INVARiance AND STRICT INVARIANCE FOR NONLINEAR EVOLUTION PROBLEMS WITH APPLICATIONS

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Abstract. Sufficient conditions for the invariance of evolution problems governed by perturbations of (possibly nonlinear) \(m\)-accretive operators are provided. The conditions for the invariance with respect to sublevel sets of a constraint functional are expressed in terms of the Dini derivative of that functional, outside the considered sublevel set in directions determined by the governing \(m\)-accretive operator. An approach for non-reflexive Banach spaces is developed and some result improving a recent paper [7] is presented. Applications to nonlinear obstacle problems and age-structured population models are presented in spaces of continuous functions where advantages of that approach are taken. Moreover, some new abstract criteria for the so-called strict invariance are derived and their direct applications to problems with barriers are shown.

1. Introduction

In the paper we study evolution problems of the form

\[
\begin{align*}
\dot{u} & \in -Au + f(u), \quad t \geq 0, \\
u(0) & = x_0 \in \overline{D}(A) \cap \Omega,
\end{align*}
\]

where \(A : D(A) \to X\) is a quasi-\(m\)-accretive operator in a Banach space \((X, \| \cdot \|)\) and \(f : \Omega \to X\), with open \(\Omega \subset X\), is a continuous map. We are interested in the behavior of the so-called integral solutions \(u\) to eq. (1.1) related to the closed set \(K \subset \Omega \cap D(A)\). In particular we look for conditions implying the invariance of \(K\) with respect to the evolution determined by (1.1), meaning that any integral solution \(u\) to (1.1) starting at \(x_0 \in K\) remains in \(K\), i.e. \(u(t) \in K\) for every \(t\) from the maximal interval of existence \([0, \tau_u)\), \(0 < \tau_u \leq \infty\). Moreover issues concerning strict invariance, when all solutions starting at \(x_0 \in K\) stay for \(t \in (0, \tau_u)\) in the interior \(\text{int}K\) of \(K\) will be of interest. Note that the concepts of invariance and strict invariance differ from the so-called viability of \(K\) with respect to (1.1) meaning that for any \(x_0 \in K\) there exists a solution \(u\) starting at \(x_0\) and staying in \(K\) (see, e.g. [1], [10] and the extensive bibliography therein).

Let us start a discussion with some particular results and assume that \(A \equiv 0\), i.e. consider the problem

\[
\begin{align*}
\dot{u} & = f(u), \quad t \geq 0, \\
u(0) & = x_0 \in X,
\end{align*}
\]

(in this case integral and \(C^1\)-solutions to (1.2) coincide) and let \(K \subset \Omega\) be closed. In view of, e.g., [10] Theorems 4.1.2, 4.1.3] \(K\) is invariant with respect to (1.2) provided it is locally viable and \(f\)
satisfies a one-sided estimate of the following form

\[(1.3) \quad [x - y, f(x) - f(y)]_+ \leq \omega(\|x - y\|) \text{ for } x \in U \setminus K \text{ and } y \in K,
\]

where \(U \subset \Omega\) is a neighborhood of \(K\), \(\omega\) is a uniqueness function (see Subsection 2.1.3) and \([\cdot, \cdot]_+\) stands for the right semi-inner product (see Subsection 2.1.2). In the proof condition (1.3) is used to compare an arbitrary solution \(u\) of (1.2) starting at some \(x_0 \in K\) and leaving \(K\) with a viable local solution \(v\) starting at \(x_0 \in K\) and then to show that \(u\) must remain in \(K\). It is easy to see that condition (1.3) does not necessarily imply the uniqueness and follows immediately if, for instance, \(f\) satisfies the local Lipschitz type condition of the form \(\|f(x) - f(y)\| \leq \omega(\|x - y\|)\) for \(x \in U \setminus K\) and \(y \in K\). It may be shown that the right semi-inner product \([\cdot, \cdot]_+\) in (1.3) can be replaced by the left semi-inner product \([\cdot, \cdot]_-\) and the assumption on \(\omega\) can be slightly relaxed.

The viability of \(K\) implies that \(f\) is tangent to \(K\), i.e.

\[(1.4) \quad f(x) \in T_K(x) \text{ for all } x \in \partial K,
\]

where \(T_K(x)\) is the contingent cone (see [2, Definition 4.1.1]). In view of [10, Theorem 3.5.7] condition (1.4) along with (1.3) imply the invariance, too. In this case a solution starting at \(x_0 \in K\) is compared with an approximate solution living in \(K\).

If the set \(K\) is proximal (see [10, Def. 2.2.3]), then in view of [27] (and comp. [10, Theorem 4.3.1]) conditions (1.4) and (1.3), even in a slightly relaxed form (see [10, Eq. (4.3.1)]) imply the so-called exterior tangency condition saying that a lower right Dini derivative of the distance function \(d_K = d(\cdot, K) := \inf_{x \in K} \| \cdot - k \|\) at \(x\) in the direction \(f(x)\) satisfies

\[(1.5) \quad D_+ d_K(x; f(x)) \leq \omega(d_K(x)) \text{ for } x \in U.
\]

In [28] it is shown that the restrictive assumption concerning proximality of \(K\) is actually superfluous if \(f\) is Lipschitz in the sense \(\|f(x) - f(y)\| \leq \omega(\|x - y\|)\) for \(x \in U \setminus K, y \in \partial K\), where \(\omega\) is a uniqueness function. Condition (1.5) actually entails the invariance (see [10, Theorem 4.2.1]).

The results described above represent two possible approaches to invariance. The first one relies on a controlled ‘monotonicity’ (or a one-sided Lipschitz) condition (1.3), which enables to compare an arbitrary solution to (1.2) with another one surviving in \(K\). The second approach does not use monotonicity. Instead the Lagrange type stability assumption (1.5) is employed. It requires that \(d_K\) plays a role of a Lyapunov function of sorts that does not allow solutions to escape from \(K\).

In the case of (1.1) with a nontrivial \(m\)-accretive operator \(A\) some important results concerning invariance are also known. They can be found for instance in [24] or [25] for linear \(A\) and [6] for nonlinear \(A\), see also [10, Theorems 10.10.1 and 10.10.2] and the references therein. Generally speaking, a closed set \(K \subset D(A) \cap \Omega\) is invariant with respect to (1.1) if \(f\) satisfies condition (1.3) above and either \(K\) is viable or \(f\) is \(A\)-tangent to \(K\). See also [9,11] for results with set-valued \(f\).

It is our aim to deal with the ‘stability’ approach rather, and to get the analogue of the exterior tangency condition (1.5) in the situation of (1.1). In this context two recently published papers [7,8] for linear \(A\) have been direct inspirations for our considerations, where conditions outside \(K\) expressed in terms of Dini derivatives in directions of \(-A + f\) where considered.

As it appears in applications, the constraint set \(K\) is represented as a sublevel set of a certain
locally Lipschitz potential $V : X \to \mathbb{R}$, i.e.

\[(1.6)\quad K = K_V := \{ x \in D(A) \mid V(x) \leq 0 \},\]

and it seems natural to look for invariance conditions in terms of the constraining functional $V$. The key ingredient of our approach relies on the analysis of behavior of $V$ along solutions of (1.1). The concept of the Dini $A$-directional derivative $D_A V(x; v)$ of $V$ at point $x \in D(A)$ in the direction $v = f(x)$ is useful. We propose the following results without the reflexivity assumption on $X$ (see Sections 2 and 3 for the terminology and notation).

**Theorem 1.1.** If for every $z \in \partial K$ there are a neighborhood $U(z) \subset \Omega$ of $z$ and a uniqueness function $\omega$ such that

\[(1.7)\quad D_A V(x; f(x)) \leq \omega(V(x)) \text{ for } x \in U(z) \setminus K,\]

then $K$ is invariant with respect to (1.1).

As the uniqueness function one often uses a linear function $\omega(t) = Ct$, where $C \geq 0$. In this case condition (1.7) has the form

$D_A V(x; f(x)) \leq CV(x)$.

We also present a version of Theorem 1.1 which is convenient in applications to partial differential equations of parabolic type (see 5.2 and 5.3).

**Theorem 1.2.** Suppose that for every $z \in \partial K$ there are a neighborhood $U(z) \subset \Omega$ of $z$ and a uniqueness function $\omega$ such that, for any integral solution $u : [0, \tau) \to X$ of (1.1) with $u(0) = z$ one has

\[(1.8)\quad D_+(V \circ u)(t) \leq \omega(V(u(t))), \text{ for a.e. } t \in (0, \tau), \text{ whenever } u(t) \in U(z) \setminus K.\]

Then $K$ is invariant with respect to (1.1).

There are situations (e.g., for first order partial differential equations) when the verification of condition (1.7) for $x$ from $D(A)$ or condition (1.8) for any integral solution is not obvious if possible. For that reason, the following result seems to be more suitable.

**Theorem 1.3.** Assume that $X$ is reflexive and:

(i) for any $x \in \partial K$, there is $\delta > 0$ and a slow function $\beta$ such that $D(x, \delta) \subset \Omega$ and

\[(1.9)\quad [u - v, f(u) - f(v)]_+ \leq \beta(\|u - v\|) \text{ for } u, v \in D(x, \delta);\]

(ii) $f$ maps bounded sets into bounded ones;

(iii) for every $z \in \partial K$ there is a neighborhood $U(z) \subset \Omega$ of $z$ such that

\[(1.10)\quad D_A V(x; f(x)) \leq \omega(V(x)) \text{ for } x \in D(A) \cap (U(z) \setminus K),\]

where $\omega$ is a nondecreasing uniqueness function.

Then $K$ is invariant with respect to (1.1).

\footnote{Note that the representation (1.6) does not restrict the generality since is applicable for any closed set $K \subset \overline{D(A)}$ with $V = d_K$ or with $V = \Delta_K := d_K - d_{X \setminus \text{int}K}$ if $\text{int}K \neq \emptyset$.}
A notion of a slow function mentioned in the above theorem is provided in Subsection 2.2.4. Note that condition (1.9) implies, in particular, that integral solutions to (1.1) starting in a neighborhood of $\partial K$ are locally unique. This is the price for relaxing condition (1.7).

If $X$ is not reflexive then the following result will be useful when applying to partial differential equations (see Section 5.4).

**Theorem 1.4.** Assume that conditions (i) and (ii) from Theorem [7.3] are satisfied and that (iii) for every $z \in \partial K$ there are a neighborhood $U(z) \subset \Omega$ of $z$ and a nondecreasing uniqueness function $\omega$ such that, for any integral solution $u : [0, \tau) \to X$ with $u(0) \in D(A)$, for a.e. $t \in (0, \tau)$ with $u(t) \in U(z) \setminus K$ one has

$$D_+ (V \circ u)(t) \leq \omega(V(u(t))).$$

Then $K$ is invariant with respect to (1.1).

**Remark 1.5.** (1) Assume, additionally, that $X$ embeds continuously into another Banach space $Y$ and that there is a quasi $m$-accretive operator $A_Y : D(A_Y) \to Y$ such that the part of $A_Y$ in $X$ is equal to $A$ and that $V$ can be extended to a differentiable functional $V_Y : Y \to \mathbb{R}$. Suppose also that, for any integral solution $u : [0, T] \to X$ of (1.1), $u \in W_{loc}^{1,1}((0, T), Y)$ (i.e. $u \in W^{1,1}([\delta, T], Y)$, for any $\delta \in (0, T)$), $u(t) \in D(A_Y)$ and

$$V_Y'(x)(-v + f(x)) \leq \omega(V(x)) \text{ for all } v \in A_Y x.$$

(2) Now assume that $X$ embeds continuously into a reflexive Banach space $Y$ and there exists a quasi $m$-accretive operator $A_Y : D(A_Y) \to Y$ such that the part of $A_Y$ in $X$ is equal to $A$ and that $V$ can be extended to a differentiable functional $V_Y : Y \to \mathbb{R}$. Moreover, suppose that $f$ is locally Lipschitz and bounded on bounded sets. Then one can prove (see Proposition 2.9) that if $u : [0, T] \to X$ is an integral solution with $u(0) \in D(A)$, then $u \in W^{1,1}([0, T], Y)$, $u(t) \in D(A_Y)$ and

$$V_Y'(u(t))(-v + f(u(t))) \leq \omega(V(u(t))) \text{ for all } v \in A_Y u(t).$$
The following results constitute analogs of the above theorems and refer to invariance of the interior of $K$. Our standing assumption is

\[(1.14) \quad K^0_V := \{x \in \overline{D(A)} \mid V(x) < 0\} = \text{int}K_V \neq \emptyset.\]

**Theorem 1.6.** If for every $z \in \partial K$ there are a neighborhood $U(z) \subset \Omega$ of $z$ and a uniqueness function $\omega$ such that

\[(1.15) \quad DA(V(x; f(x)) \leq \omega(-V(x)) \text{ for } x \in U(z) \cap K^0_V,\]

then $K^0_V$ is invariant with respect to \((1.1)\).

**Theorem 1.7.** Assume that $X$ is reflexive, assumptions (i) and (ii) from Theorem 1.3 hold and

(iv) for every $z \in \partial K$ there is a neighborhood $U(z) \subset \Omega$ of $z$ such that

\[(1.16) \quad DA(V(x; f(x)) \leq \omega(-V(x)) \text{ for } x \in D(A) \cap (U(z) \cap K^0_V),\]

where $\omega$ is a nondecreasing uniqueness function.

Then $K^0_V$ is invariant with respect to \((1.1)\).

Introducing additionally the *inwardness* condition \((1.18)\) we obtain the following strict invariance results.

**Theorem 1.8.** If for every $z \in \partial K$ there are a neighborhood $U(z) \subset \Omega$ of $z$ and a uniqueness function $\omega$ such that

\[(1.17) \quad DA(V(x; f(x)) \leq \omega(|V(x)|) \text{ for } x \in U(z),\]

\[(1.18) \quad DA(V(x; f(x)) < 0 \text{ for } x \in U(z) \cap \partial K,\]

then $K$ is strictly invariant with respect to \((1.1)\).

**Theorem 1.9.** Assume that $X$ is reflexive, assumptions (i) and (ii) of Theorem 1.3 hold,

(v) for every $z \in \partial K$ there is a neighborhood $U(z) \subset \Omega$ of $z$ such that

\[(1.19) \quad DA(V(x; f(x)) \leq \omega(|V(x)|) \text{ for } x \in D(A) \cap U(z),\]

where $\omega$ is a nondecreasing uniqueness function,

and the strong inwardness condition \((1.18)\) holds true.

Then $K$ is strictly invariant with respect to \((1.1)\).

Theorems 1.1–1.4 will be proved in Section 3 and Theorems 1.6–1.9 in Section 4.

Section 5 is devoted to applications. We discuss an evolution equation of the form \((1.1)\) under the presence of state dependent impulses – see Subsection 5.1. In Subsection 5.2 we consider a fully nonlinear double obstacle problem

\[
\begin{cases}
  u_t - \Delta_p u = f(x,u), & x \in (0,l), \ t > 0 \\
  u(0,t) = u(l,t) = 0 & t > 0, \\
  m(x) \leq u(x,t) \leq M(x), & x \in (0,l), t > 0
\end{cases}
\]

where $\Delta_p$ is the $p$-Laplace operator and functions $m$ and $M$ represent two bodies, between which solutions are expected. Since the $p$-Laplace operator is nonlinear, results from [7] do not apply.
there. In Subsection 5.3 we study also the higher dimensional obstacle problem with the Laplace operator and in Section 5.4 we deal with invariance in the McKendrick age-structured population model. Let us note that, since the reflexivity of $X$ is not required in the applied abstract results (i.e. Theorems [1.2] and [1.4]), we are able to study partial differential equations as problems (1.1) with $X$ being a space of continuous functions. Therefore we put neither growth restrictions nor global Lipschitz conditions on $f$.

2. Preliminaries

Notation: The notation used throughout the paper is standard. Given a metric space $(Y,d)$, $K \subset Y$ and $x \in Y$, $d_K(x) := \inf_{y \in K} d(x,y)$; by $\overline{K}$, int $K$ and $\partial K$ we denote the closure, the interior and the boundary of $K$; $B(x,r)$ (resp. $D(x,r)$) is the open (resp. closed) ball around $x \in Y$ of radius $r > 0$; $B(K,r)$ denotes the $r$-neighborhood of $K$, i.e. $B(K,r) = \{ y \in Y \mid d_K(x) < r \}$. In what follows $(X,\| \cdot \|)$ is a real Banach space, $X^*$ stands for the dual of $X$; $\langle \cdot , \cdot \rangle$ is the conjugation duality in $X$, i.e. if $x \in X$, $p \in X^*$, then $\langle p , x \rangle := p(x)$; by default $X^*$ is normed. The use of function spaces ($L^p$, Sobolev $W^{k,p}$, etc.), linear (unbounded in general) operators in Banach spaces, $C_0$ semigroups is standard. In particular, given real functions $u,v$, we put $u \vee v := \max\{u,v\}$, $u \wedge v := \min\{u,v\}$, $u_{\pm} = (\pm u) \vee 0$.

We collect and recall some general concepts and relevant facts concerning evolution problems involving accretive operators.

2.1. General concepts.

2.1.1. Dini derivatives. For any function $u : (a,b) \to \mathbb{R}$ one defines the Dini derivatives

$$D_{\pm}u(t) := \lim_{h \to 0^\pm} \frac{u(t+h) - u(t)}{h}, \quad D_{\pm}u(t) := \limsup_{h \to 0^\pm} \frac{u(t+h) - u(t)}{h}, \quad t \in (a,b).$$

If $f : \Omega \to \mathbb{R}$, where $\Omega \subset X$ is open, $x \in \Omega$ and $v \in X$, then the Dini directional derivatives at $x$ in the direction $v$ are given by

$$D_{\pm}f(x;v) := \lim_{h \to 0^\pm} \frac{f(x+hv) - f(x)}{h}, \quad D_{\pm}f(x;v) := \limsup_{h \to 0^\pm} \frac{f(x+hv) - f(x)}{h}.$$ 

If the function $f$ is convex, then $D_- f(x;v) = D^- f(x;v)$ and $D_+ f(x;v) = D^+ f(x;v)$.

2.1.2. Semi-inner products (see e.g. [10 Sec. 1.6]). If $x,y \in X$, then we put

$$[x,y]_{\pm} = \lim_{h \to 0^\pm} \frac{\| x + hy \| - \| x \|}{h},$$

i.e. $[x,y]_{\pm}$ is the lower right (resp. left) Dini directional derivative of $\| \cdot \|$ at $x$ in the direction of $y$ (see [10 Ex. 1.6.1] for properties of $[\cdot,\cdot]_{\pm}$).
2.1.3. **Uniqueness functions.** A continuous function \(\omega : [0, \infty) \to [0, \infty)\) such that \(\omega(0) = 0\) is a uniqueness (or a Perron) function if the only nonnegative solution to the problem \(\dot{u} = \omega(u)\) on an interval \([0, \tau), 0 < \tau \leq \infty\), such that \(u(0) = 0\) is the null function (some authors may consider different definitions, see, e.g. [30]). Examples of uniqueness functions are numerous.

In the following lemma the first result is essentially due to Perron, the second one follows as an easy consequence of the first one, and the third one is proved for the sake of completeness. By a maximality of a solution \(x\) we mean that \(u(t) \leq x(t)\) for every other solution \(u\) and every \(t\) in a common interval of existence.

**Lemma 2.1.** Let \(\omega : [0, \infty) \to [0, \infty)\) be continuous and \(\tau > 0\).

1. (see, e.g. [14] Th. 1.4.10)) If \(x\) is the maximal solution to the problem \(\dot{x} = \omega(x)\) on the interval \([0, \tau], u : [0, \tau] \to [0, \infty)\) is continuous, \(u(0) \leq x(0)\) and \(Du \leq \omega(u)\) on \((0, \tau)\), where \(D\) stands for any Dini derivative, then \(u(t) \leq x(t)\) for \(t \in [0, \tau]\). In particular if \(\omega\) is a uniqueness function and \(u(0) = 0\), then \(u \equiv 0\) on \([0, \tau]\).

2. If \(\omega\) is a uniqueness function, \(u : [-\tau, 0) \to (-\infty, 0]\) is continuous, \(u(0) = 0\) and \(Du \leq \omega(-u)\) on \((-\tau, 0]\), where \(D\) stands for any Dini derivative, then \(u \equiv 0\) on \([-\tau, 0]\).

3. If \(x\) is the maximal solution to the problem \(\dot{x} = \omega(x)\) on the interval \([0, \tau], u : [0, \tau] \to [0, \infty)\) is continuous and \(u \in W^{1,1}_{loc}((0, \tau]), u(0) \leq x(0)\) and \(\dot{u}(t) \leq \omega(u(t))\) for a.a \(t \in (0, \tau]\), then \(u \leq x\) on \([0, \tau]\). If \(\omega\) is a uniqueness function, \(u : [0, \tau] \to [0, \infty)\) is continuous and \(u \in W^{1,1}_{loc}((0, \tau]), u(0) = 0\) and \(\dot{u} \leq \omega(u)\) a.e. on \((0, \tau]\), then \(u \equiv 0\) on \([0, \tau]\).

**Proof.** We have the following property: Let \(\omega : [0, +\infty) \to [0, +\infty)\) be continuous and \(\tau > 0\). If \(x\) is the maximal solution to the problem \(\dot{x} = \omega(x)\) on \([0, \tau], u : [0, \tau] \to [0, +\infty)\) is absolutely continuous, \(\dot{u} \leq \omega(u)\) a.e. \([0, \tau]\) and \(u(0) \leq x(0)\), then \(u \leq x\) on \([0, \tau]\). This, actually, follows from [26] Theorem III.16.2.

Now let \(t_0 = \sup\{t \in [0, \tau] \mid u(s) \leq x(s)\) for \(0 \leq s \leq t\}.\) Clearly \(t_0 \geq 0\). Suppose \(t_0 < \tau\). For \(n \in \mathbb{N}\), let \(y_n : [t_0, t_n] \to [0, +\infty)\), where \(t_n \to t_0, t_0 > t_0\), be a solution to the problem \(\dot{y} = \omega(y) + \frac{1}{n}, y(t_0) = x(t_0) + \frac{1}{n}\) on \([t_0, t_n]\). Following [14] Theorem 1.4.9] or [10] Lemma 1.8.1, we can assume that there is \(\delta > 0\) such that \(t_0 + \delta \leq t_0\) for all \(n\). Moreover \(y_n \geq y_{n+1}\) on \([t_0, t_0 + \delta]\). Hence \(y_n\) converges uniformly on \([t_0, t_0 + \delta]\) to a solution \(y\) of the problem \(\dot{y} = \omega(y), y(t_0) = x(t_0)\). It is immediate to see that \(y\) is the maximal solution; hence \(y = x\) on \([t_0, t_0 + \delta]\). On the other hand we have that \(u(t_0) = x(t_0) < y_n(t_0)\) for any \(n \in \mathbb{N}\). By continuity there is \(t_0 < t_0' < t_0 + \delta\) such that \(u(t) \leq y_n(t)\) for \(t_0 \leq t \leq t_0'\). Since \(u \in W^{1,1}([t_0', \tau]), u\) is absolutely continuous on \([t_0', t_0 + \delta]\) and \(\dot{u} \leq \omega(u)\) on \([t_0', t_0 + \delta]\). Thus, by the above result, \(u \leq y_n\) on \([t_0', t_0 + \delta]\). Hence \(u \leq y_n\) on the whole interval \([t_0, t_0 + \delta]\), i.e. \(u \leq x\) on this interval, too. This contradicts the choice of \(t_0\) and proves the assertion.

2.1.4. **Slow functions.** A continuous nondecreasing function \(\beta : [0, \infty) \to [0, \infty)\) is slow if there are \(\varepsilon > 0, M > 0\) and \(\tau > 0\) such that if \(u : [0, \tau] \to [0, \infty)\) is continuous and

\[
(2.1) \quad u(t) \leq a + \int_0^t \beta(u(s))\,ds \quad \text{for} \quad t \in [0, \tau],
\]

where \(a \in [0, \varepsilon]\), then one has \(u(t) \leq aM\) for \(t \in [0, \tau]\).
Proposition 2.2. If $\beta$ is a slow function, then the growth of a local solution of the problem $\dot{u} = \beta(u), u(0) = a$, where $a$ is sufficiently small, is slow in the sense that it is ‘proportional’ to the initial value $a$. This means that the use of slow functions provide arguments similar to those relying on the Gronwall inequality. It is clear that a slow function is a uniqueness function.

The class of slow functions is large. For instance any continuous and nondecreasing function $\beta : [0, \infty) \to [0, \infty)$ such that $\beta(0) = 0$, $\beta(x) > 0$ if $x > 0$ and there is $\epsilon > 0$ such $\beta(\lambda x) \leq \lambda \beta(x)$ for all $0 < \lambda < \epsilon$ and $x \geq 0$, is slow. In particular any function $\beta$ of the form $\beta(x) = x\gamma(x)$, $x \geq 0$, where $\gamma : [0, \infty) \to [0, \infty)$ is continuous nondecreasing and $\gamma(x) > 0$ when $x > 0$, is slow. For instance slow are the following functions $\beta(x) = cx^\alpha$, where $c > 0$ and $\alpha \geq 1$, or $\beta(x) = x\ln(1 + x)$ for $x \geq 0$.

All these examples are subsumed by the following general result.

**Proposition 2.2.** If $\beta : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that

\[(2.2) \quad \liminf_{x \to 0^+} \frac{x}{\beta(x)} > 0,\]

then $\beta$ is slow.

**Proof.** For any $a > 0$, define $B_a : [1, +\infty) \to [0, B_{a,\infty})$ by

\[B_a(r) := \int_1^r \frac{a}{\beta(as)} ds, \quad r > 1,\]

where $B_{a,\infty} := \lim_{r \to +\infty} B_a(r)$. Take any $\Gamma$ such that

\[\liminf_{x \to 0^+} x/\beta(x) > \Gamma > 0.\]

In view of (2.2), there exists $\epsilon > 0$ such that

\[x/\beta(x) > \Gamma \quad \text{for all} \quad x \in (0, 2\epsilon].\]

Therefore, if $s \in (1, 2)$ and $a \in (0, \epsilon]$, then

\[\frac{a}{\beta(as)} \geq \frac{\Gamma}{s},\]

which implies

\[B_a(2) \geq \int_1^2 \Gamma s^{-1} ds \geq \frac{\Gamma}{2} \quad \text{for all} \quad a \in (0, \epsilon],\]

that is, with $M := 2$ and $\tau := \Gamma/2$,

\[(2.3) \quad B_a(M) \geq \tau \quad \text{for all} \quad a \in (0, \epsilon].\]

Assume that $a \in (0, \epsilon]$ and $u : [0, \tau] \to [0, +\infty)$ satisfies (2.1). If we define $x_a : [0, \tau] \to [0, +\infty)$ by $x_a(t) := u(t)/a$ then

\[(2.4) \quad x_a(t) \leq y_a(t) := 1 + \int_0^t a^{-1} \beta(ax_a(s)) ds.\]

By definition and the monotonicity of $\beta$, we get

\[a\dot{y}_a(t) = \beta(ax_a(t)) \leq \beta(ay_a(t)) \quad \text{for all} \quad t \in [0, \tau].\]

This yields

\[\int_0^t \frac{a\dot{y}_a(s)}{\beta(ay_a(s))} ds \leq t \quad \text{for all} \quad t \in [0, \tau],\]
which, by the change of variables and (2.3), implies
\[
B_a(y_a(t)) = \int_1^{y_a(t)} \frac{a}{\alpha(s)} ds = \int_0^t \frac{ay_a(s)}{\beta(ay_a(s))} ds \leq t \leq B_a(M) \quad \text{for all } t \in [0, \tau].
\]

Hence, by the monotonicity of \(B_a\), we get
\[
y_a(t) \leq M \quad \text{for all } t \in [0, \tau).
\]

Finally, in view of (2.4), we see that, for all \(a \in (0, \varepsilon]\),
\[
u(t)/a = x_a(t) \leq y_a(t) \leq M \quad \text{for all } t \in [0, \tau],
\]
which ends the proof.

\[\square\]

**Remark 2.3.** Observe that if a function \(\beta : [0, \infty) \to [0, \infty)\) is slow, then for any \(a, b \geq 0\), the function \([0, \infty) \ni x \mapsto a\beta(x) + bx\) is slow, too.

2.2. **Nonlinear evolution associated with accretive operators.**

2.2.1. **Accretive operators.** Let \(A : D(A) \to X\), where \(D(A) \subset X\) and \(X\) is a Banach space, be a set-valued operator, i.e. \(\emptyset \neq Ax \subset X\) for \(x \in D(A)\). Let \(Gr(A) := \{(x, u) \in X \times X \mid x \in D(A), u \in Ax\}\) be the graph of \(A\).

(a) \(A\) is **accretive** if \([x - y, u - v]_+ \geq 0\) for all \((x, u), (y, v) \in Gr(A)\). \(A\) is **\(m\)-accretive** if it is accretive and \(\text{Range}(I + \lambda A) := \{y \in X \mid y \in x + \lambda Ax\} = X\) for some (equivalently for all) \(\lambda > 0\).

(b) \(A\) is **\(\alpha\)-accretive** (resp. **\(\alpha\)-\(m\)-accretive), \(\alpha \in \mathbb{R}\), if \(\alpha I + A\) is accretive (resp. \(m\)-accretive); hence \(A\) is \(\alpha\)-accretive if and only if \([x - y, u - v]_+ \geq -\alpha \|x - y\|\) for all \((x, u), (y, v) \in Gr(A)\). \(A\) is **quasi \(m\)-accretive** if it is \(\alpha\)-accretive for some \(\alpha \in \mathbb{R}\).

(c) If \(A\) is quasi \(m\)-accretive, then \(Gr(A)\) is closed and, in particular, the set \(Ax, x \in D(A)\), is closed. If the dual \(X^*\) is uniformly convex, then \(Ax\) is convex. If both \(X\) and \(X^*\) are uniformly convex, then the closure \(\overline{D(A)}\) is convex and for each \(x \in D(A)\) and \(w \in X\) there is a unique element \((Ax - w)^0 \in Ax - w\) of minimal norm, i.e. \(\|Ax - w\|^0 = |Ax - w| := \inf_{u \in Ax} \|u - w\|\).

(d) In view of the Lumer theorem a **linear** operator \(A : D(A) \to X\) is an **\(m\)-accretive** if and only if \(-A\) is a closed densely defined generator of a strongly continuous semigroup of linear operators \(\{e^{-tA}\}_{t \geq 0}\) such that \(\|e^{-tA}\| \leq e^{\alpha t}\) for \(t \geq 0\).

(e) If \(A\) is \(\alpha\)-\(m\)-accretive, \(\lambda > 0\) with \(\lambda \alpha < 1\), then the **resolvent** \(J_\lambda = J_\lambda^A := (I + \lambda A)^{-1} : X \to D(A)\) and the **Yosida approximation** \(A_\lambda = \lambda^{-1}(I - J_\lambda) : X \to X\) are well-defined, single-valued, and
\[
\|J_\lambda x - J_\lambda y\| \leq (1 - \lambda \alpha)^{-1} \|x - y\|, \quad A_\lambda x \in AJ_\lambda x \quad \text{for all } x, y \in X,
\]
\[
\lim_{\lambda \to 0^+} J_\lambda x = x \quad \text{for } x \in \overline{D(A)}.
\]

2.2.2. **Cauchy problems.** Assume \(A\) is an \(\alpha\)-\(m\)-accretive operator. Let \(T > 0\), \(w \in L^1([0, T], X)\) and consider the problem
\[
\begin{cases}
\dot{u}(t) \in -Au(t) + w(t), \quad t \in [0, T],
\end{cases}
\]
\[
u(0) = x \in \overline{D(A)}.
\]
(a) A continuous function \( u : [0, T] \to X \) is a strong solution to (2.6) if \( u \in W^{1,1}_{\text{loc}}([0, T], X) \), \( u(t) \in D(A) \), \( u(0) = x \) and \( \dot{u}(t) - w(t) = -Au(t) \) for a.a. \( t \in (0, T) \); here \( \dot{u}(t) \) stands for the ordinary strong derivative; the formula makes sense since \( u \) is differentiable a.a.

(b) If \( X \) is reflexive, \( x \in D(A) \) and \( w \in W^{1,1}([0, T], X) \), then (2.6) has a unique strong solution \( u \in W^{1,\infty}([0, T], X) \); if both spaces \( X \) and \( X^* \) are uniformly convex and \( x \in D(A) \), then the strong solution \( u \) is right differentiable, \( \dot{u}_+ \) is right continuous and \( \dot{u}_+(t) + (Au(t) - w(t))^0 = 0 \) for a.a. \( t \in (0, T) \); if in addition \( w \) is continuous, then \( \dot{u}_+(0) = (-Ax + w(0))^0 \) (see [4] Theorems 4.5, 4.6).

(c) [4] Remarks to Corollary 4.2] A continuous function \( u : [0, T] \to X \) is an integral solution to (2.6) if \( u(0) = x \) and for any \( 0 \leq s \leq t \leq T \) and \( (y, v) \in \text{Gr}(A) \),

\[
(2.7) \quad e^{-\alpha t} \| u(t) - y \| \leq e^{-\alpha s} \| u(s) - y \| + \int_s^t e^{-\alpha z} \| \dot{u}(z) - y, w(z) - v \|_+ dz.
\]

It is known that a strong solution is an integral one. By [4] Theorem 4.2, Corollary 4.2] (2.6) admits a unique integral solution denoted by \( u = u_A(\cdot; x, w) : [0, T] \to X \) (or \( u(\cdot; x, w) \) if \( A \) is the default operator) and \( u(t) \in \overline{D(A)} \).

(d) Given \( x_1, x_2 \in \overline{D(A)} \), \( w_1, w_2 \in L^1([0, T], X) \) the Benilán inequality holds: for \( 0 \leq s \leq t \leq T \)

\[
(2.8) \quad e^{-\alpha t} \| u_1(t) - u_2(t) \| \leq e^{-\alpha t} \| u_1(s) - u_2(s) \| + \int_s^t e^{-\alpha z} \| \dot{u}_1(z) - \dot{u}_2(z), w_1(z) - w_2(z) \|_+ dz
\]

\[
\leq e^{-\alpha s} \| u_1(s) - u_2(s) \| + \int_s^t e^{-\alpha z} \| w_1(z) - w_2(z) \| dz,
\]

where \( u_i := u_A(\cdot; x_i, w_i) \), \( i = 1, 2 \), (see [4] eq. (4.14)).

(e) If \( w \equiv 0 \), then (2.6) has a unique integral solution \( u_A(\cdot; x, 0) \) defined on \( [0, \infty) \) (i.e. \( u_A(\cdot; x, 0) \) is an integral solution on \( [0, T] \) for any \( T > 0 \) and the following Crandall-Liggett formula, see [4] Theorem 4.3), holds: for any \( x \in X \) and \( t \geq 0 \),

\[
u_A(t; x, 0) = \lim_{n \to \infty} J^n_{t/n} x \text{ [3].}
\]

Let us put

\[
S_A(t)x := u_A(t; x, 0), \quad t \geq 0.
\]

Then, for any \( t \geq 0 \), \( S_A(t) : \overline{D(A)} \to D(A) \), \( \| S_A(t)x - S_A(t)y \| \leq e^{\alpha t} \| x - y \| \) or all \( x, y \in \overline{D(A)} \). The family \( \{ S_A(t) \}_{t \geq 0} \) is a (strongly continuous) semigroup of continuous maps, i.e. for any \( x \in \overline{D(A)} \) the map \( [0, \infty) \ni t \mapsto S_A(t)x \) is continuous, \( S_A(0) = I \) on \( \overline{D(A)} \) and \( S_A(t + s) = S(t) \circ S_A(s) \) for any \( t, s \geq 0 \) (see [4] Proposition 4.2)).

**Remark 2.4.** Let us derive some immediate consequences of the above facts.

(a) If \( x \in D(A) \), then \( S_A(\cdot)x : [0, \infty) \to X \) is Lipschitz continuous on every compact interval \( [0, \tau], \) \( \tau > 0 \). To see this fix \( \tau > 0 \) and take \( 0 \leq s < t \leq \tau \); then, in view of (2.8), (2.7), for any \( v \in Ax \)

\[
\| S_A(t)x - S_A(s)x \| = \| S_A(s)S_A(t - s)x - S_A(s)x \| \leq e^{\alpha s} \| S_A(t - s)x - x \|
\]

\[
\leq e^{\alpha s} \int_0^{t-s} e^{(t-s-z)\alpha} \| v \| dz \leq |Ax| \int_0^{t-s} e^{(t-s-z)\alpha} dz \leq t(t-s)
\]

\[\text{Here } J_0(x) = x, x \in X.\]
for some $\ell > 0$. The same is true for integral solutions $u_A(\cdot,x,w)$ with constant $w$.

(b) If $X$ is reflexive and $x \in D(A)$, then $u = S_A(\cdot)x \in W^{1,\infty}_{\text{loc}}([0,\infty), X)$, $u(t) \in D(A)$ and $\dot{u}(t) \in -Au(t)$ for a.a. $t \geq 0$.

(c) Suppose that $A$ is linear. Then $S_A(t) = e^{-tA}$ for any $t \geq 0$. Moreover $u = u_A(\cdot,x,w)$ is a mild solution of \((2.6)\) in the sense of the Duhamel formula, i.e.

\[
(2.9) \quad u(t) = e^{-tA}x + \int_0^t e^{-(t-s)A}w(s) \, ds, \quad t \geq 0.
\]

(d) Let $u = u_A(\cdot;x,w)$ and fix a small $h > 0$. For all $0 \leq s \leq t \leq T-h$ and $(z,v) \in \text{Gr}(A)$, one has

\[
\left\| u(t+h) - z \right\| \leq e^{(t-s)\alpha} \left\| u(s+h) - z \right\| + \frac{1}{t-h} \int_s^{t+h} e^{(t-s-h)\alpha} \left[ u(\xi) - z, w(\xi) - v \right] + d\xi
\]

where $\bar{w}(t) := w(t+h), \ t \in [0,T-h]$. Thus $y$ is an integral solution to the problem $\dot{y} \in -Ay + \bar{w}$, $y(0) = u(h)$, i.e. $y = u_A(\cdot;u(h),\bar{w})$. In other words we have the formula

\[
(2.10) \quad u_A(\cdot+h;x,w) = u_A(\cdot;u_A(h;x,w),\bar{w}(\cdot+h)), \quad t \in [0,T-h]
\]

and, in view of \((2.8)\), for any $0 \leq s \leq t \leq T-h$

\[
\left\| u(t+h) - u(t) \right\| \leq e^{(t-s)\alpha} \left\| u(s+h) - u(s) \right\| + \frac{1}{t-s} \int_s^{t} e^{(t-s-z)\alpha} \left[ u(z+h) - u(z), w(z+h) - w(z) \right] + dz
\]

\[
(2.11) \quad \leq e^{(t-s)\alpha} \left\| u(s+h) - u(s) \right\| + \frac{1}{t-s} \int_s^{t} e^{(t-s-z)\alpha} \left\| w(z+h) - w(z) \right\| dz.
\]

2.2.3. Continuous perturbations. Let $A$ be an $\alpha$-$m$-accretive operator and $f : \Omega \to X$, where $\Omega \subset X$, be continuous. A continuous $u : [0,T] \to \Omega$, where $T > 0$, is an integral (resp. strong) solution to \((1.1)\), i.e.

\[
(2.12) \quad \begin{cases} 
\dot{u} \in -Au + f(u) \\
u(0) = x \in D(A) \cap \Omega,
\end{cases}
\]

if $u$ is an integral (resp. strong) solution to \((2.6)\) with $w := f \circ u$ \(^3\). A continuous function $u : [0,\tau) \to X$, $0 < \tau \leq \infty$, is an integral solution to \((2.12)\) if for any $0 < T < \tau$, $u$ restricted to $[0,T]$ is an integral solution to \((2.12)\) on $[0,T]$. An integral solution $u : [0,\tau) \to X$ is noncontinuable if it has no extension to a solution defined on the interval $[0,\tau')$ with $\tau' > \tau$.

**Remark 2.5.** (1) Along with \((2.12)\) consider the problem

\[
(2.13) \quad \begin{cases} \dot{u} \in -Bu + g(u) \\
u(0) = x \in D(A),
\end{cases}
\]

\(^3\)This definition makes sense since here $w \in L^1([0,T],X)$. More generally (if $f$ is not assumed to be continuous), we can say that $u$ is an integral solution of \((2.12)\) if $f \circ u \in L^1([0,T],X)$ and $u$ is an integral solution to \((2.6)\) with $w = f \circ u$. 
where $B = A + \alpha I$ and $g(u) = \alpha u + f(u), u \in \Omega$. $B$ is $m$-accretive. We claim that integral solutions to (2.12) and (2.13) coincide. Clearly it is sufficient to show that if $u : [0, T] \rightarrow \Omega$ solves (2.13), then it solves (2.12). By definition $u$ is an integral solution to $\dot{u} \in -Bu + v, u(0) = x$, where $v = g \circ u$. In view of [4, Chapter 4.1] $u$ is the uniform limit of $\varepsilon$-DS-approximate solutions $u^\varepsilon$ as $\varepsilon \to 0^+$, i.e. sup$_{t \in [0, T]} \| u(t) - u^\varepsilon(t) \| < \varepsilon$. Here by an $\varepsilon$-DS-approximate solution to (2.13) we mean a step function $u^\varepsilon$ with

$$u^\varepsilon(0) = x, u^\varepsilon(t) = u_k \text{ on } (t_{k-1}, t_k \wedge T) \text{ for } k = 1, \ldots, m,$$

where $0 = t_0 < t_1 < \ldots < t_{m-1} < T \leq t_m, t_k - t_{k-1} \leq \varepsilon$ for $k = 1, \ldots, m$, and

$$\frac{u_k - u_{k-1}}{t_k - t_{k-1}} \in -Bu_k + v \text{ for } k = 1, \ldots, m,$$

with $v_1, \ldots, v_m \in X$ such that

$$\sum_{k=1}^m \int_{t_{k-1}}^{t_k \wedge T} \| v_k - v(t) \| \leq \varepsilon.$$

It is clear that for all $k = 1, \ldots, m$,

$$\frac{u_k - u_{k-1}}{t_k - t_{k-1}} \in -Au_k + v - \alpha u_k,$$

and for $t \in [t_{k-1}, t_k \wedge T],

$$\| (v_k - \alpha u_k) - w(t) \| \leq \| v_k - v(t) \| + \| \alpha u(t) - \alpha u_k \| \leq \| v_k - v(t) \| + |\alpha| \varepsilon.$$

Therefore

$$\sum_{k=1}^m \int_{t_{k-1}}^{t_k \wedge T} \| v_k - \alpha u_k - w(t) \| \leq \varepsilon \left( 1 + |\alpha| T \right).$$

This shows that $u^\varepsilon$ is an $\varepsilon \left( 1 + |\alpha| T \right)$-DS-approximate solution of (2.6) and, hence, $u$ is an integral solution to (2.12). As a conclusion we see while studying (2.12) that one can shift the part $\alpha I$ from the perturbation term to the accretive operator and vice-versa. In particular we may, without any loss of generality, consider only $m$-accretive operators $A$.

(2) Suppose $u : [0, \tau) \rightarrow \Omega$, where $\tau > 0$, is an integral solution of (2.12) and there is $M > 0$ such that $\| f(u(t)) \| \leq M$ for any $0 \leq t < \tau$. Putting $w(t) := f(u(t)), t \in [0, \tau)$, we have $w \in L^1([0, \tau], X)$. Hence there is a continuous function $\tilde{u} : [0, \tau] \rightarrow X$ being an integral solution on $[0, \tau]$ of the problem

$$\dot{u} \in -Au + w, u(0) = x.$$ 

Evidently $u = \tilde{u}$ on $[0, \tau)$. Hence $\lim_{t \to \tau^-} u(t) = \tilde{u}(\tau) \in \overline{\Omega}$ exists.

(3) Let $u : [0, T] \rightarrow X$ be a solution to (2.12), i.e. $u = u_A(\cdot, x, w)$, where $w = f \circ u$. Let $0 < h < T$. Since $w(\cdot + h) = f \circ u(\cdot + h)$ on $[0, T - h]$, we see in view of (2.10) that $\dot{y} = u(\cdot + h)$ on $[0, T - h]$ is a solution to the problem

$$\dot{y} \in -Ay + f(y), y(0) = u(h).$$

This proves a "semigroup" property of sorts of integral solutions to (2.12). Namely, putting

$$S_A(t; f)(x) := \{ u(t) \mid u \text{ is an integral solution to } (2.12), \ t \in [0, T) \}$$

we have

$$S_A(t + h; f)(x) = S_A(t; f)\left( S_A(h; f)(x) \right)$$

for $t, h \in [0, T]$ such that $t + h \leq T$. 

\qed
The following result seems to be a well-known folklore. It seems, however, to be more convenient than the corresponding result [4 Theorem 4.8]. We sketch a proof for the sake of completeness.

**Proposition 2.6.** Assume that $f : \Omega \to X$ is locally Lipschitz continuous, $A : D(A) \to X$ is $m$-accretive and $x \in D(A) \cap \Omega$. Then:

(a) there is $T > 0$ and a unique integral solution $u : [0, T] \to \Omega$ of (2.12).
(b) If $u : [0, T] \to \Omega$ is an integral solution of (2.12) with $x \in D(A) \cap \Omega$, then $u$ is Lipschitz continuous.
(c) If $X$ is reflexive, $u : [0, T] \to \Omega$ is an integral solution of (2.12) with $x \in D(A) \cap \Omega$, then $u$ is a strong solution and $u \in W^{1, \infty}([0, T], X)$.

**Proof.** (a) There is $R > 0$ such that $D := D(x, R) \subset \Omega$ and $f$ is Lipschitz with the Lipschitz constant $\ell > 0$ on $D$. Take $y \in D(A) \cap D(x, R/3)$ and $p \in Ay$. Let $M := \sup_{u \in D} \|f(u) - p\|$ and

$$
Y := \{u \in C([0, T], X) \mid u(t) \in D, t \in [0, T]\},
$$

where $T = \frac{R}{3M}$. Let $Y$ be endowed with the complete metric

$$
d(u, v) := \sup_{t \in [0, T]} e^{-\ell t} \|u(t) - v(t)\|, \quad u, v \in Y.
$$

Consider a map $N : Y \to C([0, T], X)$ given by

$$
[Nu](t) = u_A(t; x, w_u), \quad u \in Y,
$$

where $w_u := f \circ u$, i.e. $Nu$ is the integral solution to (2.12) with $w = w_u$. This map is well-defined since $w_u \in C([0, T], X) \subset L^1([0, T], X)$. Actually $N : Y \to Y$ and it is a (Banach) contraction. Indeed: for $u \in Y$ and $t \in [0, T]$

$$
\|w_u(\tau) - p\| = \|f(u(\tau)) - p\| \leq M, \quad \tau \in [0, t],
$$

and, in view of (2.7)

$$
\|\{Nu\}(t) - x\| \leq \|\{Nu\}(t) - y\| + \|x - y\| \leq 2\|x - y\| + \int_0^t \|w_u(z) - p\| dz \leq \frac{2}{3}R + MT \leq R,
$$

i.e. $Nu \in Y$. For $u, v \in Y$ in view of (2.8) we have

$$
\|\{Nu\}(t) - \{Nv\}(t)\| \leq \int_0^t \|w_u(z) - w_v(z)\| dz \leq \ell \int_0^t \|u(z) - v(z)\| dz
$$

$$
\leq \ell d(u, v) \int_0^t e^{\ell z} dz = (e^{\ell t} - 1)d(u, v)
$$

and thus $d(Nu, Nv) \leq cd(u, v)$ with $c = 1 - e^{-\ell T} < 1$. Hence there is $u \in Y$ such that $Nu = u$, i.e. $u$ is an integral solution to (2.12). Its uniqueness is straightforward.

(b) To show the Lipschitz continuity of the solution $u = u_A(\cdot; x, w) : [0, T] \to \Omega$, where $x \in D(A)$ and $w := f \circ u$, take $0 \leq t < s \leq T$ and let $h := s - t$. Arguing as in Remark 2.4 (d) we have by
Proof. Lipschitz continuous. Take \( x \) of \((2.17)\) we find \( \lambda \) for all \( z \) and let eqs. \((2.7), (2.8) \) and \((2.10)\) that for \( \{ \\}\) lem \((2.14)\) w locally Lipschitz functions. This implies that Theorem 2.7. We claim that \( T \) ake any \( n \) In view of \([23, \text{Lemma 1}]\) for any \( w \) and \( f \) maps bounded sets into bounded ones. For any \( x_0 \in D(A) \cap \Omega \) and \( 0 < r < R \) such that \( D(x_0, R) \subset \Omega \), there is \( T > 0 \) such that for any \( x \in D(x_0, r) \) the problem \((2.12)\) has a unique solution on \([0, T] \). If \( x \in D(x_0, r) \cap D(A) \), then this solution is Lipschitz continuous.

\[ \| u(s) - u(t) \| \leq c e^{\ell T} (s - t). \]

(c) Observe that \( w = f \circ u : [0, T] \to X \) is absolutely continuous as the superposition of two locally Lipschitz functions. This implies that \( w \in W^{1,1}([0, T], X) \) in view of the Komura theorem (see \([21]\)). The assertion follows from Subsection 2.2.2(b).

**Theorem 2.7.** Let \( A : D(A) \to X \) be m-accretive and suppose that \( f : \Omega \to X \) is continuous and

\[ [u - v, f(u) - f(v)]_+ \leq \beta (\| u - v \|) \quad \text{for } u, v \in \Omega, \]

where \( \beta \) is a slow function. Moreover assume that \( f \) maps bounded sets into bounded ones. For any \( x_0 \in D(A) \cap \Omega \) and \( 0 < r < R \) such that \( D(x_0, R) \subset \Omega \), there is \( T > 0 \) such that for any \( x \in D(x_0, r) \) the problem \((2.12)\) has a unique solution on \([0, T] \). If \( x \in D(x_0, r) \cap D(A) \), then this solution is Lipschitz continuous.

**Proof.** In view of \([23, \text{Lemma 1}]\) for any \( n \in \mathbb{N} \) there is a locally Lipschitz \( f_n : \Omega \to X \) such that \( \| f(x) - f_n(x) \| \leq \frac{1}{2^n} \) for \( x \in \Omega \). Then for any \( n \in \mathbb{N} \) and \( u, v \in \Omega \)

\[ [u - v, f_n(u) - f_n(v)]_+ \leq \beta (\| u - v \|) + \frac{1}{n} \leq \beta (\| u - v \|) + 1. \]

Take \( x_0 \in D(A) \cap \Omega \), \( 0 < r < R \) such that \( D(x_0, R) \subset \Omega \). Let \( \varepsilon < R - r \). Since \( J_{\lambda} x_0 \to x_0 \) as \( \lambda \to 0^+ \) we find \( \lambda_0 > 0 \) such that \( \| x_0 - y_0 \| < \varepsilon \), where \( y_0 = J_{\lambda_0} x_0 \). Let \( v_0 := A_{\lambda_0} x_0 \); then \( y_0 \in \Omega \) and \( v_0 \in Ay_0 \) (see Subsection 2.2.1(e)). Let \( \gamma := \| v_0 - f(y_0) \| + 2 \) and \( r_0 = r + \varepsilon \).

Let us consider the (scalar) problem

\[ \dot{z} = \beta(z) + \gamma, \quad z(0) = r_0, \]

and let \( z : [0, \tau) \to [0, +\infty) \), where \( 0 < \tau \leq \infty \), be the maximal solution to \((2.16)\). There exists \( 0 < T < \tau \) (\( \tau \) comes from the definition of a slow function, see Subsection 2.1.4) such that \( z(t) \leq R \) for all \( t \in [0, T] \).

Take any \( n \in \mathbb{N} \), an arbitrary \( x \in D(x_0, r) \) and a noncontinuable integral solution \( u_n \) to the problem

\[ \begin{cases} \dot{u} = -Au + f_n(u) \\ u(0) = x \in D(A) \end{cases}, \]

defined on \( [0, \tau_n) \).

We claim that \( T \leq \tau_n \) for any \( n \in \mathbb{N} \). Let

\[ g(t) := \| u_n(t) - y_0 \|, \quad 0 \leq t < \tau_n. \]
Then, by definition, for $0 \leq t < \tau_n$, 
\[
g(s) = \|u_n(s) - y_0\| \leq \|u_n(t) - y_0\| + \int_t^s [u_n(z) - y_0, f_n(u_n(z)) - v_0]_+ dz
\]
\[
= g(t) + \int_t^s [u_n(z) - y_0, f_n(u_n(z)) - f_n(y_0) + f_n(y_0) - v_0]_+ dz 
\]
\[
\leq g(t) + \int_t^s (\beta(\|u_n(z) - y_0\|) + 1 + \|f_n(y_0) - v_0\|) dz
\]
\[
\leq g(t) + \int_t^s (\beta(g(z)) + 1 + \|f(y_0) - v_0\| + \|f_n(y_0) - f(y_0)\|) dz 
\]
\[
\leq g(t) + \int_t^s (\beta(g(z)) + \gamma) dz.
\]
Hence
\[
D^+g(t) \leq \beta(g(t)) + \gamma, \quad 0 \leq t < \tau_n.
\]
Since $g(0) = \|u_n(0) - y_0\| = \|x - y_0\| \leq \|x_0 - y_0\| + \|x_0 - x\| \leq r_0$ this, in view of Lemma 2.1, implies that 
\[
g(t) \leq z(t) \quad \text{for} \quad 0 \leq t < \tau_n \land \tau.
\]
If, for some $n \in \mathbb{N}$, $\tau_n < T$, then the set \( \{u_n(t) \mid 0 \leq t < \tau_n \} \) is bounded and, hence, so is the set \( \{f(u_n(t)) \mid 0 \leq t < \tau_n \} \) as well as the set \( \{f_n(u_n(t)) \mid 0 \leq t < \tau_n \} \). This, in view of Remark 2.5 (2) implies that $\lim_{n \to \tau_n} u_n(t)$ exists., i.e. $u_n$ is continuable. The contradiction proves our claim.

Now we shall show that the functional sequence $(u_n)$ converges uniformly. In view of (2.3) we have that for any $0 \leq t \leq T$
\[
\|u_n(t) - u_m(t)\| \leq \int_0^t \left[ u_n(z) - u_m(z), f_n(u_n(z)) - f_m(u_m(z)) \right]_+ dz 
\]
\[
\leq \int_0^t \left( u_n(z) - u_m(z), f(u_n(z)) - f(u_m(z)) \right)_+ + \left( \frac{1}{2n} + \frac{1}{2m} \right) dz 
\]
\[
\leq T \left( \frac{1}{2n} + \frac{1}{2m} \right) + \int_0^t \beta(\|u_n(z) - u_m(z)\|) dz.
\]

Let $M$ come from the definition of slow function (see Subsection 2.1.4). Since $\beta$ is a slow function we get 
\[
\|u_n(t) - u_m(t)\| \leq T \left( \frac{1}{2n} + \frac{1}{2m} \right) M \to 0 \quad \text{as} \quad n, m \to \infty.
\]

Let $u(t) := \lim_{n \to \infty} u_n(t)$, $t \in [0, T]$. Then $u : [0, T] \to X$ is an integral solution to (2.12). Indeed, for any $(y, v) \in \text{Gr}(A)$, $0 \leq s \leq t \leq T$ and any $n \in \mathbb{N}$,
\[
(2.18) \quad \|u_n(t) - y\| \leq \|u_n(s) - y\| + \int_s^t [u_n(z) - y, f_n(u_n(z)) - v]_+ dz.
\]
Passing in (2.18) with $n \to \infty$, using the upper semicontinuity of $[\cdot, \cdot]_+ : X \times X \to \mathbb{R}$ and the sup-Fatou lemma we get that 
\[
\|u(t) - y\| \leq \|u(s) - y\| + \int_s^t [u(z) - y, f(u(z)) - v]_+ dz.
\]
This shows that $u$ is an integral solution to (2.12) as required. Condition (2.14) immediately implies that $u$ is a unique solution to (2.12).
To show the Lipschitz continuity of $u$ starting at $x \in D(x_0,r) \cap D(A)$ let $0 \leq t < s \leq T$, $h := s - t$ and $w(z) := f(u(z))$ for $z \in [0,T]$. As before, for $v \in Ax$,

$$
\|u(t+h) - u(t)\| \leq \|u(h) - x\| + \int_0^h \left[ \|u(z+h) - u(z)\|, f(u(z+h)) - f(u(z)) \right]_+ dz \\
\leq \int_0^h \|w(z) - v\| \, dz + \int_0^h \beta \left( \|u(z+h) - u(z)\| \right) \, dz \\
\leq ch + \int_0^h \beta \left( \|u(z+h) - u(z)\| \right) \, dz.
$$

Hence $\|u(t+h) - u(t)\| \leq chM$, where $c := \sup_{z \in [0,T]} \|w(z) - v\|$. \hfill $\square$

**Remark 2.8.** (1) Both Proposition 2.6 and Theorem 2.7 hold true for arbitrary quasi $m$-accretive operators. The first statement follows immediately from Remark 2.5 (1). The second follows from the fact that if $A$ is $\alpha$-accretive and condition (2.14) holds, then the function $f + \alpha I$ also satisfies (2.14) with the slow function $s \mapsto \beta(s) + \alpha s$ (see Remark 2.3).

(2) Assume that $X$ is reflexive. In view of Proposition 2.6, we have that for $x \in D(x_0,r)$ an integral solution $u : [0,T] \to \Omega$ to (2.12) is the uniform limit of a sequence of strong solutions to (2.17).

(3) If above $f : X \to X$ and $\beta(s) = bs$ for some $b > 0$, then $-f$ is $b$-accretive (it is even strongly $b$-accretive). Due to the Martin theorem [4, page 104], $f$ is in fact $b$-$m$-dissipative. Thus, by [4, Theorem 3.1], $A - f$ is quasi $m$-accretive. Therefore, for any $x \in D(A)$, $S_{A-f}(\cdot) x : [0,\infty) \to X$ is well-defined. Let $u : [0,\infty) \to X$ be an integral solution to (2.12), i.e. $u = u_A(\cdot;x,w)$, where $w := f \circ u$. We shall show that $u = S_{A-f}(\cdot)x$. First observe that, due to Remark 2.5 (2) without loss of generality we may assume that $b = 0$. Assuming that $f$ is strongly dissipative we have that for any $(y,v) \in \text{Gr}(A)$ and $0 \leq s < t$

$$
\|u(t) - y\| \leq \|u(s) - y\| + \int_s^t \left[ u(z) - y, f(u(z)) - v \right]_+ dz.
$$

But

$$
\left[ u(z) - y, f(u(z)) - v \right]_+ \leq \left[ u(z) - y, f(u(z)) - f(y) \right]_+ + \left[ u(z) - y, f(y) - v \right]_+ \\
\leq \left[ u(z) - y, f(y) - v \right]_+.
$$

Hence

$$
\|u(t) - y\| \leq \|u(s) - y\| + \int_s^t \left[ u(z) - y, - (v - f(y)) \right]_+ dz.
$$

Since $v \in Ay$ was arbitrary, we see that $v - f(y)$ is an arbitrary element of $(A - f)(y)$. This shows that $u = S_{A-f}(\cdot)x$.

We complete the section with the result allowing us to treat an integral solution of (1.1) as an integral (or even strong) one in a suitable greater phase space (see Remark 1.5 (2)).

**Proposition 2.9.** Suppose that $Y$, $A_Y$ and $V_Y$ are as in Remark 1.5 (1). If $u : [0,T] \to X$ is an integral solution of (1.1) with $u(0) \in D(A)$, then $u \in W^{1,1}([0,T],Y)$, $u(t) \in D(A_Y)$ and

$$
\dot{u}(t) \in -A_Y u(t) + f(u(t)),
$$

for a.e. $t \in [0,T]$. 


Proof. Observe that \( w := f \circ u \in L^1([0, T], Y) \) and that, since the part of \( A \) in \( X \) is equal to \( A \), we have \( (I + \lambda A_Y)^{-1} v = (I + \lambda A)^{-1} v \in D(A) \) for any \( v \in X \) and \( \lambda > 0 \). By the construction of integral solutions (see [4, Subsection 4.1]) it is easily seen that \( u \) is also an integral solution of

\[
\dot{V}(t) = A_Y V(t) + w(t), \quad t \in [0, T].
\]

Hence, in view of Proposition 2.6 (c) and the reflexivity of \( Y \), we infer that \( u \in W^{1, \infty}([0, T], Y) \) and \( u \) is a strong solution of (2.19). \( \square \)

$\newpage$

3. Differentiation along trajectories and proofs of Theorems 1.1–1.4

Let \( K \subset X \) be closed and \( x \in X \). Given a continuous curve \( u : [0, \tau) \to X, \tau > 0 \), such that \( u(t) = x \) for some \( t \in [0, \tau) \), the Dini derivative \( D_+(d_K \circ u)(t) \) measures the rate of changes of the distance of \( u(s) \) from \( K \) for \( s \) in a neighborhood of \( t \). It is easy to see for instance that \( D_+(d_K \circ u)(t) = 0 \) if and only if there exist sequences \( h_n \to 0^+ \) and \( v_n \to 0^+ \) such that \( d_K(u(t + h_n) + h_n v_n) \leq d_K(x) \) for all \( n \geq 1 \), i.e. \( u \) is tangential to the set \( K^a := \{ y \in X \mid d_K(y) \leq \alpha \} \), where \( \alpha := d_K(x) \). If \( u \) is (right) differentiable at \( t \), i.e. \( u'_+(t) := \lim_{h \to 0^+} h^{-1}(u(t+h) - u(t)) \) exists, then \( u \) is tangential to \( K^a \) if and only if the vector \( v = u'_+(t) \) is tangent to \( K^a \) at \( x \), i.e.

\[ D_+(d_K(x; v)) := \lim_{h \to 0^+} \frac{d_K(x + hv) - \alpha}{h} = 0; \]

in other words if and only if \( v \in T_K(x) \).

Let \( A : D(A) \to X \) be a quasi \( m \)-accretive operator, let \( V : X \to \mathbb{R} \) be a locally Lipschitz function representing \( K \) given by (1.7), let \( x \in D(A) \) and \( v \in X \). Suppose that \( u := u(\cdot; x, v) = S_{A_x}(\cdot)x : [0, \infty) \to X \), where \( A_x := A(\cdot) - v \), is the integral solution to (2.6) with \( w(\cdot) \equiv v \in X \) (see also Remark 2.8(1) with regard to the last equality). By the \( A \)-derivative of \( V \) at \( x \) in the direction \( v \) we mean the Dini type derivative

\[
D_A V(x; v) := \lim_{h \to 0^+} \frac{V(x + h u'_+(0)) - V(x)}{h} = D_+(V \circ u_A(\cdot; x, v))(0).
\]

Note that if \( x \in D(A) \), then the derivative \( D_A V(x; v) \) is finite since, by Remark 2.4(a), the function \( V \circ u \) is Lipschitz around 0. As above \( D_A V(x; v) \) measures the rate of growth of \( V \) along the integral curve \( u = u_A(\cdot; x, v) \). In particular if \( D_A V(x; v) > \alpha \), then there is \( \eta > 0 \) such that \( V(u(t)) > \alpha t + V(x) \) for \( 0 \leq t < \eta \). It is clear again that if \( \dot{u}_+(0) \) exists, then

\[
D_A V(x; v) = D_+ V(x; \dot{u}_+(0)).
\]

Indeed \( u(h) = x + h u'_+(0) + o(h) \) when \( h \to 0 \); hence

\[
h^{-1}\left|(V(u(h)) - V(x)) - (V(x + h u'_+(0)) - V(x))\right| \leq h^{-1} \ell |o(h)| \to 0 \text{ as } h \to 0,
\]

where \( \ell \) is the Lipschitz constant of \( V \) at \( x \).

For a general \( A, x \in \overline{D(A)} \) and \( v \in X \) the \( A \)-derivative \( D_A V(x; f(x)) \) is not easy to compute. The situation changes under additional assumptions on \( X \) if \( x \in D(A) \).

Proposition 3.1. (i) If \( X \) and \( X^* \) are uniformly convex, then for any \( x \in D(A) \) and \( v \in X \),

\[
D_A V(x; v) = D_+ V(x; v - y),
\]

with \( y = D(A) x + o(x) \).
where \( y \in Ax \) is such that \( y - v = (Ax - v)^0 \) is the element of the set \( Ax - v \) having minimal norm. The same holds true in an arbitrary Banach space provided \( A \) is a linear operator and then \( D_A(V(x,v) = D_+ V(x;v - Ax)\).

(ii) Assume that \( A \) is single-valued and both \( X, X^* \) are uniformly convex or \( X \) is an arbitrary Banach space but \( A \) is linear. If for \( x \in \overline{D(A)} \)

\[
(3.4) \quad V(S_A(t)x) \leq V(x) \text{ for all } t \geq 0,
\]

then

\[
(3.5) \quad D_A(V(x;v) \leq V^\circ(x;v) \text{ for any } x \in D(A), v \in X,
\]

where \( V^\circ(x;v) \) is the generalized Clarke derivative at \( x \) in the direction of \( v \) (see [13] Chapter 2.1]).

**Proof.** (i) If \( X \) and \( X^* \) are uniformly convex, then in view of Section 2.2.2 (b), \( u = u_A(\cdot;x,v) \) is a strong solution and \( \dot{u}_+(0) = (-Ax + v)^0 \) exists. Hence \( D_A(V(x;v) = D_+ V(x; -Ax + v) \) in view of \( (3.2) \).

If \( X \) is arbitrary but \( A \) is linear, then in view of Remark 2.4 (c) and (2.9)

\[
u_A(t;x,v) = e^{-tA}x + \int_0^t e^{-(t-s)A}v ds, \quad t \geq 0.
\]

Hence

\[
\dot{u}_+(0) = \lim_{t \rightarrow 0^+} \frac{e^{-tA}x - x}{t} + \frac{1}{t} \int_0^t e^{-sA}v ds = -Ax + v.
\]

and, consequently, \( D_A(V(x;v) = D_+ V(x;v - Ax) \).

(ii) By (i),

\[
D_A(V(x;v) = D_+ V(x;v - Ax) \leq V^\circ(x;v) + D_+ V(x; -Ax),
\]

where \( y \in Ax \) is as in \( (3.3) \). It is enough to see that \( D_A(V(x;0) \leq 0 \) in view of \( (3.4) \). To see this observe that if \( u := S_A(\cdot)x \), then \( u_+(0) = \lim_{t \rightarrow 0^+} h^{-1}(S_A(t)x - x) = -Ax \) exists. Hence by \( (3.4) \)

\[
D_+ V(x, -Ax) = D_A(V(x,0) = \lim_{h \rightarrow 0^+} \frac{V(S_A(h)x) - V(x)}{h} \leq 0.
\]

The following property enables to study the behavior of \( V \) along an integral solution curve to \( (1.1) \) without a prior knowledge of this solution.

**Proposition 3.2.** Let \( x_0 \in \overline{D(A)} \) and \( u : [0, \tau) \rightarrow X \) be an integral solution to \( (1.1) \). Then

\[
D_+(V \circ u)(t) = D_A(V(u(t); f(u(t)))) \quad t \in [0, \tau).
\]

**Proof.** Fix \( t \in [0, \tau) \), let \( x := u(t) \) and \( v := f(u(t)) \). Recall \( D_A(V(x;v) = D_+(V \circ u_A(\cdot;x,v))(0) \). In view of Remark 2.5 (3) \( u(t + h) = u(h;x,w) \), where \( w(h) := f(u(t + h)) \), for \( h \in [0,T - t) \). By \( (2.8) \) we have

\[
|V(u(t + h)) - V(u_A(h;x,v))| \leq \ell \|u(t + h) - u_A(h;x,v)\| = \\
\ell \|u_A(h;x,v) - u_A(h;x,v)\| \leq \ell e^{\alpha h} \int_0^h \|w(s) - v\| ds,
\]
where $\ell$ is the Lipschitz constant of $V$ around $x$. Since
\[ h^{-1} \ell e^{\alpha h} \int_0^h \| w(s) - v \| ds \to 0 \text{ as } t \to 0^+ \]
and
\[ V(u(t+h)) - V(u(t)) = V(u(t + h)) - V(u_A(h;x,v)) + V(u_A(h;x,v)) - V(x) \]
this yields that $D_+(V \circ u)(t) = D_A V(x;v) = D_A V(u(t);f(u(t)))$ as required. \hfill \Box

This gives us immediately the invariance criteria mentioned in Introduction.

**Proof of Theorem 1.2.** Suppose to the contrary that there is an integral solution $u : [0, \tau) \to X$ that leaves $K$, i.e. there is $T \in (0, \tau)$ such that $V(u(T)) > 0$. Let
\[ \bar{\tau} := \sup \{ t \in [0, T] \mid V(u(t)) \leq 0 \}. \]
Clearly $\bar{\tau} < T$, $V(u(\bar{\tau})) = 0$ and
\[ V(u(t)) > 0 \text{ for all } t \in (\bar{\tau}, T]. \]
In order to simplify the notation and without loss of generality we may assume that $\bar{\tau} = 0$ and $u(t) \in U(u(0))$ for $t \in [0, T]$. In view of Proposition 3.2 for any $t \in (0, T]$
\[ (3.6) \quad D_+(V \circ u)(t) = D_A V(u(t);f(u(t))) \leq \omega(V(u(t))). \]
This however, by Lemma 2.1 implies that $V(u(t)) = 0$ for all $t \in [0, T]$ since $\omega$ is a uniqueness function. \hfill \Box

**Proof of Theorem 1.3.** It is sufficient to slightly modify the proof of Theorem 1.1. Namely, use condition (1.8) from the assumption to have the counterpart of the relation (3.6) directly, i.e.
\[ D_+(V \circ u)(t) \leq \omega(V(u(t))) \text{ for a.e. } t \in (0, T]. \]
This allows us to use Lemma 2.1 to complete the proof. \hfill \Box

**Proof of Theorem 1.4.** Suppose to the contrary that there is an integral solution $u_0 : [0, \tau) \to X$ of (1.1) that leaves $K$, i.e. there is $T \in (0, \tau)$ such that $V(u_0(T)) > 0$. As in above proofs, we may assume that $x_0 = u_0(0) \in \partial K$ and $u_0(t) \in U := U(u_0(0))$ for all $t \in [0, T]$. Without loss of generality we may also assume that $V$ is Lipschitz on $D(x_0, \delta)$, where $\delta > 0$ is given by assumption, with the constant $\ell > 0$. Moreover, take $0 < r < \delta$ and a sequence $(x_n)$ such that $x_n \to x_0$, $x_n \in D(A) \cap (D(x_0, r) \setminus K_V)$. In view of Theorem 2.7 there is a time $T' \leq T$ such that for all (sufficiently large if necessary) $n \geq 1$ there is an integral solution $u_n : [0, T'] \to \Omega$ to the problem
\[ (3.7) \quad \begin{cases} 
\dot{u} \in -Au + f(u) \\
u(0) = x_n \in D(A),
\end{cases} \]
With no loss of generality we may assume that $T' = T$. It is clear that the sequence $(u_n)$ converges uniformly to $u_0$ on $[0, T]$ since $u_0$ is a unique integral solution to (1.1). According to Proposition 2.6 (b), (c) and Remark 2.8 (1) for each $n \geq 1$ the function $u_n$ is locally Lipschitz and, due to the reflexivity of $X$, $u_n$ is a strong solution. Therefore there is a full-measure set $S \subset [0, T]$ such that
Since \( \omega \) is the Lipschitz constant of \( V \) around \( x_0 \). Therefore

\[
(V \circ u_n)'(z) = \lim_{h \to 0^+} \frac{V(u_n(z+h)) - V(u_n(z))}{h} \\
\leq \lim_{h \to 0^+} \frac{\ell}{h} \int_0^h \|w_n(\xi) - v_n\| \, d\xi + \liminf_{h \to 0^+} \frac{V(u_n(z), v_n) - V(u_n(z))}{h} \\
= D_A V(u_n(z); f(u_n(z))) \leq \omega(V(u_n(z))).
\]

This implies that for all \( t \in [\varepsilon, T] \)

\[
(3.8) \quad V(u_n(t)) = V(u_n(\varepsilon)) + \int_{\varepsilon}^t (V \circ u_n)'(z) \, dz \leq V(u_n(\varepsilon)) + \int_{\varepsilon}^t \omega(V(u_n(z))) \, dz.
\]

Passing with \( \varepsilon \to 0^+ \) and with \( n \to \infty \) we get for all \( t \in [0, T] \)

\[
V(u_0(t)) \leq \int_0^t \omega(V(u_0(z))) \, dz.
\]

Since \( \omega \) is nondecreasing, we have

\[
\left( \int_0^t \omega(V(u_0(z))) \, dz \right)' = \omega'(V(u_0(t))) \leq \omega\left( \int_0^t \omega(V(u_0(z))) \, dz \right),
\]

which implies that \( \int_0^t \omega(V(u_0(z))) \, dz \equiv 0 \) and, in consequence, due to Lemma 2.1 we get \( V(u_0(t)) = 0 \) for all \( t \in [0, T] \): a contradiction. \[ \Box \]

**Proof of Theorem 1.4** A slight modification of the proof of Theorem 1.3 is sufficient. Without the reflexivity of \( X \) we still know that \( u_n \) are locally Lipschitz in view of Proposition 2.6 (b) and that \( V \circ u_n \) is locally Lipschitz and, hence, a.e. differentiable. In particular, for a.e. \( t \in [\varepsilon, T] \),

\[
(V \circ u_n)'(t) = D_+(V \circ u_n)(t).
\]

Applying assumption (iii)” one has (3.8), which allows us to complete the proof by following the steps of the previous one. \[ \Box \]
4. Strict invariance. Proofs of theorems 1.6–1.9

Recall that a closed \( K \subset \Omega \cap \overline{D(A)} \) is strictly invariant with respect to (1.1) if all solutions starting at \( x_0 \in K \) stay for \( t \in (0, \tau_0) \) in the interior \( \text{int} K \) of \( K \). In particular, a necessary condition for the strict invariance is that \( \text{int} K \) is invariant because solutions cannot return from \( \text{int} K \) to \( \partial K \).

At the beginning of the discussion we state conditions implying this necessary condition, i.e. the invariance of \( \text{int} K \). Recall that \( K = K_V \) for some \( V \) – see (1.6), and assume that \( K^0_V = \text{int} K \).

**Proof of Theorem 1.6.** Suppose to the contrary that there is an integral solution \( u : [0, \tau) \to X \) of (1.1) starting at \( x \in K^0 \) and leaving it, i.e. \( V(u(t)) = 0 \) for some \( t \in (0, \tau) \). Let

\[
T := \inf \{ t \in [0, \tau) \mid V(u(t)) = 0 \}.
\]

Clearly \( T > 0 \), \( V(u(t)) < 0 \) for all \( t \in [0, T) \), and \( u(T) \in \partial K \). There is \( \tilde{t} \in [0, T) \) such that \( u(t) \in U(u(T)) \) for \( t \in [\tilde{t}, T] \) and without loss of generality we may suppose that \( \tilde{t} = 0 \). In view of Proposition 3.2 and assumption (iv), for any \( t \in [0, T) \),

\[
D_+(V \circ u(t)) = D_A V(u(t); f(u(t))) \leq \omega(-V(u(t))).
\]

By Lemma 2.1(2) applied to \( v : [-T, 0] \to X \) given by \( v(s) := (V \circ u)(s + T) \) for \( -T \leq s \leq 0 \), we see that (4.1) implies that \( V(u(t)) = 0 \) for all \( t \in [0, T] \); a contradiction.

**Proof of Theorem 1.7.** Suppose to the contrary that there is an integral solution \( u_0 : [0, \tau) \to X \) of (1.1) and \( 0 < T < \tau \) such that \( V(u_0(T)) = 0 \) and \( V(u_0(t)) < 0 \) for all \( t \in [0, T) \). Without loss of generality we may assume that \( u_0(t) \in U := U(z_0) \) for all \( t \in [0, T) \), where \( z_0 = u_0(T) \). Clearly \( z_0 \in \partial K \).

There is \( \delta > 0 \) such that \( V \) is Lipschitz on \( D(z_0, \delta) \subset U \) with the constant \( \ell > 0 \). Take \( 0 < r < \delta \). In view of Theorem 2.7 there is \( 0 < T' < \tau - T \) such that all solutions to (1.1) starting in \( D(z_0, r) \) are defined on \( [0, T'] \). Take \( t_0 \in [0, T) \) such that \( u_0(t) \in D(z_0, r) \) for every \( t \in (t_0, T) \) and \( T' := T - t_0 \leq T' \). Now, take a sequence \( \{ x_n \} \) such that \( x_n \to x_0 := u_0(t_0) \), \( x_n \in D(A) \cap (D(z_0, r) \cap K^0_V) \). For all (sufficiently large if necessary) \( n \geq 1 \) there is an integral solution \( u_n : [0, T''] \to \Omega \) to problem (3.7).

It is clear that the sequence \( \{ u_n \} \) converges uniformly to \( \bar{u} := u_0(t_0 + \cdot) \) on \([0, T'']\) since \( u_0 \) is a unique integral solution to (1.1). Each of solutions \( u_n \) is Lipschitz continuous and \( u_n(t) \in D(A) \) for all \( t \in [0, T''] \). Hence there is a set \( S \subset [0, T] \) of full measure such that for all \( n \geq N \) \( u_n(t) \in D(A) \) and \( V \circ u_n \) is differentiable at \( t \in S \).

Take a small \( 0 < \varepsilon < T'' \). Since \( \{ u_n \} \) converges uniformly to \( \bar{u} \), we may assume that \( u_n(t) \in U \cap K^0_V \) for all \( n \geq 1 \) and \( t \in [0, T'' - \varepsilon) \). Let \( n \geq 1 \) and \( z \in S, 0 < z \leq T'' - \varepsilon \). Then, in view of Remark 2.4 and (2.10), \( u_n(z + \cdot) = u_A(\cdot; u_n(z), w_n) \), where \( w_n = f(u_n(z + \cdot)) \), and we can repeat arguments from the proof of Theorem 1.3 to obtain

\[
(V \circ u_n)'(z) \leq D_A V(u_n(z); f(u_n(z))) \leq \omega(-V(u_n(z))).
\]

This implies that for all \( t \in [0, T'' - \varepsilon] \)

\[
V(u_n(t)) = V(u_n(T'' - \varepsilon)) - \int_t^{T'' - \varepsilon} (V \circ u_n)'(z) \, dz \geq V(u_n(T'' - \varepsilon)) - \int_t^{T'' - \varepsilon} \omega(-V(u_n(z))) \, dz.
\]
Passing with $\varepsilon$ to 0 and with $n$ to infinity we get for all $t \in [0,T'']$

$$-V(\bar{u}(t)) \leq \int_0^t \omega \left(-V(\bar{u}(z))\right) dz$$

i.e. $0 \leq -V(u_0(t)) \leq \int_0^T \omega \left(-V(u_0(z))\right) dz$ for all $t \in [t_0,T]$. As in the proof of the previous theorem this implies that $V(u_0(t)) = 0$ for all $t \in [t_0,T]$; a contradiction. \hfill $\Box$

Finally we are in a position to prove the strict invariance results.

**Proofs of Theorem 1.8 and Theorem 1.9** From assumptions (1.17) and (1.18) (resp. (1.19) and (1.18)) and conclusions of theorems 1.1 and 1.6 (resp. 1.3 and 1.7) it follows that both $K = K_V$ and $K_V^0$ are invariant. Suppose that there is an integral solution $u$ of (1.1) such that $u(t) \in \partial K$ for some interval $[0,T]$, $T > 0$. Then $(V \circ u)(t) = 0$ for all $t \in [0,T]$, which implies (see Proposition 3.2) that

$$DAV\left(u(t);f(u(t))\right) = D+(V \circ u)(t) = 0, \text{ for } t \in [0,T);$$

a contradiction. \hfill $\Box$

5. Applications

5.1. Impulsive differential equations with state-dependent impulses. Problems with state-dependent impulses of the form

$$\tag{5.1} \begin{cases} y(t) \in F(t,y(t)), & t \in [0,T],\ t \neq \tau_j(y(t)), \ j = 1,\ldots,k, \\ y(0) = x_0, \\ y(t^+) = y(t) + I_j(y(t)) \text{ for } t = \tau_j(y(t)), \ j = 1,\ldots,k, \end{cases}$$

where $T > 0$, $F : [0, T] \times X \to X$ is a set-valued dynamics, for $j = 1,\ldots,k$, $\tau_j : X \to (0,T)$ is a barrier function and $I_j : X \to X$ an impulse function, meet a considerable interest recently. In order to characterize the suitable function space where solutions can be considered, one looks for sufficient conditions implying that every trajectory of (5.1) meets a barrier $\Gamma_j = \text{Gr}(\tau_j)$ exactly once. Note that if the global existence is achieved, then each barrier is hit at least once. One however demands that after the $j$-th jump a solution stays in the epigraph $\text{Epi}(\tau_j)$ of $\tau_j$, i.e. it immediately enters its interior and does not return to $\Gamma_j$; in other words one needs conditions implying that epigraph $\text{Epi}(\tau_j)$, $j = 1,\ldots,k$, is strictly invariant. Results from Section 4 do fit well to this problem if $F(t,y) = -Ay + f(t,y)$, where $A : D(A) \to X$ is an $m$-accretive operator.

Let us consider the following problem

$$\tag{5.2} \begin{cases} \dot{u} \in -Au + f(t,u), & t \in [0,T], \\ u(0) = x \in D(A), \end{cases}$$

where $f : \mathbb{R} \times X \to X$ is continuous. Let $\tau : X \to \mathbb{R}$ be a locally Lipschitz barrier function. By a solution to (5.2) we understand an integral solution $u : [0,T] \to X$, $T > 0$, to (2.6) with $w = f(\cdot,u(\cdot))$. \hfill $\Box$
Theorem 5.1. Assume that for every \((z, \theta) \in \text{Gr}(\tau)\) there are a neighborhood \(U = U(z, \theta)\) of \((z, \theta)\) and a uniqueness function \(\omega\) such that

\[
D_A \tau(x; f(t,x)) \leq \omega(\|\tau(x) - t\|) + 1 \text{ for } (t,x) \in U,
\]

\[
D_A \tau(x; f(t,x)) < 1 \text{ for } (t,x) \in U \cap \text{Gr}(\tau).
\]

If \(u: [0,T] \rightarrow X\) is a solution to (5.2) and \(\tau(x_0) \leq t\), then \(\tau(u(h)) < t + h\) for any \(0 < h \leq T\), i.e. \((t + h, u(h)) \in \text{Epi}(\tau)\) for \(h \in (0,T]\).

Proof. Define \(A : D(A) \rightarrow X := \mathbb{R} \times X\) by \(A(t,u) := (0, Au)\) for \((t,u) \in D(A) := \mathbb{R} \times D(A)\) and \(F : X \rightarrow X\) by \(F(t,x) := (1, f(t,x))\) for \((t,x) \in X\); \(X\) is a Banach space with the norm \(\| (t,x) \| := |t| + \|x\|\) for \((t,x) \in X\). It is immediate to see that \(A\) is \(m\)-accretive and \(F\) is continuous. A simple calculation shows that a continuous function \(u : [0,T] \rightarrow X\) is an integral solution to the problem

\[
\begin{aligned}
\dot{u} &\in -Au + w, \\
u(0) &= (t,x),
\end{aligned}
\]

with \((t,x) \in X\) and \(w = (1,w)\), where \(w \in L^1([0,T], X)\), if and only if \(u(h) = (t + h, u(h))\) for \(h \in [0,T]\), where \(u : [0,T] \rightarrow X\) is an integral solution to (2.6). In particular, \(u : [0,T] \rightarrow X\) is a solution of (5.2) if and only if \(u(h) = (t + h, u(h))\) is a solution to (5.5) with \(w = F \circ u\).

Let \(V : X \rightarrow \mathbb{R}\) be given by \(V(t,x) := \tau(x) - t, (t,x) \in X\). Clearly \(V\) is locally Lipschitz, \(K := \{(t,x) \in X \mid V(t,x) \leq 0\} = \text{Epi}(\tau), \text{int} K = \{(t,x) \in X \mid V(t,x) < 0\}\) and \(\partial K = \text{Gr}(\tau)\).

For a fixed \((t,x) \in X\) let \(\tilde{u}\) denote the solution to (5.5) with \(w(\cdot) \equiv F(t,x)\). Then \(\tilde{u} = (t + \cdot, \tilde{u}(\cdot))\) on \([0,T]\), where \(\tilde{u} = u_A (\cdot; x, f(t,x))\). Therefore

\[
D_A V((t,x); F(t,x)) = \liminf_{h \to 0^+} \frac{V(\tilde{u}(h)) - V(t,x)}{h} = \liminf_{h \to 0^+} \frac{\tau(u(h)) - t - h - \tau(x) + t}{h}
\]

\[
= \liminf_{h \to 0^+} \frac{\tau(u(h)) - \tau(x)}{h} - 1 = D_A \tau(x; f(t,x)) - 1.
\]

Hence, by (5.3) and (5.4),

\[
D_A V((t,x); F(t,x)) \leq \omega(|V(t,x)|) \text{ for each } (t,x) \in U,
\]

\[
D_A V((t,x); F(t,x)) < 0 \text{ for each } (t,x) \in U \cap \text{Gr}(\tau).
\]

In view of Theorem 1.8 the set \(K\) is strictly invariant with respect to (5.5). This completes the proof.

\[\square\]

Remark 5.2. Assumptions (5.3) and (5.4) allow to consider nonsmooth barriers and deal with integral (not strong) solutions in contrast to many other papers, see e.g. [20]. Note also that in our approach the operator \(A\) may be nonlinear and solutions need not be mild as, e.g., in [5].

Corollary 5.3. Assume that \(A\) and \(f\) satisfy the assumptions of Theorem 5.1 and let \(\tau_j : H \rightarrow (0,\infty)\) be locally Lipschitz functions such that (5.3) and (5.4) hold for \(\tau_j\) instead of \(\tau\), for every \(j = 1, \ldots, k\). We also assume standard conditions on barriers:

\((\tau 1)\) \(0 < \tau_j(x) < \tau_{j+1}(x)\) for each \(x \in X\) and \(j = 1, \ldots, k\),

\((\tau 2)\) \(\tau_j(x + I_j(x)) \leq \tau_j(x) < \tau_{j+1}(x + I_j(x))\) for each \(x \in X\) and \(j = 1, \ldots, k\).
Then any solution to (5.2) meets each barrier \( \Gamma_j \) at most once.

### 5.2. Obstacle problem for equations with one dimensional \( p \)-Laplace operator.
Consider the following nonlinear problem

\[
\begin{align*}
    u_t &= \Delta_p u + f(x,u), \quad x \in (0,l), \ t \in [0,T], \\
    u(0,t) &= u(l,t) = 0, \quad t \in [0,T],
\end{align*}
\]

where \( l > 0, \ T > 0, \ \Delta_p u := (|u_x|^{p-2}u_x)_x, \ p \geq 2 \), is the so-called \( p \)-Laplacian and \( f : [0,l] \times \mathbb{R} \to \mathbb{R} \). Suppose that functions \( m,M : [0,l] \to \mathbb{R} \) such that

\[
m \leq M, \ m(0) \leq 0 \leq M(0) \quad \text{and} \quad m(l) \leq 0 \leq M(l)
\]

represent the obstacles. In the so-called obstacle problem we look for conditions on \( f, \ m \) and \( M \) implying that for any continuous \( u_0 : [0,l] \to \mathbb{R} \) such that \( u_0(0) = u_0(l) = 0 \) and \( m(x) \leq u_0(x) \leq M(x) \) for \( x \in [0,l] \), all solutions of (5.6) starting at \( u_0 \) satisfy

\[
m(x) \leq u(x,t) \leq M(x) \quad \text{for all} \quad x \in [0,l], \ t \in [0,T].
\]

Let \( X = C_0[0,l] \) where \( C[0,l] \) stands for the space of continuous functions \( u : [0,l] \to \mathbb{R} \). Clearly \( X \) endowed with the sup-norm \( \| \cdot \|_{\infty} \) is a Banach space. Let \( A : D(A) \to X \) be given by \( Au := -\left( |u'|^{p-2}u' \right)' \) for \( u \in D(A) \), where

\[
D(A) := \left\{ u \in X \cap C^1(0,l) \mid \left( |u'|^{p-2}u' \right)' \in X \right\}.
\]

By \( \mathcal{C}_k(0,l), k \geq 1 \), we mean the space of functions from \( C[0,l] \) with continuous \( k \)-th derivative on \((0,l)\). \( D(A) \) is dense and the operator \( A \) is \( m \)-accretive (see e.g. [15, Lemma 6.1]). In addition to the above assumptions suppose that:

1. \( f : [0,l] \times \mathbb{R} \to \mathbb{R} \) is continuous, \( f(\cdot,0) \equiv 0 \) and \( f(x,\cdot) \) is locally Lipschitz continuous uniformly with respect to \( x \in [0,l] \), i.e. for any \( s \in \mathbb{R} \) there are \( L > 0 \) and \( \delta > 0 \) such that

\[
|f(x,s_1) - f(x,s_2)| \leq L|s_1 - s_2| \quad \text{for all} \quad x \in [0,l] \quad \text{and} \quad s_1, s_2 \in (s - \delta, s + \delta);
\]

2. \( m,M \in \mathcal{C}_[0,l] \cap \mathcal{C}_2(0,l), \ m \) is a subsolution and \( M \) is a supersolution of the stationary problem related to (5.6), i.e.

\[
-\Delta_p m(x) \leq f(x,m(x)), \quad x \in (0,l) \quad \text{and} \quad -\Delta_p M(x) \geq f(x,M(x)), \quad x \in (0,l).
\]

Condition (1) implies that the Nemytskii operator \( F : X \to X \) given by \( F(u)(x) := f(x,u(x)), \ u \in X \) and \( x \in [0,l] \), is well-defined and locally Lipschitz.

By a solution to (5.6) on \([0,T]\) we understand an integral solution \( u : [0,T] \to X \) of the problem

\[
\dot{u} = -Au + F(u), \quad t \in [0,T].
\]

It is clear that a solution \( u \) satisfies condition (5.8) if and only if \( u(t) \in K_m \cap K_M \) for all \( t \in [0,T] \), where

\[
K_m := \left\{ u \in X \mid u \geq m \text{ on } [0,l] \right\}, \quad K_M := \left\{ u \in X \mid u \leq M \text{ on } [0,l] \right\}.
\]

Clearly

\[
K_m = \left\{ u \in X \mid V_m(u) \leq 0 \right\}, \quad K_M = \left\{ u \in X \mid V_M(u) \leq 0 \right\},
\]

\( V_m(u) = \min\{u - m, 0\} \) and \( V_M(u) = \max\{m - u, 0\} \).
where \( V_m, V_M : X \to \mathbb{R} \) are given by
\[
(5.12) \quad V_m(u) := \frac{1}{2} \int_0^l (u-m)^2 \, dx, \quad V_M(u) := \frac{1}{2} \int_0^l (u-M)^2 \, dx \quad \text{for } u \in X.
\]

**Remark 5.4.** It can be easily verified that \( d_{K_m}(u) = \| (u-m)_- \|_{\infty} \) and \( d_{K_M}(u) = \| (u-M)_+ \|_{\infty} \).

Observe that neither \( V_m \neq d_{K_m} \) nor \( V_M = d_{K_M} \).

In order to verify condition (1.8) of Theorem 1.2 we shall follow the idea from Remark 1.5 (2) with \( Y = L^2(0,l) \) and the \( L^2 \)-realization of the \( p \)-Laplace operator. It is known (see e.g. [16, Th. 3.5]) that, for any \( u_0 \in X \), any integral solution of (5.11) has the following properties
\[
(5.13) \quad \dot{u}(t) = \Delta_p u(t) + F(u(t)) \quad \text{for a.e. } t \in [0,T],
\]
i.e.
\[
\dot{u}(t) = -A_{L^2} u(t) + F(u(t)), \quad \text{for a.e. } t \in [0,T],
\]
where \( A_{L^2} : D(A_{L^2}) \to L^2[0,l] \) is given by
\[
A_{L^2} u := -\Delta_p u, \quad D(A_{L^2}) := \{ u \in W^{1,p}(0,l) \mid \Delta_p u \in L^2[0,l] \}.
\]

It is well known that \( A_{L^2} \) is \( m \)-accretive (see e.g. [16, Lem. 3.4]) and clearly the part of \( A_{L^2} \) in \( X \) is equal to \( A \).

**Lemma 5.5.** If \( u : [0,T] \to X \) is a solution to (5.6), then \( V_m \circ u \) and \( V_M \circ u \) are absolutely continuous on compact subsets of \( (0,T] \) and for a.e. \( t \in [0,T] \), \( \Delta_p u(t) \in L^2[0,l] \) and
\[
(V_m \circ u)'(t) = -\int_0^t (u(t) - m)_- (\Delta_p u(t) + f(x,u(t))) \, dx,
\]
\[
(V_M \circ u)'(t) = \int_0^t (u(t) - M)_+ (\Delta_p u(t) + f(x,u(t))) \, dx.
\]

**Proof.** Define \( \bar{V}_m, \bar{V}_M : L^2[0,l] \to \mathbb{R} \) by (5.12). It can be easily proved that \( \bar{V}_m \) and \( \bar{V}_M \) are continuously differentiable and that, for any \( w,v \in L^2[0,l] \),
\[
(5.14) \quad \langle \bar{V}_m'(w),v \rangle = -\int_0^l (w-m)_- v \, dx, \quad \langle \bar{V}_M'(u),v \rangle = \int_0^l (w-M)_+ v \, dx;
\]
recall that \( \langle \cdot, \cdot \rangle \) stands here for the conjugation duality in \( L^2[0,l] \). These formulae follow from [22, Section 20.6] and [13, Section 7.4]. Since \( X \) is continuously embedded in \( L^2[0,l] \), \( V_m(u(t)) = \bar{V}_m(u(t)) \) and \( V_M(u(t)) = \bar{V}_M(u(t)) \) for all \( t \in [0,T] \), we infer by the chain rule that, for a.e. \( t \in [0,T] \),
\[
(V_m \circ u)'(t) = \langle \bar{V}_m'(u(t)), \dot{u}(t) \rangle \quad \text{and} \quad (V_M \circ u)'(t) = \langle \bar{V}_M'(u(t)), \dot{u}(t) \rangle.
\]
This along with (5.14) and (5.13) yields the assertion. \( \square \)
Proposition 5.6. If conditions (5.10) hold, then there exist $C > 0$ and $\delta > 0$ such that

\begin{align}
\int_0^l (u - m)_- (\Delta_p u + f(x, u)) \, dx &\leq CV_m(u) \text{ for any } u \in D(A_{L^2}) \cap (B(K_m, \delta) \setminus K_m), \\
\int_0^l (u - M)_+ (\Delta_p u + f(x, u)) \, dx &\leq CV_M(u) \text{ for any } u \in D(A_{L^2}) \cap (B(K_M, \delta) \setminus K_M).
\end{align}

Proof. In view of (5.9) and (5.10), there are $L > 0$ and $\delta > 0$ such that, for any $x \in (0, l)$ and $s \in [-\delta, 0)$,

\begin{align}
s \left( \Delta_p m(x) + f(x, m(x) + s) \right) &\leq s \left( \Delta_p m(x) + f(x, m(x)) \right) \\
+ &s \left( f(x, m(x) + s) - f(x, m(x)) \right) \leq |s| \left| f(x, m(x) + s) - f(x, m(x)) \right| \leq Ls^2.
\end{align}

Take $u \in D(A_{L^2}) \cap (B(K_m, \delta) \setminus K_m)$. We then have $0 < m(x) - u(x) < \delta$ for all $x \in \{ x \in [0, l] \mid u(x) < m(x) \} \neq \emptyset$. Observe that

\begin{align}
&- \int_0^l (u - m)_- (\Delta_p u + f(x, u)) \, dx = \\
&= - \int_0^l (u - m)_- (\Delta_p u - \Delta_p m) \, dx - \int_0^l (u - m)_- (\Delta_p m + f(x, u)) \, dx.
\end{align}

By (19), Lemme 5.1 for $s_1, s_2 \in \mathbb{R}$ one has $(|s_1| p^2 - s_1 - |s_2| p^2 s_2) \cdot (s_1 - s_2) \geq |s_1 - s_2|^p$. In view of (5.7), $(u - m)_- \in X$, so by integrating by parts, we get

\begin{align}
\int_0^l (u - m)_- (\Delta_p u - \Delta_p m) \, dx &= - \int_0^l ((u - m)_-) (|u'|^{p-2} u' - |m'|^{p-2} m') \, dx \\
&= \int_{\{u < m\}} (u' - m') (|u'|^{p-2} u' - |m'|^{p-2} m') \, dx \geq 0.
\end{align}

By applying (5.17) we get

\begin{align}
&- \int_0^l (u - m)_- (\Delta_p u + f(x, u)) \, dx \leq - \int_0^l (u - m)_- (\Delta_p m + f(x, u)) \, dx \\
&= \int_{\{u < m\}} (u - m) (\Delta_p m + f(x, u)) \, dx \leq L \int_{\{u < m\}} (u(x) - m(x))^2 \, dx \leq LV_m(u).
\end{align}

The proof of inequality (5.16) is in fact analogous. Due to (5.10) and arguing as above there is $L > 0$ and $\delta > 0$ such that, for all $x \in [0, l]$ and $s \in (0, \delta]$\n
\begin{align}
s \left( \Delta_p M(x) + f(x, M(x) + s) \right) &\leq Ls^2.
\end{align}

Since $u \in B(K_M, \delta) \setminus K_M$ we get $0 \leq (u - M)_+ < \delta$ a.e. on $[0, l]$ Now using (5.18) and reasoning as in (i) we get that

\begin{align}
&\int_0^l (u - M)_+ (\Delta_p u + f(x, u)) \, dx \leq \\
&\leq - \int_{\{u > M\}} (u' - M') (|u'|^{p-2} u' - |M'|^{p-2} M') \, dx + L \int_{\{u > M\}} (u - M)^2 \, dx \leq LV_M(u),
\end{align}

which completes the proof. \qed
As a conclusion from Lemma 5.5, Proposition 5.6 and Theorem 1.2 we get the following result.

**Theorem 5.7.** If \( m \) and \( M \) satisfy (5.10), then any integral solution of (5.6) such that \( u(0, \cdot) = u_0 \in K_m \cap K_M \) stays there for all times \( 0 \leq t \leq T \).

### 5.3. Obstacle Problem for Reaction Diffusion Equation.

Let \( \Omega \subset \mathbb{R}^N \) be open bounded with Lipschitz boundary \( \partial \Omega \) and consider the following parabolic problem

\[
\begin{aligned}
&u_t = \Delta u + f(x,u), \quad x \in \Omega, \ t > 0, \\
&u(x,t) = 0, \quad x \in \partial \Omega, \ t > 0,
\end{aligned}
\]

where \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is continuous and such that \( f(x,0) = 0 \) for all \( x \in \overline{\Omega} \). We shall deal with the related obstacle problem, i.e. we look for solutions \( u \) of (5.19) such that

\[
(5.20) \quad u(x,t) \geq m(x) \quad \text{and} \quad u(x,t) \leq M(x) \quad \text{for all} \quad x \in \Omega, \ t > 0,
\]

where obstacles \( m, M \in C^2(\Omega) \) such that

\[
(5.21) \quad m|_{\partial \Omega} \leq 0 \quad \text{and} \quad M|_{\partial \Omega} \geq 0
\]

are given. As above we look for conditions on \( f, m \) and \( M \) implying that for any \( u_0 \) such that \( m \leq u_0 \leq M \) on \( \Omega \) all solutions of (5.19) starting at \( u_0 \) satisfy (5.20).

Let \( X = C_0(\Omega) = \{ u \in C(\overline{\Omega}) : u(x) = 0 \text{ for all } x \in \partial \Omega \} \). Define \( A : D(A) \to X \) by \( Au := -\Delta u \) (here \( \Delta \) stands for the \( L^2 \)-realization of the Laplacian), where \( u \in D(A) \) and

\[
D(A) := \{ u \in X \cap H^1_0(\Omega) : \Delta u \in X \}.
\]

It is known (see e.g. [12, Prop. 2.6.7]) that \( A \) is \( m \)-accretive and its domain is dense in \( X \). Define \( F : X \to X \) by \( F(u)(x) := f(x,u(x)) \) for \( u \in X \) and \( x \in \overline{\Omega} \). It is clear that \( F \) is well-defined and continuous. By a solution to (5.19) on \([0,T]\) we mean a function

\[
(5.22) \quad u \in C([0,T], X) \cap C((0,T], H^1_0(\Omega)) \cap C^1((0,T], L^2(\Omega))
\]

such that \( \Delta u(t) \in L^2(\Omega) \) and

\[
(5.23) \quad u_t(t) = \Delta u(t) + F(u(t)), \quad \text{for each} \quad t \in (0,T],
\]

i.e. \( u \) is a strong solution to

\[
(5.24) \quad \dot{u}(t) = -A_{L^2}u(t) + F(u(t)), \quad \text{for each} \quad t \in (0,T],
\]

where \( A_{L^2} : D(A_{L^2}) \to L^2(\Omega) \) is given by

\[
A_{L^2}u := -\Delta u, \quad D(A_{L^2}) := \{ u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega) \}.
\]

It is well-known that \( A_{L^2} \) generates a strongly continuous semigroup of contractions, i.e. it is an \( m \)-accretive operator with dense domain and that the part of \( A_{L^2} \) in \( X \) is equal to \( A \) (see e.g. [12]). Observe that, since any solution to (5.19) in the above sense is, in particular, a strong solution to (5.24), each solution of (5.19) with \( u(0) = u_0 \) is also an integral solution of

\[
(5.25) \quad \dot{u}(t) = -A_{L^2}u(t) + w(t) \quad \text{on} \quad [0,T], \quad u(0) = u_0
\]

with \( w = F \circ u \). On the other hand, if \( \tilde{u} \) is an integral solution of

\[
(5.26) \quad \dot{u}(t) = -Au(t) + F(u(t)), \quad t \in [0,T],
\]

where...
with \( \tilde{u}(0) = u_0 \), i.e. \( \tilde{u} = u_A(\cdot, u_0, w) \), then \( \tilde{u} = u_{A_L}(\cdot; u_0, w) \), i.e. \( \tilde{u} \) is also an integral solution of (5.25) (see the proof of Proposition 2.9). This means that \( u = \tilde{u} \). Therefore, any solution of (5.19) is an integral solution of (5.26). Hence, we are able to use Theorem 1.2 for (5.26) and Remark 1.3 (1) to verify condition (1.8).

The obstacle conditions (5.20) can be rewritten as

\[
 u(t) \in K_m \cap K_M,
\]

where

\[
 K_m := \{ u \in X \mid u(x) \geq m(x) \text{ for } x \in \Omega \}, \quad K_M := \{ u \in X \mid u(x) \leq M(x) \text{ for } x \in \Omega \}.
\]

If we define \( V_m, V_M : X \to \mathbb{R} \) by

\[
 V_m(u) := \frac{1}{2} \int_\Omega (u - m)_-^2 \, dx, \quad V_M(u) := \frac{1}{2} \int_\Omega (u - M)_+^2 \, dx,
\]

then clearly \( K_m = \{ u \in X \mid V_m(u) \leq 0 \} \) and \( K_M = \{ u \in X \mid V_M(u) \leq 0 \} \).

**Remark 5.8.** It is well known that any solution of (5.19) with the regularity given by (5.22) is also a mild solution of (5.19), i.e. an integral solution to the problem \( \dot{u} = -Au + F(u) \) (see Remark 2.4 (c)). Conversely, if \( F \) is locally Lipschitz, then a mild solution of (5.19) satisfies (5.22) (see e.g. [10, Th. 5.2.1]). Hence, for a locally Lipschitz \( f \), in view of Proposition 2.6, we infer that, for any \( u_0 \in X \), the problem (5.19) admits a unique solution \( u : [0, \tau_{m_0}) \to X \) with the initial condition \( u(0) = u_0 \) and satisfying (5.22) for any \( 0 < T < \tau_{m_0} \). In the general case, when \( f \) is just continuous, one has only the local existence of mild solutions (see e.g. [22]).

We shall need the differentiability of \( V_m \) and \( V_M \) along trajectories of (5.19). Exactly as in Lemma 5.5 we get that

**Lemma 5.9.** If \( u : [0, T] \to X \) satisfies (5.22), then \( V_m \circ u \) and \( V_M \circ u \) are absolutely continuous on compact subsets of \( (0, T] \) and, for almost all \( t \in (0, T] \),

\[
 (V_m \circ u)'(t) = -\int_\Omega (u - m)_- (\Delta u(t) + f(x, u(t))) \, dx,
\]

\[
 (V_M \circ u)'(t) = \int_\Omega (u - M)_+ (\Delta u(t) + f(x, u(t))) \, dx.
\]

**Proposition 5.10.** Suppose that

\[
 \limsup_{s \to 0^-} \frac{\Delta m(x) + f(x, m(x) + s)}{s} < +\infty,
\]

\[
 \limsup_{s \to 0^+} \frac{\Delta M(x) + f(x, M(x) + s)}{s} < +\infty
\]

uniformly with respect to \( x \in \overline{\Omega} \). Then there are \( C > 0 \) and \( \delta > 0 \) such that, for any \( u \in D(A_L) \cap (B(K_m, \delta) \setminus K_m) \),

\[
 -\int_\Omega (u - m)_- (\Delta u + f(x, u)) \, dx \leq CV_m(u),
\]

and, for any \( u \in D(A_L) \cap (B(K_M, \delta) \setminus K_M) \),

\[
 \int_\Omega (u - M)_+ (\Delta u + f(x, u)) \, dx \leq CV_M(u).
\]
Proof. In view of (5.27) there is $C > 0$ and $\delta > 0$ such that, for any $x \in \Omega$ and $s \in [-\delta, 0)$,
\[
\frac{\Delta m(x) + f(x, m(x) + s)}{s} < C, \quad \text{i.e.} \quad s\left(\Delta m(x) + f(x, m(x) + s)\right) < C|s|^2.
\]
Note that in view of (5.21), $(u - m) \in H^1_0(\Omega)$; this together with Poincaré’s inequality and again [18, Section 7.4] gives
\[
\begin{align*}
- \int_\Omega (u - m)(\Delta u + f(x, u)) \, dx &= \int_\Omega (u - m)(\Delta (u - m) + \Delta m + f(x, u)) \, dx \\
&\leq \int \nabla [(u - m)_-] \nabla (u - m) \, dx - \int (u - m)(\Delta m + f(x, u)) \, dx \\
&\leq - \int \nabla (u - m)_-^2 \, dx + \int_{\{u < m\}} (u - m)(\Delta m + f(x, u)) \, dx \\
&\leq -2\lambda_1 V_m(u) + C \int_{\{u < m\}} |u(x) - m(x)|^2 \, dx \leq -2\lambda_1 V_m(u) + CV_m(u) = (C - 2\lambda_1)V_m(u),
\end{align*}
\]
where $\lambda_1$ stands for the first eigenvalue of the (negative) Laplacian.

Inequality (5.28) implies the existence of $C > 0$ and $\delta > 0$ such that, for all $x \in \Omega$ and $s \in (0, \delta]$
\[
\frac{\Delta M(x) + f(x, M(x) + s)}{s} < C, \quad \text{i.e.} \quad s\left(\Delta M(x) + f(x, M(x) + s)\right) < C|s|^2.
\]
Then, arguing as above, we show that
\[
\begin{align*}
\int_\Omega \left|(u - M)_+ \right| (\Delta u + f(x, u)) \, dx \leq (C - 2\lambda_1)V_M(u).
\end{align*}
\]

Remark 5.11. Observe that if $m$ (resp. $M$) is a subsolution (resp. supersolution) of the stationary problem related to (5.19), i.e.
\[
-\Delta m(x) \leq f(x, m(x)) \quad \text{(resp. $-\Delta M(x) \geq f(x, M(x))$)} \quad \text{for} \quad x \in \Omega,
\]
and
\[
\limsup_{s \to 0^-} \frac{f(x, m(x) + s) - f(x, m(x))}{s} < +\infty \quad \text{(resp. $\limsup_{s \to 0^+} \frac{f(x, M(x) + s) - f(x, M(x))}{s} < +\infty$)}
\]
uniformly with respect to $x \in \overline{\Omega}$, then (5.27) (resp. (5.28) holds. It is clear that each of the latter conditions is always satisfied whenever $f$ is locally Lipschitz.

Combining Lemma 5.9 with Proposition 5.10 yields the following.

Corollary 5.12. Assume that (5.27) and (5.28) hold and let $\delta > 0$ and $C > 0$ be as in Proposition 5.10. If $u : [0, T] \to X$ is a solution to (5.19), then
\[
(V_m \circ u)'(t) \leq CV_m(u(t)), \quad \text{for a.e.} \quad t \in (0, T], \quad \text{whenever} \quad u(t) \in B(K_m, \delta) \setminus K_m,
\]
and
\[
(V_M \circ u)'(t) \leq CV_M(u(t)), \quad \text{for a.e.} \quad t \in (0, T], \quad \text{whenever} \quad u(t) \in B(K_M, \delta) \setminus K_M.
\]

This together with Theorem 1.2 allows us to conclude that, under our assumptions on $f$, $m$ and/or $M$, if a solution of the reaction diffusion problem (5.19) starts above the obstacle $m$ and
below the obstacle $M$ then it does not cross the obstacle(s) in positive times. Formally we state it as the following.

**Theorem 5.13.** If $f$ satisfies conditions (5.27) and (5.28), and $u$ is a solution of (5.19) such that (5.20) holds for $t = 0$, then (5.20) holds for all positive times $t$. \hfill $\square$

### 5.4. Age-Structured Population Model

Consider the following first order partial differential problem

$$
\begin{cases}
  u_t(x,t) + u_x(x,t) = f(x,u(x,t)), & x \in (0,a), \ t > 0, \\
  u(0,t) = \int_0^a \beta(x) u(x,t) \, dx, & t > 0.
\end{cases}
$$

(5.29)

This is McKendrick's model for an age dependent population with the so-called birth function $\beta : [0,a] \to [0, +\infty)$ satisfying the condition

$$
\int_0^a \beta(x) \, dx < 1,
$$

(5.30)

and the so-called "migration" component $f : [0,a] \times \mathbb{R} \to \mathbb{R}$ that is locally Lipschitz in the second variable uniformly with respect to the first one, i.e. for any $s \in \mathbb{R}$ there exist $L > 0$ and $\delta > 0$ such that, for any $x \in [0,a]$ and $s_1, s_2 \in [s - \delta, s + \delta]$ one has $|f(x, s_1) - f(x, s_2)| \leq L|s_1 - s_2|.$

We provide additional conditions on $f$ and $m : [0,a] \to \mathbb{R}$ assuring that any solution $u$ of (5.29) such that

$$
u(x,0) \geq m(x) \text{ for all } x \in [0,a]$$

(5.31)

will preserve the inequality for positive times $t$, i.e.

$$
u(x,t) \geq m(x) \text{ for all } x \in [0,a] \text{ and } t \geq 0.$$

(5.32)

Let $X = C[0,a]$ and define $A : D(A) \to X$ by

$$
A u := u', \ u \in D(A) := \left\{ u \in X \cap C^1(0,a) \mid u(0) = \int_0^a \beta(x) u(x) \, dx \right\}.
$$

Condition (5.30) implies, by a straightforward calculation, that $A$ is $m$-accretive. Let $F : X \to X$ be defined by $[F(u)](x) := f(x,u(x))$ for $u \in X$ and $x \in [0,a]$. Problem (5.29) can be rewritten as

$$
u(t) = -Au(t) + F(u(t)), \ t \in [0,\tau).
$$

(5.33)

Define also $A_{L^2} : D(A_{L^2}) \to L^2[0,a]$ by

$$
A_{L^2} u := u' \text{ for } u \in D(A_{L^2}) := \left\{ u \in H^1(0,a) \mid u(0) = \int_0^a \beta(x) u(x) \, dx \right\}.
$$

For $\lambda := \frac{1}{2} \| \beta \|_{L^2}^2$, by the use of the Cauchy-Schwarz inequality, one has

$$
\left\langle u, A_{L^2} u + \lambda u \right\rangle_{L^2} = \int_0^a uu' \, dx + \lambda \| u \|_{L^2}^2 = \frac{1}{2} u^2(a) - \frac{1}{2} \left( \int_0^a \beta u \, dx \right)^2 + \lambda \| u \|_{L^2}^2
$$

$$
\geq \frac{1}{2} u^2(a) - \frac{1}{2} \| \beta \|_{L^2}^2 \| u \|_{L^2}^2 + \lambda \| u \|_{L^2}^2 = \frac{1}{2} u^2(a) \geq 0.
$$

Hence $A_{L^2} + \lambda I$ is accretive. The $m$-accretivity of $A_{L^2} + \lambda I$ may be proved in the same way as for $A$. Therefore $A_{L^2}$ is $\lambda$-$m$-accretive.

In view of Proposition 2.9 we get the following regularity result.
Lemma 5.14. For any integral solution \( u : [0, \tau) \to X \) of (5.33) with \( u(0) \in D(A) \) one has \( u \in W^{1,1}([0,T],L^2[0,a]) \), for any \( T \in (0, \tau) \), \( u(t) \in D(A_{L^2}) \) and
\[
\dot{u}(t) = -A_{L^2} u(t) + F(u(t)), \text{ for a.e. } t \in [0, \tau).
\]

This means that we are able to apply the idea from Remark 1.5 (2) in order to verify the assumption (iii)' from Theorem 1.4.

Define \( V : X \to \mathbb{R} \) by
\[
V(u) := \frac{1}{2} \int_0^a (u - m)^2 \, dx, \; u \in X.
\]

Let
\[
K := \{ u \in X \mid u(x) \geq m(x) \text{ for } x \in [0,a] \}.
\]

Proposition 5.15. If \( m \in C^1(0,a) \cap C[0,a] \),
\[
- m'(x) + f(x,m(x)) \geq 0 \quad \text{for all } x \in [0,a],
\]
and
\[
m(0) = \int_0^a \beta(x)m(x) \, dx,
\]
then there exist \( C > 0 \) and \( \delta > 0 \) such that, for any mild solution \( u : [0, \tau) \to X \) of (5.33), the function \( V \circ u \) is a.e. differentiable and, for a.e. \( t \in [0, \tau) \),
\[
(V \circ u)'(t) \leq CV(u(t)) \; \text{if } u(t) \in B(K, \delta) \setminus K.
\]

Proof. By (5.34) and the local Lipschitz condition for \( f \), there are \( L > 0 \) and \( \delta > 0 \) such that for all \( s \in (-\delta,0) \) and \( x \in [0,a] \),
\[
s \left( - m'(x) + f(x,m(x) + s) \right) = s \left( - m'(x) + f(x,m(x)) \right) + s \left( f(x,m(x) + s) - f(x,m(x)) \right) \leq L |s|^2.
\]

According to Lemma 5.14, \( u \) is almost everywhere differentiable as a function into \( L^2[0,a] \). If we define \( \widetilde{V} : L^2[0,a] \to \mathbb{R} \) by
\[
\widetilde{V}(u) := \frac{1}{2} \int_0^a (u - m)^2 \, dx, \; u \in L^2[0,a],
\]
then, by the chain rule, \( u(t) \in D(A_{L^2}) \) for a.e. \( t \in [0, \tau) \), and
\[
(V \circ u)'(t) = (\widetilde{V} \circ u)'(t) = \left[ \widetilde{V}'(u(t)) \right] (u(t)) = - \int_0^a u(t)_- (-A_{L^2} u(t) + F(u(t))) \, dx.
\]
Put \( \varphi := u(t) \) and observe that \( -\delta < \varphi(x) - m(x) \leq 0 \) if \( x \in \{ y \in [0, a] \mid (\varphi - m)(y) \neq 0 \} \). Hence
\[
-(V \circ u)'(t) = \int_0^a (\varphi - m)(-\varphi' + f(x, \varphi)) \, dx
\]
\[
= \int_0^a (\varphi - m)([\varphi - f(x, \varphi)]') \, dx + \int_0^a (\varphi - m)(-m' + f(x, \varphi)) \, dx
\]
\[
= \int_0^a (\varphi - m)([\varphi - f(x, \varphi)]') \, dx - \int_0^a (\varphi - m)(-m' + f(x, \varphi)) \, dx
\]
\[
\geq \frac{1}{2} \left( (\varphi - m)(0) - \int_{\{\varphi < m\}} (\varphi - m)^2 \, dx \right) - L \int_{\{\varphi < m\}} (\varphi - m)^2 \, dx
\]
\[
\geq -\frac{1}{2} \left( (\varphi - m)(0) - L \int_{\{\varphi < m\}} (\varphi - m)^2 \, dx \right).
\]

Note that, since \( \varphi, m \in D(A_{L^2}) \) and \( \beta \geq 0 \), one gets
\[
(\varphi - m)(0) = \int_0^a \beta (\varphi - m) \, dx = \int_0^a \beta (\varphi - m) \, dx - \int_0^a \beta (\varphi - m) \, dx,
\]
i.e.
\[
-(\varphi - m)(0) \leq \int_0^a \beta (\varphi - m) \, dx \quad \text{and thus} \quad (\varphi - m)(0) \leq \int_0^a \beta (\varphi - m) \, dx.
\]

By the Cauchy-Schwarz inequality we get
\[
(V \circ u)'(t) \leq \frac{1}{2} \left( \int_0^a \beta (\varphi - m) \, dx \right)^2 + LV(\varphi) \leq \left( \frac{1}{2} \| \beta \|^2 L^2 + L \right) V(\varphi).
\]

Combining Proposition 5.15 with Theorem 1.4 we conclude with the following.

**Theorem 5.16.** If \( f \) satisfies (5.34) and (5.35), then any mild solution \( u \) of (5.29) satisfying condition (5.31) satisfies (5.32) for all positive times \( t \).

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