LOG ENRIQUES SURFACES OF INDEX 7 AND TYPE $A_{15}$

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Dedicated to Professor Shigeyuki Kondo on the occasion of his sixtieth birthday

ABSTRACT. We show that there is only one log Enriques surface of index 7 and type $A_{15}$.

CONTENTS

1. Introduction
2. $K3$ surfaces with a non-symplectic automorphism of order 7
3. A correspondence between log Enriques surfaces and $K3$ surfaces
4. Sublattices of type $A_{15}$

References

1. Introduction

We will work over $\mathbb{C}$, the field of complex numbers, throughout this paper. Let $Z$ be a normal algebraic surface with at worst log terminal singularities. $Z$ is called log Enriques if the irregularity $\dim H^1(Z, \mathcal{O}_Z) = 0$ and a positive multiple $IK_Z$ of a canonical Weil divisor $K_Z$ is linearly equivalent to zero. The smallest integer $I > 0$ satisfying $IK_Z \sim 0$ is called the index of $Z$. Without loss of generality, we assume that a log Enriques surface has no Du Val singular points, because if $Z' \to Z$ is the minimal resolution of all Du Val singular points of $Z$ then $Z'$ is also a log Enriques surface of the same index of $Z$.

Let $Z$ be a log Enriques surface of index $I$. The Galois $\mathbb{Z}/IZ$-cover

$$
\pi : Y := \text{Spec}_{\mathbb{O}_Z} \left( \bigoplus_{i=0}^{I-1} \mathcal{O}_Z(-iK_Z) \right) \to Z
$$

is called the (global) canonical covering. Note that $Y$ is either an abelian surface or a $K3$ surface with at worst Du Val singular points, and that $\pi$ is unramified over $Z \setminus \text{Sing}(Z)$. If $Y$ is an abelian surface then $I = 3$ or 5. See also [14] for details. A log Enriques surface $Z$ is of type $A_m$ or $D_n$ if, by definition, its canonical cover $Y$ has a singular point of type $A_m$ or $D_n$, respectively.

It is interesting to consider the index $I$ of a log Enriques surface. Blache [2] proved that $I \leq 21$. Thus if $I$ is prime then $I = 2, 3, 5, 7, 11, 13, 17$ or 19.

Theorem 1.1 ([9] [10] [11] [13] [12]). The followings hold:

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(1) There is one log Enriques surface of type $D_{19}$ (resp. $A_{19}$, $D_{18}$), up to isomorphism.
(2) There are two log Enriques surfaces of type $A_{18}$, up to isomorphism.
(3) There are two log Enriques surfaces of index 5 and type $A_{17}$, up to isomorphism.

The followings do not refer to singular points. But these determine log Enriques surfaces with large prime indices:

(4) There are two maximal log Enriques surfaces of index 11, up to isomorphism.
(5) If $I = 13$, 17 or 19 then there is a unique log Enriques surface of index $I$, up to isomorphism.

Remark 1.2. If a log Enriques surface is of type $A_{19}$ (resp. $A_{18}$, $D_{18}$ or $D_{19}$) then its index is 2 (resp. 3).

Let $\omega_X$ be a nowhere vanishing holomorphic 2-form on an algebraic $K3$ surface $X$ and $\sigma$ an automorphism on $X$ of finite order $I$. It is called non-symplectic if and only if it satisfies $\sigma^* \omega_X = \zeta_I \omega_X$ where $\zeta_I$ is a primitive $I$-th root of unity. To prove Theorem [11] we studied non-symplectic automorphisms of $K3$ surfaces, because the canonical covering $\pi$ is a cyclic Galois covering of order $I$ which acts faithfully on the space $H^0(Y, \mathcal{O}_Y(K_Y))$. And we have gotten the following.

Theorem 1.3 ([9, 11, 13, 12]). Let $\sigma_I$ be a non-symplectic automorphism of order $I$ on a $K3$ surface $X_I$ and $X_I^{\sigma_I}$ be the fixed locus of $\sigma_I$; $X_I^{\sigma_I} = \{ x \in X_I | \sigma_I(x) = x \}$.

Then the followings hold:

(1) If $X_3^{\sigma_3}$ consists of only (smooth) rational curves and possibly some isolated points, and contains at least 6 rational curves then a pair $(X_3, \langle \sigma_3 \rangle)$ is unique up to isomorphism.
(2) If $X_2^{\sigma_2}$ consists of only (smooth) rational curves and contains at least 10 rational curves then a pair $(X_2, \langle \sigma_2 \rangle)$ is unique up to isomorphism.
(3) If $X_5^{\sigma_5}$ contains no curves of genus $\geq 2$, but contains at least 3 rational curves then a pair $(X_5, \langle \sigma_5 \rangle)$ is unique up to isomorphism.
(4) Put $M := \{ x \in H^2(X_{11}, \mathbb{Z})|\sigma_{11}(x) = x \}$. A pair $(X_{11}, \langle \sigma_{11} \rangle)$ is unique up to isomorphism if and only if $M = U \oplus A_{10}$.
(5) Pairs $(X_{13}, \langle \sigma_{13} \rangle)$, $(X_{17}, \langle \sigma_{17} \rangle)$ and $(X_{19}, \langle \sigma_{19} \rangle)$ are unique up to isomorphism, respectively.

These theorems miss the case of $I = 7$. The main purpose of this paper is to prove the following theorem:

Main Theorem. (1) There is, up to isomorphism, only one log Enriques surface of index 7 and type $A_{15}$.
(2) If $X_7^{\sigma_7}$ consists of only smooth rational curves and some isolated points and contains at least 2 rational curves then a pair $(X_7, \langle \sigma_7 \rangle)$ is unique up to isomorphism.

We summarize the contents of this paper. In Section 2 we study $K3$ surfaces with a non-symplectic automorphism and prove Main Theorem (2). In Section 3 we see uniqueness of the $K3$ surface with a non-symplectic automorphism of order 7 which is constructed from log Enriques surfaces of index 7 and type $A_{15}$. And we give an example of a construction for such a log Enriques surface from a $K3$ surface.
with a non-symplectic automorphism 7. In Section 3, we study a sublattice of type \( A_{15} \) in the Néron-Severi lattice of a K3 surface and give a proof of Main Theorem (1).

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2. \( K3 \) surfaces with a non-symplectic automorphism of order 7

In this section, we collect some basic results for non-symplectic automorphisms on a K3 surface. For the details, see [8] and [1], and so on.

For a K3 surface \( X \), we denote by \( S_X \) and \( T_X \) the Néron-Severi lattice and the transcendental lattice, respectively.

Lemma 2.1. Let \( \sigma \) be a non-symplectic automorphism of order \( I \) on \( X \). Then

1. The eigen values of \( \sigma^* \mid T_X \) are the primitive \( I \)-th roots of unity, hence \( \sigma^* \mid T_X \otimes \mathbb{C} \) can be diagonalized as:

\[
\begin{pmatrix}
\zeta_I E_q & 0 & \cdots & \cdots & 0 \\
0 & \ddots & & & \vdots \\
\vdots & & \ddots & & \vdots \\
0 & \cdots & & \zeta_I^q E_q & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \zeta_I^{I-1} E_q
\end{pmatrix},
\]

where \( E_q \) is the identity matrix of size \( q \) and \( 1 \leq n \leq I - 1 \) is co-prime with \( I \).

2. Let \( P_{i,j} \) be an isolated fixed point of \( \sigma \) on \( X \). Then \( \sigma^* \) can be written as

\[
\begin{pmatrix}
\zeta_i^I & 0 \\
0 & \zeta_j^I
\end{pmatrix}
\]

under some appropriate local coordinates around \( P_{i,j} \).

3. Let \( C \) be an irreducible curve in \( X^\sigma \) and \( Q \) a point on \( C \). Then \( \sigma^* \) can be written as

\[
\begin{pmatrix}
1 & 0 \\
0 & \zeta_I
\end{pmatrix}
\]

under some appropriate local coordinates around \( Q \). In particular, fixed curves are non-singular.

Lemma 2.1 (1) implies that \( \Phi(I) \) divides \( \text{rk} \, T_X \), where \( \Phi \) is the Euler function. Lemma 2.1 (2) and (3) imply that the fixed locus of \( \sigma \) is either empty or the disjoint union of non-singular curves and isolated points:

\[X^\sigma = \{ P_{i_1,j_1}^{j_1}, \ldots, P_{i_M,j_M}^{j_M} \} \sqcup C_1 \sqcup \cdots \sqcup C_N,\]

where \( P_{i_k,j_k}^{j_k} \) is an isolated fixed point and \( C_l \) is a non-singular curve.

The global Torelli Theorem gives the following.

Remark 2.2 ([5 Lemma (1.6)]). Let \( X \) be a K3 surface and \( g_i \) (\( i = 1, 2 \)) automorphisms of \( X \) such that \( g_i^* | S_X = g_2^* | S_X \) and that \( g_i^* \omega_X = g_2^* \omega_X \). Then \( g_1 = g_2 \) in Aut \( (X) \).
The Remark says that for study of non-symplectic automorphisms, the action on $S_X$ is important. Hence the invariant lattice $S_X^\sigma := \{ x \in S_X | \sigma^*(x) = x \}$ plays an essential role for the classification of non-symplectic automorphisms.

In the following, we denote $\sigma$ a non-symplectic automorphism of order 7 on a $K3$ surface $X$. The following propositions are keys for Main Theorem (2).

**Proposition 2.3** ([12]). Assume that $\sigma$ acts trivially on $S_X$, hence $S_X = S_X^\sigma$. If $\Phi(7) = 6 = \text{rk} T_X$ then such a $K3$ surface is unique.

**Proposition 2.4** ([1] Theorem 6.3). Then the fixed locus $X^\sigma$ is of the form

\[ X^\sigma = \begin{cases} 
\{ P_1, P_2, P_3 \} \amalg E & \text{if } S_X^\sigma = U \oplus K_7, \\
\{ P_1, P_2, P_3 \} & \text{if } S_X^\sigma = U(7) \oplus K_7, \\
\{ P_1, P_2, \ldots, P_8 \} \amalg E \amalg \mathbb{P}^1 & \text{if } S_X^\sigma = U \oplus E_8, \\
\{ P_1, P_2, \ldots, P_8 \} \amalg \mathbb{P}^1 & \text{if } S_X^\sigma = U(7) \oplus E_8, \\
\{ P_1, P_2, \ldots, P_{13} \} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } S_X^\sigma = U \oplus E_8 \oplus A_6.
\end{cases} \]

Here $E$ is a non-singular curve of genus 1, $A_6$ or $E_8$ are the negative-definite root lattice of type $A_6$ or $E_8$ respectively. We denote by $U$ the even indefinite unimodular lattice of rank 2 and $U(7)$ the lattice whose bilinear form is the one on $U$ multiplied by 7. The even negative-definite lattice $K_7$ is given by Gram matrix

\[
\begin{pmatrix}
-4 & 1 \\
1 & -2
\end{pmatrix}
\]

Moreover the number of isolated fixed points of type $P^{2,6}$ (resp. $P^{3,5}$ or $P^{4,4}$) is $(\text{rk } S_X^\sigma + 2)/3$ (resp. $(\text{rk } S_X^\sigma - 1)/3$ or $(\text{rk } S_X^\sigma - 4)/6$).

In the following, we treat a pair $(X, \langle \sigma \rangle)$ whose the fixed locus $X^\sigma$ consists of smooth rational curves and isolated points, and contains at least 2 rational curves. We show that the pair $(X, \langle \sigma \rangle)$ is unique up to isomorphism.

**Proposition 2.5.** The automorphism $\sigma$ acts trivially on $S_X$.

**Proof.** Since $X^\sigma$ has at least 2 rational curves, $X^\sigma = \{ P_1, P_2, \ldots, P_{13} \} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1$ and $S_X^\sigma = U \oplus E_8 \oplus A_6$ by Proposition 2.4. We know that $\text{rk} T_X \geq 6$ by Lemma 2.1 (1) and $\text{rk} S_X \geq 16$ since it contains the invariant lattice $S_X^\sigma$ which is of rank 16. This gives $\text{rk} T_X \leq 6$ so that $\text{rk} T_X = 6$ and $\text{rk} S_X = 16$, hence $S_X$ coincides with $S_X^\sigma$. This implies that the action of $\sigma$ is trivial on the $S_X$. \hfill $\square$

The following Corollary follows from Proposition 2.5 and Proposition 2.4.

**Corollary 2.6.** Under the above hypothesis, $S_X = U \oplus E_8 \oplus A_6$, $T_X = U \oplus U \oplus K_7$ and the fixed locus $\sigma$ has 2 non-singular rational curves and 13 isolated points: $X^\sigma = \{ P_1, P_2, \ldots, P_{13} \} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1$.

We recall that the dimension of a moduli space of $K3$ surfaces with a non-symplectic automorphism of order 7 is $\text{rk} T_X/\Phi(7) - 1$ (see also [3, Section 11]). In our case, its dimension is 0. Indeed we have the following.

**Theorem 2.7.** A pair $(X, \langle \sigma \rangle)$ is unique up to isomorphism, hence Main Theorem (2) holds.

**Proof.** It follows from Proposition 2.5, Proposition 2.4 and Remark 2.2. \hfill $\square$

**Example 2.8** ([1] Example 6.1 (3)). Put

\[ X_{\text{AST}} : y^2 = x^3 + \sqrt[3]{-27/4} x + t^7 - 1, \quad \sigma_{\text{AST}}(x, y, t) = (x, y, \zeta_7 t). \]
Then $X_{\text{AST}}$ is a $K3$ surface with $S_{X_{\text{AST}}} = U \oplus E_8 \oplus A_6$ and $\sigma_{\text{AST}}$ is a non-symplectic automorphism of order 7. Note that $X_{\text{AST}}$ has one singular fiber of type $I_7$ over $t = 0$, one singular fiber of type $II^*$ over $t = \infty$ and 7 singular fibers of type $I_1$ over $t^7 = 1$.

**Example 2.9** ([4, (7.5)]). Put

$$X_{\text{Ko}} : y^2 = x^3 + t^3x + t^8, \quad \sigma_{\text{Ko}}(x, y, t) = (\zeta_7^2x, \zeta_7y, \zeta_7^3t)$$

Then $X_{\text{Ko}}$ is a $K3$ surface with $S_{X_{\text{Ko}}} = U \oplus E_8 \oplus A_6$ and $\sigma_{\text{Ko}}$ is a non-symplectic automorphism of order 7. Note that $X_{\text{Ko}}$ has one singular fiber of type $III^*$ over $t = 0$, one singular fiber of type $IV^*$ over $t = \infty$ and 7 singular fibers of type $I_1$ over $4 + 27t^7 = 0$. Moreover the rank of the Mordell-Weil group is 1.

**Remark 2.10.** The local actions of a non-symplectic automorphism of order 7 at the intersection points of the rational curves appear in the following order:

$$\cdots, \begin{pmatrix} 1 & 0 \\ 0 & \zeta_7^2 \end{pmatrix}, \begin{pmatrix} \zeta_7^2 & 0 \\ 0 & \zeta_7^2 \end{pmatrix}, \begin{pmatrix} \zeta_7^2 & 0 \\ 0 & \zeta_7^2 \end{pmatrix}, \begin{pmatrix} \zeta_7^2 & 0 \\ 0 & \zeta_7^2 \end{pmatrix}, \cdots$$

3. A correspondence between log Enriques surfaces and $K3$ surfaces

Let $Z$ be a log Enriques surface of index 7 and type $A_{15}$ without Du Val singularities, $\pi : Y \to Z$ the canonical covering of $Z$ and $f : X \to Y$ the minimal resolution. Note that $X$ is uniquely determined up to isomorphism. Recall that $X$ is a $K3$ surface (see also [14, Theorem 4.1]) and $\sigma$ is a non-symplectic automorphism of order 7 induced by $\pi$.

**Lemma 3.1.** Let $\Delta$ be the exceptional divisor of the minimal resolution $f$. Then every component of $\Delta$ is $\sigma$-stable.

**Proof.** Note that $\Delta$ is $\sigma$-stable and a liner chain of Dynkin type $A_{15}$. It follows from the fact that the order of symmetry of $\Delta$ is co-prime with 7.

**Proposition 3.2.** The pair $(X, \langle \sigma \rangle)$ is unique up to isomorphism.

**Proof.** Since $\pi$ is unramified over $Z \setminus \text{Sing}(Z)$, every fixed curve by $\sigma$ in $X$ is contained in $\Delta$. Hence $X^\sigma$ contains only smooth rational curves and isolated fixed points. On the other hand, each component of $\Delta$ has two isolated fixed points or is pointwisely fixed by $\sigma$ by Lemma 3.1.

We remark that $\Delta$ consists of 15 smooth rational curves. If $X^\sigma$ has one or less smooth rational curves then $X^\sigma$ has at least 14 or more isolated fixed points. This is a contradiction by Proposition 2.4. Then $X^\sigma$ consist of exactly two smooth rational curves, and the claim follows from Theorem 2.7.

By the Proposition, if $X^\sigma$ consists of only smooth rational curves and some isolated points and contains at least 2 rational curves then a pair $(X, \langle \sigma \rangle)$ corresponds to log Enriques surfaces of index 7 and type $A_{15}$. Hence we may identify it with the pair in Example 2.8 or Example 2.9. We construct a log Enriques surface of index 7 and type $A_{15}$ from a $K3$ surface with a non-symplectic automorphism 7 given by Example 2.8.
Example 3.3. We consider the pair \((X_{\text{AST}}, \langle \sigma_{\text{AST}} \rangle)\) in Example 2.8. Let \(f : X_{\text{AST}} \to Y\) be the contraction of the following rational tree \(\Delta_{\text{AST}}\) of Dynkin type \(A_{15}\) to a point \(Q:\)
\[
\Gamma_2 - \Gamma_3 - \Gamma_4 - \Gamma_5 - \Gamma_6 - \Gamma_7 - S - \Theta_1 - \Theta_2 - \Theta_3 - \Theta_4 - \Theta_5 - \Theta_6 - \Theta_7 - \Theta_8,
\]
where \(S\) is a cross-section, \(\Gamma_i\) is a component of a singular fiber of type \(I_7\) and \(\Theta_j\) is a component of a singular fiber of type \(II^*\). Here a singular fiber of type \(I_7\) is given by \(\sum_{i=1}^{7} \Gamma_i\) which \(\Gamma_7\) meets \(S\), and a singular fiber of type \(II^*\) is given by \(\sum_{j=1}^{6} j\Theta_j + 4\Theta_7 + 2\Theta_8 + 3\Theta_9\). Hence \(\Gamma_7\) and \(\Theta_6\) are fixed curves of \(\sigma_{\text{AST}}\).

Remark 3.4. We can fined 13 isolated fixed points and 2 fixed curves of \(\sigma_{\text{AST}}\) on these singular fibers by Remark 2.10. Note that \(\Theta_6\) is pointwisely fixed by \(\sigma_{\text{AST}}\). 6 isolated fixed points of type \(P_{2,6}\) are intersection points of \(\Gamma_1\) and \(\Gamma_2\), \(\Gamma_5\) and \(\Gamma_6\), \(S\) and \(\Theta_1\), \(\Theta_4\) and \(\Theta_5\), \(\Theta_7\) and \(\Theta_8\), and a point on \(\Theta_9\). 5 isolated fixed points of type \(P_{3,5}\) are intersection points of \(\Gamma_2\) and \(\Gamma_3\), \(\Gamma_4\) and \(\Gamma_5\), \(\Theta_1\) and \(\Theta_2\), \(\Theta_3\) and \(\Theta_4\), and a point on \(\Theta_8\). 2 isolated fixed points of type \(P_{4,4}\) are intersection points of \(\Gamma_3\) and \(\Gamma_4\), and \(\Theta_2\) and \(\Theta_3\). Then \(\Gamma_7\) is a fixed curve.

Remark 3.5. In [14, Example 6.13], we constructed a log Enriques surface of index 7 and type \(A_{15}\) that do not use \(K3\) surfaces. Of course we can also see it by contracting some divisors on the minimal resolution of the quotient surface \(X/\sigma\).

4. Sublattices of type \(A_{15}\)

Assume that a pair \((X, \langle \sigma \rangle)\) corresponds to a log Enriques surface of index 7 and type \(A_{15}\). In the following we write \(\Delta = \sum_{i=1}^{15} C_i\), which is of Dynkin type \(A_{15}\) and employ the same symbol \(\Delta\) for the sublattice of \(S_X\) generated by the irreducible component of \(\Delta\).

Since the rank of the orthogonal complement \(\Delta^\perp\) of \(\Delta\) in \(S_X\) is 1, we may write \(\Delta^\perp = \mathbb{Z}H\) and assume \(H\) is a nef and big divisor. In fact if \(X \to Y\) is the contraction of \(\Delta\) then the Picard number of \(Y\) is 1, thus we can take \(H\) as the pull back of the ample generator of \(S_Y\).
Lemma 4.1. The lattice $\Delta$ is a primitive sublattice of $S_X$.

Proof. Assume that the lattice $\Delta$ is not a primitive sublattice of $S_X$. Let $\Delta$ be the primitive closure of $\Delta$ in $S_X$. Since $16 = |\det(\Delta)| = |\Delta : \Delta|^2 |\det(\Delta)|$, we have $|\det(\Delta)| = 4$ and $|\Delta : \Delta| = 2$, or $|\det(\Delta)| = 1$ and $|\Delta : \Delta| = 4$.

If $|\det(\Delta)| = 1$, that is, $\Delta$ is unimodular then we have $|H^2| = |\det(S_X)| = 7$ because of Corollary 2.6 and $S_X = \Delta \oplus ZH$. This contradicts for the fact that $S_X$ is an even lattice. Then $|\det(\Delta)| = 4$ and $|\Delta : \Delta| = 2$.

Thus we can find a non-empty subset $J$ of $\{1, 2, \ldots, 15\}$ such that $\frac{1}{2} \sum_{j \in J} C_j$ is contained in $S_X$. Since $C_i$ is a non-singular rational curve, $\sum J = 8$ or 16 by [7] Lemma 3 or [6] Lemma 3.3. But there exists an element $i \in \{1, 2, \ldots, 15\}$ such that the intersection number $\left(\frac{1}{2} \sum_{j \in J} C_j\right) C_i$ is not an integer. This is a contradiction. □

Lemma 4.2. There exists an element $h$ in $S_X$ such that $S_X = \Delta + ZH$ and the followings hold:

1. The index $[S_X : \Delta \oplus \text{ZH}] = 16$ and $H^2 = 112$.
2. There exist integers $a_i$ such that $h = H/16 + \sum_{i=1}^{15} (a_i/16)C_i$.
3. Put $h_+ := (H + 3 \sum_{i=1}^{15} iC_i)/16$. Then $h \equiv h_+ (\text{mod } \Delta)$ and $\{C_1, C_2, \ldots, C_{15}, h_+\}$ is a $Z$-basis of $S_X$.

Proof. (1) (2) Set $n := [S_X : \Delta \oplus \text{ZH}]$. Since Corollary 2.6 and $|\det(\Delta \oplus \text{ZH})| = n^2 |\det(S_X)|$, it satisfies $16H^2 = 7n^2$. After replacing $h$ by $-h$ if necessary, we can find integers $a_i$ such that $H = nh - \sum_{i=1}^{15} a_iC_i$. Note that $(a_1/n, \ldots, a_{15}/n)$ is the unique solution of the liner system:

$$\left(\frac{h - \sum_{i=1}^{15} x_iC_i}{x_iC_i}\right) C_j = 0 \quad (j = 1, \ldots, 15).$$

Since the determinant of the Gramm matrix of $\Delta$, that is, $\det(C_i, C_j) = -16$, the numbers $16a_i/n$ are integers. Hence $16H/n = 16h - \sum_{i=1}^{15} (16a_i/n)C_i = rH$ for some integer $r$, so $n$ divides 16 ($= |\det(\Delta)|$). Since $S_X$ is an even lattice, $n = 8$ ($H^2 = 28$) or $n = 16$ ($H^2 = 112$).

Note that $\sum_{i=1}^{15} (a_iC_i) C_j = n(-h.C_j) \equiv 0 \pmod{n}$ for all $j$, hence

$$-2a_1 + a_2 \equiv 0, \quad a_{i-1} - 2a_i + a_{i+1} \equiv 0 \quad (i = 2, 3, \ldots, 14), \quad a_{14} - a_{15} \equiv 0 \pmod{n}.$$  

Thus $a_i \equiv ia_1$ for all $i = 1, 2, \ldots, 15$ and

$$\left(\frac{h - \frac{1}{n} \sum_{i=1}^{15} (ia_1 + a_i)C_i}{ia_1 + a_i}\right) C_i = \frac{1}{n^2} \left(\frac{H - a_1 \sum_{i=1}^{15} iC_i}{H - a_1 \sum_{i=1}^{15} iC_i}\right) \left(\frac{H - a_1 \sum_{i=1}^{15} iC_i}{H - a_1 \sum_{i=1}^{15} iC_i}\right)^2$$

$$= \frac{1}{n^2} \left(H^2 - 16 \times 15a_1^2\right) = \frac{7}{16} \times \frac{16 \times 15a_1^2}{n^2}$$

is an integer. This implies that $n = 16$ (and $a_1 \equiv \pm 3 \pmod{16}$).

(3) It follows from the definition of $h_+$. □

In order to prove Main Theorem (1), it suffices to show that $Z$ is isomorphic to the log Enriques surface $Z_{\text{AST}}$ in Example 3.3. Hence we show that there exist an automorphism $\varphi : X_{\text{AST}} \to X_{\text{AST}}$ such that $\varphi(\Delta) = \Delta_{\text{AST}}$ and $\varphi \circ \sigma_{\text{AST}} = \sigma_{\text{AST}} \circ \varphi$. 


Lemma 4.3. Write $\Delta_{\text{AST}} = \sum_{i=1}^{15} D_i$ the same way as $\Delta$. Put $\Delta_{\text{AST}}^\perp = \mathbb{Z}H_{\text{AST}}$ and $\Delta_{\text{AST}}^\perp = \mathbb{Z}H$ in $S_{\text{AST}}$. Then there exist an isometry $\Phi$ of the lattice $S_{\text{AST}}$ such that $\Phi(\Delta) = \Delta_{\text{AST}}$, $\Phi(H) = H_{\text{AST}}$ and $\Phi$ preserves the ample cone.

Proof. By Lemma 1.2, $h_+$ is uniquely and precisely expressed $H$ and $C_j$. Two natural isometries $\Phi_1 : \Delta \rightarrow \Delta_{\text{AST}}$ and $\Phi_2 : H \rightarrow H_{\text{AST}}$ can be extended to an isometry $\Phi : S_{\text{AST}} \rightarrow S_{\text{AST}}$ such that $\Phi(\Delta) = \Delta_{\text{AST}}$ and $\Phi(H) = H_{\text{AST}}$.

We note that $H - \sum_{i=1}^{15} \alpha_i C_i$ and its image by $\Phi$, i.e., $H_{\text{AST}} - \sum_{i=1}^{15} \alpha_i D_i$ are ample for some positive rational numbers $\alpha_i$. Hence $\Phi$ preserves the ample cone. □

Proposition 4.4. There exists an automorphism $\varphi$ on $X_{\text{AST}}$ which satisfies $\varphi(\Delta) = \Delta_{\text{AST}}$ and $\varphi \circ \sigma_{\text{AST}} = \sigma_{\text{AST}} \circ \varphi$.

Proof. Since the fixed locus of $\sigma_{\text{AST}}$ is contained $\Delta$ by Lemma 3.4 (see also the proof of Proposition 3.2), we may assume that $\Delta$ consists of components of singular fiber of type $I_7$, of type $I^*$ and a cross-section. Hence $\Delta$ is either

$\Gamma_2 - \Gamma_3 - \Gamma_4 - \Gamma_5 - \Gamma_6 - \Gamma_7 - S - \Theta_1 - \Theta_2 - \Theta_3 - \Theta_4 - \Theta_5 - \Theta_6 - \Theta_7 - \Theta_8,$

namely $\Delta_{\text{AST}}$, or

$\Gamma_5 - \Gamma_4 - \Gamma_3 - \Gamma_2 - \Gamma_1 - \Gamma_7 - S - \Theta_1 - \Theta_2 - \Theta_3 - \Theta_4 - \Theta_5 - \Theta_6 - \Theta_7 - \Theta_8.$

By specifying (essentially relabeling) components of a singular fiber of type $I_7$, we can find an automorphism $\varphi$ on $X_{\text{AST}}$ satisfying $\varphi^* | S_{\text{AST}} = \Phi$ in Lemma 4.3.

We remark that $\sigma_{\text{AST}}$ acts trivially on $S_{\text{AST}}$ by Proposition 2.5. Thus each action of $\varphi \circ \sigma_{\text{AST}}$ and $\sigma_{\text{AST}} \circ \varphi$ on $H^{1,1}(X_{\text{AST}})$ is determined by $\varphi$ only. Since $H^{2,0}(X_{\text{AST}})$ and $H^{0,2}(X_{\text{AST}})$ are both 1-dimensional, $(\varphi \circ \sigma_{\text{AST}})^* = (\sigma_{\text{AST}} \circ \varphi)^*$ on $H^2(X, \mathbb{C})$. Hence $\varphi \circ \sigma_{\text{AST}} = \sigma_{\text{AST}} \circ \varphi$ by the Torelli theorem. □

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