Mpemba effect in driven granular Maxwell gas

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Mpemba effect refers to the counterintuitive result that, when quenched to a low temperature, a system at higher temperature may equilibrate faster than one at intermediate temperatures. This effect has recently been demonstrated in driven granular gases, both for smooth as well as rough hard-sphere systems based on a perturbative analysis. In this paper, we consider the inelastic driven Maxwell gas, a simplified model for a granular gas, where the rate of collision is assumed to be independent of the relative velocity. Through an exact analysis, we determine the conditions under which a Mpemba effect is present in this model. For mono-dispersed gases, we show that the Mpemba effect is present only when the initial states are allowed to be non-stationary, while for bi-dispersed gases, it is present for steady state initial states. We also demonstrate the existence of the strong Mpemba effect for bi-dispersed Maxwell gas wherein the system at higher temperature relaxes to a final steady state at an exponentially faster rate leading to smaller equilibration time.

I. INTRODUCTION

A classic lore associated to everyday experience is that under certain conditions, hot water freezes faster than cold water. Even though this intriguing effect has been known since Aristotelian times [1, 2], it has been named Mpemba effect after E. B. Mpemba, who was the first to study it systematically [3]. This out of equilibrium relaxation behavior is in a stark contrast to quasi-static relaxations, where systems with different initial conditions evolve in time through equilibrium states and therefore their temperatures never cross.

Several mechanisms were suggested to explain the Mpemba effect in water. These include conceptually simple mechanisms, as supercooling [4], convection [5] and evaporation [6], as well as more sophisticated explanations as the anomalous relaxation of the hydrogen bond [7, 8]. Others have argued that the effect does not exist in water [9]. Regardless of the true status of the effect in water, a similar effect was experimentally observed in a wide range of physical systems ranging from magnetic alloys [10] and clathrate hydrates [11] to polylactides [12].

In addition to experiments, various numerical tools and model systems were used to shed light on such anomalous relaxations. To better understand the effect in water, detailed molecular dynamic simulations were performed [7, 13, 14]. In these simulations, the system is made of up to several thousand molecules, with pairwise interactions that model the interactions between the water molecules. The initial configuration of the system is commonly sampled from the Boltzmann distribution of the hot or warm temperature, and the dynamic follows a standard molecular dynamic protocol that corresponds to the cold temperature. A different numerical approach was applied on spin systems, for both ordered [15] as well as glassy [16] models. Using a Monte Carlo simulation, the value of some order parameters were tracked during the relaxation process of systems that were sampled from the equilibrium distribution of a hot and a warm temperature, and evolved under Markovian dynamic that corresponds to a cold temperature. If, during the relaxation process the corresponding order parameters of the hot and cold systems intersect, then a Mpemba effect exists in the system.

On the analytic front, two different approaches were used to address the Mpemba effect so far. In Refs. [17–19], the Mpemba effect was defined and evaluated through the distance between probability distributions during the relaxation process. To this end, the probability distribution describing the system is initiated at the equilibrium (Boltzmann) distribution of the hot or warm temperature. The system is then quenched into an environment with a cold temperature, and thus the probability distribution of the system evolves in time towards the equilibrium of the cold temperature. By tracking the distance, in probability space, between the time dependent probability and the final equilibrium, a Mpemba effect can be identified, and the exact conditions for its existence were derived. Moreover, this framework naturally suggest a rich class of related phenomena, including the inverse Mpemba effect [17] where a cold system heats up faster than a warmer system; the strong Mpemba effect [18] where a specific initial temperature results in a jump in the relaxation rate; and non-monotonic optimal heating protocols [15] where the optimal heating protocol has a pre-cooling stage.

A different theoretical framework was used in the context of driven granular gases [20, 21]. A granular gas is a dilute composition of particles that move ballistically and interact through momentum conserving binary inelastic collisions. These dissipative systems approach a steady

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state when externally driven to compensate for the kinetic energy lost due to the inter-particle collisions. For the driven granular system with smooth mono-dispersed particles which interact via binary collisions with a rate proportional to the magnitude of the relative velocities of the colliding particles, it was shown that the total energy of a system with a higher initial energy attains the final low energy state before a similar system with an intermediate initial energy [20]. The effect was achieved by independently varying the deviation of the velocity distribution from a gaussian, characterized by the coefficient of the second Sonine polynomial $\alpha_2$. In this approach, the system was initialized in a non-stationary distribution, and relaxed towards the corresponding steady state (which is not an equilibrium distribution) associated with the parameters of the system. The Mpemba effect then exists in the system if a non-equilibrium system which is further away from equilibrium, namely has more energy in its initial state, equilibrates faster than an initial condition which is closer to the equilibrium state. Mpemba effect was also demonstrated for a rough granular gas, where a much larger range of initial energies result in anomalous relaxations [21]. In both the rough and smooth granular gas, the velocity distribution at all times was approximated by a gaussian or gaussian and first order corrections respectively, making the calculations perturbative in nature.

Do the results in the various frameworks for different systems correspond to the same effect? This is a key question, for which the answers is yet unknown. In this manuscript, we partially address a specific difference that plays an important role in both the numerical and analytical results developed so far – the initial condition of the system. In the Markovian framework, the initial condition of the hot and warm systems is an equilibrium distribution corresponding to the initial temperature [17–19]. In contrast, in the granular gas approach the initial distributions are not a steady state of the system for any (effective) temperature, but are rather transient distributions with different amount of total energy. These distributions relax towards the steady-state distribution by energy exchange between the particles as well as with some bath [20, 21]. A similar difference in the initial conditions exists between the various molecular dynamic simulations calculated for water molecules: in [14] the system was initially sampled from a hot or warm temperature and then quenched to a cold environment, whereas in [13] the initial condition is not the equilibrium distribution of any temperature, but rather an altered distribution of the final temperature.

To address this specific difference between the two approaches, we consider the inelastic Maxwell model for granular gas in which the collision rates are assumed to be independent of the relative velocity [22, 23]. The model thus presents a simpler system of granular gas keeping the essential physics intact, while allowing for exact calculation. We investigate the existence of the Mpemba effect for both mono-dispersed and bi-dispersed systems. In Maxwell gases, the equations for the relevant two-point correlations is known to form a closed set of equations [24–26]. By analyzing these equations in detail, we determine the parameter regime in which a Mpemba effect can be seen in these systems. In particular, we show that while transient initial conditions are required for the effect to be present in mono-dispersed gas, the bi-dispersed system shows Mpemba effect for initial conditions that are steady states. This allows us to use the Markovian approach for the Mpemba effect in driven granular gas, and therefore identify the existence of the strong effect in this system, where for a specific initial steady state the relaxation rate is smaller than from any other initial steady state.

The remainder of the paper is organized as follows. We define the model in Sec. II. In Sec. III, we show that the time evolution of two-point velocity-velocity correlation functions do not involve higher order correlations and form a closed set of equations. This allows for an exact solution for the time evolution of the two point correlations. In Sec. IV, we define Mpemba effect and demonstrate the possibility for the existence of Mpemba effect in driven mono-dispersed systems. Section V contains the detailed analysis for the existence of Mpemba effect in bi-dispersed granular systems. We also demonstrate the existence of inverse Mpemba effect and a stronger version of Mpemba effect in similar systems. Section VI contains the summary of results and a discussion of their implications.

II. MODEL

In this paper, we analyze both mono-dispersed as well as bi-dispersed driven inelastic Maxwell gases. We define the bi-dispersed gas, and then indicate the limits when it reduces to a mono-dispersed gas. Consider $N_A$ particles of type $A$, each of mass $m_A$, and $N_B$ particles of type $B$, each of mass $m_B$. Let $N_A + N_B = N$. Each particle has a scalar velocity $v_{i,k}$, where $i = 1, \ldots, N$ and $k \in \{A, B\}$. These velocities evolve in time through binary collisions and external driving. A pair of particles of type $k$ and $l$, where $k, l \in \{A, B\}$, collide with rate $\lambda_{kl}/N$. The factor $1/N$ in the collision rates ensures that the total rate of collisions between $N_k[N_k - 1]/2$ pairs of similar type of particles and that between $N_A N_B$ pairs of different type of particles are proportional to the system size $N$. During a collision, momentum is conserved, but energy is dissipated. Let $v_{i,k}$ and $v_{j,l}$ denote the pre-collision velocities and $v_{i,k}'$, $v_{j,l}'$ denote the post-collision velocities. Then

\begin{align*}
  v_{i,k}' &= v_{i,k} - (1 + r_{kl}) \frac{m_l}{m_k + m_l} (v_{i,k} - v_{j,l}), \\
  v_{j,l}' &= v_{j,l} + (1 + r_{kl}) \frac{m_k}{m_k + m_l} (v_{i,k} - v_{j,l}), \quad (1)
\end{align*}

where $k, l = A, B$, $r_{kl} \in [0, 1]$ is the coefficient of restitution for the collision, and $m_k$ and $m_l$ are the masses.
There are three coefficients of restitution: \( r_{AA}, r_{BB}, \) and \( r_{AB} \) depending on whether the pair of colliding particles are of type \( AA, BB, \) or \( AB. \) It is convenient to define
\[
\alpha_{kl} = \frac{1 + r_{kl}}{2}, \quad k, l = A, B, \tag{2}
\]
where \( 1/2 \leq \alpha_{kl} \leq 1. \)

In addition to collisions, the system evolves through external driving. We implement a driving scheme that drives the system to a steady state, and has been used in earlier studies\cite{24, 25, 27}. Each particle is driven at a rate \( \lambda_d. \) During such an event, the velocity of the driven particle is modified according to
\[
v'_{i,k} = -r_w v_{i,k} + \eta_k, \quad -1 < r_w \leq 1, \quad k = A, B, \tag{3}
\]
where \( r_w \in (-1, 1) \) is a parameter and \( \eta_k \) is noise drawn from a fixed distribution \( \phi_k(\eta_k). \) There is no compelling reason for \( \phi_k(\eta_k) \) to be Gaussian. However, we restrict ourselves to distributions with zero mean and finite second moment \( \sigma_k^2 \) given by
\[
\sigma_k^2 = \int_{-\infty}^{\infty} d\eta \; \eta^2 \phi_k(\eta), \quad k = A, B. \tag{4}
\]
The physical motivations for the form of driving may be found in Refs.\cite{28, 29}.

In the model, the spatial degrees of freedom have been neglected. This corresponds to the well-mixed limit where the spatial correlations between particles are ignored. In addition, we have assumed that the collision rates are independent of the relative velocity of the colliding particles. This corresponds to the so called Maxwell limit.

Let \( P_k(v, t), \) where \( k = A, B, \) denote the probability that a randomly chosen particle of type \( k \) has velocity \( v \) at time \( t. \) Its time evolution is given by:

\[
\frac{d}{dt} P_k(v, t) = \frac{\lambda k(N_k - 1)}{N} \int dv_1 dv_2 P_k(v_1, t) P_k(v_2, t) \delta[(1 - \alpha_{kk})v_1 + \alpha_{kk}v_2 - v]
+ \frac{\lambda_{kk}N_k}{N} \int dv_1 dv_2 P_k(v_1, t) P_k(v_2, t) \delta[(1 - X_{kk})v_1 + X_{kk}v_2 - v] - \frac{\lambda_{kk}(N_k - 1)}{N} P_k(v, t)
- \frac{\lambda_k N_k}{N} P_k(v, t) + \lambda_d \left[-P_k(v, t) + \int \int d\eta_k dv_1 \phi_k(\eta_k) P_k(v_1, t) \delta[-r_w v_1 + \eta_k - v]\right],
\tag{5}
\]

where
\[
\bar{k} = \begin{cases} B, & \text{if } k = A, \\ A, & \text{if } k = B, \end{cases} \tag{6}
\]
and
\[
X_k = \alpha_{AB} \mu_k \quad \text{where} \quad \mu_k = \frac{2m_k}{m_A + m_B}, \quad k = A, B, \tag{7}
\]
with \( \mu_k \in (0, 2) \) and \( \alpha_{AB} \) is defined in Eq.\,(2).

The mono-dispersed Maxwell gas is obtained by taking the limit \( N_A = N, \) \( N_B = 0, \) \( r_{AA} = r \) and setting all other coefficients of restitution to zero. The rate of inter-particle collisions is denoted by \( \lambda \) and the rate of driving for the particles by \( \lambda_d. \) If \( P(v, t) \) denote the probability that a randomly chosen particle has velocity \( v \) at time \( t \) then its time evolution, for a mono-dispersed gas is given by:

\[
\frac{d}{dt} P(v, t) = -\lambda P(v, t) - \lambda_d P(v, t)
+ \lambda \int dv_1 dv_2 P(v_1, t) P(v_2, t) \delta[(1 - \alpha)v_1 + \alpha v_2 - v]
+ \lambda_d \int \int d\eta dv_1 \phi(\eta) P(v_1, t) \delta[-r_w v_1 + \eta - v] \tag{8}
\]

III. CALCULATION OF TWO POINT CORRELATIONS

In this section, we define the relevant two point correlation functions for both mono-dispersed and bi-dispersed gases. The evolution equations for these correlation functions were derived in Refs.\cite{24, 26}. We summarize these derivations and then develop a solution that will be useful for demonstrating Mpemba effect. Sections III A and III B contain the derivation for mono-dispersed and bi-dispersed gases respectively.

A. Mono-dispersed Maxwell gas

We first discuss the case of mono-dispersed Maxwell gas. Consider the following two point correlation func-
where $E(t)$ is the mean kinetic energy of a particle, and $C(t)$ is the equal time velocity-velocity correlation between a pair of particles. In the steady state, the two point correlation function is known to be zero [25]. However, for the purpose of demonstrating Mpemba effect, we will consider non-zero correlations, which will correspond to non-stationary states. The time evolution of these correlation functions can be obtained in a straightforward manner from Eq. (8) and can be compactly represented in matrix form as [24]

$$E(t) = \frac{1}{N} \sum_{i=1}^{N} \langle v_i^2(t) \rangle,$$

$$C(t) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \langle v_i(t) v_j(t) \rangle,$$

(10)

where $E(t)$ is the mean kinetic energy of a particle, and $C(t)$ is the equal time velocity-velocity correlation between a pair of particles. In the steady state, the two point correlation function is known to be zero [25]. However, for the purpose of demonstrating Mpemba effect, we will consider non-zero correlations, which will correspond to non-stationary states. The time evolution of these correlation functions can be obtained in a straightforward manner from Eq. (8) and can be compactly represented in matrix form as [24]

$$\frac{d\Sigma(t)}{dt} = -RS(t) + D,$$

(11)

where $\Sigma(t) = [E(t), C(t)]^T$, $D = [\lambda_d \sigma^2, 0]^T$, and $R$ is given by

$$R = \begin{bmatrix} \lambda_+ (1 - r^2) + \lambda_h (1 - r_w^2) & -\lambda_c (1 - r^2) \\ \lambda_h (1 - r^2) & \lambda_0(1 + r_w) \end{bmatrix}. \quad (12)$$

Note that while $R_{11}$ and $R_{22}$ are positive, $R_{12}$ and $R_{21}$ are negative.

Equation (11) for the correlation functions forms a closed set of linear equations and does not involve higher order correlation functions. This allows for a complete solution. Equation (11) can be solved exactly by linear decomposition using the eigenvalues $\lambda_\pm$ of $R$:

$$\lambda_\pm = \frac{R_{11} + R_{22} \pm \sqrt{(R_{11} - R_{22})^2 + 4R_{12}R_{21}}}{2}. \quad (13)$$

Here, the eigenvalues $\lambda_\pm > 0$ with $\lambda_+ > \lambda_-$. The solution for $E(t)$ and $C(t)$ can then be obtained as:

$$E(t) = K_+ e^{-\lambda_+ t} + K_- e^{-\lambda_- t} + K_0,$$

$$C(t) = L_+ e^{-\lambda_+ t} + L_- e^{-\lambda_- t} + L_0,$$

(14)

where the coefficients $K_+, K_-, K_0, L_+, L_-$ and $L_0$ are given in Eq. (A1) of Appendix A.

**B. Bi-dispersed Maxwell gas**

For the case of bi-dispersed Maxwell gas, we can define two point correlation functions, $E_A$ and $E_B$ for the mean kinetic energies of type A and B particles, and three two point velocity-velocity correlation functions $C_{ij}$, where $i, j \in (A, B)$:

$$E_A(t) = \frac{1}{N_A} \sum_{i=1}^{N_A} \langle v_{i,A}^2(t) \rangle,$$

$$E_B(t) = \frac{1}{N_B} \sum_{i=1}^{N_B} \langle v_{i,B}^2(t) \rangle,$$

$$C_{AB}(t) = \frac{1}{N_A N_B} \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \langle v_{i,A}(t) v_{j,B}(t) \rangle,$$

$$C_{AB}(t) = \frac{1}{N_A N_B} \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \langle v_{i,A}(t) v_{j,B}(t) \rangle,$$

$$C_{BB}(t) = \frac{1}{N_B(N_B-1)} \sum_{i=1}^{N_B} \sum_{j=1}^{N_B} \langle v_{i,B}(t) v_{j,B}(t) \rangle.$$  

(15)

The time evolution for these correlation functions can be obtained from Eq. (9), as derived in Ref. [26] when only one type of particle is driven. We generalize these calculations to the case where both types of particles are driven. The time evolution equations are linear and form a closed set of equations as shown in Appendix (A 2).

In the steady state, in the thermodynamic limit, the inter-particle two point correlation functions $C_{ij}$, where $i, j \in (A, B)$ are zero, as shown in Ref. [26]. If in the initial state, these correlations are zero, then it remains zero for all times. We will be only considering such initial states. Unlike the mono-dispersed case, we will show that Mpemba effect is possible for initial states that are steady states. In that case, we can ignore these correlations, and write the time evolution of mean kinetic energies of A and B type particles, i.e., $E_A$ and $E_B$ respectively in a compact form (see Appendix A 2 for detailed calculations) as

$$\frac{d\Sigma(t)}{dt} = R \Sigma(t) + D,$$

(16)

where

$$\Sigma(t) = [E_A(t), E_B(t)]^T,$$

$$D = [\lambda_d \sigma^2_A, \lambda_d \sigma^2_B]^T,$$

(17)

and $R$ is a $2 \times 2$ matrix, whose entries are given by

$$R_{11} = \lambda_{AB} \nu_B X_B^2 - \lambda_{AA} \alpha_{AA}(1 - \alpha_{AA}) \nu_A,$$

$$-2 \lambda_{AB} \nu_B X_B - \lambda_d (1 - r_{wA}^2),$$

$$R_{12} = \lambda_{AB} \nu_B X_B,$$

$$R_{21} = \lambda_{AB} \nu_A X_A^2,$$

$$R_{22} = \lambda_{AB} \nu_A X_A^2 - \lambda_{BB} \alpha_{BB}(1 - \alpha_{BB}) \nu_B,$$

$$-2 \lambda_{AB} \nu_A X_A - \lambda_d (1 - r_{wB}^2).$$

(19)

We note that the linearity in the evolution equations [Eqs. (11) and (16)] for the energy arises naturally for
Maxwell gases (both mono-dispersed and bi-dispersed) when compared to the granular gas models studied earlier [20, 21], wherein the non-linear evolution equation limits analytical treatment unless linearized using perturbative methods [20]. Similar exact linear evolution equation has been analyzed in the case of Markovian Mpemba effect [17] where the vector denoting the probabilities of various states evolves according to an equation similar to Eqs. (11) and (16).

IV. MPEMBA EFFECT IN MONO-DISPERSED MAXWELL GAS

In this section, we derive the conditions for Mpemba effect to be present in the mono-dispersed driven Maxwell gas, based on an analysis of Eq. (14) for the solution of $E(t)$ and $C(t)$. More precisely, we will define Mpemba effect to mean the following. Consider two systems with two different granular temperature or kinetic energy [we will use the words to mean the same]. We let these systems evolve to a steady state at some final temperature, that is lower than the initial temperatures. If the hotter system cools faster (the energy-time plots show a crossing), then we will say that the system shows Mpemba effect.

We now proceed to find out the criteria for Mpemba effect to be present for the mono-dispersed driven Maxwell gas. Consider two systems labeled as $P$ and $Q$. Let their initial conditions be denoted by $(E^P(0), C^P(0))$ and $(E^Q(0), C^Q(0))$ with $E^P(0) > E^Q(0)$. Both the systems are driven to a common steady state. This is achieved when the systems $P$ and $Q$ are driven with the same driving strength ($\sigma$) which is chosen such that the mean kinetic energy of the common steady state is lower than the initial mean kinetic energies of $P$ and $Q$ while keeping all the other parameters of both the systems constant.

If this system shows Mpemba effect, then the trajectories $E^P(t)$ and $E^Q(t)$ must cross each other, such that there is a time $t = \tau$ at which

$$E^P(\tau) = E^Q(\tau).$$

Substituting into Eq. (14), we obtain relation:

$$K^P_+ e^{-\lambda_+ \tau} + K^P_- e^{-\lambda_- \tau} = K^Q_+ e^{-\lambda_+ \tau} + K^Q_- e^{-\lambda_- \tau},$$

whose solution is

$$\tau = \frac{1}{\lambda_+ - \lambda_-} \ln \left[ \frac{K^P_+ - K^Q_+}{K^P_- - K^Q_-} \right],$$

which in terms of the initial conditions reduce to

$$\tau = \frac{1}{\lambda_+ - \lambda_-} \ln \left[ \frac{R_{12} \Delta C - (\lambda_- - R_{11}) \Delta E}{R_{12} \Delta C - (\lambda_+ - R_{11}) \Delta E} \right],$$

where

$$\Delta E = E^P(0) - E^Q(0),$$
$$\Delta C = C^P(0) - C^Q(0).$$

For Mpemba effect to be present, we require that $\tau > 0$. Since $\lambda_+ > \lambda_-$, the argument of logarithm in Eq. (23) should be greater than one. We immediately obtain the criterion

$$(\lambda_+ - R_{11}) \Delta E < R_{12} \Delta C.$$  \quad (26)

Note that $R_{12} < 0$, and $\lambda_+ > R_{11}$ [see Eq. (12)]. Since $\Delta E > 0$ by definition, we conclude that Eq. (26) can be satisfied only if $\Delta C < 0$, i.e., the two point velocity-correlation of the hotter initial system $C^P(0)$ is sufficiently smaller than that of the cooler counterpart $C^Q(0)$. Note that if the two systems $P$ and $Q$ were initially in a steady state, then in the thermodynamic limit the correlations vanish, i.e., $\Delta C = O(1/N)$, and the inequality in Eq. (26) cannot be satisfied. Thus, for Mpemba effect to be present in the mono-dispersed gas, the initial condition of the cooler component must be different from the steady state.

In Fig. 1, we show the time evolution of the energies of the two systems for a choice of initial conditions which satisfy Eq. (26). Though the initial energy of $P$ is larger, it relaxes to the steady state faster than $Q$. The trajectories cross at the point as predicted by Eq. (23).

Keeping all other parameters fixed, and allowing only the coefficient of restitution to vary, we can identify the region of phase space (initial condition) where Mpemba effect is observable, based on Eq. (26). This is shown in Fig. 2. Clearly, as $\tau$ decreases to zero, the correlations need to be large for Mpemba effect to be present.
FIG. 2. The $\Delta E/\Delta C - r$ phase diagram showing regions where Mpemba effect is observed for mono-dispersed Maxwell gas (see Sec. IV), where $r$ is the coefficient of restitution. All other parameters are kept constant. The region below the critical line show Mpemba effect.

V. MPENBA EFFECT IN BI-DISPERSED MAXWELL GAS

In Sec. IV, we discussed the possibility of Mpemba effect in mono-dispersed gas. Mpemba effect was present provided we allowed the initial states to be different from the steady states at that corresponding temperature. We now generalize the analysis to bi-dispersed gases, based on an analysis of Eq. (16), and show the presence of Mpemba effect even when the initial states are restricted to steady states.

In a bi-dispersed gas, the temperatures of the two components are generally different [see Appendix A.2]. We denote them by $E_A$ and $E_B$. We denote the total kinetic energy of the system by $E_{tot}$, where

$$E_{tot} = E_A + E_B,$$

and the difference in energies by $E_d$:

$$E_d = E_A - E_B.$$  

We will define Mpemba effect in bi-dispersed gases similar to that for mono-dispersed gases. We consider two systems $P$ and $Q$ where $E_{tot}$ of $P$ is larger. Both $P$ and $Q$ are initially in steady states. We then quench both systems to a lower temperature and will say that Mpemba effect is present when the trajectories for $E_{tot}$ for $P$ and $Q$ cross each other.

We will consider the cases when only one component is driven and both components are driven separately. This is because there are many experiments where only one component is driven [30–35], and others where both components are driven [36–41]. The respective analysis may be found in Secs. V A and V B.

A. One component is driven

Consider a bi-dispersed driven Maxwell gas where only component $A$, is driven with driving strength $\sigma$. The time evolution equation [see Eq. (16)] for the correlation functions can be expressed in terms of $E_{tot}$ and $E_d$ as

$$\frac{dE(t)}{dt} = -\chi E(t) + D,$$  

where

$$E(t) = [E_{tot}(t), E_d(t)]^T,$$

$$D = [\lambda_d\sigma^2, \lambda_d\sigma^2]^T,$$

and $\chi$ is a $2 \times 2$ matrix with components $\chi_{11}, \chi_{12}, \chi_{21}$ and $\chi_{22}$ as given in Eq. (B2). The details of the calculation are shown in Appendix B. Equation (29) can be solved exactly by linear decomposition using the eigenvalues $\lambda_\pm$ of $\chi$:

$$\lambda_\pm = \frac{1}{2} \left( \chi_{11} + \chi_{22} \pm \sqrt{(\chi_{11} - \chi_{22})^2 + 4\chi_{12}\chi_{21}} \right).$$

It is straightforward to show that $\lambda_+ > 0$ with $\lambda_+ > \lambda_-$. The solution for $E_{tot}(t)$ and $E_d(t)$ is

$$E_{tot}(t) = K_+ e^{-\lambda_+ t} + K_- e^{-\lambda_- t} + K_0,$$

$$E_d(t) = L_+ e^{-\lambda_+ t} + L_- e^{-\lambda_- t} + L_0,$$  

where the coefficients $K_+, K_-, K_0, L_+, L_-$ and $L_0$ are as given in Eq. (B4).

We now consider two systems labeled as $P$ and $Q$ with different initial conditions ($E_{tot}(0), E_d(0)$) and ($E_{tot}(0), E_d(0)$) where $E_{tot}(0) > E_{tot}(0)$. Both the systems are quenched to a common steady state whose total energy is smaller than the initial total energies of $P$ and $Q$. This is achieved when the systems $P$ and $Q$ are now driven with the same driving strength ($\sigma$) for the component $A$, as well as keeping all the other parameters same for both the systems.

Mpemba effect is present when the two trajectories $E_{tot}^P(t)$ and $E_{tot}^Q(t)$ cross each other at some finite time $t = \tau$ at which

$$E_{tot}^P(\tau) = E_{tot}^Q(\tau).$$

Substituting into Eq. (33), we obtain

$$K_+^P e^{-\lambda_+ \tau} + K_-^P e^{-\lambda_- \tau} = K_+^Q e^{-\lambda_+ \tau} + K_-^Q e^{-\lambda_- \tau},$$

whose solution is

$$\tau = \frac{1}{\lambda_+ - \lambda_-} \ln \left[ \frac{K_+^P - K_-^Q}{K_-^P - K_+^Q} \right],$$

which in terms of the initial conditions reduce to

$$\tau = \frac{1}{\lambda_+ - \lambda_-} \ln \left[ \frac{\chi_{12}\Delta E_d - (\lambda_+ - \chi_{11})\Delta E_{tot}}{\chi_{12}\Delta E_d - (\lambda_+ - \chi_{11})\Delta E_{tot}} \right].$$
where

\[ \Delta E_{\text{tot}} = E_{\text{tot}}^P(0) - E_{\text{tot}}^Q(0), \]

\[ \Delta E_d = E_d^P(0) - E_d^Q(0). \]  

(38)

Following the same argument as for the case of mono-dispersed Maxwell gas in Sec. IV, Eq. (37) leads to the criterion for the crossing of the two trajectories as

\[ \chi_{12}\Delta E_d > (\lambda_+ - \chi_{11})\Delta E_{\text{tot}}. \]  

(39)

In Fig. 3, we consider such a situation where Eq. (39) is satisfied. The trajectories cross at the point as predicted by Eq. (37). It is clear that though \( P \) has larger initial energy, it relaxes faster.

In Fig. 4, we identify the region of phase space (initial condition) where Mpemba effect is observable, based on Eq. (39). In the figure, the variation with \( r_{AB} \) is shown. The region below the line in the phase diagram [see Fig. 4] show Mpemba effect whereas the other region does not show the effect.

In the above analysis, the systems \( P \) and \( Q \) have the same parameters once the quench is done. However, in the initial states, the parameters – reaction rates, coefficients of restitution, driving strength – could be different for \( P \) and \( Q \). These parameters, though not explicitly mentioned, enter through the initial values \( E_{\text{tot}} \) and \( E_d \). As a result, one can tune the parameters appropriately to obtain initial steady states that satisfy the condition given by Eq. (39) and hence show Mpemba effect.

We now ask a more refined question. Suppose, \( P \) and \( Q \) have the same parameters throughout (initially, as well as after the quench) except for the driving strength which is different initially and the same after the quench. We now analyze whether Mpemba effect will be present for this case, when only component \( A \) is driven? The condition for Mpemba effect to be present is the same as that derived for the more general case [see Eq. (39)]. However, when all parameters other than driving strength is kept the same, the ratio \( \Delta E_{\text{tot}}/\Delta E_d \) for initial steady states has a simple form:

\[ \frac{\Delta E_{\text{tot}}}{\Delta E_d} = \frac{1 + \lambda_{AB}\nu_AX_A^2Q}{1 - \lambda_{AB}\nu_AX_A^2Q} > 1, \]  

(40)

where \( Q \) is as defined in equation (A13). Note that \( Q > 0 \) and hence the ratio in Eq. (40) is always larger than one. On the other hand, the ratio \( \Delta E_{\text{tot}}/\Delta E_d \) should always be less than \( \chi_{12}/(\lambda_+ - \chi_{11}) \) for one to observe Mpemba effect and it can be shown that the maximum value of the quantity is one. Thus, Eq. (40) does not satisfy the required condition for the existence of Mpemba effect.

In this subsection, we discussed the possibility of Mpemba effect in bi-dispersed Maxwell gas where only one component is driven. We showed that, as compared to the mono-dispersed gas, for initial states that correspond to steady states, the bi-dispersed gas shows Mpemba effect for a wide range of parameters. However, when the two systems \( P \) and \( Q \) are identical except for the driving strength, then we showed that Mpemba effect is not possible for steady state initial conditions. We now generalize the calculations to a bi-dispersed gas where both components are driven, and show that even for systems that differ only by the driving strength, Mpemba effect can be observed.
Both components are driven

We consider bi-dispersed Maxwell gas where both the components of the gas are driven. Here, type $A$ and $B$ particles of the bi-dispersed Maxwell gas are driven at the same rate $\lambda_d$ but with different driving strengths $\sigma_A$ and $\sigma_B$ respectively. The time evolution of the quantities $E_{tot}$ and $E_d$ are given by Eq. (29) but now the column matrix $D$ takes the form

$$D = [\lambda_d(\sigma_A^2 + \sigma_B^2), \lambda_d(\sigma_A^2 - \sigma_B^2)]^T,$$

(41)

The solutions for $E_{tot}(t)$ and $E_d(t)$ are obtained in the similar way as in Eq. (33) but the coefficients $K_+, K_-, K_0, L_+, L_-$ and $L_0$ are given by Eq. (B6).

Here, the main aim is to look for the existence of Mpemba effect by considering two identical systems which are similar in their parameter space except for the different driving strengths for components of the bi-dispersed gas. We consider two such identical systems $P$ and $Q$ driven with two different pairs of noise strengths for the components $A$ and $B$ of the bi-dispersed gas such that the two systems attain different steady states with different total energy.

Let system $P$ has higher initial total energy compared to $Q$. Both the systems are driven to a common steady state having lower energy compared to the initial steady state energies of system $P$ and $Q$. This is achieved when the systems $P$ and $Q$ are driven with the same driving strengths ($\sigma_A$ and $\sigma_B$) for the components $A$ and $B$ of the bi-dispersed gas and keeping all the other parameters of both the systems constant.

The cross-over time $\tau$ for the crossing of the trajectories of $E_{tot}^P(t)$ and $E_{tot}^Q(t)$ is obtained using Eq. (37) and the criterion for the occurrence of Mpemba effect is given by Eq. (39). An example of such a crossing is shown in Fig. 5. However, the ratio $\Delta E_{tot}/\Delta E_d$ depend only on the initial steady state energies of the systems as can be seen from Eq. (38) and has a form given by Eq. (C1). These are the accessible steady state values for the ratio $\Delta E_{tot}/\Delta E_d$.

In Fig. 6, we obtain the accessible steady states of the system [given by Eq. (C1)] as a function of $r_{AB}$ while keeping all other parameters as constant. We show that the accessible steady states belong to the region of phase space (initial condition) given by Eq. (39) where Mpemba effect is observable and hence show Mpemba effect.

Note that the ratio $\Delta E_{tot}/\Delta E_d$ is also a function of the driving strengths, $\sigma_A$ and $\sigma_B$ [see Eq. (C1)]. Therefore one can tune the driving strengths appropriately to access the entire region of phase space (initial condition) in which Mpemba effect is observable.

Inverse Mpemba effect

Inverse Mpemba effect refers to the phenomenon that, when quenched to a high temperature, an initially colder system heats faster than a system at intermediate temperature. The analysis for showing Mpemba effect in Sec. VB can be generalized to show inverse Mpemba effect in the driven binary gas when both components are driven. We prepare two systems $P$ and $Q$ in steady states such that the total energy of $P$ is larger than that of $Q$. Both the systems are quenched, using suitable driv-
such that the components are driven with initial steady state conditions: two systems $P$ and $Q$ of bi-dispersed Maxwell gas where both energy compared to the initial energies of both $P$ and $Q$. The cross-over time $τ$ for the crossing of the trajectories of $E_{\text{tot}}^P(t)$ and $E_{\text{tot}}^Q(t)$ is obtained using Eq. (37) and the condition for the occurrence of inverse Mpemba effect is given by Eq. (39). An example of such a crossing is shown in Fig. 7.

The accessible steady states of the system that satisfy the condition for inverse Mpemba effect turns out to be the same as that for direct Mpemba effect and can be obtained using Eq. (C1). Thus Fig. 6 also illustrates that the valid steady states of the system given by Eq. (C1) belong to the region of phase space (initial condition) given by Eq. (39) where inverse Mpemba effect is observable.

D. Strong Mpemba effect

We now explore the possibility of strong Mpemba effect in the binary Maxwell gas. Strong Mpemba effect refers to the phenomenon wherein the system at higher temperature relaxes to a final steady state exponentially faster, namely with a larger exponential rate compared to other initial conditions. Up to now, we had only considered the case where the trajectories cross, which in general does not imply that the decay rate at large times will be different. The linear evolution equation in Eq (16) allows certain set of initial conditions to relax to the final steady state exponentially faster compared to other initial states. The effect may be realized when the coefficient $K_-$ associated with the slower relaxation rate in the time evolution of total kinetic energy, $E_{\text{tot}}(t)$ [see Eq. 33] vanishes. In what follows, we would like to probe the system of bi-dispersed gas with both type of particles driven to look for the presence of the strong Mpemba effect.

Setting the coefficient $K_-$ [given by Eq. (B6)] to zero, we obtain

$$E_{\text{tot}}(0) = \frac{\chi_{12}}{\lambda_+ - \chi_{11}} E_d(0) - c,$$

where

$$c = \frac{\lambda_+}{\lambda_-} \left[(\chi_{12} - \lambda_+ + \chi_{11}) \sigma_A^2 - (\chi_{12} + \lambda_+ - \chi_{11}) \sigma_B^2 \right].$$

For a system with all other parameters kept fixed, solution of Eq. (42) in terms of $E_{\text{tot}}(0)$ and $E_d(0)$ provides the set of initial states whose relaxation is exponentially faster than the set of generic states. Note that the set of initial states that satisfy Eq. (42) lie on a straight line.

Among these initial states one would like to determine the ones which are steady states. Remember that the steady state ratio of $E_{\text{tot}}(0)/E_d(0)$ for a system is given by

$$\frac{E_{\text{tot}}(0)}{E_d(0)} = f(\sigma_A, \sigma_B),$$

where $f(\sigma_A, \sigma_B)$ is given by Eq. (C2) and is only a function of driving strengths ($\sigma_A$ and $\sigma_B$) as all other parameters are kept constant. One observes that valid steady states with initial energies, $E_{\text{tot}}(0)$ and $E_d(0)$ that satisfy the condition for strong Mpemba effect [see Eq. (42)] can be obtained by appropriately tuning the driving strengths $\sigma_A$ and $\sigma_B$.

Thus for a system of bi-dispersed Maxwell gas where both components are driven, there exists steady state initial conditions which satisfy the condition given by Eq. (42) and hence approach the final steady state exponentially faster compared to any other similar system whose initial energies lie slightly below or above the line. An example of strong Mpemba effect is shown in Fig. 8. The figure shows the evolution of the total energy $E_{\text{tot}}$ as a function of time $t$ for two bi-dispersed systems $P$ and $Q$ that have identical parameters except for the initial driving strength after being quenched to a lower temperature by changing the driving strength to a common lower value. The initial state of the system $P$ is chosen in such a way that $[E_{\text{tot}}(0), E_d(0)]$ satisfies the condition for the strong Mpemba effect [Eq. (42)], and hence evolve to the final state with a single faster relaxation rate. On the other hand, system $Q$, with its initial state not satisfying the condition for the strong Mpemba effect, relaxes differently, and asymptotically evolves with the slower rate. Further, as the initial condition of the two systems $P$ and $Q$ happen to satisfy the relation for the existence of Mpemba effect [Eq. (39)], the trajectory of
the system $P$ with higher initial energy crosses that of $Q$ with lower initial energy. The crossing time $\tau$ could be obtained using Eq. (37) which is captured in the inset of the figure.

VI. CONCLUSION

In summary, through an exact analysis of the driven mono-dispersed and bi-dispersed Maxwell gases, we derived the conditions under which the Mpemba effect, the inverse Mpemba and the strong Mpemba effects can be observed. In Maxwell gas, the rate of collision between particles is independent of the relative velocity. In addition, the well-mixed limit is assumed such that spatial correlations are ignored. The equations for the two point correlations close among themselves resulting in a coupled set of linear equations allowing for an exact solution. This linearity happens to be natural to the model and thus does not require any approximations that has been employed in models where the collision rates are velocity dependent. To demonstrate the existence of Mpemba effect, we determine the conditions under which a hotter system relaxes faster than a cooler system when quenched to a temperature lower than both. For the case of mono-dispersed Maxwell gas, we showed that Mpemba effect is possible only if the initial states do not correspond to steady states. On the other hand, for bi-dispersed Maxwell gas, there is a range of parameters for which Mpemba effect exists, even when the states from which the quench is done are restricted to valid steady states. In a similar framework, we also demonstrated the existence of inverse Mpemba effect where a system is heated instead of cooled, i.e., a system at a lower initial temperature relaxes to a high temperature state faster than another system with an intermediate initial temperature. We also show the presence of a stronger version of Mpemba effect where a system equilibrates to a final steady state at an exponentially faster rate.

The exact analysis allows us to identify the reason behind the Mpemba effect in these systems. The state of the system is defined by two quantities for the mono-dispersed gas (energy and correlation function). In this three dimensional space (third dimension being time), two trajectories cannot cross. However, when projected onto the lower dimensional energy-time plane, trajectories may cross, leading to Mpemba effect. For bi-dispersed gases, there are two kinetic energies and three correlation functions. Since the energy-time dimensions are itself three dimensional, it is possible to observe Mpemba effect when the correlations are set to zero, as in a true steady state.

The results are analogous to those found for most realistic granular systems, but using perturbative calculations [20, 21]. In these calculations, for Mpemba effect to be seen in mono-dispersed systems, the initial conditions do not correspond to steady states, but rather to some initial states that are close to final steady states. This is achieved by choosing appropriate initial velocity distribution functions for the two systems. For the case of rough granular gas, calculations were carried out by considering states with Gaussian velocity distribution at all times which may not hold good for non-equilibrium systems. But the analysis in the present work does not make any assumption regarding the nature of velocity distribution in the steady state of the granular system. Therefore, extending the calculations of Refs. [20, 21] to smooth bi-dispersed gas would be a promising area for future study. One can also search for an exponentially faster relaxation protocols in these systems as studied in Refs. [15, 18]. Also the results in this paper, particularly the case of bi-dispersed Maxwell gas where both components are driven, suggest that driven binary gases are a good candidate for observation of Mpemba effect in granular experiments.

Further, in the aim of demonstrating the existence of the Mpemba effect in Maxwell gases, we have assumed that energies evolve monotonically, by not explicitly accounting for the possibility that when a hot system is quenched to a lower temperature, the temperature could drop below the final temperature. In order to include scenarios with non-monotonic evolution of energy into the present framework, one may have to look at the behavior of the absolute values of the coefficients $K_\alpha$. [17]. This extended case may be addressed in future, to the present and other models of granular gases. One may also look at the relation between the existence of such non-monotonic relaxation and the strong Mpemba effect, as the presence of non-monotonicity indicate a change in sign.
of the coefficient $K_-$.

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Appendix A: Time evolution for two point correlations

In this appendix, we solve for the time evolution equations of two-point correlations as defined in sections IIIA and IIB for the case of mono-dispersed and bi-dispersed Maxwell granular gas.

1. Mono-dispersed gas

For the mono-dispersed gas, the time evolution of mean kinetic energy, $E(t)$ and the two point velocity-velocity correlation function, $C(t)$ are given as in Eq. (14). The coefficients $K_+, K_-, K_0, L_+, L_-$ and $L_0$ in Eq. (14) are given by

\[ K_+ = \frac{1}{\gamma} \left[ (\lambda_+ - R_{11})\Sigma_2(0) - R_{12}\Sigma_2(0) - \frac{(\lambda_+ - R_{11})}{\lambda_+} \lambda_d \sigma^2 \right], \]

\[ K_- = \frac{1}{\gamma} \left[ (\lambda_+ - R_{11})\Sigma_1(0) + R_{12}\Sigma_2(0) + \frac{(\lambda_+ - R_{11})}{\lambda_+} \lambda_d \sigma^2 \right], \]

\[ K_0 = \frac{1}{\gamma} \left[ \lambda_+ - R_{11} \right] \lambda_d \sigma^2, \]

\[ L_+ = \frac{1}{\gamma} \left[ (\lambda_+ - R_{11})(\lambda_- - R_{11})\Sigma_1(0) - (\lambda_+ - R_{11})\Sigma_2(0) - \frac{(\lambda_+ - R_{11})(\lambda_- - R_{11})}{R_{12}\lambda_+} \lambda_d \sigma^2 \right], \]

\[ L_- = \frac{1}{\gamma} \left[ \frac{-(\lambda_+ - R_{11})(\lambda_- - R_{11})}{R_{12}} \Sigma_1(0) + (\lambda_+ - R_{11})\Sigma_2(0) + \frac{(\lambda_+ - R_{11})(\lambda_- - R_{11})}{R_{12}\lambda_-} \lambda_d \sigma^2 \right], \]

\[ \gamma = \lambda_+ - \lambda_. \]

2. Bi-dispersed gas

For the case of bi-dispersed Maxwell gas, the two point correlations are defined as in Eq. (15). First, we consider the general case where both components of the gas are driven. The results can be extended for one component driven bi-dispersed gas. We can write for the time evolution of the two point correlations using Eq. (5) in a compact matrix form as

\[ \frac{d\Sigma(t)}{dt} = R \Sigma(t) + D, \tag{A2} \]

where

\[ \Sigma(t) = [E_A(t), E_B(t), C_{AB}(t), C_{AA}(t), C_{BB}(t)]^T, \]

\[ D = [\lambda_d \sigma^2_A, \lambda_d \sigma^2_B, 0, 0, 0]^T, \tag{A4} \]

where $k=A, B$ and the matrix $R$ is given by

\[ R = \begin{bmatrix} R_{2B} - R_{1A} - R_{3B} - R_d & R_{2B} - R_{2A} - R_{3A} & -2R_{2B} + R_{3B} & 2R_{1A} & 0 \\ R_{2A} - R_{1B} - R_{3A} & R_{2A} - R_{1B} - R_{3A} & -2R_{2A} + R_{3A} & 0 & 0 \\ \frac{R_{3B}}{2N_A - 1} - R_4 & \frac{R_{3B}}{2N_A - 1} & \frac{2R_{3A}}{2N_A} & \frac{2R_{3A}}{2N_A} & -\frac{R_{1B}}{2N_B + 1} \\ \frac{R_{3B}}{N_B - 1} & \frac{R_{3B}}{N_B - 1} & \frac{R_{3B}}{2N_B} & \frac{R_{3B}}{2N_B} & -\frac{R_{1B}}{2N_B + 1} \\ R_4 & R_d & \lambda_{AB} X_A X_B & \lambda_{AB} X_A X_B & \frac{2R_{1A}}{2N_B} \end{bmatrix}. \tag{A5} \]

The constants $R_{1k}, R_{2k}, R_{3k}, R_4, R_d$ are given by:

\[ R_{1k} = \frac{\lambda_{kk}(1 - \alpha_{kk})(N_k - 1)}{N}, \quad R_{2k} = \frac{\lambda_{AB} N_k X_k^2}{N}, \]

\[ R_{3k} = \frac{2R_{2k}}{X_k}, \quad R_4 = \frac{\lambda_{AB} X_A X_B}{N}, \quad R_d = \lambda_d(1 - r_w^2). \]

In the steady state, in the thermodynamic limit, the inter-particle two point correlation functions $C_{ij}$, where $i, j \in (A, B)$ are zero, as shown in Ref. [26]. If in the initial state, these correlations are zero, then it remains zero for all times. We will be only considering such initial
where the quantities \( E \) in Sections B 1 and B 2 respectively.

In this appendix, we solve for the time evolution of the quantities \( E_{\text{tot}} \) and \( E_d \) for the case of bi-dispersed Maxwell gas. We consider two cases: when only one component of the gas is driven and another case when both the components of the gas are driven as described in Sections B 1 and B 2 respectively.

One can do similar calculations to solve for the steady state mean kinetic energies, for the case where only one component (say \( A \)) is driven with driving strength (\( \sigma \)) and at a rate (\( \lambda_d \)). In that case, the mean kinetic energies of the components are given by

\[
E_A = \frac{-\lambda_d \sigma^2 \left( -\lambda_d(1 - r_w^2) - 2\lambda_{AB} \nu_{B} X_B^2 - \lambda_{AA} \nu_{A} X_A^2 - \lambda_{BB} \nu_{B}(1 - \alpha_{BB}) \nu_{B} \right) + \lambda_{AB} \lambda_{BB} \nu_{B} X_B^2 \sigma_B^2}{\mathcal{F}},
\]

\[
E_B = \frac{-\lambda_{AB} \lambda_{BB} \nu_{B} \nu_{A} X_A^2 \sigma_A^2 - \lambda_{BB} \sigma_B^2 \left( -\lambda_A(1 - r_w^2) - 2\lambda_{AB} \nu_{B} X_B + \lambda_{BB} \nu_{B} X_B^2 - \lambda_{AA} \nu_{A}(1 - \alpha_{AA}) \nu_{A} \right) - \lambda_{AB} \nu_{A} X_A^2 - \lambda_{BB} \nu_{B}(1 - \alpha_{BB}) \nu_{BB} \right) + \lambda_{AB} \lambda_{BB} \nu_{B} X_B^2 \sigma_B^2}{\mathcal{F}},
\]

where

\[
\mathcal{F} = -\lambda_{AB} \nu_{A} X_A^2 X_B + \left( -\lambda_d(1 - r_w^2) - 2\lambda_{AB} \nu_{B} X_B + \lambda_{BB} \nu_{B} X_B^2 - \lambda_{AA} \nu_{A}(1 - \alpha_{AA}) \nu_{A} \right)
\]

\[
\left( -\lambda_d(1 - r_w^2) - 2\lambda_{AB} \nu_{A} X_A + \lambda_{BB} \nu_{B} X_B^2 - \lambda_{BB} \nu_{B}(1 - \alpha_{BB}) \nu_{BB} \right) + \lambda_{AB} \lambda_{BB} \nu_{B} X_B^2 \sigma_B^2.
\]

### Appendix B: Time evolution of \( E_{\text{tot}} \) and \( E_d \): Bi-dispersed Maxwell gas

1. **One component is driven**

We consider a bi-dispersed Maxwell gas where only one component, say \( A \), is driven with a rate, \( \lambda_d \) and with a driving strength, \( \sigma \). Using the definitions of \( E_{\text{tot}} \) and \( E_d \) as given in Eqs. (27) and (28) respectively, we can write for the time evolution of these quantities following Eq. (16) as

\[
\frac{dE_{\text{tot}}}{dt} = \chi_{11} E_{\text{tot}} + \chi_{12} E_d + \lambda_d \sigma^2,
\]

\[
\frac{dE_d}{dt} = \chi_{21} E_{\text{tot}} + \chi_{22} E_d + \lambda_d \sigma^2.
\]
where
\[\begin{align*}
\chi_{11} &= -\frac{R_{11} + R_{12} + R_{21} + R_{22}}{2}, \\
\chi_{12} &= -\frac{R_{11} - R_{12} + R_{21} - R_{22}}{2}, \\
\chi_{21} &= -\frac{R_{11} + R_{12} - R_{21} - R_{22}}{2}, \\
\chi_{22} &= -\frac{R_{11} - R_{12} - R_{21} + R_{22}}{2}.
\end{align*}\] (B2)

Equation (B1) can be represented in a compact matrix form as in Eq. (29). The solution for the time evolution of \(E_{\text{tot}}\) and \(E_d\) are given by
\[
\begin{align*}
E_{\text{tot}}(t) &= K_+ e^{-\lambda_+ t} + K_- e^{-\lambda_- t} + K_0, \\
E_d(t) &= L_+ e^{-\lambda_+ t} + L_- e^{-\lambda_- t} + L_0,
\end{align*}
\] (B3)

where the coefficients \(K_+, K_-, K_0, L_+, L_-\) and \(L_0\) are given by
\[
\begin{align*}
K_+ &= \frac{1}{\gamma} \left[\left(-\lambda_+ + \chi_{11}\right)E_{\text{tot}}(0) + \chi_{12} E_d(0) - \frac{\chi_{12} - \lambda_+ + \chi_{11}}{\lambda_+} \lambda_d \sigma^2\right], \\
K_- &= \frac{1}{\gamma} \left[\left(\lambda_+ - \chi_{11}\right)E_{\text{tot}}(0) - \chi_{12} E_d(0) + \frac{\chi_{12} - \lambda_+ + \chi_{11}}{\lambda_-} \lambda_d \sigma^2\right], \\
K_0 &= \frac{1}{\gamma} \left[\frac{\chi_{12} - \lambda_+ + \chi_{11}}{\lambda_+} \lambda_d \sigma^2, \\
L_+ &= \frac{1}{\gamma} \left[\frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12}} E_{\text{tot}}(0) + \left(\lambda_+ - \chi_{11}\right)E_d(0) - \frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12} \lambda_+} \lambda_d \sigma^2\right], \\
L_- &= \frac{1}{\gamma} \left[\frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12}} E_{\text{tot}}(0) - \left(\lambda_+ - \chi_{11}\right)E_d(0) + \frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12} \lambda_-} \lambda_d \sigma^2\right],
\end{align*}
\] (B4)

2. Both components are driven

Here, we consider a bi-dispersed Maxwell gas where both the components, \(A\) and \(B\) are driven with driving strengths \(\sigma_A\) and \(\sigma_B\) respectively. One can follow the calculations for the case of one component driven bi-dispersed gas [see subsection B1] and write the time evolution of \(E_{\text{tot}}\) and \(E_d\) in a compact representation as in Eq. (29) but the column matrix \(D\) takes the form
\[
D = \left[\lambda_d \left(\sigma_A^2 + \sigma_B^2\right), \lambda_d \left(\sigma_A^2 - \sigma_B^2\right)\right]^T.
\] (B5)

The solution for \(E_{\text{tot}}(t)\) and \(E_d(t)\) are given by Eq. (B3) with the coefficients \(K_+, K_-, K_0, L_+, L_-\) and \(L_0\) in Eq. (B3) given by
\[
\begin{align*}
K_+ &= \frac{1}{\gamma} \left[\frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12}} E_{\text{tot}}(0) + \chi_{12} E_d(0) - \frac{\chi_{12} - \lambda_+ + \chi_{11}}{\lambda_+} \lambda_d \sigma^2\right], \\
K_- &= \frac{1}{\gamma} \left[\frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12}} E_{\text{tot}}(0) - \chi_{12} E_d(0) + \frac{\chi_{12} - \lambda_+ + \chi_{11}}{\lambda_-} \lambda_d \sigma^2\right], \\
K_0 &= \frac{\lambda_d}{\gamma} \left[\left(\chi_{12} - \lambda_+ + \chi_{11}\right)\sigma_A^2 - \left(\chi_{12} - \lambda_- + \chi_{11}\right)\sigma_B^2\right], \\
L_+ &= \frac{1}{\gamma} \left[\frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12}} E_{\text{tot}}(0) + \left(\lambda_+ - \chi_{11}\right)E_d(0) - \frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\lambda_+ \chi_{12}} \lambda_d \sigma^2\right], \\
L_- &= \frac{1}{\gamma} \left[\frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\chi_{12}} E_{\text{tot}}(0) - \left(\lambda_+ - \chi_{11}\right)E_d(0) + \frac{\left(\lambda_+ - \chi_{11}\right)\lambda_-}{\lambda_- \chi_{12}} \lambda_d \sigma^2\right],
\end{align*}
\] (B6)
Appendix C: Steady state expression for $\Delta E_{tot}/\Delta E_d$

In this appendix, we calculate the steady state expression for the ratio $\Delta E_{tot}/\Delta E_d$ for a bi-dispersed Maxwell gas where both the components, $A$ and $B$ are driven at a rate $\lambda_d$ and with driving strengths $\sigma_A$ and $\sigma_B$ respectively. The quantities $\Delta E_{tot}$ and $\Delta E_d$ are defined as in Eq. (38). Using the results for the steady state mean kinetic energies for components $A$ and $B$ [see Eqs. (A8) and (A9)], one can calculate the ratio $\Delta E_{tot}/\Delta E_d$ as

\[
\frac{\Delta E_{tot}}{\Delta E_d} = \frac{\lambda_{AB}\lambda_d \left[ \nu_B X_B^2 (\sigma_B^p)^2 - \nu_A X_A^2 (\sigma_A^p)^2 \right] - \lambda_d \left[ -\lambda_d (1 - r_A^p) - 2\lambda_{AB}\nu_A X_A + \lambda_{AB}\nu_A X_A^2 - \lambda_{BB}\nu_B \alpha_{BB} (1 - \alpha_{BB}) \right] (\sigma_A^p)^2}{\lambda_{AB}\lambda_d \left[ \nu_B X_B^2 (\sigma_B^p)^2 - \nu_A X_A^2 (\sigma_A^p)^2 \right] - \lambda_d \left[ -\lambda_d (1 - r_A^p) - 2\lambda_{AB}\nu_A X_A + \lambda_{AB}\nu_A X_A^2 - \lambda_{AB}\nu_A \alpha_{AA} (1 - \alpha_{AA}) \right] (\sigma_A^p)^2}.
\]

(C1)

One can also obtain the ratio of $E_{tot}(0)/E_d(0)$ for a system $(P)$ only using Eq. (C1) as

\[
\frac{E_{tot}^P}{E_d^P} = \frac{\lambda_{AB}\lambda_d \left[ \nu_B X_B^2 (\sigma_B^p)^2 + \nu_A X_A^2 (\sigma_A^p)^2 \right] - \lambda_d \left[ -\lambda_d (1 - r_A^p) - 2\lambda_{AB}\nu_A X_A + \lambda_{AB}\nu_A X_A^2 - \lambda_{BB}\nu_B \alpha_{BB} (1 - \alpha_{BB}) \right] (\sigma_A^p)^2}{\lambda_{AB}\lambda_d \left[ \nu_B X_B^2 (\sigma_B^p)^2 - \nu_A X_A^2 (\sigma_A^p)^2 \right] - \lambda_d \left[ -\lambda_d (1 - r_A^p) - 2\lambda_{AB}\nu_A X_A + \lambda_{AB}\nu_A X_A^2 - \lambda_{AB}\nu_A \alpha_{AA} (1 - \alpha_{AA}) \right] (\sigma_A^p)^2}.
\]

(C2)

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