QUATERNIONIC REGULARITY
AND THE ∂-NEUMANN PROBLEM IN $\mathbb{C}^2$

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Abstract. Let $\Omega$ be a domain in the quaternionic space $\mathbb{H}$. We prove a differential criterion that characterizes Fueter-regular quaternionic functions $f : \Omega \to \mathbb{H}$ of class $C^1$. We find differential operators $T$ and $N$, with complex coefficients, such that a function $f$ is regular on $\Omega$ if and only if $(N - jT)f = 0$ on $\partial \Omega$ ($j$ a basic quaternion) and $f$ is harmonic on $\Omega$. As a consequence, by means of the identification of $\mathbb{H}$ with $\mathbb{C}^2$, we obtain a non-tangential holomorphicity condition which generalizes a result of Aronov and Kytmanov. We also show how the differential criterion and regularity are related to the $\overline{\partial}$-Neumann problem in $\mathbb{C}^2$.

1. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{C}^2$. Let $\mathbb{H}$ be the space of real quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where $i, j, k$ denote the basic quaternions. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by means of the mapping that associates the quaternion $q = z_1 + z_2j$ with the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$.

In this paper we give a boundary differential criterion that characterizes (left) regular functions $f : \Omega \to \mathbb{H}$ (in the sense of Fueter) among harmonic functions. We show (Corollary 3) that there exist first order differential operators $T$ and $N$, with complex coefficients, such that a harmonic function $f : \Omega \to \mathbb{H}$, of class $C^1$ on $\Omega$, is regular if and only if $(N - jT)f = 0$ on $\partial \Omega$.

In order to obtain this result we study a related space of functions that satisfy a variant of the Cauchy-Riemann-Fueter equations, the space $R(\Omega)$ of $\psi$-regular functions on $\Omega$ (see [2] for the precise definitions) for the particular choice $\psi = \{1, i, j, -k\}$ of the structural vector. These functions have been studied by many authors (see for instance [1, 2, 3, 4]). The space $R(\Omega)$ contains the identity mapping and any holomorphic mapping $(f_1, f_2)$ on $\Omega$ defines a $\psi$-regular function $f = f_1 + f_2j$. This is no more true if we replace the class of $\psi$-regular functions with that of regular functions. The definition of $\psi$-regularity is also equivalent to that of $q$-holomorphicity given by Joyce in [5], in the setting of hypercomplex manifolds.

The space $R(\Omega)$ exhibits other interesting links with the theory of two complex variables. In particular, Vasilevski and Shapiro [6] have shown that the Bochner-Martinelli kernel $U(\zeta, z)$ can be considered as a first complex component of the Cauchy-Fueter kernel associated to $\psi$-regular functions. This property was already observed by Fueter (see [7]) in the general $n$-dimensional case, by means of an imbedding of $\mathbb{C}^n$ in a real Clifford algebra. Note that regular functions are in a simple correspondence with $\psi$-regular functions, since they can be obtained from them by means of a real coordinate reflection in $\mathbb{H}$.
We prove (Theorem 2) that a harmonic function \( f \) on \( \Omega \), of class \( C^1 \) on \( \overline{\Omega} \), is \( \psi \)-regular on \( \Omega \) if and only if \( (\partial_n - jL) f = 0 \) on \( \partial\Omega \), where \( \partial_n \) is the normal part of \( \partial \) and \( L \) is a tangential Cauchy-Riemann operator.

This equation, which appeared in [8] in connection with the characterization of the traces of pluriharmonic functions, can be considered as a generalization of both the CR-tangential equation \( L(f) = 0 \) and the condition \( \partial_n f = 0 \) on \( \partial\Omega \) that distinguishes holomorphic functions among complex harmonic functions (Aronov and Kytmanov, see [9, 10, 11]).

As an application of the differential condition for \( \psi \)-regular functions, we also obtain (Theorem 7) a differential criterion for holomorphicity of functions that generalizes, for a domain with connected boundary in \( C^2 \), the result of Aronov and Kytmanov.

In §3 we give a weak formulation of the differential criterion of \( \psi \)-regularity, which makes sense also when the harmonic function \( f \) is only continuous on the closure \( \overline{\Omega} \). We obtain trace theorems for \( \psi \)-regular functions, with applications to holomorphic functions. Other results in the same vein were given by Pertici in [12], with generalizations to several quaternionic variables. Integral criteria for regularity were given also in [11 2 6], under the assumption that the trace function satisfies a Hölder condition or belongs to a \( L^p(\partial\Omega) \) space.

In §4 we study the relation between regularity and the \( \overline{\partial} \)-Neumann problem in \( C^2 \) in the formulation given by Kytmanov in [13]§14–18. We are interested in a quaternionic analogue of the Hilbert transform, which relates one of the complex components of a \( \psi \)-regular function to the boundary values of the other. We refer to [3] and [14] for generalizations of the Hilbert transform to the quaternionic setting. In these papers the functions considered are defined on plane or spatial domains, while we are interested in domains of two complex variables. In the latter case, pseudoconvexity becomes relevant, since such a domain is pseudoconvex if and only if every complex harmonic function on it is a complex component of a \( \psi \)-regular function (cf. Naser[15] and Nono[16]).

In particular we show (Corollary 10) that if \( \Omega \) is a strongly pseudoconvex domain of class \( C^\infty \) or a weakly pseudoconvex domain with real-analytic boundary, then the operator that associates to \( f = f_1 + f_2 j \) the restriction to \( \partial\Omega \) of its first complex component \( f_1 \) induces an isomorphism between the quotient spaces \( \mathcal{R}^\infty(\Omega) / A^\infty(\Omega, C^2) \) and \( C^\infty(\partial\Omega) / CR(\partial\Omega) \), where \( \mathcal{R}^\infty(\Omega) \) denotes the space of \( \psi \)-regular functions that are smooth up to the boundary and \( A^\infty(\Omega, C^2) \) is the space \( \text{Hol}(\Omega, C^2) \cap C^\infty(\overline{\Omega}, C^2) \).

Some of the results contained in the present paper have been announced in [17].
2.3. Let $H$ be the algebra of quaternions. The elements of $H$ have the form
\[ q = x_0 + ix_1 + jx_2 + kx_3, \]
where $x_0, x_1, x_2, x_3$ are real numbers and $i, j, k$ denote the basic quaternions.

We identify the space $C^2$ with the set $H$ by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j$. The commutation rule is then $aj = ja$ for every $a \in \mathbb{C}$, and the quaternionic conjugation is
\[ \bar{q} = x_0 - ix_1 - jx_2 - kx_3 = \bar{z}_1 - \bar{z}_2 j. \]
We refer to [19] and [20] for the theory of quaternionic analysis and its generalization represented by Clifford analysis. We will denote by $D$ the left Cauchy-Riemann-Fueter operator
\[ D = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3}. \]
A quaternionic $C^1$ function $f = f_1 + f_2 j$, is (left-)regular on a domain $\Omega \subseteq \mathbb{H}$ if $Df = 0$ on $\Omega$. We prefer to work with another class of regular functions, defined by the Cauchy-Riemann-Fueter operator associated with the structural vector $\psi = \{1, i, j, -k\}$:
\[ D' = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} - k \frac{\partial}{\partial x_3} = 2 \left( \frac{\partial}{\partial \bar{z}_1} + j \frac{\partial}{\partial \bar{z}_2} \right). \]

A quaternionic $C^1$ function $f = f_1 + f_2 j$, is (left-)$\psi$-regular on a domain $\Omega$, if $D' f = 0$ on $\Omega$. This condition is equivalent to the following system of complex differential equations:
\[ \frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial f_2}{\partial \bar{z}_2}, \quad \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial f_2}{\partial \bar{z}_1} \]
or to the equation \( \bar{\partial} f_1 = -\frac{1}{2} \partial (\bar{f}_2 dz_1 \wedge d\bar{z}_2) \). Note that the identity mapping is \( \psi \)-regular, and any holomorphic mapping \((f_1, f_2)\) on \( \Omega \) defines a \( \psi \)-regular function \( f = f_1 + f_2 \). This is no more true if we replace the class of \( \psi \)-regular functions with that of regular functions.

If \( \Omega \) is pseudoconvex, every complex harmonic function \( f_1 \) on \( \Omega \) is a complex component of a \( \psi \)-regular function \( f = f_1 + f_2 \), since the \((1, 2)\)-form \( \bar{\partial} f_1 \) is \( \partial \)-closed on \( \Omega \). (cf. [15] and [16] for this result and its converse). Regular and \( \psi \)-regular functions are real analytic on \( \Omega \), and they are harmonic with respect to the Laplace operator in \( \mathbb{R}^4 \). We refer, for instance, to [1, 2, 3, 4] for the properties of structural vectors and \( \psi \)-regular functions (in these references they are called \( \psi \)-hyperholomorphic functions).

**Remark 1.** Let \( \gamma \) be the transformation of \( \mathbb{C}^2 \) defined by \( \gamma(z_1, z_2) = (z_1, \bar{z}_2) \). Then a \( C^1 \) function \( f \) is regular on the domain \( \Omega \) if, and only if, \( f \circ \gamma \) is \( \psi \)-regular on \( \gamma^{-1}(\Omega) \).

2.4. A definition equivalent to \( \psi \)-regularity has been given by Joyce in [5], in the setting of hypercomplex manifolds. Joyce introduced the module of \( q \)-holomorphic functions on a hypercomplex manifold. A hypercomplex structure on the manifold \( \mathbb{H} \) is given by the complex structures \( J_1, J_2 \) defined on \( T^* \mathbb{H} \simeq \mathbb{H} \) by left multiplication by \( i \) and \( j \), and by \( J_3 = J_1J_2 \). These structures act on the basis elements in the following way:

\[
\begin{align*}
J_1 dx_0 &= -dx_1, & J_1 dx_2 &= -dx_3, & J_2 dx_0 &= -dx_2, & J_2 dx_1 &= dx_3, \\
J_3 dx_0 &= dx_3, & J_3 dx_1 &= dx_2.
\end{align*}
\]

An easy computation shows that a differentiable function \( f : \Omega \rightarrow \mathbb{H} \) is \( \psi \)-regular on \( \Omega \) if and only if it is \( q \)-holomorphic, that is

\[
df + iJ_1(df) + jJ_2(df) + kJ_3(df) = 0 \quad \text{on } \Omega.
\]

Equivalently, if \( f = f^0 + if^1 + jf^2 + kf^3 \) is the real decomposition of \( f \) with respect to the standard basis, \( f \) is \( \psi \)-regular if and only if the real equation

\[
df^0 = J_1(df^1) + J_2(df^2) + J_3(df^3)
\]

is satisfied on \( \Omega \). Returning to complex components, we can rewrite equations (11) by means of the complex structure \( J_2 \) as follows:

\[
\bar{\partial} f_1 = J_2(\partial \bar{f}_2).
\]

Note that our notations are slightly different from those of Joyce [5]. With our choices, \( \psi \)-regular functions form a right \( \mathbb{H} \)-module and a complex valued function \( f = f^0 + if^1 : \Omega \rightarrow \mathbb{C} \) is holomorphic w.r.t. a complex structure \( J \) if it satisfies the Cauchy-Riemann equations \( df^0 = J(df^1) \) on \( \Omega \) or, equivalently, \( df + iJ(df) = 0 \).

Let \( \mathbb{J}_p \) be the complex structure defined by an imaginary quaternion \( p = p_1i + p_2j + p_3k \) in the unit sphere \( S^2 \) as \( \mathbb{J}_p = p_1J_1 + p_2J_2 + p_3J_3 \). Then every \( \mathbb{J}_p \)-holomorphic function \( f = f^0 + if^1 : \Omega \rightarrow \mathbb{C} \) defines a \( \psi \)-regular function \( \tilde{f} = f^0 + pf^1 \) on \( \Omega \). The original function \( f \) can be recovered from \( \tilde{f} \) by means of the formula

\[
\tilde{f} = \text{Re}(\tilde{f}) + i \text{Re}(-p\tilde{f}).
\]

The \( \psi \)-regular functions obtained in this way can be identified with the holomorphic functions from the complex manifold \((\Omega, \mathbb{J}_p)\) to the manifold \((\mathbb{C}_p, L_p)\),
where \( \mathbb{C}_p = \langle 1, p \rangle \) is a copy of \( \mathbb{C} \) in \( \mathbb{H} \) and \( L_p \) is the complex structure defined on \( T^* \mathbb{C}_p \simeq \mathbb{C}_p \) by left multiplication by \( p \). In fact,

\[
d^*\tilde{f} = J_p(df^1) - pJ_p(df^0) = -pJ_p(df^0 + pdf^1) = -pJ_p(d\tilde{f}).
\]

More generally, holomorphic maps w.r.t. any complex structure \( J_p \) induce \( \psi \)-regular functions, since for any positive orthonormal basis \( \{1, p, q, pq\} \) of \( \mathbb{H} \) (\( p, q \in S^2 \)), \( \psi \)-regular functions are the solutions of the equation

\[
\overline{\partial}_p f_1 = J_q(\partial_p \overline{\Omega}_2),
\]

where \( f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2q, \overline{\Omega}_2 = f^2 - pf^3 \) and \( \overline{\partial}_p \) is the Cauchy-Riemann operator w.r.t. \( J_p \):

\[
\overline{\partial}_p = \frac{1}{2} (d + pJ_p \circ d).
\]

We shall return in a subsequent paper on the problem of the characterization of the (proper) submodule of holomorphic functions in the \( \mathbb{H} \)-module of \( \psi \)-regular functions.

2.5. Let’s denote by \( G \) the Cauchy-Fueter quaternionic kernel defined by

\[
G(p - q) = \frac{1}{2\pi^2} \frac{\bar{p} - \bar{q}}{|p - q|^4},
\]

and by \( G' \) the Cauchy kernel for \( \psi \)-regular functions:

\[
G'(p - q) = \frac{1}{2\pi^2} \frac{y_0 - x_0 - i(y_1 - x_1) - j(y_2 - x_2) + k(y_3 - x_3)}{|p - q|^4},
\]

where \( p = y_0 + iy_1 + jy_2 + ky_3, q = x_0 + ix_1 + jx_2 + kx_3 \).

Let \( \sigma(q) \) be the quaternionic valued 3-form

\[
\sigma(q) = dx[0] - idx[1] + jdx[2] + kdx[3],
\]

where \( dx[k] \) denotes the product of \( dx_0, dx_1, dx_2, dx_3 \) with \( dx_k \) deleted. Then the Cauchy-Fueter integral formula for left-\( \psi \)-regular functions on \( \Omega \) that are continuous on \( \overline{\Omega} \), holds true:

\[
\int_{\partial\Omega} G'(p - q)\sigma(p)f(p) = \begin{cases} f(q) & \text{for } q \in \Omega, \\ 0 & \text{for } q \notin \overline{\Omega}. \end{cases}
\]

In [6] (see also [7] and [2]) it was shown that, for a family of structural vectors, including \( \{i, j, -k\} \), the two-dimensional Bochner-Martinelli form \( U(\zeta, z) \) can be considered as a first complex component of the Cauchy-Fueter kernel associated to \( \psi \)-regular functions. Let \( q = z_1 + z_2j, p = \zeta_1 + \zeta_2j \). Then

\[
G'(p - q)\sigma(p) = U(\zeta, z) + \omega(\zeta, z)j,
\]

where \( \omega(\zeta, z) \) is the following complex \((1, 2)\)-form:

\[
\omega(\zeta, z) = \frac{1}{4\pi^2|\zeta - z|^4} ((\bar{\zeta}_1 - \bar{z}_1)d\zeta_1 + (\bar{\zeta}_2 - \bar{z}_2)d\zeta_2) \wedge \overline{d\zeta}.
\]

Here \( d\zeta = d\zeta_1 \wedge d\zeta_2 \) and we choose the orientation of \( \mathbb{C}^2 \) given by the volume form \( \frac{1}{4}dz_1 \wedge dz_2 \wedge \overline{dz_1} \wedge \overline{dz_2} \).
3. Differential criteria for regularity and holomorphicity

3.1. We now rewrite the representation formula of Cauchy-Fueter for \( \psi \)-regular functions in complex form. We use results from [2] to relate the form \( \omega(\zeta, z) \) to the tangential operator \( L_{12} \), that we will denote simply by \( L \). We show that the Bochner-Martinelli formula can then be applied to obtain a criterion that distinguishes regular functions among harmonic functions on a domain \( \Omega \) in \( \mathbb{C}^2 = \mathbb{H} \).

Let \( g(\zeta, z) = \frac{1}{4\pi}|\zeta - z|^{-2} \) be the fundamental solution of the complex laplacian on \( \mathbb{C}^2 \).

**Proposition 1.** Let \( \Omega \) be a bounded domain of class \( C^1 \) in \( \mathbb{H} \). Let \( f : \Omega \to \mathbb{H} \) be a quaternionic function, of class \( C^1 \) on \( \Omega \). Then \( f \) is (left-)\( \psi \)-regular on \( \Omega \) if, and only if, the following representation formula holds on \( \Omega \):

\[
f(z) = \int_{\partial\Omega} U(\zeta, z)f(\zeta) + 2\int_{\partial\Omega} g(\zeta, z)jL(f(\zeta))d\sigma
\]

where \( d\sigma \) is the Lebesgue measure on \( \partial\Omega \) and the tangential operator \( L \) acts on \( f = f_1 + f_2j \) as \( L(f) = L(f_1) + L(f_2)j \).

**Proof.** The integral of Cauchy-Fueter in complex form is

\[
\int_{\partial\Omega} G'(p - q)\sigma(p)f(p) = \int_{\partial\Omega} U(\zeta, z)f(\zeta) + \int_{\partial\Omega} \omega(\zeta, z)jf(\zeta).
\]

From Proposition 6.3 in [2], we get that the last integral is equal to

\[
\int_{\partial\Omega} \omega(\zeta, z)f_1j - \int_{\partial\Omega} \omega(\zeta, z)f_2 = 2\int_{\partial\Omega} g(\zeta, z)\left(L(f_1)j - L(f_2)\right)d\sigma = 2\int_{\partial\Omega} g(\zeta, z)jL(f)d\sigma.
\]

Then the result follows from the Cauchy-Fueter integral formula for \( \psi \)-regular functions. \( \square \)

If \( f = f_1 + f_2j \) is a \( \psi \)-regular function on \( \Omega \), of class \( C^1 \) on \( \overline{\Omega} \), then from equations (1) we get that it satisfies the equation

\[
(\partial_n - jL)f = 0 \quad \text{on} \ \partial\Omega,
\]

since \( \partial_n f_1 = -L(f_2), \ \partial_n f_2 = L(f_1) \) on \( \partial\Omega \).

This equation was introduced in [3]§4 in connection with the characterization of the traces of pluriharmonic functions. It can be considered as a generalization both of the CR-tangential equation \( L(f) = 0 \) (for a complex-valued \( f \)) and of the condition \( \partial_n f = 0 \) on \( \partial\Omega \) that distinguishes holomorphic functions among complex harmonic functions on \( \Omega \) (what is called the homogeneous \( \partial \)-Neumann problem for functions, see [10] and [13]§15).

**Theorem 2.** Let \( \Omega \) be a bounded domain in \( \mathbb{H} \), with boundary of class \( C^1 \). Let \( f = f_1 + f_2j : \Omega \to \mathbb{H} \) be a harmonic function on \( \Omega \), of class \( C^1 \) on \( \overline{\Omega} \). Then, \( f \) is (left-)\( \psi \)-regular on \( \Omega \) if, and only if,

\[
(\partial_n - jL)f = 0 \quad \text{on} \ \partial\Omega.
\]

**Proof.** It remains to prove the sufficiency of condition (2) for \( \psi \)-regularity of harmonic functions. For every \( z \in \Omega \), it follows from the Bochner-Martinelli integral
representation for complex harmonic functions on $\Omega$ (see for example [13], §1.1), that $f(z) = f_1(z) + f_2(z)j$ is equal to
\[
\int_{\partial \Omega} U(\zeta, z)f_1(\zeta) + 2\int_{\partial \Omega} g(\zeta, z)\overline{\partial}_n f_1(\zeta)d\sigma + \left(\int_{\partial \Omega} U(\zeta, z)f_2(\zeta) + 2\int_{\partial \Omega} g(\zeta, z)\overline{\partial}_n f_2(\zeta)d\sigma\right)j.
\]
If $\overline{\partial}_n f = jL(f)$ on $\partial \Omega$, then we obtain
\[
f(z) = \int_{\partial \Omega} U(\zeta, z)f(\zeta) + 2\int_{\partial \Omega} g(\zeta, z)\overline{\partial}_n f(\zeta)d\sigma = \int_{\partial \Omega} U(\zeta, z)f(\zeta) + 2\int_{\partial \Omega} g(\zeta, z)jL(f(\zeta))d\sigma.
\]
The result now follows from Proposition [1]

Let $N$ and $T$ be the differential operators, defined in a neighbourhood of $\partial \Omega$, as
\[
N = \frac{\partial \rho}{\partial \bar{z}_1} \frac{\partial}{\partial \bar{z}_2} + \frac{\partial \rho}{\partial \bar{z}_2} \frac{\partial}{\partial \bar{z}_1}, \quad T = \frac{\partial \rho}{\partial \bar{z}_2} \frac{\partial}{\partial \bar{z}_1} - \frac{\partial \rho}{\partial \bar{z}_1} \frac{\partial}{\partial \bar{z}_2}.
\]
$T$ is a tangential (not Cauchy-Riemann) operator to $\partial \Omega$, while $N$ is non-tangential, such that $N(\rho) = |\nabla \rho|^2$, $\text{Re}(N) = |\nabla \rho| \text{Re}(\overline{n})$. The remark made at the end of §2.3 shows that Theorem [2] gives also a boundary condition for regularity of a harmonic function on $\Omega$.

**Corollary 3.** Let $\Omega$ be a $C^1$-bounded domain in $\mathbb{H}$. Let $f = f_1 + f_2 j : \Omega \to \mathbb{H}$ be a harmonic function on $\Omega$, of class $C^1$ on $\overline{\Omega}$. Then, $f$ is (left-)regular on $\Omega$ if, and only if,
\[
(N - jT)f = 0 \quad \text{on } \partial \Omega.
\]

3.2. We now give a weak formulation of the differential criterion of $\psi$-regularity, which makes sense for example when the harmonic function $f$ is only continuous on the closure $\overline{\Omega}$.

Let $\text{Harm}^1(\Omega)$ denote the space of complex harmonic functions on $\Omega$, of class $C^1$ on $\overline{\Omega}$. By application of the Stokes’ Theorem, of the complex Green formula
\[
\int_{\partial \Omega} \bar{g}\overline{\partial}_n h d\sigma = \int_{\partial \Omega} \overline{h}\partial_n g d\sigma \quad \forall g, h \in \text{Harm}^1(\Omega),
\]
and of the equality $\overline{\partial}f \wedge d\zeta|_{\partial \Omega} = 2L(f)d\sigma$ on $\partial \Omega$, we see that the equations $\overline{\partial}_n f_1 = -L(f_2)$, $\overline{\partial}_n f_2 = L(f_1)$ on $\partial \Omega$, imply the following (complex) integral conditions: for every function $\phi \in \text{Harm}^1(\overline{\Omega})$,
\[
\int_{\partial \Omega} f_1 \overline{\partial} \phi = \frac{1}{2} \int_{\partial \Omega} f_2 \overline{\partial}(\phi d\zeta), \quad \int_{\partial \Omega} f_2 \overline{\partial} \phi = -\frac{1}{2} \int_{\partial \Omega} f_1 \overline{\partial}(\phi d\zeta).
\]
These are equivalent to one quaternionic condition, which is then necessary for the $\psi$-regularity of $f \in C^1(\overline{\Omega})$:
\[
\int_{\partial \Omega} f \left( \overline{\partial} \phi - \frac{1}{2} j \overline{\partial}(\phi d\zeta) \right) = 0 \quad \forall \phi \in \text{Harm}^1(\overline{\Omega}),
\]
that can be rewritten also as:
\[
\int_{\partial \Omega} f \left( \overline{\partial}_n - jL \right) (\phi) d\sigma = 0 \quad \forall \phi \in \text{Harm}^1(\overline{\Omega}).
\]
Now we consider the sufficiency of the integral condition (1) when \( f \) is only continuous on \( \overline{\Omega} \).

**Theorem 4.** Let \( \Omega \) be a bounded domain in \( \mathbb{H} \), with boundary of class \( C^1 \). Let \( f : \partial \Omega \to \mathbb{H} \) be a continuous function. Then, there exists a (left-)\( \psi \)-regular function \( F \) on \( \Omega \), continuous on \( \overline{\Omega} \), such that \( F|_{\partial \Omega} = f \), if and only if \( f \) satisfies the condition (4).

**Proof.** Let \( F^+ \) and \( F^- \) be the \( \psi \)-regular functions defined respectively on \( \Omega \) and on \( \mathbb{C}^2 \setminus \overline{\Omega} \) by the Cauchy-Fueter integral of \( f \):

\[
F^\pm(z) = \int_{\partial \Omega} U(\zeta, z) f(\zeta) + \int_{\partial \Omega} \omega(\zeta, z) j f(\zeta).
\]

From the equalities \( U(\zeta, z) = -2 \partial_\zeta g(\zeta, z), \omega(\zeta, z) = -\partial_\zeta (g(\zeta, z)d\overline{\zeta}) \), we get that

\[
F^-(z) = -2 \int_{\partial \Omega} f(\zeta) \overline{\partial_\zeta g(\zeta, z)} + \int_{\partial \Omega} \overline{f(\zeta)} j \overline{\partial_\zeta g(\zeta, z)}
\]

for every \( z \notin \overline{\Omega} \). If (4) holds, then \( F^- \) vanishes identically on \( \mathbb{C}^2 \setminus \Omega \). As in the complex variable case, extended to the quaternionic case in Lemma 3 of [12], this implies that also \( F^+ \) has a continuous extension on \( \overline{\Omega} \), and \( F^+ = f \) on \( \partial \Omega \). Conversely, if \( F \in C(\overline{\Omega}) \) is a \( \psi \)-regular function on \( \Omega \) with trace \( f \) on \( \partial \Omega \), and \( \Omega^\epsilon = \{ z \in \Omega : \rho < \epsilon \} \), then \( F \) satisfies (4) on \( \partial \Omega^\epsilon \) for every small \( \epsilon \). Passing to the limit as \( \epsilon \to 0 \), we obtain (4). \( \square \)

**Remark 2.** If \( f \) satisfies a Hölder condition on \( \partial \Omega \), the same result can be obtained using the Sokhotski-Plemelj formula (see [11]§3.6). In [11,2,6] other similar integral criteria were given, assuming that the trace function belongs to a Hölder or a \( L^p(\partial \Omega) \) class.

**Remark 3.** In the orthogonality condition (4) it is sufficient to consider functions \( \phi \in \text{Harm}^1(\overline{\Omega}) \) that are of class \( C^\infty \) on a neighbourhood of \( \overline{\Omega} \).

### 3.3

In Theorem 4 the boundary of \( \Omega \) is not required to be connected. If \( \partial \Omega \) is connected, we can improve the result and show that only one of the complex conditions (3) is sufficient for the \( \psi \)-regularity of the harmonic extension of \( f \).

**Theorem 5.** Let \( \Omega \) be a bounded domain in \( \mathbb{H} \), with connected boundary \( \partial \Omega \) of class \( C^1 \). Let \( f : \partial \Omega \to \mathbb{H} \) be a continuous function. Then, if \( f \) satisfies one of the conditions (3), there exists a (left-)\( \psi \)-regular function \( F \) on \( \Omega \), continuous on \( \overline{\Omega} \), such that \( F|_{\partial \Omega} = f \).

**Proof.** Assume that

\[
\int_{\partial \Omega} f_2 \overline{\partial_\zeta \phi} = -\frac{1}{2} \int_{\partial \Omega} f_1 \overline{\partial_\zeta (\phi d\zeta)} \quad \forall \phi \in \text{Harm}^1(\overline{\Omega}).
\]

We use the same notation as in the proof of Theorem 4. We get that

\[
F^-(z) = -2 \int_{\partial \Omega} f(\zeta) \overline{\partial_\zeta g(\zeta, z)} + \int_{\partial \Omega} \overline{f(\zeta)} j \overline{\partial_\zeta g(\zeta, z)}
\]

for every \( z \notin \overline{\Omega} \). Therefore, \( F^- \) is a complex-valued, \( \psi \)-regular function on \( \mathbb{C}^2 \setminus \overline{\Omega} \). The system of equations (1) then implies that \( F^- \) is a holomorphic function. Since
\(\partial \Omega\) is connected, from Hartogs’ Theorem it follows that \(F^-\) can be holomorphically continued to the whole space. Let \(F^-\) be such extension. Then \(F = F^+ - \overline{F^-}\) is a \(\psi\)-regular function on \(\Omega\), continuous on \(\overline{\Omega}\), such that \(F|_{\partial \Omega} = f\). If the first condition in (3) is satisfied, it is sufficient to consider the function \(f_j = -f_2 + f_1j\) in place of \(f\).

**Remark 4.** The hypothesis on \(f\) in the preceding theorem is satisfied, for example, when \(f\) is of class \(C^1\) on \(\overline{\Omega}\), harmonic on \(\Omega\), and one of the equations \(\partial_n f_1 = -L(f_2)\), \(\partial_n f_2 = L(f_1)\) holds on \(\partial \Omega\).

**Remark 5.** The connectedness of \(\partial \Omega\) is a necessary condition in Theorem 5, consider a locally constant function on \(\partial \Omega\). For example, if \(f_2 = 0\) on \(\Omega\) and \(f_1\) takes two distinct values on the components of \(\partial \Omega\), then \(\partial_n f_2 = L(f_1)\) on \(\partial \Omega\), but \(\partial_n f_1 \neq 0\) on \(\partial \Omega\), since otherwise \(f_1\) would be holomorphic on \(\Omega\).

The preceding result can be easily generalized in the following form:

**Corollary 6.** Let \(\Omega\) be as above. Let \(a, b \in \mathbb{C}\) be two complex numbers such that \((a, b) \neq (0, 0)\).

(i) If \(f\) is of class \(C^1\) on \(\overline{\Omega}\), harmonic on \(\Omega\), and such that

\[
a \partial_n f_1 + b \partial_n f_2 = -a \overline{L(f_2)} + b \overline{L(f_1)} \quad \text{on} \ \partial \Omega,
\]

then \(f\) is \(\psi\)-regular on \(\Omega\).

(ii) If \(f\) is continuous on \(\partial \Omega\), such that

\[
\int_{\partial \Omega} (af_1 + bf_2) \phi \ d\phi = \frac{1}{2} \int_{\partial \Omega} (af_2 - bf_1) \overline{\partial(\phi d\zeta)} \quad \forall \ \phi \in \text{Harm}^1(\overline{\Omega}),
\]

then there exists a \(\psi\)-regular function \(F\) on \(\Omega\), continuous on \(\overline{\Omega}\), such that \(F|_{\partial \Omega} = f\).

Theorem 5 can be applied, in the case of connected boundary in \(\mathbb{C}^2\), to obtain the following result of Aronov and Kytmanov (cf. [9, 10, 11]), which holds in \(\mathbb{C}^n\), \(n > 1\): if \(f\) is a complex harmonic function on \(\Omega\), of class \(C^1\) on \(\overline{\Omega}\), such that \(\partial_n f = 0\) on \(\partial \Omega\), then \(f\) is holomorphic. It is sufficient to take \(f_1 = f, f_2 = 0\).

More generally, we can deduce a differential criterion for holomorphicity of functions on a domain with connected boundary in \(\mathbb{C}^2\), analogous to those proposed in [21] and investigated in [22] and [13] §23.2.

**Theorem 7.** Let \(\Omega\) be a bounded domain in \(\mathbb{C}^2\), with connected boundary of class \(C^1\). Let \(h = (h_1, h_2) : \Omega \to \mathbb{C}^2\) be a holomorphic mapping of class \(C^1\) on \(\overline{\Omega}\), such that \(h(\zeta) \neq 0\) for every \(\zeta \in \partial \Omega\).

(i) If \(h_1 f, h_2 f \in \text{Harm}^1(\overline{\Omega})\) and \(f : \overline{\Omega} \to \mathbb{C}\) satisfies the differential condition

\[
h_1 \partial_n f = \overline{h_2 L(f)} \quad \text{on} \ \partial \Omega,
\]

then \(f\) is holomorphic on \(\Omega\).

(ii) If \(h_1 f\) and \(h_2 f\) are harmonic on \(\Omega\), \(f : \overline{\Omega} \to \mathbb{C}\) is continuous and it satisfies the integral condition

\[
\int_{\partial \Omega} h_1 f \overline{\phi d\zeta} = \int_{\partial \Omega} h_2 f \overline{\phi d\zeta} \quad \forall \ \phi \in \text{Harm}^1(\overline{\Omega}),
\]

then \(f\) is holomorphic on \(\Omega\).
Proof. It is sufficient to prove (ii). Let \( h_2' = -2h_2 \). From Theorem 5 we get that the harmonic function \( h_2'f + h_1fj \) is \( \psi \)-regular on \( \Omega \). Let \( \Omega_\epsilon = \{ z \in \Omega : \rho < \epsilon \} \) for a small \( \epsilon < 0 \). Then the following equalities hold on \( \partial \Omega_\epsilon \):

\[
\bar{\partial}_n(h_1f) = L(h_2'f), \quad \bar{\partial}_n(h_2'f) = -L(h_1f).
\]

From the holomorphicity of \( h \), we then obtain

\[
h_1\bar{\partial}_n(f) = \bar{\partial}' h_2 L(f), \quad h_2\bar{\partial}_n(f) = -\bar{\partial}' h_1 L(f)
\]

on \( \partial \Omega_\epsilon \), which implies \( \bar{\partial}_n f = L(f) = 0 \) on \( \partial \Omega_\epsilon \) for every \( \epsilon \) sufficiently small, such that \( h \neq 0 \) on \( \partial \Omega_\epsilon \). This means that there exists a holomorphic extension \( F_\epsilon \) of \( f \) on \( \Omega_\epsilon \). From the equality of the harmonic functions \( h_jF_\epsilon = h_jf \) on \( \Omega_\epsilon \), for \( j = 1, 2 \), we get \( F_\epsilon = f_{|\Omega_\epsilon} \). Then \( f \) is holomorphic on the whole domain \( \Omega \). \( \square \)

4. Regularity and the \( \overline{\partial} \)-Neumann problem

4.1. We recalled before the following fact proved by Naser and Nôno [15, 16]: \( \Omega \) is pseudoconvex if and only if every complex harmonic function on \( \Omega \) is a complex component of a \( \psi \)-regular function. Now we are interested in the boundary values of \( \psi \)-regular functions and in the quaternionic analogue of the Hilbert transform. We want to express one of the complex components of a \( \psi \)-regular function by means of the other (the two components are then a pair of ‘conjugate harmonic’ functions).

We show how this is related to the \( \overline{\partial} \)-Neumann problem in \( \mathbb{C}^2 \).

We refer, for instance, to [3] and [14] for generalizations of the Hilbert transform to the quaternionic setting. In these references the \( \psi \)-regular (or more generally, \( (\psi, \alpha) \)-hyperholomorphic) functions are defined on plane or spatial domains. Here we are interested in domains of two complex variables where, as shown before, pseudoconvexity becomes relevant. We refer to [23] for other multidimensional generalizations of the concept of conjugate harmonic functions from the complex case to the Clifford space case.

The \( \overline{\partial} \)-Neumann problem for complex functions \( \Box f = (\overline{\partial} \overline{\partial} + \overline{\partial} \overline{\partial}^\ast) f = \psi \) in \( \mathbb{C}^2 \), \( \overline{\partial}_n f = 0 \) on \( \partial \Omega \) is equivalent, in the smooth case, to the problem

\[
\overline{\partial}_n g = \phi \text{ on } \partial \Omega, \quad g \text{ harmonic in } \Omega
\]

(see [13§14]). The compatibility condition for this problem is

\[
\int_{\partial \Omega} \phi \bar{\frac{\partial}{\partial \alpha}} d\sigma = 0
\]

for every \( h \) holomorphic in a neighbourhood of \( \overline{\Omega} \). We now use the solvability of this problem in strongly pseudoconvex domains of \( \mathbb{C}^2 \) to obtain some results on regular functions.

Let \( \Omega \) be a bounded domain in \( \mathbb{C}^2 \) with connected, \( C^\infty \)-smooth boundary. We denote by \( W^s(\Omega) \) (\( s \geq 1 \)) the complex Sobolev space, and by \( G^s(\Omega) \) the space of harmonic functions in \( W^s(\Omega) \). The space \( G^s(\Omega) \) is isomorphic to \( W^{s-1/2}(\partial \Omega) \) through the restriction operator.

**Theorem 8.** Let \( \Omega \) be a bounded strongly pseudoconvex domain in \( \mathbb{C}^2 \) with connected boundary of class \( C^\infty \). Let \( f_1 \in W^{s-1/2}(\partial \Omega) \), where \( s \geq 3 \). We identify \( f_1 \) with its harmonic extension in \( G^s(\Omega) \). Then there exists a function \( f_2 \in G^{s-2}(\Omega) \) such that \( f = f_1 + f_2j \) is a \( \psi \)-regular function on \( \Omega \).
Proof. We show that the function \( \phi = \overline{L(f_1)} \in G^{s-1}(\Omega) \) satisfies the compatibility condition \( \Box \). If \( h \) is holomorphic in a neighbourhood of \( \overline{\Omega} \),
\[
\int_{\partial \Omega} L(f_1) h d\sigma = \frac{1}{2} \int_{\partial \Omega} h \overline{\partial_1 f_1} dz = 0,
\]
then \( h \) is a CR function on \( \partial \Omega \). Then we can apply a result of Kytmanov \[13]\S 18.2 and get a solution \( f_2 \in G^{s-2}(\Omega) \) of the \( \overline{\partial} \)-Neumann problem \( \overline{\partial}_n f_2 = L(f_1) \) on \( \partial \Omega \). If \( s \geq 5 \), then \( f = f_1 + f_2 \) is continuous on \( \overline{\Omega} \) by Sobolev embedding. From Theorem 5 we get that \( f \) is \( \psi \)-regular on \( \Omega \), since it satisfies the second condition in \( \Box \). In any case, \( f_2 \in L^2(\partial \Omega) \) since \( s \geq 3 \). Then the result follows from the \( L^2(\partial \Omega) \)-version of Theorem 5 that can be proved as before using the results in \[1\] §3.7. \( \square \)

Remark 6. There is a unique solution \( f_2 \) of \( \overline{\partial}_n f_2 = L(f_1) \) on \( \partial \Omega \) that is orthogonal to holomorphic functions in \( L^2(\partial \Omega) \). It is given by the bounded Neumann operator \( N_\Omega; f_2 = N_\Omega(L(f_1)) \).

Corollary 9. Suppose \( \Omega \) is a bounded strongly pseudoconvex domain in \( \mathbb{C}^2 \) with connected boundary of class \( C^\infty \). Let \( f_1 : \partial \Omega \to \mathbb{C} \) be of class \( C^\infty \). Then there exists a \( \psi \)-regular function \( f \) on \( \Omega \), of class \( C^\infty \) on \( \overline{\Omega} \), such that the first complex component of the restriction \( f|_{\partial \Omega} \) to \( \partial \Omega \) is \( f_1 \).

Remark 7. The preceding statements remain true if \( \Omega \) is a bounded weakly pseudoconvex domain in \( \mathbb{C}^2 \) with connected real-analytic boundary, since on these domains the \( \overline{\partial} \)-Neumann problem for smooth functions is solvable (cf. \[13\] §18).

4.2. We denote by \( \mathcal{R}(\Omega) \) the right \( \mathbb{H} \)-module of (left-)\( \psi \)-regular functions on \( \Omega \) and by \( \mathcal{R}^\infty(\Omega) \) the functions in \( \mathcal{R}(\Omega) \) that are of class \( C^\infty \) on \( \overline{\Omega} \). We consider the space of holomorphic maps \( \text{Hol}(\Omega, \mathbb{C}^2) \) as a real subspace of \( \mathcal{R}(\Omega) \) by identification of the map \( (f_1, f_2) \) with \( f = f_1 + f_2 \).

If \( \Omega \) is pseudoconvex, it follows from what observed in \[12,3\] that the map that associates to \( f = f_1 + f_2 \) the first complex component \( f_1 \) induces an isomorphism between the quotient real spaces \( \mathcal{R}(\Omega)/\text{Hol}(\Omega, \mathbb{C}^2) \) and \( \text{Harm}(\Omega)/\mathcal{O}(\Omega) \).

Now we are also interested in the regularity up to the boundary of the functions. Let \( A^\infty(\Omega, \mathbb{C}^2) = \text{Hol}(\Omega, \mathbb{C}^2) \cap C^\infty(\overline{\Omega}, \mathbb{C}^2) \) be identified with a \( \mathbb{R} \)-subspace of \( \mathcal{R}^\infty(\Omega) \).

Let \( C : \mathcal{R}^\infty(\Omega) \to C^\infty(\partial \Omega) \) be the linear operator that associates to \( f = f_1 + f_2 \) the restriction to \( \partial \Omega \) of its first complex component \( f_1 \). From the Corollary 9 and the remark preceding it, we get a right inverse \( R \) of \( C \). The function \( R(f_1) \) is uniquely determined by the orthogonality condition with respect to the functions holomorphic on a neighbourhood of \( \overline{\Omega} \):
\[
\int_{\partial \Omega} (R(f_1) - f_1) h d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{\Omega}).
\]

Note that \( f_1 \in CR(\partial \Omega) \cap C^\infty(\partial \Omega) \) if and only if \( R(f_1) \in A^\infty(\Omega, \mathbb{C}^2) \). Besides, the operator \( C \) has kernel
\[
\ker C = \{ f_2 j : f_2 \in A^\infty(\Omega) \} = A^\infty(\Omega) j
\]
where \( A^\infty(\Omega) \) is the space of the holomorphic functions on \( \Omega \) that are \( C^\infty \) up to the boundary. Then \( C \) induces the following isomorphism of real spaces:
\[
\tilde{C} : \mathcal{R}^\infty(\Omega) \to A^\infty(\Omega) j \to C^\infty(\partial \Omega).
\]
Corollary 10. Let $\Omega$ be a bounded domain in $\mathbb{C}^2$ with connected boundary. Suppose that $\Omega$ is a strongly pseudoconvex domain of class $C^\infty$ or a weakly pseudoconvex domain with real-analytic boundary. Then the operator $C$ induces an isomorphism of real spaces:

$$\hat{C} : \mathcal{R}^\infty(\Omega) \rightarrow C^\infty(\partial\Omega)$$

Remark 8. If $\Omega$ is a $C^\infty$-smooth bounded pseudoconvex domain, from application of Kohn’s Theorem on the solvability of the $\bar{\partial}$-problem to the equation $\ast \bar{\partial} f_1 = -\frac{1}{2} \bar{\partial} (f_2 dz_1 \wedge d\bar{z}_2)$ we can still deduce the isomorphism of Corollary 10.

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