Tunneling approach and thermalinity in dispersive models of analogue gravity

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We set up a tunneling approach to the analogue Hawking effect in the case of models of analogue gravity which are affected by dispersive effects. An effective Schroedinger-like equation for the basic scattering phenomenon $IN \to P + N^*$, where $IN$ is the incident mode, $P$ is the positive norm reflected mode, and $N^*$ is the negative norm one, signalling particle creation, is derived, aimed to an approximate description of the phenomenon. Horizons and barrier penetration play manifestly a key-role in giving rise to pair-creation. The non-dispersive limit is also correctly recovered. Drawbacks of the model are also pointed out and a possible solution ad hoc is suggested.

I. INTRODUCTION

Since the original work by Hawking, the fundamental mechanism leading to Hawking radiation has been indicated as quantum mechanical tunneling of a particle through the black hole horizon [1]. In particular, in the geometrical optics approximation, which is justified by the fact that near the horizon particle energies are severely boosted, because of an hard blueshift, a classically forbidden tunneling of a particle from inside the horizon to outside can occur, according to still different mechanisms. Together with other approaches, so-called tunneling methods appeared since the initial period in black hole evaporation calculations, and we limit ourselves to quote only some seminal papers and a fine review [2–6]. It is also worth mentioning that in the Parikh-Wilczek approach to the Hawking effect tunneling through the horizon of a particle can happen because of a quite unexpected mechanism, where the tunneling particle sets up the barrier by energy conservation, as nicely described by Parikh [7]. In this method, as well as in the Hamilton-Jacobi one, coordinates well-defined on the horizon are to be preferable because they are not involved in subtle problems of analytical continuation (see e.g. [6]). The spectrum results to be non-strictly thermal because of the backreaction which takes place explicitly in the method, and which is fundamental for its feasibility [5, 7].

As well-known, in the dispersive case geometrical concepts like metric and so on are nearly lost (it is true that one could appeal to concept like ‘rainbow metrics’ and so on, but apparently there is no gain in pursuing this route in analogue gravity). Still, geometrical optics, and in particular geometrical optics tools which occur in the aforementioned methods, can be fruitfully adopted also in this analogue gravity framework. We shall discuss this topic with a particular reference to the optical black hole case. Since the first explorations in the field [8, 9], thermality appeared to be preserved in analogous black holes, despite the presence of dispersion. See also [10]. Moreover, group velocity horizons play the role of black hole horizons in the dispersive case. We start by analyzing the most natural concept for replacing a geometrical (kinematical) horizon, i.e. the would-be event horizon of the non-dispersive analogue case: the group velocity horizon. A group horizon is defined as the geometrical locus where the group velocity of the particle falls to zero in the comoving frame, or to the locus where the group velocity reaches the limit velocity of the model (light velocity in the optical case or sound velocity in fluid-like models) in the lab frame. There is a double question which is to be posed: is the group horizon responsible of the particle creation tout-court, or it is more simply responsible of making the phenomenon of pair-creation much more efficient? Moreover, in the latter hypothesis, which is the locus deputate to be responsible of particle creation? These questions are basic questions, but we must also underline a substantial fact: in the process of rebounce of the wave, which has been identified as an unavoidable ingredient in the pair-creation process in analogous models, there occurs a somehow non negligible region where the wave is enormously distorted. If the process is diabatic, particle creation occurs, and wave distortion is even bigger. The point is that, in the aforementioned region, reasonable doubts about the feasibility of concepts

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like group horizons maybe could be questioned, and in any case it would be hard to make a sharp distinction in the dynamics of the wave between group horizon position and phase horizon and also non-dispersive horizon. We try in the following to delve into the above questions, in the attempt to gain answers in an analytical way. In this sense, it is to be noted that, only a few analytical calculations appeared [11,22], but maybe no analytical definitive calculation, showing in a clear way and in a model independent way the key-role of group velocity horizons in the analogue Hawking effect in general dispersive media, has been provided. In the case of fluid-like models, we remark that papers [16, 22] are very general in their discussion. Still, calculations are considerably (but maybe unavoidably) involved. See also [23], where nonperturbative analytical calculations are performed for the dielectric black hole case in the framework of the Hopfield model.

We propose an interesting model as an approximation and as a possible step to fill this gap. In particular, we consider a phenomenological model for optical dispersion and show that Hawking-like effect arises as tunneling by antiparticle states, as in the gravitational case [2], provided that a group horizon exists, through which a particle with energy \( \omega \) can travel. Still, in a naive approach, the numerical coefficient is not the right one, in the sense that, even in the limit of negligible dispersion, it is seemingly not recovered the correct temperature, just for a numerical coefficient.

The latter problem could be considered to arise because of the nature of our approximation, and so it could appear that it could be considered as compatible, and even tolerable. Nevertheless, we think that the limit of negligible dispersion should be correctly reproduced, and then we investigate if a different expansion point, different form the group horizon, could give us a better answer. Quite surprisingly, provided that a suitable regularization is introduced, the right answer with the right coefficient is obtained at the horizon one finds in the non-dispersive situation, which is called geometrical horizon henceforth. This would be compatible with the idea that a weak dispersion effect is not and cannot be responsible of a drastic change in the nature of the process, and that the model we propose could be an interesting bridge between the non dispersive world and the dispersive one (where a weak dispersion is taken into account). Indeed, in the limit of negligible dispersion, the usual tunneling picture result of non-dispersive and astrophysical black holes is recovered. The method itself, in this sense, could belong to the long list of methods allowing to recover Hawking radiation in the non-dispersive case. As a matter of facts, the limit as dispersive effects vanish (as \( B \rightarrow 0^+ \) in the optical model in the Cauchy approximation, and as \( \Lambda \rightarrow \infty \) in the Coutant-Parentani-Finazzi model) is a singular limit involving both vanishing quantities and diverging ones, and has to be handled with care. Furthermore, in the same limit, the group horizon is shifted to coincide with the geometrical horizon of the non-dispersive model, so it needs itself a regularization procedure which sensibly modifies the way the nondispersive limit is approached. We find again the correct result.

There is still a problem. Even if the non-dispersive limit is correct, as dispersive effects arise and the expansion is made around the group horizon for nonzero \( B \) (or finite \( \Lambda \)), a different temperature is found: a different coefficient appears, which corresponds to an enhancement of the temperature for a factor 3/2. So, if we want to identify the group horizon as the locus where pairs are created, we find a discontinuous behaviour.

An alternative locus could be the geometrical horizon, again for group horizon as the locus where pairs are created, we find a discontinuous behaviour. There is still a problem. Even if the non-dispersive limit is correct, as dispersive effects arise and the expansion is made around the group horizon for nonzero \( B \) (or finite \( \Lambda \)), a different temperature is found: a different coefficient appears, which corresponds to an enhancement of the temperature for a factor 3/2. So, if we want to identify the group horizon as the locus where pairs are created, we find a discontinuous behaviour.

Even if the dichotomy between the behaviour of the relevant tunneling amplitude at finite dispersive effects and in the limit as dispersion vanishes can be considered as a serious drawback of our model, we point out that its simplicity and its being very near to the correct answers in the dispersive case, and its providing a further path, quite unusual, to find out the well-known non-dispersive ones (which are reachable by means of several methods), make it an interesting tool for discussing the dispersive case and to find a common method for discussing both cases. Its limits are also evident: the discontinuity in the temperature one finds if one focuses on the group horizon or on the geometrical horizon, is the major one. It would be possible to reconcile all the above discussion only if it were true that a divergence would occur at the group horizon also when dispersion is present. But it is not what emerges from the present analysis.

A further possible solution of the aforementioned problems, which allows to find both the correct limit as dispersive effects vanish and continuity in the behavior of the temperature in the same limit, without invoking a special prescription for the limit itself, is provided. Under hypotheses which are seemingly interesting, particle production could be associated with a further point, which we call ‘inner horizon’, falling beyond the geometrical horizon and converging to it in the non-dispersive limit. This proposal has admittedly the drawback to be constructed ad hoc, in order to find out thermality up to corrections which vanish in the limit of negligible dispersion.
Our paper plan is the following: in sec. II we first discuss the problem of group velocity horizon, following and also modifying techniques adopted in fluid mechanics, and involving a suitable expansion of the dispersion relation around the group horizon, to be discussed in subsec. II B. Then, in sec. III we consider a more radical modification of such an expansion, by drifting our attention into the non-dispersive horizon point. A link between the two expansions in the limit as dispersive effects vanish is also provided. In sec. IV, the discontinuous behavior of the temperature expression is pointed out, with a possible solution constructed ad hoc which is the argument of sec. V. A final discussion appears in sec. VI. For the sake of completeness, in appendix A we show how concepts of geometrical optics can be applied also to the general relativistic case. In appendix B we apply a straightforward variant of the Parikh-Wilczek method to the case of analogue black holes, which displays thermality in the non-dispersive case.

II. GROUP HORIZON AND GROUP HORIZON EXPANSION

Let \( G \) stay for the dispersion relation in a 2D case, in a static situation where no explicit dependence on time \( t \) occurs: this happens e.g. in the pulse reference frame in dispersive media, and one has

\[
G(\omega, k, x) = 0.
\] (1)

\( \omega \) is constant in the given framework. By solving e.g. for \( k \), one obtains a codimension 1 submanifold. The group velocity is given by

\[
v_g = \frac{dx}{dt} = \frac{\partial k}{\partial \omega} G
\] (2)

and a group horizon occurs for

\[
\partial_k G = 0.
\] (3)

Then, putting the above equation in a system with the dispersion relation, one obtains a codimension 2 submanifold (up to exceptional configurations, which are not considered herein). In particular, we get solutions \( x_{GH} \) which depend parametrically on \( \omega \).

A. Eikonal equation and group horizon

In optical systems, we can generate a group horizon by means of a traveling perturbation of refractive index induced in a nonlinear dielectric medium by the Kerr effect \[21, 24–29\]. In this case, it is suitable (but not strictly necessary) to adopt the comoving frame of reference of the dielectric perturbation (which is assumed to be moving, within a good approximation, with constant velocity). In this framework, a seed pulse meets a group horizon when it falls to zero velocity, and this happens when it is traveling through the pulse signal. Needless to say, dispersion makes not so automatic to get a group horizon in the above sense, and suitable conditions have to be implemented. What follows is meant to complement the perturbative analysis which was carried out in \[21\].

For simplicity we take into account the case of the Cauchy approximation, which holds for frequencies much lower than the resonance one in the case of a single-resonance model. In particular, it holds \( n(\omega_{lab}) = n_0 + B_\omega^2 \cdot \omega_{lab} \), where \( n_0 \) does not depend on the lab frequency \( \omega_{lab} \). We limit ourselves to considering only the branch which is involved with group horizons. Then, in presence of the Kerr effect, we have in the comoving frame \[27\]

\[
G = 0 \iff (\omega + vk)(n(x) + B_\omega^2(\omega + vk)^2) - ck - \frac{v}{c} \omega = 0.
\] (4)

It is also useful to rewrite it as follows:

\[
B_\omega^2(\omega + vk)^3 - \left( \frac{v}{c} - n(x) \right) (\omega + vk) + \frac{c}{v} \frac{\omega}{\gamma^2} = 0.
\] (5)

We take into account that \( \omega \) is a variable separation constant in the comoving frame, so we can assume to solve the dispersion relation in \( k \), obtaining \( k_b = g_b(\omega, x) \), where \( b \) labels the different branches. The cubic equation can be solved by means of Cardano’s formulas. We are not interested in the explicit expression, which would be involved, but we are interested in the expression for the group horizon (if any), which is obtained by solving the system

\[
G = 0,
\]

\[
\partial_k G = 0.
\] (6)

(7)
As to the latter equation, we obtain
\[\partial_k G = 0 \iff 3B\gamma^2 v(\omega + vk)^2 - v \left( \frac{c}{v} - n(x) \right) = 0, \tag{8}\]
which can be solved explicitly:
\[\omega + vk = \pm \left( \frac{c}{3B\gamma^2} - n(x) \right)^{1/2}. \tag{9}\]
By substitution of the positive root in \( G = 0 \), as we mean to get the group horizon for positive norm waves (see the following section), we obtain an equation for \( n(x) \) which allows us to find out explicitly the group horizon:
\[\frac{c}{v} - n(x) = 3B\gamma^2 \left( \frac{1}{2B\gamma^4} \frac{c}{v} \right)^{2/3} \omega^{2/3} =: \zeta B \omega^{2/3}, \tag{10}\]
where \( \zeta B \propto B^{1/3} \). We also find
\[\omega + vk|_0 = \left( \frac{1}{2B\gamma^4} \frac{c}{v} \right)^{1/3} \omega^{1/3}, \tag{11}\]
where all quantities are considered at the group horizon.
It can be noted that, as \( B \to 0^+ \), the group horizon converges to the geometrical horizon of the non-dispersive model:
\[\lim_{B \to 0^+} x_{GH}(\omega) = x_{geom}, \tag{12}\]
where \( x_{geom} \) is such that \( n(x_{geom}) = \frac{c}{v} \) \[27\].
For example, in the case of a Gaussian pulse with \( n(x) = n_0 + \eta \exp \left( -\frac{x^2}{\sigma^2} \right) \), one finds
\[x_{GH}(\omega) = \pm \sqrt{2\sigma} \left[ -\log \left( \frac{1}{\eta} \left( \frac{c}{v} - n_0 - \zeta B \omega^{2/3} \right) \right) \right]. \tag{13}\]
The expression \[13\] shows explicitly that the group horizon (if allowed) depends on \( \omega \), and that for low values of \( \omega \) it is almost indistinguishable from the horizon one finds in the non-dispersive model. So, at least in that region of frequencies, the link with the non-dispersive model is quite strong. To be more precise, the geometrical horizon of the non-dispersive case is replaced, in presence of dispersion, by a 1-parameter family of group horizons, with parameter \( \omega \). If, as in the Cauchy approximation, we have \( \omega \in (0, \omega_{max}] \), then this family runs from \( x_{gh}(\omega_{max}) < x_{gh} \) up to \( x_{geom} \); \( x_{gh} \in [x_{gh}(\omega_{max}), x_{geom}) \). A different situation occurs when there exists a \( \omega_{min} > 0 \), which is such that the geometrical horizon is never approached (cf. \[28, 29\]). One could even explore what happens for \( \omega = 0 \). It is easy to show that \[9\] and \[8\] require both \( k = 0 \) and \( n = \frac{c}{v} \), the latter condition being the horizon condition of the nondispersive model. So we have undulation together with the nondispersive horizon condition. Still, there is a problem there, because of the vanishing of \( \partial_k^2 G \) at the turning point, which would require the analysis of higher order contributions.

We could as well consider the case of an incompressible fluid, where the phenomenon can be classically described as blocking of waves \[30–32\]. Analogously, in BEC \[33, 34\] and in other materials, where horizons of this kind can be as well generated. See also the following section.

### B. Approximation near the turning point TP: the quantum case and the Hawking radiation

Let us consider the tunneling of a particle from inside a group velocity horizon. We start by recalling that the presence of a turning point, to be identified with a group horizon, is known to be a problem for the eikonal approximation. As a matter of facts, we are considering wave-like phenomena in which waves are strongly distorted at the rebounce on the dielectric perturbation (comoving frame). This implies that any concept like group velocity and ideas like the precise individuation of points is to be handled with care, having in mind the implicit limits in their use. Anyway, there is a possibility to provide an useful modification for the eikonal equation in the comoving frame, such that near the turning point a wave equation still holds true, allowing a better match between eikonal solutions and the presence of turning points, in the same spirit one finds in WKB approximation the Airy equation near the turning points of
the potential in the nonrelativistic Schrödinger equation. We follow [35]. See also the discussion in [36]. We stress that the original studies are concerned only with a classical level analysis of wave propagation, and the equation derived is aimed only to provide the amplitude of the wave equation [35]. Furthermore, we do not match solutions with asymptotic ones, but simply we calculate the transmission coefficient in the aforementioned approximation. In other terms, we consider the pair creation phenomenon locally, in agreement with tunneling approach for black holes. A similar expansion at the turning point, in the case of the electromagnetic field in plasmas physics has been proposed e.g. in [37] [38]. Herein, our aim is to consider the same approximation for the dispersion relation at the turning point, with a qualifying difference: when we substitute for operators, we keep trace of the mixed term $\partial_{x}\partial_{k}G$ and we impose to get an equation involving an hermitian wave operator. Moreover, we mean to adopt the variable separation ansatz for the wave equation $\Psi(x, t) = e^{-i\omega t}\Phi(x)$, which is justified because of the static nature of the comoving frame (no explicit time-dependence of the refractive index).

The wave equation to be considered is

$$G(\omega, k, x)\Psi(x, t) = 0,$$

where $\omega = -i\partial_{t}$ and $k = -i\partial_{x}$ are considered as operators in space-time variables. In the original approach, $\Psi(x, t) = be^{i\eta}$, where $b$ is a slowly varying amplitude and $\eta$ is a rapidly varying phase [35].

We consider $\omega$ as a variable separation constant, due to the fact that $\omega$ is conserved in our static framework; terms up to the second order near the turning point $x = x_{0}$ such that $(\partial_{k}G)|_{0} = 0$ are:

$$G \sim G(\omega, k_{0}, x_{0}) + (\partial_{k}G)|_{0}(k - k_{0}) + (\partial_{x}G)|_{0}(x - x_{0})$$

$$+ \frac{1}{2}(\partial_{k}^{2}G)|_{0}(k - k_{0})^{2} + (\partial_{x}\partial_{k}G)|_{0}(x - x_{0})(k - k_{0}) + \frac{1}{2}(\partial_{x}^{2}G)|_{0}(x - x_{0})^{2}. \quad \text{(15)}$$

In this expansion, we replace $k - k_{0} \rightarrow -i\partial_{x}$ and, moreover, in view of our aim of obtaining a Schrödinger-like quantum mechanical equation, we symmetrize the term proportional to $(\partial_{x}\partial_{k}G)|_{0}$ (Weyl symmetrization rule): we get

$$\left(-\frac{1}{2}(\partial_{k}^{2}G)|_{0}\partial_{x}^{2} - i(\partial_{x}\partial_{k}G)|_{0}\frac{1}{2}((x - x_{0})(-i\partial_{x}) + (-i\partial_{x})(x - x_{0})) + (\partial_{x}G)|_{0}(x - x_{0}) + \frac{1}{2}(\partial_{x}^{2}G)|_{0}(x - x_{0})^{2}\right)\Phi(x) = 0. \quad \text{(16)}$$

Then we obtain an equation of the form

$$(\partial_{x}^{2} + 2ib_{0}(x - x_{0})\partial_{x} + (ib_{0} - 2c_{0}(x - x_{0}) - d_{0}(x - x_{0})^{2}))\Phi(x) = 0. \quad \text{(17)}$$

If we put

$$\Phi(x) = \exp\left(-\frac{b_{0}}{2}(x - x_{0})^{2}\right)f(x), \quad \text{(18)}$$

we obtain the following reduced Schrödinger-like equation where no first order term remains:

$$\frac{\partial^{2}f(x)}{\partial x^{2}} + \left[-2c_{0}(x - x_{0}) + (b_{0}^{2} - d_{0})(x - x_{0})^{2}\right]f(x) = 0. \quad \text{(19)}$$

Solutions of this equation are well-known, and are parabolic-cylinder functions $D$. Coefficients above are related to the original ones as follows:

$$b_{0} = \frac{(\partial_{x}\partial_{k}G)|_{0}}{(\partial_{k}^{2}G)|_{0}}, \quad \text{(20)}$$

$$c_{0} = \frac{(\partial_{x}G)|_{0}}{(\partial_{k}^{2}G)|_{0}}, \quad \text{(21)}$$

$$d_{0} = \frac{(\partial_{x}^{2}G)|_{0}}{(\partial_{k}^{2}G)|_{0}}. \quad \text{(22)}$$

We assume $d_{0} \ll b_{0}^{2}$, in view of the fact that, in dispersive model, we expect an efficient pair-production for high values of first $x$-derivatives on $n(x)$ (or of $v(x)$ in the fluid models; see below), much higher than second $x$-derivatives. To be more precise, it should be also be not so strong to induce a coupling with a fourth state B belonging to a monotone further branch of the dispersion relation (cf. e.g. [10] and see also Figure[1]). Then we get

$$\frac{\partial^{2}f(x)}{\partial x^{2}} + \left[-2c_{0}(x - x_{0}) + d_{0}(x - x_{0})^{2}\right]f(x) = 0, \quad \text{(23)}$$
which can easily be reduced to the following form:

$$\frac{\partial^2 f(y)}{\partial y^2} + \frac{1}{b_0^2} \left[ y^2 - \frac{c_0^2}{b_0^2} \right] f(y) = 0,$$

(24)

which is equivalent to the Schrödinger equation with energy $E = 0$ in presence of a parabolic potential barrier. We face a problem: we started with a situation where a single group horizon (turning point) was present, and we actually found out an effective potential displaying two turning points: the group horizon $x_0$ and also

$$x_1 = x_0 + 2\frac{c_0}{b_0},$$

(25)

which arises as an effect of the term $O((x - x_0)^2)$ arising from the transformation (18). The latter term should be considered as spurious, higher order, as well as $x_1$. Still, in our problem we associate a (small) scale $\epsilon_x$ to each power of $(x - x_0)$ and also a (big) scale $L_x$ to the first derivative of $n(x)$. This big scale is fundamental for the efficiency of the pair-creation process, and also it violates the hypothesis of weakly varying medium occurring e.g. in plasma physics. Then, the term $\propto b_0^2$ would be $O(\epsilon_x^2 L_x^2)$, whereas the term $\propto c_0$ would be $O(\epsilon_x L_x)$ and the term $\propto d_0$ would be only $O(\epsilon_x^2)$. E.g. for $L_x$ order of $\epsilon_x^{-2}$, one would obtain $x_1 = x_0 + O(\epsilon_x)$, and then $x_1$ would coalesce with $x_0$ in the limit as $\epsilon_x \to 0$: it would appear as a so-called secondary turning point [39]. Of course, in order to obtain $x_1$ coalescing with $x_0$ as $\epsilon_x \to 0$ is sufficient to require that $L_x$ is order of $\epsilon_x^{(1+a)}$, with $a > 0$ (so that $x_1 = x_0 + O(\epsilon_x^a)$). In looking for an approximate and effective description of the scattering process, our ansatz is to keep the term $\propto b_0^2$ and also $x_1$, as an effective model description where the further turning point $x_1$ is assumed to delimit the interaction region where the phenomenon takes place.

We assume to associate a quantum mechanical level with equation (24). This assumption is nontrivial, and is justified only by considering a microscopic quantum model associated with the same dispersion relation. We need this underlying quantum level, without which both a quantum interpretation of the states and quantum tunneling, as we see below, would be lacking.

We take into account the transmission probability rate. One could also provide a complete solution, but a WKB approximation, within the above approximation of the dispersion relation, will be enough. Then we find the following transmission coefficient:

$$\mathcal{T}_{WKB} = \exp \left( -\pi \frac{c_0^2}{|b_0|^3} \right).$$

(26)

This transmission coefficient has to be interpreted as $\Gamma$ [5] in the tunneling approach to Hawking radiation for black holes. Thus, it is related to pair-creation.

We stress that, in order to relate the above $\mathcal{T}_{WKB}$ to the Hawking effect, we should expect the emergence of a dependence on an overall factor $\omega$, which is a non-trivial requirement. Indeed, a further step is required, i.e. the possibility to achieve the following identification:

$$\exp \left( -\pi \frac{c_0^2}{|b_0|^3} \right) = \exp \left( -\beta_H \hbar \omega \right),$$

(27)

where $\beta_H$ is proportional to the inverse of the Hawking temperature $T_H$. This identification of the tunneling probability rate with a Boltzmann factor is common in the tunneling approach to the Hawking effect (see e.g. [3]). We apply our general picture to specific cases in the following.

C. Optical black hole in the Cauchy approximation

In the case of the Cauchy approximation, i.e. for [5], we have

$$b_0 = v(\partial_x n)_0 \left( \frac{2B\gamma^4}{6u^2\gamma^2} \right)^{1/3} \left( \frac{v}{c} \right)^{1/3} \omega^{-1/3},$$

(28)

$$c_0 = (\partial_x n)_0 \frac{1}{6u^2\gamma^2 \beta},$$

(29)

$$d_0 = (\partial_x^2 n)_0 \frac{1}{6u^2\gamma^2 \beta}.$$  

(30)
FIG. 1: Complete asymptotic dispersion relation for the Sellmaier dispersion relation of a diamond-like material in the lab frame (qualitative plot). A line of constant \( \omega \) is represented by a straight line. Note that a fourth state B appears, which is not actually present if the gradient of the refractive index is not too strong. We are substantially considering only the branch intersected by the straight line in states IN, P, \( N^* \).

We find also

\[
x_1 - x_0 = 32^{4/3} \gamma^{-2/3} \left( \frac{c}{\nu} \right)^{2/3} (B\omega^2)^{1/3} \frac{1}{|\partial_x n|_0},
\]

and

\[
T_{WKB} = \exp \left( -\frac{3c}{\gamma^2 \nu^2 |\partial_x n|_0} \omega \right).
\]

So we have gained an important corroboration to our approach, because we have found the overall factor \( \omega \) which was not so trivial to obtain. Then we find in our approximation

\[
k_b T_H \sim \frac{\hbar}{c} \frac{\gamma^2 \nu^2 |\partial_x n|_0}{3\pi} \omega,
\]

which, apart from the factor 3 in place of 2 at the denominator, is just the expected result. The missing factor is 1.5, which could be considered not so bad, given the approximation we adopted. Note also that \( |\partial_x n|_0 \), and then also \( T_H \), depend on \( \omega \) through \( x_{GH} (\omega) \). So thermality holds only approximately, when the latter dependence is weak. The most direct analogy is with the dependence on \( \omega \) of the temperature in the Parikh-Wilczek approach. Note that \( T_H \) is naturally involving ‘backreaction’, in the sense that there is interplay between the external field (refractive index) and the physical field (the electromagnetic field) through dispersion, which makes \( T_H \) dependent also on \( \omega \). It is a backreaction in a different sense with respect to the Parikh-Wilczek case, because there is not yet an energy balance for the total system ‘black hole+radiation’, but effects of dispersion, which influence the emitted radiation, can be as well considered backreaction, being due to an effective interaction between the radiation and the black hole which affects the effective description of the black hole itself.

We have to clarify a few very important facts: by looking at the dispersion relation which holds asymptotically in \( x \) (i.e. for \( x \to \pm \infty \), where homogeneity is recovered), one finds three states involved in the scattering process when a group horizon is present: the incident mode \( IN \), the positive norm reflected mode \( P \), and the negative norm one \( N^* \).

By looking at the dispersion curve, the group velocity \( v_g \) associated with \( IN \) is positive, whereas the group velocities of both \( P \) and \( N^* \) are negative in the comoving frame, cf. also Figure 1. Still, according to Feynman-Stueckelberg interpretation, a negative norm state is propagating backward in time, so it is natural to assign to it a group velocity which is the opposite of the one which can be deduced from the dispersion relation curve. As such, the antiparticle state is propagating forward, so it emerges as a transmitted state. The corresponding particle state \( N \), which actually enters the experimental situation, is propagating backward (i.e. is reflected; this is natural in a particle-hole picture, inspired by the Dirac sea picture, where the propagating hole is associated with a counterpropagating particle). This picture allows us to give concrete meaning to a scattering process with a transmitted state which, otherwise, would not be allowed. Notice that: a transmitted state in a scattering process is allowable only in a quantum mechanical process; moreover, we need an underlying quantum field theory model (in our case: Hopfield model) with a conserved norm for defining particle and antiparticle states.

We also point out that we could legitimately solve eqn. (17) and find again parabolic-cylinder functions as solutions.
for our problem, apart from a space-dependent phase factor. Calculations of the transmission coefficient $T_{WKB}$ would give the same value as above, and the problem would be to explain why $T_{WKB} > 0$ in this case. As a matter of facts, this result would not be so unexplainable, as we are dealing with quantum field theory in external field, and unexpected transmission coefficients different from zero are usual in all situations where e.g. Klein paradox occurs (see e.g. [10]).

As well known, dispersion tends to distort the pure thermality of the non-dispersive situation, because at the best one is able to find a temperature which depends on $\omega$. Of course, this makes the spectrum not really Planckian. Usually, in the case of weak dispersion, one assumes and verifies that, on restricted frequency intervals, it is true that $T(\omega) \sim T(\omega = 0) = T_H^{nd}$, where $T_H^{nd}$ stays for the non-dispersive temperature.

**D. Coutant-Parentani-Finazzi model**

As a matter of facts, doubts should be raised on the actual viability of this effective approach to analogous Hawking radiation in dispersive media. A positive test on a single model could be a coincidence. So we try to verify the approach on a different model. We start from Coutant-Parentani-Finazzi model [16], because, at least for a particular form for the dispersive contribution, it is possible to carry out explicit analytical calculations. Other models could be also checked, but, except for the model in the following subsection, the system $G = 0, \partial_k G = 0$ in other cases is quite tricky, so a check seems to be difficult.

The general class of models considered in [16] satisfies

$$G = (\omega - v(x) k)^2 - F^2(k) = 0,$$

(34)

where $F^2(k)$ is such that dispersive effects are involved. As in [16], we consider the following sub-case:

$$G_c := \omega - v(x) k - \left(ck + \frac{k^3}{2\Lambda^2}\right) = 0,$$

(35)

where $\Lambda \to \infty$ is the limit as the model becomes non-dispersive and where we restored momentarily $c$ (to be identified as $c_{sound}$). We put $c = 1$ henceforth. One obtains [16]

$$k|_0 = - (\omega \Lambda^2)^{1/3},$$

(36)

and

$$v|_0 = -1 - \frac{3}{2} \left(\frac{\omega}{\Lambda}\right)^{2/3}.$$  

(37)

Notice that, as $\Lambda \to \infty$, the group horizon tends to the geometrical horizon such that $v|_{geom} = -1$. We obtain

$$b_0 = \frac{\Lambda^2}{3(\omega \Lambda^2)^{1/3}} (\partial_x v)_0,$$

(38)

$$c_0 = \frac{\Lambda^2}{3} (\partial_x v)_0.$$  

(39)

As a consequence,

$$|x_1 - x_0| = \frac{6}{\Lambda^{2/3}} \frac{1}{[(\partial_x v)|_0^2]} \omega^{2/3}.$$  

(40)

From

$$T_{WKB} = \exp \left(-\pi \frac{3}{[(\partial_x v)|_0^2]_0} \omega\right),$$

(41)

we find in our approximation

$$k_b T_H \sim h \frac{[(\partial_x v)|_0^2]}{3\pi}.$$  

(42)
Thus the same error for a factor 1.5 as in the optical case is discovered. The temperature is proportional to \((\partial_x v)|_0\), as it happens in the corresponding non dispersive case. There is, also in this case, a dependence on \(\omega\) of the temperature which can be made explicit by choosing a specific velocity profile \(v(x)\).

It is remarkable that the dependence of \(\omega\) of both \(b_0\) and \(c_0\) is the same as in the optical model discussed in the previous subsection. For what we can ascertain, this seems to be an accident (we start from very different dispersion relations and dispersion mechanisms). As discussed in the following, their only apparent link is represented by the fact that both models are involved with weak dispersive contributions.

In the following subsection, we investigate a further model which can be analytically treated. We will find a very different behaviour in \(\omega\) for both \(b_0\) and \(c_0\), but still the overall \(\omega\) factor appears.

### E. Rousseaux model

We take into consideration Rousseaux model \[41\], where explicit calculations are possible. This model falls in the general class \[34\], but in the dispersion relation we are going to write we cannot find a non-dispersive limit as before. The approximate dispersion relation is \[41\]

\[
G = (\omega - v(x)k) - g|k|^{1/2} = 0. \tag{43}
\]

Assuming for simplicity \(k > 0\) (results for \(k < 0\) correspond to the case of positive velocity \(v\)), we can find that the equations \(G = 0\), \(\partial_k G = 0\) can be solved to find

\[
v(x_{GH}) = -\frac{g}{4\omega}, \tag{44}
\]

and

\[
k_0 = \frac{4\omega^2}{g}. \tag{45}
\]

The group horizon exists for \(\omega < \left|\frac{g}{4\pi}\right|\).

We can adopt the same approximation for \(G\) around the turning point as above. In particular, with reference to equation \[24\]. We have

\[
b_0 = -\frac{32}{g^2} (\partial_x v)|_0 \omega^3, \tag{46}
\]

\[
c_0 = -\frac{128}{g^3} (\partial_x v)|_0 \omega^5. \tag{47}
\]

In this case, we obtain

\[
|x_1 - x_0| = \frac{1}{4} \frac{g}{|(\partial_x v)|_0} \frac{1}{\omega}. \tag{48}
\]

Then, from

\[
\exp \left( -\pi \frac{1}{2 |(\partial_x v)|_0} \omega \right) = \exp \left( -\beta_H \hbar \omega \right), \tag{49}
\]

one infer

\[
k_b T_H = \frac{4\hbar}{\pi} |(\partial_x v)|_0, \tag{50}
\]

thus the numerical coefficient is 8 times the expected one. Not so good, an enhancement of a factor 8, but, on the other hand, maybe not so bad, given that both the model is approximate and also the equation is very approximate. It is remarkable that, apart from numerical coefficients, the dependence on gradients and on \(\omega\) are exactly the right ones. As a consequence, the method seems to be just more reliable than one could ever have expected. Moreover, it has to be pointed out that the model is essentially dispersive, in the sense that one cannot obtain a non-dispersive limit, and so it is not clear if one should obtain exactly the ‘geometrical result’ (we mean, the result inspired by the non-dispersive case).

A natural question one could put about the fluid model above, is why one should choose the above approach instead
of exact calculations. The point is that in exact calculations the explicit role of the group horizon is often hidden inside nontrivial studies of asymptotic expansions of fourth order differential equations, which are by no means an easy stuff (almost twenty years of still in progress studies quoted above).

In concluding this section, we wish to compare the results obtained in all the models we have taken under consideration and discuss them further on. Thermality arises because of the tunneling effect through the group velocity horizon. The process is most efficient as the gradient of the refractive index/the fluid velocity field increases (although it is not the only parameter). This is also associated with the distinction between adiabatic (particle number conserving) and non-adiabatic process (particle number increasing). Thermality is very near the expected result in cases where dispersion can be considered as weak (optical case, Coutant-Parentani-Finazzi model). Control over dispersive contributions is given by the coefficient $B$ serving) and non-adiabatic process (particle number increasing). Thermality is very near the expected result in cases where dispersion can be considered as weak (optical case, Coutant-Parentani-Finazzi model). Control over dispersive contributions is given by the coefficient $B$ in the optical case (as $B \to 0$ the non-dispersive case is recovered) and by $\frac{1}{\Lambda}$ in the Coutant-Parentani-Finazzi model (non dispersive case is recovered as $\Lambda \to \infty$). Both these limit are to be taken with care, because they require a suitable regularization in order to find out a reliable result. This is the topic to be discussed in the following section.

Rousseaux dispersion model is still thermal but not near the weak dispersion limit (and indeed dispersion is not weak in that case). In this case, we are not aware of any way to recover a sort of weak dispersion limit. Tunneling is involved with the presence of a secondary turning point. As a matter of fact, the presence of terms $O(x^2)$ and not simply linear in $x$ is necessary for looking for a tunnel effect: such a presence of second order terms in $x$ is non-exclusive of our model. Indeed, this presence occurs also in [16], even if in a form which is not so evident. Indeed, when the so-called $p$-representation near the horizon $x = 0$ is considered in [16], terms $O(x^2)$ appear (they correspond to terms $\propto \partial_x^2$). Consistency would also require that even $v(x)$ should contain terms $O(x^2)$. This would in turn imply to neglect higher order terms in $\partial_x$ derivatives (up to the fourth order). In view of this, the approximation adopted therein seems to be similar to ours one (in neglecting higher order terms). Of course, in [16] a much more complete analysis is performed. But we wish to remark this relevant point which maybe does not emerge so easily from [16]. A thickness for the horizon is also found as in [22], albeit with a different behaviour (and we feel that the approximation in [16] [22] is more reliable than ours one). In the following section, we deal with the non-dispersive limit and we show that some subtleties, to be handled with care, occur.

III. HOW TO HANDLE THE LIMIT OF VANISHING DISPERSION IN THE WEAK DISPERSION CASE, AND TO RESTORE THE ROLE OF NON-DISPERSIVE HORIZON

Let us consider again the expansion [15], but this time at first we focus on the non-dispersive horizon in place of the group horizon, by keeping dispersive contributions in the dispersion relation. Then we get a further term of the first order in $(k - k_0)$, which is proportional to $\partial_x G$. This modifies equation (17), so that we obtain

$$
\left( -\frac{1}{2} (\partial_x^2 G) \right) + i \partial_x (x - x_0) - \partial_x (x - x_0) - i (\partial_x G) \partial_x + \partial_x (x - x_0) + \frac{1}{2} (\partial_x^2 G) \partial_x (x - x_0)^2 \right) \Phi(x) = 0.
$$

(51)

Then the equation is of the form

$$
\left( \partial_x^2 + i (b_0 (x - x_0) + c_0) \partial_x + (i b_0 - 2 c_0 (x - x_0) - d_0 (x - x_0)^2) \right) \Phi(x) = 0,
$$

(52)

where we have defined

$$
e_0 := \frac{(\partial_x G) \partial_x (x - x_0)^2}{(\partial_x^2 G) \partial_x (x - x_0)^2}.
$$

(53)

If we choose

$$
\Phi(x) = \exp \left( -i \frac{b_0}{2} (x - x_0)^2 - i c_0 (x - x_0) \right) f(x),
$$

(54)

we obtain the following reduced Schrödinger-like equation where no first order term remains:

$$
\frac{\partial^2 f(x)}{\partial x^2} + \left[ -2 c_0 (x - x_0) + (b_0 (x - x_0) + e_0)^2 - d_0 (x - x_0)^2 \right] f(x) = 0.
$$

(55)
Solutions of this equation are again parabolic-cylinder functions $D$. Turning points of the parabolic potential are
\[ x_\pm = x_0 \mp \frac{1}{b_0^2 - d_0} \left[ c_0 - b_0 c_0 \pm \sqrt{c_0^2 - 2b_0 c_0 d_0 c_0 + d_0 c_0^2} \right], \] (56)
and the ‘thickness’ of the interaction region is
\[ x_+ - x_- = 2 \frac{1}{b_0^2 - d_0} \sqrt{c_0^2 - 2b_0 c_0 d_0 c_0 + d_0 c_0^2}. \] (57)
We have to require, in order to get real solutions,
\[ c_0^2 - 2b_0 c_0 d_0 c_0 + d_0 c_0^2 \geq 0. \] (58)
The following redefinition
\[ y := (b_0^2 - d_0)x + c_0 b_0 - c_0, \] (59)
allows us to obtain the following form for the Schroedinger-like equation:
\[ \left[ \frac{d^2}{dy^2} + \frac{1}{(b_0^2 - d_0)^2} \left( y^2 - \frac{c_0}{b_0} + 2\frac{c_0 c_0}{b_0} \right) \right] f = 0, \] (60)
which can be easily turned into the form of a Weber equation, provided that we define $z = a^{-3/4}y$. The tunneling coefficient is given by
\[ T = \exp(-\pi \alpha), \] (61)
with
\[ \alpha := \frac{c_0^2 - 2c_0 c_0 b_0 + d_0 c_0^2}{|b_0^2 - d_0|^{3/2}}. \] (62)
In terms of derivatives of the dispersion relation $G$, we get
\[ \alpha := \frac{1}{\left| (\partial_x \partial_k G)^2 - (\partial_k^2 G)(\partial^2 G) \right|^{3/2}} \left[ (\partial_x G)^2(\partial_k^2 G) - 2(\partial_x G)(\partial_k G)(\partial_x \partial_k G) \right] + \frac{1}{\left| (\partial_x \partial_k G)^2 - (\partial_k^2 G)(\partial^2 G) \right|^{3/2}} (\partial_k^2 G)(\partial^2 G). \] (63)
We consider the above expansion with coefficients whose expressions are suitably regularized: if $x_0$ identifies the geometrical horizon, we regularize it by the following shift $x_0 \mapsto x_0 - \epsilon$ (we are approaching a white hole horizon from the left, it is easy to arrange for a black hole horizon approached from the right: $x_{bh} \mapsto x_{bh} + \epsilon$). Then we perform our calculations, perform the limit as $B \to 0^+$ in the optical case and as $\Lambda \to \infty$ in the Coutant-Parentani-Finazzi case, and only then we perform also the limit as $\epsilon \to 0$. In our view, this is the correct order in which the aforementioned limits are to be taken. Indeed, in order to explore the limit in which dispersive effects vanish, we need to consider still regularized quantities (such a limit makes singular some expressions). A regularized quantity $R$ will be indicated as $R_\epsilon$.

A. The optical case: Cauchy approximation

We implement our calculations by starting from the same dispersion relation as in sec. [ITC] with the difference that we are expanding near the geometrical horizon and we are working also having in mind the limit of vanishing dispersive effects. I.e., we mean to take the limit as $B \to 0^+$. We know that, in the latter limit, singularities arise in the wave vector $k$, and then, we adopt the strategy sketched above: we first regularize and then let $B \to 0^+$. Only at the end we relax the regularization.

It is straightforward to show that
\[ \alpha_\epsilon = 2 \frac{\omega + vk_n}{v[\partial_x n]_\epsilon} \left( \frac{c}{v} - n \right)_\epsilon \] (64)
holds true; it also holds

$$\lim_{B \to 0^+} \left( \omega + v k_{\epsilon} \right) \left( \frac{c}{v} - n \right) = \frac{c}{v} \frac{\omega}{\gamma^2},$$  \hspace{1cm} (65)$$

so that

$$\lim_{\epsilon \to 0} \left( \lim_{B \to 0^+} \pi \alpha_{\epsilon} \right) = \frac{2\pi c}{\gamma^2 v^2 |\partial_x n|_0} \omega,$$  \hspace{1cm} (66)$$

which is the expected result. This is a very intriguing result, as it provides us a further version of the tunneling method in which a true effective barrier is to be overcome in order to obtain pair creation. Furthermore, quite surprisingly, one can obtain the non-dispersive result from a method devised for the dispersive case. It is also to be noted that, in the non-dispersive limit, the thickness of the barrier tends to zero, and one is left with a sort of ‘phantom barrier’ in that limit, tending to the geometrical horizon.

We also know that (12) holds true, i.e. the group horizon shifts to the geometrical horizon as $B \to 0^+$. Then, we can consistently also proceed as follows. As we are interested in the limit as $B \to 0^+$, and we know that in such a limit $x_{gh} \to x_{geom}$, also the expansion around the group horizon has to be regularized. Our choice is to shift $x_{gh} \mapsto x_{gh} - \epsilon$, (67)

in such a way that $x_{gh} - \epsilon \to x_{geom} - \epsilon$ as $B \to 0^+$. But such a shift can be coherently taken into account only at the price to restore the term $(\partial_k G)$, because it is no more valued at $x_{gh}$, where it vanishes, but at $x_{gh} - \epsilon$, where it is different from zero (albeit small). Formally, (62) holds true again. As the limit $B \to 0^+$ is consistently taken before the regularization is relaxed, and in such a limit $x_{gh} - \epsilon \to x_{geom} - \epsilon$, we obtain the same result as above. So, we recover the expected result even by starting from the group horizon, provided we understand that a regularization procedure is necessary. A regularization procedure implying a shift from the horizon coordinate is by no means a novel feature of our model: by quoting only the latest approach for deriving Hawking radiation, i.e. the (gravitational and gauge) anomaly approach introduced by [42] and then improved by [43], a regularization is required (in [43], in particular, one has $r_+ \mapsto r_+ + \epsilon$, where $r_+$ is the black hole horizon). It is also easily shown that, if a regularization is still introduced in the above sense, but no limit of vanishing dispersion is taken, when the regularization is sent to zero ($\epsilon \to 0$) the same result as for the ‘on shell’ calculation (unregularized) of section II is consistently obtained. So, putting quantities ‘on shell’ is equivalent to regularizing, computing the physical quantities and then relaxing the regularization if dispersive effects are non-vanishing: $B > 0$, and $\Lambda < \infty$.

**B. The Coutant-Parentani-Finazzi model**

In this case, by taking into account the regularization procedure devised in the previous subsection, we find

$$\alpha_{\epsilon} = 2\epsilon k_{\epsilon} \left( v + c \right) \epsilon;$$  \hspace{1cm} (68)$$

in the limit as $\Lambda \to \infty$ one has

$$k_{\epsilon} = \frac{\omega}{v + c},$$  \hspace{1cm} (69)$$

so that

$$\lim_{\epsilon \to 0} \left( \lim_{\Lambda \to \infty} \pi \alpha_{\epsilon} \right) = \frac{2\pi}{|\partial_x v|_0} \omega,$$  \hspace{1cm} (70)$$

which, again is the correct result for the non-dispersive case.

**IV. A DISCONTINUITY IN THE TEMPERATURE AS $B \to 0^+$ ($\Lambda \to \infty$)**

We face with the main drawback of our model. If we want to explore the limit $B \to 0^+$ for weak dispersion, as we are implicitly approaching the geometrical black hole, where, in absence of dispersion, we know that divergences in $k$ appear, we have to prescribe a regularization. This is by no means a disease, because also in other approaches a stretching of the horizon coordinate, to be removed at the end of the calculations, is to be introduced. But there is
the following problem: we could as well consider $T_H(x_{gh}(\omega), B)$ for $B > 0$ and find a thermality which, in the optical model in the Cauchy approximation and in the Coutant-Parentani-Finazzi model, is such that

$$ T_H(x_{gh}(\omega), B) = \frac{3}{2} T_{nd}^H, \quad (71) $$

where we use $\equiv$ for indicating that the overall functional dependences are the same in the two cases and ‘nd’ stands for non dispersive case. Still, if we wish to explore the non-dispersive limit, we have to regularize again also the group horizon coordinate and find the correct result. What happens is that

$$ \lim_{\epsilon \to 0} \left( \lim_{B \to 0^+} T_H(x_{gh}(\omega) - \epsilon, B) \right) \neq \lim_{B \to 0^+} \left( \lim_{\epsilon \to 0} T_H(x_{gh}(\omega) - \epsilon, B) \right). \quad (72) $$

The temperature is not continuous in $(x_{geom}, 0)$. Even if the regularization prescription works well and can be used as a prescription for calculating the non-dispersive result in a new way, from a physical point of view there is a unsatisfactory situation, which is even worst if one focuses on the geometrical horizon at $B > 0$. Indeed, there, when $d_0 = 0$, the temperature diverges, and thermality is still lost if $d_0 > 0$:

$$ \alpha(x_{geom}, B) \sim \frac{9\gamma^4 B^2(\omega + vk)^5}{v|\partial_x n|^3}, \quad (73) $$

where we have neglected terms proportional to $\partial^2_x n$ with respect to the ones proportional to $\partial_x n$ (or integer powers of the latter). This drawback is non-eliminable. If thermality were to be defined only as far as dispersion is very weak, one could assume that

$$ \lim_{B \to 0^+} \left( \lim_{\epsilon \to 0} T_H(x_{gh}(\omega) - \epsilon, B) \right) = T_{nd}^H. \quad (74) $$

Still, the latter definition of thermality we feel that is not appropriate, and a dichotomic behavior remains.

V. A FURTHER POINT TO RECOVER THERMALITY WITH A CONTINUOUS NONDISPERSE LIMIT

Let us consider a further hypothesis: pair creation happens at a different point $x_*$ such that a) there is thermality up to corrections which vanish as $B \to 0^+$; b) the point $x_*$ also tends to $x_{geom}$ in the non-dispersive limit. We start by discussing the optical case. Assumption (a) is really strong, but it is corroborated by the fact that the right thermality is found in the non-dispersive limit, and, moreover, by the idea that there should be a continuous behaviour of the temperature in the same limit. It is straightforward to show that, again, whichever explicit expression one can gain at a specific point, it holds

$$ \pi \alpha = \frac{2\pi}{v|\partial_x n|_*} (\omega + vk)_*(\frac{c}{v} - n)_*, \quad (76) $$

where all quantities are ‘on shell’ (i.e. satisfy the dispersion relation) at $x = x_*$. Then, we require that

$$ (\omega + vk)_*(\frac{c}{v} - n)_* = \frac{c}{v} \frac{1}{\gamma^2} \omega (1 + a(B)\omega^z), \quad (77) $$

where $z$ is a fixed non-negative real number and

$$ \lim_{B \to 0^+} a(B) = 0. \quad (78) $$

Of course, we could replace the right hand side of (77) with a series, where a number of coefficients $a_i(B)$ appear. But we assume to consider only the lowest order correction to the thermal contribution. Notice that the presence of
a correction $a(B)\omega^{\frac{1}{2}+1}$ is mandatory in order to be able to obtain a point where the spectrum is thermal (if $a(B) = 0$, then there is no solution).

Substituting in the dispersion relation (5), one finds

$$\left(\omega + \nu k\right)_* = \left(\frac{a(B)}{B^2 \gamma^2 \nu} \right)^{1/3} \omega^{(1+z)/3}, \quad (79)$$

$$\left(\frac{c}{\nu} - n\right)_* = \left(\frac{B\gamma^4}{a(B)}\right)^{1/3} \left(\frac{c}{\nu}\right)^{2/3} \frac{1}{\gamma^2} \omega^{(2-z)/3}(1 + a(B)\omega^z). \quad (80)$$

As we want that $x_* \to x_{\text{geom}}$ as $B \to 0^+$, we also require that

$$\lim_{B \to 0^+} \frac{B}{a(B)} = 0. \quad (81)$$

From the ratio

$$R := \left(\frac{c}{\nu} - n\right)_*/\left(\frac{c}{\nu} - n\right)_{gh}, \quad (82)$$

recalling the hypothesis $\partial_n n > 0$, it is easy to show that, for fixed $\omega$, if $a(B) > 0$, there is no possiblity to get $0 < R < 1$ in a right neighbourhood of $B = 0$, i.e. one finds that $x_*(\omega) < x_{gh}(\omega) < x_{\text{geom}}$. If, instead, $a(B) < 0$, one finds that $x_{gh}(\omega) < x_{\text{geom}} < x_*(\omega)$, i.e. the points falls beyond the geometrical horizon. The latter case, on the grounds of (79), appears to be more appealing under the hypothesis of positive gradient, because it would maximize the temperature, and would imply a partial wave penetration beyond both the group horizon and also the geometrical horizon, in a region of stronger gradient. We consider the latter hypothesis as a bit more appealing, and we indicate $x_*$ as ‘inner horizon’. We remark also that, for fixed $B$, it is instead possible to find $R < 1$ and then it is possible to get a point $x_*(\omega)$ coinciding with a group horizon $x_{gh}(\omega)$ for $\tilde{\omega} < \omega$.

As to the limit as $B \to 0^+$, the aforementioned overlap does not occur, and what happens is that the existence of $x_*(\omega)$ implies the existence of a corresponding $x_{gh}(\omega)$, whereas the vice-versa is not true in general, i.e. it may happen that a group horizon does not imply the existence of $x_*$ at the same $\omega$. A continuous behaviour of the temperature would emerge, with the correct result in the non-dispersive limit without explicitly requiring any regularization. The point is that, in this case, the two limits $\epsilon \to 0$ and $B \to 0^+$ commute. As a consequence, taking seriously the hypothesis of pair-creation at $x_*$, it seems that the presence of a group horizon is a necessary but not sufficient condition for thermality with the expected temperature. But we stress that, as emerges from the above discussion, the point $x_*$ is constructed ‘ad hoc’.

The same construction can be devised for the Coutant-Parentani-Finazzi model. We sketch the main results:

$$k_* = \left(-a(\Lambda)2\Lambda^2\right)^{1/3} \omega^{(1+z)/3}, \quad (83)$$

$$(v + c)_* = \left(-\frac{1}{a(\Lambda)2\Lambda^2}\right)^{1/3} \omega^{(2-z)/3}(1 + a(\Lambda)\omega^z). \quad (84)$$

We have to require $\lim_{\Lambda \to \infty} a(\Lambda)\Lambda^2 = 0$, in such a way that $x_*(\omega, \Lambda) \to x_{\text{geom}}$ as $\Lambda \to \infty$. Also in this case, for $a(\Lambda) > 0$ and for $\partial_x v > 0$, one finds $x_*(\omega) < x_{gh}(\omega) < x_{\text{geom}}$; instead, for $a(\Lambda) < 0$, one finds $x_{gh}(\omega) < x_{\text{geom}} < x_*(\omega)$. This would appear as compatible with the idea of ‘horizon thickness’ as depicted in [22].

VI. CONCLUSIONS

We have proposed an effective scattering model for the Hawking effect in analogue black holes, inferred from an expansion of the dispersion relation near the group horizon (turning point). A Schroedinger-like equation in presence of a suitable parabolic barrier is obtained. Computation of the transmission coefficient in the WKB approximation gives us a Boltzmann factor which is assumed to be a sufficient condition for thermality, as usually in standard tunneling method. Changing dispersion relation simply changes the value of the coefficients, and so our approach is universally applicable to all phenomenological models where an underlying quantum microscopic model is available. It is also remarkable that there is no free parameter at hand.

We started by studying the group horizon, which is naively the best candidate for pair-creation in a dispersive
situation. The temperature obtained taking into account the group horizon has the correct dependence on the physical parameters but it can differs from the correct one just for a numeric factor. This does not seems a fatal disease of the model (at least as far as the optical model and Coutant-Parentani-Finazzi one are concerned), because it is reasonably attributed to the approximation itself. Nevertheless, with a suitable regularization and completion of the expansion series, the correct result can be still restored in the limit as dispersive effects vanish, where the group horizon merges with the geometrical horizon. Then a unpleasant dichotomy for the behaviour of the temperature appears. Indeed, according to our model, there would be a discontinuity in the behavior of the temperature in the nondispersive limit; there is a dichotomy between the expansion series, the correct result can be still restored in the limit as dispersive effects vanish, where the group horizon, is the locus where pair creation occurs. This solution would be plausible but some more discussion is also viable.

By focusing on the geometrical horizon, in the limit of vanishing dispersion, one can recover the correct result for the non-dispersive case, by treating with care a process of regularization and the sequence of limits to be considered. It is worth pointing out that such a method, which starts from a dispersive situation, is interesting in itself, as is able to reproduce standard nondispersive results, and to corroborate in a unexpected way the idea of a barrier connected with the pair creation process by black holes. In the model, indeed, it appears a parabolic barrier whose turning points converge to the geometrical horizon as the dispersive effects are taken to be vanishing. Differently from the Parikh-Wilczek approach, where the barrier is self-tuned by backreaction, here it is induced by the presence of nonlinear interactions of the fields at hand. So our tunneling ansatz, as rough as it can be, still has elements of interest: it seems to be universal in the above sense. On the other hand, at finite dispersive effects, the serious dichotomy problem we found for the group horizon is even worse at the geometrical horizon, as discussed in the previous section.

A possible solution, which has the drawback to be constructed ad hoc, is to attribute pair creation to a further point, the ‘inner horizon’ $x_*$, which is such that it allows to find out thermality up to corrections which vanish in the limit of negligible dispersion. According to this hypothesis, thermality with a full continuous behaviour in the limit as dispersive effects vanish is possible if pair-creation occurs at a inner horizon $x_*(\omega)$, which also converges to the geometrical horizon in the limit. Moreover, in the same limit, it implies the existence of a group horizon as a necessary (but not sufficient) condition for thermality with the above characteristics. Further models and a better analysis are necessary in order to delve into problems of the locus or, more probably, the region where particle creation occurs.

As to the 4D case, we note that, if we allow variable separation also for transverse variables $y, z$ (cf. e.g. the separability in $\theta, \phi$ in the spherosymmetrical general relativistic case), there are no substantial modification to the picture described above. Indeed, two more conserved quantities $k_y, k_z$ appear in the dispersion relation, which enter in the various branches $k_B$ but do not affect the form of the differential equation.

Appendix A: group horizon: from analogous black hole to general relativistic ones

In the eikonal approximation and in the non-dispersive case, the vanishing of $v_g$ still allows to define the horizon, both in the gravitational and in the analogue cases. Let us consider the Schwarzschild case. In Painleve-Gullstrand coordinates, which are regular at the horizon, null radial geodesics correspond to the equation

$$\frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}} =: \dot{r},$$

where the plus sign holds for the outgoing geodesics. Null geodesics can be naturally assumed to represent, in geometrical optics approximation, trajectories described by light rays. We can identify $v_g = \dot{r}$, and it is evident that group horizon and horizon coincide. Note also that

$$\frac{dv_g}{dr} \mid_{2M} = \frac{1}{4M},$$

i.e. the gradient of the group velocity at the horizon coincides with the surface gravity.

We apply the definition of group horizon to the case of a non-dispersive dielectric black hole. We show that we can find the correct horizon condition even without recurring to the notion of analogous metric. In the comoving frame,
let us consider in the nondispersive case the existence condition for the GH: 
\[ v_g = c \left( 1 - \frac{\nu}{c} n(x) \right) = 0. \] (A3)
We find the condition 
\[ n(x) = \frac{c}{\nu}, \] (A4)
which is the same found in [27]. It is also interesting to consider the following quantity:
\[ \left( \frac{\partial v_g}{\partial x} \right)_{x_b} = \frac{1}{c} \gamma^2 v^2 (\partial_x n)_{x_b} \equiv \frac{1}{c} \kappa, \] (A5)
where \( \kappa \) is the surface gravity associated with the same model as in [27].

**Appendix B: A form of Parikh-Wilczek approach from geometrical optics**

Let us consider a photon which tunnels through a black hole group horizon, by passing from inside to outside. In particular, if \( x \) is the spatial coordinate in a 2D model, we have to find out, in the geometrical optics approximation, the following quantity:
\[ \text{Im} \int_{x_{in}}^{x_{out}} dx \ k(x), \] (B1)
where \( k \) stays for the usual wave vector and \( x_{in} \) and \( x_{out} \) are coordinates of points inside and outside the group horizon. Indeed, we are interested in the probability of tunneling, which is equivalent to the probability of pair creation, and it is given by the following expression:
\[ \Gamma \sim \exp \left( -2 \text{Im} \int_{x_{in}}^{x_{out}} dx \ k(x) \right). \] (B2)
In such an approximation, the effective action is replaced by the above tunneling integral.

We adopt the same trick as in the original paper by Parikh and Wilczek [5]:
\[ \int_{x_{in}}^{x_{out}} dx \ k(x) = \int_{x_{in}}^{x_{out}} dx \int_0^k dk'. \] (B3)
(Note that we are assuming that the branch is connected to \( k = 0 \). This is not true in general, but it is easy to allow also more general cases by simple shifts). Let us follow even more strictly the tricks in [5]. \( G \) plays the role of Hamiltonian for geometrical optics, as known [44]. It is also useful, for our purposes, to write
\[ G = 0 \iff \Pi_b G_b = 0, \] (B4)
where the index \( b \) indicates the branches of the dispersion relation and, more explicitly,
\[ G_b = \omega - f_b(k, x). \] (B5)
Then, we take into account that the group velocity \( (2) \) on a given branch becomes
\[ v_g = \partial_k G_b. \] (B6)
From the Hamiltonian equations we obtain
\[ \frac{dx}{dt} = v_g = \partial_k G_b \] (B7)
and then, by taking into account (B7) and

\[ (dG_b)|_{x,t,\omega} = \partial_k G_b \, dk, \tag{B8} \]

we get, for \( d\omega' \equiv (dG_b)|_{x,t,\omega} \):

\[
\int_{x_{\text{in}}}^{x_{\text{out}}} dx \int_0^k dk' = \int_{x_{\text{in}}}^{x_{\text{out}}} dx \int_0^{\omega'} \frac{1}{v_y}.
\tag{B9}
\]

We exchange the order of integration and then we assume that \( v_y \) is analytical in a neighbourhood of the group horizon for each fixed \( \omega \), and that \( \frac{\partial v_y}{\partial x} \bigg|_{x_{\text{GH}}} \neq 0 \). As a consequence, we can consider the contribution to the imaginary part of the above integral in \( x \) near the group horizon \( x_{\text{GH}} \), by means of the following series expansion:

\[
v_y(x) = \frac{\partial v_y}{\partial x} \bigg|_{x_{\text{GH}}} (x - x_{\text{GH}}) + O((x - x_{\text{GH}})^2). \tag{B10}\]

The latter assumption, as a matter of facts, eventually include only the nondispersive case, because finiteness of \( k \) in the dispersive case is the reason why \( v_y \), as a function of \( x, \omega \), cannot be analytic and has to be nonanalytic and also integrable near the group horizon.

In the non-dispersive case, the ‘usual’ logarithmic divergence, which is at the root of thermality for standard (nondispersive) black holes, occurs. As to the imaginary part of the above effective action, we don’t need a complete knowledge of the integral; it is enough a deformation of the contour in the lower half complex plane, compatibly with the request of correct decay at infinity for particles, which, by means of the so-called fractional residue theorem, applied to a semicircle encircling the simple pole at hand, leads to the following expression:

\[
\Gamma \sim \exp \left( -\frac{2\pi}{\frac{\partial v_y}{\partial x} \bigg|_{x_{\text{GH}}} \omega} \right), \tag{B11}\]

which, in light of (A5), is the correct result.

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