ON THE EXISTENCE OF EMBEDDINGS INTO MODULES OF FINITE HOMOLOGICAL DIMENSIONS

RYO TAKAHASHI, SIAMAK YASSEMI, AND YUJI YOSHINO

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Abstract. Let \( R \) be a commutative Noetherian local ring. We show that \( R \) is Gorenstein if and only if every finitely generated \( R \)-module can be embedded in a finitely generated \( R \)-module of finite projective dimension. This extends a result of Auslander and Bridger to rings of higher Krull dimension, and it also improves a result due to Foxby where the ring is assumed to be Cohen-Macaulay.

1. Introduction

Throughout this paper, let \( R \) be a commutative Noetherian local ring. All \( R \)-modules in this paper are assumed to be finitely generated.

In [1, Proposition 2.6 (a) and (d)] Auslander and Bridger proved the following.

Theorem 1.1 (Auslander-Bridger). The following are equivalent:

1. \( R \) is quasi-Frobenius (i.e. Gorenstein with Krull dimension zero).
2. Every \( R \)-module can be embedded in a free \( R \)-module.

On the other hand, in [3, Theorem 2] Foxby showed the following.

Theorem 1.2 (Foxby). The following are equivalent:

1. \( R \) is Gorenstein.
2. \( R \) is Cohen-Macaulay, and every \( R \)-module can be embedded in an \( R \)-module of finite projective dimension.

For an \( R \)-module \( C \) we denote by \( \text{add}_R C \) the class of \( R \)-modules which are direct summands of finite direct sums of copies of \( C \). The \( C \)-dimension of an \( R \)-module \( X \), \( C\text{-dim}_R X \), is defined as the infimum of nonnegative integers \( n \) such that there exists an exact sequence

\[ 0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to X \to 0 \]

of \( R \)-modules with \( C_i \in \text{add}_R C \) for all \( 0 \leq i \leq n \).

In this paper, we prove the following theorem. This result removes from Theorem 1.1 the assumption that \( R \) is Cohen-Macaulay, and it extends Theorem 1.2 to
rings of higher Krull dimension. It should be noted that our proof of this result is different from Foxby’s proof for the special case $C = R$.

**Theorem 1.3.** Let $R$ be a commutative Noetherian local ring with residue field $k$. Let $C$ be a semidualizing $R$-module of depth $t$. Then the following are equivalent:

1. $C$ is dualizing.
2. Every $R$-module can be embedded in an $R$-module of finite $C$-dimension.
3. The $R$-module $\text{Tr} \Omega^t k \otimes_R C$ can be embedded in an $R$-module of finite $C$-dimension. (Here $\text{Tr} \Omega^t k$ denotes the transpose of the $t$-th syzygy of the $R$-module $k$.)

Moreover, if one of these three conditions holds, then $R$ is Cohen-Macaulay.

**2. Proof of Theorem 1.3 and its Applications**

First of all, we recall the definition of a semidualizing module.

**Definition 2.1.** An $R$-module $C$ is called semidualizing if the natural homomorphism $R \to \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$.

Note that a dualizing module is nothing but a semidualizing module of finite injective dimension. Another typical example of a semidualizing module is a free module of rank one. Recently a considerable number of authors have studied semidualizing modules and have obtained many results concerning these modules.

We denote by $m$ the maximal ideal of $R$ and by $k$ the residue field of $R$. To prove our main theorem, we establish two lemmas.

**Lemma 2.2.** Let $C$ be a semidualizing $R$-module. Let $g : M \to X$ be an injective homomorphism of $R$-modules with $C$-$\text{dim}_R X < \infty$. If $\text{Ext}_R^i(M, C) = 0$ for any $1 \leq i \leq C$-$\text{dim}_R X$, then the natural map $\lambda_M : M \to \text{Hom}_R(\text{Hom}_R(M, C), C)$ is injective.

**Proof.** First of all we prove that $M$ can be embedded in a module $C_0$ in $\text{add}_R C$. For this we set $n = C$-$\text{dim}_R X$. If $n = 0$, then this is obvious from the assumption, since $X \in \text{add}_R C$. If $n > 0$, then there exists an exact sequence 

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} X \to 0$$

with $C_i \in \text{add}_R C$ for $0 \leq i \leq n$. Putting $X_i = \text{Im } d_i$, we have exact sequences 

$$0 \to X_{i+1} \to C_i \to X_i \to 0 \quad (0 \leq i \leq n-1).$$

Then we have $\text{Ext}_R^1(M, X_1) = 0$, since there are isomorphisms $\text{Ext}_R^1(M, X_1) \cong \text{Ext}_R^2(M, X_2) \cong \cdots \cong \text{Ext}_R^n(M, X_n) \cong \text{Ext}_R^n(M, C_n) = 0$. Hence $\text{Hom}_R(M, C_0) : \text{Hom}_R(M, C_0) \to \text{Hom}_R(M, X)$ is surjective. This implies that the homomorphism $g \in \text{Hom}_R(M, X)$ is lifted to $f \in \text{Hom}_R(M, C_0)$, i.e. $d_0 \cdot f = g$. Since $g$ is injective, $f$ is injective as well. Therefore $M$ has an embedding $f$ into $C_0$. To prove that $\lambda_M$ is injective, we note that $\lambda_{C_0}$ is an isomorphism because of $C_0 \in \text{add}_R C$. Since there is an injective homomorphism $f : M \to C_0$, the following commutative diagram forces $\lambda_M$ to be injective:

$$\begin{array}{ccc}
M & \xrightarrow{f} & C_0 \\
\lambda_M \downarrow & & \lambda_{C_0} \downarrow \cong \\
\text{Hom}_R(\text{Hom}_R(M, C), C) & \xrightarrow{\text{Hom}_R(\text{Hom}_R(f, C), C)} & \text{Hom}_R(\text{Hom}_R(C_0, C), C). \\
\end{array}$$

$\square$
Lemma 2.3. Let $C$ be a semidualizing $R$-module and let $M$ be an $R$-module. Assume that $M$ is free on the punctured spectrum of $R$. Then there is an isomorphism
$$\Ext^i_R(M,R) \cong \Ext^i_R(M \otimes_R C,C)$$
for each integer $i \leq \depth_R C$.

Proof. Set $t = \depth_R C$. Since $C$ is semidualizing, we have a spectral sequence
$$E^{p,q}_2 = \Ext^p_R(\Tor^R_q(M,C),C) \Rightarrow \Ext^{p+q}_R(M,R).$$
Note by assumption that the $R$-module $\Tor^R_q(M,C)$ has finite length for $q > 0$. By [2 Proposition 1.2.10(e)] we have $E^{p,q}_2 = 0$ if $p < t$ and $q > 0$. Hence
$$\Ext^i_R(M \otimes_R C,C) = E^{i,0}_2 \cong \Ext^i_R(M,R)$$
for $i \leq t$. □

Let $M$ be an $R$-module. Take a free resolution
$$F_* = (\cdots \to d_{n+1} F_n \to d_n F_{n-1} \to \cdots d_1 F_0 \to 0)$$
of $M$. Then for a nonnegative integer $n$ we define the $n$-th syzygy of $M$ by the image of $d_n$ and denote it by $\Omega^n_M$ or simply by $\Omega^n M$. We also define the (Auslander) transpose of $M$ by the cokernel of the map $\Hom_R(F_0, R) \to \Hom_R(F_1, R)$ and denote it by $\Tr_R M$ or simply by $\Tr M$. Note that the $n$th syzygy and the transpose of $M$ are uniquely determined up to free summand. Note also that they commute with localization; namely, for every prime ideal $p$ of $R$ there are isomorphisms $(\Omega^n_R M)_p \cong \Omega^n_{R_p} M_p$ and $(\Tr_R M)_p \cong \Tr_{R_p} M_p$ up to free summand.

Recall that for a positive integer $n$ an $R$-module is called $n$-torsionfree if
$$\Ext^i_R(M,R) = 0$$
for all $1 \leq i \leq n$. Now we can prove our main theorem.

Proof of Theorem 1.3. (1) $\Rightarrow$ (2): By virtue of [5 Theorem (3.11)], the local ring $R$ is Cohen-Macaulay. Now assertion (2) follows from [4 Theorem 1].

(2) $\Rightarrow$ (3): This implication is obvious.

(3) $\Rightarrow$ (1): We denote by $(-)^\dagger$ the $C$-dual functor $\Hom_R(-, C)$. Put $t = \depth_R C$ and set $M = \Tr \Omega^t k$. Then we have depth $R = t$ by [7]. Since
$$\grade_R \Ext^i_R(k,R) \geq i - 1$$
for $1 \leq i \leq t$, the module $\Omega^t k$ is $t$-torsionfree by [1 Proposition (2.26)]. Hence $\Ext^i_R(M,R) = 0$ for $1 \leq i \leq t$. As $M$ is free on the punctured spectrum of $R$, Lemma 2.3 implies $\Ext^i_R(M \otimes_R C,C) = 0$ for $1 \leq i \leq t$. By assumption (3), the module $M \otimes_R C$ has an embedding into a module $X$ with $C\dim_R X < \infty$. According to [7 Lemma 4.3], we have $C\dim_R X \leq t$. Lemma 2.3 shows that the natural map $\lambda_{M \otimes_R C} : M \otimes_R C \to (M \otimes_R C)^\dagger$ is injective. On the other hand, since there are natural isomorphisms
$$(M \otimes_R C)^\dagger = \Hom_R(\Hom_R(M \otimes_R C,C),C) \cong \Hom_R(\Hom_R(M,\Hom_R(C,C)),C)$$
$$\cong \Hom_R(\Hom_R(M,R),C),$$
we see from [1 Proposition (2.6)(a)] that
$$\Ker \lambda_{M \otimes_R C} \cong \Ext^1_R(\Tr M,C) \cong \Ext^1_R(\Omega^t k,C)$$
$$\cong \Ext^{t+1}_R(k,C).$$
Thus we obtain $\text{Ext}_R^{k+1}(k, C) = 0$. By [3, Theorem (1.1)], the $R$-module $C$ must have finite injective dimension.

As we observed in the proof of the implication $(1) \Rightarrow (2)$, assertion $(1)$ implies that $R$ is Cohen-Macaulay. Thus the last assertion follows. \hfill $\square$

Now we give applications of our main theorem. Letting $C = R$ in Theorem 1.3, we obtain the following result. This improves Theorem 1.2 and extends Theorem 1.1.

**Corollary 2.4.** The following are equivalent:

1. $R$ is Gorenstein.
2. Every $R$-module can be embedded in an $R$-module of finite projective dimension.

Combining Corollary 2.4 with [4, Theorem 1], we have the following.

**Corollary 2.5.** If every finitely generated $R$-module can be embedded in a finitely generated $R$-module of finite projective dimension, then every finitely generated $R$-module can be embedded in a finitely generated $R$-module of finite injective dimension.

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Department of Mathematical Sciences, Faculty of Science, Shinshu University, 3-1-1 Asahi, Matsumoto, Nagano 390-8621, Japan
E-mail address: takahasi@math.shinshu-u.ac.jp

Department of Mathematics, University of Tehran, P. O. Box 13145-448, Tehran, Iran – and – School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran
E-mail address: yassemi@ipm.ir

Graduate School of Natural Science and Technology, Okayama University, Okayama 700-8530, Japan
E-mail address: yoshino@math.okayama-u.ac.jp