THE CLOSURES OF WREATH PRODUCTS IN PRODUCT ACTION

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Let $m$ be a positive integer and let $\Omega$ be a finite set. The $m$-closure of $G \leq \text{Sym}(\Omega)$ is the largest permutation group $G^{(m)}$ on $\Omega$ having the same orbits as $G$ in its induced action on the Cartesian product $\Omega^m$. An exact formula for the $m$-closure of the wreath product in product action is given. As a corollary, a sufficient condition is obtained for this $m$-closure to be included in the wreath product of the $m$-closures of the factors.

1. MAIN RESULTS

Let $m$ be a positive integer and let $\Omega$ be a finite set. The $m$-closure of $G \leq \text{Sym}(\Omega)$ is the largest permutation group $G^{(m)}$ on $\Omega$ such that

\[ \text{Orb}_m(G^{(m)}) = \text{Orb}_m(G), \]

where $\text{Orb}_m(G^{(m)})$ and $\text{Orb}_m(G)$ are the sets of orbits in the induced actions of $G^{(m)}$ and $G$, respectively, on the Cartesian power $\Omega^m$. Wielandt [1, Thms. 5.8, 5.12] showed that

\[ G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(m)} = G^{(m+1)} = \cdots = G \] (1)

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for some $m < |\Omega|$. In this sense, the $m$-closure can be considered as a natural approximation of $G$. We can also think of $G^{(m)}$ as the full automorphism group of the family of all $m$-ary relations invariant with respect to $G$.

In general, studying the $m$-closure for $m \geq 2$ is a nontrivial problem from both theoretical and computational points of view (see, e.g., [2-7]). The usual approach here is a reduction via direct or wreath products to smaller permutation groups. Let us consider these operations in more detail.

Let $K \leq \text{Sym}(\Gamma)$ and $L \leq \text{Sym}(\Delta)$. The direct product $K \times L$ has two natural actions—on the disjoint union $\Gamma \cup \Delta$ and on the Cartesian product $\Gamma \times \Delta$. It is well known that in both cases the $m$-closure of $K \times L$ is equal to $K^{(m)} \times L^{(m)}$ (see, e.g., [8]). A similar formula holds for the wreath product $K \wr L$ acting imprimitively on the disjoint union of $|\Delta|$ copies of $\Gamma$, i.e., on $\Gamma \times \Delta$ (see [9]). However, passing to the permutation group $K \uparrow L$ induced by the product action of $K \wr L$, i.e., on the Cartesian product of $|\Delta|$ copies of $\Gamma$, causes a problem. Indeed, the equalities

$$
\begin{align*}
(Sym(2) \uparrow Alt(3))^{(2)} &= Sym(2) \uparrow Sym(3), \\
Sym(2)^{(2)} \uparrow Alt(3)^{(2)} &= Sym(2) \uparrow Alt(3)
\end{align*}
$$

show that $(K \uparrow L)^{(m)} \nleq K^{(m)} \uparrow L^{(m)}$ in general. The main goal of the present paper is to derive an exact formula for the $m$-closure of $K \uparrow L$.

Apparently, the first results related to the structure of $(K \uparrow L)^{(m)}$ were obtained in [2, Props. 3.2, 3.3] for the case where the group $K \uparrow L$ is primitive. Even in this case, however, no explicit formula was found. The inclusion $(K \uparrow L)^{(m)} \leq K^{(m)} \uparrow L^{(m)}$ was first proved in [5, Prop. 3.1] for $m = 2$ and non 2-transitive $K$ (cf. (2)), and then recently in [4, Thm. 3.3] for $m \geq 3$, for primitive $K \uparrow L$, and under some technical assumptions on the degrees of $K$ and $L$.

In order to state the main result, we need to define one more closure operator. Namely, for a group $G \leq \text{Sym}(\Omega)$, we denote by $G^{[m]}$ the largest permutation group on $\Omega$ having the same orbits as $G$ in its induced action on the ordered partitions of $\Omega$ into at most $m$ classes. This type of closure behaves similarly to the $m$-closure (cf. formulas (1) and (9)) and will be considered in more detail in Section 3.

**THEOREM 1.1.** Let $K$ and $L$ be permutation groups and $m \geq 2$ be an integer. Then

$$(K \uparrow L)^{(m)} = K^{(m)} \uparrow L^{[k]},$$

where $k = \min\{k_m, d\}$ with $k_m = |\text{Orb}_m(K)|$ and $d$ equal to the degree of $L$.

The number $k_m$ defined in Theorem 1.1 is bounded from below by the number $|\text{Orb}_m(\text{Sym}(n))|$, where $n$ is the degree of $K$, which, for $m \leq n$, is equal to the number of ordered partitions of a set of cardinality $m$ (see, e.g., [4, Example 2.1]). In particular, $k_m \geq m + 1$ if $m \geq 3$ and $n \geq 2$.

Theorem 1.1 enables us to establish a sufficient condition for the $m$-closure of $K \uparrow L$ to be included in the wreath product of the $m$-closures of $K$ and $L$.

**THEOREM 1.2.** Let $K$ and $L$ be permutation groups and $m \geq 2$. Then

$$(K \uparrow L)^{(m)} \leq K^{(m)} \uparrow L^{(m)}$$

(4)
unless \( m = 2 \) and \( K \) is 2-transitive.

**Proof.** Without loss of generality, we assume that \( K \) is of degree at least 2. Then \( k_m \geq m + 1 \) (hereinafter, we use the notation of Theorem 1.1): this follows from the above if \( m \geq 3 \), and from the fact that \( K \) is not 2-transitive if \( m = 2 \). Now if \( d \geq m + 1 \), then \( k \geq m \) and Lemma 3.2, combined with (1), yields \( L[k] \leq L^{(k-1)} \leq L^{(m)} \). On the other hand, if \( d \leq m \), then \( k = d \) and formulas (1) and (9) give \( L[k] = L[d] = L \leq L^{(m)} \). Thus, inclusion (4) holds by Theorem 1.1 in any case. □

Theorem 1.2 gives a natural generalization of the two results from [5, 4] mentioned above. The exceptional case where \( m = 2 \) and \( K \) is 2-transitive cannot be avoided (see (2)). Some other examples of primitive groups \( L \) for which \( L^{(2)} = L \) and \( L^{[2]} > L \) may be found among the groups listed in [10, Thm. 2]. We believe that there are infinitely many such examples in which \( L \) is imprimitive.

When the group \( L \) is primitive and \( m \geq 3 \), the right-hand side of the equality in Theorem 1.1 can be made more precise by using the main results of [10].

**THEOREM 1.3.** Let \( K \) and \( L \) be permutation groups and \( m \geq 3 \). Assume that \( L \) is primitive and is not an alternating group in standard action. Then

\[
(K \uparrow L)^{(m)} = K^{(m)} \uparrow L.
\]

The proof follows from Theorem 1.1 and Lemma 3.3. □

2. PRELIMINARIES

We start with some basic facts of Wielandt’s theory of \( m \)-closures. First, note that taking the \( m \)-closure is a closure operator:

\[
G \leq G^{(m)}; \quad G^{(m)} = (G^{(m)})^{(m)}; \quad G \leq H \Rightarrow G^{(m)} \leq H^{(m)};
\]

see [1, Thms. 5.4, 5.9, and 5.7] respectively. Second, there is a clear necessary and sufficient condition for a permutation to be in the \( m \)-closure.

**LEMMA 2.1** (closure argument; see [1, Thm. 5.6]). Let \( G \leq \text{Sym}(\Omega) \), \( f \in \text{Sym}(\Omega) \), and \( m \) be a natural number. Then \( f \in G^{(m)} \) if and only if for every \( \alpha \in \Omega^m \) there is \( g \in G \) such that \( \alpha f = \alpha g \).

The closure argument is crucial in finding \( m \)-closures of products of permutation groups. See, for example, a detailed proof of the following folklore result in [8, Lemma 2.4].

**THEOREM 2.2.** Let \( K \leq \text{Sym}(\Gamma) \), \( L \leq \text{Sym}(\Delta) \), and let \( K \times L \) act on the Cartesian product \( \Gamma \times \Delta \). For every integer \( m \geq 2 \),

\[
(K \times L)^{(m)} = K^{(m)} \times L^{(m)}.
\]
We use the same argument in the proof of our main result for the wreath products of permutation groups in product action. Let us take a closer look at such a product.

Let \( K \leq \text{Sym}(\Gamma) \) and \( L \leq \text{Sym}(\Delta) \). Without loss of generality, we assume that \( \Delta = \{1, \ldots, d\} \). The wreath product \( K \wr L \) induces a permutation group \( G = K \uparrow L \) on the Cartesian product

\[
\Omega = \Gamma \times \cdots \times \Gamma.
\]

(5)

Every permutation of \( G \) can be written in the form

\[
g = (g_1, \ldots, g_d; \bar{g})
\]

(6)

for some \( g_1, \ldots, g_d \in K \) and \( \bar{g} \in L \). The action of \( g \) on the point

\[
\omega = (\omega_1, \ldots, \omega_d) \in \Omega
\]

is defined as follows (see, e.g., [11, Sec. 2.7]):

\[
(\omega^g)_i = (\omega_{\bar{g}^{-1}i})^{g_{\bar{g}^{-1}i}}, \quad 1 \leq i \leq d.
\]

(7)

It is well known [12, Thm. 9.2.1] that the automorphism group of the Hamming graph is the wreath product of two symmetric groups in product action. If the vertex set of this graph is of the form (5), then its edge set is an orbit of \( \text{Sym}(\Gamma) \uparrow \text{Sym}(\Delta) \). This implies that

\[
(\text{Sym}(\Gamma) \uparrow \text{Sym}(\Delta))^{(2)} = \text{Sym}(\Gamma) \uparrow \text{Sym}(\Delta).
\]

(8)

All the undefined notation for permutation groups used in the paper is mostly standard and can be found in [11].

3. CLOSURE WITH RESPECT TO PARTITIONS

Let \( \Omega^{[m]} \) be the set of all ordered partitions \( \Pi \) of \( \Omega \) such that \(|\Pi| \leq m\). For a group \( G \leq \text{Sym}(\Omega) \), we denote by \( G^{[m]} \) the largest permutation group on \( \Omega \) having the same orbits as \( G \) in its induced action on \( \Omega^{[m]} \). Obviously,

\[
\text{Sym}(\Omega) = G^{[1]} \geq G^{[2]} \geq \cdots \geq G^{[m]} = G^{[m+1]} = \cdots = G
\]

(9)

for some \( m \leq |\Omega| \).

The groups of series (9), except for the first one, are orbit equivalent to \( G \) in the sense of [10], i.e., for all \( m \geq 2 \),

\[
\text{Orb}(G^{[m]}, 2^\Omega) = \text{Orb}(G, 2^\Omega).
\]

(10)

Indeed, let \( S \subseteq \Omega \), and let \( \Pi = (\Pi_1, \Pi_2) \) be the partition of \( \Omega \) into two classes \( \Pi_1 = S \) and \( \Pi_2 = \Omega \setminus S \). Now if \( H = G^{[m]} \), then \( \Pi^G = \Pi^H \), and hence \( S^G = \Pi_1^G = \Pi_1^H = S^H \), as required.
Consequently, \( \Pi \equiv \Pi \) verify that for every ordered partition Thm. 5.12]. On the other hand, the group \( \Pi = \text{Alt}(\Pi) \) formula (9), this implies which is possible only if \( \Pi \). Thus, \( \Pi \). It follows that

\[
\{\alpha^g\} = \Pi^g = \Pi^h = \{\alpha^h\}, \quad i = 1, \ldots, m.
\]

Thus, \( \alpha^g = \alpha^h \), and it remains to apply the closure argument. □

In general, we cannot improve Lemma 3.2 by replacing \( m + 1 \) by \( m \) in \( G_{m+1} \). Indeed, let \( G = \text{Alt}(n), n \geq 3 \). Then the pointwise stabilizer \( G_{1, \ldots, n-2} \) is trivial, and \( G^{n-1} = G \) (see [1, Thm. 5.12]). On the other hand, the group \( G \) is \( (n-2) \)-transitive. Using this fact, it is not hard to verify that for every ordered partition \( \Pi \) into at most \( n - 1 \) classes, \( G^\Pi = \Pi^\text{Sym}(n) \). Together with formula (9), this implies

\[
G^m = \begin{cases} 
\text{Sym}(n) & \text{if } m \leq n - 1, \\
G & \text{otherwise.}
\end{cases}
\]  

(11)

Hence \( G^{n-1} \not\subseteq G^{n-1} \).

**Lemma 3.3.** Let \( G \leq \text{Sym}(n) \) be a primitive group and \( m \geq 3 \). Then \( G^m = G \) unless \( G = \text{Alt}(n) \) and \( m \leq n - 1 \).

**Proof.** In view of (11), we may assume that \( G \nless \text{Alt}(n) \). The group \( G^m \geq G \) is primitive and orbit equivalent to \( G \), i.e., equality (10) holds. By [10, Cor. 3], this implies that \( G = G^m \) unless

\[
(G, G^m) \in \mathcal{S},
\]

where \( \mathcal{S} \) consists of explicitly described pairs \((H, H^*)\) of primitive groups (of degree at most 10) with \( H < H^* \).

A straightforward computation in computer package GAP [13] shows that for every pair \((H, H^*) \in \mathcal{S} \) there is an ordered partition \( \Pi = \Pi(H, H^*) \) with three classes such that \( H^* \) acts on the orbit \( \Pi^H \) regularly. Now let \( \Pi = \Pi(G, G^m) \). Since \( |\Pi| = 3 \leq m \), we have \( \Pi^{G^m} = \Pi^G \). Consequently,

\[
|G| \geq |\Pi^G| = |\Pi^{G^m}| = |G^m|,
\]

which is possible only if \( G^m = G \). □
4. PROOF OF THEOREM 1.1

Let \( K \leq \text{Sym}(\Gamma) \) and \( L \leq \text{Sym}(\Delta) \), where \( \Delta = \{1, \ldots, d\} \). Put \( G = K \uparrow L \) and \( \Omega = \Gamma^d \). Then \( G \leq \text{Sym}(\Omega) \).

Every \( m \)-tuple \( \alpha \in \Omega^m \) is written in the form \( \alpha = (\alpha^{(1)}, \ldots, \alpha^{(m)}) \), where \( \alpha^{(j)} = (\alpha^{(j)}_1, \ldots, \alpha^{(j)}_d) \in \Gamma^d \), \( j = 1, \ldots, m \). It is convenient to treat \( \alpha \) as a \( d \times m \) matrix

\[
\begin{pmatrix}
\alpha^{(1)}_1 & \alpha^{(2)}_1 & \cdots & \alpha^{(m)}_1 \\
\alpha^{(1)}_2 & \alpha^{(2)}_2 & \cdots & \alpha^{(m)}_2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^{(1)}_d & \alpha^{(2)}_d & \cdots & \alpha^{(m)}_d 
\end{pmatrix},
\]

where the \( j \)th column is \( \alpha^{(j)} \). In fact, we are rather interested in rows of this matrix which are \( m \)-tuples of points of \( \Gamma \); the \( i \)th row is denoted by \( \alpha[i] = (\alpha^{(1)}_i, \ldots, \alpha^{(m)}_i) \).

Now let \( \text{Orb}_m(K) = \{s_1, \ldots, s_a\} \), where \( a = k_m \). For the \( m \)-tuple \( \alpha \), there are uniquely determined numbers \( 1 \leq a_1 < \cdots < a_r \leq a \) such that \( \alpha[i] \in s_{a_\ell} \) for every \( i \in \Delta \) and some \( 1 \leq \ell \leq r \). Denote by \( \Pi(\alpha) \) the ordered partition of \( \Delta \) with classes

\[
\Pi_\ell(\alpha) = \{i \in \Delta : \alpha[i] \in s_{a_\ell}\}, \quad \ell = 1, \ldots, r.
\]

Note that the integer \( k \) from the statement of the theorem is equal to \( \min\{a, d\} \).

**Lemma 4.1.** The mapping \( \Omega^m \to \Delta^k, \alpha \mapsto \Pi(\alpha) \), is well defined and surjective.

**Proof.** Obviously, \( |\Pi(\alpha)| \leq d \) and \( |\Pi(\alpha)| \leq a \), and hence \( |\Pi(\alpha)| \leq k \) for all \( \alpha \). Therefore, \( \Pi(\alpha) \in \Delta^k \) and the mapping is well defined. Let \( \Pi = (\Pi_1, \ldots, \Pi_t) \in \Delta^k \), where \( t \leq k \). Then every \( i \in \Delta \) belongs to some \( \Pi_\ell \). Choose an arbitrary point \( \beta_i \in s_\ell \), which is possible because \( k \leq a \). Now let \( \alpha \) be the unique \( m \)-tuple of \( \Omega \) for which

\[
\alpha[i] = \beta_i, \quad i \in \Delta.
\]

Then \( \Pi(\alpha) = \Pi \), as required. \( \Box \)

In accordance with formula (7), a permutation \( g \in \text{Sym}(\Gamma) \uparrow \text{Sym}(\Delta) \) acts on an \( m \)-tuple \( \alpha \) as follows:

\[
\alpha^g[i] = (\alpha[i]^{\overline{\Gamma}^{-1}})^{\overline{\sigma}^{-1}}, \quad 1 \leq i \leq d.
\]

Thus, \( \overline{\sigma} \) permutes the rows of the \( d \times m \) matrix \( \alpha \), while \( g_i \) permutes elements of the \( i \)th row coordinatewise.

From (1), monotonicity of the \( m \)-closure operator, and (8), it follows that

\[
H = G^{(m)} \leq G^{(2)} \leq (\text{Sym}(\Gamma) \uparrow \text{Sym}(\Delta))^{(2)} = \text{Sym}(\Gamma) \uparrow \text{Sym}(\Delta).
\]

As in (6), every permutation of \( H \) is written in the form

\[
h = (h_1, \ldots, h_d; \overline{h})
\]
for some \( h_1, \ldots, h_d \in \text{Sym}(\Gamma) \) and \( \overline{h} \in \text{Sym}(\Delta) \).

**Lemma 4.2.** \( \Pi(\alpha)^{\overline{h}} = \Pi(\alpha^h) \) for all \( h \in K^{(m)} \uparrow \text{Sym}(\Delta) \) and \( \alpha \in \Omega^m \).

**Proof.** Let \( i \in \Delta \). There is \( \ell \in \{1, \ldots, r\} \) such that \( i \in \Pi_\ell(\alpha) \), i.e., \( \alpha[i] \in s_{a_\ell} \). By formula (12),

\[
\alpha^h[i] = (\alpha[i])^{h_{\ell}} \in (s_{a_\ell})^{h_{\ell}} = s_{a_\ell}.
\]

We emphasize that although the indices \( a_\ell, \ell = 1, \ldots, r \), have been defined for the tuple \( \alpha \), formula (13) shows that they remain the same for \( \alpha^h \). Consequently, \( i^{\overline{h}} \in \Pi_\ell(\alpha^h) \) implying \( \Pi_\ell(\alpha)^{\overline{h}} = \Pi_\ell(\alpha^h) \) for all \( \ell \). \( \square \)

Let us prove that \( K^{(m)} \uparrow L^{[k]} \leq H \). First, we note that

\[
1 \uparrow L^{[k]} \leq H.
\]

Indeed, let \( h \in 1 \uparrow L^{[k]} \) and \( \alpha \in \Omega^m \). By Lemma 3.1, there exists \( \overline{f} \in L \) such that \( \Pi(\alpha)^{\overline{h}} = \Pi(\alpha)^{\overline{f}} \). Clearly, \( f = (1, \ldots, 1; \overline{f}) \in 1 \uparrow L \). By Lemma 4.2,

\[
\Pi(\alpha^{h_f^{-1}}) = \Pi(\alpha)^{\overline{h}^{\overline{f}^{-1}}} = \Pi(\alpha).
\]

It follows that for every \( i \in \Delta \), the tuples \( \alpha^{h_f^{-1}}[i] \) and \( \alpha[i] \) belong to the same \( m \)-orbit of \( K \). Therefore,

\[
\alpha^{h_f^{-1}}[i] = \alpha[i]^{k_i}
\]

for some \( k_i \in K \). Then \( k = (k_1, \ldots, k_d) \in K^d \) and hence \( kf \in K \uparrow L \). Furthermore, formula (15) yields \( \alpha^{h_f^{-1}} = \alpha^k \) implying \( \alpha^h = \alpha^k \). Thus, \( h \in H \) by Lemma 2.1, which proves (14).

By Theorem 2.2, we have \( K^{(m)} \uparrow 1 = (K \uparrow 1)^{(m)} \leq H \). Together with (14), this shows that

\[
K^{(m)} \uparrow L^{[k]} = (K^{(m)} \uparrow 1, 1 \uparrow L^{[k]}) \leq H = (K \uparrow L)^{(m)}.
\]

To prove the reverse inclusion, let \( h \in H \). For an arbitrary \( m \)-orbit \( s \) of \( K \), the set

\[
X_s = \{ \alpha \in \Omega^m : \alpha[i] \in s, \ i = 1, \ldots, d \}
\]

is invariant with respect to the group \( G \). Consequently, \( (X_s)^h = X_s \). By formula (12), this yields

\[
\alpha[i]^{h_i} = \alpha^h[i^{\overline{h}}] \in s
\]

for all \( \alpha \in X_s \) and \( 1 \leq i \leq d \). Since \( \alpha[i] \in s \), it follows that \( s^{h_i} = s \). Thus, \( h_i \) preserves each \( m \)-orbit of \( K \). Hence \( h_i \in K^{(m)} \) for all \( i \).

It remains to prove that \( \overline{h} \in L^{[k]} \) for all \( h \in H \). To this end, let \( \Pi \in \Delta^{[k]} \). By Lemma 4.1, there is \( \alpha \in \Omega^m \) such that \( \Pi = \Pi(\alpha) \). By the closure argument, there is \( g \in K \uparrow L \) for which \( \alpha^g = \alpha^\theta \).

Since \( h_i \in K^{(m)} \) for all \( i \), Lemma 4.2 yields

\[
\Pi^{\overline{h}} = \Pi(\alpha)^{\overline{h}} = \Pi(\alpha^h) = \Pi(\alpha^g) = \Pi(\alpha)^{\overline{\theta}} = \Pi^{\overline{\theta}}.
\]

Thus, \( \overline{h} \in L^{[k]} \) by Lemma 3.1. Hence \( h \in K^{(m)} \uparrow L^{[k]} \).

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