Abstract. We show that the hyperbolization of polyhedra pulls back regular neighborhoods of pl submanifolds. Applying this to the Riemannian version of the hyperbolization due to Ontaneda gives open complete manifolds of pinched negative curvature that are homotopy equivalent to closed smooth manifolds but contain no smooth spines. We also find open complete negatively pinched manifolds that are homotopy equivalent to closed non-smoothable manifolds.

1. Introduction

In [Gro87] Gromov introduced several procedures that turn a simplicial complex $K$ into a polyhedron with piecewise-Euclidean metric of nonpositive curvature. Roughly speaking, every simplex of $K$ is replaced with a hyperbolized simplex of the same dimension, which are then assembled in the combinatorial pattern given by $K$. The procedures were further elaborated and developed in [DJ91, CD95], and in particular, the latter paper introduced the strict hyperbolization that converts a finite simplicial complex $K$ into a finite piecewise-hyperbolic locally $\text{CAT}(-1)$ polyhedron $K$ of the same dimension such that the following holds:

1. The procedure is functorial, i.e., an inclusion $L \subset K$ of a subcomplex induces an isometric embedding $i: L \to K$ onto a locally convex subset. Hence given $p \in L$, the map $i_*$ embeds $\pi_1(L, p)$ onto a quasiconvex subgroup of the hyperbolic group $\pi_1(K, p)$.

2. The link of a simplex $L$ in $K$ is pl homeomorphic to the link of $L$ in $K$. Hence if $L, K$ are pl manifolds, then so are $L, K$, and furthermore, $L$ is locally flat in $K$ if and only if $L$ is locally flat in $K$.

3. There is a continuous map $h: K \to K$ that

   (3a) pulls back rational Pontryagin classes,

   (3b) restricts to a homeomorphism on the one-skeleton, and hence induces a bijection $\pi_0(K) \to \pi_0(K)$.

   (3c) is surjective on homology with coefficients in any commutative ring $R$ with identity. Hence if $K$ is a closed $R$-oriented manifold, then so is $K$, and $h$ is injective on cohomology with coefficients in $R$.

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In particular, if $K$ is a connected closed oriented manifold, then so is $K$. We add to the above list with the following.

**Theorem 1.1.** Let $K$ be a closed PL manifold, and $L$ be a boundaryless (not necessarily locally flat) PL submanifold of $K$. If $h: K \to K$ is a hyperbolization map, and $L = h^{-1}(L)$, then $h$ pulls the regular neighborhood of $L$ in $K$ back to the regular neighborhood of $L$ in $K$.

Even though Theorem 1.1 refers to the strict hyperbolization, it also holds for other hyperbolization procedures, see Section 6. The proof of Theorem 1.1 hinges on the stratified block transversality due to Stone [Sto72], which reduces to the work of Rourke and Sanderson [RS68b] when $\dim(K) - \dim(L) > 2$.

To make sense of Theorem 1.1 let us review properties of regular neighborhoods, which are also known as thickenings; the two terms will be used interchangeably, see Section 2 for details. Basically, thickenings and regular neighborhoods are related in the same way as vector bundles and normal bundles of smooth submanifolds. Recall that a regular neighborhood of a smooth codimension $q$ submanifold $M$ is a linear disk bundle, and isomorphism classes of such bundles are classified by homotopy classes of maps from $M$ to $BO_q$. A similar theory for (compact boundaryless) PL submanifolds $M$ was developed by Rourke and Sanderson for $q \geq 3$ [RS68a], and Cappell and Shaneson for $q = 2$ [CS76], and a unified treatment for any $q \geq 1$ can be found in [McC77]. The only known thickenings of codimension $q = 1$ are the $I$-bundles, and the yet unresolved Schoenflies conjecture predicts that there is no other examples. We refer to $M$ as a spine of the thickening; more generally, any closed manifold that is a deformation retract of a manifold is called a spine. Thickenings can be pullbacked via continuous maps, and concordance classes of codimension $q$ thickenings of $M$ bijectively correspond to the homotopy classes of maps of $M$ into a certain classifying space. Concordant thickenings of $M$ are PL homeomorphic rel $M$, and the converse is true if $q \geq 3$.

Ontaneda [Ont20] established a Riemannian version of strict hyperbolization: for every $\varepsilon > 0$ and a smoothly triangulated closed $k$-manifold $K$ with $k \geq 2$ there is a hyperbolized $k$-simplex as in [CD95] such that the corresponding strict hyperbolization $K$ admits a Riemannian metric $g_{\varepsilon,K}$ with sectional curvature within $[-1 - \varepsilon, -1]$. Here is a smooth version of Theorem 1.1.

**Corollary 1.2.** If $L$ is a boundaryless smooth submanifold of closed smooth manifold $K$ of dimension $k \geq 2$ that is smoothly triangulated with $L$ as a subcomplex, then there is a hyperbolized $k$-simplex and a smooth structure on $K$ such that

(a) $L$ is a smooth submanifold of $K$ whose normal bundle is isomorphic as a linear disk bundle to the pullback via $h$ of the normal bundle to $L$ in $K$,

(b) $K$ admits a Riemannian metric with sectional curvature within $[-1 - \varepsilon, -1]$. 
For example, if $L$ is any smoothly embedded 2-sphere in an oriented 4-manifold $K$ with normal Euler number $e$, then Corollary 1.2 gives a genus $g$ hyperbolic surface $L$ with the same normal Euler number $e$ in the negatively pinched closed Riemannian 4-manifold $K$. Here $g$ depends on a smooth triangulation of $K$ that contains $L$ as a subcomplex, and in particular, $g$ depends on $e$.

Here is a way to produce open complete negatively curved Riemannian manifolds. If $L \to K$ is a PL-embedding of closed manifolds, then the inclusion $L \to K$ is $\pi_1$-injective, and hence the subgroup $\pi_1(L)$ corresponds to a covering space $\overline{K}$ of $K$. Since both $L$ and $K$ are aspherical, $\overline{K}$ deformation retracts onto a (PL-embedded) lift of $L$. If $K$ is smoothable, the pullback of Ontaneda’s negatively pinched Riemannian metric on $K$ to $\overline{K}$ is convex-cocompact, i.e., $\overline{K}$ deformation retracts onto a compact codimension zero locally convex subset whose interior is diffeomorphic to $\overline{K}$, see Remark 6.2. To avoid confusion we stress that $L$ need not be locally convex in Ontaneda’s Riemannian metric on $K$, and in fact, the submanifold $L$ may not even be smoothable, which is the main theme of this paper. By contrast, the strict hyperbolization ensures that $L$ is locally convex in the locally CAT($-1$) metric on $K$.

Previously known open manifolds that are homotopy equivalent to closed manifolds and admit complete negatively pinched metrics are as follows:

- The total space of any vector bundle over a closed negatively curved manifold admits a complete pinched negatively curved metric [And87].

- In [GLT88, Kui88, Kap89, Bel97, GKL01, AGG11] one finds complete locally symmetric metrics of negative curvature that are interiors of codimension 2 thickenings of real hyperbolic manifolds of dimensions $\leq 3$. Some of these have smooth spines, while others have non-locally-flat PL spines.

- If $Y$ is a totally geodesic submanifold in a symmetric space $X$ of negative curvature, and $\Gamma$ is a discrete torsion-free isometry group of $X$ that stabilizes $Y$, then the nearest point projection $X \to Y$ is a $\Gamma$-equivariant vector bundle, and hence $X/\Gamma$ is the total space of a vector bundle over $Y/\Gamma$.

The following theorem exploits the differences between smoothable PL thickenings and linear disk bundles. Here $[q/2]$ is the largest integer that is $\leq q/2$.

**Theorem 1.3.** Let $l, q \in \mathbb{Z}$ such that either $l \geq 2 = q$ or $l \geq 4[q/2] > 0$. Then for any $\varepsilon > 0$ there is a closed Riemannian $(q + l)$-manifold $K$ of sectional curvature in $[-1 - \varepsilon, -1]$, a closed smooth locally CAT($-1$) $l$-manifold $L$ that is PL-embedded into $K$, and a convex-cocompact covering space $\overline{K}$ of $K$ such that

(i) $\overline{K}$ deformation retracts onto a PL-embedded lift of $L \subset K$,

(ii) $\overline{K}$ has no deformation retraction onto a smooth $l$-dimensional submanifold. Moreover, if $l \geq 4$, then any finitely-sheeted cover of $\overline{K}$ deformation retracts onto
a PL-embedded closed \(l\)-manifold, but admits no deformation retraction a smoothly embedded closed \(l\)-manifold.

The conclusion of Theorem 1.3 for \(l \geq 4\) hinges on properties of certain rational characteristic classes which live in cohomology of the classifying space of oriented codimension \(q\) thickenings. The task is to construct thickenings whose characteristic classes satisfy an identity that fails in \(H^*(BSO_q; \mathbb{Q})\). The classes are preserved by the canonical homotopy equivalence between different spines, and also survive under \(h\) (because it is injective on rational cohomology). Hence there is no smooth spine.

If \(l\) is 2 or 3, the above-mentioned characteristic classes vanish, and instead we argue that there are thickenings of \(L = S^2\) and \(L = S^2 \times S^1\) with nontrivial normal invariant, and the same is true for \(L\) because \(h\) is surjective on homology. The canonical homotopy equivalence between between different spines is homotopic to a PL homeomorphism (because \(l \leq 3\)), and therefore has trivial normal invariant. By a result of Cappell and Shaneson [CS76] the two normal invariants have to agree, hence there is no smooth spine.

By Haefliger’s metastable range embedding theorem [Hae61] any homotopy equivalence from a closed smooth \(l\)-manifold to an open smooth \((l + q)\)-manifold is homotopic to a smooth embedding if \(l < 2q - 2\). Thus the dimension bound in Theorem 1.3 is sharp for odd \(q\), and close to being sharp for even \(q\).

Finally, here is a version of Theorem 1.3 with non-smoothable \(L\). The proof does not use Theorem 1.1.

**Theorem 1.4.** If \(\varepsilon > 0\), \(k \in \mathbb{Z}\) and \(l \in 4\mathbb{Z}\) with \(k \geq 2l - 1 \geq 15\), then there is a \(\pi_1\)-injective PL embedding \(L \to K\) of closed manifolds with \(\dim(L) = l\) and \(\dim(K) = k\), and such that

- \(L\) is locally CAT\((-1)\), and not homotopy equivalent to a smooth \(l\)-manifold,
- \(K\) has Riemannian metric of sectional curvature within \([-1 - \varepsilon, -1]\),
- \(K\) has a convex-cocompact covering space \(\overline{K}\) that deformation retracts onto a PL-embedded lift of \(L\).

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**Structure of the paper** Theorem 1.1 and Corollary 1.2 are proved in Section 6 which is independent of the rest of the paper. Section 2 gives background on thickenings whose classifying spaces are discussed in Sections 3 and 5. Theorem 1.3 for \(q > 2\) is proved in Section 4. The case \(q = 2\) is treated in Section 5 where we also establish Theorem 1.4. The three appendices contain some technical lemmas.
Notations and conventions Let $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote positive and nonnegative integers, respectively. Set $I^q = [-1,1]^q$, $I = [-1,1] = I^1$, and $I = [0,1]$. Unless stated otherwise we work in the category of polyhedra and PL maps, and in particular, all manifolds, homeomorphisms, isotopies, and embeddings are PL; see [Sta67, RS82] for PL topology background. A submanifold is a subcomplex homeomorphic to a manifold; thus a submanifold need not be locally flat.

2. Background on thickenings

Notation In this section $M$ is a closed connected $m$-manifold, $q \in \mathbb{Z}_{>0}$, and $V$ is an open $(m+q)$-manifold.

If $M$ is a subpolyhedron of a compact $(m+q)$-manifold $W$ that collapses onto $M$ [RS82, Chapter 3], then $W$ is a codimension $q$ thickening of $M$. Two thickenings of $M$ are equivalent if they are homeomorphic rel $M$. Any thickening $W$ of $M$ is equivalent to the star of $M$ in the second-derived subdivision in (any triangulation of) $W$, and conversely, for any manifold that contains $M$ as a subpolyhedron the star of $M$ in the second-derived subdivision is a thickening of $M$.

Thickenings with locally flat $M$ are precisely the block bundles with fiber $I^q$ [RS68a, Corollary 4.6]. By Zeeman’s unknotting theorem $M$ is locally flat if $q \geq 3$ [RS82].

If a codimension 1 thickening with non-locally flat $M$ exists, it would give a counterexample to the PL Schoenflies conjecture, which is true when $m \leq 2$ and open for $m > 2$. Thus all known codimension one thickenings are block bundles. Also every codimension one thickening is topologically (rather than PL) homeomorphic to a linear $I$-bundle [Bro62, Mic64].

By contrast, thickenings of codimension 2 often contain $M$ as a non-locally-flat submanifold, such as, e.g., a regular neighborhood of embedded 2-sphere in $S^4$ obtained as the suspension a PL knotted circle in $S^3$.

If $q \leq 2$, every block bundle with fiber $I^q$ has a structure of a linear disk bundle, which is unique up to an isomorphism [Wal99, p.127]. Thus block bundles and thickenings differ for $q = 2$, and possibly for $q = 1$.

Two thickenings of $M$ are concordant if they appear as boundaries of a thickening of $M \times I$, see [CS76, p.172] for details. Concordant thickenings are equivalent, i.e., homeomorphic rel $M$. Equivalent thickenings are concordant if $q \geq 3$ [RS68a, Corollary 1.8]. If $q = 2$, it is not easy to decide if two given non-concordant thickenings are equivalent.

As we discuss later, the concordance classes of codimension $q$ thickenings form a classifying space, which was constructed in [RS68a] for block bundles, in [CS76] for codimension 2 thickenings, and a unified construction for all $q \geq 1$ was given in [McC77]. The homotopy type of the classifying space is fairly well-understood, except when $q = 1$ again due to unresolved Schoenflies conjecture. The classifying space for block bundles of codimension $q \leq 2$ is homotopy equivalent to $BO_q$. 
Here is a common way of how thickenings arise. Consider a homotopy equivalence
\( f : M \to V \).

- If \( M, V \) are smooth and \( m \leq 2q - 3 \), then \( f \) is homotopic to a smooth embedding [Hae61], whose normal disk bundle is a thickening.
- If \( q \geq 3 \), then \( f \) is homotopic to an embedding [DV09, Theorem 5.2.1], and in particular, any regular neighborhood of \( f(M) \) is a codimension \( q \) thickening, which we call the normal block bundle of \( f \).
- If \( q = 2 \) and \( V \) is the interior of a compact manifold with boundary, then [CS76, Theorem 6.1] any homotopy equivalence \( f : M \to V \) is homotopic to an embedding if one of the following is true:
  - \( m \) is odd and \( f \) is orientation-true (i.e. \( f \) preserves the first Stiefel-Whitney class of the tangent microbundles \( TM, TW \)),
  - \( m \) is even, \( m \neq 2 \), and \( M \) is simply connected.

If \( q = 2 \), the above restrictions cannot be dropped: There are so-called totally spineless manifolds \( V \), which are interiors of compact manifolds, such that no homotopy equivalence \( M \to V \) is homotopic to an embedding, where \( M = T^2 \) [Mat75], \( M = S^2 \) [LL19], or any \( M \) with even \( m \geq 4 \) and \( H_1(M) \neq 0 \) [CS78]. If we do not assume that \( V \) is the interior of a compact manifold, then more can be said: For every \( m \geq 4 \) there is an open smooth \((m + 2)\)-dimensional manifold \( V \) that is homotopy equivalent to \( S^2 \times S^{m-2} \) but does not deformation retract to any compact subset, and there is a similar example with \( m = 2 \) and \( S^2 \times S^{m-2} \) replaced by \( S^2 \) [Ven98].

If \( W \) is a thickening of \( M \) of codimension \( q \geq 3 \), then the inclusion \( \partial W \to W \) followed by a deformation retraction \( W \to M \) is (homotopy equivalent over \( M \) to) a spherical fibration with fibers homotopy equivalent to \( S^{q-1} \) [RS68a, Corollary 5.9]. If this spherical fibration is orientable, its Thom class generates \( H^q(W, \partial W) \cong \mathbb{Z} \), and we call the generator an orientation of \( W \).

Define an orientation of a codimension 2 thickening \( W \) of \( M \) as the orientation of the spherical fibration \( \partial W' \to M \), where \( W' := W \times I \supset M \times \{0\} = M \). The Künneth formula for \((W, \partial W) \times (I, \partial I)\) gives a natural isomorphism \( H^2(W, \partial W) \cong H^2(W', \partial W') \), hence this definition agrees with [CS76, p.172]. Thus an orientation of a codimension 2 thickening can also be thought of as a generator in \( H^2(W, \partial W) \), cf. [Spi67].

A thickening is orientable if it has an orientation. It is easy to see that a thickening is orientable if and only if the inclusion \( M \to W \) is orientation-true. An equivalence of oriented thickenings is a homeomorphism rel \( M \) that preserves the orientation.

Define the Euler class \( e \) of an oriented codimension \( q \) thickening \( W \) of \( M \) as the image of its orientation under the homomorphism \( H^q(W, \partial W) \to H^q(M) \) induced
by the inclusion. If \( q = 2 \), the definition appears in [CS76, p.178]. If \( q \geq 3 \), then \( e \) is precisely the Euler class of the spherical fibration \( \partial W \to M \).

In fact, one can define the Euler class \( e(f) \) of any continuous map \( f: M \to V \) of oriented manifolds as the image of the fundamental class \([M]\) under the composite

\[
H_n(M) \to H^f_n(V) \cong H^q(V) \to H^q(M)
\]

where the middle map is the Poincaré duality isomorphism between the homology based on locally finite chains and the singular cohomology. (More generally, the definition works if \( f \) is orientation-true and the homology groups in (2.1) have twisted coefficients, which we avoid here).

Here is a sketch on why the above definitions of the Euler class agree when \( f \) is the inclusion and \( V \) is the interior of a thickening: The Thom class can be thought of as the element in \( H^q(V, V \setminus M) \) that is Poincaré dual to the fundamental class in \( H_m(M) \), see [Dol72, Chapter VIII, section 11], and this Poincaré duality isomorphism fits in a commutative square as in [Bre97, Theorem 5.9.3] whose other arrows are the Poincaré duality isomorphism in (2.1) and the inclusion-induced maps \( H_n(M) \to H^f_n(V) \) and \( H^q(V, V \setminus M) \to H^q(V) \).

One can define the total rational Pontryagin class of a continuous map \( f: M \to V \) via \( \pi(f) = \pi(\nu_M \oplus f^*\tau_V) \) by mimicking the Whitney sum formula, where \( \nu_M \) and \( \tau_V \) are the stable normal and tangent microbundles of \( M \), \( V \), respectively. If \( f \) is an embedding, then its stable normal microbundle \( \nu_f \) is \( \nu_M \oplus f^*\tau_V \) [Mil64], and \( \pi(f) \) is the Pontryagin class of \( \nu_f \). It then follows from [RW] that \( \pi(f) = \pi(\nu_M) \cup f^*\pi(TV) \).

If \( f \) is an inclusion, \( e(f) \) and \( \pi(f) \) are called the normal Euler class and the normal rational Pontryagin class of \( M \) in \( V \), respectively.

For the remainder of this section we discuss manifolds that deformation retract to two different compact boundaryless submanifolds, which is the setting of Theorem 1.3.

**Lemma 2.2.** If two thickenings \( W, W' \) of \( M, M' \), respectively, share the same interior \( U \), then there is a fiber homotopy equivalence of the corresponding spherical fibrations that covers the composite of the inclusion \( f': M' \to U \) followed by a deformation retraction \( \pi: U \to M \).

**Proof.** This a minor modification of the same assertion for vector bundles proved in [BKS11, Proposition 4.1]. The only change is to replace smooth tubular neighborhoods by regular neighborhoods, and the point is that \( U \) is the union of a countable family of nested regular neighborhoods of \( M \) obtained by removing small open collars of \( \partial W \) in \( W \), and the same is true for \( M', W' \). \( \square \)

**Proposition 2.3.** Let \( M, M' \) be closed \( m \)-manifolds embedded into an open manifold \( V \) such that there are deformation retractions \( \pi: V \to M \) and \( V \to M' \).
(1) If \( \dim(V) \geq m + 3 \), then there is a fiber homotopy equivalence of spherical fibrations of the normal block bundles to \( M, M' \) that covers \( \pi|_{M'} \).

(2) If \( M, M', V \) are oriented and \( \pi|_{M'}: M' \to M \) orientation-preserving, then \( \pi|_{M'} \) preserves the normal Euler classes to \( M \) and \( M' \) in \( V \).

(3) If \( \pi|_{M'} \) maps \( p(\nu_M) \to p(\nu_{M'}) \), then \( \pi|_{M'} \) preserves the normal rational Pontryagin classes to \( M \) and \( M' \) in \( V \).

Proof. (1) We can assume \( \dim(V) \geq 5 \). If \( R \) is a regular neighborhood of \( M \), then the inclusion \( R \to V \) is a homotopy equivalence. Since \( m \geq 3 \), Stallings engulfing [DV09, Theorem 3.1.3] gives an ambient isotopy that moves \( M \) to a submanifold \( L \subset \text{Int}(R) \). If \( W \) is a regular neighborhood of \( L \) in \( \text{Int}(R) \), then the inclusion \( W \to \text{Int}(R) \) is a homotopy equivalence, and hence \( R \setminus \text{Int}(W) \) is an h-cobordism [Sie69, p.213], so \( \text{Int}(R) \setminus \text{Int}(W) \) is homeomorphic to \( [0,1] \times \partial W \) rel \( \{0\} \times \partial W \). Hence \( R, W \) have homeomorphic interiors. Lemma 2.2 gives desired fiber homotopy equivalence of the associated sphere bundles.

(2) Since \( \pi \) is homotopic to the identity of \( V \), and \( \pi|_{M'} \) is orientation-preserving, the images of the fundamental classes \( [M], [M'] \) in \( \mathcal{H}_n(V) \) are equal, and after taking the Poincare dual and restricting to \( M', M' \) we get Euler classes that are taken to each other by \( \pi|_{M'} \).

(3) Since \( \pi \) is homotopic to the identity of \( V \), the map \( \pi|_{M'} \) takes \( \tau_{V}|_{M} \) to \( \tau_{V}|_{M'} \). By assumption it also takes \( p(\nu_M) \) to \( p(\nu_{M'}) \), and hence sends the normal rational Pontryagin class to \( M \) to the normal rational Pontryagin class to \( M' \). \( \square \)

Remark 2.4. We shall apply (3) when \( M, M' \) are closed locally \( \text{CAT}(-1) \) manifolds, in which case any homotopy equivalence preserves the rational Pontryagin class of the stable normal microbundle. (If \( \dim(M) \geq 5 \) this follows from the solution of the Borel conjecture for \( \text{CAT}(0) \) groups [BL12] and topological invariance of rational Pontryagin classes. If \( \dim(M) = 4 \), one can either apply the argument of the previous sentence to the product of the homotopy equivalence \( M \to M' \) and the identity map of \( S^1 \), or more classically, note that the only relevant class is \( p_1 \), and if \( M \) is oriented, \( p_1(\tau_M) \) is proportional to the signature \( \sigma(M) \) which is an oriented homotopy invariant, and the non-orientable case easily follows by passing to orientable 2-fold covers using that finitely-sheeted covering maps are injective on rational cohomology).

Remark 2.5. In Section 3 we shall define for every odd \( q \geq 3 \) a characteristic class \( o_q \) of oriented spherical fibrations. If in Proposition 2.3(1) the fiber homotopy equivalence preserves orientation of the fiber, then \( \pi|_{M'} \) preserves \( o_q \).

If \( W \) is a thickening of \( M \) and \( \pi: \overline{W} \to W \) is a finitely-sheeted covering map, then \( \overline{W} \) is a thickening of \( \overline{M} := p^{-1}(M) \), e.g. because the \( \pi \)-preimage of the star of \( M \) in a second-derived subdivision of \( W \) is clearly the star of \( \overline{M} \) in a second-derived subdivision of \( \overline{W} \).
Proposition 2.6. The classes \( e, p, o_q \) of \( \mathbb{W} \) are \( \pi^* \)-images of the corresponding classes \( e, p, o_q \) of \( W \).

Proof. If \( W \) is a block bundle over \( M \), then the definition of a pullback in [RS68a] easily implies that the pullback \( \pi^*W \) of \( W \) via \( p: \overline{M} \to M \) is a covering space of \( W \) that extends \( p \), and uniqueness of the covering space shows that \( \mathbb{W} \) and \( \pi^*W \) are equivalent, which proves the claim for \( q \geq 3 \).

The pullback of a codimension \( q < 3 \) thickening is defined in a less explicit way, see [CS76], and here we merely observe that \( e \) and \( p \) pullback via \( \pi \). For the Euler class the claim follows by observing that the Thom class in \( H^q(W, W \setminus M) \) visibly maps to the Thom class in \( H^q(W, W \setminus \overline{M}) \) because it is represented by a \( q \)-disk transverse to \( M \) at a locally flat point. For the Pontryagin class the claim holds because covering maps pullback tangent microbundles. \( \square \)

3. Classifying spaces for block bundles and spherical fibrations

This section reviews some results in [Mil68, RS68a, RS68b, RS68c, CS76, MM79].

Let \( G_q \) be the space \( G_q \) of homotopy self-equivalences of \( S^{q-1} \). Then \( BG_q \) is the classifying space for fibrations with fiber \( S^{q-1} \) [MM79, Chapter 3A].

Let \( B\tilde{P}L_q \) be a classifying space for block bundles with fiber \( I^{q} \), it is a locally finite simplicial complex [RS68a, Section 2].

Taking boundary of a block bundle defines a canonical map of \( B\tilde{P}L_q \) to \( B\tilde{G}_q \), the block version of \( BG_q \) which deformation retracts onto \( BG_q \) [RS68a, Corollaries 5.8-5.9].

If \( q \leq 2 \), the canonical maps \( BO_q \to B\tilde{P}L_q \to B\tilde{G}_q \) are homotopy equivalences [Wal99, p.127], which is why for the rest of this section we assume \( q \geq 3 \).

The set of equivalence classes of thickenings of \( M \) of codimension \( q \geq 3 \) is bijective to the set of homotopy classes of maps from \( M \) into \( B\tilde{P}L_q \) [RS68a, Corollary 4.6].

We wish to enumerate oriented block bundles up to finite ambiguity in terms of their characteristic classes in the same way as an oriented vector bundle is determined up to finite ambiguity by its Euler and Pontryagin classes. (The orientability assumption is to avoid twisted Euler class). To this end we consider the classifying space \( B\tilde{S}PL_q \) for oriented thickenings of codimension \( q \geq 3 \). Its basic properties are folklore, and not treated in the literature, so for completeness we derive them below. Define \( B\tilde{S}PL_q \) as the universal cover of \( B\tilde{P}L_q \); as we show below the covering projection is 2-fold, and coincides with the forget orientation map.

Following [MM79, Chapter 3A] we review the properties of \( SG_q \), the space of degree one homotopy self-equivalences of \( S^{q-1} \), where we assume \( q \geq 3 \):
• The evaluation map $SG_q \to S^{q-1}$ is a fibration whose fiber can be identified with a path-component of $\Omega^{q-1}S^{q-1}$. All components of $\Omega^{q-1}S^{q-1}$ are homotopy equivalent, and for any choice of a basepoint $\pi_i(\Omega^{q-1}S^{q-1}) \cong \pi_{i+q-1}(S^{q-1})$.

• $SG_q$ is the identity component in the space $G_q$ of homotopy self-equivalences of $S^{q-1}$; the space $G_q$ has two path-components.

The space $BSG_q$ is the classifying space for oriented fibrations with fiber $S^{q-1}$. Since $SG_q$ is path-connected, $BSG_q$ is simply-connected. The fiberwise join with the trivial $S^0$ bundle defines the stabilization map $BSG_q \to BSG_{q+1}$ with direct limit $BSG$, which is a simply-connected space with finite homotopy groups.

Similarly, $BG_q$ is the classifying space for fibrations with fiber $S^{q-1}$, and $BG$ is the direct limit of $BSG_q$ under stabilization. The spaces $BG_q$, $BG$ are path-connected, and since $G_q$ has two path-components, the stabilization indices an isomorphism $\pi_1(BG_q) \cong \pi_1(BG) \cong \mathbb{Z}_2$.

Product with $I$ defines the stabilization map $B\tilde{P}L_q \to B\tilde{P}L_{q+1}$ with direct limit $B\tilde{P}L$, which by [RS68c, Corollary 5.5] is homotopy equivalent to $BPL$, the classifying space for stable PL microbundles.

To unclutter notations we sometimes suppress differences between classifying spaces and their homotopy equivalent block versions, e.g., $BG_q$, $B\tilde{P}L$ and $BG_q$, $BPL$, respectively.

The homotopy fiber $PL/O$ of the canonical map $G/O \to G/PL$ is 6-connected [MM79, Remark 4.21] and $G/O$ is simply-connected [MM79, p.43]. Thus $\pi_1(G/PL) = 0$, or equivalently, the pair $(BG, B\tilde{P}L)$ is 2-connected. It follows that $BPL \to BG$ is a $\pi_1$-isomorphism.

For $q \geq 3$ it is proved in [RS68c, Theorem 1.10] that the inclusion

$$(BG_q, B\tilde{P}L_q) \to (BG_{q+1}, B\tilde{P}L_{q+1})$$

is an isomorphism on homotopy groups, and then the previous paragraph gives 2-connectedness of the pair $(BG_q, B\tilde{P}L_q)$ for $q \geq 3$. Hence every map in the square below induces a $\pi_1$-isomorphism

$\begin{align*}
\begin{array}{ccc}
B\tilde{P}L_q & \to & B\tilde{G}_q \\
\downarrow & & \downarrow \\
BPL & \to & BG
\end{array}
\end{align*}$
where all fundamental groups have order two, and therefore, we get the corresponding square of the 2-fold (universal) covers:

\[
\begin{array}{ccc}
BS\tilde{PL}_q & \longrightarrow & BSG_q \\
\downarrow & & \downarrow \\
BS\tilde{PL} & \longrightarrow & BSG
\end{array}
\]

where \(BS\tilde{PL}\) is homotopy equivalent to the classifying space \(BSPL\) of stable oriented PL bundles.

The canonical map \(BSO \to BSPL\) is a rational homotopy equivalence, as a map of simply-connected spaces whose homotopy fiber \(PL/O\) has finite homotopy groups.

A thickening of \(M\) of codimension \(q \geq 3\) is orientable if and only if its classifying map \(f: M \to B\tilde{PL}_q\) postcomposed with \(B\tilde{PL}_q \to B\tilde{G}_q\) lifts to \(BS\tilde{G}_q\). Thus the thickening is orientable if and only if \(f\) lifts to \(BS\tilde{PL}_q\).

For \(q \geq 3\) it is proved in [RS68c, Theorem 1.11] that the above map

\[(B\tilde{PL}_{q+1}, B\tilde{PL}_q) \to (B\tilde{G}_{q+1}, B\tilde{G}_q)\]

is an isomorphism on homotopy groups, and hence the same is true for the corresponding map \((BS\tilde{PL}_{q+1}, BS\tilde{PL}_q) \to (BS\tilde{G}_{q+1}, BS\tilde{G}_q)\) of 2-fold covers. In other words, by e.g. [MV15, Remark 3.3.19], the square

\[
\begin{array}{ccc}
BS\tilde{PL}_q & \longrightarrow & BSG_q \\
\downarrow & & \downarrow \\
BS\tilde{PL} & \longrightarrow & BSG
\end{array}
\]

is homotopy Cartesian. For Cartesian squares [MV15, Example 3.3.14] the arrows from the upper left corner give a fibration \(BSPL_q \to BSPL \times BSG_q\) whose fiber \(\Omega BSG\) has finite homotopy groups [MM79, Corollary 3.8], so that the fibration is a rational homotopy equivalence.

The classifying spaces \(BS\tilde{PL}_q, BSPL, BSG_q\) are simply-connected with finitely generated homotopy groups (as can be seen from the long exact homotopy sequences of the above fibrations together with the fact that \(\pi_i(BO)\) and \(\pi_i(S^m)\) are finitely generated for all \(m, i\)). Hence \(BS\tilde{PL}_q, BSPL, BSG_q\) have finitely generated homology groups (by the Hurewicz theorem for the Serre class of finitely generated abelian groups).

In what follows we need to compare the cohomology of these classifying spaces with coefficients in \(\mathbb{Z}, \mathbb{Z}[\frac{1}{2}], \mathbb{Q}\), for which we recall the following.
Lemma 3.1. If $X$ be a space with finitely generated homology groups, then for any short exact sequence $1 \to A \to B \to G \to 1$ with torsion group $G$ the homomorphism $H^i(X; A) \to H^i(X; B)$ in the corresponding long exact cohomology sequence becomes an isomorphism after tensoring with $\mathbb{Q}$.

Proof. For any abelian group $G$ the kernel $\text{Ext}(H_{i-1}(X), G)$ of the natural surjection $H^i(X; G) \to \text{Hom}(H_i(X), G)$ is torsion, because $H_{i-1}(X)$ is finitely generated. Since $H_i(X)$ is finitely generated and $G$ is a torsion, the group $\text{Hom}(H_i(X), G)$ is torsion. Thus $H^i(X; G)$ is torsion. The exactness of the long exact cohomology sequence corresponding to $1 \to A \to B \to G \to 1$ finishes the proof. \qed

The cohomology algebra $H^*(BSPL; \mathbb{Q})$ is a polynomial algebra over $\mathbb{Q}$ on the PL Pontryagin classes $p_i$, which correspond to the usual Pontryagin classes under the rational homotopy equivalence $BSO \to BP SL$ [MM79, 4.20]. The PL Pontryagin classes are generally not integral. The discussion after [MM79, Theorem 11.14] gives explicit classes $2^iR_{4i} \in H^{4i}(BSPL)/\text{Tors}$, where $i \in \mathbb{Z}$, $i > 0$, whose representatives in $\rho_i \in H^{4i}(BSPL)$ generate $H^*(BSPL; \Lambda)$ as a polynomial algebra over $\Lambda := \mathbb{Z}[\frac{1}{2}]$. By Lemma 3.1 the coefficient homomorphisms

$H^*(BSPL; \mathbb{Z}) \to H^*(BSPL; \Lambda) \to H^*(BSPL; \mathbb{Q})$

(3.2) become isomorphisms after tensoring with $\mathbb{Q}$. Then the images of $\rho_i$ under the composite (3.2) generate $H^*(BSPL; \mathbb{Q})$ as a polynomial algebra over $\mathbb{Q}$.

We keep the notations $p_i$ and $\rho_i$ for their images in $H^*(BSPL)$ under the stabilization $BSPL_q \to BSPL$.

Milnor in [Mil68] showed that $H^*(BSG_q; \mathbb{Q})$ is a polynomial algebra over $\mathbb{Q}$ on a single generator $o_q$ whose degree $|o_q|$ equals $q$ is even, and $2q - 2$ if $q$ is odd. By Lemma 3.1 the coefficient homomorphism $H^*(BSG_q; \mathbb{Z}) \to H^*(BSG_q; \mathbb{Q})$ becomes an isomorphism after tensoring with $\mathbb{Q}$. Hence we can (and will) pick $o_q$ to be an integral class, and if $q$ is even we normalize $o_q \in H^q(BSG_q; \mathbb{Q})$ to be the integral Euler class of the universal spherical fibration under the coefficient homomorphism $H^q(BSG_q) \to H^q(BSG_q; \mathbb{Q})$. For any $k \in \mathbb{Z}_{>0}$ we have:

1. The canonical map $BSO_{2k+1} \to BSG_{2k+1}$ takes $o_{2k+1}$ to a nonzero multiple of the Pontryagin class $p_k$ [Mil68, pp.73-74].

2. The stabilization map $BSG_{2k} \to BSG_{2k+1}$ takes $o_{2k+1}$ to a nonzero rational multiple of $o_{2k}^2$ [Mil68, pp.73-74].

3. The stabilization map $BSG_{2k+1} \to BSG_{2k+2}$ takes $o_{2k+2}$ to zero. (This is obvious by degree reasons except when $k = 1$ which is excluded as follows: if the stabilization $\sigma: BSG_3 \to BSG_4$ is nonzero on $H^4(\cdot; \mathbb{Q})$, then $\sigma$ is a rational homotopy equivalence of rationally 3-connected spaces, and hence $\sigma^*o_4$ is nonzero on some $f: S^4 \to BSG_3$, so that the Euler class $o_4$ is nonzero on $\sigma \circ f$, which is a sphere bundle with a section). Thus the double stabilization $BSG_{2k+1} \to BSG_{2k+3}$ takes $o_{2k+3}$ to zero.
For $\xi \colon M \to B\widetilde{PL}_q$ let $S(\xi)$ be the spherical fibration obtained by composing $\xi$ with $B\widetilde{PL}_q \to BSG_q$ and define $o_q(\xi) := o_q(S(\xi))$. It follows that (2) and (3) hold with $G$ replaced by $\widetilde{PL}$.

The Euler class $e$ of an oriented thickening defined in Section 2 is sent by the coefficient homomorphism $H^q(M) \to H^q(M; \mathbb{Q})$ to $o_q$ if $q$ is even, and to 0 if $q$ is odd because in this case $H^q(BSG_q; \mathbb{Q}) = 0$.

The sequence $(o_q, \rho_1, \rho_2, \ldots)$ defines the map

$$BS\widetilde{PL}_q \to K_q := K(\mathbb{Z}, |o_q|) \times \prod_{i \in \mathbb{Z}_{>0}} K(\mathbb{Z}, 4i).$$

where the co-domain is topologized as a weak product [Whi78, p.28]. The map is a rational homotopy equivalence (because the rational cohomology of the domain and the co-domain are polynomial algebras over $\mathbb{Q}$ and generators are sent to generators).

**Theorem 3.4.** Given a closed manifold $M$, an integer $q \geq 3$, and a sequence of cohomology classes $(\beta_i)_{i \in \mathbb{Z}_{>0}}$ with $\beta_0 \in H^{4|\beta_0|}(M)$ and $\beta_i \in H^{4i}(M)$ for $i \in \mathbb{Z}_{>0}$, there exists an $n \in \mathbb{Z}_{>0}$ and an oriented codimension $q$ thickening $W$ of $M$ with $o_q(W) = n\beta_0$ and $\rho_i(W) = n\beta_i$.

**Proof.** Realize the sequence $(\beta_i)$ as a map from $M$ into $K_q$ and apply Theorem B.1 to lift the map into $BS\widetilde{PL}_q$. \qed

**Remark 3.5.** The classes $\rho_i$ and the Pontryagin classes $p_j$ are related by universal polynomials, so to some extent Theorem 3.4 lets us specify the Pontryagin classes, albeit in a non-explicit way. Still, if we insist that $p_i = 0$ for $i \neq j$, then $\rho_i$ and $p_i$ are proportional, and then $o_q$, $p_i$ can be prescribed arbitrarily up to multiplicative constant.

Since the fiber of (3.3) is rationally contractible, standard obstruction theory arguments imply:

**Theorem 3.6.** For a closed manifold $M$ and an integer $q \geq 3$, there are at most finitely many equivalence classes of codimension $q$ thickenings of $M$ with the same classes $o_q$, $p_i$, $i \in \mathbb{Z}_{>0}$.

Let us highlight some differences between $BSO_q$ and $BS\widetilde{PL}_q$:

- The Pontryagin class $p_i$ in $H^*(BSO_q; \mathbb{Q})$ vanishes $i > \frac{q}{2}$ while the PL Pontryagin class $p_i$ in $H^*(BS\widetilde{PL}_q; \mathbb{Q})$ is nonzero for any $i$.
- The relation $o_{2k}^2 = p_k$ holds in $H^*(BSO_{2k})$ while in $H^{4k}(BS\widetilde{PL}_q; \mathbb{Q})$ the classes $o_{2k}^2$, $p_k$ are linearly independent.
• The canonical map $BSO_{2k+1} \to BS\widetilde{PL}_{2k+1}$ takes $o_{2k+1}$ to a nonzero multiple of the Pontryagin class $p_k$. Thus for a linear $(2k+1)$-disk bundle $p_k = 0$ if and only if $o_{2k+1} = 0$.

4. Smoothable thickenings of codimension $\geq 3$

Theorem 3.4 allows us to construct thickenings whose characteristic classes are prescribed up to a multiplicative constant. More work is required to produce thickenings that are smoothable. By smoothing theory [HM74] a manifold $W$ is smoothable if and only if its tangent microbundle $\tau_W: W \to BPL$ lifts to $BO$.

**Theorem 4.1.** If in Theorem 3.4 the manifold $M$ is smoothable, then the integer $n$ can be chosen so that the thickening $W$ is smoothable.

**Proof.** Apply Theorem B.1 for $X = M$ and

- $\alpha_1: BSP\widetilde{L}_q \to BSG_q \times BSPL$ is the above rational homotopy equivalence,
- $\alpha_2$ is the product of the identity map of $BSG_q$ with $BSO \to BSPL$,
- $f: M \to K_q$ represents the homotopy class given by the sequence $(\beta_0, \beta_1, \ldots)$,
- $\beta: BSG_q \times BSPL \to K_q$ represents the homotopy class given by $(o_q, \rho_1, \rho_2, \ldots)$,

where $\alpha_1 \circ f_1$, $\alpha_2 \circ f_2$ are homotopic. The homotopy class $[f_1]$ classifies a thickening $W$ whose stabilization is the composite of $\alpha_1 \circ f_1$ with the coordinate projection to $BSPL$. By commutativity the stabilization of $[f_1]$ lifts to $BSO$. Thus the tangent microbundle $\tau_W$ of $W$ is such that $\tau_W \oplus \nu_M$ has a structure of a stable vector bundle, and hence so does $\tau_W \oplus \nu_M \oplus \tau_M$ which is stably isomorphic to $\tau_W$. Here $\nu_M$ is the stable normal bundle of $M$. $\square$

**Proof of Theorem 1.3 for $q \geq 3$.** The construction depends on the parity of $q$.

$q$ is even. Fix $k, l \in \mathbb{Z}$ with $l \geq 4k \geq 8$, set $q = 2k$, and let $L$ be any smooth closed $l$-manifold such that $H^{2q}(L)$ contains an infinite order element. By Theorem 4.1 and Remark 3.5 there is a smoothable thickening $W$ of $L$ with $p_k \neq 0$ and $o_q = 0 = p_i$ for $i \neq k$.

$q$ is even Fix $k, l \in \mathbb{Z}$ with $l \geq 4k \geq 4$, set $q = 2k + 1$, and let $L$ be any smooth closed $l$-manifold such that $H^{4k}(L)$ contains an infinite order element. By Theorem 4.1 and Remark 3.5 there is a smoothable thickening $W$ of $L$ such that $o_q \neq 0 = p_i$ for all $i$.

Regardless of the parity of $q$ let $K$ be the double of $W$ along the boundary. Apply Ontaneda’s Riemannian hyperbolization to a smooth triangulation of $K$ to get $L \subset K$ satisfying (i)-(ii). Let $V$ be a covering space of $K$ corresponding to the image of $\pi_1(L)$ in $\pi_1(K)$, and let $f$ denote a lift of the inclusion $L \to K$ to $V$. By Theorem 1.1 and Section 2 the classes $o_q$, $p_i$ of the regular neighborhood of $f(L)$ in $V$ satisfy the same identities (depending on parity of $q$) as those of $W$, and hence so does the normal disk bundle of the smooth submanifold that is a deformation
retract of \( V \). The existence of such a smooth submanifold leads to a contradiction as follows:

**q is even.** Linear disk bundles satisfy \( o_q^2 = p_k \) which gives \( 0 = o_q^2 = p_k \neq 0 \) in our case.

**q is odd.** The map \( BSO_q \to B\tilde{P}\ell_q \) sends the PL Pontryagin class to the usual Pontryagin class \( p_k \), and takes \( o_q \) to a multiple of \( p_k \). Thus \( o_q \) of a linear disk bundle with \( p_k = 0 \) must be zero, which contradicts \( o_q \neq 0 \).

Proposition 2.6 proves the claim about the finitely-sheeted covers of \( \mathbf{K} \) because finitely-sheeted covers are injective on rational cohomology. \( \square \)

5. **Classifying spaces of codimension 2 thickenings**

A classifying space \( BRN_2 \) for codimension 2 thickenings was introduced in [CS76] by considering them up to concordance. Concordant thickenings are homeomorphic rel \( M \), and the set of concordance classes of codimension 2 thickenings is bijective to the set of homotopy classes of maps from \( M \) into a classifying space \( BRN_2 \), see [CS76] and also [McC77].

A **concordance of oriented thickenings** is defined in [CS76, p.173], where it is shown that the set of concordance classes of oriented codimension 2 thickenings is bijective to the set of homotopy classes of maps from \( M \) into a classifying space \( BSRN_2 \), which was extensively discussed in [CS76, CS78]. The space \( BSRN_2 \) is a path-connected CW complex [CS76, Theorem 1.4].

To get smoothable codimension 2 thickenings with prescribed Euler and Pontryagin classes consider the diagram

\[
\begin{array}{ccc}
BSPL & \xleftarrow{\oplus} & BSO \times BSPL \\
\sigma \downarrow & & \sigma(*) \times BSO \\
BSRN_2 & \xrightarrow{(e,\eta)} & BSO \times G/PL \\
\end{array}
\]

Here \( * \) is a basepoint in \( BSO_2 \), \( \oplus \) is the Whitney sum, the map \( \iota \) is the product of the stabilization \( \sigma \): \( BSO_2 \to BSO \) and the standard fibration \( \pi \): \( G/PL \to BSPL \) with fiber \( SG \). The rightmost upward arrow is the standard fibration with fiber \( SG \). The rightmost top arrow is the product of the basepoint inclusion with the forget-SO-structure map. The rightmost bottom arrow is the product of the basepoint inclusion with the standard fibration \( G/O \to G/PL \) which is a rational homotopy equivalence (whose fiber \( PL/O \) is 6-connected and \( \pi_i(PL/O) \) is a finite group of h-cobordism classes of oriented homotopy \( i \)-spheres). The map \( e \) corresponds to a boundary-preserving homology equivalence that sends a thickening to a 2-disk bundle with the same Euler class, and \( \eta \) corresponds to its normal invariant.

The rightmost square commutes by construction. Corollary 4.9 on [CS76, p.208] says that \( (e,\eta) \) has a section \( s \) that extends the map \( BSO_2 \times * \to BSRN_2 \) that
considers a 2-disk bundle as a thickening. Proposition 1.8 on [CS76, p.183] gives commutativity of the leftmost square, i.e., \( \sigma = \oplus \circ l \circ (e, \eta) \), which also implies \( \sigma \circ s = \oplus \circ l \circ (e, \eta) \circ s = \oplus \circ l . \)

Every pair of continuous maps \( f: M \to BSO_2 \) and \( g: M \to G/PL \) defines a thickening \( W \) classified by \( s \circ (f \times g): M \to BSRN_2 \) whose stabilization
\begin{equation}
(\sigma \circ f) \oplus (\pi \circ g): M \to BSPL
\end{equation}
lifts to \( BSO \) if \( g \) lifts to \( G/O \). Therefore, if \( g \) lifts to \( G/O \) and \( M \) is smoothable, then \( W \) is smoothable. Now we are ready to prove the following.

**Theorem 5.2.** Given a closed smooth manifold \( M \) and a sequence \((\beta_i)_{i \geq 0}\) with \( \beta_0 \in H^2(M) \) and \( \beta_i \in H^{4i}(M) \) for \( i \in \mathbb{Z}_{\geq 0} \) there is a positive integer \( n \) and a smoothable oriented codimension 2 thickening \( W \) over \( M \) with \( e(W) = \beta_0 \) and \( p_i(W) = n\beta_i \).

**Proof.** The standard fibration \( G/O \to BSO \) with fiber \( SG \) is a rational homotopy equivalence. Integral Pontryagin classes define a rational homotopy equivalence \( BSO \to K \) where \( K \) is the product of Eilenberg-MacLane space in positive degrees divisible by 4. Realize the sequence \((\beta_i)\) as a map \( M \to K \), and apply Theorem B.1 to the composite \( G/O \to BSO \to K \), which gives a lift \( g: M \to G/O \) whose composite with \( G/O \to BSO \) has Pontryagin classes \( p_i = n\beta_i \). Any element in \( H^2(M) \) is an Euler class of a vector bundle \( f: M \to BSO_2 \). Applying \( s \) as in the above diagram gives a smoothable thickening with desired Euler and Pontryagin classes.

**Proof of Theorem 1.3 for \( q = 2 \) and \( \ell \geq 4 \).** Let \( L \) be any smooth closed oriented \( \ell \)-manifold such that \( H^4(L) \) contains an infinite order element. Use Theorem 5.2 to find a codimension 2 smoothable thickening with \( e = 0 \neq p_1 \). If \( K \) deformation retracts to a smooth \( \ell \)-dimensional submanifold, its normal bundle also has \( e = 0 \neq p_1 \), which contradicts \( e^2 = p_1 \). Proposition 2.6 proves the claim about the finitely-sheeted covers of \( K \) because finitely-sheeted covers are injective on rational cohomology.

**Proof of Theorem 1.3 for \( q = 2 \) and \( \ell \in \{2, 3\} \).** Recall that \( \pi_2(G/PL) \cong \pi_2(G/O) \cong \mathbb{Z}_2 \). The map \((\eta, e): BSRN_2 \to BSO_2 \times G/PL \) has a cross-section, so there is a codimension 2 oriented thickening \( R \) of \( S^2 \) with non-trivial normal invariant and Euler class \( e \). The composite \( S^2 \to BSRN_2 \to G/PL \) is surjective on \( \pi_2 \), and hence on \( H_2 \) by the Hurewicz theorem.

If \( m = 2 \), let \( L = S^2 \) and \( K \) be the double of \( R \) along the boundary.

If \( m = 3 \), let \( L = S^2 \times S^1 \) and \( K \) be the double of \( R \times S^1 \) along the boundary.

Note that \( R \times S^1 \) is the regular neighborhood of \( L \) that is the pullback of \( R \) via the coordinate projection \( S^2 \times S^1 \to S^2 \), which is surjective on \( H_2 \).
Apply Ontaneda’s Riemannian hyperbolization to smooth triangulation of $K$ for which $L$ is a subcomplex. Lift the inclusion $L \to K$ to a covering of $K$ corresponding to $\pi_1(L)$. Since $\pi_1(L)$ is quasiconvex, the covering space is the interior of the compact manifold $W$ with boundary that is the $\varepsilon$-neighborhood of the convex core, and we still denote by $L$ its lift in $W$.

The regular neighborhood of $L$ in $W$ is equivalent to the regular neighborhood of $L$ in $K$, which by Theorem 1.1 is the pullback via $h$ of the regular neighborhood of $L$ in $K$. Since $h$ is surjective on $H^2$, so is the composite $L \to S^2 \to BSRN_2 \to G/PL$, where $L \to S^2$ is the coordinate projection if $m = 3$. Therefore, the regular neighborhood of $L$ in $W$ has a nontrivial normal invariant. Similarly, if $m = 2$, then the normal invariant of $L \times S^1$ in $W \times S^1$ is nontrivial (again because the coordinate projection $L \times S^1 \to L$ is $H_2$-surjective).

Set $(W', L') = (W, L)$ if $m = 3$ and $(W', L') = (W \times S^1, L \times S^1)$ if $m = 2$. Thus $W'$ is a compact 5-manifold that deformation retracts onto a PL embedded closed 3-manifold $L'$ whose regular neighborhood has nontrivial normal invariant.

Arguing by contradiction suppose that $W'$ deformation retracts to a smooth submanifold $M'$. The restriction to $L'$ of the deformation retraction $W' \to M'$ is homotopic to a diffeomorphism $g: L' \to M'$ (which uses the geometrization if $m = 3$). Hence the normal invariant of $g^{-1}$ is trivial.

As we explain in Appendix C, the pullback via $g^{-1}$ of the Poincaré embedding given by the inclusion $L' \subset W'$ is isomorphic to the Poincaré embedding of the inclusion $M' \subset W'$, By [CS76, Theorem 6.2] the Poincaré embedding for $M' \subset W'$ can be realized by a locally flat embedding if and only if the normal invariants of $g^{-1}$ equals the normal invariant of the Poincaré embedding $L' \subset W'$, which is not the case. □

Proof of Theorem 1.4. Let $L$ be any closed connected PL $l$-manifold $L$ with some non-integer Pontryagin number. For any hyperbolized simplex the strict hyperbolization $L$ has the same property. Any homotopy equivalence from $L$ to a closed manifold preserves Pontryagin classes, see Remark 2.4, and hence Pontryagin numbers, which are integers for smooth manifolds. Thus $L$ is not homotopy equivalent to a smooth manifold.

Fix a PL embedding of $L$ into a high-dimensional sphere $K$. Let $(K, L)$ be the result of applying Ontaneda’s strict hyperbolization to a PL triangulation of the pair $(K, L)$.

Closed PL manifolds with non-integer Pontryagin number exist in all dimensions $l \geq 8$ divisible by 4, see [BLW10, section 5]. There are such simply-connected manifolds, see [Bro72, V.2.9], and any closed simply-connected PL $l$-manifold with $l \geq 4$ admits a PL embedding into the $(2l - 1)$-sphere [Irw65]. □
6. Hyperbolization Pulls Back Regular Neighborhood

Recall that unless stated otherwise all maps, manifolds, embeddings, isotopies, bundles, submanifolds are PL.

To simplify notations for a map $f: X \to \triangle$ and $J \subseteq \triangle$ we set $X_J := f^{-1}(J)$.

An $n$-dimensional hyperbolized simplex is a map $f: (X, \partial X) \to (\triangle, \partial \triangle)$ such that $\triangle$ is an $n$-simplex, $X$ is a compact connected $n$-manifold with boundary, $f$ has degree one mod 2, and for every $k$-dimensional face $\alpha$ of $\triangle$ the set $X_\alpha$ is a $k$-dimensional submanifold of $X$ whose boundary equals $X_{\partial \alpha}$. This definition corresponds to conditions C0, C1, C2 in [DJ91, Section 1c]. We always assume that the hyperbolized simplex is proper as defined in Appendix A. Then the restriction $f$ to a component of $X_\alpha$ is a proper hyperbolized simplex.

Remark 6.1. In our applications $X$ is oriented, the map $f: (X, \partial X) \to (\triangle, \partial \triangle)$ has degree one, and the tangent bundle to $X$ has zero rational Pontryagin classes.

For $0 \leq l \leq k$ let $\triangle^l$ denote the convex combination of $e_1, \ldots, e_{l+1}$ in $\mathbb{R}^{k+1}$; thus $\triangle^l$ is an $l$-dimensional face of the standard $k$-simplex $\triangle^k$.

Let $K$ be a $k$-dimensional simplicial complex. Denote its first barycentric subdivision by $K'$, and let $p_K: K' \to \triangle^k$ be the simplicial map defined on vertices by sending the barycenter of every $i$-dimensional simplex of $K$ to $e_{i+1}$. The map $p_K$ is injective on every simplex, and thus it “folds” $K'$ onto $\triangle^k$. Then

$K_f := \{(s, x) \in K \times X: p_K(s) = f(x)\}$

is the fiber product of $p_K$ and $f$. We refer to $h_K: K_f \to K$ given by $h_K(s, x) = s$ as the hyperbolization map. If $K$ is a closed manifold, then so is $K_f$.

Let $L$ be a full $l$-dimensional subcomplex of $K$. Then $p_L$ is the restriction of $p_K$ to $L$. Let $f_l$ be the restriction of $f$ to a component $X^l$ of $f^{-1}(\triangle^l)$; recall that $f_l$ is a hyperbolized $l$-simplex. The fiber product $L_{f_l}$ of $p_L$ and $f_l$ can be also described as

$L_{f_l} = \{(s, x) \in L \times X: p_L(s) = f(x)\}$

because the equation $p_L(s) = f(x)$ has no solutions outside $L \times X^l$.

Proof of Theorem 1.1. We are going to quote results that require $K$ to stay away from the boundary of the ambient manifold. To this end we attach a collar to $K \times X$. Let $X$ be the result of attaching $\partial X \times [0, 1]$ along the boundary of $X$. Let $b$ be the barycenter of $\triangle^k$, and let $\triangle^k$ be the image of $\triangle^k$ under the self-map of $\mathbb{R}^{k+1}$ given by $x \mapsto b + 2(x - b)$; thus $\triangle^k \supset \triangle^k$ are concentric simplices in the affine $k$-plane spanned by $\triangle^k$. We extend $f$ to the map $\tilde{f}: \tilde{X} \to \triangle^k$ as follows: pass to subdivisions in which $f$ is simplicial, take an arbitrary simplex $\sigma$ in the subdivision of $\partial X$, and send the prism $\sigma \times [0, 1]$ linearly to the portion of the cone over $f(\sigma)$ with tip $b$ that lies in $\triangle^k \setminus \text{Int}(\triangle^k)$. 
Thus $K$ is a locally flat submanifold of $\mathcal{K}$ with trivial normal bundle $\nu^K_L$, cf. Appendix A. Note that $K$ lies in $K \times X$ and intersects its boundary, and we attach collars precisely to keep $K$ away from the boundary. Recall that $L$ is the intersection of $K$ and $\mathcal{L} := L \times X$. By Appendix A we can (and will) choose $\nu^K_L$ so that its fibers over $L$ contains the fibers of the normal bundle $\nu^L_L$ of $L$ in $\mathcal{L}$.

Suppose $k - l > 2$. The normal block bundle $\nu^L_L$ of $\mathcal{L}$ in $\mathcal{K}$ is the pullback of the normal block bundle $\nu^K_L$ of $L$ in $K$ via the coordinate projection $\pi_L: \mathcal{L} \to L$. The hyperbolization map $h_L$ factors as the inclusion $L \to \mathcal{L}$ followed by $\pi_L$, and therefore, $h_L$ pulls $\nu^K_L$ back to $\nu^K_L|_L$. By [RS68b, Theorem 1.2] there is a small isotopy of $\mathcal{K}$ that takes $K$ to a submanifold $K^1$ that is block transverse to $\mathcal{L}$, i.e., $K^1 := K^1 \cap \mathcal{L}$ is a submanifold, and the intersection of $K^1$ and $\nu^K_L$ equals $\nu^K_L|_{K^1}$, the normal block bundle of $L^1$ in $K^1$. (Here we do not distinguish between a block bundle and its total space). Since the above isotopy is small, the projection of $\nu^K_L$ restricts to a homeomorphism $(K^1, L^1) \to (K, L)$ identifying the normal block bundle of $L$ in $K$ with $\nu^K_L|_{L^1}$ which is isomorphic to $\nu^K_L|_L$, as claimed.

If $k - l \leq 2$, then the same argument works except that one has to replace block bundles structures on $\nu^L_L$, $\nu^K_L$ by stratifications in the sense of Stone [Sto72] of the regular neighborhoods of $\mathcal{L}$ in $\mathcal{K}$, and $L$ in $K$, respectively, and appeal to a more general transversality theorem of Stone, see [Sto72, Corollary on p.97]; also cf. [CS76, pp.173–175], [McC75, p.285] and [McC77, p.162].

**Proof of Corollary 1.2.** By [Ont20] the manifold $K$ has a smooth structure that admits a Riemannian metric $\varepsilon,K$ as in (b), and induces the given PL structure. By Theorem 1.1 the PL submanifold $L$ has a linear disk bundle neighborhood in $K$. Therefore, by [LR65, Theorem 7.3] there is a piecewise-differentiable homeomorphism $d$ of $M$ that moves $L$ to a smooth submanifold $h(L)$ of $K$. Hence $L$ is a smooth submanifold in the pullback, via $d$, of the given smooth structure on $M$. Then $d$ is a diffeomorphism of these smooth structures, and the pullback metric $d^* \varepsilon,K$ satisfies (b).

**Remark 6.2.** Let us justify the claim made in the introduction that the cover of the Riemannian hyperbolization $K$ corresponding to $\pi_1(L)$ is convex-cocompact. Since $L$ is locally convex in the CAT(−1) metric on $K$, the subgroup $\pi_1(L)$ of $\pi_1(K)$ is quasiconvex. Hence $\pi_1(L)$ is a hyperbolic group with ideal boundary equivariantly homeomorphic to its limit set in the ideal boundary of $\pi_1(K)$. Every hyperbolic group acts cocompactly on the triple space of its boundary. A group of isometries of a negatively pinched Hadamard manifold is convex-cocompact if and only if the group acts cocompactly on the space of triples of the limit set [Bow95, Bow99].

**Appendix A. Transversality and hyperbolization**

The purpose of this appendix is to prove that $K$, $L$ are submanifolds of $\mathcal{K}$, $\mathcal{L}$ that have trivial normal bundles $\nu^K_K$, $\nu^L_L$ such that each fiber of $\nu^K_K$ lies in a fiber of $\nu^L_L$.
We follow notations and conventions of Section 6, and in particular, work in the PL category. We also assume that \( K, L \) are closed manifolds, and identify \( \Delta^k \) with a \( k \)-simplex \( \triangle \) in \( \mathbb{R}^k \). Then \( p_k \circ f \) becomes a map from \( \mathcal{K} \to \mathbb{R}^k \) whose zero set is \( K \). It is stated in [DJ91, 1f.3] that the map is “transverse to 0”, and hence “\( K \) is a submanifold of \( \mathcal{K} \) with trivial normal bundle”. No justification is given, and in fact, as the author learned from Pedro Ontaneda, the claim is not true without the extra assumption that \( f \) is “transverse to every face of \( \Delta^k \)”, which probably is implicit in [DJ91, CD95].

Two submanifolds are transverse if in a local chart they become hyperplanes in general position; the same definition works at a boundary point [AZ67, p.436]. Clearly, if \( M, N \) are transverse, then \( M \cap N \) is a locally flat submanifold of \( M \).

Let \( \Gamma_k \subset \mathcal{K} \times \mathbb{R}^{2k} \) be the graph of \( p_k \times f \), let \( P_k \) be the diagonal in \( \mathbb{R}^k \times \mathbb{R}^k = \mathbb{R}^{2k} \), and let \( P_k = \mathcal{K} \times P_k \). The coordinate projection \( \Gamma_k \to \mathcal{K} \) is a homeomorphism that takes \( \Gamma_k \cap P \) to \( K \).

By Lemma A.4 below, \( \Gamma_k \) and \( P_k \) are transverse, and in particular, \( \Gamma_k \cap P_k \) is a locally flat submanifold of \( \Gamma_k \) of codimension \( k \). The normal bundle of \( \Gamma_k \cap P_k \) in \( \Gamma_k \) is trivialized by the composite of the coordinate projection \( \mathcal{K} \times \mathbb{R}^{2k} \to \mathbb{R}^{2k} \) followed by the orthogonal projection \( \mathbb{R}^{2k} \to P_k \) onto the orthogonal complement of \( P_k \) in \( \mathbb{R}^{2k} \).

The restriction of the orthogonal projection \( \mathbb{R}^{2k} \to P_k \) to \( \mathbb{R}^{2l} \) is the orthogonal projection \( \mathbb{R}^{2l} \to P_k \). Thus one can choose the trivialization of the normal bundle of \( \Gamma_k \cap P_k \) in \( \Gamma_k \) so that each of its fibers lies in the fiber of the normal bundle of \( \Gamma_k \cap P_k \) in \( \Gamma_k \).

To prove Lemma A.4 we need some terminology. If \( z \) is a point in the subspace \( S \) of \( Z \), and \( U \) is a neighborhood of \( z \) in \( Z \) such that there is a map \( U \to U \cap S \) that is a disk bundle over a disk then \( U \) is a square at \( z \in S \subset Z \). A map of squares is a product if after composing with trivializations of the disk bundles the map becomes the product of two maps of disks that correspond under the trivialization to bases and fibers.

**Example A.1.** Let \( Z \) be a triangulated compact manifold (possibly with boundary). Give \( Z \) the path-metric \( d \) induced by making every simplex isometric to the standard simplex. Given \( z \in Z \) let \( \sigma \) be the simplex of \( Z \) whose relative interior \( \hat{\sigma} \) contains \( z \). Let \( B \) be a metric ball about \( z \) in \( \hat{\sigma} \). If \( \delta > 0 \) is sufficiently small, then the nearest point projection \( \pi \) onto \( \sigma \) is defined on the \( \delta \)-neighborhood of \( B \), and \( U = \{ z \in Z \mid d(z, \sigma) \leq \delta, \pi(z) \in B \} \) is a square lying in the star of \( \sigma \). Moreover, \( \pi : U \to B \) is a disk bundle whose fiber is the cone on the link of \( \sigma \) in \( Z \). Each fiber (or, rather, its intersection with any simplex that meets \( \hat{\sigma} \)) is orthogonal to \( \sigma \). Thus the base and the fibers are orthogonal.

**Example A.2.** The folding map \( p_k \) takes every simplex \( \sigma \) of \( K' \) homeomorphically onto \( p_k(\sigma) \). By Example A.1 at every point \( z \in \hat{\sigma} \) there is a square sent by \( p_k \) to a
square at $p_{\kappa}(z) \in p_{\kappa}(\sigma)$, and moreover, $p_{\kappa}$ is a product. The factor corresponding to the map of bases is a bijection, being the restriction of $p_{\kappa}|\sigma$. Since $K$ is a boundaryless manifold, the factor corresponding to the map of fibers is bijective only if $z \in \triangle$, in which case the fiber is a point.

**Example A.3.** Let $f : X \to \triangle$ be a hyperbolized simplex. Denote the relative interior of a face $\alpha$ of $\triangle$ by $\hat{\alpha}$. We say that $f$ is proper if for every face $\alpha$ of $\triangle$ with $\alpha \neq \triangle$ and every $z \in X_{\hat{\alpha}}$ there is a square at $z \in X_{\hat{\alpha}}$ in $X_{\hat{\alpha} \cup \Delta}$ whose disk bundle is the pullback via $f$ of the disk bundle of a square at $f(z) \in \hat{\alpha}$ in $\hat{\alpha} \cup \Delta$ as in Example A.1. Thus $f$ is a product such that the factor corresponding to the map of the fibers is a bijection. The factor corresponding to the bases need not be a bijection because $f : X_{\hat{\alpha}} \to \hat{\alpha}$ is not locally bijective if $\dim(X) > 1$.

**Lemma A.4.** If $f$ is proper, then $\Gamma$ and $\mathcal{P}$ are transverse.

**Proof.** Fix $(s_o, x_o) \in K$, and use the disk bundle structures of the squares at $s_o \in K$, $x_o \in X$ to represent points $s = (b,v)$, $x = (\beta,\nu)$ of $K$ near $s_o$, $x_o$, respectively. Here $b, \beta$ are in the bases, while $v, \nu$ are in the fibers of the disk bundles.

By working in a chart we will assume that both squares are in $\mathbb{R}^k$ with fibers and the bases parallel to coordinate planes, and moreover, $s_o$, $x_o$ become the origin. Denote the product of the squares in $\mathbb{R}^k \times \mathbb{R}^k$ by $\Pi$.

Also we rotate and translate $\triangle$ in $\mathbb{R}^k$ so that $p_{\kappa}(s_o) = f(x_o)$ is the origin, and the simplex $\tau$ of $\triangle$ whose relative interior contains the origin is a coordinate plane whose dimension equals $\dim(\tau)$. Then write $p_{\kappa}(b,v) = (p_b(b), p_n(v))$ and $f(\beta,\nu) = (f_b(\beta), f_n(\nu))$ where $p_b$, $f_b$ are the base-coordinates, $p_n$, $f_n$ are the fiber-coordinates, and by construction the bases and the fibers lie in coordinate planes. Recall that $p_b$ and $f_n$ are injective.

Group the coordinates as follows

$$a = (v, \beta), \quad b = (b,\nu), \quad I(b) = (p_b(b), f_n(\nu)), \quad J(a) = (f_b(\beta), p_n(\nu))$$

so that $\Gamma$ and $\mathcal{P}$ look in these coordinates in $\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k$ as

$$\Gamma' = \{(a,b, I(b), J(a)) | (a,b) \in \Pi\} \quad \mathcal{P}' = \{(a,b, u, u) : a, b, u \in \mathbb{R}^k\}$$

The map $(x,y,z,w) \mapsto (x,y,z+w,z-w)$, where $x,y,z,w \in \mathbb{R}^k$, takes $\Gamma'$, $\mathcal{P}'$ to

$$\overline{\Gamma} = \{(a,b, I(b) + J(a), I(b) - J(a)) | (a,b) \in \Pi\}$$

$$\overline{\mathcal{P}} = \{(a,b, u, 0) | a, b, u \in \mathbb{R}^k\}.$$
(q, p) → (q, q + h(p)), q ∈ D, p ∈ \mathbb{R}^{2k} \text{ preserves } P \text{ and takes } D \times \{0\} \text{ to the graph of } h. \text{ Hence the graph of } h \text{ is transverse to } \overline{P}. \text{ Thus } \Gamma \text{ and } P \text{ are transverse.} \square

**Appendix B. Lifting and power maps**

This appendix presents an obstruction theory argument that is used to prove the existence of smoothable thickenings with prescribed rational characteristic classes. It is based on [BK03, Appendix B] where linear disk bundles were treated.

Let $K$ be the weak product [Whi78, p.28] of the family \{$(G_i, i)\}_{i \in \mathcal{I}}$ of pointed Eilenberg-MacLane spaces such that each $G_i$ is infinite cyclic. For $S \subset \mathcal{I}$ let $K_S := \{ (x_i) \in K : x_i = *_{i} \text{ if } i \notin S \}$, where $*_{i} \in K(G_i, i)$ is the basepoint. For an integer $n$ let $n_i$ denote a cellular basepoint-preserving self-map of $K(G_i, i)$ induced by the “multiplication by $n$” on $G_i$, where we let $n_i$ be the identity map if $n = 1$. The product of the maps $n_i$ over $i \in S$ is denoted by $\pi^n: K_S \rightarrow K_S$ and called an $n$-power map of $K_S$. Let $\pi: K \rightarrow K$ be the map that sends the coordinate $x_i$ of $(x_i)$ to $n_i(x_i)$ if $i \in S$ and to $x_i$ if $i \in I \setminus S$. Thus $\pi|_{K_S} = n$ and we call $\pi$ the extension of an $n$-power map of $K_S$.

Consider the following diagram of CW complexes and cellular maps

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\alpha_1} & B \\
\downarrow{f_1} & \downarrow{\beta} & \downarrow{\alpha_2} \\
K & \xrightarrow{\pi} & A_2 \\
\uparrow{f_2}
\end{array}
\]

where $X$ is a finite complex $X$, $f: X \rightarrow K$ is a cellular map, $K$ is 2-connected, $A_1$, $B$, $A_2$ are simply-connected, and $\beta$, $\alpha_1$, $\alpha_2$ are fibrations which are rational homotopy equivalences. It follows that the fibers of $\beta$, $\beta \circ \alpha_1$, $\beta \circ \alpha_2$ are simply-connected spaces with finite homotopy groups.

Since $f(X)$ is compact, the definition of a weak product places $f(X)$ inside some $K_J$ for a finite subset $J$ of $I$. Hence $K_J$ is the product of finitely many $K(G_i, i)$-factors indexed by $J$. In this setting we have the following:

**Theorem B.1.** There is a positive integer $n$ and maps $f_1$, $f_2$ such that the extension $\pi^n: K \rightarrow K$ of an $n$-power map of $K_J$ makes the diagram commute.

**Proof.** First, we show that if $A$ is a finite abelian group, then there is a power map that induces the zero map on $H^n(K_J; A)$ for every $n > 0$.

We can assume that $A$ cyclic because the factorization of $A = \bigoplus_l \mathbb{Z}_{n_l}$ as a sum of finitely many cyclic groups gives rise via long exact coefficient sequences to a natural factorization $H^n(K_J; A) = \bigoplus_l H^n(K_J; \mathbb{Z}_{n_l})$, and if for every $l$ there is a power map
that annihilates $H^n(K_J;\mathbb{Z}_{m_1})$ for all $n > 0$, then their composite sends $H^n(K_J;A)$ to zero for all $n > 0$.

Induct by the number of $K(\mathbb{Z}, i)$-factors of $K_J$. If $K_J = K(\mathbb{Z}, i)$ and $A = \mathbb{Z}_{m_1}$, then an $n_1$-power map works [HQ18, Lemma 4]. For the induction step write $J$ as the disjoint union of the subsets $S, T$ so that $K_J = K_S \times K_T$. By Hurewicz theorem $K_S$, $K_T$ have finitely generated homology groups, and since $A$ is cyclic we have $A \otimes A \cong A$ and Tor$(A, A) = 0$, which gives the natural Künneth short exact sequence [Dol72, Proposition VI.12.16]

$$0 \to \bigoplus_{i+j=n} H^i(K_S;A) \otimes H^j(K_T;A) \to H^n(K_J;A) \to \bigoplus_{i+j=n+1} \text{Tor}(H^i(K_S;A), H^j(K_T;A)) \to 0.$$

By induction there are two power maps $s$, $t$ that respectively annihilate $H^i(K_S;A)$, $H^j(K_T;A)$ for any $i, j > 0$, and hence $st$ sends all these groups to zero. Consider $st$ as a self-map of $K_J$, and note that it sends to zero the kernel and the quotient in the above Künneth sequence. A diagram chase on the three copies of the Künneth sequence stacked on top of each other implies that the power map $st \circ st$ of $K_J$ annihilates $H^n(K_J;A)$.

Fix $r \in \{1, 2\}$. The obstruction classes to lifting the identity map $\iota: K_J \to K_J$ as a section of $\beta \circ \alpha_r$ over $K_J$ lie in the groups $H^{j+1}(K_J, \pi_j(F_{\beta \circ \alpha_r}))$ where $F_g$ denotes the homotopy fiber of $g$. Fix a power map $k$ that annihilates this cohomology group for all $j$. Naturality of obstruction classes gives a map $\sigma_r: K_J \to A_r$ such that $\beta \circ \alpha_r \circ \sigma_r$ is homotopic to $k = \iota \circ k$.

The cohomology classes of the difference cochains of $\alpha_1 \circ \sigma_1$, $\alpha_1 \circ \sigma_2$ lie in the groups $H^j(K_J, \pi_j(F_\beta))$. Let $l$ be a power map that annihilates this cohomology group for all $j$. By naturality of the difference cochains the maps $\alpha_1 \circ \sigma_1 \circ l$, $\alpha_2 \circ \sigma_2 \circ l$ are homotopic. Setting $f_r := \sigma_r \circ l \circ f$ and $n := kl$, we conclude that $\beta \circ \alpha_r \circ f_r$ is homotopic to $k \circ l \circ f = n \circ f = n! \circ f$. \hfill \qed

**Remark B.2.** The proof of Theorem B.1 implies that $n$ can be replaced by any positive integer multiple of $n$.

**Remark B.3.** Theorems 3.4 and 5.2 use the case when $\alpha_1 = \alpha_2$ is the identity map of $B$ and $f_1 = f_2$. The full generality of the theorem is needed in Section 4.

**Remark B.4.** In Theorem B.1, if the homotopy class $[f]$ of $f$ is thought of as $(\alpha_i)$ in $\bigoplus_{i \in \mathcal{J}} H^i(X)$ with $\alpha_i \in H^i(X)$, then $[n \circ f]$ corresponds to $n \alpha_i$.

**Appendix C. Codimension two Poincaré embeddings**

In this appendix we review a result in [CS76] about compact manifolds with two codimension 2 spines one of which is locally flat.

As usual, we work in the PL category. Throughout this section $M$ be a closed oriented manifold and $W$ is a compact oriented manifold with boundary with
\text{dim}(W) - \text{dim}(M) = 2. For a vector bundle \(\xi\) over \(M\) let \(p_\xi: D_\xi \to M\) and \(S_\xi = \partial D_\xi\) denote the associated disk and sphere bundles, respectively.

An oriented \(h\)-Poincaré embedding of \(M\) into \(W\) consists of an oriented 2-plane bundle \(\xi\) over \(M\), a finite Poincaré pair \((E, S_\xi \coprod F)\), and a homotopy equivalence \(h: (W, \partial W) \to (E \cup S_\xi D_\xi, F)\) that maps the fundamental class \([W, \partial W]\) to the class that corresponds to the fundamental class of \((D_\xi, S_\xi)\) after excising \(E \setminus S_\xi\).

Any embedding of \(M\) into \(W\) gives rise to a \(h\)-Poincaré embedding as follows, see [CS76, p.210]. Let \(\xi\) be the 2-plane bundle over \(M\) whose Euler class is the normal Euler class of \(M\) in \(W\). Let \(R\) be a regular neighborhood of \(M\) in \(W\); thus \(C := W \setminus \text{Int}(R)\) deformation retracts onto \(\partial R\). By [CS76, Proposition 1.6] there is a homology isomorphism \(h: (R, \partial R) \to (D_\xi, S_\xi)\) such that \(p_\xi \circ h \mid_M\) is homotopic to the identity of \(M\). Let \(E\) be the quotient space of the disjoint union of \(C\) and \(S_\xi\) by the equivalence relation that identifies each \(x \in \partial R\) with \(h(x) \in S_\xi\), let \(q: C \coprod S_\xi \to E\) denote the corresponding quotient map, and set \(F = \partial W\). Since \(C\) and \(S_\xi\) are compact, \(q \mid_C\) is a quotient map onto \(E\), and \(q \mid S_\xi\) is a homeomorphism onto its image. Then \(q \mid_{\partial R}\) can be identified with \(h \mid_{\partial R}\). Gluing \(h: R \to D_\xi\) and \(q \mid C: C \to E\) along \(h \mid_{\partial R}\) gives the map \(h\) as in the previous paragraph.

The paper [CS76] studies when a Poincaré embedding is induced by an embedding.

If \(M \to W\) is a locally flat embedding, then its regular neighborhood \(R\) is a linear disk bundle [Wal99, p.127], and the corresponding \(h\)-Poincaré embedding can be described by identifying \(R\) with \(D_\xi\), and \(E\) with \(W \setminus \text{Int}(R)\), and taking \(h\) to be the identity map of \(W\).

A map of \(h\)-Poincaré embeddings \(\alpha, \beta\) of \(M\) into \(W\) is a map of triples

\[(E_\alpha, F_\alpha, D_{\xi_\alpha}) \to (E_\beta, F_\beta, D_{\xi_\beta})\]

such that \(D_{\xi_\alpha} \to D_{\xi_\beta}\) is a bundle map, and the composite of \(h_\alpha\) and the map obtained by gluing \(D_{\xi_\alpha} \to D_{\xi_\beta}\) and \(E_\alpha \to E_\beta\) is homotopic to \(h_\beta\) as a map of pairs with domain \((W, \partial W)\).

We say that \(h\)-Poincaré embeddings \(\alpha, \beta\) of \(M\) into \(W\) are isomorphic if there are maps \(\alpha \to \beta\) and \(\beta \to \alpha\) of \(h\)-Poincaré embeddings such that both composites are homotopic to the identity through maps of \(h\)-Poincaré embedding.

If \(g: M' \to M\) is a homotopy equivalence of closed manifolds, and \(\alpha\) is an \(h\)-Poincaré embedding of \(M\), then the pullback \(g^*\alpha\) of \(\alpha\) is an \(h\)-Poincaré embedding of \(M'\) for which \(\xi_{g^*\alpha} = g^*\xi_\alpha\), \(F_{g^*\alpha} = F_\alpha\), and \(E_{g^*\alpha}\) is the union of \(E_\alpha\) with the mapping cylinder of the bundle map \(S_{\xi_{g^*\alpha}} \to S_{\xi_\alpha}\), and \(h_{g^*\alpha}\) is the composite of \(h_\alpha\) with a bundle map covering a homotopy inverse of \(g\).

The following result is implicit in [CS76, Example 6.2.1].

\textbf{Lemma C.1.} Let \(M_1, M_2\) be closed manifolds embedded into \(W\), and let \(\alpha_1, \alpha_2\) be corresponding \(h\)-Poincaré embeddings. Suppose \(W\) deformation retracts both onto
If $M_2$ is locally flat, and the restriction of a deformation retraction $W \to M_2$ to $M_1$ is homotopic to a homeomorphism $g: M_1 \to M_2$, then $g^*\alpha_2$ and $\alpha_1$ are isomorphic, or equivalently, $\alpha_2$ is isomorphic to the pullback of $\alpha_1$ via $g^{-1}$.

Proof. Label the objects related to the h-Poincaré embedding of $M_i$ into $W$ with the subscript $i \in \{1, 2\}$. Set $B_i = \partial C_i \setminus \partial W$. Let $r_i^1: C_i \to B_i$ be a deformation retraction with $t \in [0, 1]$, where $r_i^0$ is the identity map of $C_i$, and $r_i^1(C_i) = B_i$. Denote the mapping cylinder of $r_i^1|_{\partial W}$ by $T_i$. The map $(x, t) \to r_i^1(x)$, where $x \in \partial W$, descends to a map $\rho_i: T_i \to C_i$. Sending the equivalence class of $(x, t)$ to itself defines a map $\tau: T_1 \to T_2$.

Let $\bar{g}: D_{\xi_1} \to D_{\xi_2}$ be the bundle map covering the homeomorphism $g$. Let $S_1$ be the mapping cylinder of $h_1: \partial R_1 \to S_{\xi_1}$, and $S_2$ be the mapping cylinder of $\bar{g}^{-1}: S_{\xi_1} \to S_{\xi_2}$, where $\partial R_2 = S_{\xi_2}$. Represent the equivalence class of a point in these mapping cylinders as a pair: a point in $\partial R_i$ and $s \in [0, 1]$. The map $(y, s) \to (\bar{g}(h_1(y)), s)$, where $y \in \partial R_1$, descends to a map $\sigma: S_1 \to S_2$, which is well-defined because $(y, 1) \sim h_1(y) = \bar{g}(h_1(y))) \sim (\bar{g}(h_1(y)), 1)$.

If we glue $S_1$ to $T_i$ along $\partial R_i$, then the above map $\rho_i: T_i \to C_i$ extends to a map $T_1 \cup_{\partial R_1} S_1 \to E_i$ which is the identity on $S_{\xi_1} \coprod \partial W$, and is also a homotopy equivalence because both $T_1 \cup_{\partial R_1} S_1$ and $E_i$ deformation retract onto $S_{\xi_1}$. Similarly, the map $T_1 \cup_{\partial R_1} S_1 \to T_2 \cup_{\partial R_2} S_2$ glued from $\tau$ and $\sigma$ is a homotopy equivalence that is the identity on $S_{\xi_1} \coprod \partial W$. By [Hat02, Proposition 0.19] there are homotopy equivalences $T_i \cup_{\partial R_i} S_1 \to E_i$ and $T_1 \cup_{\partial R_1} S_1 \to T_2 \cup_{\partial R_2} S_2$ rel $S_{\xi_1} \coprod \partial W$. This yields a homotopy equivalence $E_1 \to E_2$ rel $S_{\xi_1} \coprod \partial W$ which easily gives rise to an isomorphism $g^*\alpha_2 \cong \alpha_1$. \hfill \ensuremath{\Box}

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Igor Belegradek, School of Mathematics, Georgia Tech, Atlanta, GA, USA 30332
Email address: ib@math.gatech.edu