On Drury’s solution of Bhatia & Kittaneh’s question✩

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Abstract

Let $A, B$ be $n \times n$ positive semidefinite matrices. Bhatia and Kittaneh asked whether it is true

$$\sqrt{\sigma_j(AB)} \leq \frac{1}{2} \lambda_j(A + B), \quad j = 1, \ldots, n$$

where $\sigma_j(\cdot)$, $\lambda_j(\cdot)$, are the $j$-th largest singular value, eigenvalue, respectively. The question was recently solved by Drury in the affirmative. This article revisits Drury’s solution. In particular, we simplify the proof for a key auxiliary result in his solution.

Keywords: AM-GM inequality, singular value, eigenvalue.

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1. Introduction

Bhatia has made many fundamental contributions to Matrix Analysis. One of his favorite topics is matrix inequalities. Roughly speaking, matrix inequalities are noncommutative versions of the corresponding scalar inequalities. To get a glimpse of this topic, let us start with a simple example. The simplest AM-GM inequality says that

$$a, b > 0 \implies \frac{a + b}{2} \geq \sqrt{ab}.$$
Now it is known that [3, p. 107] its most “direct” noncommutative version is

$$A, B$$ are $$n \times n$$ positive definite matrices $$\Rightarrow \frac{A + B}{2} \geq A^\sharp B, \quad (1)$$

where $$A^\sharp B := A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}$$ is called the geometric mean of $$A$$ and $$B$$. For two Hermitian matrices $$A$$ and $$B$$ of the same size, in this article, we write $$A \geq B$$ (or $$B \leq A$$) to mean that $$A - B$$ is positive semidefinite.

If we denote $$S := A^\sharp B$$, then $$B = S A^{-1} S$$. Thus a variant of (1) is the following

$$A, S$$ are $$n \times n$$ positive definite matrices $$\Rightarrow A + S A^{-1} S \geq 2S. \quad (2)$$

There is a long tradition in matrix analysis of comparing eigenvalues or singular values. To proceed, let us fix some notation. The $$j$$-th largest singular value of a complex matrix $$A$$ is denoted by $$\sigma_j(A)$$. If all the eigenvalues of $$A$$ are real, then we denote its $$j$$-th largest one by $$\lambda_j(A)$$. By Weyl’s Monotonicity Theorem [2, p. 63], (1) readily implies

$$\lambda_j(A + B) \geq 2 \lambda_j(A^\sharp B), \quad j = 1, \ldots, n.$$  

As far as the eigenvalues or singular values are considered, there are other versions of “geometric mean”. Bhatia and Kittaneh studied this kind of inequalities over a twenty year period [4, 5, 6]. Their elegant results include the following: If $$A, B$$ are $$n \times n$$ positive semidefinite matrices, then

$$\lambda_j(A + B) \geq 2 \sqrt{\lambda_j(AB)} = 2 \sigma_j(A^{\frac{1}{2}} B^{\frac{1}{2}}); \quad (3)$$

$$\lambda_j(A + B) \geq 2 \lambda_j(A^{\frac{1}{2}} B^{\frac{1}{2}}) \quad (4)$$

for $$j = 1, \ldots, n$$.

To complete the picture in (3)-(4), they asked whether it is true

$$\lambda_j(A + B) \geq 2 \sqrt{\sigma_j(AB)}, \quad j = 1, \ldots, n?$$

This question was recently answered in the affirmative by Drury in his very brilliant work [7]. The purpose of this expository article is to revisit Drury’s
solution. Hopefully, some of our arguments would shed new insights into the beautiful result, which is now a theorem.

**Theorem 1.1.** If $A, B$ are $n \times n$ positive definite semidefinite matrices, then

$$\lambda_j(A + B) \geq 2\sqrt{\sigma_j(AB)}, \quad j = 1, \ldots, n. \quad (5)$$

2. Drury’s reduction in proving (5)

Our presentation here is just slightly different from that in [7]. Assume without loss of generality that $A, B$ are positive definite (the general case is by a standard perturbation argument). Fix $r$ in the range $1 \leq r \leq n$ and normalize so that $\sigma_r(AB) = 1$. Our goal is to show that $\lambda_r(A + B) \geq 2$.

Note that $\sigma_r(AB) = 1$ is the same as $\lambda_r(AB^2A) = 1$. Consider the spectral decomposition

$$AB^2A = \sum_{k=1}^{n} \lambda_k(AB^2A)P_k,$$

where $P_1, P_2, \ldots, P_n$, are orthogonal projections. Then $\lambda_k(AB^2A) \geq 1$ for $k = 1, \ldots, r$. Define a positive semidefinite

$$B_1 := A^{-1}\left(\sum_{k=1}^{r} P_k\right)A^{-1/2}.$$

It is easy to see (indeed, from $B^2 \geq B_1^2$) that

$$B = A^{-1}\left(\sum_{k=1}^{r} \lambda_k(AB^2A)P_k\right)A^{-1/2} \geq B_1.$$

So we are done if we can show

$$\lambda_r(A + B_1) \geq 2. \quad (6)$$

As $B_1$ has rank $r$, split the underlying space as the direct sum of image and kernel of $B_1$, we may partition conformally $B_1$ and $A$ in the following form

$$B_1 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}. \quad (7)$$
Note $AB_1^2A$ is an orthogonal projection of rank $r$, the same is true for $B_1A^2B_1$. Therefore,

$$B_1A^2B_1 = \begin{pmatrix} X(A_{11}^2 + A_{12}A_{12}^*)X & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow X(A_{11}^2 + A_{12}A_{12}^*)X = I_r$$

where $I_r$ is the $r \times r$ identity matrix.

Finally, observe that

$$X(A_{11}^2 + A_{12}A_{12}^*)X = I_r$$

Therefore, $A$ would follow from

$$\lambda_r (A_1 + B_1) \geq 2.$$  \hspace{1cm} (7)

Thus, the remaining effort is made to show (7), which we formulate as a proposition.

**Proposition 2.1.** Let $A_{11}$ and $X$ be $r \times r$ positive definite matrices and $A_{12}$ is an $(n-r) \times (n-r)$ matrix such that $X(A_{11}^2 + A_{12}A_{12}^*)X = I_r$. Then

$$\lambda_r \begin{pmatrix} A_{11} + X & A_{12} \\ A_{12}^* & A_{12}^*A_{11}^{-1}A_{12} \end{pmatrix} \geq 2.$$  \hspace{1cm} (8)

3. The mystified part

In order to prove (8), Drury made the following key observations.

**Proposition 3.1.** [7, Proposition 2] Let $M$ and $N$ be $r \times r$ positive definite matrices. Then

$$\lambda_r \begin{pmatrix} M & (M_N)^{-1} \\ (M_N)^{-1} & N \end{pmatrix} \geq 2.$$  \hspace{1cm} (9)

**Proposition 3.2.** [7, Theorem 7] Let $L$ and $M$ be $r \times r$ positive definite matrices, and let $Z$ be an $r \times r$ matrix such that $ML(I + ZZ^*)LM = I_r$. Then

$$\lambda_r \begin{pmatrix} L + M & LZ \\ Z^*L & Z^*LZ \end{pmatrix} \geq 2.$$  \hspace{1cm} (9)
The way that Drury proved (9) is by showing that
\[ T := \begin{pmatrix} L + M & LZ \\ Z^*L & Z^*LZ \end{pmatrix} \]
and
\[ R := \begin{pmatrix} M & (M^*N)^{-1} \\ (M^*N)^{-1} & N \end{pmatrix} \]
have the same characteristic polynomial, and so the eigenvalues of \( R \) and \( T \) coincide. As explained in [8], this connection (between \( R \) and \( T \)) is mystified. Formally, the mystified part also comes from \( R \) and \( T \) themselves, indeed, \( T \) is always positive semidefinite while \( R \) is not!

In order to apply Proposition 3.2 to Proposition 2.1, Drury discussed three possible relations between the size \( n \) and \( r \). Our proof of Proposition 2.1 in the next section allows us to skip this discussion on the size.

4. Proof of Proposition 2.1

The following lemma slightly generalizes Proposition 3.1 in form.

**Lemma 4.1.** Let \( X \) be a \( r \times r \) positive definite matrix and let \( S \) be a \( r \times r \) nonsingular matrix. Then
\[ \lambda_r \left( \begin{pmatrix} SX^{-1}S^* & (S^{-1})^* \\ S^{-1} & X \end{pmatrix} \right) \geq 2. \]

**Proof.** Consider the polar decomposition of \( S \), \( S = U|S| \), where \( U \) is unitary and \( |S| = (S^*S)^{\frac{1}{2}} \). The matrix \( \begin{pmatrix} SX^{-1}S^* & (S^{-1})^* \\ S^{-1} & X \end{pmatrix} \) is unitarily similar to
\[ \begin{pmatrix} U^*SX^{-1}S^*U & U^*(S^{-1})^* \\ S^{-1}U & X \end{pmatrix} = \begin{pmatrix} |S|X^{-1}|S| & |S|^{-1} \\ |S|^{-1} & X \end{pmatrix}. \]

As \( P := \frac{1}{2} \begin{pmatrix} I_r \\ I_r \end{pmatrix} \) is a partial isometry,
\[
\lambda_r \begin{pmatrix} SX^{-1}S^* & (S^{-1})^* \\ S^{-1} & X \end{pmatrix} = \lambda_r \begin{pmatrix} |S|X^{-1}|S| & |S|^{-1} \\ |S|^{-1} & X \end{pmatrix} 
\]
\[
\geq \lambda_r \begin{pmatrix} P^* \begin{pmatrix} |S|X^{-1}|S| & |S|^{-1} \\ |S|^{-1} & X \end{pmatrix} P \\ X + |S|X^{-1}|S| + |S|^{-1} \end{pmatrix} 
\]
\[
= \lambda_r \left( \frac{X + |S|X^{-1}|S| + |S|^{-1}}{2} \right) \geq \lambda_r (|S| + |S|^{-1}) \geq 2, \quad \text{by (2)}
\]

The required result follows.

Now we are ready to give a simpler proof of Proposition 2.1.

**Proof.** Consider the factorization

\[
\begin{pmatrix} A_{11} + X & A_{12} \\ A_{12}^* & A_{12}^*A_{11}^{-1}A_{12} \end{pmatrix} = \begin{pmatrix} A_{11}^\frac{1}{2} & X^{\frac{1}{2}} \\ A_{12}^\frac{1}{2}A_{11}^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} A_{11}^\frac{1}{2} & X^{\frac{1}{2}} \\ A_{12}^\frac{1}{2}A_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}^*.
\]

Clearly, \( \begin{pmatrix} A_{11} + X & A_{12} \\ A_{12}^* & A_{12}^*A_{11}^{-1}A_{12} \end{pmatrix} \) is unitarily similar to

\[
\begin{pmatrix} A_{11}^\frac{1}{2} & X^{\frac{1}{2}} \\ A_{12}^\frac{1}{2}A_{11}^{-\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} A_{11}^\frac{1}{2} & X^{\frac{1}{2}} \\ A_{12}^\frac{1}{2}A_{11}^{-\frac{1}{2}} & 0 \end{pmatrix} = \begin{pmatrix} A_{11} + A_{11}^\frac{1}{2}A_{12}A_{12}^*A_{11}^{-\frac{1}{2}} & A_{11}^\frac{1}{2}X^{\frac{1}{2}} \\ X_{11}^\frac{1}{2}A_{11}^{-\frac{1}{2}} & X \end{pmatrix} = \begin{pmatrix} A_{11}^\frac{1}{2}X^{-\frac{1}{2}}A_{11}^{-\frac{1}{2}} & A_{11}^\frac{1}{2}X^{\frac{1}{2}} \\ X_{11}^\frac{1}{2}A_{11}^{-\frac{1}{2}} & X \end{pmatrix}.
\]

Now setting \( S = A_{11}^{-\frac{1}{2}}X^{-\frac{1}{2}} \) in Lemma 4.1 yields the desired result.

**5. A conjecture**

A weighted version of (3) is known. That is, if \( A, B \) are \( n \times n \) positive semidefinite matrices, then for any \( t \in [0,1] \) and \( j = 1, \ldots, n \)

\[
\lambda_j((1-t)A + tB) \geq \sqrt{\lambda_j(A^{2(1-t)}B^{2t})} = \sigma_j(A^{1-t}B^t). \quad (10)
\]
Inequality (10) is due to Ando [1]. With (10), it is not hard to present a weighted version of (10).

**Proposition 5.1.** If $A, B$ are $n \times n$ positive semidefinite matrices, then for any $t \in [0, 1]$ and $j = 1, \ldots, n$

$$\lambda_j((1 - t)A + tB) \geq \lambda_j(A^{1-t}B^t).$$

(11)

**Proof.** By (10) and the matrix convexity of the square function,

$$\lambda_j(A^{1-t}B^t) = \sigma_j^2(A^{(1-t)/2}B^{t/2})$$

$$\leq \lambda_j((1 - t)A^{1/2} + tB^{1/2})^2$$

$$\leq \lambda_j((1 - t)A + tB).$$


We conclude the paper with the following conjecture

**Conjecture 5.2.** If $A, B$ are $n \times n$ positive definite semidefinite matrices, then for any $t \in [0, 1]$

$$\lambda_j((1 - t)A + tB) \geq \sqrt{\sigma_j(A^{2(1-t)}B^{2t})}, \quad j = 1, \ldots, n.$$

The present method of proof does not seem to lead to a solution of this conjecture.

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