Genuine Lie semigroups
and semi-symmetries of PDEs

Elemér E. Rosinger

Department of Mathematics
and Applied Mathematics
University of Pretoria
Pretoria
0002 South Africa
eerosinger@hotmail.com

Abstract

Any Lie group $G$ acting on a Euclidean nonvoid open subset $M$ can be seen as a subgroup of the smooth diffeomorphisms $\text{Diff}^\infty(M, M)$ of $M$ into itself. Thus actions by such Lie groups $G$ correspond to smooth coordinate transforms on $M$ which, in particular, have smooth inverses.

In Rosinger [1, chap. 13], the study of Lie semigroups $G$ in the vastly larger semigroup $C^\infty(M, M)$ of smooth maps of $M$ into itself was initiated. Such semigroups were named genuine Lie semigroups, or in short, GLS, since they are no longer contained in $\text{Diff}^\infty(M, M)$, thus they correspond to smooth coordinate transforms which need not have smooth inverses.

Genuine Lie semigroups, or GLS, have a major interest since they still can transform solutions of linear or nonlinear PDEs into other solutions of the respective equations, thus leading to the vastly larger class of semi-symmetries of such equations.

Certain Lie semigroups have earlier been studied in the literature, Hilgert, et al. However, such semigroups have always been contained in suitable Lie groups, thus they have been contained in $\text{Diff}^\infty(M, M)$ as sub-semigroups. In particular, the coordinate transforms defined by them were always invertible, unlike in the case of genuine Lie semigroups, or GLS, studied here.
1. Motivation

Given an open subset $M \subseteq \mathbb{R}^l$, it is obvious that a Lie group $G$ acting on $M$, that is

$$(1.1) \quad G \times M \ni (g, x) \mapsto gx \in M$$

can be identified with a subgroup of all the smooth diffeomorphism of $M$. Namely, we have the injective group homomorphism

$$(1.2) \quad G \ni g \mapsto f_g \in \text{Diff}^\infty(M, M)$$

where $f_g$ is defined by

$$(1.2^*) \quad M \ni x \mapsto f_g(x) = gx \in M$$

Here the noncommutative group structure on $\text{Diff}^\infty(M, M)$ is defined by the usual composition of mappings, and thus the neutral element is $e = id_M$, that is, the identity mapping of $M$ onto itself.

In this way, in terms of the Euclidean domain $M$, the group homomorphism (1.2) is but a group of smooth coordinate transforms which have smooth inverses.

Given a linear or nonlinear PDE

$$(1.3) \quad T(x, D) U(x) = 0, \quad x \in \Omega$$

where $\Omega$ is an open subset in $\mathbb{R}^n$, one of the major interests in Lie groups - according to Lie’s original aim - is in the study of the symmetries of solutions $U : \Omega \rightarrow \mathbb{R}$ of (1.3), which therefore, transform them into other solutions of (1.3). This can be done as follows. One takes $M = \Omega \times \mathbb{R}$ and finds the corresponding Lie group actions (1.1) which, when extended to the solutions $U \in C^\infty(\Omega, \mathbb{R})$ of (1.3), will transform them into solutions of the same equation, Appendix, or Olver, Rosinger [1].
We present here a significant extension of this classical symmetry method. For that purpose we note the following four facts:

- In view of (1.2), for every Lie group $G$ acting on any nonvoid open subset $M \subseteq \mathbb{R}^l$, we have the inclusions

$$(1.4) \quad G \subseteq \text{Diff}^\infty(M, M) \subset C^\infty(M, M)$$

- Even in the simplest one-dimensional case when $M = \mathbb{R}$, the set $C^\infty(M, M)$ is considerably larger than $\text{Diff}^\infty(M, M)$. Indeed, only those functions $f : M \rightarrow M$ in $C^\infty(M, M)$ belong to $\text{Diff}^\infty(M, M)$ which are bijective and have their derivative either everywhere strictly positive, or everywhere strictly negative on $M$.

- With respect to the usual composition of mappings, $C^\infty(M, M)$ is a noncommutative semigroup with identity, which is not a group since it contains a vast amount of non-invertible mappings, while on the other hand, $\text{Diff}^\infty(M, M)$ is a noncommutative subgroup of it, and as such, the largest one.

- As seen in the sequel, the property that solutions $U \in C^\infty(\Omega, \mathbb{R})$ of PDEs in (1.3) are transformed by suitable Lie group actions $g \in G$ - thus in view of (1.4), by diffeomorphism in $\text{Diff}^\infty(M, M)$ - into other solutions of (1.3), is by no means restricted to diffeomorphisms alone, but it can also be valid for certain smooth and not necessarily invertible transformations which are elements in the considerably larger set $C^\infty(M, M)$.

2. Genuine Lie Semigroups

Our main aim is to extend the usual Lie group actions (1.1), (1.2), that is

$$G \times M \ni (g, x) \mapsto gx \in M, \quad G \ni g \mapsto f_g \in \text{Diff}^\infty(M, M)$$

in such a way that we are no longer restricted to Lie groups, see (1.4)
\[ G \subseteq \text{Diff}^\infty(M, M) \]

and instead we can now deal with the \textit{vastly larger} class of \textit{semigroups}

\[ G \subseteq \mathcal{C}^\infty(M, M) \]

In other words, in terms of the Euclidean domains \( M \), we are expanding into the vastly larger class of semigroups of smooth coordinate transforms which \textit{need not} have smooth inverses, and in fact, \textit{need not} be surjective either.

Consequently, we are interested in exploring the structure of semigroups \( G \) in \( \mathcal{C}^\infty(M, M) \) which have minimal overlap with \( \text{Diff}^\infty(M, M) \).

Here we note two facts:

- Since we are interested in semigroups \( G \) with neutral element \( e \in G \), it follows that a certain overlap between such semigroups and \( \text{Diff}^\infty(M, M) \) is inevitable. Indeed, in the overall semigroup \( \mathcal{C}^\infty(M, M) \), the neutral element is \( e = \text{id}_M \), that is, the identity mapping of \( M \) into itself. And obviously, we have \( e = \text{id}_M \in \text{Diff}^\infty(M, M) \).

- Similar with the classical Lie group theory, we may start with exploring the structure of \textit{one-dimensional} semigroups \( G \) in \( \mathcal{C}^\infty(M, M) \).

In this regard, a first clarification follows from

\textbf{Lemma 2.1}

Let \((X, \circ)\) be any semigroup with neutral element \( e \in X \). Let

\[ [0, \infty) \ni t \mapsto x_t \in X \]

be a semigroup homomorphism, where \([0, \infty)\) is considered with its usual additive semigroup structure.

If for a certain \( t > 0 \), the element \( x_t \in X \) has an inverse in \( X \), then every element \( x_s \in X \), with \( s \in [0, \infty) \), has an inverse in \( X \).
Consequently, the above semigroup homomorphism can be extended to a group homomorphism

\[ \mathbb{R} \ni t \mapsto x_t \in X \]

where \( \mathbb{R} \) is considered with its usual additive group structure.

In particular, every element \( x_t \in X \), with \( t \in \mathbb{R} \), will have an inverse.

**Proof**

Let \( x'_t \in X \) be the inverse element of \( x_t \). Let \( 0 \leq s < t \), then the commutativity of the additive semigroup on \([0, \infty)\) and the semigroup homomorphism \([0, \infty) \ni t \mapsto x_t \in X\) sends \( s + (t - s) \) to \( x_s \circ x_{t-s} \), and \( (t-s) + s \) to \( x_{t-s} \circ x_s \), both being equal with \( x_t \), since \( s + (t - s) = (t - s) + s = t \). In this way

\[ x_s \circ x_{t-s} = x_t, \quad x_{t-s} \circ x_s = x_t \]

Hence, by multiplying with \( x'_t \) on the right the first relation, and on the left the second one, we obtain

\[ e = x_t \circ x'_t = x_s \circ (x_{t-s} \circ x'_t), \quad e = x'_t \circ x_t = (x'_t \circ x_{t-s}) \circ x_s \]

Let us denote

\[ x'_s = x_{t-s} \circ x'_t, \quad x''_s = x'_t \circ x_{t-s} \]

then

\[ x_s \circ x'_s = x''_s \circ x_s = e \]

hence

\[ x''_s = x''_s \circ e = x''_s \circ (x_s \circ x'_s) = (x''_s \circ x_s) \circ x'_s = e \circ x'_s = x'_s \]

Therefore \( x_s \in X \) has indeed the inverse \( x'_s = x''_s \in X \).

Now, let \( t < s < \infty \). Let \( n \in \mathbb{N} \) be such that \( s < nt \). Then obviously \( x_{nt} \in X \) has the inverse \( x_t' \circ \ldots \circ x'_t \in X \), where the composition \( \circ \)
is applied \(n - 1\) times. Taking now \(t' = nt\), we obtain \(s < t'\), and the proof is reduced to the previous case.

**Remark 2.1**

In the above Lemma it is not necessary that the semigroup \((X, \circ)\) be commutative, nor that \(x_0 = e \in X\).

\[\square\]

In view of Lemma 2.1, we are led to, Rosinger [1, chap. 13]

**Definition 2.1 (Genuine Lie Semigroups, or GLS)**

We call one-dimensional genuine Lie semigroup on \(M\), every semigroup homomorphism

\[(2.1) \quad [0, \infty) \ni t \mapsto g_t \in C^\infty(M, M)\]

which has the properties

\[(2.2) \quad g_0 = id_M\]

\[(2.3) \quad g_t \in C^\infty(M, M) \setminus Diff^\infty(M, M), \text{ for } t > 0\]

**Remark 2.2**

1) Actually, every genuine Lie semigroup, or GLS, in the above definition is given by the image of the semigroup homomorphism (2.1), namely

\[(2.4) \quad G = \{ g_t \mid t \in [0, \infty) \} \subseteq C^\infty(M, M)\]

And we call it genuine, since we obviously have

\[(2.5) \quad G \cap Diff^\infty(M, M) = \{ id_M \}\]

In other words, none of the elements of \(G\), except for \(id_M\), are invertible in the overall semigroup \(C^\infty(M, M)\).
2) In view of Lemma 2.1, we have the following *dichotomy* with respect to arbitrary semigroup homomorphisms

\[(2.6) \quad [0, \infty) \ni t \mapsto g_t \in C^\infty(M, M), \quad \text{with} \quad g_0 = id_M\]

namely:

Either

- None of the smooth coordinate transforms $g_t$ has a smooth inverse, except for $g_0 = id_M$.

Or

- All the smooth coordinate transforms $g_t$ have a smooth inverse, and then the above semigroup homomorphism (2.6) can be extended to a group homomorphism

\[(2.7) \quad \mathbb{R} \ni t \mapsto g_t \in Diff^\infty(M, M)\]

In view of this dichotomy, and within the realm of semigroups of transformations, the concept of genuine one-dimensional Lie semigroup proves to be the *natural* alternative to that of one-dimensional Lie group.

3) Here it should be recalled that Lie semigroups have been studied in Hilgert et.al., for instance. However, so far, such studies have only concerned Lie semigroups which are sub-semigroups of Lie groups, or with the above notation, are sub-semigroups of $Diff^\infty(M, M)$.

Therefore, they cannot be genuine Lie semigroups.

In the sequel, it will be shown that there are *plenty* of one-dimensional genuine Lie semigroups. Namely, it will among others be shown that
∀ \( f \in \mathcal{C}^\infty(M, M) : \)
\[
\exists \ [0, \infty) \ni t \mapsto g_t \in \mathcal{C}^\infty(M, M) \text{ one-dimensional GLS : }
\]
\[
f = g_1
\]
In other words, when taken all together, the one-dimensional genuine Lie semigroups cover the whole of \( \mathcal{C}^\infty(M, M) \).
Needless to say, any given one-dimensional genuine Lie semigroup is but a path in \( \mathcal{C}^\infty(M, M) \).

3. Applications to PDEs, Semi-symmetries

Very simple examples can already show that genuine Lie semigroups, or in short, GLS, can give semi-symmetries of PDEs which cannot be obtained by smooth coordinate transforms which are invertible. Therefore, they are not within the reach of Lie group theory.

Let us start with an example, before giving the precise definition.

One of the simplest linear PDEs is
\[
U_t = U_x, \quad (t, x) \in \Omega = \mathbb{R}^2
\]
whose smooth solutions are given by
\[
U(t, x) = h(t + x), \quad (t, x) \in \Omega
\]
where \( h \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \) is arbitrary.

Let us take now \( M = \Omega \times \mathbb{R} = \mathbb{R}^3 \), and any smooth function \( f \in \mathcal{C}^\infty(M, M) \) which is of the form
\[
M \ni (t, x, u) \mapsto (t, x, g(u)) \in M
\]
where \( g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \) is an arbitrary non-injective function.
Then clearly, the smooth function \( f : M \rightarrow M \) is not a coordinate transform on \( M \), since it is not injective, thus it fails to be invertible,
and consequently \( f \notin \mathcal{D}iff^\infty(M, M) \).

We note, nevertheless, that the function \( f \) transforms solutions of the PDE in (3.1) into solutions of the same PDE, thus it is a semi-symmetry of that equation. Indeed, the action of the function \( f \) upon the solution \( U \) in (3.2) is given by, Appendix

\[
\tilde{U}(t, x) = (fU)(t, x) = g(h(t + x)), \quad (t, x) \in \Omega
\]

thus \( \tilde{U} = fU \) is again a solution of the PDE in (3.1).

The above example of a semi-symmetry which is not a usual symmetry is no doubt very simple. However, it can already clearly illustrate the main issue. Namely, given any linear or nonlinear PDE

\[
T(x, D) U(x) = 0, \quad x \in \Omega
\]

where \( \Omega \) is an open subset in \( \mathbb{R}^n \), and \( U : \Omega \rightarrow \mathbb{R} \) is the unknown solution. If we denote \( M = \Omega \times \mathbb{R} \), then the standard Lie symmetries of that equation are given by functions \( f \in \mathcal{D}iff^\infty(M, M) \) which by their actions, turn solutions \( U \) into solutions \( \tilde{U} = fU \) of that equation.

However, as seen above in (3.4), we can define such actions not only for functions \( f \in \mathcal{D}iff^\infty(M, M) \), but also for the much larger class of functions \( f \in \mathcal{C}^\infty(M, M) \). And such actions \( \tilde{U} = fU \) are called semi-symmetries of the PDEs in (3.5), if they turn solutions \( U \) of those equations into solutions \( \tilde{U} = fU \) of the respective equations.

Here it should be noted that in order to define the actions \( \tilde{U} = fU \) for arbitrary solutions \( U \in \mathcal{C}^\infty(M, \mathbb{R}) \) and functions \( f \in \mathcal{C}^\infty(M, M) \), one has to use the full power of the parametric representation of functions and actions, introduced and developed in Rosinger [1, chapters 1-5]. For convenience, a brief review of these issues is presented in the Appendix.

Also, in Rosinger [1, chap. 13], other more involved examples of semi-symmetries of PDEs are presented.
4. "Enforcing" : How to Generate One-Dimensional Genuine Lie Semigroups

Our interest is to generate one-dimensional genuine Lie semigroups in $\mathcal{C}^\infty(M, M)$, for arbitrary nonvoid open sets $M \subseteq \mathbb{R}^l$.

What we shall do in this section is to show how to solve the following particular case of that general problem. Namely, let us take arbitrary smooth functions

$$f \in \mathcal{C}^\infty(M, M)$$

and use them in a simple "enforcing" method, in order to generate one-dimensional genuine Lie sub-semigroups

$$G^\# \subset \mathcal{C}^\infty(M^\#, M^\#)$$

such that

$$f^\# = (1, f) \in G^\#$$

where

$$M^\# = (0, \infty) \times M \subseteq \mathbb{R}^{l+1}$$

In this way, we obtain the following general result

$$\forall \ M \subseteq \mathbb{R}^l\ \text{nonvoid open, } f \in \mathcal{C}^\infty(M, M) :$$

( dim + 1 )$$\exists \ G^\# \subset \mathcal{C}^\infty(M^\#, M^\#) \text{ a genuine Lie semigroup :}$$

$$f^\# = (1, f) \in G^\#$$

where $M^\# = (0, \infty) \times M \subseteq \mathbb{R}^{l+1}$.

**Open Problem.** The ultimately general result regarding GLS-s, namely
∀ \( M \subseteq \mathbb{R}^l \) nonvoid open, \( f \in C^\infty(M, M) \) :

\[
( \text{GLS} ) \quad \exists \ G \subset C^\infty(M, M) \text{ a genuine Lie semigroup :} \quad f \in G
\]
is still open.

**Short Review of the Classical Case of Lie Groups.** When generating one-dimensional genuine Lie semigroups, we shall try as much as possible to follow the classical way in Lie group theory which generates the one-dimensional Lie groups. There are a number of well known reasons why that classical way is important and useful, and therefore, its possible extension to the generation of one-dimensional genuine Lie semigroups is worth exploring. Indeed, in that classical way, several fundamental mathematical ideas, constructions and properties come together in a fruitful interaction. Among them are:

- Lie Groups,
- Actions,
- Infinitesimal Generators,
- Autonomous ODEs,
- Flows,
- Evolution Operators.

Let us start, therefore, by recalling here in short that classical construction.

**Autonomous ODEs Generating Lie Groups.** Let \( M \subseteq \mathbb{R}^l \) be any nonvoid open subset and \( F \in C^\infty(M, \mathbb{R}^l) \) any smooth function. We consider the autonomous nonlinear system of ODEs with the respective initial conditions.
\[
\frac{d}{dt}Y(t) = F(Y(t)), \quad t \in \mathbb{R}
\]
(4.1)
\[
Y(0) = y \in M
\]
Then the unique solution - for convenience, assumed to exist globally - namely
\[(4.2) \quad \mathbb{R} \ni t \longmapsto Y(t) \in M\]
defines through each point \(y \in M\) a flow on \(M\), which corresponds to the one-dimensional Lie group \(G = (\mathbb{R}, +)\) that acts on \(M\) according to
\[(4.3) \quad \mathbb{R} \times M \ni (t, y) \longmapsto Y(t) \in M\]
In this case \(F\) is called the infinitesimal generator of the Lie group action (4.3).

**Lie Groups Generating Autonomous ODEs.** A basic fact in Lie group theory is that the converse of the above construction also holds. Namely, given the one-dimensional Lie group action (4.3) on \(M\), then one can obtain an infinitesimal generator \(F \in C^\infty(M, \mathbb{R})\) defined by
\[(4.4) \quad F(y) = \frac{d}{dt}Y(t)|_{t=0}, \quad y \in M\]
and the corresponding ODE and initial value problem (4.1) will always have a global solution. In this case, the steps (4.1) - (4.3) will give us back the initial Lie group action (4.3) on \(M\), with which we started.

**Evolution Operators.** The above in (4.1) - (4.4) can be described in terms of evolution operators \(E\) as well. Namely, we define
\[(4.5) \quad \mathbb{R} \ni t \longmapsto E(t) : M \rightarrow M\]
\[
E(t)(y) = Y(t), \quad t \in \mathbb{R}, \ y \in M
\]
and then the above one-dimensional Lie group action (4.3) can be written in the form
These evolution operators $E$ have the important group property

$$E(0) = id_M$$

$$E(t) \circ E(s) = E(t + s), \quad t, s \in \mathbb{R}$$

thus we have the group homomorphism, see (4.5)

$$\mathbb{R} \ni t \mapsto E(t) \in Diff^\infty(M, M)$$

"Enforcing" : How to Find Infinitesimal Generators for One-Dimensional Genuine Lie Semigroups. Our aim - according to the most general program (GLS) above - is to find on $M$ one-dimensional semigroup actions, see (4.3), (4.6)

$$[0, \infty) \times M \ni (t, y) \mapsto S(t, y) \in M$$

or equivalently

$$[0, \infty) \ni t \mapsto S(t, .) : M \ni y \mapsto S(t, y) \in M$$

which give one-dimensional genuine Lie semigroups, that is, with the properties, see (2.1) - (2.3)

$$S(0, .) = id_M$$

$$S(t, .) \circ S(s, .) = S(t + s, .), \quad t, s \in [0, \infty)$$

For that purpose, and in order to become more familiar with the new one-dimensional genuine Lie semigroup situation, we shall proceed step by step, analyzing cases which are more and more general, and in the process we shall eliminate those which are not suited.

**Autonomous Nonsingular ODEs with Global Solutions.** First we note that a one-dimensional GLS in (4.9) - (4.11) cannot be generated by an autonomous ODE of the type (4.1).
Indeed, if the respective ODE in (4.1) has global solutions (4.2) for every $y \in M$, then as seen above, it generates a one-dimensional Lie group (4.8) acting on $M$, which obviously is not a genuine Lie semigroup.

**Autonomous Nonsingular ODEs with Local Solutions.** On the other hand, in case the ODE in (4.1) does not have such a global solution property, then since the respective $F$ is assumed to be smooth on the whole of $M$, a classical result on the existence of solutions for ODEs, Coddington & Levinson, states that for every initial condition $y \in M$, there exists a largest nonvoid open interval $0 \in I_y \subseteq \mathbb{R}$, such that a unique solution $Y : I_y \rightarrow M$ exists, which satisfies the initial condition $Y(0) = y \in M$.

Here it is important to note that such a unique solution $Y$ will exist on an open neighbourhood of $t = 0 \in \mathbb{R}$, that is, both for strictly positive and strictly negative values of $t$, possibly limited accordingly. And as seen next, this is enough in order to prevent such a solution $Y$ from generating a genuine Lie semigroup on $M$.

Indeed, the above unique solution property gives

\[(4.12) \quad M \ni y \mapsto E : I_y \ni t \mapsto E(t)(y) = Y(t) \in M\]

Let us denote for $t \in \mathbb{R}$

\[(4.13) \quad M_t = \{ y \in M \mid t \in I_y \}\]

then (4.12) gives

\[(4.14) \quad \mathbb{R} \ni t \mapsto E(t) : M_t \ni y \mapsto E(t)(y) = Y(t) \in M\]

and we have the generalized group property of the evolution operators $E$, Rosinger [3, pp. 56,57], given by the commutative diagram
where \( t, s \in \mathbb{R} \).

What happens now in case the ODE in (4.1) does not have global solutions (4.2) for every \( y \in M \), is that \( M_t = \emptyset \), or at least \( M_t \subset M \), \( M_t \neq M \), for certain \( t \in \mathbb{R} \). Consequently, the evolution operators \( E \) are not defined on the whole of \( M \), thus we cannot possibly obtain (4.10), where \( S(t, \cdot) \) are supposed to be defined everywhere on \( M \).

**Autonomous Singular ODEs.** A next level of generality is to consider the autonomous ODEs in (4.1) with \( F \) no longer smooth all over \( M \), but having certain *singularities*, for instance

\[
F \in C^\infty(M \setminus \Sigma, \mathbb{R}^l) \setminus C^\infty(M, \mathbb{R}^l)
\]

for suitable nonovid subsets \( \Sigma \subset M \). Such an approach, however, need not always lead to one-dimensional genuine Lie semigroups, as seen from the following simple example. Let \( M = \mathbb{R} \), and consider the ODE

\[
\frac{d}{dt} Y(t) = 1/(Y(t))^2, \quad t \in \mathbb{R}
\]

\[
Y(0) = y \in M \setminus \{0\}
\]

Here we have \( F(y) = 1/y^2 \), for \( y \in M \setminus \Sigma \), where \( \Sigma = \{0\} \subset M \), thus (4.16) is satisfied. However, the unique solution is

\[
Y(t) = (3t + y^3)^{1/3}, \quad y \in M, \ y \neq 0, \ t \in \mathbb{R}, \ t \neq -y^3/3
\]

And we note that this function \( Y \) can in fact be extended to
\[ Y(t) = (3t + y^3)^{1/3}, \quad y \in M, \ t \in \mathbb{R} \]

which for every \( y \in M \) satisfies the singular autonomous ODE

\[ \frac{d}{dt} Y(t) = 1/(Y(t))^2, \quad t \in \mathbb{R}, \ t \neq -y^3/3 \]

Thus instead of (4.10), we have

\[ \mathbb{R} \ni t \mapsto S(t, .) : M \ni y \mapsto Y(t) \in M \]

and this leads to a Lie group action on \( M \), since obviously

\[ S(t, S(s, y)) = S(t + s, y), \quad t, s \in \mathbb{R}, \ y \in M \]

**Non-autonomous Singular ODEs.** In view of the above, the next step is to consider *non-autonomous* ODEs of the form

\[ \frac{d}{dt} Y(t) = F(t, Y(t)), \quad t \in \mathbb{R} \]

(4.17)

\[ Y(t_0) = y_0 \in M \]

where \( F : \mathbb{R} \times M \longrightarrow \mathbb{R}^l \) and

(4.18) \quad \( F \in C^\infty \)

except for certain possible singularities in its domain \( \mathbb{R} \times M \).

The idea here is twofold, namely:

- **( Reduct )** To use the standard *reduction* method of such non-autonomous ODEs to autonomous ones.

- **( Sing )** To include certain *singularities* in the non-autonomous ODEs, so that, when reduced to autonomous ODEs, the solutions of these latter ODEs do *not* give one-dimensional Lie groups, but only one-dimensional genuine Lie semigroups.

It follows that the only problem here is to find out what kind of *singularities* the non-autonomous ODEs (4.17), and more precisely, their
right hand terms (4.18), must have in the very least, in order to secure
the above property (Sing).

Remark 4.1.

For the sake of clarity, let us recall in short the standard way non-
autonomous ODEs can be reduced to autonomous ones. The further
details needed will be presented in section 5.
Given an explicit non-autonomous ODE with a respective initial value
problem

\[ \frac{d}{dt} Y(t) = F(t, Y(t)), \quad t \in \mathbb{R} \]
\[ Y(t_0) = y_0 \in M \]

or more generally, an implicit non-autonomous ODE with an associated initial value problem

\[ F(t, Y(t), \frac{d}{dt} Y(t)) = 0, \quad t \in \mathbb{R} \]
\[ Y(t_0) = y_0 \in M \]

there is a well known standard procedure in Control Theory to reduce it to an autonomous ODE. This is done simply by increasing with 1 the dimension of the system of ODEs (4.20), namely, from \( l \) to \( l + 1 \). For that purpose, we replace the \( l \)-dimensional solution vector \( Y : \mathbb{R} \rightarrow M \) with the \((l + 1)\)-dimensional solution vector

\[ Y^\# : \mathbb{R} \rightarrow M^\# \]

where

\[ M^\# = \mathbb{R} \times M, \quad Y^\#(t) = (t, Y(t)), \quad t \in \mathbb{R} \]

In this case (4.20) obviously becomes the implicit autonomous ODE

\[ F^\#(Y^\#(t), \frac{d}{dt} Y^\#(t)) = 0, \quad t \in \mathbb{R} \]
\[ Y^\#(t_0) = (t_0, y_0) \in M^\# \]
where the equation \( F^#(Y^#(t), \frac{d}{dt} Y^#(t)) = 0 \) given by

\[
\frac{d}{dt} t = 1, \quad t \in \mathbb{R}
\]

(4.24)

\[
F(t, Y(t), \frac{d}{dt} Y(t)) = 0, \quad t \in \mathbb{R}
\]

Clearly, in the particular case of (4.19), this procedure leads to an autonomous ODE system which again is explicit, namely

\[
(4.23^*) \quad \frac{d}{dt} Y^#(t) = F^#(Y^#(t)), \quad t \in \mathbb{R}
\]

However, it is important to note that in (4.23) care has to be taken with the initial condition. Indeed, if as in (4.20), the initial condition is given at \( t_0 \in \mathbb{R} \), then in the case of the extended ODE in (4.23), this will become

(4.25) \quad Y^#(t_0) = (t_0, y_0) \in M^#

in other words, the right hand term in (4.25) is not completely arbitrary in \( M^# \), since it must have the same \( t_0 \) as in the left hand term.

Now the point of interest for us in the above reduction of non-autonomous ODEs to autonomous ones is in the following two facts:

- The resulting autonomous ODE in (4.23) has solutions \( Y^# \) defined on the same \( t \)-interval \( I \subseteq \mathbb{R} \) with the solutions \( Y \) of the non-autonomous ODE (4.20), as this follows easily from (4.24).

- The solutions \( Y^# \) of the autonomous ODE in (4.23) may under rather general conditions have a one-dimensional group property similar with (4.5) - (4.7).

- In case there are appropriate singularities involved in the ODE in (4.23), the solutions \( Y^# \) will have a one-dimensional semigroup property, namely, they lead to an evolution operator

\[
[0, \infty) \ni t \mapsto E^#(t) : M^# \rightarrow M^#
\]

(4.26)

\[
E^#(t)(y^#) = Y^#(t), \quad t \in [0, \infty), \quad y^# \in M^#
\]
with the semigroup property

\[ E^\#(0) = id_{M^\#} \]

\[ E^\#(t) \circ E^\#(s) = E^\#(t + s), \quad t, s \in [0, \infty) \]

(4.27)

In this way, we shall be able to generate one-dimensional genuine Lie semigroups in \( C^\infty(M^\#, M^\#) \), rather than in the initial \( C^\infty(M, M) \).

**A First Simple Example of ”Enforcing” : A Singular Non-autonomous ODE and its Solution.** We note that we can start, so to say, backwards. Namely, we can start with an action on \( M \) of the form, see (4.9) - (4.10), given by

\[ (0, \infty) \times M \ni (t, y) \mapsto H(t, y) \in M \]

(4.28)

which need not be a semigroup, and then find the non-autonomous singular ODE which it satisfies.

Now the enforcing part comes when on purpose we choose this action (4.28) so that it cannot be extended to an action

\[ (\mathbb{R}, \infty) \times M \ni (t, y) \mapsto H(t, y) \in M \]

(4.28*)

which in certain cases may possibly give a Lie group action on \( M \).

Consequently, by such an enforcing, we may obtain an insight into the kind of singularities possessed by the non-autonomous ODE satisfied by such an action (4.28).

As mentioned, such an action (4.28) from which we start need not in general be a semigroup action on \( M \), and even less a genuine Lie semigroup action. However, when the respective non-autonomous ODE satisfied by that action is reduced to an autonomous one, we shall inevitably obtain a semigroup action, this time on the \((l+1)\)-dimensional open subset \( M^\# = \mathbb{R} \times M \), see Remark 4.1, above.

And now, we choose one of the simplest possible such actions (4.28),
namely

\begin{equation}
(4.29) \quad H(t, y) = y + \sqrt{t} y^2, \quad t \in [0, \infty), \ y \in M = \mathbb{R}
\end{equation}

Then obviously (4.29) cannot be extended to a smooth action (4.28*) owing to the presence of \( \sqrt{t} \). Also, since \( y^2 \) appears in (4.29), the corresponding action (4.28) is not injective. Furthermore, we have

\[
H(0, y) = y, \quad y \in M
\]

\[
H(t, .) \notin \text{Diff}^\infty(M, M), \quad t \in (0, \infty)
\]

\[
(4.30) \quad H \notin C^1([0, \infty) \times M, M)
\]

\[
H \in C^0([0, \infty) \times M, M) \cap C^\infty((0, \infty) \times M, M)
\]

Let us now find the non-autonomous explicit or implicit ODE satisfied by \( H \) in (4.29). By partial derivation of that relation with respect to \( t \), we obtain

\begin{equation}
(4.31) \quad \frac{\partial}{\partial t} H(t, y) = \frac{y^2}{2\sqrt{t}}, \quad t \in (0, \infty), \ y \in M
\end{equation}

Then we solve (4.29) as a quadratic equation in \( y \), and obtain

\begin{equation}
(4.32) \quad y_1, y_2 = \frac{-1 \pm \sqrt{1 + 4\sqrt{t} H(t, y)}}{2\sqrt{t}}, \quad t \in (0, \infty), \ 1 + 4\sqrt{t} H(t, y) \geq 0
\end{equation}

Further, we return to (4.29), replace \( y \) with its values from (4.32), while we replace \( y^2 \) with its value from (4.31). In this way, we obtain the two non-autonomous singular explicit ODEs

\begin{equation}
(4.33) \quad \frac{\partial}{\partial t} H(t, y) = \frac{1 + 2\sqrt{t} H(t, y) \pm \sqrt{1 + 4\sqrt{t} H(t, y)}}{4t\sqrt{t}},
\end{equation}

\quad \quad \quad \quad t \in (0, \infty), \ 1 + 4\sqrt{t} H(t, y) \geq 0

If we now consider \( H \) given in (4.29) as a function of \( t \in (0, \infty) \), and with \( y \in M \) being a fixed parameter, then substituting \( H \) into (4.33),
a simple computation shows that $H$ satisfies the ODE in (4.33) with the sign "+" in front of the large radical, when

\begin{equation}
(4.34) \quad t \in (0, \infty), \; y \in M, \; 1 + 2\sqrt{t}y \leq 0
\end{equation}

and alternatively, satisfies the ODE in (4.33) with the sign "−" in front of the large radical, when

\begin{equation}
(4.35) \quad t \in (0, \infty), \; y \in M, \; 1 + 2\sqrt{t}y \geq 0
\end{equation}

Moreover, in both of these cases, $H$ as defined in (4.29) will obviously satisfy the limit type initial condition

\begin{equation}
(4.36) \quad \lim_{t \searrow 0} Y(t) = y
\end{equation}

since in fact we have $H(0, y) = y$, for $y \in M$, as for every $y \in M$, $H$ is continuous at $t = 0$ from the right, see (4.30).

Here we note that, since it is necessary that $t > 0$ for the ODEs in (4.33) to be defined, one cannot in general ask for them to satisfy a usual initial condition $Y(0) = y$, but only of the limit type one in (4.36).

**A Type of Implicit Singular Non-autonomous ODEs Leading to Genuine Lie Semigroups.** In view of the above example, we are led to consider explicit non-autonomous singular ODEs and associated limit type initial condition of the form, see (4.17), (4.18)

\begin{equation}
(4.37) \quad \frac{d}{dt}Y(t) = F(t, Y(t)), \quad t \in (0, \infty)
\end{equation}

\begin{equation}
\lim_{t \searrow 0} Y(t) = y \in M_0
\end{equation}

for a suitable subset $M_0 \subseteq M$, and with

\begin{equation}
(4.38) \quad F \in \mathcal{C}^\infty((0, \infty) \times M, \mathbb{R}^l)
\end{equation}

And we expect to have unique solutions $Y$ such that
\[ Y \in C^\infty((0, \infty), M) \]

\[ Y \in C^0([0, \infty), M) \setminus C^1([0, \infty), M) \]

\[ (0, \infty) \text{ is the largest interval in } \mathbb{R} \]

Needless to say, we may have to go one level of generality higher, namely, to consider \textit{implicit} non-autonomous singular ODEs and associated limit type initial conditions, of the form

\[ F(t, Y(t), \frac{d}{dt}Y(t)) = 0, \quad t \in (0, \infty) \]

\[ \lim_{t \to 0} Y(t) = y \in M_0 \]

instead of the explicit ones in (4.37).

We present a first result on the existence of solutions (4.39) of implicit singular non-autonomous ODEs of the type (4.40). This result shows that there are so many such solutions as to \textit{cover} the whole of \( C^\infty(M, M) \), at least in the case when \( M = \mathbb{R}^l \).

\textbf{Lemma 4.1. "Enforcing"}

Let \( f \in C^\infty(\mathbb{R}^l, \mathbb{R}^l) \setminus Diff^\infty(\mathbb{R}^l, \mathbb{R}^l) \). Then there exist actions on \( \mathbb{R}^l \)

\[ [0, \infty) \times \mathbb{R}^l \ni (t, y) \mapsto H(t, y) \in \mathbb{R}^l \]

such that

\[ H(0, .) = \text{id}_{\mathbb{R}^l}, \quad H(1, .) = f \]

and \( H \) as a function of \( t \), and with \( y \in \mathbb{R}^l \) considered as a fixed parameter, is a solution of an implicit singular non-autonomous ODE and associated limit type initial condition in (4.40), which satisfies (4.39).

\textbf{Proof}
Let us take any function \( g \in C^0([0, \infty), \mathbb{R}) \cap C^\infty((0, \infty), \mathbb{R}) \) such that
\[
(4.43) \quad g(0) = 0, \quad g(1) = 1
\]
\( g \not\in C^1([0, \infty), \mathbb{R}) \)
\( g'(t) \neq 0, \quad t \in (0, \infty) \)
for instance, we can consider \( g(t) = \sqrt{t}, \) for \( t \in [0, \infty). \)

Let us then define
\[
(4.44) \quad H \in C^0([0, \infty) \times \mathbb{R}, \mathbb{R}) \cap C^\infty((0, \infty) \times \mathbb{R}, \mathbb{R})
\]
as the homotopic deformation of \( \text{id}_{\mathbb{R}} \) into \( f, \) mediated by \( g, \) namely
\[
(4.45) \quad H(t, y) = (1 - g(t))y + g(t)f(y), \quad t \in [0, \infty), \ y \in \mathbb{R}
\]
Then clearly
\[
(4.46) \quad H(0, .) = \text{id}_{\mathbb{R}}, \quad H(1, .) = f \in C^\infty(\mathbb{R}, \mathbb{R}) \setminus \text{Diff}^\infty(\mathbb{R}, \mathbb{R})
\]
Let us now find the ODE satisfied by \( H \) when it is considered a function of \( t, \) while \( y \) is taken as a parameter. The relation (4.45) and its partial derivative in \( t \) give the two linear algebraic equations in \( y \) and \( f(y), \) namely
\[
(4.47) \quad H = (1 - g(t))y + g(t)f(y)
\]
\[
\frac{\partial}{\partial t} H = g'(t)(f(y) - y)
\]
for \( t \in (0, \infty), \ y \in M. \) Since in view of (4.43) we have \( g'(t) \neq 0 \) for \( t \in (0, \infty), \) we can solve (4.47) for \( y \) and \( f(y), \) and obtain
\[
(4.48) \quad y = \frac{g'H - gH_t}{g'}, \quad f(y) = \frac{(1 - g)H_t + g'h}{g'}, \quad t \in (0, \infty)
\]
Substituting in (4.45) these values for \( y \) and \( f(y), \) we obtain the implicit singular non-autonomous ODE in \( Y, \) namely
(4.49) \((1-g(t)) \frac{d}{dt} Y(t) + g'(t)Y(t) = g'(t)f \left( \frac{g'(t)Y(t) - g(t) \frac{d}{dt} Y(t)}{g'(t)} \right)\)

for \(t \in (0, \infty)\). Indeed, in view of (4.43), \(g'(t)\) may be singular at \(t = 0\).

Finally, in view of (4.46), the solution \(Y\) of the ODE in (4.49) does satisfy the initial condition in (4.40).

\[\square\]

It is easy to see, Rosinger [1, pp. 199,200], that a lot more general examples than above in (4.41) - (4.49), (4.40) can be constructed for arbitrary \(f \in C^\infty(\mathbb{R}^l, \mathbb{R}^l) \setminus \text{Diff} \infty(\mathbb{R}^l, \mathbb{R}^l)\).

The conclusion is that, as indicated in (4.40), a class of implicit non-autonomous ODEs which are singular at \(t = 0\), is needed in order to be able, through the standard construction in section 5, to obtain genuine Lie semigroups, or GLS-s.

**Nonremovable Singularities**

As seen in section 5, the property of the actions \(H\), see (4.30), and in particular (4.29), (4.45)

(4.50) \(H \in C^0([0, \infty) \times M, M) \setminus C^1([0, \infty) \times M, M)\)

is needed in order to obtain genuine Lie semigroups, or GLS-s. This property, as seen above, corresponds to a singularity at \(t = 0\) of the non-autonomous ODEs satisfied by such actions \(H\). Let us further note that such singularities of the actions \(H\) must, therefore, be non-removable, and the example below shows that this is possible to attain.

Returning to the action \(H\) in (4.29) with \(M = \mathbb{R}\), we can associate with it the action \(K \in C^\infty(\mathbb{R} \times M, M)\), given by

(4.51) \(K(s, y) = y + sy^2, \quad s \in \mathbb{R}, \quad y \in M\)

and then we have
for $s \in \mathbb{R}$, $t \in [0, \infty)$.

However, even if $K$ has no singularity for $(s, y) \in \mathbb{R} \times M$, the fact remains that in the sense of $\star \star \star$) in (4.39), $H$ is singular at $t = 0$, as it fails to be $C^1$-smooth in a neighbourhood of $t = 0$.

5. Standard Reduction to Autonomous ODEs and Evolution

Here we present the needed details on the standard way mentioned briefly in Remark 4.1., and according to which non-autonomous ODEs can be reduced to autonomous ones, with the all important corresponding reduction of non-autonomous evolutions to an autonomous ones. This standard reduction is fundamental in obtaining the general result in (dim + 1) in section 4 above, which is a stepping stone towards the ultimate result in ( GLS ), a result still open.

**Autonomous Evolution.** Let be given the autonomous nonlinear system of ODEs

\begin{equation}
\frac{d}{dt}Y(t) = F(Y(t)), \quad t \in \mathbb{R} \\
Y(t_0) = y_0
\end{equation}

with $F \in C^\infty(\mathbb{R}^l, \mathbb{R}^l)$, $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^l$, where the sought after solution is $Y \in C^\infty(\mathbb{R}, \mathbb{R}^l)$. For convenience, we assume that the unique solution $Y$ exists globally on $\mathbb{R}$ for every initial condition $y_0 \in \mathbb{R}^l$. Thus we can associate with (5.1) the evolution operator

\begin{equation}
\mathbb{R} \ni t \mapsto E(t) : \mathbb{R}^l \to \mathbb{R}^l \\
E(t - t_0)(y) = Y(t), \quad t_0, t \in \mathbb{R}, \quad y \in \mathbb{R}^l
\end{equation}

which, as mentioned, defines the one dimensional Lie group action on $\mathbb{R}^l$
Thus the evolution operator $E$ has the group properties

$$E(0) = \text{id}_{\mathbb{R}^l}$$

(5.4)

$$E(t + s) = E(t)E(s), \quad t, s \in \mathbb{R}$$

in other words, we have the group homomorphism

$$\mathbb{R} \ni t \mapsto E(t) \in \text{Diff}^\infty(\mathbb{R}^l)$$

(5.5)

So much for a recapitulation of needed well known properties of autonomous nonlinear systems of ODEs.

**Non-autonomous Evolution.** And now, let us consider the non-autonomous nonlinear system of ODEs

$$\frac{d}{dt}Y(t) = F(t, Y(t)), \quad t \in \mathbb{R}$$

(5.6)

$$Y(t_0) = y_0 \in \mathbb{R}^l$$

where $F \in C^\infty(\mathbb{R}^{l+1}, \mathbb{R}^l)$, $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^l$, and for convenience, the unique solution $Y \in C^\infty(\mathbb{R}, \mathbb{R}^l)$ is supposed to exist for all $t \in \mathbb{R}$. This time, the associated non-autonomous evolution operator $E \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^l, \mathbb{R}^l)$ has the property

$$\mathbb{R} \times \mathbb{R} \ni (t_0, t) \mapsto E(t_0, t) : \mathbb{R}^l \rightarrow \mathbb{R}^l$$

(5.7)

$$E(t_0, t)(y_0) = Y(t), \quad t_0, t \in \mathbb{R}, \quad y_0 \in \mathbb{R}^l$$

However, it is important to note that if we solve only for a given fixed $t_0 \in \mathbb{R}$ the non-autonomous ODE system in (5.6) and with all the initial conditions $y_0 \in \mathbb{R}^l$, then (5.7) will in general not give the full information on $E$, and even less will give it in an explicit manner. This is unlike the case with autonomous ODEs, see (5.2).

Nevertheless, as seen in (5.22), the full information on the non-autonomous evolution operator $E$ can be recovered from the knowledge of (5.7) even for one single $t_0 \in \mathbb{R}$, and moreover, it can be recovered without
having to solve the ODE system (5.6) for every other $t_0 \in \mathbb{R}$, but only by solving some additional algebraic equations. Indeed, this becomes possible, once we transform the non-autonomous ODE system in (5.6) into the autonomous ODE system (5.10), and then we use the corresponding autonomous evolution operator $E_A$.

In the case of (5.6), (5.7), the relations (5.4) take the following more general, *non-autonomous form of group property*, namely

$$E(t, t) = \text{id}_{\mathbb{R}^l}, \quad t \in \mathbb{R}$$

(5.8)

$$E(s, r)E(t, s) = E(t, r), \quad t, s, r \in \mathbb{R}$$

**Reduction to Autonomous Evolution.** Now we recall the standard way the non-autonomous system (5.6) can be *reduced* to an autonomous one, such as for instance in (5.1).

Namely, let us augment the function $Y \in C^\infty(\mathbb{R}, \mathbb{R}^l)$ in (5.6) to the function $Y_A \in C^\infty(\mathbb{R}, \mathbb{R}^{l+1})$, given by

$$(5.9) \quad Y_A(t) = (t, Y(t)), \quad t \in \mathbb{R}$$

then clearly, the $l$-dimensional non-autonomous ODE system (5.6) is *equivalent* with the $l + 1$-dimensional autonomous ODE system

$$\frac{d}{dt}Y_A(t) = (1, F(Y_A(t))), \quad t \in \mathbb{R}$$

(5.10)

$$Y_A(t_0) = (t_0, y_0) \in \mathbb{R}^{l+1}$$

Two facts should be noted here.

First, the *non-autonomous infinitesimal generator* $F$ in (5.6), becomes associated by (5.10) with the *autonomous infinitesimal generator* $F_A \in C^\infty(\mathbb{R}^{l+1}, \mathbb{R}^{l+1})$ defined by

$$F_A(y_A) = (1, F(y_A)), \quad y_A \in \mathbb{R}^{l+1}$$

(5.11)

Second, in the initial condition $(t_0, y_0)$ in (5.10), the first coordinate
$t_0 \in \mathbb{R}$ must be the *same* with that in the left hand term. Clearly, this is a direct consequence of the definition of $Y_A$ in (5.9).

Let now $E_A \in C^\infty(\mathbb{R} \times \mathbb{R}^{l+1}, \mathbb{R}^{l+1})$ be the autonomous evolution operator associated with (5.10), then

\begin{equation}
E_A(t - t_0)(t_0, y_0) = Y_A(t), \quad t_0, t \in \mathbb{R}, \ y_0 \in \mathbb{R}^l
\end{equation}

and it is easy to see that it can be decomposed as follows

\begin{equation}
E_A(s)(t, y) = (E_{A1}(s)(t, y), E_{A2}(s)(t, y)), \quad t, s \in \mathbb{R}, \ y \in \mathbb{R}^l
\end{equation}

with the $C^\infty$-smooth functions

\begin{equation}
\begin{align*}
\mathbb{R} \ni s & \mapsto E_{A1}(s) : \mathbb{R}^{l+1} \to \mathbb{R} \\
\mathbb{R} \ni s & \mapsto E_{A2}(s) : \mathbb{R}^{l+1} \to \mathbb{R}^l
\end{align*}
\end{equation}

In view of (5.7), (5.9), (5.12), (5.13), it follows that, for $t_0, t \in \mathbb{R}, \ y_0 \in \mathbb{R}^l$, we have

\begin{equation}
E(t_0, t)(y_0) = Y(t) = E_{A2}(t - t_0)(t_0, y_0)
\end{equation}

Also, $E_A$ satisfies the group properties corresponding to the autonomous case (5.4), namely

\begin{equation}
\begin{align*}
E_A(0) & = id_{\mathbb{R}^{l+1}} \\
E_A(t + s) & = E_A(s)E_A(t), \quad t, s \in \mathbb{R}
\end{align*}
\end{equation}

which result in the group homomorphism

\begin{equation}
\mathbb{R} \ni t \mapsto E_A(t) \in Diff(\mathbb{R}^{l+1})
\end{equation}

Therefore, for $t, s, r \in \mathbb{R}$ and $y \in \mathbb{R}^l$, we obtain the relations

\begin{equation}
\begin{align*}
E_{A1}(r + s)(t, y) & = E_{A1}(r)(E_{A1}(s)(t, y), E_{A2}(s)(t, y)) \\
E_{A2}(r + s)(t, y) & = E_{A2}(r)(E_{A1}(s)(t, y), E_{A2}(s)(t, y))
\end{align*}
\end{equation}
And in view of (5.7) - (5.12), we obtain for $t, s \in \mathbb{R}, y \in \mathbb{R}^l$

$$E_{A1}(s)(t, y) = t + s$$

(5.19)

$$E_{A2}(s)(t, y) = E(t, t + s)(y)$$

In particular, we can check for $E$ the non-autonomous form of the semigroup property (5.8). Namely, for $t, s, r \in \mathbb{R}, y \in \mathbb{R}^l$, we have

$$E(t, t + s + r)(y) =$$

$$= E_{A2}(s + r)(t, y) =$$

$$= E_{A2}(r)(t + s, E_{A2}(s)(t, y)) =$$

$$= E_{A2}(r)(t + s, E(t, t + s)(y)) =$$

$$= E(t + s, t + s + r)E(t, t + s)(y)$$

(5.20)

**Recovering the Non-autonomous Evolution.** Let us show now how we can recover the full non-autonomous evolution operator $E$ in (5.8), from anyone of its particular cases in (5.7), which corresponds merely to a certain $t_0 \in \mathbb{R}$ fixed. For that purpose, we shall use the autonomous extension $E_A$ of $E$.

Indeed, (5.15) gives

$$E(t, s)(y) = E_{A2}(s, t)(t, y), \quad t, s \in \mathbb{R}, y \in \mathbb{R}^l$$

(5.21)

We fix now $t_0 \in \mathbb{R}$. Given any $t \in \mathbb{R}$, if we can compute the mapping

$$\mathbb{R}^l \ni y \mapsto y_\ast \in \mathbb{R}^l$$

where $y_\ast$ is the solution of the algebraic equation

$$E(t_0, t)(y_\ast) = y$$

then it is clear that we thus obtain $(t, y) = E_A(t - t_0)(t_0, y_0)$, hence

$$E(t, s)(y) = E_{A2}(s - t)E_A(t - t_0)(t_0, y_\ast) = E(t_0, s)(y_\ast)$$

Therefore
\[ E(t, s)(y) = E(t_0, s)(y_\ast), \quad t, s \in \mathbb{R}, \ y \in \mathbb{R}^l \]

(5.22)

where \( y_\ast \) is a solution of \( E(t_0, t)(y_\ast) = y \)

We shall also assume that

(5.23) \( y_\ast \) depends \( C^\infty \) smoothly on \( t, y \)

Let us illustrate the above with a simple example. We consider the non-autonomous ODE

\[
\frac{d}{dt} Y(t) = 2t, \quad t \in \mathbb{R}
\]

\[
Y(t_0) = y_0 \in \mathbb{R}
\]

for which obviously

\[
Y(t) = E(t_0, t)(y_0) = E_{A_2}(t-t_0)(t_0, y_0) = t^2 - t_0^2 + y_0
\]

\[
Y_A(t) = (t, E_{A_2}(t-t_0)(t_0, y_0))
\]

\[
E_{A_1}(s)(t, y) = t + s
\]

\[
E_{A_2}(s)(t, y) = E(t, t+s)(y) = s^2 + 2st + y
\]

We now check the group properties (5.15). First, we note that the relation

\[ E_A(0) = id_{\mathbb{R}^{l+1}} \]

results immediately from the expressions of \( E_{A_1} \) and \( E_{A_2} \) above. Then

\[
E_A(r)E_A(s) = E_A(s + r), \quad s, r \in \mathbb{R}
\]

holds since the previous relations give \( E_A(s)(t, y) = (t+s, s^2+2st+y) \), hence \( E_A(r)E_A(s)(t, y) = E_A(r)(t+s, s^2+2st+y) = (t+s+r, r^2+2r(t+s)+s^2+2st+y) \), while \( E_A(s+r)(t, y) = (t+s+r, (s+r)^2+2(s+r)t+y) \).

In other words, the non-autonomous version (5.8) of the group prop-
erty works as follows, for \( t, s, r \geq 0, \ y \in M \), we have

\[
E(t, s)(y) = s^2 - t^2 + y
\]

hence clearly \( E(t, t)(y) = y \), while

\[
E(s, r)E(t, s)(y) = r^2 - s^2 + (s^2 - t^2 + y) = r^2 - t^2 + y = E(t, r)(y)
\]

The autonomous version, which corresponds to the non-autonomous one, according to (5.9) - (5.22), will have the group property (5.16) working as follows. For \( t, s, r \geq 0, \ y \in M \), we have

\[
E_A(s)(t, y) = (t + s, s^2 + 2st + y)
\]

thus obviously \( E_A(0)(t, y) = (t, y) \), while

\[
E_A(r)E_A(s)(t, y) = (t + s + r, r^2 + 2r(t + s) + s^2 + 2st + y) =
\]

\[
= (t + s + r, (s + r)^2 + 2(s + r)t + y) = E_A(s + r)(t, y)
\]

Finally, in the case of this example, the relation (5.22) works as follows. Given a fixed \( t_0 \in \mathbb{R} \), the algebraic equation

\[
t^2 - t_0^2 + y_* = y
\]

can obviously be solved in \( y_* \) for every \( t, y \in \mathbb{R} \), and it gives \( y_* = y - t^2 + t_0^2 \), which also satisfies (5.23), therefore

\[
E(t, s)(y) = E(t_0, s)(y_*) = s^2 - t_0^2 + y_* = s^2 - t_0^2 + y
\]

6. Examples of Genuine Lie Semigroups

We return to the examples in section 4, and show the way in which the respective actions can be associated with genuine Lie semigroup actions, by applying to them the method of reduction in section 5.

First we note that the results in section 5, where the ODEs are defined on the whole of \( \mathbb{R} \), have to be adapted since, in general, and
as seen with the examples in section 4, one no longer has \( t \in \mathbb{R} \), but only \( t \in (0, \infty) \), or at most \( t \in [0, \infty) \), see (4.29), (4.43) - (4.46), or Rosinger [1, p. 199, (13.2.36) - (13.2.38)]. As noted in section 4, this restriction of the domain of \( t \) is due to the nonremovable singularities of the respective ODEs, and more specifically, of their solutions of interest.

Concerning the autonomous case (5.1) - (5.5), such a restriction of the domain of \( t \) means that the respective evolution operator \( E \) in (5.2) will only be defined for \( t \in [0, \infty) \), and instead of the group property (5.4), will only have the semigroup property

\[
(6.1) \quad E(t + s) = E(s)E(t), \quad t, s \in [0, \infty)
\]

In particular, \( E(t) \), with \( t \in (0, \infty) \), may fail to be invertible, since \( E(-t) \) need not exist, and thus, we could not always obtain from (5.4) the relations \( E(t)E(-t) = E(-t)E(t) = id_{\mathbb{R}^l} \).

In the non-autonomous case (5.6) - (5.8), the evolution operator \( E \) in (5.7) will only be defined for \( t_0, t \in [0, \infty) \), and will have the following semigroup version of property (5.8)

\[
(6.2) \quad E(r, s)E(t, s) = E(t, r), \quad t, s, r \in [0, \infty)
\]

This however allows for the existence of its inverses, since we have for any \( t, s \in [0, \infty) \)

\[
(6.3) \quad E(t, s)E(s, t) = E(s, s) = id_{\mathbb{R}^l}
\]

\[
E(s, t)E(t, s) = E(t, t) = id_{\mathbb{R}^l}
\]

Let us now transform, more precisely reduce, the thus restricted non-autonomous version of (5.6) - (5.8) into the corresponding restricted autonomous version of (5.9) - (5.22). Then clearly, for \( t, s \in [0, \infty), y \in \mathbb{R}^l \), we obtain

\[
(6.4) \quad E_A(s)(t, y) = (t + s, E(t, t + s)(y))
\]

or equivalently
Therefore, in the case of such singular non-autonomous ODEs and of their corresponding evolution operators $E$, after the transformation into the autonomous case, except for the trivial situation of $s = t$, we need no longer be able to benefit from the existence of the inverses in (6.3), when we deal with the associated autonomous evolution operator $E_A$.

This is precisely at the basis of our construction of genuine Lie semigroup actions.

And in particular, this is how we shall associate such genuine Lie semigroup actions with the examples in section 4.

Before going further, let us note that in the case of both non-autonomous examples in section 4, we do have the corresponding versions of (6.2), namely

(6.6) \[ E(s, r)E(t, s) = E(t, r), \quad t, s, r \in [0, \infty) \]

Indeed, for the example in (4.29) - (4.36), this follows from the fact that $H$ in (4.29) is a solution on $(0, \infty)$ of (4.31) - (4.36), while in addition, see (4.37), $H$ is such that $H(0, y) = y$, for all $y \in M$, and also $H \in C^0([0, \infty) \times M, M)$.

A similar argument will apply to the more general examples in (4.43) - (4.46), or Rosinger [1, p. 199, (13.2.36) - (13.2.38)].

In view of (6.6), we obtain for the example in (4.29) - (4.36), the relation

(6.7) \[ E(0, t)(y) = y + \sqrt{t}y^2, \quad t \in [0, \infty), \ y \in M \]

while for the example in (4.43) - (4.49), we shall have

(6.8) \[ E(0, t)(y) = (1 - g(t))y + g(t)f(y), \quad t \in [0, \infty), \ y \in \mathbb{R}^d \]
These two relations will help us in fully computing the respective non-autonomous evolution operators \( E \) for the mentioned examples.

Indeed, let us determine \( E(t, s)(y) \) for the first example, and do so for all \( t, s \in (0, \infty), \ y \in M = \mathbb{R} \). From (6.6) we have

\[
E(t, s)(y) = E(0, s)E(t, 0)(y) = E(0, s)(E(0, t))^{-1}(y)
\]

Hence, proceeding for \( t_0 = 0 \) as in (5.22), let us assume that

\[
(E(0, t))^{-1}(y) = y_* \in M
\]

then clearly

\[
y = E(0, t)(y_*) = y_* + \sqrt{t}y_*^2
\]

and we note the important consequence that

\[
\lim_{t \to 0} y_* = y \in M
\]

provided that \( y_* \) is bounded.

Now computing \( y_* \) from the above quadratic equation, we obtain

\[
y_* = \frac{-1 \pm \sqrt{1 + 4\sqrt{t}y}}{2\sqrt{t}}, \quad t \in (0, \infty), \ y \in M, \ 1 + 4\sqrt{t}y \geq 0
\]

and in order to secure the above limit, it follows that we must choose

\[
y_* = \frac{-1 + \sqrt{1 + 4\sqrt{t}y}}{2\sqrt{t}} = \frac{2y}{1 + \sqrt{1 + 4\sqrt{t}y}}, \quad t \in (0, \infty), \ y \in M, \ 1 + 4\sqrt{t}y \geq 0
\]

In this way (5.23) is satisfied, while
\[ E(t, s)(y) = E(0, s)(y_*) = \]
\[ = \frac{2y}{1 + \sqrt{1 + 4\sqrt{ty}}} + \sqrt{s} \frac{4y^2}{\left(1 + \sqrt{1 + 4\sqrt{ty}}\right)^2} \]
\[ t, \ s \in (0, \infty), \ y \in M, \ 1 + 4\sqrt{ty} \geq 0 \]

and we have obtained the full expression of the non-autonomous evolution operator \( E \) for the first example (4.29) - (4.36) in section 4.

The genuine Lie semigroup of actions generated by this example in (4.29) - (4.36) will now be given by the autonomous evolution operator \( E_A \), which corresponds to \( E \) above, according to (5.9), namely, see also (6.4)

\[ E_A(s)(t, y) = (t + s, E(t, t + s)(y)) \]
\[ t, \ s \in [0, \infty), \ y \in M, \ 1 + 4\sqrt{ty} \geq 0 \]

The genuine Lie semigroup actions generated by the example in (4.43) - (4.49) can be obtained in a similar manner, provided that one can solve in \( y_0 \in \mathbb{R}^l \) the corresponding algebraic equations

\[ y = (1 - g(t))y_0 + g(t)f(y_0) \]

for given \( (t, y) \in (0, \infty) \times \mathbb{R}^l \).

**Milder Singularities.** We show with an example that the condition

\[ (\text{SING}) \quad H(t, .) \notin \mathcal{Diff}^\infty(M), \quad t \in (0, \infty) \]

is not necessary, in order to obtain genuine Lie group actions by using the above method. Indeed, let \( M = \mathbb{R} \) and \( f \in \mathcal{C}^\infty(M, M) \) be given by

\[ f(y) = \frac{1}{y^2 + 1}, \quad y \in M \]

We define \( H \in \mathcal{C}^0([0, \infty) \times M, M) \cap \mathcal{C}^\infty((0, \infty) \times M, M) \) such that
(6.12) \[ H(t, y) = (1 - \sqrt{t})y + \sqrt{t}f(y) = (1 - \sqrt{t})y + \frac{\sqrt{t}}{y^2 + 1}, \]
\[ t \in [0, \infty), \; y \in M \]

Then it follows that

\[ H(1, .) = f \notin Diff^\infty(M) \]
(6.13)

\[ H(t, .) \in Diff^\infty(M) \quad \text{for} \; t \in [0, 4/9) \cup (4, \infty) \]

However, it is obvious that the above procedure applied to the examples in section 4, is equally applicable to (6.11) - (6.13), and again, it will lead to genuine Lie semigroup actions.

7. Singularity, Continuity, Smoothness and Domains of Action

In view of sections 4 - 6, one possible method to obtain genuine Lie semigroups is that given by the evolution operators \( E_A \) of the autonomous singular ODEs, which are associated in the above standard manner with the non-autonomous singular ODEs in (4.37) - (4.40).

In other words, such genuine Lie semigroup actions on suitable subsets \( \tilde{M} \) in Euclidean spaces are given by mappings

\[ (7.1) \quad E_A \in (C^0([0, \infty) \times \tilde{M}, \tilde{M}) \cap C^\infty((0, \infty) \times \tilde{M}, \tilde{M})) \backslash C^1([0, \infty) \times \tilde{M}, \tilde{M}) \]

With respect to the domains of action of such genuine Lie semigroups, as constructed in sections 4 - 6, we have to note the following. We have started with certain open subsets \( M \) in Euclidean spaces, and see for instance (4.29), with singular actions

\[ (7.2) \quad H \in (C^0([0, \infty) \times M, M) \cap C^\infty((0, \infty) \times M, M)) \backslash C^1([0, \infty) \times M, M) \]

which clearly were not any kind of semigroup actions on the respective open subsets \( M \).
Then, we found non-autonomous singular ODEs which were satisfied by these singular actions \( H \), and associated with them in the standard manner autonomous singular ODEs.

Finally, the evolution operators \( E_A \) of these associated autonomous singular ODEs gave us the genuine Lie semigroup actions.

However, such an \( E_A \) is no longer acting on \( M \), but on the set with one dimension higher, namely

\[
(7.3) \quad \tilde{M} = [0, \infty) \times M
\]

Thus we are led to the Open Problem ( GLS ) formulated at the beginning of section 4.

8. Remark on Singularities

Let us note that in order to obtain genuine Lie semigroups, we can use milder forms of singularities than those in section 4. For instance, instead of \( H \) given by (4.29), let us consider it defined as follows

\[
(8.1) \quad H(t, y) = y + ty^2, \quad t \in \mathbb{R}, \quad y \in M = \mathbb{R}
\]

Then (4.30) becomes replaced with

\[
H(0, y) = y, \quad y \in M
\]

\[
(8.2) \quad H(t, .) \notin Diff^\infty(M), \quad t \in \mathbb{R} \setminus \{0\}
\]

\[
H \in C^\infty(\mathbb{R} \times M, M)
\]

while the action in (4.29) limited to \( t \in [0, \infty) \), extends now to the following one defined for all \( t \in \mathbb{R} \), namely

\[
(8.3) \quad \mathbb{R} \times M \ni (t, y) \mapsto H(t, y) \in M
\]

However, this extended action still cannot be part of a group or local
group action on $M$, since in view of (8.2), $H(t, \cdot)$, with $t \in \mathbb{R} \setminus \{0\}$, is not a $\mathcal{C}^\infty$-smooth diffeomorphism of $M$.

Proceeding now in a manner similar with that in (4.31) - (4.36), it follows that $H$ in (8.1), as a function of $t \in \mathbb{R}$, and for any given fixed $y \in M$, will satisfy the ODE

$$
\frac{\partial}{\partial t} Y(t) = \frac{2Y(t)^2}{1 + 2tY(t) + \sqrt{1 + 4tY(t)}},
$$

(8.4) 

$$
t \in \mathbb{R}, \ 1 + 4tY(t) \geq 0
$$

if

(8.5) 

$$
1 + 2ty \geq 0
$$

while it will satisfy the ODE

$$
\frac{\partial}{\partial t} Y(t) = \frac{1 + 2tY(t) + \sqrt{1 + 4tY(t)}}{2t^2},
$$

(8.6) 

$$
t \in \mathbb{R} \setminus \{0\}, \ 1 + 4tY(t) \geq 0
$$

if

(8.7) 

$$
1 + 2ty \leq 0
$$

Furthermore, both these ODEs will for $H$ specified above be associated with the initial condition

(8.8) 

$$
\lim_{t \to 0} Y(t) = y \in M
$$

We can note in the above example (8.1) - (8.8) that in the case of (8.5), the corresponding ODE in (8.4) which is satisfied by $H$, seen as a function of $t \in \mathbb{R}$, and with $y \in M$ fixed, is \textit{not} singular at $t = 0$. Nevertheless, the respective solution $H$, for $t \in \mathbb{R} \setminus \{0\}$, still \textit{cannot} be part of a group or local group of transformations on $M$, in view of (8.2).
9. Evolution PDEs and Genuine Lie Semigroups

In this section, as a possible alternative to the method in section 4, we give an indication about another way one may try to generate genuine Lie semigroup actions. This alternative way is suggested by the well established literature on solving initial value problems for evolution PDEs through the associated semigroups of operators acting on the respective initial values. The early basic result in the case of linear evolution PDEs is the celebrated Hille-Yoshida theorem, which was followed by a rather large body of more recent results, including non-linear developments, see for instance Pazy and the references cited there.

The important point to note here is that, in general, the solutions of the initial value problems for evolution PDEs will be given by semigroups, rather than groups, of such operators. A well known class of evolution PDEs for which, typically, one can only obtain such semigroups of operators is that of parabolic equations.

The idea suggested in this section is to try to use such semigroups acting on initial values, in order to generate the genuine Lie semigroups which are the object of our interest in this paper.

For simplicity, we shall again consider the case when $M = \mathbb{R}^l$ is the open set on which we want to define a genuine Lie semigroup action.

Let us therefore take any evolution PDE of the form

\begin{equation}
(9.1) \quad D_t U(t, x) = T(x, D)U(t, x), \quad t \in [0, \infty), \ x \in M
\end{equation}

where $U \in C^\infty([0, \infty) \times M, \mathbb{R})$ is the unknown function, while $T(x, D)$ is a partial differential operator in $x$ alone.

Associated with (9.1), we consider the initial value problem

\begin{equation}
(9.2) \quad U(0, x) = f(x), \quad x \in M
\end{equation}

where $f$ belongs to a suitable class of functions, namely
We shall assume that there exists a semigroup of operators acting on the typically *infinite* dimensional vector space $F(M)$, namely

\[ (9.4) \quad [0, \infty) \ni t \mapsto E(t) : F(M) \to F(M) \]

such that, given any $f \in F(M)$, if we define

\[ (9.5) \quad U(t, x) = (E(t)f)(x), \quad t \in [0, \infty), \; x \in M \]

then $U$ is a solution of (9.1), (9.2).

So far, we have been moving within the well established framework of the mentioned literature, provided that we work with suitable evolution equations (9.1) and initial values (9.2).

Now the idea which is the subject of this section is to make the following further assumption. Namely, suppose that there exists a family of functions

\[ (9.6) \quad V_{a,b} \in F(M), \quad a \in M, \; b \in B \]

where $B \subseteq \mathbb{R}^k$ is a suitable open subset, such that

\[ (9.7) \quad E(t)V_{a,b} = V_{\alpha(t,a,b),\beta(t,a,b)}, \quad t \in [0, \infty), \; a \in M, \; b \in B \]

for appropriate functions

\[ (9.8) \quad \alpha \in C^\infty([0, \infty) \times M \times B, M), \; \beta \in C^\infty([0, \infty) \times M \times B, B) \]

In this case, the assumed semigroup property

\[ (9.9) \quad E(s)E(t) = E(t + s), \quad t, s \in [0, \infty) \]

together with (9.6), (9.7) will result in

40
\[\alpha(t+s,a,b) = \alpha(s,\alpha(t,a,b),\beta(t,a,b))\] (9.10)
\[\beta(t+s,a,b) = \beta(s,\alpha(t,a,b),\beta(t,a,b))\]
for \(t,s \in [0,\infty), \ a \in M, \ b \in B.\)

It follows that if, for instance
\[\beta(t,a,b) = b, \ t \in [0,\infty), \ a \in M, \ b \in B\] (9.11)
then (9.10) yields
\[\alpha(t+s,a,b) = \alpha(s,\alpha(t,a,b),b), \ t,s \in [0,\infty), \ a \in M, \ b \in B\] (9.12)
hence for any fixed \(b \in B,\) the mapping
\[\beta(t,a,b) = b, \ t \in [0,\infty), \ a \in M, \ b \in B\] (9.13)

is a semigroup action on \(M,\) and it is a priori not impossible that it may indeed be a genuine Lie semigroup action.

It is easy to see that the assumptions (9.6) - (9.8) can be satisfied by a large variety of soliton solutions, for instance. Indeed, for simplicity, let us consider the one dimensional case, when \(M = \mathbb{R},\) and assume that (9.1) has a soliton solution
\[U(t,x) = W(x - ct), \ t \in [0,\infty), \ x \in M\] (9.14)
for \(c \in \mathbb{R}.\) In this case (9.5) becomes
\[(E(t)W)(x) = W(x - ct), \ t \in [0,\infty), \ x \in M\] (9.15)
For further more detailed illustration, let us consider the well known Burgers equation
\[U_t(t,x) + U(t,x)U_x(t,x) = \mu U_{xx}(t,x), \ t,x \in \mathbb{R}\] (9.16)
where \(\mu \in (0,\infty).\) This equation has a soliton solution given by
\[(9.17) \quad U(t, x) = c - \sqrt{c^2 + d} \tanh \left( \frac{\sqrt{c^2 + d}}{2\mu} (x - x_0 - ct) \right), \quad t, x \in \mathbb{R} \]

where \(x_0, c, d \in \mathbb{R}\) are arbitrary fixed, and \(c^2 + d > 0\). For this soliton it is clear that (9.6) - (9.8), (9.11) - (9.13) are satisfied, if we take

\[(9.18) \quad a = x_0 \in M = \mathbb{R}, \quad b = (c, d) \in B \subset \mathbb{R}^2 \]

where \(B = \{(c, d) \in \mathbb{R}^2 \mid c^2 + d > 0\}\), in which case

\[(9.19) \quad \alpha(t, a, b) = x_0 + ct \quad \beta(t, a, b) = b \quad \text{for } t \in [0, \infty), \ a \in M, \ b \in B. \]

10. Other Instances of Semigroups of Actions

In a private correspondence, P J Olver mentioned further instances in which semigroups of actions appear in a natural way. Not all of them, however, need be genuine Lie semigroup actions.

A first such example happens in the framework of (1.1), when a certain subset \(S \subset M\) is given, and we are interested in the set of actions which invariate it, namely

\[ G_S = \{ g \in G \mid gS \subseteq S \} \]

For instance, let

\[ S = (-1, 1) \times \mathbb{R} \subset \mathbb{R}^2 = M \]

and \(G = (\ (0, \infty), \ . \ )\) be the usual Lie group, which is supposed to act on \(M\) according to

\[ G \times M \ni (g, (x, y)) \mapsto (gx, y) \in M \]
Then clearly

\[ G_S = (0, 1] \]

which is a semigroup action. However, \( G_S \) is not a genuine Lie semigroup action, since it is a subsemigroup of the Lie group action \( G \).

A second example is given by ODEs with inequality constraints. Let us, for instance, consider the differential equation

\[ \frac{d}{dx} U(x) = 0, \quad x \in \mathbb{R} \]

with the inequality constraint

\[ U(x) > 0, \quad x \in \mathbb{R} \]

Then the Lie group actions

\( (x, u) \mapsto (x, u + c) \)

are symmetries, only if \( c \geq 0 \). Needless to say, in view of a large class of applications, such as control theory or differential games, for instance, where ODEs with inequality constraints play a crucial role, the study of semigroups of symmetries of such equations can present a special interest.

Another class of examples, this time related to PDEs, is given by generalized symmetries, see Olver [1, chap. 5]. Indeed, the evolution PDEs governing the flow of a generalized symmetry often only define a semigroup. A good example is the symmetry

\[ V = U_{xx} \partial U \]

which is a symmetry of any linear constant coefficient PDE. Its flow is

\[ U_t = U_{xx} \]

43
Appendix

In Rosinger [1] the *global* approach to arbitrary Lie group actions on smooth functions was introduced and developed. And it was shown that for such a purpose, the use of a *parametric* representation of functions upon which the Lie groups are supposed to act is particularly appropriate. Here we present the essentials in this regard, as needed in this paper.

Let us consider linear or nonlinear PDEs of the general form

\[(A.1) \quad T(x, D) U(x) = 0, \quad x \in \Omega\]

where \(\Omega\) is an open subset in \(\mathbb{R}^n\).

Lie group theory deals, among others, with those symmetries of solutions \(U : \Omega \rightarrow \mathbb{R}\) of any given PDE in (A.1) which lead to other solutions of the same equation. For that purpose, one takes \(M = \Omega \times \mathbb{R}\) and finds the corresponding Lie groups \(G\) and their actions on \(M\), namely

\[(A.2) \quad G \times M \ni (g, (x, u)) \mapsto g(x, u) = (g_1(x, u), g_2(x, u)) \in M\]

where

\[(A.3) \quad G \times M \ni (g, (x, u)) \mapsto g_1(x, u) \in \Omega
g \times M \ni (g, (x, u)) \mapsto g_2(x, u) \in \mathbb{R}\]

actions which, when extended to the solutions \(U \in C^\infty(\Omega, \mathbb{R})\) of the PDE in (A.1), will transform them into solutions of the same equation.

The well known difficulty here related to *global* actions of (A.2) on functions in \(C^\infty(\Omega, \mathbb{R})\) is the following.

In general, the Lie group actions (A.2) defined on the Euclidean domains \(M\) cannot so easily be extended to act on the functions \(U : \Omega \rightarrow \mathbb{R}\) as well. And the only problem here is that such extended actions *cannot* be defined so easily *globally*, that is, for the functions
$U : \Omega \rightarrow \mathbb{R}$ considered on the *whole* of their domain of definition $\Omega$, Rosinger [1, chapters 1,2]. The reason for that is rather simple, namely, the lack of invertibility of certain functions involved, Rosinger [1, pp. 14,15].

For further clarity about the mentioned difficulty facing global Lie group actions on functions, we recall here a simple example, namely, the rotation in plane of a parabola.

Let us take the function $U : \Omega \rightarrow \mathbb{R}$ given by

$$U(x) = x^2, \quad x \in \Omega = \mathbb{R}$$

and let us consider the Lie group $G$ on $M = \Omega \times \mathbb{R} = \mathbb{R}^2$ given by the rotations of the plane around the origin $(0,0) \in \mathbb{R}^2$. Then it is obvious that, unless it is an integer multiple of $\pi$, every such rotation, when applied to all of the parabola, will turn it into a curve in plane which is *no longer* the graph of any function in $V : \Omega \rightarrow \mathbb{R}$.

Of course, bounded parts of the parabola can be rotated with sufficiently small angles, and one again obtains the graph of a function.

In other words, arbitrary Lie group actions (A.2) *cannot* be extended to actions

$$G \times C^\infty(\Omega, \mathbb{R}) \rightarrow C^\infty(\Omega, \mathbb{R})$$

This difficulty can, however, be easily overcome by the use of parametric representation of the respective functions $U : \Omega \rightarrow \mathbb{R}$, Rosinger [1, chapters 3-5]. In this way Lie group actions (A.2) can act *globally* on the functions $U : \Omega \rightarrow \mathbb{R}$ which are solutions of the rather general type of PDEs in (A.1).

For that purpose, we proceed as follows. Given any smooth function

(A.4) \[ U : \Omega \rightarrow \mathbb{R} \]

we associate with it its *graph*
\( \gamma_U = \{ (x, U(x)) \mid x \in \Omega \} \subseteq M = \Omega \times \mathbb{R} \)

Then by definition, a *parametric* representation of the function \( U \) is given by any smooth function

\[ V : \Lambda \longrightarrow M \]

where \( \Lambda \subseteq \mathbb{R}^n \) is nonvoid and open, such that

\[ V(\Lambda) = \gamma_U \]

We note the following well known *advantage* of such parametric representations. Namely, the set of functions in (A.6) is *larger* than that in (A.4). In other words, not every function \( V \) in (A.6) is the parametric representation of a function \( U \) in (A.4). For instance, a nontrivially rotated parabola in the plane can easily be written as a function in (A.6), but not as a function in (A.4).

Therefore, we denote by

\[ C_\infty^n(M) \]

the set of all smooth functions in (A.6), and call them *n-dimensional parametric representations in* \( M \).

Clearly, we have the following embedding which associates with each function \( U \) in (A.4) its *canonical* parametric representation \( V_U \) in (A.6), (A.7), namely

\[ C_\infty^\infty(M, \mathbb{R}) \ni U \longmapsto V_U \in C_\infty^n(M) \]

where for \( U : \Omega \longrightarrow \mathbb{R} \), we define \( V_U : \Omega \longrightarrow M \) by \( V_U(x) = (x, U(x)) \), with \( x \in \Omega \). It follows that with the notation in (A.6), we have in this particular case \( \Lambda = \Omega \), therefore (A.7) holds, which means that indeed \( V_U \in C_\infty^n(M) \).

The important property of parametric representations is that for every Lie group \( G \) acting on \( M \), see (A.2), one can naturally define the
global Lie group actions on each of the functions in $C_*^\infty(M)$, namely

\[(A.10)\quad G \times C_*^\infty(M) \longrightarrow C_*^\infty(M)\]

as follows. Given $g \in G$ and $V : \Lambda \longrightarrow M$ in $C_*^\infty(M)$, we define

\[(A.11)\quad gV = g \circ V\]

where in the right hand term, $g$ denotes the mapping, see (A.2)

\[(A.12)\quad g : M \ni (x, u) \longmapsto g(x, u) \in M\]

while $\circ$ in (A.11) is the usual composition of mappings.

In other words, we define the action $gV$ in (A.11) by the commutative diagram

\[\begin{array}{c}
\Lambda \\
\downarrow \\
gV
\end{array} \xrightarrow{V} \begin{array}{c} M \\
\downarrow \\
g
\end{array} \xrightarrow{g} \begin{array}{c} M \\
\downarrow \\
gV
\end{array}\]

\[(A.13)\]

And then, in view of (A.9), the simple construction in (A.13) allows the action (A.2) of every Lie group on $M = \Omega \times \mathbb{R}$ to be extended globally to every smooth function $U : \Omega \longrightarrow \mathbb{R}$.

**Remark A**

From the point of view of genuine Lie semigroups, the essential feature of the definition of action on functions in (A.11), (A.13) is that it is valid not only for Lie group elements $g$, which therefore generate bijections (A.12), thus elements of $\text{Diff}^\infty(M, M)$. Indeed, (A.11), (A.13) make also sense for all smooth mappings in the far larger $C^\infty(M, M)$, thus for mappings generated by $g$ which need no longer be elements of Lie groups, and instead can belong to genuine
Lie semigroups as well.

The essence of the above definition (A.10) - (A.13) of global action on functions is very simple when seen in categorial terms, that is, in terms of most general properties of the usual composition of functions. Indeed, initially, the functions of interest on which the actions are supposed to be defined are, see (A.4)

\[(A.14)\quad U : \Omega \rightarrow \mathbb{R}\]

while the actions operate according to, see (A.2), (A.3)

\[(A.15)\quad M \xrightarrow{g} M, \quad \text{with} \quad g \in G\]

where \(M = \Omega \times \mathbb{R}\). In this way, the extension of the actions (A.15) to functions (A.14) leads to having to deal with the inversion of certain functions which may fail to exist, Rosinger [1, pp. 14,15].

However, if the functions (A.14) are embedded into the larger set of functions, see (A.6)

\[(A.16)\quad V : \Lambda \rightarrow M\]

where \(\Lambda \subseteq \mathbb{R}^n\) is nonvoid and open, and this embedding is done according to, see (A.9)

\[(A.17)\quad \mathcal{C}^\infty(M, \mathbb{R}) \ni U \mapsto V_U \in \mathcal{C}^\infty_n(M)\]

then the mappings (A.15) and (A.16) can trivially be composed with one another, thus yielding (A.11), (A.13). And obviously, such a composition of mappings does not require the mappings \(g\) in (A.15) to be bijections, that is, to belong to \(\text{Diff}^\infty(M, M)\). Instead, they can belong to the far larger \(\mathcal{C}^\infty(M, M)\).

It appears that the above definition (A.10) - (A.13) of a global action on all smooth functions by arbitrary Lie groups was presented for the first time in Rosinger [1, chapters 1-5], based on the above simple de-
vice of parametric representation of functions.

References

[1] Coddington E A, Levinson N : Theory of Ordinary Differential Equations. McGraw-Hill, New York, 1955

[2] Hilgert, J, Hofmann K H, Lawson, J D : Lie Groups, Convex Cones, and Semigroups. Oxford Univ. Press, Oxford, 1989

[3] Oberguggenberger M B, Rosinger E E : Solution of Continuous Nonlinear PDEs through Order Completion. North-Holland Mathematics Studies, Vol. 181. North-Holland, Amsterdam, 1994

[4] Olver P J [1] : Applications of Lie Groups to Differential Equations. Springer, New York, 1986

[5] Olver P J [2] : Equivalence, Invariants and Symmetry. Cambridge Univ. Press, 1995

[6] Olver P J [3] : Nonassociative local Lie groups. J. Lie Theory, Vol. 6, 1996, 23-51

[7] Rosinger E E [1] : Parametric Lie Group Actions on Global Generalized Solutions of Nonlinear PDEs, including a Solution to Hilbert’s Fifth Problem, (234 pages). Kluwer, Dordrecht, 1998

[8] Rosinger E E [2] : Arbitrary Global Lie Group Actions on Generalized Solutions of Nonlinear PDEs and an Answer to Hilbert’s Fifth Problem. In (Eds. Grosser M, Hörmann G, Kunzinger M, Oberguggenberger M B) Nonlinear Theory of Generalized Functions, 251-265, Research Notes in Mathematics, Chapman & Hall / CRC, London, New York, 1999

[9] Rosinger E E [3] : Nonlinear Equivalence, Reduction of PDEs to ODEs, and Fast Convergent Numerical Methods. Research Notes in Mathematics, Vol. 77, Pitman, Boston, 1982
[10] Rosinger E E, Rudolph M : Group invariance of global generalised solutions of nonlinear PDEs : A Dedekind order completion method. Lie Groups and their Applications, Vol. 1, No. 1, July-August 1994, 203-215

[11] Rosinger E E, Walus E Y [1] : Group invariance of generalized solutions obtained through the algebraic method. Nonlinearity, Vol. 7, 1994, 837-859

[12] Rosinger E E, Walus E Y [2] : Group invariance of global generalised solutions of nonlinear PDEs in nowhere dense algebras. Lie Groups and their Applications, Vol. 1, No. 1, July-August 1994, 216-225