Recurrence relations for symplectic realization of (quasi)-Poisson structures

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Abstract
It is known that any Poisson manifold can be embedded into a bigger space which admits a description in terms of a global symplectic structure. Such a procedure is known as a symplectic realization and has a number of important applications like the quantization of the original Poisson manifold. In the present paper we extend the above idea to the case of quasi-Poisson structures which should not necessarily satisfy the Jacobi identity. For any given quasi-Poisson structure $\Theta$ we provide a closed recursive formula for local embedding functions and Darboux coordinates. Our construction is illustrated for the examples of the constant $R$-flux algebra, quasi-Poisson structure isomorphic to the commutator algebra of imaginary octonions and the non-geometric M-theory $R$-flux background. In all cases we derive explicit formulas for the symplectic realization and the corresponding expression for Darboux coordinates.

Keywords: symplectic realizations, non-associative algebras, non-geometric backgrounds

1. Introduction

Symplectic realizations are an important mathematical tool for the investigation of Poisson and quasi-Poisson manifolds with a number of important applications, from the study of classical dynamics [1] and the algebra of symmetries [2] to a quantization; see, e.g. [1, 3–5]. To be more precise let us recall that a symplectic realization of a Poisson structure $\omega$ on a manifold $M$ is a symplectic manifold $(S, \Omega)$ together with a surjective subsersion $p : S \rightarrow M$ which preserves the Poisson structures: $p_* \Omega^{-1} = \omega$. It is a fundamental result in Poisson geometry that any Poisson manifold admits a symplectic realization. The original local construction
for $M = \mathbb{R}^d$ is due to [6]; it proceeds by taking $S = T^*M$ to be the phase space of $M$, with the canonical projection $p : T^*M \to M$, and $\Omega$ to be the integrated pullback of the canonical symplectic structure $dp_j \wedge dx^i$ on $T^*M$ by the flow of the vector field $\omega^j(x) p_i \partial_j$, where $(x, p) \in T^*M = \mathbb{R}^d \times (\mathbb{R}^d)^*$. The early global constructions based on integrating symplectic groupoids are due to [7–9]. The extension to almost symplectic realizations of twisted Poisson structures is established globally by [10].

In the previous work [1] we have proposed a symplectic realization for the monopole algebra, $\{\pi_i, \pi_j\} = e \varepsilon_{ijk} B^k(\vec{x})$, where $\pi_i$ are the covariant momenta and $B^k$ is the magnetic field, which can be produced by the magnetic charges, i.e. in general, $\vec{\nabla} \cdot \vec{B} \neq 0$. The latter means violation of the Jacobi identity by the brackets $\{\pi_i, \pi_j\}$, which however can be restored after introducing auxiliary degrees of freedom. The obtained formulation was used for the investigation of classical dynamics and quantization of the electric charge in a field of monopole. The aim of the present work is to extend the construction of [1] to the case of an arbitrary quasi-Poisson structure $\Theta = \frac{1}{4} \Theta^{ij}(x) \partial_i \wedge \partial_j$ on a manifold $M$ of dimension $d$.

The paper is organized as follows. In section 2 we address the problem of the construction of the symplectic realization of a Poisson structure which satisfies the Jacobi identity. Then in section 3 we extend the obtained results to the case of quasi-Poisson structures. Our main technical result is explicit recursive formulas for the embedding coordinates. Examples are discussed in section 4. In particular we construct the symplectic realization of the quasi-Poisson structure isomorphic to the commutator algebra of octonions. In the appendix, for the convenience of the reader, we briefly review the algebra of octonions. The possible applications of the proposed construction to the $L_\infty$ algebras and non-associative deformation quantization are discussed in the conclusion.

Throughout the text we will use different notations for the brackets: $\{\cdot, \cdot\}_p$ denotes the arbitrary Poisson (satisfying Jacobi identity) bracket, $\{\cdot, \cdot\}$ stands for the canonical Poisson bracket, and $\{\cdot, \cdot\}_Q$ indicates the quasi-Poisson bracket, when the Jacobi identity can be violated.

2. Symplectic embeddings of Poisson structures

As a warm-up we start with a more familiar case of symplectic realizations of Poisson manifolds. Note that for the case of a two-dimensional Poisson manifold this problem was solved in [11]. Some elements of the construction in this paper first appeared in [12, 13].

Suppose that coordinates $x^i, i = 1, ..., d$ satisfy the algebra of Poisson brackets

$$\{x^i, x^j\}_p = \alpha \omega^{ij}(x), \quad (2.1)$$

where $\alpha$ is a deformation parameter and $\omega^{ij}(x)$ is a Poisson bi-vector. Our aim here is to describe a recursive procedure of the construction of a $2d$-dimensional symplectic manifold $S$ with coordinates $(x^i, \tilde{x}_j)$, satisfying the algebra

$$\{x^i, x^j\}_p = \alpha \omega^{ij}(x),$$

$$\{x^i, \tilde{x}_j\}_p = \delta^i_j(x, \tilde{x}) + \alpha \delta^{(1)}(x, \tilde{x}) + \mathcal{O}(\alpha^2),$$

$$\{\tilde{x}_i, \tilde{x}_j\}_p = \alpha \varphi_{ij}(x, \tilde{x}), \quad (2.2)$$

where $\delta^i_j$ is a Kronecker delta and the functions $\delta^i_j(x, \tilde{x})$ and $\varphi_{ij}(x, \tilde{x})$ should be found from the Jacobi identity for the complete algebra (2.2). Note that the structure defined in (2.2) is symplectic in the sense of a formal decomposition in $\alpha$, since it starts from the canonical Poisson brackets.
To solve the above problem we will employ the Darboux coordinates on the symplectic manifold $\mathcal{S}$. Let the coordinates $(y^i, \pi_j)$, $i = 1, \ldots, d$ satisfy the canonical Poisson brackets,
\[
\{y^i, \pi_j\} = \delta^i_j, \quad \{y^i, y^j\} = \{\pi_i, \pi_j\} = 0. \tag{2.3}
\]
We will look at the expression for the original coordinates $x^i$ in terms of the Darboux coordinates $(y^i, \pi_j)$ in the form of a generalized Bopp shift,
\[
x^i(y, \pi) = y^i + \sum_{n=1}^{\infty} \alpha^n \Gamma^{(n)}(y) \pi^n, \tag{2.4}
\]
where for simplicity we use the abbreviation $\Gamma^{(n)}(y) = \Gamma^{y_1 \ldots y_n}(y)$ $\pi_j \ldots \pi_{j_n}$. By construction the coefficient functions $\Gamma^{y_1 \ldots y_n}(y)$ should be symmetric in the last $n$ indices, since they are contracted with the commuting vectors $\pi_j$. Also, it is better to keep the totally symmetric part of $\Gamma^{y_1 \ldots y_n}(y)$ vanishing. This requirement is necessary for the stability of unity condition, $f \star 1 = 1 \star f = f$, for the corresponding star product $\star$ after quantization; see [12] for more details. The latter implies that under the permutations of indices the tensor $\Gamma^{y_1 \ldots y_n}(y)$ transforms according to the Young tableau:
\[
\begin{array}{c}
\bar{j}_1 \\
\bar{j}_2 \\
\ddots \\
\bar{j}_n
\end{array}
\] \tag{2.5}
To find the coefficient functions $\Gamma$ we substitute (2.4) in (2.1) and obtain the equation
\[
\left\{ y^i + \sum_{n=1}^{\infty} \alpha^n \Gamma^{(n)}(y) \pi^n, y^j + \sum_{n=1}^{\infty} \alpha^n \Gamma^{(n)}(y) \pi^n \right\} = \alpha \omega^{ij} \left( y^i + \sum_{n=1}^{\infty} \alpha^n \Gamma^{(n)}(y) \pi^n \right). \tag{2.6}
\]
Comparing the coefficients on the left- and right-hand sides of (2.6) in each order in $\alpha$, one obtains algebraic equations on the coefficients $\Gamma^{(n)}(y)$ in terms of $\omega^{ij}$ and lower order coefficients $\Gamma^{(m)}(y)$, $m < n$. A solution of equation (2.6) can be formulated as follows:

**Theorem 1.** Provided that $\Gamma^{y_i}(y) = -\omega^{y_i}(y)/2$, and the recurrence relation for the definition of the coefficient functions $\Gamma^{y_1 \ldots y_n}(y)$, $m \geq 2$ is given by,
\[
\Gamma^{y_1 \ldots y_n}(y) = -\frac{1}{m(m+1)} \left( G^{y_1 \ldots y_{m-1} y_m} + G^{y_1 \ldots y_{m-2} y_{m-1} y_m} + \ldots + G^{y_1 \ldots y_{m-1} y_{m-2}} \right), \tag{2.7}
\]
where the tensors $G^{y_1 \ldots y_n}(y)$ are defined by the relation,
\[
G^{y_1 \ldots y_n}(y) \pi_{j_1} \ldots \pi_{j_n} = \frac{1}{(m-1)!} \left[ \frac{d^{m-1}}{d\alpha^{m-1}} \omega^{ij}(x(y, \pi)) \right]_{\alpha = 0} - \sum_{k=1}^{m-1} \left\{ \Gamma^{y_{m-k}}(y) \pi^{m-k}, \Gamma^{y_i}(y) \pi^k \right\}, \tag{2.8}
\]
then the expression for $x^i(y, \pi)$ defined in (2.4) satisfies equation (2.6).
The theorem will be proved by induction. Before starting the proof we note that once expression (2.4) for the original coordinates $x'$ in terms of the Darboux coordinates is found, for the double coordinates $\tilde{x}_i$ we may write

$$\tilde{x}_i = \pi_i - \alpha j_i(y, \pi),$$

(2.9)

where $j_i(y, \pi, \alpha)$ is an arbitrary differentiable function. Calculating the Poisson brackets between coordinates (2.4) and double coordinates (2.9) one finds functions $\delta^j_i(y, \pi)$ and $\pi_j(y, \pi)$ which together with $\omega^j_i(x)$ determine the symplectic structure (2.3). Using the inverse expressions $y^i = y_i(x, \tilde{x})$ and $\pi_i = \pi_i(x, \tilde{x})$ we calculate the functions $\delta^j_i(x, \tilde{x})$ and $\pi_j(x, \tilde{x})$ in (2.2):

$$\delta^j_i(x, \tilde{x}) = \delta^j_i + \alpha \left( \tilde{\partial}_i j_j - \frac{1}{2} \partial_j \omega^j_i \tilde{x}_j \right) + O(\alpha^2),$$

(2.10)

$$\pi_j(x, \tilde{x}) = \alpha (\tilde{\partial}_i j_j - \partial_j j_i) + O(\alpha^2),$$

where $\tilde{\partial}_i = \partial_i / \partial \tilde{x}_i$. Setting for simplicity $j_i(y, \pi, \alpha) = 0$, which implies that $\tilde{x}_i = \pi_i$, one finds $\delta^j_i(x, \tilde{x}) = \delta^j_i - \alpha \partial_j \omega^j_i / 2 + O(\alpha^2)$, and $\pi_j(x, \tilde{x}) = 0$.

In the first order in $\alpha$ we have from (2.6):

$$\Gamma^{[j]} := \Gamma^{[j]} - \Gamma^{j} = \omega^{[j]},$$

with a solution $\Gamma^{[j]} = -\omega^{[j]} / 2 + s^{[j]}$, where $s^{[j]}$ is an arbitrary symmetric matrix. We chose $s^{[j]} = 0$, according to (2.5), such that $\Gamma^{[j]} = -\omega^{[j]}$. In the order $O(\alpha^2)$, the equation (2.6) results in

$$2 \Gamma^{[j]} k \delta^k = G^{jk} \delta^k,$$

(2.11)

$$G^{jk} = -\frac{1}{2} \omega^{jk} \partial_j \omega^l + \frac{1}{4} \omega^{jk} \partial_l \omega^m - \frac{1}{4} \partial_j \omega^m \partial_k \omega^m.$$

The symmetry of $\Gamma^{jk}$ in the last two indices implies the identity $\Gamma^{[j]} k \delta^k \equiv 0$, which in turn means the consistency condition for the solution of (2.11),

$$G^{jk} + G^{kj} + G^{ki} = 0.$$  

(2.12)

We call this condition cyclicity and it is satisfied due to the Jacobi identity

$$\{x^i, \{x^i, x^k\}_p\}_p + \{x^i, \{x^j, x^k\}_p\}_p + \{x^i, \{x^j, x^k\}_p\}_p = 0.$$  

(2.13)

A solution of equation (2.11) is given by

$$\Gamma^{jk} = -\frac{1}{6} (G^{jk} + G^{kj}) = \frac{1}{24} \omega^{km} \partial_m \omega^j + \frac{1}{24} \omega^{jm} \partial_m \omega^k.$$  

(2.14)

It is symmetric in $j$ and $k$ by the construction and using (2.12) one can make sure that it satisfies (2.11). The resulting expression for $x'$ becomes

$$x' = y' - \frac{\alpha}{2} \omega^j \pi_j + \frac{\alpha}{12} \omega^{km} \partial_m \omega^j \pi_k + O(\alpha^3).$$

Now let us discuss how to construct the solution in the higher orders. It is convenient to represent the right-hand side of (2.6) as

$$\omega^j_i = \omega^j_i + O(\alpha^{n+1}),$$

$$\omega^j = \omega^j(y), \quad \omega^j_1 = \omega^j + \alpha \partial \omega^j \Gamma_{jk} \delta^k.$$  

(2.15)
We also introduce corresponding notations for $x'$,
$$x' = x'_n + \mathcal{O} (\alpha^{n+1}), \quad x'_{n+1} = x'_n + \alpha^{n+1} \Gamma^{j_1 \ldots j_{n+1}} \pi_{j_1} \ldots \pi_{j_{n+1}}. \quad (2.16)$$

One may easily check that for any analytic function $f(x)$ one has
$$f' (x_n) = f_n + \mathcal{O} (\alpha^{n+1}). \quad (2.17)$$

Suppose that the expansion (2.4) is known up to the $n$th order, i.e. the equation
$$\{x'_n, x'_m\} = \alpha \omega^n_{n-1} + \mathcal{O} (\alpha^{n+1}) \quad \text{is true} \quad (2.18)$$
holds true. In order to construct the $(n + 1)$th order in the decomposition we have to solve the next order equation
$$\{x'_{n+1}, x'_{n+1}\} = \alpha \omega^n_{n+1} + \mathcal{O} (\alpha^{n+2}). \quad (2.19)$$

Using (2.16) we represent it in the form
$$\alpha^{n+1} (n + 1) \Gamma^{j_1 \ldots j_n} \pi_{j_1} \ldots \pi_{j_n} = G^n_{n+1} + \mathcal{O} (\alpha^{n+2}), \quad (2.20)$$

where
$$G^n_{n+1} = \alpha \omega^n_{n+1} - \{x'_n, x'_n\} + \mathcal{O} (\alpha^{n+2}). \quad (2.21)$$

The above equation defines $G^n_{n+1}$ up to the terms $\mathcal{O} (\alpha^{n+2})$, and we do not include any higher-order terms in $G^n_{n+1}$. Taking into account (2.18), one writes
$$\alpha^{-(n+1)} G^n_{n+1} = \frac{1}{n!} \left[ \frac{d^n}{dx^n} \omega^n (x_n) \right]_{\alpha = 0} - \sum_{m=1}^n \left\{ \Gamma^{n+1-m} \pi_{n+1-m} \Gamma^m \pi^m \right\} \quad (2.22)$$

One can also represent it in the form
$$G^n_{n+1} = \alpha^{n+1} G^{j_1 \ldots j_n} (y) \pi_{j_1} \ldots \pi_{j_n} \quad \text{(2.23)}$$

where the coefficient functions $G^{j_1 \ldots j_n}$ are antisymmetric in first two indices and symmetric in the last $n$ by the construction. The formulas (2.22) and (2.23) imply (2.8) in the formulation of the theorem 1.

Thus, (2.19) implies the algebraic equation
$$(n + 1) \Gamma^{j_1 \ldots j_n} = G^{j_1 \ldots j_n}. \quad (2.24)$$

Like in the case of equation (2.11), the symmetry of the tensors $\Gamma^{j_1 \ldots j_n}$ in the last $n + 1$ indices yields the consistency condition on the right-hand side of (2.24), namely the cyclicity relation
$$G^{j_1 \ldots j_n} + G^{j_1 j_n \ldots j_{n+1}} + G^{j_2 j_3 \ldots j_n} = 0. \quad (2.25)$$

The condition (2.12) holds true as a consequence of the Jacobi identity (2.13). The following lemma shows that the same is valid for (2.25).

**Lemma 2.** The functions $G^{j_1 \ldots j_n}$ defined in equations (2.22) and (2.23) satisfy the cyclicity relation (2.25).

**Proof.** The Jacobi identity (2.13) can be written as
$$\{x', \omega^j (x)\}_p + \{x', \omega^k (x)\}_p + \{x', \omega^k (x)\}_p = 0. \quad (2.26)$$

Since $x' = x'_n + \mathcal{O} (\alpha^{n+1})$, and $\omega^j (x_n) = \omega^j_n + \mathcal{O} (\alpha^{n+1})$, the above equation implies...
\[ \{x^k_n, \omega^{ij}_n \} + \{x'_n, \omega^{ij}_n \} + \{x''_n, \omega^{ij}_n \} = \mathcal{O} \left( \alpha^{n+1} \right), \quad (2.27) \]

or equivalently,

\[ \{x^k_n, \alpha \omega^{ij}_n \} + \text{cycl.}(kij) = \mathcal{O} \left( \alpha^{n+2} \right). \quad (2.28) \]

Now using (2.21), one gets

\[ \{x^k_n, G^i_{n+1} \} + \{x'_n, x'_n \} + \text{cycl.}(kij) = \mathcal{O} \left( \alpha^{n+2} \right). \quad (2.29) \]

Since the Jacobi identity,

\[ \{x^k_n, \{x^i_n, x^j_n \} \} + \text{cycl.}(kij) = 0 \quad (2.30) \]

holds true at all orders of \( \alpha \), including the order \( \alpha^{n+1} \), from (2.29) one obtains

\[ \{x^k_n, G^i_{n+1} \} + \text{cycl.}(kij) = \mathcal{O} \left( \alpha^{n+2} \right). \quad (2.31) \]

meaning that

\[ \{x^k, G^{ij \ldots j_k}(y)\pi_{j_1} \ldots \pi_{j_n} \} + \text{cycl.}(kij) = 0. \quad (2.32) \]

Next one calculates the Poisson bracket, and uses the symmetry of \( G^{ij \ldots j_k} \) in the last \( n \) indices to prove the condition (2.25). The symmetry of \( G^{ij \ldots j_k} \) in the last \( n \) indices also implies that the cyclic condition holds for permutations of \((i,j,i_k)\) for any \( k = 1, \ldots, n \). □

As long as the consistency condition (2.25) is satisfied, the solution of the equation (2.24) is provided by the following:

**Lemma 3.** The tensors

\[ \Gamma^{ij \ldots j_k}_{n+1} = -\frac{1}{(n+1)(n+2)} \left( G^{ij \ldots j_k}_{n+1} + G^{ij j_k j_{k-1}} + \ldots + G^{j_k j_{k-1} \ldots j_i}_{n+1} \right) \quad (2.33) \]

are symmetric in the last \( n + 1 \) indices and satisfy the equation (2.24).

**Proof.** The symmetry follows by the construction. Now let us consider the right-hand side of (2.7) and calculate

\[ \Gamma^{ij \ldots j_k}_{n+1} = \Gamma^{j_k j_{k-1} \ldots j_i}_{n+1} \]

Since \( G^{ij \ldots j_k}_{n+1} \) is antisymmetric in the first two indices one has

\[ G^{ij j_k j_{k-1}}_{n+1} = G^{j_k j_{k-1} \ldots j_i}_{n+1} = -2G^{j_k j_{k-1} \ldots j_i}_{n+1}. \]

The cyclic condition (2.25) in \( i, j_1, j_2 \) implies

\[ G^{j_k j_{k-1} \ldots j_i}_{n+1} - G^{j_k j_{k-1} \ldots j_i}_{n+1} = -G^{j_k j_{k-1} \ldots j_i}_{n+1}. \]

The remaining \( n-1 \) combinations are treated similarly using (2.25), and the assertion follows immediately. □
This lemma implies that the tensors (2.33) are indeed the coefficient functions of the expansion (2.4) and the functions \(x'(y, \pi)\) satisfy the algebra of Poisson brackets (2.1).

### 3. Generalization to quasi-Poisson structures

Let us now consider an algebra of quasi-Poisson brackets

\[
\{x^i, x^j\}_Q = \alpha \Theta^{ij}(x), \tag{3.37}
\]

where an antisymmetric bi-vector \(\Theta^{ij}(x)\) should not necessarily satisfy the Jacobi identity,

\[
\Pi^{jk} = \frac{1}{3} \left( \Theta^{il} \partial_l \Theta^{jk} + \Theta^{kl} \partial_l \Theta^{ij} + \Theta^{jl} \partial_l \Theta^{ki} \right) \neq 0. \tag{3.38}
\]

Because of (3.38) we no longer require that the original algebra (3.37) should form a sub-algebra of a bigger symplectic algebra. Instead we formulate the problem in the following way. For a given algebra of quasi-Poisson brackets (3.37) to construct a 2d-dimensional symplectic manifold \(S\) with coordinates \((x^i, \tilde{x}^i)\), satisfying the Poisson brackets

\[
\begin{align*}
\{x^i, x^j\}_p &= \alpha \omega^{ij}(x, \tilde{x}), \\
\{x^i, \tilde{x}^j\}_p &= \delta^i_j(x, \tilde{x}) = \delta^i_j + \alpha \delta^{(1)i}_j(x, \tilde{x}) + O(\alpha^2), \\
\{\tilde{x}^i, \tilde{x}^j\}_p &= \omega^{ij}(x, \tilde{x}),
\end{align*} \tag{3.39}
\]

such that

- a restriction of (3.39) on the subspace generated by \(x^i\) should reproduce the original quasi-Poisson structure (3.37), i.e. \(\{x^i, x^j\}_p|_{\tilde{x}}=0 = \{x^i, x^j\}_Q\), meaning that

\[
\omega^{ij}(x, \tilde{x}) = \sum_{n=0}^{\infty} \alpha^n \Theta^{ij(n)}(x) \tilde{x}^n = \Theta^{ij}(x) + \alpha \Theta^{ij}(x) \tilde{x}_k + \ldots \tag{3.40}
\]

- If (3.38) vanishes and (3.37) becomes a Poisson bracket, then (3.39) should restore the symplectic realization of the Poisson structure previously defined in (2.2). In other words, in this case the tensors \(\Theta^{ij(n)}(x), n \geq 1\) should vanish, which in turn means that they should be proportional to the Jacobiator \(\Pi^{jk}\) and its derivatives.

By the definition \(\Theta^{ij(n)}(x)\) are antisymmetric in first two indices and symmetric in the last \(n\) indices. There is no \textit{a priori} symmetry between the first pair and the rest of the indices, but we impose the requirement that \(\Theta^{ij(k_1 \ldots k_n)}\) transforms under the permutations according to the following Young tableau.

\[
\begin{array}{c}
\begin{array}{cccc}
k_1 & k_2 & \ldots & k_n \\
i & \_ & \_ & \_ \\
j & \_ & \_ & \_ \\
\end{array}
\end{array} \tag{3.41}
\]

Like in the previous section the formulated problem will be solved by employing the Darboux coordinates \((y^i, \pi_i)\) and writing the generalized Bopp shift (2.4) together with \(\tilde{x}^i = \pi_i\). To define the coefficient functions \(\Gamma^{(n)}\) and \(\Theta^{ij(n)}\) of the series (2.4) and (3.40) correspondingly, one should solve the equation
\[
\left\{ y^i + \sum_{n=1}^{\infty} \alpha^n \Gamma_i^{(n)}(y) \pi^n, y^j + \sum_{n=1}^{\infty} \alpha^n \Gamma_j^{(n)}(y) \pi^n \right\} = \alpha \sum_{n=0}^{\infty} \alpha^n \Theta_{ij}^{(n)} \left( y^i + \sum_{m=1}^{\infty} \alpha^m \Gamma_i^{(m)}(y) \pi^m \right) \pi^n. 
\]

(3.42)

In the first order in \( \alpha \), as before, one obtains

\[ \Gamma_{ij}^{[1]} = \Theta_{ij}, \]

with the solution \( \Gamma_{ij}^{[1]} = -\Theta_{ij}/2 \). The order \( O(\alpha^2) \) gives

\[ 2\Gamma_{ij}^{[2]} = -\frac{1}{2} \Theta_{ik} \partial_k \Theta_{jl} + \frac{1}{4} \Theta_{il} \partial_k \Theta_{jk} - \frac{1}{4} \Theta_{jl} \partial_k \Theta_{ik} + \Theta_{ijk}. \]

(3.43)

Similarly to (2.12) the consistency condition for the equation (3.43) implies the relation

\[ 3 \Pi_{ijk} + \Theta_{ijk} + \Theta_{jki} = 0, \]

(3.44)

which in turn can be interpreted as an equation on the coefficient function \( \Theta_{ijk} \). A solution is

\[ \Theta_{ijk} = -\Pi_{ijk}. \]

(3.45)

Since \( \Theta_{ijk} \) is antisymmetric in all indices it will not contribute to a solution of the equation (3.43) such that

\[ \Gamma_{ij}^{[n]} = \frac{1}{24} \Theta_{lmn} \partial_m \Theta_{ij} + \frac{1}{24} \Theta_{ijm} \partial_m \Theta_{lk}. \]

(3.46)

The crucial difference of the current situation with respect to the Poisson case is that the consistency condition (3.44) of the algebraic equation (3.43) is not satisfied automatically, but instead provides the equation for the definition of the correction \( \alpha \Theta_{ij}(x, \tilde{x}) \) to the given bi-vector \( \Theta_{ij}(x) \) in (3.40), which is needed to satisfy the Jacobi identity for the algebra (3.39) up to the order \( O(\alpha^3) \).

The strategy of the solution of the equation (3.42) in higher orders is similar. In the order \( O(\alpha^n) \) equation (3.42) results in an algebraic equation on the coefficients \( \Gamma_i^{(n)} \) in terms of lower-order coefficients \( \Gamma_i^{(m)} \), \( m < n \), and tensors \( \Theta_{ij}^{(l)} \) with \( l < n \). The consistency condition to these equations, i.e. the cyclicity relation similar to (2.12), will result in algebraic equations on the tensors \( \Theta_{ij}^{(n-1)} \) in terms of lower-order tensors \( \Theta_{ij}^{(l)} \) with \( l < n-1 \), as well as coefficients \( \Gamma_i^{(m)} \), \( m < n \). Once the cyclicity relation in the order \( O(\alpha^n) \) is satisfied and the tensors \( \Theta_{ij}^{(n-1)} \) are determined, the coefficients \( \Gamma_i^{(n)} \) can be constructed according to the theorem 1, where now in (2.8) instead of the Poisson bi-vector \( \omega_{ij}(x) \) one takes \( \omega_{ij}(x, \tilde{x}) \) defined in (3.40).

**Theorem 4.** The recurrence relations for the definition of tensors \( \Theta_{ij}^{(n)} \) satisfying equation (3.42) are defined as \( \Theta_{ij}^{[n]} = -\Pi_{ij}^{[n]} \), for \( n = 1 \), and for \( n > 1 \),

\[ \Theta_{ij}^{[n]} = -\frac{1}{n(n+2)} \left( F_{ij}^{l_2-l_1} + F_{ij}^{l_3-l_1} + ... + F_{ij}^{l_n-l_{n-1}} \right), \]

(3.47)
where

\[
\alpha^n \mathcal{F}^{ijkl}_{lm}(y) \pi_l \ldots \pi_n = \left\{ x^l(y, \pi), \sum_{m=0}^{n-1} \alpha^n \Theta^{ijkl}_{lm}(x(y, \pi)) \pi_l \ldots \pi_n \right\} + \sum_{m=0}^{n-1} \alpha^n \Theta^{ijkl}_{lm}(x(y, \pi)) \pi_l \ldots \pi_n + \mathcal{O}\left(\alpha^{n+1}\right).
\]  

(3.48)

To prove the theorem it is first convenient to introduce here the following notations:

\[
\omega^{ij}(x, \bar{x}) = \tilde{\omega}^{ij}(x, \bar{x}) + \mathcal{O}\left(\alpha^{n+1}\right),
\]

(3.49)

\[
\tilde{\omega}^{ij}_{n+1}(x, \bar{x}) = \tilde{\omega}^{ij}(x, \bar{x}) + \alpha^{n+1} \Theta^{ij(n+1)}(x) \bar{x}^{n+1}.
\]

Here, as in the previous section, we introduce

\[
\omega^{ij} \left( y^1 + \sum_{n=1}^{\infty} \alpha^n \Gamma^{(n)}(y) \pi^n, \pi \right) = \omega^{ij}_0 + \mathcal{O}\left(\alpha^{n+1}\right).
\]

(3.50)

The difference between \( \tilde{\omega}^{ij}_n \) and \( \omega^{ij}_n \) is that the first is a truncation of the series (3.40) written in terms of the original phase space coordinates \( x \) and \( \bar{x} \), while the second is a truncation of the corresponding series expressed in Darboux coordinates \( y \) and \( \pi \). We also stress that the structure of the function \( \omega^{ij}_0 \) in (3.50) is different from the corresponding expression (2.15) in the Poisson case, because of the presence of the terms \( \alpha^n \Theta^{ij(n)}(\bar{x}) \bar{x}^n \) in the expression for \( \omega^{ij}(x, \bar{x}) \). In particular,

\[
\omega^{ij}_0 = \Theta^{ij}(y), \quad \omega^{ij}_1 = \Theta^{ij} + \alpha \left( \partial_i \Theta^{jk} \pi_k + \Theta^{ik} \pi_k \right),
\]

etc. Taking into account (2.16), (2.17) and (3.49) one may see that

\[
\omega^{ij}_n = \tilde{\omega}^{ij}_n (x_n, \pi) + \mathcal{O}\left(\alpha^{n+1}\right)
\]

\[= \Theta^{ij}(x_n) + \alpha \Theta^{jk}(x, \pi) \pi_k + \cdots + \alpha^{n-1} \Theta^{ij(n-1)}(x_1) \pi^{n-1} + \alpha^n \Theta^{ij(n)}(y) \pi^n + \mathcal{O}\left(\alpha^{n+1}\right).
\]

(3.51)

Also, it is useful to write

\[
\omega^{ij}_n = \tilde{\omega}^{ij}_{n-1}(x_n, \pi) + \alpha^n \Theta^{ij(n)}(y) \pi^n + \mathcal{O}\left(\alpha^{n+1}\right).
\]

(3.52)

Following the logic of the previous section, supposing we know the solution of equation (3.42) up to the \( m \)th order in \( \alpha \), it means that the expressions for \( x_n^l \) and \( \omega^{ij}_{n-1} \) are known such that the equation

\[
\{x_n^l, x_n^j\} = \alpha \omega^{ij}_{n-1} + \mathcal{O}\left(\alpha^{n+1}\right)
\]

holds true. Taking into account (3.51) the consistency condition for the above equation can be written as

\[
\{x_n^l, x_{n-1}^j\} + \{x_{n-1}^l, \tilde{\omega}^{ij}_{n-1}(x_{n-1}, \pi)\} + \{x_{n-1}^l, \tilde{\omega}^{ik}_{n-1}(x_{n-1}, \pi)\} = \mathcal{O}\left(\alpha^n\right).
\]

(3.54)

To obtain the next order contributions in the expansions (2.4) and (3.40) one needs to solve the equation
\[ \{ x^i_{n+1}, x^j_{n+1} \} = \alpha \omega_n^i + \mathcal{O} \left( \alpha^{n+2} \right). \] (3.55)

Like in the Poisson case discussed in the previous section, the above equation is equivalent to the equations (2.20) and (2.21), with the difference that now \( \omega_n^i \) contains the corrections \( \alpha^n \Theta^{ij(n)} \tilde{x}^i \). In particular

\[ \omega_n^i = \tilde{\omega}_{n-1}^i (x_n, \pi) + \alpha^n \Theta^{ij(n)} (y) \pi^n + \mathcal{O} \left( \alpha^{n+2} \right), \] (3.56)

where the function \( \Theta^{ij(n)} (y) \) is still unknown and should be found solving the consistency condition for equation (3.55):

\[ \{ x^i_n, \omega_n^j \} + \{ x^j_n, \omega_n^i \} = \mathcal{O} \left( \alpha^{n+1} \right). \] (3.57)

Taking into account (3.56) the above equation can be written as

\[ \{ x^i_n, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \{ x^j_n, \tilde{\omega}_{n-1}^i (x_n, \pi) \} + \{ x^i_n, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \{ x^j_n, \tilde{\omega}_{n-1}^i (x_n, \pi) \} + n \alpha^n \left( \Theta^{jk(n-1)} + \Theta^{ij(n-1)} + \Theta^{ik(n-1)} \right) \pi^{n-1} = \mathcal{O} \left( \alpha^{n+1} \right). \] (3.58)

That is, the coefficient function \( \Theta^{ij(n)} \) should satisfy the equation

\[ n \alpha^n \left( \Theta^{jk(n-1)} + \Theta^{ij(n-1)} + \Theta^{ik(n-1)} \right) \pi^{n-1} + F_{n}^{ijk} = 0, \] (3.59)

where

\[ F_{n}^{ijk} = \{ x^i_n, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \{ x^j_n, \tilde{\omega}_{n-1}^i (x_n, \pi) \} + \{ x^i_n, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \{ x^j_n, \tilde{\omega}_{n-1}^i (x_n, \pi) \} + \mathcal{O} \left( \alpha^{n+1} \right). \] (3.60)

In particular, \( F_{n}^{ijk} = 3 \Pi^{ijk} \), etc. The above definition and the relation (3.54) imply that

\[ F_{n}^{ijk} = \alpha^n F^{ijk(n-1)} (y) \pi^{n-1}. \] (3.61)

The coefficient functions \( F^{ijk(n-1)} \) by the definition (3.48) are antisymmetric in the first three indices \( ijk \) and symmetric in the last \( n-1 \) indices. One may easily check that the equations (3.60) and (3.61) are equivalent to (3.48) in the theorem 4.

3.1. Next to leading order

The solution of equation (3.59) in the first order in \( \alpha \) is given by (3.45). However, as seen with equation (2.24), its non-trivial nature and the corresponding consistency condition can be seen in the next to leading order. That is why in this subsection we discuss in detail equation (3.59) in the second order in \( \alpha \), where it is equivalent to the algebraic equation on the coefficient function \( \Theta^{ijkl} \):

\[ 2 \left( \Theta^{ijkl} + \Theta^{ijlk} + \Theta^{iklj} \right) - F^{ijkl} = 0, \] (3.62)

with

\[ - F^{ijkl} = \Theta^{ml} \partial_m \Theta^{ijl} + \Theta^{ml} \partial_m \Theta^{ijkl} \] (3.63)

\[ \Theta^{ijm} \partial_m \Theta^{kl} + \Theta^{ijkl} \partial_m \Theta^{ijkl} + \frac{1}{2} \Theta^{ijm} \partial_m \Theta^{kl} + \frac{1}{2} \Theta^{ijkl} \partial_m \Theta^{jkl} + \frac{1}{2} \Theta^{ijkl} \partial_m \Theta^{jkl}. \]

Since \( \Theta^{ijkl} \) should be antisymmetric in the first two indices and symmetric in the last two indices one obtains the consistency condition for the solution of the algebraic equation (3.62):
\[ F^{ijkl} - F^{jikl} + F^{klij} - F^{ijkl} = 0. \]  

(3.64)

Taking into account (3.45) and (3.63) one may check that the condition (3.63) is satisfied automatically for any given antisymmetric bi-vector \( \Theta^j(x) \).

There is a more simple way to see it. To this end we remind that the bracket \( \{ x', x' \} \) defined in (3.37) is bilinear and antisymmetric, but in general the Jacobi identity can be violated, meaning that the three-bracket

\[ \{ f, g, h \} := \frac{1}{3} \left( \{ f, \{ g, h \} \} + \{ h, \{ f, g \} \} + \{ g, \{ h, f \} \} \right). \]  

(3.65)

can be different from zero for three arbitrary functions \( f, g \) and \( h \). By the definition (3.65) is antisymmetric, tri-linear and can be written as

\[ \{ f, g, h \} = \alpha^2 \Pi^{ij} \partial_i f \partial_j g \partial_k h. \]  

(3.66)

Using the definition and the properties of the brackets (3.37) and (3.65) one may check that the following combination of the two and three brackets is identically zero\(^1\):

\[ - \{ f, g, \{ h, k \} \} + \{ h, \{ f, g \} \} - \{ k, \{ f, g \} \} = 0. \]  

(3.67)

Since (3.67) should be valid for any functions \( f, g \) and \( h \) one obtains the following identity involving \( \Theta \) and \( \Pi \):

\[ \Pi^{i j m} \partial_m \Theta^{k l} - \Pi^{j k m} \partial_m \Theta^{i l} + \Pi^{i k m} \partial_m \Theta^{j l} \]  

(3.68)

\[ - \Pi^{l i m} \partial_m \Theta^{k j} - \Pi^{k j m} \partial_m \Theta^{i l} + \Pi^{j i m} \partial_m \Theta^{k l} \]  

\[ + \Theta^{i m} \partial_m \Pi^{j k} - \Theta^{j m} \partial_m \Pi^{i k} + \Theta^{i j} \partial_m \Pi^{k m} = 0. \]

The left-hand side of (3.68) coincides with (3.64).

The following combination

\[ \Theta^{i j k} = \frac{1}{8} \left( F^{i j k} + F^{i k j} \right), \]  

(3.69)

is symmetric in \( kl \) and antisymmetric in \( ij \) by the construction, and also satisfies the equation (3.62) due to (3.64). One writes

\[ \Theta^{i j k} = \frac{3}{16} \Pi^{i j m} \partial_m \Theta^{k l} + \frac{3}{16} \Pi^{j k m} \partial_m \Theta^{i l} - \frac{3}{16} \Pi^{i k m} \partial_m \Theta^{j l} - \frac{3}{16} \Pi^{j i m} \partial_m \Theta^{k l} \]

\[ - \frac{1}{8} \Theta^{i m} \partial_m \Pi^{j k} - \frac{1}{8} \Theta^{j m} \partial_m \Pi^{i k}. \]  

(3.70)

Now using theorem 4 we construct the third-order expression for \( x' \). From the equation

\[ \{ x'_3, x'_5 \} = \alpha \omega^j_2 + \mathcal{O} \left( \alpha^4 \right), \]  

(3.71)

where

\[ \omega^j_2 = \Theta^j(x_2) + \alpha \Theta^{j k}(x_1) \pi_k + \alpha^2 \Theta^{j kl}(y) \pi_k \pi_l + \mathcal{O} \left( \alpha^3 \right), \]

\(1\) In fact, this expression relates two different ways of rebracketing the expression \( \{ f, \{ g, h \} \} \); see [14] for details.
we find that
\[ G^{ijmn} = \partial_i \Theta^{ij} \Gamma^{kmn} + \frac{1}{8} \partial_0 \partial_\alpha \Theta^{ij} \partial^\alpha \Theta^{kmn} \]  
(3.72)
\[ + \frac{1}{4} \partial_i \Pi^{ijm} \Theta^{kn} + \frac{1}{4} \partial_i \Pi^{ijm} \Theta^{km} + \Theta^{ijmn} \]
\[ - \frac{1}{2} \Theta^{ij} \partial_i \Gamma^{kmn} + \frac{1}{2} \Theta^{ij} \partial_i \Gamma^{kjm} - \frac{1}{2} \Gamma^{ijmn} \partial_i \Theta = \frac{1}{2} \Gamma^{ijmn} \partial_i \Theta. \]

After simplifications one obtains the expression
\[ \Gamma^{ijmn} = - \frac{1}{12} (G^{ijmn} + G^{ijmn} + G^{ijmn}) \]
\[ = - \frac{1}{12} (2 \Gamma^{ijmn} \partial_i \Theta^{ij} + 2 \Gamma^{ijmn} \partial_i \Theta^{km} + 2 \Gamma^{ijmn} \partial_i \Theta^{kn}) \]
\[ + \frac{1}{12} \Theta^{lm} \Theta^{kn} + \frac{1}{12} \Theta^{ij} \Theta^{km} \partial_i \Theta^{kn} + \frac{1}{12} \Theta^{ij} \Theta^{kn} \partial_i \Theta^{kn} \]  
(3.73)
That is,
\[ x'_i = y'_i - \frac{\alpha}{2} \Theta^{ij} \pi_j + \frac{\alpha^2}{12} \Theta^{ijm} \partial_i \Theta^{km} \pi_j \pi_k - \frac{\alpha_3}{48} \Theta^{ijm} \Theta^{km} \partial_i \Theta^{km} \pi_j \pi_k. \]

Note that the expression for the generalized Bopp shift is absolutely identical to the Poisson case. The reason for that is the fact that although the tensors \( \Theta^{(n)} \) for \( n \geq 1 \) enter the right-hand side of the equation (3.55), i.e. in the definition of the tensors \( \Theta^{(n)} \), they are constructed as a linear combination of the tensors \( F^{ijkl(n-1)} \), which are antisymmetric in the first three indices. According to theorem 4, the solution of the equation (3.55), i.e. the expression for \( x'_n(n+1) \) is constructed by the symmetrization of the last \( (n+1) \) indices of the tensors \( G^{(n)} \), which annihilates any contribution from the tensors \( F^{ijkl(n-1)} \), and consequently from \( \Theta^{(n)} \), for \( n \geq 1 \).

3.2. Higher-order terms \( \Theta^{(n)} \), \( n \geq 2 \)

Once we know the structure of the consistency condition for equation (3.59), we may proceed to the higher orders and prove the following:

**Lemma 5.** The tensors \( F^{ijkl(n-2)} \), defined in (3.61), satisfy the relation
\[ F^{ijkl(n-2)} - F^{ijkl(n-2)} + F^{ijkl(n-2)} - F^{ijkl(n-2)} = 0. \]  
(3.75)

**Proof.** To prove the above statement first we observe that the relation (3.75) is equivalent to the condition
\[ \{ x'^{i}_n, F^{jk}_{n} \} - \{ x'^{i}_n, F^{ij}_{n} \} + \{ x'^{i}_n, F^{ik}_{n} \} - \{ x'^{i}_n, F^{ji}_{n} \} = O (\alpha^{n+1}) \]  
(3.76)
Using the definition (3.48) we may write the left-hand side of this equation as
\[ \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} + \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} + \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} \]
\[ - \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} - \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} - \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} \]
\[ + \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} + \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} + \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} \]
\[ - \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} - \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} - \{ x'^{i}_n, \{ x'^{j}_n, \pi \} \} ] + O (\alpha^{n+1}). \]  
(3.77)
The Jacobi identity for the functions \( x^i_n, x^k_n \) and \( \tilde{\omega}_{n-1}^j (x_n, \pi) \) reads
\[
\{ x^i_n, \{ x^k_n, \tilde{\omega}_{n-1}^j (x_n, \pi) \} \} + \{ \tilde{\omega}_{n-1}^j (x_n, \pi), \{ x^k_n, x^i_n \} \} + \{ x^k_n, \{ \tilde{\omega}_{n-1}^j (x_n, \pi), x^i_n \} \} = 0,
\]
and holds in all orders in \( \alpha \). Since by (3.53) the functions \( x^i_n \) satisfy the equation,
\[
\{ x^i_n, x^k_n \} = \alpha \omega_{n-1}^{jk} + \mathcal{O} (\alpha^{n+1}),
\]
from the above identity we conclude that
\[
\{ x^i_n, \{ x^k_n, \tilde{\omega}_{n-1}^j (x_n, \pi) \} \} - \{ x^i_n, \tilde{\omega}_{n-1}^j (x_n, \pi) \} = \alpha \omega_{n-1}^{jk} (x_n, \pi) + \mathcal{O} (\alpha^{n+1}) .
\]
The latter means that the expression (3.77) can be rewritten as
\[
\{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} = \{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} - \{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \mathcal{O} (\alpha^{n+1}) .
\]
And finally, we observe that by (3.51), \( \alpha \tilde{\omega}_{n-1}^{jk} (x_n, \pi) = \alpha \omega_{n-1}^{jk} + \mathcal{O} (\alpha^{n+1}) \), such that
\[
\{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} = \{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} - \{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \mathcal{O} (\alpha^{n+1}) = \{ \alpha \omega_{n-1}^{jk}, \tilde{\omega}_{n-1}^j (x_n, \pi) \} + \mathcal{O} (\alpha^{n+1}) .
\]
That is, (3.78) is \( \mathcal{O} (\alpha^{n+1}) \), and the condition (3.76) holds true.

Since the consistency condition (3.75) for the equation (3.59) is satisfied due to the above lemma, the solution can be constructed according to the logic of the lemma 3, i.e. taking the symmetrization of the tensors \( F^{ijk(n-1)} \) in the last \( n \) indices. The corresponding expression is given by equation (3.47).

4. Examples

4.1. Constant \( R \)-flux algebra

As a first example we consider the phase space algebra which appeared in the context of closed string theory in the presence of the non-geometric constant \( R \)-flux [19–21]. This is possibly the most simple physically motivated example of the quasi-Poisson structure which reads
\[
\{ x^i, x^k \} = \Theta^{ij} (x) = \begin{pmatrix} \ell^i & R^{ik} p_k & \delta^i_j \\ -\delta^i_j & 0 & 0 \end{pmatrix}
\]
with \( x = (x^i) = (x^p) \) (4.81)
where \( R^{ik} = R \varepsilon^{ijk} \), and \( \varepsilon^{123} = +1 \). Note that making \( p \rightarrow x \) and \( x \rightarrow -p \), one obtains the monopole algebra corresponding to a constant magnetic charge distribution \( \rho (x) = R \) [22].

In this case the Jacobiator,
\[
\Pi^{ijk} = \begin{pmatrix} \ell^i & R^{ik} & 0 \\ -R^{ik} & 0 & 0 \end{pmatrix},
\]
does not vanish, but is constant. Consequently the only non-vanishing \( \Theta^{ij(n)} \) with \( n \geq 1 \) is \( \Theta^{ijk} = -\Pi^{ijk} \). The corresponding Bopp shift reads
\[ x' = y' - \frac{1}{2} \Theta^{IJ} (y) \pi_J. \] (4.83)

Finally for the symplectic realization one finds
\[ \{ x^I, x^J \}_p = \Theta^{IJ} (x) - \Pi^{IK} \tilde{x}_K, \]
\[ \{ x^I, \tilde{x}_J \}_p = \delta^I_J + \frac{1}{2} (\partial_J \Theta^{IK}) \tilde{x}_K, \]
\[ \{ \tilde{x}_I, \tilde{x}_J \}_p = 0. \] (4.84)

The letter is also convenient to write in components. The non-vanishing Poisson brackets between the extended phase space coordinates \((x, p, \tilde{x}, \tilde{p})\) are given by
\[ \{ x^i, p_j \}_\eta = \{ x^i, \tilde{p}_j \}_\eta = - \{ \tilde{p}_j, p_i \}_\eta = \delta^i_j, \] (4.85)
\[ \{ x^i, x^j \}_\eta = \frac{\ell^3}{\hbar^2} R^{ijk} (p_k - \tilde{x}_k), \]
\[ \{ \tilde{x}^i, \tilde{x}^j \}_\eta = 0. \] (4.86)

4.2. Quasi-Poisson structure isomorphic to the Malcev algebra of octonions

Consider the algebra of classical brackets on the coordinate algebra \(C[\tilde{\xi}]\) which is isomorphic to the commutator algebra of imaginary octonions, see (A.5) in the appendix,
\[ \{ \xi_A, \xi_B \}_\eta = 2 \epsilon_{ABC} \xi_C, \] (4.87)
where \(\tilde{\xi} = (\xi_A)\) with \(\xi_A \in \mathbb{R}, A = 1, \ldots, 7\). This bracket is bilinear, antisymmetric and satisfies the Leibniz rule by definition. Note that although the algebra (4.86) is isomorphic to the Malcev algebra of octonions (A.5) it does not satisfy Malcev identity, [23]. Moreover, in [24] a stronger statement was given: the quasi-Poisson structure satisfies the Malcev identity only if it is Poisson, i.e. satisfies the Jacobi identity.

Introducing \(\sigma^i := \xi_{i+3}\) for \(i = 1, 2, 3\) and \(\sigma^4 := \xi_7\), one may rewrite (4.86) in components as
\[ \{ \xi_i, \xi_j \}_\eta = 2 \epsilon_{ijk} \xi_k \quad \text{and} \quad \{ \sigma^4, \xi_i \}_\eta = 2 \sigma^i, \]
\[ \{ \sigma^i, \sigma^j \}_\eta = -2 \epsilon_{ijk} \xi_k \quad \text{and} \quad \{ \sigma^A, \sigma^B \}_\eta = -2 \xi_A, \]
\[ \{ \sigma^i, \xi_j \}_\eta = -2 (\delta^i_j \sigma^4 - \epsilon^i_{jk} \sigma^k). \] (4.88)

Using (A.10) the non-vanishing Jacobiators can be written as
\[ \{ \xi_i, \xi_j, \sigma^k \}_\eta = -4 (\epsilon^{i}_{jk} \sigma^4 + \delta^k_j \sigma_i - \delta^k_i \sigma_j), \]
\[ \{ \xi_i, \sigma^j, \sigma^k \}_\eta = 4 (\delta^j_i \xi_k - \delta^k_i \xi_j), \]
\[ \{ \sigma^i, \sigma^j, \sigma^k \}_\eta = 4 \epsilon^{ijk} \sigma^4, \]
\[ \{ \xi_i, \sigma^j, \sigma^k \}_\eta = 4 \epsilon_{ijk} \xi_k, \]
\[ \{ \sigma^i, \sigma^j, \sigma^k \}_\eta = -4 \epsilon^{ijk} \sigma^4. \]
The calculation of the first orders of the generalized Bopp shift indicate the ansatz
\[ \xi_A = y_A - \eta_{ABC} \pi_B y_C - (y_A \pi^2 - \pi_A y_B \pi_B) \chi(\pi^2), \tag{4.89} \]
where \( \chi(\pi^2) \) is a function to be determined. One calculates
\[ \{\xi_A, \xi_B\} - 2 \eta_{ABC} \xi_C = (y_A \pi_B - y_B \pi_A) \left[ 2\pi^2 \chi' + 3\chi - 1 - \pi^2 \chi^2 \right] + 4 \eta_{ABCD} \pi_C y_D. \tag{4.90} \]
From this we get an equation on \( \chi(t) \):
\[ 2\pi' + 3\chi - 1 - \pi^2 = 0, \quad \chi(0) = \frac{1}{3}, \tag{4.91} \]
with the solution
\[ \chi(t) = -\frac{1}{t} \left( \sqrt{t} \cot \sqrt{t} - 1 \right). \tag{4.92} \]
The perturbative calculation for \( \omega_{AB}(\xi, \tilde{\xi}) \) suggests the ansatz
\[ \omega_{AB}(\xi, \tilde{\xi}) = 2 \eta_{ABC} \xi_C + \eta_{ABCD} \tilde{\xi}_C \xi_D \phi \left( \tilde{\xi}^2 \right) + \eta_{ABCD} \eta_{DEF} \tilde{\xi}_C \xi_D \xi_F \psi \left( \tilde{\xi}^2 \right), \tag{4.93} \]
where the functions \( \phi \left( \tilde{\xi}^2 \right) \) and \( \psi \left( \tilde{\xi}^2 \right) \) can be found from the relation
\[ 4 \eta_{ABCD} \pi_C y_D = \eta_{ABCD} \tilde{\xi}_C \xi_D \phi \left( \tilde{\xi}^2 \right) + \eta_{ABCD} \eta_{DEF} \tilde{\xi}_C \xi_D \xi_F \psi \left( \tilde{\xi}^2 \right). \tag{4.94} \]
Using the expression for the generalized Bopp shift (4.89), as well as the choice \( \tilde{\xi} = \pi \), contraction identity (A.7) and the antisymmetry of \( \eta_{ABCD} \) we find the relations
\[ -\phi + \psi \left( 1 - \pi^2 \chi(\pi^2) \right) = 0, \]
\[ \phi \left( 1 - \pi^2 \chi(\pi^2) \right) + \pi^2 \psi = 4, \]
implying that
\[ \phi = 2 \frac{\sin 2 \sqrt{\pi^2}}{\sqrt{\pi^2}}, \quad \text{and} \quad \psi = 4 \frac{\sin^2 \sqrt{\pi^2}}{\pi^2}. \tag{4.95} \]
Finally we conclude that the symplectic realization of the quasi-Poisson structure (4.86) is given by
\[ \{\xi_A, \xi_B\}_\rho = 2 \eta_{ABC} \xi_C + 2 \frac{\sin 2 \sqrt{\tilde{\xi}^2}}{\sqrt{\xi^2}} \eta_{ABCD} \tilde{\xi}_C \xi_D + 4 \frac{\sin^2 \sqrt{\tilde{\xi}^2}}{\xi^2} \eta_{ABCD} \eta_{DEF} \tilde{\xi}_C \xi_D \xi_F, \]
\[ \{\xi_A, \tilde{\xi}_B\}_\rho = \delta_{AB} + \eta_{ABC} \tilde{\xi}_C + \left( \delta_{AB} \tilde{\xi}^2 - \xi_A \tilde{\xi}_B \right) \frac{\sqrt{\tilde{\xi}^2} \cot \sqrt{\tilde{\xi}^2} - 1}{\xi^2}, \]
\[ \{\tilde{\xi}_A, \tilde{\xi}_B\}_\rho = 0 \tag{4.96} \]
while the expression for the generalized Bopp shift is
\[ \xi_A = y_A - \eta_{ABC} \pi_B y_C + (y_A \pi^2 - \pi_A y_B \pi_B) \frac{\sqrt{\pi^2} \cot \sqrt{\pi^2} - 1}{\pi^2}, \]
\[ \tilde{\xi}_A = \pi_A. \tag{4.97} \]
Note that the restriction of the quasi-Poisson structure (4.86) to the three-dimensional space with coordinates $\xi_i$, $i = 1, 2, 3$, results according to (4.87) in the Poisson structure $\{\xi_i, \xi_j\}_\epsilon = 2 \epsilon_{ijk} \xi_k$, isomorphic to the $\mathfrak{su}(2)$ Lie algebra. Since in three dimensions the totally antisymmetric tensor $\epsilon_{ABCD}$ of the rank four automatically vanishes, from (4.96) one obtains immediately the symplectic realization of the $\mathfrak{su}(2)$-like Poisson structure,

$$\{\xi_i, \xi_j\}_\epsilon = \xi_i \xi_j.$$

Substituting formally in the above expression the canonical phase space momenta $\pi_i$ by the derivative operators, $-i \partial_i$, the canonical phase space coordinates $y_i$ by the multiplication operators $x_i$, and supposing normal ordering, one recovers the expression for the operators $\xi_i$ giving the polydifferential representation of the algebra, $[\xi_i, \xi_j] = i \epsilon_{ijk} \xi_k$, which was obtained in [25] and used for the derivation of the $\mathfrak{su}(2)$-like star product.

### 4.3. R-flux in M-theory

To relate the quasi-Poisson structure (4.86) to the constant $R$-flux algebra (4.81) let us follow [26] and introduce the $7 \times 7$ matrix

$$\Lambda = (\Lambda^{AB}) = \frac{1}{2h} \begin{pmatrix} 0 & \sqrt{\lambda \ell_s^3} R & 0 \\ 0 & 0 & \sqrt{\lambda \ell_s^3} R \\ -\lambda h & 0 & 0 \end{pmatrix}$$

with $\mathbb{I}_3$ the $3 \times 3$ identity matrix. The matrix $\Lambda$ is non-degenerate as long as all parameters are non-zero, but it is not orthogonal. Using it we define new coordinates

$$\tilde{x} = (x^A) = (x, x^4, p) := \Lambda \tilde{\xi} = \frac{1}{2h} \left( \sqrt{\lambda \ell_s^3} R \sigma, \sqrt{\lambda \ell_s^3} R \sigma^4, -\lambda h \xi \right).$$

From the classical brackets (4.86) one obtains the quasi-Poisson algebra

$$\{x^A, x^B\}_\lambda = 2 \lambda^{ABC} x^C$$

with $\lambda^{ABC} := \Lambda^{AA'} \Lambda^{BB'} \eta_{AA'BC'} \Lambda^{-1}_{CC'}$.

which can be written in components as

$$\{x^i, x^j\}_\lambda = \frac{\ell_s^3}{R^2} R^{ik} p_k$$

and

$$\{x^4, x^4\}_\lambda = \frac{\lambda \ell_s^3}{R^2} R^{44} p^4,$$

$$\{x^i, p_j\}_\lambda = \delta^{ij} x^4 + \lambda \epsilon^{ijk} x^k$$

and

$$\{x^4, p_i\}_\lambda = \lambda^2 x_i,$$

$$\{p_i, p_j\}_\lambda = -\lambda \epsilon_{ijk} p_k.$$

Now taking the contraction limit $\lambda \to 0$ we observe that the element $x^4$ becomes a central element and can be taken to be identity, while the phase space coordinates $x^i$ and $p_i$ form the algebra of the constant $R$-flux algebra (4.81). The main conjecture of [26] is that the quasi-Poisson structure
brackets (4.103) provide the uplift of the string $R$-flux algebra to M-theory. In this sense $\lambda$ plays the role of the M-theory radius.

The corresponding Jacobiators are

$$\{x^A, x^B, x^C\}_\lambda = -4 \lambda^{ABCD} x^D$$

with the components

$$\{x^i, x^j, x^k\}_\lambda = \frac{\ell^3}{\hbar^2} R^{i}{}_{j}{}^{k} x^i,$$

$$\{x^i, x^j, x^k\}_\lambda = -\frac{\lambda}{\ell^2} \frac{\ell^3}{\hbar^2} R^{i}{}_{j}{}^{k} x^i,$$

$$\{p_i, x^j, x^k\}_\lambda = \frac{\lambda}{\ell^2} \frac{\ell^3}{\hbar^2} R^{i}{}_{j}{}^{k} p_i,$$

$$\{p_i, p_j, x^k\}_\lambda = -\lambda \varepsilon^{ijk} k^2 - \lambda (\delta^i_j x_k - \delta^i_k x_j),$$

$$\{p_i, p_j, p_k\}_\lambda = 0.$$

Making transformation (4.101) in (4.96) and (4.97) we obtain

$$\{x_a, x_b\}_p = 2 \lambda_{ABC} x_c + 2 \sin \sqrt{\frac{(\Lambda x)^2}{(\Lambda x)^2}} \lambda_{ABCD} \bar{x}_c x_d + 4 \frac{\sin^2 \sqrt{\frac{(\Lambda x)^2}{(\Lambda x)^2}} \lambda_{ABCD} \lambda_{DEF} \bar{x}_c \bar{x}_e x_f},$$

$$\{x_a, \bar{x}_b\}_p = \delta_{AB} + \lambda_{ABC} \bar{x}_c + (\delta_{AB} (\Lambda x)^2 - \Lambda_{AV} \bar{x}_A \Lambda_{BB'} \bar{x}_B) \sqrt{\frac{(\Lambda x)^2}{(\Lambda x)^2}} \cot \sqrt{\frac{(\Lambda x)^2}{(\Lambda x)^2}} - 1,$$

$$\{\bar{x}_A, \bar{x}_B\}_p = 0,$$

while the expression for the generalized Bopp shift reads

$$x_A = y_A - \lambda_{ABC} \pi_B y_C + (y_A (\Lambda \pi)^2 - \Lambda_{AV} \pi_A \Lambda_{BB'} \pi_B) \sqrt{\frac{(\Lambda \pi)^2}{(\Lambda \pi)^2}} \cot \sqrt{\frac{(\Lambda \pi)^2}{(\Lambda \pi)^2}} - 1,$$

$$\bar{x}_A = \pi_A.$$ (4.106)

Following the logic of [22] that the magnetic monopole algebra can be obtained from the non-geometric $R$-flux quasi-Poisson structure by swapping the phase space coordinates and momenta, in [27] a magnetic analog of the M-theory $R$ flux algebra (4.103) was introduced, the smeared Kaluza–Klein monopole. In [1] we used the symplectic realization of the mono-

5. Conclusions

In the conclusion we would like to mention some possible applications of the proposed construction. It is notable that if in the $L_{\infty}$ bootstrap program [15] one selects the quasi-Poisson bracket (3.37) as the initial setup, i.e. $\ell_2(f, g) = \{f, g\}_Q$, for $f, g \in X_0$, then the structure of the coefficient functions (3.47) is exactly the same as the one which determines the brackets, $\ell_{n+2}(f, g, A^n) = \Theta^{\beta_1 \ldots \beta_n} \partial_{\beta_1} f \partial_{\beta_2} g A_{\beta_1} \ldots A_{\beta_n} \in X_0$, with $A \in X_{-1}$. The latter
is needed to compensate the violation of the Jacobi identity by the bracket \( \{ f, g \} \). The products \( \ell_{n+1}(f, A^n) \in X_{n+1} \) carry one vector index and that is why they are different from the expression \( \Gamma^{ij_1 \ldots j_n} \partial_i f A_{j_1} \ldots A_{j_n} \). However, they can be constructed using similar logic as in lemmas 2 and 3; see [28] for more details. It would be interesting to understand the precise relation between the symplectic realizations and the \( L_\infty \) construction proposed in [15].

Another interesting application is the non-associative deformation quantization. In the same way as the symplectic realization of the Poisson manifolds were used for the construction of the associative star products representing the quantization of a given Poisson structure [3–5], the obtained symplectic realizations of the quasi-Poisson structures can be useful as a starting point for the construction of the non-associative star products compatible with the topological limit [16–18]. The latter, in fact, is the requirement that the non-associative star product \( \star \) quantizing the quasi-Poisson structure \( \Theta^{ij}(x) \) should become associative if \( \Theta^{ij}(x) \) satisfies the Jacobi identity. In other words, the star associator should be proportional to the Jacobiator \( \Pi^{ijk} \) and its derivatives. A similar requirement was imposed in section 3 on the symplectic realization of the quasi-Poisson brackets.

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Appendix. Octonions

The algebra \( \mathbb{O} \) of octonions is the best known example of a non-associative but alternative algebra. Every octonion \( X \in \mathbb{O} \) can be written in the form

\[
X = k^0 \mathbb{I} + k^A e_A \tag{A.1}
\]

where \( k^0, k^A \in \mathbb{R}, A = 1, \ldots, 7 \), while \( \mathbb{I} \) is the identity element and the imaginary unit octonions \( e_A \) satisfy the multiplication law

\[
e_A e_B = -\delta_{AB} \mathbb{I} + \eta_{ABC} e_C. \tag{A.2}
\]

Here \( \eta_{ABC} \) is a completely antisymmetric tensor of rank three with non-vanishing values

\[
\eta_{ABC} = +1 \quad \text{for} \quad ABC = 123, 435, 471, 457, 516, 572, 624, 673. \tag{A.3}
\]

Introducing \( f_i := e_{i+3} \) for \( i = 1, 2, 3 \), the algebra (A.2) can be rewritten as

\[
e_i e_j = -\delta_{ij} \mathbb{I} + \varepsilon_{ijk} e_k, \tag{A.4}
e_i f_j = \delta_{ij} e_7 - \varepsilon_{ijk} f_k, f_i f_j = \delta_{ij} \mathbb{I} - \varepsilon_{ijk} e_k, e_7 e_i = f_i \quad \text{and} \quad f_i e_7 = e_i,
\]

which emphasises a sub-algebra \( \mathbb{I} \) of quaternions generated by \( e_i \); we will use this component form of the algebra \( \mathbb{O} \) frequently in what follows.

The algebra \( \mathbb{O} \) is neither commutative nor associative. The commutator algebra of the octonions is given by

\[
[e_A, e_B] := e_A e_B - e_B e_A = 2 \eta_{ABC} e_C. \tag{A.5}
\]
which can be written in components as

\[
[e_i, e_j] = 2 \varepsilon_{ijk} e_k \quad \text{and} \quad [e_7, e_i] = 2 f_i, \\
[f_i, f_j] = -2 \varepsilon_{ijk} e_k \quad \text{and} \quad [e_7, f_i] = -2 e_i, \\
[e_i, f_j] = 2 (\delta_{ij} e_7 - \varepsilon_{ijk} f_k).
\]

(A.6)

The structure constants \( \eta_{ABC} \) satisfy the contraction identity

\[
\eta_{ABC} \eta_{DEC} = \delta_{AD} \delta_{BE} - \delta_{AE} \delta_{BD} + \eta_{ABDE},
\]

(A.7)

where \( \eta_{ABDE} \) is a completely antisymmetric tensor of rank four with non-vanishing values

\[
\eta_{ABDE} = +1 \quad \text{for} \quad ABDE = 1267, 1346, 1425, 1537, 3247, 3256, 4567.
\]

One may also represent the rank four tensor \( \eta_{ABDE} \) as the dual of the rank three tensor \( \eta_{FGH} \) through

\[
\eta_{ABDE} = \frac{1}{6} \varepsilon_{ABDEFGH} \eta_{FGH},
\]

(A.8)

where \( \varepsilon_{ABDEFGH} \) is the alternating symbol in seven dimensions normalized as \( \varepsilon_{1234567} = +1 \).

Together they satisfy the contraction identity

\[
\eta_{AEF} \eta_{ABCD} = \delta_{EB} \eta_{FCD} - \delta_{FB} \eta_{ECD} + \delta_{EC} \eta_{BFD} - \delta_{FC} \eta_{BED} + \delta_{ED} \eta_{BCF} - \delta_{FD} \eta_{BCE}.
\]

(A.9)

Taking into account (A.7), for the Jacobiator we get

\[
[e_A, e_B, e_C] := \frac{1}{3} \left( [e_A, [e_B, e_C]] + [e_C, [e_A, e_B]] + [e_B, [e_C, e_A]] \right) = -4 \eta_{ABCD} e_D,
\]

(A.10)

and the alternative property of the algebra \( \mathbb{O} \) implies that the Jacobiator is proportional to the associator, i.e. \([X, Y, Z] = 6 ((XY)Z - X(YZ))\) for any three octonions \(X, Y, Z \in \mathbb{O}\).

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**References**

[1] Kupriyanov V G and Szabo R J 2018 Symplectic realization of electric charge in fields of monopole distributions Phys. Rev. D 98 045005

[2] Gracia-Bondia J M, Lizzi F, Varilly J C and Vitale P 2018 The Kirillov picture for the Wigner particle J. Phys. A: Math. Theor. 51 255203

[3] Hammou A B, Lagraa M and Sheikh-Jabbari M M 2002 Coherent state induced star product on \(\mathbb{R}^3(\lambda)\) and the fuzzy sphere Phys. Rev. D 66 025025

[4] Gracia-Bondia J M, Lizzi F, Marmo G and Vitale P 2002 Infinitely many star products to play with J. High Energy Phys. JHEP04(2002)026

[5] Pachol A and Vitale P 2015 \(\kappa\)-Minkowski star product in any dimension from symplectic realization J. Phys. A: Math. Theor. 48 445202

[6] Weinstein A 1983 The local structure of Poisson manifolds J. Differ. Geom. 18 523–57

[7] Weinstein A 1987 Symplectic groupoids and Poisson manifolds Bull. Am. Math. Soc. 16 101–4

[8] Karasev M V 1986 Analogues of objects of the theory of Lie groups for nonlinear poisson brackets Izv. Akad. Nauk SSSR Ser. Mat. 50 638 (Russian)
[9] Coste A, Dazord P and Weinstein A 1987 *Groupoides Symplectiques* (*Publications du Département de mathématiques A*, 87-2) (Lyon: The Université Claude Bernard Lyon 1) pp 1–62 (French)

[10] Cattaneo A S and Xu P 2004 Integration of twisted Poisson structures *J. Geom. Phys.* 49 187–96

[11] Gomes M and Kupriyanov V G 2009 Position-dependent noncommutativity in quantum mechanics *Phys. Rev. D* 79 125011

[12] Kupriyanov V G and Vassilevich D V 2008 Star products made (somewhat) easier *Eur. Phys. J. C* 58 627–37

[13] Kupriyanov V G 2013 Quantum mechanics with coordinate dependent noncommutativity *J. Math. Phys.* 54 112105

[14] Mylonas D, Schupp P and Szabo R J 2014 Non-geometric fluxes, quasi-Hopf twist deformations and nonassociative quantum mechanics *J. Math. Phys.* 55 122301

[15] Blumenhagen R, Brunner I, Kupriyanov V and Lust D 2018 Bootstrapping non-commutative gauge theories from $L_\infty$ algebras *J. High Energy Phys.* JHEP05(2018)097

[16] Cornalba L and Schiappa R 2002 Nonassociative star product deformations for D-brane world volumes in curved backgrounds *Commun. Math. Phys.* 225 33

[17] Herbst M, Kling A and Kreuzer M 2001 Star products from open strings in curved backgrounds *J. High Energy Phys.* JHEP09(2001)014

[18] Herbst M, Kling A and Kreuzer M 2004 Cyclicity of nonassociative products on D-branes *J. High Energy Phys.* JHEP03(2004)003

[19] Blumenhagen R and Plauschinn E 2011 Nonassociative gravity in string theory? *J. Phys. A: Math. Theor.* 44 015401

[20] Lüst D 2010 T-duality and closed string noncommutative (doubled) geometry *J. High Energy Phys.* JHEP12(2010)084

[21] Blumenhagen R, Deser A, Lust D, Plauschinn E and Rennecke F 2011 Non-geometric fluxes, asymmetric strings and nonassociative geometry *J. Phys. A: Math. Theor.* 44 385401

[22] Bakas I and Lust D 2014 3-cocycles, non-associative star-products and the magnetic paradigm of $R$-flux string vacua *J. High Energy Phys.* JHEP01(2014)171

[23] Kupriyanov V G and Szabo R J 2017 $G_2$-structures and quantization of non-geometric M-theory backgrounds *J. High Energy Phys.* JHEP02(2017)099

[24] Vassilevich D and Oliveira F M C 2018 Nearly associative deformation quantization *Lett. Math. Phys.* 108 2293

[25] Kupriyanov V G and Vitale P 2015 Noncommutative $R^d$ via closed star product *J. High Energy Phys.* JHEP08(2015)024

[26] Günyaydin M, Lust D and Malek E 2016 Nonassociativity in non-geometric string and M-theory backgrounds, the algebra of octonions, and missing momentum modes *J. High Energy Phys.* JHEP11(2016)027

[27] Lust D, Malek E and Szabo R J 2017 Non-geometric Kaluza–Klein monopoles and magnetic duals of M-theory R-flux backgrounds *J. High Energy Phys.* JHEP10(2017)144

[28] Kupriyanov V G 2019 $L_\infty$-bootstrap approach to non-commutative gauge theories *Fortschritte der Physik* accepted (https://doi.org/10.1002/prop.201910010)