On a Diophantine inequality with five prime variables

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Abstract: Let $N$ be a sufficiently large real number. In this paper, it is proved that, for $1 < c < \frac{665576}{319965}$, $c \neq 2$, the following Diophantine inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c + p_5^c - N| < \log^{-1} N$$

is solvable in prime variables $p_1, p_2, p_3, p_4, p_5$, which improves the result of Baker and Weingartner [Monatsh. Math. 170 (2013), no. 3–4, 261–304], a result of Shi and Liu [Monatsh. Math. 169 (2013), no. 3–4, 423–440], and more earlier result of Zhai and Cao [Monatsh. Math. 150 (2007), no. 2, 173–179].

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1 Introduction and main result

Let $k \geq 1$ be a fixed integer and $N$ a sufficiently large integer. The famous Waring–Goldbach problem is to study the solvability of the following Diophantine equality

$$N = p_1^k + p_2^k + \cdots + p_r^k$$ (1.1)

in prime variables $p_1, p_2, \ldots, p_k$. For $k = 2$, in 1938, Hua [5] proved that the equation (1.1) is solvable for $r = 5$ and sufficiently large integer $N$ satisfying $N \equiv 5$ (mod 24).

In 1952, Piatetski-Shapiro [7] studied the following analogue of the Waring–Goldbach problem: Suppose that $c > 1$ is not an integer, $\varepsilon$ is a small positive number, and $N$ is a sufficiently large real number. Denote by $H(c)$ the smallest natural number $r$ such that the following Diophantine inequality

$$|p_1^c + p_2^c + \cdots + p_r^c - N| < \varepsilon$$ (1.2)

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is solvable in primes \( p_1, p_2, \ldots, p_r \), then it was proved in [7] that
\[
\limsup_{c \to +\infty} \frac{H(c)}{c \log c} \leq 4.
\]

In [7], Piatetski-Shapiro considered the case \( r = 5 \) in (1.2) and proved that \( H(c) \leq 5 \) for \( 1 < c < 3/2 \). Later, the upper bound \( 3/2 \) for \( H(c) \leq 5 \) was improved successively to
\[
\frac{14142}{8923} = 1.584892 \cdots, \quad \frac{1 + \sqrt{5}}{2} = 1.618033 \cdots, \quad \frac{81}{40} = 2.025, \quad \frac{108}{53} = 2.037735 \cdots, \quad 2.041
\]
by Zhai and Cao [13], Garaev [3], Zhai and Cao [14], Shi and Liu [9], Baker and Weingartner [1], respectively. Especially, the results in [14, 9, 1] satisfy \( c > 2 \), which can be regarded as an analogue of Hua’s theorem on sums of five squares of primes. By noting the fact that, for \( c > 2 \), the sequence \( p^c \) is sparser than the sequence \( p^2 \), thus the solvability of (1.2) becomes more difficult when the range of \( c \), which satisfies \( c > 2 \), becomes larger.

In this paper, we shall continue to improve the result of Baker and Weingartner [1] and establish the following theorem.

**Theorem 1.1** Suppose that \( 1 < c < \frac{665576}{319965}, c \neq 2 \), then for any sufficiently large real number \( N \), the following Diophantine inequality
\[
|p_1^c + p_2^c + p_3^c + p_4^c + p_5^c - N| < \log^{-1} N
\]
is solvable in primes \( p_1, p_2, p_3, p_4, p_5 \).

**Remark 1.** In order to compare our result with the results of Baker and Weingartner [1], Shi and Liu [9], and Zhai and Cao [14], we list the numerical result as follows
\[
\frac{665576}{319965} = 2.0801525 \cdots, \quad 2.041; \quad \frac{108}{53} = 2.037735 \cdots, \quad \frac{81}{40} = 2.025.
\]

**Remark 2.** Since several authors listed above have showed the solvability of (1.3) for \( 1 < c < 2 \), we only need to show that, for \( 2 < c < \frac{665576}{319965} \), the Diophantine inequality (1.3) is solvable in primes \( p_1, p_2, p_3, p_4, p_5 \). Therefore, in this paper, we only focus on the case \( 2 < c < \frac{665576}{319965} \).

**Notation.** Throughout this paper, we suppose that \( 1 < c < \frac{665576}{319965}, c \neq 2 \). Let \( p \), with or without subscripts, always denote a prime number. \( \eta \) always denotes an arbitrary small positive constant, which may not be the same at different occurrences; \( N \) always denotes a sufficiently large real number. As usual, we use \( \Lambda(n) \) to denote
von Mangoldt’s function; \( e(x) = e^{2\pi i x}; \) \( f(x) \ll g(x) \) means that \( f(x) = O(g(x)) \); \( f(x) \asymp g(x) \) means that \( f(x) \ll g(x) \ll f(x) \).

We also define
\[
X = \frac{1}{2} \left( \frac{3N}{10} \right)^{\frac{1}{4}}, \quad \varepsilon = \log^{-4} X, \quad K = \log^{10} X, \quad \tau = X^{1-c-n},
\]

\[
\mathcal{T}(x) = \sum_{X<n \leq 2X} e(n^c x), \quad S(x) = \sum_{X<p \leq 2X} (\log p) e(p^c x), \quad I(x) = \int_{X}^{2X} e(t^c x) dt.
\]

## 2 Preliminary Lemmas

In this section, we shall give some preliminary lemmas, which are necessary in the proof of Theorem 1.1.

**Lemma 2.1** Let \( a, b \) be real numbers, \( 0 < b < a/4 \), and let \( r \) be a positive integer. Then there exists a function \( \phi(y) \) which is \( r \) times continuously differentiable and such that
\[
\begin{aligned}
\phi(y) &= 1, & \text{if } |y| \leq a - b, \\
0 < \phi(y) < 1, & \text{if } a - b < |y| < a + b, \\
\phi(y) &= 0, & \text{if } |y| \geq a + b,
\end{aligned}
\]
and its Fourier transform
\[
\Phi(x) = \int_{-\infty}^{+\infty} e(-xy) \phi(y) dy
\]
satisfies the inequality
\[
|\Phi(x)| \leq \min \left( 2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left( \frac{r}{2\pi|x|b} \right)^r \right).
\]

**Proof.** See Piatetski–Shapiro [7] or Segal [10].

**Lemma 2.2** Let \( M, Q \geq 1 \) and \( z_m \) be complex numbers. Then we have
\[
\left| \sum_{M<m \leq 2M} z_m \right|^2 \leq \left( 2 + \frac{M}{Q} \right) \sum_{|q|<Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{M<m+q, m-q \leq 2M} z_{m+q} z_{m-q}.
\]

**Proof.** See Lemma 2 of Fouvry and Iwaniec [2].

**Lemma 2.3** Suppose that \( f(x) : [a, b] \to \mathbb{R} \) has continuous derivatives of arbitrary order on \([a, b]\), where \( 1 \leq a < b \leq 2a \). Suppose further that
\[
|f^{(j)}(x)| \asymp \lambda_j a^{1-j}, \quad j \geq 1, \quad x \in [a, b].
\]
Then for any exponential pair \((\kappa, \lambda)\), we have
\[
\sum_{a < n < b} e(f(n)) \ll \lambda_1^a \kappa + \lambda_1^{-1}.
\]

**Proof.** See (3.3.4) of Graham and Kolesnik [4].

**Lemma 2.4** Suppose \(Y > 1, \gamma > 0, c > 1, c \notin \mathbb{Z}\). Let \(\mathcal{A}(Y; c, \gamma)\) denote the number of solutions of the inequality
\[
|n_1^c + n_2^c - n_3^c - n_4^c| < \gamma, \quad Y < n_1, n_2, n_3, n_4 \leq 2Y,
\]
then there holds
\[
\mathcal{A}(Y; c, \gamma) \ll (\gamma Y^{4-c} + Y^2)Y^\eta.
\]

**Proof.** See Theorem 2 of Robert and Sargos [8].

**Lemma 2.5** For \(1 < c < \frac{665576}{319965}, c \neq 2\), we have
\[
\int_{-\tau}^{+\tau} |S(x)|^4dx \ll X^{4-c} \log^5 X, \quad (2.1)
\]
\[
\int_{-\tau}^{+\tau} |I(x)|^4dx \ll X^{4-c} \log^5 X. \quad (2.2)
\]

**Proof.** We only prove (2.1), and (2.2) can be proved likewise. It is easy to see that
\[
\int_{-\tau}^{+\tau} |S(x)|^4dx = \sum_{X < p_1, \ldots, p_4 \leq 2X} \left(\log p_1\right) \cdots \left(\log p_4\right) \int_{-\tau}^{+\tau} e\left((p_1^c + p_2^c - p_3^c - p_4^c)x\right)dx
\]
\[
\ll \sum_{X < p_1, \ldots, p_4 \leq 2X} \left(\log p_1\right) \cdots \left(\log p_4\right) \cdot \min\left(\tau, \frac{1}{|p_1^c + p_2^c - p_3^c - p_4^c|}\right)
\]
\[
\ll U \tau \log^4 X + V \log^4 X, \quad (2.3)
\]

where
\[
U = \sum_{X < n_1, n_2, n_3, n_4 \leq 2X} \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|}, \quad V = \sum_{X < n_1, n_2, n_3, n_4 \leq 2X} \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|}.
\]

On one hand, we have
\[
U \ll \sum_{X < n_1 \leq 2X} \sum_{X < n_2 \leq 2X} \sum_{n_3 \leq 2X} \sum_{X < n_4 \leq 2X} \frac{1}{(n_1^c + n_2^c - n_3^c - n_4^c)^{4-c}} \ll \sum_{X < n_4 \leq 2X} \frac{1}{(n_1^c + n_2^c - n_3^c - n_4^c)^{4-c}}.
\]
\[
\sum_{X < n_1, n_2, n_3 \leq 2X \atop n_1 + n_2 - n_3 \leq X} \left( 1 + (n_1^c + n_2^c - n_3^c + 1/\tau)^{1/c} - (n_1^c + n_2^c - n_3^c - 1/\tau)^{1/c} \right),
\]

and by the mean–value theorem we get
\[
U \ll X^3 + \frac{1}{\tau}X^{4-c}.
\] (2.4)

On the other hand, we have \( V \leq \sum \mathcal{V}_\ell \), where
\[
\mathcal{V}_\ell = \sum_{X < n_1, n_2, n_3, n_4 \leq 2X \atop \ell < n_1^c + n_2^c - n_3^c - n_4^c \leq 2\ell} \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|}
\] (2.5)

and \( \ell \) takes the values \( 2^k \tau^{-1}, k = 0, 1, 2, \ldots \), with \( \ell \ll X^c \). Then, we deduce that
\[
\mathcal{V}_\ell \ll \frac{1}{\ell} \sum_{X < n_1, n_2, n_3, n_4 \leq 2X \atop (n_1^c + n_2^c - n_3^c + \ell)^{1/c} \leq n_4^c \leq (n_1^c + n_2^c - n_3^c + 2\ell)^{1/c} \atop n_1^c + n_2^c - n_3^c \leq X^c} 1.
\]

For \( \ell \gg 1/\tau \) and \( X < n_1, n_2, n_3 \leq 2X \) with \( n_1^c + n_2^c - n_3^c > X^c \), it is easy to see that
\[
(n_1^c + n_2^c - n_3^c + 2\ell)^{1/c} - (n_1^c + n_2^c - n_3^c + \ell)^{1/c} > 1.
\]

Therefore, by the mean–value theorem, there holds
\[
\mathcal{V}_\ell \ll \frac{1}{\ell} \sum_{X < n_1, n_2, n_3 \leq 2X \atop n_1^c + n_2^c - n_3^c \leq X^c} \left( (n_1^c + n_2^c - n_3^c + 2\ell)^{1/c} - (n_1^c + n_2^c - n_3^c + \ell)^{1/c} \right) \ll X^{4-c}.
\] (2.6)

Thus, the conclusion (2.1) follows from (2.3)–(2.6).

**Lemma 2.6** For \( 1 < c < 3, c \neq 2 \), \(|x| \leq \tau \), we have
\[
S(x) = I(x) + O \left( X e^{-(\log X)^{1/5}} \right).
\]

**Proof.** The proof of Lemma 2.6 is similar to that of Lemma 14 in Tolev [12].

**Lemma 2.7** For \( 1 < c < 3, c \neq 2 \), we have
\[
\int_{-\infty}^{+\infty} I^5(x) \Phi(x)e(-Nx)dx \gg \varepsilon X^{5-c}.
\]

**Proof.** We denote the above integral by \( \mathcal{H} \). We have
\[
\mathcal{H} := \int_X^{2X} \cdots \int_X^{2X} \int_{-\infty}^{+\infty} e((t_1^c + \cdots + t_5^c - N)x) \Phi(x) dx dt_1 \cdots dt_5.
\]
The change of the order of integration is legitimate because of the absolute convergence of the integral. From Lemma 2.1 with \(a = 9\varepsilon/10, b = \varepsilon/10\), by using the Fourier inversion formula we obtain

\[
H = \int_X^{2X} \cdots \int_X^{2X} \phi(t_1^c + t_2^c + \cdots + t_5^c - N)dt_1 \cdots dt_5.
\]

From the property of \(\phi(y)\) we derive that

\[
H \geq \int_X^{2X} \cdots \int_X^{2X} |t_1^c + \cdots + t_5^c - N| < \varepsilon d t_1 \cdots d t_5.
\]

where \(\mu\) and \(\lambda\) are real numbers satisfying

\[
1 < 2 \cdot \left( \frac{7}{12} \right)^\frac{1}{2} < \mu < \lambda < 2 \cdot \left( \frac{5}{6} - \frac{1}{2c+2} \right)^\frac{1}{2} < 2
\]

and

\[
\mathfrak{M} = [X, 2X] \cap \left[ \left( N - \frac{4\varepsilon}{5} - t_1^c - \cdots - t_4^c \right)^{1/c}, \left( N + \frac{4\varepsilon}{5} - t_1^c - \cdots - t_4^c \right)^{1/c} \right].
\]

Therefore, from the mean–value theorem we deduce that

\[
H \gg \varepsilon \int_{\mu X}^{\lambda X} \cdots \int_{\mu X}^{\lambda X} (\xi_{t_1,t_2,t_3,t_4})^{\frac{1}{c} - 1} dt_1 \cdots dt_4,
\]

where \(\xi_{t_1,t_2,t_3,t_4} \asymp X^c\), and thus \(H \gg \varepsilon X^{5-c}\), which completes the proof.

**Lemma 2.8** Suppose that

\[
L(Q) = \sum_{i=1}^{n} A_i Q_{ai} + \sum_{j=1}^{m} B_j Q_{aj},
\]

where \(A_i, B_j, a_i\) and \(b_j\) are positive. Assume further that \(Q_1 \leq Q_2\). Then there exists some \(Q\) with \(Q \in [Q_1, Q_2]\) such that

\[
L(Q) \ll \sum_{i=1}^{n} A_i Q_{ai} + \sum_{j=1}^{m} B_j Q_{bj} + \sum_{i=1}^{n} \sum_{j=1}^{m} (A_i B_j)^{(a_i+b_j)/(a_i+b_j)},
\]

where the implied constant depends only on \(n\) and \(m\).

**Proof.** See Graham and Kolesnik [4], Lemma 2.4.

\(\square\)
Lemma 2.9 Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0,1,2$, $\beta \neq 0,1,2,3, |a_m| \ll 1, |b_l| \ll 1$. For $F \gg ML^2$, we have

$$(ML)^{-\epsilon} \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a_m b_{l\ell} \left( F \frac{n^\alpha \ell^\beta}{M^\alpha L^\beta} \right) \ll_{\alpha, \beta, \epsilon} M^7 L^{13} F^{\frac{1}{16}} + M^{\frac{5}{12}} L^{34} F^{\frac{1}{16}} + M^{\frac{34}{12}} L^{31} F^{\frac{1}{16}} + M^{\frac{53}{12}} L^{22} F^{\frac{1}{16}}

$$

Proof. We follow the process of the proof of Theorem 1 of Baker and Weingartner [1] step by step until p. 267 in [1] and get

$$(ML)^{-\epsilon} \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a_m b_{l\ell} \left( F \frac{n^\alpha \ell^\beta}{M^\alpha L^\beta} \right) \ll M^{14} L^{13} F + M^{14} L^{12} Q^{\frac{13}{12}} F + M^{\frac{53}{12}} L^{12} Q^{\frac{25}{25}} F + M^{\frac{53}{12}} L^{13} Q^{5} F + M^{16} L^{14} Q^{\frac{1}{16}}

$$

Next, according to the arguments on p. 267 of Baker and Weingartner [1], we use Lemma 2.8 to optimize $Q$ over $[1, M^{1/4}]$ and deduce that

$$(ML)^{-\epsilon} \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a_m b_{l\ell} \left( F \frac{n^\alpha \ell^\beta}{M^\alpha L^\beta} \right) \ll M^{14} L^{13} F + M^{\frac{53}{12}} L^{22} F^{\frac{1}{16}} + M^{\frac{53}{12}} L^{14} F^{\frac{1}{16}} + M^{\frac{53}{12}} L^{22} F^{\frac{1}{16}}

$$

which implies the desired result. 

Lemma 2.10 Let $3 < U < V < Z < X$ and suppose that $Z - \frac{1}{2} \in \mathbb{N}$, $X \gg Z^2 U$, $Z \gg U^2$, $V^3 \gg X$. Assume further that $f(n)$ is a complex-valued function such that $|f(n)| \leq 1$. Then the sum

$$\sum_{X < n \leq 2X} \Lambda(n)f(n)$$
can be decomposed into \( O(\log^{10} X) \) sums, each of which either of Type I:

\[
\sum_{M < m \leq 2M} a(m) \sum_{L < \ell \leq 2L} f(m\ell)
\]

with \( L \gg Z \), where \( a(m) \ll m^\varepsilon \), \( ML \asymp X \), or of Type II:

\[
\sum_{M < m \leq 2M} a(m) \sum_{L < \ell \leq 2L} b(k)f(m\ell)
\]

with \( U \ll M \ll V \), where \( a(m) \ll m^\varepsilon \), \( b(\ell) \ll \ell^\varepsilon \), \( ML \asymp X \).

**Proof.** See Lemma 3 of Heath–Brown [6].

In the rest of this paper, we always suppose that \( 2 < c < \frac{0.65576}{0.319965} \), and set \( F = |x|^c \) for \( \tau < |x| < K \). Trivially, we have \( X^{1-\eta} \ll F \ll KX^c \).

**Lemma 2.11** Suppose that \( \tau < |x| < K, F \ll ML^2, L \gg X^{\frac{51571}{191979}}, a(m) \ll m^\eta, ML \asymp X \), then we have

\[
S_I(M, L) := \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m)e(xm^\varepsilon \ell^\varepsilon) \ll X^{\frac{627107}{639930}+\eta}.
\]

**Proof.** By Lemma 2.3 with the exponential pair \((\kappa, \lambda) = AB(0, 1) = (\frac{1}{62}, \frac{57}{62})\), we deduce that

\[
S_I(M, L) \ll X^\eta \left| \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} e(xm^\varepsilon \ell^\varepsilon) \right|
\ll X^\eta \left( (|x|^cL^{-1})^{\frac{1}{62}} L^{\frac{57}{62}} + \frac{1}{|x|^cL^{-1}} \right)
\ll X^\eta \left( M(FL^{-1})^{\frac{1}{62}} L^{\frac{57}{62}} + \tau^{-1}X^{1-c} \right)
\ll MF^{\frac{1}{62}} L^{\frac{57}{62}} X^\eta \ll M(ML^2)^{\frac{1}{62}} L^{\frac{57}{62}} X^\eta
\ll M^{\frac{63}{62}} L^{\frac{29}{62}} X^\eta \ll X^{\frac{627107}{639930}+\eta},
\]

which completes the proof of Lemma 2.11.

**Lemma 2.12** Suppose that \( \tau < |x| < K, F \gg ML^2, M \ll X^{\frac{5170}{191979}}, a(m) \ll m^\eta, ML \asymp X \), then we have

\[
S_I(M, L) := \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m)e(xm^\varepsilon \ell^\varepsilon) \ll X^{\frac{627107}{639930}+\eta}.
\]

**Proof.** For \( M \ll X^{\frac{5170}{191979}} \), by Lemma 2.3 with the exponential pair \((\kappa, \lambda) = AB(0, 1) = (\frac{1}{62}, \frac{57}{62})\), we derive that

\[
S_I(M, L) \ll M^\eta \left| \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} e(xm^\varepsilon \ell^\varepsilon) \right|
\]
\[ \ll M^\eta \sum_{M < m \leq 2M} \left( \frac{(|x|X^cL^{-1})^{\frac{\tau}{2}}L^\frac{\tau}{2}}{|x|X^cL^{-1}} \right) \]
\[ \ll X^\eta \left( K^{\frac{\tau}{2}}X^\frac{\tau}{2}L^\frac{\tau}{2}M + \tau^{-1}X^{1-c} \right) \]
\[ \ll M^\tau X^{\frac{\tau}{2} + \frac{\eta}{2}} \ll X^{\frac{627107}{639930} + \eta}, \]

which completes the proof of Lemma 2.12. ■

Lemma 2.13 Suppose that \( \tau < |x| < K, F \gg ML^2, X^{\frac{51170}{44728}} \ll M \ll X^{\frac{332788}{332788}}, a(m) \ll m^\eta, ML \asymp X, \) then we have
\[ S_1(M, L) := \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m)e(xm^\ell \ell^c) \ll X^{\frac{627107}{639930} + \eta}. \]

Proof. It follows from Lemma 2.9 with \((m, \ell) = (m, \ell)\) that \( S_1(M, L) \ll X^{\frac{627107}{639930} + \eta}, \)
which derives the desired result. ■

Lemma 2.14 Suppose that \( \tau < |x| < K, X^{\frac{12823}{106655}} \ll M \ll X^{\frac{44728}{44728}}, a(m) \ll m^\eta, b(\ell) \ll \ell^\eta, ML \asymp X, \) then we have
\[ S_{II}(M, L) := \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m)b(\ell)e(xm^\ell \ell^c) \ll X^{\frac{627107}{639930} + \eta}. \]

Proof. Taking \( Q = X^{\frac{12823}{106655}}(\log X)^{-1}, \) if \( X^{\frac{12823}{106655}} \ll M \ll X^{\frac{44728}{44728}}, \) by Cauchy’s inequality and Lemma 2.2, we deduce that
\[ S_{II}(M, L) \ll \left| \sum_{L < \ell \leq 2L} b(\ell) \sum_{M < m \leq 2M} a(m)e(xm^\ell \ell^c) \right| \]
\[ \ll \left( \sum_{L < \ell \leq 2L} |b(\ell)|^2 \right)^{\frac{1}{2}} \left( \sum_{M < m \leq 2M} \left| a(m)e(xm^\ell \ell^c) \right|^2 \right)^{\frac{1}{2}} \]
\[ \ll L^{\frac{\tau}{2} + \eta} \left( \sum_{L < \ell \leq 2L} \frac{M}{Q} \sum_{0 < q < Q} \left( 1 - \frac{q}{Q} \right) \right) \]
\[ \times \sum_{M + q < m \leq 2M - q} a(m + q)a(m - q)e(x \ell^c((m + q)^c - (m - q)^c)) \]
\[ \ll L^{\frac{\tau}{2} + \eta} \left( \sum_{L < \ell \leq 2L} \frac{M}{Q} \sum_{1 < q < Q} \left( 1 - \frac{q}{Q} \right) \right) \]
\[ \times \sum_{M + q < m \leq 2M - q} a(m + q)a(m - q)e(x \ell^c((m + q)^c - (m - q)^c)) \]
\[ \ll X^\eta \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 < q < Q} \sum_{M < m \leq 2M} \left| \sum_{L < \ell \leq 2L} e(x \ell^c((m + q)^c - (m - q)^c)) \right|^2 \right)^{\frac{1}{2}}. \quad (2.7) \]
Therefore, it suffices to give the estimate of the following sum
\[ S_0 := \sum_{L < \ell \leq 2L} e\left(x\ell^c ((m + q)^c - (m - q)^c)\right). \]

From Lemma 2.3 with the exponential pair \((\kappa, \lambda) = A^3 B(0, 1) = (\frac{1}{30}, \frac{26}{30})\), we have
\[ S_0 \ll (|x|X^{c-1}q)^{\frac{1}{30}} L^{\frac{26}{30}} + \frac{1}{|x|X^{c-1}q}. \]

Inserting the above estimate into (2.7), we derive that
\[ S_{II}(M, L) \ll X^{\eta} \left(\frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q \leq Q} \sum_{M < m \leq 2M} \left((|x|X^{c-1}q)^{\frac{1}{30}} L^{\frac{26}{30}} + \frac{1}{|x|X^{c-1}q}\right)\right)^\frac{1}{2} \]
\[ \ll X^{\eta} \left(\frac{X^2}{Q} + \frac{X}{Q} \left(K^{\frac{1}{30}} X^{\frac{1}{2}} M^{\frac{26}{30}} Q^{\frac{26}{30}} + \tau^{-1} X^{1-c} M \log Q\right)\right)^\frac{1}{2} \]
\[ \ll (X^{2+\eta}Q^{-1})^{\frac{1}{2}} \ll X^{627107 / 639930 + \eta}, \]
which completes the proof of Lemma 2.14. \[ \square \]

**Lemma 2.15** Suppose that \(2 < c < \frac{665576}{319965}\), for \(\tau < |x| < K\), there holds
\[ S(x) \ll X^{627107 / 639930 + \eta}. \]

**Proof.** First, we have
\[ S(x) = \mathcal{U}(x) + O(X^{1/2}), \tag{2.8} \]
where
\[ \mathcal{U}(x) = \sum_{X < n \leq 2X} \Lambda(n) e(n^c x). \]

Taking \(U = X^{12321 / 14472}, V = X^{44728 / 55311}, Z = \left[X^{153571 / 189963}\right] + \frac{1}{2}\) in Lemma 2.10, it is easy to see that the sum
\[ \sum_{X < n \leq 2X} \Lambda(n) e(n^c x) \]

can be decomposed into \(O(\log^{10} X)\) sums, each of which either of Type I:
\[ S_I(M, L) = \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m) e(xm^c \ell^c) \]
with \(L \gg Z, a(m) \ll m^\eta, ML \sim X\), or of Type II:
\[ S_{II}(M, L) = \sum_{M < m \leq 2M} \sum_{L < \ell \leq 2L} a(m)b(\ell) e(xm^c \ell^c) \].
with \( U \ll M \ll V, a(m) \ll m^n, b(\ell) \ll \ell^n, ML \asymp X \). For the Type I sums, if \( F \ll ML^2 \), then from Lemma 2.11, we have \( S_I(M, L) \ll X^{627107/639930 + \eta} \); if \( F \gg ML^2 \) with \( M \ll X^{1/109999} \), then from Lemma 2.12, we have \( S_I(M, L) \ll X^{627107/639930 + \eta} \); if \( F \gg ML^2 \) with \( X^{1/109999} \ll M \ll X^{332788/639930} \), then from Lemma 2.13, we have \( S_I(M, L) \ll X^{627107/639930 + \eta} \).

For the Type II sums, from Lemma 2.14, we get \( S_{II}(M, L) \ll X^{627107/639930 + \eta} \). Thus, we derive that
\[
\sum_{X < n \leq 2X} \Lambda(n)e(n^\epsilon x) \ll X^{627107/639930 + \eta}. \tag{2.9}
\]

By (2.8) and (2.9), we complete the proof of Lemma 2.15.

3 Proof of Theorem 1.1

In this section, we denote by \( \phi(y) \) and \( \Phi(x) \) the functions which appear in Lemma 2.1 with parameter \( a = \frac{9}{10} \epsilon, b = \frac{1}{10} \epsilon, r = [\log X] \). Define
\[
B_5(N) = \sum_{X < p_1, p_2, p_3, p_4, p_5 \leq 2X} (\log p_1)(\log p_2) \cdots (\log p_5).
\]

By the property of \( \phi(y) \), we have \( B_5(N) \geq C_5(N) \), where
\[
C_5(N) = \sum_{X < p_1, p_2, p_3, p_4, p_5 \leq 2X} (\log p_1) \cdots (\log p_5) \phi(p_1^\epsilon + \cdots + p_5^\epsilon - N).
\]

From the Fourier transformation formula, we derive that
\[
C_5(N) = \sum_{X < p_1, \ldots, p_5 \leq 2X} (\log p_1) \cdots (\log p_5) \int_{-\infty}^{+\infty} e\left((p_1^\epsilon + \cdots + p_5^\epsilon - N)y\right) \Phi(y)dy = \int_{-\infty}^{+\infty} S_5(x)\Phi(x)e(-Nx)dx
\]
\[
= \int_{|x| \leq \tau} + \int_{\tau < |x| < K} + \int_{|x| \geq K} S_5(x)\Phi(x)e(-Nx)dx
\]
\[
= C_5^{(1)}(N) + C_5^{(2)}(N) + C_5^{(3)}(N), \text{ say.} \tag{3.1}
\]

3.1 The Estimate of \( C_5^{(1)}(N) \)

Define
\[
H(N) = \int_{-\infty}^{+\infty} I_5(x)\Phi(x)e(-Nx)dx, \quad H_{\tau}(N) = \int_{-\tau}^{+\tau} I_5(x)\Phi(x)e(-Nx)dx.
\]
From Lemma 2.1, we derive that
\[
\left| \mathcal{H}(N) - \mathcal{H}_r(N) \right| \ll \int_{-\tau}^{\infty} |I(x)|^5 \Phi(x) \, dx \ll \varepsilon \int_{-\tau}^{\infty} \left( \frac{1}{|x| X^{c-1}} \right)^5 \, dx \ll \varepsilon X^{5-c} - \eta, \tag{3.2}
\]
where we use the estimate
\[
I(x) \ll \frac{1}{|x| X^{c-1}},
\]
which follows from Lemma 4.2 in Titchmarsh \[11\]. By Lemma 2.5 and Lemma 2.6, we deduce that
\[
\left| \mathcal{C}_r(1) \right| \ll \int_{-\tau}^{+\tau} \left| S(x) - I(x) \right| \Phi(x) \, dx \ll \varepsilon \int_{-\tau}^{+\tau} \left| S(x) \right|^4 + |I(x)|^4 \, dx
\ll \varepsilon \cdot X \exp \left( - (\log X)^{1/5} \right) \left( \int_{-\tau}^{+\tau} |S(x)|^4 \, dx + \int_{-\tau}^{+\tau} |I(x)|^4 \, dx \right)
\ll \varepsilon X^{5-c} \exp \left( - (\log X)^{1/6} \right). \tag{3.3}
\]

It follows from Lemma 2.7, (3.2) and (3.3) that
\[
\mathcal{C}_r(1)(N) = \left( \mathcal{C}_r(1)(N) - \mathcal{H}_r(N) \right) + \mathcal{H}_r(N) - \mathcal{H}(N) \gg \varepsilon X^{5-c}. \tag{3.4}
\]

### 3.2 The Estimate of \( \mathcal{C}_5^{(2)}(N) \)

In order to evaluate \( \mathcal{C}_5^{(2)}(N) \), we first need to estimate the following integral
\[
\int_{-\tau}^{K} |S(x)|^4 \Phi(x) \, dx
\]
under the condition \( c > 2 \). We have
\[
\int_{-\tau}^{K} |S(x)|^4 \Phi(x) \, dx \ll \varepsilon \int_{-\tau}^{K} |S(x)|^4 \, dx
\ll \varepsilon \sum_{X < p_1, \ldots, p_4 \leq 2X} (\log p_1) \cdots (\log p_4) \int_{-\tau}^{K} e\left( (p_1^c + p_2^c - p_3^c - p_4^c) x \right) \, dx
\ll \varepsilon (\log X)^4 \sum_{X < p_1, \ldots, p_4 \leq 2X} \min \left( K, \frac{1}{|p_1^c + p_2^c - p_3^c - p_4^c|} \right)
\ll \sum_{X < n_1, \ldots, n_4 \leq 2X} \min \left( K, \frac{1}{n_1^c + n_2^c - n_3^c - n_4^c} \right). \tag{3.5}
\]

Set \( u = n_1^c + n_2^c - n_3^c - n_4^c \), then by Lemma 2.4, we know that the contribution of \( K \) to (3.5) is
\[
\ll K \cdot A(X; c, K^{-1}) \ll K \left( K^{-1} X^{4-\epsilon} + X^2 \right) X^\eta
\]
\[
\ll (X^{4-c} + KX^2)X^\eta \ll X^{2+\eta}.
\]

By a splitting argument, the contribution of \( u \) with \(|u| > K^{-1} \) to (3.5) is
\[
\ll (\log X) \max_{K^{-1} \ll U \ll X^c} \sum_{X < n_1, n_2, n_3, n_4 \ll 2X} \frac{1}{|u|}
\]
\[
\ll (\log X) \max_{K^{-1} \ll U \ll X^c} U^{-1} \cdot \mathcal{A}(X; c, 2U)
\]
\[
\ll (\log X) \max_{K^{-1} \ll U \ll X^c} U^{-1} (UX^{4-c} + X^2)X^\eta
\]
\[
\ll (X^{4-c} + KX^2)X^\eta \ll X^{2+\eta}.
\]

Combining the above two cases, we deduce that, for \( c > 2 \), there holds
\[
\int_{\tau}^{K} |S(x)|^4 |\Phi(x)|dx \ll X^{2+\eta}.
\] (3.6)

By the definition of \( \varepsilon_5^{(2)}(N) \), we obtain
\[
|\varepsilon_5^{(2)}(N)| = \left| \sum_{X < p \ll 2X} (\log p) \int_{\tau < |x| < K} e(p^c x)S^4(x)\Phi(x)e(-Nx)dx \right|
\]
\[
\ll \sum_{X < p \ll 2X} (\log p) \left| \int_{\tau < |x| < K} e(p^c x)S^4(x)\Phi(x)e(-Nx)dx \right|
\]
\[
\ll (\log X) \sum_{X < n \ll 2X} \left| \int_{\tau < |x| < K} e(n^c x)S^4(x)\Phi(x)e(-Nx)dx \right|.
\]

From Cauchy’s inequality, we derive that
\[
|\varepsilon_5^{(2)}(N)| \ll X^{\frac{1}{2}}(\log X) \left( \sum_{X < n \ll 2X} \left| \int_{\tau < |x| < K} e(n^c x)S^4(x)\Phi(x)e(-Nx)dx \right|^2 \right)^{\frac{1}{2}}
\]
\[
= X^{\frac{1}{2}}(\log X) \left( \sum_{X < n \ll 2X} \int_{\tau < |x| < K} e(n^c x)S^4(x)\Phi(x)e(-Nx)dx \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_{\tau < |y| < K} \frac{e(n^c y)}{S^4(y)\Phi(y)e(-Ny)}dy \right)^{\frac{1}{2}}
\]
\[
= X^{\frac{1}{2}}(\log X) \left( \int_{\tau < |y| < K} \frac{S^4(y)\Phi(y)e(-Ny)}{S^4(y)\Phi(y)e(-Ny)\Xi(x-y)}dx \right)^{\frac{1}{2}}
\]
\[
\ll X^{\frac{1}{2}}(\log X) \left( \int_{\tau < |y| < K} |S^4(y)\Phi(y)|dy \int_{\tau < |x| < K} |S^4(x)\Phi(x)\Xi(x-y)|dx \right)^{\frac{1}{2}}.
\] (3.7)

For the inner integral in (3.7), we get
\[
\int_{\tau < |x| < K} |S^4(x)\Phi(x)\Xi(x-y)|dx
\]
\[
\ll \int_{\tau < |x| < K} \frac{\tau}{|x-y| \leq X^{-c}} |S^4(x)\Phi(x)\Sigma(x-y)| dx + \int_{\tau < |x| < K} \frac{\tau}{X^{-c} < |x-y| < 2K} |S^4(x)\Phi(x)\Sigma(x-y)| dx. \tag{3.8}
\]

By the trivial estimate, we have \(\Sigma(x-y) \ll X\), which combines with Lemma 2.1 and Lemma 2.15 to obtain
\[
\int_{\tau < |x| < K} \frac{\tau}{|x-y| \leq X^{-c}} |S^4(x)\Phi(x)\Sigma(x-y)| dx \ll \varepsilon X \cdot \sup_{\tau < |x| < K} |S(x)|^4 \times \int_{\tau < |x| < K} \frac{\tau}{|x-y| \leq X^{-c}} dx \ll \varepsilon X \cdot X^{\frac{1254214}{119966} - c} \ll \varepsilon X^{\frac{1574179}{119966} - c}. \tag{3.9}
\]

From Lemma 2.3, for \(|x| > X^{-c}\), we have
\[
\Sigma(x) \ll \left(|x|^{-1}X^c\right)^{\kappa} X^\lambda + \frac{1}{|x|X^{-c-1}} \ll |x|^\kappa X^{\kappa c + \lambda - 1} + \frac{1}{|x|X^{-c-1}}. \tag{3.10}
\]

Taking
\[
(\kappa, \lambda) = ABA^2BABA^2BA^2BABABA^2BAB(0, 1) = \begin{pmatrix} 19369 & 105283 \\ 150298 & 150298 \end{pmatrix}
\]
in (3.10), we derive that, for \(|x| > X^{-c}\), there holds
\[
\Sigma(x) \ll |x|^{19369} X^{105283} X^{-c} + \frac{1}{|x|X^{-c-1}}. \tag{3.11}
\]

From (3.6), (3.11) and Lemma 2.15, we derive that
\[
\ll \int_{\tau < |x| < K} \frac{\tau}{X^{-c} < |x-y| < 2K} |S^4(x)\Phi(x)\Sigma(x-y)| dx \ll \int_{\tau < |x| < K} \frac{\tau}{X^{-c} < |x-y| < 2K} |S^4(x)\Phi(x)| \left(|x-y|^{19369} X^{105283} X^{-c} + \frac{1}{|x-y|X^{-c-1}} \right) dx
\]
\[
\ll X^{\frac{19369}{150298} + \frac{193255}{119966} + \eta} \int_{\tau < |x| < K} |S^4(x)\Phi(x)| dx + \varepsilon X^{1-c} \cdot \sup_{\tau < |x| < K} |S(x)|^4 \times \int_{\tau < |x| < K} \frac{\tau}{X^{-c} < |x-y| < 2K} \frac{dx}{|x-y|} \ll X^{\frac{1574179}{119966} - c + \eta}. \tag{3.12}
\]

From (3.8), (3.9) and (3.12), we deduce that
\[
\int_{\tau < |x| < K} |S^4(x)\Phi(x)\Sigma(x-y)| dx \ll \varepsilon X^{\frac{1574179}{119966} - c + \eta},
\]
from which and (3.6), we can conclude that
\[
|\vartheta^{(2)}_0(N)| \ll X^{\frac{1}{4}}(\log X) \left(X^{2+\eta} \cdot \varepsilon X^{\frac{1574179}{119966} - c + \eta} \right)^{\frac{1}{2}} \ll \varepsilon X^{5-c-\eta}. \tag{3.13}
\]
3.3 The Estimate of \( \mathcal{G}_5^{(3)}(N) \)

By Lemma 2.1, we have

\[
|\mathcal{G}_5^{(3)}(N)| \ll \int_K |S(x)|^5 |\Phi(x)| \, dx \ll X^5 \int_K \frac{1}{\pi|x|} \left( \frac{r}{2\pi|x|b} \right)^r \, dx \\
\ll X^5 \left( \frac{r}{2\pi b} \right)^r \int_K \frac{dx}{x^{r+1}} \ll X^5 \frac{r}{r} \left( \frac{r}{2\pi K b} \right)^r \\
\ll X^5 \frac{1}{\log X} \left( \frac{1}{2\pi \log^5 X} \right) \log X \ll X^5 \frac{1}{X^{5 \log \log X + \log(2\pi)(\log X)}} \ll 1. \tag{3.14}
\]

3.4 Proof of Theorem 1.1

From (3.1), (3.4), (3.13) and (3.14), we deduce that

\[
\mathcal{G}_5(N) = \mathcal{G}_5^{(1)}(N) + \mathcal{G}_5^{(2)}(N) + \mathcal{G}_5^{(3)}(N) \gg \varepsilon X^{5-c},
\]

and thus

\[
\mathcal{B}_5(N) \gg \mathcal{G}_5(N) \gg \varepsilon X^{5-c} \gg \frac{X^{5-c}}{\log^4 X},
\]

which finishes the proof of Theorem 1.1.

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