Self-Similar Extrapolation of Asymptotic Series and Forecasting for Time Series

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Abstract

The method of extrapolating asymptotic series, based on the Self-Similar Approximation Theory, is developed. Several important questions are answered, which makes the foundation of the method unambiguous and its application straightforward. It is shown how the extrapolation of asymptotic series can be reformulated as forecasting for time series. The probability measure is introduced characterizing the ensemble of forecasted scenarios. The way of choosing the complete family of data bases is advanced.
1 Introduction

The problem of reconstructing functions from their asymptotic series is constantly met in all branches of physics and applied mathematics [1]. Not less ubiquitous is the problem of forecasting for time series [2]. Here we show that these two problems have much in common and can be solved in the same way. The solution stems from the Self-Similar Approximation Theory [3–13]. The basics of the particular techniques to be considered here have been suggested earlier and the feasibility of applying them to analyzing time series has been discussed [14,15]. However several important points remained unclear. The aim of the present paper is to develop further the theory, so that to remove ambiguities left, to provide a firm mathematical foundation, and to advance some fresh ideas permitting one to extend the applicability of the method. More concretely, I aim at providing answers to the following questions:

(1) What presentation of asymptotic series is the most correct and general for their extrapolation involving self-similar exponential approximants?

(2) What is the result of the fractal transformation, that is, of the multiplicative power-law transformation?

(3) What should be the power of the fractal transformation?

(4) How to define control functions characterizing the effective time of motion with respect to the approximation number?

(5) What is the correct presentation of time series, as a forward or backward recursion, in order that forecasting for time series could be formulated as extrapolation of asymptotic series?

(6) How to select among several scenarios forecasted by using different data bases?

(7) How to construct a complete family of data bases, that is, how many terms from the past-history data base are to be taken and how to vary the related past-time scales?

2 Extrapolation and Forecasting

Let us start with the problem of extrapolating asymptotic series. Assume, for simplicity, that the sought function $f(x)$ is real and depends on a real variable $x$. Suppose that all we know is the asymptotic behaviour of this function, $f(x) \simeq f_k(x)$, in the vicinity of $x = 0$. The index $k = 0, 1, 2, \ldots$ here enumerates approximations. The most common form of such asymptotic approximants is the power series

$$f_k(x) = \sum_{n=0}^{k} a_n x^{\alpha_n},$$

(1)

in which $a_0 \neq 0$ and the powers $\alpha_n$ are arranged in the ascending order

$$\alpha_n < \alpha_{n+1} \quad (n = 0, 1, \ldots, k).$$
One usually has $\alpha_n = n$, but, for generality, we may imply that the powers $\alpha_n$ are arbitrary real numbers, that even can be negative, provided the ascending order is retained.

The problem of extrapolation of asymptotic series sounds as follows: How to define the function $f(x)$ at finite $x$ from the knowledge of its asymptotic expansions (1) valid only in the asymptotic vicinity of $x = 0$?

In order that mathematical manipulations be independent from the choice of physical units which the sought function is measured in, it is more appropriate to deal with a dimensionless function. This can be easily achieved by factorizing out the zero term $f_0(x) = a_0 x^{\alpha_0}$, with $a_0 \neq 0$, and introducing the dimensionless function

$$\varphi_k(x) \equiv \frac{f_k(x)}{f_0(x)} .$$

This point is rather trivial and well understood in physics, where one always prefers to deal with scale-invariant quantities.

Clearly, the variable $x$ is also to be dimensionless. When the variable pertains to a finite interval, it always can be scaled so that to be in the unitary interval

$$0 \leq x \leq 1 .$$

This requirement will be assumed in what follows. And it will be shown that condition (3) is not merely nontrivial but principal. If the initial physical variable is given on an infinite interval, one can change the variables so that the new variable be defined on a finite interval, and then, by the appropriate scaling, one can reduce the latter interval to the unitary one.

Thus, we consider the asymptotic series

$$\varphi_k(x) = \sum_{n=0}^{\infty} b_n x^{\beta_n} ,$$

in which

$$b_n \equiv \frac{\alpha_n}{\alpha_0} , \quad \beta_n \equiv \alpha_n - \alpha_0 \geq \beta_0 = 0 .$$

The sequence $\{\varphi_k(x)\}_{k=0}^{\infty}$ has the zero radius of convergence, being divergent for any finite $x$. It converges solely at one point $x = 0$. How it would be possible to extrapolate the asymptotic expansion (4), having sense only for $x \to 0$, to the finite interval (3)?

Such an extrapolation can be done by employing the Self-Similar Approximation Theory [3–13]. The main idea of the latter is to present a relation between subsequent approximants as the motion with respect to an effective time whose role is played by the approximation number $k$. However, it is meaningless to look for a relation between the terms of a divergent sequence. Hence, the first thing one has to do is to transform the divergent sequence $\{\varphi_k(x)\}$ to a convergent form, which can be accomplished by a transformation involving control functions [3]. The name of the latter comes from their role in controlling convergence. In the case of the power series (4), a convenient transformation is the multiplicative power-law transformation [10–12], which is simply the multiplication of $\varphi_k(x)$ by a power-law factor $x^s$, where $s$ is a control function.
Since the power laws are generic for fractals [16], this kind of multiplication can be termed the fractal transformation. In this way, we introduce the fractal transform

$$\Phi_k(x, s) \equiv x^s \varphi_k(x) ,$$  \hspace{1cm} (5)

whose inverse is

$$\varphi_k(x) = x^{-s} \Phi_k(x, s) .$$  \hspace{1cm} (6)

The resulting fractal transform is

$$\Phi_k(x, s) = \sum_{n=0}^{k} b_n x^{s+\beta_n} .$$  \hspace{1cm} (7)

This can be considered as asymptotic series, for all $|x| < 1$, provided that

$$s \to \infty .$$  \hspace{1cm} (8)

The value of expansion (7) at $x = 1$ can always be defined as the limit from the left. Thus the sequence $\{\Phi_k(x, s)\}$ can be treated as convergent on the whole unitary interval (3) under condition (8).

For a convergent sequence, it becomes meaningful to try to find a relation between subsequent terms. Such a relation, according to the Self-Similar Approximation Theory [3-9], can be formulated as the motion with respect to the approximation number $k$. In mathematical parlance, the motion can be expressed as the semigroup property for a family of endomorphisms. To this end, we define the function $x(\varphi, s)$ by the equation

$$\Phi_0(x, s) = \varphi , \quad x = x(\varphi, s) .$$  \hspace{1cm} (9)

Then, we introduce

$$y_k(\varphi, s) \equiv \Phi_k(x(\varphi, s), s) ,$$  \hspace{1cm} (10)

which is an endomorphism of $\mathbb{R}$. The semigroup property for an endomorphism $y_k$ reads as $y_{k+p} = y_k \cdot y_p$. This is equivalent to the property of group self-similarity

$$y_{k+p}(\varphi, s) = y_k(y_p(\varphi, s), s) .$$  \hspace{1cm} (11)

The family $\{y_k| k = 0, 1, 2, \ldots\}$ of the endomorphisms $y_k$, with the semigroup property (11), forms a cascade, that is, a dynamical system in discrete time $k$. Then Eq. (11) is the evolution equation of the cascade, with the initial condition

$$y_0(\varphi) = \varphi ,$$  \hspace{1cm} (12)

which results from Eqs. (9) and (10). The functional equation (11) does not uniquely define a solution $y_k(\varphi)$. Among the wide class of solutions of Eq. (11) there are, in particular, the power-law solutions $y_k(\varphi, s) = \varphi^{\alpha_k(s)}$ provided that their powers satisfy the condition $\alpha_{k+p}(s) = \alpha_k(s) + \alpha_p(s)$. But the power-law expressions form only a narrow class of possible solutions of Eq. (11). Hence the group self-similarity (11) is essentially more general property than the simple geometric self-similarity resulting in power-law solutions [16].
To find a solution of the functional equation (11), corresponding to a fixed point, we can proceed as follows [4–9]. Let us embed the cascade \( \{y_k | k = 0, 1, 2, \ldots \} \) into a flow \( \{y_\tau | \tau \geq 0 \} \), which implies that the latter satisfies the same self-similarity relation (11) with the initial condition (12). This group relation for the flow yields the Lie equation, which is the differential equation with respect to the continuous effective time \( \tau \). The differential equation may be integrated from the effective time \( \tau = n \) to the time \( n + \tau_n \) that is necessary for reaching a quasifixed point \( y_n^* \) which is an approximate fixed point. The resulting evolution integral is

\[
\int_{y_n^*}^{y_n} \frac{d\varphi}{v_n(\varphi, s)} = \tau_n ,
\]

where

\[
v_n(\varphi, s) \equiv y_n(\varphi, s) - y_{n-1}(\varphi, s) = b_n \varphi^{1+\beta_n/s}
\]

is the cascade velocity. From the evolution integral (13), we get the quasifixed point \( y_n^* = y_n^*(\varphi, s) \), which, according to Eqs. (9) and (10), defines

\[
\Phi_k^*(x, s) \equiv y_k^*(x^s, s) .
\]

Then we have to realize the inverse fractal transformation (6), with the limit (8), which gives

\[
\varphi_k^*(x) \equiv \lim_{s \to \infty} x^{-s} \Phi_k^*(x, s) .
\]

This procedure is to be accomplished so many times as necessary for reorganizing all series entering \( \varphi_k^*(x) \). Such a repeated procedure has been called [11] self-similar bootstrap. Here we suggest the fastest way of accomplishing this bootstrap, when the \( k \)-order series \( \varphi_k(x) \) requires just \( k \) iterations. To this end, we may present the asymptotic expansion (4) as

\[
\varphi_k(x) \equiv 1 + x_1 ,
\]

with the sequence of the iterative terms

\[
x_1 = \frac{b_1}{b_0} x^{\beta_1-\beta_0} (1 + x_2) , \quad x_2 = \frac{b_2}{b_1} x^{\beta_2-\beta_1} (1 + x_3) , \quad \ldots
\]

\[
x_n = \frac{b_n}{b_{n-1}} x^{\beta_n-\beta_{n-1}} (1 + x_{n+1}) , \quad \ldots , \quad x_k = \frac{b_k}{b_{k-1}} x^{\beta_k-\beta_{k-1}} .
\]

Then we may consider each expression \( 1 + x_n \) as asymptotic series with respect to small \( x_n \rightarrow 0 \). The self-similar renormalization described above results in transforming \( 1 + x_n \) to \( \exp(x_n \tau_n) \). Starting with the form (17), we accomplish \( k \) steps of this self-similar interactive procedure. Then, with the notation

\[
c_n \equiv \frac{a_n}{a_{n-1}} \tau_n , \quad \nu_n \equiv \alpha_n - \alpha_{n-1} ,
\]

where \( n = 1, 2, \ldots , k \), we come to the exponential self-similar approximant

\[
\varphi_k^*(x) = \exp (c_1 x^{\nu_1} \exp (c_2 x^{\nu_2} \ldots \exp (c_k x^{\nu_k}))) .
\]
What is yet left undefined in the approximant (20) is the effective time \( \tau_n \), which play the role of control functions. The latter may be determined from fixed-point conditions [3,12]. Here we advance a more general way of defining control functions \( \tau_n \). This way is common for optimal control theory, where control functions are defined from the minimization of a cost functional. To construct the latter, one has to formulate the requirements imposed on the sought control functions. In our case, we would like to reach a quasifixed point as fast as possible. The minimal effective time for this corresponds to one step of the iterative procedure. If we reach an answer in \( n \) steps, then the minimal effective time \( \tau_n \) should be close to \( 1/n \). Another requirement characteristic of the motion near a quasifixed point is the smallness of the distance \( v_n \tau_n \) passed at the \( n \)-th step. This suggests to construct the time-distance cost functional

\[
F = \frac{1}{2} \sum_n \left[ \left( \tau_n - \frac{1}{n} \right)^2 + (v_n \tau_n)^2 \right]. \tag{21}
\]

Minimizing \( F \) with respect to \( \tau_n \) yields

\[
\tau_n = \frac{1}{n(1 + v_n^2)}. \tag{22}
\]

Here the \( x \)-representation of the cascade velocity \( v_n(x) \equiv x^{-s}v_n(x^s, s) \) results in

\[
v_n(x) = \varphi_n(x) - \varphi_{n-1}(x) = b_n x^{\beta_n}. \tag{23}
\]

Summarizing, for expressions (19) we have

\[
c_n(x) = \frac{a_n}{a_{n-1}} \tau_n(x), \quad \nu_n = \alpha_n - \alpha_{n-1},
\]

\[
\tau_n(x) = \frac{1}{n[1 + v_n^2(x)]}, \quad v_n(x) = \frac{a_n}{a_0} x^{\alpha_n - \alpha_0}. \tag{24}
\]

The functions \( c_n(x) \), being proportional to the control functions \( \tau_n(x) \), can be called controllers.

Finally, for the asymptotic series (1), we obtain the self-similar extrapolation

\[
f_k^*(x) = f_0(x) \exp \left( c_1 x^{\nu_1} \exp \left( c_2 x^{\nu_2} \ldots \exp \left( c_k x^{\nu_k} \right) \right) \right), \tag{25}
\]

where \( c_n = c_n(x) \) and \( \nu_n \) are given in Eq. (24), and which extrapolates the series (1), valid only for \( x \to 0 \), to the whole interval (3).

Now we shall show how the extrapolation of asymptotic series can be reformulated as forecasting for time series. This reformulation, as will be evident from what follows, requires that the considered time series be presented as a backward recursion, with the related moments of time being arranged in the backward order

\[
t_{n+1} < t_n \quad (n = 0, 1, 2, \ldots, k). \tag{26}
\]

The time series for a quantity of interest is an ordered set \( \{ f_n \} \) of the values \( f_n \) of this quantity, measured at the corresponding times \( t_n \). The past time horizon is the
time interval \([t_k, 0]\) when the series is observed. The present time is chosen to be at \(t_0 = 0\). Let the considered time series be described by the past-history data base

\[
\mathbb{D}_k \equiv \{f_k, f_{k-1}, \ldots, f_0 \mid t_k < t_{k-1} < \ldots < 0\}.
\]  

(27)

The problem to be attacked is how, on the grounds of the given data, to predict the behaviour of the time series in future times \(t > 0\)?

In order to employ the technique developed for asymptotic series, we need, first, to construct a sequence of functions approximating the behaviour of the time series at asymptotically small time \(t \to 0\). Similarly to the case of asymptotic series, we have to work with the quantities reduced to the scale-invariant dimensionless form. And the time variable is to be normalized with respect to the length of the prediction horizon so that this be a unitary interval

\[0 \leq t \leq 1.\]  

(28)

For the given data base (27), one may construct an interpolating function \(f_k(t)\) such that

\[f_k(t_n) = f_n \quad (n = 0, 1, \ldots, k).\]  

(29)

This interpolation can be uniquely realized e.g. by means of the Lagrange interpolation formula [17] as the sum

\[f_k(t) = \sum_{n=0}^{k} f_n l^k_n(t) \quad (k \geq 1)\]  

(30)

over the Lagrange polynomials

\[l^k_n(t) \equiv \prod_{m \neq n}^{k} \frac{t - t_m}{t_n - t_m} \quad (n \leq k).\]

Formula (30) can be rewritten as the power series

\[f_k(t) = \sum_{n=0}^{k} a_n t^n.\]  

(31)

The coefficients \(a_n\) depend on the given data base (27), but an additional index \(k\) is not explicitly shown here for brevity. Expressions (30) or (31), by construction, satisfy Eq. (29), including the moment of time \(t_0 = 0\), where

\[f_k(0) = f_0.\]  

(32)

Therefore, the interpolative formula (31) can be considered as an asymptotic series with respect to \(t \to +0\).

In the same way as asymptotic series, expression (31) has no sense for finite \(t > 0\), that is, cannot be directly employed for predicting future. But, being just a particular kind of asymptotic series, Eq. (31) can be extrapolated to the whole interval (28) following the general method of self-similar extrapolation. Then, analogously to Eq. (25), we obtain the self-similar forecast

\[f_k^*(t) = f_0 \exp(c_1 t \exp(c_2 t \ldots \exp(c_k t)\ldots)),\]  

(33)
where $c_n = c_n(t)$ are the controllers

$$c_n(t) = \frac{a_n}{a_{n-1}} \tau_n(t) \quad (n = 1, 2, \ldots, k) ,$$

$$\tau_n(t) = \frac{1}{n[1 + v_n^2(t)]} \quad v_n(t) = \frac{a_n}{f_0} t^n . \quad (34)$$

In this way, for each given past-history data base (27), we may construct the forecast (33) valid for the future time horizon (28). Since the coefficients $a_n \equiv a_{nk}$ depend on the choice of the data base, according to the conditions (29), the controllers $c_n(t) \equiv c_{nk}(t)$ also depend on the data base. This means that, for another data base, we shall obtain another value of the forecast. There are two things that may be varied for data bases: the number of points $k$ and the set $\{t_n\}$ of the time moments. Denoting a particular choice of the latter set with the index $j$, we have $\{t^{(j)}_n\}$. Thus, each data base has to be labelled by two indices

$$D_{k}(j) \equiv \{f^*_k, f^*_{k-1}, \ldots, f_0 | t^{(j)}_k < t^{(j)}_{k-1} < \ldots < 0 \} . \quad (35)$$

Varying these indices produces a family $\{D_{k}(j)\}$ of the data bases and, respectively, results in an ensemble $\{f^*_k(j, t)\}$ of the self-similar forecasts. How could we classify this ensemble of possible scenarios?

The problem of classifying scenarios in forecasting is analogous to the problem of pattern selection in the case of nonunique solutions of nonlinear partial differential equations [18]. A probabilistic approach to the problem of pattern selection has been recently advanced [19], which can also be used for defining a probability measure for the ensemble of forecasted scenarios. Following the approach [19], we treat the map $\{f^*_k(j, t) | k = 0, 1, 2, \ldots\}$ as the image of a dynamical cascade, where $k$ plays the role of discrete time, and for the probability of a forecast $f^*_k(j, t)$ we have

$$p_k(j, t) = \frac{1}{Z_k(t)} \exp \{-\Delta S_k(j, t)\} , \quad (36)$$

where $Z_k(t)$ is a normalization factor and $\Delta S_k(j, t)$ is an entropy variation. Let us introduce the mapping multiplier

$$m_k(j, t) \equiv \frac{\delta f^*_k(j, t)}{\delta f^*_1(j, t)} = \frac{\partial f^*_k(j, t)/\partial t}{\partial f^*_1(j, t)/\partial t} . \quad (37)$$

and the average multiplier $\overline{m}_k(t)$, such that

$$\frac{1}{|\overline{m}_k(t)|} = \sum_j \frac{1}{|m_k(j, t)|} . \quad (38)$$

Then the scenario probability (36) can be reduced [19] to the form

$$p_k(j, t) = \frac{\overline{m}_k(t)}{|m_k(j, t)|} . \quad (39)$$
The most probable scenario is, by definition, that one corresponding to the maximal probability \( (39) \).

It may happen that the data-base variation is limited by the information available. Then there is no choice as to deal with the given family \( \{ \mathbb{D}_k(j) \} \). But if this variation can be done in a very wide range, the question remains as when should one stop varying data bases? A brief answer to this question is: Vary data bases until reaching numerical convergence. More precisely, the variation procedure can be realized as follows. Let us fix a time scale, labelled by \( j \), and increase the data-base order \( k \), that is the number of points from the past history. The increase of \( k \) is to be stopped at the saturation number \( N_j \), after which the forecasts \( f_k^*[j, t] \), for \( k \geq N_j \), practically (within a given accuracy) do not change. Then the forecasts \( (33) \) as well as the scenario probability \( (39) \) become dependent only on the time scale, labelled by \( j \),

\[
 f^*[j, t] \equiv \lim_{k \to N_j} f_k^*[j, t], \quad p[j, t] \equiv \lim_{k \to N_j} p_k[j, t].
\]  

(40)

Varying the data-base time scale, it looks convenient to deal with equidistant steps \( \Delta_j \equiv t^{(j)}_n - t^{(j)}_{n+1} \), where \( n = 0, 1, \ldots, N_j \), although the equal distance between the time moments is, in general, not obligatory. It is natural to start with the step \( \Delta_j = 1 \), coinciding with the prediction horizon. Then we may increase as well as decrease the time grid, which can be done in different ways, say by following the law \( \Delta_{2j} = 2^{-j}, \quad \Delta_{2j+1} = 2^j \), with \( j = 0, 1, 2, \ldots \). Again, the further variation of the time mesh is to be stopped at \( j = j_{\text{max}} \) as soon as we reach numerical convergence. In this way, the complete data family is a set \( \{ \mathbb{D}_{N_j}(j) \mid j = 0, 1, 2, \ldots, j_{\text{max}} \} \) corresponding to the ensemble \( \{ f^*[j, t] \mid j = 0, 1, 2, \ldots, j_{\text{max}} \} \) that can be called the scenario spectrum. As far as this ensemble of forecasts is weighted with the probability measure \( p[j, t] \), it is straightforward to define the expected forecast, the dispersion, and other quantities typical of probabilistic ensembles.

3 Conclusion

The possibility of forecasting the main trends of time series is of great importance and can have various applications. One of such exciting application would be with respect to economics and financial time series. The behaviour of markets is nowadays studied by different methods imported from physics [20]. However the problem of quantitative prediction for markets remains yet a puzzle. I would think that the novel approach, developed in this paper, is a practical step for solving the problem of forecasting for time series, including financial time series. The aim of this communication has been to present a mathematical foundation for the approach, whose applications will be considered in separate publications. The content of this paper answers the questions posed in the Introduction, and these answers are:

(1) The most general and correct presentation of asymptotic series, appropriate for the self-similar extrapolation is in a dimensionless scale-invariant form, but with the domain of a variable reduced to the unitary interval.
(2) The fractal transformation extends the region of validity of the initial asymptotic series.

(3) The power of the fractal transformation must asymptotically tend to infinity in order to yield series with the unitary radius of convergence.

(4) The effective time of motion with respect to the approximation number is to be defined from the minimum of a cost functional.

(5) In order that extrapolation of asymptotic series could be correctly reformulated as forecasting for time series, the latter are to be presented as a backward recursion.

(6) All forecasted scenarios, derived by employing different data bases, compose a statistical ensemble equipped with a probability measure characterizing the probabilities of particular patterns.

(7) The complete data family is a set of data bases, with time scales varying in the range limited by the saturation of calculations, and with the numbers of past-history terms sufficient for providing numerical convergence for each given time grid.

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