BOUNDARY CROSSING PROBLEMS AND FUNCTIONAL TRANSFORMATIONS FOR ORNSTEIN-UHLENBECK PROCESSES

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Abstract

We are interested in the law of the first passage time of an Ornstein-Uhlenbeck process to time-varying thresholds. We show that this problem is connected to the laws of the first passage time of the process to members of a two-parameter family of functional transformations of a time-varying boundary. For specific values of the parameters, these transformations appear in a realisation of a standard Ornstein-Uhlenbeck bridge. We provide three different proofs of this connection. The first one is based on a similar result for Brownian motion, the second uses a generalisation of the so-called Gauss-Markov processes and the third relies on the Lie group symmetry method. We investigate the properties of these transformations and study the algebraic and analytical properties of an involution operator which is used in constructing them. We also show that these transformations map the space of solutions of Sturm-Liouville equations into the space of solutions of the associated nonlinear ordinary differential equations. Lastly, we interpret our results through the method of images and give new examples of curves with explicit first passage time densities.

Keywords: First passage times; Lie algebras; Sturm-Liouville equations; Ornstein-Uhlenbeck; Fokker Planck equation; Ornstein-Uhlenbeck bridge; Brownian motion.

2020 Mathematics Subject Classification: Primary 35K05, 60J50, 60J60.

1 Introduction

Let $U := (U_t)_{t \geq 0}$ be a one-dimensional Ornstein-Uhlenbeck (OU for short) process defined on a filtered probability space $(\Omega, (\mathcal{F})_{t \geq 0}, \mathcal{F}, \mathbb{P})$ as the unique solution to the following stochastic differential equation (SDE)

$$dU_t = -kU_t \, dt + dB_t, \quad U_0 = 0,$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion (BM) starting at 0 and $k \in \mathbb{R}$ is a constant. The OU process is a Gauss-Markov process with transition density function given by

$$p_t(x, y) := \frac{\partial}{\partial y} \mathbb{P}(U_t \leq y | U_0 = x) = \frac{e^{kt}}{\sqrt{r(t)}} \phi \left( \frac{ye^{kt} - x}{\sqrt{r(t)}} \right), \quad x, y \in \mathbb{R},$$

where $\phi(z) = e^{-z^2/2} / \sqrt{2\pi}$, $z \in \mathbb{R}$, is the probability density function of the standard normal distribution and

$$r(t) = (e^{2kt} - 1) / 2k, \quad t \geq 0,$$

$$s(t) = \ln \left( 2kt + 1 \right) / 2k, \quad t \leq \zeta(k),$$

arXiv:2210.01658v3 [math.PR] 25 Mar 2024
where
\[ \zeta^{(k)} = \begin{cases} -\frac{1}{2k} & \text{if } k < 0; \\ +\infty & \text{otherwise}. \end{cases} \]

It is well known by the Dambis, Dubins-Schwarz theorem (see, e.g., Theorem V.1.6 in [34]), that the OU process can be written in terms of a time changed BM \((W_t)_{t \geq 0}\) as
\[ U_t = e^{-kt}W_{r(t)}, \quad t \geq 0. \tag{3} \]

Let \( f \in \mathcal{C}([0, \infty), \mathbb{R}) \) be such that \( f(0) \neq 0 \), with \( \mathcal{C}(I, K) \) denoting the space of continuous functions from \( I \) into \( K \) for some intervals \( I \) and \( K \subseteq \mathbb{R} \). We are interested in the first passage time (FPT) of the OU to \( f \) given by
\[ T^f_k = \inf\{t > 0; \ U_t = f(t)\}, \]
with \( \inf\{\emptyset\} = \infty \). The main goal of this paper is to derive, through different methods, an explicit analytical expression linking the distribution of \( T^f_k \) with inf \( S^\alpha_0 \). The two-parameter family of curves \( \{S^\alpha_0 f; \alpha \neq 0, \beta \in \mathbb{R}\} \) is defined by
\[ S^\alpha_0 f(t) = \left( \frac{1 + \alpha \beta r(t)}{\alpha} \right) \left( 2k \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} + 1 \right)^{1/2} e^{-kt} f \left( s \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right) \right), \quad t < \zeta_{k,\alpha,\beta}, \tag{4} \]
where
\[ \zeta_{k,\alpha,\beta} = \begin{cases} s \left( -\frac{1}{\alpha \beta} \right) & \text{if } \alpha \beta < 0, \ k \geq 0; \\ s \left( -\frac{1}{\alpha \beta} \right) & \text{if } 0 < \frac{2k}{\alpha \beta + 2k^2} < 1, \ k < 0; \\ +\infty & \text{otherwise.} \end{cases} \]

By doing so, we generalize the results obtained for a BM in [3], which can be immediately recovered from ours by letting \( k \to 0 \), i.e. \( T^f_0, S^\alpha_0 \) and \( \zeta_{0,\alpha,\beta} \). To simplify the notation and for consistency with [3], we drop the subscript 0 when referring to the BM.

To the best of our knowledge, explicit results for the FPT of the OU to \( f \) only exist for constants [130] or hyperbolic type boundaries [131]. Further results for the boundary crossing problem of Gauss-Markov processes to moving boundaries have been obtained in, e.g., [14, 17, 18, 31]. We also refer to [8, 39] for applications of Lie symmetries to FPT problems. Our main result, stated in Theorem 3.1, allows to map those results for the law of the FPT of \( T^f_k \) to that of \( T^f_k S^\alpha_0 \) for the OU process. Such problems are of great interest, as the OU process has been used in many applications to model objects such as interest rates in finance or the evolution of the neuronal membrane voltages in neuroscience, see e.g. [1] and citations therein. In this paper, we focus on the OU without drift, as the results for the OU process with drift can be directly obtained from our results after some transformations, as discussed in Remark 3.2.

The paper is organised as follows. In Subsection 2.1, we introduce some notations and provide the key results for the \( S^\alpha_0 \) operator for the BM. The OU functional setting is presented in Subsection 2.2. In Subsection 2.3, we recall the different constructions of OU bridges and their properties. In particular, the process \( (S^\alpha_0 \{U_t, 0 < t \leq T\}) \) has the same law as an OU bridge of length \( T \) from 0 to 0, for some \( T > 0 \). Section 3 is devoted to the statement of Theorem 3.1 which contains the main result of the paper, and two examples of its application. In Section 4, we discuss the properties of the \( S^\alpha_0 \) transformation with its connection to a certain nonlinear differential equation (Lemma 4.1), while in Section 5, we prove Theorem 3.1 in three different ways. In the first proof, we use the relationship between the FPT of an OU and that of a BM. In the second one, we use a generalisation of the Gauss-Markov processes introduced in Section 3.2 of [3] and find an analogue version of that proof in our case. In the third one, we use the
Lie algebra to find the symmetries of the Fokker-Planck equation or the Kolmogorov forward differential equation

$$\frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial^2 h}{\partial x^2} + kx \frac{\partial h}{\partial x} + kh.$$  \hspace{1cm} (5)

Then, we use these symmetries to construct the function $h^{\alpha,\beta}_k$ of equation (30), derive our transformation $S^{\alpha,\beta}_k$ and relate the FPT distribution of $T^f_k$ to that of $T^{S^{\alpha,\beta}_k}_k$, in Section 5.3. In Section 6, we discuss the asymptotic distribution of $T^{S^{\alpha,\beta}_k}_k$ and the transience of the transformed curves $S^{\alpha,\beta}_k f$. We provide the analogue of the Kolmogorov–Erdős–Petrovski transience test [19] in the OU case and show its connection to the asymptotic behaviour of the FPT. Lastly, in Section 7, we use the method of images to obtain new classes of boundaries yielding explicit FPT distributions and use our $S^{\alpha,\beta}_k$ transformation (4) to produce new examples. A limitation of this method is that it only works for boundaries with certain properties given in Lemma 7.1.

As we were finalising the paper, we discovered that the $S^{\alpha,\beta}_k$ transformation, a variant of the boundary crossing identity (18) in Theorem 3.1 and the Lie symmetries (31) had previously appeared in [28] (our (4) can be obtained by setting $A = k^2$ and $B = 0$ in equation (39) therein), using the Lie approach. When comparing our results, we found misprints in one of their Lie symmetries and boundary crossing identity, as discussed in Section 5.3.

2 Notation and preliminaries

We first introduce some functional spaces, transformations and related results for the BM, as in [3], in Subsection 2.1, and then define the corresponding functional objects for the OU in Subsection 2.2. We end this section by providing different representations of OU bridges and highlighting their connection to our functional transformations.

2.1 Brownian motion setting

We start by introducing a nonlinear operator $\tau$ defined on the space of functions whose reciprocals are square integrable in some (possibly infinite) interval of $\mathbb{R}^+ = [0, \infty)$ by

$$\tau f(t) = \int_0^t f^{-2}(z)dz,$$

and use it to define

$$A_\infty = \bigcup_{a>0} \bigcup_{b>0} A(a,b) \text{ and } A(a,b) = \{ f \in C([0,a],\mathbb{R}^+) : \tau f(a) = b \},$$

where $a,b \in \mathbb{R}^+$. In [3], the authors derived the following relationship between the laws of the FPTs of $T^f$ and $T^{S^{\alpha,\beta}_f}$,

$$\mathbb{P}(T^{S^{\alpha,\beta}_f} \in dt) = \alpha^3 (1 + \alpha \beta t)^{-\frac{3}{2}} e^{-\frac{\alpha^2 t}{2(1 + \alpha \beta t)}} S^{\alpha,\beta} \mathbb{P}(T^f \in dt), \quad t < \zeta^{\alpha,\beta},$$  \hspace{1cm} (6)

where $S^{\alpha,\beta}$ is the two-parameter family of transformations $S^{\alpha,\beta} : A_\infty \rightarrow A_\infty$ given by

$$S^{\alpha,\beta} f(t) = \left( \frac{1 + \alpha \beta t}{\alpha} \right) f \left( \frac{\alpha^2 t}{1 + \alpha \beta t} \right), \quad \alpha \neq 0, \beta \in \mathbb{R}.$$  \hspace{1cm} (7)

Equation (6) extends a previous result by the same authors for a relationship between the laws of the FPT of $T^f$ and $T^{S^{1,\beta}_f}$, with the one-parameter family of transformations $\{S^{1,\beta}_f, \beta \in \mathbb{R}\}$ obtained using the construction of Brownian bridges, see [2]. The connection to Brownian bridges naturally appears as $(S^{1,\beta}_{-1/T}(B)_t, 0 \leq t < T)$ is a Brownian bridge of length $T$ from 0 to 0, see
page 64 of [11]. Besides deriving (6) in [3], the authors showed also that the transformation $S^{\alpha, \beta}$ can be obtained as

$$S^{\alpha, \beta} = \Sigma \circ \Pi^{\alpha, -\beta} \circ \Sigma$$  \hspace{1cm} (8)

where $\circ$ denotes the composition operator. Here, $\Sigma : A_{\infty} \to A_{\infty}$ is the involution operator, i.e., $\Sigma \circ \Sigma = Id$, specified by

$$\Sigma f(t) = \frac{1}{f(\rho \circ \tau f(t))},$$

where $\rho$ is the inversion operator acting on the space of continuous monotone functions i.e., $\rho f \circ f(t) = t$. For $\alpha \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R}$, $\Pi^{\alpha, \beta} : A_{\infty} \to A_{\infty}$ is the family of nonlinear operators given by

$$\Pi^{\alpha, \beta} f(t) = f(t)(\alpha + \beta \tau f(t)).$$  \hspace{1cm} (9)

As explained in the beginning of Section 2 of [3] and Appendix 8 of [34], the operators (9) are closely related to the Sturm-Liouville equation

$$\phi'' = \mu \phi,$$  \hspace{1cm} (10)

where $\mu$ denotes a positive Radon measure on $\mathbb{R}^+$ and $\phi''$ is the second derivative in the sense of distributions. In fact, if $\phi$ solves (10), then the vectorial space $\{\Pi^{\alpha, \beta} \phi; \alpha \neq 0, \beta \in \mathbb{R}\}$ is the set of other solutions to the same equation. Moreover, all positive solutions are convex and described by the set $\{\Pi^{\alpha, \beta} \varphi; \alpha > 0, \beta \geq 0\}$, where $\varphi$ is the unique, positive, decreasing solution such that $\varphi(0) = 1$.

Moreover, it was noted in [3] that $S^{\alpha, \beta} \circ S^{\alpha', \beta'} = S^{\alpha + \alpha', \beta + \beta'}$, for all couples $(\alpha, \beta)$ and $(\alpha', \beta') \in \mathbb{R}^* \times \mathbb{R}$, and that $(S^{1, \beta})_{\beta \geq 0}$ is a semi-group, while $(S^{1, \beta})_{\beta \in \mathbb{R}}$ and $(S^{\alpha, \beta})_{\alpha, \beta \in \mathbb{R}}$ are groups.

### 2.2 Ornstein-Uhlenbeck setting

We shall now define the operators of interest on the space of functions

$$A_{k, \infty} = \bigcup_{a > 0} \bigcup_{b > 0} A_k(a, b),$$  \hspace{1cm} (11)

where $a, b \in \mathbb{R}^+$, and

$$A_k(a, b) = \left\{ f \in C([0, a], \mathbb{R}^+) : \tau f \left( \text{sgn}(k) s \left( \text{sgn}(k) a \right) \right) = \frac{b}{1 - \mathbb{I}_{k < 0} 2kb} \right\},$$  \hspace{1cm} (12)

with $\text{sgn}(k)$ being the sign of $k$. Note that $A_{k, \infty}$ is the set of continuous functions which are of constant sign on some nonempty interval $[0, l]$, $l > 0$. Let the isomorphic nonlinear operators $\Lambda_k$ and $\Sigma_k$ be defined, on $A_{k, \infty}$, by

$$\Lambda_k f(t) = e^{kt} f(s(t))$$  \hspace{1cm} (13)

and

$$\Sigma_k = \Lambda_k^{-1} \circ \Sigma \circ A_k,$$  \hspace{1cm} (14)

respectively. Note that the inverse of $\Lambda_k$ is $\Lambda_k^{-1} f(t) = e^{-kt} f(r(t))$. Here, $\Lambda_k$ maps the curves from the OU setting to the corresponding curves in the BM setting, with limits $\lim_{k \to 0} \Lambda_k = Id$, and $\lim_{k \to 0} \Sigma_k = \Sigma$. From (14), we can immediately see that $\Sigma_k$ can also be represented as

$$\Sigma_k f(t) = \frac{e^{-k \rho \tau f(r(t)) - kt}}{f(\rho \circ \tau f(r(t)))}.$$  \hspace{1cm} (15)

Inspired by (8) in the BM setting, for $\alpha \in \mathbb{R}^*$ and $\beta \in \mathbb{R}$, define $S^{\alpha, \beta} : A_{k, \infty} \to A_{k, \infty}$ by

$$S_k^{\alpha, \beta} = \Sigma_k \circ \Pi^{\alpha, -\beta} \circ \Sigma_k.$$  \hspace{1cm} (16)
Alternatively, $S_{k}^{\alpha,\beta}$ can be also obtained as $\Lambda_{k}^{-1} \circ S_{\alpha,\beta} \circ \Lambda_{k}$, as shown in the proof of Lemma 4.1. This result provides us with an intuitive interpretation of this family of transformations. First, we map the boundary $f$ for the OU process to its corresponding boundary for the standard BM using the $\Lambda_{k}$ transformation. Then, we use $S_{\alpha,\beta}$ on this curve for the standard BM, and finally we revert it back to the OU problem via $\Lambda_{k}^{-1}$, as illustrated in the following figure.

![Figure 1: Flow chart of $S_{k}^{\alpha,\beta}$](image)

2.3 Ornstein-Uhlenbeck bridges

As mentioned in the introduction, $S_{k}^{\alpha,\beta}$ for specific values of $\alpha$ and $\beta$ relates to a representation of the standard OU bridge. We shall now define this process and then recall its different representations, using the detailed analysis of both Wiener and OU bridges provided in [6, 11].

**Definition 2.1.** An OU bridge $U_{br} = \{U_{br}^{t} : t \in [0, T]\}$ from $a$ to $b$, of length $T$, is characterised by the following properties:

(i) $U_{br}^{0} = a$ and $U_{br}^{T} = b$ (each with probability 1).

(ii) $U^{br}$ is a Gaussian process.

(iii) $E[U_{br}^{t}] = a \frac{\sinh (k(T-t))}{\sinh (kt)} + b \frac{\sinh (kt)}{\sinh (kT)}$.

(iv) $\text{Cov}(U_{br}^{s}, U_{br}^{t}) = \frac{\sinh (ks) \sinh (k(T-t))}{k \sinh (kT)}$, $0 \leq s \leq t < T$.

(v) The paths of $U^{br}$ are almost surely continuous.

In what follows, we present three different representations of OU bridges. First, consider the following linear SDE

$$dU_{br}^{t} = \left( -k \coth (k(T-t))U_{br}^{t} + k \frac{b}{\sinh (k(T-t))} \right) dt + dB_{t}, \quad 0 \leq t < T,$$

with initial condition $U_{br}^{0} = a$. This has a unique strong solution given by

$$U_{ir}^{t} := \begin{cases} a \frac{\sinh (k(T-t))}{\sinh (kt)} + b \frac{\sinh (kt)}{\sinh (kT)} \frac{\sinh (k(T-t))}{\sinh (kT)} dB_{t} & \text{if } 0 \leq t < T, \\ b & \text{if } t = T. \end{cases}$$

This is referred to as the integral representation (ir) of the OU bridge. The following anticipative (av) and the space-time (st) versions can be obtained by using the different representations of the Wiener bridge,

$$U_{av}^{t} = a \frac{\sinh (k(T-t))}{\sinh (kT)} + b \frac{\sinh (kt)}{\sinh (kT)} + \left( U_{t} - \frac{\sinh (kt)}{\sinh (kT)} U_{T} \right), \quad 0 \leq t < T,$$

and...
Theorem 3.1. Let \( T \) be the distribution of \( F_k^{1/3} \), i.e. it belongs to the space of continuously differentiable functions from \((0, \infty)\) to \( \mathbb{R} \). Now by setting \( a = b = 0 \), we see that \( U_t = S_k^{1/3} U_t \). One thing to notice is that the anticipative and space-time versions are only weak solutions to [17], since the former requires information about the random variable \( U_T \) and the latter is adapted to a different filtration, see Section 1 of [3]. Of course, \( U^{br} \) can be thought of as \((U_t, t \leq T)\) conditioned on \( U_T = b \). For conditioned processes and Markovian bridges, see [20] and [34].

3 Main result and examples

We are now ready to state the main result of this paper which relates the distributions of the family of stopping times \( (T_k^{\alpha, \beta})_{\alpha \in \mathbb{R}^+, \beta \in \mathbb{R}} \) to that of \( T_k \). We assume that \( f \in C^1((0, \infty), \mathbb{R}^+) \), i.e. it belongs to the space of continuously differentiable functions from \((0, \infty)\) into \( \mathbb{R}^+ \). This assumption is linked to Strassen’s result for the BM in [11] and is crucial in ensuring that the distribution of \( T_k \) has a continuous density with respect to the Lebesgue measure on \((0, \infty)\).

**Theorem 3.1.** Let \( f \in C^1([0, \infty), \mathbb{R}) \) be such that \( f(0) \neq 0 \). Let \( \alpha \in \mathbb{R}^+, \beta \in \mathbb{R} \). Then, for \( t < \zeta_k, \alpha, \beta \), we have the following relationship

\[
\mathbb{P}(T_k^{\alpha, \beta} \in dt) = e^{3kt} \alpha^3 (1 + \alpha\beta t)^{-5/2} \left( 2k \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} + 1 \right)^{-3/2} e^{-\frac{\alpha^2 \beta}{2(1 + \alpha \beta r(t))} (S_k^{\alpha, \beta})^2} e^{2kt} S_k^{\alpha, \beta} \left( \mathbb{P}(T_k \in dt) \right),
\]

where \( S_k^{\alpha, \beta} \) is given by [4].

**Remark 3.2.** As mentioned in the introduction, focusing on the driftless OU is not restrictive. Consider the process \( U^\mu := (U_t^\mu)_{t \geq 0} \), defined for each fixed \( t > 0 \) by

\[ U_t^\mu := U_t + \mu (1 - e^{-kt}), \]

is called an OU process with drift \( \mu \), see Remark 2.5 of [1]. For \( f : [0, \infty) \to \mathbb{R} \) define \( f^*(t) := f(t) - \mu (1 - e^{-kt}) \), \( t \geq 0 \). Then, the FPT of \( U^\mu \) hitting the curve \( f \) equals to the FPT of \( U \) hitting the curve \( f^* \).

**Example 3.3.** Theorem 3.1 in [11] gives an expression for the FPT density of an OU process starting at 0 hitting a constant threshold \( f(t) = a > 0 \). Writing \( p_k^a(t)dt = \mathbb{P}(T_k^a \in dt) \), we have

\[ p_k^a(t) = -ke^{-\nu_a t^2/2} \sum_{j=1}^{\infty} \frac{D_{\nu_{j,j-a\nu}}(0)}{D_{\nu_{j,j-a\nu}}(-a\sqrt{2k})} \exp(-\nu_{j,j-a\nu}^2/2), \]

where \( D_{\nu}(.) \) is the parabolic cylinder function with index \( \nu \in \mathbb{R} \), \( \nu_{j,h} \) the ordered sequence of positive zeros of \( D_{\nu}(b) \), \( D_{\nu_{j,j-a\nu}}(0) = 2^{\nu_{j,j-a\nu}} \frac{\Gamma(1/2)}{\Gamma((1-\nu_{j,j-a\nu})/2)} \) and \( D_{\nu_{j,h}}(b) = \frac{\partial D_{\nu}(b)}{\partial \nu} \big|_{\nu=\nu_{j,h}} \). Here, \( D_{\nu}(z) = 2^{-\nu/2} e^{-z^2/2} H_{\nu}(z/\sqrt{2}) \), where \( H_{\nu}(.) \) is the Hermit function, as found on page 285 of [23].

Now, we use \( S_k^{\alpha, \beta} \) to get the following family of curves

\[
S_k^{\alpha, \beta} a(t) = a \left( 1 + \frac{\alpha \beta r(t)}{\alpha} \right) \left( 2k \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} + 1 \right)^{1/2} e^{-kt} = \frac{ae^{-kt}}{2k\alpha} \left( (\alpha \beta e^{2kt} - \alpha \beta + 2k) \left( 2k\alpha^2 + \alpha \beta \right) e^{2kt} - 2k\alpha^2 - \alpha \beta + 2k \right)^{1/2}.
\]
Such curves for $\beta = 1, k = 1/2, a = 1$ and different values of $\alpha$ are plotted in Figure 2. By our Theorem 3.1, we can write

\[
p_k^{S,\alpha,\beta}(t) = -\frac{k e^{2kt} \alpha^2 (1 + \alpha \beta r(t))^{-5/2}}{2\alpha^2 r(t) + \alpha \beta r(t) + 1} \exp \left(-\frac{\alpha \beta}{2(1 + \alpha \beta r(t))} (S_k^{\alpha,\beta, a(t)} e^{2kt} - ka^2/2)\right)
\]

\[
\times \sum_{j=1}^{\infty} \frac{D_{\nu_j, -a \sqrt{2k}}(0)}{D'_{\nu_j, -a \sqrt{2k}}(-a \sqrt{2k})} \exp \left(-k \nu_j, -a \sqrt{2k} \right) \left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right). \quad (19)
\]

In the special case of $\alpha \beta = 2k$, we get the family of curves

\[S_k^{\alpha,\beta, f(t)} = a \sqrt{(\alpha^2 + 1)e^{2kt} - \alpha^2},\]

with FPT density given by

\[p_k^{S,\alpha,\beta, f(t)} = -\frac{k a^2 e^{-k^2 \alpha^2 (1 + \alpha \beta r(t))^{-5/2}}}{(2\alpha^2 r(t) + \alpha \beta r(t) + 1)^{2/3}} e^{2kt} \sum_{j=1}^{\infty} \frac{D_{\nu_j, -a \sqrt{2k}}(0)}{D'_{\nu_j, -a \sqrt{2k}}(-a \sqrt{2k})} \exp \left(-k \nu_j, -a \sqrt{2k} \right) \left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right).\]

An alternative expression for (19) can be obtained by using Theorem 3.1 with the semi-explicit expression for $p_k^{S,\alpha,\beta, f(t)}$ provided in [36].

**Example 3.4.** By Theorem 2.1 in [27], the law of the first time when a BM hits a parabolic curve of the form $f_a,b(t) = a + bt^2$, for $a, b > 0$, is given by

\[P(T^{f_a,b} \in dt) = e^{-2/3 b^2 t^3} h_{b,a}(t) dt, \quad t > 0,\]

where $h_{b,a} : \mathbb{R}^+ \to \mathbb{R}^+$ is characterised by the Laplace transform

\[\int_0^{\infty} e^{-\lambda u} h_{b,a}(u) du = \frac{Ai \left( (4b)^{1/3} a + (2b^2)^{-1/3} \lambda \right)}{Ai \left( (2b^2)^{-1/3} \lambda \right)}, \quad \lambda > 0,\]
and $Ai$ denotes the Airy function of the first kind, as introduced on page 136 of [24]. By the scaling property of the BM, one has

$$T^{f_{a,b}} = a^2 T^{f_{1,ba^3}};$$

and so we can use the series expansion given in example 4.1.2 of [2] for the density of $T^{f_{1,ba^3}}$ to get

$$\mathbb{P}(T^{f_{a,b}} \in dt) = 2(cba^3)^2 e^{-2/3b^2t^3} \sum_{j=0}^{\infty} \frac{Ai(z_j + 2cba^3)}{Ai'(z_j)} e^{-z_j t/a^3} dt/a^2,$$  \hspace{1cm} (20)

where $(z_j)_{j \geq 0}$ is the decreasing sequence of negative zeros of the function $Ai(.)$ and $c = (2ba^3)^{-1/3}$. Now, the FPT problem for this curve is equivalent to the FPT problem of the OU hitting the curve $g(t) := \Lambda_k^{-1} f_{a,b}(t) = e^{-kt}(a + br^2(t))$, with its FPT law given by

$$\mathbb{P}(T^g_k \in dt) = 2(cba^3)^2 e^{-2/3b^2r(t)^3} \sum_{j=0}^{\infty} \frac{Ai(z_j + 2cba^3)}{Ai'(z_j)} e^{-z_j r(t)/a^3} e^{2kt} dt/a^2,$$

which is obtained using [20] and [23]. Applying the transformation $S^{\alpha,\beta}_k$ to $g$, we get the family of curves

$$S^{\alpha,\beta}_k g(t) = \alpha^{-1}(1 + \alpha \beta r(t)) e^{-kt} \left[ a + b \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right)^2 \right].$$

Recalling the notation $p_k^g(t) dt = \mathbb{P}(T^g_k \in dt)$, applying Theorem 3.1 and simplifying, we obtain

$$p_k^{S^{\alpha,\beta}_k g}(t) = 2e^{2kt}e^{2(cba^3)^2(1 + \alpha \beta r(t))^{-3/2}e^{-t/(1+\alpha \beta r(t))}(S^{\alpha,\beta}_k g(t))^2} e^{2kt} e^{-2/3b^2 \left( \frac{r^2(t)}{1 + \alpha \beta r(t)} \right)^3} \times \sum_{k=0}^{\infty} \frac{Ai(z_k + 2cba^3)}{Ai'(z_k)} e^{-z_k (\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)})/a^3}.$$

### 4 Properties of transformations

We now state a result that gives an insight into our $S^{\alpha,\beta}_k$ transformation.

**Lemma 4.1.** (1) Let $(\alpha, \beta)$ and $(\alpha', \beta') \in \mathbb{R}^* \times \mathbb{R}$ and $A_{k,\infty}$ be defined as in [11]. The mapping $S^{\alpha,\beta}_k: A_{k,\infty} \to A_{k,\infty}$ defined by (16) admits representation (4) and satisfies

$$S^{\alpha,\beta}_k \circ S^{\alpha',\beta'}_k = S^{\alpha\alpha',\beta\beta'}_k.$$

In particular, $(S^{1,\beta}_k)_{\beta \in \mathbb{R}}$ and $(S^{0,0}_k)_{\alpha \in \mathbb{R}}$ are groups. Furthermore, for $a, b > 0$ and $A_k(a, b)$ defined as in [12], if $f \in A_k(a, b)$, then $S^{\alpha,\beta}_k f \in A_k(a, b, b^f_{a,\beta})$, where we set

$$a_{\alpha, \beta} = \begin{cases} \frac{a^2 - \alpha \beta a}{\alpha^2 - \alpha \beta a} & \text{if } \alpha^2 - \alpha \beta a > 0, k \geq 0; \\ \frac{a}{\alpha^2 - (\alpha \beta + 2ka^2) a} & \text{if } 0 < \frac{2ka}{(\alpha \beta + 2ka^2) a} < 1, k < 0; \\ +\infty & \text{otherwise}, \end{cases}$$

and we wrote for $f \in A_{k,\infty}$

$$b^f_{a, \beta} = \begin{cases} b & \text{if } \alpha^2 - \alpha \beta b > 0, k \geq 0; \\ \frac{b}{1 - 2kb} & \text{if } 0 < \frac{2kb}{(\alpha \beta + 2ka^2) b - \alpha^2} < 1, k < 0; \\ \rho \circ \tau f \left( s \left( \frac{\alpha}{a} \right) \right) & \text{otherwise}. \end{cases}$$

Observe that $a_{\alpha, \beta} \to r^*(\xi_{k,\alpha, \beta})$ as $a \to \infty$. 

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(2) Let \( \mu \) be a positive Radon measure on \( \mathbb{R}^+ \). Then, there exists a unique positive and differentiable function \( f \) with \( f(0) = 1 \) and \( f' + k f + \frac{k}{(2k+1)f} > 0 \), which satisfies the following nonlinear differential equation

\[
f'' - k^2 f^4 = -\mu(s(t f)) + k^2 \frac{2k(t f) + 1}{2k(t f) + 1},
\]

where \( f'' \) is the second derivative in the sense of distributions. Moreover, \( \{S_{k}^{\alpha,\beta}f; \alpha > 0, \beta \in \mathbb{R} \} \) spans the set of positive solutions of (22).

Before proving Lemma 4.1 we start by discussing some of the properties of \( \Sigma_k \) and \( \Pi^{\alpha,\beta} \).

**Proposition 4.2.** For any \( a, b > 0 \) and \( f \in A_{k,\infty} \), we have the following assertions.

1. \( \Sigma_k \) is an involution operator, that is \( \Sigma_k \circ \Sigma_k = \text{Id} \) on \( A_{k,\infty} \).
2. \( \tau f(t) = r(\rho \circ \tau \circ \Sigma_k)(r(t)) \).
3. \( \Sigma_k(A_k(a,b)) = A_k(b,a) \).
4. \( \Pi^{\alpha,\beta} \Lambda_k = \Lambda_k \Pi^{\alpha,\beta} \).

**Proof.** (1) The involution property follows from (14) and the fact that \( \Sigma \) is an involution.

(2) By using (15), we obtain

\[
\tau \circ \Sigma_k f(t) = \int_0^t f^2(\rho \circ \tau f(r(y)))e^{2ky+2k\rho r f(r(y))}dy = \int_0^{\rho \circ \tau f(r(t))} e^{2kz}dz = r(\rho \circ \tau f(r(t))),
\]

where we used a change of variables in the second equality. The assertion follows using the fact that \( \Sigma_k \) is an involution.

(3) First, consider \( k < 0 \) and take \( f \in A_k(a,b) \). Observe that \( \rho \circ \tau f(b/(1-2kb)) = -s( -a) \). Now, by part (2), \( \tau \circ \Sigma_k f(-s(-b)) = r(\rho \circ \tau f(b/(1-2kb)) = a/(1-2ka) \) and so \( \Sigma_k f \in A_k(b,a) \). Similarly, the result holds for \( k > 0 \).

(4) \( \tau \Lambda_k f(t) = f^t_0 [e^{2ks(y)}f^2(s(y))]^{-1} dy = \tau f(s(t)) \), which follows by a change of variable. The identity can easily be derived now. \( \square \)

**Proof of Lemma 4.1.** (1) Starting from the definition of \( S_{k}^{\alpha,\beta} \) in (16) and using the decomposition of \( \Sigma_k \) in (14), identity (4) from Proposition 4.2. and the decomposition of \( S^{\alpha,\beta} \) given in (5), we can write

\[
S_{k}^{\alpha,\beta} = \Sigma_k \circ \Pi^{\alpha,\beta} \circ \Sigma_k = \Lambda_k^{-1} \circ (\Sigma \circ \Pi^{\alpha,\beta} \circ \Sigma) \circ \Lambda_k = \Lambda_k^{-1} \circ S^{\alpha,\beta} \circ \Lambda_k.
\]

Now, if we use the functional definitions of the operators, we get the representation (4), as illustrated in the previously reported Figure 1. Next, property (21) follows from the similar group property of the family of operators \( (S^{\alpha,\beta})_{(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}} \), as stated in Section 2.1, see also Proposition 2.3 of [3]. Lastly, using the identity \( \tau S_{k}^{\alpha,\beta} f(t) = \tau f \left( s \left( \frac{a^2 r(t)}{1+a^2 r(t)} \right) \right) \) which follows readily by a change of variables, we get that if \( f \in A_k(a,b) \), then \( S_{k}^{\alpha,\beta} f \in A_k(\alpha,\beta, b') \).
(2) As discussed in Section 2.2, if \( \phi \) solves the Sturm-Liouville equation \((10)\), then the vectorial space \([\Pi^{\alpha,\beta}\phi; \alpha, \beta \in \mathbb{R}]\) is the set of solutions to the same equation. Moreover, all positive solutions are convex and are described by the set \([\Pi^{\alpha,\beta}\varphi; \alpha > 0, \beta \geq 0]\), where \( \varphi \) is the unique, positive, decreasing solution such that \( \varphi(0) = 1 \). Now, we need to show that the image of \((22)\) by \( \Sigma_k \) is \((10)\) and vice-versa. Let \( \phi(t) = \Sigma_k f(t) \). Then, we obtain

\[
\phi(s(t f(t))) = e^{-ks(t f(t))} = e^{-k\alpha t f(t) - kt} \left( k f(t) + f'(t) + \frac{k^2}{(2k t f(t) + 1)f(t)} \right).
\]

Differentiating both sides once, we get

\[
\phi'(s(t f(t))) = -e^{-ks(t f(t))} - k t f(t) + f'(t) + \frac{k^2}{(2k t f(t) + 1)f(t)}.
\]

Differentiating again, we obtain

\[
\phi''(s(t f(t))) = -(2k t f(t) + 1)^2 e^{-ks(t f(t))} - k^2 f'(t) - \frac{k^2}{(2k t f(t) + 1)^2 f(t)}.
\]

So if \( \phi \) satisfies \((10)\), we get \((22)\) and vice versa. Now, by letting \( f = \Sigma_k \varphi \), we see that it satisfies the required properties.

5 Proofs of Theorem 3.1

We provide three different proofs of Theorem 3.1. The first method is a direct approach that uses results from [3]. The second approach uses a generalisation of the so-called Gauss-Markov processes and gives an insight into some of the results given in Lemma 4.1. The third proof relies on the Lie group techniques applied to the Fokker-Planck equation \((5)\). We also mention that, similarly as in the BM case, it is sufficient to only consider the case where \( \alpha > 0 \). This is because, by the symmetry of the BM and the OU process starting at 0 without drift, hitting for the first time negative or positive valued thresholds has the same probability. Hence, for any \( (\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R} \), we have \( T_k^{S_{\alpha,\beta} f} \overset{d}{=} T_k^{S_{\alpha,\beta} \varphi t(\alpha)} f \), where \( \overset{d}{=} \) denotes equality in distribution.

5.1 First proof of Theorem 3.1

We aim to use \((6)\) which connects the FPT distribution of \( S_{\alpha,\beta} f \) to that of \( f \). We can connect the FPTs of OU and BM in the following manner,

\[
T_k^f = \inf \{ t > 0; U_t = f(t) \} = \inf \{ s(t) > 0; W_t = e^{ks(t)} f(s(t)) \} = s(T_k^{\Lambda_k f}). \tag{23}
\]

Note that \( T_n^{S_{\alpha,\beta} f} = s(T^{S_{\alpha,\beta} \Lambda_k f}) \). Writing \( p_k^f(t)dt = P(T_k^f \in dt) \) and \( p(t)dt = P(T^f \in dt) \), we get

\[
p_k^{S_{\alpha,\beta} f}(t) = e^{2k t} p^{S_{\alpha,\beta} \Lambda_k f}(r(t)). \tag{24}
\]

Now, using \((6)\) and \((7)\), we can rewrite the right hand side as follows:

\[
p^{S_{\alpha,\beta} \Lambda_k f}(r(t)) = \alpha^2(1 + \alpha \beta r(t))^{-5/2} e^{-\alpha^2 \beta r(t) / 2(1 + \alpha \beta r(t))} \left( S_{\alpha,\beta} \Lambda_k f(r(t)) \right) \left( S_{\alpha,\beta} \Lambda_k f(r(t)) \right) \left( p^{\Lambda_k f}(r(t)) \right) = \frac{\alpha^2(1 + \alpha \beta r(t))^{-3/2} e^{-\alpha^2 \beta r(t) / 2(1 + \alpha \beta r(t))} \left( S_{\alpha,\beta} \Lambda_k f(r(t)) \right) \left( p^{\Lambda_k f}(r(t)) \right)}{1 + \alpha \beta r(t)}. \tag{25}
\]

By using \((23)\), we get

\[
p_k^{\Lambda_k f} \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right) = \frac{1}{2k} \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right) + 1 \left( p_k^f \left( s \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right) \right) \right). \tag{26}
\]
Notice that \( S_{k}^{α,β}A_k f(r(t)) = e^{kt} S_{k}^{α,β} f(t) \). Now, plugging (26) into (25) and then into (24), we get
\[
p_k f(t) = e^{2kt} (1 + αβr(t))^{-3/2} \left( 2k \frac{α^2 r(t)}{1 + αβr(t)} + 1 \right)^{-3/2} e^{-\frac{α^2 r(t)}{1 + αβr(t)} S_k^{α,β} \left( p_k^t \right)} ,
\]
where we used (4) to obtain the last equation.

\[\square\]

5.2 Second proof of Theorem 3.1

We take \( φ ∈ A_k(a, b) \cap AC([0, b]) \), where \( AC([0, b]) \) is the space of absolutely continuous functions on \([0, b)\). We introduce the process \( X := (X_t)_{0 ≤ t < b} \) to be the generalised Gauss-Markov process of OU type with parameters \( (φ, k) \), which is defined as the unique strong solution to the following SDE
\[
dX_t = \left( \frac{φ'(t)}{φ(t)} + k \right) X_t dt + e^{kt} dB_t, \quad 0 ≤ t < b,
\]
with \( X_0 = x ∈ \mathbb{R} \), i.e.,
\[
X_t = φ(t) e^{kt} \left( x + \int_0^t \frac{1}{φ(s)} dB_s \right), \quad 0 ≤ t < b.
\]

We also denote by \( P^{(φ, k)} = \left( P^{(φ, k)}_x \right)_{x ∈ \mathbb{R}} \) the family of probability measures corresponding to the process \( X \). We assume throughout that \( φ(0) = 1 \). Notice that when \( φ(t) = e^{-kt} \), we get \( X_t = W_r(t) \), where \( (W_t, t ≥ 0) \) is another BM.

**Lemma 5.1.** For any \( y ∈ \mathbb{R} \), set \( T_y = \inf \{0 < t < b; φ(t) e^{kt} \int_0^t \frac{1}{φ(s)} dB_s = y\} \). Then, for any \( f ∈ A_k(a, b) \), setting \( φ = \Sigma_k f \), the identity
\[
T_k^f = s(τφ(T_1))
\]
holds almost surely.

**Proof.** Using (3), we have a.s.
\[
T_1 = \inf \{ t > 0; φ(t) e^{kt} W_rφ(t) = 1\} = \inf \left\{ t > 0; U_{s(τφ(t))} = \frac{e^{-kt - ks(τφ(t))}}{φ(t)} \right\} = p \circ τφ(r(T_k^f))
\]
and the result follows.

\[\square\]

Next, we introduce the following notation
\[
H^f_t(x) = \sqrt{\frac{αφ(t)}{π^{α,β}φ(t)}} \exp \left( \frac{β}{2} \frac{x^2 e^{-2kt}}{φ(t)π^{α,β}φ(t)} \right).
\]

Our aim now is to show that the parametric families of distributions \( \left( P^{(π^{α,β}φ, k)} \right)_{(α, β) ∈ R^+ × R} \) of generalised Gauss-Markov processes are related by some space-time harmonic transformations. The proof of the following proposition is similar to that of Lemma 3.2 in [3]. However, the proof therein has typos, so we give the full proof here.
Proposition 5.2. For \((\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R}\) and \(\phi\) as above, the process \((H^k_t(X_t))_{0 \leq t < a^\phi_{\alpha, \beta}}\) is a \(\mathbb{P}^{(\phi, k)}\)-martingale. Furthermore, the absolute-continuity relationship
\[
d\mathbb{P}^{(\Pi^\alpha, \beta \phi, k)}_{x|F_t} = \frac{H^k_t(X_t)}{H^k_0(x)} \, d\mathbb{P}^{(\phi, k)}_{x|F_t}
\]
holds for all \(x \in \mathbb{R}\) and \(t < a^\phi_{\alpha, \beta}\). Consequently, for any reals \(x\) and \(y\), we have
\[
\mathbb{P}^{(\Pi^\alpha, \beta \phi, k)}_x(T_y \in dt) = \frac{H^k_t(y)}{H^k_0(x)} \mathbb{P}^{(\phi, k)}_x(T_y \in dt), \quad t < a^\phi_{\alpha, \beta}.
\]

Proof. From the Itô formula, we have
\[
\frac{\beta}{2} X^2_t e^{-2kt} = \frac{\beta}{2} \int_0^t X_s e^{-ks} dB_s - \frac{\beta^2}{2} \int_0^t \left( \frac{X_s e^{-ks}}{\phi(s) \Pi^\alpha, \beta \phi(s)} \right)^2 ds + \frac{1}{2} \ln \left( \frac{\alpha + \beta \tau \phi(t)}{\alpha} \right).
\]
Now, as \(E[e^{-\lambda B^2_t}] = (1 + \lambda t)^{-1/2}\), \(\lambda > -1/t\), we deduce that, for all \(t < a^\phi_{\alpha, \beta}\), we have
\[
E[H^k_t(X_t)] = E \left[ \sqrt{\frac{\alpha}{\alpha + \beta \tau \phi(t)}} e^{-\frac{\beta^2}{2 \alpha + \beta \tau \phi(t)}} \right] = 1
\]
Hence, it is a true martingale. Next, notice that
\[
d \left( \beta \int_0^t \frac{X_s e^{-ks}}{\phi(s) \Pi^\alpha, \beta \phi(s)} dB_s, X_t \right) = \beta \frac{X_t}{\phi(t) \Pi^\alpha, \beta \phi(t)} dt,
\]
where \(\langle \cdot, \cdot \rangle\) is the quadratic variation process. Also,
\[
\frac{(\Pi^\alpha, \beta \phi(t))'}{\Pi^\alpha, \beta \phi(t)} = \frac{\phi'(t)}{\phi(t)} + \frac{\beta}{\phi(t) \Pi^\alpha, \beta \phi(t)},
\]
and hence the absolute continuity relationship follows by an application of Girsanov’s theorem. Now, Doob’s optional stopping theorem implies that
\[
\mathbb{P}^{(\Pi^\alpha, \beta \phi, k)}_x(T_y \leq t) = \mathbb{P}^{(\phi, k)}_x \left[ 1_{\{T_y \leq t\}} \frac{H^k_t(X_t)}{H^k_0(x)} \right] = \mathbb{P}^{(\phi, k)}_x \left[ 1_{\{T_y \leq t\}} \mathbb{P}^{(\phi, k)}_x \left( H^k_t(X_t) \middle| \mathcal{F}_t \cap T_y \right) \right] = \mathbb{P}^{(\phi, k)}_x \left[ 1_{\{T_y \leq t\}} \frac{H^k_t(y)}{H^k_0(x)} \right],
\]
where \(1_A\) is the indicator function of the set \(A\). The result follows by differentiation. \(\square\)

Second proof of Theorem 3.1. Let \(\phi = \Sigma_k f, f^\alpha, \beta_k = \phi^\alpha, \beta_k f\) and thus, by definition, \(f^\alpha, \beta_k = \Sigma_k \circ \Pi^\alpha, \beta \phi\). Since \(\Sigma_k\) is an involution, from Lemma 5.1 we get \(T^{f^\alpha, \beta_k}_k = s(\tau \circ \Pi^\alpha, \beta \phi(T_1))\) a.s. Using identity (2) from Proposition 4.2, we get
\[
r(\rho \circ \tau \circ \Pi^\alpha, \beta \phi(r(t))) = r(\rho \circ \tau \circ \Sigma_k f^\alpha, \beta_k(r(t))) = r(f^\alpha, \beta_k(t)).
\]
Now, for \(t < \zeta_{\alpha, \beta}\), writing \(p^{f^\alpha, \beta_k}_k(t)dt = \mathbb{P}(T_k^{f^\alpha, \beta_k} \in dt)\) and \(p^{(\phi, k)}(t)dt = \mathbb{P}^{(\phi, k)}(T_1 \in dt)\), we get
\[
p^{f^\alpha, \beta_k}_k(t) = \frac{1}{(2k(\tau f^\alpha, \beta_k) + 1)(f^\alpha, \beta_k)^2} \mathbb{P}^{(\Pi^\alpha, \beta \phi, k)}(s(\tau f^\alpha, \beta_k(t)))
\]
\[
= \frac{1}{(2k(\tau f^\alpha, \beta_k) + 1)(f^\alpha, \beta_k)^2} \mathbb{P}^{(\Pi^\alpha, \beta \phi, k)}\left( s \left( \tau f \left( s \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right) \right) \right) \right),
\]
\(\rho \circ \tau \circ \Pi^\alpha, \beta \phi(r(t)) = r(\rho \circ \tau \circ \Sigma_k f^\alpha, \beta_k(r(t))) = r(f^\alpha, \beta_k(t))\).
where we used the identity $\tau f_k^{\alpha,\beta}(t) = \tau f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right)$ which follows readily by a change of variable. Now, using Proposition 5.2 with the identities

$$\left(\phi\left(s \circ \tau f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right)\right)\right)^{-1} = \left(\phi\left(\rho \circ \tau \phi\left(r \circ s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right)\right)\right)^{-1}$$

$$= e^{k_{p\rho r}\Sigma_k f\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right) + ks\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)} f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right)$$

$$= e^{ks(\tau f_k^{\alpha,\beta}(t)) + ks\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)} f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right),$$

and

$$(\Pi^{\alpha,\beta}\phi(s(\tau f_k^{\alpha,\beta}(t))))^{-1} = (\Pi^{\alpha,\beta}\phi(\rho \circ \tau \circ \Pi^{\alpha,\beta}\phi(r(t))))^{-1} = e^{k_{p\rho r} \Pi^{\alpha,\beta}\phi(r(t)) + k t f_k^{\alpha,\beta}(t)}$$

yields, for any $t < \zeta_{\alpha,\beta}$,

$$p^{(\Pi^{\alpha,\beta}\phi, k)}\left(s\left(\tau f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right)\right)\right) = \sqrt{1 + \alpha \beta r(t)} e^{-\frac{\alpha^2}{2(1 + \alpha \beta r(t))}f_k^{\alpha,\beta}(t)^2 e^{2 kt}} p(\phi, k)\left(s\left(\tau f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right)\right)\right).$$

Hence,

$$f_k^{\alpha,\beta}(t) = e^{-\frac{\alpha^2}{2(1 + \alpha \beta r(t))}f_k^{\alpha,\beta}(t)^2 e^{2 kt}} \sqrt{1 + \alpha \beta r(t)} p(\phi, k)\left(s\left(\tau f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right)\right)\right).$$

Using Lemma 5.1 again, we finally get (18).

\[ \square \]

### 5.3 Third proof of Theorem 3.1 via the Lie group symmetries

We now provide the last proof for Theorem 3.1 using the Lie group symmetries approach. For Lie Group theory, we refer to [9, 32]. In general, this technique can be used to find solutions to differential equations with new boundary conditions from known ones. For example, the Lie point symmetries of the heat equation

$$\frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial^2 h}{\partial x^2}$$

can be found in Section 3.3 of [3]. Before going through the proof, we first discuss the connections between the below boundary value problems (27) corresponding to the heat equation and the OU Fokker-Planck equation. Set

$$D^f = \{(x, t) \in \mathbb{R} \times \mathbb{R}^+: x \leq f(t)\}.$$  

We introduce the following boundary value problem,

$$\mathcal{H}_k(f) := \begin{cases} \frac{\partial}{\partial t} h(x, t) = k \frac{\partial}{\partial x} (x h(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} h(x, t) & \text{on } D^f; \\
h(f(t), t) = 0 & \text{for all } t > 0; \\
h(., 0) = \delta_0(\cdot) & \text{on } (-\infty, f(0)). \end{cases}$$

(27)

Note that the first equation in $\mathcal{H}_k(f)$ is the OU Fokker-Planck equation [6]. As $k \to 0$, we recover the heat equation and obtain the corresponding boundary value problem for the BM
\( \mathcal{H}(f) \). By Proposition 5.4.3.1 of \([7]\), solutions to \( \mathcal{H}(f) \) and \( \mathcal{H}_k(f) \) admit the following probabilistic representations:

\[
\begin{align*}
&h(x, t)dx = \mathbb{P}(W_t \in dx, \ t < T^f) \quad \text{and} \quad h_k(x, t)dx = \mathbb{P}(U_t \in dx, \ t < T^f_k).
\end{align*}
\]

Using \([3]\) and \((23)\), we connect the two solutions directly in the following way:

\[
\begin{align*}
\mathbb{P}(W_t < x, \ t < T^{\Lambda_kf}) &= \mathbb{P}(e^{-ks(t)}W_t < e^{-ks(t)}x, \ s(t) < s(T^{\Lambda_kf})) = \mathbb{P}(U_s(t) < e^{-ks(t)}x, \ s(t) < T^f_k)
\end{align*}
\]

and so

\[
\begin{align*}
h(x, t) &= e^{-ks(t)}h_k(e^{-ks(t)}x, s(t)) \quad \text{and} \quad h_k(x, t) = e^{kt}h(e^{kt}x, r(t)),
\end{align*}
\]

where \( h \) now denotes a solution to \( \mathcal{H}(\Lambda_kf) \). This shows that the solutions to the Fokker-Plank equation of the OU on \( D^f \) are directly connected to the solutions to the heat equation on \( D^{\Lambda_kf} \). As will become clear in the third proof of Theorem 3.1 at the end of this section, the aim is to find a solution to the Fokker-Plank equation of the OU such that it vanishes on our desired transformed boundary \( S_k^{\alpha,\beta}f \). To do this, we first present a proposition for the OU which resembles Proposition 3.5 in \([3]\) for the Brownian motion.

**Proposition 5.3.** Let \( h_k \) be the solution to the boundary value problem \( \mathcal{H}_k(f) \) in \([27]\). Then, for any \( \alpha > 0, \beta \in \mathbb{R} \), the following mapping

\[
\begin{align*}
h^{\alpha,\beta}_k(x, t) &= \frac{\alpha e^{kt}}{\sqrt{1 + \alpha \beta r(t)}} \frac{\alpha^2 t}{1 + \alpha \beta r(t)} \left(1 + \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right) h_k(x, t, s) \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)},
\end{align*}
\]

for \( t < \zeta_{\alpha,\beta} \), is the solution to the boundary value problem \( \mathcal{H}_k(S_k^{\alpha,\beta}f) \).

**Proof.** Firstly, if \( f \) is an infinitely continuously differentiable function, then so is its transformation \( S_k^{\alpha,\beta}f \). From Proposition 3.5 in \([3]\), the following function

\[
\begin{align*}
h^{\alpha,\beta}_k(x, t) &= \frac{\alpha e^{kt}}{\sqrt{1 + \alpha \beta r(t)}} e^{-\frac{\alpha^2 t}{1 + \alpha \beta r(t)}} h_k(x, t, s) \frac{\alpha^2 t}{1 + \alpha \beta r(t)},
\end{align*}
\]

is a solution to \( \mathcal{H}(S_k^{\alpha,\beta}f) \) whenever \( h \) solves \( \mathcal{H}(\Lambda_kf) \). Now, using relation \((29)\), we define \( \tilde{h}^{\alpha,\beta} \) as \( \tilde{h}^{\alpha,\beta} = e^{kt}h^{\alpha,\beta}(e^{kt}x, r(t)) \). Then, this expression is a solution to the boundary value problem \( \mathcal{H}_k(S_k^{\alpha,\beta}f) \). Using relation \((29)\) again, we can write

\[
\begin{align*}
h \frac{\alpha e^{kt}x}{1 + \alpha \beta r(t)} \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} &= e^{-ks} \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} h_k \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)},
\end{align*}
\]

and so we get that \( \tilde{h}^{\alpha,\beta} \) and \( h^{\alpha,\beta}_k \) coincide, which implies that \( h^{\alpha,\beta}_k \) is indeed a solution to \( \mathcal{H}_k(S_k^{\alpha,\beta}f) \). Moreover, we have that \( h^{\alpha,\beta}_k(x, t) = 0 \iff x = S_k^{\alpha,\beta}f(t) \), because by assumption \( h_k \) is a solution to \( \mathcal{H}_k(f) \), hence \( h_k(f(\cdot), \cdot) = 0 \). Also, \( h^{\alpha,\beta}_k(x, 0) = e^{-\frac{\alpha^2 t}{2}} h^{\alpha,0}_k(x, 0) \). Now, let us investigate \( h^{\alpha,0}_k \). Using relation \((29)\) again, we get

\[
\begin{align*}
h^{\alpha,0}_k(x, t) &= \alpha e^{kt-ks(\alpha^2 r(t))} h_k(\alpha e^{-ks(\alpha^2 r(t))} + kt x, a(\alpha^2 r(t)))
\end{align*}
\]

\[
\begin{align*}
&= \alpha e^{kt-ks(e^{-2(-ln(\alpha))}r(t))} h_k(e^{-ks(e^{-2(-ln(\alpha))}r(t)) + kt - (-ln(\alpha))} x, s(e^{-2(-ln(\alpha))}r(t))
\end{align*}
\]

\[
\begin{align*}
&= e^{kt} h(e^{-e^{kt}x} e^{-2e^{rt}} t) = e^{kt} h^{(4)}(e^{kt} x, r(t)),
\end{align*}
\]

where \( \epsilon = -\ln(\alpha) \). Now, the latter \( h^{(4)}_k \) is the fourth symmetry of the heat equation listed in Section 3.3 of \([3]\), meaning that it satisfies the heat equation. By using relation \((29)\), we see that
\[ e^{kt}h_k^{(4)}(e^{kt}x, r(t)) \] indeed satisfies the Fokker-Plank equation of the OU and so \( h_k^{\alpha,0} \) is a solution to \( \mathcal{H}_k(S_k^{\alpha,0} f) \) and thus in particular \( h_k^{\alpha,0}(\cdot, 0) = \delta_0(\cdot) \) on \((-\infty, \frac{f(0)}{\alpha})\). Hence, we get
\[ h_k^{\alpha,\beta}(x, 0) = e^{-\frac{\alpha x^2}{2}} h_k^{\alpha,0}(x, 0) = \delta_0(x), \]
which concludes the proof. \( \square \)

### 5.3.1 Lie symmetries of OU Fokker-Planck equation

In this section, we provide a direct construction of the function \( h_k^{\alpha,\beta} \) given in (30) from the Lie-symmetries of the OU Fokker-Planck equation (5). Using similar techniques discussed in Section 2.4 in [32], after some lengthy and tedious calculations, one finds that the Lie algebra of infinitesimal symmetries of the OU Fokker-Planck equation is spanned by six vector fields, where \( x, t \) are the two independent variables and \( h \) is the dependent variable,

\[
\begin{align*}
\mathbf{v}_1 &= e^{-kt} \frac{\partial}{\partial x}, & \mathbf{v}_2 &= \frac{\partial}{\partial t}, & \mathbf{v}_3 &= h \frac{\partial}{\partial h}, & \mathbf{v}_4 &= e^{kt} \frac{\partial}{\partial x} - 2khe^{kt} \frac{\partial}{\partial h}, \\
\mathbf{v}_5 &= -kxe^{-2kt} \frac{\partial}{\partial x} + e^{-2kt} \frac{\partial}{\partial t} + kh e^{-2kt} \frac{\partial}{\partial h}, & \mathbf{v}_6 &= kxe^{2kt} \frac{\partial}{\partial x} + e^{2kt} \frac{\partial}{\partial t} - 2k^2x^2h e^{2kt} \frac{\partial}{\partial h}
\end{align*}
\]

and by the infinite-dimensional sub-algebra \( \mathbf{v}_a = u(x, t) \frac{\partial}{\partial x} \), where \( u(x, t) \) is an arbitrary solution of the OU Fokker-Planck equation. Now, by exponentiating the basis, as in page 89 of [32], we can produce the one-parameter group of transformations leaving invariant \( \mathcal{H}_k(f) \). Doing this procedure for each of the vector fields, we obtain the one-parameter groups \( \mathcal{G}_i \) generated by the \( \mathbf{v}_i \). Since each group \( \mathcal{G}_i \) is a symmetry group, if \( h \) is a solution to (5), then the following are also solutions to (5):

\[
\begin{align*}
\mathit{h}_{k,\epsilon}^{(1)}(x, t) &= h(x - \epsilon e^{-kt}, t) \\
\mathit{h}_{k,\epsilon}^{(2)}(x, t) &= h(x, t - \epsilon) \\
\mathit{h}_{k,\epsilon}^{(3)}(x, t) &= \epsilon \mathit{h}(x, t) \\
\mathit{h}_{k,\epsilon}^{(4)}(x, t) &= e^{-2kxe^{kt} + k^2e^{2kt}} h(x - \epsilon e^{kt}, t) \\
\mathit{h}_{k,\epsilon}^{(5)}(x, t) &= \frac{e^{kt}}{\sqrt{e^{2kt} - 2k\epsilon}} h \left( \frac{x e^{kt}}{\sqrt{e^{2kt} - 2k\epsilon}} - \frac{\ln (e^{2kt} - 2k\epsilon)}{2k} \right) \\
\mathit{h}_{k,\epsilon}^{(6)}(x, t) &= \frac{e^{-x^2e^{2kt}/2k\epsilon}}{\sqrt{e^{-2kt} + 2k\epsilon}} h \left( \frac{x e^{-kt}}{\sqrt{e^{-2kt} + 2k\epsilon}} - \frac{\ln (e^{-2kt} + 2k\epsilon)}{2k} \right).
\end{align*}
\]

Notice that these symmetry groups provide an explanation of the invariance of the law of the OU process under some specific transformations. More precisely, \( \mathit{h}_{k,\epsilon}^{(1)} \) and \( \mathit{h}_{k,\epsilon}^{(2)} \) show the space and time invariance of the law of the OU, respectively, \( \mathit{h}_{k,\epsilon}^{(3)} \) is the trivial multiplication by a constant, \( \mathit{h}_{k,\epsilon}^{(4)} \) represents the Girsanov transform connecting the law of the OU with different exponential drifts, while compositions of \( \mathit{h}_{k,\epsilon}^{(5)} \) and \( \mathit{h}_{k,\epsilon}^{(6)} \) are deeply connected to the change of measure connecting the law of the OU with its bridges.

In Proposition 5.3 we proved that the function \( h_k^{\alpha,\beta} \) is a solution to the boundary value problem \( \mathcal{H}_k(S_k^{\alpha,\beta} f) \) by directly using the relation between the two boundary value problems in [27]. We can also construct this function directly using the symmetries above.

**Remark 5.4.** Proposition 5 of [28] gives a general class of symmetries for SDEs with a drift term specified in Proposition 4 therein. The expression therein contains misprints in \( P_{1,2} \) and \( T_{p_{1,2}} \).
Using the notation in the original article, the correct expressions are

\[
P_{1,2}^r : \quad \frac{u(x, t)}{U(X, T)} = \frac{\theta_{F_2}(x)}{\theta_{F_2}(X)} e^{\pm 2\sqrt{\mathcal{A}}(\nu - \frac{1}{4} \pm \frac{1}{4})t \mp \frac{2\alpha}{\beta^2(x + 2B/A)^2}}
\]

\[
T_{P_{1,2}} = \mp \ln(e^\pm 2\sqrt{\mathcal{A}} t + \epsilon)/(2\sqrt{\mathcal{A}}).
\]

If we set \( A = k^2, B = 0, \nu = 0 \) and \( \theta_{F_2}(x) = e^{-kx^2/2} \), we recover the above symmetries \((31)\) for the OU process.

**Lemma 5.5.**

\[
h_k^{\alpha, \beta} = \left( h_k^{(2)}(\frac{k}{\ln(\alpha)} \circ h_k^{(5)})(\frac{\alpha^2 - 1}{2\pi}) \right) \circ \left( h_k^{(6)}(\frac{\beta}{2k(2k\alpha - \beta)}) \circ h_k^{(2)}(\frac{1}{\ln(\frac{2k\alpha - \beta}{2k})} \circ h_k^{(5)}(\frac{\beta}{2k(2k\alpha - \beta)}) \right).
\]

Although the calculation is tedious, it can be easily shown that the expression given in Lemma 5.5 holds. Now, because the function \( h_k^{\alpha, \beta} \) is a composition of symmetries of the OU Fokker-Planck equation, then it is itself a solution to the OU Fokker-Planck equation. Lemma 5.5 can then be used to shorten the first part of the proof given in Proposition 5.3. Now, we give the third proof of Theorem 3.1.

**Third proof of Theorem 3.1.** Let \( h_k \) be the solution to \( \mathcal{H}_k(f) \). Then, by (28), we have \( \mathbb{P}(T_k^f \leq t) = 1 - \int_{-\infty}^{f(t)} h_k(x, t)dx \). Setting \( p_k^f(t)dt = \mathbb{P}(T_k^f \in dt) \), we get

\[
p_k^f(t) = -\frac{d}{dt} \left( \int_{-\infty}^{f(t)} h_k(x, t)dx \right) = -h_k(f(t), t)f'(t) - \int_{-\infty}^{f(t)} \frac{\partial}{\partial t} h_k(x, t)dx = -\int_{-\infty}^{f(t)} \frac{\partial}{\partial t} h_k(x, t)dx,
\]

because \( h_k \) vanishes on the boundary \( f \). Using the OU Fokker-Planck equation \((5)\), we get

\[
p_k^f(t) = -\int_{-\infty}^{f(t)} -\frac{\partial}{\partial x}[-kxh_k(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} h_k(x, t) dx = -\frac{1}{2} \frac{\partial}{\partial x} h_k(x, t)|_{x=f(t)}.
\]

Now, by Proposition 5.3 and Lemma 5.5, we deduce that \( h_k^{\alpha, \beta} \) in \((30)\) is a solution to the boundary value problem \( \mathcal{H}_k(S_k^{\alpha, \beta} f) \). Hence, using \( h_k^{\alpha, \beta} \) with its corresponding boundary \( S_k^{\alpha, \beta} f \) in \((32)\) gives us expression \((18)\) from Theorem 3.1.

**Remark 5.6.** The algebraic proof based on the Lie symmetry approach yields uniqueness of the family of transformations \( S_k^{\alpha, \beta} \) and thus of the relationship \((18)\). Indeed, the function \( h_k^{\alpha, \beta} \) is the unique solution to \( \mathcal{H}_k(S_k^{\alpha, \beta} f) \), and in order to satisfy the corresponding third condition of \((27)\), we can only construct further solutions by composing it with symmetry groups \((31)\) such that the time component is 0 whenever \( t = 0 \). This only leads to a change in constants \( \alpha \) or \( \beta \), making \( S_k^{\alpha, \beta} \) the only non-trivial transformation leading to \((18)\). Note that, one could use symmetry \( h_k^{(4)} \) to add an extra trend to the \( S_k^{\alpha, \beta} \) transformation, but the form of the transformation would stay the same.

**Remark 5.7.** When we take the limit as \( k \) goes to 0 of the symmetries in \((31)\), we only recover the first three symmetries of the heat equation, found in Section 3.3 of \([3]\). The rest of the symmetries of the heat equation can be recovered by using a combination of the symmetries in \((31)\).

**Remark 5.8.** This Lie approach led to a variant of our boundary crossing identity \((15)\) in \([28]\) (their equation 40) up to some misprints therein. In particular, by setting \( \beta' = \beta/2k\alpha, \alpha' = 1/\alpha^2 - \beta/2k\alpha - 1, A = k^2, B = 0, \nu = 0, x_0 = 0 \) and \( \theta_{F_2}(x) = e^{-kx^2/2} \), we noted that the first term in \((40)\) should be \( T_{-1}^{-1/2}T_{+1}^{1/2} \sqrt{\alpha' + \beta' + 1} \) instead of \( T_{+1}^{1/2} / \sqrt{\alpha' + \beta' + 1} \).
6 Asymptotic behaviour of the FPT densities

Here, we discuss the asymptotic behaviour of the FPT distribution of $T_k^{S_{\alpha,\beta}^f}$ and the transience of the transformed curve $S_{\alpha,\beta}^f$, i.e., $\mathbb{P}(T_k^{S_{\alpha,\beta}^f} < \infty) < 1$. As before, we assume that our boundary $f \in C^1([0,\infty),\mathbb{R}^+)$, $f(0) \neq 0$ and only consider $\alpha > 0$, as discussed before Section 5.1. Write $\mathbb{P}(T_k^f \in dt) = p_k(t)dt$. If $\zeta_{\alpha,\beta} < \infty$, from Theorem 3.1 we get the following asymptotic identity

$$p_k^{S_{\alpha,\beta}^f}(t) \sim e^{2kt} \alpha^2 (\alpha r(t) + 1)^{-1/2} \frac{e^{-\frac{2\alpha r(t)}{\alpha + 2}} (S_k^{S_{\alpha,\beta}^f}(t))^2 e^{2kt}}{2k\alpha^2 r(t) + \alpha r(t) + 1} p_k(s(\Gamma_{\alpha,\beta})) \quad as \ t \to \infty,$$

where

$$\Gamma_{\alpha,\beta} := \begin{cases} \frac{2}{\beta} & \text{if } \beta > 0, \ k \geq 0; \\ \frac{2}{\beta} \frac{\alpha^2}{\alpha - \beta} & \text{if } \beta > \frac{2k}{\alpha} - 2k\alpha, \ k < 0, \end{cases}$$

and $h(t) \sim I(t)$ as $t \to \zeta$ denotes that $h(t)/I(t) \to 1$ as $t \to \zeta$, for some $\zeta \in [0,\infty]$. We now give a necessary and sufficient condition for a curve to be transient for the OU process.

**Theorem 6.1.** $S_{\alpha,\beta}^f$ is transient in the following two cases

(i) $\zeta_{\alpha,\beta} < \infty$;

(ii) $\zeta_{\alpha,\beta} = +\infty$ and $0 < (\beta - \frac{2k}{\alpha} 1_{\{k < 0\}}) \sqrt{2k\Gamma_{\alpha,\beta} + 1} f(s(\Gamma_{\alpha,\beta})) < \infty$.

**Proof.** In case (i), there is always a positive probability of the OU process never hitting the curves, making them transient. We now proceed with proving the result for case (ii). By relation [23], we deduce that

$$\mathbb{P}(T_k^f \in dt) = \mathbb{P}(s(T_k^{\Lambda_k}f) < \infty) = \mathbb{P}(T_k^f < \infty).$$

Now, by the classical Kolmogorov-Erős-Petrovskii theorem reported in [19], if $t^{-1/2} \Lambda_k f(t)$ (i.e. $f(t)e^{kt}/\sqrt{r(t)}$) is increasing for sufficiently large $t$, then $\Lambda_k f$ is a transient curve for the Brownian motion if and only if

$$\int_1^\infty t^{-3/2} \Lambda_k f(t) e^{-(\Lambda_k f)^2/2t} dt = \int_1^\infty \frac{e^{3kt} f(t)}{r(t)^{3/2}} e^{-e^{2kt} f(t)^2/2r(t)} dt < \infty. \quad (34)$$

Since $\zeta_{\alpha,\beta} = +\infty$, note that

$$g(t) := e^{kt} S_k^{\alpha,\beta} f(t)/r(t) \sim \left(\beta - \frac{2k}{\alpha} 1_{\{k < 0\}}\right) \sqrt{2k(\Gamma_{\alpha,\beta}) + 1} f(s(\Gamma_{\alpha,\beta})) > 0 \quad as \ t \to \infty. \quad (35)$$

Furthermore, because of (35), $e^{kt} S_k^{\alpha,\beta} f(t)/r(t)$ is increasing for sufficiently large $t$, and under the condition $0 < (\beta - \frac{2k}{\alpha} 1_{\{k < 0\}}) \sqrt{2k\Gamma_{\alpha,\beta} + 1} f(s(\Gamma_{\alpha,\beta})) < \infty$, we have

$$\int_1^\infty \frac{e^{3kt} S_k^{\alpha,\beta} f(t)}{r(t)^{3/2}} e^{-2kt(S_k^{\alpha,\beta} f(t))^2/2r(t)} dt = \int_1^\infty \frac{1}{\sqrt{z}} g(s(z)) e^{-g(s(z))^2 z^{1/2}} dz < \int_1^\infty g(s(z)) e^{-g(s(z))^2 z^{1/2}} dz < \infty.$$

The last integral is finite because $g$ eventually stabilises and the assumptions ensure that $g$ never blows up or vanishes, so its minimum and the maximum are always attained. Hence by the integral test (34), $S_{\alpha,\beta}^f$ is transient. 

Now, we derive asymptotics for when $\zeta_{\alpha,\beta} < \infty$. Notice that in this case, $s \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right) \to +\infty$ as $t \to \zeta_{\alpha,\beta}$ and whenever $f$ is transient and satisfies suitable conditions, we can derive the asymptotic expression of the FPT density of the OU hitting curves $S_{\alpha,\beta}^f$ by making use of the corresponding result for the BM by Anderson and Pitt [4], as proved in the following Proposition.

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Proposition 6.2. Assume that $f$ is transient and satisfies the following conditions

(i) $\Lambda_k f$ is increasing, concave, twice differentiable on $(0, \infty)$ and of regular variation at $\infty$ with index $a \in [1/2, 1)$,

(ii) $\Lambda_k f(t)/\sqrt{t}$ is increasing at $\infty$, and $\Lambda_k f(t)/t$ is convex and decreases to $0$ for sufficiently large $t$,

(iii) There exist positive constants $c < 1$ and $c'$ such that the inequalities

$$tf'(s(t)) \leq (c(2kt + 1) - kt)f(s(t)) \quad \text{and} \quad |t^2/(2kt + 1)^2(f''(s(t)) - k^2f(s(t)))| \leq c'f(s(t))$$

are met for sufficiently large $t$.

Then, if $\zeta_{k, \alpha, \beta} < \infty$, we have

$$p^S_{k, \alpha, \beta} f(t) \sim \left(\frac{2k}{\alpha \beta} + 1\right)(1 - r)\bar{f}(t) \quad \text{as} \quad t \to \zeta_{k, \alpha, \beta},$$

where

$$\bar{f}(t) = \left(\frac{k \alpha^2 r(t)}{1 + \alpha \beta r(t)} + 1\right) f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right) - \left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)f\left(s\left(\frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)}\right)\right).$$

Proof. By Theorem 1 of [4], if conditions (i)-(iii) are satisfied, we can write

$$p^\Lambda_{k, f}(t) \sim (1 - r)\Lambda_k f(t) - t(\Lambda_k f(t))' \frac{e^{-\Lambda_k f(t)/2t}}{\sqrt{2\pi t^{3/2}}} \quad \text{as} \quad t \to \infty.$$

Using (23), we get $p^f_{k}(t) = e^{2kt}p^{\Lambda_k f}(r(t))$. We then simplify the expression and combine it with (18). The result follows by noticing that the assumption $\Lambda_k f(t)/t \downarrow 0$ as $t \to \infty$ forces the exponential terms to vanish. \qed

Remark 6.3. We noted a misprint in Section 4.3 of [2]. For the Brownian motion, when $\beta > 0$, the one parameter family of transformed functions $S^{1, \beta} f$ is increasing for sufficiently large $t$, $0 < \beta f(\alpha/\beta) < \infty$ and $f(0) > 0$.

7 Interpretation via the method of images

In [25], the author did a thorough investigation of the method of images for the standard BM. Here, we apply the method of images to the OU process and use it to produce new examples of curves with explicit FPT densities. As in the BM case, we would like to construct a function $h_k$ satisfying the OU Fokker-Planck equation. Then, by the uniqueness of such solutions, as seen in Section 5.3, $h_k$ would also satisfy (28) and be the solution to $\mathcal{H}_k(f)$. We proceed as follows. First, we assume to have a positive $\sigma$-finite measure $F$ with $\int_0^\infty \phi(\sqrt{\epsilon \theta})F(d\theta) < \infty$ for all $\epsilon > 0$.

Then, for $a > 0$, we define the $h_k$ function by

$$h_k(x, t) := p_t(0, x) - \frac{1}{a} \int_0^\infty p_t(x, \theta)F(d\theta) = \frac{e^{kt}}{\sqrt{r(t)}} \phi\left(\frac{xe^{kt}}{\sqrt{r(t)}}\right) - \frac{1}{a} \int_0^\infty \frac{e^{kt}}{\sqrt{r(t)}} \phi\left(\frac{xe^{kt} - \theta}{\sqrt{r(t)}}\right)F(d\theta).$$

We know that $h_k$ vanishes on the boundary $f$, and so by simplifying, we get that $h_k(x, t) = 0$ is equivalent to

$$l\left(\frac{xe^{kt}}{r(t)} \cdot \frac{1}{r(t)}\right) = a,$$

where $l(y, s) = \int_0^\infty e^{\theta y - \frac{1}{2} \theta^2 s}F(d\theta)$.

The following lemma gives us a characterisation of the boundaries obtained through the method of images.
Lemma 7.1 (Characterisation of boundaries). The boundaries $f$ obtained through the method of images have the following properties:

1. $f$ is infinitely often continuously differentiable;
2. $\Lambda_k f(t)/t$ (i.e. $f(t)e^{kt}/r(t)$) is monotone decreasing;
3. $f''(t) - k^2 f(t) \leq 0$.

Proof. In Lemma 1.1 of [25], it is shown that the curves $\eta(t)$ that satisfy the equation $l(\eta(t), \frac{1}{t}) = a$ are infinitely continuously differentiable and concave, with $\eta(t)/t$ monotone decreasing. As in our case we have $a = l \left( \frac{f(t)e^{kt}}{r(t)}, \frac{1}{r(t)} \right)$, by changing $t \to s(t)$, we get $a = l \left( \frac{\Lambda_k f(t)}{t}, \frac{1}{t} \right)$. So $\eta(t) := \Lambda_k f(t)$ must satisfy the three properties given in Lemma 1.1 of [25], so (1) and (2) follow directly. For (3),

$$f''(t) = k^2 e^{-kt} \eta(r(t)) + e^{3kt} \eta''(r(t)) = k^2 f(t) + e^{3kt} \eta''(r(t))$$

and the result follows as $\eta(t)$ is concave. 

Remark 7.2. From Lemma 4.1 if $\mu(s(\tau f)) > k^2$ and $f \geq 0$, then we get that the solutions to nonlinear ordinary differential equations [22] satisfy $f'' - k^2 f \leq 0$.

The following theorem gives us a way of calculating the density of $T_k^f$ explicitly.

Theorem 7.3. The FPT density of $T_k^f$ is given by

$$\mathbb{P}(T_k^f \in dt) = \frac{e^{2kt}}{2r(t)^{3/2}} \phi \left( \frac{f(t)e^{kt}}{\sqrt{r(t)}} \right) E(\theta|(f(t), t)) dt,$$

where

$$E(\theta|(f(t), t)) = \frac{\int_0^\infty \theta \phi \left( \frac{f(t)e^{kt}-\theta}{\sqrt{r(t)}} \right) F(d\theta)}{\int_0^\infty \phi \left( \frac{f(t)e^{kt}-\theta}{\sqrt{r(t)}} \right) F(d\theta)}.$$

Proof. We just use relation [32] to get the result. 

We now consider some examples of the boundaries that arise for specific measures $F$ via Theorem 7.3.

Example 7.4. Consider $F(d\theta) = \delta_{2z}$ for some $z > 0$. Using $h_k(f(t), t) = 0$ and simplifying, we get

$$f(t) = \frac{\ln(a)}{2zk} \sinh(kt) + ze^{-kt}.$$

By Theorem 7.3

$$\mathbb{P}(T_k^f \in dt) = \frac{ze^{2kt}}{r(t)^{3/2}} \phi \left( \frac{\frac{\ln(a)}{2zk} r(t) + z}{\sqrt{r(t)}} \right) dt.$$

Remark 7.5. In [13], the FPT of a mean-reverting OU with parameter $\mu$ hitting a hyperbolic type boundary of the form $\mu + Ae^{kt} + Be^{-kt}$ is studied. This reduces to the FPT of a standard OU hitting a curve of the form $Ae^{kt} + Be^{-kt}$ for arbitrary constants $A$ and $B$ which is the curve given in this example. Also, applying the $S_k$ transformation to these curves gives the same family of curves with different constant coefficients.
If the support of the measure $F$ is on $\mathbb{R}$ and $F(\{0\}) = 0$, then we define

$$h_k(x, t) := p_t(0, x) - \frac{1}{a} \int_{-\infty}^{\infty} p_t(\theta, x) F(d\theta).$$

Then, there exist positive and negative valued functions $f_+$ and $f_-$, with $f_- < f_+$ and the properties $h_k(f_+(t), t) = 0$ and $h_k(f_-(t), t) = 0$ for all $t < t_a$ for a certain $t_a \leq \infty$. Then, the stopping time can also be defined as the first exit time from the region $(f_-(t), f_+(t))$, i.e.

$$T^f_k = \inf\{0 < t < t_a; \; U_1 \notin (f_-(t), f_+(t))\}.$$ (37)

Moreover, if $F$ is symmetric, then $f_+ = -f_-$, and the FPT distribution of $T^{S_{\alpha, \beta} f_\pm}_k$ can be obtained using our main identity (18) in Theorem 3.1, replacing $T^f_k$ and $T^{S_{\alpha, \beta} f}$ with $T^{f_\pm}_k$ and $T^{S_{\alpha, \beta} f_\pm}_k$, respectively. This follows directly by replacing the processes $B$ and $U$ with $|B|$ and $|U|$, respectively. We now see an example of this.

**Example 7.6.** Consider $F(d\theta) = \frac{d\theta}{\sqrt{2\pi}}$ on $\mathbb{R}$. Then,

$$f_\pm(t) = \pm e^{-kt} \sqrt{r(t) \ln \left( \frac{a^2}{r(t)} \right)}, \quad 0 < t \leq t_a = \begin{cases} s(a^2) & \text{if } k \geq 0 \text{ or } k < 0, a^2 < -1/(2k), \\ +\infty & \text{if } k < 0, a^2 \geq -1/(2k). \end{cases}$$

This is a two sided boundary, with exit time $T^{f_\pm}_k$ defined in (37) and FPT density given by

$$P(T^{f_\pm}_k \in dt) = \frac{e^{2kt}}{2r(t)} \phi \left( \ln \left( \frac{a^2}{r(t)} \right) \right) \sqrt{\ln \left( \frac{a^2}{r(t)} \right)} dt.$$

Applying the $S_k$ transformation [3] to this curve, we get

$$S^{\alpha, \beta}_k f_\pm(t) = \pm \sqrt{r(t)} \sqrt{1 + \alpha \beta r(t)} e^{-kt} \left( \ln \left( \frac{a^2(1 + \alpha \beta r(t))}{\alpha^2 r(t)} \right) \right), \quad t < \zeta_{\alpha, \beta, a},$$

where

$$\zeta_{\alpha, \beta, a} = \begin{cases} s \left( \frac{-a^2}{a^2 + \alpha^2} \right) & \text{if } a^2 \beta < \alpha; \\ +\infty & \text{otherwise}. \end{cases}$$

The FPT density is given by

$$P^{S_{\alpha, \beta} f_\pm}_k(t) = e^{2kt} \alpha^2 \frac{(1 + \alpha \beta r(t))^{-3/2}}{(2k + a^2 r(t))^{1 + \alpha \beta r(t)}} e^{-\frac{a^2}{2(1 + \alpha \beta r(t))} (S^{\alpha, \beta}_k f_\pm(t))^2} e^{2kt} p^{f_\pm}_k \left( s \left( \frac{\alpha^2 r(t)}{1 + \alpha \beta r(t)} \right) \right)$$

$$= e^{2kt} \sqrt{8\pi} \left[ \ln \left( \frac{a^2(1 + \alpha \beta r(t))}{\alpha^2 r(t)} \right) \right] e^{-\frac{1}{2} \ln \left( \frac{a^2(1 + \alpha \beta r(t))}{\alpha^2 r(t)} \right) (\alpha \beta r(t) + 1)}.$$ (38)

Now, for example, by setting $a = 1, \alpha = 2, \beta = k$, we get

$$S^{2, k}_k f_\pm(t) = \pm \sqrt{r(t)} \sqrt{\ln \left( \frac{k}{2(1 - e^{-2kt})} \right)},$$

with FPT density given by

$$P^{S^{2, k} f_\pm}_k(t) = e^{kt} \sqrt{8\pi} \left[ \ln \left( \frac{k}{2(1 - e^{-2kt})} \right) \right] e^{-\frac{2kt}{2} \ln \left[ \frac{k}{2(1 - e^{-2kt})} \right]}.$$
Figure 3: Transformed curves \( \alpha < 2 \) (left panel) and \( \alpha > 2 \) (right panel). In particular, we choose: \( \beta = 1, k = 1, a = 1 \) with \( \alpha = 2, 3, 4 \) in the left panel, and \( k = 1, a = 1, \alpha = 1 \) and \( \beta = 2, 3, 4 \) in the right panel.

In Figure 3 we report the transformed curves (38) for different values of \( \alpha \) and \( \beta \). In particular, for \( \alpha^2 > \alpha \), we get closed shaped curves defined on \( t \in (0, \zeta_{\alpha,\beta,a}) \), approaching 0 as \( t \to 0 \) or \( t \to \zeta_{\alpha,\beta}, \) see the left panel. Otherwise, we get open curves that diverge to \( \pm \infty \), see the right panel.

Remark 7.7. Back to our \( S_{\alpha,\beta}^{k} \) transformation, using (36) with the time-transformation \( t \to s \left( \frac{\alpha^2 r(t)}{1+\alpha^2 r(t)} \right) \), we get

\[
a = \int_{0}^{\infty} e^{\frac{\beta^2}{2} F_{\alpha,\beta}(d\theta)} e^{\frac{1}{2} \left( \frac{\theta}{\alpha} \right)^2} F_{\alpha,\beta}^{0}(d\theta),
\]

where \( F_{\alpha,\beta}(d\theta) = e^{-\frac{\theta^2}{2} F_{\alpha,\beta}^{0}(d\theta)} \) is the measure corresponding to the curve \( S_{\alpha,\beta}^{k} f(t) \), for \( t < \zeta_{\alpha,\beta} \).

Acknowledgements

We are grateful to Dmitry Muravey for a discussion on the Lie symmetries while finalising the paper. AA was funded by EPSRC grant [EP/V520226/1] as part of the Warwick CDT in Mathematics and Statistics. For the purpose of open access, the authors have applied a Creative Commons Attribution (CC BY) licence to any Author Accepted Manuscript version arising from this submission.

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