Solutions of the Klein-Gordon equation on manifolds with variable geometry including dimensional reduction

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We develop the recent proposal to use dimensional reduction from the four-dimensional space-time \( D = (1 + 3) \) to the variant with a smaller number of space dimensions \( D = (1 + d), \ d < 3 \) at sufficiently small distances to construct a renormalizable quantum field theory. We study the Klein-Gordon equation on a few toy examples ("educational toys") of a space-time with variable special geometry, including a transition to a dimensional reduction. The examples considered contain a combination of two regions with a simple geometry (two-dimensional cylindrical surfaces with different radii) connected by a transition region. The new technique of transforming the study of solutions of the Klein-Gordon problem on a space with variable geometry into solution of a one-dimensional stationary Schrödinger-type equation with potential generated by this variation is useful. We draw the following conclusions: (1) The signal related to the degree of freedom specific to the higher-dimensional part does not penetrate into the smaller-dimensional part because of an inertial force inevitably arising in the transition region (this is the centrifugal force in our models). (2) The specific spectrum of scalar excitations resembles the spectrum of the real particles; it reflects the geometry of the transition region and represents its "fingerprints". (3) The parity violation due to the asymmetric character of the construction of our models could be related to violation of the CP symmetry.

Keywords: dimensional reduction, space with variable geometry, Klein-Gordon equation, spectrum of scalar excitations, violation of CP symmetry

I. INTRODUCTION

The main problem of standard quantum gravity with the classical Einstein-Hilbert action is related to the fact that the Newton constant \( G_N \), has a negative mass dimension \( |G_N| = |M^{2-D}| \) (in terms of a proper mass scale \( M \)), where \( D = 1 + d \) is the space-time dimension. Quantum gravity is not perturbatively renormalizable for \( D > 2 \). The same is true for the electroweak theory without the Higgs field. This research is connected with [1], where a simple palliative was proposed for temporarily neutralizing of the nonrenormalizability problem. The point is to use dimensional reduction (also see [2]) from the manifold with dimensionality \((1 + 3)\equiv 4\) to one with the smaller dimensionality \((1 + d), \ d < 3\) at sufficiently small distances (large momentum transfer).

Our approach does not assume any modification of the concept of time. Instead, we have in mind some smooth enough reduction of spatial dimensions. As a result, the physical space is continuous, but may be not infinitely smooth manifold, and consists of parts with different topological dimension. Below, we use some toy models of space-time with variable geometry. To acquire some physical intuition and experience, we start with scalar wave solutions of the Klein-Gordon equation (KGE).

We consider the KGE for a complex scalar field \( \varphi(x) \) on a \((1 + d)\)-dim space-time with the signature \( \{+, -, \ldots, -\} \), \( \mu, \nu = 0, 1 \ldots d, \ x = \{x^0, x^1, \ldots, x^d\} = \{x^0, \mathbf{x}\} \), where \( x^0 = t \) and \( d \geq 1 \) is an integer dimension of the space:

\[
\Box \varphi - M^2 \varphi = 0; \quad \Box \equiv -\frac{1}{\sqrt{|g|}} \partial_{\mu} \left( \sqrt{|g|} g^{\mu \nu} \partial_{\nu} \varphi \right). \tag{I.1}
\]

Commonly, after quantization [3] (see also Chap. 1 in [4] and [5]), a complex field \( \varphi(x) \neq \bar{\varphi}(x) \) corresponds to charged particles with electric charge \( Q = \pm 1 \) and (real) mass \( M \) with current \( j_\mu = \bar{\varphi} \partial_\mu \varphi - \varphi \partial_\mu \bar{\varphi} \) satisfying the continuity equation yielding the charge conservation law.

The simplest possibility for choosing a reference system is to use Gaussian coordinates in which the \((1+d)\)-dimensional operator \( \Box \) (at least locally) takes the form

\[
\Box = -\partial_0^2 + \Delta_d, \quad \Delta_d = \frac{1}{\sqrt{|g|}} \partial_m \left( \sqrt{|g|} g^{mn} \partial_n \right), \quad m, n = 1, \ldots, d. \tag{I.2}
\]
with the $d$-dimensional Laplacian $\Delta_d$ standardly defined by the $d$-dimensional metric $\gamma_{mn}$. From the very beginning we work in the framework of the $(1 + d)$-formalism with a global time that is obligatory for constructing a correct physical picture and, especially, a quantum field theory. As a consequence, the frequency $\omega$ of our solutions does not change on the parts with different topological dimension.

II. THE KGE IN A CYLINDRICAL SPACE GEOMETRY

A. The KGE in (1+2)-dimensions with cylindrical space symmetry

We consider the KGE

$$\Box \varphi - M^2 \varphi = - \partial_{\phi}^2 \varphi + \Delta_2 \varphi - M^2 \varphi = 0 \quad (\text{II.1})$$

on a two-dimensional space manifold $M^2_{\phi z}$ with cylindrical symmetry. In the Cartesian coordinates it is defined by a shape function $\rho(z)$ as a surface of rotation in the three-dimensional Euclidean manifold $\mathbb{R}^3_{X^1, X^2, X^3}$:

$$X^1 = \rho(z) \cos \phi, \quad X^2 = \rho(z) \sin \phi, \quad X^3 = z.$$  

The restriction of the three-dimensional interval $(dL)^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2$ on the two-dimensional manifold $M^2_{\phi z}$ is written as

$$(dL)^2 = \gamma_{mn} dx^m dx^n = \rho^2(z) d\phi^2 + \left(1 + \rho'^2\right) dz^2, \quad \rho' = d\rho/\rho.$$

The Laplacian in explicit form

$$\Delta_2 = \frac{1}{\rho^2} \left( \partial_{\phi \phi}^2 + \frac{\rho}{\sqrt{1 + \rho'^2}} \partial_z \frac{\rho}{\sqrt{1 + \rho'^2}} \partial_z \right) \quad (\text{II.3})$$

admits the separation of the variables $\varphi(t, \phi, z) = T(t) \Phi(\phi) Z(z)$, yielding a system of ordinary differential equations (ODEs). Two of them are simple: $T'' + \omega^2 T = 0 \Rightarrow T(t) = e^{-i\omega t}$ and $\Phi'' + m^2 \Phi = 0 \Rightarrow \Phi(\phi) = e^{im\phi}, \ m = 0, \pm 1, \pm 2, \ldots.$ The only nontrivial equation is for the function $Z(z)$,

$$\frac{1}{\rho \sqrt{1 + \rho'^2}} \partial_z \left( \frac{\rho}{\sqrt{1 + \rho'^2}} \partial_z Z \right) + \left( \omega^2 - M^2 - \frac{m^2}{\rho(z)^2} \right) Z = 0, \quad \Rightarrow \quad Z(z) = Z(z; \omega, m). \quad (\text{II.4})$$

It contains the remarkable term $m^2/\rho(z)^2$ with the form of the potential energy of centrifugal force acting for $m \neq 0$. We note that this term presents the simplest example of an inertial force with a transparent physical meaning. Such forces are inevitable for motion in a curved space-times. They arise in the transition regions between the parts of space with different topological dimensions. The hope therefore arises that we can learn something about the possible physical effects of the inertial forces acting in the class of curved space-times with variable dimensions described above.

B. Transformation of the problem to solution of Schrödinger-type ODE.

A useful mathematical property for studying solutions of Eq. (II.4) is contained in the following theorem.

**Theorem.** Stationary equation (II.4) can be transformed into the stationary Schrödinger-type equation

$$U''(u) + (E - V(u)) U(u) = 0. \quad (\text{II.5})$$

We note that the possibility of reducing solutions of the KGE to solutions of a Schrödinger-type equation arises because of the separation of variables in the KGE. Moreover, it uses the fact that any two-dimensional metric $\gamma_{mn}$ is conformally flat. Transforming the coordinate $z$ to $u$ and the shape function $\rho(z)$ to $\varrho(u) = \rho(z(u))$,

$$z \mapsto u : \quad u(z) = \int \sqrt{1 + \rho'(z)^2} \frac{dz}{\rho(z)}, \quad u \mapsto z : \quad z(u) = \int \sqrt{\varrho(u)^2 - \varrho'(u)^2} \, du,$$  

one obtains the Laplacian (II.3) in the form $\Delta_2 = \varrho^{-2} \left( \partial_{\phi \phi}^2 + \partial_{uu}^2 \right)$, which shows that the function $\varrho(u)^2$ in our problem is precisely the two-dimensional conformal factor. In terms of variable $u$, Eq. (II.4) takes the form (II.5):

$$U''(u) + \left( - (M^2 - \omega^2) \varrho(u)^2 - m^2 \right) U(u) = 0,$$  

(II.7)
with identification \( E = 0, V(u) = (M^2 - \omega^2) \rho(u)^2 + m^2, Z(z) = U(u(z)) \).

Studying the KGE on the considered curved manifolds thus reduces to solving the Schrödinger-type equation with the potential \( V(u) \) defined by the geometry. The broad literature devoted to the properties of such equations and their solutions for concrete potentials \( V(u) \) (see, e.g., [3, 4]) can now be used for our problem. We also note that there is no additive spectral parameter in Eq. (II.7) like the usual quantum energy \( E \) in (II.5) contained in the term \( E - V(u) \). Instead, the specific spectral parameter \( \omega \) of our problem appears in the factor \( M^2 - \omega^2 \) in the potential function \( V(u) \).

This result can be considered a realization of the old idea by Hertz [5] about a geometrical description of the physical forces, but now in the wave language. It can also be applied in the theory of wave guides in acoustics and optics, for radio-waves, and in other branches of wave physics.

C. The scalar wave equation in (1+1)-dimensional spacetime

For a one-dimensional case with the coordinate \( z \) and Laplacian \( \Delta_1 = \partial^2_{zz} \), the wave function has the simple form

\[
\varphi_1^{Qz}(t, z) = \text{const} \times e^{-i\omega t} e^\pm i\sqrt{\omega^2 - M^2}z : Q = +1 \Rightarrow \text{for particles},
\]

\[
\varphi_1^{Qz}(t, z) = \text{const} \times e^{i\omega t} e^\mp i\sqrt{\omega^2 - M^2}z : Q = -1 \Rightarrow \text{for antiparticles}.
\]

The additional superscript \( \pm \) denotes the sign of the momentum \( p_z = \pm k = \pm \sqrt{\omega^2 - M^2} \), or complex conjugated momentum \( \star p_z = \pm \star k = \pm \sqrt{\omega^2 - M^2} \). With the conventional \( \Re(\omega) \geq 0 \) we have the standard relativistic combination \(-i(\omega t - p_z z) = -ipx \) in the solutions for particles and \( i(\omega t - p_z \star z) = (-ipx)^* \) in the conjugate solutions for antiparticles.

III. SOME EXPLICIT EXAMPLES

A. Two cylinders of constant radii \( R \) and \( r < R \), connected by a part of cone

We turn to the simple case of the two-dimensional space \( \mathbb{R} \): the surface of two cylinders of radii \( R \) and \( r < R \) continuously connected by part of a cone (see Fig. 1). Let the symmetry axis be the horizontal axis \( Oz \) with the origin at the cone vertex. The shape function \( \rho(z) \) (see Fig. 2) then has the form

\[
\rho(z) = \begin{cases} 
R = \text{const} : & \text{for } z \in [z_R, +\infty), \\
\tan \alpha : & \text{for } z \in [z_r, z_R], \\
r = \text{const} : & \text{for } z \in (-\infty, z_r]. 
\end{cases}
\]

(III.1)

Here \( \alpha \in (0, \pi/2) \) is half of the (fixed) cone vertex angle, \( z_R = R \cot \alpha > 0 \), and \( z_r = r \cot \alpha > 0 \). The functions \( z(u) \) and \( \rho(u) \) (see Figs. 3 and 4) are obtained from (II.6):

\[
z(u; R, r, \alpha) = \begin{cases} 
R u : & \text{for } u \geq u_R, \\
R u_R \exp(u \sin \alpha - \cos \alpha) : & \text{for } u \in [u_r, u_R], \\
r(u + \ln(R/r)/\sin \alpha) : & \text{for } u \leq u_r,
\end{cases}
\]

(III.2)

\[
\rho(u; R, r, \alpha) = \begin{cases} 
R = \text{const} : & \text{for } u \geq u_R, \\
R \exp(u \sin \alpha - \cos \alpha) : & \text{for } u \in [u_r, u_R], \\
r = \text{const} : & \text{for } u \leq u_r,
\end{cases}
\]

(III.3)
where \( u_R = \cot \alpha \) and \( u_r = \cot \alpha - \ln(R/r)/\sin \alpha \).

For waves on the surface of a cylinder with an arbitrary radius \( \rho \) we have the dispersion relation:

\[
k_\rho = \sqrt{\omega^2 - M^2 - \frac{m^2}{\rho^2}}. \quad (\text{III.4})
\]

Under normalization: \( \varphi_{\omega,m}^Q = (a/\sqrt{2}) e^{-iQ \omega t} e^{i m \phi} e^{\pm i k_\rho z} \), the complex solutions of KGE with the charge \( Q = \pm 1 \) and mass \( M \) on the cylinder with radius \( \rho \) generate a conserved current with the components

\[
\begin{align*}
  j^t &= Q |a|^2 \Re(\omega), \quad j^\rho = m |a|^2, \quad j^z = \pm |a|^2 \Im(k_\rho).
\end{align*}
\]

(III.5)

As can be seen, for \( m \neq 0 \) the wave rotates on the surface of the cylinder because \( J^\phi \neq 0 \) in Eq. (III.5). This rotation of the scalar wave generates a centrifugal force. It becomes clear that for \( m \neq 0 \) in the limit \( \rho \to \rho_{\text{crit}} = |m|/\sqrt{\omega^2 - M^2} \) the height of the centrifugal potential barrier increase without bounded and stops all physical signals. In contrast, no such obstacle exists for a nonrotational movement along the axis Oz with \( m = 0 \). The physical signals can propagate without obstacles on the common physical degree of freedom of the two parts of the physical space with different topological dimension.

Using the standard continuity conditions for the function \( Z(z) \) and its derivative \( Z'(z) \) we obtain (see the notation in Appendix A)

\[
\begin{align*}
\mathcal{Z}(z; \omega, m; R, r, \alpha) &= -\Theta^+ e^{ik_r z} \begin{cases}
  \frac{2ik_r z}{\Delta} e^{ik_R(z-z_R)} : & z \geq z_R, \\
  2ik_R \left( Y_R^{-} J_{\nu} (k_c z) - J_R^{-} Y_\nu (k_c z) \right) : & z_R \leq z \leq z_R, \\
  -e^{ik_r(z-z_R)} + \frac{\Delta \omega}{\Delta} e^{-ik_r(z-z_R)} : & z \leq z_R,
\end{cases} \\
\mathcal{Z}(z; \omega, m; R, r, \alpha) &= -\Theta^- e^{-ik_R z} \begin{cases}
  \frac{2ik_R z}{\Delta} e^{-ik_R(z-z_R)} : & z \geq z_R, \\
  2ik_R \left( Y_R^{+} J_{\nu} (k_c z) - J_R^{+} Y_\nu (k_c z) \right) : & z_R \leq z \leq z_R, \\
  -e^{-ik_r(z-z_R)} + \frac{\Delta \omega}{\Delta} e^{ik_r(z-z_R)} : & z \leq z_R.
\end{cases}
\end{align*}
\]

(III.6) (III.7)
for waves going to the left from \( z = +\infty \) to \( z = -\infty \). We consider the limit \( r \to +0 \), \( R = \text{const}, \alpha = \text{const} \) in relation (III.1) when the two-dimensional cylinder of radius \( r \) with \( z \leq z_r \) transforms into the one-dimensional manifold, an infinitely thin thread stretched along the negative part of the axis \( OZ \). From a geometrical standpoint, this is a very singular limit (see Figs. 3 and 4). In this limit \( z_r = R \cot \alpha \) does not change, \( z_r \to 0 \) and the third piece of the functions \( z(u; R, r, \alpha) \) and \( \phi(u; R, r, \alpha) \) disappears because \( u_r \) goes to \( -\infty \). As a result, the shape function \( g(u; R, r, \alpha) \), or the corresponding conformal factor \( g(u; R, r, \alpha)^2 \) is unsuitable for describing the one-dimensional part of the emergent continuous (but not smooth) manifold with variable dimension because the infinite interval \( u \in (-\infty, \infty) \) is mapped only to the semi-infinite interval \( z \in [0, \infty) \) when \( r = 0 \).

1. The states of the continuum spectrum

If \( J_R^- \neq 0 \), then for real frequencies \( \omega \geq M \) we obtain

\[
S(\omega, m; R, \alpha) = \lim_{r \to 0} S(\omega, m; R, \alpha) = -\Theta^+_R e^{-2ik_n z_n} \left( J_R^+ / J_R^- \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).
\]

(III.8)

Several important consequences in the case \( m \neq 0 \) become obvious.

i) In the limit \( r \to +0 \) the transition coefficients are \( |S_{12}|^2 = |S_{21}|^2 \equiv 0 \). Hence, the communication by wave signals with an azimuthal \( m \neq 0 \) between the domains with different dimension is impossible.

ii) In fact, the limit \( r \to +0 \) of the solutions incoming from \( z = -\infty \) is trivial for any \( m \neq 0 \):

\[
\vec{Z}(z; \omega, m; R, \alpha) = \lim_{r \to 0} \vec{Z}(z; \omega, m; R, r, \alpha) \equiv 0
\]

everywhere because the factor \( \Theta^+_R = \Theta \left( \omega - \sqrt{M^2 + m^2 / r^2} \right) \equiv 0 \) for \( r < \sqrt{\omega^2 - M^2 / |m|} \) in (III.6). In addition \( S_{11} \to 0, S_{12} \to 0 \).

iii) In the two-dimensional domain, the modulus of the reflection coefficient for real frequencies \( \omega \) is \( |S_{22}| \equiv 1 \), i.e., we have a total reflection on the cone of the waves incoming from \( z = +\infty \), accompanied by a change of the phase of the scattered wave, according to Eq. (III.8). Hence, now we obtain the nontrivial solutions with \( m \neq 0 \):

\[
\vec{Z}(z; \omega, m; R, \alpha) = \lim_{r \to 0} \vec{Z}(z; \omega, m; R, r, \alpha) = -\Theta^-_R e^{-2ik_n z_n}
\]

\[
\left\{ \begin{array}{ll}
-e^{-ik_n (z-z_n)} + \left( J_R^+ / J_R^- \right) e^{ik_n (z-z_n)}, & z \geq z_n, \\
(2ik_n z_n / J_R^-) J_\nu(k_c z), & 0 \leq z \leq z_n, \\
0, & z \leq 0.
\end{array} \right.
\]

(III.9)

We note the following features of these solutions:

1. They are regular and finite everywhere, including the singular point \( z = 0 \) where they vanish.
2. They do not penetrate nor propagate into the one-dimensional part of the space.
3. There are no solutions with nonnegative real \( \omega < \sqrt{M^2 + m^2 / R^2} \) because the factor \( \Theta^-_R \) vanishes. This is obviously a correct physical result because the group velocity of the waves vanishes for \( \omega = \sqrt{M^2 + m^2 / R^2} \). In addition, the wavelength becomes infinite in this case.
4. The waves propagating on the cone junction have a complicated continuous spectrum of momenta \( p_z \). The spectrum can be obtained by the Fourier transform of solution (III.9) with respect to the variable \( z \). This spectrum is a sort of "fingerprint" characterizing the geometry of the transition region (a cone in this case). By studying this spectrum, we can hope to reconstruct the geometry of the junction, at least to some extent. An analogous problem was first posed in acoustics as early as 1877 by Lord Rayleigh, then substantially advanced in 1911 by Hermann Weyl and later by many others (see the recent review article [3] and the references therein).

2. The resonant states

Up to now, we have worked far from the poles of the scattering matrix \( S \) defined as zeros of the denominator in (III.1) or Eq. (III.9). For some \( m \neq 0 \) we consider a discrete (infinite) sequence of frequencies \( \omega_{n,m}, n = 0, 1, \ldots, \) satisfying the condition \( J_R^- = 0 \). Using the dimensionless variable \( \Omega = k_c z \) we write the spectral condition in the form \( \Omega J'_n(\Omega) / J_n(\Omega) = \cos \alpha \sqrt{\nu^2 - \Omega^2} \). A typical example of the absence of real roots and a representative sequence of complex roots is shown in Figs. 5 and 6. As can be seen, there is an infinite number of complex resonant frequencies

\[
\omega^Q_{n,m} = \sqrt{M^2 + \frac{\sin^2 \alpha}{R^2} \left( \Omega_{n,m}^Q \right)^2},
\]

(III.10)
which correspond to the complex roots $\Omega_{n,m}$ and depend to some extent on the mass $M$ of the scalar field (see Figs. 5, 6). The spectrum $\omega_{Q,n,m}^2$ characterizes the geometry of the conic form of the junction and the continuous but not smooth transition between the two cylinders.

Using the asymptotic expansion of the Bessel functions $J_\nu(\Omega)$ for large $|\Omega| \gg \nu = |m|/\sin \alpha$ one obtains the asymptotic form of the complex roots:

$$\Omega_{n,m}^Q \approx \pi \left( n + \frac{|m|}{2 \sin \alpha} - \frac{1}{4} \right) - i Q \text{artanh}(\cos \alpha), \quad n = 0, 1, 2, \ldots; \, m = 0, \pm 1, \pm 2, \ldots. \quad (\text{III.11})$$

The choice of the sign of the imaginary part of $\Omega_{n,m}^Q$ in (III.11) ensures the exponential decay of the field excitations as $t \to +\infty$ for both particles $Q = +1$ and antiparticles $Q = -1$.

The simple formula (III.11) to some extent reflects the influence of the geometry of the junction from the two-dimensional space to the one-dimensional space on the physics of resonant excitations of the Klein-Gordon field.

When the divergence angle $\alpha$ of the cone tends to the upper limit $\alpha \to \pi/2$ and $n \gg |m|$, we have the relation

$$\omega_{n,m}^Q \to \sqrt{M^2 + \frac{\pi^2}{R^2} \left( n + \frac{|m|}{2} - \frac{1}{4} \right)^2}.$$ 

In this limit the halflive of the resonant excitation increases without bounded, $\tau_{n=0,m} = 1/|\Im(\omega_{n=0,m}^Q)| \to \infty$, i.e., the resonant excitations are preserved for a very long time.

In the opposite, case when the angle $\alpha \to 0$, approximation (III.11) itself is unapplicable because $|\Omega_{n,m}^Q| \sim \pi \nu \to \infty$ and the condition $|\Omega_{n,m}^Q| \gg \nu$ is not satisfied as needed, because $|\Omega_{n,m}^Q|/\nu \sim \pi > 1$ but $\pi \gg 1$. The corresponding more accurate estimates can be found in [10].

Hence, we observe that the change in the topological dimension of space yields a new physical effect. Effective mass-spectrum (III.10) of excitations and their lifetime depend on the junction geometry between the two-dimensional and one-dimensional spaces.

Using the simplest normalization we write down two equivalent representations of the relativistic resonance wave functions:

$$Z_{\text{res}}(z; \omega_{n,m}, m; R, \alpha) = \begin{cases} e^{ikR(z-z_R)}, & z \geq z_R, \\ Z_{\text{res,c}}(z; \omega_{n,m}, m; R, \alpha), & z_L \leq z \leq z_R, \end{cases} \quad \text{where} \quad z_L = z_{r_R},$$

$$Z_{\text{res,c}}(z; \omega_{n,m}, m; R, \alpha) = \frac{\pi}{2} Y_R^{-1} J_\nu(k_c z) \equiv \frac{2ikRzR}{J_R^2(z)} J_\nu(k_c z),$$

obtaining an identity for Bessel functions and their derivatives as a byproduct.
B. A conelike junction with smooth transition to the cylinder

We consider the special case of the function \( \rho(u) = R \exp(pu)/(1 + \exp(pu))^1 \). In the variables \( \rho \) and \( z \), the corresponding form function of the surface \( \rho(z) \) is given by

\[
\frac{z(\rho)}{R} = \frac{1}{p} \ln 1 + \sqrt{1 - p^2(1 - \rho/R)^2} - \frac{1}{p} \sqrt{1 - p^2(1 - \rho/R)^2} + \text{const.} \tag{III.13}
\]

implicitly defining a two-dimensional surface (See Fig. 7). The possible forms of its sections are shown in Fig. 8. At the vertex, as before, this is a conelike surface with the angle \( \alpha = \arctan(p/\sqrt{1 - p^2}) \). Therefore \( p = \sin \alpha \).

![FIG. 7: A conelike junction with a smooth transition to the cylinder](image)

![FIG. 8: The sections of the two-dimensional surface \( \rho(z) \) for \( R = 1 \) and \( p = \sin \alpha \).](image)

For \( E = 0 \) the Schrödinger-like equation (II.5) with effective potential \( V(u) = V_0/(1 + e^{-pu})^2 + m^2 \), where \( V_0 = R^2(M^2 - \omega^2) \), has two independent solutions:

\[
U^\pm(u) = A^\pm(u)e^{\pm mu} = \frac{\Gamma(c_\pm)\Gamma(b_\pm - a_\pm)}{\Gamma(b_\pm)\Gamma(c_\pm - a_\pm)}B^\pm(u)e^{-u\sqrt{V_0 + m^2}} + \frac{\Gamma(c_\pm)\Gamma(a_\pm - b_\pm)}{\Gamma(a_\pm)\Gamma(c_\pm - b_\pm)}C^\pm(u)e^{+u\sqrt{V_0 + m^2}}, \tag{III.14}
\]

(see the notation in Appendix B). In addition we have \( A^\pm(u) \to 1 \) for \( u \to -\infty \), and \( B^\pm(u), C^\pm(u) \to 1 \) for \( u \to +\infty \).

If we substitute \(|m|\) instead of \( \pm m \) in all places in Eq. (III.14), then we obtain a solution \( \bar{U}(u; \omega, m; R, a) \), analogous to the solution (III.9) and also vanishing at the cone vertex.

The poles of the \( S \)-matrix are defined by the equations \( b_+ = -n, n = 0, 1, 2, \ldots \), or \( c_+ - a_+ = -n, n = 0, 1, 2, \ldots \). As a result, we obtain the sequence of frequencies:

\[
\omega_{n,m} = \sqrt{M^2 + m^2 + \frac{m^2 - p^2/4}{R^2}} \bar{Z}(2n + 1/p/2 + |m|) + \sqrt{m^2 - p^2/4}, \quad n = 0, 1, 2, \ldots, \tag{III.15}
\]

where we use the Zhukovsky function \( \bar{Z}(x) = (x + 1/x)/2 \). It can be seen that for \( M > 0 \) between frequencies (III.15), we have a finite number of real positive frequencies, for \( n \in [0, n_{\text{max}}] \), where \( n_{\text{max}} = \lfloor N \rfloor \) is the integer part of the

\footnote{Other examples of analytic solutions of the KGE on two-dimensional manifolds with cylindrical symmetry can be seen in [10].}
number
\[ N = \sqrt{\left(\frac{MR}{p}\right)^2 + \left(\frac{m}{p}\right)^2 - \frac{|m|}{p} + \sqrt{\left(\frac{MR}{p}\right)^2 + \left(\frac{1}{2}\right)^2 - \frac{1}{2}}} > 0. \]

Because the real frequencies are less than \( \sqrt{M^2 + m^2/R^2} \), the corresponding momenta \( \sqrt{\omega_{n,m}^2 - M^2 - m^2/R^2} \) are purely imaginary. Such solutions correspond to series of bound states with different \( n \) and \( m \), which decay exponentially at space infinity.

For \( n > n_{\text{max}} \), formula (III.15) gives an infinite series of purely imaginary frequencies.

IV. SUMMARY AND OUTLOOK

We generalize our results.

1. We have proved a useful theorem that relates studying the Klein-Gordon equation on spaces with variable geometry to solving a Schrödinger-type equation with an effective potential generated by the geometry variation. This result is based on separation of variables in the KGE and on the fact that two-dimensional spaces are conformally flat. We showed that in the case of space dimension \( d = 2 \), the conformal factor of the metric enters the effective potential in the Schrödinger-type equation because of the corresponding changes of variables. The generalization to space dimensions \( d > 2 \) was studied in [11].

2. As a corollary, we conclude that the obtained field excitation spectra could serve as "fingerprints" of the form \( \rho(z) \) of the junction region. The obtained nontrivial spectra for scalar excitations qualitatively resemble some actual spectra of resonances of elementary particles.

3. Signals related to the degrees of freedom of only the higher-dimensional part of space do not propagate freely through the junction region (in particular, these signals do not penetrate into the smaller-dimensional part). In our toy models, the reason is the centrifugal force acting on the solutions with nonzero angular momenta. In the limit case of dimensional reduction, this force grows infinitely and blocks all the solutions but one with azimuthal number \( m = 0 \). We can say that signals that penetrate from higher dimensions into lower ones are only the signals associated with common degrees of freedom.

Studying variable geometry in more realistic \( (d = 3) \) spaces is definitely interesting for further research. This case is close to problems in wave physics (acoustics, wave guides, etc.).

The parity violation (P-violation) due to the asymmetry of the space geometry is obvious. Hence, spaces with variable geometry can perhaps provide a simple basis for describing the real situation with the C, P and T properties of the particles.

The second result listed above suggests a new idea. We can try to reproduce the observed spectra of elementary particles using an appropriate geometry of the junction region between parts of the space-time with different topological dimension.

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Appendix A: Brief notation for the formulas in Sec. 3

In the problem with a conelike junction domain, we use the standard notation for the Bessel functions \( J_\nu(x) \) and \( Y_\nu(x) \). Sometimes only \( J_\nu(x) \) are called the Bessel functions and \( Y_\nu(x) \) are called the Neumann functions \( (N_\nu(x) \equiv Y_\nu(x)) \) [12, 13].
The wave numbers along the $z$-axis on the corresponding cylinders are $k_{z} = \sqrt{\omega^2 - M^2 - m^2/R^2}$ ($\Re(k_{z}) \geq 0$) and $k_{r} = \sqrt{\omega^2 - M^2 - m^2/r^2}$ ($\Re(k_{r}) \geq 0$). We use the notation $k = k_{c} \cos \alpha = \sqrt{\omega^2 - M^2}$ ($\Re(k) \geq 0$).

We use the brief notation

$$J_{\nu}^{\pm}(x, \alpha) = \left( x \frac{d}{dx} \pm i \kappa_{\nu}(x, \alpha) \right) J_{\nu}(x), \quad Y_{\nu}^{\pm}(x, \alpha) = \left( x \frac{d}{dx} \pm i \kappa_{\nu}(x, \alpha) \right) Y_{\nu}(x), \quad (A.1)$$

$$\kappa_{\nu}(x, \alpha) = \cos \alpha \sqrt{x^2 - \nu^2}, \quad \kappa_{z} = \kappa_{\nu}(k_{c} z_{r}, \alpha) = k_{r} z_{r}, \quad \kappa_{r} = \kappa_{\nu}(k_{c} z_{r}, \alpha) = k_{r} z_{r}, \quad (A.2)$$

$$J_{\nu}^{\pm}_{R} = J_{\nu}^{\pm}(k_{c} z_{r}, \alpha), \quad J_{\nu}^{\pm}_{r} = J_{\nu}^{\pm}(k_{c} z_{r}, \alpha), \quad Y_{\nu}^{\pm}_{R} = Y_{\nu}^{\pm}(k_{c} z_{r}, \alpha), \quad Y_{\nu}^{\pm}_{r} = Y_{\nu}^{\pm}(k_{c} z_{r}, \alpha), \quad (A.3)$$

$$\Delta = J_{\nu}^{+}_{R} Y_{\nu}^{-}_{R} - J_{\nu}^{+}_{r} Y_{\nu}^{-}_{r}, \quad \Delta_{11} = J_{\nu}^{+}_{R} Y_{\nu}^{-}_{R} - J_{\nu}^{+}_{r} Y_{\nu}^{-}_{r}, \quad \Delta_{22} = J_{\nu}^{+}_{R} Y_{\nu}^{-}_{R} - J_{\nu}^{+}_{r} Y_{\nu}^{-}_{r}, \quad (A.4)$$

This notation is useful for a compact description of the S-matrix in formulas (III.6)-(III.9), and (III.12).

**Appendix B: Brief notation in the formula (III.14)**

In formula (III.14) we use the brief notation

$$a_{\pm} = \frac{1}{2} + \frac{1}{p} \left( \sqrt{V_{0} + m^2} - \sqrt{\frac{p^2}{4} + V_{0} \pm m} \right),$$

$$b_{\pm} = \frac{1}{2} + \frac{1}{p} \left( -\sqrt{V_{0} + m^2} - \sqrt{\frac{p^2}{4} + V_{0} \pm m} \right), \quad c_{\pm} = 1 \pm \frac{2}{p},$$

$$A_{\pm}(u) = (1 + e^{pu})^{1/2 - \sqrt{1/4V_{0}/p^{2}}} F_{1}(a_{\pm}, b_{\pm}; c_{\pm}; -e^{pu}),$$

$$B_{\pm}(u) = (1 + e^{-pu})^{1/2 - \sqrt{1/4V_{0}/p^{2}}} F_{1}(a_{\pm}, a_{\pm} - c_{\pm} + 1; a_{\pm} - b_{\pm} + 1; -e^{-pu}),$$

$$C_{\pm}(u) = (1 + e^{-pu})^{1/2 - \sqrt{1/4V_{0}/p^{2}}} F_{1}(b_{\pm}, b_{\pm} - c_{\pm} + 1; b_{\pm} - a_{\pm} + 1; -e^{-pu}).$$

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