Towards non-iterative closest point: Exact recovery of pose for rigid 2D/3D registration using semidefinite programming

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Abstract—We describe a convex programming framework for pose estimation in 2D/3D point-set registration with unknown point correspondences. We give two mixed-integer nonlinear program (MINP) formulations of the 2D/3D registration problem when there are multiple 2D images, and propose convex relaxations for both of the MINPs to semidefinite programs (SDP) that can be solved efficiently by interior point methods. Furthermore, these convex programs can readily incorporate feature descriptors of points to enhance registration results. We prove that the convex programs exactly recover the solution to the original nonconvex 2D/3D registration problem under noiseless condition. We apply these formulations to the registration of 3D models of coronary vessels to their 2D projections obtained from multiple intra-operative fluoroscopic images. For this application, we experimentally corroborate the exact recovery property in the absence of noise and further demonstrate robustness of the convex programs in the presence of noise.

Index Terms—Rigid registration, 2D/3D registration, iterative-closest-point, convex relaxation, semidefinite programming.

I. INTRODUCTION

RIGID registration of two point sets is a classical problem in computer vision and medical imaging of finding a transformation that aligns these two sets. Typically, a point-set registration problem consists of two intertwined subproblems, pose estimation and point correspondence, where solving one is often the pre-condition for solving the other. A canonical formulation of the rigid point-set registration problem for two point clouds is the following. Let \( X, Y \in \mathbb{R}^{d \times m} \) be two point sets in dimension \( d \), we want to solve:

\[
\min_{P \in \Pi^m_{d \times m}, R \in SO(d)} \|RX - YP\|^2_F, \tag{1}
\]

where \( \Pi^m_{d \times m} \) is the set of \( m \times m \) permutation matrices, and \( R \in SO(d) \) is the rotation matrix from the special orthogonal group in \( d \) dimension. Finding a solution to this problem is difficult as it is a nonconvex, integer nonlinear problem. Another close relative of this problem, namely the 2D/3D point-set registration, assumes a 3D point-set and 2D projections of the 3D point-set. The objective is to find out the pose of the 3D model that gives rise to the 2D degenerate point-set upon projections. This adds an additional complication to that of the regular point-set registration problem, namely, the loss of depth information. We propose global optimizers for 2D/3D point-set registration problems with guarantees using our semidefinite programs conreg1 and conreg2, and demonstrate their usefulness in the application of coronary vessel imaging.

From a broader perspective, 2D/3D point-set registration problem arises in numerous medical imaging applications in fields such as neurology \([1]\), orthopaedics \([2]\), and cardiology \([3]\). The associated body of literature on 2D/3D registration is expanding rapidly, as is apparent in the thorough review of techniques recently published by Markelj et al. \([4]\). Amongst the vast literature on the 2D/3D registration problem, we briefly discuss the techniques most closely related to the one presented in this paper, namely, methods for registration of point-sets.

As mentioned earlier, a typical point-set registration problem consists of two mutually interlocked subproblems, pose estimation and point correspondence. The key idea in seminal iterative Closest Point (ICP) \([5]\) algorithm is to alternatively solve the two subproblems starting from an initial estimate of pose. For the correspondence subproblem, it uses closeness in terms of Euclidean distance between two points to determine whether they correspond to each other. There are many variants proposed to enhance the original ICP, either to make fast correspondence \([6]\) or to include more features to obtain high quality correspondence \([7]\). Variants for 2D/3D registration include \([8]–[11]\). Though popular, these methods suffer for common drawbacks, namely, they are all local methods and do not guarantee global convergence. Their performances all rely on a good initialization and the spatial configuration (distribution) of 3D points. In 2D/3D point-set registration, for this type of local optimizers several strategies have been proposed to increase the capture range by the use of multi-resolution pyramids \([12]\), use of stochastic global optimization methods such as differential evolution \([9]\) or direct search wherein the parameter space is systematically and deterministically searched by dividing it into smaller and smaller hyperrectangles \([9]\). In these cases, except in direct search, the guarantees on finding the correct global minima are very weak. In the case of direct search one requires the parameter space to be finite.

Another line of work \([13]–[15]\) in point-set registration focus on using soft or inexact correspondences to enhance the search for global optimizers. In a recent work \([16]\), the Oriented Gaussian Mixture Model (OGMM) method is proposed...
to extend the successful Gaussian mixture based nonrigid registration algorithm (gmmreg) [15] to the 2D/3D case. These methods model the point configuration as Gaussian mixtures, and they intend to find a transformation that maximizes the overlap between these distributions. The structure of Gaussian distribution encapsulates the idea of soft correspondence and enables fast implementation. Empirically, as more fuzziness is allowed for point correspondences (larger variance for the Gaussians), the target function of optimization is smoother and hence it is less likely to find a local optimum. Nevertheless, a good initialization is still crucial for these types of algorithms due to the nonconvex nature of the cost.

We take a different approach to jointly and globally solve the pose and correspondence problem in 2D/3D point-set registration using convex programming. Unlike local methods, it is possible to find a global optimum solution, regardless of initialization, due to convexity. Our effort to find global optimizers to problems of the same type as [1], is similar in spirit to the work by Li and Hartley [17]. However, the key difference in the two methods is in the approach adopted of solving the joint pose and correspondence problem. While, they solve their mixed integer problem by the method of branch and bound, we propose a convex relaxation to a semi-definite program that can be solved efficiently by techniques such as interior point methods.

Our algorithm currently only supports rigid registration. However, our algorithm can be of help in places where classic algorithms such as ICP and Gaussian mixture based algorithms have difficulties, as we aim to provide a global solution. This could be used as high quality initialization for rigid or non-rigid local methods that need a starting point in the neighborhood of the true transformation.

A. Our contributions

The main contributions of this paper can be summarized as the following:

Algorithm: The original 2D/3D point-set registration of a 3D model and multiple 2D images is formulated as nonconvex MINPs that are difficult to solve. We propose two convex relaxations for these MINPs. The programs jointly optimize the correspondences and transformation, and the convex nature of these programs enable efficient search of global optimum regardless of initialization using standard off-the-shelf conic programming software. Furthermore, one of the convex programs conreg2 gives solution to a variant of 2D/3D registration problem where we simultaneously estimate the point correspondences between multiple 2D images while respecting the epipolar constraint. This could be utilized in finding the corresponding image points for 3D reconstruction purposes. Another natural extension of these programs is the incorporation of local descriptors of points as additional terms in the objective of both the programs. For our clinical application, we use tangency of the vessel in the local neighborhood of the point to illustrate the use of point descriptors.

Exact recovery analysis: We prove exactness of the convex programs, that is, under certain conditions the proposed relaxed convex programs will give a solution to the original MINPs. We prove that under noiseless situation the relaxed convex programs are in fact able to exactly recover the rotation and permutation matrices that match the projected points to the 3D model. Our simulations show that algorithms’ global convergence results also hold with noise when the error in recovery grows nearly proportionally to the added noise. Real-data examples corroborate the theoretical results, and suggest potential applications in coronary tree matching.

Here we outline the organization of this paper. In Section II we summarize the notation used in the paper. In Section III we present the mixed integer programming formulation of the 2D/3D registration problem when there are multiple images. In Section IV we present the convexly relaxed versions of the 2D/3D registration problem in terms of tractable semidefinite programs. In Section V we prove that achieving global optimality is possible under certain situations. In Section VI-A we mention how to incorporate additional features to the convex programs to enhance registration results, in particular in the application of coronary vessel imaging. Lastly, in Section VI-B we empirically verify the exact recovery property, and demonstrate the robustness of the algorithm on simulated and real medical datasets for the registration of 2D coronary angiogram with the 3D model of the coronaries.

II. Notation

We use upper case letters such as $A$ to denote matrices, and lower case letters such as $t$ for vectors. We use $I_d$ to denote the identity matrix of size $d \times d$. We denote the diagonal of a matrix $A$ by diag($A$). We use $A^n$ for integer $n \geq 1$ to denote the multiplication of $A$ with itself $n$ times. We will frequently use block matrices built from smaller matrices, in particular when we deal with problem (REG2). For some block matrix $A$, we will use $A_{ij}$ to denote its $(i,j)$-th block, and $A(p,q)$ to denote its $(p,q)$-th entry. We also use $A_i$ to denote the $i$-th column of $A$. When we have an index set $s = \{s_1, \ldots, s_n\}$ where each $s_i$ is an integer, we use $A_s$ to denote the matrix $[A_{s_1}, \ldots, A_{s_n}]$. We use $A \succeq 0$ to mean that $A$ is positive semidefinite. We use $\|x\|_2$ to denote the Euclidean norm of $x \in \mathbb{R}^n$. We denote the trace of a square matrix $A$ by Tr($A$). We use the following matrix norms. The Frobenius, mixed $\ell_2/\ell_1$ and entry-wise $\ell_1$ norms are defined as:

$$\|A\|_F = \text{Tr}(A^T A)^{1/2}, \quad \|A\|_{2,1} = \sum_i \|A_i\|_2, \quad \text{and}$$

$$\|A\|_1 = \sum_i \sum_j |A(i,j)|.$$  \(2\)

The Kronecker product between matrices $A$ and $B$ is denoted by $A \otimes B$. The all-ones vector is denoted by $1$. We use $|s|$ to denote the number of elements in a set $s$. For a set $s$ we use conv($s$) to denote the convex hull of $s$.

We introduce the following sets,

$$\Pi^{a \times b}_d \equiv \{ A \in \{0,1\}^{a \times b} : \sum_{i=1}^a A(i,j) = 1, \sum_{j=1}^b A(i,j) = 1 \},$$

$$\Pi^{a \times b}_t \equiv \{ A \in \{0,1\}^{a \times b} : \sum_{i=1}^a A(i,j) = 1, \sum_{j=1}^b A(i,j) \leq 1 \},$$
\[ \Pi^{a \times b} \equiv \{ A \in \{0,1\}^{a \times b} : \sum_{i=1}^{a} A(i, j) \leq 1, \sum_{j=1}^{b} A(i, j) \leq 1 \}, \]

and we frequently call them the permutation, left permutation and sub-permutation matrices.

### III. Problem Statement

A 3D centerline representation of a coronary artery tree, segmented from a preoperative CTA volume, is to be registered to \(N\) fluoroscopic images. Our observed 3D model with \(m\) points is described by matrix \(X \in \mathbb{R}^{3 \times m}\). The projection operators between the 3D space and the \(i\)-th image, represented by \(\Psi_i\), are known from the calibration of the X-ray apparatus.

The projection operator maps the 3D model to a degenerate point cloud \(I_i \in \mathbb{R}^{3 \times n_i}\) that represents \(i\)-th projection image.

By degenerate we mean the affine rank of \(I_i\) is two. We will assume that \(n_i \geq m\), \(1 \leq i \leq N\). The 2D/3D registration problem is to find an alignment matrix \(R\), in the special orthogonal group \(SO(3)\), that matches some permutation of the observed 3D model with each of the degenerate projections.

We propose to solve the multiple images 2D/3D registration problem as:

\[
(\text{REG1}) \quad \min_{P_1 \in \Pi_1^{n_1 \times m}, \ldots, P_N \in \Pi_N^{n_N \times m}} \sum_{i=1}^{N} \| \Psi_i RX - I_i P_i \|_{2,1},
\]

where \(\| \cdot \|_{2,1}\) is the mixed \(\ell_2/\ell_1\) norm. We do not know a priori the correspondences between points in the 3D model and projection image (which are encoded in \(P_i\) for each of \(N\) given images), and the rotation \(R\). These are found by solving (REG1). Intuitively, the minimization of the cost in (REG1) simply ensures by subsampling and permuting \(I_i\), the image points should as close as possible to the projections of the 3D model \(X\) posed by some rotation \(R\). We employ \(\| \cdot \|_{2,1}\) norm in order to alleviate the costs due to the outliers. If we consider a maximum likelihood estimation framework with Gaussian type noise, a squared Frobenius norm \(\| \cdot \|_F^2\) could be replaced instead.

Next we consider a variant of problem (REG1) where the correspondences of the points between the two images are available. Such correspondences could come from feature matching, or by exploiting epipolar constraints. This basically means \(P_i\) need to be optimized dependently to preserve correspondences between the images. Let the coordinates of points between \(I_a\) and \(I_b\) be denoted by \(P_{ab}\), where \(P_{ab}(i, j) = 1\) if point \(i\) in \(I_a\) corresponds to point \(j\) in \(I_b\) and zero otherwise. We do not require one-to-one correspondence between points such as each row or column of \(P_{ab}\) can have more than one nonzero entries. In the presence of such correspondence information, \(R\) can be obtained as the solution to the following optimization problem:

\[
(\text{REG2}) \quad \min_{R \in SO(3), P_1 \in \Pi_1^{n_1 \times m}, \ldots, P_N \in \Pi_N^{n_N \times m}} \sum_{i=1}^{N} \| \Psi_i RX - I_i P_i \|_{2,1} + \sum_{a=1}^{N} \sum_{b>a}^{N} \| P_{ab} - \tilde{P}_{ab} \|_1
\]

where \(P_{ab}\) is the permutation matrix that relates the points of \(I_a\) and \(I_b\) for \(N(N-1)/2\) image pairs, and \(\| \cdot \|_1\) is the entry-wise \(\ell_1\) norm. The domain of optimization for the \(S\) will be explained in the next subsection. We note that solving this problem could also be useful if we are interested in estimating the correspondence between the \(N(N-1)/2\) image pairs directly for reconstruction purpose.

For the remainder of this paper we only consider two images to simplify notation. It should be obvious the solution we propose in the subsequent sections can be easily extended to the multiple images case. Problems (REG1) and (REG2) have nonconvex domains which consist of integer and rotation matrices, as thus are very difficult to solve. In subsequent sections, we formally define these domains and present convex relaxation that can recover the exact solution under certain conditions.

#### A. Domains for permutation matrices

A more formal way to understand the problems (REG1), (REG2) is the following: Suppose there is a ground truth 3D model with \(m\) points which is described by the coordinate matrix \(X_{gt} \in \mathbb{R}^{3 \times m}\). Our observed 3D model is described by matrix \(X \in \mathbb{R}^{3 \times m}\). Assume \(m \leq n_1, n_2 \leq m\). In this case, we have

\[
I_1 = \Psi_1 RX_{gt} Q_1, \quad I_2 = \Psi_2 RX_{gt} Q_2, \quad X = X_{gt} Q_3,
\]

where \(R, \Psi_1, \Psi_2, I_1, I_2\) are matrices as introduced as before, \(Q_1 \in \Pi_1^{n_1 \times n_1}, Q_2 \in \Pi_2^{n_2 \times n_2}\) and \(Q_3 \in \Pi_3^{m \times m}\). One can intuitively regard \(Q_1, Q_2, Q_3\) as operators that generate the images and the observed model by sub-sampling and permuting the points in ground truth 3D model \(X_{gt}\). For example, if the \(i\)-th row of \(Q_1\) is zero, then it means the \(i\)-th point of \(X_{gt}\) (or more precisely, \(i\)-th column of \(X_{gt}\)) is not selected to be in \(I_1\).

Here we make an assumption, that if a point \(i\) in \(X_{gt}\) (the \(i\)-th column of \(X_{gt}\)) is contained in \(X\), then the projections of point \(i\) must correspond to some columns of \(I_1, I_2\). Loosely speaking, it means the observed 3D model \(X\) is a subset of the point clouds \(I_1, I_2\). In this case we know \(P_1 = Q_1^T Q_3 \in \Pi_1^{n_1 \times m}\) and \(P_2 = Q_2^T Q_3 \in \Pi_2^{n_2 \times m}\). Furthermore, if a point \(i\) of \(X_{gt}\) is not selected by \(Q_1\) (meaning \(\sum_j Q_1(i, j) = 0\)), then \(Q_1Q_3^T \in \mathbb{R}^{n \times n}\) is almost an identity matrix \(I_n\), except it is zero for the \(i\)-th diagonal entry (similarly for \(Q_2Q_3^T\)). We use these facts after multiplying the equations in (3) from the right by \(Q_1^T Q_3, Q_2^T Q_3\) to get

\[
\Psi_1 RX = I_1 P_1, \quad \Psi_2 RX = I_2 P_2.
\]

When there is no noise in the image, the equations in (3) have to be satisfied. If not we turn to solve the optimization problem (REG1).

To simultaneously estimate the correspondence \(P_{12}\), we use a similar construction as \([18]\) and \([19]\). Let \(P_{12} = Q_1^T Q_2\). When having exact correspondence matrix \(P_{12}\) without ambiguity, we require

\[
P_{12} = \hat{P}_{12}.
\]

We relate \(P_{12}\) through a new variable

\[
P = [Q_1 Q_2 Q_3]^T [Q_1 Q_2 Q_3] \in \mathbb{R}^{(n_1 + n_2 + m) \times (n_1 + n_2 + m)}.
\]
In this context, the variables \( P_1, P_2 \) used in (REG1), along with \( P_{12} \) are simply the off diagonal blocks of the variable \( P \). Then the domain of \((P_1, P_2, P_{12})\) is simply
\[
\mathcal{S} = \{(P_1, P_2, P_{12}) : P \succeq 0, \text{rank}(P) \leq n, P_{ii} = I_{n_i}, P_1 \in \Pi_{1}^{n_1 \times m}, P_2 \in \Pi_{2}^{n_1 \times m}, P_{12} \in \Pi_{n_1 \times n_2}\}.
\] (8)
\( P_{ii} \) should be understood as the diagonal blocks of \( P \). Again, we solve the optimization (REG2) when images or correspondences are noisy. This formulation ensures that the left permutation and sub-permutation matrices \( P_1, P_2, P_{12} \) are cycle-consistent. This basically means the relative permutation matrices \( P_1, P_2, P_{12} \) all can be constructed by some underlying global “sampling” matrices \( Q_1, Q_2, Q_3 \) for each image and the 3D model (see [19] for a detail description).

IV. CONVEX RELAXATION

In this section, we propose two programs, which are the convex relaxations of problem (REG1) and (REG2). We replace the non-convex constraints that prevents these problems from being convex to looser convex constraints. These relaxations enables the efficient search of global optimum using standard convex programming packages. To solve the convex problems we use CVX, a package based on interior point methods [20] for specifying and solving convex programs [21], [22]. Before looking at each of the two problems, we introduce a result from [23] to deal with the nonconvexity of \( \mathbb{SO}(3) \) manifold. In [23] the authors give a characterization of the convex hull of \( \mathbb{SO}(d) \) for any dimension \( d \), in terms of positive semidefinite matrices. The \( d \) version will be restated in the following theorem.

**Theorem 1** (Thm 1.3 of [23]), \( \text{conv}(\mathbb{SO}(3)) = \{ X \in \mathbb{R}^{3 \times 3} : \]
\[
\begin{bmatrix}
1 - x_{12} - x_{23} + x_{31} & x_{13} + x_{31} & x_{12} - 2x_{23} + x_{31} \\
-x_{12} - x_{23} + x_{31} & 1 + x_{13} - 2x_{23} - x_{31} & x_{23} - x_{32} \\
x_{12} + x_{23} - x_{32} & 1 + x_{12} - 2x_{23} - x_{32} & 1 + x_{12} + x_{23} + x_{32}
\end{bmatrix}
\] is positive semidefinite \}

Such linear matrix inequality is convex and can be included in a semidefinite program. For both problems (REG1) and (REG2), we relax the domain of \( R \), which is \( \mathbb{SO}(3) \) into \( \text{conv}(\mathbb{SO}(3)) \). However, this relaxation alone is not sufficient to turn (REG1) and (REG2) into convex problems, and we lay out the details of the fully relaxed problem in the next two subsections. We present the relaxations in the case of two images, and the generalization to more than two images should be obvious to the reader.

A. Relaxing (REG1)

We remind readers that in problem (REG1) we find a rotation to register the 3D model with the images. The points in the 3D model \( X \) and \( \mathcal{I}_1, \mathcal{I}_2 \) should be matched after a permutation operation through the variables \( P_1, P_2, P_1, P_2 \) have their entries on the domain \([0, 1]\). It is difficult to optimize on such discrete domain. A typical way to bypass such difficulty is by the following discrete-to-continuous relaxation,
\[
P_i \in [0, 1]^{n_1 \times m} \rightarrow P_i \in [0, 1]^{n_1 \times m}
\] (9)

for \( i = 1, 2 \). The relaxed matrices \( P_1 \) and \( P_2 \) can be interpreted as probabilistic map between the points in the image and the points in the 3D model. For example, the size of an entry \( P_i(i, j) \) resembles the probability of point \( i \) in \( \mathcal{I}_1 \) being corresponded to point \( j \) in the 3D model. Such relaxation has been used for problems such as graph isomorphism [24].

We now introduce the convexly relaxed problem \( \text{conreg1} \),
\[
\begin{align*}
\min_{R, P_1, P_2} & \quad \| \Psi_1 RX - \mathcal{I}_1 P_1 \|_{2,1} + \| \Psi_2 RX - \mathcal{I}_2 P_2 \|_{2,1} \\
\text{s.t} & \quad R \in \text{conv}(\mathbb{SO}(3)), \quad P_i \in [0, 1]^{n_1 \times m}, P_1 \leq 1, 1^T P_i = 1, \text{ for } i \in \{1, 2\}.
\end{align*}
\]

In case when there are \( N > 2 \) images, there should be \( N \) matrix variables \( P_1, \ldots, P_N \) for \( i \in \{1, \ldots, N\} \) with constraints as indicated above.

B. Relaxing (REG2)

In problem (REG2) we include the estimation of \( P_{12} \), the correspondence between image \( \mathcal{I}_1, \mathcal{I}_2 \) on top of pose estimation. In this case, we relax the \( \mathbb{SO}(3) \) and the integrality constraints just like the case of problem (REG1). However, different from (REG1), the set \( S \) has an extra rank requirement that prevents the convexity of (REG2). A popular way of dealing with the rank constraint is by simply dropping it. Such relaxation is common in many works, for example in the seminal paper of low rank matrix completion [25]. We now state the convex problem \( \text{conreg2} \),
\[
\begin{align*}
\min_{R, P_1, P_2, P_{12}} & \quad \| \Psi_1 RX - \mathcal{I}_1 P_1 \|_{2,1} + \| \Psi_2 RX - \mathcal{I}_2 P_2 \|_{2,1} \\
\quad + \| P_{12} - \hat{P}_{12} \|_1 \\
\text{s.t} & \quad R \in \text{conv}(\mathbb{SO}(3)), \quad P_i \in [0, 1]^{n_1 \times m}, \quad P_1 \leq 1, 1^T P_i = 1, \text{ for } i \in \{1, 2\},
\end{align*}
\]

where \( \hat{P}_{12} \) is an identity matrix. In case when there are \( N > 2 \) images, there should be \( N \) matrix variables \( P_1, \ldots, P_N \) and \( N(N - 1)/2 \) variables \( P_{ij} \). The relevant constraint that encodes cycle consistency in such case should be
\[
\begin{bmatrix}
P_{i1} & P_{i2} & \cdots & P_{iN} & P_i \\
P_{1j}^T & I_{n_2} & \cdots & P_{2N} & P_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
P_{1N}^T & P_{2N}^T & \cdots & I_{n_N} & P_N \\
P_{1i}^T & P_{2i}^T & \cdots & P_{Ni}^T & I_m
\end{bmatrix} \succeq 0.
\] (10)
C. Connection to classical registration problem in 3D

It is obvious that similar relaxation can be derived for the classical rigid registration problem of (1). For the sake of completeness, we state the convex solution to the 3D registration problem in (1).

\[
\min_{R, P} \|RX - YP\|_F^2 \\
\text{s.t.} \ R \in \text{conv}(SO(3)), \\
P \in [0, 1]^{m \times m}, P1 = 1, 1^TP = 1.
\]
(11)

Again, in applications where robustness is a requirement, a mixed \(\ell_2/\ell_1\) can always be replaced instead. The discussion of this program is out of the scope of this paper and will be included in a future publication of the authors.

D. Projection and local optimization

Although in section V we prove that under certain situations the two convex programs give solution to (REG1) and (REG2), this will not always be the case as the domain of search has been made less restrictive. As we are concerned about posing the 3D model in this paper, we describe a strategy in this section to obtain an approximate solution from conreg1 and conreg2 for the rotation in general cases. Denote the optimal “rotation” in the domain \(\text{conv}(SO(3))\) of both programs as \(\tilde{R}\). As \(\tilde{R}\) is not necessarily in \(SO(3)\), we need a strategy to project \(\tilde{R}\) to a point \(R^*\) on \(SO(3)\). The strategy we use is simply by finding the closest orthogonal matrix to \(\tilde{R}\) in terms of Frobenius norm, namely

\[R^* = \arg\min_{R \in SO(3)} \|R - \tilde{R}\|_F^2\]  
(12)

where \(SO(3)\) is the orthogonal group in 3D. For completeness we provide the analytical expression for \(R^*\). Let the singular value decomposition (SVD) of \(\tilde{R}\) be of the form

\[\tilde{R} = U\Sigma V^T.\]  
(13)

Then

\[R^* = UV^T.\]  
(14)

The reader may be curious that the special orthogonality constraint is not enforced in the rounding procedure. This is indeed unnecessary since an element \(\tilde{R}\) in \(\text{conv}(SO(3))\) has positive determinant \([23]\), hence \(UV^T\) obtained from the above procedure necessarily is in \(SO(3)\).

As we see in the case when the solutions of conreg1, conreg2 are not feasible for (REG1) and (REG2), a rounding procedure is required to project \(R^*\) to a nearest point in \(SO(3)\). Although it is possible to prove \(\|R^* - R_0\|\) being small (as in other applications of convex relaxation, see \([26]\) for example), \(R^*\) in general is still suboptimal in terms of the cost of the original nonconvex problem (REG1) and (REG2), as the rounding procedure does not consider optimizing any cost. It is thus quite typical to use the approximate solution from convex relaxation as a initialization and search locally for a more optimal solution, for example in \([27]\). In this paper, we use the OGM algorithm, which searches locally by optimizing a nonconvex cost using gradient-descent based method.

V. Exact recovery

In this section, we will give proof that the two relaxed convex programs exactly recover the solution to the original problems (REG1) and (REG2) under certain situations. We will show that the solutions return by our algorithm conreg1 and conreg2 lies in the domain of (REG1) and (REG2), hence showing the attainment of global optimizers of (REG1) and (REG2), as the domain of (REG1) and (REG2) are properly contained in the domain of conreg1 and conreg2. By this we mean exact recovery. The proofs of exact recovery for conreg1 and conreg2 are almost the same. We will go through proving exact recovery for conreg1 in detail, and just state the results for exact recovery of conreg2.

A. Our assumptions

We summarize the assumptions for our proof.

(1) We consider the situation when \(n_1 = n_2 = m = m'\), which means that all the image points \(I_1, I_2\) come from the projection of the 3D model. In such case \(P_1, P_2\) are permutation matrices instead of left-permutation matrices.

(2) We consider the noiseless situation when the equations

\[I_1\tilde{P}_1 = \Psi_1\tilde{R}X, \quad I_2\tilde{P}_2 = \Psi_2\tilde{R}X,\]  
(15)

can be satisfied, where \(\tilde{P}_1, \tilde{P}_2 \in \Pi_{n \times n}^m\) and \(\tilde{R} \in SO(3)\). We will call these variables the ground truth. In the case of conreg2, we further assume

\[\tilde{P}_{12} = \tilde{P}_{12}.\]  
(16)

In order to show exact recovery, we want to show that the solution \(\tilde{R}, \tilde{P}_1, \tilde{P}_2\) to conreg1, is in fact \(\tilde{R}, \tilde{P}_1, \tilde{P}_2\). For conreg2 we in addition show \(\tilde{P}_{12} = \tilde{P}_{12}\).

(3) Without lost of generality we can consider \(\tilde{R} = I_3\) and \(\tilde{P}_1, \tilde{P}_2 = I_m\). Such consideration is backed by lemma \([1]\)

(4) The \(m\) points are in generic positions. A set \(S = \{s_1, s_2, \ldots, s_n\}\) of \(n\) real numbers are called generic if there is no polynomial \(h(x_1, \ldots, x_n)\) with integer coefficients such that \(h(s_1, \ldots, s_n) = 0\) \([28]\). When we say the points are in generic positions, we mean the set of \(3m\) real numbers from \(X\) is generic. This assumption excludes the possibilities that three points can lie on a line or four points can lie on a plane. It also implies the centroid of any subset of the points is nonzero. Furthermore, no two subsets can share the same centroid.

(5) There are at least four points, i.e. \(m \geq 4\).

(6) Without lost of generality, we consider \(\Psi_1[0 0 1]^T = \Psi_2[0 0 1]^T = [0 0 1]^T\) (the directions of projection are perpendicular to \(z\)-axis).

We note that although we only give proof of exact recovery under these assumptions, in practice it is possible for our algorithm to recover the pose exactly when \(n_1, n_2 \geq m\).

B. Exact recovery property of conreg1

Let the outcome of the optimization conreg1 be \(\tilde{R}, \tilde{P}_1, \tilde{P}_2\).

Theorem 2. Under our assumptions, the solution \(\tilde{R}, \tilde{P}_1, \tilde{P}_2\) to conreg1 is unique, and \(\tilde{R} = \tilde{R}, \tilde{P}_1 = \tilde{P}_1, \tilde{P}_2 = \tilde{P}_2\).
When we have exact recovery, $R^*$ obtained from the projection of $R$ to $SO(3)$ is simply $R$, as $R$ is already in $SO(3)$. To prove the theorem, we first state a couple lemmas.

**Lemma 1.** If we make the following change of data

$$X \rightarrow X', \ T_1 \rightarrow T_1', \ T_2 \rightarrow T_2',$$

(17)

where

$$X' = R X, \ T_1' = T_1 P_1, \ T_2' = T_2 P_2,$$

(18)

then the solutions of conreg1 are changed according to the invertible maps

$$\hat{R} \rightarrow \hat{R} R^T, \ \hat{P}_1 \rightarrow \hat{P}_1 P_1, \ \hat{P}_2 \rightarrow \hat{P}_2 P_2.$$  

(19)

** Proof.**

$$\begin{align*}
\|T_1 P_1 - \Psi \hat{R} X\|_{2,1} + \|T_2 P_2 - \Psi \hat{R} X\|_{2,1} & = \|T_1 P_1 P_1^T P_1 - \Psi \hat{R} R^T X\|_{2,1} + \|T_2 P_2 P_2^T P_2 - \Psi \hat{R} R^T X\|_{2,1} \\
& = \|T_1 P_1 P_1^T P_1 - \Psi \hat{R} R^T X\|_{2,1} + \|T_2 P_2 P_2^T P_2 - \Psi \hat{R} R^T X\|_{2,1}.
\end{align*}$$

(20)

The last second equality is due to the fact that $\hat{P}_1 P_1^T = I_m, P_2 P_2^T = I_m, \hat{R}^T \hat{R} = I_3$, since permutation matrices are necessarily orthogonal matrices. Therefore, a solution after the change of data in (17) is $P_1^T \hat{P}_1, P_2^T \hat{P}_2$ and $\hat{R} R^T$. □

Since the map presented in (19) is invertible, we know the solutions to conreg1 with 3D model $X$ and image $T_1, T_2$ are in one-to-one correspondence with solutions to conreg1 with 3D model $RX$ and image $T_1 P_1, T_2 P_2$. Therefore without lost of generality, using the change of variable introduced in (17), we can let the 3D model being $X' = R X$, the images being $T_1' = T_1 P_1 = \Psi \hat{R} X = \Psi X'$ and $T_2' = \Psi X'$. We thus consider solving the problem conreg1 in the following form

$$\begin{align*}
\min_{R, P_1, P_2} & \ \| \Psi_1 RX' - \Psi_1 X' P_1\|_{2,1} + \| \Psi_2 RX' - \Psi_2 X' P_2\|_{2,1} \\
\text{s.t.} & \ \ R \in \text{conv}(SO(3)), \\
& \ \ P_1 \in [0,1]^{n \times m}, \ \\
& \ \ P_1 1 \leq 1, 1^T P_1 = 1, \text{for } i \in \{1,2\}.
\end{align*}$$

(21)

In this case, instead of showing $\hat{R} = \hat{R}, \hat{P}_1 = \hat{P}_1, \hat{P}_2 = \hat{P}_2, \hat{R}$, it suffices to show $\hat{R} = I_3$ and $\hat{P}_1, \hat{P}_2 = I_m$ being the unique solution to conreg1.

**Lemma 2.**

$$\begin{align*}
\Psi_1 \hat{R} X' &= \Psi_1 X' \hat{P}_1 \\
\Psi_2 \hat{R} X' &= \Psi_2 X' \hat{P}_2
\end{align*}$$

and

$$\begin{align*}
\Psi_1 \hat{R}^n X' &= \Psi_1 X' \hat{P}_1^n \\
\Psi_2 \hat{R}^n X' &= \Psi_2 X' \hat{P}_2^n
\end{align*}$$

for any $n \geq 1$.

** Proof.** See appendix A □

**Lemma 3.** $\hat{P}_1, \hat{P}_2$ are doubly stochastic matrices and it suffices to consider $\| \hat{P}_1 - I_m \|_1 < 1$ and $\| \hat{P}_2 - I_m \|_1 < 1$. Then

$$\lim_{n \rightarrow \infty} \hat{P}_1^n = \lim_{n \rightarrow \infty} \hat{P}_2^n = A,$$

(26)

where $A$ is partitioned by index sets $a_1, \ldots, a_k$ and

$$A(a_i, a_i) = (1/|a_i|) 1^{T_1}, \quad A(a_i, a_j) = 0 \text{ for } i \neq j.$$  

(27)

** Proof.** See appendix B □

We are now ready to prove theorem 2.

** Proof.** We first want to show $\hat{R} = I_3$. Taking limits of (24), (25) and using lemma 3 we get

$$\Psi_1 BX' = \Psi_2 X'A, \quad \Psi_2 BX' = \Psi_2 X'A,$$

(28)

where $B = \lim_{n \rightarrow \infty} \hat{R}^n$ and $A = \lim_{n \rightarrow \infty} \hat{P}_1^n = \lim_{n \rightarrow \infty} \hat{P}_2^n$. We saw in lemma 3 A has multiple irreducible components, each being an averaging operator for the points of relevant indices. The equations lead to

$$BX' = X'A.$$  

(29)

Since $B = \lim_{n \rightarrow \infty} \hat{R}^n = \hat{R}$ and it follows

$$\hat{R}X'A = X'A.$$  

(30)

Now we show $\hat{R} = I_3$ for each of the following cases.

Case 1: Suppose $A$ has three or more than three irreducible components. Let the centroid of any three components be $c_1, c_2, c_3 \in R^{3 \times 3}$. By equation (30), $R[c_1, c_2, c_3] = [c_1, c_2, c_3]$. By the assumption that columns of $X'$ are generic, $[c_1, c_2, c_3]$ has full rank hence invertible. Thus $\hat{R} = I_3$.

Case 2: Suppose $A$ has two irreducible components. This means $R[c_1, c_2] = [c_1, c_2]$ where $c_1$ and $c_2$ are independent by the assumption that points are generic. Let $\hat{c}_1$ and $\hat{c}_2$ denote the normalized version of $c_1, c_2$. $\hat{R} = \Psi_{c_1, c_2} + \alpha \hat{c}_1 \wedge \hat{c}_2$, where $\Psi_{c_1, c_2}$ denotes the projection operator to the plane containing $c_1, c_2$, $\wedge$ denotes the vector cross product and $\alpha$ some scalar constant with $|\alpha| \leq 1$. We know $|\alpha| \leq 1$ since the largest singular value of $\hat{R}$ is less than 1, as

$$\text{conv}(SO(3)) \subset \text{conv}(SO(3)) = \{ O \in R^{3 \times 3} | OTO \preceq I_3 \}$$

(31)

where $\text{conv}(SO(3))$ is the convex hull of the orthogonal group in 3D. Such characterization of the convex hull of the orthogonal group can be found in [29]. If $|\alpha| = 1$, then we indeed have proven $\hat{R} = I_3$. If $|\alpha| < 1$, we will arrive at a contradiction. In this situation $B = \lim_{n \rightarrow \infty} \hat{R}^n = \Psi_{c_1, c_2}$. By the assumption that $m \geq 4$ and equation (29), there exists four points with coordinates $Y = [Y_1, Y_2, Y_4, Y_3] \in R^{3 \times 4}$ such that either

$$\Psi_{c_1, Y} = [c_1, c_1, c_1, c_2], \text{ or } \Psi_{c_1, Y} = [c_1, c_1, c_2, c_2].$$

(32)

This means either $Y_1, Y_2, Y_4$ form a line, or $Y_1, Y_2, Y_3, Y_4$ form a plane and both cases violate the assumption that points are in generic positions.

Case 3: Suppose $A$ has a single irreducible component. In this case $BX' = [c_1, \ldots, c_1]$, where $c_1$ is the center of the point cloud. It is easy to show that $B = \Psi_{c_1}$, where $\Psi_{c_1}$ is the projection onto the line spanned by $c_1$. However, $\Psi_{c_1, X'} = \Psi_{c_1, X'} = \Psi_{c_1, X'} = \Psi_{c_1, X'} = \Psi_{c_1, X'} = \Psi_{c_1, X'}$. □
Again from the generic positions assumption, under our assumptions, the solution to \( \text{CTA} \) with blood pool contrast injection that enables the occlusions causing calcifications to be clearly distinguished. The benefit of using pre-operative injection is that it helps our proof at this point, in practice in the case of \( \text{REG1} \) when considering matching patches become registration algorithm presented in [35] and fluoroscopic images using [36], [37] to get the respective point-sets. Thus, the clinical alignment transformation between pre-operative and intra-operative images is reduced to (REG1) or (REG2).

A. Incorporating additional features

In the context of finding an alignment transformation between two point clouds, point descriptors may be valuable by favoring transformation that matches the descriptors as well as the coordinates. Our proposed formulations easily allow the incorporation of descriptors by adding additional terms in the cost of (REG1), (REG2) that encourage matching of transformation invariant features, or features transforming according to rotation. To match transformation invariant features, we simply add the following terms:

\[
\|X^F - T_1^F P_1\|_{2,1} + \|X^F - T_2^F P_2\|_{2,1},
\]

(34)
to our cost in (REG1) and (REG2), \( X^F \in \mathbb{R}^{d \times m} \), \( I_1^F \in \mathbb{R}^{d \times n_1} \), and \( I_2^F \in \mathbb{R}^{d \times n_2} \) denote some \( d \) dimensional feature vectors for the points in 3D model and the images, respectively. To deal with features that transform with rotation, such as intensity gradient, the following terms can be added to (REG1) and (REG2):

\[
\|\Psi_1 R X^F - T_1^F P_1\|_{2,1} + \|\Psi_2 R X^F - T_2^F P_2\|_{2,1}. 
\]

(35)

B. Constructing features for coronary vessel

To incorporate features for the specific application of registering coronary vessel, we will leverage the idea of patch [38]. We note that the typical features for vessels, such as the tangent of the vessel, are in general extracted from the local neighborhood of the particular point [39]. Therefore, we simply combine the coordinates of neighborhood points around each point, which we denote as patch, as the descriptor for each point. For the coordinate of point \( i \), we concatenate it with the coordinate of subsequent points \( i+1, i+2, \ldots, i+n_p-1 \) to form a patch of size \( n_p \). We denote the result of such concatenation as \( X^P_i, T_1^P_i \), and we store it as the \( i \)-th column of the matrices \( X^P \in \mathbb{R}^{3n_p \times m} \), \( I_1^P \in \mathbb{R}^{3n_p \times n_1} \), \( I_2^P \in \mathbb{R}^{3n_p \times n_2} \). Now instead of registering point, our goal is then to register these patches of the 3D model to the images. The cost in (REG1) when considering matching patches become

\[
\| (I_n_p \otimes \Psi_1 R) X^P - T_1^P P_1 \|_{2,1} + \| (I_n_p \otimes \Psi_2 R) X^P - T_2^P P_2 \|_{2,1}, 
\]

(36)

and for (REG2) the cost is

\[
\| (I_n_p \otimes \Psi_1 R) X^P - T_1^P P_1 \|_{2,1} + \| (I_n_p \otimes \Psi_2 R) X^P - T_2^P P_2 \|_{2,1} + \| P_12 - \hat{P}_{12} \|_1.
\]

(37)
As a reminder, the convex reformulation of problem (REG1) and (REG2) is known in the case of synthetic data, and for the case of \( \text{conreg}_1 \) = 1 \( \text{RMSD} \) enables attaining global optimality. To measure the quality of observations are also verified when working with real data.

In our experiments, we found \( n_p = 3 \) suffices to improve the solution without introducing too many variables into the optimization. A practical issue is that one needs to take care of the sampling density of the points on the centerline. If the point spacing is very different between \( X \) and \( \mathcal{I}_1, \mathcal{I}_2 \), the patches cannot be well matched. To deal with this issue, we first project the randomly posed 3D model using \( \Psi_1 \) and \( \Psi_2 \), calculate the average spacing between two subsequent points in the projection, and sample the image points \( \mathcal{I}_1, \mathcal{I}_2 \) according to this spacing.

### VII. Experiments

In this section we evaluate the performance of our algorithm using synthetic data of the heart vessels, and real CT data of six different patients. In particular, we demonstrate through simulation that regardless of the initial pose of the 3D model, our algorithm exactly recovers the ground truth pose when there is no noise. In the presence of image noise, our algorithm is able to return a solution close to the ground truth. These observations are also verified when working with real data. The convex reformulation of problem (REG1) and (REG2) enables attaining global optimality. To measure the quality of registration, we use the following measure

\[
\text{RMSD} = \frac{1}{2\sqrt{n}} (\| \Psi_1 R^* X - \Psi_1 \hat{R} X \|_F + \| \Psi_2 R^* X - \Psi_2 \hat{R} X \|_F ).
\]  

(38)

As a reminder, \( R^* \) is the solution after rounding from \( \text{conreg}_1, \text{conreg}_2 \), and \( \hat{R} \) is the ground truth rotation. \( \hat{R} \) is known in the case of synthetic data, and for the case of real data it is manually given by medical experts. The unit of RMSD will be in millimeter.

#### A. Synthetic data

We use synthetic data to demonstrate the ability of our algorithm to exactly recover the pose of the 3D model when there is no noise in the image. We also add bounded noise to each point in the image in the following way

\[
\mathcal{I}_{1i} = \Psi_1 (RX_i + \varepsilon_{1i}), \quad \mathcal{I}_{2j} = \Psi_2 (RX_j + \varepsilon_{2j}),
\]

\( i \in 1, ..., n_1 \), \( j \in 1, ..., n_2 \), \hspace{1cm} (39)

where \( \varepsilon_{1i}, \varepsilon_{2j} \sim \mathcal{U}[-\varepsilon, \varepsilon]^3 \) and \( \mathcal{U}[-\varepsilon, \varepsilon]^3 \) is the uniform distribution in the cube \([-\varepsilon, \varepsilon]^3\). In order to run \( \text{conreg}_2 \), we simply get \( \hat{P}_{12} \) from the epipolar constraints. We perform simulations for two different cases (figure[1]), namely, when the 2D images are exactly the projections of the 3D model, and when there are extra centerlines in the 2D images. In the latter case, the 3D model is a proper subset. In our simulations, we first rotate the 3D model of coronary vessels into some arbitrary posture and project it in two fixed orthogonal directions. We tested our algorithm systematically when rotation \( R \) in (39) is within each of the ranges \([0^\circ, 10^\circ], [10^\circ, 20^\circ], [20^\circ, 40^\circ], [40^\circ, 60^\circ], \) and \([60^\circ, 90^\circ]\), using 10 different noise realizations in each case.

We observed exact recovery in both programs when the noise level is zero, which confirms our proof in Section V. Since we are guaranteed to achieve global optimum in convex programs, regardless of the position of the 3D model pose, the ground truth rotation can be recovered. Figure[2] shows the results of both programs with no noise added to the points with respect to the initial pose of the 3D model. In our experiments, we set the tolerance level for the convex solver to \( 10^{-3} \).

Fig. [3] shows the results for both programs with bounded noise added to each point in the image. In this case the RMSD increases proportionally as the noise \( \varepsilon \) increases from 0 to 2 mm for both input conditions. We observe that the latter input condition of extra centerlines in the image may be a better approximation of a typical clinical scenario as often non vascular objects such as catheters are mis-segmented as vessels.

We compare our algorithm with the recent 2D/3D registration algorithm \( \text{OGMM} \) [16], [40]. In our experiments, we use our
MATLAB implementation of [16] for two images with each individual gaussian distribution being isotropic. If we are to use OGMM, using the identity as an initialization, the recovery of the pose is not guaranteed unless we are in the $[0^\circ, 10^\circ]$ range. In fact for most experiment we cannot find solution within 25mm RMSD. We show the results from OGMM in figure 2 where we plot the RMSD against different 3D poses. We note that there are many ways one can improve on our implementation of OGMM in terms of radius of convergence, such as annealing to zero the variance of Gaussian distributions starting from a large value as proposed in [15]. Nevertheless, it still remains that registration methods based on Gaussian mixtures method [15] can get stuck in a local optimum due to the nature of the cost.

B. Real data

To establish a clinically meaningful quantitative requirement for registration accuracy, we provided a prototype to medical experts at 5 hospitals worldwide that allowed them to manually/semi-automatically register 3D centerlines models to 2D fluoroscopic images. It has been used (under appropriate approvals) in more than 100 Chronic Total Occlusion (CTO) cases treated using PCI. The feedback from several operators confirms that the registration between 3D centerlines models from CT and 2D fluoroscopic images is clinically most useful if the registration error is less than 10 mm. A side-by-side use of 3D centerlines with 2D fluoroscopic images is acceptable for registration errors up to 15 mm. An error beyond 25 mm should be considered a failed registration attempt.

We test out algorithm on six sets of clinical data in which a medical expert has aligned the 3D model to 2D fluoroscopic images. We consider this the ground truth to compare against our results. To characterize the recovery of our algorithm, we perturb the aligned 3D model by an arbitrary rotation within the ranges $[0^\circ, 10^\circ], [10^\circ, 20^\circ], [20^\circ, 40^\circ], [40^\circ, 60^\circ]$, and $[60^\circ, 90^\circ]$. For each rotation range, we generate 10 different instances. The results are detailed in figure 4 and 5. Again for comparison we use results from OGMM. We see that we are able to recover the pose to within 10 mm for most cases. Further, the recovery of pose in consistent irrespective of the initial pose. Whereas, in general we cannot obtain RMSD within 25 mm with the OGMM method. The effect of additional point consistency terms between the images in conreg2 is more prominent in real patient data. We note that clinical data typically may have more number of ambiguous 3D model to projection point matches that cannot be resolved adequately using conreg1 without enforcing explicit matching between images. In such cases, the point consistency terms in conreg2 may aid in resolution of such ambiguities. We present the images of the 3D model after the 2D/3D registration in figure 6.

Typically for convex relaxation techniques such as one adopted in this paper, it is not unusual to prove the convex relaxed solutions to the noisy data lie in the vicinity of the ground truth. We in fact demonstrate via simulation experiments that the solutions of algorithms conreg1 and conreg2 lie within certain radius (proportional to the noise magnitude) of the ground truth. Thus, the use of local optimizers such as ICP or GMM based methods after conreg1 and conreg2 could be used to further refine the results. As local search based registration method often fails to reach the global optimum due to non-convex nature of the cost, this combined approach ensures a starting point for local search in a neighborhood that is already close to the ground truth. Results using algorithms conreg1 and conreg2 with OGMM as a local search method are presented in table 1. Though conreg2 has some instances with larger RMSD error, overall its performance is better than conreg1 in terms of median RMSD error. Both methods outperform OGMM alone for robustness to initial starting point and median RMSD.

### Table I

| Stage | OGMM | conreg1 + OGMM | conreg2 + OGMM |
|-------|------|----------------|----------------|
| Good  | 85 (4.57) | 300 (3.80) | 291 (2.31) |
| Acceptable | 101 (5.14) | 292 (2.31) | |
| Fail  | 165 (25.18,737.64) | 4 (26.63,28.40) | |

The number of experiments with RMSD within “Good”, “Acceptable” and failed attempts on 6 real data sets. Each patient data set was simulated with a total of 50 different poses, for 6 patients. The number in bracket is the median of the RMSD for solutions within respective thresholds. The range of RMSD is reported for attempts with RMSD greater than 25 mm.
Fig. 6. Each row is the registration results for a single patient presented in two different viewing directions. The black lines are the segmented vessel centerline in the X-ray images. Column 1 and 3: Red lines denote the 3D model posed by $R^*$ from $\text{conreg1}$. Blue lines denote the 3D model posed by $R^*$ from $\text{conreg2}$. Column 2 and 4: Red lines denote the 3D model posed by $R^*$ from $\text{conreg1} + \text{OGMM}$. Blue lines denote the 3D model posed by $R^*$ from $\text{conreg2} + \text{OGMM}$. 
VIII. DISCUSSION AND CONCLUSION

A new formulation for 2D/3D registration based on convex optimization programs has been proposed, and applied to the problem of registering a 3D centerline model of the coronary arteries with a pair of fluoroscopic images. The proposed optimization programs jointly optimize the correspondence between points and their projections, and the relative transformation. Since the optimization is convex, it enables an efficient search of global optima regardless of initialization using standard off-the-shelf conic programming software.

In the first program presented, we find the correspondence and transformation simultaneously between two degenerate point clouds obtained by projection of a 3D model and the model itself. In the second program presented, we solve a variant of the first program where a-priori information on the radiography frames have also been presented. The validation was an explicit correspondence match. We hope to bridge the gap in such cases, in addition to providing the global optimum due to non-convex nature of the cost. We registration methods based on local search may fail to reach a neighborhood patch.

That there are ways one can improve on the implementations of local optimizers in terms of range of convergence by using methods such as annealing. Nevertheless, it still remains that registration methods based on local search may fail to reach the global optimum due to non-convex nature of the cost. We hope to bridge the gap in such cases, in addition to providing an explicit correspondence match.

Finally, experiments using multiple X-ray biplane angiography frames have also been presented. The validation was performed by perturbing the original pose by a random pose in a wide range of values. The pose and the correspondence can be recovered to within 10 mm RMSD error in most cases. Our algorithm can be used as a pre-processing step to provide a high quality starting point for a local registration algorithm such as OGM or alone to provide recovery of transformation and correspondence.

APPENDIX A

Proof. The equations (22) and (25) simply follows from the fact that

\[ 0 \leq \| \Psi_1 \tilde{R}X' - \Psi_1 X' \tilde{P}_1 \|_{1,1} + \| \Psi_2 \tilde{R}X' - \Psi_2 X' \tilde{P}_2 \|_{1,1} \]
\[ \leq \| \Psi_1 I_3 X' - \Psi_1 X'I_m \|_{1,1} + \| \Psi_2 I_3 X' - \Psi_2 X'I_m \|_{1,1} \]
\[ = 0. \]

The second inequality follows from the optimality of \( \tilde{P}_1, \tilde{P}_2 \) and \( \tilde{R} \).

Now
\[ \tilde{R}X' = X' \tilde{P}_1 + \eta_1 \]
\[ \tilde{R}X' = X' \tilde{P}_2 + \eta_2 \]

where \( \eta_1 \in \mathbb{R}^{3 \times m}, \eta_2 \in \mathbb{R}^{3 \times m} \) satisfies \( \Psi_1 \eta_1 = \Psi_2 \eta_2 = 0 \).

By induction, we have
\[ \tilde{R}^n X' = X' \tilde{P}_1^n + \eta_1 (\tilde{P}_1^{n-1} + \ldots + \tilde{P}_1 + 1) \]
\[ \tilde{R}^n X' = X' \tilde{P}_2^n + \eta_2 (\tilde{P}_2^{n-1} + \ldots + \tilde{P}_2 + 1). \]

Multiplying these two equations by \( \Psi_1 \) and \( \Psi_2 \) respectively we get (24) and (25).

APPENDIX B

Proof. Our goal is to show the solution \( \tilde{P}_1, \tilde{P}_2, \tilde{R} \) being \( I_m, I_m, I_3 \) respectively. It suffices to prove uniqueness of \( I_m, I_m, I_3 \) as solution to conreg1 in a local neighborhood, since for a convex program local uniqueness implies global uniqueness. Therefore we will assume \( \| \tilde{P}_1 - I_m \|_1 < 1 \) and \( \| \tilde{P}_2 - I_m \|_1 < 1 \). We now show \( \tilde{P}_1, \tilde{P}_2 \) are doubly stochastic matrices. The constraints \( 1^T \tilde{P}_1 = 1^T \tilde{P}_2 = 1^T \) in conreg1 indicates

\[ \sum_{i=1}^{m} (\sum_{j=1}^{m} \tilde{P}_1(i,j)) = \sum_{i=1}^{m} (\sum_{j=1}^{m} \tilde{P}_2(i,j)) = m. \]

Since for every \( i = 1, \ldots, m, \sum_{j=1}^{m} P_1(i,j), \sum_{j=1}^{m} P_2(i,j) \leq 1 \), it has to be that

\[ \sum_{j=1}^{m} \tilde{P}_1(i,j), \sum_{j=1}^{m} \tilde{P}_2(i,j) = 1 \]

for every \( i = 1, \ldots, m, \) or else it violates (43).

Based on these facts and regarding \( \tilde{P}_1, \tilde{P}_2 \) as transition matrices of a Markov chain, we will use the fundamental theorem of Markov chain to prove the lemma. Before stating the theorem, we need to give two standard definitions of stochastic processes.

Definition 1. A Markov chain with transition matrix \( P \in \mathbb{R}^{m \times m} \) is aperiodic if the period of every state is 1, where the period of a state \( i \), \( i = 1, \ldots, m \) is defined as

\[ \gcd\{k : P^k(i,i) > 0\} \]
Theorem 4 (Fundamental Theorem of Markov Chain). If a Markov chain with transition matrix $P \in \mathbb{R}^{m \times m}$ is irreducible and aperiodic, then

$$\lim_{n \to \infty} P^n(i, j) = \pi_j \quad \forall i, j = 1, \ldots, m,$$  

(46)

where the limiting distribution $\pi = [\pi_1, \ldots, \pi_m]$ is the unique solution to the equation

$$\pi = \pi P.$$  

(47)

We are now ready to prove the lemma. From the assumption $\|\hat{P}_1 - I_m\| < 1$ and $\|\hat{P}_2 - I_m\| < 1$, we have

$$\text{diag}(\hat{P}_1) > 0, \quad \text{diag}(\hat{P}_2) > 0.$$  

(48)

This means the two doubly stochastic matrices $\hat{P}_1, \hat{P}_2$ are aperiodic. Furthermore we can decompose the matrices $\hat{P}_1, \hat{P}_2$ into irreducible components. Such decomposition essentially amounts to decomposing a graph into connected components, if we regard a doubly stochastic matrix as weighted adjacency matrix of a graph with $m$ nodes. Now denote $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_l$ as index sets of the irreducible components of $\hat{P}_1$ and $\hat{P}_2$ respectively. For such decomposition, we know $\hat{P}_1(a_i, a_j) = 0$ for $i \neq j$ (similarly for $\hat{P}_2$). Therefore for each aperiodic and irreducible block $\hat{P}_1(a_i, a_i)$ of $\hat{P}_1$ and $\hat{P}_2(b_i, b_i)$ of $\hat{P}_2$, we have

$$\frac{1}{|a_i|} 1^T \hat{P}_1(a_i, a_i) = \frac{1}{|b_i|} 1^T \hat{P}_2(b_i, b_i) = \frac{1}{|a_i|} 1^T.$$  

(49)

Applying fundamental theorem of Markov chain to each irreducible and aperiodic block along with equation (49), we have

$$\lim_{n \to \infty} \hat{P}_1(a_i, a_i) = \frac{1}{|a_i|} 1 1^T, \quad \lim_{n \to \infty} \hat{P}_2(b_i, b_i) = \frac{1}{|b_i|} 1 1^T.$$  

(50)

and zero for all other indices. We note that the multiplication $(1/|a_i|) 1 1^T X_{a_i}$ results $|a_i|$ copies of the centroid of $X_{a_i}$. Hence the limit of each irreducible component of $\hat{P}_1, \hat{P}_2$ is an averaging operator.

We now take limit of equations (24) and (25). Multiplying both (24) and (25) from the right with $[0 \ 0 \ 1]$, and using the fact that $[0 \ 0 \ 1] \Psi_1 = [0 \ 0 \ 1] \Psi_2 = [0 \ 0 \ 1]$ we get

$$[0 \ 0 \ 1] X \lim_{n \to \infty} \hat{P}_1 = [0 \ 0 \ 1] X \lim_{n \to \infty} \hat{P}_2 = A.$$  

(51)

Now assuming the point set $X$ contains generic coordinates, we know that no two set of points from $X$ have centers with same $z$ coordinates. Combining this fact with (50) and (51), we get

$$\{(a_1, a_2, \ldots, a_k) \mid (b_1, b_2, \ldots, b_l) \} \text{ and } \lim_{n \to \infty} P^n_1 = \lim_{n \to \infty} P^n_2 = A.$$  

Definition 2. A Markov chain with transition matrix $P \in \mathbb{R}^{m \times m}$ is irreducible if for all $i, j$, there exists some $k$ such that $P^k(i, j) > 0$. Equivalently, if we regard $P$ as the adjacency matrix of a directed graph, it means the graph corresponding $P$ is strongly connected.

We then state the fundamental theorem of Markov chain.

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