Hidden 12-dimensional Structures in
AdS$_5 \times S^5$ and $M^4 \times R^6$ Supergravities

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Abstract

It is shown that AdS$_5 \times S^5$ supergravity has hitherto unnoticed supersymmetric properties that are related to a hidden 12-dimensional structure. The totality of the AdS$_5 \times S^5$ supergravity Kaluza-Klein towers is given by a single superfield that describes the quantum states of a 12-dimensional supersymmetric particle. The particle has super phase space $(X^M, P^M, \Theta)$, with $(10, 2)$ signature and 32 fermions. The worldline action is constructed as a generalization of the supersymmetric particle action in Two-Time Physics. SU$(2, 2|4)$ is a linearly realized global supersymmetry of the 2T action. The action is invariant under the gauge symmetries Sp$(2, R)$, SO$(4, 2)$, SO$(6)$, and fermionic kappa. These gauge symmetries insure unitarity and causality while allowing the reduction of the 12-dimensional super phase space to the correct super phase space for AdS$_5 \times S^5$ or $M^4 \times R^6$ with 16 fermions and one time, or other dually related one time spaces. One of the predictions of this formulation is that all of the SU$(2, 2|4)$ representations that describe Kaluza-Klein towers in AdS$_5 \times S^5$ or $M^4 \times R^6$ supergravity universally have vanishing eigenvalues for all the Casimir operators. This prediction has been partially verified directly in AdS$_5 \times S^5$ supergravity. This suggests that the supergravity spectrum supports a hidden $(10, 2)$ structure. A possible duality between AdS$_5 \times S^5$ and $M^4 \times R^6$ supergravities is also indicated. Generalizations of the approach applicable 10-dimensional super Yang Mills theory and 11-dimensional M-theory are briefly discussed.
I.  INTRODUCTION

A standard way of describing the motion of a particle on a sphere $S^5$ is to consider the six Euclidean coordinates $X^I(\tau)$ with the constraint $X^I(\tau)X_I(\tau) = R_1^2$, where the radius $R_1$ is a constant in $\tau$. Similarly the motion of a particle on AdS$_5$ space is described by six coordinates $X^m(\tau)$ with (4, 2) signature with the constraint $X^mX_m = -R_2^2$. When the constant radii are the same $R_1 = R_2 = R$, the combined 12-dimensional space $X^M = (X^m, X^I)$ with two constraints describes the motion of the particle on AdS$_5 \times S^5$. One of the constraints on the 12-dimensional dynamical space $X^M(\tau)$ with (10, 2) signature can be rewritten as a SO(10, 2) invariant on the light-cone

$$X^M X_M = X^m X_m + X^I X_I = 0.$$ 

The second constraint (which requires the AdS$_5 \times S^5$ radius $R$ to be a constant in $\tau$) will soon be argued to arise from fixing a local ($\tau$-dependent) gauge symmetry that acts on $X^M(\tau)$.

In AdS$_5 \times S^5$ supergravity the spectrum is described by demanding that the eigenvalues of the AdS$_5$ Laplacian are correlated with those of the $S^5$ Laplacian, $\nabla^2_{\text{AdS}_5} \psi = -\nabla^2_{S^5} \psi$, such that the total AdS$_5 \times S^5$ Laplacian vanishes on the supergravity states $\nabla^2_{\text{AdS}_5 \times S^5} \psi = 0$. In a previous paper and in an appendix in this paper it is shown that these properties follow directly from quantizing the 12-dimensional particle $X^M(\tau)$ with the constraints described above, such that the 12 momenta $P_M(\tau) = (P_m(\tau), P_I(\tau))$ satisfy an additional SO(10, 2) invariant constraint on the light-cone

$$P_M P^M = P_m P^m + P_I P^I = 0.$$ 

Under quantization, the $P_M P^M = 0$ condition reduces to the Laplace equation $\nabla^2_{\text{AdS}_5 \times S^5} \psi = 0$. The Poisson brackets of the two constraints $[X^M X_M, P_M P^M]$ shows that there must be a third SO(10, 2) invariant constraint

$$X^M P_M = X^m P_m + X^I P_I = 0,$$ 

This becomes a differential operator in the quantum version which fixes the overall dimension of the field in 12D.
These three first class constraints $X^2, P^2, (X \cdot P + P \cdot X)$ close under commutation at the quantum level to form the algebra of Sp(2, R) which is the fundamental gauge symmetry in two-time physics (2T-physics) \[1\]-\[10\]. The reader is also reminded of Dirac’s approach \[11\] to conformal symmetry which consists of differential equations that are equivalent to imposing these three constraints (indeed AdS$_5 \times$S$^5$ is conformal invariant \[12\]). It is the Sp(2) gauge symmetry that allows the constant radius $R$ of AdS$_5 \times$S$^5$ to be taken independent of $\tau$ as a gauge choice, thus creating the AdS$_5 \times$S$^5$ phase space out of the flat 12D phase space (this was shown in detail in the third reference in \[1\] and will be discussed again below). The gauge can also be fixed to create other 1-time 10-dimensional phase spaces related to the 12-dimensional phase space, but in this paper we will concentrate mainly on the AdS$_5 \times$S$^5$ and $M^4 \times$R$^6$ gauges$^1$.

The argument above begins to show that AdS$_5 \times$S$^5$ or other forms of supergravity may have a hidden 12-dimensional structure. In this paper we extend these considerations to 12-dimensional super phase space $(X^M, P_M, \Theta)$ with 32 fermionic coordinates $\Theta$. This will be done by studying a 12-dimensional superparticle in the 2T-physics formalism. The results will reveal that AdS$_5 \times$S$^5$ or $M^4 \times$R$^6$ supergravity indeed have hitherto unnoticed supersymmetric properties that are related to the 12-dimensional structure and associated hidden symmetries.

The standard massless superparticle actions in $d = 3, 4, 5, 6$ dimensions with $N$ supersymmetries have hidden superconformal symmetries, OSp($N|4$), SU(2, 2|$N$), F(4), OSp(8$^*$|$N$) respectively, whose non-linear realizations are rather intricate. These hidden symmetries

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$^1$ The two times are not arbitrarily introduced in 2T-physics. They are a consequence of the fundamental Sp(2, R) phase space local symmetry. The Sp(2, R) is a generalization of $\tau$-reparametrization. For a free particle, with only $\tau$-reparametrization of the worldline, one obtains the constraint $p^2 = 0$. This constraint is nontrivial and physical only when there is one timelike dimension. To see this, consider the worldline formalism for any signature. For Euclidean signature the solution of $p^2 = 0$ is trivial. If the signature has more than one time the theory is non-unitary and non-causal since $\tau$-reparametrization alone is insufficient to remove ghosts for more than one time. Hence in a theory with only $\tau$-reparametrization the target spacetime must have one time, no more and no less. Similarly, Sp(2, R) local symmetry, in flat space, requires the three constraints $X^2 = P^2 = X \cdot P = 0$, which indicate that the physical phase space is realized only by Sp(2, R) gauge invariants. Euclidean or one-time spacetimes have trivial content, while more than two times lead to ghosts. Therefore the only non-trivial and physical solution of the Sp(2, R) constraints requires two timelike dimensions. These constraints have generalizations in the presence of interactions with arbitrary background fields, including curved spacetimes \[1\]-\[3\]. 2T-physics has the virtue of making hidden symmetries manifest while unifying various dynamics in 1T-physics as explained in footnote\[2\].
were initially discovered for \( d = 3, 4 \) a long time ago in [13] while the additional cases \( d = 5, 6 \) were first discovered using the two-time physics formulation (2T-physics) [1]-[10] of the superparticle [3]. The non-linear superconformal symmetries for all cases above are understood much more simply as linearly realized supersymmetries in \( d + 2 \) dimensions in 2T-physics. The non-linear version emerges after choosing Sp(2, \( R \)) gauges and expressing the 2T theory in terms of fewer (gauge fixed) degrees of freedom suitable for one-time physics (1T-physics).

In the 2T approach the theory is formulated in terms of phase space degrees of freedom in \( d + 2 \) dimensions \((X^M(\tau), P^M(\tau))\) and a supergroup element \( g(\tau) \subset G \). The supergroup \( G \) contains the bosonic subgroup \( \text{SO}(d, 2) \) in the spinor representation. A worldline action that is invariant under linearly realized global supersymmetry \( G \), local Sp(2, \( R \)), local \( \text{SO}(d, 2) \), and local kappa supersymmetries, was constructed [3]. It was shown that for the special cases of \((d, G)\) dimensions and supergroups \((d = 3, \text{OSp}(N|4)), (d = 4, \text{SU}(2, 2|N)), (d = 5, \text{F}(4)), (d = 6, \text{OSp}(8^*|N))\), the 2T action is re-expressed as the standard superparticle action in 1T-physics after choosing some special gauges.

In this paper the SU(2, 2|4) 2T superparticle action will be extended to a \((10, 2)\) superparticle action by adding 6 more coordinates. The 12 coordinates \((X^M(\tau), P^M(\tau))\) have a special coupling to \( g(\tau) \subset \text{SU}(2, 2|4) \) that respect the global symmetry \( \text{SU}(2, 2|4) \). In a special gauge the system describes a superparticle moving in the 10-dimensional AdS\(_5\)×S\(_5\) superspace. The quantum states of this superparticle correspond to the complete set of the Kaluza-Klein towers that emerge in the AdS\(_5\)×S\(_5\) compactification of 10D type IIB supergravity.

In section 2, we present the action and discuss its symmetries. In section 3 first class constraints that follow from the gauge symmetries are used to make gauge invariant statements about the SU(2, 2|4) representation content of the physical states at the quantum level. It is shown that the eigenvalues of all Casimir operators must vanish in the physical sector. In section 4 the gauge choice that corresponds to AdS\(_5\)×S\(_5\) or M\(_4\)×R\(_6\) is discussed and the quantum states are given as a superfield that corresponds to the Kaluza-Klein spectrum of compactified type IIB supergravity. We now outline the essential ideas in these sections.

The global symmetry of the total action is SU(2, 2|4) and the local symmetries are SO(4, 2)×SO(6), fermionic kappa, and Sp(2, \( R \)). These arise as follows. We start with the standard 2T-physics action in flat spacetime that contains the Sp(2, \( R \)) doublets
and the Sp(2, R) gauge fields $A^{ij}(\tau)$ that insure a local Sp(2, R) symmetry. As usual in 2T-physics, this structure exists non-trivially and is physically meaningful only when the 12-dimensional spacetime has (10,2) signature. Hence, this purely bosonic part of the action (first term in Eq.(2.2) below) necessarily has a global SO(10, 2) symmetry that acts on the spacetime indices $M$.

Next, to introduce fermions, we consider the supergroup element $g(\tau) \subset SU(2, 2|4)$ that contains fermions $\Theta$. All the bosons in $g(\tau)$ can be removed by the gauge symmetries discussed below, but in the presence of the bosons in $g(\tau)$ the formulation of the theory is more transparent and more elegant. On the left and right sides of $g(\tau)$ we define the supergroup transformations $SU(2, 2|4)_L \times SU(2, 2|4)_R$. On the left side of $g(\tau)$, and on $(X^M(\tau), P^M(\tau))$, we gauge the bosonic subgroup $SO(4, 2) \times SO(6) = SU(2, 2) \times SU(4)$ that sits at the intersection of SO(10, 2) and SU(2, 2|4). This couples $(X^M, P^M)$ to $g$ (second term in Eq.(2.2) below) and thus breaks the 12D symmetry of the first term down to the $SO(4, 2) \times SO(6)$ gauge symmetry of the total action. The action also has a subtle fermionic local kappa supersymmetry embedded in $SU(2, 2|4)_L$ acting on the left side of $g(\tau)$ and on the Sp(2, R) gauge fields $A^{ij}(\tau)$.

In addition to the local symmetries, there is a global $SU(2, 2|4)_R$ supersymmetry of the system that acts on the right side of $g(\tau)$. Initially the global $SU(2, 2|4)_R$ does not transform $(X^M, P^M)$. However, after a gauge fixing that removes the bosons in $g(\tau)$ the $SU(2, 2|4)_R$ induces a transformation on $(X^M, P^M)$ as follows. The gauged fixed $g$ takes the form

$$g = \exp \left( \begin{array}{cc} 0 & \Theta \\ \bar{\Theta} & 0 \end{array} \right). \tag{1.4}$$

The $\Theta$ are in $(4, \bar{4})$ spinor representation of $SO(4, 2) \times SO(6)$. A global $SU(2, 2|4)_R$ transformation of the gauged fixed $g$ must be compensated from the left by $\Theta(\tau)$-dependent local $SO(4, 2) \times SO(6)$ transformations in order to maintain the gauge fixed form. However, the $\Theta$-dependent local $SO(4, 2) \times SO(6)$ rotates also $(X^M, P^M)$. This combination of global and local transformations become the expected spacetime supersymmetry in 12D super phase space $(X, P, \Theta)$, but it is important to understand that its origin is the much simpler $SU(2, 2|4)_R$ acting only on $g(\tau)$.

Thus the local symmetries are Sp(2, R), SO(4, 2) × SO(6), and the fermionic kappa, while the global symmetry is $SU(2, 2|4)_R$, all of which are linearly realized in the 2T theory de-
scribed by the deceptively simple invariant action in Eq.(2.2) below.

In section 3 an interesting set of constraints is obtained on the Noether charges $J$ of the global SU(2, 2|4)$_R$ given in Eq.(2.6). These constraints imply that only certain special representations of SU(2, 2|4) can be realized on physical states. In particular all the SU(2, 2|4) Casimir eigenvalues Str($J^n$) must vanish for all the physical states that satisfy the Sp(2, $R$) 12-dimensional constraints $X^2 = X \cdot P = P^2 = 0$. The same zero Casimirs must occur in AdS$_5 \times $S$^5$ supergravity since the physical states coincide with the Kaluza-Klein towers that emerge in the AdS$_5 \times $S$^5$ compactification spectrum of type-IIB supergravity. The same conclusion applies to other forms of supergravity spectra in super phase spaces (such as on $M^4 \times R^6$) obtainable via gauge fixing of the 12-dimensional action.

This striking 2T-physics prediction about AdS$_5 \times $S$^5$ supergravity has actually been verified purely within the context of supergravity. Although it has been known for many years that the standard oscillator approach for unitary representations of SU(2, 2|4) [16] describes the Kaluza-Klein towers of supergravity [14][15], the quadratic Casimir of these infinite set of representations was not correctly computed. After the universal zero Casimir eigenvalues was predicted by 2T-physics (in this paper), the direct computation of the quadratic Casimir Str($J^2$) within supergravity [17] verified the expected zero result.

The universal zero eigenvalues found here and [17] are not automatic consequences of representations of SU(2, 2|4). Indeed, as shown in [17], among the infinite number of lowest level (singular) representations of SU(2, 2|4), most representations have non-zero quadratic Casimir eigenvalues. Our universal zero is a nontrivial property of the representations selected by the Sp(2, $R$) gauge invariance conditions $X^2 = X \cdot P = P^2 = 0$, and is explained fundamentally by the 12-dimensional structure of the theory presented here.

It is important to emphasize that the physical constraints on the SU(2, 2|4) representations are fully covariant under all symmetries of the model, and are independent of any gauge fixing. AdS$_5 \times $S$^5$ supergravity happens to emerge in one of the gauge choices, and therefore it shares the same SU(2, 2|4) properties with any other 1T-physics model that would emerge from a different choice of gauge.

In section 4 we show how the AdS$_5 \times $S$^5$ superparticle action emerges by a series of gauge choices. First using the SO(4, 2) $\times$SO(6) gauge symmetry we remove all the bosons from $g(\tau)$ and keep only the fermions $\Theta(\tau)$ as in Eq.(1.4). In this gauge the remaining degrees of freedom are described by the 12-dimensional phase space superspace $(X^M, P^M, \Theta)$ with 32
real (or 16 complex) fermionic components in $\Theta$. Before choosing such a gauge the 2T action
had the global supersymmetry $SU(2,2|4)_R$ as a linearly realized supersymmetry acting on
the right side of $g(\tau)$, but it did not act on $(X^M, P^M)$. After choosing the gauge in Eq.\(^{[14]}\)
the global $SU(2,2|4)$ supersymmetry becomes non-linearly realized and correctly gives the
supertransformations of the 12D superspace $(X^M, P^M, \Theta)$ as already mentioned above.

Next, one can make gauge choices for Sp(2, $R$) and the fermionic local kappa supersym-
metries that are still local symmetries of the action constructed from the 12-D superspace
$(X^M, P^M, \Theta)$. These gauge symmetries reduce the 2T degrees of freedom from the 12D
superspace to 10-dimensional AdS$_5 \times$ S$^5$ or $M^4 \times R^6$ superspace with one time. In principle,
there are many possible gauge choices that produce various 10D superspaces, but only two
of these, the AdS$_5 \times$ S$^5$ or $M^4 \times R^6$ gauges will be explored in this paper. These choices
produce the superspaces for a 10D super particle moving on AdS$_5 \times$ S$^5$ or $M^4 \times R^6$. After
quantization, the physical states of these 1T systems coincide with the spectrum of the cor-
responding form of supergravity. The spectrum is simply described by a single superfield
that provides a basis for a nonlinearly realized unitary representation of the global super-
symmetry $SU(2,2|4)_R$. The non-linearly realized generators of $SU(2,2|4)_R$ that act on the
superfield are explicitly given in terms of the canonical degrees of freedom in a 1T formal-
isim. They are fairly complicated expressions, but they are derived directly from the 12D
action as the Noether charges $J$ that have a very simple and compact expression in the 2T
formalism.

Other gauge fixed forms of the same 2T action can be considered\(^2\). As is usual in 2T

\(^2\) The 2T formulation of the theory is more than a trivial embedding in higher dimensions that obey
constraints. The non-triviality comes from the fact that the 2T theory describes not only one, but many
one-time dynamical systems. The one-time systems form a family of dual theories that holographically
represent the same $d + 2$ theory in many ways in $d$ dimensions. The different 1T dynamics come about
through Sp(2, $R$) gauge fixing that identifies “time”. Choices of gauges in 2T-physics usually identify the
proper time $\tau$ with some combination of the two timelike coordinates $X^0(\tau)$, $X^0'(\tau)$ in target spacetime.
Each such gauge choice corresponds to a different rearrangement of the time evolution of the remaining
target space degrees of freedom as a function of the chosen “time” in target space. Therefore a family
of different Hamiltonians (hence different 1T-dynamics) are obtained by gauge fixing a given 2T-physics
theory. There are measurable consequences of the relationships among the 1T systems that emerge from
the same 2T-theory. The simplest example of such relations is the same Casimirs and representations of the
global symmetry, as already mentioned in this paper, but there is much more content that can be tested.
The success of such tests for a few simple quantum mechanical systems \([1]\) have already established that
the 2T approach has measurable physical consequences. This higher dimensional unification of apparently
physics, there are many gauges that give different 1T dynamical systems which form a family of dual systems. This provides new realizations of SU(2, 2|4) in 1T systems that are expected to be “dual” to the AdS$_5 \times$S$^5$ supergravity system. We will make comments about more general gauge choices at various points in the paper, but the only gauges that will be considered in detail in the present paper are the AdS$_5 \times$S$^5$ or $M^4 \times R^6$ gauges.

II. ACTION AND SYMMETRIES

We define 12 coordinates $X^M$ in flat spacetime with $(10, 2)$ signature and divide them into two sets $X^M = (X^m, X^I)$. The $(10, 2)$ signature is not chosen “by hand”, rather it is required as a consequence of the local Sp(2, $R$) symmetry. Similarly we define the 12 momenta $P^M = (P_m, P_I)$. We indicate $X_i^M \equiv X^M$ and $X_2^M = P^M$, and use the notation $X_i^M$ with $i = 1, 2$ to indicate that the 12 coordinates and momenta form doublets of Sp(2, $R$) for every spacetime index $M$. The orbital “angular momenta” in the 12-dimensional space are Sp(2, $R$) gauge invariant singlets

$$L^{MN} = X^M P^N - X^N P^M = \varepsilon^{ij} X_i^M X_j^N. \tag{2.1}$$

The $L^{MN}$ are the generators of SO(10, 2) that acts on the spacetime indices $M$. The subset $L^{mn}, L^{IJ}$ are the generators of the subgroup SO(4, 2) $\times$SO(6) which will later be related to the Killing symmetry of the space AdS$_5 \times$S$^5$.

We also introduce the supergroup element $g(\tau) \in$SU(2,2|4). This contains bosonic and fermionic degrees of freedom. The bosons are in the adjoint representation of SO(4, 2) $\times$SO(6) =SU(2,2) $\times$SU(4); these will later be eaten away by a SO(4, 2) $\times$SO(6) gauge symmetry that sits at the intersection SO(10, 2) $\cap$SU(2,2|4). The fermions $\Theta(\tau)$ are in the (4, $\bar{4}$) representation of SU(2,2) $\times$SU(4) (i.e. spinor $\otimes$ spinor of SO(4, 2) $\times$SO(6)).

The 2T Lagrangian with an invariant coupling among these degrees of freedom is

$$\mathcal{L} = \left( \dot{X}_i^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN} - \frac{1}{16} \text{Str} \left( L \left( \partial_\tau g g^{-1} \right) \right), \tag{2.2}$$

where

$$L = \begin{pmatrix} L^{mn} \Gamma_{mn} & 0 \\ 0 & -L^{IJ} \Gamma_{IJ} \end{pmatrix}. \tag{2.3}$$

different 1T-physics dynamics (or “dually” related 1T systems), that really describe the same 2T-system, is one of the advantages of 2T-physics.
$\eta_{MN}$ is the flat metric with $(10, 2)$ signature. The gamma matrices $(\Gamma_{mn}, \Gamma_{IJ})$ are given in detail in Appendix-A. They form the spinor representation of the generators of $\text{SO}(4,2) \times \text{SO}(6) = \text{SU}(2,2) \times \text{SU}(4)$. The negative sign in the lower block in (2.3) is multiplied by another negative sign when the supertrace in Eq.(2.2) is evaluated. Thus the coupling consists of projecting the Cartan connection $\partial_{\tau} g^{-1}$ to the subgroup $\text{SU}(2,2) \times \text{SU}(4)$ and then coupling it to the $\text{SO}(4,2) \times \text{SO}(6)$ orbital “angular momenta” $(L^m, L^I)$. Note that the projected Cartan connection is not a pure gauge since it includes contributions from the coset parameters $\Theta$.

This action is a generalization of the superparticle action in $d = 4$ with $\text{SO}(4,2)$ superconformal symmetry in the 2T formalism [3]. In that case we had only the subset of 6 coordinates and momenta $(X^m, P^m)$ with a coupling to $g(\tau)$ that corresponds to the upper block in the matrix in (2.3). In the present case we have the additional 6 coordinates and momenta $X^I, P^I$ with a coupling to $g(\tau)$ via the lower block in the matrix.

In the absence of $g(\tau)$ the purely bosonic Lagrangian is invariant under a global $\text{SO}(10,2)$. This $\text{SO}(10,2)$ is unavoidable because the correct normalization of the kinetic term $X \cdot P$ imposes it automatically and the non-Abelian symmetry $\text{Sp}(2, R)$ forces the same $\eta_{MN}$ on the quadratic dot products $X_i \cdot X_j$ that are the coefficients of the gauge field $A^{ij}$. Thus the dot product cannot be split into two parts with different coefficients for the $(4, 2)$ and $(6, 0)$ parts. Therefore the 12-dimensional constraints

$$X^2 = P^2 = X \cdot P = 0 \quad (2.4)$$

that follow from the $A^{ij}$ equations of motion are necessarily invariant under $\text{SO}(10,2)$.

Although the first term in the Lagrangian has manifest $\text{SO}(10,2)$ symmetry, the second term breaks it to manifest subgroup symmetry $\text{SO}(4,2) \times \text{SO}(6)$, which is also the bosonic subgroup in $\text{SU}(2,2|4)_{L}$. We will see below that the common subgroup that sits at the intersection of $\text{SU}(2,2|4) \cap \text{SO}(10,2)$ is a local $\text{SO}(4,2) \times \text{SO}(6)$ space-time Lorentz symmetry. The relative coefficient ($-1/16$) between the two terms in $L$ is fixed by this local symmetry. We see that there are no free parameters in this action.

The action has a number of global and local symmetries. The brief description below summarizes the essential aspects of more detailed symmetry discussions given in [3].

(1) From the extensive analysis in [1]-[10] we already know that the Lagrangian above has local symmetry under $\text{Sp}(2, R)$ separately for both terms in the Lagrangian ($X^M_i$ is a
doublet, $A^{ij}$ is a triplet gauge field, $g$ is a singlet, $L^{mn}$ and $L^{IJ}$ are invariant). This is the basic local symmetry of 2T-physics; it is responsible for the constraints $X_i \cdot X_j = 0$ that follow as the equations of motion for the gauge field $A^{ij}$. The solution of these constraints requires two time-like coordinates. This local symmetry is also responsible for removing the ghosts associated with two timelike dimensions. No more and no less than two timelike coordinates are possible for a non-trivial unitary system consistent with local Sp(2, $R$). This symmetry allows one to recast the 2T system in fixed gauges in the form of many different looking 1T dynamical systems (different Hamiltonians  [4]) that are related to each other by duality type Sp(2, $R$) transformations. One of these gauge choices will lead to the AdS$_5 \times $ S$^5$ Kaluza-Klein towers.

(2) The Cartan connection $\partial_\tau gg^{-1}$ is invariant under the transformation of $g(\tau)$ by right multiplication $g(\tau) \to g(\tau)g_R$, for global $g_R \subset \text{SU}(2, 2|4)_R$. Therefore, with $X_i^M$ and $A^{ij}$ taken as singlets under SU(2, 2) the Lagrangian is invariant under

$$g \to gg_R.$$  \hspace{1cm} (2.5)

Via Noether’s theorem, the Lagrangian yields the conserved charges $J(\tau)$ which can be written in the form of a $8 \times 8$ supermatrix that belongs to the Lie algebra of SU(2, 2) \text{R}

$$J = i\frac{1}{4}g^{-1}Lg, \quad \partial_\tau J = 0. \hspace{1cm} (2.6)$$

The conservation of the charges $\partial_\tau J = 0$ is verified by using the equations of motion. This global supersymmetry is physical since it is gauge invariant, and cannot be altered by making gauge choices. Therefore, no matter what gauge is chosen to express the theory in terms of canonical variables, the SU(2, 2) \text{R} bosonic and fermionic charges described by $J$ must remain conserved, and must form the Lie superalgebra of SU(2, 2) \text{R} under Poisson brackets at the classical level, or under supercommutators when the theory is quantized and operators properly ordered. This observation will play a crucial role in understanding the supersymmetry SU(2, 2) in terms of AdS$_5 \times $ S$^5$ superspace, supertwistors, or other degrees of freedom arrived at by a variety of gauge fixing choices.

(3) There is a bosonic local Lorentz symmetry SO(4, 2) $\times$ SO(6) defined by the intersection of SO(10, 2) $\cap$ SU(2, 2) \text{L}. The local group element $g_L(\tau) \in \text{SU}(2, 2|4)_L$ acts on $g(\tau)$ by left multiplication $g \to g_Lg$. The local symmetry has infinitesimal parameters $\omega^{mn}(\tau), \omega^{IJ}(\tau)$.
which act on \( g \) in the spinor representation and on \( X_i^m, X_i^j \) in the vector representation

\[
\delta_\omega X_i^m = \omega^{mn} (X_i)_n, \quad \delta_\omega X_i^j = \omega^{IJ} (X_i)_J, \tag{2.7}
\]

\[
\delta_\omega g = \frac{1}{4} \begin{pmatrix} 
\omega^{mn} \Gamma_{mn} & 0 \\
0 & \omega^{IJ} \Gamma_{IJ} 
\end{pmatrix} g, \quad \delta_\omega A^{ij} = 0. \tag{2.8}
\]

The Lagrangian \( \mathcal{L} \) and the physical \( \text{SU}(2, 2\mid 4)_R \) current \( J \) are both invariant. The time derivatives \( \partial_\tau \omega^{mn} (\tau), \partial_\tau \omega^{IJ} (\tau) \) produced by the two terms in the Lagrangian cancel each other. This is what fixes the relative coefficient \(-1/16\) in the Lagrangian. It is possible to display this symmetry more clearly by rewriting the Lagrangian in the form (up to a total \( \tau \) derivative)

\[
\mathcal{L} = \frac{1}{2} (D_\tau X_i) \cdot X_j \varepsilon^{ij}, \tag{2.9}
\]

where \( D_\tau X_i^M \) is the covariant derivative under both \( \text{Sp}(2, R) \) and \( \text{SO}(4, 2) \times \text{SO}(6) \) local transformations

\[
D_\tau X_i^M = \partial_\tau X_i^M - \varepsilon_{ik} A^{kj} X_j^M - \Omega_{MN} X_i^N \tag{2.10}
\]

constructed by using the Cartan connection \( \Omega^{MN} \) for the bosonic subgroup in \( \text{SU}(2, 2\mid 4)_L \)

\[
\Omega^{MN} = \frac{1}{16} \text{Str} \left( \Gamma^{MN} \partial_\tau g \right), \tag{2.11}
\]

\[
\Gamma^{MN} = \begin{pmatrix} \Gamma^{mn} & 0 \\
0 & 0 
\end{pmatrix}, \begin{pmatrix} 0 & 0 \\
0 & -\Gamma^{IJ} 
\end{pmatrix}. \tag{2.12}
\]

Note that \( \Omega^{mI} \) or \( \Gamma^{mI} \) are absent.

(4) There is also a local fermionic kappa supersymmetry embedded in \( \text{SU}(2, 2\mid 4)_L \). The fermionic coset elements in \( \text{SU}(2, 2\mid 4)_L \) act on \( g \) from the left for infinitesimal \( K \) as follows

\[
\delta_\kappa g = K g, \quad \delta_\kappa A^{ij} \neq 0, \quad \delta_\kappa X_i^M = 0, \tag{2.13}
\]

\[
K = \begin{pmatrix} 0 & \xi (\tau) \\
\bar{\xi} (\tau) & 0 
\end{pmatrix}. \tag{2.14}
\]

Here \( \delta_\kappa A^{ij} \) is non-zero as specified below. The local fermionic parameter \( \xi (\tau) \), which is classified as \((4, \bar{4})\) under the subgroup \( \text{SU}(2, 2) \times \text{SU}(4) \) must take the form (for reasons explained below)

\[
\xi (\tau) = \varepsilon^{ij} X_i^m X_j^I (\Gamma_m \kappa \Gamma_I) = L^{mI} (\Gamma_m \kappa \Gamma_I), \tag{2.15}
\]
where the local $\kappa(\tau)$ are the unrestricted local fermionic parameters also classified as $(4, \bar{4})$. Under such a transformation we obtain after some simplification

$$\delta_\kappa L = -\frac{1}{2} (\delta_\kappa A^{ij}) X_i \cdot X_j + \frac{1}{8} \text{Str} \left( \begin{pmatrix} 0 & \psi(\xi) \\ \bar{\psi}(\xi) & 0 \end{pmatrix} (\partial_\tau g g^{-1}) \right),$$

(2.16)

where $\psi(\xi) \equiv L^{mn} (\Gamma_{mn} \xi) + L^{I\bar{J}} (\xi \Gamma_{I\bar{J}})$. The only way to achieve kappa invariance $\delta_\kappa L = 0$ is to arrive at a $\psi(\xi)$ that is proportional to $X_i \cdot X_j$ so that $\delta_\kappa A^{ij}$ can be chosen to cancel the contribution from the second term. Indeed, this property is satisfied for arbitrary $\kappa(\tau)$ provided $\xi(\tau)$ is of the form given in (2.15). To prove this we insert the form of $\xi(\tau)$ in $\psi(\xi)$, work out the algebra of the gamma matrices by using $\Gamma_{MN} \Gamma_R = \Gamma_{MNR} + \eta_{NR} \Gamma_M - \eta_{MR} \Gamma_N$ (for $M = m, I$ etc.) and note that the antisymmetric $\Gamma_{MNR}$ term forces antisymmetry in the indices $[mnr]$ or $[IJK]$ for the expressions $X_i^m X_j^n X_k^r$ or $X_i^I X_j^J X_k^K$ that appear in $\psi(\xi)$. These terms vanish due to the fact that the $\text{Sp}(2, \mathbb{R})$ indices $i, j, k$ take only two possible values. The remaining terms in $\psi(\xi)$ generate dot products $X_i^m X_j^n \eta_{mn}$ or $X_i^I X_j^J \delta_{IJ}$ in such a way that they assemble to the total

$$X_i \cdot X_j \equiv X_i^m X_j^n \eta_{mn} + X_i^I X_j^J \delta_{IJ},$$

(2.17)

with the correct relative coefficient consistent with $\text{SO}(10, 2)$. Therefore $\psi(\xi)$ is proportional to the $\text{SO}(10, 2)$ invariant $X_i \cdot X_j$ when $\xi(\tau)$ is of the form (2.15). This last point is essential to have kappa supersymmetry since the coefficient of $(\delta_\kappa A^{ij})$ in Eq.(2.16) is necessarily invariant under $\text{SO}(10, 2)$ as noted in the paragraph following Eq.(2.2). We can then choose $\delta_\kappa A^{ij}$ to cancel the coefficient of $X_i \cdot X_j$ produced by the second term in $\delta_\kappa L$ in Eq.(2.16) and obtain kappa supersymmetry, off-shell. The physical current $J$ in (2.3) is also invariant under local kappa transformations in the physical sector $\delta_\kappa J = 0$. This is shown by using the same arguments, but now imposing the on shell condition $X_i \cdot X_j \equiv 0$, which is true for the $\text{Sp}(2, \mathbb{R})$ gauge invariant sector (physical states).

III. GAUGE INVARIANT CONSTRAINTS

There are three constraint equations $X_i \cdot X_j \equiv X_i^M X_j^N \eta_{MN} = 0$ that follow from the equations of motion for $A^{ij}$

$$X_i \cdot X_j = X_i^m X_j^n \eta_{mn} + X_i^I X_j^J \delta_{IJ} = 0.$$

(3.1)
It will be useful to write them more explicitly in the form

\[ X_m X^m = -X_I X^I, \quad P_m P^m = -P_I P^I, \quad X_m P^m = -X_I P^I \]  

(3.2)

The expressions \( X_i \cdot X_j \) are the three generators of \( \text{Sp}(2, \mathbb{R}) \) as seen by commuting them. Their vanishing implies that the physical phase space or the physical states of the theory are \( \text{Sp}(2, \mathbb{R}) \) singlets.

Let us examine the gauge invariant \( \text{SU}(2, \mathbb{2})_R \) charges \( J \). The square of this 8×8 supermatrix gives (taking into account quantum ordering of operators)

\[ (J)^2 = -\frac{1}{16} g^{-1} (L)^2 g = \frac{1}{4} g^{-1} \begin{pmatrix} C^{(4,2)}_2 & 0 \\ 0 & C^{(6,0)}_2 \end{pmatrix} g - 2\hbar (J) \]  

(3.3)

where \( C^{(4,2)}_2, C^{(6,0)}_2 \) are the Casimir operators of the orbital \( \text{SO}(4, 2) \times \text{SO}(6) \) subgroup

\[ C^{(4,2)}_2 = \frac{1}{2} L_{mn} L^{mn}, \quad C^{(6,0)}_2 = \frac{1}{2} L_{IJ} L^{IJ} \]  

(3.4)

If we examine the constraints (3.2) we find that these two Casimirs must be the same (respecting order of quantum operators)

\[ C^{(4,2)}_2 = X_m (P_n P^m) X^m - (X_m P^m) (P^n X_n) \]

\[ = X_I (P_J P^I) X^I - (X_I P^I) (P^J X_J) \]

\[ = C^{(6,0)}_2. \]  

(3.5) (3.6) (3.7)

Therefore the first term in \((J)^2\) is proportional to the identity supermatrix \( 1 \), yielding

\[ (J)^2 = \frac{1}{4} C^{(6,0)}_2 1 - 2\hbar (J) = \frac{\hbar^2}{4} l (l + 4) - 2\hbar (J) \]  

(3.8)

Higher powers \((J)^n\) are now easily computed by repeatedly using the formula for \((J)^2\). The eigenvalues of \( C^{(6,0)}_2 = \hbar^2 l (l + 4) \) will increase with \( l = 0, 1, 2, \cdots \), where \( l \) labels the harmonics on \( S^5 \) (see Appendix-B). The representation of \( \text{SU}(2, \mathbb{2})_R \) changes as \( l \) changes, therefore this theory describes an infinite number of representations of the supersymmetry \( \text{SU}(2, \mathbb{2})_R \) (corresponding to the supergravity Kaluza-Klein towers).

Since the left side of Eq.(3.8) (and similarly \((J)^n\)) is an 8×8 supermatrix, this equation expresses rather strong constraints on the generators of \( \text{SU}(2, \mathbb{2})_R \), such that only certain representations of \( \text{SU}(2, \mathbb{2})_R \) can be realized in the physical sector of this theory. In particular the quadratic and all higher Casimir operators of \( \text{SU}(2, \mathbb{2})_R \) must vanish in these
realizations since the supertrace of $1$ and the supertrace of $J$ are zero

$$C_n^{(2,2|4)} = \text{Str} (J^n) = 0. \quad (3.9)$$

These SU$(2,2|4)_R$ properties arose through the 12-dimensional constraints $X^2 = P^2 = X \cdot P = 0$, hence the representations of SU$(2,2|4)_R$ that are selected through these physical state constraints reflect the underlying 12-dimensional structure.

Using the local gauge symmetries for kappa, Sp$(2,R)$, SO$(4,2) \times$SO$(6)$ described in the previous section, both the Lagrangian and the SU$(2,2|4)_R$ charges $J$ can be gauge fixed to various forms and expressed in terms of fewer (physical) degrees of freedom. The details of a specific gauge fixing will be given in the next section, but since the model can be reduced to many possible 1T-physics versions by different gauge choices, it is useful to first argue in general terms about the effect of any gauge fixing on the global supersymmetry SU$(2,2|4)_R$ that commutes with all the gauge symmetries. Since both $L$ and $J$ are gauge invariant, the gauge fixed Lagrangian still has the same SU$(2,2|4)_R$ global supersymmetry after gauge fixing, but then it is realized nonlinearly on the remaining fewer degrees of freedom. The conserved charges $J$ for the nonlinearly realized global supersymmetry SU$(2,2|4)_R$ are still given by the same gauge invariant $J$ that appears in Eq.(2.6). But after gauge fixing, $J$ is expressed in terms of the fewer degrees of freedom that results from inserting the gauge fixed $X, P, g$ into the original expression for $J$. This gauge fixed form of $J$ is still conserved $\partial_\tau J(\tau) = 0$ since this is a gauge invariant equation. Therefore, in fixed gauges, although the realizations are in terms of various degrees of freedom, they all represent the same irreducible representation of SU$(2,2|4)_R$ that satisfy Eqs.(3.8,3.9). This symmetry is one of the observables that ties together the different looking 1T systems.

Typically the SU$(2,2|4)$ symmetry is non-linearly realized on the remaining superspace coming both from the gauge fixed $X_i^M$ and $g$. By contrast, initially the global supersymmetry was linearly realized on the full $g \rightarrow gg_R$, and it did not transform $X^M$ before gauge fixing. The reason that $X^M$ must transform after the gauge fixing is understood as follows: for the gauge fixed $g(\tau)$, the global transformation from the right $g \rightarrow gg_R$ must be followed by a compensating local transformation SO$(4,2) \times$SO$(6)$ from the left in order to maintain its gauge fixed form, but the compensating local transformation SO$(4,2) \times$SO$(6)$ must also act on $X_i^M$; therefore the supersymmetry generated by the global parameters $g_R$ induces transformations on the entire physical superspace defined after any gauge fixing.
The infinitesimal non-linear transformation rules for $\text{SU}(2,2|4)$ in the gauge fixed sectors are automatically obtained by commuting the gauge fixed $J$ with the remaining canonical degrees of freedom. These nonlinear transformations are guaranteed to be the global $\text{SU}(2,2|4)_R$ symmetries of the gauge fixed action (for similar simpler bosonic examples see the third reference in [1] for detailed computations).

The expressions that we will use below for the $\text{AdS}_5 \times S^5$ gauge fixed versions of the charges $J$ or $L^{mn}$ will be needed in this paper at the classical level, so we will not care about quantum ordering of non-linear expressions for most of our discussion. However, quantum ordering of operators must be done such that the vanishing of the quadratic Casimirs $C_n(2,2|4) = 0$ must be obeyed as a gauge invariant condition on the quantum states. We have indeed verified that $C_2(2,2|4) = 0$ is satisfied at the quantum level by the $\text{AdS}_5 \times S^5$ Kaluza-Klein towers in supergravity [17].

IV. $\text{AdS}_5 \times S^5$ KALUZA-KLEIN TOWERS

We will choose a gauge that reduces the 2T system above to a 1T system that describes the $\text{AdS}_5 \times S^5$ supersymmetric Kaluza-Klein towers. We will express the model in the fixed gauge in terms of supercoordinates on the worldline (positions, momenta, theta). Our approach provides a fundamental description of the system and explains the existence of a multiplet structure for the entire tower system. This is in agreement with the Kaluza-Klein towers that result from the $\text{AdS}_5 \times S^5$ compactification of 10D type IIB supergravity [14][15], therefore we obtain a hidden 12-dimensional interpretation of the supergravity spectrum of states. We will choose the gauges to reveal the symmetry structures in stages.

A. 12D superspace or twistor gauge (local Lorentz gauge fixing)

Using the local $\text{SO}(4,2) \times \text{SO}(6)$ symmetry contained in the intersection of $\text{SO}(10,2) \cap G_L$ we can eliminate all the bosons in $g(\tau)$ and keep only the bosons in $X^M, P^M$. In this gauge we have $g(\Theta)$ as given in Eq.(1.4), where $\bar{\Theta} = \Theta^\dagger \gamma^0$. The fermion $\Theta$ is classified as $(4,\bar{4})$ under $\text{SU}(2,2) \times \text{SU}(4) = \text{SO}(4,2) \times \text{SO}(6)$ has 16 complex or 32 real fermionic components. The action and the conserved charges $J = g^{-1}Lg$ are now expressed only in terms of the 12D superspace variables $(X,P,\Theta)$. 

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The action, with this form of \( g \) still has local kappa symmetry that was embedded SU(2,2|4)\(_L\), but now the naive kappa transformation must be followed by a \( \Theta \)-dependent SO(4,2) \( \times \) SO(6) gauge transformation to keep the gauge fixed form of \( g \) unchanged. Similarly, there still is a global SU(2,2|4)\(_R\) symmetry, but this transformation must be compensated by a local SO(4,2) \( \times \) SO(6) gauge transformation. Therefore now both kappa and global supersymmetry transformations also act on \((X,P)\) via the \( \Theta \)-dependent SO(4,2) \( \times \) SO(6) compensating gauge transformations.

There is another way of fixing the local SO(4,2) \( \times \) SO(6) symmetry. We can completely gauge away \((X,P)\) and shift all degrees of freedom to \( g(\tau) \). Then the first term in the action can be completely eliminated by satisfying the constraints \( X^2 = P^2 = X \cdot P = 0 \). The action reduces to only the second term with \( L \) a fixed matrix oriented in a particular direction in SO(4,2) \( \times \) SO(6) space. With an appropriate parametrization of \( g \) this form of the action can be written in terms of twistors by using the techniques of the third reference in [3]. Hence 12D superspace \((X,P,\Theta)\) can be converted to supertwistor space and vice versa.

**B. Kappa gauge fixing**

Half of the fermions in \( \Theta \) can be eliminated by choosing the kappa gauge \( \Gamma^+ \Theta = 0 \), i.e.

\[
\Theta = \begin{pmatrix} \theta^{\alpha a} \\ 0 \end{pmatrix}, \quad \bar{\Theta} = \begin{pmatrix} 0 & \bar{\theta}^{\dot{\alpha}} \end{pmatrix}.
\]  

(4.1)

Here \( a = 1, \cdots, 4 \) is an SU(4) index for the \( \bar{4} \) (raised) or 4 (lowered) representations, and \( \alpha = 1, 2 \) or \( \dot{\alpha} = 1, 2 \) are SL(2,C) =SO(3,1) spinor indices. The \( \theta^{aa}, \bar{\theta}^{\dot{\alpha}} \), are related to each other by Hermitian conjugation and multiplication by the SL(2,C) invariant Levi-Civita tensors \( \varepsilon^{\ddot{\alpha} \ddot{\beta}} \) or \( \varepsilon_{\ddot{\alpha} \ddot{\beta}} \). This gauge fixing breaks the manifest SO(4,2) \( \times \) SO(6) classification of the fermions. Then \( g \) contains only 8 complex fermions or 16 real fermions that are non-linearly coupled to the orbital \( \text{AdS}_5 \times S^5 \) operators \( L^{mn}, L^{IJ} \) (which themselves are also non-linearly realized as given in [4.12-4.16]). The expansion of the exponential yields the following form for \( g \)

\[
g = \begin{pmatrix} 1_2 & \frac{i}{2} \theta \bar{\theta} & \theta \\ 0 & 1_2 & 0 \\ 0 & \bar{\theta} & 1_4 \end{pmatrix}
\]  

(4.2)
With this form of \( g \) both the action and the global SU(2, 2|4) \(_R\) charges \( J = g^{-1}Lg \) simplify dramatically.

C. \( \text{Sp}(2, R) \) gauge fixing and \( \text{AdS}_5 \times S^5 \) versus \( M^4 \times R^6 \)

The \( \text{AdS}_5 \times S^5 \) curved background is created from the flat 12-dimensional background in a special \( \text{Sp}(2, R) \) gauge. Thus, using the local \( \text{Sp}(2, R) \) symmetry we choose two gauges: the component \( P^{+'} = (P^{0'} + P^{1'}) / \sqrt{2} \) of the SO(4, 2) vector vanishes for all \( \tau, P^{+'}(\tau) = 0 \), and the magnitude of the SO(6) vector \( X^I(\tau) \) is taken as a \( \tau \) independent constant \( |X^I(\tau)| = R \). The remaining components of the 12 dimensional vectors \( X^M_i = (X^M, P^M) \) are then parameterized as follows after solving explicitly the two 12-dimensional constraints \( X^2 = X \cdot P = 0 \)

\[
M = ( +', -', \mu, I)
\]

\[
X^M = \frac{R}{|y|} \left( R, \frac{x^2 + y^2}{2R}, x^\mu, y^I \right)
\]

\[
P^M = \frac{|y|}{R} \left( 0, \frac{1}{R} (x \cdot p + y \cdot k), p^\mu, k^I \right).
\]  

(4.3)

(4.4)

Here the \( I \) are SO(6) indices as before, the \( \mu \) are SO(3, 1) indices, and the \( m = (+', -', \mu) \) are SO(4, 2) indices in a lightcone type basis in the extra dimensions \( m = (0', 1') \leftrightarrow (+', -') \).

In this gauge the 12-dimensional flat (10,2) metric becomes the 10 dimensional \( \text{AdS}_5 \times S^5 \) metric after expressing \((y^I, k^I)\) in terms of radial and angular variables (see Appendix-B)

\[
ds^2 = dX^M dX_M = \frac{R^2}{y^2} \left[ (dx^\mu)^2 + (dy)^2 \right] + (d\Omega)^2.
\]

(4.5)

The boundary of the \( \text{AdS}_5 \) space at \( y \to 0 \) is Minkowski space \( x^\mu \) in 4-dimensions.

In the following we work in units of \( R = 1 \) to simplify our expressions. Inserting the gauge fixed form of \( X^M_i \) into the Lagrangian (2.2), and using the definitions of radial and angular variables given in Appendix-B at the classical level, gives

\[
\mathcal{L} = p \cdot \dot{x} + k \cdot \dot{y} - \frac{1}{2} A^{22} \left( p^2 + k^2 \right) y^2 + \frac{1}{2} L^{IJ} L_{IJ} \right) \right) - \frac{1}{16} \text{Str} \left( L \left( \partial_\tau gg^{-1} \right) \right).
\]

(4.6)

For the gauged fixed \( g \) in Eq.(1.2) the second term simplifies to

\[
- \frac{1}{16} \text{Str} \left( L \left( \partial_\tau gg^{-1} \right) \right) = \frac{1}{2} \left( \partial_\sigma \bar{\sigma}_\mu \theta^\mu - \bar{\sigma}_\sigma \sigma_\mu \partial_\tau \theta^\mu \right) p^\mu
\]

(4.7)
The term multiplying $A^{22}$ in Eq.(4.6), after quantum ordering, $P^2 = y (p^2 + k^2) y + \frac{1}{2} L^{IJ} L_{IJ} \sim 0$, is the remaining 12-dimensional constraint. In Appendix-B we show in detail that it has the interpretation of the Laplacian on $\text{AdS}_5 \times \text{S}^5$ when applied on physical states

$$P^2 \phi = -\hbar^2 \nabla^2_{(\text{AdS}_5 \times \text{S}^5)} \phi \sim 0. \quad (4.8)$$

Another way of seeing this is to use the relations in Eqs.(3.5-3.7) that followed from the $\text{Sp}(2, \mathbb{R})$ constraints, including $P^2 = 0$,

$$[-C_2 (4, 2) + C_2 (6)] \phi = 0. \quad (4.9)$$

These too reflect the underlying 12 dimensions in the 2T formalism. Indeed, we have

$$-\hbar^2 \nabla^2_{\text{S}^5} = C_2 (6) = \hbar^2 I (l + 4),$$

and if we compute $C_2 (4, 2) = \frac{1}{2} L_{mn} L^{mn} = \frac{1}{2} L_{\mu \nu} L^{\mu \nu} - (L^{+ -})^2 - L^{+ \mu} L^{- \mu} - L^{- \mu} L^{+ \mu}$ by inserting an appropriately quantum ordered version (see third reference in [1]) of the non-linear $L_{mn}$ in Eqs.(4.12-4.16), we find precisely $C_2 (4, 2) \to +\hbar^2 \nabla^2_{\text{AdS}_5}$.

It is worth noting another 1T-physics gauge choice, namely the relativistic particle gauge given by $X^{+'} = 1$ and $P^{+'} = 0$ used in many previous applications of 2T-physics. Again, solving explicitly the constraints $X^2 = X \cdot P = 0$ the 12 dimensional vectors $X_i^M = (X^M, P^M)$ are parameterized as follows

$$M = ( +', -', \mu, I)$$

$$X^M = \begin{pmatrix} 1, & \frac{1}{2} (x^2 + x^2), & x^\mu, & x^I \end{pmatrix} \quad (4.10)$$

$$P^M = \begin{pmatrix} 0, & (x \cdot p + x \cdot p), & p^\mu, & p^I \end{pmatrix}. \quad (4.11)$$

where the surviving 10 coordinates $(x^\mu, x^I)$ now are in flat 10-dimensional Minkowski space, however the interactions with the 16 $\theta'$s break this space to $M^4 \times R^6$ with linearly realized $\text{SO}(3, 1) \times \text{SO}(6)$ symmetry. The gauge fixed action looks just like Eq.(4.6) except for replacing $(y^I, k^I)$ by $(x^I, p^I)$ and modifying the coefficient of $A^{22}$ to the form $-\frac{1}{2} A^{22} (p^2 + p^2)$.

The two 1T-physics superparticle actions obtained by the above gauge fixing represent the same 2T-physics system, but with different 1T Hamiltonians. In this sense they are duals. They both have the same gauge invariant global symmetry $\text{SU}(2, 2|4)$ realized in the same representations with vanishing Casimirs as in Eq.(3.9) at the quantum level.
Therefore we see signs of a possible duality between $\text{AdS}_5 \times S^5$ supergravity and $M^4 \times R^6$ supergravity which must be true at the level of the kinetic term, but remains to be examined in the presence of interactions.

D. Global $SU(2,2|4)_R$ supersymmetry generators

Using the equations of motion that follow from the gauge fixed Lagrangian, one may easily verify that there is a conserved $SU(2,2|4)_R$ current that is none other than the gauge fixed form of the global current given in (2.6), $J = g^{-1}Lg$. This is expected since the global symmetry $SU(2,2|4)_R$ in the original action, and the corresponding symmetry current $J$, commute with all the local symmetries $\text{Sp}(2,R), \text{SO}(4,2) \times \text{SO}(6)$ and kappa. Any fixing of the gauge symmetries cannot destroy the physical global symmetry. However, the transformation rules of the surviving degrees of freedom become complicated because the naive $SU(2,2|4)_R$ transformation of the gauge fixed $g$ must be compensated by a local $(x,p,\theta$-dependent) kappa, $\text{SO}(4,2) \times \text{SO}(6)$ and $\text{Sp}(2,R)$ gauge transformations in order to maintain the form of the gauge fixed $g,X,P$.

The complicated nonlinear $SU(2,2|4)_R$ transformations are all taken into account automatically in the gauge fixed expression of the global $SU(2,2|4)_R$ charges $J = g^{-1}Lg$. To generate the correct transformations we only need to write these charges in terms of the canonical variables of the surviving degrees of freedom and commute them with any quantity that needs to be transformed. At the classical level orders of factors of canonical variables are neglected.

To construct the gauge fixed charges $J$ we begin with the orbital $L^{MN}$. The non-linear $\text{SO}(10,2)$ generators are constructed in the $\text{AdS}_5 \times S^5$ background by inserting the gauge fixed form of $(X^M,P^M)$ in $L^{MN} = X^M P^N - X^N P^M$. We get the classical expressions (not
watching orders of factors\(^3\)

\[
L^{IJ} = y^I k^J - y^J k^I, \quad L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \tag{4.12}
\]

\[
L^{+\cdot-} = y \cdot k + x \cdot p, \quad L^{+\mu} = p^\mu, \quad L^{+I} = k^I, \tag{4.13}
\]

\[
L^{-\mu} = \frac{1}{2} (x^2 + y^2) p^\mu - (x \cdot p + y \cdot k) x^\mu, \tag{4.14}
\]

\[
L^{-I} = \frac{1}{2} (x^2 + y^2) k^I - (x \cdot p + y \cdot k) y^I, \tag{4.15}
\]

\[
L^\mu_I = x^\mu k^I - p^\mu y^I. \tag{4.16}
\]

These are easily expressed in terms of radial/angular variables as in Appendix-B. The components of the SO(10, 2) angular momenta \(L^{IJ}, L^{mn}, L^{mI}\) that appear in the last term of the gauge fixed Lagrangian in \((4.6)\), the gauge fixed current \(J\) in \((2.6)\), and the gauge fixed kappa transformation rules in \((2.13, 2.16)\) now contain the non-linear forms given above.

For the \(M^4 \times R^6\) gauge the expressions for the \(L^{MN}\) are identical to those of AdS\(_5 \times S^5\) expressions above at the classical level (except for replacing \((y^I, k^I)\) by \((x^I, p^I))\), but at the quantum level the ordering of operators is different, and is probably the same as flat 10D space (see the first reference in [1]).

The various components of the 8\times8 matrix \(J\) can be identified as the superconformal charges \(P^\mu, J^{\mu\nu}, D, K^\mu, Q, S\) and R-symmetry charges \(J^{IJ}\) as follows

\[
J = \begin{pmatrix}
J^\beta_\beta + \frac{1}{2} \delta^\beta_\beta D & K^{\alpha\beta} & S^{ab} \\
\sigma_\alpha^{\beta\gamma} D & J^\gamma_\alpha & Q^b_a \\
\sigma_\beta^{\alpha\gamma} S & S^\gamma_\alpha & J^b_a
\end{pmatrix}, \tag{4.17}
\]

where, after using the standard definitions that convert \(SL(2, C)\) and SU(4) spinor indices to spacetime indices,

\[
P^{\alpha\beta}_\beta = (\sigma_\mu)_{\dot{\alpha}\dot{\beta}} P^\mu, \quad K^{\alpha\beta}_\beta = (\sigma_\mu)_{\dot{\alpha}\dot{\beta}} K^\mu, \tag{4.18}
\]

\[
J^\alpha_\beta = \frac{1}{2} (\sigma_{\mu\nu})^\alpha_\beta J^{\mu\nu}, \quad J^\beta_\alpha = \frac{1}{2} (\sigma_{\mu\nu})^{\dot{\alpha}\dot{\beta}} J^{\mu\nu}, \tag{4.19}
\]

\[
J^b_a = (\tau I)_{\alpha}^{\beta} J^{IJ}, \quad \bar{Q} = Q^\dagger (i\sigma_2), \quad \bar{S} = S^\dagger (i\sigma_2), \tag{4.20}
\]

\(^3\) These are the classical expressions needed in the classical transformation rules. Correct quantum ordering is not needed in the classical action in this paper, but is needed in other contexts. See the third reference [1] (hep-th/9810025) for the correct quantum ordering and verification of the full SO(10, 2) symmetry of the quantum ordered Laplacian. Compared to this reference we have redefined \(y^I = \frac{u^I}{\sqrt{2}}\).
the generators of SU(2, 2|4) are given in terms of the dynamical variables \((x^\mu, y, \Omega)\) and their canonical conjugates \((p^\mu, k, L^{IJ})\) as follows

\[
P^\mu = p^\mu, \quad Q_{a\alpha} = \bar{\theta}_a^\beta \partial_{\beta \alpha}, \tag{4.21}
\]

\[
S^{a\alpha} = x^{\alpha \beta} \bar{Q}_\beta^a + \theta^{ab} L_b^a + \theta^{ab} Q_{b\beta} \theta^{\beta \alpha}, \tag{4.22}
\]

\[
J^{IJ} = L^{IJ} + \frac{1}{2} (\theta \Gamma^{IJ} Q + \bar{Q} \Gamma^{IJ} \bar{\theta}), \tag{4.23}
\]

\[
J^{\mu \nu} = L^{\mu \nu} + \frac{1}{2} (Q_a \sigma^{\mu \nu} \theta^a + \bar{\theta} a \bar{\sigma}^{\mu \nu} \bar{Q}^a), \tag{4.24}
\]

\[
D = y k + x \cdot p + \frac{1}{2} (Q_a \theta^a + \bar{\theta} a \bar{Q}^a), \tag{4.25}
\]

\[
K^\mu = L^{-\mu} + \frac{1}{2} x^{\alpha \beta} \bar{Q}_\beta^a \bar{\theta}_a^{\gamma}(\bar{\sigma}^\mu)_{\gamma \alpha} + \frac{1}{2} (\bar{\sigma}^\mu)_{\gamma \alpha} \theta^{\alpha \alpha} Q_a \bar{x}_\alpha \bar{\sigma}^{\beta \gamma} + \theta^{ab} L_b^a \bar{\theta}_a^{\gamma}(\bar{\sigma}^\mu)_{\gamma \alpha} + \frac{1}{4} \theta^{ab} Q_{b\beta} \theta^{\alpha \alpha} \bar{\theta}_a^{\gamma}(\bar{\sigma}^\mu)_{\gamma \alpha}, \tag{4.26}
\]

where \(L^{\mu \nu}, L^{IJ}, L^{-\mu}\) and \(L^b_a = (\Gamma_{IJ})^b_a L^{IJ}\) are given in (4.12-4.16). The case for the \(M^4 \times R^6\) gauge is similar.

To find the non-linear action of the SU(2, 2|4) symmetry on the 10D super phase space variables we only need to commute the generators given above with the remaining canonical variables, or any other operator \(O\) constructed from the canonical variables

\[
\delta_\varepsilon O = -i \left[ \text{Str} (\varepsilon J), O \right], \tag{4.27}
\]

where \(\varepsilon\) is the infinitesimal global parameters of the original SU(2, 2|4). For details of similar discussions of nonlinearly realized symmetries see [1]. In the present problem we also need to know the commutation rules for fermions. From Eq.(4.17) we see that the canonical conjugate to \(\bar{\theta}_a^a\) is \(\pi^a_a = \frac{1}{2} (\bar{\sigma}^\mu)_{\gamma \alpha} \theta^{\alpha \alpha} p^\mu\). This is a second class constraint which can be solved (since \(p^2 \neq 0\), by taking the fundamental non-zero (anti)commutation rules for fermions to be

\[
\{ \bar{\theta}_a^a, \theta^{\beta b} \} = \hbar \frac{2p^\mu (\sigma_{\mu})_{\alpha \beta} \delta^{b}_{\beta}}{p^2} \delta^a_\alpha. \tag{4.28}
\]

After this, obtaining the \(\delta_\varepsilon O\) for any \(O\) is a straightforward computation. In particular, it can be shown that the gauge fixed action is invariant under the nonlinear transformations generated by these SU(2, 2|4) charges, as should be expected by construction.

Note that the gauge invariant constraints \((J)^2 = -4C_2 (6) + 8i \hbar (J)\) and vanishing Casimir conditions \(C_n (2, 2|4) = 0\) of Eqs.(3.8,3.9) are automatically satisfied by the generators given above (at either the classical level, or quantum level, after appropriate quantum ordering in any gauge).
E. Spectrum and the $\text{AdS}_5 \times S^5$ supersymmetric Kaluza-Klein towers

The gauge fixed action describes the entire $\text{AdS}_5 \times S^5$ supersymmetric Kaluza-Klein tower given in previous literature [14][15]. To see this, it is worth noticing the following simple observations. First and foremost, the global symmetry of the theory is $\text{SU}(2, 2|4)_R$, therefore all states must fall into infinite dimensional unitary representations of this group. Such representations are fully characterized by their lowest states labelled by the compact bosonic subgroup $\text{SU}(2)_1 \times \text{SU}(2)_2 \times \text{SU}(4)$. The specific representations that are realized are built from the canonical degrees of freedom given above. Once the graviton Kaluza-Klein tower is determined, the $\text{SU}(2, 2|4)$ supersymmetry guarantees that the full super multiplet structure is also present at all Kaluza-Klein levels. Such simple facts are also the determining factor of the representation content of supergravity solutions [14], but here they occur in the context of the superparticle. Hence the representations are identical once the supergravity multiplet is determined.

The quantum physical states are represented by a superfield with 8 complex $\theta'$s and the position coordinates (actually there is also dependence on $\bar{\theta}$ like in a chiral superfield). In the following we deal with the $\text{AdS}_5 \times S^5$ case (the $M^4 \times R^6$ case is similar and simpler). Thus, we consider the physical states in the form of the superfield

$$\Phi \left( x^\mu, y^I, \theta \right) = \Phi \left( x^\mu, y, \Omega^I, \theta \right).$$ (4.29)

It is a simple exercise to identify the $2^8$ fields corresponding to 128 bosons and 128 fermions as the 10-dimensional $\text{AdS}_5 \times S^5$ supergravity fields. We have already constructed the generators of $\text{SU}(2, 2|4)$ acting on this basis, and therefore shown that this superfield is a basis for a non-linear representation of $\text{SU}(2, 2|4)$. Thus the supergravity multiplet is evident, while the Kaluza-Klein towers are simply the expansion of the superfield in harmonics of $S^5$.

---

This can be understood by going to the “rest frame” of $p^\mu$ in which spin is identified as the SO(3) subgroup. Then, with $\alpha \leftrightarrow \dot{\alpha}$ doublet indices identified as SO(3) = SU(2) spin indices, the 8 $\theta$'s behave like spin=1/2 creators and the 8 $\bar{\theta}$'s like annihilators, according to Eq.(4.28). Applying all possible powers of 8 fermionic creators on a vacuum gives 128 bosons and 128 fermions. The components of the superfield identified in this “rest” frame are in one-to-one correspondence to the supergravity states. This is verified by the SO(3) reduction of the $\text{SU}(2) \times \text{SU}(2) = \text{SO}(4)$ list of states provided in [17]. There $\text{SO}(3) \subset \text{SU}(2) \times \text{SU}(2) \subset \text{SU}(2, 2) \subset \text{SU}(2, 2|4)$, while here the same SO(3) occurs in the chain $\text{SO}(3) \subset \text{SO}(3, 1) \subset \text{SU}(2, 2) \subset \text{SU}(2, 2|4)$, which is is why these supergravity fields, labelled differently, can be put into one to one correspondence. Actually, the analytic continuation of $\text{SU}(2) \times \text{SU}(2) = \text{SO}(4)$ to $\text{SL}(2, C) = \text{SO}(3, 1)$ allows the rewriting of the list in covariant SO(3, 1) notation as outlined in [17].
This superfield is subject to the remaining constraint that corresponds to the vanishing of the AdS$_5 \times $S$^5$ Laplacian that arises as follows. The physical states in our AdS$_5 \times $S$^5$ gauge must be consistent with the Sp(2, $R$) gauge invariance, implying that they should satisfy the constraints $X_i \cdot X_j \sim 0$. Although the constraints $X^2 = X \cdot P = 0$ have been explicitly solved in the chosen Sp(2, $R$) gauge, the remaining constraint $P^2 = 0$, or equivalently the AdS$_5 \times $S$^5$ Laplacian, takes the following form on the graviton tower

$$\text{Casimir of SO}(4,2) - \text{Casimir of SO}(6) = 0.$$

This is the same as the AdS$_5 \times $S$^5$ Laplacian as shown in Appendix-B. Thus, the mass of the field as given by the Casimir of SO(4, 2) is now fully determined by the Casimir of SO(6). In turn, the Casimir of SO(6) is determined by the SO(6) representations that can be constructed from the traceless symmetrized products of the six $y^I$. The traceless tensor with $l$ indices has the SO(6) Casimir eigenvalue $l(l + 4)$. Thus, the mass of the Kaluza-Klein modes of the graviton supermultiplet and all of its AdS$_5 \times $S$^5$ partners are determined by the integers $l = 0, 1, 2, \cdots$.

For the SU(2, 2|4) classification of the Kaluza-Klein towers, the orbital part is expressed in angular momentum space SO(4, 2) $\times$ SO(6) instead of 10D configuration space $(x^\mu, y^I)$. The total SO(6) = SU(4) representation of each member of the SU(2, 2|4) supermultiplet is then determined by combining the orbital SO(4, 2) $\times$ SO(6) quantum numbers of the state with those SO(4, 2) $\times$ SO(6) quantum numbers coming from the 128$_B + 128_F$ multiplet. This was done in [15]. Such representations are more conveniently described in the oscillator formalism [16, 14, 17].

V. DISCUSSION

The 2T-physics approach to AdS$_5 \times $S$^5$ superspace introduced a natural underlying 12-dimensional superspace and its associated symmetries. Evidence of the hidden 12-dimensional structure may be expected in supergravity. In particular this approach predicts that the Casimir operators of SU(2, 2|4), i.e. Str($J^n$) for any $n$, applied on the superfield must vanish as a consequence of the 12D constraints $X^2 = P^2 = (X \cdot P + P \cdot X) = 0$ applied on physical states. This is true either in the fully covariant 12D quantization or the partially gauge fixed 10D quantization on AdS$_5 \times $S$^5$ or $M^4 \times R^6$ or others. Therefore, it
must be true in the supergravity spectrum.

The representations of SU(2, 2|4) that arise here are highly limited by the algebraic constraints on its generators \( (J) \) written as an \( 8 \times 8 \) supermatrix

\[
(J)^2 = \frac{\hbar^2}{4} l (l + 4) 1 - 2 \hbar (J), \tag{5.1}
\]
as derived in Eq.(3.8). In particular the fields of supergravities on AdS\( _5 \times S^5 \) and M\( _4 \times R^6 \) or other 2T-dual versions, which are equivalent to the quantum states of our 12D superparticle, must be consistent with these algebraic constraints. We conjecture that any SU(2, 2|4) representation that satisfies these constraints must be uniquely equivalent to the 12-dimensional superparticle representation presented in this paper.

Indeed direct computation within AdS\( _5 \times S^5 \) supergravity, using the independent oscillator formalism, has already shown that all Casimir operators do have a universal zero eigenvalue for all Kaluza-Klein towers \([17]\), consistent with Eq.(5.1). In further work on the oscillator formalism \([18]\) we have shown that the oscillator states that satisfy the algebraic constraints above indeed form a unique subset within the vast Fock space, such that they uniquely give the SU(2, 2|4) representations that correspond to all the Kaluza-Klein towers of AdS\( _5 \times S^5 \) supergravity. Additional evidence of the 12D structure related to Eq.(5.1) is likely to emerge. In particular the suggested duality between AdS\( _5 \times S^5 \) and M\( _4 \times R^6 \) supergravity would be an interesting avenue to explore. According to the results of this paper this duality must exist at the level of the spectra (or the kinetic terms) and it would be interesting to investigate it including the supergravity interactions.

We remind the reader that there are multiple advantages of the 2T formulation. First it displays explicitly the \( d+2 \) higher dimensional spacetime supersymmetries that are hidden in 1T-physics. Second, by different gauge choices, it provides a family of 1T-physics dynamical systems (different Hamiltonians\(^5\) in \( d \) dimensions) that are dual to each other in the sense that they can be related to each other under the gauge transformations in the enlarged

\(^5\) The reason that there are many 1T theories under the umbrella of the same 2T theory, is because there is a single worldline parameter \( \tau \) but there are two timelike dimensions in target spacetime \( X^M(\tau) \). The two times \( X^0(\tau), X^0(\tau) \) are not introduced arbitrarily, rather their presence is required as a consequence of the local Sp(2, \( R \)) symmetry. In the gauge fixing process one must decide which combination of the two times will be identified with \( \tau \). Each such choice corresponds to a different rearrangement of the evolution of the 2T system (in terms of fewer 1T degrees of freedom) and therefore results in a different Hamiltonian that represents a 1T dynamical system. Thus, a rich set of dualities is established among the members of a family of 1T dynamical systems through the 2T formalism. These ideas were illustrated with simple
These gauge transformations become duality type transformations among the
gauge fixed 1T dynamical systems. Each member in the 1T family of dynamical systems in
d dimensions holographically represents the same 2T system in \(d + 2\) dimensions. We have
not explored the multiple 1T systems in this paper, except for the two cases \(\text{AdS}_5 \times S^5\) and
\(M^4 \times R^6\) but it would be very interesting to pursue this avenue in the future to find multiple
dual systems related to supergravity.

Are there generalizations of the superparticle action considered in this paper? For cases,
such as \(\text{AdS}_4 \times S^7\), \(\text{AdS}_7 \times S^4\), \(\text{AdS}_6 \times S^2\), etc., with symmetries \(\text{OSp}(8^*|4)\), \(F_4\), etc., one is
tempted to extend the superparticle actions based on \(\text{OSp}(8^*|4)\) and \(F_4\) in dimensions \(d = 3, 5, 6\) mentioned in the introduction, by adding additional coordinates \(X^I\) and coupling them
to the supergroup element \(g(\tau)\) as we did for \(\text{SU}(2, 2|4)\). However, the resulting theory does
not seem to describe the spectrum of supergravity compactified on \(\text{AdS}_7 \times S^4\) etc., but rather
some other spectrum which is interesting anyway.

For example, consider the case of \(G = \text{OSp}(8^*|4)\). The action is the same as Eqs. (2.2)
with \((11, 2)\) dimensions \((X^M, P^M)\), and Eq. (2.3) is replaced by

\[
L = \begin{pmatrix}
\frac{1}{2} L^{mn} \Gamma_{mn} & 0 \\
0 & -L^{IJ} \Gamma_{IJ}
\end{pmatrix}
\]  

The extra factor of \(\frac{1}{2}\) in the first entry of the matrix \(L\) is required, due to the 8 dimensions
of the spinor in \((6, 2)\) dimensions, to maintain the local \(\text{SO}(6, 2) \times \text{SO}(5)\) symmetry. This
action has global symmetry \(\text{OSp}(8^*|4)\) and local symmetry \(\text{Sp}(2, R) \times \text{SO}(6, 2) \times \text{SO}(5)\), but
it fails to have the kappa supersymmetry. The reason is that the extra factor of \(\frac{1}{2}\) needed
for local \(\text{SO}(6, 2) \times \text{SO}(5)\) spoils the correct relative coefficient needed for the automatic
\(\text{SO}(11, 2)\) in the steps that led to Eq. (2.17). Therefore, there is no local kappa symmetry,
which means the number of \(\Theta\)'s cannot be cut down from 32 real components to 16 real
components. Therefore the spectrum is described by a superfield \(\Phi(x^\mu, y, \Omega, \Theta)\) with 32 real
\(\Theta\)'s. It has \(2^{15}\) bosonic and \(2^{15}\) fermionic fields as functions of \(\text{AdS}_7 \times S^4\) (or \(M^6 \times R^5\), with
quantum numbers that correspond to the first massive level of the 11D supermembrane.  

---

\[\text{examples in [1]-[10]}\]. The existence of previously unknown hidden symmetries and duality relations among
simple systems (such as free particle, hydrogen atom, harmonic oscillator, particle in AdSxS backgrounds,
etc.) is part of the evidence for the validity and effectiveness of the underlying 2T structure.

\[\text{Generally the full factor in front of } L^{mn} \text{ (similarly } L^{IJ} \text{), including the } 1/16 \text{ in the action of Eq. (2.2), is}
\frac{1}{4s_D} \Gamma_{mn} L^{mn}, \text{ where } s_D \text{ is the size of the spinor in D-dimensions.}\]
This supermultiplet is much larger than the supergravity multiplet with 128 bosons and 128 fermions. This superfield may be interesting for investigations in 11-dimensional $M$-theory, but is not useful to investigate $\text{AdS}_4 \times S^7$ or $\text{AdS}_7 \times S^4$ supergravity.

There are supergroups and appropriate $d + 2$ spacetime dimensions that would parallel the structure of $SU(2,2|4)$ action of this paper, including the local kappa supersymmetry. The crucial point is the size of the spinor representations for the dimensions associated with $X^m$ and $X^I$. When the spinor is of the same size the kappa supersymmetry is automatically present. A straightforward case is the supergroup $SU(1,1|2)$ applied to $\text{AdS}_2 \times S^2$ (or $R^1 \times R^3$) as exact analog of the present paper, but in 4+2 dimensions instead of 10+2 dimensions. Additional cases include

- The supergroup $SU(1,1|2) \times SU(1,1|2)$ with $X^M = (10,2)$ such that $X^m = (2,1) + (2,1)$ and $X^I = (3,0) + (3,0)$ are coupled via the $L^{MN}$ to the four bosonic subgroups. After eliminating all the bosons from $g(\tau)$, the remaining super phase space has $(X^M, P^M, \Theta)$ with 8 complex or 16 real $\Theta$’s. The kappa supersymmetry cuts down the fermions to 8 real components and $\text{Sp}(2)$ reduces the bosonic space to 10 dimensions with one time. Therefore the physical states are described by a superfield with 8 bosons and 8 fermions in 10 dimensions. This spectrum must correspond to a compactified version of 10-dimensional super Yang Mills theory with a global symmetry of $SU(1,1|2) \times SU(1,1|2)$.

- The supergroup $SU(1,1|2) \times SU(2|2)$ with $X^M = (10,2)$ and $X^m = (2,1)$ and $X^I = (3,0) + (3,0) + (3,0)$ coupled via the $L^{MN}$ to the bosonic subgroups. This must correspond again to another form of compactified super Yang Mills theory with a global symmetry of $SU(1,1|2) \times SU(2|2)$.

Other supergroups provide generalizations in a different direction, by having additional bosonic coordinates in $g(\tau)$ that cannot be gauged away. In appropriate cases the extra bosonic coordinates can be interpreted precisely as collective coordinates (providing charges) for D-branes. In particular the supergroup $G = \text{OSp}(1|64)$ gives an interesting “Toy M-model” described briefly in previous publications [3].

A similar situation can arise with alternative schemes in constructing models using $\text{OSp}(8|4)$ (or $F_4$, etc.). As mentioned above, if we insist on $\text{SO}(6,2) \times \text{SO}(5)$ (for $F_4$:
SO(5, 2) × SO(3)) local symmetry to gauge away all bosons in g(\tau), then we cannot have kappa supersymmetry because the spinor dimensions of these subgroups are different from one another. However, there is another alternative: we can insist on kappa supersymmetry, which is guaranteed for g(\tau) ∈ OSp(8|4) (or F_4, etc.), if we take the same coefficient in the blocks of L (i.e. when L has the form of Eq.(2.3) instead of Eq.(5.2)). Then, with the factor 1/32 in the second term of the action Eq.(2.4), there is local SO(6, 2) symmetry (or SO(5, 2) for F_4), but only global SO(5) (for F_4: SO(3)) symmetry. This implies that the Sp(4) (for F_4: SU(2)) bosons in g(\tau) remain as additional bosonic degrees of freedom. The extra bosons correspond to D-brane collective degrees of freedom. Of course, now there are half as many physical Θ’s thanks to the local kappa symmetry. With 16 remaining physical fermions (8 for F_4), one obtains the 11D supergravity states (10D Yang-Mills states for F_4). Therefore the quantum states of the model must correspond to the AdS_7 × S^5 compactified version of 11-dimensional supergravity in the presence of D-branes, with nontrivial charges in the 5-dimensions (for F_4: AdS_6 × S^2 × S^2 compactification of 10D super Yang Mills).

For all the generalized cases of the type described in the previous paragraph, the global supersymmetry G has charges (J) written in the form of a supermatrix J = \frac{i}{2} g^{-1} L g. By the same arguments as SU(2, 2|4) that we discussed in Eqs.(3.1-3.8), we obtain the form

\[(J)^2 = \frac{2\hbar}{4} l (l + n - 2) \left[ \frac{\hbar^2}{16} (d^2 - n^2) \right] g^{-1} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) g \]  (5.3)

where l = 0, 1, 2, · · · is the angular momentum of the harmonics on S^n as discussed in Appendix-B, and d, n correspond to the AdS_d × S^n described by the model. The last term arises from reordering operators in Eqs.(3.3-3.7) before the constraints in Eq.(3.2) can be applied. This equation is an algebraic constraint on the representations of G, which distinguishes the ones relevant for the superparticle described by the corresponding action. It determines all properties of the representation, including the physical states, their representation properties, and eigenvalues of all the Casimir operators. In particular, since the supertace of the right hand side does not vanish, these Casimir eigenvalues do not vanish.

7 Similarly, for the AdS_4 × S^7 model, with the analytically continued g(\tau) ∈ OSp(8, 4), we take the factor 1/16 to have local SO(3, 2) and global SO(8) symmetry. Also, for AdS_2 × S^6 model, with the analytically continued g(\tau) ∈ F_4, we take the factor 1/8 to have local SO(1, 2) and global SO(7) symmetry.
They are predicted to be certain numbers that depend on the representation labels, i.e.

\[ C_2 = \frac{1}{2} \text{Str} \left( J^2 \right) = \frac{\hbar^2}{8} i (l + n - 2) \text{Str} \left( 1 \right) - \frac{\hbar^2}{16} (d^2 - n^2) \text{Str} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right), \quad (5.4) \]

while higher Casimirs are computed by repeatedly applying Eq. (5.3). Thus, these 2T-physics models make rather strong predictions which can be tested by analyzing the representation content of the corresponding field theories. The verification of such predictions would imply the existence of the hidden higher dimensional structures in the corresponding field theories.

These and other generalizations of interest will be further studied in future publications.

1. Appendix-A (gamma matrices)

The gamma matrices for SO(4, 2) = SU(2, 2) connect the 4 and 4 spinor representations. They may be taken as \( \Gamma_m = (1, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5) \) for \( 4 \times 4 \) or \( \bar{\Gamma}_m = (-1, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5) \) for \( 4 \times 4 \), which differ in the representation of \( \Gamma_0 \). These satisfy \( \Gamma_0 (\Gamma_m)^\dagger \Gamma_0 = \Gamma_m \) and \( \Gamma_0 (\Gamma_{mn})^\dagger \Gamma_0 = \Gamma_{mn} \). Similarly one constructs the fifteen SO(6) = SU(4) generators by using the analytic continuation of the two timelike directions that give \( \Gamma_I = (i, \gamma_1, \gamma_2, \gamma_3, \gamma_5) \) or \( \bar{\Gamma}_I = (-i, \gamma_1, \gamma_2, \gamma_3, \gamma_5) \), with \( \gamma_4 = i \gamma_0 \). In this case \( \Gamma_I = \Gamma_I^\dagger \) and \( (\Gamma_{IJ})^\dagger = -\Gamma_{IJ} \). In this notation the inverse of the SU(2,2|4) group element \( g \) is given by \( g^{-1} = C^{-1} g^\dagger C \), where

\[
C = \begin{pmatrix} \Gamma_0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Another basis for SO(4, 2) gamma matrices is the one used in [3]. We use \( \Gamma^{\pm} = \pm \tau^\pm \times 1 \), \( \Gamma^0 = \tau_3 \times i \sigma_2 \), \( \Gamma^1 = \tau_3 \times \sigma_1 \), \( \Gamma^3 = \tau_3 \times \sigma_3 \), which are purely real, and the last one proportional to the identity which is purely imaginary \( \Gamma^2 = i 1 \times 1 \). Then define \( \Gamma^m = (\Gamma^+, \Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3) \) and \( \bar{\Gamma}^m = (\Gamma^+, \Gamma^0, \Gamma^1, -\Gamma^2, \Gamma^3) \). These satisfy \( \Gamma^m \bar{\Gamma}^m + \Gamma^m \bar{\Gamma}^m = 2 \eta^{mn} \). The SU(2, 2) generators \( \frac{1}{2} \Gamma_{mn} \) are given in terms of \( \Gamma_{mn} = \frac{1}{2} (\Gamma_m \bar{\Gamma}_n - \bar{\Gamma}_m \Gamma_n) \). These satisfy \( u^{-1} (\Gamma^m)^\dagger u = \bar{\Gamma}^m \) and \( u^{-1} (\Gamma^{mn})^\dagger u = -\Gamma^{mn} \), where \( u = \tau_1 \times i \sigma_2 \) is the charge conjugation matrix. Similarly, define SO(6) gamma matrices \( \Gamma^I \) by analytic continuation of the above, \( \Gamma^6 = i \Gamma^0 \) and \( \Gamma^4 = i \Gamma^0 \). Then \( (\Gamma^I)^\dagger = \Gamma^I \) (i.e. \( \Gamma^2 \) changes sign) and the generators
are antihermitian \((\Gamma^{IJ})^\dagger = -\Gamma^{IJ}\). Then the group element satisfies \(g^{-1} = C^{-1}g^\dagger C\) where
\[
C = \begin{pmatrix}
u & 0 \\
0 & 1
\end{pmatrix}.
\]

2. Appendix-B (12D constraints and Laplacian on \(\text{AdS}_5 \times S^5\))

In this appendix we relate the 12-dimensional constraint \(P^2 = 0\) to the Laplacian on \(\text{AdS}_5 \times S^5\)
\[
\nabla^2_{(\text{AdS}_5 \times S^5)} \sim 0. \tag{5}
\]
We have seen that the \(\text{AdS}_5 \times S^5\) gauge in Eqs.(4.3,4.4) already solves the 12D constraints \(X^2 = X \cdot P = 0\). After quantum ordering, the remaining constraint takes the form \(P^2 = y\left(p^2 + k^2\right)y\), where \(y = |y|\) is the radial component of the six dimensional vector \(y^I\). It will be handled quantum mechanically by applying it on physical states \(\Psi(x^\mu, y)\)
\[
P^2\Psi(x^\mu, y) = y\left(p^2 + k^2\right)y\Psi(x^\mu, y) = 0. \tag{6}
\]

To proceed, we define the six Cartesian variables \(y^I\) in terms of angular and radial variables \(y^I = y\Omega^I\), where \(\Omega\) is a unit vector that describes \(S^5\), \(\Omega^2 = 1\), and \(y\) is the radial variable. The corresponding momenta are then
\[
k^I = y\Omega^I - \frac{1}{2y}\left(L^{IJ}\Omega^J + \Omega^J L^{IJ}\right), \tag{7}
\]
where \((y, k)\) are canonical radial variables defined by \(y = |y|\) and \(k = (k \cdot \Omega + \Omega \cdot k)/2\). Evidently the radial variables \((y, k)\) commute with the \(\text{SO}(6)\) generators \(L^{IJ}\). The \(\text{SO}(6)\) generators \(L^{IJ} = y^I k^J - y^J k^I\) are expressed purely in terms of angular variables and their derivatives on \(S^5\), while the radial variables \((y, k)\) together with the 4-dimensional Minkowski variables \((x^\mu, p^\mu)\) make up the 5 canonical pairs on \(\text{AdS}_5\). The canonical operators in this appendix are quantum ordered. In particular, one computes
\[
k^2 = k^2 + \frac{1}{2y}\left(L^{IJ} L_{IJ} + \frac{(n-1)(n-3)}{4}\hbar^2\right). \tag{8}
\]
where we wrote the quantum ordering term more generally as \((n-1)(n-3)\hbar^2/4\) in \(n\) dimensions, but here we need \(n = 6\), which yields \(\frac{15}{4}\hbar^2\). In the classical case one simply drops the term \(\frac{15}{4}\hbar^2\) and does not care about orders of non-commuting factors.
Now, the field \( \Psi (x^\mu, y) \) is expanded in spherical harmonics on \( S^n \)

\[
\Psi (x^\mu, y) = \sum y^{(1-n)/2} f_1 (x^\mu, y) T^{l_1 \cdots l_l} (\Omega),
\]

where the rank \( l \) traceless tensor \( T^{l_1 \cdots l_l} (\Omega) \), constructed by symmetrizing products of \( \Omega^I \), is the harmonic with angular momentum \( l \) on \( S^n \). The angular momentum \( \frac{1}{2} L^I J L_{IJ} \) has eigenvalue \( \hbar^2 l (l + n - 2) \) on the tensor \( T^{l_1 \cdots l_l} (\Omega) \). Then \( k^2 \) reduces to \( k^2 = k^2 + \frac{l_n (l_n+1) \hbar^2}{y^2} \) with \( l_n = l + (n-3)/2 \). The \( f_1 (x^\mu, y) \) are the radial wavefunctions that are normalized according to

\[
\int d^4 x d^m y |\psi (x, y)|^2 = 1,
\]

which yields \( \int_0^\infty d^4 x dy |f_1 (x, y)|^2 = 1 \) after integrating over the orthonormal harmonics \( T^{l_1 \cdots l_l} (\Omega) \). On \( f_1 (x^\mu, y) \) the radial momentum \( k \) acts as a simple derivative according to \( k f_1 (y) = -i \hbar \partial_y f_1 (y) \).

For \( n = 6 \) the physical state condition for the field in Eq. (3) reduces to the following differential operator on the radial wavefunction \( f_1 (x^\mu, y) \)

\[
\left[ -\hbar^2 y \left( \partial_\mu^2 + \partial_y^2 \right) y + \left( l + \frac{3}{2} \right) \left( l + \frac{5}{2} \right) \hbar^2 \right] f_1 = 0.
\]

This equation can be rewritten as

\[
y^{-5/2} \left[ -\hbar^2 \nabla^2_{AdS_5} + l (l + 4) \hbar^2 \right] \left[ y^{5/2} f_1 \right] = 0.
\]

In the last line \( \nabla^2_{AdS_5} = \left( y^2 \partial_\mu^2 + y^5 \partial_y^2 - y^{-3} \partial_y \right) \) is the Laplacian \( \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \right) \) for \( AdS_5 \) in our parametrization of the \( AdS_5 \) metric in Eq. (4.5). The wavefunction \( \phi_1 (x^\mu, y) = y^{5/2} f_1 (x^\mu, y) \) is normalized according to the \( AdS_5 \) metric \( \int d^4 x d^m y \sqrt{-g} |\phi_1 (x^\mu, y)|^2 = \int d^4 x d^m y |f_1 (x^\mu, y)|^2 \), which is consistent with the above normalization of the radial wavefunction. We see that the last equation now is precisely the \( AdS_5 \times S^5 \) Laplace equation since \( l (l + 4) \hbar^2 \) is the eigenvalue of the Laplace operator on \( S^5 \)

\[
\nabla^2_{(AdS_5 \times S^5)} \phi_1 (x^\mu, y) = 0.
\]

This is precisely equivalent to the Casimir relations in Eqs. (3.5, 3.7)

\[
\left[ -C_2^{(4,2)} + C_2^{(6,0)} \right] \phi_1 = 0
\]

that followed from the gauge invariant \( Sp(2, R) \) constraints directly in the 2T formalism in 12 dimensions. Indeed, we already have \( C_2^{(6,0)} = \hbar^2 l (l + 4) \), and if we compute \( C_2^{(4,2)} = \)
\[ \frac{1}{2} L_{mn} L^{mn} = \frac{1}{2} L_{\mu\nu} L^{\mu\nu} - (L_{\mu'\nu'})^2 - L_{\mu'} L_{\nu'} - L_{\mu} L_{\nu'} + L_{\mu'} L_{\nu} \]

by inserting an appropriately quantum ordered version of the non-linear forms in Eqs. (4.12-4.16), we find precisely \( C_2^{(4,2)} = \hbar^2 \nabla_{AdS_5}^2 \).

[1] I. Bars, C. Deliduman and O. Andreev, Phys. Rev. D58 (1998) 066004, hep-th/9803188; I. Bars, Phys. Rev. D58 (1998) 066006, hep-th/9804028; Phys. Rev. D59 (1999) 045019, hep-th/9810023; S. Vongehr, hep-th/9907077.

[2] I. Bars and C. Deliduman, Phys. Rev. D58 (1998) 106004, hep-th/9806085.

[3] I. Bars, C. Deliduman and D. Minic, “Supersymmetric Two-Time Physics”, Phys. Rev. D59 (1999) 125004, hep-th/9812161; “Lifting M-theory to Two-Time Physics”, Phys. Lett. B457 (1999) 275, hep-th/9904063.

I. Bars, “2T formulation of superconformal dynamics relating to twistors and supertwistors”, Phys. Lett. B483 (2000) 248, hep-th/0004090; “A Toy M-model”, in preparation (partial results in hep-th/9904063 and [10]).

[4] I. Bars, C. Deliduman and D. Minic, “String, Branes and Two-Time Physics”, Phys. Lett. B466 (1999) 135, hep-th/9906223.

[5] I. Bars, “Two-Time Physics with Gravitational and Gauge Field Backgrounds”, Phys. Rev. D62 (2000) 085015, hep-th/0002140.

[6] I. Bars, “Two-Time Physics in Field Theory”, Phys. Rev. D62 (2000) 046007, hep-th/0003100.

[7] I. Bars and C. Deliduman, “High spin gauge fields and Two-Time Physics As The Foundation of 2T-Physics in Field Theory”, hep-th/0103042.

[8] I. Bars and S.J. Rey, “Noncommutative Sp(2,R) Gauge Theories - A Field Theory Approach to Two-Time Physics”, hep-th/0104133.

[9] I. Bars, “U\( _\star \) (1,1) Noncommutative Gauge Theory of 2T-Physics ”, hep-th/0105013.

[10] I. Bars, reviews in conference proceedings: “Two-Time Physics”, hep-th/9809034; “Survey of Two-Time Physics”, hep-th/0008164; “2T-Physics 2001”, hep-th/0106021.

[11] P. A.M. Dirac, Ann. Math 37 (1936) 429.

[12] J. Maldacena, hep-th/9711200.

[13] J. Schwarz, Nucl. Phys. B185 (1981) 221.
[14] M. Günaydin and N. Marcus, Class. and Quant. Grav. 2 (1985) L11; M. Günaydin, D. Minic and M. Zagermann, Nucl. Phys. B534 (1998) 96; Nucl. Phys. B544 (1999) 737. M. Günaydin and D. Minic, Nucl. Phys. B523 (1998) 145.

[15] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, Nucl. Phys. B242 (1984) 377.

[16] I. Bars and M. Günaydin, Comm. Math. Phys. 91 (1983) 31.

[17] I. Bars, “A Mysterious Zero in AdS_5×S^5 Supergravity”, hep-th/0205194.

[18] I. Bars, in preparation.

[19] I. Bars, “First massive level and anomalies in the supermembrane”, Nucl. Phys. B308 (1988) 462; “Stringy evidence for d=11 structure in strongly coupled type IIA superstring, Phys. Rev. D2 (1995) 3567 (hep-th/9503228).