NILPOTENT VARIETIES IN SYMMETRIC SPACES AND TWISTED AFFINE SCHUBERT VARIETIES

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ABSTRACT

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(Under the direction of Hong, Jiuzu)

Let $G$ be a simply-connected simple algebraic group, $\mathfrak{g}$ its Lie algebra. Let $\text{Gr}_G$ be the affine Grassmannian of $G$, and $\text{Gr}_0^-$ the opposite open Schubert cell in $\text{Gr}_G$. Achar and Henderson defined a map $\pi : \text{Gr}^-_0 \to \mathfrak{g}$ and used it to exhibit the relation between Schubert varieties in $\text{Gr}_G$ and nilpotent cone in $\mathfrak{g}$. In our work, we consider a twisted analogue. We relate the geometry of Schubert varieties in twisted affine Grassmannian and the nilpotent varieties in symmetric spaces. This extends some results of Achar-Henderson in the twisted setting. We also get some applications for the geometry and topology of the order 2 nilpotent varieties in certain classical symmetric spaces.

In the appendix, we describe how nilpotent cone in $\mathfrak{gl}_n$ is embedded into the affine Grassmannian of type $A$ via the Lusztig map. We regard the nilpotent cone as the nilpotent cyclic quiver varieties of a single vertex. In the twisted setting with an automorphism $\sigma$ of order $n$, the nilpotent cone in the $\exp(\frac{2\pi i}{n})$-eigenspace can be regarded as the nilpotent cyclic quiver varieties of $n$ vertices which is mapped to the twisted affine Grassmannian. Besides, the dimensions of nilpotent orbits of highest weight vectors are computed. As a consequence, the minimal orbit is locally isomorphic to a quasi-minuscule twisted Schubert variety.
To my family.
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CHAPTER 1

Introduction

Let $G$ be a reductive group over $\mathbb{C}$. Let $N$ denote the nilpotent cone of the Lie algebra $\mathfrak{g}$ of $G$. Let $\text{Gr}_G$ be the affine Grassmannian of $G$. Each spherical Schubert cell $\text{Gr}_\lambda$ is parametrized by a dominant coweight $\lambda \in X_*(T)^+$. When $G = \text{GL}_n$, in [Lu] Lusztig defined an embedding from $N$ to $\text{Gr}_G$, and showed that each nilpotent variety in $\text{gl}_n$ can be openly embedded into certain affine Schubert variety $\text{Gr}_\lambda$. This embedding is called the Lusztig map whose definition and basis properties are given in the Appendix A.2. The Lusztig map identifies the geometry of nilpotent varieties and certain affine Schubert varieties in type $A$. However, there is no direct generalization for general reductive groups.

In [AH], Achar-Henderson took a different idea for a general algebraic simple group $G$. Let $\text{Gr}_G^{-}$ be the opposite open Schubert cell in $\text{Gr}_G$. One can naturally define a map $\pi : \text{Gr}_G^{-} \to \mathfrak{g}$. Achar-Henderson showed that $\pi(\text{Gr}_G^{-} \cap \text{Gr}_\lambda)$ is contained in $N$ if and only if $\lambda$ is small in the sense of Broer [Br] and Reeder [Re], i.e. $\lambda \not\geq 2\gamma_0$, where $\gamma_0$ is the highest short coroot of $G$. They also proved that $\pi : \text{Gr}_{\text{sm}} \cap \text{Gr}_G^{-} \to \pi(\text{Gr}_{\text{sm}} \cap \text{Gr}_G^{-})$ is a finite map whose fibers admits transitive $\mathbb{Z}_2$-actions, where $\text{Gr}_{\text{sm}}$ is the union of all $\text{Gr}_\lambda$ such that $\lambda$ is small. Moreover, with respect to $\pi$, Achar-Henderson [AH, AH2] related the geometric Satake correspondence and springer correspondence.

In this work, we consider a twisted analogue. Let $\sigma$ be a diagram automorphism of order 2, and let $\sigma$ acts on the field $K = \mathbb{C}((t))$ via $\sigma(t) = -t$ and $\sigma|_{\mathbb{C}} = \text{Id}_{\mathbb{C}}$. Then, we may define a twisted affine Grassmannian $\mathcal{G}r := G(K)^{\sigma}/G(O)^{\sigma}$, where $O = \mathbb{C}[[t]]$. Each twisted Schubert cell $\mathcal{G}r_\lambda$, i.e. a $G(O)^{\sigma}$-orbit, is parametrized by the image $\tilde{\lambda}$ of a dominant coweight $\lambda$ in the coinvariant lattice $X_*(T)^{\sigma}$ with respect to the induced action of $\sigma$. In fact, $X_*(T)^{\sigma}$ can be regarded as the weight lattice of a reductive group $H := (\tilde{\mathcal{G}})^{\sigma}$, where $\tilde{\mathcal{G}}$ is the Langlands dual group of $G$. In the Appendix B, we compute the dimension of certain minimal $K$-orbits. We also define the embedding from the nilpotent cone in the (-1)-eigenspace $\mathfrak{p}$ of $\sigma$ in $\mathfrak{g}$ into $\mathcal{G}r$ of the adjoint group. This embedding provides that each $K$-orbit can be open embedded into certain twisted Schubert cell. This inspire us to find the relationship between other $K$-orbits and twisted Schubert cells.

On the other hands, we may naturally define a map $\pi : \mathcal{G}r_G^{-} \to \mathfrak{p}$, where $\mathcal{G}r_G^{-}$ is the opposite open Schubert
cell in $Gr$. This map $\pi$ plays the important roles to prove the following main result of our work, and it can follow from Proposition 2.2.3 in Chapter 2.1 and Theorem 4.1.2 in Chapter 4.

**Theorem 1.0.1.** Assume that $G$ is of type $A_\ell$ or $D_{\ell+1}$. The image $\pi(Gr_\lambda \cap Gr_0^-)$ is contained in the nilpotent cone $N_\mathfrak{p}$ of $\mathfrak{p}$, if and only if $\lambda$ is a small dominant weight with respect to $H$.

Actually, in Theorem 4.1.2 we describe precisely $\pi(Gr_\lambda \cap Gr_0^-)$ as a union of nilpotent orbits in $\mathfrak{p}$. We also determine all $\lambda$ such that $\pi(Gr_\lambda \cap Gr_0^-)$ is a nilpotent orbit and $\pi : Gr_\lambda \cap Gr_0^- \to \pi(Gr_\lambda \cap Gr_0^-)$ is an isomorphism. Furthermore, We describe all fibers of $\pi : Gr_{\text{sm}} \cap Gr_0^- \to N_\mathfrak{p}$, where $Gr_{\text{sm}}$ is the union of all small twisted Schubert cells. In particular, we show that when $G$ is of type $A_{2\ell-1}$ (resp. $D_{\ell+1}$) the fiber $\pi^{-1}(0)$ is actually the minimal (resp. maximal) order 2 nilpotent variety in $\mathfrak{sp}_{2\ell}$ (resp. $\mathfrak{so}_{2\ell+1}$). This is a very different phenomenon with the untwisted setting in the work of Achar-Henderson [AH], and it actually makes the twisted setting more challenging.

For general simple Lie algebra $\mathfrak{g}$ and general diagram automorphism $\sigma$, it was proved in [HLR, Appendix C] by Haines-Lourenço-Richarz that, when $\lambda$ is quasi-miniscule and $\overline{O}$ is the minimal nilpotent variety in $\mathfrak{p}$, the map $\pi : \overline{Gr}_\lambda \cap \overline{Gr}_0^- \to \overline{O}$ is an isomorphism. In fact, we have also obtained this result independently, cf. [Ko]. Further detail is contained in the Appendix B. Also, under the same assumption as in Theorem 1.0.1, this isomorphism is a special case of our Theorem 4.2.1, Theorem 4.3.1, and Theorem 4.4.2.

The geometric Satake correspondence for $Gr$ was proved by Zhu [Zh], and it exactly recovers the Tannakian group $H$. On the other hand, the springer correspondence for symmetric spaces is more sophisticated than the usual Lie algebra setting, see a survey on this subject [Sh]. It would be interesting to relate these two pictures as was done in [AH, AH2].

From Theorem 1.0.1, we can deduce some applications for the order 2 nilpotent varieties in classical symmetric spaces. Let $\langle , \rangle$ be a symmetric or symplectic non-degenerate bilinear form on a vector space $V$. Let $\mathcal{A}$ be the space of self-adjoint linear maps with respect to $\langle , \rangle$. We consider $\mathfrak{sp}_{2n}$-action on $\mathcal{A}$ when $\langle , \rangle$ is symplectic and $\dim V = 2n$, and $\mathfrak{so}_n$-action when $\langle , \rangle$ is symmetric and $\dim V = n$. In Chapter 5, we obtain the following results.

**Theorem 1.0.2.**

1. If $\langle , \rangle$ is symmetric and $\dim V$ is odd, then any order 2 nilpotent variety in $\mathcal{A}$ is normal.

2. If $\langle , \rangle$ is symplectic, then there is a bijection of order 2 nilpotent varieties in $\mathfrak{so}_{2n+1}$ and in $\mathcal{A}$, such that they have the same cohomology of stalks of IC-sheaves.

3. If $\langle , \rangle$ is symplectic, the smooth locus of any order 2 nilpotent variety in $\mathcal{A}$ is the open nilpotent orbit.
It is known that when \((, )\) is symplectic, any nilpotent variety in \(\mathcal{A}\) is normal, but it is not always true when \((, )\) is symmetric, cf. [Oh]. Using our method, we can also prove that there is a bijection of order 2 nilpotent varieties in \(\mathfrak{sp}_{2n}\) and in the space of symmetric \((2n + 1) \times (2n + 1)\) matrices, such that they have the same cohomology of stalks of IC-sheaves. This was already proved earlier by Chen-Vilonen-Xue [CVX] using different methods. Also, Part 3) of Theorem 1.0.2 is not true when \((, )\) is symmetric and \(\dim V\) is odd, see more detailed discussions in Chapter 5.
CHAPTER 2

Notation and Preliminaries

In this chapter, we provide the definitions and basic properties of root datum and twisted affine Grassmannian. We construct the map $\pi$ and $\iota$ which play important roles in our work, especially in the Chapter 4. The definition of small dominant weight is also given. We give the proof of Proposition 2.2.3 which is a part of the main Theorem 1.0.1.

2.1 Root Datum

Let $G$ be a simply-connected simple algebraic group over $\mathbb{C}$, and let $\mathfrak{g}$ be its Lie algebra. Let $\sigma$ be a diagram automorphism of $G$ of order $r$, preserving a maximal torus $T$ and a Borel subgroup $B$ containing $T$ in $G$. Then $G$ has a root datum $(X_\ast(T), X^\ast(T), \langle \cdot, \cdot \rangle, \check{\alpha}_i, \alpha_i, i \in I)$ with the action of $\sigma$, where

- $X_\ast(T)$ (resp. $X^\ast(T)$) is the coweight (resp. weight) lattice;
- $I$ is the set of vertices of the Dynkin diagram of $G$;
- $\alpha_i$ (resp. $\check{\alpha}_i$) is the simple root (resp. coroot) for each $i \in I$;
- $\langle \cdot, \cdot \rangle : X_\ast(T) \times X^\ast(T) \to \mathbb{Z}$ is the perfect pairing.

The automorphism $\sigma$ of this root datum satisfies

- $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ and $\sigma(\check{\alpha}_i) = \check{\alpha}_{\sigma(i)}$;
- $\langle \sigma(\check{\lambda}), \sigma(\mu) \rangle = \langle \check{\lambda}, \mu \rangle$ for any $\check{\lambda} \in X_\ast(T)$ and $\mu \in X^\ast(T)$.

As a diagram automorphism on $G$, $\sigma$ also preserves a pinning with respect to $B$ and $T$, i.e. there exists root subgroups $x_i, y_i$ associated to $\alpha_i, -\alpha_i$ for each $i \in I$, such that

$$\sigma(x_i(a)) = x_{\sigma(i)}(a), \quad \sigma(y_i(a)) = y_{\sigma(i)}(a), \quad \text{for any } a \in \mathbb{C}.$$
Let \( I_\sigma \) be the set of \( \sigma \)-orbits in \( I \). Denote \( X'(T)^\sigma = \{ \lambda \in X'(T) \mid \sigma \lambda = \lambda \} \) and \( X_\sigma(T)^\sigma = X_\sigma(T)/(\text{Id} - \sigma)X_\sigma(T) \). For each \( i \in I_\sigma \), define \( \gamma_i = \tilde{\alpha}_i \in X_\sigma(T)^\sigma \) for any \( i \in t \), and define \( \tilde{\gamma}_i \in X'(T)^\sigma \) by

\[
\tilde{\gamma}_i = \begin{cases} 
\sum_{i \in t} \alpha_i & \text{if no pairs in } t \text{ is adjacent,} \\
2 \sum_{i \in t} \alpha_i & \text{if } t = \{i, \sigma(i)\} \text{ and } i \text{ and } \sigma(i) \text{ are adjacent,} \\
\alpha_i & \text{if } t = \{i\}.
\end{cases}
\]

Let \( \check{G} \) denote the Langlands dual group of \( G \), and we still denote the induced diagram automorphism on \( \check{G} \) by \( \sigma \). Denoted by \( H = (\check{G})^\sigma \) the \( \sigma \)-fixed subgroup of \( \check{G} \). Then, \( H \) has the root datum \( (X^*(T)^\sigma, X_\sigma(T)^\sigma, \tilde{\gamma}_i, \gamma_i, t \in I_\sigma) \), cf. [HS, Section 2.2]. For \( \lambda, \bar{\mu} \in X_\sigma(T)^\sigma \), define the partial order \( \bar{\mu} \leq \bar{\lambda} \) if \( \bar{\lambda} - \bar{\mu} \) is a sum of positive roots of \( H \). Let \( X_\sigma(T)^\sigma_+ \) be the set of dominant weight of \( H \). In fact, \( X_\sigma(T)^\sigma_+ \) is the image of the quotient map \( X_\sigma(T)^+ \to X_\sigma(T)^\sigma \), where \( X_\sigma(T)^+ \) is the set of dominant weights of \( G \).

### 2.2 Twisted Affine Grassmannian

Let \( \sigma \) be a diagram automorphism of \( G \) of order \( r \). Let \( O \) denote the set of formal power series in \( t \) with coefficients in \( \mathbb{C} \) and denote \( K \) the set of Laurent series in \( t \) with coefficients in \( \mathbb{C} \). Denote the automorphism \( \sigma \) of order \( r \) on \( K \) and \( O \) given by \( \sigma \) acts trivially on \( \mathbb{C} \) and maps \( t \to et \) where we fix the primitive \( r \)-root of unity \( \epsilon \). We consider the following twisted affine Grassmannian attached to \( G \) and \( \sigma \),

\[
\mathcal{Gr}_G = G(K)^\sigma / G(O)^\sigma.
\]

This space has been studied intensively in [BH, HR, PR, Ri]. The ramified geometric Satake correspondence [Zh] asserts that there is an equivalence between the category of spherical perverse sheaves on \( \mathcal{Gr}_G \) and the category of representations of the algebraic group \( H = (\check{G})^\sigma \). If there is no confusion, we write \( \mathcal{Gr} \) for convenience.

Let \( e_0 \) be the based point in \( \mathcal{Gr} \). For any \( \lambda \in X_\sigma(T) \), we attach an element \( t^1 \in T(K) \) naturally and define the norm \( n^1 \in T(K)^\sigma \) of \( t^1 \) by

\[
n^1 := \prod_{i=0}^{r-1} \sigma^i(t^1) = \epsilon^{\sum_{i=0}^{r-1} i \sigma^i(\lambda)t^1 \sigma^i \lambda}.
\]  

(2.1)

Let \( \tilde{\lambda} \) be the image of \( \lambda \in X_\sigma(T)^\sigma \). Set \( e_{\tilde{\lambda}} = n^1 \cdot e_0 \in \mathcal{Gr} \). Then \( e_{\tilde{\lambda}} \) only depends on \( \tilde{\lambda} \). Following [BH, Zh],

5
Gr admits the following Cartan decomposition

\[ Gr = \bigsqcup_{\lambda \in X(T)_r^+} Gr_{\lambda}, \tag{2.2} \]

where \( Gr_{\lambda} = G(O)\sigma \cdot e_{\lambda} \) is a Schubert cell. Let \( \overline{Gr}_{\lambda} \) be the closure of \( Gr_{\lambda} \). Then

\[ \overline{Gr}_{\lambda} = \bigsqcup_{\bar{\mu} \preceq \lambda} Gr_{\bar{\mu}}, \]

and \( \dim \overline{Gr}_{\lambda} = \langle 2\rho, \lambda \rangle \), where \( \rho \) is the half sum of all positive coroots of \( H \).

By abuse of notation, we still use \( \sigma \) to denote the induced automorphism on \( g \) of order \( r \). Then there is a grading on \( g \),

\[ g = g_0 \oplus g_1 \oplus \cdots \oplus g_{r-1} \]

where \( g_i \) is the \( e^i \)-eigenspace. Set

\[ p = g_1. \]

Set \( O^- = \mathbb{C}[t^{-1}] \). Consider the evaluation map \( ev_{\infty} : G(O^-) \to G \). Let \( G(O^-)_0 \) denote its kernel. The map \( ev_{\infty} \) factors through \( G(\mathbb{C}[t^{-1}]/(t^{-2})) \to G \). Note that the kernel of \( G(\mathbb{C}[t^{-1}]/(t^{-2})) \to G \) is canonical identified with the vector space \( g \otimes t \) with respect to the adjoint action of \( G \) and \( \sigma \). It induces a \( G \times \langle \sigma \rangle \)-equivariant map

\[ G(O^-)_0 \to g \otimes t^{-1}. \]

Taking \( \sigma \)-invariance, we get a \( K \)-equivariant map

\[ G(O^-)_0^\sigma \to p, \tag{2.3} \]

where \( K := G^\sigma \). Note that \( K \) is a connected simply-connected simple algebraic group, as \( G \) is simply-connected.

Set \( \overline{Gr}_0^- := G(O^-)^\sigma \cdot e_0 = G(O^-)_0^\sigma \). Then \( \overline{Gr}_0^- \) is the open opposite Schubert cell in \( Gr \). From (2.3), we have the following \( K \)-equivariant map

\[ \pi : \overline{Gr}_0^- \to p. \tag{2.4} \]
Lemma 2.2.1. \( \mathcal{G} \triangledown \cap \mathcal{G} \triangledown \) is nonempty for any \( \lambda \in X_\ast(T)_{\triangledown \sigma}. \)

Proof. It suffices to show that \( G(O)^{\sigma \ast} n^1 G(O)^{\sigma \ast} \cap G(O^-)^{\sigma \ast} \neq \emptyset. \) Let \( I \) be the Iwahori subgroup contained in \( G(O)^{\sigma \ast} \) and let \( I^- \) be the opposite Iwahori subgroup contained in \( G(O^-)^{\sigma \ast}. \) It suffices to show that \( I n^1 I \cap I^- \neq 0. \)

Let \( \mathcal{G} \) be the Kac-Moody group associated to the twisted loop group \( G(K)^{\sigma \ast}, \) see a construction in [BH, p.14]. There is a projection map \( p: \mathcal{G} \rightarrow G(K)^{\sigma \ast}. \) Let \( I \) (resp. \( I^- \)) be the preimage of \( I \) (resp. \( I^- \)) via the projection \( p. \) Then we are reduced to show that \( I w I \cap I^- \neq 0, \) where \( w \) is an element of the Weyl group \( \mathcal{W} \) of \( \mathcal{G}. \) This is true, since \( I s I \cap I^- \neq 0 \) for any simple reflection \( s \in \mathcal{W} \), and for any \( y \in \mathcal{W} \) and simple reflection \( s, I y I s I = I y s I \) if \( \ell(y s) = \ell(y) + 1, \) cf. [Ku, 5.1.3 (d)]. \( \square \)

Following [Br, Re, AH], an element \( \tilde{\lambda} \) of \( X_\ast(T)_{\triangledown \sigma} \) is called small, if \( \tilde{\lambda} \not\in 2\gamma_0, \) where \( \gamma_0 \) is the highest short root of \( H. \) The set of all small dominant weights is a lower order ideal of \( X_\ast(T)_{\triangledown \sigma}, \) i.e., if \( \tilde{\mu} \preceq \tilde{\lambda} \) and \( \tilde{\lambda} \) is small, then \( \tilde{\mu} \) is also small. Let \( \mathcal{G} \triangledown \) be the union of \( \mathcal{G} \triangledown \) for small dominant weights \( \tilde{\lambda}. \) Set

\[
\mathcal{M} = \mathcal{G} \triangledown \cap \mathcal{G} \triangledown. 
\]

For each small dominant weight \( \tilde{\lambda}, \) set

\[
\mathcal{M}_{\tilde{\lambda}} = \mathcal{G} \triangledown \cap \mathcal{G} \triangledown. 
\]

Let \( \mathcal{N}_p \) denote the nilpotent cone of \( p. \) We shall prove in Chapter 4 that \( \pi(\mathcal{M}) \) is contained \( \mathcal{N}_p, \) when \( G \) is of type \( A_n \) and \( D_n \) and \( \sigma \) is of order 2.

Recall that \( \gamma_0 \) is the highest short root of \( H. \) The following lemma is a twisted analogue of [AH, Lemma 3.3].

Lemma 2.2.2. If \( \sigma \) is a diagram automorphism of order \( r, \) then \( \pi(\mathcal{G} \triangledown \cap \mathcal{G} \triangledown) \not\in \mathcal{N}_{\gamma_1}. \)

Proof. Let \( X_N \) be the Dynkin diagram of \( G. \) Following [Ka, p.128-129], we choose the following root of \( G,
\[
\theta_0 = \begin{cases} 
\alpha_1 + \cdots + \alpha_{2\ell-2}, & (X_N, r) = (A_{2\ell-1}, 2); \\
\alpha_1 + \cdots + \alpha_{2\ell}, & (X_N, r) = (A_{2\ell}, 2); \\
\alpha_1 + \cdots + \alpha_{\ell}, & (X_N, r) = (D_{\ell+1}, 2); \\
\alpha_1 + \alpha_2 + \alpha_3, & (X_N, r) = (D_4, 3); \\
\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, & (X_N, r) = (E_6, 2). 
\end{cases}
\]
where the label of simple roots $\alpha_i$ follows from [Ka, TABLE Fin, p.53]. Recall from the Chapter 2.1 that for each $i \in I_\sigma$, we define simple roots of $H$, $\gamma_i = \tilde{\alpha}_i \in X_*(T)_\sigma$.

Let $\tilde{\theta}_0$ be the coroot of $\theta_0$. Then,

$$\tilde{\theta}_0 = \begin{cases} 
\gamma_0 & \text{if } (X_N, r) \neq (A_2 \ell, 2) \\
2\gamma_0 & \text{if } (X_N, r) = (A_2 \ell, 2) 
\end{cases}.$$

Suppose $(X_N, r) \neq (A_2 \ell, 2)$. Note that $\theta_0 \in X^*(T)$ and $\tilde{\theta}_0 : \mathbb{C}^\times \to T$. Each $a \in \mathbb{C}^\times$ can be identified with

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2. \text{ For each } i = 0, ..., r - 1, \text{ define a homomorphism } \phi_{\sigma^i(\tilde{\theta}_0)} : SL_2 \to G \text{ given by }$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_{\sigma^i(\tilde{\theta}_0)}(a), \quad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mapsto y_{\sigma^i(\tilde{\theta}_0)}(a), \quad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \sigma^i(\tilde{\theta}_0)(a).$$

Let $S$ be the product of $r$ copies of $SL_2$. Then $\tilde{\theta}_0$ can be extended to $\phi : S \to G$ given by

$$\phi(g_0, ..., g_{r-1}) = \prod_{i=0}^{r-1} \phi_{\sigma^i(\tilde{\theta}_0)}(g_i).$$

This $\phi$ can extend scalar to $\mathcal{K}$. Abusing notation, define $\sigma : \prod_{i=1}^r (SL_2(\mathcal{K})) \to \prod_{i=1}^r (SL_2(\mathcal{K}))$ by

$$\sigma(g_1(t), g_2(t), ..., g_r(t)) = (g_r(\epsilon t), g_1(\epsilon t), ..., g_{r-1}(\epsilon t)).$$

There exists an isomorphism

$$\varphi : SL_2(\mathcal{K}) \to (\prod_{i=1}^r (SL_2(\mathcal{K})))^r = \{(g(t), g(\epsilon t), ..., g(\epsilon^{r-1} t)) | g(t) \in SL_2(\mathcal{K})\}.$$

Hence

$$\phi \circ \varphi : \begin{pmatrix} t \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} t \epsilon t \\ 0 \end{pmatrix} \mapsto \prod_{i=0}^{r-1} (\epsilon^t)^{\sigma^i(\tilde{\theta}_0)} = n^{\tilde{\theta}_0}.$$

Let $s$ be the product of $r$ copies of $sl_2$. Define $\sigma : s \to s$ by

$$\sigma(x_1, ..., x_{r-1}, x_r) = (\epsilon x_r, \epsilon x_1, ..., \epsilon x_{r-1}).$$
Since \( \sigma \) has order \( r \), we have \( s = \oplus_{i=0}^{r-1} s_i \) where \( s_i \) is the eigenspace of eigenvalue \( \epsilon^i \). Then \( s_1 = \{(x, \epsilon x, \ldots, \epsilon^{r-1} x) \mid x \in s_{12}\} \cong s_{12} \). The derivative of \( \phi \) is \( \frac{d\phi}{dt} : s \rightarrow \mathfrak{g} \) which induces \( s_1 \rightarrow \mathfrak{g}_1 \). Hence we have the map \( \Psi : sl_2 \rightarrow \mathfrak{g}_1 \).

Consider the matrix \( g(t) \in SL_2(O^-) \),
\[
g(t) = \begin{pmatrix} 1 + t^{-1} & t^{-2} \\ t^{-1} & 1 - t^{-1} + t^{-2} \end{pmatrix}
= \begin{pmatrix} 0 & 1 \\ -1 & t^2 - t + 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^2 + t & 1 \end{pmatrix}.
\]

Then \( (\phi \circ \varphi)(g(t)) \in G(O) \cap G(O)^\sigma \). Since \( G \) is not type \( A_2 \), \( \tilde{\theta}_0 = \gamma_0 \) and then \( (\phi \circ \varphi)(g(t)) \cdot e_0 \in Gr_{2\gamma_0} \cap Gr_{\gamma_0} \).

We have the commutative diagram
\[
\begin{array}{ccc}
Gr_{SL_2,0} & \xrightarrow{g(t)-L_0} & Gr_{S,0} \\
\downarrow \pi_{SL_2} & & \downarrow \pi \\
sl_2 & \xrightarrow{a \mapsto (x, \epsilon x, \ldots, \epsilon^{r-1} x)} & \mathfrak{sl}_2 \\
\end{array}
\]

where \( Gr_{SL_2,0} := SL_2(O^-) \cdot e_0 \subset Gr_{SL_2} \), and \( Gr_{S,0} \) is defined similarly. The commutativity follows from
\[
\pi((\phi \circ \varphi)(g(t)) \cdot e_0) = \Psi(\pi_{SL_2}(g(t) \cdot e_0)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where the latter is not nilpotent. It follows that, \( \pi(Gr_{2\gamma_0}) \not\subseteq N_p \).

Suppose that \( (X_N, r) = (A_2, 2) \). In this case, \( \tilde{\theta}_0 = 2\gamma_0 \) and \( \sigma(\tilde{\theta}_0) = \tilde{\theta}_0 \). Then \( \tilde{\theta}_0 \) can be extended to \( \phi : SL_2 \rightarrow G \) defined by
\[
\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_{\theta_0}(a), \quad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mapsto y_{\theta_0}(a), \quad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \tilde{\theta}_0(a).
\]

\( \phi \) can extend the scalar to \( K \). Define a group homomorphism \( \sigma : SL_2(K) \rightarrow SL_2(K) \) by
\[
\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \mapsto \begin{pmatrix} a(-t) & -b(-t) \\ -c(-t) & d(-t) \end{pmatrix}
\]

where \( a(t) \in K \). Then \( \phi : SL_2(K) \rightarrow G(K) \) is \( \sigma \)-equivariant. The induced homomorphism \( \sigma : sl_2 \rightarrow sl_2 \) is
given by

\[
\begin{pmatrix}
  a & b \\
  c & -a
\end{pmatrix} \mapsto \begin{pmatrix}
  a & -b \\
  -c & -a
\end{pmatrix}.
\]

The derivative \(d\phi: sl_2 \to g\) induces the map \(\Psi: (sl_2)_1 \to g_1\). Similarly to the above argument, we have the commutative diagram

\[
\begin{array}{ccc}
Gr_{\SL_2,0} & \longrightarrow & Gr_{G,0} \\
\pi_{\SL_2} \downarrow & & \downarrow \pi \\
(sl_2)_1 & \longrightarrow & g_1
\end{array}
\]

where \((sl_2)_1\) is the eigenspace of eigenvalue \(-1\) under \(\sigma\). Now consider \(g(t) \in \SL_2(O)^\sigma\)

\[
g(t) = \begin{pmatrix}
  1 & t^{-1} \\
  t^{-1} & 1 + t^{-2}
\end{pmatrix} = \begin{pmatrix}
  0 & t \\
  -t^{-1} & 1 + t^2
\end{pmatrix} \begin{pmatrix}
  t^2 & 0 \\
  0 & t^2
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}.
\]

Then \(\phi(g(t)) \in G(O)^\sigma n_{R_\lambda} G(O)^\sigma\) and \(\phi(g(t)) \cdot e_0 \in Gr_{2\gamma_0} \cap Gr_0^-\). The result follows from

\[
\pi(\phi(g(t)) \cdot e_0) = \Psi(\pi_{\SL_2}(g(t) \cdot e_0)) = \Psi \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}
\]

where the latter is not nilpotent. It also follows that, \(\pi(Gr_{2\gamma_0}) \notin N_p\). \(\square\)

**Proposition 2.2.3.** For \(\bar{\lambda} \in X_*(T)_+\), if \(\pi(Gr_{\lambda} \cap Gr_0^-) \subset N_p\), then \(\bar{\lambda}\) is small.

**Proof.** Since \(Gr_0^-\) is an open subset of \(Gr\), \(\pi(Gr_{\lambda} \cap Gr_0^-) \subset N_p\). By Lemma 2.2.2, \(Gr_{2\gamma_0} \notin \overline{Gr_{\lambda}}\) which means \(\bar{\lambda} \notin 2\gamma_0\). \(\square\)

Define the following anti-involution

\[
\iota: G(K) \to G(K), \quad g(t) \mapsto g(-t)^{-1}.
\]

It can be checked that \(\iota\) commutes with \(\sigma\), and \(\iota\) preserves \(G(K)^\sigma, G(O)^\sigma\) and \(K^-\). This induces the map

\[
\iota: Gr_0^- \to Gr_0^-\], \quad g(t) \cdot e_0 \mapsto g(-t)^{-1} \cdot e_0.
\]
The following lemma will be used in Chapter 4.

**Lemma 2.2.4.** For $\lambda \in X_*(T)^+_\tau$, $\iota(M_{\lambda}) \subset M_{\lambda}$.

**Proof.** It suffices to prove $\iota(n^\lambda) \in M_{\lambda}$ for each $\lambda \in X_*(T)^+$. 

\[
\iota(n^\lambda) = \iota(e^{(\sigma \lambda + 2\sigma^2 \lambda + \ldots + (r-1)\sigma^{r-1} \lambda \sum_{i=0}^{r-1} \sigma^i \lambda)}
= \epsilon^{-((\sigma \lambda + 2\sigma^2 \lambda + \ldots + (r-1)\sigma^{r-1} \lambda)(-1)\sum_{i=0}^{r-1} \sigma^i \lambda)}
= (-1)^{\sum_{i=0}^{r-1} \sigma^i \lambda} n^{-\lambda}.
\]

Since $(-1)^{\sum_{i=0}^{r-1} \sigma^i \lambda}$ is fixed by $\sigma$, $\iota(n^\lambda) \in G(\mathcal{O})^\sigma n^{-\lambda} G(\mathcal{O})^\sigma$. Let $W$ be the Weyl group of $G$ with respect to the maximal torus $T$ and $\omega_0$ the longest element of $W$. We can choose a representative $\dot{\omega}_0 \in G$ of $\omega_0$ such that $\sigma(\dot{\omega}_0) = \dot{\omega}_0$, cf. [HS, Section 2.3].

When $G$ is of type $D_{2\ell}$ with $\ell \geq 2$, $w_0 = -1$; otherwise, $w_0 = -\sigma$ and $\sigma$ is of order 2, cf. [Hu2, Ex 5, p.71]. If $w_0 = -1$, it is easy to see that $n^{-\lambda} = w_0 n^\lambda w_0^{-1}$. If $w_0 = -\sigma$ and $\sigma$ has order 2,

\[
n^{-\lambda} = (-1)^{-\sigma \lambda} (-1)^{\lambda + \sigma \lambda} = (-1)^{w_0 \lambda} (-1)^{\lambda + \sigma \lambda} \left( w_0 n^\lambda \right) w_0^{-1} = w_0 n^\lambda w_0^{-1}.
\]

In any case, $\iota(n^\lambda) \in G(\mathcal{O})^\sigma n^\lambda G(\mathcal{O})^\sigma$. $\square$
CHAPTER 3

Nilpotent Orbits in the Space of Self-adjoint Maps

In this chapter, we will review some facts on the nilpotent orbits in certain symmetric spaces. These results are known, cf. [Se]. We provide proofs here, as the proofs in [Se] are omitted.

3.1 Classification of Nilpotent Orbits

Let \( B = \langle \cdot, \cdot \rangle \) be a nondegenerate symmetric or skew-symmetric bilinear form on a vector space \( V = \mathbb{C}^m \) and \( \mathcal{A} \) the set of self-adjoint linear maps under the bilinear form. In this chapter, we describe the classification of nilpotent orbits in the space \( \mathcal{A} \) in Theorem 3.1.3, Theorem 3.1.4 and Theorem 3.1.6.

The isometry group of the form \( B \) is

\[
I_B = \{ g \in \text{GL}(V) \mid \langle gu, gv \rangle = \langle u, v \rangle \text{ for all } u, v \in V \},
\]

whose Lie algebra is

\[
\mathfrak{g}_B := \{ X \in \text{sl}(V) \mid \langle Xu, v \rangle + \langle u, Xv \rangle = 0 \text{ for all } u, v \in V \}. \tag{3.1}
\]

When \( B \) is symplectic, \( \dim V \) is even, \( I_B \cong \text{Sp}_{2n} \) and \( \mathfrak{g}_B \cong \mathfrak{sp}_{2n} \) where \( m = 2n \). When \( B \) is symmetric, \( I_B \cong O_m \) and \( \mathfrak{g}_B \cong \mathfrak{so}_m \).

The group \( I_B \) acts on the space of self-adjoint linear maps

\[
\mathcal{A} = \{ X \in \text{End}(V) \mid \langle Xu, v \rangle = \langle u, Xv \rangle \text{ for all } u, v \in V \} \tag{3.2}
\]

by conjugation. The orbit is called nilpotent if it is the orbit of a nilpotent element of \( \mathcal{A} \).

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra. Suppose that \( \mathfrak{g} \) has \( \mathbb{Z}_m \)-grading

\[
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i
\]

so that \( [\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell} \). We have the following graded version of Jacobson-Morozov theorem.
Lemma 3.1.1. Let $X$ be a nonzero nilpotent element in $\mathfrak{g}$. There exists an $\mathfrak{sl}_2$-triple $H, X, Y$ such that $H \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_{-i}$.

Proof. This Lemma follows from the usual Jacobson-Morozov Theorem, and the proof is similar to [EK, Lemma 1.1]. □

For each $A \in \mathfrak{sl}(V)$, as a linear map, we denote its adjoint by $A^*$ under the form $B$. We define an involution $\sigma$ on $\mathfrak{g} = \mathfrak{sl}(V)$ by

$$\sigma(A) = -A^*.$$ (3.3)

Then $\mathfrak{g}$ is the direct sum of eigenspaces, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Thus $\mathfrak{g}_0 = \mathfrak{g}_B$ and $\mathfrak{g}_1 = \mathfrak{A}$. Fix a nonzero nilpotent element $X \in \mathfrak{A}$. By Lemma 3.1.1, there exists $Y \in \mathfrak{A}$ and $H \in \mathfrak{g}_B$, such that $X, Y, H$ is an $\mathfrak{sl}_2$-triple. This induces a representation of $\mathfrak{sl}_2$ on $V$ and hence we have a decomposition

$$V = \bigoplus_{r \geq 0} M(r)$$ (3.4)

where $M(r)$ is a finite direct sum of irreducible representation of $\mathfrak{sl}_2$ of highest weight $r$. For $r \geq 0$, let $H(r)$ be the highest weight space in $M(r)$. Define a new bilinear form $(\cdot, \cdot)_r$ on $H(r)$ by

$$(u, v)_r = \langle u, Y^r v \rangle.$$ 

Lemma 3.1.2. For any $r \geq 0$, $(\cdot, \cdot)_r$ is symplectic (resp. symmetric) if $B$ is symplectic (resp. symmetric).

Proof. We assume $B$ is symplectic. The proof is similar when $B$ is symmetric. It is easy to see that $(\cdot, \cdot)_r$ is skew-symmetric. It remains to show that $(\cdot, \cdot)_r$ is nondegenerate. Let $V_r$ be an $r$-weight space in $\mathbb{C}^{2n}$. For any $u \in V_r$, $v \in V_s$ with $s \neq -r$,

$$(r + s)(u, v) = \langle ru, v \rangle + \langle u, sv \rangle = \langle Hu, v \rangle + \langle u, Hv \rangle = 0$$

This implies that $V_r$ and $V_s$ are $(\cdot, \cdot)$-orthogonal. Let

$$W = \text{Span}\{u \in V_r \mid u = Yv \text{ for some } v \in \mathbb{C}^{2n}\}.$$
It can be seen that $V_r = H(r) \oplus W$. For $u \in H(r)$ and $v \in W$, write $v = Yv'$.

$$(u, v)_r = \langle u, Y^r v \rangle = \langle u, Y^{r+1} v' \rangle = \langle Y^{r+1} u, v' \rangle = 0.$$  

Hence $H(r)$ is $(\cdot, \cdot)_r$-orthogonal to $W$.

We claim that $\langle \cdot, \cdot \rangle : (Y^r \cdot H(r)) \times H(r) \rightarrow \mathbb{C}$ is nondegenerate. Let $u = Y^r u' \in Y^r \cdot H(r)$ be such that $\langle u, v \rangle = 0$ for all $v \in H(r)$. For each $w \in \mathbb{C}^{2n}$, write $w = \sum_s w_s$ where each $w_s$ belongs to $V_s$. Since $u \in V_r$, $\langle u, w_s \rangle = 0$ for $s \neq r$. Write $w_r = w_1 + w_2$ where $w_1 \in H(r)$ and $w_2 = Yw'_2 \in W$. By the assumption $\langle u, w_1 \rangle = 0$ and hence

$$\langle u, w_r \rangle = \langle u, w_2 \rangle = \langle Y^r u', Yw'_2 \rangle = \langle Y^{r+1} u', w'_2 \rangle = 0.$$  

We obtain $\langle u, w \rangle = 0$ for any $w$ and hence $u = 0$. This claim implies that $(\cdot, \cdot)_r$ is nondegenerate. □

A partition of a positive integer is denoted by a tuple $[d_1, d_2, \ldots, d_k]$ of positive integers. We use the exponent notation $[a_1^{i_1}, \ldots, a_r^{i_r}]$ to denote a partition where $a_j^{i_j}$ means there are $i_j$ copies of $a_j$. For example, $[3^2, 1^4] = [3, 3, 1, 1, 1, 1]$ is a partition of 10. Put $r_i = |\{ j \mid d_j = i \}|$ and $s_i = |\{ j \mid d_j \geq i \}|$. In fact, each partition can be illustrated by Young diagram and then $s_i$ is the $i$-th part of the dual diagram. The following Theorem gives the parametrization of nilpotent $I_B$-orbits in $\mathbb{A}$.

**Theorem 3.1.3.** There exists one-to-one correspondences

$$\{\text{nilpotent } \text{Sp}_{2n}\text{-orbits in } \mathbb{A}\} \leftrightarrow \left\{ \begin{array}{l} \text{partitions of } 2n \text{ such that} \\ \text{every part occurs with even multiplicity} \end{array} \right\}.$$  

and

$$\{\text{nilpotent } \text{O}_m\text{-orbits in } \mathbb{A}\} \leftrightarrow \{\text{partitions of } m\}.$$  

**Proof.** The proof is similar to [CM, Lemma 5.1.17]. For the case that $B$ is symplectic, it suffices to show that any nilpotent element in $\mathbb{A}$ gives arise the partition of $2n$ such that every part occurs with even multiplicity. Given nilpotent $X \in \mathbb{A}$, a number of Jordan blocks of size $r + 1$ equals to the multiplicity of $M(r)$ in $\mathbb{C}^{2n}$ which is exactly $\dim H(r)$. By Lemma 3.1.2, $\dim H(r)$ is even for every $r$. 

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If $B$ is symmetric, there are no constraints on $\dim H(r)$ which means there are no conditions on partitions of $m$. □

**Theorem 3.1.4.** There exists one-to-one correspondence

\[
\{\text{nilpotent } SO_{2n+1}-\text{orbits in } \mathcal{A}\} \leftrightarrow \{\text{partitions of } 2n+1\}.
\]

**Proof.** Since $O_{2n+1} = SO_{2n+1} \times \{\pm I_{2n+1}\}$, the orbits under $O_{2n+1}$ and $SO_{2n+1}$ coincide. The results immediately follows from Theorem 3.1.3 □

Consider the case that $B$ is symmetric and $m = 2n$. Given nilpotent elements $X, X' \in \mathcal{A}$ whose partitions are the same and have at least one odd part. Say that they are conjugated by an element $g \in O_{2n}$. If $\det g = 1$, we conclude that $X, X'$ are in the same $SO_{2n}$-orbits. Suppose that $\det g = -1$. We modify this $g$ so that it has determinant 1. Note that either $X$ or $X'$ gives the same decomposition (3.4). An odd part in the partition corresponds to an odd dimensional irreducible representation $S$ of $sl_2$ in $\mathbb{C}^{2n}$. We put $h = g$ except that $h(v) = -g(v)$ for $v \in S$. Therefore, $\det h = 1$, and $X$ and $X'$ are conjugated by $h$. If there is no odd parts, we need the following Lemma.

**Lemma 3.1.5.** Let $X$ be a nilpotent element in $\mathcal{A}$ whose partition contains only even parts, and $k \in O_{2n}$ such that $k \cdot X = kXk^{-1} = X$. Then $\det k = 1$.

**Proof.** Let $O^X_{2n}$ be the stabilizer group of $O_{2n}$ at $X$. Then $k \in O^X_{2n}$. By multiplicative Jordan decomposition, cf. [Bo, Theorem 4.4, p.83], let $k_s \in O^X_{2n}$ be the semisimple part of $k$. Then $\det k_s = \det k$. Hence we may assume that $k$ is semisimple. Let $\sigma$ be an automorphism on $\mathfrak{g} = sl(V)$ defined by (3.3). Then $\sigma$ commutes with $Adk$ on $\mathfrak{g}$, as $k \in O_{2n}$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and we have the decomposition of $k$-stabilizers $\mathfrak{g}^k = \mathfrak{g}_0^k \oplus \mathfrak{g}_1^k$ where $\mathfrak{g}_0^k = \mathfrak{g}_0 \cap \mathfrak{g}^k$. Since $\mathfrak{g}^k$ is reductive and $X \in \mathfrak{g}_1^k$, by Lemma 3.1.1, there exists an $sl_2$-triple $H, X, Y$ such that $X, Y \in \mathfrak{g}_1^k$ and $H \in \mathfrak{g}_0^k$ and hence we have the decomposition (3.4). It is easy to see that $k(M(r)) \subset M(r)$, and also, $k$ stabilizes each weight space of $M(r)$.

Recall that $\langle \cdot, \cdot \rangle$ is a nondegenerate symmetric form on $V = \mathbb{C}^{2n}$ and a form on $H(r)$ given by $(u, v)_r = \langle u, Y^r v \rangle$ is also symmetric for any $r \geq 0$. For any $u, v \in H(r)$,

\[
(ku, kv)_r = \langle ku, Y^r(kv) \rangle = \langle ku, kY^r(v) \rangle = \langle u, Y^r(v) \rangle = (u, v)_r.
\]
Hence \( k \mid_{H(r)} \in O(H(r)) \) for any \( r \). In particular \( \det \left( k \mid_{H(r)} \right) = \pm 1 \).

Let \( M(r) \ell \) be an \( \ell \)-weight space, and \( L(r) \) the lowest weight space in \( M(r) \). Observe that \( X \mid_{M(r) \ell} \) is an isomorphism from \( M(r) \ell \) to \( M(r) \ell+2 \) and the diagram

\[
\begin{array}{ccc}
L(r) & \longrightarrow & M(r) \ell \\
\downarrow k \mid_{L(r)} & & \downarrow k \mid_{M(r) \ell} \\
L(r) & \longrightarrow & M(r) \ell \\
\end{array}
\begin{array}{ccc}
& X \mid_{M(r) \ell} & \longrightarrow M(r) \ell+2 \\
& \downarrow k \mid_{M(r) \ell+2} & \downarrow k \mid_{M(r) \ell+2} \\
& \longrightarrow H(r) & \longrightarrow H(r)
\end{array}
\]

commutes. Then \( k \mid_{M(r) \ell} \) has the same determinant for all \( \ell \). Since \( X \) has only even parts, a number of weight spaces in \( M(r) \) is even for each \( r \). Then

\[
\det k = \prod_r \det \left( k \mid_{M(r)} \right) = \prod_r \prod_\ell \det \left( k \mid_{M(r) \ell} \right) = \prod_r \left( \det \left( k \mid_{H(r)} \right) \right) = 1
\]

as desired. \( \square \)

Now suppose that \( X, X' \in A \) have the same partition and contain only even parts. If \( \det g = 1 \), they are in the same \( SO_{2n} \)-orbits. Suppose that \( \det g = -1 \) and they are conjugated by another element \( h \in SO_{2n} \). Say \( g \cdot X = X' = h \cdot X \) and let \( k = g^{-1} h \). Then \( \det k = (\det g^{-1}) (\det h) = -1 \) but this contradicts to Lemma 3.1.5. In this case, it means \( X, X' \) are conjugated by an element in \( O_{2n} \) of determinant -1 only. We have the following theorem:

**Theorem 3.1.6.** Nilpotent \( SO_{2n} \)-orbits in \( A \) are parametrized by partitions of \( 2n \) except that the partitions with only even parts correspond to two orbits.

### 3.2 Dimensions and Closure Ordering of Nilpotent Orbits

For each nilpotent element \( X \) having the partition \([d_1, d_2, ..., d_k]\), we denote the \( I_B \)-orbits of \( X \) by \( O_X, O_{[d_1, d_2, ..., d_k]} \), or simply \([d_1, d_2, ..., d_k]\). We are now ready to compute the dimension of nilpotent \( I_B \)-orbits.

**Theorem 3.2.1.** Let \( X \) be a nilpotent element in \( A \). Then the dimension of \( I_B \)-orbit of \( X \) is

\[
\dim O_X = \frac{1}{2} \left( m^2 - \sum_i s_i^2 \right)
\]

**Proof.** Suppose that \( B \) is symplectic on \( \mathbb{C}^m \), \( m = 2n \). Recall that we have the decomposition (3.4). For each
Therefore the set of all linear maps from $L$ and hence $Z \in Z_1 \in \mathfrak{sl}_2$ commute, by theory of representations of $\mathfrak{sl}_2$. $Z$ is uniquely determined by a linear map $L(d) \to M(e)$ where $L(d)$ is the lowest weight space in $M(d)$. In this case, $Z$ sends $L(e)$ to $M(d)$. Thus we can assume $d < e$. For $v \in L(d)$, $X^{d+1}v = 0$ and then $ZX^{d+1}v = X^{d+1}Zv = 0$ is the constraint of $Zv = 0$. Note that $r_{d+1} = \dim L(d)$. Therefore the set of all linear maps from $L(d)$ to $M(e)$ forms a vector space of dimension $(d + 1)r_{d+1}e_{d+1}$.

Now consider the case $Z$ sends $M(d)$ to itself. We consider where $Z$ sends $L(d)$ to. Suppose that $Z$ sends $L(d)$ to $H(d)$. We define a new bilinear form $(\cdot, \cdot)_d$ on $L(d)$ given by $(u, v)_d = \langle u, Zv \rangle$. It can be checked $(\cdot, \cdot)_d$ is symmetric and completely determine the map $Z$. The set of all such $(\cdot, \cdot)_d$ forms a vector space of dimension $\frac{1}{2}r_{d+1}(r_{d+1} + 1)$. If $Z$ sends $L(d)$ to $(d - 2)$-weight space in $M(d)$, we define the new form by $(u, v)_{d-2} = \langle u, XZv \rangle$. Again, this form is symmetric and completely determine the map $Z$. Continue this process up to the case $Z$ sends $L(d)$ to itself. We obtain

$$\dim t^X = \sum_{d \geq 0} \left( (d + 1) \left( \sum_{c > d} r_{d+1}e_{d+1} \right) + \frac{d + 1}{2} r_{d+1}(r_{d+1} + 1) \right)$$

$$= \left[ r_1(r_2 + r_3 + \cdots) + \frac{1}{2} r_1(r_1 + 1) \right] + \left[ 2r_2(r_3 + r_4 + \cdots) + \frac{2}{2} r_2(r_2 + 1) \right]$$

$$\quad + \left[ 3r_3(r_4 + r_5 + \cdots) + \frac{3}{2} r_3(r_3 + 1) \right] + \cdots$$

$$= \left[ \frac{1}{2} r_1(r_1 + 2r_2 + 2r_3 + \ldots) + \frac{1}{2} r_1 \right] + \left[ \frac{2}{2} r_2(r_2 + 2r_3 + 2r_4 + \ldots) + \frac{2}{2} r_2 \right]$$

$$\quad + \left[ \frac{3}{2} r_3(r_3 + 2r_4 + 2r_5 + \ldots) + \frac{3}{2} r_3 \right] + \cdots$$

$$= \left[ \frac{1}{2} (s_1 - s_2)(s_1 + s_2) + \frac{1}{2} r_1 \right] + \left[ \frac{2}{2} (s_2 - s_3)(s_2 + s_3) + \frac{2}{2} r_2 \right]$$

$$\quad + \left[ \frac{3}{2} (s_3 - s_4)(s_3 + s_4) + \frac{3}{2} r_3 \right] + \cdots$$

$$= \frac{1}{2} \sum_i s_i^2 + \frac{1}{2} (r_1 + 2r_2 + 3r_3 + \cdots) + \cdots$$

$$= \frac{1}{2} \sum_i s_i^2 + \frac{1}{2} \sum_i s_i.$$  

and hence

$$\dim O_X = \dim g_B - \dim g_B^X = (2n^2 + n) - \left( n + \frac{1}{2} \sum_i s_i^2 \right) = \frac{m^2}{2} - \frac{1}{2} \sum_i s_i^2.$$  

If $B$ is symmetric, the argument is similar except that the form $(u, v)_d$ is symplectic and hence the vector
space consisting of such forms $(\cdot, \cdot)_d$ has dimension $\frac{1}{2} r_{d+1}(r_{d+1} - 1)$. □

**Remark 3.2.2.** The dimension of $I_B$-orbits can also be obtained from [Se, 3.1.c, 3.2.b], where the formulae are not uniform and the proofs are also omitted.

The closure relation on the set of nilpotent orbits in $\mathcal{A}$ is given by

$$O_Y \preceq O_X \text{ if } O_Y \subset \overline{O}_X$$

for nilpotent elements $X, Y \in \mathcal{A}$. Given two partitions $\vec{d} = [d_1, ..., d_N], \vec{f} = [f_1, ..., f_N]$ of $N$ (put some $d_i, f_i = 0$ if needed). We say that $\vec{d}$ dominates $\vec{f}$, denoted by $\vec{d} \succeq \vec{f}$ if

$$d_1 \geq f_1$$
$$d_1 + d_2 \geq f_1 + f_2$$
$$\vdots$$
$$d_1 + ... + d_N \geq f_1 + ... + f_N.$$

**Theorem 3.2.3.** Let $X, Y$ be nilpotent elements in $\mathcal{A}$ having partition $\vec{d}, \vec{f}$, respectively. Then $O_{\vec{d}} \succeq O_{\vec{f}}$ if and only if $\vec{d}$ dominates $\vec{f}$.

**Proof.** See [Oh, Theorem 1]. □

**Example 3.2.4.** All nilpotent $\text{Sp}_{10}$-orbits in $\mathcal{A} \subset \mathfrak{sl}_{10}$ are

$$O_{[5^2]} \succeq O_{[4^2, 1^2]} \succeq O_{[3^2, 2^2]} \succeq O_{[3^2, 1^4]} \succeq O_{[2^4, 1^2]} \succeq O_{[2^2, 1^6]} \succeq O_{[1^{10}]}.$$

The dimensions are 40, 36, 32, 28, 24, 16, 0, respectively.
CHAPTER 4

The Connection Between Schubert Cells and Nilpotent \( K \)-orbits

The goal of this Chapter is to show that for any small \( \bar{\lambda} \), \( M_{\bar{\lambda}} \) is sent to the nilpotent cone \( N_p \) by the map \( \pi \), and show that how each \( M_{\bar{\lambda}} \) is sent to nilpotent \( K \)-orbits in the \( N_p \). Theorem 4.1.2 describes the image \( \pi(M_{\bar{\lambda}}) \) where the proofs are provided by case-by-case consideration in this Chapter.

4.1 Small Dominant Weights

Let \( X_N \) be the type of Dynkin diagram of \( G \) and \( \sigma \) the diagram automorphism on \( G \) of order \( r \), denoted by the pair \((X_N, r)\). We consider the cases \((X_N, r) = (A_{2\ell}, 2), (A_{2\ell-1}, 2), \) and \((D_{\ell+1}, 2)\). Then \( H = (\hat{G})^\sigma \) is either of type \( B_{\ell} \) or \( C_{\ell} \). We make the following labelling for simple roots \( \gamma_i \):

\[
\begin{align*}
\gamma_1 &= \gamma_{\{1,2\ell-1\}}, \ldots, \gamma_{\ell} = \gamma_{\{2\ell-1,2\ell\}}, \gamma_{\ell+1} = \gamma_{\{\ell\}} & (X_N, r) = (A_{2\ell-1}, 2); \\
\gamma_1 &= \gamma_{\{1,2\ell\}}, \ldots, \gamma_{\ell-1} = \gamma_{\{\ell-1,\ell+1\}}, \gamma_{\ell} = \gamma_{\{\ell,\ell+1\}} & (X_N, r) = (A_{2\ell}, 2); \\
\gamma_1 &= \gamma_{\{1\}}, \ldots, \gamma_{\ell-1} = \gamma_{\{\ell-1\}}, \gamma_{\ell} = \gamma_{\{\ell,\ell+1\}} & (X_N, r) = (D_{\ell+1}, 2).
\end{align*}
\]

This labelling of vertices of type \( B_{\ell} \) and \( C_{\ell} \) agrees with the labelling in [Ka, TABLE Fin, p.53]. Then the highest short root \( \gamma_0 \) of \( H \) can be described in the following table.

| \((X_N, r)\) | \(G\) | \(H\) | Simple roots of \( H \) | Highest short root \( \gamma_0 \) of \( H \) |
|-------------|------|------|-----------------|------------------|
| \((A_{2\ell}, 2)\) | \(SL_{2\ell+1}\) | \(PSO_{2\ell+1}\) | \(\gamma_i = \bar{\alpha}_i = \bar{\delta}_{2\ell+1}, \ i = 1, \ldots, \ell\) | \(\gamma_1 + \gamma_2 + \ldots + \gamma_{\ell}\) |
| \((A_{2\ell-1}, 2)\) | \(SL_{2\ell}\) | \(PSp_{2\ell}\) | \(\gamma_i = \bar{\alpha}_i = \bar{\delta}_{2\ell+1}, \ i = 1, \ldots, \ell\) | \(\gamma_1 + 2\gamma_2 + \ldots + 2\gamma_{\ell-1} + \gamma_{\ell}\) |
| \((D_{\ell+1}, 2)\) | \(Spin_{2\ell+2}\) | \(PSO_{2\ell+1}\) | \(\gamma_i = \bar{\alpha}_i = 1, \ldots, \ell - 1\) \(\gamma_\ell = \bar{\delta}_{\ell+1}\) | \(\gamma_1 + \gamma_2 + \ldots + \gamma_{\ell}\) |

Table 4.1: Simple roots and highest short root of \( H \)

From Chapter 2.1, we can identify the weight lattice of \( H \) with \( X_s(T)_{\sigma^r} \), and the set of dominant weights of \( H \) with \( X_s(T)^+_r \). Then, from the construction of root system of classical Lie algebras given in [Hu2, §12].
we can make the following identifications:

\[
X_\sigma(T)^+ \equiv \begin{cases} \mathbb{Z}^\ell & \text{if } H = B_\ell; \\
\{(a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell \mid a_1 + \cdots + a_\ell \in 2\mathbb{Z}\} & \text{if } H = C_\ell
\end{cases}
\]

and

\[
X_\sigma(T)^+ \equiv \{(a_1, \ldots, a_\ell) \in X_\sigma(T) \mid a_1 \geq \cdots \geq a_\ell \geq 0\}
\]

for any cases of \( H \). This identification preserves the relation on \( X_\sigma(T)^+ \) and the dominance relation on \( \{(a_1, \ldots, a_\ell) \in X_\sigma(T) \mid a_1 \geq \cdots \geq a_\ell \geq 0\} \).

In the Table 4.2, we can further make the following identifications for simple roots and fundamental weights of \( H \). Those fundamental weights follows from [Hu2, Table 1., p.69].

The following lemma is well-known, cf.[AH]. We give a self-contained proof here.

**Lemma 4.1.1.** All small dominant weights of \( H \) are

1. \( \omega_j = (1^j0^{\ell-j}), j = 0, \ldots, \ell - 1 \), \( 2\omega_\ell = (1, 1, \ldots, 1) \), if \( H \) has the type \( B_\ell \).

2. \( \omega_1 + \omega_{2j+1} = (2^j1^{\ell-2j-1}), j = 0, \ldots, \lfloor \frac{\ell-1}{2} \rfloor \)
   
   \[ \omega_2j = (1^j0^{\ell-2j}), j = 0, \ldots, \lfloor \frac{\ell}{2} \rfloor \], if \( H \) has the type \( C_\ell \).

**Proof.** Suppose that \( H \) has the type \( B_\ell \). The highest short root is \( \gamma_0 = (1, 0, \ldots, 0) \). By definition, a dominant weight \( (a_1, \ldots, a_\ell) \in X_\sigma(T)^+ \) is small if and only if \( (a_1, \ldots, a_\ell) \not\in (2, 0, \ldots, 0) \) which is equivalent to \( a_1 \leq 1 \). This proves the first part.
Now assume that $H$ has the type $C_\ell$. The highest short root is $\gamma_0=(1,1,...,0)$. Let $(a_1,...,a_\ell) \in X_\ast(T)^\ast$ be a small dominant weight. Then $(a_1,...,a_\ell) \not\equiv (2,2,...,0)$ and so $a_1 \leq 2$. If $a_1=1$, then $(a_1,...,a_\ell)=(1^2/0^{\ell-2})$. If $a_1=2$, then $a_2<2$ and hence $(a_1,...,a_\ell)=(21^2/0^{\ell-2}-1)$.

Let $\bar{\mu}$ be the maximal element among all small dominant weights of $H$, then

$$\mathcal{G}_{r_{sm}} = \bigsqcup_{\lambda \leq \bar{\mu}, \lambda \text{ small}} \mathcal{G}_{r_{\lambda}} = \mathcal{G}_{r_{\bar{\mu}}}.$$

Since $\mathcal{G}_{r_{sm}}$ is irreducible and $M$ is an open subset of $\mathcal{G}_{r_{sm}}$, $M$ is irreducible.

The following theorem is the main result of this Chapter.

**Theorem 4.1.2.** If $\bar{\lambda}$ is small, then $\pi(M_{\bar{\lambda}})$ is contained in $N_\ast$. Moreover, the image $\pi(M_{\bar{\lambda}})$ can be described as the union of nilpotent orbits as the following table where the nilpotent orbit $[a_1^\ell,...,a_\ell^\ell]$ in the table means empty set if the associated partition is invalid for some small $\ell$.

| $(X_N,r)$ | Small dominant weight $\bar{\lambda}$ of $H$ | Orbits in $\pi(M_{\bar{\lambda}})$ |
|-----------|------------------------------------------|----------------------------------|
| $(A_{2\ell},2)$ | $(1^00^{\ell-2j}), j=0,1,...,\ell$ | $[2^{\ell-2j}]+1$ |
| | $(1^20^{\ell-2j}), j=0,1,...,\lfloor \frac{\ell}{2} \rfloor$ | $[2^{\ell-4j}]$ |
| | $(20^{\ell-1})$ | $0,[2^{2\ell-4}]$ |
| | $(21^20^{\ell-3})$ | $[2^{2\ell-4}],[2^{4\ell-8}],[3^{2\ell-6}]$ |
| | $(21^{2j}/0^{\ell-2j-1}), j=2,...,\lfloor \frac{\ell-3}{2} \rfloor$ | $[2^{2\ell-4j}],[2^{2\ell-4j-4}],[3^{2\ell-6j-2}],[3^{2\ell-6j-4}]$ |
| | $(21^{2\ell-2j}/0^{\ell-2j},j=2,3,...,\ell-1)$ | $[2^{\ell-2j+1}], [2^{\ell-4j+1}], [3^{2\ell-6j+2}], [3^{2\ell-8j+4}]$, if $\ell$ is even $[2^{\ell-1}], [3^{2\ell-6j}],[3^{2\ell-8j+4}]$, if $\ell$ is odd |
| | $(D_{\ell+1},2)$ | $(1^00^{\ell-2j}), j=0,1,...,\ell$ | $0$, if $j=0$ $[31^{2\ell-1}], [31^{2\ell-1}]$, if $j$ is odd $[31^{2\ell-1}], [31^{2\ell-1}]$, if $j$ is even, $j \geq 2$ |

Table 4.3: Small dominant weights $\bar{\lambda}$ of $H$ and orbits in $\pi(M_{\bar{\lambda}})$

This theorem follows from Theorem 4.2.1, Theorem 4.3.1, Theorem 4.3.5, and Theorem 4.4.2, which will be proved separately case by case.

The partial orders of small dominant weights of $H$ are shown in the below picture, where the partial order is compatible with the height.
We first recall a crucial lemma from [AH, Lemma 4.3].

**Lemma 4.1.3.** Let \( g = \sum_{i=1}^{\infty} x_i t^i \in SL_n(K) \), where \( x_N \neq 0 \). Let \( \lambda = (a_1, a_2, ..., a_n) \) be a tuple of integers such that \( a_1 \geq a_2 \geq ... \geq a_n \) and \( \sum a_i = 0 \), and \( g(t) \in SL_n(O) t^i SL_n(O) \). Then

1. \( N = a_n \).

2. The rank of \( x_N \) equals to the number of \( j \) such that \( a_j = a_n \).

3. For any \( s \geq 1 \),

\[
\text{rk}\left( \begin{array}{cccc} x_N & x_{N+1} & \cdots & x_{N+s-2} & x_{N+s-1} \\ 0 & x_N & \cdots & x_{N+s-3} & x_{N+s-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_N & x_{N+1} \\ 0 & 0 & \cdots & 0 & x_N \end{array} \right) = \sum_{j=1}^{n} \max\{s - (a_j - a_n), 0\}.
\]

We have the following lemma for the twisted version.
Lemma 4.1.4. Let \( g(t) = \sum_{i=N}^{\infty} x_i t^i \in G(K)^{\sigma} \) where \( x_i \in \text{Mat}_{m \times m} \), \( x_N \neq 0 \). Let \( \tilde{\lambda} = (a_1, ..., a_\ell) \in X_*(T)_\sigma^\vee \) be such that \( g(t) \in G(O)^{\sigma} n^k G(O)^{\sigma} \). Then

1. 
   \[
   N = \begin{cases} 
   -a_1 & \text{if } (X_N, r) = (A_{2\ell}, 2), (A_{2\ell-1}, 2); \\
   -2a_1 & \text{if } (X_N, r) = (D_{\ell+1}, 2).
   \end{cases}
   \]

2. The rank of \( x_N \) is equal to the number of \( j \) such that \( a_j = a_1 \).

Proof. We can write
   \[
   \lambda = (a_1, ..., a_\ell) = \sum_{i=1}^{\ell} \left( \sum_{j=1}^{i} a_j \right) \gamma_i
   \]
   where \( \gamma_i \) are simple roots of \( H \) as labelled by (4.1). We choose a representative \( \lambda \in X_*(T) \) of \( \tilde{\lambda} \) by

   \[
   \lambda = \begin{cases} 
   \sum_{i=1}^{\ell} \left( \sum_{j=1}^{i} a_j \right) \tilde{a}_i & \text{if } (X_N, r) = (A_{2\ell}, 2), (D_{\ell+1}, 2); \\
   \sum_{i=1}^{\ell-1} \left( \sum_{j=1}^{i} a_j \right) \tilde{a}_i + \frac{1}{2} \left( \sum_{j=1}^{\ell} a_j \right) \tilde{a}_\ell & \text{if } (X_N, r) = (A_{2\ell-1}, 2)
   \end{cases}
   \]
   so that

   \[
   \lambda + \sigma \lambda = \begin{cases} 
   \sum_{i=1}^{\ell} \left( \sum_{j=1}^{i} a_j \right) \tilde{a}_i + \sum_{i=\ell+1}^{2\ell} \left( \sum_{j=1}^{2\ell-1} a_j \right) \tilde{a}_i & \text{if } (X_N, r) = (A_{2\ell}, 2); \\
   \sum_{i=1}^{\ell-1} \left( \sum_{j=1}^{i} 2a_j \right) \tilde{a}_i + \sum_{i=\ell+1}^{2\ell-1} \left( \sum_{j=1}^{2\ell-1} a_j \right) \tilde{a}_i & \text{if } (X_N, r) = (A_{2\ell-1}, 2); \\
   \sum_{i=1}^{\ell-1} \left( \sum_{j=1}^{i} 2a_j \right) \tilde{a}_i + \left( \sum_{j=1}^{\ell} a_j \right) (\tilde{a}_\ell + \tilde{a}_{\ell+1}) & \text{if } (X_N, r) = (D_{\ell+1}, 2).
   \end{cases}
   \]

The simple coroots of \( G \) are identified with tuples of integers through the construction of root system given from [Hu2, §12].

Let \( \rho : G \to \text{GL}(V) \) be the standard representation of \( G \). We will determine the double \( \text{SL}(V_O) \)-coset in \( \text{SL}(V_K) \) that \( \rho(g(t)) \) belongs to.

If \( G \) is of the type \( A_m \), then \( \tilde{a}_i, i = 1, ..., m \), are identified with the following \((m + 1)\)-tuples

\[
\tilde{a}_1 = (1, -1, 0, 0, ..., 0) \\
\tilde{a}_2 = (0, 1, -1, 0, ..., 0) \\
\vdots
\]

\[
\tilde{a}_m = (0, 0, 0, ..., 1, -1)
\]

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and hence, as the coweight of $\text{SL}_m$, $\lambda + \sigma \lambda$ corresponds to the following tuples

$$\lambda + \sigma \lambda = \begin{cases} (a_1, a_2, \ldots, a_\ell, 0, -a_\ell, \ldots, -a_2, -a_1) & \text{if } (X_N, r) = (A_{2\ell}, 2); \\ (a_1, a_2, \ldots, -a_\ell, \ldots, -a_2, -a_1) & \text{if } (X_N, r) = (A_{2\ell-1}, 2). \end{cases}$$

Assume that $G$ has the type $D_{\ell+1}$. Then $\tilde{\alpha}_i, i = 1, \ldots, \ell + 1$, are identified with following $(\ell + 1)$-tuples

$$\tilde{\alpha}_1 = (1, -1, 0, 0, \ldots, 0)$$
$$\tilde{\alpha}_2 = (0, 1, -1, 0, \ldots, 0)$$

$$\vdots$$

$$\tilde{\alpha}_\ell = (0, 0, 0, \ldots, 1, -1)$$
$$\tilde{\alpha}_{\ell+1} = (0, 0, 0, \ldots, 1, 1).$$

Then $\lambda + \sigma \lambda = (2a_1, 2a_2, \ldots, 2a_\ell, 0)$ as the coweight of $G = \text{Spin}_{2\ell+2}$. Choose an appropriate maximal torus and a positive root system in $G$. Composing with $\rho : G \to \text{SL}_{2\ell+2}$, as the coweight of $\text{SL}_{2\ell+2}$, $\lambda + \sigma \lambda$ corresponds to the following tuple

$$(2a_1, 2a_2, \ldots, 2a_\ell, 0, 0, -2a_\ell, \ldots, -2a_2, -2a_1).$$

We write $g(t) = A(t)n^\lambda B(t)$, where $n^\lambda$ is a norm of $t^\lambda$ defined by (2.1) and $A(t), B(t) \in G(O)^\sigma$. Hence $\rho(g(t)) \in \text{SL}_m(O)\rho(t^{\lambda+\sigma \lambda})\text{SL}_m(O)$. By above descriptions of $\rho(t^{\lambda+\sigma \lambda})$, this lemma follows from Lemma 4.1.3. □
4.2 Case $(X_N, r) = (A_2f, 2)$

Let $(\cdot, \cdot)$ be a nondegenerate symmetric bilinear form on $V = \mathbb{C}^{2f+1}$ whose matrix is

$$ J = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 \end{pmatrix}. $$

The diagram automorphism $\sigma$ on $g$ given by (3.3) becomes $\sigma(A) = -JA^TJ^{-1}$ and the diagram automorphism $\sigma$ on $G$ is given by

$$ \sigma(A) = JA^{-T}J^{-1}. $$

This $\sigma$ gives the decomposition $g = \mathfrak{t} \oplus \mathfrak{p}$ to 1 and -1 eigenspaces $\mathfrak{t}$ and $\mathfrak{p}$, respectively. Let $K := (\operatorname{SL}_{2f+1})^T = \operatorname{SO}_{2f+1}$. The classification of nilpotent $K$-orbits in $\mathfrak{p}$ and their dimensions follow from the Theorem 3.1.4 and 3.2.1.

Set

$$ N_{p, 2} = \{ x \in N_p | x^2 = 0 \}. $$

**Theorem 4.2.1.** $\pi$ maps $M$ isomorphically onto $N_{p, 2}$. Moreover, $\pi$ maps $M_{(1/0^{f-j})}$ isomorphically to $[2j1^{2f-2j+1}]$.

**Proof.** Let $g(t) \cdot e_0 \in M$. Then $g(t) \cdot e_0 \in M_{(1/0^{f-j})}$ for some $j$. By Lemma 4.1.4, $g(t) = I + x t^{-1}$ for some $x \in \operatorname{Mat}_{2f+1 \times 2f+1}$. By Lemma 2.2.4, $\iota(g(t) \cdot e_0) \in M_{(1/0^{f-j})}$. Hence $\iota(g(t)) = (I - x t^{-1})^{-1} = I + z t^{-1}$ for some $z \in \operatorname{Mat}_{2f+1 \times 2f+1}$, and so $x^2 = 0$. Conversely, let $x \in N_p$ be such that $x^2 = 0$. Then $I + x t^{-1} \in G(K)^\sigma$. By (2.2), $(I + x t^{-1}) \cdot e_0 \in \mathcal{G}_{\mathcal{R}_{\lambda}}$ for some $\lambda = (a_1, \ldots, a_f) \in X_\ell(T)_p^\ell$. By lemma 4.1.4, $\lambda = (1^{j/0^{f-j}})$. We have proved that $M = \{(I + x t^{-1}) \cdot e_0 | x \in p, x^2 = 0 \}$.

Let $(I + x t^{-1}) \cdot e_0 \in M_{(1/0^{f-j})}$. Then $x^2 = 0$ and $x$ has the Jordan blocks of size at most 2. By Lemma 4.1.4, $\operatorname{rk} x = j$ and then $x$ has the partition $[2j1^{2f-2j+1}]$. It is obvious that $\pi$ is injective. To prove surjectivity, let $x \in N_{p, 2}$ having the partition $[2j1^{2f-2j+1}]$. Then, $(I + x t^{-1}) \cdot e_0 \in M$ and hence $(I + x t^{-1}) \cdot e_0 \in M_{(1/0^{f-j})}$.
for some $k$. In fact, $k = j$ since $\pi((I + xt^{-1}) \cdot e_0) = x \in [2^k 1^{2\ell - 2k + 1}]$. \hfill \Box

**Example 4.2.2.** Consider the case $(X_N, r) = (A_8, 2)$. In this case, $G = \text{SL}_9$ and $\mathfrak{g} = \mathfrak{sl}_9 = \mathfrak{t} \oplus \mathfrak{v}$. The diagram shown in Figure 4.2 describes the image of $M_\lambda$ for each small dominant weight $\bar{\lambda}$.

![Diagram](image)

Figure 4.2: $M_\lambda$ for small dominant weight $\bar{\lambda}$ and their image under the map $\pi$ in type $(X_N, r) = (A_8, 2)$

### 4.3 Case $(X_N, r) = (A_{2\ell - 1}, 2)$

Let $\langle \cdot, \cdot \rangle$ be a symplectic bilinear form on $V = \mathbb{C}^{2\ell}$ whose matrix is

$$J = \begin{pmatrix}
1 & -1 & 1 & 1 & \\
-1 & 1 & -1 & 1 & \\
1 & -1 & 1 & -1 & \\
-1 & 1 & -1 & 1 & \\
1 & -1 & 1 & -1 & \\
-1 & 1 & -1 & 1 & \\
1 & -1 & 1 & -1 & \\
-1 & 1 & -1 & 1 & \\
1 & -1 & 1 & -1 & \\
-1 & 1 & -1 & 1 & \\
\end{pmatrix}.$$

The diagram automorphism $\sigma$ on $\mathfrak{g}$ given by (3.3) becomes $\sigma(A) = -JA^TJ^{-1}$ and the action on $G$ is given by

$$\sigma(A) = JA^{-T}J^{-1}.$$ (4.4)
This $\sigma$ gives the decomposition $g = t \oplus p$ to 1 and -1 eigenspaces $t$ and $p$, respectively. Let $K := (\text{SL}_2)'' = \text{Sp}_2$.

The classification of nilpotent $K$-orbits in $p$ and their dimensions follow from the Theorem 3.1.3 and 3.2.1.

Define constructible sets

$$M' := \bigcup_{j=0}^{\lfloor \ell/2 \rfloor} M_{(2j/0^{\ell-2j})}, \quad M'' := \bigcup_{j=0}^{\lfloor (\ell-1)/2 \rfloor} M_{(2j+1/0^{\ell-2j-1})}.$$  

By Lemma 4.1.4, the element of $M'$ is of the form $(I + xt^{-1}) \cdot e_0$ and the element of $M''$ is of the form $(I + xt^{-1} + yt^{-2}) \cdot e_0$. Let $N_{\ell,2}$ be the order 2 nilpotent cone defined as in (4.3).

**Theorem 4.3.1.** $\pi$ maps $M'$ isomorphically onto $N_{\ell,2}$. Moreover, $\pi$ maps $M_{(2j/0^{\ell-2j})}$ isomorphically to $[2^j 1^{2j-4}]$.

**Proof.** The proof is the same as the proof of Theorem 4.2.1, where in this case we use Lemma 4.1.4 for $(A_{2\ell-1}, 2)$. □

Before we describe elements of $M''$, we need the following lemma.

**Lemma 4.3.2.** Let $g(t) \cdot e_0 = (I + xt^{-1} + yt^{-2}) \cdot e_0 \in M'$. Then

1. $\iota(g(t)) \neq g(t)$

2. If $g(t) \cdot e_0 \in M_{(2j+1/0^{\ell-2j-1})}$, then $\text{rk} \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} = 2j + 2$.

**Proof.** Note that $\iota(g(t)) = g(t)$ if and only if

$$(I + xt^{-1} + yt^{-2})(I - xt^{-1} + yt^{-2}) = I$$

which is equivalent to $y = \frac{1}{2} x^2$ and $x^4 = 0$. Suppose that $\iota(g(t)) = g(t)$. Observe that $\text{rk} x^3 \leq \text{rk} x^2 = \text{rk} y = 1$.

If $\text{rk} x^3 = 1$, then $\text{rk} x^4 = \text{rk} x^3 = 1$ which is impossible. Hence $x^3 = 0$. Since $\text{rk} x^2 = 1$, $x \in p$ is nilpotent having the partition $[32^1 1^{2j-3}]$ but this contradicts to Theorem 3.1.3. This proves the first part.

Assume that $g(t) \cdot e_0 \in M_{(2j+1/0^{\ell-2j-1})}$. We write

$$g(t) = A^t \cdot \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \cdot (-1)^{j-2} \cdot B$$
where \( A = \sum_{i=0} A_i t^i, B = \sum_{i=0} B_i t^i \in G(O)^\sigma \). In particular, \( g(t) \in SL_{2\ell}(O)t^1SL_{2\ell}(O) \) where \( \lambda = (2, 1^{2j}, 0^{2\ell-4j-2}, (-1)^{2j}, -2) \).

By Lemma 4.1.3,

\[
\text{rk} \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} = \sum_{j=1}^{2\ell} \max[-a_j, 0] = 2j + 2
\]

as desired. \( \square \)

Let \( g(t) = I + xt^{-1} + yr^{-2} \). By Lemma 2.2.4, \( \sigma(g(t)) = I + x't^{-1} + y't^{-2} \) for some matrices \( x', y' \). Hence

\[
(I - xt^{-1} + yr^{-2})(I + x't^{-1} + y't^{-2}) = I = (I + x't^{-1} + y't^{-2})(I - xt^{-1} + yr^{-2})
\]

which implies

\[
x' = x, \quad x^2 = y + y', \quad xy = y'x, \quad xy' = yx, \quad yy' = y'y = 0. \quad (4.5)
\]

By Lemma 4.1.4 and Lemma 4.3.2, \( \text{rk} y = \text{rk} y' = 1 \) and \( y' \neq y \). Since \( \sigma(g(t)) = g(t), y' = J_y T J^{-1} \) which means that \( y \) and \( y' \) are adjoint each other. We set

\[
\mathcal{M}^I_{(21^{2j}(0^{\ell-j-1})} := \{(I + xt^{-1} + yr^{-2}) \cdot e_0 \in \mathcal{M}_{(21^{2j}(0^{\ell-j-1})} \mid L = L'\},
\]

\[
\mathcal{M}^I_{(21^{2j}(0^{\ell-j-1})} := \{(I + xt^{-1} + yr^{-2}) \cdot e_0 \in \mathcal{M}_{(21^{2j}(0^{\ell-j-1})} \mid L \neq L'\},
\]

where \( L = \text{Im} y \) and \( L' = \text{Im} y' \).

The following lemma will be used in the proofs of Lemma 4.3.4 and Theorem 4.4.2.

**Lemma 4.3.3.** Let \((\cdot, \cdot)\) be a nondegenerate bilinear form on a vector space \( V \) over a field \( \mathbb{C} \) and let \( T \) a linear map on \( V \). Denote the adjoint of \( T \) by \( T^* \). Assume that \( \text{Im} T = \text{Im} T^* \) and \( \text{rk} T = 1 \). Then \( T \) is self-adjoint or skew-adjoint.

**Proof.** It is easy to see that \( \ker T = (\text{Im} T^*)^\perp \) and \( \ker T^* = (\text{Im} T)^\perp \). Say that \( \text{Im} T = \mathbb{C} v \) and \( \text{Im} T^* = \mathbb{C} v' \) for some \( v, v' \in V \). Then \( Tw = v \) for some \( w \in V \). Since \( \text{Im} T = \text{Im} T^* \), we have \( T^* w = \lambda v \) for some \( \lambda \in \mathbb{C} \). Let \( u \in V \). Then \( Tu = kv \) for some \( k \in \mathbb{C} \). Since \( T(u - kw) = 0, u - kw \in \ker T = (\text{Im} T^*)^\perp = (\text{Im} T)^\perp = \ker T^* \). Hence \( T^* u = T^*(kw) = k\lambda v = \lambda Tu \). Since \( u \) is arbitrary, \( T^* = \lambda T \). Consider \( T + T^* = (1 + \lambda)T \). Then

\[
(1 + \lambda)T = T + T^* = (T + T^*)^* = (1 + \lambda)T^*.
\]
Therefore $1 + \lambda = 0$ or $T = T^*$ which means that $T$ is skew-adjoint or self-adjoint. \hfill \Box

**Lemma 4.3.4.** If $(I + xt^{-1} + yt^{-2}) \cdot e_0 \in \mathcal{M}_1^{l_{(2^i2(j-2,2,1)}}$, then $y' = -y$.

**Proof.** We know that $y \neq y'$ are adjoint each other, they have the same images, and $\text{rk} \ y = 1$. By Lemma 4.3.3, $y$ is skew-adjoint, i.e., $y' = -y$. \hfill \Box

**Theorem 4.3.5.**

1. If $\ell$ is even, then
   \[ \pi(\mathcal{M}_1^{l_{(2^i2(j-2,2,1)}}) = [2^j 1^{2\ell-4j}] \cup [2^j 1^{2\ell-4j-4}] \]
   for $j = 0, 1, \ldots, \frac{\ell-2}{2}$. If $\ell$ is odd, then
   \[ \pi(\mathcal{M}_1^{l_{(2^i2(j-2,2,1)}}) = \begin{cases} [2^j 1^{2\ell-4j}] \cup [2^j 1^{2\ell-4j-4}] & \text{if } \ell \geq 3, 0 \leq j \leq \frac{\ell-3}{2}; \\ [2^j 1^{2\ell-4j}] & \text{if } j = \frac{\ell-1}{2}. \end{cases} \]

2. When $\ell \geq 3$, for $j = 1, \ldots, \lfloor \frac{\ell-1}{2} \rfloor$, we have
   \[ \pi(\mathcal{M}_1^{l_{(2^i2(j-2,2,1)}}) = \begin{cases} [3^2 1^{2\ell-6}] & \text{if } j = 1; \\ [3^2 1^{2\ell-4j-2}] \cup [3^2 1^{2\ell-4j+2}] & \text{if } \ell \geq 4, 2 \leq j \leq \lfloor \frac{\ell-1}{2} \rfloor. \end{cases} \]

Moreover, for any $\ell \geq 1$, $\mathcal{M}_1^{l_{(2^i2(j-2,2,1)}}$ is empty.

**Proof.** Let $g(t) \cdot e_0 = (I + xt^{-1} + yt^{-2}) \cdot e_0 \in \mathcal{M}_1^{l_{(2^i2(j-2,2,1)}}$. By Lemma 4.3.2,
\[
\text{rk} \ x \leq \text{rk} \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} = 2j + 2 \leq 2 \text{rk} \ y + \text{rk} \ x = 2 + \text{rk} \ x.
\]
Since $x^2 = y + y' = 0$, $x$ is nilpotent whose partition is $[2^k 1^{2\ell-4k}]$ so that $\text{rk} \ x = 2k$. Hence $k = j$ or $j + 1$. If $\ell$ is odd and $j = \frac{\ell-1}{2}$, then $k = j$.

Let $E_{ij} \in \text{Mat}_{2\ell\times 2\ell}$ be the matrix which has 1 at the entry $i, j$ and 0 elsewhere. For each $j = 0, 1, \ldots, \lfloor \frac{\ell-1}{2} \rfloor$, let
\[
x_j = \text{diag}(0, J_2, \ldots, J_2, 0_{2\ell-4j-2}, -J_2, \ldots, -J_2, 0)
\]
where there are $j$ blocks of $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $j$ blocks of $-J_2$, and $0_{2\ell-4j-2}$ is the square zero matrix of size $2\ell - 4j - 2$. Then $x_j \in \mathfrak{p}$ is nilpotent and has the partition $[2^j 1^{2\ell-4j}]$. It is easy to check that $g(t) = I + xt^{-1} + E_{1,2t^{-2}} \in G(\mathcal{K})^r$ and $g(t) = I + xt^{-1} - E_{1,2t^{-2}}$. By (2.2), $g(t) \cdot e_0 \in \mathcal{G}_\mathfrak{p}$ for some \( \bar{\lambda} = (a_1, \ldots, a_\ell) \in X_\ell(T)_\sigma^+ \) with $a_1 \geq a_2 \geq \cdots \geq a_\ell \geq 0$ and $\sum a_i$ is even. By Lemma 4.1.4, since $\text{rk} E_{1,2\ell} = 1$, $\bar{\lambda} = (21^k 0^{\ell-2k-1})$ for some $k$. By Lemma 4.3.2,

$$2k + 2 = \text{rk} \begin{pmatrix} E_{1,2\ell} & x_j \\ 0 & E_{1,2\ell} \end{pmatrix} = 2j + 2.$$ 

Then $(I + xt^{-1} + E_{1,2t^{-2}}) \cdot e_0 \in \mathcal{M}_{(2^{j+2}/2^{j-1})}$. For $j = 0, 1, \ldots, \lfloor \ell/2 \rfloor$, let $x_j' = x_j + E_{1,2j+2} - E_{2\ell-2j-1,2\ell}$. Then $x_j' \in \mathfrak{p}$ is nilpotent and has the partition $[2^{j+2}/2^{j-1}]$. Similarly, one can show that $(I + x'_j t^{-1} + E_{1,2t^{-2}}) \cdot e_0 \in \mathcal{M}_{(2^{j+2}/2^{j-1})}$. Since $\pi$ is $K$-invariant, this proves the first part.

Now, we prove the second part. Let $g(t) \cdot e_0 = (I + xt^{-1} + yt^{-2}) \cdot e_0 \in \mathcal{M}_{(2^{j+2}/2^{j-1})}$. Set $U = L + L'$. Since $y \neq y'$, $\text{dim} U = 2$ and $U = \text{Im} x^2$. Assume that $L = \mathbb{C}v, L' = \mathbb{C}v'$. By (4.5), we have $xy = y'x$ and $xy' = yx$. Hence $xv = bv'$ and $xv' = av$ for some $a, b \in \mathbb{C}, v, v' \in \mathbb{C}^{2l}$. Then

$$x^2|_U = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad x^2|_U = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix}$$

Suppose that $ab \neq 0$. To show that $\langle \cdot, \cdot \rangle|_{U \times U}$ is nondegenerate, let $u \in \mathbb{C}^{2l}$ be such that $\langle x^2u, x^2v \rangle = 0$ for all $v \in \mathbb{C}^{2l}$. Since $x^2$ is self-adjoint, $\langle x^4u, v \rangle = 0$ for all $v$. Therefore, $x^4u = 0$ and so $x^2u \in \ker x^2|_U = 0$. Since $y, y'$ are adjoint each other and $yy' = 0 = y'y$, $L' \subset \ker y = (L')^\perp$ and $L \subset \ker y' = L'^\perp$. This implies $\langle v, v \rangle = \langle v', v' \rangle = 0$ and $\langle v, v' \rangle \neq 0$. Observe that

$$ab \langle v, v' \rangle = \langle v, x^2v' \rangle = \langle xv, xv' \rangle = \langle bv', av \rangle = -ab \langle v, v' \rangle$$

which implies $ab = 0$, a contradiction. This proves $x^4 = 0$. Since $x \in \mathfrak{p}$ and $\text{rk} x^2 = 2$, by Theorem 3.1.3, $x$ is
nilpotent having the partition \([3^22^k1^{2^k-4^k-6}]\). By Lemma 4.3.2,

\[
\text{rk } x \leq \text{rk } \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} = 2j + 2 \leq 2 \text{rk } y + \text{rk } x = 2 + \text{rk } x.
\]

Since \(\text{rk } x = 2k + 4\), \(k = j - 1\) or \(j - 2\). Here we see that \(j \neq 0\) and hence \(M^I_{(2^0-1)}\) is empty. When \(j = 1\), we see that \(k = 0\). Let

\[
J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

For each \(j = 1, \ldots, \lfloor \frac{2^k - 1}{2} \rfloor\), let

\[
x_{j-1} = \text{diag}(J_3, J_2, \ldots, J_2, 0, -J_2, \ldots, -J_2, -J_3)
\]

where there are \(j - 1\) blocks of \(J_2\), and \(j - 1\) blocks of \(-J_2\). Then \(x_{j-1} \in p\) is nilpotent having the partition \([3^22^{j-2}1^{2^j-4^j-2}]\). Note that \(g(t) \equiv 1 + x_{j-1} t^{-1} + E_{13} t^{-2} \in G(K)^\sigma\) and \(i(g(t)) = 1 + x_{j-1} t^{-1} + E_{22} t^{-2}\). By (2.2), \(g(t) \cdot e_0 \in G r_{\lambda}\) for some \(\lambda = (a_1, \ldots, a_{2^j}) \in X_0(T')^\sigma\) with \(a_1 \geq a_2 \geq \cdots \geq a_{2^j} \geq 0\) and \(\sum a_i\) is even. By Lemma 4.1.4, since \(\text{rk } E_{13} = 1\), \(\bar{\lambda} = (2^k0^j - 2k - 1)\) for some \(k\). By Lemma 4.3.2,

\[
2k + 2 = \text{rk } \begin{pmatrix} E_{13} & x_{j-1} \\ 0 & E_{13} \end{pmatrix} = 2j + 2.
\]

Then \((I + x_{j-1} t^{-1} + E_{13} t^{-2}) \cdot e_0 \in M^I_{(2^j0^j - 2j - 1)}\). For \(j = 2, \ldots, \lfloor \frac{2^k - 1}{2} \rfloor\), let

\[
x_{j-2}' = \text{diag}(0, J_3, J_2, \ldots, J_2, 0, -J_2, \ldots, -J_2, -J_3, 0)
\]

where there are \(j - 2\) blocks of \(J_2\), and \(j - 2\) blocks of \(-J_2\). Then \(x_{j-2}' \in p\) is nilpotent having the partition \([3^22^{j-4}1^{2^{j-4}j+2}]\). One can check that \(h(t) := 1 + x_{j-2}' t^{-1} + (E_{24} + E_{12}) t^{-2} \in G(K)^\sigma\) and \(i(h(t)) = 1 + x_{j-2}' t^{-1} + (E_{22} t^{-3} + E_{22} t^{-1} - E_{12}) t^{-2}\). Similarly, \(h(t) \cdot e_0 \in G r_{\bar{\lambda}}\) where \(\bar{\lambda} = (2^k0^j - 2k - 1)\) for some \(k\). By Lemma 4.3.2,

\[
2k + 2 = \text{rk } \begin{pmatrix} E_{24} + E_{12} & x_{j-2}' \\ 0 & E_{24} + E_{12} \end{pmatrix} = 2j + 2.
\]
Then \((I + x'_{j-2}t^{-1} + (E_{24} + E_{1,2t})t^{-2}) \cdot e_0 \in M^{\Pi}_{(21^2/(j^{-2}j^{-1})�)}\). □

In the following proposition, we describe the fibers of \(\pi : M \rightarrow \pi(M)\).

**Proposition 4.3.6.** For any \(x \in \pi(M)\),

\[
\pi^{-1}(x) \cong \{z \in \mathfrak{sp}_{2t} \mid xz + zx = 0, z^2 = 0, \text{rk}(z + \frac{1}{2}x^2) \leq 1\}.
\]

In particular, \(\pi^{-1}(0)\) is isomorphic to the closure of nilpotent orbit \([21^2\ell^2] \) in \(\mathfrak{sp}_{2t}\) and \(\dim \pi^{-1}(0) = 2\ell + 1\).

**Proof.** Note that \((1 + xt^{-1} + yt^{-2}) \cdot e_0 \in M\) if and only if \(\det(1 + xt^{-1} + yt^{-2}) = 1\), \(\text{rk} y \leq 1\) and

\[
x^T J - Jx = 0, \quad -x^T Jx + y^T J + Jy = 0, \quad x^T Jy - y^T Jx = 0, \quad y^T Jy = 0.
\]

(4.6)

Set \(z = y - \frac{1}{2}x^2\), (4.6) is equivalent to

\[
x \in \mathfrak{p}, \quad z \in \mathfrak{t}, \quad xz + zx = 0, \quad z^2 = 0.
\]

(4.7)

If \(xz + zx = 0\), then \(\det(1 + xt^{-1} + (z + \frac{1}{2}x^2)t^{-2}) = 1\). Since \(\pi^{-1}(0) = \{(1 + yt^{-2}) \cdot e_0 \mid y \in \mathfrak{sp}_{2t}, y^2 = 0, \text{rk} y \leq 1\}\), the dimension, cf. [CM, Corollary 6.1.4], is

\[
\dim \pi^{-1}(0) = \dim [21^{2\ell - 2}] = (2\ell^2 + \ell) - \frac{1}{2}((2\ell - 1)^2 + 1^2) - \frac{1}{2}(2\ell - 2) = 2\ell + 1.
\]

(4.8)

\[
\frac{1}{2}\left(\frac{1}{2}(2\ell - 2)ight) = 2\ell + 1.
\]

□

In [AH, Theorem 1.2], they proved that there are finitely many \(G\)-orbits in \(\text{Gr}_{\lambda} \cap \text{Gr}_{0-\ell}\) for small dominant coweight \(\lambda\). In the case of \((A_{2\ell}, 2)\), it is easy to see that \(K\) acts transitively on \(M_{(t^1/t^0)}\) and hence there are finitely many \(K\)-orbits in \(M\). For the case \((A_{2(\ell-1)}, 2)\), it is not obvious to determine if there are finitely many \(K\)-orbits in \(M_{(21^2/(j^{-2}j^{-1})�)}\). If \(g(t) = 1 + xt^{-1} + (z + \frac{1}{2}x^2)t^{-2} \in M_{(21^2/(j^{-2}j^{-1})�)}\), then \(g(t)\) satisfies (4.7). If the action of \(K\) on the following anti-commuting nilpotent variety

\[
\{(x, z) \in \mathcal{N}_p \times \mathcal{N}_t \mid xz + zx = 0\}
\]

by diagonal conjugation has finitely many orbits, then there are finitely many \(K\)-orbits in \(M_{(21^2/(j^{-2}j^{-1})�)}\).
Figure 4.3: $M_{\lambda}$ for small dominant weight $\lambda$ and their image under the map $\pi$ in type $(X, r) = (A_9, 2)$

**Example 4.3.7.** Consider the case $(X, r) = (A_9, 2)$. In this case, $G = SL_{10}$ and $\mathfrak{g} = sl_{10} = \mathfrak{t} \oplus \mathfrak{p}$. The diagram as shown in Figure 4.3 describes the image of $M_{\lambda}$ for each small dominant weight $\lambda$. For instance, $M_{(21^20^2)}$ consists of two parts, $M_{(21^20^2)}$ and $M_{(21^20^2)}$. By Theorem 4.3.5, $\pi(M_{(21^20^2)})$ is precisely the union of two nilpotent orbits $[2^41^2]$ and $[2^41^6]$ in $\mathfrak{p}$ while $\pi(M_{(21^20^2)})$ is the single nilpotent orbit $[3^21^4]$ in $\mathfrak{p}$. Since $(21^20^2) \succeq (1^40)$, $M_{(21^20^2)} \succeq M_{(1^40)}$. Similarly, $M_{(21^20^2)} \succeq M_{(20^2)}$. According to the table in Theorem 4.1.2, the image of certain $M_{\lambda}$ is a union of 4 nilpotent orbits. It does not happen in this case since $\ell = 5$ is not large enough. In the case of $(X, r) = (A_{13}, 2)$, $\pi(M_{(21^40^2)})$ is a union of nilpotent orbits $[2^41^6], [2^61^2], [3^22^21^4], [3^21^8]$ in $\mathfrak{p} \subset sl_{14}$.

4.4 **Case** $(X, r) = (D_{\ell+1}, 2)$

In this case, it is more convenient to work with $G = SO_{2\ell+2}$ and $\sigma$ is a diagram automorphism on $G$. It is known that $G^\sigma = SO_{2\ell+1} \times \{\pm I\}$. Let $G(O)^{\sigma, o}$ denote the identity component of the group $G(O)^{\sigma}$. Then, the action of $Spin_{2\ell+2}(O)^{\sigma}$ on the twisted affine Grassmanian $Gr$ of $Spin_{2\ell+2}$ factors through $G(O)^{\sigma, o}$. Let
$G(O^-)^{\sigma}_0$ be the kernel of the evaluation map $G(O^-)^{\sigma} \to G^{\sigma}$. The action of $\text{Spin}_{2\ell+2}(O)^{-}_{0}$ on $G$ factors through $G(O^-)^{\sigma}_0$. Hence, the opposite open Schubert cell $Gr^{-}_{0}$ is a $G(O^-)^{\sigma}_0$-orbit. In fact, $Gr$ is naturally the neutral component of the twisted affine Grassmannian associated to $(G, \sigma)$, whose definition is a bit more sophisticated.

We can realize the group $G$ as \( \{ g \in \text{SL}_{2\ell+2} \mid gJg^T = J \} \), and the Lie algebra of $G$ as \( \mathfrak{g} = so_{2\ell+2}(J) = \{ x \in gl_{2n} \mid Jx + x^TJ = 0 \} \) where

\[
J = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}.
\]

The diagram automorphism $\sigma$ of order 2 on $\mathfrak{g}$ can be given by $\sigma(x) = wxw$ where

\[
w = \text{diag} \left( I_{\ell}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{\ell} \right).
\]

The diagram automorphism $\sigma$ on $G$ is also defined the same. We also have the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. Let $K$ be the identity component of $G^{\sigma}$. $K$ has Lie algebra $\mathfrak{t}$ and acts on $\mathfrak{p}$ by conjugation. It can be checked that $J = A^T A$ where

\[
A = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{\sqrt{2}} \\
0 & \ddots & \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\
\vdots & \ddots & \frac{1}{\sqrt{2}} & \cdots & \cdots & \cdots & \ddots & \vdots \\
\vdots & \ddots & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \cdots & \cdots & \ddots & \vdots \\
0 & \ddots & \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\
\frac{i}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{i}{\sqrt{2}}
\end{pmatrix}.
\]

Another realization of $so_{2\ell+2}$ is $so_{2\ell+2}(I) = \{ x \in gl_{2\ell} \mid x + x^T = 0 \}$. There exists an isomorphism from $so_{2\ell+2}(J)$ to $so_{2\ell+2}(I)$ given by $x \mapsto AxA^{-1}$. Under $so_{2\ell+2}(I)$, the diagram automorphism $\sigma_0$ is defined by $\sigma_0(x) = w_0 x w_0$ where $w_0 = (PA)w(PA)^{-1} = \text{diag}(-1, 1, 1, ..., 1)$ and $P$ is some matrix of change of basis.

**Proposition 4.4.1.**
1. If $x$ is a nonzero nilpotent element in $p$, then $x$ has the partition $[31^{2\ell-1}]$.

2. There are exactly 2 nilpotent $K$-orbits in $p$: $\{0\}$ and $N_p \setminus \{0\}$.

Proof. Since $w_0 x w_0 = -x$, $x$ has the form

$$x = \begin{pmatrix} 0 & -u' \\ u & 0 \end{pmatrix}$$

where $u \in \mathbb{C}^{2\ell+1}$ is a nonzero column vector. Then

$$x^2 = \begin{pmatrix} -uu' & 0 \\ 0 & -uu' \end{pmatrix}.$$ 

If $x^2 = 0$, then $uu' = 0$ which implies $u = 0$, a contradiction. Since $\text{rk} \ x = 2$ and $x^2 \neq 0$, $x$ has the partition $[31^{2\ell-1}]$. This proves the first part.

The element of $K$ has the form

$$k = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

where $g \in \text{SO}_{2\ell+1}$ and $k$ acts on $x \in p$ by

$$k \cdot x = k x k^{-1} = \begin{pmatrix} 1 & -gu' \\ gu & 0 \end{pmatrix}.$$ 

Hence the action of $K$ on $p$ is the same as the action of $\text{SO}_{2\ell+1}$ on $\mathbb{C}^{2\ell+1}$. Note that for every $k \geq 3$, $x^k$ has the scalar $u^T u$ on every nonzero entry. Since $x$ is nilpotent, $u^T u = 0$. The result immediately follows since the action of $\text{SO}_{2\ell+1}$ on $\{z \in \mathbb{C}^{2\ell+1} \mid z^T z = 0\}$ has two orbits.

\[\square\]

Theorem 4.4.2. For $j = 0, 1, \ldots, \ell$, we have

$$\pi(M_{(1/0^{\ell-j})}) = \begin{cases} 
N_p \setminus \{0\} & \text{if } j \text{ is odd}; \\
\{0\} & \text{if } j = 0; \\
N_p & \text{if } j \text{ is even and } j \geq 2.
\end{cases}$$

Moreover, $M_{(10^{\ell-1})} = \{(1 + xr^{-1} + \frac{1}{2} x^2 r^{-2}) \cdot e_0 \mid x \in N_p \setminus \{0\}\}$. Consequently, $\pi$ maps $M_{(10^{\ell-1})}$ isomorphically
onto \( N_p \setminus \{0\} \).

**Proof.** By Lemma 4.1.4, let \( g(t) \cdot e_0 = (1 + xt^{-1} + yt^{-2}) \cdot e_0 \in M_{(1/0^{\ell-j})} \). We work under \( so_{2\ell+2}(I) \) and \( SO_{2\ell+2}(I) \). Since \( g(t) \) is fixed by \( \sigma_0 \), \( w_0xw_0 = -x \) and \( w_0yw_0 = y \). Similarly to the proof of Proposition 4.4.1, \( x, y \) are in the form

\[
\begin{pmatrix}
0 & -u^T \\
u & 0
\end{pmatrix},
\begin{pmatrix}
y_0 & 0 \\
0 & D
\end{pmatrix}
\]

where \( u \in \mathbb{C}^{2\ell+1} \) is a column vector, \( y_0 \in \mathbb{C} \), and \( D \in \text{Mat}_{(\ell-1)\times(\ell-1)} \). Since \( g^T g = I \), the following equations hold:

\[
x^T + x = 0, \quad x^T x + y^T y = 0, \quad x^T y + y^T x = 0, \quad y^T y = 0.
\]

Then \( y_0 = 0 \) and \( u^T u = 0 \) which implies \( x^3 = 0 \). Hence \( x \in N_p \).

Suppose that \( j \) is odd and \( x = 0 \). We have \( y + y^T = 0 \) and \( y^T y = 0 \). Then \( y_0 = 0 \) and \( D \) is a nilpotent element of \( so_{2\ell+1} \) with \( D^2 = 0 \). Since \( \text{rk} \ D = \text{rk} \ y = j \), \( D \) has the partition \([2j^{12\ell-j+2}]\). This contradicts to the classification of nilpotent orbits of type B, [CM, Theorem 5.1.2]. Hence \( x \neq 0 \). Now, consider the matrix

\[
x_0 = \begin{pmatrix}
0 & \cdots & 0 & -1 & -i \\
& & & \vdots & \\
0 & & & 1 & \end{pmatrix},
\]

Let \( N \) be a nilpotent element in \( so_{2\ell-1}(I) \) having the partition \([2j^{-1}^{12\ell-j+1}]\). Such a matrix \( N \) exists in view of [CM, Theorem 5.1.2]. Then the matrices \( x_0 \) and

\[
y_0 := \text{diag}(0, \ldots, 0, N, 0, 0) + \frac{1}{2} x_0^2
\]

satisfy the relations in (4.9). Hence \( g(t) := 1 + x_0 t^{-1} + y_0 t^{-2} \in G(O)^\pi \). By (2.2), \( g(t) \cdot e_0 \in G r_\lambda \) for some \( \lambda = (a_1, a_2, \ldots, a_\ell) \in X_*(T)_{\sigma_0}^+ \) with \( a_1 \geq \cdots \geq a_\ell \). Since \( \text{rk} \ y_0 = j \), by Lemma 4.1.4, \( \lambda = (1/0^{\ell-j}) \) and hence \( g(t) \cdot e_0 \in M_{(1/0^{\ell-j})} \). Since \( \pi \) is \( K \)-equivariant, the first part is done.

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Suppose that $j$ is even. For each $j = 0, 2, 4, \ldots, 2\lfloor \frac{\ell}{2} \rfloor$, consider the $\ell \times \ell$ matrix
\[
\begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & -1 \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\]
where there are $\frac{j}{2}$ copies of each 1 and -1. Denote $z_j$ the square zero matrix of size $2\ell + 2$ whose $\ell \times \ell$ submatrix on the right top is replaced by the above matrix. Now we work under $so_{2\ell+2}(J)$ and $SO_{2\ell+2}(J)$.

Since $wz_jw = z_j$ and $rk z_j = j$, we have $(1 + z_j r^{-2}) \cdot e_0 \in M_{(10^\ell)^\circ}$ and then $\pi((1 + z_j r^{-2}) \cdot e_0) = 0$. Let $x_0$ be the square zero matrix of size $2\ell + 2$ whose $4 \times 4$ submatrix at the center is replaced by
\[
\begin{pmatrix}
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Then $x_0 \in N_p \setminus \{0\}$. Set $y_0 = \frac{1}{2} x_0^2 + z_j$. Then $rk y_0 = j$. It can be checked that $x_0, y_0$ satisfy $wx_0 w = -x_0, wy_0 w = y_0$, and
\[
x_0^T J + Jx_0 = 0, \quad x_0^T Jx_0 + y_0^T J + Jy_0 = 0, \\
x_0^T Jy_0 + y_0^T Jx_0 = 0, \quad y_0^T Jy_0 = 0.
\] (4.10)

Hence $h(t) := 1 + x_0 r^{-1} + y_0 r^{-2} \in G(O)^p$. Similarly, one can show that $h(t) \cdot e_0 \in M_{(10^\ell)^\circ}$. This proves the second part.

To prove the last part, let $x$ be a nonzero nilpotent element in $\mathfrak{p}$. Since $x$ has the partition $[31^{2\ell}]$, $rk x^2 = 1$. It is easy to check that $(1 + x t^{-1} + \frac{1}{2} x^2 t^{-2}) \cdot e_0 \in M_{(10^\ell)^\circ}$. Conversely, let $g(t) \cdot e_0 = (1 + x t^{-1} + y t^{-2}) \cdot e_0 \in M_{(10^\ell)^\circ}$. Let $t(g(t)) = 1 + x t^{-1} + y t^{-2}$. Since $g(t) = g(t)^{-T} = (t(g(-t)))^T$, $y = (y')^T$. Then $y$ and $y'$ are adjoint each other under the symmetric form whose matrix is $I$. Note that $rk y = rk y' = 1$. If $Im y \neq Im y'$, then
rk \ x^2 = \rk y + \rk y' = 2, a contradiction. Hence Im \ y = \Im y'. By Lemma 4.3.3, y' = y or y' = -y. By (4.5), \ x^2 = y + y' and hence y' = y. By (4.9), x^T + x = 0 and x^T x + y^T + y = 0. Then y + y' = x^2 = y + y^T, so y = y' = y^T. Therefore, g(t) = 1 + xt^{-1} + yr^{-2} = 1 + xt^{-1} + \frac{1}{2} x^2 r^{-2}. \quad \square

Proposition 4.4.3. For x ∈ N_p, write x as in (4.8). Then

\[ \pi^{-1}(x) \cong \{ D \in \mathfrak{so}_{2\ell+1} \mid Du = 0, D^2 = 0 \}. \]

In particular, π^{-1}(0) is isomorphic to the maximal order 2 nilpotent variety in \mathfrak{so}_{2\ell+1}, and

\[ \dim \pi^{-1}(0) = \begin{cases} \ell^2 & \text{if } \ell \text{ is even;} \\ \ell^2 - 1 & \text{if } \ell \text{ is odd.} \end{cases} \]

Proof. Under the realization \mathfrak{so}_{2\ell+2}(I) and \SO_{2\ell+2}(I), and the diagram automorphism \sigma_0, we have that 1 + xt^{-1} + yr^{-2} ∈ M if and only if w_0yw_0 = y and the conditions (4.9) hold. Set \ z = y - \frac{1}{2} x^2, these conditions are equivalent to

\[ z = \begin{pmatrix} 0 \\ D \end{pmatrix}, \quad D \in \mathfrak{so}_{2\ell+1}, \quad D^2 = 0, \quad Du = 0. \quad (4.11) \]

Hence π^{-1}(0) ≅ \{ D \in \mathfrak{so}_{2\ell+1} \mid D^2 = 0 \} which is \mathcal{O}_{[2^{\ell}]2^{\ell-1+1}} in \mathfrak{so}_{2\ell+1} where k is the maximal even integer. By the dimension formula, cf. [CM, Corollary 6.1.4],

\[ \dim \pi^{-1}(0) = \begin{cases} \dim \mathcal{O}_{[2^{\ell}]} = \ell^2 & \text{if } \ell \text{ is even;} \\ \dim \mathcal{O}_{[2^{\ell-1}1]} = \ell^2 - 1 & \text{if } \ell \text{ is odd} \end{cases} \]

as desired. \quad \square

Similar to the case (A_{2\ell-1}, 2), it is not obvious to see if there are finitely many K-orbits in M_{(1/0^{\ell-1})}. If \ g(t) = 1 + xt^{-1} + (z + \frac{1}{2} x^2)r^{-2} \in M_{(1/0^{\ell-1})}, then g(t) satisfies (4.11). If the action of K on the following anti-commuting nilpotent variety

\[ \{(x, z) \in \mathfrak{so}_{2\ell+2}(I) \times \mathfrak{so}_{2\ell+2}(I) \mid xz + zx = 0, x, z \text{ nilpotent}\} \]

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by diagonal conjugation has finitely many orbits, then there are finitely many $K$-orbits in $\mathcal{M}_{(1/0^{\ell-j})}$.

**Example 4.4.4.** Consider the case $(X_N, r) = (D_5, 2)$. In this case, we work with $G = SO_{10}$ and $\mathfrak{g} = \mathfrak{so}_{10} = \mathfrak{f} \oplus \mathfrak{v}$.

The diagram as shown in Figure 4.4 describes the image of $\mathcal{M}_{\lambda}$ for each small dominant weight $\lambda$. By Theorem 4.4.2, $\pi$ maps $\mathcal{M}_{(1^{10})}$ isomorphically to the nilpotent orbit $[31^7]$ in $\mathfrak{v}$.

Figure 4.4: $\mathcal{M}_{\lambda}$ for small dominant weight $\lambda$ and their image under the map $\pi$ in type $(X_N, r) = (D_5, 2)$.
CHAPTER 5

Applications

In this Chapter, we describe some applications to the geometry of order 2 nilpotent varieties in the certain classical symmetric spaces.

Let \( \langle \cdot, \cdot \rangle \) be a symmetric or symplectic non-degenerate bilinear form on a vector space \( V \). Recall that \( \mathcal{A} \) is the space of all self-adjoint linear maps with respect to \( \langle \cdot, \cdot \rangle \). Set \( N_{\mathcal{A}, 2} \) denote the space of all nilpotent operators \( x \) in \( \mathcal{A} \) such that \( x^2 = 0 \). If \( \langle \cdot, \cdot \rangle \) is symmetric and \( \dim V = 2n + 1 \), then SO\(_{2n+1}\)-orbits in \( N_{\mathcal{A}, 2} \) are classified by the partitions \( [2^j1^{2n+1-2j}] \) with \( 0 \leq j \leq n \); if \( \langle \cdot, \cdot \rangle \) is symplectic and \( \dim V = 2n \), then Sp\(_{2n}\)-orbits in \( N_{\mathcal{A}, 2} \) are classified by the partitions \( [2^j1^{2n-2j}] \) with \( 0 \leq j \leq \lfloor \frac{n}{2} \rfloor \).

**Theorem 5.0.1.** Assume that \( \langle \cdot, \cdot \rangle \) is symplectic or symmetric and \( \dim V \) is odd. Then any order 2 nilpotent variety in \( \mathcal{A} \) is normal.

**Proof.** By Theorem 4.2.1 and Theorem 4.3.1, for any order 2 nilpotent variety \( \overline{O} \) in \( \mathcal{A} \), \( \overline{O} \) is isomorphic to \( \overline{M}_\lambda := \overline{Gr}_3 \cap Gr_0 \) for a small dominant weight \( \lambda \) of \( H \). Note that \( \overline{M}_\lambda \) is an open subset of the twisted Schubert variety \( \overline{Gr}_\lambda \) and \( \overline{Gr}_3 \) is a normal variety. It follows that \( \overline{O} \) is also normal. \( \square \)

In fact, when \( \langle \cdot, \cdot \rangle \) is symplectic, any nilpotent variety in \( \mathcal{A} \) is normal, see [Oh]. In *loc.cit.*, Ohta also showed that not all nilpotent varieties are \( N_p \) is normal, when \( \langle \cdot, \cdot \rangle \) is symmetric. When \( \langle \cdot, \cdot \rangle \) is symmetric and \( \dim V \) is odd, this theorem seems to be new.

**Remark 5.0.2.** Theorem 5.0.1 is true for any field \( k \) of characteristic \( p > 2 \), as one can easily see that the classification theorem in Chapter 3 still holds for order 2 nilpotent orbits, and the arguments in Theorem 4.2.1, Theorem 4.3.1 applies as well. The same remark applies to the following Theorem 5.0.3 and Theorem 5.0.4.

For any variety \( X \), let \( IC_X \) denote the intersection cohomology sheaf on \( X \). The perverse sheaf \( IC_X \) captures the singularity of the variety \( X \). For any \( x \in X \), we denote by \( H^k_{IC}(IC_X) \) the k-th cohomology of the stalk of \( IC_X \) at \( x \).
Theorem 5.0.3. 1. When \( \langle , \rangle \) is symmetric and \( \dim V = 2n + 1 \), for any \( 0 \leq j \leq n \), let \( O_j \) denote the nilpotent orbit in \( A \) associated to the partition \( [2^j1^{2n+1-2j}] \) and let \( O_j' \) denote the nilpotent orbit in \( \mathfrak{sp}_{2n} \) associated to the partition \( [2^j1^{2n-2j}] \), we have

\[
\dim O_j = \dim O_j' = j(2n + 1 - j).
\]

Moreover, for any \( x \in O_{[2^j1^{2n+1-2j}]} \) and \( x' \in O_{[2^j1^{2n-2j}]}' \) and for any \( k \in \mathbb{Z} \),

\[
\dim \mathcal{H}^k_x(I\!C_{\widetilde{O}_j}) = \dim \mathcal{H}^k_x(I\!C_{\widetilde{O}_j}').
\]

2. When \( \langle , \rangle \) is symplectic and \( \dim V = 2n \), for any \( 0 \leq j \leq \lfloor \frac{n}{2} \rfloor \), let \( O_{2j} \) denote the nilpotent orbit in \( A \) associated to the partition \( [2^j1^{2n-4j}] \) and let \( O_{2j}' \) denote the nilpotent orbit in \( \mathfrak{so}_{2n+1} \) associated to the partition \( [2^j1^{2n+1-4j}] \), we have

\[
\dim O_{2j} = \dim O_{2j}' = 4j(n - j).
\]

Moreover, for any integer \( 0 \leq i \leq j \), \( x \in O_{2i} \), \( x' \in O_{2i}' \), and for any \( k \in \mathbb{Z} \), we have

\[
\dim \mathcal{H}^k_x(I\!C_{\widetilde{O}_{2j}}) = \dim \mathcal{H}^k_x(I\!C_{\widetilde{O}_{2j}}').
\]

**Proof.** We first prove part 1). By Theorem 3.2.1 and [CM, Corollary 6.1.4], it is easy to verify \( \dim O_j = \dim O_j' = j(2n + 1 - j) \). By Theorem 4.2.1, \( \widetilde{O}_j \) can be embedded into an open subset in the twisted affine Schubert variety \( \overline{G}_{\widetilde{\alpha}_j} \) associated to \( (\text{SL}_{2n+1}, \sigma) \). On the other hand, in view of [AH], \( \widetilde{O}_j \) can be embedded into the untwisted affine Schubert variety \( \overline{G}_{\alpha_j}^{\text{aff}} \) in the affine Grassmannian \( \overline{G}_{\text{Sp}_{2n}}^{\text{aff}} \) of \( \text{Sp}_{2n} \). Set

\[
F = I\!C_{\widetilde{O}_j}[-\dim \widetilde{O}_j], \quad \text{and} \quad F' = I\!C_{\widetilde{O}_j'}[-\dim \widetilde{O}_j'].
\]

By purity vanishing property of intersection cohomology sheaf of Schubert varieties (cf. [KL]), \( \mathcal{H}^k_x(F) = \mathcal{H}^k_x(F') = 0 \) when \( k \) is odd. Equivalently,

\[
\mathcal{H}^k_x(I\!C_{\widetilde{O}_j}) = \mathcal{H}^k_x(I\!C_{\widetilde{O}_j'}) = 0
\]

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for any odd integer $k$, as $\dim \overline{O}_j = \dim \overline{O}'_j$ is even.

Note that the affine Grassmannian $\text{Gr}_{\text{Sp}_{2n}}$ and the twisted affine Grassmannian $\text{Gr}_{\text{SL}_{2n+1}}$ have the same underlying affine Weyl group. Applying the results in [KL], the polynomials $\sum \dim H^2_k(F)q^k$ and $\sum \dim H^2_k(F')q^k$ are both equal to the same Kazhdan-Lusztig polynomial $P_{\omega_i, \omega_j}(q)$ for the affine Weyl group of $\mathfrak{so}_{2n+1}$. It follows that

$$\dim \mathcal{H}_x^k(\text{IC}_{\overline{O}_j}) = \dim \mathcal{H}_x^k(\text{IC}_{\overline{O}'_j})$$

for all even integer $k$. Alternatively, one can see these two polynomials are equal, as they both coincide with the jump polynomial of the Brylinsky-Kostant filtration on the irreducible representation $V_{\omega_j}$ of $H$, see [Bry, Zh].

For the second part of the theorem, the proof is almost the same, except that by Theorem 4.3.1, $\overline{O}_{2j}$ can be openly embedded into the twisted affine Schubert variety $\overline{Gr}_{\omega_{2j}}$ associated to $(\text{SL}_{2n}, \sigma)$, and $\overline{O}_{2j}$ can be openly embedded into the affine Schubert variety $\overline{Gr}_{\text{Spin}_{2n+1}}$. □

Part 1) of this theorem was due to Chen-Xue-Vilonen [CVX] by different methods. This theorem shows that there is a natural bijection between order 2 nilpotent varieties in $\mathcal{A}$ and order 2 nilpotent varieties in its dual classical Lie algebras, such that they share similar geometry and singularities. However, under such bijection the associated two nilpotent varieties are not necessarily isomorphic. For example, when $\langle \cdot, \cdot \rangle$ is symmetric and $\dim V = 2n + 1$, $\overline{O}_{[21^{2n-1}]}$ is the minimal nilpotent variety in $\mathcal{A}$. In fact $\overline{O}_{[21^{2n-1}]}$ is smooth, since the quasi-miniscule twisted affine Schubert variety $\overline{Gr}_{\omega_1}$ is smooth, cf. [HR, Section 5.1]. On the other hand, the minimal nilpotent variety $\overline{O}_{[21^{2n-2}]}$ in $\mathfrak{sp}_{2n}$ is not smooth, as the smooth locus of the quasi-miniscule affine Schubert variety $\overline{Gr}_{\omega_1}$ is the open cell $\text{Gr}_{\omega_1}$.

We now describe another application.

**Theorem 5.0.4.** If $\langle \cdot, \cdot \rangle$ is symplectic, then the smooth locus of any order 2 nilpotent variety in $\mathcal{A}$ is the open nilpotent orbit.

**Proof.** Let $\overline{O}$ be any order 2 nilpotent variety in $\mathcal{A}$. By Theorem 4.3.1, $\overline{O}$ can be openly embedded into a twisted Schubert variety $\overline{Gr}_{\lambda}$ with $\lambda$ small, in the twisted affine Grassmannian $\text{Gr}_{\text{SL}_{2n}}$. Then this theorem follows from [BH, Theorem 1.2]. □
In this appendix, we summarize definitions and properties, without proof, of Lusztig map as we introduce in the Chapter 1. For further detail, see [Lu] and [Lu2]. We also give the reformulation of the Lusztig map.

A.1 Lusztig Map

Let $K = \mathbb{C}((t))$ be the set of Laurent series, $O = \mathbb{C}[[t]]$ the set of formal power series. Let $V$ be a vector space over $K$ with a fixed basis $e_1, ..., e_n$. Lattices in $V$ by the definition means free $O$-submodules in $V$ of rank $n$. Denote $L_0 = \text{Span}_O(e_1, ..., e_n)$ the standard lattice. The affine Grassmannian $\text{Gr}_{GL_n}$ of type A is the set of all lattices in $V$. It can be identified with the coset space

$$\text{Gr}_{GL_n} = GL_n(K)/GL_n(O).$$

Let $T$ be the maximal torus subgroup of $GL_n$. We make the usual identification

$$X_*(T)^+ = \{(a_1, ..., a_n) \in \mathbb{Z} | a_1 \geq \cdots \geq a_n \geq 0\}.$$

For each dominant coweight $\lambda = (a_1, ..., a_n) \in X_*(T)^+$,

$$t^\lambda = \begin{pmatrix} t^{a_1} & 0 & \cdots & 0 \\ 0 & t^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{a_n} \end{pmatrix}.$$ 

Let $e_0$ be the base point in $\text{Gr}_{GL_n}$. Set $e_\lambda = t^\lambda \cdot e_0 \in \text{Gr}_{GL_n}$. Note that $e_0$ can be identified with the standard lattice, and $e_\lambda$ can be identified with the lattice $\text{Span}_O(t^{a_1}e_1, ..., t^{a_n}e_n)$. Then, $\text{Gr}_{GL_n}$ admits the Cartan decomposition

$$\text{Gr}_{GL_n} = \bigsqcup_{\lambda \in X_*(T)^+} \text{Gr}_{GL_n}^\lambda,$$

where $\text{Gr}_{GL_n}^\lambda = GL_n(O) \cdot e_\lambda$. 

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Let \( \mathcal{N} \) be the set of nilpotent matrices in \( \mathfrak{gl}_n(\mathbb{C}) \). The **Lusztig map** \( \mathcal{N} \to \text{Gr}_{\text{GL}_n} \) is given by

\[
x \mapsto (tI_n - x) \cdot L_0.
\]

**Proposition A.1.1.**

1. The Lusztig map is \( \text{GL}_n(\mathbb{C}) \)-equivariant.

2. \( \mathcal{N} \) appears as an open subset of \( \mathcal{L}_n^+ = \{ \text{lattices } L \mid \dim_k(L_0/L) = n \} \).

3. For any partition \( \lambda = (\lambda_1, ..., \lambda_n) \) of \( n \), the nilpotent class \( O_\lambda \) can be identified with the set of lattices in \( \mathcal{L}_n^+ \) such that

\[
t : L_0/L \to L_0/L
\]

has Jordan blocks of size \( \lambda_1, ..., \lambda_n \).

4. Under Lusztig map, \( O_\lambda \) is open embedded in an affine Schubert cell \( \text{Gr}^\lambda_{\text{GL}_n} \).

**A.2 A Generalization of Lusztig map**

Lusztig also gives the generalization of the map. Let \( Q \) be a cyclic quiver

\[
\begin{array}{ccccccc}
1 & \leftrightarrow & 2 & \leftrightarrow & 3 & \cdots & \cdots & n-1 & \leftrightarrow & n
\end{array}
\]

with \( n \) vertices. Let \( d = (d_1, ..., d_n) \) be a dimension vector and \( N = d_1 + ... + d_n \). Let \( V \) be a \( \mathcal{K} \)-vector space of dimension \( N \) with the fixed basis \( B = B_1 \cup ... \cup B_n \) where \( B_i = \{ e^1_i, ..., e^d_i \} \). For each \( i = 1, ..., n \), fix lattices in \( V \)

\[
L_i = (\mathbb{C}B_i \oplus ... \oplus \mathbb{C}B_n) \oplus (t) \cdot B
\]

where \((t)\) is the ideal generated by \( t \). Fix the vector spaces \( P_i = \mathbb{C}B_i \). The standard flag of lattices is

\[
L_1 \supset L_2 \supset ... \supset L_n.
\]

Let \( \text{Fl}_d \) be the set of flags of lattices \( M_1 \supset M_2 \supset ... \supset M_n \) in \( V \) such that

\[
M_n \supset tM_1, \quad \dim(M_i/M_{i+1}) = d_i, \quad \dim(M_n/tM_1) = d_n.
\]
In fact, $\text{Fl}_d$ can be embedded to $GL_{N}(\mathcal{K})/P$ for some parabolic subgroup $P$. Let $Z$ be the set of flags of lattices $M_1 \supset M_2 \supset \cdots \supset M_n$ in $V$ such that for all $i$,

$$M_n \supset tM_1, \quad M_i \subset L_i, \quad \dim L_i/M_i = d_i.$$ 

Note that $L_2 \supset L_3 \supset \cdots \supset L_n \supset tL_1$ is the element of $Z$. Let $Z'$ be the subset of $Z$ with the additional condition $M_i \oplus P_i = L_i$ for all $i$. One can show that $Z'$ is a open subvariety of $Z$.

The group 

$$GL_d(\mathbb{C}) = \prod_{i=1}^{n} GL_n(\mathbb{C})$$

act on the set $\text{Rep}(Q_d)$ of representations of the cyclic quiver $Q$ of the dimension vector $d$. Let $\text{Rep}(Q_d)^{nil}$ be the set of representations in $\text{Rep}(Q_d)$ such that the composition is nilpotent. Note that $GL_d(\mathbb{C})$ also acts on $\text{Rep}(Q_d)^{nil}$. The generalized Lusztig map $\text{Rep}(Q_d)^{nil} \to Z'$ is given by

$$(f_1, \ldots, f_n) \mapsto \theta(L_2 \supset \cdots \supset L_n \supset tL_1)$$

where $f_{i+1,i} : V_{i+1} \to V_i$ for $i = 2, \ldots, n-1$, $f_{1n} : V_1 \to V_n$ and $\theta$ is a $\mathcal{K}$-linear map

$$\theta = \begin{pmatrix}
I_{d_1} & -f_2 & 0 & \cdots & 0 \\
0 & I_{d_2} & -f_3 & \cdots & 0 \\
0 & 0 & I_{d_4} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & -f_n \\
-t^{-1} f_1 & 0 & 0 & \cdots & I_{d_n}
\end{pmatrix}$$

The generalized Lusztig map is bijective and $GL_d(\mathbb{C})$-equivariant. If $n = 1$, it is exactly the Lusztig map $\mathcal{N} \to \text{Gr}_{GL_n}$ as defined before.

Let $\mathfrak{g} = gl_N(\mathbb{C})$ and $G = GL_N(\mathbb{C})$. Define Lie algebra isomorphism , $\sigma : \mathfrak{g} \to \mathfrak{g}$ of order $n$ by $\sigma(x) = JxJ^{-1}$,
where

\[
J = \begin{pmatrix}
I_{d_1} & 0 & 0 & \cdots & 0 \\
0 & \xi^{-1}I_{d_2} & 0 & \cdots & 0 \\
0 & 0 & \xi^{-2}I_{d_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \xi^{-(n-1)}I_{d_n}
\end{pmatrix}
\]

and \(\xi\) is a primitive \(n\)-th root of unity. Then there is a decomposition on \(\mathfrak{g}\).

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}
\]

where \(\mathfrak{g}_i\) is the \(\xi^i\)-eigenspace. There exists a bijection from \(\text{Rep}(Q_d)^{nil}\) to \(N_{\mathfrak{g}_1}\) given by

\[
(f_1, f_2, \ldots, f_n) \mapsto \begin{pmatrix}
0 & f_2 & 0 & \cdots & 0 \\
0 & 0 & f_3 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

\(\sigma\) induces the map, using the same notation, \(\sigma : \text{GL}_N(K) \to \text{GL}_N(K)\) by

\[
\sigma(x(t)) = Jx(\xi^t)J^{-1}.
\]

Define \(\varphi : \text{GL}_N(K) \to \text{GL}_N(K)^{\sigma}\) by

\[
\varphi(x(t)) = J(t)x(t^n)J(t)^{-1}
\]

where

\[
J(t) = \begin{pmatrix}
I_{d_1} & 0 & 0 & \cdots & 0 \\
0 & tI_{d_2} & 0 & \cdots & 0 \\
0 & 0 & t^2I_{d_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & t^{n-1}I_{d_n}
\end{pmatrix}
\]
Since $\varphi(P) = GL_N(O)^r$, the map $\varphi$ induces a bijection

$$\tilde{\varphi} : GL_N(K)/P \rightarrow GL_N(K)^r/GL_N(O)^r.$$ 

The below commutative diagram shows reformulation of the generalized Lusztig map:

$$\begin{align*}
\text{Rep}(Q_d)^{nil} & \overset{\approx}{\longrightarrow} N_{\theta_1} \overset{x \mapsto (1-t^{-1})xG(O)^r}{\longrightarrow} GL_N(K)^r/GL_N(O)^r \\
Z' & \overset{\approx}{\longleftarrow} Z & \overset{\approx}{\longleftarrow} \text{Fl}_{d} & \overset{\approx}{\longrightarrow} GL_N(K)/P.
\end{align*}$$
Let $g$ be a simple Lie algebra of type $X_N$ and $\sigma$ its diagram automorphism of order $r$, denoted by the pair $(X_N, r)$. Then there is a grading on $g$,

\[ g = g_0 \oplus g_1 \oplus \cdots \oplus g_{r-1} \]

where $g_i$ is the $\xi^i$-eigenspace and $\xi$ is a primitive $r$-th root of unity. Let $K$ be a Lie group corresponding to the Lie algebra $g_0$. In this appendix, we consider the case $(X_n, r) = (A_{2\ell-1}, 2), (D_{\ell+1}, 2), (E_6, 2), (D_4, 3)$ or $(A_{2\ell}, 2)$ and the dimension of the $K$-orbits of highest weight vector in $g_1$. Further details can be found in [Ka, 6.7, 7.9, 7.10, 8.3].

Let $v_1, ..., v_n$ be the standard basis of $\mathbb{R}^n$, and $(\cdot \mid \cdot)$ a bilinear form on $\mathbb{R}^n$ given by

\[ (v_i \mid v_j) = \delta_{ij}. \]

Let $Q(X_N)$ be the root lattice of type $X_N$. The set of all roots is

\[ \Phi' = \{ \alpha \in Q(X_N) \mid (\alpha \mid \alpha) = 2 \} \]

and we denote the set of all simple roots by $\Delta'$. Then for $\alpha_i, \alpha_j \in \Delta'$,

\[ (\alpha_i \mid \alpha_j) = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases} \]

For each $X_N$, the root lattice, the set of all roots, and the set of all simple roots can be described below

\[ Q(X_N) = \begin{cases} \{ k_1v_1 + ... + k_{\ell+1}v_{\ell+1} \in \mathbb{R}^{\ell+1} \mid k_i \in \mathbb{Z}, \sum k_i = 0 \} & \text{if } X_N = A_\ell; \\
\{ k_1v_1 + ... + k_\ell v_\ell \in \mathbb{R}^\ell \mid k_i \in \mathbb{Z}, \sum k_i \in 2\mathbb{Z} \} & \text{if } X_N = D_\ell; \\
\{ k_1v_1 + ... + k_6 v_6 + \sqrt{2}k_7 v_7 \in \mathbb{R}^7 \mid \text{all } k_i \in \mathbb{Z}, \text{or all } k_i \in \frac{1}{2} + \mathbb{Z}, k_1 ... + k_6 = 0 \} & \text{if } X_N = E_6, \end{cases} \]
\[ \Phi' = \begin{cases} 
\{ v_i - v_j | i, j \in \{1, \ldots, \ell + 1\}, i \neq j \} & \text{if } X_N = A_\ell; \\
\{ \pm v_i \pm v_j | i, j \in \{1, \ldots, \ell\}, i \neq j \} & \text{if } X_N = D_\ell; \\
\{ v_i - v_j | i, j \in \{1, \ldots, 6\}, i \neq j \} \cup \{ \pm \sqrt{2}v_7 \} & \text{if } X_N = E_6; \\
\{ \frac{1}{2}(\varepsilon_1v_1 + \ldots + \varepsilon_6v_6) \pm \sqrt{2}v_7 \} \cup \{ \varepsilon_i = \pm 1, \sum \varepsilon_i = 0 \} & \text{if } X_N = E_6. 
\end{cases} \]

\[ \Delta' = \begin{cases} 
\{ \alpha_1 = v_1 - v_2, \alpha_2 = v_2 - v_3, \ldots, \alpha_\ell = v_\ell - v_{\ell+1} \} & \text{if } X_N = A_\ell; \\
\{ \alpha_1 = v_1 - v_2, \alpha_{\ell-1} = v_{\ell-1} - v_\ell, \alpha_\ell = v_\ell - v_1 \} & \text{if } X_N = D_\ell; \\
\{ \alpha_1 = v_1 - v_2, \alpha_2 = v_2 - v_3, \alpha_3 = v_3 - v_4, \alpha_4 = v_4 - v_5, \alpha_5 = v_5 - v_6, \alpha_6 = \frac{1}{2}(-v_1 - v_2 + v_4 + v_5 + v_6 + \sqrt{2}v_7) \} & \text{if } X_N = E_6. 
\end{cases} \]

The labelling of the Dynkin diagram of each type \( X_N \) follows from [Ka, Table Fin].

Define a function \( \epsilon : Q \times Q \to \{ \pm 1 \} \) as follow: choose an orientation of the Dynkin diagram, let

\[ \epsilon(\alpha_i, \alpha_j) = \begin{cases} 
-1 & \text{if } i = j \text{ or } i \text{ points } j; \\
1 & \text{otherwise} 
\end{cases} \]

for \( \alpha_i, \alpha_j \in \Delta' \), and then extend by the bimultiplicativity condition

\[ \begin{align*}
\epsilon(\alpha + \alpha', \beta) &= \epsilon(\alpha, \beta)\epsilon(\alpha', \beta) \\
\epsilon(\alpha, \beta + \beta') &= \epsilon(\alpha, \beta)\epsilon(\alpha, \beta')
\end{align*} \]

for \( \alpha, \alpha', \beta, \beta' \in Q \). Note that for \( \alpha \in Q \),

\[ \epsilon(\alpha, \alpha) = (-1)^{\frac{1}{2}(\alpha | \alpha)}. \]

Let \( \mathfrak{h}' \) be the complex hull of \( Q(X_N) \) and then extend \( (\cdot | \cdot) \) from \( Q \) to \( \mathfrak{h}' \) by bilinearity. For each case of \( Q(X_N) \), denote

\[ g(X_N) = \mathfrak{h}' \bigoplus_{\alpha \in \Phi'} \mathbb{C}X_\alpha \]
where $X_\alpha$ is a basis of 1-dimensional vector space $\mathbb{C}X_\alpha$. Define a bracket $[\cdot, \cdot]$ on $g$ by

\[
\begin{align*}
[h, h'] &= 0 \quad \text{if } h, h' \in \mathfrak{h}', \\
[h, X_\alpha] &= (h | \alpha)X_\alpha \quad \text{if } h \in \mathfrak{h}', \alpha \in \Phi', \\
[X_\alpha, X_{-\alpha}] &= -\alpha \quad \text{if } \alpha \in \Phi', \\
[X_\alpha, X_\beta] &= 0 \quad \text{if } \alpha, \beta \in \Phi', \alpha + \beta \notin \Phi' \cup \{0\}, \\
[X_\alpha, X_\beta] &= \epsilon(\alpha, \beta)X_{\alpha + \beta} \quad \text{if } \alpha, \beta \in \Phi', \alpha + \beta \in \Phi'.
\end{align*}
\]

Hence $g(X_N)$ is a Lie algebra of type $X_N$.

Consider the root lattices of type $X_N$ and their automorphism $\sigma$ order 2 or 3: for $\alpha_i \in \Delta'$,

- $(A_{2\ell-1}, 2): \sigma(\alpha_i) = \alpha_{2\ell-i}$;
- $(D_{\ell+1}, 2): \sigma(\alpha_i) = \alpha_i$ for $i = 1, \ldots, \ell - 1$, $\sigma(\alpha_\ell) = \alpha_{\ell+1}$, $\sigma(\alpha_{\ell+1}) = \alpha_\ell$;
- $(E_6, 2): \sigma(\alpha_1) = \alpha_5$, $\sigma(\alpha_2) = \alpha_4$, $\sigma(\alpha_3) = \alpha_3$, $\sigma(\alpha_6) = \alpha_6$;
- $(D_4, 3): \sigma(\alpha_1) = \alpha_3$, $\sigma(\alpha_2) = \alpha_4$, $\sigma(\alpha_3) = \alpha_1$, $\sigma(\alpha_4) = \alpha_2$;
- $(A_{2\ell}, 2): \sigma(\alpha_i) = \alpha_{2\ell-i+1}$.

We fix an orientation of the Dynkin diagrams of each type so that it is invariant under $\sigma$, except the type $(A_{2\ell}, 2)$, as shown in Figure B.1.

The automorphism $\sigma$ on $Q(X_N)$ induces an automorphism on $g(X_N)$, also denoted by $\sigma$, defined by for each $\alpha \in \Phi'$,

\[
\sigma(X_\alpha) = \begin{cases} X_{\sigma(\alpha)} & \text{if } (X_N, r) \neq (A_{2\ell}, 2); \\
(-1)^{1+\text{ht}\alpha} X_{\sigma(\alpha)} & \text{if } (X_N, r) = (A_{2\ell}, 2), \end{cases}
\]

where $\text{ht}\alpha$ is the height of $\alpha$.

Denote the following notations:

- $\Phi_\ell = \{\alpha \in \Phi' \mid \sigma(\alpha) = \alpha\}$,
- $\Phi_s = \left\{\frac{1}{2}(\alpha + \sigma(\alpha)) \mid \alpha \in \Phi', \sigma(\alpha) \neq \alpha\right\}$,
- $\Phi = \Phi_\ell \cup \Phi_s$.
Let $(X_N, r) = (A_{2\ell-1}, 2), (D_{\ell+1}, 2), (E_6, 2), (D_4, 3)$ or $(A_{2\ell}, 2)$. Then

$$g(X_N) = \begin{cases} 
    \mathfrak{t} \oplus p & \text{if } r = 2; \\
    \mathfrak{t} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 & \text{if } r = 3,
\end{cases}$$

where $\mathfrak{t}$ is the fixed point set of $\sigma$, $p$ is the eigenspace of $\sigma$ with eigenvalue -1, and $\mathfrak{g}_j$ is the eigenspace of $\sigma$ with eigenvalue $\exp \frac{2j\pi i}{3}$, $j = 1, 2$. In fact, $\mathfrak{t}$ is a Lie algebra of type $C_\ell, B_\ell, F_4$ or $G_2$, respectively. For the case $(X_N, r) = (A_{2\ell}, 2)$, $g(X_N) = \mathfrak{t} \oplus p$ where $\mathfrak{t}$ is a Lie algebra of type $B_\ell$. $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{t}$. Moreover, $\Phi$ is the set of roots of $\mathfrak{t}$. We adopt $\theta_0$ from the proof of the Lemma 2.2.2 and denote $X \in g$ by

$$X = \begin{cases} 
    X_{\theta_0} - X_{\sigma(\theta_0)} & \text{if } (X_N, r) = (A_{2\ell-1}, 2), (D_{\ell+1}, 2), (E_6, 2); \\
    X_{\theta_0} + e^2 X_{\sigma(\theta_0)} + eX_{\sigma^2(\theta_0)} & \text{if } (X_N, r) = (D_4, 3); \\
    X_{\theta_0} & \text{if } (X_N, r) = (A_{2\ell}, 2)
\end{cases}$$
where \( \epsilon = \exp \frac{2\pi i}{3} \). Then \( X \) is a highest weight vector of the \( t \)-module \( \mathfrak{p} \) or \( \mathfrak{g}_1 \). Let \( K \) be a Lie group corresponding to \( t \). Then \( K \) acts on \( \mathfrak{p} \) and \( \mathfrak{g}_1 \). Let \( a_0, \ldots, a_\ell \) be a number labelled in the affine type of the Dynkin diagram, cf.[Ka, Table Aff 2], and \( h = \sum_{i=0}^\ell a_i \) the Coxeter number.

**Theorem B.0.1.** \( \dim K \cdot X = \frac{2}{a_0} (h - 1) \). In fact,

| \((X_N, r)\) | \( \dim K \cdot X \) | \( h \) | \( a_0 \) |
|---|---|---|---|
| \((A_{2\ell-1}, 2)\) | \(4\ell - 4\) | \(2\ell - 1\) | 1 |
| \((D_{\ell+1}, 2)\) | \(2\ell\) | \(\ell + 1\) | 1 |
| \((E_6, 2)\) | 16 | 9 | 1 |
| \((D_4, 3)\) | 6 | 4 | 1 |
| \((A_{2\ell}, 2)\) | \(2\ell\) | \(2\ell + 1\) | 2 |

Table B.1: Dimension of \( K \)-orbits of highest weight vectors

**Proof.** The dimension of the \( K \)-orbit of \( X \) is

\[
\dim K \cdot X = \dim t - \dim t^X.
\]

One can check that

\[
\dim t^X = \dim \mathfrak{b}^X + ||\{\alpha \in \Phi \mid [E_\alpha, X] = 0\}||.
\]

We consider case by case.

**Case** \((A_{2\ell-1}, 2)\): It can be checked that \(\{\alpha_1 + \alpha_{2\ell-1}, \alpha_3 + \alpha_{2\ell-3}, \alpha_4 + \alpha_{2\ell-4}, \ldots, \alpha_{\ell-1} + \alpha_{\ell+1}, \alpha_\ell\}\) is a basis of \( \mathfrak{b}^X \). We list all \( \alpha \in \Phi \) such that \( [E_\alpha, X] \neq 0 \):

- \(-(\alpha_2 + \ldots +, \alpha_{2\ell-2})\),
- \(-(\alpha_1 + \ldots +, \alpha_{2\ell-1})\),
- \(-\frac{1}{2}(\alpha_2 + \alpha_{2\ell-2})\),
- \(\frac{1}{2}\left(-\sum_{i=1}^k \alpha_i - \sigma(\sum_{i=1}^k \alpha_i)\right)\), \(\frac{1}{2}\left(-\sum_{i=2}^{k+1} \alpha_i - \sigma(\sum_{i=2}^{k+1} \alpha_i)\right)\) for \( k = 2, \ldots, 2\ell - 4 \),
- \(\frac{1}{2}\left(-\sum_{i=1}^{2\ell-3} \alpha_i - \sigma(\sum_{i=1}^{2\ell-3} \alpha_i)\right)\),
- \(\frac{1}{2}(\theta - \sigma(\theta))\).
Then
\[ \dim t^X = \dim b^X + \|\{\alpha \in \Phi \mid [E_\alpha, X] = 0\} \| = (\ell - 1) + (2\ell^2 - 4\ell + 5) = 2\ell^2 - 3\ell + 4, \]
and hence
\[ \dim K \cdot X = \dim t - \dim t^X = (2\ell^2 + \ell) - (2\ell^2 - 3\ell + 4) = 4\ell - 4. \]

**Case** \((D_{\ell+1}, 2)\): It can be checked that \(\{\alpha_2, \alpha_3, \ldots, \alpha_{\ell-1}, \alpha_\ell + \alpha_{\ell+1}\}\) is a basis of \(b^X\). We list all \(\alpha \in \Phi\) such that \([E_\alpha, X] \neq 0\):

- \(- (\alpha_1 + \ldots + \alpha_k)\), for \(k = 1, \ldots, \ell - 1\),
- \(- (\alpha_1 + \ldots + \alpha_{j-1} + 2\alpha_j + 2\alpha_{j+1} + \ldots + 2\alpha_{\ell-1} + \alpha_\ell + \alpha_{\ell+1})\), for \(j = 2, \ldots, \ell - 1\),
- \(-(\alpha_1 + \ldots + \alpha_\ell + \alpha_{\ell+1})\),
- \(- \frac{1}{2} (\theta^0 + \sigma\theta^0)\).

Then
\[ \dim t^X = \dim b^X + \|\{\alpha \in \Phi \mid [E_\alpha, X] = 0\} \| = (\ell - 1) + (2\ell^2 - 2\ell + 1) = 2\ell^2 - \ell, \]
and hence
\[ \dim K \cdot X = \dim t - \dim t^X = (2\ell^2 + \ell) - (2\ell^2 - \ell) = 2\ell. \]

**Case** \((E_6, 2)\): It can be checked that \(\{\alpha_2 + \alpha_4, \alpha_3, \alpha_6\}\) is a basis of \(b^X\). We list all \(\alpha \in \Phi\) such that \([E_\alpha, X] \neq 0\):

- \(- (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\),
- \(- (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6)\),
- \(- (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)\),
- \(- (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)\),
- \(- (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\),
- \(- (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\),
- \(- (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\),
- \(- \frac{1}{2} (\alpha_1 + \alpha_5)\).
\[ \cdot -\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5), \]
\[ \cdot -\frac{1}{2}(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5), \]
\[ \cdot -\frac{1}{2}(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5), \]
\[ \cdot -\frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6), \]
\[ \cdot -\frac{1}{2}(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + \alpha_5 + 2\alpha_6), \]
\[ \cdot -\frac{1}{2}(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6), \]
\[ \cdot -\frac{1}{2}(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6), \]
\[ \cdot -\frac{1}{2}(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_6). \]

Then
\[ \dim t^X = \dim h^X + |\{\alpha \in \Phi \mid [E_\alpha, X] = 0\}| = 3 + 33 = 36, \]
and hence
\[ \dim K \cdot X = \dim t - \dim t^X = 52 - 36 - 16. \]

**Case** \((D_4, 3):\) It can be checked that \(\{\alpha_2\}\) is a basis of \(h^X.\) We list all \(\alpha \in \Phi\) such that \([E_\alpha, X] \neq 0:\)
\[ \cdot -(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4), \]
\[ \cdot -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \]
\[ \cdot -\frac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4), \]
\[ \cdot -\alpha_2 - \frac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4), \]
\[ \cdot -\alpha_2 - \frac{2}{3}(\alpha_1 + \alpha_3 + \alpha_4). \]

Then
\[ \dim t^X = \dim h^X + |\{\alpha \in \Phi \mid [E_\alpha, X] = 0\}| = 1 + 7 = 8, \]
and hence
\[ \dim K \cdot X = \dim t - \dim t^X = 14 - 8 = 6. \]
Case \((A_2, 2)\): It can be checked that \(\{\alpha_2 + \alpha_{2\ell - 1}, \alpha_3 + \alpha_{2\ell - 2}, \ldots, \alpha_\ell + \alpha_{\ell + 1}\}\) is a basis of \(\mathfrak{h}^X\). All \(\alpha \in \Phi\) such that \([E_\alpha, X] \neq 0\) are

\[-(\alpha_1 + \ldots + \alpha_i), \quad i = 1, \ldots, 2\ell - 1.

Then

\[
\dim t^X = \dim \mathfrak{h}^X + ||\alpha \in \Phi | [E_\alpha, X] = 0|| = (\ell - 1) + (2\ell^2 - 2\ell + 1) = 2\ell^2 - \ell,
\]

and hence

\[
\dim K \cdot X = \dim t - \dim t^X = (2\ell^2 + \ell) - (2\ell^2 - \ell) = 2\ell.
\]

\[\square\]

Let \(G\) be the adjoint group of \(\mathfrak{g}\), and \(G\mathcal{r}\) the twisted affine Grassmannian attached to \(G\) and \(\sigma\). Recall that \(G\mathcal{r}\) admits the Cartan decomposition 2.2, and \(\mathfrak{p}\) is the \(\epsilon\)-eigenspace of \(\sigma\). Let \(e_0\) be the base point of \(G\mathcal{r}\).

Define the \(K\)-equivariant map \(L : \mathcal{N}_p \to G\mathcal{r}\) by

\[
L(x) = \exp(t^{-1}\text{ad}x) \cdot e_0
\]

**Proposition B.0.2.** The map \(L\) is injective.

**Proof.** Let \(x, y \in \mathcal{N}_p\) be such that \(\exp(t^{-1}\text{ad}x) \cdot e_0 = \exp(t^{-1}\text{ad}y) \cdot e_0\). By Baker-Campbell-Hausdorff formula,

\[
\exp(t^{-1}\text{ad}x) \exp(-t^{-1}\text{ad}y) = \exp(Z)
\]

where \(Z = t^{-1}\text{ad}x - t^{-1}\text{ad}y - \frac{t^{-2}}{2} [\text{ad}x, \text{ad}y] + \cdots\). Since \(\exp(Z) \in G(O)^\sigma\), \(\text{ad}x = \text{ad}y\) and so \(x - y\) belongs to the center of the Lie algebra of \(G\). Since \(x\) and \(y\) are both nilpotent, \(x = y\). \(\square\)

**Proposition B.0.3.** Let \(\gamma_0\) be the highest short root of \(H = (\check{G})^\sigma\).

1. \(\dim K \cdot X = \dim \mathcal{G}r_{\gamma_0} \cap \mathcal{G}r_0^-\).

2. The orbit \(K \cdot X\) is open embedded by the map \(L\) into \(\mathcal{G}r_{\gamma_0} \cap \mathcal{G}r_0^-\).
Proof. One can check that

\[
\gamma_0 = \begin{cases} 
\gamma_1 + 2\gamma_2 + \ldots + 2\gamma_{\ell-1} + \gamma_\ell & \text{if } (X_N, r) = (A_{2\ell-1}, 2); \\
\gamma_1 + \ldots + \gamma_\ell & \text{if } (X_N, r) = (D_{\ell+1}, 2), (A_{2\ell}, 2); \\
\gamma_1 + 2\gamma_2 + 3\gamma_3 + 2\gamma_4 & \text{if } (X_N, r) = (E_6, 2); \\
\gamma_1 + 2\gamma_2 & \text{if } (X_N, r) = (D_4, 3). 
\end{cases}
\]

where \(\gamma_i\) is a simple root of \(H\) defined by (4.1). By Theorem (B.0.1),

\[
\dim K \cdot X = \langle 2\rho, \gamma_0 \rangle = \dim \mathcal{G}r_{\gamma_0}
\]

where \(\rho\) is the sum of all fundamental coweights of \(H\). This proves the first part.

For any \(a \in \mathbb{C}(t)\),

\[
\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{-1} & 1 \end{pmatrix} \quad (B.1)
\]

For each root vector \(X_\alpha\), we attach an \(sl_2\)-triple \([X_\alpha, Y_\alpha] := -X_\alpha, \alpha\). Then the equation B.1 induces

\[
\exp(a \ad X_\alpha) = \exp(a^{-1} \ad Y_\alpha) a^2 \dot{X}_\alpha \exp(a^{-1} \ad Y_\alpha) \in G(K) \quad (B.2)
\]

where

\[
\dot{X}_\alpha = \exp(-\ad Y_\alpha) \exp(\ad X_\alpha) \exp(-\ad Y_\alpha).
\]

Assume that \((X_N, r) = (A_{2\ell-1}, 2), (D_{\ell+1}, 2), (E_6, 2)\). Recall that \(X = X_{\theta_0} - X_{\sigma(\theta_0)}\). By (B.2),

\[
\exp(t^{-1} \ad X) = \exp(t^{-1} \ad X_{\theta_0}) \exp(-t^{-1} \ad X_{\sigma(\theta_0)})
\]

\[
= (\exp(\ad Y_{\theta_0}) t^{-\dot{\theta}_0} \dot{X}_{\theta_0} \exp(\ad Y_{\theta_0})) (\exp(-\ad Y_{\sigma(\theta_0)}) (-t)^{\sigma \dot{\theta}_0} \dot{X}_{\sigma(\theta_0)} \exp(-\ad Y_{\sigma(\theta_0)}))
\]

\[
= \exp(\ad(Y_{\theta_0} - Y_{\sigma(\theta_0)})) (\exp(-\ad Y_{\theta_0}) (-1)^{-\sigma \dot{\theta}_0} t^{-\dot{\theta}_0} \dot{X}_{\theta_0} \dot{X}_{\sigma(\theta_0)} \exp(t^{-1} \ad(Y_{\theta_0} - Y_{\sigma(\theta_0)})))
\]

\[
= \exp(\ad(Y_{\theta_0} - Y_{\sigma(\theta_0)})) n^{\dot{\theta}_0} t^{-2\dot{\theta}_0 - 2\sigma \dot{\theta}_0} \dot{X}_{\theta_0} \dot{X}_{\sigma(\theta_0)} \exp(t^{-1} \ad(Y_{\theta_0} - Y_{\sigma(\theta_0)}))
\]

where \(n^{\dot{\theta}_0}\) is defined by (2.1). It is easy to check that \(\exp(\ad(Y_{\theta_0} - Y_{\sigma(\theta_0)})) t^{-2\dot{\theta}_0 - 2\sigma \dot{\theta}_0}, \dot{X}_{\theta_0}, \dot{X}_{\sigma(\theta_0)}\) are fixed by \(\sigma\). Hence \(\exp(t^{-1} \ad X) \in G(O)^\sigma n^{\dot{\theta}_0} G(O)^\sigma\).
For the case $(D_4, 3)$, $X = X_{\theta_0} + \epsilon^2 X_{\sigma(\theta_0)} + \epsilon X_{\sigma^2(\theta_0)}$. Set $Y = Y_{\theta_0} + \epsilon X_{\sigma(\theta_0)} + \epsilon^2 Y_{\sigma^2(\theta_0)}$. Similarly, we have

$$\exp(t^{-1}\text{ad}X) = \exp(t\text{ad}Y) = \exp(t\text{ad}Y) t^{-\delta_0}(et)^{-\sigma\delta_0}(e^2t)^{-\sigma^2\delta_0} s X_{\sigma(\theta_0)} s X_{\sigma^2(\theta_0)} \exp(t\text{ad}Y)$$

$$= \exp(t\text{ad}Y) n_{\delta_0} e^{\sigma\delta_0 + 2\sigma^2\delta_0 - 2\delta_0 - 2\sigma^2\delta_0} s X_{\sigma(\theta_0)} s X_{\sigma^2(\theta_0)} \exp(t\text{ad}Y)$$

Since $\exp(t\text{ad}Y), e^{\sigma\delta_0 + 2\sigma^2\delta_0 - 2\delta_0 - 2\sigma^2\delta_0}$, and $s X_{\sigma(\theta_0)} s X_{\sigma^2(\theta_0)}$ are fixed by $\sigma$, $\exp(t^{-1}\text{ad}X) \in G(O)^{\sigma} n_{\delta_0} G(O)^{\sigma}$. From the proof Lemma 2.2.2, $\tilde{\theta}_0 = \gamma_0$ if $(X_N, r) \neq (A_2, 2)$. In this case, $\exp(t^{-1}\text{ad}X) \cdot e_0 \in G r_{\gamma_0}$.

For the case $(A_2, 2)$, we have $X = X_{\theta_0}$. By (B.2),

$$\exp(t^{-1}\text{ad}X) \in G(O)^{\sigma} n_{\delta_0} G(O).$$

Recall that $\gamma_0$ is the highest short root of $H = (\check{G})^\sigma$. There exists $\lambda \in X_\sigma(T)^*$ such that $\lambda + \sigma\lambda = \tilde{\theta}_0$. Then $\lambda = \gamma_0$. Therefore, $G(O)^{\sigma} n_{\delta_0} G(O)^{\sigma}$ is the only double $G(O)^{\sigma}$-orbit that is contained in $G(O)^{\delta_0} G(O)$. Hence $\exp(t^{-1}\text{ad}X) \in G(O)^{\sigma} n_{\delta_0} G(O)^{\sigma}$ and so $\exp(t^{-1}\text{ad}X) \cdot e_0 \in G r_{\gamma_0}$.

We have proved that $L(X) = \exp(t^{-1}\text{ad}X) \cdot e_0 \in G r_{\gamma_0}$ for any cases of $(X_N, r)$. By the first part, $L(K \cdot X)$ contains an open subset. Since $L$ is $K$-equivariant, $L(K \cdot X)$ is an open subset of $G r_{\gamma_0} \cap G r_{-\gamma_0}$. This proves the second part.

Note that $G r_{\gamma_0}$ is a quasi-minuscule twisted Schubert variety and $K \cdot X$ is the minimal orbit, which is denoted it by $O_{\min}$. We prove that $O_{\min} \equiv G r_{\gamma_0} \cap G r_{-\gamma_0}$. This asserts the result from [HLR, C.4]. To prove this result, we need the following Lemma.

**Lemma B.0.4.** Let $\pi : G r_{-\gamma} \rightarrow \mathfrak{p}$ be defined in the Chapter 2. Then $\pi \circ L$ is the identity map on $N_\mathfrak{p}$.

**Proof.** By the adjoint representation, $G \hookrightarrow \text{GL}_n$. Recall that $G r_{-\gamma} \cong G(O^-)^{\sigma}$. We have the commutative diagram

$$\begin{array}{cccc}
N_\mathfrak{p} & \xrightarrow{\pi} & N_{\mathfrak{gl}_n} & \pi^1 \\
& \searrow & & \searrow \\
& L & \searrow & \text{GL}_n(O^-) \\
& & \downarrow & \downarrow \\
G(O^-)^{\sigma} & \xrightarrow{\pi} & \text{GL}_n(O^-) \\
& \downarrow & & \downarrow \\
& \mathfrak{p} & \rightarrow & \mathfrak{gl}_n
\end{array}$$
where \( L(x) = \exp(t^{-1}x) \) for nilpotent \( x \in \mathfrak{gl}_n \), and \( \pi^\dagger \) is defined by [AH]. Since \( \pi^\dagger \circ L \) is the identity map on \( \mathcal{N}_{\mathfrak{gl}_n} \), the result immediately follows. \( \square \)

**Theorem B.0.5.** \( \overline{O}_{\text{min}} \equiv \overline{Gr}_{\gamma_0} \cap \overline{Gr}_0^- \).

**Proof.** Since \( \exp \) is a closed immersion, \( \overline{O}_{\text{min}} \) is closed embedded into \( \overline{Gr}_{\gamma_0} \cap \overline{Gr}_0^- \). Since both spaces are irreducible, by Proposition B.0.2 and B.0.3, they are bijective. Note that \( \pi \) restricted to \( \overline{Gr}_{\gamma_0} \cap \overline{Gr}_0^- \) is a surjective morphism from \( \overline{Gr}_{\gamma_0} \cap \overline{Gr}_0^- \) onto \( \overline{O}_{\text{min}} \). By Lemma B.0.4, they are isomorphic. \( \square \)
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