Homotopy Theory of Topological Defects in Spinor Condensates

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We investigate the topological defects in atomic spin-1 and spin-2 Bose-Einstein condensates by applying the homotopy group theory. With this rigorous approach we clarify the previously controversial identification of symmetry groups and order parameter spaces for the spin-1 case, and show that the spin-2 case provides a rare example of a physical system with non-Abelian line defects, and the possibility to have winding numbers of 1/3 and its multiples.

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The all-optical trapping of Bose-Einstein condensates (BEC) has opened up a new direction in the study of dilute atomic gases, i.e., the spinor condensates with degenerate internal degrees of freedom of the hyperfine spin \(F\). For alkali atoms with \(F = 1\), both experiments and theories have shown two possible kinds of spin correlations in the atom species, namely ferromagnetic (e.g. \(^{87}\)Rb) or antiferromagnetic (e.g. \(^{23}\)Na). With the experimental success of condensing alkali bosons with \(F > 1\) such as \(^{85}\)Rb and \(^{133}\)Cs, and the unusual stability of the \(F = 2\) state (against spin-exchange) in \(^{87}\)Rb, one expects that defects with much richer structure can be created in the future. A remarkable feature here is that not only the gauge symmetry U(1) but also the spin symmetry SO(3) are involved, a situation similar to superfluid \(^{3}\)He where three different continuous symmetries (orbital, spin and gauge) are broken either independently or in a connected fashion.

Topological defects and excitations in the spinor BECs have been studied theoretically by several groups \cite{1,2,3,4,5,6,7}. Most of the work on this subject is based on the original identifications of the order parameter spaces by Ho \cite{2}. After the original studies it was also claimed by Zhou that a discrete symmetry of \(Z_2\) type was missed in the case of antiferromagnetic spin-1 condensate \cite{10} and therefore the topological defects would manifest totally different structures. In this Letter we present a rigorous topological study that both solves this spin-1 controversy, and reveals interesting aspects of spin-2 systems. The phases of spin-2 spinor condensates are characterized by a pair of order parameters \((\mathbf{F})\) and \(|\Theta|\) describing the ferromagnetic order and the formation of singlet pairs, respectively \cite{11,12,13}. It turns out that for the so called cyclic phase the fundamental group that determines the nature of possible stable topological defects is non-Abelian. The only known physical example of such a system so far has been the biaxial nematic liquid crystal.

Outline of the homotopy theory of defects: We sketch out the procedure which has been widely used in the study of topological defects in ordered media such as liquid crystals, superfluid \(^{3}\)He and heavy-fermion superconductors \cite{16}. The central feature of the classification scheme of the defects emerges from examining the mappings of closed curves in physical space into the order-parameter space (OPS). The order parameter of a spinor BEC, \(\Psi(r) = \sqrt{n(r)}\zeta(r)\), where \(n(r)\) is the density and \(\zeta(r)\) is a normalized spinor \(\zeta^\dagger(r)\cdot\zeta(r) = 1\), has associated with it a group of transformations \(G\), i.e., all spinors related to each other by gauge transformation \(e^{i\theta}\) and spin rotations \(e^{-iF_2}\zeta e^{-iF_0}\zeta e^{-iF_2}\gamma\) are energetically degenerate in zero external magnetic field, where \((\alpha, \beta, \gamma)\) are the Euler angles. The set of all transformations in \(G\) that leave the reference order parameter (chosen arbitrarily but thereafter fixed) unchanged is known as the isotropy group \(H\). The OPS can then be taken to be the space of cosets of \(H\) in \(G\): \(M = G/H\). In terms of broken symmetry, the fact that the ordering breaks the underlying symmetry is expressed in the fact that \(H\) is only a subgroup of \(G\). Let \(G\) be a connected, simply connected continuous group and \(H_0\) be the set of points in \(H\) that can be connected to the identity by a continuous path lying entirely in \(H\). The fundamental theorems assert that

\[\pi_1(M) = H/H_0, \quad \pi_2(M) = \pi_1(H_0).\]

where the fundamental (second) homotopy group \(\pi_1(\pi_2)\) characterizes the singular line (point) defects in three-dimensional space.

For the spinor condensate it seems natural to identify the underlying symmetry group as \(U(1) \times SO(3)\), the groups in the direct product representing the gauge and spin degrees of freedom respectively. This group is not simply connected, i.e., \(\pi_1(U(1) \times SO(3)) \neq 0\). To apply the theorems, however, it is essential that one chooses the group \(G\) to be simply connected (i.e., \(\pi_1(G) = 0\)). This can always be done because any continuous group can be imbedded in its universal covering group. We proceed by specifying the symmetry group as \(R \times SU(2)\), with the group of real numbers \(R\) representing any translation \(\theta \in (-\infty, +\infty)\) in the phase of the condensate. For \(F = 1\), we use the 3D representation of the group
SU(2) in order to obtain the isotropy group, e.g., a rotation \(u(z,\alpha)\) around axis \(z\) by angle \(\alpha\) takes the form of a diagonal matrix \(\text{Diag}(e^{-i\alpha}, 1, e^{i\alpha})\). The element \(-u(z,\alpha)\) is represented by the same matrix, i.e., \(u(z,\alpha + 2\pi) = -u(z,\alpha)\) while \(u(z,\alpha + 4\pi) = u(z,\alpha)\).

**Calculation of the homotopy groups:** There are two possible ground states in \(F = 1\) case. For the ferromagnetic state, the isotropy group \(H\) is constructed by the set of transformations which leave the reference order parameter \((1,0,0)^T\) invariant. From the ground state spinor 
\[
\zeta = e^{i(\theta - \gamma)} \left( e^{-i\alpha} \cos^2 \frac{\beta}{2}, \sqrt{2} \cos \frac{\beta}{2} \sin \frac{\beta}{2}, e^{i\alpha} \sin^2 \frac{\beta}{2} \right)^T
\]
we know immediately that the angles should satisfy
\[
\beta = 0, \theta - \alpha - \gamma = 2n\pi
\]
with \(n\) an integer. The elements in group \(H\) are the combination of a translational part and a rotational part \(H = \{(\theta, u(z,\theta)), (\theta, u(z,\theta + 2\pi))\} = \{(\theta, \pm u(z,\theta))\}\). Evidently this group includes two disconnected components—the connected component of the identity \(H_0 = \{(\theta, u(z,\theta))\}\) is isomorphic to \(R\). The group \(H/H_0\) is isomorphic to the integers modulo 2, i.e., \(Z_2\). The second homotopy group \(\pi_2\) is trivial and we arrive at the same result as that in Ref. 3.

\[
\pi_1(M) = Z_2, \quad \pi_2(M) = 0, \text{ (spin-1 FM state).} \quad (3)
\]

A ferromagnetic spin-1 condensate may have therefore only singular vortices with winding number one while the point-like defects are topologically unstable.

The polar state emerges if the atoms in the condensate interact anti-ferromagnetically. In the ground state 
\[
\zeta = e^{i\theta} / \sqrt{2} \left( -e^{-i\alpha} \sin \beta, \sqrt{2} \cos \beta, e^{i\alpha} \sin \beta \right)^T
\]
the reference parameter \((0,1,0)^T\) is left invariant for just those elements with
\[
\beta = 0, \theta = 2n\pi \quad \text{or} \quad \beta = \pi, \theta = (2n+1)\pi. \quad (4)
\]

Thus the isotropy group \(H\) must now be expanded to include not only transformations in which both the rotation and the translation leave the spinor unchanged, but also those in which the rotation takes the reference spinor \((0,1,0)^T\) to \((-1,0,0)^T\) and the translation takes it back, i.e., a \(\pi\) rotation about arbitrary axis perpendicular to \(\hat{z}\) combined with a \(\pi\) translation in \(\theta\) (or any odd multiples of \(\pi\)). The full isotropy group is the union of these two sets, \(H = \{(2n\pi, u(z,\alpha)), ((2n+1)\pi, gu(z,\alpha))\} \) where \(g = u(y,\pi)\). There are infinitely many discrete components in \(H\), while the connected component of the identity \(H_0 = \{(0, u(z,\alpha))\}\) is isomorphic to \(U(1)\). The elements with an even translation parity are of the form \((2n\pi, I)H_0\), and those with an odd parity are of the form \(((2n+1)\pi, g)H_0\). The group \(H/H_0\) is therefore isomorphic to the group of integers \(Z\) through the isomorphism \(((2n+1)\pi, g)H_0 \mapsto 2n + j\) for \(j = 0,1\). We recover the conclusion that line and point defects in spin-1 polar state can be classified by integer winding numbers,
\[
\pi_1(M) = Z, \quad \pi_2(M) = Z, \text{ (spin-1 Polar state).} \quad (5)
\]

Thus the \(Z_2\) term does not appear in the homotopy group. As we shall see the identification of the OPS in Ref. 10 is also incorrect.

**Spin-2 Bose condensate:** We next apply the same approach to the BEC of spin-2 bosons. There are three possible phases for a spin-2 spinor Bose condensate, i.e., one more compared to the spin-1 case. This extra phase comes from the additional interaction parameter describing the singlet formation, and is referred to as the cyclic phase \([12, 13, 14]\). The defects which may be created in spin-2 condensate exhibit even more elaborate structures due to quantum correlations among bosons. For \(F = 2\) we have to use the 5D representation of \(SU(2)\), e.g., the rotation \(u(z,\alpha)\) is represented by matrix \(\text{Diag}(e^{-2i\alpha}, e^{-i\alpha}, 1, e^{i\alpha}, e^{2i\alpha})\). The calculations of the degenerate family of the ground state spinors and the corresponding homotopy groups are straightforward and details will be presented elsewhere [15]. Here we only pick up some interesting features in our results, focusing on the symmetry properties of the defects in comparison with those in other ordered media.

Equating the general expression for the ground state spinor of the ferromagnetic state \(F\)
\[
\zeta = e^{-i(\theta - 2\gamma)} \left( e^{-2i\alpha} \cos^4 \frac{\beta}{2}, \frac{e^{-i\alpha} \sin \beta \cos^2 \frac{\beta}{2}}{\sqrt{2}}, \frac{e^{i\alpha} \sin \beta \sin^2 \frac{\beta}{2}}{\sqrt{2}}, \frac{e^{2i\alpha} \sin^4 \frac{\beta}{2}}{\sqrt{2}} \right)
\]
with the reference spinor \((1,0,0,0,0)^T\) leads to the requirement for the isotropy group \(H\)
\[
\beta = 0, \theta - 2\alpha - 2\gamma = 2n\pi. \quad (7)
\]

We see that taking \(n = 0,1,2,3\) is enough for all possible transformations, with the translational part being arbitrary and the rotational part containing the rotations around axis \(z\) by \(\theta/2, \theta/2 + \pi, \theta/2 + 2\pi, \theta/2 + 3\pi\) respectively. Hence the group \(H\) is composed of 4 pieces \(H = \{(\theta, u(z,\theta/2))\}\). Here it is important to show that the 4 components are not connected: there does not exist a continuous path in \(H\) which connects one component to another, though the rotational parts themselves are connected. The connected component of the identity \(H_0 = \{(\theta, u(z,\theta/2))\}\) is again isomorphic to \(R\). If we define an element \(g\) of the group \(R \times SU(2)\) by \((0, u(z,\pi))\), we see that the quotient group \(H/H_0\) has the same structure as the cyclic group of order 4, i.e., \(\{e, g, g^2, g^3\}\) and we conclude that
\[
\pi_1(M) = Z_4, \quad \pi_2(M) = 0, \text{ (spin-2 F state).} \quad (8)
\]

It is interesting to check how the group \(Z_4\) characterizes vortices for state \(F\). In spin-1 case there is only
one topologically stable line defect, that is, a vortex with
winding number one. Equation (8) shows that there
are 3 stable vortices for spin-2 condensates. We can
set \( \theta - 2 \gamma = 2m\varphi, -\alpha = m\varphi, \beta = \pi t \)
in the ground state for \( F \) state, Eq. (6), which leads to a family of
spinor states parametrized by a parameter \( t \) between 0
and 1. Here \( m > 0 \) is an integer, \( \varphi \) is the azimuthal
angle. When \( t \) evolves from 0 to 1, the \( 4m\varphi \) vortex state
\( \zeta(t = 0) = (e^{i4m\varphi}, 0, 0, 0, 0)^T \) evolves continuously
to the vortex free state \( \zeta(t = 1) = (0, 0, 0, 0, 1)^T \).
This shows that vortices with winding number \( 4m \) are
topologically unstable. Similarly, by multiplying factors
\( e^{ik\varphi} (k = 1, 2, 3) \) one obtains the following correspon-
dences
\[
e^{i(4m+k)\varphi} (1, 0, 0, 0, 0)^T \rightarrow e^{ik\varphi} (0, 0, 0, 0, 1)^T
\] (9)
i.e., the vortices with winding numbers \( 4m+k \) may evolve
into vortices with winding numbers \( k \), respectively. There
are thus 3 classes of topologically stable line defects.
Together with the uniform state, they form the fundamental
group \( Z_4 \). Straightforwardly for condensates with spin \( F \),
the fundamental group \( \pi_1(M) = Z_{2F} \) characterizes
\( (2F - 1) \) classes of stable line defects.

Non-Abelian homotopy groups: Media with non-
Abelian fundamental groups are especially interesting
from the topological point of view. The only illustrative
example in ordered media so far have been biaxial
nematic liquid crystals [17]. If \( G \) is taken to be \( SO(3) \)
then the isotropy group \( H \) is the four-element group, \( D_2 \),
the symmetry group of a rectangular box. If, however,
we take \( G \) to be \( SU(2) \), then \( H \) is expanded to the non-
Abelian quaternion group \( Q \) (known as the lift or double
group) with eight elements, the multiplication table of
which has been verified experimentally [18].

We have found that the cyclic state \( C \) provides an-
other physically realistic example in which the funda-
mental group is non-commutative. The reference spinor
\( \frac{1}{2} (e^{i\phi}, 0, \sqrt{2}, 0, -e^{-i\phi})^T \) is left invariant by the elements in
the isotropy group
\[
H = \{ \pm I, \pm a, \pm b, \pm c, \\
\pm d, \pm e, \pm f, \pm g \}.
\] (10)
The spin rotations \( a = u(z, \pi), b = u(y, \pi)u(z, \phi + \pi/2) \)
and \( c = ba \) satisfy \( a^2 = b^2 = c^2 = -I \), while
\( d = u(z, \pi/4 + \phi/2)u(y, \pi/2)u(z, \pi/4 - \phi/2), e = -da, \)
\( f = -ad \) and \( g = -ada \) satisfy \( d^3 = e^3 = f^3 = g^3 = -I \).
Each element in the first, second, third row is associated
with an additional phase change \( 2n\pi, 2\pi/3 + 2n\pi, 4\pi/3 + 
2n\pi \) respectively. It is a discrete group, and \( H_0 \) consists
of the identity \( (0, I) \) only. The fundamental theorems
identify that
\[
\pi_1(M) = H, \quad \pi_2(M) = 0, \text{ (spin-2 C state)}. \] (11)
The elements in the fundamental group are non-
commutative, for example \( ab = -c \neq ba \). The criterion
for the topological equivalence of defects applies in the
most general case in terms of conjugacy classes of the
fundamental group. It is thus necessary to classify the
group into the following conjugacy classes: \( \{ I \}, \{ -I \}, \\
\{ \pm a, \pm b, \pm c \}, \{ \pm d, \pm e, \pm f, \pm g \} \) with
the subscripts standing for the winding numbers of the
defects. Physically this indicates the feasibility of creating
not only vortices with any integer winding number but
also fractional quantum vortices. The homology theory
assumes the conjugacy classes further into 3 sets for
each \( n \), in which the defects are labeled by the winding
numbers \( n, n+1/3, n+2/3 \) respectively [19].

Interesting features of this non-Abelian fundamental
group include the topological instability of the defects
and their interaction, i.e., entanglement when two of
them are brought to cross with each other [18]. Two de-
fects in the same conjugacy classes can be continuously
converted into one another by local surgery. In addition,
the defects in the same homology set can be transformed
into each other via a catalysis process consisting of
splitting a line singularity into two and recombining them
beyond a third one [18].

Like the quaternion group \( Q \) for biaxial nematics, the fundamental group [19] is
the lift of a point group in \( R \times SU(2) \). To find the remaining discrete symmetry
group for the cyclic state, and, in addition, to clarify the
controversial identification of the OPS for spin-1 case,
in the remaining of this paper we turn to describe the
system in terms of rotations in \( SO(3) \), e.g., two elements
\( \pm u(z, \alpha) \) in \( SU(2) \) are mapped into one \( R(z, \alpha) \) in \( SO(3) \) with
\( R(z, \alpha + 2\pi) = R(z, \alpha) \).

Order Parameter Spaces: The OPS for \( F = 1 \) polar
state was identified as \( U(1) \times S^2 \) in Refs. [3, 4]. An extra
\( Z_2 \) symmetry was claimed in Ref. [11] so the author
concluded the OPS is \( U(1) \times S^2 / Z_2 \). Here we show that
previous studies are incorrect. Taking the group \( G \) as
\( U(1) \times SO(3)_S \) where the subscripts stand for the gauge
and spin symmetries respectively, we see that the isotropy
group \( H \) consists of two separate parts, \( \{ e^{i\theta}, R(y, \pi)R(z, \alpha) \} \)
and \( \{ e^{i\theta}, R(y, \pi)R(z, \alpha) \} \). The rotations in the first
part constitute the group \( SO(2) \), while the elements in
the second part are just those in the group \( O(2) \) but
not in \( SO(2) \) with determinants \(-1 \). The combination
of these two parts gives the full isotropy group as \( O(2) \)
where both gauge and spin symmetries are involved. The
OPS is the quotient \( G/H = (U(1)_G \times SO(3)_S) / O(2)_{G+S} \) and
here it is not possible to apply the fundamental theo-
rem for \( G \) is not any more simply connected.

For the ferromagnetic state the group \( H \) may be
obtained if one notices that the 2\( \pi \) difference in the
rotational angle does not give another component as it did
in the case of \( SU(2) \). We have \( H = \{ e^{i\theta}, R(z, \theta) \} \)
which is isomorphic to \( U(1)_{G+S} \). This means that
there is a remaining symmetry U(1) in the symmetry broken system. The OPS is thus factorized as 
\((U(1)_G \times SO(3)_S)/U(1)_{G+S} = SO(3)_{S+C_G}\).

The discrete symmetry group of defects in the spin-2 cyclic state \(C\) can be shown to be isomorphic to the tetrahedral group \(T\). We continue to represent \(G\) as \(U(1)_G \times SO(3)_S\). The isotropy group \(U(1)\) is shrunk to a group of 12 elements if one understands the rotation in the sense of \(SO(3)\) (i.e., \(a = R(z, \pi)\)),

\[
H = \{I, a, b, c, \varepsilon d, \varepsilon e, \varepsilon f, \varepsilon g, \varepsilon^2 d^2, \varepsilon^2 e^2, \varepsilon^2 f^2, \varepsilon^2 g^2\},
\]

where \(\varepsilon = \exp(2\pi i/3)\) comes from the gauge transformation and \(ed\), for instance, is an abbreviation for the element \((\varepsilon, d)\). Three 2-fold rotational axes are \(x\) and \(y\) with \(\varepsilon d, \varepsilon e, \varepsilon f, \varepsilon g\) are four 3-fold axes. The symmetries remaining in the symmetry broken states for biaxial nematics and spin-2 cyclic state are shown in Figure 1. The OPS for state \(C\) can be identified as \((U(1)_G \times SO(3)_S)/T_{G+S}\).

In summary, we have determined the nature of the topological defects in spin-1 and spin-2 condensates. The order parameter spaces are identified as the spaces of the coset of the isotropy group \(H\) in the transformation group \(G\). Topologically stable vortices with winding numbers larger than unity may be created in the ferromagnetic state for condensates with \(F > 1\), up to the value \((2F - 1)\). The line defects in the spin-2 cyclic state \(C\) exhibit non-commutative features, resulting e.g. in line defects with winding numbers of \(1/3\) and its multiples.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{symmetries.png}
\caption{Symmetries of the defects in biaxial nematics (\(D_2\)) and cyclic state \(C\) in spin-2 condensate(T). The dot in the center of the rectangle stands for axis \(z\). The dashed lines represent 2-fold axes, except that with a triangle for 3-fold axis.}
\end{figure}

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