An exact characterization of saturation for permutation matrices

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A 0-1 matrix $M$ contains a 0-1 matrix pattern $P$ if we can obtain $P$ from $M$ by deleting rows and/or columns and turning arbitrary 1-entries into 0s. The saturation function $\text{sat}(P, n)$ for a 0-1 matrix pattern $P$ indicates the minimum number of 1s in an $n \times n$ 0-1 matrix that does not contain $P$, but changing any 0-entry into a 1-entry creates an occurrence of $P$. Fulek and Keszegh recently showed that each pattern has a saturation function either in $O(1)$ or in $\Theta(n)$. We fully classify the saturation functions of permutation matrices.

1. Introduction

In this paper, all matrices are 0-1 matrices. For cleaner presentation, we write matrices with dots ($\cdot$) instead of 1s and spaces instead of 0s, for example:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\cdot & \cdot & \cdot
\end{pmatrix}
\]

In line with this notation, we call a row or column empty if it only contains 0s. Furthermore, we refer to changing an entry from 0 to 1 as adding a 1-entry, and to the reverse as removing a 1-entry.

A pattern is a matrix that is not all-zero. A matrix $M$ contains a pattern $P$ if we can obtain $P$ from $M$ by deleting rows and/or columns, and removing arbitrary 1-entries. If $M$ does not contain $P$, we say $M$ avoids $P$. Matrix pattern avoidance can be seen as a generalization of two other well-known areas in extremal combinatorics. Pattern avoidance in permutations (see, e.g., Vatter’s survey [Vat14]) corresponds to the case where both $M$ and $P$ are permutation matrices; and forbidden subgraphs in bipartite graphs correspond to avoiding a pattern $P$ and all other patterns obtained from $P$ by

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permutation of rows and/or columns.\footnote{For this, we interpret the $M$ and $P$ as adjacency matrices of bipartite graphs.} There are also close connections to the extremal theory of ordered graphs \cite{PT06} and posets \cite{GNPV21}.

A classical question in extremal graph theory is to determine the maximum number of edges in an $n$-vertex graph avoiding a fixed pattern graph $H$. The corresponding problem in forbidden submatrix theory is determining the maximum weight (number of 1s) of an $m \times n$ matrix avoiding the pattern $P$, denoted by $\text{ex}(P,m,n)$. We call $\text{ex}(P,n) = \text{ex}(P,n,n)$ the extremal function of the pattern $P$. The study of the extremal function originates in its applications to (computational) geometry \cite{Mit87, Fü90, BG91}. A systematic study initiated by Füredi and Hajnal \cite{FH92} has produced numerous results \cite{Kla00, Kla01, MT04, Tar05, Kesz09, Ful09, Gen09, Pet11a, Pet11b}, and further applications in the analysis of algorithms have been discovered \cite{Pet10, CGK+15}.

Clearly, for non-trivial patterns, $\text{ex}(P,n)$ is at least linear and at most quadratic. Large classes of patterns with linear and quasi-linear extremal functions have been identified \cite{Kesz09, Pet11a}. On the other hand, there are patterns with nearly quadratic extremal functions \cite{ARSz99}.

A natural counterpart to the extremal problem is the saturation problem. A matrix $M$ is saturating for a pattern $P$, or $P$-saturating if it avoids $P$ and is maximal in this respect, i.e., turning any 0-entry of $M$ into a 1 creates an occurrence of $P$. Clearly, $\text{ex}(P,m,n)$ can also be defined as the maximum weight of an $m \times n$ matrix that is $P$-saturating. The function $\text{sat}(P,m,n)$ indicates the minimum weight of an $m \times n$ matrix that is $P$-saturating. We focus on square matrices and the saturation function $\text{sat}(P,n) = \text{sat}(P,n,n)$.

The saturation problem for matrix patterns was first considered by Brualdi and Cao \cite{BC20} as a counterpart of saturation problems in graph theory.\footnote{We refer to \cite{FK20} for references to graph saturation results.} Fulek and Keszegh \cite{FK20} started a systematic study. They proved that, perhaps surprisingly, every pattern $P$ satisfies $\text{sat}(P,n) \in \mathcal{O}(1)$ or $\text{sat}(P,n) \in \Theta(n)$, where the hidden constants depend on $P$. This is in stark contrast to the extremal problem, where a wide range of different orders of magnitude is attained by various patterns. Fulek and Keszegh also present large classes of patterns with linear saturation functions. For our purposes, their most important result is that every decomposable pattern has linear saturation function. We call a pattern $P$ decomposable if it has the form

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\]

for two matrices $A, B \neq 0$, where 0 denotes an all-0 matrix of the appropriate size. Otherwise, we call $P$ indecomposable. Also, patterns of the first form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are called sum decomposable, and patterns not of that form are called sum indecomposable.\footnote{These terms are derived from the theory of permutation patterns (see, e.g., Vatter \cite{Vat14}). We are not aware of a standard term for this property in the context of 0-1 matrices.}

Fulek and Keszegh also found a single non-trivial pattern with bounded saturation function ($Q$, pictured in Figure 1), and conjectured that there are many more. Geneson \cite{Gen20} recently confirmed this by proving that almost all permutation matrices have
bounded saturation function. A permutation matrix is matrix with exactly one 1-entry in each row and each column. A different class of matrices with bounded saturation function, containing both permutation matrices and non-permutation matrices where found recently by the author [Ber20].

In this paper, we show that, in fact, all indecomposable permutation matrices have bounded saturation function. This completes the characterization of permutation matrices in terms of their saturation function.

Theorem 1.1. A permutation matrix has linear saturation function if and only if it is decomposable.

A simple generalization of the technique that Fulek and Keszegh used to prove that $\text{sat}(Q, n) \in O(1)$ implies the following: To prove Theorem 1.1, it is sufficient to find a vertical witness for every indecomposable permutation matrix $P$, where we define a vertical witness for $P$ to be a matrix $M$ (of arbitrary size) that avoids $P$, has an empty row, and adding a 1-entry in that empty row creates an occurrence of $P$ in $M$.

We therefore construct vertical witnesses for all permutation matrices. Our constructions are based on the fact that indecomposable permutation matrices contain a certain substructure which we call spanning oscillation.

We also generalize a partial result to a class that contains non-permutation patterns:

Theorem 1.2. Let $P$ be a pattern that contains four 1-entries $x_1, x_2, x_3, x_4$ such that for each $i \in [4]$, there are no other 1-entries in the same row or column as $x_i$, and $x_i$ is in the first or last row or column, and $x_1, x_2, x_3, x_4$ form one of the two patterns

\[
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix},
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}.
\]

Then $\text{sat}(P, n) \in O(1)$.

In Section 1.1 we define (vertical) witnesses, and in Section 1.2, we define spanning oscillations. In Section 1.4 we introduce an alternative characterization of pattern containment that simplifies our proofs. In Sections 2 to 4, we construct vertical witnesses for all permutation matrices, based on different types of spanning oscillations, which proves Theorem 1.1. We also prove Theorem 1.2 in Section 2.

We now introduce conventions and notation used throughout the paper. Some more definitions that are only needed for Sections 2 to 4 will be introduced in Section 1.4.

4These results have been incorporated into this paper in Sections 1.1 and 2.
We identify 1-entries in an $m \times n$ matrix $M$ as their positions $(i, j) \in [m] \times [n]$, where $i$ is the row of the 1-entry (from top to bottom), and $j$ is its column (from left to right). $E(M)$ denotes the set of 1-entries in $M$. For two 1-entries $x = (i, j) \in E(M)$ and $x' = (i', j') \in E(M)$, we write $x <_v x'$ if $i < i'$ and $x <_h x'$ if $j < j'$. Define $x \leq_v x'$ and $x \leq_h x'$ analogously. We also say $x$ is above $x'$ if $x <_v x'$, and use below, to the right, and to the left similarly.

In a permutation matrix $P$, we denote the leftmost (rightmost, topmost, bottommost) 1-entry of $P$ by $\ell_P$ ($r_P$, $t_P$, $b_P$). Note that in an indecomposable $k \times k$ permutation matrix with $k \geq 2$, these four 1-entries are pairwise distinct.

Let $M$ be an arbitrary matrix. Denote by rot($M$) the matrix obtained by 90-degree clockwise rotation of $M$, denote by rev($M$) the matrix obtained by reversing all rows of $P$, and denote by trans($M$) the transpose of $M$, i.e., the matrix obtained by swapping the roles of rows and columns.\(^5\)

### 1.1. Witnesses

Let $P$ be a matrix pattern without empty rows or columns. An explicit witness\(^6\) for $P$ is a matrix $M$ that is $P$-saturating and contains at least one empty row and at least one empty column. If sat($P, n$) ∈ $O(1)$, then $P$ has an explicit witness: assume sat($P, n$) ≤ $c_P$, then there exists a $(c_P+1) \times (c_P+1)$ $P$-saturating matrix $M$ with at most $c_P$ 1-entries. Clearly, $M$ has an empty row and an empty column.

Fulek and Keszegh note that the reverse is also true: We can replace an empty row (column) in a $P$-saturating matrix by an arbitrary number of empty rows (columns), and the resulting arbitrarily large matrix will still be $P$-saturating. As such, an $m_0 \times n_0$ explicit witness for $P$ of weight $w$ implies that sat($P, m, n$) ≤ $w$ for each $m \geq m_0$ and $n \geq n_0$. Note that it is critical here that $P$ has no empty rows or columns. Otherwise, inserting empty rows or columns into $M$ might create an occurrence of $P$.

We call a row (column) of a matrix $M$ $P$-expandable if the row (column) is empty and adding a single 1-entry anywhere in that row (column) creates a new occurrence of $P$ in $M$. An explicit witness for $P$ is thus a saturating matrix with at least one $P$-expandable row and an $P$-expandable column. We define a witness for $P$ (used implicitly by Fulek and Keszegh) as a matrix that avoids $P$ and has at least one $P$-expandable row and at least one $P$-expandable column. Clearly, an explicit witness is a witness. The following lemma shows that finding a witness is sufficient to show that sat($P, n$) ∈ $O(1)$.

**Lemma 1.3.** If a pattern $P$ without empty rows or columns has an $m_0 \times n_0$ witness, then $P$ has an $m_0 \times n_0$ explicit witness.

**Proof.** Let $M$ be an $m_0 \times n_0$ witness for $P$. If $M$ is $P$-saturating, then we are done. Otherwise, there must be a 0-entry $(i, j)$ in $M$ that can be changed to 1 without creating an occurrence $P$. Choose one such 0-entry and turn it into 1. Note that $(i, j)$ cannot be contained in an expandable row or column of $M$, so the resulting matrix is still a

\(^5\)We do not use the common superscript $^t$, as it will later be used with the meaning “top”.

\(^6\)An explicit witness is what Fulek and Keszegh [FK20] call a witness.
witness. Thus, we obtain an explicit witness after repeating this step at most \( m_0 \cdot n_0 \) times.

1.1.1. Vertical and horizontal witnesses

Fulek and Keszegh also considered the asymptotic behavior of the functions \( \text{sat}(P, m_0, n) \) and \( \text{sat}(P, m, n_0) \), where \( m_0 \) and \( n_0 \) are fixed. The dichotomy of \( \text{sat}(P, n) \) also holds in this setting:

**Theorem 1.4** ([FK20, Parts of Theorem 1.3]). For every pattern \( P \), and constants \( m_0, n_0 \),

(i) either \( \text{sat}(P, m_0, n) \in O(1) \) or \( \text{sat}(P, m_0, n) \in \Theta(n) \);

(ii) either \( \text{sat}(P, m, n_0) \in O(1) \) or \( \text{sat}(P, m, n_0) \in \Theta(m) \).

We can adapt the notion of witnesses in order to classify \( \text{sat}(P, m_0, n) \) and \( \text{sat}(P, m, n_0) \). The following two lemmas are a generalization of the technique used by Fulek and Keszegh to prove that \( \text{sat}(Q, n) \in O(1) \) for the pattern \( Q \) depicted in Figure 1.

**Lemma 1.5.** Let \( P \) be a matrix pattern without empty rows or columns, and only one 1-entry in the last row or column. Let \( W \) be a horizontal (vertical) witness for \( P \). Then, appending an empty row (column) to \( W \) again yields a horizontal (vertical) witness.

**Proof.** We prove the lemma for horizontal witnesses, and appending a row. The other case follows by symmetry. Let \( W \) be an \( m_0 \times n_0 \) horizontal witness for \( P \), where the \( j \)-th column of \( W \) is expandable. Let \( W' \) be the matrix obtained by appending a row to \( W \). Clearly, \( W' \) still does not contain \( P \). Moreover, adding an entry in \( W' \) at \((i, j)\) for any \( i \neq n_0 + 1 \) creates a new occurrence of \( P \). It remains to show that adding an entry at \((n_0 + 1, j)\) creates an occurrence of \( P \).

We know that adding an entry at \((n_0, j)\) in \( W' \) creates an occurrence of \( P \). Let \( I \) the set of positions of 1-entries in \( W(P) \) that form the occurrence of \( P \). Since \( P \) has only one entry in the last row, all positions \( (i', j') \in I \setminus \{(n_0, j)\} \) satisfy \( i' < n_0 + 1 \). Thus, adding a 1-entry at \((n_0 + 1, j)\) instead of \((n_0, j)\) creates an occurrence of \( P \) at positions \( I \setminus \{(n_0, j)\} \cup \{(n_0 + 1, j)\} \), which implies that \( W' \) is a horizontal witness.

**Lemma 1.6.** Let \( P \) be a indecomposable pattern without empty rows or columns, and with only one 1-entry in the last row and one 1-entry in the last column. Then \( \text{sat}(P, n) \in O(1) \).

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A horizontal witness can be expanded horizontally, a vertical witness can be expanded vertically.
\( \mathcal{O}(1) \) if and only if there exist constants \( m_0, n_0 \) such that \( \text{sat}(P, m_0, n) \in \mathcal{O}(1) \) and \( \text{sat}(P, m, n_0) \in \mathcal{O}(1) \).

**Proof.** Suppose that \( \text{sat}(P, n) \in \mathcal{O}(1) \). Then \( P \) has an \( m_0 \times n_0 \) witness \( M \), and thus \( \text{sat}(P, m_0, n) \) is at most the weight of \( M \), for every \( n \geq n_0 \). Similarly, \( \text{sat}(P, m, n_0) \in \mathcal{O}(1) \).

Now suppose that \( \text{sat}(P, m_0, n) \in \mathcal{O}(1) \) and \( \text{sat}(P, m, n_0) \in \mathcal{O}(1) \). Then, for some \( m_1, n_1 \), there exists an \( m_0 \times n_1 \) horizontal witness \( W_H \) and an \( m_1 \times n_0 \) vertical witness \( W_V \). Consider the following \((m_0 + m_1) \times (n_0 + n_1)\) matrix, where \( 0_{m \times n} \) denotes the all-0 \( m \times n \) matrix:

\[
W = \begin{pmatrix}
0_{m_0 \times n_0} & W_H \\
W_V & 0_{m_1 \times n_1}
\end{pmatrix}
\]

We first show that \( W \) does not contain \( P \). Suppose it does. Since \( P \) is contained neither in \( W_H \) nor in \( W_V \), an occurrence of \( P \) in \( W \) must contain 1-entries in both the bottom left and top right quadrant. But then \( P \) is decomposable, a contradiction.

By Lemma 1.5, \( W_V' = (W_V, 0_{m_1 \times n_1}) \) is a vertical witness, and \( W_H' = (W_H, 0_{m_1 \times n_1}) \) is a horizontal witness. The expandable row in \( W_V' \) and the expandable column in \( W_H' \) are both also present in \( W \). This implies that \( W \) is a witness for \( P \), so \( \text{sat}(P, n) \in \mathcal{O}(1) \).

Figure 2 shows an example of a witness, constructed with Lemma 1.6, using vertical/horizontal witnesses presented later in Section 2, and an explicit witness constructed using Lemma 1.3.

Observe that the transformations rev, rot, and trans all preserve witnesses. However, the latter two change vertical witnesses to horizontal witnesses, and vice versa. Formally:

**Observation 1.7.** Let \( P \) be a matrix with a vertical witness \( W \). Then \( \text{rev}(W) \) is a vertical witness of \( \text{rev}(P) \), \( \text{rot}(W) \) is a horizontal witness of \( \text{rot}(P) \), and \( \text{trans}(W) \) is a horizontal witness of \( \text{trans}(P) \).

Recall that our goal is to show that every indecomposable permutation matrix has a witness. Since indecomposable permutation matrices are closed under transposition, Lemma 1.6 and Observation 1.7 imply that it suffices to find a vertical witness for each
indecomposable permutation matrix. The same is true for every class of permutation matrices satisfying the conditions of Lemma 1.6 that is closed under transposition or 90-degree clockwise rotation. This is useful to prove Theorem 1.2.

**Lemma 1.8.** Let \( \mathcal{P} \) be a class of indecomposable patterns without empty rows or columns, and with only one 1-entry in the last row and one 1-entry in the last column. If \( \mathcal{P} \) is closed under transposition or 90-degree clockwise rotation and each pattern in \( \mathcal{P} \) has a vertical witness, then \( \text{sat}(P,n) \in \mathcal{O}(1) \) for each \( P \in \mathcal{P} \).

**Proof.** Suppose that \( \mathcal{P} \) is closed under transposition and each \( P \in \mathcal{P} \) has a vertical witness. By Lemma 1.6, it suffices to show that each pattern in \( \mathcal{P} \) has a horizontal witness. Let \( P \in \mathcal{P} \). Then \( \text{trans}(P) \in \mathcal{P} \) has a vertical witness \( W \). By Observation 1.7, \( \text{trans}(W) \) is a horizontal witness for \( \text{trans}(\text{trans}(P)) = P \).

The case that \( \mathcal{P} \) is closed under 90-degree rotation can be handled analogously. \( \square \)

### 1.2. Spanning oscillations

We now introduce *spanning oscillations*, a class of substructures that characterizes indecomposable permutation matrices.

For a permutation matrix \( P \), the *permutation graph* \( G_P \) of the underlying permutation can be defined as follows: The vertex set is \( E(P) \), and two 1-entries \( x, y \in E(P) \) have an edge between them if \( x \) is below and to the left of \( y \) (or vice versa).

An *oscillation* in a permutation matrix \( P \) is a sequence \( X = (x_1, x_2, \ldots, x_m) \) of distinct 1-entries in \( P \) such that \( X \) forms an induced path in \( G_P \), i.e., there is an edge between \( x_i \) and \( x_{i+1} \) for each \( i \in [m - 1] \), and no other edges between 1-entries in \( X \). Oscillations have been studied before in several contexts [Pra73, BRV08, Vat11]. Vatter showed that a permutation matrix \( P \) is sum indecomposable if and only if it has an oscillation that starts with \( \ell_P \) and ends with \( r_P \) [Vat11, Propositions 1.4, 1.7]. Our characterization of indecomposable permutations is very similar. Call an oscillation \( X = (x_1, x_2, \ldots, x_m) \) spanning if \( \{x_1, x_2\} = \{\ell_P, t_P\} \) and \( \{x_{m-1}, x_m\} = \{b_P, r_P\} \).

**Lemma 1.9.** Let \( P \) be a sum indecomposable permutation matrix such that \( t_P \) is to the left of \( b_P \) or \( \ell_P \) is above \( r_P \). Then \( P \) has a spanning oscillation.

**Proof.** We write \( \ell, t, b, r \) for \( \ell_P, t_P, b_P, r_P \). By symmetry, we can assume that \( t \) is to the left of \( b \) (otherwise, replace \( P \) by \( \text{trans}(P) \), noting that \( G_P = G_{\text{trans}(P)} \)). Recall that \( \ell, t, b, r \) are pairwise distinct, as \( P \) is indecomposable and not \( 1 \times 1 \).

Since \( P \) is sum indecomposable, it has an oscillation \( X' = (x'_1, x'_2, \ldots, x'_m) \) with \( x'_1 = \ell, x'_m = r \). Suppose first that \( t \) occurs in \( X' \). Since \( G_P \) has an edge between \( \ell \) and \( t \), and \( X \) is an induced path in \( G_P \), this means that \( x'_2 = t \). Otherwise, note that \( t \) is connected in \( G_P \) to precisely those 1-entries that are to the left of \( t \). Let \( i \) be maximal such that \( x_i \) is to the left of \( t \). If \( i = 1 \), then \( (t, \ell, x_2', \ldots, x_m') \) is an induced path in \( G_P \). Otherwise, \( \ell, t, x'_1, \ldots, x'_m \) is an induced path in \( G_P \). In either case, we have an oscillation \( X'' = (x''_1, x''_2, \ldots, x''_m) \) that starts with \( \{\ell, t\} \) and ends with \( r \).

It remains to make sure that \( b \) is among the last two 1-entries in the oscillation. If \( b \) occurs in \( X'' \), then \( X_{m-1} = b \), as with \( t \). Otherwise, let \( j \) be minimal such that \( x_j \) is
to the right of $b$. If $j = m$, then $X = (x'_1, x'_2, \ldots, x'_{m-1}, r, b)$ is an induced path in $G_P$. Otherwise, $X = (x''_1, x''_2, \ldots, x''_b, b, r)$ is an induced path in $G_P$. Since $\ell, t$ are both to the left of $b$, we have $j \geq 2$, so $X$ is a spanning oscillation.

We obtain the following characterization of indecomposable permutation matrices.

**Corollary 1.10.** A permutation matrix $P$ is indecomposable if and only if $P$ or $\text{rev}(P)$ has a spanning oscillation or $P$ is the $1 \times 1$ permutation matrix.

**Proof.** First, assume $P$ is indecomposable. If $t_P$ is to the left of $b_P$, then Lemma 1.9 implies that $P$ has a spanning oscillation. If $t_P$ is to the right of $b_P$, then Lemma 1.9 implies that $\text{rev}(P)$ has a spanning oscillation. If $t_P = b_P$, then $P$ is $1 \times 1$.

Second, assume $P$ has a spanning oscillation. Then $P$ is sum indecomposable. Suppose $P$ is decomposable, then $P$ has the form \(( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \)\), so $t$ is to the right of $b$ and $\ell$ is below $r$. But then $\ell, b, t, r$ form the complete bipartite graph $K_{2,2}$ in $G_P$, implying that $P$ has no spanning oscillation, a contradiction. A symmetric argument shows that $P$ is indecomposable if $\text{rev}(P)$ has a spanning oscillation.

(Spanning) oscillations have a very rigid structure, which we now describe more concretely, in terms of relative positions of 1-entries. Let $P$ be a permutation matrix and $X = (x_1, x_2, \ldots, x_m)$ be a spanning oscillation of $P$. For $2 \leq i \leq m - 1$, call $x_i$ an upper 1-entry if $x_i$ is above and to the right of $x_{i-1}$ and $x_{i+1}$, and call $x_i$ a lower 1-entry if $x_i$ is below and to the left of $x_{i-1}$ and $x_{i+2}$. Since $G_P$ contains the edges $\{x_{i-1}, x_i\}$ and \{\{x_i, x_{i+1}\}$, but not the edge $\{x_i, x_{i+2}\}$, every 1-entry (except $x_1, x_m$) is either upper or lower. Clearly, upper and lower 1-entries alternate, i.e., $x_i$ is upper if and only if $x_{i+1}$ is lower, for $2 \leq i < m - 1$. It is convenient to also call $\ell_P, b_P$ lower 1-entries and $t_P, r_P$ upper 1-entries. We then have:

**Observation 1.11.** Let $P$ be a permutation matrix and $X = (x_1, x_2, \ldots, x_m)$ be a spanning oscillation of $P$. If $x_1 = \ell_P$, then all $x_i$ with odd $i$ are lower 1-entries, and all $x_i$ with even $i$ are upper 1-entries. If $x_1 = t_P$, then all $x_i$ with odd $i$ are upper 1-entries, and all $x_i$ with even $i$ are lower 1-entries.

It is easy to see that, if $x_1 = \ell_P$, then $x_3, x_4$ must be below and to the right of $x_1$. By induction, and by considering symmetric cases, we can prove:

**Observation 1.12.** Let $P$ be a permutation matrix and $X = (x_1, x_2, \ldots, x_m)$ be a spanning oscillation of $P$. Then $x_i$ is above and to the left of $x_j$ for each $i \in [m - 2]$ and $i + 2 \leq j \leq m$.

This leaves us with only two possible spanning oscillations for each length $m$, see Figure 3. Observe that spanning oscillations are preserved by transposition and 180-degree rotation, in the following sense. Let $P$ be a permutation matrix and $X$ be a spanning oscillation of $P$. Let $P' = \text{trans}(P)$ (resp., $P' = \text{rot}^2(P) = \text{rot(}\text{rot}(P)))$. Then $P'$ has a spanning oscillation $X'$ that corresponds to the transpose (resp., the 180-degree rotation) of $X$. With slight abuse of notation we write $X' = \text{trans}(X)$ (resp., $X' = \text{rot}^2(X)$).
A spanning oscillation \( X = (x_1, x_2, \ldots, x_m) \) is tall if the following two properties are satisfied for each \( 2 \leq i \leq m - 2 \) where \( x_i \) is an upper 1-entry.

(i) \( P \) has no 1-entry that is below \( x_{i+1} \) and to the left of \( x_i \).

(ii) \( P \) has no 1-entry that is above \( x_i \) and to the right of \( x_{i+1} \).

A spanning oscillation \( X \) is wide if \( \text{trans}(X) \) is tall. We now show that we can always assume that a minimum-length spanning oscillation is tall (or wide).

**Lemma 1.13.** Let \( P \) be a permutation matrix and \( X = (x_1, x_2, \ldots, x_m) \) be a spanning oscillation of \( P \) of minimum length \( m \). Then \( P \) has a tall spanning oscillation of length \( m \) that starts with \( x_1, x_2 \) and ends with \( x_{m-1}, x_m \).

**Proof.** Suppose \( X \) is not tall, so it violates (i) or (ii) at some index \( i \) with \( 2 \leq i \leq m - 2 \). We now show how to construct a spanning oscillation \( X' \) of length \( m \) that starts with \( x_1, x_2 \), ends with \( x_{m-1}, x_m \), and violates (i) or (ii) less often than \( X \). Repeating this, we eventually obtain a tall spanning oscillation.

Suppose first that \( X \) violates (i) at index \( i \). Then \( x_i \) is an upper 1-entry, and there is a \( y \in E(P) \) such that \( y \) is below \( x_{i+1} \) and to the left of \( x_i \). Assume \( y \) is the bottommost such 1-entry. Note \( y \notin \{\ell_P, b_P\} \), and that \( x_{i+2} \) is above \( x_{i+1} \) by Observation 1.11.

Let \( j \) be minimal such that \( x_j \) is to the right of \( y \). Since \( \ell_P <_h y <_h x_i \), we have \( 2 \leq j \leq i \). Let \( k \) be maximal such that \( x_k \) is above \( y \). Since \( x_{i+2} <_v y <_v b_P \), we have \( i + 2 \leq k \leq m - 1 \).

Consider the sequence \( X' = (x_1, x_2, \ldots, x_j, y, x_k, x_{k+1}, \ldots, x_m) \). We want to show that \( X' \) is a spanning oscillation of \( P \). Let \( j' < j \). By definition of \( j \), we know that \( x_{j'} \) is to the left of \( y \). By Observation 1.12, \( x_{j'} \) is above \( x_{i+1} \), implying that \( x_{j'} \) is above \( y \). Thus, \( G_P \) has no edge between \( x_{j'} \) and \( y \). Similarly, we can prove that there is no edge between \( y \) and \( x_{k'} \) for each \( k' > k \). This means that \( X' \) is an oscillation. Since \( 2 \leq j \)

Figure 3: The spanning oscillations of length \( m \), for \( m = 4, 5, 6, 7 \). The dashed line segments indicate the edges of the permutation graph. The borders indicate the possible positions for other 1-entries if the spanning oscillation is tall (top row) or wide (bottom row).
and $k \leq m - 1$, we know that $X'$ starts with $x_1, x_2$ and ends with $x_{m-1}, x_m$, implying that $X'$ is a spanning oscillation.

By assumption, $P$ has no spanning oscillation shorter than $P$, so $X$ must have length $m$, implying that $j = i$ and $k = i + 2$. Further, $X'$ does not violate (i) at index $i$, since, by choice of $y$, there are no 1-entries below $y$ and to the left of $x_j = x_i$. Thus, $X'$ has strictly less overall violations of (i) or (ii) than $X$.

The second case, where $X$ violates (ii), can be proven symmetrically.

Clearly, the statement of Lemma 1.13 is also true when replacing “tall” with “wide”, using the same proof on $\text{trans}(P)$.

1.3. Structure of the main proof

We divide the proof of Theorem 1.1 into three cases, proven in Sections 2 to 4. In Section 2, we handle the special case of length-4 spanning oscillations:

**Lemma 1.14.** Each permutation matrix with a spanning oscillation of length 4 has a vertical witness.

In Section 3, we prove:

**Lemma 1.15.** Each permutation matrix $P$ with a wide spanning oscillation of length $m \geq 5$ that starts with $t_P$ has a vertical witness.

The final and most involved case is treated in Section 4:

**Lemma 1.16.** Each permutation matrix $P$ with a tall spanning oscillation of even length $m \geq 6$ that starts with $\ell_P$ has a vertical witness.

It is not immediately obvious that Lemmas 1.14 to 1.16 cover all indecomposable permutation matrices. We now show that this is the case.

**Corollary 1.17.** Every indecomposable permutation matrix has a vertical witness.

*Proof.* Let $P$ be an indecomposable permutation matrix. If $P$ is $1 \times 1$, any all-zero matrix is a witness of $P$. Otherwise, one of $P$ and $\text{rev}(P)$ has a spanning oscillation $X$ by Corollary 1.10. By Observation 1.7, it suffices to find a vertical witness for either $P$ or $\text{rev}(P)$, so without loss of generality, assume that $X$ is a spanning oscillation of $P$, and that $X$ has minimum length $m$. If $m = 4$, we can apply Lemma 1.14. If $m \geq 5$ and $X$ starts with $t_P$, then Lemma 1.13 implies that $P$ also has a wide spanning oscillation of size $m$ that starts with $t_P$, so we can apply Lemma 1.15.

Now assume $m \geq 5$ and $X$ starts with $\ell_P$. If $m$ is even, we can apply Lemma 1.16, since by Lemma 1.13 we can assume that $X$ is tall. Otherwise, if $m$ is odd, Observation 1.11 implies that $X$ ends with $b_P$. This means that the spanning oscillation $\text{rot}^2(X)$ of $\text{rot}^2(P)$ starts with $t_{\text{rot}^2(P)}$, so we can apply Lemma 1.15 to obtain a witness $W'$ of $\text{rot}^2(P)$. Observation 1.7 implies that $\text{rot}^2(W')$ is a witness of $P$. \qed
1.4. Embeddings

In the following sections, we use an alternative definition of pattern containment based on sets of 1-entries. Let $P$ be a pattern and $M$ be a matrix. We say a function $\phi : E(P) \to E(M)$ is an embedding of $P$ into $M$ if for $x, y \in E(P)$ we have $x <_{h} y \iff \phi(x) <_{h} \phi(y)$ and $x <_{v} y \iff \phi(x) <_{v} \phi(y)$.

Note that if we allow empty rows or columns in $P$, then $E(P)$ does not determine $P$, since appending an empty row or column to $P$ does not change $E(P)$. This means that the existence of an embedding of $P$ into $M$ does not necessarily imply that $P$ is contained in $M$. However, we only consider patterns without empty rows or columns in this paper, and in that case, equivalence holds.

**Lemma 1.18.** Let $P$, $M$ be matrices, and let $P$ have no empty rows or columns. Then $P$ is contained in $M$ if and only if there is an embedding of $P$ into $M$.

A proof of Lemma 1.18 is provided in Appendix A. We now introduce some notation used in the following sections.

Let $x = (i, j)$, $y = (i', j')$ be two 1-entries. The horizontal distance between $x$ and $y$ is $d_{h}((i, j), (i', j')) = |i - i'|$, and the vertical distance between $x$ and $y$ is $d_{v}((i, j), (i', j')) = |j - j'|$. The width $\text{width}(A)$ (resp. height $\text{height}(A)$) of a set $A \subseteq E(M)$ is the maximum horizontal (resp. vertical) distance between 1-two entries in $A$.

Let $\phi$ be an embedding of $P$ into $M$, and let $x, y \in E(M)$. We define variants of the above notions that only “count” 1-entries of $M$ that are hit by $\phi$. This will be useful if we have some, but not full information about $\phi$. Let $d_{h}^{\phi}(x, y)$ be the number of 1-entries $z \in E(P)$ such that $x <_{h} \phi(z) \leq_{h} y$, and let $d_{v}^{\phi}(x, y)$ be the number of 1-entries $z \in E(P)$ such that $x <_{v} \phi(z) \leq_{v} y$. For $A \subseteq E(M)$, let $\text{width}_{\phi}(A) = \max_{x, y \in A} d_{h}^{\phi}(x, y)$, and $\text{height}_{\phi}(A) = \max_{x, y \in A} d_{v}^{\phi}(x, y)$.

**Observation 1.19.** Let $\phi$ be an embedding of $P$ into $M$, let $x, y \in E(P)$, and let $\phi(x), \phi(y) \in A \subseteq E(M)$. Then
\[
\begin{align*}
d_{h}^{\phi}(x, y) &= d_{h}^{\phi}(\phi(x), \phi(y)) \leq d_{h}^{\phi}(\phi(x), \phi(y)) \leq \text{width}(A); \\
d_{v}^{\phi}(x, y) &= d_{v}^{\phi}(\phi(x), \phi(y)) \leq d_{v}^{\phi}(\phi(x), \phi(y)) \leq \text{height}(A).
\end{align*}
\]

2. Spanning oscillations of length 4

In this section, we show Theorem 1.2, which immediately implies Lemma 1.14.

**Theorem 1.2.** Let $P$ be a pattern that contains four 1-entries $x_{1}, x_{2}, x_{3}, x_{4}$ such that for each $i \in [4]$, there are no other 1-entries in the same row or column as $x_{i}$, and $x_{i}$ is in the first or last row or column, and $x_{1}, x_{2}, x_{3}, x_{4}$ form one of the two patterns
\[
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{pmatrix},
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{pmatrix}.
\]

Then $\text{sat}(P, n) \in O(1)$.
Figure 4: Construction of $S(Q_1)$ from $Q_1$. The small dots indicate the expandable row.

Let $P$ denote the class of patterns defined in Theorem 1.2. Note that $P$ is closed under transposition. Thus, by Lemma 1.8, it is sufficient to prove that each $P \in P$ has a vertical witness.

Let $P'$ be the subset of patterns $P \in P$ where the unique leftmost 1-entry $\ell$ of $P$ is above the unique rightmost 1-entry $r$ of $P$. It is easy to see that $P$ has the following form, where the boxes contain arbitrarily many 1-entries:

\[
\begin{array}{c}
\ell \\
\end{array} 
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array} 
\begin{array}{c}
\ell \\
\end{array} 
\begin{array}{c}
\bullet \\
\end{array} 
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array} 
\begin{array}{c}
r \\
\end{array} 
\begin{array}{c}
b \\
\end{array}
\]

Since for each $P \in P \setminus P'$, we have rev$(P) \in P'$, Observation 1.7 implies that it is sufficient to prove that each $P \in P'$ has a vertical witness.

**Lemma 2.1.** Each $P \in P'$ has a vertical witness.

**Proof.** Let $P \in P'$ be a $k_1 \times k_2$ pattern, let $\ell = (i, j)$ be the unique leftmost 1-entry in $P$, and let $r = (i', j')$ be the unique rightmost 1-entry in $P$. Note that $i < i'$.

Let $P_L$ and $P_R$ be the submatrices of $P$ obtained by removing the rightmost, resp. leftmost, column. Note that in $P_L$, the $i'$-th row is empty, and in $P_R$, the $i$-th row is empty. We place a copy of $P_L$ to the left of $P_R$, so that the two empty rows coincide. Formally, obtain $L$ from $P_L$ by appending $i' - i > 0$ rows (at the bottom), obtain $R$ from $P_R$ by prepending $i' - i > 0$ rows (at the top), and define $S(P)$ as the concatenation $(L, R)$. Note that $S(P)$ is a $(k_1 + i' - i) \times (2k_2 - 2)$ matrix, and that the $i'$-th row of $S(P)$ is empty. Figure 4 shows an example of the construction. In the following, we will use $L$ and $R$ interchangeably with the corresponding subsets of $E(S(P))$.

We claim that the $i'$-th row is $P$-expandable. Indeed, adding a 1-entry in the $i'$-th row in the first $k - 1$ columns (to the left of $R$) completes an occurrence of $P$ with $R$, and adding a 1-entry in the last $k - 1$ columns (to the right of $L$) completes an occurrence of $P$ with $L$.

It remains to show that $S(P)$ avoids $P$. Suppose $S(P)$ contains $P$, so there is an embedding $\phi$ of $P$ into $S(P)$. Let $t, b \in E(P)$ be the unique topmost, respectively bottommost, 1-entry in $P$.

Suppose first that $\phi(b) \in L$. Since height$(L) = d^v(t, b) = k - 1$, and the $i'$-th row of $P$ is empty, we have height$_\phi(L) < d^v(t, b)$. This implies that $\phi(t)$ is above $L$. But $S(P)$ has no 1-entries above $L$, a contradiction.
Otherwise, $\phi(b) \in R$. Since $t$ is to the right of $b$, this implies that $\phi(t) \in R$. But a similar argument as above shows that $\height_{q}(R) < d'(t, b)$, a contradiction.

Thus, $S(P)$ avoids $P$ and has a $P$-expandable row, implying that $S(P)$ is a vertical witness of $P$. \hfill $\Box$

3. Spanning oscillations starting with $t$

In this section, we prove:

Lemma 1.15. Each permutation matrix $P$ with a wide spanning oscillation of length $m \geq 5$ that starts with $t_P$ has a vertical witness.

In Section 3.1 we present a construction of (possible) witnesses, which we first use for the case $m = 5$ in Section 3.2, and then for the case $m \geq 5$ in Section 3.3.

3.1. Witness construction

Let $P$ be an $k \times k$ permutation matrix such that $\ell = \ell_P$ is above $r = r_P$, and let $q = (i_q, j_q) \in E(P)$, such that $q$ is above $\ell$. We first construct a matrix $S'(P,q)$ with a $P$-expandable row, and then modify $S'(P,q)$ to obtain the matrix $S(P,q)$, which retains the expandable row and will be shown to avoid $P$ if $P$ has a wide spanning oscillation $(t_P, \ell_P, x_3, x_4, \ldots, x_m)$ with $m \geq 5$ and we choose $q = x_3$.

Let $P^L_R (P^L_L)$ be the submatrix of $P$ obtained by removing the leftmost (rightmost) column. Both $P^L_R$ and $P^L_L$ have an empty row. To start the construction of $S'(P,q)$, we place a copy of $P^L_R$ to the left of a copy of $P^L_L$, such that the two copies do not intersect, and the empty rows are aligned. We denote the copy of $P^L_R$ in the construction with $R$ and the copy of $P^L_L$ with $L$. Note that, compared to the construction in Section 2, $L$ and $R$ switch places.

Let $P^L_R$ consist of all columns to the left of $q$, and $P^L_R$ consist of all columns to the right of $q$. To finish the construction of $S'(P,q)$, we place a copy of $P^L_L$ to the left of $R$ and a copy of $P^L_R$ to the right of $L$, such that the empty $i_q$-th rows of $P^L_L$ and $P^L_R$ are aligned with the empty row in $R$ and $L$. Denote the copies of $P^L_L$ and $P^L_R$ as $L'$ and $R'$ and let $P' = L' \cup R'$.

Clearly, the empty row in $S'(P,q)$ is expandable: Adding a 1-entry to the left of $R$ will complete the partial occurrence $R$ of $P$, adding a 1-entry to the right of $L$ will complete $L$, and adding a 1-entry inside $R$ or $L$ will complete $P'$.

We modify $S'(P,q)$ to obtain $S(P,q)$ as follows.\footnote{This modification resembles the principle of Geneson’s construction. [Gen20]} Let $B$ be the set of entries in $P' = L' \cup R'$ that are below the leftmost 1-entry in $P'$ (the copy of $\ell$ in $P'$). Move $B$ down by a fixed number of rows, such that each 1-entry in $B$ is lower than all 1-entries in $R \cup L$. Clearly, the expandable row stays expandable after this change.

Figure 5 sketches the constructions. In the following sections, we denote the 1-entries in $S(P,q)$ as follows. If $x$ is a 1-entry in $P$, then let $x^R$ be the copy of $x$ in $R$, let $x^L$ be the copy of $x$ in $L$, and let $x'$ be the copy of $x$ in $P'$. For subsets $X \subseteq E(P)$, we use $X^R$, $X^L$, and $X'$ similarly.
We now show a property of $S(P,q)$ that is useful in both of the following subsections.

Lemma 3.1. Let $P$ be a $k \times k$ permutation matrix and $q \in E(P)$ such that $q < v_\ell < v_r$ and $t$ is to the left of $b$. If $\phi$ is an embedding of $P$ in $S(P,q)$, then $\phi(t) \notin L'$ and $\phi(b) \in R'$.

Proof. Let $L'_2$ denote the portion of $L'$ below $\ell'$, and let $L'_1 = L' \setminus L'_2$.

We first show that $\phi(t) \notin L'$. Suppose $\phi(t) \in L'$. Then also $\phi(\ell) \in L'$. Since height$(L'_2) < d^v(\ell,b)$, and there are no nonempty rows below $L'_2$, we know that $\phi(\ell) \notin L'_2$, and thus $\phi(t), \phi(\ell) \in L'_1$. But height$\phi(L'_1) \leq d^v(t,l) - 1$, a contradiction.

$\phi(t) \notin L'$ already shows that $\phi(b) \notin L'$, since $b$ is to the right of $t$. It remains to show that $\phi(b) \notin R \cup L$. First, suppose that $\phi(b) \in L$. Then there are at most $k - 2$ nonempty rows above $\phi(b)$, but $d^v(t,b) = k - 1$, a contradiction.

Second, suppose that $\phi(b) \in R$. Then $\phi(t) \in L' \cup R$, because $t$ is to the left of $b$. Since $q$ is above $\ell$, we have $d^v(t,q) < d^v(t,\ell)$, so $t^R$ is above $t'$, and thus $t^R$ is the highest 1-entry in $L' \cup R$. But then $d^v_\phi(\phi(t),\phi(b)) \leq d^v_\phi(t^R,b^R) \leq d^v(t,b) - 1$, a contradiction. \]

3.2. Length-5 spanning oscillations

Lemma 3.2. Let $P$ be a permutation matrix with a spanning oscillation $X = (t_P, x_2, x_3, x_4, x_5)$. Then $S(P,x_3)$ avoids $P$. 

Figure 5: $P$ and the two witness constructions $S'(P,q)$ and $S(P,q)$. The expandable row and the distance between $R, L$ and $P'$ are exaggerated.

Figure 6: $P$ and $S(P,q)$ in the case of Lemma 3.2.
Proof. Let \( q = x_3 \), and write \( \ell,t,b,r \) for \( \ell_P,t_P,b_P,r_P \). Note that \( x_2 = \ell \) and \( x_4 = b \), so \( q \) is above \( \ell \) and to the right of \( b \). Figure 6 sketches \( P \) and \( S(P,q) \). Suppose \( \phi \) is an embedding of \( P \) into \( S(P,q) \). By Lemma 3.1, \( \phi(b) \in R' \). But \( \text{width}(R') = d^h(q,r) - 1 < d^b(b,r) \), a contradiction. \( \square \)

3.3. Longer spanning oscillations

We now consider the case where \( P \) has a wide spanning oscillation \((t_P,x_2,\ldots,x_m)\) of length greater than five. We first prove a general statement on spanning oscillations starting with \( t_P \).

Lemma 3.3. Let \( P \) be a permutation matrix and \( X = (t_P,x_2,\ldots,x_m) \) be a spanning oscillation of \( P \) with \( m \geq 6 \). Then, removing \( t = t_P \), the columns to the left of \( t \), and the rows above \( x_3 \) (as well as all newly created rows or columns) does not make \( P \) decomposable.

Proof. Suppose it does, and let \( P_0 \) be the resulting decomposable pattern. Since \( x_3 \) is the highest 1-entry in \( P_0 \) (slightly abusing notation), and \( x_3 \) is above \( r = r_P \) and to the left of \( b = b_P \), we know that \( P_0 \) has the form \( \begin{pmatrix} A & 0 \\ B \end{pmatrix} \), where \( x_3 \) lies in \( A \) and \( r, b \) lie in \( B \). This means that \( x_4 \) lies in \( A \), since \( t <_h x_4 <_h x_3 \). Let \( P_1 \) be the matrix obtained from \( P_0 \) by further removing all columns to the right of \( x_4 \). Clearly, \( P_1 \) is decomposable, but \((x_3,x_4,\ldots,x_m)\) is a spanning oscillation of \( P_1 \), a contradiction. \( \square \)

We are now ready to prove the main result of this subsection.

![Figure 7: P and S(P,q) in the case of Lemma 3.4.](image)

Lemma 3.4. Let \( X = (t_P,x_2,\ldots,x_m) \) be a wide spanning oscillation of \( P \) with \( m \geq 6 \). Then \( P \) has a vertical witness.

Proof. We write \( \ell,t,b,r \) for \( \ell_P,t_P,b_P,r_P \) in the following. Let \( q = x_3 \), and let \( P_0 \) be the set of 1-entries of \( P \) that are to the right of \( t \) and not above \( q \). By Lemma 3.3, \( P_0 \) does not correspond to a decomposable pattern. Let \( A \) denote the set of 1-entries to the right of \( q \). Note that \( b, r \in A \), and, by wideness of \( X \), all 1-entries in \( A \) are below \( \ell \). Let \( x \) be the highest 1-entry in \( A \), and let \( B \) be the set of 1-entries below \( x \), to the left of \( q \) and to the right of \( t \). Then \( B \neq \emptyset \), otherwise \( P_0 \) would be decomposable. Finally,
$C = P_0 \setminus (A \cup B)$ consists of the 1-entries to the right of $t$, not above $q$, and above $x$. Figure 7 shows a sketch of $P$ and $S(P,X)$. Note that $A' = R'$.

Suppose $\phi$ is an embedding of $P$ into $S(P,q)$. By Lemma 3.1, $\phi(b) \in R'$ and $\phi(t) \notin L'$. Since all 1-entries in $B$ are to the right of $t$, this implies $\phi(y) \notin L'$ for each $y \in B$. Moreover, width($R'$) = $d^h(q,r) - 1 < d^h(y,r)$ for each $y \in B$, so we have $\phi(B) \subseteq L \cup R$.

Let $L'_2$ denote the portion of $L'$ below $\ell$ and let $L'_1 = L' \setminus L'_2$. Note that $L'_2$ is below all 1-entries in $L \cup R$. Since all 1-entries in $C$ are above all 1-entries in $B$, and all 1-entries in $A$ are to the right of all 1-entries in $B$, we have $\phi(R_0) = \phi(A \cup B \cup C) \subseteq L'_1 \cup L \cup R \cup R'$. Since $R' = A'$, all 1-entries in $R'$ are to the right and below all 1-entries in $L'_1 \cup L \cup R$, so $L'_1 \cup L \cup R \cup R'$ can be decomposed into the two blocks $L'_1 \cup L \cup R$ and $R'$. Further, $\phi(b) \in R'$ by Lemma 3.1, and since height($R'$) < $d^v(q,b)$, we have $\phi(q) \notin R'$.

This means that $P_0$ is decomposable, a contradiction. \qed

4. Even-length spanning oscillations starting with $\ell$

In this section, we prove:

Lemma 1.16. Each permutation matrix $P$ with a tall spanning oscillation of even length $m \geq 6$ that starts with $\ell_P$ has a vertical witness.

For our witness construction to work, we need to define a substructure that generalizes (all) spanning oscillations of even length that start with $\ell$. We call that substructure a traversal. Defining our witness construction for traversals instead of spanning oscillations will allow us to make a maximality assumption that is required for the proof that the witness avoids $P$.

4.1. Traversals

Let $P$ be a permutation matrix and let $m \geq 4$. A traversal of $P$ is a sequence $X$ of distinct 1-entries $x_1, x_2, \ldots, x_m$ such that

(i) $x_1 = \ell_P$, $x_2 = t_P$, $x_{m-1} = b_P$, $x_m = r_P$;
(ii) $x_1 <_h x_3 <_h x_2 <_h x_5 <_h x_4 <_h \ldots <_h x_{m-1} <_h x_{m-2} <_h x_m$;
(iii) $\ell_P <_v x_4 <_v x_6 <_v \ldots <_v x_m$;
(iv) $x_3 <_v x_5 <_v \ldots <_v x_{m-3} <_v r_P$; and
(v) $x_s$ is below $x_{s+1}$ for each odd $s \in [m-1]$.

Intuitively, property (ii) keeps the horizontal order of its 1-entries fixed, in the same way it is fixed in an even-length spanning oscillation starting with $\ell_P$. Vertically, however, we allow to arrange the 1-entries more freely. There are still upper (even) and lower (odd) 1-entries as in Observation 1.11 (this is implied by (iii), (iv), (v)), and we keep the order within the upper, resp. lower, 1-entries with (iii), (iv). But we drop the condition that $x_i$ is above $x_j$ for each odd $i \leq m - 3$ and even $j \geq i + 3$. This means that we are allowed to “move” some upper 1-entries upwards, and some lower 1-entries
downwards, as long as the vertical order among upper (lower) 1-entries is kept intact. (iii), (iv) additionally ensure that we cannot move any 1-entries above \( \ell_P \) or below \( r_P \). Figure 8 shows the shortest traversal that is not an oscillation.

We say a traversal \((x_1, x_2, \ldots, x_m)\) is tall if it satisfies the following two properties for each even \( 2 \leq i \leq m - 2 \).

(i) \( P \) has no 1-entry that is below \( x_{i+1} \) and to the left of \( x_i \).

(ii) \( P \) has no 1-entry that is above \( x_i \) and to the right of \( x_{i+1} \).

It is easy to see that each tall spanning oscillation of even length that starts with \( \ell \) is a tall traversal.

4.2. Maximality assumption

Let \( P \) be a permutation matrix with a tall traversal \( X \). We can assume that \( X \) is maximal in the sense that no tall traversal of \( P \) has \( X \) as a proper subsequence. We now show that such a maximally tall traversal also cannot be extended to a larger non-tall traversal in the following sense. Call a traversal \((x_1, x_2, \ldots, x_m)\) extendable if there is an odd \( s \) with \( 5 \leq s \leq m - 5 \), and two 1-entries \( y_1, y_2 \) in \( P \) such that \((x_1, x_2, \ldots, x_s, y_1, y_2, x_{s+1}, \ldots, x_m)\) is a traversal of \( P \).  

**Lemma 4.1.** Let \( X = (x_1, x_2, \ldots, x_m) \) be a maximally tall traversal of the permutation matrix \( P \). Then \( X \) is non-extendable.

**Proof.** Suppose \( X \) is extendable. Then there exists an odd \( s \) with \( 5 \leq s \leq m - 5 \) and 1-entries \( y_1, y_2 \in E(P) \) such that \( Y = (x_1, x_2, \ldots, x_s, y_1, y_2, x_{s+1}, \ldots, x_m) \) is a traversal of \( P \). We show that then \( P \) has an tall traversal of length \( m + 2 \) with \( X \) as a subsequence. This contradicts our assumption that \( X \) is maximally tall.

Note that property (v) of \( X \) implies that \( x_{s+1} \) is above \( x_s \). Further using properties (ii), (iii), (iv) of \( Y \), it follows that the relative positions of \( x_{s-1}, x_s, y_1, y_2, x_{s+1}, \) and \( x_{s+2} \) are fixed as shown in Figure 9.

Let \( y'_1 \) and \( y'_2 \) be 1-entries in \( P \) such that

(a) \( y'_2 \) is to the left of \( y'_1 \);

(b) \( y'_1 \) is above or equal to \( y_1 \) and \( y'_2 \) is below or equal to \( y_2 \); and
Figure 9: Arrangement of $x_{s-1}, x_s, y_1, y_2, x_{s+1}, x_{s+2}$ in Lemma 4.1. The shaded areas must be empty, since $X$ is tall.

(c) $d^y(y'_1, y'_2)$ is maximal under the previous two conditions.

and let $Y' = (x_1, x_2, \ldots, x_s, y_1', y_2', x_{s+1}, \ldots, x_m)$. We first show that $Y'$ is a traversal. $Y'$ clearly satisfies (i). Since $y'_1$ is not below $y_1$, it is above $x_{s+1}$, so tallness of $X$ implies that $y'_1$ is to the left of $x_{s+2}$. Symmetrically, $y'_2$ is to the right of $x_{s-1}$, so (a) implies $x_{s-1} < h y'_2 < h y'_1 < h x_{s+2}$, and thus $Y'$ satisfies (ii).

Since $x_{s-1}$ is to the left of $y'_1$, tallness of $X$ implies that $y'_1$ is below $x_{s-1}$. We already observed that $y'_1$ is above $x_{s+1}$, so we have $x_{s-1} < v y'_1 < v x_{s+1}$. Similarly, we have $x_s < v y'_s < v x_{s+2}$. Together with $x_{s+1} < v x_s$, this implies the remaining traversal properties (iii), (iv), (v).

It remains to show that $Y'$ is tall. Suppose $Y$ violates tallness property (vi). Since $X$ is tall, the only way this can happen is if there is a 1-entry $z$ below $y'_2$ and to the left of $y'_1$. Then $z$ is also below $y_2$, but $d^y(y'_1, z) > d^y(y'_1, y'_2)$, violating our assumption (c). A symmetric argument shows that $Y$ satisfies (vii).  

\[\square\]

4.3. Construction

Fix a $k \times k$ permutation matrix $P$. Throughout this subsection, we write $\ell, b, t, r$ for $\ell_P, b_P, t_P, r_P$. For a 1-entry $x = (i, j) \in E(P)$, denote by $P^L_x$ the submatrix of $P$ consisting of all columns to the left of $x$ (i.e., the leftmost $j - 1$ columns), and denote by $P^R_x$ the submatrix of $P$ consisting of all columns to the right of $x$ (i.e., the rightmost $k - j$ columns). Note that in both $P^L_x$ and $P^R_x$, the $i$-th row is empty. Also note that the constructions in Sections 2 and 3 implicitly used $P^L_x, P^R_x$, with $x \in \{\ell, r, q\}$.

Let $X = (x_1, x_2, \ldots, x_m)$ be a traversal of $P$ with $m \geq 6$, and write $(i_s, j_s) = x_s$ for $s \in [m]$. Then the $(2k - 1) \times (m - 2)k$ matrix $S(P, X)$ is constructed as follows. Let $L'_s$ be the $(2k - 1) \times (j_s - 1)$ matrix consisting of a copy of $P^L_{x_s}$ that is shifted down by $k - i_s$ rows (i.e., we prepend $k - i_s$ rows and append $i_s - 1$ rows to $P^L_{x_s}$). Similarly, let $R'_s$ be the $(2k - 1) \times (k - j_s)$ matrix consisting of a copy of $P^R_{x_s}$ that is shifted down by $k - i_s$ rows. Note that the empty $i_s$-th ($j_s$-th) row of $P^L_{x_s} (P^R_{x_s})$ corresponds to the $k$-th row of $L'_s (R'_s)$. Finally, we define $S'(P, X)$ as the following horizontal concatenation of matrices:

$$S'(P, X) = (L'_1, R'_1, L'_2, R'_3, L'_3, R'_4, \ldots, L'_{m-3}, R'_{m-4}, L'_{m-2}, R'_{m-3}, R'_m, L'_m, R'_{m-2}).$$
for an example. can be seen as a special case of both shows the rough structure of move them down by. similarly, let a copy of notation by writing construction that preserves the expandable row. in the following, we will slightly abuse notation by writing \( L_s' \) (\( R_s' \)) for the subsets of \( E(S'(P,X)) \) that correspond to \( L_s' \) (\( R_s' \)).

Let \( S(P,X) \) be a \( ((2m-6)k+1) \times (m-2)k \) matrix, constructed as follows. Start with a copy of \( S'(P,X) \), shifted down by \( (m-4)k \) rows, such that the expandable \( k \)-th row of \( S(P,X) \) corresponds to the \( (m-3)k \)-th row of \( S'(P,X) \). Now, for each \( s \in \{5,6,\ldots,m-1,m-2,m\} \), take all 1-entries in \( L_s' \cup R_s' \) that are above the \( (m-3)k-1 \)-th row (i.e., at least two rows above the expandable row), and move them up by \( (s-4)k \) rows. Similarly, for each \( s \in \{1,3,4,\ldots,m-4\} \), take all 1-entries in \( L_s' \cup R_s' \) that are below the \( (m-3)k+1 \)-th row (i.e., at least two rows below the expandable row), and move them down by \( (m-s-3)k \) rows. Figure 11 shows the rough structure of \( S(P,X) \) when \( m = 12 \) and \( X \) is tall.

Let \( L_s' \) (\( R_s' \)) denote the modified set of entries in \( S(P,X) \) corresponding to \( L_s' \) (\( R_s' \)). Clearly, \( L_s' \) and \( R_s' \) still form a partial occurrence of \( P \) with a single 1-entry missing between them in the \( (m-3)k \)-th row, and \( R_1' \), \( L_m' \) similarly form occurrences when adding a 1-entry in the left- or rightmost part of that row. Thus:

**Lemma 4.2.** If \( X \) is a traversal of \( P \), then \( S(P,X) \) has an expandable row.

Note that the construction used in Section 2 can be seen as a special case of both \( S(P,X) \) and \( S'(P,X) \) when \( m = 4 \).

The rest of this section is dedicated to the proof that if \( X \) is a non-extendable tall traversal of a permutation matrix \( P \), then \( S(P,X) \) avoids \( P \), implying that \( S(P,X) \) is a vertical witness of \( P \). We first fix some notation and make a few observations about \( S(P,X) \). Let \( T \) denote the set of 1-entries that are above row \( (m-3)k-1 \) (at least two rows above the expandable row). Similarly, let \( B \) denote the set of 1-entries that are below row \( (m-3)k+1 \), and let \( M \) denote the remaining 1-entries. For a subset
A \subseteq E(S(P, X))$, let $A^T = A \cap T$, let $A^B = A \cap B$ and let $A^M = A \cap M$. For a 1-entry $p \neq x_s$, let $p^s$ denote the copy of $p$ in $L_s \cup R_s$.

**Observation 4.3.** Let $s, u \in \{1, 3, 4, \ldots, m - 3, m - 2, m\}$ with $s < u$. If $u \geq 5$, then every 1-entry in $L^T_s \cup R^T_s$ is below every 1-entry in $L^T_u \cup R^T_u$. Moreover, if $s \leq m - 4$, then every 1-entry in $L^B_s \cup R^B_s$ is below every 1-entry in $L^B_u \cup R^B_u$. \hfill \Box

Since $X$ is tall, there are no 1-entries below and to the left of $x_s$ if $s$ is odd, or above and to the right of $x_s$ if $s$ is even. This implies:

**Observation 4.4.** $L^B_s = \emptyset$ and $R^T_s = \emptyset$ for each odd $s$ with $3 \leq s \leq m - 3$. \hfill \Box

We now consider the width and height of relevant parts of $S(P, X)$.

**Observation 4.5.** For each $s \in \{1, 3, 4, \ldots, m - 3, m - 2, m\}$,

- width($L_s$) = $d^h(\ell, x_s) - 1$;
- width($R_s$) = $d^h(x_s, r) - 1$;
- height($L^T_s \cup R^T_s$) = $d^v(t, x_s) - 2$, if $L^T_s \cup R^T_s \neq \emptyset$;
- height($L^M_s \cup R^M_s$) \leq 1; and
- height($L^B_s \cup R^B_s$) = $d^v(x_s, b) - 2$, if $L^B_s \cup R^B_s \neq \emptyset$. \hfill \Box

Let $3 \leq s \leq m - 3$ be odd. Since $X$ is tall, there are no 1-entries in $P$ above $x_{s-1}$ and to the right of $x_s$. Thus, $x_{s-1}$ is the topmost 1-entry in $R_s$. Similarly, $x_{s+1}$ is the bottommost 1-entry in $L_{s+1}$. This implies the following improved bounds:

**Observation 4.6.** For each odd $s \in \{3, 4, \ldots, m - 2\}$:

- height($R^T_{s}$) \leq $d^v(x_{s-1}, x_s) - 2$, if $R^T_s \neq \emptyset$; and
- height($L^B_{s+1}$) \leq $d^v(x_{s+1}, x_{s+2}) - 2$, if $L^B_{s+1} \neq \emptyset$. \hfill \Box

### 4.4. $S(P, X)$ avoids $P$

In this section, we show:

**Lemma 4.7.** Let $P$ be a permutation matrix, $m \geq 6$ be even and let $X = (x_1, x_2, \ldots, x_m)$ be a non-extendable tall traversal of $P$. Then $S(P, X)$ avoids $P$.

Together with Lemmas 4.1 and 4.2, this implies Lemma 1.16. For the remainder of this section, fix $P$ and $X$ as in Lemma 4.7, and write $\ell, b, t, r$ for $\ell_P, b_P, t_P, r_P$. We use the same notation for parts of $S(P, X)$ as defined in Section 4.3. Suppose $\phi$ is an embedding of $P$ into $S(P, X)$. Our overall strategy is to distinguish cases based on the location of $\phi(t)$, and derive a contradiction in each case.

Note that we make no further assumptions on $P, X, \phi$, so each lemma or corollary in this section holds on its own for every choice of $P, X, \phi$ (we only fix $P, X, \phi$ for brevity). This allows us to make use of the following symmetry argument. Note that $S(P, X)$ is
usually not symmetric, in the sense that its 180-degree rotation \( \text{rot}^2(S(P, X)) \) is equal to \( S(P, X) \). However, it is easy to see that \( \text{rot}^2(S(P, X)) \) is equal to \( S(\text{rot}^2(P), \text{rot}^2(X)) \).

Now, in Lemma 4.8, for example, we show that \( \phi(t) \notin L_3 \) for each choice of \( P, X, \phi \), in particular for \( \text{rot}^2(P), \text{rot}^2(X) \) and every embedding \( \phi' \) of \( \text{rot}^2(P) \) into \( \text{rot}^2(S(P, X)) \). Since \( L_3 \) in \( S(\text{rot}^2(P), \text{rot}^2(X)) = \text{rot}^2(S(P, X)) \) corresponds to \( R_{m-2} \) in \( S(P, X) \), \( t \) in \( \text{rot}^2(P) \) corresponds to \( b \) in \( P \), and \( \phi \) corresponds to some embedding \( \phi' \), we also have \( \phi(b) \notin R_{m-2} \).

### 4.4.1. \( \phi(t) \) in the front or the back

We first consider some special cases, showing that \( \phi(t) \) cannot lie in the leftmost few “blocks” of \( S(P, X) \), and symmetric statements for \( \phi(b) \). The proofs in this section also serve as a warm-up for what follows.

**Lemma 4.8.** \( \phi(t) \notin L_3 \) and \( \phi(b) \notin R_{m-2} \).

**Proof.** By symmetry, it suffices to show \( \phi(t) \notin L_3 \). Suppose \( \phi(t) \in L_3 \). Then also \( \phi(\ell) \in L_3 \), since \( S(P, X) \) contains no 1-entries to the left of \( L_3 \). But \( \text{width}(L_3) = d^b(\ell, x_3) - 1 < d^b(\ell, t) - 1 \), thus we cannot have both \( \phi(\ell) \) and \( \phi(t) \) in \( L_3 \), a contradiction. \( \square \)

**Lemma 4.9.** \( \phi(t) \notin R_1 \) and \( \phi(b) \notin L_m \).

**Proof.** By symmetry, it suffices to show \( \phi(t) \notin R_1 \). Suppose \( \phi(t) \in R_1 \). Note that height\( \phi(R_1^T \cup R_1^M) \leq d^r(t, \ell) < d^r(t, x_3) \), so \( \phi(x_3) \in B \). Since \( x_3 \) is to the right of \( t \) and \( L_3^B = \emptyset \), we have \( \phi(x_3) \in R_1^B \). Since \( r \) is below \( x_3 \), and all 1-entries in \( S(P, X) \) that are to the right of \( R_1 \) are above \( R_1^B \), we have \( \phi(r) \in R_1^B \). Since \( \text{width}(R_1) < d^b(\ell, r) \), this implies that \( \phi(\ell) \) is to the left of \( R_1 \). But since \( t \in R_1 \) and the highest 1-entry \( t^1 \) of \( R_1 \) is at most \( d^b(t, \ell) \) rows above the expandable row, \( \phi(\ell) \) must be below the expandable
row. This is a contradiction, since there are no 1-entries to the left of $R_1$ and below the expandable row.

If $m = 6$ (see Figure 12), then the only remaining possibility is $\phi(t), \phi(b) \in L_4 \cup R_3$, which implies $\phi(t) \in L_4$ or $\phi(b) \in R_3$ (since $t$ is to the left of $b$). Thus, the following lemma concludes the case $m = 6$.

**Lemma 4.10.** If $m = 6$, then $\phi(t) \notin L_4$ and $\phi(b) \notin R_3$.

**Proof.** By symmetry, it suffices to show $\phi(t) \notin L_4$. This can be done with essentially the same argument as in the proof of Lemma 2.1. Suppose $\phi(t) \in L_4$. Then $\phi(t)$ is not above $t^4 \in L_4$. By Lemmas 4.8 and 4.9, $\phi(b) \in L_4 \cup R_3$. The lowest 1-entry in $L_4 \cup R_3$ is $b^4$, so $\phi(b)$ is not below $b^4$. But $d^v_\phi(t^4, b^4) < d^v(t, b)$ (note the empty expandable row), a contradiction.

We now continue with the case $m \geq 8$.

**Lemma 4.11.** If $m \geq 8$, then $\phi(t) \notin L_4$ and $\phi(b) \notin R_{m-3}$.

**Proof.** By symmetry, it suffices to show $\phi(t) \notin L_4$. Suppose $\phi(t) \in L_4$. We have $\text{height}_{\phi}(L_4^T \cup L_4^M) \leq d^v(t, x_4) < d^v(t, x_3)$, implying $x_3 \in B$. Since $x_3$ is to the left of $t$, this means that $\phi(x_3) \in R_3^B \cup L_4^B$. If $\phi(x_3) \in R_3^B$, then also $\phi(r) \in R_3^B$, but then $\phi(r)$ is to the left of $\phi(t)$, which is impossible. Thus, we have $\phi(x_3) \in L_4^B$.

This implies that $\phi(r) \in L_4^B \cup R_3^B \cup R_{m-3}^B$. As such, $\phi$ maps no 1-entry of $P$ to the right of $R_4$, and thus $\phi$ maps no 1-entry into the rows below $M$ and above $L_4^B \cup R_3^B$. If now $\phi(x_3)$ is above or equal to $x_3^4$, then $d^v_{\phi}(\phi(t), \phi(x_3)) \leq d^v_{\phi}(t^4, x_3^4) < d^v(t, x_3)$ (note the empty expandable row), a contradiction. Thus, $\phi(x_3)$ must be below $x_3^4$. By tallness of $X$, this also implies that $x_3$ is to the right of $x_3^4$, and $\phi(t)$ is to the right of $t^4$, since $t$ is to the right of $x_3$ and $d^b(t^4, x_3^4) = d^b(t, x_3)$. Similarly, $\phi(x_4)$ is to the right of $L_4$.

Consider now $\phi(x_3)$. Since $\phi(x_3)$ is below $x_3^4$ and $x_3^4$ is the bottommost 1-entry in $L_4^B$, we know that $\phi(x_3)$ is below $L_4^B$. This implies $\phi(x_5) \in R_4^B \cup R_3^B$. Suppose $\phi(x_5) \in R_4^B$. Then also $\phi(r) \in R_4^B$. But $\text{width}(R_4) = d^b(x_4, r) - 1 < d^b(x_5, r)$, a contradiction. Thus, $\phi(x_5) \in R_3^B$. This implies $\phi(b), \phi(r) \in R_3^B$, and $\phi(x_4) \in R_3$.

Recall that $\phi(t) \in L_4 \setminus \{t^4\}$, so $t$ is below $t^4$. As $\text{height}_{\phi}(L_4^T \cup M) \leq d^v(t, x_4)$, this means that $\phi(x_4) \in R_3^B$. But $\text{height}(R_3^B) = d^v(x_3, b) - 2 < d^v(t, b)$, a contradiction.
Lemma 4.12. Let \( m \geq 8 \). If \( \phi(t) \in R_3 \), then \( \phi(b) \) is to the right of \( R_4 \). Moreover, if \( \phi(b) \in L_{m-2} \), then \( \phi(t) \) is to the left of \( L_{m-3} \).

Proof. By symmetry, proving the first statement suffices. Let \( \phi(t) \in R_3 \) and suppose \( \phi(b) \) is not to the right of \( R_4 \). The portion of \( R_3 \) above the expandable row has height at most \( d^v(t, x_3) - 1 \), so \( \phi(x_3) \) must lie below the expandable row. Let \( q_3 \) be the 1-entry directly below \( x_3 \) in \( P \). Clearly, \( \phi(q_3), \phi(b), \phi(r) \in B \), and since \( \phi(b) \) is to the right of \( R_4 \), we have \( \phi(b) \in R_3^B \cup R_4^B \). We separately consider three cases.

Case 1: \( \phi(r) \in R_3^B \). Since \( X \) is tall, \( q_3 \) is to the right of \( t \), so \( \phi(q_3) \in R_3^B \). But \( \text{height}(R_3^B) = d^v(x_3, b) - 2 = d^v(q_3, b) - 1 \), a contradiction.

Case 2: \( \phi(r) \in R_4^B \). Consider \( x_5 \). Since \( x_5 \) is below \( x_3 \), we have \( \phi(x_5) \in B \). Since \( x_5 \) is to the right of \( t \), and above and to the left of \( r \), we have \( \phi(x_5) \in R_4^B \). But \( \text{width}(R_4) = d^b(x_4, r) - 1 < d^b(x_5, r) \), a contradiction.

Case 3: \( \phi(r) \) is to the right of \( R_4 \). Then \( \phi(r) \) is also above \( L_4^B \cup R_4^B \). Consider again \( x_5 \). We know that \( \phi(x_5) \) is below \( M \) and above \( L_4^B \cup R_4^B \). Since \( x_5 \) is to the left of \( b \), we also know that \( \phi(x_5) \) is not to the right of \( R_4 \). But there are no such 1-entries in \( S(P, X) \).

We proceed with some more special cases, showing that \( \phi(t) \) also cannot lie in the rightmost few blocks of \( S(P, X) \).

Lemma 4.13. Let \( m \geq 8 \). Then, \( \phi(t) \) lies to the left of \( L_{m-2} \), and \( \phi(b) \) lies to the right of \( R_3 \).

Proof. By symmetry, it suffices to prove that \( \phi(t) \) lies to the left of \( L_{m-2} \). If \( \phi(b) \) lies to the left of \( L_{m-2} \), then \( \phi(t) \) does, too. \( \phi(b) \notin R_{m-3} \cup L_m \cup R_{m-2} \) by Lemmas 4.8, 4.9, and 4.11. The only remaining possibility is that \( \phi(b) \in L_{m-2} \), where Lemma 4.12 implies that \( \phi(t) \) lies to the left of \( L_{m-3} \), and thus to the left of \( L_{m-2} \).

To show that \( \phi(t) \notin L_{m-3} \cup R_{m-4} \), we use the following more general lemma, to be used in later sections. Figures 11 and 13 are useful to visualize the proof.

Lemma 4.14. Let \( s \) be odd with \( 5 \leq s \leq m - 3 \). If \( \phi(t) \in L_s \cup R_{s-1} \), then \( \phi(b) \) lies to the right of \( R_{s-1} \).

Proof. Suppose not. Then, \( \phi(b) \in L_s \cup R_{s-1} \).

Case 1: \( \phi(\ell) \notin L_s \cup R_{s-1} \). Since \( \ell \) is to the left of \( t \), this means that \( \phi(\ell) \) is to the left of \( L_s \). This implies that \( \phi(\ell) \) is also below \( L_s^T \), and thus \( \phi(x_4) \) is below \( L_{s}^T \). Since \( x_4 \) is to the right of \( t \), we have \( \phi(x_4) \in M \cup B \), which implies \( \phi(x_5) \in B \), as \( \text{height}_M(M) \leq 1 < d^v(x_4, x_5) \). Since \( x_5 \) is to the right of \( t \) and to the left of \( b \), we further know \( \phi(x_5) \in R_{s-1}^B \). Since \( \text{width}(R_{s-1}) < d^b(x_5, r) \), this implies that \( \phi(r) \) is to the right of \( R_{s-1} \). But then \( \phi(r) \) is above \( \phi(x_3) \), a contradiction.
Case 2: \( \phi(\ell) \in L_s \cup R_{s-1} \). Then \( \phi \) maps no 1-entry to the left of \( L_s \). It is easy to see that there must be some \( y \in E(P) \) such that \( \phi(y) \in R_3^T \), otherwise \( P \) is decomposable (more precisely, \( P = \left( \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) \), where \( \phi(E(A)) \subseteq L_s \)). Since \( \phi(b) \in L_s \cup R_{s-1} \), we know that \( b \) is to the left of \( y \). Tallness of \( X \) implies that \( y \) is not above \( x_{m-2} \). Now consider \( x_{s-1} \). We know \( x_{s-1} \leq_h x_{m-4} <_h b \) and \( x_{s-1} \leq_v x_{m-4} \leq_v x_{m-2} \leq_v y \). Thus, \( \phi(x_{s-1}) \in L_3^T \). But \( \text{width}(L_3^T) < d^v(\ell, x_s) < d^v(\ell, x_{s-1}) \), a contradiction.

**Corollary 4.15.** If \( m \geq 8 \), then \( \phi(t) \notin L_{m-3} \cup R_{m-4} \) and \( \phi(b) \notin L_5 \cup R_4 \).

**Proof.** By symmetry, it suffices to prove that \( \phi(t) \notin L_{m-3} \). Suppose \( \phi(t) \in L_{m-3} \). By Lemmas 4.8, 4.9, 4.11, and 4.12, \( \phi(b) \) cannot lie in \( L_{m-2} \) or further left. This contradicts Lemma 4.14.

We now consolidate and reformulate the above results. For the more involved proofs in Sections 4.4.2 and 4.4.3, it will be convenient to organize the “middle” blocks \( L_i, R_i \) of \( S(P, X) \) into two sets of groups, as follows. For each odd \( s \) with \( 5 \leq s \leq m - 5 \), let \( G_s = L_s \cup R_{s-1} \cup L_{s+1} \cup R_s \), and let \( H_s = L_{s+1} \cup R_s \cup L_{s+2} \cup R_{s+1} \). A sketch of \( G_s \) and \( H_s \) can be found in Figure 13. Combining Lemmas 4.8, 4.9, 4.11, and 4.13 and Corollary 4.15 yields:

**Corollary 4.16.** If \( m \geq 8 \), then:

- \( \phi(t) \) lies to the right of \( L_3 \) and to the left of \( L_{m-3} \). In other words, \( \phi(t) \in R_3 \) or \( \phi(t) \in G_s \) for some odd \( s \) with \( 5 \leq s \leq m - 5 \); and
- \( \phi(b) \) lies to the right of \( R_3 \) and to the left of \( R_{m-3} \). In other words, \( \phi(b) \in L_{m-2} \) or \( \phi(b) \in H_s \) for some odd \( s \) with \( 5 \leq s \leq m - 5 \).

At this stage, we cannot easily show that both \( \phi(t) \notin R_3 \) and \( \phi(b) \notin L_{m-2} \), but we can show that at least one of the two must be true.

**Lemma 4.17.** If \( m \geq 8 \), then \( \phi(t) \notin R_3 \) or \( \phi(b) \notin L_{m-2} \).

**Proof.** Suppose \( \phi(t) \in R_3 \) and \( \phi(b) \in L_{m-2} \). Since \( \text{height}(R_3^T) < d^v(t, x_3) \), we have \( \phi(x_3) \in M \), implying \( \phi(x_5) \in B \). More precisely, since \( b \in L_{m-2} \), we have \( \phi(x_3) \in R_{m-2}^B \cup R_{m-3}^B = R_{m-3}^B \cup R_{m-3}^B \cup R_{m-2}^B \). Similarly, \( \phi(x_{m-2}) \in T \cup M \), implying \( \phi(x_{m-4}) \in L_3^T \cup R_1^T \cup L_3^T \cup R_1^T \). In particular, \( \phi(x_5) \) is to the right of \( \phi(x_{m-4}) \). But \( x_5 <_h x_4 \leq_h x_{m-4} \), a contradiction.

Note that Corollary 4.16 and Lemma 4.17 completely resolve the case \( m = 8 \).

In the following two subsections, we show that the remaining possibilities also lead to a contradiction. In Section 4.4.2 we treat the easier case, where \( \phi(t) \in G_s \) for some odd \( s \) with \( 5 \leq s \leq m - 5 \), and \( \phi(b) \) is to the right of \( R_{s+1} \) (i.e., to the right of \( H_s \)). This also handles the symmetric case where \( \phi(b) \in H_s \) and \( \phi(t) \) is to the left of \( G_s \). In Section 4.4.3, we consider the case where \( \phi(t) \in G_s \) and \( \phi(b) \in H_s \).
4.4.2. \( \phi(t), \phi(b) \) in the middle and far from each other

The following lemma is central to this subsection.

**Lemma 4.18.** For each odd \( s \) with \( 5 \leq s \leq m - 5 \), if \( \phi(t) \in G_s \) and \( \phi(b) \) is to the right of \( R_{s+1} \), then \( \phi(x_s) \) is below the expandable row.

**Proof.** We consider the following cases:

**Case 1:** \( \phi(t) \notin L_s \cup L_{s+1} \cup R_s \). Then \( \phi(t) \in R_{s-1} \), so \( \phi(t) \) is below the expandable row, implying the same for \( \phi(x_s) \).

**Case 2:** \( \phi(t), \phi(b) \notin L_s \cup L_{s+1} \cup R_s \). Then \( \phi(t) \) is below \( L^T_s \cup R^T_s \). Since \( x_s \) is to the right of \( t \) and below \( \ell \), this implies that \( x_s \in M \cup B \), and thus \( \phi(x_s) \) is below the expandable row.

**Case 3:** \( \phi(t), \phi(b) \in L_s \cup R_s \). Since \( \phi \) does not map any 1-entry to a position below \( L^T_s \cup R^T_s \) and above \( M \), we have \( \text{height}(L^T_s \cup L^M_s \cup R^T_s \cup R^M_s) \leq d^v(t, x_s) \). Thus, \( \phi(x_s) \) is either below \( M \) or in the bottom row of \( M \), so \( \phi(x_s) \) is below the expandable row.

**Case 4:** \( \phi(t) \in L_{s+1} \) and \( \phi(b) \in L_s \). Since \( x_s \) is below \( \ell \) and to the right of \( t \), we have \( \phi(x_s) \in M \cup B \) or \( \phi(x_s) \in R^T_s \). In the former case, we are done, as above. Otherwise, note that \( \phi \) does not map any 1-entry of \( P \) into a row below \( R^T_s \) and above \( M \), thus \( \text{height}(R^T_s \cup R^M_s) = d^v(x_{s-1}, x_s) \leq d^v(x_4, x_s) \). This implies that \( \phi(x_s) \) is below the expandable row.

**Case 5:** \( \phi(t), \phi(b) \in L_{s+1} \) and \( \phi(x_s) \notin L_{s+1} \). Since \( x_s \) is to the right of \( t \), this means that \( \phi(x_s) \) is to the right of \( L_{s+1} \). Suppose \( x_s \) is above the expandable row. Tallness of \( X \) implies that there are no 1-entries in \( P \) that are below and to the
left of \( x_s \), so \( \phi \) maps no 1-entry to \( L_{s+1}^M \cup L_{s+1}^B \). But then \( \phi \) maps every 1-entry of \( P \) either to \( L_{s+1}^T \) or to the left and below \( L_{s+1} \), and both possibilities occur (e.g., with \( t \) resp. \( b \)). This means that \( P \) is decomposable, a contradiction.

**Case 6:** \( \phi(t), \phi(t), \phi(x_s) \in L_{s+1} \). Suppose \( \phi(x_s) \) is above the expandable row. Since \( \text{height}(L_{s+1}^T) < d^v(t, x_s) \), we know that \( \phi(x_s) \) is below \( L_{s+1} \), so \( \phi(x_s) \in L_{s+1}^M \).

\( x_{s+1} \) is above and to the right of \( x_s \), implying that \( \phi(x_{s+1}) \in L_{s+1}^T \cup R_s^T \). Further, \( x_{s+2} \) is below \( x_s \), so \( \phi(x_{s+2}) \in M \cup B \), and \( x_s < h x_{s+2} < h x_{s+1} \), so \( \phi(x_{s+2}) \in L_{s+1} \cup R_s \). Since \( \phi(b) \) is to the right of \( R_{s+1} \), we know that \( \phi \) maps no 1-entry to \( L_{s+1}^B \cup R_{s+1}^B \). Thus, \( \phi(x_{s+2}) \in L_{s+1}^M \cup R_{s+1}^M \).

But now \( \phi(x_s), \phi(x_{s+2}) \in M \), so \( \phi \) maps no further 1-entries to \( M \). Therefore, \( \phi \) maps every 1-entry either to \( A = L_{s+1}^T \cup L_{s+1}^M \cup R_{s+1}^T \cup R_{s+1}^M \), or below and to the right of \( A \) (and \( \phi(t) \in A, \phi(b) \notin A \)). This means \( P \) is decomposable, a contradiction. \( \square \)

We first consider a simple special case.

**Lemma 4.19.** If \( \phi(t) \in G_s \) for some odd \( s \) with \( 5 \leq s \leq m - 5 \), then \( \phi(b) \notin L_{m-2} \).

Moreover, if \( \phi(b) \in H_s \) for some odd \( s \) with \( 5 \leq s \leq m - 5 \), then \( \phi(t) \notin R_3 \).

**Proof.** By symmetry, it suffices to show the first statement. Suppose \( \phi(b) \in L_{m-2} \). Then, \( \phi(b) \) is to the right of \( R_{m-4} \), and thus to the right of \( R_{s+1} \), so Lemma 4.18 implies that \( \phi(x_s) \) is below the expandable row. Since \( x_s \) is to the left and above \( b \), we have \( \phi(x_s) \in L_{m-2} \). Since \( x_s < h x_{s+1} < h b \), and \( x_{s+1} \) is below \( t \), we have \( \phi(x_{s+1}) \in L_{m-2}^T \cup L_{m-2}^B \). But \( \text{height}(L_{m-2}^M \cup L_{m-2}^B) \leq d^v(x_{m-2}, b) < d^v(x_{m-4}, b) \leq d^v(x_{s+1}, b) \), a contradiction. \( \square \)

We proceed with the case that \( \phi(t) \in G_s \) and \( \phi(b) \in H_u \) for some \( 5 \leq s < u \leq m - 5 \).

**Lemma 4.20.** Let \( s, u \) be odd such that \( 5 \leq s < u \leq m - 5 \). If \( \phi(t) \in G_s \), then \( \phi(b) \notin H_u \).

**Proof.** Suppose \( \phi(b) \in H_u \). Note that this means that \( \phi \) maps no 1-entry to \( G_s \cap B \) or \( H_u \cap T \). We start by establishing a few facts about \( \phi(x_s), \phi(r), \) and \( \phi(x_{u+1}) \).

Lemma 4.18 implies that \( \phi(x_s) \) is below the expandable row. Since \( x_u \) is below \( x_s \), we have \( \phi(x_u) \in B \).

We claim that \( \phi(r) \in H_u \). Suppose not, then \( \phi(r) \) must be to the right of \( H_u \), and thus above \( L_{u+1}^B \cup R_{u+1}^B \). Since \( x_s \) is above \( r \) and to the left of \( b \), we have that \( \phi(x_u) \) is above \( L_{u+1}^B \cup R_{u+1}^B \) and not to the right of \( H_u \). But then \( \phi(x_u) \in T \cup M \), contradicting our previous observation.

Further, \( \phi(x_u) \in B \) and \( \phi(b) \in H_u \) imply \( \phi(x_u) \in H_u \), and thus \( \phi(x_{u-1}) \in H_u \). This means that \( \phi(x_{u-1}) \in M \cup B \) (as \( \phi \) maps nothing to \( H_u \cap T \)), and thus \( \phi(x_{u+1}) \) is below the expandable row.

We distinguish between the following cases:

**Case 1:** \( \phi(b), \phi(r) \in L_{u+1} \cup R_{u+1} \). Note that \( \phi \) does not map any 1-entries to the rows between \( M \) and \( L_{u+1}^B \cup R_{u+1}^B \). Thus, \( \text{height}(L_{u+1}^M \cup L_{u+1}^B \cup R_{u+1}^M \cup R_{u+1}^B) \leq d^v(x_{u+1}, b) \), contradicting the fact that \( \phi(x_{u+1}) \) is below the expandable row.

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Case 2: $\phi(b), \phi(r) \in R_u$. Since $\text{height}(R^B_u) < d^v(x_u, b) < d^v(x_{u+1}, b)$, we have $\phi(x_{u+1})$ above $R^B_u$.

Suppose first that $\phi(x_{u+1}) \in L^B_{u+1} \cup L^M_{u+1}$. Note that no 1-entry in $M$ is below $\phi(x_{u+1})$. Thus, tallness of $X$ implies that $\phi$ maps no 1-entry to $R^M_u$. But then $\phi$ maps all 1-entries to $R^B_u$ or above and to the left of $R^B_u$ (and $\phi(b) \in R^B_u$, $\phi(t) \notin R^B_u$), so $P$ is decomposable, a contradiction.

Second, suppose that $\phi(x_{u+1}) \in R^M_u$. Since $x_{u-1}$ is above $x_{u+1}$, this also implies that $\phi(x_{u-1}) \in L^M_{u+1} \cup R^M_u$. Note that $\phi$ maps no further 1-entries to $M$, and $\phi(x_{u-1}), \phi(x_{u+1}) \in L_{u+1} \cup R_u$. But this means that $\phi$ maps all 1-entries either to $L^B_{u+1} \cup L^M_{u+1} \cup R^B_u \cup R^M_u$ or above and to the left of that entry set, again contradicting that $P$ is indecomposable.

Case 3: $\phi(b) \in R_u$ and $\phi(r) \in R_{u+1}$. Since $x_{u+2}$ is above $r$ and to the left of $b$, we know that $\phi(x_{u+2})$ is above and not to the right of $R^B_u$. Since $\phi(x_s)$ is below the expandable row and $x_{u+2}$ is below $x_s$, we have $\phi(x_{u+2}) \in B$, implying $\phi(x_{u+2}) \in L^B_{u+1}$. Since $\phi(r) \in R_{u+1}$, we know that $\phi$ maps no 1-entries into the rows between $M$ and $L^B_{u+1} \cup R^B_u$. This implies that $\text{height}_P(L^B_{u+1} \cup R^B_u) \leq d^v(x_{u+1}, x_{u+2})$. But $\phi(x_{u+1})$ is below the expandable row, a contradiction. \qed

Lemmas 4.19 and 4.20 together with Corollary 4.16 and Lemma 4.17 imply:

**Corollary 4.21.** There is some odd $s$ with $5 \leq s \leq m - 5$ such that $\phi(t) \in G_s$ and $\phi(b) \in H_s$.

**Proof.** Suppose first that $\phi(t) \in R_3$. Then Corollary 4.16 and Lemma 4.17 imply that $\phi(b) \in H_s$ for some odd $s$ with $5 \leq s \leq m - 5$. But then Lemma 4.19 implies that $\phi(t) \notin R_3$, a contradiction. A similar argument shows that $\phi(b) \notin L_{m-2}$.

As such, there are odd $s, u$ with $5 \leq s, u \leq m - 5$ such that $\phi(t) \in G_s$ and $\phi(b) \in H_u$. Clearly, $s \leq u$, and Lemma 4.20 implies that $s \geq u$, so $s = u$. \qed

**4.4.3. $\phi(t), \phi(b)$ in the middle and close to each other**

In this subsection, we show that Corollary 4.21 also leads to a contradiction, which shows that our assumption that $S(P, X)$ contains $P$ must have been false. Figure 13 will be useful throughout this subsection. The next two lemmas treat the case that $\phi(t) \in L_s$ (or, symmetrically, $\phi(b) \in R_{s+1}$).

**Lemma 4.22.** Let $s$ be odd and $5 \leq s \leq m - 5$. If $\phi(t) \in L_s$, then $\phi(\ell) \in L^T_s$. Moreover, if $\phi(b) \in R_{s+1}$, then $\phi(r) \in R_{s+1}$.

**Proof.** By symmetry, it suffices to prove the first statement. Suppose $\phi(t) \in L_s$ and $\phi(\ell) \notin L^T_s$. Then $\phi(\ell)$ lies below $L^T_s$, and not to the right of $L_s$. Since $x_4$ is to the right of $t$ and below $\ell$, we have $\phi(x_4) \in M \cup B$. This directly implies that $\phi(x_3) \in M \cup B$, since $x_3$ is below $x_4$. Further, $x_3$ is to the left of $t$, and $\phi(b)$ is not to the left of $\phi(t) \in L_s$. This implies $\phi(x_3) \in M$, and thus $x_4 \in M$. Note that, since $\text{height}_P(M) \leq 1$, no other 1-entries are mapped to $M$.

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Clearly, $\phi(b) \in B$, and by Corollary 4.21, $\phi(b) \in H_s$. We now consider the possible locations of $\phi(b)$.

**Case 1:** $\phi(b) \in L_{s+1}^B \cup R_{s+1}^B$. Then $\phi(x_{s+1})$ is above $R_{s+1}^B$ (because height$(L_{s+1}^B \cup R_{s+1}^B) = d'(x_{s+1}, b) - 2$ and not to the right of $R_{s+1}$ (since $x_{s+1}$ is to the left of $b$). Thus, $\phi(x_{s+1}) \in T \cup M$. But $x_{s+1}$ is below $x_4$, so $x_{s+1} \in B$, a contradiction.

**Case 2:** $\phi(b) \in R_{s+1}^B$. First, suppose that $\phi(x_4)$ is to the left of $R_s$. Then tallness of $X$ implies that $\phi$ maps no 1-entry to $R_1^T$. But then $\phi$ maps all 1-entries either to $R_s$ or to the left and above $R_s^B$, and $\phi(b) \in R_s^B$, $\phi(t) \notin R_s^B$, contradicting the fact that $P$ is indecomposable.

Second, suppose that $\phi(x_4)$ is not to the left of $R_s$. Since $\phi(b) \in R_s$, and $x_4$ is to the left of $b$, we have $\phi(x_4) \in R_s$. Moreover, since $x_4 < h < x_6 < h$ and $x_4 < v < x_6 < v$, we have $\phi(x_6) \in R_s^B$. But height$(R_{s+1}^B) \leq d'(x_a, b) \leq d'(x_5, b) < d'(x_6, b)$, a contradiction. □

**Lemma 4.23.** $\phi(t) \notin L_s$ and $\phi(b) \notin R_{s+1}$ for each odd $s$ with $5 \leq s \leq m - 5$.

**Proof.** By symmetry, it suffices to prove the first statement. Suppose $\phi(t) \in L_s$. Then Lemma 4.22 implies that $\phi(t) \in L_{s+1}^T$. This means that $\phi$ does not map any 1-entry below $L_{s+1}^T$ and above $M$, so height$_\phi(L_{s+1}^T \cup L_{s+1}^B) \leq d'(t, x_s)$, implying that $\phi(x_s)$ is below the expandable row and thus to the right of $L_s$.

We consider several possibilities for the location of $\phi(b)$ and $\phi(r)$. Corollary 4.21 implies that $\phi(b) \in H_s$. Since $\phi(x_s)$ is below the expandable row, $\phi(x_{s+2}), \phi(b) \in B$, and thus $\phi(x_{s+2}), \phi(b) \in H_s \cup B = L_{s+1}^B \cup R_{s+1}^B$. Since $x_{s+2} \leq v < m - 3 < v$, this also implies $\phi(r) \in H_s \cup B$. This means that $\phi$ does not map any 1-entry to the rows between $M$ and $L_{s+1}^B \cup R_{s+1}^B$, so height$_\phi(L_{s+1}^M \cup L_{s+1}^B) \leq d'(x_{s+1}, x_{s+2})$ and height$_\phi(R_{s+1}^M \cup R_{s+1}^B) \leq d'(x_{s+1}, b)$.

**Case 1:** $\phi(b) \in L_{s+1}^B$. Then $\phi(r) \in L_{s+1}^B \cup R_{s+1}^B$. Since height$_\phi(L_{s+1}^M \cup L_{s+1}^B) \leq d'(x_{s+1}, x_{s+2}) < d'(x_{s+1}, b)$, we have $\phi(x_{s+1}) \in T$. Since $x_{s+1}$ is to the left of $b$, we have $\phi(x_{s+1}) \in L_{s+1}^T$. But width$(L_{s+1}^T) = d(b, x_{s+1}) - 1 < d(b, x_{s+1})$, a contradiction.

**Case 2:** $\phi(b) \in R_{s+1}^B$. Then $\phi(r) \in R_{s+1}^B$. Since height$_\phi(R_{s+1}^M \cup R_{s+1}^B) \leq d'(x_{s+1}, b)$, we know that $\phi(x_{s+1})$ is above the expandable row, and therefore to the left of $R_{s+1}$.

Since $x_{s-1}$ is above $x_{s+1}$, we have $\phi(x_{s-1}) \in T$, implying $\phi(x_{s-1}) \in L_{s-1}^T \cup R_{s-1}^T$. Further, width$(L_{s-1}^T) < d(b, x_{s-1})$, so $\phi(x_{s-1}) \in R_{s-1}^T$.

Finally, since $\phi(x_{s+2}) \in B$ and $x_{s+2}$ is to the left of $x_{s+1}$, we have $\phi(x_{s+2}) \in L_{s+1}^B$. But then $\phi(x_{s+2})$ is to the left of $\phi(x_{s+1}) \in R_{s+1}^T$, while $x_{s+2}$ is to the right of $x_{s-1}$, a contradiction.

**Case 3:** $\phi(b), \phi(r) \in R_{s+1}^B$. We consider the location of $\phi(x_{s-1})$. 28
First suppose that \( \phi(x_{s-1}) \in R_s \). Let \( q_s \) be the 1-entry of \( P \) in the row below \( x_s \). We have \( \phi(q_s) \in B \). Since \( X \) is tall, \( q_s \) is to the right of \( x_{s-1} \), thus \( \phi(q_s) \in R^B_s \). But height(\( R^B_s \)) \( \leq d^v(x_s, b) - 2 = d^v(q_s, b) - 1 \), a contradiction.

Second, suppose \( \phi(x_{s-1}) \in L_{s+1} \), and \( \phi(x_{s-1}) \) is below the expandable row. By tallness of \( X \), there are no 1-entries in \( P \) that are above and to the right of \( x_{s-1} \), so \( \phi \) does not map any 1-entry to \( R^T_s \cup R^M_s \). But then \( \phi \) maps all 1-entries to \( R^B_s \) or above and to the left of \( R^B_s \) (and \( \phi(b) \in R^B_s, \phi(t) \notin R^B_s \)). Thus, \( P \) is decomposable, a contradiction.

Third, suppose \( \phi(x_{s-1}) \in L_{s+1} \), and \( \phi(x_{s-1}) \) is above the expandable row. Since \( \phi(t) \) is below \( L^T_{s+1} \), we know that \( \phi(x_{s-1}) \) must be mapped to the row directly above the expandable row. Since \( X \) is tall, \( \phi \) does not map any 1-entry to \( R^T_s \). Note that \( L^M_s \) consists of only one row, which is already occupied by \( \phi(x_{s-1}) \), so \( \phi \) also maps no 1-entry to \( L^M_s \). But then \( \phi \) maps all 1-entries either to \( L^T_s \) or below and to the right of \( L^T_s \) (and \( \phi(t) \in L^T_s \), \( \phi(b) \notin L^T_s \)), so \( P \) is decomposable, a contradiction.

Fourth, suppose \( \phi(x_{s-1}) \in R^M_{s-1} \). Then \( \phi(x_s) \in B \), because \( x_s \) is below \( x_{s-1} \). But \( \phi(x_s) \) also lies to the left of \( \phi(x_{s-1}) \) and above \( \phi(b) \in R_s \), a contradiction.

Finally, suppose \( \phi(x_{s-1}) \in L_s \). Since width(\( L_s \)) \( \leq d^h(\ell, x_{s-1}) \), this is impossible.

Case 4: \( \phi(b) \in R^B_s \) and \( \phi(r) \in R^B_{s+1} \). Then \( \phi(x_{s+2}) \) is above and not to the right of \( R^B_{s+1} \). Together with the fact that \( \phi(x_{s+2}) \in B \), this implies \( \phi(x_{s+2}) \in L^B_{s+1} \). Since height(\( L^B_{s+1} \)) \( \leq d^v(x_{s+1}, x_{s+2}) \), we know that \( \phi(x_{s+1}) \) is above the expandable row, and thus \( \phi(x_{s-1}) \in T \), implying \( \phi(x_{s-1}) \in R^T_s \). Since width(\( L^T_s \)) \( \leq d^h(\ell, x_{s-1}) \), we have \( \phi(x_{s-1}) \in R^T_s \). Now \( \phi(x_{s-1}) \in R^T_s \) is to the right of \( \phi(x_{s+2}) \) in \( L^B_{s+1} \), but \( x_{s-1} \) is to the left of \( x_{s+2} \), a contradiction. \( \square \)

The next three lemmas deal with the case that \( \phi(t) \in L_{s+1} \).

**Lemma 4.24.** Let \( 5 \leq s \leq m - 5 \). If \( \phi(t) \in L_{s+1} \), then \( \phi(b) \notin L_{s+1} \).

**Proof.** Suppose \( \phi(t), \phi(b) \in L_{s+1} \). First note that, since \( t \leq h x_{s+1} \), we have \( \phi(x_{s+1}) \in L_{s+1} \). Further, \( \text{height}(L^T_{s+1}) < d^v(t, x_{s+1}) \) and \( \text{height}(L^B_{s+1}) < d^v(x_{s+1}) \) implies that \( \phi(x_{s+1}) \in L^M_{s+1} \).

Let \( t \in E(P) \) be the 1-entry in the column directly left of \( x_{s+1} \). Note that \( y^{s+1} \) is the rightmost 1-entry in \( L_{s+1} \), and that \( d^h(t, y^{s+1}) = d^h(t, b) \). Since \( \phi(b) \) is not to the right of \( y^{s+1} \), this implies that \( \phi(t) \) is to the left of \( t^{s+1} \). By a similar argument, \( \phi(x_3) \) is to the left of \( x_3^{s+1} \).

Since \( t^{s+1} \) is the topmost 1-entry in \( L_{s+1} \), this also means that \( \phi(t) \) is below \( t^{s+1} \). Further, \( d^v(t^{s+1}, x_3^{s+1}) = d^v(t, x_3) \leq d^v(\phi(t), \phi(x_3)) \), so \( \phi(x_3) \) is below \( x_3^{s+1} \).

Tallness of \( X \) implies that \( L_{s+1} \) contains no 1-entries to the left and below \( x_3^{s+1} \), so \( \phi(x_3) \) is to the left of \( L_{s+1} \). Since \( \phi(b) \in L_{s+1} \), this means that \( \phi(x_3) \in T \cup M \).

Since \( \phi(x_3) \) is to the left of \( L_{s+1} \), we also know that \( \phi(\ell) \) is to the left of \( L_{s+1} \), implying that \( \phi(\ell) \) is below \( L^T_{s+1} \). Since \( t < x_4 < b \) and \( \ell < x_4 < x_3 \), we have \( \phi(x_4) \in L^M_{s+1} \).
Since $x_3$ is below $x_4$, this also means that $\phi(x_3) \in M$. But now $\phi(x_3), \phi(x_4), \phi(x_{s+1}) \in M$, while $M$ consists of only two nonempty rows, a contradiction.

**Lemma 4.25.** Let $s$ be odd with $5 \leq s \leq m - 5$. If $\phi(t) \in L_{s+1}$ and $\phi(b) \in R_s$, then $\phi(x_{s-1}) \in L_{s+1}$ and $\phi(x_{s+2}) \in R_s$.

**Proof.** By symmetry, it suffices to show that $\phi(x_{s-1}) \in L_{s+1}$. Suppose not. Since $t < h$, $x_{s-1} < h$, this means that $\phi(x_{s-1}) \in R_s$. Let $q_s \in E(P)$ be the 1-entry of $P$ in the row directly below $x_s$.

We claim that $\phi(x_s)$ is below the expandable row, and thus $\phi(q_s) \in B$. If $\phi(x_{s-1}) \in M \cup B$, then $\phi(x_s)$ is below the expandable row, since $x_s$ is below $x_{s-1}$. Otherwise, $\phi(x_{s-1}) \in R_s^T$, which implies that $\phi(\ell) \in L_s \cup L_{s+1}$, so $\phi$ maps no 1-entry into the rows between $L_s^T \cup R_s^T$ and $M$. Thus, $\text{height}_\phi(R_s^T \cup R_s^M) \leq d^\vee(x_{s-1}, x_s)$, implying that $\phi(x_s)$ is below the expandable row. This proves the claim.

$$\text{height}(R_s^B) \leq d^\vee(x_s, b) - 2 = d^\vee(q_s, b) - 1$$ implies that $\phi(q_s) \notin R_s^B$. Since $X$ is tall, $q_s$ is to the right of $x_{s-1}$ and thus $\phi(q_s)$ is not to the left of $R_s$. Since $\phi(q_s) \in B \setminus R_s^B$, this implies that $\phi(q_s)$ is to the right of $R_s$, so $\phi(r)$ is to the right of $R_s$, and thus above $R_s^B$.

Consider now $x_{s+2}$. First, $x_{s-1} < h$, $x_{s+2} < h$, $b$ implies that $\phi(x_{s+2}) \in R_s$. Since $x_s$ is below the expandable row, $\phi(x_{s+2}) \in R_s^B$. But then $\phi(x_{s+2})$ is below $\phi(r)$, a contradiction.

**Lemma 4.26.** Let $s$ be odd with $5 \leq s \leq m - 5$. If $\phi(t) \in L_{s+1}$, then $\phi(b) \in R_s$ and $\phi(E(P)) \subseteq L_{s+1} \cup R_s$.

**Proof.** Assume $\phi(t) \in L_{s+1}$. By Corollary 4.21, we have $\phi(b) \in H_s$. Lemmas 4.23 and 4.24 imply that $\phi(b) \notin L_{s+1} \cup R_{s+1}$. If $\phi(b) \in L_{s+2}$, then $\phi(b)$ is above the expandable row. But then $\phi(r)$ is to the right of $R_s$, below $L_{s+2}$ and in $T$, which is impossible. The only remaining possibility is that $\phi(b) \in R_s$.

To show $\phi(E(P)) \subseteq L_{s+1} \cup R_s$, it is enough to prove that $\phi(\ell) \in L_{s+1}$ and $\phi(r) \notin R_s$, and by symmetry, we only have to prove $\phi(\ell) \in L_{s+1}$. Suppose $\phi(\ell) \notin L_{s+1}$. Then $\phi(\ell)$ is below $L_{s+1}^T$. Since $x_{s-1}$ is below $\ell$ and $\phi(x_{s-1}) \in L_{s+1}$ by Lemma 4.25, we have $\phi(x_{s-1}) \in L_{s+1}^T \cup L_{s+1}^B$.

Since $\text{height}(L_{s-1}^B) < d^\vee(x_{s+1}, x_{s+2}) < d^\vee(x_{s+1}, r)$, we know that $\phi(r)$ is below $L_{s+1}^B$, so $\phi(r) \in R_s^B$. Moreover, tallness of $X$ implies that $\phi$ maps no 1-entry to $R_s^T \cup R_s^M$. But then $\phi$ maps all 1-entries either to $R_s^B$ or above and to the left of $R_s^B$ (and $\phi(b) \in R_s$, $\phi(\ell) \notin R_s^B$), so $P$ is decomposable, a contradiction.

**Lemma 4.27.** $\phi(t) \notin L_{s+1}$ and $\phi(b) \notin R_s$ for each odd $s$ with $5 \leq s \leq m - 5$.

**Proof.** By symmetry, it suffices to show the first statement. Suppose $\phi(t) \in L_{s+1}$. By Lemmas 4.25 and 4.26, we have $\phi(x_{s-1}) \in L_{s+1}$ as well as $\phi(x_{s+2}), \phi(b) \in R_s$ and $\phi(P) \subseteq L_{s+1} \cup R_s$.

Since $t < h$, $x_s < h$, we have $\phi(x_s) \in L_{s+1}$. Symmetrically, $\phi(x_{s+1}) \in R_s$. Let $p_{s+1} \in E(P)$ be the 1-entry of $P$ in the row directly above $x_{s+1}$, and let $q_s \in E(P)$ be the 1-entry in the row directly below $x_s$. Since $\text{height}(L_{s+1}^T) = d^\vee(t, p_{s+1}) - 1$, we
know that \( \phi(p_{s+1}) \) and \( \phi(x_{s+1}) \) are below \( L^T_{s+1} \), and, symmetrically, \( \phi(q_s), \phi(x_s) \) are above \( R^B_s \). All in all, we have \( \phi(x_s) \) and \( \phi(q_s) \) are above \( L^M_{s+1} \cup L^B_{s+1} \cup R^I_s \cup R^B_s \).

Our strategy for the remainder of the proof is to find two 1-entries \( y_1, y_2 \in E(P) \) such that the sequence \( Y = x_1, x_2, \ldots, x_s, y_1, y_2, x_{s+1}, \ldots, x_m \) is a traversal of \( P \). For this, we have to show that \( x_{s-1} < h y_2 < h y_1 < h x_{s+1} \), as well as \( y_1 < v x_{s+1} \) and \( x_s < v y_2 \) (note that \( x_{s-1} < v y_1 \) and \( y_2 < v x_{s+2} \) then follow from tallness of \( X \)). The existence of such a traversal implies that \( X \) is extendable, contradicting Lemma 4.41.

We consider two cases. First, assume that \( q_s \) is to the left of \( p_{s+1} \). Then we simply choose \( y_1 = p_{s+1} \) and \( y_2 = q_s \). By definition, \( p_{s+1} \) is above \( x_{s+1} \) and \( q_s \) is below \( x_s \). By assumption, \( q_s < h p_{s+1} \), and tallness implies that \( x_{s-1} < h q_s \) and \( p_{s+1} < h x_{s+2} \).

Otherwise, \( p_{s+1} \) is to the left of \( q_s \). Then either \( \phi(p_{s+1}) \in L_{s+1} \) or \( \phi(q_s) \in R_s \). By symmetry, we can assume the former, which implies \( \phi(p_{s+1}) \in L^M_{s+1} \cup L^B_{s+1} \). Since \( \phi(x_{s+1}) \) is below \( p_{s+1} \), we have \( \phi(p_{s+1}), \phi(x_{s+1}) \in M \). More precisely, \( \phi(p_{s+1}) = p_{s+1}^{s+1} \in L^M_{s+1} \) and \( \phi(x_{s+1}) = q_s^{s+1} \in R^B_s \).

Since \( \phi(\ell) \in L_{s+1} \), we know that \( \phi \) maps no 1-entry into the rows between \( M \) and \( L^B_{s+1} \). Since \( \phi \) also maps no 1-entry into the expandable row, we have \( d^v(\ell, p_{s+1}, x_{s+1}^{s+1}) = d^v(p_{s+1}, x_{s+1}) - 1 \). This implies that \( \phi(x_{s+1}) \) is below \( x_s^{s+1} \).

Since \( X \) is tall, all 1-entries in \( L_{s+1} \) below \( x^{s+1}_s \), including \( \phi(x_s) \), must be to the right of \( x^{s+1}_s \) (note that \( x^{s+1}_s \in L_{s+1} \)). Since \( x_{s-1} \) is above \( x_{s+1} \) and \( \phi(x_{s+1}) = q_s^{s+1} \), we have \( \phi(x_{s-1}) \in L_{s+1} \cup \{ p_{s+1}^{s+1} \} \). Thus, \( x_{s-1} = p_{s+1} \) is possible. Tallness of \( X \) implies that \( \text{width}(L^T_{s+1} \cup \{ p_{s+1}^{s+1} \}) < d^h(\ell, x_{s+1}) \), so \( \phi(x_{s-1}) \) is to the left of \( x_{s+1}^{s+1} \).

Putting everything together, we have \( x^{s+1}_{s-1} < h \phi(x_{s-1}) < h x^{s+1}_{s+1} \), and \( \phi(x_{s-1}) \in L_{s+1} \) above the expandable row, and \( \phi(x_s) \) is below \( x^{s+1}_{s+1} \). Now choose \( y_1, y_2 \in E(P) \) such that \( y_1^{s+1} = \phi(x_{s-1}) \) and \( y_2^{s+1} = \phi(x_s) \). Then \( x_{s-1} < h y_2 < h y_1 < h x_{s+2} \), as well as \( y_1 < v x_{s+1} \) and \( x_s < v y_2 \). This implies that \( Y \) is a traversal of \( P \).

The last remaining cases are now easy:

**Lemma 4.28.** \( \phi(t) \notin R_{s-1} \cup R_s \) and \( \phi(b) \notin L_{s+1} \cup L_{s+2} \) for each odd \( s \) with \( 5 \leq s \leq m - 5 \).

**Proof.** By symmetry, it suffices to show the first statement. Suppose \( \phi(t) \in R_{s-1} \cap R_s \). By Corollary 4.21 and Lemmas 4.23 and 4.27, we have \( \phi(b) = L_{s+1} \cup L_{s+2} \).

Suppose first that \( \phi(t) \in R_{s-1} \). Then \( \phi(t) \) is below the expandable row, meaning that \( \phi(t) \) is below \( M \), but not to the right of \( R_{s-1} \). So \( \phi(b) \) is above \( R^B_{s-1} \), implying that \( \phi(t) \) is also above \( R^B_{s-1} \), a contradiction.

Second, if \( \phi(t) \in R_s \), then \( \phi(b) \in L_{s+2} \), since \( b \) is to the right of \( t \). A symmetric argument shows that \( \phi(r) \) is above \( M \) and below \( L^T_{s+2} \), a contradiction.

Lemmas 4.23, 4.27, and 4.28 imply that \( \phi(t) \notin G_s \), contradicting Corollary 4.21. As such, our assumption that \( \phi \) is an embedding of \( P \) into \( S(P, X) \) must be false. This concludes the proof of Lemma 4.7.
5. Conclusion and open problems

We showed that each decomposable permutation matrix has bounded saturation function, thereby completing the classification of saturation functions of permutation matrices. Our proofs imply the upper bound $\text{sat}(P, n) \leq 9k^4$ for an indecomposable $k \times k$ permutation matrix $P$ (note that the largest witness $S(P, X)$ is not larger than $2k^2 \times k^2$, and Lemma 1.6 combines it with its 90-degree rotation, resulting in a $3k^2 \times 3k^2$ matrix). It would be interesting to improve this bound, especially if a simpler construction for patterns satisfying the conditions of Lemma 4.7 can be found. Note that for general patterns with bounded saturation functions, no upper bound for $\text{sat}(P, n)$ in terms of $P$ is known, as noted by Fulek and Keszegh [FK20].

We also characterized a large class of non-permutation patterns with bounded saturation function, including very dense matrices (Theorem 1.2). Still, a full characterization of the saturation functions of all matrices remains out of reach. Note that there are indecomposable patterns without spanning oscillations, see, e.g., Figure 14. Thus, new techniques are likely required to fully resolve this problem.

Our results trivially imply that every permutation matrix with a vertical witness also has a horizontal witness. It would be interesting to determine whether this is true for arbitrary patterns.

It is also possible consider the saturation functions of sets of patterns. If $\mathcal{P}$ is a set of patterns, let a matrix $M$ be $\mathcal{P}$-saturating if $M$ avoids each $P \in \mathcal{P}$, and adding a single 1-entry in $M$ creates an occurrence of some $P \in \mathcal{P}$ in $M$. Let $\text{sat}(\mathcal{P}, n)$ be the minimum weight of $\mathcal{P}$-saturating matrices. Since our witnesses for $k \times k$ permutation matrices have size at most $3k^2 \times 3k^2$, and thus avoid all patterns with one side of side length more than $3k^2$. Thus, if $\mathcal{P}$ contains one permutation matrix, and arbitrarily many much larger patterns, our results imply that $\text{sat}(\mathcal{P}, n) \in O(1)$.

It would be interesting to determine the saturation functions for, say, all pairs of two permutation matrices of the same size. Gerbner, Nagy, Patkós, and Vizer [GNPV21] observed that certain saturation problems for two-dimensional posets can be reduced to saturation problems for sets of matrix patterns. However, these sets usually contain both permutation matrices and non-permutation matrices (of similar size).

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A. Proof of Lemma 1.18

Lemma 1.18. Let $P$, $M$ be matrices, and let $P$ have no empty rows or columns. Then $P$ is contained in $M$ if and only if there is an embedding of $P$ into $M$.

Proof. Say $P$ is $q \times s$ and $M$ is $m \times n$. Suppose $P = (p_{i,j})_{i,j}$ is contained in $M$. Then there are rows $r_1 < r_2 < \cdots < r_q$ and columns $c_1 < c_2 < \cdots < c_r$ such that $p_{i,j} \leq m_{r_i,c_j}$ for each $i \in [q], j \in [r]$. Now simply define $\phi(i,j) = (r_i, c_j)$. Clearly, $\phi(E(P)) \subseteq E(M)$. Moreover, consider $(i,j), (i', j') \in E(P)$. We have $i < i'$ if and only if $r_i < r_{i'}$, and $j < j'$ if and only if $r_j < r_{j'}$. Thus $\phi$ is an embedding of $P$ into $M$.

Now suppose $\phi : E(P) \to E(M)$ is an embedding of $P$ into $M$. Note that $x, y \in E(P)$ are in the same row (resp. column) if and only if $\phi(x), \phi(y)$ are in the same row (resp. column). Thus, $\phi(E(P))$ intersects exactly $q$ rows and $s$ columns. Let $r_1 < r_2 < \cdots < r_q$ be those rows and $c_1 < c_2 < \cdots < c_r$ be those columns. We show that $\phi(i,j) = (r_i, c_j)$ for each $(i,j) \in E(P)$. Let $x_1, x_2, \ldots, x_m \in E(P)$ such that $x_i$ is in the $i$-th row for each $i \in [m]$, and let $r'_i$ be the row of $M$ containing $\phi(x_i)$. Clearly $r'_i \geq r_i$. By induction, we further have $r'_i \geq r_i$ for each $i \in [m]$. Similarly, $r'_m \leq r_m$, and, again by induction, $r'_i \leq r_i$ for each $i \in [m]$. This implies that $\phi(i,j)$ is in the $r_i$-th row of $M$ for every $(i,j) \in E(P)$. An analogous argument shows that $\phi(i,j)$ is in the $c_j$-th column of $M$.

Since $\phi$ is an embedding, we have $(r_i, c_j) = \phi(i,j) \in E(M)$ for each $(i,j) \in E(P)$. Thus, $p_{i,j} \leq m_{r_i,c_j}$ for each $(i,j) \in [q] \times [s]$, so $P$ is contained in $M$. \qed