The Fermat-Torricelli Problem in Projective Space

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Abstract. We pose and study the Fermat-Torricelli problem for a triangle in the projective plane. Our main finding is that if every side of the triangle has length greater than $60^\circ$, then the Fermat-Torricelli point is a vertex. For such triangles, the Fermat-Torricelli point is unique if and only if the triangle has a unique side of maximal length. Our proof relies on a complete characterization of the equilateral case together with a deformation argument.

1. Introduction

Almost four centuries ago, motivated by a question of Descartes, Fermat posed the problem of finding the point $P$ in the plane that minimizes the sum of its Euclidean distances to three non-collinear points $A, B, C$. Fermat communicated the problem to Torricelli, who was the first to solve it using properties of ellipses, while he also gave a geometric construction of $P$. When all angles of the triangle $\triangle ABC$ are less than $120^\circ$, $P$ is the first isogonic center of the triangle $ABC$, that is the interior point of $\triangle ABC$ such that all angles $\angle APB, \angle APC, \angle BPC$ are $120^\circ$. When an angle of $\triangle ABC$ is equal to or larger than $120^\circ$, $P$ coincides with the corresponding vertex. As such, the Fermat-Torricelli point is unique. Since Torricelli’s original proof, many other proofs have been discovered over the years, e.g., see [Eri97] for a modern geometric treatment and a concise historical account, [Rou12] for an algebraic approach and [KR17] for an analytic approach.

By the first half of the 19th century, the study of Fermat-Torricelli points had already been extended from planar to spherical triangles, and the isogonic property had been observed [P29]. In the second half of the 20th century Cockayne [Coc67, Coc72] showed that Fermat-Torricelli points of spherical triangles are in principle non-unique and, in the non-coplanar case, they are either vertex points or they satisfy the isogonic condition. A complete characterization of the equilateral case was provided, and conditions for the uniqueness of the Fermat-Torricelli point of general spherical triangles were given. Later, Gallieh & Hajja [GH96] observed that uniqueness holds for spherical triangles with sides of arc length less than $90^\circ$. Fermat-Torricelli points have also been considered in Minkowski spaces [CG85, MSW02], Riemannian manifolds [Yan10, Afs11] and more recently even in a tropical geometry setting [LY18]. Of great practical significance, especially in operations and location research, is the generalization of Fermat-Torricelli points to so-called Steiner-Weber points, where one seeks the point that minimizes the sum of the weighted distances to $n$ other points, e.g., see [KM97, ZZ08, HZ17].
Even though a large literature exists for the above family of problems, it is usually very difficult to give a closed form characterization of the location of the Fermat-Torricelli or Steiner-Weber points beyond the planar case \cite{Bar00}; instead one is typically content with developing algorithms for finding that point, e.g., as in \cite{ALST03, HAT11}.

The Fermat-Torricelli point can be thought of as a geometric median, as opposed to the geometric mean. This latter relies on minimizing the sum of the squared distances and, as such, it is much easier to deal with analytically and computationally. On the other hand, geometric medians are known to be robust to outliers and thus are very important for contemporary applications in statistics and adjacent fields. Recently, in the machine learning problem of hyperplane clustering, where one is concerned with clustering a given set of points $\mathcal{X} \subset \mathbb{R}^D$ sampled from an unknown arrangement of $n$ hyperplanes $\mathcal{H}_1, \ldots, \mathcal{H}_n$ of $\mathbb{R}^D$, Tsakiris & Vidal showed in \cite{TV17} that fitting a hyperplane $\mathcal{H}$ to the points $\mathcal{X}$ so that the sum of the distances of the points to $\mathcal{H}$ is minimized, can be seen as a discrete version of a Steiner-Weber problem in the projective space $\mathbb{P}^{D-1}$. This Steiner-Weber problem involves $n$ points in $\mathbb{P}^{D-1}$ associated to the normal vectors of the $n$ hyperplanes $\mathcal{H}_1, \ldots, \mathcal{H}_n$, while the weights capture the number of points in $\mathcal{X}$ contributed by each hyperplane. Of great significance for the learning problem, was the identification of conditions on the data $\mathcal{X}$ and the hyperplane arrangement that would allow the fitted hyperplane $\mathcal{H}$ to coincide with one of the underlying hyperplanes $\mathcal{H}_1, \ldots, \mathcal{H}_n$; in the language of Fermat-Torricelli problems, useful solutions are vertices. Inspired by that work, we are here concerned with the following special case:

**PROBLEM 1** (Fermat-Torricelli points of a triangle in the projective plane). For $P_1, P_2 \in \mathbb{P}^2$ let $\varphi_{P_1P_2}$ be the smallest angle between the lines $\ell_{P_1}, \ell_{P_2}$ through the origin in $\mathbb{R}^3$ represented by $P_1$ and $P_2$ respectively and set $d_{P_2}(P_1, P_2) = \sin \varphi_{P_1P_2}$. Given $A, B, C \in \mathbb{P}^2$ such that the corresponding lines $\ell_A, \ell_B, \ell_C$ are not coplanar in $\mathbb{R}^3$, characterize the solutions $\mathcal{P}_{ABC} \subset \mathbb{P}^2$ to the problem

$$\min_{P \in \mathbb{P}^2} J_{ABC}(P) = d_{P_2}(A, P) + d_{P_2}(B, P) + d_{P_2}(C, P).$$

Problem 1 is fundamental and of intrinsic mathematical interest. Within the Fermat-Torricelli and Steiner-Weber literature, it represents a natural progression from planar to spherical to projective triangles. Beyond its aforementioned connection with hyperplane clustering in machine learning, it also relates to the old and recently growing literature concerning algebraic, geometric and combinatorial properties of line arrangements, e.g. see \cite{GKMS16, KT19, CHS21}, and their applications to signal processing, quantum information and coding theory.

**Remark 1.** The choice of the sine distance in Problem 1 is because it this distance that appears in the machine learning problem of hyperplane clustering mentioned above, which motivated this work. If instead one considers the angular distance in $\mathbb{P}^2$, that is $\varphi_{P_1P_2}$ for $P_1, P_2 \in \mathbb{P}^2$, then one can view $\mathbb{P}^2$ as a complete Riemannian manifold and the results of Afsari \cite{Afs11} and Arnaudon & Milco \cite{AM14} apply. In particular, it follows from Theorem 1 in \cite{Afs11}, that the Fermat-Torricelli point of a projective triangle $\triangle ABC$ is unique, as soon as the lines $\ell_A, \ell_B, \ell_C$ pass through the interior of a spherical cap of the unit sphere of radius equal to $45^\circ$. Uniqueness also follows from \cite{AM14}, whenever $A, B, C$ are randomly chosen under the Lebesque measure on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. 

The main contribution of this paper is a rather remarkable property of the Fermat-Torricelli point of a triangle in the projective plane, placing it on one or more of the vertices of the triangle, providing that all sides of the triangle are larger than $\sqrt{3}/2$ (equivalently, larger than 60° in the angular distance). In particular, under the said condition, the Fermat-Torricelli point is unique if and only if there is a unique side of maximal length. Perhaps surprisingly, this result complements the result of Afsari mentioned in Remark 1, which guarantees uniqueness for small triangles. Embedded in our proof is another interesting result, a complete characterization of the equilateral case. More precisely:

**Theorem 1.** Let $A, B, C \in \mathbb{P}^2$ be ordered such that $\varphi_{AB} \leq \varphi_{AC} \leq \varphi_{BC}$.

1. If $60^\circ \leq \varphi_{AB} \leq \varphi_{AC} < \varphi_{BC}$, then $\mathcal{P}_{ABC} = \{A\}$.
2. If $60^\circ \leq \varphi_{AB} < \varphi_{AC} = \varphi_{BC}$, then $\mathcal{P}_{ABC} = \{A, B\}$.
3. If $\varphi_{AB} = \varphi_{AC} = \varphi_{BC}$, denote by $E$ the point of $\mathbb{P}^2$ that represents the centroid $\ell_E$ in $\mathbb{R}^3$ of the three equiangular lines $\ell_A, \ell_B, \ell_C$ associated to $A, B, C$; see Figure 1. We have:
   a. If $\varphi > 60^\circ$, then $\mathcal{P}_{ABC} = \{A, B, C\}$.
   b. If $\varphi = 60^\circ$, then $\mathcal{P}_{ABC} = \{A, B, C, E\}$.
   c. If $\varphi < 60^\circ$, then $\mathcal{P}_{ABC} = \{E\}$.

![Figure 1. An equilateral triangle $\triangle ABC$ in $\mathbb{P}^2$ or equivalently an equiangular arrangement of three lines through the origin $\ell_A, \ell_B, \ell_C$ in $\mathbb{R}^3$. The angle of the arrangement is $\varphi$ and $E$ is the centroid.](image)

The paper is organized as follows. In §2 we reduce Problem 1 to the unit sphere $S^2 \subset \mathbb{R}^3$ and establish basic properties that will be used repeatedly in the sequel. In §3 we discuss a special type of projective triangles, that we call big triangles, and prove that for such triangles the Fermat-Torricelli point is always a vertex. In §4 we prove part (3) of Theorem 1, the equilateral case. This is used in §5 together with a deformation argument to prove part (2) of Theorem 1, the isosceles case. Finally, §6 uses these two cases to prove part (1) of Theorem 1, the general case.

The statement of the equilateral case was already contained in the expository paper [TV17]. A proof has only been published in the PhD thesis [Tsa17]; the proof here is an improved version. We are grateful to Bijan Afsari for valuable
2. Reduction to the Sphere and First Properties

Let $A, B$ be two points in the real projective plane $\mathbb{P}^2$ and let $a, b$ be points on the unit sphere $S^2$ of $\mathbb{R}^3$ that respectively represent $A, B$. Denote by $\ell_A$ the line in $\mathbb{R}^3$ that passes through the origin and $a$ and similarly for $\ell_B$. We denote by $\varphi_{ab}$ the angle between the vectors $a, b$ and by $\varphi_{AB} \leq 90^\circ$ the smallest angle between $\ell_A$ and $\ell_B$. With $a^T b$ the standard inner product of $a$ and $b$, we have that $\varphi_{AB} = \arccos(|a^T b|)$ and $\varphi_{ab} = \arccos(a^T b)$. Hence, the function $d_{\mathbb{P}^2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{R}_{\geq 0}$ that appears in Problem $\square$ induces a function $d_{\mathbb{S}^2} : S^2 \times S^2 \to \mathbb{R}_{\geq 0}$ given by

$$d_{\mathbb{S}^2}(a, b) = \sqrt{1 - (a^T b)^2} = \sin \varphi_{AB} = d_{\mathbb{P}^2}(A, B).$$

Note that $d_{\mathbb{S}^2}(a, b)$ is the Euclidean distance of the point $a$ to the line $\ell_B$.

Problem $\square$ can be equivalently cast as a problem on the sphere

$$(2.1) \quad \min_{p \in \mathbb{S}^2} \ J_{abc}(p) := d_{\mathbb{S}^2}(a, p) + d_{\mathbb{S}^2}(b, p) + d_{\mathbb{S}^2}(c, p),$$

where now $c \in \mathbb{S}^2$ represents $C \in \mathbb{P}^2$. Following the assumption in Problem $\square$ the points $a, b, c$ are linearly independent.

The function $d_{\mathbb{S}^2}$ satisfies the triangle inequality and thus both $d_{\mathbb{S}^2}, d_{\mathbb{P}^2}$ are genuine distance functions:

**Lemma 1.** For any $v_1, v_2, v_3 \in S^2$ we have $d_{\mathbb{S}^2}(v_1, v_2) \leq d_{\mathbb{S}^2}(v_1, v_3) + d_{\mathbb{S}^2}(v_2, v_3)$, with the equality achieved if and only if $v_3 = \pm v_1$ or $v_3 = \pm v_2$.

**Proof.** If $v_1, v_2$ are colinear then the statement is trivial. So assume otherwise and let $\mathcal{H}$ be the plane $\text{span}(v_1, v_2)$. Let $\varphi_{ij} = \arccos(v_i^T v_j)$ for $i, j \in \{1, 2, 3\}$. Suppose first that $v_3 \in \mathcal{H}$. The points $v_1, -v_1, v_2, -v_2$ partition the unit circle $S^1 \subset \mathcal{H}$ into four arcs. The point $v_3$ lies in one of these arcs, say $\alpha$, whose endpoints we denote by $v_1 \in \{v_1, -v_1\}$ and $v_2 \in \{v_2, -v_2\}$. The length of $\alpha$ is either $\varphi_{12}$ or $\pi - \varphi_{12}$. The point $v_3$ partitions $\alpha$ into two arcs $\alpha_1$ and $\alpha_2$ with respective endpoints $v_1, v_3$ and $v_3, v_2$. The length of $\alpha_1$ is either $\varphi_{13}$ or $\pi - \varphi_{13}$. Similarly, the length of $\alpha_2$ is either $\varphi_{23}$ or $\pi - \varphi_{23}$. Now,

$$d_{\mathbb{S}^2}(v_1, v_2) = \sin(\varphi_{12})$$

$$= \sin(\text{length}(\alpha))$$

$$= \sin(\text{length}(\alpha_1) + \text{length}(\alpha_2))$$

$$\leq \sin(\text{length}(\alpha_1)) + \sin(\text{length}(\alpha_2))$$

$$= \sin(\varphi_{13}) + \sin(\varphi_{23})$$

$$= d_{\mathbb{S}^2}(v_1, v_3) + d_{\mathbb{S}^2}(v_2, v_3).$$

The equality is achieved if and only if $\text{length}(\alpha_1) = 0$ or $\text{length}(\alpha_2) = 0$, which means $v_3 = v_1$ or $v_3 = v_2$. This proves the statement for $v_3 \in \mathcal{H}$.

To finish, consider any $v_3 \in S^2 \setminus \mathcal{H}$ and let $\ell_{v_3}$ be the line in $\mathbb{R}^3$ spanned by $v_3$. Let $\pi_{\mathcal{H}}(v_3)$ be the orthogonal projection of $v_3$ onto the plane $\mathcal{H}$. Let $v_3'$ be any of the intersection points of $\pi_{\mathcal{H}}(v_3)$ with the unit circle $S^1 \subset \mathcal{H}$. By elementary arguments one sees that $|v_1^T v_3'| > |v_1^T v_3|$ and $|v_2^T v_3'| > |v_2^T v_3|$. Hence

$$d_{\mathbb{S}^2}(v_1, v_3) + d_{\mathbb{S}^2}(v_2, v_3) > d_{\mathbb{S}^2}(v_1, v_3') + d_{\mathbb{S}^2}(v_2, v_3').$$
By what we proved in the first paragraph, we have
\[ d_{S^2}(v_1, v'_1) + d_{S^2}(v_2, v'_2) \geq d_{S^2}(v_1, v_2), \]
which concludes the proof. \( \square \)

Next, we establish some first properties of the set \( \mathcal{P}_{abc} \) of solutions to problem 2.1 that we will need towards proving Theorem 2.

**Lemma 2.** Let \( p \in \mathcal{P}_{abc} \) and \( v_1, v_2 \) two distinct points in \( \{a, b, c\} \). Then
\[ d_{S^2}(v_1, p) \leq d_{S^2}(v_1, v_2). \]

**Proof.** Write \( \{v_3\} = \{a, b, c\} \setminus \{v_1, v_2\} \). By Lemma 1 we have
\[ d_{S^2}(v_2, p) + d_{S^2}(v_3, p) \geq d_{S^2}(v_2, v_3). \]
If \( d_{S^2}(v_1, p) > d_{S^2}(v_1, v_2) \), then
\[ J_{abc}(p) = d_{S^2}(v_1, p) + d_{S^2}(v_2, p) + d_{S^2}(v_3) > d_{S^2}(v_1, v_2) + d_{S^2}(v_2, v_3) = J_{abc}(v_2). \]
This contradicts the fact that \( p \in \mathcal{P}_{abc} \). \( \square \)

As in the proof of Lemma 1 let us continue using \( \mathcal{H} \) for the plane \( \text{span}(a, b) \) and \( \pi_{\mathcal{H}} \) for the orthogonal projection of \( \mathbb{R}^3 \) onto \( \mathcal{H} \).

**Lemma 3.** Let \( p \in \mathcal{P}_{abc} \) and set \( p_{\mathcal{H}} = \pi_{\mathcal{H}}(p) \) and \( p_{\perp} = p - p_{\mathcal{H}} \). Then
\[ (c^\top p_{\mathcal{H}})(c^\top p_{\perp}) \geq 0. \]

**Proof.** For the sake of a contradiction suppose that \( (c^\top p_{\mathcal{H}})(c^\top p_{\perp}) < 0 \). Let \( p' = p_{\mathcal{H}} - p_{\perp} \in S^2 \) be the reflection of \( p \) with respect to \( \mathcal{H} \). Then \( a^\top p' = a^\top p \), \( b^\top p' = b^\top p \) and
\[ |c^\top p'| = |c^\top p_{\mathcal{H}} - c^\top p_{\perp}| < |c^\top p_{\mathcal{H}} + c^\top p_{\perp}| = |c^\top p|. \]
Therefore, \( d_{S^2}(a, p') = d_{S^2}(a, p) \), \( d_{S^2}(b, p') = d_{S^2}(b, p) \) and \( d_{S^2}(c, p') < d_{S^2}(c, p) \), that is \( J_{abc}(p') < J_{abc}(p) \). This contradicts the fact that \( p \in \mathcal{P}_{abc} \). \( \square \)

**Lemma 4.** Let \( p \in \mathcal{P}_{abc} \) and let \( \{v_1, v_2, v_3\} = \{a, b, c\} \). Let \( n \in \mathbb{R}^3 \) be a non-zero normal vector to the plane \( \text{span}(v_1, v_2) \). Then \( v_3^\top p \) and \( (v_3^\top n)(p^\top n) \) are both either non-negative or non-positive.

**Proof.** It is enough to prove the statement for \( v_1 = a, v_2 = b, v_3 = c \). By Lemma 3 \( (c^\top p_{\mathcal{H}})(c^\top p_{\perp}) \geq 0 \). If \( c^\top p = c^\top p_{\mathcal{H}} + c^\top p_{\perp} \geq 0 \), we have \( c^\top p_{\mathcal{H}} \geq 0 \) and \( c^\top p_{\perp} \geq 0 \). Moreover \( p_{\perp} = (p^\top \hat{n})\hat{n}, \) where \( \hat{n} = n/\|n\|_2 \). Taking inner product of both sides of this equality with \( c \), we get \( c^\top p_{\perp} = (c^\top \hat{n})(p^\top \hat{n}) \). Thus \( (c^\top n)(p^\top n) \geq 0 \). The case \( c^\top p \leq 0 \) follows similarly. \( \square \)

**Lemma 5.** Let \( p \in \mathcal{P}_{abc} \) and suppose that \( a^\top b, b^\top c, c^\top a \) are all positive. Then \( a^\top p, b^\top p, c^\top p \) are either all non-positive or all non-negative.

**Proof.** At least two of \( a^\top p, b^\top p, c^\top p \) are both non-negative or non-positive. Without loss of generality we assume that \( a^\top p \) and \( b^\top p \) are non-negative. We may assume that not both of them are zero, since otherwise we are done. We will prove that \( c^\top p \geq 0 \).
In the notation of Lemma 3 we have \( p_H = \lambda_a a + \lambda_b b \) for some \( \lambda_a, \lambda_b \in \mathbb{R} \). We will prove that \( \lambda_a, \lambda_b \geq 0 \). By Lemma 2 we have
\[
1 \geq a^\top p_H = \lambda_a + \lambda_b a^\top b = a^\top p \geq a^\top b, \\
1 \geq b^\top p_H = \lambda_a a^\top b + \lambda_b = b^\top p \geq a^\top b,
\]
from which we extract
\[
1 \geq \lambda_a + \lambda_b a^\top b \geq a^\top b \\
a^\top b \leq \lambda_a a^\top b + \lambda_b \leq 1.
\]
Subtracting the first row of inequalities from the second row we get
\[
a^\top b - 1 \leq (1 - a^\top b)(\lambda_b - \lambda_a) \leq 1 - a^\top b,
\]
which in turn gives \(-1 \leq \lambda_b - \lambda_a \leq 1\). Combining the inequalities \( a^\top b \leq \lambda_a + \lambda_b a^\top b \) and \( \lambda_b \leq 1 + \lambda_a \) we have
\[
a^\top b \leq \lambda_a + \lambda_b a^\top b \leq \lambda_a + (1 + \lambda_a) a^\top b,
\]
from which we read \((1 + a^\top b)\lambda_a \geq 0\) and thus \( \lambda_a \geq 0 \). Similarly, we have \( \lambda_b \geq 0 \).

As a consequence,
\[
c^\top p_H = \lambda_a a^\top c + \lambda_b b^\top c \geq 0.
\]
Since by hypothesis \( a^\top b > 0 \), (2.2) gives that not both \( \lambda_a \) and \( \lambda_b \) are zero. Thus the hypothesis \( a^\top c, b^\top c > 0 \) gives \( c^\top p_H > 0 \). Now Lemma 3 gives \( c^\top p_{H^+} \geq 0 \), whence \( c^\top p = c^\top p_H + c^\top p_{H^+} > 0 \).

Next, we develop a first order optimality condition. For convenience, with \( p \in S^2 \) we define a function \( \tau_p : S^2 \setminus \{\pm p\} \to \mathbb{R}^3 \) as
\[
\tau_p(x) = \frac{(I - pp^\top)x}{\| (I - pp^\top)x \|_2} = \frac{(I - pp^\top)x}{\sqrt{1 - (p^\top x)^2}},
\]
where \( I \) is the \( 3 \times 3 \) identity matrix. The condition is:

**Lemma 6.** If \( p \in \mathcal{P}_{abc} \) and \( p \notin \{a, b, c\} \), then
\[
(a^\top p)\tau_p(a) + (b^\top p)\tau_p(b) + (c^\top p)\tau_p(c) = 0.
\]

**Proof.** The formula for \( \mathcal{J}_{abc} : S^2 \to \mathbb{R} \) given in (2.1) can also be used to define a function \( \mathcal{J}_{abc} : C \to \mathbb{R} \), where \( C \) is the convex polytope
\[
C = \{ \xi \in \mathbb{R}^3 \mid |a^\top \xi| \leq 1, |b^\top \xi| \leq 1, |c^\top \xi| \leq 1 \}.
\]
This function is smooth everywhere except at the boundary of the polytope
\[
\partial C = \{ \xi \in C : |a^\top \xi| = 1 \text{ or } |b^\top \xi| = 1 \text{ or } |c^\top \xi| = 1 \}.
\]
Its gradient at a point \( q \in C \setminus \partial C \) is given by
\[
\nabla_q \mathcal{J}_{abc} = \frac{(a^\top q) a}{\sqrt{1 - (a^\top q)^2}} - \frac{(b^\top q) b}{\sqrt{1 - (b^\top q)^2}} - \frac{(c^\top q) c}{\sqrt{1 - (c^\top q)^2}}.
\]
Now, \( S^2 \subset C \) is a smooth Riemannian manifold and the punctured sphere \( S^2_{abc} := S^2 \setminus \partial C \cap S^2 = S^2 \setminus \{\pm a, \pm b, \pm c\} \) is a smooth open submanifold. Hence \( \mathcal{J}_{abc} \) is smooth on \( S^2_{abc} \) with Riemann gradient at a point \( q \in S^2_{abc} \) given by
\[
\nabla_q \mathcal{J}_{abc} = (I - qq^\top)\nabla_q \mathcal{J}_{abc}.
\]
Let us call $\mathcal{J}''_{abc}$ the restriction of $\mathcal{J}_{abc}$ to $\mathcal{S}^2_{abc}$. Since $p \in \mathcal{P}_{abc} \setminus \{\pm a, \pm b, \pm c\}$, it is a global minimum of the smooth function $\mathcal{J}''_{abc}$ and thus basic facts from Riemannian optimization imply that $p$ must satisfy the equation

$$\nabla_p \mathcal{J}''_{abc} = \nabla_p \mathcal{J}_{abc} = (I - pp^\top) \nabla_p \mathcal{J}'_{abc} = 0,$$

which is exactly the relationship in the statement of the lemma. \hfill $\Box$

### 3. Big Triangles

As in the previous section, let $a, b, c \in \mathbb{S}^2$ be representatives of $A, B, C \in \mathbb{P}^2$ and observe that the sign of the quantity $(a^\top b)(b^\top c)(a^\top c)$ is independent of the choice of representatives. The following notion will turn out to be convenient:

**Definition 7.** We call the projective triangle $\triangle ABC$ a big triangle, if

$$(a^\top b)(b^\top c)(a^\top c) \leq 0.$$ 

If the inequality in Definition 7 is strict, the corresponding line arrangement was called coherent in [Tay77]. Instead, the attribute big here is justified by:

**Proposition 8.** If $\triangle ABC$ is a big triangle, then $\varphi_{AB} + \varphi_{AC} + \varphi_{BC} > \pi$.

**Proof.** Recall our convention $\varphi_{AB} \leq \varphi_{AC} \leq \varphi_{BC}$. If $\varphi_{AC} = \pi/2$, then $\varphi_{BC} = \pi/2$ and we are done. We may thus assume $\varphi_{AB} \leq \varphi_{AC} < \pi/2$. We may also assume that $a^\top b = \cos \varphi_{AB}$ and $a^\top c = \cos \varphi_{AC}$. Hence $a^\top b, a^\top c > 0$ and so necessarily $b^\top c \leq 0$. With $v_1, v_2 \in \mathbb{S}^2$ we denote by $\varphi_{v_1v_2}$ the angle between the vectors $v_1, v_2$. We have $\varphi_{bc} = \pi - \varphi_{BC}$ and the triangular inequality for arc lengths

$$\varphi_{ab} + \varphi_{ac} \geq \varphi_{bc},$$

gives the inequality in the statement with $\geq 0$. Note that the equality is achieved if and only if $a \in bc$, which would imply that $a, b, c$ are coplanar. \hfill $\Box$

The main result of this section is:

**Proposition 9.** If $\triangle ABC$ is a big triangle, then $\mathcal{P}_{ABC} \subset \{A, B, C\}$.

Towards proving this fact, we let $p$ be a representative of $P \in \mathcal{P}_{ABC}$ and assume without loss of generality that $a, b, c$ are such that $a^\top p, b^\top p, c^\top p \geq 0$. Again without loss of generality we assume

$$a^\top p \geq \max\{b^\top p, c^\top p\} \iff d_{\mathbb{S}^2}(a, p) \leq \min\{d_{\mathbb{S}^2}(b, p), d_{\mathbb{S}^2}(c, p)\}. \tag{3.1}$$

**Lemma 10.** We have $a^\top p > 1/\sqrt{2}$; in particular $\varphi_{ap} = \varphi_{AP} < \pi/4$.

**Proof.** If $(a^\top p)^2 \leq 1/2$, then $d_{\mathbb{S}^2}(a, p) \geq \sqrt{2}/2$. By the assumption

$$\mathcal{J}_{abc}(p) \geq 3d_{\mathbb{S}^2}(a, p) \geq \frac{3}{2}\sqrt{2} > 2 \geq \sin \varphi_{AB} + \sin \varphi_{AC} = \mathcal{J}_{abc}(a),$$

which contradicts the fact that $p \in \mathcal{P}_{abc}$. \hfill $\Box$

**Lemma 11.** There is a choice of representatives additionally satisfying $b^\top c \leq 0$.

**Proof.** Recall the assumption $a^\top p \geq \max\{b^\top p, c^\top p\}$ and suppose first that equality is achieved. So suppose that $a^\top p = b^\top p$; the case $a^\top p = b^\top p$ follows similarly. From Lemma 11 we know that $\varphi_{ap} = \varphi_{bp} < \pi/4$, and so $\varphi_{ab} \leq \varphi_{ab} + \varphi_{bp} < \pi/2$. Hence, $\varphi_{ab} = \varphi_{AB}$ and $a^\top b > 0$. Since the triangle is big, either $a^\top c \leq 0$ or $b^\top c \leq 0$, and we can choose $c$ such that $b^\top c \leq 0$. 

We may thus assume that \(a^\top p > \max\{b^\top p, c^\top p\}\). If \(b^\top c \leq 0\) we are done, so assume \(b^\top c > 0\). Since the triangle is big, we have either \(a^\top b \geq 0\), \(a^\top c \leq 0\) or \(a^\top b \leq 0\), \(a^\top c \geq 0\). We consider only the first case; the second case is similar.

By the assumption, \(a^\top p > b^\top p \geq 0\). Let \(d = a - b\), then \(d^\top p > 0\) and \(d^\top c = a^\top c - b^\top c < 0\). Let \(\mathcal{V}_d = \text{span}(d)\) and \(\mathcal{V}_{d^\perp}\) the orthogonal complement of \(\mathcal{V}_d\). Then we have the decomposition \(p = p_d + p_{d^\perp}\), where \(p_d \in \mathcal{V}_d\) and \(p_{d^\perp} \in \mathcal{V}_{d^\perp}\). Since \((a - b)^\top (a + b) = 0\), we have

\[
\begin{align*}
  a^\top p_d &= (a + b - b)^\top p_d = -b^\top p_d, \\
  a^\top p_{d^\perp} &= (a - b + b)^\top p_{d^\perp} = b^\top p_{d^\perp}.
\end{align*}
\]

From \(p_d = (dd^\top)p\) we obtain \(c^\top p_d = (d^\top c)(d^\top p) < 0\). Now \(c^\top p = c^\top p_d + c^\top p_{d^\perp}\) and \(c^\top p \geq 0\) by hypothesis, so \(c^\top p_{d^\perp} \geq 0\). Let \(p^\prime = p_{d^\perp} - p_d\). Then

\[
\begin{align*}
  a^\top p^\prime &= a^\top p_{d^\perp} - a^\top p_d = b^\top p_{d^\perp} + b^\top p_d = b^\top p, \\
  b^\top p^\prime &= b^\top p_{d^\perp} - b^\top p_d = a^\top p_{d^\perp} + a^\top p_d = a^\top p, \\
  c^\top p^\prime &= c^\top p_{d^\perp} - c^\top p_d > c^\top p_{d^\perp} + c^\top p_d = c^\top p.
\end{align*}
\]

But this implies the contradiction \(J_{abc}(p^\prime) < J_{abc}(p)\).

We can now finish the proof of Proposition 9.

By Lemma 11 we have \(b^\top c \leq 0\). From the assumption \(\varphi_{AP} \leq \min(\varphi_{BP}, \varphi_{CP})\), we have \(p \neq \pm b\) and \(p \neq \pm c\). Since \(a^\top p \geq 0\), we also have \(p \neq -a\). We assume that \(p \neq a\) and reach a contradiction. Lemma 6 gives

\[
(a^\top p)\tau_p(a) + (b^\top p)\tau_p(b) + (c^\top p)\tau_p(c) = 0.
\]

With \(u = (b^\top p)\tau_p(b) + (c^\top p)\tau_p(c)\) the above equation can be written as \((a^\top p)\tau_p(a) + u = 0\). In what follows we show that \(|u|_2 < a^\top p\); this is a contradiction because by definition \(|\tau_p(a)|_2 = 1\). From the definition of \(u\), we have

\[
u^\top u = (b^\top p)^2 + (c^\top p)^2 + 2(b^\top p)(c^\top p)\tau_p(b)^\top \tau_p(c).
\]

Using \(b^\top c \leq 0\) and the definition of the function \(\tau_p\), one easily verifies that

\[
\tau_p(b)^\top \tau_p(c) \leq -(b^\top p)(p^\top c).
\]

In turn, this gives

\[
u^\top u \leq (b^\top p)^2 + (c^\top p)^2 - 2(b^\top p)^2(c^\top p)^2 = (b^\top p)^2 + (c^\top p)^2[1 - 2(b^\top p)^2].
\]

From Lemma 10 \((a^\top p)^2 > 1/2\). If both \((b^\top p)^2 \leq 1/2\) and \((c^\top p)^2 \leq 1/2\), then

\[
u^\top u \leq (b^\top p)^2 + \frac{1}{2}(1 - 2(b^\top p)^2) = \frac{1}{2} < (a^\top p)^2,
\]

and we are done. Suppose then that either \((b^\top p)^2 > 1/2\) or \((c^\top p)^2 > 1/2\). If \((b^\top p)^2 > 1/2\), then

\[
u^\top u \leq (b^\top p)^2 \leq (a^\top p)^2,
\]

where the last inequality is due to the hypothesis \((3.1)\). If equality is achieved everywhere, \((3.2)\) gives \(c^\top p = 0\), and we also have \(b^\top p = a^\top p\). Now the triangular inequality (Lemma 1) gives the contradiction

\[
J_{abc}(p) = d_{g^2}(a, p) + d_{g^2}(b, p) + 1 > \sin \varphi_{AB} + 1 \geq J_{abc}(a).
\]
4. Equilateral Triangles

In this section we will prove part (3) of Theorem 1 which we will use in later sections to prove parts (1) and (2). This is the equilateral case where \( \varphi_{AB} = \varphi_{BC} = \varphi \). If \( \triangle ABC \) is a big triangle in the sense of Definition 7 then Proposition 8 gives \( \varphi > 60^\circ \). Then Proposition 9 together with the fact that \( J_{ABC}(A) = J_{ABC}(B) = J_{ABC}(C) \) shows that \( \mathcal{P}_{ABC} = \{ A, B, C \} \). We may thus assume in the rest of this section that the triangle \( \triangle ABC \) is not big, that is

\[
(a^\top b)(a^\top c)(b^\top c) > 0,
\]

for any representatives \( a, b, c \). We may also assume that \( b, c \) are conveniently chosen such that \( a^\top b = a^\top c = \cos \varphi \). From the above inequality \( b^\top c > 0 \) and so

\[
a^\top b = a^\top c = b^\top c = \cos \varphi.
\]

A first consequence of the symmetry of the configuration is:

**Lemma 12.** Let \( p \in \mathcal{P}_{abc} \) and set

\[
x_1 = a^\top p, \ x_2 = b^\top p, \ x_3 = c^\top p, \ \text{and} \ y_i = \sqrt{1 - x_i^2}, \ \text{for} \ i = 1, 2, 3.
\]

Then either \( y_1 = y_2 \) or \( y_1 = y_3 \) or \( y_2 = y_3 \).

**Proof.** If \( p \) is one of \( \pm a, \pm b, \pm c \), then the statement clearly holds, since if say \( p = a \), then \( y_2 = y_3 = \sin(\psi) \). So suppose that \( p \notin \{ \pm a, \pm b, \pm c \} \). Lemma 9 gives

\[
(a^\top p)\tau_p(a) + (b^\top p)\tau_p(b) + (c^\top p)\tau_p(c) = 0.
\]

Setting \( z = \cos \psi \), and taking inner product of both sides of the above equation with \( a, b, c \), we respectively obtain

\[
\begin{align*}
x_1y_1 + x_2y_2 + x_3y_3 &= \left( \frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} \right)x_1 = 0, \\
x_1y_1 + x_2y_2 + x_3y_3 &= \left( \frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} \right)x_2 = 0, \\
x_1y_1 + x_2y_2 + x_3y_3 &= \left( \frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} \right)x_3 = 0.
\end{align*}
\]

Let \( T = \mathbb{R}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z] \) be a polynomial ring in seven variables over the real numbers. Then for \( i = 1, 2, 3 \) the \( x_i, y_i \) and \( z \) are roots of the following polynomials of \( T \):

\[
\begin{align*}
F_1 &= X_1Y_1Y_2Y_3 + X_2Y_3(Z - X_1X_2) + X_3Y_2(Z - X_1X_3), \\
F_2 &= X_1Y_3(Z - X_1X_2) + X_2Y_2Y_3 + X_3Y_1(Z - X_2X_3), \\
F_3 &= X_1Y_2(Z - X_1X_3) + X_2Y_1(Z - X_2X_3) + X_3Y_1Y_2Y_3, \\
F_4 &= X_1^2 + Y_1^2 - 1, \\
F_5 &= X_2^2 + Y_2^2 - 1, \\
F_6 &= X_3^2 + Y_3^2 - 1.
\end{align*}
\]

Let \( I \) be the ideal of \( T \) generated by the \( F_i, i = 1, \ldots, 6 \). Then the polynomial

\[
F = (1 - Z)(Y_1 - Y_2)(Y_2 - Y_3)(Y_3 - Y_1)(Y_1 + Y_2 + Y_3) \in T
\]

is in the ideal \( I \). That is,

\[
F = C_1F_1 + C_2F_2 + C_3F_3 + C_4F_4 + C_5F_5 + C_6F_6,
\]
for polynomials $C_i \in \mathcal{T}$. The expression for the $C_i$'s is as follows:
\[
C_1 = X_2Y_1Z - X_3Y_1Z + X_1Y_2Z - X_1Y_3Z - X_1Y_2 + X_1Y_3 \\
C_2 = -X_2Y_1Z - X_1Y_2Z + X_3Y_2Z + X_2Y_3Z + X_2Y_1 - X_2Y_3 \\
C_3 = X_3Y_1Z - X_2Y_2Z + X_1Y_3Z - X_2Y_3Z - X_3Y_1 + X_3Y_2 \\
C_4 = X_2^2Y_1Y_2Z - X_3^2Y_2^2Z - X_2^2Y_3ZZ - X_1X_2Y_2Y_3Z + X_1X_3Y_2Y_3Z \\
- Y_1Y_2^2Y_3Z - Y_3^2Y_3Z + X_2^2Y_2^2Z + Y_1Y_2Y_2^2Z + Y_2^2Y_3^2Z - X_3^2Y_1Z \\
+ X_1^2Y_2^2 + X_2^2Y_1Y_3 + Y_1Y_2^2Y_3 + Y_2^2Y_3^2 - X_3^2Y_3^2 - Y_1Y_2Y_3^2 - Y_2^3Y_3^2 \\
+ Y_2^2Z - Y_3^2Z - Y_2^2 + Y_3^2 \\
C_5 = X_2^3Y_1^2Z - X_3^2Y_1Y_2Z + X_1X_2Y_1Y_3Z - X_2X_3Y_1Y_3Z + Y_3^2Y_3Z \\
+ X_1^2Y_2Y_3Z + Y_2^2Y_2Y_3Z - X_3^2Y_2^2Z - Y_1Y_2^2Y_3Z - Y_3^2Y_3Z - X_3^2Y_2^2 \\
+ X_1^2Y_1Y_2 - Y_1^2Y_3 - X_2^2Y_2Y_3 - Y_3^2Y_2Y_3 + Y_3^2Y_3^2 + Y_1Y_2Y_3^2 + Y_1Y_3^2 \\
- Y_2^2Z + Y_3^2Z + Y_2^2 - Y_3^2 \\
C_6 = -X_2^2Y_1^2Z - X_1X_3Y_1Y_2Z + X_2X_3Y_1Y_2Z - Y_1^2Y_2Z + X_1^2Y_3^2Z \\
+ Y_1^2Y_2^2 + X_2^2Y_2Y_3Z - X_3^2Y_2Y_3Z - Y_2^2Y_2Y_3Z + Y_1^2Y_2Y_3Z + X_2^2Y_3^2 \\
+ Y_1Y_2 - X_1^2Y_2^2 - Y_1Y_3^2 - X_2^2Y_1Y_3 - X_2^2Y_2Y_3 + Y_1^2Y_3^2 + Y_2^2Y_3^2 \\
- Y_1Y_2Y_2^2 + Y_1^2Z - Y_2^2Z - Y_1^2 + Y_2^2
\]
Since the $x_i, y_i, z$ are roots of the $F_1, \ldots, F_6$, we see from (4.1) that they are roots of $F$. Hence,
\[
(1 - z)(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)(y_1 + y_2 + y_3) = 0.
\]
Now, $\psi > 0$ and so $z < 1$. Also, by definition $y_1, y_2, y_3$ are non-negative and by the linear independence assumption on $a, b, c$ not all of them can be zero. That is,
\[
(y_1 - y_2)(y_1 - y_3)(y_2 - y_3) = 0,
\]
which is exactly the statement of the lemma. \hfill \square

We are now ready to prove part (3) of Theorem 11.

**Proposition 13.** Set $e = (a + b + c)/\|a + b + c\|_2 \in \mathbb{S}^2$ and let $E$ be the point of $\mathbb{F}^2$ represented by $e$. Then $\mathcal{P}_{ABC}$ satisfies the following phase transition:

1. If $\varphi > 0^\circ$, then $\mathcal{P}_{ABC} = \{A, B, C\}$.
2. If $\varphi = 60^\circ$, then $\mathcal{P}_{ABC} = \{A, B, C, E\}$.
3. If $\varphi < 60^\circ$, then $\mathcal{P}_{ABC} = \{E\}$.

**Proof.** Recall that by definition $0 < \varphi < 90^\circ$. We can parametrize $a, b, c$ as
\[
a = \mu \begin{bmatrix} \alpha & 1 & 1 \end{bmatrix}^T, \\
b = \mu \begin{bmatrix} 1 & \alpha & 1 \end{bmatrix}^T, \\
c = \mu \begin{bmatrix} 1 & 1 & \alpha \end{bmatrix}^T,
\]
with $\mu := (2 + \alpha^2)^{-\frac{1}{2}}$. The parameter $\alpha$ encodes the value of the angle $\varphi$ as follows:
\[
\cos(\varphi) = \frac{1 + 2\alpha}{2 + \alpha^2}, \quad \alpha = \frac{1}{\cos \varphi} \pm \sqrt{\frac{1}{\cos^2 \varphi} + \frac{1}{\cos \varphi} - 2}
\]
From the second formula above, we select the largest of these values whenever \( \varphi \in (0^\circ, 90^\circ) \) gives a 1-1 correspondence with \( \alpha \in (1, +\infty) \). With this convention \( \alpha \) ranges in the interval \( (1, +\infty) \) with

- \( \alpha \in (4, +\infty) \) if and only if \( \varphi \in (60^\circ, 90^\circ) \),
- \( \alpha = 4 \) if and only if \( \varphi = 60^\circ \), and
- \( \alpha \in (1, 4) \) if and only if \( \varphi \in (0^\circ, 60^\circ) \).

By the above parametrization and the hypothesis that \( \varphi < 90^\circ \), we see that \( a^\top b, a^\top c, b^\top c \) are all positive. Thus by Lemma 5 we have that \( a^\top p, b^\top p, c^\top p \) are either all non-negative or all non-positive. Hence, by Lemma 12 at least two among \( a^\top p, b^\top p, c^\top p \) are equal. Which two, depends on \( P \). Let us assume for a moment that \( P \) is such that \( a^\top p = b^\top p \). This equation places \( p \) inside the plane \( V_{ab} = \text{span} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \).

The plane \( V_{ab} \) consists of all vectors that have equal angles from \( a \) and \( b \); in particular both \( c \) and \( e \) are in \( V_{ab} \). In the rest of the proof we will show that, depending on the value of \( \alpha \), it can only be that \( P \) is \( C \) or \( E \) or either of the two. The cases \( p \in V_{ac} \) or \( p \in V_{bc} \) are treated in an identical manner and this concludes the proof because \( J_{ABC}(A) = J_{ABC}(B) = J_{ABC}(C) \).

Since \( p \in V_{ab} \cap S^2 \), we have the parametrization

\[
p = \frac{1}{\sqrt{2v^2 + w^2}} \begin{pmatrix} v \\ v \\ w \end{pmatrix},
\]

with \( v, w \in \mathbb{R} \). The choice \( v = 0 \), corresponding to \( p = e_3 \) (the third standard basis vector) can be excluded, since moving \( p \) from \( e_3 \in V_{ab} \cap S^2 \) to \( c \in V_{ab} \cap S^2 \) while staying in \( V_{ab} \cap S^2 \), results in decreasing angles of \( p \) to \( a, b, c \) and thus in lower values of \( J_{abc}(p) \). Consequently, we can assume \( v = 1 \), and write

\[
p = \frac{1}{\sqrt{2 + w^2}} \begin{pmatrix} 1 \\ 1 \\ w \end{pmatrix},
\]

with \( w \in \mathbb{R} \). Problem 2.1 is now reduced to minimizing the function

\[
J(w) = \frac{2 \left[ (2 + w^2)(2 + \alpha^2) - (1 + \alpha + w)^2 \right]^{1/2}}{\left[ (2 + w^2)(2 + \alpha^2) \right]^{1/2}} + \sqrt{2} |\alpha - w|.
\]

It is easy to check that:

- The following quantity is always positive:

\[
u := (2 + w^2)(2 + \alpha^2) - (1 + \alpha + w)^2
\]

- The choice \( w = \alpha \) corresponds to \( p = c \), and that is the only point where \( J(w) \) is non-differentiable.

- The choice \( w = 1 \) corresponds to \( p = e = (a + b + c)/\|a + b + c\|_2 \).

- \( J(c) = J(e) \) when \( \alpha = 4 \).

We consider the three intervals of \( \alpha \) mentioned above corresponding to statements (1), (2) and (3) of the proposition.
First we consider \( \alpha \in (4, +\infty) \). We will show that for \( w \neq \alpha \), it is always the case that \( J(w) > J(\alpha) \) so that \( P = C \). Expanding the inequality \( J(w) > J(\alpha) \), we obtain

\[
\frac{2u^{1/2} + \sqrt{2}|\alpha - w|}{[(2 + w^2)(2 + \alpha^2)]^{1/2}} > \frac{2\sqrt{(2 + \alpha^2)^2 - (1 + 2\alpha)^2}}{2 + \alpha^2}.
\]

The quantity inside the square root on the right-hand-side is always non-negative:

\[
(2 + \alpha^2)^2 - (1 + 2\alpha)^2 = \alpha^4 - 4\alpha + 3 = (\alpha - 1)^2(\alpha^2 + 2\alpha + 3)
\]

Thus squaring (4.4) we have that \( J(w) > J(\alpha) \) if and only if

\[
4\sqrt{2u^{1/2}|\alpha - w|(2 + \alpha^2)} > p_1
\]

where

\[
p_1 := 4(2 + w^2)[(2 + \alpha^2)^2 - (1 + 2\alpha)^2] - (2 + \alpha^2)[4u + 2(\alpha - w)^2].
\]

By squaring (4.5) we have that \( J(w) > J(\alpha) \) if

\[
p_2 := 32u(\alpha - w)^2(2 + \alpha^2)^2 - p_1^2 > 0.
\]

Now \( p_2 \) admits the factorization

\[
p_2 = 4(w - \alpha)^2p_3, \quad p_3 := (8\alpha^6 - 9\alpha^4 - 112\alpha^3 + 32)w^2 + (-30\alpha^5 - 88\alpha^4 + 40\alpha^3 + 240\alpha^2 + 64\alpha - 64)w
\]

\[
+ (7\alpha^6 - 24\alpha^5 + 64\alpha^4 + 48\alpha^3 + 48\alpha^2 - 256\alpha + 32),
\]

where \( p_3 \) as a quadratic in \( w \) has discriminant

\[
\Delta(p_3) = -32\alpha(\alpha - 4)(\alpha - 1)^2(\alpha^2 + 2)(\alpha^2 + 2\alpha + 3)(7\alpha^2 + 4\alpha + 16).
\]

One checks directly that \( \Delta(p_3) < 0 \) for \( \alpha \in (4, +\infty) \). Moreover, for \( \alpha \in (4, +\infty) \) we have

\[
8\alpha^6 - 9\alpha^4 - 112\alpha^3 + 32 > 128\alpha^4 - 9\alpha^4 - 112\alpha^4 + 32 > 0,
\]

Hence for \( \alpha \in (4, +\infty) \) the leading coefficient \( 8\alpha^6 - 9\alpha^4 - 112\alpha^3 + 32 \) of \( p_3 \) is always positive. Therefore, for \( \alpha \in (4, +\infty) \) we have that \( p_3 \) is always positive, thus \( p_2 \) is positive when \( w \neq \alpha \). That is \( J(w) > J(\alpha) \) for any \( w \neq \alpha \), that is \( P = C \).

Moving on to the case \( \alpha = 4 \), we have

\[
p_2|_{\alpha=4} = 93312(w - 4)^2(w - 1)^2.
\]

Since \( p_2 > 0 \) implies that \( J(w) > J(4) \), we see that the minimum of \( J(w) \) can only be attained for \( w \in \{1, 4\} \). Since \( J(1) = J(4) = 3\), we conclude that \( w = 1, 4 \) are the only minima of \( J(w) \).

For \( \alpha \in (1, 4) \) we proceed in a similar manner to show that \( J(w) > J(1) \) for any \( w \neq 1 \). Expanding the inequality \( J(w) > J(1) \) we have

\[
\frac{2u^{1/2} + \sqrt{2}|\alpha - w|}{[(2 + w^2)(2 + \alpha^2)]^{1/2}} > \frac{3\sqrt{2}|\alpha - 1|}{\sqrt{3(2 + \alpha^2)}}.
\]

Both sides are non-negative and so after squaring them we have that \( J(w) > J(1) \) if and only if

\[
q_1 < 4\sqrt{2}u^{1/2}|\alpha - w|,
\]
where \( q_1 \) is defined as
\[
q_1 := 6(2 + w^2)(\alpha - 1)^2 - [4u + 2(\alpha - w)^2].
\]
If \( q_1 < 0 \), then we are done. So suppose in the rest of the proof that \( q_1 \geq 0 \). By squaring 4.6 we have \( J(w) > J(1) \) if
\[
q_2 > 0, \quad q_2 := 32u(a - w)^2 - q_1^2.
\]
Define the following quantity
\[
q_3 = q_2 - 4\alpha(4 - \alpha)(w - 1)^2 q_1.
\]
Because for \( \alpha \in (1, 4) \) the second term appearing in \( q_3 \) is always non-negative, to show \( q_2 > 0 \) it is enough to show that \( q_3 > 0 \). Now \( q_3 \) admits the factorization
\[
q_4 = 4(w - 1)^2 q_4,
\]
\[
q_4 = (\alpha^4 - 8\alpha^3 + 20\alpha^2 + 8)w^2 + (-2\alpha^4 + 8\alpha^3 - 32\alpha^2 - 16\alpha)w + (5\alpha^4 - 8\alpha^3 + 24\alpha^2),
\]
where \( q_4 \) as a quadratic in \( w \) has discriminant
\[
\Delta(q_4) = -16\alpha^2(\alpha - 4)^2(\alpha - 1)^2(\alpha^2 + 2).
\]
It is clear that \( \Delta(q_4) < 0 \) for \( \alpha \in (1, 4) \). Also, for \( \alpha \in (1, 4) \) the leading coefficient of \( q_4 \) is positive:
\[
\alpha^4 - 8\alpha^3 + 20\alpha^2 + 8 = \alpha^4 - 8\alpha^3 + 16\alpha^2 + 4\alpha^2 + 8 = \alpha^2(\alpha - 4)^2 + 4\alpha^2 + 8 > 0.
\]
Therefore, for every \( w \neq 1 \) we have \( q_4 > 0 \), hence \( q_3 > 0 \), thus \( q_2 > 0 \). That is \( J(w) > J(1) \) for \( w \neq 1 \). □

5. Isosceles Triangles

In this section we will prove part (2) of Theorem 1 which is the case
\[
60^\circ \leq \varphi_{AB} < \varphi_{AC} = \varphi_{BC}.
\]
When \( \varphi_{AC} = \varphi_{BC} = 90^\circ \), we are done by Proposition 9 together with \( J_{ABC}(A) = J_{ABC}(B) = \sin \varphi_{AB} + 1 < 2 = J_{ABC}(C) \). We will thus assume in the rest of this section that \( \varphi_{AC} = \varphi_{BC} < 90^\circ \). For the same reason as in the beginning of §4 we can choose representatives such that \( a^\top b, a^\top c, b^\top c > 0 \).

We may assume without loss of generality that
\[
a = (\cos \alpha, \sin \alpha, 0)
b = (\cos \alpha, -\sin \alpha, 0)
c = (\cos \beta, 0, \sin \beta),
\]
where
\[
\beta = \arccos(\cos \varphi_{AC} / \cos \alpha) \in (0, \pi/2]
\]
is the angle from \( c \) to the plane \( \mathcal{H} = \mathcal{H}_{ab} = \text{span}(a, b) \). Set
\[
O_{xy}^+ = \{ (p_x, p_y, p_z) \in \mathbb{S}^2 | p_x \geq 0, p_y \geq 0 \}.
\]
We first prove:

**Lemma 14.** Suppose that \( \mathcal{P}_{abc} \cap O_{xy}^+ = \{ a \} \). Then Theorem 1(2) is true.
Proof. We introduce an additional notation:

\[ \mathcal{O}_x^+ = \{(p_x, p_y, p_z) \in \mathbb{S}^2 \mid p_x \geq 0\} \]

Let \( M = \text{diag}(1, -1, 1) \in \text{SO}(3) \) be the reflection matrix with respect to the \( xz \)-plane. Since \( Ma = b \) and \( Mc = c \), we have that \( \mathcal{J}_{abc}(Mv) = \mathcal{J}_{abc}(v) \) for any \( v \in \mathbb{S}^2 \). Since \( \mathcal{O}_x^+ = \mathcal{O}_{xy}^+ \cup \mathcal{M}\mathcal{O}_{xy}^+ \) and \( \mathcal{P}_{abc} \cap \mathcal{M}\mathcal{O}_{xy}^+ = M(\mathcal{P}_{abc} \cap \mathcal{O}_{xy}^+) \), we have

\[ \mathcal{P}_{abc} \cap \mathcal{O}_x^+ = (\mathcal{P}_{abc} \cap \mathcal{O}_{xy}^+) \cup M(\mathcal{P}_{abc} \cap \mathcal{O}_{xy}^+) = \{a\} \cup M\{a\} = \{a, b\}, \]

which concludes the proof. \( \square \)

Thus in the rest of this section we will show that \( \mathcal{P}_{abc} \cap \mathcal{O}_{xy}^+ = \{a\} \). We need the following fact:

**Lemma 15.** Let \( p = (p_x, p_y, p_z) \in \mathcal{P}_{abc} \cap \mathcal{O}_{xy}^+ \). Then \( p_x, a^\top p, b^\top p, c^\top p \geq 0 \).

**Proof.** By hypothesis \( p_x, p_y \geq 0 \) and so \( a^\top p = p_x \cos \alpha + p_y \sin \alpha \geq 0 \). If \( a^\top p = 0 \), then \( p_x = p_y = 0 \) and so \( \mathcal{J}_{abc}(p) = 1 + 1 + d_{O(2)}(c, p) > \mathcal{J}_{abc}(a) \). Hence \( a^\top p > 0 \) and Lemma 4 gives \( b^\top p, c^\top p \geq 0 \). Now \( e_z = (0, 0, 1) \) is normal to \( H_{ab} \), thus by Lemma 4 we have \( p_z \sin \beta = (p^\top e_z)(c^\top e_z) \geq 0 \), which means \( p_z \geq 0 \). \( \square \)

In what follows, we take \( p \in \mathcal{P}_{abc} \cap \mathcal{O}_{xy}^+ \) with \( p \neq a \) and proceed in several steps to derive a contradiction by showing that \( \mathcal{J}_{abc}(p) > \mathcal{J}_{abc}(a) \).

**Step 1.** Define the angle

\[ \beta^* = \arccos(\cos(\varphi_{AB})/\cos \alpha) < \arccos(\cos \varphi_{AC}/\cos \alpha) = \beta, \]

and the point

\[ c^* = (\cos \beta^*, 0, \sin \beta^*). \]

Let \( C^* \) be the point in \( \mathbb{P}^2 \) represented by \( c^* \). One immediately checks that \( \varphi_{AB} = \varphi_{AC}^* = \varphi_{BC}^* \). Geometrically, the point \( c^* \) is obtained as follows:

**Construction 16.** Let \( V_{ab} \) be the plane consisting of all points that have equal angles from \( a, b \). By hypothesis \( c \in V_{ab} \cap \mathbb{S}^2 \) and so is the point \( e_x = (1, 0, 0) \) that cuts in half the arc \( \hat{ab} \) and whose angle from \( a, b \) is \( \varphi_{AB}/2 \). Since \( \varphi_{AC} = \varphi_{BC} > \varphi_{AB}/2 \), \( c^* \) is the unique point on the arc \( \hat{ce}_x \) that has angle from \( a, b \) equal to \( \varphi_{AB} \).

**Step 2.** We define an auxiliary point \( c_p \). We start with the following property:

**Lemma 17.** The point \( p \) lies in the spherical triangle of \( \mathbb{S}^2 \) defined by \( a, c, e_x \).

**Proof.** Any two vertices of a spherical triangle, say \( v_1 \) and \( v_2 \), together with the origin, determine a plane \( \mathcal{H}_{v_1v_2} \) which cuts the space into two half-spaces. A point \( q \in \mathbb{S}^2 \) lies in the spherical triangle if and only if for any two vertices \( v_1 \) and \( v_2 \), the third vertex \( v_3 \) and \( q \) lie in the same half-space of \( \mathcal{H}_{v_1v_2} \).

By Lemma 4 we have that \( b \) and \( p \) lie in the same half-space of \( \mathcal{H}_{ac} \). Similarly, \( c \) and \( p \) lie in the same half-space of \( \mathcal{H}_{ab} = \mathcal{H}_{ac}^* \). Moreover, \( e_y = (0, 1, 0) \) is normal to \( \mathcal{H}_{ce_x} \) and since \( p \in \mathcal{O}_{xy}^+ \), we have \( p^\top e_y = p_y \geq 0 \). Also, \( a^\top e_y = \sin \alpha \geq 0 \) and so \( p \) and \( a \) lie in the same half-space of \( \mathcal{H}_{ce_x} \). \( \square \)

Our auxiliary point \( c_p \) is defined as follows:

**Construction 18.** By Lemma 17 the point \( p \) lies in the spherical triangle defined by \( a, c, e_x \). Extend the arc \( \hat{ap} \) in the direction from \( a \) to \( p \) so that it intersects the opposite arc \( \hat{ce}_x \) of the spherical triangle at a point \( c' \). From Construction 10 we have that \( c^* \in \hat{ce}_x \). If \( c' \in c^* \hat{e}_x \), we let \( c_p = c^* \). Otherwise, we let \( c_p = c' \).
We next describe the coordinates of $c_p$. Note that the arc $e_t$ can be written parametrically as

$$g(\psi) = (\cos \psi, 0, \sin \psi)$$

with $\psi \in [0, \beta]$. Thus there are unique $\beta', \beta^* \in [0, \beta]$ such that $c' = g(\beta')$ and $c^* = g(\beta^*)$. With $\beta_p = \max(\beta', \beta^*)$ we have

$$c_p = (\cos \beta_p, 0, \sin \beta_p).$$

In what follows, $C_p$ will be the point in $\mathbb{P}^2$ represented by $c_p$.

**Step 3.** In step 4 we will prove the inequalities

$$J_{abc}(p) - J_{abc_p}(p) \geq J_{abc}(a) - J_{abc_p}(a),$$

$$J_{abc_p}(p) \geq J_{abc}(a),$$

with at least one of them being strict. This will establish that $J_{abc}(p) > J_{abc}(a)$ and thus conclude this section. Here we prove certain auxiliary facts that we need.

**Lemma 19.** Write $p = (p_x, p_y, p_z)$, then $p_x \sin \alpha - p_y \cos \alpha > 0$.

**Proof.** From Lemma 15 we know that $b^\top p \geq 0$. With $n = a \times c$ Lemma 3 gives $(b^\top n)(n^\top p) \geq 0$. Since $b^\top n = \frac{1}{2} \sin \beta \sin 2\alpha > 0$, we thus have $n^\top p \geq 0$. In terms of coordinates this reads

$$p_x \sin \alpha \sin \beta - p_y \cos \alpha \sin \beta - p_z \sin \alpha \cos \beta \geq 0.$$

Again from Lemma 15 we know that $p_z \geq 0$ and so

$$\sin \beta(p_x \sin \alpha - p_y \cos \alpha) \geq p_z \sin \alpha \cos \beta \geq 0.$$

We conclude that $p_x \sin \alpha - p_y \cos \alpha \geq 0$. If $p_x \sin \alpha - p_y \cos \alpha = 0$, then the above inequality gives $p_z = 0$. But this would imply that $p = a$, a contradiction. □

**Lemma 20.** Define function $f : [0, \pi/2] \to \mathbb{R}$ as

$$f(\psi) = (p_x \sin \alpha - p_y \cos \alpha) \sin \psi - p_z \sin \alpha \cos \psi.$$

Then $f(\beta') = 0$ and $f$ is strictly ascending on $[0, \pi/2]$.

**Proof.** By construction $a, c', p$ are coplanar and so $(a \times c')^\top p = 0$, which in coordinates is the same as $f(\beta') = 0$. From Lemma 19 we have $p_x \sin \alpha - p_y \cos \alpha > 0$, while from Lemma 15 we have $p_z \geq 0$, thus $f$ is strictly ascending. □

**Lemma 21.** For $\psi \in [\beta', \beta]$ we have $g(\psi)^\top p \geq g(\psi)^\top a \geq 0$.

**Proof.** Note that $g(\psi)^\top a = \cos \psi \cos \alpha \geq 0$ for $\psi \in [\beta', \beta]$, which proves the second inequality in the statement.

For the first inequality, we first prove it for $\psi = \beta, \beta'$. Lemma 15 gives $c^\top p \geq 0$, while Lemma 22 gives $c^\top a \geq 0$. By the construction of $c'$, we have $p \in ac'$ and so length $c^\top p \leq$ length $ac'$. Hence $c^\top p \geq c^\top a \geq 0$ and so $g(\beta')^\top (p-a) \geq 0$, where we recall that $c' = g(\beta')$.

We next consider $\psi \in (\beta', \beta)$. Define function $h : [0, \beta - \beta'] \to \mathbb{R}$ as

$$h(\delta) = g(\delta + \beta')^\top (p-a).$$

It is enough show that $h(\delta) \geq 0$ for any $\delta \in (0, \beta - \beta')$. A direct calculation gives

$$h(\delta) = \cos \delta \left(g(\beta')^\top (p-a)\right) + \sin \delta \left(g(\beta' + \pi/2)^\top (p-a)\right).$$

We already have $g(\beta')^\top (p-a) \geq 0$. If $g(\beta' + \pi/2)^\top (p-a) \geq 0$, then $h(\delta) \geq 0$. Otherwise, $h(\delta)$ is decreasing and so $h(\delta) \geq h(\beta - \beta') = g(\beta)^\top (p-a) \geq 0$. □
For convenience, we will be using Newton notation \((g(\psi)^\top p)\)' and \((g(\psi)^\top a)\) for the derivatives
\[
\frac{d}{d\psi} (g(\psi)^\top p) = -p_x \sin \psi + p_z \cos \psi,
\frac{d}{d\psi} (g(\psi)^\top a) = -\cos \alpha \sin \psi.
\]

**Lemma 22.** For \(\psi \in (\beta', \beta)\), we have \((g(\psi)^\top p)' < 0\), \((g(\psi)^\top a)' < 0\), and
\[
-\sin \alpha (g(\psi)^\top p)' > -p_y (g(\psi)^\top a)' \geq 0.
\]

**Proof.** By immediate inspection \((g(\psi)^\top a)' = -\cos \alpha \sin \psi < 0\). From Lemma 20 \(f(\beta') = 0\) and \(f\) is strictly ascending on \(\psi \in [\beta', \beta]\). Then \(f(\psi) > f(\beta') = 0\) for \(\psi \in (\beta', \beta)\). In terms of coordinates this reads
\[p_x \sin \psi - p_y \cos \psi \sin \psi - p_z \sin \alpha \cos \psi > 0,
\]
which can be equivalently written as
\[-\sin \alpha (g(\psi)^\top p)' > -p_y (g(\psi)^\top a)' \geq 0,
\]
and the statement follows. \(\square\)

**Lemma 23.** For every \(\psi \in (\beta', \beta)\) we have
\[
\frac{-(g(\psi)^\top p)'}{\sqrt{1 - (g(\psi)^\top p)^2}} > \frac{-(g(\psi)^\top a)'}{\sqrt{1 - (g(\psi)^\top a)^2}} > 0.
\]

**Proof.** We first show that both sides of \((5.3)\) are well-defined. By Lemma 22 both \(g(\psi)^\top p\) and \(g(\psi)^\top a\) are strictly descending on \((\beta', \beta)\). Since these are continuous functions, they are also strictly descending on \([\beta', \beta]\). For \(\psi \in (\beta', \beta)\)
\[g(\psi)^\top p < g(\beta')^\top p \leq 1, \quad g(\psi)^\top a < g(\beta')^\top a \leq 1.
\]
One readily verifies that \(g(\psi)^\top a \geq 0\), and similarly \(g(\psi)^\top p \geq 0\) by Lemma 15 and the hypothesis that \(p \in \mathcal{O}_{xy}^+\). Thus the expressions in \((5.3)\) are well-defined. The second inequality in \((5.3)\) follows from the assertion \((g(\psi)^\top a)' < 0\) of Lemma 22.

In view of that, the first inequality is equivalent to
\[
\frac{(g(\psi)^\top p)'}{1 - (g(\psi)^\top p)^2} > \frac{(g(\psi)^\top a)'}{1 - (g(\psi)^\top a)^2}.
\]
A simple calculation gives
\[
1 - (g(\psi)^\top p)^2 = p_y^2 + ((g(\psi)^\top p)')^2,
1 - (g(\psi)^\top a)^2 = \sin^2 \alpha + ((g(\psi)^\top a)')^2.
\]
Substituting into \((5.4)\) we get the equivalent expression
\[
\sin^2 \alpha ((g(\psi)^\top p)')^2 > p_y^2 ((g(\psi)^\top a)')^2,
\]
which is true from Lemma 22. \(\square\)

**Lemma 24.** For any \(\psi \in (\beta', \beta)\) we have
\[
\frac{d\sqrt{1 - (g(\psi)^\top p)^2}}{d\psi} > \frac{d\sqrt{1 - (g(\psi)^\top a)^2}}{d\psi}.
\]
Proof. By Lemmas 21 and 23, \(- (g(\psi) \cdot p)' > -(g(\psi) \cdot a) > 0\). By Lemma 21, \[
\frac{g(\psi) \cdot p}{\sqrt{1 - (g(\psi) \cdot p)^2}} > \frac{g(\psi) \cdot a}{\sqrt{1 - (g(\psi) \cdot a)^2}}.
\]
Multiplying these two inequalities gives the inequality in the statement. \(\square\)

Step 4. We are now ready to prove inequalities (5.1) and (5.2). The following lemma proves (5.1).

Lemma 25. We have \(\mathcal{J}_{abc}(p) - \mathcal{J}_{abc}(p) \geq \mathcal{J}_{abc}(a) - \mathcal{J}_{abc}(a)\) with equality if and only if \(\beta_p = \beta\).

Proof. If \(\beta_p = \beta\), then \(c = c_p\) and the equality is trivial. Otherwise, we have the equivalences

\[
\mathcal{J}_{abc}(p) - \mathcal{J}_{abc}(p) > \mathcal{J}_{abc}(a) - \mathcal{J}_{abc}(a)
\]

\[
\Leftrightarrow \quad d_{g2}(c, p) - d_{g2}(c, p) > d_{g2}(c, a) - d_{g2}(c, a)
\]

\[
\Leftrightarrow \quad d_{g2}(g(\beta), p) - d_{g2}(g(\beta), p) > d_{g2}(g(\beta), a) - d_{g2}(g(\beta), a)
\]

\[
\Leftrightarrow \quad \int_{\beta_p}^{\beta} d\psi \sqrt{1 - (g(\psi) \cdot p)^2} > \int_{\beta_p}^{\beta} d\psi \sqrt{1 - (g(\psi) \cdot a)^2},
\]

the last of them true in view of Lemma 24, continuity and the fact that \(\beta_p \geq \beta'\). \(\square\)

For (5.2) we first need a lemma:

Lemma 26. Let \(\chi\) be the angle of \(b\) from \(\mathcal{H} = \text{span}(a, c_p)\). Then \(\cos^2 \chi < \frac{1}{2}\).

Proof. Set

\[
d = \frac{c_p - a a^\top c_p}{\|c_p - a a^\top c_p\|} = \frac{c_p - a a^\top c_p}{\sin \varphi_{AC_p}}.
\]

and note that \(\{a, d\}\) is an orthonormal basis of \(\mathcal{H}\). Then the orthogonal projection of \(b\) onto \(\mathcal{H}\) is \(b_\mathcal{H} = (a^\top b)a + (d^\top b)d\). Now

\[
\cos \chi = b_\mathcal{H}^\top b_\mathcal{H} = \frac{(a^\top b)^2 + (d^\top b)^2}{\sqrt{(a^\top b)^2 + (d^\top b)^2}} = \frac{(a^\top b)^2 + (d^\top b)^2}{(a^\top b)^2 + (d^\top b)^2}.
\]

Recalling that \(60^\circ \leq \varphi_{AB} \leq \varphi_{AC_p} = \varphi_{BC_p}\), we have

\[
(d^\top b)^2 = \frac{(\cos \varphi_{BC_p} - \cos \varphi_{AB} \cos \varphi_{AC_p})^2}{\sin^2 \varphi_{AC_p}} = \frac{\cos^2 \varphi_{AC_p}(1 - \cos \varphi_{AB})^2}{\sin^2 \varphi_{AC_p}}
\]

\[
\leq \cos^2 \varphi_{AB} \frac{(1 - \cos \varphi_{AB})^2}{\sin^2 \varphi_{AC_p}} < \cos^2 \varphi_{AB} \frac{(1 - \cos \varphi_{AB})(1 + \cos \varphi_{AB})}{\sin^2 \varphi_{AC_p}}
\]

\[
= \cos^2 \varphi_{AB} \leq \frac{1}{4}.
\]

Now \(\cos^2 \chi = (a^\top b)^2 + (d^\top b)^2 < \cos^2 \varphi_{AB} + \frac{1}{4} \leq \frac{1}{2}\). \(\square\)

Finally, we prove inequality (5.2), which concludes this section:

Lemma 27. We have \(\mathcal{J}_{abc}(p) \geq \mathcal{J}_{abc}(a)\) with strict inequality if \(\beta_p > \beta^*\).
Here and so \( \phi \) where we used \( \cos \varphi_{AC_p} = a^T c_p = \cos \alpha \cos \beta_p \in [\cos \alpha \cos \beta, \cos \alpha \cos \beta^*] = [\cos \varphi_{AC}, \cos \varphi_{AB}] \), and so \( \varphi_{AC_p} \in (\varphi_{AB}, \varphi_{AC}] \). We need a new coordinate system which brings \( a, c_p \) and \( p \) in the \( xOy \) plane as follows:
\[
\begin{align*}
a &= (1, 0, 0) \\
b &= (\cos \chi \cos \theta, \cos \chi \sin \theta, \sin \chi) \\
c_p &= (\cos \varphi_{AC_p}, \sin \varphi_{AC_p}, 0) \\
p &= (\cos \varphi_{AP}, \sin \varphi_{AP}, 0).
\end{align*}
\]
Here \( \chi \) is the angle of \( b \) from the plane span\((a, c_p)\) and \( \theta \in (-\pi, \pi] \). Then
\[
J_{abc}(p) = \sin \varphi_{AP} + \sin(\varphi_{AC_p} - \varphi_{AP}) + \sqrt{1 - \cos^2 \chi \cos^2(\varphi_{AP} - \theta)}.
\]
Consider the function
\[
J(\delta) = \sin \delta + \sin(\varphi_{AC_p} - \delta) + \sqrt{1 - \cos^2 \chi \cos^2(\delta - \theta)}.
\]
We have
\[
\begin{align*}
J_{abc}(a) &= J(0) = \sin \varphi_{AC_p} + \sqrt{1 - (a^T b)^2} = \sin \varphi_{AC_p} + \sin \varphi_{AB}, \\
J_{abc}(p) &= J(\varphi_{AC_p}) = \sin \varphi_{AC_p} + \sqrt{1 - (c_p^T b)^2} = 2 \sin \varphi_{AC_p},
\end{align*}
\]
where we used \( \varphi_{AC_p} = \varphi_{BC_p} \). Since \( p \in \overline{ac_p} \), we have \( 0 \leq \varphi_{AP} \leq \varphi_{AC_p} \). By Lemma 16 \( \cos^2 \chi \leq 1/2 \). Consider the second derivative of \( J(\delta) \) for \( \delta \in [0, \varphi_{AC_p}] \):
\[
J''(\delta) = -\sin \delta - \sin(\varphi_{AC_p} - \delta) - \cos^2 \chi \sin^2(\theta - \delta) (1 - \cos^2 \chi \cos^2(\theta - \delta))^{-3/2}
+ (1 - \cos^2 \chi \cos^2(\theta - \delta))^{-1/2} \cos^2 \chi \cos^2(\theta - \delta)
\leq -\sin \delta - \sin(\varphi_{AC_p} - \delta) + (1 - \cos^2 \chi \cos^2(\theta - \delta))^{-1/2} \cos^2 \chi \cos^2(\theta - \delta)
\leq -\sin \varphi_{AC_p} + (1 - \cos^2 \chi)^{-1/2} \cos^2 \chi
\leq -\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} < 0
\]
We see that \( J(\delta) \) is strictly concave on \( [0, \varphi_{AC_p}] \). Then we have
\[
J(\varphi_{AP}) \geq \min\{J(0), J(\varphi_{AC_p})\} = J(0),
\]
and the equality is achieved if and only if \( \varphi_{AP} = 0 \), which means \( p = a \). Hence, if \( p \neq a \), we get \( J(\delta) > J(0) \), that is \( J_{abc}(p) > J_{abc}(a) \). \( \square \)

6. General Triangles

In this section we complete the proof of Theorem 11 by proving part (1), which is concerned with the general configuration
\[
60^\circ \leq \varphi_{AB} \leq \varphi_{AC} < \varphi_{BC}.
\]
It is immediate that \( J_{ABC}(A) < J_{ABC}(B) \leq J_{ABC}(C) \) and thus in the case of a big triangle \( \mathcal{P}_{ABC} = \{A\} \) by Proposition 12. We may thus assume that the projective triangle \( \triangle ABC \) is not big. Moreover, as in the beginning of §4 we may assume...
representatives $a, b, c$ such that $a^\top b, a^\top c, b^\top c > 0$. Without loss of generality, we assume coordinates
\[
a = (0, 0, 1)
b = (\sin \varphi_{AB}, 0, \cos \varphi_{AB})
c = (\sin \varphi_{AC} \cos \alpha, \sin \varphi_{AC} \sin \alpha, \cos \varphi_{AC}),
\]
where $\alpha \in (0, \pi]$ is given by
\[
\alpha = \arccos \frac{\cos \varphi_{BC} - \cos \varphi_{AB} \cos \varphi_{AC}}{\sin \varphi_{AB} \sin \varphi_{AC}}.
\]

Let $P \in \mathcal{P}_{ABC}$ and $p \in S^2$ a representative of $P$ such that $a^\top p \geq 0$. We assume $P \neq A$, write $p = (\sin \omega \cos \theta, \sin \omega \sin \theta, \cos \omega)$ with $\omega \in (0, \pi/2]$ and $\theta \in (-\pi, \pi]$, and we exhibit a contradiction. First we determine the range of $\theta$:

**Lemma 28.** We have $0 \leq \theta \leq \alpha$.

**Proof.** Since by hypothesis $a^\top p \geq 0$, Lemma 6 gives $b^\top p, c^\top p \geq 0$. Let $n_{ab} = a \times b$ be the normal vector to the plane spanned by $a, b$. Writing $n_{ab}$ in coordinates gives
\[
n_{ab}^\top c = \sin \varphi_{AB} \sin \varphi_{AC} \sin \alpha > 0.
\]
This, together with $c^\top p \geq 0$ and Lemma 4 give
\[
n_{ab}^\top p = \sin \varphi_{AB} \sin \omega \sin \theta \geq 0.
\]
This implies $\sin \theta \geq 0$ and so $\theta \geq 0$. A similar argument with $n_{ca} = c \times a$ gives $\sin(\alpha - \theta) \geq 0$, that is $\alpha - \theta \geq 0$.

Next, we define two auxiliary points:

**Construction 29.** With angle $\alpha' \in [0, \pi]$ given by
\[
\alpha' = \arccos \frac{\cos \varphi_{AC} - \cos \varphi_{AB} \cos \varphi_{AC}}{\sin \varphi_{AB} \sin \varphi_{AC}},
\]
we define points $b', c' \in S^2$, respectively representing points $B', C' \in P^2$, as
\[
b' = (\sin \varphi_{AB} \cos(\alpha - \alpha'), \sin \varphi_{AB} \sin(\alpha - \alpha'), \cos \varphi_{AB}),
c' = (\sin \varphi_{AC} \cos \alpha', \sin \varphi_{AC} \sin \alpha', \cos \varphi_{AC}).
\]
One verifies easily that $a^\top c' = b^\top b' = b^\top c = \cos \varphi_{AC}$ and $a^\top b' = \cos \varphi_{AB}$.

We also need:

**Lemma 30.** With $\alpha'$ as in Construction 29 we have $\alpha' \in (\alpha/2, \alpha)$.

**Proof.** By hypothesis $\varphi_{AC} < \varphi_{BC}$ and so by definition $\alpha' < \alpha$. We are going to prove $2\alpha' > \alpha$. First we have $\alpha < 2\pi/3$, since $\varphi_{AB}, \varphi_{AC} \geq 60^\circ$ and so
\[
\cos \alpha > -\frac{\cos \varphi_{AB} \cos \varphi_{AC}}{\sin \varphi_{AB} \sin \varphi_{AC}} = -\cot \varphi_{AB} \cot \varphi_{AC} > -\frac{1}{3} > -\frac{1}{2}.
\]
Moreover, since
\[
\frac{\cos \varphi_{AC}}{\sin \varphi_{AC}} < \frac{\cos \varphi_{AB}}{\sin \varphi_{AB}},
\]
we get $\alpha' > \pi/3$ from
\[
\cos \alpha' = \frac{\cos \varphi_{AC} - \cos \varphi_{AB}}{\sin \varphi_{AC} \sin \varphi_{AB}} < \frac{1 + \cos \varphi_{AB}}{2 \sin \varphi_{AB}} \frac{1 - \cos \varphi_{AB}}{\sin \varphi_{AB}} = \frac{1}{2}.
\]
Thus $2\alpha' > 2\pi/3 > \alpha$ from which we read $\alpha' > \alpha/2$. \hfill \Box

Let us consider the arrangements $ABC'$ and $AB'C$. We have:

**Lemma 31.** \{(A, B) \in P_{ABC'} \text{ and } (A, B') \in P_{AB'C}\}.

**Proof.** By Construction 29 we have

$$A, B, C', \phi_{AB} \leq \phi_{AC'} = \phi_{BC'},$$

$$A, B', C : \phi_{AB'} \leq \phi_{AC} = \phi_{B'C}.$$ 

Hence the arrangements $ABC'$ and $AB'C$ fall either in part (2) or part (3) of Theorem 1 which have already been proved. \hfill \Box

Now, from Lemma 31 we have

$$J_{abc}(p) \geq J_{abc'}(a),$$

$$J_{ab'c}(p) \geq J_{ab'c}(a).$$

From Lemma 28 and Lemma 30 we have that either $\theta \in [0, \alpha']$ or $\theta \in [\alpha - \alpha', \alpha]$. If $\theta \in [0, \alpha']$, and since $\alpha \in (0, \pi]$, we have $0 \leq \alpha' - \theta < \alpha - \theta$ and so

$$0 \leq c^\top p = \sin \phi_{AC} \sin \omega \cos(\alpha - \theta) + \cos(\omega) \cos(\phi_{AC})$$

$$< \sin \phi_{AC} \sin \omega \cos(\alpha' - \theta) + \cos(\omega) \cos(\phi_{AC}) = c'^\top p.$$ 

Hence $d_{c^2}(c, p) > d_{c'^2}(c', p)$ and we get the contradiction

$$J_{abc}(p) > J_{abc'}(p) \geq J_{ab'c}(a) = J_{abc}(a).$$

By a similar argument, if $\theta \in [\alpha - \alpha', \alpha]$ we get $0 \leq b^\top p < b'^\top p$ and thus arrive at the contradiction $J_{abc}(p) > J_{ab'c}(p) \geq J_{ab'c}(a) = J_{abc}(a)$, concluding the proof of Theorem 1.

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