TAME IDEALS, CLUTTERS AND THEIR REES ALGEBRAS

ABBAS NASROLLAH NEJAD, ASHKAN NIKSERESHT, ALI AKBAR YAZDAN POUR, RASHID ZAARE-NAHANDI

Abstract. A tame ideal is an ideal $I$ such that the blowup of the affine space $\mathbb{A}_k^n$ along $I$ is regular. In this paper, we give a combinatorial characterization of tame squarefree monomial ideals. More precisely, we show that a squarefree monomial ideal is tame if and only if the corresponding clutter is a union of some isolated vertices and a complete $d$-partite $d$-uniform clutter. It turns out that a squarefree monomial ideal is tame, if and only if the facets of its Stanley-Reisner complex have mutually disjoint complements. Also, we characterize all monomial ideals generated in degree at most 2 which are tame. Finally, we prove that tame squarefree ideals are of fiber type.

Introduction

The blowup of a scheme $X$ along the closed subscheme $Z \subset X$ is the scheme $\text{Bl}_Z(X)$ together with a proper rational map $\pi : \text{Bl}_Z(X) \rightarrow X$ which is an isomorphism outside of $Z$. The closed subscheme $Z$ is called the center of the blowup. It is well known that the blowup of a regular scheme in a regular center is regular. A natural question which arises from this statement is: "when the blowup of a regular scheme in non-regular center remains regular".

The main interest in this article is the question of determining squarefree monomial ideals such that the blowup of the affine space along these ideals is regular. This work is inspired by a paper of E. Faber and D.B. Westra [4]. They give a smoothness criterion, based on convex geometry, for a monomial ideal such that the blowing up $\mathbb{A}_k^n$ along this ideal is regular. These ideals have been dubbed tame. They also find some classes of tame monomial ideals such as monomial building sets and permutohedra.

Throughout this paper, the base field $k$ will be considered as an algebraically closed field. Let $R = k[x, y, z]$ be a polynomial ring and let $Z$ stand for the closed subscheme of affine space $\mathbb{A}_k^3 = \text{Spec}(R)$ defined by the monomial ideal $I = (x, yz)$. The blowup of $\mathbb{A}_k^3$ along $Z$ (or with center $Z$) is singular. In fact, the affine chart corresponding with the ring extension $R \hookrightarrow R \left[\frac{yz}{x}\right] \simeq k[x, y, z, t]/(yz - xt)$, defines a singular variety. Assume that $X \subset \mathbb{A}_k^3$ is an affine scheme such that the singular subscheme of $X$ is defined by $I = (x, yz)$ (see e.g. [4]). If we want to resolve $X$ by one blowup, we have to use the singular subscheme as a center. Then the blowup of $X$ is embedded in a singular ambient scheme, because $I$ is not tame. In this case,
we can not speak of an embedded resolution of singularity of \( X \) \cite{3} (see also \cite{5}). But note that the blowup of \( \mathbb{A}^n_k \) in a center defined by the ideal \((x, yz)(x, y)(x, z)\) is regular and the latter ideal is tame \cite{6}.

The blowup of an affine space \( \mathbb{A}^n_k \) in a center defined by a monomial ideal in \( k[x_1, \ldots, x_n] \) is a toric variety. Therefore, we may restate the tameness property in combinatorics. In this paper, we determine tame monomial ideals in \( k[x_1, \ldots, x_n] \) which correspond to clutters and graphs with loops.

The outline of the paper is as follows. In section 1, we state definition of blowup via the Rees algebra. For a complete discussion and introduction to blowup and resolution of singularity we refer to \cite{8} and \cite{10}. We focus on blowup of \( \mathbb{A}^n_k \) along monomial ideals and recall some definitions in convex geometry. We give an algebraic description of the smoothness criterion presented in \cite[Theorem 12(ii)]{4}.

In section 2, a full combinatorial characterization of tame squarefree monomial ideals is given. A clutter \( \mathcal{C} \) is called tame if the circuit ideal \( I(\mathcal{C}) \) is tame. One of the main results of this paper is that \( \mathcal{C} \) is tame if and only if \( \mathcal{C} \) is a union of some isolated vertices and a complete \( d \)-partite \( d \)-uniform clutter (Theorem 2.8). It turns out that, a squarefree monomial ideal is tame if and only if the facets of the Stanley-Reisner complex of \( I(\mathcal{C}) \) have mutually disjoint complements (Proposition 2.11). Also in this section, it is shown that if the polarization of a monomial ideal \( I \) is tame then \( I \) is tame, but the converse is not true in general.

In section 3, we present a characterization of tame monomial ideals generated in degree at most two. In particular, if \( G \) is a graph without isolated vertices (possibly with loops) then the edge ideal \( I(G) \) is tame if and only if \( G \) is a looped star, looped complete or simple complete bipartite graph (Corollary 3.6 and Theorem 3.8).

Finally, we give an explicit description of the defining ideal of affine charts \( U_i \) of blowup \( \text{Proj} (\mathcal{R}_R(I)) \) where \( I \) is the circuit ideal of a complete \( d \)-partite \( d \)-uniform clutter. By using this, we find the defining equations of the Rees algebra of a tame squarefree monomial ideal. In particular, it is proved that the circuit ideal of a tame squarefree monomial ideal is of fiber type.

## 1. Blowup along monomial ideals

Let \( R \) be a Noetherian ring and \( I \subset R \) be an ideal. The Rees algebra of \( I \) is defined to be the graded algebra \( \mathcal{R}_R(I) = R[\mathfrak{t}] = \oplus_{i \geq 0} I^i \mathfrak{t} \subset R[\mathfrak{t}] \). Assume \( I \) is generated by \( f_1, \ldots, f_m \in R \). Consider the polynomial ring \( S = R[T_1, \ldots, T_m] \), where \( T_i \) are indeterminates. Then there is a natural ring homomorphism \( \varphi : S \to \mathcal{R}_R(I) \) that sends \( T_i \) to \( f_i \mathfrak{t} \). Let \( \mathcal{J} = \ker \varphi \) be the defining ideal of \( \mathcal{R}_R(I) \). Then \( \mathcal{R}_R(I) \simeq S/\mathcal{J} \) and \( \mathcal{J} = \bigoplus_{i=1}^{\infty} \mathcal{J}_i \) is a graded ideal. A minimal generating set for \( \mathcal{J} \) is called the defining equation set of the Rees algebra. Also, \( \mathcal{J}_1 \) is known as the defining ideal of the symmetric algebra of \( I \), in fact \( \mathcal{J}_1 = I_1(T, \psi) \) is the ideal generated by one minors of the product of the variable matrix \( T = [T_1 \ T_2 \ \ldots \ T_n] \) by the first syzygy matrix \( \psi \) of \( I \).

Let \( (R, \mathfrak{m}) \) be a Noetherian local ring and \( I \subset R \) an ideal, the special fiber of \( I \) is defined to be \( \mathcal{F}(I) = \text{gr}_R(I) \otimes R/\mathfrak{m} \), where \( \text{gr}_R(I) = \mathcal{R}_R(I)/I \mathcal{R}_R(I) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1} \). In the case that \( R \) is a polynomial ring over a field \( k \) and \( I = (f_1, \ldots, f_m) \) the
special fiber \( F(I) \) is isomorphic to \( k[f_1, \ldots, f_m] \). Then there is a homomorphism \( \Psi : k[T_1, \ldots, T_m] \to F(I) \) that maps \( T_i \) to \( f_i \). Set \( \mathcal{H} = \ker \Psi \). The ideal \( I \) is called of fiber type if \( \mathcal{J} = S\mathcal{J}_1 + S\mathcal{H} \).

Suppose that \( X = \text{Spec} (R) \) is an affine scheme and let \( Z = \text{Spec} (R/I) \) be a closed subscheme of \( X \) defined by an ideal \( I \) of \( R \). The blowup of \( X \) along \( Z \) is the scheme \( \text{Bl}_Z (X) = \text{Proj}(\mathcal{R}_R(I)) \) together with the morphism \( \pi : \text{Bl}_Z (X) \to X \) given by the natural ring homomorphism \( R \to \mathcal{R}_R(I) \). The subscheme \( E = \pi^{-1}(Z) \) of \( \text{Bl}_Z (X) \) is called the exceptional divisor of the blowup. The blowup \( \text{Bl}_Z (X) \) can be embedded into \( \mathbb{P}^{m-1}_R \) as the closed subscheme defined by the defining ideal \( \mathcal{J} \) of the Rees algebra of \( I \), where \( m \) is the size of a generating set for \( I \). The following statement can be found in standard textbooks, like [3].

**Proposition 1.1.** Let \( X = \text{Spec} (R) \) be an affine scheme and let \( Z \) be a closed subscheme defined by an ideal \( I = (f_1, \ldots, f_m) \) of \( R \). The following statements hold:

(i) The blowup of \( X \) along \( Z \) can be covered by affine charts

\[
\text{Spec} \left( R \left[ \frac{f_1}{f_i}, \ldots, \frac{f_m}{f_i} \right] \right),
\]

for \( i = 1, \ldots, m \).

(ii) The defining ideal of \( \text{Bl}_Z (X) \) in the \( i \)'th affine chart \( \text{Spec}(R[f_1/f_i, \ldots, f_m/f_i]) \) is given by dehomogenization of the defining ideal \( \mathcal{J} \) of the Rees algebra of \( I \) with respect to variable \( T_i \).

**Proof.** For \( f \in I \), let \( R[I^{-1}f] \) be the subalgebra of \( R_f \) which is generated by elements of the form \( x/f^d \) with \( x \in I, d \in \mathbb{N} \). Such an element may be also considered as a degree zero element of \( R_R(I)_f \). In fact, there is an \( R \)-algebra isomorphism \( R[I^{-1}] \simeq R_R(I)_f \). The inverse map of this isomorphism is given by considering \( y/f^i \) for \( y \in I^i \) as an element of \( R[I^{-1}] \). We can see that if \( f \) runs through a generating set of \( I \), then we have

\[
\text{Bl}_Z X = \bigcup_{i=1}^m \text{Spec} \left( [I/f^i] \right) = \bigcup_{i=1}^m \text{Spec} \left( R \left[ \frac{f_j}{f_i} : 1 \leq j \leq m \right] \right).
\]

Let \( \mathcal{J} \) stand for the defining ideal of the affine chart \( R[f_2/f_1, \ldots, f_m/f_1] \), then

\[
\frac{R \left[ \frac{T_2}{T_1}, \ldots, \frac{T_m}{T_1} \right]}{\mathcal{J}} \simeq R \left[ \frac{f_2}{f_1}, \ldots, \frac{f_m}{f_1} \right].
\]

Assume that \( G_1(T_1, \ldots, T_m), \ldots, G_s(T_1, \ldots, T_m) \) with \( \text{deg } G_i = d_i \) are the defining equations of the Rees algebra of \( I \). We claim that \( \mathcal{J} = (g_1, \ldots, g_s) \) where

\[
g_i \left( \frac{T_2}{T_1}, \ldots, \frac{T_m}{T_1} \right) = T_1^{-d_i} G_i.
\]

Note that \( g_i \) is the dehomogenization of \( G_i \) with respect to the variable \( T_1 \). We first prove that \( g_i \in \mathcal{J} \). It suffices to prove that \( f_1^{d_i} g_i (f_2/f_1, \ldots, f_m/f_1) = 0 \). However \( f_1^{d_i} g_i (f_2/f_1, \ldots, f_m/f_1) = G_i(f_1, \ldots, f_m) = 0 \).
Now let $g \in \overline{J}$. Setting $d = \deg g$. Denote by $G$ the homogenization of $g$ with respect to $T_1$, that is $G(T_1, \ldots, T_m) = T_1^d g(T_2/T_1, \ldots, T_m/T_1)$. Then

$$G(f_1, \ldots, f_m) = f_1^d g \left( \frac{f_2}{f_1}, \ldots, \frac{f_m}{f_1} \right) = 0.$$ 

So that $G(T_1, \ldots, T_m)$ is a homogeneous polynomial in $\overline{J}$. Thus there are $H_1, \ldots, H_s$ with $\deg H_i = d - d_i$ in $R[T_1, \ldots, T_m]$ such that $G = \sum_{i=1}^s G_i H_i$. Hence

$$g = T_1^{-d} G(T_1, \ldots, T_m) = \sum_{i=1}^s T_1^{d_i - d} H_i(T_1, \ldots, T_m) g_i \left( \frac{T_2}{T_1}, \ldots, \frac{T_m}{T_1} \right)$$

and then letting $h_i(T_2/T_1, \ldots, T_m/T_1) = T_1^{d_i - d} H_i(T_1, \ldots, T_m)$, one has $g = \sum_{i=1}^s g_i h_i$. \hfill $\square$

Assume that $A_k^n = \text{Spec}(k[x_1, \ldots, x_n])$ and let $Z$ be the subscheme defined by the ideal of coordinate $I = (x_1, \ldots, x_m)$. Then, for $i = 1, \ldots, m$

$$k[x] \left[ \frac{x_1}{x_1}, \ldots, \frac{x_m}{x_i} \right] \simeq k[x] \left[ \frac{T_1}{T_1}, \ldots, \frac{T_{i-1}}{T_i}, \frac{T_{i+1}}{T_i}, \ldots, \frac{T_m}{T_i} \right].$$

The latter ring is isomorphic with the polynomial ring with $n$ variables. Therefore, we can see that in each affine chart, the blowup $\text{Bl}_Z(A_k^n)$ is a regular affine variety. More generally, if the center $Z$ given by the ideal $I$ is regular, $I$ is generated by a regular system of parameters say $f_1, \ldots, f_m$, then the defining ideal of the Rees algebra of $I$ is generated by $2 \times 2$-minors of the matrix

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ T_1 & T_2 & \cdots & T_m \end{bmatrix}.$$ 

Thus, like above in each affine chart $\text{Bl}_Z(A_k^n)$ is a polynomial ring and hence regular.

In general, as in the example in the introduction, blowing up $A_k^n$ in a non-regular subscheme may produce singularity. One class of ideals which define non-regular schemes is the class of monomial ideals which have at least one generator of degree at least 2. In this paper, we focus on such ideals. In the sequel, we set $R = k[x_1, \ldots, x_n]$ and denote the blowup of $A_k^n$ along a subscheme defined by a monomial ideal $I$ by $\text{Bl}_I(R)$.

For an starting point to characterize squarefree tame monomial ideals, we need some results of [4] which uses convex geometry. Set $\text{Supp}(I) = \{ a \in \mathbb{N}^n : x^a \in I \}$ (here $\mathbb{N}$ denotes the set of non-negative integers), where $I$ is a monomial ideal, and let $N(I)$ be the convex hull of $\text{Supp}(I)$. A point $p$ in $N(I)$ is said to be a vertex when $p$ cannot be written as $\lambda_1 p_1 + \lambda_2 p_2$, for some $0 < \lambda_i < 1$ with $\lambda_1 + \lambda_2 = 1$ and $p_1 \neq p_2 \in N(I)$. Suppose that $x^a \in I$ for some $a \in \mathbb{N}^n$. We say that $x^a$ is a vertex of $I$, when $a$ is a vertex of $N(I)$.

Let $a \in \text{Supp}(I)$, $u = x^a$ and $G(I)$ be the minimal set of generators of $I$. Then by the $a$-chart (or as sometimes we call it, the $u$-chart) of the blowup $\text{Bl}_I(R)$ of $A_k^n$ along $I$, we mean $\text{Spec}(k[U])$ where $U = \{ x_1, \ldots, x_n \} \cup \{ u : u \neq u' \in G(I) \}$. Thus by Proposition 1.1, $\text{Bl}_I(R)$ is covered by the $u$-charts ($u \in G(I)$). Faber and Westra presented a method to check the tameness of a monomial ideal $I$ of $R$ in [4,
Theorem 12] in terms of convex geometry. Here we restate their result in a more algebraic language.

**Proposition 1.2.** Suppose that \( u \) is a vertex of the monomial ideal \( I \) of \( R \). Then,

(i) the \( u \)-chart of \( \text{Bl}_I(R) \) is regular if and only if there is a \( U' \subseteq U \), with \( |U'| = n \) such that \( k[U] = k[U'] \), where \( U = \{x_1, \ldots, x_n\} \cup \{u, u' \in G(I)\} \).

(ii) \( I \) is tame if and only if every chart of \( \text{Bl}_I(R) \) which correspond to a vertex of \( I \) is regular.

**Proof.** (i) Since \( u \) is a vertex of \( I \), it follows [4, Theorem 12(ii)] that the \( u \)-chart (= Spec \( (k[U]) \)) is regular if and only if the minimal set of monomials \( U' \subseteq S \) such that \( U' \) generates \( k[U] \) as a \( k \)-algebra, has exactly cardinality \( n \). Note that this set always has at least \( n \) elements by [4, Lemma 9]. Also this minimal set is unique by Lemmata 6 and 7 of [4]. Consequently, we can obtain \( U' \) from \( U \) by deleting monomials \( u \in U \) which can be written as a product of monomials in \( U \) different from \( u \). In particular, \( U' \subseteq U \) as required.

Statement (ii) follows from [4, Theorem 12(i)]. \( \square \)

2. TAME CLUTTERS

In this section, we present a full combinatorial characterization of tame squarefree monomial ideals. Recall that a clutter \( \mathcal{C} \) on vertex set \( [n] = \{1, \ldots, n\} \), is a family of incomparable subsets of \( [n] \) (ordered by inclusion). Throughout this paper, \( \mathcal{C} \) denotes a clutter on \( [n] \). The elements of \( \mathcal{C} \) are called circuits of \( \mathcal{C} \) and \( \mathcal{C} \) is called \( d \)-uniform when all of its circuits have cardinality \( d \). Let \( F \subseteq [n] \). By \( x_F \) we mean \( \prod_{i \in F} x_i \) in the polynomial ring \( R \). Also the circuit ideal of \( \mathcal{C} \) is \( I(\mathcal{C}) = (x_F : F \in \mathcal{C}) \). Thus there is a one-to-one correspondence between squarefree monomial ideals of \( R \) and clutters on \( [n] \). We say that \( \mathcal{C} \) is tame when \( I(\mathcal{C}) \) is so. Moreover, for a subset \( A \subseteq [n] \) we call the set \( N(A) = N_{\mathcal{C}}(A) = \{v \in [n] \setminus A : \exists e \in \mathcal{C} s.t. A \cup \{v\} \subseteq e\} \), the open neighborhood of \( f \) in \( \mathcal{C} \).

Let \( d' = d \) be two positive integers. Following [2], we say that a \( d \)-uniform clutter \( \mathcal{C} \) is \( d' \)-partite, if the set of vertices can be written as the union of mutually disjoint subsets \( V_1, \ldots, V_{d'} \), such that each circuit of \( \mathcal{C} \) meets each \( V_i \) in at most one vertex. If moreover, \( \mathcal{C} \) contains all \( d \)-subsets of \( [n] \) which intersect each \( V_i \) in at most one vertex, we say that \( \mathcal{C} \) is complete \( d' \)-partite. The partition \( \{V_i : i \in [d']\} \) as above is called a \( d' \)-partition of \( \mathcal{C} \). Here we prove that \( \mathcal{C} \) is tame if and only if \( \mathcal{C} \) a complete \( d \)-uniform clutter.

Not all the minimal generators of a monomial ideal \( I \) need be vertices. For example if \( I = (x_1^2, x_2^2, x_1x_2) \), then obviously \( x_1x_2 \) is not a vertex of \( I \). But as the following lemma shows, in the squarefree case, the vertices of \( I \) are exactly the minimal generators of \( I \).

**Lemma 2.1.** Suppose that \( I \) is a squarefree monomial ideal of \( R \) with minimal generating set \( \mathcal{G}(I) = \{x_i^a : i \in [s]\} \) and let \( a \in N(I) \). Then \( a \) is a vertex of \( N(I) \) if and only if \( x^a \in \mathcal{G}(I) \).

**Proof.** \( (\Rightarrow) \) : Clearly \( a \in \text{Supp}(I) \). If \( x^a \notin \mathcal{G}(I) \), then \( a = a_s + b \), for some \( i \in [s] \) and some non-zero vector \( b \in \mathbb{N}^n \). Assume that \( b_i > 0 \). Then \( a = \frac{1}{b_i}(a_i + b + e_i) + \)}
\[ \frac{1}{2}(a_i + b - e_1) \text{ where } e_i's \text{ are the standard basis of } \mathbb{Z}^n. \] Thus \( a \) is not a vertex, a contradiction.

\( \Leftarrow \): We show that for example \( a_1 \) is a vertex of \( N(I) \). On the contrary, assume that \( a_1 = \lambda_1 p_1 + \lambda_2 p_2 \), for some \( 0 < \lambda_i < 1 \) with \( \lambda_1 + \lambda_2 = 1 \) and \( p_1 \neq p_2 \in N(I) \). Replacing \( p_i \)'s with their convex combinations of elements of \( \text{Supp}(I) \), it follows that \( a_1 = \sum_{j=1}^{t} \lambda_j (a_{ij} + a'_{ij}) \) where \( t \geq 2, 0 < \lambda_j < 1, i_j \in [s] \) and \( a'_{ij} \in \mathbb{N}^n \) for all \( j \), \( \sum_{j=1}^{t} \lambda_j = 1 \) and the points \( p_j = (a_{ij} + a'_{ij}) \) are mutually distinct.

If for all \( j \), we have \( i_j = 1 \), then it easily follows that \( a'_j = 0 \) for all \( j \), contradicting the mutually distinctness of \( p_j \)'s. Thus there exists \( j \in [t] \), with \( i_j \neq 1 \), say \( j = 1 \) and \( i_1 = 2 \). There is \( d \in \mathbb{N} \) such that \( d\lambda_1 > 1 \). Then for each \( i \in [n] \), the \( i \)'th component of \( da_1 \) is at least equal to the \( i \)'th component of \( a_2 \). So \( (x^{a_1})^d \in Rx^{a_2} \) and \( x^{a_1} \in \sqrt{Rx^{a_2}} = Rx^{a_2} \), for \( x^{a_2} \) is squarefree. But this is in contradiction with the minimality of the generating set \( G(I) \) and the result is concluded.

When \( e_0 \in \mathcal{C} \), by the \( e_0 \)-chart of \( Bl_{I(\mathcal{C})}(R) \) we mean the \( x_{-e_0} \)-chart, that is, \( \text{Spec}(k[U]) \) where \( U \) is as in the notes above Lemma 1.2 with \( u = x_{e_0} \). An immediate consequence of Lemmata 1.2 and 2.1 is the following.

**Corollary 2.2.** Assume that \( e_0 \) and \( U \) are as above and \( S = k[U] \). Then \( \text{Spec}(S) \) is regular if and only if there is a subset \( U' \subseteq U \) with \( |U'| = n \) such that \( S = k[U'] \).

Now we can apply Corollary 2.2 to find a combinatorial characterization of tame clutters. But first we need some lemmas.

**Lemma 2.3.** Let \( e_0 \in \mathcal{C} \), \( U_1 = \{x_1, \ldots, x_n\}, U_2 = \{x_{-e_0}: e_0 \neq e \in \mathcal{C}\} \) and \( U = U_1 \cup U_2 \). Suppose that the \( e_0 \)-chart of \( Bl_{I(\mathcal{C})}(R) \) is regular and \( U' \) as in Corollary 2.2. Then,

(i) for each \( x_j \in U_1 \setminus U' \), there exists a nonempty set \( \pi(j) \subseteq e_0 \) such that \( U' = (U_1 \cap U') \cup \{x_{\pi(j)}: x_j \in U_1 \setminus U'\} \);

(ii) for each \( j \in e_0 \), one has \( x_j \in U' \).

**Proof.** (i): It is sufficient to show that for each \( x_j \in U_1 \setminus U' \), there is a \( \pi(j) \subseteq e_0 \) such that \( x_{\pi(j)} \in U' \), since then \( U'' = (U_1 \cap U') \cup \{x_{\pi(j)}: x_j \in U_1 \setminus U'\} \subseteq U'' \) and \( |U''| = |U_1| = n = |U'| \) (by Corollary 2.2) and the result follows.

Assume that \( x_j \notin U' \) for some \( j \in [n] \). Then as \( k[U] = k[U'] \), we can write \( x_j \) as a product

\[
(1) \quad x_j = u_1 u_2 \cdots u_t x_{e_1} \cdots x_{e_v},
\]

where \( u_i = \frac{x_{e_i}}{x_{e_0}} \in U' \cup U_2 \). If \( x_s \mid x_{e_i} \), then we should have \( s \in e_0 \cup \{j\} \), else \( x_s \) does not cancel out and should appear in the left hand side of (1), which is not the case. A similar argument shows that at most one of the \( e_i \)'s contain \( j \), and since none of the \( e_i \)'s is contained in \( e_0 \), we get \( t \leq 1 \). Clearly \( t \geq 1 \), hence \( t = 1 \) and \( u_1 = \frac{x_j}{x_{e_1} \cdots x_{e_v}} \).

Set \( \pi(j) = \{i_1, \ldots, i_r\} \) which is clearly nonempty. Since \( u_1 \in U' \subseteq U \), we deduce that \( x_{\pi(j)} x_{e_0} \) which means \( \pi(j) \subseteq e_0 \), as claimed.

(ii): Note that, if \( j \in e_0 \) and \( x_j \notin U' \), then in (1) all \( e_i \)'s should be contained in \( e_0 \) which is not possible (since \( \mathcal{C} \) is a clutter). Thus for \( j \in e_0 \), \( x_j \in U' \). \( \square \)
In what follows, when we say that a monomial $u$ of $R = k[x_1, \ldots, x_n]$ divides the numerator [resp. denominator] of $q = \frac{p_1}{p_2}$ where $p_1$ and $p_2$ are monomials of $R$, we mean that $u$ divides the numerator [resp. denominator] of $q$ when $q$ is written in the simplest form. Also, we consider $k(x_1, \ldots, x_n)$ in its standard grading, that is, $\deg q = \deg p_1 - \deg p_2$. Furthermore, an isolated vertex of a clutter $\mathcal{C}$ means a vertex which has not appeared in any circuit of $\mathcal{C}$.

**Lemma 2.4.** Suppose that the conditions of Lemma 2.3 holds and for simplicity assume $e_0 = [d]$. Moreover, assume that $\mathcal{C}$ is $d$-uniform and without any isolated vertex. Then for each $d < j \leq n$, there exists a $v(j) \leq d$ such that $U' = \{x_1, \ldots, x_d\} \cup \{\frac{x_i}{x_{v(j)}} : d < j \leq n\}$.

**Proof.** We use the notations of Lemma 2.3. First note that for each $x_j \in U_1 \setminus U'$, we have $u = \frac{x_j}{x_{v(j)}} \in U' \subseteq U$. Since $\mathcal{C}$ is uniform, we conclude that the degree of every element of $U$, including $u$, is non-negative. Consequently, $\deg u = 0$ and $|\pi(j)| = 1$, say $\pi(j) = \{v(j)\}$.

Therefore, according to Lemma 2.3, we just need to prove that if $j > d$, then $x_j \notin U'$. On the contrary suppose that $x_j \in U'$ for some $j > d$. As $j$ is not an isolated vertex of $\mathcal{C}$, there is an $e \in \mathcal{C}$ with $j \in e$. Thus $u = \frac{x_j}{x_{e_0}} \in U$ should be a product of monomials in $U'$, say $u = u_1 \cdots u_t \ (u_i \in U')$. Since $j \notin e_0$, $x_j$ divides the numerator of $u$. But by Lemma 2.3, the only monomial in $U'$ with numerator divisible by $x_j$, is $x_j$ itself. Hence we can assume that $u_1 = x_j$. If $t = 1$ then $e = e_0 \cup \{j\}$ which contradicts $\mathcal{C}$ being a clutter. So $t > 1$ and we deduce that $u' = \frac{x_j}{x_{e_0}} = u_2 \cdots u_t$. Now all $u_i$'s have degree $\geq 0$ but $\deg u' = -1$ (because $\mathcal{C}$ is uniform), a contradiction from which the result follows.

**Theorem 2.5.** Let $\mathcal{C}$ be a $d$-uniform clutter without isolated vertices and $e_0 = [d] \in \mathcal{C}$. The $e_0$-chart of $B_{\Pi(\mathcal{C})}(R)$ is regular if and only if $\mathcal{C}$ is a $d$-partite clutter with $d$-partition $\{N(e_0 \setminus \{i\}) : i \in [d]\}$.

**Proof.** ($\Rightarrow$): Assume that $U$, $U'$ and $v(j)$'s are as in Lemma 2.4. Also for $i \leq d$ set $v(i) = i$. Let $V_i = \{j \in [n] : v(j) = i\}$ for each $i \in [d]$. Clearly $V_i$'s form a partition of $[n]$. Suppose that for some $e \in \mathcal{C}$ and $i \in [d]$, $e \cap V_i > 1$, say $r \neq s \in e \cap V_i$. Because $e \neq e_0$ and $u = \frac{x_j}{x_{e_0}} \in U$ and since $\deg u = 0$, we deduce that $u$ is a product of monomials in $U'$ with zero degree. That is, $u = \frac{x_{j_1}}{x_{v(j_1)}} \cdots \frac{x_{j_r}}{x_{v(j_r)}}$ for some $j_i \in [n]$ (note that we are using the convention $v(j) = j$ for $j \leq d$). Since $x_r, x_s|e_0$, we can assume that $j_1 = r$ and $j_2 = s$. So $x_{v(r)}x_{v(s)} = x_1^2|x_{e_0}$, a contradiction.

From this contradiction, we conclude that $V_i$'s form a $d$-partition of the $d$-uniform clutter $\mathcal{C}$. Whence $N(e \setminus \{j\})$ is contained in $V_{v(j)}$ for each $j \in e \in \mathcal{C}$. In particular, $N(e_0 \setminus \{i\}) \subseteq V_i$ for each $i \in [d]$. Now as $\frac{x_j}{x_{v(j)}} \in U$, there should exist $e \in \mathcal{C}$ with $\frac{x_j}{x_{e_0}} = \frac{x_j}{x_{v(j)}}$, that is, $e = (e_0 \setminus \{v(j)\}) \cup \{j\} \in \mathcal{C}$. Therefore, $j \notin N(e_0 \setminus \{v(j)\})$ and the union of $N(e_0 \setminus \{i\})$'s $(i \in [d])$ is the whole $[n]$. It follows that $V_i = N(e_0 \setminus \{i\})$, as required.

($\Leftarrow$): Let $V_i = N(e_0 \setminus \{i\})$. Thus $e = (e_0 \setminus \{i\}) \cup \{j\} \in \mathcal{C}$ for each $j \in V_i$. So if we set $v(j) = i$, then $\frac{x_j}{x_{v(j)}} = \frac{x_j}{x_{e_0}} \in U$. Let $U' = \{x_1, \ldots, x_d\} \cup \{\frac{x_j}{x_{v(j)}} : d < j \leq n\}$. 

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**TAME GRAPHS, CLUTTERS AND THEIR REES ALGEBRAS**

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Obviously $x_i \in k[U']$ for each $i \in [n]$. If $e' \in \mathcal{C}$, then $e' = \{j_1, \ldots, j_d\}$ with $j_i \in V_i$, because $\mathcal{C}$ is $d$-partite with $d$-partition $V_i$’s. Hence $\frac{x_{e'}}{x_{e_0}} = \frac{x_{j_1}}{x_{e_0(j_1)}} \cdots \frac{x_{j_d}}{x_{e_0(j_d)}} \in k[U']$. Consequently, $k[U] = k[U']$ and the result follows from Lemma 2.2. \hfill \blacksquare

**Corollary 2.6.** Assume that $\mathcal{C}$ is a $d$-uniform clutter without isolated vertices. Then $\mathcal{C}$ is tame if and only if $\mathcal{C}$ is complete $d$-partite.

**Proof.** $(\leftarrow)$: Immediate consequence of Theorem 2.5.

$(\rightarrow)$: Let $e = [d] \in \mathcal{C}$. Then by Theorem 2.5, $\mathcal{C}$ is a $d$-partite clutter with $d$-partition $\{V_i = N(e \setminus \{i\}) | i \in [d]\}$. First note that the $d$-partition of $\mathcal{C}$ is unique, because clearly vertices $1, \ldots, d$ should be in different partitions and each $v \in N(e \setminus \{i\})$ should be in the same partition as $i (i \in [d])$.

Now assume that there is an $e' \subseteq [n] \setminus \mathcal{C}$, $|e'| = d$ and $|e' \cap V_i| = 1$ for each $i \in [d]$. Choose an $e_0 \in \mathcal{C}$ such that $|e' \cap e_0|$ is the maximum possible. If $a = e' \cap e_0$, then we can assume that $e' = a \cup \{i_1, \ldots, i_r\}$ and $e_0 = a \cup \{\bar{i}_1, \ldots, \bar{i}_r\}$ for some $i \neq \bar{i}_j \in V_i$. Since the $e_0$-chart of $\text{Bl}_{\mathcal{C}}(R)$ is regular it follows Theorem 2.5 that $i_1 \in V_1 = N(e_0 \setminus \{j_1\})$ and so $e'_0 = e_0 \setminus \{j_1\} \cup \{i_1\} \in \mathcal{C}$. But $|e' \cap e'_0| > |e' \cap e_0|$ contradicting the choice of $e_0$. Consequently no $e'$ with the above properties exists, that is, $\mathcal{C}$ is complete $d$-partite. \hfill \blacksquare

Now that we have a characterization of tame uniform clutters, let’s pay attention to non-uniform clutters.

**Lemma 2.7.** Let $\mathcal{C}$ be a clutter and $e_0 = [d] \in \mathcal{C}$. Also assume that $d \leq |e|$ for each $e \in \mathcal{C}$ and set $\mathcal{C}' = \{e \in \mathcal{C}: |e| = d\}$.

(i) If the $e_0$-chart of $\text{Bl}_{\mathcal{C}}(R)$ is regular, then the $e_0$-chart of $\text{Bl}_{\mathcal{C}}(e') (R)$ is regular.

(ii) If $e_0$-chart of $\text{Bl}_{\mathcal{C}}(e') (R)$ is regular, $\mathcal{C}'$ has no isolated vertices and $U'$ is as in Corollary 2.2, then for each $d < j \leq n$, there is a $\nu(j) \leq d$ such that $U' = \{x_1, \ldots, x_d\} \cup \{\frac{x_{e_0}}{x_{e_0(j)}}: d < j \leq n\}$.

**Proof.** Suppose that $U$, $U'$ and $S$ are as in Lemma 2.2 and the notes before it. Also let $U'' = \{x_1, \ldots, x_n\} \cup \{x_{e_0}| e_0 \neq e \in \mathcal{C}'\}$. Thus the $e_0$-chart of $\mathcal{C}'$ is $\text{Spec}(k[U''])$.

We just need to show that $U' \subseteq U''$. According to Lemma 2.3, the monomials in $U'$ which are not a variable are of the form $u = \frac{x_{e_0}}{x_{e_0(j)}}$, for some $\pi(j) \subseteq [d]$. So there is an $e \in \mathcal{C}$ with $u = \frac{x_{e_0}}{x_{e_0}}$, which means $e = (e_0 \setminus \pi(j)) \cup \{j\}$. Since $|e| \geq |e_0|$ we conclude that $|\pi(j)| = 1$ and hence $|e| = |e_0|$, that is $e \in \mathcal{C}'$ and $u \in U''$. The final statement follows by applying Lemma 2.4 on the $e_0$-chart of $\mathcal{C}'$. \hfill \blacksquare

Now we can present a full characterization of tame clutters.

**Theorem 2.8.** Suppose that $\mathcal{C}$ is a clutter. Then $\mathcal{C}$ is tame if and only if $\mathcal{C}$ is a union of some isolated vertices and a complete $d$-partite $d$-uniform clutter, for some positive integer $d$.

**Proof.** $(\leftarrow)$: Proved in Corollary 2.6. (Note that adding or removing isolated vertices does not affect tameness.)

$(\rightarrow)$: Let $d = \min\{|e|: e \in \mathcal{C}\}$ and $\mathcal{C}' = \{e \in \mathcal{C}: |e| = d\}$. Then by Lemma 2.7, $\mathcal{C}'$ is tame. We may assume that $m \leq n$ is such that the isolated vertices of $\mathcal{C}'$ are
m + 1, . . . , n. Applying Corollary 2.6 on ℂ′|_{[m]} (that is, viewing ℂ′ as a clutter on vertex set [m]), we get that ℂ′|_{[m]} is a complete d-partite d-uniform clutter. Let e ∈ ℂ and e ̸= e_0 = [d] ∈ ℂ′. Then it follows from Lemma 2.7 that, the coordinate ring of the e_0-chart of ℂ′ is generated over k by U′ = \{x_1, . . . , x_d\} ∪ \{(x_{v(j)}) : d < j ≤ m\} ∪ \{x_{m+1}, . . . , x_n\} where each v(j) ≤ d. In particular, 
\[ u = \frac{x_e}{x_{e_0}} = \frac{x_{j_1}}{x_{v(j_1)}} \cdots \frac{x_{j_t}}{x_{v(j_t)}} 1_{\bar{e}_t} \cdots 1_{x_t}, \]
where d < j_1, . . . , j_t ≤ m. As for each i ∈ [d] \setminus e the denominator of u is divisible by x_i, we deduce that such an i should appear as some v(j_i), say v(j_i). It follows that j_i ∈ Ne_e(e_0 \setminus \{i\}) which is the i’th partition of ℂ′|_{[m]}. Since x_j_i’s divide the numerator of u, we have e′ = \{j_i : i ∈ e_0 \setminus e\} ∪ (e \cap e_0) ⊆ e. But e′ meets each partition of ℂ′|_{[m]} in exactly one vertex and so e′ ∈ ℂ′ ⊆ ℂ which is in contradiction with ℂ being a clutter, unless e = e′. Therefore, |e| = d and as e was arbitrary we conclude that ℂ = ℂ′ is a union of some isolated vertices and a complete d-partite d-uniform clutter. □

Let F ⊆ [n]. By P_F we mean the prime ideal of R generated by x_i’s with i ∈ F. Using this notation, the algebraic restatement of the above theorem is:

**Corollary 2.9.** Suppose that I is a proper squarefree monomial ideal of R. Then the following are equivalent:

(i) the blowup of \( \mathbb{A}_k^n \) along I is regular;

(ii) there exist mutually disjoint nonempty subsets F_1, . . . , F_d of [n] such that I = P_{F_1}P_{F_2}\cdots P_{F_d} = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_d}.

**Tameness via Stanley-Reisner complexes.** Let R = k[x_1, . . . , x_n] be the polynomial ring over a field k and m = (x_1, . . . , x_n) be its irredundant (homogeneous) maximal ideal. Squarefree monomial ideals I ⊆ m^2 are in one-to-one correspondence to simplicial complexes on [n] = \{1, . . . , n\}, via Stanley-Reisner ideals. The interaction between combinatorial behaviour of simplicial complexes and algebraic (geometric) properties of corresponding ideals (varieties) is wide area of research in combinatorial commutative algebra (algebraic geometry). In the first part of this section, it was observed that, for a squarefree monomial ideal I, the property of being tame is eventually a combinatorial property. In the following, we state an equivalent condition on a simplicial complex, such that the corresponding Stanley-Reisner ideal is tame.

**Definition 2.10** (Simplicial complex). A simplicial complex \( \Delta \) over a set of vertices \( V = \{v_1, . . . , v_n\} \), is a collection of subsets of V, with the property that:

(a) \{v_i\} ∈ \Delta, for all i;

(b) if F ∈ \Delta, then all subsets of F are also in \( \Delta \) (including the empty set).

An element of \( \Delta \) is called a face of \( \Delta \) and a non-face of \( \Delta \) is a subset F of V with F \notin \Delta. The maximal faces of \( \Delta \) under inclusion are called facets of \( \Delta \). Let \( \mathcal{F}(\Delta) = \{F_1, . . . , F_q\} \) be the facet set of \( \Delta \). It is clear that \( \mathcal{F}(\Delta) \) determines \( \Delta \) completely and we write \( \Delta = \langle F_1, . . . , F_q \rangle \).
Let \( \Delta \) be a simplicial complex over \( n \) vertices labelled \( v_1, \ldots, v_n \). The non-face ideal or the Stanley-Reisner ideal of \( \Delta \), denoted by \( I_\Delta \), is the ideal of \( R \) generated by squarefree monomials \( \{x_F: F \in \mathcal{N}(\Delta)\} \). One may notice that there exists a one-to-one correspondence between squarefree monomial ideals in \( m^2 \) and Stanley-Reisner ideals of simplicial complexes. It is well-known that \( I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F \) is a prime decomposition of \( I_\Delta \), where \( P_F \) denotes the (prime) ideal generated by all \( \{x_i: v_i \notin F\} \) (see e.g. [9], for more details). By virtue of Corollary 2.9, we may characterize tame squarefree monomial ideals in terms of combinatorics of the associated simplicial complexes.

**Proposition 2.11.** Let \( I \) be a squarefree monomial ideal and \( \Delta \) be a simplicial complex on vertex set \([n]\) such that \( I_\Delta = I \). The following are equivalent.

(i) \( I \) is a tame ideal.

(ii) For any two distinct facets \( F, G \in \mathcal{F}(\Delta) \), we have \( F \cup G = [n] \).

**Proof.** Let \( \mathcal{F}(\Delta) = \{G_1, \ldots, G_s\} \). Then

\[
\text{(2)} \quad I = I_\Delta = P_{G_1} \cap \cdots \cap P_{G_s}
\]

is the minimal prime decomposition of \( I \).

(i) \( \Rightarrow \) (ii): Since \( I \) is a tame squarefree monomial ideal, it follows from Corollary 2.9 that:

\[
I = P_{F_1} \cap \cdots \cap P_{F_r},
\]

where \( F_1, \ldots, F_r \) are mutually disjoint subsets of \([n]\). Since the minimal prime decomposition of a squarefree monomial ideal is unique (up to a permutation of prime components), we conclude that, for all \( i \in [r] \) there exits a (unique) \( j \in [s] \), such that \( F_i = G_j \). The assertion now follows from the fact that \( F_1, \ldots, F_r \) are mutually disjoint.

(ii) \( \Rightarrow \) (i): Our assumption in (ii) implies that, \( G_i \)'s are mutually disjoint subset of \([n]\). Thus the desired conclusion follows from (2) and Corollary 2.9. \( \square \)

**Remark 1.** The statement in Proposition 2.11, provides a simple algorithm for detecting tameness of squarefree monomial ideals. To be more precise, for a given a squarefree monomial ideal \( I = (x_{F_1}, \ldots, x_{F_r}) \), let

\[
P_{F_1} \cap \cdots \cap P_{F_r} = (x_{[n]\setminus T_1}, \ldots, x_{[n]\setminus T_s}),
\]

and \( \Delta = \langle T_1, \ldots, T_s \rangle \). Then, \( I_\Delta = I \) and the equivalent condition on \( I \) to be tame is to show that \( T_i \cup T_j = [n], \) for all \( i \neq j \).

**Polarization and tameness.** Polarization is a technique which corresponds to an arbitrary monomial ideal, a squarefree monomial ideal in a new set of variables. The construction is as follows:

Let \( I \) be a monomial ideal in the polynomial ring \( R = k[x_1, \ldots, x_n] \) with the (unique) minimal set of generators \( \mathcal{G}(I) = \{u_1, \ldots, u_r\} \), where \( u_i = \prod_{j=1}^n x_j^{\alpha_{i,j}} \). Let \( \alpha_i \) be the maximum exponent of the variable \( x_i \) appearing in elements of \( \mathcal{G}(I) \). Without loss of generality, we may assume that \( \alpha_i \) are all positive. Let \( S = R[Y_{i,j}: i = 1, \ldots, n \text{ and } j = 2, \ldots, \alpha_i] \). For each monomial \( u_i \), we define

\[
u_i^\varphi = x_1^{\min\{\alpha_{i,1}:1\}}Y_{1,2} \cdots Y_{1,\alpha_{i,j}} \cdots x_n^{\min\{\alpha_{i,n}:1\}}Y_{n,2} \cdots Y_{n,\alpha_{i,j}}.
\]
By the choice of $\alpha_i$, $u_i^{\alpha_i} \in S$. The polarization $I^{\varphi}$ of $I$ is the ideal in $S$ generated by $\{u_1^{\alpha_1}, \ldots, u_i^{\alpha_i}\}$.

A monomial ideal $I$ and its polarization $I^{\varphi}$ share many homological and algebraic properties. Thus, by polarization, many questions concerning monomial ideals can be reduced to squarefree monomial ideals. For example the graded Betti numbers of $I$ and $I^{\varphi}$ are the same (c.f. [7, Corollary 1.6.3]). In the following we show that if $I^{\varphi}$ is a tame ideal, then so is $I$, but the converse is not true as the following example shows:

**Example 2.12.** Assume that $I = (x_1^2, x_1 x_2, x_2^2) \subset k[x_1, x_2]$. Then, $I^{\varphi} = (x_1 y_1, x_1 x_2, x_2 y_2) \subseteq k[x_1, x_2, y_1, y_2]$ is not tame by Theorem 2.8 (or Corollary 3.6). However, it follows from Theorem 3.8 that, the ideal $I$ is tame.

**Proposition 2.13.** Let $I \subset R$ be a monomial ideal and $I^{\varphi} \subset S$ be the polarization of $I$. The following are equivalent:

(i) $I^{\varphi}$ is a tame squarefree monomial ideal in $S$;

(ii) there exist a monomial $u \in R$ and a tame squarefree monomial ideal $I' \subset R$ such that $I = u I'$.

In particular, if $I^{\varphi}$ is tame, then so is $I$.

**Proof.** (i) $\Rightarrow$ (ii): Let $\mathcal{C}$ be the clutter associated to $I^{\varphi}$. Since $I^{\varphi}$ is a tame squarefree monomial ideal, it follows from Theorem 2.8 that, $\mathcal{C}$ is a uniform clutter, say $d$-uniform, and hence $I$ is a monomial ideal generated by monomials of the same degree $d$.

Using the same notation as above, let $\alpha_i$ be the maximum exponent of the variable $x_i$ appearing in elements of $\mathcal{G}(I)$. Again, without loss of generality, we may assume that $\alpha_i$ are all positive. Since $I^{\varphi}$ is tame, we conclude from Theorem 2.8 that, $\mathcal{C}$ is complete $d$-partite and $\{Y_{i,2}, \ldots, Y_{i,\alpha_i}\}$ are some of the partitions of $\mathcal{C}$, for all $i$. So $Y_{i,2} \cdots Y_{i,\alpha_i}$ divides all generators of $I^{\varphi}$, for all $i$. Equivalently, $x_i^{\alpha_i-1}$ divides all generators of $I$, for all $i$. Let $u = x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1}$ and $I' = (f/u: f \in \mathcal{G}(I))$. Then, by the above discussion, $I'$ is a squarefree monomial ideal in $R$ and $I = u I'$. Let $\mathcal{C}'$ be the clutter associated to $I'$. Then,\

$$\mathcal{C}' = \{F \setminus \{Y_{1,2}, \ldots, Y_{1,\alpha_1}, \ldots, Y_{n,2}, \ldots, Y_{n,\alpha_n}\}: F \in \mathcal{C}\}.$$\

Since $\mathcal{C}$ is tame, we may apply Theorem 2.8, to observe that $\mathcal{C}'$ is again tame.

(ii) $\Rightarrow$ (i): In this case, $I^{\varphi} = u^{\alpha_i} I'$. Hence, $\mathcal{G}(I^{\varphi}) = \{u^{\alpha_i} f: f \in \mathcal{G}(I')\}$. So that, for every $f \in \mathcal{G}(I)$, the $f$-chart of $I'$ coincide with $u^{\alpha_i} f$-chart of $I^{\varphi}$. Since $I'$ is a tame ideal, the $f$-chart of $I'$ is regular, for every $f \in \mathcal{G}(I')$. Hence, $f'$-chart of $I^{\varphi}$ is also regular, for all $f' \in \mathcal{G}(I^{\varphi})$. This means that $I^{\varphi}$ is a tame ideal.

To obtain the last statement, we use the same argument as in the proof of (ii) $\Rightarrow$ (i). □

3. Tame Monomial Ideals Generated in Degree at Most Two

Squarefree monomial ideals are not the only class of monomial ideals whose vertices can be classified exactly. Another such class is the class of monomial ideals generated in degree at most 2. In this section we investigate tameness of these ideals.
Our first simple observation is that if \( x_i^n \) is in \( G(I) \) for a monomial ideal \( I \) and \( a = (a_1, \ldots, a_n) \in \text{Supp}(I) \) (or even in \( N(I) \)) with \( a_i = 0 \) for all \( 1 \neq i \), then \( a_1 \geq d \).

Now if \( p = (d, 0, 0, \ldots, 0) \) is a convex combination of different points in \( N(I) \), then clearly all these points should have zero on all entries except the first one. So indeed all these points should be \( p \) itself, that is, \( p \) is a vertex of \( N(I) \). A similar argument, which we leave the details to the reader, proves the following lemma.

**Lemma 3.1.** If \( x_i^\alpha \in G(I) \) for some \( \alpha > 0 \), then \( x_i^\alpha \) is a vertex of \( I \). Also if degree of all elements of \( G(I) \) is at most two, then \( u \in G(I) \) is not a vertex of \( I \), if and only if \( u = x_ix_j \) for some \( i \neq j \in [n] \) such that \( x_i^2, x_j^2 \in G(I) \).

Suppose that \( I \) is a monomial ideal and \( u \in G(I) \) is a vertex of \( I \) such that the \( u \)-chart of \( B_1(R) \) is regular. If we set \( U_1 = \{x_1, \ldots, x_n\}, U_2 = \{\frac{x_i}{u}: u \neq u' \in G(I)\} \) and \( U = U_1 \cup U_2 \), then by Lemma 1.2 there is a \( U' \subseteq U \) with \( |U'| = n \) and \( k[U] = k[U'] \). The next lemma states the form of \( U' \) in some special cases. Here by \( \text{deg}_i(f) \) for a \( f \in R \), we mean the degree of \( f \) in the variable \( x_i \). Also \( \text{Supp}(u) \) means \( \{i: x_i|u\} \) for a monomial \( u \) of \( R \).

**Lemma 3.2.** Using the above notations and assumptions, if

1. \( \text{deg} \leq \text{deg} u' \) for all \( u' \in G(I) \)
2. \( \text{deg}_i u \geq \text{deg}_i u' \) for all \( i \in \text{Supp}(u) \) and all \( u' \in G(I) \),

then for each \( j \) with \( x_j \notin U' \), there is a \( v(j) \in \text{Supp}(u) \) such that \( U' = (U_1 \cap U') \cup \{\frac{x_i}{x_{v(j)}}: x_j \notin U'\} \).

**Proof.** Suppose that \( x_j \notin U' \). Then

\[
(3) \quad x_j = x_{i_1} \cdots x_{i_r} z_1 \cdots z_s
\]

where \( z_i = \frac{w_i}{u_i} \), with all terms of the right hand side in \( U' \). Note that the assumption (ii) implies that when we write \( z_i \)’s in the simplest form, if \( x_i \) divides the numerator of some \( z_i \), then \( j \notin \text{Supp}(u) \). Hence the in the product \( z_1 \cdots z_s \) the denominators of \( z_i \)’s do not cancel out. This forces \( r \) to be at least one. On the other hand, since degree of the left hand side of (3) is one and \( \text{deg} z_i \geq 0 \) (by (i)), we see that \( r = 1 \).

Again since the denominators of \( z_i \)’s do not cancel out in their product, we conclude that \( s \leq 1 \), so \( s = 1 \). Therefore, \( z_1 = \frac{x_i}{x_{v(j)}} \in U' \) and \( i_1 \in \text{Supp}(u) \). Thus if we set \( v(j) = i_1 \), then the result is established. \qed

**Proposition 3.3.** Suppose that \( G(I) = \{x_1, x_2, \ldots, x_{t-1}, u_1, \ldots, u_s\} \), where \( \text{deg} u_i \geq 2 \).

If \( t, s \geq 2 \), then the \( x_1 \)-chart of \( B_1(R) \) is not regular for \( i \in [t] \).

**Proof.** We assume \( i = 1 \) and that the \( x_1 \)-chart is regular. According to Lemma 3.1, \( x_1 \) is a vertex and clearly satisfies the conditions of Lemma 3.2. Thus using the notations of Lemma 3.2, it easily follows that \( U' = \{x_1, \frac{x_2}{x_1}, \ldots, \frac{x_t}{x_1}, x_{t+1}, \ldots, x_{t+s}\} \).

Now \( \frac{x_i}{x_1} \) should be a product of elements of \( U' \). Clearly this product contains a term of the form \( \frac{x_i}{x_1} \) for \( 1 < j \leq t \) which means \( x_j | u_1 \), a contradiction with the minimality of \( G(I) \). \qed

**Corollary 3.4.** Suppose that \( I \) is a tame monomial ideal generated in degree at most 2. Then either all elements of \( G(I) \) have degree 1 or all of them have degree 2.
Corollary 3.4 together with Theorem 2.8 imply that if \( I \) is a monomial ideal and either is squarefree or generated in degree at most 2, then it is generated in one degree. This poses the question “Does there exist a tame monomial ideal not generated in one degree?” The answer is yes, as the following example shows.

**Example 3.5.** Let \( I = \langle x^2, y^2, xy \rangle \). Then it is easy to see that the three generators of \( I \) are vertices. Also the charts of \( \text{Bl}_I(R) \) corresponding to \( x^2, y^2 \) and \( xy \) are \( k[x, \frac{x}{2}] \), \( k[y, \frac{y}{2}] \) and \( k[\frac{x^2}{2}, \frac{y^2}{2}] \), respectively. Hence according to Lemma 1.2, \( I \) is tame.

Note that any ideal generated with monomials of degree one is clearly tame. Thus we focus on ideals \( I \) where \( \mathcal{G}(I) \) consists of monomials of degree 2. Such an ideal could be represented by the edge ideal of an undirected graph with loops (but no multiple edges). For such graphs \( G \), the edge ideal \( I(G) \) is generated by \( x_e \)'s where \( e \) is an edge of \( G \). Here if \( e \) is a loop on \( i \), we set \( x_e = x_i^2 \). Note that if \( G \) is simple, then it is a clutter and by setting \( d = 2 \) in Theorem 2.8, we get:

**Corollary 3.6.** Let \( G \) be a simple graph. Then \( G \) is tame if and only if \( G \) is the disjoint union of a complete bipartite graph and some isolated vertices.

Thus, in the rest of this section we use the following notation.

**Notation 1.** In the sequel, \( G \) is assumed to be a graph on \([n]\) where the vertices \(1, \ldots, r\) have loops and \(r + 1, \ldots, n\) do not have loops with \( r \geq 1\).

Note that since 1 has a loop we have \( 1 \in N_G(1) \).

**Lemma 3.7.** Using Notation 1 and for \( i \in [r] \), the \( x_i^2 \)-chart of \( \text{Bl}_{I(G)}(R) \) is regular if and only if for all edges \( e \) of \( G \) (including loops), we have \( e \subseteq N_G(i) \).

**Proof.** (\( \Rightarrow \)): We assume \( i = 1 \). By Lemma 3.1, \( x_1^2 \) is a vertex of \( I(G) \) and clearly satisfies the conditions of Lemma 3.2. Let \( U \) be as in the statement right before Lemma 3.2. It follows that \( k[U] \) is generated over \( k \) by a set \( U' \) of some variables and some monomials of the form \( u_j = \frac{x_i}{x_1} \in U \). So \( u_j = \frac{x_j}{x_1} \) for some edge \( e' \) of \( G \) which should be \( e' = \{1, j\} \). Therefore \( j \in N(1) \). Now for each edge \( e \) of \( G \), \( \frac{x_j}{x_1} \) should be a product of elements of \( U' \) and since \( \deg \frac{x_j}{x_1} = 0 \) we should have \( \frac{x_j}{x_1} = \frac{x_i}{x_1} \frac{x_i}{x_1} \) for some \( i, j \in N(1) \), and \( e \subseteq N(1) \).

(\( \Leftarrow \)): It is easy to see that the coordinate ring is generated by \( \frac{x_j}{x_1} \) for \( j \in N(i) \) together with \( x_l \) with \( l \notin N(i) \) or \( l = i \). \( \square \)

In the following by a **looped star graph** we mean an star graph with an additional loop on the central vertex and by a **looped complete graph**, we mean a complete graph with loops on each vertex.

**Theorem 3.8.** Suppose that \( G \) is a graph with at least one loop. Then \( G \) is tame if and only if \( G \) is a disjoint union of some isolated vertices with either a looped star graph or a looped complete graph.

**Proof.** We use Notation 1. Clearly we can assume that \( G \) has no isolated vertices. (\( \Rightarrow \)): By Lemma 3.7, we see that the induced subgraph of \( G \) on \([r]\) (denoted \( G_1 \)) is
a looped complete graph. Also since for each $r < j$ the vertex $j$ is in some edge and again by Lemma 3.7, $j \in N(i)$ for each $i \leq r$.

Assume that there are vertices $r < i \neq j \leq n$ such that $e = \{i, j\}$ is an edge of $G$. Then by Lemmata 3.1 and 3.2, the coordinate ring $S$ of the $e$-chart of $\text{Bl}_{I(G)}(R)$ is generated as a $k$ algebra by a set $U'$ of some variables and some monomials of the form $\frac{x_i}{x_{ij}}$ for some $l \in [n]$ and $v(l) \in e$. Since the only elements of $U'$ with degree zero are in the latter form and $u = \frac{x_i^2}{x_ix_j} \in S$, we conclude that $u = \frac{x_i^2 x_{ij}}{x_ix_j}$ for some $\frac{x_i}{x_{ij}}, \frac{x_j}{x_{ij}} \in U'$. But then $l = l' = 1$ and hence $v(l) = i = j$ a contradiction. Thus the induced subgraph of $G$ on vertices $r + 1, \ldots, n$ has no edges.

Consequently, $G$ is the union of a looped complete graph $G_1$ and all edges $\{i,j\}$ with $i \leq r < j \leq n$. If $r = n$, we are finished. Hence assume that $r < n$ and consider the chart corresponding to $e = \{1,n\}$. One can readily check that its coordinate ring is

$$S' = k \left[ \{x_1, \ldots, x_n\} \cup \left\{ \frac{x_i x_j}{x_1 x_n} : i \in [r], j \in [n] \right\} \right]$$

$$= k \left[ \{x_n\} \cup \left\{ \frac{x_j}{x_n} : 1 \leq j \leq n - 1 \right\} \cup \left\{ \frac{x_i}{x_1} : 1 < i \leq r \right\} \right] .$$

Note that the second set of generators $U'$ of $S'$ is minimal and as $x_e$ is a vertex of $I(G)$ by Lemma 3.1, we should have $1 + (n - 1) + (r - 1) = |U'| = n$ or $r = 1$. This means that $G$ is a looped star graph as required.

$(\Leftarrow)$: If $G$ is looped complete graph, then the according to Lemma 3.1 the only vertices of $I(G)$ are $x_i^2$ for $i \in [r]$ and the corresponding charts are regular by Lemma 3.7. If $G$ is a looped star graph, then the $x_1^2$-chart is regular by Lemma 3.7. For the other charts, say corresponding to $x_1 x_n$, the minimal set of generators found above for $S'$ has cardinality $n$ and hence the result follows by Lemma 1.2.

The following summarizes our results in this section. Note that in this result, (i) corresponds to ideals generated in degree 1, (ii) to looped complete graphs, (iii) to looped star graphs and (iv) to simple graphs.

**Corollary 3.9.** Assume that $I$ is an ideal of $R$ generated by monomials of degree at most two. Then the blowup of $A^m_k$ along $I$ is regular if and only if one of the following holds.

1. There exists a nonempty $F \subseteq [n]$ such that $I = P_F$.
2. There exists a nonempty $F \subseteq [n]$ such that $I = P_2^m$.
3. There exist nonempty $F \subseteq [n]$ and $i \in F$ such that $I = x_i P_F$.
4. There exist nonempty $F_1, F_2 \subseteq [n]$ with $F_1 \cap F_2 = \emptyset$ and $I = P_{F_1} P_{F_2}$.

4. Rees algebra of tame clutters

In this section we find a generating set for the presentation ideal or toric ideal of the coordinate ring of an affine chart of the blowup $A^m_k$ along the circuit ideal of a complete $d$-partite $d$-uniform clutter. By using this presentation, we look closely at the defining ideal of the Rees algebra of a tame complete $d$-partite $d$-uniform clutter.
If \( I \) is the edge ideal of a graph, R. H. Villarreal shows that \( I \) is of fiber type [11, Theorem 3.1]. He gives an explicit description of the defining ideal of the Rees algebra of any squarefree monomial ideal generated in degree 2. Villarreal give an example of a clutter to show that his arguments do not extend for monomial ideals generated in higher degree than 2 [11, Example 3.1]. If \( I \) is the edge ideal of a complete bipartite graph he finds the defining equations of the Rees algebra of \( I \) in terms of 2-minors of a ladder [12, Proposition 2.1]. Here we generalize the latter result of Villarreal. More precisely, we find the defining ideal of the Rees algebra of a tame squarefree monomial ideal. This result shows that such ideals are of fiber type.

**Notation 2.** We let \( \mathcal{C} \) be a complete \( d \)-partite \( d \)-uniform clutter with the \( d \)-partition \( \{ V_i : i \in [d] \} \) and \( e \in \mathcal{C} \). Also \( S_e \) stands for the coordinate ring of the \( e \)-chart of \( \text{Bl}_I(R) \). Consider the ring homomorphism

\[
\phi_e : S = R[\{ T_e' : e \neq e' \in \mathcal{C} \}] \to S_e
\]

that sends \( T_e \) to \( \frac{x_e}{x_e} \). Set \( J_e = \ker \phi_e \). Moreover, for \( e \neq e' \in \mathcal{C} \) we fix a vertex \( v(e, e') \in e \setminus e' \) such that \( v(e, e') \) and \( v(e', e) \) lie in the same partition and by \( v_e(j) \) we mean the only vertex of \( e \) in the same partition as the vertex \( j \). Finally we denote the circuit of \( \mathcal{C} \) obtained from \( e \) by replacing \( j \) instead of \( v_e(j) \) by \( e(j) \).

Assume that \( j = v(e, e') \in V_i \). Then since \( j' = v(e', e) \) is in \( V_i \cap e' \) we have \( j' = v_e(j) \) and similarly \( j = v_e(j') \). In this case, \( e(j') \) and \( e'(j) \) are the circuits obtained from \( e \) and \( e' \) respectively, by “swapping” those vertices of \( e \) and \( e' \) which lie in the \( i \)’th partition. For example, \( e(j') = (e \cup \{ j' \}) \setminus \{ v_e(j') \} = (e \cup \{ j' \}) \setminus \{ j \} \).

**Proposition 4.1.** Using the above notations and assumptions, for each \( e \in \mathcal{C} \), \( J_e \) is generated by \( G = G_1 \cup G_2 \) where \( G_1 \) is the set of all binomials of the form \( x_i - x_j T_{e(i)} \) with \( i \in [n] \setminus e \) and \( r = v_e(i) \) and \( G_2 \) is the set of all binomials of the form \( T_{e(j')} - T_{e(j)} T_{e(j')} \) where \( j = v(e, e') \) and \( j' = v(e', e) \), \( e \in \mathcal{C} \) with \( |e' \setminus e| > 1 \).

**Proof.** For simplicity we assume that \( e = [d] \). It is routine to check that the stated binomials map to zero under \( \phi_e \), that is, \( G \subseteq \ker \phi_e \). We use \( T \) to denote the set of indeterminates \( \{ T_{e'} : e \neq e' \in \mathcal{C} \} \) and set \( T|_l = \{ T_{e'} : e \neq e' \in \mathcal{C}, |e' \setminus e| \leq l \} \). Also by \( G_{2l} \) we mean the subset of \( G_2 \) consisting of those binomials with \( |e' \setminus e| \leq l \). In particular, \( G_{21} = 0 \). Note that for a binomial \( T_{e'} - T_{e(j')} T_{e(j)} \) in \( G_2 \), \( |e(j') \setminus e| = 1 \) and \( |e' \setminus e| = |e' \setminus e| - 1 \).

Let \( J \) be the ideal of \( S \) generated by the stated binomials, \( J' \) the ideal of \( S' = k[x_1, \ldots, x_d][T] \) generated by \( G_2 \) and \( J'_1 \) the ideal of \( S'_1 = k[x_1, \ldots, x_d][T][|i|] \) generated by \( G_{2|i|} \). Then it is easy to see that the homomorphism \( S \to S'/J' \) which maps \( x_i \) to \( x_i T_{e(i)} \) with \( i \) and \( r \) as in the statement of the theorem, induces an isomorphism \( S/J \to S'_1/J'_1 \).

Similarly:

\[
\frac{S}{J} \cong \frac{S'}{J'} = \frac{S'_{d-1}}{J'_{d-1}} \cong \frac{S'_{d-2}}{J'_{d-2}} \cong \cdots \cong \frac{S'_{1}}{J'_{1}} = S'_{1}.
\]
Because $S_1'$ is a polynomial ring with $n$ indeterminates, $J$ is a prime ideal with $\dim J = n$. But $\ker \phi_e$ is also a prime ideal with $\frac{S}{\ker \phi_e}$ having dimension $n$ and $J \subseteq \ker \phi_e$. Consequently, $J = \ker \phi_e$. \hfill \Box

**Theorem 4.2.** Let $C$ be a complete $d$-partite $d$-uniform clutter with $d$-partition \( \{V_i : i \in [d]\} \). Then

\[
R_R(I(C)) \simeq \frac{R[T_e : e \in C]}{A}
\]

where $A$ is generated by the set of all binomials of the form $T_e x_i - x_i T_{e(i)}$ with $e \in C$, $i \in [n] \setminus e$ and $r = v_e(i)$ together with those of the form $T_e x_{e'} - T_{e'(j)} T_{e'(j)}$ where $j = v(e, e')$ and $j' = v(e', e)$, for $e \neq e' \in C$ with $|e' \setminus e| > 1$. In particular, the circuit ideal of a complete $d$-partite $d$-uniform clutter is of fiber type.

**Proof.** First note that the set of generators of $A$ is obtained by homogenization of the generators of affine chart $J_e$ in the variable $T_e$ and taking union over all $e \in C$. By Proposition 4.1 and the proof of Proposition 1.1(iii), clearly $A$ belongs to the defining ideal $J$ of $R_R(I(C))$. Conversely, let $F \in J$. Then the dehomogenizing of $F$ with respect to a variable $T_e$, which has appeared in $F$, lies in at least one affine chart. Thus again homogenizing it with respect to $T_e$, we get the assertion. \hfill \Box

**Example 4.3.** Let $R = k[x_1, x_2, y_1, y_2, z]$. By Theorem 2.5 the ideal $I = (x_1, x_2)(y_1, y_2)(z)$ is tame. By Theorem 4.2, the defining ideal of the Rees algebra of $I$ is generated by

\[
x_2 T_1 - x_1 T_3, \, x_2 T_2 - x_1 T_4, \, y_1 T_4 - y_2 T_3, \, y_2 T_1 - y_1 T_2, \, T_2 T_3 - T_1 T_4.
\]

Note that the defining ideal of the Rees algebra of ideals in cases (i) and (iv) in Corollary 3.9, can be determined by Theorem 4.2. Also in case (iii), the defining equation of the $x_i P_F$ is the same as those of $P_F$. Finally, in the case (ii), the defining ideal of the Rees algebra of $P_F^2$ can be found in [1, Theorem 1]. Consequently, tame monomial ideals generated in degree at most 2 are of fiber type.

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DEPARTMENT OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES (IASBS), P.O.BOX 45195-1159, ZANJAN, IRAN

E-mail address: abbasnn@iasbs.ac.ir
E-mail address: ashkan_nikseresht@yahoo.com
E-mail address: yazdan@iasbs.ac.ir
E-mail address: rashidzn@iasbs.ac.ir