Jordan classes and Lusztig strata in disconnected reductive groups

Martina Costa Cesari
Dipartimento di Matematica “Tullio Levi-Civita”
Torre Archimede - via Trieste 63 - 35121 Padova - Italy
email: martina.costacesari@math.unipd.it

Abstract

Let $G$ be a non-connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ and let $D$ be a connected component of $G$. We investigate Jordan classes of $D$ and we obtain a description of the regular part of the closure of a Jordan class in terms of induction of $G^\circ$-orbits. We use this result to show that Lusztig strata in a non-connected reductive algebraic group are locally closed.

Introduction

Reductive non-connected groups appear frequently in the study of algebraic groups, for example as centralizers of semisimple elements in non-simply connected semisimple groups. Let $G$ be a non-connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic and let $D$ be a connected component of $G$. In [11] G. Lusztig defined a partition of $D$ in finitely many strata, generalizing the partition defined in [10] for $G^\circ$. There is an action of $G/G^\circ$ on the Weyl group $W$: we will denote by $W^D$ the fixed point set for the action of $D$ on $W$. The subgroup $W^D$ is a Weyl...
group. Using a variant of the Springer’s representation, it is possible to define a map $E$ from $D$ to the set of irreducible representations of $W^D$. Lusztig’s strata are the fibers of $E$.

It was proved in [1] that the strata in $G^o$ are locally closed. G. Lusztig suggested in [11] that also the strata of $D$ are locally closed. In this paper we prove this assertion. From the definition of strata it is not immediate that they have geometrical properties, so thanks to this result, we can treat them as geometrical objects. For this purpose, we study the partition of $D$ into Jordan classes, i.e., the equivalence classes defined in [7]. Strata of $D$ are union of finitely many Jordan classes of elements whose $G^o$-orbits have same dimension [11]. We show that a stratum is a union of the regular part of the closure of Jordan classes. For this reason we need to study Jordan classes and their closures.

Jordan classes for $G^o$ were defined in [6]. Our aim is to generalize some properties of Jordan classes of $G^o$ to Jordan classes of $G$. We describe a procedure of induction of an orbit from a connected component of the normalizer of a Levi subgroup of $G^o$ to a $G^o$-orbit in $D$. This allows us to investigate the closure and the regular closure of a Jordan class. In particular, similarly to the connected case ([2]), the regular closure of a Jordan class is a union of induced $G^o$-orbits.

The paper is structured as follows. In the first section we give the definition of Jordan classes and recall relevant properties. Furthermore we describe the induction of $G^o$-orbits in $D$, and investigate the regular closure of a Jordan class. In the second section, using the description of regular closure of Jordan classes, we prove that a stratum $X \in D$ is locally closed.

The last section is devoted to examples: we describe the Jordan classes of $G = SL(n) \rtimes \langle \tau \rangle$, where $\tau$ is the exterior automorphism reversing the Dynkin diagram of type $A_{n-1}$ and $\text{char } \mathbb{K} = 2$. 

2
**Notation**

Let $G$ be a reductive algebraic group, not necessarily connected, and let $G^o$ be its identity connected component. If $x$ is an element of $G$, and $H$ a closed subgroup of $G$, then

- $H^o$ is the identity component of $H$;
- $N_G H$ is the normalizer of $H$ in $G$;
- $Z(H)$ is the center of $H$;
- $x_s, x_u$ are, respectively, the semisimple and the unipotent part of $x$;
- if $G$ acts on a variety $X$ and $x \in X$, we denote by $\text{Stab}(x)$ the stabilizer of $x$ in $G$, i.e. $\text{Stab}(x) = \{h \in G \mid h \cdot x = x\}$; when $X = G$ and the action is conjugation, we will denote by $G^x$ or with $C_G(x)$ the centralizer of $x$ in $G$. If $H$ is a subgroup of $G$, we will denote by $H^x$ the centralizer of $x$ in $H$;
- If $X$ is a $G$-variety, then $X_{(d)} = \{x \in X \mid \dim(G \cdot x) = d\}$. Let $Y \subseteq X$ and let $m$ be the maximum integer such that $Y \cap X_{(m)} \neq \emptyset$. We will denote by $Y^{\text{reg}}$ the set of regular elements of $Y$, i.e., the elements $y \in Y$ such that $\dim(G \cdot y) = m$;
- $D$ is a connected component of $G$.

**1 Jordan classes**

In this section, we recall the definition of Jordan classes and their properties and study the closure and the regular closure of a Jordan class. In order to do that, we will describe the induction of $G^o$-orbits in $D$. 

3
1.1 Preliminars on Jordan classes and isolated elements

In this section, we give the definition of a Jordan class and an isolated Jordan class. Furthermore, we recall the analogue of parabolic subgroups in the disconnected case, namely the normalizers in $G$ of a parabolic subgroup of $G^\circ$, and their properties.

Let $a$ be an element of $G$ with Jordan decomposition $a = a_s a_u$. We set

$$T(a) = (Z((C_G(a_s))^\circ) \cap C_G(a_u))^\circ = (Z((C_{G^\circ}(a_s))^\circ) \cap C_{G^\circ}(a_u))^\circ$$

We consider the equivalence relation on $G$:

$$a \sim h \text{ if } \exists x \in G^\circ \text{ such that } T(xhx^{-1}) = T(a) \text{ and } xhx^{-1} a^{-1} \in T(a).$$

A Jordan class is an equivalence class for this relation, and we denote the Jordan class containing $a$ by $J(a)$.

By [17, Corollary 9.4], $C_{G^\circ}(a_s)^\circ$ is a reductive group, so $Z(C_{G^\circ}(a_s))^\circ$ is a torus. Hence $T(a)$ is a closed connected subgroup of a torus, so $T(a)$ is a torus.

Remark 1.1. By construction elements in a Jordan class, up to conjugation, have the same unipotent part. In fact, if $a \sim h$, then, up to conjugation, $h \in T(a)a$ with $T(h) = T(a)$ so $h = za$, where $z \in T(h)$ is a semisimple element, because $T(h)$ is a torus, and it commutes with $h_s, h_u, a_s$ and $a_u$, so $h_s = za_s$ and $h_u = a_u$.

Remark 1.2. Let $h \in T(a)a$. By Remark 1.1, $h_s = za_s \in T(a)a_s$, with $z \in T(a) \subset Z(C_G(a_s)^\circ)$. Let $x \in C_G(a_s)^\circ$. Then $z$ commutes with $x$, hence $h_s = za_s = zxa_s x^{-1} = xza_s x^{-1} = xh_s x^{-1}$. Thus $C_G(a_s)^\circ \subset C_G(h_s)^\circ$. So $\dim(C_G(a_s)^\circ) \leq \dim(C_G(h_s)^\circ)$ for all $h \in T(a)a_s$. Therefore

$$(T(a)a_s)^{\text{reg}} = \{ h_s \in T(a)a_s \mid C_G(h_s)^\circ = C_G(a_s)^\circ \}.$$
Remark 1.3. Let $J$ be a Jordan class, and $a = a_s a_u$, $h = h_s h_u \in J$. Then, up to conjugation, $C_G(h_s)^{\circ} = C_G(a_s)^{\circ}$. Indeed, by definition of Jordan classes, up to conjugation, $h \in T(a) a$ and $a \in T(h) h$, so by Remark 1.2 $C_G(a_s)^{\circ} = C_G(h_s)^{\circ}$.

Also, if $a \sim h$ then $C_G(a)^{\circ}$ and $C_G(h)^{\circ}$ are $G^\circ$-conjugated (see [7, Lemma 3.4]). As a consequence, $J \subset G(d)$ for some $d$.

By Remark 1.1 and Remark 1.3 the semisimple parts of the elements in the Jordan class $J(a)$ are contained in $G^\circ \cdot (T(a)a_s)^{\text{reg}}$.

We denote

$$
(T(a)a)^\bullet = \{ h \in T(a) a \mid T(h) = T(a) \}.
$$

Lemma 1.4. With notation as above there holds

$$(T(a)a)^\bullet = (T(a)a_s)^{\text{reg}} a_u$$

Proof. Let $h \in T(a) a$ such that $T(h) = T(a)$. Thus $h = z a_s a_u$ with $z \in T(a)$. By definition of $T(a)$, $z$ is semisimple and commutes with $a_s$ and $a_u$, so $h_s = za_s$ and $h_u = a_u$. By Remark 1.3 $C_G(h_s)^{\circ} = C_G(a_s)^{\circ}$, so, by Remark 1.2 $x \in (T(a)a_s)^{\text{reg}} a_u$.

Conversely, let $y \in (T(a)a_s)^{\text{reg}} a_u$. Then $y_s = xa_s$ and $y_u = h_u$ with $C_G(xa_s)^{\circ} = C_G(a_s)^{\circ}$. So

$$Z((C_G(xa_s)^{\circ}) = Z((C_G(a_s)^{\circ}),$$

thus

$$T(y) = (Z((C_G(xa_s)^{\circ}) \cap C_G(a_u))^{\circ} = (Z((C_G(a_s)^{\circ}) \cap C_G(a_u)))^{\circ} = T(a).$$

and $y \in (T(a)a)^\bullet$. \hfill \qed

By Lemma 1.4 we have

$$J(a) = G^{\circ} \cdot ((T(a)a_s)^{\text{reg}} a_u).$$
We set

\[ L(a) = C_G(T(a)). \]

Since \( L(a) \) is a centralizer of the torus \( T(a) \), it is a Levi subgroup of \( G^\circ \). Hence, there exists a parabolic subgroup \( P \) of \( G^\circ \) that has \( L(a) \) as a Levi subgroup. Furthermore one may choose \( P \) so that \( a \in N_GP \) ([7, 2.1 (a)]).

By [7, 2.2] the following conditions for \( a \in G \) are equivalent:

(i) \( L(a) = G^\circ \);

(ii) \( T(a) \subset Z(G^\circ) \);

(iii) \( T(a) = Z(G^\circ) \cap C_G(a) \);

(iv) there is no proper parabolic subgroup \( P \) of \( G^\circ \) with a Levi subgroup \( L \)
     such that \( a \in N_GP \cap N_GL \) and \( C_G(a)^\circ \subset L \)

(v) there is no proper parabolic subgroup \( P \) of \( G^\circ \) such that \( a \in N_GP \) and
     \( C_G(a)^\circ \subset P \).

An element \( a \in G \) satisfying any of the above conditions is called isolated. An isolated Jordan class is a Jordan class in which every element or equivalently some element is isolated ([7, 3.3]). By [7, 2.2 (b)], the element \( a \in G \) is isolated in \( N_G(L(a)) \), so, by (i), \( L(a) = (N_G(L(a)))^\circ \).

**Remark 1.5.** It can be useful to notice that if \( a = a_u \), then \( T(a) = (Z(G^\circ) \cap C_G(a))^\circ \). So \( J(a) \) is isolated and it is a \( G^\circ \)-orbit translated by a subgroup of \( Z(G^\circ) \), in particular \( J(a) = (Z(G^\circ) \cap C_G(a))^\circ(G^\circ \cdot a) \).

If \( G^\circ \) is a simple group, then \( T(a) \) is trivial. So the Jordan class of \( a \) is the unipotent orbit \( G^\circ \cdot a \).

Now we recall some facts from [7] that will be useful in the sequel.

**Remark 1.6.** By construction of \( T(a) \) and \( L(a) \), the following properties hold:
(a) $C_G(a_s) \circ \subset L(a),$
(b) $T(a) = T_{N_G(L(a))}(a)$ ([7] 2.1(d)),
(c) $T_{N_G(L(a))}(a) = (Z(L(a)) \circ \cap C_{N_G(L(a))}(a)) \circ$ ([7] 3.9).
(d) $T(a) = (Z(L(a)) \circ \cap C_{N_G(L(a))}(a)) \circ$, by (b),(c).

Following [7, 1.22], we prove the next proposition.

**Proposition 1.7.** Let $H$ be an algebraic group, let $D'$ be a connected component of $H$, $h \in D'$, and $x \in \overline{H^o \cdot h}$. Then $x_s \in \overline{H^o \cdot h_s}$.

**Proof.** Let $C = \{ y \in D' \mid y_s \in H^o \cdot h_s \}$ and let $(H^o \cdot h)_s = \{ f_s \mid f \in H^o \cdot h \}$. We observe that $(H^o \cdot h)_s$ is the orbit $H^o \cdot h_s$, so, by [7] 1.4 (e), is closed. Let, for some $n$, $H \subset GL(n)$ be an embedding of algebraic groups. Let $Y$ be the semisimple class of $GL(n)$ containing $H^o \cdot h_s$ and let $Y' = \{ a \in GL(n) \mid a_s \in Y \}$. Then $Y'$ is the set in $GL(n)$ of the matrices that have characteristic polynomial equal to that of a fixed matrix in $Y$, so $Y'$ is closed. We consider the morphism

$$\rho : Y' \longrightarrow Y$$

$$a \mapsto a_s.$$ 

Then $Y' \cap H = \rho^{-1}(H)$ is also closed in $H$ and the restriction of $\rho$ from $Y' \cap H$ to $Y \cap H$ is a morphism of varieties. Hence $C = \rho^{-1}(H^o \cdot h_s) \cap D'$ is closed in $D'$. So $H^o \cdot h \subset C$ implies $H^o \cdot h \subset C$.

Let $Q = MU_Q$ be the Levi decomposition of a parabolic subgroup $Q \subset G^o$. Note that $U_Q$ is normalized by $N_GQ$ because $U_Q$ is characteristic in $Q$. Furthermore $U_Q$ acts simply transitively on Levi subgroups of $Q$, hence $U_Q \cap (N_GQ \cap N_G M) = 1$ and the group $(N_GQ \cap N_G M)U_Q$ is isomorphic to the semidirect product of $N_GQ \cap N_G M$ and $U_Q$. By standard arguments, for all $x \in N_GQ \cap N_G M$ the coset $xU_Q$ is $U_Q$-stable.

Following the proof of [7] Proposition 3.15, we show the next proposition.
Proposition 1.8. Let $Q$ be a parabolic subgroup of $G^o$ with Levi decomposition $Q = MU_Q$. Let $h \in N_GQ \cap N_GM$. Then the semisimple parts of the elements in $hU_Q$ are all $U_Q$-conjugate.

Proof. Let $a = hu \in hU_Q$ and let $a = a_s a_u$ be its Jordan decomposition. Then, by the properties of Jordan decomposition, $a_s \in N_GQ$ and by \[3, Proposition 1.6\] it normalizes a Levi subgroup $M'$ of $Q$. Now, $hU_Q$ is $U_Q$-stable, so, since $M'$ is $U_Q$-conjugated to $M$, conjugating $a$ by some element in $U_Q$, we may assume $a_s \in N_GQ \cap N_GM$. Let $\pi$ be the projection of the semidirect product $(N_GQ \cap N_GM)U_Q$ onto $N_GQ \cap N_GM$ (a homomorphism of algebraic groups). Then $h = \pi(a)$ and $h_s = \pi(a_s) = \pi(a_s)$. Since $a_s \in N_GQ \cap N_GM$, we have $\pi(a_s) = a_s$ so $a_s = h_s$.

Let $D$ be a connected component of $G$. Let $Q = MU_Q$ be the Levi decomposition of a parabolic subgroup $Q$ of $G^o$ such that $N_GQ \cap N_GM \cap D \neq \emptyset$. By \[3, I\] $(N_GQ \cap N_GM)^o = N_GQ \cap N_GM = M$. Let $h \in N_GQ \cap N_GM \cap D$, so $D = G^o h$. Then

$$Mh \subseteq N_GQ \cap N_GM \cap G^o h = (N_GQ \cap N_GM \cap G^o)h = Mh.$$ 

So $N_GQ \cap N_GM \cap D = Mh$ is a connected component of $N_GQ \cap N_GM$.

From now on $g$ will always denote an element of $D$ and its Jordan decomposition will be $su$, $L$ will always denote $L(g)$. We will denote by $P$ a parabolic subgroup of $G^o$ with Levi subgroup $L$ and such that $gPg^{-1} = P$, whose existence is ensured by \[7, 2.1 (a)\]. By \[7, 2.2 (b)\] $g$ is isolated in $N_GL$. We will denote by $S$ the (isolated) $N_GL$-Jordan class of $g$. Then $S \subset N_GP$ (\[7, Proof of lemma 3.6\]), and $S$ is an isolated $N_GP \cap N_GL$-Jordan class. We will denote the connected component $N_GP \cap N_GL \cap D = Lg$ of $N_GP \cap N_GL$ by $D_L$. Let $S^* = \{ h \in S \mid C_G(h) \cap L \subseteq L \}$. By \[7, 3.9\], if $h \in S^*$ then $T_G(h) = (Z(L)^o \cap C_L(h))^o = T_{N_GL}(h)$. 

8
1.2 Induction of orbits

In this section we recall the induction procedure of $G^\circ$-orbits in $G$. The connected case is described in [12]. The induction of unipotent orbits in a disconnected group is reported in [14 II, 3].

**Definition 1.9.** Let $X$ be a $G$-variety, $H$ a subgroup of $G$, and $Y$ an $H$-variety. We set

$$G \times^H Y = G \times Y / \sim$$

where $(a, y) \sim (a', y')$ if $\exists h \in H$ such that $ah = a'$ and $h^{-1} \cdot y = y'$. We denote the elements of $G \times^H Y$ by $[(a, y)]$.

There is an action of $G$ on $G \times^H Y$ given by

$$b \ast [(a, y)] = [(ba, y)] \quad \text{for } b \in G, \quad [(a, y)] \in G \times^H Y.$$ 

Let now $Y$ be a $H$-stable subvariety of $X$. Let

$$Z = \{(aH, z) \in G/H \times G \cdot Y \mid a^{-1} \cdot z \in Y\} \subset G/H \times X.$$ 

It is a $G$-variety with action

$$b \ast (aH, z) = (baH, b \cdot z) \quad \text{for } b \in G, \quad (aH, z) \in Z.$$ 

The following lemma is well-known and can be proved by direct verification.

**Lemma 1.10.** Let $X$ be a $G$-variety, let $H \leq G$ and $Y$ an $H$-stable subvariety of $X$. Then

$$\psi : G \times^H Y \longrightarrow Z$$

$$[(a, y)] \mapsto (aH, a \cdot y).$$

is a well defined $G$-equivariant isomorphism of varieties.

With notation of Subsection 1.1, observe that $(L \cdot g)U_P$ is contained in the semidirect product

$$(N_G P \cap N_G L)U_P,$$ 

and $L \cdot g$ is $L$-stable, so $(L \cdot g)U_P$ is $P$-stable. We consider $G^\circ \times^P (L \cdot g)U_P$. 


Proposition 1.11. Let \( \phi \) be the following morphism:

\[
\phi : G^o \times P \to G
\]

\[
[(x, y)] \mapsto xyx^{-1}.
\]

Then the image of \( \phi \) is the closure of a single orbit for the action of \( G^o \).

Proof. Let \( X = \{(xP, h) \in G^o/P \times G \mid x^{-1}hx \in \overline{L \cdot gU_P}\} \), and let \( \pi : X \to G \) be the projection to the second factor. Then for \( \psi : G^o \times P \overline{L \cdot gU_P} \sim \to X \) the \( G^o \)-equivariant isomorphism as in Lemma 1.10 we have \( \pi \circ \psi = \phi \). We show that the image of \( \pi \) is the closure of a \( G^o \)-orbit. The variety \( G^o/P \) is complete because \( P \) is a parabolic group of \( G^o \), and \( (L \cdot g)U_P \) is a closed \( P \)-stable subset of \( N_{G^o}P \), so the map \( \pi \) is proper. Therefore the image of \( \pi \) is a closed subvariety of \( G \). Furthermore \( G^o \times P \overline{L \cdot gU_P} \) is a quotient of a product of irreducible varieties, so its image through \( \phi \) is irreducible. Since the latter is also \( G^o \)-stable, it is a union of \( G^o \)-orbits. We claim that these orbits are finitely many. Each of these orbits is represented in \( \overline{L \cdot gU_P} \) by construction. So we take \( x \in \overline{L \cdot gU_P} \) and let \( y \) be the unique element in \( \overline{L \cdot g} \) such that \( x \in yU_P \). By Proposition 1.8 the elements \( x_s \) and \( y_s \) are \( U_P \)-conjugated, and \( y_s \in \overline{L \cdot g} \). By Proposition 1.7 \( y_s \in L \cdot s \), where \( s \) is the semisimple part of \( g \). Thus the \( G^o \)-orbits of the elements in \( \overline{L \cdot gU_P} \), are in bijection with the unipotent orbits of \( C_{G^o}(s) \), and the unipotent orbits in \( C_{G^o}(s) \) are finitely many ([7, 1.15]). Since the image of \( \pi \) is closed, \( G^o \)-stable and irreducible, it is the closure of a single \( G^o \)-orbit. \( \square \)

Then the image of \( \phi \) from equation (3) is the closure of a single orbit, we call this orbit the induced orbit from \( D_L \) to \( D \) of the orbit \( L \cdot g \), and we denote it by \( \text{Ind}_{D_L}^D(L \cdot g) \). By construction

\[
\text{Ind}_{D_L}^D(L \cdot g) = G^o \cdot (gU_P)^{\text{reg}}.
\]
Our next goal is to describe the induced orbits in terms of unipotent induced orbits. Recall that we fixed $g = su \in D$.

By [7, 1.12] $P \cap G^{s^0}$ is a parabolic subgroup of $G^{s^0}$. Hence $P^{s^0} = P \cap G^{s^0}$ and it has Levi decomposition $L^{s^0}U^*_P$. So we can consider $\text{Ind}^{G^{s^0}}_{L^{s^0}h}(L^{s^0} \cdot h)$ for the $L^{s^0}$-orbit of an element $h \in N_{G^s}P^{s^0} \cap N_{G^s}L^{s^0}$. Note that $G^{s^0}h$ is the connected component of $G^s$ containing $h$, and $L^{s^0}h$, as observed in [2], is the connected component of $N_{G^s}P^{s^0} \cap N_{G^s}L^{s^0}$ containing $h$.

We will need the following lemma.

**Lemma 1.12.** With notation as above

$$\text{Ind}^{D^L}_{D^L}(L \cdot su) \cap suU^*_P \neq \emptyset.$$ 

**Proof.** We proceed as in [2, Lemma 4.4].

Let $\rho : U^*_P \times suU^*_P \rightarrow U^*_P \cdot (suU^*_P)$ be the dominant morphism mapping $(v, x)$ to $v xv^{-1}$. There exists an open subset $V$ of $U^*_P \cdot (suU^*_P)$ such that, $\dim U^*_P \cdot (suU^*_P) = \dim U^*_P + \dim U^*_P - \dim \rho^{-1}(y)$ for all $y \in V$. Using $U^*_P$-equivariance of $\rho$ we can choose $V$ in such way $V \cap suU^*_P \neq \emptyset$. Let $y \in V \cap suU^*_P$. Then $\rho^{-1}(y) = \{(v, x) \in U^*_P \times (suU^*_P) \mid v \cdot x = y\}$. Let us consider the Levi decomposition $P^{s^0} = L^{s^0}U^*_P$. Observe that $su \in N_{G^s}P^{s^0} \cap N_{G^s}L^{s^0}$, so, by Proposition 1.8, the elements in $suU^*_P$ have semisimple part conjugated to $s$ by an element of $U^*_P$, so $y_s = s$. If $(v, x) \in \rho^{-1}(y)$, then $vsv^{-1} = v x_s v^{-1} = y_s = s$, so $v \in U^*_P$ and then $\rho^{-1}(y) \cong U^*_P$.

Therefore $\dim(U^*_P \cdot (suU^*_P)) = \dim U^*_P$ and $(U^*_P \cdot (suU^*_P)) \subseteq suU^*_P$ implies the equality. By [16, 1], $(suU^*_P)^{\text{reg}}$ is dense in $(suU^*_P)$. Thus $U^*_P \cdot (suU^*_P) \cap (suU^*_P)^{\text{reg}} \neq \emptyset$.

\[\blacksquare\]

**Proposition 1.13.** With notation as above

$$\text{Ind}^{D^L}_{D^L}(L \cdot su) = G^s \cdot s(\text{Ind}^{G^{s^0}}_{L^{s^0}u}(L^{s^0} \cdot u)).$$

**Proof.** We proceed as in [2, Proposition 4.5]. We give the proof for completeness.
Since $\text{Ind}^D_{DL}(L \cdot su) \cap suU_P^s \neq \emptyset$, there holds $(suU_P^s)^{\text{reg}} \subset (suU_P)^{\text{reg}}$, so
\[ s(\text{Ind}_{L^s \cdot u}^G(L^s \cdot u)) = G^{s_0} \cdot (suU_P^s)^{\text{reg}} \subset G^o \cdot (suU_P)^{\text{reg}} = \text{Ind}^D_{DL}(L \cdot su). \]
Thus $\text{Ind}^D_{DL}(L \cdot su) = G^o \cdot s(\text{Ind}_{L^s \cdot u}^G(L^s \cdot u))$.

\[ \square \]

1.3 Closure and regular closure

In [7] G. Lusztig described the closure of a Jordan class. Building on this, we describe the closure and the regular closure of a Jordan class in terms of induced orbits.

We retain notation of Section 1.2. We observe that, since the set $\overline{SU}_P$ is contained in the semidirect product $(N_G P \cap N_G L)U_P$, and $\overline{S}$ is $L$-stable, then $\overline{SU}_P$ is $P$-stable.

This allows us to consider the variety
\[ \tilde{X} = \{ (xP, h) \in G^o / P \times G \mid x^{-1}hx \in \overline{SU}_P \}. \]

By [7] Lemma 3.14 the image of $\tilde{X}$ through the projection $\pi$ on the second factor of $\tilde{X}$ is $\overline{J}(g)$, so
\[ \pi(\tilde{X}) = \bigcup_{x \in G^o} x\overline{SU}_P x^{-1} = G^o \cdot \overline{SU}_P = \overline{J}(g), \]

and also it is an union of Jordan classes. Hence by the identification in Lemma 1.10, the image of the map
\[ (4) \quad \tilde{\phi} : G^o \times^P \overline{SU}_P \longrightarrow G \]
\[ [(x, y)] \mapsto xyx^{-1} \]
is $\overline{J}(g)$.

Now, we look at the structure of isolated Jordan classes.
By [7, 3.3 (a)], any isolated Jordan class of a reductive group $H$ is a single orbit for the action of $Z(H^o)^\circ \times H^o$ on $H$:

\[(5) \quad (Z(H^o)^\circ \times H^o) \times H \rightarrow H \]

\[
((z,x),y) \mapsto xyzx^{-1}.
\]

We apply it to the case $H = (N_G P \cap N_G L)$, so the isolated Jordan class $S$ in $N_G P \cap N_G L$ considered above, is an orbit for $Z(L)^o \times L$. Then

\[ S = Z(L)^o(L \cdot g). \]

We need the following lemma.

**Lemma 1.14.** With notation as above

(a) $S = Z(L)^o([L,L] \cdot g)$;

(b) $\overline{S} = Z(L)^o(L \cdot g) = Z(L)^o([L,L] \cdot g)$;

(c) $S = T(g)(L \cdot g)$;

(d) $\overline{S} = T(g)(L \cdot g)$.

**Proof.**

(a) Obviously $Z(L)^o([L,L] \cdot g) \subset Z(L)^o(L \cdot g) = S$. Conversely, the subgroup $Z(L)^o$ is a characteristic subgroup of $L$, so, since $g \in N_G L$, $g \in N_G(Z(L)^o)$. In addition $L$ is reductive, so $L = Z(L)^o[L,L]$. Hence, if $x \in Z(L)^o(L \cdot g)$, there exist $z_1, z_2 \in Z(L)^o$ and $l \in [L,L]$ such that $x = z_1z_2lg^{-1}z_2^{-1}$. Since $z_2^{-1} \in Z(L)^o$ and $lg^{-1} \in N_G(Z(L)^o)$, $x = z_3lg^{-1}$,

for some $z_3 \in Z(L)^o$. Therefore $Z(L)^o(L \cdot g) \subset Z(L)^o([L,L] \cdot g)$ and $S = Z(L)^o(L \cdot g) = Z(L)^o([L,L] \cdot g)$.
(b) The derived subgroup \([L, L]\) is characteristic in \(L\). So \(g \in N_G([L, L])\), then \([L, L] \cdot g \subset [L, L]g\).

The morphism \(m : Z(L)^\circ \times [L, L]g \longrightarrow Lg\) given by the multiplication, is finite ([?, Prop 2.3.2, Theorem 2.4.9]) hence closed. Thus \(Z(L)^\circ([L, L] \cdot g) = m(Z(L)^\circ \times [L, L] \cdot g)\) is closed in \(Lg\), since \(Z(L)^\circ \times [L, L] \cdot g \subset Z(L)^\circ \times [L, L]g\) is closed.

Clearly \([L, L] \cdot g \subset Z(L)^\circ([L, L] \cdot g) = \overline{S}\), so by \(Z(L)^\circ\)-stability of \(\overline{S}\) we obtain

\[
S = Z(L)^\circ([L, L] \cdot g) \subset Z(L)^\circ([L, L] \cdot g) = \overline{Z(L)^\circ([L, L] \cdot g)} = \overline{S}.
\]

Since \(Z(L)^\circ([L, L] \cdot g)\) is closed, thus \(Z(L)^\circ([L, L] \cdot g) = \overline{S}\).

By point (a), \(\overline{S} = Z(L)^\circ([L, L] \cdot g)\). Furthermore, from the \(Z(L)^\circ\)-stability of \(\overline{S}\), we get

\[
Z(L)^\circ([L, L] \cdot g) \subset Z(L)^\circ([L, L] \cdot g) \subset \overline{S} = Z(L)^\circ([L, L] \cdot g).
\]

So the equality holds.

(c) By [7, 1.21(d)], since \(S \in D_L\), it is an orbit for the action (5) restricted to \((Z(L)^\circ \cap C_{N_G L}(g)) \times L\). By Remark 2.6(d) there holds \(Z(L)^\circ \cap C_{N_G L}(g) = T(g)\), so

\[
S = T(g)(L \cdot g).
\]

(d) By point (b), \(Z(L)^\circ L \cdot g = \overline{S}\). Hence, since \(T(g) \subset Z(L)^\circ\),

\[
T(g)L \cdot g \subset Z(L)^\circ L \cdot g = \overline{S},
\]

By [7, 1.21 (b)], for a \(L\)-stable subset \(X\) of \(D_L\), we have that if \(T(g)X \subset X\) then \(Z(L)^\circ X \subset X\). Let \(X = T(g)L \cdot g\). So obviously \(T(g)X \subset X\), then \(Z(L)^\circ X \subset X\), that is \(Z(L)^\circ T(g)L \cdot g = Z(L)^\circ L \cdot g = \overline{S} \subset T(g)L \cdot g\). Thus \(\overline{S} = T(g)L \cdot g\).
Proposition 1.15. With notation as above

\[ J(g) = \bigcup_{z \in T(g)} \text{Ind}_{DL}^D (L \cdot zg). \]

Proof. Let \( z \in T(g) \) and let \( \phi_z \) be the map

\[ \phi_z : G^o \times P \text{Mult} \cdot zg U_P \rightarrow D \]

\[ [(a, x)] \mapsto axa^{-1}. \]

By Proposition 1.11, the image of \( \phi_z \) is \( \text{Ind}_{DL}^D (L \cdot zg) \). By Lemma 1.14 (d)

\[ G^o \times P SU_P = G^o \times P \bigcup_{z \in T(g)} \text{Mult} \cdot zg U_P = \bigcup_{z \in T(g)} G^o \times P \text{Mult} \cdot zg U_P \]

Then \( \text{Im} \tilde{\phi} = \bigcup_{z \in T(g)} \text{Im} \phi_z \), where \( \tilde{\phi} \) is the map defined in (4), so \( J(g) = \bigcup_{z \in T(g)} \text{Ind}_{DL}^D (L \cdot zg) \).

We study now the regular closure of \( J(g) \), that is \( J(g)^{\text{reg}} \).

Let \( d \in \mathbb{N} \) be such that \( J(g) \subset G(d) \). Then \( J(g)^{\text{reg}} \subset G_{(d)} \subset \bigcup_{k \leq d} G_{(k)} \), so \( J(g) \subset J(g)^{\text{reg}} = G_{(d)} \cap J(g) \).

Let \( w \) be a representative of \( \text{Ind}_{L \cdot su}^{G^o \cdot w}(L^o \cdot u) \). By Proposition 1.13

\[ \text{Ind}_{DL}^D (L \cdot su) = G^o \cdot sw. \]

Hence

\[ \text{codim}_D \text{Ind}_{DL}^D (L \cdot su) = \text{codim}_D G^o \cdot sw = \dim(G^o \cap G^w) = \]

\[ = \dim G^o - \dim G^o \cdot w = \text{codim}_{G^o \cdot u} G^o \cdot w. \]

(6)

Proposition 1.16. With notation as above

\[ \text{codim}_{DL}(L \cdot g) = \text{codim}_D \text{Ind}_{DL}^D (L \cdot g) \]
Proof. There holds
\[
\text{codim}_D \text{Ind}^D_{D_L}(L \cdot su) = \text{codim}_{G^{so} u} \text{Ind}^{G^{so} u}_{L^{so} u}(L^{so} \cdot u) = \\
= \text{codim}_{L^{so} u} L^{so} \cdot u = \dim L^{so} u - \dim L^{so} \cdot u = \\
= \dim L^{so} - \dim L^{so} \cdot u = \dim L^{so} - \dim L^{so} + \dim(L^{so} \cap L^{uo}) = \\
= \dim(L^{so} \cap L^{uo}) = \text{codim}_{D_L} L \cdot su.
\]
where the first equality is (6), the second is [14, Proposition 3.2], and the others follow from properties of the Jordan decomposition and the equality \( \dim D_L = \dim L \).

Remark 1.17. By [7, 3.9], \( G^{so} \subseteq L \). Then \( L^{zo} = G^{so} \), for any \( z \in Z(L)^0 \). Indeed \( G^{so} \subseteq L \cap G^{zs} \subseteq L^{zs} \), and then if \( x \in L^{zo} \subseteq L \cap G^{zs} \), then \( x \) commutes with \( zs \) and with \( z \), so \( x \) also commutes with \( s \), so \( x \in G^{s} \), thus \( L^{zo} \subseteq G^{so} \).

Proposition 1.18. With notation as above
\( (a) \ \overline{J(g)}^{\text{reg}} = \bigcup_{z \in T(g)} \text{Ind}^D_{D_L}(L \cdot zg), \)
\( (b) \ \overline{J(g)}^{\text{reg}} = \bigcup_{z \in T(g)} G^o \cdot zs \text{Ind}^{G^{zo} u}_{L^{zo} u}(G^{so} \cdot u). \)

Proof.

(a) Let \( \mathcal{O} \) be a \( G^o \)-orbit in \( \overline{J(g)} \) of maximal dimension. By Proposition 1.15, \( \exists z \in T(g) \) such that \( \mathcal{O} \subseteq \text{Ind}^D_{D_L}(L \cdot zg) \), so, by maximality, \( \mathcal{O} = \text{Ind}^D_{D_L}(L \cdot zg) \). So there exists an induced orbit in \( \overline{J(g)} \) that has dimension \( d \), where \( d \in \mathbb{N} \) is such that \( J(g) \subseteq G^{(d)} \). We claim that all the induced \( G^o \)-orbits from \( L \)-orbits in \( S \) have the same dimension. Since \( S \) is a Jordan class, the dimension of its \( L \)-orbits coincide. The claim follows from Proposition 1.16.

(b) Let \( z \in T(g) \). Then \( z \) is semisimple and lies in \( G^s \cap G^u \), so \( (zg)_s = zs \) and \( (zg)_u = u \). Therefore, by Proposition 1.13 and point (a)
\[
\overline{J(g)}^{\text{reg}} = \bigcup_{z \in T(g)} \text{Ind}^D_{D_L}(L \cdot zg) = \bigcup_{z \in T(g)} G^o \cdot zs \text{Ind}^{G^{zo} u}_{L^{zo} u}(L^{zo} \cdot u) =
\]
\[
\bigcup_{z \in T(g)} G^o \cdot zs \text{Ind}_{G^O u}^{G^o u}(G^o \cdot u),
\]

where the last equality follows from Remark 1.17.

\[\Box\]

1.4 The poset of Jordan classes

Let \( J_1, J_2 \) be Jordan classes. In [9, 7.2 (c)] G. Lusztig described when \( J_1 \subset J_2 \).

In this section we describe when \( J_1 \subset \overline{J_2}^{reg} \) in terms of induced orbits.

We define a partial order on the set of Jordan classes:

\[ J_1 \leq J_2 \text{ if and only if } J_1 \subset \overline{J_2}^{reg}. \]

Let \( g_i \in J_i \). Set \( L_i = L(g_i) \) and let \( P_i \) be a parabolic subgroup of \( G^o \) with Levi \( L_i \) and such that \( g_i \in N_G P_i \). Let \( S_i \) be the isolated Jordan class in \( N_G P_i \cap N_G L_i \) of \( g_i \), for \( i = 1, 2 \).

Let \( g_1 = tv \) and \( g_2 = su \) be respectively the Jordan decomposition of \( g_1 \) and \( g_2 \). By construction of \( S_i \) we have \( tv \in S_i^* \) and \( su \in S_i^* \), where, as we saw in the first Section, \( S_i^* = \{ h \in S_i \mid C_G(h_s)^o \subset L_i \} \). We recall that, by [7] 3.9, if \( h_i \in S_i^* \) then \( T_G(h_i) = T_{N_G L}(h_i) \) and \( L(h_i) = L_i \). Therefore

\[ \bullet T_G(tv) = T_{N_G L_1}(tv) \text{ and } T_G(su) = T_{N_G L_2}(su), \]

\[ \bullet L(tv) = L_1 \text{ and } L(su) = L_2. \]

Proposition 1.19. With the notation above if \( J_1 \leq J_2 \), then \( \exists y \in G^o \) such that

(a) \( v \in y \cdot (\text{Ind}_{G^O u}^{G^o u}(G^o \cdot u)) \);

(b) \( T(tv)t \subset y \cdot (T(su)s) \);

(c) \( y \cdot (G^o) \subset G^o \).
Proof. Since $J_1 \leq J_2$, by Proposition 1.18

$$J_1 \subset \bigcup_{z \in T(su)} G^o \cdot (zs \text{Ind}_{G^{z_0}u}^{G^o} (G^{z_0} \cdot u)).$$

Then $\exists z \in T(su)$ and $\exists y \in G^o$ such that $tv \in y \cdot (zs \text{Ind}_{G^{z_0}u}^{G^o} (G^{z_0} \cdot u))$. By Lemma 1.12, we can choose an element in $\text{Ind}_{G^{z_0}u}^{G^o} (G^{z_0} \cdot u)$ of the form $uu' \in uU_{P_2}$. Therefore $tv = y \cdot zsuu'$. We observe that, since $z \in T(su)$, it is semisimple and commutes with $s$, so $zs$ is semisimple. And also $zs = uu'zs$ with $uu'$ unipotent element. So $t = (tv)_s = y \cdot (zs (uu'))_s = y \cdot zs$ and $v = y \cdot (tv)_u = y \cdot (zsuu')_u = y \cdot uu'$, giving (a).

Now we show that $T(zsuu') \subset T(su)$. We observe that, since $L^{z_0}$ is a Levi subgroup of $G^{z_0}$, then $Z(G^{z_0}) \subset Z(L^{z_0})$, so

$$(7) \quad T(zsuu') = (Z(G^{z_0}) \cap C_G(uu'))^o$$

and

$$(Z(G^{z_0}) \cap C_G(uu'))^o \subset (Z(L^{z_0}) \cap C_G(uu'))^o = (Z(L^{z_0}) \cap C_{L_2}(uu'))^o.$$

Let $l \in C_{L_2}(uu')$. Then $uu' = luu'l^{-1} = lul^{-1}lu'l^{-1}$, where $lul^{-1} \in N_GL_2$ and $lu'l^{-1} \in U_{P_2}$. By the uniqueness of the decomposition in the semidirect product $N_GL_2U_{P_2}$, $lul^{-1} = u$. Thus

$$C_{L_2}(uu') \subset C_{L_2}(u) \subset C_G(u).$$

Therefore, combining (7) with Remark 1.17, we get

$$T(zsuu') \subset (Z(L^{z_0}) \cap C_{L_2}(uu'))^o \subset (Z(G^{z_0}) \cap C_G(u))^o = T(su).$$

Furthermore $z \in T(su)$, so $T(zsuu')zs \subset T(su)s$. Thus $T(tv)t = y \cdot (T(zsuu')zs) \subset y \cdot (T(su)s)$, proving (b).

By Remark 1.17, $y \cdot G^{z_0} = y \cdot L^{z_0} \subset y \cdot G^{z_0} = G^o$, then (c).
2 Lusztig strata

In this section we show that the strata of a disconnected reductive algebraic group defined by Lusztig in [11] are locally closed using the strategy from [1]. We recall the definition of the strata.

Let $W$ be the Weyl group of $G^\circ$, viewed as the set of $G^\circ$-orbits on $B \times B$, where $B$ is the flag manifold of $G^\circ$. The group $G$ acts by inner automorphisms on $G^\circ$ and this induces an action of $G/G^\circ$ on $W$. In particular, $D$, viewed as an element of $G/G^\circ$, defines an automorphism $[D] : W \rightarrow W$ whose fixed point set is denoted by $W^D$. It is the Weyl group of some root system ([8, Appendix]). We set $D_{un} = \{ a \in D \mid a = a_u \}$.

Let $E$ be the map from $D$ to the set of isomorphism classes of irreducible representations of the group $W^D$ defined in [11] so

$$E : D \rightarrow \text{Irr}(W^D)$$

$$a \mapsto E(a)$$

where $E(a)$ is constructed according to the following rules

- if $a = a_u$, then $E(a)$ is the unique irreducible representation of $W^D$ such that $\pi_l(\mathbb{Q}_l[\dim D]|_E)|_{D_{un}}$ is (up to shift) the intersection cohomology complex of $G^\circ \cdot a$ with coefficients in $\mathbb{Q}_l$, where $\pi$ is defined as in [11, 1.2].

- if $a_s$ is central in $G$, let $D_{a_u}$ be the connected component of $G$ containing $a_u$. We observe that $W^{D_{a_u}} = W^D$, because $D_{a_u} = G^\circ a_u$ and $D = G^\circ a_s a_u = a_s G^\circ a_u = a_s D_{a_u}$ with $a_s$ central. Then $E(a) := E(a_u)$.

- if $a_s \notin Z(G)$, then $a_s$ is central in $C_G(a_s)$, and we let $D'$ be the connected component of $C_G(a_s)$ containing $a$. We denote by $E'(a) \in \text{Irr}(W'(C_G(a_s)^\circ D'))$ the image of $a$ through the map $E$ referred to the group $C_G(a_s)$. Then we set $E(a) = j_{W'(C_G(a_s)^\circ D')}E'(a)$, where $j$ is the truncated induction as defined in [12].
The fibers of $E$ are called the Lusztig strata of $D$. By [11, 1.16 (e)] Lusztig strata are union of $G^o$-orbits of the same dimension, so if $X$ is a Lusztig stratum contained in $D$, then there exists $d \in \mathbb{N}$ such that $X \subset G(d) \cap D$.

We recall a basic property of $E$.

**Proposition 2.1.** If $J$ is a Jordan class, and $a = a_s a_u, h = h_s h_u \in J$, then $E(a) = E(h)$, so strata are unions of Jordan classes.

**Proof.** By Remark 1.3, up to conjugation $C_G(a_s) = C_G(h_s)$. Furthermore $h = za$ where $z$ is in $C_G(a_s)^o$, so $a$ and $h$ are in the same connected component of $C_G(a_s)$. Hence the connected component $D'_h$ of $C_G(h_s)$ containing $h$ and the connected component $D'_a$ of $C_G(a_s)$ containing $a$ coincide, as well as the unipotent parts of $a$ and $h$. Thus $E(h) = E(a)$.

By Proposition 1.18 (b), if $x \in J(su)^{reg}$, then $\exists z \in T(su)$ and $v \in \text{Ind}_{G^{zsuo}}^{zsuo}(G^{suo} \cdot u)$ such that, up to conjugation, $x = zsv$.

To simplify notation, we set:

- $D' := sG^{suo}$ the connected component of $G^s$ containing $su$,
- $D'_u := G^{suo}$ the connected component of $G^s$ containing $u$,
- $\widetilde{D}$ the connected component of $G^{zsuo}$ containing $x$,
- $\widetilde{D}_u := G^{zsuo}$ the connected component of $G^{zsuo}$ containing $u$,
- $\widetilde{D}'_u := L^{zsuo}$ the connected component of $N_{G^{zsuo}} P^{zsuo} \cap N_{G^{zsuo}} L^{zsuo}$ containing $u$.

Since $x = zsv$ and $v \in \text{Ind}_{D'_u}^{\widetilde{D}_u}(G^{suo} \cdot u)$ then $v \in \widetilde{D}_u$, so $W(G^{zsuo})\widetilde{D} = W(G^{zsuo})\widetilde{D}_u$. Observe also that, by Remark 1.17 $\widetilde{D}'_u = D'_u$. Then $W(G^{suo})D' = W(G^{suo})\widetilde{D}_u$.

The statement of the following result was suggested by G. Lusztig in [11, 1.12 (b)].
Proposition 2.2. A stratum $X$ is locally closed.

Proof. We first prove that if $J(su) \subset X$ then $\overline{J(su)}^{\text{reg}} \subset X$. Let $x \in \overline{J(su)}^{\text{reg}}$. We show that $E(x) = E(su)$. With notation as before 

$\exists z \in T(su)$ and $v \in \text{Ind}_{\overline{D_u}^D}^{\overline{D_u}}(G^{so} \cdot u)$ such that $x = zsv$.

By [11, 1.9 (iii)] $E(u)$ is good, so we can apply $j$-induction from $W(G^{so})\overline{D_u}^D$ to $W(G^{so})\overline{D_u}$ obtaining an irreducible representation. By [13, Corollary 2.10],

$$E(v) = j_{W(G^{so})}^{W(G^{so})}E(u).$$

By definition of $E$,

$$E(x) = j_{W(G^{so})}^{W(G^{so})}E(v).$$

So putting together these relations and using transitivity of the truncated induction, we have

$$E(x) = j_{W(G^{so})}^{W(G^{so})}E(v) = j_{W(G^{so})}^{W(G^{so})}j_{W(G^{so})}^{W(G^{so})}E(u) = j_{W(G^{so})}^{W(G^{so})}E(u).$$

By definition of $E$, there holds $E(su) = j_{W(G^{so})}^{W(G^{so})}E(u)$, so $E(x) = E(su)$ and $x \in X$.

By Proposition 2.1 $X$ is a finite union of Jordan classes (11), therefore $X$ is a finite union of regular closures of Jordan classes. Let $I$ be the set parametrizing them and let $X \subset G_{(d)}$. Then

$$X = \bigcup_{i \in I} J(g_i)^{\text{reg}} = \bigcup_{i \in I} J(g_i) \cap G_{(d)} = \left( \bigcup_{i \in I} J(g_i) \right) \cap \left( \bigcup_{i \geq \overline{d}} G_{(i)} \right),$$

is the intersection of a closed subset and an open subset of $D$, so $X$ is locally closed.

$\Box$

3 Examples

Let $\mathbb{K}$ be an algebraic closed field of characteristic 2. In this section we will analize the Jordan classes in the semidirect product of $SL(n)$ with the group generated by a non-inner automorphism of $SL(n)$. 

21
Let $\tau$ be the involution of $SL(n)$, defined as follows

$$\tau : SL(n) \rightarrow SL(n)$$

$$X \mapsto J^t \cdot X^{-1} \cdot J,$$

with $J = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, and $^tX^{-1}$ the transposed of the inverse of $X$.

Let $T = \begin{pmatrix} \ast \\ \vdots \\ \ast \end{pmatrix}$ be the maximal torus in $SL(n)$ consisting of diagonal matrices, and $B = \begin{pmatrix} \ast & \cdots & \ast \\ \vdots & \ddots & \vdots \\ \ast & \cdots & \ast \end{pmatrix}$ the Borel subgroup of upper triangular matrices. Let $U$ be the subgroup of unipotent elements in $B$.

Then $\tau$ preserves $B$, $T$ and $U$. So $\tau$ induces an automorphism of the Dynkin diagram of $SL(n)$. We consider the non connected algebraic group $G = SL(n) \rtimes \langle \tau \rangle$, so $G^o = SL(n)$.

Since $|\tau| = \text{char } \mathbb{K}$, the element $\tau$ is a unipotent element of $G$. Furthermore the automorphism $\tau$ is such that $\tau \cdot T = T$ and $\tau \cdot B = B$, so it is a quasi semisimple element ([7, 1.4]). Then we can apply the result in [17] 8.2 pp 52, and we obtain that $G^\tau$ is connected and reductive. In [5] Theorem 1.2 (iii)] there is the description of the reductive part of the centralizer of a unipotent element. Using this result and the fact that $G^\tau$ is reductive, we can conclude that $C_{G^\circ}(\tau) = (G^o)^\tau = Sp(2n [\frac{n}{2}])$.

We analyze the Jordan classes of elements of $G^o \tau$.

**Remark 3.1.** Let $H$ be an algebraic group in a field of characteristic $p$, let $\sigma$ be an automorphism of $H^o$. Then, by [17, pp.51], $\sigma$ is semisimple if and only if there exists an integer $l$ coprime with $p$ such that $\sigma^l$ is a semisimple inner automorphism of $H^o$, i.e., the action of $\sigma^l$ is conjugation by a semisimple
element of $H^\circ$.
Therefore in $G^\circ \tau$ there are no semisimple elements.

**Remark 3.2.** By [15, Lemma 5], every element in $G^\circ \tau$ is $G^\circ$-conjugated to an element $tu\tau$ with $t \in T^\tau$ and $u \in U$.

Let $B' = T^\tau \ltimes U$ and let $h \in B' \tau$. By Remark 3.1, $h_s \in B'$ and $h_u = h'u'\tau \in B' \tau$. Observe that $B'$ is a solvable group. Then ([4, 19.3]), all maximal tori in $B'$ are conjugated by elements in $U$. Hence, there exists $v \in U$ such that $vh_s v^{-1} = z \in T^\tau$. Thus $vhv^{-1} = vh_s v^{-1}vh'_u \tau (v^{-1})\tau = zvh'_u \tau (v^{-1})\tau$. So, up to conjugation, every element $x \in G^\circ \tau$ is of the form $tu\tau$ with $t \in T^\tau$ and $tu = ut$.

Let $x = tu\tau \in B' \tau$, with $t \in T^\tau$ and $u \in U$, with $tu = ut$, and let $k = [\frac{n}{2}]$.

An element in $T^\tau$ is of the form:

- if $n$ is even, then
  \[
  t = \text{diag}(a_1, \ldots, a_k, a_k^{-1}, \ldots, a_1^{-1}), \quad \text{for } a_1, \ldots, a_k \in K^\ast,
  \]

- if $n$ is odd, then
  \[
  t = \text{diag}(a_1, \ldots, a_k, 1, a_k^{-1}, \ldots, a_1^{-1}), \quad \text{for } a_1, \ldots, a_k \in K^\ast.
  \]

Observe that $t$ is regular if and only if $a_i \neq a_j$, $a_i \neq a_j^{-1}$ and $a_i \neq 1$ for all $i \neq j \in \{1, \ldots, k\}$. So, if $t$ is regular, the condition $tu = ut$ implies $u = \text{Id}$, and $x = t\tau$.

We describe the Jordan class $J(t\tau)$ for $t$ a regular semisimple element of $G$ contained in $T^\tau$.

It is well known that in this case $C_G(t)^\circ = T$. So

Therefore $T(t\tau) = (T \cap C_G(\tau))^\circ = (T^\tau)^\circ = T^\tau$. So $z \in T(t\tau)^\text{reg}$ is of the form
• if \( n = 2k + 1 \),

\[
z = \text{diag}(z_1, \ldots, z_k, 1, z_k^{-1}, \ldots, z_1^{-1}) \quad \text{where } z_i \neq 0, \ z_i \neq z_j^\pm 1 \forall i, j \in \{1, \ldots, k\},
\]

• if \( n = 2k \),

\[
z = \text{diag}(z_1, \ldots, z_k, z_k^{-1}, \ldots, z_1^{-1}) \quad \text{where } z_i \neq 0, \ z_i \neq z_j^\pm 1 \forall i, j \in \{1, \ldots, k\}.
\]

Then \( J(t\tau) = SL(n) \cdot (T(t\tau)^{\text{reg}}\tau) \).

Now we consider \( x = tu\tau \in B'\tau \) with \( tu = ut \), \( t \) not regular, and \( t \) as in \( \square \) or \( \square \).

Then there exist coefficients of \( t \) that coincide. We can assume that the coinciding elements are among \( a_1, \ldots, a_k \), because the other cases are conjugated to this one by a monomial matrix in \((G^2)^\tau\), so they produce the same Jordan class.

So suppose that there exist \( a_1, \ldots, a_{r-1} \in \mathbb{K}^* \) such that \( a_i \neq a_j^\pm 1 \) for all \( i, j = 1, \ldots, r \), and, up to conjugation,

\[
t = \begin{pmatrix}
a_1 \text{Id}_{h_1} &  &  \\
 & \ddots &  \\
 &  & a_{r-1} \text{Id}_{h_{r-1}} \\
 &  &  \\
 &  &  \text{Id}_{h_r} \\
 &  &  \\
\end{pmatrix},
\]
with \( n \) even, \( h_i \in \mathbb{N} \) and \( h_r \geq 0 \) even, and

\[
t = \begin{pmatrix}
  a_1 \text{Id}_{h_1} \\
  \vdots \\
  a_{r-1} \text{Id}_{h_{r-1}} \\
  \text{Id}_{h_r} \\
  a_{r-1}^{-1} \text{Id}_{h_{r-1}} \\
  \vdots \\
  a_1^{-1} \text{Id}_{h_1}
\end{pmatrix},
\]

with \( n \) odd, \( h_i \in \mathbb{N} \) and \( h_r \geq 1 \) odd, where \( \text{Id}_{h_i} \) is the \( h_i \times h_i \)-identity matrix.

Let \( z \in T(x) \). By [3, Theorem 1.8], the subgroup \( T \cap C_G(t)^o \) is a maximal torus of \( C_G(t)^o \). This implies \( Z(C_G(t)^o) \subseteq T \). So \( z \in T \cap C_G(u\tau) \). Since \( zu\tau(z^{-1}) = u \in U \) and \( z \in T \), looking at the projection from \( TU \) to \( T \), we see that \( z \in T^r \).

Furthermore, from the condition \( z \in Z(C_G(t)^o) \) there exist \( z_1, \ldots, z_{r-1} \in \mathbb{K}^* \) such that

\[
z = \text{diag}(z_1 \text{Id}_{h_1}, \ldots, z_{r-1} \text{Id}_{h_{r-1}}, \text{Id}_{h_r}, z_{r-1}^{-1} \text{Id}_{h_{r-1}}, \ldots, z_1^{-1} \text{Id}_{h_1}),
\]

if \( n \) is even, where \( h_i \in \mathbb{N} \) and \( h_r \geq 0 \) even, and

\[
z = \text{diag}(z_1 \text{Id}_{h_1}, \ldots, z_{r-1} \text{Id}_{h_{r-1}}, \text{Id}_{h_r}, z_{r-1}^{-1} \text{Id}_{h_{r-1}}, \ldots, z_1^{-1} \text{Id}_{h_1}),
\]

if \( n \) is odd, where \( h_i \in \mathbb{N} \) and \( h_r \geq 1 \) odd.

Therefore, looking at \( (T(x)t)^{\text{reg}} \), for \( n \) even

\[
z \text{ is such that } z_i \neq z_j^{\pm 1} \text{ if } i \neq j, \; z_i \neq 1, \; h_i \in \mathbb{N}, \; h_r \geq 0 \text{ even },
\]

and for \( n \) odd

\[
z \text{ is such that } z_i \neq z_j^{\pm 1} \text{ if } i \neq j, \; z_i \neq 1, \; h_i \in \mathbb{N}, \; h_r \geq 1 \text{ odd}.
\]

Then \( J(x) = SL(n) \cdot ((T(x)t)^{\text{reg}}u\tau) \).

Lastly, as we saw in Remark [1.5], the class \( J(u\tau) \) with \( u\tau \) unipotent is the orbit \( C^o \cdot u\tau \).
3.1 The case $n = 3$

In this section we analyze the case $n = 3$ and compute explicitly the poset of Jordan classes with respect to the partial order defined in Subsection 1.4.

Every element $x' \in SL(3)\tau$ is $SL(3)$-conjugate to an element $x = tu\tau$ with $t \in T^{\tau}$ and $u \in U$ such that $tu = ut$. Observe that a non-trivial element in $T^{\tau}$ is always regular when $n = 3$. Since if $t$ is regular then $u = Id$, we have only two possibilities for $x$, namely $x = t\tau$ or $x = u\tau$.

Let $x = t\tau$ with $t \in T^{\tau}$. Then, as we have seen in the general case,\(^(*)\)

\[ J(t\tau) = SL(3) \cdot \left\{ \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix} \mid a \neq 0, 1 \right\} \tau. \]

We analyze the isolated strata associated to $J(t\tau)$.

The torus $T(t\tau) = T^{\tau}$, so in this case $L(t\tau) = C_{SL(3)}(T(t\tau)) = T$, $N_GT = T \rtimes \langle \tau \rangle$, $P = B$, $N_GP = B \rtimes \langle \tau \rangle$. Thus the isolated stratum $S$ of $t\tau$ in $N_GL$ is

\[ S = T(T \cdot t\tau) = T\tau. \]

Let $y\tau = \begin{pmatrix} \alpha \\ \beta \\ \alpha^{-1}\beta^{-1} \end{pmatrix} \tau \in T\tau$. Then its Jordan decomposition is

\[ (y\tau)_s = \begin{pmatrix} \alpha\sqrt{\beta} \\ 1 \\ \alpha^{-1}\sqrt{\beta}^{-1} \end{pmatrix} \text{ and } (y\tau)_u = \begin{pmatrix} \sqrt{\beta}^{-1} \\ \beta \\ \sqrt{\beta}^{-1} \end{pmatrix} \tau. \]

Thus the set $S^* = \{ y\tau \in T\tau \mid C_G((y\tau)_s) \subseteq T \}$ is

\[ \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha^{-1}\beta^{-1} \end{pmatrix} \tau \mid \alpha, \beta \in \mathbb{K}^*, \alpha \neq \sqrt{\beta}^{-1} \right\}. \]
We consider now $x = u\tau$, with $u \in U$. By Remark 1.5, the torus $T(u\tau) = \{\text{Id}\}$, so the Jordan class of $x$ is just the $G^o$-orbit of $u\tau$. In this case the Jordan class is isolated.

The following Lemma lists all the possible Jordan classes $J(x)$ in $G\tau$ that are unipotent $G^o$-orbits.

**Lemma 3.3.** There are only two unipotent $SL(3)$-orbits represented in $U\tau$:

- the orbit of $\tau$,
- the orbit $SL(3) \cdot (u_1\tau)$ where $u_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

**Proof.** Let $y \in G^o$ be the matrix with entries $y_{i,j}$ and such that $y\tau \in G^o \cdot \tau$, so there exists $x \in G^o$ such that $y = x\tau(x^{-1})$. Computing the product $x\tau(x^{-1})$, we find that $y_{1,2} = y_{2,3}$, giving a necessary condition for $y\tau$ to lie in the $G^o$-orbit of $\tau$.

Let $u_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $u_1\tau \notin G^o \cdot \tau$.

Let $\pi = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} \in U$. We distinguish two cases.

- if $a_1 = a_3$, it possible to find a matrix $\tilde{b} \in U$ such that $\tilde{b}\pi\tau(\tilde{b}^{-1}) = \text{Id}$. So $\tilde{b}\pi\tau \in G^o \cdot \tau$.
- if $a_1 \neq a_3$, there exists $\tilde{b}' \in U$ such that $\tilde{b}'\pi\tau(\tilde{b}'^{-1}) = u_1$. Then $\tilde{b}'\pi\tau \in G^o \cdot u_1\tau$.

Therefore the only unipotent orbits represented in $U\tau$ are $G^o \cdot \tau$ and $G^o \cdot u_1\tau$. □

The next theorem summarizes the results of this section.
**Teorema 3.4.** The Jordan classes of $SL(3) \rtimes \langle \tau \rangle$ are

- $J(t\tau)$ as in (*) , with $t \in T^{\tau}$, $t \neq \text{Id}$;

- $J(u_{1}\tau) = SL(3) \cdot u_{1}\tau$, where $u_{1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

- $J(\tau) = SL(3) \cdot \tau$.

We analyze the poset of the Jordan classes.

By [14, II 2.21], there exists a unique minimal unipotent quasi-semisimple orbit in $G^{\circ} \cdot \tau$, that is $G^{\circ} \cdot \tau$.

Thus we have $\dim G^{\circ} \cdot u_{1}\tau > \dim G^{\circ} \cdot \tau$. So $\tau \notin G^{\circ} \cdot (\tau U)^{\text{reg}}$, then the unipotent induced orbit $\text{Ind}^{G_{T^{\tau}}(T \cdot \tau)}_{T^{\tau}}(T \cdot \tau)$ is $G^{\circ} \cdot u_{1}\tau$. By Proposition 1.18, $\text{Ind}^{G_{T^{\tau}}(T \cdot \tau)}_{T^{\tau}}(T \cdot \tau) \subset \overline{J(t\tau)^{\text{reg}}}$, so $J(u_{1}\tau) = G^{\circ} \cdot u_{1}\tau \subset \overline{J(t\tau)^{\text{reg}}}$. Therefore

- $\overline{J(t\tau)^{\text{reg}}} = J(t\tau) \cup J(u_{1}\tau)$ and $\overline{J(t\tau)} = J(t\tau) \cup J(u_{1}\tau) \cup J(\tau) = G^{\circ} \cdot \tau$;

- $\overline{J(u_{1}\tau)^{\text{reg}}} = J(u_{1}\tau) = G^{\circ} \cdot u_{1}\tau$;

- $\overline{J(\tau)^{\text{reg}}} = J(\tau) = G^{\circ} \cdot \tau$.

Then the only confrontable elements are $J(u_{1}\tau) \leq J(t\tau)$.

Now we look at the Lusztig strata in $G$. By Theorem 2.2 any stratum is a union of regular closures of Jordan classes whose elements have $G^{\circ}$-orbits of the same dimension. Thus there are two strata

$$X_{1} = \overline{J(t\tau)^{\text{reg}}}$$

$$X_{2} = \overline{J(\tau)^{\text{reg}}}.$$

The reader can compare this result with the example in [11, 3.4].
Acknowledgments

It is a pleasure to thank Giovanna Carnovale and Francesco Esposito for proposing the problem and for helpful suggestions. The author would like to thank Mauro Costantini for his help. The author acknowledges support by: DOR2207212/22 “Algebre di Nichols, algebre di Hopf e gruppi algebrici” and BIRD203834 “Grassmannians, flag varieties and their generalizations.” funded by the University of Padova. She is a member of the INdAM group GNSAGA.

References

[1] G. Carnovale. Lusztig’s strata are locally closed. Archiv der Mathematik, 115, 03 2020.

[2] G. Carnovale and F. Esposito. On sheets of conjugacy classes in good characteristic. International Mathematics Research Notices, 2012:810–828, 2012.

[3] F. Digne and J. Michel. Groupes réductifs non connexes. Annales scientifiques de l’École Normale Supérieure, 4e série, 27(3):345–406, 1994.

[4] J. E. Humphreys. Linear Algebraic Groups. Springer New York, NY, 1975.

[5] R. Lawther, M. W. Liebeck, and G. M. Seitz. Outer unipotent classes in automorphism groups of simple algebraic groups. Proceedings of the London Mathematical Society, 109(3):553–595, 2014.

[6] G. Lusztig. Intersection cohomology complexes on a reductive group. Inventiones mathematicae, 75, pages 205–272, 1984.

[7] G. Lusztig. Character sheaves on disconnected groups I. Represent. Th. 7, pages 374–403, 2003.
[8] G. Lusztig. Hecke algebras with unequal parameters. American Mathematical Soc., 2003.

[9] G. Lusztig. Character sheaves on disconnected groups II. *Represent. Theory 10* (2006), pages 353–379, 2006.

[10] G. Lusztig. *On conjugacy classes in a reductive group*, pages 333–363. Springer International Publishing, Cham, 2015.

[11] G. Lusztig. Strata of a disconnected reductive group. *Indagationes Mathematicae*, 32(5):968–986, 2021. Special issue to the memory of T.A. Springer.

[12] G. Lusztig and N. Spaltenstein. Induced unipotent classes. *Journal of the London Mathematical Society*, s2-19(1):41–52, 1979.

[13] G. Malle and K. Sorlin. Springer correspondence for disconnected groups. *Mathematische Zeitschrift*, 246:291–319, 01 2004.

[14] N. Spaltestein. *Classes Unipotentes et Sous-groupes de Borel*. Springer-Verlag Berlin Heidelberg, 1982.

[15] T. Springer. Twisted conjugacy in simply connected groups. *Transformation Groups*, 11:539–545, 2006.

[16] R. Steinberg. Regular elements of semi-simple algebraic groups. *Publications Mathématiques de l’IHÉS*, 25:49–80, 1965.

[17] R. Steinberg. *Endomorphisms of Linear Algebraic Groups*. American Mathematical Soc., 1968.

[18] Springer T.A. *Linear algebraic groups*. 1998.