Incidence Matrices of Directed Graphs of Groups and their up-down Pregroups

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ABSTRACT: The aim of this work is to give a definition of the incidence matrices of the directed graph of groups, construct an up-down pregroup of the incidence matrices of the directed graph of groups and then give an algorithm for the up-down pregroup of the directed graph of groups.

Keywords: Incidence matrix of X-labeled graph; up-down pregroup; directed graph of groups and incidence matrix of a directed graph of groups.

1. Introduction

In [1] we gave the definition of the incidence Matrices of X-Labeled graphs. In [2], [3] we gave the definition of the directed graph of groups, constructed graph of groups for pregroups directly from the ordered tree of pregroups, and from that directed graph of groups we constructed the up-down pregroups, and then we showed those two pregroups are isomorphic. In [4] Rimlinger gave an example of a pregroup \( P \) of finite height; he said “but Jim Shearer and I spent a very long evening with the computer and verified the pregroup axioms”. I bear this point in mind. In [2], [3] we have a direct method to obtain examples of pregroups in the form of up-down pregroups from any directed graph of groups, but sometimes those graphs of groups are large, and then will take a long time to find those up-down pregroups. In [1] we defined the incidence matrices of X-labeled graphs. The main aim of this work is to represent the directed graph of finite groups in terms of the incidence matrices of X-labeled graphs, so that by adding certain conditions to allow the incidence matrices of the X-labeled graph to be more confident with the definition of the directed graph of groups; we can then write a computer program to record all elements of the up-down pregroup of that directed graph of groups, as an application of the incidence matrices of X-labeled graph. Therefore, this paper is divided into six sections. In section 2, we give the basic concepts of graphs, pregroups and incidence matrices of X-labeled graphs. In section 3, we give the definition of incidence matrices of directed graphs of groups. In section 4, we construct the up-down pregroup of the incidence matrices of the directed graph of groups. In section 5, we define an algorithm on the incidence matrices of the directed graph of groups, so we can then write a computer program for this algorithm.

2. Basic concepts

2.1 Pregroups

The idea of pregroups goes back to Baer [5] and the definition of pregroup was given independently by Stallings [6] in 1971. The theory of pregroups has been developed by [4], Stallings [6], Hoare [7] and Hoare – Jassim [3] and others. We now return to the original definition of pregroups [6].

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Let \( P \) be a set with an element \( 1 \in P \) and a mapping of a subset \( D \) of \( P \times P \) into \( P \), denoted by \((x, y) \mapsto xy\). We shall say that \( xy \) is defined instead of \((x, y) \in D\). Suppose that there is an involution on \( P \) denoted by \( x \mapsto x^{-1} \), such that the following axioms hold:

**P1:** \( x1 = 1x \) for all \( x \in P \),

**P2:** \( xx^{-1} = 1 = x^{-1}x \) for all \( x \in P \),

**P3:** If \( xy \) is defined, then \( y^{-1}x^{-1} \) is defined and \((xy)^{-1} = y^{-1}x^{-1}\).

**P4:** If \( xy \) and \( yz \) are defined then \((xy)z\) is defined if and only if \( x(yz) \) is defined, in which case the two are equal and we will say \( xyz \) is defined.

**P5:** For any \( w, x, y \) and \( z \) in \( P \), if \( wx, xy \) and \( yz \) are defined, then either \( wxy \) or \( xyz \) is defined.

Hoare [7] showed that we could prove axiom P3 above by using the following proposition, and axioms P1, P2 and P4.

**Definition 2.2.** [7]: For any \( x \in P \), put \( L(x) = \{a \in P : ax \text{ is defined} \} \). We write \( x \leq y \) if \( L(y) \subseteq L(x) \), \( x < y \) if \( L(y) \subset L(x) \) and \( L(x) \neq L(y) \), and \( x \sim y \) if \( L(x) \neq L(y) \). It is clear that \( \sim \) is an equivalence relation compatible with \( \leq \).

The following results are taken from Stallings [6] and Rimlinger [4]. (See [7] for shorter proofs).

**Proposition 2.3.**

(i) If \( x \leq y \) or \( y \leq x \), then \( x^{-1}y \) and \( y^{-1}x \) are defined.

(ii) If \( xa \) and \( a^{-1}y \) are defined, then \((xa)(a^{-1}y)\) is defined if and only if \( xy \) is defined, in which case they are equal.

By using axiom P5 above (which will be denoted by P5(i)) Rimlinger [4] proved conditions P5(ii) and P5(iii) of Lemma 2.4 below.

**Lemma 2.4** [7]. The following conditions on elements of \( P \) are equivalent:

**P5(i):** If \( w, x, y \) and \( z \) are defined, then either \( wxy \) or \( xyz \) is defined.

**P5(ii):** If \( x^{-1}a \) and \( a^{-1}y \) are defined but \( x^{-1}y \) is not, then \( a < x \) and \( a < y \).

**P5(iii):** If \( x^{-1}y \) is defined, then \( x \leq y \) or \( y \leq x \).

Therefore, we will say \( P \) is a pregroup if it satisfies axioms P1, P2, P4, and the conditions of Lemma 2.4, above. The universal group of a pregroup \( P \) [13] is denoted by \( U(P) \) and has the following presentation \( \langle P \rangle, x \equiv x \) whenever \( xy \) is defined, for \( x, y \in P \). Now if \( P \) is a pregroup, then \( \langle P, \leq \rangle \) is a partial ordering; that is \( P/\leq \) has a minimum element and, for any \( x, y \) and \( z \) in \( P \), \( x \leq y \) and \( y \leq z \) we have \( x \leq y \) or \( y \leq x \). Moreover Rimlinger in [4] defined that for any element \( x \) in \( P \), we say that \( x \) has \textit{finite height} \( n \geq 0 \), if there exists a maximal totally ordered subset \( \{x_0, x_1, \ldots, x_n\} \) of \( P \) such that \( 1 = x_0 < x_1 < \cdots < x_n = x \). He also showed that the elements of \( P \) form an \textit{order tree} (denoted by \( O \)) whose vertices, \([x]\), are the equivalence classes of the elements of \( P \) under \( \sim \), and whose edges \( e \), are formed by joining each vertex \([x]\) of height \( n \geq 0 \) to the unique vertex \([y]\) of height \( n - 1 \) satisfying \([y] < [x]\) and all edges \( e \) of \( O \) are directed away the base vertex \([x_0]\) of height 0. In [8] Stallings constructed an up-down pregroup for a free group \( F \) generated by \( X = \{a, b\} \) of infinite height, and he showed that \( U(F) \) the universal group of a pregroup \( P \) is isomorphic to \( F \). In [23] we gave the definition of a \textit{directed graph of groups} which consists of a directed graph \( Y \), with a base vertex \( v \) and a spanning tree \( T \), whose edges are directed away from the base vertex \( v \), together with a group \( G_v \) for each vertex \( v \) and for each directed edge \( e \in Y \), a subgroup \( G_e \) of \( G_v \) which is embedded in \( G_{e(i)} \) by \( \psi_e \) which is defined by \( \psi_e(a) = y^{-1}a y \), where \( a \in G_v \) and \( y \) is the labeled of the edge \( e \). It is denoted by \( \langle G_v, G_e, Y, T, v, \psi_e \rangle \). We also constructed a directed graph of groups of \( P \) directly from the order tree \( O \) of \( P \) and then showed that the fundamental group of a graph of groups \( \pi_1(\langle G_v, G_e, Y, T, v, \psi_e \rangle) \) is isomorphic to \( U(P) \). We constructed an up-down pregroup \( Q \) directly from the directed graph of groups \( \langle G_v, G_e, Y, T, v, \psi_e \rangle \) of a pregroup \( P \) and we showed that \( U(Q) \) is isomorphic to \( \pi_1(\langle G_v, G_e, Y, T, v, \psi_e \rangle) \) and then that \( U(Q) \equiv U(P) \).
2.5 Incidence Matrices of $X$ – Labeled Graphs

In [1] we gave the definition of the incidence matrices of $X$ – Labeled graphs (where an $X$-labeled graph is a directed graph with each edge labeled by an element $x$ of the subset $X$ of the group $F$ and $X$ generating the group $F$), and some definitions and results related to it. Recall that from graph theory the directed graphs $\Gamma$ are without loops, because we cannot define the incidence matrices of directed graphs $\Gamma$. The incidence matrices of directed graphs $\Gamma$ are with $n$ vertices and $m$ edges (i.e. it is $n \times m$ matrices $[x_{ij}]$, where $1 \leq i \leq n$, $1 \leq j \leq m$) such that:

$$x_{ij} = \begin{cases} 
1 & \text{if } v_i = i(e_j) \\
0 & \text{if } v_i \text{ is not incidence with } e_j \\
-1 & \text{if } v_i = \tau(e_j) 
\end{cases}$$

Since all edges $e$ in $X$ – Labeled graphs are labeled $x \in X \cup X^{-1}$ and the incidence matrices of the directed graphs do not deal with the labeling of edges, we will put more conditions on the incidence matrices of directed graphs as below to obtain the definition of the incidence matrices of the $X$- Labeled graphs.

**Definition 2.6:** Let $\Gamma$ be any $X$ – Labeled graph without loops (where $X = \{a, b\}$), then the incidence matrix of the $X$ – Labeled graph $\Gamma$ is an $n \times m$ incidence matrix $[x_{ij}]$, where $1 \leq i \leq n$ and $1 \leq j \leq m$ with $x_{ij}$ entries such that

$$(*) \quad \begin{cases} 
1 & \text{if } v_i \text{ is incidence with } e_j \\
0 & \text{if } v_i \text{ is not incidence with } e_j \\
-1 & \text{if } v_i = \tau(e_j) 
\end{cases}$$

$X$-Labeled graphs might have loops which are labeled $a$ or $b$, therefore in the rest of this work we will assume that all $X$ – Labeled graphs $\Gamma$ are without loops.

**Definition 2.7:** Let $M^X(\Gamma)$ be an incidence matrix of $X$ – Labeled graph $\Gamma$. If $M^X(\Gamma)$ doesn’t contain any row $r_i$ with non zero entries $x_{ij}$ and $x_{ik}$ in $X \cup X^{-1}$ such that $x_{ij} = x_{ik}$, then $M^X(\Gamma)$ is called a folded incidence matrix of $X$ – Labeled graph $\Gamma$.

Now we give the basic definitions and some results on the incidence matrix of $X$ – Labeled graph $M^X(\Gamma)$, as given in [1].

Let $M^X(\Gamma)$ be an $n \times m$ incidence matrix $[x_{ij}]$ of $X$ – Labeled graphs $\Gamma$, and let $r_i$ and $c_j$ be a row and a column in $M^X(\Gamma)$ respectively. If $x_{ij}$ is a non – zero entry in the row $r_i$, then $r_i$ is called an incidence row with the column $c_j$ at the non – zero entry $x_{ij} \in X \cup X^{-1}$, and if the non – zero entry $x_{ij} \in X$, then the row $r_i$ is called the starting row (denoted by $s(c_j)$) of the column $c_j$, and the row $r_i$ is called the ending row (denoted by $e(c_j)$) of the column $c_j$ if $x_{ij} \in X^{-1}$. If the rows $r_i$ and $r_k$ are incident with column $c_j$ at the non – zero entries $x_{ij}$ and $x_{kj}$ respectively, then we say that the rows $r_i$ and $r_k$ are adjacent. If $c_j$ and $c_k$ are two distinct columns in $M^X(\Gamma)$ such that the row $r_i$ is incidence with the columns $c_j$ and $c_h$ at the non – zero entries $x_{ij}$ and $x_{ih}$ respectively (where $x_{ij}, x_{ih} \in X \cup X^{-1}$), then we say that $c_j$ and $c_k$ are adjacent columns. For each column $c$ there is an inverse column denoted by $\overline{c}$ such that $s(\overline{c}) = e(c), e(\overline{c}) = s(c)$ and $\overline{\overline{c}} = c$. The degree of a row $r_i$ of
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$M_x(\Gamma)$ is the number of the columns incident to $r_i$ and is denoted by $\deg(r_i)$. If the row $r_i$ is incident with at least three distinct columns $c_{j1}, c_{j2}, c_{j3}$ at the non-zero entries, then the row $r_i$ is called a branch row. If the row $r_i$ is incident with only one column $c_j$ at the non-zero entry $x_{ij} \in X \cup X^{-1}$ and all other entries of $r_i$ are zero, then the row $r_i$ is called an isolated row. A scale in $M_x(\Gamma)$ is a finite sequence of form $S = r_1, r_2, r_3, \ldots, r_s, r_k$, where $k \geq 1$, $\in \mathbb{T}$, $s(c_j^{k}) = r_j$, and $e(c_j^{k}) = r_{j+1} = s(c_{j+1})$, $1 \leq j \leq k$.

The starting row of a scale $S = r_1, r_2, r_3, \ldots, r_s, r_k$ is the starting row $r_1$ of the column $c_1$ and the ending row of the scale $S$ is the ending row $r_k$ of the column $c_{k-1}$ and we say that $S$ is a scale from $r_1$ to $r_k$ and $S$ is a scale of length $k$ for $1 \leq j \leq k - 2$. If $s(S) = e(S)$, then the scale $S$ is called a closed scale. If the scale $S$ is reduced and closed, then $S$ is called a circuit or a cycle. If $M_x(\Gamma)$ has no cycle, then $M_x(\Gamma)$ is called a forest incidence matrix of $X$-Labeled graph $\Gamma$. The rows and columns of $M_x(\Gamma)$ are subsets of $M_x(\Gamma)$ as they do in $M_x(\Gamma)$. If $M_x(\Omega) \neq M_x(\Gamma)$, then $M_x(\Omega)$ is a proper subincidence matrix of $M_x(\Gamma)$. A component of $M_x(\Gamma)$ is a maximal connected subincidence matrix of $M_x(\Gamma)$. Let $\Omega$ be a subgraph of $\Gamma$, then $M_x(\Omega)$ is called a subincidence matrix of $M_x(\Gamma)$, and if $c$ is a column in $M_x(\Gamma)$ and $c$ is a column in $M_x(\Omega)$, then $s(c), e(c)$ and $\mathcal{T}$ have the same meaning in $M_x(\Omega)$ as they do in $M_x(\Gamma)$. If $M_x(\Omega) \neq M_x(\Gamma)$, then $M_x(\Omega)$ is called a spanning incidence matrix of $M_x(\Gamma)$ and $M_x(\Omega)$ is called a spanning tree of $M_x(\Gamma)$ if $M_x(\Omega)$ is a spanning and tree incidence matrix. The inverse of $M_x(\Gamma)$ is an incidence matrix of $X^{-1}$-Labeled graph $\Gamma$.

Now by direct calculations and the definitions above, we can prove the following results.

**Lemma 2.8:** If $M_x(\Gamma)$ is a tree incidence matrix of $X$-Labeled graph $\Gamma$ with $n$ rows, then $M_x(\Gamma)$ has $n - 1$ columns.

3. Incidence matrices of directed graphs of finite groups.

**Definition 3.1:** An incidence matrix of a directed graph of finite groups consists of an incidence matrix of $X$-labeled graph $M_x(\Gamma)$ with a spanning tree matrix of $X$-labeled graph $M_x(\Gamma)$, and a base row $r^* = r_1$, together with a finite group $G_r$ for each row $r$, and a finite group $G_c$ for each column $c$, such that:

1) The columns of $M_x(\Gamma)$ are directed away from $r^* = r_1$;
2) Each column group $G_c$ is a subgroup of $G_{(c)}$;
3) Each column group $G_c$ is embedded in $G_{(c)}$ by a fixed monomorphism $\psi_c$, defined by $\psi_c(a) = y^{-1}a y_c$, $a \in G_c$, and $y_c = s(c_j)$ is the non-zero entrance of $M_x(\Gamma) / M_x(\Gamma)$. It is denoted by $(G_r, G_c, M_x(\Gamma), M_x(\Gamma), r^*, \psi_c)$.

**N.B.:** Any incidence matrix of a graph of groups may be made into an incidence matrix of a directed graph of groups, that by choosing $M_x(\Gamma)$, a base row $r^* = r_1$, an orientation on columns and then identifying $G_c$ with the image of $G_{(c)}$ under $M_x(\Gamma)$ relevant monomorphism.
For each directed column $c_j = (x_{ij}, x_{ij})$ in $M_X(T)$, let $c^+ = c$, and let $c^{-1} = (x_{ij}, x_{ij})$ be the inverse column with starting row $s(c_j) = e(c_j)$, and $e(c_j) = s(c_j)$, where, $x_{ij} = s(c_j) = y_j$, $x_{ij} = e(c_j) = y_j^{-1}$ and $y_j$ are the entries of the column $c_j$, such that $y_j = 1$, if $c_j = (x_{ij}, x_{ij})$ is in $M_X(T)$.

N.B. We will denote $y_j$, by $y_j^c$, (where, $y_j^c$ are nonzero entries of $c_j = (y_{c_j}, y_{c_j})$, $y_{c_j} \in X$, $y_{c_j} \in X^{-1}$), such that $y_j^c$ is equal to 1 or -1, if $c_j \in M_X(T)$.

**Example:** In this example, we will give a directed graph of groups and then, construct the incidence matrix of this directed graph of groups $(G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$.

Let the directed graph of groups $(G_r, G_c, Y, T, v^*, \psi_c)$ be as follows:

![Directed Graph](image)

Figure 1. The directed graph of groups constructed in [3] p72 of the pregroup given in [4] p41.

The incidence matrix of the above directed graph of groups $(G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c)$ is as below:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Figure 2. The incidence matrix of the directed graph of groups given in Figure 1 above.
4. The up-down pregroup of an incidence matrix of a directed graph of finite groups.

In this section we construct the up-down pregroup of the incidence matrix of a directed graph of groups as below:

Let \( M_x(\Gamma) \) be the incidence matrix of a directed graph of groups \((G_x, G_c, M_x(Y), M_x(T), r^*, \psi_e)\).

The fundamental group \( \Gamma \) of \( M_x(\Gamma) \) has the following presentation:

\[
\left< G_x, \psi_e(a) \mid \forall a \in G_x, \psi_e = 1 \forall c \in M_x(\Gamma), \psi_e \neq 1 \forall c \in M_x(Y) / M_x(T) \right>.
\]

Now for each directed column \( c \in M_x(Y) \), let \( c \) be also denoted by \( c^{-1} \) and let \( c^{-1} \) denote the inverse column with \( s(c^{-1}) = e(c) \) and \( e(c^{-1}) = s(c) \), also let \( y_c \) be of form \( w = g_0 y_1 y_1 \cdots y_n g_n \), where \( e_i = \pm 1 \) and \( c_i \) is a circuit at \( r^* \), with rows \( r^* = r_1, r_2, \ldots, r_{n+1} = r^* \) say, and where each \( g_i \) is in \( G_i \). A word of this form and any subword of it is reduced if it contains no subword \( y_c \) or \( y_c \psi_e(a) \), where \( a \in G_e \). If it does contain such a subword, we can, using the relations, substitute \( \psi_e(a) \) or \( \psi_e(a) \) respectively to obtain a shorter word of the same group presentation.

Let \( M_x(\Gamma) \) be the incidence matrix of the directed graph of groups \((G_x, G_c, M_x(Y), M_x(T), r^*, \psi_e)\) and let \( q = c_1, c_2, \ldots, c_n \) be an upward scale in \( M_x(\Gamma) \) which is a finite sequence of columns directed away of the base row \( r^* \). Let the rows of the scale \( q \) be \( r^* = r_1, r_2, \ldots, r_n \). A word of type \( q \) is a word \( w = g_0 y_1 y_1 \cdots y_n g_n \), where \( g_i \in G_i \), \( 1 \leq i \leq n+1 \), and every word \( w \) must be reduced and \( y_i \) is the non-zero entry of the starting row \( s(c_i) \) of the column \( c_i \). Now let \( q = c_1, c_2, \ldots, c_k \) and \( q' = c_1', c_2', \ldots, c_{k'} \) be upward scales in \( M_x(\Gamma) \) both starting at \( r^* \). Let \( w = g_1 y_1 y_2 y_3 \cdots y_k y_k g_k \) and \( w = g_1' y_1' y_2' y_3' \cdots y_{k-1}' y_{k-1}' g_{k-1}' \) be words of type \( q \) and \( q' \) respectively, where \( k \leq k' \). The word \( w \) is called an initial subword of the word \( w' \), written \( w \ll w' \), if \( c_j = c_j' \), and hence \( y_c = y_{c_j} \), for \( 1 \leq j \leq k \), and if \( g_j y_j y_{j+1} \cdots y_{j-1} y_{j-1} g_{j-1} y_{j-1} \cdots g_j y_j \) is an element of \( G_j \), for each \( j \).

**Lemma 4.1:** The relation \( " \) is an initial subword of \( " \) is both transitive and tree incidence matrix like, that is \( w \ll w' \) and \( w' \ll w'' \), then \( w \ll w'' \), and if \( w' \ll w \), \( w' \ll w \), then either \( w \ll w' \) or \( w'' \ll w' \).

**Proof:** The result follows directly from the definition.

Now let \( q = c_1, c_2, \ldots, c_k \) and \( q' = c_1', c_2', \ldots, c_{k'} \) be upward scales in \( M_x(\Gamma) \) both of them starting at \( r^* \) and ending at the same row \( r \), and let \( w = g_1 y_1 y_2 y_3 \cdots y_k y_k g_k \) and \( w' = g_1' y_1' y_2' y_3' \cdots y_{k-1}' y_{k-1}' g_{k-1}' \) be words of type \( q \) and \( q' \) respectively, such that the elements \( g_k \) and \( g_{k}' \) are in \( G_r \), and then such a word \( w'' \) is called an up-down word.

**For example,** the word \( 1, b, x, 1, y, 1, 1, 1 \) is an up-down word, from Figure 2 derived from the upward scales \( q \) with rows \( r^* \) and \( q' \), with rows \( r^* \) and \( r_r \).

Let \( Q(M_x(\Gamma)) \) be the set of all up-down words of the incidence matrix of a directed graph of groups \( M_x(\Gamma) \).

Reducing an up-down word in \( M_x(\Gamma) \) gives another up-down. Therefore, we assume that such a word is reduced.

We use \( w' \) to denote an up-down word.

**Lemma 4.2:** Let \( w' \) and \( z' \) be reduced up-down words, then \( (w' \Rightarrow z') \) is in \( Q(M_x(\Gamma)) \) if and only if \( w' \) is an initial segment of \( z' \) or \( z' \) is an initial segment of \( w' \).

**Proof:** Since the words \( w' \) and \( z' \) are both reduced, so reduction can only take place in the word \( (w' \Rightarrow z') \) between the last \( y \) of \( w' \) and the first one of \( z' \).
Moreover $(w'w^{-1})^{-1}(z'z^{-1})$ reduces to an up-down word if and only if either all $y_c$ in $w'$ or all $y_c$ in $z'$ are eliminated when putting $(w'w^{-1})^{-1}(z'z^{-1})$ in reduced form. This happens if and only if $w'$ is an initial segment of $z'$ or $z'$ is an initial segment of $w'$ respectively.

Now we show that $Q(M_x(\Gamma))$ is a pregroup. Since $Q(M_x(\Gamma))$ is a subset of $\pi_1(M_x(\Gamma))$, so $Q(M_x(\Gamma))$ satisfies conditions $P_1$, $P_2$ and $P_4$. It remains to be shown that $Q(M_x(\Gamma))$ satisfies condition $P_5$.

**Lemma 4.3:** Let $w'w^{-1}$ and $z'z^{-1}$ be reduced up-down words, then $w' \ll z'$, implies that $w'w^{-1} \leq z'z^{-1}$.

**Proof:** Suppose that $w' \ll z'$. To show $w'w^{-1} \leq z'z^{-1}$, we must show that $L(z'z^{-1})$ is a subset of $L(w'w^{-1})$. If $(u'u^{-1})^{-1}(z'z^{-1})$ is in $Q(M_x(\Gamma))$, for some $u'u^{-1} \in Q(M_x(\Gamma))$, then, by Lemma 4.2, $z'$ is an initial segment of $u'$ or $u'$ is an initial segment of $z'$ (i.e. $z' \ll u'$ or $u' \ll z'$) respectively. Since $w' \ll z'$, in either case, by Lemma 4.1, we have $w' \ll u'$ or $u' \ll w'$ and then by Lemma 4.2 again, $(u'u^{-1})^{-1}(w'w^{-1})$ is defined in $Q(M_x(\Gamma))$. Therefore $L(z'z^{-1})$ is a subset of $L(w'w^{-1})$. Hence $w'w^{-1} \leq z'z^{-1}$.

**Theorem 4.4:** $Q(M_x(\Gamma))$ is a pregroup.

**Proof:** To show $Q(M_x(\Gamma))$ is a pregroup, we will show that $Q(M_x(\Gamma))$ satisfies condition $P_5$ of Lemma 2.4. Therefore let $w'w^{-1}$ and $z'z^{-1}$ be reduced up-down words in $Q(M_x(\Gamma))$, and suppose that $(w'w^{-1})^{-1}(z'z^{-1})$ is defined in $Q(M_x(\Gamma))$. Hence by Lemma 4.2, we have $z' \ll w'$ or $w' \ll z'$. Thus by Lemma 4.3, $w'w^{-1} \leq z'z^{-1}$ or $z'z^{-1} \leq w'w^{-1}$. Therefore condition $P_5$ of Lemma 2.4 holds in $Q(M_x(\Gamma))$.

**Definition 4.5:** The set of all up – down words of the incidence matrix of a directed graph of groups $Q(M_x(\Gamma))$ is called the up-down pregroup of $(G_r,G_c,M_x(Y),M_x(T),r^*,\psi_c)$ the incidence matrix of a directed graph of groups. where $Q$ is the up- down pregroup of the directed graph of groups, as shown in [2] and [1].

**Theorem 4.6:** The Universal group of $Q(M_x(\Gamma))$ (it is denoted by $\bar{U}(Q(M_x(\Gamma)))$ is isomorphic to the fundamental group $\pi_1 = \pi_1(M_x(\Gamma))$.

**Proof:** Since every element in $Q(M_x(\Gamma))$ is an element in $\pi_1 = \pi_1(M_x(\Gamma))$ and since the tree incidence matrix $M_x(T)$ spans $M_x(Y)$ and is directed away from $r^*$, every element of $\pi_1(M_x(T))$ can be written as a product of elements of $Q(M_x(\Gamma))$. Moreover the partial multiplication in $Q(M_x(\Gamma))$ implies the relations of $\pi_1 = \pi_1(M_x(\Gamma))$.

5. An algorithm for the up-down Pregroup of incidence matrices of directed graphs of groups.

Let $M_x(\Gamma) = (G_r,G_c,M_x(Y),M_x(T),r^*,\psi_c)$ be the incidence matrix of a directed graph of groups, then we use the representation of the directed graph of groups of an up-down pregroup, to write down all the elements of the up-down pregroup of that graph of groups by applying the following algorithm. The steps are given below:

I) Find all up words $w = g_{i_1}y_{i_1}\cdot g_{i_2}y_{i_2}\cdot \ldots \cdot g_{i_k}y_{i_k} \cdot g_{i_{k+1}}$ of type upward scales $q = (r^* = r_{i_1},c_{i_1},r_{i_2},c_{i_2},\ldots ,r_{i_k},c_{i_k},r_{i_{k+1}})$, where $g_{i_k} \in G_{i_k}$, $1 \leq k \leq n$ and $y_{i_k}$ is the non- zero entrance of the row $r_{i_k}$ which is the starting of the column $c_{i_k}$ as defined above and then proceed step II;

II) 1) If two up words $w = g_{i_1}y_{i_1}\cdot g_{i_2}y_{i_2}\cdot \ldots \cdot g_{i_k}y_{i_k} \cdot g_{i_{k+1}}$ and $w' = g'_{i_1}y'_{i_1}\cdot g'_{i_2}y'_{i_2}\cdot \ldots \cdot g'_{i_k}y'_{i_k} \cdot g'_{i_{k+1}}$, ending at the same row $r_{i_k}$, (i.e. row $r_{i_k}$ contains non-zero entrances of forms $x_{ij}^{-1}$ and $x_{ik}^{-1}$), then one makes one of them an up word, say $w' = g'_{i_1}y'_{i_1}\cdot g'_{i_2}y'_{i_2}\cdot \ldots \cdot g'_{i_k}y'_{i_k} \cdot g'_{i_{k+1}}$, and the other up word: $w = g_{i_1}y_{i_1}\cdot g_{i_2}y_{i_2}\cdot \ldots \cdot g_{i_k}y_{i_k} \cdot g_{i_{k+1}}$, down word by changing the direction of all $g_{i_k}$ columns and its entrance $y_{i_k}$ to be $w = g_{i_1}^{-1}y_{i_1}^{-1}\cdot g_{i_2}^{-1}y_{i_2}^{-1}\cdot \ldots \cdot g_{i_k}^{-1}y_{i_k}^{-1} \cdot g_{i_{k+1}}^{-1}$, and by, identifying them we get an up-down word $w'w^{-1} = g'_{i_1}y'_{i_1}\cdot g'_{i_2}y'_{i_2}\cdot \ldots \cdot g'_{i_k}y'_{i_k} \cdot g'_{i_{k+1}}^{-1}y_{i_k}^{-1} \cdot g_{i_{k+1}}^{-1}$, (where, $g'_{i_{k+1}}^{-1}y_{i_{k+1}}^{-1} \cdot g_{i_{k+1}}^{-1} \in G_{i_{k+1}}$). Then proceed to step III;
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2) If the up words \[ w = g_{i_1} y_{j_1} g_{i_2} \cdots g_{i_k} y_{j_k} g_{i_{k+1}} \] end at an isolated row, then change the direction of all columns and its label to be \[ w^{-1} = g_{i_{k+1}}^{-1} y_{j_k}^{-1} g_{i_k}^{-1} \cdots g_{i_2}^{-1} y_{j_1}^{-1} g_{i_1}^{-1} \], and by identifying them with the row \( r \); that both of them end with \( r \), then we get an up-down word \[ w \cdot w^{-1} = g_{i_1} g_{i_{k+1}}^{-1} \cdots g_{i_k} g_{i_2}^{-1} y_{j_1} y_{j_k}^{-1} \cdots y_{j_2} y_{j_1}^{-1} \].

III; III If there is no other up-down word, then stop.

Proposition 5.1: All up words \[ w = g_{i_1} y_{j_1} g_{i_2} \cdots g_{i_k} y_{j_k} g_{i_{k+1}} \] of type upward scales in \[ M_X(\Gamma) = (G_r, G_c, M_X(Y), M_X(T), r^*, \psi_c) \] are same as all up words \[ w = g_{i_1} y_{j_1} g_{i_2} \cdots g_{i_k} y_{j_k} g_{i_{k+1}} \] of type upward paths in \[ (G_r, G_c, Y, T, v^*, \psi_c) \].

Proof: Since all vertices \( v \) and edges \( e \) in \[ M_X(\Gamma) \] are represented by rows \( r \) and columns \( c \) and associated vertex groups \( G_r \) and edge groups \( G_c \) are represented by row groups \( G_r \) and columns groups \( G_c \) respectively, with entrances \( x_{j_v} \) of the labeled \( y_e \) of the edges of \( (G_r, G_c, Y, T, v^*, \psi_c) \) such that \( y_e = 1 \) if \( e \in T \) and \( y_e \neq 1 \) if \( e \in Y - T \). Therefore the direction and the labeling of columns of \( M_X(\Gamma) \) are same as in \( (G_r, G_c, Y, T, v^*, \psi_c) \). Hence all up words \[ w = g_{i_1} y_{j_1} g_{i_2} \cdots g_{i_k} y_{j_k} g_{i_{k+1}} \] of type upward scales in \[ M_X(\Gamma) \] are same as all up words \[ w = g_{i_1} y_{j_1} g_{i_2} \cdots g_{i_k} y_{j_k} g_{i_{k+1}} \] of type upward paths in \[ (G_r, G_c, Y, T, v^*, \psi_c) \].

Proposition 5.2: The algorithm must stop.

Proof: Since the size of \( M_X(\Gamma) \) is \( n \times m \) and all vertex groups and edge groups are finite, so \( M_X(\Gamma) \) is finite incidence matrix. By step I, we get all reduced up-words; by step II we get all up- down reduced words; and then by step III, we will get all up- down reduced words. Since the origin \( X \)-labeled graph does not contain loops, so the set of all reduced up-down words is finite and then the algorithm must be stop after a finite time.

6. Conclusion

We have given a new application for the incidence matrices of \( X \)-labeled graphs. This application is the incidence matrices of directed graph of finite groups. Therefore, we have added certain conditions to allow the incidence of \( X \)-labeled graphs to be more confident with the definition of the directed graph of finite groups. By this way we can write a computer program to record all elements of the up- down pregroups of that the directed graphs of finite groups.

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