Counting Labelled Trees with Given Indegree Sequence

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Mar 27, 2008

Abstract. For a labelled tree on the vertex set \([n] := \{1, 2, \ldots, n\}\), define the direction of each edge \(ij\) to be \(i \to j\) if \(i < j\). The indegree sequence of \(T\) can be considered as a partition \(\lambda \vdash n - 1\). The enumeration of trees with a given indegree sequence arises in counting secant planes of curves in projective spaces. Recently Ethan Cotterill conjectured a formula for the number of trees on \([n]\) with indegree sequence corresponding to a partition \(\lambda\). In this paper we give two proofs of Cotterill’s conjecture: one is “semi-combinatorial” based on induction, the other is a bijective proof.

Keywords: Labelled tree, indegree sequence, partition, bijection, lattice.

AMS Classification: 05A15, 05C07, 05A18.

1 Introduction

For a labelled tree on the vertex set \([n] := \{1, 2, \ldots, n\}\), define the direction of each edge \(ij\) as \(i \to j\) if \(i < j\). The indegree sequence of \(T\) can be considered as a partition \(\lambda \vdash n - 1\). The problem of counting labelled trees with a given indegree sequence was encountered by Ethan Cotterill [2] when counting secant planes of curves in projective spaces. Write \(\lambda = \langle 1^{m_1}2^{m_2} \cdots \rangle\) if \(\lambda\) has \(m_i\) parts equal to \(i\). Given \(\lambda = \langle 1^{m_1}2^{m_2} \cdots \rangle \vdash n - 1\), let \(k\) be the number of parts of \(\lambda\), and \(a_\lambda\) be the number of trees on \([n]\) with indegree sequence corresponding to \(\lambda\). Cotterill [2, Page 29] conjectured the following result:

\[
a_\lambda = \frac{(n - 1)!^2}{(n - k)!1^{m_1}2^{m_2} \cdots m_1!m_2! \cdots}.
\]

(1.1)

Note that the above formula can also be written as

\[
a_\lambda = \frac{(n - 1)!}{(n - k)!} \cdot \frac{(n - 1)!}{1^{m_1}2^{m_2} \cdots m_1!m_2! \cdots},
\]

(1.2)

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in which the second factor on the right hand side counts the number of partitions $\pi$ of an $(n-1)$-element set of type $\lambda$, i.e., the block sizes of $\pi$ are $\lambda_1, \lambda_2, \ldots$. This suggests that it may help to prove (1.1) if we can find a map $\phi : T_\lambda \to \Pi_\lambda$ for any $\lambda \vdash n-1$, where $T_\lambda$ is the set of trees on $[n]$ with indegree sequence $\lambda$, and $\Pi_\lambda$ is the set of partitions of $[2,n] := \{2,3,\ldots,n\}$ of type $\lambda$. Richard Stanley (personal communication) suggested that such a map $\phi$ can be defined as follows.

Given $\lambda \vdash n-1$ and $T \in T_\lambda$, we can consider $T$ as a rooted tree on $[n]$ with the root 1 “hung up” (See Figure 1). Now we label the edges of $T$ such that each edge has the same label as the vertex right below it. It is obvious that during the labelling each number in $[2,n]$ is used exactly once. Putting the labels of those edges which point to the same vertex into one block, we get a partition $\pi \in \Pi_\lambda$. Figure 1 shows a tree $T \in T_{3221}$, and $\phi(T) = \pi = 8/569/37/24 \in \Pi_{3221}$. We put a bar over the label of each edge to avoid confusion.

While the map $\phi$ gives a natural interpretation of the second factor in equation (1.2), one can easily check that the preimage of $\phi$ is not unique: we can get the same partition by applying $\phi$ to different trees. Let $T_\pi$ be the set of preimages of $\pi \in \Pi_\lambda$ under the map $\phi$, i.e., $T_\pi = \phi^{-1}(\pi)$, and let $f(\pi) := |T_\pi|$. Then $T_\lambda = \bigcup_{\pi \in \Pi_\lambda} T_\pi$. Our main task is to prove the following theorem.

**Theorem 1.1** Given $\lambda \vdash n-1$ and $\pi \in \Pi_\lambda$, we have

$$f(\pi) = |T_\pi| = \frac{(n-1)!}{(n-|\pi|)!},$$

where $|\pi|$ is the number of blocks of $\pi$.

In the remainder of this paper we give proofs of this result using two different approaches. In Section 2, we give a “semi-combinatorial” proof based on induction on $n$. In Section 3 and Section 4, we give a bijective proof. Finally in section 5, some further problems are raised.
2 A Semi-combinatorial Proof

In this section, we will give an inductive proof of Theorem 1.1.

Lemma 2.2 The value $f(\pi)$ is independent of $\pi \in \Pi_\lambda$, i.e., for any $\pi_1, \pi_2 \in \Pi_\lambda$, we have $f(\pi_1) = f(\pi_2)$.

Proof. Since the symmetric group of $[2, n]$ is generated by adjacent transpositions $\{s_i : 2 \leq i \leq n - 1\}$, where $s_i = (i, i+1)$ is the function that swaps two elements $i$ and $i+1$, it suffices to show that $f(\pi_1) = f(\pi_2)$ for any $\pi_1, \pi_2 \in \Pi_\lambda$ such that by switching $i$ and $i+1$ in $\pi_2$ we will get $\pi_1$ ($2 \leq i \leq n - 1$). If $i$ and $i+1$ are in the same block of $\pi_1$, then $\pi_1 = \pi_2$. The assertion is trivial in this case. In the following, we will assume that $i$ and $i+1$ are in different blocks of $\pi_1$.

In order to prove $f(\pi_1) = f(\pi_2)$, we construct an involution $\varphi_i : T_{\pi_1} \cup T_{\pi_2} \rightarrow T_{\pi_1} \cup T_{\pi_2}$. For any tree $T \in T_{\pi_1} \cup T_{\pi_2}$, consider the two vertices labelled $i$ and $i+1$.

If vertices $i$ and $i+1$ are not adjacent, exchanging the labels of these two vertices will give us a new tree $T'$. Let $\varphi_i(T) = T'$.

If vertices $i$ and $i+1$ are adjacent, let $T_i$ (resp. $T_{i+1}$) be the largest subtree containing vertex $i$ but not $i+1$ (resp. containing $i+1$ but not $i$). For $j = i, i+1$, let $T_j = \{j\} \cup A_j \cup B_j$, where $A_j$ (resp. $B_j$) is the sub-forest such that every edge between itself and vertex $j$ is pointing away from $j$ (resp. pointing to $j$). (See Figure 2.)

![Figure 2: A partition of the tree T.](image)

Considering the position of vertex 1, there are three cases:

Case 1: If vertex 1 is in either $A_i$ or $A_{i+1}$, make all edges from $B_i$ to vertex $i$ point to vertex $i+1$ instead, make all edges from $B_{i+1}$ to vertex $i+1$ point to vertex $i$ instead, and switch the vertex labels $i$ and $i+1$ (at the same time the direction of the edge between $i$ and $i+1$ will be changed automatically). Then we will get a new tree $T'$. Let $\varphi_i(T) = T'$. (See Figure 3)

Case 2: If vertex 1 is in $B_i$, let $B'_i$ be the maximum subtree of $B_i$ which contains vertex 1, and let $B''_i$ be $B_i \setminus B'_i$. Make all edges from $B'_i$ to vertex $i$ point to vertex $i+1$ instead, and switch the vertex labels $i$ and $i+1$ (at the same time the direction of the edge between $i$ and $i+1$
Inductive Step: Assume that the theorem is true for $n - 1$. Then we will get a new tree $T'$. Let $\varphi_i(T) = T'$. (See Figure 4(1)).

Case 3: If vertex 1 is in $B_{i+1}$, both edges labelled $i$ and $i + 1$ are pointing to vertex $i + 1$, i.e., $i$ and $i + 1$ are in the same block of $\pi_1$ or $\pi_2$, then we have a contradiction to the assumption. (See Figure 4(2)).

From the definition of the map, we can easily check that $\phi(T)$ and $\phi(T')$ only differ in the positions of $i$ and $i + 1$, i.e., $\phi(T')$ is the same as $\phi(T)$ after switching $i$ and $i + 1$. Since $\varphi_i(T) = T'$, we have $\varphi_i : \mathcal{T}_{\pi_1} \cup \mathcal{T}_{\pi_2} \rightarrow \mathcal{T}_{\pi_1} \cup \mathcal{T}_{\pi_2}$ is well-defined, and $\varphi_i(\mathcal{T}_{\pi_1}) \in \mathcal{T}_{\pi_2}, \varphi_i(\mathcal{T}_{\pi_2}) \in \mathcal{T}_{\pi_1}.$ And by applying $\varphi_i$ again, we have $\varphi_i(\varphi_i(T)) = T$. Hence, $\varphi_i$ is an involution with no fixed points. Hence, we have $|\mathcal{T}_{\pi_1}| = |\mathcal{T}_{\pi_2}|$, i.e., $f(\pi_1) = f(\pi_2)$.

**Proof of Theorem 1.1:** Now with Lemma 2.2 we can prove Theorem 1.1 by induction on $n$, the number of vertices.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n - 1$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Then what we need to show is that for any $\pi \in \Pi_\lambda$, we have $f(\pi) = (n-1)!(n-k)!$.

**Base case:** If $n = 1$, we have $k = 0, \lambda = \emptyset, \pi = \emptyset$, and $f(\pi) = 1 = (n-1)!(n-k)!$.

**Inductive Step:** Assume that the theorem is true for $n - 1$ ($\geq 1$). Then consider the case for $n$.

If $\lambda_1 = 1$, then $\lambda = (1^{n-1}), \pi = n/n-1/\ldots/2$ and $k = n - 1$. In this case, each $T \in \mathcal{T}_\pi$ is an increasing tree, i.e., the label of any vertex is bigger than the label of its parent, i.e., the directions of edges are pointing away from the root 1. Otherwise, there is at least one vertex

![Figure 3: Map $\varphi_i$ (left: 1 in $A_i$, right: 1 in $A_{i+1}$).](image-url)
with indegree at least 2, contradicting that $\lambda$ is the indegree sequence. Hence, we can do the
bijection as in [3, §1.3] by mapping $T$ to a permutation of $[2, n]$. Or we can use the bijection
between labelled trees and Pr"ufer codes (see, for example, [1, §2.4], or a more generalized forest
version [4, §5.3]). But while doing this bijection, what we will get is a subset of all possible
Pr"ufer codes, i.e., a subset of $[n - 1] \times [n - 2] \times \cdots \times [2] \times [1]$. Both methods show that
$f(\pi) = (n - 1)! = (n - 1)!/(n - k)!$.

Now suppose that $\lambda_1 \geq 2$. By Lemma 2.2, we can assume without loss of generality, that
both $n$ and $n - 1$ are in the same block $B_1$ of $\pi = \{B_1, B_2, \ldots, B_k\}$. Pick $T \in \mathcal{T}_\pi$. Since $n$
is the largest label, by the definition of $\pi$ and $\mathcal{T}_\pi$, we know that vertices $n$ and $n - 1$ are adjacent.
By merging the edge between $n$ and $n - 1$ in $T$, and deleting the label $n$, we get a new tree $\tilde{T}$
with $n - 1$ vertices. There are two possible cases:

Case 1: If the indegree of vertex $n - 1$ in $T$ is 0, then $\phi(\tilde{T}) = \{B_1 \setminus \{n\}, B_2, \ldots, B_k\} =: \tilde{\pi}_1$.

Case 2: If the indegree of vertex $n - 1$ in $T$ is not 0, then there exists $j \in [2, k]$ such that
$\phi(\tilde{T}) = \{B_1 \cup B_j \setminus \{n\}, B_2, \ldots, B_{j-1}, B_{j+1}, \ldots, B_k\} =: \tilde{\pi}_j$.

One can easily check that this is a bijection. Thus, $f(\pi) = \sum_{j=1}^{k} f(\tilde{\pi}_j)$. By the induction
hypothesis we have

$$f(\pi) = \sum_{j=1}^{k} f(\tilde{\pi}_j) = \frac{(n - 1)!}{(n - 1)!} + (k - 1) \frac{(n - 1)!}{((n - 1) - (k - 1))!} = \frac{(n - 1)!}{(n - k)!},$$

which proves the case for $n$. 

Figure 4: Map $\varphi_1$ (left: 1 in $B_i$, right: 1 in $B_{i+1}$ (impossible)).
Hence it follows by induction that Theorem 1.1 is true for all possible \( n \).

\[ \square \]

3 An “Almost” Bijective Proof

The inductive proof in the former section makes Corterill’s conjecture a theorem, but it does not explain combinatorially why there is such a simple factor \( (n - 1)!/(n - k)! \). In this section, we will try to give a bijective proof to explain this fact.

First we will give some terminology and notation related to posets. Let \( S \) be a finite set. We use \( \Pi_S \) to denote the poset (actually a geometric lattice) of all partitions of \( S \) ordered by refinement (\( \sigma \preceq \pi \) in \( \Pi_S \) if every block of \( \sigma \) is contained in a block of \( \pi \)). In the following discussion we will consider the case that \( S = [2, n] \).

Second, we will state the basic definitions. Given \( \pi \in \Pi_{[2, n]} \), recall that \( T_\pi \) is the set of labelled trees with preimage \( \pi \) under the map \( \phi \). Let \( B, B' \) be two subsets of \( [2, n] \). We say that \( B \preceq B' \) (resp. \( B < B' \)) if and only if \( \min B \leq \min B' \) (resp. \( \min B < \min B' \)). Given \( T \in T_\pi \) and \( \pi = \phi(T) \), let \( B = \{b_1, b_2, \ldots, b_t\} \) be a subset of one of the blocks of \( \pi \). We define the Star corresponding to \( B \) to be the subset of \( T \) that contains all vertices and edges with labels in the set \( B \), and denote it as \( \text{Star}(B) \). Induced by the ordering of the subsets of \( [2, n] \), we will also get an ordering of the stars. For \( \text{Star}(B) \), there exists a unique vertex of \( T \) with some label, say \( c \), such that the vertex \( c \) is attached to one of the edges in \( \text{Star}(B) \), but \( c \not\in B \). We call the vertex \( c \) the cut point of \( B \), and denote it by \( c(B) \).

Let \( T \in T_\pi \) and \( \sigma \preceq \pi \), we define the decomposition of \( T \) with respect to \( \sigma = \{B_1, B_2, \ldots, B_k\} \) to be \( T = (\bigcup_{j = 1}^{k} \text{Star}(B_j)) \cup \{\text{vertex } 1\} \), where \( \text{Star}(B_j) \) are the stars corresponding to \( B_j \) in \( T \). In this decomposition, the leaf-stars are the stars that don’t contain any cut points, i.e., if you remove a leaf-star from \( T \), what’s left is still a connected tree.

For example, for the tree \( T \) in Figure 1 we have \( \phi(T) = \pi = 8/569/37/24 \). \( \text{Star}([3, 7]) \), \( \text{Star}([2, 4]) \) and \( \text{Star}([8]) \) are all leaf-stars of \( T \), and we have \( c([3, 7]) = 1, c([2, 4]) = 5, c([8]) = 5 \) and \( c([5, 6, 9]) = 1 \).

Now we define a variant of the map \( \phi \), which turns out to be a bijection. For any \( \sigma = \{B_1, B_2, \ldots, B_k\} \in \Pi_{[2, n]} \), let \( T_{\sigma} = \bigcup_{\pi \supseteq \sigma} T_\pi \). We define \( \phi_\sigma : T_{\sigma} \rightarrow [n]^{\sigma - 1} \) as follows.

1. Let \( T_0 = T \).

2. For \( i = 1, 2, \ldots, k \), let \( \text{Star}(B(i)) \) be the largest leaf-star in the decomposition of \( T_{i-1} \) with respect to \( \sigma \setminus \{B(1), B(2), \ldots, B(i - 1)\} \). Then we remove \( \text{Star}(B(i)) \) and keep a record of the vertex it is attached to, i.e., let \( \omega_i = c(B(i)) \), \( T_i = T_{i-1} \setminus \text{Star}(B(i)) \).

Let \( \phi_\sigma(T) = \omega := \omega_1 \omega_2 \ldots \omega_{k - 1} \in [n]^{\sigma - 1} \). (We don’t need to include \( \omega_k \) since it is always 1.)

**Theorem 3.3** For any \( \sigma \in \Pi_{[2, n]} \), the map \( \phi_\sigma \) is a bijection between \( T_{\sigma} \) and \( [n]^{\sigma - 1} \).

**Proof.** We now define the reverse procedure. Given \( \sigma = \{B_1, B_2, \ldots, B_k\} \in \Pi_{[2, n]} \) and \( \omega = \omega_1 \omega_2 \ldots \omega_{k - 1} \in [n]^{k - 1} \), set \( \omega_k = 1 \). Define the inverse map \( \phi_\sigma^{-1} : [n]^{k - 1} \rightarrow T_{\sigma} \) as follows. For \( i = 1, 2, \ldots, k \):
1. Let $B(i) = \{b_1, b_2, \ldots, b_t\}$ be the largest block of $\sigma \setminus \{B(1), B(2), \ldots, B(i-1)\}$ such that $B(i)$ does not contain any number in $\{\omega_1, \omega_{i+1}, \ldots, \omega_{k-1}\}$.

2. Attach the vertices in $B(i)$ to $\omega_i$ according to the following two cases:

   - **Case 1:** If $b_l > \omega_i$, we connect vertices $b_1, b_2, \ldots, b_t$ and $\omega_i$ such that the edges between $b_1, b_2, \ldots, b_t, \omega_i$ and $b_l$ are all pointing to $b_l$ (See Figure 5 (1));
   - **Case 2:** If $b_l < \omega_i$ we simply connect $b_1, b_2, \ldots, b_t$ and $\omega_i$ such that all edges between $b_1, b_2, \ldots, b_t$ and $\omega_i$ are all pointing to $\omega_i$ (See Figure 5 (2)).

![Figure 5: Two cases when attaching $B(i)$ to $\omega_i$.](image)

It is easy to see that after all $k$ steps, we get a tree $T := \phi_\sigma^{-1}(\omega) \in T_{\omega, \sigma}$. One can easily check that $\phi_\sigma$ is a bijection.

**Example 3.4** For the tree $T$ in Figure 6, let $\sigma = 8/7/6/59/3/24$. We then have $B(1) = \{8\}$, $\omega_1 = c(B(1)) = 5$; $B(2) = \{6\}$, $\omega_2 = c(B(2)) = 9$; $B(3) = \{3\}$, $\omega_3 = c(B(3)) = 7$; $B(4) = \{7\}$, $\omega_4 = c(B(4)) = 1$; $B(5) = \{2,4\}$, $\omega_5 = c(B(5)) = 5$; $B(6) = \{5,9\}$, $\omega_6 = c(B(6)) = 1$ (which we don’t write). Thus we have $\phi_{8/7/6/59/3/24}(T) = 59715 \in [9]^5$.

![Figure 6: A tree $T \in T_{3221}$, with $\phi(T) = 8/569/37/24 \in \Pi_{3221}$, $\sigma = 8/7/6/59/3/24 \prec \pi$, and $\phi_\sigma(T) = 59715$.](image)

**Proof of Theorem 1.1.** Let $g(\sigma) = |T_{\omega, \sigma}|$. From the bijection $\phi_\sigma : T_{\omega, \sigma} \rightarrow [n]|\sigma|^{-1}$ we know that $g(\sigma) = n|\sigma|^{-1}$. Recall from Section 2 that $f(\pi) = |T_\pi|$. Since $T_{\omega, \sigma} = \bigcup_{n \geq \sigma} T_\pi$ is a disjoint
union, we have
\[ \sum_{\pi \geq \sigma} f(\pi) = n^{k-1}, \text{ for any } \sigma \in \Pi_{\{2, n\}}. \] (3.1)

It is now sufficient to prove that the unique solution of the above equations is \( f(\pi) = (n - 1)!/(n - |\pi|)! \), for any \( \pi \in \Pi_{\{2, n\}} \).

First, since equation (3.1) holds for any \( \pi, \sigma \in \Pi_{\{2, n\}} \) such that \( \pi \geq \sigma \), we have, by the poset structure of \( \Pi_{\{2, n\}} \), that the solution \( f \) to equation (3.1) (valid for all \( \sigma \in \Pi_{\{2, n\}} \)) is unique.

Second, let \( \sigma = \{B_1, B_2, \ldots, B_k\} \). Then the interval \([\sigma, \hat{1}_{\{2, n\}}]\) is isomorphic in an obvious way to the lattice of partitions of the set \( \{B_1, B_2, \ldots, B_k\} \). Hence \([\pi, \hat{1}_{\{2, n\}}] \cong \Pi_{\{k\}}\), where \( \hat{1}_{\{2, n\}} \) is the maximum element of \( \Pi_{\{2, n\}} \). Thus we have
\[
\sum_{\pi \in \Pi_{\{k\}}} \frac{(n - 1)!}{(n - |\pi|)!} = \sum_{\pi \in \Pi_{\{k\}}} \frac{(n - 1)!}{(n - |\tau|)!}
= \sum_{j=1}^{k} S(k, j) \frac{(n - 1)!}{(n - j)!}
= \frac{1}{n} \sum_{j=1}^{k} S(k, j)n(n - 1) \cdots (n - j + 1)
= n^{k-1},
\]
where \( S(k, j) \) is the Stirling number of the second kind, i.e., the number of partitions of a \( k \)-set into \( j \) blocks. The last equation follows from a standard Stirling number identity, see e.g., identity (24d) in [3, §1.4]. Thus, \((n - 1)!/(n - |\pi|)!\) is a possible solution to equations (3.1).

Hence, by uniqueness, we have \( f(\pi) = (n - 1)!/(n - |\pi|)! \).

Remark: In fact, given equation (3.1), we can solve for \( f \) by using the dual form of the Möbius inversion formula:
\[ f(\pi) = \sum_{\sigma \geq \pi} \mu(\pi, \sigma)g(\sigma), \]
where the coefficient \( \mu(\pi, \sigma) \) is the Möbius function of \( \Pi_{\{2, n\}} \), which can be calculated explicitly, [3, Exam 3.10.4].

4 The Real Bijective Map

Although we gave a bijection \( \phi_{\pi} \) in Section 3, we needed to prove Theorem 1.1 by solving equations, and we still don’t have a very good bijection that maps \( T_{\lambda} \) to a set of cardinality \( a_{\lambda} \) for any \( \lambda \vdash n - 1 \).

Let \( \phi': T_{\lambda} \times [n]^{k-1}, T \mapsto (\pi, \phi_{\pi}(T)) \), where \( \pi = \phi(T) \). Since \( \phi_{\pi} \) is a bijection, we have that \( \phi' \) is an injection. Let \( \Omega_{\pi} := \phi_{\pi}(T_{\pi}) \). Then \( \phi'(T_{\lambda}) = \{\pi\} \times \Omega_{\pi} : \pi \in \Pi_{\lambda} \} =: (\Pi \times \Omega_{\lambda})_{\lambda} \). Thus, \( \phi': T_{\lambda} \times (\Pi \times \Omega_{\lambda})_{\lambda} \) is the bijection we are looking for.

Example 4.5 Assume \( \pi = \{B_1, B_2\} \). For any \( T \in T_{\pi} \), we have \( \phi'(T) = (\pi, \max\{c(B_1), c(B_2)\}) \), and \( \Omega_{\pi} = [n - 1] \), \( f(\pi) = n - 1 \).
Example 4.6 When \( \lambda = (1^{n-1}) \), \( \Pi_\lambda \) contains only the partition \( \hat{0}_{[2,n]} = n/n - 1/\ldots/2 \). As also pointed out in the proof in Section 2, \( T_{[2,n]} \) is the set of all increasing trees on \([n]\), in this case we have \( \Omega_{[2,n]} = [n - 1] \times [n - 2] \times \ldots \times [1] \), and for each \( T \in T_{[2,n]} \) and \( \phi'(T) = (\pi, \omega) \), \( \omega \) is the Pr"ufer code of \( T \).

Though it seems quite hard to find what \( \Omega_\pi \)'s are, there still exists a very good relation among them.

**Theorem 4.7** For any \( \pi_1, \pi_2 \in \Pi_{[2,n]} \), if \( \pi_2 \succ \pi_1 \), we have that for any \( T \in T_{\geq \pi_2} \), \( \phi_{\pi_2}(T) \) is a subsequence of \( \phi_{\pi_1}(T) \). In particular, if \( \pi_2 = \phi(T) \), we have \( \phi'(T) = (\pi_2, \omega) \) and \( \omega \) is a subsequence of \( \phi_{\pi_1}(T) \).

**Proof.** It suffices to prove the assertion for all covering pairs. Assume that \( \pi_2 \succ \pi_1 \). Thus there exist two blocks \( B \) and \( B' \) of \( \pi_1 \) which become one block in \( \pi_2 \).

Assume \( \phi_{\pi_1}(T) = \omega = \omega_1 \omega_2 \ldots \omega_{k-1} \), \( \phi_{\pi_2}(T) = \omega' = \omega'_1 \omega'_2 \ldots \omega'_{k-2} \) and \( \omega_k = \omega'_{k-1} = 1 \). Then there exist \( 1 \leq r, s \leq k \) such that \( B \) and \( B' \) are removed from \( T \) at steps \( r \) and \( s \), respectively, in process \( \phi_{\pi_1} \). Assume, without loss of generality, that \( r < s \). Then it is easy to see that \( \omega'_l = \omega_l \) for \( 1 \leq l < r \).

For step \( r \) in process \( \phi_{\pi_2} \), there are two cases:

*Case 1:* If \( B < B' \) and Star\((B \cup B')\) is a leaf-star, then it must be that \( s = r + 1 \). At this step, we remove Star\((B \cup B')\). Then \( \omega'_r = \omega_s \), \( \omega'_l = \omega_l \) for \( r < l \leq k - 2 \).

*Case 2:* If \( B > B' \), or Star\((B \cup B')\) is not a leaf-star, then we will remove Star\((B \cup B')\) at the step that we remove Star\((B')\) in the process \( \phi_{\pi_1} \), i.e., \( \omega'_l = \omega_{l+1} \) for \( r \leq l \leq k - 2 \).

In both cases, we have that \( \omega' \) is a subsequence of \( \omega \), i.e., \( \phi_{\pi_2}(T) \) is a subsequence of \( \phi_{\pi_1}(T) \).

**Example 4.8** Let \( \pi = 8/569/37/24 \) and \( \sigma = 8/7/6/59/3/24 \), so \( \pi \succ \sigma \). For the tree \( T \) in Figure 6, we have \( T \in T_{\geq \sigma} \), and \( \phi_\pi(T) = 515 \), which is a subsequence of \( \phi_\sigma(T) = 59715 \).

By the proof of Theorem 4.7, for any \( \sigma \in \Pi_{[2,n]} \) we can define a bijection from \( \bigcup_{\pi \succ \sigma} \Omega_\pi \) to \([n]^{k-1} \backslash \Omega_\sigma \) such that each sequence will be a subsequence of its image. Inductively using this bijection, we can find out all \( \Omega_\sigma \)'s. But when \( |\sigma| \) gets larger and larger, it will become more and more difficult to find out what this bijection is explicitly.

### 5 Remarks

We want to remark that the bijection we defined in Section 3 can be considered as a generalization of the Pr"ufer codes for labelled trees: instead of deleting (attaching) vertices one by one, we are dealing with groups of vertices with respect to a partition of \([2,n]\). Moreover, the bijection \( \phi' \) together with Theorem 4.7 suggests a structure on the set of labelled trees \( \{ T_\pi : \pi \in \Pi_{[2,n]} \} \) as a lattice isomorphic to \( \Pi_{[2,n]} \) under the map \( T_\pi \mapsto \pi \).

The following problems are still interesting to consider.
1. Given $\pi \in \Pi_{[2,n]}$, Theorem 4.7 shows how to find $\Omega_\pi$ explicitly, i.e., $\Omega_\pi$ is the subset of $[n]^{[\pi]-1}$ with sequences corresponding to its subsequences from $\Omega_\sigma$ deleted, for any $\sigma \succ \pi$. For example, let $\pi = 45/3/2$, we have:

$$
\Omega_\pi = \begin{bmatrix}
11 & 12 & \mathcal{X} & 14 & \mathcal{X} \\
21 & 22 & \mathcal{X} & 24 & \mathcal{X} \\
31 & 32 & \mathcal{X} & 34 & \mathcal{X} \\
41 & 42 & 43 & \mathcal{X} & \mathcal{X} \\
\mathcal{X} & \mathcal{X} & \mathcal{X} & \mathcal{X} & \mathcal{X}
\end{bmatrix},
$$

where 13, 23, 33, 44 correspond to its subsequences 1, 2, 3, 4 in $\Omega_{45/23}$, 15, 25, 53, 45 correspond to its subsequences 1, 2, 3, 4 in $\Omega_{3/245}$, 51, 52, 53, 54 correspond to its subsequences 1, 2, 3, 4 in $\Omega_{345/2}$, and 55 correspond to its subsequence $\emptyset$ in $\Omega_{2345}$.

However, the “corresponding relationship”, between sequences and its subsequences described inductively in the proof of Theorem 4.7, depends highly on the set $\{\sigma \in \Pi_{[2,n]} : \sigma \succ \pi\}$, and it is not easy to describe in general. Hence, it would be nice if one can give a simple description of this relationship, and use it to characterize $\Omega_\pi$.

2. In the proof of Theorem 1.1 in Section 2, we mentioned that when $\lambda = \langle 1^{n-1} \rangle$, we can map an increasing tree to a permutation of $[2,n]$ (§1.3). Is it possible to generalize this bijection to any $\lambda$ by mapping a tree in $T_\lambda$ to $(\phi(T), w)$, where $w$ is a length $k-1$ permutation of an $(n-1)$-element set?

Acknowledgments. The authors want to thank Prof. Richard P. Stanley for his inspiring suggestions and helpful discussions, and also thank the referee for helpful comments on an earlier version of this paper. The first author is supported by the National Science Foundation of China under Grant No. 10726048 and No. 10801053, Shanghai Educational Development Foundation under the Chenguang Project No. 2007CG29, and Shanghai Leading Academic Discipline Project No. B407.

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