HOMOTOPY CATEGORIES AND IDEMPOTENT COMPLETENESS, WEIGHT STRUCTURES AND WEIGHT COMPLEX FUNCTORS

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Abstract. This article provides some basic results on weight structures, weight complex functors and homotopy categories. We prove that the full subcategories $K(A)^{w \leq n}$, $K(A)^{w \geq n}$, $K(A)^-$ and $K(A)^+$ (of objects isomorphic to suitably bounded complexes) of the homotopy category $K(A)$ of an additive category $A$ are idempotent complete, which confirms that $(K(A)^{w \leq 0}, K(A)^{w \geq 0})$ is a weight structure on $K(A)$.

We discuss weight complex functors and provide full details of an argument sketched by M. Bondarko, which shows that if $w$ is a bounded weight structure on a triangulated category $T$ that has a filtered triangulated enhancement $\tilde{T}$ then there exists a strong weight complex functor $T \to K(\vee(w))^{anti}$. Surprisingly, in order to carry out the proof, we need to impose an additional axiom on the filtered triangulated category $\tilde{T}$ which seems to be new.

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1. Introduction

The aim of this article is to provide and review some foundational results on homotopy categories, weight structures and weight complex functors.

Weight structures were independently introduced by M. Bondarko in [Bon10], and D. Pauksztello in [Pau08] where they are called co-t-structures. Their definition is formally very similar to that of a t-structure. In both cases there is an axiom demanding that each objects fits into a triangle of a certain form. In the case of a t-structure these “truncation triangles” are functorial whereas in the case of a weight structure these “weight decomposition triangles” are not functorial. This is a technical issue but the theory remains amazingly rich. Weight structures have applications in various branches of mathematics, for example in algebraic geometry (theory of motives), algebraic topology (stable homotopy category) and representation theory, see e.g. the work of M. Bondarko (e.g. [Bon10]), J. Wildeshaus (e.g. [Wil09]) and D. Pauksztello ([Pau08, Pau10]) and references therein; other references are [AT08], [IY08], [WW09], [AI10], [KN10], [MSSSS10], [AK11].

A crucial observation due to M. Bondarko is that, in the presence of a weight structure \( w = (\mathcal{T}^{w \leq 0}, \mathcal{T}^{w \geq 0}) \) on a triangulated category \( \mathcal{T} \), there is a weak weight complex functor \( WC : \mathcal{T} \to K_{\text{weak}}(\heartsuit) \) where \( \heartsuit = \mathcal{T}^{w \leq 0} \cap \mathcal{T}^{w \geq 0} \) is the heart of \( w \) and the weak homotopy category \( K_{\text{weak}}(\heartsuit) \) is a certain quotient of the homotopy category \( K(\heartsuit) \). M. Bondarko explains that in various natural settings this functor lifts to a “strong weight complex functor” \( \mathcal{T} \to K(\heartsuit)^{\text{anti}} \) (the upper index does not appear in [Bon10]; it will be explained below). We expect that this strong weight complex functor will be an important tool.

The basic example of a weight structure is the “standard weight structure” \( (K(A)^{w \leq n}, K(A)^{w \geq n}) \) on the homotopy category \( K(A) \) of an additive category \( A \); here \( K(A)^{w \leq n} \) (resp. \( K(A)^{w \geq n} \)) is the full subcategory of \( K(A) \) consisting of complexes \( X = (X^i, d^i : X^i \to X^{i+1}) \) that are isomorphic to a complex concentrated in degrees \( \leq n \) (resp. \( \geq n \)) (for fixed \( n \in \mathbb{Z} \)). The subtle point to confirm this example (which appears in [Bon10] and [Pau08]) is to check that \( K(A)^{w \leq 0} \) and \( K(A)^{w \geq 0} \) are both closed under retracts in \( K(A) \).

This basic example was our motivation for the first part of this article where we discuss the idempotent completeness of (subcategories of) homotopy categories. Let \( A \) be an additive category as above. Then a natural question is whether \( K(A) \) is idempotent complete. Since \( K(A) \) is a triangulated category this question can be rephrased as follows:
Question 1.1. Given any idempotent endomorphism \( e : X \to X \) in \( K(\mathcal{A}) \), is there an isomorphism \( X \cong E \oplus F \) such that \( e \) corresponds to \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} : E \oplus F \to E \oplus F \)?

We do not know the answer to this question in general. We can show that certain full subcategories of \( K(\mathcal{A}) \) are idempotent complete:

**Theorem 1.2** (see Thm. 3.1). The full subcategories \( K(\mathcal{A})^{w \leq n} \) and \( K(\mathcal{A})^{w \geq n} \) of \( K(\mathcal{A}) \) are idempotent complete. In particular \( K(\mathcal{A})^- := \bigcup_{n \in \mathbb{Z}} K(\mathcal{A})^{w \leq n} \), \( K(\mathcal{A})^+ := \bigcup_{n \in \mathbb{Z}} K(\mathcal{A})^{w \geq n} \) and \( K(\mathcal{A})^{bw} := K(\mathcal{A})^- \cap K(\mathcal{A})^+ \) are idempotent complete.

Possibly this result is known to the experts but we could not find a proof in the literature. Our proof of this result is constructive and based on work of R. W. Thomason [Tho97] and ideas of P. Balmer and M. Schlichting [BS01]. Theorem 1.2 implies that \( (K(\mathcal{A})^{w \leq 0}, K(\mathcal{A})^{w \geq 0}) \) is a weight structure on \( K(\mathcal{A}) \) (see Prop. 4.6).

Another approach to Question 1.1 is to impose further assumptions on \( \mathcal{A} \). We can show (see Thm. 3.4): \( K(\mathcal{A}) \) is idempotent complete if \( \mathcal{A} \) has countable coproducts (this follows directly from results of M. Bökstedt and A. Neeman [BN93] or from a variation of our proof of Theorem 1.2) or if \( \mathcal{A} \) is abelian (this is an application of results from [BR07] and [LC07]). If \( \mathcal{A} \) itself is idempotent complete then projectivization (see [Kra08]) shows that the full subcategory \( K(\mathcal{A})^b \) of \( K(\mathcal{A}) \) of complexes that are isomorphic to a bounded complex is idempotent complete.

It may seem natural to assume that \( \mathcal{A} \) is idempotent complete and additive in Question 1.1. However, if \( \mathcal{A}^{ic} \) is the idempotent completion of \( \mathcal{A} \), then \( K(\mathcal{A}) \) is idempotent complete if and only if \( K(\mathcal{A}^{ic}) \) is idempotent complete (see Rem. 3.6).

Results of R. W. Thomason [Tho97] indicate that it might be useful to consider the Grothendieck group \( K_0(K(\mathcal{A})) \) of the triangulated category \( K(\mathcal{A}) \) for additive (essentially) small \( \mathcal{A} \). We show that the Grothendieck groups \( K_0(K(\mathcal{A})), K_0(K(\mathcal{A})^-) \) and \( K_0(K(\mathcal{A})^+) \) all vanish for such \( \mathcal{A} \) (see Prop. 4.12).

The second part of this article concerns weight complex functors. In the example of the standard weight structure \( (K(\mathcal{A})^{w \leq 0}, K(\mathcal{A})^{w \geq 0}) \) on \( K(\mathcal{A}) \), the heart \( \heartsuit \) is the idempotent completion of \( \mathcal{A} \) (see Cor. 4.11) and the weak weight complex functor \( WC : T \to K(\heartsuit) \) naturally and easily lifts to a triangulated functor \( \widehat{WC} : K(\mathcal{A}) \to K(\heartsuit) \) (see Prop. 5.9); here the triangulated categories \( K(\heartsuit) \) and \( K(\heartsuit) \) coincide as additive categories with translation but a candidate triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} [1]X \) is a triangle in \( K(\heartsuit) \) if and only if \( X \xrightarrow{-u} \).
$Y \xrightarrow{v} Z \xrightarrow{w} [1]X$ is a triangle in $K(\heartsuit)$. This functor $\widehat{WC}$ is an example of a “strong weight complex functor”.

Let us return to the general setup of a weight structure $w$ on a triangulated category $\mathcal{T}$ with heart $\heartsuit$. Assume that $\tilde{T}$ is a filtered triangulated category over $\mathcal{T}$ in the sense of [Be˘ ı87, App.]. The main result of the second part of this article is a complete proof of the following Theorem.

**Theorem 1.3** (see Thm. [7.1] and cf. [Bon10, 8.4]). Assume that $w$ is bounded and that $\tilde{T}$ satisfies axiom (fcat7) stated in Section 7.2. Then there is a strong weight complex functor

$$\widehat{WC} : \mathcal{T} \to K^b(\heartsuit)^{anti}.$$  

This means that $\widehat{WC}$ is a triangulated functor whose composition with $K^b(\heartsuit)^{anti} \to K_{weak}(\heartsuit)$ is isomorphic to the weak weight complex functor (as a functor of additive categories with translation).

Our proof of this theorem relies on the ideas of A. Beilinson and M. Bondarko sketched in [Bon10, 8.4]. We explain the idea of the proof in Section 7.1. The additional axiom (fcat7) imposed on $\tilde{T}$ in Theorem 1.3 seems to be new. It states that any morphism gives rise to a certain $3 \times 3$-diagram; see Section 7.2 for the precise formulation. It is used in the proof of Theorem 1.3 at two important points; we do not know if this axiom can be removed.

We hope that this axiom is satisfied for reasonable filtered triangulated categories; it is true in the basic example of a filtered triangulated category: If $\mathcal{A}$ is an abelian category its filtered derived category $DF(\mathcal{A})$ is a filtered triangulated category (see Prop. 6.3) that satisfies axiom (fcat7) (see Lemma 7.4).

In the short third part of this article we prove a result which naturally fits into the context of weight structures and filtered triangulated categories: Given a filtered triangulated category $\tilde{T}$ over a triangulated category $\mathcal{T}$ with a weight structure $w$, there is a unique weight structure on $\tilde{T}$ that is compatible with $w$ (see Prop. 8.3).

**Plan of the article.** We fix our notation and gather together some results on additive categories, idempotent completeness, triangulated categories and homotopy categories in Section 2 we suggest skimming through this section and coming back as required. Sections 3 and 4 constitute the first part of this article - they contain the results on idempotent completeness of homotopy categories and some basic results on weight structures. The next two sections lay the groundwork for the proof of Theorem 1.3. In Section 5 we construct the weak weight
complex functor in detail. We recall the notion of a filtered triangulated category in Section 6 and prove some important results stated in [Bei87, App.] as no proofs are available in the literature. We prove Theorem 1.3 in Section 7. Section 8 contains the results on compatible weight structures.

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2. Preliminaries

For the definition of an additive category (with translation (= shift)), a functor of additive categories (with translation), and of a triangulated category see [KS06].

2.1. (Additive) categories. Let $\mathcal{A}$ be a category and $X$ an object of $\mathcal{A}$. An object $Y \in \mathcal{A}$ is a retract of $X$ if there are morphisms $p : X \to Y$ and $i : Y \to X$ such that $pi = \text{id}_Y$. Then $ip : X \to X$ is an idempotent endomorphism. A subcategory $\mathcal{B} \subset \mathcal{A}$ is closed under retracts (= Karoubi-closed) if it contains all retracts in $\mathcal{A}$ of any object of $\mathcal{B}$. In this case $\mathcal{B}$ is a strict subcategory of $\mathcal{A}$, i.e. it is closed under isomorphisms.

Let $\mathcal{B}$ be a subcategory of an additive category $\mathcal{A}$. We say that $\mathcal{B}$ is dense in $\mathcal{A}$ if each object of $\mathcal{A}$ is a summand of an object of $\mathcal{B}$. We define full subcategories $\perp^\mathcal{B}, \mathcal{B}^\perp \subset \mathcal{A}$ by

\[\perp^\mathcal{B} = \{ Z \in \mathcal{A} \mid \mathcal{A}(Z, \mathcal{B}) = 0 \},\]
\[\mathcal{B}^\perp = \{ Z \in \mathcal{A} \mid \mathcal{A}(\mathcal{B}, Z) = 0 \} .\]
2.2. **Idempotent completeness.** Let \( \mathcal{A} \) be a category and \( X \) an object of \( \mathcal{A} \). An idempotent endomorphism \( e \in \text{End}(X) \) splits if there is a splitting of \( e \), i.e., there are an object \( Y \in \mathcal{A} \) and morphisms \( p : X \to Y \) and \( i : Y \to X \) such that \( ip = e \) and \( pi = \text{id}_Y \). A splitting of \( e \) is unique up to unique isomorphism. If every idempotent endomorphism splits we say that \( \mathcal{A} \) is **idempotent complete** (= Karoubian). If \( \mathcal{A} \) is additive, an idempotent \( e : X \to X \) has a splitting \((Y, p, i)\) and \( 1 - e \) has a splitting \((Z, q, j)\), then obviously \((i, j) : Y \oplus Z \to X\) is an isomorphism with inverse \( [p\ q] \). In particular in an idempotent complete additive category any idempotent endomorphism of an object \( X \) induces a direct sum decomposition of \( X \).

Any additive category \( \mathcal{A} \) has an **idempotent completion** (= Karoubi completion) \((\mathcal{A}^{ic}, i)\), i.e., there is an idempotent complete additive category \( \mathcal{A}^{ic} \) together with an additive functor \( i : \mathcal{A} \to \mathcal{A}^{ic} \) such that any additive functor \( F : \mathcal{A} \to \mathcal{C} \) to an idempotent complete additive category \( \mathcal{C} \) factors as \( F = i \circ F^{ic} \) for an additive functor \( F^{ic} : \mathcal{A}^{ic} \to \mathcal{C} \) which is unique up to unique isomorphism; see e.g. \([\text{BS01}]\) for an explicit construction. Then \( i \) is fully faithful and we usually view \( \mathcal{A} \) as a full subcategory of \( \mathcal{A}^{ic} \); it is a dense subcategory. Conversely if \( \mathcal{A} \) is a full dense additive subcategory of an idempotent complete additive category \( \mathcal{B} \), then \( \mathcal{B} \) together with the inclusion \( \mathcal{A} \hookrightarrow \mathcal{B} \) is an idempotent completion of \( \mathcal{A} \).

2.3. **Triangulated categories.** Let \( \mathcal{T} \) (more precisely \((\mathcal{T}, [1])\)) together with a certain class of candidate triangles) be a triangulated category (see \([\text{KS06} \text{ Ch. 10]}, \text{[Nee01]}, \text{[BBD82]}\). We follow the terminology of \([\text{Nee01}]\) and call candidate triangle (resp. triangle) what is called triangle (resp. distinguished triangle) in \([\text{KS06}]\). We say that a subcategory \( \mathcal{S} \subset \mathcal{T} \) is **closed under extensions** if for any triangle \( X \to Y \to Z \to [1]X \) in \( \mathcal{T} \) with \( X \) and \( Z \) in \( \mathcal{S} \) we have \( Y \in \mathcal{S} \).

2.3.1. **Basic statements about triangles.**

**Lemma 2.1.** Let \( X \xrightarrow{\delta} Y \xrightarrow{\varepsilon} Z \xrightarrow{\delta} [1]X \) be a triangle in \( \mathcal{T} \). Then

1. \( h = 0 \) if and only if there is a morphism \( \varepsilon : Y \to X \) such that \( \varepsilon f = \text{id}_X \).

Let \( \varepsilon : Y \to X \) be given satisfying \( \varepsilon f = \text{id}_X \). Then:

2. There is a unique morphism \( \delta : Z \to Y \) such that \( \varepsilon \delta = 0 \) and \( g\delta = \text{id}_Z \).
(3) The morphism

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \text{id}_X & & \downarrow [g] \\
X & \xrightarrow{[1]} & X \oplus Z \\
\end{array}
\begin{array}{ccc}
Z & \xrightarrow{h=0} & [1]X \\
\downarrow \text{id}_Z & & \downarrow [1]X \\
Z & \xrightarrow{0} & [1]X
\end{array}
\]

is an isomorphism of triangles and \([g] \) is invertible with inverse \([f \delta] \). Under the isomorphism \([g] : Y \sim X \oplus Z \) the morphism \(\varepsilon \) corresponds to \([1 0] : X \oplus Z \to X \) and the morphism \(\delta \) to \([0] : Z \to X \oplus Z \).

Proof. If \(\varepsilon f = \text{id}_X \) then \(h = \text{id}_{[1]X} \circ h = [1] \varepsilon \circ [1] f \circ h = 0 \). If \(h = 0 \) then use the cohomological functor \(\text{Hom}(),X \) to find \(\varepsilon \).

Let \(\varepsilon : Y \to X \) be given satisfying \(\varepsilon f = \text{id}_X \). The morphism (2.1) is a morphism of candidate triangles and even of triangles since coproducts of triangles are triangles (e.g. [Nec01, Prop. 1.2.1]). Hence \([g] \) is an isomorphism.

For any \(\delta : Z \to Y \) satisfying \(\varepsilon \delta = 0 \) and \(g \delta = \text{id}_Z \) we have \([\varepsilon g][f \delta] = [1 0] \). Hence \(\delta \) is unique if it exists.

Let \([a b] \) be the inverse of \([g] \). Then \(\text{id}_Y = a \varepsilon + b g \) and hence \(f = \text{id}_Y f = a \varepsilon f + b g f = a \); on the other hand \([1 0] = [g] [a b] = [\varepsilon f \, eb] = [1 eb] \). Hence \(b \) satisfies the conditions imposed on \(\delta \). □

We say that a triangle \(X \xrightarrow{i} Y \xrightarrow{g} Z \xrightarrow{h} [1]X \) splits if it is isomorphic (by an arbitrary isomorphism of triangles) to the triangle \(X \xrightarrow{[b]} X \oplus Z \xrightarrow{[0 1]} Z \xrightarrow{0} [1]X \). This is the case if and only if \(h = 0 \) as we see from Lemma 2.1.

Corollary 2.2 (cf. [LC07, Lemma 2.2]). Let \(e : X \to X \) be an idempotent endomorphism in \(T \). Then \(e \) splits if and only if \(1 - e \) splits. In particular, an object \(Y \) is a retract of an object \(X \) if and only if \(Y \) is a summand of \(X \).

This corollary shows that the question of idempotent completeness of a triangulated category is equivalent to the analog of Question 1.

Proof. Let \((Y,p,i)\) be a splitting of \(e \). Complete \(i : Y \to X \) into a triangle \(Y \xrightarrow{i} X \xrightarrow{q} Z \to [1]Y \). Lemma 2.1 (2) applied to this triangle and \(p : X \to Y \) yields a morphism \(j : Z \to X \) and then Lemma 2.1 (3) shows that \((Z,q,j)\) is a splitting of \(1 - e \). □
Proposition 2.3 ([BBD82, Prop. 1.1.9]). Let \((X, Y, Z)\) and \((X', Y', Z')\) be triangles and let \(g : Y \rightarrow Y'\) be a morphism in \(\mathcal{T}\):

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
| & f \downarrow & | \\
X' & \xrightarrow{u'} & Y' \\
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{v} & Z \\
| & g \downarrow & | \\
Y' & \xrightarrow{v'} & Z' \\
\end{array}
\quad
\begin{array}{ccc}
Z & \xrightarrow{d} & [1]X \\
| & h \downarrow & | \\
Z' & \xrightarrow{d'} & [1]X' \\
\end{array}
\]

The following conditions are equivalent:

1. \(v'gu = 0\);
2. there is a morphism \(f\) such that (1) commutes;
3. there is a morphism \(h\) such that (2) commutes;
4. there is a morphism \((f, g, h)\) of triangles as indicated in diagram (2.2).

If these conditions are satisfied and \(\text{Hom}([1]X, Z') = 0\) then \(f\) in (2) and \(h\) in (3) are unique.

Proof. See [BBD82, Prop. 1.1.9]. □

Proposition 2.4 ([BBD82, 1.1.11]). Every commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
| & & | \\
X' & \xrightarrow{u'} & Y' \\
\end{array}
\]

can be completed to a so called \(3 \times 3\)-diagram

\[
\begin{array}{ccc}
[1]X' & \xrightarrow{u} & [1]Y' \\
\uparrow & & \uparrow \\
X'' & \xrightarrow{u''} & Y'' \\
\uparrow & & \uparrow \\
X & \xrightarrow{u} & Y \\
\uparrow & & \uparrow \\
X' & \xrightarrow{u'} & Y' \\
\end{array}
\quad
\begin{array}{ccc}
[1]Y' & \xrightarrow{v} & [1]Z' \\
\uparrow & & \uparrow \\
Y'' & \xrightarrow{v''} & Z'' \\
\uparrow & & \uparrow \\
Y & \xrightarrow{v} & Z \\
\uparrow & & \uparrow \\
Y' & \xrightarrow{v'} & Z' \\
\end{array}
\quad
\begin{array}{ccc}
[2]X' & \xrightarrow{d} & [1]X \\
\uparrow & & \uparrow \\
[1]X' & \xrightarrow{d'} & [1]X' \\
\end{array}
\]

i.e. a diagram as above having the following properties: The dotted arrows are obtained by translation \([1]\), all small squares are commutative except the upper right square marked with \(\varnothing\) which is anti-commutative, and all three rows and columns with solid arrows are triangles. (The column/row with the dotted arrows becomes a triangle if an odd number of its morphisms is multiplied by \(-1\).)
Variation: Any of the eight small commutative squares in the above diagram can be completed to a $3 \times 3$-diagram as above. It is also possible to complete the anti-commutative square to a $3 \times 3$ diagram.

Proof. See [BBD82, 1.1.11]. To obtain the variations remove the column on the right and add the $[-1]$-shift of the “Z-column” on the left. Modify the signs suitably. Iterate this and use the diagonal symmetry. □

2.3.2. Anti-triangles. Following [KS06, 10.1.10] we call a candidate triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} [1]X$ an anti-triangle if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} [1]X$ is a triangle. Then $(\mathcal{T}, [1])$ with the class of all anti-triangles is again a triangulated category that we denote by $(\mathcal{T}^{\text{anti}}, [1])$ or $\mathcal{T}^{\text{anti}}$. The triangulated categories $\mathcal{T}$ and $\mathcal{T}^{\text{anti}}$ are equivalent as triangulated categories (cf. [KS06, Exercise 10.10]): Let $F = \text{id}_\mathcal{T} : \mathcal{T} \to \mathcal{T}$ be the identity functor (of additive categories) and $\tau : F[1] \xrightarrow{\sim} [1]F$ the isomorphism given by $\tau_X = -\text{id}_{[1]X} : [1]X = F[1]X \xrightarrow{\sim} [1]X = [1]FX$. Then

\begin{equation}
(\text{id}_\mathcal{T}, \tau) : \mathcal{T} \xrightarrow{\sim} \mathcal{T}^{\text{anti}}
\end{equation}

is a triangulated equivalence.

2.3.3. Adjoints are triangulated. Assume that $(G, \gamma) : \mathcal{T} \to \mathcal{S}$ is a triangulated functor and that $F : \mathcal{S} \to \mathcal{T}$ is left adjoint to $G$. Let $X \in \mathcal{S}$. The morphism $[1]X \to [1]GFX \xrightarrow{\gamma_X} G[1]FX$ (where the first morphism is the translation of the unit of the adjunction) corresponds under the adjunction to a morphism $\varphi_X : F[1]X \to [1]FX$. This construction is natural in $X$ and defines a morphism $\varphi : F[1] \to [1]F$ (which in fact is an isomorphism). We omit the tedious proof of the following Proposition.

**Proposition 2.5** ([KV87, Prop. 1.6] (without proof); cf. [KS06, Exercise 10.3]). Let $F$ be left adjoint to a triangulated functor $(G, \gamma)$. Then $(F, \varphi)$ as defined above is a triangulated functor.

Similarly the right adjoint to a triangulated functor is triangulated.

2.3.4. Torsion pairs and t-structures. The notion of a t-structure [BBD82, 1.3.1] and that of a torsion pair [BR07] on a triangulated category essentially coincide, see [BR07, Prop. I.2.13]: A pair $(\mathcal{X}, \mathcal{Y})$ is a torsion pair if and only if $(\mathcal{X}, [1]\mathcal{Y})$ is a t-structure. We will use both terms.
2.4. **Homotopy categories and variants.** Let $\mathcal{A}$ be an additive category and $C(\mathcal{A})$ the category of (cochain) complexes in $\mathcal{A}$ with cochain maps as morphisms: A morphism $f : X \to Y$ in $C(\mathcal{A})$ is a sequence $(f^n)_{n \in \mathbb{Z}}$ of morphisms $f^n : X^n \to Y^n$ such that $d_Y^n f^n = f^{n+1} d_X^n$ for all $n \in \mathbb{Z}$ (or in shorthand notation $df = fd$).

Let $f, g : X \to Y$ be morphisms in $C(\mathcal{A})$. Then $f$ and $g$ are **homotopic** if there is a sequence $h = (h^n)_{n \in \mathbb{Z}}$ of morphisms $h^n : X^n \to Y^{n-1}$ in $\mathcal{A}$ such that $f^n - g^n = d_Y^n h^n + h^{n+1} d_X^n$ for all $n \in \mathbb{Z}$ (or in shorthand notation $f - g = dh + hd$). The **homotopy category** $K(\mathcal{A})$ has the same objects as $C(\mathcal{A})$, but

$$\text{Hom}_{K(\mathcal{A})}(X,Y) := \frac{\text{Hom}_{C(\mathcal{A})}(X,Y)}{\{\text{morphisms homotopic to zero}\}}.$$ 

Let $f, g : X \to Y$ be morphisms in $C(\mathcal{A})$. Then $f$ and $g$ are **weakly homotopic** (see [Bon10, 3.1]) if there is a pair $(s,t)$ of sequences $s = (s^n)_{n \in \mathbb{Z}}$ and $t = (t^n)_{n \in \mathbb{Z}}$ of morphisms $s^n, t^n : X^n \to Y^{n-1}$ such that

$$f^n - g^n = d_Y^n s^n + t^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$ (or in shorthand notation $f - g = ds + td$). The **weak homotopy category** $K_{\text{weak}}(\mathcal{A})$ has the same objects as $C(\mathcal{A})$, but

$$\text{Hom}_{K_{\text{weak}}(\mathcal{A})}(X,Y) := \frac{\text{Hom}_{C(\mathcal{A})}(X,Y)}{\{\text{morphisms weakly homotopic to zero}\}}.$$ 

**Remark 2.6.** Let $h$ and $(s,t)$ be as in the above definition (without asking for $f - g = dh + hd$ or $f - g = ds + td$). Then $dh + hd : X \to Y$ is homotopic to zero. But $ds + td : X \to Y$ is not necessarily weakly homotopic to zero: It need not be a morphism in $C(\mathcal{A})$. It is weakly homotopic to zero if and only if $dss = dttd$ (which is of course the case if $f - g = ds + td$).

Note that weakly homotopic maps induce the same map on cohomology.

All categories $C(\mathcal{A}), K(\mathcal{A})$ and $K_{\text{weak}}(\mathcal{A})$ are additive categories. Let $[1] : C(\mathcal{A}) \to C(\mathcal{A})$ be the functor that maps an object $X$ to $[1]X$ where $([1]X)^n = X^{n+1}$ and $d_{[1]X}^n = -d_X^{n+1}$ and a morphism $f : X \to Y$ to $[1]f$ where $([1]f)^n = f^{n+1}$. This is an automorphism of $C(\mathcal{A})$ and induces automorphisms $[1]$ of $K(\mathcal{A})$ and $K_{\text{weak}}(\mathcal{A})$. The categories $C(\mathcal{A}), K(\mathcal{A})$ and $K_{\text{weak}}(\mathcal{A})$ become additive categories with translation. Sometimes we write $\Sigma$ instead of $[1]$. Obviously there are canonical functors

$$C(\mathcal{A}) \to K(\mathcal{A}) \to K_{\text{weak}}(\mathcal{A})$$

of additive categories with translation.
The category $K(A)$ has a natural structure of triangulated category: Given a morphism $m : M \to N$ in $C(A)$ we define its mapping cone $\text{Cone}(m)$ of $m$ to be the complex $\text{Cone}(m) = N \oplus [1]M$ with differential $\begin{bmatrix} d_N & m \\ 0 & d_{[1,M]} \end{bmatrix} = \begin{bmatrix} d_N & m \\ 0 & -d_M \end{bmatrix}$. It fits into the following mapping cone sequence

\begin{equation}
M \xrightarrow{m} N \xrightarrow{\text{Cone}(m)} [1]M
\end{equation}

in $C(A)$. The triangles of $K(A)$ are precisely the candidate triangles that are isomorphic to the image of a mapping cone sequence (2.5) in $K(A)$; this image is called the mapping cone triangle of $m$. We will later use: If we rotate the mapping cone triangle for $-m$ twice we obtain the triangle

\begin{equation}
[-1]N \xrightarrow{-1} [-1] \text{Cone}(-m) \xrightarrow{0-1} M \xrightarrow{-m} N.
\end{equation}

In this setting there is apart from (2.3) another triangulated equivalence between $K(A)$ and $K(A)^{\text{anti}}$: The functor $S : C(A) \xrightarrow{\sim} C(A)$ which sends a complex $(X^n, d^n_X)$ to $(X^n, -d^n_X)$ and a morphism $f$ to $f$ descends to a functor $S : K(A) \xrightarrow{\sim} K(A)$. Then

\begin{equation}
(S, \text{id}) : K(A) \xrightarrow{\sim} K(A)^{\text{anti}}
\end{equation}

(where $\text{id} : S[1] \xrightarrow{\sim} [1]S$ is the obvious identification) is a triangulated equivalence: Observe that $S$ maps (2.5) to

\begin{equation}
S(M) \xrightarrow{S(m)=m} S(N) \xrightarrow{1} \text{Cone}(-S(m)) \xrightarrow{0 1} [1]S(M)
\end{equation}

which becomes an anti-triangle in $K(A)$.

We introduce some notation: Any functor $F : A \to B$ of additive categories obviously induces a functor $F_C : C(A) \to C(B)$ of additive categories with translation and a functor $F_K : K(A) \to K(B)$ of triangulated categories. If $F : A \to A$ is an endofunctor we denote these functors often by $F_{C(A)}$ and $F_{K(A)}$.

### 3. Homotopy categories and idempotent completeness

Let $K(A)$ be the homotopy category of an additive category $A$. We define full subcategories of $K(A)$ as follows. Let $K^{\leq n}(A)$ consist of all objects that are zero in all degrees $> n$ (where $w$ means “weights”; the terminology will become clear from Proposition 4.6 below). The union of all $K^{\leq n}(A)$ for $n \in \mathbb{Z}$ is $K^-(A)$, the category of all bounded above complexes.
Similarly we define $K_{w \leq n}(\mathcal{A})$ and $K^+(\mathcal{A})$. Let $K^b(\mathcal{A}) = K^-\mathcal{A} \cap K^+(\mathcal{A})$ be the full subcategory of all bounded complexes.

For $* \in \{+, -, b, w \leq n, w \geq n\}$ let $K^*(\mathcal{A})$ be the closure under isomorphisms of $K^*(\mathcal{A})$ in $K(\mathcal{A})$. Then all inclusions $K^*(\mathcal{A}) \subset K(\mathcal{A})^*$ are equivalences of categories.

We define $K(\mathcal{A})^{bw} := K(\mathcal{A})^- \cap K(\mathcal{A})^+$ (where $bw$ means “bounded weights”). In general the inclusion $K(\mathcal{A})^b \subset K(\mathcal{A})^{bw}$ is not an equivalence (see Rem. 3.2).

**Theorem 3.1.** Let $\mathcal{A}$ be an additive category and $n \in \mathbb{Z}$. The following categories are idempotent complete:

1. $K_{w \leq n}(\mathcal{A}), K(\mathcal{A})^{w \leq n}, K^-(\mathcal{A}), K^-(\mathcal{A})$;
2. $K_{w \geq n}(\mathcal{A}), K(\mathcal{A})^{w \geq n}, K^+(\mathcal{A}), K^+(\mathcal{A})$;
3. $K(\mathcal{A})^{bw}$.

**Remark 3.2.** In general the (equivalent) categories $K^b(\mathcal{A})$ and $K(\mathcal{A})^b$ are not idempotent complete (and hence the inclusion into the idempotent complete category $K(\mathcal{A})^{bw}$ is not an equivalence): Let mod$(k)$ be the category of finite dimensional vector spaces over a field $k$ and let $\mathcal{E} \subset$ mod$(k)$ be the full subcategory of even dimensional vector spaces. Note that $K(\mathcal{E}) \subset K(\text{mod}(k))$ is a full triangulated subcategory. Then $X \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(X)$ is well defined on the objects of $K(\text{mod}(k))^b$. It takes even values on all objects of $K(\mathcal{E})^b$; hence $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} : k^2 \rightarrow k^2$ cannot split in $K(\mathcal{E})^b$.

**Remark 3.3.** Let us indicate the ideas behind the rather explicit proof of Theorem 3.1 which might perhaps be seen as a variation of the Eilenberg swindle.

Let $\mathcal{T}'$ be an essentially small triangulated category. R. W. Thomason shows that taking the Grothendieck group establishes a bijection between dense strict full triangulated subcategories of $\mathcal{T}'$ and subgroups of its Grothendieck group $K_0(\mathcal{T}')$, see [Tho97, Section 3].

Now let $\mathcal{T}^{ic}$ be the idempotent completion of an essentially small triangulated category $\mathcal{T}$, cf. Section 2.2 it carries a natural triangulated structure [BS01, Thm. 1.5]. The previous result applied to $\mathcal{T}' = \mathcal{T}^{ic}$ shows that the vanishing of $K_0(\mathcal{T}^{ic})$ implies that $\mathcal{T}$ is idempotent complete; this was observed by P. Balmer and M. Schlichting [BS01, 2.2-2.5] where they also provide a method that sometimes shows this vanishing condition.

These results show directly that $K^-(\mathcal{A}), K(\mathcal{A})^-, K^+(\mathcal{A}), K(\mathcal{A})^+$ and $K(\mathcal{A})^{bw}$ are idempotent complete if $\mathcal{A}$ is an essentially small additive category.
A careful analysis of the proofs of these results essentially gives our proof of Theorem 3.1 below.

**Proof.** We prove (1) first. It is obviously sufficient to prove that $T^{\leq n} := K^{\leq n}(A)$ is idempotent complete. Let $T := K^-(A)$ and consider the endofunctor

$$S : T \to T,$$

$$X \mapsto \bigoplus_{n \in \mathbb{N}} [2n]X = X \oplus [2]X \oplus [4]X \oplus \ldots$$

(it is even triangulated by [Nee01, Prop. 1.2.1]). Note that $S$ is well defined: Since $X$ is bounded above, the countable direct sum has only finitely many nonzero summands in every degree. There is an obvious isomorphism of functors $S \sim - \to \text{id} \oplus [2]S$. The functor $S$ extends to a (triangulated) endofunctor of the idempotent completion $T^{\text{ic}}$ of $T$, denoted by the same symbol, and we still have an isomorphism $S \sim - \to \text{id} \oplus [2]S$.

Now let $M$ be an object of $T^{w\leq n}$ with an idempotent endomorphism $e : M \to M$. In $T^{\text{ic}}$ we obtain a direct sum decomposition $M \cong E \oplus F$ such that $e$ corresponds to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} : E \oplus F \to E \oplus F$. We have to show that $E$ is isomorphic to an object of $T^{w\leq n}$.

Since $S$ preserves $T^{w\leq n}$, we obtain that

$$SM \cong SE \oplus SF \sim (E \oplus [2]SE) \oplus SF \in T^{w\leq n},$$

where we use the convention just to write $X \in T^{w\leq n}$ if an object $X \in T^{\text{ic}}$ is isomorphic to an object of $T^{w\leq n}$. The direct sum of the triangles

$$
\begin{aligned}
0 &\to SE \xrightarrow{1} SE \to 0, \\
SE &\to 0 \to [1]SE \xrightarrow{1} [1]SE, \\
SF &\xrightarrow{1} SF \to 0 \to [1]SF
\end{aligned}
$$

in $T^{\text{ic}}$ is the triangle

$$
SE \oplus SF \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} SE \oplus SF \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} SE \oplus [1]SE \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} [1](SE \oplus SF).
$$

The first two vertices are isomorphic to $SM \in T^{w\leq n}$. The mapping cone of a map between objects of $T^{w\leq n}$ is again in $T^{w\leq n}$. We obtain that

$$SE \oplus [1]SE \in T^{w\leq n}.$$
Applying \([1]\) (which preserves \(\mathcal{T}^{w \leq n}\)) to the same statement for \(F\) yields
\[(3.3) \quad [1]SF \oplus [2]SF \in \mathcal{T}^{w \leq n}.\]
Taking the direct sum of the objects in \((3.1), (3.2)\) and \((3.3)\) shows that
\[E \oplus [2]SE \oplus SF \oplus SE \oplus [1]SE \oplus [1]SF \oplus [2]SF \]
\[\cong E \oplus [SM \oplus [1]SM \oplus [2]SM] \in \mathcal{T}^{w \leq n}.\]
Define \(R := SM \oplus [1]SM \oplus [2]SM\) which is obviously in \(\mathcal{T}^{w \leq n}\). Then the “direct sum” triangle
\[R \to R \oplus E \to E \to [1]R\]
shows that \(E\) is isomorphic to an object of \(\mathcal{T}^{w \leq n}\).

Now we prove \((2)\). (The proof is essentially the same, but one has to pay attention to the fact that the mapping cone of a map between objects of \(K^{w \geq n}(\mathcal{A})\) is only in \(K^{w \geq n-1}(\mathcal{A})\).) Again it is sufficient to show that \(\mathcal{T}^{w \geq n}(\mathcal{A})\) is idempotent complete. Let \(\mathcal{T} := K^+(\mathcal{A})\) and consider the (triangulated) functor \(S : \mathcal{T} \to \mathcal{T}\), \(X \mapsto \bigoplus_{n \in \mathbb{N}} [-2n]X = X \oplus [-2]X \oplus \ldots\). It is well defined, extends to the idempotent completion \(\mathcal{T}^{ic}\) of \(\mathcal{T}\), and we have an isomorphism \(S \cong \text{id} \oplus [-2]S\) of functors. Let \(M\) in \(\mathcal{T}^{w \geq n}\) with an idempotent endomorphism \(e : M \to M\). In \(\mathcal{T}^{ic}\) we have a direct sum decomposition \(M \cong E \oplus F\) such that \(e\) corresponds to \(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} : E \oplus F \to E \oplus F\). We have to show that \(E\) is isomorphic to an object of \(\mathcal{T}^{w \geq n}\).

Since \(S\) preserves \(\mathcal{T}^{w \geq n}\), we obtain (with the analog of the convention introduced above) that
\[(3.4) \quad [SM \cong SE \oplus SF] \cong (E \oplus [-2]SE) \oplus SF \in \mathcal{T}^{w \geq n}.\]
As above we have a triangle
\[SE \oplus SF \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} SE \oplus SF \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} SE \oplus [1]SE \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} [1](SE \oplus SF).\]
The first two vertices are isomorphic to \(SM \in \mathcal{T}^{w \geq n}\); hence we get
\(SE \oplus [1]SE \in \mathcal{T}^{w \geq n-1}\) or equivalently
\[(3.5) \quad [-1]SE \oplus SE \in \mathcal{T}^{w \geq n}.\]
Applying \([-1]\) (which preserves \(\mathcal{T}^{w \geq n}\)) to the same statement for \(F\) yields
\[(3.6) \quad [-2]SF \oplus [-1]SF \in \mathcal{T}^{w \geq n}.\]
Taking the direct sum of the objects in (3.4), (3.5) and (3.6) shows that
\[
E \oplus [-2]SE \oplus SF \oplus [-1]SE \oplus SE \oplus [-2]SF \oplus [-1]SF \\
\cong E \oplus [SM \oplus [-1]SM \oplus [-2]SM] \in \mathcal{T}^{w \geq n}.
\]
Define \( R := SM \oplus [-1]SM \oplus [-2]SM \) which is obviously in \( \mathcal{T}^{w \geq n} \).
Then the “direct sum” triangle
\[
E \to E \oplus R \to R \\ 0 \to [1]E
\]
shows that \([1]E\) is isomorphic to an object of \( \mathcal{T}^{w \geq n-1} \). Hence \( E \) is isomorphic to an object of \( \mathcal{T}^{w \geq n} \).

The statement (3) that \( K(A) \) is idempotent complete is a consequence of (1) and (2).

\[\Box\]

**Theorem 3.4.** Let \( A \) be an additive category.

1. If \( A \) is abelian then \( K(A) \) is idempotent complete.
2. If \( A \) has countable coproducts then \( K(A) \) is idempotent complete.
3. If \( A \) is idempotent complete then \( K^b(A) \) and \( K(A)^b \) are idempotent complete.

**Remark 3.5.** We do not know whether \( K(A) \) is idempotent complete for additive \( A \) (cf. Question 1.1).

**Remark 3.6.** If \( A^{ic} \) is the idempotent completion of an additive category \( A \) then \( K(A) \) is idempotent complete if and only if \( K(A^{ic}) \) is idempotent complete: This follows from R. W. Thomason’s results cited in Remark 3.3 (note that \( K(A) \subset K(A^{ic}) \) is dense) and Proposition 4.12 below. Hence in Remark 3.5 one can assume without loss of generality that \( A \) is idempotent complete.

**Proof.** We prove (1) in Corollary 3.9 below.

Let us show (2). Assume that \( A \) has countable coproducts. Then \( K(A) \) has countable coproducts; hence any idempotent endomorphism of an object of \( K(A) \) splits by [Nee01 Prop. 1.6.8]. Another way to see this is to use the strategy explained in Remark 3.3. More concretely, adapt the proof of Theorem 3.1 (1) in the obvious way: Note that the functor \( X \mapsto \bigoplus_{n \in \mathbb{N}} [2n]X \) is well-defined on \( K(A) \).

For the proof of (3) assume now that \( A \) is idempotent complete. Since \( K^b(A) \subset K(A)^b \) is an equivalence it is sufficient to show that \( K^b(A) \) is idempotent complete.

Let \( C \) be an object of \( K^b(A) \). Let \( X := \bigoplus_{i \in \mathbb{Z}} C^i \) be the finite direct sum over all nonzero components of \( C \). Let \( \text{add} X \subset A \) be the full subcategory of \( A \) that contains \( X \) and is closed under finite direct
splits.

In the rest of this section we assume that $\mathcal{A}$ is abelian. We use torsion pairs/t-structures in order to prove that $K(\mathcal{A})$ is idempotent complete.

Let $K^h \leq 0 \subset K(\mathcal{A})$ be the full subcategory of objects isomorphic to complexes $X$ of the form

$$\cdots \to 0 \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{} \cdots$$

with $X^0$ in degree zero and $d^{-2}$ the kernel of $d^{-1}$. (Here $K$ stands for “kernel” and “$h \geq 0$” indicates that the cohomology is concentrated in degrees $\geq 0$.)

Let $C^h \leq 0 \subset K(\mathcal{A})$ be the full subcategory of objects isomorphic to complexes $X$ of the form

$$\cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{} 0 \xrightarrow{} \cdots$$

with $X^0$ in degree zero and $d^1$ the cokernel of $d^0$. (Here $C$ stands for “cokernel” and “$h \leq 0$” indicates that the cohomology is concentrated in degrees $\leq 0$.)

Define $C^{h \leq n} := [-n]C^h \leq 0$ and $K^{h \geq n} := [-n]K^h \geq 0$.

**Lemma 3.7** (cf. Example after [BR07, Prop. I.2.15]). Let $\mathcal{A}$ be abelian. Then

$$k := (K(\mathcal{A})^{w \leq 0}, K^{h \geq 1}) \quad \text{and} \quad c := (C^{h \leq -1}, K(\mathcal{A})^{w \geq 0})$$

are torsion pairs on $K(\mathcal{A})$.

**Remark 3.8.** It is easy to see that the torsion pair $k$ of Lemma 3.7 coincides with the torsion pair defined in the Example after [BR07, Prop. I.2.15]: An object $X \in K(\mathcal{A})$ is in $K(\mathcal{A})^{w \leq n}$ (resp. $K^{h \geq n}$) if and only if the complex $\text{Hom}(A, X)$ of abelian groups is exact in all degrees.
> n (resp. < n) for all \( A \in \mathcal{A} \). The categories \( \mathcal{C}^{h \leq n} \) and \( K(\mathcal{A})^{w \geq n} \) can be characterized similarly.

**Proof.** Let \((\mathcal{X}, \mathcal{Y})\) be the pair \( k \) or the pair \([1]c := (\mathcal{C}^{h \leq -2}, K(\mathcal{A})^{w \geq -1})\).

Let \( f \) represent a morphism \( X \to Y \) with \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \). Then the diagram

\[
\begin{array}{cccccccccc}
X : & \cdots & X^{-2} & \cdots & X^{-1} & d^{-1} & X^{0} & 0 & 0 & \cdots \\
& \downarrow f & & & & \downarrow d^{-1} & & & & \\
Y : & \cdots & 0 & \cdots & 0 & Y^{-1} & d^{-1} & Y^{0} & d^{0} & Y^{1} & d^{1} & Y^{2} & \cdots \\
\end{array}
\]

shows that \( f \) is homotopic to zero; hence \( \text{Hom}_{K(\mathcal{A})}(\mathcal{X}, \mathcal{Y}) = 0 \). It is obvious that \( \mathcal{X} \) is stable under \([1]\) and \( \mathcal{Y} \) is stable under \([-1]\).

We need to show that any object \( A = (A^0, A^1) \) of \( K(\mathcal{A}) \) fits into a triangle \( X \to A \to Y \to [1]X \) with \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \).

- Case \((\mathcal{X}, \mathcal{Y}) = k\): Let \( f : M \to A^0 \) be the kernel of \( d^0 : A^0 \to A^1 \); then there is a unique morphism \( g : A^{-1} \to M \) such that \( d^{-1} = fg \).
- Case \((\mathcal{X}, \mathcal{Y}) = [1]c\): Let \( g : A^{-1} \to M \) be the cokernel of \( d^{-2} : A^{-2} \to A^{-1} \); then there is a unique morphism \( f : M \to A^0 \) such that \( d^{-1} = fg \).

The commutative diagram

\[
\begin{array}{cccccccccc}
X : & \cdots & A^{-2} & \cdots & A^{-1} & g & M & 0 & 0 & \cdots \\
& \downarrow 1 & & & 1 & \downarrow f & & & & \\
A : & \cdots & A^{-2} & \cdots & A^{-1} & d^{-1} & A^0 & d^0 & A^1 & d^1 & A^2 & \cdots \\
& \downarrow g & & & 1 & \downarrow 1 & & & & \\
Y : & \cdots & 0 & \cdots & M & f & A^0 & d^0 & A^1 & d^1 & A^2 & \cdots \\
\end{array}
\]

defines a candidate triangle \( X \to A \to Y \to [1]X \) in \( K(\mathcal{A}) \). It is easy to check that it is in fact a triangle. \(\square\)

**Corollary 3.9.** Let \( \mathcal{A} \) be an abelian category and \( n \in \mathbb{Z} \). Then \( K(\mathcal{A}) \) is idempotent complete, and the same is true for \( K^{h \geq n} \) and \( \mathcal{C}^{h \leq n} \).

**Proof.** Let \((\mathcal{X}, \mathcal{Y})\) be one of the torsion pairs of Lemma 3.7.

Let \( e : A \to A \) be an idempotent endomorphism in \( K(\mathcal{A}) \). The truncation functors \( \tau_{\mathcal{X}} : K(\mathcal{A}) \to \mathcal{X} \) and \( \tau_{\mathcal{Y}} : K(\mathcal{A}) \to \mathcal{Y} \) yield a
Morphism of triangles

\[
\begin{array}{ccc}
\tau_X(A) & \xrightarrow{u} & A \\
\downarrow \tau_X(e) & & \downarrow e \\
\tau_X(A) & \xrightarrow{u} & A
\end{array}
\quad \begin{array}{ccc}
\tau_Y(A) & \xrightarrow{v} & \tau_Y(A) \\
\downarrow \tau_Y(e) & & \downarrow [1] \tau_Y(e) \\
\tau_Y(A) & \xrightarrow{v} & \tau_Y(A)
\end{array}
\quad \begin{array}{ccc}
\tau_X(A) & \xrightarrow{w} & [1] \tau_X(A) \\
\downarrow & & \downarrow \\
\tau_X(A) & \xrightarrow{w} & [1] \tau_X(A)
\end{array}
\]

All morphisms \(\tau_X(e), e, \tau_Y(e)\) are idempotent and \(\tau_X(e)\) and \(\tau_Y(e)\) split by Theorem 3.1 (since \(K^h_{\geq n} \subset K(A)^{w \geq n-2}\) and \(C^h_{\leq n} \subset K(A)^{w \leq n+2}\)). Hence \(e\) splits by [LC07, Prop. 2.3]. This shows that \(K(A)\) is idempotent complete.

Since \(\mathcal{X} = \perp \mathcal{Y}\) and \(\mathcal{Y} = \perp \mathcal{X}\) for any torsion pair this implies that \(\mathcal{X}\) and \(\mathcal{Y}\) are idempotent complete. \(\square\)

4. Weight structures

The following definition of a weight structure is independently due to M. Bondarko [Bon10] and D. Pauksztello [Pau08] who calls it a co-t-structure.

**Definition 4.1.** Let \(\mathcal{T}\) be a triangulated category. A weight structure (or \(w\)-structure) on \(\mathcal{T}\) is a pair \(w = (\mathcal{T}^w_{\leq 0}, \mathcal{T}^w_{\geq 0})\) of two full subcategories such that:

- \((ws1)\) \(\mathcal{T}^w_{\leq 0}\) and \(\mathcal{T}^w_{\geq 0}\) are additive categories and closed under retracts in \(\mathcal{T}\);
- \((ws2)\) \(\mathcal{T}^w_{\leq 0} \subset \mathcal{T}^w_{\leq 1}\) and \(\mathcal{T}^w_{\geq 1} \subset \mathcal{T}^w_{\geq 0}\);
- \((ws3)\) \(\text{Hom}_\mathcal{T}(\mathcal{T}^w_{\geq 1}, \mathcal{T}^w_{\leq 0}) = 0\);
- \((ws4)\) For every \(X\) in \(\mathcal{T}\) there is a triangle

\[
A \to X \to B \to [1]A
\]

in \(\mathcal{T}\) with \(A\) in \(\mathcal{T}^w_{\geq 1}\) and \(B\) in \(\mathcal{T}^w_{\leq 0}\).

A weight structure \(w = (\mathcal{T}^w_{\leq 0}, \mathcal{T}^w_{\geq 0})\) is **bounded above** if \(\mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^w_{\leq n}\) and **bounded below** if \(\mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^w_{\geq n}\). It is **bounded** if it is bounded above and bounded below.

The **heart** of a weight structure \(w = (\mathcal{T}^w_{\leq 0}, \mathcal{T}^w_{\geq 0})\) is

\[
\mathcal{H}(w) := \mathcal{T}^w_{=0} := \mathcal{T}^w_{\leq 0} \cap \mathcal{T}^w_{\geq 0}.
\]

A weight category (or \(w\)-category) is a pair \((\mathcal{T}, w)\) where \(\mathcal{T}\) is a triangulated category and \(w\) is a weight structure on \(\mathcal{T}\). Its heart is the heart of \(w\).

A triangle \(A \to X \to B \to [1]A\) with \(A\) in \(\mathcal{T}^w_{\geq n+1}\) and \(B\) in \(\mathcal{T}^w_{\leq n}\) (cf. \([ws4]\)) is called a weight decomposition of \(X\), or more precisely
a \((w \geq n + 1, w \leq n)\)-weight decomposition or a weight decomposition of type \((w \geq n + 1, w \leq n)\) of \(X\). It is convenient to write such a weight decomposition as \(w_{\geq n+1}X \rightarrow X \rightarrow w_{\leq n}X \rightarrow [1]w_{\geq n+1}X\) where \(w_{\geq n+1}X\) and \(w_{\leq n}X\) are just names for the objects \(A\) and \(B\) from above. If we say that \(w_{\geq n+1}X \rightarrow X \rightarrow w_{\leq n}X \rightarrow [1]w_{\geq n+1}X\) is a weight decomposition without specifying its type explicitly, this type is usually obvious from the notation.

The heart \(\bigcirc(w)\) is a full subcategory of \(\mathcal{T}\) and closed under retracts in \(\mathcal{T}\) by (ws1).

We will use the following notation (for \(a, b \in \mathbb{Z}\)): \(\mathcal{T}^{w \in [a,b]} := \mathcal{T}^{w \leq b} \cap \mathcal{T}^{w \geq a}\). Note that \(\mathcal{T}^{w \in [a,b]} = 0\) if \(b < a\) by (ws3).

For \(X \in \mathcal{T}^{w \in [a,b]}\) we have \(\text{id}_X = 0\).

**Definition 4.2.** Let \(\mathcal{T}\) and \(\mathcal{S}\) be weight categories. A triangulated functor \(F : \mathcal{T} \rightarrow \mathcal{S}\) is called weight-exact (or weight-exact) if \(F(\mathcal{T}^{w \leq 0}) \subset \mathcal{S}^{w \leq 0}\) and \(F(\mathcal{T}^{w \geq 0}) \subset \mathcal{S}^{w \geq 0}\).

4.1. First properties of weight structures. Let \(\mathcal{T}\) be a triangulated category with a weight structure \(w = (\mathcal{T}^{w \leq 0}, \mathcal{T}^{w \geq 0})\).

**Lemma 4.3** (cf. [Bon10] Prop. 1.3.3) for some statements).

1. Let \(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} [1]X\) be a triangle in \(\mathcal{T}\). If \(Z \in \mathcal{T}^{w \geq n}\) and \(X \in \mathcal{T}^{w \leq n}\) then this triangle splits.

   In particular any triangle \(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} [1]X\) with all objects \(X, Y, Z\) in the heart \(\bigcirc(w)\) splits.

Let

\[
(4.1) \quad w_{\geq n+1}X \xrightarrow{f} X \xrightarrow{g} w_{\leq n}X \xrightarrow{h} [1]w_{\geq n+1}X
\]

be a \((w \geq n + 1, w \leq n)\)-weight decomposition of \(X\), for some \(n \in \mathbb{Z}\).

2. If \(X\) is in \(\mathcal{T}^{w \leq n}\), then \(w_{\leq n}X \cong X \oplus [1]w_{\geq n+1}X\)

3. If \(X\) is in \(\mathcal{T}^{w \geq n+1}\), then \(w_{\geq n+1}X \cong X \oplus [-1]w_{\leq n}X\).

4. For every \(n \in \mathbb{Z}\) we have

\[
(4.2) \quad (\mathcal{T}^{w \geq n+1})^\perp = \mathcal{T}^{w \leq n},
\]

\[
(4.3) \quad \perp (\mathcal{T}^{w \leq n}) = \mathcal{T}^{w \geq n+1}.
\]

In particular \(\mathcal{T}^{w \leq n}\) and \(\mathcal{T}^{w \geq n+1}\) are closed under extensions.

5. Assume that \(a \leq n < b\) (for \(a, b \in \mathbb{Z}\)) and that \(X \in \mathcal{T}^{w \in [a,b]}\). Then \(w_{\leq n}X \in \mathcal{T}^{w \in [a,n]}\) and \(w_{\geq n+1}X \in \mathcal{T}^{w \in [n+1,b]}\).

More precisely: If \(a \leq n\) then \(X \in \mathcal{T}^{w \geq a}\) implies \(w_{\leq n}X \in \mathcal{T}^{w \in [a,n]}\) (and obviously \(w_{\geq n+1}X \in \mathcal{T}^{w \geq n+1}\subset \mathcal{T}^{w \geq a}\)). If \(n < b\) then \(X \in \mathcal{T}^{w \leq b}\) implies (obviously \(w_{\leq n}X \in \mathcal{T}^{w \leq n} \subset \mathcal{T}^{w \leq b}\) and)

\(w_{\geq n+1}X \in \mathcal{T}^{w \in [n+1,b]}\).
(6) Let $a, b, n \in \mathbb{Z}$. For $X \in \mathcal{T}^{w \geq a}$ (resp. $X \in \mathcal{T}^{w \leq b}$ or $X \in \mathcal{T}^{w \in [a, b]}$) there is a $(w \geq n + 1, w \leq n)$-weight decomposition (4.1) of $X$ such that both $w_{\geq n+1} X$ and $w_{\leq n} X$ are in $\mathcal{T}^{w \geq a}$ (resp. in $\mathcal{T}^{w \leq b}$ or $\mathcal{T}^{w \in [a, b]}$).

**Proof.** By (ws3) we have $h = 0$ in (1), $f = 0$ in (2) and $g = 0$ in (3), use Lemma 2.1.

We prove (4). Axiom (ws3) shows that the inclusions $\supset \subseteq \cap$ in (4.2) and (4.3) are true. Let $X \in \mathcal{T}$ and take a weight decomposition (4.1) of $X$. If $X \in (\mathcal{T}^{w \geq n+1})^\perp$ then $f = 0$ by (ws3), hence $X$ is a summand of $w_{\leq n} X \in \mathcal{T}^{w \leq n}$ and hence in $\mathcal{T}^{w \leq n}$ by (ws1). Similarly $X \in (\mathcal{T}^{w \leq n})^\perp$ implies $g = 0$ so $X$ is a summand of $w_{\geq n+1} X \in \mathcal{T}^{w \geq n+1}$ and hence in $\mathcal{T}^{w \geq n+1}$.

Let us prove (5): Since $X \in \mathcal{T}^{w \geq a}$ and $[1]w_{\geq n+1} X \in \mathcal{T}^{w \geq n} \subset \mathcal{T}^{w \geq a}$ and $\mathcal{T}^{w \geq a}$ is closed under extensions by (4) we have $w_{\leq n} X \in \mathcal{T}^{w \in [a, n]}$. Turning the triangle we see that $w_{\geq n+1} X$ is an extension of $[-1]w_{\leq n} X \in \mathcal{T}^{w \leq n+1} \subset \mathcal{T}^{w \leq b}$, hence $w_{\geq n+1} X \in \mathcal{T}^{w \in [n+1, b]}$.

We prove (6): Assume $X \in \mathcal{T}^{w \geq a}$. If $a \leq n$ any such weight decomposition does the job by (5): if $a > n$ take $X \xrightarrow{\text{id}} X \to 0 \to [1] X$.

Assume $X \in \mathcal{T}^{w \leq b}$. If $n < b$ use (5); if $b \leq n$ take $0 \to X \xrightarrow{\text{id}} X \to 0$.

Assume $X \in \mathcal{T}^{w \in [a, b]}$. The case $a > b$ is trivial since then $X = 0$. So assume $a \leq b$. If $a \leq n < b$ use (4); if $a > n$ take $X \xrightarrow{\text{id}} X \to 0 \to [1] X$ if $b \leq n$ take $0 \to X \xrightarrow{\text{id}} X \to 0$.

**Corollary 4.4.** Let $(\mathcal{D}^{w \leq 0}, \mathcal{D}^{w \geq 0})$ and $(\mathcal{T}^{w \leq 0}, \mathcal{T}^{w \geq 0})$ be two weight structures on a triangulated category. If

\[ \mathcal{D}^{w \leq 0} \subset \mathcal{T}^{w \leq 0} \text{ and } \mathcal{D}^{w \geq 0} \subset \mathcal{T}^{w \geq 0}, \]

then these two weight structures coincide.

**Proof.** Our assumptions and (4.2) give

\[ \mathcal{T}^{w \leq 0} = (\mathcal{T}^{w \geq 1})^\perp \subset (\mathcal{D}^{w \geq 1})^\perp = \mathcal{D}^{w \leq 0} \]

Similarly we obtain $\mathcal{T}^{w \geq 0} \subset \mathcal{D}^{w \geq 0}$. □

The following Lemma is the analog of [BBD82, 1.3.19].

**Lemma 4.5.** Let $\mathcal{T}'$ be a full triangulated subcategory of a triangulated category $\mathcal{T}$. Assume that $w = (\mathcal{T}'^{w \leq 0}, \mathcal{T}'^{w \geq 0})$ is a weight structure on $\mathcal{T}$. Let $w' = (\mathcal{T}' \cap \mathcal{T}^{w \leq 0}, \mathcal{T}' \cap \mathcal{T}^{w \geq 0})$. Then $w'$ is a weight structure on $\mathcal{T}'$ if and only if for any object $X \in \mathcal{T}'$ there is a triangle

\[ w_{\geq 1} X \xrightarrow{w_{\geq 1}} X \xrightarrow{[1]} w_{\leq 0} X \xrightarrow{[1]} w_{\geq 1} X \]

in $\mathcal{T}'$ that is a weight decomposition of type $(w \geq 1, w \leq 0)$ in $(\mathcal{T}, w)$.
If $w'$ is a weight structure on $\mathcal{T}'$ it is called the induced weight structure.

Proof. If $w'$ is a weight structure on $\mathcal{T}'$ weight decompositions in $(\mathcal{T}', w')$ are triangles and yield weight decompositions in $(\mathcal{T}, w)$.

Conversely let us show that under the given condition $w'$ is a weight structure on $\mathcal{T}'$. This condition obviously says that $w'$ satisfies (ws4).

If $Y$ is a retract in $\mathcal{T}'$ of $X \in \mathcal{T}' \cap \mathcal{T}^{w \leq 0}$, it is a retract of $X$ in $\mathcal{T}$ and hence $Y \in \mathcal{T}^{w \leq 0}$. This proves that $\mathcal{T}' \cap \mathcal{T}^{w \leq 0}$ is closed under retracts in $\mathcal{T}'$, cf. (ws1). The remaining conditions for $w'$ being a weight structure are obvious. $\square$

4.2. Basic example. Let $\mathcal{A}$ be an additive category and $K(\mathcal{A})$ its homotopy category. We use the notation introduced in Section 3.

Proposition 4.6 (cf. [Bon10], [Pan08]). The pair

$$(K(\mathcal{A})^{w \leq 0}, K(\mathcal{A})^{w \geq 0})$$

is a weight structure on $K(\mathcal{A})$.

It induces (see Lemma 4.5) weight structures on $K^*(\mathcal{A})$ for $* \in \{+, -, b\}$ and on $K^b(\mathcal{A})$ for $* \in \{+, -, b, bw\}$.

All these weight structures are called the standard weight structure on the respective category.

Remark 4.7. The triangulated equivalence (2.7) between $K(\mathcal{A})$ and $K(\mathcal{A})^{\text{anti}}$ allows us to transfer the weight structure from Proposition 4.6 to $K(\mathcal{A})^{\text{anti}}$. This defines the standard weight structure

$$(K(\mathcal{A})^{\text{anti}, w \leq 0}, K(\mathcal{A})^{\text{anti}, w \geq 0})$$

on $K(\mathcal{A})^{\text{anti}}$. We have $K(\mathcal{A})^{\text{anti}, w \leq 0} = K(\mathcal{A})^{w \leq 0}$ and $K(\mathcal{A})^{\text{anti}, w \geq 0} = K(\mathcal{A})^{w \geq 0}$. Similarly one can transfer the induced weight structures.

Proof. Condition (ws1) It is obvious that both $K(\mathcal{A})^{w \leq 0}$ and $K(\mathcal{A})^{w \geq 0}$ are additive. Since they are strict subcategories of $K(\mathcal{A})$ and idempotent complete by Theorem 3.1, they are in particular closed under retracts in $K(\mathcal{A})$. Conditions (ws2) and (ws3) are obvious.

We verify condition (ws4) explicitly. Let $X = (X^i, d^i : X^i \to X^{i+1})$ be a complex. We give $(w \geq n + 1, w \leq n)$-weight decompositions of $X$ for any $n \in \mathbb{Z}$. The following diagram defines complexes $\underline{w}_{\leq n}(X)$, $\underline{w}_{\geq n + 1}(X)$ and a mapping cone sequence $[-1]\underline{w}_{\leq n}(X) \to \underline{w}_{\geq n + 1}(X) \to$
Passing to $K(\mathcal{A})$ and rotation of the triangle yields the weight decomposition we need:

\[(4.5)\]

The statements about the induced weight structures on $K^*(\mathcal{A})$ and $K(\mathcal{A})^*$ for $* \in \{+, -, b\}$ are obvious from (4.5) and Lemma 4.3. For $K(\mathcal{A})^{bw}$ use additionally Lemma 4.3 (6).

We continue to use the maps $w_{\leq n}$ and $w_{\geq n+1}$ on complexes introduced in the above proof. Define $w_{\geq n} := w_{\geq n+1}$, $w_{\leq n} := w_{\leq n+1}$, $w_{[a,b]} = w_{\geq a}w_{\leq b} = w_{\leq b}w_{\geq a}$, and $w_{\leq} = w_{[0,\infty]}$. The triangle (4.3) will be called the $w$-weight decomposition of $X$.

Remark 4.8. Note that $w_{\geq}$ and $w_{\leq}$ are functorial on $C(\mathcal{A})$ but not at all on $K(\mathcal{A})$ (if $\mathcal{A} \neq 0$): Take an object $M \in \mathcal{A}$ and consider its mapping cone $\text{Cone(id}_M)$. All objects $\text{Cone(id}_M)$ (for $M \in \mathcal{A}$) are isomorphic to the zero object in $K(\mathcal{A})$. If $w_{\leq 0}$ were functorial, all $w_{\leq 0}(\text{Cone(id}_M)) = M$ were isomorphic, hence $\mathcal{A} = 0$. 

\[
X \to w_{\leq n}(X) \text{ in } C(\mathcal{A}):\n\]

\[
[-1]w_{\leq n}(X) : \quad \cdots \to X^{n-2} \xrightarrow{-d^{n-2}} X^{n-1} \xrightarrow{-d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \cdots
\]

\[
w_{\geq n+1}(X) : \quad \cdots \to 0 \to 0 \xrightarrow{X^{n+1}} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \cdots
\]

\[
X : \quad \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \cdots
\]

\[
w_{\leq n}(X) : \quad \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \cdots
\]

\[
[1]w_{\geq n+1}(X) : \quad \cdots \to 0 \xrightarrow{X^{n+1}} X^{n+1} \xrightarrow{-d^{n+1}} X^{n+2} \xrightarrow{-d^{n+2}} X^{n+3} \cdots
\]
Remark 4.9. Assume that $\mathcal{A}$ is an abelian category and consider the following four subcategories of $K(\mathcal{A})$:

$$(C^{h \leq 0}, K(\mathcal{A})^w \geq 1, K(\mathcal{A})^w \leq 0, K^h \geq 1).$$

The two outer pairs are torsion pairs on $K(\mathcal{A})$ by Lemma 3.7; the pair in the middle is (up two a swap of the two members and a translation) the standard w-structure on $K(\mathcal{A})$ from Proposition 4.6. In any pair of direct neighbors there are no morphisms from left to right; more precisely the left member is the left orthogonal of the right member and vice versa.

In the terminology of [Bon10, Def 4.4.1], the t-structure $(K(\mathcal{A})^w \leq 0, K^h \geq 0)$ and the (standard) w-structure $(K(\mathcal{A})^w \leq 0, K(\mathcal{A})^w \geq 0)$ on $K(\mathcal{A})$ are left adjacent (i.e. their left aisles coincide) to each other. Similarly, the t-structure $(C^{h \leq 0}, K(\mathcal{A})^w \geq 0)$ and the (standard) w-structure $(K(\mathcal{A})^w \leq 0, K(\mathcal{A})^w \geq 0)$ on $K(\mathcal{A})$ are right adjacent to each other.

Lemma 4.10. Let $X \in K(\mathcal{A})$ and $a, b \in \mathbb{Z}$. If $X \in K(\mathcal{A})^w \in [a,b]$ then $X$ is a summand of $\underline{w}_{a,b}(X)$ in $K(\mathcal{A})$.

Proof. Let $X \in K(\mathcal{A})^w \in [a,b]$. If $a > b$ then $X \cong 0 = \underline{w}_{a,b}(X)$. Assume $a \leq b$. Lemma 4.3 (2) gives $\omega_{\leq b}(X) \cong X \oplus [1]w_{>b}(X)$. Observe that $[1]w_{>b}(X) \in K(\mathcal{A})^w > b \subset K(\mathcal{A})^w \geq a$. Hence $\omega_{\leq b}(X) \in K(\mathcal{A})^w \geq a$. Now Lemma 4.3 (3) shows that $\underline{w}_{a,b}(X) \cong \omega_{\leq b}(X) \oplus [-1]w_{\geq a}(X)$.

We view $\mathcal{A}$ as a full subcategory of $K(\mathcal{A})$, namely given by all complexes concentrated in degree zero.

Corollary 4.11. The heart $\heartsuit$ of the standard weight structure on $K(\mathcal{A})$ is the idempotent completion of $\mathcal{A}$.

The same statement is true for the standard weight structure on $K(\mathcal{A})^* \ {\heartsuit}$ for $* \in \{+, -, bw\}$ (and for that on $K^*(\mathcal{A})$ for $* \in \{+, -\}$).

The heart of the standard weight structure on $K(\mathcal{A})^b$ (resp. $K^b(\mathcal{A})$) is the closure under retracts of $\mathcal{A}$ in $K(\mathcal{A})^b$ (resp. $K^b(\mathcal{A})$).

Proof. Since $K(\mathcal{A})^{bw}$ is idempotent complete (Thm. 3.1) and the heart $\heartsuit$ is contained in $K(\mathcal{A})^{bw}$ and closed under retracts in $K(\mathcal{A})$ it follows that $\heartsuit$ is idempotent complete. Furthermore any object $X \in \heartsuit$ is a summand of $X^0 = \underline{w}_0(X) \in \mathcal{A}$ by Lemma 4.10. Since $\mathcal{A}$ is a full subcategory of $\heartsuit$ the claim follows from the results at the end of Section 2.2. The proof of the second statement is similar.

For the proof of the last statement let $\heartsuit$ be the heart of the standard weight structure on $\mathcal{T} = K(\mathcal{A})^b$ (resp. $\mathcal{T} = K^b(\mathcal{A})$). Since $\mathcal{A} \subset \heartsuit$ and the heart $\heartsuit$ is closed under retracts in $\mathcal{T}$, any retract of an object of $\mathcal{A}$
Proposition 4.12. Let \( \mathcal{A} \) be a small additive category. Then the Grothendieck group of its homotopy category \( K(\mathcal{A}) \) is trivial:

\[
K_0(\mathcal{A}) = 0.
\]

The Grothendieck groups of \( K^{-}(\mathcal{A}), K^{+}(\mathcal{A}), K^{+}(\mathcal{A}) \) and \( K(\mathcal{A})^{+} \) vanish as well.

This result is presumably well-known to the experts (cf. [Sch06] or [Miy06]).

Proof. We prove that \( K_0(\mathcal{A}) = 0 \). We write \([X]\) for the class of an object \( X \) in the Grothendieck group. Let \( X \in K(\mathcal{A}) \). Let \( A = \mathcal{w}_{\geq 0}(X) \) and \( B = \mathcal{w}_{\leq 0}(X) \). The \( \mathcal{w} \)-weight decomposition \( A \to X \to B \to [1]A \) gives \([X] = [A] + [B]\) in the Grothendieck group. Since \( B \) is a bounded above complex

\[
T(B) := B \oplus [2]B \oplus [4]B \oplus \cdots = \bigoplus_{n \in \mathbb{N}} [2n]B
\]

is a well-defined object of \( K(\mathcal{A}) \). There is an obvious isomorphism \( T(B) = B \oplus [2]T(B) \) which gives

\[
[T(B)] = [B] + [[2]T(B)] = [B] + [T(B)]
\]

implying \([B] = 0\). Considering \( \bigoplus_{n \in \mathbb{N}} [-2n]A \) we similarly find \([A] = 0\). Hence \([X] = [A] + [B] = 0\).

The proof that the other Grothendieck groups mentioned in the proposition vanish is now obvious.

Remark 4.13. Assume that we are in the setting of Proposition 4.12. If we can show that the Grothendieck group of the idempotent completion of \( K(\mathcal{A}) \) vanishes then \( K(\mathcal{A}) \) is idempotent complete by the results cited in Remark 3.3.

Example 4.14. Let \( \{0\} \subset \Lambda \subset \mathbb{N} \) be a subset that is closed under addition, e.g. \( \Lambda = 17\mathbb{N} + 9\mathbb{N} \). Let \( \text{mod}(k) \) be the category of finite dimensional vector spaces over a field \( k \) and let \( \text{mod}_{\Lambda}(k) \subset \text{mod}(k) \) be the full subcategory of vector spaces whose dimension is in \( \Lambda \). We claim that \( K(\text{mod}_{\Lambda}(k)) \) is idempotent complete.

Obviously \( K(\text{mod}_{\Lambda}(k)) \subset K(\text{mod}(k)) \) is a full triangulated subcategory. It is easy to see that any object of \( K(\text{mod}(k)) \) is isomorphic to a stalk complex, i.e. a complex in \( \text{mod}(k) \) with all differentials \( d = 0 \). Hence \( K(\text{mod}(k)) \) is idempotent complete (alternatively, this follows
from Theorem 3.4 (1) and $K(\text{mod}_k(k))$ is dense in $K(\text{mod}(k))$. In particular $K(\text{mod}(k))$ is the idempotent completion of $K(\text{mod}_k(k))$.

The Grothendieck groups of $K(\text{mod}_k(k))$ and $K(\text{mod}(k))$ vanish by Proposition 4.12. Hence results of R. W. Thomason [Tho97, Section 3] (as explained in Remark 3.3) show that the closure of $K(\text{mod}_k(k))$ under isomorphisms in $K(\text{mod}(k))$ equals $K(\text{mod}(k))$. In particular $K(\text{mod}_k(k))$ is idempotent complete.

5. Weak weight complex functor

In this section we construct the weak weight complex functor essentially following [Bon10, Ch. 3] where it is just called weight complex functor. We repeat the construction in detail since we need this for the proof of Theorem 7.1.

Let $(\mathcal{T}, w = (\mathcal{T}^{w \leq 0}, \mathcal{T}^{w \geq 0}))$ be a weight category. We fix for every object $X$ in $\mathcal{T}$ and every $n \in \mathbb{Z}$ a weight decomposition

\begin{equation}
T^n_X : \quad w_{\geq n+1}X \xrightarrow{g^n_{X}} X \xrightarrow{k^n_X} w_{\leq n}X \xrightarrow{v^n_X} [1]w_{\geq n+1}X;
\end{equation}

as suggested by the notation we assume $w_{\geq n+1}X \in \mathcal{T}^{w \geq n+1}$ and $w_{\leq n} \in \mathcal{T}^{w \leq n}$. For any $n$ there is a unique morphism of triangles

\begin{equation}
T^n_X : \quad w_{\geq n+1}X \xrightarrow{g^n_{X}} X \xrightarrow{k^n_X} w_{\leq n}X \xrightarrow{v^n_X} [1]w_{\geq n+1}X
\end{equation}

extending $\text{id}_X$ (use Prop. 2.3 and (ws3)). More precisely $h^n_X$ (resp. $l^n_X$) is the unique morphism making the square $\Delta$ (resp. $\nabla$) commutative. We use the square marked with $\Delta$ as the germ cell for the octahedral
axiom and obtain the following diagram:

\[ (5.3) \]

\[ O^n_X := \]

\[
\begin{array}{c}
\text{The octahedral axiom says that after fitting } h^n_X \text{ into the triangle } T^\geq_X \text{ the dotted arrows exist such that } T^\leq_X \text{ is a triangle and everything commutes. The lower dotted arrow is in fact } l^n_X \text{ by the uniqueness statement given above. We fix such octahedral diagrams } O^n_X \text{ for all objects } X \text{ and all } n \in \mathbb{Z}.
\end{array}
\]

\[ \text{The triangles } T^\leq_X \text{ and } T^\geq_X \text{ and the fact that } T^{w \leq n} \text{ and } T^{w \geq n} \text{ are closed under extensions show that } w_n X \in T^{w=n} \text{ and } [n]w_n X \in \triangle := \bigtriangleup(w).
\]

\[ \text{We define the (candidate) weight complex } WC_c(X) \in C(\bigtriangleup) \text{ of } X \text{ as follows (the index } c \text{ stands for "candidate": Its } n\text{-th term is }
\]

\[ WC_c(X)^n := [n]w_n X
\]

and the differential \( d^n_{WC_c(X)} : [n]w_n X \to [n+1]w_{n+1} X \) is defined by

\[ (5.4) \quad d^n_{WC_c(X)} := [n](b^n_{X} \circ a^n_X) = [n([1]e^n_{X} + 1) \circ v^n_X \circ a^n_X) = [n](e^n_{X} + 1) \circ c^n_X).
\]

Note that \( d^n_{WC_c(X)} \circ d^{n-1}_{WC_c(X)} = 0 \) since the composition of two consecutive maps in a triangle is zero (apply this to (5.3)). Hence \( WC_c(X) \) is in fact a complex in \( \triangle \).
Now let \( f : X \to Y \) be a morphism in \( \mathcal{T} \). We can extend \( f \) to a morphism of triangles (use Prop. 2.3 and (ws3))

(5.5)

\[
\begin{align*}
T^n_X & : \quad w_{\geq n+1}X \xrightarrow{g^n_{X,Y}} X \xrightarrow{k^n_X} w_{\leq n}X \xrightarrow{v^n_X} [1]w_{\geq n+1}X, \\
(f_{w_{\geq n+1}}, f_{w_{\leq n}}) & \\
T^n_Y & : \quad w_{\geq n+1}Y \xrightarrow{g^n_{Y,Z}} Y \xrightarrow{k^n_Y} w_{\leq n}Y \xrightarrow{v^n_Y} [1]w_{\geq n+1}Y.
\end{align*}
\]

This extension is not unique in general; this will be discussed later on. Nevertheless we fix now such an extension \( (f_{w_{\geq n+1}}, f_{w_{\leq n}}) \) for any \( n \in \mathbb{Z} \).

Consider the following diagram in the category of triangles (objects: triangles; morphisms: morphisms of triangles):

(5.6)

\[
\begin{align*}
T^n_X & \xrightarrow{(f_{w_{\geq n+1}}, f_{w_{\leq n}})} T^n_Y \\
(h^n_X, \text{id}_X, h^n_Y) & \quad (h^n_Y, \text{id}_Y, h^n_Y) \\
T^{n-1}_X & \xrightarrow{(f_{w_{\leq n}}, f_{w_{\leq n-1}})} T^{n-1}_Y
\end{align*}
\]

This square is commutative since a morphism of triangles \( T^n_X \to T^n_Y \) extending \( f \) is unique (use Prop. 2.3 and (ws3)).

In particular we have \( f_{w_{\leq n-1}}l^n_X = l^n_Y f_{w_{\leq n}} \), so we can extend the partial morphism \( (f_{w_{\leq n}}, f_{w_{\leq n-1}}) \) by a morphism \( f^n : w_nX \to w_nY \) to a morphism of triangles

(5.7)

\[
\begin{align*}
T^{n\leq} X & : \quad w_nX \xrightarrow{a^n_X} w_{\leq n}X \xrightarrow{l^n_X} w_{\leq n-1}X \xrightarrow{b^n_X} [1]w_nX \\
(f^n, f_{w_{\leq n}}, f_{w_{\leq n-1}}) & \\
T^{n\leq} Y & : \quad w_nY \xrightarrow{a^n_Y} w_{\leq n}Y \xrightarrow{l^n_Y} w_{\leq n-1}Y \xrightarrow{b^n_Y} [1]w_nY
\end{align*}
\]

as indicated. Again there might be a choice, but we fix for each \( n \in \mathbb{Z} \) such an \( f^n \). The commutativity of the squares on the left and right in (5.7) shows that the sequence \( WC_c(f) := ([n]f^n)_{n \in \mathbb{Z}} \) defines a morphism of complexes

(5.8)

\[
\begin{align*}
WC_c(X) : & \quad \cdots \xrightarrow{[n]w_nX} w_{n+1}X \xrightarrow{[n+1]w_{n+1}X} \cdots \\
WC_c(f) & \\
WC_c(Y) : & \quad \cdots \xrightarrow{[n]w_nY} w_{n+1}Y \xrightarrow{[n+1]w_{n+1}Y} \cdots
\end{align*}
\]
Since some morphisms existed but were not unique we cannot expect that $WC_c$ defines a functor $\mathcal{T} \to C(\triangle)$, cf. Example 5.5 below.

Let $WC$ be the composition of $WC_c$ with the canonical functor $C(\triangle) \to K_{\text{weak}}(\triangle)$ (cf. (2.4)), i.e. the assignment mapping an object $X$ of $\mathcal{T}$ to $WC(X) := WC_c(X)$ and a morphism $f$ of $\mathcal{T}$ to the class of $WC_c(f)$ in $K_{\text{weak}}(\triangle)$. The complex $WC(X)$ is called a **weight complex** of $X$.

Recall that the assignment $X \mapsto WC(X)$ depends on the choices made in (5.1) and (5.3). For morphisms we have:

**Proposition 5.1.** Mapping a morphism $f$ in $\mathcal{T}$ to $WC(f)$ does not depend on the choices made in (5.5) and (5.7).

**Proof.** By considering appropriate differences it is easy to see that it is sufficient to consider the case that $f = 0$. Consider (5.5) now for $f = 0$ (but we write $f_{w \leq n}$ instead of $0_{w \leq n}$).

(5.9)

\[
\begin{align*}
\begin{array}{c}
T^n_X : \\
(f_{w \geq n+1}, 0, f_{w \leq n})
\end{array}
\end{align*}
\]

\[
\begin{array}{cccccccc}
 & w_{\geq n+1}X & g_{X}^{n+1} & k_{X}^{n} & v_{X}^{n} & [1]w_{\geq n+1}X, \\
& & f_{w \geq n+1} & f = 0 & f_{w \leq n} & [1]f_{w \geq n+1} \\
T^n_Y : \\
& w_{\geq n+1}Y & g_{Y}^{n+1} & k_{Y}^{n} & v_{Y}^{n} & [1]w_{\geq n+1}Y
\end{array}
\]

Since $f_{w \leq n} \circ k_{X}^{n} = 0$ there exists $s^{n} : [1]w_{\geq n+1}X \to w_{\leq n}Y$ such that $f_{w \leq n} = s^{n}v_{X}^{n}$. Then in the situation

\[
\begin{array}{cccccccc}
[1]w_{\geq n+2}X & [1]h_{X}^{n+1} & [1]w_{\geq n+1}X & [1]e_{X}^{n+1} & [1]w_{n+1}X \\
& a_{Y}^{n} & s^{n} & & p_{Y}^{n+1} &
\end{array}
\]

(where both rows are part of triangles in (5.3), the upper row comes up to signs from a rotation of $T_{X}^{\geq n+1}$, the lower row is from $T_{Y}^{\leq n}$) the indicated factorization

(5.10)

\[s^{n} = a_{Y}^{n}r^{n+1}([1]e_{X}^{n+1})\]
exists by \([\text{ws3}]\). Now (5.7) takes the form (the dotted diagonal arrow will be explained)

\[
\begin{array}{cccc}
T_X^{\leq n} : & w_nX & & [1]w_nX \\
(f^n.fw_{\leq n}.fw_{\leq n-1}) & a^n_X & w_{\leq n}X & b^n_X \\
T_Y^{\leq n} : & w_nY & & [1]w_nY \\
& a^n_Y & w_{\leq n}Y & b^n_Y \\
\end{array}
\]

Equation (5.10) and \((1)c^n_{\leq 1}v^n_X = b^n_{\leq 1}X\) (which follows from the octahedron \(O_{\leq 1}X\), cf. (5.3)) yield

\[
s^n v^n_X = a^n_{\leq 1}\tau^{n+1}((1)c^n_{\leq 1}v^n_X) = a^n_{\leq 1}\tau^{n+1}b^n_{X}.
\]

This shows that the honest triangle marked \(\nabla\) commutes. Consider \(f^n - \tau^{n+1}b^n_{X}a^n_{X}\): Since

\[
a^n_{Y}(f^n - \tau^{n+1}b^n_{X}a^n_{X}) = a^n_{Y}f^n - s^n v^n_a^n_X = 0
\]

there is a morphism \(\nu^n : w_nX \to [-1]w_{\leq n-1}Y\) such that

\[
(5.11)\quad - ([1]b^n_{Y})\nu^n = f^n - \tau^{n+1}b^n_{X}a^n_{X}.
\]

Now consider the following diagram

\[
\begin{array}{cccc}
[-1]w_{n-1}Y & [-1]w_{\leq n-2}Y & [-1]w_{n-1}Y \\
\sigma^n & \nu^n & \end{array}
\]

where the lower row is the triangle obtained from \(T_Y^{\leq n-1}\) by three rotations. The composition \((1)b^n_{Y}\nu^n\) vanishes by \([\text{ws3}]\) hence there is a morphism \(\sigma^n : w_nX \to [-1]w_{n-1}Y\) as indicated such that \(-([1]a^n_{Y-1})\sigma^n = \nu^n\). If we plug this into (5.11) we get

\[
f^n - \tau^{n+1}b^n_{X}a^n_{X} = -([1]b^n_{Y})\nu^n = ([1]b^n_{Y})([-1]a^n_{Y-1})\sigma^n.
\]

Applying \([n]\) to this equation yields (using (5.4))

\[
(5.12)\quad WC_c(f)^n = [n]f^n = ([n]\tau^{n+1})d_{WC_c(X)}^n + d_{WC_c(Y)}^{n-1}([n]\sigma^n).
\]

This shows that \(WC_c(f)\) is weakly homotopic to zero proving the claim (since we assumed that \(f = 0\)).

\[\Box\]

**Theorem 5.2** (cf. [Bon10 Ch. 3]). The assignment \(X \mapsto WC(X), f \mapsto WC(f)\), depends only on the choices made in (5.1) and (5.3) and defines an additive functor

\[WC : \mathcal{T} \to K_{\text{weak}}(\nabla).\]
This functor is in a canonical way a functor of additive categories with translation: There is a canonical isomorphism of functors \( \varphi : \Sigma \circ WC \cong WC \circ [1] \) such that \((WC, \varphi)\) is a functor of additive categories with translation (the translation on \(K_{\text{weak}}(\bigtriangledown)\) is denoted \(\Sigma\) here for clarity).

Proof. It is obvious that \(WC(id_X) = id_{WC(X)}\) and \(WC(f \circ g) = WC(f) \circ WC(g)\). Hence \(WC\) is a well-defined functor which is obviously additive.

We continue the proof after the following Remark 5.3.

Remark 5.3. Let \(WC_1 := WC\) be the additive functor from Theorem 5.2 (we do not know yet how it is compatible with the respective translations) and let \(WC_2\) be another such functor constructed from possibly different choices in (5.1) and (5.3). For each \(X \in T\) the identity \(id_X\) gives rise to a well-defined morphism \(\psi_{21,X} : WC_1(X) \to WC_2(X)\) in \(K_{\text{weak}}(\bigtriangledown)\) which is constructed in the same manner as \(WC(f)\) was constructed from \(f : X \to Y\) above. The collection of these \(\psi_{21,X}\) in fact defines an isomorphism \(\psi_{21} : WC_1 \cong WC_2\). If there is a third functor \(WC_3\) of the same type all these isomorphisms are compatible in the sense that \(\psi_{32} \circ \psi_{21} = \psi_{31}\) and \(\psi_{ii} = id_{WC_i}\) for all \(i \in \{1, 2, 3\}\).

Proof of Thm. 5.2 continued: Our aim is to define \(\varphi\). Let \(U^n_{[1],X}\) be the triangle obtained by three rotations from the triangle \(T^n_{X+1}\) (see (5.1)):

\[
U^n_{[1],X} : \quad [1] \xrightarrow{w_{\geq n+2}} X \xrightarrow{[1]w_{\leq n+1}} X \xrightarrow{[2]w_{\geq n+2}} X;
\]
Note that it is a \((w \geq n + 1, w \leq n)\)-weight decomposition of \([1]X\). Using (5.3) it is easy to check that

\[(5.14)\]

\[
U_{[1]X}^{\leq n} : \\
U_{[1]X}^{n-1} : [1]e_{X}^{n+1} \quad [1]e_{X}^{n+1} \quad [2]w_{\geq n+2}X \\
U_{[1]X}^{n} : [1]w_{\geq n+1}X \quad [1]w_{\leq n+1}X \quad [1]w_{\leq n+1}X \\
U_{[1]X}^{\geq n} : [1]w_{\geq n+2}X \quad [1]w_{\leq n+1}X \quad [1]w_{\leq n+1}X \\
\]

is an octahedron. In the same manner in which the choices (5.1) and (5.3) gave rise to the functor \(WC : T \rightarrow K_{weak}(\heartsuit)\), the choices (5.13) and (5.14) give rise to an additive functor \(WC' : T \rightarrow K_{weak}(\heartsuit)\). As seen in Remark 5.3 there is a canonical isomorphism \(\psi : WC' \sim WC\).

We have

\[
WC'([1]X)^{n} = [n][1]w_{n+1}X = WC(X)^{n+1} = \Sigma(WC(X))^{n}
\]

and

\[
d_{WC'([1]X)}^{m} = [n]((-1)[b_{X}^{n+2}] \circ [1]a_{X}^{n+1}) = -[n + 1](b_{X}^{n+2} \circ a_{X}^{n+1}) \\
= -d_{WC(X)}^{m+1} = d_{\Sigma(WC(X))}^{m}
\]

This implies that \(\Sigma(WC(X)) = WC'([1]X)\) and it is easy to see that even \(\Sigma \circ WC = WC' \circ [1]\). Now define \(\varphi\) as the composition

\[
\Sigma \circ WC = WC' \circ [1] \xrightarrow{\psi_{[1]}} WC' \circ [1].
\]

\[\square\]

**Definition 5.4.** The functor \(WC\) (together with \(\varphi\)) of additive categories with translation from Theorem 5.2 is called a **weak weight complex functor**.
A weak weight complex functor depends on the choices made in [5.1] and [5.3]. However it follows from the proof of the Theorem 5.2 (and Remark 5.3) that any two weak weight complex functors are canonically isomorphic. Hence we allow ourselves to speak about the weak weight complex functor.

**Example 5.5** (cf. [Bon10, Rem. 1.5.2.3]). Let \( \text{Mod}(R) \) be the category of \( R \)-modules for \( R = \mathbb{Z}/4\mathbb{Z} \) and consider the standard weight structure on \( K(\text{Mod}(R)) \) (see Prop. 4.6). Let \( X \) be the complex \( \cdots \to 0 \to R \xrightarrow{2} R \to 0 \to \cdots \) with \( R \) in degrees 0 and 1. If we use the \( w \)-weight decompositions from the proof of Proposition 4.6, the only interesting weight decomposition is \( T^0_X \) of type \( (w \geq 1, w \leq 0) \) and has the following form (where we draw the complexes vertically and give only their components in degrees 0 and 1, and similar for the morphisms):

\[
\begin{align*}
 & w_{\geq 1}X \xrightarrow{[1]} w_{\leq 0}X \xrightarrow{[0]} \cdots \to \to 0 \\
R & \xrightarrow{1} R \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \to \cdots
\end{align*}
\]

We can choose \( w_1X = w_{\geq 1}X \) and \( w_0X = w_{\leq 0}X \) and then one checks that \( WC(X) \) is given by the connecting morphism of this triangle, more precisely

\[
WC(X) = (\cdots \to 0 \to R \xrightarrow{2} R \to 0 \to \cdots)
\]

concentrated in degrees 0 and 1. Consider the morphism \( 0 = 0_X = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] : X \to X \) and extend it to a morphism of triangles (cf. [5.5] or [5.9])

\[
\begin{align*}
 & w_{\geq 1}X \xrightarrow{[y]} X \xrightarrow{[0]} w_{\leq 0}X \xrightarrow{[0]} \cdots \to \to 0 \\
 & w_{\geq 1}X \xrightarrow{[y]} X \xrightarrow{[0]} w_{\leq 0}X \xrightarrow{[0]} \cdots \to \to 0 \\
 & w_{\geq 1}X \xrightarrow{[y]} X \xrightarrow{[0]} w_{\leq 0}X \xrightarrow{[0]} \cdots \to \to 0 \\
 & w_{\geq 1}X \xrightarrow{[y]} X \xrightarrow{[0]} w_{\leq 0}X \xrightarrow{[0]} \cdots \to \to 0
\end{align*}
\]

It is an easy exercise to check that the dotted arrows complete \( 0_X \) to a morphism of triangles for any \( x, y \in \{0, 2\} \). Now one checks that all four morphism \( (0, 0), (0, 2), (2, 0), (2, 2) : WC(X) \to WC(X) \) are weakly homotopic whereas for example \( (0, 0) \) and \( (2, 0) \) are not homotopic.
In particular, this example confirms that mapping an object $X$ to $WC_c(X)$ and a morphism $f$ to $WC_c(f)$ (or its class in $K(\triangledown)$) is not a well-defined functor: We have to pass to the weak homotopy category.

(But one can easily find a preferred choice for the morphisms $f^w$ in this example which defines a functor to $K(\triangledown)$, see Section 5.1 below.)

**Lemma 5.6** (cf. [Ron10] Thm. 3.3.1.IV]). Let $a, b \in \mathbb{Z}$. If $X \in T^{w \geq a}$ (resp. $X \in T^{w \leq b}$ or $X \in T^{w \in [a,b]}$) then $WC(X) \in K(\triangledown)^{w \geq a}$ (resp. $WC(X) \in K(\triangledown)^{w \leq b}$ or $WC(X) \in K(\triangledown)^{w \in [a,b]}$).

In particular, if the weight structure is bounded, then $WC(X) \in K(\triangledown)^b$ for all $X \in T$. 

**Proof.** Let $X \in S$ where $S$ is one of the categories $T^{w \geq a}$, $T^{w \leq b}$, $T^{w \in [a,b]}$. Lemma 4.3 (6) shows that we can assume that in all weight decompositions $T^w_X$ (see (5.1)) of $X$ the objects $w_{n+1}X$ and $w_{n}X$ are in $S$. Choose octahedra (5.3) and let $WC''(X)$ be constructed using these choices.

Consider the octahedron (5.3) again. We claim that $w_nX \in S$. We already know that $w_nX \in T^{w=n}$. In particular the triangle $T^{\geq n}_X$ is a $(w \geq n+1, w \leq n)$-weight decomposition of $w_{n+1}X$ and the triangle $T^{\leq n}_X$ is a $(w \geq n, w \leq n-1)$-weight decomposition of $w_{n}X$. We obtain

- **Case $S = T^{w \geq a}$:** If $a \leq n$ then the weight decomposition $T^{\geq n}_X$ and Lemma 4.3 (6) show $w_nX \in T^{w \geq a} = S$. If $a > n$ the triangle $T^{\leq n}_X$ shows that $w_nX$ is an extension of $w_{n-1}X \in T^{w \geq a}$ and $[1]w_{n-1}X \in T^{w \geq a+1} \subset T^{w \geq a}$ and hence in $T^{w \geq a} = S$.

- **Case $S = T^{w \leq b}$:** If $n-1 < b$ the weight decomposition $T^{\leq n}_X$ and Lemma 4.3 (5) show $w_nX \in T^{w \leq b} = S$. If $n-1 \geq b$ the triangle $T^{\leq n}_X$ shows that $w_nX$ is an extension of $w_{n-1}X \in T^{w \leq b}$ and $[1]w_{n-1}X \in T^{w \leq b-1} \subset T^{w \leq b}$ and hence in $T^{w \leq b} = S$.

- **Case $S = T^{w \in [a,b]}$:** Follows from the above two cases.

This proves the claim $w_nX \in S$. Let $I$ be the one among the intervals $[a, \infty)$, $(-\infty, b]$, $[a, b]$ that satisfies $S = T^{w \in I}$. If $n \not\in I$ then $w_nX \in T^{w \in I} \cap T^{w = n} = 0$ and hence $WC'(X)^n = [n]w_nX = 0$. This shows $WC'(X) \in K^{w \in I}(\triangledown)$ and $WC(X) \in K(\triangledown)^{w \in I}$ since $WC'(X) \approx WC(X)$. (Here the categories $K^{w \in I}(\triangledown)$ and $K(\triangledown)^{w \in I}$ are defined in the obvious way, cf. beginning of Section 3.)

In the following definition the triangulated category $K(\triangledown(w))^{\text{anti}}$ appears (see Section 2.3 for the definition of $T^{\text{anti}}$ for a triangulated category $T$). This happens naturally as can be seen from Proposition 5.9 below. Let us however remark that we could avoid its appearance by replacing the definition of a weak weight complex functor $WC$ above
with its composition with the functor induced by \((S, \text{id})\) (see (2.7)) which just changes the signs of all differentials.

**Definition 5.7** (cf. [Bon10, Conj. 3.3.3]). A **strong weight complex functor** is a triangulated functor \(\widetilde{WC} : \mathcal{T} \to K(\heartsuit)^\text{anti}\) such that the obvious composition

\[
\mathcal{T} \xrightarrow{\widetilde{WC}} K(\heartsuit)^\text{anti} \to K(\heartsuit)
\]

is isomorphic to the/a weak weight complex functor as a functor of additive categories with translation.

Recall the standard weight structure on \(K(\heartsuit)^\text{anti}\) from Remark 4.7.

**Lemma 5.8.** Any strong weight complex functor \(\widetilde{WC} : \mathcal{T} \to K(\heartsuit)^\text{anti}\) is weight-exact.

**Proof.** This follows immediately from Lemma 5.6. □

### 5.1. Strong weight complex functor for the standard weight structure.

Consider the standard weight structure \(w\) on the homotopy category \(K(\mathcal{A})\) of an additive category \(\mathcal{A}\) from Proposition 4.6. Given \(X \in K(\mathcal{A})\) the \(w\)-weight decomposition (4.5) is a preferred choice for the weight decomposition \(T_X^w\) in (5.1). Then there is also an obvious preferred choice for the octahedron \(O_X^w\) in (5.3) in which \(w_nX\) is just \([-n]X^n\), the \(n\)-th term \(X^n\) of the complex \(X\) shifted into degree \(n\).

With this choices the complex \(WC_c(X) = WC(X)\) is obtained from \(X\) by multiplying all differentials by \(-1\), i.e. \(WC_c(X) = S(X)\) where \(S\) is the functor defined in Section 2.4; here we view \(\mathcal{A} \subset \heartsuit(w)\) as a full subcategory (see Cor. 4.11 for a more precise statement).

Similarly there are preferred choices for morphisms: Let \(f : X \to Y\) be a morphism in \(K(\mathcal{A})\). Let \(\hat{f} : X \to Y\) be a morphism in \(C(\mathcal{A})\) representing \(f\). The morphisms \(\overline{f}_{\geq n+1}, \overline{f}_{\leq n} \hat{f}\) and \(\overline{f}_n\hat{f}\) gives rise to preferred choices for the morphisms \(f_{w \geq n+1}, f_{w \leq n}\) and \(f_n\) in (5.5) and (5.7). If we identify \(\mathcal{A} \subset \heartsuit(w)\) as above the morphism \(WC_c(f)\) (see (5.8)) of complexes is just \(\hat{f} = S(\hat{f}) : WC_c(X) = S(X) \to WC_c(Y) = S(Y)\). Obviously its class in the homotopy category is \(f = S(f)\) and hence does not depend on the choice of the representative for \(f\).

**Proposition 5.9.** The composition

\[
K(\mathcal{A}) \xrightarrow{(S, \text{id})} K(\mathcal{A})^{\text{anti}} \subset K(\heartsuit(w))^{\text{anti}}
\]

of the triangulated equivalence \((S, \text{id})\) (cf. (2.7)) and the obvious inclusion is a strong weight complex functor.

**Proof.** Clear from the above arguments. □
6. Filtered triangulated categories

We very closely follow [Be˘ı87, App.]. Let us recall the definition of a filtered triangulated category.

**Definition 6.1.** (1) A **filtered triangulated category**, or **f-category** for short, is a quintuple \((\tilde{T}, \tilde{T}(\leq 0), \tilde{T}(\geq 0), s, \alpha)\) where \(\tilde{T}\) is a triangulated category, \(\tilde{T}(\leq 0)\) and \(\tilde{T}(\geq 0)\) are strict full triangulated subcategories, \(s: \tilde{T} \sim \rightarrow \tilde{T}\) is a triangulated automorphism (called “shift of filtration”) and \(\alpha: \text{id}_{\tilde{T}} \rightarrow s\) is a morphism of triangulated functors, such that the following axioms hold (where \(\tilde{T}(\leq n) := s^n(\tilde{T}(\leq 0))\) and \(\tilde{T}(\geq n) := s^n(\tilde{T}(\geq 0))\)):

- (fcat1) \(\tilde{T}(\geq 1) \subset \tilde{T}(\geq 0)\) and \(\tilde{T}(\leq 0) \subset \tilde{T}(\leq 1)\)
- (fcat2) \(\tilde{T} = \bigcup_{n \in \mathbb{Z}} \tilde{T}(\leq n) = \bigcup_{n \in \mathbb{Z}} \tilde{T}(\geq n)\).
- (fcat3) \(\text{Hom}(\tilde{T}(\geq 1), \tilde{T}(\leq 0)) = 0\).
- (fcat4) For any \(X\) in \(\tilde{T}\) there is a triangle
  \[A \rightarrow X \rightarrow B \rightarrow A[1]\]
  with \(A\) in \(\tilde{T}(\geq 1)\) and \(B\) in \(\tilde{T}(\leq 0)\).
- (fcat5) For any \(X \in \tilde{T}\) one has \(\alpha_{s(X)} = s(\alpha_X)\) as morphisms \(s(X) \rightarrow s^2(X)\).
- (fcat6) For all \(X\) in \(\tilde{T}(\leq 0)\) and \(Y\) in \(\tilde{T}(\geq 1)\), the map
  \[
  \text{Hom}(X, s^{-1}(Y)) \xrightarrow{\sim} \text{Hom}(X, Y)
  \]
  \[f \mapsto \alpha_{s^{-1}(Y)} \circ f\]
  is bijective (equivalently one can require that
  \[
  \text{Hom}(s(X), Y) \xrightarrow{\sim} \text{Hom}(X, Y)
  \]
  \[g \mapsto g \circ \alpha_X\]
  is bijective). As diagrams these equivalent conditions look as follows:

By abuse of notation we then say that \(\tilde{T}\) is an f-category.
(2) Let \( T \) be a triangulated category. A **filtered triangulated category over** \( T \) (or f-category over \( T \)) is an f-category \( \tilde{T} \) together with an equivalence 
\[
i : T \to \tilde{T}((\leq 0)) \cap \tilde{T}((\geq 0))
\]
of triangulated categories.

Let \( \tilde{T} \) be an f-category. We will use the shorthand notation 
\[
\tilde{T}(\leq b) \cap \tilde{T}(\geq a)
\]
and abbreviate \([a, a] \) by \([a] \). Similarly \( \tilde{T}(<a) \) etc. have the obvious meaning. For \( a < b \) we have \( \tilde{T}(\leq a) \cap \tilde{T}(\geq b) = 0 \): If \( X \) is in this intersection then axiom \([\text{fcat3}]\) implies that \( \text{id}_X = 0 \) hence \( X = 0 \). Note that \( \tilde{T} \) together with the identity functor \( \text{id} : \tilde{T}(\lfloor 0 \rfloor) \to \tilde{T}(\lfloor 0 \rfloor) \) is an f-category over \( \tilde{T}(\lfloor 0 \rfloor) \).

**Remark 6.2.** Let \( \tilde{T} \) be a filtered triangulated category. Define \( D^{t \leq 0} := \tilde{T}(\geq 1) \) and \( D^{t \geq 0} := \tilde{T}(\leq 0) \). Then \( (D^{t \leq 0}, D^{t \geq 0}) \) defines a t-structure on \( \tilde{T} \).

Note that in this example all \( D^{t \leq i} \) coincide since \( \tilde{T}(\geq 1) \) is a triangulated subcategory; similarly, all \( D^{t \geq i} \) are equal. The heart of this t-structure is zero.

Of course we can apply the shift of filtration to this example and obtain t-structures \( (\tilde{T}(\geq n + 1), \tilde{T}(\leq n)) \) for all \( n \in \mathbb{Z} \).

Similarly, define \( E^{w \leq 0} := \tilde{T}(\leq 0) \) and \( E^{w \geq 0} := \tilde{T}(\geq 1) \). Then \( (E^{w \leq 0}, E^{w \geq 0}) \) is a weight structure on \( \tilde{T} \). Note that \([\text{ws1}]\) is satisfied since \( (\tilde{T}(\geq 1), \tilde{T}(\leq 0)) \) is a t-structure. Again all \( E^{w \leq i} \) (resp. \( E^{w \geq i} \)) coincide and the heart is zero.

**6.1. Basic example.** We introduce the filtered derived category of an abelian category, following [Ill71, V.1]. The reader who is not interested in this basic example of an f-category can skip this section and continue with 6.2.

Let \( A \) be an abelian category and \( CF(A) \) the category whose objects are complexes in \( A \) with a finite decreasing filtration and whose morphisms are morphisms of complexes which respect the filtrations. If \( L \) is an object of \( CF(A) \) and \( i, j \in \mathbb{Z} \) we denote the component of \( L \) in degree \( i \) by \( L^i \) and by \( F^jL \) the \( j \)-the step of the filtration, and by \( F^jL^i \) the component of degree \( i \) in \( F^jL \). Pictorially an object \( L \) looks as

\[
\cdots \subset F^{-1}L \subset F^0L \subset F^1L \subset F^2L \subset \cdots
\]

1 We do not ask here for a functor \( \tilde{T} \to T \) as suggested by the word “over”; Proposition 6.6 will yield such a functor.
or, if we also indicate the differentials, as

\[
\begin{array}{ccccccc}
\cdots & F^{-1}L & F^0L & F^1L & \cdots \\
\uparrow & \cdots & \uparrow & \cdots & \uparrow \\
L^1 : & \cdots & F^{-1}L & F^0L & F^1L & \cdots \\
\uparrow & \cdots & \uparrow & \cdots & \uparrow \\
L^0 : & \cdots & F^{-1}L & F^0L & F^1L & \cdots \\
\uparrow & \cdots & \uparrow & \cdots & \uparrow \\
L^{-1} : & \cdots & F^{-1}L^{-1} & F^0L^{-1} & F^1L^{-1} & \cdots \\
\end{array}
\]

This is an additive category having kernels and cokernels, but it is not abelian (if \( A \neq 0 \)). There is an obvious translation functor \([1]\) on \( CF(A)\).

A morphism \( f : L \to M \) between objects of \( CF(A) \) is called a \textit{quasi-isomorphism} if one of the following equivalent conditions is satisfied:

1. \( F^n f : F^nL \to F^nM \) is a quasi-isomorphism for all \( n \in \mathbb{Z} \).
2. \( \text{gr}^n(f) : \text{gr}^n(L) \to \text{gr}^n(M) \) is a quasi-isomorphism for all \( n \in \mathbb{Z} \).

We localize \( CF(A) \) with respect to the class of all quasi-isomorphisms and obtain the \textit{filtered derived category} \( DF(A) \) of \( A \). This category can equivalently be constructed as the localization of the filtered homotopy category. The latter category is triangulated with triangles isomorphic to mapping cone triangles; this structure of a triangulated category is inherited to \( DF(A) \).

Morphisms in \( DF(A) \) are equivalence classes of so-called roofs: Any morphism \( f : L \to M \) in \( DF(A) \) in \( DF(A) \) can be represented as

\[
gs^{-1} : L \leftarrow s L' \rightarrow M
\]

where \( s \) and \( g \) are morphisms in \( CF(A) \) and \( s \) is a quasi-isomorphism. Similarly, one can also represent \( f \) as

\[
t^{-1}h : L \rightarrow M' \leftarrow h M
\]

where \( t \) and \( h \) are morphisms in \( CF(A) \) and \( t \) is a quasi-isomorphism.

Let \( D(A) \) be the derived category of \( A \). The functor \( \text{gr}^n : CF(A) \to C(A) \) passes to the derived categories and yields a triangulated functor \( \text{gr}^n : DF(A) \to D(A) \).

Define (strict) full subcategories

\[
DF(A)(\leq n) := \{ L \in DF(A) \mid \text{gr}^i(L) = 0 \text{ for all } i > n \},
\]

\[
DF(A)(\geq n) := \{ L \in DF(A) \mid \text{gr}^i(L) = 0 \text{ for all } i < n \}.
\]

Let \( s : CF(A) \to CF(A) \) the functor which shifts the filtration: Given an object \( L \) the object \( s(L) \) has the same underlying complex but
filtration \( F^n(s(L)) = F^{n-1}L \). It induces a triangulated automorphism \( s : DF(A) \to DF(A) \). Let \( \alpha : \text{id}_{DF(A)} \to s \) be the obvious morphism of triangulated functors: We include a picture of \( \alpha_L \) where we indicate the 0-th part of the filtration by a box:

\[
\begin{array}{ccccccc}
L : & \cdots & F^{-1}L & \supset (F^0L) & \subset F^1L & \supset \cdots \\
\downarrow \alpha_L & & & & & & \\
S(L) : & \cdots & F^{-2}L & \supset (F^{-1}L) & \subset F^0L & \supset \cdots
\end{array}
\]

Note that \( s^n(DF(A)(\leq 0)) = DF(A)(\leq n) \) and \( s^n(DF(A)(\geq 0)) = DF(A)(\geq n) \).

We define a functor \( i : D(A) \to DF(A) \) by mapping an object \( L \) of \( D(A) \) to \( i(L) = (L, \text{Tr}) \), where \( \text{Tr} \) is the trivial filtration on \( L \) defined by \( \text{Tr}^iL = L \) for \( i \leq 0 \) and \( \text{Tr}^iL = 0 \) for \( i > 0 \). We often consider \( i \) as a functor to \( DF(A)([0]) \).

**Proposition 6.3** (cf. [Beï87, Example A 2]). The datum

\[
(DF(A), DF(A)(\leq 0), DF(A)(\geq 0), s, \alpha)
\]

defines a filtered triangulated category \( DF(A) \), and the functor \( i : D(A) \to DF(A)([0]) \) makes \( DF(A) \) into a filtered triangulated category over \( D(A) \).

**Proof.** We first check that \( (6.1) \) defines a filtered triangulated category. Since all \( \text{gr}^i : DF(A) \to D(A) \) are triangulated functors, \( DF(A)(\leq 0) \) and \( DF(A)(\geq 0) \) are strict full triangulated subcategories of \( DF(A) \). The conditions [fcat1], [fcat2] (we use finite filtrations) and [fcat5] are obviously satisfied.

**Condition [fcat4]** Let \( X \) be any object in \( D(A) \). We define objects \( X(\geq 1) \) and \( X/(X(\geq 1)) \) and (obvious) morphisms \( X(\geq 1) \xrightarrow{i} X \xrightarrow{\rho} X/(X(\geq 1)) \) in \( CF(A) \) by the following diagram:

\[
\begin{array}{ccccccc}
X(\geq 1) : & \cdots & F^{1}X & = (F^{1}X) & = F^{1}X & \supset F^{2}X & \supset \cdots \\
\downarrow \iota & & & & & & \\
X : & \cdots & F^{-1}X & \supset (F^{0}X) & \subset F^{1}X & \supset F^{2}X & \supset \cdots \\
\downarrow \rho & & & & & & \\
X/(X(\geq 1)) : & \cdots & F^{-1}X/F^{1}X & \supset (F^{0}X/F^{1}X) & \subset 0 & = 0 & = \cdots
\end{array}
\]

There is a morphism \( X/(X(\geq 1)) \xrightarrow{[1]} (X(\geq 1)) \) in \( DF(A) \) such that

\[
(6.3) \quad X(\geq 1) \xrightarrow{i} X \xrightarrow{\rho} X/(X(\geq 1)) \xrightarrow{[1]} (X(\geq 1))
\]
is a triangle in $DF(A)$: Use the obvious quasi-isomorphism from the mapping cone of $i$ to $X/(X(\geq 1))$. Since $X(\geq 1) \in DF(A)(\geq 1)$ and $X/(X(\geq 1)) \in DF(A)(\leq 0)$ by definition this proves condition $(fcat4)$.

**Observation:** Application of the triangulated functors $gr^i$ to the triangle (6.3) shows: If $X$ is in $DF(A)(\geq 1)$ then $X(\geq 1) \to X$ is an isomorphism in $DF(A)$. Similarly, if $X$ is in $DF(A)(\leq 0)$, then $X \to X/(X(\geq 1))$ is an isomorphism. We obtain:

- Any object in $DF(A)(\geq a)$ is isomorphic to an object $Y$ with $Y = F^{-\infty}Y = \ldots = F^{a-1}Y = F^aY$.
- Any object in $DF(A)(\leq b)$ is isomorphic to an object $Y$ with $0 = F^{b+1}Y = F^{b+2}Y = \ldots$.
- Any object in $DF(A)([a, b])$ is isomorphic to an object $Y$ with $Y = F^{-\infty}Y = \ldots = F^aY \supset \cdots \supset 0 = F^{b+1}Y = \ldots$.

**Condition $(fcat3)$** Let $X$ in $DF(A)(\geq 1)$ and $Y$ in $DF(A)(\leq 0)$. By the above observation we can assume that $0 = F^1Y = F^2Y = \ldots$. Let a morphism $f : X \to Y$ be represented by a roof $X \xleftarrow{h} Z \xrightarrow{t} X$ where $s$ a quasi-isomorphism. Then the obvious morphism $\iota : Z(\geq 1) \to Z$ is a quasi-isomorphism and the roof $X \xleftarrow{s} Z(\geq 1) \xrightarrow{g} Y$. Since $F^1Y = 0$ and $F^1(Z(\geq 1)) = Z(\geq 1)$ we obtain $g \iota = 0$. Hence $f = 0$.

**Condition $(fcat6)$** Let $X$ in $DF(A)(\geq 1)$ and $Y$ in $DF(A)(\leq 0)$ as before. As observed above we can assume that $X = \cdots = F^0X = F^1X$ and that $0 = F^1Y = F^2Y = \ldots$.

We prove that $\Hom(sY, X) \to \Hom(Y, X)$

$$g \mapsto g \circ \alpha_Y$$

is bijective. Here is a picture of $\alpha_Y$:

$$\begin{array}{ccccccc}
Y : & \cdots & \supset F^{-1}Y & \supset \{F^0Y\} & \supset 0 & = 0 & = \cdots \\
\downarrow \alpha_Y & & & & & & \\
s(Y) : & \cdots & \supset F^{-2}Y & \supset \{F^{-1}Y\} & \supset F^0Y & \supset 0 & = \cdots \\
\end{array}$$

We define a map $\Hom(Y, X) \to \Hom(s(Y), X)$ which will be inverse to the above map. Let $f : Y \to X$ be a morphism, represented by a roof $Y \xrightarrow{h} Z \xleftarrow{t} X$ where $h$ and $t$ are morphisms in $CF(A)$ and $t$ is a quasi-isomorphism. We define an object $\tilde{Z}$ and a morphism $Z \to \tilde{Z}$ by
the following picture in which we include $Z \xleftarrow{t} X$:

$$
\begin{array}{c}
\tilde{Z} : \cdots = Z = \left( \tilde{Z} \right) = Z \supset F^2Z \supset \cdots \\
\xrightarrow{s} \\
Z : \cdots \supset F^{-1}Z \supset \left( F^0Z \right) \supset F^1Z \supset F^2Z \supset \cdots \\
\xrightarrow{t} \\
X : \cdots = X = \left( X \right) = X \supset F^2X \supset \cdots
\end{array}
$$

Since $t$ is a quasi-isomorphism, all $F^i t : F^i X \to F^i Z$ are quasi-isomorphisms. For $i$ small enough we have $Z = F^i Z$; this implies that $X \to Z$ is a quasi-isomorphism in $C(A)$; hence $st : X \to \tilde{Z}$ is a quasi-isomorphism in $CF(A)$. Hence we get the following diagram

$$
\begin{array}{c}
& & \tilde{Z} \\
& & \xrightarrow{sh} \\
Y \xrightarrow{h} Z \xrightarrow{t} X \\
\xrightarrow{\alpha_Y} \\
& & s(Y)
\end{array}
$$

Because of the special form of the filtrations on $\tilde{Z}$ and on $Y$ it is obvious that $sh : Y \to \tilde{Z}$ comes from a unique morphism $\lambda : s(Y) \to \tilde{Z}$ in $CF(A)$ such that $\lambda \alpha_Y = sh$. We map $g$ to the class of the roof $(st)^{-1} \lambda$; it is easy to check that this is well-defined and inverse to the map $g \mapsto g \circ \alpha_Y$.

Now we check that $i$ makes $DF(A)$ into an f-category over $D(A)$. It is obvious that $i : D(A) \to DF(A)([0])$ is triangulated. Our observation shows that it is essentially surjective. Since $\text{gr}^0 \circ i = \text{id}_{D(A)}$, our functor $i$ is faithful. It remains to prove fullness: Let $X$ and $Y$ be in $D(A)$ and let $f : i(X) \to i(Y)$ be a morphism in $DF(A)$, represented by a roof $i(X) \xleftarrow{s} Z \xrightarrow{g} i(Y)$ with $s$ a quasi-isomorphism. Consider the morphism

$$
\begin{array}{c}
i(F^0Z) : \cdots = F^0Z = \left( F^0Z \right) \supset 0 \supset \cdots \\
\xrightarrow{t} \\
Z : \cdots \supset F^{-1}Z \supset \left( F^0Z \right) \supset F^1Z \supset \cdots
\end{array}
$$

Since $F^0 s : F^0 Z \to F^0 i(X) = X$ is a quasi-isomorphism, $st$ is a quasi-isomorphisms (and so is $t$). But the roof $i(X) \xleftarrow{st} i(F^0Z) \xrightarrow{st} i(Y)$ comes from a roof $X \xleftarrow{F^0 Z} Y$; hence $f$ is in the image of $i$. □
6.2. First properties of filtered triangulated categories. We will make heavy use of some results of [Bei87, App.] in Section 7. As no proofs have appeared we give proofs for the more difficult results we need.

**Proposition 6.4** (cf. [Bei87, Prop. A 3] (without proof)). Let \( \tilde{T} \) be a filtered triangulated category and \( n \in \mathbb{Z} \).

1. The inclusion \( \tilde{T}(\geq n) \subset \tilde{T} \) has a right adjoint \( \sigma_{\geq n} : \tilde{T} \to \tilde{T}(\geq n) \), and the inclusion \( \tilde{T}(\leq n) \subset \tilde{T} \) has a left adjoint \( \sigma_{\leq n} : \tilde{T} \to \tilde{T}(\leq n) \).

We fix all these adjunctions.

2. For any \( X \) in \( \tilde{T} \) there is a unique morphism \( v^n_X : \sigma_{\leq n}X \to [1]\sigma_{\geq n+1}X \) in \( \tilde{T} \) such that the candidate triangle

\[
\sigma_{\geq n+1}X \xrightarrow{g^{n+1}_X} X \xrightarrow{k^n_X} \sigma_{\leq n}X \xrightarrow{v^n_X} [1]\sigma_{\geq n+1}X
\]

is a triangle where the first two morphisms are adjunction morphisms. From every triangle \( A \to X \to B \to [1]A \) with \( A \) in \( \tilde{T}(\geq n) \) and \( B \) in \( \tilde{T}(\leq n) \) there is a unique isomorphism of triangles to the above triangle extending \( \text{id}_X \). We call (6.4) the \( \sigma \)-truncation triangle (of type \((\geq n+1, \leq n)\)).

3. We have

\[
\tilde{T}(\leq n) = (\tilde{T}(\geq n))^\perp \quad \text{and} \quad \tilde{T}(\geq n) = \perp(\tilde{T}(\leq n)).
\]

In particular if in a triangle \( X \to Y \to Z \to [1]X \) two out of the three objects \( X, Y, Z \) are in \( \tilde{T}(\leq n) \) (resp. \( \tilde{T}(\geq n) \)) then so is the third.

4. Let \( a, b \in \mathbb{Z} \). All functors \( \sigma_{\leq n} \) and \( \sigma_{\geq n} \) are triangulated and preserve all subcategories \( \tilde{T}(\leq a) \) and \( \tilde{T}(\geq b) \). There is a unique morphism

\[
\sigma^{[b, a]} : \sigma_{\leq a}\sigma_{\geq b} \to \sigma_{\geq b}\sigma_{\leq a}
\]

(which is in fact an isomorphism) such that the diagram

\[
\begin{array}{ccc}
\sigma_{\geq b}X & \xrightarrow{g^b_X} & X \\
\downarrow{k^a_X} & & \downarrow{g^b_{\sigma_{\leq a}X}} \\
\sigma_{\leq a}\sigma_{\geq b}X & \xrightarrow{\sigma^{[b, a]}} & \sigma_{\geq b}\sigma_{\leq a}X
\end{array}
\]

commutes for all \( X \) in \( \tilde{T} \).
Our proof of this theorem gives some more canonical isomorphisms, see Remark 6.3 below. If we were only interested in the statements of Proposition 6.4 we could do without the 3 × 3-diagram (6.7).

Proof. The statements (1) and (2) follow from the fact that \( \tilde{T}(\geq n + 1), \tilde{T}(\leq n) \) is a t-structure for all \( n \in \mathbb{Z} \) (see Rem. 6.2 and [BBD82 1.3.3]). For (3) use [BBD82 1.3.4] and the fact that \( \tilde{T}(\geq n) \) and \( \tilde{T}(\leq n) \) are stable under [1].

We prove (4): The functors \( \sigma_{\leq a} \) and \( \sigma_{\geq b} \) are triangulated by Proposition 2.5. Let \( X \in \tilde{T} \) and \( a, b \in \mathbb{Z} \). Consider the following 3 × 3-diagram (6.7)

\[
\begin{array}{ccc}
\sigma_{\geq b+1}\sigma_{\geq a+1}X & \xrightarrow{[1]\sigma_{\geq b+1}(g_{X}^{a+1})} & \sigma_{\geq b+1}\sigma_{\geq a+1}X \\
\sigma_{\leq b}\sigma_{\geq a+1}X & \xrightarrow{[1]\sigma_{\geq b+1}(k_{X}^{a})} & \sigma_{\leq b}\sigma_{\leq a}X \\
\sigma_{\geq a+1}X & \xrightarrow{[1]\sigma_{\leq b}(g_{X}^{a})} & \sigma_{\leq a}X \\
\sigma_{\geq b+1}\sigma_{\geq a+1}X & \xrightarrow{[1]\sigma_{\leq b}(k_{X}^{a})} & \sigma_{\leq b}\sigma_{\leq a}X \\
\sigma_{\geq b+1}\sigma_{\geq a+1}X & \xrightarrow{[1]\sigma_{\leq b}(g_{X}^{a})} & \sigma_{\leq a}X \\
\end{array}
\]

constructed as follows: All morphisms \( g \) and \( k \) are adjunction morphisms. We start with the third row which is the \( \sigma \)-truncation triangle of \( X \) of type \( (\geq a + 1, \leq a) \). We apply the triangulated functors \( \sigma_{\leq b} \) and \( \sigma_{\geq b+1} \) to this triangle and obtain the triangles in the second and fourth row. The adjunctions give the morphisms of triangles from fourth to third and third to second row. Then extend the first three columns to \( \sigma \)-truncation triangles; they can be uniquely connected by morphisms of triangles extending \( g_{X}^{a+1} \) and \( k_{X}^{a} \) respectively using Proposition 2.3. Similarly (multiply the last arrow in the fourth column by \(-1\) to get a triangle) we obtain the morphism between third and fourth column.

We prove that \( \sigma_{\geq b+1} \) and \( \sigma_{\leq b} \) preserve the subcategories \( \tilde{T}(\geq a + 1) \) and \( \tilde{T}(\leq a) \).

- **Case \( a \geq b \):** Then in the left vertical triangle the first two objects are in \( \tilde{T}(\geq b + 1); \) hence \( \sigma_{\leq b}\sigma_{\geq a+1}X \in \tilde{T}(\geq b + 1) \cap \tilde{T}(\leq b) \) is zero (use (3)). (This shows that \( g_{\sigma_{\geq a+1}X}^{b+1} \) and \( \sigma_{\leq b}(k_{X}^{b}) \) are isomorphisms.)
- \( X \in \tilde{T}(\geq a+1) \): Then \( g^a_{X} \) is an isomorphism and the first two vertical triangles are isomorphic. This shows \( \sigma_{\leq b} X = 0 \in \tilde{T}(\geq a+1) \) and that all four morphisms of the square (1) are isomorphisms; hence \( \sigma_{\geq b+1} X \in \tilde{T}(\geq a+1) \).
- \( X \in \tilde{T}(\leq a) \): Then \( \sigma_{\leq b} X \in \tilde{T}(\leq b) \subset \tilde{T}(\leq a) \). Hence in the second vertical triangle two objects are in \( \tilde{T}(\leq a) \); hence the third object \( \sigma_{\geq b+1} X \) is in \( \tilde{T}(\leq a) \).

- Case \( a \leq b \): Then in the third vertical triangle the second and third object are in \( \tilde{T}(\leq b) \); hence \( \sigma_{\geq b+1} \sigma_{\leq a} X \in \tilde{T}(\leq b) \cap \tilde{T}(\geq b+1) \) is zero. (This shows that \( k^b_{\sigma_{\leq a}} \) and \( \sigma_{\geq b+1}(g^a_{X}) \) are isomorphisms.)

- \( X \in \tilde{T}(\leq a) \): Then \( k^a_X \) is an isomorphism and the second and third vertical triangles are isomorphic. This shows \( \sigma_{\geq b+1} X = 0 \in \tilde{T}(\leq a) \) and that all four morphisms of the square (2) are isomorphisms; hence \( \sigma_{\leq b} X \in \tilde{T}(\leq a) \).
- \( X \in \tilde{T}(\geq a+1) \): Then \( \sigma_{\geq b+1} X \in \tilde{T}(\geq b+1) \subset \tilde{T}(\geq a+1) \). Hence in the second vertical triangle two objects are in \( \tilde{T}(\geq a+1) \); hence the third object \( \sigma_{\leq b} X \) is in \( \tilde{T}(\geq a+1) \).

For the last statement consider the diagram (6.6) without the arrow \( \sigma^a_X \) and with \( b \) replaced by \( b+1 \). The vertical arrows are part of \( \sigma \)-truncation triangles. Note that we already know that \( \sigma_{\geq b+1} \sigma_{\leq a} X \in \tilde{T}(\leq a) \) and \( \sigma_{\leq a} \sigma_{\geq b+1} X \in \tilde{T}(\geq b+1) \). Appropriate cohomological functors give the following commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
\tilde{T}(\sigma_{\leq a} \sigma_{\geq b+1} X, \sigma_{\leq a} X) & \xrightarrow{g^{b+1}_{\sigma_{\leq a} X} \circ 0^a_X} & \tilde{T}(\sigma_{\leq a} \sigma_{\geq b+1} \sigma_{\leq a} X) \\
\sim & & \sim \\
\tilde{T}(\sigma_{\leq a} \sigma_{\geq b+1} X, \sigma_{\leq a} X) & \xrightarrow{0^a_{\sigma_{\geq b+1} X} \circ g^{b+1}_{\sigma_{\leq a} X}} & \tilde{T}(\sigma_{\leq a} \sigma_{\geq b+1} X, \sigma_{\leq a} X)
\end{array}
\]

It shows that there is a unique morphism

\[ \sigma^a_X \]: \( \sigma_{\leq a} \sigma_{\geq b+1} X \rightarrow \sigma_{\geq b+1} \sigma_{\leq a} X \]

such that \( g^{b+1}_{\sigma_{\leq a} X} \circ \sigma^a_X \circ k^a_{\sigma_{\geq b+1} X} = k^a_X \circ g^{b+1}_{\sigma_{\leq a} X} \). We have to prove that \( \sigma^a_X \) is an isomorphism.
Similarly it is easy to see that $X$ is uniquely isomorphic to the corresponding $\sigma$-truncation triangle:

$$
\sigma_{\geq b+1}\sigma_{\geq a+1}X \xrightarrow{\sigma_{\geq b+1}(g^{b+1}_{X})} \sigma_{\geq b+1}\sigma_{\leq a}X \xrightarrow{\sigma_{\geq b+1}(v^{a}_{X})}[1]\sigma_{\geq b+1}\sigma_{\geq a+1}X
$$

In combination with (6.7) this diagram yields $g^{b+1}_{\sigma_{\leq a}X} \circ h \circ k^{a}_{\sigma_{\geq b+1}X} = k^{a}_{X} \circ g^{b+1}_{X}$. This shows that $\sigma_{X}^{[b+1,a]}$ is hence an isomorphism. Similarly it is easy to see that $X \mapsto \sigma_{X}^{[b,a]}$ in fact defines an isomorphism (6.8) of functors.

**Remark 6.5.** Let us just mention some consequences one can now easily deduce from the $3 \times 3$-diagram (6.7).

- (As already mentioned in the proof:) If $a \geq b$ the object $\sigma_{\leq b}\sigma_{\geq a+1}X$ is zero providing two isomorphisms

  \begin{equation}
  \sigma_{\leq b}(k^{a}_{X}) : \sigma_{\leq b}X \xrightarrow{\sim} \sigma_{\leq b}\sigma_{\leq a}X \quad \text{(for } a \geq b), \quad \text{and}
  \end{equation}
  
  \begin{equation}
  g^{b+1}_{\sigma_{\geq a+1}X} : \sigma_{\geq b+1}\sigma_{\geq a+1}X \xrightarrow{\sim} \sigma_{\geq a+1}X \quad \text{(for } a \geq b).
  \end{equation}

  Similarly $\sigma_{\geq b+1}\sigma_{\leq a}X$ vanishes for $a \leq b$ and provides two isomorphisms

  \begin{equation}
  \sigma_{\geq b+1}(g^{a+1}_{X}) : \sigma_{\geq b+1}\sigma_{\geq a+1}X \xrightarrow{\sim} \sigma_{\geq b+1}X \quad \text{(for } a \leq b), \quad \text{and}
  \end{equation}

  \begin{equation}
  k^{b}_{\sigma_{\leq a}X} : \sigma_{\leq a}X \xrightarrow{\sim} \sigma_{\leq b}\sigma_{\leq a}X \quad \text{(for } a \leq b).
  \end{equation}

- In case $a = b$ the two squares marked (1) and (2) consist of isomorphisms. Hence

  \begin{equation}
  g^{a+1}_{\sigma_{\geq a+1}X} = \sigma_{\geq a+1}(g^{a+1}_{X}) \quad \text{and} \quad k^{a}_{\sigma_{\leq a}X} = \sigma_{\leq a}(k^{a}_{X}).
  \end{equation}

- Proposition 2.3 gives several uniqueness statements, e.g. it shows that the morphisms connecting the horizontal triangles are also unique extending $g^{b+1}_{X}$, $k^{b}_{X}$ and $v^{b}_{X}$ respectively (in the last case one has to change the sign of the third morphism of the top sequence to make it into a triangle).

- Application of $\sigma_{\geq a+1} \xrightarrow{g^{a+1}_{X}} \text{id} \xrightarrow{k^{a}_{X}} \sigma_{\leq a} \xrightarrow{v^{a}_{X}} [1]\sigma_{\geq a+1}$ to the second vertical triangle in (6.7) yields a similar $3 \times 3$-diagram which is uniquely isomorphic to the $3 \times 3$-diagram (6.7) by an isomorphism extending $\text{id}_{X}$ (and this is functorial in $X$). The four isomorphisms in the corners are the isomorphism $\sigma_{X}^{[b+1,a]}$. 


from the Proposition, the (inverse of) the isomorphism \( \sigma^b \),
and the isomorphisms \( \sigma_{\leq a} \sigma_{\leq b} \sim \sigma_{\leq b} \sigma_{\leq a} \) and \( \sigma_{\leq a+1} \sigma_{\leq b+1} \sim \sigma_{\leq b+1} \sigma_{\leq a+1} \).

We use the isomorphisms (6.8) and (6.9) and those from the last point
sometimes tacitly in the following and write them as equalities.

We introduce some shorthand notation: For \( a,b \in \mathbb{Z} \) define \( \sigma_{[a,b]} := \sigma_{\leq b} \sigma_{\geq a} \) (which equals \( \sigma_{\geq a} \sigma_{\leq b} \) by the above convention) and \( \sigma_a := \sigma_{[a,a]} \).

We give some commutation formulas: Applying the triangulated
functor \( s \) to the triangle \( (\sigma_{\geq a+1} X, X, \sigma_{\leq a} X) \) yields a triangle that is
uniquely isomorphic to \( (\sigma_{\geq a+2} s(X), s(X), \sigma_{\leq a+1} s(X)) \). Hence we ob-
tain isomorphisms (that we write as equalities)
\[
\begin{align*}
\sigma_{\geq a+1} s & = \sigma_{\geq a+2} s, \\
\sigma_{\leq a+1} s & = \sigma_{\leq a+2} s, \\
\sigma_a & = \sigma_{a+1}.
\end{align*}
\]

Let \( (\widetilde{T}, i) \) be an f-category over a triangulated category \( T \). Define
\[
\text{gr}^n := i^{-1} s^{-n} \sigma_n : \widetilde{T} \to T
\]
where \( i^{-1} \) is a fixed quasi-inverse of \( i \). Note that \( \text{gr}^n \) is a triangulated
functor. From \( \text{gr}^{n+1} s = i^{-1} s^{-n-1} \sigma_{n+1} = i^{-1} s^{-n} \sigma_n = \text{gr}^n \) we obtain
\[
\text{gr}^{n+1} s = \text{gr}^n.
\]

Given \( X \in \widetilde{T} \) we define its support by
\[
\text{supp}(X) := \{ n \in \mathbb{Z} \mid \sigma_n(X) \neq 0 \}.
\]

Note that \( \text{supp}(X) \) is a bounded subset of \( \mathbb{Z} \) by axiom \([\text{fcat2}]\) and
Proposition 6.4, (4). It is empty if and only if \( X = 0 \). The range \( \text{range}(X) \) of \( X \) is defined as the smallest interval (possibly empty) in
\( \mathbb{Z} \) containing \( \text{supp} X \). It is the smallest interval \( I \) such that \( X \in \widetilde{T}(I) \).

The length \( \text{l}(X) \) of \( X \) is the number of elements in \( \text{range}(X) \).

6.3. Forgetting the filtration – the functor \( \omega \).

**Proposition 6.6** (cf. [Be˘ı87] Prop. A 3) (without proof). Let \( (\widetilde{T}, i) \)
be an f-category over a triangulated category \( T \). There is a unique (up
to unique isomorphism) triangulated functor
\[
\omega : \widetilde{T} \to T
\]
such that
\[
(\text{om1}) \text{ The restriction } \omega|_{\widetilde{T}(\leq 0)} : \widetilde{T}(\leq 0) \to T \text{ is left\footnote{typo in [Be˘ı87] Prop. A 3} \text{ adjoint to } i : T \to \widetilde{T}(\leq 0).}
\]
(om2) The restriction $\omega|_{\tilde{T}(\geq 0)}: \tilde{T}(\geq 0) \to T$ is right adjoint to $i: T \to \tilde{T}(\geq 0)$.

(om3) For any $X$ in $\tilde{T}$, the arrow $\alpha_X: X \to s(X)$ is mapped to an isomorphism $\omega(\alpha_X): \omega(X) \tilde{\sim} \omega(s(X))$.

(om4) For all $X$ in $\tilde{T}(\leq 0)$ and all $Y$ in $\tilde{T}(\geq 0)$ we have an isomorphism

\[ \omega: \text{Hom}_{\tilde{T}}(X, Y) \tilde{\sim} \text{Hom}_T(\omega(X), \omega(Y)). \]

In fact $\omega$ is uniquely determined by the properties (om1) and (om3) (or by the properties (om2) and (om3)).

The functor $\omega$ is often called the "forgetting of the filtration functor"; the reason for this name is Lemma 6.7 below. Note that $\omega|_{\tilde{T}(\{0\})}$ is left (and right) adjoint to the equivalence $i: T \tilde{\sim} \tilde{T}(\{0\})$ and hence a quasi-inverse of $i$.

**Proof. Uniqueness:** Assume that $\omega: \tilde{T} \to T$ satisfies the conditions (om1) and (om3).

Let $f: X \to Y$ be a morphism in $\tilde{T}$. By (fcat2) there is an $n \in \mathbb{Z}$ such that this is a morphism in $\tilde{T}(\leq n)$. By (fcat1) we can assume that $n \geq 0$. Consider the commutative diagram

\[
\begin{array}{cccccccccc}
s^{-n}(X) & \xrightarrow{\alpha} & s^{-n+1}(X) & \xrightarrow{\alpha} & \cdots & \cdots & \xrightarrow{\alpha} & s^{-1}(X) & \xrightarrow{\alpha} & X \\
& & \downarrow{s^{-n}(f)} & & \downarrow{s^{-n+1}(f)} & & \cdots & \downarrow{s^{-1}(f)} & & \\
s^{-n}(Y) & \xrightarrow{\alpha} & s^{-n+1}(Y) & \xrightarrow{\alpha} & \cdots & \cdots & \xrightarrow{\alpha} & s^{-1}(Y) & \xrightarrow{\alpha} & Y
\end{array}
\]

where we omit the indices $s^{-i}(X)$ and $s^{-i}(Y)$ at the various maps $\alpha$. If we apply $\omega$, all the horizontal arrows become isomorphisms; note that $s^{-n}(f)$ is a morphism in $\tilde{T}(\leq 0)$. This shows that $\omega$ is uniquely determined by its restriction to $\tilde{T}(\leq 0)$ and knowledge of all isomorphisms $\omega(\alpha_Z): \omega(Z) \tilde{\sim} \omega(s(Z))$ for all $Z$ in $\tilde{T}$.

If we have two functors $\omega_1$ and $\omega_2$ whose restrictions are both adjoint to the inclusion $i: T \to \tilde{T}(\leq 0)$ (i.e. satisfy (om1)), these adjunctions give rise to a unique isomorphism between $\omega_1|_{\tilde{T}(\leq 0)}$ and $\omega_2|_{\tilde{T}(\leq 0)}$. If both $\omega_1$ and $\omega_2$ satisfy in addition (om3) this isomorphism can be uniquely extended to an isomorphism between $\omega_1$ and $\omega_2$ (use the above diagram after application of $\omega_1$ and $\omega_2$ respectively).

**Existence:** Let $X$ in $\tilde{T}$. We define objects $X_l$ and $X_r$ as follows:

If $X = 0$ let $X_l = X_r = 0$. If $X \neq 0$ let $a = a^X, b = b^X \in \mathbb{Z}$ such that range$(X) = [a, b]$ (and $l(X) = b - a + 1$); let $X_l := s^{-b}(X)$
and $X_r := s^{-a}(X)$ (the indices stand for left and right); observe that $X_l \in \widetilde{T}(\leq 0) \not\ni s(X_l)$ and $X_r \in \widetilde{T}(\geq 0) \not\ni s^{-1}(X_r)$.

We denote the composition $X_l \xrightarrow{\alpha_{X_l}} s(X_l) \rightarrow \ldots \xrightarrow{\alpha_{s^{-1}(X_r)}} X_r$ by $\alpha^X_{il} : X_l \rightarrow X_r$. Our first goal is to construct for every $X$ in $\widetilde{T}$ an object $\Omega_X$ in $\mathcal{T}$ and a factorization of $\alpha^X_{il}$

\[
\begin{array}{ccc}
x_l & \overset{\alpha^X_{il}}{\longrightarrow} & X_l \\
\downarrow{\varepsilon_X} & & \downarrow{\delta_X} \\
i(\Omega_X) & \overset{\delta_X}{\longrightarrow} & X_r
\end{array}
\]

such that

\begin{align*}
(6.14) \quad \text{Hom}(i(\Omega_X), \tilde{A}) & \xrightarrow{\sim} \text{Hom}(X_l, \tilde{A}) \quad \text{for all } \tilde{A} \text{ in } \widetilde{T}(\geq 0); \\
(6.15) \quad \text{Hom}(\tilde{B}, i(\Omega_X)) & \xrightarrow{\sim} \text{Hom}(\tilde{B}, X_r) \quad \text{for all } \tilde{B} \text{ in } \widetilde{T}(\leq 0).
\end{align*}

Here is a diagram to illustrate this:

\[
\begin{array}{ccc}
& & \tilde{B} \\
& b \downarrow & \\
x_l \xrightarrow{\varepsilon_X} i(\Omega_X) & \xrightarrow{\delta_X} & X_r \\
& \downarrow{a'} & \downarrow{a} \\
& A \\
\end{array}
\]

If $X$ is in $\widetilde{T}(\leq 0)$ the composition $X \xrightarrow{\alpha^X_l} s(X) \rightarrow \ldots \xrightarrow{\alpha_{s^{-1}(X_l)}} X_l$ is denoted $\alpha^X_{il} : X \rightarrow X_l$. If (6.14) holds then (6.14) shows:

\begin{align*}
(6.16) \quad \text{Hom}(i(\Omega_X), \tilde{A}) & \xrightarrow{\sim} \text{Hom}(X, \tilde{A}) \quad \text{for all } X \text{ in } \widetilde{T}(\leq 0) \\
& \text{ and } \tilde{A} \text{ in } \widetilde{T}(\geq 0).
\end{align*}

If $X$ is in $\widetilde{T}(\geq 0)$ and (6.15) holds, we similarly have a morphism $\alpha^X_r : X_r \rightarrow X$ and obtain:

\begin{align*}
(6.17) \quad \text{Hom}(\tilde{B}, i(\Omega_X)) & \xrightarrow{\sim} \text{Hom}(\tilde{B}, X) \quad \text{for all } X \in \widetilde{T}(\geq 0) \\
& \text{ and } \tilde{B} \text{ in } \widetilde{T}(\leq 0).
\end{align*}

We construct the triples $(\Omega_X, \varepsilon_X, \delta_X)$ by induction over the length $l(X)$ of $X$. The case $l(X) = 0$ is trivial.
Base case $l(X) = 1$. Then $X_l = X_r \in \tilde{T}([0])$. Let $\kappa$ be a quasi-inverse of $i$ and fix an isomorphism $\theta : \text{id} \sim i\kappa$. We define $\Omega_X := \kappa(X_l)$ and $X_l \xrightarrow{\varepsilon_X := \theta X_l} i(\Omega_X) \xrightarrow{\delta_X := \theta X_r^{-1}} X_r$. This is a factorization of $\alpha_{rl}^X = \text{id}_X$ and the conditions (6.14) and (6.15) are obviously satisfied.

Now let $X \in \tilde{T}$ be given with $l(X) > 1$ and assume that we have constructed $(\Omega_Y, \varepsilon_Y, \delta_Y)$ as desired for all $Y$ with $l(Y) < l(X)$. Let $L := l(X) - 1$. Let $P := \sigma_{\geq 0}(X_l)$ and $Q := \sigma_{\leq -1}(X_l)$ and let us explain the following diagram:

\[
\begin{array}{ccccccc}
P = P_l & \xrightarrow{\varepsilon_P} & X_l & \xrightarrow{\varepsilon_X} & Q & \xrightarrow{h} & [1]P \\
\sim & & & & & & \\
\sim & & & & & & \\
\sim & & & & & & \\
P = P_r & \xrightarrow{\delta_P} & s^L(X_l) = X_r & \xrightarrow{\delta_Q} & Q_r & \xrightarrow{s^L(h)} & [1]s^L(P)
\end{array}
\]

The first row is the $\sigma$-truncation triangle of type $(\geq 0, \leq -1)$ of $X_l \in \tilde{T}([-L, 0])$. The last row is the image of this triangle under the triangulated automorphism $s^L$. There is a morphism between these triangles given by $\alpha^L$. Observe that $\text{range}(P) = [0]$ and $\text{range}(Q) = [-L, b_Q] \subset [-L, -1]$; in particular the triples $(\Omega_P, \varepsilon_P, \delta_P)$ and $(\Omega_Q, \varepsilon_Q, \delta_Q)$ are constructed. Since $\delta \circ \varepsilon$ is a factorization of $\alpha_{rl}$, the first, third and fourth column are components of this morphism $\alpha^L$ of triangles. By (6.16) there is a unique morphism $h'$ as indicated making the upper right square commutative, and then (6.16) again shows that the lower right square commutes. We can find some $\Omega_X \in \mathcal{T}$ and a completion of $h'$ to a triangle as shown in the middle row of the above diagram. Then we complete the partial morphism of the two upper triangles by $\varepsilon_X$ to a morphism of triangles.

Let us construct the wiggly morphism $\delta_X$. Take an arbitrary object $\tilde{A}$ in $\tilde{T}(\geq 0)$ and apply $\text{Hom}(?, \tilde{A})$ to the morphism of the upper two triangles. The resulting morphism of long exact sequences and the five
lemma show that (6.14) holds for $\varepsilon_X$. This shows in particular that the morphism $\alpha^{X}_{rl} : X \to X_r$ factorizes uniquely over $\varepsilon_X$ by some $\delta_X$. Using (6.14) again we see that $\delta_X$ defines in fact a morphism of the two lower triangles.

Finally we apply for any $\tilde{B}$ in $\tilde{T}(\leq 0)$ the functor $\text{Hom}(\tilde{B}, ?)$ to the morphism of the two lower triangles; the five lemma again shows that $\delta_X$ satisfies (6.15). This shows that the triple $(\Omega_X, \varepsilon_X, \delta_X)$ has the properties we want.

Now we define the functor $\omega : \tilde{T} \to T$. On objects we define $\omega(X) := \Omega_X$. Let $f : X \to Y$ be a morphism. Then for some $N \in \mathbb{N}$ big enough we have $s^{-N}(X), s^{-N}(Y) \in \tilde{T}(\leq 0)$, hence $s^{-N}(f)$ is a morphism in $\tilde{T}(\leq 0)$. Similarly for some $M \in \mathbb{N}$ big enough $s^M(f)$ is a morphism in $\tilde{T}(\geq 0)$. Consider the diagram (6.18)

$$
\begin{array}{ccccccc}
  s^{-N}(X) & \xrightarrow{\alpha_X} & X_l & \xrightarrow{\varepsilon_X} & i(\omega(X)) & \xrightarrow{\delta_X} & X_r & \xrightarrow{\alpha_Y} & s^M(X) \\
  \downarrow{s^{-N}(f)} & & \downarrow{\Omega_f} & & \downarrow{\delta_Y} & & \downarrow{s^M(f)} \\
  s^{-N}(Y) & \xrightarrow{\alpha_Y} & Y_l & \xrightarrow{\varepsilon_Y} & i(\omega(Y)) & \xrightarrow{\delta_Y} & Y_r & \xrightarrow{\alpha_Y} & s^M(Y).
\end{array}
$$

The morphism from $s^{-N}(X)$ to $i(\omega(Y))$ factors uniquely to the dotted arrow by (fcat6) and (6.14). Similarly the morphism from $i(\omega(X))$ to $s^M(Y)$ factors to the dotted arrow. These (a priori) two dotted arrows coincide, since

$$
\text{Hom}(i(\omega(X)), i(\omega(Y)) \sim \text{Hom}(s^{-N}(X), s^M(Y))
$$

by (fcat6) and (6.14) and (6.15) again and since both compositions $\delta \circ \varepsilon$ are a factorization of $\alpha_{rl}$. Note that $\Omega_f$ does not depend on the choice of $N$ and $M$. We define $\omega(f)$ to be the unique arrow $\omega(X) \to \omega(Y)$ that is mapped under the equivalence $i$ to $\Omega_f$. This respects compositions and identity morphisms and hence defines a functor $\omega$.

We verify (om1)-(om4) and that $\omega$ can be made into a triangulated functor.

Let us show (om3) first: We can assume that $X \neq 0$. We draw the left part of diagram (6.18) for the morphism $\alpha_X : X \to s(X)$ (note
that $b_X + 1 = b_{s(X)}$ and hence $X_t = (s(X))_t$:

$$
\begin{array}{c}
\xymatrix{ s^{-N}(X) \ar[r]^{\alpha} & X_t = s^{-b_X}(X) \ar[r]^{\varepsilon_X} & i(\omega(X)) \\
 s^{-N}(\alpha_X) \ar[r]_{\alpha^{-1}} & (s(X))_t = s^{-b_{s(X)}}(X) \ar[r]_{\varepsilon_{s(X)}} & i(\omega(s(X))) }
\end{array}
$$

By $(\text{fcat5})$ we have $s^{-N}(\alpha_X) = \alpha s^{-N}(X)$ and hence the left square becomes commutative with the dotted arrow. Since $\varepsilon_X$ and $\varepsilon_{s(X)}$ have the same universal property $(6.14)$ the morphism $\Omega_{\alpha X}$ must be an isomorphism.

Let us show $(\text{om1})$. Given $X$ in $\widetilde{T}(\leq 0)$ and $A$ in $T$ replace $\tilde{A}$ by $i(A)$ in $(6.16)$ and use $i: \text{Hom}(\omega(X), A) \cong \text{Hom}(i(\omega(X)), i(A))$. This gives

$$
\text{Hom}_T(\omega(X), A) \xrightarrow{i(?)|_{\omega X}} \text{Hom}_{\tilde{T}}(X, i(A)).
$$

From $(6.18)$ it is easy to see that this isomorphism is compatible with morphisms $f: X \to Y$ in $\widetilde{T}(\leq 0)$ and $g: A' \to A$ in $T$.

The proof of $(\text{om2})$ is similar.

Proposition 2.5 shows that $\omega|_{\widetilde{T}(\leq 0)}$ is triangulated for a suitable isomorphism $\varphi: \omega|_{\widetilde{T}(\leq 0)}[1] \cong [1]\omega|_{\widetilde{T}(\leq 0)}$. Using the above techniques it is easy to find an isomorphism $\varphi: \omega[1] \cong [1]\omega$ such that $(\omega, \varphi)$ is a triangulated functor. We leave the details to the reader.

We finally prove $(\text{om4})$. Let $X$ in $\widetilde{T}(\leq 0)$ and $Y$ in $\widetilde{T}(\geq 0)$. The composition

$$
\text{Hom}_T(\omega(X), \omega(Y)) \xrightarrow{i(?)|_{\omega X}} \text{Hom}_{\tilde{T}}(i(\omega(X)), i(\omega(Y)))
$$

$$
\xrightarrow{6.19} \text{Hom}_{\tilde{T}}(s^{-N}(X), s^M(Y))
$$

$$
\xleftarrow{\alpha M \circ \alpha N} \text{Hom}_{\tilde{T}}(X, Y).
$$

of isomorphisms (the last isomorphism comes from $(\text{fcat6})$) is easily seen to be inverse to $(6.13)$. \qed

6.4. Omega in the basic example. Let $\mathcal{A}$ be an abelian category and consider the basic example of the $\text{f}$-category $DF(\mathcal{A})$ over $D(\mathcal{A})$ (as described in Section 6.1, Proposition 6.3). Let $\omega: CF(\mathcal{A}) \to C(\mathcal{A})$ be the functor mapping a filtered complex $X = (X, F)$ to its underlying non-filtered complex $X$. This functor obviously induces a triangulated functor $\omega: DF(\mathcal{A}) \to D(\mathcal{A})$. 
Lemma 6.7. The functor $\omega : DF(A) \to D(A)$ satisfies the conditions (om3) and (om1) of Proposition 6.6.

Proof. Condition (om3) is obvious, so let us show condition (om1).

We abbreviate $\omega' := \omega \mid_{DF(A)(\leq 0)}$ and consider $i$ as a functor to $DF(A)(\leq 0)$. We first define a morphism $\varepsilon : \text{id}_{DF(A)(\leq 0)} \to i \circ \omega'$ of functors as follows. Let $X$ be in $DF(A)(\leq 0)$. We have seen that the obvious morphism $X \to X/(X(\geq 1))$ (cf. (6.2)) is an isomorphism in $DF(A)$. As above we denote the filtration of $X$ by $\cdots \supset F_i \supset F_{i+1} \supset \cdots$ and its underlying non-filtered complex $\omega(X)$ by $X$. Define $\varepsilon_X$ by the commutativity of the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varepsilon_X} & i(\omega(X)) = i(X) \\
\downarrow & & \downarrow \\
X/X(\geq 1) & \xrightarrow{i(\omega(X/X(\geq 1))) = i(X/F^1)} & X/F^1 \\
\end{array}
\]

where the lower horizontal map is the obvious one. This is compatible with morphisms and defines $\varepsilon$. Let $\delta : \omega' \circ i \to \text{id}_{D(A)}$ be the identity morphism. We leave it to the reader to check that $\delta \omega' \circ \omega' \varepsilon = \text{id}_{\omega'}$ and $i \delta \circ \varepsilon i = \text{id}_i$. This implies that $\varepsilon$ and $\delta$ are unit and counit of an adjunction $(\omega', i)$. \hfill \Box

6.5. Construction of the functor $c$. Let $(\tilde{T}, i)$ be an f-category over a triangulated category $T$. Our aim in this section is to construct a certain functor $c : \tilde{T} \to C^b(T)$. The construction is in two steps. It can be found in a very condensed form in [Bei87, Prop. A 5] or [Bon10, 8.4].

6.5.1. First step. We proceed similar as in the construction of the weak weight complex functor (see Section 5). This may seem a bit involved (compare to the second approach explained later on) but will turn out to be convenient when showing that the strong weight complex functor is a lift of the weak one.

For every object $X$ in $\tilde{T}$ we have functorial $\sigma$-truncation triangles (for all $n \in \mathbb{Z}$)

\[
(6.20) \quad S^n_X : \sigma_{\geq n+1}X \xrightarrow{g^n_{X+1}} X \xrightarrow{k^n_X} \sigma_{\leq n}X \xrightarrow{v^n_X} [1] \sigma_{\geq n+1}X.
\]
For any $n$ there is a unique morphism of triangles $S^n_X \to S^{n-1}_X$ extending $\text{id}_X$ (use Prop. 2.3 and \text{(fcat3)}):

\begin{equation}
S^n_X : \quad \sigma_{\geq n+1}X \xrightarrow{g^n_X} X \xrightarrow{k^n_X} \sigma_{\leq n}X \xrightarrow{v^n_X} [1]\sigma_{\geq n+1}X
\end{equation}

\[\begin{array}{c}
S^{n-1}_X : \\
\end{array}\]

More precisely $h^n_X$ (resp. $l^n_X$) is the unique morphism making the square $\Delta$ (resp. $\nabla$) commutative. It is easy to see that $h^n_X$ corresponds under (6.9) to the adjunction morphism $g_{\sigma_{\geq n}X} : \sigma_{\geq n+1}\sigma_{\geq n}X \to \sigma_{\geq n}X$. Hence we see from Proposition 6.4 (2) that there is a unique morphism $c^n_X$ such that

\begin{equation}
S'_n : \quad \sigma_{\geq n+1}X \xrightarrow{h^n_X} \sigma_{\geq n}X \xrightarrow{e^n_X = k^n_{\sigma_{\geq n}X}} \sigma_{\leq n}X \xrightarrow{c^n_X} [1]\sigma_{\geq n+1}X
\end{equation}

is a triangle. We use the tree triangles in (6.21) and (6.22) and the square marked with $\Delta$ in (6.21) as the germ cell and obtain (from the octahedral axiom) the following octahedron:

\begin{equation}
O^n_X :=
\end{equation}

Note that the lower dotted morphism is in fact $l^n_X$ by the uniqueness of $l^n_X$ observed above. From Proposition 6.4 (4) we obtain that the upper dotted morphism labeled $a^n_X$ is unique: It is the morphism $g^n_{\sigma_{\leq n}X}$ (more precisely it is the morphism $g^n_{\sigma_{\leq n}X} \circ \sigma^{[n,n]}_X$ in the notation of (6.6)). We
see that the triangle $S^{\sigma_{\leq n} X}_{\sigma_{\leq n} X}$ can be constructed completely analogous as triangle (6.22) above.

It is now easy to see that $X \mapsto \tilde{O}^X$ is in fact functorial.

Let $c'(X)$ be the following complex in $\tilde{T}$: Its $n$-th term is
\[ c'(X)^n := [n] \sigma_n X \]
and the differential $d^n_{c'(X)} : [n] \sigma_n X \to [n+1] \sigma_{n+1} X$ is defined by
\begin{align*}
(6.24) \quad d^n_{c'(X)} := [n](b^{n+1}_X \circ a^n_X) &= [n](([1] e^{n+1}_X) \circ v^n_X \circ a^n_X) \\
&= [n](([1] e^{n+1}_X) \circ c^n_X).
\end{align*}

Note that $d^n_{c'(X)} \circ d^{n-1}_{c'(X)} = 0$ since the composition of two consecutive maps in a triangle is zero (apply this to (6.23)).

Since everything is functorial we have in fact defined a functor
\[ c' : \tilde{T} \to C(\tilde{T}). \]

Let $C^b_{\Delta}(\tilde{T}) \subset C(\tilde{T})$ be the full subcategory consisting of objects $A \in C(\tilde{T})$ such that $A^n \in \tilde{T}([n])$ for all $n \in \mathbb{Z}$ and $A^n = 0$ for $n \gg 0$.

Axiom (ict2) and $c'(X)^n = [n] \sigma_n (X) \in \tilde{T}([n])$ show that we can view $c'$ as a functor
\[ c' : \tilde{T} \to C^b_{\Delta}(\tilde{T}). \]

In the proof of the fact that a weak weight complex functor is a functor of additive categories with translation we have used an octahedron (5.14). A similar octahedron with the same distribution of signs yields a canonical isomorphism
\begin{align*}
(6.25) \quad c' \circ [1] s^{-1} \cong \Sigma \circ (s^{-1})_{c^b(\tilde{T})} \circ c'
\end{align*}
where we use notation introduced at the end of section 2.4. We leave the details to the reader. Note that $\Sigma$ and $(s^{-1})_{c^b(\tilde{T})}$ commute.

We describe two other ways to obtain the functor $c'$: Let $X \in \tilde{T}$. Apply $\sigma_{\leq n+1}$ to the triangle (6.22). This gives the triangle
\begin{align*}
(6.26) \quad \sigma_{n+1} X \longrightarrow \sigma_{[n,n+1]} X \longrightarrow \sigma_{\leq n+1} \sigma_n X \xrightarrow{\sigma_{\leq n+1}(e^n_X)} [1] \sigma_{n+1} X.
\end{align*}
The following commutative diagram is obtained by applying the transformation $k^{n+1} : \text{id} \to \sigma_{\leq n+1}$ to its upper horizontal morphism:
\begin{align*}
(6.27) \quad \sigma_{n} X \xrightarrow{e^n_X} [1] \sigma_{\geq n+1} X \quad \sim \quad \sigma_{n} X \xrightarrow{k^{n+1}_n} [1] e^{n+1}_X \quad \sigma_{\leq n+1} \sigma_n X \xrightarrow{\sigma_{\leq n+1}(e^n_X)} [1] \sigma_{n+1} X.
\end{align*}
We use the isomorphism $k_{\sigma_n X}^{n+1} : \sigma_n X \sim \sigma_{\leq n+1} \sigma_n (X)$ in order to replace the third term in (6.26) and obtain (using (6.27)) the following triangle, where $\tilde{d}_X^n := [1] e_{\sigma_n}^{n+1} \circ c_X^n$:

$$
\sigma_{n+1} X \longrightarrow \sigma_{[n,n+1]} X \longrightarrow \sigma_n X \overset{\tilde{d}_X^n}{\longrightarrow} [1] \sigma_{n+1} X
$$

(6.28)

Completely analogous $\sigma \geq n$ applied to the triangle $S_{\sigma_{\leq n+1} X}$ and the isomorphism $g_{\sigma_{n+1} X}^n : \sigma_{\geq n} \sigma_{n+1} X \sim \sigma_{n+1} X$ provide a triangle of the form (6.28) with the same third morphism $\tilde{d}_X^n = b_{\sigma_n}^{n+1} \circ a_X^n$ (cf. (6.24)) which presumably is (6.28) under the obvious identifications. Note that

$$
\cdots \longrightarrow [n] \sigma_n (X) \overset{[n]d_n}{\longrightarrow} [n+1] \sigma_{n+1} (X) \longrightarrow \cdots
$$

is the complex $c'(X)$. Hence we have described two slightly different (functorial) constructions of the functor $c'$.

We will use the construction described after (6.28) later on and refer to it as the second approach to $c'$.

6.5.2. Second step. In the second step we define the functor $c$ via a functor $c'' : C^b_{\Delta}(\tilde{T}) \rightarrow C^b(\tilde{T}([0])).$ There is a shortcut to $c$ described in Remark 6.8 below.

Let $A = (A^n, d_A^n) \in C^b_{\Delta}(\tilde{T})$. We draw this complex horizontally in the following diagram:

$$
\begin{array}{cccccc}
\cdots & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \cdots \\
 & s(A^{-1}) & & d_A^0 & & d_A^1 & & d_A^2 & \\
 & \alpha & & & & & & & \\
 & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \cdots \\
 & \alpha & & \alpha & & \alpha & & \alpha & \\
 & s^{-1}(A^1) & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & s^{-2}(A^2) & & & & & & & \\
\end{array}
$$

(6.29)

The dotted arrows making everything commutative are uniquely obtained using (fcat6). The diagonal is again a bounded complex (i.e. $d^2 = 0$ again by (fcat6)), now even in $\tilde{T}([0])$. We denote this complex by $c''(A)$. This construction in fact defines a functor $c'' : C^b_{\Delta}(\tilde{T}) \rightarrow C^b(\tilde{T}([0])).$

It is easy to check that there is a canonical isomorphism

$$
c'' \circ \Sigma \circ (s^{-1})_{C^b(\tilde{T})} \cong \Sigma \circ c''.
$$

(6.30)
6.5.3. Definition of the functor $c$. By fixing a quasi-inverse $i^{-1}$ to $i$ we now define $c$ to be the composition

$$c : \tilde{T} \xrightarrow{\varepsilon'} C^b(\tilde{T}) \xrightarrow{\varepsilon''} C^b(\tilde{T}([0])) \xrightarrow{i^{-1}} C^b(T).$$

This functor maps an object $X \in \tilde{T}$ to the complex

$$\cdots \rightarrow [-1] \text{gr}^{-1}(X) \rightarrow \text{gr}^0(X) \rightarrow [1] \text{gr}^1(X) \rightarrow [2] \text{gr}^2(X) \rightarrow \ldots .$$

Remark 6.8. An equivalent shorter definition of $c$ would be to define it as $\omega_{C^b} \circ c'$.

Combining the above canonical isomorphisms (6.25) and (6.30) we obtain:

**Proposition 6.9.** The functor $c$ constructed above is a functor

$$c : (\tilde{T}, [1]s^{-1}) \rightarrow (C^b(T), \Sigma)$$

of additive categories with translation: On objects we have a canonical isomorphism

$$c([1]s^{-1}(X)) \cong \Sigma c(X).$$

7. Strong weight complex functor

Our aim in this section is to show the following Theorem:

**Theorem 7.1** (cf. [Bon10, 8.4]). Let $\mathcal{T}$ be a triangulated category with a bounded weight structure $w = (\mathcal{T}^{w \leq 0}, \mathcal{T}^{w \geq 0})$. Let $(\tilde{T}, i)$ be an f-category over $\mathcal{T}$. Assume that $\tilde{T}$ satisfies axiom (fcat7) stated below. Then there is a strong weight complex functor

$$\tilde{WC} : \mathcal{T} \rightarrow K^b(\mathcal{C}w)^\text{anti}.$$

In particular $\tilde{WC}$ is a functor of triangulated categories.

A proof of (a stronger version of) this theorem is sketched in [Bon10, 8.4], where M. Bondarko attributes the argument to A. Beilinson. When we tried to understand the details we had to impose the additional conditions that $\tilde{T}$ satisfies axiom (fcat7) and that the weight structure is bounded. We state axiom (fcat7) in Section 7.2 and show in Section 7.3 that it is satisfied in the basic example of a filtered derived category. Our proof of Theorem 7.1 is an elaboration of the ideas of A. Beilinson and M. Bondarko; we sketch the idea of the proof in Section 7.4 and give the details in Section 7.5.
7.1. Idea of the proof. Before giving the details let us explain the strategy of the proof of Theorem 7.1.

Let \((\tilde{T}, i)\) be an f-category over a triangulated category \(T\) and assume that \(w = (T^{w \leq 0}, T^{w \geq 0})\) is a weight structure on \(T\). Its heart \(\mathcal{H}(w) = T^{w = 0}\) is a full subcategory of \(T\), and hence \(C^b(\mathcal{H}(w)) \subseteq C^b(T)\) and \(K^b(\mathcal{H}(w)) \subseteq K^b(T)\) are full subcategories. Let \(\tilde{T}^s\) be the full subcategory of \(\tilde{T}\) consisting of objects \(X \in \tilde{T}\) such that \(c(X) \in C^b(\mathcal{H}(w))\) where \(c\) is the functor constructed in Section 6.5. We have a “pull-back” diagram

\[
\begin{array}{ccc}
\tilde{T} & \xrightarrow{c} & C^b(T) \\
\cup & & \cup \\
\tilde{T}^s & \xrightarrow{c} & C^b(\mathcal{H}(w))
\end{array}
\]

of categories where we denote the lower horizontal functor also by \(c\).

Note that (6.33) shows that \(\tilde{T}^s\) is stable under \(s^{-1}[1] = [1]s^{-1}\).

We expand diagram (7.1) to

\[
\begin{array}{ccc}
\tilde{T} & \xrightarrow{h} & C^b(T) \\
\cup & & \cup \\
\tilde{T}^s & \xrightarrow{c} & C^b(\mathcal{H}(w))
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{T} & \xrightarrow{h} & C^b(T) & \xrightarrow{\text{can}} & K^b(T)_{\text{anti}} \\
\cup & & \cup & & \cup \\
\tilde{T}^s & \xrightarrow{c} & C^b(\mathcal{H}(w)) & \xrightarrow{\text{can}} & K^b(\mathcal{H}(w))_{\text{anti}}
\end{array}
\]

and define \(h := \text{can} \circ c\) as indicated. We have seen in Proposition 6.9 that \(c\) is a functor of additive categories with translation (where \(\tilde{T}\) (or \(\tilde{T}^s\)) are equipped with the translation \([1]s^{-1}\)); the same is obviously true for \(h\). For this statement the homotopy categories on the right are just considered as additive categories with translation. In the following however we view them as triangulated categories with the class of triangles described in Sections 2.3.2 and 2.4.

From now on we assume that \(\tilde{T}\) satisfies axiom \((\text{fcat7})\) (stated below) and that the weight structure is bounded.
Consider the following diagram whose dotted arrows will be explained:

\[(7.3)\]

\[
\begin{array}{c}
\mathcal{T} \xleftarrow{\omega} \tilde{T} \xrightarrow{h} K^b(\mathcal{T})^{\text{anti}} \\
\mathcal{T} \xleftarrow{\omega|\tilde{T}_s} \tilde{T}_s \xrightarrow{h} K^b(\heartsuit(w))^{\text{anti}} \\
\mathcal{T} \xleftarrow{\sim} \tilde{T}_s \xrightarrow{\text{can}} K^b(\heartsuit(w))^{\text{anti}} \\
\mathcal{T} \xleftarrow{\omega|\tilde{T}_s} \tilde{T}_s \xrightarrow{\text{can}} K^b(\heartsuit(w))^{\text{anti}} \\
\mathcal{T} \xleftarrow{\sim} \tilde{T}_s \xrightarrow{\text{can}} K^b(\heartsuit(w))^{\text{anti}} \\
\end{array}
\]

We will prove below: The restriction \(\omega|\tilde{T}_s\) factors over some quotient functor \(\tilde{T}_s \xrightarrow{\text{can}} Q\) and induces an equivalence \(\overline{\omega} : Q \xrightarrow{\sim} \mathcal{T}\) of additive categories with translation (see Prop. 7.10) where the translation functor of \(Q\) is induced by \([1]s^{-1}\). Transfer of structure turns \(Q\) into a triangulated category; its class of triangles can be explicitly described (cf. Lemma 7.11).

On the other hand the functor \(h : \tilde{T}_s \rightarrow K^b(\heartsuit(w))^{\text{anti}}\) factors over \(Q\) to a functor \(h\) of triangulated categories (see Cor. 7.15).

Let \(\tilde{WC} : \mathcal{T} \rightarrow K^b(\heartsuit(w))^{\text{anti}}\) be the composition \(h \circ \overline{\omega}^{-1}\), where \(\overline{\omega}^{-1}\) is a quasi-inverse of \(\overline{\omega}\). (In diagram (7.3) \(\tilde{WC}\) is the composition of the dotted arrows.) Then \(\tilde{WC}\) will turn out to be a strong weight complex functor, proving Theorem 7.1.

7.2. An additional axiom for filtered triangulated categories.

Let \(\tilde{T}\) be a filtered triangulated category \(\mathcal{T}\). Let \(Y\) in \(\tilde{T}\) be an object and consider the \(\sigma\)-truncation triangle

\[
S_Y^0 : \quad \sigma_{\geq 1} Y \xrightarrow{g_Y^1} Y \xrightarrow{k_Y^0} \sigma_{\leq 0} Y \xrightarrow{v_Y^0} [1] \sigma_{\geq 1} Y.
\]

Applying the morphism \(\alpha\) we obtain a morphism of triangles

\[
s(S_Y^0) : \quad s(\sigma_{\geq 1} Y) \xrightarrow{s(g_Y^1)} s(Y) \xrightarrow{s(k_Y^0)} s(\sigma_{\leq 0} Y) \xrightarrow{s(v_Y^0)} [1] s(\sigma_{\geq 1} Y)
\]

\[
\begin{array}{c}
\sigma_{\geq 1} Y \xrightarrow{g_Y^1} Y \xrightarrow{k_Y^0} \sigma_{\leq 0} Y \xrightarrow{v_Y^0} [1] \sigma_{\geq 1} Y
\end{array}
\]

where we tacitly identify \(s([1] \sigma_{\geq 1}(Y)) = [1] s(\sigma_{\geq 1}(Y))\) and \(\alpha_{[1] \sigma_{\geq 1}(Y)} = [1] \alpha_{\sigma_{\geq 1}(Y)}\).
Given a morphism \( f : X \to Y \) in \( \tilde{T} \), the morphism of triangles
\[
S_Y^0 : \begin{array}{cccc}
\sigma_{\geq 1} Y & \xrightarrow{s(g^Y)} & Y & \xrightarrow{k^Y} & \sigma_{\leq 0} Y & \xrightarrow{[1]\sigma_{\geq 1} Y} \\
\sigma_{\geq 1}(f) & & f & & \sigma_{\leq 0}(f) & & [1]\sigma_{\geq 1}(f)
\end{array}
\]
\[
S_X^0 : \begin{array}{cccc}
\sigma_{\geq 1} X & \xrightarrow{s(g_X)} & X & \xrightarrow{k_X} & \sigma_{\leq 0} X & \xrightarrow{[1]\sigma_{\geq 1} X} \\
\sigma_{\geq 1}(f) & & f & & \sigma_{\leq 0}(f) & & [1]\sigma_{\geq 1}(f)
\end{array}
\]
is the unique morphism of triangles extending \( f \) (use Prop. 2.3 and (fcat3)).

We denote the composition of this two morphisms of triangles by \( \alpha \circ f : S_X^0 \to s(S_Y^0) \):
\[
(7.4)
\]
\[
\begin{array}{c}
s(S_Y^0) : \\
\sigma_{\geq 1} Y \xrightarrow{s(g^Y)} Y \xrightarrow{k^Y} \sigma_{\leq 0} Y \xrightarrow{[1]s(\sigma_{\geq 1} Y)} \\
\alpha_{\sigma_{\geq 1}(Y) \sigma_{\geq 1}(f)} \alpha_Y \circ f \sigma_{\leq 0}(f) \xrightarrow{[1](\alpha_{\sigma_{\geq 1}(Y) \sigma_{\geq 1}(f)} \alpha_Y \circ f \sigma_{\leq 0}(f))} [1](\alpha_{\sigma_{\geq 1}(Y) \sigma_{\geq 1}(f)})
\end{array}
\]
\[
\begin{array}{c}
S_X^0 : \\
\sigma_{\geq 1} X \xrightarrow{s(g_X)} X \xrightarrow{k_X} \sigma_{\leq 0} X \xrightarrow{[1]\sigma_{\geq 1} X} \\
\sigma_{\geq 1}(f) \sigma_{\leq 0}(f) \xrightarrow{[1]\sigma_{\geq 1}(f) \sigma_{\leq 0}(f)} [1]s(\sigma_{\geq 1} X)
\end{array}
\]

(We don’t see a reason why this morphism of triangles extending \( \alpha_Y \circ f \) should be unique; Proposition 2.3 does not apply. If it were unique axiom ((fcat7) below would be satisfied automatically.)

Now the additional axiom can be stated:

(fcat7) For any morphism \( f : X \to Y \) in \( \tilde{T} \) the morphism (7.4) of triangles \( \alpha \circ f : S_X^0 \to s(S_Y^0) \) explained above can be extended to a \( 3 \times 3 \)-diagram
\[
(7.5)
\]
\[
\begin{array}{cccc}
[1]\sigma_{\geq 1} X & \xrightarrow{[1]g_X} & [1]X & \xrightarrow{[1]k^Y} & [1]\sigma_{\leq 0} X & \xrightarrow{[1]v_X} & [2]\sigma_{\geq 1} X \\
\sigma_{\geq 1}(f) & \xrightarrow{\alpha_Y \circ f} & \sigma_{\leq 0}(f) & \xrightarrow{[1](\alpha_{\sigma_{\geq 1}(Y) \sigma_{\geq 1}(f)})} & [1]\sigma_{\geq 1}(f) \sigma_{\leq 0}(f) & \xrightarrow{[1\alpha_{\sigma_{\geq 1}(Y) \sigma_{\geq 1}(f)}]} & [1]\sigma_{\geq 1}(f)
\end{array}
\]

having the properties described in Proposition 2.4.

Instead of taking the morphism \( S_X^0 \xrightarrow{\alpha \circ f} s(S^0_Y) \) at the bottom of this \( 3 \times 3 \)-diagram we get similar diagrams with morphisms \( s(S_X^0) \xrightarrow{\alpha \circ f} s(S_Y^0) \) at the bottom (use the functor \( s \) of triangulated categories).
Remark 7.2. We first expected that axiom \((\text{fcat7})\) is a consequence of Proposition 2.4. In fact this proposition implies that there is a diagram \((7.5)\) with nearly all the required properties: If we start from the small commutative square in the lower left corner, the only thing that is not clear to us is why one can assume that the morphism from \(\sigma_{\leq 0}X\) to \(s(\sigma_{\leq 0}Y)\) is \(\alpha_{\sigma_{\leq 0}(Y)} \circ \sigma_{\leq 0}(f)\).

Remark 7.3. Some manipulations of diagram \((7.5)\) (mind the signs!) show that axiom \((\text{fcat7})\) gives: For any morphism \(f : X \to Y\) and any \(n \in \mathbb{Z}\) there is a \(3 \times 3\)-diagram of the following form:

\[
\begin{array}{ccc}
[-1]s(\sigma_{\geq n+1}(Y)) & \longrightarrow & A' \\
\downarrow & & \downarrow \\
[-1]s(g^n_{\geq 1}) & \longrightarrow & \sigma_{\geq n+1}(X) \\
\downarrow & & \downarrow \\
[-1]s(Y) & \longrightarrow & X \\
\downarrow & & \downarrow \\
[-1]s(k^n_Y) & \longrightarrow & \sigma_{\leq n}(X) \\
\downarrow & & \downarrow \\
[-1]s(\sigma_{\leq n}(Y)) & \longrightarrow & B' \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow s(\sigma_{\geq n+1}(Y)) & \longrightarrow & [1]A' \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\end{array}
\]

\[
\begin{array}{ccc}
\sigma_{\geq n+1}(Y) & \longrightarrow & s(\sigma_{\geq n+1}(Y)) \\
\downarrow & & \downarrow \\
\alpha_{\sigma_{\geq n+1}(Y)} \circ \sigma_{\geq n+1}(f) & \longrightarrow & s(Y) \\
\downarrow & & \downarrow \\
\alpha_{Y \circ f} & \longrightarrow & s(Y) \\
\downarrow & & \downarrow \\
k^n_Y & \longrightarrow & s(k^n_Y) \\
\downarrow & & \downarrow \\
\sigma_{\leq n}(X) & \longrightarrow & s(\sigma_{\leq n}(Y)) \\
\downarrow & & \downarrow \\
\alpha_{\sigma_{\leq n}(Y)} \circ \sigma_{\leq n}(f) & \longrightarrow & s(\sigma_{\leq n}(Y)) \\
\downarrow & & \downarrow \\
v^n_X & \longrightarrow & s(v^n_X) \\
\downarrow & & \downarrow \\
\sigma_{\geq n+1}(X) & \longrightarrow & [1]s(\sigma_{\geq n+1}(Y)) \\
\downarrow & & \downarrow \\
\alpha_{\sigma_{\geq n+1}(Y)} \circ \sigma_{\geq n+1}(f) & \longrightarrow & [1]s(\sigma_{\geq n+1}(Y)) \\
\end{array}
\]

7.3. The additional axiom in the basic example. Let \(\mathcal{A}\) be an abelian category and consider the basic example \(DF(\mathcal{A})\) of a filtered triangulated category (as described in Section 6.1 Prop. 6.3).

Lemma 7.4. Axiom \((\text{fcat7})\) is true in \(DF(\mathcal{A})\).

Proof. Let \(f : X \to Y\) be a morphism in \(DF(\mathcal{A})\). We can assume without loss of generality that \(f\) is (the class of) a morphism \(f : X \to Y\)

---

\(^3\)We assume here and in similar situations in the following that \([1][-1] = \text{id}\).
in $CF(A)$. We explain the following diagram:

\[
\begin{align*}
F^n[1]L_X & \to F^n[1]X \\
F^n[1]X & \to F^n[1]X \oplus F^n[2]L_X \\
F^n[2]L_X & \to F^n[2]L_X
\end{align*}
\]

This diagram is the $n$-th filtered part of the $3 \times 3$ diagram we need. For simplicity we will in the rest of this description not distinguish between a morphism and its $n$-th filtered part.

The lowest row is constructed as follows: Let $L_X := X(\geq 1)$ be as defined in (6.2) and let $g_X : L_X \to X$ be the obvious morphism (called $i$ there). The lowest row is the $(n$-th filtered part of) the mapping cone triangle of this morphism. This triangle is isomorphic to the triangle (6.3) and hence a possible choice for the $\sigma$-truncation triangle of $X$, cf. the proof of Proposition 6.3.

The second row from below is the corresponding triangle for $s(Y)$.

The lower right “index” at each object indicates its differential, e.g. the differential of $X$ is $x$.

The morphism of triangles between the lower two rows is constructed as described before (7.4), e.g. $\gamma = \alpha_Y \circ f : X \to s(Y)$.

Then we fit the morphisms $\beta$, $\gamma$ and $[\begin{smallmatrix} \gamma & 0 \\ 0 & \beta \end{smallmatrix}]$ into mapping cone triangles. Then consider the horizontal arrows in the second row from above: They are morphisms of complexes and make all small squares (anti-)commutative as required.

We only need to show that this second row is a triangle. This is a consequence of the following diagram which gives an isomorphism of
this row to the mapping cone triangle of $\begin{bmatrix} g_y & 0 \\ 0 & g_x \end{bmatrix}$.

\[
\begin{array}{cccccccc}
F^{n-1}L_Y \oplus F^n[1]L_X & \xrightarrow{\begin{bmatrix} g_y & 0 \\ 0 & g_x \end{bmatrix}} & F^{n-1}Y \oplus F^n[1]X & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} & F^n-1Y \oplus F^n[1]L_Y & \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} & F^n[2]L_X & \xrightarrow{\begin{bmatrix} y & \gamma \\ 0 & -x \end{bmatrix}} & F^{n-1}[1]L_Y \oplus F^n[2]L_X \\
\begin{bmatrix} y & \beta \\ 0 & -x \end{bmatrix} & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \sim & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \xrightarrow{\begin{bmatrix} -y & -\beta \\ 0 & x \end{bmatrix}} & \begin{bmatrix} -y & -\beta \\ 0 & x \end{bmatrix}
\end{array}
\]

\[\square\]

7.4. Existence of a strong weight complex functor. Let $\mathcal{T}$ be a triangulated category with a bounded weight structure $w = (\mathcal{T}^{w \leq 0}, \mathcal{T}^{w \geq 0})$ and let $(\widetilde{T}, i)$ be an $f$-category over $\mathcal{T}$.

An object $X \in \widetilde{T}$ is by definition in $\widetilde{T}^s$ if and only if $c(X) \in C^b(\varpi(w))$. Using (6.31) we obtain:

\[\begin{aligned}
X \in \widetilde{T}^s & \iff \forall a \in \mathbb{Z}: [a] \text{gr}^a(X) \in \varpi(w) = \mathcal{T}^{w = 0} \\
& \iff \forall a \in \mathbb{Z}: \text{gr}^a(X) = i^{-1}s^{-a}\sigma_a(X) \in [-a]\mathcal{T}^{w = 0} = \mathcal{T}^{w = a} \\
& \iff \forall a \in \mathbb{Z}: \sigma_a(X) \in s^a(i(\mathcal{T}^{w = a})),
\end{aligned}\]

where “$\cong$” stands for “is isomorphic to some object in”.

Remark 7.5. Note the difference between $\widetilde{T}^s$ and the heart (8.3) of the unique compatible $w$-structure on $\widetilde{T}$ described in Proposition 8.3.

Observe that $\widetilde{T}^s$ is not a triangulated subcategory of $\widetilde{T}$: It is not closed under the translation $[1]$.

However $\widetilde{T}^s$ is closed under $[1]s^{-1}$ (use (6.33)) and under extensions: If $(A, X, B)$ is a triangle with $A, B \in \widetilde{T}^s$, apply the triangulated functor $\text{gr}^a$ and obtain the triangle $(\text{gr}^a(A), \text{gr}^a(X), \text{gr}^a(B))$; now use that $\mathcal{T}^{w = a}$ is closed under extensions (Lemma 4.3 (1)).

It is obvious from (7.7) that $\widetilde{T}^s$ is stable under all $\sigma$-truncations.

Lemma 7.6. Let $X \in \widetilde{T}^s([a, b])$ for $a, b \in \mathbb{Z}$. Then $\omega(X) \in \mathcal{T}^{w \in [a, b]}$. 

Proof. We can build up $X$ as indicated in the following diagram in the case $[a, b] = [-2, 1]$:

(7.8) $X = \sigma_{\leq 1}X \rightarrow \sigma_{\leq 0}X \rightarrow \sigma_{\leq -1}X \rightarrow \sigma_{\leq -2}X \rightarrow \sigma_{\leq -3}X = 0$

All triangles are isomorphic to $\sigma$-truncation triangles with the wiggly arrows of degree one. Since $\sigma_n(X) \cong s^n(i(\text{gr}^n(X)))$ Proposition 6.6 (om3) yields $\omega(\sigma_n(X)) \cong \text{gr}^n(X)$. If we apply $\omega$ to diagram (7.8) we obtain a diagram that is isomorphic to

$$
\begin{array}{ccc}
\omega(X) & \rightarrow & \omega(\sigma_{\leq 0}X) \\
\text{gr}^1(X) & \rightarrow & \text{gr}^0(X) \\
\end{array}
$$

Since $\text{gr}^n(X) \in \mathcal{T}^{w=n}$ and all $\mathcal{T}^{w\leq n}$, $\mathcal{T}^{w\geq n}$ are closed under extensions (Lemma 4.3 (1)) we obtain the claim. □

Remark 7.7. The converse statement of Lemma 7.6 is not true in general (but cf. Prop. 7.9): Take a non-zero object $X \in \tilde{T}^s([0])$ (so $X \cong \sigma_0(X)\tilde{\in} i(\mathcal{T}^{w=0})$) and fit the morphism $\alpha_X$ into a triangle

$$
[-1]s(X) \rightarrow E \rightarrow X \xrightarrow{\alpha_X} s(X).
$$

Then $E \in \tilde{T}^s$. Since $\omega(\alpha_X)$ is an isomorphism we have $\omega(E) = 0$ but $\text{range}(E) = [0, 1]$.

In the rest of this section we additionally assume: The weight structure $w = (\mathcal{T}^{w\leq 0}, \mathcal{T}^{w\geq 0})$ is bounded, and $\tilde{T}$ satisfies axiom $[\text{fcat7}]$

Lemma 7.8 ([Bon10, Lemma 8.4.2]). For all $M, N \in \tilde{T}^s$ the map

$$(\alpha_N)_* = (\alpha_N \circ ?) : \text{Hom}_{\tilde{T}}(M, N) \rightarrow \text{Hom}_{\tilde{T}}(M, s(N))$$

is surjective and for all $a > 0$ the map

$$(\alpha_{s^a(N)})_* = (\alpha_{s^a(N)} \circ ?) : \text{Hom}_{\tilde{T}}(M, s^a(N)) \rightarrow \text{Hom}_{\tilde{T}}(M, s^{a+1}(N))$$

is bijective.

Proof. We complete $\alpha_N$ to a triangle

(7.9) $N \xrightarrow{\alpha_N} s(N) \rightarrow Q \rightarrow [1]N$, 
apply $s^a$ and obtain the triangle

$$s^a(N) \xrightarrow{s^a(\alpha_N) = \alpha s^a(N)} s^{a+1}(N) \xrightarrow{s^a(Q)} [1]s^a(N).$$

Applying $\tilde{T}(M, ?)$ to this triangle yields an exact sequence

$$\tilde{T}(M, [-1]s^a(Q)) \rightarrow \tilde{T}(M, s^a(N)) \rightarrow \tilde{T}(M, s^{a+1}(N)) \rightarrow \tilde{T}(M, s^a(Q))$$

Hence we have to prove:

$$\tilde{T}(M, s^a(Q)) = 0 \quad \text{for } a \geq 0,$n
$$\tilde{T}(M, [-1]s^a(Q)) = 0 \quad \text{for } a > 0.$$

This is clearly implied by

$$\tilde{T}(M, [b]s^a(Q)) = 0 \quad \text{for all } a, b \in \mathbb{Z} \text{ with } a + b \geq 0.$$

We claim more generally that

$$\tilde{T}(M', [c]Q) = 0 \quad \text{for all } c \in \mathbb{N} \text{ and all } M' \in \tilde{T}^*;$$

the above special case is obtained by setting $M' := s^{-a}[a]M$ (note that $\tilde{T}^*$ is $[1]s^{-1}$-stable) and $c = a + b$ using $\tilde{T}(M, [b]s^a(Q)) \xrightarrow{s^{-a}[a]} \tilde{T}(M', [c]Q)$.

We claim that we can assume in (7.10) that the support of $M'$ and $N$ (the object $N$ determines $Q$ up to isomorphism) is a singleton (or empty): For $M'$ this is obvious; for $N$ we use axiom [fcat7] for $\text{id}_N : N \rightarrow N$ and obtain the following $3 \times 3$-diagram:

$$\begin{array}{ccc}
[1]s_{d+1}(N) & \xrightarrow{1} & [1]s(s_{d+1}(N)) & \xrightarrow{1} & [1]Q' & \xrightarrow{1} & [2]s_{d+1}(N) \\
\downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} \\
\sigma_{d+1}(N) & \xrightarrow{\alpha_{d+1}} & s(s_{d+1}(N)) & \xrightarrow{\alpha_{d+1}} & Q'' & \xrightarrow{\alpha_{d+1}} & [1]s_{d+1}(N) \\
\downarrow \alpha_N & & \downarrow \alpha_N & & \downarrow \alpha_N & & \downarrow \alpha_N & & \downarrow \alpha_N \\
N & \xrightarrow{\alpha_N} & s(N) & \xrightarrow{\alpha_N} & Q & \xrightarrow{\alpha_N} & [1]N \\
\downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} \\
\sigma_{d+1}(N) & \xrightarrow{\alpha_{d+1}} & s(s_{d+1}(N)) & \xrightarrow{\alpha_{d+1}} & Q' & \xrightarrow{\alpha_{d+1}} & [1]s_{d+1}(N) \\
\downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} & & \downarrow \alpha_{d+1} \\
\sigma_{d+1}(N) & \xrightarrow{\alpha_{d+1}} & s(s_{d+1}(N)) & \xrightarrow{\alpha_{d+1}} & Q' & \xrightarrow{\alpha_{d+1}} & [1]s_{d+1}(N)
\end{array}$$

This shows that knowing (7.10) for $Q'$ and $Q''$ implies (7.10) for $Q$, proving the claim.
Assume now that the support of $M'$ is $[x]$ and that of $N$ is $[y]$ for some $x, y \in \mathbb{Z}$. This means that we can assume (cf. (7.7)) that

$$M' = s^x i(X) \quad \text{for some } X \in \mathcal{T}^{w=x}, \text{ and}$$

$$N = s^y i(Y) \quad \text{for some } Y \in \mathcal{T}^{w=y}.$$  

Since $M'$ is in $\mathcal{T}(\geq x)$, the triangle $(\sigma_{\geq x}(c|Q), [c|Q], \sigma_{\leq x-1}(c|Q))$ and (fcat3) show the first isomorphism in

$$\mathcal{T}(M', \omega M) \overset{\sim}{\longrightarrow} \mathcal{T}(\omega M', \omega(\sigma_{\geq x}(c|Q))),$$

the second isomorphism is a consequence of (om4) and $M' \in \mathcal{T}(\leq x)$. Note that $Q$ and $[c|Q]$ is in $\mathcal{T}([y, y+1])$ by (7.9). In order to show that $\mathcal{T}(\omega M', \omega(\sigma_{\geq x}(c|Q)))$ vanishes, we consider three cases:

1. $x > y + 1$: Then $\sigma_{\geq x}(c|Q) = 0$.
2. $x \leq y$: Then $\sigma_{\geq x}(c|Q) = [c|Q]$. Applying the triangulated functor $\omega$ to (7.9) and using (om3) shows that $\omega(\sigma_{\geq x}(c|Q)) = \omega([c|Q]) = 0$.
3. $x = y + 1$: Applying the triangulated functor $\sigma_{\geq x}$ to (7.9) shows that $\sigma_{\geq x}(c|Q) \cong [c]s(N) = [c]s^{y+1}i(Y) = [c]s^x i(Y)$. Hence we have

$$\mathcal{T}(\omega M', \omega(\sigma_{\geq x}(c|Q))) = \mathcal{T}(\omega(s^x i(X)), \omega([c]s^x i(Y))) \cong \mathcal{T}(X, [c]Y)$$

where we use (om3) ($i(X)$ and $s^x i(X)$ are connected by a sequence of morphisms $\alpha_{s^x i(X)}$, and similarly for $i(Y)$) and the fact that $\omega|_{\mathcal{T}([0])}$ is a quasi-inverse of $i$. Since $X \in \mathcal{T}^{w\geq x} = \mathcal{T}^{w\geq y+1}$ and $[c]Y \in \mathcal{T}^{w\leq y-c} \subset \mathcal{T}^{w\leq y}$ we have $\mathcal{T}(X, [c]Y) = 0$ by (ws3).

**Proposition 7.9.** The restriction $\omega|_{\mathcal{T}_s} : \tilde{\mathcal{T}}_s \to \mathcal{T}$ of $\omega : \tilde{\mathcal{T}} \to \mathcal{T}$ is full (i.e. induces epimorphisms on morphism spaces) and essentially surjective (i.e. surjective on isoclasses of objects).

More precisely $\omega|_{\mathcal{T}_s([a, b])} : \tilde{\mathcal{T}}_s([a, b]) \to \mathcal{T}^{w \in [a, b]}$ (cf. Lemma 7.6) has these properties.

**Proof.** We first prove that $\omega|_{\mathcal{T}_s}$ induces epimorphisms on morphism spaces. Let $M, N \in \tilde{\mathcal{T}}_s$. By (fcat2) we find $m, n \in \mathbb{Z}$ such that $M \in \tilde{\mathcal{T}}(\leq m)$ and $N \in \tilde{\mathcal{T}}(\geq n)$. Choose $a \in \mathbb{Z}$ satisfying $a \geq m - n$.  


We give a pictorial proof: Consider the following diagrams

(7.11) \[
\begin{align*}
\xymatrix{
N & \omega(s^a(N)) \\
\downarrow\downarrow & \downarrow\downarrow \\
\alpha_N & \omega(N)
}
\end{align*}
\]

\[
\xymatrix{
M & s(N) \\
\downarrow f_0 & \downarrow \omega(s(N)) \\
N & \omega(N)
}
\]

\[
\xymatrix{
M & f_0 \\
\downarrow f_1 & \downarrow g_0 \\
\omega(M) & \omega(N)
}
\]

Assume that we are given a morphism \(g_0\). Since \(\omega\) maps every \(\alpha\) to an isomorphism (cf. (om3)), \(g_0\) uniquely determines \(g_1, \ldots, g_a\) such that the diagram on the right commutes.

Since \(M \in \tilde{T}(\leq m)\) and \(s^a(N) \in \tilde{T}(\geq n + a) \subset \tilde{T}(\geq m)\), (om4) implies that there is a unique \(f_a\) satisfying \(\omega(f_a) = g_a\). This \(f_a\) yields \(f_{a-1}, \ldots, f_1\) (uniquely) and \(f_0\) (possibly non-uniquely) such that the diagram on the left commutes (use Lemma 7.8). Taking \(\omega\) of the diagram on the left it is clear that \(\omega(f_0) = g_0\).

Now we prove that \(\omega|_{\tilde{T}_\leq}\) is surjective on isoclasses of objects. Let \(X \in \mathcal{T}\) be given. Since the given weight structure is bounded there are \(a, b \in \mathbb{Z}\) such that \(X \in \mathcal{T}^{w \in [a,b]}\). We prove the statement by induction on \(b - a\). If \(a > b\) then \(X = 0\) and the statement is obvious. Assume \(a = b\). Then \(s^a(i(X))\) is in \(\tilde{T}^s([a])\) by (7.7) and we have \(\omega(s^a(i(X))) \cong \omega(i(X)) \cong X\).

Now assume \(a < b\). Choose \(c \in \mathbb{Z}\) with \(a \leq c < b\) and take a weight decomposition

(7.12) \[
\begin{align*}
w_{\geq c+1}X & \rightarrow X \rightarrow w_{\leq c}X \rightarrow [1]w_{\geq c+1}X.
\end{align*}
\]

Lemma 4.3 (5) shows that \(w_{\leq c}X \in \mathcal{T}^{w \in [a,c]}\) and \(w_{\geq c+1}X \in \mathcal{T}^{w \in [c+1,b]}\). By induction we can hence lift \(w_{\leq c}X \in \mathcal{T}^{w \in [a,c]}\) and \([1]w_{\geq c+1}X \in \mathcal{T}^{w \in [c,b-1]}\) to objects \(\tilde{A} \in \tilde{T}^s([a,c]), \tilde{B} \in \tilde{T}^s([c,b-1])\). This shows that the triangle (7.12) is isomorphic to a triangle

(7.13) \[
[-1]\omega(\tilde{B}) \rightarrow X \rightarrow \omega(\tilde{A}) \xrightarrow{\varphi} \omega(\tilde{B}).
\]

Since we have already proved fullness there exists \(f : \tilde{A} \rightarrow \tilde{B}\) such that \(\omega(f) = g\). We complete the composition \(\tilde{A} \xrightarrow{f} \tilde{B} \xrightarrow{\alpha \circ g} s(\tilde{B})\) to a triangle

\[
[-1]s(\tilde{B}) \rightarrow \tilde{X} \rightarrow \tilde{A} \xrightarrow{\alpha \circ f} s(\tilde{B}).
\]
The image of this triangle under $\omega$ is isomorphic to triangle \((7.13)\). In particular $X \cong \omega(\tilde{X})$. Since $[-1]s(\tilde{B}) \in \tilde{T}^s([c+1, b])$ and $\tilde{A} \in \tilde{T}^s([a, c])$ we have $\tilde{X} \in \tilde{T}^s([a, b])$ since $\tilde{T}^s([a, b])$ is closed under extensions. □

In the following we view $\tilde{T}^s$ as an additive category with translation $[1]s^{-1}$. Proposition 6.6, (om3) and the fact that $\omega$ is triangulated turn $\omega|_{\tilde{T}^s} : \tilde{T}^s \to \mathcal{T}$ into a functor of additive categories with translation.

Recall (for example from [ASS06, A.3.1]) that a two-sided ideal $\mathcal{I}$ in an additive category $\mathcal{A}$ is a subclass of the class of all morphism satisfying some obvious properties. Then the quotient $\mathcal{A}/\mathcal{I}$ has the same objects as $\mathcal{A}$ but morphisms are identified if their difference is in the ideal. Then $\mathcal{A}/\mathcal{I}$ is again an additive category and the obvious quotient functor $\text{can} : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ has an obvious universal property. If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor of additive categories, its kernel $\ker F$ is the two-sided ideal given by

$$(\ker F)(A, A') = \ker(F : \mathcal{A}(A, A') \to \mathcal{B}(FA, FA'))$$

for $A, A' \in \mathcal{A}$.

Define $Q := \tilde{T}^s/(\ker \omega|_{\tilde{T}^s})$ and let $\text{can} : \tilde{T}^s \to Q$ be the quotient functor.

The functor $[1]s^{-1} : \tilde{T}^s \to \tilde{T}^s$ descends to a functor $Q \to Q$ denoted by the same symbol: $(Q, [1]s^{-1})$ is an additive category with translation and $\text{can}$ is a functor of such categories.

**Proposition 7.10.** The functor $\omega|_{\tilde{T}^s}$ factors uniquely to an equivalence

$$\overline{\omega} : (Q, [1]s^{-1}) \xrightarrow{\sim} (\mathcal{T}, [1])$$

of additive categories with translation (as indicated in diagram \((7.3)\)).

**Proof.** This is clear from Proposition 7.9. □

In any additive category with translation we have the notion of candidate triangles and morphisms between them. Let $\Delta_Q$ be the class of all candidate triangles in $Q$ that are isomorphic to the image of a sequence

$$[-1]s(X) \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$$

in $Q$ of a sequence

$$[-1]s(X) \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$$

in $\tilde{T}^s$ such that

$$[-1]s(X) \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\alpha X \circ h} s(X)$$

\footnote{We assume that $[1]s^{-1}[-1]s = \text{id}$.}
Lemma 7.11. \((Q,[1]s^{-1},\Delta_Q)\) is a triangulated category and \(\mathcal{W}: Q \to \mathcal{T}\) is a functor of triangulated categories.

Proof. Since we already know that \(\mathcal{W}\) is an equivalence of additive categories with translation it is sufficient to show that a candidate triangle in \(Q\) is in \(\Delta_Q\) if and only if its image under \(\mathcal{W}\) is a triangle in \(\mathcal{T}\). This is an easy exercise left to the reader. \(\square\)

Remark 7.12. There is another description of \(\Delta_Q\): Let \(\Delta'_Q\) be the class of all candidate triangles in \(Q\) that are isomorphic to the image

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} [1]s^{-1}(X)
\]

in \(Q\) of a sequence

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} [1]s^{-1}(X)
\]

in \(\tilde{T}^s\) such that

\[
s^{-1}(X)\xrightarrow{f \circ \alpha_{s^{-1}(X)}} Y \xrightarrow{g} Z \xrightarrow{h} [1]s^{-1}(X)
\]

is a triangle in \(\tilde{T}\). Lemma 7.11 is also true for the class \(\Delta'_Q\) of triangles with essentially the same proof. This shows that \(\Delta_Q = \Delta'_Q\).

Recall that \(c\) and \(h\) are functors of additive categories with translation if we equip \(\tilde{T}\) (or \(\tilde{T}^s\)) with the translation \([1]s^{-1}\). This is used implicitly in the next proposition.

Proposition 7.13. Let \(m : M \to N\) be a morphism in \(\tilde{T}^s\). Assume that \(\alpha_N \circ m\) appears as the last morphisms in a triangle

\[
[-1]s(N) \xrightarrow{u} Q \xrightarrow{v} M \xrightarrow{\alpha_N \circ m} s(N)
\]

in \(\tilde{T}\). Then \(Q \in \tilde{T}^s\), the functor \(c : \tilde{T}^s \to C^b(\heartsuit(w))\) maps the sequence

\[
[-1]s(N) \xrightarrow{u} Q \xrightarrow{v} M \xrightarrow{m} N
\]

in \(\tilde{T}^s\) to a sequence that is isomorphic to

\[
\Sigma^{-1}(c(N)) \xrightarrow{[0 \ 1]} \Sigma^{-1}(\text{Cone}(c(m))) \xrightarrow{c(M) \ c(m)} c(N)
\]

in \(C^b(\heartsuit(w))\), and the functor \(h : \tilde{T}^s \to K^b(\heartsuit(w))^{\text{anti}}\) maps \((7.15)\) to a triangle in \(K^b(\heartsuit(w))^{\text{anti}}\).

In the particular case that \(m = \text{id}_N : N \to N\) we have \(h(Q) = 0\) in \(K^b(\heartsuit(w))^{\text{anti}}\), i.e. \(\text{id}_{c(Q)}\) is homotopic to zero.
Proof. Since $\mathcal{T}^s$ is stable under $[-1]s$ and closed under extensions, the first three objects in the triangle \((7.14)\) are in $\mathcal{T}^s$.

When computing $c = c'' \circ c'$ in this proof we use the second approach to $c'$ described at the end of Section 6.5.1.

**First case: $m$ is a morphism in $\mathcal{T}^s([a])$:** We start with the case that $m : M \to N$ is in $\mathcal{T}^s([a])$ for some $a \in \mathbb{Z}$. Assume that we are given a triangle

\[(7.17) \quad [-1]s(N) \xrightarrow{u'} L \xrightarrow{\nu'} M \xrightarrow{\alpha_{N\text{om}}} s(N).\]

in $\mathcal{T}$ (cf. \((7.14)\); we write $L$ instead of $Q$ here for notational reasons to become clear later on). Note that range($L$) $\subset [a, a + 1]$ and range($[-1]s(N)$) $\subset [a + 1]$. Let

\[
\sigma_{a+1}(L) \xrightarrow{u''} L \xrightarrow{\nu''} \sigma_a(L) \xrightarrow{d_L^a} [1]\sigma_{a+1}(L).
\]

be the triangle constructed in the second approach to $c'(L)$, cf. \((6.28)\). This triangle is uniquely isomorphic to the triangle \((7.17)\) by an isomorphism extending $\text{id}_L$ (use Prop. \((2.3)\)):

\[
(7.18) \quad \sigma_{a+1}(L) \xrightarrow{u''} L \xrightarrow{\nu''} \sigma_a(L) \xrightarrow{d_L^a} [1]\sigma_{a+1}(L)
\]

\[
\sim \quad \quad \quad \quad \quad \quad \sim \quad \quad \quad \quad \quad \quad \sim \quad \quad \quad \quad \quad \quad \sim [1]p
\]

\[
[-1]s(N) \xrightarrow{u'} L \xrightarrow{\nu'} M \xrightarrow{\alpha_{N\text{om}}} s(N).
\]

This proposition even characterizes $p$ as the unique morphism making the square on the left commutative (and similarly for $p^{-1}$), and $q$ as the unique morphism making the square in the middle commutative.

In order to compute the differential of $c'(L)$ in terms of $m$ and to identify $p$ and $q$ we follow the second approach to $c'$ described in Section 6.5. We apply to the triangle \((7.17)\) the sequence of functors in the left column of the following diagram (cf. \((6.28)\)) and obtain in this

---

5 We assume in this proof without loss of generality that $\sigma_{\geq n}$ (more precisely $g^n : \sigma_{\geq n} \to \text{id}$) is the identity on objects of $\mathcal{T}(\geq n)$, and similarly for $k^n$. This gives for example $\sigma_{[a,a+1]}(L) = L$. 
way the rest of the diagram:

\[(7.19)\]

\[
\begin{array}{ccccccc}
\sigma_{a+1} & \downarrow & \sigma_{a+1}(u') & \downarrow & \sigma_{a+1}(L) & \downarrow & 0 \longrightarrow s(N) \\
\sigma_{[a,a+1]} & \downarrow & u' & \downarrow & \sigma_{a}(L) & \downarrow & \alpha_{N}\circ m \\
\sigma_{a} & \downarrow & v' & \downarrow & \sigma_{a}(v') & \downarrow & M \longrightarrow 0 \\
\tilde{d} & \downarrow & \tilde{d}_{L} & \downarrow & [1] \sigma_{a+1} & \downarrow & [1] s(N) \\

\end{array}
\]

From the above remarks we see that \(\sigma_{a+1}(u') = p^{-1}\) and \(\sigma_{a}(v') = q\) (this can also be seen directly from (7.18) using the adjunctions but we wanted to include the above diagram). If we also use the commutativity of the square on the right in (7.18) we obtain

\[(7.20)\]

\[
([1] \sigma_{a+1}(u'))^{-1} \circ \tilde{d}_{L} \circ (\sigma_{a}(v'))^{-1} = ([1]p) \circ \tilde{d}_{L} \circ q^{-1} = \alpha_{N} \circ m.
\]

We will need precisely this formula later on. It means that (up to unique isomorphisms) the differential \(\tilde{d}_{L}\) is \(\alpha_{N} \circ m\).

Let us finish the proof of the proposition in this special case in order to convince the reader that we got all signs correct.

These results show that the image of the sequence

\[(7.21)\]

\[
[-1]s(N) \longrightarrow L \longrightarrow M \longrightarrow N.
\]

in \(\mathcal{T}^{s}\) under the functor \(c'\) is the sequence

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \sigma_{a}(L) & \longrightarrow & M & \longrightarrow & N \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \tilde{d}_{L} & & M & & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & [1]p^{-1} & & [1] \sigma_{a+1}(L) & & 0 \\
\end{array}
\]

in \(C_{\Delta}^{h}(\mathcal{T})\) where we draw the complexes vertically and only draw their components of degree \(a\) in the upper and of degree \(a+1\) in the lower row (all other components are zero). These sequence is isomorphic by the isomorphism \([id, [1]p], id, id\) to the sequence (use (7.20))

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow M & \longrightarrow & N \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \alpha_{N} \circ m & & \longrightarrow & & \longrightarrow 0 \\
\end{array}
\]
If we now apply the functor \( c'' \) we see that the image of \((7.21)\) under \( c \) is isomorphic to
\[
\begin{array}{ccc}
0 \rightarrow s^{-a}(M) \xrightarrow{id} s^{-a}(M) & \xrightarrow{s^{-a}(m)} & s^{-a}(N) \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
s^{-a}(N) \xrightarrow{id} s^{-a}(N) & \xrightarrow{0} & 0
\end{array}
\]
which is precisely \((7.16)\). If we pass to the homotopy category we obtain a candidate triangle that has the same shape as \((2.6)\) but all morphisms are multiplied by \(-1\): It is a triangle in \( K^b(\mathcal{D}(u))^{\text{anti}} \). This finishes the proof in the case that \( m : M \rightarrow N \) is in \( \mathcal{T}^s([a]) \) for some \( a \in \mathbb{Z} \).

**Observation:** Recall that the first three objects of triangle \((7.14)\) are in \( \mathcal{T}^a \). If we apply the triangulated functor \( \sigma_b \) (for any \( b \in \mathbb{Z} \)), we obtain a triangle with first three terms isomorphic to objects in \( s^b(i(\mathcal{T}^a)) \) by \((7.7)\); since there are only trivial extensions in the heart of a weight structure (see Lemma 4.3), this triangle is isomorphic to the obvious direct sum triangle. We will construct explicit splittings.

**The general case.** Let \( m : M \rightarrow N \) be a morphism in \( \mathcal{T}^a \).

Let \( a \in \mathbb{Z} \) be arbitrary. Axiom \([\text{cact7}]\) (we use the variant \((7.6)\) for the morphism \( \sigma_{\leq a}(m) : X = \sigma_{\leq a}(M) \rightarrow Y = \sigma_{\leq a}(N) \) and \( n = a - 1 \)) gives a \( 3 \times 3 \)-diagram
\[
\begin{array}{ccc}
[-1]s(\sigma_a(N)) & \xrightarrow{u_a'} & L_a & \xrightarrow{v_a'} & \sigma_a(M) & \xrightarrow{\alpha_{\sigma_a(N)} \circ \sigma_a(m)} & s(\sigma_a(N)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[-1]s(g^a) & \xrightarrow{f_a'} & Q_a & \xrightarrow{v_a} & \sigma_{\leq a}(M) & \xrightarrow{\alpha_{\sigma_a(N)} \circ \sigma_{\leq a}(m)} & s(g^a) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[-1]s(k_{N}^{a-1,a}) & \xrightarrow{f_{a-1,a}} & Q_{a-1} & \xrightarrow{v_{a-1}} & \sigma_{\leq a-1}(M) & \xrightarrow{\alpha_{\sigma_{a-1}(N)} \circ \sigma_{\leq a-1}(m)} & s(k_{N}^{a-1,a}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[-1]s(\sigma_{\leq a}(N)) & \xrightarrow{u_{a-1}} & Q_{a-1} & \xrightarrow{v_{a-1}} & \sigma_{\leq a-1}(M) & \xrightarrow{\alpha_{\sigma_a(N)} \circ \sigma_{\leq a-1}(m)} & s(\sigma_{\leq a-1}(N))
\end{array}
\]

where we write \( k_{N}^{a-1,a} \) for the adjunction morphism \( k_{\sigma_{\leq a}}^{a-1,a} \) and \( g^a \) for the adjunction morphisms \( g^a_{\sigma_{\leq a}M} \) and \( g^a_{\sigma_{\leq a}N} \). We fix such a \( 3 \times 3 \)-diagram for any \( a \in \mathbb{Z} \).

By \([\text{cact2}]\) we find \( A \in \mathbb{Z} \) such that \( M, N \in \mathcal{T}(\leq A) \). Then the top and bottom row in diagram \((7.22)\) are zero for \( a > A \), and the two rows in the middle are connected by an isomorphism of triangles. It is
easy to see (first define $f_A$, then $f_{A-1} = f_{A-1,A}f_A$ etc.) that there are morphisms $f_a : Q \to [1]Q_a$ (for any $a \in \mathbb{Z}$) such that $f_{a-1,a}f_a = f_{a-1}$ holds and such that

\[
(7.23) \quad [-1]s(N) \xrightarrow{u} Q \xrightarrow{v} M \xrightarrow{\alpha_{N \circ m}} s(N)
\]

\[
[-1]s(k^a_M) \xrightarrow{f_a} \sigma_a(M) \xrightarrow{k^a_M} s(k^a_N)
\]

\[
[-1]s(\sigma_{\leq a}(N)) \xrightarrow{u_a} \sigma_{\leq a}(M) \xrightarrow{v_a} \sigma_{\leq a}(N)
\]

is a morphism of triangles, where the top triangle is (7.14), the bottom triangle is the second horizontal triangle from above in (7.22) and $k^a_A$ are the adjunction morphisms. Since $k^a_{A-1,A}k^a_A = k^a_{A-1}$ holds (under the usual identifications, cf. (6.7)), the morphisms of triangles (7.23) (for $a \in \mathbb{Z}$) are compatible with the morphisms between the two middle rows of (7.22).

We claim that $\sigma_b(f_a)$ is an isomorphism for all $b \leq a$: Applying the triangulated functor $\sigma_b$ to (7.23) gives a morphism of triangles with two components isomorphisms by (6.8) (and (6.11)); hence it is an isomorphisms and $\sigma_b(f_a)$ is an isomorphism. This shows that any $f_a$ induces an isomorphism between the parts of the complexes $c(Q)$ and $c(Q_a)$ in degrees $\leq a$: Apply $\sigma_b : [1]s_{b+1}$ to $f_a$ for all $b < a$.

Note that the three horizontal triangles in (7.22) are of the type considered in the observation. Applying $\sigma_b$ (any $b \in \mathbb{Z}$) to them yields triangles with vanishing connecting morphisms. Hence we omit these morphisms in the following diagram which is $\sigma_a$ applied to the nine upper left entries of (7.22) (we use again some canonical identifications)

\[
(7.24) \quad 0 \xrightarrow{0} \sigma_a(L_a) \xrightarrow{\sigma_a(v'_a)} \sigma_a(M) \xrightarrow{\sigma_a(v_a)} \sigma_a(Q_a) \xrightarrow{\sigma_a(f'_a)} \sigma_a(M) \xrightarrow{\sigma_a(f_{a-1,a})} \sigma_a(Q_{a-1}) \xrightarrow{0}
\]

and where the dotted arrows are defined by

\[
\varepsilon_a := (\sigma_a(u_{a-1}))^{-1} \circ \sigma_a(f_{a-1,a}),
\]

\[
\delta_a := \sigma_a(f'_a) \circ (\sigma_a(v'_a))^{-1}.
\]
So the four honest triangles with one dotted arrow commute, in particular \( \varepsilon_a \circ \sigma_a(u_a) = \text{id} \) and \( \sigma_a(v_a) \circ \delta_a = \text{id} \). Note that \( \varepsilon_a \circ \delta_a = 0 \) since the middle column in (7.24) is part of a triangle (this comes from axiom (fcat7) used above). Lemma 2.1 gives an explicit splitting of the middle row of (7.24):

\[
\begin{align*}
\sigma_a([-1]s(N)) & \overset{\sigma_a(u_a)}{\longrightarrow} \sigma_a(Q_a) \overset{\sigma_a(v_a)}{\longrightarrow} \sigma_a(M) \\
\sigma_a([-1]s(N)) & \overset{[1]}{\sim} [\varepsilon_a \sigma_a(v_a)] \overset{[0 1]}{\sim} \sigma_a(M)
\end{align*}
\]

and states that \([\sigma_a(u_a) \delta_a]\) is inverse to \([\sigma_a(v_a)]\).

Our aim now is to compute the morphisms which will yield the differential of \(c(Q)\) using this explicit direct sum decomposition.

We explain the following diagram (which is commutative without the dotted arrows).

\[
\begin{align*}
\sigma_{a-1}([-1]s(N)) & \overset{\sigma_{a-1}(u_{a-1})}{\longrightarrow} \sigma_{a-1}(Q_{a-1}) \overset{\sigma_{a-1}(v_{a-1})}{\longrightarrow} \sigma_{a-1}(M) \\
\sigma_{a-1}([-1]s(N)) & \overset{\varepsilon_{a-1}}{\sim} \sigma_{a-1}(f_{a-1,a}) \\
\sigma_{a-1}([-1]s(N)) & \overset{\delta_{a-1}^{-1}}{\sim} \sigma_{a-1}(M) \\
[1] \sigma_a([-1]s(N)) & \overset{[1]\varepsilon_a}{\longrightarrow} [1] \sigma_a(Q_a) \overset{[1]\sigma_a(v_a)}{\longrightarrow} [1] \sigma_a(M)
\end{align*}
\]

The first two rows are (the horizontal mirror image of) \(\sigma_{a-1}\) applied to the two middle rows of (7.22). The last row is \([1]\) applied to the middle row of (7.24). The morphism between second and third row is that from (6.28). Note that we have split the first and third row explicitly before.

This diagram shows that

\[\tilde{\partial}_Q^{a-1} : \sigma_{a-1}(Q) \rightarrow [1]\sigma_a(Q)\]
has the form

\[(7.27) \quad \begin{bmatrix} d_{[-1]s(N)}^{\sigma -1} & \kappa_{\sigma -1}^\alpha \\ 0 & d_{M}^{\sigma -1} \end{bmatrix} \]

if we identify

\[\sigma_{a-1}(Q) \sim_{\sigma_{a-1}(f_{a-1})} \sigma_{a-1}(Q_{a-1}) \sim_{\sigma_{a-1}(f_{a-1})} \sigma_{a-1}([-1]s(N)) \oplus \sigma_{a-1}(M)\]

and

\[\sigma_{a}(Q) \sim_{\sigma_{a}(f_{a})} \sigma_{a}(Q_{a}) \sim_{\sigma_{a}(f_{a})} \sigma_{a}([-1]s(N)) \oplus \sigma_{a}(M)\]

along (7.25), for some morphism

\[\kappa_{\sigma -1}^\alpha : \sigma_{a-1}(M) \rightarrow \sigma_{a}([-1]s(N));\]

which we will determine now; our aim is to prove (7.29) below.

We apply to the commutative diagram

\[L_{a-1} \xrightarrow{f_{a-1}'} Q_{a-1} \xrightarrow{\sigma_{a-1}(f_{a-1})} Q_{a} \xrightarrow{\sigma_{a}} Q\]

the morphism \(d_{a-1} : \sigma_{a-1} \rightarrow \sigma_{a}\) of functors and obtain the middle part of the following commutative diagram (the rest will be explained below):

\[\sigma_{a-1}(M) \xrightarrow{\delta_{a-1}} \sigma_{a-1}(Q_{a-1}) \xrightarrow{\sigma_{a-1}(f_{a-1})} \sigma_{a-1}(Q_{a}) \xrightarrow{\sigma_{a-1}(f_{a})} \sigma_{a-1}(Q)\]

\[\sim \quad \sim \quad \sim \quad \sim \quad \sim \]

The honest triangles with diagonal sides \(\delta_{a-1}\) and \([1] \varepsilon_{a}\) respectively are commutative by (7.24). The lower left square commutes since it is
(up to rotation) $[1] \sigma_a$ applied to the upper left square in (7.22) (after substituting $a - 1$ for $a$ there). Note that $[1] \sigma_a(u'_a - 1)$ is an isomorphism since $\sigma_a(\sigma_{a-1}(M)) = 0$. It is immediate from this diagram that $\kappa^{a-1}$ is the downward vertical composition in the left column of this diagram, i.e.

\[(7.28) \quad \kappa^{a-1} = ([1] \sigma_a(u'_a - 1))^{-1} \circ \tilde{d}^{-1}_{a-1} \circ (\sigma_{a-1}(u'_a))^{-1}.
\]

Recall that the considerations in the first case gave formula (7.20); if we apply them to the morphism $\sigma_{a-1}(m) : \sigma_{a-1}(M) \rightarrow \sigma_{a-1}(N)$ in $\tilde{T}^s([a - 1])$ and the top horizontal triangle in (7.22) (with $a$ replaced by $a - 1$) (which plays the role of (7.17)), this formula describes the right hand side of (7.28) and hence yields

\[(7.29) \quad \kappa^{a-1} = \alpha_{\sigma_{a-1}(N)} \circ \sigma_{a-1}(m).
\]

Let us sum up what we know: We use from now on tacitly the identifications

\[\sigma_a(Q) \sim \sigma_a([-1]s(N)) \oplus \sigma_a(M)\]

given by $\sigma_a(f_a)$ and (7.25). Then the morphism $\tilde{d}_Q^a : \sigma_a(Q) \rightarrow [1] \sigma_{a+1}(Q)$ is given by the matrix

\[\tilde{d}_Q^a = \begin{bmatrix} \tilde{d}^a_{[-1]s(N)} & \alpha_{\sigma_a(N)} \circ \sigma_a(m) \\ 0 & \tilde{d}^a_M \end{bmatrix}.
\]

Shifting accordingly this describes the complex $c'(Q)$ completely. Moreover, the morphisms $c'(u) : c'([-1]s(N)) \rightarrow c'(Q)$ and $c'(v) : c'(Q) \rightarrow c'(M)$ become identified with the inclusion $[\tilde{c}]$ of the first summand and the projection $[\tilde{d}]$ onto the second summand respectively.

Then it is clear that $c(Q) = c''(c'(Q))$ is given by (we assume that $i$ is just the inclusion $\tilde{T}(0) \subset \tilde{T}$):

\[c(Q)^a = c([-1]s(N))^a \oplus c(M)^a,
\]

\[d^a_{c(Q)} = \begin{bmatrix} d^a_{c([-1]s(N))} & c(m)^a \\ 0 & d^a_{c(M)} \end{bmatrix}.
\]

Using the canonical identification $c([-1]s(N)) \cong \Sigma^{-1} c(N)$ we see that $c$ maps (7.15) to (7.16). As before this becomes a triangle in $K^b(\mathcal{O}(w))^{\text{anti}}$, cf. (2.6).

**Corollary 7.14.** For all $M, N \in \tilde{T}^s$,

\[h : \text{Hom}_{\tilde{T}^s}(M, N) \rightarrow \text{Hom}_{K^b(\mathcal{O}(w))^{\text{anti}}}(h(M), h(N))
\]

vanishes on the kernel of $\omega|_{\tilde{T}^s} : \text{Hom}_{\tilde{T}^s}(M, N) \rightarrow \text{Hom}_{T}(\omega(M), \omega(N))$. \qed

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Proof. (We argue similarly as in the proof of Proposition 7.9.) Let $M, N \in \tilde{T}$. By (fcat2) we find $m, n \in \mathbb{Z}$ such that $M \in \tilde{T}((\leq m))$ and $N \in \tilde{T}((\geq n))$. Choose $a \in \mathbb{Z}$ satisfying $a \geq m - n$. Consider the following commutative diagram where the epimorphisms and isomorphisms in the upper row come from Lemma 7.8, the isomorphisms in the lower row from Proposition 6.6 (om3), and the vertical morphisms are application of $\omega$; the vertical morphism on the right is an isomorphism by Proposition 6.6 (om4):

$$
\begin{array}{ccc}
\tilde{T}(M, N) & \xrightarrow{\alpha_N \circ ?} & \tilde{T}(M, s(N)) \\
\downarrow{\omega} & & \downarrow{\omega} \\
\mathcal{T}(\omega(M), \omega(N)) & \sim & \mathcal{T}(\omega(M), \omega(s(N)))
\end{array}
$$

This diagram shows that the kernel of the vertical arrow $\omega$ on the left coincides with the kernel of the horizontal arrow $(\alpha_N \circ ?)$. Let $f : M \to N$ be a morphism in $\tilde{T}$ and assume that $\omega(f) = 0$. Since $\alpha_N \circ f = 0$ the morphism $f$ factors to $f'$ as indicated in the following commutative diagram

$$
\begin{array}{ccc}
& M & \xrightarrow{f'} & \tilde{T}(M, N) \\
[-1]s(N) & \xrightarrow{u} & Q & \xrightarrow{v} & N & \xrightarrow{\alpha_N} & s(N),
\end{array}
$$

where the lower horizontal row is a completion of $\alpha_N$ into a triangle. Proposition 7.13 shows that $h(Q) = 0$ and hence $h(f) = 0$. □

**Corollary 7.15.** As indicated in diagram (7.3), the functor $h$ of additive categories with translation factors uniquely to a triangulated functor

$$
\overline{h} : (Q, [1]s^{-1}, \Delta_Q) \to K^b(\nabla(w))^{anti}.
$$

**Proof.** Corollary 7.14 shows that $h$ factors uniquely to a functor $\overline{h}$ of additive categories with translation. Proposition 7.13 together with the description of the class of triangles $\Delta_Q$ before Lemma 7.11 show that $\overline{h}$ is a triangulated functor. □

**Proof of Thm. 7.7.** We first use Proposition 7.9. For any object $X \in \mathcal{T}$ we fix an object $\tilde{X} \in \tilde{T}$ and an isomorphism $X \cong \omega(\tilde{X})$. Then we fix for any morphism $f : X \to Y$ in $\mathcal{T}$ a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ in $\tilde{T}$ such that $f$ corresponds to $\omega(\tilde{f})$ under the isomorphisms $X \cong \omega(\tilde{X})$ and $Y \cong \omega(\tilde{Y})$. Mapping $X$ to $\tilde{X}$ and $f$ to the class of $\tilde{f}$ in $Q$ defines a quasi-inverse $\overline{\omega}^{-1}$ to $\overline{\omega}$ (and any quasi-inverse is of this form). We
claim that \( \widetilde{WC} := \widetilde{h} \circ \omega^{-1} \) (cf. diagram (7.3)) is a strong weight complex functor. Lemma 7.11 and Corollary 7.15 show that it is a triangulated functor. We have to show that its composition with the canonical functor \( K(\triangledown(w))^{\text{anti}} \to K_{\text{weak}}(\triangledown(w)) \) is isomorphic to a weak weight complex functor.

Observe that the constructions of the weak weight complex functor (see Section 5) and of \( c' \) (see Section 6.5.1) are almost parallel under \( \omega \): Lemma 7.6 shows that \( \omega \) maps \( \sigma \)-truncation triangles of objects \( \widetilde{X} \in \widetilde{T}^s \) to weight decompositions. This means that we can take the image \( \omega(S^n_{\widetilde{X}}) \) of \( S^n_{\widetilde{X}} \) (see (6.20)) under \( \omega \) as a preferred choice for the triangle \( T^n_{\widetilde{X}} \) (see (5.1)); more precisely we have to replace \( \omega(\widetilde{X}) \) in \( \omega(S^n_{\widetilde{X}}) \) by \( X \).

Similarly the octahedron \( \widetilde{O}^n_{\widetilde{X}} \) (see (6.23)) yields \( \omega(\widetilde{O}^n_{\widetilde{X}}) \) as a preferred choice for the octahedron \( O^n_{\widetilde{X}} \) (see (5.3)): We have \( w_n X = \omega(\sigma_n(X)) \). Since the octahedron \( \widetilde{O}^n_{\widetilde{X}} \) is functorial we immediately get preferred choices for the morphisms \( f^n \) in (5.7), namely \( \omega(\sigma_n(f)) \). These preferred choices define the assignment \( X \mapsto WC^{c}(X), f \mapsto WC^{c}(f) \) = \( ([n] \omega(\sigma_n(f)))_{n \in \mathbb{Z}} \). Passing to the weak homotopy category defines a weak weight complex functor \( WC : \mathcal{T} \to K_{\text{weak}}(\triangledown(w)) \) (see Thm. 5.2).

Let \( \text{can} : C^b(\triangledown) \to K^b(\triangledown(w))^{\text{anti}} \) be the obvious functor. Then we have (using Rem. 6.8)

\[
\text{can} \circ WC_{c} \circ \omega = \text{can} \circ \omega_{C^b} \circ c' \cong \text{can} \circ c'' \circ c' = \text{can} \circ c = h
\]
on objects and morphisms and, since \( h \) is a functor, as functors \( \widetilde{T}^s \to K^b(\triangledown(w))^{\text{anti}} \). This implies that \( \text{can} \circ WC_{c} \cong \overline{h} \circ \omega^{-1} = \widetilde{WC} \). The composition of \( \text{can} \circ WC_{c} \) with \( K^b(\triangledown(w))^{\text{anti}} \to K_{\text{weak}}(\triangledown(w)) \) is the weak weight complex functor \( WC \) from above. Hence \( WC \) is a strong weight complex functor. \( \square \)

**Remark 7.16.** At the beginning of the proof of Theorem 7.1 we chose some objects \( \widetilde{X} \). Let us take some more care there: For \( X = 0 \) choose \( \widetilde{X} = 0 \). Assume \( X \neq 0 \). Then let \( a, b \in \mathbb{Z} \) be such that \( X \in \mathcal{T}^{w \in [a,b]} \) and \( b - a \) is minimal. Then Proposition 7.9 allows us to find an object \( \widetilde{X} \in \widetilde{T}^s([a,b]) \) and an isomorphism \( X \cong \omega(\widetilde{X}) \). Taking these choices and proceeding as in the above proof it is obvious that \( \widetilde{WC} : \mathcal{T} \to K^b(\triangledown(w))^{\text{anti}} \) maps \( \mathcal{T}^{w \in [a,b]} \) to \( K^{[a,b]}(\triangledown(w)) \).

8. **Lifting weight structures to f-categories**

We show some statements about compatible weight structures. For the corresponding and motivating results for t-structures see [Bei87, Prop. A 5 a].
Definition 8.1. Let $(\widetilde{T}, i)$ be an f-category over a triangulated category $T$. Assume that both $\widetilde{T}$ and $T$ are weight categories (see Def. 4.1), i.e., they are equipped with weight structures. Then these weight structures are compatible if:

1. $i : T \to \widetilde{T}$ is w-exact and
2. $s(\widetilde{T}^{w \leq 0}) = \widetilde{T}^{w \leq 1}$.

Note that the at first sight asymmetric condition implies its counterpart

3. $s(\widetilde{T}^{w \leq 0}) = \widetilde{T}^{w \leq 1}$,

as we prove in Remark 8.2 below.

Remark 8.2. Assume that $\widetilde{T}$ together with $i : T \to \widetilde{T}$ is an f-category over the triangulated category $T$, and that both $\widetilde{T}$ and $T$ are equipped with compatible w-structures. Then (4.3) and (wcomp-ft2) show that $\widetilde{T}^{w \geq 2} = \perp(\widetilde{T}^{w \leq 0}) = \perp(s(\widetilde{T}^{w \leq 0})) = s(\widetilde{T}^{w \geq 1})$.

The statement of the following proposition appears independently in [AK11, Prop. 3.4 (1)] where the proof is essentially left to the reader.

Proposition 8.3. Let $(\widetilde{T}, i)$ be an f-category over a triangulated category $T$. Given a w-structure $w = (T^{w \leq 0}, T^{w \geq 0})$ on $T$, there is a unique w-structure $\widetilde{w} = (\widetilde{T}^{w \leq 0}, \widetilde{T}^{w \geq 0})$ on $\widetilde{T}$ that is compatible with the given w-structure on $T$. It is given by

\[
\begin{align*}
\widetilde{T}^{w \leq 0} &= \{ X \in \widetilde{T} \mid \text{gr}^j(X) \in T^{w \leq -j} \text{ for all } j \in \mathbb{Z} \}, \\
\widetilde{T}^{w \geq 0} &= \{ X \in \widetilde{T} \mid \text{gr}^j(X) \in T^{w \geq -j} \text{ for all } j \in \mathbb{Z} \}.
\end{align*}
\]

From (8.1) we get

\[
\begin{align*}
\widetilde{T}^{w \leq n} &:= [-n, \widetilde{T}^{w \leq 0}] = \{ X \in \widetilde{T} \mid \text{gr}^j(X) \in T^{w \leq n-j} \text{ for all } j \in \mathbb{Z} \}, \\
\widetilde{T}^{w \geq n} &:= [-n, \widetilde{T}^{w \geq 0}] = \{ X \in \widetilde{T} \mid \text{gr}^j(X) \in T^{w \geq n-j} \text{ for all } j \in \mathbb{Z} \}.
\end{align*}
\]

The heart of the w-structure of the proposition is

\[
\bigtriangleup(\widetilde{w}) = \widetilde{T}^{w=0} = \{ X \in \widetilde{T} \mid \text{gr}^j(X) \in T^{w=-j} \text{ for all } j \in \mathbb{Z} \}.
\]

\[\text{Condition } \text{(wcomp-ft1)} \text{ is natural whereas } \text{(wcomp-ft2)} \text{ is perhaps a priori not clear. Here is a (partial) justification. Take an object } 0 \neq X \in \bigtriangleup(w) \text{ (where } w \text{ is the w-structure on } T \text{) and denote } i(X) \text{ also by } X. \text{ Then we have the nonzero morphism } \alpha_X : X \to s(X). \text{ Now } X \in \widetilde{T}^{w \geq 0} \text{ and } s(X) \in s(\widetilde{T}^{w \leq 0}). \text{ So we cannot have } s(\widetilde{T}^{w \leq 0}) = \widetilde{T}^{w \leq -1}.\]
Proof. 

**Uniqueness:** We assume that we already know that (8.1) is a compatible w-structure. Let $X$ be any object in $\tilde{T}$. Then $X$ is in $\tilde{T}([a,b])$ for some $a,b \in \mathbb{Z}$. We can build up $X$ from its graded pieces as indicated in the following diagram in the case $[a,b] = [-2,1]$:

(8.4)

\[
\begin{array}{cccccc}
X &=& \sigma_{\leq 1}X & \longrightarrow & \sigma_{\leq 0}X & \longrightarrow & \sigma_{\leq -1}X & \longrightarrow & \sigma_{\leq -2}X & \longrightarrow & \sigma_{\leq -3}X = 0 \\
& & s(i(\text{gr}^1(X))) & \longrightarrow & i(\text{gr}^0(X)) & \longrightarrow & s^{-1}(i(\text{gr}^{-1}(X))) & \longrightarrow & s^{-2}(i(\text{gr}^{-2}(X))) \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
\]

All triangles are isomorphic to $\sigma$-truncation triangles with the wiggly arrows of degree one.

Assume that $(\mathcal{C}^{w\leq 0}, \mathcal{C}^{w\geq 0})$ is a compatible w-structure on $\tilde{T}$. We claim that

\[
\tilde{T}^{w\leq 0} \subset \mathcal{C}^{w\leq 0} \quad \text{and} \quad \tilde{T}^{w\geq 0} \subset \mathcal{C}^{w\geq 0}
\]

which implies equality of the two w-structures (see Lemma 4.4). Assume that $X$ is in $\tilde{T}^{w\leq 0}$. Then $\text{gr}^j(X) \in \tilde{T}^{w\leq -j}$ implies $i(\text{gr}^j(X)) \in \mathcal{C}^{w\leq -j}$ and we obtain $s^j(i(\text{gr}^j(X))) \in \mathcal{C}^{w\leq 0}$. Now an induction using (8.4) (respectively its obvious generalization to arbitrary $[a,b]$) shows that $X$ is in $\mathcal{C}^{w\leq 0}$. Analogously we show $\tilde{T}^{w\geq 0} \subset \mathcal{C}^{w\geq 0}$.

**Existence:** If (8.1) defines a w-structure compatibility is obvious: (wcomp-ft1) is clear and (wcomp-ft2) follows from $\text{gr}^j \circ s = \text{gr}^{j-1}$ (see (6.12)).

We prove that (8.1) defines a w-structure on $\tilde{T}$.

Condition (ws1) holds: All functors $\text{gr}^j$ are triangulated and in particular additive. Since all $\mathcal{T}^{w\leq j}$ and $\mathcal{T}^{w\geq j}$ are additive categories and closed under retracts in $\mathcal{T}$, $\tilde{T}^{w\leq 0}$ and $\tilde{T}^{w\geq 0}$ are additive categories and closed under retracts in $\tilde{T}$.

Condition (ws2) follows from (8.2).

Condition (ws3): Let $X \in \tilde{T}^{w\geq 1}$ and $Y \in \tilde{T}^{w\leq 0}$. It is obvious from (8.2) that all $\mathcal{T}^{w\geq n}$ and $\mathcal{T}^{w\leq n}$ are stable under all $\sigma$-truncations. Hence we can first $\sigma$-truncate $X$ and reduce to the case that $l(X) \leq 1$ and then similarly reduce to $l(Y) \leq 1$. So it is sufficient to prove $\text{Hom}(X,Y) = 0$ for $X \in \tilde{T}([a])$ and $Y \in \tilde{T}([b])$ for arbitrary $a,b \in \mathbb{Z}$. Let $f : X \to Y$ be a morphism.

- If $b < a$ then $f = 0$ by (fcat3).
- Let $b = a$. Then $f = \sigma_a(f)$ (under the obvious identification) and $\sigma_a(f) \cong s^a(i(\text{gr}^a(f)))$. But $\text{gr}^a(f) : \text{gr}^a(X) \to \text{gr}^a(Y)$ is zero since $\text{gr}^a(X) \in \mathcal{T}^{w\geq 1-a}$ and $\text{gr}^a(Y) \in \mathcal{T}^{w\leq -a}$.
• Let $b > a$. Then $f$ can be factorized as $X \to s^{-(b-a)}(Y) \to Y$ for a unique $f'$ using (fcat6). Then $s^{-(b-a)}(Y) \in s^{-(b-a)}(\mathcal{T}^{w\leq 0}([b])) = \mathcal{T}^{w\leq a-b}([a]) \subset \mathcal{T}^{w\leq 0}([a])$

and the case $b = a$ imply that $f' = 0$ and hence $f = 0$.

Condition (ws4) By induction on $b - a$ we prove the following statement (which is sufficient by (fcat2)): Let $X$ be in $\mathcal{T}([[a, b]])$. Then for each $n \in \mathbb{Z}$ there are triangles

\begin{equation}
(8.5) \quad w_{\geq n+1}X \to X \to w_{\leq n}X \to [1]w_{\geq n+1}X
\end{equation}

with $w_{\geq n+1}X \in \mathcal{T}^{w\geq n+1}([a, b])$ and $w_{\leq n}X \in \mathcal{T}^{w\leq n}([a, b])$ and satisfying

\begin{equation}
(8.6) \quad \omega(w_{\geq n+1}X) \in \mathcal{T}^{w\geq n+1-b},
\omega(w_{\leq n}X) \in \mathcal{T}^{w\leq n-a}.
\end{equation}

For $b - a < 0$ we choose everything to be zero. Assume $b - a = 0$. Then the object $s^{-a}(X)$ is isomorphic to $i(Y)$ for some $Y$ in $\mathcal{C}$. Let $m \in \mathbb{Z}$ and let

$$w_{\geq m+1}Y \to Y \to w_{\leq m}Y \to [1]w_{\geq m+1}Y$$

be a $(w \geq m+1, w \leq m)$-weight decomposition of $Y$ in $\mathcal{C}$ with respect to $w$. Applying the triangulated functor $s^a \circ i$ we obtain (using an isomorphism $s^{-a}(X) \cong i(Y)$) a triangle

\begin{equation}
(8.7) \quad s^a(i(w_{\geq m+1}Y)) \to X \to s^a(i(w_{\leq m}Y)) \to [1]s^a(i(w_{\geq m+1}Y)).
\end{equation}

Since $i(w_{\geq m+1}Y) \in \mathcal{T}^{w\geq m+1}([0])$ we have $s^a(i(w_{\geq m+1}Y)) \in \mathcal{T}^{w\geq m+1+a}([a])$, and similarly $s^a(i(w_{\leq m}Y)) \in \mathcal{T}^{w\leq m+a}([a])$. Take now $m = n - a$ and define the triangle (8.5) to be (8.7). Then (8.6) is satisfied by (om3)

Now assume $b > a$. Choose $a \leq c < b$. In the diagram

\begin{equation}
(8.8) \quad \begin{array}{cccc}
[1]w_{\geq n+1}\sigma_{\geq c+1}X & \to & [1]\sigma_{\geq c+1}X & \to [1]\sigma_{\geq c+1}X \\
\downarrow & & \downarrow & \downarrow \\
\quad w_{\geq n+1}\sigma_{\leq c}X & \to & \quad \sigma_{\leq c}X & \to \quad w_{\leq n}\sigma_{\leq c}X \\
\quad & \quad & \quad & \quad \\
& \quad & \quad & \quad [1]w_{\geq n+1}\sigma_{\leq c}X
\end{array}
\end{equation}

the vertical arrow in the middle is the last arrow in the $\sigma$-truncation triangle $\sigma_{\geq c+1}X \to X \to \sigma_{\leq c}X \to [1]\sigma_{\geq c+1}X$. Using induction the lower row is (8.5) for $\sigma_{\leq c}X$ and the upper row is [1] applied to (8.5) for $\sigma_{\geq c+1}X$ where we also multiply the last arrow by $-1$ as indicated by $\ominus$; then this row is a triangle. In order to construct the indicated completion to a morphism of triangles we claim that

\begin{equation}
(8.9) \quad \text{Hom}([\varepsilon]w_{\geq n+1}\sigma_{\leq c}X, [1]w_{\leq n}\sigma_{\geq c+1}X) = 0 \text{ for } \varepsilon \leq 2.
\end{equation}
Since the left entry is in \( \tilde{T}(\leq c) \) and the right entry is in \( \tilde{T}(\geq c + 1) \), it is sufficient by [om4] to show that
\[
\text{Hom}(\varepsilon, \omega(\sigma_{\geq c+1}(X))), [\omega(\sigma_{\leq c}(X))), [\omega(\sigma_{\geq c+1}(X))]) = 0 \text{ for } \varepsilon \leq 2.
\]
But this is true by axioms [ws3] since the left entry is in \( \mathcal{T}^{w_{n+1-c-c} \leq c} \subset \mathcal{T}^{w_{n-c-1} \leq c} \) and the right entry is in \( \mathcal{T}^{w_{n-c} \leq (c+1)-1} = \mathcal{T}^{w_{n-c} \leq c} \) by [8.6]. Hence claim [8.9] is proved.

This claim for \( \varepsilon \in \{0,1\} \) and Proposition 2.3 show that we can complete the arrow in (8.8) uniquely to a morphism of triangles as indicated by the squiggly arrows.

Now multiply the last morphism in the first row of (8.8) by \(-1\); this makes the square on the right anti-commutative. Proposition 2.3 and the uniqueness of the squiggly arrows show that this modified diagram fits as the first two rows into a \( 3 \times 3 \)-diagram

\[
\begin{array}{ccc}
[1]w_{n+1}\sigma_{c+1}X & \rightarrow & [1]\sigma_{c+1}X \\
\downarrow & & \downarrow \\
[1]w_{n}\sigma_{c+1}X & \rightarrow & [2]w_{n+1}\sigma_{c+1}X
\end{array}
\]

\[
\begin{array}{ccc}
w_{n+1}\sigma_{c}X & \rightarrow & \sigma_{c}X \\
\downarrow & & \downarrow \\
w_{n}\sigma_{c}X & \rightarrow & [1]w_{n+1}\sigma_{c}X
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
A & \rightarrow & X \\
\downarrow & & \downarrow \\
w_{n+1}\sigma_{c+1}X & \rightarrow & \sigma_{c+1}X \\
\downarrow & & \downarrow \\
w_{n}\sigma_{c+1}X & \rightarrow & [1]w_{n+1}\sigma_{c+1}X
\end{array}
\]

where we can assume that the second column is the \( \sigma \)-truncation triangle of \( X \).

We claim that we can define \( w_{n+1}X := A \) and \( w_{n}X := B \) and that we can take the horizontal triangle \((A, X, B)\) as \( \omega(\sigma_{c+1}(X)) \). Applying the triangulated functor \( \text{gr}^{j} \) to the vertical column containing \( A \). For \( j \leq c \) this yields an isomorphism
\[
\text{gr}^{j}A \rightarrow \text{gr}^{j}w_{n+1}\sigma_{c+1}X \in \mathcal{T}^{w_{n+1-c-j}}
\]
and for \( j > c \) we obtain isomorphisms
\[
\text{gr}^{j}A \rightarrow \text{gr}^{j}w_{n+1}\sigma_{c+1}X \in \mathcal{T}^{w_{n+1-c-j}}
\]
This shows \( A \in \mathcal{T}^{w_{n+1}([a, b])} \) where the statement about the range comes from the fact that the first isomorphism is zero for \( j < a \) and the second one is zero for \( j > b \). Furthermore, if we apply \( \omega \) to the column containing \( A \), we obtain a triangle
\[
\omega(w_{n+1}(\sigma_{c+1}(X))) \rightarrow \omega(A) \rightarrow \omega(w_{n+1}(\sigma_{c}(X))) \rightarrow [1]\omega(w_{n+1}(\sigma_{c+1}(X)))
\]
in which the first term is in $T_{w \geq n+1-b}$ and the third term is in $T_{w \geq n+1-c} \subset T_{w \geq n+1-b}$. Hence $\omega(A) = \omega(w_{\geq n+1}X)$ is in $T_{w \geq n+1-b}$ by Lemma 4.3. Similarly we treat $B$. □

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