Cosmological Perturbations Through a General Relativistic Bounce

Christopher Gordon and Neil Turok
DAMTP, Centre for Mathematical Sciences, Cambridge University, Cambridge, CB3 0WA, United Kingdom

The ekpyrotic and cyclic universe scenarios have revived the idea that the density perturbations apparent in today's universe could have been generated in a 'pre-singularity' epoch before the big bang. These scenarios provide explicit mechanisms whereby a scale invariant spectrum of adiabatic perturbations may be generated without the need for cosmic inflation, albeit in a phase preceding the hot big bang singularity. A key question they face is whether there exists a unique prescription for following perturbations through the bounce, an issue which is not yet definitively settled. This goal of this paper is more modest, namely to study a bouncing Universe model in which neither General Relativity nor the Weak Energy Condition is violated. We show that a perturbation which is pure growing mode before the bounce does not match to a pure decaying mode perturbation after the bounce. Analytical estimates of when the comoving curvature perturbation varies around the bounce are given. It is found that in general it is necessary to evaluate the evolution of the perturbation through the bounce in detail rather than using matching conditions.

I. INTRODUCTION

In the inflationary Universe scenario, large scale structure is generated by the stretching of subatomic scale quantum fluctuations to macroscopic length scales, during an epoch of superluminal expansion supposed to have occurred prior to the radiation dominated era. Indeed such expansion appears to be essential to the creation of correlations on super-Hubble radius scales, a feat prohibited by causality in the standard hot big bang era.

However, cosmic inflation does not resolve the problem of the initial singularity, and it seems clear that a more complete theory than inflation is needed to deal with it. But if time existed before the the initial singularity, it is a logical possibility that the large scale structure and density perturbations we see today were generated during this pre-singularity epoch. The ekpyrotic and cyclic scenarios [1,2] provide explicit realisations of this idea (pre-figured in the 'pre-big bang' scenario of Veneziano et al. [3]). In the ekpyrotic and cyclic scenarios, scale invariant large scale density perturbations are generated during a phase which is contracting (in the Einstein frame) before the bounce to expansion [1]. The task of matching these perturbations across the bounce to the expanding epoch is highly challenging, since general relativity must break down there, and the use of string theory methods will ultimately be essential. There are indications that for the particular type of singularity involved here (namely the collapse of a single extra dimension) the divergences are relatively weak [4], but the matching issue remains unsettled at the present time [5–7].

In a contracting universe, the growing mode adiabatic density perturbation corresponds to a shift in the time to the 'big crunch'. A perturbation in the space curvature, however, is a decaying mode perturbation because it becomes increasingly irrelevant as the scale factor shrinks. (These statements both hold in Einstein frame and comoving gauge). However, in an expanding universe, the situation is reversed. A time delay is now a decaying mode perturbation and the curvature perturbation is the growing mode. This has led several authors to suggest that in the ekpyrotic and cyclic scenarios, where scale invariant density perturbations are generated in the collapsing phase growing mode, that these perturbations would match on to pure decaying mode perturbations in the subsequent expanding phase. Indeed this result is obtained if one insists on matching the curvature perturbation across the bounce [5].

However, this behaviour would appear surprising from a more physical viewpoint. In the models being considered, gravity is attractive throughout. The growing mode instability is driven by gravitational attraction and it is physically implausible that it should precisely reverse at a bounce so as to dissipate after the bounce and render the final universe perfectly homogeneous.

In this paper we address this issue by considering a model in which the issue can be definitively settled within conventional general relativity. It is well known that a massive scalar field in a closed Friedman-Robertson-Walker (FRW) Universe can lead to a bounce without a singularity, and without violating the Weak Energy Condition. This model has a long history starting with Schrödinger [8], however he did not include the back-reaction of the scalar field on the background scale factor. It was also employed in semi-classical studies of quantum effects (see for example [9]), and of course is a common model in quantum cosmology and studies of the no boundary proposal (see for example [10], and more recently [11]). The classical dynamics for the spatially homogeneous case have also been investigated quite extensively, see for example [12].

A preliminary study of cosmological perturbations in such a Universe (and in other bouncing models) was performed by Hwang and Noh [13], with inconclusive results since the usual large scale approximations break down around the
bounce. We develop new approximations that do not break down around the bounce, and using numerical simulations, accurately determine the propagation of linear perturbations through the bounce. We show that growing mode perturbations developed in the collapsing phase do not match to pure decaying mode perturbations in the expanding phase, thus there is no contradiction with the physical argument given above. Of course, this does not at all prove that the same is true in the ekpyrotic/cyclic models: as the original papers made clear, an unambiguous matching condition is still required. But the present work does show that propagation of a growing mode perturbation across a bounce is possible at least in this toy model.

We would also like to briefly mention recent work on the ekpyrotic/cyclic scenarios. Durrer and Vernizzi [14] showed how matching across the bounce depends on what surface the matching is done on and the intrinsic stress energy of that surface. They showed that only for the special case of matching on constant energy density surfaces with no surface tension that the growing mode in the collapsing phase matches completely onto the decaying mode in the expanding phase. Generally, some of the growing mode in the collapsing phase will go to the growing mode in the expanding phase. We should also mention a recent study of the possibility of isocurvature perturbations in the Ekpyrotic model [15]. Peter and Pinto-Neto [16] studied the matching problem in a hydro-dynamical fluid, and in a second publication employed a scalar field with negative kinetic energy in order to obtain a regular bounce [17]. The problem of matching quantum fields across a bouncing universe of the type relevant to the ekpyrotic/cyclic scenarios has been considered by Tolley and Turok [18], and a number of studies have been made of analogous bouncing models within string theory [19].

II. MODEL

The equation for the scale factor in a FRW Universe is given by

\[ \ddot{a} = -\frac{1}{6M_{Pl}^2}(\rho + 3p)a \] (1)

where \( a \) is the scale factor, dot denotes a derivative with respect to time, \( \rho \) is the density, \( p \) is the pressure and \( M_{Pl} = 1/\sqrt{8\pi G} \) is the reduced Planck mass. As the scale factor is always positive and \( \dot{a} \) is negative for a collapsing Universe, it follows from Eq. (1) that the Strong Energy Condition \( (\rho + 3p \geq 0) \) must be violated for \( \ddot{a} > 0 \). Which is what is needed for a bounce at a finite scale factor to be possible. This is the same as the condition for inflation. It is satisfied by a scalar field that is potential dominated as

\[ \rho = \frac{1}{2}\dot{\phi}^2 + V \] (2)

and

\[ p = \frac{1}{2}\dot{\phi}^2 - V \] (3)

where \( \phi \) is the value of the scalar field and \( V \) is its potential. For simplicity we shall only consider a quadratic potential

\[ V = \frac{1}{2}m^2\phi^2 \] (4)

where \( m \) is the constant mass of the scalar field.

Using the Friedman equation

\[ H^2 = \frac{1}{3M_{Pl}^2}\rho - \frac{K}{a^2} \] (5)

it can be seen that the background curvature \( (K) \) must be positive for the Universe to bounce. This is because at the bounce \( H = 0 \) and \( \rho > 0 \). As \( \rho + 3p \geq 0 \), it follows that the Weak Energy Condition is never violated by this model.

Although the above arguments show that it is possible for a closed Universe with a scalar field to bounce, it does depend on the initial conditions. The scalar field evolution is given by the Klein-Gordon equation

\[ \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \] (6)

where \( V_{,\phi} = dV/d\phi \). In a expanding Universe \( H > 0 \) and so the \( 3H\dot{\phi} \) acts as a friction term. While in a collapsing Universe \( H < 0 \) and so the \( 3H\dot{\phi} \) acts as an anti-friction term.
So, it is possible that the system will always be kinetic energy dominated and go to $\phi = \pm \infty$ and $a = 0$ instead of bouncing. It is only for a small subset of initial conditions that such a Universe would bounce [12]. The bounce will be symmetric in the background quantities if $\dot{\phi} = 0$ when $\dot{a} = 0$. We use this case to study the evolution of perturbations through the bounce which is taken to occur at $t = 0$. Figure 1 summarises the sequence of events.

![Collapsing Expanding](image)

**FIG. 1.** A sketch of the background dynamics of the model investigated in this paper. The scalar field is represented by the ball and $\ln(a)$ is proportional to the size of the ball. The scale factor changed by approximately 15 e-folds from its minimum to maximum size in the numerical integration. First the scalar field rolls up the potential as the Universe collapses. Then when the Universe reaches a non-zero scale factor $a_0$, it starts to expand again and roll down the potential. The background dynamics are time-symmetric around the bounce.

Around the bounce we can make the approximation $\dot{\phi} \approx 0$ and $\phi = \phi_0$ where subscript 0 is taken to the value of the quantity at the bounce ($t = 0$). Using Eqs. (1), (2) and (3) this gives

$$a \approx a_0 \cosh(t/a_0).$$

(7)

This is the same as the equation for a closed de Sitter space. If we choose $K = 1$ then from Eqs. (4) and (5)

$$a_0 = M_{Pl} \sqrt{\frac{3}{V_0}} = \frac{\sqrt{6} M_{Pl}}{m |\phi_0|}$$

(8)

The Hubble parameter around the bounce is then given by

$$H = \frac{1}{a_0} \tanh(t/a_0).$$

(9)

So the time scale around the bounce is $a_0$. In order to approximate the evolution of the scalar field around the bounce we rewrite Eq. (6) as

$$\dot{\phi} a^3 = -V_{,\phi} a^3.$$  

(10)

Integrating both sides gives

$$\dot{\phi} a^3 = -\int_0^t V_{,\phi} a^3 \, dt$$

(11)

where $(\dot{\phi})_0 = 0$ was used. Using $V_{,\phi} \approx (V_{,\phi})_0$ and Eqs. (4) and (7) we get

$$\phi \approx \frac{1}{3} a_0^2 m^2 \phi_0 \left( \frac{1}{\cosh^2(t/a_0)} - \ln(\cosh(t/a_0)) - 1 \right) + \phi_0.$$  

(12)
Away from the bounce \((t \gg a_0)\), this equation can be approximated by

\[
\phi \approx \phi_0 \left( 2 \frac{M_{\text{Pl}}^2}{\phi_0^2} \left( -1 + \ln(2) - \frac{|t|}{a_0} \right) + 1 \right).
\] (13)

This is the same equation that is obtained by using the slow roll approximations to Eqs. (5) and (6) (see for example [20])

\[
3H\dot{\phi} + V,\phi \approx 0
\] (14)

and

\[
H^2 \approx \frac{1}{3M_{\text{Pl}}^2} V.
\] (15)

Which will be a good approximation when the slow roll parameters [20]

\[
\epsilon = \frac{M_{\text{Pl}}^2}{2} \left( \frac{V,\phi}{V} \right)^2
\] (16)

and

\[
|\eta| = M_{\text{Pl}}^2 \frac{1}{V} \left| \frac{d^2V}{d\phi^2} \right|
\] (17)

are small and there has been enough expansion to make the \(K/a^2\) in Eq. (5) negligible. For the quadratic potential given in Eq. (4) both conditions become

\[
\phi^2 \gtrsim 2M_{\text{Pl}}^2.
\] (18)

For \(|t| \gtrsim a_0\), a good approximation for \(H\) is obtained by substituting Eq. (13) into Eq. (15)

\[
H(|t| \gtrsim a_0) \approx \frac{\text{sign}(t)}{a_0} \left( 2 \left( \frac{M_{\text{Pl}}^2}{\phi_0^2} \right) \left( -1 + \frac{|t|}{a_0} + \ln(2) \right) + 1 \right)
\] (19)

This can then be integrated to obtain an expression for \(a(t)\) and the integration constant can be found by matching to the bounce approximation Eq. (7)

\[
a(|t| \gtrsim a_0) \approx \frac{a_0}{2} \exp \left( \frac{|t|}{a_0} \left( \ln(4) - 2 - \frac{|t|}{a_0} \right) \frac{M_{\text{Pl}}^2}{\phi_0^2} + \frac{|t|}{a_0} \right).
\] (20)

If we further assume \(|\phi_0|/M_{\text{Pl}} \gg |t|/a_0\) then Eqs. (13), (19) and (20) can be simplified to

\[
\phi(|t| \gtrsim a_0) \approx \phi_0, \quad H(|t| \gtrsim a_0) \approx \frac{\text{sign}(t)}{a_0}, \quad \frac{dH}{dt} \bigg|_{|t| \gtrsim a_0} \approx -2M_{\text{Pl}}^2 \frac{1}{\phi_0^2}, \quad a(|t| \gtrsim a_0) \approx \frac{a_0}{2} \exp \left( \frac{|t|}{a_0} \right).
\] (21)

In which case \(a_0\) is also the time scale away from the bounce.

### III. PERTURBATION EQUATIONS

For studying the evolution of the perturbations we use the Bardeen formulism [21,22]. For purely scalar perturbations in the zero-shear (conformal Newtonian) gauge, the perturbed metric is given by

\[
ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)\gamma_{ij}dx^i dx^j
\] (22)

where \(x^i\) is the comoving spatial coordinate and for a closed Universe

\[
\gamma_{ij} = \delta_{ij} \left( 1 + \frac{1}{4}Kx^k x_k \right)^{-2}.
\] (23)
where Latin indices range from 1 to 3, $\Phi$ and $\Psi$ are the metric perturbations and $\delta_{ij}$ is the Kronecker delta. For a scalar field there is no anisotropic stress and so there is the further simplification $\Psi = \Phi$ [22].

The equation for the evolution of $\Phi$ can be derived from the Einstein equations [22,23] and is

$$\ddot{\Phi} + \left( H - 2\frac{\dot{\Phi}}{\phi} \right) \Phi + \frac{1}{a^2} (-\nabla^2 - 4K) \Phi + 2 \left( \dot{H} - H \frac{\dot{\Phi}}{\phi} \right) \Phi = 0 \tag{24}$$

The usual harmonic decomposition is performed on the perturbations so that in a closed Universe the eigenvalues of $-\nabla^2$ are $(k^2 = [n^2 - 1]K)$ [24] where $n \geq 3$ is an integer. The $n = 1$ mode is homogenous and the $n = 2$ mode is a pure gauge mode [21]. We will be looking at the evolution of an individual mode with time. So the dependence of $\Phi$ will be in terms of $n$ and $t$.

An approximation for $\Phi$ around the bounce ($|t| \lesssim a_0$) can be found by substituting Eqs. (12) and (7) and $K = 1$ into Eq. (24) and working with the harmonic components of $\Phi$

$$\ddot{\Phi} + \left( \frac{\tanh(t/a_0)}{a_0} - 2\frac{\dot{\Phi}}{\phi} \right) \Phi + \frac{n^2 - 5}{a_0^2 \cosh^2(t/a_0)} \Phi + 2 \left( \frac{1}{a_0^2} - \frac{\tanh^2(t/a_0)}{a_0^2} - \frac{\tanh(t/a_0) \dot{\Phi}}{a_0 \phi} \right) \Phi = 0 \tag{25}$$

The equation can be reduced to a solvable form by a non-linear transform of the time coordinate

$$\nu = \ln \left( \tanh \left( \frac{1}{2} \frac{t}{a_0} - \frac{\pi}{4} i \right) \right) \tag{26}$$

where $i = \sqrt{-1}$. Then Eq. (25) becomes

$$\frac{\partial^2 \Phi}{\partial \nu^2} + \frac{6 \left( -1 + e^{2\nu} \right) \left( 1 + e^{4\nu} \right)}{\left( -1 + 4e^{2\nu} - e^{4\nu} \right) \left( 1 + e^{2\nu} \right)} \frac{\partial \Phi}{\partial \nu} + \left( -n^2 + 5 - \frac{4(1 + e^{2\nu})^2}{-1 + 4e^{2\nu} - e^{4\nu}} \right) \Phi = 0. \tag{27}$$

Which has the solution

$$\Phi(|t| \lesssim a_0) \approx C_1 \Phi_1 + C_2 \Phi_1^* \tag{28}$$

where $C_1$ and $C_2$ are integration constants, superscript $*$ denotes the complex conjugate and

$$\Phi_1 = e^{(n^2 - 5)\nu} \left[ -((n - 1)(n - 2)e^{6\nu} + 3(n + 1)(n - 2)e^{4\nu} + 3(n + 2)(n - 1)e^{2\nu} - (n + 2)(n + 1) \right]. \tag{29}$$

We will use this solution to check our numerical calculations in Section IV.

For $|t| \gtrsim a_0$ the Universe is effectively flat ($K = 0$) and we can use the the background approximations given in Eqs. (13), (19) and (20). As seen from Eq. (24), we can also treat the perturbations as if we were in a flat Universe provided $n^2 - 1 \gg 4$. In which case the usual transformation can be used [22,25]

$$u = \Phi/\dot{\phi} \tag{30}$$

which has the equation

$$u'' + (k^2 - \theta''/\theta)u = 0 \tag{31}$$

where prime indicates the derivative with respect to conformal time $\tau$ with $a d\tau = dt$ and

$$\theta = \frac{H}{a\dot{\phi}}. \tag{32}$$

The ‘large scale condition’ is

$$k^2 \ll \theta''/\theta. \tag{33}$$
A simple expression for the time spans that the large scale condition is satisfied can be obtained by assuming that $|\phi_0|/M_{\text{Pl}} \gg |t|/a_0$ and using Eqs. (21), (32) and (33) which gives

$$\frac{|t|}{a_0} \gtrsim \ln \left( k \frac{|\phi_0|}{M_{\text{Pl}}} \right) .$$

(34)

As is expected the larger $k$ the further from the bounce the large scale approximation becomes valid. Also, the large scale condition is always violated for $|t| > a_0$ as $k^2 \geq 3^2 - 1$ and $|\phi_0| > M_{\text{Pl}}$. When the large scale condition is not satisfied the solutions of Eq. (30) and therefore Eq. (24) are clearly oscillatory with frequency $k$.

If the large scale condition is satisfied then we can approximate the solution to Eq. (31) by a truncated series of $k^2$:

$$u = \sum_{j=0}^{\infty} C_{2j}(\tau) k^{2j} .$$

(35)

Substituting this into Eq. (31) and matching coefficients on the left and right side gives the zeroth order approximation

$$u = C_3 \theta + C_4 \theta \int \frac{1}{\theta^2} d\tau .$$

(36)

Using Eqs. (13), (19), (20), (30), (32) and (36) and redefining the constants $C_3$ and $C_4$ gives

$$\Phi(|t| \gtrsim a_0) \approx C_3 \frac{H}{a} + C_4 \left( 1 - \sqrt{\pi} \psi \text{erfcx}(\psi) \right)$$

(37)

where

$$\psi = \frac{1}{2} \frac{|\phi|}{M_{\text{Pl}}}$$

(38)

and

$$\text{erfcx}(\psi) = e^{\psi^2} \text{erfc}(\psi)$$

(39)

is the scaled complementary error function. If we further assume $|\phi_0|/M_{\text{Pl}} \gg |t|/a_0$, then substituting Eq. (21) into Eq. (37) gives [26]

$$\Phi(|t|/a_0 \gtrsim 1) \approx 2 \left( C_3 \text{sign}(t) \frac{1}{a_0^2} \exp \left( -|t|/a_0 \right) + C_4 \frac{M_{\text{Pl}}^2}{\phi_0^2} \right) .$$

(40)

Although the form of the solution (Eq. (37)) is the same before and after the bounce, it is not necessary that the integration constants ($C_3$ and $C_4$) should be the same. The slow-roll approximation for Eq. (24) is not the same before and after the bounce due to the presence of the absolute value signs in Eqs. (13) and (19). So we will have four integration constants:

$$C_3 = \begin{cases} 
C_{\text{grow}}^{-}, & t \lesssim -a_0 \\
C_{\text{decay}}^{+}, & t \gtrsim a_0 
\end{cases}$$

(41)

and

$$C_4 = \begin{cases} 
C_{\text{decay}}^{-}, & t \lesssim -a_0 \\
C_{\text{grow}}^{+}, & t \gtrsim a_0 
\end{cases}$$

(42)

The + and − superscript indicate after and before the bounce. The ‘grow’ and ‘decay’ subscripts indicate that the constant is the amplitude of the growing and decaying mode respectively.

In principle there should be two constraint equations relating the four integration constants as Eq. (24) is second order and so should only have two integration constants. That is, there should be a matching rule expressing $C_{\text{decay}}^{+}$ and $C_{\text{grow}}^{+}$ in terms of $C_{\text{decay}}^{-}$ and $C_{\text{grow}}^{-}$.
IV. NUMERICAL SOLUTIONS

Although Eqs. (28) and (37) provide good approximations around and away from the bounce. They do not overlap sufficiently to provide a solution for the whole domain of interest for all initial conditions. In principle, the stage before the global curvature becomes important, \(|t| \gtrsim a_0\), and when the large scale condition is violated, Eq. (34), could also be included in the matching. But the analytical approximations are found still not to be sufficiently accurate in the overlap regions for general initial conditions. Therefore it is also necessary to solve Eq. (24) numerically.

Before doing a numerical integration it is important to identify if there are any singularities in the coefficients of Eq. (24). A series solution for Eqs. (1) and (6) is

\[ \phi = \phi_0 \left[ 1 - 3 \frac{M_{Pl}}{\phi_0^3} \left( \frac{t}{a_0} \right)^2 + O(4) \right] \]  \hspace{1cm} (43)

and

\[ a = a_0 \left[ 1 + \frac{1}{2} \left( \frac{t}{a_0} \right)^2 + O(4) \right]. \]  \hspace{1cm} (44)

Substituting Eqs. (43) and (44) and \( \tau = t/a_0 \) into Eq. (24) gives

\[ \frac{\partial^2 \Phi}{\partial \tau^2} - \frac{4}{(2 + \tau^2) \tau} \frac{\partial \Phi}{\partial \tau} - 4 \frac{-n^2 + 5 + 2\tau^2}{(2 + \tau^2)^2} \Phi = 0. \]  \hspace{1cm} (45)

As can be seen there is a singularity in the coefficient for \( \partial \Phi/\partial \tau \) at \( \tau = 0 \). This means that it is not possible to numerically integrate Eq. (24) through the bounce. However, we can solve Eq. (45) to give an analytical solution very close to the bounce

\[ \Phi = (2 + \tau^2)^{-N} \{ C_5 F (-N + \frac{1}{2}(-1 + \sqrt{33}), -N - \frac{1}{2}(1 + \sqrt{33}), -\frac{1}{2}, -\frac{1}{2}\tau^2) \]
\[ + C_6 \tau^3 F (-N + \frac{1}{2}(5 - \sqrt{33}), -N + \frac{1}{2}(5 + \sqrt{33}), \frac{2}{3}, -\frac{1}{2}\tau^2) \} \]  \hspace{1cm} (46)

where \( F(\cdot) \) is the hyper-geometric function, \( N = \sqrt{(n^2 - 1)/2} \) and \( C_5 \) and \( C_6 \) are integration constants. A series approximation can be used to evaluate Eq. (46) around the origin

\[ \Phi = C_{\text{even}} \Phi_{\text{even}} + C_{\text{odd}} \Phi_{\text{odd}} \]  \hspace{1cm} (47)

where \( C_{\text{even}} \) and \( C_{\text{odd}} \) are integration constants and

\[ \Phi_{\text{even}} = \frac{1}{2} \left( n^2 - 5 \right) \left( \frac{t}{a_0} \right)^2 + 1 + O(4) \]  \hspace{1cm} (48)

and

\[ \Phi_{\text{odd}} = \left( \frac{t}{a_0} \right)^3 + O(5). \]  \hspace{1cm} (49)

Eqs. (47), (48) and (49) can then be used to generate initial conditions on either side of the bounce for an adaptive step-length Runge-Kutta numerical integration routine [27]. We set these initial conditions for the numerical simulation at \( t/a_0 = \pm 10^{-4} \). The relative accuracy for the Runge-Kutta routine was set to \( 10^{-14} \). The numerical results were found to be insensitive to the precise values used.

The parameters used for the simulations were \( \phi_0 = -50 M_{Pl} \) and \( m = 10^{-6} M_{Pl} \). As the system of equations Eqs. (1), (6) and (24) only depends on \( m \) through the combination \( nt \), changing \( m \) will just change the scaling of time. As can be seen from Eqs. (26), (28) and (29), around the bounce the only dependence of \( \phi_0 \) is through the dependence on \( a_0 \), see Eq. (8). So around the bounce changing \( \phi_0 \) will just change the time scale of the problem. From Eqs. (21) and (40), it can be seen that increasing \( \phi_0 \) will effect the time scale through its effect on \( a_0 \) (Eq. (8)) and the normalisation of the \( C_3 \) and \( C_4 \) modes.

The only other parameters that need to be specified are the initial values for \( \Phi \) and \( \dot{\Phi} \). The numerical solution for these can be found by choosing \( C_{\text{even}} \) and \( C_{\text{odd}} \) in Eq. (47) so that the numerical extrapolation has the required values of \( \Phi \) and \( \dot{\Phi} \) at the initial time. This technique is known as shooting [27].
Figure 2 shows a plot of $\Phi$ for the wave-number corresponding to $n = 3$. The initial conditions at $t/a_0 = -20$ were set to be a pure growing mode $\Phi = H/a$ using the shooting method. The decaying mode in the collapsing phase and the growing mode in the expanding phase was evaluated by matching Eq. (37) at the initial and final times. They are plotted for those times for which the large scale condition, Eq. (34), is satisfied. This corresponds to $|t|/a_0 \gtrsim 5$. Due to the finite accuracy of the shooting algorithm, there is still a finite amplitude of decaying mode $a t/a_0 = -20$. However, by the time the large scale condition, Eq. (34), is violated, the decaying mode is negligible relative to the growing mode.

As can be seen from Eqs. (47), (48) and (49), $\Phi$ and its first and second time derivatives are finite at $t = 0$. This shows that there are initial conditions that lead to linear perturbations around the bounce [7,16].

Figures 3 and 4 show examples of numerical solutions and the bounce approximation, Eq. (28), for small and large wave numbers. The initial conditions for Figure 3 are the same as those for Figure 2. In Figure 4, the initial conditions are set so that at $t/a_0 = -20$ there is a pure decaying mode, Eq. (40), $\Phi = M_{Pl}^2/\phi_0^2$. It is an important confirmation of the numerical integrations that they match the bounce approximations so well for $|t|/a_0 \lesssim 1$. 

![Figure 2: A plot of the absolute value of the zero shear curvature perturbation $\Phi$ for the wave number corresponding to $n = 3$. The solid lines are for positive $\Phi$ and the dotted line is for where $\Phi$ is negative. The collapsing phase decaying mode and expanding phase growing mode (Eq. (37)) are plotted as dashed lines.](image)
FIG. 3. A plot of the absolute value of the zero shear curvature perturbation ($\Phi$) and its approximation around the bounce, Eq. (28), for $n = 3$. The solid lines are for positive $\Phi$ and the dotted line is for where $\Phi$ is negative. The absolute value of the approximation is plotted as a dashed line.

FIG. 4. A plot of $\Phi$ (solid line) and its approximation using Eq. (28) (dotted line) for $n = 10$.

V. COMOVING CURVATURE

In inflation, the use of the curvature on constant density hyper-surfaces ($\zeta_{\text{BST}}$) [28] or the curvature on comoving hyper-surfaces ($\mathcal{R}$) [29] is very useful because in an expanding flat Universe with no decaying mode or entropy perturbation they are constant on scales larger than the Hubble horizon ($1/H$). We will be examining whether this can be used for the present model.

The comoving curvature perturbation for any background curvature ($K$) is [30]
\[ R = \Psi + 2M_{Pl}^2 \frac{H\Phi + \dot{\Psi}}{\rho + p} \]  

(50)

where for a scalar field \( \Psi = \Phi \). Note that the formula for the comoving curvature giving in Mukhanov et. al. [22] \( (\zeta_{\text{MFB}}) \) only applies for the flat case \( (K = 0) \). Alternatively the curvature on constant density hyper-surfaces [28] could be used. However, away from the bounce, when the decaying mode is sub-dominant, they are all approximately equal up to a change in sign and so they will all serve the same purpose below.

Using Eqs. (43), (44), (47), (48) and (49) in Eq. (50) gives

\[ R(0) = C_{\text{even}}(2n^2 - 7) \]  

(51)

which shows that \( R \) is finite at the bounce.

Using Eqs. (2), (3), (4), (13), (19) and (37) into Eq. (50) gives

\[ R \approx C_4. \]  

(52)

From Eq. (52) it appears that \( \dot{R} = 0 \) when the large scale condition, Eq. (33), holds. However, Eq. (52) was derived using the zeroth order truncation of \( k^2 \), Eqs. (35), (36) and (37). As is well known [22,25,20], \( R \) will not be constant if the series expansion, Eq. (35), is extended to order \( k^2 \). Alternatively, the time change of \( R \) can be found using the general equation [20]

\[ \dot{R} = H\frac{\delta p_{\text{comov}}}{\rho + p} \]  

(53)

where \( \delta p_{\text{comov}} \) is the pressure perturbation on comoving hyper-surfaces. For a single scalar field this becomes [31]

\[ \dot{R} = \frac{H}{a^2} \frac{k^2}{\Phi}. \]  

(54)

As \( a \) shrinks exponentially as the bounce is approached, Eq. (20), \( R \) may no longer be approximately constant. Thus, \( R \) can vary on large scales as the pressure perturbation on comoving hyper-surfaces grows exponentially as the bounce is approached.

The pressure perturbation can be decomposed as [20]

\[ \delta p = \frac{\dot{\rho}}{\rho} \delta \rho + \delta p_{\text{nad}} \]  

(55)

where \( \delta \rho \) is the density perturbation and \( \delta p_{\text{nad}} \) is the gauge invariant non-adiabatic component of the pressure perturbation. For the quadratic potential in this paper, both the adiabatic and non-adiabatic components of the pressure perturbation on comoving hyper-surfaces grow exponentially as the bounce is approached. Where as for a massless field, such as in the stage just before the bounce in the Ekpyrotic scenario [1] there is no non-adiabatic component to the pressure perturbation [31] and it is only the adiabatic part of the pressure perturbation that grows and causes \( R \) to vary [7].

Once the large scale condition, Eq. (33), is violated then \( \Phi \) and hence \( R \) will become oscillatory. In order to quantify whether \( R \) varies before this, we look at whether there is a significant relative change in \( R \) in one Hubble time which is approximately given by the time when

\[ \bigg| \frac{1}{H} \frac{\dot{R}}{R} \bigg| \geq 1. \]  

(56)

Assuming the \( C_4 \) mode is dominant in Eq. (40) and substituting Eqs. (21), (40), (52) and (54) into Eq. (56) gives the condition for significant variation for \( R \) to be

\[ 4k^2|R|\exp(-2|t|/a_0) \geq |R| \]  

(57)

which implies there will be significant variation in \( R \) due to the comoving pressure perturbation even on large scales for

\[ \frac{|t|}{a_0} \leq \ln(2k). \]  

(58)
However, comparing this with the large scale condition, Eq. (34) and remembering that we have assumed $|\phi_0| \gg M_{Pl}$, it is clear that the large scale condition is always violated before the condition in Eq. (58). So, we can conclude that if the $C_4$ mode is dominant in Eq. (40), then the large scale condition will be violated before $R$ is effected by the comoving pressure perturbation.

Assuming the $C_3$ mode is dominant in Eq. (40) and assuming $|\phi_0|/M_{Pl} \gg |t|/a_0$, substituting Eqs. (21) and (40) into Eq. (54) gives

$$\frac{1}{H} \dot{R} = -2\Phi_* k^2 \frac{\phi_0^2}{M_{Pl}^2} \exp \left( \frac{|t_*|}{a_0} - 3 \frac{|t|}{a_0} \right)$$

(59)

where $\Phi_* = \Phi(t_*)$ and $t_*$ is arbitrary except for satisfying the large scale condition Eq. (33). Substituting Eqs. (59) and (52) into Eq. (56) gives the time period for which $R$ will not vary on large scales due to the comoving pressure perturbation

$$\frac{|t|}{a_0} \geq \frac{1}{3} \left[ \ln \left( 2k^2 \frac{\phi_0^2}{M_{Pl}^2} \frac{\Phi_*}{C_4} \right) + \frac{|t_*|}{a_0} \right].$$

(60)

If $t_*$ is taken to be the time when the large scale approximation breaks down, Eq. (34), then using Eqs. (40), (52) and (60) implies the condition for the comoving pressure perturbation to cause $R$ to vary before the large scale condition, Eq. (34) is violated is

$$\frac{\Phi_3}{\Phi_4} > \frac{1}{4} \left( \frac{|\phi_0|}{M_{Pl}} \right)^2 k^2$$

(61)

where $\Phi_3$ and $\Phi_4$ are the $C_3$ and $C_4$ modes in Eq. (40). It follows that provided the $\Phi_3$ mode is sufficiently larger than the $\Phi_4$ mode, $R$ can vary significantly even while the large scale condition, Eq. (34), holds. However, for large enough $k$ the modes will always violate the large scale condition before the comoving pressure effects $R$.

Eq. (60) can be tested against the numerical results in Figures 2 and 5. Separate values of $\Phi_*$ and $C_4$ are needed for before and after the bounce. The values of $\Phi_*$ can be read off for some $|t_*|$ of Figure 2 which satisfies the large scale condition, Eq. (34) and is where the growing mode is dominating. The values of $C_4$ can be evaluated from the $R$ constant parts of Figure 5. These values can be substituted into Eq. (60) to determine the region in which the entropy perturbation causes $R$ to vary. These limits are drawn in with dashed vertical lines in Figure 5. As can be seen they accurately delimit the region in which $R$ changes. From Eq. (34), it can be seen that the large scale condition holds for $|t|/a_0 \geq 5$ and so $R$ is varying in the collapsing phase even when the large scale condition holds.

FIG. 5. A plot of the absolute value of the comoving curvature perturbation ($R$) corresponding to the zero shear curvature perturbation ($\Phi$) in Figure 2 which was for $n = 3$. The solid line is positive values of $R$ and the dotted line is negative values. The vertical dashed lines enclose the time domain in which $R$ is expected to vary according to Eq. (60).
Eq. (52) can be used to write Eq. (37) as

$$
\Phi(|t| > |t|) \approx C_3 \frac{H}{a} + R \left[ 1 - \sqrt{\pi} \text{erfc}(\psi) \right].
$$

(62)

Where $|t|$ satisfies both Eqs. (34) and (60), i.e. it delimits the time on which $R$ can vary. The even and odd modes obtained from the numerical solution in Section IV can be decomposed into growing and decaying parts as in Eqs. (37), (41) and (42)

$$
\Phi_{\text{even}} \approx \begin{cases} 
C_{\text{even, grow}} |H/a + R_{\text{even}}(-|t|) (1 - \sqrt{\pi} \text{erfc}(\psi))|, & t < |t| \\
C_{\text{even, decay}} |H/a + R_{\text{even}}(|t|) (1 - \sqrt{\pi} \text{erfc}(\psi))|, & t > |t| 
\end{cases}
$$

(63)

A similar equation holds for the odd mode

$$
\Phi_{\text{odd}} \approx \begin{cases} 
C_{\text{odd, grow}} |H/a + R_{\text{odd}}(-|t|) (1 - \sqrt{\pi} \text{erfc}(\psi))|, & t < |t| \\
C_{\text{odd, decay}} |H/a + R_{\text{odd}}(|t|) (1 - \sqrt{\pi} \text{erfc}(\psi))|, & t > |t| 
\end{cases}
$$

(64)

The growing and decaying mode also have even and odd properties:

$$
\left. \frac{H}{a} \right|_{-t} = -\left. \frac{H}{a} \right|_{-t}
$$

(65)

and

$$
(1 - \sqrt{\pi} \text{erfc}(\psi))|_{-t} = (1 - \sqrt{\pi} \text{erfc}(\psi))|_{-t}
$$

(66)

The fact that the growing and decaying mode have even and odd properties does not imply they do not both contribute to $\Phi_{\text{even}}$ and $\Phi_{\text{odd}}$. It does mean that it is possible to reduce the number of unknown coefficients used to describe the even and odd modes. From Eq. (63)

$$
C_{\text{even, grow}} |\frac{H}{a} - t| + R_{\text{even}} |1 - |t|| = C_{\text{even, decay}} |\frac{H}{a} + t| + R_{\text{even}} |1 - |t||
$$

(67)

from which it follows that

$$
C_{\text{even, grow}} |\frac{H}{a} - t| = C_{\text{even, decay}} |\frac{H}{a} + t|
$$

(68)

which when combined with Eq. (65) implies

$$
C_{\text{even, grow}} = -C_{\text{even, decay}}.
$$

(69)

Also from Eq. (67) it follows that

$$
R_{\text{even}} |1 - |t|| = R_{\text{even}} |1 - |t||
$$

(70)

which when combined with Eq. (66) implies

$$
R_{\text{even}} |1 - |t|| = R_{\text{even}} |1 - |t||.
$$

(71)

Similarly, using Eqs. (64), (65) and (66) it follows that

$$
C_{\text{odd, grow}} = C_{\text{odd, decay}}
$$

(72)

and

$$
R_{\text{odd}} |1 - |t|| = -R_{\text{odd}} |1 - |t||.
$$

(73)

An initially pure growing mode in the collapsing phase could be constructed as follows

$$
\Phi_{\text{grow}} = \frac{\Phi_{\text{odd}}}{R_{\text{odd}}(|t|)} - \frac{\Phi_{\text{even}}}{R_{\text{even}}(|t|)}
$$

(74)
provided $R_{\text{odd}}(-|t|)$ and $R_{\text{even}}(-|t|)$ are non-zero. Using Eqs. (63), (64), (71) and (73), Eq. (74) becomes

$$\Phi_{\text{grow}} = \begin{cases} 
\frac{H}{a} \left( \frac{C_{\text{odd, grow}}}{R_{\text{odd}}(-|t|)} - \frac{C_{\text{even, grow}}}{R_{\text{even}}(-|t|)} \right), & t < -|t| \\
\frac{H}{a} \left( \frac{C_{\text{odd, grow}}}{R_{\text{odd}}(-|t|)} + \frac{C_{\text{even, grow}}}{R_{\text{even}}(-|t|)} \right) - 2(1 - \sqrt{\pi} \text{erfcx}(\psi)), & t > |t| 
\end{cases}$$

(75)

Substituting Eq. (74) into Eq. (50) gives

$$R_{\text{grow}} = \frac{R_{\text{odd}}}{R_{\text{odd}}(-|t|)} - \frac{R_{\text{even}}}{R_{\text{even}}(-|t|)}$$

(76)

and substituting Eq. (75) into Eq. (50) gives

$$R_{\text{grow}} = \begin{cases} 
0, & t < -|t| \\
-2, & t > |t| 
\end{cases}$$

(77)

As we are free to multiply Eq. (74) by a $k$ dependent constant, it follows that $R_{\text{grow}}(t > |t|)$ may be some other value besides minus two. But it is only zero if $\Phi_{\text{grow}} = 0$, i.e. no perturbation at all. Therefore we have shown analytically that an ingoing pure growing mode leads to a non-zero outgoing growing mode provided both the odd and even modes have a $C_4$ component. However, we can not say what the scale dependence will be without knowing the scale dependence $\Phi_{\text{grow}}$. Figure 6 shows a plot of the various quantities in Eq. (76), where $-|t|$ is taken as the minimum time in the figure.

![Figure 6](https://via.placeholder.com/150)

FIG. 6. A plot of the comoving curvature perturbation $R$ for $n = 3$. The initially pure growing mode as constructed in Eq. (76) is the solid line. The normalised (as in Eq. (76)) even (dotted line) and odd (dashed line) are also displayed.

This numerical integration is in agreement with Eq. (77).

**VI. CONCLUSION**

In this article we have examined the way perturbations evolve across a bounce in a scalar field dominated closed Universe. The background model naturally bounces without violating General Relativity or the Weak Energy Condition. Analytical approximations were found for the background and perturbations. The perturbation approximations were found to work well around the bounce $|t| \lesssim a_0$ and away from the bounce in the slow-roll regime $|t| \gtrsim a_0$. However the overlap between the approximations was not sufficient to use the analytical approximations alone for arbitrary initial conditions. But they were useful in checking and interpreting the analytical solutions. The numerical
solutions were complicated by the ordinary differential equation for $\Phi$ having a singular coefficient at the bounce. Thus a Taylor approximation had to be used around the bounce and shooting methods used to obtain the numerical solution for particular initial conditions for a wider domain. The Taylor expansion around the bounce could also be used to show that the perturbations and their first and second time derivatives are finite at the bounce and so non-linearities in the Einstein tensor are not inevitable, but depend on the choice of initial conditions.

It was shown that all wave number modes violate the large scale condition for $|t| \gtrsim a_0$ and so become oscillatory around the bounce. We also found during the collapsing phase the growing mode could cause the comoving pressure to increase sufficiently to make $R$ vary significantly as the bounce is approached. The same is true for the decaying mode as the bounce is approached from the expanding phase side. Thus, in general we do not expect $R$ to be constant around the bounce. A formula predicting when $R$ starts to vary was given. We also showed numerically that, for this model, a pure ingoing growing mode ($R^- = 0$) did not lead to a pure outgoing decaying mode ($R^+ = 0$).

This model shows that in general the dynamics of the bounce need to be taken into account when evaluating how perturbations change across a bounce.

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