Compositional Performance Certification of Interconnected Systems using ADMM

Chris Meissen  Laurent Lessard  Murat Arcak  Andrew Packard

Abstract

A compositional performance certification method is presented for interconnected systems using dissipativity properties of the subsystems along with the interconnection structure. A large-scale optimization problem is formulated to search for the most relevant dissipativity properties of each subsystem. The alternating direction method of multipliers (ADMM) is employed to decompose and solve this problem. The effectiveness of the algorithm is demonstrated on several practical examples.

1 Introduction

In this paper, compositional analysis is applied to performance certification of an interconnection of subsystems as depicted in Figure 1. The $G_i$ blocks are known subsystems mapping $u_i \mapsto y_i$ and $M$ is a static matrix that characterizes the interconnection topology. The overarching goal of compositional analysis is to establish properties of the interconnected system using only properties of the subsystems and their interconnection. Henceforth, the term “local” is used to refer to properties or analysis of individual subsystems in isolation. Likewise, “global” refers to the entire interconnected system.

![Figure 1: Interconnected system with exogenous input $d$ and performance output $e$.](image)

In this paper, local behavior and global performance is cast and quantified in the framework of dissipative systems [25]; specifically the case with quadratic supply rates [26]. The global supply rate is specified by the user and dictates the system performance that is to be verified. For example, different supply rates can be used to verify $L_2$-gain properties, passivity, output-strict passivity, etc., with respect to the exogenous input $d$ and performance output $e$.

A conventional approach to compositional analysis, as presented for example in [1, 4, 20, 25], is to establish individual supply rates (and storage functions) for which each subsystem is dissipative. Then, a storage function certifying dissipativity of the interconnected system is sought as a linear combination of the subsystem storage functions.

The method presented here is less conservative because the local supply rates (and storage functions) are treated as decision variables, and are optimized with regards to
their particular suitability in certifying desired properties of the interconnected system. Thus, global properties are certified via local certificates that have been automatically generated, as opposed to having been preselected.

The idea of optimizing over the local supply rates (and storage functions) to certify stability of an interconnected system was introduced in [22], with the individual supply rates constrained to be diagonally-scaled induced $L_2$-norms. This perspective, coupled with dual decomposition, gave rise to a distributed optimization algorithm using subgradient techniques to certify stability of the interconnected system. We generalize this approach in several ways: certifying dissipativity (rather than stability) of the interconnected system with respect to a quadratic supply rate; searching over arbitrary quadratic supply rates for the local subsystems; and employing ADMM [7] to decompose and solve the resulting problem. The ADMM algorithm exposes the distributed certification as a convergent negotiation between parallelizable, local problems for each subsystem, and a global problem. The roles of the local and global problems are as follows:

- Each local problem receives a proposed supply rate from the global problem and solves an optimization problem to find a supply rate that certifies dissipativity of the corresponding subsystem and is close to the proposed supply rate.
- The global problem, with knowledge of the interconnection $M$ and the updated supply rates, solves an optimization problem to certify dissipativity of the interconnected system and proposes new supply rates to the local problems.

In [15] the method presented here was applied to linear systems and compared to other distributed optimization methods. In [14] this method was extended to nonlinear systems using sum-of-squares (SOS) optimization. Additionally, equilibrium-independent dissipativity was used to certify the local subsystem properties where the equilibrium point depended nontrivially on the input.

This paper unifies and expands on the conference papers [14, 15]. A new theorem is presented that establishes the proposed method is equivalent to searching for an additively separable storage function for interconnections of linear subsystems. Furthermore, it is shown that for large nonlinear interconnections the proposed method can be more computationally efficient and applied to much larger systems. A detailed description of the proposed method using integral quadratic constraints, instead of dissipativity, to quantify the individual subsystem properties is included. This allows frequency dependent properties of the subsystems to be used to certify the performance of the interconnected system. New examples, including a skew-symmetric interconnection structure common in communication networks and multiagent systems, are presented. The convergence properties and conditions of the ADMM algorithm are described in general and shown to hold for this application if a solution exists.

2 Preliminaries

**Dissipative dynamical systems** [25] Consider a time-invariant, continuous-time dynamical system described by

\[
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)), \quad f(0, 0) = 0 \\
y(t) &= h(x(t), u(t)), \quad h(0, 0) = 0
\end{aligned}
\]  

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. A supply rate is a function $w : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$. A system of the form (1) is dissipative with respect to a supply rate $w$ if there exists a differentiable and nonnegative function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

\[
\nabla V(x)^T f(x, u) - w(u, y) \leq 0
\]
for all \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \), and \( y = h(x, u) \). Equation (2) is referred to as the Dissipation Inequality Equation (DIE) and \( V \) as a storage function.

**Equilibrium independent dissipativity (EID)** [11] EID can be used to analyze systems where the equilibrium point depends nontrivially on the system input. Consider a system of the form
\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t), u(t))
\end{align*}
\]
(3)
such that for each equilibrium point \( x^* \in \mathbb{R}^n \) there exists a unique \( u^* \in \mathbb{R}^m \) such that \( f(x^*, u^*) = 0 \). The equilibrium state-input map is then defined as
\[
k_u(x) : \mathbb{R}^n \to \mathbb{R}^m \text{ such that } u^* = k_u(x^*)
\]
and the equilibrium state-output map is defined as
\[
k_y(x) : \mathbb{R}^n \to \mathbb{R}^p \text{ such that } y^* = k_y(x^*) = h(x^*, k_u(x^*))
\]
A system of the form (3) is EID with respect to the supply rate, \( w \), if there exists a nonnegative storage function, \( V : \mathbb{R}^{2n} \to \mathbb{R}_+ \), such that \( V(x^*, x^*) = 0 \) and
\[
\nabla_x V(x, x^*)^T f(x, u) - w(u - u^*, y - y^*) \leq 0
\]
(4)
for all equilibrium points \( x^* \), for all \( x \in \mathbb{R}^n \) and for all \( u \in \mathbb{R}^m \) where \( y = h(x, u) \), \( y^* = k_y(x^*) \), and \( u^* = k_u(x^*) \).

Note that the definition given here is slightly different than in [11] and follows the convention in [8, 9]. We assume the existence of an equilibrium state-input map whereas in [11] it is assumed that an equilibrium input-state map exists. The definition presented here allows systems such as an integrator to be EID.

**Integral Quadratic Constraints (IQC)**s [13] IQCs are a generalization of the dissipativity framework that allows frequency dependent properties of a system to be certified. Let \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\) be the realization of a stable LTI system \( \Psi \) with state \( \eta \). Then a system as described by (1) satisfies the IQC defined by \( \Pi = \Psi^* X \Psi \) if there exists a positive semidefinite storage function \( V(x, \eta) \) such that
\[
\nabla_x V(x, \eta)^T f(x, u) + \nabla_\eta V(x, \eta)^T \left( \bar{A} \eta + \bar{B} \begin{bmatrix} u \\ y \end{bmatrix} \right) \leq \left( \bar{C} \eta + \bar{D} \begin{bmatrix} u \\ y \end{bmatrix} \right)^T \bar{X} \left( \bar{C} \eta + \bar{D} \begin{bmatrix} u \\ y \end{bmatrix} \right)
\]
(5)
for all \( x, u \), and \( y = h(x, u) \). Traditional dissipativity is recovered in the special case where \( \Psi \) is static.

**SOS programming** [19] Let \( \mathbb{R}[x] \) be the set of all polynomials in \( x \) with real coefficients. \( \Sigma[x] := \{ \pi \in \mathbb{R}[x] : \pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_n^2, \ \pi_1, \ldots, \pi_n \in \mathbb{R}[x] \} \) is the subset of \( \mathbb{R}[x] \) containing the SOS polynomials. A polynomial, \( p(x) \), being a sum of squares polynomial is equivalent to the existence of a positive semidefinite matrix \( Q \) such that
\[
p(x) = m(x)^T Q m(x)
\]
for a properly chosen vector of monomials, \( m(x) \). Therefore checking the nonnegativity of a polynomial can be relaxed to a SOS program and then solved as a semidefinite program (SDP).
Polynomial dynamics Suppose that $f$ and $h$ in (1) are polynomials. Then certification of dissipativity of the system with respect to a given polynomial supply rate, $w$, can be relaxed to a SOS feasibility program:

$$V(x) \in \Sigma[x]$$

$$-\nabla V(x)^T f(x, u) + w(u, y) \in \Sigma[x, u].$$

(6)

Similarly, as presented in [11], certification of EID for polynomial systems can be relaxed to a SOS feasibility program:

$$V(x, x^*) \in \Sigma[x, x^*]$$

$$r(x, x^*, u, u^*) \in \mathbb{R}[x, u, x^*, u^*]$$

$$-\nabla_x V(x, x^*)^T f(x, u) + w(u - u^*, y - y^*) + r(x, x^*, u, u^*) f(x^*, u^*) \in \Sigma[x, u, x^*, u^*].$$

(7)

Thus, for polynomial dynamics, a storage function that satisfies (2) or (4) can be found by solving (6) or (7), respectively.

Rational polynomial dynamics If each state of the system dynamics, $f_i$, are described by rational polynomials of the form $f_i(x, u) = \frac{p_i(x, u)}{q_i(x, u)}$ for $i = 1, \ldots, n$ where $p_i \in \mathbb{R}[x, u]$ and $q_i - \epsilon \in \Sigma[x, u]$ for some positive $\epsilon$, then certifying dissipativity of the system with respect to a polynomial supply rate, $w$, can be relaxed to a SOS feasibility program:

$$-\sum_{i=1}^{n} \nabla_x V(x)p_i(x, u) \prod_{j \neq i} q_j(x, u) + \prod_{i=1}^{n} q_i(x, u) w(u, y) \in \Sigma[x, u]$$

(8)

Thus, for rational polynomial dynamics, a storage function satisfying (8) can be found by solving (2). Similarly to the polynomial case, certification of EID for rational polynomial systems can also be formulated as an SOS feasibility program.

3 Problem statement

Consider the interconnected system in Figure 1 where the subsystems $G_1, \ldots, G_N$ are each known and have dynamics of the form (1). Each subsystem may be different, so $G_i$ is characterized by $(f_i, h_i)$ with $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $y_i(t) \in \mathbb{R}^{p_i}$. Define $n := n_1 + \cdots + n_M$ and similarly for $m$ and $p$. The static interconnection $M \in \mathbb{R}^{m \times p}$ is known, and satisfies

$$[u \ \epsilon] = M \begin{bmatrix} v \\ d \end{bmatrix}.$$

(9)

We assume that the interconnected system is well-posed. Thus, for any $d \in L_{2e}$ and any initial condition $x_0 \in \mathbb{R}^n$ there exists unique $\epsilon, u, y \in L_{2e}$ that causally depend on $d$.

The global and local supply rates are assumed to be quadratic forms. In particular, the global supply rate, which is part of the problem specification, is

$$\begin{bmatrix} d \\ e \end{bmatrix}^T W \begin{bmatrix} d \\ e \end{bmatrix}.$$

(10)
and the local supply rates, which are decision variables in our optimization, are

\[
\begin{bmatrix}
    [u_i] \\
    [y_i]
\end{bmatrix}^T X_i \begin{bmatrix}
    [u_i] \\
    [y_i]
\end{bmatrix}
\quad \text{for } i = 1, \ldots, N
\tag{11}
\]

where \(W\) and \(X_i\) are real symmetric matrices. Dissipativity certification of the interconnected system with respect to (10) can be posed as an optimization problem of the following form:

\[
\begin{align*}
\text{minimize} & \quad X_1: N \\
\text{subject to} & \quad X_i \in L_i \quad \text{for } i = 1, \ldots, N \\
& \quad (X_1, \ldots, X_N) \in G.
\end{align*}
\tag{12}
\]

Each \(L_i\) constraint set is local because it depends only on the local supply rate \(X_i\). The \(G\) constraint is global because it depends on all the supply rates \(\{X_1, \ldots, X_N\}\). If a supply rate for a subsystem is known, it can be fixed in (12) by the user. This could happen, for example, when some of the subsystems are static nonlinearities or integrators composed with static nonlinearities. This is demonstrated in the vehicle platoon example in Section 5.

Before defining the \(L_i\) and \(G\) sets, we first introduce the following conformal block partitions

\[
W = \begin{bmatrix}
    W_{11} & W_{12} \\
    W_{21} & W_{22}
\end{bmatrix}, \quad
X_i = \begin{bmatrix}
    X_{i11} & X_{i12} \\
    X_{i21} & X_{i22}
\end{bmatrix}
\]

and the following block-diagonal matrices

\[
X^{jk} = \begin{bmatrix}
    X_i^{jk} & \cdots \\
    \cdots & \cdots \\
    X_N^{jk}
\end{bmatrix}
\quad \text{for all } j, k \in \{1, 2\}.
\tag{13}
\]

Recall that \(W\) is specified, while \(X_1, \ldots, X_N\) are to be determined. The local and global constraint sets are defined as follows:

\[
L_i := \left\{X_i \mid \text{the } i\text{-th subsystem is dissipative w.r.t.} \quad \begin{bmatrix}
    [u_i] \\
    [y_i]
\end{bmatrix}^T X_i \begin{bmatrix}
    [u_i] \\
    [y_i]
\end{bmatrix} \right\}
\tag{14}
\]

\[
G := \left\{X_{1:N} \mid \sum_{i=1}^N H_i X_i H_i^T - H_0 W H_0^T \preceq 0 \right\}.
\tag{15}
\]

In (15) the constant matrices \(H_0, \ldots, H_N\) are defined such that the following identity holds.

\[
\sum_{i=1}^N H_i X_i H_i^T - H_0 W H_0^T = \begin{bmatrix}
    M & X_{11} & 0 & X_{12} & 0 \\
    0 & -W_{22} & 0 & -W_{21} & 0 \\
    X_{21} & 0 & X_{22} & 0 & 0 \\
    0 & -W_{12} & 0 & -W_{11}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    M \\
    I
\end{bmatrix}
\tag{16}
\]

The following proposition shows that if (12) is feasible, then the interconnected system is dissipative with respect to the specified global supply rate (10). Furthermore, the storage function certifying global dissipativity is of the form

\[
V(x_1, \ldots, x_N) = \sum_{i=1}^N V_i(x_i)
\tag{17}
\]

where each \(V_i(x_i)\) is a storage function certifying local dissipativity according to (14). Special cases of this result have appeared previously, for example in [18, 23].
Proposition 1. Consider the interconnected system described in Section 2, with a global 
supply rate of the form (10). Suppose that $X_1, \ldots, X_N$ are a solution to (12). Then the 
interconnected system is dissipative with respect to the global supply rate.

Proof. Multiplying the right-hand side of (16) on the left by $[y^T \ d^T]$ and on the right 
by $[y^T \ d^T]^T$ and making use of (9) and (15), we obtain

$$
\begin{bmatrix}
u \\
e \\
y \\
d
\end{bmatrix}^T 
\begin{bmatrix}
X_{11} & 0 & X_{12} & 0 \\
0 & -W_{22} & 0 & -W_{21} \\
X_{21} & 0 & X_{22} & 0 \\
0 & -W_{12} & 0 & -W_{11}
\end{bmatrix} 
\begin{bmatrix}
u \\
e \\
y \\
d
\end{bmatrix} \leq 0 \tag{18}
$$

Using this and the block diagonal structure of $X^{jk}$, we can rewrite (18) as

$$
\sum_{i=1}^{N} \begin{bmatrix}
u_i \\
y_i
\end{bmatrix}^T X_i \begin{bmatrix}
u_i \\
y_i
\end{bmatrix} - \begin{bmatrix}
d \\
e
\end{bmatrix}^T W \begin{bmatrix}
d \\
e
\end{bmatrix} \leq 0 \tag{19}
$$

For each subsystem, since $X_i \in \mathcal{L}_i$ there exists a storage function $V_i$ that satisfies

$$
\nabla V_i(x_i)^T f_i(x_i, u_i) - \begin{bmatrix}
d \\
e
\end{bmatrix}^T W \begin{bmatrix}
d \\
e
\end{bmatrix} \leq 0 \tag{20}
$$

for all $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i = h_i(x_i, u_i)$. Adding to (19) the local dissipativity inequalities (20) for each subsystem, we obtain

$$
\sum_{i=1}^{N} \nabla V_i(x_i)^T f_i(x_i, u_i) - \begin{bmatrix}
d \\
e
\end{bmatrix}^T W \begin{bmatrix}
d \\
e
\end{bmatrix} \leq 0 \tag{21}
$$

which certifies the system is dissipative with respect to the global supply rate matrix $W$. Moreover, the global storage function is the sum of the local storage functions $V_i$, as in (17).

The benefit of the formulation in (12) is that verifying feasibility of a candidate point 
may be carried out in an efficient manner.

1. Verifying a local constraint (14) only depends on the supply rate for the associated 
subsystem. Therefore, all the local storage functions can be searched for separately 
and in parallel.

2. Verifying the global constraint (15) requires solving a global LMI. While this con- 
straint cannot be decoupled, it is a constraint merely on the supply rates, and does 
not require searching for storage functions.

Note that the formulation of (12) can only be used to certify dissipativity properties 
consistent with storage functions that are additively separable in the subsystem states, 
as in (17).

The following theorem states that for a interconnection of linear subsystems the exis- 
tence of a separable quadratic storage function is equivalent to the existence of supply 
rates satisfying (12). Thus, the conservatism of this approach is equivalent to that of 
searching for a separable storage function for the interconnected system. First, we make 
a mild assumption on the interconnection:

Assumption 2. We denote $M^i$ as the rows of $M$ that map $[y^T \ d^T]$ to $u_i$. We assume that 
the block diagonal elements of $M$ are zero and for each $i$ the rows of $M^i$ are linearly 
independent.
The first part of Assumption 2 states that the subsystems don’t have any self-loops. If they did, the feedback could be removed from the interconnection and included in the individual subsystem model. The second part states that each element of \( u_i \) is nonzero and unique for some \( [u_i] \).

**Theorem 3.** Consider \( N \) linear subsystems

\[
\begin{align}
\dot{x}_i &= A_ix_i + B_iu_i & (22a) \\
y_i &= C_ix_i & (22b)
\end{align}
\]

interconnected according to (9) by a matrix \( M \) that satisfies Assumption 2. Suppose a global supply rate of the form (10) is given. The following statements are equivalent.

(i) The optimization (12) is feasible using local storage functions of the form \( V_i(x_i) = x_i^TP_ix_i \) with \( P_i \succeq 0 \).

(ii) There exists a separable quadratic storage function of the form (17) certifying dissipativity of the interconnected system with respect to the global supply rate. In other words, there exists \( P_i \succeq 0 \) for \( i = 1, \ldots, N \) such that

\[
\sum_{i=1}^{N} \left[ \begin{array}{c} x_i^T \\ u_i \\ y_i \\
\end{array} \right] \left[ \begin{array}{ccc} A_i^TP_i + P_iA_i & P_iB_i & 0 \\ B_i^TP_i & 0 & 0 \\
\end{array} \right] \left[ \begin{array}{c} x_i \\ u_i \\ y_i \\
\end{array} \right] \preceq \left[ \begin{array}{c} d^T \\ e \\
\end{array} \right] W \left[ \begin{array}{c} d \\ e \\
\end{array} \right] \quad \text{for all } x \text{ and } d. \quad (23)
\]

Note that in (23), \( u \) and \( e \) are expressed in terms of \( x \) and \( d \) using (9) and (22b).

If the above statements hold, then they hold for the same \( P_i \) matrices.

**Proof.** (i) \( \Rightarrow \) (ii) follows by specializing Proposition 1 to linear subsystems and quadratic storage functions. In particular, the dissipation inequality equation (21) is then equivalent to (23).

We now prove the converse (ii) \( \Rightarrow \) (i). Suppose (23) holds. If we define \( V_i(x_i) = x_i^TP_ix_i \), the \( i \)th summand on the left-hand side of (23) is simply \( \dot{V}(x_i) \). We will now prove that the \( V_i \) may be used as storage functions to certify local dissipativity of the subsystems. We may assume without loss of generality that for each subsystem, \( C_i = [\mathbf{0}_{m_i \times (n_i-m_i)} \quad \mathbf{I}_{m_i \times m_i}] \). This allows each \( x_i \) to be partitioned as \( x_i = [z_i^T \ y_i]^T \), where \( z_i \in \mathbb{R}^{n_i-m_i} \) and \( y_i \in \mathbb{R}^{m_i} \). By eliminating the \( x_i \) from (23) and rearranging the terms, we obtain

\[
\sum_{i=1}^{N} \left( z_i^TQ_iz_i + 2z_i^TR_iu_i + u_i^TS_iu_i \right) \leq d^TWd \quad \text{for all } z, y, d. \quad (24)
\]

for appropriately chosen \( Q_i, R_i, \) and \( S_i \). Since (24) holds for all \( z, y, d \), it holds in particular when \( y = d = 0 \). We then have from (9) that \( u = e = 0 \), and we conclude that \( Q_i \succeq 0 \). Using a similarity transform, we may assume, again without loss of generality, that \( z_i \) is decomposed as:

\[
z_i = [\xi_i^T \ \hat{z}_i], \quad \text{where } z_i^TQ_iz_i = \hat{z}_i^T\hat{Q}_i\hat{z}_i \text{ and } \hat{Q}_i \prec 0
\]

and the dimensions of \( \hat{z}_i \) and \( \hat{Q}_i \) correspond to the number of nonzero eigenvalues of \( Q_i \).

Rewriting

\[
z_i^TR_iu_i = \xi_i^TV_iy_i + \xi_i^TU_iu_i + \hat{z}_i^T\hat{R}_i \begin{bmatrix} u_i \\ y_i \end{bmatrix}
\]
where \( Y_i, U_i, \) and \( \hat{R}_i \) are appropriately defined matrices, the summands in (24) take the form

\[
\dot{V}_i(z_i, \xi_i, y_i) = \ddot{z}_i^T Q_i \ddot{z}_i + 2\xi^T Y_i y_i + 2\xi^T U_i u_i + 2z_i^T \hat{R}_i \begin{bmatrix} u_i \\ y_i \end{bmatrix} + \begin{bmatrix} u_i \\ y_i \end{bmatrix}^T S_i \begin{bmatrix} u_i \\ y_i \end{bmatrix}. \tag{25}
\]

Because (24) holds for all \( z, y, d \) it must also hold if we maximize over \( \ddot{z}_i \). Performing the maximization,

\[
\dot{V}_i(\ddot{z}^*_i, \xi_i, y_i) = \begin{bmatrix} u_i \\ y_i \end{bmatrix}^T (S_i - \hat{R}_i^T \tilde{Q}_i^{-1} \hat{R}_i) \begin{bmatrix} u_i \\ y_i \end{bmatrix} + 2\xi^T Y_i y_i + 2\xi^T U_i u_i \tag{26}
\]

where \( \ddot{z}^*_i := \text{arg max}_{\ddot{z}_i} \dot{V}_i(\ddot{z}_i, \xi_i, y_i) \). If we further define \( X_i := S_i - \hat{R}_i^T \tilde{Q}_i^{-1} \hat{R}_i \), we can write

\[
\sum_{i=1}^{N} \dot{V}_i(\ddot{z}^*_i, \xi_i, y_i) = \begin{bmatrix} u \\ y \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + 2\xi^T \hat{Y} y + 2\xi^T \hat{U} u \tag{27}
\]

where \( X^{jk} \) is defined in (13), and we have defined \( \hat{Y} := \text{diag}(Y_1, \ldots, Y_N) \) and similarly for \( \hat{U} \). In summary, we have deduced that

\[
\sum_{i=1}^{N} \dot{V}_i(\ddot{z}_i, \xi_i, y_i) \leq \sum_{i=1}^{N} \dot{V}_i(\ddot{z}^*_i, \xi_i, y_i) \leq \begin{bmatrix} d \\ e \end{bmatrix}^T W \begin{bmatrix} d \\ e \end{bmatrix}. \tag{28}
\]

for all \( \ddot{z}, \xi, y, d \) where \( e, u \) satisfy (9). Note that the the right-hand side of (28) does not depend on \( \xi \), yet its lower bound (27) is linear in \( \xi \). The only way this inequality can be true for all \( \xi \) is if

\[
2\xi^T \hat{Y} y + 2\xi^T \hat{U} u = 0 \quad \text{for all } \xi, y, d.
\]

From (9), we have \( u = M_{11} y + M_{12} d \) and so

\[
\hat{Y} + \hat{U} M_{11} = 0 \tag{29a}
\]

\[
\hat{U} M_{12} = 0. \tag{29b}
\]

By denoting \( M_{11j}^{ij} \) as the submatrix of \( M \) mapping \( y_j \to u_i \) and \( M_{12}^{ij} \) as the submatrix of \( M \) mapping \( d \to u_i \), then for each \( i \), (29a) simplifies to

\[
Y_i + U_i M_{11}^{ij} = 0
\]

\[
U_i M_{12}^{ij} = 0 \quad \text{for } j \neq i
\]

and (29b) simplifies to

\[
U_i M_{12}^{ij} = 0 \quad \text{for } j = 1, \ldots, N
\]

Assumption 2 implies that \( U_i = Y_i = 0 \). Therefore, (25)–(26) simplifies to

\[
\dot{V}_i(x_i) \leq \dot{V}_i(\ddot{z}^*_i, \xi_i, y_i) = \begin{bmatrix} u_i \\ y_i \end{bmatrix}^T X_i \begin{bmatrix} u_i \\ y_i \end{bmatrix} \tag{30}
\]

and hence, the storage function \( V_i \) certifies dissipativity of the \( i \)th local subsystem with respect to the supply rate matrix \( X_i \). Combining (28) and (30), we conclude that

\[
\sum_{i=1}^{N} \begin{bmatrix} u_i \\ y_i \end{bmatrix}^T X_i \begin{bmatrix} u_i \\ y_i \end{bmatrix} \leq \begin{bmatrix} d \\ e \end{bmatrix}^T W \begin{bmatrix} d \\ e \end{bmatrix}. \tag{31}
\]

It follows from (30)–(31) that (12) is feasible, as required. \( \blacksquare \)
**Extension to EID Systems**  We make the additional assumption that there exists an equilibrium point $x^*$ for which there is a unique $u^* = k_u(x^*), y^* = k_y(x^*),$ and $e^* = M_{21} y^*.$

The global and local supply rates must be modified to depend on $u^*, y^*,$ and $e^*$. Specifically, the global supply rate becomes

$$
\begin{bmatrix}
  d \\
  e - e^*
\end{bmatrix}^T W
\begin{bmatrix}
  d \\
  e - e^*
\end{bmatrix}
$$

and the local supply rates are

$$
\begin{bmatrix}
  u_i - u^*_i \\
  y_i - y^*_i
\end{bmatrix}^T X_i
\begin{bmatrix}
  u_i - u^*_i \\
  y_i - y^*_i
\end{bmatrix}
$$

where $W$ and all $X_i$ are real symmetric matrices.

For each subsystem we must determine a supply rate $X_i$ such that the subsystem is EID. Therefore, the local constraint sets $\mathcal{L}_i$ become

$$
\mathcal{L}_i := \{ X_i \mid \text{the $i$-th subsystem is EID w.r.t. } \begin{bmatrix}
  u_i - u^*_i \\
  y_i - y^*_i
\end{bmatrix}^T X_i
\begin{bmatrix}
  u_i - u^*_i \\
  y_i - y^*_i
\end{bmatrix} \} \quad (34)
$$

for all $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m,$ and $y_i = h_i(x_i, u_i).$ The global constraint set $\mathcal{G}$ is unchanged.

Proposition 1 is directly applicable to this formulation, thus a feasible solution to (12) with the local constraint sets defined as in (34) implies that the interconnected system is dissipative with respect to the global supply rate.

**Extension to IQCs**  We extend this approach to IQCs by redefining the local constraint sets in (14) as

$$
\mathcal{L}_i := \{ X_i \mid \text{the $i$th subsystem satisfies the IQC defined by } \Pi_i = \Psi_i^* X_i \Psi_i \} \quad (35)
$$

where $\Psi_i$ is a stable linear system specified by the user and $X_i$ is the decision variable. We modify the inequality given in (16) that describes the global constraint set (15) to

$$
\begin{bmatrix}
  M \\
  I
\end{bmatrix}^T
\begin{bmatrix}
  \Pi_{11} & 0 & \Pi_{12} & 0 \\
  0 & -W_{22} & 0 & -W_{21} \\
  \Pi_{21} & 0 & \Pi_{22} & 0 \\
  0 & -W_{12} & 0 & -W_{11}
\end{bmatrix}
\begin{bmatrix}
  M \\
  I
\end{bmatrix} \preceq -\varepsilon I \quad (36)
$$

where $W_{jk}$ is the global IQC specified by the user and $\varepsilon \in \mathbb{R}$ is positive. This constraint is frequency dependent and must hold for all $\omega \in \mathbb{R}.$

Proposition 1 can easily be extended to this case. Specifically, if there exists $X_i$ satisfying the local (35) and global (36) constraint sets then the interconnected system satisfies the global IQC $W$.

4 **ADMM**

The ADMM algorithm [7] allows us to decompose (12) into individual local subproblems, which require searching for a potentially high order storage function, and a global problem that only involves the supply rates and interconnection.
ADMM is used to solve problems of the form

\[
\begin{align*}
\text{minimize} & \quad f(x) + g(z) \\
\text{subject to} & \quad Ax + Bz = c 
\end{align*}
\]

where \(x\) and \(z\) are vector decision variables. The scaled ADMM algorithm is then given by

\[
\begin{align*}
x^{k+1} &= \arg \min_x f(x) + \frac{\rho}{2} \|Ax + Bz^k - c + v^k\|_2^2 \\
z^{k+1} &= \arg \min_z g(z) + \frac{\rho}{2} \|Ax^{k+1} + Bz - c + v^{k}\|_2^2 \\
v^{k+1} &= Ax^{k+1} + Bz^{k+1} - c + v^k 
\end{align*}
\]

where \(v\) is a scaled dual variable.

**Application of ADMM** Our problem (12) may be put into this form by defining the following indicator functions:

\[
\begin{align*}
\mathbb{I}_{\mathcal{L}_i}(X_i) &= \begin{cases} 
0 & X_i \in \mathcal{L}_i \\
\infty & \text{otherwise}
\end{cases} \\
\mathbb{I}_G(X_{1:N}) &= \begin{cases} 
0 & (X_1, \ldots, X_N) \in \mathcal{G} \\
\infty & \text{otherwise}
\end{cases}
\end{align*}
\]

Introducing the auxiliary variable \(Z_i\) for each subsystem allows us to write (12) as

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^N \mathbb{I}_{\mathcal{L}_i}(X_i) + \mathbb{I}_G(Z_1, \ldots, Z_N) \\
\text{subject to} & \quad X_i - Z_i = 0 \quad \text{for } i = 1, \ldots, N
\end{align*}
\]

so that it is in the canonical form (37). The first term in the objective and the constraints are separable for each subsystem, so the ADMM algorithm takes on the following parallelized form.

1. **X-updates:** for each \(i\), solve the local problem:

\[
X_i^{k+1} = \arg \min_{X \in \mathcal{L}_i} \|X - Z_i^k + V_i^k\|_F^2
\]

2. **Z-update:** if \(X_{1:N}^{k+1} \in \mathcal{G}\), then we have found a solution to (12), so terminate the algorithm. Otherwise, solve the global problem:

\[
Z_{1:N}^{k+1} = \arg \min_{Z_{1:N} \in \mathcal{G}} \sum_{i=1}^N \|X_i^{k+1} - Z_i + V_i^k\|_F^2
\]

3. **V-update:** Update \(U\) and return to step 1.

\[
V_i^{k+1} = X_i^{k+1} - Z_i^{k+1} + V_i^k
\]

Figure 2 depicts the parallelizable nature of the ADMM algorithm. The local problems are solved independently to determine the supply rate \(X_i \in \mathcal{L}_i\) that is closest, in the Euclidian norm sense, to \(V_i - Z_i\). These supply rates are then passed to the global problem to determine the supply rates \(\{Z_1, \ldots, Z_N\} \in \mathcal{G}\) that are closest to \(X_i + V_i\). The \(V\)-update then plays a role similar to integral control to drive \(X_i - Z_i\) to 0.
ADMM Convergence  If the extended real-valued functions $f$ and $g$ in (37) are closed, proper, and convex and the Lagrangian has a saddle point then as $k \to \infty$

- the objective $f(x^k) + g(z^k)$ converges to the optimal value,
- the dual variable $v^k$ converges to the dual optimal point $v^*$, and
- the residual $Ax + Bz - c$ converges to zero [7].

Furthermore, if $A$ and $B$ are full column rank then the decision variables $x^k$ and $z^k$ are guaranteed to converge to $x^*$ and $z^*$ as $k \to \infty$ [17].

An extended real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is closed, convex, and proper if and only if the epigraph of $f$ is closed, convex, and nonempty. The epigraph of $f$ is defined as

$$\text{epi}(f) = \{(x, t) \in \mathbb{R}^{n+1} | f(x) \leq t\}.$$ 

Since the local and global constraint sets in (12) are convex and assumed to be nonempty the epigraph of the indicator functions $I_{L_i}$ and $I_G$ in (38) satisfy these requirements. If the intersection of the local and global constraint sets is non-empty then a feasible point $(X^*, Z^*)$ exists. Additionally, by Slater’s condition strong duality holds since the constraints are affine equality constraints [6]. Then, by the saddle point theorem there exists a dual optimal point $V^*$ such that the Lagrangian has a saddle point [16]. Therefore, for this application ADMM is guaranteed to find a feasible point in the limit as $k \to \infty$.

In practice a feasible point is typically found in a finite number of iterations, but if the interior of the feasible set is empty then the algorithm may asymptotically approach a feasible point and only reach it in the limit.

5 Examples

The first example certifies the stability of a skew-symmetric interconnection structure that commonly arises in communication networks and multiagent systems. The second example shows the potential of this approach for large polynomial and rational polynomial systems. The third example is based on a nonlinear model of vehicle platoons and the fourth example demonstrates the benefits of IQCs.

Skew-symmetric interconnection structure  For this example 50 LTI subsystems of the following form were generated.

$$H_i : \left\{\begin{array}{l}
\dot{x}_i = \begin{bmatrix} -\epsilon_i & 1 \\ -1 & -\epsilon_i \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i \\
y_i = \begin{bmatrix} 0 & 1 \end{bmatrix} x_i
\end{array} \right.$$

Figure 2: The parallelizable nature of ADMM where the local supply rates $X_i$ are updated individually based on the subsystem properties described by the local constraint sets $L_i$ and the global supply rates are updated simultaneously based on the global constraint set $G$. 
The decay rate $\epsilon_i$ was chosen from a uniform distribution over $[0, 0.1]$. Each subsystem is passive, but has large $L_2$ gain due to the small decay rates $\epsilon_i$. The interconnection matrix $G$ is skew-symmetric with the following form:

$$G = \begin{bmatrix} 0 & G_0 \\ -G_0^T & 0 \end{bmatrix}$$

where each element of $G_0$ was chosen randomly from a standard normal distribution. This interconnection structure arises commonly in communication networks and multi-agent systems [2, 3, 5, 24]. The skew-symmetry of $G$ along with the passivity of the subsystems guarantees stability of the network without any restriction on the $L_2$ gain of the subsystems or the size of the interconnection. In contrast, the large interconnection size and the large subsystem gains in this example suggest that stability cannot be ascertained by the small-gain theorem.

The ADMM algorithm found passivity of the subsystems (i.e. $X_i = [0 \ 1 \ 1 \ 0]$) as a feasible solution for 100 random instances of the skew-symmetric interconnected system. Figure 3 shows the solution was found in less than 65 iterations for all cases, and 90% of systems tested required fewer than 47 iterations.

![Figure 3: Cumulative plot showing the fraction of 100 total trials that required at most a given number of iterations to find a feasible point using ADMM.](image)

**Large-scale rational polynomial system** A system consisting of $N$, 2-state rational polynomial subsystems was generated. The subsystems, modified from [12], are described by

$$H : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{-ax_2 - bx_1^3 + u}{1 + cx_2^2} \\ y = x_2 \end{cases}$$

where $a, b, c > 0$ are parameters of the subsystem. The positive-definite storage function $V(x) := \frac{ab}{2} x_1^4 + \frac{ac}{2} x_2^4 + ax_2^2$ and supply rate $w(u, y) := u^2 - a^2 y^2$ certify that the $L_2$-gain of $H$ is less than or equal to $a^{-1}$. 

12
Clearly, any well-posed interconnection involving many instances of these systems, along with an interconnection matrix whose spectral norm is less than \( \gamma \) results in a dynamical system whose \( L_2 \) gain is less than 1. This insight allows us to construct large-scale examples as described in the following steps:

1. Choose \( \{a_i, b_i, c_i\}_{i=1}^{N} \) uniformly distributed in \((1, 2) \times (0, 1) \times (0.5, 2)\). These constitute the parameters of system \( H_i \). Denote \( \gamma := \max_i a^{-1}_i \).

2. Choose each entry of \( S \in \mathbb{R}^{(N+d) \times (N+d)} \) from a standard normal distribution. Overlay any desired sparsity pattern by selectively zeroing out particular entries.

3. Compute \( \beta := \inf_B \sigma(BMB^{-1}) \) where \( B = \text{diag}(b_1, \ldots, b_N, I_d), b \in \mathbb{R}^N_{++} \). Redefine \( S := \frac{0.99}{\gamma \beta} S \) to guarantee the spectral norm of the interconnection is less than \( \gamma \).

4. Choose random nonzero, diagonal scalings \( \Psi = \text{diag}(\Psi_1, \ldots, \Psi_N) \) and \( \Phi = \text{diag}(\Phi_1, \ldots, \Phi_N) \).

5. Define \( G_i := \Phi_i H_i \Psi_i \), and

\[
M := \begin{bmatrix} \Psi^{-1} & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & I_d \end{bmatrix} S \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & I_d \end{bmatrix}
\]

The scalings introduced in step 4 alter the gain properties of the subsystems and interconnection disguising the simple construction that guarantees the \( L_2 \)-gain of the interconnected system is less than 1. Figure 4 below illustrates the interconnection that the algorithm must attempt to certify.

![Figure 4: Scaling of the interconnected system of Figure 1 that leaves the closed-loop map unchanged.](image)

We generated 100 random instances of the interconnected system described above were generated, each with \( N = 100 \). The ADMM algorithm was applied to these instances, attempting to certify the \( L_2 \)-gain of the interconnected system is less than or equal to \( \gamma \). SOS programming was used to search for quartic storage functions that certified dissipativity for each local problem. Figure 5 shows the result of this test. The algorithm succeeded for all 100 tests, requiring at most 48 iterations and less than 15 iterations for 90% of the tests.

Using this example we also were able to test the performance of our method compared to directly searching for a separable storage function to certify performance of the interconnected system. Both polynomial and rational polynomial systems with different numbers of subsystems were tested. The systems where generated as described above except for the polynomial systems \( c \) in the denominator of (5) was set to 0. Since for each example the number of iterations required for the ADMM algorithm to find a solution may be different, 100 test for each system size were performed. Figure 6 shows the average time for the ADMM algorithm to find a solution and the time to directly search for a separable storage function.
Figure 5: Cumulative plot displaying the fraction of 200 total tests that required at most a given number of iterations to certify the $L_2$-gain property of the interconnected system.

Figure 6: Time comparison of our method using ADMM and directly searching for a separable storage function.

Only moderately sized systems were tested because directly searching for a separable storage function became computationally intractable for systems with more than 16 polynomial subsystems and more than 6 rational polynomial subsystems.

**Vehicle platooning** In this example we analyze the $L_2$-gain properties of a vehicle platoon model[8, 9]. A possible configuration is shown in Figure 7. Each vehicle measures its distance to a subset of vehicles (indicated by dashed lines), and adjusts its throttle according to a control law. We would like to certify that under a broad range of control laws, bounds on the $L_2$-gain properties can be certified. For the purpose of this...
Figure 7: Vehicle platoon example. Each vehicle measures the relative distance of all vehicles connected to it by a dotted line. The objective is that all vehicles eventually move at a common speed.

For a platoon with $N$ vehicles the dynamics of the $i^{th}$ vehicle can be described by

$$
\Sigma_i : \begin{cases} 
\dot{v}_i = -v_i + v_{i}^{\text{nom}} + u_i \\
y_i = v_i 
\end{cases} \quad i = 1, \ldots, N
$$

where $v_i(t)$ is the vehicle velocity and $v_{i}^{\text{nom}}$ is the nominal velocity. In the absence of a control input $u_i(t)$, each vehicle tends to its nominal velocity.

Each vehicle uses the relative distance between itself and a subset of the other vehicles to control its velocity. The subsets are represented by a connected, bidirectional, acyclic graph with $L$ links interconnecting the $N$ vehicles. In Figure 7, the links are shown as dotted lines. Letting $p_\ell$ be the relative displacement between the vehicles connected by link $\ell$ gives $\dot{p}_\ell = v_i - v_j$ where $x_i$ is the leading node and $x_j$ is the trailing node. We define $D \in \mathbb{R}^{N \times L}$ as

$$
D_{i\ell} = \begin{cases} 
1 & \text{if } i \text{ is the leading node of edge } \ell \\
-1 & \text{if } i \text{ is the trailing node of edge } \ell \\
0 & \text{otherwise.}
\end{cases}
$$

Thus, $D$ maps the individual velocities of the vehicles to the relative velocities across each link. That is, $\dot{\mathbf{p}} = D^T \mathbf{v}$.

Control strategies for velocity regulation of the platoon were presented in [8, 9]. We consider a more general set of control laws that encompass those in [8, 9]. Specifically, we will analyze control laws of the form

$$
u_i = - \sum_{\ell=1}^{L} D_{i\ell} \phi_\ell(p_\ell)
$$

where $\phi_\ell : \mathbb{R} \to \mathbb{R}$ can be any function that is increasing and surjective, ensuring the existence of an equilibrium point [8]. Defining $\Phi := \text{diag}(\phi_1, \ldots, \phi_L)$, we may represent the system as the block diagram in Figure 8.
The map $\Lambda$ from $\dot{p}$ to $\Phi(p)$, indicated by a dashed box in Figure 8, is diagonal; each $\dot{p}_\ell$ is separately integrated and then the corresponding $\phi_\ell$ is applied. Thus, we may write

$$\Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_L),$$

where $\Lambda_\ell$ is described by

$$\Lambda_\ell : \quad \dot{p}_\ell = \eta_\ell \quad z_\ell = \phi_\ell(p_\ell)$$

where $\eta_\ell$ is the input and $z_\ell$ is the output. The interconnection can be transformed into the standard form of Figure 1 by diagonally concatenating $\Sigma := \text{diag}(\Sigma_1, \ldots, \Sigma_N)$ with $\Lambda := \text{diag}(\Lambda_1, \ldots, \Lambda_L)$ as is shown in Figure 9.

Although an equilibrium $(v^*, p^*)$ is guaranteed to exist it depends on the unknown functions $\phi_\ell$. Therefore we will exploit the EID properties of the subsystems and establish the desired global property without explicit knowledge of the equilibrium.

For each $\Lambda_\ell$ subsystem the dissipativity properties depend on the unknown $\phi_\ell$ function. However, it is not difficult to show that $\Lambda_\ell$ is EID with respect to the following supply rate (also known as equilibrium independent passivity)

$$\begin{bmatrix} \eta_\ell - \eta^*_\ell \\ z_\ell - z^*_\ell \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \eta_\ell - \eta^*_\ell \\ z_\ell - z^*_\ell \end{bmatrix}$$

This property can be proven by using the storage function

$$V_\ell(p_\ell) = 2 \int_{p_\ell^*}^{p_\ell} \left[ \phi_\ell(\theta) - \phi_\ell(p_\ell^*) \right] d\theta$$

and the property $(p_\ell - p_\ell^*)[\phi_\ell(p_\ell) - \phi_\ell(p_\ell^*)] \geq 0$ which follows because $\phi_\ell$ is increasing. Therefore, instead of searching over supply rates for the $\Lambda_\ell$ subsystems in the ADMM algorithm, we fix (39) as the supply rate. The algorithm does not need to know the $\phi_\ell$ functions and their associated equilibrium points; it is enough to know the supply rate (39).

For the simulation, we used $N = 20$, and each vehicle’s nominal velocity was randomly chosen. A linear topology was used as in Figure 7. That is, each vehicle measures the distance to the vehicle in front of it and the vehicle behind it. We investigated how a force disturbance applied to the trailing vehicle would affect the velocity of the lead vehicle. Specifically, we augmented the interconnection matrix $M$ (see Figure 1) such that the disturbance $d$ is applied to the last vehicle:

$$\dot{v}_N = -v_N + v_N^{\text{nom}} + u_N + d$$

and the output $e$ is the velocity of the first vehicle $v_1$. We then attempted to certify the $L_2$ gain from $d$ to $e$ is no greater than $\gamma$ using a supply rate of the form

$$\begin{bmatrix} d \\ e - e^* \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^{-2} \end{bmatrix} \begin{bmatrix} d \\ e - e^* \end{bmatrix}.$$
A bisection search was used to find that $\gamma_{\min} = 0.71$ was the smallest value that could be certified. Since our method searches over a restricted class of possible storage functions it may be conservative. In an effort to bound this conservatism, we performed an ad-hoc search over linear $\phi_\ell$ functions, seeking a worst-case $L_2$ gain. The result was that $\gamma_{\min} \geq 0.49$.

In this problem the interior of the feasible set is empty. Therefore, the ADMM algorithm will only converge in the limit as $k \to \infty$. This is a potential pitfall of our method that is common to many iterative optimization algorithms. Certain provisions were made to achieve satisfactory convergence of the ADMM routine. The local subproblems have the constraints $X_i^{1i} \geq 0$ for $i = 1, \ldots, N$ (these are scalar variables), while the global problem has the constraint $D_1^T \text{diag}(X_1^{11}, \ldots, X_N^{11}) D_1 \leq 0$. The only solution that satisfies both of these constraints is $X_i^{11} = 0$ for all $i$. This further implies that $X_i^{12} = X_i^{21} = 1$ for all $i$. Therefore, even though each of the $\mathcal{G}$ and $\mathcal{L}_i$ sets defined in (12) have a nonempty interior, their intersection does not. The result is that the ADMM algorithm oscillates between $X_i^{11} > 0$ for the local problems and $Z_i^{11} < 0$ for the global problem, leading to slow convergence and only finding a feasible solution (i.e. $X_i^{11} = Z_i^{11} = 0$) in the limit.

We addressed this issue by setting $X_i^{11} = 0$ and $X_i^{12} = X_i^{21} = 1$, effectively removing those variables from the optimization. The feasible region of the resulting problem has a nonempty interior, and the ADMM algorithm converged in a few iterations.

**IQC Example** This example, while very simple, demonstrates the advantage of using IQCs instead of dissipativity to quantify the properties of the subsystems. Consider two subsystems $G_1$ and $G_2$ described by the following transfer functions.

\[ G_1(s) = \frac{1}{s+1} \quad G_2(s) = \frac{2}{5s+1} \]

These subsystems are interconnected in negative feedback by the matrix

\[ M = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \]

The $L_2$-gain for $G_1$ and $G_2$ is 1 and 2, respectively, while the $L_2$-gain of the interconnected system is less than 1, specifically 0.861.

It is not possible to certify that the $L_2$-gain of the interconnected system is finite using only dissipative properties of the individual subsystems. However, the $L_2$ gain of the interconnected system can be bounded using the IQC given by

\[ \Psi(s) = \begin{bmatrix} \psi_0(s) \\ \vdots \\ \psi_K(s) \end{bmatrix} \quad \text{where} \quad \psi_k(s) = \left( \frac{1}{s+2} \right)^k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

with $X_1$ and $X_2$ restricted to be block diagonal. With $K = 3$ the ADMM algorithm certified the $L_2$-gain of the interconnected system is less than or equal to 0.864. The global constraint (36), which must hold for all $\omega \in \mathbb{R}$, was checked on a logarithmically spaced frequency grid of 1000 points from $10^{-3}$ to $10^3$.

All simulations performed were implemented in MATLAB using the SOSOPT toolbox [21] to solve SOS programs and the CVX toolbox [10] to solve SDP problems.
References

[1] J. Anderson, A. Teixeira, H. Sandberg, and A. Papachristdoulou. Dynamical system decomposition using dissipation inequalities. In \textit{IEEE Conference on Decision and Control}, pages 211–216, 2011.

[2] M. Arcak. Passivity as a design tool for group coordination. \textit{IEEE Transactions on Automatic Control}, 52(8):1380–1390, 2007.

[3] M. Arcak. \textit{Passivity approach to network stability analysis and distributed control synthesis}, volume Control System Advanced Methods, chapter 76, pages 1–18. CRC Press, second edition, 2010.

[4] M. Arcak and E. D. Sontag. A passivity-based stability criterion for a class of biochemical reaction networks. \textit{Mathematical Biosciences and Eng.}, 5(1):1–19, 2008.

[5] H. Bai, M. Arcak, and J. Wen. \textit{Cooperative Control Design: A Systematic, Passivity-Based Approach}. Communications and Control Eng. Springer, New York, 2011.

[6] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. \textit{Nonlinear Programming: Theory and Algorithms}. Wiley-InterScience, 3rd edition, 2006.

[7] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. \textit{Foundations and Trends in Machine Learning}, 3(1):1–122, 2011.

[8] M. Bürger, D. Zelazo, and F. Allgöwer. Duality and network theory in passivity-based cooperative control. In \textit{arXiv preprint arXiv:1301.3676}, 2013.

[9] S. Coogan and M. Arcak. A dissipativity approach to safety verification for interconnected systems. In \textit{IEEE Transactions on Automatic Control, Submitted}, 2014.

[10] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.0 beta. \texttt{http://cvxr.com/cvx}, Sept. 2013.

[11] G. Hines, M. Arcak, and A. Packard. Equilibrium-independent passivity: a new definition and numerical certification. In \textit{Automatica}, pages 1949–1956, 2011.

[12] H. Khalil. \textit{Nonlinear Systems}. Prentice Hall, 3rd edition, 2002.

[13] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. \textit{IEEE Transactions on Automatic Control}, 42(6):819–830, 1997.

[14] C. Meissen, L. Lessard, M. Arcak, and A. Packard. Performance certification of interconnected nonlinear systems using admm. In \textit{IEEE Conference on Decision and Control, Accepted}, 2014.

[15] C. Meissen, L. Lessard, and A. Packard. Performance certification of interconnected systems using decomposition techniques. In \textit{American Control Conference}, pages 5030–5036, 2014.

[16] J. F. C. Mota, J. M. F. Xavier, P. M. Q. Aguiar, and M. Püschel. Distributed ADMM for model predictive control and congestion control. In \textit{IEEE Conference on Decision and Control}, pages 5110–5115, 2012.

[17] J. F. C. Mota, J. M. F. Xavier, P. M. Q. Aguiar, and M. Püschel. A proof of convergence for the alternating direction method of multipliers applied to polyhedral-constrained functions. In \textit{arXiv preprint: 1112.2295}, 2012.

[18] P. Moylan and D. Hill. Stability criteria for large-scale systems. \textit{IEEE Transactions on Automatic Control}, 23:143–149, 1978.

[19] P. Parrilo. \textit{Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization}. PhD thesis, California Institute of Tech., 2000.

[20] N. Sandell, Jr., P. Varaiya, M. Athans, and M. Safonov. Survey of decentralized control methods for large scale systems. \textit{IEEE Transactions on Automatic Control}, 23(2):108–128, 1978.
[21] P. Seiler. Sosopt: A toolbox for polynomial optimization. In *arXiv preprint arXiv:1308.1889*, 2013.

[22] U. Topcu, A. Packard, and R. M. Murray. Compositional stability analysis based on dual decomposition. In *IEEE Conference on Decision and Control*, pages 1175–1180, 2009.

[23] M. Vidyasagar. *Input-output analysis of large-scale interconnected systems: decomposition, well-posedness, and stability*. Springer-Verlag, 1981.

[24] J. Wen and M. Arcak. A unifying passivity framework for network flow control. *IEEE Transactions on Automatic Control*, 49(2):162–174, 2004.

[25] J. C. Willems. Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis*, 45(5):321–351, 1972.

[26] J. C. Willems. Dissipative dynamical systems part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):352–393, 1972.