early universe and examine the stability conditions for its cosmological solutions. It is demonstrated that the bouncing behavior with an exponential as well as a power-law scale factor naturally occurs in modified Gauss-Bonnet gravity. We also derive the $F(G)$ gravity model to produce the ekpyrotic scenario. Furthermore, we construct the bounce with the scale factor composed of a sum of two exponential functions and show that not only the early-time bounce but also the late-time cosmic acceleration can occur in the corresponding modified Gauss-Bonnet gravity. Also, the bounce and late-time solutions in this unified model is explicitly analyzed.

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\textbf{Introduction} – As a cosmological model to describe the early universe, the matter bounce scenario \cite{1} is known. In this scenario, in the contraction phase the universe is dominated by matter, and a non-singular bounce occurs. Also, the density perturbations whose spectrum is consistent with the observations can be produced (for a review, see \cite{2}). In addition, after the contracting phase, the so-called BKL instability \cite{3} happens, so that the universe will be anisotropic. The way of avoiding this instability \cite{4} and issues of the bounce \cite{5} in the Ekpyrotic scenario \cite{6} has been investigated \cite{7,8}. Moreover, the density perturbations in the matter bounce scenario with two scalar fields has recently been examined \cite{9}.

On the other hand, various cosmological observations support the current cosmic accelerated expansion. To explain this phenomenon in the homogeneous and isotropic universe, it is necessary to assume the existence of dark energy, which has negative pressure, or propose that gravity is modified on large scales (for recent reviews on issues of dark energy and modified gravity theories, see, e.g., \cite{10,11}). Regarding the latter approach, there have been proposed a number of modified gravity theories such as $F(R)$ gravity. The bouncing behavior has been investigated in $F(R)$ gravity \cite{12,13}, string-inspired gravitational theories \cite{14}, non-local gravity \cite{15}. A relation between the bouncing behavior and the anomalies on the cosmic microwave background (CMB) radiation has also been discussed \cite{16}.

In this Letter, we explore bounce cosmology in $F(G)$ gravity with $F(G)$ an arbitrary function of the Gauss-Bonnet invariant $G = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, where $R_{\mu\nu}$ is the Ricci tensor and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. Such $F(G)$ theory has been proposed as gravitational alternative for dark energy and inflation in Ref. \cite{19} and its application to the late-time cosmology \cite{20} has been studied. Moreover, cosmology in a theory with a dynamical dilaton coupling to the Gauss-Bonnet invariant has also been studied \cite{21}. We use units of $k_B = c = \hbar = 1$, where $c$ is the speed of light, and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi / M_{Pl}^2$ with the Planck mass of $M_{Pl} = G^{-1/2} = 1.2 \times 10^{19}$ GeV.

In the following, we first explain $F(G)$ gravity and its reconstruction method. We also investigate the stability of the solutions in the reconstructed $F(G)$ gravity model. As more concrete examples, we study an exponential model and a power-law model, in which the bouncing behavior happens. In addition, the $F(G)$ gravity model to make the ekpyrotic scenario \cite{22} is build. Next, we examine a sum of two exponential functions model of the scale factor. We explicitly show that in this model the bouncing behavior in the early universe and the late-time cosmic acceleration can be realized in a unified way. Furthermore, we make the stability analysis of the bounce and late-time solutions in the unified model. Finally, our results are summarized.

\textbf{$F(G)$ theory of gravity} – The action of $F(G)$ gravity model is described as \cite{19}

\begin{equation}
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + F(G) \right) + S_{\text{matter}},
\end{equation}

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $S_{\text{matter}}$ is the matter action. It follows from this action that
the gravitational field equation reads

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} g_{\mu\nu} F(\mathcal{G}) - (-2R_{\mu\nu} + 4R_{\mu}\rho_{\nu} - 2R_{\mu}^{\rho\sigma\tau} R_{\nu\rho\sigma\tau} + 4g^{\alpha\rho}g^{\beta\sigma} R_{\mu\alpha\nu\beta} R_{\rho\sigma}) F'(\mathcal{G}) \]

\[ - 2(\nabla_{\mu} \nabla_{\nu} F'(\mathcal{G})) R + 2g_{\mu\nu} (\nabla^2 F'(\mathcal{G})) - 4 (\nabla^2 F'(\mathcal{G})) R_{\mu\nu} + 4 (\nabla_{\rho} \nabla_{\mu} F'(\mathcal{G})) R_{\rho\nu} + 4 (\nabla_{\rho} \nabla_{\nu} F'(\mathcal{G})) R_{\rho\mu} \]

\[ - 4g_{\mu\nu} (\nabla_{\rho} \nabla_{\sigma} F'(\mathcal{G})) R_{\rho\sigma} + 4 (\nabla_{\rho} \nabla_{\sigma} F'(\mathcal{G})) g^{\alpha\rho}g^{\beta\sigma} R_{\mu\alpha\nu\beta} = \kappa^2 T_{\mu\nu}^{\text{matter}}. \]  

(2)

Here, the prime denotes the derivative with respect to \( \mathcal{G} \), \( \nabla_{\mu} \) is the covariant derivative, \( \Box \equiv g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \) is the covariant d'Alembertian, and \( T_{\mu\nu}^{\text{matter}} = \text{diag} (-\rho_{\text{matter}}, p_{\text{matter}}, p_{\text{matter}}, p_{\text{matter}}) \) is the energy-momentum tensor of matter, where \( \rho_{\text{matter}} \) and \( p_{\text{matter}} \) are the energy density and pressure of matter, respectively. We take the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, given by

\[ ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} (dx^i)^2, \]  

(3)

where \( a \) is the scale factor, \( H = \dot{a}/a \) is the Hubble parameter, and the dot shows the time derivative. In this background, we have \( \dot{R} = 6\dot{H} + 12H^2 \) and \( \mathcal{G} = 24H^2 (H + H') \). The gravitational field equations become

\[ 6H^2 + F(\mathcal{G}) - \mathcal{G} F'(\mathcal{G}) + 4H^3 \mathcal{G} F''(\mathcal{G}) = 2\kappa^2 \rho_{\text{matter}}, \]  

(4)

\[ 4\dot{H} + 6H^2 + F(\mathcal{G}) - \mathcal{G} F'(\mathcal{G}) + 16H \mathcal{G} \left( H + H' \right) F''(\mathcal{G}) + 8H^2 \mathcal{G} F''(\mathcal{G}) + 8H^2 \mathcal{G}^2 F'''(\mathcal{G}) = -2\kappa^2 p_{\text{matter}}. \]  

(5)

In what follows, we investigate only gravity part of the action in Eq. (1) without its matter part. For the case that the scale factor \( a(t) \) has the form of linear combination of two exponential terms as

\[ a(t) = \sigma \exp(\lambda t) + \tau \exp(-\lambda t), \]  

(6)

where \( \lambda \neq 0 \), \( \sigma \), and \( \tau \) are constants. In this case, we find

\[ H(t) = \frac{\dot{a}}{a} = \lambda \frac{\sigma \exp(\lambda t) - \tau \exp(-\lambda t)}{\sigma \exp(\lambda t) + \tau \exp(-\lambda t)}, \quad \mathcal{G}(t) = 24\lambda^4 \left( \frac{\exp(2\lambda t)\sigma - \tau}{\exp(2\lambda t)\sigma + \tau} \right)^2. \]  

(7)

Reconstruction method of \( F(\mathcal{G}) \) gravity—Next, we reconstruct \( F(\mathcal{G}) \) gravity models by using the method \( \mathcal{G} \). Introducing proper functions \( P(t) \) and \( Q(t) \) of a scalar field \( t \), which is interpreted as the cosmic time, the action in Eq. (1) without matter is described as

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + P(t) \mathcal{G} + Q(t) \right). \]  

(8)

By varying this action with respect to \( t \), we obtain \( (dP(t)/dt) \mathcal{G} + (dQ(t)/dt) = 0 \). Solving this equation in terms of \( t \), we get \( t = t(\mathcal{G}) \). The substitution of \( t = t(\mathcal{G}) \) into Eq. (8) yields \( F(\mathcal{G}) = P(t) \mathcal{G} + Q(t) \). Using this equation and Eq. (8), we find

\[ Q(t) = -6H^2(t) - 24H^3(t) \frac{dP(t)}{dt}. \]  

(9)

With this equation and the relation \( F(\mathcal{G}) = P(t) \mathcal{G} + Q(t) \), we acquire

\[ 2H^2(t) \frac{d^2P(t)}{dt^2} + 2H(t) \left( 2\dot{H}(t) - H^2(t) \right) \frac{dP(t)}{dt} + \dot{H}(t) = 0. \]  

(10)

Suppose the scale factor is given by Eq. (10), we present the general solution of Eq. (10) for the following two cases.

In Case 1: \( \lambda > 0 \), \( \sigma > 0 \), \( \tau > 0 \).

The general solution of Eq. (10) becomes

\[ P(t) = c_1 + \left( c_2 - \frac{1}{4\lambda^2 \sqrt{\sigma \tau}} \arctan \left( e^{\lambda t} \sqrt{\sigma / \tau} \right) \right) T - \frac{1}{2\lambda^2} \ln \left( 1 - \frac{2\tau}{e^{2\lambda t} \sigma + \tau} \right), \]  

(11)

\[ T \equiv \frac{e^{3\lambda t} \sigma^2 - 6e^{2\lambda t} \sigma \tau + e^{-\lambda t} \tau^2}{e^{2\lambda t} \sigma - \tau}, \]  

(12)
where $c_1$ and $c_2$ are constants. From Eq. (11), we obtain

$$Q(t) = -6\lambda^2 e^{-\lambda t} \left( e^{2\lambda} \sigma - \tau \right) \left( 4c_2\lambda^2 - \frac{1}{\sqrt{-\sigma \tau}} \right) \arctan \left( e^{\lambda t} \sqrt{-\frac{\sigma}{\tau}} \right).$$

(13)

Plugging this expression with Eq. (9), we have

$$t_{\pm} = \frac{1}{2\lambda} \ln \left( \frac{\left( \mathcal{G} \pm 4\sqrt{6}\sqrt{\mathcal{G}\lambda^4 + 24\lambda^4} \right) \tau}{\left( \mathcal{G} - 24\lambda^4 \right) \sigma} \right), \quad 0 \leq \mathcal{G} < 24\lambda^4. \quad (14)$$

Accordingly, by solving $F(\mathcal{G}) = P(t)\mathcal{G} + Q(t)$, we find the most general form of $F(\mathcal{G})$ as

$$F^{(1)}_{\pm}(\mathcal{G}) = c_1\mathcal{G} + c_2\sqrt{\mathcal{G}}(24\lambda^4 - \mathcal{G}) - \frac{1}{2\lambda^2} \ln \left( \frac{\mathcal{G} + \sqrt{\mathcal{G}} + 24\lambda^4}{\mathcal{G} - 24\lambda^4} \right) \arctan \left( \frac{\sqrt{\mathcal{G}} + 2\sqrt{6}\lambda^2}{\sqrt{\mathcal{G}} - 24\lambda^4} \right).$$

(15)

where the superscription $(1)$ of $F^{(1)}_{\pm}(\mathcal{G})$ means “Case 1”. Note that the functions $F^{(1)}_{+}(\mathcal{G})$ and $F^{(1)}_{-}(\mathcal{G})$ are defined for $0 \leq \mathcal{G} < 24\lambda^4$ and the function $F^{(1)}_{-}(\mathcal{G})$ takes only complex values. Furthermore, we see that

$$\lim_{\mathcal{G} \to 0^+} F^{(1)}_{+}(\mathcal{G}) = 0, \quad \lim_{\mathcal{G} \to 24\lambda^4^-} F^{(1)}_{+}(\mathcal{G}) = 24c_1\lambda^4. \quad (16)$$

In Case 2: $\lambda > 0$, $\sigma > 0$, $\tau < 0$.

The general solution of Eq. (10) is derived as

$$P(t) = c_1 + c_2\lambda \left( e^{\lambda^2} - e^{2\lambda^2} \sigma - \tau \right) \sqrt{-\sigma \tau} \left( e^{\lambda t} \sqrt{-\frac{\sigma}{\tau}} \right) + \frac{1}{\sqrt{-\sigma \tau}} \arctan \left( e^{\lambda t} \sqrt{-\frac{\sigma}{\tau}} \right).$$

(17)

Using Eq. (9), we get

$$Q(t) = -6\lambda^2 e^{-\lambda t} \left( e^{2\lambda} \sigma - \tau \right) \left( 4c_2\lambda^2 + \frac{1}{\sqrt{-\sigma \tau}} \right) \arctan \left( e^{\lambda t} \sqrt{-\frac{\sigma}{\tau}} \right).$$

(18)

With the equation $(dP(t)/dt) \mathcal{G} + (dQ(t)/dt) = 0$, we observe that for $\mathcal{G} > 24\lambda^4$, the expression of $t_{\pm}$ is given by Eq. (14). It follows from solving the equation $F(\mathcal{G}) = P(t)\mathcal{G} + Q(t)$ that the most general form of $F(\mathcal{G})$ reads

$$F^{(2)}_{\pm}(\mathcal{G}) = c_1\mathcal{G} + c_2\sqrt{\mathcal{G}}(24\lambda^4 - \mathcal{G}) - \frac{1}{2\lambda^2} \sqrt{\mathcal{G}} \arctan \left( \frac{\sqrt{\mathcal{G}} + 2\sqrt{6}\lambda^2}{\sqrt{\mathcal{G} - 24\lambda^4}} \right).$$

(19)

However, it is clear that the function $F^{(2)}_{\pm}(\mathcal{G})$ has no real values for $\mathcal{G} > 24\lambda^4$.

**Stability of the solutions**— We define $H^2(t) \equiv \ddot{g}(N)$, where the number of e-fold is defined by $N \equiv \ln(a/a_*) \geq 0$ with $a_* \equiv a(t_*)$ the scale factor at a fiducial time $t_*$. Using $\dot{g}(N)$, the Friedmann equation (1) is rewritten to

$$6\ddot{g}(N) + F(\mathcal{G}) - 12\dot{g}(N)(\ddot{g}'(N) + 2\ddot{g}(N)) F'(\mathcal{G}) + 288\ddot{g}^2(N) \left( \ddot{g}'(N) \right)^2 + \dot{g}(N)\dddot{g}'(N) + 4\dot{g}(N)\ddot{g}(N) \right) F''(\mathcal{G}) = 0. \quad (20)$$

Here, the prime stands the derivative with respect to $N$, we have used $\mathcal{G} = 12\ddot{g}(N)(\ddot{g}'(N) + 2\ddot{g}(N))$, and only the gravity part has been considered (i.e., $\rho_{\text{matter}} = 0$). We express the solution of Eq. (14) as $\ddot{g} = \ddot{g}(N)$. To examine the stability of this background solution, we describe $\ddot{g}(N) = \ddot{g}_0(N) + \ddot{g}(N)$, where $\ddot{g}(N)$ is the perturbation around the background solution. By substituting the above expression of $\ddot{g}(N)$ into Eq. (20), we obtain

$$\mathcal{J}_1 \ddot{g}_0''(N) + \mathcal{J}_2 \dddot{g}_0'(N) + \mathcal{J}_3 \ddot{g}_0(N) = 0,$$  

(21)

with

$$\mathcal{J}_1 = 288\ddot{g}_0^3(N) F''(\mathcal{G}_0), \quad (22)$$

$$\mathcal{J}_2 = 432\ddot{g}_0^2(N) \left\{ (2\ddot{g}_0(N) + \dddot{g}_0(N)) F''(\mathcal{G}_0) + 8\ddot{g}_0(N) \left[ (\ddot{g}_0'(N))^2 + \ddot{g}_0(N)(4\dddot{g}_0(N) + \dddot{g}_0'(N)) \right] F''''(\mathcal{G}_0) \right\}, \quad (23)$$

$$\mathcal{J}_3 = 6 \left[ 1 + 24\ddot{g}_0(N) \left\{ -8\ddot{g}_0(N) + 3(\ddot{g}_0(N))^2 + 6\ddot{g}_0(N)(3\dddot{g}_0(N) + \dddot{g}_0'(N)) \right\} F''(\mathcal{G}_0) \right].$$

(24)
\[
\frac{J_2}{J_1} = \frac{3}{2} (1 + \coth(N)) + 48\lambda^4 \exp(-2N) \frac{F'''(G_0)}{F''(G_0)} > 0.
\]
\[
\frac{J_3}{J_1} = \frac{4 - 18e^{2N} + 18e^{4N} - 4e^{6N}}{(-1 + e^{2N})^3} + \frac{e^{6N}}{48\lambda^6 (-1 + e^{2N})^3} \frac{1}{F'''(G_0)} + \frac{96\lambda^2 e^{-2N} (-1 + e^{2N})}{-1 + e^{2N}} \frac{F'''(G_0)}{F''(G_0)} > 0.
\]

Consequently, we observe that
\[
\lim_{N \to +\infty} \frac{J_2}{J_1} = 6, \quad \lim_{N \to +\infty} \frac{J_3}{J_1} = 8.
\]

In most cases, it is easy to find \(N_0 = N_0(c_2, \lambda)\) and for all \(N > N_0\) both stability conditions will be executed. Thus, for the model of \(F(G) = F^{(1)}_+(G)\) in Eq. (26) with the scale factor in Eq. (25), the background solution is stable.

Here, we mention finite-time future singularities \(\frac{22}{26}\) in \(F(G)\) gravity by following the observations in Ref. 27.

In the limit of a finite time \(t_\ast\) (constant \(> t\)) in the future, cosmological quantities of the scale factor \(a(t)\), the effective (namely, total) energy density \(\rho_{\text{eff}}\), and pressure \(P_{\text{eff}}\) of the universe, and the higher derivative of the Hubble parameter would diverge. The finite-time future singularities can be classified into four types \(\frac{26}{26}\). In the limit of \(t \to t_\ast\), (a) Type I (“Big Rip”): \(a \to \infty, \rho_{\text{eff}} \to \infty,\) and \(|P_{\text{eff}}| \to \infty\). This type enforces the case that \(\rho_{\text{eff}}\) and \(P_{\text{eff}}\) at \(t_\ast\) are finite. (b) Type II (“sudden”): \(a \to a_\ast, \rho_{\text{eff}} \to \rho_\ast,\) and \(|P_{\text{eff}}| \to \infty,\) where \(a_\ast\) and \(\rho_\ast\) are constants. (c) Type III: \(a \to a_\ast, \rho_{\text{eff}} \to \infty,\) and \(|P_{\text{eff}}| \to \infty\). (d) Type IV: \(a \to a_\ast, \rho_{\text{eff}} \to 0, |P_{\text{eff}}| \to 0,\) but higher derivatives of \(H\) diverge. This type also includes the case that \(\rho_{\text{eff}}\) and/or \(|P_{\text{eff}}|\) are finite at \(t = t_\ast\).

If \(H = \frac{\dot{a}}{a} (t_\ast - t)\) with \(\dot{a}(\tau) > 0\) a constant, the reconstructed form of \(F(G)\) to produce the Big Rip singularity is given by \(F(G) = \frac{1}{\sqrt{6\dot{a}(1 + \frac{\dot{a}}{\ddot{a}}) (1 - \frac{\dot{a}}{\ddot{a}})}} \sqrt{G} + d_1G^{(0+1)/4} + d_2G,\) where \(d_1\) and \(d_2\) are constants. When \(\ddot{a} = 1,\) we find \(F(G) = (\sqrt{3}/2) \sqrt{G} \ln (\zeta G),\) where \(\zeta(> 0)\) is a positive constant. In the case of large values of \(G, F(G) \sim \xi \sqrt{G} \ln (\zeta G)\) with \(\xi(> 0)\) a positive constant, and eventually the Big Rip singularity happens. The same consequence is obtained for \(F(G) \sim \xi \sqrt{G} \ln (\zeta G^w + c),\) where \(u(> 0)\) and \(c\) are constants.

Moreover, when \(H = \frac{\dot{a}}{a} (t_\ast - t)\) with \(\dot{a}(\tau) > 0\) a constant, for \(v > 1,\) if the value of \(G\) is large, \(F(G) \sim -l_2 \sqrt{G}\) with \(l_2 > 0,\) there occurs the Type I singularity. For \(0 < v < 1,\) when \(G\) is large, we acquire \(F(G) \sim l_3G^w\) with \(l_3(> 0)\) a positive constant and \(v = 2v/(3v - u),\) where \(0 < v < 1/2,\) \(v = 2v/(3v - u),\) and therefore \(-1/3 < v < 0.\) Thus, there emerges the Type II (i.e., sudden) singularity. Additionally, when \(\zeta \to 0, F(G) \sim l_4G^w,\) where \(l_4(> 0)\) a negative constant and \(1 < v < \infty,\) we get \(-1 < v < -1/3.\) As a result, the Type II singularity appears. If for \(G \to 0, F(G) \sim l_5G^w,\) where \(2/3 < v < 1\) and \(v \neq 2n/(3m - 1)\) with \(m = 1, 2, 3, \ldots\) a natural number, we see that \(-\infty < v < -1.\) Consequently, there occurs the Type IV singularity. We also remark that for \(H = \frac{\dot{a}}{a} (t_\ast - t)^{-1/3},\) there can appear any kind of the Type II singularity. For it, we have \(G = 24\dot{h}^3 + 24\dot{h}^4 (t_\ast - t)^{1/3} < 0.\) Eventually, for \(F(G) = j_1\sqrt{G} + r_1 + j_2\sqrt{G} + r_2\) with where \(j_1(> 0), j_2(> 0)\) and \(r_1(> 0)\) are positive constants, the Type II singularity happens.

**Examples of \(F(G)\) gravity realizing bounce cosmology**—We present several simple examples of bounce cosmology.

**Case (i): Exponential model**

We examine the following form of the scale factor
\[
a(t) = \exp (\tilde{\alpha} t^2).
\]
In this case, the functions \( P(t) \) and \( Q(t) \) are represented as

\[
P(t) = s_1 + s_2 \left( -\frac{e^{2\hat{\alpha}}}{t} + \sqrt{\pi} \sqrt{-\hat{\alpha}} \text{Erf} \left( t \sqrt{-\hat{\alpha}} \right) \right) + \frac{\sqrt{\pi} \sqrt{-\hat{\alpha}} \text{MeijerG} \left[ \{0\}, \{1\}, \{0,0\}, \{-\frac{1}{2}\}, -t^2 \hat{\alpha} \right]}{16(-\hat{\alpha})^{3/2}},
\]

\[
Q(t) = -24t\hat{\alpha} \left[ \hat{\alpha} \left( t + 8s_2 e^{2\hat{\alpha}} \hat{\alpha} \right) + e^{2\hat{\alpha}} \sqrt{\pi} \sqrt{-\hat{\alpha}} \text{Erf} \left( t \sqrt{-\hat{\alpha}} \right) \right],
\]

with \( s_1 \) and \( s_2 \) constants. Here, \( \hat{\alpha} < 0 \) is a negative constant, \( \text{Erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt \), and \( \text{MeijerG}[\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_q\}, z] \) is the Meijer G function \( G_{p,q}^{m,n} \left( z \bigg| a_1, \ldots, a_p \bigg| b_1, \ldots, b_q \right) \). Thus, we find

\[
F(\mathcal{G}) = s_1 \mathcal{G} - s_2 \left( \frac{48}{\sqrt{3}} \hat{\alpha}^2 \mathcal{X}_1 \mathcal{X}_2 + 4\sqrt{3} \hat{\alpha} \mathcal{X}_1 \mathcal{X}_2 + \sqrt{\pi} \mathcal{G} \sqrt{-\hat{\alpha}} \mathcal{X}_3 \right) + \frac{1}{2} (-\mathcal{X}_2^2) - \sqrt{\pi} \sqrt{-\hat{\alpha}} \mathcal{X}_2 \mathcal{X}_1 \sqrt{\pi} \mathcal{X}_3
\]

\[
+ \frac{\sqrt{\pi} \mathcal{G}}{16\hat{\alpha}} \text{MeijerG} \left[ \{0\}, \{1\}, \{0,0\}, \{-\frac{1}{2}\}, \frac{1}{4}, \frac{\sqrt{\mathcal{G} + 24\hat{\alpha}^2}}{8\sqrt{6}\hat{\alpha}} \right].
\]

with

\[
\mathcal{X}_1 = \exp \left( -\frac{1}{4} \frac{\sqrt{\mathcal{G} + 24\hat{\alpha}^2}}{8\sqrt{6}\hat{\alpha}} \right),
\]

\[
\mathcal{X}_2 = \sqrt{-12\hat{\alpha} + \sqrt{6} \sqrt{\mathcal{G} + 24\hat{\alpha}^2}},
\]

\[
\mathcal{X}_3 = \text{Erf} \left( \frac{-\sqrt{-12\hat{\alpha} + \sqrt{6} \sqrt{\mathcal{G} + 24\hat{\alpha}^2}}}{4\sqrt{3} \sqrt{-\hat{\alpha}}} \right).
\]

The form in Eq. \( (34) \) is defined for \( \hat{\alpha} < 0 \) and \( \mathcal{G} \geq -24\hat{\alpha}^2 \). It is evident that with this choice of the scale factor, there does not arise any kind of singularities, that is, the scale factor, the Hubble parameter, and its derivative are finite. A similar situation takes place for the metric in Eq. \( (6) \), because the parameters \( \tau \) and \( \sigma \) have the same sign with each other. For the reconstruction of \( F(R) \) gravity, there exists a solution with different signs of \( \tau \) and \( \sigma \). In this case, there exists a situation in which the scale factor vanishes, and the Hubble parameter and its derivative tends to infinity (the type III singularities). However, we can construct an example of cosmology with singularities in \( F(\mathcal{G}) \) gravity.

**Case (ii): Power-law model**

We investigate the scale factor with its form

\[
a(t) = \beta t^{2n},
\]

where \( \beta \) is a constant and \( n \) is an integer. In this case, the functions \( P(t) \) and \( Q(t) \) become

\[
P(t) = y_1 + y_2 \frac{t^{3+2n}}{3 + 2n} - \frac{t^2}{8n(1 + 2n)},
\]

\[
Q(t) = \frac{24n^2 \left[ 1 - 2n - 8y_2 n (1 + 2n) t^{1+2n} \right]}{(1 + 2n)^2},
\]

where \( y_1 \) and \( y_2 \) are integration constants.

The cosmic time is expressed in terms of the Gauss-Bonnet invariant as

\[
t = \pm 2\sqrt{2} 3^{1/4} \left[ \frac{n^3(-1 + 2n)}{\mathcal{G}} \right]^{1/4}.
\]

Thus, the final form of \( F(\mathcal{G}) \) can be written as

\[
F(\mathcal{G}) = y_1 \mathcal{G} + y_2 \mathcal{G}^{(1-2n)/4} - \frac{2\sqrt{3} \sqrt{n^3(-1 + 2n)} \mathcal{G}}{n (1 + 2n)}.
\]

It is clearly seen that the parameter \( n \) can be both positive and negative values. For example, if \( n = 1 \), we have

\[
F_1(\mathcal{G}) = -\frac{y_2}{G^{1/4}} - \frac{2\sqrt{\mathcal{G}}}{\sqrt{3}} + y_1 \mathcal{G},
\]
whereas for \( n = -1 \), we find

\[
F_2(\mathcal{G}) = -6\sqrt{\mathcal{G}} - y_2\mathcal{G}^{3/4} + y_1\mathcal{G}.
\]  

(44)

In this model, we acquire the type III singularity. We mention that it is possible to examine more general case that the scale factor is described as \( a(t) = z_1 t^{2n} + z_2 \) with \( z_1 \) and \( z_2 \) constants, but that for this model, \( F(\mathcal{G}) \) would become quite complicated.

**Case (iii): Ekpyrotic scenario**

The so-called ekpyrotic universe is an alternative explanation to the inflationary paradigm proposed one decade ago in Ref. [22]. It can provide a realistic picture of the universe evolution (for a confrontation between both models, see [28]). In the same way as the inflationary scenario, the ekpyrotic cosmological models can also predict the origin of primordial inhomogeneities that leads to the formation of large structures and the anisotropies observed in the CMB radiation. In addition, this model does not require initial conditions in comparison with the standard inflationary scenario due to its cyclic nature (see, for example, [29]).

Let us now consider a model that may reproduce a entire cycle of an ekpyrotic universe

\[
H(t) = H_0 - H_1 e^{-\beta t}, \quad H_0 > 0, \quad \beta > 0,
\]

(45)

where \( H_0, H_1, \) and \( \beta \) are constants. In this case, the scale factor takes the form

\[
a(t) = a_0 e^{-\frac{H_1}{H_0} t + H_0 t},
\]

(46)

where \( a_0 \) is a constant of integration. Moreover, the functions \( P(t) \) and \( Q(T) \) can only be found in the form of integrals

\[
P(t) = c_1 + \int_1^t dz_2 \left( \frac{e^{H_1 t} - e^{-\beta z_2 + H_0 z_2} c_2}{(H_0 - e^{-\beta z_2 H_1})^2} \left( 1 - \int_1^{z_2} dz_1 e^{-\frac{H_1}{H_0} z_1 - H_0 z_1 + \beta z_1} (H_0 - e^{-\beta z_1 H_1})^2 \frac{\beta}{2 (e^{\beta z_1 H_0} - H_1)^2} \right) \right),
\]

(47)

\[
Q(t) = -6e^{-2\beta} (e^{H_0 t} H_0 - H_1) \left( 2e^{H_0 t + \frac{H_0}{H_1} t} e^{-\beta H_0 t + \beta t} \left( 2c_2 - \int_1^t d\tau e^{-\frac{H_1}{2H_0} H_0 - z_1 (H_0 + \beta) H_1 \beta} + e^{\beta H_0 - H_1} \right) \right).
\]

(48)

These expressions have been obtained under conditions \( H_0 > 0 \) and \( \beta > 0 \), and \( c_1 \) and \( c_2 \) are constants of integration. We now find the time dependence of \( G \) as

\[
t = \ln \left[ \frac{12H_0^2}{\sqrt{9G + 12H_0^2}} \right].
\]

(49)

This expression is defined for \( G > 24H_0^4 \).

\[
F(\mathcal{G}) = c_1 \mathcal{G} + c_2 \left( 24e^{\frac{e^{-4H_0 H_1}}{4H_0} t + H_0 t} (e^{-4H_0 H_1} H_1 - H_0) + \mathcal{G} \int_1^t \frac{e^{\frac{e^{-4H_0 H_1} H_1}{4H_0} t + 9H_0 z_2}}{(-e^{4H_0 z_2 H_1} H_0 + H_1)^2} d\tau \right)
\]

- \( 6(e^{-4H_0 H_1} H_1 - H_0)^2 - 48H_0 H_1 e^{\frac{e^{-4H_0 H_1} H_1}{4H_0} + H_0 t} (e^{-4H_0 H_1} H_1 - H_0) \int_1^t e^{\frac{-e^{-4H_0 z_1 H_1}}{4H_0} t - 5H_0 z_1 d\tau \right) \]

- \( \mathcal{G} \int_1^t \left( \int_1^{z_2} \frac{2H_0 H_1}{(e^{4H_0 z_2 H_1} H_0 - H_1)} e^{\frac{-e^{-4H_0 z_2 H_1} + 9H_0 z_2}{4H_0} t - \frac{e^{-4H_0 z_1 H_1}}{4H_0} t - 5H_0 z_1} d\tau \right) d\tau \).

(50)

Thus, the ekpyrotic cosmology may be realized naturally in \( F(\mathcal{G}) \) gravity.

**Sum of multiple exponential functions model**– We explore a sum of multiple exponential functions model of the scale factor

\[
a(t) = \exp(\alpha t^2) + \exp(\alpha^2 t^4), \quad \alpha > 0,
\]

(51)

with \( \alpha \) a positive constant. From this expression, we have

\[
H(t) = \frac{2\alpha t (1 + 2\alpha^2 t^2 \exp(\alpha^2 t^4 - \alpha t^2))}{1 + \exp(\alpha^2 t^4 - \alpha t^2)}.
\]

(52)

Here, we note that \( \mathcal{G}(t) = 24H^2 \left( \dot{H} + H^2 \right) \geq 0 \) for \( t \in \mathbb{R} \). We have the following equation

\[
p_2(\mathcal{G}) \frac{d^2 F(\mathcal{G})}{d\mathcal{G}^2} + p_1(\mathcal{G}) \frac{dF(\mathcal{G})}{d\mathcal{G}} + F(\mathcal{G}) = b(\mathcal{G}),
\]

(53)
where the coefficients are presented in the form of series in powers of $\mathcal{G}$ in the neighborhood of $\mathcal{G} = 0$

\[
p_2(\mathcal{G}) = 2\mathcal{G}^2 + \frac{5}{8\alpha^2}\mathcal{G}^3 - \frac{113}{144\alpha^4}\mathcal{G}^4 + o(\mathcal{G}^4), \tag{54}
\]

\[
p_1(\mathcal{G}) = -\mathcal{G}, \tag{55}
\]

\[
b(\mathcal{G}) = -\frac{1}{4\alpha}\mathcal{G} + \frac{17}{192\alpha^3}\mathcal{G}^2 - \frac{379}{4608\alpha^5}\mathcal{G}^3 + o(\mathcal{G}^3). \tag{56}
\]

We seek a solution of equation (58) in the neighborhood of $\mathcal{G} = 0$. First of all, we construct a fundamental system of solutions of the corresponding homogeneous equation

\[
d\frac{d^2F(\mathcal{G})}{d\mathcal{G}^2} + q_1(\mathcal{G})\frac{dF(\mathcal{G})}{d\mathcal{G}} + q_2(\mathcal{G})F(\mathcal{G}) = 0. \tag{57}
\]

Since the coefficients $q_k(\mathcal{G})$ for $k = 1, 2$ has a pole of order not higher than $k$ at $\mathcal{G} = 0$, then $\mathcal{G} = 0$ is regular singular point of equation (57). Therefore, we can obtain

\[
q_1(\mathcal{G}) = \bar{q}_1(\mathcal{G})/\mathcal{G}, \quad q_2(\mathcal{G}) = \bar{q}_2(\mathcal{G})/\mathcal{G}^2. \tag{58}
\]

Here, $\bar{q}_1(\mathcal{G})$ and $\bar{q}_2(\mathcal{G})$ are holomorphic functions in a neighborhood of $\mathcal{G} = 0$. We construct a fundamental system of solutions of Eq. (57) in the neighborhood of $\mathcal{G} = 0$. Solutions will be found in the form of a generalized series

\[
F_1(\mathcal{G}) = \mathcal{G}^\gamma \sum_{k=0}^{+\infty} c_k \mathcal{G}^k. \tag{59}
\]

By combining this expression with Eq. (57), we acquire

\[
\gamma (\gamma - 1) + \bar{q}_1(0)\gamma + \bar{q}_2(0) = 0. \tag{60}
\]

We find the following solutions of this equation: $\gamma_1 = 1/2$ and $\gamma_2 = 1$. From Eq. (57), we represent

\[
F_1(\mathcal{G}) = \mathcal{G}\varphi_1(\mathcal{G}), \quad F_2(\mathcal{G}) = \sqrt[3]{\mathcal{G}}\varphi_2(\mathcal{G}), \tag{61}
\]

where $\varphi_k(\mathcal{G})$ for $k = 1, 2$ are holomorphic functions in $\mathcal{G} = 0$ at that $\varphi_k(\mathcal{G}) \neq 0$. Substituting (60) into the homogeneous equation, we obtain a recurrent system from which we consistently find the coefficients $c_1, c_2, \cdots$. Thus, the fundamental system of the homogeneous equation has the form

\[
F_1(\mathcal{G}) = \mathcal{G}, \tag{62}
\]

\[
F_2(\mathcal{G}) = \sqrt[3]{\mathcal{G}} \left( 1 + \frac{5}{32\alpha^2}\mathcal{G} - \frac{2483}{55296\alpha^4}\mathcal{G}^2 + o(\mathcal{G}^3) \right). \tag{63}
\]

Solving the inhomogeneous equation (53) by the method of variation of constants, we obtain the particular solution of the inhomogeneous equation

\[
F_3(\mathcal{G}) = \frac{1}{2\alpha} \left( 1 - \frac{1}{2} \ln \mathcal{G} \right) \mathcal{G} + \frac{47}{576\alpha^3}\mathcal{G}^2 - \frac{1753}{46080\alpha^5}\mathcal{G}^3 + o(\mathcal{G}^3). \tag{64}
\]

As a result, an approximate solution of Eq. (53) has the form

\[
F(\mathcal{G}) = c_1\mathcal{G} + c_2\sqrt[3]{\mathcal{G}} \left( 1 + \frac{5}{32\alpha^2}\mathcal{G} - \frac{2483}{55296\alpha^4}\mathcal{G}^2 \right) + \frac{1}{2\alpha} \left( 1 - \frac{1}{2} \ln \mathcal{G} \right) \mathcal{G} + \frac{47}{576\alpha^3}\mathcal{G}^2 + o(\mathcal{G}^3). \tag{65}
\]

Unification of bounce with the late-time cosmic acceleration—We study the reconstruction of an $F(\mathcal{G})$ gravity theory where both the bouncing behavior in the early universe and the late-time accelerated expansion of the universe at the dark energy dominated stage can occur within a unified model.
In Eq. \((54)\), we take \(\alpha = 1/t_*^2\) with \(t_*\) a fiducial time. We also have \(N = \ln a/a_*\) with \(a_* = 1\) and hence \(H = \dot{N}\). From Eq. \((52)\), we see that for \(t < 0\), \(H < 0\), whereas for \(t > 0\), \(H > 0\), so that around \(t = 0\), namely, in the early universe, the bouncing behavior of the universe can be realized. In Fig. 1, we show the behavior of the Hubble parameter in Eq. \((52)\) with \(\alpha = 1/t_*^2 = 1\), where \(t_* = 1\), around the bouncing time \(t = 0\). Clearly, it can be observed that the value of \(H\) evolves from negative to positive as the cosmic time \(t\) does.

On the other hand, when \(\alpha t^2 \gg 1\), the universe can be considered to be at the dark energy dominated stage, because for \(\alpha t^2 \gg 1\), we find \(a(t) \approx \exp(\alpha t^2)\). Eventually, we obtain

\[
\ddot{a}(t) \approx 4\alpha^2 t^2 (3 + 4\alpha^2 t^4) \exp(\alpha^2 t^4) > 0.
\]

This implies the accelerated expansion of the universe happens. As a result, we see that in the case that the scale factor is given by Eq. \((51)\), which is expressed as a sum of two exponential functions, the late-time cosmic acceleration as well as the bouncing behavior in the early universe can be realized in a unified manner. For \(a(t)\) in Eq. \((51)\), the form of \(F(G)\) is represented as in Eq. \((55)\).

In addition, if \(a(t)\) is given by Eq. \((51)\), the stability conditions \(J_2/J_1 > 0\) and \(J_3/J_1 > 0\) with Eqs. \((22)-(24)\) can be satisfied. For \(\alpha t^2 \ll 1\), namely, around the bounce in the early universe, we have \(a(t) \approx \exp(\alpha t^2)\), \(H \approx 2\alpha t\), \(N \approx \alpha t^2 \ll 1\), \(\bar{g}_0(N) \approx 4N\), and \(G_0 \approx 192N (1 + 2N) \approx 192N\). Hence, for \(F(G_0)\) in Eq. \((65)\), we find

\[
\frac{J_2}{J_1} \approx \frac{3}{2} \left[ 2 + \frac{1}{N} + 128(1 + 4N) \frac{F'''(G_0)}{F'''(G_0)} \right] \approx 3 > 0,
\]

\[
\frac{J_3}{J_1} \approx \frac{1}{3072N^2} \left\{ \frac{1}{F''(G_0)} + 1536N \left[ (3 + 18N - 8N^2) + 384N (1 + 4N)^2 \frac{F'''(G_0)}{F'''(G_0)} \right] \right\}.
\]

Regarding \(J_3/J_1\) in Eq. \((68)\), only for the value of the content within the brackets \([\ ]\) is positive, we have \(J_3/J_1 > 0\), so that the bouncing solution can be stable. While for \(\alpha t^2 \gg 1\), namely, in the late-time universe, we have \(a(t) \approx \exp(\alpha t^2)\), \(H \approx 4\alpha^2 t^3\), \(N \approx \alpha^2 t^4 \gg 1\), \(\bar{g}_0(N) \approx 16\alpha N^{3/2}\), and \(G_0 \approx 1536\alpha^2 N^2 (4N + 3)\). Thus, for \(F(G_0)\) in Eq. \((65)\), we obtain

\[
\frac{J_2}{J_1} \approx 3N^{1/2} \left[ N + \frac{3}{4} + 3072\alpha^2 N^2 (2N + 1) \frac{F'''(G_0)}{F'''(G_0)} \right] \approx 3N^{1/2} \left( N + \frac{3}{4} + \frac{2N + 1}{4N + 3} \right) > 0,
\]

\[
\frac{J_3}{J_1} \approx \frac{1}{196608\alpha^3 N^3} \left[ \frac{1}{F''(G_0)} + 24576\alpha^3 N^{5/2} (-32N^2 + 108N + 45) \right.
\]

\[
+ 902500664\alpha^5 N^{9/2} (8N + 3) (2N + 1) \frac{F'''(G_0)}{F'''(G_0)} \right]
\]

\[
\approx \frac{-32N^2 + 108N + 45}{8N^{1/2}}.
\]
Here, the last approximate equality in Eq. (70) follows from the approximate relations \(1/F''(G_0) \approx -221184a^4/[2483(8-c_2)] G_0^{1/2} \) and \(F''(G_0)/F''(G_0) = (3c_2 + 1)/[2(8-c_2)] G_0^{-1} \). That is, since \(G_0 \gg 1\), the second term in the brackets [ ] on the right-hand side of the first approximate equality in Eq. (70) would be the dominant term. If \(0 < N < 15/4\), we see that \(\mathcal{J}_0/\mathcal{J}_1 > 0\), and therefore the solution in the late-time universe can be stable.

We mention that among three terms in the brackets [ ], when the two or three terms would be comparable with each other, by taking into consideration the contributions from not only the second term but also the other ones, even for larger \(N\), the condition \(\mathcal{J}_0/\mathcal{J}_1 > 0\) could be met.

**Summary**—We have studied bounce cosmology in \(F(G)\) gravity. We have reconstructed \(F(G)\) gravity model with the bouncing behavior in the early universe. Also, we have analyzed the stability of the solutions in the reconstructed model. Moreover, we have explored an exponential model and a power-law model and found that in these models the bouncing behavior can happen. Furthermore, the \(F(G)\) gravity theory with the ekpyrotic scenario has been investigated. In addition, it has been verified that in a sum of two exponential functions model of the scale factor, the bouncing behavior in the early universe. Also, we have analyzed the stability of the solutions in the reconstructed models. Our unified model of the early-time bounce with the late-time deceleration, jerk, snap, and lerk parameters are examined, our unified model of the early-time bounce with the late-time acceleration can be realized. In this unified model, we have further examined the stability of the bounce and late-time solutions.

We here mention the comparison of this unified scenario with the observations by following the discussions in Ref. [11]. It is known that the cosmography can be adopted to test modified gravity theories such as \(F(R)\) gravity. Therefore, by analogy with this fact, it is considered that with the cosmographical procedure in which the Hubble, deceleration, jerk, snap, and lerk parameters are examined, our unified model of the early-time bounce with the late-time cosmic acceleration in \(F(G)\) gravity can also be checked whether it can be consistent with the recent observational data. This point is regarded as the strong advantage of the cosmography.

The novel significant ingredient observed in this work is that we have explicitly derived the \(F(G)\) gravity model analytically, where the early-time bouncing behavior and the late-time cosmic acceleration can occur within the framework of the single model. Such behaviour supports the first proposal on the unification of early-time and late-time accelerations within modified gravity as given in Ref. [30]. This analysis is considered to be a useful clue for building models to represent the early universe, which should be described by the high-energy theories, and for seeking for the cosmological mechanism of the current accelerated expansion of the universe.

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