Moduli of Admissible Pairs for Arbitrary Dimension, II: Functors and Moduli

N.V. Timofeeva

December 29, 2020

Abstract. Morphisms between the moduli functor of admissible semistable pairs and the Gieseker – Maruyama moduli functor (of semistable coherent torsion-free sheaves) with the same Hilbert polynomial on a non-singular N-dimensional projective algebraic variety, are constructed. It is shown that these functors are isomorphic, and the moduli scheme for semistable admissible pairs $((\tilde{S}, \tilde{L}), \tilde{E})$ is isomorphic to the Gieseker – Maruyama moduli scheme. The considerations involve all components of moduli functors and corresponding moduli scheme as they exist.

Bibliography: 17 items.

Keywords: moduli space, semistable coherent sheaves, semistable admissible pairs, vector bundles, N-dimensional algebraic variety.

1 Introduction

Present paper is a continuation of [1] where the standard resolution of a flat family of torsion-free coherent sheaves on a projective non-singular variety was developed.

We construct the scheme of moduli for semistable admissible pairs of arbitrary dimension. In the series of articles [2] – [10] the construction was done in dimensional 2: admissible schemes had dimension 2 and they were obtained from nonsingular projective algebraic surface $S$ by some transformation. The final result was an isomorphism between moduli functor of Gieseker-semistable torsion-free coherent sheaves of rank $r$ and Hilbert polynomial $rp(n)$ on the surface $S$ with fixed polarisation $L$, and moduli functor of semistable admissible pairs $((\tilde{S}, \tilde{L}), \tilde{E})$. Each pair consists of admissible scheme $\tilde{S}$ with distinguished polarisation $\tilde{L}$ and semistable locally free sheaf $\tilde{E}$ of rank $r$ and with Hilbert polynomial $rp(n)$. The advantage of such moduli functor is that its moduli scheme is isomorphic to Gieseker–Maruyama moduli scheme for same rank and Hilbert polynomial. In particular this can be interpreted as a approach to compactify the moduli space of stable vector bundles by vector bundles on some special (admissible) schemes instead of classical compactification by attaching non-locally free coherent sheaves. Compactifications
which do not involve non-bundles can be of use for extending of Kobayashi – Hitchin correspondence to compact case. Since Kobayashi – Hitchin correspondence holds not only in dimension 2, we extend the "locally free" compactification to arbitrary dimension. Kobayashi – Hitchin correspondence operates with coefficient field \( \mathbb{C} \) but we will work with arbitrary algebraically closed field \( k \) of zero characteristic because the isomorphism of functors is not subject to special properties of \( \mathbb{C} \).

Gieseker – Maruyama moduli schemes for sheaves on arbitrary non-singular algebraic variety are projective and they can consist of several connected components. Their components can have non-equal dimensions and can carry non-reduced scheme structures. Also some components can contain no locally free sheaves. These phenomena (reducibility, non-reducedness, non-equidimensionality and presence of non-bundle components) occur for some combinations of numerical invariants even in dimension 2 case.

Let \( S \) be a nonsingular (e.g. irreducible) projective algebraic variety over an algebraically closed field \( k \) of zero characteristic, \( \mathcal{O}_S \) its structure sheaf, \( \mathcal{E} \) coherent torsion-free \( \mathcal{O}_S \)-module, \( \mathcal{E}^\vee := \text{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S) \) its dual \( \mathcal{O}_S \)-module. A locally free sheaf and its corresponding vector bundle are canonically identified and both terms are used as synonyms.

Let \( L \) be very ample invertible sheaf on \( S \); it is fixed and is used as a polarisation. The symbol \( \chi(\cdot) \) denotes Euler – Poincaré characteristic of a coherent sheaf, \( c_i(\cdot) \) its \( i \)-th Chern class.

Now discuss notions and objects to be of use.

Recall the definition of a sheaf of 0-th Fitting ideals known from commutative algebra. Let \( X \) be a scheme, \( F \mathcal{O}_X \)-module of finite presentation

\[
\tilde{F}_1 \overset{\varphi}{\longrightarrow} \tilde{F}_0 \to F \to 0.
\]

Without loss of generality we assume that \( \text{rank} \tilde{F}_1 \geq \text{rank} \tilde{F}_0 \).

**Definition 1.** The sheaf of 0-th Fitting ideals of \( \mathcal{O}_X \)-module \( F \) is defined as \( \text{Fitt}^0 F = \text{im} (\Lambda^{\text{rank} \tilde{F}_0} \tilde{F}_1 \otimes \Lambda^{\text{rank} \tilde{F}_0} \tilde{F}_0^\vee \overset{\varphi'}{\longrightarrow} \mathcal{O}_X) \), where \( \varphi' \) is a morphism of \( \mathcal{O}_X \)-modules induced by \( \varphi \).

**Remark 1.** We will work with invertible sheaves \( L \) and \( \tilde{L} \) where expression for \( \tilde{L} \) involves appropriate tensor power of \( L \). In further considerations we replace \( L \) by its big enough tensor power, if necessary for \( \tilde{L} \) to be very ample. This power can be chosen uniform and fixed, as shown in [6] for the case \( \text{hd} E = 1 \). The same reasoning is true in arbitrary dimension of \( S \) and in arbitrary homological dimension of sheaves. All Hilbert polynomials are compute according to new \( L \) and \( \tilde{L} \) respectively.

First we remind the definition for \( S \)-stable and \( S \)-semistable pair for dimension 2 case. The analogous notions and whole of the construction for arbitrary dimension is natural generalisation of it. The essential condition is that \( S \) is necessarily non-singular.

**Definition 2.** \( S \)-stable (respectively, \( S \)-semistable) pair \((\tilde{S}, \tilde{L})\), \( E \)) for \( \text{dim} S = 2 \) is the following data:

- \( \tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i \) – admissible scheme, \( \sigma : \tilde{S} \to S \) morphism which is called canonical, \( \sigma_i : \tilde{S}_i \to S \) its restrictions on components \( \tilde{S}_i, i \geq 0 \);
- \( \tilde{E} \) vector bundle on the scheme \( \tilde{S} \);
- \( \tilde{L} \in \text{Pic} \tilde{S} \) distinguished polarisation;
such that

- $\chi(\tilde{E} \otimes \tilde{L}^n) = rp(n)$, the polynomial $p(n)$ and the rank $r$ of the sheaf $\tilde{E}$ are fixed;
- the sheaf $\tilde{E}$ on the scheme $\tilde{S}$ is stable (respectively, semistable) due to Gieseker,
- on each of additional components $\tilde{S}_i$, $i > 0$, the sheaf $\tilde{E}_i := \tilde{E}|_{\tilde{S}_i}$ is quasi-ideal, i.e. admits a description of the form

$$\tilde{E}_i = \sigma_i^* \ker q_0/tors_i,$$

for some $q_0 \in \bigcup_{i \leq c_2} \text{Quot} \bigoplus^n \mathcal{O}_S$.

The definition of the subsheaf $\text{tors}_i$ will be given below. Coefficients of the Hilbert polynomial $rp(n)$ depend on Chern classes. In particular, $c_2$ is 2nd Chern class of a sheaf with Hilbert polynomial equal to $rp(n)$.

Pairs $((\tilde{S}, \tilde{L}), \tilde{E})$ such that $(\tilde{S}, \tilde{L}) \cong (S, L)$ will be called $S$-pairs.

In the series of articles of the author [2] — [6] a projective algebraic scheme $\tilde{M}$ is built up as reduced moduli scheme of $S$-semistable admissible pairs and in [7] it is constructed as possibly non-reduced moduli space.

The scheme $\tilde{M}$ contains an open subscheme $\tilde{M}_0$ which is isomorphic to the subscheme $M_0$ of Gieseker-semistable vector bundles in the Gieseker – Maruyama moduli scheme $\mathcal{M}$ of torsion-free semistable sheaves whose Hilbert polynomial is equal to $\chi(E \otimes L^n) = rp(n)$.

Let $E$ be a semistable locally free sheaf. Then the sheaf $I = \mathcal{Fitt}^0 E xt^1(E, \mathcal{O}_S)$ is trivial and $\tilde{S} \cong S$. In this case $(\tilde{S}, \tilde{L}, \tilde{E}) \cong ((S, L), E)$ and we have a bijective correspondence $\tilde{M}_0 \cong M_0$. It is a scheme isomorphism.

Let $E$ be a semistable nonlocally free coherent sheaf; then the scheme $\tilde{S}$ contains reduced irreducible component $\tilde{S}_0$ such that the morphism $\sigma_0 := \sigma|_{\tilde{S}_0} : \tilde{S}_0 \to S$ is a morphism of blowing up of the scheme $S$ in the sheaf of ideals $I = \mathcal{Fitt}^0 E xt^1(E, \mathcal{O}_S)$. Formation of a sheaf $I$ is an approach to the characterisation of singularities of the sheaf $E$ i.e. its difference from a locally free sheaf. Indeed, the quotient sheaf $\pi := E^{\vee \vee}/E$ is Artinian of length not greater then $c_2 = c_2(E)$, and $E xt^1(E, \mathcal{O}_S) \cong E xt^2(\pi, \mathcal{O}_S)$. Then $\mathcal{Fitt}^0 E xt^2(\pi, \mathcal{O}_S)$ is a sheaf of ideals of (in general case non-reduced) subscheme $Z$ of bounded length [7] supported at finite set of points on the surface $S$. As it is shown in [5], others components $\tilde{S}_i, i > 0$ of the scheme $\tilde{S}$ in general case can carry non-reduced scheme structures.

Each semistable coherent torsion-free sheaf $E$ corresponds to a pair $((\tilde{S}, \tilde{L}), \tilde{E})$ where $(\tilde{S}, \tilde{L})$ defined as described.

Now we describe the construction of the subsheaf tors in $\bigcup_{i > 0} \tilde{S}_i$. Let $U$ be Zariski-open subset in one of components $\tilde{S}_i$, $i \geq 0$, and $\sigma^* E|_{\tilde{S}_i}(U)$ corresponding group of sections. This group is $\mathcal{O}_{\tilde{S}_i}(U)$-module. Sections $s \in \sigma^* E|_{\tilde{S}_i}(U)$ annihilated by prime ideals of positive codimensions in $\mathcal{O}_{\tilde{S}_i}(U)$, form a submodule in $\sigma^* E|_{\tilde{S}_i}(U)$. This submodule is denoted as $\text{tors}_{i}(U)$. The correspondence $U \mapsto \text{tors}_{i}(U)$ defines a subsheaf $\text{tors}_{i} \subset \sigma^* E|_{\tilde{S}_i}$. Note that associated primes of positive codimensions which annihilate sections $s \in \sigma^* E|_{\tilde{S}_i}(U)$, correspond to subschemes supported in the preimage $\sigma^{-1}(\text{Supp } x) = \bigcup_{i \geq 0} \tilde{S}_i$. Since by the construction the scheme $\tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i$ is connected [5], subsheaves $\text{tors}_{i}, i \geq 0$, allow to construct a subsheaf $\text{tors} \subset \sigma^* E$. The former subsheaf is defined as follows. A section $s \in \sigma^* E|_{\tilde{S}_i}(U)$ satisfies the condition $s \in \text{tors}|_{\tilde{S}_i}(U)$ if and only if
• there exist a section \( y \in \mathcal{O}_S(U) \) such that \( ys = 0 \),
• at least one of the following two conditions is satisfied: either \( y \in \mathfrak{p} \), where \( \mathfrak{p} \) is prime ideal of positive codimension; or there exist Zariski-open subset \( V \subset \tilde{S} \) and a section \( s' \in \sigma^*E(V) \) such that \( V \supset U \), \( s'|_U = s \), and \( s'|_{V \cap \tilde{S}_0} \in \text{tors}(\sigma^*E|_{\tilde{S}_0})(V \cap \tilde{S}_0) \). In the former expression the torsion subsheaf \( \text{tors}(\sigma^*E|_{\tilde{S}_0}) \) is understood in usual sense.

The role of the subsheaf \( \text{tors} \subset \sigma^*E \) in our construction is analogous to the role of torsion subsheaf in the case of reduced and irreducible base scheme. Since no confusion occur, the symbol tors is understood everywhere in described sense. The subsheaf tors is called a torsion subsheaf (in modified sense).

In [6] it is proven that sheaves \( \sigma^*E/\text{tors} \) are locally free. The sheaf \( \tilde{E} \) include in the pair \( ((\tilde{S}, \tilde{L}), \tilde{E}) \) is defined by the formula \( \tilde{E} = \sigma^*E/\text{tors} \). In this circumstance there is an isomorphism \( H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}) \cong H^0(S, E \otimes L) \).

In the same article it was proven that the restriction of the sheaf \( \tilde{E} \) to each of components \( \tilde{S}_i, i > 0 \), is given by the quasi-ideality relation (1.1) where \( q_0 : \mathcal{O}_{\tilde{S}} \rightarrow \mathcal{X} \) is an epimorphism defined by the exact triple \( 0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow \mathcal{X} \rightarrow 0 \) in view of local freeness of the sheaf \( E^{\vee\vee} \).

Resolution of singularities of a semistable sheaf \( E \) can be globalised in a flat family by means of the construction developed in various versions in [7]. Let \( T \) be a scheme, \( \mathbb{E} \) a sheaf of \( \mathcal{O}_{T \times S} \)-modules, \( L \) invertible \( \mathcal{O}_{T \times S} \)-sheaf very ample relative to \( T \) and such that \( L_{|T \times S} = L \), and \( \chi(\mathbb{E} \otimes L^n_{|T \times S}) = rp(n) \) for all closed points \( t \in T \). \( \mathbb{E} \) and \( L \) were assumed to be flat relative to \( T \) and \( T \) were assumed to contain nonempty open subset \( T_0 \) such that \( \mathbb{E}|_{T_0 \times S} \) is locally free \( \mathcal{O}_{T_0 \times S} \)-module. Then following objects were defined:

- \( \pi : \tilde{\Sigma} \rightarrow \tilde{T} \) flat family of admissible schemes with invertible \( \mathcal{O}_{\tilde{S}} \)-module \( \tilde{L} \) such that \( \tilde{L}_{|T \times S} \) distinguished polarisation of the scheme \( \pi^{-1}(t) \),
- \( \tilde{\mathbb{E}} \) locally free \( \mathcal{O}_{\tilde{S}} \)-module and \( ((\pi^{-1}(t), \tilde{L}_{|\pi^{-1}(t)}), \tilde{\mathbb{E}}_{|\pi^{-1}(t)}) \) is \( S \)-semistable admissible pair.

In this situation there is a blowup morphism \( \Phi : \tilde{\Sigma} \rightarrow \tilde{T} \times S \) and the scheme \( \tilde{T} \) is birational to the initial base scheme \( T \). In [9] the procedure of standard resolution is modified so as \( \tilde{T} \cong T \).

The mechanism described was called a standard resolution.

In the present paper we admit that open subset \( T_0 \) such that \( \mathbb{E}|_{T_0 \times S} \) is locally free, can be empty.

In [11] the following result was proven.

**Theorem 1.** Let \( T \) be an algebraic scheme of finite type over \( k \), \( \Sigma = T \times S \), \( L \) (big enough) very ample invertible sheaf on \( S \), \( \mathbb{E} \) \( T \)-flat coherent \( \mathcal{O}_{\Sigma} \)-module of rank \( r \) and such that \( \mathbb{E}|_{T \times S} \) is torsion-free coherent sheaf, fibrewise Hilbert polynomial \( \chi(\mathbb{E}|_{T \times S} \otimes L^n) = rp(n) \) is independent of the choice of \( t \in T \). Then there are

- flat family \( \pi : \tilde{\Sigma} \rightarrow T \) together with a morphism \( \sigma : \tilde{\Sigma} \rightarrow \Sigma \) of \( T \)-schemes,
- invertible \( \mathcal{O}_{\tilde{S}} \)-sheaf \( \tilde{L} \),

such that

- \( \tilde{\mathbb{E}} = \sigma^*\mathbb{E}/\text{tors} \) is locally free \( \mathcal{O}_{\tilde{S}} \)-module,
• $\chi(\tilde L^n|_{\pi^{-1}(t)})$ is uniform over $t \in T$,
• $\chi(\tilde E \otimes \tilde L^n|_{\pi^{-1}(t)}) = rp(n)$.

The resolution described in [1] takes any coherent torsion-free $O_S$-sheaf $E$ on a polarised projective scheme $S$ to admissible pair of the form ($\tilde (S, \tilde L, \tilde E)$) which is a generalisation of one in dimension 2 (resp., homological dimension 1) case. Particularly, $\tilde S$ is admissible scheme; its structure depends on the structure of the sheaf $E$ under resolution. In the forthcoming paper we will enhance stability (semistability) notion for such pairs and provide functorial approach to their moduli scheme.

In the present article we apply results of [1] to prove the following theorem.

**Theorem 2.** (i) There is a natural transformation $\kappa : f^{GM} \to f$ of Gieseker–Maruyama moduli functor for non-singular projective algebraic variety $S$ to moduli functor of admissible semistable pairs with same rank and Hilbert polynomial.

(ii) There is a natural transformation $\tau : f \to f^{GM}$ of the moduli functor of admissible semistable pairs to Gieseker–Maruyama moduli functor for sheaves with same rank and Hilbert polynomial.

(iii) Natural transformations $\kappa$ and $\tau$ are mutually inverse. Hence both morphisms of non-reduced moduli functors $\kappa : f^{GM} \to f$ and $\tau : f \to f^{GM}$ are isomorphisms.

**Corollary 1.** The non-reduced moduli scheme $\tilde M$ for $f$ is isomorphic to the non-reduced Gieseker–Maruyama scheme $M$ for sheaves with same rank and Hilbert polynomial.

In section 2 we give an outline of multidimensional version of standard resolution developed in [1]. It leads to the functorial morphism $\kappa : f^{GM} \to f$ and proves part i) of the Theorem 2. Section 3 is devoted to the notion of (semi)stability of admissible pair and its relation to Gieseker-semistability of coherent sheaf $E$ when admissible pair arises from standard resolution of $E$. The class of semistable admissible pairs in dimension greater then 2 is much wider then one for dimension 2. In Section 4 we introduce and examine M-equivalence for admissible pairs and its relation to S-equivalence of coherent sheaves. Sections 5 and 6 are devoted to the natural transformation $\tau : f \to f^{GM}$ taking the functor $f$ of moduli for admissible pairs $((\pi : \tilde \Sigma \to T, \tilde L), \tilde E)$ with (possibly, non-reduced) base scheme $T$ to a family $E$ of coherent torsion-free semistable sheaves with the same base $T$. The transformation provides a morphism of the functor $\tau$ of admissible semistable pairs $f$ to Gieseker–Maruyama functor $f^{GM}$ and proves part ii) of Theorem 2. This completes the proof of Theorem 2. In section 7 we show that morphisms of functors $\kappa : f^{GM} \to f$ and $\tau : f \to f^{GM}$ we constructed are mutually inverse.

Principally the constructions done in this paper are very similar to [10]. But in details there are essential difference in details of argument caused by dimension, iteration of scheme morphism leading to reducible nonreduced structures, and wider classes of equivalent objects.

**Acknowledgements.** This work was carried out within the framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher
Education of the Russian Federation (Agreement No. 075-02-2020-1514/1 additional to the agreement on provision of subsidies from the federal budget No. 075-02-2020-1514).

2 Outline of the standard resolution and GM-to-Pairs transformation

The aim of this section is to adopt the transformation of the family \((T, \mathbb{L}, \mathbb{E})\) of semistable coherent torsion-free sheaves to the family \(((T, \pi : \tilde{\Sigma} \to T, \tilde{\mathbb{L}}), \mathbb{E})\) of admissible semistable pairs as it is done in previous papers, to \(N\)-dimensional ground variety \(S\). The initial family of sheaves \(\mathbb{E}\) can contain no locally free sheaves and codimension of singular locus \(\text{Sing}\mathbb{E}\) in \(\Sigma = T \times S\) is more than or equal to 2. Hence if the blowing up \(\sigma : \tilde{\Sigma} \to \Sigma\) of the sheaf of ideals \(\mathcal{F}itt^0\mathcal{E}xt^1(\mathbb{E}, \mathcal{O}_\Sigma)\) (or any other sheaf of ideals of some subscheme with support equal to the singular locus of \(\mathbb{E}\)) is considered then the fibres of the composite \(p \circ \sigma\) are not obliged to be equidimensional. Such a blowing up cannot produce family of admissible schemes. Equidimensionality is needed because admissible schemes must include in flat families.

To overcome this difficulty we perform the trick (base enhancing) done in [11]. Consider the product \(\Sigma' = \Sigma \times \mathbb{A}^1\) and fix a closed immersion \(i_0 : \Sigma \hookrightarrow \Sigma'\) which identifies \(\Sigma\) with zero fibre \(\Sigma \times 0\). Now let \(Z \subset \Sigma\) be a subscheme defined by the sheaf of ideals \(\mathbb{I} = \mathcal{F}itt^0\mathcal{E}xt^1(\mathbb{E}, \mathcal{O}_\Sigma)\). Then consider the sheaf of ideals \(\mathbb{I}' := \ker(\mathcal{O}_{\Sigma'} \to i_0^*\mathcal{O}_Z)\) and the blowup morphism \(\sigma' : \tilde{\Sigma}' \to \Sigma'\) defined by the sheaf \(\mathbb{I}'\). Denote the projection onto the product of factors \(\Sigma' \to T \times \mathbb{A}^1 =: T'\) by \(p'\) and the composite \(p' \circ \sigma'\) by \(\tilde{\pi}'\). We are interested in the induced morphism \(\pi : \Sigma := i_0(\Sigma) \times_{\Sigma'} \tilde{\Sigma}' \to T\). Under the identification \(\Sigma \cong i_0(\Sigma)\) we denote by \(\sigma\) the induced morphism \(\Sigma \to \Sigma\). Set \(\mathbb{L}' := \mathbb{L} \boxtimes \mathcal{O}_{\mathbb{A}^1}\). Obviously, there exists \(m \gg 0\) such that the invertible sheaf \(\tilde{\mathbb{L}}' := \sigma'^*\mathbb{L}'^m \otimes (\sigma'^{-1}\mathbb{I}') \cdot \mathcal{O}_\Sigma\) is ample relatively to \(\tilde{\pi}'\). For brevity of notations we fix this \(m\) and replace \(L\) by its \(m\)-th tensor product throughout further text. Denote \(\tilde{\mathbb{L}} := \tilde{\mathbb{L}}|_{\tilde{\Sigma}}\). The following proposition is formulated and proven analogously to Proposition 1 in [11] but has wider range of applicability.

**Proposition 1.** [11] Proposition 1/ The morphism \(\pi : \tilde{\Sigma} \to T\) is flat and fibrewise Hilbert polynomial compute with respect to \(\mathbb{L}\), i.e. \(\chi(\mathbb{L}^n|_{\pi^{-1}(t)})\), is uniform over \(t \in T\).

We denote \(\sigma := \sigma'|_{\tilde{\Sigma}} : \tilde{\Sigma} \to \Sigma\).

Let \(T\) be arbitrary (possibly nonreduced) \(k\)-scheme of finite type. We assume that its reduction \(T_{\text{red}}\) is irreducible. If \(S\) is a surface and \(\mathbb{E}\) is a family of coherent torsion-free sheaves with base \(T\) on \(S\) then homological dimension of \(\mathbb{E}\) as \(\mathcal{O}_{T \times S}\)-module is not greater then 1. In our case \(S\) has bigger dimension and we have to work with locally free resolution of higher length. Start with locally free resolution of the family of sheaves \(\mathbb{E}\) and cut the corresponding exact \(\mathcal{O}_\Sigma\)-sequence of length \(\ell\)

\[
0 \to \hat{E}_\ell \to \hat{E}_{\ell-1} \to \cdots \to \hat{E}_0 \to \mathbb{E} \to 0
\]

with locally free \(\mathcal{O}_\Sigma\)-modules \(\hat{E}_\ell, \ldots \hat{E}_0\), into triples:

\[
0 \to W_i \to \hat{E}_{i-1} \to W_{i-1} \to 0.
\]  

(2.1)

Here \(W_\ell = \hat{E}_\ell\), \(W_1 = \ker(\hat{E}_0 \to \mathbb{E})\) and \(W_i = \ker(\hat{E}_{i-1} \to \hat{E}_{i-2}) = \text{coker}(\hat{E}_{i+1} \to \hat{E}_i)\) for \(i = 2, \ldots, \ell - 1\). Also we keep in mind that all \(W_i\) except \(W_\ell\) are not locally free (otherwise
the resolution can be shorter). Since $S$ is assumed to be regular then $\mathbb{E}$ possesses a locally free resolution of length not bigger then $\dim S$. Also we need not only $1^{st}$ $\mathcal{E}xt$-module but all $\mathcal{E}xt^i(\mathbb{E}, \mathcal{O}_\Sigma)$, $i = 1, \ldots, \ell$.

We resolve singularities of the sheaf $\mathbb{E}$ by consequent blowing ups in sheaves of ideals $\mathbb{I}'_i = \mathbb{I}_i[t_i] + (t_i)$, $i = 1, \ldots, \ell$, for sheaves of ideals

$$
\mathbb{I}_1 = \mathcal{F}itl^0 \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma), \quad \mathbb{I}_2 = \mathcal{F}itl^0 \mathcal{E}xt^1(\sigma^*_1 W_{\ell-2}, \mathcal{O}_{\Sigma_1}), \\
\mathbb{I}_3 = \mathcal{F}itl^0 \mathcal{E}xt^1(\sigma^*_2 \sigma^*_1 W_{\ell-3}, \mathcal{O}_{\Sigma_2}), \ldots, \quad \mathbb{I}_\ell = \mathcal{F}itl^0 \mathcal{E}xt^0(\sigma^*_\ell-1 \ldots \sigma^*_1 W_0, \mathcal{O}_{\Sigma_\ell}).
$$

Each morphism $\sigma_i : \Sigma_i \to \Sigma_{i-1}$ is induced by the sheaf of ideals $\mathbb{I}_i$, $i = 1, \ldots, \ell$, and these morphisms form a sequence

$$
\tilde{\Sigma} := \Sigma_\ell \circ \ldots \circ \sigma_1 := \Sigma_0.
$$

Each segment (2.1) is resolved by the morphism $\sigma_{\ell-i+1}$ at $(\ell - i + 1)^{st}$ step of the process.

**Remark 2**. If the scheme $X = \Sigma_1$ has irreducible reduction then this lemma is applicable immediately and we conclude that $\text{hd} \sigma^*_1 \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma) = 1$. If $\Sigma_1$ has reducible reduction then there is a natural decomposition $\Sigma_1 = \Sigma^0_1 \sqcup \Sigma^1_1$ where $\Sigma^0_1 \cong \text{Proj} \bigoplus_{s \geq 0} \mathbb{I}^s_1$ is isomorphic to the scheme obtained by blowing up $\Sigma$ in the sheaf of ideals $\mathbb{I}_1$ but $\tilde{\Sigma}^0_1$ is an exceptional divisor of the blowing up morphism $\sigma'_1 : \Sigma'_1 \to \Sigma \times \mathbb{A}^1$. Their scheme-theoretic intersection equals the exceptional divisor $\tilde{\Sigma}^1_0$ of the blowing up morphism $\sigma^*_1 : \Sigma^0_1 \to \Sigma$:

$$
\Sigma^0_1 \cap \tilde{\Sigma}^1_0 = \tilde{\Sigma}^1_0,
$$

and one comes to the decomposition of the morphism $\sigma_1$:

$$
\Sigma^0_1 \xrightarrow{\delta_1} \Sigma^0_1 \xrightarrow{\sigma^*_1} \Sigma
$$

where $\delta_1$ acts identically on $\Sigma^0_1$ and its action on $\tilde{\Sigma}^1_0$

$$
\delta_1|_{\tilde{\Sigma}^1_0} : \tilde{\Sigma}^1_0 \to \Sigma^0_1
$$

factors through the exceptional divisor $\tilde{\Sigma} = \text{Proj} \bigoplus_{s \geq 0} \mathbb{I}^s_1$ of the morphism $\sigma^*_1$ and is defined by the structure of $\bigoplus_{s \geq 0} \mathbb{I}^s_1$-algebra on the graded ring $\bigoplus_{s \geq 0} \mathbb{I}^s_1/\mathbb{I}^{s+1}_1$.

**Convention 1**. For uniformity of notation we use decomposition $\sigma_i = \sigma^*_i \circ \delta_i$ for all $i = 1, \ldots, \ell$ with no reference whenever $\delta_i$ is non-identity or identity morphism.

Composites $\delta_i \circ \sigma^*_i$ are « interchangeable » in the sense of the commutative diagram

$$
\Sigma^0_{i+1} \xrightarrow{\sigma^*_i} \Sigma_i \xrightarrow{\delta_i} \Sigma^0_i
$$

(2.2)

**Remark 3**. We use double indexing with lower and upper indices for schemes and morphisms. For example, $-1$ in notation $\delta_i^{-1}$ or $\sigma_i^{-1}$ is just index but neither sign of inverse of the map nor notation for inverse image of a sheaf.
The resolution of singularities of the sheaf $E$ is done by the sequence of morphisms $\sigma_i = \delta_i \circ \sigma_i^0$, $i = 1, \ldots, \ell$. Their composite can be decomposed into the following diagram by iterating square (2.2):

\[
\begin{array}{c}
\Sigma_{\ell}^0 \\
\delta_{\ell}^0 \downarrow \\
\Sigma_{\ell - 1}^0 \sigma_{\ell - 1}^0 \\
\Sigma_{\ell - 2}^0 \sigma_{\ell - 2}^0 \downarrow \\
\Sigma_{\ell - 3}^0 \sigma_{\ell - 3}^0 \\
\vdots
\end{array}
\]

Each square of this diagram has a view

\[
\begin{array}{c}
\Sigma_{i - j}^0 \\
\delta_{i - j}^0 \downarrow \\
\Sigma_{i - j - 1}^0 \\
\sigma_{i - j - 1}^0 \\
\Sigma_{i - j - 2}^0 \sigma_{i - j - 2}^0 \downarrow \\
\Sigma_{i - j - 3}^0 \sigma_{i - j - 3}^0 \\
\vdots
\end{array}
\]

for $i = 0, \ldots, \ell$, $j = -1, \ldots, i - 1$, where $\Sigma_0 := \Sigma$, $\Sigma_i := \Sigma_i$, $\delta_i^0 := \delta_i$.

In this diagram the bottom horizontal row is composite of consequent blowups and but left vertical column $\delta_{1}^{\ell+1} \circ \cdots \circ \delta_i^0$ provides flatness of the scheme $\Sigma_{\ell} = \Sigma$ over its base $T$ by consequent «implanting» additional components of several levels and $(\delta_{1}^{\ell+1} \circ \cdots \circ \delta_i^0 \sigma_{\ell}^0)$ produces just an inverse image of locally free sheaf. Each square of the diagram is an analog of (2.2).

To generalise this recipe to arbitrary homological dimension we act inductively and do interchanging in the inductive step.

Now we pass to the next segment

\[
0 \to W_{\ell - 1} \to \hat{E}_{\ell - 2} \to W_{\ell - 2} \to 0
\]

and to its «inverse image» under $\sigma_1^0$:

\[
0 \to W'_{\ell - 1} \to \sigma_1^0 \hat{E}_{\ell - 2} \to \sigma_1^0 W_{\ell - 2} \to 0.
\]

The «inverse image» under $\sigma_1 = \sigma_1^0 \circ \delta_i^0$ is exact because of local freeness of the kernel $W'_{\ell - 1}$. In this and other further segments the cokernel sheaf contains torsion (in modified
sense). This does not cause any problem for the procedure of resolution because, as we will see later, the resolution leads to the factoring out of the torsion.

Next steps are similar to each other and involve inverse images of consequent segments

\[
0 \to W'_{\ell - 2} \to \sigma_2^0 \sigma_1^* \hat{E}_{\ell - 3} \to \sigma_2^0 \sigma_1^* W_{\ell - 3} \to 0,
\]

\[
0 \to W'_{\ell - i} \to \sigma_i^0 \sigma_i^* \hat{E}_{\ell - i - 1} \to \sigma_i^0 \sigma_i^* W_{\ell - i - 1} \to 0, \tag{2.4}
\]

\[
0 \to W'_1 \to \sigma_1^* \hat{E}_0 \to \sigma_1^* W_0 \to 0,
\]

where \( W_0 = \mathbb{E} \) and the kernel sheaf \( W'_{\ell - i} \) in the next triple is locally free \( \mathcal{O}_{\Sigma_{\ell - i}} \)-module which is produced by standard resolution of \( \sigma_i^* \ldots \sigma_i^* W_{\ell - i} \) in the previous triple.

As it is proven in \([1]\) the composite \( \sigma_i := \sigma_i \circ \ldots \circ \sigma_i \)

\[
\sigma_i[i] := \sigma_1 \circ \ldots \circ \sigma_i
\]

can be rewritten as

\[
\sigma_i[i] = \sigma_{i - 1} \circ \sigma_i = \sigma_{i - 1} \circ \sigma_i^0 \circ \delta_i^0 = \sigma_i^0 \circ \delta_i^0
\]

where we have introduced notations

\[
\sigma_i^0 := \sigma_i^0 \circ \sigma_i^0 \circ \ldots \circ \sigma_i^0, \quad \delta_i^0 := \delta_i^0 \circ \delta_i^0 \circ \ldots \circ \delta_i^0, \quad \delta_i^{-1} := \delta_i^{-1} \circ \delta_i^{-1} \circ \ldots \circ \delta_i^{-1}.
\]

The resolution leads to locally free \( \mathcal{O}_{\Sigma_{\ell + 1}} \)-sheaf \( \hat{E} := \sigma_i^0 \mathbb{E}/\text{tors} \) and \( \mathcal{O}_{\Sigma} \)-sheaf \( \tilde{E} := \delta_i^0 \hat{E} \).

We will use notation \( \tilde{\Sigma} := \Sigma_{\ell} \) analogous to one used for family of admissible schemes in previous papers.

**Remark 4.** Since \( \tilde{E} \) is locally free as \( \mathcal{O}_{\Sigma} \)-module and \( \mathcal{O}_{\Sigma} \) is \( \mathcal{O}_T \)-flat then \( \tilde{E} \) is also flat over \( T \).

The transformation of families we constructed, has a form

\[
(T, L, \mathbb{E}) \mapsto (\pi : \tilde{\Sigma} \to T, \tilde{L}_i, \tilde{E}_i)
\]

and is defined by the commutative diagram

\[
\begin{array}{ccc}
T & \to & \{(T, L, \mathbb{E})\} \\
\downarrow & & \downarrow \\
T & \to & \{\pi : \tilde{\Sigma} \to T, \tilde{L}_i, \tilde{E}_i\}
\end{array}
\]

The right vertical arrow is the map of sets. Their elements are families of objects to be parametrised. The map is determined by the procedure of resolution as it developed in \([1]\).

**Remark 5.** As we conclude below, the resolution as it is constructed now defines a morphism of functors \( \kappa : f^{GM} \to f \) where \( f \) is the functor of moduli for admissible pairs (to be described below) and \( f^{GM} \) the functor of moduli for semistable torsion free sheaves in the
setting of Gieseker and Maruyama. The morphism of functors \( \kappa : f^{GM} \rightarrow f \) is defined by the class of commutative diagrams

\[
\begin{array}{ccc}
T & \xrightarrow{f^{GM}} & \tilde{\mathcal{S}}_{T} / \sim \\
\downarrow{\gamma(T)} & & \downarrow{\sim} \\
\mathcal{S}_{T} / \sim
\end{array}
\]

where \( T \in \text{Ob}(\text{Schemes})_{k} \), \( \kappa(T) : (\tilde{\mathcal{S}}_{T} / \sim) \rightarrow (\mathcal{S}_{T} / \sim) \) is a morphism in the category of sets (mapping).

**Remark 6.** The procedure of resolution involves a choice of locally free resolution of the sheaf under resolution. But it is proven \([1]\) that the sequence of morphisms does not depend on this choice.

The structure of fibres of the morphism \( \tilde{\Sigma} \rightarrow T \) is produced directly from the composite of morphisms of \( T \)-schemes \( \pi_{i} : \Sigma_{i} \rightarrow T \), \( i = 0, \ldots, \ell \):

\[
\begin{array}{cccccccc}
\tilde{\Sigma} & \xrightarrow{\sigma_{\ell}} & \Sigma_{\ell-1} & \xrightarrow{\sigma_{\ell-1}} & \cdots & \xrightarrow{\sigma_{2}} & \Sigma_{1} & \xrightarrow{\sigma_{1}} & \Sigma_{0} = \Sigma \\
\downarrow{\delta_{\ell}} & & \downarrow{\delta_{\ell-1}} & & \cdots & & \downarrow{\delta_{2}} & & \downarrow{\delta_{1}} \\
\Sigma_{\ell} & & \Sigma_{\ell-1} & & \cdots & & \Sigma_{2} & & \Sigma_{1} \\
\end{array}
\]

When restricted to fibres over any fixed closed point \( t \in T \) (or, equivalently, one can set \( T \) to be reduced point) this chain gives rise to the chain of morphisms among fibres \( S_{i} = \pi_{i}^{-1}(t) \):

\[
\begin{array}{cccccccc}
\tilde{S} & \xrightarrow{\sigma_{\ell}} & S_{\ell-1} & \xrightarrow{\sigma_{\ell-1}} & \cdots & \xrightarrow{\sigma_{2}} & S_{1} & \xrightarrow{\sigma_{1}} & S_{0} = S \\
\downarrow{\delta_{\ell}} & & \downarrow{\delta_{\ell-1}} & & \cdots & & \downarrow{\delta_{2}} & & \downarrow{\delta_{1}} \\
S_{\ell} & & S_{\ell-1} & & \cdots & & S_{2} & & S_{1} \\
\end{array}
\]

Each of \( \sigma_{i}^{0} \) is a blowup morphism and each of \( \delta_{i}^{0} \) contracts additional component onto exceptional divisor of \( \sigma_{i}^{0} \). Moving against arrows we can say that \( \sigma_{i}^{0} \)'s blow up and \( \delta_{i}^{0} \)'s grow additional components. Since these two types of morphisms alternate next blowup is applied to the scheme which consist of several connected components.

**Convention 2.** We use notations for a single fibre which are completely parallel to the rules for families in \([2,3]\). The fibrewise version for \([2,3]\) can be obtained when one replaces \( \Sigma \)'s by \( S \)'s with all indices preserved for schemes and double letters \( \sigma, \delta \) by respective usual single letters \( \sigma, \delta \) with all indices preserved for morphisms. The reader should keep in mind that each \( \delta \) projects its source scheme to its component.

We start with the initial non-singular variety \( S \). When passing to \( S^{0} \) and after that to \( S_{1} \) we see the scheme \( S_{1} \) consisting of principal component \( S_{1}^{0} \) and additional component \( S_{1}^{\text{add}} \). Principal component \( S_{1}^{0} = (\sigma_{1}^{0})^{-1}S \) is algebraic variety which is obtained by blowing up \( S \). Additional component \( S_{1}^{\text{add}} \) can carry non-reduced scheme structure and can have reducible reduction. In previous papers this closed subscheme appeared as union of additional components of admissible scheme. In general \( S_{1}^{\text{add}} \) can consist of several connected components.
Passing to $S_2^0$ leads to transformation of both $S_1^0$ and $S_2^{add}$. We come to algebraic variety $S_1^0 = (\sigma_{i-1})^{-1}S_1^{odd} = (\sigma_1^{-1})^{-1}(\sigma_1^1)^{-1}S$ obtained by blowup of principal component $S_1^0$ of $S_1$ and the scheme $(\sigma_0^{add})^{-1}S_1^{add}$. Passing to $S_2$ against $\sigma_2^0$ implants additional component $S_2^{add}$, and we can write $S_2 = S_2^0 \cup (\sigma_2^{add})^{-1}S_1^{add} \cup S_2^{add}$.

Analogously, on $\ell$-th step we have the following scheme

$$\tilde{S} := S_\ell = (\sigma_1^\ell \cdots \cdots \sigma_1^\ell+1)^{-1}S \cup (\sigma_2^\ell \cdots \cdots \sigma_2^\ell+2)^{-1}S_1^{add} \cup \cdots \cup (\sigma_\ell^\ell)^{-1}S_{\ell-1} \cup S_\ell^{add}.$$  

Depending on the structure of the initial $O_S$-sheaf $E$ several morphisms $\sigma_i$ can turn to be identities and hence actual length of the chain $S_1, \ldots, S_\ell$ can vary from 0 (for the case when $E$ is locally free) to maximal value equal to $\ell$.

To measure numerical invariants of objects we obtained in the standard resolution we need to fix appropriate ample invertible sheaf $\tilde{L}$ on each $\tilde{S}$. The sheaf of analogous role was called distinguished polarisation in previous papers where hd-one case was developed. This class of invertible sheaves provides, in particular, fibrewise uniform Hilbert polynomials in families of admissible schemes. Strictly speaking, if $\tilde{L}$ is invertible $O_{\tilde{S}}$-sheaf which is very ample relatively to the base $T$ then for any closed point $t \in T$ and for any integer $n \gg 0$ Hilbert polynomial of the fibre $\pi_1(t)$ compute as $\chi(\tilde{L}^n|_{\pi_1(t)})$ is independent of the choice of the $t \in T$.

Let $\Sigma$ carries an invertible sheaf $L$ which is very ample relatively to $T$. In hd-one case there is one-step resolution by $T$-morphism $\sigma : \tilde{\Sigma} \to \Sigma$ associated to the sheaf of ideals $I \subset O_\Sigma$. In this case is was shown that distinguished polarisation can be chosen as $\tilde{L} = \sigma^*L^m \otimes \sigma^I \cdot O_{\tilde{\Sigma}}$ whenever $m$ is sufficiently big to provide amplenness relatively to $T$.

In the case of bigger homological dimension $\ell$ this step of constructing family of polarisations on $T$-scheme $\tilde{\Sigma}$ is iterated $\ell$ times until one comes to

$$\tilde{L} := L_\ell = [\sigma_1^* \cdots [\sigma_2^*L^m_1 \otimes (\sigma_1)^{-1}I_1 \cdot O_{\Sigma_1}]^m_2 \otimes (\sigma_2)^{-1}I_2 \cdot O_{\Sigma_2}] \cdots \otimes (\sigma_\ell)^{-1}I_\ell \cdot O_{\Sigma_\ell}.$$  

(2.5)

Corresponding $T$-scheme $\Sigma_\ell$ has fibrewise constant Hilbert polynomial with respect to $L_\ell$ after $i$-th step of the resolution.

Distinguished polarisation on a single admissible scheme $\tilde{S}$ has a view

$$\tilde{L} := L_\ell = [\sigma_1^* \cdots [\sigma_2^*[\sigma_1^*L^m_1 \otimes (\sigma_1)^{-1}I_1 \cdot O_{\Sigma_1}]^m_2 \otimes (\sigma_2)^{-1}I_2 \cdot O_{\Sigma_2}] \cdots \otimes (\sigma_\ell)^{-1}I_\ell \cdot O_{\Sigma_\ell}.$$  

(2.6)

Distinguished polarisation $\tilde{L}$ is assumed to be fixed for each admissible scheme $\tilde{S}$. If $\tilde{S} = S$ then $L = L$ and polarisation’s $\tilde{L}$ for all possible $\tilde{S}$ are chosen in such a way that for all admissible schemes $(\tilde{S}, \tilde{L})$ their Hilbert polynomials are uniform:

$$\chi(\tilde{L}^n) = \chi(L^n), \quad n \gg 0.$$  

Now we re-denote $L^m_1 \cdots \cdots m_\ell$ as $L$ and $L^{m_1 \cdots m_\ell}$ as $L$ and from now we work with these new polarisations. Also we come to following shorthand notations for (2.5) and (2.6) respectively:

$$\tilde{L} = \sigma^*L \otimes \text{Exc};$$  

(2.7)

$$\text{Exc} := [\sigma_1^* \cdots [\sigma_2^*[(\sigma_1)^{-1}I_1 \cdot O_{\Sigma_1}]^m_2 \otimes (\sigma_2)^{-1}I_2 \cdot O_{\Sigma_2}] \cdots \otimes (\sigma_\ell)^{-1}I_\ell \cdot O_{\Sigma_\ell};$$  

(2.8)

$$\text{Exc}_{S_\ell} := [\sigma_1^* \cdots [\sigma_2^*[(\sigma_1)^{-1}I_1 \cdot O_{\Sigma_1}]^m_2 \otimes (\sigma_2)^{-1}I_2 \cdot O_{\Sigma_2}] \cdots \otimes (\sigma_\ell)^{-1}I_\ell \cdot O_{\Sigma_\ell};$$  

(2.9)

$$\tilde{L} = \sigma^*L \otimes \text{Exc}_{S_\ell};$$  

(2.10)
3 (Semi)stability

Definition 3. \( \mathcal{O}_{\tilde{S}} \)-sheaf \( \tilde{E} \) is obtained from \( E \) by standard resolution if for some locally free \( \mathcal{O}_{\tilde{S}} \)-resolution \( \tilde{E}_* \to E \to 0 \) with intermediate quotients \( W_i, i = 0, \ldots, \ell, W_0 = E, W_\ell = \tilde{E}_\ell \) and for a sequence of morphisms \( \tilde{S} = S_\ell \xrightarrow{\sigma_\ell} S_{\ell - 1} \xrightarrow{\sigma_{\ell - 1}} \ldots \xrightarrow{\sigma_1} S_1 \xrightarrow{\sigma_1} S \) there is an isomorphism \( \tilde{E} = \sigma_\ell^* \ldots \sigma_1^* E / \text{tors} \). Morphisms \( \sigma_i, i = 1, \ldots, \ell \), are specified as in Sec. 2: \( S_i = \text{Proj } \bigoplus_{s \geq 0} (I_i[t] + (t))^{s/(t)^{s+1}} \) and \( \sigma_i : S_i \to S_{i-1} \) is structure morphism.

Definition 4. [12] The \( \mathcal{O}_{\tilde{S}} \)-sheaf \( \tilde{E} \) is Gieseker-stable (resp., Gieseker-semistable) if for any subsheaf \( \tilde{F} \subset \tilde{E} \) and for any proper subsheaf \( \tilde{F} \subset \tilde{E} \) for \( m \gg 0 \) one has

\[
\frac{h^0(\tilde{F} \otimes \tilde{L}^m)}{\text{rank} \tilde{F}} < \frac{h^0(\tilde{E} \otimes \tilde{L}^m)}{\text{rank} \tilde{E}} \quad \text{(respectively,} \quad \frac{h^0(\tilde{F} \otimes \tilde{L}^m)}{\text{rank} \tilde{F}} \leq \frac{h^0(\tilde{E} \otimes \tilde{L}^m)}{\text{rank} \tilde{E}} \).
\]

When \( \text{hd} \ E > 1 \) characterising of sequence of centres for blowups in the standard resolution rises a difficult problem if we want to eliminate the data of initial sheaf \( E \) under resolution to formulate moduli problem for pairs independently of Gieseker–Maruyama moduli problem. The way to overcome this problem is to make the class of pairs wider without direct restrictions on blowing up morphisms. This step brings additional profit: this approach includes some other ways to compactify moduli of stable vector bundles on prescribed algebraic variety. Let \( \sigma_i : S_i \to S_{i-1} \) birational morphism such that

- \( U \subset S_{i-1} \) biggest open subset such that \( \sigma_i|_{(\sigma_i)^{-1}(U)} : (\sigma_i)^{-1}(U) \to U \) is an isomorphism. Denote \( H := S_{i-1} \setminus U \).
- Denote \( S_i^0 := (\sigma_i)^{-1}(U) \) and call it a principal component of \( S_i \). Also denote \( \sigma_i^0 := \sigma_i|_{S_i^0} : S_i^0 \to S_{i-1} \). The restriction \( \sigma_i|_{S_i^0} : S_i^0 \to S_{i-1} \) is blowup morphism with exceptional divisor \( \sigma_i^{-1}(H) \).
- The restriction \( \sigma_i|_{S_i \setminus S_i^0} : S_i \setminus S_i^0 \to S_{i-1} \) factors through the closed immersion \( H \hookrightarrow S_{i-1} \). This implies that
- there is a decomposition of the morphism \( \sigma_i \) as \( \sigma_i = \sigma_i^0 \circ \delta_i \) where \( \delta_i : S_i \to S_i^0 \) is such that \( \delta_i|_{S_i^0} : S_i^0 \to S_i^0 \) is identity morphism and there is a decomposition \( \delta_i|_{S_i \setminus S_i^0} : S_i \setminus S_i^0 \to H \hookrightarrow S_{i-1} \).

Admissible scheme is obtained as

\[
\tilde{S} = S_\ell \xrightarrow{\sigma_\ell} S_{\ell - 1} \longrightarrow \cdots \longrightarrow S_1 \xrightarrow{\sigma_1} S = S_0
\]

where \( \sigma_i, i = 1, \ldots, \ell \) are as described.

We well consider \( S_i = \text{Proj } \bigoplus_{s \geq 0} (I_i[t] + (t))^{s/(t)^{s+1}} \) for \( I_i \subset \mathcal{O}_{S_{i-1}} \) and \( \sigma_i \) as its structure morphism to \( S_{i-1} \). We assume no specification for \( I_i, i = 1, \ldots, \ell \).

Very ample invertible sheaf \( L_i \) on \( S_i \) is fixed by the equality

\[
L_i = [\sigma_i^* \ldots [\sigma_i^* L_i^m \otimes (\sigma_1)^{-1} I_1 \cdot \mathcal{O}_{S_1}]^{m_2} \otimes (\sigma_2)^{-1} I_2 \cdot \mathcal{O}_{S_2}]^{m_3} \ldots ]^{m_i} \otimes (\sigma_i)^{-1} I_\ell \cdot \mathcal{O}_{S_i}
\]

and after re-denoting \( L := L^{m_1 \ldots m_i} \) it satisfies \( h^0(S_i, L^m) = h^0(S, L^m) \) for \( m \gg 0 \). \( \tilde{L} = L_\ell \) is called distinguished polarisation on \( \tilde{S} \).
If $E_i = \tilde{E}$ is obtained from $\mathcal{O}_S$-sheaf $E = E_0$ then $E_i = \delta_i^* E_i|_{S^i_0}/\text{tors}$ and $E_i = \sigma_i^* E_{i-1}/\text{tors}$. Torsion is understood in modified sense. In case of a sheaf on integral scheme modified torsion becomes usual torsion.

All considerations about the structure of a sheaf and transformations of a scheme in the process of standard resolution and about interchanging the order of morphisms $\sigma_i^j$ and $\delta_i^j$ (cf. Sec. 2 and [1] for details) hold for a single sheaf $E$ on $S$ and this leads to the parallel procedure with consequent $E_i \cong \delta_i^j E_i^0$, $E_i^0 = E_i|_{S^i_0}$. Also interchanging the order of morphisms $\sigma_i^j$ and $\delta_i^j$ holds for arbitrary ideals $I_i$ being not only Fitting ideals and leads to the resolution for subsequent short exact sequences analogous to (2.4). Hence we rewrite the sequence of morphisms $\sigma_i$ by $\delta_i$'s followed by $\sigma_i^0$'s: $\sigma := \sigma_i \circ \cdots \circ \sigma_1 = \sigma_i^0 \circ \delta_i^0$. In this case principal component $\tilde{S}^0$ of the whole of the scheme $\tilde{S}$ is naturally distinguished: this is unique component $\tilde{S}^0 = (\sigma_0^0)^{-1} S$. Now $\tilde{E} = (\delta_0^0)|^* \tilde{E}|_{\tilde{S}^0}$.

**Definition 5.** $S$-(semi)stable pair $( (\tilde{S}, \tilde{L}), \tilde{E} )$ is the following data:

- $\tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i$ – admissible scheme, $S_i$, $i \geq 0$ its components, $\sigma : \tilde{S} \rightarrow S$ – canonical morphism;
- $\tilde{E}$ – vector bundle on the scheme $\tilde{S}$;
- $\tilde{L} \in \text{Pic } \tilde{S}$ – distinguished polarisation;

such that

- $\chi(\tilde{E} \otimes \tilde{L}^n) = rp_E(n)$;
- the sheaf $\tilde{E}$ is Gieseker-(semi)stable on the scheme $\tilde{S}$.
- the sheaf $\tilde{E}$ is quasi-ideal sheaf, namely it has a description of the form $\tilde{E} = (\delta_0^0)|^* \tilde{E}|_{\tilde{S}^0}$.

**Remark 7.** If $\tilde{S} \cong S$, then (semi)stability of a pair $(\tilde{S}, \tilde{E})$ is equivalent to Gieseker-(semi)stability of vector bundle $\tilde{E}$ on the variety $\tilde{S}$ with respect to the polarisation $\tilde{L} \in \text{Pic } \tilde{S}$.

To investigate the relation of $S$-(semi)stability of the sheaf $\tilde{E}$ to Gieseker-(semi)stability of the corresponding sheaf $E$ on the variety $S$ note that for $m \gg 0$ $rp_E(m) = h^0(E \otimes L^m)$. For the Gieseker-stability the behaviour of the Hilbert polynomial under $m \gg 0$ is important. Therefore we assume that $m$ is big enough.

Now we regret for a time to discussion of the standard resolution procedure. Firstly if we consider a family $\Sigma = T \times S$ whose base $T$ is a projective curve, and this family is supplied with a sheaf $E$ which is flat over $T$ and gives torsion-free sheaf when restricted on any fibre. Also let $\Sigma$ carries invertible sheaf $\underline{L}$ which is very ample relatively to $T$ such that fibrewise Hilbert polynomial $\chi((E \otimes \underline{L}^u)|_{t \times S})$ is uniform over $T$ and equals to $rp(u)$. Let general fibre $t \times S$ over $t \in T$ carry locally free sheaf $E|_{t \times S}$ and special fibre carry nonlocally free sheaf $E|_{t \times S}$. Applying standard resolution to this family we come to the family $\pi : \Sigma \rightarrow T$ supplied with locally free sheaf $\tilde{E}$. Since any projective morphism with base being a projective curve is necessarily flat then $\pi$ is to be flat. Since general fibres do not overcome any transformation Hilbert polynomials of restrictions of $\tilde{E}$ with respect to $\underline{L}$ remain equal to the initial Hilbert polynomial $\chi((\tilde{E} \otimes \underline{L}^u)|_{\pi^{-1}(t)}) = \chi((E \otimes \underline{L}^u)|_{t \times S}) = rp(u)$. 

13
If we regret to iterative process of standard resolution then each step is a standard resolution of a sheaf of homological dimension equal to 1 by the morphism $\sigma_i = \sigma_i^0 \circ \delta_i : \Sigma_i \to \Sigma_{i-1}$ and this step transforms flat family of schemes $\Sigma_{i-1}$ with distinguished polarisation $\mathbb{L}_{i-1}$ to flat family of schemes $\Sigma_i$ with distinguished polarisation $\mathbb{L}_i$. In the triple

$$0 \to \delta_i^*W_{\ell-1}^i \to \sigma_i^* \mathbb{S}_{i-1} \to \sigma_i^* \hat{\mathbb{S}}_{i-1} \to \sigma_i^* W_{\ell-1}^i \to 0,$$

$\delta_i^*W_{\ell-1}^i$ being locally free on $\Sigma_i$ turns to be $T$-flat and hence cokernel of injective morphism of flat sheaves is also necessarily flat. Since uniformity of fibrewise Hilbert polynomial holds for distinguished polarization $\mathbb{L}_i$ for kernel and extension the same holds for the cokernel.

Now we get confirmed that standard resolution takes flat families to flat families with uniformity of fibrewise Hilbert polynomial compute with respect to distinguished polarization.

**Proposition 2.** Standard resolution of the family $((\Sigma, \mathbb{L}), \mathbb{E})$ induces fixed isomorphism of $k$-vector spaces $H^0(\tilde{S}, \mathbb{E} \otimes \tilde{\mathbb{L}}^m) \cong H^0(S, \mathbb{E} \otimes \mathbb{L}^m)$, for all $m \gg 0$, for $\tilde{E} = \tilde{E}|_{\pi^{-1}(t)}$, $E = \mathbb{E}|_{t \times S}$ and for each closed point $t \in T$.

**Remark 8.** In case when $\text{hd} \mathbb{E} = 1$ this result was proven in [5] using the fact that in this case $\mathbb{E}$ is reflexive sheaf. But reflexivity is not guaranteed when $\text{hd} \mathbb{E} > 1$.

**Proof.** Start with a commutative diagram

$$\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\sigma} & \Sigma \\
\downarrow \pi & & \downarrow \rho \\
T & \xrightarrow{p} & T
\end{array}$$

and consider sheaf morphisms

$$\mathbb{E} \otimes \mathbb{L}^m \hookrightarrow \sigma_* \mathbb{E}^* (\mathbb{E} \otimes \mathbb{L}^m) = \sigma_* (\mathbb{E}^* \otimes \mathbb{E}^* \mathbb{L}^m) \to \sigma_* (\mathbb{E}^*/\text{tors} \otimes \mathbb{E}^* \mathbb{L}^m) = \sigma_* \tilde{\mathbb{E}} \otimes \mathbb{L}^m$$

The first composite is injective. Now apply direct image $p_*$ and by $p_* \sigma_* = \pi_*$ we come to two monomorphisms

$$p_*(\mathbb{E} \otimes \mathbb{L}^m) \hookrightarrow p_*(\sigma_* \tilde{\mathbb{E}} \otimes \mathbb{L}^m) \hookrightarrow \pi_*(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m).$$

Now we are to prove that images of both locally free sheaves $p_*(\mathbb{E} \otimes \mathbb{L}^m)$ and $\pi_*(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m)$ coinide in $p_*(\sigma_* \tilde{\mathbb{E}} \otimes \mathbb{L}^m)$. We will prove it by restrictions on fibres. Since $\pi$ is projective morphism of Noetherian schemes, the sheaf $\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m$ is flat over $T$ and functions $t \mapsto H^0(\pi^{-1}(t), (\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m)|_{\pi^{-1}(t)})$ for $m \gg 0$ are constant in $t \in T$ then by [13] ch.III, Corollary 12.9] there is an isomorphism

$$\pi_*(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m) \otimes k_t \cong H^0(\pi^{-1}(t), (\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m)|_{\pi^{-1}(t)}).$$

Also by analogous reason

$$p_*(\mathbb{E} \otimes \mathbb{L}^m) \otimes k_t \cong H^0(t \times S, (\mathbb{E} \otimes \mathbb{L}^m)|_{t \times S}).$$

Note that the morphism $\sigma_t : \pi^{-1}(t) \to t \times S$ which is induced by restriction of $\sigma$ to the fibre over $t \in T$ is an isomorphism on nonempty open subset, i.e. for $\tilde{S}_t := \pi^{-1}(t)$ there are
open subschemes $\widetilde{S}_{0t} \subset \widetilde{S}_t$ and $S_{0t} \subset t \times S$ such that $\sigma|_{\widetilde{S}_{0t}} : \widetilde{S}_{0t} \to S_{0t}$ is an isomorphism. Also for $\widetilde{E}|_{\pi^{-1}(t)} = \widetilde{E}_t$, $\widetilde{L}_t = \widetilde{L}|_{\pi^{-1}(t)}$ and for $E_t = E|_{t \times S}$ corresponding restrictions are isomorphic locally free sheaves: $\widetilde{E}_t|_{\widetilde{S}_{0t}} \cong \sigma_{0t}^* E_t|_{S_{0t}}$. Here $S_{0t}$ is precisely the maximal subset where $E_t$ is locally free. Restriction maps $H^0(t \times S, E_t \otimes L^m) \hookrightarrow H^0(S_{0t}, (E_t \otimes L^m)|_{S_{0t}})$ and $H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m) \hookrightarrow H^0(\widetilde{S}_{0t}, (\widetilde{E}_t \otimes \widetilde{L}_t^m)|_{\widetilde{S}_{0t}})$ together with coincidence $H^0(S_{0t}, (E_t \otimes L^m)|_{S_{0t}}) = H^0(\widetilde{S}_{0t}, (\widetilde{E}_t \otimes \widetilde{L}_t^m)|_{\widetilde{S}_{0t}})$ lead to the bijective correspondence $H^0(t \times S, E_t \otimes L^m) \sim H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m)$ also as images in $\sigma_* \widetilde{E} \otimes L^m \otimes k_t$.

**Proposition 3.** Let the locally free $O_{\widetilde{S}}$-sheaf $\widetilde{E}$ be obtained from a coherent $O_S$-sheaf $E$ by its standard resolution. The sheaf $\widetilde{E}$ is (semi)stable on the scheme $\widetilde{S}$ if and only if the sheaf $E$ is (semi)stable.

**Proof.** Let $E$ be Gieseker-semistable on $(S, L)$ and $\widetilde{E}$ be the locally free sheaf on the scheme $\widetilde{S}$. Let $\widetilde{E}$ be obtained from $E$ by standard resolution. Consider a proper subsheaf $\widetilde{F} \subset \widetilde{E}$. Since $m \gg 0$ we assume that both sheaves $\widetilde{E} \otimes \widetilde{L}^m$ and $\widetilde{F} \otimes \widetilde{L}^m$ are globally generated. Fix an epimorphism $H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m) \otimes \widetilde{L}(-m) \twoheadrightarrow \widetilde{E}$. The subsheaf $\widetilde{F}$ is generated by a subspace of global sections $V_\widetilde{F} = H^0(\widetilde{S}, \widetilde{F} \otimes \widetilde{L}^m) \subset H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m)$. Then a subspace $V_\widetilde{F} \subset H^0(S, E \otimes L^m)$ which is isomorphic to $V_\widetilde{F}$ and generates some subsheaf $F \subset E$, is given by the distinguished isomorphism $\nu : H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m) \xrightarrow{\sim} H^0(S, E \otimes L^m)$ by the equality $V_\widetilde{F} = \nu(V_\widetilde{F})$. Since sheaves $\widetilde{F}$ and $F$ are canonically isomorphic on the corresponding open subsets of schemes $\widetilde{S}$ and $S$, then their ranks are equal. Clearly, $V_\widetilde{F} = H^0(S, F \otimes L^m)$ and

$$h^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m) = h^0(S, E \otimes L^m) > (\geq)0.$$  

This implies the semistability of $\widetilde{E}$. The opposite implication is proven similarly.

**Remark 9.** This shows that there is a bijection among subsheaves of $O_{\widetilde{S}}$-sheaf $E$ and subsheaves of the corresponding $O_S$-sheaf $\widetilde{E}$. This bijection preserves Hilbert polynomials.

## 4 Equivalence of semistable admissible pairs

In this section we introduce and examine the equivalence relation for admissible pairs of arbitrary dimension. This equivalence generalises S-equivalence for semistable coherent sheaves in Gieseker – Maruyama moduli problem and is an analogue of M-equivalence for $\text{hd} E = 1$ developed in [6]. Most of proofs concerning the equivalence are very similar to ones in [3]. Since class of admissible schemes is sufficiently wide the notion of S- (or M-) equivalence is also strengthened.

Now we need some additional construction. Let $\sigma : \widetilde{S} \to S$ and $\sigma' : \widetilde{S}' \to S$ be two admissible schemes with their structure morphisms.

In this section we investigate the behaviour of Jordan – Hölder filtration for semistable coherent sheaf under the standard resolution. Also the notion of M-equivalence for semistable pairs is introduced and relation of M-equivalence of resolutions to S-equivalence.
for initial semistable coherent sheaves is examined. In particular it is proven that S-equivalent coherent sheaves on the variety $S$ are resolved in M-equivalent pairs of the form $((\tilde{S}, \tilde{L}), \tilde{E})$.

Remind some classical notions from the theory of semistable coherent sheaves.

**Definition 6.** [14, definition 1.5.1] The *Jordan – H"older filtration* for semistable sheaf $E$ with reduced Hilbert polynomial $p_E(n)$ on the polarised projective scheme $(X, L)$ is a sequence of subsheaves

$$0 = F_0 \subset F_1 \subset \cdots \subset F_s = E,$$

such that quotient sheaves $\text{gr}_i(E) = F_i/F_{i-1}$, $i = 1, \ldots, s$, are stable with reduced Hilbert polynomials equal to $p_E(n)$.

Denote $\text{gr}(E) := \bigoplus \text{gr}_i(E_i)$. Well-known theorem [14, Prop. 1.5.2] claims that the isomorphism class of the sheaf $\text{gr}(E)$ has no dependence on a choice of Jordan – H"older filtration of $E$.

**Definition 7.** [14, definition 1.5.3] Semistable sheaves $E$ and $E'$ are called *S-equivalent* if $\text{gr}(E) = \text{gr}(E')$.

**Remark 10.** Obviously, S-equivalent stable sheaves are isomorphic.

Define Jordan-H"older filtration for $S$-semistable sheaf on reducible admissible polarised scheme $((\tilde{S}, \tilde{L}), \tilde{E})$. This definition will be completely analogous to the classical definition for Gieseker-semistable sheaf.

**Definition 8.** *Jordan – H"older filtration* for a sheaf $\tilde{E}$ on the polarised projective reducible scheme $((\tilde{S}, \tilde{L}), \tilde{E})$ such that a pair $((\tilde{S}, \tilde{L}), \tilde{E})$ is semistable in the sense of definition [5] and with reduced Hilbert polynomial $p_E(n)$, is a sequence of subsheaves

$$0 = \tilde{F}_0 \subset \tilde{F}_1 \subset \cdots \subset \tilde{F}_\ell = \tilde{E},$$

such that quotients $\text{gr}_i(\tilde{E}) = \tilde{F}_i/\tilde{F}_{i-1}$ are Gieseker-stable with reduced Hilbert polynomials equal to $p_E(n)$.

Example 2 in [6] shows that S-equivalent coherent sheaves can have different associated sheaves of Fitting ideals leading to non-isomorphic schemes $\tilde{S}$ (even in case of $\text{hd} E = 1$).

Example 3 in [6] shows that fibred product cannot be used to construct the notion of equivalence for semistable pairs.

Since resolution can consist of several morphisms $\sigma_i : S_i \to S_{i-1}$, $i = 1, \ldots, \ell$, followed by each other we develop the notion of equivalence which is analogue of S-equivalence in classical sheaf moduli theory. Also our notion of equivalence is a straightforward generalisation of M-equivalence of semistable admissible pairs in the case of $\text{hd} E = 1$. It is enough to analyse one step of resolution i.e. one morphism $\sigma_i$. The construction of admissible scheme provides all $S_i$ being equidimensional schemes.

Now we restrict ourselves by one step but all the construction holds when $\sigma$’s are iterated because it does not depend of nonsingularity, irreducibility and reducedness of the scheme $S$ (it can be replaced by $S_i$, $i = 2, \ldots, \ell - 1$.)

Now consider schemes

$$\tilde{S}_1 = \text{Proj} \bigoplus_{s \geq 0} (I_1[t] + (t))^s/(t)^{s+1},$$ $$\tilde{S}_2 = \text{Proj} \bigoplus_{s \geq 0} (I_2[t] + (t))^s/(t)^{s+1}.$$
with their canonical morphisms \( \sigma_1 : \tilde{S}_1 \to S \) and \( \sigma_2 : \tilde{S}_2 \to S \) to the scheme \( S \). Form inverse images of sheaves of ideals \( I'_2 = \sigma_1^{-1}I_2 \cdot \mathcal{O}_{\tilde{S}_1} \subseteq \mathcal{O}_{\tilde{S}_1} \) and \( I'_1 = \sigma_2^{-1}I_1 \cdot \mathcal{O}_{\tilde{S}_2} \subseteq \mathcal{O}_{\tilde{S}_2} \), and corresponding projective spectra

\[
\tilde{S}_{12} = \text{Proj} \left( \bigoplus_{s \geq 0} (I'_2[t] + (t))^s/(t)^{s+1} \right),
\]

\[
\tilde{S}_{21} = \text{Proj} \left( \bigoplus_{s \geq 0} (I'_1[t] + (t))^s/(t)^{s+1} \right).
\]

There are canonical morphisms \( \sigma'_2 : \tilde{S}_{12} \to \tilde{S}_1 \) and \( \sigma'_1 : \tilde{S}_{21} \to \tilde{S}_2 \).

**Proposition 4.** \( \tilde{S}_{12} \) and \( \tilde{S}_{21} \) are equidimensional schemes. Moreover, \( \tilde{S}_{12} \cong \tilde{S}_{21} \).

**Proof.** First we prove that \( \tilde{S}_{12} \cong \tilde{S}_{21} \), and that these schemes can be include into flat families with general fibre isomorphic to \( S \), or to \( \tilde{S}_1 \), or to \( \tilde{S}_2 \). This implies that all components of the scheme \( \tilde{S}_{12} \) have dimension not bigger then \( \dim S \). Then we will give the scheme-theoretic characterisation of schemes \( \tilde{S}_{12} \). It proves that \( \tilde{S}_{12} \) is equidimensional scheme, namely, all reduced schemes corresponding to its components have dimension equal to \( \dim S \).

Let \( T = \text{Spec} \ k[t] \). Turn to the trivial 2-parameter family of schemes \( T \times T \times S \) with projections \( T \times S \xrightarrow{\tilde{p}_{12}} T \times T \xrightarrow{\tilde{p}_{23}} T \times S \). Introduce the notations \( \mathbb{I}_1 := \mathcal{O}_T \otimes I_1 \subseteq \mathcal{O}_{T \times S} \), \( \mathbb{I}_2 := \mathcal{O}_T \otimes I_2 \subseteq \mathcal{O}_{T \times S} \). Form inverse images \( p_{13}^*\mathbb{I}_1 \) and \( p_{23}^*\mathbb{I}_2 \). These are sheaves of ideals on the scheme \( T \times T \times S \). Consider the morphism \( \sigma_1 \times \text{id}_T : \tilde{S}_1 \times T \to T \times T \times S \) with identity map on the second factor. Also consider a preimage \( (\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathbb{I}_2 \cdot \mathcal{O}_{\tilde{S}_1 \times T} \) on the scheme \( \tilde{S}_1 \times T \), and the corresponding morphism of blowing up \( \sigma_{12} : \Sigma_{12} \to \tilde{S}_1 \times T \). Now restrict the sheaf \( (\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathbb{I}_2 \cdot \mathcal{O}_{\tilde{S}_1 \times T} \) on the fibre of the composite map \( \tilde{S}_1 \times T \xrightarrow{\sigma_{12}^{-1} \text{id}_T} T \times T \times S \xrightarrow{p_{12}} T \times T \) at the point \((t_1, t_2)\). Let \( \tilde{i} : \tilde{S}_1 \hookrightarrow \tilde{S}_1 \times T \) be the morphism of the embedding of this fibre. The commutativity of the diagram

\[
\begin{array}{ccc}
\tilde{S}_1 \times T & \xrightarrow{\sigma_1 \times \text{id}_T} & T \times T \times S \\
\uparrow \tilde{i} & & \uparrow i \\
\tilde{S}_1 & \xrightarrow{\sigma_1} & S \\
\end{array}
\]

leads to \( \tilde{i}^{-1}(\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathbb{I}_2 \cdot \mathcal{O}_{\tilde{S}_1 \times T} \cdot \mathcal{O}_{\tilde{S}_1} = \sigma_1^{-1}i^{-1}(p_{23}^*\mathbb{I}_2) \cdot \mathcal{O}_{\tilde{S}_1} = \sigma_1^{-1}I_2 \cdot \mathcal{O}_{\tilde{S}_1} \).

Now consider the embedding of the line \( j_T : T \hookrightarrow T \times T \) fixed by the equation \( at_1 + bt_2 + c = 0 \), \( a, b, c \in k \). The corresponding fibred diagram

\[
\begin{array}{ccc}
\Sigma_{12} & \xrightarrow{\sigma_{12}} & \tilde{S}_1 \times T \\
\uparrow j_{12} & & \uparrow j_1 \\
\Sigma_{12j} & \xrightarrow{\sigma'_{1j}} & T \times S \\
\end{array}
\]

fixes notations. If the embedding \( j_T \) does not correspond to the case \( b = 0 \) then \( \Sigma_{1j} \simeq \tilde{S}_1 \) and \( j_1^{-1}(\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathbb{I}_2 \cdot \mathcal{O}_{\tilde{S}_1 \times T} \cdot \mathcal{O}_{\Sigma_{1j}} = \sigma_1^{-1}\mathbb{I}_2 \cdot \mathcal{O}_{\Sigma_{1j}} \). Otherwise (for \( b = 0 \)) we have \( \Sigma_{1j} \cong T \times S \).
The morphism $\sigma_{ij} : \widehat{\Sigma}_{ij} \to \Sigma_{ij}$ of the blowing up of the sheaf of ideals $\sigma_{ij}^{-1} \mathbb{I}_2 \cdot \mathcal{O}_{\Sigma_1}$ is included into the commutative diagram

$$
\begin{array}{ccc}
\Sigma_{12} & \xrightarrow{\sigma_{12}} & \Sigma_1 \times T \\
\downarrow & & \downarrow j_1 \\
\widehat{\Sigma}_{1j} & \xrightarrow{\sigma_{1j}} & \Sigma_{1j}
\end{array}
$$

By the universal property of the left fibred product in [1.1], there is a morphism $u : \widehat{\Sigma}_{1j} \to \Sigma_{12}$. The morphism of blowing up $\sigma_1' : \widehat{\Sigma}_{12} \to \widehat{\Sigma}_1$ of the sheaf of ideals $\sigma_1^{-1} \mathbb{I}_2 \cdot \mathcal{O}_{\Sigma_1}$ is included into the commutative diagram

$$
\begin{array}{ccc}
\widehat{\Sigma}_{12} & \xrightarrow{\sigma_1'} & \widehat{\Sigma}_2 \\
\downarrow & & \downarrow \sigma_2 \\
\widehat{\Sigma}_1 & \xrightarrow{\sigma_2} & T \times S
\end{array}
$$

Note that in this diagram $\sigma_1'$ is a morphism of blowing up of the sheaf of ideals $\sigma_2^{-1} \mathbb{I}_1 \cdot \mathcal{O}_{\Sigma_2}$ and it follows that $\widehat{\Sigma}_{21} = \widehat{\Sigma}_{12}$. Also $\widehat{\Sigma}_1$, $\widehat{\Sigma}_2$, $\widehat{\Sigma}_{12}$ are schemes which are obtained from $\Sigma$ by one or two consequent blowing ups and then each of them consists of the same number of components as $\Sigma$. Also if $\Sigma_i'$ denotes the scheme which arises when the fibre containing exceptional divisor of corresponding $\sigma_i$, $i = 1, 2$, removed, then $\widehat{\Sigma}_i := \Sigma_i'$ where closure is taken in $T$-based relative projective space with $T$-morphism $\mathbb{P}_{T,i} \leftarrow \widehat{\Sigma}_i$ respecting fibres. Then $\widehat{\Sigma}_i$ is $T$-flat. The same reasoning and conclusion is true for $\widehat{\Sigma}_{12}$. Each of schemes $\widehat{\Sigma}_1$, $\widehat{\Sigma}_2$, $\widehat{\Sigma}_{12}$ is fibred over the regular one-dimensional base $T$ with fibres isomorphic to the projective schemes. Hence [13] ch. III, Proposition 9.8 schemes $\widehat{\Sigma}_1$, $\widehat{\Sigma}_2$, $\widehat{\Sigma}_{12}$ are flat families of projective schemes over $T$. Each of these families has fibre which is isomorphic to the scheme $S$, at general enough point of $T$. This implies that each fibre of the family $\widehat{\Sigma}_{12}$ has a form of projective spectrum $\text{Proj} \left( \bigoplus_{s \geq 0} (I[t])^s / (t)^{s+1} \right)$ for an appropriate sheaf of ideals $I \subset \mathcal{O}_S$. Fibres of flat family of projective schemes carry polarisations with following property. Hilbert polynomials of fibres compute with respect to these polarisations, remain constant over the base. By the construction, such polarisations on fibres of schemes $\widehat{\Sigma}_1$, $\widehat{\Sigma}_2$, $\widehat{\Sigma}_{12}$ are exactly the same as polarisations compute in (2.5), (2.6).

Now we prove that $\Sigma_{12}$ is family of schemes which is flat over $T$. Consider the exact $\mathcal{O}_{\widehat{\Sigma}_{12}}$-triple induced by the sheaf of ideals $(\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 \cdot \mathcal{O}_{\widehat{\Sigma}_{12}}$:

$$
0 \to (\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 \cdot \mathcal{O}_{\widehat{\Sigma}_{12}} \to \mathcal{O}_{\widehat{\Sigma}_{12}} \to \mathcal{O}_Z \to 0
$$

for an appropriate closed subscheme $Z$. Apply the functor $j_1^*$ and note that the sheaf of ideals $j_1^*((\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 \cdot \mathcal{O}_{\widehat{\Sigma}_{12}})$, is isomorphic to the quotient sheaf $j_1^*(\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 \cdot \mathcal{O}_{\widehat{\Sigma}_{12}} / \text{tors}$, for the torsion subsheaf given by the equality

$$
\text{tors} = \mathcal{T} \text{or}_1^{j_1^{-1} \mathcal{O}_{\widehat{\Sigma}_{12}}} (j_1^{-1} \mathcal{O}_Z, \mathcal{O}_{\Sigma_1 j}).
$$

Note that $\Sigma_{1j} \cong \widehat{\Sigma}_1$, and $j_1^* \mathcal{O}_{\widehat{\Sigma}_{12}} \cong \mathcal{O}_{\Sigma_{1j}}$. With the last two isomorphisms taken into account we have $\mathcal{T} \text{or}_1^{j_1^{-1} \mathcal{O}_{\widehat{\Sigma}_{12}}} (j_1^{-1} \mathcal{O}_Z, \mathcal{O}_{\Sigma_1 j}) = \mathcal{T} \text{or}_1^{\sigma_{1j}} (j_1^{-1} \mathcal{O}_Z, \mathcal{O}_{\Sigma_{1j}}) = 0$. Then

$$
j_1^*(\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 \cdot \mathcal{O}_{\widehat{\Sigma}_{12}} = j_1^{-1}((\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 \cdot \mathcal{O}_{\widehat{\Sigma}_{12}}) \cdot \mathcal{O}_{\Sigma_{1j}} = \sigma_{1j}^{-1} \mathbb{I}_2 \cdot \mathcal{O}_{\Sigma_{1j}}.
$$
Also for blowups one has
\[ \Sigma_{12j} = \text{Proj} \bigoplus_{s \geq 0} (j_1^*(\sigma_1 \times \text{id}_T))^{-1} p_{23}^* \mathbb{I}_2 \cdot \mathcal{O}_{\tilde{\Sigma}_1} \] \[ \text{s.t.} \mathcal{O}_{\Sigma_1}^s = \tilde{\Sigma}_{12}. \]

Since \( \tilde{\Sigma}_{12} \) is a flat family over \( T \) then the scheme \( \Sigma_{12j} \) is also flat over \( T \).

Any two points on \( T \times T \) can be connected by a chain of two lines satisfying the condition \( b \neq 0 \). Then Hilbert polynomials of fibres of the scheme \( \Sigma_{12} \to T \times T \) are constant over the base \( T \times T \). Hence the scheme \( \Sigma_{12} \) is flat over the base \( T \times T \).

To characterise the scheme structure of the special fibre of the scheme \( \Sigma_{12} \) (and consequently the corresponding fibre of the scheme \( \Sigma_{12} \)) it is enough to consider the embedding \( j_T \) defined by the equation \( t_2 = 0 \), and a subscheme \( \tilde{\Sigma}_1 = j_1(\Sigma_{1j}) \). It is a flat family of subschemes with fibre \( \tilde{S}_1 = \text{Proj} \bigoplus_{s \geq 0}(I_1^*[t] + (t))^s/(t)^{s+1} \). As proven before, the preimage \( \tilde{\Sigma}_{12} = \sigma_{12}^{-1}(\tilde{\Sigma}_1) \) is also flat over \( j_T(T) \cong T \) with generic fibre isomorphic to \( \tilde{S}_1 = \text{Proj} \bigoplus_{s \geq 0}(I_1^*[t] + (t))^s/(t)^{s+1} \). Applying in this situation the reasoning of the article \( \text{[2]} \), we obtain that the special fibre \( \tilde{S}_{12} \) of the scheme \( \tilde{\Sigma}_{12} \) has the following scheme-theoretic characterisation: \( \tilde{S}_{12} = \text{Proj} \bigoplus_{s \geq 0}(I_2^*[t] + (t))^s/(t)^{s+1} \) for the sheaf of ideals \( I'_2 \subset \mathcal{O}_{\tilde{S}} \) defined as \( I'_2 = \sigma_1^{-1}I_1 \cdot \mathcal{O}_{\tilde{S}} \).

Hence, for any two schemes
\[ \tilde{S}_1 = \text{Proj} \bigoplus_{s \geq 0}(I_1^*[t] + (t))^s/(t)^{s+1} \]
and
\[ \tilde{S}_2 = \text{Proj} \bigoplus_{s \geq 0}(I_2^*[t] + (t))^s/(t)^{s+1} \]
the scheme \( \tilde{S}_{12} = \text{Proj} \bigoplus_{s \geq 0}(I_1^*[t] + (t))^s/(t)^{s+1} \) is defined together with morphisms \( \tilde{S}_1 \xleftarrow{\sigma_1^*} \tilde{S}_{12} \xrightarrow{\sigma_2^*} \tilde{S}_2 \), such that the diagram
\[
\begin{array}{ccc}
\tilde{S}_{12} & \xrightarrow{\sigma_2^*} & \tilde{S}_2 \\
\sigma_1^* \downarrow & & \downarrow \sigma_2 \\
\tilde{S}_1 & \xrightarrow{\sigma_2} & \tilde{S}
\end{array}
\]
commutes. Since in our (higher-dimensional) case resolution consists of several consequent morphisms \( \sigma_i \) then the proposition is to be applied iteratively. Binary operation \( (\tilde{S}_1, \tilde{S}_2) \mapsto \tilde{S}_1 \circ \tilde{S}_2 \) defined by this way, is obviously associative. Moreover, for any admissible morphism \( \sigma : \tilde{S} \to S \) there are equalities \( \tilde{S} \circ S = S \circ \tilde{S} = \tilde{S} \), then admissible morphisms of each class \( [E] \) of \( S \)-equivalent semistable coherent sheaves generate a commutative monoid \( \Diamond \) with binary operation \( \circ \) and neutral element \( \text{id}_S : S \to S \).

Note that by proposition \( \text{[3]} \), there is a bijective correspondence among subsheaves of coherent \( \mathcal{O}_S \)-sheaf \( E \) and subsheaves of the corresponding locally free \( \mathcal{O}_{\tilde{S}} \)-sheaf \( \tilde{E} \). This correspondence preserves Hilbert polynomials. Let there is a fixed Jordan-Hölder filtration in \( E \) formed by subsheaves \( F_i \). Then there is a sequence of semistable subsheaves \( \tilde{F}_i \) with the same reduced Hilbert polynomial and rank \( \tilde{F}_i = \text{rank} \tilde{F}_i \) distinguished in \( \tilde{E} \) by the described correspondence.
Let $X$ be a projective scheme, $L$ be ample invertible $\mathcal{O}_X$-sheaf, $E$ be a coherent $\mathcal{O}_X$-sheaf. Let the sheaf $E \otimes L^m$ is globally generated, namely, there is an epimorphism $q : H^0(X, E \otimes L^m) \otimes L^{(-m)} \rightarrow E$. Fix a subspace $H \subset H^0(X, E \otimes L^m)$. The subsheaf $F \subset E$ is said to be generated by the subspace $H$ if it is an image of the composite map $H \otimes L^{(-m)} \subset H^0(X, E \otimes L^m) \otimes L^{(-m)} \rightarrow E$.

**Proposition 5.** The transformation $E \mapsto \sigma^*E/\text{tors}$ is compatible on all subsheaves $F \subset E$ with the isomorphism $\nu$ for all $m \gg 0$.

**Proof.** Take an arbitrary subsheaf $F \subset E$ of rank $r'$. It is necessary to check that the subsheaf $\tilde{F} \subset \tilde{E} = \sigma^*E/\text{tors}$ generated in $\tilde{E}$ by the subspace $\nu^{-1}H^0(S, F \otimes L^m)$, coincides with the subsheaf $\sigma^*F/\text{tors}$.

It is known that the sheaf $\sigma^*E/\text{tors}$ is generated by the vectorspace $H^0(\tilde{S}, \tilde{E} \otimes L^m) = \nu^{-1}H^0(S, E \otimes L^m)$ i.e. there is an epimorphism of $\mathcal{O}_S$-modules

$$\nu^{-1}H^0(S, E \otimes L^m) \otimes \tilde{L}^{-m} \twoheadrightarrow \sigma^*E/\text{tors}.$$ 

Also we know that the subsheaf $\sigma^*F/\text{tors} \subset \sigma^*E/\text{tors}$ is include into the commutative diagram

$$\nu^{-1}H^0(S, E \otimes L^m) \otimes \sigma^*L^{-m} \twoheadrightarrow \sigma^*E/\text{tors}$$

as well as the subsheaf $\tilde{F} \subset \tilde{E}$ is include into the commutative diagram

$$\nu^{-1}H^0(S, F \otimes L^m) \otimes \tilde{L}^{-m} \twoheadrightarrow \tilde{F}$$

Combining (4.3),(4.4) together with inclusion of invertible sheaves $\sigma^*L^{-m} \subset \tilde{L}^{-m}$ we come to the inclusion $\sigma^*F/\text{tors} \subset \tilde{F}$ and the commutative diagram

$$\nu^{-1}H^0(S, E \otimes L^m) \otimes \sigma^*L^{-m} \twoheadrightarrow \sigma^*E/\text{tors}$$

Now consider global sections of sheaves $\sigma^*F/\text{tors} \otimes \tilde{L}^s$ and $\tilde{F} \otimes \tilde{L}^s$ for $s \gg 0$. They coincide along open subscheme $\tilde{U} \subset \tilde{S}$ where $\sigma : \tilde{S} \rightarrow S$ is an isomorphism $\sigma|_{\tilde{U}} : \tilde{U} \sim \rightarrow U$. Then we have an isomorphism $H^0(\tilde{S}', \sigma^*F/\text{tors} \otimes \tilde{L}^s\big|_{\tilde{S}')} \cong H^0(\tilde{S}', \tilde{F} \otimes \tilde{L}^s\big|_{\tilde{S}'})$ for all $s \gg 0$. 

20
Corollary 2. Sheaves $\tilde{F}_i = \sigma^* F_i/tors$ are semistable of rank $r_i = \text{rank } F_i$ with reduced Hilbert polynomial equal to $p_E(n)$.

Proof. The equality of ranks follows from the equality $\tilde{F}_i = \sigma^* F_i/tors$ and from the fact that the morphism $\sigma$ is an isomorphism on open subscheme in $\widetilde{S}$. The Hilbert polynomial

$s \gg 0$ and hence inclusion $\sigma^0 F/tors \subset \tilde{F}|_{\widetilde{S}_0}$ is the inclusion of sheaves with equal Hilbert polynomials. From this we conclude that they are equal along principal component $\widetilde{S}_0$ i.e. $\sigma^0 F/tors = \tilde{F}|_{\widetilde{S}_0}$. Also we have an analog of commutative diagram (4.5) for principal component $\widetilde{S}_0$:

$$
\begin{array}{ccc}
\nu^{-1}H^0(S, E \otimes L^m) \otimes \sigma^0 L^{-m} & \to & \sigma^0 E/tors \\
\nu^{-1}H^0(S, E \otimes L^m) \otimes \tilde{L}^{-m}|_{\widetilde{S}_0} & \to & \tilde{E}|_{\widetilde{S}_0} \\
\nu^{-1}H^0(S, F \otimes L^m) \otimes \sigma^0 L^{-m} & \to & \sigma^0 F/tors \\
\nu^{-1}H^0(S, F \otimes L^m) \otimes \tilde{L}^{-m}|_{\widetilde{S}_0} & \to & \tilde{F}|_{\widetilde{S}_0}
\end{array}
$$

(4.6)

Now we observe that $\tilde{E} = \delta^* \tilde{E}|_{\widetilde{S}_0}$, $\nu^{-1}H^0(S, E \otimes L^m) \otimes \tilde{L}^{-m} \to \tilde{E}$ and $\delta^* \sigma^0 F/tors = \sigma^*/tors$ then we come from (4.6) to the diagram:

$$
\begin{array}{ccc}
\nu^{-1}H^0(S, E \otimes L^m) \otimes \sigma^* L^{-m} & \to & \tilde{E} \\
\nu^{-1}H^0(S, E \otimes L^m) \otimes \tilde{L}^{-m} & \to & \tilde{E} \\
\nu^{-1}H^0(S, F \otimes L^m) \otimes \sigma^* L^{-m} & \to & \sigma^* F/tors \\
\nu^{-1}H^0(S, F \otimes L^m) \otimes \tilde{L}^{-m} & \to & \delta^* \tilde{F}|_{\widetilde{S}_0}
\end{array}
$$

(4.7)

From the left rectangle of (4.7) it follows that $\delta^* \tilde{F}|_{\widetilde{S}_0}$ is the image of $\nu^{-1}H^0(S, F \otimes L^m) \otimes \tilde{L}^{-m}$ under the evaluation morphism generating $\tilde{E}$ but this image is exactly $\tilde{F}$, i.e. $\delta^* \tilde{F}|_{\widetilde{S}_0} = \tilde{F}$. This leads to the equality $\tilde{F} = \sigma^* F/tors$ as required. This completes the proof.

We return to Jordan – Hölder filtration

$$
0 = F_0 \subset F_1 \subset \cdots \subset F_s = E
$$

of a semistable sheaf $E$.

From now $\sigma$ is understood as a composite $\sigma = \sigma_1 \circ \ldots \sigma_\ell$, i.e. it is structure morphism of admissible scheme $\tilde{S}$.

Corollary 2. Sheaves $\tilde{F}_i = \sigma^* F_i/tors$ are semistable of rank $r_i = \text{rank } F_i$ with reduced Hilbert polynomial equal to $p_E(n)$.

Proof. The equality of ranks follows from the equality $\tilde{F}_i = \sigma^* F_i/tors$ and from the fact that the morphism $\sigma$ is an isomorphism on open subscheme in $\tilde{S}$. The Hilbert polynomial
for all $n \gg 0$ is fixed by the equalities $\chi(\tilde{F}_i \otimes \tilde{L}^n) = h^0(S, \tilde{F}_i \otimes \tilde{L}) = h^0(S, F_i \otimes L^n) = \chi(F_i \otimes L^n) = r p_E(n)$. □

For $\tilde{E} = \sigma^* E/\text{tors}$ consider epimorphisms $\tilde{q} : H^0(S, \tilde{E} \otimes \tilde{L}^m) \otimes \tilde{L}(-m) \to \tilde{E}$ and $q : H^0(S, E \otimes L^m) \otimes L(-m) \to E$.

**Definition 9.** Subsheaves $\tilde{F} \subset \tilde{E}$ and $F \subset E$ are called $v$-**corresponding** if there exist subspaces $\tilde{V} \subset H^0(S, \tilde{E} \otimes L^m)$ and $V = v(\tilde{V}) \subset H^0(S, E \otimes L^m)$ such that $\tilde{q}(\tilde{V} \otimes \tilde{L}^{-m}) = \tilde{F}$, $q(V \otimes L^{-m}) = F$. Notation: $F = v(\tilde{F})$. The corresponding quotient sheaves $\tilde{E}/\tilde{F}$ and $E/F$ will be also called $v$-**corresponding** and denoted $E/F = v(\tilde{E}/\tilde{F})$.

**Proposition 6.** The transformation $E \mapsto \sigma^* E/\text{tors}$ takes saturated subsheaves to saturated subsheaves.

**Proof.** Let $F_{i-1} \subset F_i$ be a saturated subsheaf. Assume that the quotient sheaf $\tilde{F}_i/\tilde{F}_{i-1}$ has a subsheaf of torsion $\tau$. This subsheaf is generated by vector subspace $\tilde{T} \subset H^0(S, \tilde{F}_i \otimes \tilde{L}^m)/H^0(S, \tilde{F}_{i-1} \otimes \tilde{L}^m)$. Let $\tilde{T}'$ be its preimage in $H^0(S, \tilde{F}_i \otimes \tilde{L}^m)$ and $T' \subset \tilde{F}_i$ be a subsheaf generated by subspace $\tilde{T}'$. Then there is a sheaf epimorphism $T' \to \tau$ with kernel $T' \cap \tilde{F}_{i-1}$. Let $\tilde{K} = H^0(S, (T' \cap \tilde{F}_{i-1}) \otimes \tilde{L}^m) \subset H^0(S, \tilde{F}_{i-1} \otimes \tilde{L}^m)$ be its generating subspace. Then the isomorphism $v$ leads to the exact diagram of vector spaces

$$
\begin{array}{c}
0 \to v(\tilde{K}) \xrightarrow{i} v(\tilde{T}') \xrightarrow{v} v(\tilde{T}')/v(\tilde{K}) \to 0 \\
\downarrow v \quad \downarrow v \quad \downarrow v \quad \downarrow \pi \\
0 \to \tilde{K} \xrightarrow{i} \tilde{T}' \xrightarrow{\pi} \tilde{T}'/\tilde{K} \to 0
\end{array}
$$

with morphism $\pi$ induced by the morphism $v$. Also there are exact sequences of $v$-corresponding coherent sheaves

$$
\begin{array}{c}
0 \to v(T' \cap \tilde{F}_{i-1}) \to v(T') \xrightarrow{v(\tau)} v(\tau) \to 0, \\
0 \to T' \cap \tilde{F}_{i-1} \to T' \to \tau \to 0.
\end{array}
$$

Sheaves $T' \cap \tilde{F}_{i-1}$, $v(T' \cap \tilde{F}_{i-1})$, $T'$ and $v(T')$ coincide under restriction on open subsets $W$ and $\sigma(W)$ respectively. Then if $\tau$ is a torsion sheaf, then $v(\tau)$ is also torsion sheaf. This contradicts saturatedness of the subsheaf $F_{i-1}$. □

**Corollary 3.** There are isomorphisms $\sigma^*(F_i/F_{i-1})/\text{tors} \cong \tilde{F}_i/\tilde{F}_{i-1}$.

**Proof.** Take an exact triple

$$
0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \to 0
$$

and apply the functor $\sigma^*$. This yields

$$
0 \to \sigma^* F_{i-1}/\tau \to \sigma^* F_i \to \sigma^* F_i/F_{i-1} \to 0
$$

22
where the symbol \( \tau \) denotes the subsheaf of torsion violating exactness. Factoring first two sheaves by torsion and applying the proposition 5 one has an exact diagram

\[
\begin{array}{cccc}
0 & N & \text{tors}(\sigma^*F_i) & \tau' & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \sigma^*F_{i-1}/\tau & \sigma^*F_i & \sigma^*(F_i/F_{i-1}) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \tilde{F}_{i-1} & \tilde{F}_i & \tilde{F}_i/F_{i-1} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where the sheaf \( N \) is defined as \( \ker (\sigma^*F_{i-1}/\tau \to \tilde{F}_{i-1}) \). It rests to note that the sheaf \( \tau' \) is torsion sheaf. Also since the subsheaf \( F_{i-1} \subset F_i \) is saturated then due to the proposition 6 the sheaf \( \tilde{F}_i/F_{i-1} \) has no torsion. Then \( \tilde{F}_i/F_{i-1} \cong \sigma^*(F_i/F_{i-1})/\text{tors} \).

**Corollary 4.** Quotient sheaves \( \tilde{F}_i/F_{i-1} \) are stable and their reduced Hilbert polynomial is equal to \( p_E(n) \).

**Proof.** Consider a subsheaf \( \tilde{R} \subset \tilde{F}_i/F_{i-1} \) and the space of global sections

\[
H^0(\tilde{S}, \tilde{R} \otimes \tilde{L}^m) \subset H^0(\tilde{S}, (\tilde{F}_i/F_{i-1}) \otimes \tilde{L}^m) = H^0(\tilde{S}, \tilde{F}_i \otimes \tilde{L}^m)/H^0(\tilde{S}, \tilde{F}_{i-1} \otimes \tilde{L}^m).
\]

We assume as usually \( m \) to be as big as higher cohomology groups vanish. Let \( \tilde{H} \) be the preimage of subspace \( H^0(\tilde{S}, \tilde{R} \otimes \tilde{L}^m) \) in \( H^0(\tilde{S}, \tilde{F}_i \otimes \tilde{L}^m) \), and \( H := v(\tilde{H}) \). Denote by \( \mathcal{H} \) a subsheaf in \( \tilde{F}_i \) if this subsheaf is generated by the subspace \( \tilde{H} \). It is clear that \( \mathcal{H}/F_{i-1} \subset F_i/F_{i-1} \). From the chain of obvious equalities

\[
h^0(\tilde{S}, \tilde{R} \otimes \tilde{L}^m) = \dim \tilde{H} - h^0(\tilde{S}, \tilde{F}_{i-1} \otimes \tilde{L}^m) = \dim H - h^0(S, F_{i-1} \otimes L^m)
\]

\[
= h^0(S, \mathcal{H} \otimes L^m) - h^0(S, F_{i-1} \otimes L^m) = h^0(S, (\mathcal{H}/F_{i-1}) \otimes L^m)
\]

it follows that

\[
\frac{h^0(\tilde{S}, (\tilde{F}_i/F_{i-1}) \otimes \tilde{L}^m)}{\text{rank } (\tilde{F}_i/F_{i-1})} - \frac{h^0(\tilde{S}, \tilde{R} \otimes \tilde{L}^m)}{\text{rank } \tilde{R}} = \frac{h^0(S, (\mathcal{H}/F_{i-1}) \otimes L^m)}{\text{rank } (\mathcal{H}/F_{i-1})} > 0.
\]

This proves stability of the quotient sheaf \( \tilde{F}_i/F_{i-1} \). \( \Box \)

Now consider the exact triple \( 0 \to E_1 \to E \to \text{gr}_1(E) \to 0 \) and the corresponding triple of spaces of global sections

\[
0 \to H^0(S, E_1 \otimes L^m) \to H^0(S, E \otimes L^m) \to H^0(S, \text{gr}_1(E) \otimes L^m) \to 0.
\]
It is exact for $m \gg 0$. The transition to the corresponding $O_{\tilde{S}}$-sheaf $\tilde{E}$, to its subsheaf $\tilde{E}_1$, to global sections, and application of the isomorphism $\nu$, lead to the commutative diagram of vector spaces

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(S, E_1 \otimes L^m) \\
& \downarrow_{\nu} & \downarrow_{\nu} \\
0 & \longrightarrow & H^0(S, E \otimes L^m)
\end{array}
\begin{array}{ccc}
& \longrightarrow & H^0(S, \text{gr}_1(E) \otimes L^m) \\
& \downarrow_{\pi} & \downarrow_{\pi} \\
& \longrightarrow & 0
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & H^0(\tilde{S}, \tilde{E}_1 \otimes \tilde{L}^m) \\
& \downarrow_{\nu} & \downarrow_{\nu} \\
0 & \longrightarrow & H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m)
\end{array}
\begin{array}{ccc}
& \longrightarrow & H^0(\tilde{S}, \text{gr}_1(\tilde{E}) \otimes \tilde{L}^m) \\
& \downarrow_{\pi} & \downarrow_{\pi} \\
& \longrightarrow & 0
\end{array}
$$

where the isomorphism $\pi$ is induced by the isomorphism $\nu$. Continuing the reasoning inductively for rest subsheaves of Jordan–Hölder filtration of the sheaf $E$, we get that bijective correspondence of subsheaves is continued onto quotients of filtrations. Then the transition from quotient sheaves $E_i/E_{i-1}$ to $\tilde{E}_i/\tilde{E}_{i-1}$ preserves Hilbert polynomials and stability.

**Definition 10.** Jordan–Hölder filtration of semistable vector bundle $\tilde{E}$ with Hilbert polynomial equal to $rp(n)$, is a sequence of semistable subsheaves $0 \subset F_1 \subset \cdots \subset F_s \subset \tilde{E}$ with reduced Hilbert polynomials equal to $p(n)$, such that quotient sheaves $\text{gr}_i(\tilde{E}) = \tilde{F}_i/\tilde{F}_{i-1}$ are stable.

The sheaf $\bigoplus_i \text{gr}_i(\tilde{E})$ will be called as associated polystable sheaf for the bundle $\tilde{E}$.

Then it follows from the results of propositions and corollaries that the transformation $E \mapsto \sigma^* E/\text{tors}$ takes the Jordan–Hölder filtration of the sheaf $E$ to Jordan–Hölder filtration of the bundle $\sigma^* E/\text{tors}$.

Let $(\tilde{S}, \tilde{E})$ and $(\tilde{S}', \tilde{E}')$ be semistable pairs.

**Definition 11.** Semistable pairs $(\tilde{S}, \tilde{E})$ and $(\tilde{S}', \tilde{E}')$ are called $M$-equivalent (monoidally equivalent) if for morphisms of $\diamond$-product $\tilde{S} \diamond \tilde{S}'$ to factors $\sigma' : \tilde{S} \diamond \tilde{S}' \to \tilde{S}$ and $\sigma : \tilde{S} \diamond \tilde{S}' \to \tilde{S}'$ and for associated polystable sheaves $\bigoplus_i \text{gr}_i(\tilde{E})$ and $\bigoplus_i \text{gr}_i(\tilde{E}')$ there are isomorphisms

$$
\sigma^* \bigoplus_i \text{gr}_i(\tilde{E})/\text{tors} \cong \tilde{\sigma}' \bigoplus_i \text{gr}_i(\tilde{E}')/\text{tors}.
$$

**Proposition 7.** $S$-equivalent semistable coherent sheaves $E$ and $E'$ correspond to $M$-equivalent semistable pairs $(\tilde{S}, \tilde{E})$ and $(\tilde{S}', \tilde{E}')$.

**Proof.** Resolution takes semistable coherent sheaf $E$ to semistable pair $(\tilde{S}, \tilde{E})$. Jordan–Hölder filtration of the sheaf $E$ is taken to Jordan–Hölder filtration of the bundle $\tilde{E}$. Then the polystable sheaf $\bigoplus_i \text{gr}_i(E)$ is taken to the associated polystable sheaf $\bigoplus_i \text{gr}_i(\tilde{E})$. Hence we have for the sheaf $E$

$$
\sigma^* \bigoplus_i \text{gr}_i(E)/\text{tors} = \tilde{\sigma}' \bigoplus_i \text{gr}_i(\tilde{E})/\text{tors} = \sigma^* \sigma^* \bigoplus_i \text{gr}_i(E)/\text{tors}.
$$

Analogously for a sheaf $E'$ which is $S$-equivalent to the sheaf $E$ one has

$$
\sigma^* \bigoplus_i \text{gr}_i(E')/\text{tors} = \sigma^* \sigma^* \bigoplus_i \text{gr}_i(E')/\text{tors}.
$$
Right hand sides of (4.8) and (4.9) are isomorphic by the isomorphism of polystable \(O_S\)-sheaves \(\bigoplus_i \text{gr}_i(E) \cong \bigoplus_i \text{gr}_i(E')\) and by commutativity of diagram

\[
\begin{array}{c}
\tilde{S} \circ \tilde{S}' \xrightarrow{\sigma} \tilde{S}' \\
\downarrow \sigma \quad \downarrow \sigma' \\
\tilde{S} \xrightarrow{\sigma} S
\end{array}
\]

for \(\circ\)-product. The proposition is proven. \(\square\)

We turn again to the polystable sheaf \(\text{gr}(E) = \bigoplus_i \text{gr}_i(E)\) in the given \(S\)-equivalence class. Let \(\sigma_{\text{gr}} : \tilde{S}_{\text{gr}} \to S\) be the corresponding canonical morphism for standard resolution defined by the sequence of sheaves of ideals \((I_i)_i, i = 1, \ldots, \ell\).

**Proposition 8.** For all \(E \in \mathcal{E}\) sheaves \(\sigma^*E/\text{tors}\) are locally free.

**Proof.** The sheaf \(\sigma^*\text{gr}(E)/\text{tors}\) is locally free. This implies that all direct summands \(\sigma^*\text{gr}_1(E)/\text{tors}\) are also locally free. Consider inductively following exact sequences

\[
\begin{align*}
0 & \to \sigma^*F_1/\text{tors} = \sigma^*\text{gr}_1(E)/\text{tors} \to 0, \\
0 & \to \sigma^*F_2/\text{tors} \to \sigma^*F_2/\text{tors} \to \sigma^*\text{gr}_2(E)/\text{tors} \to 0, \\
& \vdots \notag \\
0 & \to \sigma^*F_{l-1}/\text{tors} \to \sigma^*E/\text{tors} \to \sigma^*\text{gr}_l(E)/\text{tors} \to 0,
\end{align*}
\]

induced by Jordan – Hölder filtration of any semistable sheaf \(E\) of the given \(S\)-equivalence class. It follows that the sheaf \(\sigma^*E/\text{tors}\) is locally free. \(\square\)

Now come to the global version of \(M\)-equivalence. Prove that on the set of all \(T\)-flat families of admissible schemes of the view \((\pi : \tilde{\Sigma} \to T, \tilde{\mathbb{L}})\) there is a commutative binary operation

\[
\diamond : ((\pi : \tilde{\Sigma} \to T, \tilde{\mathbb{L}}), (\pi' : \tilde{\Sigma}' \to T, \tilde{\mathbb{L}}')) \mapsto (\pi_\Delta : \tilde{\Sigma}_\Delta \to T, \tilde{\mathbb{L}}_\Delta)
\]

- there is a commutative diagram of schemes

\[
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\tilde{\sigma}} & \tilde{\Sigma}' \\
\downarrow \pi & \downarrow \pi' \quad \downarrow \pi' \\
\tilde{\Sigma}_\Delta & \xrightarrow{\tilde{\sigma}_\Delta} & \tilde{\Sigma}'_{\Delta}
\end{array}
\]

- \((\pi_\Delta : \tilde{\Sigma}_\Delta \to T, \tilde{\mathbb{L}}_\Delta)\) is flat over \(T\);  
- if \(\chi(\tilde{\mathbb{L}}^n|_{\pi^{-1}(t)}) = \chi(\tilde{\mathbb{L}}^n|_{\pi'^{-1}(t)})\) then \(\chi(\tilde{\mathbb{L}}^m|_{\pi^{-1}(t)}) = \chi(\tilde{\mathbb{L}}^m|_{\pi^{-1}(t)})\);  
- if \(\chi(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^n|_{\pi^{-1}(t)}) = \chi(\tilde{\mathbb{E}}' \otimes \tilde{\mathbb{L}}^m|_{\pi'^{-1}(t)})\) then \(\chi(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m|_{\pi^{-1}(t)}) = \chi(\tilde{\mathbb{E}}' \otimes \tilde{\mathbb{L}}^n|_{\pi'^{-1}(t)})\);  
- if \(\chi(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^n|_{\pi^{-1}(t)}) = \chi(\tilde{\mathbb{E}}' \otimes \tilde{\mathbb{L}}^m|_{\pi'^{-1}(t)})\) then \(\chi(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m|_{\pi^{-1}(t)}) = \chi(\tilde{\mathbb{E}}' \otimes \tilde{\mathbb{L}}^n|_{\pi'^{-1}(t)})\).

Let \(T, S\) be schemes over a field \(k\), \(\pi : \tilde{\Sigma} \to T\) a morphism of \(k\)-schemes. We remind the following
Definition 12. [9, definition 5] The family of schemes $\pi : \tilde{\Sigma} \to T$ is birationally $S$-trivial if there exist isomorphic open subschemes $\tilde{\Sigma}_0 \subset \tilde{\Sigma}$ and $\Sigma_0 \subset T \times S$ and there is a scheme equality $\pi(\tilde{\Sigma}_0) = T$.

The former equality in particular means that all fibres of the morphism $\pi$ have nonempty intersections with the open subscheme $\tilde{\Sigma}_0$.

In particular, if $T = \text{Spec} \ k$ then $\pi$ is a constant morphism and $\tilde{\Sigma}_0 \cong \Sigma_0$ is open subscheme in $S$.

Since in the present paper we consider only birationally $S$-trivial families, they will be referred to as birationally trivial families.

Since we are interested in birationally trivial families of admissible schemes we can assume that there are commutative diagrams

\[
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\sigma} & \Sigma \\
\downarrow{\pi} & & \downarrow{\pi} \\
T & = & T \\
\end{array}
\quad
\begin{array}{ccc}
\tilde{\Sigma}' & \xrightarrow{\sigma'} & \Sigma' \\
\downarrow{\pi'} & & \downarrow{\pi'} \\
T & = & T \\
\end{array}
\]

where morphisms $\sigma$ and $\sigma'$ are birational and projective.

Now we concentrate on one point case when $T = \text{Spec} \ k$ and $\tilde{\Sigma} = \tilde{S}$, $\tilde{\Sigma}' = \tilde{S}'$ are admissible schemes. Then they can be obtained as zero fibres by consequent blowups of one-parameter family $\Sigma = \Sigma_0 = T \times S$ for $T = \text{Spec} \ k[t]$:

$\Sigma_0 \xleftarrow{\sigma_1} \Sigma_1 \xleftarrow{\sigma_2} \Sigma_2 \ldots \xleftarrow{\sigma_s} \Sigma_s = \tilde{\Sigma}$, $\sigma_i = b_1 \mathbb{I}_i$, $\mathbb{I}_i \subset \mathcal{O}_{\Sigma_{i-1}}$, $i = 1, \ldots, s$;

$\Sigma_0 \xleftarrow{\sigma'_1} \Sigma_1' \xleftarrow{\sigma'_2} \Sigma_2' \ldots \xleftarrow{\sigma'_{s'}} \Sigma_{s'}' = \tilde{\Sigma}'$, $\sigma'_i = b_1' \mathbb{I}_i'$, $\mathbb{I}_i' \subset \mathcal{O}_{\Sigma_{i-1}}$, $i = 1, \ldots, s'$.

For the first step consider morphisms $\sigma_1$ and $\sigma_1'$ and inverse images

$\mathbb{I}_1 = \sigma_1^{-1} \mathbb{I}_1 \cdot \mathcal{O}_{\Sigma_1}$, $\mathbb{I}_1' = \sigma_1'^{-1} \mathbb{I}_1 \cdot \mathcal{O}_{\Sigma_1}$

and corresponding blown up schemes

$\widehat{\Sigma}_{11} = \text{Bl}_{\Sigma_1} \mathbb{I}_1 \xrightarrow{\sigma_1} \Sigma_1$, $\widehat{\Sigma}'_{11} = \text{Bl}_{\Sigma_1'} \mathbb{I}_1' \xrightarrow{\sigma_1'} \Sigma_1'$.

By universality of blowups these schemes are supplied with morphisms

$\widehat{\Sigma}_{11} \to \Sigma_1'$, $\widehat{\Sigma}'_{11} \to \Sigma_1$.

Schemes $\widehat{\Sigma}_{11}$ and $\widehat{\Sigma}'_{11}$ are irreducible and reduced and have morphisms to $\Sigma_1 \times_T \Sigma_1'$ and both schemes are realised as a closure of diagonal immersion of the open set $\Sigma_1 \supset U \subset \Sigma_1$, $U = (T \setminus 0) \times S$. Hence $\widehat{\Sigma}_{11} = \widehat{\Sigma}'_{11}$ and hence we come to the commutative square

\[
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\sigma_1'} & \Sigma_{11} \\
\downarrow{\sigma_1} & & \downarrow{\sigma_1} \\
\Sigma_0 & \xrightarrow{\sigma_1} & \Sigma_1 \\
\end{array}
\]

and projection morphism $\pi_{11} : \widehat{\Sigma}_{11} \xrightarrow{\sigma_1 \circ \sigma_1'} \Sigma_0 \xrightarrow{\pi} T$. If $\Sigma_1$ has fibrewise uniform Hilbert polynomial $\chi(\mathbb{L}_1|_{\pi^{-1}(t)})$ if it is computed with respect to $\mathbb{L}_1 = \sigma_1^* \mathbb{L} \otimes \sigma_1^{-1} \mathbb{I}_1 \cdot \mathcal{O}_{\Sigma_1}$ and
\( \Sigma'_i \) has fibrewise uniform Hilbert polynomial \( \chi(\mathbb{L}'_i|_{\pi'_i(t)}) \) if it is computed with respect to \( \mathbb{L}'_i = \sigma'_i^*\mathbb{L} \otimes \sigma'_i^{-1}\mathbb{P}'_i \cdot \mathcal{O}_{\Sigma'_i} \) then \( \Sigma_{11} \) has also fibrewise uniform Hilbert polynomial \( \chi(\mathbb{L}'_{11}|_{\pi_{11}'(t)}) \) if it is computed with respect to

\[
\mathbb{L}'_{11} = \sigma'_{11}^*\mathbb{L} \otimes \sigma'_{11}^{-1}\sigma'_{11}^{-1}\mathbb{P}'_{11} \cdot \mathcal{O}_{\Sigma_{11}} \cdot \sigma'_{11}^*\mathbb{L} \otimes \sigma'_{11}^{-1}\mathbb{P}'_{11} \cdot \mathcal{O}_{\Sigma_{11}} \cdot \sigma'_{11}^*\mathbb{L} \otimes \sigma'_{11}^{-1}\mathbb{P}'_{11} \cdot \mathcal{O}_{\Sigma_{11}}.
\]

Iterating the process one comes to the scheme \( \tilde{\Sigma}_{ij} \) included in the commutative square

![Diagram](image)

with projection morphism \( \pi_{ij} = p \circ \sigma_{ij} : \tilde{\Sigma}_{ij} \to T \) for any pair \( i, j \). Also morphism \( \pi_i : \Sigma_i \to T \) is flat with fibrewise Hilbert polynomial being uniform over \( t \in T \) if it is computed with respect to the polarisation

\[
\mathbb{L}_i = \sigma_i^* \cdots \sigma_i^*\mathbb{L} \otimes \sigma_i^{-1} \cdots \sigma_i^{-1}\mathbb{I}_1 \otimes \cdots \otimes \sigma_i^{-1}\mathbb{I}_1 \cdot \mathcal{O}_{\Sigma_i},
\]

(4.10)

morphism \( \pi'_j : \Sigma'_j \to T \) is also flat with fibrewise Hilbert polynomial being uniform over \( t \in T \) if it is computed with respect to the polarisation

\[
\mathbb{L}'_j = \sigma'_j^* \cdots \sigma'_j^*\mathbb{L} \otimes \sigma'_j^{-1} \cdots \sigma'_j^{-1}\mathbb{I}_1 \otimes \cdots \otimes \sigma'_j^{-1}\mathbb{I}_1 \cdot \mathcal{O}_{\Sigma'_j},
\]

(4.11)

and as well the morphism \( \pi_{ij} : \Sigma_{ij} \to T \) is flat with fibrewise Hilbert polynomial being uniform over \( t \in T \) if it is computed with respect to the polarisation

\[
\mathbb{L}_{ij} = \sigma_{ij}^* \mathbb{L} \otimes (\sigma'_{i} \circ \cdots \circ \sigma'_{j})^{-1}((\sigma_i^{-1} \cdots \sigma_1^{-1}\mathbb{I}_1) \otimes \cdots \otimes \sigma_i^{-1}\mathbb{I}_1) \\
\otimes (\sigma_1 \circ \cdots \circ \sigma_i^{-1}((\sigma_j^{-1} \cdots \sigma_j^{-1}\mathbb{I}_1) \otimes \cdots \otimes \sigma_j^{-1}\mathbb{I}_1) \cdot \mathcal{O}_{\Sigma_{ij}}.
\]

(4.12)

If continue on \( i, j \) we come to \( \widetilde{\mathbb{L}}_\Delta \) as fibrewise polarisation for \( \tilde{\Sigma}_\Delta \).

**Remark 11.** If necessary, as in [2.5] there are mentioned some degrees of ideals \( \mathbb{L}_i, i = 1, \ldots, \ell, \mathbb{P}'_j, j = 1, \ldots, \ell \), in (4.10), (4.11), (4.12).

Now we address to the preimages on \( \tilde{\Sigma}_\Delta \) of locally free sheaves of \( \mathcal{O}_{\Sigma} \)-modules \( \widetilde{\mathbb{E}} \) and of \( \mathcal{O}_{\Sigma'} \)-modules \( \mathbb{E}' \). Since \( \widetilde{\mathbb{E}} \) is locally free then \( \sigma'^*\widetilde{\mathbb{E}} \) is also locally free; \( \pi_\Delta : \tilde{\Sigma}_\Delta \to T \) is flat morphism and hence \( \sigma'^*\mathbb{E}' \) is flat over \( T \), by [15] Proposition 7.9.14 \( \pi_\Delta*(\sigma'^*\widetilde{\mathbb{E}} \otimes \mathbb{L}'_n) \) is locally free sheaf and by [15] Corollary 7.9.13 fibrewise Hilbert polynomial \( \chi(\sigma'^*\widetilde{\mathbb{E}} \otimes \mathbb{L}'_n|_{\pi_\Delta'(t)}) \) is constant over \( t \in T \). The sheaf of \( \mathcal{O}_{\pi_\Delta'(t)} \)-modules \( \sigma'^*\widetilde{\mathbb{E}} \otimes \mathbb{L}'_n|_{\pi_\Delta'(t)} \) can be obtained as a zero fibre of the appropriately blown up family Spec \( k[t'] \times \pi^{-1}(t) \leftrightarrow \Sigma \) (the centre of blowing up is concentrated in zero fibre) with inverse image \( \sigma'^*\mathbb{E}' \mathcal{O}_{\Sigma'|_{\pi_\Delta'(t)}} \). The blowup morphism \( \pi \) is assumed to be a composite of consequent blowups in sequence of sheaves of ideals \( I_1[t'] + (t'), \ldots, I_s[t'] + (t') \). Distinguished polarisation \( \widetilde{\mathbb{E}} \) for flat family \( \pi : \Sigma \to \text{Spec} k[t'] \) takes a view \( \mathbb{E} = \sigma^*pr_2^*\mathbb{L}_{\pi^{-1}(t)} \otimes \sigma^{-1} [I_1, \ldots, I_s] \cdot \mathcal{O}_{\Sigma|_{\pi^{-1}(t)}} \) where \( \sigma^{-1}[I_1, \ldots, I_s] \) is some product of inverse images of \( I_i[t'] + (t'), i = 1, \ldots, s \). The restriction of \( \mathbb{E} \) to zero fibre takes the view \( \mathbb{E}|_{\pi^{-1}(0)} = \mathbb{E}_{\pi^{-1}(0)} \). The inverse image of \( \sigma^*\mathbb{E}|_{\pi^{-1}(t)} \) has Hilbert polynomial

\[
\chi(\sigma^*\mathbb{E} \otimes \mathbb{L}'_n|_{\pi^{-1}(t)}) = \chi(\sigma^*\mathbb{E}|_{\pi^{-1}(t)} \otimes \mathbb{L}'_n|_{\pi(u)}) = \chi(\omega|_{\pi^{-1}(t)} \otimes \mathbb{L}'_n|_{\pi(u)})
\]
where $u \neq 0$. By the construction the binary operation is commutative and leads to $M$-equivalence of pairs which are obtained by resolutions of the same sheaf $E$ or of two $S$-equivalent sheaves $E, E'$ on $S$.

## 5 Moduli functors

Following [14] ch. 2, sect. 2.2, we recall some definitions. Let $C$ be a category, $C^o$ its dual, $C' = \text{Funct}(C^o, \text{Sets})$ category of functors to the category of sets. By Yoneda’s lemma, the functor $C \to C': F \mapsto (F': X \mapsto \text{Hom}_C(X,F))$ includes $C$ into $C'$ as full subcategory.

**Definition 13.** [14] ch. 2, definition 2.2.1] The functor $f \in \text{Ob}C'$ is *corepresented by the object* $M \in \text{Ob}C$, if there exist a $C'$-morphism $\psi: f \to M$ such that any morphism $\psi': f \to E'$ factors through the unique morphism $\omega: M \to E'$.

**Definition 14.** The scheme $\widetilde{M}$ is a *coarse moduli space* for the functor $f$ if $f$ is corepresented by the scheme $\widetilde{M}$.

We consider sets of families of semistable pairs

$$\mathfrak{F}_T = \left\{ \begin{array}{l}
\pi: \tilde{\Sigma} \to T \text{ birationally } S\text{-trivial}, \\
\tilde{L} \in \text{Pic}\tilde{\Sigma} \text{ flat over } T, \\
\text{for } m \gg 0 \text{ } \tilde{L}^m \text{ very ample relatively } T, \\
\forall t \in T \text{ } \tilde{L}_t = \tilde{L}|_{\pi^{-1}(t)} \text{ ample;} \\
(\pi^{-1}(t), \tilde{L}_t) \text{ admissible scheme with distinguished polarisation;} \\
\chi(\tilde{L}_t^n) \text{ does not depend on } t, \\
\tilde{E} \text{ locally free } \mathcal{O}_\Sigma - \text{sheaf flat over } T; \\
\chi(\tilde{E} \otimes \tilde{L}_t^n)|_{\pi^{-1}(t)} = rp(n); \\
(\pi^{-1}(t), \tilde{L}_t, \tilde{E}|_{\pi^{-1}(t)}) \text{ - semistable pair}
\end{array} \right\}$$

(5.1)

and a functor

$$f: (\text{Schemes}_k)^o \to (\text{Sets})$$

(5.2)

from the category of $k$-schemes to the category of sets. It attaches to any scheme $T$ the set of equivalence classes of families of the form $(\mathfrak{F}_T/\sim)$.

The equivalence relation $\sim$ is defined as follows. Families $((\pi: \tilde{\Sigma} \to T, \tilde{L}), \tilde{E})$ and $((\pi': \tilde{\Sigma} \to T, \tilde{L}'), \tilde{E}')$ from the class $\mathfrak{F}_T$ are said to be equivalent (notation: $((\pi: \tilde{\Sigma} \to T, \tilde{L}), \tilde{E}) \sim ((\pi': \tilde{\Sigma} \to T, \tilde{L}'), \tilde{E}')$) if

1) there exist an isomorphism $\iota: \tilde{\Sigma} \sim \tilde{\Sigma}'$ such that the diagram

$$\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\iota} & \tilde{\Sigma}' \\
\pi \downarrow & & \pi' \\
T & & T
\end{array}$$

commutes.

2) There exist line bundles $L', L''$ on the scheme $T$ such that $\iota^*\tilde{E}' = \tilde{E} \otimes \pi^*L'$, $\iota^*\tilde{L}' = \tilde{L} \otimes \pi^*L''$.

Now discuss what is the "size" of $\widetilde{\Sigma}_0$ which is maximal under inclusion of those open subschemes in a family of admissible schemes $\tilde{\Sigma}$, which are isomorphic to appropriate
open subschemes in $T \times S$ in the definition. The set $F = \tilde{\Sigma} \setminus \Sigma_0$ is closed. If $T_0$ is open subscheme in $T$ whose points carry fibres isomorphic to $S$, then $\Sigma_0 \supseteq \pi^{-1}T_0$ (inequality is true because $\pi(\Sigma_0) = T$ in the definition). The subscheme $\Sigma_0$ which is open in $T \times S$ and isomorphic to $\tilde{\Sigma}_0$, is such that $\Sigma_0 \supseteq T_0 \times S$. If $\pi : \tilde{\Sigma} \to T$ is a family of admissible schemes then $\tilde{\Sigma}_0 \cong \tilde{\Sigma} \setminus F$, and $F$ is (set-theoretically) the union of additional components of fibres which are non-isomorphic to $S$. Particularly, this means that $\text{codim}_{T \times S}(T \times S) \setminus \Sigma_0 \geq 2$.

The Gieseker – Maruyama functor

$$f^{GM} : (\text{Schemes}_k)^{\circ} \to \text{Sets},$$

attaches to any scheme $T$ the set of equivalence classes of families of the following form $\mathfrak{S}_T^{GM} / \sim$, where

$$\mathfrak{S}_T^{GM} = \left\{ \begin{array}{l}
\mathbb{E} \text{ sheaf of } \mathcal{O}_{T \times S} - \text{modules flat over } T;
\mathbb{L} \text{ invertible sheaf of } \mathcal{O}_{T \times S} - \text{modules},
\text{ample relatively to } T
\end{array} \right\}$$

and such that $L_t := \mathbb{L}|_{t \times S} \cong L$ for any point $t \in T$; $E_t := \mathbb{E}|_{t \times S}$ torsion-free and Gieseker-semistable; $\chi(E_t \otimes L^n) = rp(n)$.

Families $(\mathbb{E}, \mathbb{L})$ and $(\mathbb{E}', \mathbb{L}')$ from the class $\mathfrak{S}_T^{GM}$ are said to be equivalent (notation: $(\mathbb{E}, \mathbb{L}) \sim (\mathbb{E}', \mathbb{L}')$), if there exist line bundles $L', L''$ on the scheme $T$ such that $\mathbb{E}' = \mathbb{E} \otimes p^*L'$, $\mathbb{L}' = \mathbb{L} \otimes p^*L''$ where $p : T \times S \to T$ is projection onto the first factor.

**Remark 12.** Since $\text{Pic}(T \times S) = \text{Pic}T \times \text{Pic}S$, our definition of the moduli functor $f^{GM}$ is equivalent to the standard definition which can be found, for example, in [14]: the difference in choice of polarisations $\mathbb{L}$ and $\mathbb{L}'$ having isomorphic restrictions on fibres over the base $T$, is avoided by the equivalence which is induced by tensoring by inverse image of an invertible sheaf $L''$ from the base $T$.

**Remark 13.** The procedure of standard resolution developed in [1] and outlined in Sec.2 provides a natural transformation $\kappa : f^{GM} \to f$ mentioned in part (i) of Theorem 2.

### 6 Pairs-to-GM transformation

Further we show that there is a morphism of the nonreduced moduli functor of admissible semistable pairs to the nonreduced Gieseker – Maruyama moduli functor. Namely, for any scheme $T$ we build up a correspondence $((\pi : \tilde{\Sigma} \to T, \tilde{\mathbb{L}}, \tilde{\mathbb{E}}) \mapsto (\mathbb{L}, \mathbb{E}))$. It leads to a set mapping $\left\{ ((\pi : \tilde{\Sigma} \to T, \tilde{\mathbb{L}}, \tilde{\mathbb{E}}) / \sim) \right\} \to \left\{ (\mathbb{L}, \mathbb{E}) / \sim \right\}$. This means that the family of semistable coherent torsion-free sheaves $\mathbb{E}$ with the same base $T$ can be constructed by any family $((\pi : \tilde{\Sigma} \to T, \tilde{\mathbb{L}}, \tilde{\mathbb{E}})$ of admissible semistable pairs which is birationally trivial and flat over $T$.

First we construct a $T$-morphism $\phi : \tilde{\Sigma} \to T \times S$. Since the family $\pi : \tilde{\Sigma} \to T$ is birationally trivial there is a fixed isomorphism $\phi_0 : \tilde{\Sigma}_0 \sim \Sigma_0$ of maximal open subschemes $\tilde{\Sigma}_0 \subset \tilde{\Sigma}$ and $\Sigma_0 \subset T \times S$. Define an invertible $\mathcal{O}_{T \times S}$-sheaf $\mathbb{L}$ by the equality

$$\mathbb{L}(U) := \tilde{\mathbb{L}}(\phi_0^{-1}(U \cap \Sigma_0)).$$
Identifying \( \tilde{\Sigma}_0 \) with \( \Sigma_0 \) by the isomorphism \( \phi_0 \) one comes to the conclusion that sheaves \( \mathbb{L}|_{\Sigma_0} \) and \( \mathbb{L}|_{\tilde{\Sigma}_0} \) are also isomorphic.

For each closed point \( t \in T \) there is a canonical morphism of the fibre \( \sigma_t : \tilde{S}_t \to S \) where \( \tilde{S}_t = \pi^{-1}(t) \).

**Proposition 9.** For any closed point \( t \in T \) and for any open \( V \subset S \)

\[
\mathbb{L} \otimes (k_t \boxtimes \mathcal{O}_S)(V) = \tilde{L}_t(\sigma_t^{-1}(V) \cap \tilde{\Sigma}_0).
\]

In particular, \( \mathbb{L} \otimes (k_t \boxtimes \mathcal{O}_S) = L \).

**Proof.** The restriction \( \mathbb{L}|_{t \times S} \) is the sheaf associated to a presheaf

\[
V \mapsto \mathbb{L}(U) \otimes_{\mathcal{O}_{T \times S}(U)} (k_t \boxtimes \mathcal{O}_S)(U \cap t \times S)
\]

for any open \( U \subset T \times S \) such that \( U \cap (t \times S) = V \). Since \( \text{codim} T \times S \setminus \Sigma_0 \geq 2 \),

\[
\mathcal{O}_{T \times S}(U) = \mathcal{O}_{T \times S}(U \cap \Sigma_0)
\]

and

\[
(k_t \boxtimes \mathcal{O}_S)(U \cap t \times S) = (k_t \boxtimes \mathcal{O}_S)(U \cap \Sigma_0 \cap t \times S) = \mathcal{O}_{\tilde{S}_t}(\phi_0^{-1}(U \cap \Sigma_0) \cap \pi^{-1}(t)).
\]

Hence \( \mathbb{L}|_{t \times S} \) is associated to the presheaf

\[
V \mapsto \tilde{L}_t(\phi_0^{-1}(U \cap \Sigma_0)) \otimes_{\mathcal{O}_{\tilde{S}_t}(\phi_0^{-1}(U \cap \Sigma_0))} \mathcal{O}_{\pi^{-1}(t)}(\phi_0^{-1}(U \cap \Sigma_0) \cap \pi^{-1}(t)),
\]

or, equivalently,

\[
V \mapsto \tilde{L}_t(\phi_0^{-1}(U \cap \Sigma_0) \cap \tilde{S}_t) = L(U \cap \Sigma_0 \cap t \times S) = L(U \cap t \times S).
\]

We keep in mind that \( \phi_0^{-1}(U \cap \Sigma_0) \cap \tilde{S}_t = \sigma_t^{-1}(V) \cap \tilde{\Sigma}_0 \) what completes the proof. \( \square \)

Define a sheaf \( \mathbb{L}' \) by the correspondence \( U \mapsto \tilde{\mathbb{L}}(U \cap \tilde{\Sigma}_0) \) for any open \( U \subset \tilde{\Sigma} \). It carries a natural structure of invertible \( \mathcal{O}_{\tilde{\Sigma}} \)-module. This structure is induced by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{\tilde{\Sigma}}(U) \times \mathbb{L}'(U) & \xrightarrow{\text{res}} & \mathbb{L}'(U) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\tilde{\Sigma}}(U \cap \tilde{\Sigma}_0) \times \tilde{\mathbb{L}}(U \cap \tilde{\Sigma}_0) & \xrightarrow{\text{res}} & \tilde{\mathbb{L}}(U \cap \tilde{\Sigma}_0)
\end{array}
\]

where vertical arrow is induced by the natural restriction map in \( \mathcal{O}_{\tilde{\Sigma}} \). Compare direct images \( p_* \mathbb{L} \) and \( \pi_* \mathbb{L}' \); for any open \( V \subset T \)

\[
p_* \mathbb{L}(V) = \mathbb{L}(p^{-1}V) = \mathbb{L}(p^{-1}V \cap \Sigma_0)
\]

By the definition of \( \mathbb{L}' \)

\[
\mathbb{L}(p^{-1}V \cap \Sigma_0) = \tilde{\mathbb{L}}(\pi^{-1}V \cap \tilde{\Sigma}_0) = \mathbb{L}'(\pi^{-1}V) = \pi_* \mathbb{L}'(V).
\]

Now \( \pi_* \mathbb{L}' = p_* \mathbb{L} \).
The invertible sheaf $\mathbb{L}'$ induces a morphism $\phi' : \tilde{\Sigma} \to \mathbb{P}(\pi_*\mathbb{L})^\vee$ which includes into the commutative diagram of $T$-schemes

\[
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\phi'} & \mathbb{P}(\pi_*\mathbb{L})^\vee \\
\downarrow \Sigma_0 = \tilde{\Sigma}_0 & & \downarrow \\
T \times S' & \xrightarrow{i_*} & \mathbb{P}(p_*\mathbb{L})^\vee
\end{array}
\]

where $i_*\Sigma$ is a closed immersion induced by $\mathbb{L}$ and $\phi'|_{\tilde{\Sigma}_0}$ is also immersion. From now we identify $\mathbb{P}(\pi_*\mathbb{L})^\vee$ and $\mathbb{P}(p_*\mathbb{L})^\vee$ and use common notation $\mathbb{P}$ for these bundles. Formation of scheme closures of images of $\tilde{\Sigma}_0$ and $\Sigma_0$ in $\mathbb{P}$ leads to $\phi'(\tilde{\Sigma}_0) = i_*\Sigma = T \times S$. Also by the definition of the sheaf $\mathbb{L}'$ for any open $U \subset \tilde{\Sigma}$ and $V \subset T \times S$ such that $U \cap \tilde{\Sigma}_0 \cong V \cap \Sigma_0$ the following chain of equalities holds:

\[\mathbb{L}'(U) = \mathbb{L}'(U \cap \tilde{\Sigma}_0) = \mathbb{L}(V \cap \Sigma_0) = \mathbb{L}(V).\]  \hfill (6.1)

Now for a moment we suppose that $T$ is affine: $T = \text{Spec} A$ for some commutative algebra $A$, $\mathbb{P} = \text{Proj} A[x_0 : \cdots : x_N]$ where $x_0, \ldots, x_N \in H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))$ generate $\mathcal{O}_\mathbb{P}(1)$. Images $\phi'^*x_i = s'_i$, $i = 0, \ldots, N$, generate $\mathbb{L}'$ along $\tilde{\Sigma}_0$ and they are not obliged to generate $\mathbb{L}'$ along the whole of $\tilde{\Sigma}$. Images $i_*^\Sigma x_i = s_i$, $i = 0, \ldots, N$, generate $\mathbb{L}$ along the whole of $T \times S$ and provide that $i_*\Sigma$ is closed immersion.

We pass to standard affine covering by $\mathbb{P}_i = \text{Spec} A[x_0, \ldots, \hat{x}_i, \ldots, x_N]$, $i = 0, \ldots, N$, and $\hat{}$ means omitting the symbol below. Denote $(i_*\Sigma)(T \times S)_i := i_*\Sigma(T \times S) \cap \mathbb{P}_i$ and $(\phi'(\tilde{\Sigma}))_i := \phi'(\tilde{\Sigma}) \cap \mathbb{P}_i$. Set also $(T \times S)_i := i_*^{-1}(i_*\Sigma(T \times S)_i)$ and $\tilde{\Sigma}_i := \phi'^{-1}(\phi'(\tilde{\Sigma}))_i$. Now we have mappings

\[A[x_0, \ldots, \hat{x}_i, \ldots, x_N] \to \Gamma(\tilde{\Sigma}_i, \mathbb{L}') : x_j \mapsto s'_j\]

and

\[A[x_0, \ldots, \hat{x}_i, \ldots, x_N] \to \Gamma((T \times S)_i, \mathbb{L}) : x_j \mapsto s_j\]

which fit into triangular diagram

\[
\begin{array}{ccc}
A[x_0, \ldots, \hat{x}_i, \ldots, x_N] & \xrightarrow{\phi} & \Gamma((T \times S)_i, \mathbb{L}) \\
\downarrow \Gamma(\tilde{\Sigma}_i, \mathbb{L}') & & \downarrow \Gamma(\tilde{\Sigma}_i, \mathbb{L}') \\
\end{array}
\]

where the vertical sign of equality means bijection (6.1) rewritten for the covering under the scope. Commutativity of (6.2) implies that $\phi'$ factors through $i_*\Sigma(T \times S)$, i.e. $\phi'(\tilde{\Sigma}) = i_*\Sigma(T \times S)$.

Now identifying $i_*\Sigma(T \times S)$ with $T \times S$ by means of obvious isomorphism we arrive to the $T$-morphism

\[\phi : \tilde{\Sigma} \to T \times S.\]

It coincides with $\phi_0 : \tilde{\Sigma}_0 \to \Sigma_0$ when restricted to $\tilde{\Sigma}_0$.

For $n > 0$ consider an invertible $\mathcal{O}_{S \times T}$-sheaf $U \mapsto \mathbb{L}^n(\phi_0^{-1}(U \cap \Sigma_0))$. It coincides with $\mathbb{L}^n$ on $\Sigma_0$ and hence it coincides with it in total.
Now there is a commutative triangle

\[
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\phi} & T \times S \\
& \searrow & \downarrow \pi \\
& & T
\end{array}
\]

Firstly note that \( T \) contains at least one closed point, say \( t \in T \); let \( \tilde{S}_t = \pi^{-1}(t) \) be the corresponding closed fibre and \( \tilde{L}_t = \tilde{L}|_{\tilde{S}_t} \) and \( \tilde{E}_t = \tilde{E}|_{\tilde{S}_t} \) restrictions of sheaves to it. By the definition of admissible scheme there is a canonical morphism \( \sigma_t : \tilde{S}_t \rightarrow T \). Then \( (\sigma_t \tilde{L}_t)^n = L \). Indeed, both sheaves \((\sigma_t \tilde{L}_t)^n\) are reflexive on nonsingular \( S \). Also \( \sigma_t \tilde{L}_t/\text{tors} \) is a sheaf of ideals tensored by some invertible sheaf, hence \((\sigma_t \tilde{L}_t)^n\) is invertible. Both sheaves coincide off closed subset of codimension \( \geq 2 \). Hence they coincide on the whole of \( S \).

Secondly, the family \( \pi : \tilde{\Sigma} \rightarrow T \) is birationally trivial, i.e. there exist isomorphic open subschemes \( \tilde{\Sigma}_0 \subset \tilde{\Sigma} \) and \( \Sigma_0 \subset T \times S \). Note that the "boundary" \( \Delta = S \times T \setminus \Sigma_0 \) has codimension \( \geq 2 \) and that for any closed point \( t \in T \) \( \text{codim} \Delta \cap (t \times S) \geq 2 \).

Thirdly, the morphism of multiplication of sections

\[
(\sigma_t \tilde{L}_t)^n \rightarrow \sigma_t \tilde{L}_t^n
\]

induces the morphism of reflexive hulls

\[
((\sigma_t \tilde{L}_t)^n)^{\vee \vee} \rightarrow (\sigma_t \tilde{L}_t^n)^{\vee \vee}
\]

which are normal sheaves on a nonsingular variety and coincide apart from a subset of codimension \( \geq 2 \). Hence they are equal on the whole of \( S \). Also the sheaf \((\sigma_t \tilde{L}_t)^n\) coincides with them by the analogous reason. Then for all \( n > 0 \)

\[
((\sigma_t \tilde{L}_t)^n)^{\vee \vee} = L^n.
\]

Now take a product \( \Sigma = A^1 \times S \overset{p}{\rightarrow} A^1, A^1 = \text{Spec} k[u] \). Let \( \sigma : \tilde{\Sigma} \rightarrow \Sigma \) be the sequence of blowups whose loci do not dominate \( A^1 \) and \( \tilde{S} \) be the zero fibre \( \tilde{S} = (p \circ \sigma)^{-1}(0) \). In this case general fibre is isomorphic to \( S \), \( \tilde{\Sigma} \overset{\pi}{\rightarrow} A^1 \) is flat family of admissible schemes and hence \( H^0(\tilde{S}, \tilde{L}^n) = h^0(S, L^n) \) for \( n \gg 0 \).

**Proposition 10.** There are morphisms of \( \mathcal{O}_{T \times S} \)-sheaves

\[
\sigma_\ast \tilde{L}^n \rightarrow L^n
\]

for all \( n > 0 \).

**Proof.** For any open \( U \in T \times S \) and any \( n > 0 \) there is a restriction map of sections \( \text{res} : (\sigma_\ast \tilde{L}^n)(U) \rightarrow (\sigma_\ast \tilde{L}^n)(U \cap \Sigma_0) \). Denoting as usual the preimage \( \sigma^{-1}(\Sigma_0) \) by \( \tilde{\Sigma}_0 \) (recall that \( \sigma|_{\tilde{\Sigma}_0} \) is an isomorphism) one arrives to the chain of equalities:

\[
(\sigma_\ast \tilde{L}^n)(U \cap \Sigma_0) = \tilde{L}^n(\sigma^{-1}(U \cap \Sigma_0)) = L^n(U).
\]

\( \square \)
Corollary 5. For \( n > 0 \) morphisms \( \sigma_{[i]} \tilde{L}_i^n \to \mathbb{L}^n \) induce isomorphisms of \( \mathcal{O}_T \)-sheaves
\[
\pi_* \tilde{L}_i^n \to p_* \mathbb{L}^n.
\]

Proof. Both sheaves \( \pi_*\tilde{L}_i^n \) and \( p_*\mathbb{L}^n \) are locally free and have equal ranks. Passing to fibrewise consideration one gets \( \pi_*\tilde{L}_i^n \otimes k_t \to p_*\mathbb{L}^n \otimes k_t \) or, equivalently, \( H^0(\tilde{S}_i, \mathbb{L}_n) \to H^0(t \times S, \mathbb{L}^n) \). This map is an isomorphism and hence \( \pi_*\tilde{L}_i^n \to p_*\mathbb{L}^n \).

We will need sheaves
\[
\tilde{V}_m = \pi_*(\tilde{E} \otimes \tilde{L}^m)
\]
for \( m \gg 0 \) such that \( \tilde{V}_m \) are locally free of rank \( rp(m) \) and \( \tilde{E} \otimes \tilde{L}^m \) are fibrewise globally generated in such sense that the canonical morphisms
\[
\pi^*\tilde{V}_m \to \tilde{E} \otimes \tilde{L}^m
\]
are surjective for those \( m \)'s.

Let also for \( m \gg 0 \)
\[
\mathcal{E}_m = \sigma_*(\tilde{E} \otimes \tilde{L}^m),
\]
now
\[
p_*\mathcal{E}_m = p_*\sigma_*(\tilde{E} \otimes \tilde{L}^m) = \pi_*(\tilde{E} \otimes \tilde{L}^m) = \tilde{V}_m.
\]

We intend to confirm ourselves that sheaves \( p_*(\mathcal{E}_m \otimes \mathbb{L}^n) \) are locally free of rank \( rp(m + n) \) for all \( m, n \gg 0 \). This implies \( T \)-flatness of \( \mathcal{E}_m \).

To proceed further we need morphisms \( \tilde{L}^n \to \sigma^*\mathbb{L}^n, n > 0 \).

Proposition 11. For all \( n > 0 \) there are injective morphisms \( \iota_n : \tilde{L}^n \to \sigma^*\mathbb{L}^n \) of invertible \( \mathcal{O}_{\tilde{S}} \)-sheaves.

Proof. For \( n > 0 \) and for any open \( U \subset \tilde{S} \) there is a restriction map on sections
\[
\tilde{L}^n(U) \to \tilde{L}^n(U \cap \tilde{S}_0) = \mathbb{L}^n(\sigma_0(U \cap \tilde{S}_0)) = \mathbb{L}^n(\sigma_0(U) \cap \sum_0) = \mathbb{L}^n(\sigma(U)).
\]
Since \( \sigma \) is projective and hence takes closed subsets to closed subsets (resp., open to open), this implies the sheaf morphism \( \tilde{L}^n \to \sigma^{-1}\mathbb{L}^n \). Combining it with multiplication by unity section \( 1 \in \mathcal{O}_{\tilde{S}}(U) \) leads to the morphism \( \iota_n : \tilde{L}^n \to \sigma^*\mathbb{L}^n \) of invertible \( \mathcal{O}_{\tilde{S}} \)-modules.

Proposition 12. \( \mathcal{E}_m \) are \( T \)-flat for \( m \gg 0 \).

Proof. Consider the morphism of multiplication of sections
\[
p_*\sigma_*(\tilde{E} \otimes \tilde{L}^m) \otimes p_*\mathbb{L}^n \to p_*\sigma_*(\tilde{E} \otimes \tilde{L}^m) \otimes \mathbb{L}^n
\]
which is surjective for \( m, n \gg 0 \). By the projection formula
\[
p_*\sigma_*(\tilde{E} \otimes \tilde{L}^m) \otimes \mathbb{L}^n = p_*\sigma_*(\tilde{E} \otimes \tilde{L}^m \otimes \sigma^*\mathbb{L}^n).
\]
Also for the projection $\pi$ we have another morphism of multiplication of sections

$$\pi_*(\widetilde{E} \otimes \widetilde{L}^m) \otimes \pi_*\widetilde{L}^n \to \pi_*(\widetilde{E} \otimes \widetilde{L}^{m+n})$$

Injective $\mathcal{O}_\Sigma$-morphism $\widetilde{L}^n \hookrightarrow \sigma^*\mathbb{L}^n$ in Proposition [11] after tensoring by $\widetilde{E} \otimes \widetilde{L}^m$ and applying $\pi_*$ leads to

$$\pi_*(\widetilde{E} \otimes \widetilde{L}^{m+n}) \hookrightarrow \pi_*(\widetilde{E} \otimes \widetilde{L}^m \otimes \sigma^*\mathbb{L}^n)$$

Taking into account the isomorphism $p_*\mathbb{L}^n = \pi_*\widetilde{L}^n$ and Proposition [11] we gather these mappings into the commutative diagram

$$\pi_*(\widetilde{E} \otimes \widetilde{L}^m) \otimes p_*\mathbb{L}^n \to \pi_*(\sigma_*(\widetilde{E} \otimes \widetilde{L}^m \otimes \sigma^*\mathbb{L}^n))$$

(6.3)

By commutativity of this diagram we conclude that

$$\pi_*(\widetilde{E} \otimes \widetilde{L}^{m+n}) = p_*\sigma_*(\widetilde{E} \otimes \widetilde{L}^m \otimes \mathbb{L}^n)$$

(6.4)

or, in our notation, $p_*(\mathcal{E}_m \otimes \mathbb{L}^n) = \widetilde{V}_{m+n} = p_*\mathcal{E}_{m+n}$ for $m,n \gg 0$. This guarantees that $\mathcal{E}_m$ are $T$-flat for $m \gg 0$.

We intend to confirm ourselves that $\mathcal{E}_m \otimes \mathbb{L}^{-m}$ are families of semistable sheaves on $S$ as we need. First we prove the following

**Proposition 13.** $\mathcal{E}_{m+n} = \mathcal{E}_m \otimes \mathbb{L}^n$ for any $m \gg 0, n > 0$.

**Proof.** Consider the sheaf inclusion $i_n : \widetilde{L}^n \hookrightarrow \sigma^*\mathbb{L}^n$ valid for any $n > 0$. Tensoring by locally free $\mathcal{O}_\Sigma$-module $\widetilde{E} \otimes \widetilde{L}^m$, formation of direct image under $\sigma$ and projection formula yield in the inclusion

$$i_{m,n} : \mathcal{E}_{m+n} \hookrightarrow \mathcal{E}_m \otimes \mathbb{L}^n.$$  

(6.5)

By (6.3) we come to the epimorphism $p_*\mathcal{E}_m \otimes p_*\mathbb{L}^n \twoheadrightarrow p_*\mathcal{E}_{m+n}$. Now apply $p^*$:

$$p^*(p_*\mathcal{E}_m \otimes p_*\mathbb{L}^n) \twoheadrightarrow p^*p_*\mathcal{E}_{m+n}$$

and combine this morphism with natural morphisms

$$p^*(p_*\mathcal{E}_m \otimes p_*\mathbb{L}^n) \rightarrow \mathcal{E}_m \otimes \mathbb{L}^n,$n

$$p^*p_*\mathcal{E}_{m+n} \rightarrow \mathcal{E}_{m+n}$$

and with (6.3):

$$p^*(p_*\mathcal{E}_m \otimes p_*\mathbb{L}^n) \twoheadrightarrow p^*p_*\mathcal{E}_{m+n}$$

(6.6)

By commutativity of (6.6) we conclude that $i_{m,n}$ is surjective as well as injective and hence $\mathcal{E}_m \otimes \mathbb{L}^n = \mathcal{E}_{m+n}$ whenever $m \gg 0, n \gg 0$. This proves the proposition.  

34
We can introduce the goal sheaf of our construction
\[ \mathcal{E} := \mathcal{E}_m \otimes \mathcal{L}^{-m}. \]
By the proposition proved this definition is independent of \( m \) at least in case when \( m \gg 0 \). The sheaf \( \mathcal{E} \) is \( T \)-flat.

**Proposition 14.** The sheaf \( \mathcal{E} \) with respect to the invertible sheaf \( \mathcal{L} \) has fibrewise Hilbert polynomial equal to \( rp(n) \), i.e. for \( n \gg 0 \)
\[ \text{rank } p_*(\mathcal{E} \otimes \mathcal{L}^n) = rp(n). \]

**Proof.** For \( n \gg m \gg 0 \) by (6.4) we have chain of equalities
\[ p_*(\mathcal{E} \otimes \mathcal{L}^n) = p_*(\mathcal{E}_m \otimes \mathcal{L}^{-m}) = p_*(\sigma_*\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m) \otimes \mathcal{L}^n = \pi_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^n). \]
The latter sheaf of the chain has rank equal to \( rp(n) \). \( \square \)

**Proposition 15.** For any closed point \( t \in T \) the sheaf
\[ E_t := \mathcal{E}|_{t \times S} \]
is torsion-free and Gieseker-semistable with respect to
\[ L_t := \mathcal{L}|_{t \times S} \cong L. \]

**Proof.** The isomorphism \( \mathcal{L}|_{t \times S} \cong L \) is the subject of Proposition 9. Now for \( E_t \) one has
\[ E_t = \mathcal{E}|_{t \times S} = (\mathcal{E}_m \otimes \mathcal{L}^{-m})|_{t \times S} = \mathcal{E}_m|_{t \times S} \otimes \mathcal{L}^{-m} = \sigma_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m)|_{t \times S} \otimes \mathcal{L}^{-m}. \]
Denoting by \( i_t : t \times S \hookrightarrow T \times S \) and \( \tilde{i}_t : \tilde{S}_t \hookrightarrow \tilde{\Sigma} \) morphisms of closed immersions of fibres we come to
\[ \sigma_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m)|_{t \times S} \otimes \mathcal{L}^{-m} = (i_t^*\sigma_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m)) \otimes \mathcal{L}^{-m} \]
and base change morphism
\[ \beta_t : i_t^*\sigma_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m) \rightarrow \sigma_{t*}\tilde{i}_t^*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m) \quad (6.7) \]
in the fibred square
\[ \begin{array}{ccc}
  t \times S & \xrightarrow{i_t} & T \times S \\
  \sigma & \downarrow & \sigma \\
  \tilde{S}_t & \xrightarrow{\tilde{i}_t} & \tilde{\Sigma}
\end{array} \]
The following lemma will be proven below.

**Lemma 6.1.** The sheaf \( \sigma_{t*}\tilde{E}_t \) is torsion-free.

Both sheaves in (6.7) coincide along \( (t \times S) \cap \Sigma_0 \). Now consider corresponding map of global sections:
\[ H^0(\beta_t) : H^0(t \times S, i_t^*\sigma_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m)) \rightarrow H^0(t \times S, \sigma_{t*}\tilde{i}_t^*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m)). \]
It is injective. Left hand side takes the view
\[ H^0(t \times S, i_t^*\sigma_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m)) \otimes k_t = i_t^*p_*\sigma_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{L}}^m) = k_t^{\oplus rp(m)}. \]
Also for right hand side one has

\[ H^0(t \times S, \sigma_{ts} \xi^*(\widetilde{E} \otimes \widetilde{L}^m)) \otimes k_t = H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m) \otimes k_t = k_t^{\otimes \text{rp}(m)}. \]

This implies that \( H^0(\beta_t) \) is bijective and there is a commutative diagram

\[
\begin{array}{ccc}
H^0(t \times S, \sigma_{ts} (\widetilde{E}_t \otimes \widetilde{L}_t^m)) \otimes \mathcal{O}_S & \longrightarrow & \sigma_{ts} (\widetilde{E}_t \otimes \widetilde{L}_t^m) \\
\downarrow H^0(\beta_t) & & \downarrow \beta_t \\
i^*p^* \mathcal{V}_m & \longrightarrow & i^*_t \mathcal{E}_m
\end{array}
\] (6.8)

Observe that

\[ \sigma_{ts} (\widetilde{E}_t \otimes \widetilde{L}_t^m) = \sigma_{ts} (\widetilde{E}_t \otimes \sigma_t^* L^m \otimes (\text{Exc}_{\widetilde{S}_t})^m) = \sigma_{ts} (\widetilde{E}_t \otimes (\text{Exc}_{\widetilde{S}_t})^m) \otimes L^m, \]

and for \( m \gg 0 \) the latter sheaf is globally generated. We have used (2.9), (2.10). This implies surjectivity of upper horizontal arrow in (6.8). It follows from (6.8) that \( \beta_t \) is surjective. Since \( \ker H^0(\beta_t) = H^0(t \times S, \ker \beta_t) = 0 \), then \( \beta_t \) is isomorphic.

Now take a subsheaf \( F_t \subset E_t \). Now for \( m \gg 0 \) there is a commutative diagram

\[
\begin{array}{ccc}
H^0(t \times S, E_t \otimes L^m) \otimes \mathcal{O}_S & \longrightarrow & E_t \otimes L^m \\
\downarrow H^0(t \times S, F_t \otimes L^m) \otimes \mathcal{O}_S & & \downarrow \mathcal{O}_S \\
F_t \otimes L^m & \longrightarrow & E_t \otimes L^m
\end{array}
\]

The isomorphism \( E_t \otimes L^m = \sigma_{ts} (\widetilde{E}_t \otimes \widetilde{L}_t^m) \) proven above fixes bijection on global sections

\[ H^0(t \times S, E_t \otimes L^m) \simeq H^0(t \times S, \sigma_{ts} (\widetilde{E}_t \otimes \widetilde{L}_t^m)) = H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m). \]

Let \( \widetilde{V}_t \subset H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m) \) be the subspace corresponding to \( H^0(t \times S, F_t \otimes L^m) \subset H^0(t \times S, E_t \otimes L^m) \) under this bijection. Now one has a commutative diagram

\[
\begin{array}{ccc}
H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m) \otimes \mathcal{O}_{\widetilde{S}_t} & \longrightarrow & \widetilde{E}_t \otimes \widetilde{L}_t^m \\
\downarrow \gamma & & \downarrow \gamma \\
\widetilde{V}_t \otimes \mathcal{O}_{\widetilde{S}_t} & \longrightarrow & \widetilde{F}_t \otimes \widetilde{L}_t
\end{array}
\]

where \( \widetilde{F}_t \otimes \widetilde{L}_t^m \subset \widetilde{E}_t \otimes \widetilde{L}_t^m \) is defined as a subsheaf generated by the subspace \( \widetilde{V}_t \) by means of the morphism \( \varepsilon \circ \gamma \) and \( \gamma \) is the morphism induced by the inclusion \( \widetilde{V}_t \subset H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m) \).

The associated map of global sections

\[ H^0(\varepsilon') : \widetilde{V}_t \rightarrow H^0(\widetilde{S}_t, \widetilde{F}_t \otimes \widetilde{L}_t^m) \]

includes into the commutative triangle

\[
\begin{array}{ccc}
\widetilde{V}_t & \xrightarrow{H^0(\varepsilon')} & H^0(\widetilde{S}_t, \widetilde{F}_t \otimes \widetilde{L}_t^m) \\
\downarrow H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m) & & \downarrow H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m) \\
H^0(\widetilde{S}_t, \widetilde{E}_t \otimes \widetilde{L}_t^m)
\end{array}
\]

what implies that \( H^0(\varepsilon') \) is injective. Since each section from \( H^0(\widetilde{S}_t, \widetilde{F}_t \otimes \widetilde{L}_t^m) \) corresponds to a section in \( H^0(t \times S, F_t \otimes L^m) \subset H^0(t \times S, E_t \otimes L^m) \) then \( H^0(\varepsilon') \) is surjective. Hence \( h^0(t \times S, F_t \otimes L^m) = h^0(\widetilde{S}_t, \widetilde{F}_t \otimes \widetilde{L}_t^m) \) for all \( m \gg 0 \) and stability (resp., semistability) for \( \widetilde{E}_t \) implies stability (resp., semistability) for \( E_t \).  

\[ \square \]
Proof of Lemma 6.1. Since sheaves $\tilde{E}_t$ and $\sigma_t \tilde{E}_t$ coincide along identified open subschemes $\tilde{S}_t \cap \Sigma_0 \simeq t \times S \cap \Sigma_0$, it is enough to confirm that there is no torsion subsheaf concentrated on $t \times S \cap (T \times S \setminus \Sigma_0)$ in $\sigma_t \tilde{E}_t$. Assume that $T \subset \sigma_t \tilde{E}_t$ is such a torsion subsheaf i.e. $T \neq 0$ and for any open $U \subset t \times S \cap \Sigma_0$ $T(U) = 0$. Let $A = \text{Supp } T \subset t \times S$ and $U \subset t \times S$ be such an open subset that $T(U) \neq 0$, i.e. $U \cap A \neq 0$. Now

$$T(U) \subset \sigma_t \tilde{E}_t(U) = \tilde{E}_t(\sigma^{-1}U),$$

and any nonzero section $s \in T(U)$ is supported in $U \cap A$ and comes from the section $\tilde{s} \in \tilde{E}_t(\sigma^{-1}U)$ with support in $\sigma^{-1}(U \cap A)$. This means that $\tilde{s}$ is supported in some additional component $\tilde{S}_{t,j}^{\text{add}}$ of the admissible scheme $\tilde{S}_t$. Hence $\tilde{s} \in \text{tors}_j$. But on additional components $\tilde{S}_{t,j}^{\text{add}}$ of $\tilde{S}_t$, $\text{tors}_j = 0$. This implies that $T = 0$. \hfill \square

7 Functor isomorphism

In this section we prove that natural transformations $\mathbf{E} : \mathcal{F}^\text{GM} \to \mathcal{F}$ and $\mathbf{Z} : \mathcal{F} \to \mathcal{F}^\text{GM}$ are inverse to each other and hence they provide the isomorphism between the functor of moduli of admissible semistable pairs and the functor of moduli in the sense of Gieseker and Maruyama. As a corollary, we get the isomorphism of moduli schemes for these moduli functors, with no dependence on number and geometry of their connected components, on reducedness of scheme structure and on presence of locally free sheaves (respectively, $S$-pairs) in each component.

As in 2-dimensional case, the proof has two aspects.

1. Pointwise. a) For any torsion-free semistable $O_S$-sheaf the composite of transformations

$$E \mapsto (\langle \tilde{S}, \tilde{L} \rangle, \tilde{E}) \mapsto E'$$

returns $E' = E$.

b) Conversely, for any admissible semistable pair $\langle (\tilde{S}, \tilde{L}), \tilde{E} \rangle$ the composite of transformations

$$\langle (\tilde{S}, \tilde{L}), \tilde{E} \rangle \mapsto E \mapsto \langle (\tilde{S}', \tilde{L}'), \tilde{E}' \rangle$$

returns $\mathcal{M}$-equivalent pairs $\langle (\tilde{S}', \tilde{L}'), \tilde{E}' \rangle$ and $\langle (\tilde{S}, \tilde{L}), \tilde{E} \rangle$.

2. Global. a) For any family of semistable torsion-free sheaves $\mathcal{E}$ with base scheme $T$ the composite

$$(\mathcal{E}, \mathcal{L}) \mapsto \langle (\pi : \tilde{\Sigma} \to T, \tilde{\mathcal{L}}), \tilde{\mathcal{E}} \rangle \mapsto (\mathcal{E}', \mathcal{L}')$$

returns such $(\mathcal{E}', \mathcal{L}') \sim (\mathcal{E}, \mathcal{L})$ in the sense of the description of the functor $\mathcal{F}^\text{GM}$ (5.3) (5.4).

b) Conversely, for any family $\langle (\pi : \tilde{\Sigma} \to T, \tilde{\mathcal{L}}), \tilde{\mathcal{E}} \rangle$ of admissible semistable pairs with base scheme $T$ the composite of transformations

$$\langle (\pi : \tilde{\Sigma} \to T, \tilde{\mathcal{L}}), \tilde{\mathcal{E}} \rangle \mapsto (\mathcal{E}, \mathcal{L}) \mapsto \langle (\pi' : \tilde{\Sigma}' \to T, \tilde{\mathcal{L}}'), \tilde{\mathcal{E}}' \rangle$$

returns family $\langle (\pi' : \tilde{\Sigma}' \to T, \tilde{\mathcal{L}}'), \tilde{\mathcal{E}}' \rangle$ such that for $\tilde{\Sigma} \xleftarrow{\varphi} \Sigma_\Delta \xrightarrow{\varphi'} \tilde{\Sigma}'$ inverse images of $\tilde{E}$ and of $\tilde{E}'$ on $\Sigma_\Delta$ are equivalent (generally $S$-equivalence is enough but there is an isomorphism) in the sense of the description of the functor $\mathcal{F}$ (5.2) (5.1).
Observe that $\kappa$ operates by standard resolution and its target is a class of admissible pairs which are obtained by standard resolutions of coherent sheaves. At the same time the source of $\tau$ is essentially wider.

We begin with 2 a); it will be specialised to pointwise version 1 a) when $T = \text{Spec } k$.

Families of polarisations $\mathbb{L}$ and $\mathbb{L}'$ coincide along the open subset (locally free locus for sheaves $\mathbb{E}$ and $\mathbb{E}'$) $\Sigma_0$ where $\text{codim}_{T \times S} T \times S \setminus \Sigma_0 \geq 2$. Since $\mathbb{L}$ and $\mathbb{L}'$ are locally free this implies that $\mathbb{L} = \mathbb{L}'$.

Now consider three locally free $\mathcal{O}_T$-sheaves of ranks equal to $r_p(m)$:

$$V_m = p_*(E \otimes \mathbb{L}^m), \quad \tilde{V}_m = \pi_*(\mathbb{E} \otimes \mathbb{L}^m), \quad V'_m = p_*(E' \otimes \mathbb{L}^m).$$

**Lemma 7.1.** $V_m \cong \tilde{V}_m \cong V'_m$.

**Proof.** Start with the epimorphism $\sigma E \twoheadrightarrow \tilde{E}$ where $\sigma : \tilde{\Sigma} \rightarrow \Sigma$ is well defined as morphism of standard resolution. Tensoring it by $\mathbb{L}^m$ and direct image $\sigma_*$ yield in the morphism of $\mathcal{O}_T$-sheaves $\sigma_*(\sigma E \otimes \mathbb{L}^m) \rightarrow \sigma_*(\tilde{E} \otimes \mathbb{L}^m)$.

Turn our attention to the following result.

**Lemma 7.2.** [10] Let $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces such that $f_* \mathcal{O}_X = \mathcal{O}_Y$, $\mathcal{E}$ $\mathcal{O}_Y$-module of finite presentation, $\mathcal{F}$ $\mathcal{O}_X$-module. Then there is a monomorphism $E \otimes f_* \mathcal{F} \rightarrow f_*[f^* \mathcal{E} \otimes \mathcal{F}]$.

**Remark 15.** In general case we have only the morphism $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ in our disposal. Applying the inverse image $f^*$, tensoring by $\otimes \mathcal{F}$ and direct image $f_*$ to a finite presentation $E_1 \rightarrow E_0 \rightarrow \mathcal{E} \rightarrow 0$ we come to the commutative diagram

$$
\begin{array}{ccc}
E_1 \otimes f_* \mathcal{F} & \rightarrow & E_0 \otimes f_* \mathcal{F} \\
| & & \downarrow \\
\mathcal{E} \otimes f_* \mathcal{F} & \rightarrow & f_*[f^* \mathcal{E} \otimes \mathcal{F}]
\end{array}
$$

where right hand side is a morphism of interest

$$\mathcal{E} \otimes f_* \mathcal{F} \rightarrow f_*[f^* \mathcal{E} \otimes \mathcal{F}].$$

Setting $\mathcal{E} = \mathbb{E}$, $\mathcal{F} = \mathbb{L}^m$ and $f = \sigma$ we get

$$
\begin{array}{c}
\mathbb{E} \otimes \mathbb{L}^m \otimes \sigma_*(\sigma^{-1} \mathbb{I} \cdot \mathcal{O}_\tilde{E})^m \\
\downarrow \\
\sigma_*(\sigma^* \mathbb{E} \otimes \mathbb{L}^m) \rightarrow \sigma_*(\tilde{\mathbb{E}} \otimes \mathbb{L}^m)
\end{array}
$$

and after taking direct image $p_*$

$$
\begin{array}{c}
p_*(\mathbb{E} \otimes \mathbb{L}^m \otimes \sigma_*(\sigma^{-1} \mathbb{I} \cdot \mathcal{O}_\tilde{E})^m) \\
\downarrow \eta \\
p_*(\mathbb{E} \otimes \mathbb{L}^m) \rightarrow \mathbb{p}*(\tilde{\mathbb{E}} \otimes \mathbb{L}^m)
\end{array}
$$
where upper horizontal arrow is natural immersion into reflexive hull. Since target sheaf of the composite map \( \eta \) is reflexive, \( \eta \) factors through \( p_*(E \otimes \mathbb{L}^m) \) as reflexive hull of the source. This yields in existence of the morphism of locally free sheaves of equal ranks

\[
\tilde{\eta} : p_*(E \otimes \mathbb{L}^m) \to \pi_*(\tilde{E} \otimes \tilde{L}^m).
\]

The morphism of sheaves is an isomorphism iff it is stalkwise isomorphic. Fix an arbitrary closed point \( t \in T \); till the end of this proof we omit subscript \( t \) in notations corresponding to the sheaves corresponding to \( t \): \( E_t =: E, \tilde{E}_t =: \tilde{E}, \tilde{S}_t =: \tilde{S}, \sigma_t =: \sigma : \tilde{S} \to S \). There is an epimorphism of \( \mathcal{O}_S \)-modules

\[
\sigma^*E \otimes \tilde{L}^m \twoheadrightarrow \tilde{E} \otimes \tilde{L}^m.
\]

The analog of projection formula as in global case leads to

\[
E \otimes L^m \otimes \sigma_*(\sigma^{-1}I \cdot \mathcal{O}_S)^m \\
\downarrow \hspace{3cm} \downarrow
\sigma_*(\sigma^*E \otimes \tilde{L}^m) \\
\sigma_*(\tilde{E} \otimes \tilde{L}^m)
\]

and taking global sections

\[
H^0(S, E \otimes L^m \otimes \sigma_*(\sigma^{-1}I \cdot \mathcal{O}_S)^m) \twoheadrightarrow H^0(S, E \otimes L^m) \\
\downarrow \quad \eta \quad \downarrow \tilde{\eta}
\]

\[
H^0(S, \sigma_*(\sigma^*E \otimes \tilde{L}^m)) \twoheadrightarrow H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m)
\]

Here \( \tilde{\eta} \) is included into the commutative diagram

\[
H^0(S, E \otimes L^m) \twoheadrightarrow H^0(t \times S \cap \Sigma_0, E \otimes L^m) \\
\downarrow \tilde{\eta} \downarrow
H^0(S \cap \Sigma_0, E \otimes L^m)
\]

where both horizontal arrows are restriction maps and upper restriction map is injective. Hence \( \tilde{\eta} \) is also injective. Since it is a monomorphism of vector spaces of equal dimensions is is an isomorphism. Then \( \eta : \mathbb{V}_m \to \mathbb{V}'_m \) is also an isomorphism. The isomorphism \( \mathbb{V}'_m \cong \mathbb{V}_m \) has been proven in previous section (proof of Proposition [14]).

Identifying locally free sheaves \( \mathbb{V}_m = \mathbb{V}'_m \) consider relative Grothendieck’ scheme \( \text{Quot}_{T \times S}^{r_p(n)}(p^*\mathbb{V}_m \otimes \mathbb{L}^{-m}) \) and two \( T \)-morphisms of closed immersion

\[
T \times S \xrightarrow{\tilde{ev}} \text{Quot}_{T \times S}^{r_p(n)}(p^*\mathbb{V}_m \otimes \mathbb{L}^{-m}) \times S \xleftarrow{ev'} T \times S.
\]

Morphism \( \tilde{ev} \) is induced by the morphism \( ev : p^*\mathbb{V}_m \otimes \mathbb{L}^{-m} \to E \) and \( ev' \) by the morphism \( ev' : p^*\mathbb{V}_m \otimes \mathbb{L}^{-m} \to E' \).

Since both morphisms \( \tilde{ev} \) and \( \tilde{ev}' \) are proper and they coincide along \( \Sigma_0 \) such that \( \text{codim}_{T \times S}(T \times S) \setminus \Sigma_0 \geq 2 \), then \( \tilde{ev} = ev' \) and \( \tilde{ev}(T \times S) = ev'(T \times S) \) in scheme sense. Hence by universality of Quot-scheme \( E = E' \) as inverse images of universal quotient sheaf over \( \text{Quot}_{T \times S}^{r_p(n)}(p^*\mathbb{V}_m \otimes \mathbb{L}^{-m}) \) under morphisms \( \tilde{ev} = \tilde{ev}' \).
For global version of 2,b) consider families \((\pi : \tilde{\Sigma} \to T, \tilde{\mathbb{L}})\) and \((\pi' : \tilde{\Sigma}' \to T, \tilde{\mathbb{L}}')\) and epimorphisms \(\pi^*\pi_*\tilde{\mathbb{L}} \to \tilde{\mathbb{L}}, \pi'^*\pi'_*\tilde{\mathbb{L}}' \to \tilde{\mathbb{L}}'\). Generally families \(\tilde{\Sigma}\) and \(\tilde{\Sigma}'\) are not obliged to be isomorphic. We can assume that \(\pi_*\tilde{\mathbb{L}} = \pi'_*\tilde{\mathbb{L}}'\) and then identify corresponding projective bundles \(\mathbb{P}(\pi_*\tilde{\mathbb{L}})^\vee = \mathbb{P}(\pi'_*\tilde{\mathbb{L}}')^\vee\). We introduce shorthand notations \(\mathbb{P} := \mathbb{P}(\pi_*\tilde{\mathbb{L}})^\vee\) and \(\mathbb{P}' := \mathbb{P}(\pi'_*\tilde{\mathbb{L}}')^\vee\) and consider fibred product together with diagonal embedding

\[
\begin{array}{ccc}
\mathbb{P} \times_T \mathbb{P}' & \xrightarrow{\pi} & \mathbb{P}' \\
\downarrow & & \downarrow \\
\mathbb{T} & \xleftarrow{\pi} & \mathbb{T}
\end{array}
\]

Closed immersions of \(T\)-schemes \(i : \tilde{\Sigma} \hookrightarrow \mathbb{P}\) and \(i' : \tilde{\Sigma}' \hookrightarrow \mathbb{P}'\) complete it to the commutative diagram

\[
\begin{array}{ccc}
\tilde{\Sigma} \times_T \tilde{\Sigma}' & \xrightarrow{i} & \mathbb{P} \\
\downarrow & & \downarrow \\
\tilde{\Sigma} & \xrightarrow{i} & \mathbb{P}' \\
\downarrow & & \downarrow \\
\mathbb{T} & \xrightarrow{\pi} & \mathbb{T}
\end{array}
\]

and now set \(\tilde{\Sigma}_\Delta := (\tilde{\Sigma} \times_T \tilde{\Sigma}') \cap \mathbb{P}_\Delta; i_\Delta : \tilde{\Sigma}_\Delta \hookrightarrow \mathbb{P}_\Delta\). Using standard notations for ample invertible sheaves on projective spaces we write \(\tilde{\mathbb{L}} = i^*\mathcal{O}_\mathbb{P}(1), \tilde{\mathbb{L}}' = i'^*\mathcal{O}_{\mathbb{P}'}(1)\) and \(\tilde{\mathbb{L}}_\Delta = i_\Delta^*\mathcal{O}_{\mathbb{P}_\Delta}(1)\) for polarisation on \(\tilde{\Sigma}_\Delta\).

Regretting to one point case (or reducing to a single fibre) we come to the following diagram for principal components of the fibres:

\[
\begin{array}{ccc}
\tilde{\Sigma}_{\Delta 0} & \xrightarrow{\pi} & \tilde{\mathbb{S}}_0 \\
\downarrow & & \downarrow \\
\tilde{\mathbb{S}}_0 & \leftarrow & \tilde{\mathbb{S}}_0 \times \tilde{\mathbb{S}}'_0
\end{array}
\]

Here \(\tilde{\Sigma}_{\Delta 0}\) is closure of the image of the diagonal embedding \(U \hookrightarrow U \times U' \subset \tilde{\mathbb{S}}_0 \times \tilde{\mathbb{S}}'_0\) of open subset \(U \subset \tilde{\mathbb{S}}_0\) where \(U\) is maximal open subset where \(\sigma : \tilde{\mathbb{S}} \to S\) is isomorphic, \(U'\) maximal open subset where \(\sigma' : \tilde{\mathbb{S}}' \to S\) is isomorphic and \(U \cong U'\).

Now switch to immersions into relative Grassmannians. Let \(\mathbb{G} := \text{Grass}(\tilde{\mathbb{V}}_m, r)\) and \(\mathbb{G}' := \text{Grass}(\tilde{\mathbb{V}}'_m, r)\) be relative Grassmannians of \(r\)-dimensional quotient spaces in vector bundles \(\tilde{\mathbb{V}}_m \cong \tilde{\mathbb{V}}'_m\). Let \(\mathcal{Q}\) be universal quotient sheaf for \(\mathbb{G}\) and \(\mathcal{Q}'\) be the same for \(\mathbb{G}'\) then \(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m = j^*\mathcal{Q}, \tilde{\mathbb{E}}' \otimes \tilde{\mathbb{L}}'^m = j'^*\mathcal{Q}'\) for closed immersions \(j : \tilde{\Sigma} \hookrightarrow \mathbb{G}\) and \(j' : \tilde{\Sigma}' \hookrightarrow \mathbb{G}'\) respectively.
Considering fibred products of Grassmannians analogous to ones of projective spaces we come to analogous commutative diagram with diagonal $G_\Delta \hookrightarrow G \times_T G'$

Also $\tilde{\Sigma}_\Delta = (\tilde{\Sigma} \times_T \tilde{\Sigma}') \cap G_\Delta$ (with same $\tilde{\Sigma}_\Delta$), $j_\Delta : \tilde{\Sigma}_\Delta \hookrightarrow G_\Delta$ the corresponding closed immersion and the equality $\tilde{E}_\Delta \otimes \tilde{L}_m = j_\Delta^* \mathcal{O}_\Delta$ defines locally free sheaf $\tilde{E}_\Delta$ on $\tilde{\Sigma}_\Delta$.

Now we came to the family $\pi_\Delta : \tilde{\Sigma}_\Delta \to T$ included into commutative diagram

Start with $\tilde{E}$ and prove that $\tilde{\sigma}^* \tilde{E} \cong \tilde{\sigma}'^* \sigma'^* (E_m \otimes L^{-m})_{/\text{tors}}$ for $m \gg 0$. We use obvious notation $\tilde{E}' = \sigma'^* (E_m \otimes L^{-m})_{/\text{tors}}$ and symbol $U_\Delta$ for open subset of $\tilde{\Sigma}_\Delta$. If restricted to irreducible component, $\tilde{\sigma}$ and $\tilde{\sigma}'$ take open subsets to open subsets. If any of schemes $\tilde{\Sigma}$, $\tilde{\Sigma}'$ or $\tilde{\Sigma}_\Delta$ has more then one component we replace it by its principal component (principal component of, say, $\tilde{\Sigma}$ is one containing union of principal components of fibres of the morphism $\pi : \tilde{\Sigma} \to T$). Now we can write down sheaves of interest explicitly.

For $\tilde{\sigma}^* \tilde{E}$ we use the fact that $\tilde{\sigma} \circ \tilde{\sigma} = \tilde{\sigma}' \circ \tilde{\sigma}'$, symbol $\tilde{V}'$ for open subset in $\tilde{\Sigma}'$, $\tilde{V}$ for
open in $\tilde{\Sigma}$, $V$ for open in $\Sigma$.

$$\tilde{\sigma}^n \tilde{E}' = (U_\Delta \mapsto \lim_{V \supset \sigma(U_\Delta)} \tilde{E}'(\tilde{V}') \otimes_{\lim_{W \supset \sigma(U_\Delta)} c_{\Sigma(W)}(\tilde{W}'')} O_{\tilde{\Sigma}_{\Delta}}(U_\Delta)^+)$$

$$= (U_\Delta \mapsto \lim_{V \supset \sigma(U_\Delta)} E(V) \otimes (\sigma^{-1} c_{\Sigma})_{U_\Delta} O_{\tilde{\Sigma}_{\Delta}}(U_\Delta)^+$$

$$= (U_\Delta \mapsto \lim_{V \supset \sigma(U_\Delta)} \tilde{E}(V) \otimes \tilde{L}^m((V) \otimes c_{\Sigma(V)} \tilde{L}^{-m}(V) \otimes (\sigma^{-1} c_{\Sigma})_{U_\Delta} O_{\tilde{\Sigma}_{\Delta}}(U_\Delta)^+)$$

$$= (U_\Delta \mapsto \lim_{V \supset \sigma(U_\Delta)} \tilde{E}(\sigma^{-1} V) \otimes c_{\Sigma(V)} \tilde{L}^m(\sigma^{-1} V) \otimes (\sigma^{-1} c_{\Sigma})_{U_\Delta} O_{\tilde{\Sigma}_{\Delta}}(U_\Delta)^+)$$

Refining $U_\Delta$ one comes to $\tilde{V} \subset \sigma^{-1} V$ and to the restriction map $\tilde{E}(\sigma^{-1} V) \to \tilde{E}(\tilde{V})$. This leads to the morphism of inductive limits

$$\lim_{\tilde{V} \supset \sigma(U_\Delta)} \tilde{E}(\tilde{V}) \otimes (\sigma^{-1} c_{\Sigma})_{U_\Delta} O_{\tilde{\Sigma}_{\Delta}}(U_\Delta)$$

which yields in the morphism of presheaves

$$(U_\Delta \mapsto \lim_{\tilde{V} \supset \sigma(U_\Delta)} \tilde{E}(\tilde{V}) \otimes (\sigma^{-1} c_{\Sigma})_{U_\Delta} O_{\tilde{\Sigma}_{\Delta}}(U_\Delta))$$

and, consequently, the morphism of corresponding associated locally free $O_{\tilde{\Sigma}_{\Delta}}$-sheaves $\tilde{\sigma}^n \tilde{E} \to \tilde{\sigma}^n \tilde{E}'$. Let $U_0 \subset \tilde{\Sigma}_{\Delta}$ be the maximal open subscheme where both morphisms $\sigma'$ and $\sigma$ become isomorphisms. Restriction of the morphism $\tilde{\sigma}^* \tilde{E} \to \tilde{\sigma}^* \tilde{E}'$ to $U_0$ is identity isomorphism. Hence when the sheaf morphism restricted to maximal reduced subscheme $\tilde{\Sigma}_{\Delta \text{red}}$ it remains the morphism of locally free sheaves but has torsion subsheaf as a kernel. It is impossible. We proved that both sheaves $\tilde{\sigma}^* \tilde{E}$ and $\tilde{\sigma'}^* \tilde{E}'$ have equal Hilbert polynomials when restricted to fibres over the base $T$.

From this we conclude the $O_{\tilde{\Sigma}_{\Delta}}$-isomorphism $\tilde{\sigma}^* \tilde{E} \cong \tilde{\sigma}^* \tilde{E}'$.

**Proposition 16.** [10, ch. 1, Proposition 2.5] Let $B$ be flat $A$-algebra and $b \in B$. If the image of $b$ in $B/mB$ is not a zero divisor for any maximal ideal $m$ in $A$ then $B/(b)$ is flat $A$-algebra.

Take a section $(s, s)$ of $O_{\mathbb{P}(\pi, \tilde{L})} \otimes O_{\mathbb{P}(\pi', \tilde{L})}$ and let $b = (s', s'')$ be its image in $O_{\tilde{\Sigma}} \otimes T O_{\Sigma}$. In our situation $m = m_1$, and $b$ has an image in $O_{\pi^{-1}(t)} \otimes_k O_{\pi'-1(t)}$ which is not zero divisor. Iterating the usage of the Proposition 16 in regular sequence [10] ch. 1, Remark 2.6 d one comes to conclusion that $\tilde{\Sigma}_{\Delta}$ is flat over $T$. It rests to compare Hilbert polynomials of fibres over $t$ in the exact triple

$$0 \to I_{\tilde{\Sigma}_{\Delta}, \tilde{\Sigma}} \to O_{1} \to O_{i_{\Delta}(\tilde{\Sigma}_{\Delta})} \to 0$$
when isomorphic projective bundles $\mathbb{P}_\Delta \cong \mathbb{P}(\pi_*\widetilde{L})^\vee = \mathbb{P}$ are identified under the composite map $\mathbb{P}_\Delta \hookrightarrow \mathbb{P} \times_T \mathbb{P}^r \to \mathbb{P}$ in (7.1). Since $O_{\Sigma}$ and $O_{\tilde{\Sigma}}$ are $T$-flat then $I_{\tilde{\Sigma},\Sigma}$ is also $T$-flat. By infinitesimal criterion of flatness [17] fibrewise Hilbert polynomials $\chi(I_{\tilde{\Sigma},\Sigma} \otimes \widetilde{L}^n|_{j(\pi^{-1}(t))})$ do not depend on closed point $t \in T$.

We have $\chi(O(n)|_{i(\pi^{-1}(t))}) = \chi(O(n)|_{\Delta}(\pi^{-1}(t)))$ and hence we conclude that $\chi(I_{\tilde{\Sigma},\Sigma} \otimes \widetilde{L}^n|_{i(\pi^{-1}(t))}) = 0$. Now $i$ and $i_\Delta$ are identified under isomorphism $\mathbb{P}(\pi_*\widetilde{L})^\vee = \mathbb{P}_\Delta$. By the same reason $i'$ and $i_\Delta$ are identified under isomorphism $\mathbb{P}(\pi'_*\widetilde{L}')^\vee = \mathbb{P}_\Delta$ and hence $\tilde{\Sigma} \cong \tilde{\Sigma}'$ and under this identification also $\widetilde{L} = i^*O(1) = i'^*O(1) = \widetilde{L}'$.

To confirm that also $\tilde{E} = \tilde{E}'$ we reason in similar way and consider closed immersions of $T$-schemes $\tilde{\Sigma}$ and $\tilde{\Sigma}'$ into Grassmann varieties

$$\tilde{\Sigma} \hookrightarrow \text{Grass}(\tilde{V}_m, r) = \text{Grass}(\tilde{V}_m', r) \xrightarrow{j'} \tilde{\Sigma}' .$$

We use shorthand notations $G := \text{Grass}(\tilde{V}_m, r)$ and $G' := \text{Grass}(\tilde{V}_m', r)$ and form a fibred product $G \times_T G'$ together with the diagonal $G_\Delta \hookrightarrow G \times_T G'$; and form the subscheme $\tilde{\Sigma}_\Delta = (\tilde{\Sigma} \times_T \tilde{\Sigma}') \cap G_\Delta$. As previously, there is a commutative square

$$\begin{array}{ccc}
\tilde{\Sigma}_\Delta & \xrightarrow{j_\Delta} & G_\Delta \\
\downarrow & & \downarrow j \\
\tilde{\Sigma} & \xrightarrow{j} & G
\end{array}$$

from which $\tilde{\Sigma}_\Delta \hookrightarrow \tilde{\Sigma}$ as closed subscheme. Applying Proposition [16] we conclude that $\tilde{\Sigma}_\Delta$ is flat over $T$. Since fibres of schemes $\tilde{\Sigma}_\Delta$ and $\tilde{\Sigma}$ over same closed point $t \in T$ coincide they have equal Hilbert polynomials as subschemes in Grassmann variety $G_\Delta \cong G(rp(m), r)$. Hence $j_\Delta(\tilde{\Sigma}_\Delta) = j(\tilde{\Sigma})$ under identification $G_\Delta = G$ and also $j_\Delta(\tilde{\Sigma}_\Delta) = j'(\tilde{\Sigma}')$ under identification $G_\Delta = G'$. Now let $\pi^*\tilde{V}_m \to Q$ be the universal quotient bundle on $G = G'$. Then $\tilde{E} \otimes \tilde{L}^m = j^*Q = j'^*Q = \tilde{E}' \otimes \tilde{L}'^m$ and hence $\tilde{E} \cong \tilde{E}'$.

References

[1] N.V. Timofeeva Admissible pairs for arbitrary dimension, I: Resolution. arXiv:2012.11194 [math.AG] http://arxiv.org/abs/2012.11194

[2] N.V. Timofeeva, Compactification in Hilbert scheme of moduli scheme of stable 2-vector bundles on a surface, Math. Notes, 82:5(2007), 677 – 690.

[3] N.V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface, Sb. Math., 199:7(2008), 1051–1070.

[4] N.V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface. II, Sb. Math. 200:3(2009), 405–427.

[5] N.V. Timofeeva, On degeneration of surface in Fitting compactification of moduli of stable vector bundles, Math. Notes, 90(2011), 142–148.

[6] N.V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface. III: Functorial approach, Sb. Math., 202:3(2011), 413 – 465.
[7] N. V. Timofeeva, *On a new compactification of the moduli of vector bundles on a surface. IV: Nonreduced moduli*, Sb. Math., **204:1**(2013), 133–153.

[8] N. V. Timofeeva, *On a new compactification of the moduli of vector bundles on a surface. V: Existence of universal family*, Sb. Math., **204:3**(2013), 411–437.

[9] N. V. Timofeeva, *On a morphism of compactifications of moduli scheme of vector bundles* / N. V. Timofeeva // Siberian Electronic Mathematical Reports (SEMR) – 2015. – V. 12. – P. 577–591.

[10] N. V. Timofeeva, *Isomorphism of compactifications of moduli of vector bundles*: arXiv:1411.7872v1 [math.AG]. Russian original in: Modelirovanie i Analiz Informatsionnykh Sistem (Modeling and Analysis of Information Systems). 2015. V. 22. No. 5, P. 629 - 647.

[11] N. V. Timofeeva, *Admissible pairs vs Gieseker-Maruyama* // Sbornik Math., 2019, V. 210. No. 5. P. 731–755

[12] D. Gieseker, *On the moduli of vector bundles on an algebraic surface* / D. Gieseker // Annals of Math. – 1977. – V. 106. – P. 45 – 60.

[13] R. Hartshorne, *Algebraic Geometry* / R. Hartshorne // Graduate Texts in Mathematics, 52. New York – Heidelberg – Berlin: Springer-Verlag, 1977.

[14] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves* / D. Huybrechts, M. Lehn // Aspects Math., E31. Braunschweig: Vieweg, 1997.

[15] A. Grothendieck, *Éléments de Géométrie Algébrique III: Étude Cohomologique des Faîceaux Cohérents, Première Partie* / Rédigés avec la collaboration de J. Dieudonné, Inst. Hautes Études Sci. Publ. Mathématiques, No. 11, IHÉS, Paris, 1961.

[16] J. Milne, *Étale cohomology*, Princeton Univ. Press, Princeton – New Jersey, 1980.

[17] N. V. Timofeeva, *Infinitesimal criterion for flatness of projective morphism of schemes*, S.-Petersburg Math. J. (Algebra i Analiz), **26-1**(2014), 185–195.