THE CLASSIFICATION OF ENDS OF PROPERLY CONVEX REAL PROJECTIVE ORBIFOLDS II: PROPERLY CONVEX RADIAL ENDS AND TOTALLY GEODESIC ENDS.

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Abstract. Real projective structures on $n$-orbifolds are useful in understanding the space of representations of discrete groups into $SL(n+1, \mathbb{R})$ or $PGL(n+1, \mathbb{R})$. A recent work shows that many hyperbolic manifolds deform to manifolds with such structures not projectively equivalent to the original ones. The purpose of this paper is to understand the structures of properly convex ends of real projective $n$-dimensional orbifolds. In particular, these have the radial or totally geodesic ends. For this, we will study the natural conditions on eigenvalues of holonomy representations of ends when these ends are manageably understandable. The main techniques are the Vinberg duality and a generalization of the work of Goldman, Labourie, and Margulis on flat Lorentzian 3-manifolds. Finally, we show that a noncompact strongly tame properly convex real projective orbifold with generalized admissible ends satisfying some topological conditions always has a strongly irreducible holonomy group.

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1. Introduction

1.1. Preliminary definitions.

1.1.1. Topology of orbifolds and their ends. An orbifold $O$ is a topological space with charts modeling open sets by quotients of Euclidean open sets or half-open sets by finite group actions and compatible patching maps with one another. The boundary $\partial O$ of an orbifold is defined as the set of points with only half-open sets as models. Orbifolds are stratified by manifolds. Let $O$ denote an $n$-dimensional orbifold with finitely many ends where end-neighborhoods are homeomorphic to closed $(n-1)$-dimensional
orbifolds times an open interval. We will require that $O$ is strongly tame; that is, $O$ has a compact suborbifold $K$ so that $O - K$ is a disjoint union of end-neighborhoods homeomorphic to closed $(n - 1)$-dimensional orbifolds multiplied by open intervals. Hence $\partial O$ is a compact suborbifold. This is a strong assumption; however, we note that the mathematicians have great difficulty understanding the topology of the ends of manifolds presently. (We apologize for going through definitions for a few pages. See [22] for an introduction to the geometric orbifold theory.)

An orbifold covering map is a map so that for a given modeling open set as above, the inverse image is a union of modeling open sets that are quotients as above. We say that an orbifold is a manifold if it has a subatlas of charts with trivial local groups. We will consider good orbifolds only, i.e., covered by a simply connected manifold. In this case, the universal covering orbifold $\tilde{O}$ is a manifold with an orbifold covering map $p_O : \tilde{O} \to O$. The group of deck transformations will be denote by $\pi_1(O)$ or $\Gamma$. They act properly discontinuously on $\tilde{O}$ but not necessarily freely.

By strong tameness, $O$ has only finitely many ends $E_1, \ldots, E_m$, and each end has an end-neighborhood diffeomorphic to $\Sigma_{E_i} \times (0, 1)$. Let $\Sigma_{E_i}$ here denote the compact orbifold diffeomorphism type of the end $E_i$, which is uniquely determined. Such end-neighborhoods of these types are said to be of the product types.

Each end-neighborhood $U$ diffeomorphic to $\Sigma_E \times (0, 1)$ of an end $E$ lifts to a connected open set $\tilde{U}$ in $\tilde{O}$ where a subgroup of deck transformations $\Gamma_{\tilde{U}}$ acts on $\tilde{U}$ where $p_O^{-1}(U) = \bigcup_{g \in \pi_1(O)} g(\tilde{U})$. Here, each component of $\tilde{U}$ is said to a proper pseudo-end-neighborhood.

- A pseudo-end sequence is a sequence of proper pseudo-end-neighborhoods $U_1 \supset U_2 \supset \cdots$ so that for each compact subset $K$ of $\tilde{O}$ there exists an integer $N$ so that $K \cap U_i = \emptyset$ for $i > N$.
- Two pseudo-end sequences are compatible if an element of one sequence is contained eventually in the element of the other sequence.
- A compatibility class of a pseudo-end sequence is called a pseudo-end of $\tilde{O}$. Each of these corresponds to an end of $O$ under the universal covering map $p_O$.
- For a pseudo-end $\tilde{E}$ of $\tilde{O}$, we denote by $\Gamma_{\tilde{E}}$ the subgroup $\Gamma_{\tilde{U}}$ where $U$ and $\tilde{U}$ is as above. We call $\Gamma_{\tilde{E}}$ is called a pseudo-end fundamental group.
- A pseudo-end-neighborhood $U$ of a pseudo-end $\tilde{E}$ is a $\Gamma_{\tilde{E}}$-invariant open set containing a proper pseudo-end-neighborhood of $\tilde{E}$.

(See Section 2.2.1 of [24] for more detail.)
\(x\). The general linear group \(\text{GL}(n+1, \mathbb{R})\) acts on \(\mathbb{R}^{n+1}\) and \(\text{PGL}(n+1, \mathbb{R})\) acts faithfully on \(\mathbb{RP}^n\).

Denote by \(\mathbb{R}_+ = \{ r \in \mathbb{R} | r > 0 \}\). The real projective sphere \(S^n\) is defined as the quotient of \(\mathbb{R}^{n+1} - \{O\}\) under the quotient relation \(\vec{v} \sim \vec{w}\) iff \(\vec{v} = s\vec{w}\) for \(s \in \mathbb{R}_+\). We will also use \(S^n\) as the double cover of \(\mathbb{RP}^n\). The projective automorphism group \(\text{Aut}(S^n)\) is isomorphic to the subgroup \(\text{SL}_+ (n+1, \mathbb{R})\) of \(\text{GL}(n+1, \mathbb{R})\) of determinant \(\pm 1\), double-covers \(\text{PGL}(n+1, \mathbb{R})\) and acts as a group of projective automorphisms of \(S^n\). A projective map of a real projective orbifold to another is a map that is projective by charts to \(\mathbb{RP}^n\). Let \(\mathcal{P} : \mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n\) be a projection and let \(\mathcal{S} : \mathbb{R}^{n+1} - \{O\} \to S^n\) denote one for \(S^n\). An infinite subgroup \(\Gamma\) of \(\text{PGL}(n+1, \mathbb{R})\) (resp. \(\text{SL}_+ (n+1, \mathbb{R})\)) is strongly irreducible if every finite-index subgroup is irreducible. A subspace \(S\) of \(\mathbb{RP}^n\) (resp. \(S^n\)) is the image of a subspace with the origin removed under the projection \(\mathcal{P}\) (resp. \(\mathcal{S}\)). Also, given any subspace \(V\) of \(\mathbb{R}^{n+1}\) we denote \(\mathcal{P}(V)\) the image of \(V - \{O\}\) under \(\mathcal{P}\) (rest. \(\mathcal{S}(V)\) the image of \(V - \{O\}\) under \(\mathcal{S}\)).

A line in \(\mathbb{RP}^n\) or \(S^n\) is an embedded arc in a 1-dimensional subspace. A projective geodesic is an arc developing into a line in \(\mathbb{RP}^n\) or to a one-dimensional subspace of \(S^n\). An affine subspace \(A^n\) can be identified with the complement of a codimension-one subspace \(\mathbb{RP}^{n-1}\) so that the geodesic structures are same up to parameterizations. A convex subset of \(\mathbb{RP}^n\) is a convex subset of an affine subspace in this paper. A properly convex subset of \(\mathbb{RP}^n\) is a precompact convex subset of an affine subspace. \(\mathbb{R}^n\) identifies with an open hemisphere in \(S^n\) defined by a linear function on \(\mathbb{R}^{n+1}\). (In this paper an affine space is either embedded in \(\mathbb{RP}^n\) or \(S^n\).)

An \(i\)-dimensional complete affine subspace is a subset of a projective manifold projectively diffeomorphic to an \(i\)-dimensional affine subspace in some affine subspace \(A^n\) of \(\mathbb{RP}^n\) or \(S^n\).

Again an affine subspace in \(S^n\) is a lift of an affine space in \(\mathbb{RP}^n\), which is the interior of an \(n\)-hemisphere. Convexity and proper convexity in \(S^n\) are defined in the same way as in \(\mathbb{RP}^n\).

We will consider an orbifold \(\mathcal{O}\) with a real projective structure: This can be expressed as

- having a pair \((\text{dev}, h)\) where \(\text{dev} : \tilde{\mathcal{O}} \to \mathbb{RP}^n\) is an immersion equivariant with respect to
- the homomorphism \(h : \pi_1 (\mathcal{O}) \to \text{PGL}(n+1, \mathbb{R})\) where \(\tilde{\mathcal{O}}\) is the universal cover and \(\pi_1 (\mathcal{O})\) is the group of deck transformations acting on \(\tilde{\mathcal{O}}\).

\((\text{dev}, h)\) is only determined up to an action of \(\text{PGL}(n+1, \mathbb{R})\) given by

\[g \circ (\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1})\]  for \(g \in \text{PGL}(n+1, \mathbb{R})\).

We will use only one pair where \(\text{dev}\) is an embedding for this paper and hence identify \(\tilde{\mathcal{O}}\) with its image. A holonomy is an image of an element under \(h\). The holonomy group is the image group \(h(\pi_1 (\mathcal{O}))\).
Let $x_0, x_1, \ldots, x_n$ denote the standard coordinates of $\mathbb{R}^{n+1}$. The interior $B$ in $\mathbb{R}^n$ or $\mathbb{S}^n$ of a standard ball that is the image of the positive cone of $x_0^2 > x_1^2 + \cdots + x_n^2$ in $\mathbb{R}^{n+1}$ can be identified with a hyperbolic $n$-space. The group of isometries of the hyperbolic space equals the group $\text{Aut}(B)$ of projective automorphisms acting on $B$. Thus, a complete hyperbolic manifold carries a unique real projective structure and is denoted by $B/\Gamma$ for $\Gamma \subset \text{Aut}(B)$.

We also have lifts $\tilde{O} \to \mathbb{S}^n$ and $\pi_1(O) \to \text{SL}_\pm(n+1, \mathbb{R})$ again denoted by $\text{dev}$ and $h$ and are also called developing maps and holonomy homomorphisms. The discussions below apply to $\mathbb{R}^n$ and $\mathbb{S}^n$ equally. This pair also completely determines the real projective structure on $O$. Fixing $\text{dev}$, we can identify $\tilde{O}$ with $\text{dev}(\tilde{O})$ in $\mathbb{S}^n$ when $\text{dev}$ is an embedding. This identifies $\pi_1(O)$ with a group of projective automorphisms $\Gamma$ in $\text{Aut}(\mathbb{S}^n)$. The image of $h'$ is still called a holonomy group.

An orbifold $O$ is convex (resp. properly convex and complete affine) if $\tilde{O}$ is a convex domain (resp. a properly convex domain and an affine subspace).

A totally geodesic hypersurface $A$ in $\tilde{O}$ or $O$ is a subset where each point $p$ in $A$ has a neighborhood $U$ projectively diffeomorphic to an open or half-open ball where $A$ corresponds to a subspace of codimension-one.

Remark 1.1. A summary of the deformation spaces of real projective structures on closed orbifolds and surfaces is given in [22] and [15]. See also Marquis [59] for the end theory of 2-orbifolds. The deformation space of real projective structures on an orbifold loosely speaking is the space of isotopy equivalent real projective structures on a given orbifold. (See [26] also.)

1.2. The classification of ends. We will now try to describe our classification methods. Two oriented geodesic starting from a point $x$ of $\mathbb{S}^n$ is equivalent if they agree on small open neighborhood of $x$. A direction of a geodesic starting a point $x$ of $\mathbb{S}^n$ is an equivalence class of geodesic segments starting from $x$.

Radial ends: We will assume that our real projective orbifold $O$ is a strongly tame orbifold and some of the ends are radial. This means that the end $E$ has a neighborhood $U$, and a component $\tilde{U}$ of the inverse image $p_O^{-1}(U)$ has a $\Gamma_{\tilde{E}}$-invariant foliation by properly embedded projective geodesics ending at a common point $v_{\tilde{U}} \in \mathbb{R}P^n$ where $\tilde{E}$ is a pseudo-end corresponding to $E$ and $\tilde{U}$. We call such a point a pseudo-end vertex.

- The space of directions of oriented projective geodesics through $v_{\tilde{E}}$ forms an $(n-1)$-dimensional real projective space. We denote it by $\mathbb{S}^{n-1}_{v_{\tilde{E}}}$, called a linking sphere.
- Let $\Sigma_{\tilde{E}}$ denote the space of equivalence classes of lines from $v_{\tilde{E}}$ in $\tilde{U}$. $\Sigma_{\tilde{E}}$ projects to a convex open domain in an affine space in $\mathbb{S}^{n-1}_{v_{\tilde{E}}}$ by the convexity of $\tilde{O}$.
• The subgroup $\Gamma_{\tilde{E}}$, a pseudo-end fundamental group, of $\Gamma$ fixes $v_{\tilde{E}}$ and acts on as a projective automorphism group on $S^n_{v_{\tilde{E}}}$. Thus, the quotient $\tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}}$ admits a real projective structure of dimension $n - 1$.

• We denote by $\Sigma_{\tilde{E}}$ the real projective $(n - 1)$-orbifold $\tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}}$. Since we can find a transversal orbifold $\tilde{\Sigma}_{\tilde{E}}$ to the radial foliation in a pseudo-end-neighborhood for each pseudo-end $\tilde{E}$ of $O$, it lifts to a transversal surface $\tilde{\Sigma}_{\tilde{E}}$ in $\tilde{U}$.

• We say that a radial pseudo-end $\tilde{E}$ is convex (resp. properly convex, and complete affine) if $\tilde{\Sigma}_{\tilde{E}}$ is convex (resp. properly convex, and complete affine).

The real projective structure on $\Sigma_{\tilde{E}}'$ is independent of $\tilde{E}'$ as long as $\tilde{E}'$ correspond to a same end $E'$ of $O$. We will just denote it by $\Sigma_{E'}$ sometimes.

Totally geodesic ends: An end is totally geodesic if an end-neighborhood $U$ has as the closure an orbifold $Cl(U)$ in an ambient orbifold where

• $Cl(U) = U \cup \Sigma_{E}$ for a totally geodesic suborbifold $\Sigma_{E}$ and
• where $Cl(U)$ is homeomorphic to $\Sigma_{E} \times I$ for an interval $I$.

$\Sigma_{E}$ is said to be the ideal boundary component of $E$. Two compactifications are equivalent if some respective neighborhoods of the ideal boundary components in ambient orbifolds are projectively diffeomorphic. If $\Sigma_{E}$ is properly convex, then the end is said to be properly convex. (One can see in [19] two inequivalent ways to compactify for a real projective elementary annulus.)

Note that the diffeomorphism types of end orbifolds are determined for radial or totally geodesic ends. From now on, we will say that a radial end is an R-end and a totally geodesic end is a T-end.

In this paper, we will only consider the properly convex radial ends and totally geodesic ends.

1.2.1. Horospherical domains, lens domains, lens-cones, and so on. If $A$ is a domain of subspace of $\mathbb{RP}^n$ or $S^n$, we denote by $bdA$ the topological boundary in the subspace. The closure $Cl(A)$ of a subset $A$ of $\mathbb{RP}^n$ or $S^n$ is the topological closure in $\mathbb{RP}^n$ or in $S^n$. Define $\partial A$ for a manifold or orbifold $A$ to be the manifold or orbifold boundary. Also, $A^o$ will denote the manifold or orbifold interior of $A$.

**Definition 1.2.** Given a convex set $D$ in $\mathbb{RP}^n$, we obtain a connected cone $C_D$ in $\mathbb{R}^{n+1} - \{O\}$ mapping to $D$, determined up to the antipodal map. For a convex domain $D \subset S^n$, we have a unique domain $C_D \subset \mathbb{R}^{n+1} - \{O\}$.

A *join* of two properly convex subsets $A$ and $B$ in a convex domain $D$ of $\mathbb{RP}^n$ or $S^n$ is defined

$$A \ast B := \{[tx + (1 - t)y] | x, y \in C_D, [x] \in A, [y] \in B, t \in [0, 1]\}$$
where $C_D$ is a cone corresponding to $D$ in $\mathbb{R}^{n+1}$. The definition is independent of the choice of $C_D$ but depends on $D$.

**Definition 1.3.** Let $C_1, \ldots, C_m$ be cone respectively in a set of independent vector subspaces $V_1, \ldots, V_m$ of $\mathbb{R}^{n+1}$. In general, the *sum* of convex sets $C_1, \ldots, C_m$ in $\mathbb{R}^{n+1}$ in independent subspaces $V_i$ is defined as

$$C_1 + \cdots + C_m := \{ v | v = c_1 + \cdots + c_m, c_i \in C_i \}.$$

A *strict join* of convex sets $\Omega_i$ in $S^n$ (resp. in $\mathbb{RP}^n$) is given as

$$\Omega_1 \ast \cdots \ast \Omega_m := \prod(C_1 + \cdots + C_m)$$

where each $C_i - \{O\}$ is a convex cone with image $\Omega_i$ for each $i$.

In the following, all the sets are required to be inside an affine subspace $A^n$ and its closure either is in $\mathbb{RP}^n$ or $S^n$.

- $K$ is *lens-shaped* if it is a convex domain and $\partial K$ is a disjoint union of two smoothly strictly convex embedded open $(n-1)$-cells $\partial_{\pm} K$ and $\partial K_-$.

- A *cone* is a domain $D$ in $A^n$ whose closure in $\mathbb{RP}^n$ has a point in the boundary, called an *end vertex* $v$ so that every other point $x \in D$ has a properly convex segment $l, l^o \subset D$, with endpoints $x$ and $v$.

- A *cone* $\{p\} * L$ over a lens-shaped domain $L$ in $A^n, p \notin \mathcal{C}(L)$ is a convex domain so that $\{p\} * L = \{p\} * \partial_{+} L$ for one boundary component $\partial_{+} L$ of $L$. A *lens* is the lens-shaped domain $L$, not determined uniquely by the lens-cone itself.

- We can allow $L$ to have non-smooth boundary that lies in the boundary of $p * L$.
  - One of two boundary components of $L$ is called *top* or *bottom* hypersurfaces depending on whether it is further away from $p$ or not. The top component is denoted by $\partial_{+} L$ which can be not smooth. $\partial_{-} L$ is required to be smooth.
  - A cone over $L$ where $\partial(\{p\} * L - \{p\}) = \partial_{+} L, p \notin \mathcal{C}(L)$ is said to be a *generalized lens-cone* and $L$ is said to be a *generalized lens*.
  - A quasi-lens cone is a properly convex cone of form $p * S$ for a strictly convex open hypersurface $S$ so that $\partial(\{p\} * S - \{p\}) = S$ and $p \in \mathcal{C}(S) - S$ and the space of directions from $p$ to $S$ is a properly convex domain in $S^{n-1}$.

- A *totally-geodesic domain* is a convex domain in a hyperspace. A *cone-over* a totally-geodesic domain $D$ is a union of all segments with one endpoint a point $x$ not in the hyperspace and the other in $D$. We denote it by $\{x\} * D$.

Let the radial pseudo-end $\tilde{E}$ have a pseudo-end-neighborhood of form $\{p\} * L - \{p\}$ that is a generalized lens-cone $p * L$ over a generalized lens $L$ where $\partial(p * L - \{p\}) = \partial_{+} L$ and let $\Gamma_{\tilde{E}}$ acts on $L$. A *concave pseudo-end-neighborhood* of $\tilde{E}$ is the open pseudo-end-neighborhood in $\tilde{O}$ contained in
a radial pseudo-end-neighborhood in $\tilde{O}$ that is a component of $\{p\} \ast L - \{p\} - L$ containing $p$ in the boundary. As it is defined, such a pseudo-end-neighborhood always exists for a generalized lens pseudo-end.

From now on, we will replace the term “pseudo-end” with “p-end” everywhere.

For a domain $A$ of $S^n$ or $\mathbb{RP}^n$, we denote by $\partial A$ the topological boundary. For imbedded manifold $A$, $\partial A$ denotes the manifold boundary. For lower-dimensional open submanifolds $\partial A$ is defined as $\text{Cl}(A) - A$.

**Lens-shaped R-end:** An R-end is lens-shaped (resp. totally geodesic cone-shaped, generalized lens-shaped, quasi-lens shaped) if it has a pseudo-end-neighborhood that is a lens-cone (resp. a cone over a totally-geodesic domain, a concave pseudo-end-neighborhood, or a quasi-lens cone.) Here, we require that $\Gamma_{\tilde{E}}$ acts on the lens of the lens-cone.

**Lens-shaped T-end:** A pseudo-T-end $\tilde{E}$ of $\tilde{O}$ is of lens-type if it has a $\Gamma_{\tilde{E}}$-invariant lens neighborhood $L$ in an ambient orbifold of $\tilde{O}$. Here a p-end neighborhood closed in $\tilde{O}$ is compactified by a totally geodesic hypersurface in a hyperplane $P$. $\partial L \cap \tilde{O}$ is smooth and strictly convex and $\partial L \subset P$ for the hyperspace $P$ containing the ideal boundary of $\tilde{E}$. A T-end of $O$ is of lens-type if the corresponding pseudo-T-end is of lens-type.

A lens-type p-end neighborhood of a p-T-end $\tilde{E}$ of lens-type is a component $C_1$ of $L - P$ in $\tilde{O}$.

**1.3. Main results.** Let $\tilde{E}$ be a p-end and $\Gamma_{\tilde{E}}$ the associated p-end fundamental group. If every subgroup of finite index of a group $\Gamma_{\tilde{E}} \subset \Gamma$ has a finite center, we say that $\Gamma_{\tilde{E}}$ is a virtual center-free group or a vcf-group. An admissible group is a finite extension of a finite product group $\mathbb{Z} \times \Gamma_1 \times \cdots \times \Gamma_k$ for trivial or infinite hyperbolic groups $\Gamma_i$ in the sense of Gromov. (See Section 2.3.1 of [24] for details. In this paper, we will simply use $\mathbb{Z} \times \Gamma_1 \times \cdots \Gamma_k$ to denote the subgroup in $\Gamma_{\tilde{E}}$ corresponding to it.) We say that $\tilde{E}$ is virtually non-factorable if any finite index subgroup has a finite center or $\Gamma_{\tilde{E}}$ is virtually center-free; otherwise, $\tilde{E}$ is virtually factorable.

Let $\Gamma$ be generated by finitely many elements $g_1, \ldots, g_m$. Let $w(g)$ denote the minimum word length of $g \in G$ written as words of $g_1, \ldots, g_m$. The conjugate word length $\text{cwl}(g)$ of $g \in \pi_1(\tilde{E})$ is

$$\min\{w(\gamma g c^{-1}) | c \in \pi_1(\tilde{E})\}.$$ 

Let $\Omega$ be a convex domain in an affine space $A$ in $\mathbb{RP}^n$ or $S^n$. Let $[o, s, q, p]$ denote the cross ratio of four points as defined by

$$\frac{\bar{o} - \bar{q} \bar{s} - \bar{p}}{\bar{s} - \bar{q} \bar{o} - \bar{p}}$$

where

$$o = [\bar{o}, 1], p = [\bar{p}, 1], q = [\bar{q}, 1], s = [\bar{s}, 1]$$
for homogeneous coordinates of a line or a great circle containing $o, s, p, q$. Define the Hilbert metric
\[ d_\Omega(p, q) = \log |[o, s, q, p]| \]
where $o$ and $s$ are endpoints of the maximal segment in $\Omega$ containing $p, q$ where $o, q$ separate $p, s$. The metric is one given by a Finsler metric provided $\Omega$ is properly convex. (See [53].) Given a properly convex real projective structure on $\Omega$, the cover $\tilde{\Omega}$ carries a Hilbert metric which we denote by $d_{\tilde{\Omega}}$. This induces a metric on $\tilde{\Omega}$. (Note that even if $\tilde{\Omega}$ is not properly convex, $d_{\tilde{\Omega}}$ is still a pseudo-metric.)

Let $d_K$ denote the Hilbert metric of the interior $K^o$ of a properly convex domain $K$ in $\mathbb{R}P^n$ or $\mathbb{S}^n$. Suppose that a projective automorphism group $\Gamma$ acts on $K$ properly. Let $\text{length}_K(g)$ denote the infimum of $\{d_K(x, g(x))|x \in K^o\}$, compatible with $	ext{cwl}(g)$.

An ellipsoid is a subset in an affine space defined as a zero locus of a positive definite quadratic polynomial in term of the affine coordinates. A projective conjugate $H_v$ of a parabolic subgroup of $SO(i_0 + 1, 1)$ acting co-compactly on $E - \{v\}$ for an $i_0$-dimensional ellipsoid $E$ containing the point $v$ is called an $i_0$-dimensional cusp group. Let $H_v$ denote the Lie group acting as the translations on the $i_0$-dimensional ellipsoid containing $v$. $H_v$ is isomorphic to $\mathbb{R}^{i_0}$ and is Zariski closed. If the horospherical $p$-end neighborhood with the $p$-R-end vertex $v$ has the $p$-end fundamental group that is a discrete cocompact subgroup in $H_v$, then we call the $p$-R-end to be of cusp type.

Recall from [24] the following:

**Theorem 1.4.** Let $O$ be a properly convex real projective $n$-orbifold with radial or totally geodesic ends satisfying (IE). Let $\tilde{E}$ be a $p$-R-end of its universal cover $\tilde{O}$ and $\Gamma_{\tilde{E}}$ denote the end fundamental group. Then $\tilde{E}$ is a complete affine $p$-R-end if and only if $\tilde{E}$ is a cusp $p$-R-end.

We will learn that the norm of the eigenvalues $\lambda_i(g)$ equals 1 for every $g \in \Gamma_{\tilde{E}}$ if and only if $\tilde{E}$ is horospherical by Proposition 5.2 and Theorem 5.3 of [24].

A subset $A$ of $\mathbb{R}P^n$ or $\mathbb{S}^n$ spans a subspace $S$ if $S$ is the smallest subspace containing $A$.

**Definition 1.5.** Let $v_{\tilde{E}}$ be a $p$-end vertex of a $p$-R-end $\tilde{E}$. We assume that $\Gamma_{\tilde{E}}$ is admissible and the associated real projective orbifold $\Sigma_{\tilde{E}}$ is properly convex. We assume that $\Gamma_{\tilde{E}}$ acts on a strict join $\mathcal{C}(\Sigma_{\tilde{E}}) = K := K_1 \ast \cdots \ast K_{l_0}$ in $\mathcal{S}_{\Sigma_{\tilde{E}}}^{n-1}$ where $K_j$ is a properly convex compact domain in a projective sphere $\mathcal{S}^b$ of dimension $i_j \geq 0$. Thus, $\Gamma_{\tilde{E}}$ restricts to a semisimple hyperbolic group $\Gamma_j$ acting on $K_j$ for some $j = 1, \ldots, l_0$ and also contains the central abelian group $\mathbb{Z}^{b-1}$. The admissibility implies that $\Gamma_j$ is a hyperbolic group and we may assume without the loss of generality
\[ \Gamma_{\tilde{E}} \cong \mathbb{Z}^{b-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}. \]
Let $\hat{K}_i$ denote the subspace spanned by $v_{\tilde{E}}$ and the segments from $v_{\tilde{E}}$ in the direction of $K_i$. The p-end fundamental group $\Gamma_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition if each $g \in \Gamma_{\tilde{E}}$ satisfies for a uniform $C > 0$ independent of $g$

\begin{equation}
C^{-1}{\text{length}}_K(g) \leq \log \left( \frac{\lambda(g)}{\lambda_{v_{\tilde{E}}}(g)} \right) \leq C{\text{length}}_K(g),
\end{equation}

- for $\bar{\lambda}(g)$ equal to the largest norm of the eigenvalues of $g$, and
- the eigenvalue $\lambda_{v_{\tilde{E}}}(g)$ of $g$ at $v_{\tilde{E}}$.

If we require only

$$\bar{\lambda}(g) \geq \lambda_{v_{\tilde{E}}}(g)$$

and the uniform middle eigenvalue condition for each hyperbolic $\Gamma_i$, then we say that $\Gamma_{\tilde{E}}$ satisfies the weakly uniform middle-eigenvalue conditions.

The definition of course applies to the case when $\Gamma_{\tilde{E}}$ has the finite index subgroup with the above properties.

Remark 1.6. We remark that choice of $\Gamma_i$ as a factor of $\Gamma_{\tilde{E}}$ is not unique. We can choose a central element $h$ so that $h$ has the largest eigenvalue at $K_i$. Multiplying such central elements to generators of $\Gamma_i$, we can choose each hyperbolic factor $\Gamma_i$ so that the largest norm eigenvalue of $g \in \Gamma_i$ occurs in $K_i$. We will assume that $\Gamma_i$ is chosen to satisfy this condition when ever $\Gamma_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition.

We give a dual definition:

Definition 1.7. Suppose that $\tilde{E}$ is a properly convex p-T-end. Then let $\Gamma_{\tilde{E}}^*$ acts on a point $v_{\tilde{E}}^* \in \mathbb{R}P^n$ corresponding to $\tilde{\Sigma}_{\tilde{E}}$ with the eigenvalue to be denoted $\lambda_{v_{\tilde{E}}^*}$. Let $g^* : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the dual transformation of $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Assume that $\Gamma_{\tilde{E}}$ acts on a properly convex compact domain $K = \text{Cl}(\tilde{\Sigma}_{\tilde{E}})$ and $K$ is a strict join $K := K_1 \ast \cdots \ast K_6$. Defining $\Gamma_i$ as above, the p-end fundamental group $\Gamma_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition if

- the equation (1) for the largest norm $\bar{\lambda}(g)$ of the eigenvalues of $g$
- and
- the eigenvalue $\lambda_{v_{\tilde{E}}^*}(g)$ of $g^*$ in the vector in the direction of $v_{\tilde{E}}^*$.

Here $\Gamma_{\tilde{E}}$ will act on a properly convex domain $K^o$ of lower dimension and we will apply the definition here. This condition is similar to ones studied by Guichard and Wienhard [48], and the results also seem similar. Our main tools to understand these questions are in Appendix A.

We will see that the condition is an open condition; and hence a “structurally stable one.” (See Corollary 4.3.)

For a strongly tame orbifold $O$, 

O or $\pi_1(O)$ satisfies the infinite-index end fundamental group condition if $\pi_1(\tilde{E})$ is of infinite index in $\pi_1(O)$ for the fundamental group $\pi_1(\tilde{E})$ of each p-end $\tilde{E}$.

(NA) Let $\mathcal{E}$ denote the set of all conjugates of end fundamental group of $\pi_1(O)$. Also, if $\Gamma_{E_1} \cap \Gamma_{E_2}$ is finite for any pair of distinct end fundamental groups $\Gamma_{E_1}$ and $\Gamma_{E_2}$, we say that $O$ or $\pi_1(O)$ satisfies no essential annuli condition or (NA).

Our main result is:

**Theorem 1.8.** Let $O$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and satisfy (IE). Each end fundamental group is virtually isomorphic to a direct product of hyperbolic groups and infinite cyclic groups. Assume that the holonomy group of $O$ is strongly irreducible.

• Let $\tilde{E}$ be a properly convex p-R-end.
  – Suppose that the p-end holonomy group satisfies the uniform middle-eigenvalue condition. Then $\tilde{E}$ is a generalized lens-type p-R-end.
  – Suppose that the p-each end holonomy group satisfies the weakly uniform middle-eigenvalue condition. Then $\tilde{E}$ is a generalized lens-type p-R-end or is a quasi-lens-type p-R-end.

• If $O$ satisfies the triangle condition (see Definition 3.12) or $\tilde{E}$ is virtually factorable or is a totally geodesic R-end, then we can replace the word “generalized lens-type” to “lens-type” in each of the above statements.

In Chapter 8 of [26], there are two examples given by S. Tillman and myself with above types of ends. Later, Gye-Seon Lee and I computed more examples starting from hyperbolic Coxeter orbifolds (These are not published results.) Assume that these structures are properly convex. In these cases, they have only lens-type R-end by Proposition 4.5 in [24].

Recently in 2014, G. Lee has found exactly computed one-parameter families of real projective structures deformed from a complete hyperbolic structure on the figure eight knot complement and from one on the figure-eight sister knot complement. These have radial ends only. Assume that these structures are properly convex. The ends will correspond to lens-type R-ends or cusp R-ends by Corollary 1.9. since the computations shows that the end satisfies the unit eigenvalue condition of the corollary.

The proper convexity of these types real projective orbifolds of examples will be proved in [26].

We will explain the quasi-joined type in Section 3.3 and prove these.

**Corollary 1.9.** Let $O$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and satisfy (IE). Each end fundamental group is virtually isomorphic to a direct product of hyperbolic groups
and infinite cyclic groups. Assume that the holonomy group of $O$ is strongly irreducible.

- Let $\tilde{E}$ be a p-R-end whose end orbifold $\Sigma_{\tilde{E}}$ has the p-end holonomy group with eigenvalue 1 at the p-end vertex. Suppose that $\tilde{E}$ is not NPCC. Then $\tilde{E}$ is a generalized lens-type p-R-end or a horospherical (cusp) R-end.

- Let $\tilde{E}$ be a p-T-end and have the 1-form defining the p-end of $\tilde{E}$ has eigenvalue 1. Then $\tilde{E}$ is a lens-type p-T-end.

Examples are end orbifolds that have Coxeter groups as the fundamental groups.

Our work is a “classification” since we will show how to construct lens-type R-ends (Theorem 3.13), quasi-lens-type R-ends (Propositions 3.18, 3.19), lens-type T-ends (Theorem 3.16). (See also Example 10.1 in [25].) (Of course, provided that we know how to compute certain cohomology groups.)

We will also show that lens-shaped ends are stable (see Theorem 4.1) and that we can always approximate the whole universal cover with lens-shaped end neighborhoods. (See Lemma 4.10.)

Let $\mathbb{RP}^n = P(\mathbb{R}^{n+1})$ be the dual real projective space of $\mathbb{RP}^n$. In Section 2, we define the projective dual domain $\Omega^*$ in $\mathbb{RP}^n$ to a properly convex domain $\Omega$ in $\mathbb{RP}^n$ where the dual group $\Gamma^*$ to $\Gamma$ acts on. We show that a lens-type R-end corresponds to a lens-type T-end and vice versa. (See Section 2.2.3.)

Above orbifold $O = \tilde{O}/\Gamma$ has a diffeomorphic dual orbifold $O^*$ defined as the quotient of the dual domain $\tilde{O}^*$ by the dual group $\Gamma^*$ of $\Gamma$ by Theorem 3.5. The ends of $O$ and $O^*$ are in a one-to-one correspondence. Horospherical ends are dual to themselves, i.e., “self-dual types”, and properly convex R-ends and T-ends are dual to one another. (See Proposition 2.8.) We will see that properly convex R-ends of generalized lens-type are always dual to T-ends of lens-type by Proposition 2.10, Theorem 3.13, and Theorem 3.16.

Remark 1.10 (Self-dual reducible ends). A generalized lens-type virtually factorable properly convex p-R-end $\tilde{E}$ of $\tilde{O}$ is always totally geodesic and of lens-type by Theorem 2.15 and Theorem 1.8. Let $\tilde{O}^*$ be the dual domain of $\tilde{O} \subset S^*$. The dual p-end $\tilde{E}^*$ of $\tilde{O}^*$ is a T-end of lens-type by Corollary 4.4. Hence, $\tilde{E}$ has a properly convex totally geodesic hypersurface $S_{\tilde{E}}$ in $\tilde{O}$. The p-end $\tilde{E}^*$ can be made into a totally geodesic radial one since the p-end holonomy group of $\tilde{E}^*$ fixes a unique point in $S^*$ dual to the hyperspace containing $S_{\tilde{E}}$ of $\tilde{E}$ and by taking a cone over that point. Thus, the virtually factorable properly convex ends are “self-dual” up to some modifications. Thus, we consider these our best type cases.

Admissible ends are ends that are lens-type R-ends or lens-type T-ends or horospherical type R-ends. Generalised admissible ends are ends that are generalized lens-type, lens-type R-ends or lens-type T-ends or horospherical type R-ends.
Our final main result of this paper is the following:

**Theorem 1.11.** Let $O$ be a noncompact strongly tame properly convex real projective orbifold with horospherical, generalized lens-type $R$-ends or lens-type $T$-ends, and satisfies (IE) and (NA). Then the holonomy group is strongly irreducible and is not contained in a proper parabolic subgroup of $\text{PGL}(n+1, \mathbb{R})$ (resp. $\text{SL}_\pm(n+1, \mathbb{R})$).

For closed properly convex real projective orbifold, this was shown by Benoist [4].

1.4. **Outline.** In Section 2, we start to study the end theory. First, we discuss the holonomy representation spaces. Tubular actions and the dual theory of affine actions are discussed. We show that distanced actions and asymptotically nice actions are dual. We prove that the uniform middle eigenvalue condition implies the existence of the distanced action.

In Section 2.3, we discuss the properties of lens-shaped ends. We show that if the holonomy is strongly irreducible, the lens shaped ends have concave neighborhoods. If the lens-shaped end is virtually factorable, then it can be made into a totally-geodesic $R$-end of lens-type, which is a surprising result in the author’s opinion.

In Section 3, we show that the uniform middle-eigenvalue condition of a properly convex end is equivalent to the lens-shaped property of the end under some assumptions. In particular, this is true for virtually factorable properly convex ends. This is a major section with numerous central lemmas.

In Section 4, we prove many results we need in another paper [26], not central to this paper. Also, we show that the lens-shaped property is a stable property under the change of holonomy representations. We obtain the exhaustion by a sequence of $p$-end-neighborhoods of $\tilde{O}$.

In Section 4.6, we will discuss the totally geodesics ends. In Theorem 4.13, we discuss how the lens-shaped totally geodesic end can be extended by a compact totally geodesic hypersurface. We obtain the duality between the $T$-ends of lens-type and $R$-ends of generalized lens-type.

In Section 5, we will define limits sets of ends and discuss the properties in Proposition 4.11. Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with generalized admissible ends and satisfy (IE) and (NA). We prove the strong irreducibility of $\mathcal{O}$; that is, Theorem 1.11.

In Appendix A, we show that the affine action of strongly irreducible group $\Gamma$ acting cocompactly on a convex domain $\Omega$ in the boundary of the affine space is asymptotically nice if $\Gamma$ satisfies the uniform middle-eigenvalue condition. This was needed in Section 2.

**Remark 1.12.** Note that the results are stated in the space $S^n$ or $\mathbb{R}P^n$. Often the result for $S^n$ implies the result for $\mathbb{R}P^n$. In this case, we only prove for $S^n$. In other cases, we can easily modify the $S^n$-version proof to one for the $\mathbb{R}P^n$-version proof. We will say this in the proofs.
1.5. **Acknowledgements.** We thank David Fried for helping me understand the issues with the distanced nature of the tubular actions and duality. We thank Yves Benoist with some initial discussions on this topic, which were very helpful for Section 2.1 and thank Bill Goldman and François Labourie for discussions resulting in Appendix A.3.1. We thank Daryl Cooper and Stephan Tillmann for explaining their work and help and we also thank Mickael Crampon and Ludovic Marquis also. Their works obviously were influential here. The study was begun with a conversation with Tillmann at “Manifolds at Melbourne 2006” and I began to work on this seriously from my sabbatical at Univ. Melbourne from 2008. We also thank Craig Hodgson and Gye-Seon Lee for working with me with many examples and their insights. The idea of R-ends comes from the cooperation with them.

2. **The end theory**

In this section, we discuss the properties of lens-shaped radial and totally geodesic ends and their duality also.

2.1. **The holonomy homomorphisms of the end fundamental groups: the fibering.** We will discuss for $\mathbb{S}^n$ only here but the obvious $\mathbb{R}P^n$-version exists for the theory. Let $\tilde{E}$ be a p-R-end of $\tilde{O}$. Let $\mathrm{SL}_+(n + 1, \mathbb{R})_{\tilde{v}_E}$ be the subgroup of $\mathrm{SL}_+(n + 1, \mathbb{R})$ fixing a point $\mathbf{v}_E \in \mathbb{S}^n$. This group can be understood as follows by letting $\mathbf{v}_E = [0, \ldots, 0, 1]$ as a group of matrices: For $g \in \mathrm{SL}_+(n + 1, \mathbb{R})_{\tilde{v}_E}$, we have

$$
\left( \begin{array}{cc}
\lambda_{\tilde{v}_E}(g)^{1/n} \hat{h}(g) & 0 \\
\tilde{v}_E & \lambda_{\tilde{v}_E}(g)
\end{array} \right)
$$

where $\hat{h}(g) \in \mathrm{SL}_+(n, \mathbb{R})$, $\tilde{v} \in \mathbb{R}^n$, $\lambda_{\tilde{v}_E}(g) \in \mathbb{R}_+$, is the so-called linear part of $h$. Here,

$$
\lambda_{\tilde{v}_E} : g \mapsto \lambda_{\tilde{v}_E}(g) \text{ for } g \in \mathrm{SL}_+(n + 1, \mathbb{R})_{\tilde{v}_E}
$$

is a homomorphism so it is trivial in the commutator group $[\Gamma_{\tilde{E}}, \Gamma_{\tilde{E}}]$. There is a group homomorphism

$$
\mathcal{L}' : \mathrm{SL}_+(n + 1, \mathbb{R})_{\tilde{v}_E} \to \mathrm{SL}_+(n, \mathbb{R}) \times \mathbb{R}_+, \\
g \mapsto (\hat{h}(g), \lambda_{\tilde{v}_E}(g))
$$

(2)

with the kernel equal to $\mathbb{R}^n$, a dual space to $\mathbb{R}^n$. Thus, we obtain a diffeomorphism

$$
\mathrm{SL}_+(n + 1, \mathbb{R})_{\tilde{v}_E} \to \mathrm{SL}_+(n, \mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}_+
$$

We note the multiplication rules

$$
(A, \tilde{v}, \lambda)(B, \tilde{w}, \mu) = (AB, \frac{1}{\mu^{1/n}} \tilde{v}B + \lambda \tilde{w}, \lambda \mu).
$$

(We denote by $\mathcal{L}_1$ the further projection to $\mathrm{SL}_+(n, \mathbb{R})$.)
Let $\Sigma_\tilde{E}$ be the end $(n-1)$-orbifold. Given a representation $\hat{h} : \pi_1(\Sigma_\tilde{E}) \to \SL_{\pm}(n,\mathbb{R})$ and $\lambda : \pi_1(\Sigma_\tilde{E}) \to \mathbb{R}_+$, we denote by $\mathbb{R}^n_{\hat{h},\lambda}$ the $\mathbb{R}$-module with the $\pi_1(\Sigma_\tilde{E})$-action given by

$$g \cdot \tilde{v} = \frac{1}{\lambda(g)^{1/n}} \hat{h}(g)(\tilde{v}).$$

And we denote by $\mathbb{R}^{\alpha*}_{\hat{h},\lambda}$ will the dual vector space with the right dual action given by

$$g \cdot \tilde{v} = \frac{1}{\lambda(g)^{1/n}} \hat{h}(g)^*(\tilde{v}).$$

Let $H^1(\pi_1(\tilde{E}), \mathbb{R}^{\alpha*}_{\hat{h},\lambda})$ denote the cohomology space of 1-cocycles $\tilde{v}(g) \in \mathbb{R}^{\alpha*}_{\hat{h},\lambda}$.

As $\text{Hom}(\pi_1(\Sigma_\tilde{E}), \mathbb{R}_+)$ equals $H^1(\pi_1(\Sigma_\tilde{E}), \mathbb{R})$, we obtain:

**Theorem 2.1.** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and let $\tilde{\mathcal{O}}$ be its universal cover. Let $\Sigma_\tilde{E}$ be the end orbifold associated with a $p$-$R$-end $\tilde{E}$ of $\tilde{\mathcal{O}}$. Then the space of representations

$$\text{Hom}(\pi_1(\Sigma_\tilde{E}), \SL_{\pm}(n+1,\mathbb{R})_{\tilde{v}_\tilde{E}})/\SL_{\pm}(n,\mathbb{R})_{\tilde{v}_\tilde{E}}$$

is the fiber space $B$ over

$$\text{Hom}(\pi_1(\Sigma_\tilde{E}), \SL_{\pm}(n,\mathbb{R}))/\SL_{\pm}(n,\mathbb{R}) \times H^1(\pi_1(\Sigma_\tilde{E}), \mathbb{R})$$

with the fiber isomorphic to $H^1(\pi_1(\Sigma_\tilde{E}), \mathbb{R}^{\alpha*}_{\hat{h},\lambda})$ for each $([\hat{h}], \lambda)$.

We remark that we don’t really understand the fiber dimensions and their behavior as we change the base points. A similar idea is given by Mess [60]. In fact, the dualizing these matrices gives us a representation to $\text{Aff}(\text{Aff}^n)$. In particular if we restrict ourselves to linear parts to be in $\text{SO}(n,1)$, then we are exactly in the cases studied by Mess. (See the concept of the duality in Section 2.2.1 and Appendix A.)

Thus, one interesting question that Benoist and we talked about is how to compute the dimension of $H^1(\pi_1(\Sigma_\tilde{E}), \mathbb{R}^{\alpha*}_{\hat{h},\lambda})$ under some general conditions on $\hat{h}$.

### 2.1.1. Tubular actions.

Let us give a pair of antipodal points $v$ and $v_-$. If a group $\Gamma$ of projective automorphisms fixes a pair of fixed points $v$ and $v_-$, then $\Gamma$ is said to be tubular. There is a projection $\Pi_v : \mathbb{S}^n - \{v, v_-\} \to \mathbb{S}^{n-1}_v$ given by sending every great segment with endpoints $v$ and $v_-$ to the sphere of directions at $v$. (We denote by $\mathbb{RP}^{n-1}_v$ the quotient of $\mathbb{S}^{n-1}_v$ under the antipodal map given by the change of directions. We use the same notation $\Pi_v : \mathbb{RP}^n - \{v\} \to \mathbb{RP}^{n-1}_v$ the induced projection.)

A tube in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) is the closure of the inverse image of a convex domain $\Omega$ in $\mathbb{S}^{n-1}$ (resp. in $\mathbb{RP}^{n-1}$). Given a p-$R$-end $\tilde{E}$ of $\tilde{\mathcal{O}}$, let $v := v_{\tilde{E}}$. The end domain is $R_v(\tilde{\mathcal{O}})$. If a p-$R$-end $\tilde{E}$ has the end domain $\Sigma_\tilde{E} = R_v(\tilde{\mathcal{O}})$, $h(\pi_1(\tilde{E}))$ acts on the tube domain $\mathcal{T}_v$ associated with $\Sigma_\tilde{E}$. 
We will now discuss for the $S^n$-version but the $\mathbb{R}P^n$ version is obviously clearly obtained from this by a minor modification.

Letting $\mathbf{v}$ have the coordinates $[0, \ldots, 0, 1]$, we obtain the matrix of $g$ of $\pi_1(\tilde{E})$ of form

$$
\begin{pmatrix}
\frac{1}{\lambda_v(g)} & \hat{h}(g) & 0 \\
\vec{b}_g & \lambda_v(g)
\end{pmatrix}
$$

where $\vec{b}_g$ is an $n \times 1$-vector and $\hat{h}(g)$ is an $n \times n$-matrix of determinant $\pm 1$ and $\lambda_v(g)$ is a positive constant.

Note that the representation $\hat{h} : \pi_1(\tilde{E}) \to \text{SL}_\pm(n, \mathbb{R})$ is given by $g \mapsto \hat{h}(g)$.

Here we have $\lambda_v(g) > 0$. If $\tilde{\Sigma}$ is properly convex, then the convex tubular domain and the action are properly tubular.

2.2. Duality map. The Vinberg duality diffeomorphism induces a one-to-one correspondence between p-ends of $\tilde{O}$ and $\tilde{O}^*$ by considering the dual relationship $\Gamma_{\tilde{E}}$ and $\Gamma_{\tilde{E}'}$ for each pair of p-ends $\tilde{E}$ and $\tilde{E}'$ with dual p-end fundamental groups. (See Section 3 of [24].)

Given a properly convex domain $\Omega$ in $S^n$ (resp. $\mathbb{R}P^n$), we recall the augmented boundary of $\Omega$

$$
\text{bd}^{A\Omega} := \{(x, h) | x \in \text{bd}\Omega, x \in h,
$$

$h$ is an oriented supporting hyperplane of $\Omega \subset S^n \times S^{n*}$.

Each $x \in \text{bd}\Omega$ has at least one supporting hyperspace, an oriented hyperspace is an element of $S^{n*}$ since it is represented as a linear functional, and an element of $S^n$ represent an oriented hyperspace in $S^{n*}$.

We recall a duality map

$$
D_{\Omega} : \text{bd}^{A\Omega} \leftrightarrow \text{bd}^{A\Omega^*}
$$
given by sending $(x, h)$ to $(h, x)$ for each $(x, h) \in \text{bd}^{A\Omega}$. This is a homeomorphism.

For later purposes, we need

**Lemma 2.2.** Let $\Omega^*$ be the dual of a properly convex domain $\Omega$ in $S^n$ or $\mathbb{R}P^n$. Then

(i) $\text{bd}\Omega$ is $C^1$ and strictly convex at a point $p \in \text{bd}\Omega$ if and only if $\text{bd}\Omega^*$ is $C^1$ and strictly convex at the unique corresponding point $p^*$.

(ii) $\Omega$ is an ellipsoid if and only if so is $\Omega^*$.

(iii) $\text{bd}\Omega^*$ contains a properly convex domain $D = P \cap \text{bd}\Omega^*$ open in a totally geodesic hyperplane $P$ if and only if $\text{bd}\Omega$ contains a vertex $p$ with $R_p(\Omega)$ a properly convex domain. In this case, $D$ sends the pair of $p$ and the associated supporting hyperplanes of $\Omega$ to the pairs of the totally geodesic hyperplane containing $D$ and points of $D$. Moreover, $D$ and $R_p(\Omega)$ are properly convex and are projectively diffeomorphic to dual domains.
Proof. (i) bdΩ near p is a graph of a function \( f : B \rightarrow bdΩ \) where \( B \) is an open set in a hyperspace supporting \( Ω \) at \( p \). (i) follows from this.

(ii) This is trivial.

(iii) We consider the set of hyperplanes supporting \( Ω \) at \( p \). This forms a properly convex domain as we can see them as linear functionals supporting \( C(Ω) \). The converse is also easy.

Let \( v \) be the vector in \( \mathbb{R}^{n+1} \) in the direction of \( p \). Then the set of supporting linear functionals of \( C(Ω) \). Let \( V \) be a complementary space of \( v \) in \( \mathbb{R}^{n+1} \). Let \( A \) be given as \( V + v \). We choose \( V \) so that \( C_v := C(Ω) \cap A \) is a bounded convex domain in \( A \). We give \( A \) a linear structure so that \( v \) corresponds to the origin. Let \( A^* \) denote the dual linear space. The set of linear functionals positive on \( C(Ω) \) and \( 0 \) at \( v \) is identical with that of linear functionals on the linearized \( A \) positive on \( C_v \): we define

\[
C(D) := \{ f \in \mathbb{R}^{n+1} | f|C(Ω) > 0, f(v) = 0 \}
\]

(6) \( \hat{C}_v^* := \{ g \in A^* | g|C_v > 0 \} \).

The equality follows by the decomposition \( \mathbb{R}^{n+1} = \{ tv | t \in \mathbb{R} \} \oplus V \). Define \( R'_v(C_v) \) as the equivalence classes of properly convex segments in \( C_v \) ending at \( v \) where two segments are equivalent if they agree in an open neighborhood of \( v \). \( R'_v(Ω) \) is identical with \( R'_v(C_v) \) by projectivization \( \mathbb{R}^{n+1} \rightarrow \mathbb{S}^n \). Hence \( R'_v(C_v) \) is a properly convex domain in \( S(A) \). Since \( R'_v(C_v) \) is properly convex, the interior of the spherical projectivization \( S(\hat{C}_v^*) \subset S(A^*) \) is dual to the properly convex domain \( R'_v(C_v) \subset S(A) \).

Define \( D := S(C(D)) \). Since \( R'_v(C_v) \) corresponds to \( R'_v(Ω) \), and \( S(\hat{C}_v^*) \) corresponds to \( D \), the conclusion follows.

We will need this later.

**Corollary 2.3.** The following holds

- Let \( L \) be a lens and \( v \notin L \) so that \( v * L \) is a properly convex lens-cone. Suppose the smooth strictly convex boundary component \( A \) of \( L \) is tangent to a segment from \( v \) at each point of \( bdA \) and \( v * L = v * A \). The dual domain of \( Cl(v * L) \) is the closure of a component \( L_1 \) of \( L' - P \) where \( L' \) is a lens and \( P \) is a hyperspace meeting \( L^o \) but not meeting the boundary of \( L' \) and \( bd\partial L_1 \subset P \).

- Conversely, we are given a lens \( L' \) and \( P \) is a hyperspace meeting \( L^o \) but not meeting the boundary of \( L' \). Let \( L_1 \) be a component of \( L' - P \) with smooth strictly convex boundary \( \partial L_1 \) so that \( bd\partial L_1 \subset P \). The dual of the closure of a component \( L_1 \) of \( L' - P \) is the closure of \( v * L \) for a lens \( L \) and \( v \notin L \) so that \( v * L \) is a properly convex lens-cone. The outer boundary component \( A \) of \( L \) is tangent to a segment from \( v \) at each point of \( bdA \) and \( v * L = v * A \). Moreover, \( v \notin Cl(A) \).

**Proof.** Let \( A \) denote the boundary component of \( L \) so that \( v * L = v * A \). We will determine \( Cl(v * L) \) by finding the boundary of the dual domain \( D \) using the duality map \( D \). The set of hyperplanes supporting \( Cl(v * L) \) at \( v \) forms a
properly totally geodesic domain $D_1$ in $\mathbb{R}^n$ contained in a hyperplane $P$ dual to $v$. The set of hyperplanes supporting $Cl(v + L)$ at $A$ goes to the strictly convex hypersurface $A'$ in $\partial D_1$ since $D$ is a homeomorphism. The points of $bd(v + A) - A$ is a union $S$ of segments from $v$. The supporting hyperplanes containing the segments go to points in $\partial D$. Each point of $Cl(A') - A'$ is a limit of a sequence $\{p_i\}$ of points of $A'$, corresponding to a sequence of supporting hyperspheres $\{h_i\}$ to $A$. By the tangency condition of $A$ and $bdA$, it follows that the limit hypersphere contains the segment in $S$ from $v$. Thus, $Cl(A') - A'$ equals the set of hyperspheres containing the segments in $S$ from $v$. Thus, it goes to a point of $\partial D_1$. Thus, $bdA' = \partial D_1$. Let $P$ be the unique hyperplane containing $D_1$. Then $\partial D = A' \cup D_1$. The points of $bdA$ goes to a supporting hyperplane at points of $bdA'$ distinct from $P$. Taking intersection of the closed $n$-hemispheres bounded by all of these supporting hyperplanes at points of $Cl(A')$, we obtain $L'$. Moreover, $A' = \partial L_1$ for a component $L_1$ of $L - P$.

The second item is proved similarly to the first. Then $\partial L_1$ goes to a hypersurface $A$ in the boundary of the dual domain $D'$ of $Cl(L_1)$. Again $A$ is smooth strictly convex boundary. $\partial L_1 - A$ is a totally geodesic domain $D_1$. Let $P$ be the hyperspace containing $D_1$ and let $v$ be the dual point. Then the space of supporting hyperplanes at $\partial D_1$ goes to segments from $v$ in the boundary of $D'$. Every supporting hyperplane $P'$ to $L_1$ at points of $A$ can be deformed to the one $P_1$ containing $D_1$ by paths of hyperplanes not meeting $L_1^o$. We simply consider ones sharing $P' \cap P_1$ and outside $L_1^o$. Each of the path is a geodesic segment in $S^\infty$. Hence, $D'$ is a cone over $A$. □

2.2.1. **Affine actions dual to tubular actions.** The automorphism in $S^n$ acting on a codimension-one subspace $S^\infty_{-1}$ of $S(\mathbb{R}^{n+1})$ and the components of the complement acts on an affine space $A^n$, a component of the complement of $S^\infty_{-1}$. The subgroup of projective automorphisms preserving $S^\infty_{-1}$ and the components equals the affine group $Aff(A^n)$.

By duality, a great $(n-1)$-sphere $S^\infty_{-1}$ corresponds to a point $v_{S^\infty_{-1}}$. Thus, for a group $\Gamma$ in $Aff(A^n)$, the dual groups $\Gamma^*$ acts on $S(\mathbb{R}^{n+1,*})$ fixing $v_{S^\infty_{-1}}$. (See Proposition 3.4 also.)

Suppose that $\Gamma$ acts on a properly convex open domain $U$ where $\Omega := bdU \cap S^\infty_{-1}$ is a properly convex domain. We call $\Gamma$ a properly convex affine action.

**Proposition 2.4.** Suppose that $\Gamma$ acts on a properly convex open domain $\Omega \subset S^\infty_{-1}$ cocompactly. Then the dual group $\Gamma^*$ acts on a properly tubular domain $B$ with vertices $v := v_{S^\infty_{-1}}$ and $v_- := v_{S^\infty_{-1,*}}$ dual to $S^\infty_{-1}$. The domain $\Omega^\circ$ and domain $R_0(B)$ in the linking sphere $S^\circ_{-1}$ from $v$ in direction of $B^\circ$ are projectively diffeomorphic to a pair of dual domains.

**Proof.** Let $S^\infty_{-1,*}$ be the dual sphere of $S^\infty_{-1}$. Given $\Omega^\circ \subset S^\infty_{-1}$, we obtain the properly convex open dual domain $\Omega^\circ$ in $S^\infty_{-1,*}$. A supporting $n-1$-hemisphere in $S^\infty_{-1}$ of $\Omega$ corresponds to a point of $bd\Omega^\circ$ and vice versa.
An open $n$-hemisphere supporting $\Omega^o$ contains an open $n-1$-hemisphere in $S^{n-1}$ supporting $\Omega^o$. The set of $n$-hemispheres containing a fixed supporting $n-1$-hemisphere of $S^{n-1}$ and supporting $\Omega^o$ forms a great open segment in $S^{n*}$ with endpoints $v$ and $v_-$. The set $\text{bd} \Omega^{o*}$ parametrizes the space of such open segments. Let $I_x$ for $x \in \text{bd} \Omega^{o*}$ denote such a segment. $\bigcup_{x \in \text{bd} \Omega^{o*}} C(I_x)$ is the boundary of a convex tube $B := \mathcal{T}(\Omega^{o*})$ with vertices $v$ and $v_-$. Thus, there is a one-to-one correspondence between the set of open $n$-hemispheres supporting $\Omega^o$ and the set of $\text{bd} \mathcal{T}(\Omega^{o*}) - \{v, v_-\}$. Also, $R_\Omega(B) = \Omega^{o*}$ by $B := \mathcal{T}(\Omega^{o*})$. Thus, $\Gamma$ acts on $\Omega^o$ if and only if $\Gamma^*$ acts on $B$. □

Given a convex open subset $U$ of $A^n$, an asymptotic hyperspace $H$ of $U$ at a point $x \in \text{bd} A^n \cap \text{Cl}(\text{bd} U)$ is a hyperspace so that a component of $A^n - H$ contains $U$.

2.2.2. Distanced tubular actions and asymptotically nice affine actions.

**Definition 2.5.** A properly tubular action is said to be *distanced* if the tubular domain contains a properly convex compact $\Gamma$-invariant subset disjoint from the vertices. A properly convex affine action of $\Gamma$ is said to be *asymptotically nice* if $\Gamma$ acts on a properly convex open domain $U'$ in $A^n$ with boundary in $\Omega \subset S^{n-1}_{\infty}$ so that there exists a closed set

$$\{H | H \text{ is a supporting hyperspace at } x \in \text{bd} \Omega, H \not\subset S^{n-1}_{\infty}\}$$

where each supporting hyperspace in $S^{n-1}_{\infty}$ in $\text{bd} \Omega$ is in one of the element of the set.

Let $d_H$ denote the Hausdorff metric of $S^n$ with the spherical metric $d$.

**Proposition 2.6.** Let $\Gamma$ and $\Gamma^*$ be dual groups where $\Gamma$ has an affine action on $A^n$ and $\Gamma^*$ is tubular with the vertex $v = v_{S^{n-1}}$ dual to the boundary $S^{n-1}_{\infty}$ of $A^n$. Let $\gamma = (\Gamma^*)^*$ acts on a convex open domain $\Omega$ with compact $\Omega/\Gamma$. Then $\Gamma$ acts asymptotically nicely if and only if $\Gamma^*$ acts on a properly tubular domain $B$ and is distanced.

**Proof.** For each point $x$ of $\text{bd} \Omega$, an open hemisphere in $S^n$ at $x$ supports $\Omega$ uniformly bounded at a distance in $d_H$-sense from the open hemisphere $A^n$ with boundary $S^{n-1}_{\infty}$ or $S^n - A^n$. We choose the set so that $\Gamma$ acts on it. (Otherwise, $U$ would become empty.)

The dual points of the supporting hyperplanes of $U$ at $\text{bd} \Omega$ are points on $\text{bd} B$ for a tube domain $B$ with vertex $v$ dual to $S^{n-1}_{\infty}$. Since the hyperspheres of form $H$ supporting $U$ at $x \in \text{bd} \Omega$, are bounded at a distance from $S^{n-1}_{\infty}$ in the $d_H$-sense, the dual points are uniformly bounded at a distance from the vertices $v$ and $v_-$. We take the closure of the set of hyperplanes in the dual space of $S^{n*}$. Let us call this compact set $K$. Let $\Omega^* \subset S^{n-1}_{\infty}$ be the dual domain of $\Omega$. Then for every point of $\text{bd} \Omega^*$, we have a point of $K$ in the corresponding great segment from $v$ to $v_-$. $K$ is uniformly bounded at a distance from $v$ and $v_-$. Since $\Omega^*$ is a compact convex set bounded at a uniform distance from $v$ and $v_-$. Since
the tube domain is properly convex. Since $K$ is $\Gamma$-invariant, so is the convex hull in $\text{Cl} (\hat{O})$.

Conversely, every compact convex subset $K$ of the tubular domain $B$ bounded away from $v$ and $v_-$ meets a great segment from $v$ to $v_-$ at a point bounded away from the endpoints. Let $A'$ denote the set $\partial B - \{v, v_-\}$. Then $K \subset A'$ is a compact convex and $\Gamma$-invariant and bounded away from $v, v_-$. Each point $x$ of $K$ is dual to a hypersphere $P$ in $S^n$ bounded at a distance from $S^{n-1}_\infty$ since $x$ is bounded at a distance from $v, v_-$. Since $x \in \text{bd} B$, $P$ must be a supporting plane to a convex domain $\Omega$ in $S^{n-1}_\infty$; by Proposition 2.4, $P \cap A^n$ is a complete hyperplane with a point of $\text{bd} \Omega$ its boundary in $S^n$. The intersection of the corresponding half-spaces in $A^n$ is not empty and is a properly convex open domain since there is no sequence of hemispheres from $K$ converging to one in $S^{n-1}$.

Theorem 2.7. Let $\Gamma$ be a nontrivial properly convex tubular action at vertex $v = v_{\infty}$ on $S^n$ (resp. in $\mathbb{R} P^n$) and acts on a properly convex tube $B$ and satisfies the uniform middle-eigenvalue conditions. Then $\Gamma$ is distanced inside the tube $B$ where $\Gamma$ acts on and the minimal distanced $\Gamma$-invariant compact set $K$ in $B$ is uniquely determined. Furthermore, $K$ meets each open boundary great segment in $\partial B$ at a unique point. Also, $K$ is contained in a hypersphere disjoint from $v, v_-$ when $\Gamma$ is virtually factorable.

Proof. Let $v$ be the vertex of $B$. First assume that $\Gamma$ is virtually non-factorable. $\Gamma$ induces a strongly irreducible action on the link sphere $S^r_\infty$. The dual group $\Gamma^*$ acts on a properly convex domain $U^* \subset S^n$ dual to $U$ and on $S^{n-1}_\infty$. Let $\Omega$ denote the convex domain in $S^{n-1}_\infty$ corresponding to $B^n$. Then the closure of $U^*$ meets $S^{n-1}_\infty$ corresponding to a domain $\Omega^*$ dual to $\Omega$. By Theorem A.1, $\Gamma^*$ is asymptotically nice. Proposition 2.6 implies the result. The uniqueness part of Theorem A.1 implies the uniqueness of the minimal set and the last statement.

Suppose that $\Gamma$ acts virtually reducibly on $S^{n-1}_\infty$ on a properly convex domain $\Omega$. Then $\Gamma$ is isomorphic to $\mathbb{Z}^{k-1} \times \Gamma_1 \times \cdots \times \Gamma_s$ where $\Gamma_i$ is nontrivial hyperbolic for $i = 1, \ldots, s$ and trivial for $s + 1 \leq i \leq l_0$ where $s \leq l_0$. By [5], $\Gamma$ acts on

$$K := K_1 \ast \cdots \ast K_0 = \text{Cl} (\Omega) \subset S^{n-1}_\infty$$

where $K_i$ denotes the properly convex compact set in $S^{n-1}_\infty$ where $\Gamma_i$ acts on for each $i$. Here, $K_i$ is 0-dimensional for $i = 1, \ldots, s$. Let $B_i$ be the convex tube with vertices $v$ and $v_-$ corresponding to $K_i$. Each $\Gamma_i$ for $i = 1, \ldots, s$ acts on a nontrivial tube $B_i$ with vertices $v$ and $v_-$ in a subspace.

For each $i, s + 1 \leq i \leq r$, $B_i$ is a great segment with endpoints $v$ and $v_-$. A point $v_i$ corresponds to $B_i$ in $S^{n-1}_\infty$.

Recall that a nontrivial element $g$ of the center acts trivially on the subspace $K_i$ of $S^{n-1}_\infty$; that is, $g$ has only one associated eigenvalue in points of $K_i$ by Proposition 2.4 of [24]. There exists a nontrivial element $g$ of the
center with the largest norm eigenvalue in $K_i$. Since the action of $\Gamma_E$ on $\tilde{\Sigma}_E$ is compact.

By the middle eigenvalue condition, for each $i$, we can find $g$ in the center so that $g$ has a hyperspace $K_i' \subset B_i$ with largest norm eigenvalues. Since $\Gamma_i$ acts on $K_i'$ and commutes with $g$, $\Gamma_i$ also acts on $K_i'$.

The convex hull of $K_1' \cup \cdots \cup K_l'$ in $\text{Cl}(B)$ is a distanced $\Gamma$-invariant compact convex set. □

2.2.3. The duality of T-ends and properly convex R-ends. Let $\Omega$ be the properly convex domain covering $O$. For T-end $E$, the totally geodesic ideal boundary $\Sigma_E$ of $E$ is covered by a properly convex open domain in $\text{bd} \Omega$ corresponding to a p-T-end $\tilde{E}$. We denote it by $S_{\tilde{E}}$. We call it the ideal boundary of $\tilde{E}$.

**Proposition 2.8.** Let $O$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends. Then the dual real projective orbifold $O^*$ is also strongly tame and has the same number of ends so that

- there exists a one-to-one correspondence $C$ between the set of ends of $O$ and the set of ends of $O^*$.
- $C$ restricts to such a one between the subset of horospherical ends of $O$ and the subset of horospherical ones of $O^*$.
- $C$ restricts to such a one between the subset of totally geodesic ends of $O$ with the subset of ends of properly convex radial ones of $O^*$. The ideal boundary $\tilde{S}_E$ is projectively diffeomorphic to the properly convex open domain dual to the domain $\Sigma_E^*$ for the corresponding end $\tilde{E}^*$ of $\tilde{E}$.
- $C$ restricts to such a one between the subset of all properly convex R-ends of $O$ and the subset of all T-ends of $O^*$. Also, $\tilde{S}_E$ is projectively dual to the ideal boundary $\tilde{S}_{\tilde{E}}^*$ for the corresponding dual end $\tilde{E}^*$ of $\tilde{E}$.

**Proof.** We prove for the $S^n$-version. By the Vinberg duality diffeomorphism of Theorem 3.5 of [24], $O^*$ is also strongly tame. Let $\tilde{O}$ be the universal cover of $O$. Let $\tilde{O}^*$ be the dual domain. The first item follows by the fact that this diffeomorphism sends pseudo-ends neighborhoods to pseudo-end neighborhoods.

A p-R-end $\tilde{E}$ of $\tilde{O}$ has a p-end vertex $v_{\tilde{E}}$. $\tilde{S}_{\tilde{E}}$ is a properly convex domain in $S_{w_{\tilde{E}}}^{n-1}$. The space of supporting hyperplanes of $\tilde{O}$ at $v_{\tilde{E}}$ forms a properly convex domain of dimension $n - 1$ since they correspond to hyperplanes in $S_{w_{\tilde{E}}}^{n-1}$ not intersecting $\tilde{S}_{\tilde{E}}$. Under the duality map $D_{\tilde{O}}: (v_{\tilde{E}}, h)$ for a supporting hyperplane $h$ is sent to $(h^*, v_{\tilde{E}})$. Lemma 2.2 shows that $h^*$ is a point in a properly convex $n - 1$-dimensional domain $\text{bd}\tilde{O}^* \cap P$ for $P = v_{\tilde{E}}$, a
hyperplane. Thus, $\tilde{E}^*$ is a totally geodesic end with $\tilde{\Sigma}_{\tilde{E}}$, dual to $S_{\tilde{E}}$. This proves the third item. The fourth item follows similarly.

Let $\tilde{E}$ be a horospherical p-R-end with $x$ as the end vertex. Since there is a subgroup of a cusp group acting on $Cl(\tilde{O})$ with $x$ fixed by [24], the intersection of the unique supporting hyperspace $h$ with $Cl(\tilde{O})$ is a singleton $\{x\}$. The dual subgroup is also a cusp group and acts on $Cl(\tilde{O}^*)$ with $h$ fixed. So the corresponding $\tilde{O}^*$ has the dual hyperspace $x^*$ of $x$ as the unique intersection at $h^*$ dual to $h$ at $Cl(\tilde{O}^*)$. Hence $x^*$ is a horospherical end.

□

An NPCC p-R-end may have a dual end that is not radial nor totally geodesic. We will not talk about this in the article.

Remark 2.9. We also remark that the map induced on the set of pseudo-ends of $\tilde{O}$ to that of $\tilde{O}^*$ by $D_{\tilde{O}}$ is compatible with the Vinberg diffeomorphism. This easily follows by Proposition 6.7 of [43] and the fact that the level set $S_x$ of the Koszul-Vinberg function is asymptotic to the boundary of $\tilde{O}$. (See Chapter 6 of Goldman [43].)

Proposition 2.10. Let $O$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends and satisfy (IE). The following conditions are equivalent:

(i) A properly convex R-end of $O$ satisfies the uniform middle-eigenvalue condition.

(ii) The corresponding totally geodesic end of $O^*$ satisfies this condition.

Proof. The items (i) and (ii) are equivalent by considering equation (1). This follows from Corollary 4.4. □

Moreover, $C$ restricts to a correspondence between the lens-type R-ends with lens-type T-ends.

2.3. The properties of lens-shaped ends. One of the main results of this section is that a generalized lens-type end has a “concave end-neighborhood” that actually covers a p-end-neighborhood.

We will need a preliminary lemma on actions. Given three sequences of projectively independent points $\{p_{ij}\}$ with $j = 1, 2, 3$ so that $\{p_{ij}\} \rightarrow p^{(j)}$ where $p^{(1)}, p^{(2)}, p^{(3)}$ are independent points in $S^n$. Then a simple matrix computation shows that a uniformly bounded sequence $\{r_i\}$ of elements of $Aut(S^n)$ or $PGL(n+1, \mathbb{R})$ acts so that $r_i(p_{ij}) = p^{(j)}$ for every $i$ and $j = 1, 2, 3$.

A convex arc is an arc in a totally geodesic subspace where an arc projectively equivalent to a graph of a convex function $I \rightarrow \mathbb{R}$ for a connected interval in $\mathbb{R}$.

Find the tube $B_{\tilde{E}}$ of great open segments with endpoints $v_{\tilde{E}}$ and $v_{\tilde{E}^-}$ corresponding to elements of $\tilde{\Sigma}_{\tilde{E}}$. The union is a convex domain not properly convex with distinguished vertices $v_{\tilde{E}}$ and $v_{\tilde{E}^-}$.
Lemma 2.11. Let $O$ be a properly convex real projective $n$-orbifold with radial or totally geodesic ends. Suppose that the universal cover $\hat{O}$ is in $S^n$. Suppose that $g_i \in SL_\pm(n+1, \mathbb{R})$ be a sequence of automorphisms so that $g_i(v_E) = v_E$ for an end vertex $v_E$ and $l$ is a maximal segment in a generalized lens with endpoints in $O$. (See Figure 1.) Let $g_i'$ denote the induced projective automorphisms on $S^\alpha_{v_E}$. \{g_i'(l') \subset \Sigma_v\} converges geometrically to $l'$ where $l'$ is the projection of $l$ to the linking sphere $S^\alpha_{v_E}$ of $v_E$. Let $P$ be the 2-dimensional subspace containing $v_E$ and $l$. Furthermore, we suppose that

- In $P$, $l$ is in the disk $D$ bounded by two segments $s_1$ and $s_2$ from $v_E$ and a convex curve $\alpha$ with endpoints $q_1$ and $q_2$ that are endpoints of $s_1$ and $s_2$ respectively.
- $\beta$ is another convex curve with $\beta^0 \subset D^0$ and endpoints in $S^\alpha_1 - \{v_E\}$ and $S^\alpha_2 - \{v_E\}$ so that $\alpha$ and $\beta$ and parts of $s_1$ and $s_2$ bound a convex disk in $D$.
- There is a sequence of points $\tilde{q}_i \in \alpha$ converging to $q_1$ and $g_i(\tilde{q}_i) \in F$ for a fixed fundamental domain $F$ of $\hat{O}$.
- The sequences $g_i(D)$, $g_i(\alpha)$, $g_i(\beta)$, $g_i(s_1)$, and $g_i(s_2)$ converge to $D, \alpha, \beta, s_1$, and $s_2$ respectively.

Then

- If the endpoints of $\alpha$ and $\beta$ do not coincide at $s_1$ or $s_2$, then $\alpha$ and $\beta$ must be geodesics from $q_1$ or $q_2$.
- Suppose that the pairs of endpoints of $\alpha$ and $\beta$ coincide and they are distinct curves. Then no segment in $Cl(\hat{O})$ extends $s_1$ or $s_2$ properly.

Proof. By the geometric convergence conditions, we obtain a bounded sequence of elements $r_i \in SL_\pm(n+1, \mathbb{R})$ so that $r_i(g_i(s_1)) = s_1$ and $r_i(g_i(s_2)) = s_2$ and \{r_i\} → 1. Then $r_i \circ g_i$ is represented as an element of $SL_\pm(3, \mathbb{R})$ in the projective plane $P$ containing $D$. Using $v_E$ and $q_1$ and $q_2$ as standard basis points, $r_i \circ g_i$ is represented as a diagonal matrix. Moreover \{r_i \circ g_i(\alpha)\} is still converging to $\alpha$ as \{r_i\} → 1. Hence, this implies that the diagonal elements of each $r_i \circ g_i$ are of form $\lambda_i, \mu_i, \tau_i$ where \{$\lambda_i$\} → 0, \{$\tau_i$\} → +∞ as $i \to \infty$ and $\lambda_i$ is associated with $q_1$ and $\mu_i$ is associated with $v_E$ and $\tau_i$ is associated with $q_2$. (Thus, $r_i \circ g_i$ is diagonalizable with fixed points $q_1, q_2, v_E$.)

We have that \{r_i \circ g_i(\beta)\} also converges to $\beta$. Suppose that the endpoint of $\beta$ at $s_1$ is different from that of $\alpha$. Then we claim

$$1/C < \left\{ \log \left| \frac{\lambda_i}{\mu_i} \right| \right\} < C \text{ for a constant } C > 0 :$$

Otherwise, for the endpoint $\partial_1 \beta$, we obtain a contradiction

$$r_i \circ g_i(\partial_1 \beta) \to q_1 \text{ or } v_E \text{ for } i \to \infty.$$
Since $r_i \circ g_i(\partial_1 \beta) \to \partial_1 \beta$, it follows that $\lambda_i/\mu_i \to 1$. In this case, $\beta$ has to be a geodesic from $q_2$ since $\{r_i \circ g_i(\beta)\} \to \beta$. And so is $\alpha$. The similar argument holds for the case involving $s_2$.

For the second item, $\{\mu_i/\tau_i\} \to 0, +\infty$ and $\{\lambda_i/\mu_i\} \to 0, +\infty$ also since otherwise we can show that $\beta$ and $\alpha$ have to be geodesics with distinct endpoints as above. If a segment $s'_2$ in $\text{Cl}(\tilde{\Omega})$ extends $s_2$, then $\{r_i \circ g_i(s'_2)\}$ converges to a great segment and so does $\{g_i(s'_2)\}$ as $i \to \infty$ or $i \to -\infty$. This contradicts the proper convexity of $\mathcal{O}$.

A trivial one-dimensional cone is an open half space in $\mathbb{R}^1$ given by $x > 0$ or $x < 0$.

Recall that if $\pi_1(\tilde{E})$ is an admissible group, then $\pi_1(\tilde{E})$ has a finite index subgroup isomorphic to $\mathbb{Z}^{k-1} \times \Gamma_1 \times \cdots \times \Gamma_k$ for some $k \geq 0$ where each $\Gamma_i$ is hyperbolic or trivial.

Let us consider $\Sigma_{\tilde{E}}$ the real projective $(n-1)$-orbifold associated with $\tilde{E}$ and consider $\tilde{\Sigma}_{\tilde{E}}$ as a domain in $\mathbb{S}^{n-1}$ and $h(\pi_1(\tilde{E}))$ induces $\hat{h} : \pi_1(\tilde{E}) \to \text{SL}_+(n, \mathbb{R})$ acting on $\tilde{\Sigma}_{\tilde{E}}$. We denote by $\text{bd} \tilde{\Sigma}_{\tilde{E}}$ the boundary of $\tilde{\Sigma}_{\tilde{E}}$ in $\mathbb{S}^{n-1}$.

**Definition 2.12.** A (generalized) lens-shaped $p$-R-end with the $p$-end vertex $v_{\tilde{E}}$ is strictly generalized lens-shaped if we can choose a (generalized) lens domain $D$

- with the top hypersurfaces $A$ and the bottom one $B$ so that
- each great open segment in $\mathbb{S}^n$ from $v_{\tilde{E}}$ in the direction of $\text{bd} \tilde{\Sigma}_{\tilde{E}}$ meets $\text{Cl}(D) - A - B$ at a unique point.

As a consequence $\text{Cl}(A) - A = \text{Cl}(B) - B$ and $\text{Cl}(A) \cup \text{Cl}(B) = \partial D$.

**Theorem 2.13.** Let $\mathcal{O}$ be a strongly tame properly convex $n$-orbifold with radial or totally geodesic ends and satisfy (IE). Assume that the universal cover $\tilde{\mathcal{O}}$ is a subset of $\mathbb{S}^n$ (resp. in $\mathbb{R}P^n$). Let $\tilde{E}$ be a $p$-R-end of $\tilde{\mathcal{O}}$ with a
generalized lens p-end-neighborhood. Let $v_E$ be an associated with a p-end vertex. Assume that $\pi_1(\tilde{E})$ is hyperbolic.

(i) The complement of the manifold boundary of the generalized lens-shaped domain $D$ is a nowhere dense set in $bdD$. Moreover, $bdD - \partial D$ is independent of the choice of $D$, and $D$ is strictly generalized lens-shaped. Moreover, each element $g \in \Gamma_{\tilde{E}}$ has an attracting fixed point in $bdD$ in a great segment from $v_{\tilde{E}}$ in $bd\tilde{\Sigma}_{\tilde{E}}$. The set of attracting fixed points is dense in $bdD - A - B$ for the top and the bottom hypersurfaces.

(ii) Let $l$ be a segment $l \subset bd\tilde{O}$ with $l \cap Cl(U) \neq \emptyset$ for any concave p-end-neighborhood $U$ of $v_{\tilde{E}}$. Then $l$ is in closure in $Cl(V)$ of a concave or proper p-end-neighborhood $V$ of $v_{\tilde{E}}$. Moreover, the set $S(v_{\tilde{E}})$ of maximal segments from $v_{\tilde{E}}$ in $Cl(V)$ is independent of a concave or proper p-end neighborhood $V$, and

$$\bigcup S(v_{\tilde{E}}) = Cl(V) \cap bd\tilde{O}.$$ 

(iii) $S(g(v_{\tilde{E}})) = g(S(v_{\tilde{E}}))$ for $g \in \pi_1(\tilde{E})$.

(iv) Any concave p-end-neighborhood $U$ of $v_{\tilde{E}}$ under the covering map $\tilde{O} \to O$ covers the p-end-neighborhood of $\tilde{E}$ of form $U/\pi_1(\tilde{E})$. That is, a concave p-end-neighborhood is a proper p-end-neighborhood.

(v) Assume that $w$ is the p-end vertex of a p-R-end with hyperbolic end-fundamental group. Then

$$S^o(v_{\tilde{E}}) \cap S(w) = \emptyset \text{ or } S(v_{\tilde{E}}) = S(w) \text{ (with v}_{\tilde{E}} = w)$$

for p-end vertices $v_{\tilde{E}}$ and $w$ where we defined $S^o(v_{\tilde{E}})$ to denote the relative interior of $\bigcup S(v_{\tilde{E}})$ in $bd\tilde{O}$. 

Proof. The proof is done for $S^n$ but the result implies the $\mathbb{R}P^n$-version. Here the closure is independent of the ambient spaces.

(i) By Fait 2.12 [6], we obtain that $\pi_1(\tilde{E})$ is virtually center free and acts irreducibly on a proper convex cone and the cone has to be strictly convex by Theorem 1.1 of [5].

Let $C_{\tilde{E}}$ be a concave end. Since $\Gamma_{\tilde{E}}$ acts on $C_{\tilde{E}}$, $C_{\tilde{E}}$ is a component of the complement of a generalized lens domain $D$ in a generalize R-end by definition.

We have a domain $D$ with boundary components $A$ and $B$ transversal to the lines in $R_{v_{\tilde{E}}}$. We can assume that $B$ is strictly concave and smooth as we have a concave end-neighborhood. $\Gamma_{\tilde{E}}$ acts on both $A$ and $B$. We define

$$A_1 := Cl(A) - A \text{ and } B_1 := Cl(B) - B.$$ 

By Theorem 1.2 of [4], the geodesic flow on the real projective $(n - 1)$-orbifold $\tilde{\Sigma}_{\tilde{E}}/\Gamma_{\tilde{E}}$ is topologically mixing, i.e., recurrent since $\Gamma_{\tilde{E}}$ is hyperbolic. Thus, each geodesic $l$ in $\tilde{\Sigma}_{\tilde{E}} \subset S_{v_{\tilde{E}}}^{n-1}$, we can find a sequence $\{g_i \in \Gamma_{\tilde{E}}\}$ that satisfies the conditions of Lemma 2.11. The two arcs in $bdD$ corresponding
to \( l \) share endpoints. Since this is true for all geodesics, we obtain \( A_1 = B_1 \) and \( A \cup B \) is dense in \( \partial D \).

Hence, \( \partial D = A \cup B \). Thus, \( \partial D \) is the closure of the set of the attracting and repelling fixed points of \( h(\pi_1(\hat{E})) \) since the set of fixed points is dense in \( A_1 = B_1 \) by Theorem 1.1 of [4]. Therefore this set is independent of the choice of \( D \).

(ii) Consider any segment \( l \) in \( \partial D \) with \( l^o \) meeting \( Cl(U_1) \) for a concave p-end-neighborhood \( U_1 \) of \( v_E \). Let \( T \) be the open tube corresponding to \( \hat{E} \). In the closure of \( U_1 \), an endpoint in \( l \) is in a component \( T_1 \) of \( bdT - B \) where \( T_1 \subset Cl(U_1) \cap \partial D \) by the definition of concave p-end neighborhoods. Then \( l^o \subset bdT \) since some part of \( l^o \) is in \( T_1 \). For any convex segment \( s \) in \( Cl(\hat{O}) \) from \( v_E \) to any point of \( l \) must be in \( bdT \). Thus, \( s \) is in \( \partial D \) since \( bdT \cap Cl(\hat{O}) \subset \partial D \). Therefore, the segment \( l \) is contained in the union of segments in \( \partial D \) from \( v_E \).

We suppose that \( l \) is a segment from \( v_E \) containing a segment \( l_0 \) in \( Cl(U_1) \cap \partial D \) from \( v_E \), and we will show that \( l \) is in \( Cl(U_1) \cap \partial D \). This will be sufficient to prove (ii).

A point of \( \partial D \) is a p-end vertex of a recurrent geodesic by Lemma 2.14. Suppose that the interior of \( l \) contains a point \( p \) of \( \partial D - A - B \) that is in the direction of a p-end vertex of a recurrent geodesic \( m \) in \( \hat{E} \). Lemma 2.11 again applies. Thus, \( l^o \) does not meet \( \partial D - A - B \).

Suppose that \( l' \) is a segment from \( v_E \) in \( \partial D \) where

\[
\mathcal{O} \cap (bdCl(D) - A - B) = \emptyset.
\]

Then the maximal segment \( l'' \) containing \( l \) from \( v_E \) in \( \partial D \) meets \( bdCl(D) - A - B \) at the end. Thus,

\[
l \subset Cl(U_1) \cap \partial D.
\]

Let \( U' \) be any proper p-end-neighborhood associated with \( v_E \). Let \( s \) be a segment in \( U' \) from \( v_E \). Then since each \( g \in \Gamma_E \) has an attracting fixed point and the repelling fixed point on \( \partial D - A - B \), \( \{g^i(s)\} \) converges to an element of \( S(v_E) \). The set of the attracting and the repelling fixed points of elements of \( \Gamma_E \) is dense in the directions of \( \partial D \). Thus, every segment of \( S(v_E) \) is in the closure \( Cl(U') \). We have

\[
\bigcup S(v_E) \subset Cl(U') \cap \partial D.
\]

We can form \( S'(v_E) \) as the set of maximal segments from \( v_E \) in \( Cl(U') \cap \partial D \). Then no segment \( l \) in \( S'(v_E) \) has interior points in \( \partial D - A - B \) as above. Thus,

\[
S(v_E) = S'(v_E).
\]

Also, since every points of \( Cl(U') \cap \partial D \) has a segment in the direction of \( \partial D \), we obtain

\[
\bigcup S(v_E) = Cl(U') \cap \partial D.
\]
(iii) By the proof above, we now characterize $S(v_E)$ as the maximal segments in $\text{bd} \tilde{\Omega}$ from $v_E$ ending at points of $\text{bd} \tilde{D} - A - B$. Since $g(D)$ is the generalized lens for the the generalized lens neighborhood $g(U)$ of $g(v_E)$, we obtain $g(S(v_E)) = S(g(v_E))$ for any p-end vertex $v_E$.

(iv) Given a concave-end-neighborhood $C_E$ of a p-end vertex $v_E$, we show that

$$g(C_E) = C_E \text{ or } g(C_E) \cap C_E = \emptyset \text{ for } g \in \Gamma :$$

Suppose that

$$g(C_E) \cap C_E \neq \emptyset, g(C_E) \not\subset C_E, \text{ and } C_E \not\subset g(C_E).$$

Since $C_E$ is concave, each point of $\text{bd} C_E \cap \tilde{\Omega}$ is contained in a supporting totally geodesic hypersurface $D$ so that

- a component $C_{E,1}$ of $C_E - D$ is in $C_E$
- $C(C_{E,1}) \ni v_{C_E}$ for the p-end vertex $v_{C_E}$ of $C_E$.

Similar statements hold for $g(C_E)$.

Since $g(C_E) \cap C_E \neq \emptyset$, one is not a subset of the other, we have

$$\text{bd} g(C_E) \cap C_E \neq \emptyset \text{ or } g(C_E) \cap \text{bd} C_E \neq \emptyset.$$ 

Then by above a set of form of $C_{E,1}$ for some boundary point of $C_{E,1}$ and $g(C_{E})$ meet. Since $C_{E,1}$ is a component of a separating hypersurface in $\tilde{\Omega}$, the interiors of some segments in $S(v_E)$ meet the interiors of segments of $S(g(v_E))$ or vice versa.

Suppose that $v_E \neq g(v_E)$. Let $l$ be such a segment in $S(v_E)$. Then $l^0$ must be inside $S(g(v_E))^0$ by (ii). Since $\tilde{\Sigma}_E$ is strictly convex, $l$ must agree with a segment in $S(g(v_E))$ in an interval. By Lemma 2.11, then $l$ agrees with a segment in $S(g(v_E))$ and have vertices $v_E$ and $g(v_E)$. For any nearby segment $l'$ to $l$, this must be true also. This implies a contradiction that $S(g(v_E))$ is a singleton.

We obtain $v_E = g(v_E)$. Hence, $g \in \Gamma_E$, and thus, $C_E = g(C_E)$ as $C_E$ is a concave neighborhood. Therefore, this is a contradiction. We obtain three possibilities

$$g(C_E) \cap C_E = \emptyset, g(C_E) \subset C_E \text{ or } C_E \subset g(C_E).$$

In the last two cases, it follows that $g(C_E) = C_E$ since $g$ fixes $v_E$, i.e., $g \in \Gamma_E$. This implies that $C_E$ is a proper p-end-neighborhood.

(v) If $S(v_E) \circ \cap S(w) \neq \emptyset$, then the above argument applies with in this situation to show that $v_E = w$.

□

Lemma 2.14. Let $\tilde{E}$ be a p-end that can be virtually factorable or nonfactorable. Every point of $\text{bd} \tilde{\Sigma}_E$ is an end point of an oriented geodesic $l$ that is recurrent in that direction when projected to $\Sigma_{\tilde{E}}$.

Proof. We will prove for $S^n$-version but this implies the version for $\mathbb{R}P^n$. If $\pi_1(\tilde{E})$ is a hyperbolic group, then the conclusion follows from Theorem 1.2 [4].
Assume now that $\pi_1(\tilde{E})$ is a virtual product of the hyperbolic groups and the abelian group in the center. Let $D$, $D \subset S^{n-1}_E$, be a properly convex compact set so that $D^o = \tilde{\Sigma}_E$. Then as in Section 2.3.1 of [24], we obtain $D$ is a strict join $D_1 \ast \cdots \ast D_k$ for some $k$, $k \geq 2$ where the virtual center isomorphic to $\mathbb{Z}^{k-1}$ acts trivially and each $D_i$ is a compact properly convex domain. For any subset $J \subset \{1, \ldots, k\}$, we denote by

$$D_J := \ast_{i \in J} D_i, \mathbb{Z}^J := \oplus_{i \in J} \mathbb{Z}, \text{ and } \mathbb{R}^J := \oplus_{i \in J} \mathbb{R}.$$ 

Let $x \in \text{bd}D$. Then $x = \left[\sum_{i=1}^{k} \lambda_i x_i\right]$ for $[x_i] \in D_i$ and $\lambda_i \geq 0$. Let $J_x$ denote the set where $\lambda_i > 0$. $J_x$ is a proper subset of $\{1, \ldots, k\}$. Let $J'_x \subset J_x$ denote the set of indices where $[x_i]$ is not in a boundary of $D_i$. We choose a geodesic $l_i$ for each $i \in J_x - J'_x$. $l_i$ projects to a recurrent geodesic in $D_i^o/\Gamma_i$ since $\Gamma_i$ is hyperbolic. Let $J''_x = \{1, \ldots, k\} - J'_x$. Then we choose a geodesic $l$ in $D''_x$ ending at $\left[\sum_{i \in J''_x} \lambda_i x_i\right]$ and at an interior point of $D_{\{1,\ldots,k\}-J''_x}$. $l$ projects to a recurrent geodesic in $D''_x/\mathbb{Z}^{J''_x}$ since $\mathbb{Z}^{J''_x}$ is a lattice acting cocompactly on $\mathbb{R}^{J''_x}$. Then we let $l_i$ for $i \in J''_x$ to be the ones obtained by projection of $l$ to each subspace corresponding to $D_i$. We lift $l_i$ to an affine line $\tilde{l}_i$ in $\mathbb{R}^{n+1}$ with unit speed parameters and $\tilde{l}_i(0) = x_i$. Then we let $\tilde{l}$ denote the affine geodesic obtained by $\tilde{l}(t) = \sum_{i=1}^{k} \lambda_i \tilde{l}_i(t)$. The projection of $\tilde{l}$ to $D$ gives us the desired recurrent geodesic passing $D^o$ since the factor groups commute with one another.

Now we go to the cases when $\pi_1(\tilde{E})$ has more than two nontrivial factors abelian or hyperbolic. The following theorem shows that concave ends are totally geodesic and of lens-type in these cases. The author obtained the proof of (i-3) from Benoist.

**Theorem 2.15.** Let $O$ be a strongly tame $n$-orbifold with radial or totally geodesic ends and satisfy (IE). Suppose that

- $\mathcal{C}(O)$ is not a strict join, or
- the holonomy group $\Gamma$ is strongly irreducible.

Let $\tilde{E}$ be a $p$-R-end of the universal cover $\tilde{O}$, $\tilde{O} \subset S^n$ (resp. $\subset \mathbb{R}P^n$), with a generalized lens $p$-end-neighborhood. Let $v_E$ be the $p$-end vertex and $\tilde{\Sigma}_E$ the $p$-end domain of $\tilde{E}$. Suppose that the $p$-end fundamental group $\Gamma_E$ is admissible. Then the following statements hold:

(i) For $S^{n-1}_{v_E}$, we obtain

(i-1) Under a finite-index subgroup of $\hat{h}(\pi_1(\tilde{E}))$, $\mathbb{R}^o$ splits into $V_1 \oplus \cdots \oplus V_b$ and $\tilde{\Sigma}_E$ is the quotient of the sum $C_1 + \cdots + C_b$ for properly convex or trivial one-dimensional cones $C_i \subset V_i$ for $i = 1, \ldots, b$.

(i-2) The Zariski closure of a finite index subgroup of $\hat{h}(\pi_1(\tilde{E}))$ is isomorphic to the product $G = G_1 \times \cdots \times G_b \times \mathbb{R}^{b-1}$ where $G_i$ is
a semisimple subgroup of $\text{Aut}(\mathcal{S}(V_i))$ with identity components isomorphic to $\text{SO}(\dim V_i - 1, 1)$ or $\text{SL}(\dim V_i, \mathbb{R})$.

(i-3) Let $D_i$ denote the image of $C_i'$ in $S_{v_E}^{n-1}$. Each hyperbolic group factor of $\pi_1(\tilde{E})$ divides exactly one $D_i$ and acts on trivially on $D_j$ for $j \neq i$.

(i-4) A finite index subgroup of $\pi_1(\tilde{E})$ has a rank $l_0 - 1$ free abelian group center corresponding to $\mathbb{Z}^{b-1}$ in $\mathbb{R}^{b-1}$.

(ii) $g$ in the center is diagonalizable with positive eigenvalues. For a nonidentity element $g$ in the center, the eigenvalue $\lambda_{v_E}$ of $g$ at $v_{E_i}$ is strictly between its largest norm and smallest norm eigenvalues.

(iii) The $p$-R-end is totally geodesic. $D_i \subset S_{v_E}^{n-1}$ is projectively diffeomorphic by the projection $\prod_{v_E}$ to totally geodesic convex domain $D'_i$ in $S^n$ (resp. in $\mathbb{R}P^n$) of dimension $\dim V_i - 1$ disjoint from $v_{E_i}$, and the actions of $\Gamma_i$ are conjugate by $\prod_{v_E}$.

(iv) The $p$-R-end is strictly lens-shaped, and each $C_i'$ corresponds to a cone $C_i^*$ in $S^n$ (resp. in $\mathbb{R}P^n$) over a totally geodesic $(n - 1)$-dimensional domain $D'_i$ with $v_{E_i}$. The $p$-R-end has a $p$-end-neighborhood equal to the interior of

$$v_{E_i} \ast D$$

for $D := \text{Cl}(D'_1) \ast \cdots \ast \text{Cl}(D'_b)$

where the interior of $D$ forms the boundary of the $p$-end neighborhood in $\tilde{O}$.

(v) The set $S(v_{E_i})$ of maximal segments in $\text{bd} \tilde{O}$ from $v_{E_i}$ in the closure of a $p$-end-neighborhood of $v_{E_i}$ is independent of the $p$-end-neighborhood.

$$S(v_{E_i}) = \bigcup_{i=1}^{j} v_{E_i} \ast \text{Cl}(D'_1) \ast \cdots \ast \text{Cl}(D'_{i-1}) \ast \text{Cl}(D'_{i+1}) \ast \cdots \ast \text{Cl}(D'_b).$$

(vi) A concave $p$-end-neighborhood of $\tilde{E}$ is a proper $p$-end-neighborhood. Finally, the statements (iii) and (v) of Theorem 2.13 also hold.

Proof. Again the $S^n$-version is enough. (i) Since each hyperbolic factor of $\Gamma_{E_i}$ contains no nontrivial normal subgroup, it goes to one of $\Gamma_i$ isomorphically to a finite index subgroup. Hence, each $\Gamma_i$ is hyperbolic or trivial. Now the proof follows from Proposition 2.4 in [24].

(ii) If $\lambda_{v_E}(g)$ is the largest norm of eigenvalue with multiplicity one, then $\{g^n(x)\}$ for a point $x$ of a generalized lens converges to $v_{E_i}$ as $n \to \infty$. Since the closure of a generalized lens is disjoint from the point, this is a contradiction. Therefore, the largest norm $\lambda_1(g)$ of the eigenvalues of $g$ is greater than or equal to $\lambda_{v_E}(g)$.

Let $U$ be a concave $p$-end-neighborhood of $\tilde{E}$ in $\tilde{O}$. Let $S_1, ..., S_b$ be the projective subspaces in general position meeting only at the $p$-end vertex $v_{E_i}$ on which the factor groups $\Gamma_1, ..., \Gamma_b$ act irreducibly. Let $C_i$ denote the union of great segments from $v_{E_i}$ corresponding to the invariant cones in $S_i$. 

where $\Gamma_i$ acts irreducibly for each $i$. The abelian center isomorphic to $\mathbb{Z}^{b-1}$ acts as the identity on $C_i$ in the projective space $\mathbb{S}^n_{v^*E}$.

Let $g \in \mathbb{Z}^{b-1}$. $g|C_i$ can have more than two eigenvalues or just single eigenvalue by the last item of Proposition 2.4 of [24]. In the second case $g|C_i$ could be represented by a matrix with eigenvalues all 1 fixing $v^*_E$. Since a generalized lens $L$ meets it, $g|C_i$ has to be identity by the proper convexity of $\tilde{O}$. Otherwise, $g^n|C$ will send $x \in L \cap C_i$ to $v^*_E$ and to $v^*_E- \text{ as } i \to \pm \infty$. This contradicts the proper convexity of $\tilde{O}$.

We have one of the two possibilities for $g$ in the center and $C_i$:

(a) $g|C_i$ fixes each point of a hyperspace $P_i \subset S_i$ not passing through $v^*_E$ and $g$ has a representation as a nontrivial scalar multiplication in the affine subspace $S_i - P_i$ of $S_i$. Since $g$ commutes with every element of $\Gamma_i$ acting on $C_i$, $\Gamma_i$ acts on $P_i$ as well.

(b) $g|C_i$ is an identity.

We denote $h_1 := \{i|\exists g \in \mathbb{Z}^{b-1}, g|C_i \neq 1\}$ and $h_2 := \{i|\forall g \in \mathbb{Z}^{b-1}, g|C_i = 1\}$.

By the cocompactness of $F^*_E$, we can choose an element $g \in \mathbb{Z}^{b-1}$ so that $g|C_i$ for each $i \in h_2$ has the submatrix with the largest norm eigenvalues in the unit norm matrix representation of $g$. Thus, $h_2$ cannot have more than one elements. Hence, $h_1 \neq \emptyset$.

Suppose that $h_2 \neq \emptyset$. For each $C_i$, we can find $g_i \in \mathbb{Z}^{b-1}$ with the largest norm eigenvalue associated with it. By multiplying with some other element of the virtual center, we can show that if $i \in h_1$, then $C_i \cap P_i$ has a sequence $\{g_{i,j}\}$ with $i$ fixed and if $i \in h_2$, then $C_i$ has such a sequence $\{g_{i,j}\}$ so that the premises of Proposition 2.18 are satisfied.

By Proposition 2.18, this implies that $\text{Cl}(\tilde{O})$ is a join

$$ *_{i \in h_1} K_i * *_{i \in h_2} K_i $$

where $K_i$, $i \in h_1$, for a properly convex domain in $C_i \cap P_i$, and $K_i$, $i \in h_2$, is a properly convex domain in $C_i$ containing $v^*_E$.

This contradicts the assumptions that $\text{Cl}(\tilde{O})$ is not a join or that $\Gamma$ is not virtually reducible by Proposition 2.17. Thus, $h_2 = \emptyset$.

(iii) By (ii), for all $C_i$, every $g \in \mathbb{Z}^{b-1} - \{1\}$ acts as nonidentity. Then the strict join of all $P_i$ gives us a hyperspace $P$ disjoint from $v^*_E$. We will show that it forms a totally geodesic p-R-end for $\tilde{E}$.

From above, we obtain that every nontrivial $g \in \mathbb{Z}^{b-1}$ is clearly diagonalizable with positive eigenvalues associated with $P_i$ and $v^*_E$, and the eigenvalue at $v^*_E$ is smaller than the maximal ones at $P_i$.

Let us choose $C_i$. We can find at least one $g' \in \mathbb{Z}^{b-1}$ so that $g'$ has the largest norm eigenvalue $\lambda_1(g')$ with respect to $C_i$ as an automorphism of $\mathbb{S}^n_{v^*E}$. We have $\lambda_1(g') > \lambda_{v^*E}(g')$.

Each $C_i \cap P_i$ has an attracting fixed point of some $g_i \in \Gamma_i$ restricted to $P_i$ if $\Gamma_i$ is hyperbolic. We can choose $g_i$ so that the largest norm eigenvalue $\lambda_i$ of $g_i|P_i$ is sufficiently large. This follows since $\Gamma_i$ is linear on $S_i - P_i$ where we know that this is true for strictly convex cones by the theories of Koszul.
and so on. If $\Gamma_i$ is a trivial group, then we choose $g_i$ to be the identity. Then by taking $k$ sufficiently large, $g_i^k g_i$ has an attracting fixed point in $C'_i \cap P_i$. This must be in $\overline{\text{Cl}}(\hat{O})$. Since the set of attracting fixed points in $C'_i$ is dense in $\text{bd} C'_i \cap P_i$ by Benoist [4], we obtain $C'_i \cap P_i \subset \overline{\text{Cl}}(\hat{O})$.

Let $D'_i$ denote $C_i \cap P_i$. Then the strict join $D'$ of $\overline{\text{Cl}}(D'_1), \ldots, \overline{\text{Cl}}(D'_b)$ equals $P \cap \overline{\text{Cl}}(\hat{O})$, which is $h(\pi_1(\hat{E}))$-invariant. And $D'^o$ is a properly convex subset. If any point of $D'^o$ is in $\text{bd} \hat{O}$, then $D'$ is a subset of $\text{bd} \hat{O}$ by Lemma 3.8. Then $\hat{O}$ is a contained in $v_{\hat{E}} + D'$ where $D'$ is a strict join of $D'_1, \ldots, D'_b$. Some or none could be 0-dimensional. Then $\Gamma$ acts on a strict join. By Proposition 2.17, $\Gamma$ is virtually reducible, a contradiction. Therefore, $D'^o \subset \hat{O}$, and $\hat{E}$ is a totally geodesic end.

(iv) Let $P$ be the minimal totally geodesic subspace containing all of $P_1, \ldots, P_b$. The hyperspace $P$ separates $\hat{O}$ into two parts, ones in the p-end-neighborhood $U$ and the subspace outside it. Clearly $U$ covers $\Sigma_{\hat{E}}$ times an interval by the action of $h(\pi_1(\hat{E}))$ and the boundary of $U$ goes to a compact orbifold projectively diffeomorphic to $\Sigma_{\hat{E}}$.

Let $L$ be a generalized lens for $\hat{E}$. Let $A$ and $B$ be the upper and the lower boundary components of $L$. Each point of $\text{bd} D'$ is an end point of a geodesic projecting to a recurrent geodesic $l$ in $D'^o/\Gamma_{\hat{E}}$. Let $P_i$ be the 2-dimensional subspace containing $v_{\hat{E}}$ and $l$. Lemma 2.11 shows that $A \cap P_i, B \cap P_i$ are either convex arcs sharing endpoints or both are geodesics. Lemma 2.19 and the proof of Lemma 2.11 show that the first case holds. This implies that $\text{bd} A, \text{bd} B \subset \text{bd} D$ and $L$ is a strict lens.

(v) Let $U$ be the p-end-neighborhood of $v_{\hat{E}}$ obtained in (iv). For each $i$, we can find a sequence $g_i$ in the virtual center so that

$$g_i \overline{\text{Cl}}(D'_1) * \cdots * \overline{\text{Cl}}(D'_{i-1}) * \overline{\text{Cl}}(D'_{i+1}) * \cdots * \overline{\text{Cl}}(D'_b)$$

converges to the identity. Therefore, we obtain

$$v_{\hat{E}} * \overline{\text{Cl}}(D'_1) * \cdots * \overline{\text{Cl}}(D'_{i-1}) * \overline{\text{Cl}}(D'_{i+1}) * \cdots * \overline{\text{Cl}}(D'_b) = \text{bd} \hat{O} \cap \overline{\text{Cl}}(U)$$

by the eigenvalue conditions of the virtual center obtained in (iii) and Lemma 2.19. Hence, (v) follows easily now.

(vi) follows by an argument similar to the proof of Theorem 2.13.

$$\square$$

**Corollary 2.16.** Assume as above Theorem 2.15 Given two concave p-end neighborhoods $U$ and $V$, either

- $U$ and $V$ have the same p-end and $U \cap V$ is another concave p-end neighborhood or
- $U \cap V = \emptyset$ when they have distinct p-ends.

Finally a concave p-end neighborhood is always a proper p-end neighborhood.

**Proof.** Let $\tilde{E}_1$ and $\tilde{E}_2$ be the p-end associated with $U$ and $V$ respectively. If $U \cap V \neq \emptyset$, then $S^o(v_{\tilde{E}_1})$ intersect $S^o(v_{\tilde{E}_2})$ since the lens for $U$ is supported
by a totally geodesic hyperspace and so is $V$. Thus, the conclusion follows by Theorems 2.13(iv) and 2.15(vi).

2.3.1. Technical propositions.

**Proposition 2.17.** If a group $G$ of projective automorphisms acts on a strict join $A = A_1 \ast A_2$ for two compact convex sets $A_1$ and $A_2$, then $G$ is virtually reducible.

**Proof.** We prove for $S^n$. Let $x_1, \ldots, x_{n+1}$ denote the homogeneous coordinates. There is at least one set of strict join sets $A_1, A_2$. We choose a maximal number collection of compact convex sets $A'_1, \ldots, A'_m$ so that $A$ is a strict join $A'_1 \ast \cdots \ast A'_m$. Here, we have $A'_i \subset S_i$ for a subspace $S_i$ corresponding to a subspace $V_i \subset \mathbb{R}^{n+1}$ that form independent set of subspaces.

We claim that $g \in G$ permutes the collection $\{A'_1, \ldots, A'_m\}$: Suppose not. We give coordinates so that $A'_i$ satisfies $x_j = 0$ for $j \in I_i$ for some indices and $x_i \geq 0$ for elements of $A$. Then we form a new collection of nonempty sets

$$J' := \{A'_i \cap g(A'_j) \mid 0 \leq i, j \leq n, g \in G\}$$

with more elements. Since

$$A = g(A) = g(A'_1) \ast \cdots \ast g(A'_n),$$

using coordinates we can show that each $A'_i$ is a strict join of nonempty sets in

$$J'_i := \{A'_i \cap g(A'_j) \mid 0 \leq j \leq n, g \in G\}.$$

$A$ is a strict join of the collection of the sets in $J'$, a contraction to the maximal property.

Hence, by taking a finite index subgroup $G'$ of $G$ acting trivially on the collection, $G'$ is reducible. □

**Proposition 2.18.** Suppose that a set $G$ of projective automorphisms in $S^n$ (resp. in $\mathbb{R}P^n$) acts on subspaces $S_1, \ldots, S_b$ and a properly convex domain $\Omega \subset S^n$ (resp. $\subset \mathbb{R}P^n$), corresponding to subspaces $V_1, \ldots, V_b$ so that $V_i \cap V_j = \{0\}$ for $i \neq j$ and $V_1 \oplus \cdots \oplus V_b = \mathbb{R}^{n+1}$. Let $\Omega_i := \text{Cl}(\Omega) \cap S_i$. We assume that

- for each $S_i$, $G_i := \{g|S_i \mid g \in G\}$ forms a bounded set of automorphisms and
- for each $S_i$, there exists a sequence $\{g_{i,j} \in G\}$ with largest norm eigenvalue $\lambda_{i,j}$ restricted at $S_i$ has the property $\{\lambda_{i,j}\} \rightarrow \infty$ as $j \rightarrow \infty$.

Then we obtain $\text{Cl}(\Omega) = \Omega_1 \ast \cdots \ast \Omega_b$ for $\Omega_j \neq \emptyset, j = 1, \ldots, b_0$.

**Proof.** We will prove for $S^n$ but the proof for $\mathbb{R}P^n$ is identical. First, $\Omega_i \subset \text{Cl}(\Omega)$ by definition. Since the element of a strict join has a vector that is a linear combination of elements of the vectors in the directions of $\Omega_1, \ldots, \Omega_b$, Hence, we obtain

$$\Omega_1 \ast \cdots \ast \Omega_b \subset \text{Cl}(\Omega)$$

since $\text{Cl}(\Omega)$ is convex.
Let \( z = [\vec{v}_j] \) for a vector \( \vec{v}_j \) in \( \mathbb{R}^{n+1} \). We write \( \vec{v}_j = \vec{v}_1 + \cdots + \vec{v}_l, \vec{v}_j \in V_j \) for each \( j, j = 1, \ldots, l_0 \), which is a unique sum. Then \( z \) determines \( z_i = [v_i] \) uniquely.

Let \( z \) be any point. We choose a subsequence of \( \{g_{i,j}\} \) so that \( \{g_{i,j}|S_i\} \) converges to a projective automorphism \( g_{i,\infty} : S_i \to S_i \) and \( \lambda_{i,j} \to \infty \) as \( j \to \infty \). Then \( g_{i,\infty} \) also acts on \( \Omega_i \). And \( g_{i,j}(z_i) \to g_{i,\infty}(z_i) = z_{i,\infty} \) for a point \( z_{i,\infty} \in S_i \). We also have

\[
    z_i = g_{i,\infty}^{-1}(g_{i,\infty}(z_i)) = g_{i,\infty}^{-1}(\lim_{j \to \infty} g_{i,j}(z_i)) = g_{i,\infty}^{-1}(z_{i,\infty}).
\]

Now suppose \( z \in \text{Cl}(\Omega) \). We have \( g_{i,j}(z) \to z_{i,\infty} \) by the eigenvalue condition. Thus, we obtain \( z_{i,\infty} \in \Omega_i \) as \( z_{i,\infty} \) is the limit of a sequence of orbit points of \( z \). Hence we also obtain \( z_i \in \Omega_i \) by equation (7) and \( \Omega_i \neq \emptyset \). This shows that \( \text{Cl}(\Omega) \) is a strict join.

**Lemma 2.19.** Assume as in Theorem 2.15. Assume \( \tilde{O} \subset S^n \) (resp. \( \tilde{O} \subset \mathbb{R}P^n \)). Suppose that \( \tilde{E} \) is either of strict lens-type \( R \)-end, and \( \tilde{E} \) is virtually factorable. Then for every sequence \( \{g_j\} \) of distinct elements of the virtual center \( \mathbb{Z}^{k-1} \), we have

\[
    \frac{\lambda_1(g_j)}{\lambda_{v_E}(g_j)} \to \infty, \\
    \frac{\lambda_n(g_j)}{\lambda_{v_E}(g_j)} \to 0.
\]

for the largest norm \( \lambda_1(g) \) of the eigenvalues of \( g \) and the least norm \( \lambda_n(g) \) of those of \( g \).

**Proof.** Since \( \tilde{E} \) is virtually factorable, it has an invariant totally geodesic surface \( S \) as in Theorem 2.15. Suppose that for a sequence \( g_j \) of \( \mathbb{Z}^{l} - \{1\} \), we have

\[
    \begin{bmatrix}
        \lambda_1(g_j) \\
        \lambda_n(g_j) \\
        \lambda_{v_E}(g_j)
    \end{bmatrix}
\]

is bounded above. If the subsequence converges to 0, then \( g_j(x) \) for some \( x \in L \) converges to \( v_E \). This contradicts the invariance and disjointness of \( L \). Thus, we assume that the sequence converges to a positive constant.

We assume without loss of generality that \( \lambda_1(g_j) \) occurs for a fixed collection \( C_i, i \in I \), by taking a subsequence of \( \{g_j\} \) if necessary. Then \( \{g_j\} \) acts as a bounded set of projective automorphisms of \( \ast_{i \in I} C_i \). Since \( g_j \) acts trivially on each \( D_j \) for each \( j \) for all \( j \neq i \) by Theorem 2.15(i). Again by Proposition 2.18, \( \text{Cl}(\Omega) \) is a nontrivial strict join.

By the eigenvalue estimates, there is a convex domain with nonempty interior in \( \ast_{i \in I} C_i \) where a subsequence of \( \{g_j(x)\} \) converges. Thus, the geometric limit of a subsequence of \( \{g_j(L)\} \) is then \( n \)-dimensional and is a strict join as well by Proposition 2.18 by the eigenvalue condition. This contradicts the strictness of \( L \).
The second limit follows by considering $g_j^{-1}$.

3. THE CHARACTERIZATION OF LENS-SHAPED REPRESENTATIONS

The main purpose of this section is to characterize the lens-shaped representations in terms of eigenvalues. This is a major result of this paper and is needed for understanding the duality of the ends.

First, we prove the inequality result for virtually non-factorable and hyperbolic ends. Next, we give the definition of uniform middle-eigenvalue conditions. We show that the uniform middle-eigenvalue conditions imply the existence of limits. Finally, we show the equivalence of the lens condition and the uniform middle-eigenvalue condition in Theorem 3.13 for both R-ends and T-ends under very general conditions.

We will now be working on $S^n$. However, the arguments easily apply to $\mathbb{RP}^n$ versions, which we omit.

3.1. The uniform middle-eigenvalue conditions. Let $O$ be a properly convex real projective orbifold with radial ends and $\hat{O}$ be the universal cover in $S^n$. Let $\hat{E}$ be a p-R-end of $\hat{O}$ and $v_{\hat{E}}$ be the p-end vertex. Let $h : \pi_1(\hat{E}) \to \text{SL}_\pm(n+1, \mathbb{R})_{v_{\hat{E}}}$ be a homomorphism and suppose that $\pi_1(\hat{E})$ is hyperbolic.

Assume that for each nonidentity element of $\pi_1(\hat{E})$, the eigenvalue of $g$ at the vertex $v_{\hat{E}}$ of $\hat{E}$ has a norm strictly between the maximal and the minimal norms of eigenvalues of $g$ —(*). In this case, we say that $h$ satisfies the middle-eigenvalue condition.

In this article, we assume the middle eigenvalue condition for every non-trivial $g$. We denote by the norms of eigenvalues of $g$ by

$$
\lambda_1(g), ..., \lambda_n(g), \lambda_{v_{\hat{E}}}(g), \text{where } \lambda_1(g) \cdot \ldots \cdot \lambda_n(g) \lambda_{v_{\hat{E}}}(g) = \pm 1.
$$

Recall $L_1$ from the beginning of Section 2. We denote by $\hat{h} : \pi_1(\hat{E}) \to \text{SL}_\pm(n, \mathbb{R})$ the homomorphism $L_1 \circ h$. Since $\hat{h}$ is a holonomy of a closed convex real projective $(n - 1)$-orbifold, and $\Sigma_{\hat{E}}$ is assumed to be properly convex, $\hat{h}(\pi_1(\hat{E}))$ divides a properly convex domain $\hat{\Sigma}_{\hat{E}}$ in $S^{n-1}_{v_{\hat{E}}}$.

We denote by $\tilde{\lambda}_1(g), ..., \tilde{\lambda}_n(g)$ the norms of eigenvalues of $\hat{h}(g)$ so that

$$
\tilde{\lambda}_1(g) \geq ... \geq \tilde{\lambda}_n(g), \tilde{\lambda}_1(g) ... \tilde{\lambda}_n(g) = \pm 1
$$

hold. These are called the relative norms of eigenvalues of $g$. We have

$$
\lambda_i(g) = \tilde{\lambda}_i(g)/\lambda_{v_{\hat{E}}}(g)^{1/n} \text{ for } i = 1, ..., n.
$$

Note here that eigenvalues corresponding to

$$
\lambda_1(g), \tilde{\lambda}_1(g), \lambda_n(g), \tilde{\lambda}_n(g), \lambda_{v_{\hat{E}}}(g)
$$

are all positive by Benoist [9]. We define the

$$
\text{length}(g) := \log \left( \frac{\tilde{\lambda}_1(g)}{\lambda_n(g)} \right) = \log \left( \frac{\lambda_1(g)}{\lambda_n(g)} \right).
$$
This equals the infimum of the Hilbert metric lengths of the associated closed curves in $\tilde{\Sigma}/\hat{h}(\pi_1(\tilde{E}))$ as first shown by Kuiper. (See [9] for example.)

We recall the results in [9] and [8].

**Definition 3.1.** Each element $g \in \text{SL}_\pm(n+1, \mathbb{R})$

- that has the largest and smallest norms of the eigenvalues which are distinct and
- the largest or the smallest norm correspond to the eigenvectors with positive eigenvalues respectively

is said to be *bi-semiproximal*. Each element $g \in \text{SL}_\pm(n+1, \mathbb{R})$

- that has the largest and smallest norms of the eigenvalues which are distinct and
- the largest or the smallest norm correspond to the eigenvector of positive eigenvalue unique up to scalars respectively

is said to be *biproximal*.

All infinite order elements of $\hat{h}(\pi_1(\tilde{E}))$ are bi-semiproximal and a finite index subgroup has only bi-semiproximal elements and the identity. Note also when $\Gamma$ acts on a properly convex domain divisibly, an element is *semiproximal* if and only if it is bi-semiproximal (see [5]).

When $\pi_1(\tilde{E})$ is hyperbolic, all infinite order elements of $\hat{h}(\pi_1(\tilde{E}))$ are biproximal and a finite index subgroup has only biproximal elements and the identity. When $\Gamma_{\tilde{E}}$ is a hyperbolic group, an element is *proximal* if and only if it is biproximal.

Assume that $\Gamma_{\tilde{E}}$ is hyperbolic. Suppose that $g \in \Gamma_{\tilde{E}}$ is proximal. We define

$$\alpha_g := \frac{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_n(g)}{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_{n-1}(g)}, \beta_g := \frac{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_n(g)}{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_2(g)},$$

and denote by $\mathcal{R}_\tilde{E}$ the set of proximal elements. We define

$$\beta_{\mathcal{R}_\tilde{E}} := \sup \beta_g, \alpha_{\mathcal{R}_\tilde{E}} := \inf \alpha_g.$$

Proposition 20 of [47] shows that we have

$$1 < \alpha_{\mathcal{Z}_\tilde{E}} \leq \alpha_{\mathcal{R}} \leq 2 \leq \beta_{\mathcal{R}} \leq \beta_{\mathcal{Z}_\tilde{E}} < \infty$$

for constants $\alpha_{\mathcal{Z}_\tilde{E}}$ and $\beta_{\mathcal{Z}_\tilde{E}}$ depending only on $\mathcal{Z}_\tilde{E}$ since $\mathcal{Z}_\tilde{E}$ is properly and strictly convex.

Here, it follows that $\alpha_{\mathcal{R}_\tilde{E}}, \beta_{\mathcal{R}_\tilde{E}}$ depends on $\hat{h}$, and they form positive-valued functions on the union of components of

$$\text{Hom}(\pi_1(\tilde{E}), \text{SL}_\pm(n+1, \mathbb{R}))/\text{SL}_\pm(n+1, \mathbb{R})$$

consisting of convex divisible representations with the algebraic convergence topology as given by Benoist [6].
**Theorem 3.2.** Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends. Let $\tilde{\mathcal{E}}$ be a properly convex $p$-$R$-end of the universal cover $\tilde{\mathcal{O}}$, $\tilde{\mathcal{O}} \subset S^n$, $n \geq 2$. Let $\Gamma_{\tilde{\mathcal{E}}}$ be a hyperbolic group. Then

$$\frac{1}{n} \left( 1 + \frac{n-2}{\beta_{\Gamma_{\tilde{\mathcal{E}}}}} \right) \text{length}(g) \leq \log \tilde{\lambda}_1(g) \leq \frac{1}{n} \left( 1 + \frac{n-2}{\alpha_{\Gamma_{\tilde{\mathcal{E}}}}} \right) \text{length}(g)$$

for every proximal element $g \in \hat{h}(\pi_1(\tilde{\mathcal{E}}))$.

**Proof.** Since there is a biproximal subgroup of finite index, we concentrate on biproximal elements only. We obtain from above that

$$\log \tilde{\lambda}_1(g) \lambda_{n}(g) \leq \log \tilde{\lambda}_2(g) \lambda_{n}(g) \leq \beta_{\Sigma_{\tilde{\mathcal{E}}}} \lambda_{n}(g).$$

We deduce that

$$\tilde{\lambda}_1(g) \geq \left( \frac{\lambda_1(g) \lambda_{n}(g)}{\lambda_2(g) \lambda_{n}(g)} \right)^{1/\beta_{\Sigma_{\tilde{\mathcal{E}}}}} = \exp \left( \frac{\text{length}(g)}{\beta_{\Sigma_{\tilde{\mathcal{E}}}}} \right).$$

Since we have $\tilde{\lambda}_i \leq \tilde{\lambda}_2$ for $i \geq 2$, we obtain

$$\tilde{\lambda}_1(g) \geq \left( \frac{\lambda_1(g) \lambda_{n}(g)}{\lambda_2(g) \lambda_{n}(g)} \right)^{1/\beta_{\Sigma_{\tilde{\mathcal{E}}}}}.$$

and since $\tilde{\lambda}_1 \cdots \tilde{\lambda}_n = 1$, we have

$$\tilde{\lambda}_1(g)^n = \frac{\lambda_1(g) \cdots \lambda_1(g)}{\lambda_2(g) \cdots \lambda_{n-1}(g) \lambda_n(g)} \geq \left( \frac{\lambda_1(g) \lambda_{n}(g)}{\lambda_2(g) \lambda_{n}(g)} \right)^{\frac{n-2}{\beta_{\Sigma_{\tilde{\mathcal{E}}}}}}.$$

We obtain

$$\log \tilde{\lambda}_1(g) \geq \frac{1}{n} \left( 1 + \frac{n-2}{\beta_{\Gamma_{\tilde{\mathcal{E}}}}} \right) \text{length}(g).$$

By similar reasoning, we also obtain

$$\log \tilde{\lambda}_1(g) \leq \frac{1}{n} \left( 1 + \frac{n-2}{\alpha_{\Gamma_{\tilde{\mathcal{E}}}}} \right) \text{length}(g).$$

\[\square\]

**Remark 3.3.** Under the assumption of Theorem 3.2, if we do not assume that $\pi_1(\tilde{\mathcal{E}})$ is hyperbolic, then we obtain

$$\frac{1}{n} \text{length}(g) \leq \log \tilde{\lambda}_1(g) \leq C \frac{n-1}{n} \text{length}(g)$$

for every semiproximal element $g \in \hat{h}(\pi_1(\tilde{\mathcal{E}}))$.  

Proof. Let $\tilde{\lambda}_i(g)$ denote the norms of $\hat{h}(g)$ for $i = 1, 2, \ldots, n$.

$$\log \tilde{\lambda}_1(g) \geq \cdots \geq \log \tilde{\lambda}_n(g), \log \tilde{\lambda}_1(g) + \cdots + \log \tilde{\lambda}_n(g) = 0$$

hold. We deduce

\[
\log \tilde{\lambda}_n(g) = -\log \lambda_1 - \cdots - \log \tilde{\lambda}_{n-1}(g) \\
\geq -(n-1) \log \tilde{\lambda}_1 \\
\log \tilde{\lambda}_1(g) \geq -\frac{1}{n-1} \log \tilde{\lambda}_n(g) \\
\left(1 + \frac{1}{n-1}\right) \log \tilde{\lambda}_1(g) \geq \frac{1}{n-1} \log \frac{\tilde{\lambda}_1(g)}{\lambda_n(g)} \\
\log \tilde{\lambda}_1(g) \geq \frac{1}{n} \text{length}(g).
\] (15)

We also deduce

\[
-\log \tilde{\lambda}_1(g) = \log \tilde{\lambda}_2(g) + \cdots + \log \tilde{\lambda}_n(g) \\
\geq (n-1) \log \tilde{\lambda}_n(g) \\
-(n-1) \log \tilde{\lambda}_n(g) \geq \log \tilde{\lambda}_1(g) \\
(n-1) \log \frac{\tilde{\lambda}_1(g)}{\lambda_n(g)} \geq n \log \tilde{\lambda}_1(g) \\
\frac{n-1}{n} \text{length}(g) \geq \log \tilde{\lambda}_1(g).
\] (16)

□

Remark 3.4. We cannot show that the middle-eigenvalue condition implies the uniform middle-eigenvalue condition. This could be false. For example, we could obtain a sequence of elements $g_i \in \Gamma$ so that $\lambda_1(g_i) / \lambda_{\tilde{E}}(g_i) \to 1$ while $\Gamma$ satisfies the middle-eigenvalue condition. Certainly, we could have an element $g$ where $\lambda_1(g) = \lambda_{\tilde{E}}(g)$. However, even if there is no such element, we might still have a counter-example. For example, suppose that we might have

\[
\frac{\log(\lambda_1(g_i) / \lambda_{\tilde{E}}(g_i))}{\text{length}(g)} \to 0.
\]

(If the orbifold were to be homotopy-equivalent to the end orbifold, this could happen by changing $\lambda_v$ considered as a homomorphism $\pi_1(\Sigma_{\tilde{E}}) \to \mathbb{R}^+$. Such assignments are not really understood globally but see Benoist [9]. Also, an analogous phenomenon seems to happen with the Margulis space-time and diffused Margulis invariants as investigated by Charette, Drumm, Goldman, Labourie, and Margulis recently.)

3.1.1. The uniform middle-eigenvalue conditions and the orbits. Let $\tilde{E}$ be a p-R-end of the universal cover $\tilde{O}$ of a properly convex real projective orbifold $O$ with radial ends. Assume that $\Gamma_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition. There exists a $\Gamma_{\tilde{E}}$-invariant convex set $K$ distanced
from \( \{v_E, v_{E_-}\} \) by Theorem 2.7. For the corresponding tube \( T_{v_E} : K \cap \text{bd} T_{v_E} \) is a compact subset distanced from \( \{v_E, v_{E_-}\} \). We call \( K \) the \( \Gamma_E \)-invariant boundary distanced set. Let \( C_1 \) denote the convex hull of \( K \) in the tube \( T_E \) obtained by Theorem 2.7. Then \( C_1 \) is a \( \Gamma_E \)-invariant subset of \( T_{v_E} \).

Also, \( K \cap \text{bd} T_{v_E} \) contains all attracting and repelling fixed points of \( \gamma \in \Gamma_E \) by invariance and the middle-eigenvalue condition.

**Lemma 3.5.** Let \( O \) be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends. Let \( \tilde{E} \) be a properly convex \( p \)-R-end of the universal cover \( \tilde{O} \), \( \tilde{O} \subset \mathbb{S}^n \). Assume that \( \Gamma_{\tilde{E}} \) is not virtually factorable and hyperbolic and satisfies the uniform middle eigenvalue conditions.

- Suppose that \( \gamma_i \) is a sequence of elements of \( \Gamma_{\tilde{E}} \) acting on \( T_{v_{\tilde{E}}} \).
- The sequence of attracting fixed points \( a_i \) and the sequence of repelling fixed points \( b_i \) are so that \( a_i \to a_\infty \) and \( b_i \to b_\infty \) where \( a_\infty, b_\infty \) are not in \( \{v_E, v_{E_-}\} \).
- Suppose that the sequence \( \{\lambda_i\} \) of eigenvalues where \( \lambda_i \) corresponds to \( a_i \) converges to \( +\infty \).

Then for

\[
M := T_{v_{\tilde{E}}} - \text{Cl}(\bigcup_{i=1}^{\infty} \overline{b_i v_E} \cup \overline{b_i v_{E_-}}),
\]

the point \( a_\infty \) is the limit of \( \{\gamma_i(K)\} \) for any compact subset \( K \subset M \).

**Proof.** There exists a totally geodesic sphere \( \mathbb{S}^{n-1}_i \) at \( b_i \) supporting \( T_{v_{\tilde{E}}} \). \( a_i \) is uniformly bounded away from \( \mathbb{S}^{n-1}_i \) for \( i \) sufficiently large. \( \mathbb{S}^{n-1}_i \) bounds an open hemisphere \( H_i \) containing \( a_i \) where \( a_i \) is the attracting fixed point so that for a euclidean metric \( d_{E_i} : \gamma_i H_i : H_i \to H_i \) is a contraction by the inverse \( k_i \) of the factor

\[
\min \left\{ \frac{\hat{\lambda}_1(\gamma_i)}{\lambda_2(\gamma_i)}, \frac{\lambda_1(\gamma_i)}{\hat{\lambda}_2(\gamma_i)} \right\} k_i^{\frac{n+1}{n}}.
\]

Also, \( k_i \to 0 \) by the uniform middle eigenvalue condition and and by equation (11). Note that \( \{\text{Cl}(H_i)\} \) converges geometrically to \( \text{Cl}(H) \) for an open hemisphere containing \( a \) in the interior.

Actually, we can choose a Euclidean metric \( d_{E_i} \) on \( H_i \) so that \( \{d_{E_i}: J \times J\} \) is uniformly convergent for any compact subset \( J \) of \( H_\infty \). This implies that since \( \{a_i\} \to a \), if \( d_{E}(a_i, K) \leq \varepsilon \) for sufficiently small \( \varepsilon > 0 \), then \( d(a_i, K) \leq C' \varepsilon \) for a positive constant \( C' \).

Since \( \Gamma_{\tilde{E}} \) is hyperbolic, the domain \( \Omega \) corresponding to \( T_{v_{\tilde{E}}} \) in \( \mathbb{S}^{n-1}_i \) is strictly convex. For any compact subset \( K \) of \( M \), the equation \( K \subset M \) is equivalent to

\[
K \cap \text{Cl}(\bigcup_{i=1}^{\infty} \overline{b_i v_E} \cup \overline{b_i v_{E_-}}) = \emptyset.
\]
Since the boundary sphere $bdH_\infty$ meets $\text{Cl}(T_\mathcal{E})$ in this set only by the strict convexity of $\Omega$, we obtain $K \cap bdH_\infty = \emptyset$. And $K \subset H_\infty$ since $\text{Cl}(T_\mathcal{E}) \subset \text{Cl}(H_\infty)$.

We have $d(K, bdH_\infty) > \epsilon_0$ for $\epsilon_0 > 0$. Thus, the distance $d(K, bdH_\infty)$ is uniformly bounded by a constant $\delta$. $d(K, bdH_\infty) > \delta$ implies that $d_E(a_i, K) \leq C/\delta$ for a positive constant $C > 0$. Acting by $g_i$, we obtain $d_E(g_i(K), a_i) \leq k_iC/\delta$, which implies $d(g_i(K), a_i) \leq C'k_iC/\delta$. Since $\{k_i\} \to 0$ and $\{a_i\} \to a$, we imply that $\{g_i(K)\}$ geometrically converges to $a$. □

For the following, $\Gamma_\mathcal{E}$ can be virtually factorable.

**Proposition 3.6.** Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends. Let $\tilde{\mathcal{E}}$ be a properly convex $p$-$R$-end of the universal cover $\tilde{\mathcal{O}}$, $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. Assume that $\Gamma_\mathcal{E}$ satisfies the uniform middle eigenvalue condition. Let $v_\mathcal{E}$ be the $R$-end vertex and $z \in \mathcal{T}_\mathcal{E}$. Then a $\Gamma_\mathcal{E}$-invariant distanced compact set $K$ in $\text{Cl}(T_\mathcal{E}) - \{v_\mathcal{E}, v_\mathcal{E}^-\}$ satisfies the following properties:

(i) Limit points of orbit elements of $z$ are in $K^b := K \cap \partial T_\mathcal{E} - \{v_\mathcal{E}, v_\mathcal{E}^\pm\}$.

(ii) Each point of $K^b$ is a limit of $g_i(x)$ for a point $x \in \tilde{\mathcal{O}}$ for a sequence $g_i \in \Gamma_\mathcal{E}$.

(iii) For each segment $s$ in $\partial T_\mathcal{E}$ with an endpoint $v_\mathcal{E}$, the great segment containing $s$ meets $K^b$ at a point other than $v_\mathcal{E}$. That is, there is a one-to-one correspondence between $bd\Sigma_\mathcal{E}$ and $K^b$.

**Proof.** (i) Let $K$ be any given $\Gamma_\mathcal{E}$-invariant distanced compact set in $\text{Cl}(T_\mathcal{E}) - \{v_\mathcal{E}, v_\mathcal{E}^\pm\}$ by Theorem 2.7.

Consider first when $\Gamma_\mathcal{E}$ is not virtually factorable and hyperbolic. Let $z \in \mathcal{T}_\mathcal{E} - \{v_\mathcal{E}, v_\mathcal{E}^-\}$. Let $[z]$ denote the corresponding element in $\Sigma_\mathcal{E}$. Let $\{\gamma_i\}$ be any sequence in $\Gamma_\mathcal{E}$ so that the corresponding sequence $\{\gamma_i([z])\}$ in $\Sigma_\mathcal{E} \subset \mathbb{S}^{n-1}$ converges to a point $z'$ in $bd\Sigma_\mathcal{E} \subset \mathbb{S}^{n-1}$.

Clearly, a fixed point of $g \in \Gamma_\mathcal{E} - \{1\}$ in $bd\mathcal{T}_\mathcal{E} - \{v_\mathcal{E}, v_\mathcal{E}^-\}$ is in $K^b$ since $g$ has at most one fixed point on each open segment in the boundary. We can assume that for the attracting fixed points $a_i$ and $r_i$ of $\gamma_i$, we have

$$\{a_i\} \to a, \{r_i\} \to r$$

for $a, r \in K^b$ by the closedness of $K^b$. Assume $a \neq r$ first. By Lemma 3.5, we have $\{\gamma_i(z)\} \to a$ and hence the limit $z_\infty = a$.

However, it could be that $a = r$. In this case, we choose $\gamma_0 \in \Gamma_\mathcal{E}$ so that $\gamma_0(a) \neq r$. Then $\gamma_0\gamma_i$ has the attracting fixed point $a_i'$ so that we obtain $\{a_i'\} \to a\gamma_0(a)$ and repelling fixed points $r_i'$ so that $\{r_i'\} \to r$ holds by Lemma 3.7.

Then as above $\{\gamma_0\gamma_i(z)\} \to \gamma_0(a)$ and we need to multiply by $\gamma_0^{-1}$ now to show $\{\gamma_i(z)\} \to a$. Thus, (i) for $K^b$ is proved.

Since $\Gamma_\mathcal{E}$ is hyperbolic, any point $y$ of $bd\Sigma_\mathcal{E} \subset \mathbb{S}^{n-1}$ is a limit point of some sequence $\{g_i\}$. Thus, at least one point of $K$ in the segment $l_y$ containing
with endpoints $v_E$ and $v_{E-}$ is a limit point of some sequence $\{g_i\}$. Also, $l_y \cap K^b$ is a unique since otherwise we can apply $\{g_i^{-1}\}$ and obtain that $K^b$ is not uniformly bounded away from $v_E$ and $v_{E-}$ using the argument of the proof of Lemma 3.5 in reverse. Thus, (ii) and (iii) holds for $K^b$.

Suppose that $\Gamma_E$ is virtually factorable. Then a totally geodesic hyper-space $H$ is disjoint from $\{v_E, v_{E-}\}$ and meets $\hat{O}$ by the proof of Theorem 2.7. Then for any sequence $g_i$ so that $g_i(x) \to x_0$, let $x'$ denote the corresponding point of $\hat{S}_E$. Then $g_i(x')$ converges to a point $y \in S^{n-1}_E$. Let $\bar{x}$ be the vector in the direction of $x'$. We write\[\bar{x} = \bar{x}_E + \bar{x}_H\]where $\bar{x}_H$ is in the direction of $H$ and $\bar{x}_E$ is in the direction of $v_E$. By the uniform middle eigenvalue condition, we obtain $g_i(x') \to x''$ for $x'' \in H$. Hence, $x'' \in H \cap K$.

(ii) We can take any $a \in bd \Sigma_E$, and find a sequence $g_i \in \Gamma_E$ so that $g_i(z) \to a$ for $z \in \hat{S}_E$. We can use a point $z'$ in $\hat{O}$ in the direction of $z$ and this implies the result.

(iii) $K$ meets at a unique point the every segment in $bd T_{v_E}$ from $v_E$ to $v_{E-}$ by Theorem 2.7.

□

We obtain from the above that $K^b$ is homeomorphic to $S^{n-2}$.

**Lemma 3.7.** Let $\{g_i\}$ be a sequence of projective automorphisms acting on a strictly convex domain $\Omega$ in $S^n$ (resp. $\mathbb{R}P^n$). Suppose that the sequence of attracting fixed points $\{a_i \in bd \Omega\} \to a$ and the sequence of repelling fixed points $\{r_i \in bd \Omega\} \to r$. Assume that the corresponding sequence of eigenvalues of $a_i$ limits to $+\infty$ and that of $r_i$ limits to $0$. Let $g$ be any projective automorphism of $\Omega$. Then $\{gg_i\}$ has the sequence of attracting fixed points $\{a'_i\}$ converging to $g(a)$ and the sequence of repelling fixed points converging to $r$.

**Proof.** Recall that $g$ is a quasi-isometry. Given $\epsilon > 0$ and a compact ball $B$ disjoint from a ball around $r$, we obtain that $gg_i(B)$ is in a ball of radius $\epsilon$ of $g(a)$ for sufficiently large $i$. For a choice of $B$ and a sufficiently large $i$, we obtain $gg_i(B) \subset B^0$. Since $gg_i(B) \subset B^0$, we obtain

$$\left(gg_i\right)^n(B) \subset \left(gg_i\right)^m(B)^0$$

for $n > m$ by induction. There exists an attracting fixed point $a'_i$ of $gg_i$ in $gg_i(B)$. Since the diameter of $gg_i(B)$ is converging to 0, we obtain that $\{a'_i\} \to g(a)$.

Also, given $\epsilon > 0$ and a compact ball $B$ disjoint from a ball around $g(a)$, $g_i^{-1}g^{-1}(B)$ is in the ball of radius $\epsilon$ of $r$. Similarly to above, we obtain the needed conclusion. □
3.1.2. Convex cocompact actions of the p-end fundamental groups. In this section, we will prove Proposition 3.9 obtaining a lens.

Lemma 3.8. Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends. Let $\tilde{E}$ be a properly convex p-R-end of the universal cover $\tilde{\mathcal{O}}$, and $\tilde{\mathcal{O}}$ is a subset of $\mathbb{S}^n$. Suppose that $\mathcal{O}$ is properly convex. Let $\sigma$ be a convex domain in $\text{Cl}(\tilde{\mathcal{O}}) \cap P$ for a subspace $P$. Then either $\sigma \subset \text{bd} \tilde{\mathcal{O}}$ or $\sigma^o$ is in $\tilde{\mathcal{O}}$.

Proof. Suppose that $\sigma^o$ meets $\text{bd} \tilde{\mathcal{O}}$ and is not contained in it entirely. We can find a segment $s \subset \sigma^o$ with a point $z$ so that a component $s_1$ of $s - \{z\}$ is in $\text{bd} \tilde{\mathcal{O}}$ and the other component $s_2$ is disjoint from it. We may perturb $s$ in the subspace containing $s$ and $vE_\tilde{\mathcal{O}}$ so that the new segment $s'$ meets $\text{bd} \tilde{\mathcal{O}}$ only in its interior. This contradicts the fact that $\tilde{\mathcal{O}}$ is convex by Theorem A.2 of [16].

Proposition 3.9. Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and with admissible end fundamental groups. Assume that the universal cover $\tilde{\mathcal{O}}$ is a subset of $\mathbb{S}^n$.

- Let $\Gamma_{\tilde{E}}$ be the holonomy group of a properly convex p-R-end $\tilde{E}$,
- Let $\mathcal{T}_{v_\tilde{E}}$ be an open tube corresponding to $R(v_\tilde{E})$,
- Suppose that $\Gamma_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition, and acts on a distanced compact convex set $K$ in $\text{Cl}(\mathcal{T}_{v_\tilde{E}})$ with $K \cap \mathcal{T}_{v_\tilde{E}} \subset \tilde{\mathcal{O}}$.

Then any open p-end-neighborhood containing $K \cap \mathcal{T}_{v_\tilde{E}}$ contains a lens-cone p-end-neighborhood of the p-R-end $\tilde{E}$.

Proof. By assumption, $\tilde{\mathcal{O}} - K$ has two components since $K \cap \mathcal{T}_{v_\tilde{E}}$ has two boundary components closer and farther away from $v_\tilde{E}$. Let $K^b$ denote $\text{bd} \mathcal{T}_{v_\tilde{E}} \cap K$. Let us choose finitely many points $z_1, \ldots, z_m \in U - K$ in the two components of $\tilde{\mathcal{O}} - K$.

Proposition 3.6 shows that the orbits of $z_i$ for each $i$ accumulate to points of $K^b$ only. Hence, a totally geodesic hypersphere separates $v_\tilde{E}$ with these orbit points and another one separates $v_{\tilde{E}^-}$ and the orbit points. Define the convex hull $C_2 := C(\Gamma_{\tilde{E}}(\{z_1, \ldots, z_m\} \cup K))$. Thus, $C_2$ is a compact convex set disjoint from $v_{\tilde{E}^-}$ and $v_{\tilde{E}^-}$ and $C_2 \cap \text{bd} \mathcal{T}_{v_\tilde{E}} = K'$.

Continuing to assume as above.

Lemma 3.10. We are given a distanced compact convex set $K$ in $\text{Cl}(\mathcal{T}_{v_\tilde{E}})$ where $\Gamma_{\tilde{E}}$ acts on, where $(K \cap \mathcal{T}_{v_\tilde{E}})/\Gamma_{\tilde{E}}$ is compact. Let $U$ be a p-end-neighborhood of $v_\tilde{E}$. Then we can choose $z_1, \ldots, z_m$ in $U$ so that for $C_2 := C(\Gamma_{\tilde{E}}(\{z_1, \ldots, z_m\} \cup K))$, $\text{bd} C_2 \cap \tilde{\mathcal{O}}$ is disjoint from $K$ and $C_2 \subset U$.

Proof. We can cover a compact fundamental domain of $\text{bd} K \cap \mathcal{T}_{v_\tilde{E}}$ by the interior of $n$-balls in $\tilde{\mathcal{O}}$ that are convex hulls of finite set $F$ of points in $U$. Since $(K \cap \tilde{\mathcal{O}})/\Gamma_{\tilde{E}}$ is compact, there exists a positive lower bound of
\{d_{O}(x, \text{bd}U) | x \in K\}. We can choose \(\epsilon > 0\) so that the \(\epsilon\cdot d_{O}\)-neighborhood \(U'\) of \(K\) in \(\hat{O}\) is a subset of \(U\). Moreover \(U'\) is convex by Lemma 1.8 of [35].

The convex hull \(C_{2}\) is a union of simplicies with vertices in \(\Gamma_{E}(F)\). If we choose \(F\) to be in \(U'\), then by convexity \(C_{2}\) is in \(U'\) as well.

The disjointness of \(\text{bd}C_{2}\) can be obtained by taking sufficiently many points in \(U\).

We continue:

**Lemma 3.11.** Let \(C\) be a \(\Gamma_{E}\)-invariant distanced compact convex set with boundary in \(T_{E}\) where \((C \cap T_{E})/\Gamma_{E}\) is compact. There are two components \(A\) and \(B\) of \(\text{bd}C \cap T_{E}\) meeting every great segment in \(T_{E}\). Suppose that \(A\) (resp. \(B\)) be disjoint from \(C\). Then \(A\) (resp. \(B\)) contains no line ending in \(\text{bd}\hat{O}\).

**Proof.** It is enough to prove for \(A\). Suppose that there exists a line \(l\) in \(A\) ending at a point of \(\text{bd}T_{E}\). Assume \(l \subset A\). The line \(l\) project to a line \(l'\) in \(\hat{E}\).

Let \(C_{1} = C \cap T_{E}\). Since \(A/\Gamma_{E}\) and \(B/\Gamma_{E}\) are both compact, and there exists a fibration \(C_{1}/\Gamma_{E} \to A/\Gamma_{E}\) induced from \(C_{1} \to A\) using the foliation by great segments from \(v_{E}\).

Since \(A/\Gamma_{E}\) is compact, we choose a compact fundamental domain \(F\) in \(A\) and choose a sequence \(\{x_{i} \in l\}_{i=1,2,...}\) whose image sequence converging to the endpoint of \(l'\) in \(\text{bd}\Sigma_{E}\). We choose \(\gamma_{i} \in \Gamma_{E}\) so that \(\gamma_{i}(x_{i}) \in F\) where \(\{\gamma_{i}(l')\}\) geographically converges to a segment \(l'_{\infty}\) with both endpoints in \(\text{bd}\Sigma_{E}\). Hence, \(\{\gamma_{i}(l)\}\) geographically converges to a segment \(l_{\infty}\) in \(A\). We can assume that for the endpoint \(z\) of \(l\) in \(A\), \(\gamma_{i}(z)\) converges to the endpoint \(p_{1}\) of \(l_{\infty}\) in \(K^{b}\) also. Let \(t\) be the endpoint of \(l\) not equal to \(z\). Then \(t \in K^{b}\) since \(t\) is in the boundary \(A\) with limit points in \(K^{b}\) by Proposition 3.6. Thus, \(\gamma_{i}(t)\) converges to a point of \(K^{b}\) and both endpoints of \(l_{\infty}\) are in \(K^{b}\) and hence \(l_{\infty} \subset C_{1}\). \(l \subset A\) implies that \(l_{\infty} \subset A\). As \(A\) is disjoint from \(C_{1}\), this is a contradiction. The similar conclusion holds for \(B\).

Since \(A\) and analogously \(B\) do not contain any geodesic ending at \(\text{bd}\hat{O}\), \(\text{bd}C_{1}' - \text{bd}T_{E}\) is a union of compact \(n - 1\)-dimensional simplicies meeting one another in strictly convex dihedral angles. By choosing \(\{z_{1}, ..., z_{m}\}\) sufficiently close to \(\text{bd}C_{1}\), we may assume that \(\text{bd}C_{1}' - \text{bd}T_{E}\) is in \(\hat{O}\). Now by smoothing we obtain two boundary components of a lens. (Actually the condition can replace the definition of the lens condition.)

This completes the proof of Proposition 3.9.

3.2. **The uniform middle-eigenvalue conditions and the lens-shaped ends.** A *radially foliated end-neighborhood system* of \(O\) is a collection of end-neighborhoods of \(O\) that are radially foliated where
• each great segment from the end vertex meets the boundary of the end-neighborhoods uniquely and
• the complement of their union is a compact suborbifold with the boundary the union of boundary components of the end-neighborhoods.

Definition 3.12. We say that $\mathcal{O}$ satisfies the triangle condition if for any fixed radially foliated end-neighborhood system of $\mathcal{O}$, every triangle $T \subset \text{Cl}(\tilde{\mathcal{O}})$, if $\partial T \subset \text{bd} \tilde{\mathcal{O}}$, $T^o \subset \tilde{\mathcal{O}}$, then $T^o$ is a subset of a radially foliated $p$-end-neighborhood $U$ in $\tilde{\mathcal{O}}$.

In [26], we will show that this condition is satisfied if $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to the end fundamental groups. We will prove this in [26] since it is a global result and not a result on ends only.

Theorem 3.13. Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and admissible end fundamental groups and satisfy (IE). Assume the following conditions.

• The universal cover $\tilde{\mathcal{O}}$ is a subset of $S^n$ (resp. in $\mathbb{R}P^n$).
• The holonomy group $\Gamma$ is strongly irreducible.

Let $\Gamma_{\tilde{E}}$ be the holonomy group of a properly convex $R$-end $\tilde{E}$. Then the following are equivalent:

(i) $\Gamma_{\tilde{E}}$ is a generalized lens-type $R$-end.
(ii) $\Gamma_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition.

Furthermore, if $\mathcal{O}$ furthermore satisfies the triangle condition or, alternatively, assume that $\tilde{E}$ is virtually factorable, then the following are equivalent.

• $\Gamma_{\tilde{E}}$ is of lens-type if and only if $\Gamma_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition.

Proof. (ii) $\Rightarrow$ (i): Let $\mathcal{T}_{v_{\tilde{E}}}$ denote the tube domain with the $p$-end vertex $v_{\tilde{E}}$ and $v_{E^\emptyset}$. Let $K^b$ denote the intersection of $\text{bdCl}(\mathcal{T}_{v_{\tilde{E}}})$ with the distanced compact $\Gamma_{\tilde{E}}$-invariant convex set $K$ by Theorem 2.7. We may assume that $K$ is the closure of the convex hull of an orbit of points in $\tilde{\mathcal{O}}$ by Proposition 3.6. We assume that $K \subset \text{Cl}(\tilde{\mathcal{O}})$ since $\text{Cl}(\tilde{\mathcal{O}})$ is properly convex.

$\text{bd}K \cap \mathcal{T}_{v_{\tilde{E}}}$ has two components. One closer to $v_{\tilde{E}}$ is called the inner boundary component and the other one is called the outer boundary component. $\mathcal{T}_{v_{\tilde{E}}} - K$ has two components. The one containing $v_{\tilde{E}}$ in the boundary is called the inner component. The other one is called the outer component. Since $K$ is the convex hull of $K^b \subset \text{bd} \mathcal{T}_{v_{\tilde{E}}}$, we obtain that $\text{bd}K \cap \mathcal{T}_{v_{\tilde{E}}}$ is a union of the interiors of simplices with boundary in $K^b$.

Let $C_1$ be the convex hull of $K$ and the finite number of points in the inner component of $\mathcal{T}_{v_{\tilde{E}}} - K$ so that $\text{bd}C_1 \cap \mathcal{T}_{v_{\tilde{E}}}$ is disjoint from $K$. By Lemma 3.11, the component $\text{bd}C_1 \cap \mathcal{T}_{v_{\tilde{E}}}$ contains no line $l$ with endpoints $x, y$ in $K$, and hence can be isotoped to be strictly convex and smooth as above. Thus, a component of $\mathcal{T}_{v_{\tilde{E}}} - \text{bd}C_1$ is a concave end neighborhood of $\tilde{E}$. 

\textbf{(i)} \Rightarrow \textbf{(ii)}: First, the generalized lens condition implies that $\gamma_{\tilde{E}}$ satisfies the middle-eigenvalue condition that $\lambda_1(g)/\lambda_{\nu_{\tilde{E}}}(g) > 1$ for every $g$ since the proof of Theorem 2.15 (ii) shows for virtually factorable $\gamma_{\tilde{E}}$ as well.

There is a map
\[ \gamma_{\tilde{E}} \to H_1(\gamma_{\tilde{E}}, \mathbb{R}) \]
obtained by taking a homology class. The above map $g \to \log \lambda_{\nu_{\tilde{E}}}(g)$ induces homomorphism
\[ \Lambda^h : H_1(\gamma_{\tilde{E}}, \mathbb{R}) \to \mathbb{R} \]
that depends on the holonomy homomorphism $h$.

By the generalized lens-domain, there is a lower boundary component $B$ of $\mathcal{D} \cap T_{\nu_{\tilde{E}}}$ closer to $v_{\tilde{E}}$ that is strictly convex and transversal to every radial great segment from $v_{\tilde{E}}$ in $\tilde{\Sigma}_{\tilde{E}}$.

If $\gamma_{\tilde{E}}$ satisfies the middle-eigenvalue condition, then so does its factors. Suppose that $\gamma_{\tilde{E}}$ does not satisfy the uniform middle-eigenvalue condition. Then there exists a sequence of elements $g_i$ so that
\[ \log \left( \frac{\lambda_h^1(g)}{\lambda_{\nu_{\tilde{E}}}(g)} \right) \frac{\text{length}(g_i)}{\text{length}(g_i)} \to 0 \] as $i \to \infty$.

Note that we can change $h$ by only changing the homomorphism $\Lambda^h$ and still obtain a representation. Let $[g_\infty]$ denote a limit point of $\{[g_i]/\text{length}(g_i)\}$. By a small change of $h$ so that $\Lambda^h(k)$ becomes strictly bigger at $[g_\infty]$. From this, we obtain that
\[ \log \left( \frac{\lambda_h^1(g_i)}{\lambda_{\nu_{\tilde{E}}}(g)} \right) < 0 \] for some $g_i \in \Gamma$.

We know that a small perturbation of a lower boundary component of a generalized lens-shaped end remains strictly convex and in particular distanced since we are changing the connection by a small amount which does not change the strict convexity. (See the proof of Theorem 4.1.) We obtain that $\lambda_h^1(g) < \lambda_{\nu_{\tilde{E}}}(g)$ for some $g$ for the largest eigenvalue $\lambda_h^1(g)$ of $h(g)$ and that $\lambda_{\nu_{\tilde{E}}}(g)$ at $v_{\tilde{E}}$. However, as above the proof of Theorem 2.15 (iii) contradicts.

The final part follows by Lemma 3.14. \hfill \Box

**Lemma 3.14.** Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective manifold with radial or totally geodesic ends and satisfies (IE) and the triangle condition or, alternatively, assume that $\tilde{E}$ is virtually factorable. Then an R-end is of generalized lens-type if and only if it is of lens-type.

**Proof.** If $\tilde{E}$ is virtually factorable, this follows by Theorem 2.15 (iv).

Now assume the triangle condition. By Lemma 3.15, every triangle $T$ with $\partial T$ in $\mathrm{bd} \tilde{\mathcal{O}}$ that is a subset of a p-end-neighborhood in $\tilde{\mathcal{O}}$ in a p-end-neighborhood system does not have the corresponding p-end vertex as its vertex.
Thus, given a generalized lens $L$, let $L^b$ denote $\text{Cl}(L) \cap \text{Cl}(T_{\tilde{v}_E})$. We obtain the convex hull $K$ of $L^b$. $K$ is a subset of $\text{Cl}(L)$. The lower boundary component of $L$ is a smooth convex surface.

Let $K_1$ be the outer component of $\text{bd}K \cap T_{\tilde{v}_E}$. Suppose that $K_1$ meets $\text{bd}\tilde{O}$. $K_1$ is a union of the interior of simplices. By Lemma 3.8, a simplex is either in $\text{bd}\tilde{O}$ or disjoint from it. Hence, there is a simplex $\sigma$ in $K_1 \cap \text{bd}\tilde{O}$. Taking the convex hull of $v_{\tilde{E}}$ and an edge in $\sigma$, we obtain a triangle $T$ with $\partial T \subset \text{bd}\tilde{O}$ and $T^o \subset \tilde{O}$. This contradicts the triangle condition. Thus, $K_1 \subset \tilde{O}$. By Proposition 3.9, we obtain a lens-cone in $\tilde{O}$.

Lemma 3.15. Suppose that $O$ is a strongly tame properly convex real projective manifold with radial or totally geodesic ends and satisfies (IE) and the triangle condition. Assume that the universal cover $\tilde{O}$ is a subset of $\mathbb{S}^n$. Then every triangle $T$ with $\partial T \subset \text{bd}\tilde{O}$ and $T^o \subset \tilde{O}$ contains a vertex equal to a $p$-end vertex.

Proof. Let $v_{\tilde{E}}$ be a $p$-end vertex. Choose a fixed radially foliated $p$-end-neighborhood system. Suppose that a triangle $T$ with $\partial T \subset \text{bd}\tilde{O}$ contains a vertex equal to a $p$-end vertex. Let $U$ be an inverse image of a radially foliated $p$-end-neighborhood $\hat{U}$ in the $p$-end-neighborhood system corresponding to $\tilde{E}$ with a $p$-end vertex $v_{\tilde{E}}$. The end orbifold $\Sigma_{\tilde{E}}$ is a properly convex end of an orbifold $\tilde{O}$.

Choose a maximal line $l$ in $T$ with endpoints $v_{\tilde{E}}$ and $w$ in the interior of an edge of $T$ not containing $v_{\tilde{E}}$. Then this line has to pass a point of the boundary of $U$ and in $T^o$ by definition of the radial foliations of the $p$-end-neighborhoods. This implies that $T^o$ is not a subset of a $p$-end-neighborhood and contradicts the assumption. \hfill \Box

We now prove the dual to Theorem 3.13. For this we do not need the triangle condition or the reducibility of the end.

Theorem 3.16. Let $O$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends and (IE). Assume that the holonomy group is strongly irreducible. Assume that the universal cover $\tilde{O}$ is a subset of $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$). Let $S_{\tilde{E}}$ be a totally geodesic ideal boundary of $\tilde{O}$. Then the following conditions are equivalent:

(i) $\tilde{E}$ satisfies the uniform middle-eigenvalue condition.

(ii) $S_{\tilde{E}}$ has a lens neighborhood in an ambient open manifold containing $\tilde{O}$ and hence $\tilde{E}$ has a lens-type $p$-end-neighborhood in $\tilde{O}$.

Proof. It suffices to prove for $\mathbb{S}^n$. Assuming (i), the existence of a lens neighborhood follows from Theorem A.10.

Assuming (ii), we obtain a totally geodesic $(n-1)$-dimensional properly convex domain $S_{\tilde{E}}$ in a subspace $\mathbb{S}^{n-1}$ where $\Gamma_{\tilde{E}}$ acts on. Let $U$ be the two-sided properly convex neighborhood of it where $\Gamma_{\tilde{E}}$ acts on. Then since $U$ is a two-sided neighborhood, the supporting hemisphere at each point of
\( \text{Cl}(S_{E}) - S_{E} \) is now transversal to \( S^{n-1} \). The dual \( U^* \) of \( U \) is contained in a lens-cone by Corollary 2.3. \( \Gamma_{E} \) acts on the lens-cone again. We apply the part (i) \( \Rightarrow \) (ii) of Theorem 3.13.

**Proof of Corollary 1.9.** Let \( E \) be an R-end. Under the premise, \( \lambda_{v_{E}}(g) = 1 \) for p-end vertex \( v_{E} \) of \( E \). Suppose that \( \hat{E} \) is properly convex. By Theorem 3.2 and Remark 3.3, \( \hat{E} \) satisfies the uniform middle eigenvalue condition. Theorem 3.13 implies the result.

If \( \hat{E} \) is complete affine, then the result follows by Theorem 1.4. If \( E \) is a T-end, Theorem 3.16 implies the result similarly.

**3.3. The characterization of quasi-lens p-R-end-neighborhoods.** This is the last remaining case for the properly convex ends with weak uniform middle eigenvalue conditions. We will only prove for \( S^n \).

**Definition 3.17.** Let \( U \) be a totally geodesic lens cone p-end-neighborhood of a p-R-end in a subspace \( S^{n-1} \) with vertex \( v \). Let \( G \) denote the p-end fundamental group satisfying the weak uniform middle eigenvalue condition.

- Let \( D \) be the totally geodesic \( n-2 \)-dimensional domain so that \( U = D \star v \).
- Let \( S^1 \) be a great circle meeting \( S^{n-1} \) at \( v \).
- Extend \( G \) to act on \( S^1 \) as a nondiagonalizable transformation fixing \( v \).
- Let \( \zeta \) be a projective automorphism acting on \( U \) and \( S^1 \) so that \( \zeta \) commutes with \( G \) and restrict to a diagonalizable transformation on \( \text{Cl}(D) \) and act as a nondiagonalizable transformation on \( S^1 \) fixing \( v \) and with largest norm eigenvalue at \( v \).

Every element of \( G \) and \( \zeta \) can be written as a matrix

\[
\begin{pmatrix}
S(g) & 0 \\
0 & \lambda_v(g) & \lambda_v(g)v(g) & \lambda_v(g)
\end{pmatrix}
\]

where \( v = [0, \ldots, 1] \). Note that \( g \mapsto v(g) \in \mathbb{R} \) is a well-defined map inducing a homomorphism

\[
\langle G, \zeta \rangle \to H_1(\langle G, \zeta \rangle) \to \mathbb{R}
\]

and hence

\[
|v(g)| \leq C\text{cw}(g)
\]

for a positive constant \( C \).

We assume that \( \zeta \) has the largest eigenvalue associated with \( S^1 \) and acts trivially on \( D \). Again, we may assume that \( G \) has the largest norm eigenvalue and the smallest norm eigenvalue occur in \( D \). (See Remark 1.6.) Hence \( \lambda_v(g) \) for \( g \in G \) is not the eigenvalue with largest or smallest norms.

**Positive translation condition:** We choose an affine coordinate on a component \( I \) of \( S^1 - \{v, v_-\} \). We assume that for each \( g \in \langle G, \zeta \rangle \),
• if \( \lambda_{v}(g) > \lambda_{2}(g) \) for the largest eigenvalue \( \lambda_{2} \) associated with \( \text{Cl}(D) \), then \( v(g) > 0 \) in equation (17),

- For \( g \) satisfying \( \lambda_{v}(g) > \lambda_{2}(g) \), there exists a constant \( c_{1} \) independent of \( g \)

\[
\frac{v(g)}{\log \frac{\lambda_{v}(g)}{\lambda_{2}(g)}} > c_{1} > 0.
\]

Clearly, this type of construction can be done easily by choosing \( G \) and \( \zeta \) satisfying the above properties by essentially choosing \( \zeta \) well. The converse to this construction is the following:

**Proposition 3.18.** Suppose that \( \langle G, \zeta \rangle \) satisfies the positive translation condition. Then the above \( U \) is in the boundary of a properly convex p-end open neighborhood \( V \) of \( v \) and \( \langle G, \zeta \rangle \) acts on \( V \).

**Proof.** Let \( I \) be the segment in \( S^{1} \) bounded by \( v \) and \( v_{-} \). Take \( D \ast I \) is a tube with vertices \( v \) and \( v_{-} \).

Taking the interior of the convex hull of an orbit and \( U \) will give us \( V \).

Let \( x \) be an interior point of the tube. Given a sequence \( g_{i} \in G \), then we will show that \( g_{i}(x) \) accumulates to points uniformly bounded away from \( v_{-} \) by the positive translation conditions as we can show by using estimates.

Suppose not. Then there exists a sequence \( q_{i} \in G \) with \( \{g_{i}(x)\} \) accumulates to \( v_{-} \). Given any sequence \( g_{i} \in \langle G, \zeta \rangle \), we write as \( g_{i} = \zeta^{j} g'_{i} \) for \( g'_{i} \in G \). We write

\[
x = [v], v = v_{1} + v_{2}, [v_{1}] \in D, [v_{2}] \in I - \{v\} \subset S^{1},
\]

\[
g_{i}(x) = [g_{i}(v_{1}) + g_{i}(v_{2})].
\]

Since we can always extract a subsequence for any converging subsequence, we consider only three cases:

\[
\frac{\lambda_{v}(g_{i})}{\lambda_{2}(g_{i})} \to \infty,
\]

\[
\frac{1}{C} < \frac{\lambda_{v}(g_{i})}{\lambda_{2}(g_{i})} < C \text{ for some } C > 1
\]

\[
\frac{\lambda_{v}(g_{i})}{\lambda_{2}(g_{i})} \to 0.
\]

If \( \lambda_{v}(g_{i})/\lambda_{2}(g_{i}) \to \infty \), then \( ||g_{i}(v_{1})||/||g_{i}(v_{2})|| \to 0 \) and \( g_{i}(x) \) converges to the limit of \( [g_{i}(v_{2})] \), i.e., \( v \), since \( v(g_{i}) \to \infty \). Suppose that \( \lambda_{v}(g_{i})/\lambda_{2}(g_{i}) \) is uniformly bounded from 0 and \( \infty \). Then we multiply by \( \zeta^{k} \) for uniformly bounded \( |j| \) so that \( \lambda_{v}(\zeta^{j} g_{i}) > \lambda_{2}(\zeta^{j} g_{i}) \) but the ratio

\[
\left| \log \frac{\lambda_{v}(\zeta^{j} g_{i})}{\lambda_{2}(\zeta^{j} g_{i})} \right|
\]

is uniformly bounded. Then \( |v(\zeta^{j} g_{i})| < C' \) for a constant by the positive translation condition. This also implies that \( |v(g_{i})| \) is uniformly bounded as
\(|j|\) is uniformly bounded. This implies \(g_i(x)\) lies in a \((\pi - \epsilon)\cdot d\)-neighborhood of \(v_\tilde{E}\) for a uniform constant \(\epsilon\).

Suppose now that \(\lambda_v(g_i)/\lambda_2(g_i) \to 0\). Then by estimating with the equation (18), we obtain
\[
||g_i(v_2)||/||g_i(v_1)|| \to 0,
\]
and \(g_i(x)\) converges to a point of \(D\).

We showed in all cases that the accumulation points of any orbit is outside a small ball at \(v_-\). This contradicts our assumption that \(\{g_i(x)\}\) accumulates to \(v_-\). Thus, these orbit points are inside the properly convex tube and outside a small ball at \(v_-\). The convex hull of the orbit of \(x\) is a properly convex open domain as desired above.

This generalizes the quasi-hyperbolic annulus discussed in [20]. We give a more concise condition at the end of the subsection.

Conversely, we obtain:

**Proposition 3.19.** Let \(\mathcal{O}\) be a strongly tame properly convex real projective manifold with radial or totally geodesic ends and satisfy (IE). Assume that the universal cover \(\tilde{\mathcal{O}}\) is a subset of \(\mathbb{S}^n\) (resp. \(\mathbb{R}P^n\)). Suppose that holonomy group of \(\pi_1(\mathcal{O})\) is strongly irreducible. Let \(\tilde{E}\) be a properly convex radial end satisfying the weak uniform middle eigenvalue conditions but not the uniform middle eigenvalue condition. Then \(\tilde{E}\) has a quasi-lens type \(p\)-end-neighborhood.

**Proof.** If \(\tilde{E}\) is not virtually factorable and hyperbolic, then it satisfies the uniform middle eigenvalue condition by definition. We recall a part of the proof of Theorem 2.15.

Now assume that \(\tilde{E}\) is virtually factorable. Let \(U\) be a \(p\)-end-neighborhood of \(\tilde{E}\) in \(\tilde{\mathcal{O}}\). Let \(S_1, \ldots, S_0\) be the projective subspaces in general position meeting only at the \(p\)-end vertex \(v_\tilde{E}\) on which the factor groups \(\Gamma_1, \ldots, \Gamma_0\) act irreducibly. Let \(C_i\) denote the union of great segments from \(v_\tilde{E}\) corresponding to the invariant cones in \(S_i\) where \(\Gamma_i\) acts irreducibly for each \(i\). The abelian center isomorphic to \(\mathbb{Z}^{l_0-1}\) acts as the identity on \(C_i\) in the projective space \(\mathbb{S}^n_{v_\tilde{E}}\). Let \(g \in \mathbb{Z}^{b-1}\). \(g|C_i\) can have more than two eigenvalues or just single eigenvalue. In the second case \(g|C_i\) could be represented by a matrix with eigenvalues all 1 fixing \(v_\tilde{E}\).

(a) \(g|C_i\) fixes each point of a hyperspace \(P_i \subset S_i\) not passing through \(v_\tilde{E}\) and \(g\) has a representation as a nontrivial scalar multiplication in the affine subspace \(S_i - P_i\) of \(S_i\). Since \(g\) commutes with every element of \(\Gamma_i\) acting on \(C_i\), \(\Gamma_i\) acts on \(P_i\) as well. We let \(D'_i = C_i \cap P_i\).

(b) \(g|C_i\) is represented by a matrix with eigenvalues all 1 fixing \(v_\tilde{E}\).

We denote \(l_1 := \{i|\exists g \in \mathbb{Z}^{b-1}, g|C_i \neq 1\}\) and \(l_2 := \{i|\forall g \in \mathbb{Z}^{b-1}, g|C_i\text{ has only one eigenvalue}\}\).
Let \( D_i \subset S^{n-1} \) denote the convex compact domain that is the space of great segments in \( C_i \) from \( \nu_+ \) to \( \nu_- \). Then
\[
\tilde{\Sigma}_E = D_1 \ast \cdots \ast D_b
\]
by Theorem 2.15. Also, \( D'_i \) is projectively diffeomorphic to \( D_i \) by projection for \( i \in I_1 \).

Suppose that hyperbolic \( \Gamma_i \) acts on \( C_i \). Then it satisfies the uniform middle eigenvalue condition by Definition 1.5. We recall Remark 1.6. Hence by Theorem 3.13, \( \Gamma_i \) acts on a lens domain \( D_i \). For the central \( g \) in \( \Gamma_E \), \( g \) acts on each great segment from \( \nu_+ \) to \( \nu_- \). If \( i \in I_2 \), then \( g|C_i \) must be the identity; otherwise, we again obtain a violation of the proper convexity condition. Hence, \( \Gamma_E \) satisfies the uniform middle eigenvalue condition, a contradicting the assumption.

For \( i \in I_2 \), \( \Gamma_i \) is not hyperbolic as above and hence must be a trivial group and \( C_i \) is a segment. Consider \( C_{h} := \ast_{i \in I_2} C_i \). Then \( g|C_i \) for \( g \in \mathbb{Z}^{b-1} \) has only eigenvalue \( \lambda_{\nu_+} \) associated with it so that we don't have two distinct eigenvalues for \( C_i \). Since \( \dim C_i = 1 \), \( g|C_i \) is a translation in an affine coordinate system. Therefore, \( \mathbb{Z}^{b-1} \) acts trivially on the space of great segments in \( C_h \). Thus, \( \dim C_h = 1 \) since otherwise we cannot obtain the compact quotient \( \tilde{\Sigma}_E/\Gamma_E \).

Therefore, we obtain \( D = \ast_{i=1}^{n-1} D_i \) is a totally geodesic plane disjoint from \( \nu_+ \). Let \( \nu_+ = [0, \ldots, 0, 1] \subset S^n \). Let \( I'_2 = \{ n \} = I_2 \). We write \( g \in \Gamma_E \) in coordinates as:
\[
g = \begin{pmatrix}
S_g & 0 \\
0 & \lambda_\nu(g)
\end{pmatrix}
\]
where \( S_g \) is a \( n \times n \) matrix representing coordinates \( \{ 1, \ldots, n-1 \} \). Then \( V : g \in \Gamma_E \to \nu(g) \in \mathbb{R} \) is a linear function.

The proper convexity of \( \tilde{\Omega} \) implies that \( \nu(g) \geq 0 \) if \( \lambda_\nu(g)/\lambda_\nu(g) > 1 \); otherwise, we obtain a great segment in \( S^1 \) by a limit of \( \gamma_i(s) \) for a segment \( s \subset U \) from \( \nu_+ \).

Suppose that we have a sequence \( \gamma_i \) so that \( \lambda_\nu(g_i)/\lambda_\nu(g_i) \to \infty \), and
\[
0 \leq \nu(g_i) < C \quad \text{for a uniform constant } C.
\]
Given a segment \( s \subset U \) with an endpoint \( \nu_+, \gamma_i(s) \) then converges to a segment \( s_\infty \) in \( S^1 \). If \( \nu(g) > 0 \) for any \( g \in \Gamma_E \), we can apply \( \nu_+ \) to obtain a great segment in the limit for \( i \to \pm \infty \). Therefore, \( \nu(g) = 0 \) for all \( g \in \Gamma_E \).

We obtain an element \( \eta_i \) so that \( \lambda_\nu(\eta_i)/\lambda_\nu(\eta_i) \to \infty \) and \( \eta_i|D \) is uniformly bounded using the fact that \( \tilde{\Sigma}_E \) is projectively diffeomorphic to the interior of
the cone \( \{ \rho \} \ast D \). We have \( v(\eta_i) = 0 \) for all \( i \). Then we can apply Propositions 2.17 and 2.18 to obtain a contradiction to the strong irreducibility of \( \Gamma \).

Since \( \Sigma_{\tilde{E}} \) is a join with a factor equal to a vertex corresponding to \( S^1 \), we can choose a generator \( \zeta \) of the virtual center so that \( \lambda_v(\zeta) > \lambda_2(\zeta) \). \( \langle \zeta \rangle \) is a factor of the center. Let \( G \) be the product of other factors of \( \Gamma_{\tilde{E}} \). The above paragraph shows \( v(\zeta) > 0 \).

Every element \( g \) with \( \lambda_v(g) > \lambda_2(g) \) is of form \( \zeta^j g' \) for \( \lambda_v(g') \) uniformly bounded above. For such a set \( A \) of \( g' \) we have \( v(g') \) are uniformly bounded below since otherwise the orbit of a point under \( A \) has a subsequence converging to \( v_{E_\to} \). We can verify the positive translation condition.

By Proposition 3.18, we obtain a quasi-lens \( p \)-end-neighborhood. \( \Box \)

Remark 3.20. To explain the positive translation condition more, \( \log \lambda_v(\tilde{E}) \) and \( v(g) \) give us homomorphisms \( \log \lambda_v, V : H_1(\Gamma_{\tilde{E}}) \to \mathbb{R} \). Restricted to \( Z_{l_0-1} \subset H_1(\Gamma_{\tilde{E}}) \), we obtain \( \log \lambda_i : Z^{b-1} \to \mathbb{R} \) given by taking the log of the eigenvalues restricted to \( D_i \) above. The condition restricts to the positivity of \( V \) on the cone \( C \) in \( Z^{b-1} \) defined by

\[ \log \lambda_v(\tilde{E}) > \log \lambda_i(\tilde{E}), i = 1, \ldots, l_0 - 1. \]

Since \( \lambda_v(g) \) is less than largest norm of the eigenvalues in \( \text{Cl}(D) \) for \( g \in \Gamma_i - \{1\}, i < l_0 \) by the uniform middle eigenvalue conditions, this cone condition is equivalent to the full conditions by our choice in Remark 1.6.

Also, we remark that S. Ballas [1] has found such \( R \)-ends on a strongly tame manifold with real projective structures deformed from a complete hyperbolic structure on a figure-eight knot complement.

4. The openness of the lens properties, and expansion and shrinking of end neighborhoods

We will list a number of properties that we will need later. (These are not essential in this paper itself.) We show the openness of the lens properties, i.e., the stability for properly convex radial ends and totally geodesic ends. We can find an increasing sequence of horoball \( p \)-end-neighborhoods, lens-type \( p \)-end-neighborhoods for radial or totally geodesic \( p \)-ends that exhausts \( \tilde{E}^{\circ} \). We also show that the \( p \)-end-neighborhood always contains a horoball \( p \)-end-neighborhood or a concave \( p \)-end neighborhood.

4.1. The openness of lens properties. A radial affine connection is an affine connection on \( \mathbb{R}^{n+1} - \{O\} \) invariant under the radial dilatation \( S_t : \tilde{v} \to t\tilde{v} \) for every \( t > 0 \).

As conditions on representations of \( \pi_1(\tilde{E}) \), the condition for generalized lens-shaped ends and one for lens-shaped ends are the same. Given a holonomy group of \( \pi_1(\tilde{E}) \) acting on a generalized lens-shaped cone \( p \)-end neighborhood, the holonomy group satisfies the uniform middle eigenvalue condition by Theorem 3.13. We can find a lens cone by choosing our orbifold to be \( \mathcal{T}_{\tilde{E}} / \pi_1(\tilde{E}) \) and using the last step of Theorem 3.13.
Theorem 4.1. Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends and satisfy (IE). Assume that the holonomy group is strongly irreducible. Assume that the universal cover $\tilde{\mathcal{O}}$ is a subset of $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$). Let $\tilde{E}$ be a properly convex $p$-$R$-end of the universal cover $\tilde{\mathcal{O}}$. Let $\text{Hom}_E(\pi_1(\tilde{E}), \text{SL}_+(n+1, \mathbb{R}))$ (resp. $\text{Hom}_E(\pi_1(\tilde{E}), \text{PGL}(n+1, \mathbb{R}))$) be the space of representations of the fundamental group of an $(n-1)$-orbifold $\Sigma_{\tilde{E}}$ with an admissible fundamental group. Then

(i) $\tilde{E}$ is a generalized lens-type $R$-end if and only if $\tilde{E}$ is a strictly generalized lens-type $R$-end.

(ii) The subspace of generalized lens-shaped representations of an $R$-end is open.

Finally, if $\mathcal{O}$ satisfies the triangle condition or every end is virtually factorable, then we can replace the word generalized lens-type to lens-type in each of the above statements.

Proof. (i) If $\pi_1(\tilde{E})$ is hyperbolic, then the equivalence is given in Theorem 2.13 (i), and if $\pi_1(\tilde{E})$ is a virtual product of hyperbolic groups and abelian groups, then it is in Theorem 2.15 (iv).

(ii) Let $\mu$ be a representation $\pi_1(\tilde{E}) \to \text{SL}_+(n+1, \mathbb{R})$ associated with a generalized lens-cone. By Theorem 3.13 (ii) $\Rightarrow$ (i), $\tilde{E}$ satisfies the uniform middle eigenvalue condition. Conversely, by Theorem 3.13 (i) $\Rightarrow$ (ii), we obtain a lens domain $K$ in $T_v\ddot{\mathcal{E}}$ with smooth convex boundary components $A \cup B$ since $T_v\ddot{\mathcal{E}}$ itself satisfies the triangle condition although it is not properly convex.

We note that $K/\mu(\pi_1(\tilde{E}))$ is a compact manifold whose boundary is the union of two closed $n$-orbifold components $A/\mu(\pi_1(\tilde{E})) \cup B/\mu(\pi_1(\tilde{E}))$. Suppose that $\mu'$ is sufficiently near $\mu$.

Given $K$, we can find a convex $(n+1)$-domain $K' \subset K^\circ$ bounded by two smooth open $n$-cells $A'$ and $B'$ in $K^\circ$.

Since $K$ is properly convex, we choose $K'$ as above. The linear cone $C(K) \subset \mathbb{R}^{n+1} = \Pi_1(K)$ over $K$ has a smooth strictly convex hessian function $V$ by Vey's work [70]. Let $C(K')$ denote the linear cone over $K'$. We extend the group $\mu'(\pi_1(\tilde{E}))$ by adding a transformation $\gamma : \dddot{v} \mapsto 2\dddot{v}$ to $C(K')$. For the fundamental domain $F'$ of $C(K')$ under this group, the hessian matrix of $V$ restricted to $F \cap C(K')$ has a lower bound. Also, the boundary $\partial C(K')$ is strictly convex in any affine coordinates in any transversal subspace to the radial directions at any point.

Let $M$ be a compact orbifold $C(K')/\langle \mu(\pi_1(\tilde{E})), \gamma \rangle$ with a flat affine structure. Note that $S_t$, $t \in \mathbb{R}_+$, becomes an action of a circle on $M$. The change of representation $\mu$ to $\mu' : \pi_1(\tilde{E}) \to \text{Aut}(\mathbb{S}^n)_{v\ddot{E}}$ is realized by a change of holonomy representations of $M$ and hence by a change of affine connections on $C(K')$. Since $S_t$ commutes with the images of $\mu$ and $\mu'$, $S_t$ still gives us a circle action on $M$ with a different affine connection. We may assume
without loss of generality that the circle action is fixed and $M$ is invariant under this action.

Thus, $M$ is a union of $B_1, \ldots, B_m$ that are $n$-ball times circles foliated by circles that are flow arcs of $S_t$. We can change the affine structure on $M$ to a one with the holonomy group $\langle \mu'(\pi_1(\tilde{E})), \gamma \rangle$ by local regluing $B_1, \ldots, B_m$ as in [15]. We assume that $S_t$ still gives us a circle affine action since $\gamma$ is not changed. We may assume that $M$ and $\partial M$ are foliated by circles that are flow curves of the circle action. The change corresponds to a sufficiently small $C^r$-change in the affine connection for $r \geq 2$ as we can see from [15].

Now, the strict positivity of the hessian of $V$ in the fundamental domain, and the boundary convexity are preserved. Let $C(K'')$ denote the universal cover of $M$ with the new affine connection. Thus, $C(K'')$ is also a properly convex affine cone by Koszul’s work [56]. Also, it is a cone over a properly convex domain $K''$ in $\mathbb{S}^n$.

Thus $K''$ is a properly convex domain with strictly convex boundary $A''$ and $B''$. The complement $\Lambda = C((K'') - A'' - B''$ is a closed subset. Then by Theorems 2.13 and 2.15, we obtain that the end is also strictly lens-shaped.

We may assume that $v_{\tilde{E}}$ is fixed by conjugating $\mu'$ by a bounded projective transformation. A segment is radial if it is in the radial segment from $v_{\tilde{E}}$. By considering the radial segments in $K$, we obtain a foliation by radial lines in $K$ also. Thus, we obtain a lens-cone in $T_{v_{\tilde{E}}}$. This completes the proof of (ii).

The final statement follows by Lemma 3.14.

A strict lens $p$-end-neighborhood of a $p$-T-end $\tilde{E}$ is a lens $p$-end-neighborhood so that for its boundary component $A$, $C(A) - A$ is a subset of $\partial S_{\tilde{E}}$ for the ideal boundary component $S_{\tilde{E}}$ of $\tilde{E}$.

**Theorem 4.2.** Let $O$ be a strongly tame properly convex real projective orbifold with radial ends or totally geodesic ends and satisfy (IE). Assume that the holonomy group is strongly irreducible. Assume that the universal cover $\tilde{O}$ is a subset of $\mathbb{S}^n$ (resp. of $\mathbb{R}P^n$). Let $\tilde{E}$ be a $p$-T-end of the universal cover $\tilde{O}$. Let $\text{Hom}_E(\pi_1(\tilde{E}), \text{SL}_+(n+1, \mathbb{R}))$ (resp. $\text{Hom}_E(\pi_1(\tilde{E}), \text{PGL}(n+1, \mathbb{R}))$) be the space of representations of the fundamental group of an $n$-orbifold $\Sigma_{\tilde{E}}$ with an admissible fundamental group. Then

(i) $\tilde{E}$ is a lens-type $p$-T-end if and only if $\tilde{E}$ is a strictly lens-type $p$-T-end.

(ii) The subspace of lens-shaped representations of a $p$-T-end is open.

**Proof.** We are proving for $\mathbb{S}^n$ only. The $\mathbb{R}P^n$-version follows from this. First assume that $\Gamma_{\tilde{E}}$ is not virtually factorable and hyperbolic.

(i) Let $L$ be a lens $p$-end-neighborhood of the totally geodesic domain $S_{\tilde{E}}$ corresponding to $\tilde{E}$. Similarly to the proof of Proposition 3.6, we obtain the result using Lemma A.11.
\((i)\) follows as in the proof of Theorem 4.1 for the virtually non-factorable and hyperbolic \(\Gamma_E\) since the lens neighborhood of the end totally geodesic orbifold in the ambient space containing \(O\) has smooth strictly convex boundary.

In the virtually factorable case for \(\Gamma_\bar{E}\), \(\Gamma_\bar{E}\) is dual to the holonomy group of a lens-shaped R-end by Corollary 2.3. By Theorem 3.13, the dual is a generalize lens-shaped R-end. Since \(\Gamma_\bar{E}\) is virtually factorable, it is a totally geodesic R-end acting on a hyperspace \(S\) by Theorem 2.15. Thus, \(\Gamma_\bar{E}\) acts fixing a point \(S^*\) dual to \(S\). Then the proof of Theorem 4.1 for virtually factorable \(\Gamma_E\) applies for this case. \(\Box\)

**Corollary 4.3.** We are given a properly convex end \(\bar{E}\) of a strongly tame properly convex orbifold \(O\) with radial or totally geodesic ends and satisfying (IE). Assume that \(\bar{O} \subset S^n\) (resp. \(\bar{O} \subset \mathbb{R}P^n\)). Then the subset of

\[
\text{Hom}_E(\pi_1(\bar{E}), \text{SL}_\pm(n + 1, \mathbb{R})) \ (\text{resp. Hom}_E(\pi_1(\bar{E}), \text{PGL}(n + 1, \mathbb{R})))
\]

consisting of representations satisfying the uniform middle-eigenvalue condition is open.

**Proof.** For p-R-ends, this follows by Theorems 3.13 and 4.1. For p-T-ends, this follows by dual results: Theorem 3.16 and Theorems 4.2. \(\Box\)

**Corollary 4.4.** Let \(O\) be a strongly tame properly convex real projective manifold with radial or totally geodesic ends and satisfy (IE). Moreover, \(\mathcal{C}\) of Proposition 2.8 restricts to a correspondence between the generalized lens-type R-ends with lens-type T-ends.

**Proof.** Let \(\bar{O}\) be the universal cover of \(O\). By Theorem 4.2, a lens-type T-ends is a strict lens-type T-end. By Corollary 2.3, a lens-type T-end neighborhood in \(\bar{O}\) is dual to a lens-cone. Since the lens-cone contains \(\bar{O}^*\) by equation (3) of [24], the lens-cone contains a generalized lens-type R-end under \(\mathcal{C}\).

Conversely, given a generalized lens-type R-end neighborhood of a p-R-end \(\bar{E}\) of \(\bar{O}\), there is a lens-cone containing \(\bar{O}\). The lens \(L\) of the cone is dual to a lens \(L'\) containing the ideal boundary of the dual p-end \(\bar{E}'\). By Theorem 3.13, the R-end satisfies the uniform middle eigenvalue condition. By Theorem 4.2, the lens \(L'\) can be chosen to be a strict lens. Then \(L'\) gives us the strict lens p-end neighbourhood in \(\bar{O}^*\) of \(\bar{E}'\). \(\Box\)

4.2. The end and the limit sets.

**Definition 4.5.** • Define the limit set \(L(\bar{E})\) of a p-R-end \(\bar{E}\) with a generalized p-end-neighborhood to be \(\partial D - \partial D\) for a generalized lens \(D\) of \(\bar{E}\) in \(S^n\) (resp. \(\mathbb{R}P^n\)).

• The limit set \(L(\bar{E})\) of a p-T-end \(\bar{E}\) of lens type to be \(\partial(S_\bar{E}) - S_\bar{E}\) for the ideal totally geodesic boundary component \(S_\bar{E}\) of \(\bar{E}\).

• The limit set of a horospherical end is the set of the end vertex.
Corollary 4.6. Let $O$ be a noncompact strongly tame $n$-orbifold with radial or totally geodesic ends and satisfy (IE) and the holonomy group is strongly irreducible. Let $U$ be a p-end-neighborhood of $\tilde{E}$ where $\tilde{E}$ is a lens-type p-T-end or a generalized lens-type or lens-type or horospherical p-R-end. Then $\text{Cl}(U) \cap \text{bd}\tilde{O}$ equals $\text{Cl}(S_{\tilde{E}})$ or $\text{Cl}(S(v_{\tilde{E}}))$ or $\{v_{\tilde{E}}\}$ depending on whether $\tilde{E}$ is a lens-type p-T-end or a generalized lens-type or lens-type or horospherical p-R-end, this set is independent of the choice of $U$ and so is the limit set $\Lambda(\tilde{E})$ of $\tilde{E}$.

Proof. Let $\tilde{E}$ be a generalized lens-type p-R-end. Then by Theorem 3.13, $\tilde{E}$ satisfies the uniform middle eigenvalue condition. Suppose that $\pi_1(\tilde{E})$ acts irreducibly. Let $K^{b}$ denote $\text{bd}\mathcal{T}_{v_{\tilde{E}}} \cap K$ for a distanced minimal compact convex set $K$ where $\Gamma_{\tilde{E}}$ acts on. Proposition 3.6 shows that the limit set is determined by a set $K^{b}$ in $\bigcup_{v_{\tilde{E}}}S(v_{\tilde{E}})$ since $S(v_{\tilde{E}})$ is an $h(\pi_1(\tilde{E}))$-invariant set. We deduce that $\text{Cl}(U) \cap \text{bd}\tilde{O} = \bigcup_{v_{\tilde{E}}}S(v_{\tilde{E}})$.

Also, $\Lambda(\tilde{E}) \supset K^{b}$ since $\Lambda(\tilde{E})$ is a $\pi_1(\tilde{E})$-invariant compact set in $\text{bd}\mathcal{T}_{v_{\tilde{E}}}$ $\bigcup_{v_{\tilde{E}}}$ for $x \in D$ for a generalized lens. Since $D$ is $\pi_1(\tilde{E})$-invariant compact set, $K^{b} \subset \Lambda(\tilde{E})$.

Suppose now that $\pi_1(\tilde{E})$ acts reducibly. Then by Theorem 2.15, $\tilde{E}$ is a totally geodesic p-R-end. Proposition 3.6 again implies the result.

Let $\tilde{E}$ be a p-T-end. By Theorem 4.2(i),

$\text{Cl}(A) - A \subset \text{Cl}(S_{\tilde{E}})$ for $A = \text{bd}L \cap \tilde{O}$

for a lens neighborhood $L$ by the strictness of the lens. Thus, $\text{Cl}(U) \cap \text{bd}\tilde{O}$ equals $\text{Cl}(S_{\tilde{E}})$.

For horospherical, we simply use the definition to obtain the result. □

Definition 4.7. An SPC-structure or a stable irreducible properly-convex real projective structure on an $n$-orbifold (with radial or totally geodesic end of lens-type) is a real projective structure so that the orbifold with stable and strongly irreducible holonomy. That is, it is projectively diffeomorphic to a quotient orbifold of a properly convex domain in $\mathbb{R}P^n$ by a discrete group of projective automorphisms that is stable and strongly irreducible.

Definition 4.8. Suppose that $O$ has an SPC-structure. Let $\tilde{U}$ be the inverse image in $\tilde{O}$ of the union $U$ of some choice of a collection of disjoint end neighborhoods of $O$. If every straight arc in the boundary of the domain $\tilde{O}$ and every non-$C^1$-point is contained in the closure of a component of $\tilde{U}$ for some choice of $U$, then $O$ is said to be strictly convex with respect to the collection of the ends. And $O$ is also said to have a strict SPC-structure with respect to the collection of ends.

Corollary 4.9. Suppose that $O$ is a noncompact strongly tame strictly SPC-orbifold with generalized admissible ends and satisfy (IE). Let $\tilde{O}$ is a properly
convex domain in $\mathbb{R}P^n$ (resp. in $S^n$) covering $\mathcal{O}$. Choose any disjoint collection of end neighborhoods in $\mathcal{O}$. Let $U$ denote their union. Let $p_\mathcal{O}: \tilde{\mathcal{O}} \to \mathcal{O}$ denote the universal cover. Then any segment or a non-$C^1$-point of $bd\tilde{\mathcal{O}}$ is contained in the closure of a component of $p_\mathcal{O}^{-1}(U)$ for any choice of $U$.

Proof. By the definition of a strict SPC-orbifold, any segment or a non-$C^1$-point has to be in the closure of a p-end neighborhood. Corollary 4.6 proves the claim. □

4.3. Expansion of admissible p-end-neighborhoods.

Lemma 4.10. Let $\mathcal{O}$ have a noncompact strongly tame properly convex real projective structure $\mu$ with admissible ends and satisfy (IE). Assume that the holonomy group is strongly irreducible. Let $U_1$ be a p-end neighborhood of a horospherical or a lens-type p-R-end $\tilde{E}$ with the p-end vertex $v$ in $\tilde{\mathcal{O}}$ that is foliated by segments from $v$; or $U_1$ is a lens-type p-end neighborhood of a p-T-end $\tilde{E}$. Let $\Gamma_{\tilde{E}}$ denote the p-end fundamental group corresponding to $\tilde{E}$. Then we can construct a sequence of lens-cone or lens p-end neighborhoods $U_i$, $i = 1, 2, \ldots$, where $U_i \subset U_j$ for $i < j$ where the following hold:

- Given a compact subset of $\tilde{\mathcal{O}}$, there exists an integer $i_0$ such that $U_i$ for $i > i_0$ contains it.
- The Hausdorff distance between $U_i$ and $\tilde{\mathcal{O}}$ can be made as small as possible, i.e.,
  \[ \forall \epsilon > 0, \exists J > 0, \text{ so that } d_H(U_i, \tilde{\mathcal{O}}) < \epsilon \text{ for } i > J. \]
- There exists a sequence of convex open p-end neighborhoods $U_i$ of $\tilde{E}$ in $\tilde{\mathcal{O}}$ so that $(U_i - U_j)/\Gamma_{\tilde{E}}$ for a fixed $j$ and $i > j$ is homeomorphic to a product of an open interval with the end orbifold.
- We can choose $U_i$ so that $bdU_i \cap \tilde{\mathcal{O}}$ is smoothly embedded and strictly convex with $Cl(bdU_i) - \tilde{\mathcal{O}} \subset \Lambda(\tilde{E})$.

Proof. First, we study the p-R-end case. The p-end-neighborhood $U_1$ is foliated by segments from $v$. The foliation leaves are geodesics concurrently ending at a vertex $v$ corresponding to the p-end of $U_1$. We take a union of finitely many geodesic leaves $L$ from $v_\tilde{E}$ of finite $d_{\tilde{\mathcal{O}}}$-length outside $U_1$, and take the convex hull of $U_1$ and $\Gamma_{\tilde{E}}(L)$ in $\tilde{\mathcal{O}}$.

Suppose that $\tilde{E}$ is a lens-type R-end first. Let $U_1$ be a lens-cone. Take a union of finitely many geodesic leaves $L$ from $v_\tilde{E}$ in $\tilde{\mathcal{O}}$ of $d_{\tilde{\mathcal{O}}}$-length $t$ outside the lens-cone $U_1$ and $\Gamma_{\tilde{E}}(L)$ in $\tilde{\mathcal{O}}$. Denote the result by $\Omega_t$. Thus, the endpoints of $L$ not equal to $v_\tilde{E}$ are in $\tilde{\mathcal{O}}$.

We claim that

- $bd\Omega_t \cap \tilde{\mathcal{O}}$ is a connected $(n - 1)$-cell,
- $bd\Omega_t \cap \tilde{\mathcal{O}}/\Gamma_{\tilde{E}}$ is a compact $(n - 1)$-orbifold homeomorphic to $\Sigma_{\tilde{E}}$, and
- $bdU_1 \cap \tilde{\mathcal{O}}$ bounds a compact orbifold homeomorphic to the product of a closed interval with $(bd\Omega_t \cap \tilde{\mathcal{O}})/\Gamma_{\tilde{E}}$.
First, each leaf of \( g(l), g \in \Gamma_E \) for \( l \) in \( L \) is so that any converging subsequence of \( \{g_i(l)\}, g_i \in \Gamma_E \), converges to a segment in \( S(v) \) for an infinite collection of \( g_i \). This follows since a limit is a segment in \( \partial \hat{O} \) with an endpoint \( v \) and must belong to \( S(v) \) by Proposition 5.2 of [24].

Let \( S_1 \) be the set of segments with endpoints in \( \Gamma_E(L) \cup \bigcup S(v) \). We define inductively \( S_i \) to be the set of simplices with sides in \( S_{i-1} \). Then the convex hull of \( \Gamma_E(L) \) in \( \text{Cl}(\hat{O}) \) is a union of \( S_1 \cup \cdots \cup S_n \).

We claim that for each maximal segment \( s \) in \( \text{Cl}(\hat{O}) \) from \( v \) not in \( S(v) \), \( s^o \) meets \( \partial \Omega_t \cap \hat{O} \) at a unique point: Suppose not. Then let \( v' \) be its other endpoint of \( s \) in \( \partial \hat{O} \) with \( s^o \cap \partial \Omega_t \cap \hat{O} = \emptyset \). Thus, \( v' \) in \( \partial \hat{O} \).

Now, \( v' \) is contained in the interior of a simplex \( \sigma \) in \( S_i \) for some \( i \). Since \( \sigma^o \cap \partial \hat{O} \neq \emptyset \), \( \sigma \subset \partial \hat{O} \) by Lemma 3.8. Since the endpoints \( \Gamma_E(L) \) are in \( \partial \hat{O} \), the only possibility is that the vertices of \( \sigma \) are in \( \bigcup S(v) \). Since \( U_i \) is convex and contains \( \bigcup S(v) \) in its boundary, \( \sigma \) is in the lens-cone \( \text{Cl}(U_i) \).

We claim that for each maximal segment \( s \) from \( v \) meets the boundary \( \partial \Omega_t \cap \hat{O} \) exactly once.

As in Lemma 3.11, \( \partial \Omega_t \cap \hat{O} \) contains no line segment ending in \( \partial \hat{O} \). The strictness of convexity of \( \partial \Omega_t \cap \hat{O} \) follows as by smoothing as in the proof of Proposition 3.9. By taking sufficiently many leaves for \( L \) with \( d_{\hat{O}} \)-lengths \( t \) sufficiently large, we can show that any compact subset is inside \( \Omega_t \). From this, the final item follows. The first three items now follow if \( \tilde{E} \) is an R-end.

Suppose now that \( \tilde{E} \) is horospherical and \( U_i \) is a horospherical \( p \)-end neighborhood. We can smooth the boundary to be strictly convex. Call the set \( \Omega_t \) where \( t \) is a parameter \( \to \infty \) measuring the distance from \( U_i \). \( \Gamma_E \) is in a parabolic subgroup of a conjugate of \( \text{SO}(n, 1) \) by Theorem 5.3 of [24]. By taking \( L \) sufficiently densely, we can choose similarly to above a sequence \( \Omega_i \) of strictly convex horospherical open sets at \( v \) so that eventually any compact subset of \( \hat{O} \) is in it for sufficiently large \( i \).

Suppose now that \( \tilde{E} \) is totally geodesic. Now we use the dual domain \( \hat{O}^* \) and the group \( \Gamma^*_E \). Let \( v^*_E \) denote the vertex dual to \( S^*_E \). By the homeomorphism induced by great segments with endpoints \( v^*_E \), we obtain

\[
(\partial \hat{O}^* - \bigcup S(v^*_E)) / \Gamma^*_E \cong \Sigma_E / \Gamma^*_E,
\]

a compact orbifold. Then we obtain \( U_i \) containing \( \hat{O}^* \) in \( \mathcal{T}_E \) by taking finitely many hypersphere \( F_i \) disjoint from \( \hat{O}^* \) but meeting \( \mathcal{T}_E \). Let \( H_i \) be the open hemisphere containing \( \hat{O}^* \) bounded by \( F_i \). Then we form \( U_i := \bigcap_{g \in \Gamma_E} g(H_i) \). By taking more hyperspheres, we obtain a sequence

\[
U_1 \supset U_2 \supset \cdots \supset U_i \supset U_{i+1} \supset \cdots \supset \hat{O}^*
\]

so that \( \text{Cl}(U_{i+1}) \subset U_i \) and

\[
\bigcap_i \text{Cl}(U_i) = \text{Cl}(\hat{O}^*).
\]
That is for sufficiently large hyperplanes, we can make $U_i$ disjoint from any compact subset disjoint from $\text{Cl}(\tilde{O})$. Now taking the dual $U^*_i$ of $U_i$ and by equation 3 we obtain

$$U^*_1 \subset U^*_2 \subset \cdots \subset U^*_i \subset U^*_i+1 \subset \cdots \subset \tilde{O}.$$ 

Then $U^*_i \subset \tilde{O}$ is an increasing sequence eventually containing all compact subset of $\tilde{O}$. This completes the proof for the first three items.

The fourth item follows from Corollary 4.6.

4.4. **Convex hulls of ends.** Here we will be working on $\mathbb{R}P^n$ exclusively from now on but since we are working in $\text{Cl}(\tilde{O})$, there are no differences in theory.

We will sharpen Corollary 4.6 and the convex hull part in Lemma 4.10.

One can associate a *convex hull* of a p-end $E$ of $\tilde{O}$ as follows:

- For horospherical p-ends, the convex hull of each is defined to be the set of the end vertex actually.
- The convex hull of a totally geodesic p-end $\tilde{E}$ of lens-type is the closure $\text{Cl}(S_{\tilde{E}})$ the totally geodesic ideal boundary component $S_{\tilde{E}}$ corresponding to $\tilde{E}$.
- For a generalised lens-type p-end $\tilde{E}$, the convex hull $I(\tilde{E})$ of $\tilde{E}$ is the convex hull of $\bigcup S(v_{\tilde{E}})$.

They equal $\text{Cl}(U) \cap \text{bd} \tilde{O}$ for any p-end neighborhood $U$ of $\tilde{E}$ by Corollary 4.6.

Corollary 4.6 and Proposition 4.11 imply that the convex hull of an end is a well-defined.

For a lens-shaped p-end $E$ with a p-end vertex $v$, the *convex hull* $I(E)$ is defined as

$$CH(\bigcup S(v)) \cap \tilde{O}.$$ 

We can also characterize it as the intersection

$$\bigcap_{U_i \in \mathcal{U}} CH(\text{Cl}(U_i)) \cap \tilde{O}$$

for the collection $\mathcal{U}$ of p-end neighborhoods $U_i$ of $v$ by (iv) and (v) of Proposition 4.11.

**Proposition 4.11.** Let $O$ be a strongly tame properly convex real projective orbifold with radial ends or totally geodesic ends of lens-type and satisfies (IE) and (NA). Assume that the holonomy group of $\pi_1(O)$ is strongly irreducible. Let $\tilde{E}$ be a radial lens-shaped p-end and $\tilde{E}$ an associated p-end vertex. Let $I(\tilde{E})$ be the convex hull of $\tilde{E}$.

(i) $\text{bd}I(\tilde{E}) \cap \tilde{O}$ is contained in the union of a lens part of a lens-shaped p-end neighborhood.

(ii) $I(\tilde{E})$ contains any concave p-end-neighborhood of $E$ and $I(\tilde{E}) \cap \tilde{O} = CH(\text{Cl}(U)) \cap \tilde{O}$
for a concave p-end neighborhood U of v. Thus, I(\tilde{E}) has a nonempty interior.

(iii) Each segment from v maximal in \tilde{O} meets the set bdI(\tilde{E}) \cap \tilde{O} at most once and bdI(\tilde{E}) \cap \tilde{O}/Γ_v is an orbifold isotopic to E for the end fundamental group Γ_v of v.

(iv) There exists a nonempty interior of the convex hull I(\tilde{E}) of \tilde{E} where Γ_v acts so that I(\tilde{E}) \cap \tilde{O}/Γ_v is diffeomorphic to the end orbifold times an interval.

Proof. (i) We define S_1 as the set of 1-simplices with endpoints in segments in \bigcup S(v) and we inductively define S_i to be the set of i-simplices with faces in S_{i-1}. Then I(\tilde{E}) is a union \bigcup_{σ \in S_1U S_2U...U S_n} σ. Notice that bdI(\tilde{E}) is the union \bigcup_{σ : ε \in S_1U S_2U...U S_n, ε \subset bdI(\tilde{E})} σ since each point of bdI(\tilde{E}) is contained in the interior of a simplex which lies in bdI(\tilde{E}) by the convexity of I(\tilde{E}).

If σ ∈ S_1 with σ ⊂ bdI(\tilde{E}), then its endpoint must be in an endpoint of a segment in \bigcup S(v); otherwise, σ^o is in the interior of I(\tilde{E}). If an interior point of σ is in a segment in S(v), then the vertices of σ are in \bigcup S(v) by the convexity of Cl(Rv(\tilde{O})). Hence, if σ^o ⊂ bdI(\tilde{E}) ∩ \tilde{O} meets \tilde{O}, then σ^o is contained in the lens-shaped domain L as the vertices of σ is in bdL − ∂L by the convexity of L. Now by induction on S_i, i > 1, we can verify (i) since any simplex with boundary in the union of subsimplices in the lens-domain is in the lens-domain by convexity.

(ii) Since I(\tilde{E}) contains the segments in S(v) and is convex, and so does a concave p-end neighborhood U, we obtain bdU ⊂ I(\tilde{E}): Otherwise, let x be a point of bdU ∩ bdI(\tilde{E}) ∩ \tilde{O} where some neighborhood in bdU is not in I(\tilde{E}). Then since bdU is a union of a convex hyper surface bdU ∩ \tilde{O} and S(v), each supporting hyperspace at x of the convex set bdU ∩ \tilde{O} meets a segment in S(v) in its interior. This is a contradiction since x must be then in I(\tilde{E})^o. Thus, U ⊂ I(\tilde{E}). Thus, CH(Cl(U)) ⊂ I(\tilde{E}). Conversely, since Cl(U) ⊃ \bigcup S(v) by Theorems 2.13 and 2.15, we obtain that CH(Cl(U)) ⊂ I(\tilde{E}).

(iii) bdI(\tilde{E}) ∩ \tilde{O} is a subset of a lens part of a p-end neighborhood by (iii). Each point of it meets a maximal segment from v in the end but not in S(x) at exactly one point since a maximal segment must leave the lens cone eventually. Thus bdI(\tilde{E}) ∩ \tilde{O} is homeomorphic to an (n − 1)-cell and the result follows.

(iv) This follows from (iii) since we can use rays from x meeting bdI(\tilde{E}) ∩ \tilde{O} at unique points and use them as leaves of a fibration.

□

4.5. Shrinking of generalized admissible p-end-neighborhoods. We now discuss the “shrinking” of p-end-neighborhoods. These repeat some results.
Corollary 4.12. Suppose that $O$ is a strongly tame properly convex real projective orbifold with radial or totally geodesic ends and satisfy (IE) and let $\hat{O}$ be a properly convex domain in $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$) covering $O$. Assume that the holonomy group is strongly irreducible. Then the following statements hold:

(i) If $\tilde{E}$ is a horospherical $p$-R-end, every $p$-end-neighborhood of $\tilde{E}$ contains a horospherical $p$-end-neighborhood.

(ii) Suppose that $\tilde{E}$ is a generalized lens-shaped or lens-shaped $p$-R-end. Let $I(\tilde{E})$ be the convex hull of $\bigcup S(v_{\tilde{E}})$, and let $V$ be a $p$-end-neighborhood $V$ where $(bd V \cap \tilde{O})/\pi_1(\tilde{E})$ is a compact orbifold. If $V^o \supset I(\tilde{E}) \cap \tilde{O}$, $V$ contains a lens-cone $p$-end neighborhood of $\tilde{E}$, and a lens-cone contains $\tilde{O}$ properly.

(iii) If $\tilde{E}$ is a generalized lens-shaped $p$-R-end or satisfies the uniform middle eigenvalue condition, every $p$-end-neighborhood of $\tilde{E}$ contains a concave $p$-end-neighborhood.

(iv) Suppose that $\tilde{E}$ is a $p$-T-end of lens type or satisfies the uniform middle eigenvalue condition. Then every $p$-end-neighborhood contains a strictly lens-type $p$-end-neighborhood $L$ with strictly convex boundary in $\tilde{O}$.

Proof. Let us prove for $\mathbb{S}^n$.

(i) Let $v_{\tilde{E}}$ denote the $p$-R-end vertex corresponding to $\tilde{E}$. By Theorem 5.3, we obtain a conjugate $G$ of a parabolic subgroup of $SO(n,1)$ as the finite index subgroup of $h(\pi_1(\tilde{E}))$ acting on $U$, a $p$-end-neighborhood of $\tilde{E}$. We can choose a $G$-invariant ellipsoid of diameter $\leq \epsilon$ for any $\epsilon > 0$ in $U$ containing $v_{\tilde{E}}$.
(ii) This follows from Proposition 3.9 since the convex hull of $\bigcup S(v_E)$ has the right properties.

(iii) Suppose that we have a lens-cone $V$ that is a p-end-neighborhood equal to $L \ast v_E \cap \tilde{O}$ where $L$ is a generalized lens bounded away from $v_E$.

By taking smaller $U$ if necessary, we may assume that $U$ and $L$ are disjoint. Since $bdU/h(\pi_1(\tilde{E}))$ and $L/h(\pi_1(\tilde{E}))$ are compact, $\epsilon > 0$. Let

$$L' := \{ x \in V | d_V(x, L) \leq \epsilon \}.$$ 

Then we can show that $L'$ is a generalized lens since a lower component of $\partial L'$ is strictly convex by Lemma 1.8 of [35]. (Given $u, v \in N_\epsilon$, we find $w, t \in \Omega$ so that $d_V(u, w) < \epsilon, d_V(v, t) < \epsilon$.

Then $\overline{wv}$ is within $\epsilon$ of $\overline{wt} \subset \Omega$ in the $d_V$-sense.) Clearly, $h(\pi_1(\tilde{E}))$ acts on $L'$.

We choose sufficiently large $\epsilon'$ so that $bdU \cap \tilde{O} \subset L'$, and hence $V - L' \subset U$ form a concave p-end-neighborhood as above.

(iv) The existence of lens-type p-end neighborhood of $S_{\tilde{E}}$ follows from Theorem A.10. By Theorem 4.2, the lens p-end neighborhood is a strict one. Then we apply similar reasoning to the proof of Proposition 3.9. □

4.6. Totally geodesic ends and the ideal boundary. We discuss more on totally geodesic ends. For totally geodesic ends, by the lens condition, we only consider the ones that have lens neighborhoods in some ambient orbifolds, i.e., admissible ones. First, we discuss the extension to bounded orbifolds.

**Theorem 4.13.** Suppose that $O$ is a noncompact strongly tame properly convex real projective orbifold with generalized admissible ends and satisfy (IE). Assume that the holonomy group of $\pi_1(O)$ is strongly irreducible. Let $E$ be a lens-shaped totally geodesic end, and let $\Sigma_E$ be a totally geodesic hyper-surface that is the ideal boundary corresponding to $E$. Let $L$ be a lens-shaped end neighborhood of $\Sigma_E$ in an ambient real projective orbifold containing $O$. Then

- $L \cup O$ is a properly convex real projective orbifold and has a strictly convex boundary component corresponding to $E$.
- Furthermore if $O$ is strictly SPC and $\tilde{E}$ is a hyperbolic end, then so is $L \cup O$ which now has one more boundary component and one less totally geodesic ends.

**Proof.** It is sufficient to prove for $S^n$ cases here. Let $\tilde{O}$ be the universal cover of $O$ which we can identify with a properly convex bounded domain in an affine subspace. Then $\Sigma_E$ corresponds to a p-end $\tilde{E}$ and to a totally geodesic surface $S = S_{\tilde{E}}$. And $L$ is covered by a lens $\tilde{L}$ containing $S$. The p-end fundamental group $\pi_1(\tilde{E})$ acts on $\tilde{O}$ and $\tilde{L}_1$ and $\tilde{L}_2$ the two components of $\tilde{L} - S_{\tilde{E}}$ in $\tilde{O}$ and outside $\tilde{O}$ respectively.
Definition 4.14. Let \( \mathbb{R}^n \) denote the affine subspace in \( S^n \) with boundary \( S_{\infty}^{n-1} \). Suppose that \( \Omega \) is a properly convex open domain in \( S_{\infty}^{n-1} \). Let \( \Omega_1 \) be a properly convex open domain with \( \partial \Omega_1 \supset \partial \Omega \) in \( \mathbb{R}^n \). The supporting hyperplanes at \( p \in \Lambda = \partial \Omega \cap \Omega \) contains the unique hyperplane of codimension-two supporting \( \Omega \). Let

\[ A_p := \{ H \mid H \text{ is a supporting hyperspace of } \Omega_1 \text{ at } p \text{ in } \mathbb{R}^n \}. \]

An asymptotic supporting hyperplane \( h \) at a point \( p \) of \( \Lambda \) is a supporting hyperplane at \( p \) so that there exists no other element \( h' \) of \( A_p \) with \( \partial \Omega \cap S_{\infty}^{n-1} = \partial \Omega' \cap S_{\infty}^{n-1} \) closer to \( \Omega_1 \) from a point of \( \partial \Omega_1 - \partial \Omega \) (using minimal distance between a point and a set).

Lemma 4.15. Suppose that \( S_{\tilde{E}} \) is the totally geodesic ideal boundary of a lens-type totally geodesic end \( \tilde{E} \) of a strongly tame real projective orbifold \( O \) and \( \pi_1(\tilde{E}) \) is hyperbolic.

- Given a \( \pi_1(\tilde{E}) \)-invariant properly convex open domain \( \Omega_1 \) containing \( S_{\tilde{E}} \) in the boundary, at each point of \( \Lambda \), there exists a unique asymptotic supporting hyperplane.
- At each point of \( \Lambda \), the hyperspace supporting any \( \pi_1(\tilde{E}) \)-invariant properly convex open set \( \Omega \) containing \( S_{\tilde{E}} \) is unique.
- We are given two \( \pi_1(\tilde{E}) \)-invariant properly convex open domains \( \Omega_1 \) containing \( S_{\tilde{E}} \) in the boundary and \( \Omega_2 \) containing \( S \) in the boundary from the other side. Then \( \Omega_1 \cup \Omega_2 \) is a convex domain with

\[ \partial \Omega_1 \cap \partial \Omega_2 = \partial (S_{\tilde{E}}) \]

and their asymptotic supporting hyperplanes at each point of \( \Lambda \) coincide.

Proof. Let \( A \) denote the affine subspace that is the complement in \( S^n \) of the hyperspace containing \( S_{\tilde{E}} \). Because \( \pi_1(\tilde{E}) \) acts on a lens-type domain, the dual group of \( h(\pi_1(\tilde{E})) \) is the holonomy group of a lens-type radial p-end by Corollary 2.3. By Theorem 3.13, \( h(\pi_1(\tilde{E})) \) satisfies the uniform middle eigenvalue condition.

If \( \Omega_1 \) has an asymptotic supporting half-space \( H(x) \) for each \( x \in \Lambda \) containing \( \Omega_1 \). \( H(x) \) is uniquely determined by \( \pi_1(\tilde{E}) \) and \( x \) by Lemma A.9 and its proof.

The third item follows since the asymptotically supporting hyperplane at each point of \( \partial (S_{\tilde{E}}) - S_{\tilde{E}} \) to \( \Omega_1 \) and \( \Omega_2 \) have to agree by Lemma A.9(ii). The convexity follows easily from this. Also, the second item follows.

We continue with the proof of Theorem 4.13. Suppose that \( \pi_1(\tilde{E}) \) is hyperbolic. By Lemma 4.15, \( \tilde{L}_2 \cup S \cup \tilde{O} \) is a convex domain. If \( \tilde{L}_2 \cup \tilde{O} \) is not properly convex, then it is a union of two cones over \( S_{\tilde{E}} \) over of
the irreducibility of \( h(\pi_1(O)) \). Hence, it follows that \( \tilde{L}_2 \cup \tilde{O} \) is properly convex.

Suppose that \( O \) is strictly SPC and \( \pi_1(\tilde{E}) \) is hyperbolic. Then every segment in \( \text{bd}\tilde{O} \) or a non-C\(^1\)-point in \( \text{bd}\tilde{O} \) is in the closure of one of the p-end neighborhoods. \( \text{bd}\tilde{L}_2 \cap \text{Cl}(S_{\tilde{E}}) \) does not contain any segment in it or a non-C\(^1\)-point. \( \text{bd}\tilde{O}-\text{Cl}(S_{\tilde{E}}) \) does not contain any segment or a non-C\(^1\)-point outside the union of the closures of p-end neighborhoods. By Lemma 4.15, \( \text{bd}(\tilde{O} \cup \tilde{L}_2 \cup S_{\tilde{E}}) \) is C\(^1\) at each point of \( \Lambda(\tilde{E}) := \text{Cl}(S_{\tilde{E}}) - S_{\tilde{E}} \) by the uniqueness of the supporting hyperplanes by Lemma 4.15.

Recall that \( S_{\tilde{E}} \) is strictly convex for \( \pi_1(\tilde{E}) \) is a hyperbolic group. (See Theorem 1.1 of [4].) Thus, \( \Lambda \) does not contain a segment, and hence, \( \text{bd}(\tilde{O} \cup \tilde{L}_2 \cup S_{\tilde{E}}) \) does not contain one. Therefore, \( \tilde{L}_2 \cup \tilde{O} \) is strictly convex relative to the ends.

Suppose now that \( \pi_1(\tilde{E}) \) is a product of hyperbolic and abelian groups. Then the dual of the totally geodesic p-end is a radial p-end by Proposition 2.8. The dual radial p-end has a p-end neighborhood that is contained in a strict join with a vertex \( x \) with a properly convex open domain \( K \) in a hyperplane \( V \). \( \text{Cl}(K) \) is a strict join \( C_1 \cdots C_k \) for properly compact convex domains \( C_i \), for \( i = 1, \ldots, k \) by Theorem 2.15.

Recall that \( \tilde{O} \) contains an open one-sided properly convex p-end neighborhood \( D \) of \( S_{\tilde{E}} \). By equation (3) of [24], the dual \( D^* \) of \( D \) contains the dual \( \tilde{O}^* \) of \( \tilde{O} \). Let \( x \) be a dual point to the hyperplane containing ideal boundary component \( S_{\tilde{E}} \). \( D^* \) is the interior of a lens-cone with end vertex \( x \) by Corollary 2.3. By Theorem 2.15, \( D^* \) is a totally geodesic lens-cone with end vertex \( x \). \( D^* \) is contained in the union \( U \) of two strict joins \( x \ast K \cup x_- \ast K \). Thus, \( \tilde{O}^* \subset x \ast K \cup x_- \ast K \). However, \( D^* \) contains \( x \ast K \).

The set of supporting hyperspaces at the vertex \( x \) is projectively isomorphic to the dual \( K \) of \( \text{Cl}(S_{\tilde{E}}) \) by Proposition 2.8. Let \( V \) be the hyperspace containing \( K \). Since \( D^* \) contains \( x \ast K \), \( D \) is contained in \( (x \ast K)^* = a \ast \text{clo}(S_{\tilde{E}}) \) for the point \( a \) dual to the hyperplane \( V \) by equation 4 of [24]. Therefore, the dual \( \tilde{O} \) of \( \tilde{O}^* \) is contained in the the cone \( \text{Cl}(S_{\tilde{E}}) \ast a \) for some point \( a \) dual to the hyperplane \( V \).

Now, \( \tilde{L}_2 \) is a subset of \( \text{Cl}(S_{\tilde{E}}) \ast a \) sharing boundary \( \text{Cl}(S_{\tilde{E}}) \) with \( \tilde{O} \) since we can treat \( \tilde{L}_2 \) as \( \tilde{O} \) in the above arguments. Since both share \( S_{\tilde{E}} \) and are in \( S_{\tilde{E}} \ast a \cup S_{\tilde{E}} \ast a_- \), the convexity of the union \( \tilde{L}_2 \cup \tilde{O} \) follows. The proper convexity follows also as above.

Since \( \tilde{L}_2 \cup \tilde{O} \) has a Hilbert metric, the action is properly discontinuous. \( \square \)

5. The Strong Irreducibility of the Real Projective Orbifolds.

The main purpose of this section is to prove Theorem 1.11, the irreducibility result. In particular, we don’t assume the holonomy group of \( \pi_1(O) \) is...
strongly irreducible for results from now on. But we will discuss the convex hull of the ends first.

**Definition 5.1.** Let \( \tilde{E} \) be a lens-type radial end. Let \( L \) be the lens-cone p-end neighborhood of \( \tilde{E} \). Let \( CH(\Lambda(\tilde{E})) \) denote the convex hull of \( \Lambda(\tilde{E}) \). Let \( U' \) be any p-end neighborhood \( U' \) of \( \tilde{E} \) containing \( CH(\Lambda(\tilde{E})) \cap \tilde{O} \). We define a maximal concave p-end neighborhood or mc-p-end-neighborhood \( U \) to be the component of \( U' - CH(\Lambda(\tilde{E})) \) containing a p-end neighborhood of \( \tilde{E} \). The closed maximal concave p-end neighborhood is \( Cl(U) \cap \tilde{O} \). An \( \epsilon-d \)-neighborhood \( U'' \) of a maximal concave p-end neighborhood is called an \( \epsilon-mc-p-end-neighborhood \).

In fact, these are independent of choices of \( U' \).

**Lemma 5.2.** Let \( D \) be an \( i \)-dimensional totally geodesic compact convex domain, \( i \geq 1 \). Let \( \tilde{E} \) be a generalized lens-type p-end with the p-end vertex \( v_{\tilde{E}} \). Suppose \( \partial D \subset \bigcup S(v_{\tilde{E}}) \). Then \( D \subset Cl(V) \) for a maximal concave p-end neighborhood \( V \), and for sufficiently small \( \epsilon > 0 \), an \( \epsilon-d \)-neighborhood of \( D_0 \subset V' \) for any \( \epsilon-mc-p-end \)-neighborhood \( V' \).

**Proof.** Assume that \( U \) is a generalized lens-cone of \( v_{\tilde{E}} \). Then \( \Lambda \) is the set of endpoints of segments in \( S_{v_{\tilde{E}}} \) with \( v_{\tilde{E}} \) removed. Let \( P \) be the spanning subspace of \( D \) and \( v_{\tilde{E}} \). Since \( \partial D, \Lambda \cap P \subset \bigcup S(v_{\tilde{E}}) \cap P \), and \( \partial D \cap P \) is closer than \( \Lambda \cap P \) from \( v_{\tilde{E}} \), it follows that \( P \cap Cl(U) = D \) has a component \( C_1 \) containing \( v_{\tilde{E}} \) and a component \( C_2 \) contains \( \Lambda \cap P \). Hence \( Cl(C_2) \supset CH(\Lambda) \cap P \) by the convexity of \( Cl(C_2) \). This implies that

- \( D \) is disjoint from \( CH(\Lambda)_0 \) or
- \( D \) contains \( CH(\Lambda) \cap P \).

Let \( V \) be an mc-p-end neighborhood of \( U \). Since \( Cl(V) \) contains the closure of the component of \( U - CH(\Lambda) \) whose closure contains \( v_{\tilde{E}} \), it follows that \( Cl(V) \) contains \( D \).

Since \( D \) is in the mc-p-end neighborhood \( V \), the boundary \( bd Cl(V') \cap \tilde{O} \) of the \( \epsilon-mc-p-end \)-neighborhood \( V' \) do not meet \( D \). Hence \( D_0 \subset V' \).

**Corollary 5.3.** Let \( O \) be a properly convex real projective manifold with generalized admissible ends and satisfies (IE) and (NA). Let \( \tilde{E} \) be a generalized lens-type radial end. Then

(i) Also, a concave p-end neighborhood of \( \tilde{E} \) is always a subset of an mc-p-end-neighborhood of the same p-end.

(ii) The mc-p-end-neighborhood of \( \tilde{E} \) is a union of all concave end neighborhoods of \( \tilde{E} \).

(iii) The mc-p-end-neighborhood of \( \tilde{E} \) is a proper p-end neighborhood, and covers an end-neighborhood with compact boundary in \( O \).

(iv) An \( \epsilon-mc-p-end \)-neighborhood of \( \tilde{E} \) for sufficiently small \( \epsilon > 0 \) is a proper p-end neighborhood.

(v) We can choose \( \epsilon-mc-p-end \)-neighborhoods of p-ends so that their image end-neighborhoods in \( O \) are mutually disjoint.
Proof. (i) Since the limit set $\Lambda(\tilde{E})$ is in any generalized lens by Corollary 4.6, a generalized lens-cone p-end neighborhood $U$ of $\tilde{E}$ contains $CH(\Lambda) \cap \tilde{O}$. Hence, a concave end neighborhood is contained in an mc-p-end-neighborhood.

(ii) Let $V$ be an mc-p-end neighborhood of $\tilde{E}$. Then define $S$ to be the set of endpoints in $Cl(\tilde{O})$ of maximal segments in $V$ from $v_{E}$ in directions of $S_{E}$. Thus, $S/\pi_{1}(\tilde{E})$ is a compact set since $S$ is contractible and $S_{E}/\pi_{1}(\tilde{E})$ is a $K(\pi_{1}(\tilde{E}))$-space.

We can $d_{S}$-approximate $S$ by the smooth boundary component $S$, outwards of a generalized lens. A component $U - S$ is a concave p-end neighborhood.

(iii) Since a concave p-end neighborhood is a proper p-end neighborhood by Corollary 2.16, we obtain $g(V) \cap V = \emptyset$ or $g(V) = V$ for $g \in \pi_{1}(O)$ by (ii).

Suppose that $g(Cl(V) \cap \tilde{O}) \cap Cl(V) \neq \emptyset$. Then $g(V) = V$ and $g \in \pi_{1}(\tilde{E})$; Otherwise, $g(V) \cap V = \emptyset$, and $g(Cl(V) \cap \tilde{O})$ meets $Cl(V)$ in a totally geodesic hypersurface $S$ equal to $CH(\Lambda)^{o}$ by the concavity of $V$. Hence for every $g \in \pi_{1}(O)$, $g(S) = S$, and $g(V) \cup S \cup V = \tilde{O}$ since these are subsets of a properly convex domain $\tilde{O}$. Then $\pi_{1}(O)$ acts on $S$ and $S/G$ is homotopy equivalent to $\tilde{O}/G$ for a finite index torsion free subgroup $G$ of $\pi_{1}(O)$ by Selberg’s lemma. This contradicts the condition (IE). Hence, we conclude that $g(V \cup S) \cap V \cup S = \emptyset$ or $g(V \cup S) = V \cup S$ for $g \in \pi_{1}(O)$.

Now suppose that $S \cap bd\tilde{O} \neq \emptyset$. Let $S'$ be a maximal totally geodesic domain in $Cl(V)$ supporting $S$. Then $S' \subset bd\tilde{O}$ by convexity and Lemma 3.8, meaning that $S' = S \subset bd\tilde{O}$. In this case, $\tilde{O}$ is a cone over $S$ and the end vertex $v_{E}$ of $\tilde{E}$. For each $g \in \pi_{1}(O)$, $g(V) \cap V \neq \emptyset$ meaning $g(V) = V$ since $g(v_{E})$ is on $Cl(S)$. Thus, $\pi_{1}(O) = \pi_{1}(\tilde{E})$. This contradicts the infinite index condition of $\pi_{1}(\tilde{E})$.

We showed that $Cl(V) \cap \tilde{O} = V \cup S$. Thus, an mc-p-end-neighborhood $Cl(V) \cap \tilde{O}$ is a proper end neighborhood of $\tilde{E}$ with compact imbedded boundary $S/\pi_{1}(\tilde{E})$. Therefore we can choose positive $\epsilon$ so that an $\epsilon$-mc-p-end-neighborhood is a proper p-end neighborhood also. This proves (iv).

(v) For two mc-p-end neighborhoods $U$ and $V$ for different p-ends, we have $U \cap V = \emptyset$ by (iii).

We showed that $Cl(V) \cap \tilde{O}$ for an mc-p-end-neighborhood $V$ covers an end neighborhood in $O$. Suppose that $U$ is another mc-p-end neighborhood different from $V$ and $Cl(U) \cap Cl(V) \cap \tilde{O} \neq \emptyset$. Since $U \cap V = \emptyset$, we have $Cl(U) \cap Cl(V)$ are in the boundary of $U$ and $V$ in a properly convex domain $\tilde{O}$, and $bdU \cap \tilde{O}$ and $bdV \cap \tilde{O}$ equal a tangent maximal hyperspace $CH(\Lambda)^{o}$ in $\tilde{O}$ and hence they are equal. As above, this is a contradiction. Hence $Cl(U) \cap Cl(V) \cap \tilde{O} = \emptyset$.

Since the closures of mc-p-end neighborhoods with different p-ends are disjoint, and these have compact boundary components, the final item follows. □
For the following, we need a stronger condition of lens-type ends to obtain the disjointness of the closures of p-end neighborhoods.

**Corollary 5.4.** Let $\mathcal{O}$ be a properly convex real projective manifold with generalized admissible ends and satisfies (IE) and (NA). Let $\mathcal{U}$ be the collection of the components of the inverse image in $\tilde{\mathcal{O}}$ of the union of disjoint collection of end neighborhoods of $\mathcal{O}$. Now replace each of the p-end neighborhoods of radial lens-type of collection $\mathcal{U}$ by a concave p-end neighborhood by Corollary 4.12 (iii). Then the following statements hold:

(i) Given horospherical, concave, or one-sided lens p-end-neighborhoods $U_1$ and $U_2$ contained in $\bigcup \mathcal{U}$, we have $U_1 \cap U_2 = \emptyset$ or $U_1 = U_2$.

(ii) Let $U_1$ and $U_2$ be in $\mathcal{U}$. Then $\text{Cl}(U_1) \cap \text{Cl}(U_2) \cap \text{bd} \tilde{\mathcal{O}} = \emptyset$ or $U_1 = U_2$.

**Proof.** (i) Suppose that $U_1$ and $U_2$ are p-end neighborhoods of radial ends. Let $U_1'$ be the interior of the associated generalized lens-cone of $U_1$ in $\text{Cl}(\tilde{\mathcal{O}})$ and $U_2'$ be that of $U_2$. Let $U_i''$ be the concave p-end-neighborhood of $U_i'$ for $i = 1, 2$ that covers an end neighborhood in $\mathcal{O}$ by Corollary 4.12 (iii). Since the neighborhoods in $\mathcal{U}$ are mutually disjoint,

- $\text{Cl}(U_1'') \cap \text{Cl}(U_2'') \cap \tilde{\mathcal{O}} = \emptyset$ or
- $U_1'' = U_2''$

since we can choose these to cover disjoint or identical p-end neighborhoods in $\mathcal{O}$.

(ii) Assume that $U_1'' \in \mathcal{U}$, $i = 1, 2$, and $U_1'' \neq U_2''$. Suppose that the closures of $U_1''$ and $U_2''$ intersect in $\text{bd} \tilde{\mathcal{O}}$. Suppose that they are both radial p-end neighborhoods. Then the respective convex hulls $h_1$ and $h_2$ as obtained by Proposition 4.11 intersect as well. Take a point $z \in \text{Cl}(U_1'') \cap \text{Cl}(U_2'') \cap \text{bd} \tilde{\mathcal{O}}$. Let $p_1$ and $p_2$ be the respective p-end vertices of $U_1'$ and $U_2'$. Then $\overline{p_1 z} \in S(p_1)$ and $\overline{p_2 z} \in S(p_2)$ and these segments are maximal since otherwise $U_1'' \cap U_2'' \neq \emptyset$. The segments intersect transversally at $z$ since otherwise we violated the maximality in Theorems 2.13 and 2.15. We obtain a triangle $\Delta(p_1p_2z)$ in $\text{Cl}(\tilde{\mathcal{O}})$ with vertices $p_1, p_2, z$. We assume that $\overline{p_1 p_2} \subseteq \tilde{\mathcal{O}}$.

If this is not true, we need to perturb $p_1$ and $p_2$ by a small amount. We may not have the geodesic triangle but will have a totally geodesic disk bounded by three arcs. However, the disk has an angle $\leq \pi$ at $z$ since $\text{Cl}(\tilde{\mathcal{O}})$ is properly convex. We will denote the disk by $\Delta(p_1p_2z)$ still.

We define a convex curve $\alpha_i := \Delta(p_1p_2z) \cap \text{bd} \tilde{l}_i$ with an endpoint $z$ for each $i, i = 1, 2$. Let $\tilde{E}_i$ denote the p-end corresponding to $p_i$. Since $\alpha_i$ maps to a geodesic in $R_{p_i}(\tilde{\mathcal{O}})$, there exists a foliation $\mathcal{T}$ of $\Delta(p_1p_2z)$ by maximal segments from the vertex $p_1$. There is a natural parametrization of the space of leaves by $\mathbb{R}$ as the space is projectively equivalent to an open interval using the Hilbert metric of the interval. We parameterize $\alpha_i$ by these parameters as $\alpha_i$ intersected with a leaf is a unique point. They give the geodesic length parameterizations under the Hilbert metric of $R_{p_i}(\tilde{\mathcal{O}})$ for $i = 1, 2$. 


We now show that an infinite-order element of $\pi_1(\tilde{E}_1)$ is the same as one in $\pi_1(\tilde{E}_2)$: By convexity, either $\alpha_2$ goes into $l_1$ and not leave again or $\alpha_2$ is disjoint from $l_1$. Suppose that $\alpha_2$ goes into $l_1$ and not leave it again. Since $l_2/\pi_1(\tilde{E}_2)$ is compact, there is a sequence $t_i$ so that the image of $\alpha_2(t_i)$ converges to a point of $l_1/\pi_1(\tilde{E}_1)$. Hence, by taking a short path between $\alpha_2(t_i)$s, there exists an essential closed curve $\gamma_2$ in $l_2/\pi_1(\tilde{E}_2)$ homotopic to an element of $\pi_1(\tilde{E}_1)$. In fact $\gamma_2$ is in a lens-cone end neighborhood of the end corresponding to $\tilde{E}_1$. This contradicts (NA). (The element is of infinite order since we can take a finite cover of $O$ so that $\pi_1(O)$ is torsion-free by Selbert’s lemma.)

Suppose now that $\alpha_2$ is disjoint from $l_1$. Then $\alpha_1$ and $\alpha_2$ have the same endpoint $z$ and by the convexity of $\alpha_2$. We parameterize $\alpha_i$ so that $\alpha_1(t)$ and $\alpha_2(t)$ is on a line segment in the triangle with endpoints in $p_1$, $p_2$ and $z$.

We obtain $d_O(\alpha_2(t), \alpha_1(t)) \leq C$ for a uniform constant $C$: We define $\beta(t) := \alpha_2(t)\alpha_1(t)$. Let $\gamma(t)$ denote the full extension of $\beta(t)$ in $\Delta(p_1, p_2 z)$. One can project the space of lines through $z$, a one-dimensional projective space. The image of $\beta(t)$ are so that the image of $\beta(t')$ is contained in that of $\beta(t)$ if $t < t'$. Also, the image of $\gamma(t)$ contains that of $\gamma(t')$ if $t < t'$. Thus, we can show by computation that the Hilbert-metric length of the segment $\beta(t)$ is bounded above by the uniform constant.

We have a sequence $t_i \to \infty$ so that $p_O \circ \alpha_2(t_i) \to x$, $d_O(p_O \circ \alpha_2(t_{i+1}), p_O \circ \alpha_2(t_i)) \to 0$, $x \in O$.

So we obtain a closed curve $\gamma_{2,i}$ in $O$ obtained by taking a short path jumping between the two points. By taking a subsequence, the image of $\beta(t_i)$ in $O$ geometrically converges to a segment of Hilbert-length $\leq C$. As $i \to \infty$, we have $d_O(p_O \circ \alpha_1(t_i), p_O \circ \alpha_1(t_{i+1})) \to 0$ by extracting a subsequence. There

Figure 3. The diagram of the quadrilateral bounded by $\beta(t_i), \beta(t_{i+1}), \alpha_1, \alpha_2$. 

\[ p_1 \quad \beta(t_i) \quad \alpha_1 \quad \beta(t_{i+1}) \quad \alpha_2 \quad p_2 \]
exists a closed curve $c_{1,i}$ in $\mathcal{O}$ again by taking a short jumping path. We see that $c_{1,i}$ and $c_{2,i}$ are homotopic in $\mathcal{O}$ since we can use the image of the disk in the quadrilateral bounded by $\alpha_2(t_i)\alpha_2(t_{i+1})$, $\alpha_1(t_i)\alpha_1(t_{i+1})$, $\beta(t_i)$, $\beta(t_{i+1})$ and the connecting thin strips between the images of $\beta_t$ and $\beta_{t+1}$ in $\mathcal{O}$. This again contradicts (NA).

(ii) Now, consider when $U_1$ is a one-sided lens-neighborhood of a totally geodesic p-end and let $U_2$ be a concave p-end neighborhood of a radial p-end of $\bar{\mathcal{O}}$. Let $z$ be the intersection point in $\text{Cl}(U_1) \cap \text{Cl}(U_2)$. We can use the same reasoning as above by choosing any $p_1$ in $S_{E_1}$ so that $\overline{p_1z}$ passes the interior of $\tilde{E}_1$. Let $p_2$ be the p-end vertex of $U_2$. Now we obtain the triangle with vertices $p_1$, $p_2$, and $z$ as above. Then the arguments are analogous and obtain infinite order elements in $\pi_1(\tilde{E}_1) \cap \pi_1(\tilde{E}_2)$.

(iv) Finally, consider when $U_1$ and $U_2$ are one-sided lens-neighborhoods of totally geodesic p-ends respectively. Using the intersection point $z$ of $\text{Cl}(U_1) \cap \text{Cl}(U_2) \cap \bar{\mathcal{O}}$ and we choose $p_i$ in $\text{bd} \tilde{E}_i$ so that $\overline{zp_i}$ passes the interior of $S_{E_i}$ for $i = 1, 2$. Again, we obtain a triangle with vertex $p_1$, $p_2$, and $z$, and find a contradiction as above.

(v) We now consider horospherical p-ends. Since $\text{Cl}(U) \cap \text{bd} \bar{\mathcal{O}}$ is a unique point, (iii) of Proposition 5.2 of [24] implies the result.

\[\square\]

5.1. The strong irreducibility and stability of the holonomy group of properly convex strongly tame orbifolds. First, we modify Theorem 2.15 by replacing some conditions. In particular, we don’t assume $h(\pi_1(\mathcal{O}))$ is strongly irreducible.

Lemma 5.5. Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with generalized admissible ends and satisfies (IE) and (NA). Let $\tilde{E}$ be a virtually factorable p-end of $\bar{\mathcal{O}}$ of generalized lens-type. Then

- there exists a totally geodesic hyperspace $P$ on which $h(\pi_1(\tilde{E}))$ acts,
- $D := P \cap \bar{\mathcal{O}}$ is a properly convex domain,
- $D^o \subset \bar{\mathcal{O}}$, and
- $D^o/\pi_1(\tilde{E})$ is a compact orbifold.
- Also, each element of $g \in \pi_1(\tilde{E})$ acts as nonidentity on a subspace properly containing $v$.

Proof. The proof of Theorem 2.15 shows that

- either $\text{Cl}(\bar{\mathcal{O}})$ is a strict join or
- the conclusion of Theorem 2.15 holds.

In both cases, $\pi_1(\tilde{E})$ acts on a totally geodesic convex compact domain $D$ of codimension 1. $D$ is the intersection $P_{\tilde{E}} \cap \text{Cl}(\bar{\mathcal{O}})$ for a $\pi_1(\tilde{E})$-invariant subspace $P_{\tilde{E}}$. Suppose that $D^o$ is not a subset of $\bar{\mathcal{O}}$. Then by Lemma 3.8, $D \subset \text{bd} \text{Cl}(\bar{\mathcal{O}})$. In the former case, we can show that $\text{Cl}(\bar{\mathcal{O}})$ is the join $v_{\tilde{E}} \ast D$.

For each $g \in \pi_1(\tilde{E})$ satisfying $g(v_{\tilde{E}}) \neq v_{\tilde{E}}$, we have $g(D) \neq D$ since $g(v_{\tilde{E}}) \ast g(D) = v_{\tilde{E}} \ast D$. $g(D) \cap D$ is a proper compact convex subset of $D$
and \( g(D) \). Moreover, \( \text{Cl}(\tilde{O}) = \nu_E * g(\nu_E) * (D \cap g(D)) \). We can continue as many times as there is a mutually distinct collection of vertices of form \( g \) and \( h \),\n
### Theorem 1.11

The proof of Theorem 1.11. We need to prove for \( \text{PGL}(n + 1, \mathbb{R}) \) only for strong irreducibility. Let \( h : \pi_1(O) \to \text{PGL}(n + 1, \mathbb{R}) \) be the holonomy homomorphism. Suppose that \( h(\pi_1(O)) \) is virtually reducible. Then we can choose a finite cover \( O_1 \) so that \( h(\pi_1(O_1)) \) is reducible.

We denote \( O_1 \) by \( O \) for simplicity. Let \( S \) denote a proper subspace where \( \pi_1(O) \) acts on. Suppose that \( S \) meets \( \tilde{O} \). Then \( \pi_1(\tilde{E}) \) acts on a properly convex open domain \( S \cap \tilde{O} \) for each p-end \( \tilde{E} \). Thus, \( (S \cap \tilde{O})/\pi_1(\tilde{E}) \) is a compact orbifold homotopy equivalent to one of the end orbifolds. However, \( S \cap \tilde{O} \) is \( \pi_1(\tilde{E}) \)-invariant and cocompact for each p-end \( \tilde{E} \). Each p-end fundamental group \( \pi_1(\tilde{E}) \) is virtually identical to any other p-end fundamental group. This contradicts (IE). Therefore,

\[
K := S \cap \text{Cl}(\tilde{O}) \subset \text{bd} \tilde{O}.
\]

(A) We show that \( K := \text{Cl}(\tilde{O}) \cap S \neq \emptyset \): Let \( \tilde{E} \) be a p-end. If \( \tilde{E} \) is horospherical, \( \pi(\tilde{E}) \) acts on a great sphere \( \tilde{S} \) tangent to an end vertex. \( S \) has to be a subspace in \( \tilde{S} \) containing the end vertex by Proposition 5.1(iii) of [24].

Suppose that \( \tilde{E} \) is a radial end of generalized lens-type. Then by the existence of attracting subspaces of some elements of \( \Gamma_{\tilde{E}} \), we have

- either \( S \) passes the end vertex \( \nu_E \) or
- there exists a subspace \( S' \) containing \( S \) and \( \nu_E \) that is \( \pi_1(\tilde{E}) \)-invariant.

We need to consider the second case only. Hence \( S' \) corresponds to a proper-invariant subspace in \( S'_E \) or \( S \) is a hyperspace of dimension \( n - 1 \) disjoint from \( \nu_E \). By considering a hyperbolic factor, we obtain some attracting fixed points in the limit sets of \( \pi_1(\tilde{E}) \) in both cases. Considering when \( \pi_1(\tilde{E}) \) has nontrivial diagonalizable elements, we obtain \( S \cap \text{Cl}(L) \neq \emptyset \) for a lens or generalized lens \( L, L \subset \tilde{O} \). The existence of the attracting fixed points of some elements of \( \pi_1(\tilde{E}) \) implies that \( S \cap \text{Cl}(L) \neq \emptyset \) for a lens \( L, L \subset \tilde{O} \). Hence, \( S \cap \text{Cl}(\tilde{O}) \neq \emptyset \). (This follows from Lemma 5.5 and Proposition 1.1 of [8] and the uniform middle eigenvalue condition.)

If \( \tilde{E} \) is totally geodesic of lens-type, we can apply a similar argument using the attracting fixed points. Therefore, \( S \cap \text{Cl}(\tilde{O}) \) is a subset \( K \) of \( \text{bd} \tilde{O} \) of \( \dim K \geq 0 \) and is not empty. In fact, we showed that the closure of each p-end neighborhood meets \( K \).

By taking dual orbifold if necessary, we assume without loss of generality that there exists a radial end of generalized lens-type with a radial p-end vertex \( \nu \).
(B) Suppose that a horospherical p-end vertex $v$ is in $K$. By Condition (IE), each $g(v)$ for $g \in \pi_1(O)$ is in $K$ also. Hence, $\dim K \geq 1$. Now, $v$ cannot be a horospherical p-end vertex by Proposition 5.2(iii) of [24].

(C) From now on, we assume that there is no horospherical p-end. Now let $v$ be a p-end vertex of a generalized lens-shaped end $\tilde{E}$.

As above in (A), suppose that $v_\tilde{E} \in K$. There exists $g \in \pi_1(O)$, $g(v_\tilde{E}) \neq v_\tilde{E}$, and $g(v_\tilde{E}) \in K \subset \text{bd}\tilde{O}$. Since $g(v_\tilde{E})$ is outside the lens-cone or the generalized lens-cone of $\tilde{E}$, $K$ meets $\text{Cl}(L)$ for the lens or generalized lens $L$ of $\tilde{E}$.

If $v_\tilde{E} \notin K$, then again $K \cap \text{Cl}(L) \neq \emptyset$ as in (A) using attracting fixed points of some elements of $\pi_1(\tilde{E})$. Hence, we conclude $K \cap \text{Cl}(L) \neq \emptyset$ for the lens $L$ of $\tilde{E}$.

Let $\Sigma_\tilde{E}$ denote $D^\circ$ from Lemma 5.5. Since $K \subset \text{bd}\tilde{O}$, $K$ cannot contain $\Sigma_\tilde{E}$. Thus, $K \cap \text{Cl}(\Sigma_\tilde{E})$ is a proper subspace of $\text{Cl}(\Sigma_\tilde{E})$, $\tilde{E}$ must be a virtually factorable end.

By Lemma 5.5, there exists a totally geodesic domain $\Sigma_\tilde{E}$ in the lens-part so that the p-end neighborhood of $v$ equals $U_v := v \star \Sigma_\tilde{E}$. Since $\pi_1(\tilde{E})$ acts reducibly, $\text{Cl}(\Sigma_\tilde{E})$ is a join $\text{Cl}(\Sigma_v)$. $K \cap \text{Cl}(\Sigma_v)$ contains a join $D_j := \ast_{i \in J} D_i$ for a proper subcollection $J$ of $\{1, \ldots, n\}$. Moreover, $K \cap \text{Cl}(\Sigma_\tilde{E}) = D_j$.

Since $g(U_v)$ is a p-end neighborhood of $g(v)$, we obtain $g(U_v) = U_{g(v)}$. Since $g(v) \in K$ for $g \in \pi_1(\tilde{O})$ and $g(K) = K$, we obtain that $K \cap g(\text{Cl}(\Sigma_\tilde{E})) = g(D_j)$.

Lemma 5.5 implies that

$$U_{g(v)} \cap U_v = \emptyset \text{ for } g \notin \pi_1(\tilde{E}) \text{ or } \quad U_{g(v)} = U_v \text{ for } g \in \pi_1(\tilde{E})$$

by the similar properties of $S(g(v))$ and $S(v)$ and the fact that $\text{bd} U_v \cap \tilde{O}$ and $\text{bd} U_{g(v)} \cap \tilde{O}$ are totally geodesic domains.

Let $\lambda_j(g)$ denote the $(\dim D_j + 1)$-th root of the norm of the determinant of the submatrix of $g$ associated with $D_j$ for the unit norm matrix of $g$. Since the strict lens-type ends satisfy the uniform middle eigenvalue condition by Theorem 2.15, Lemma 2.19 and the reducibility give us a sequence of elements $\gamma_i \in \pi_1(\tilde{E})$ so that

$$\gamma_i | D_J \to 1, \quad \gamma_i | D_J \to 1 \text{ for the complement } J^c := \{1, 2, \ldots, n\} - J,$$

$$\lambda_j(\gamma_i) \to \infty, \quad \lambda_j(\gamma_i) \to 0, \quad \lambda_j(\gamma_i) \to \infty.$$

Since $v, D_J \subset K$, the eigenvalue condition implies that either

$$K = D_J, \quad K = v \ast D_J \text{ or } K = v \ast D_J \cup v_- \ast D_J$$
Consider the second case. Let $g$ be an arbitrary element of $\pi_1(\mathcal{O})$. Since $D_J \subset K$, we obtain $g(D_J) \subset K$. Recall that $U_v \cap S(v)^o$ is a neighborhood of points of $S(v)^o$. Thus, $g(U_v \cap S(v)^o)$ is a neighborhood of points of $g(S(v)^o)$. $D_J^o$ is in the closure of $U_v$. If $D_J^o$ meets $g(v \ast D_J - D_J) \subset g(S(v)^o)$, then $U_v \cap g(U_v) \neq \emptyset$, and hence, $v = g(v)$ by Theorems 2.13 and 2.15. Finally, we obtain $D_J = g(D_J)$ as $K = v \ast D_J = g(v) \ast g(D_J)$.

If $D_J^o$ is disjoint from $g(v \ast D_J - D_J)$, then $g(D_J) \subset D_J$. By taking $g^{-1}$, we obtain $g(D_J) = D_J$. In either case, we conclude $g(D_J) = D_J$ for $g \in \pi_1(\mathcal{O})$.

Consider the first case: Clearly, $g(D_J) = D_J$ for all $g \in \pi_1(\mathcal{O})$. This implies $g(D_J) = D_J$ for $g \in \pi_1(\mathcal{O})$. Since $v$ and $g(v)$ are not equal for $g \in \pi_1(\mathcal{O}) - \pi_1(\mathcal{E})$, we obtain a triangle $\Delta$ with vertices $v, g(v), x \in D_J$. Then as in the part (ii) of the proof of Corollary 5.4, we obtain the existence of essential annulus. (For this argument, we did not need the assumption on strong irreducibility of $h(\pi_1(\mathcal{O}))$.)

Therefore, we deduced that the $h(\pi_1(\mathcal{O}))$-invariant subspace $S$ does not exist.

\[ \square \]

APPENDIX A. THE AFFINE ACTION DUAL TO THE TUBULAR ACTION

Let $\Gamma$ be an affine group acting on the affine space $A^n$ with boundary $\text{bd} A^n$ in $S^n$, i.e., an open hemisphere. Let $U'$ be a properly convex invariant $\Gamma$-invariant domain with boundary in a properly convex domain $\Omega \subset \text{bd} A^n$. $\Gamma$ is asymptotically nice if and only if we can find such $U'$ that is the intersection of all half-spaces $H, H \neq A^n$, supporting $U'$ at all point of $\text{bd} \Omega$.

In this section, we will work with $S^n$ only, while the $\mathbb{R} P^n$ versions are clear enough.

Each element of $g \in \Gamma$ is of the form

\[
\begin{pmatrix}
\frac{1}{\lambda_E(g)^{1/n}} \hat{h}(g) \\
0 \\
\lambda_E(g)
\end{pmatrix}
\]

where $\vec{b}_g$ is $1 \times n$-vector and $\hat{h}(g)$ is an $n \times n$-matrix of determinant $\pm 1$ and $\lambda_E(g) > 0$. In the affine coordinates, it is of the form

\[
x \mapsto \frac{1}{\lambda_E(g)^{1+\frac{1}{n}}} \hat{h}(g)x + \frac{1}{\lambda_E(g)} \vec{b}_g.
\]

Recall that if there exists a uniform constant $C > 0$ so that

\[
C^{-1} \text{length}(g) \leq \log \frac{\lambda_1(g)}{\lambda_E(g)} \leq C \text{length}(g), \quad g \in \Gamma_E - \{1\},
\]

then $\Gamma$ is said to satisfy the uniform middle-eigenvalue condition.
In this appendix, it is sufficient for us to prove when \( \Gamma \) is a hyperbolic group.

**Theorem A.1.** We assume that \( \Gamma \) is a hyperbolic group. Let \( \Omega \) be a properly convex domain in \( \text{bd}A^n \). Let \( \Gamma \) have a properly convex affine action on the affine space \( A^n, A^n \subset S^n \), acting on a properly convex domain \( U \subset A^n \) with boundary in the convex domain \( C(\Omega) \). Suppose that \( \Omega/\Gamma \) is a closed \((n-1)\)-dimensional orbifold and \( \Gamma \) satisfies the uniform middle-eigenvalue condition. Then \( \Gamma \) is asymptotically nice with the properly convex open domain \( U \), and the asymptotic hyperspace at each boundary point of \( \Omega \) is uniquely determined and is transversal to \( \text{bd}A^n \).

Note here \( C(U) \cap \text{bd}A^n = C(\Omega) \).

In the case when the linear part of the affine maps are unimodular, Theorem 8.2.1 of Labourie [57] shows that such a domain \( U \) exists but without showing the asymptotic niceness. In general, we think that the existence of the domain \( U \) can be obtained but the proof is much longer. (See Appendix of [30] in the special case that can be extended here.) Here, we are in an easier case when a domain \( U \) is given without the properties.

(It is fairly easy to show that this holds also for virtual products of hyperbolic and abelian groups as well by Proposition 2.10 and Theorem 2.7.)

### A.1. The Anosov flow.

We apply the work of Goldman-Labourie-Margulis [45]: Assume as in the premise of Theorem A.1. Since \( \Omega \) is properly convex, \( \Omega \) has a Hilbert metric. Let \( U\Omega \) denote the unit tangent bundle over \( \Omega \). This has a smooth structure as a quotient space of \( T\Omega - O/\sim \) where

- \( O \) is the image of the zero-section, and
- \( \vec{v} \sim \vec{w} \) if \( \vec{v} \) and \( \vec{w} \) are over the same point of \( \Omega \) and \( \vec{v} = s\vec{w} \) for a real number \( s > 0 \).

Assume \( \Gamma \) as above. Since \( \Sigma := \Omega/\Gamma \) is a properly convex real projective orbifold, \( U\Sigma := U\Omega/\Gamma \) is a compact smooth orbifold again. A geodesic flow on \( U\Omega/\Gamma \) is Anosov and hence topologically mixing. Hence, the flow is nonwondering everywhere. (See [4].) \( \Gamma \) acts irreducibly on \( \Omega \), and \( \text{bd}\Omega \) is \( C^1 \).

Let \( h : \Gamma \to \text{Aff}(A^n) \) denote the representation as described in equation (25). We form the product \( U\Omega \times A^n \) that is an affine bundle over \( U\Omega \). We take the quotient \( \tilde{A} := U\Omega \times A^n \) by the diagonal action

\[
g(x, \vec{u}) = (g(x), h(g)\vec{u}) \text{ for } g \in \Gamma, x \in U\Omega, \vec{u} \in A^n
\]

We denote the quotient by \( \tilde{A} \) fiber over the smooth orbifold \( U\Omega/\Gamma \) with fiber \( A^n \).

Let \( V^n \) be the vector space associated with \( A^n \). Then we can form \( \tilde{V} := U\Omega \times V^n \) and take the quotient under the diagonal action:

\[
g(x, \vec{u}) = (g(x), L \circ h(g)\vec{u}) \text{ for } g \in \Gamma, x \in U\Omega, \vec{u} \in V^n
\]

where \( L \) is the homomorphism taking the linear part of \( g \). We denote by \( \tilde{V} \) the fiber bundle over \( U\Omega/\Gamma \) with fiber \( V^n \).
We recall the trivial product structure. $U\Sigma \times A^n$ is a flat $A^n$-bundle over $U\Sigma$ with a flat affine connection $\nabla^A$, and $U\Omega \times V^n$ has a flat linear connection $\nabla^V$. The above action preserves the connections. We have a flat affine connection $\nabla^A$ on the bundle $A$ over $U\Sigma$ and a flat linear connection $\nabla^V$ on the bundle $V$ over $U\Sigma$.

We give a decomposition of $\tilde{V}$ into three parts $\tilde{V}_+, \tilde{V}_0, \tilde{V}_-$: For each vector $\tilde{u} \in U\Omega$, we find the maximal oriented geodesic $l$ ending at two points $\partial_+l, \partial_-l$. They correspond to the 1-dimensional vector subspaces $V_+(\tilde{u})$ and $V_-(\tilde{u})$. Recall that $bd\Omega$ is $C^1$ since $\Omega$ is strictly convex (see [4]). There exists a unique pair of supporting hyperspheres $H_+$ and $H_-$ in $bdA^n$ at each of $\partial_+l$ and $\partial_-l$. We denote by $H_0 = H_+ \cap H_-$. It is a codimension 2 great sphere in $bdA^n$ and corresponds to a vector subspace $V_0$ of codimension-two in $V$. For each vector $\tilde{u}$, we find the decomposition of $V$ as $V_+(\tilde{u}) \oplus V_0(\tilde{u}) \oplus V_-(\tilde{u})$ and hence we can form the subbundles $\tilde{V}_+, \tilde{V}_0, \tilde{V}_-$ over $U\Omega$ where

$$\tilde{V} = \tilde{V}_+ \oplus \tilde{V}_0 \oplus \tilde{V}_-.$$

The map $U\Omega \to bd\Omega$ by sending a vector to the endpoint of the geodesic tangent to it is $C^1$. The map $bd\Omega \to H$ sending a boundary point to its supporting hyperspace in the space $H$ of hyperspaces in $S^n$ is continuous. Hence $\tilde{V}_+, \tilde{V}_0, \tilde{V}_-$ are $C^0$-bundles. Since the action preserves the decomposition of $\tilde{V}$, $\tilde{V}$ also decomposes as

$$V = V_+ \oplus V_0 \oplus V_-.$$

We can identify $bdA^n = S(V^n)$ where $g$ acts by $\mathcal{L}(g) \in GL(n, \mathbb{R})$.

For each complete geodesic $l$ in $\Omega$, let $\tilde{l}$ denote the set of unit vectors on $l$ in one-directions. $\tilde{V}|\tilde{l} = \tilde{V}_+|\tilde{l} \oplus \tilde{V}_0|\tilde{l} \oplus \tilde{V}_-|\tilde{l}$ and these are of form $\tilde{l} \times V_+(\tilde{u}), \tilde{l} \times V_0(\tilde{u}), \tilde{l} \times V_-(\tilde{u})$ for a vector $\tilde{u}$ tangent to $l$. That is, the bundle is constant.

If $g \in \Gamma$ acts on a complete geodesic $l$ with a unit vector $\tilde{u}$, then $V_+(\tilde{u})$ and $V_-(\tilde{u})$ corresponding to endpoints of $l$ are eigenspaces of the largest norm $\lambda_1(g)$ of the eigenvalues and the smallest norm $\lambda_0(g)$ of the eigenvalues of the linear part $\mathcal{L}(g)$ of $g$. Hence on $V_+(\tilde{u})$, $g$ acts by expending by $\lambda_1(g)$ and on $V_-(\tilde{u})$, $g$ acts by contracting by $\lambda_0(g)$.

There exists a flow $\tilde{\Phi}_t : U\Omega \to U\Omega$ for $t \in \mathbb{R}$ given by sending $\tilde{v}$ to the unit tangent vector to $\alpha(t)$ where $\alpha$ is a geodesic tangent to $\tilde{v}$ with $\alpha(0)$ equal to the base point of $\tilde{v}$.

We define a flow on $\tilde{\Phi}_t : A \to A$ by considering a unit speed geodesic flow line $\tilde{l}$ in $U\Omega$ and considering $\tilde{l} \times E$ and acting trivially on the second factor as we go from $\tilde{v}$ to $\tilde{\Phi}_t(\tilde{v})$. (See remarks in the beginning of Section 3.3 and equations in Section 4.1 of [45].) Each flow line in $U\Sigma$ lifts to a flow line on $A$ from every point in it. This induces a flow $\Phi_t : A \to A$.

We define a flow on $\tilde{\Phi}_t : \tilde{V} \to \tilde{V}$ by considering a unit speed geodesic flow line $\tilde{l}$ in $U\Omega$ and and considering $\tilde{l} \times V$ and acting trivially on the second factor as we go from $\tilde{v}$ to $\tilde{\Phi}_t(\tilde{v})$ for each $t$. (This generalizes the flow on
Also, \( \tilde{\Phi}_t \) preserves \( \tilde{\mathcal{V}}_+ \), \( \tilde{\mathcal{V}}_0 \), and \( \tilde{\mathcal{V}}_- \) since on the line \( l \), the endpoint \( \partial_{\pm}l \) does not change. Again, this induces a flow

\[
\Phi_t : \mathcal{V} \to \mathcal{V}, \mathcal{V}_+ \to \mathcal{V}_+, \mathcal{V}_0 \to \mathcal{V}_0, \mathcal{V}_- \to \mathcal{V}_-.
\]

We let \( \| \cdot \|_S \) denote some metric on these bundles over \( U\Sigma/\Gamma \) defined as a fiberwise inner product: We chose a cover of \( \Omega/\Gamma \) by compact sets \( K_i \) and choosing a metric over \( K_i \times A^n \) and use the partition of unity. This induces a fiberwise metric on \( \mathcal{V} \) as well. Pulling the metric back to \( \tilde{\mathcal{A}} \) and \( \tilde{\mathcal{V}} \), we obtain a fiberwise metrics to be denoted by \( \| \cdot \|_S \).

As in Section 4.4 of [45], \( \mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_0 \oplus \mathcal{V}_- \). By the uniform middle-eigenvalue condition, \( \mathcal{V} \) has a fiberwise Euclidean metric \( g \) with the following properties:

- the flat linear connection \( \nabla^\mathcal{V} \) is bounded with respect to \( g \).
- hyperbolicity: There exists constants \( \tilde{C}, k > 0 \) so that

\[
\| \Phi_t(\vec{v}) \|_S \geq \frac{1}{\tilde{C}} \exp(kt) \| \vec{v} \|_S \quad \text{as} \quad t \to \infty
\]

for \( \vec{v} \in \mathcal{V}_+ \) and

\[
\| \Phi_t(\vec{v}) \|_S \leq C \exp(-kt) \| \vec{v} \|_S \quad \text{as} \quad t \to \infty
\]

for \( \vec{v} \in \mathcal{V}_- \).

Proposition A.2 proves this property by taking \( C \) sufficiently large according to \( t_1 \), which is a standard technique.

A.2. The proof of the Anosov property. We can apply this to \( \mathcal{V}_- \) and \( \mathcal{V}_+ \) by possibly reversing the direction of the flow. The Anosov property follows from the following proposition.

Let \( \mathcal{V}_{-1} \) denote the subset of \( \mathcal{V}_- \) of the unit length under \( \| \cdot \|_S \).

**Proposition A.2.** Let \( \Omega/\Gamma \) be a closed real projective orbifold with hyperbolic group. Then there exists a constant \( t_1 \) so that

\[
\| \Phi_t(v) \|_S \leq \tilde{C} \| v \|_S, v \in \mathcal{V}_- \quad \text{and} \quad \| \Phi_{-t}(v) \|_S \leq \tilde{C} \| v \|_S, v \in \mathcal{V}_+
\]

for \( t \geq t_1 \) and a uniform \( \tilde{C}, 0 < \tilde{C} < 1 \).

**Proof.** It is sufficient to prove the first part of the inequalities since we can substitute \( t \to -t \) and switching \( \mathcal{V}_+ \) with \( \mathcal{V}_- \) as the direction of the vector changed to the opposite one.

Let \( \mathcal{V}_{-1} \) denote the subset of \( \mathcal{V}_- \) of the unit length under \( \| \cdot \|_S \). By following Lemma A.3, the uniform convergence implies that for given \( 0 < \epsilon < 1 \), for every vector \( v \) in \( \mathcal{V}_{-1} \), there exists a uniform \( T \) so that for \( t > T \), \( \Phi_t(v) \) is in an \( \epsilon \)-neighborhood \( U_\epsilon(S_0) \) of the image \( S_0 \) of the zero section. \( \square \)

The line bundle \( \mathcal{V}_- \) lifts to \( \tilde{\mathcal{V}}_- \) where each unit vector \( u \) on \( \Omega \) one associates the line \( \mathcal{V}_-u \) corresponding to the starting point in \( \text{bd}\Omega \) of the
Let $\Phi: U\Omega \to \Omega$ be a projection of the unit tangent bundle to the base space.

**Lemma A.3.** $||\Phi_t||_S \to 0$ uniformly as $t \to \infty$.

**Proof.** Let $F$ be a fundamental domain of $U\Omega$ under $\Gamma$. It is sufficient to prove this for $\hat{\Phi}_t$ on the fibers of over $F$ of $U\Omega$ with a fiberwise metric $||\cdot||_S$.

We choose an arbitrary sequence $\{x_i\}$, $\{x_i\} \to x$ in $F$. For each $i$, let $v_{-i}$ be a Euclidean unit vector in $V_{-i} := V_-(x_i)$ for the unit vector $x_i \in U\Omega$. That is, $v_{-i}$ is in the 1-dimensional subspace in $\mathbb{R}^n$, corresponding to the endpoint of the geodesic determined by $x_i$ in $\partial \Omega$.

We will show that $||\Phi_t((x_i, v_{-i}))||_S \to 0$ for any sequence $t_i \to \infty$. This is sufficient to prove the uniform convergence to 0 by the compactness of $V_{-1}$. (Here, $[v_{-i}]$ is an endpoint of $l_i$ in the direction given by $x_i$.)

For this, we just need to show that any sequence of $\{t_i\} \to \infty$ has a subsequence $\{t_j\}$ so that $||\Phi_{t_j}((x_i, v_{-j}))||_S \to 0$ converging to 0. Otherwise, we can always extract a subsequence converging to nonzero or to $\infty$.

Let $y_i := \hat{\Phi}_t(x_i)$ for the lift of the flow $\hat{\Phi}$. By construction, we recall that each $P(y_i)$ is in the geodesic $l_i$. Since $x_i \to x$, $x_i, x \in F$, we obtain that $l_i$ geometrically converges to a line $l_\infty$ passing $P(x)$ in $\Omega$. Let $y_+$ and $y_-$ be
the endpoints of \(l_\infty\) where \(\{P(y_i)\} \to y_-\). Hence,
\[
[v_+, i] \to y_+, [v_-, i] \to y_-
\]

Find a deck transformation \(g_i\) so that \(g_i(y_i) \in F\) and \(g_i\) acts on the line bundle \(\tilde{\mathcal{Y}}_\infty\) by the linearization of the matrix of form of equation (24):
\[
g_i : \mathcal{Y}_\infty \to \mathcal{Y}_\infty \text{ given by } (y_i, v) \to (g_i(y_i), L(g_i)(v)) \text{ where }
\]

(28) \[
L(g_i) := \frac{1}{\lambda_E(g_i)^{1+\frac{1}{n}}} \hat{h}(g_i) : \mathcal{Y}_\infty(y_i) = \mathcal{Y}_\infty(x_i) \to \mathcal{Y}_\infty(g_i(y_i)).
\]

(Goal) We will show \(\{(g_i(y_i), L(g_i)(v_i-))\} \to 0\) under \(|| \cdot ||_S\). This will complete the proof since \(g_i\) acts as isometries on \(\mathcal{Y}_\infty\) with \(|| \cdot ||_S\).

Since \(g_i(l) \cap F \neq \emptyset\), we choose a subsequence of \(g_i\) and relabel it \(g_i\) so that \(\{g_i(l_i)\}\) converges to a nontrivial line in \(\Omega\).

We choose a subsequence of \(\{g_i\}\) so that the sequences \(\{a_i\}\) and \(\{r_i\}\) are convergent for the attracting fixed point \(a_i \in Cl(\Omega)\) and the repelling fixed point \(r_i \in Cl(\Omega)\) of each \(g_i\). Then
\[
\{a_i\} \to a_\ast \text{ and } r_i \to r_\ast \text{ for } a_\ast, r_\ast \in bd\Omega
\]

(See Figure 4.) Also, it follows that for every compact \(K \subset Cl(\Omega) - \{r_\ast\},\)
\[
g_i|K \to \{a_\ast\}
\]

uniformly as in the proof of Theorem 5.7 of [26].

Suppose that \(a_\ast = r_\ast\). Then we choose an element \(g \in \Gamma\) so that \(g(a_\ast) \neq r_\ast\) and replace the sequence by \(\{gg_i\}\) and replace \(F\) by \(F \cup g(F)\). The above uniform convergence condition still holds. Then the new attracting fixed points \(a_i' = g(a_i)\) and the sequence \(\{r_i\}\) of repelling fixed point \(r_i'\) of \(gg_i\) converges to \(r_\ast\) also by Lemma 3.7. Hence, we may assume without loss of generality that
\[
a_\ast \neq r_\ast
\]

by replacing our sequence \(g_i\).

Suppose that both \(y_+, y_- \neq r_\ast\). Then \(\{g_i(l_i)\}\) converges to a singleton \(\{a_\ast\}\) by equation (29) and this cannot be. If
\[
r_\ast = y_+ \text{ and } y_- \in bd\Omega - \{r_\ast\},
\]
then \(g_i(y_i) \to a_\ast\) by equation (29) again. Since \(g_i(y_i) \in F\), this is a contradiction. Therefore
\[
r_\ast = y_- \text{ and } y_+ \in bd\Omega - \{r_\ast\}.
\]

Let \(d_i\) denote the other endpoint of \(l_i\) from \([v_-, i]\).

- Since \([v_-, i] \to y_-\) and \(l_i\) converges to a nontrivial line \(l_\infty\), it follows that \(\{d_i\}\) is in a compact set in \(bd\Omega - \{y_-\}\).
- Then \(\{g_i(d_i)\} \to a_\ast\) as \(\{d_i\}\) is in a compact set in \(bd\Omega - \{y_-\}\).
- Thus, \(\{g_i([v_-, i])\} \to y' \in bd\Omega\) where \(a_\ast \neq y'\) holds since \(\{g_i(l_i)\}\) converges to a nontrivial line in \(\Omega\).
Also, \( g_i \) has an invariant great sphere \( S_i^{n-2} \subset \text{bd} A^n \) containing the attracting fixed point \( a_i \) and supporting \( \Omega \) at \( a_i \). Thus, \( r_i \) is uniformly bounded at a distance from \( S_i^{n-2} \) as \( \{ r_i \} \to y_- = r_* \).

Let \( \| \cdot \|_E \) denote the standard Euclidean metric of \( \mathbb{R}^{n+1} \).

- Since \( y_i \to y_- \), it follows that \( y_i \) is also uniformly bounded away from \( a_i \) and the tangent sphere \( S_i^{n-1} \) at \( a_i \).
- Since \( \{ \varphi_{-i} \} \to y_- \), the vector \( \varphi_{-i} \) has the component \( \eta_i^p \) parallel to \( r_i \) and the component \( \eta_i^s \) in the direction of \( S_i^{n-2} \) where \( \varphi_{-i} = \eta_i^p + \eta_i^s \).
- Since \( r_i \to r_* = y_- \) and \( \{ \varphi_{-i} \} \to y_- \), we obtain \( \eta_i^s \to 0 \) and that \( \eta_i^p \) is uniformly bounded in \( \| \cdot \|_E \).
- \( g_i \) acts by preserving the directions of \( S_i^{n-2} \) and \( r_i \).

Since \( \{ g_i(\varphi_{-i}) \} \) converging to \( y' \) is bounded away from \( S_i^{n-2} \) uniformly, we have that

\[
\text{the Euclidean norm of } \frac{L(g_i)(\eta_i^s)}{\|L(g_i)(\eta_i^p)\|_E}
\]

is bounded above uniformly.

As \( r_i \) is a repelling fixed point of \( g_i \) and \( \| \eta_i^p \|_E \) is uniformly bounded above, we have \( \{ L(g_i)(\eta_i^p) \} \to 0 \).

\[
\{ L(g_i)(\eta_i^p) \} \to 0 \implies \{ L(g_i)(\eta_i^s) \} \to 0
\]

for \( \| \cdot \|_E \). Hence, we obtain \( \{ L(g_i)(\varphi_{-i}) \} \to 0 \) under \( \| \cdot \|_E \).

Recall that \( \Phi_f \) is the identity map on the second factor of \( U \Omega \times V_- \).

\[
g_i(\Phi_f(x, \varphi_{-i})) = (g_i(y_i), L(g_i)(\varphi_{-i}))
\]

is a vector over the compact fundamental domain \( F \) of \( U \Omega \). Since

\[
(g_i(y_i), L(g_i)(\varphi_{-i}))
\]

is a vector over the compact fundamental domain \( F \) of \( U \Omega \) with

\[
\|L(g_i)(\varphi_{-i})\|_E \to 0,
\]

we conclude that \( \{ \| \Phi_f(x, \varphi_{-i}) \|_S \} \to 0 \): For the compact fundamental domain \( F \), the Euclidean metric \( \| \cdot \|_S \) and the Riemannian metric \( \| \cdot \|_S \) of \( \tilde{V}_- \) are related by a bounded constant on the compact set \( F \). \( \square \)

A.3. The neutralized section. A section \( \sigma : U \Sigma \to \mathbb{A} \) is neutralized if

\[
(30) \quad \nabla^{\mathbb{A}}_s \in V_0.
\]

We denote by \( \Gamma(V) \) the space of sections \( U \Sigma \to V \) and by \( \Gamma(\mathbb{A}) \) the space of sections \( U \Sigma \to \mathbb{A} \).

Recall from [45] the one parameter-group of bounded operators \( D\Phi_{t, *} \) on \( \Gamma(V) \) and \( \Phi_{t, *} \) on \( \Gamma(\mathbb{A}) \). We denote by \( \phi \) the vector field generated by this flow on \( U \Sigma \). Recall Lemma 8.3 of [45] also
Lemma A.4. If $\psi \in \Gamma(A)$, and $$t \mapsto D\Phi_{t*}(\psi)$$ is a path in $\Gamma(V)$ that is differentiable at $t = 0$, then $$\frac{d}{dt} |_{t=0}(D\Phi_{t*}(\psi)) = \nabla^{s}_{\phi}(\psi).$$

Recall that $U\Sigma$ is a recurrent set under the geodesic flow.

Lemma A.5. A neutralized section exists on $U\Sigma$. This lifts to a map $\tilde{\xi}_0 : U\Omega \to A$ so that $\tilde{\xi}_0 \circ \gamma = \gamma \circ \tilde{\xi}_0$.

Proof. Let $s$ be a continuous section $U\Sigma \to A$. We decompose $$\nabla^s_{\phi}(s) = \nabla^{s+}_{\phi}(s) + \nabla^{s0}_{\phi}(s) + \nabla^{s-}_{\phi}(s) \in V$$ so that $\nabla^{s\pm}_{\phi}(s) \in V_{\pm}$ and $\nabla^{s0}_{\phi}(s) \in V_0$ hold. By the uniform convergence property of equations (26) and (27), the following integrals converge to smooth functions over $U\Sigma$. Again

$$s_0 = s + \int_0^\infty (D\Phi_{t*}(\nabla^{s-}_{\phi}(s)) dt - \int_0^\infty (D\Phi_{-t*}(\nabla^{s+}_{\phi}(s)) dt$$

is a continuous section and $\nabla^{s}_{\phi}(s_0) = \nabla^{s0}_{\phi}(s_0) \in V_0$ as shown in [45].

Since $U\Sigma$ is connected, there exists a fundamental domain $F$ so that we can lift $s_0$ to $\tilde{s}_0'$ defined on $F$ mapping to $A$. We can extend $\tilde{s}_0'$ to $U\Omega \to \Omega \times E$. \hfill \qed

Let $N_2(A^n)$ denote the space of codimension two affine spaces of $A^n$. We denote by $G(\Omega)$ the space of maximal oriented geodesics in $\Omega$. We use the quotient topology on both spaces. There exists a natural action of $\Gamma$ on both spaces.

For each element $g \in \Gamma - \{1\}$, we define $N_2(g)$: Now, $g$ acts on $bdA^n$ with invariant subspaces corresponding to invariant subspaces of the linear part $L(g)$ of $g$. Since $g$ and $g^{-1}$ are positive proximal,

- a unique fixed point in $bdA^n$ corresponds to the largest norm eigenvector, an attracting fixed point in $bdA^n$, and
- a unique fixed point in $bdA^n$ corresponds to the smallest norm eigenvector, a repelling fixed point

by [4] or [9]. There exists an $L(g)$-invariant vector subspace $V^0_g$ complementary to the join of the subspace generated by these eigenvectors. (This space equals $V_0$ for the unit tangent vector tangent to the unique maximal geodesic $l_g$ in $\Omega$ on which $g$ acts.) It corresponds to a $g$-invariant subspace $M(g)$ of codimension two in $bdA^n$.

Let $\tilde{c}$ be the geodesic in $U\Sigma$ that is $g$-invariant for $g \in \Gamma$. $\tilde{s}_0(\tilde{c})$ lies on a fixed affine space parallel to $V^0_g$ by the neutrality, i.e., Lemma A.5. There exists a unique affine subspace $N_2(g)$ of codimension two in $A^n$ whose containing $\tilde{s}_0(\tilde{c})$. Immediate properties are $N_2(g) = N_2(g^m)$, $m \in \mathbb{Z}$ and that $g$ acts on $N_2(g)$. 

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Definition A.6. We define $S'(bd\Omega)$ the space of $(n - 1)$-dimensional hemi-spheres with interiors in $A^n$ each of whose boundary in $bdA^n$ is a supporting hypersphere in $bdA^n$ to $\Omega$. We denote by $S(\Omega)$ the space of pairs $(x, H)$ where $H \in S'(bd\Omega)$ and $x$ is in the boundary of $H$ and $bd\Omega$.

Define $\Delta$ to be the diagonal set of $bd\Omega \times bd\Omega$. Denote by $\Lambda^* = bd\Omega \times bd\Omega - \Delta$. Let $G(\Omega)$ denote the space of maximal oriented geodesics in $\Omega$. $G(\Omega)$ is in a one-to-one correspondence with $\Lambda^*$ by the map taking the maximal oriented geodesic to the ordered pair of its endpoints.

Proposition A.7. 

- There exists a continuous function $\tilde{s} : U\Omega \rightarrow N_2(A^n)$ equivariant with respect to $\Gamma$-actions.
- Given $g \in \Gamma$ and for the unique unit speed geodesic $\tilde{l}_g$ in $U\Omega$ lying over a geodesic $l_g$ where $g$ acts on, $\tilde{s}(\tilde{l}_g) = \{N_2(g)\}$.
- This gives a continuous map

$$\tilde{s}' : bd\Omega \times bd\Omega - \Delta \rightarrow N_2(A^n)$$

again equivariant with respect to the $\Gamma$-actions. There exists a continuous function

$$\tau : \Lambda^* \rightarrow S(bd\Omega).$$

Proof. Given a vector $\tilde{u} \in U\Omega$, we find $\bar{s}_0(\tilde{u})$. There exists a lift $\tilde{d}_t : U\Omega \rightarrow U\Omega$ of the geodesic flow $\phi_t$. Then $\bar{s}_0(\tilde{d}_t(\tilde{u}))$ is in an affine subspace $H^{n-2}$ parallel to $V_0$ for $\tilde{u}$ by the neutrality condition equation (30). We define $\bar{s}(\tilde{u})$ to be this $H^{n-2}$.

For any unit vector $\tilde{u}'$ on the maximal (oriented) geodesic in $\Omega$ determined by $\tilde{u}$, we obtain $\bar{s}(\tilde{u}') = H^{n-2}$. Hence, this determines the continuous map $\bar{s} : G(\Omega) \rightarrow N_2(A^n)$. The $\Gamma$-equivariance comes from that of $\bar{s}_0$.

For $g \in \Gamma$, $\tilde{u}$ and $g(\tilde{u})$ lie on the $g$-invariant geodesic $l_g$ provided $\tilde{u}$ is tangent to $l_g$. Since $\bar{s}_0(g(\tilde{u})) = \bar{s}_0(g(\tilde{u}))$ by equivariance, $g(\bar{s}_0(\tilde{u}))$ lies on $\bar{s}(\tilde{u}) = \bar{s}(g(\tilde{u}))$. For $g(\bar{s}(\tilde{u})) = \bar{s}(l_g)$.

The map $\tilde{s}'$ is defined since $bd\Omega \times bd\Omega - \Delta$ is in one-to-one correspondence with the space $G(\Omega)$. The map $\tau$ is defined by taking for each pair $(x, y) \in \Lambda^*$

- we take the geodesic $l$ with endpoints $x$ and $y$, and
- taking the hyperspace in $A^n$ containing $\bar{s}(l)$ and its boundary containing $x$.

\[\square\]

A.3.1. The asymptotic niceness. We denote by $h(x, y)$ the $(n-1)$-dimensional hemisphere part in $\tau(x, y) = (x, h(x, y)))$.

Lemma A.8. Let $U$ be a $\Gamma$-invariant properly convex open domain in $\mathbb{R}^n$ so that $bdU \cap bdA^n = Cl(\Omega)$. Suppose that $x$ and $y$ are fixed points of an element $g$ of $\Gamma$ in $bd\Omega$. Then $h(x, y)$ is disjoint from $U$.

Proof. Suppose not. $h(x, y)$ is a $g$-invariant hemisphere, and $x$ is an attracting fixed point of $g$ in it. (We can choose $g^{-1}$ if necessary.) Then $U \cap h(x, y)$ is a $g$-invariant properly convex open domain containing $x$ in its boundary.
Suppose first that \( h(x, y) \) has a fixed point \( z \) of \( g \) with the smallest eigenvalue in \( h(x, y)^0 \). Then the associated eigenvalue to \( z \) is strictly less than that of \( x \) by the uniform middle-eigenvalue condition and hence \( z \) is in the closure of the convex open domain \( U \cap h(x, y) \). \( g \) acts on the 2-sphere \( P \) containing \( x, y, z \). Then the \( g \) acts on \( P \cap U \) intersecting \( xz^0 \). This set \( P \cap U \) cannot be properly convex due to the fact that \( z \) is a saddle-type fixed point. Hence, there exists no fixed point \( z \).

The alternative is as follows: \( h(x, y) \) contains a \( g \)-invariant affine subspace \( A \) of codimension at least 2 in \( A^0 \), and the fixed point of the smallest eigenvalue in \( h(x, y) \) is associated with a point of \( \partial A \). \( g|h(x, y) \) has the largest norm eigenvalue at \( x, x_\infty \). Therefore, we act by \( \langle g \rangle \) on a generic point \( z \) of \( h(x, y) \cap U \). We obtain an arc in \( h(x, y) \) with endpoints \( x \) or \( x_\infty \) and an endpoint \( y' \) in \( \partial A' \subset \partial A \). Here \( y' \) is a fixed point in \( h(x, y) \) different from \( y \) as \( y \notin h(x, y) \) and \( y' \in \partial h(U) \). It follows \( y' \in \partial h(x, y) \). \( x \in \partial h(x, y) \) implies \( x_\infty \notin \partial h(x, y) \) by the proper convexity. \( x, y' \in \partial h(x, y) \) implies \( xy' \subset \partial h(x, y) \subset C^0 \). Finally, \( xy' \subset \partial h(x, y) \) for the supporting subspace \( \partial h(x, y) \) of \( \partial h(x, y) \) violates the strict convexity of \( \partial h(x, y) \). (See Benoist [4].)

\[ \square \]

The proof of the following lemma is slightly different from that of Theorem 9.1 in [31] since we can use an invariant properly convex domain \( U \). In Theorem 4.2, we will obtain that this also give us strict lens p-end neighborhoods.

**Lemma A.9.** Let \((x, y) \in \Lambda^* \). Then

- \( \tau(x, y) \) does not depend on \( y \) and is unique for each \( x \).
- \( h(x, y) \) contains \( \bar{\mathbb{S}}(xy) \) but is independent of \( y \).
- \( h(x, y) \) is never a hemisphere in \( \partial A^0 \) for every \((x, y) \in \Lambda^* \).
- \( \tau : \partial A^0 \to S(\partial A^0) \) is continuous.

**Proof.** We claim that for any \( x, y \) in \( \partial A^0 \), \( h(x, y) \) is disjoint from \( U \): By Theorem 1.1, the geodesic flow on \( \Omega/\Gamma \) is Anosov, and hence closed geodesics in \( \Omega/\Gamma \) is dense in the space of geodesics by the basic property of the Anosov flow. Since the fixed points are in \( \partial A^0 \), we can find a sequence \( x_\tau \to x \) and \( y_\tau \to y \) where \( x_i \) and \( y_i \) are fixed points of an element \( g_i \in \Gamma \) for each \( i \). If \( h(x, y) \cap U \neq \emptyset \), then \( h(x_i, y_i) \cap U \neq \emptyset \) for \( i \) sufficiently large by the continuity of the map \( \tau \). This is a contradiction by Lemma A.8

Also \( \partial A^0 \) does not contain \( h(x, y) \) since \( h(x, y) \) contains the \( \bar{\mathbb{S}}(xy) \) while \( y \) is chosen \( y \neq x \).

Let \( H(x, y) \) denote the half-space bounded by \( h(x, y) \) containing \( U \). \( h(x, y') \) is supporting \( \partial A^0 \) and hence is independent of \( y' \) as \( \partial A^0 \) is \( C^1 \). So, we have

\[ H(x, y) \subset H(x, y') \quad \text{or} \quad H(x, y) \supset H(x, y'). \]

For each \( x \), we define

\[ H(x) := \bigcap_{y \in \partial A^0 - \{x\}} H(x, y). \]
Define \( h(x) \) as the boundary \((n - 1)\)-hemisphere of \( H(x) \).

Now, \( U' \coloneqq \bigcap_{x \in \partial \Omega} H(x) \) contains \( U \) by the above disjointness. Since \( \partial \Omega \) has at least \( n + 1 \) points in general position and tangent hemispheres, \( U' \) is properly convex. Let \( U'' \) be the properly convex open domain \( \bigcap_{x \in \partial \Omega} \left( E - \text{Cl}(H(x)) \right) \).

It has the boundary \( A(\text{Cl}(\Omega)) \) in \( \partial A^n \) for the antipodal map \( A \). Since the antipodal set of \( \partial \Omega \) has at least \( n + 1 \) points in general position, \( U'' \) is a properly convex domain. Note that \( U' \cap U'' = \emptyset \).

If for some \( x, y \), \( h(x, y) \) is different from \( h(x) \), then \( h(x, y) \cap U'' \neq \emptyset \). This is a contradiction by the above part of the proof where \( U \) is replaced by \( U'' \).

Thus, we obtain \( h(x, y) = h(x) \) for all \( y \in \partial \Omega - \{x\} \).

We show the continuity of \( x \mapsto h(x) \): Let \( x_i \in \partial \Omega \) be a sequence converging to \( x \in \partial \Omega \). Then choose \( y_i \in \partial \Omega \) so that \( y_i \rightarrow y \) and we have \( \{h(x_i) = h(x, y_i)\} \) converges to \( h(x, y) = h(x) \) by the continuity of \( \tau \). Therefore, \( h \) is continuous. \( \square \)

(Proof of Theorem A.1). For each point \( x \in \partial \Omega \), an \((n - 1)\)-dimensional hemisphere \( h(x) \) passes \( A^n \) with \( \partial h(x) \subset \partial A^n \) supporting \( \Omega \) by Lemma A.9. Then a hemisphere \( H(x) \subset A^n \) is bounded by \( h(x) \) and contains \( \Omega \). The properly convex open domain \( \bigcap_{x \in \partial \Omega} H(x) \) contains \( U \). Since \( \partial \Omega \) is \( C^1 \) and strictly convex, the uniqueness of \( h(x) \) in the proof of Lemma A.9 gives us the unique asymptotic totally geodesic hypersurface. \( \square \)

The following is another version of Theorem A.1. We do not assume that \( \Gamma \) is hyperbolic here.

**Theorem A.10.** Let \( \Gamma \) be a discrete group in \( \text{SL}_+(n + 1, \mathbb{R}) \) acting on \( \Omega \), \( \Omega \subset \partial A^n \), so that \( \Omega / \Gamma \) is a compact orbifold. \( \Gamma \) satisfies the uniform middle eigenvalue condition.

- Suppose that \( \Omega \) has a \( \Gamma \)-invariant properly convex open domain \( U \) forming a one-sided neighborhood of \( \Omega \) in \( A^n \).
- Suppose that \( \Gamma \) satisfies the uniform middle eigenvalue condition.
- Let \( P \) be the hyperplane containing \( \Omega \).

Then \( \Gamma \) acts on a properly convex open domain \( L \) in \( S^n \) containing \( \Omega \) and contained in \( U \) and having strictly convex boundary in \( S^n - P \). That is, \( L \) is a lens-shaped neighborhood of \( \Omega \).

**Proof.** Suppose that \( \Gamma \) is virtually non-factorable and hyperbolic. Define a half-space \( H(x) \subset A^n \) bounded by \( h(x) \) and containing \( \Omega \) in the boundary. For each \( H(x), x \in \partial \Omega \), in the proof of Theorem A.1, an open \( n \)-hemisphere \( H'(x) \subset S^n \) satisfies \( H'(x) \cap A^n = H(x) \). Then we define

\[ V := \bigcup_{x \in \partial \Omega} H'(x) \subset S^n \]

is a convex open domain containing \( \Omega \) as in the proof of Lemma A.9.
Suppose that $\Gamma$ is virtually factorable. By Theorem 2.7 and Proposition 2.6, $\Gamma$ acts on a closed set

$$\mathcal{H} := \{ h \mid h \text{ is a supporting hyperspace at } x \in \text{bd} \Omega, h \not\subset S_n^{\text{def}} \}$$

Let $\mathcal{H}'$ denote the set of hemispheres bounded by an element of $\mathcal{H}$ and containing $\Omega$. Then we define

$$V := \bigcap_{H \in \mathcal{H}'} H \subset S_n$$

is a convex open domain containing $\Omega$. Here again the set of supporting hyperspaces is closed and bounded away from $S_n^{\text{def}}$.

First suppose that $V$ is properly convex. Then $V$ has a $\Gamma$-invariant Hilbert metric $d_V$ that is also Finsler. (See [43] and [53].) Then

$$N_\epsilon = \{ x \in V \mid d_V(x, \Omega) < \epsilon \}$$

is a convex subset of $V$ by Lemma 1.8 of [35].

A compact tubular neighborhood $M$ of $\Omega / \Gamma$ in $V / \Gamma$ is diffeomorphic to $\Omega / \Gamma \times [-1, 1]$. (See Section 4.4.2 of [22].) Since $\Omega$ is compact, the regular neighborhood has a compact closure. Thus, $d_V(\Omega / \Gamma, \text{bd} M / \Gamma) > \epsilon_0$ for some $\epsilon_0 > 0$. If $\epsilon < \epsilon_0$, then $N_\epsilon \subset M$. We obtain that $\text{bd} N_\epsilon / \Gamma$ is compact.

Clearly, $\text{bd} N / \Gamma$ has two components in the respective components of $(V - \Omega) / \Gamma$. Let $F_1$ and $F_2$ be the fundamental domains of both components. We procure finitely many open hemispheres $H_i, H_i \supset \Omega$, so that open sets $(S_n - \text{Cl}(H_i)) \cap N_\epsilon$ cover $F_1 \cup F_2$. Define

$$W := \bigcap_{g \in \Gamma} g(H_i) \cap V.$$ 

Since any path from $\Omega$ to $\text{bd} N_\epsilon$ must meet $(\text{bd} W - P) \cap V$ first, $N_\epsilon$ contains $W$ and $\text{bd} W$. A collection of compact totally geodesic polyhedrons meet in angles $< \pi$ and comprise $\text{bd} W / \Gamma$. We can smooth $\text{bd} W$ to obtain a strictly convex lens neighborhood $W' \subset W$ of $\Omega$ in $N_\epsilon$.

Since the closed set $\text{bd} U \cap V / \Gamma$ does not meet the compact $\Sigma$,

$$d_V(\Omega, \text{bd} U \cap V) > \epsilon_0$$

for $\epsilon_0 > 0$. We choose $\epsilon > 0$ smaller than this number. Then $W'$ is a subset of $U$ that we construct for $N_\epsilon$ as above.

Suppose that $V$ is not properly convex. Then $\text{bd} V$ contains $v, v_\perp$. $V$ is a tube. We take any two open hemispheres $S_1$ and $S_2$ containing $\text{Cl}(\Omega)$ so that $\{ v, v_\perp \} \cap S_1 \cap S_2 = \emptyset$. Then $\bigcap_{g \in \Gamma} g(S_1 \cap S_2) \cap V$ is a properly convex open domain containing $\Omega$. and we can apply the same argument as above.

Let $L$ be $\text{Cl}(W) \cap \tilde{O}$. Then $\partial L$ has boundary only in $\text{bd} A^0$ by Lemma A.11 since $\Gamma$ satisfies the uniform middle eigenvalue condition.

**Lemma A.11.** Let $\Gamma$ be a discrete group in $\text{SL}_+ (n + 1, \mathbb{R})$ acting on $\Omega$, $\Omega \subset \text{bd} A^0$, so that $\Omega / \Gamma$ is a compact orbifold. Suppose that $\Gamma$ satisfies the uniform middle eigenvalue condition. The supporting hyperspheres are at uniformly bounded distances from the hypersphere containing $\Omega$. 


Suppose that $\gamma_i$ is a sequence of elements of $\Gamma$ acting on $\Omega$.
The sequence of attracting fixed points $a_i$ and the sequence of repelling fixed points $b_i$ are so that $a_i \to a_\infty$ and $b_i \to b_\infty$ where $a_\infty, b_\infty$ are in $C(\Omega) - \Omega$.
Suppose that the sequence $\{\lambda_i\}$ of eigenvalues where $\lambda_i$ corresponds to $a_i$ converges to $+\infty$.

Then for a lens domain $U$ containing $\Gamma$ of the affine action the point $\{a_\infty\}$ is the limit of $\{\gamma_i(J)\}$ for any compact subset $J \subset U$.

Proof. The proof is similar to that of Lemma 3.5. Here we can use the fact that the supporting hyperspheres are at uniformly bounded distances from the hypersphere containing $\Omega$. The eigenvalue estimations are similar. \(\square\)

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