Cancellation of long-range forces in Einstein-Maxwell-dilaton system

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Abstract
We examine cancellation of long-range forces in Einstein-Maxwell-Dilatonic system. Several conditions of the equilibrium of two charged masses in general relativity is found by many authors. These conditions are altered by taking account of dilatonic field. Under the new condition, we show cancellation of \(1/r^2\) potential using Feynman diagrams.

1 Introduction
The interaction between two charged massive particles is described by the Newton and the Coulomb potential:
\[
V(r) = -G \frac{M m}{r} + \frac{1}{4\pi} \frac{Q q}{r},
\]
where \(G\) is the Newton constant. If \(Q q = 4\pi G M m\), \(V(r) = 0\) and long-range forces are canceled each other at the classical level. In general relativity, the static exact solution for the equilibrium of two charged masses was found by Majumdar and Papapetrou [1]. The solution requires a condition:
\[
\sqrt{4\pi G M_i} = Q_i.
\]
The static exact solution with dilaton was found by Shiraishi [2]. The solution stands for the case with the balance condition: \((M_i : Q_i : \Sigma_i) = (1 : \sqrt{1 + a^2} : a)\), where \(\Sigma\) is a dilatonic charge. In this work, we examine cancellation of long-range forces in the Einstein-Maxwell-Dilatonic system using Feynman diagrams under the balance condition, \(Q = \sqrt{1 + a^2} M\).

2 Feynman rules for dilaton
We start with the Lagrangian including a dilaton field \(\phi\):
\[
\mathcal{L} = \frac{\sqrt{-g}}{4} (R - e^{-2\phi} F^2 + 2g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi),
\]
where \(4\pi G = 1\), \(R\) is Ricci scalar, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\), \(\nabla_\mu \phi = \partial_\mu \phi\), and \(a\) is the coupling constant between the dilaton field and other fields. In order to extract the interactions coupled with the dilaton field, we use a complex scalar boson \(\varphi\) as a probe. The Lagrangian of complex Klein-Gordon field is [3]
\[
\mathcal{L}_{KG} = \sqrt{-g} \left[ e^{-a\phi} g^{\mu\nu} (D_\mu \varphi)^* D\nu \varphi - m^2 e^{a\phi} \varphi^* \varphi \right],
\]
where \(D_\mu \varphi = \partial_\mu \varphi + i q A_\mu \varphi\). We decompose the metric \(g_{\mu\nu}\) into the flat background field \(\eta_{\mu\nu}\) and the graviton field \(h_{\mu\nu}\):
\[
g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu},
\]
where \(\kappa = \sqrt{32\pi G}\) and \(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). In this decomposition, we read
\[
g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda} h_\lambda{}^\nu - \kappa^3 h^{\mu\lambda\gamma} h_\lambda{}^\gamma h_\gamma{}^\nu + \cdots,
\]
and
\[
\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} = 1 + \frac{\kappa}{2} h + \frac{\kappa^2}{8} (h^2 - 2h^{\mu\nu} h_{\mu\nu}) + \frac{\kappa^3}{48} (h^3 - 6hh^{\mu\nu} h_{\mu\nu} + 8h^{\mu}_\nu h^\lambda_\mu h^\lambda_\nu) + \cdots ,
\]
(6)
where \( h^{\mu\nu} \equiv \eta^{\mu\alpha} h_{\alpha\beta} \eta^{\beta\nu} \) and \( h \equiv \eta^{\mu\nu} h_{\mu\nu} \). Using these expansions, we obtain the Einstein-Hilbert action:
\[
\mathcal{L}_{\text{EH}} = \frac{1}{16\pi G} \sqrt{-g} R - \frac{2}{\kappa^2} \sqrt{-g} R
\]
(7)
\[
= \frac{1}{2} \left( \partial^\mu h^{\mu\lambda} \partial_\mu h_{\nu\lambda} - \frac{1}{2} \partial^\mu h \partial_\mu h \right) + \kappa \left( \frac{1}{2} h^2 \partial^\mu h^\beta_\alpha \partial^\nu h^\alpha_\beta h_{\mu\nu} - \frac{1}{2} h^2 \partial^\beta h^\beta_\nu \partial^\mu h^\alpha_\beta - h^2 h_{\mu\nu} \partial^\mu h^\beta_\nu \right)
\]
where we use the de Donder gauge, \( \partial^\mu h^\beta_\nu = \frac{1}{2} \partial^\nu h \). This expression involves a kinetic term as well as terms for an infinite number of interactions among gravitons. The Lagrangian of Maxwell theory coupled with gravitons becomes
\[
\mathcal{L}_M = -\frac{\sqrt{-g}}{4} F^2 - \frac{\sqrt{-g}}{4} g^\alpha\beta g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}
\]
(9)
\[
= -\frac{1}{4} \eta^{\alpha\beta} \eta^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} - \frac{\kappa}{2} h^{\mu\nu} \left( -\eta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \eta_{\mu\nu} \left( \frac{1}{4} \eta^{\gamma\delta} F_{\gamma\lambda} F_{\delta\sigma} \right) \right) + \kappa^2 \left[ \frac{1}{4} \left( h^2 - 2h^{\mu\nu} h_{\mu\nu} \right) \left( -\frac{1}{4} \eta^{\gamma\delta} F_{\gamma\lambda} F_{\delta\sigma} \right) F_{\alpha\beta} F_{\mu\nu} (hh^{\alpha\mu} \eta^{\beta\nu} - 2h^{\alpha\lambda} h^\lambda_\mu \eta^{\beta\nu} - h^{\alpha\mu} h^{\beta\nu} \right) \right] + \cdots ,
\]
(10)
where we choose the Lorenz gauge, \( \partial^\mu A_\mu = 0 \). We also find the coupling between the massive scalar boson and the dilaton as follows:
\[
\mathcal{L}_{\text{KG}} = \mathcal{L}_0 - \alpha (D_\mu \phi)^* D^\mu \phi - \frac{\kappa}{2} h^{\mu\nu} (\partial_\mu \phi^* \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L}_0) + \frac{\kappa^2}{4} \left[ \frac{1}{4} (h^2 - 2h^{\mu\nu} h_{\mu\nu}) \mathcal{L}_0 + \left( h^\mu_\lambda h^\lambda_\nu - \frac{1}{2} hh^{\mu\nu} \right) \partial_\mu \phi^* \partial_\nu \phi \right] + \cdots ,
\]
(11)
where \( \mathcal{L}_0 \) is the Lagrangian in flat spacetime,
\[
\mathcal{L}_0 = \frac{1}{2} \left( \eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - m^2 \phi^* \phi \right).
\]
(12)

3 Elementary processes : gravitation

Following Ref. [4], we use a classical source of gravitation: \( T^{\mu\nu}(x) = M \delta_0^x \delta^{(3)}(x) \). According to the equation of motion, this source creates an external field \( h^{ext}_{\mu\nu}(k) \):
\[
h^{\mu\nu}(k) = \frac{\kappa M}{4k^2} (\eta^{\mu\nu} - 2\eta_{\mu0} \eta_{\nu0}) 2\pi \delta(k_0) = h^{ext}_{\mu\nu}(k) 2\pi \delta(k_0)
\]
(13)
where \( k = p' - p \), \( k_0 = 0 \) and \( h^{ext}_{\mu\nu}(k) \) is shown in Fig. 1 (a). We call amplitudes including the external field as “semiclassical” amplitudes. The “Newton” potential is evaluated from the semiclassical amplitude including the graviton-scalar boson interaction. Up to \( \mathcal{O}(1/r^2) \), the potential becomes
\[
V(r)_{\text{Newton}} \approx -\frac{GMm}{r} - \frac{3GMp^2}{2mr^2} + \frac{G^2 M^2 m}{2r^4} + \cdots .
\]
(14)
The first term in (14) is recognized as the attractive Newton potential.
4 Elementary process: electromagnetism

We use a static charge as a classical source: \( J^\mu(x) = Q\delta^\mu_0\delta^{(3)}(x) \), as usual. This source creates an external field \( A^\mu_{\text{ext}} \):

\[
A^\mu(k) = \frac{Q}{k^2}\delta_0^\mu 2\pi\delta(k_0) \equiv A^\text{ext\ }^\mu(k)2\pi\delta(k_0),
\]

where \( k = p' - p \), \( k_0 = 0 \) and \( A^\text{ext\ }^\mu \) is shown in Fig. 1 (b). We obtain “Coulomb” potential from the semiclassical amplitude, where several interactions between photons, scalar bosons and gravitons, as well as between photons and scalar bosons, are included. Up to \( \mathcal{O}(1/r^2) \), the potential becomes

\[
V(r)_{\text{Coulomb}} \approx \frac{Qq}{4\pi r} + \frac{GmQ^2}{8\pi r^2} - \frac{GMQq}{4\pi r^2} + \cdots .
\]

Note that the first term in (16) is the Coulomb potential with no dependence on the momentum \( p \).

5 Elementary process: Dilaton

Now, we introduce a dilaton field and employ a dilatonic charge \( \rho_{\Sigma} = \Sigma \delta^{(3)}(x) \), as a classical source. This source creates an external field \( \phi^\text{ext}(k) \), which is shown in Fig. 1 (c):

\[
\phi^\text{ext}(k) = -\frac{aM}{k^2} \equiv -\frac{\Sigma}{k^2}.
\]

The diagrams for the first and second order of interactions are gathered in Fig. 2. These amplitudes include the gravitational, the electrical and the dilatonic sources.

We finally evaluate the semiclassical amplitudes of the scalar boson including the dilatonic force and interactions between gravitons, photons and dilatons. The sum of the contributions of the diagrams gives the additional potential due to the dilatonic interaction. The potential of the dilatonic force can be read as

\[
V(r)_{\text{dilaton}} \approx -\frac{a^2Mm}{4\pi r} + \frac{a^2Mp^2}{8\pi mr^2} + \frac{a^2Q^2m}{2(4\pi)^2r^2} + \frac{a^2M^2m}{(4\pi)^2r^2} - \frac{a^2MQq}{(4\pi)^2r^2} + \frac{a^4M^2m}{2(4\pi)^2r^2}.
\]

6 Cancellation of potential

We consider a static case. If we set \( p = 0 \) and impose balance conditions, \( Q = \sqrt{1 + a^2M} \), \( q = \sqrt{1 + a^2m} \), in (14), (16) and (18), the static potentials are cancelled each other up to \( \mathcal{O}(1/r^2) \):

\[
V_{\text{Newton}} + V_{\text{Coulomb}} + V_{\text{dilaton}} = 0 .
\]

7 Summary and Outlook

We evaluated the potential of long-range forces coupled with the dilaton field from Feynman diagrams. Up to \( \mathcal{O}(1/r^2) \), we showed cancellation of the static potential under the balance condition, \( Q = \sqrt{1 + a^2M} \). In future, we wish to study the following subjects; higher-derivative theories, calculation on the classical curved background which realizes a three-body system, loop corrections as in Ref. [5, 6], higher-dimensional theories, ··· and much more !

Acknowledgements

The authors would like to thank K. Shinoda for comments, and also the organizers of JGRG18.
Figure 1: The external field of graviton, photon and dilaton with momentum $k$ in (a), (b) and (c), respectively.

Figure 2: The diagrams for the first and second order of interactions between gravitons, photons and dilatons. These diagrams include the classical sources shown in Fig. 1.

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