Three-dimensional braids and their descriptions

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Abstract
The notion of a braid is generalized into two and three dimensions.
Two-dimensional braids are described by braid monodromies or graphics
called charts. In this paper we introduce the notion of curtains, and
show that three-dimensional braids are described by braid monodromies
or curtains.

1 Introduction
Throughout this paper, we work in the PL category ([8, 15]) and assume that
all manifolds are oriented and m-manifolds embedded in (m + 2)-manifolds are
locally flat. We denote by $D^2$ the 2-disk and by $B^m$ the m-disk. Let $d$ be a
positive integer and $X_d$ a fixed set of $d$ interior points of the 2-disk $D^2$.

For the product space $D^2 \times \Sigma^m$ of $D^2$ and an $m$-manifold $\Sigma^m$, we denote by
$pr_1 : D^2 \times \Sigma^m \to D^2$ the first factor projection, and by $pr_2 : D^2 \times \Sigma^m \to \Sigma^m$
the second factor projection.

First we introduce the notion of a 3-dimensional braid.

**Definition 1** (1) A 3-dimensional braid in $D^2 \times B^3$ (or over $B^3$) of degree
$d$ is a 3-manifold $M$ embedded in $D^2 \times B^3$ such that (i) the restriction map
$pr_2|_M : M \to B^3$ is a simple branched covering map of degree $d$ branched along
a link in $B^3$ and (ii) $\partial M = M \cap \partial(D^2 \times B^3) = X_d \times \partial B^3$.

(2) A 3-dimensional braid in $D^2 \times S^3$ (or over $S^3$) of degree $d$ is a 3-manifold
$M$ embedded in $D^2 \times S^3$ such that (i) the restriction map $pr_2|_M : M \to S^3$ is
a simple branched covering map of degree $d$ branched along a link in $S^3$.

When we refer to a link, it may be the empty set. Refer to [1][2] for simple
branched coverings.

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More generally, we introduce the notion of a braided 3-manifold as follows. Let $\Sigma^3$ be a 3-manifold.

**Definition 2** A *braided 3-manifold* in $D^2 \times \Sigma^3$ (or over $\Sigma^3$) of degree $d$ is a 3-manifold $M$ embedded in $D^2 \times \Sigma^3$ such that the restriction map $pr_2|M : M \to \Sigma^3$ is a simple branched covering map of degree $d$ and $\partial M = M \cap \partial (D^2 \times \Sigma^3) \subset \text{int} D^2 \times \partial \Sigma^3$.

A 3-dimensional braid in $D^2 \times B^3$ is a braided 3-manifold in $D^2 \times B^3$ such that $\partial M = X_d \times \partial B^3$ and the branch set is a link in $B^3$. A 3-dimensional braid in $D^2 \times S^3$ is a braided 3-manifold in $D^2 \times S^3$ such that the branch set is a link in $S^3$.

Since any closed 3-manifold can be presented as a simple branched covering of $S^3$ branched along a link [7, 13], our assumption that the branch set is a link is not so restrictive.

In this paper, we study how to describe 3-dimensional braids. We consider two methods, one is braid monodromies and the other is curtain descriptions. The idea of the curtain description was introduced in [3], and some examples were shown in [3, 4]. However, existence of a curtain for any 3-dimensional braid was not shown. The main purpose of this paper is to show how to construct a curtain.

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2 2-dimensional braids, braid monodromies and charts

Before going to the case of 3-dimension in the next section, we quickly recall the notions of 2-dimensional braids, braid monodromies and charts. For the precise definitions and details, refer to [5, 12]. The reader who is familiar with these notions may skip this section.

Let $\Sigma^2$ be a surface.

**Definition 3** A *braided surface* in $D^2 \times \Sigma^2$ (or over $\Sigma^2$) of degree $d$ is a surface $S$ embedded in $D^2 \times \Sigma^2$ such that the restriction map $pr_2|S : S \to \Sigma^2$ is a simple branched covering map of degree $d$ and $\partial S = S \cap \partial (D^2 \times \Sigma^2) \subset \text{int} D^2 \times \partial \Sigma^2$.

(1) A *2-dimensional braid* in $D^2 \times B^2$ (or over $B^2$) is a braided surface in $D^2 \times B^2$ such that $\partial S = X_d \times \partial B^2$.

(2) A *2-dimensional braid* in $D^2 \times S^2$ (or over $S^2$) is a braided surface in $D^2 \times S^2$. 
**Definition 4** Two 2-dimensional braids $S$ and $S'$ in $D^2 \times B^2$ are said to be equivalent if there is an ambient isotopy $\{h_s : D^2 \times B^2 \to D^2 \times B^2\}_{s \in [0,1]}$ such that

1. $h_0 = \text{id}$ and $h_1(S) = S'$,
2. there is an ambient isotopy $\{h_s : B^2 \to B^2\}_{s \in [0,1]}$ with $h_s \circ pr_2 = pr_2 \circ h_s$ for each $s \in [0,1]$, and
3. for each $s \in [0,1]$, the restriction map of $h_s$ to $D^2 \times \partial B^2$ is the identity map.

Moreover, if $h_s = \text{id} : B^2 \to B^2$ for each $s \in [0,1]$, then we say that $S$ and $S'$ are isomorphic.

We assume that the points of $X_d$ are arranged on a straight line and identify the fundamental group $\pi_1(C_d,X_d)$ of the unordered configuration space $C_d$ of degree $d$ interior points of $D^2$ with base point $X_d$ with Artin’s braid group $B_d$ (cf. [12]).

Let $S$ be a 2-dimensional braid in $D^2 \times B^2$ of degree $d$. Take a point $q_0$ in $\partial B^2$. Let $\Delta(S)$ be the set of branch values of the branched covering $S \to B^2$.

**Definition 5** The braid monodromy of $S$ is the homomorphism

$$\rho_S : \pi_1(B^2 \setminus \Delta(S), q_0) \to \pi_1(C_d, X_d) = B_d$$

sending the homotopy class of a path $\alpha : ([0,1],\{0,1\}) \to (B^2 \setminus \Delta(S), q_0)$ to the braid presented by the path $([0,1],\{0,1\}) \to (C_d, X_d)$ with $t \mapsto pr_1(S \cap pr_2^{-1}(\alpha(t)))$.

**Definition 6** A chart of degree $d$ is a labeled and oriented graph $\Gamma$ in $\Sigma^2$ such that $\Gamma \cap \partial \Sigma^2 = \emptyset$ and that each edge is labeled in $\{1,\ldots,d-1\}$ and each vertex is as in Figure 1. We call a vertex a black vertex, a crossing or a white vertex if the valency of the vertex is 1, 4 or 6, respectively. The arrow at a black vertex in this figure is suppressed since it may either be incoming or outgoing.

When $\partial \Sigma^2 \neq \emptyset$, a chart with external boundary is a chart for which we allow the case $\Gamma \cap \partial \Sigma^2 \neq \emptyset$ such that the intersection $\Gamma \cap \partial \Sigma^2$ consists of degree-1 vertices of $\Gamma$. We call the degree-1 vertices on $\partial \Sigma^2$ the boundary vertices or external boundary vertices of $\Gamma$. (Note that degree-1 vertices in $\text{int} \Sigma^2$ are called black vertices.)

Let $\sigma_i$ $(i = 1,\ldots,d-1)$ be the standard generators of the braid group $B_d$. Then $B_d$ has a group presentation

$$B_d = \langle \sigma_1, \ldots, \sigma_{d-1} | \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad (|i-j| = 1), \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| > 1) \rangle.$$ 

Let $\Gamma$ be a chart in $B^2$ of degree $d$. Take a point $q_0$ in $\partial B^2$. Let $\Delta(\Gamma)$ be the set of black vertices of $\Gamma$. 


For a path $\alpha : ([0, 1], \{0, 1\}) \to (B^2 \setminus \Delta(\Gamma), q_0)$, up to homotopy, we assume that $\alpha$ intersects with $\Gamma$ in general position. We associate with a letter $\sigma^i_k$ the $k$th intersection of $\alpha$ with $\Gamma$, where $i_k$ is the label on the edge of $\Gamma$ and $\epsilon_k$ is $+1$ or $-1$ determined by the orientations of $\alpha$ and the edge. Reading these letters along $\alpha$ we have a word on the standard generators of $B_d$. The word is called the intersection word of $\alpha$ with respect to $\Gamma$.

**Definition 7** The braid monodromy of $\Gamma$ is the homomorphism

$$
\rho_\Gamma : \pi_1(B^2 \setminus \Delta(\Gamma), q_0) \to B_d
$$

sending the homotopy class of a path $\alpha : ([0, 1], \{0, 1\}) \to (B^2 \setminus \Delta(\Gamma), q_0)$ to the braid presented by the intersection word of $\alpha$ with respect to $\Gamma$.

Note that, by the correspondence $i \leftrightarrow \sigma_i \in B_d$, the labels of a chart are assumed to present the standard generators in $B_d$. The left of Figure 2 is an example of a chart of degree 4. For the path $\alpha$ illustrated in Figure 2 as a dotted line, the intersection word is $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_2\sigma_1\sigma_2$, and $\rho_\Gamma([\alpha]) = \sigma_1 \in B_4$.

**Theorem 8 ([10], cf. [12])** For any 2-dimensional braid $S$ in $D^2 \times B^2$, there exists a chart $\Gamma$ with $\rho_S = \rho_\Gamma$. Conversely, for any chart $\Gamma$, there exists a 2-dimensional braid $S$ in $D^2 \times B^2$ with $\rho_S = \rho_\Gamma$. 

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In the situation of this theorem, we call $\Gamma$ a chart description of $S$.

The local moves on charts illustrated in Figure 3 are called chart moves.

**Theorem 9** ([10][11], cf. [6][12][16]) Let $S$ and $S'$ be 2-dimensional braids in $D^2 \times B^3$, and let $\Gamma$ and $\Gamma'$ be chart descriptions of them, respectively.

1. $S$ and $S'$ are isomorphic if and only if $\Delta(\Gamma) = \Delta(\Gamma')$ and $\Gamma$ is related to $\Gamma'$ by a finite sequence of ambient isotopies of $B^2$ and chart moves keeping $\Delta(\Gamma)$ fixed.

2. $S$ and $S'$ are equivalent if and only if $\Gamma$ is related to $\Gamma'$ by a finite sequence of ambient isotopies of $B^2$ and chart moves.

### 3-dimensional braids

First we introduce the notion of equivalence on 3-dimensional braids.

**Definition 10** Two 3-dimensional braids $M$ and $M'$ in $D^2 \times B^3$ are said to be equivalent if there is an ambient isotopy $\{h_s : D^2 \times B^3 \to D^2 \times B^3\}_{s \in [0,1]}$ such that

1. $h_0 = \text{id}$ and $h_1(M) = M'$,

2. there is an ambient isotopy $\{h_s : B^3 \to B^3\}_{s \in [0,1]}$ with $h_s \circ pr_2 = pr_2 \circ h_s$ for each $s \in [0,1]$, and
(3) for each \( s \in [0, 1] \), the restriction map of \( h_s \) to \( D^2 \times \partial B^3 \) is the identity map.

Moreover, if \( h_s = \text{id} : B^3 \to B^3 \) for each \( s \in [0, 1] \), then we say that \( M \) and \( M' \) are isomorphic.

Let \( M \) and \( M' \) be 3-dimensional braids in \( D^2 \times B^3 \) of degree \( d \), and let \( L = \Delta(M) \) and \( L' = \Delta(M') \) be the set of branch values of the branched coverings \( M \to B^3 \) and \( M' \to B^3 \), respectively. If \( M \) and \( M' \) are equivalent, then \( L \) is ambient isotopic to \( L' \) in \( B^3 \). If \( M \) and \( M' \) are isomorphic, then \( L = L' \).

The notion of a braid monodromy is also defined for 3-dimensional braids.

Let \( M \) be a 3-dimensional braid in \( D^2 \times B^3 \) of degree \( d \). Take a point \( q_0 \) in \( \partial B^3 \). Let \( \Delta(M) \) be the set of branch values of the branched covering \( M \to B^3 \).

**Definition 11** The braid monodromy of \( M \) is the homomorphism

\[
\rho_M : \pi_1(B^3 \setminus \Delta(M), q_0) \to \pi_1(C_d, X_d) = B_d
\]

sending the homotopy class of a path \( \alpha : ([0, 1], \{0, 1\}) \to (B^3 \setminus \Delta(M), q_0) \) to the braid presented by the path \( ([0, 1], \{0, 1\}) \to (C_d, X_d) \) with \( t \mapsto \text{pr}_1(M \cap \text{pr}_2^{-1}(\alpha(t))) \).

**Example 12** Let \( L \) be a trefoil in \( B^3 \) as in Figure 4. There is a homomorphism \( \rho : \pi_1(B^3 \setminus L, q_0) \to B_3 \) with \( \rho(x_1) = \sigma_1 \) and \( \rho(x_2) = \sigma_2 \). There is a 3-dimensional braid \( M \) in \( D^2 \times B^3 \) of degree 3 with \( \rho_M = \rho \). We will show how to construct such a 3-dimensional braid \( M \) later.

![Figure 4: A branch set \( L = \Delta(M) \)](image)

Two braid monodromies \( \rho : \pi_1(B^3 \setminus \Delta, q_0) \to B_d \) and \( \rho' : \pi_1(B^3 \setminus \Delta', q_0) \to B_d \) are said to be equivalent if there is a homeomorphism \( h : (B^3, \Delta) \to (B^3, \Delta') \) rel \( \partial B^3 \) such that \( \rho = \rho' \circ h_s \), where \( h_s : \pi_1(B^3 \setminus \Delta, q_0) \to \pi_1(B^3 \setminus \Delta', q_0) \) is the isomorphism induced from \( h \).

**Lemma 13** Let \( M \) and \( M' \) be 3-dimensional braids in \( D^2 \times B^3 \) of degree \( d \).
(1) If $M$ and $M'$ are equivalent, then $\rho_M$ and $\rho_{M'}$ are equivalent.

(2) If $M$ and $M'$ are isomorphic, then $\rho_M = \rho_{M'}$.

Proof. Let $\{h_s : D^2 \times B^3 \to D^2 \times B^3\}_{s \in [0,1]}$ and $\{h_s : B^3 \to B^3\}_{s \in [0,1]}$ be ambient isotopies as in Definition 10. Put $h = h_1 : B^3 \to B^3$, which is a homeomorphism rel $\partial B^3$. Since $h_1(M) = M'$ and $h \circ pr_2|_M = pr_2|_{M'} \circ h_1|_M$, we have $h(\Delta(M)) = \Delta(M')$ and $\rho = \rho' \circ h^*$.

Lemma 14 Let $\rho_M : \pi_1(B^3 \setminus \Delta(M), q_0) \to B_d$ be the braid monodromy of a 3-dimensional braid $M$ of degree $d$. Then, for any meridian element $x$ of $\pi_1(B^3 \setminus \Delta(M), q_0)$, $\rho_M(x)$ is a conjugate of a standard generator of $\sigma_k$ or $\sigma_k^{-1}$.

Proof. Let $D$ be a meridian disk of the link $\Delta(M)$ in $B^3$ such that $x$ is represented by $\beta^{-1} \cdot \partial D \cdot \beta$, where $\beta$ is a path in $B^3 \setminus \Delta(M)$ connecting $q_0$ and a point of $\partial D$. The preimage $(pr_2|_M)^{-1}(D)$ is the disjoint union of $d-1$ 2-disks in $M$ and one of them, say $D'$, is mapped onto $D$ as a 2-fold branched covering. The boundary $\partial D'$ is a $(2,1)$-torus knot in the solid torus $D^2 \times \partial D = pr_2^{-1}(\partial D)$, and $\rho_M(x)$ is a conjugate of a standard generator of $\sigma_k$ or $\sigma_k^{-1}$ (cf. Lemma 16.12 of [12]).

For a chart in $B^2$, an edge both of whose endpoints are black vertices is called a free edge. An oval nest is a free edge together with some concentric simple loops ([12]). See Figure 5; the left is a free edge and the right is an oval nest.

![Figure 5: A free edge and an oval nest](image)

We identify $B^3$ with $B^2 \times [0,1]$. For a subset $X$ in $B^3 = B^2 \times [0,1]$, the motion picture of $X$ means the one-parameter family $\{X_t\}_{t \in [0,1]}$ with $X \cap B^2 \times \{t\} = X_t \times \{t\}$ for $t \in [0,1]$. When $X$ is an oriented 2-simplex, $X_t$ is an oriented interval, a point, or the empty set in $B^2$.

Definition 15 A curtain in $B^3$ of degree $d$ is a 2-complex $C$ in $B^3$ such that $C \cap \partial B^3 = \emptyset$ and that each face is labeled in $\{1, \ldots, d-1\}$ and oriented, and after a slight perturbation, the motion picture $\{C_t\}_{t \in [0,1]}$ of $C$ satisfies the following conditions:
(1) The slices $C_t$ are charts of degree $d$ for all but a finite number of $t$. Let $t_1, \ldots, t_k$ be the exceptional values. And put $t_0 = 0, t_{k+1} = 1$.

(2) For each interval $(t_i, t_i + 1)$ ($i = 0, \ldots, k$), $C_t$ is deformed by an ambient isotopy of $B^2$ rel $\partial B^2$.

(3) For each exceptional value $t_i$ ($i = 1, \ldots, k$), a chart move, insertion of some free edges or deletion of some free edges occurs.

A curtain with external boundary is a curtain for which we allow the case $C \cap \partial B^3 \neq \emptyset$ such that the slices $C_t$ are charts with external boundary. $(C_0$ and $C_1$ may be nonempty charts with external boundary.)

Let $C$ be a curtain (possibly with external boundary) and let $\{C_t\}_{t \in [0,1]}$ be the set of exceptional values $\{t_1, \ldots, t_k\}$. We call the subset $\partial^{\text{int}}(C) := (\bigcup_{t \in [0,1]} \partial^{\text{int}}(C_t)) \cup (\bigcup_{t \in E} \text{ free edges in } C_t \text{ that are inserted or deleted at } t)$ the internal boundary of $C$, and the subset $\partial^{\text{ext}}(C) := \bigcup_{t \in [0,1]} \partial^{\text{ext}}(C_t)$ the external boundary of $C$, where $\partial^{\text{int}}(C_t)$ is the set of black vertices of $C_t$ and $\partial^{\text{ext}}(C_t)$ is the set of boundary vertices of $C_t$. Note that the internal boundary is a link in $B^3$ if $C_0$ and $C_1$ are the empty charts. We also denote the internal boundary $\partial^{\text{int}}(C)$ by $\Delta(C)$.

Let $C$ be a curtain in $B^3$ of degree $d$. Take a point $q_0$ in $\partial B^3$. Let $\Delta(C) = \partial^{\text{int}}(C)$ be the internal boundary of $C$.

For a path $\alpha : ([0,1], \{0,1\}) \to (B^3 \setminus \Delta(C), q_0)$, up to homotopy, we assume that $\alpha$ intersects with $C$ in general position, namely, each intersection is a transverse intersection at an interior point of a face of $C$. We associate with a letter $\sigma_{i_k}^{\epsilon_k}$ the $k$th intersection of $\alpha$ with $C$, where $i_k$ is the label on the face of $C$ and $\epsilon_k$ is +1 or −1 determined by the orientations of $\alpha$ and the face. Reading these letters along $\alpha$ we have a word on the standard generators of $B_d$. The word is called the intersection word of $\alpha$ with respect to $C$.

**Definition 16** The braid monodromy of $C$ is the homomorphism

$$\rho_C : \pi_1(B^3 \setminus \Delta(C), q_0) \to B_d$$

sending the homotopy class of a path $\alpha : ([0,1], \{0,1\}) \to (B^3 \setminus \Delta(C), q_0)$ to the braid presented by the intersection word of $\alpha$ with respect to $C$.

**Theorem 17** Let $L$ be a link in $B^3$. For any homomorphism $\rho : \pi_1(B^3 \setminus L, q_0) \to B_d$ sending each meridian element to a conjugate of $\sigma_k$ or $\sigma_k^{-1}$, there exists a curtain $C$ of degree $d$ such that $\Delta(C) = L$ and $\rho_C = \rho$.

We will prove this theorem in Section 4.

Combining this theorem and Lemma 14, we have the following.

**Theorem 18** For any 3-dimensional braid $M$ in $D^2 \times B^3$ of degree $d$, there exists a curtain $C$ of degree $d$ such that $\Delta(C) = \Delta(M)$ and $\rho_C = \rho_M$.

In the situation of this theorem, we call $C$ a curtain description or a chart description of $M$. 
4 Construction of a curtain

In this section we prove Theorem 17. For simplicity of the argument, we identify $B^3$ with $B^2 \times [-2, 2]$ (not $B^2 \times [0, 1]$) and assume $B^2 = [-1, 1] \times [0, 1] \subset \mathbb{R}^2$. Put $B^- := [-1, 0] \times [0, 1]$ and $B^+ := [0, 1] \times [0, 1]$ so that $B^2 = B^- \cup B^+$. Take the base point $q_0 = (q_0^+, 1/2) \in B^3 = B^2 \times [-2, 2]$ where $q_0^* = (0, 1) \in \partial B^2 \subset \mathbb{R}^2$.

Let $L$ be a link in $B^3$ and $\rho : \pi_1(B^3 \setminus L, q_0) \to B_\kappa$ a homomorphism sending each meridian element to a conjugate of $\sigma_\kappa$ or $\sigma_\kappa^{-1}$.

Assume that $L$ can be presented as a closed braid of degree $n$ for some positive integer $n$. (By the Alexander theorem, there exists such an $n$. Here we do not assume that $n$ is the minimum among such integers.)

Take $n$ real numbers $q_1, \ldots, q_n$ with $0 < q_1 < q_2 < \cdots < q_n < 1$, and let $q_i^- = (-1/2, q_i) \in B^-$ and $q_i^+ = (1/2, q_i) \in B^+$ for $i = 1, \ldots, n$. See Figure 6. (The arcs $a_i^-$ and $a_i^+$ in the figure will be used later.) Let $A_i$ ($i = 1, \ldots, n$) be the straight segment $|q_i^- q_i^+|$.

By an ambient isotopy of $B^3 = B^2 \times [-2, 2]$, deform the link $L$ to a link $L'$ in a braid form of degree $n$ satisfying the following.

(1) $L' \cap B^2 \times \{t\} = \emptyset$ for $t$ with $|t| > 1$.
(2) $L' \cap B^2 \times \{t\} = \bigcup_{i=1}^n A_i \times \{t\}$ for $t$ with $|t| = 1$.
(3) $L' \cap B^2 \times \{t\} = \bigcup_{i=1}^n \{q_i^+, q_i^-\} \times \{t\}$ for $t$ with $1/2 < |t| < 1$.
(4) $L' \cap B^+ \times [0, 1/2]$ is an $n$-braid.
(5) $L' \cap B^+ \times [-1/2, 0]$ is the trivial $n$-braid $\bigcup_{i=1}^n \{q_i^+\} \times [-1/2, 0]$.
(6) $L' \cap B^- \times [-1/2, 1/2]$ is the trivial $n$-braid $\bigcup_{i=1}^n \{q_i^-\} \times [-1/2, 1/2]$.

Let $f : B^3 \to B^3$ be a homeomorphism with $f(L') = L$ and $f|_{\partial B^3} = \text{id}$, and let $\rho' = \rho \circ f_* : \pi_1(B^3 \setminus L', q_0) \to B_\kappa$, where $f_* : \pi_1(B^3 \setminus L', q_0) \to \pi_1(B^3 \setminus L, q_0)$ is the isomorphism induced from $f$. If there exists a curtain $C'$ with $\rho_{C'} = \rho'$, then $C := f(C)$ is a curtain with $\rho_C = \rho$. 

\begin{figure}[h]
\centering
\includegraphics{figure6.png}
\caption{Points $q_i^-$ and $q_i^+$ and arcs $a_i^-$ and $a_i^+$}
\end{figure}
Therefore, without loss of generality, we may assume that \( L \) is in a braid form as \( L' \) stated above. Now we will construct a desired curtain \( C \).

Let \( \{L_t\}_{t \in [-1, 2]} \) be the motion picture of \( L \subset B^2 \times [-2, 2] \). At the level of \( t = 1/2 \), \( L_{1/2} = \cup_{i=1}^n \{q_i^+, q_i^-\} \) and \( \pi_1(B^2 \setminus L_{1/2}, q_0) \) is a free group generated by meridian elements. Consider a Hurwitz arc system \( (a_1^-, a_2^-, \ldots, a_n^-, a_n^+, \ldots, a_2^+, a_1^+) \) for \( L_{1/2} \), where \( a_i^- \) and \( a_i^+ \) are arcs as in Figure 8. Let \( (x_1, x_2, \ldots, x_n, y_n, \ldots, y_2, y_1) \) be the corresponding generating system of \( \pi_1(B^2 \setminus L_{1/2}, q_0) \). Let \( \iota : \pi_1(B^2 \setminus L_{1/2}, q_0) \to \pi_1(B^3 \setminus L, q_0) \) be the homomorphism induced from the inclusion map. Since \( L \cap B^2 \times [1/2, 2] \) is a trivial \( n \)-tangle, we have \( \iota(x_i) = (\iota(y_i))^{-1} \) for each \( i = 1, \ldots, n \). By the assumption of \( \rho \), each \( \rho(\iota(x_i)) \) is a conjugate of \( \sigma_k \) or \( \sigma_k^{-1} \). Therefore, there exists a chart \( \Gamma \) in \( B^2 \) with \( \Delta(\Gamma) = L_{1/2} \) and \( \rho_1 = \rho \circ \iota : \pi_1(B^2 \setminus L_{1/2}, q_0) \to B_d \). Moreover, by the argument in [12], this chart can be taken as a disjoint union of oval nests \( U_1, \ldots, U_n \) such that the free edge of the oval nest \( U_i \) is \( A_i \). (Such a chart is called a ribbon chart ([10], [12]).)

We assume that each oval nest \( U_i \) is symmetric with respect to the \( y \)-axis. We define \( C \cap B^2 \times [1/2, 2] \) and \( C \cap B^2 \times [-2, -1/2] \) as follows. Let \( \{C_t\}_{t \in [-2, 2]} \) denote the motion picture for \( C \).

For \( t \) with \( 1/2 \leq |t| \leq 1 \), define \( C_t = \cup_{i=1}^n U_i \). For \( t \) with \( 1 < |t| \leq 3/2 \), define \( C_t = \cup_{i=1}^n U_i \setminus A_i \). Since \( C_{3/2} = \cup_{i=1}^n U_i \setminus A_i \), consists of simple loops, we can remove the loops one by one by chart moves to obtain the empty chart. By this sequence, we construct \( \{C_t\}_{t \in [3/2, 2]} \) with \( C_{-2} = \emptyset \). Similarly, we construct \( \{C_t\}_{t \in [-3/2, -2]} \) with \( C_{-2} = \emptyset \).

Now we construct \( C \cap B^2 \times [0, 1/2] \). From the configuration of \( L \), there is an ambient isotopy of \( B^2 \) rel \( \partial B^+ \cup B^- \) such that the trace of the \( 2n \) point \( \cup_{i=1}^n \{q_i^+, q_i^-\} \) forms the braid \( L \cap B^2 \times [0, 1/2] \). Using this isotopy, we can construct \( C \cap B^2 \times [0, 1/2] \) such that the motion picture \( \{C_t\}_{t \in [0, 1/2]} \) is a deformation of \( C_{1/2} \) by an ambient isotopy of \( B^2 \) rel \( \partial B^+ \cup B^- \). Then \( C_0 \) is a chart which is ambient isotopic to \( C_{1/2} \) in \( B^2 \) rel \( \partial B^+ \cup B^- \). Note that \( \Delta(C_0) = L_0 = L_{1/2} = \cup_{i=1}^n \{q_i^+, q_i^-\} \).

The braid monodromy \( \rho_{C_0} \) is equal to \( \rho_{C_{-1/2}} \). Therefore the chart \( C_0 \) is related to the chart \( C_{-1/2} \) by a finite sequence of ambient isotopies of \( B^2 \) and chart moves keeping \( L_0 \) fixed. Using this sequence, we can construct a motion picture \( \{C_t\}_{t \in [-1/2, 0]} \) connecting \( C_0 \) and \( C_{-1/2} \).

Now we have constructed a curtain \( C \) in \( B^2 = B^2 \times [-2, 2] \) with \( \Delta(C) = L \). By the construction, \( \rho_{C} = \rho \).

**Example 19** Figure 7 is an example of a curtain constructed by the proof of Theorem 17. Let \( L \) be a trefoil in \( B^3 = B^2 \times [-2, 2] \) as in Figure 4. Consider a homomorphism \( \rho : \pi_1(B^2 \setminus L, q_0) \to B_3 \) with \( \rho(x_1) = \sigma_1 \) and \( \rho(x_2) = \sigma_2 \). Then the curtain \( M \) illustrated in Figure 7 satisfies that \( \Delta(M) = L \) and \( \rho_C = \rho \).

Here is an explanation on the motion picture \( \{C_t\}_{t \in [-2, 2]} \). For \( t \in (1, 2] \), \( C_t \) is the empty chart, see (1) of Figure 7. For \( t \in [1/2, 1] \), \( C_t \) is a chart consisting of two free edges \( A_1 \) and \( A_2 \) as in (2) of the figure. For \( t \in [0, 1/2] \), the motion picture \( \{C_t\} \) is constructed by using an ambient isotopy of \( B^2 \). See (2)–(5). \( C_0 \) is depicted in (5). We prepare the chart \( C_{-1/2} \) as the same as \( C_{1/2} \). This is (10)
of the figure. Since the chart $C_0$ and $C_{-1/2}$ describe the same braid monodromy, there is a finite sequence of chart moves changing $C_0$ to $C_{-1/2}$. This process is depicted from (5) to (10). For $t \in [-1, -1/2]$, $C_t$ is the same with $C_{1/2}$. For $t \in [-2, -1)$, $C_t$ is the empty chart.

![Figure 7: A curtain](image)

**Example 20** Here is another example. Let $L$ be a trefoil in $B^3 = B^2 \times [-2, 2]$ as in Figure 4. Consider a homomorphism $\rho : \pi_1(B^3 \setminus L, q_0) \to B_3$ with $\rho(x_1) = \sigma_2$ and $\rho(x_2) = \sigma_2^{-1} \sigma_1 \sigma_2$. Then the curtain $M$ illustrated in Figure 8 satisfies that $\Delta(C) = L$ and $\rho(C) = \rho$.

Here is an explanation on the motion picture $\{C_t\}_{t \in [-2, 2]}$. For $t \in (1, 3/2)$, $C_t$ is a chart consisting of a simple loop as in (1) of the figure. The motion picture of $C \cap B^2 \times [3/2, 2]$ has a chart move removing the loop of $C_{3/2}$. For $t \in [1/2, 1]$, $C_t$ is a chart consisting of a free edge and an oval nest as in (2) of the figure. For $t \in [0, 1/2]$, the motion picture $\{C_t\}$ is a deformation of $C_{1/2}$ by an ambient isotopy of $B^2$. $C_0$ is depicted in (5) of the figure. We prepare the chart $C_{-1/2}$ as the same as $C_{1/2}$. This is (12) of the figure. Since the chart $C_0$ and $C_{-1/2}$ describe the same braid monodromy, there is a finite sequence of chart moves changing $C_0$ to $C_{-1/2}$. This process is depicted from (5) to (12) in the figure. For $t \in [-1, -1/2]$, $C_t$ is the same with $C_{1/2}$. For $t \in [-3/2, -1)$, $C_t$ is the same as $C_{3/2}$, which is illustrated in (13). Finally, the motion picture for $C \cap B^2 \times [-2, -3/2]$ is the inverse of the motion picture for $C \cap B^2 \times [3/2, 2]$.

5 From a curtain to a 3-dimensional braid

We have seen that a 3-dimensional braid determines the braid monodromy, and the braid monodromy is described by a curtain. In this section, we explain how
to recover a 3-dimensional braid, up to isomorphism, from its curtain description.

Let $C$ be a curtain in $B^3 = B^2 \times [0,1]$ of degree $d$. Let $\{C_t\}_{t \in [0,1]}$ be the motion picture of $C$ and $E = \{t_1, \ldots, t_k\}$ the set of exceptional values. For a regular value $t$, $C_t$ is a chart of degree $d$ describing a 2-dimensional braid $S_t$ in $D^2 \times B^2$. When a chart is deformed by an ambient isotopy of $B^2$, the corresponding 2-dimensional braid is deformed by an ambient isotopy of $D^2 \times B^2$. So we have a one-parameter family of equivalent 2-dimensional braids. For an exceptional value $t_j$ where a chart move occurs, the braid monodromy determined from the chart does not change, and the corresponding 2-dimensional braids are isomorphic. For an exceptional value $t_j$ where free edges are inserted, the corresponding 2-dimensional braid is transformed into a 2-dimensional braid by surgery along 1-handles compatible with the structure of 2-dimensional braids (cf. Chapter 20 of [3].) For an exceptional value $t_j$ where free edges are deleted, the inverse of surgery along 1-handles (or equivalently, surgery along 2-handles) occurs. Considering the trace of these 2-dimensional braids with surgeries along
1-handles and 2-handles, we have a motion picture of a 3-dimensional braids in $D^2 \times B^3 = (D^2 \times B^2) \times [0, 1]$. This is a desired 3-dimensional braid.

**Remark 21** Assume that $S^3 = B^3 \cup S^2 \times [0, 1] \cup B^3$ where $\partial B^3_\pm$ is identified with $S^2 \times \{0\}$ and $\partial B^3_\pm$ is identified with $S^2 \times \{1\}$.

Let $C$ be a curtain in $B^3 = B^2 \times [0, 1]$ of degree $d$. Let $\{C_t\}_{t \in [0, 1]}$ be the motion picture of $C$ and $E = \{t_1, \ldots, t_k\}$ the set of exceptional values. Assume that $B^2$ is contained in the 2-sphere $S^2$ and then $B^3 = B^2 \times [0, 1] \subset S^2 \times [0, 1]$. For a regular value $t$, $C_t$ is regarded as a chart in $S^2$ describing a 2-dimensional braid $S_t$ in $D^2 \times S^2$. By the same argument as above, we have a one-parameter family of 2-dimensional braids in $D^2 \times S^2$ in which surgeries along 1-handles or 2-handles may occur. Since $S_0$ and $S_1$ are described by the empty charts $C_0$ and $C_1$ respectively, they are the trivial 2-dimensional braid $X_d \times S^2$ in $D^2 \times S^2$. The trace of this one-parameter family is a 3-manifold embedded in $D^2 \times (S^2 \times [0, 1])$. Taking the union of this 3-manifold with $X_d \times (B^3_\pm \cup B^3_\pm)$, we have a 3-dimensional braid in $D^2 \times S^3$. We call it a 3-dimensional braid in $D^2 \times S^3$ described by the curtain $C$. Conversely, any 3-dimensional braid in $D^2 \times S^3$ is equivalent to one obtained from a curtain this way.

**Remark 22** Let $M$ be a 3-dimensional braid in $D^2 \times S^3 = D^2 \times (B^3_\pm \cup S^2 \times [0, 1] \cup B^3_\pm)$ and let $\Delta(M)$ be the set of branch values of the branched covering $M \to S^3$. Deforming $M$ up to equivalence, we assume that $M$ is obtained as in Remark 21 from a curtain $C$ in $B^3 = B^2 \times [0, 1]$. Suppose that $\Delta(M)$ is a braid form as in the proof of Theorem 17 (Here we replace the values $t = -2, -1, 0, 1, 2$ in the proof with $t = 0, 1/4, 1/2, 3/4, 1$, respectively.) Put $M_- = (M \cap D^2 \times (S^2 \times [0, 1/2])) \cup X_d \times B^3$ and $M_+ = (M \cap D^2 \times (S^2 \times [1/2, 1])) \cup X_d \times B^3$. Then $M = M_- \cup M_+$. This splitting gives a Heegaard splitting of the 3-manifold $M$.

**Remark 23** In [3, 1] the notion of 3-dimensional braids are extended to immersed 3-manifolds. For immersed 3-dimensional braids, braid monodromies and curtain descriptions can be also considered. Further detailed research on these descriptions is expected.

**Remark 24** In this paper we assume that the branch set is a link in $B^3$. For a definition of a 4-dimensional braid, it seems to be natural to consider the branch set is (i) an embedded surface or (ii) an immersed surface with transverse double points (cf. [3, 14]). An example of a chart description for a 4-dimensional braid is given in [4].

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