Integrable structures of dispersionless systems and differential geometry

A. Odesskii

Brock University, St. Catharines, Canada

Abstract

We develop the theory of Whitham type hierarchies integrable by hydrodynamic reductions as a theory of certain differential-geometric objects. As an application we construct Gibbons-Tsarev systems associated to moduli space of algebraic curves of arbitrary genus and prove that the universal Whitham hierarchy is integrable by hydrodynamic reductions.

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Address: Brock University, Niagara Region, 500 Glenridge Ave., St. Catharines, Ont., L2S 3A1 Canada

E-mail: aodesski@brocku.ca
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1 Introduction

1.1 Integrability of quasilinear systems

Consider a (2+1)-dimensional quasilinear system of PDEs of the form
\[
\sum_{j=1}^{n} a_{ij}(\vec{u}) \frac{\partial u_j}{\partial t} + \sum_{j=1}^{n} b_{ij}(\vec{u}) \frac{\partial u_j}{\partial y} + \sum_{j=1}^{n} c_{ij}(\vec{u}) \frac{\partial u_j}{\partial x} = 0, \quad i = 1, \ldots, D,
\] (1.1)

where \( u_j = u_j(t, x, y), \; j = 1, \ldots, n \) are dependent variables and \( D \geq n \). A number of important integrable equations (the dispersionless KP equation, the Boyer-Finley equation to name a few) can be written in this form.

There exist (at least) two approaches to integrability theory of such systems: via a pseudo-potential representation (also known as a dispersionless zero-curvature representation) [1, 2] and via hydrodynamic reductions [3, 4, 5, 6].

**Definition 1.1.** A system (1.1) admits a pseudo-potential representation
\[
\Phi_y = A(\Phi_x, u_1, \ldots, u_n), \quad \Phi_t = B(\Phi_x, u_1, \ldots, u_n),
\] (1.2)

if (1.1) is equivalent to the compatibility conditions for (1.2). Here \( \Phi(t, x, y) \) is an auxiliary function.

**Example 1.1.** The system
\[
u_y = v_x, \quad v_y = u_t - uu_x.
\] (1.3)
admits the pseudo-potential representation
\[
\Phi_y = \frac{\Phi_x^2}{2} + u, \quad \Phi_t = \frac{\Phi_x^3}{3} + u\Phi_x + v.
\] (1.4)

Indeed, computing \((\Phi_y)_t - (\Phi_t)_y\) by virtue of (1.4) we get
\[
(v_x - u_y)\Phi_x + u_t - uu_x - v_y = 0.
\]

Splitting by \( \Phi_x \) we get (1.3).

The approach via hydrodynamic reductions is more technical. Roughly speaking, a system (1.1) is integrable if for every \( N > 0 \) it admits “sufficiently many” so-called hydrodynamic reductions of the form \( u_i = u_i(r^1, \ldots, r^N), \; i = 1, \ldots, n \) were \( r^j = r^j(t, x, y) \) satisfy to a pair of compatible (1+1)-dimensional systems of hydrodynamic type
\[
\lambda^i(r^1, \ldots, r^N) r^i_x, \quad \mu^i(r^1, \ldots, r^N) r^i_x, \quad i = 1, \ldots, N,
\] (1.5)
Notice that compatibility conditions for the system (1.5) read
\[
\frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} = \frac{\partial_i \mu^j}{\mu^i - \mu^j}, \quad i \neq j = 1, \ldots, N, \quad \partial_i = \frac{\partial}{\partial r^i}. \tag{1.6}
\]

The main technical complication here is to explain what “sufficiently many hydrodynamic reductions” exactly means. It turns out that the set of such reductions should be parametrized by solutions of yet another compatible system of PDEs called Gibbons-Tsarev system.

**Definition 1.2.** A Gibbons-Tsarev system is a compatible overdetermined system of the form
\[
\begin{align*}
\partial_i p_j &= f(p_i, p_j, u_1, \ldots, u_n) \partial_i u_1, \quad i \neq j, \quad i, j = 1, \ldots, N, \\
\partial_i u_m &= g_m(p_i, u_1, \ldots, u_n) \partial_i u_1, \quad m = 2, \ldots, n, \quad i = 1, \ldots, N, \\
\partial_i \partial_j u_1 &= h(p_i, p_j, u_1, \ldots, u_n) \partial_i u_1 \partial_j u_1, \quad i \neq j, \quad i, j = 1, \ldots, N.
\end{align*} \tag{1.7}
\]

Here, \(p_1, \ldots, p_N\) and \(u_1, \ldots, u_n\) are functions of \(r^1, \ldots, r^N\), \(N \geq 3\), and \(\partial_i = \partial/\partial r^i\).

Now we can rigorously explain the second approach to integrability of (1.1).

**Definition 1.3.** A system (1.1) admits sufficiently many hydrodynamic reductions if there exists a Gibbons-Tsarev system (1.7) and functions \(F(p, u_1, \ldots, u_n), G(p, u_1, \ldots, u_n)\) such that \(u_i = u_i(r^1, \ldots, r^N), \quad i = 1, \ldots, n\) satisfy (1.1) by virtue of (1.5) with \(\lambda^i = F(p_i, u_1, \ldots, u_n), \mu^i = G(p_i, u_1, \ldots, u_n), \quad i = 1, \ldots, N\) and by virtue of (1.7).

Notice that compatibility conditions of (1.7) constrain functions \(f, g, h\) and compatibility conditions (1.6) constrain functions \(F, G\).

**Example 1.2.** Consider a system
\[
\begin{align*}
\partial_i p_j &= \frac{\partial_i u}{p_i - p_j}, \quad \partial_i v = p_i \partial_i u, \quad \partial_i \partial_j u = \frac{2 \partial_i u \partial_j u}{(p_i - p_j)^2}, \quad i, j = 1, \ldots, N, \quad i \neq j. \tag{1.8}
\end{align*}
\]

One can check that this system is compatible and therefore gives an example of Gibbons-Tsarev system. Set \(\lambda^i = u + p_i^2, \mu^i = p_i\) in (1.3). One can check by straightforward computation that (1.6) holds by virtue of (1.8), and (1.3) holds by virtue of (1.5), (1.8). In other words, Gibbons-Tsarev system (1.8) and functions \(F(p, u, v) = u + p^2, \quad G(p, u, v) = p\) give sufficiently many hydrodynamic reductions of the system (1.3).

Assume that the system (1.1) admits both pseudo-potential representation (1.2) and sufficiently many hydrodynamic reductions. It was shown in [7] that in this case the following equations holds
\[
f(p_1, p_2, u_1, \ldots, u_n) = \sum_{k=1}^n A_{u_k} (p_2, u_1, \ldots, u_n) g_k (p_1, u_1, \ldots, u_n) \frac{A_{p_1} (p_1, u_1, \ldots, u_n) - A_{p_2} (p_2, u_1, \ldots, u_n)}{A_{p_1} (p_1, u_1, \ldots, u_n)}, \tag{1.9}
\]

where \(g_1 = 1\) and similar equation holds for \(B\). Introducing parametric representation for pseudo-potentials
\[A(\Phi_x, u_1, \ldots, u_n) = h_2 (p, u_1, \ldots, u_n), \quad B(\Phi_x, u_1, \ldots, u_n) = h_3 (p, u_1, \ldots, u_n), \quad \Phi_x = h_1 (p, u_1, \ldots, u_n)\]

\]
one can write (1.9) as follows

\[ f(p_1, p_2) = \sum_{k=1}^{n} \left( \frac{h'_i(p_1)h_j(p_2)v_k - h'_j(p_1)h_i(p_2)v_k}{h'_j(p_1)h'_j(p_2) - h'_j(p_2)h'_i(p_1)} \right) g_k(p_1). \] (1.10)

Here we omit arguments \( u_1, \ldots, u_n \) and prime stands for \( p_i \)-derivatives.

Therefore, construction and/or classification of integrable systems (1.1) can be done in two steps [7]

- Construct/Classify Gibbons-Tsarev systems (1.7).
- For each Gibbons-Tsarev system (1.7) construct/classify functions \( A(p, u_1, \ldots, u_n) \) satisfying (1.9).

It turns out that the key point here is classification/construction of Gibbons-Tsarev systems. It was shown in [12, 7] that there exist only a few universal Gibbons-Tsarev systems and all known dispersionless integrable equations are connected with these universal Gibbons-Tsarev systems.

Notice that given a pair of functions \( A, B \) it is a straightforward computation to construct the corresponding system (1.1) admitting pseudo-potential representation (1.2).

### 1.2 Gibbons-Tsarev systems and differential geometry

Assume that \( f(p_1, p_2) \) has a pole of order one at \( p_1 = p_2 \). In this case (after redefining \( f, g_i \)) the Gibbons-Tsarev system (1.7) can be written in the form

\[
\partial_i p_j = \frac{f(p_i, p_j, u_1, \ldots, u_n)}{g_1(p_i, u_1, \ldots, u_n)} \partial_i u_1, \quad i \neq j, \quad i, j = 1, \ldots, N,
\]

\[
\frac{\partial_i u_1}{g_1(p_i, u_1, \ldots, u_n)} = \frac{\partial_i u_j}{g_j(p_i, u_1, \ldots, u_n)}, \quad j = 2, \ldots, n, \quad i = 1, \ldots, N,
\]

\[
\partial_i \partial_j u_1 = q(p_i, p_j, u_1, \ldots, u_n) \partial_i u_1 \partial_j u_1, \quad i \neq j, \quad i, j = 1, \ldots, N
\]

where \( f(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1) \).

Define a family of vector fields by

\[ g(p) = \sum_{i=1}^{n} g_i(p, u_1, \ldots, u_n) \frac{\partial}{\partial u_i}. \]

In this paper we prove that compatibility conditions for the system (1.11) is equivalent to the following commutations relations for this family of vector fields

\[ [g(p_1), g(p_2)] = f(p_2, p_1)g'(p_1) - f(p_1, p_2)g'(p_2) + 2f(p_2, p_1)g(p_1) - 2f(p_1, p_2)p_2 g(p_2), \] (1.12)

[5]
\[ g(p_2)(f(p_1,p_3)) - g(p_1)(f(p_2,p_3)) = f(p_1,p_2)f(p_2,p_3)p_2 - f(p_2,p_1)f(p_1,p_3)p_1 + \]
\[ + f(p_1,p_3)f(p_2,p_3)p_3 - f(p_1,p_3)f(p_1,p_3)p_3 + 2f(p_2,p_3)f(p_1,p_2)p_2 - 2f(p_1,p_3)f(p_2,p_1)p_1. \]  

A family of vector fields \( g(p) \) satisfying (1.12), (1.13) is called a local GT structure. Notice that point transformations of the form

\[ p_i \rightarrow \lambda(p_i, u_1, ..., u_n), \quad u_j \rightarrow \mu_j(u_1, ..., u_n) \]

do not change the form of (1.7). Using this observation we can promote local GT structures to global differential-geometric object, see Definition 3.4.

Notice that (1.13) is equivalent to Jacobi identity for (1.12) if vector fields \( g(p), g'(p) \) are linearly independent in three generic points \( p = p_1, p_2, p_3 \).

Equation (1.10) can be written in terms of \( g(p) \) as

\[ g(p_1)(h(p_2)) = \lambda(p_1, p_2)h'(p_1) - f(p_1, p_2)h'(p_2). \]

where \( h = h_1, h_2, h_3 \) and \( \lambda \) must satisfy the relations

\[ g(p_1)(\lambda(p_2, p_3)) = \lambda(p_1, p_3)\lambda(p_2, p_1) - \lambda(p_2, p_3)f(p_1, p_2)p_2 - f(p_1, p_2)\lambda(p_2, p_3)p_2 - f(p_1, p_3)\lambda(p_2, p_3)p_3, \]

\[ \lambda(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1). \]

### 1.3 Paper composition and main results

In Section 2 we recall definition of Whitham type hierarchies [8, 2, 9, 10]. Essentially these are just systems (1.1) admitting pseudo-potential representation but instead of \( t, x, y \) we have an arbitrary set of times \( t_1, ..., t_M \).

In Section 3 we introduce the main object of this paper which we call GT structure. Locally a GT structure is given by a family of vector fields \( g(p) \) and by a function \( f(p_1, p_2, u_1, ..., u_n) \) satisfying relations (1.12), (1.13). We explain how to promote a local GT structure to a global differential geometric object, see Proposition 3.4 and Definition 3.4. We also explain how to construct new GT structures from a given one (see Propositions 3.1 and 3.2) and how to construct potentials of integrable Whitham type hierarchies associated with a given GT structure (Proposition 3.3). We also explain a relation between GT structures and Lie algebroids of a certain type.
In Section 4 we prove the first main result of this paper: compatibility conditions of a Gibbons-Tsarev system are equivalent to commutation relations (1.12), (1.13) of the corresponding GT structure, see Proposition 4.1.

In Section 5 we explain how to express integrability of Whitham type hierarchies in terms of GT structures (Proposition 5.2).

In Section 6 we construct a GT structure on the moduli space of algebraic curves of arbitrary genus (see Proposition 6.2, formulas (6.41) - (6.43)) which is the second main result of this paper.

In Section 7 we recall the definition of the universal Whitham hierarchy [8, 2, 11] and prove our third main result: the universal Whitham hierarchy is integrable in all genera via hydrodynamic reductions (Proposition 7.2).

2 Whitham type hierarchies

Given a set of independent variables \( t_1, \ldots, t_M \) called times, a set of dependent variables \( v_1, \ldots, v_m \) called fields and a set of functions \( h_i(z, v_1, \ldots, v_m), \ i = 1, \ldots, M \) called potentials we define a Whitham type hierarchy as compatibility conditions of the following system of PDEs:

\[
\frac{\partial \psi}{\partial t_i} = h_i(z, v_1, \ldots, v_m), \ i = 1, \ldots, M. \tag{2.14}
\]

Here \( \psi, v_1, \ldots, v_m \) are functions of times \( t_1, \ldots, t_M \) and \( z \) is a parameter. The system (2.14) is understood as a parametric way of defining \( M - 1 \) relations between partial derivatives \( \frac{\partial \psi}{\partial t_i} \) obtained by eliminating \( z \) from these equations. Let us assume that the system (2.14) is compatible. Compatibility conditions can be written as

\[
\sum_{l=1}^{m} \left( \frac{\partial h_i}{\partial z} \frac{\partial h_j}{\partial v_l} \frac{\partial v_l}{\partial t_k} + \frac{\partial h_j}{\partial z} \frac{\partial h_k}{\partial v_l} \frac{\partial v_l}{\partial t_i} + \frac{\partial h_k}{\partial z} \frac{\partial h_i}{\partial v_l} \frac{\partial v_l}{\partial t_j} \right) = 0 \tag{2.15}
\]

where \( i, j, k = 1, \ldots, M \) are pairwise distinct. Let \( V_{i,j,k} \) be the linear space of functions in \( z \) spanned by \( \frac{\partial h_i}{\partial z} \frac{\partial h_j}{\partial v_l} - \frac{\partial h_j}{\partial z} \frac{\partial h_i}{\partial v_l} \), \( \frac{\partial h_j}{\partial z} \frac{\partial h_k}{\partial v_l} - \frac{\partial h_k}{\partial z} \frac{\partial h_j}{\partial v_l} \), \( \frac{\partial h_k}{\partial z} \frac{\partial h_i}{\partial v_l} - \frac{\partial h_i}{\partial z} \frac{\partial h_k}{\partial v_l} \), \( l = 1, \ldots, m \).

**Proposition 2.1.** Let \( V_{i,j,k} \) be finite dimensional and \( \dim V_{i,j,k} = D \). Then (2.15) is equivalent to a hydrodynamic type system of \( D \) linearly independent equations of the form

\[
\sum_{l=1}^{m} \left( a_{rl}(v_1, \ldots, v_m) \frac{\partial v_l}{\partial t_i} + b_{rl}(v_1, \ldots, v_m) \frac{\partial v_l}{\partial t_j} + c_{rl}(v_1, \ldots, v_m) \frac{\partial v_l}{\partial t_k} \right) = 0, \ r = 1, \ldots, D. \tag{2.16}
\]

**Proof.** Let \( \{ S_1(z), \ldots, S_D(z) \} \) be a basis in \( V_{i,j,k} \) and

\[
\frac{\partial h_i}{\partial z} \frac{\partial h_j}{\partial v_l} - \frac{\partial h_j}{\partial z} \frac{\partial h_i}{\partial v_l} = \sum_{r=1}^{D} a_{rl} S_r, \ \frac{\partial h_i}{\partial z} \frac{\partial h_k}{\partial v_l} - \frac{\partial h_k}{\partial z} \frac{\partial h_i}{\partial v_l} = \sum_{r=1}^{D} a_{rl} S_r, \ \frac{\partial h_i}{\partial z} \frac{\partial h_j}{\partial v_l} - \frac{\partial h_j}{\partial z} \frac{\partial h_i}{\partial v_l} = \sum_{r=1}^{D} b_{rl} S_r.
\]

Substituting these expressions into (2.15) and equating to zero coefficients at \( S_1, \ldots, S_D \) we obtain (2.16). \( \square \)
Remark 2.1. In the theory of integrable systems of hydrodynamic type the system (2.14) is often referred to as a pseudo-potential representation of the system (2.16).

Remark 2.2. In all known examples of integrable Whitham type hierarchies we have $m \leq D \leq 2m - 1$. Therefore, this inequality can be regarded as a criterion of integrability. However, in this paper we explore another criterion of integrability given by the so-called hydrodynamic reduction method.

3 GT structures

Let $g(p) = \sum_{i=1}^{m} g_i(p, v_1, \ldots v_m) \frac{\partial}{\partial v_i}$ be a family of vector fields parametrized by $p$ and let $f(p_1, p_2, v_1, \ldots, v_m)$ be a function.

Definition 3.1. A local GT structure is a family $g(p)$ and a function $f(p_1, p_2)$ satisfying the following relations:

$$[g(p_1), g(p_2)] = f(p_2, p_1)g'(p_1) - f(p_1, p_2)g'(p_2) + 2f(p_2, p_1)p_1g(p_1) - 2f(p_1, p_2)p_2g(p_2),$$  

(3.17)

$$g(p_2)(f(p_1, p_3)) - g(p_1)(f(p_2, p_3)) = f(p_1, p_2)f(p_2, p_3)p_2 - f(p_2, p_1)f(p_1, p_3)p_1 +$$  

$$+ f(p_1, p_3)f(p_2, p_3)p_3 - f(p_2, p_3)f(p_1, p_3)p_3 + 2f(p_2, p_3)f(p_1, p_2)p_2 - 2f(p_1, p_3)f(p_2, p_1)p_1,$$

(3.18)

$$f(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1).$$  

(3.19)

Here and in the sequel we often omit additional arguments $v_1, \ldots v_m$, indices stand for partial derivatives and $g'(p) = \frac{\partial g(p, v_1, \ldots v_m)}{\partial p}$.

Given a GT structure we can construct new GT structures in different ways.

Proposition 3.1. Let $g(p)$, $f(p_1, p_2)$ satisfy relations (3.17), (3.18) and

$$\hat{g}(p) = f(p, u_1) \frac{\partial}{\partial u_1} + \ldots + f(p, u_n) \frac{\partial}{\partial u_n} + g(p).$$

(3.20)

Then $\hat{g}(p)$, $f(p_1, p_2)$ also satisfy relations (3.17), (3.18).

Proof. Equation (3.17) is verified by direct computation for $n = 1$ and through induction by $n$ for $n > 1$. Equation (3.18) remains the same because $f(p_1, p_2)$ does not depend on $u_1, \ldots, u_n$. □

We say that a GT structure given by $\hat{g}(p)$, $f(p_1, p_2)$ is obtained from a GT structure $g(p)$, $f(p_1, p_2)$ by adding $n$ points $u_1, \ldots, u_n$. This procedure corresponds to a regular fields extension.
of a Gibbons-Tsarev system [7]. In the case of GT structures corresponding to the moduli space $M_{g,n}$ of algebraic curves of genus $g$ with $n$ (see Section 6) punctures this procedure corresponds to increasing the number of punctures.

**Proposition 3.2.** Let $g(p), f(p_1, p_2)$ satisfy relations (3.17), (3.18) and

$$
\hat{g}^{(n_1, \ldots, n_k)}(p) = \sum_{1 \leq j \leq k, \atop 0 \leq l \leq n_j} (i_{j,1} + 2i_{j,2} + \ldots + n_{j,i_{j,n_j}})! \frac{\partial^{i_{j,1} + \ldots + i_{j,n_j}} f(p, u_{j,0})}{\partial u_{j,0}^{i_{j,1} + \ldots + i_{j,n_j}}} + g(p).
$$

Then $\hat{g}^{(n_1, \ldots, n_k)}(p), f(p_1, p_2)$ also satisfy relations (3.17), (3.18).

**Proof.** Let us start with the following local GT structure

$$
\hat{g}(p) = \sum_{1 \leq j \leq k, \atop 0 \leq l \leq n_j} f(p, v_{j,l}) \frac{\partial}{\partial v_{j,l}} + g(p).
$$

We make the following change of coordinates

$$
v_{j,0} = u_{j,0},
$$

$$
v_{j,1} = u_{j,0} + \epsilon u_{j,1},
$$

$$
v_{j,2} = u_{j,0} + 2\epsilon u_{j,1} + \epsilon^2 u_{j,2},
$$

$$
\ldots
$$

$$
v_{j,n_j} = u_{j,0} + n_j \epsilon u_{j,1} + \frac{n_j(n_j-1)}{2} \epsilon^2 u_{j,2} + \ldots + \epsilon^{n_j} u_{j,n_j}.
$$

In new coordinates we have

$$
\hat{g}(p) = \sum_{1 \leq j \leq k} \left( f(p, u_{j,0}) \frac{\partial}{\partial u_{j,0}} + \frac{1}{\epsilon} (f(p, u_{j,0} + \epsilon u_{j,1}) - f(p, u_{j,0})) \frac{\partial}{\partial u_{j,1}} +
\right.

$$

$$
+ \frac{1}{\epsilon^2} (f(p, u_{j,0} + 2\epsilon u_{j,1} + \epsilon^2 u_{j,2}) - 2f(p, u_{j,0} + \epsilon u_{j,1}) + f(p, u_{j,0})) \frac{\partial}{\partial u_{j,2}} +
$$

$$
+ \frac{1}{\epsilon^{n_j}} (f(p, u_{j,0} + n_j \epsilon u_{j,1} + \ldots + \epsilon^{n_j} v_{j,n_j}) - \ldots + (-1)^{n_j} f(p, u_{j,0})) \frac{\partial}{\partial u_{j,n_j}}) + g(p).
$$

Taking the limit $\epsilon \to 0$ we obtain (3.21). \(\square\)

We say that the GT structure (3.21) is obtained from the GT structure (3.22) by colliding points $v_{j,0}, v_{j,1}, \ldots, v_{j,n_j}$ for each $j$. 9
Remark 3.1. Equation (3.18) is equivalent to Jacobi identity for (3.17) provided that vector fields $g(p_1)$, $g(p_2)$, $g(p_3)$, $g'(p_1)$, $g'(p_2)$, $g'(p_3)$ are linearly independent for generic $p_1$, $p_2$, $p_3$.

Remark 3.2. A local GT structure can be regarded as a certain Lie algebroid. Let

$$g(p) = e_2 + (p - z)e_3 + (p - z)^2e_4 + ....$$

In other words, let $e_{i+2} = i! g^{(i)}(z)$. Let $e_1 = \frac{\partial}{\partial z}$ and

$$f(p_1, p_2) = \frac{1}{p_1 - p_2} + \sum_{i,j=0}^{\infty} f_{i,j}(z) (p_1 - z)^i(p_2 - z)^j.$$ 

Then we have $[e_1, e_i] = (i - 1)e_{i+1}$ and equation (3.17) is equivalent to

$$[e_i, e_j] = (j - i)e_{i+j} + \sum_{r=0}^{i-1} (i + r - 1)f_{j-2,r}e_{i-r+1} - \sum_{r=0}^{j-1} (j + r - 1)f_{i-2,r}e_{j-r+1}.$$ 

In particular, if $f(p_1, p_2) = \frac{1}{p_1 - p_2}$, then we get $[e_i, e_j] = (j - i)e_{i+j}$ for $e_1, e_2, ....$ Note that (3.18) always holds for $f(p_1, p_2) = \frac{1}{p_1 - p_2}$. Therefore, a local GT structure can be regarded as a certain deformation of a Lie algebra with basis $e_1, e_2, ...$ and bracket $[e_i, e_j] = (j - i)e_{i+j}$ in the class of Lie algebroids.

Given a local GT structure one wants to classify all Whitham type hierarchies that are integrable by hydrodynamic reductions and that correspond to a given Gibbons-Tsarev system. It turns out that in order to do this one needs to find all functions $\lambda(p_1, p_2, v_1, ..., v_m)$ satisfying a certain condition. This can be formalized in the following way:

Definition 3.2. An enhanced local GT structure is a family of vector fields $g(p)$, a function $f(p_1, p_2)$ and an additional function $\lambda(p_1, p_2, v_1, ..., v_m)$ satisfying the relations (3.17), (3.18), (3.19) and

$$g(p_1)(\lambda(p_2, p_3)) = \lambda(p_1, p_3)\lambda(p_2, p_1)_{p_1} - \lambda(p_2, p_3)f(p_1, p_2)_{p_2} - f(p_1, p_2)\lambda(p_2, p_3)_{p_2} - f(p_1, p_3)\lambda(p_2, p_3)_{p_3},$$

$$\lambda(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1).$$

Given an enhanced local GT structure one wants to find a vector space of all potentials of the corresponding Whitham type hierarchy. In all known examples these spaces are spaces of solutions of linear systems of PDEs. However, in the general case we can define this vector space as a space of solutions of a linear functional equation.

Definition 3.3. Given an enhanced local GT structure we define the corresponding vector space of potentials as the space of solutions of the following functional equation for a function $h(p, v_1, ..., v_m)$:
\[ g(p_1)(h(p_2)) = \lambda(p_1, p_2)h'(p_1) - f(p_1, p_2)h'(p_2). \] (3.25)

Note that expanding (3.25) near diagonal \( p_2 = p_1 \) we obtain for \( h(p, v_1, ..., v_m) \) a system of linear PDEs equivalent to (3.25).

The following procedure gives a standard way to obtain solutions of (3.25):

**Proposition 3.3.** Let \( \gamma \) be a path in \( \mathbb{C} \) such that \[ \int_\gamma \frac{\partial(\lambda(t, p_2)f(p_1, t))}{\partial t}dt = 0. \]

Then
\[ h(p) = \int_\gamma \lambda(t, p)dt \]
is a solution of (3.25).

**Proof.** Substitute this expression for \( h(p) \) into (3.25) and use (3.24). Direct computation shows that the difference between the r.h.s and the l.h.s. of (3.25) is
\[ \int_\gamma \frac{\partial(\lambda(t, p_2)f(p_1, t))}{\partial t}dt. \]

Let us promote local GT structures to differential-geometric ones.

**Proposition 3.4.** Relations (3.17), (3.18), (3.19) are invariant with respect to arbitrary transformations of the form
\[ p_i = \mu(\tilde{p}_i, v_1, ..., v_m), \quad \tilde{g}(\tilde{p}) = \mu'((\tilde{p})^2 g(\mu(\tilde{p})), \quad \tilde{f}(\tilde{p}_1, \tilde{p}_2) = \frac{\mu'(\tilde{p}_1)}{\mu'(\tilde{p}_2)} \left( f(\mu(\tilde{p}_1), \mu(\tilde{p}_2)) - g(\mu(\tilde{p}_1))(\mu(\tilde{p}_2)) \right). \] (3.26)

Let \( \pi : M \rightarrow B \) be a bundle with one dimensional fiber \( F \) and \( m \) dimensional base \( B \).

**Definition 3.4.** A GT structure on \( \pi \) is a local GT structure on each trivialization for each \( U \subset B \) such that for different trivializations these local GT structures are connected by (3.26).

Here \( v_1, ..., v_m \) stands for coordinates on \( B \) and \( p \) is a coordinate on \( F \).

**Proposition 3.5.** Relations (3.24) are invariant with respect to an arbitrary transformations of the form (3.26) provided that \( \lambda \) is transformed as
\[ \tilde{\lambda}(\tilde{p}_1, \tilde{p}_2) = \mu'(\tilde{p}_1)\lambda(\mu(\tilde{p}_1), \mu(\tilde{p}_2)) \] (3.27)

**Definition 3.5.** An enhanced GT structure on \( \pi \) is an enhanced local GT structure on each trivialization for each \( U \subset B \) such that for different trivializations these enhanced local GT structures are connected by (3.26), (3.27).

**Example 3.1.** It is clear from (6.41), (6.43) that \( g(p) = G(p), f(p_1, p_2) = F(p_1, p_2) \) constructed in Section 6 is a GT structure on the bundle \( M_{g,1} \rightarrow M_g \).

Similar GT structures exist for \( g = 0, 1 \). In the case \( g = 0 \) we consider the moduli space \( M_{0,n+3} \) of complex structures on \( \mathbb{C}P^1 \) with punctures in \( n + 3 \) points. We fix 3 points at 0, 1, \( \infty \)

\[ ^1 \text{The corresponding Gibbons-Tsarev systems in the cases } g=0,1,2 \text{ where constructed previously [12, 7] in ad hoc way.} \]
and move other points. The formulas for the corresponding GT structures read

\[ f(p_1, p_2) = \frac{p_2(p_2 - 1)}{(p_1 - p_2)p_1(p_1 - 1)}, \quad g(p) = \sum_{i=1}^{n} \frac{u_i(u_i - 1)}{(p - u_i)p(p - 1)} \frac{\partial}{\partial u_i}. \] (3.28)

In the case \( g = 1 \) we consider the moduli space \( M_{1,n+1} \) of complex structures on an elliptic curve with punctures in \( n + 1 \) points. We fix one point at 0 and move other points. We also deform the complex structure on our elliptic curve. The space of complex structures is one dimensional in this case. We use the modular parameter \( \tau \) with \( \text{Im} \tau > 0 \) as a coordinate on the moduli space of elliptic curves. The formulas for the corresponding GT structures read

\[ f(p_1, p_2) = \rho(p_1 - p_2, \tau) - \rho(p_1), \quad g(p) = 2\pi i \frac{\partial}{\partial \tau} + \sum_{j=1}^{n} (\rho(p - u_j, \tau) - \rho(p, \tau)) \frac{\partial}{\partial u_j}. \] (3.29)

where \( \rho(p, \tau) = \frac{\partial}{\partial p} \ln(\theta(p, \tau)) \) and \( \theta(p, \tau) = \sum_{k \in \mathbb{Z}} (-1)^k e^{2\pi i (kp + k(k-1)\tau)}. \)

**Remark 3.3.** In these GT structures we can also collide points and obtain new GT structures. Moreover, in the case \( g = 0 \) (resp. \( g = 1 \)) we can collide points with 0, 1, \( \infty \) (resp. with 0) by doing a substitution similar to (3.23). In the case \( g = 0 \) we can also make an arbitrary fractional linear transformation with constant coefficients sending 0, 1, \( \infty \) to \( a, b, c \) and collide some of \( a, b, c \).

**Remark 3.4.** Consider an enhanced local GT structure with \( g(p) \) given by (3.22). Colliding points \( v_{j,0}, v_{j,1}, \ldots, v_{j,n_j} \) by substitution (3.23) and taking the limit \( \epsilon \to 0 \) we can do the same substitution and limit in the function \( \lambda \) and obtain a new enhanced local GT structure.

## 4 Gibbons–Tsarev systems

Gibbons-Tsarev systems are the main ingredient of the approach to integrability of Whitham type hierarchies and, more generally, to integrability of quasi-linear systems of the form (2.16) based on hydrodynamic reductions. In this approach hydrodynamic reductions of a given hierarchy are parametrized by solutions of a Gibbons-Tsarev system. In this Section we explain a connection between Gibbons-Tsarev systems and GT structures.

Let \( p_1, \ldots, p_N, v_1, \ldots, v_m \) be functions of auxiliary variables \( r^1, \ldots, r^N \) and \( \partial_i = \frac{\partial}{\partial r^i} \).

**Definition 4.1.** A Gibbons–Tsarev system is a compatible system of partial differential equations of the form

\[ \partial_i p_j = f(p_i, p_j, v_1, \ldots, v_m) \partial_i v_1, \quad i \neq j, \quad i, j = 1, \ldots, N, \]
\[ \partial_i v_j = g_j(p_i, v_1, \ldots, v_m) \partial_i v_1, \quad j = 2, \ldots, m, \quad i = 1, \ldots, N, \]
\[ \partial_i \partial_j v_1 = q(p_i, p_j, v_1, \ldots, v_m) \partial_i v_1 \partial_j v_1, \quad i \neq j, \quad i, j = 1, \ldots, N. \] (4.30)
Remark 4.1. It follows from the compatibility assumption that the space of solutions of a Gibbons-Tsarev system is locally parametrized by $2N$ functions in one variable. Note that $f, g, q$ do not depend on $N$ and therefore $N$ can be arbitrary large for a given Gibbons-Tsarev system.

We say that a Gibbons-Tsarev system is non-degenerate if $f(p_1, p_2, v_1, ..., v_m)$ has a pole of order one on the diagonal $p_2 = p_1$. Assume in the sequel that all Gibbons-Tsarev systems are non-degenerate.

Proposition 4.1. There exists a one-to-one correspondence between non-degenerate Gibbons-Tsarev systems and local GT structures.

Proof. Redefining $f, g_i$ from (4.30) we write a Gibbons-Tsarev system in the form

$$\partial_i p_j = \frac{f(p_i, p_j, v_1, ..., v_m)}{g_i(p_i, v_1, ..., v_m)} \partial_i v_1, \quad i \neq j, \quad i, j = 1, ..., N,$$

$$\frac{\partial_i v_1}{g_i(p_i, v_1, ..., v_m)} = \frac{\partial_j v_j}{g_j(p_j, v_1, ..., v_m)}, \quad j = 2, ..., m, \quad i = 1, ..., N, \quad (4.31)$$

$$\partial_i \partial_j v_1 = q(p_i, p_j, v_1, ..., v_m) \partial_i v_1 \partial_j v_1, \quad i \neq j, \quad i, j = 1, ..., N$$

where $f(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1)$. Indeed, $\frac{1}{g_i(p_i)}$ is the residue of $f(p_i, p_j)$ from (4.30) at $p_j = p_i$.

Write

$$g(p) = \sum_{i=1}^{m} g_i(p, v_1, ..., v_m) \frac{\partial}{\partial v_i}.$$ Compatibility of the system (4.31) implies $\partial_1 \partial_2 \phi(p_3, v_1, ..., v_m) = \partial_2 \partial_1 \phi(p_3, v_1, ..., v_m)$ for an arbitrary function $\phi$. This can be written as

$$\left( f(p_1, p_2) \frac{\partial}{\partial p_2} + f(p_1, p_3) \frac{\partial}{\partial p_3} + g(p_2) \right) \left( f(p_2, p_3) \frac{\partial}{\partial p_3} + g(p_2) \phi(p_3) \right) \frac{\partial u_1}{g_1(p_2)} \frac{\partial u_1}{g_1(p_1)} =$$

$$\left( f(p_2, p_1) \frac{\partial}{\partial p_1} + f(p_2, p_3) \frac{\partial}{\partial p_3} + g(p_1) \right) \left( f(p_1, p_3) \frac{\partial}{\partial p_3} + g(p_1) \phi(p_3) \right) \frac{\partial u_1}{g_1(p_1)} \frac{\partial u_2}{g_1(p_2)}.$$

Expanding this equation and equating coefficients at $\phi$ and $\phi_{p_3}$ we get

$$f(p_1, p_2) f(p_2, p_3) p_2 - f(p_2, p_1) f(p_1, p_3) p_1 + f(p_1, p_3) f(p_2, p_3) p_3 - f(p_2, p_3) f(f(p_1, p_3) p_3 +$$

$$+ g(p_1)(f(p_2, p_3)) - g(p_2)(f(p_1, p_3)) +$$

$$f(p_2, p_3) \left( g_1(p_1) \frac{\partial_1 \partial_2 u_1}{\partial_1 u_1 \partial_2 u_1} - \frac{1}{g_1(p_2)} \left( f(p_1, p_2) \frac{\partial}{\partial p_2} + g(p_1) \right) \left( g_1(p_2) \right) \right) -$$

$$f(p_1, p_3) \left( g_1(p_2) \frac{\partial_1 \partial_2 u_1}{\partial_1 u_1 \partial_2 u_1} - \frac{1}{g_1(p_1)} \left( f(p_2, p_1) \frac{\partial}{\partial p_1} + g(p_2) \right) \left( g_1(p_1) \right) \right) = 0,$$

$$f(p_1, p_2) g'(p_2) - f(p_2, p_1) g'(p_1) + \left[ g(p_1), g(p_2) \right] +$$

$$+ \left[ g(p_2), g(p_1) \right] = 0.$$
In this Section we explain a relation between integrable Whitham type hierarchies and enhanced local GT structures. Under this correspondence the space of integrable Whitham type hierarchies and enhanced local GT structures. 

Proposition 5.1. A Whitham type hierarchy with potentials \( h_i(p, v_1, ..., v_m), \ i = 1, ..., M \) is integrable by hydrodynamic reductions if and only if there exists a Gibbons-Tsarev system (4.31) such that

\[
h'_j(p_1) \partial_1 (h_i(p_2)) = h'_i(p_1) \partial_1 (h_j(p_2)), \quad i, j = 1, ..., M
\]  

by virtue of (4.31).

Proof. The equation (5.32) can be written as

\[
f(p_1, p_2) = \sum_{k=1}^{m} \left( h'_i(p_1) h_j(p_2) v_k - h'_j(p_1) h_i(p_2) v_k \right) g_k(p_1)
\]  

and, therefore, coincides with the formula (77) from [7]. It is proven in [7] that the equation (5.33) is equivalent to the integrability of a given Whitham type hierarchy. 

Proposition 5.2. There exists a one-to-one correspondence between integrable Whitham type hierarchies and enhanced local GT structures. Under this correspondence the space of
potentials of a Whitham type hierarchy coincides with the space of solutions of the linear system (3.25).

Proof. Write (3.32) as \( \frac{\partial h_i(p_2)}{h'_i(p_1)} = \frac{\partial h_j(p_2)}{h'_j(p_1)} \). By executing \( \partial_1 \) in numerators we get

\[
\frac{f(p_1, p_2)h'_i(p_2) + g(p_1)(h_i(p_2))}{h'_i(p_1)} = \frac{f(p_1, p_2)h'_j(p_2) + g(p_1)(h_j(p_2))}{h'_j(p_1)}.
\]

Let \( \lambda(p_1, p_2) = \frac{f(p_1, p_2)h'_i(p_2) + g(p_1)(h_i(p_2))}{h'_i(p_1)} \), this function does not depend on \( i \). Therefore, we get

\[
g(p_1)(h_i(p_2)) = \lambda(p_1, p_2)h'_i(p_1) - f(p_1, p_2)h'_i(p_2)
\]

which coincides with (3.25). Applying the relation (3.17) to \( h_i(p_3) \) we can write

\[
g(p_1)g(p_2)(h_i(p_3)) - g(p_2)g(p_1)(h_i(p_3)) = f(p_2, p_1)b'(p_1)(h_i(p_3)) - f(p_1, p_2)b'(p_2)(h_i(p_3)) + 2f(p_2, p_1)b(p_1)(h_i(p_3)) - 2f(p_1, p_2)b(p_2)(h_i(p_3)).
\]

Computing the l.h.s. and the r.h.s. of this relation by virtue of (3.25) we obtain (3.24). \( \square \)

6 Holomorphic objects on Riemann surfaces and deformations of complex structures

Let \( \mathcal{E} = \mathbb{D}/\Gamma \) be a compact Riemann surface of genus \( g > 1 \), \( \mathbb{D} \subset \mathbb{C} \) its universal covering and \( \Gamma = \pi_1(\mathcal{E}) \). Denote by \( a_\alpha, b_\alpha, \alpha = 1, ..., g \) a canonical basis in the homology group \( H_1(\mathcal{E}, \mathbb{Z}) \). Let us choose a coordinate in \( \mathbb{D} \) and use the same symbols for holomorphic objects on \( \mathcal{E} \) and their lifting on \( \mathbb{D} \). We will also use the same symbol for a point in \( \mathcal{E} \), its lifting in \( \mathbb{D} \) and its coordinate. Let \( \omega_\alpha(z)dz \) be the basis of holomorphic 1-forms on \( \mathcal{E} \) normalized by \( \int_{a_\alpha} \omega_\beta dz = \delta_{\alpha\beta} \). Choose a basepoint \( z_0 \) and define the Abel map \( q_\alpha(z) = \int_{z_0}^z \omega_\alpha(z)dz \). Note that \( \omega_\alpha = \omega'_\alpha \). Denote the prime form\(^2\) by \( E(x, y)(dx)^{-1/2}(dy)^{-1/2} \). Let \( B_{\alpha\beta} = \int_{b_\alpha} \omega_\beta dz \) be the matrix of \( b \)-periods. Details on holomorphic objects on Riemann surfaces are given in \([13][14][15]\). Recall that

\[
E(v, u) = -E(u, v), \quad E(u, v) = u - v - \frac{1}{12}S(u)(u - v)^3 + O((u - v)^4), \quad (6.34)
\]

where \( S(p) \) is the Bergman projective connection on \( \mathcal{E} \). Note that \( E(u, v) \) is multivalued. If \( u \) or \( v \) is moved by \( a_\alpha \), it remains invariant. If \( u \) moves by \( b_\alpha \) to \( \bar{u} \) or \( v \) moves by \( b_\alpha \) to \( \bar{v} \), then

\[
E(\bar{u}, v) = E(u, v) \exp \left(-\pi i B_{\alpha\alpha} + 2\pi i (q_\alpha(v) - q_\alpha(u))\right), \quad (6.35)
\]

\(^2\)In this paper we represent differential-geometric objects as functions with prescribed transformation laws with respect to an arbitrary change of coordinates. For example if \( x = \mu(\bar{x}), \ y = \mu(\bar{y}), \) then the prime form transforms as \( E(\bar{x}, \bar{y}) = \mu'(\bar{x})^{-1/2}\mu'(\bar{y})^{-1/2}E(\mu(\bar{x}), \mu(\bar{y})) \).
\[ E(u, \bar{v}) = E(u, v) \exp \left( -\pi i B_{\alpha\alpha} - 2\pi i (q_\alpha(v) - q_\alpha(u)) \right). \]

Let \( W(u, v) = (\ln(E(u, v)))_{uv} \) be the Bergman kernel. Recall that

\[
\int_{a_i} W(u, v) du = 0, \quad \int_{b_\alpha} W(u, v) du = 2\pi i \omega_\alpha(v), \quad \int_{b_\beta} \int_{b_\beta} W(u, v) dudv = 2\pi i B_{\alpha\beta}. \quad (6.36)
\]

Recall a description of the tangent space\(^3\) to the moduli space \( M_g \) of Riemann surfaces at the point corresponding to \( E \) [16, 17]. Let \( p \in \mathcal{E} \) be the center of a small disc \( D \subset \mathcal{E} \). Let \( L \) be the Lie algebra of holomorphic vector fields on \( D \setminus \{p\} \) and \( L_p, L_{\text{out}} \) be subalgebras of \( L \) consisting of vector fields holomorphic at \( p \) and holomorphic on \( \mathcal{E} \setminus \{p\} \) correspondingly. It is known that the tangent space to the moduli space \( M_g \) is isomorphic to the quotient \( L/(L_p \oplus L_{\text{out}}) \). Let \( M_{g,1} \) be the moduli space of Riemann surfaces with a puncture at \( u \in \mathcal{E} \). The tangent space to \( M_{g,1} \) is isomorphic to the quotient \( L/(L_p \oplus L_{\text{out},u}) \) where \( L_{\text{out},u} \subset L_{\text{out}} \) consists of vector fields with zero at \( u \). Let us construct vector spaces dual to these tangent spaces using the Serre duality theorem [18]. There exists a non-degenerate pairing between the space \( L \) and the space \( Q \) of quadratic differentials holomorphic on \( D \setminus \{p\} \). This pairing is given by \((v, q) = \text{Res}_p(vq)\). The space dual to the tangent space of \( M_g \) is equal to \((L_p \oplus L_{\text{out}})^\perp \subset Q \) and consists of quadratic differentials holomorphic on \( \mathcal{E} \). Similarly, the space dual to the tangent space of \( M_{g,1} \) is equal to \((L_p \oplus L_{\text{out},u})^\perp \subset Q \) and consists of quadratic differentials holomorphic on \( \mathcal{E} \setminus \{u\} \) with pole of order less or equal to one at \( u \). More generally, the space dual to the tangent space of \( M_{g,n} \) of the moduli space of Riemann surfaces with punctures at \( u_1, ..., u_n \) consists of quadratic differentials holomorphic on \( \mathcal{E} \setminus \{u_1, ..., u_n\} \) with poles of order less or equal to one at \( u_1, ..., u_n \).

Let \( v_1, ..., v_{3g-3} \) be local coordinates on moduli space \( M_g \). Let \( \frac{\partial}{\partial v_1}, ..., \frac{\partial}{\partial v_{3g-3}} \) be the corresponding basis in the tangent space and \( g_1(p)dp^2, ..., g_{3g-3}(p)dp^2 \) be the dual basis in the space of quadratic differentials. The object\(^4\)

\[
G(p)dp^2 = \sum_{i=1}^{3g-3} g_i(p)dp^2 \frac{\partial}{\partial v_i}
\]

do not depend on the choice of coordinates. A similar construction for \( M_{g,n} \) gives the object

\[
\hat{G}(p)dp^2 = \sum_{i=1}^{n} F(p, u_i)dp^2 \frac{\partial}{\partial u_i} + \sum_{j=1}^{3g-3} g_j(p)dp^2 \frac{\partial}{\partial v_j}
\]

where \( u_1, ..., u_n \) are coordinates of \( n \) points in \( \mathcal{E} \) and

\[
F(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1). \quad (6.37)
\]

\(^3\)Various approaches to deformation theory of complex structures can be found in [19, 20, 21]

\(^4\)Note that the functions \( G, g_i, F \) etc. depend also on \( v_1, ..., v_{3g-3} \). We will often omit these arguments in order to simplify formulas.
Proposition 6.1. Under an arbitrary change of coordinates of the form
\[ p = \mu(\tilde{p}, v_1, ..., v_{3g-3}), \ u_i = \mu(\tilde{u}_i, v_1, ..., v_{3g-3}) \] (6.38)
the objects \( G(p), F(p_1, p_2) \) obey the following transformation rules
\[ \tilde{G}(\tilde{p}) = \mu'(\tilde{p})^2 G(\mu(\tilde{p})), \] (6.39)
\[ \tilde{F}(\tilde{p}_1, \tilde{p}_2) = \frac{\mu'(\tilde{p}_1)^2}{\mu'(\tilde{p}_2)} \left( F(\mu(\tilde{p}_1), \mu(\tilde{p}_2)) - G(\mu(\tilde{p}_1))(\mu(\tilde{p}_2)) \right) \] (6.40)

Proof. The relation (6.39) means that \( G(p) \) is a quadratic differential in \( p \) (with values in vector fields in \( v_1, ..., v_{3g-3} \)). In order to obtain (6.40) we perform an arbitrary change of coordinates of the form \( p = \mu(\tilde{p}, v_1, ..., v_{3g-3}), \ u_i = \mu(\tilde{u}_i, v_1, ..., v_{3g-3}), \ v_j = \tilde{v}_j \) and require that the object \( \tilde{G}(\tilde{p}) \) transforms as a vector field in \( u_1, ..., u_n, v_1, ..., v_{3g-3} \). The relation (6.40) is a consequence of this requirement. □

Proposition 6.2. The following identities hold
\[
\begin{align*}
[G(p_1), G(p_2)] &= F(p_2, p_1)G'(p_1) - F(p_1, p_2)G'(p_2) + 2F(p_2, p_1)p_1G(p_1) - 2F(p_1, p_2)p_2G(p_2), \\
[\tilde{G}(p_1), \tilde{G}(p_2)] &= F(p_2, p_1)\tilde{G}'(p_1) - F(p_1, p_2)\tilde{G}'(p_2) + 2F(p_2, p_1)p_1\tilde{G}(p_1) - 2F(p_1, p_2)p_2\tilde{G}(p_2),
\end{align*}
\] (6.41, 6.42)

\[
\begin{align*}
G(p_2)(F(p_1, p_3)) - G(p_1)(F(p_2, p_3)) &= F(p_1, p_2)F(p_2, p_3)p_2 - F(p_2, p_1)F(p_1, p_3)p_1 + F(p_1, p_3)F(p_2, p_3)p_3 - F(p_2, p_3)F(p_1, p_3)p_3 + 2F(p_2, p_3)F(p_1, p_2)p_2 - 2F(p_1, p_3)F(p_2, p_1)p_1, \\
\frac{G(p_1)(E(p_2, p_3))}{E(p_2, p_3)} &= \frac{1}{2}F(p_1, p_2)p_2 + \frac{1}{2}F(p_1, p_3)p_3 - F(p_1, p_2)^2E(p_2, p_3) - F(p_1, p_3)^2E(p_2, p_3) - \frac{1}{2} \left( \frac{E(p_1, p_2)p_1}{E(p_2, p_3)} - \frac{E(p_1, p_3)p_3}{E(p_2, p_3)} \right)^2, \\
G(p_1)\left( \int_{p_2}^{p_3} \omega_i \right) &= F(p_1, p_2)\omega_i(p_2) - F(p_1, p_3)\omega_i(p_3) - \frac{E(p_1, p_2)p_1}{E(p_1, p_2)}\omega_i(p_1) + \frac{E(p_1, p_3)p_3}{E(p_1, p_3)}\omega_i(p_1),
\end{align*}
\] (6.43, 6.44, 6.45)
\[ G(p)(B_{jk}) = 2\pi i \omega_j(p)\omega_k(p). \]  

(6.46)

**Proof.** Notice that (6.42) is a formal consequence of (6.41) and (6.43) (see Proposition 3.1).

Consider the difference of the l.h.s. and the r.h.s. of each of (6.41), (6.43), (6.44), (6.45). Expanding these expressions on each diagonal \( p_i = p_j, \ i \neq j \) and using (6.34) and (6.37) one can check that each of these expressions is holomorphic on all diagonals. Making an arbitrary change of coordinates of the form \( p_i = \mu(p_i, v_1, ..., v_{3g-3}), \ i = 1, 2, 3 \) one can check that all these differences are transformed as tensor fields in \( p_1, p_2, p_3 \). In particular, the difference between the l.h.s. and the r.h.s. of (6.43) is a holomorphic quadratic differential in \( p_1, p_2 \) and holomorphic vector field in \( p_3 \). This proves (6.43) because any holomorphic vector field vanishes. Similarly, the differences between the l.h.s. and the r.h.s. of (6.41), (6.45) are holomorphic quadratic differentials in \( p_1 \) and holomorphic functions in \( p_2, p_3 \). Moreover, these functions vanish on the diagonal \( p_2 = p_3 \). This would prove (6.44), (6.45) (any holomorphic function is a constant) provided that we prove that the differences between the l.h.s. and the r.h.s. are single valued.

Taking the second derivative of the equation (6.41) we get

\[ G(p_1)(W(p_2, p_3)) = \]  

(6.47)

where \( (W(p_2, p_3) = (\ln(E(p_2, p_3))_{p_2p_3} \) is the Bergman kernel. Let us prove this identity. Let \( \Delta(p_1, p_2, p_3) \) be the difference of the l.h.s. and the r.h.s. of (6.47). It is a quadratic differential in \( p_1 \) and 1-form in both \( p_2, p_3 \). Using transformation properties (6.35) we see that \( \Delta(p_1, p_2, p_3) \) is single valued. Therefore, \( \Delta(p_1, p_2, p_3) = \sum_{\alpha, \beta = 1}^{g} r_{\alpha\beta}(p_1) \omega_\alpha(p_2)\omega_\beta(p_3) \) where \( \omega_\alpha(p_i) \) are some holomorphic quadratic differentials. Computing \( \int_{b_\alpha} \int_{b_\beta} \Delta(p_1, p_2, p_3) dp_2 dp_3 \) we obtain \( r_{\alpha\beta}(p_1) = 0 \) which proves (6.47). Computing \( \int_{b_\alpha} \int_{b_\beta} \delta_\alpha dp_2 dp_3 \) of the l.h.s. and the r.h.s. of (6.47) and using (6.36) and (6.37) we obtain (6.46). The difference between the l.h.s. and the r.h.s. of (6.45) is single valued by virtue of (6.46). This proves (6.45). Equation (6.44) is proven in a similar way. Note that the difference between the l.h.s. and the r.h.s. of (6.44) is single valued by virtue of (6.45). Equation (6.41) is proven by applying its l.h.s. and the r.h.s. to \( B_{jk} \). For example, on the l.h.s. we have \( G(p_1)(G(p_2)(B_{jk})) = G(p_2)(G(p_1)(B_{jk})) \). Computing by virtue of (6.46), (6.45) we prove (6.41). □

**Remark 6.1.** Recall that the Riemann theta-function is defined by

\[ \theta(z_1, ..., z_g) = \sum_{m \in \mathbb{Z}^g} \exp(2\pi i m \cdot z + \pi i m B m^t). \]

Here we use bold symbols for the corresponding vectors: \( \mathbf{m} = (m_1, ..., m_g) \), \( \mathbf{z} = (z_1, ..., z_g) \), \( \mathbf{m} \cdot \mathbf{z} = m_1 z_1 + ... + m_g z_g \), and \( B \) is the period matrix. We have

\[ G(p)(\theta(z_1, ..., z_g)) = \sum_{\alpha, \beta = 1}^{g} \frac{\partial \theta(z_1, ..., z_g)}{\partial B_{\alpha\beta}} G(p)(B_{\alpha\beta}) = \frac{1}{2} \sum_{\alpha, \beta = 1}^{g} \frac{\partial^2 \theta(z_1, ..., z_g)}{\partial z_\alpha \partial z_\beta} \omega_\alpha(p) \omega_\beta(p) \]

(6.48)
where we used heat equation for \( \theta \) and (6.46).

**Remark 6.2.** Expanding (6.44) on diagonal \( p_2 = p_3 \) we obtain

\[
G(p_1)(S(p_2)) + F(p_1, p_2)p_3^2 + 2S(p_2)F(p_1, p_2)p_2 + S(p_2)p_2F(p_1, p_2) - 6W(p_1, p_2)^2 = 0.
\]

**Remark 6.3.** Equations (6.41), (6.43) show that \( G, F \) define a GT structure on the moduli space \( M_g \).

**Example 6.1.** Let \( g = 2 \). Represent \( E \) as a 2-fold covering of \( \mathbb{CP}^1 \). Let \( x \) be an affine coordinate in \( \mathbb{CP}^1 \) and let branch points of the covering be at \( x = 0, 1, \infty, a, b, c \). The curve \( E \) is given by \( y^2 = x(x-1)(x-a)(x-b)(x-c) \). One can check that

\[
G(p) = \frac{1}{2p(p-1)} \left( \frac{a(a-1)}{p-a} \frac{\partial}{\partial a} + \frac{b(b-1)}{p-b} \frac{\partial}{\partial b} + \frac{c(c-1)}{p-c} \frac{\partial}{\partial c} \right),
\]

\[
F(p_1, p_2) = \frac{(p_1-a)(p_1-b)(p_1-c)p_2(p_2-1) + q_1q_2}{2(p_1-p_2)p_1(p_1-1)(p_1-a)(p_1-b)(p_1-c)}
\]

where \( p, p_1, p_2 \) are affine coordinates in \( \mathbb{CP}^1 \) and \( q_i^2 = p_i(p_i-1)(p_i-a)(p_i-b)(p_i-c) \).

Let us compare our variational formulas with Rauch ones. The equation (6.46) reads

\[
\frac{1}{2p(p-1)} \left( \frac{a(a-1)}{p-a} \frac{\partial}{\partial a} + \frac{b(b-1)}{p-b} \frac{\partial}{\partial b} + \frac{c(c-1)}{p-c} \frac{\partial}{\partial c} \right)(B_{jk})dp^2 = 2\pi i \omega_j(p)dp \cdot \omega_k(p)dp. \tag{6.48}
\]

Let \( \tau \) be a local coordinate near branch point \( a \), we have \( p = a + \tau^2, dp = 2\tau d\tau \). Expanding the l.h.s. of (6.48) we get \((2\partial B_{jk} + O(\tau))d\tau^2 \). Therefore \( \frac{\partial B_{jk}}{\partial a} = \pi i \omega_j(p)dp \bigg|_{p=a} \cdot \frac{\omega_k(p)dp}{dp} \bigg|_{p=a} \) and we arrive at a Rauch formula.

In general, if \( E \) is represented as a branched covering of \( \mathbb{CP}^1 \) ramified at \( a_k \in \mathbb{CP}^1 \) with ramification indices \( r_k, k = 1, 2, \ldots, \) then \( G(p)dp^2 = \left( \frac{1}{r_k(p-a_k)} \frac{\partial}{\partial a_k} + o((p-a_k)^{-1}) \right)dp^2 \) and Rauch formulas can be derived from ours in a similar way.

**Example 6.2.** Let us choose \( B_{j_1, k_1}, \ldots, B_{j_{3g-3}, k_{3g-3}} \) as local coordinates in \( M_g \). Applying (6.46) to \( B_{j_1, k_1} \) we get

\[
G(p) = 2\pi i \sum_{l=1}^{3g-3} \omega_{j_1}(p)\omega_{k_1}(p) \frac{\partial}{\partial B_{j_1, k_1}}.
\]

Applying again (6.46) to an arbitrary \( B_{j_k} \) we obtain quadratic relations between normalized differentials. Namely, if \( S(B_{j_1}, B_{j_2}, \ldots, B_{j_{3g}}) = 0 \) is a relation between the entries of the period matrix (there are \( \frac{g(g+1)}{2} - 3g + 3 \) functionally independent ones), then

\[
\sum_{j, k=1}^{g} \frac{\partial S}{\partial B_{j_k}} \omega_j(p)\omega_k(p) = 0.
\]

See [22] for a similar formula for quadratic relations between normalized differentials.
7 The universal Whitham hierarchy

In this Section we use notations introduced in Section 6, including \( G(p) \) and \( F(p_1, p_2) \).

According to [2] the universal Whitham hierarchy is given by potentials obtained by integration of meromorphic differentials on a Riemann surface. We are going to construct such an hierarchy explicitly and prove that it is integrable by hydrodynamic reductions.

**Proposition 7.1.** Fix constants \( s_1, ..., s_m \) such that \( s_1 + ... + s_m = 1 \) (the simplest possibility is \( m = 1 \) and \( s_1 = 1 \)). The following formulas define an enhanced GT structure:

\[
g(p) = \sum_{j=1}^{n} F(p, u_j) \frac{\partial}{\partial u_j} + \sum_{k=1}^{m} F(p, w_k) \frac{\partial}{\partial w_k} + G(p), \quad f(p_1, p_2) = F(p_1, p_2),
\]

\[
\lambda(p_1, p_2) = \frac{E(p_1, p_2)}{E(p_1, p_2)} - \sum_{k=1}^{m} s_k \frac{E(p_1, w_k)}{E(p_1, w_k)}.
\]

Moreover, the following functions belong to the space of potentials of this enhanced GT structure:

\[
h_j(p) = h_1(p), \quad j = 2, ..., n, \quad q_\alpha(p) - \sum_{k=1}^{m} s_k q_\alpha(w_k), \quad \alpha = 1, ..., g
\]

where

\[
h_j(p) = \ln(E(p, u_j)) - \sum_{k=1}^{m} s_k \ln(E(u_j, w_k)).
\]

**Proof.** We need to prove identities (3.24) and (3.25) for given \( \lambda(p_1, p_2) \) and potentials. This can be done by straightforward computation using identities from Proposition 2.2. The simplest way is to start from identity (3.25) for \( h_j(p) - h_1(p) \) and check it using identity (6.44). It is clear from the proof of the Proposition 6.2 that (3.24) is a consequence of (3.25). It follows from Proposition 3.3 that \( \frac{1}{2\pi i} \int_{b_\alpha} \lambda(t, p) dt \) are also potentials of our hierarchy. Computing these integrals by virtue of (6.35) we conclude that the functions \( q_\alpha(p) - \sum_{k=1}^{m} s_k q_\alpha(w_k), \quad \alpha = 1, ..., g \) belong to the space of potentials. □

**Proposition 7.2.** The universal Whitham hierarchy is integrable by hydrodynamic reductions.

**Proof.** It is clear that the vector space spanned by derivatives with respect to \( p \) of potentials described in the previous Proposition coincides with the space of meromorphic differentials on \( \mathcal{E} \) holomorphic outside \( u_1, ..., u_n \) and with poles of order less or equal to one in these points. Therefore, we obtain a part of the universal Whitham hierarchy. In order to obtain the full hierarchy we apply the procedure of colliding point, see Proposition 3.2 and Remark 3.4. This proves the Proposition in the case \( g > 1 \).

\footnote{We need to choose constants of integrations carefully in order to obtain an integrable hierarchy.}
In the case $g = 0$ we define an enhanced GT structure by (3.28) and set $\lambda(p_1, p_2) = \frac{1}{p_1 - p_2}$. The space of potentials contains the functions $h_j(p) - h_1(p), j = 2, ..., n + 2$ where $h_j(p) = \ln(p - u_j), j = 1, ..., n, h_{n+1}(p) = \ln(p)$ and $h_{n+2}(p) = \ln(p - 1)$. This gives a part of the universal Whitham hierarchy corresponding to meromorphic differentials on $\mathbb{C}P^1$ with poles of order less or equal to one in $u_1, ..., u_n, 0, 1$. To obtain the full hierarchy we collide these points by a procedure similar to the one in the proof of Proposition 3.2, see also Remarks 3.3 and 3.4.

In the case $g = 1$ we define an enhanced GT structure by (3.29) and set

$$\lambda(p_1, p_2) = \rho(p_1 - p_2, \tau) - \rho(p_1) - 2\pi i.$$ 

The space of potentials contains $p - \tau$ and the functions $h_j(p) - h_1(p), j = 2, ..., n$ where $h_j(p) = \ln(\theta(p - u_j, \tau)) - \ln(\theta(u_j, \tau))$. This gives a part of the universal Whitham hierarchy corresponding to meromorphic differentials on $\mathcal{E}$ with poles of order less or equal to one in $u_1, ..., u_n$. To obtain the full hierarchy we collide some of these points by a procedure similar to one in the proof of Proposition 3.2, see also Remark 3.4. □

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