ON SEMIGROUP MAXIMAL OPERATORS ASSOCIATED WITH DIVERGENCE-FORM OPERATORS WITH COMPLEX COEFFICIENTS

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ABSTRACT. Let \( L_A = -\text{div} (A \nabla) \) be an elliptic divergence form operator with bounded complex coefficients subject to mixed boundary conditions on an arbitrary open set \( \Omega \subseteq \mathbb{R}^d \). We prove that the maximal operator \( M^A_f = \sup_{t > 0} |\exp(-tL_A)f| \) is bounded in \( L^p(\Omega) \) whenever \( A \) is \( p \)-elliptic in the sense of [10]. The relevance of this result is that, in general, the semigroup generated by \(-L_A\) is neither contractive in \( L^\infty \) nor positive, therefore neither the Hopf–Dunford–Schwart maximal ergodic theorem [20, Chap. VIII] nor Akcoglu’s maximal ergodic theorem [1] can be used. We also show that if \( d \geq 3 \) and the domain of the sesquilinear form associated with \( L_A \) embeds into \( L^{2^*}(\Omega) \) with \( 2^* = \frac{2d}{d-2} \), then the range of \( L^p \)-boundedness of \( M^A \) improves to \((\frac{rd}{(r-1)d+2}, \frac{rd}{d-2})\), where \( r \geq 2 \) is such that \( A \) is \( r \)-elliptic. With our method we are also able to study the boundedness of the two-parameter maximal operator \( \sup_{s,t > 0} |T^s_A T^t_A f| \).

1. Principal result of the paper

Let \( \Omega \subseteq \mathbb{R}^d \) be an arbitrary open set. Denote by \( A(\Omega) \) the family of all complex uniformly strictly accretive (also called elliptic) \( d \times d \) matrix functions on \( \Omega \) with \( L^\infty \) coefficients. That is, \( A(\Omega) \) is the set of all measurable \( A : \Omega \to \mathbb{C}^{d \times d} \) for which there exist \( \lambda, \Lambda > 0 \) such that for almost all \( x \in \Omega \) we have

\[ \text{Re} \langle A(x) \xi, \xi \rangle \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{C}^d; \tag{1} \]

\[ |\langle A(x) \xi, \eta \rangle| \leq \Lambda |\xi| |\eta|, \quad \forall \xi, \eta \in \mathbb{C}^d. \tag{2} \]

For any \( A \in A(\Omega) \) denote by \( \lambda(A) \) the largest admissible \( \lambda \) in (1) and by \( \Lambda(A) \) the smallest \( \Lambda \) in (2).

Given \( A \in A(\Omega) \) and \( p \in (1, \infty) \), we say that \( A \) is \( p \)-elliptic [10] if \( \Delta_p(A) > 0 \), where

\[ \Delta_p(A) := \text{ess inf} \min_{x \in \Omega} \text{Re} \left( \langle A(x) \xi, \xi + |1 - 2/p|\xi \rangle \right)_{\mathbb{C}^d}. \tag{3} \]

It follows straight from (3) that \( \Delta_2(A) = \lambda(A) \), and that \( \Delta_p \) is invariant under conjugation of \( p \), meaning that \( \Delta_p(A) = \Delta_{p'}(A) \), where \( 1/p + 1/p' = 1 \). Denote by \( A_p(\Omega) \) the class of all \( p \)-elliptic matrix functions on \( \Omega \). Then \( A_p(\Omega) = A_{p'}(\Omega) \). It is known [10] that \( A \in A_p(\Omega) \) if and only \( A^* \in A_{p^*}(\Omega) \).
where $A^*$ denotes the conjugate transpose of $A$. Moreover, $\{A_p(\Omega) : p \in [2, \infty)\}$ is a decreasing chain of matrix classes such that

\[
\begin{align*}
\{\text{elliptic matrices on } \Omega\} &= A_2(\Omega), \\
\{\text{real elliptic matrices on } \Omega\} &= \bigcap_{p \in [2, \infty)} A_p(\Omega).
\end{align*}
\]

We will refer to $\lambda(A)$, $\Lambda(A)$ and $\Delta_p(A)$ collectively as the $p$-ellipticity constants of $A$.

The reader interested in the genesis of $p$-ellipticity and its properties should consult [9, 10, 11]. Here we mention that M. Dindoš and J. Pipher [18], while studying reverse Hölder inequalities for weak solutions of complex elliptic operators, discovered (independently of the authors of the present paper) a condition equivalent to $p$-ellipticity which can be considered a strengthening of [12, (2.25)]. Since then the same authors have continued the study of $p$-ellipticity, applying it successfully in different contexts; see their recent papers [17, 19].

Suppose that either

(a) $\mathcal{V} = W^{1,2}_0(\Omega)$,

(b) $\mathcal{V} = W^{1,2}(\Omega)$, or

(c) $\mathcal{V}$ is the closure in $W^{1,2}(\Omega)$ of the set of restrictions $\{f|_{\Omega} : f \in C^\infty_c(\mathbb{R}^d \setminus D)\}$, where $D$ is a (possibly empty) closed subset of $\partial \Omega$.

For every $A \in \mathcal{A}(\Omega)$ we denote by $L_A$ the maximal accretive Kato-sectorial operator on $L^2(\Omega)$ associated with the densely defined, closed and sectorial form $a(f, g) := \int_\Omega \langle A \nabla f, \nabla g \rangle_{\mathbb{C}^d}$, $f, g \in \mathcal{V}$.

We denote by $(T^A_t)_{t>0}$ the associated contractive analytic semigroup on $L^2(\Omega)$; see [27, Chapter VI], [3] and [32, Chapters I and IV]. It was proven in [10, Theorem 1.3], [22, Theorem 2] and [9, Lemma 17] that $(T^A_t)_{t>0}$ extends to a contractive analytic semigroup on $L^p(\Omega)$ whenever $A \in A_p(\Omega)$.

The maximal operator associated with $(T^A_t)_{t>0}$ is defined by the rule

\[\mathcal{M}^A f(x) := \sup_{t>0} \left| T^A_t f(x) \right| .\]

To the best of our knowledge, boundedness of $\mathcal{M}^A$ for general complex elliptic $A$ is unknown.

The following is our principal result.

**Theorem 1.** Let $p \in (1, \infty)$ and $A \in A_p(\Omega)$. There exists $C > 0$, that depends only on the $p$-ellipticity constants of $A$, such that

\[\left\| \mathcal{M}^A f \right\|_p \leq C \|f\|_p, \quad \forall f \in L^p(\Omega). \tag{4}\]

Unless the form $a$ is given by real coefficients, the semigroup $(T^A_t)_{t>0}$ is neither bounded in $L^\infty(\Omega)$ nor positive [32, Theorems 4.14, 4.2]. The novelty of Theorem 1 is that the relevant literature known to us deals with semigroups of sub-positive contraction whose generators have a bounded holomorphic functional calculus in $L^p$; see [14] and [33, Chapter III].
The known methods allow proving (4) only in the case where A is a complex rotation of a real matrix (see Section 2.2). Namely, [14, 33] exploit the Hopf–Dunford–Schwartz maximal ergodic theorem and its generalizations, that is to say, the boundedness of the maximal ergodic operator (defined in (16) for the case of divergence-form operators) associated with a positive contractive semigroup [20, Chap. VIII], [13, Chap. 4], [1] and [23]; see also [25, Section 10.7.d] for a summary of the known results in this direction. However, in the case of complex matrices A there is no general result that replaces the Hopf–Dunford–Schwartz maximal ergodic theorem. Instead, the boundedness of the maximal ergodic operator associated with \((T_t^A)_{t>0}\) will be an immediate consequence of our Theorem 1.

1.1. Principal ideas for the proof of Theorem 1. It turns out that for real elliptic matrices B the maximal operator \(\mathcal{M}^B\) is bounded in \(L^p(\Omega)\) for all \(p \in (1, \infty)\) (see Corollary 4).

The first idea for proving Theorem 1 comes from afar (for instance, see [34]): by triangular inequality,

\[
\mathcal{M}^A f(x) \leq \mathcal{M}^B f(x) + \sup_{t>0} \left| T_t^A f(x) - T_t^B f(x) \right|,
\]

therefore we reduce the problem to proving that the second maximal operator on the right-hand side is bounded in \(L^p(\Omega)\). The point is that the maximal operator

\[
\mathcal{M}^{A,B} f(x) = \sup_{t>0} \left| T_t^A f(x) - T_t^B f(x) \right|
\]

is expected to behave much better than \(\mathcal{M}^A\). As an example, consider the case of the two semigroups \(\exp(-tzL)\) and \(\exp(-tL)\), where \(\text{Re} z > 0\) and \(L = -\Delta\) is the positive Euclidean Laplacian. The Fourier multiplier \(m_z(\xi) := \exp(-z|\xi|^2) - \exp(-|\xi|^2)\) satisfies the estimates \(|m_z(\xi)| \lesssim \min(|\xi|, |\xi|^{-1})\) and \(|\xi \cdot \nabla m_z(\xi)| \lesssim 1\), from which one can prove [4, 5, 6] that the square function

\[
f(x) = \left( \sum_{n \in \mathbb{Z}} \left| \exp(-2^n zL)f - \exp(-2^n L)f \right|^2 \right)^{1/2}
\]

and, consequently, the maximal operator

\[
f(x) = \sup_{t>0} \left| \exp(-tL)f - \exp(-tzL)f \right|
\]

are bounded in \(L^p(\mathbb{R}^d)\) for all \(p \in (1, \infty)\).

The second idea used in the proof of Theorem 1 comes from [14], where the author developed a subordination method for studying the maximal operator \(\sup_{t>0} [m(t\mathcal{A})f]\) associated with, say, a generator \(\mathcal{A}\) of a holomorphic semigroup with bounded imaginary powers on \(L^p\) and a multiplier \(m \in L^1(\mathbb{R}_+, d\lambda/\lambda)\): at least formally we have

\[
m(t\mathcal{A})f = \frac{1}{2\pi} \int_{\mathbb{R}} t^{iu}[Mm](u)\mathcal{A}^{iu}f \, du,
\]

where \(Mm\) denotes the Mellin transform of \(m\), which is defined by rule

\[
[Mm](u) = \int_0^\infty m(\lambda)\lambda^{-iu} \frac{d\lambda}{\lambda}, \quad u \in \mathbb{R}.
\]
It follows from (6) that
\[ \left\| \sup_{t>0} |m(t \mathcal{A} f)| \right\|_p \leq \frac{1}{2\pi} \int_{\mathbb{R}} |[Mm](u)| \left\| \mathcal{A}^{iu} f \right\|_p \, du. \tag{8} \]
This reduces the estimate of the $L^p$-norm of maximal function on the left-hand side of (6) to finding pointwise estimates of $Mm$ and $L^p$-estimates of imaginary powers of $\mathcal{A}$.

If we apply this method to the multiplier
\[ m_{\pm\theta}(\lambda) := \exp(-e^{\pm i \theta} \lambda) - \exp(-\lambda), \quad \lambda > 0, \quad 0 < \theta < \pi/2, \tag{9} \]
and the generator $L_B$ where $B \in \mathcal{A}(\Omega)$ is real, then it follows from [9, Theorem 3] that $\mathcal{A} e^{\pm i \theta} B$ is bounded in $L^p(\Omega)$ whenever the matrix $A = e^{\pm i \theta} B$ is $p$-elliptic (see Section 2.2 for details). This argument breaks down when the complex rotations of $B$ are replaced by a general $A \in \mathcal{A}_p(\Omega)$, because the two generators $L_A$ and $L_B$, in general, are not related via functional calculus (see also the comments in Section 3). Let us explain how we get around this.

Let $A, B \in \mathcal{A}(\Omega)$, where $B$ is real. We have the Duhamel’s formula:
\[ T^A_t f - T^B_t f = \int_0^t T^B_s (L_B - L_A) T^A_{t-s} f \, ds, \quad \forall f \in (L^2 \cap L^p)(\Omega). \tag{10} \]
Duhamel’s formula is typically used for showing that the difference of the two semigroups is small (with respect to some topology) whenever $A - B$ is small, for example, with respect to the $L^\infty$-norm. So it would seem futile to use the estimate
\[ |T^A_t f - T^B_t f| \leq \int_0^t L_B T^B_s T^A_{t-s} f \, ds + \int_0^t T^B_s L_A T^A_{t-s} f \, ds, \tag{11} \]
but actually it works.

For simplicity consider the first term on the right-hand side (11). Our idea is to transfer a fractional power of $L_B$ from the first semigroup to the second one as follows:
\[ L_B T^B_s T^A_{t-s} = L_B^{1-\alpha} T^B_s L_B^\alpha T^A_{t-s}, \quad \alpha \in (0, 1). \]
Now we would like to transform the fractional power of $L_B$ into a fractional power of $L_A$, more precisely we aim at the decomposition
\[ L_B^\alpha = U^{B,A}_\alpha L_A^\alpha, \]
where $U^{B,A}_\alpha$ is a bounded operator on $L^p(\Omega)$. Yet this is not possible for all $A, B, p, \alpha$, because for, say, $1/2 < \alpha < 1$ and $p = 2$, the domains of $L_A^\alpha$ and $L_B^\alpha$ are either unrelated or unknown. This procedure with $B = I$, $\alpha = 1/2$ and $p = 2$ would require a solution to the Kato problem for the square root of $L_A$ which is still an open problem in the general setting we consider in this paper.

However, by combining a classic result of Kato [26] with [9, Theorem 3] and a classic complex interpolation argument, we are able to make the procedure described above work for a small positive power $\alpha = \alpha(p) < 1/2$.
whenever $A \in A_p(\Omega)$ (see Proposition 8). This gives
\[
\int_0^t L_B T_s B \hat{T}_t \alpha f \, ds = \int_0^t \psi_{1-\alpha}(sL_B) U_{\alpha}^{\beta,A}(t-s)^\alpha f \, d\mu_{\alpha}(s)
\]
with $U_{\alpha}^{\beta,A}$ as above,
\[
d\mu_{\alpha}(s) = s^{\alpha-1}(t-s)^{-\alpha} 1_{[0,t]}(s) \, ds, \quad t > 0,
\]
and
\[
\psi_{\beta}(\lambda) = \lambda^\beta e^{-\lambda}, \quad \lambda > 0, \quad \beta > 0.
\]
The advantage of this decomposition is that $\mu_{\alpha}(\mathbb{R})$ is a finite measure on $\mathbb{R}$ with total mass independent of $t$. More precisely,
\[
\mu_{\alpha}(\mathbb{R}) = B(\alpha,1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}, \quad \forall t > 0.
\]
Then, by using Cowling’s subordination [14] twice in combination with our functional calculus result in [9, Proposition 3], we obtain that the two-variable maximal operator
\[
\sup_{0<s<t} \left| \psi_{1-\alpha}(sL_B) U_{\alpha}^{\beta,A}(t-s)^\alpha f \right|
\]
is bounded in $L^p(\Omega)$, whenever $A \in A_p(\Omega)$ (see Section 3.1). In this way we can prove that $\mathcal{M}_{A,B}$ — and thus $\mathcal{M}_A$ — is bounded in $L^p(\Omega)$ whenever $A \in A_p(\Omega)$.

2. More notation and preliminaries

Let $1 < p < 2$ and $A \in A_p(\Omega)$. With a slight abuse of notation, we maintain the symbol $L_A$ for denoting the negative generator of $(T_t^A)_{t>0}$ in $L^q(\Omega)$, $p \leq q \leq p'$. We have $L^q(\Omega) = N_q(L_A) \oplus \overline{N_q(L_A)}$ and
\[
Q_q f = \lim_{t \to +\infty} T_t^A f
\]
is a consistent family of contractive projections onto $N_q(L_A)$ for $p \leq q \leq p'$ [15]. Hence, $N_q(L_A) = \{0\}$ if and only if $N_2(L_A) = \{0\}$. A simple calculation based on the definition of $L_A$ by means of the sesquilinear form $a$ shows that $N_2(L_A) = N_2(L_I)$ consists of the vector space of all $L^2$ functions which are constant on each connected component of $\Omega$ of finite measure whose boundary does not intersect $D$ in a set of positive relative 2-capacity [2], and are 0 on all other connected components of $\Omega$. In order to simplify our proofs, in this paper we shall always assume that $N_2(L_I) = \{0\}$.

Under this assumption $L_A$ is an injective sectorial operator of angle $< \pi/2$ on $L^q(\Omega)$, whenever $p \leq q \leq p'$. In this range of $q$’s the complex powers $L_A^z$ are well defined on $L^q(\Omega)$ for every $z \in \mathbb{C}$. Also, for every $z \in \mathbb{C}$, the closed densely defined operators $L_A^z$ on $L^q(\Omega)$, $p \leq q \leq p'$, are consistent [24,36].

**Proposition 2** ([9]). Let $p > 1$ and $A \in A_p(\Omega)$. Then $L_A$ has bounded $H^\infty$-calculus of angle $< \pi/2$ on $L^p(\Omega)$ in the sense of [15]. It particular, $L_A$ has bounded imaginary powers on $L^p(\Omega)$ and there exist $\theta_p \in (0,\pi/2)$ and $C_p > 0$ such that
\[
\|L_A^{iu}\|_p \leq C_p e^{\theta_p|u|}, \quad \forall u \in \mathbb{R}.
\]
The angle $\theta_p$ and the constant $C_p$ only depend on $p$ and the $p$-ellipticity constants of $A$ [10, Corollary 5.17 and Proposition 5.23].
2.1. Cowling’s subordination technique. It is by now a classic result in semigroup theory that for a semigroup \((S_t)_{t>0}\) of linear transformations on some Lebesgue spaces \(L^p\), the boundedness of the associated maximal ergodic operator in \(L^p\) together with a \(H^\infty\)-calculus of angle \(<\pi/2\) in \(L^p\) for the negative generator implies the boundedness of the associated maximal operator in \(L^p\). One way to prove this result is to use Cowling’s subordination technique [14] (see also [7, 31]) which we briefly describe below in the specific case we are interested in here (see (17), Proposition 3 and Corollary 4). Note that Cowling originally formulated his argument only for symmetric semigroups. Note also that as an alternative to Cowling’s subordination method, we can slightly modify an argument of Stein [33, Chapter III] that exploits complex interpolation and fractional integration.

Fix \(A \in \mathcal{A}(\Omega)\). Define the associated maximal ergodic operator by the rule

\[
\mathcal{E}^A f(x) = \sup_{t>0} \left| \frac{1}{t} \int_0^t T_s^A f(x) \, ds \right|, \quad f \in L^2(\Omega). \tag{16}
\]

A rapid calculation shows that for all \(f \in (L^2 \cap L^p)(\Omega)\) and \(t > 0\) we have

\[
T_t^A f = \frac{1}{t} \int_0^t T_s^A f \, ds - \frac{1}{t} \int_0^t \psi(tL_A)f \, ds,
\]

where \(\psi_\alpha, \alpha > 0\), is defined by (13) and the bounded operator \(\psi(tL_A)\) on \(L^2(\Omega)\) is defined by means of McIntosh’s \(L^2\)-functional calculus [30]. Therefore,

\[
\sup_{t>0} \left| T_t^A f(x) \right| \leq \mathcal{E}^A f(x) + \sup_{t>0} \left| \psi(tL_A)f(x) \right|, \quad \forall f \in (L^2 \cap L^p)(\Omega). \tag{17}
\]

**Proposition 3.** Let \(p > 1\). Suppose that \(A \in \mathcal{A}_p(\Omega)\). For every \(\alpha > 0\) there exists \(C > 0\), which depends only on \(\alpha\), \(p\) and the \(p\)-ellipticity constants of \(A\), such that

\[
\left\| \sup_{t>0} \left| \psi_\alpha(tL_A)f \right| \right\|_p \leq C \|f\|_p, \quad \forall f \in L^p(\Omega).
\]

**Proof.** We use Cowling’s subordination (8). The Mellin transform (7) of \(\psi_\alpha\) is

\[
[M \psi_\alpha](u) = \Gamma(\alpha - iu), \quad \text{for all } u \in \mathbb{R}, \tag{18}
\]

where \(\Gamma\) denotes the Euler gamma function. Stirling’s formula gives

\[
|\Gamma(\alpha - iu)| \sim \sqrt{2\pi} |u|^{\alpha-1/2} e^{-\frac{\pi}{2}|u|}, \quad \text{for } |u| \to \infty. \tag{19}
\]

Therefore,

\[
\left\| \sup_{t>0} \left| \psi_\alpha(tL_A)f \right| \right\|_p \lesssim \int_{\mathbb{R}} (1 + |u|)^{\alpha-1/2} e^{-\frac{\pi}{2}|u|} \left\| L_A^u f \right\|_p \, du.
\]

Now the desired result follows from Proposition 2. \(\Box\)

**Corollary 4.** Let \(B \in \mathcal{A}(\Omega)\) be a real matrix. Then \(\mathcal{M}^B\) is bounded in \(L^p(\Omega)\), for every \(p \in (1, +\infty]\).
Proof. As remarked before, when $B$ is real elliptic, $(T^B_t)_{t>0}$ is positive and contractive on $L^\infty(\Omega)$ and on $L^1(\Omega)$ by the duality $(T^B_t)^* = T_t^{B^*}$ [32]. Hence, by the Hopf–Dunford–Schwartz theorem, $\mathcal{E}^B$ is bounded in $L^p(\Omega)$ for all $p > 1$. Moreover, $B \in A_p(\Omega)$ for all $p > 1$ [10, p. 3179], therefore by Proposition 3 the maximal operator $f \mapsto \sup_{t>0}|\psi_1(tL_B)f|$ is bounded in $L^p(\Omega)$ for all $p > 1$. The desired result for $p \neq \infty$ now follows from (17), while the boundedness of the maximal operator in $L^\infty(\Omega)$ trivially follows from the contractivity of $(T^B_t)_{t>0}$ on $L^\infty(\Omega)$.

$\square$

2.2. Complex rotations of real elliptic matrices. In the case of a symmetric contraction semigroup $(S_t)_{t>0}$ on some $\sigma$-finite measure space, Cowling [14] observed that the subordination technique explained in Section 2.1 can be used for proving the boundedness on $L^p(\Omega)$, $1 < p < \infty$, of the maximal operator

$$f \mapsto \sup_{|\arg(\varepsilon)| < \omega_p - \varepsilon} |S_\varepsilon f|,$$

where $\omega_p = \pi/2 - \omega_p^*$ and $\omega_p^*$ denotes the angle of the $H^\infty$-calculus in $L^p$ of the negative generator of $(S_t)_{t>0}$. Recall that by [7], for every symmetric contraction semigroup we have $\omega_p^* \leq \phi_p^* := \arcsin |1 - 2/p|$. Despite the fact that for a real elliptic matrix $B$, in general, the semigroup $(T^B_t)_{t>0}$ is not symmetric, Cowling’s argument can be easily adapted and in combination with Proposition 2 gives the following extension of Corollary 4.

Proposition 5. Let $B \in A(\Omega)$ be a real matrix. Suppose that $1 < p < \infty$ and $0 < \theta < \pi/2$ are such that $e^{i\theta} B \in A_p(\Omega)$. Then $\mathcal{M}^{e^{i\theta} B}$ is bounded on $L^p(\Omega)$.

Proof. Recall the definition (9) of $m_{\pm \theta}$. Then,

$$\mathcal{M}^{e^{i\theta} B} f \leq \mathcal{M}^B f + \sup_{t>0} |m_{\pm \theta}(tL_B)f|.$$

By Corollary 4 the maximal operator $\mathcal{M}^B$ is bounded on $L^p(\Omega)$.

As for the maximal operator associated with $m_{\pm \theta}$, we have

$$|Mm_{\pm \theta}|(u) = i\theta \left( \frac{e^{i\theta} u - 1}{\theta u} \right) \Gamma(1 - iu), \quad u \in \mathbb{R}.$$

Hence by (19),

$$|Mm_{\pm \theta}|(u) \lesssim (1 + |u|)^{1/2} e^{(\theta - \pi/2)|u|}, \quad \forall u \in \mathbb{R}.$$

Now use the assumptions, the fact that $A_p(\Omega)$ is invariant under conjugation [10, Corollary 5.17], the identity $L_{e^{i\theta} B} = e^{i\theta} L_B$, Cowling’s subordination (8) applied with $\mathcal{A} = L_B$ and $m = m_{\pm \theta}$, and Proposition 2 applied with $A = e^{i\theta} B$.

$\square$

Remark 6. Let $B$ be a real elliptic matrix. By [10, Propositions 5.13 and 5.21] $e^{\pm i\theta} B$ is $p$-elliptic precisely when $\theta < \pi/2 - \vartheta_p^*$, where $\vartheta_p^* = \vartheta_p^*(B)$ is given by [8, (3.7)]. This is consistent with the general result for symmetric contractions [7] explained above, because $T^{e^{i\theta} B}_t = T^B_{te^{i\theta}}$ and when the real matrix $B$ is symmetric we have $\vartheta_p^* = \vartheta_p^* = \arcsin |1 - 2/p|$.
3. Proof of Theorem 1

The proof of Proposition 5 is based on the fact that \( T_t^{e^{it\theta}B} - T_t^B \) coincides with the multiplier \( m_{\pm \theta}(tL_B) \) of the generator \( L_B \): this is clearly false when we replace the complex rotations of the real matrix \( B \) by a general complex \( p \)-elliptic matrix \( A \), since \( T_t^A \) and \( T_t^B \) may not commute and the generators \( L_A \) and \( L_B \) may not have a joint functional calculus. As we already explained in Section 1.1, despite this difficulty, we prove Theorem 1 by combining Corollary 4 with Proposition 2 and using Cowling’s subordination (6) appropriately. To do that, another ingredient is needed: a classic result of Kato [26] concerning the domain of the fractional powers of Kato-sectorial operators, which we state below in the particular case we will use in this paper.

For every \( \alpha \in [0, 1/2] \) denote by

\[
\mathcal{Y}_\alpha = \left[ L^2(\Omega), \mathcal{Y} \right]_{2\alpha}
\]

the complex interpolation space of index \( 2\alpha \) between \( L^2(\Omega) \) and the domain \( \mathcal{Y} \) of the sesquilinear form associated with \( L_A, A \in \mathcal{A}(\Omega) \).

**Lemma 7** (Kato [26] and Lions [28]). Let \( A, B \in \mathcal{A}(\Omega) \) and \( 0 < \alpha < 1/2 \). Then

\[
\text{D}_2(L_A^\alpha) = \text{D}_2(L_B^\alpha) = \mathcal{Y}_\alpha,
\]

and there exist \( C_\alpha, C'_\alpha > 0 \) depending only on \( \alpha \) and the ellipticity constants of \( A \) and \( B \) such that

\[
C_\alpha \| L_A^\alpha f \|_2 \leq \| L_B^\alpha f \|_2 \leq C'_\alpha \| L_A^\alpha f \|_2, \quad \forall f \in \mathcal{Y}_\alpha. \tag{20}
\]

Moreover, for every \( \alpha_0 \in [0, 1/2) \) we have \( \sup_{0 \leq \alpha \leq \alpha_0} \left( |C_\alpha|^{-1} + |C'_\alpha| \right) < +\infty \).

Recall that for every \( A \in \mathcal{A}(\Omega) \) we have \( N_2(L_A) = N_2(L_I) \) and that for simplicity in this paper we always assume that \( \Omega \) and \( \mathcal{Y} \) are such that \( N_2(L_I) = \{0\} \). Therefore \( \mathcal{R}_2(L_A) = L^2(\Omega) \) and (20) is equivalent to the fact that

\[
U_{\alpha, A, B} := L_B^\alpha L_A^{-\alpha}
\]

extends to a bounded (invertible) operator in \( L^2(\Omega) \), for all \( \alpha \in (0, 1/2) \) and all \( A, B \in \mathcal{A}(\Omega) \). This together with a standard complex interpolation argument based on (15) gives the next proposition which extends the \( L^2 \)-boundedness of \( U_{\alpha, A, B} \) to the \( L^p \)-boundedness in the range of \( p \)-ellipticity of \( A \) and \( B \).

**Proposition 8.** Let \( p > 1 \). Suppose that \( A, B \in \mathcal{A}_p(\Omega) \). Then there exists \( \alpha = \alpha(p) \in (0, 1/2) \) such that \( U_{\alpha, A, B} \) extends to a bounded operator in \( L^p(\Omega) \).

**Proof.** For simplicity, suppose that \( 1 < p < 2 \). For every \( \alpha > 0 \) define the strip

\[
\Sigma_\alpha = \{ z \in \mathbb{C} : 0 < \text{Re} z < \alpha \}.
\]

Fix \( \alpha_0 \in (0, 1/2) \) and consider the family of operators

\[
S_z = e^{z \text{Im} z} L_B^{1 \text{Im} z} U_{\text{Re} z, A, B} L_A^{-i \text{Im} z}, \quad z \in \Sigma_{\alpha_0}.
\]

The imaginary powers of \( L_A \) and \( L_B \) are bounded in \( L^2(\Omega) \) (this is a consequence of [16], but it also follows from (15) applied with \( p = 2 \)) and, by
Lemma 7, the operators \( U_{Re_x}^{B,A} \) are bounded in \( L^2(\Omega) \) uniformly in \( z \in \Sigma_{\alpha_0} \).

Hence,
\[
\|S_z\|_2 \leq C(\alpha_0), \quad \forall z \in \Sigma_{\alpha_0}.
\]

Moreover, by (15) we have
\[
\|S_{z\sigma}\|_p \leq C_p, \quad \forall \sigma \in \mathbb{R}.
\]

Suppose we have already proved that for all \( f \in L^2(\Omega) \) the function
\[
z \mapsto S_z f \in L^2(\Omega)
\]
is continuous and bounded on \( \Sigma_{\alpha_0} \), and holomorphic in the interior of the strip. Then, by Stein’s complex interpolation theorem, for every \( q \in (p,2) \) there exist \( \alpha(q) \in (0, \alpha_0) \) and \( C_q > 0 \) such that
\[
\|U_{\alpha(q)}^{B,A}\|_q \leq C_q.
\]

This proves the proposition, because the interval of \( p \)-ellipticity of \( A \) and \( B \) is open \([10]\).

It remains to show that \( z \mapsto S_z f \in L^2(\Omega) \) is continuous and bounded on \( \Sigma_{\alpha_0} \), and holomorphic in the interior of the strip. We first show that
\[
S_z f = e^{z^2 L_B^0 L_A^{-1}} f, \quad \forall z \in \Sigma_{1/2}, \quad \forall f \in R_2(L_A).
\]

Indeed, since the imaginary powers of \( L_B \) are bounded in \( L^2(\Omega) \), we have \( D(L_B^{\alpha}) = D(P_B^{\alpha}) \) and \( L_B^z = L_B^{\text{Im} z} L_B^{\text{Re} z} \); see, for example, \([24, \text{Proposition 3.2.1 (c)}]\) applied with \( \alpha = i \text{Im} z \) and \( \beta = \text{Re} z \). Therefore, by Lemma 7 we have
\[
L_B^z = L_B^{\text{Im} z} L_B^{\text{Re} z} L_A^{-1} = L_B^{\text{Im} z} U_{\text{Re} z}^{B,A} L_A^{-1}.
\]

Using once again \([24, \text{Proposition 3.2.1 (c)}]\) we deduce that
\[
L_A^{\text{Re} z} L_A^{-1} \subset L_A^{\text{Re} z} = L_A^{\text{Re} z} L_A^{-1} = L_A^{1-\text{Re} z} = L_A L_A^{-1} - z,
\]

which implies
\[
R_2(L_A) \subseteq R_2(L_A), \quad \forall z \in \Sigma_{1/2}.
\]

Now (22) follows by combining (23) with (24) and (25).

We are now ready to study the regularity of \( z \mapsto S_z f \). Fix \( f \in D_2(L_A) \cap R_2(L_A) \) and \( g \in D_2(L_B^*) \cap R_2(L_B^*) \). Let \( z \in \mathbb{C} \) be such that \( 0 < \text{Re} z < 1/2 \). By Lemma 7 we have \( D_2(L_A\lambda) = D_2(L_B\lambda) \), which together with (25) gives
\[
\left\langle L_B^0 L_A^{-1} f, g \right\rangle_{L^2} = \left\langle L_A^{-1} f, L_B^* g \right\rangle_{L^2}.
\]

By \([24, \text{Proposition 3.2.1 (f)}]\) (applied with \( \alpha_0 = \alpha_1 = 1/2 \)) the maps \( z \mapsto L_A^{-1} f \) and \( z \mapsto L_B^* g \) are analytic in the strip \(-1/2 < \text{Re} z < 1/2\).

We now observe that, by assumption, \( L^2(\Omega) = \overline{R_2(L_B^*)} \). Hence \( D_2(L_B^*) \cap R_2(L_B^*) \) is dense in \( L^2(\Omega) \). In order to see this, consider \( g_n \in D(L_B^*) \) such that \( L_B^* g_n \rightarrow f \) and set \( f_n = n \int_0^{1/n} T_s^* L_B^* g_n \, ds \). It follows form (22) and the previous considerations that for every \( f \in D_2(L_A) \cap R_2(L_A) \) the map \( z \mapsto S_z f \in L^2(\Omega) \) is continuous in \( \Sigma_{\alpha_0} \) and holomorphic in the interior of the strip. By the uniform boundedness of the family \( \{S_z : z \in \Sigma_{\alpha_0}\} \) (see (21)) and the density of \( D_2(L_A) \cap R_2(L_A) \) in \( L^2(\Omega) \), we deduce that the same is true for every \( f \in L^2(\Omega) \). \( \square \)
3.1. Proof of Theorem 1. Fix a real elliptic matrix $B$ (for example, $B = I$). By Corollary 4, the maximal operator associated with $(T^B_t)_{t > 0}$ is bounded in $L^p(\Omega)$. Hence by (5) it suffices to prove that the maximal operator $\mathcal{M}^{A,B}$ is bounded in $L^p(\Omega)$. Fix $f \in (L^2 \cap L^p)(\Omega)$. By Duhamel’s formula (11) for every $t > 0$ we have

$$|T^A_t f - T^B_t f| \leq \left| \int_0^t L_B T_s^B T^A_t f \, ds \right| + \left| \int_0^t T_s^B L_A T^A_t f \, ds \right| = |I_t(f)| + |II_t(f)|.$$  

3.1.1. The maximal operator $\sup_{t > 0} |I_t(f)|$. Let $\alpha = \alpha(p) \in (0, 1/2)$ be as in Proposition 8. Set $\beta = 1 - \alpha$. Then we have

$$I_t(f) = \int_0^t \psi_\beta(sL_B) U_\alpha^{B,A} \psi_\alpha((t - s)L_A) f \, d\mu^i_\alpha(s),$$

(26)

where the finite measures $\mu^i_\alpha$, $t > 0$, are defined by (12). Hence by (14) we have

$$\sup_t |I_t(f)| \leq \frac{\pi}{\sin(\alpha\pi)} \sup_{0 < s < t} \left| \psi_\beta(sL_B) U_\alpha^{B,A} \psi_\alpha((t - s)L_A) f \right|.$$ 

We now subordinate the operators $\psi_\beta(sL_B)$ and $\psi_\alpha((t - s)L_A)$ to imaginary powers by means of (6), and by using (18) we obtain

$$\sup_{0 < s < t} \left| \psi_\beta(sL_B) U_\alpha^{B,A} \psi_\alpha((t - s)L_A) f \right| \leq \frac{1}{\pi^2} \int_{\mathbb{R}^2} |\Gamma(\beta - iv)| |\Gamma(\alpha - iv)| \left| L_B^i U_\alpha^{B,A} L_A^i f \right| \, du \, dv.$$ 

The boundedness of $f \mapsto \sup_t |I_t(f)|$ now follows by combining the Stirling’s formula (19) with Propositions 8 and 2.

3.1.2. The maximal operator $\sup_{t > 0} |II_t(f)|$. For every $\alpha \in (0, 1/2)$ set

$$V_\alpha^{B,A} = (\mathcal{T}_\alpha^{A^* B'})^*.$$ 

A rapid calculation shows that

$$V_\alpha^{B,A} \mathcal{Y}_\alpha \subseteq \mathcal{Y}_\alpha \quad \text{and} \quad L_B^\alpha V_\alpha^{B,A} f = L_A^\alpha f, \quad \forall f \in \mathcal{Y}_\alpha.$$ 

Fix $\alpha = \alpha(p^*)$ as in Proposition 8, but for the pair $(A^*, B^*)$ instead of the pair $(B, A)$. Set $\beta = 1 - \alpha$. Then $V_\alpha^{B,A}$ extends to a bounded operator on $L^p(\Omega)$ and

$$II_t(f) = \int_0^t L_B^\alpha T_s^B V_\alpha^{B,A} L_A^\beta T^A_t f \, ds = \int_0^t \psi_\beta(sL_B) V_\alpha^{B,A} \psi_\beta((t - s)L_A) f \, d\mu^i_\beta(s).$$ 

Proceeding as we did in the previous section, we obtain the boundedness on $L^p(\Omega)$ of the maximal operator $f \mapsto \sup_{t > 0} |II_t(f)|$ that completes the proof of Theorem 1.

4. Range improvement in Theorem 1

In this section we consider an open set $\Omega \subseteq \mathbb{R}^d$, $d > 3$. We fix a subspace $\mathcal{Y}$ of the type described on page 2 and we assume that it satisfies the homogeneous embedding property:

$$\|u\|_{2^*} \lesssim \|\nabla u\|_2, \quad \forall u \in \mathcal{Y},$$

(27)
where $2^* = 2d/(d - 2)$ is the usual Sobolev upper exponent. For every $p \geq 2$ we define the two exponents

$$p_0 = \frac{pd}{d-2}, \quad p_0 = \frac{p^o}{p^o - 1}.$$ 

Note that $2^o = 2^*$.

**Lemma 9** ([22] and [35]). Assume that $\mathcal{V}$ satisfies (27). Let $p > 2$ and $A \in A_p(\Omega)$. Then for all $r \in [p_0, p^o]$ the semigroup $(T_t^A)_{t > 0}$ extends to a uniformly bounded analytic semigroup in $L^r(\Omega)$. Moreover, the analyticity angle does not depend on $r \in [p_0, p^o]$.

**Remark 10.** The homogeneous Sobolev embedding (27) implies $N_2(L_I) = \{0\}$ and the very same considerations of Section 2 show that, under the assumptions of Lemma 9, we have

$$\{0\} = N_2(L_I) = N_2(L_A) = N_r(L_A), \quad \forall r \in [p_0, p^o].$$

In the next result $\vartheta_2$ denotes the sectoriality angle of $L_A$ in $L^2(\Omega)$. We always have $0 \leq \vartheta_2 \leq \arccos(\lambda/\Lambda) < \pi/2$.

**Lemma 11** ([21, 22]). Under the assumptions of Lemma 9, for every $r \in [p_0, p^o]$ the generator $L_A$ has a bounded $H^\infty(S_\vartheta)$-calculus in $L^r(\Omega)$, whenever $\vartheta > \vartheta_2$. In particular, there exists $\vartheta_0 \in (0, \pi/2)$ such that

$$\|L^u_A\|_r \leq C(r, d)e^{\vartheta_0|u|}, \quad \forall u \in \mathbb{R},$$

for all $r \in [p_0, p^o]$.

**Theorem 12.** Assume that $\mathcal{V}$ satisfies (27). Let $p > 2$ and $A \in A_p(\Omega)$. Then the maximal operator $M_A$ is bounded on $L^p(\Omega)$.

**Proof.** Exactly as we did in the proof of Theorem 1, we fix a real matrix $B$ and reduce the proof to showing that the maximal operator $M^{A, B}$ is bounded in $L^r(\Omega)$. By using (28) it is easy to see that Proposition 8 extends to the range $[p_0, p^o]$ and, using again (28), we can now repeat the arguments in Sections 3.1.1 and 3.1.2. □

5. Two-parameter maximal operator

Suppose that $A_1, A_2 \in A_p(\Omega)$. Define the associated two-parameter maximal operator by the rule

$$\mathcal{M}^{A_1, A_2} f = \sup_{w, t > 0} \left| T_w^{A_1} T_t^{A_2} f \right|.$$ 

**Theorem 13.** Let $p \in (1, \infty)$ and $A_1, A_2 \in A_p(\Omega)$. Then $\mathcal{M}^{A_1, A_2}$ is bounded on $L^p(\Omega)$.

**Proof.** In the special case when $A_1$ is real-valued, the associated semigroup is nonnegative and, therefore, we have

$$\left| T_w^{A_1} T_t^{A_2} f \right| \leq T_w^{A_1} M^{A_2} f \leq M^{A_1, A_2} f.$$ 

(29)

It follows from Theorem 1 that $\mathcal{M}^{A_1, A_2}$ is bounded on $L^p(\Omega)$. 

When $A_1$ is not real-valued, the domination in (29) does not necessarily hold and we need another argument for proving the boundedness of $\mathcal{M}^{A_1,A_2}$. It is a variant of the proof of Theorem 1.

Fix $f \in L^p(\Omega)$ and write
\[ T_w^{A_1}T_t^{A_2}f = T_{t+w}^{A_1}f + T_w^{A_1}(T_t^{A_2}f - T_t^{A_1}f). \]
We clearly have
\[ \mathcal{M}^{A_1,A_2}f \leq \mathcal{M}^{A_1}f + \sup_{w,t>0} \left| T_w^{A_1}(T_t^{A_2}f - T_t^{A_1}f) \right| \]
By Theorem 1 the maximal operator $\mathcal{M}^{A_1}$ is bounded in $L^p(\Omega)$. As for the second term in the right hand side, we use Duhamel’s (10) formula and write
\[ T_{t-w}^{A_1}(T_t^{A_2}f - T_t^{A_1}f) = \int_0^t T_{t-w+s}^{A_1}(L_{A_1} - L_{A_2})T_{s}^{A_2}f \, ds = I_{w,t}(f) - J_{w,t}(f), \]
where
\[ I_{t,w}(f) = \int_0^t L_{A_1}T_{t-w+s}^{A_1}T_{s}^{A_2}f \, ds; \quad J_{t,w}(f) = \int_0^t T_{t-w+s}^{A_1}L_{A_2}T_{s}^{A_2}f \, ds. \]
We now proceed as we did in Section 3.1.1. We write the details only for the first integral.

Let $\alpha = \alpha(p) \in (0,1/2)$ be as in Proposition 8. Then,
\[ I_{t,w}(f) = \int_0^t \psi_{1-\alpha}((w+s)L_{A_1})U_{\alpha}^{A_1,A_2}\psi_{\alpha}((t-s)L_{A_2})f \, d\mu_{\alpha}^{w,t}(s), \]
where
\[ d\mu_{\alpha}^{w,t}(s) = (w+s)^{\alpha-1}(t-s)^{-\alpha}1_{[0,t]}(s) \, ds. \]
By (14) we have
\[ \mu_{\alpha}^{w,t}([0,t]) \leq \mu_{\alpha}^{t}([0,t]) = \frac{\pi}{\sin(\alpha \pi)}, \quad \forall w, t > 0. \]

Therefore,
\[ \sup_{w,t>0} \left| I_{t,w}(f) \right| \leq \frac{\pi}{\sin(\alpha \pi)} \sup_{s,t>0} \left| \psi_{1-\alpha}(sL_{A_1})U_{\alpha}^{A_1,A_2}\psi_{\alpha}(tL_{A_2})f \right|. \]
By Cowling’s subordination, the maximal operator on the right-hand side is dominated by the sublinear operator
\[ \frac{1}{\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |\Gamma(1 - \alpha - iu)| |\Gamma(\alpha - iv)| \left| L_{A_1}^{iu}U_{\alpha}^{A_1,A_2}L_{A_2}^{iv}f \right| \, du \, dv. \]
It follows from Stirling’s formula (19) in combination with Propositions 8 and 2 that the sub-linear operator above is bounded in $L^p(\Omega)$. \hfill \Box

A. PROOF OF KATO’S THEOREM (LEMMA 7) VIA SQUARE FUNCTIONS

In this appendix we give a proof of Lemma 7 which somewhat differs from the original one by Kato [26]. We find it likely that this proof may already be known, yet we where not able to find references that would confirm this.

Let $H$ be a complex Hilbert space. Let $a_1,a_2$ be two densely defined, closed and sectorial (thus continuous) sesquilinear forms on $H$ [24, 27, 32].
Denote by $\mathcal{L}_a_1$, and $\mathcal{L}_a_2$, respectively, the two associated Kato-sectorial operators on $H$. Suppose further that
\[ D(a_1) = D(a_2) = \mathcal{V} \quad \text{and} \quad \Re a_1(h) \sim \Re a_2(h), \]
uniformly in $h \in \mathcal{V}$. This is equivalent to $\|h\|_{a_1} \sim \|h\|_{a_2}$ uniformly in $h \in \mathcal{V}$, where
\[ \|h\|_{a_j}^2 = \|h\|_H^2 + \Re a_j(h), \quad h \in H, \quad j = 1, 2. \]
Moreover, we assume that $\mathcal{L}_a_1$, and thus also $\mathcal{L}_a_2$, are injective.

Kato’s theorem on fractional powers [26], which is a part of Lemma 7, can be rephrased by saying that, for every $\beta \in (0, 1/2)$ we have
\[ \mathcal{V}_\beta := D(\mathcal{L}_a^\beta) = D(\mathcal{L}_a_2) \quad \text{and} \quad \norm{\mathcal{L}_a^\beta h}_2 \sim \norm{\mathcal{L}_a_2 h}_2, \quad \forall h \in \mathcal{V}_\beta. \tag{30} \]
Once we have this result, Lions’s improvement, $\mathcal{V}_\beta = [H, \mathcal{V}]_{2\beta}$ for all $\beta \in (0, 1/2)$, follows from the modern theory of complex interpolation of domains of operators (see, for example [29, Theorem 11.6.1]). Indeed, in (30) fix $a_1 = a$ and take $a_2 = b = (a_1 + a_1^*)/2$ (the real part of $a$). The operator $\mathcal{H}$ associated with $b$ is nonnegative and self-adjoint. Therefore, $D(\mathcal{H}^{1/2}) = D(b) = \mathcal{V}$ and $\mathcal{V}_{a/2} = D(\mathcal{H}^{a/2}) = [H, \mathcal{V}]_{a}$ for all $a \in [0, 1]$, because $\mathcal{H}$ has bounded imaginary powers in $H$.

We now prove (30) by means of McIntosh’s square functions [30]. Recall the following well-known result.

**Lemma 14.** Let $a$ be a closed and densely defined sectorial form on a complex Hilbert space $H$. Let $\mathcal{L}_a$ denote the associated operator. Then for every $\gamma > 0$ the square function
\[ G_{a, \gamma}(x) := \left( \int_0^\infty \norm{\psi_\gamma(t\mathcal{L}_a)x}^2 \frac{dt}{t} \right)^{1/2}, \quad x \in H \]
is bounded on $H$.

**Proof.** By [16] the operator $\mathcal{L}_a$ has a bounded $H^\infty$-calculus on $H$, thus the result follows from [30, Section 8]. See also [15] and [24, Corollary 7.1.17]. $\square$

Define $T_t^{a_j} = \exp(-t\mathcal{L}_a)$, $j = 1, 2$. Fix $\beta \in (0, 1/2)$, $x \in D(\mathcal{L}_a^\beta)$ and $y \in D(\mathcal{L}_a^\beta)$. We have
\[
\left\langle x, \mathcal{L}_a^\beta y \right\rangle = \left| \int_0^\infty -\frac{d}{dt} \left\langle T_t^{a_1}x, T_t^{a_2} \mathcal{L}_a^\beta y \right\rangle \right| dt \\
\leq \left| \int_0^\infty \left\langle \mathcal{L}_a T_t^{a_1}x, T_t^{a_2} \mathcal{L}_a^\beta y \right\rangle \right| dt + \int_0^\infty \left| \left\langle T_t^{a_1}x, \mathcal{L}_a^{\beta+1/2} T_t^{a_2} y \right\rangle \right| dt \\
= I + J.
\]
It follows from Lemma 14 that
\[ I = \int_0^\infty \left| \left\langle \psi_1-\beta(t\mathcal{L}_a) \mathcal{L}_a^\beta x, \psi_\beta(t\mathcal{L}_a)y \right\rangle \right| \frac{dt}{t} \]
\[ \leq \left( \int_0^\infty \norm{\psi_1-\beta(t\mathcal{L}_a) \mathcal{L}_a^\beta x}^2 \frac{dt}{t} \right)^{1/2} \left( \int_0^\infty \norm{\psi_\beta(t\mathcal{L}_a)y}^2 \frac{dt}{t} \right)^{1/2} \]
\[ \lesssim_\beta \norm{\mathcal{L}_a^\beta x}_2 \norm{y}. \]
As for $J$, we have
\[ J = \int_0^\infty \left| \left( \int_t^\infty \mathcal{L}_a T_s^a x \, ds, \mathcal{L}_a^{\beta+1} T_s^a y \right) \right| \, dt \]
\[ \leq \int_1^\infty \int_0^t \left| \left( \mathcal{L}_a T_s^a x, \mathcal{L}_a^{\beta+1} T_s^a y \right) \right| \, dt \, ds \]
\[ \leq \int_1^\infty s^{\beta-1} \int_0^\infty \left| \left( \psi_{1-\beta}(\mathcal{L}_a^\beta x, \mathcal{L}_a^{\beta+1} T_s^a t y) \right) \right| \, dt \, ds. \]

By analyticity of the semigroup $(T_t^a)_{t>0}$ we have
\[ \psi_{1-\beta}(\mathcal{L}_a^\beta x) \in D(\mathcal{L}_a^\beta) \subseteq D(a_1) = D(a_2) = D(a'_2). \quad (31) \]

It follows that
\[ \left\langle \psi_{1-\beta}(\mathcal{L}_a^\beta x, \mathcal{L}_a^{\beta+1} T_s^a t y) \right\rangle = a_2 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta y), \psi_{1-\beta}(\mathcal{L}_a^\beta x) \right) \]
\[ = a_2 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta x, \psi_{1-\beta}(\mathcal{L}_a^\beta y) \right). \]

Since $a_2$ is sectorial, there exists $C > 0$, that does not depend on $x$, $y$, and $\beta$, such that
\[ \left| a_2 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta x, \psi_{1-\beta}(\mathcal{L}_a^\beta y) \right) \right| \leq C \sqrt{\Re a_2 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta x, \psi_{1-\beta}(\mathcal{L}_a^\beta y) \right).} \]

Recall that we are assuming that $\Re a_2(h) \sim \Re a_1(h)$ and note that $\Re a'_2(h) = \Re a_2(h)$. Hence, by Cauchy-Schwartz inequality,
\[ J \lesssim \int_1^\infty s^{\beta-1} \left( \int_0^\infty \Re a_1 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta x) \right) \, dt \right)^{1/2} \left( \int_0^\infty \Re a'_2 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta y) \right) \, dt \right)^{1/2} \, ds. \]

Then a change of variable in the first integral with respect to the variable $t$ gives
\[ J \lesssim \frac{1}{1/2 - \beta} \left( \int_0^\infty \Re a_1 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta x) \right) \, dt \right)^{1/2} \left( \int_0^\infty \Re a'_2 \left( \psi_{1-\beta}(\mathcal{L}_a^\beta y) \right) \, dt \right)^{1/2}. \]

We now observe that, by (31),
\[ a_1 \left( \psi_{1-\beta}(t, \mathcal{L}_a^\beta x) \right) \, dt = \left( \int \mathcal{L}_a \psi_{1-\beta}(t, \mathcal{L}_a^\beta x, \psi_{1-\beta}(t, \mathcal{L}_a^\beta x) \right) \, dt \]
\[ a'_2 \left( \psi_{1-\beta}(t, \mathcal{L}_a^\beta y) \right) \, dt = \left( \int \mathcal{L}_a^\beta \psi_{1-\beta}(t, \mathcal{L}_a^\beta y, \psi_{1-\beta}(t, \mathcal{L}_a^\beta y) \right) \, dt. \]

We now use Cauchy-Schwartz inequality and Lemma 14. It follows that
\[ J \lesssim \frac{1}{1/2 - \beta} \left\| \mathcal{L}_a^\beta x \right\| \left\| y \right\| \]
and combining the estimate above with the similar estimate for $I$, we conclude that
\[ \left| \left\langle x, \mathcal{L}_a^\beta y \right\rangle \right| \lesssim \left\| \mathcal{L}_a^\beta x \right\| \left\| y \right\|, \quad \forall x \in D(\mathcal{L}_a^\beta), \quad y \in D(\mathcal{L}_a^\beta), \quad \forall \beta \in (0, 1/2) \]
which is equivalent to
\[ D(\mathcal{L}_a^\beta) \subseteq D(\mathcal{L}_a^\beta) \quad \text{and} \quad \left\| \mathcal{L}_a^\beta h \right\| \lesssim \left\| \mathcal{L}_a^\beta h \right\| \quad \forall h \in D(\mathcal{L}_a^\beta). \]
The reverse inclusion and inequality (and thus (30)) follow by reversing the role of $a_1$ and $a_2$ in the previous proof.
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