Note on gauge invariance of second order cosmological perturbations

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Abstract: We study the gauge invariant cosmological perturbations up to second order. We show that there are infinite families of gauge invariant variables at both of the first and second orders. The conversion formulae among different families are shown to be described by a finite number of bases that are gauge invariant. For the second order cosmological perturbations induced by the first order scalar perturbations, we explicitly represent the equations of motion of them in terms of the gauge invariant Newtonian, synchronous and hybrid variables, respectively.

1 Introduction

The gauge invariance is known as an important concept in the cosmological perturbation theory. At first order, the gauge invariant variables were introduced for the first time by Bardeen in 1980s [1]. Since then, different gauge-invariant formalisms were studied [2–7]. They have been widely utilized to study the cosmological fluctuations [8], for example, the anisotropies in the cosmic microwave background and the large scale structures, which have been precisely measured in the past decades [9]. For the second order cosmological perturbations, the gauge invariance has also been studied recently, and a quantity of gauge invariant variables have been constructed [5–7, 10–17]. In addition, it is believed that the gauge invariance is available to study the higher order cosmological perturbations [18].

However, the things changed recently when one studied the second order cosmological perturbations, including but not limited to the second order gravitational waves [19–23], which are induced by the first order scalar perturbations [24–42]. The energy density spectrum of the second order gravitational waves as a physical observable have recently been involved in gauge issue [43–49]. One possible explanation is related to the gauge fixing, which may give rise to unknown fictitious perturbations [45]. Therefore, one may resolve this problem by constructing the gauge invariant variables [2–7, 10–13]. The other explanation is related to the definition of the physical observable, which has been suggested to be defined in the synchronous frame [48]. However, in such a framework, the concept of gauge invariance was suggested to be necessarily abandoned, since it is impossible to truly construct the gauge invariant second order synchronous variables [50].

However, we would show that the gauge invariance could be preserved when one studies the second order cosmological perturbations, in particular, the second order gravitational waves. On the one hand, the power spectrum as the physical observable should be gauge invariant. Otherwise, it could take an arbitrary value if we choose an appropriate gauge fixing, e.g., the synchronous gauge [51], as reviewed in Ref. [50]. On the other hand, the gauge invariant synchronous variables can be reasonably defined for the induced gravitational waves at second order, though they are ill-defined for the second order scalar and vector perturbations. In this sense, the energy density spectrum of the second order gravitational waves could be well defined.

In this work, we will investigate the gauge invariance of the second order cosmological perturbations by following the Lie derivative method [5–7, 10, 11]. The gauge invariant variables of any order can be systematically constructed in such a method. We will study the gauge invariant Newtonian (i.e., Bardeen’s) and synchronous variables at first and second orders. In particular, for the first time, we will construct the so-called gauge invariant hybrid variables, i.e., Newtonian at first order while synchronous at second order, and vice versa. Further, we will find the conversion formulae among different families of gauge invariant variables. Finally, we will derive the gauge invariant equations of motion for the second order cosmological perturbations by adopting the scalar-vector-tensor decomposition [52]. To accomplish it, we will decompose the gauge invariant perturbed Einstein...
field equations into the scalar, vector and tensor components.

The remainder of the paper is arranged as follows. In Section 2, we introduce the gauge transformations and the gauge invariant variables following the Lie derivative method. In Section 3, we revisit the gauge invariant first order variables in such a method. In Section 4, we present explicit expressions of the gauge invariant second order variables and study the conversion formulæ among different families of gauge invariant variables. In Section 5, the equations of motion of the cosmological perturbations are presented in terms of the gauge invariant Newtonian, synchronous and hybrid variables. Finally, the conclusions and discussions are summarized in Section 6.

2 Gauge transformation and Gauge invariant variables

2.1 Gauge transformation of an arbitrary tensor

A gauge transformation for a perturbed quantity in space-time starts from an infinitesimal transformation of coordinate \( x^\mu \to \tilde{x}^\mu \). The expansion of \( \tilde{x}^\mu \) up to second order is given by

\[
\tilde{x}^\mu = x^\mu + \xi^{(1)\mu} + \frac{1}{2!} \xi^{(2)\mu} + \mathcal{O}(\xi^{(3)}),
\]

where \( \xi^{(1)\mu} \) and \( \xi^{(2)\mu} \) are the first and second order expansions of \( \tilde{x}^\mu \), respectively. For a generic tensor \( Q \), to second order, the infinitesimal transformations are shown to be [5]

\[
\begin{align*}
\hat{Q}^{(1)} &= Q^{(0)} + \mathcal{L}_{\xi_1} Q^{(0)}, \\
\hat{Q}^{(2)} &= Q^{(2)} + 2 \mathcal{L}_{\xi_1} Q^{(1)} + \left( \mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2 \right) Q^{(0)},
\end{align*}
\]

where we have let \( \xi_1^\mu \equiv \xi^{(1)\mu} \) and \( \xi_2^\mu \equiv \xi^{(2)\mu} - \xi^{(1)\mu} \partial_\nu \xi^{(1)\nu} \), \( Q^{(n)} \) denotes the \( n \)-th order perturbation of the tensor \( Q \), and \( \mathcal{L}_\xi \) is Lie derivative along the infinitesimal vector \( \xi \). The degree of freedom of the transformation is the same for the first and the second order perturbations. The \( \xi_1 \) determines the \( \hat{Q}^{(1)} \). Once the \( \xi_1 \) is fixed, thus the \( \xi_2 \) determines the \( \hat{Q}^{(2)} \). These formulæ are based on that the \( \xi_1 \) is set to be the same order of tensor perturbation \( Q^{(1)} \). It could simplify the formalism of perturbation theory. For a general infinitesimal transformation, there is no relevance between \( \xi_1 \) and \( Q^{(1)} \) on orders. In Appendix A, an introduction to the Lie derivative is shown. We briefly summarize the derivations of Eqs. (2) in Appendix B.

2.2 Gauge invariant metric perturbations

Based on Eqs. (2a) and (2b), the infinitesimal transformations of the first and second order metric perturbations are presented as

\[
\begin{align*}
\hat{g}^{(1)}_{\mu\nu} &= g^{(1)}_{\mu\nu} + \mathcal{L}_{\xi_1} g^{(0)}_{\mu\nu}, \\
\hat{g}^{(2)}_{\mu\nu} &= g^{(2)}_{\mu\nu} + 2 \mathcal{L}_{\xi_1} g^{(1)}_{\mu\nu} + \left( \mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2 \right) g^{(0)}_{\mu\nu},
\end{align*}
\]

If we introduce the gauge invariant metric perturbations via \( g^{(GI)}_{\mu\nu} \equiv g^{(i)}_{\mu\nu} - C^{(i)}_{\mu\nu} \) \( (i = 1, 2 \) for example), by making use of Eqs. (3a) and (3b), we could rewrite the counter terms \( C^{(i)}_{\mu\nu} \) in terms of the infinitesimal vectors \( X^\mu \) and \( Y^\mu \), namely,

\[
\begin{align*}
\hat{g}^{(GI,1)}_{\mu\nu} &= g^{(1)}_{\mu\nu} - \mathcal{L}_X g^{(0)}_{\mu\nu}, \\
\hat{g}^{(GI,2)}_{\mu\nu} &= g^{(2)}_{\mu\nu} - 2 \mathcal{L}_X g^{(1)}_{\mu\nu} - \left( \mathcal{L}_Y - \mathcal{L}_X^2 \right) g^{(0)}_{\mu\nu},
\end{align*}
\]

where

\[
\begin{align*}
\hat{X}^\mu &= X^\mu + \xi_1^\mu, \\
\hat{Y}^\mu &= Y^\mu + \xi_2^\mu + [\xi_1, X]^\mu.
\end{align*}
\]

We present the derivations of Eqs. (4a)–(5b) in Appendix C. These formulæ could also be found in Refs. [7, 53, 54]. Recently, they have been used in cosmology [7, 54] and in the post-Newtonian formalism [55].

Based on Eqs. (5a) and (5b), the infinitesimal vectors \( X^\mu \) and \( Y^\mu \) could be independent of the metric perturbations in principle. At first order, Bardeen has constructed the gauge invariant variables in terms of the metric perturbations [1]. Therefore, we limit our investigations to the case that both \( X^\mu \) and \( Y^\mu \) are expressed in terms of the metric perturbations in the following.

2.3 Gauge invariant perturbations of the energy-momentum tensor

In the aspect of matter perturbations, we adopt the infinitesimal transformation in Eqs. (2a) and (2b) to obtain the perturbed energy-momentum tensors, i.e.,

\[
\begin{align*}
\hat{T}^{(1)}_{\mu\nu} &= T^{(1)}_{\mu\nu} + \mathcal{L}_{\xi_1} T^{(0)}_{\mu\nu}, \\
\hat{T}^{(2)}_{\mu\nu} &= T^{(2)}_{\mu\nu} + 2 \mathcal{L}_{\xi_1} T^{(1)}_{\mu\nu} + \left( \mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2 \right) T^{(0)}_{\mu\nu}.
\end{align*}
\]

The gauge invariant perturbed energy-momentum tensors can be given by

\[
\begin{align*}
\hat{T}^{(GI,1)}_{\mu\nu} &= T^{(1)}_{\mu\nu} - \mathcal{L}_X T^{(0)}_{\mu\nu}, \\
\hat{T}^{(GI,2)}_{\mu\nu} &= T^{(2)}_{\mu\nu} - 2 \mathcal{L}_X T^{(1)}_{\mu\nu} - \left( \mathcal{L}_Y - \mathcal{L}_X^2 \right) T^{(0)}_{\mu\nu}.
\end{align*}
\]

In the above formulæ, it should be noticed that the energy-momentum tensors are not in a special status.
In general, any of the gauge invariant tensors could be formulated in the same form as Eqs. (7a) and (7b), i.e.,

\[
Q^{(G1,1)} = Q^{(1)} - L_X Q^{(0)}, \\
Q^{(G1,2)} = Q^{(2)} - 2L_X Q^{(1)} - (L_Y - L_X^2) Q^{(0)},
\]

(8a)

(8b)

2.4 Gauge invariant Einstein field equations

We have already known that the gauge invariant perturbed Einstein field equations have the same form as the conventional ones [8, 54]. This can be explicitly shown by expanding the Einstein field equations, \( G_{\mu\nu} = \kappa T_{\mu\nu} \), namely,

\[
0 = G_{\mu\nu} - \kappa T_{\mu\nu} \\
\approx G_{\mu\nu}^{(0)} - \kappa T_{\mu\nu}^{(0)} + G_{\mu\nu}^{(1)} - \kappa T_{\mu\nu}^{(1)} + \frac{1}{2} (G_{\mu\nu}^{(2)} - \kappa T_{\mu\nu}^{(2)}) \\
= G_{\mu\nu}^{(0)} - \kappa T_{\mu\nu}^{(0)} \\
+ (G_{\mu\nu}^{(G1,1)} - \kappa T_{\mu\nu}^{(G1,1)}) + L_X (G_{\mu\nu}^{(0)} - \kappa T_{\mu\nu}^{(0)}) \\
+ \frac{1}{2} (G_{\mu\nu}^{(G1,2)} - \kappa T_{\mu\nu}^{(G1,2)}) + 2L_X (G_{\mu\nu}^{(1)} - \kappa T_{\mu\nu}^{(1)}) \\
+ (L_Y - L_X^2) (G_{\mu\nu}^{(0)} - \kappa T_{\mu\nu}^{(0)}),
\]

(9)

where \( G_{\mu\nu} \) is Einstein tensor, \( \kappa \equiv 8\pi G \), and \( G \) is the gravitational constant. We could separate Eq. (9) order by order to obtain

\[
G_{\mu\nu}^{(0)} = \kappa T_{\mu\nu}^{(0)}, \\
G_{\mu\nu}^{(G1,1)} = \kappa T_{\mu\nu}^{(G1,1)}, \\
G_{\mu\nu}^{(G1,2)} = \kappa T_{\mu\nu}^{(G1,2)}.
\]

(10a)

(10b)

(10c)

The gauge invariant perturbed Einstein tensor \( G_{\mu\nu}^{(G1,n)} \) takes the same form as the conventional one \( G_{\mu\nu}^{(n)} \). It can be obtained by substituting the metric perturbations \( g_{\mu\nu}^{(n)} \) with the gauge invariant ones \( g_{\mu\nu}^{(G1,n)} \). The same situation is also true for the perturbed energy-momentum tensor \( T_{\mu\nu}^{(G1,n)} \).

In fact, the above conclusion is obvious. It can be understood as follows. As known, the perturbed Einstein tensor \( G_{\mu\nu}^{(n)} \) are defined with the metric perturbations \( g_{\mu\nu}^{(n)} \). Upon the gauge transformation, the transformed perturbed Einstein tensor \( \tilde{G}_{\mu\nu}^{(n)} \) should be written in terms of the transformed metric perturbations \( \tilde{g}_{\mu\nu}^{(n)} \). Based on this, we notice that \( g_{\mu\nu}^{(G1,n)} \) in Eqs. (4a) and (4b) formally take the same form as the gauge transformations, i.e., \( \tilde{g}_{\mu\nu}^{(G1,n)} \) in Eqs. (3a) and (3b), the gauge invariant perturbed Einstein tensor \( G_{\mu\nu}^{(G1,n)} \) should be defined with the gauge invariant metric perturbations \( g_{\mu\nu}^{(G1,n)} \).

3 Gauge invariant variables for the first order cosmological metric perturbations

The gauge invariant first order variables were first proposed by Bardeen [1]. As was known, Bardeen’s gauge invariant variables have the same form as the metric perturbations in Newtonian gauge. In this section, we will reproduce Bardeen’s formulae and further show that there are infinite families of gauge invariant variables that are allowed.

3.1 Gauge transformations of the first order metric perturbations

In flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time, the metric takes the form of

\[
g_{\mu\nu}^{(0)} dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + \delta_{ij} dx^i dx^j),
\]

(11)

where \( \eta \) and \( a(\eta) \) are the conformal time and the scale factor of the Universe, respectively. The spatial curvature of the space-time is zero. The metric perturbations of \( n \)-th order can take the form of

\[
\partial_i b^{(n)} + v^{(n)}_i = -2\partial_i \phi^{(n)} + 2\partial_i \delta_{ij} e^{(n)} + \partial_i c^{(n)} + \partial_i c^{(n)} + h^{(n)}_{ij},
\]

(12)

where \( \phi^{(n)} \), \( \psi^{(n)} \), \( b^{(n)} \), \( e^{(n)} \), \( c^{(n)} \) and \( h^{(n)}_{ij} \) are scalar perturbations, \( \nu^{(n)}_i \) and \( \epsilon^{(n)}_j \) are vector perturbations and \( h^{(n)}_{ij} \) are tensor perturbations. For tensor and vector perturbations, the transverse or traceless conditions should be satisfied as

\[
\begin{align*}
\partial_i \nu^{(n)} &= 0, \\
\partial_i e^{(n)} &= 0, \\
\delta^{ik} \partial_k h^{(n)}_{ij} &= 0, \\
\delta^{ij} h^{(n)}_{ij} &= 0.
\end{align*}
\]

(13a)

(13b)

(13c)

(13d)

These variables can be introduced via scalar-vector-
tensor decomposition, which is summarized in Appendix D.

Following Eq. (3a), the gauge transformations of the scalar, vector, and tensor perturbations are explicitly shown as

\[
\begin{align*}
\tilde{g}_{00}^{(1)} &= -2a^2 \tilde{\phi}^{(1)} \\
&= -2a^2 \phi^{(1)} - 2a^2 (\partial_0 + \frac{\dot{a}}{a}) \xi_1^0, \\
\tilde{g}_{0i}^{(1)} &= a^2 (\partial_i \tilde{b}^{(1)} + \tilde{\nu}_i^{(1)}) \\
&= a^2 (\partial_i b^{(1)} + \nu_i^{(1)}) + a^2 (\delta_{ij} \partial_j \xi_1^i - \partial_i \xi_1'^0), \\
\tilde{g}_{ij}^{(1)} &= a^2 (-2 \tilde{\psi}^{(1)} \delta_{ij} + 2 \partial_i \partial_j \tilde{c}^{(1)} + \partial_j \tilde{c}_i^{(1)} + \partial_j \tilde{c}_i^{(1)} + \tilde{h}_{ij}^{(1)}) \\
&= a^2 (-2 \psi^{(1)} \delta_{ij} + 2 \partial_i \partial_j \psi^{(1)} + \partial_j \psi_i^{(1)} + \partial_j \psi_i^{(1)} + \tilde{h}_{ij}^{(1)}) \\
&+ a^2 ((\delta_{ik} \partial_j + \delta_{jk} \partial_i) (\xi_1^k + \delta^{k}_{is} \partial_i \xi_1^s) \\
&+ \frac{2a}{a} \delta_{ij} \xi_1^0),
\end{align*}
\]

Here, the spatial part of $\xi_1^0$ has been decomposed as $\xi_1^0 = \xi_{1,T} + \delta^{ij} \partial_j \xi_1^i$, where we have the transverse part $\xi_{1,T}$ and the longitudinal part $\xi_1^l$. Using Eqs. (14a)–(14c) and the scalar-vector-tensor decomposition, we rewrite the gauge transformations of the variables of metric perturbations as follows

\[
\begin{align*}
\tilde{\phi}^{(1)} &= \phi^{(1)} + \left( \partial_0 + \frac{\dot{a}}{a} \right) \xi_1^0, \\
\tilde{b}^{(1)} &= b^{(1)} + \delta_0 \xi_{1,S} - \xi_1'^0, \\
\tilde{\nu}_i^{(1)} &= \nu_i^{(1)} + \delta_{ij} \delta_0 \xi_1^j, \\
\tilde{\psi}^{(1)} &= \psi^{(1)} - \frac{\dot{a}}{a} \xi_1^0, \\
\tilde{c}_i^{(1)} &= c_i^{(1)} + \delta_{ik} \xi_{1,T}, \\
\tilde{h}_{ij}^{(1)} &= h_{ij}^{(1)}.
\end{align*}
\]

In the following, we will introduce the gauge invariant variable of metric perturbations by making use of Eq. (15). Since the gauge invariant metric perturbations could take the same form as Newtonian (or synchronous) gauge, we call them the gauge invariant Newtonian (or synchronous) metric perturbations for the sake of presentations.

### 3.2 Gauge invariant first order Newtonian variables

Based on Eq. (4a), we could obtain the gauge invariant Newtonian metric perturbations, i.e.,

\[
\begin{align*}
g_{00}^{(GI,1)} &= -2a^2 \Phi^{(1)}, \\
g_{0i}^{(GI,1)} &= a^2 V_i^{(1)}, \\
g_{ij}^{(GI,1)} &= a^2 (-2 \Psi^{(1)} \delta_{ij} + H_{ij}^{(1)}),
\end{align*}
\]

where the gauge invariant variables are defined as

\[
\begin{align*}
\Phi^{(1)} &= \phi^{(1)} - (\partial_0 + \frac{\dot{a}}{a}) X^0, \\
\Psi^{(1)} &= \psi^{(1)} - \frac{\dot{a}}{a} X^0, \\
V_i^{(1)} &= \nu_i^{(1)} - \delta_{ij} \partial_j X^0, \\
H_{ij}^{(1)} &= h_{ij}^{(1)},
\end{align*}
\]

and we have decomposed $X^i = X_T^i + \delta^{ij} \partial_j X_S^i$, i.e., into the transverse part and the longitudinal one. As expected that $X^\mu$ is expressed in terms of the metric perturbations $g_{\mu\nu}^{(1)}$, we can derive its formula from Eq. (16b),

\[
\begin{align*}
a^2 V_i^{(1)} &= g_{0i}^{(1)} - \mathcal{L}_X g_{0i}^{(0)} \\
&= a^2 (\partial_i (b^{(1)} - \partial_0 X_S^i + X^0) \\
&+ (\delta^{ij} \partial_i - \delta_{ij} \partial_0 X_S^j)).
\end{align*}
\]

Since the vector perturbation is transverse, it leads to

\[
\dot{b}^{(1)} - \partial_0 X_S^i + X^0 = 0.
\]

By making use of Eq. (16c), namely,

\[
\begin{align*}
a^2 (-2 \Psi^{(1)} \delta_{ij} + H_{ij}^{(1)}) &= g_{ij}^{(1)} - \mathcal{L}_X g_{ij}^{(0)} \\
&= a^2 (-2 \delta_{ij} (\psi^{(1)} + \frac{\dot{a}}{a} X^0) + 2 \partial_i \partial_j (\epsilon^{(1)} - X_S) \\
&+ (\delta_{ij} \partial_i + \delta_{ij} \partial_j) (\epsilon_{ik}^{(1)} - \delta_{ik} X_T^k) + h_{ij}^{(1)}),
\end{align*}
\]

and the scalar-vector-tensor decomposition, we obtain

\[
\begin{align*}
\epsilon^{(1)} - X_S &= 0, \\
\epsilon_{ik}^{(1)} - \delta_{ik} X_T^k &= 0.
\end{align*}
\]

Based on Eqs. (19), (21a) and (21b), $X^\mu$ can be given by

\[
\begin{align*}
X^0 &= \partial_0 e^{(1)} - b^{(1)}, \\
X^i &= \delta^{ik} (\epsilon_{ik}^{(1)} + \partial_0 e^{(1)}).
\end{align*}
\]

Therefore, we rewrite the gauge invariant variables as

\[
\begin{align*}
\Phi^{(1)} &= \phi^{(1)} - \frac{1}{a} \partial_0 (a (\partial_0 e^{(1)} - b^{(1)})), \\
\Psi^{(1)} &= \psi^{(1)} - \frac{\dot{a}}{a} (\partial_0 e^{(1)} - b^{(1)}), \\
V_i^{(1)} &= \nu_i^{(1)} - \partial_0 c_i^{(1)}, \\
H_{ij}^{(1)} &= h_{ij}^{(1)}.
\end{align*}
\]

This implies that the gauge invariant variables proposed by Bardeen [1] can be reproduced by making use of Lie derivative method [54].
3.3 Gauge invariant first order synchronous variables

Based on Eq. (4a), the gauge invariant synchronous metric perturbations take the form of

\[
\begin{align*}
g^{(GL,1)}_{00} & = 0, \quad (24a) \\
g^{(GL,1)}_{0i} & = 0, \quad (24b) \\
g^{(GL,1)}_{ij} & = a^2 (-2\Psi^{(1)} \delta_{ij} + 2\partial_i \partial_j E^{(1)}) \\
 & \quad + \partial_i C^{(1)}_j + \partial_j C^{(1)}_i + H^{(1)}_{ij}, \quad (24c)
\end{align*}
\]

where the gauge invariant variables are defined as

\[
\begin{align*}
\Psi^{(1)} & = \psi^{(1)} + \hat{\alpha} X^a, \quad (25a) \\
E^{(1)} & = e^{(1)} - X_S, \quad (25b) \\
C^{(1)}_i & = c^{(1)}_i - \delta_{ik} X^k, \quad (25c) \\
H^{(1)}_{ij} & = h^{(1)}_{ij}. \quad (25d)
\end{align*}
\]

In this case, \(X^\mu\) can be determined by the form \(g^{(GL,1)}_{0a} = 0\) and the scalar-vector-tensor decomposition. The result is given by

\[
\begin{align*}
X^0 & = \frac{1}{a} \int d\eta \{a\phi^{(1)}\}, \quad (26a) \\
X^j & = \delta^{ji} \int d\eta \{\phi^{(1)} + \partial_i b^{(1)} + \frac{1}{a} \int d\eta' \{a\partial_i \phi^{(1)}\}\}. \quad (26b)
\end{align*}
\]

Then we rewrite the gauge invariant variables in the form of

\[
\begin{align*}
\Psi^{(1)} & = \psi^{(1)} + \hat{\alpha} \int d\eta \{a\phi^{(1)}\}, \quad (27a) \\
E^{(1)} & = e^{(1)} + \int d\eta \{b^{(1)} + \frac{1}{a} \int d\eta' \{a\phi^{(1)}\}\}, \quad (27b) \\
C^{(1)}_i & = c^{(1)}_i - \int \nu^{(1)} d\eta, \quad (27c) \\
H^{(1)}_{ij} & = h^{(1)}_{ij}. \quad (27d)
\end{align*}
\]

The above approach has been used to study the scalar cosmological perturbations in Ref. [56]. As suggested in Ref. [48], the gauge invariant synchronous variables are not unique due to the indefinite integral in Eq. (26). In this sense, it might be difficult to define observables with the gauge invariant synchronous variables in Eqs. (27a)–(27d).

3.4 Conversion among different families of
gauge invariant first order variables

In general, the gauge invariant variables are not limited to the forms in Eqs. (16a)–(16c) and (24a)–(24c). There are other families of gauge invariant variables (see reviews in Refs. [11, 57]). For two different families of the gauge invariant first order variables of metric perturbations \(g^{(GL,A,1)}_{\mu\nu}\) and \(g^{(GL,B,1)}_{\mu\nu}\), the conversion between them can be derived from

\[
\begin{align*}
g^{(GL,A,1)}_{\mu\nu} - g^{(GL,B,1)}_{\mu\nu} & = (g^{(1)}_{\mu\nu} - \mathcal{L}_X B^{(1)}) - (g^{(1)}_{\mu\nu} - \mathcal{L}_X A^{(1)}) \\
& = \mathcal{L}(X^A - X^B) g_\mu^{(0)} g^{(0)}_{\nu\nu}. \quad (28)
\end{align*}
\]

If we let \(Z^{AB}_1 \equiv X^A - X^B\), Eq. (28) can be rewritten as

\[
g^{(GL,A,1)}_{\mu\nu} = g^{(GL,B,1)}_{\mu\nu} - \mathcal{L}_{Z^{AB}_1} g_\mu^{(0)} g^{(0)}_{\nu\nu}. \quad (29)
\]

The variable \(Z^{AB}_1\) was mentioned by Nakamura [54]. It relates two different families of gauge invariant first order variables of metric perturbations. In this work, we further show that the infinitesimal vector \(Z_1\) can be expressed as a linear combination of the gauge invariant variables \(A^{(1)}, B^{(1)}\) and \(C^{(1)}\), namely,

\[
Z^{AB}_1 = \hat{N}_1 A^{(1)} + \hat{N}_2 B^{(1)} + \hat{N}_3 C^{(1)}; \quad (30)
\]

where \(\hat{N}_1, \hat{N}_2\) and \(\hat{N}_3\) are arbitrary linear operators that are irrelative to any perturbations, and we have the gauge invariant variables,

\[
\begin{align*}
A^{(1)} & \equiv \partial_\mu (\frac{a^2}{\hat{\alpha}} \psi^{(1)}) + a \phi^{(1)}, \quad (31a) \\
B^{(1)} & \equiv \partial_\mu e^{(1)} - b^{(1)} + \frac{a}{\hat{\alpha}} \psi^{(1)}, \quad (31b) \\
C^{(1)} & \equiv \nu^{(1)} - \partial_\mu c^{(1)}. \quad (31c)
\end{align*}
\]

The above expressions of \(A^{(1)}, B^{(1)}\) and \(C^{(1)}\) can be obtained by making use of Eq. (15).

The existence of infinite families of gauge invariant variables can also be indicated by the infinite number of choices of \(X^\mu\). To be specific, we can extend the expression of Eq. (22) to be

\[
\begin{align*}
X^0 & = \partial_\mu (\frac{a^2}{\hat{\alpha}} \psi^{(1)}) - b^{(1)} + Z^0, \quad (32a) \\
X^j & = \delta^{jk} (c^{(1)}_k + \partial_k e^{(1)}) + Z^j. \quad (32b)
\end{align*}
\]

In this way, the gauge invariant metric perturbations could take a general form of

\[
\begin{align*}
g^{(GL,1)}_{00} & = -2a^2 \Phi^{(1)}, \quad (33a) \\
g^{(GL,1)}_{0i} & = a^2 (\partial_i B^{(1)} + V^{(1)}_i), \quad (33b) \\
g^{(GL,1)}_{ij} & = a^2 (-2\Psi^{(1)} \delta_{ij} + 2\partial_i \partial_j E^{(1)}) \\
& \quad + \partial_i C^{(1)}_j + \partial_j C^{(1)}_i + H^{(1)}_{ij}. \quad (33c)
\end{align*}
\]
where the gauge invariant variables are defined as

\[
\Phi^{(1)} = \phi^{(1)} - \frac{1}{a} \partial_0 (a(\partial_0 e^{(1)} - b^{(1)} + Z_1^{(1)})],
\]

\[
B^{(1)} = Z_0^0 - \partial_0 \Delta^{-1} \partial_j Z_1^{(1)} \tag{34b}
\]

\[
\Psi^{(1)} = \psi^{(1)} + \frac{\dot{a}}{a} (\partial_b c^{(1)} - h^{(1)} + Z^0),
\]

\[
E^{(1)} = \Delta^{-1} \partial_j Z_1^{(1)},
\]

\[
V_i^{(1)} = \nu_i^{(1)} - \partial_0 c_i^{(1)} - (\delta_{ik} - \partial_i \Delta^{-1} \partial_k) \partial_0 Z_1^{(1)},
\]

\[
C_i^{(1)} = - (\delta_{ik} - \partial_i \Delta^{-1} \partial_k) Z_1^{(1)},
\]

\[
H_{ij}^{(1)} = h_{ij},
\]

and \(\Delta^{-1}\) is the inverse Laplacian operator on the background.

As an example, we consider that \(g_{\mu \nu}^{(0) GL, A, I}\) and \(g_{\mu \nu}^{(0) GL, B, I}\) are synchronous in Eqs. (27a)–(27d) and Newtonian in Eqs. (23a)–(23d) variables, respectively. In this case, we obtain the explicit expression of \(Z_1^{j, AB}\) as follows

\[
Z_1^{j, AB} = \frac{1}{a} \int A^{(1)} d\eta - B^{(1)},
\]

\[
Z_1^{AB} = \delta^{ik} \partial_k \int d\eta \{ \frac{1}{a} \int A^{(1)} d\eta' \}
- \delta^{ik} \partial_k \int B^{(1)} d\eta + \delta^{ij} \int c_i^{(1)} d\eta. \tag{35b}
\]

Therefore, these two families of gauge invariant variables of metric perturbations are related via the expressions of the linear operators \(N^i_a\), \(N^i_\alpha\) and \(N^i_\nu\).

4 Gauge invariant variables for the second order cosmological metric perturbations

In this section, the gauge invariant second order variables in cosmology will be derived in the framework of Lie derivative method and scalar-vector-tensor decomposition. Previous similar studies can be found in Ref. [11, 54]. Further, we will show the conversion formulae among different families of gauge invariant variables. For illustrations, we will consider the gauge invariant metric perturbations that are Newtonian and synchronous, respectively.

4.1 Gauge transformations of the second order metric perturbations

Based on Eq. (3b), the gauge transformation of the second order metric perturbations is presented explicitly as follows

\[
\begin{align*}
\tilde{g}_{00}^{(2)} &= -2a^2 \tilde{\delta}^{(2)} \\
&= -2a^2 \delta^{(2)} + 2\mathcal{L}_{\xi_1}g_{00}^{(1)} + (\mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1})g_{00}^{(0)} ,
\end{align*}
\]

\[
\begin{align*}
\tilde{g}_{0i}^{(2)} &= a^2 (\partial_0 B^{(2)} + \dot{\nu}_i^{(2)}) \\
&= g_{0i}^{(2)} + 2\mathcal{L}_{\xi_1}g_{0i}^{(1)} + (\mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1})g_{0i}^{(0)} \\
&= a^2 (\partial_0 (b^{(2)} + \Delta^{-1} \partial^i \Xi_0) + \nu_i^{(2)} + (\delta_i - \partial_i \Delta^{-1} \partial^i) \Xi_0) \\
&= a^2 (\partial_0 (b^{(2)} + \Delta^{-1} \partial^i \Xi_0) + \nu_i^{(2)} + T_i \Xi_0) ,
\end{align*}
\]

\[
\begin{align*}
\tilde{g}_{ij}^{(2)} &= a^2 (-2\delta_{ij} \tilde{\psi}^{(2)} + 2\partial_0 \partial_j \tilde{e}^{(2)} + \partial_i \tilde{c}_j^{(2)} + \partial_j \tilde{c}_i^{(2)} + h_{ij}^{(2)}) \\
&= a^2 (\partial_i \psi^{(2)} + 2\partial_j \partial_0 e^{(2)} + \partial_i c_j^{(2)} + \partial_j c_i^{(2)} + h_{ij}^{(2)}) \\
&\quad + \frac{1}{2} \delta_{ij} (\delta^k \Delta^{-1} \partial^l \Xi_{kl}) \\
&\quad + \partial_0 \partial_i \Delta^{-1} \partial^j (\delta^k - \partial_k \Delta^{-1} \partial^k) \Xi_{kl} \\
&\quad + (\delta_i - \partial_i \Delta^{-1} \partial^k) (\delta_j - \partial_j \Delta^{-1} \partial^j) \Xi_{kl} \\
&\quad + (\delta_i - \partial_i \Delta^{-1} \partial^k) (\delta_j - \partial_j \Delta^{-1} \partial^j) - \frac{1}{2} (\delta_{ij} - \partial_i \Delta^{-1} \partial_j) \Xi_{kl} \\
&= a^2 (\partial_i \psi^{(2)} + 2\partial_j \partial_0 e^{(2)} + \partial_i c_j^{(2)} + \partial_j c_i^{(2)} + h_{ij}^{(2)}) \\
&\quad + \frac{1}{2} \delta_{ij} T_i \Xi_{kl} + \partial_0 \partial_i \Delta^{-1} (\partial^k \Delta^{-1} \partial^l - \frac{1}{2} T_{kl}) \Xi_{kl} \\
&\quad + \partial_0 \partial_i \Delta^{-1} \partial^l T_i \Xi_{kl} + \partial_0 \partial_i \Delta^{-1} \partial^k T_i \Xi_{kl} + (T_i T_j - \frac{1}{2} T_{ij} T_{kl}) \Xi_{kl} ,
\end{align*}
\]
where $T^i_l \equiv \delta^i_j - \partial_j \Delta^{-1} \partial^i$ is a transverse operator, and we define
\[ \Xi_{\mu \nu} = \frac{2 \mathcal{L}_\xi g^{(1)}_{\mu \nu} - (\mathcal{L}_\zeta + \mathcal{L}_\xi^2) g_{\mu \nu}^{(0)}}{a^2}. \] (37)

In Eqs. (36b) and (36c), we have decomposed the gauge transformation into the scalar, vector and tensor components. Therefore, the gauge transformation of each component can be rewritten as
\[ \tilde{\phi}^{(2)} = \phi^{(2)} - \frac{1}{2} \Xi_{00}, \] (38a)
\[ \tilde{b}^{(2)} = b^{(2)} + \Delta^{-1} \partial^j \Xi_{0j}, \] (38b)
\[ \tilde{\nu}_i^{(2)} = \nu_i^{(2)} + T_i^j \Xi_{0j}, \] (38c)
\[ \tilde{\psi}^{(2)} = \psi^{(2)} - \frac{1}{4} T^{kl} \Xi_{kl}, \] (38d)
\[ \tilde{c}^{(2)} = c^{(2)} + \Delta^{-1} \partial^j T_j^l \Xi_{kl}, \] (38e)
\[ \tilde{h}_{ij}^{(2)} = h_{ij}^{(2)} + \left( T_i^k T_j^l - \frac{1}{2} T_{ij} T^{kl} \right) \Xi_{kl}. \] (38f)

In particular, the second order tensor perturbation is no longer invariant upon the gauge transformation. Based on Eqs. (3b), (4b) and (38), we obtain the gauge invariant second order variables as follows
\[ \Phi^{(2)} = \phi^{(2)} - \left( \partial_0 + \frac{\dot{a}}{a} \right) Y^0 + \mathcal{X}_{00}, \] (39a)
\[ B^{(2)} = b^{(2)} - \partial_0 Y_S + Y^0 - \Delta^{-1} \partial^i \mathcal{X}_{0i}, \] (39b)
\[ V_i^{(2)} = \nu_i^{(2)} - \delta_{il} \partial_0 Y^l - \Delta^{-1} \partial^l \mathcal{X}_{0l}, \] (39c)
\[ \Psi^{(2)} = \psi^{(2)} + \frac{\dot{a}}{a} Y^0 + \frac{1}{4} T^{kl} \mathcal{X}_{kl}, \] (39d)
\[ E^{(2)} = e^{(2)} - \partial_0 \mathcal{X}_S - \frac{1}{2} \Delta^{-1} \left( \frac{\partial^k \Delta^{-1} \partial^i - \frac{1}{2} T_{kl} \right) \mathcal{X}_{i(l)} \right), \] (39e)
\[ C_j^{(2)} = c_j^{(2)} - \delta_{jk} \partial_0 Y^k - \Delta^{-1} \partial^k \mathcal{X}_{kl}, \] (39f)
\[ H_{ij}^{(2)} = h_{ij}^{(2)} + \left( T_i^k T_j^l - \frac{1}{2} T_{ij} T^{kl} \right) \mathcal{X}_{kl}. \] (39g)

where we define
\[ \mathcal{X}_{\mu \nu} = \frac{2 \mathcal{L}_\chi g^{(1)}_{\mu \nu} - \mathcal{L}_\chi^2 g_{\mu \nu}^{(0)}}{a^2}. \] (40)

and we have decomposed $Y^i =: Y^i_S + \delta^i_0 \partial_0 Y_S$, i.e., into the transverse part and the longitudinal one. We find that $\mathcal{X}_{\mu \nu}$ depends on $X^\mu$ which determines the gauge invariant first order variables. In Appendix E, we will present an explicit expression of $\mathcal{X}_{\mu \nu}$. As shown in Eq. (39), all the gauge invariant variables are expressed in terms of $X^\mu$ and $Y^\mu$, except that the gauge invariant second order tensor perturbation $H_{ij}^{(2)}$ only depends on $X^\mu$. As the explicit expression of $X^\mu$ has been known, only $Y^\mu$ is undetermined in Eq. (39). Therefore, we will show how to express $Y^\mu$ in terms of the first and second order metric perturbations in the following.

### 4.2 Gauge invariant second order Newtonian variables

To obtain the gauge invariant Newtonian variables, we derive the expression of $Y^\mu$ from $B^{(2)} = E^{(2)} = C_j^{(2)} = 0$. The explicit expression of $Y^\mu$ is given by
\[ Y^0 = \partial_0 e^{(2)} - b^{(2)} + \Delta^{-1} \partial^i \mathcal{X}_{0i} - \frac{1}{2} (\partial^k \Delta^{-2} \partial^j \partial_0 \mathcal{X}_{kl} \mathcal{X}_{kl}, \] (41a)
\[ Y^i = \delta^i_0 (e^{(2)} + c_j^{(2)}) + \left( \frac{1}{4} \partial^k \partial^0 \Delta^{-2} \partial^j \right) \mathcal{X}_{kl} + \frac{1}{4} \delta^k \partial^0 \Delta^{-1} - \delta^i_0 \Delta^{-1} \partial^k \partial^l \mathcal{X}_{kl}. \] (41b)

We find that $Y^\mu$ depends on a choice of the first order variable $X^\mu$. Therefore, the gauge invariant second order metric perturbations turn to be
\[ g^{(GI, 2)}_{00} = -2a^2 \Phi^{(2)}, \] (42a)
\[ g^{(GI, 2)}_0 = a^2 V_i^{(2)}, \] (42b)
\[ g^{(GI, 2)}_{ij} = -2a^2 \delta_{ij} \Psi^{(2)} + a^2 H_{ij}^{(2)}, \] (42c)

where the gauge invariant variables are defined as
\[ \Phi^{(2)} = \phi^{(2)} - \left( \frac{\dot{a}}{a} + \partial_0 \right) (b^{(2)} - b^{(2)}) + X_{00}, \] (43a)
\[ \Psi^{(2)} = \psi^{(2)} - \frac{\dot{a}}{a} (b^{(2)} - b^{(2)}) \] (43a)
\[ \mathcal{X}_S + 3 \Delta^{-2} \partial^0 \partial^j \mathcal{X}_{kl} + \delta^0 \Delta^{-1} \mathcal{X}_{kl}, \] (43b)
\[ V_i^{(2)} = \nu_i^{(2)} - \partial_0 c_j^{(2)} + \Delta^{-1} \partial^k \mathcal{X}_{kl}, \] (43c)
\[ H_{ij}^{(2)} = h_{ij}^{(2)} - \left( T_i^k T_j^l - \frac{1}{2} T_{ij} T^{kl} \right) \mathcal{X}_{kl}. \] (43d)

### 4.3 Gauge invariant second order synchronous variables

For gauge invariant synchronous variables, the $Y^\mu$ is determined by making use of $\Phi^{(2)} = 0$, $B^{(2)} = 0$ and $V_i^{(2)} = 0$. The explicit expression of the $Y^\mu$ takes the
from the gauge invariant Newtonian variables, except the 
The gauge invariant synchronous variables are different 
first order variables. To be specific, for two different fam-
cinated, since the gauge invariant second order variables 
gauge invariant second order variables is more compli-
car perturbations in the form of
form of
\[
Y^0 = \frac{1}{a} \int \text{d}\eta \{ a\phi(2) + \frac{1}{2} a\chi_{00} \}, \quad (44a)
\]
\[
Y^k = \delta^k_l \int \text{d}\eta \{ \nu_l^{(2)} + \partial_l \phi^{(2)} + \frac{1}{a} \int \text{d}\eta \{ a\phi^{(2)} \} \}
- \int \text{d}\eta \{ \delta^k_0 \chi_{0j} - \frac{1}{2a} \int \text{d}\eta \{ a\phi^k \chi_{00} \} \}. \quad (44b)
\]

Therefore, we have the gauge invariant second order met-
ic perturbations in the form of
\[
g^{(GI,2)}_{00} = 0, \quad (45a)
\]
\[
g^{(GI,2)}_{0i} = 0, \quad (45b)
\]
\[
g^{(GI,2)}_{ij} = a^2 (-2\eta^{(2)} \delta_{ij} + 2\partial_i \partial_j \eta^{(2)})
+ \partial_i C_j^{(2)} + \partial_j C_i^{(2)} + H^{(2)}_{ij}, \quad (45c)
\]
where the gauge invariant variables are defined with
\[
\Psi^{(2)} = \psi^{(2)} + \frac{\dot{a}}{a^2} \int \text{d}\eta \{ a\phi^{(2)} + \frac{1}{2} a\chi_{00} \} + T^{(1)} \chi_{0i}, \quad (46a)
\]
\[
E^{(2)} = \epsilon^{(2)} - \int \text{d}\eta \{ b^{(2)} + \frac{1}{a} \int \text{d}\eta \{ a\phi^{(2)} \} \}
+ \int \text{d}\eta \{ \Delta^{-\frac{1}{2}} \partial^i \chi_{0i} - \frac{1}{2a} \int \text{d}\eta \{ a\chi_{00} \} \}
- \frac{1}{4} (3\Delta^{-2} \partial^k \partial^i - \delta^{ki} \Delta^{-1}) \chi_{kl}, \quad (46b)
\]
\[
C_j^{(2)} = c_j^{(2)} - \nu_j^{(2)} \text{d}\eta + T_j^{(1)} \chi_{0i} \text{d}\eta
- \Delta^{-\frac{1}{2}} \partial^j T_i^{(1)} \chi_{kl}, \quad (46c)
\]
\[
H^{(2)}_{ij} = h^{(2)}_{ij} - (T_i^{(1)} T_j^{(1)} - \frac{1}{2} T_{ij} T^{(1)}) \chi_{kl}. \quad (46d)
\]
The gauge invariant synchronous variables are different 
from the gauge invariant Newtonian variables, except the 
tensor perturbations. In both of the two cases, \( H^{(2)} \) is 
completely determined by the choice of the gauge invariant 
first order variables.

4.4 Conversion among different families of 
gauge invariant second order variables

Compared with the first order case in the previous 
section, a conversion between two different families of 
gauge invariant second order variables is more complic-
cated, since the gauge invariant second order variables 
are also dependent of the choice of the gauge invariant 
first order variables. To be specific, for two different fam-
ilies of gauge invariant second order metric perturbations 
\( g^{(GL,1,2)}_{\mu\nu} \) and \( g^{(GI,B,2)}_{\mu\nu} \), the conversion between them can 
be derived from
\[
g^{(GL,A,2)}_{\mu\nu} - g^{(GI,B,2)}_{\mu\nu} = \begin{cases} 
\frac{2}{\Delta} (\mathcal{L}^{A} - \mathcal{L}^{B} - \mathcal{L}^{C}) g^{(0)}_{\mu\nu}, \\
2\mathcal{L}^{(X_{A} - X_{B})} g^{(GI,B,1)}_{\mu\nu} + \mathcal{L}^{X_{A} \mu\nu}, \\
(L_{Y_{A} - Y_{B}} - \mathcal{L}^{X_{A}}) g^{(0)}_{\mu\nu}.
\end{cases}
\]

\( \text{Eq. (47)} \)

This conversion was also mentioned by Nakamura \[ 54 \] in 
a different formula. One can easily check that both \( Z^{A}_{AB} \) 
and \( Z^{A}_{AB} \) are gauge invariant by making use of Eqs. \((5a)\) 
and \((5b)\). The above formula is generic. It is obvious 
that the gauge invariant second order variables depend 
on the gauge invariant first order variables, since in gen-
eral we have \( Z^{A}_{AB} \neq 0 \), namely, two different families of 
gauge invariant first order variables. This generic case 
will be studied later in this section. When we take the 
same family of the gauge invariant variables at first or-
der, i.e., \( Z^{A}_{1AB} = 0 \), Eq. \((48)\) can be reduced to a simpler 
form \( g^{(GL,A,2)}_{\mu\nu} = g^{(GI,B,2)}_{\mu\nu} - \mathcal{L}^{X_{A} - X_{B}} g^{(0)}_{\mu\nu} \), where \( Z^{A}_{2AB} = Y^{A} - Y^{B} \). 
For this simple case, the formula is as similar as that of 
Eq. \((29)\).

We also have infinite families of gauge invariant second 
order variables. Similar to \( Z^{A}_{1} \) in Eq. \((30)\), the 
infinite vector \( Z^{A}_{2AB} \) can also be expressed as a linear 
combination of the gauge invariant second order variables 
\( A^{(2)} \), \( B^{(2)} \), \( C^{(2)} \) and \( D^{(2)} \):
\[
Z^{2}_{AB} = \begin{cases} 
\tilde{M}^{0}_{A} A^{(2)} + \tilde{M}^{0}_{B} B^{(2)} + \tilde{M}^{0}_{C} C^{(2)} \\
\tilde{M}^{0}_{D} D^{(2)}
\end{cases}, \quad (49a)
\]
\[
Z^{2}_{AB} = \begin{cases} 
\tilde{M}^{0}_{A} A^{(2)} + \tilde{M}^{0}_{B} B^{(2)} + \tilde{M}^{0}_{C} C^{(2)} \\
\tilde{M}^{0}_{D} D^{(2)}
\end{cases}, \quad (49b)
\]
where \( \tilde{M}^{0}_{A} \), \( \tilde{M}^{0}_{B} \), \( \tilde{M}^{0}_{C} \) and \( \tilde{M}^{0}_{D} \) are four arbitrary linear 
operators that are relative to any perturbations, and
the gauge-invariant variables are defined as
\[ A^{(2)} = \partial_\theta \left( \frac{a^2}{a} \psi^{(2)} \right) + a \partial_\theta \left( \frac{a^2}{a} \mathcal{T}^{k l} X_{k l} \right) + \frac{1}{2} \theta \delta_{00}, \]  
(50a)
\[ B^{(2)} = \partial_\theta e^{(2)} - b^{(2)} + \frac{a}{a} \psi^{(2)} + \Delta^{-1} \partial^j X_{0j}, \]  
(50b)
\[ C^{(2)} = \nu^{(2)} - \partial_\theta \nu^{(2)} + \Delta^{-1} \partial^j T_{j}^k \partial_\theta X_{k l} - T_{k l}^k X_{0l}, \]  
(50c)
\[ D^{(2)} = \frac{1}{a^2} \left( 2 \mathcal{L}_{\mu \nu} \mathcal{G}_{\mu \nu} - C_{\mu \nu}^{(2)} \right). \]  
(50d)

Here, we express \( A^{(2)}, B^{(2)} \), and \( C^{(2)} \) in terms of \( X_{\mu \nu} \), which is completely determined by \( X^B \). Based on Eqs. (49a) and (49b), we can also obtain a new family of the gauge invariant second order variables by a conversion from a given family of the gauge invariant second order variables. This prediction is similar to that of the first order case.

We consider three typical cases in the following. In the case that \( g_{\mu \nu}^{(G, A, n)} \) and \( g_{\mu \nu}^{(G, B, n)} \) \( (n = 1, 2) \) are synchronous and Newtonian, respectively, we find that \( Z_i^{(2), AB} \) takes the form of Eqs. (35a) and (35b), and \( Z_i^{(2), AB} \) is shown to be
\[ Z_i^{0, AB} = \frac{1}{a} \int A^{(2)} d\eta - B^{(2)} + \frac{1}{2a} \int d\eta \left( a D^{(2)} \right), \]  
(51a)
\[ Z_i^{2, AB} = \delta^{ik} \partial_k \int d\eta \left( \frac{1}{a} \int A^{(2)} d\eta' \right) - \delta^{ik} \partial_k \left( B^{(2)} \right) + \delta^{ik} \int C^{(2)} d\eta \]  
\[- \int d\eta \left( \delta^{ik} D^{(2)} \right) - \frac{1}{2a} \int d\eta \left( a \partial^j D^{(2)} \right) \]  
(51b)
Since the vectors \( X^\mu \) and \( Y^\mu \) are independent, we can choose, e.g., the gauge invariant Newtonian variables at the first order and the gauge invariant synchronous variables at the second order, and vice versa. First, in the case that \( g_{\mu \nu}^{(G, A, 1)} \) and \( g_{\mu \nu}^{(G, B, 1)} \) are synchronous and Newtonian, respectively, we obtain \( Z_i^{0, AB} = 0 \) and
\[ Z_i^{0, AB} = \frac{1}{a} \int A^{(2)} d\eta - B^{(2)}, \]  
(52a)
\[ Z_i^{2, AB} = \delta^{ik} \partial_k \int d\eta \left( \frac{1}{a} \int A^{(2)} d\eta' \right) - \delta^{ik} \partial_k \left( B^{(2)} \right) + \delta^{ik} \int C^{(2)} d\eta. \]  
(52b)
Second, in the case that \( g_{\mu \nu}^{(G, A, 1)} \) and \( g_{\mu \nu}^{(G, B, 1)} \) are synchronous and Newtonian, respectively, while both \( g_{\mu \nu}^{(G, A, 2)} \) and \( g_{\mu \nu}^{(G, B, 2)} \) are Newtonian, we find that \( Z_i^{(2), AB} \) takes the same form as Eqs. (35a) and (35b), and \( Z_i^{(2), AB} \) turns to be
\[ Z_i^{0, AB} = \Delta^{-1} \partial^j D^{(2)}_{0j}, \]  
(53a)
\[ Z_i^{2, AB} = \left( \frac{1}{4} \partial^j \partial^k \Delta^{-1} \partial^l \partial_\theta - \delta^{kl} \Delta^{-1} \partial^j \right) D^{(2)}_{kl}, \]  
(53b)
which is expressed in terms of the square of the first order metric perturbations only. The above two cases are called the gauge invariant hybrid variables.

5 Gauge invariant equations of motion for the second order cosmological perturbations

In this section, we will derive the equations of motion of the second order cosmological perturbations, which are sourced from the first order scalar perturbations in the gauge invariant framework. For simplicity, we will consider the gauge invariant Newtonian, synchronous, and hybrid variables that have been introduced in the previous sections.

5.1 Gauge invariant energy-momentum tensor up to second order

On the side of matter, we expand the energy-momentum tensor of the perfect fluid up to second order, i.e.,
\[ T_{\mu \nu}^{(G)} = T_{\mu \nu}^{(0)} + T_{\mu \nu}^{(G, 1)} + \frac{1}{2} T_{\mu \nu}^{(G, 2)} + \mathcal{O}(T_{\mu \nu}^{(G, 3)}), \]  
(54)
where
\[ T^{(0)}_{\mu\nu} = u^{(0)}_{\mu} u^{(0)}_{\nu} (\rho^{(0)} + P^{(0)}) + \delta^{(0)}_{\mu\nu} P^{(0)}, \]
\[ T^{(G,1)}_{\mu\nu} = u^{(G,1)}_{\mu} u^{(G,1)}_{\nu} (\rho^{(0)} + P^{(0)}) + u^{(0)}_{\mu} u^{(G,1)}_{\nu} (\rho^{(0)} + P^{(0)}) + u^{(0)}_{\mu} u^{(0)}_{\nu} (\rho^{(G,1)} + P^{(G,1)}) + g^{(G,1)}_{\mu\nu} P^{(G,1)}, \]
\[ T^{(G,2)}_{\mu\nu} = u^{(G,1)}_{\mu} u^{(G,1)}_{\nu} (\rho^{(0)} + P^{(0)}) + u^{(G,1)}_{\mu} u^{(G,1)}_{\nu} (\rho^{(G,1)} + P^{(G,1)}) + u^{(G,1)}_{\mu} u^{(G,1)}_{\nu} (\rho^{(G,1)} + P^{(G,1)}) + u^{(0)}_{\mu} u^{(G,1)}_{\nu} (\rho^{(G,1)} + P^{(G,1)}) + g^{(G,1)}_{\mu\nu} P^{(G,1)} + g^{(G,2)}_{\mu\nu} P^{(G,2)}. \]

Here, \( \rho^{(0)}, P^{(0)} \) denote the background density and pressure, respectively. The \( \rho^{(G,1)}, u^{(G,1)}_\mu \) and \( P^{(G,1)} \) denote the gauge invariant \( n \)-th order density, pressure and velocity perturbations, respectively. As has been suggested in Eqs. (8a) and (8b), the gauge invariant metric perturbations are formulated as
\[ \rho^{(G,1)} = \rho^{(1)} - \mathcal{L}_X \rho^{(0)}, \]
\[ P^{(G,1)} = P^{(1)} - \mathcal{L}_X P^{(0)}, \]
\[ u^{(G,1)}_{\mu} = u^{(1)}_{\mu} - \mathcal{L}_X u^{(0)}_{\mu}, \]
\[ \rho^{(G,2)} = \rho^{(2)} - 2 \mathcal{L}_X \rho^{(1)} - (\mathcal{L}_Y - \mathcal{L}_X^2) P^{(0)}, \]
\[ P^{(G,2)} = P^{(2)} - 2 \mathcal{L}_X P^{(1)} - (\mathcal{L}_Y - \mathcal{L}_X^2) P^{(0)}, \]
\[ u^{(G,2)}_{\mu} = u^{(2)}_{\mu} - 2 \mathcal{L}_X u^{(1)}_{\mu} - (\mathcal{L}_Y - \mathcal{L}_X^2) u^{(0)}_{\mu}. \]

The velocity field of the perfect fluid can be redefined as \( (u^{(1)}(G,1)) \equiv a (u^{(1)}(G,1)) \), where \( (u^{(0)})(G,1) \) could be determined via \( g_{\mu\nu} u^\mu u^\nu = -1 \). The equation of state \( w \) and the speed of sound \( c_s \) (and \( c_s^{(2)} \)) are defined as
\[ P^{(0)} = w \rho^{(0)}, \]
\[ P^{(G,1)} = c_s^{(1)} \rho^{(G,1)}, \]
\[ P^{(G,2)} = (c_s^{(2)})^2 \rho^{(G,2)}. \]

All of them are gauge invariant. In principle, one could choose \( c_s^{(2)} \) to be equal to \( c_s \) for the adiabatic perturbation with a constant speed of sound \( [58] \).

### 5.2 Second order cosmological perturbations induced by the first order Newtonian variables

As a first step, we study the gauge invariant Newtonian metric perturbations at first order and their equations of motion. The gauge invariant metric up to first order is given by
\[ g^{(G,1)}_{\mu\nu} dx^\mu dx^\nu = -a^2 (1 + 2 \Phi^{(1)}) d\eta^2 + 2a^2 V^{(1)}_i d\eta dx^i + a^2 \delta_{ij} (1 - 2 \Psi^{(1)}) dx^i dx^j, \]
where the gauge invariant first order Newtonian variables have been shown in Eq. (23a)–(23c). Here, we neglect the first order tensor perturbation, i.e., \( H'_{ij} = 0 \). Differed from the neglect of the scalar or vector perturbations, the neglect of \( H'_{ij} \) does not violate the gauge invariance that is defined with all the diffeomorphisms (namely, arbitrary \( \xi^i \)). At second order, we will consider that the gauge invariant metric perturbations are Newtonian and synchronous, respectively.

Based on Eqs. (10a) and (10b), we express the temporal derivative of conformal Hubble parameter, the density and velocity perturbations in terms of the gauge invariant first order metric perturbations, i.e.,
\[ \dot{H} = \frac{1}{2} (1 + 3w) H^2, \]
\[ \rho^{(0)} = \frac{3 H^2}{\kappa a^2}, \]
\[ \rho^{(G,1)} = -6 H^2 \Phi^{(1)} - 6 \mathcal{H} \partial_\eta \Psi^{(1)} + 2 \Delta \Psi^{(1)}, \]
\[ \rho^{(G,2)} = \frac{\mathcal{H}^2 \Phi^{(1)} - 4 \mathcal{H} \partial_\eta \Phi^{(1)} + 4 \delta_{ij} (\mathcal{H}^2 \Phi^{(1)} - c_s^2 \Delta \Phi^{(1)})}{6 (1 + w) H^2}, \]
where \( \rho^{(G,1)} \equiv \delta_{ij} (V_i)^{(G,1)} \). By substituting the above equations into the spatial part of the first order Einstein field equation, we obtain
\[ \Phi^{(1)} = \Psi^{(1)}, \]
\[ V^{(1)} = 0, \]
and the equation of motion of \( \Phi^{(1)} \), i.e,
\[ \partial_\eta \Phi^{(1)} + 3 (1 + c_s^2) \mathcal{H} \partial_\eta \Phi^{(1)} + 3 (c_s^2 - w) H^2 \Psi^{(1)} - c_s^2 \Delta \Phi^{(1)} = 0. \]
Here, we disregard the decaying mode of the first order vector perturbations. In fact, the above results about the first order cosmological perturbations are well known \([8]\).
The explicit expressions of the gauge invariant second order variables $\Phi^{(2)}$, $\Psi^{(2)}$, $V_i^{(2)}$ and $H_{ij}^{(2)}$ are presented in Eqs. (43a)–(43d) with the expression of $A_{\mu\nu}^{\rm s}$ in Appendix E. Based on the temporal components of the second order Einstein field equations in Eq. (59a)–(60b), we can express the gauge invariant second order density perturbations in terms of the gauge invariant metric perturbations, i.e.,

$$
\rho^{\rm (GI,2)} = \frac{2}{3(1+w)\kappa a^2 H^2}(9(1+w)H^2(4(\Phi^{(1)})^2 - \Phi^{(2)}) - 9(1+w)H^2 \partial_0 \Psi^{(2)} \\
+ H^2(9(1+w)\partial_0 \Phi^{(1)})^2 + 24(1+w)\Phi^{(1)} \Delta \Phi^{(1)} + 3(1+w)\Delta \Psi^{(2)} + (5+9w)\partial_i \Phi^{(1)} \partial^i \Phi^{(1)}) \\
- 8H \partial_0 \partial_j \Phi^{(1)} \partial^j \Phi^{(1)} - 4 \partial_j \partial_0 \Phi^{(1)} \partial^0 \Phi^{(1)}). 
$$

(63)

Using Eqs. (59a)–(60b) and substituting the expression of $\rho^{\rm (GI,2)}$ into the spatial part of the gauge invariant second order Einstein field equations, we can rewrite the second order Einstein field equations in Eq. (10c) to be

$$
G_{ij} + S_{ij} = 0, \quad (64)
$$

$$
S_{ij} = \delta_{ij} \left( \frac{4(c_s^{(2)})^2}{3(1+w)} \partial_0 \Phi^{(1)} \partial^k \Phi^{(1)} - 12(c_s^{(2)})^2 - 2w) H^2 (\Phi^{(1)})^2 - 4(5+3c_s^{(2)}) \Phi^{(1)} \partial_0 \Phi^{(1)} \\
+ \frac{8(c_s^{(2)})^2}{3(1+w)} \partial_0 \partial_0 \Phi^{(1)} \partial^k \Phi^{(1)} - (1 + 3(c_s^{(2)})^2) (\partial_0 \Phi^{(1)})^2 - 4 \Phi^{(1)} \partial_0^2 \Phi^{(1)} - 4 \Phi^{(1)} \Delta \Phi^{(1)} - 3 \partial_0 \Phi^{(1)} \partial^k \Phi^{(1)} \\
- 8(c_s^{(2)})^2 \Phi^{(1)} \Delta \Phi^{(1)} - 3(c_s^{(2)})^2 \partial_0 \Phi^{(1)} \partial^k \Phi^{(1)} + \frac{4(c_s^{(2)})^2}{3(1+w)H^2} \partial_0 \partial_0 \Phi^{(1)} \partial^k \partial^0 \Phi^{(1)} + 4c_s^{(2)} \Phi^{(1)} \Delta \Phi^{(1)} \right) \\
+ 4 \Phi^{(1)} \partial_i \partial_j \Phi^{(1)} - \frac{4}{3(1+w)} (\partial_0 \Phi^{(1)} \partial_j \Phi^{(1)} + \partial_i \Phi^{(1)} \partial_0 \Phi^{(1)} \\
+ (2 - \frac{4}{3(1+w)}) \partial_j \Phi^{(1)} \partial_0 \Phi^{(1)} - \frac{4}{3(1+w)H^2} \partial_0 \Phi^{(1)} \partial_0 \Phi^{(1)}), 
$$

(65)

$$
G_{ij} = \frac{1}{4} \partial_0^2 H_{ij}^{(2)} + \frac{1}{2} H \partial_0 H_{ij}^{(2)} - \frac{1}{4} \Delta H_{ij}^{(2)} \\
+ \delta_{ij} \left( 3((c_s^{(2)})^2 - w) H^2 \Phi^{(2)} + H \partial_0 \Phi^{(2)} + \frac{1}{2} \Delta \Phi^{(2)} \\
+ \partial_0^2 \Psi^{(2)} + (2 + 3(c_s^{(2)})^2) H \partial_0 \Psi^{(2)} - \frac{1}{2} (1 + 2(c_s^{(2)})^2) \Delta \Psi^{(2)} \\
- \frac{1}{2} H \partial_0 V_j^{(2)} - \frac{1}{2} \partial_0 V_j^{(2)} - \frac{1}{2} \partial_0 V_i^{(2)} - \frac{1}{2} \partial_j \partial_0 V_i^{(2)} - \frac{1}{2} \partial_i \partial_0 V_j^{(2)} + \frac{1}{2} \partial_0 \partial_j \partial_i \Phi^{(2)} \right). 
$$

(66)

We can decompose Eq. (64) into the tensor, vector and scalar components. For illustrations, we decompose $G_{ij}$ as a first step. The decomposition of $G_{ij}$ is explicitly given by

$$
\Lambda_{kl}^{(2)} G_{ij} = \frac{1}{4} \partial_0^2 H_{kl}^{(2)} + \frac{1}{2} H \partial_0 H_{kl}^{(2)} - \frac{1}{4} \Delta H_{kl}^{(2)}, 
$$

(67a)

$$
\Delta^{-1} \partial^i T^k_{ji} G_{kl} = \frac{1}{4} (\partial_0 + 2H) V_i^{(2)}, 
$$

(67b)

$$
\frac{1}{2} \Delta^{-1} (\partial^k \Delta^{-1} \partial_j - \frac{1}{2} T^{kli}) G_{kl} = \frac{1}{4} (\Phi^{(2)} - \Phi^{(2)}), 
$$

(67c)

$$
\frac{1}{4} T^{ij} G_{ij} = \frac{1}{2} (3((c_s^{(2)})^2 - w) H^2 \Phi^{(2)} + H \partial_0 \Phi^{(2)} + \frac{1}{2} \Delta \Phi^{(2)} \\
+ \partial_0^2 \Psi^{(2)} + (2 + 3(c_s^{(2)})^2) H \partial_0 \Psi^{(2)} - \frac{1}{2} (1 + 2(c_s^{(2)})^2) \Delta \Psi^{(2)}), 
$$

(67d)
where $\Lambda_{ij}^{kl} \equiv T_i^k T_j^l - (1/2) T_{ij} T^{kl}$. The equations of motion of the gauge invariant second order cosmological perturbations can be written as

\[
\begin{align*}
\partial_i^2 H_{ij}^{(2)} + 2 \mathcal{H} \partial_i H_{ij}^{(2)} - \Delta H_{ij}^{(2)} &= -4 \Lambda_{ij}^{kl} S_{kl}, \\
(\partial_i + 2 \mathcal{H}) \dot{V}_{ij}^{(2)} &= -4 \Delta^{-1} T_i^k \partial^l S_{kl}, \\
\Psi^{(2)} - \Phi^{(2)} &= -2 \Delta^{-1} (\partial^i \Delta^{-1} \partial^j - \frac{1}{2} T^{ij}) S_{ij}, \\
\partial_0^2 \Psi^{(2)} + (2 + 3(c_s^{(2)})^2) \mathcal{H} \partial_0 \Psi^{(2)} - \frac{1}{2} (1 + 2(c_s^{(2)})^2) \Delta \Psi^{(2)} &+ 3((c_s^{(2)})^2 - w) \mathcal{H}^2 \Phi^{(2)} + \mathcal{H} \partial_0 \Phi^{(2)} + \frac{1}{2} \Delta \Phi^{(2)} = - \frac{1}{2} T^{ij} S_{ij},
\end{align*}
\]

where we have the following expressions of $S_{ij}$,

\[
\begin{align*}
\Lambda_{ij}^{kl} S_{ij} &= \Lambda_{ij}^{kl} \left( 2 + \frac{4}{3(1+w)} \right) \Phi^{(1)} \partial_i \partial_j \Phi^{(1)} + \frac{8}{3(1+w) \mathcal{H}} \Phi^{(1)} \partial_i \partial_j \partial_c \Phi^{(1)} \\
&- \frac{4}{3(1+w) \mathcal{H}^2} \partial_c \partial_i \Phi^{(1)} \partial_j \partial_c \Phi^{(1)}), \\
\Delta^{-1} T_i^k \partial^l S_{kl} &= \Delta^{-1} T_i^k \partial^l \left( 2 - \frac{4}{3(1+w)} \right) \Phi^{(1)} \partial_i \partial_j \partial_c \Phi^{(1)} - \frac{4}{3(1+w) \mathcal{H}} \left( \partial_i \partial_j \partial_c \partial_k \Phi^{(1)} \partial_l \Phi^{(1)} + \partial_k \Phi^{(1)} \partial_i \partial_j \partial_l \Phi^{(1)} \right) \\
&+ 4 \Phi^{(1)} \partial_i \partial_j \partial_l \Phi^{(1)} - \frac{4}{3(1+w) \mathcal{H}^2} \partial_i \partial_j \partial_k \Phi^{(1)} \partial_l \Phi^{(1)} \\
&+ \frac{1}{4} T^{ij} S_{ij} = \frac{1}{2} \left( 4(c_s^{(2)})^2 \mathcal{H}^2 \Phi^{(1)} - 12(c_s^{(2)})^2 \mathcal{H}^2 \Phi^{(1)} \right) - 2(c_s^{(2)})^2 \mathcal{H}^2 \Phi^{(1)} \right), \\
\end{align*}
\]

The above equations of motion can be derived in an easier way, by adopting the scalar-vector-tensor decomposition operators onto both sides of Eq. (64). These operators are defined on the background and thus gauge invariant. Therefore, the equations of motion in Eqs. (68a)-(68d) should also be gauge invariant. Here, the operator $\Lambda_{ij}^{kl}$ is defined in configuration space. In Appendix F, we express $\Lambda_{ij}^{kl}$ in momentum space and show that it can be rewritten in terms of the polarization tensors.

### 5.2.2 Gauge invariant second order synchronous variables

At second order, the gauge invariant synchronous metric perturbations are given by

\[
\begin{align*}
g_{\mu
u}^{(2)}(x) &\equiv \mathcal{A}(\partial_\mu \dot{\Psi}^{(2)} + 2 \partial_\mu \partial_j E_j^{(2)} \\
&+ \partial_i C_i^{(2)} + \partial_i C_i^{(2)} + H_j^{(2)}) dx^i dx^j. 
\end{align*}
\]
The explicit expressions of the gauge invariant second order variables, $E^{(2)}$, $\Psi^{(2)}$, $C^{(2)}_i$ and $H^{(2)}_{ij}$ are presented in Eqs. (46a)--(46d) with the expression of $\mathcal{A}_{\mu\nu}^{(2)}$, in Appendix E. Based on the temporal components of the second order Einstein field equations and Eqs. (59a)--(60b), the gauge invariant second order density perturbations in terms of the metric perturbations is given by

\[
\rho^{(2)} = \frac{2}{3(1+w)\kappa a^2 H^2} \left(36(1+w)H^4(\Phi^{(1)})^2 - 3(1+w)H^3(3\partial_0\Psi^{(2)} - \Delta\partial_0 E^{(2)})
+ H^2(9(1+w)(\partial_0\Phi^{(1)})^2 + 24(1+w)\Phi^{(1)}\Delta\Phi^{(1)} + 3(1+w)\Delta\Psi^{(2)} + (5+9w)\partial_i\Phi^{(1)}\partial_i\Phi^{(1)})
- 8H\partial_0\partial_0\Phi^{(1)}\partial_0\Phi^{(1)} - 4\partial_0\partial_0\Phi^{(1)}\partial_0\Phi^{(1)}\right).
\] (74)

Using Eqs. (59a)--(60b) and substituting the expression of $\rho^{(2)}$ into the spatial parts of the gauge invariant second order Einstein field equations, we can also rewrite the second order Einstein field equations in the terms of the tensors $G_{ij}$ and $S_{ij}$, i.e.,

\[G_{ij} + S_{ij} = 0,\] (75)

where the $S_{ij}$ takes the same form as Eq. (65) and

\[G_{ij} = \frac{1}{4}\partial_0^2 H_{ij}^{(2)} + \frac{1}{2}H\partial_0 H_{ij}^{(2)} - \frac{1}{4}\Delta H_{ij}^{(2)}
+ \delta_{ij} \left(\partial_0^2 \Psi^{(2)} + (2 + 3(c_s^{(2)})^2)H\partial_0 \Psi^{(2)} - \frac{1}{2}(1 + 2(c_s^{(2)})^2)\Delta \Psi^{(2)} - \Delta \left(\frac{1}{2}(\partial_0 + 2H)\partial_0 E^{(2)} + (c_s^{(2)})^2 H\partial_0 E^{(2)}\right)\right)
+ \frac{1}{2}H\partial_0 \partial_0 C_{ij}^{(2)} + \frac{1}{4}\partial_0 \partial_0^2 C_{ij}^{(2)} + \frac{1}{2}H\partial_0 \partial_0 C_{ij}^{(2)} + \frac{1}{4}\partial_0 \partial_0^2 C_{ij}^{(2)} + \Delta \partial_0 \partial_0 E^{(2)} + \frac{1}{2}\partial_0 \partial_0 E^{(2)} + \frac{1}{2}\partial_0 \partial_0^2 E^{(2)} + \frac{1}{2}\partial_0 \partial_0^2 E^{(2)} + \frac{1}{2}\partial_0 \partial_0 \Psi^{(2)}.\] (76)

By adopting the scalar-vector-tensor decomposition to Eq. (75), the equations of motion of the gauge invariant second order cosmological perturbations are obtained to be

\[\partial_0^2 H_{ij}^{(2)} + 2H\partial_0 H_{ij}^{(2)} - \Delta H_{ij}^{(2)} = -4\Lambda^{(1)}_{kl} S_{kl},\] (77a)
\[(\partial_0 + 2H)\partial_0 C_{ij}^{(2)} = -4\Delta^{-1} T^{(1)}_k \partial^k S_{kl},\] (77b)
\[(\partial_0 + 2H)\partial_0 E^{(2)} + \Psi^{(2)} = -2\Delta^{-1} (\partial^i \Delta^{-1} \partial^j - \frac{1}{2} T^{(1)}_i) S_{ij},\] (77c)
\[\partial_0^2 \Psi^{(2)} + (2 + 3(c_s^{(2)})^2)H\partial_0 \Psi^{(2)} - \frac{1}{2}(1 + 2(c_s^{(2)})^2)\Delta \Psi^{(2)}
- \Delta \left(\frac{1}{2}(\partial_0 + 2H)\partial_0 E^{(2)} + (c_s^{(2)})^2 H\partial_0 E^{(2)}\right) = \frac{1}{2} T^{(1)}_i S_{ij}.\] (77d)

The right hand sides of the above equations are as same as those in Eqs. (69)--(72), since the source term $S_{ij}$ is uniquely determined by the gauge invariant first order Newtonian variables. In particular, the equation of motion of the tensor perturbation $H^{(2)}_{ij}$ in Eq. (77a) is as same as that in Eq. (68a). For the vector perturbation, the evolution of $\partial_0 C^{(2)}_i$ in Eq. (77b) is shown to be as same as that of $V_i^{(2)}$ in Eq. (68b).

**5.3 Second order cosmological perturbations induced by the first order synchronous variables**

In the subsection, we turn to discuss the gauge invariant synchronous metric perturbations at first order and their equations of motion. For the second order equations of motion, we will also consider the gauge invariant variables that are Newtonian and synchronous, respectively.
The gauge invariant synchronous metric up to first order is given by

$$g^{(G1)}_{\mu \nu} dx^\mu dx^\nu = -a^2 d\eta^2 + a^2 (\delta_{ij}(1-2\Psi^{(1)}) + 2 \partial_i \partial_j E^{(1)})$$

$$+ \partial_i C^{(1)}_{\nu} + \partial_j C^{(1)}_{\mu} dx^i dx^j , \quad (78)$$

where we also neglect the first order tensor perturbation, i.e., $H^{(1)} = 0$. Here, the gauge invariant first order variables $\Psi^{(1)}$, $E^{(1)}$ and $C^{(1)}$ have been shown in Eqs. (27a)–(27d). Based on the first-order Einstein field equations and Eqs. (59a) and (59b), the density and velocity perturbations are expressed in terms of the gauge invariant first order metric perturbations, i.e.,

$$\rho^{(G1,1)} = -6 \mathcal{H} \partial_i \Psi^{(1)} + 2 \mathcal{H} \Delta \partial_i E^{(1)} + 2 \Delta \Psi^{(1)}$$

$$\mathcal{H} \partial_i C^{(1)}_{\nu} + 4 \partial_i \partial_j \Psi^{(1)}$$

$$\frac{6(1+w)\mathcal{H}^2}{\kappa a^2} \quad (79a)$$

$$v^{(G1,1)}_i = -\frac{\Delta \partial_\alpha C^{(1)}_{\alpha} + 4 \partial_\alpha \partial_\beta \Psi^{(1)} \partial^\alpha \partial^\beta \Psi^{(1)}}{6(1+w)\mathcal{H}^2} \quad (79b)$$

By substituting Eqs. (59a) and (59b) and the equations above into the spatial parts of the first-order Einstein field equations, we can obtain

$$C^{(1)}_{\nu} = 0, \quad (80)$$

and the equations of motion of $E^{(1)}$ and $\Psi^{(1)}$, i.e.,

$$\Psi^{(1)} + 2 \mathcal{H} \partial_i E^{(1)} + \partial_0^2 E^{(1)} = 0, \quad (81a)$$

$$\partial_\nu \Psi^{(1)} + \mathcal{H} (2+3c_s^2) \partial_\nu \Psi^{(1)}$$

$$-c_s^2 \Delta E^{(1)} - \partial_i \partial_j \partial_\nu \partial^{\nu} E^{(1)} = 0. \quad (81b)$$

Here, we also disregard the decaying mode of the first order vector perturbations.

5.3.1 Gauge invariant second order Newtonian variables

At second order, the gauge invariant Newtonian metric perturbations are given by

$$g^{(G2)}_{\mu \nu} dx^\mu dx^\nu = 2 a^2 \Phi^{(2)} d\eta^2 + 2 a^2 \Phi^{(2)} d\eta dx^i$$

$$+ a^2 (\mathcal{H} \partial_\nu \Psi^{(1)} + \mathcal{H} (2+3c_s^2) \partial_\nu \Psi^{(1)}$$

$$-c_s^2 \Delta E^{(1)} - \partial_i \partial_j \partial_\nu \partial^{\nu} E^{(1)}) \quad (82)$$

The explicit expressions of the gauge invariant second order variables, $\Phi^{(2)}$, $\Psi^{(2)}$, $V^{(2)}_{ij}$ and $H^{(2)}_{ij}$ are presented in Eqs. (43a)–(43d) with the expression of $\lambda^S_{\nu \mu}$ in Appendix E. Based on the temporal components of the second order Einstein field equations and Eqs. (59a), (59b), (79a)–(80), we can express the gauge invariant second order density perturbation in terms of metric perturbations, namely,

$$\rho^{(G2)} = -\frac{1}{3(1+w)\kappa a^2 \mathcal{H}^2} \left(18(1+w)\mathcal{H}^3 \partial_\nu \partial_\mu \Psi^{(1)} + \partial_\nu \partial_\mu \Psi^{(1)} \partial^\nu \partial^\mu \Psi^{(1)} + 6 \partial_\nu \partial_\mu \Psi^{(1)} \partial^\nu \partial_\mu \Psi^{(1)}ight.$$

$$\left. + 4 \partial_\nu \partial_\mu \Psi^{(1)} \partial^\nu \partial_\mu \Psi^{(1)} - 4 \partial_\nu \partial_\mu \Psi^{(1)} \partial^\nu \partial_\mu \Psi^{(1)} + 6 \partial_\nu \partial_\mu \Psi^{(1)} \partial^\nu \partial_\mu \Psi^{(1)}ight) \quad (83)$$

Using Eqs. (59a), (59b), (79a)–(80) and substituting the expression of $\rho^{(G2)}$ into the spatial parts of the gauge invariant second order Einstein field equations, we can also obtain

$$G_{ij} + S_{ij} = 0, \quad (84)$$

where the form of $G_{ij}$ is the same as Eq. (66) and
\[ S_{ij} = \delta_{ij} \left( -12(c_s^2 - (c_s^2)^2)H\Psi^1 \partial_0 \Psi^1 + (1 - 3(c_s^2)^2)(\partial_0 \Psi^1)^2 - 2(1 - 2c_s^2 + 4(c_s^2)^2)\Psi^1 \Delta \Psi^1 \right) \]
\[ - (2 + 3(c_s^2)^2)\partial_0 \Psi^1 \partial_0 \Psi^1 + \frac{4(c_s^2)^2}{3(1 + w)H^2} \partial_0 \partial_0 \Psi^1 \partial_0 \Psi^1 - (1 + (c_s^2)^2)\Delta \partial_0 E^1 \Delta \partial_0 E^1 \]
\[ + (1 + (c_s^2)^2)\partial_0 \Delta E^1 \partial_0 \Delta E^1 + 8(1 + (c_s^2)^2)H\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 + 4\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 \]
\[ + (3 + (c_s^2)^2)\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 - (1 + (c_s^2)^2)\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 \]
\[ - 2\partial_0 \Psi^1 \partial_0 \Psi^1 - (1 - 2(c_s^2)^2)\partial_0 \Psi^1 \partial_0 \Psi^1 + (1 + 2(c_s^2)^2)\partial_0 \Delta E^1 \partial_0 \Delta E^1 \]
\[ + 2(1 + (c_s^2)^2)\Delta E^1 \Delta \Psi^1 + 2(c_s^2)^2 \partial_0 \partial_0 \Psi^1 \partial_0 \partial_0 \Psi^1 \]
\[ - 2 - 4H(\partial_0 \Psi^1 (1 + (c_s^2)^2)\Delta E^1 + \Psi^1 ((c_s^2)^2 - c_s^2)\Delta \partial_0 E^1) \right) \]
\[ + 3\partial_0 \Psi^1 \partial_0 \Psi^1 \frac{4}{3(1 + w)H^2} \partial_0 \partial_0 \Psi^1 \partial_0 \Psi^1 + 2\Psi^1 \partial_0 \partial_0 \Psi^1 \]
\[ - 2\partial_0 \partial_0 \Psi^1 \partial_0 \partial_0 \Psi^1 - \Delta \partial_0 E^1 \partial_0 \partial_0 E^1 - (1 + (c_s^2)^2)\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 - (1 + (c_s^2)^2)\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 \]
\[ - 2\partial_0 \partial_0 \Psi^1 \partial_0 \partial_0 \Psi^1 - \Delta \partial_0 E^1 \partial_0 \partial_0 E^1 - (1 + (c_s^2)^2)\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 \]
\[ - 2\partial_0 \partial_0 \Psi^1 \partial_0 \partial_0 \Psi^1 - \Delta \partial_0 E^1 \partial_0 \partial_0 E^1 - (1 + (c_s^2)^2)\partial_0 \partial_0 E^1 \partial_0 \partial_0 E^1 \]
\[ + 6\partial_0 \Psi^1 \partial_0 \partial_0 \Psi^1 - 4(1 + c_s^2)\Delta \Psi^1 \partial_0 \partial_0 \Psi^1 + \partial_0 \Psi^1 \partial_0 \partial_0 \Psi^1 - 2\Delta E^1 \partial_0 \partial_0 \Psi^1 \partial_0 \partial_0 \Psi^1. \]  

Following the approach in the previous subsection, the equations of motion of the gauge invariant second order cosmological perturbations can be written as

\[ \partial_0^2 H_{ij} + 2\partial_0 H_{ij} - \Delta H_{ij} = -4\Lambda_{ij} S_{kl}, \]  

\[ (\partial_0 + 2H)V_{i}^{(2)} = -4\Delta^{-1} T_{i}^{kl} S_{kl}, \]

\[ \Psi^{(2)} - \Phi^{(2)} = -2\Delta^{-1} (\partial_{i} \Delta^{-1} \partial^{i} - \frac{1}{2} T_{ij} S_{ij}), \]

\[ \partial_0^2 \Psi^{(2)} + (2 + 3(c_s^2)^2)\partial_0 \Psi^{(2)} - \frac{1}{2} (1 + 2(c_s^2)^2)\Delta \Psi^{(2)} \]

\[ + 3((c_s^2)^2 - w)H^2 \Phi^{(2)} + \partial_0 \Phi^{(2)} + \frac{1}{2} \Delta \Phi^{(2)} = -\frac{1}{2} T_{ij} S_{ij}, \]
where

\[
\Lambda^{ij}_{\delta \partial} S_{ij} = \Lambda^{ij}_{\delta \partial} \left( -\Psi^{(1)} \partial \delta \partial \Psi^{(1)} + \frac{4}{3(1+w)H^2} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
+ (-5 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - 4(1 - c_s^2) \partial \delta \partial \Psi^{(1)} - 2(1 - c_s^2) \partial \delta \partial \Psi^{(1)} - 2\Psi^{(1)} \Delta) \partial \delta \partial \Psi^{(1)} \\
+ 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + \Delta \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
- \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + 2c_s^2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \right),
\]

\[
\Delta^{-1} T_{\delta \partial} S_{ij} = \Delta^{-1} T_{\delta \partial} \partial \left( \frac{3}{3(1+w)H^2} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
- 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
+ 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
\right),
\]

\[
\frac{1}{4} T^{ij} S_{ij} = \frac{1}{2} \left( -12 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + (1 - 3(c_s^2)^2) \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
- 2(1 - c_s^2 + 4(c_s^2)^2) \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + \frac{4}{3(1+w)H^2} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
+ (1 - c_s^2)^2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
\right),
\]

\[
\frac{1}{2} \Delta^{-1} \left( \partial \delta \partial \Psi^{(1)} - \frac{1}{2} T^{ij} S_{ij} \right) = \frac{1}{2} \Delta^{-1} \left( \partial \delta \partial \Psi^{(1)} - \frac{1}{2} T^{ij} S_{ij} \right) \left( \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - \frac{4}{3(1+w)H^2} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
+ 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} - \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
+ \Delta \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} + 2 \partial \delta \partial \Psi^{(1)} \partial \delta \partial \Psi^{(1)} \\
\right),
\]
It is not surprising that the left hand sides of the equations of motion are as same as those of Eqs. (68a)–(68d). As suggested in Ref. [47], we show that the source terms in Eq. (86b) include the first order perturbation $E^{(1)}$ without temporal derivative. This is different from the formulae in the other previous works [13, 44, 48].

5.3.2 Gauge invariant second order synchronous variables

For the gauge invariant synchronous metric perturbations at second order, we also have

$$g^{(G1,2)}_{\mu\nu}dx^\mu dx' = a^2(-2\delta_{ij}\Psi^{(2)} + 2\partial_i\partial_j E^{(2)} + \partial_i C^{(2)}_{ij} + \partial_j C^{(2)}_{ij} + H^{(2)}_{ij})dx'dx \tag{91}$$

The explicit expressions of the gauge invariant second order variables, $E^{(2)}$, $\Psi^{(2)}$, $C^{(2)}_i$ and $H^{(2)}_{ij}$ are presented in Eqs. (46a)–(46d) with the expression of $X^{(2)}_{\mu\nu}$ in Appendix E. Based on the temporal components of the second order Einstein field equations and Eqs. (59a), (59b), (79a)–(80), the gauge invariant second order density perturbation is obtained to be

$$\rho^{(G1,2)} = -\frac{1}{3(1+w)\kappa a^2H^2}(8\partial_i\partial_j\Psi^{(1)}\partial^\mu\partial^\nu\Psi^{(1)} + 6(1+w)H^2(3\partial_i\Psi^{(2)} - \Delta\partial_i E^{(2)} - 4\partial_i\Psi^{(1)}\Delta E^{(1)} + 4\Psi^{(1)}(3\partial_i\Psi^{(1)} - \Delta\partial_i E^{(1)})) + 4\partial_i\partial_j\partial^\mu\partial^\nu E^{(1)} - 3(1+w)H^2(6\partial_i\Psi^{(1)})^2 - 4\partial_i\Psi^{(1)}\Delta\partial_i E^{(1)} + 16\Psi^{(1)}\Delta\Psi^{(1)} + 2\Delta\Psi^{(2)} + 6\partial_i\Psi^{(1)}\partial^\mu\partial_j\Psi^{(1)} - 4\partial_i\Delta E^{(1)}\partial^\mu\partial^\nu E^{(1)} + \Delta\partial_i E^{(1)}\partial^\mu\partial_j E^{(1)} - 4\Delta E^{(1)}\Delta E^{(1)} - \partial_i\Delta E^{(1)}\partial^\mu\partial_j E^{(1)} - 4\partial_i\partial_j\Psi^{(1)}\partial^\mu\partial^\nu E^{(1)}) \tag{92}$$

Substituting Eqs. (59a), (59b), (79a)–(80) and (92) into the spatial parts of the gauge invariant second order Einstein field equations, we also obtain

$$\mathcal{G}_{ij} + S_{ij} = 0 \tag{93}$$

$$\partial_i^2 H^{(2)}_{ij} + 2\mathcal{H}\partial_i H^{(2)}_{ij} - \Delta H^{(2)}_{ij} = -4\Lambda^{(1)}_i S_{ij}, \tag{94a}$$

$$\partial_i C^{(2)}_{ij} = -4\Delta^{-1}T^k_i \partial^k S_{kl}, \tag{94b}$$

$$\partial_i^2 \Psi^{(2)} + (2 + 3(c^{(2)}_s)^2)\mathcal{H}\partial_i \Psi^{(2)} - \frac{1}{2}(1 + 2(c^{(2)}_s)^2)\Delta \Psi^{(2)} - \Delta \left(\frac{1}{2}(\partial_i + 2\mathcal{H})\partial_i E^{(2)} + (c^{(2)}_s)^2\mathcal{H}\partial_i E^{(2)}\right) = -\frac{1}{2}T^{ij} S_{ij} \tag{94c}$$

$$\partial_i^2 \Psi^{(2)} + (2 + 3(c^{(2)}_s)^2)\mathcal{H}\partial_i \Psi^{(2)} - \frac{1}{2}(1 + 2(c^{(2)}_s)^2)\Delta \Psi^{(2)} - \Delta \left(\frac{1}{2}(\partial_i + 2\mathcal{H})\partial_i E^{(2)} + (c^{(2)}_s)^2\mathcal{H}\partial_i E^{(2)}\right) = -\frac{1}{2}T^{ij} S_{ij} \tag{94d}$$

where $\mathcal{G}_{ij}$ is shown in Eq. (66) and $S_{ij}$ in Eq. (85). Therefore, we derive the equations of motion of the gauge invariant second order cosmological perturbations as

The right hand sides of the above equations are as same as that in Eqs. (87)–(90), since the source term $S_{ij}$ is uniquely determined by the gauge invariant first order synchronous variables. In particular, the equation of motion of the tensor perturbation $H^{(G1,2)}_{\mu\nu}$ in Eq. (94a) is as same as that in Eq. (86a). For the vector perturbation, the evolution of $\partial_i C^{(2)}_{ij}$ in Eq. (94b) is shown to be as same as that of $V^{(2)}_i$ in Eq. (86b).

6 Conclusions and discussions

In this paper, we investigated the gauge invariance of the cosmological perturbations up to second order by fol-
lowing the Lie derivative method. We showed that there are infinite families of gauge invariant variables for the cosmological perturbations up to second order. For different families, we found their conversion formulae which belong to a linear space spanned by a finite number of bases that were also shown to be gauge invariant. In particular, we have focused on the Newtonian, synchronous, and hybrid variables, respectively. We presented the explicit conversions between these different families of gauge invariant variables. In contrast to the first order, the second order gravitational waves were shown to be mixed with the first order cosmological perturbations. Therefore, the gauge invariance is important in studies of the second order gravitational waves.

We derived the equations of motion of the gauge invariant second order cosmological perturbations, which are sourced from the gauge invariant first order scalar perturbations. It was found that the choices of gauge invariant variables at different orders are independent, e.g., Newtonian at first order while synchronous at second order, and vice versa. In this work, we have studied both of the above four typical cases. To obtain the gauge invariant equations of motion, we decomposed the gauge invariant perturbed Einstein field equations into the scalar, vector and tensor components.

In principle, one could generalize the above method to explore the higher order cosmological perturbations. A formal derivation of the gauge invariant higher order cosmological perturbations has been shown previously [18]. Following the same approach in this work, it is straightforward to obtain the explicit expressions for the gauge invariant higher order scalar, vector and tensor perturbations, as well as the equations of motion of them.

This study is based on the method developed by [5, 7]. We further focus on conversion between different gauge invariant variables. Since there is no uniqueness in the framework, it might show that one can not determine which gauge-invariant tensor perturbations \( H^{(2)}_{ij} \) correspond to the energy density spectrum of gravitational waves. Perhaps, physical insights or arguments should be explored in future studies.

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A Expansion of a generic tensor with Lie derivative

For a generic tensor \( Q \), its Lie derivative along a vector \( \zeta^\mu \) is defined as

\[
\mathcal{L}_\zeta Q \equiv \lim_{\epsilon \to 0} \frac{\varphi_\epsilon^* Q(x) - Q(x)}{\epsilon},
\]

(95)

where \( \varphi_\epsilon^* Q \) is an one-parameter coordinate transformation that acts on the tensor \( Q \). In the regime of \( \epsilon \to 0 \), it turns to be the infinitesimal transformation in the form of Eq. (1), where \( \xi^{(1)} = \epsilon \zeta^\mu \). For a scalar function, contravariant vector and covariant vector, the expressions of \( \varphi_\epsilon^* \) are given by

\[
\begin{align}
\varphi_\epsilon^* f(x) &= f(\tilde{x}), \\
\varphi_\epsilon^* w_\mu(x) &= \frac{\partial \tilde{x}^\nu}{\partial x^\mu} w_\nu(\tilde{x}), \\
\varphi_\epsilon^* A^\mu(x) &= \frac{\partial \tilde{x}_\nu}{\partial x^\mu} A^\nu(\tilde{x}).
\end{align}
\]

(96a)–(96c)

Based on the first order expansions of Eqs. (96a)–(96c), we obtain

\[
\begin{align}
\mathcal{L}_\zeta f &= \zeta^\nu \partial_\nu f, \\
\mathcal{L}_\zeta w_\mu &= \zeta^\nu \partial_\nu w_\mu + w_\nu \partial_\nu \zeta^\mu, \\
\mathcal{L}_\zeta A^\mu &= \zeta^\nu \partial_\nu A^\mu - A^\nu \partial_\nu \zeta^\mu.
\end{align}
\]

(97a)–(97c)

The Lie derivative in Eq. (97c) is also symbolized as \( \mathcal{L}_X A^\mu \equiv [X, A]^\mu \), where \([,] \) is Lie bracket. For a tensor \( S_{\mu\nu} \), its Lie derivative is given by

\[
\mathcal{L}_\zeta S_{\mu\nu} = \zeta^\lambda \partial_\lambda S_{\mu\nu} + S_{\lambda\nu} \partial_\lambda \zeta^\mu + S_{\mu\lambda} \partial_\lambda \zeta^\nu.
\]

(98)

Higher order expansions of any tensor upon the infinitesimal transformation have been constructed in terms of Lie derivatives [5–7, 10, 11]. In the following, we briefly review the formulae up to second order.

For the scalar function \( f(x) \) upon the infinitesimal transformation, up to second order, we expand it in
terms of $\xi^{(1)}$ and $\xi^{(2)}$, namely,
\[ f(x) \to f(\tilde{x}) \approx f(x + \xi^{(1)} + \frac{1}{2} \xi^{(2)}) \]
\[ = f + \xi^{(1)} \nu \partial_{\nu} f + \frac{1}{2} \xi^{(2)} \nu \partial_{\nu} f + \frac{1}{2} \xi^{(1)} \rho \partial_{\rho} f \]
\[ = f + \xi^{(1)} \nu \partial_{\nu} f + \frac{1}{2} (\xi^{(2)} \nu \partial_{\nu} f + \xi^{(1)} \rho \partial_{\rho} f) \]
\[ + \xi^{(1)} \nu \partial_{\nu} \xi^{(1)} \rho \partial_{\rho} f \]
\[ = f + \mathcal{L}_{\xi(1)} f + \frac{1}{2} (\mathcal{L}_{\xi(2) - \xi(1)} w_{\nu} + \mathcal{L}^2_{\xi(1)}) f. \quad (99) \]

For simplicity, we denote
\[ \xi^{(1)} \mu \equiv \xi^{(1)} \mu, \quad (100a) \]
\[ \xi^{(2)} \mu \equiv \xi^{(2)} \mu - \xi^{(1)} \nu \partial_{\nu} \xi^{(1)} \mu. \quad (100b) \]

It leads to
\[ f(\tilde{x}) = f + \mathcal{L}_{\xi(1)} f + \frac{1}{2} (\mathcal{L}_{\xi(2)} + \mathcal{L}^2_{\xi(1)}) f + \mathcal{O}(\xi^{(3)}). \quad (101) \]

For the contravariant vector $w_{\nu}$, up to second order, we expand it in terms of $\xi^{(1)}$ and $\xi^{(2)}$, namely,
\[ w_{\mu}(x) \to \frac{\partial \tilde{x}_{\mu}}{\partial x^{\nu}} w_{\nu}(\tilde{x}) \]
\[ \approx (\delta_{\mu}^{\lambda} + \partial_{\mu} \xi^{(1)} \lambda + \frac{1}{2} \partial_{\mu} \xi^{(2)} \lambda) \left( w_{\lambda}(x) \right) \]
\[ + (\xi^{(1)} \nu + \frac{1}{2} \xi^{(2)} \nu) \partial_{\nu} w_{\lambda} \]
\[ + (\xi^{(1)} \rho + \frac{1}{2} \xi^{(2)} \rho) \left( \xi^{(1)} \nu \partial_{\nu} \partial_{\rho} w_{\lambda} \right) \]
\[ = w_{\nu}(x) + \xi^{(1)} \nu \partial_{\nu} w_{\mu} + w_{\lambda} \partial_{\mu} \xi^{(1)} \lambda + \frac{1}{2} w_{\lambda} \partial_{\mu} \xi^{(2)} \lambda \]
\[ + \partial_{\mu} \xi^{(1)} \nu \partial_{\nu} w_{\lambda} + \frac{1}{2} \left( \xi^{(2)} \nu \partial_{\nu} w_{\mu} + \xi^{(1)} \nu \partial_{\nu} \partial_{\mu} w_{\lambda} \right) \]
\[ = w_{\nu} + \mathcal{L}_{\xi(1)} w_{\mu} + \frac{1}{2} L_{\xi(2)} w_{\mu} \]
\[ + \frac{1}{2} (L_{\xi(2)} - \xi^{(1)} \nu \partial_{\nu} \xi^{(1)} \mu) \partial_{\mu} w_{\nu} \]
\[ - w_{\mu} \partial_{\nu} \left( (\xi^{(1)} \nu \partial_{\nu} \xi^{(1)} \mu) \right) \]
\[ = w_{\nu} + \mathcal{L}_{\xi(1)} w_{\mu} + \frac{1}{2} (L_{\xi(2) - \xi(1)} \nu \partial_{\nu} \xi^{(1)} + \mathcal{L}^2_{\xi(1)}) w_{\mu} \]
\[ = w_{\nu} + \mathcal{L}_{\xi(1)} w_{\mu} + \frac{1}{2} (L_{\xi(2)} + \mathcal{L}^2_{\xi(1)}) w_{\mu}. \quad (102) \]

Similarly, for the covariant vector $A^{\nu}$, up to second order, we expand it in terms of $\xi^{(1)}$ and $\xi^{(2)}$, namely,
\[ A^{\nu}(x) \to \frac{\partial x^{\mu}}{\partial x^{\nu}} A^{\nu}(\tilde{x}) \]
\[ \approx \left( \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}} \right)^{-1} A^{\nu}(x + \xi^{(1)} + \frac{1}{2} \xi^{(2)}) \]
\[ \approx A^{\nu} + \xi^{(1)} \nu \partial_{\nu} A^{\mu} - A^{\nu} \partial_{\nu} \xi^{(1)} \mu - \xi^{(1)} \nu \partial_{\nu} A^{\mu} \]
\[ + \frac{1}{2} (\partial_{\nu} \xi^{(2)} \mu - 2 \partial_{\lambda} \xi^{(1)} \mu \partial_{\nu} \xi^{(1)} \lambda) A^{\nu} \]
\[ + \frac{1}{2} ((\xi^{(2)} \nu \partial_{\nu} + \xi^{(1)} \nu \partial_{\nu} \partial_{\mu}) A^{\mu}) \]
\[ = A^{\nu} + \mathcal{L}_{\xi(1)} A^{\mu} + \frac{1}{2} (L_{\xi(2) - \xi(1)} \nu \partial_{\nu} \xi^{(1)} A^{\nu} + \mathcal{L}^2_{\xi(1)} A^{\nu}) \]
\[ = A^{\nu} + \mathcal{L}_{\xi(1)} A^{\mu} + \frac{1}{2} (L_{\xi(2)} A^{\mu} + \mathcal{L}^2_{\xi(1)} A^{\mu}), \quad (103) \]

where the inverse Jacobi matrix $\partial x^{\mu}/\partial \tilde{x}^{\nu}$ is given by
\[ \left( \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}} \right)^{-1} = \delta_{\nu}^{\mu} - \partial_{\nu} \xi^{(1)} \mu \frac{1}{2} (\partial_{\nu} \xi^{(2)} \mu - 2 \partial_{\lambda} \xi^{(1)} \mu \partial_{\nu} \xi^{(1)} \lambda) \mathcal{O}(\xi^{(3)}). \quad (104) \]

It is derived from $\left( \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}} \right)^{-1} = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}} \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\nu}} = \delta_{\nu}^{\mu}$. Finally, we consider the tensor $S_{\mu \nu}$ which is expanded as
For simplicity, the above equation can be rewritten as

\[ S_{\sigma\rho}(x) \rightarrow \frac{\partial \tilde{\varphi}^\mu}{\partial x^\mu} \frac{\partial \tilde{\varphi}^\nu}{\partial x^\nu} S_{\mu\nu}(\tilde{x}) \]

\[ \approx S_{\mu\nu}(x + \xi(1) + \frac{1}{2} \xi(2)) \frac{\partial}{\partial \varphi} + \frac{1}{2} \xi(2)) \frac{\partial}{\partial \varphi} \]

\[ \approx S_{\sigma\rho}(x + \xi(1) + \frac{1}{2} \xi(2)) \frac{\partial}{\partial \varphi} + \frac{1}{2} \xi(2)) \frac{\partial}{\partial \varphi} \]

\[ \approx S_{\sigma\rho} + \xi(1) \frac{\partial}{\partial \varphi} + \frac{1}{2} \xi(2) \frac{\partial}{\partial \varphi} \]

\[ = S_{\sigma\rho} + \xi(1) \frac{\partial}{\partial \varphi} + \frac{1}{2} \xi(2) \frac{\partial}{\partial \varphi} \]

Therefore, the order of the \( \xi(n) \) is determined by the power of \( \varepsilon \) in the transformation (Eq. (106)), and has no relevance with metric perturbations or matter perturbations in principle.

### B Gauge transformations in the language of Lie derivative

For an arbitrary tensor \( Q(x^{\mu})^{i_1i_2\ldots} \) upon the infinitesimal transformation, it can be expanded as [5]

\[ Q(x^{\mu})^{i_1i_2\ldots} \rightarrow \frac{\partial \tilde{x}^i}{\partial x^i} \frac{\partial \tilde{x}^j}{\partial x^j} \frac{\partial \tilde{x}^k}{\partial x^k} \]

\[ \approx \left( \delta^i_{i_1} + \delta^i_{i_2} \xi(1) + \frac{1}{2} \xi(2) \right) \left( \delta^j_{i_1} + \delta^j_{i_2} \xi(1) + \frac{1}{2} \xi(2) \right) \left( \delta^k_{i_1} + \delta^k_{i_2} \xi(1) + \frac{1}{2} \xi(2) \right) \]

\[ = Q(x^{\mu})^{i_1i_2\ldots} + \xi(1) \frac{\partial}{\partial \varphi} Q(x^{\mu})^{i_1i_2\ldots} + \frac{1}{2} \xi(2) \frac{\partial}{\partial \varphi} Q(x^{\mu})^{i_1i_2\ldots} + \frac{1}{2} \xi(3) \frac{\partial}{\partial \varphi} Q(x^{\mu})^{i_1i_2\ldots} \]

In order to study the gauge transformation of \( Q(0) \), one can choose a type of infinitesimal transformation where the order of \( \xi(1) \) is the same order of \( Q(1) \). Combining Eqs. (110) with (111), we obtain

\[ \tilde{Q} = Q(0) + (Q(1) + \xi(1) Q(0)) \]

\[ + \frac{1}{2} \xi(2) + Q(0) \xi(1) = (112) \]

\[ = Q(0) + Q(1) + \frac{1}{2} Q(2) + O(\xi(3)). \]

In Appendix A, we have presented four examples of Eq. (110), namely, Eqs. (99), (102), (103) and (105).

In the following, we consider the infinitesimal transformation of perturbations up to second order. The \( n \)-th order perturbations \( Q(n) \) of the tensor \( Q \) can be expanded as follows

\[ Q = Q(0) + Q(1) + \frac{1}{2} Q(2) + O(\xi(3)). \]

In order to study the gauge transformation of \( Q(\xi) \), one can choose a type of infinitesimal transformation where the order of \( \xi(n) \) is the same order of \( Q(n) \). Combining Eqs. (110) with (111), we obtain

\[ \tilde{Q} = Q(0) + (Q(1) + \xi(1) Q(0)) \]

\[ + \frac{1}{2} \xi(2) + Q(0) \xi(1) = (112) \]

\[ = Q(0) + Q(1) + \frac{1}{2} Q(2) + O(\xi(3)). \]

In order to study the gauge transformation of \( Q(n) \), one can choose a type of infinitesimal transformation where the order of \( \xi(n) \) is the same order of \( Q(n) \). Combining Eqs. (110) with (111), we obtain

\[ \tilde{Q} = Q(0) + (Q(1) + \xi(1) Q(0)) \]

\[ + \frac{1}{2} \xi(2) + Q(0) \xi(1) = (112) \]

\[ = Q(0) + Q(1) + \frac{1}{2} Q(2) + O(\xi(3)). \]
Therefore, based on Eqs. (112) and (113), we conclude the infinitesimal transformations of the zeroth, first and second order perturbations in Eqs. (2).

C  Brief derivation of Eqs. (5a) and (5b)

For the first order metric perturbations $g^{(1)}_{\mu\nu}$, it could be split into a gauge invariant part $g^{(GI,1)}_{\mu\nu}$ and a gauge variant counter term $C^{(1)}_{\mu\nu}$, i.e.,

$$g^{(1)}_{\mu\nu} = g^{(GI,1)}_{\mu\nu} + C^{(1)}_{\mu\nu}. \quad (114)$$

Based on the gauge transformation of $g^{(1)}_{\mu\nu}$ in Eq. (3a), we could rewrite $g^{(GI,1)}_{\mu\nu}$ to be

$$g^{(GI,1)}_{\mu\nu} = \tilde{g}^{(1)}_{\mu\nu} = g^{(1)}_{\mu\nu} + \mathcal{L}_X g^{(0)}_{\mu\nu} - \tilde{C}^{(1)}_{\mu\nu}. \quad (115)$$

Based on Eqs. (114) and (115), we obtain

$$\tilde{C}^{(1)}_{\mu\nu} = C^{(1)}_{\mu\nu} + \mathcal{L}_X g^{(0)}_{\mu\nu}. \quad (116)$$

If we let $C^{(1)}_{\mu\nu} \equiv \mathcal{L}_X g^{(0)}_{\mu\nu}$, Eq. (116) is reduced to $\mathcal{L}_X g^{(0)}_{\mu\nu} = \mathcal{L}_X g^{(0)}_{\mu\nu}$. We can obtain

$$\tilde{X}^\mu - X^\mu = \xi^\mu. \quad (117)$$

Here, $X^\mu$ is rewritten in the form independent on the background metric. Therefore, we can rewrite Eq. (114) in the form of

$$g^{(1)}_{\mu\nu} = g^{(GI,1)}_{\mu\nu} + \mathcal{L}_X g^{(0)}_{\mu\nu}. \quad (118)$$

For the second order metric perturbations $g^{(2)}_{\mu\nu}$, we could also split it as

$$g^{(2)}_{\mu\nu} = g^{(GI,2)}_{\mu\nu} + C^{(2)}_{\mu\nu}, \quad (119)$$

where $g^{(GI,2)}_{\mu\nu}$ are the gauge invariant second order metric perturbations. Based on Eqs. (3b) and (119), we can also express $g^{(GI,2)}_{\mu\nu}$ as

$$g^{(GI,2)}_{\mu\nu} = \tilde{g}^{(2)}_{\mu\nu} = g^{(2)}_{\mu\nu} - \tilde{C}^{(2)}_{\mu\nu} = 2\mathcal{L}_{\xi_1} g^{(1)}_{\mu\nu} + (\mathcal{L}_{\xi_2} + \mathcal{L}_X) g^{(0)}_{\mu\nu} - \tilde{C}^{(2)}_{\mu\nu}. \quad (120)$$

Based on Eqs. (119) and (120), we have

$$\tilde{C}^{(2)}_{\mu\nu} - C^{(2)}_{\mu\nu} = 2\mathcal{L}_{\xi_1} g^{(1)}_{\mu\nu} + (\mathcal{L}_{\xi_2} + \mathcal{L}_X) g^{(0)}_{\mu\nu} = 2\mathcal{L}_X g^{(1)}_{\mu\nu} - \mathcal{L}_{\xi_1} g^{(0)}_{\mu\nu} + (\mathcal{L}_{\xi_2} + \mathcal{L}_X) g^{(0)}_{\mu\nu} = \mathcal{L}_X g^{(1)}_{\mu\nu} - \mathcal{L}_{\xi_1} g^{(0)}_{\mu\nu} = \mathcal{L}_X g^{(1)}_{\mu\nu}. \quad (121)$$

where we have introduced the infinitesimal vector $Y^\mu$ satisfying

$$\tilde{Y}^\mu - Y^\mu = \xi^\mu + [\xi_1, X]^\mu. \quad (122)$$

Therefore, we can rewrite Eq. (119) in the form of

$$g^{(2)}_{\mu\nu} = g^{(GI,2)}_{\mu\nu} + 2\mathcal{L}_X g^{(1)}_{\mu\nu} + (\mathcal{L}_X - \mathcal{L}^2) g^{(0)}_{\mu\nu}. \quad (123)$$

D  Scalar-vector-tensor decomposition

By making use of the transverse operators, namely, $T^i_j \equiv \delta^i_j - \partial^i \partial^j \Delta^{-1} \partial_j$, we could decompose an arbitrary spatial vector $U_i$ to be

$$U_i = U_i^{(T)} + \partial_i U_i^{(S)}, \quad (124)$$

where the transverse part $U_i^{(T)}$ and the longitudinal part $U_i^{(S)}$ are defined as

$$U_i^{(T)} \equiv \mathcal{T}^k_i U_k, \quad \mathcal{T}^k_i \equiv \Delta^{-1} \partial^k U_i, \quad (125a)$$

and $\Delta^{-1}$ is the inverse Laplacian operator.

For an arbitrary spatial tensor $S_{ij}$, we could decompose it as [60, 61]

$$S_{ij} = S_{ij}^{(H)} + 2\delta_{ij} S^{(F)} + 2\partial_i \partial_j S^{(E)} + \partial_i S_i^{(C)} + \partial_j S_j^{(C)} + \partial_j S_j^{(F)}. \quad (126)$$

where we can define the tensor part $S_{ij}^{(H)}$, the vector part $S_i^{(C)}$ and the scalar parts $S^{(F)}$, $S^{(E)}$ as

$$S_{ij}^{(H)} \equiv (T^k_i T^j_k - \frac{1}{2} T^k_i T^k_j) S_{kl}, \quad (127a)$$

$$S_i^{(C)} \equiv \Delta^{-1} \partial^k S_k i^l, \quad \Delta \equiv \mathcal{T}^k k^l S_{kl}, \quad (127b)$$

$$S^{(F)} \equiv \frac{1}{4} T^k k^l S_{kl}, \quad (127c)$$

$$S^{(E)} \equiv \frac{1}{2} \Delta^{-1} (\partial^k \Delta^{-1} \partial^l - \frac{1}{2} T^k T^l) S_{kl}. \quad (127d)$$

Here, $(T^k_i T^j_k - T^k_i T^k_j)$ denotes the transverse and traceless operator. Other kinds of decomposition can be found in Refs. [52, 62].

E  Explicit expressions of $X_{\mu\nu}$

For a given $X^\mu$, we express $X_{\mu\nu}$ in terms of the first order metric perturbations by following Eq. (40). The results are given by
\[ X_{\mu\nu} = \frac{4}{a^2} H X^\sigma g^{(1)}_{\mu\nu} + 2 X^\sigma \partial_\sigma \left( \frac{1}{a^2} g^{(1)}_{\mu\nu} \right) + \frac{2}{a^2} \partial_\mu X^\nu + \frac{2}{a^2} \partial_\nu X^\mu - \eta_{\mu\nu} \left( (4H^2 + 2\dot{H})(X^0)^2 + 2HX^\nu \partial_\nu X^0 - 2\eta_{\mu\nu} \partial_\mu X^0 \partial_\nu X^\nu \right) \]

\[ - \eta_{\mu\nu} \left( X^\mu \partial_\nu X^\sigma + \partial_\nu X^\sigma \partial_\mu X^\rho + 4H^2 \partial_\nu X^\sigma \right) \]

\[ - \eta_{\mu\nu} \left( X^\nu \partial_\mu X^\sigma + \partial_\nu X^\sigma \partial_\mu X^\rho + 4H^2 \partial_\nu X^\sigma \right) \]

For \( X^\mu \) in Eq. (22), we obtain \( X_{\mu\nu} \) to be

\[ X^{N}_{\mu} = -2 \left( \frac{2}{a} (\partial_\mu (a(\partial_0 e^{(1)} - b^{(1)}))) + (\partial_\mu e^{(1)} - b^{(1)})) \partial_0 + \delta^{ik}(c_k^{(1)} + \partial_k e^{(1)}) \partial_i \right) (2\phi^{(1)}) \]

\[ - \frac{1}{a} \partial_\mu (a(\partial_0 e^{(1)} - b^{(1)}))) + 2\delta^{ik}(\partial_0 (c_k^{(1)} + \partial_k e^{(1)}))(\partial_i b^{(1)} + 2\nu^{(1)} - \partial_i c^{(1)}), \]

\[ X^{N}_{0j} = \left( \delta^i_j \left( \frac{2a}{a} (\partial_0 e^{(1)} - b^{(1)})) + (\partial_0 e^{(1)} - b^{(1)})) \partial_0 + \delta^{ik}(c_k^{(1)} + \partial_k e^{(1)}) \partial_i + (\partial_0 (\partial_0 e^{(1)} - b^{(1)}))) \right) \]

\[ + \delta^{ik}(\partial_0 (c_k^{(1)} + \partial_k e^{(1)}))(\partial_i b^{(1)} + 2\nu^{(1)} - \partial_i c^{(1)} - 2(\partial_0 (\partial_0 e^{(1)} - b^{(1)}))(2\phi^{(1)}) \]

\[ - \frac{1}{a} \partial_0 (a(\partial_0 e^{(1)} - b^{(1)}))) + \delta^{ik}(\partial_0 (c_k^{(1)} + \partial_k e^{(1)})) \left( - 2\delta_{jk} (2\phi^{(1)} + \frac{a}{a} (\partial_0 e^{(1)} - b^{(1)})) \right) \]

\[ + 2\delta_{jk} \partial_0 e^{(1)} + \partial_j c^{(1)} + \partial_k c^{(1)} + 2h_{jk}^{(1)} \],

\[ X^{N}_{kl} = \left( \delta^i_k \delta^j_l \left( \frac{2a}{a} (\partial_0 e^{(1)} - b^{(1)})) + (\partial_0 e^{(1)} - b^{(1)})) \partial_0 + \delta^{ik}(c_k^{(1)} + \partial_k e^{(1)}) \partial_i \right) \]

\[ + ((\delta^i_k \delta^j_l \partial_0 + \delta^i_k \delta^j_l \partial_i)(c_k^{(1)} + \partial_k e^{(1)}))( - 2\delta_{ij} (2\phi^{(1)} + \frac{a}{a} (\partial_0 e^{(1)} - b^{(1)})) + 2\partial_i \partial_j e^{(1)} \]

\[ + \partial_i c^{(1)} + \partial_j c^{(1)} + 2h_{ij}^{(1)} \) + ((\delta^i_k \partial_0 + \delta^i_k \partial_i)(\partial_0 e^{(1)} - b^{(1)}))(\partial_i b^{(1)} + 2\nu^{(1)} - \partial_c c^{(1)}). \]
For $X^a$ in Eq. (26), we have $X^a_{\mu\nu}$ in the form of

$$X^a_{\mu\nu} = -2 \left( 2\phi^{(1)} + \frac{1}{a} \int d\eta \{a \phi^{(1)}\} \partial_\mu + \delta^{ij} \int d\eta \{\nu^{(1)}_i + \partial_i b^{(1)} + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\}\} \partial_j \phi^{(1)} + 2\delta^{ij} (\nu^{(1)}_j + \partial_j b^{(1)} + \frac{1}{a} \int d\eta' \{a \partial_j \phi^{(1)}\}) (\partial_i b^{(1)} + \nu^{(1)}_i) \right)$$

$$+ 2 \left( \int d\eta' \{a \partial_i \phi^{(1)}\} \right) (\partial_i \phi^{(1)} + \nu^{(1)}_i) + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\} (\partial_i \phi^{(1)} + \nu^{(1)}_i) + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\} (\partial_i \phi^{(1)} + \nu^{(1)}_i)$$

$$+ 2 \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\} \nu^{(1)}_i + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\} (\nu^{(1)}_i + \partial_i b^{(1)} + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\}) (\partial_i b^{(1)} + \nu^{(1)}_i)$$

$$+ 2 \partial_i \phi^{(1)} + 2 \partial_i c^{(1)} + 2 h^{(1)} \frac{2a}{a^2} \int d\eta' \{a \phi^{(1)}\}$$

$$+ (\delta^i_j \partial_i + \delta^i_j \partial_i) \int d\eta \{\nu^{(1)}_i + \partial_i b^{(1)} + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\}\},$$

$$X^a_{\nu i} = \delta^i_j \left( \frac{2a}{a^2} \int d\eta' \{a \phi^{(1)}\} + \frac{1}{a} \int d\eta \{a \phi^{(1)}\} \partial_\nu + \delta^{ij} \int d\eta \{\nu^{(1)}_i + \partial_i b^{(1)} + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\}\} \right)$$

$$+ \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\} \partial_j + \left( (\delta^i_j \partial_i + \delta^i_j \partial_i) \int d\eta \{\nu^{(1)}_i + \partial_i b^{(1)} + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\}\} \right)$$

$$(\nu^{(1)}_i + \partial_i b^{(1)} + \frac{1}{a} \int d\eta' \{a \partial_i \phi^{(1)}\} - \frac{2a}{a^2} \int d\eta' \{a \phi^{(1)}\}$$

$$+ \frac{1}{a} (\partial_i b^{(1)} + \nu^{(1)}_i) \int d\eta \{a (\delta^i_j \partial_i + \delta^i_j \partial_i) \phi^{(1)}\}.$$

$$X^a_{\nu i} = \left( \delta^i_j \left( \frac{k_i k_j}{|k|^2} \right) \delta^{lm} - \frac{k^m k_j}{|k|^2} \right)$$

$$- \frac{1}{2} \left( \delta^i_j - \frac{k_i k_j}{|k|^2} \right) \left( \delta^{lm} - \frac{k^l k^m}{|k|^2} \right)$$

$$= (\delta^i_j - n^i n_j) (\delta^{lm} - n^m n^l)$$

$$- \frac{1}{2} (\delta^l_j - n^l n_j) (\delta^{lm} - n^i n^m),$$

where we have let $n^i \equiv k^i / |k|$ for simplicity.

The 3-dimensional momentum space can be spanned by a set of normalized orthogonal bases $\{\epsilon_i, \tilde{\epsilon}_i, n_i\}$ which satisfy

$$\epsilon_i \epsilon^i = \tilde{\epsilon}_i \tilde{\epsilon}^i = n_i n^i = 1, \quad \epsilon_i \tilde{\epsilon}^i = \epsilon_i n^i = \tilde{\epsilon}_i n^i = 0.$$

The indices are raised and lowered via Kronecker delta $\delta^i_j$, which can be written as

$$\delta^i_j = \epsilon_i \epsilon^j + \tilde{\epsilon}_i \tilde{\epsilon}^j + n_i n^j.$$

Substituting Eq. (134) into Eq. (132), we can obtain

$$\Lambda^m_{lj} = \epsilon_i \epsilon^l + \tilde{\epsilon}_i \tilde{\epsilon}^l + n_i n^l.$$
\[ \Lambda_{ij}^{lm} = (\epsilon_i \epsilon^l + \epsilon_j \epsilon^m)(\epsilon_j \epsilon^m + \epsilon_i \epsilon^l) - \frac{1}{2} (\epsilon_i \epsilon_j + \epsilon_j \epsilon_i)(\epsilon^l \epsilon^m + \epsilon^m \epsilon^l) \]

\[ \begin{align*}
&= \epsilon_i \epsilon_j \epsilon^l \epsilon^m + \epsilon_i \epsilon_j \epsilon^m \epsilon^l + \epsilon_i \epsilon_j \epsilon^l \epsilon^m + \epsilon_i \epsilon_j \epsilon^m \epsilon^l - \frac{1}{2} (\epsilon_i \epsilon_j \epsilon^l \epsilon^m + \epsilon_i \epsilon_j \epsilon^m \epsilon^l + \epsilon_j \epsilon_i \epsilon^l \epsilon^m + \epsilon_j \epsilon_i \epsilon^m \epsilon^l) \\
&= \frac{1}{2} (\epsilon_i \epsilon_j \epsilon^l \epsilon^m + \epsilon_i \epsilon_j \epsilon^m \epsilon^l + \epsilon_i \epsilon_j \epsilon^l \epsilon^m + \epsilon_i \epsilon_j \epsilon^m \epsilon^l) + \frac{1}{2} (\epsilon_i \epsilon_j \epsilon^l \epsilon^m - \epsilon_i \epsilon_j \epsilon^m \epsilon^l + \epsilon_j \epsilon_i \epsilon^l \epsilon^m - \epsilon_j \epsilon_i \epsilon^m \epsilon^l) \\
&= \frac{1}{\sqrt{2}} (\epsilon_i \epsilon_j + \epsilon_j \epsilon_i \frac{1}{\sqrt{2}} (\epsilon^l \epsilon^m + \epsilon^m \epsilon^l) + \frac{1}{\sqrt{2}} (\epsilon_i \epsilon_j - \epsilon_j \epsilon_i) \frac{1}{\sqrt{2}} (\epsilon^l \epsilon^m - \epsilon^m \epsilon^l) \\
&=: \epsilon^x_{ij} \epsilon^x_{lm} + \epsilon^+_{ij} \epsilon^+_{lm}, \quad (135)
\end{align*} \]

where we define

\[ \begin{align*}
\epsilon^x_{ij} &= \frac{1}{\sqrt{2}} (\epsilon_i \epsilon_j + \epsilon_j \epsilon_i), \quad (136a) \\
\epsilon^+_{ij} &= \frac{1}{\sqrt{2}} (\epsilon_i \epsilon_j - \epsilon_j \epsilon_i). \quad (136b)
\end{align*} \]

Here, \( \epsilon^x_{ij} \) and \( \epsilon^+_{ij} \) are called the transverse and traceless polarization tensors. They are widely used to study the gravitational waves [61]. We note that the third equal in Eq. (135) is established when \( \Lambda_{ij}^{lm} \) act on the rank-two symmetric tensors.

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