About one aspect of effective building of bus traffic schedule with an approximate algorithm

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Abstract. The assignment problem belongs to the main tasks of combinatorial optimization. This is a special case of the transport problem. Special methods that allow finding the optimal solution from the set of possible ones have been developed for solving the transport problem. The assignment problem is used in transport to optimize the arrangement of the cars along the routes, in the process of distributing the drivers among the cars, and in the process of distributing freights along the production lines. This article considers the case when \( n < m \) during the search of optimal solutions in the assignment problem of a transport type. The authors offer an algorithm that allows us to build a bus schedule taking into account the current restrictions in the presence of which the passenger traffic is carried out. A theoretical assessment of time complexity of an offered procedure for constructing all optimal solutions to the assignment problem is presented. An example illustrating the operation of the constructed algorithm, which allows finding the whole set of optimal solutions in polynomial time is given. Recommendations on the use of the algorithm in practice are given.

1. Introduction

Activity largely depends on the level of management, which provides for the adoption of optimal decisions for a certain time.

The development of computer technology has made it possible to successfully implement and introduce into production quite complex, but effective algorithms for solving the tasks of operational planning and control.

An approximate method is offered for solving a class of problems related to the optimization of the transport process, which allows building schedules efficiently taking into account the constraints that characterize real conditions.

2. Analysis of publications

The assignment problem is one of the fundamental problems of combinatorial optimization in the field of mathematical optimization.

Combinatorial optimization problems are usually considered in a single criterion setting. The tasks in which the set of acceptable solutions correspond to the classical version, but the choice of the
The optimal solution involves taking into account more than one criterion are less studied but important in applications. The latter include the multicriteria assignment problem. Multicriteria optimization methods for such tasks can be divided into two groups. The methods of the first group reduce the multicriteria problem to a single criterion one by folding the vector criterion into a super criterion, which is then optimized by one of the methods of single criterion optimization [1, 2, 3]. The second group includes such methods as the method of compromise, the method of subsequent concessions, the method of analysis of hierarchies [3, 4, 5], etc.

The Hungarian algorithm is one of the many algorithms designed to solve the linear task of assignment in polynomial time upon the number of works.

The assignment task is a special case of the transport problem, which is a special case of the problem of finding the minimum cost flow, and in turn, is a special case of the linear programming task. Any of these tasks can be solved by the simplex method, but each specialization has its own more efficient algorithm, based on the structural features of the problem.

The solution of discrete optimization problems is connected with fundamental difficulties. A complete enumeration of the points of an admissible set, as a rule, is not feasible because of too much computational work. Due to the discreteness of the admissible set, many methods developed in mathematical programming are not applicable. For example, moving in the direction of the gradient, moving from one vertex of a polyhedral set to another, etc.

To solve the assignment problem, we can use the branch and bound method. This method is based on the sequential partitioning of an admissible set into subsets (branching) and calculation of estimates (boundaries) of the values of the objective function on these subsets, which allows us to discard subsets that obviously do not contain solutions.

In the case when \( n > m \), the task of constructing a schedule of minimum length belongs to the class of NP-hard problems, even when the second level of the system is represented by identical machines [6]. In the case when \( n = m \), the stated problem is one of the generalizations of the assignment problem and has been effectively solved [7].

In this article, it is offered to consider the case when \( n < m \). An algorithm for solving this problem was considered previously [8].

3. Formulation of the problem

The subject of study is a model for the efficient organization of passenger traffic, the construction of which is based on the solution of the well-known assignment problem. This problem has a wide range of practical applications in transport.

It is proved that the assignment problem is effectively solvable. Moreover, it is assumed that in order to achieve the optimum of its objective function, it is sufficient to find a unique solution to the problem. However, in practical situations, there is a need to find the set of optimal solutions with given properties, under various conditions. This problem has been repeatedly considered by a number of authors, for example, in [9, 10, 11].

The results of the study of these properties comprise the contents of this article. The condition under which it is necessary to minimize the total time that it takes the buses to make the shuttle routes between two points 1 and 2, contains an additional requirement. It involves the fact that the transportation between these points must be provided by \( m \) buses. They include \( m_1 \) buses of the car company located in point 1 and \( m - m_1 \) buses of the car company located in point 2. From the service schedule between bus stations, the departure time for each run from point 1 to point 2 and vice versa from point 2 to point 1, is known. Any bus of the car company, located in point \( k \), \( k = 1, 2 \), starts and ends the route according to the schedule in this point.

The run \( i \) from point 1 to point 2 starts at a point of time \( t_{1i}, \ i = 1, m \), and its duration is \( \tau_{1i} \). The run \( j \) from point 2 to point 1 starts at a point of time \( t_{2j}, \ j = 1, m \), and is completed in \( \tau_{2j} \) time. For a bus setting off by the run \( i \) from point 1 to point 2 and returning by the run \( j \) from point 2 to point 1,
the time on duty is defined as \( \gamma_{ij} = t_{2j} - t_{li} + r_{2j}, \quad t_{2j} - t_{li} \leq r_{li} \).

Similarly, the duration of the shuttle route of the bus, performing the run \( j \) at first, and then the run \( j \) equals \( \beta_{ji} = t_{ui} - t_{2j} + r_{ui}, \quad t_{ui} - t_{2j} \leq r_{2j} \).

Let \( Y = [y_{ij}]_m \) and \( Z = [z_{ij}]_m \) denote \((0, 1)\) are the square matrices of \( m \) order meeting the following conditions: \( y_i = \sum_{j=1}^{m} y_{ij} = 1 \neq 0, \quad y_j = \sum_{i=1}^{m} y_{ij} = 1 \neq 0, \quad \sum_{i=1}^{m} \sum_{j=1}^{m} y_{ij} = m_1, \quad 0 < m_1 < m; \)
\[
z_i = \sum_{j=1}^{m} z_{ij} = 1 \neq 0, \quad z_j = \sum_{i=1}^{m} z_{ij} = 1 \neq 0, \quad \sum_{i=1}^{m} \sum_{j=1}^{m} z_{ij} = m - m_1; \quad Y + Z = X. \quad X = [x_{ij}] \) is a permutation matrix.

Let’s consider the \([\gamma_{ij}]_m\) and \([\beta_{ij}]_m\) square matrices of \( m \) order connected with \( Y, Z \) and \( X \) matrices by the functional
\[
T(\pi_1, \pi_2) = \sum_{i=1}^{m} \sum_{j=1}^{m} y_{ij} \gamma_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{m} z_{ij} \beta_{ij},
\]
where \( \gamma_{ij}, \beta_{ij} \) are non-negative integers, \( \pi_1 \) is a subset of \( m_1 \) row indexes of the \( Y \) matrix, and \( \pi_2 \) is respectively a non-empty set of \( m - m_1 \) row indexes of the \( Z \) matrix. The \( X \) permutation matrix corresponds to the sequence \( \pi = (\pi_1, \pi_2) \) represented by the partition into the \( \pi_1 \) and \( \pi_2 \) subsets: \( \pi = \pi_1 \cup \pi_2, \quad \pi_1 \cap \pi_2 = \emptyset, \quad |\pi_1| = m_1, \quad |\pi_2| = m - m_1. \)

Let’s set the problem of finding such \( \pi^* \) permutation that its partition \((\pi_1^*, \pi_2^*)\) \[\pi_1^* = m_1, \quad |\pi_2^*| = m - m_1, \quad m_1 > 0, \] gives a value of \( T(\pi_1^*, \pi_2^*) = \min_{\pi=(\pi_1, \pi_2)} T(\pi_1, \pi_2) \) to functional \((1)\).

Analysis of the problem begins with the compilation of \([\gamma_{ij}]_m\) and \([\beta_{ij}]_m\) matrices. Each element \( \gamma_{ij} \) of the first table is equal to the run time of the shuttle route, including the run 1 from point 1 to point 2, and then the run \( j \) from point 2 to point 1.

The \([\beta_{ij}]_m\) matrix contains the durations of all routes that begin with the run \( j, \quad j = 1, \ldots, m, \) from point 2 to point 1, and end with the run \( i, \quad i = 1, \ldots, m, \) from point 1 to point 2. In other words, the \([\gamma_{ij}]_m\) matrix is made on the \( m_1 = m \) assumption, and the \([\beta_{ij}]_m\) matrix contains information about the duration of the routes, provided that \( m_1 = 0. \)

By overlaying the \([\gamma_{ij}]_m\) and \([\beta_{ij}]_m\) matrices we form a \([\gamma_{ij}, \beta_{ij}]_m\) table of \( m^2 \) ordered pairs \( (\gamma_{ij}, \beta_{ij}), \quad i, j = 1, \ldots, m. \) Any \( X \) permutation matrix of \((\gamma_{ij}, \beta_{ij})\) elements determines the \( \pi = (\pi[1], \pi[2], \ldots, \pi[k], \ldots, \pi[l], \ldots, \pi[m]) \) sequence of row indices of the resulting configuration, as well as the partition of \( \pi \) into subsets \( \pi_1 \) and \( \pi_2, \quad |\pi_1| = m_1, \quad |\pi_2| = m - m_1, \quad 0 < m_1 < m, \) which is estimated by the value of the functional
\[
T(\pi_1, \pi_2) = T(\pi_1) + T(\pi_2),
\]
where \( T(\pi_1) = \sum_{k \in \pi_1} \gamma_{k}, \quad T(\pi_2) = \sum_{l \in \pi_2} \beta_{l}. \)

Thus, an acceptable solution to the problem of finding the minimum \((1)\) includes the \( \pi \)
permutation of the indices from the $m$ rows of the $([\gamma_{ij}, \beta_{ij}])_m$ table and the $(\pi_1, \pi_2)$ partition of the $\pi$ permutation into $\pi_1$ and $\pi_2$ subsets with a predetermined number of elements $m_1, 0 < m_1 < m,$ in the $\pi_1$ subset. Let's denote $\pi = (\pi_1, \pi_2)$ as the acceptable solution to a problem (1).

The task under consideration contains all the features of the assignment problem. In the absence of restrictions on the number of elements in the $\pi_1$ subset, its optimal solution $\pi^*$ is the solution of the assignment problem with the initial data in the form of a $[a_{ij}]_m$ matrix, where $a_{ij} = \min\{\gamma_{ij}, \beta_{ij}\}$.

4. The results of research
The minimization problem (1) has a visual representation in terms of the graph theory.

We associate the $([\gamma_{ij}, \beta_{ij}])_m$ table with a complete bipartite directed graph $G = (V \cup W, E)$, $|V| = |W| = m, |E| = 2m^2$, where each pair of vertices $\{v_i, w_j\}, v_i \in V, w_j \in W$ forms two arcs $(v_i, w_j), (w_j, v_i) \in E$ with the weights $\gamma(v_i, w_j) = \gamma_{ij}$ and $\beta(w_j, v_i) = \beta_{ij}$. Then the $\pi = (\pi[1], \pi[2], \ldots, \pi[m])$ sequence of elements of the $([\gamma_{ij}, \beta_{ij}])_m$ table can be represented as a perfect combination of a complete bipartite undirected graph $G_0 = (V \cup W, E_0)$, where $E_0$ is the set of edges $\{v_i, w_j\}, v_i \in V, w_j \in W, |E_0| = m^2$. Splitting $\pi = (\pi_1, \pi_2)$ sets the orientation of the edge from $v_i \in V$ to $w_j \in W$ and its weight equals $\gamma_{ij}$, if $\{v_i, w_j\} \in \pi_1, |\pi_1| = m_1$. If $\{v_i, w_j\} \in \pi_2, |\pi_2| = m - m_1$, then the edge is oriented from $w_j \in W$ to $v_i \in V$ and takes the weight equal to $\beta_{ij}$. Let us define the weight of the obtained oriented subgraph of the $G = (V \cup W, E)$ graph as the sum of the weights of its arcs. It is required to construct the $\pi^* = (\pi_1^*, \pi_2^*)$ subgraph with the least weight on the set of all such subgraphs.

Let us estimate the potency of the region of feasibility $\pi = (\pi_1, \pi_2)$. The table $([\gamma_{ij}, \beta_{ij}])_m$ generates $m!$ sequences $\pi$, and the number of all possible partitions of $\pi$ elements into two such subsets as $\pi_1$ and $\pi_2$, that $|\pi_1| = m_1, |\pi_2| = m - m_1, 0 < m_1 < m$, is equal to $m!/m_1(m - m_1)$. Therefore, the partition search area as $\pi^* = (\pi_1^*, \pi_2^*)$ contains $(m!)^2/[m_1(m - m_1)]$ of acceptable solutions $\pi = (\pi_1, \pi_2)$.

Let us present an intermediate result that allows us to construct $\pi^* = (\pi_1^*, \pi_2^*)$ on the set of $m!$ permutations $\pi = (\pi_1, \pi_2)$.

For this purpose, let us consider the problem of splitting $m$ ordered pairs $(\gamma_j, \beta_j), j = 1, m$, into two subsets $\sigma^0_1$ and $\sigma^0_2$ with a given number $m_1$ of pairs in $\sigma^0_1, 0 < m_1 < m$, and with the smallest value

$$T(\sigma_1^0, \sigma_2^0) = \min_{\sigma = (\sigma_1, \sigma_2)} \left( \sum_{\sigma_1 = 1}^{m_1} \gamma_j + \sum_{\sigma_2 = 1}^{m - m_1} \beta_j \right),$$

among all partitions $(\sigma_1, \sigma_2)$ of the set $\sigma = \{(\gamma_j, \beta_j) / 1 \leq j \leq m\}$.

Statement. The value $T(\sigma_1^0, \sigma_2^0)$ is achieved by arranging the set $\sigma = \{(\gamma_j, \beta_j) / 1 \leq j \leq m\}$ upon non-decreasing values $\delta_j = \gamma_j - \beta_j$ and by including to the subset $\sigma^0_1$ the first elements on the left of the obtained sequence.

Proof. For partition of the set $\sigma = \{(\gamma_j, \beta_j) / 1 \leq j \leq m\}$ into the subsets $\sigma_1$ and $\sigma_2, |\sigma_1| = m_1$, $|\sigma_2| = m - m_1$,
\[ |\sigma_2| = m - m_1, \quad m_1 > 0, \text{ we have } T = (\sigma_1, \sigma_2) = \sum_{j=1}^{m} \gamma_j + \sum_{j=1}^{m} \beta_j = \sum_{j=1}^{m} \gamma_j + m \beta_1 - \sum_{j=1}^{m} \beta_j + \]
\[
+ \sum_{j=1}^{m} (\gamma_j - \beta_j) = \text{const} + \sum_{j=1}^{m} \delta_j.
\]

It follows that to find the \( T(\sigma_1, \sigma_2) \) minimum it is enough to obtain a \( \sigma^0 \) sequence of components \( (\gamma_j, \beta_j) \), \( 1 \leq j \leq m \), in which \( \delta_{\sigma^0[1]} \leq \delta_{\sigma^0[2]} \leq \ldots \leq \delta_{\sigma^0[m]} \), and a \( \sigma^0_1 \) subsequence consisting of the \( m_1 \) first components \( \sigma^0 \) on the left. The statement is proved.

Thus, the established result makes it easy to find the only \( \pi = (\pi^0_1, \pi^0_2) \) partition for which the value (2) reaches its minimum among all \( m!/[m_1!(m-m_1)!] \) partitions generated by the set \{\( (\gamma_{1j}, \beta_{1j}) \)|\( 1 \leq j \leq m \)} upon a given sequence \( \pi = (\pi[1], \pi[2], \ldots, \pi[m]) \), and the corresponding \( X \) permutation matrix of the \( [(\gamma_{ij}, \beta_{ij})]_m \) table elements.

Let’s describe an algorithm that does not guarantee the construction of an exact solution, but for typical individual examples of tasks it defines a schedule that is close to optimal.

S1. The \( [\gamma_{ij}]_m \) matrix is made under the \( m_1 = m \) assumption, and the \( [\beta_{ij}]_m \) matrix contains information on the duration of the routes, provided that \( m_1 = 0 \).

\( [\gamma_{ij}, \beta_{ij}]_m \) matrix determined by overlapping of \( [\gamma_{ij}]_m \) on \( [\beta_{ij}]_m \); \( m_1 \) and \( m-m_1 \) – the given number of buses in points 1 and 2.

S2. To define a \([\alpha_{ij}]_m \) matrix where \( \alpha_{ij} = \gamma_{ij} + \beta_{ij} \).

S3. To find the set of all solutions that minimize the objective function of the assignment problem for the \([\alpha_{ij}]_m \) matrix.

S4. To associate the \( m \) components from the \([\gamma_{ij}, \beta_{ij}]_m \) matrix and put them in order using the \( S^1 \) – \( S^4 \) steps of the algorithm [7] provided that \( n = m \) to each optimal solution to the assignment problem.

S5. To define, respectively, the \( m_1 \) of the first components on the left of the \([\gamma_{ij}]_m \) matrix and the \( m-m_1 \) components of the \([\beta_{ij}]_m \) matrix.

The accuracy of the acceptable solution depends on the input length specified by the difference between \([\gamma_{ij}]_m \) and \([\beta_{ij}]_m \) matrices, and on the extent to which the values of their elements change. Since the meaningful formulation of the problem excludes the presence of extremely large numbers at the input, that is, there is every reason to expect the solutions with acceptable errors in practice at the output of the algorithm. Having determined the \( \pi^0 = (\pi^0(1), \pi^0(2), \ldots, \pi^0(m)) \) permutation that provides the minimum of the objective function of the assignment problem for the initial data in the form of a matrix \([\alpha_{ij}]_m \), \( \alpha^0_{ij} = \min(\gamma_{ij}, \beta_{ij}) \), we obtain the inequalities
\[
\sum_{j=1}^{m} \alpha^0_{ij} \leq T(\pi^*_1, \pi^*_2) \leq T(\pi^*_1, \pi^*_2),
\]
that establish the search range for the value \( T(\pi^*_1, \pi^*_2) \).

Let us estimate the complexity of constructing an acceptable solution \((\pi^1, \pi^2)\). When performing S1 and S2 steps to determine \([\gamma_{ij}, \beta_{ij}]_m \) and \([\alpha_{ij}]_m \) matrices, \( O(m^2) \) elementary actions are needed. Used at the S3 step procedure for constructing all the optimal solutions to the assignment problem is performed in \( O(N_{max} m^4 \log_2 m) \) time, where \( N_{max} \) is the maximum number.
of optimal local solutions obtained during the procedure.

Let us consider an example. Passenger transportation between points 1 and 2 is carried out by two buses of the car company, located in point 1, and by three buses of the car company, located in point 2.

Based on the bus schedule between the bus stations, we determine: the time of departure of the buses from the first point is \( \tau_{1j} = \{6.00, 8.00, 12.00, 16.00, 18.00\} \), with corresponding \( \tau_{1j} = \{5, 5, 6, 6, 5\} \); the departure time of the buses from the second point is \( \tau_{2j} = \{7.00, 9.00, 12.00, 15.00, 19.00\} \), with corresponding \( \tau_{2j} = \{6, 6, 6, 5, 5\} \); \( m_1 = 2 \); \( m_2 = 5 - 2 = 3 \).

In accordance with the given data, we determine the \([\gamma_{ij}]_m\) and \([\beta_{ij}]_m\) tables:

\[
[\gamma_{ij}]_5 = \begin{bmatrix}
29 & 27 & 24 & 14 & 18 \\
29 & 31 & 27 & 12 & 16 \\
25 & 27 & 30 & 28 & 12 \\
21 & 23 & 26 & 28 & 32 \\
19 & 21 & 24 & 27 & 30 \\
\end{bmatrix},
[\beta_{ij}]_5 = \begin{bmatrix}
28 & 30 & 23 & 15 & 16 \\
26 & 28 & 30 & 13 & 15 \\
23 & 25 & 26 & 30 & 11 \\
20 & 22 & 26 & 31 & 32 \\
16 & 18 & 22 & 26 & 28 \\
\end{bmatrix}.
\]

By overlaying the \([\gamma_{ij}]_m\) and \([\beta_{ij}]_m\) matrices we form a \([\gamma_{ij}, \beta_{ij}]_m\) table:

\[
[\gamma_{ij}, \beta_{ij}]_5 = \begin{bmatrix}
(29, 28) & (27, 30) & (24, 23) & (14, 15) & (18, 16) \\
(29, 26) & (31, 28) & (27, 30) & (12, 13) & (16, 15) \\
(25, 23) & (27, 25) & (30, 26) & (28, 30) & (12, 11) \\
(21, 20) & (23, 22) & (26, 26) & (28, 31) & (32, 32) \\
(19, 16) & (21, 18) & (24, 22) & (27, 26) & (30, 28) \\
\end{bmatrix}.
\]

We obtain a \([a_{ij}]_m\) table where \( a_{ij} = \gamma_{ij} + \beta_{ij} \):

\[
[a_{ij}]_5 = \begin{bmatrix}
57 & 57 & 47 & 29 & 34 \\
55 & 59 & 57 & 25 & 31 \\
48 & 52 & 56 & 58 & 23 \\
41 & 45 & 52 & 59 & 64 \\
35 & 39 & 46 & 53 & 58 \\
\end{bmatrix}.
\]

Let us find all solutions that provide the minimum of the objective function of the assignment problem for the initial data represented by the \([a_{ij}]_m\) matrix:

\[
\pi_1 = (a_{13}, a_{24}, a_{35}, a_{41}, a_{52}) , \quad T(\pi_1) = 47 + 25 + 23 + 41 + 39 = 175 ;
\]

\[
\pi_2 = (a_{13}, a_{24}, a_{35}, a_{42}, a_{51}) , \quad T(\pi_2) = 47 + 25 + 23 + 45 + 35 = 175 .
\]

For each \( \pi_1 \) permutation element, we find the corresponding pair of \([\gamma_{ij}, \beta_{ij}]_5\), resulting in a set of five pairs: \( (\gamma_{13}, \beta_{13}) = (24, 23) \), \( (\gamma_{24}, \beta_{24}) = (12, 13) \), \( (\gamma_{35}, \beta_{35}) = (12, 11) \), \( (\gamma_{41}, \beta_{41}) = (21, 20) \), \( (\gamma_{52}, \beta_{52}) = (21, 18) \).

Now we determine \( \sigma_j \) for \( \pi_1 \): \( \sigma_1 = 24 - 23 = 1 \), \( \sigma_2 = 12 - 13 = -1 \), \( \sigma_3 = 12 - 11 = 1 \), \( \sigma_4 = 21 - 20 = 1 \), \( \sigma_5 = 21 - 18 = 3 \).

Having ordered the \( \sigma_j \) values in non-decreasing order, we obtain a \( \sigma_2 \leq \sigma_1 \leq \sigma_3 \leq \sigma_4 \leq \sigma_5 \) sequence, and \( \pi_{11} \) and \( \pi_{21}^+ \) subsets, in which we select the \( m_1 \) first components on the left, for which
\[
\sum_{i,j \in \pi_1^+} \gamma_{ij} = \gamma_{13} + \gamma_{24} = 24 + 12 = 36, \quad \sum_{i,j \in \pi_2^+} \beta_{ij} = \beta_{35} + \beta_{41} + \beta_{52} = 11 + 20 + 18 = 49, \quad T(\pi_1) = 85.
\]

Similarly, we find permutation the corresponding pair of \([(\gamma_{ij}, \beta_{ij})]_5^*\) for each element of the \(\pi_2\) as a result of which we obtain a set of five pairs: \((\gamma_{13}, \beta_{13}) = (24, 23), (\gamma_{24}, \beta_{24}) = (12, 13), (\gamma_{35}, \beta_{35}) = (12, 11), (\gamma_{42}, \beta_{42}) = (23, 22), (\gamma_{51}, \beta_{51}) = (19, 16)\).

Now we define \(\sigma_j\) for \(\pi_2\): \(\sigma_1 = 24 - 23 = 1, \sigma_2 = 12 - 13 = -1, \sigma_3 = 12 - 11 = 1, \sigma_4 = 23 - 22 = 1, \sigma_5 = 19 - 16 = 3\).

Having ordered the \(\sigma_j\) values in non-decreasing order, we obtain a \(\sigma_2 \leq \sigma_1 \leq \sigma_3 \leq \sigma_4 \leq \sigma_5\) sequence, and \(\pi_{12}^+\) and \(\pi_{22}^+\) subsets, in which we select the \(m_1\) first components on the left, for which
\[
\sum_{i,j \in \pi_{12}^+} \gamma_{ij} = \gamma_{13} + \gamma_{24} = 24 + 12 = 36, \quad \sum_{i,j \in \pi_{22}^+} \beta_{ij} = \beta_{35} + \beta_{42} + \beta_{51} = 11 + 22 + 16 = 49, \quad T(\pi_2) = 85.
\]

All obtained partitions provide the objective function of the problem a value of 85.

The perfect matching corresponding to the \(\pi_1 = (\pi_{11}, \pi_{21})\) partition is shown in figure 1.

![Figure 1. The perfect matching corresponding.](image)

Let us evaluate the accuracy of the constructed solutions. To do this, we calculate the \(T(\pi_1^*, \pi_2^*)\) lower bound which equals \(T(\pi_0) = \sum_{j=1}^n \min(\gamma_{ij}, \beta_{ij})\).

From \([(\gamma_{ij}, \beta_{ij})]_5\) we define \([(\gamma_{ij}, \beta_{ij})]_5^* = \begin{bmatrix} 28 & 30 & 13 & 14 & 17 \\ 27 & 29 & 12 & 13 & 16 \\ 22 & 24 & 29 & 31 & 11 \\ 19 & 21 & 28 & 29 & 32 \\ 17 & 19 & 24 & 26 & 29 \end{bmatrix} \).

The solution of the assignment problem gives a \(\pi_0 = (a_{13}, a_{24}, a_{35}, a_{41}, a_{52})\) sequence for which \(T(\pi_0) = 23 + 12 + 11 + 20 + 18 = 84\). Thus, the \(T(\pi_1, \pi_2)\) value is limited below by a \(T(\pi_0) = 84\) value, and above – by a \(T(\pi_1, \pi_2) = 85\) value.

5. Conclusions

Thus, the proposed algorithm allows us to build a bus schedule taking into account the real restrictions under which passenger transportation are carried out. In polynomial time, the algorithm finds all the set of optimal solutions. This assessment of the time complexity of the algorithm makes it possible to use it on modern computers. The algorithm is easily programmable, after which it can be applied in practice.
References

[1] Baeva N B, Bondarenko Yu V 2009 Fundamentals of theory and computational schemes of vector optimization: Tutorial (Voronezh: VSU) p 96

[2] Nikonov O Ya, Podolyaka O A, Podolyaka A N, Skakalina Ye V 2011 Mathematical methods for solving the multicriteria assignment problem Bulletin of Kharkov National Automobile and Highway University. 55 103-12

[3] Steuer R 1992 Multicriteria optimization. Theory, computing, applications (Moscow: Radio and Communication) p 504

[4] Kashirina I L, Semenov B A 2007 Genetic algorithm for solving the multicriteria assignment problem Information Technologies. 5 62-8

[5] Podinovsky V V, Nogin V D 1982 Pareto-optimal solutions to multicriteria problems (Moscow: Nauka) p 256

[6] Panishev A V, Varakin A S 1992 Optimal streamlining of two-stage work in the FCM Cybernetics and Systems Analysis. 2 85-93

[7] Panishev A V, Podolyaka O A, Skakalina Ye V 1999 Efficient parallelization algorithm for non-identical machines Aerospace technic and technology. 13 136-46

[8] Panishev A V, Skripina I V, Skakalina Ye V 2000 Effective construction of optimal solutions in the assignment problem of a transport type Automobile transport: Collection of scientific works. 4 63-5

[9] Panishev A V, Podolyaka O O 1999 One of the generalizations of the assignment problem with restrictions The Journal of Zhytomyr State Technological University. 11 139-44

[10] Medvedeva O A, Poletaev A Yu 2016 Solving the assignment problem with additional requirements Proceedings of Voronezh State University, Series: Systems analysis and information technologies. 1 77-81

[11] Podolyaka A N, Nikonov O Ya, Timonin V A 2011 Finding of balanced solutions to the assignment problem Information Processing Systems. 2 (92) 46-8