The rotation problem

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October 24, 2017, 00:25

Abstract. On a large scale, inertial frames seem not to rotate relative to the average matter distribution in the universe. Without absolute space or finely tuned initial conditions, it is difficult to explain the lack of relative rotation. Two classical (non-quantum) arguments have been proposed to explain why: (1) that gravitational fields (including inertial fields) be completely determined by the matter distribution, with no independent degrees of freedom for the gravitational field (AKA Mach’s principle). (2) inflation. Although either of these is a possible explanation, a more likely explanation comes from considering reasonable forms of quantum gravity. A semi-classical approximation to quantum gravity shows that phase interference would cancel out cosmologies with significant relative rotation. A generic general estimate for a perfect fluid cosmology with a realistic variation of average vorticity with cosmological scale factor shows that only cosmologies with an average present relative rotation smaller than about \(L^*H^2 \approx 10^{-71}\) radians per year could contribute significantly to a measurement of relative rotation rate, where \(L^*\) is the Planck length and \(H\) is the present value of the Hubble parameter.

PACS numbers: 98.80.Qc, 04.60.-m, 98.80.Jk

Keywords: rotation problem, quantum gravity, Mach principle, cosmology
To be Submitted to: Class. Quantum Grav.

1. Introduction

Although there are solutions of Einstein’s field equations that allow relative rotation of matter and inertial frames, it has long been known that in our universe inertial frames seem not to rotate with respect to the visible stars. The “Rotation Problem” is to explain “If the universe can rotate, why does it rotate so slowly?” [1]. The rotation problem can easily be seen by comparing the rotation of the plane of a Foucault pendulum with the movement of the stars relative to the Earth. More accurate estimates of this effect are
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now available, mostly because of the availability of isotropy measurements on the cosmic microwave background radiation.

For example, Hawking [2] showed that if the universe contains a large-scale homogeneous vorticity, then the rotation rate corresponding to that vorticity cannot be larger than somewhere between $7 \times 10^{-17}$ rad yr$^{-1}$ and $10^{-14}$ rad yr$^{-1}$ if the universe is closed and about $2 \times 10^{-46}/(\text{present density in g cm}^{-3})$ if it is open. Many studies have been done since then resulting in progressively decreasing estimates of the allowed rotation rate, e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In General Relativity, gravitation (including inertia) (as expressed by the metric tensor) is determined not only by the distribution of matter (in terms of the stress-energy tensor), but also by initial and boundary conditions. There are many solutions of Einstein’s field equations for General Relativity that have large-scale rotation of matter and inertial frames, e.g. [17, 18, 19, 20, 21]. It is difficult to explain the absence of relative rotation classically without absolute space (as proposed by Newton) or without assuming very finely tuned initial conditions for the universe.

References [1] and [22] suggest that inflation might lead to very small shear and rotation rates for our present universe because shear and rotation rate decrease as the universe expands. Although that is a possible explanation, it does not seem likely that inflation could completely explain the rotation problem. Even if inflation were sufficient to explain the rotation problem, it would not be necessary because a semi-classical approximation to quantum gravity is sufficient to explain the observed lack of rotation, as will be shown.

Ernst Mach [23, 24, 25, 26] suggested that inertia might be determined by distant matter. Various versions of that proposal have come to be known as Mach’s principle. Since we now know (from General Relativity) that inertia is a gravitational force, such an implementation of Mach’s principle would require that the gravitational field (or at least part of it) be determined only by its sources (matter) rather than having independent degrees of freedom (in terms of initial and boundary conditions).

If the many proposals to implement exactly that for General Relativity, e.g. [27, 28, 29, 30, 31, 32] were correct, then gravitation would behave very differently from the electromagnetic interaction, in that electric and magnetic fields are determined not only from sources (charges and currents), but also from initial and boundary conditions.

A more likely explanation for the apparent lack of relative rotation of the average inertial frame and the average matter distribution comes from quantum gravity. Although we do not have a final theory of quantum gravity, and therefore, no universally accepted theory of quantum cosmology, we have some speculations for a theory of quantum gravity, e.g. [33, 34, 35], and we have a rough idea for what the action should be. If the action dominates such a calculation (as we expect), it might be possible to estimate the upper bounds on the rotation of the universe in terms of a calculation using a semi-classical approximation to quantum cosmology, based on a sum-over-histories
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approach \cite{36,37,38,39,40}. We would expect the result of such a calculation to show that the only 4-geometries contributing significantly to such a calculation would have very small relative rotation rates. In that case, any measurement of relative rotation rate would yield a very small value.

Section 2 gives an overview. Section 3 makes a saddlepoint approximation to the path integral. Section 4 discusses the validity of the saddlepoint approximation. Section 5 discusses possible problems.

Appendix A sets up a calculation of the amplitude for measuring a rotation of the universe in terms of a path-integral calculation. Appendix B calculates the action for a perfect fluid. Appendix C gives a generalized Friedmann equation that includes vorticity and shear. Appendix D discusses the radiation, matter, and dark-energy eras. Appendix E considers the effect of vorticity and shear on the action, and uses a realistic variation of vorticity with cosmological scale factor to show that only very small rotation rates contribute significantly to a path integral.

We take the speed of light $c$ and Newton’s gravitational constant $G$ to be 1 throughout.

2. Overview

We start by giving an overview of the calculation. The details are in the appendices. Using a semi-classical representation for quantum gravity allows an estimate of the amplitude for measuring the relative rotation between the average matter distribution and the average inertial frame. The calculation uses some general formulas for the action integral. It is seen that the final result is insensitive to details of the Lagrangian used. The main effect depends mostly on the form for a generalized Friedmann equation that includes vorticity.

In generalizing Feynman’s path integral method to quantum cosmology, it is usual to estimate the amplitude for a given 3-geometry to evolve into another given 3-geometry. In this case, the “path” is a 4-geometry that connects the two given 3-geometries. The contribution of each 4-geometry to the amplitude is weighted by the action for that 4-geometry, which is proportional to the phase for that contribution. In general, the paths (4-geometries) are not restricted to classical 4-geometries (that is, not restricted to solutions to Einstein’s field equations).

However, in many situations, classical 4-geometries give the main contributions to calculations of the amplitude. In particular, in the present situation, where we want to explain why we observe nearly zero relative rotation between the average inertial frame and the average matter distribution, it makes sense to restrict the calculation to 4-geometries that are solutions to Einstein’s field equations.

In addition, for the present situation, it makes sense to further restrict the 4-geometries we consider to those that look essentially like our present universe (roughly a spatially flat Robertson-Walker universe), except with vorticity. Further, there are \cite{41,42,43,44,45,46,47,48,49,50,51,52} also called decoherent histories or consistent histories.
estimates for how we would expect vorticity to vary with time if there were any vorticity. Vorticity might not be homogeneous, however. It might vary from one place to another. However, we could consider the spatially averaged vorticity as a parameter. More specifically, we could consider the present value of the spatially averaged vorticity as a parameter. Because we have a good idea for how vorticity would vary with time, we would have a good idea for how the spatially averaged vorticity would vary with time. Then, the present value of the spatially averaged vorticity would make a good parameter to specify a 4-geometry with vorticity. Call that parameter $\omega_3$. So now, when we perform a path integral, we are performing an integral over $\omega_3$.

Strictly speaking, that integration over $\omega_3$ should go from $-\infty$ to $+\infty$. However, extending the integration to infinite limits could introduce technical difficulties. So, instead, of infinite limits, we choose one rotation of the universe in a Planck time. That is close enough to $\infty$ for practical calculations. In addition, we expect the main contribution to the path integral to come from nearly zero vorticity.

As mentioned above, we do not have a theory of quantum gravity, and therefore no theory of quantum cosmology. However, we have some reasonable estimates for what the action should be, and if the action dominates the calculation (as we expect), then we should be able to make reasonable estimates for a path integral calculation.

The formula for the action involves an integral of the Lagrangian over 4-dimensional spacetime plus an integral over the three-dimensional hypersurface that bounds the 4-dimensional spacetime. The surface term is necessary to insure consistency if the action integral is broken into parts. Here, we shall assume that the surface term will not be such as to change significantly the main calculation. That is, we include the surface term, but neglect it’s effect on the vorticity calculation. As we shall see, the main result is so overwhelming that it is unlikely the surface term could change the result.

Strictly speaking, the integral of the Lagrangian over the 4-dimensional spacetime should involve an infinite integration over space since we believe our universe to be open. However, let us consider the significance of that choice. We are trying to explain why inertial frames seem not to rotate relative to the average matter distribution (except for possible local frame dragging). When we make the path integral calculation, we are considering physical processes in which the distribution of matter in the universe affects inertial frames here. If we were to integrate over all matter in the universe we would be including matter outside of our past light cone (that is, matter that should have no significant effect on inertial frames here). A correct quantum gravity calculation would presumably show negligible effect from matter outside of our past light cone. However, for now, it seems sufficient to simply restrict the volume integration of the Lagrangian for the action calculation to the past light cone. With the above considerations, we can write the effective action as

$$I = \int V(t) L \, dt + \text{surface term},$$

where $L$ is the Lagrangian, $t$ is cosmological time, $V(t)$ is the spatial volume, and all
quantities are considered to be spatial averages.

It is useful to change the integration variable from time $t$ to cosmological scale factor $a$, where $a$ is related to $t$ by the generalized Friedmann equation (to include vorticity $\omega$):

$$\frac{1}{a} \frac{d a}{d t} = \sqrt{H(a)^2 + H_\omega^2} = \sqrt{H(a)^2 + f_1(a)\omega_3^2}, \quad (2)$$

where $H(a)$ is the Hubble parameter without vorticity, and $H_\omega^2 = f_1(a)\omega_3^2$ is the vorticity term, $f_1(a)$ is defined in (E.3), and $\omega_3$ is the present value of the vorticity. Changing the integration variable in (1) gives

$$I = \int \frac{V(a)L(a)}{a\sqrt{H(a)^2 + f_1(a)\omega_3^2}} d a + \text{surface term}, \quad (3)$$

where

$$V(a) = \frac{4}{3} \pi a^3 r_3^2 \quad (4)$$

is the spatial volume, and $r_3$ is the present radius of the visible universe.

As a first approximation, we split the integral in (3) into two parts, depending on which of the two terms in the radical is larger, and we then assume the larger term is dominant. The error introduced by this approximation will be discussed later. This gives

$$I \approx I_0 - \frac{\omega_3^2}{2} \int_{H(a)^2/f_1(a) > \omega_3^2} \frac{V(a)L(a)f_1(a)}{a|H|^3} d a + \int_{H(a)^2/f_1(a) < \omega_3^2} \frac{V(a)L(a)}{a} \left( \frac{1}{\omega_3 \sqrt{f_1(a)}} - \frac{1}{H(a)} \right) d a, \quad (5)$$

where $I_0$ is the action without vorticity, but including the surface term. Notice that the approximation in (5) is the crucial approximation in the calculation of the action, as will be seen in the following.

It is convenient to write (5) as

$$I \approx I_0 + \hbar \left( \frac{\omega_3^2}{\omega_m} \right)^2 f_2(\omega_3) + \hbar A f_3(\omega_3), \quad (6)$$

where

$$\omega_m = \left( \frac{\hbar H_3}{r_3^2} \right)^{1/2} = \frac{L^*}{r_3} \sqrt{\frac{H_3}{r_3}} \approx L^* H_3^2 \approx 10^{-89} \text{cm}^{-1} \approx 10^{-71} \text{rad yr}^{-1}, \quad (7)$$

$$A = \frac{r_3^2 H_3}{6 \hbar} = \frac{r_3^3 H_3^2}{6(L^*)^2} \approx \frac{1}{6(L^* H_3)^2} \approx 10^{121}, \quad (8)$$

$H_3$ is the present value of the Hubble parameter, $r_3$ is the present radius of the universe (which we approximate by the inverse of the Hubble parameter), $L^*$ is the Planck length,

$$f_2(\omega_3) \equiv -\frac{H_3}{2r_3^2} \int_{H(a)^2/f_1(a) > \omega_3^2} \frac{V(a)L(a)f_1(a)}{a|H|^3} d a, \quad (9)$$
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\[ f_3(\omega_3) \equiv -\frac{6}{H_3 \epsilon_3^2} \int H(a)^2/f_1(a) < \omega_3^2 \int V(a)L(a)\left(\frac{1}{\omega_3^3 f_1(a)} - \frac{1}{H(a)}\right) d\omega_3. \] (10)

The important point here is that the value of (7) is so small and that \( f_2(\omega_3) \) is a quantity of order unity (within a factor of ten or so).

In calculating the values of \( f_2(\omega_3) \), a detailed calculation in section 3 shows that the value of \( f_2(\omega_3) \) is of order unity in any regime for which it has a significant effect on the integral. Section 4 shows that because the factor \( A \) in (11) is so large, there is no significant contribution to the integral from \( f_3(\omega_3) \) for the stationary-phase path as long as the integral stops at \( \omega_{\text{max}} = 1/L^* \) rather than continuing to infinity.

The path integral to give the amplitude for a 3-geometry \((3)G_f\) with matter fields \( \phi_f \) is then

\[ \psi_f((3)G_f, \phi_f) \approx \int \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} \psi_i((3)G_i, \phi_i) \exp(iI_0/\hbar) \exp[i f_2(\omega_3)(\omega_3/\omega_m)^2] \exp[i Af_3(\omega_3)] d\omega_3 D(3)G_i D\phi_i, \] (11)

where the path integral is the integral over \( \omega_3 \), and we also integrate over the initial 3-geometry \((3)G_i\) with matter fields \( \phi_i \). We assume that the initial wave function is broad enough that the integral is dominated by the exponential factor.

### 3. Saddlepoint approximation

The saddlepoint at \( \omega_3 = 0 \) in (11) is isolated from the branch points, as pointed out in section 4. The integral in (11) can be approximated by a saddlepoint integration to give

\[ \psi_f((3)G_f, \phi_f) \approx \int \psi_i((3)G_i, \phi_i)|_{\omega_3=0} \psi[(4)G, \phi; \omega_3]|_{\omega_3=0} \frac{\omega_m \sqrt{\pi}}{\sqrt{f_2(0)}} e^{i\pi/4} D(3)G_i D\phi_i \]

for \( |\omega_3| < L^* H_3^2 \approx 10^{-71} \text{ rad/year}. \) (12)

\[ \psi_f((3)G_f, \phi_f) \approx 0 \text{ otherwise, where now} \]

\[ f_2(0) \approx \frac{\alpha}{3a_2} + \frac{1}{2a_2} + \frac{1}{3a_2} - \frac{1}{3} \approx \frac{4}{9} \alpha + \frac{7}{9}, \] (13)

where \( a_2 \) is given by (D.3), and is the value of the cosmological scale factor when dark energy begins to dominate over matter, and we have recognized that except for the \( \omega_3 \) dependence in (11), the integrand is approximately the 4-geometry without shear or vorticity. Although there should be an additional slowly varying part in the integrand, it can be neglected for the purposes here.

The calculation of \( f_2(0) \) in (13) is done by using different approximations in the integrand in (9) in each cosmological era. The error introduced by that approximation should not be more than a factor of two or three, but even a factor of 10 error would not change the main result.

\[ \text{§ We can take } \alpha = 0 \text{ [55] or } \alpha = 1 \text{ [56]. In either case, } f_2 \text{ is a number of order unity.} \]
\( \omega_3 \) represents the relative rotation rate at the present time. The main significance of (7) and (11) is that only relative rotations less than \( \omega_m \approx L^*H_3^2 \approx 10^{-71} \text{rad per year} \) contribute significantly to the integration in (11). Thus, any experiment that can measure relative rotation will see an effective 4-geometry that is consistent with a relative rotation rate smaller than (7), and therefore will measure a relative rotation rate that is less than \( \omega_m \approx L^*H_3^2 \approx 10^{-71} \text{rad per year} \).

Equation (11) requires that \( \psi_f(\mathcal{G}_f, \phi_f) \) be sharply peaked at the 3-geometry that is determined by the 4-geometry \( \psi(\mathcal{G}, \phi; \omega_3) \bigg|_{\omega_3=0} \).

Although the calculation here should be sufficient to convince us that any reasonable theory of quantum gravity would explain why we seem to observe no rotation of our inertial frame relative to the average matter distribution, it would be useful if we had an actual cosmological solution that had relative rotation that reduced to the Robertson-Walker solution continuously as the relative rotation approached zero. Unfortunately, however, I have not been able to find such a solution. For example, all of the solutions given by [19] have zero vorticity.

4. Validity of the saddlepoint approximation

There are several conditions that must apply (in addition to those already discussed) for the saddlepoint approximation to (11) to be valid. Basically, the functions \( f_2(\omega_3) \) and \( f_3(\omega_3) \) must be such that there are no significant contributions to the integral other than at the saddlepoint at \( \omega_3 = 0 \).

There are six considerations: First, the value of \( f_2(\omega_3) \) should be of order unity. Second, the value of \( f_3(\omega_3) \) should be such that it makes no significant contributions to the integral in any regime for which it has a significant value. Third, the saddlepoint must be isolated from the branch points (which are also the places where we switch from one approximation to another). Fourth, the approximations used in the integrations that introduce non-analytic points must be taken care of. Fifth, Contributions from endpoints in the integration may require special handling. Sixth is the semi-classical approximation to quantum gravity.

4.1. Behavior of \( f_2(\omega_3) \)

In calculating the values of \( f_2(\omega_3) \), a detailed calculation shows that the value of \( f_2(\omega_3) \) is of order unity in any regime for which it has a significant effect on the integral.
4.2. Behavior of $f_3(\omega_3)$

Because the factor $A$ in (11) is so large, there is no significant contribution to the integral from $f_3(\omega_3)$ for the stationary-phase path as long as the integral stops at $\omega_{\text{max}} = 1/L^*$ rather than continuing to infinity.

A detailed calculation shows that $f_3(\omega_3)$ varies from about \( [1 - a^2_3 - (3/5)(1 - a^2_5)]/\pi \) to about \( (1 - a^3_3)/\pi \) as $\omega_3$ varies from $\omega_3 = \sqrt{\Lambda/2}$ to $\omega_{\text{max}} = 1/L^*$ for the realistic model of vorticity in Appendix E.

4.3. The branch points in the Friedmann equation must be isolated from any saddlepoints.

Appendix E shows that the branch points in the Friedmann equation are isolated from the saddlepoints.

4.4. Analyticity requirements

The integrations involved in calculating the action, that will be necessary to evaluate the integrand in (11) usually cannot be done in closed form, requiring approximations, that then must be justified because of the analyticity requirements of saddlepoint approximations.

The analyticity requirements are satisfied because the non-analytic points are isolated from any saddlepoints.

4.5. Contributions from endpoints in a path integral must be considered, especially if the endpoints are infinite.

By choosing the endpoints for the $\omega_3$ integration to be finite (only one rotation of the universe in a Planck time), we were able to avoid significant contributions from the endpoints in the integration. Although the contributions from leaving the endpoints infinite would still not be significant, it might be more difficult to demonstrate it.

4.6. Validity of a semi-classical approximation to quantum gravity

Although formulating a correct implementation of a semi-classical approximation to quantum gravity is difficult without a correct formulation of quantum gravity, it is difficult to imagine a formulation that would give results that differ significantly from those obtained here.

5. Discussion

There are several questions to consider in this calculation of the rotation of the universe:
5.1. How general is the calculation?

It would be useful to find closed-form models that have relative rotation of matter and inertial frames and that approach the Robertson-Walker metric continuously as the relative rotation approaches zero.

The contribution of vorticity to the calculation in (11) enters not through the Lagrangian, but through a generalized Friedmann equation. This has the advantage that the main results are mainly independent of the Lagrangian (within limits).

Although these calculations do not consider possible effects of the surface term in the action, nor parameters other than vorticity in the path integral, this is appropriate because the purpose of the present calculation is to show why measurements of the relative rotation of the universe are so small.

5.2. That the dominance of various terms in the Friedmann equation depends on the cosmological era adds complications.

Because the dominance of various terms depends on cosmological era, some integrals need to be separated into contributions from each era with different approximations used in each era. The approximations are good enough to give the value of the integral valid within a factor of two or three. However, even an error of a factor of 10 or so would be good enough, considering the main result. The non-analyticity introduced at the boundaries of the eras does not cause a problem because the non-analytic points are significantly isolated from any saddlepoints.

5.3. Infinite action for an open universe?

Formally, $V(a)$ should be infinite for an open universe to give an infinite action [57]. However it is not physically realistic to consider contributions to the action outside of the past light cone. A correct theory of quantum gravity may make that clear.

However, if we actually did take the action to be infinite, then the parameter $f_2$ in (9) would also be infinite, and no longer of order unity. In that case, the parameter $\omega_m$ in (7) would no longer be the limiting value for relative rotation of inertial frames and matter. The allowed relative rotation would be even more sharply peaked than the calculation given here. The relative rotation rate would be exactly zero, with zero width. That is, if the action were really infinite, the explanation for why we do not observe a relative rotation of the average inertial frame and the average matter distribution would be even stronger.

5.4. Euclidean versus Lorentzian path integral

Does it matter if the path integral is specified with a Euclidean signature versus a Lorentzian signature?
Revising the Lorentzian path integral in (A.7) by a Euclidean path integral \[58\] would give
\[
\psi[(4)G, \phi; \omega_3] \approx e^{-I/\bar{h}},
\] (14)
where the saddlepoint would still be at \(\omega_3 = 0\), the steepest descent path would be along the real \(\omega_3\) axis, and it would be necessary to take the opposite root for \(H_\omega\) in (E.12) to keep the steepest-descent path along the complete real \(\omega_3\) axis. As pointed out by references \[59\] and \[60\], the value of the path integral is independent of the regime, Lorentzian or Euclidean, that one is using to calculate it.

5.5. Dependence on the choice of Lagrangian
How sensitive do the results depend on the choice of Lagrangian?

Using a standard form for the Lagrangian for General Relativity with two choices \[55, 56\] for the matter Lagrangian shows no significant difference on the main result for how only a narrow range of possible values of relative rotation near zero contribute significantly to a measurement of global relative rotation. It is difficult to imagine a reasonable Lagrangian that would give a significantly different result.

The contribution of vorticity to the calculation in (11) enters not through the Lagrangian, but through a generalized Friedmann equation. This has the advantage that the main results are mainly independent of the Lagrangian (within limits).

Acknowledgments
I thank David Peterson, David Bartlett, and Andrew Hamilton for useful discussion.

Appendix A. Amplitude for measuring a rotation of the universe

The amplitude for measuring a particular value for some quantity is equal to the amplitude for measuring that value given a particular 4-geometry times the amplitude for that 4-geometry, and then we sum over all 4-geometries.

For example, following \[58\], the amplitude for the 3-geometry and matter field to be fixed at specified values on two spacelike hypersurfaces is
\[
\langle (3) G_f, \phi_f | (3) G_i, \phi_i \rangle = \int \psi[(4) G, \phi] D(4) G D\phi,
\] (A.1)
where the integral is over all 4-geometries and field configurations that match the given values on the two spacelike hypersurfaces, and
\[
\psi[(4) G, \phi] \equiv \exp (iI[(4) G, \phi]/\bar{h})
\] (A.2)
is the contribution of the 4-geometry \((4) G\) and matter field \(\phi\) on that 4-geometry to the path integral, where \(I[(4) G, \phi]\) is the action. The proper time between the two hypersurfaces is not specified. A correct theory of quantum gravity would be necessary to specify the measures \(D(4) G\) and \(D\phi\), but that will not be necessary for the purposes
here. Hartle and Hawking [58] restricted the integration in (A.1) to compact (closed) 4-geometries, but (A.1) can be applied to open 4-geometries if that is done carefully.

Equation (A.1) is a path integral. In this case, the “path” is the sequence of 3-geometries that form the 4-geometry $(4)G$. Thus, each 4-geometry is one “path.” The space in which these paths exist is often referred to as superspace, e.g. [61]. As pointed out by Hajicek [59], there are two kinds of path integrals: those in which the time is specified at the endpoints, and those in which the time is not specified. The path integral in (A.1) is the latter. References [59] and [60] consider refinements to the path integral in (A.1), but such refinements are not necessary here.

Because of diffeomorphisms, a given 4-geometry can be specified by different metrics that are connected by coordinate transformations. This makes it difficult to avoid duplications when making path integral calculations. We avoid that difficulty here by considering only simple models.

Let $\psi_i^{(3)G_i, \phi_i}$ be the amplitude that the 3-geometry was $(3)G_i$ on some initial space-like hypersurface and that the matter fields on that 3-geometry were $\phi_i$. Let $\psi_f^{(3)G_f, \phi_f}$ be the amplitude that the 3-geometry is $(3)G_f$ on some final space-like hypersurface and that the matter fields on that 3-geometry are $\phi_f$. Then, we have

$$
\psi_f^{(3)G_f, \phi_f} = \int \psi_i^{(3)G_i, \phi_i} D^{(3)G_i} D\phi_i \psi_i^{(3)G_i, \phi_i}.
$$

(A.3)

The condition that there are not finely tuned initial conditions is equivalent to $\psi_i^{(3)G_i, \phi_i}$ being a broad wave function.

Substituting (A.1) into (A.3) gives

$$
\psi_f^{(3)G_f, \phi_f} = \int \psi^{(4)G, \phi} D^{(4)G} D\phi \psi_i^{(3)G_i, \phi_i} D^{(3)G_i} D\phi_i.
$$

(A.4)

Although in (A.4), the integration is over all possible 4-geometries, not just classical 4-geometries, the main contribution to the integral (in most cases) comes from classical 4-geometries, e.g. [62, 60]. Thus, we shall now restrict (A.4) to be an integration over classical 4-geometries. This is appropriate for our purposes, in any case, since we are trying to explain why we do not measure relative rotation of matter and inertial frames in what appears to be a classical universe.

In principle, the idea is very simple. Any measurement to determine the inertial frame will give a result that depends on the 4-geometry. If several 4-geometries contribute significantly to an amplitude, such as in (A.4), then any measurement to determine an inertial frame might give the inertial frame corresponding to any one of those 4-geometries. However, the probability for the result being a particular inertial frame will depend on the contribution of the corresponding 4-geometry to calculations such as that in (A.4).

In this calculation, we consider 4-geometries that have an average vorticity (rotation rate) that depends only on time (i.e., on the cosmological scale factor). We take these 4-geometries to be characterized by a parameter $\omega_3$ that is proportional to the spatially averaged relative rotation rate, which we take to be the average rotation rate
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(or vorticity) at the present time. Thus, we can rewrite (A.4) for our purposes as

\[ \psi_f(\mathcal{G}_f, \phi_f) = \int \int_{-\infty}^{\infty} \psi_i(\mathcal{G}_i, \phi_i) \psi^{(4)}(\mathcal{G}, \phi; \omega_3) d\omega_3 D^{(3)}G_i D\phi_i . \]  

The integral in (A.5) is still a path integral. In this case, each value of \( \omega_3 \) specifies one “path”, in that it specifies one 4-geometry, and that specifies one sequence of 3-geometries. The space of “paths” in this case is often referred to as a mini-superspace because it is restricted to a much smaller space of 4-geometries. The parameter \( \omega_3 \), classically determined by initial conditions on the 4-geometry, represents an independent degree-of-freedom of the gravitational field.

Actually, taking the \( \omega_3 \) integration from \(-\infty \) to \( \infty \) in (A.5) is not physically realistic, and might lead to problems if the infinite endpoints contribute significantly to the integration. The largest relative rotation that could possibly be considered without having a theory of quantum gravity would be one rotation of the universe in a Planck time. This would correspond to taking the maximum value of \( \omega_3 \) to be the reciprocal of the Planck time, \( T^* \), or \( \omega_{\text{max}} \approx 10^{44} \text{sec}^{-1} \). Thus, we can rewrite (A.5) as

\[ \psi_f(\mathcal{G}_f, \phi_f) \approx \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} \psi_i(\mathcal{G}_i, \phi_i) \psi^{(4)}(\mathcal{G}, \phi; \omega_3) d\omega_3 D^{(3)}G_i D\phi_i . \]  

We anticipate that the properties of \( \psi^{(4)}(\mathcal{G}, \phi; \omega_3) \) will dominate the integral in (A.6), so we shall start with

\[ \psi^{(4)}(\mathcal{G}, \phi; \omega_3) \approx e^{iI(\omega_3)/\hbar} , \]  

where \( I(\omega_3) \) is the action.

Either a stationary-phase path or a steepest-descent path could be used when making the saddlepoint approximation \([63, 64, 65]\), but here, we use a stationary-phase path. Halliwell \([66]\) gives an example of a more detailed path-integral calculation of quantum gravity.

Appendix B. Action for a perfect fluid

We can take the action in (A.7) to be

\[ I = \int (-g^{(4)})^{1/2} L d^4x + \frac{1}{8\pi} \int (g^{(3)})^{1/2} K d^3x , \]  

where

\[ L = L_{\text{geom}} + L_{\text{matter}} \]  

is the Lagrangian, and the surface term is necessary to insure consistency if the action integral is broken into parts \([53, 54]\). The quantity

\[ K = g^{(3)ij} K_{ij} = -\frac{1}{2} g^{(3)ij} \frac{\partial g^{(3)}_{ij}}{\partial t} \]  

is the trace of the extrinsic curvature, where \( g^{(3)}_{ij} \) is the 3-metric. In this example, we take the Lagrangian for the geometry as

\[ L_{\text{geom}} = \frac{R^{(4)} - 2\Lambda}{16\pi} , \]  

where \( R^{(4)} \) is the 4-curvature and \( \Lambda \) is the cosmological constant.
where \( R^{(4)} \) is the four-dimensional scalar curvature and \( \Lambda \) is the cosmological constant.

For a perfect fluid, the energy momentum tensor is

\[
T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + p g^{\mu\nu}, \tag{B.5}
\]

where \( p \) is the pressure, \( \rho \) is the density, and \( u \) is the 4-velocity. For solutions to Einstein’s field equations for a perfect fluid, \((\ref{eq:geom})\) becomes

\[
L_{\text{geom}} = \frac{1}{2} \rho - \frac{3}{2} p + \frac{\Lambda}{8\pi}, \tag{B.6}
\]

and we can take the Lagrangian for the matter as

\[
L_{\text{matter}} = \rho + \alpha (p - \rho), \tag{B.7}
\]

where \( \alpha \) is a constant, and we can take \( \alpha = 0 \) \cite{55}, \( \alpha = 1 \) \cite{67, 56}, or \( \alpha = \frac{3}{2} \) (from combining \((\ref{eq:geom})\) with \cite{61} eq. 21.33a). However, as we shall see, the final result is insensitive to the exact form of the Lagrangian. Substituting \((\ref{eq:geom})\) and \((\ref{eq:matter})\) into \((\ref{eq:action})\) gives

\[
L = (\alpha - \frac{3}{2})(p - \rho) + \frac{\Lambda}{8\pi}, \tag{B.8}
\]

For some cosmological models, it is possible to represent \( K \) as a time-integral of some quantity. If so, then the effect of \( K \) on the action could be represented as an integral over a 4-volume, allowing us to combine the two terms in \((\ref{eq:action})\). This would give

\[
I = \int (-g^{(4)})^{1/2} \tilde{L} d^4x, \tag{B.9}
\]

where \( \tilde{L} \) can be considered to be an effective Lagrangian. Sometimes we can assume the effective Lagrangian to take the form

\[
\tilde{L} = \alpha_2 (p - \rho) + \alpha_3 \Lambda, \tag{B.10}
\]

where \( \alpha_2 \) and \( \alpha_3 \) are constants of order unity.

We know that the universe is not homogeneous. However, we usually approximate the universe as a spatially homogeneous fluid. This procedure is commonly referred to as course-graining. If we now consider a spatially averaged homogeneous universe, \((\ref{eq:action})\) can be written as

\[
I = \int V(t) \tilde{L} dt, \tag{B.11}
\]

where \( V(t) \) is the spatial volume, and all quantities are now considered to be spatial averages. The question whether \( V \) should be infinite because this is an open cosmology \cite{57} is considered in section \ref{sec:open}.
Appendix C. Generalized Friedmann equation

Derivation of a generalization of the Friedmann equation that includes relative rotation of matter and inertial frames (specifically, shear and vorticity), starts with the Raychaudhuri equation \[ [68, 69], [70, \text{eq. (1.3.4)}], [71, \text{eq. (36)}], [4, 5, \text{eq. (4.12)}], [72, 17, 18, 73, 74, 75] \], or the Raychaudhuri-Ehlers equation \[ [76, \text{eq. 6.4}] \]. Specifically, following \[ [76, \text{eq. 6.5 to eq. 6.12}] \] gives

\[
\left( \frac{1}{a} \frac{da}{dt} \right)^2 = H^2 + H^2_\omega + H^2_\sigma + H^2_a
\]  

(C.1)

for the generalized Friedmann equation, where \( a \) is the cosmological scale factor,

\[ H \equiv \sqrt{\frac{\Lambda}{3} + \frac{8\pi \rho}{3} - \frac{k}{a^2}} \]  

(C.2)

is the Hubble parameter without vorticity, shear, or acceleration,

\[ H^2_\omega \equiv \frac{4}{3a^2} \int a \omega^2 \, da \]  

(C.3)

is the vorticity term, \( \omega \) is vorticity,

\[ H^2_\sigma \equiv -\frac{4}{3a^2} \int a \sigma^2 \, da \]  

(C.4)

is the shear term, \( \sigma \) is shear, and

\[ H^2_a \equiv -\frac{2}{3a^2} \int a \dot{u}^a_{,a} \, da \]  

(C.5)

is the acceleration term.

Strictly speaking, (C.1) applies only to a homogeneous universe. However, it is usual to represent our universe on the large scale as a spatially homogeneous fluid as a reasonable approximation. We do not expect vorticity to be the same everywhere. However, it seems reasonable that only the average vorticity will be important in the calculations we are making. Therefore, we shall assume that (C.1) applies to averaged quantities, and from here on, we shall assume that vorticity is a spatially averaged vorticity.

Appendix D. Radiation, matter, and dark-energy eras

There are three cosmological eras to consider. In the early universe, radiation dominates over matter to determine the density \( \rho \) in the radical in (C.2). When the cosmological scale factor \( a \) reaches a certain size (which we define as \( a_1 \)), matter begins to dominate over radiation to determine the density \( \rho \). When the cosmological scale factor \( a \) gets even larger (to a size we define as \( a_2 \)), the density of matter has fallen low enough that the cosmological constant \( \Lambda \) begins to dominate over the density term in (C.2).

For an equation of state, we take \( p = w \rho \), where \( w = 1/3 \) in the radiation-dominated era, and \( w = 0 \) in the matter-dominated era. The variation of density \( \rho \) with cosmological scale factor \( a \) is given by \[ [76, \text{Table 6.1}] \]

\[ \rho = \rho_1 (a/a_1)^{-3(1+w)} \]  

(D.1)
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where \( \rho_1 \) is the value of \( \rho \) at the boundary between the radiation era and the matter era where \( a = a_1 \).

We can take for the present fraction of radiation density \( \Omega_{\text{rad}} = 5.4 \times 10^{-5} \) [77, p. 78]. Otherwise, we take [78]

\[
a_1 = \frac{\Omega_{\text{rad}}}{\Omega_{\text{mat}}} \approx \frac{5.4 \times 10^{-5}}{0.3089} \approx 1.7 \times 10^{-4} ,
\]

and

\[
a_2 = \left( \frac{\Omega_{\text{mat}}}{\Omega_{\Lambda}} \right)^{1/3} \approx \left( \frac{0.3089}{0.6911} \right)^{1/3} \approx 0.76 ,
\]

where \( \Omega_{\text{mat}} \) is the present fraction of matter density (including dark matter), and \( \Omega_{\Lambda} \) is the present fraction of dark energy. From the Hubble parameter \( H_3 = 67.74 \text{ km s}^{-1} \text{ Mpc}^{-1} \) [78], we can calculate the critical density, and combining with \( \Omega_{\Lambda} \), we get the value of the density of matter and dark energy when they were equal, which is

\[
\rho_2 = \frac{\Lambda}{8\pi} \approx 4.4 \times 10^{-58} \text{ cm}^{-2} ,
\]

which gives

\[
\Lambda \approx 1.1 \times 10^{-56} \text{ cm}^{-2} .
\]

We also have

\[
\hbar \rightarrow \frac{\hbar G}{c^2} = L^2 \approx 2.616 \times 10^{-66} \text{ cm}^2 .
\]

Appendix E. Effect of vorticity and shear

The main effect of vorticity and shear on the action is through the generalized Friedmann equation (C.1). We know that both vorticity and shear decrease with time as the universe expands, [e.g., 19].

We can take vorticity to depend on the cosmological scale factor \( a \) as

\[
\omega \propto a^{-m} ,
\]

and we can take \( m = 1 \) in the radiation era and \( m = 2 \) in the matter era [76, Table 6.1]. If we put the time-variation of vorticity into (C.3), then we get

\[
H^2 \omega = f_1(a) \omega_3^2 ,
\]

where \( \omega_3 \) is the present value of spatially averaged vorticity,

\[
f_1(a) = \frac{2}{3a_1^4} \left( \frac{2a_1^2}{a^2} \ln \frac{a_1}{a} - 1 \right) \text{ for } a \leq a_1
\]

\[
= \frac{2}{3a_1^4} \text{ for } a \geq a_1 .
\]

Similarly, we can assume that shear depends on the cosmological scale factor as

\[
\sigma \propto a^{-n} ,
\]

where \( n \) might have different values in the radiation era and the matter era.
We return to (B.11), and change the $t$ integration to an integration over the cosmological scale factor $a$. We consider only the effect of vorticity and shear to give
\[ I = \int \int \frac{V(a) \bar{L} \, da}{a \sqrt{H^2 + H^2 + H^2}}. \tag{E.5} \]
Substituting (E.5) into (A.7) gives
\[ \psi^{(4)} G, \phi; \omega_3 \approx \exp \left( \int \int \frac{iV(a) \bar{L} \, da}{a \sqrt{H^2 + H^2 + H^2}} \right), \tag{E.6} \]
Substituting (E.6) into (A.6) gives
\[ \psi_f^{(3)} G_f, \phi_f = \int \int_{\omega_{\text{max}}}^{\omega_{\text{max}}} \psi_i^{(3)} G_i, \phi_i \left[ \exp \left( \int \int \frac{iV(a) \bar{L} \, da}{a \sqrt{H^2 + H^2 + H^2}} \right) \right] \, d\omega_3 \, D^{(3)} G_i \, D\phi_i. \tag{E.7} \]
We suspect that the main effect of the rotation (vorticity) term $H^2$ on the calculation of the $\omega_3$ integral in (E.7) is at the saddlepoint where $\omega_3 = 0$.

We assume that the vorticity term and the shear term are both small near the saddlepoint. As an approximation, we expand the radical in the denominator of (E.7) by assuming that $|H^2|$ is small whenever $|H^2| < H^2$, or by assuming that $H^2$ is small whenever $|H^2| > H^2$. For now, we ignore the shear term $H^2$. This gives
\[ \psi_f^{(3)} G_f, \phi_f \approx \int \int_{\omega_{\text{max}}}^{\omega_{\text{max}}} \psi_i^{(3)} G_i, \phi_i \exp \left( \frac{iI_0}{\hbar} \right) \exp \left[ i f_2(\omega_3)(\omega_3/\omega_m)^2 \right] \exp \left[ iAf_3(\omega_3) \right] \, d\omega_3 \, D^{(3)} G_i \, D\phi_i \tag{E.8} \]
for the main contribution of vorticity to (E.7), where $I_0$ is the action for zero vorticity and zero shear,
\[ A = \frac{r_3^3 H_3}{6\hbar} = \frac{r_3^3 H_3}{6(L^*)^2} \approx \frac{1}{6(L^* H_3)^2} \approx 10^{121}, \tag{E.9} \]
\[ \omega_m = \left( \frac{\hbar H_3}{r_3^3} \right)^{1/2} = \frac{L^*}{r_3} \sqrt{\frac{H_3}{r_3}} \approx L^* H_3^2 \approx 10^{-89} \text{cm}^{-1} \approx 10^{-71} \text{rad yr}^{-1}. \tag{E.10} \]

$H_3$ is the present value of the Hubble parameter, $r_3$ is the present radius of the universe (which we approximate by the inverse of the Hubble parameter), and $L^*$ is the Planck length. The quantity
\[ f_2(\omega_3) = -\frac{H_3}{2r_3^3} \int_{r_0/r_3}^{a_{11}(\omega_3)} \frac{V(a) \bar{L} H_3^2}{\omega_3^2 H^3 a} \, da - \frac{H_3}{2r_3^3} \int_{a_{12}(\omega_3)}^{1} \frac{V(a) \bar{L} H_3^2}{\omega_3^2 H^3 a} \, da, \tag{E.11} \]
is a slowly varying function of $\omega_3$ of order unity, and $r_0/r_3$ is a value of the cosmological scale factor that is much smaller than one, but large enough that a semi-classical approximation is valid.
\[ f_3(\omega_3) = -\frac{6}{H_3 r_3^3} \int_{a_{11}(\omega_3)}^{a_{12}(\omega_3)} \left[ \frac{1}{H_3} - \frac{1}{H} \right] \frac{V(a) \bar{L}}{a} \, da, \tag{E.12} \]

\[ \text{As pointed out by [76, Section 6.2.2], a shear-free solution to Einstein’s field equations does not generally exist when there is vorticity, but the shear term is probably small whenever the vorticity term is small.} \]
where $a_{t1}(\omega_3)$ and $a_{t2}(\omega_3)$ are the values of $a$ for which the magnitude of the rotation term $|H_2^2|$ in the radical in (E.7) equals $H^2$. Because $H_2^2$ is negative in this case, there are branch points on the real $a$ axis at $a = a_{t1}(\omega_3)$ and $a = a_{t2}(\omega_3)$. However, for the calculations here, we can neglect the branch-point contributions.

When $\omega_3 < \sqrt{a_1 a_3^3 \Lambda/2}$, there are no branch points because $|H_2^2|$ is always smaller than $H^2$. Because of that, the saddlepoint at $\omega_3 = 0$ is isolated from the branch points. However, for the calculations here, we can take $a_{t1}(\omega_3) = a_{t2}(\omega_3)$ for the purposes of (E.11) and (E.12).

Also, for the purposes of (E.11) and (E.12), we take the maximum value of $a_{t2}(\omega_3)$ to be 1, and we take the minimum value of $a_{t1}(\omega_3)$ to be $r_0/r_3$.

In the radiation era (where $a_{t1} < a_1$), $a_{t1}(\omega_3)$ is determined by the equation

$$\omega_3^2 \left[ \ln \frac{a_{t1}}{a_1} - 1/2 \left( \frac{a_{t1}}{a_1} \right)^2 \right] = \frac{\Lambda a_3^3}{4 a_{t1}^2}.$$  \hspace{1cm} (E.13)

In the matter era (where $a_1 < a_{t2} < a_2$), $a_{t2}(\omega_3)$ is given by

$$a_{t2}(\omega_3) = \frac{2}{a_3^2} \frac{\omega_3^2}{\Lambda}.$$  \hspace{1cm} (E.14)

In the dark-energy era (where $a_2 < a_{t2} < 1$), $a_{t2}(\omega_3)$ is given by

$$a_{t2}(\omega_3) = \left[ \frac{2\omega_3^2}{\Lambda} \right]^{1/4}.$$  \hspace{1cm} (E.15)

The $\omega_3$ integration in (E.8) is dominated by a 4-geometry that has a relative rotation rate less than that given by (E.10), and therefore any measurement of relative rotation will yield a value no bigger than that given by (E.10) with large probability.

References

[1] John Ellis and Keith A. Olive. “Inflation can solve the rotation problem”. Nature, 303:679–681, 1983.
[2] Stephen W. Hawking. “On the rotation of the universe”. Mon. Not. R. Astron. Soc., 142:129–141, 1969.
[3] A. M. Wolfe. “New limits on the shear and rotation of the the universe from the x-ray background”. The Astrophysical Journal, 159:L61–L66, 1970.
[4] G. F. R. Ellis. “Relativistic cosmology”. In R. K. Sachs, editor, General relativity and cosmology, pages 104–182. Academic Press, New York, 1971.
[5] G. F. R. Ellis. “Republication of: Relativistic cosmology”. Gen. Relativ. Gravit., 41:581–660, 2009.
[6] C. B. Collins and S. W. Hawking. “Why is the universe isotropic?”. The Astrophysical Journal, 180:317–334, 1973.
[7] C. B. Collins and S. W. Hawking. “The rotation and distortion of the universe”. Mon. Not. R. Astron. Soc., 162:307–320, 1973.
[8] A. J. Fennelly. “Effects of a rotation of the universe on the number counts of radio sources: Gödel’s universe”. The Astrophysical Journal, 207:693–699, 1976.
[9] S. S. Bayin and F. I. Cooperstock. “Rotational perturbations of Friedmann universes”. Phys. Rev. D, 22:2317–2322, 1980.
[10] D. J. Raine and E. G. Thomas. “Mach’s principle and the microwave background”. Astrophysical Letters, 23:37–45, 1982.
The rotation problem

[11] John D. Barrow, R. Juszkiewicz, and D. H. Sonoda. “Universal rotation: how large can it be?”.
Mon. Not. R. Astron. Soc., 213:917–943, 1985.
[12] G. F. R. Ellis and J. Wainwright. “Cosmological observations”. In J. Wainwright and G. F. R.
Ellis, editors, Dynamical Systems in Cosmology, pages 65–83. The University Press, Cambridge,
1997.
[13] T. R. Jaffe, A. J. Banday, H. K. Eriksen, K. M. Górski, and F. K. Hansen. “Evidence of vorticity
and shear at large angular scales in the WMAP data: A violation of cosmological isotropy?”.
Astrophysical Journal Letters, 629:L1–L4, 2005.
[14] T. R. Jaffe, A. J. Banday, H. K. Eriksen, K. M. Górski, and F. K. Hansen. “Fast and efficient
template fitting of deterministic anisotropic cosmological models applied to WMAP data”.
Astrophysical Journal, 643:616–629, 2006.
[15] G. F. R. Ellis. “The Bianchi models: Then and now”. Gen. Relativ. Gravit., 38:1003–1015, 2006.
[16] S.-C. Su and M.-C. Chu. “Is the universe rotating?”. The Astrophysical Journal, 703:354–361,
2009.
[17] Dietrich Kramer, Hans Stephani, Malcolm MacCallum, and Eduard Herlt. Exact Solutions of
Einstein’s Field Equations. VEB Deutscher Verlag der Wissenschaften, Berlin, 1980.
[18] Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, and Eduard
Herlt. Exact Solutions of Einstein’s Field Equations. Cambridge University Press, Cambridge,
England, 2nd edition, 2003.
[19] George F. R. Ellis and Malcolm A. H. MacCallum. “A class of homogeneous cosmological models”.
Comm. Math. Phys., 12:108–141, 1969.
[20] V. A. Korotky and Y. N. Obukhov. On Cosmic Rotation. In P. Pronin and G. Sardanashvily,
editors, Gravity, Particles and Space-time, pages 421–439, 1996.
[21] L. M. Chechin. “On the modern status of the universe rotation problem”. Journal of Modern
Physics, 4(8A):126–132, 2013.
[22] A. Braccesi. “Inflation and Mach’s principle”. Astronomy and Astrophysics, 194:1–2, 1988.
[23] Ernst Mach. Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit. J.G.
Calvelsche K. u. K. Universitäts Buchhandlung, Prag, 1872.
[24] Ernst Mach. History and Root of the Principle of the Conservation of Energy. Open Court
Publishing Company, Chicago, 1911. English translation by Philip B. Jourdain.
[25] Ernst Mach. Die Mechanik in ihrer Entwicklung. F. A. Brockhaus, Leipzig, 9th edition, 1933.
[26] Ernst Mach. The Science of Mechanics. Open Court Publishing Company, LaSalle, Illinois, 6th
Edition with revisions through the 9th German edition, 1960. English translation by Thomas
J. McCormack.
[27] Dennis W. Sciama. “On the origin of inertia”. Mon. Not. R. Astron. Soc., 113:34–42, 1953.
[28] Dennis W. Sciama, P. C. Waylen, and Robert C. Gilman. “Generally covariant integral formulation
of Einstein’s field equations”. Phys. Rev., 187:1762–1766, 1969.
[29] Robert C. Gilman. “Machian theory of inertia and gravitation.”. Phys. Rev. D, 2:1400–1410,
1970.
[30] Derek J. Raine. “Mach’s principle in General Relativity”. Mon. Not. R. Astron. Soc., 171:507–528,
1975.
[31] D. J. Raine. “Mach’s principle and space-time structure”. Rep. Prog. Phys., 44:1151–1195, 1981.
[32] Derek J. Raine. “The integral formulation of Mach’s principle”. In Julian Barbour and Herbert
Pfister, editors, Einstein Studies, vol. 6: Mach’s Principle: From Newton’s Bucket to Quantum
Gravity, pages 274–292, Boston, 1995. Birkhäuser Boston, Inc.
[33] Bryce S. DeWitt. Quantum theory of gravity. I. the canonical theory. Phys. Rev., 160:1113–1148,
Aug 1967.
[34] Bryce S. DeWitt. Quantum theory of gravity. II. the manifestly covariant theory. Phys. Rev.,
162:1195–1239, Oct 1967.
[35] Bryce S. DeWitt. Quantum theory of gravity. III. applications of the covariant theory. Phys.
Rev., 162:1239–1256, Oct 1967.
The rotation problem

[36] Jonathon J. Halliwell and Miguel E. Ortiz. “Sum-over-histories origin of the composition laws of relativistic quantum mechanics and quantum cosmology”. *Phys. Rev. D*, 48:748–768, 1993.

[37] Murray Gell-Mann and James Hartle. “Classical equations for quantum systems”. *Phys. Rev. D*, 47:3345–3382, 1993.

[38] Murray Gell-Mann and James Hartle. “Quantum mechanics in the light of quantum cosmology”. In Wojciech H. Zurek, editor, *Complexity, Entropy, and the Physics of Information, Volume VIII of Santa Fe Institute Studies in the Sciences of Complexity*, Addison-Wesley, Readwood City, California, 1990.

[39] James Hartle. “The spacetime approach to quantum mechanics”. UCSBTH-92-12 <http://arXiv.org/abs/gr-qc/9210004>, 26 pages, 1992.

[40] James Hartle. “Quasiclassical realms in a quantum universe”. <http://arXiv.org/abs/gr-qc/9404017>, 8 pages, 1994.

[41] Jonathon J. Halliwell. Quantum cosmology: An introductory review. Preprint NSF-ITP-88-131, Institute for Theoretical Physics, University of California, Santa Barbara, 1988. 22 pp.

[42] Jonathon J. Halliwell. “Information dissipation in quantum cosmology and the emergence of classical spacetime”. In Wojciech H. Zurek, editor, *Complexity, Entropy, and the Physics of Information, Volume VIII of Santa Fe Institute Studies in the Sciences of Complexity*, Addison-Wesley, Readwood City, California, 1990.

[43] Jonathon J. Halliwell. “Decoherent histories and the emergent classicality of local densities”. *Phys. Rev. Lett.*, 83:2428–2485, 1999.

[44] Jonathon J. Halliwell. “Some recent developments in the decoherent histories approach to quantum theory”. Imperial/TP/2-03/9 <http://arXiv.org/abs/quant-ph/0301117v1>, 26 pages, 2003.

[45] Jonathon J. Halliwell. “Decoherence of histories and hydrodynamic equations for a linear oscillator chain”. *Phys. Rev. D*, 68:025018, 2003.

[46] Jonathon J. Halliwell. “Decoherent histories analysis of models without time”. *Brazilian Journal of Physics*, 35:300–306, 2005.

[47] Jonathon J. Halliwell. “How the quantum world became classical”. *Contemporary Physics*, 46:93–104, 2005.

[48] Jonathon J. Halliwell. “Glafla 2004: The decoherent histories approach to the quantization of cosmological models”. *International Journal of Theoretical Physics*, 45:1471–1485, 2006.

[49] Robert B. Griffiths. “Consistent histories and the interpretation of quantum mechanics”. *Journal of Statistical Physics*, 36:219–272, 1984.

[50] Robert B. Griffiths. *Consistent Quantum Theory*. University Press, Cambridge, 2002.

[51] Roland Omnès. “Consistent interpretations of quantum mechanics”. *Rev. Mod. Phys.*, 64:339–382, 1992.

[52] Roland Omnès. *Understanding Quantum Mechanics*. Princeton University Press, Princeton, 1999.

[53] James W. York. “Role of conformal three-geometry in the dynamics of gravitation”. *Phys. Rev. Lett.*, 28:1082–1085, 1972.

[54] Stephen W. Hawking. “The path integral approach to quantum gravity”. In Stephen W. Hawking and Werner Israel, editors, *General Relativity, an Einstein Centenary Survey*, pages 746–789. The University Press, Cambridge, 1979.

[55] Bernard F. Schutz and Rafael Sorkin. “Variational aspects of relativistic fluid theories, with application to perfect fluids”. *Annals of Physics*, 107:1–43, 1977.

[56] Bernard F. Schutz, Jr. “Perfect fluids in General Relativity: velocity potentials and a variational principle”. *Phys. Rev. D*, 2:2762–2773, 1976.

[57] James Hartle. “The action is infinite for an open cosmology”. private communication at the conference, “Spacetime in action, 100 years of relativity,” 31 March 2005, Pavia, Italy, 2005.

[58] James Hartle and Stephen W. Hawking. “Wave function of the Universe”. *Phys. Rev. D*, 28:2960–2975, 1983.

[59] P. Hajicek. “Elementary properties of a new kind of path integral”. *J. Math. Phys.*, 27:1800–1805, 1986.
The rotation problem

[60] Claus Kiefer. “On the meaning of path integrals in quantum cosmology”. *Annals of Physics*, 207:53–70, 1991.

[61] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation*. W. H. Freeman and Company, San Francisco, 1973.

[62] Jonathon J. Halliwell and James B. Hartle. “Integration contours for the no-boundary wave function of the universe”. *Phys. Rev. D*, 41:1815–1834, 1990.

[63] K. G. Budden. *Radio waves in the ionosphere*. University Press, Cambridge, 1961.

[64] Leopold B. Felsen and Nathan Marcuvitz. *Radiation and Scattering of Waves*. Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

[65] Jonathon J. Halliwell and Jorma Louko. “Steepest-descent contours in the path-integral approach to quantum cosmology I. The de Sitter minisuperspace model”. *Phys. Rev. D*, 39:2206–2215, 1989.

[66] Jonathon J. Halliwell. “Derivation of the Wheeler-DeWitt equation from a path integral for minisuperspace models”. *Phys. Rev. D*, 38:2468–2481, 1988.

[67] M. A. H. MacCallum and A. H. Taub. “Variational principles and spatially-homogeneous universes, including rotation”. *Commun. Math. Phys.*, 25:173–189, 1972.

[68] Amalkumar Raychaudhuri. “Relativistic cosmology I”. *Phys. Rev.*, 98:1123–1126, 1955.

[69] Amalkumar Raychaudhuri. “Relativistic and Newtonian cosmology”. *Zeitschrift für Astrophysik*, 43:161–164, 1957.

[70] Jürgen Ehlers. “Beiträge zur Relativistischen Mechanik Kontinuierlicher Medien”. *Akademie der Wissenschaften und Literatur (Mainz), Abhandlungen der Mathematisch-Naturwissenschaftlichen Klasse [Proceedings of the Mathematical-Natural Science Section of the Mainz Academy of Science and Literature]*, (11):791–837, 1961.

[71] Jürgen Ehlers. “Contributions to the Relativistic Mechanics of Continuous Media”. *Gen. Relativ. Gravit.*, 25(12):1225–1266, 1993. Notice that the sign of the second term on the right of the = in equations (15) and (16) should be plus instead of minus. See the corresponding equations (1.1.15) and (1.1.16) in the original 1961 publication.

[72] Morimasa Kubo. “Perfect fluids expanding with both vorticity and shear”. *Publ. Astron. Soc. Japan*, 30:327–336, 1978.

[73] G. F. R. Ellis, S. T. C. Siklos, and J. Wainwright. “Geometry of cosmological models”. In J. Wainwright and G. F. R. Ellis, editors, *Dynamical Systems in Cosmology*, pages 11–50. The University Press, Cambridge, 1997.

[74] Włodzimierz Godłowski, Marek Szydłowski, Piotr Flin, and Monika Biernacka. “Rotation of the universe and the angular momenta of celestial bodies”. *Gen. Relativ. Gravit.*, 35:907–913, 2003.

[75] Włodzimierz Godłowski and Marek Szydłowski. “Dark energy and global rotation of the universe”. *Gen. Relativ. Gravit.*, 35:2171–2187, 2003.

[76] George F. R. Ellis, Roy Maartens, and Malcolm A. H. MacCallum, editors. *Relativistic Cosmology*. Cambridge University Press, Cambridge, England, 2012.

[77] Andrew Liddle. *An Introduction to Modern Cosmology*. Wiley, Chichester, England, 3rd. edition, 2015.

[78] Planck_Collaboration. “Planck 2015 results. XIII. Cosmological parameters”. <http://arXiv.org/abs/1502.01589v2>[astro-ph.CO], February 2015.