Compact smallest eigenvalue expressions in Wishart–Laguerre ensembles with or without a fixed trace

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Abstract. The degree of entanglement of random pure states in bipartite quantum systems can be estimated from the distribution of the extreme Schmidt eigenvalues. For a bipartition of size $M \geq N$, these are distributed according to a Wishart–Laguerre ensemble (WL) of random matrices of size $N \times M$, with a fixed-trace constraint. We first compute the distribution and moments of the smallest eigenvalue in the fixed-trace orthogonal WL ensemble for arbitrary $M \geq N$. Our method is based on a Laplace inversion of the recursive results for the corresponding orthogonal WL ensemble given by Edelman. Explicit examples are given for fixed $N$ and $M$, generalizing and simplifying earlier results. In the microscopic large $N$ limit with $M - N$ fixed, the orthogonal and unitary WL distributions exhibit universality after a suitable rescaling and are therefore independent of the constraint. We prove that very recent results given in terms of hypergeometric functions of matrix argument are equivalent to more explicit expressions in terms of a Pfaffian or determinant of Bessel functions. While the latter were mostly known from the random matrix literature on the QCD Dirac operator spectrum, we also derive some new results in the orthogonal symmetry class.

Keywords: rigorous results in statistical mechanics, matrix models, entanglement in extended quantum systems (theory), random matrix theory and extensions

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1. Introduction

The theory of random matrices finds applications to the most diverse physical situations, and Wigner and Dyson are usually referred to as the pioneers of this field for their works on nuclear spectra. However, many years before their seminal papers, John Wishart had already introduced random covariance matrices in his studies on multivariate populations [1]. The Wishart ensemble (also named Laguerre or chiral, and called WL hereafter) contains random $N \times N$ covariance matrices of the form $W = X^\dagger X$ where $X$ is an $M \times N$ matrix with i.i.d. Gaussian entries (real, complex or quaternions labeled by the Dyson indices $\beta = 1, 2, 4$ respectively). The joint probability density function (jpdf) of its $N$ non-negative eigenvalues $\{\lambda_i\}$ is given in equation (2.8). A related ensemble of random matrices is the fixed-trace Wishart–Laguerre (FTWL) ensemble, which contains WL matrices with a prescribed trace $\text{Tr}(W) = t > 0$ and whose eigenvalue jpdf is given in (2.10). Recent results concerning the density of eigenvalues of FTWL matrices can be found in [2]–[4]. The study of ensembles of non-chiral random matrices with fixed and bounded traces has however a longer history [5]. In [6, 7] the spectral density of the
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related, fixed-trace Gaussian unitary ensemble was computed explicitly for finite $N$, while for works on non-chiral fixed-trace $\beta$-ensembles we refer the reader to [8, 9].

Two recent applications of WL and FTWL ensembles that motivate our study are effective theories of quantum chromodynamics (QCD) and QCD-like theories, where we refer the reader to [10] for a most recent review and references, and the statistical theory of entangled random pure states in bipartite systems; see [11] for an excellent review.

Entanglement is indeed one of the most distinctive features of quantum systems, and it is a crucial resource in quantum computation issues, as performances of quantum computers will heavily rely on the possibility of producing states with large entanglement [12, 13]. In order to quantify the degree of entanglement of a given quantum state, it is useful to introduce an unbiased benchmark of states with the lowest degree of built-in information: pure bipartite systems (defined below) with a Hamiltonian of size $M \cdot N$ constitute a typical example where well-behaved entanglement quantifiers can be defined, such as the von Neumann or Rényi entropies of either subsystem [13], the so-called concurrence for two-qubit systems [14] and other entanglement monotones [15, 16].

The main focus of this paper is on the cumulative distribution and density of the smallest FTWL eigenvalue and its relation to the corresponding quantity for the unconstrained WL ensemble: these two distributions arise naturally in the two settings described above (entanglement and QCD, respectively). The upshot is that the smallest Schmidt eigenvalue (a relevant quantity in the entanglement setting discussed below) is precisely given by the smallest eigenvalue of one of the FTWL ensembles with $\text{Tr}(W) = t = 1$. Which of these ensembles to choose depends on the global symmetry of the quantum system, and we will report on the orthogonal ($\beta = 1$) and unitary ($\beta = 2$) symmetry classes, but not on the symplectic ($\beta = 4$) class. Another application of the smallest FTWL eigenvalue is related to the Demmel condition number [17], which quantifies how hard certain numerical problems concerning matrix inversion are.

The results of this paper can be grouped into two parts.

(i) First, we study the full distribution of the smallest eigenvalue of the FTWL orthogonal ensemble ($\beta = 1$) at finite $N, M$.

(ii) Next, we take the large $N, M$ limit of the smallest eigenvalue with $\nu = M - N \geq 0$ fixed (the so-called microscopic limit) for both $\beta = 1, 2$.

Let us summarize first which results were previously known for both FTWL and unconstrained WL.

- For the FTWL ensemble at finite $N, M$, the full distribution of the smallest eigenvalue was derived for $\beta = 1, 2$ and $M = N$ in [18], and for $\beta = 4$ in [11]. In [18], a conjecture by Znidaric [22] was proven for the first moment. For general real $\beta > 0$ and $m \in \mathbb{N}$, where $m = (\beta/2)(M - N + 1) - 1$, Chen et al [23] reported formal expressions for the smallest eigenvalue cumulative distribution and density in terms of cumbersome sums of Jack polynomials. More explicit expressions are given for the case $\beta = 2$ in terms of finite sums.

- For the WL ensemble at finite $N, M$, the first results for the smallest eigenvalues of the orthogonal WL ensemble go back to Edelman [24, 25] who gave a recursive
Curiously, analogous recurrence relations for $\beta = 2, 4$ have not been worked out to date, while for $\beta = 2$ a closed expression exists in terms of determinants [26]. These results were extended by Forrester [27] to the case of real $\beta > 0$ for special values of $\nu = M - N$. Forrester again waived some limitations on $\nu$ and further generalized these results [28]. The most up-to-date and general formula for the smallest eigenvalue distribution involves a hypergeometric function of matrix argument (HFMA); see equation (2.13b) in [28], as well as our appendix A. Note that for $\beta = 1$ this requires $\nu$ to be odd. For $\beta = 1, 4$, expressions in terms of Pfaffians were reported [29], with the same restriction for $\beta = 1$, and where for $\beta = 4$ the values of $\nu > 0$ are half-integers.

The next natural step is to take the large $N, M$ limit for the smallest eigenvalue. For real $\beta > 0$ keeping $\nu$ fixed, this was done in [28] for the WL ensembles and subsequently in [23] for FTWL ensembles, expressing them in terms of HFMA. It turned out that in this large $N$ limit (after a suitable rescaling) the fixed-trace constraint does not matter, and in that sense the results are universal (for a different limit, namely $c = N/M$ fixed, see the recent paper [30]).

The issue of universality in the ensembles with fixed (or bounded) traces had been addressed earlier in the literature, mainly for the non-chiral ensembles. It was shown in [6] that in the macroscopic large $N$ limit where the oscillations of the density are smoothed, the semi-circle or its generalizations can be matched in the constrained and unconstrained ensembles. In contrast, the higher order connected correlators become non-universal when adding the constraint to the non-Gaussian generalized WL ensembles, with a potential in $W$ in the exponent. On the other hand, for the microscopic limit in the bulk of the spectrum, where the oscillatory behavior of the density is zoomed into, the constraint was found to be immaterial and the universality of the sine-kernel was established in [31]. The same result was provided later in a mathematically more rigorous way for non-invariant generalizations of WL in [32, 33].

Because of this universality (i.e. constraint independence), it is extremely useful to recall the large $N$ results derived independently for the smallest eigenvalue distributions in the application of unconstrained WL ensembles to QCD [34–36]. Here also non-Gaussian generalizations of WL were considered, and determinantal and Pfaffian expressions for Bessel functions were derived for $\beta = 2$ [35] and $\beta = 1$ [36], respectively. In fact much more general results were derived there, including an arbitrary number of characteristic polynomials of random matrices (so-called mass terms) in the weight function. In the second part of this paper we will show how these results in the QCD literature and the aforementioned ones in terms of HFMA are related.

The presentation of the paper is organized as follows.

In order to give a self-contained presentation, in section 2 we provide some background material on the relation between bipartite entanglement and FTWL (section 2.1). In section 2.2 we give definitions and notation used for WL and FTWL ensembles. The reader familiar to either or both of these topics may skip the corresponding subsection(s).

In section 3 we derive our new results for FTWL ensembles at $\beta = 1$ for arbitrary finite $N$ and $M$, both for odd and for even $\nu$. There we briefly recall the results for standard WL given by Edelman on which we heavily rely. This section also contains our results for arbitrary moments in section 3.3 and numerical checks in section 3.4.
Section 4 brings us to the universal microscopic limit for fixed $\nu = M - N$. It provides proofs of equivalence between the known results from the QCD Dirac spectrum literature, given in section 4.2, and the very recent results in terms of HFMA, given in section 4.1. In section 4.3 we explicitly compute the large $N$ limit of the scaled smallest eigenvalue for $\beta = 1$ and $\nu = 0, 2$ from their finite $N$ expressions and confirm the universality results. For $\beta = 2$ the above mentioned equivalence is established in section 4.4, and for $\beta = 1$, in section 4.5, including new results for the cumulative distribution and for even $\nu = 2$. We also report on the general scaling for moments in section 4.6 before concluding.

Some technical details concerning the definition of HFMA and a universality check for $\beta = 1$ and $\nu = 0$ are deferred to the appendices.

2. Bipartite entanglement and FTWL ensembles

2.1. Bipartite entanglement

Let $H^{(NM)}$ be an $N \cdot M$-dimensional Hilbert space which is bipartite as $H^{(NM)} = H^{(N)}_A \otimes H^{(M)}_B$, where $N \leq M$ without loss of generality. For example, $A$ may be taken as a given subsystem (say a set of spins) and $B$ may represent the environment (e.g., a heat bath). Let $\{|i^A\rangle\}$ and $\{|\alpha^B\rangle\}$ be two complete bases of $H^{(N)}_A$ and $H^{(M)}_B$ respectively. Then, any quantum state $|\psi\rangle$ of the composite system can be decomposed as

$$|\psi\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} |i^A\rangle \otimes |\alpha^B\rangle$$

(2.1)

where the coefficients $x_{i,\alpha}$ form the entries of a rectangular $(N \times M)$ matrix $X$.

In the following, we will consider entangled random pure states $|\psi\rangle$. This means that:

(i) $|\psi\rangle$ cannot be expressed as a direct product of two states belonging to the two subsystems $A$ and $B$;

(ii) the expansion coefficients $x_{i,\alpha}$ are random variables drawn from a certain probability distribution (see below);

(iii) the density matrix of the composite system is simply given by $\rho = |\psi\rangle \langle \psi |$ with the constraint $\text{Tr}[\rho] = 1$, or equivalently $\langle \psi | \psi \rangle = 1$.

We will not consider statistically mixed states here, and we refer the reader to [37] and references therein for recent results on mixed states.

The density matrix $\rho$ of a quantum state is a very important quantity as it allows one to compute expectation values for observables. For an entangled pure state $|\psi\rangle$ of a bipartite quantum system it can then be straightforwardly written as

$$\rho = \sum_{i,\alpha} \sum_{j,\beta} x_{i,\alpha} x_{j,\beta}^* \langle i^A | j^A \rangle \otimes |\alpha^B\rangle \langle \beta^B |,$$

(2.2)

where the italic indices $i$ and $j$ run from 1 to $N$ and the Greek indices $\alpha$ and $\beta$ run from 1 to $M$.

In some applications, it is useful to separate the contribution of the subsystem $A$ under consideration from the environment $B$. Expectation values of observables $\Theta_A$ for
the system $A$ can be obtained by ‘tracing out’ the environmental degrees of freedom (i.e., those of subsystem $B$) and defining the reduced density matrix $\rho_A = \text{Tr}_B[\rho]$ as

$$\rho_A = \text{Tr}_B[\rho] = \sum_{\alpha=1}^{M} \langle \alpha^B | \rho | \alpha^B \rangle. \quad (2.3)$$

Using the expansion in equation (2.2) one gets

$$\rho_A = \sum_{i,j=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} x_{j,\alpha}^* |i^A \rangle \langle j^A| = \sum_{i,j=1}^{N} W_{ij} |i^A \rangle \langle j^A| \quad (2.4)$$

where the $W_{ij}$ are the entries of the $N \times N$ covariance matrix $W = XX^\dagger$.

Proceeding further, one could also obtain the reduced density matrix $\rho_B = \text{Tr}_A[\rho]$ of the subsystem $B$ in terms of the $M \times M$ matrix $W' = X^\dagger X$ and find that $W$ and $W'$ share the same set of nonzero (positive) real eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, called Schmidt eigenvalues.

In the basis of eigenvectors of $W$, $\rho_A$ can be expressed as

$$\rho_A = \sum_{i=1}^{N} \lambda_i |\lambda_i^A \rangle \langle \lambda_i^A| \quad (2.5)$$

where the $|\lambda_i^A \rangle$ are the normalized eigenvectors of $W = XX^\dagger$. The original composite state $|\psi\rangle$ in this diagonal basis reads

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} |\lambda_i^A \rangle \otimes |\lambda_i^B \rangle. \quad (2.6)$$

Equation (2.6) is known as the Schmidt decomposition, and the normalization condition $\langle \psi | \psi \rangle = 1$, or equivalently $\text{Tr}[\rho_A] = 1$, imposes a constraint on the eigenvalues, $\sum_{i=1}^{N} \lambda_i = 1$.

In the Schmidt decomposition (2.6), each state $|\lambda_i^A \rangle \otimes |\lambda_i^B \rangle$ is separable, but their linear combination $|\psi\rangle$ (depending on the set of Schmidt eigenvalues) cannot, in general, be written as a direct product $|\psi\rangle = |\phi^A \rangle \otimes |\phi^B \rangle$ of two states of the respective subsystems, i.e. it is entangled. Knowledge of the Schmidt eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of the matrix $W$ is therefore essential in providing information about how entangled a pure state is.

For random pure states, the expansion coefficients in equation (2.1) can be typically drawn from an unbiased (so-called Hilbert–Schmidt) distribution

$$\text{Prob}[X] \propto \delta \left( \text{Tr}(XX^\dagger) - 1 \right). \quad (2.7)$$

The meaning of equation (2.7) is clear: all normalized density matrices are sampled with equal probability, which corresponds to having minimal a priori information about the quantum state under consideration. This in turn induces nontrivial correlations among the Schmidt eigenvalues (which are now real random variables between 0 and 1 whose sum is 1) and makes the investigation of several statistical quantities concerning such states quite interesting. The jpdf of Schmidt eigenvalues for a Hilbert–Schmidt distribution of coefficients was derived in [38] and turns out to be exactly of the FTWL form (2.10) with $t = 1$, where the Dyson indices $\beta = 1, 2$ correspond respectively to real and complex $X$.
matrices\textsuperscript{4}. The delta constraint there indeed guarantees a proper normalization of the reduced density matrix $\text{Tr}[\rho_A] = 1$.

Why is the smallest Schmidt eigenvalue distribution interesting at all? We first note that due to the constraint $\sum_{i=1}^N \lambda_i = 1$ and the fact that all eigenvalues are non-negative, it follows that $1/N \leq \lambda_{\text{max}} \leq 1$ and $0 \leq \lambda_{\text{min}} \leq 1/N$. Now consider the following limiting situations. Suppose that the smallest eigenvalue $\lambda_{\text{min}} = \min_i \{\lambda_i\}$ takes its maximum allowed value $1/N$. Then it follows immediately that all the remaining $(N-1)$ eigenvalues must be also identically equal to $\lambda_i = 1/N$. In this situation, equation (2.6) tells us that $|\psi\rangle$ is maximally entangled. On the other hand, if $\lambda_{\text{min}} = 0$ (i.e., it takes its lowest allowed value) or is close to 0, while this will not provide much information about the degree of entanglement of $|\psi\rangle$, it actually tells us that one component in the Schmidt decomposition (2.6) can be safely ignored. In other words, the ‘effective’ dimension of the Hilbert space $\mathcal{H}_A$ has been reduced from $N$ to $N-1$. The proximity of the smallest eigenvalue to zero, therefore, gives information about the efficiency of this dimensional reduction process.

For more references on entangled random pure states we refer the reader to \cite{11}.

2.2. WL ensembles with and without a fixed trace

The joint probability density function (jpdf) of the non-negative eigenvalues of the unconstrained WL ensemble is given by

$$p^{(WL)}(\lambda_1, \ldots, \lambda_N) = K^{(WL)}_{N,M} \exp \left[ -\frac{\beta}{2} \sum_{i=1}^N \lambda_i \right] \prod_{i=1}^N \frac{\Gamma((\beta/2)(1+M-N)-1)}{\Gamma(1+(\beta/2)j) \Gamma((\beta/2)(M-N+j))} |\Delta(\vec{\lambda})|^\beta$$

(2.8)

where $K^{(WL)}_{N,M}$ is a known normalization constant:

$$K^{(WL)}_{N,M} = \left( \frac{\beta}{2} \right)^{\beta/2} \prod_{j=1}^N \frac{\Gamma((1+(\beta/2)j)}{\Gamma((\beta/2)(M-N+j))}.$$ (2.9)

and $\Delta(\vec{\lambda}) = \prod_{j<k}(\lambda_j - \lambda_k)$ is the Vandermonde determinant.

On the other hand, the jpdf of the eigenvalues $\lambda_i \in [0,t]$ of the FTWL ensemble is given by

$$p^{(FT)}(\lambda_1, \ldots, \lambda_N; t) = C^{(FT)}_{N,M}(t) \delta \left( \sum_{i=1}^N \lambda_i - t \right) \frac{\Gamma((\beta/2)(M-N+1)-1)}{\prod_{j=1}^N \Gamma((\beta/2)(M-N+j))} |\Delta(\vec{\lambda})|^\beta.$$ (2.10)

For $t = 1$ it coincides with the distribution of Schmidt eigenvalues, where the delta function guarantees that $\text{Tr}[\rho] = \sum_{i=1}^N \lambda_i = t$. The normalization constant for $t = 1$ reads \cite{20}

$$C^{(FT)}_{N,M} \equiv C^{(FT)}_{N,M}(1) = \frac{\Gamma((\beta/2)(1+\beta/2))}{\prod_{j=0}^{N-1} \Gamma((\beta/2)(M-N+j))}.$$ (2.11)

The presence of a fixed-trace constraint is known to have crucial consequences for (connected) spectral correlation functions, both for finite $N$ and in the macroscopic large $N$ limit \cite{6,39}. However, in the microscopic large $N$ limit the correlations become

\textsuperscript{4} These two cases in turn correspond to quantum systems whose Hamiltonians preserve ($\beta = 1$) or break ($\beta = 2$) time-reversal symmetry.
universal, and we shall exploit this in section 3. Let us first introduce the crucial quantities for this paper, the cumulative distributions \( q_{N,\nu}(x) = \text{Prob}[\lambda_{\text{min}} > x] \) (also called gap probabilities), and the corresponding densities for the smallest eigenvalues of both ensembles:

\[
q^{(\text{FT})}_{N,\nu}(x; t) = C^{(\beta)}_{N,M}(t) \prod_{i=1}^{N} \lambda_i \delta \left( \sum_{i=1}^{N} \lambda_i - t \right) \prod_{i=1}^{N} \lambda_i^{(\beta/2)(\nu+1)-1} |\Delta(\vec{\lambda})|^\beta, \tag{2.12}
\]

\[
p^{(\text{FT})}_{N,\nu}(x; t) = -\frac{\partial}{\partial x} q^{(\text{FT})}_{N,\nu}(x; t), \tag{2.13}
\]

\[
q^{(\text{WL})}_{N,\nu}(x) = K^{(\beta)}_{N,M} \prod_{i=1}^{N} \lambda_i e^{-\frac{1}{2} \sum_{i=1}^{N} \lambda_i} \prod_{i=1}^{N} \lambda_i^{(\beta/2)(\nu+1)-1} |\Delta(\vec{\lambda})|^\beta, \tag{2.14}
\]

\[
p^{(\text{WL})}_{N,\nu}(x) = -\frac{d}{dx} q^{(\text{WL})}_{N,\nu}(x), \tag{2.15}
\]

where the superscript distinguishes fixed-trace (FT) and ordinary WL. We have the following normalization conditions:

\[
\int_{0}^{\infty} dx p^{(\text{WL})}_{N,\nu}(x) = \int_{0}^{t/N} dx p^{(\text{FT})}_{N,\nu}(x; t) = 1, \tag{2.16}
\]

\[
q^{(\text{WL})}_{N,\nu}(0) = q^{(\text{FT})}_{N,\nu}(0; t) = 1. \tag{2.17}
\]

The \( \ell \)th moments of the smallest Schmidt and WL eigenvalues are then given by

\[
\langle \lambda_{\text{min}}^{\ell}_{N,\nu,\beta} \rangle^{(\text{FT})} = \int_{0}^{1/N} dx \int_{0}^{t/N} dx p^{(\text{FT})}_{N,\nu}(x), \tag{2.18}
\]

\[
\langle \lambda_{\text{min}}^{\ell}_{N,\nu,\beta} \rangle^{(\text{WL})} = \int_{0}^{\infty} dx \int_{0}^{t/N} dx p^{(\text{WL})}_{N,\nu}(x), \tag{2.19}
\]

where we define \( p^{(\text{FT})}_{N,\nu}(x) := p^{(\text{FT})}_{N,\nu}(x; 1) \) for later convenience. All density correlation functions and thus also the smallest eigenvalue distribution in ensembles with and without a fixed trace can be related via an inverse Laplace transform; see e.g. [31]. For the smallest eigenvalue distribution this relation reads

\[
\mathcal{L}[p^{(\text{FT})}_{N,\nu}(x; t)](s) = \int_{0}^{\infty} dt p^{(\text{FT})}_{N,\nu}(x; t) e^{-st} = \frac{C^{(\beta)}_{N,M}}{K^{(\beta)}_{N,M}} \left( \frac{1}{2s} \right)^{-1+MN(\beta/2)+((1-\beta)/2)N(N-1)} p^{(\text{WL})}_{N,\nu}(2sx). \tag{2.20}
\]

This is the main technical tool that we use in section 3. For finite \( N, M \) we are going to use known explicit expressions for WL smallest eigenvalue densities in order to derive the sought density for FTWL via the inverse Laplace transform of equation (2.20).

\[5\text{For this reason we have kept the fixed trace as a free parameter } t \text{ in the jpdf (2.10).}\]
3. New results for the FTWL ensemble at finite $N, M$ and $\beta = 1$

In this section we will focus on the $\beta = 1$ ensemble only. Our strategy is very simple. We will take the known results for the standard $\beta = 1$ WL ensemble at any finite $N, M$ derived by Edelman [25] and invert the Laplace transform in equation (2.20). The same strategy was followed by Chen et al [23]; however, they applied the inverse Laplace transform directly to the HFMA result of Forrester [28], valid for real $\beta > 0$. This led to expressions that are formally exact, but not very transparent or manageable. Furthermore, in contrast to [23, 28], our work will not be restricted to odd \( \nu = M - N \) for $\beta = 1$. The special case for $\nu = 0$ was derived independently in [19] and [18].

3.1. Odd $\nu = M - N$

We begin with the simpler case. Edelman [25] states that the smallest eigenvalue distribution for a WL ensemble with $\beta = 1$ and $\nu = M - N$ odd can be written as follows:

\[
    p^{(\text{WL})}(x) = 2^{(N/2)-1} c_{N,\nu} \rho^{(\nu-1)/2} e^{-N\rho/2} h_{N,\nu}(x) \tag{3.1}
\]

where $h_{N,\nu}(x)$ is a polynomial of degree $(N-1)(\nu-1)/2$ and $c_{N,\nu}$ is the following constant:

\[
    c_{N,\nu} = \frac{N^{2-N\nu/2} \Gamma((N+1)/2)}{\sqrt{\pi}} \prod_{j=1}^{\nu} \frac{\Gamma(j/2)}{\Gamma((N+j)/2)} \tag{3.2}
\]

As a result, $h_{N,\nu}(x)$ can be written as

\[
    h_{N,\nu}(x) = \sum_{k=0}^{(N-1)(\nu-1)/2} q_k x^k \tag{3.3}
\]

with some rational coefficients $q_k$, which depend also on $N, \nu$. The polynomial $h_{N,\nu}$ is determined via a simple recursion relation explicitly given in [25]. In addition, a simple Mathematica® code is given in appendix A of [25] for generating the $h_{N,\nu}(x)$. The first two polynomials read

\[
    h_{N,\nu=1}(x) = 1, \quad h_{N,\nu=3}(x) = \frac{2^N \Gamma((N/2) + 1) \Gamma((N + 3)/2)}{N(N + 1)\Gamma(3/2)} L_N^{(3)}(-x), \tag{3.4}
\]

and so on using the given recursion. In (3.4), $L_n^{(\alpha)}(x)$ is a generalized Laguerre polynomial\(^7\).

Now, it is easy to observe that (no matter what the coefficients $q_k$ are) the form (3.1) lends itself to a very friendly Laplace inversion. More precisely, take the inverse Laplace transform of the fundamental relation (2.20) after inserting equation (3.3) into (3.1). One obtains

\[
    p^{(\text{FT})}(x; t) = \sum_{k=0}^{(N-1)(\nu-1)/2} q_k x^k e^{-(N+\nu)/2} e^{-N\rho x} (t). \tag{3.5}
\]

\(^6\) Compared to the notation in [25], ours spells out the parameter $\rho$ explicitly contained in $c_{N,\nu}$ there.

\(^7\) Note the typographical error in lemma 4.5 in [25] in the representation of $L_n^{(\alpha)}(-x)$. 

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Computing the inverse Laplace transform and setting $t = 1$, one gets the final general formula:

$$
p_{N,\nu}^{(\text{FT})}(x) = \frac{C_{N,M}^{(1)}}{K_{N,M}^{(1)}} \theta^{(N/2)-1} c_{N,\nu}^{(N-1)(\nu-1)/2} \sum_{k=0} q_k^2 \frac{\Gamma((N(N+\nu) - 2k - \nu - 1)/2)}{(N(N+\nu)-2k-\nu-1)/2-1} \theta(1-Nx),
$$

(3.6)

where $\theta(z)$ is the Heaviside step function. This equation is our first main result of this section. The algorithm for computing $p_{N,\nu}^{(\text{FT})}(x)$ for odd $\nu$ works as follows.

(i) Compute the polynomial $h_{N,\nu}(x)$ for the sought $(N, \nu)$ using Edelman’s recursion relation [25].

(ii) Extract the coefficients $q_k$ of the polynomial from (3.3).

(iii) Insert these coefficients in the general formula (3.6).

To give some explicit examples we have worked out in full detail the cases $\nu = 1, 3$, which are based on equation (3.4):

$$
p_{N,\nu=1}^{(\text{FT})}(x) = \frac{NC_{N,N+1}^{(1)} 2^{-N(N+1)/2} (1-Nx)^{-2+N(N+1)/2}}{K_{N,N+1}^{(1)} \Gamma(-1+N(N+1)/2)} \theta(1-Nx),
$$

(3.7)

$$
p_{N,\nu=3}^{(\text{FT})}(x) = \frac{C_{N,N+3}^{(1)} \Gamma(3+N)}{K_{N,N+3}^{(1)} 2(N+1)(N-1)!} \theta(1-Nx) \sum_{k=0}^{N-1} \frac{(1-N)(-1)^k}{\Gamma(4+k)k!}
\times \frac{2^{2+k-N(N+3)/2}}{\Gamma(-2-k+N(N+3)/2)} x^{1+k}(1-Nx)^{-3-k+N(N+3)/2}.
$$

(3.8)

We stress, however, that the general formula (3.6) and the algorithm above provide an explicit and user-friendly solution to the problem of computing $p_{N,\nu}^{(\text{FT})}(x)$ for any desired value of $(N, \nu)$. The results obtained are much more explicit and manageable than those expressed in terms of a finite sum over partitions in [23]. To illustrate our algorithm we have generated examples with higher $\nu$.

We wrote a simple Mathematica® code that generates explicit expressions for the smallest eigenvalue density (the code is available on request). A plot for different values of odd $\nu$ and $N = 9$ is provided in figure 1.

3.2. Even $\nu = M - N$

Now, we turn to the more complicated case of $\nu = M - N \geq 0$ even. Also in this case, we can derive a general formula for $p_{N,\nu}^{(\text{FT})}(x)$ involving the same polynomial coefficients as in Edelman’s recursion. To the best of our knowledge, apart from for $\nu = 0$ [18], no such expressions were previously known for the FTWL ensemble.

Edelman [25] states that the smallest eigenvalue distribution for a WL ensemble with $\beta = 1$ and $\nu$ even can be written as follows:

$$
p_{N,\nu}^{(\text{WL})}(x) = 2^{-1/2} c_{N,\nu} x^{(\nu-1)/2} e^{-N x/2} [f_{N,\nu}(x) U_1(x) + g_{N,\nu}(x) U_2(x)],
$$

(3.9)

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$P_{N,\nu}^{(FT)}(x)$ for $N = 9$ and $\nu = 1, 3, 5, \ldots, 13$, odd, obtained using formula (3.6) and a simple Mathematica® implementation of our algorithm above. The running time for the full evaluation was less than 10 s. on a standard laptop.

where $f_{N,\nu}(x)$ and $g_{N,\nu}(x)$ are polynomials of degree at most $(N - 1)\nu/2$, and $c_{N,\nu}$ is given in equation (3.2). The two related functions $\mathcal{U}_1(x)$ and $\mathcal{U}_2(x)$ are defined as follows:

$$
\mathcal{U}_1(x) \equiv U \left( \frac{N - 1}{2}, \frac{1}{2}, \frac{x}{2} \right),
\mathcal{U}_2(x) \equiv -\frac{N - 1}{4} U \left( \frac{N + 1}{2}, \frac{1}{2}, \frac{x}{2} \right) = \mathcal{U}_1(x)',
$$

(3.10)

where $U(a, b, z)$ is a Tricomi confluent hypergeometric function given by

$$
U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty d\tau e^{-\tau} \tau^{a-1} (\tau + 1)^{-a+b-1}.
$$

(3.11)

Let us now write the polynomials $f_{N,\nu}(x)$ and $g_{N,\nu}(x)$ as

$$
\begin{align*}
&f_{N,\nu}(x) = \sum_{k=0}^{(\nu/2)(N-1)} f_k x^k, \\
g_{N,\nu}(x) = \sum_{k=0}^{(\nu/2)(N-1)} g_k x^k,
\end{align*}
$$

(3.12, 3.13)

with some rational coefficients $f_k$ and $g_k$, which depend also on $N, \nu$. As in the odd $\nu$ case they follow from a recurrence relation given in [25] along with a Mathematica® code for generating them. The first two examples read

$$
\begin{align*}
f_{N,\nu=0}(x) &= \frac{\Gamma(N) \sqrt{\pi}}{2^{N-1} \Gamma(N/2) \Gamma((N+1)/2)}, & g_{N,\nu=0}(x) &= 0, \\
f_{N,\nu=2}(x) &= \frac{2^N \Gamma((N + 1)/2) \Gamma((N + 2)/2)}{N \sqrt{\pi}} L_{N-1}^{(2)}(-x), \\
g_{N,\nu=2}(x) &= -\frac{2^{N+1} \Gamma((N + 1)/2) \Gamma((N + 2)/2)}{N(N - 1) \sqrt{\pi}} x L_{N-2}^{(3)}(-x).
\end{align*}
$$

(3.14, 3.15)

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Applying now equation (2.20)–(3.9), we get

\[ \mathcal{L} \left[ \tilde{p}^{(FT)}_{N,\nu}(x; t) \right] (s) = \frac{C^{(1)}_{N,M}}{K^{(1)}_{N,M}} 2^{-1/2} c_{N,\nu} \sum_{k=0}^{(\nu/2)(N-1)} \left[ f_k \Phi_k(x, s) - g_k \Psi_k(x, s) \right], \]  

where

\[
\Phi_k(x, s) \equiv x^{k+(\nu-1)/2} e^{-sNx} (2s)^{k+1+(\nu-1-N(N+\nu))/2} \mathcal{U}_1(2sx) \\
\equiv \frac{x^{k+(\nu-1)/2}}{\Gamma \left( (N-1)/2 \right)} \int_0^\infty d\tau \frac{\tau^{(N-3)/2} (\tau + 1)^{-(N/2)-1} (2s)^{k+(\nu+1-N(N+\nu))/2} e^{-sx(N+\tau)}}{\Gamma(\nu)}, \\
\Psi_k(x, s) \equiv \frac{N-1}{4} x^{k+(\nu-1)/2} e^{-sNx} (2s)^{k+1+(\nu-1-N(N+\nu))/2} \mathcal{U}_2(2sx) \\
= \frac{N-1}{4} x^{k+(\nu-1)/2} \frac{\Gamma \left( (N+1)/2 \right)}{\Gamma(\nu)} \int_0^\infty d\tau \frac{\tau^{(N-1)/2} (\tau + 1)^{-(N/2)-1} (2s)^{k+(\nu+1-N(N+\nu))/2} e^{-sx(N+\tau)}}{\Gamma(\nu)}, \\
\]

Here we have inserted the integral representation (3.11).

Taking the inverse Laplace transform of the quantities in square brackets, and setting \( t = 1 \) we can eventually write

\[ p^{(FT)}_{N,\nu}(x) = \frac{C^{(1)}_{N,M}}{K^{(1)}_{N,M}} 2^{-1/2} c_{N,\nu} \sum_{k=0}^{(\nu/2)(N-1)} \frac{2(\nu+2k-N(N+\nu))/2}{\Gamma(N(N+\nu) - \nu - 1 - 2k)/2} \]

\[ \times \left[ f_k \Theta_k(x) - g_k \Xi_k(x) \right], \]  

where

\[
\Theta_k(x) \equiv \frac{x^{k+(\nu-1)/2}}{\Gamma \left( (N-1)/2 \right)} \int_0^\infty d\tau \frac{\tau^{(N-3)/2} (\tau + 1)^{(N/2)+1}}{\Gamma(\nu)(N(N+\nu) - \nu - 1 - 2k)/2} \\
\times \left( 1-x(N+\tau) \right)^{(N(N+\nu) - \nu - 3 - 2k)/2} \theta(1-x(N+\tau)) \\
= \frac{x^{k+(\nu-N)/2}(1-Nx)^{(N(N+\nu) - \nu - N-4-2k)/2}}{\Gamma(\nu)(N(N+\nu) + N - \nu - 2 - 2k)/2} \\
\times \frac{\Gamma \left( (N(N+\nu) - \nu - 1 - 2k)/2 \right)}{\Gamma \left( (N-1)/2 \right)} \int_0^\infty d\tau \frac{\tau^{(N-1)/2} (\tau + 1)^{-(N/2)-1}}{\Gamma(\nu)(N(N+\nu) - \nu - 1 - 2k)/2} \\
\times 2F_1 \left( \frac{N-1}{2}, \frac{N}{2}+1; \frac{-2k+(N-1)(N+\nu+2)}{2}; \frac{N-1}{x} \right) \]  

\[
\Xi_k(x) \equiv \frac{N-1}{4} x^{k+(\nu-1)/2} \frac{\Gamma \left( (N+1)/2 \right)}{\Gamma(\nu)(N(N+\nu) + N - \nu - 2 - 2k)/2} \\
\times \left( 1-x(N+\tau) \right)^{(N(N+\nu) - \nu - 3 - 2k)/2} \theta(1-x(N+\tau)) \\
= \frac{N-1}{4} x^{k+(\nu-N)/2}(1-Nx)^{(N(N+\nu) - \nu - N-4-2k)/2} \Gamma(\nu)(N(N+\nu) + N - \nu - 1 - 2k)/2} \\
\times \frac{\Gamma \left( (N(N+\nu) - \nu - 1 - 2k)/2 \right)}{\Gamma \left( (N+1)/2 \right)} \int_0^\infty d\tau \frac{\tau^{(N-1)/2} (\tau + 1)^{-(N/2)-1}}{\Gamma(\nu)(N(N+\nu) - \nu - 1 - 2k)/2} \\
\times 2F_1 \left( \frac{N+1}{2}, \frac{N}{2}+1; \frac{-2k-\nu+(N+1)(N+\nu+2)}{2}; \frac{N-1}{x} \right). \]
In the second step we have performed the integrations and applied some simple algebra, and $2F_1(a, b; c; z)$ is a standard hypergeometric function.

Equation (3.17) together with equations (3.18) and (3.19) is the second result of this section. Again, the algorithm for computing $p_{N,\nu}^{(\text{FT})}(x)$ for even $\nu$ works as follows.

(i) Compute the polynomials $f_{N,\nu}(x)$ and $g_{N,\nu}(x)$ for the sought $(N, \nu)$ using Edelman’s recursion [25].

(ii) Extract the coefficients $f_k$ and $g_k$ of the polynomial from (3.12) and (3.13).

(iii) Insert these coefficients in the general formula (3.17).

To provide explicit examples we give the expressions for $\nu = 0, 2$ based on equations (3.14) and (3.15), where the result for $\nu = 0$ was previously derived in [18]:

\[
p_{N,\nu=0}^{(\text{FT})}(x) = \frac{N\Gamma(N)\Gamma(N^2/2)}{2^{N-1}N!\Gamma(N/2)\Gamma((N^2 + N - 2)/2)} x^{-N/2}(1 - Nx)^{(N^2+N-4)/2} \\
\times 2F_1\left(\frac{N+2, N-1}{2}; \frac{N^2+N-2}{2}; -\frac{1}{x}\right)
\]

and

\[
p_{N,\nu=2}^{(\text{FT})}(x) = \frac{C_{N,N+2}^{(1)}}{K_{N,N+2}^{(0)}} \Gamma((N + 1)/2) \sqrt{2\pi} \sqrt{x} \left(\phi_N(x) + \psi_N(x)\right),
\]

where

\[
\phi_N(x) = 2^{(3-N(N+2)/2)} x^{(1-N)/2} (1 - N x)^{-3+3N/2+N^2/2} \sum_{k=0}^{N-1} \frac{2^k(N+1)!}{k!(N-1-k)!(k+2)!} \\
\times \left(\frac{x}{1-Nx}\right)^k 2F_1\left(\frac{N-1}{2}, 1 + \frac{N}{2}; -\frac{N^2}{2} - 2 - k; -\frac{1}{x}\right),
\]

\[
\psi_N(x) = 2^{(3-N(N+2)/2)} x^{(1-N)/2} (1 - N x)^{-3+3N/2+N^2/2} \sum_{k=0}^{N-2} \frac{2^k(N+1)!}{k!(N-2-k)!(k+3)!} \\
\times \left(\frac{x}{1-Nx}\right)^k 2F_1\left(\frac{N+1}{2}, 1 + \frac{N}{2}; -\frac{N^2}{2} - 2 - k; -\frac{1}{x}\right).
\]

In order to illustrate our algorithm above we again wrote a simple Mathematica® code that generates explicit expressions for the smallest eigenvalue density. A plot for different even $\nu > 0$ and $N = 9$, as an example, is provided in figure 2.

3.3. Moments for odd $\nu$

Arbitrary moments $\langle \chi_{\text{min}}^{\ell} \rangle_{N,\nu,\beta=1}^{(\text{FT})}$ can be computed easily in a closed form for odd $\nu$ from equation (3.6), using the following integral formula:

\[
\int_0^{1/N} dx x^{\nu-1}(1 - Nx)^{\xi-1} = N^{-\nu} B(\omega, \xi).
\]

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Figure 2. Smallest eigenvalue density $p_{N,\nu}^{(\text{FT})}(x)$ for $N = 9$ and $\nu = 2, 4, 6, \ldots, 12$, even, using formula (3.17) and a simple Mathematica® implementation of the algorithm above. The running time for the full evaluation was less than 50 s. on a standard laptop.

Here $B(\omega, \xi)$ is Euler’s Beta function. We thus obtain

$$
\langle \lambda_{\min}^{(\text{FT})} \rangle_{N,\nu=2m+1,\beta=1} = C_{N,M}^{(1)} \frac{2^{(N/2)-1}c_{N,2m+1}}{K_{N,M}^{(1)}} \sum_{k=0}^{(N-1)m} \frac{q_k 2^{m+1+(2k-N(N+\nu))/2}}{\Gamma((N(N + 2m + 1))/2 - k - m - 1)}
\times B\left(\ell + k + 1 + m, \frac{N(N + 2m + 1)}{2} - k - m - 1\right).
$$

(3.25)

To give an example, the $\ell$th moment for $\nu = 1$ is given as follows:

$$
\langle \lambda_{\min}^{(\text{FT})} \rangle_{N,\nu=1,\beta=1} = C_{N,N+1}^{(1)} \frac{2^{N(N+1)/2}}{K_{N,N+1}^{(1)}} \frac{\Gamma(1 + \ell)}{N^\ell \Gamma(\ell + N(N+1)/2)}.
$$

In particular for the first moment ($\ell = 1$) or average value we get the following answer:

$$
\langle \lambda_{\min}^{(\text{FT})} \rangle_{N,\nu=1,\beta=1} \sim \frac{2}{N^2(N+1)} \sim 2/N^3 \quad \text{for } N \gg 1.
$$

(3.26)

For comparison, at $\nu = 0$ the large $N$ behavior is as follows [18]:

$$
\langle \lambda_{\min}^{(\text{FT})} \rangle_{N,\nu=0,\beta=1} \sim \frac{c}{N^3} \quad \text{for } N \gg 1,
$$

(3.27)

where

$$
c = 2 \left[ 1 - \sqrt{\frac{\pi e}{2}} \operatorname{erfc}(1/\sqrt{2}) \right] \approx 0.688641 \cdots,
$$

(3.28)

(compare with section 4.6). Although the computation of moments for even $\nu$ is possible, based on our explicit expressions given previously, we did not find short closed expressions as in equation (3.25) above.

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Figure 3. The density $p^{(\text{FT})}_{N=7,\nu}(x)$ of the smallest Schmidt eigenvalue from numerical simulations compared with our predictions, equations (3.7), (3.21), and (3.8) for $\nu = 1, 2, 3$ respectively (solid lines).

3.4. Numerical checks

In figure 3 we compare a few theoretical densities from the previous sections 3.1 and 3.2 for finite $N$ with the corresponding numerical results. These are obtained as follows [20,21].

(i) We generate $n \approx 10^5$ real Gaussian $M \times N$ matrices $X$ (where $N = 7, M = N + \nu$).
(ii) For each instance we construct the Wishart matrix $W = X^TX$.
(iii) We diagonalize $W$ and collect its $N$ real and non-negative eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$.
(iv) We define a new variable $0 \leq \mu_1 \leq 1$ as $\mu_1 = \lambda_{\text{min}} / \sum_{i=1}^N \lambda_i$.
(v) We construct a normalized histogram of $\mu_1$.

The agreement between theory and simulations is excellent.

4. Equivalence proofs and new results in the large $N, M$ limit

We now turn to the microscopic large $N, M$ limit for the orthogonal and unitary ensembles ($\beta = 1, 2$), with $\nu = M - N \geq 0$ fixed. More precisely, let us define the following microscopic scaling limits at the origin:

$$Q^{(\text{FT})}_{\nu}(y) = \lim_{N,M \to \infty} q^{(\text{FT})}_{N,\nu} \left( \frac{y}{4N^3} \right), \quad P^{(\text{FT})}_{\nu}(y) = \lim_{N,M \to \infty} \frac{1}{4N^3} p^{(\text{FT})}_{N,\nu} \left( \frac{y}{4N^3} \right);$$ (4.1)

$$Q^{(\text{WL})}_{\nu}(y) = \lim_{N,M \to \infty} q^{(\text{WL})}_{N,\nu} \left( \frac{y}{4N^3} \right), \quad P^{(\text{WL})}_{\nu}(y) = \lim_{N,M \to \infty} \frac{1}{4N^3} p^{(\text{WL})}_{N,\nu} \left( \frac{y}{4N^3} \right).$$ (4.2)

Notice that the mean level spacing at the origin (the $N$-dependent rescaling factor) is different for FTWL and standard WL ensembles. This fact was also observed when comparing the corresponding macroscopic densities in the Gaussian ensembles with and without constraint [6].

It has been shown that in the microscopic limit (i) the smallest eigenvalue distribution in non-Gaussian generalizations of WL ensembles is universal [35, 36], and (ii) for all real $\beta > 0$, the microscopic limit for the smallest eigenvalue distribution is the same for WL and
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FTWL ensembles in all cases where a representation in terms of HFMA exists [23]. We also recall that universality (ii), that is the independence of the constraint, was shown earlier for all microscopic density correlation functions in the bulk of the spectrum for unitary non-chiral ensembles for monic even potentials [31] or i.i.d. matrix elements [32, 33].

Building on [23], from now on we shall assume that

\[ Q_{\nu}^{(WL)}(y) = Q_{\nu}^{(FT)}(y), \]
\[ P_{\nu}^{(WL)}(y) = P_{\nu}^{(FT)}(y), \]

for all \( \beta \) and \( y \). Therefore, there is no reason to keep the superscripts \((WL)\) and \((FT)\). We will thus use the unified notation \( Q_{\nu}(\beta, y) \) and \( P_{\nu}(\beta, y) \), labeled by \( \beta \) only\(^{8}\). Note that one still has

\[ P_{\nu}(\beta, y) = -Q_{\nu}(\beta, y)', \]
\[ \int_0^\infty dy \, P_{\nu}(\beta, y) = 1, \]
\[ Q_{\nu}(\beta, 0) = 1. \]

What are the known results for \( Q_{\nu}(\beta, y) \) and \( P_{\nu}(\beta, y) \)? It turns out that two communities have derived different formulas for these very same quantities using different languages: HFMA on the one hand, and determinants or Pfaffians of Bessel functions on the other (see the next two subsections). It is one of the goals of this paper to highlight this connection, which is not widely appreciated, and to actually prove the equivalence of these formulas by simple algebraic methods. As a result of this (and of section 3) we will derive some new compact results in the second language in cases where only HFMA or no results were previously known.

We should also mention here that there is a third way to compute the cumulative distribution. Formally it is given by the Fredholm determinant of the corresponding Bessel kernel. This has been mapped to expressions containing a transcendent of the solution of the Painlevé V for \( \beta = 2 \) [40] and \( \beta = 1, 4 \) [41]. In particular, the distributions for different values of \( \beta \) can be related to each other. However, this relation does not allow us to generate the more explicit expressions in terms of determinants and Pfaffians below.

4.1. Hypergeometric functions of matrix argument (HFMA)

In order to establish the claimed equivalence we first need to state the results in the first language. Forrester [28] and Chen et al [23] independently gave expressions for \( P_{\nu}(\beta, y) \) and \( Q_{\nu}(\beta, y) \) valid for all \( \beta > 0 \) and integer \( m = (\beta/2)(\nu + 1) - 1 \in \mathbb{N} \), in terms of HFMA, as

\[ Q_{\nu}(\beta, y) = e^{-\beta y/8} F_1^{(\beta/2)}(-; 2m/\beta; (y/4)^3 m), \]
\[ P_{\nu}(\beta, y) = A_{m, \beta} y^m e^{-\beta y/8} F_1^{(\beta/2)}(-; 2 + 2m/\beta; (y/4)^3 m), \]

\(^{8}\) We note in passing that the large \( y \gg 1 \) behavior of \( Q_{\nu}(\beta, y) \) is known from [40]; see also [42] for the calculation of the leading term using a Coulomb gas method.
where the constant $A_{m,\beta}$ is given by\(^9\)

$$A_{m,\beta} = \frac{1}{4^{m+1}(\beta/2)^{2m+1}} \frac{\Gamma(\beta/2 + 1)}{\Gamma(m + 1)\Gamma(m + 1 + \beta/2)},$$

and $I_m$ is the identity matrix $m \times m$. For definitions and properties of HFMA we refer the reader to appendix A. We note that for $\beta = 1$, integer $m$ implies that $\nu = M - N$ must be odd.

The formulas above were specialized in [23] to the two cases (a) $m = 1$ for all $\beta > 0$, and (b) for $\beta = 2$, yielding respectively

(a) $Q_{\nu=1}(y) = 2^{1-1/\beta}\Gamma(2/\beta)e^{-\beta y/2} y^{1/2-1/\beta} I_{(2/\beta)-1}(\sqrt{y})$

(b) $Q_{\nu=2}(y) = e^{-y/4} (I_0(\sqrt{y})^2 - I_1(\sqrt{y}))$

where $I_{s}(x)$ is a modified Bessel function of the first kind. The appearance of Bessel functions for special instances of formulas (4.8) and (4.9) is not at all accidental, as we will see now.

4.2. Determinants and Pfaffians of Bessel functions

In the context of random matrix theory applied to effective theories of quantum chromodynamics (QCD) or QCD-like theories, the large $N$ distribution of the smallest Dirac operator eigenvalue has been studied intensively for the symmetry classes $\beta = 1, 2$ (and 4); see [10] for references. It turns out that the non-zero Dirac eigenvalues $y_i$ (occurring in $\pm$ pairs due to chiral symmetry) are precisely distributed according to the WL jpdf (2.8) upon carrying out the mapping $y_i^2 = \lambda_i$. The relation between the smallest Dirac eigenvalue $\Psi_{\nu}^{(\beta)}(s)$, and the smallest WL eigenvalue distribution trivially follows:

$$\Psi_{\nu}^{(\beta)}(s) = 2s P_{\nu}^{(\beta)}(s^2)$$

$$\Omega_{\nu}^{(\beta)}(s) = Q_{\nu}^{(\beta)}(s^2),$$

where we also give the relation for the cumulative distribution $\Omega_{\nu}^{(\beta)}(s)$ in the Dirac picture. We still have $\Psi_{\nu}^{(\beta)}(s) = -\Omega_{\nu}^{(\beta)'}(s)$ and the standard normalization $\int_0^\infty ds \Psi_{\nu}^{(\beta)}(s) = 1$ is ensured.

The following results hold for our symmetry classes:

$$\Psi_{\nu=2}(s) = \frac{s}{2} e^{-s^2/4} \det [I_{i-j+2}(s)_{i,j=1,\ldots,\nu}], \quad \nu \in \mathbb{N};$$

$$\Psi_{\nu=2m+1}(s) = C_m s^{1-m} e^{-s^2/8} \text{Pf} \left[ (i-j+3)(s)_{i,j=-m+1/2, \ldots, m-1/2} \right], \quad m \in \mathbb{N};$$

$$\Psi_{\nu=0}(s) = \frac{1}{4} (2 + s) e^{-s^2/8 - s/2};$$

$$\Psi_{\nu=2}(s) = \frac{1}{4} ((2 + s) I_2(s) + s I_3(s)) e^{-s^2/8 - s/2},$$

where Pf($A$) = $\sqrt{\det A}$ stands for the Pfaffian of the antisymmetric matrix $A$.

\(^9\) Note that the expression for the constant $A_{m,\beta}$, equation (2.15c) in [28] (and subsequently in [23]), and equation (4.11) as it appears in [23] contain misprints.
Most of these results were known from the literature, apart from the normalization constant \( C_m \) given in equation (4.41), and the new equation (4.18) to be derived below. For \( \beta = 2 \) the distribution of equation (4.15) was derived in [26,34,35]. For \( \beta = 1 \) and \( \nu = 0 \) equation (4.17) was shown in [27], and then extended to all odd \( \nu \) in [36] up to normalization.

We note in passing that for \( \beta = 4 \) the distribution for \( \nu = 0 \) is known explicitly from [27] (see equation (4.11) and take its derivative), but the results quoted in [29] for \( \beta = 4 \) only hold for \( m = \beta (\nu + 1)/2 -1 \) even, that is for half-integer values of \( \nu \). Finally we mention that for small \( \nu \) all distributions for \( \beta = 1, 2, 4 \) can be very well approximated by using finite \( N = 2 \) results [43], like in the Wigner surmise.

Turning to the corresponding cumulative distributions the following results hold:

\[
\Omega_\nu^{(\beta=2)}(s) = e^{-s^2/4} \det [I_{i-j}(s)]_{i,j=1,\ldots,\nu}, \quad \nu \in \mathbb{N}
\]  
(4.19)

for \( \beta = 2 \) [35], and for \( \beta = 1 \) and odd \( \nu = 2m + 1 \) our new result reads

\[
\Omega_\nu^{(\beta=1)}(s) = 2^m s^{-m} e^{-s^2/8} \text{Pf} [(j - k)I_{1+k+j}(s)]_{j,k=-m+1/2,\ldots,-m-1/2}, \quad m \in \mathbb{N}.
\]  
(4.20)

Before providing a general algebraic proof of the equivalence of the two languages in sections 4.4 and 4.5, we can quickly check the agreement between the two different formulations above, starting from the special HFMA cases in equations (4.11) and (4.12). Using the map (4.14), equation (4.19) obviously contains equation (4.12) as a special case. Specifying \( \beta \) in equation (4.11) leads to the special cases \( \beta = 2, \nu = 1 \) and \( \beta = 1, \nu = 3 \) in equations (4.19) and (4.20) respectively.

Additionally one can differentiate these and arrive at corresponding special cases in equations (4.15) and (4.17) respectively, after using some Bessel identities.

### 4.3. Results for the scaled smallest eigenvalue distribution for \( \beta = 1 \) and \( \nu = 0,2 \)

In this subsection, we give two results for the scaled smallest eigenvalue distribution at even \( \nu = 0 \) and 2. In principle the universality equation (4.4), that is the agreement between the WL and FTWL distributions in the microscopic large \( N \) limit, has not been shown for \( \beta = 1 \) with even \( \nu \), although there is little doubt that it extends from the odd \( \nu \) case. We explicitly check here that this is indeed the case by taking the microscopic large \( N \) limit for FTWL at \( \nu = 0 \) starting from the known equation (3.20) for finite \( N \) and getting to the same result as in WL. For \( \nu = 2 \) we derive a new result that was unknown even for the microscopic limit of the unconstrained WL ensemble.

For \( \nu = 0 \) we first recall how the known microscopic result for WL is obtained. Starting from the finite \( N \) expression of equation (3.9) together with equation (3.14) we need to know the asymptotic limit of the Tricomi confluent hypergeometric function, in the scaling limit of equation (4.2). It was derived in corollary 3.1 of [24]; alternatively it follows from equation (13.3.3) of [44]:

\[
\lim_{n \to \infty} \frac{2}{\sqrt{n}} \Gamma \left( \frac{n}{2} + 1 \right) U \left( \frac{n-1}{2}, -\frac{1}{2}, \frac{x}{2n} \right) = (1 + \sqrt{x}) e^{-\sqrt{x}}.
\]  
(4.21)

Collecting all prefactors we thus obtain

\[
\lim_{N \to \infty} \frac{1}{4N} P_{N,\nu}^{(WL)} (y/4N) = \frac{1}{8} \left( 1 + \frac{2}{\sqrt{y}} \right) e^{-y/8 - \sqrt{y}/2}.
\]  
(4.22)

doi:10.1088/1742-5468/2011/05/P05020
The relation (4.13) maps this to equation (4.17). This result has to be compared with the limit equation (4.1) for equation (3.20) in the FTWL ensemble. Because the limit of the hypergeometric function is more involved we defer the derivation to appendix B, finding complete agreement as in equation (4.4):

\[ P_{\nu=0}^{(\beta=1)}(y) \equiv P_{\nu=0}^{(\text{FT})}(y) = \lim_{N \to \infty} \frac{1}{4N^2} P_{N,\nu}^{(\text{FT})} \left( \frac{y}{4N^2} \right) = P_{\nu=0}^{(\text{WL})}(y) = \lim_{N \to \infty} \frac{1}{4N^2} P_{N,\nu}^{(\text{WL})} \left( \frac{y}{4N} \right). \]  

(4.23)

This extends the expected universality to the case of even \( \nu = 0 \).

For \( \nu = 2 \) the corresponding microscopic limit of (3.21) for FTWL is already rather cumbersome, involving the asymptotic form of a sum of hypergeometric functions. We therefore restrict ourselves to computing the corresponding microscopic limit in the WL ensemble—which is also new—and conjecture that the universal relation of equation (4.23) extends to \( \nu = 2 \) and in fact to all higher even \( \nu \) values. Combining equations (3.9) and (3.15) we have to compute

\[ \lim_{N \to \infty} \frac{1}{4N} p_{N,\nu=2}^{(\text{WL})} \left( \frac{y}{4N} \right) = \lim_{N \to \infty} \frac{\Gamma \left( (N+1)/2 \right)}{\sqrt{2\pi}} \left( \frac{y}{4N} \right)^{1/2} e^{-y/8} \times \left[ L_{N-1}^{(2)} \left( \frac{-y}{4N} \right) U \left( \frac{N-1}{2}, -\frac{1}{2}, \frac{y}{8N} \right) + \frac{y}{8N} L_{N-2}^{(3)} \left( \frac{-y}{4N} \right) \frac{N+1}{4} U \left( \frac{N+1}{2}, \frac{1}{2}, \frac{y}{8N} \right) \right]. \]

Equation (4.21) together with the Laguerre asymptotic form for negative argument,

\[ \lim_{m \to \infty} m^{-a} I_{m}(-x/m) = x^{-a/2} I_{a}(2\sqrt{x}), \]

(4.24)

yields the following final answer:

\[ P_{\nu=2}^{(\beta=1)}(y) \equiv P_{\nu=2}^{(\text{FT})}(y) = P_{\nu=2}^{(\text{WL})}(y) = \frac{1}{8} \left( 1 + 2 / \sqrt{y} \right) I_{2}(\sqrt{y}) + I_{3}(\sqrt{y}) e^{-y/8 - \sqrt{y}/2}. \]  

(4.25)

Using equation (4.13), this is mapped to equation (4.18) as claimed.

4.4. Equivalence proofs for \( \beta = 2 \)

In this subsection and in the following, we provide an algebraic link between the HFMA and the Bessel determinant languages in the spirit of earlier works [41,45]. We start from the formula in equation (4.9) which expresses the scaled smallest eigenvalue distribution (with or without a fixed-trace constraint) in terms of HFMA:

\[ P_{\nu=2m}^{(\beta=2)}(y) = A_{m,2} y^{m} e^{-y/4} Q_{1}^{(1)} \left( -; m + 2; \frac{y}{4} \right). \]  

(4.26)

Next, we use the following integral representation due to Forrester [28] valid for integer values of \( \lambda \) and \( c \):

\[ \hat{B}_{m}(c, \lambda) = \frac{1}{2\pi} m^{(c-1) m/2} \left( \frac{1}{2\pi} \right)^{m} \times \int_{[-\pi,\pi]^{m}} d\theta_{j} e^{2\sqrt{\tau} \cos \theta_{j}} e^{i(c-1) \theta_{j}} \left| \Delta(e^{i\theta}) \right|^{\lambda}, \]

(4.27)

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where we have defined

\[ \hat{B}_m(c, \lambda) = \prod_{j=1}^{m} \frac{\Gamma(1+\lambda/2)\Gamma(c+\lambda(j-1)/2)}{\Gamma(1+\lambda j/2)}, \]  

(4.28)

and the Vandermonde determinant of the angles

\[ \Delta(e^{i\hat{\theta}}) \equiv \prod_{j<k} (e^{i\theta_k} - e^{i\theta_j}). \]  

(4.29)

Comparison between (4.27) and (4.26) yields \( c = 3 \) and \( \lambda = 2 \). Denoting by * complex conjugation, we can use the Andréeif identity [46]

\[ \int \prod_{j=1}^{m} d\theta_j \omega(\theta) \Delta(e^{i\hat{\theta}}) \Delta^*(e^{i\hat{\theta}}) = m! \det \left[ \int d\theta e^{i\hat{\theta}(\ell-k)} \omega(\theta) \right]_{\ell,k=1,...,m}. \]  

(4.30)

We thus obtain for the \( m \)-fold integral in the second line of equation (4.27)

\[ m! \det \left[ \int d\theta e^{i\hat{\theta}(\ell-k+2)} e^{2\sqrt{x}\cos \theta} \right]_{\ell,k=1,...,m} = m!(2\pi)^m \det [I_{\ell-k+2}(2\sqrt{x})]_{\ell,k=1,...,m}, \]  

(4.31)

where we have used the following integral representation for the modified Bessel function (valid for integer index only):

\[ I_n(z) = \frac{1}{\pi} \int_0^\pi dt e^{zt} \cos n(t), \quad n \in \mathbb{N}. \]  

(4.32)

Setting \( x = y/4 \) in (4.31) and simplifying all prefactors, we eventually get from (4.26)

\[ P_{\nu \equiv m}^{(2)}(y) = \frac{1}{4} e^{-y/4} 0_F^{(1)}(-; m; (y/4)I_m) = e^{-y/4} \det [I_{\ell-j+2}(\sqrt{y})]_{i,j=1,...,\nu}, \quad \nu \in \mathbb{N}. \]  

(4.33)

This is identical to equation (4.15) after switching to the Dirac picture of equation (4.13). The proof for the cumulative distribution goes along the same lines, merely changing the coefficient in front and the index shift of the Bessel function. We obtain the following from equation (4.8) by identifying \( c = 1 \) and \( \lambda = 2 \) in the representation of equation (4.27) and collecting all prefactors:

\[ Q_{\nu}^{(\beta=2)}(y) = e^{-y/4} 0_F^{(1)}(-; m; (y/4)I_m) = e^{-y/4} \det [I_{\ell-j}((\sqrt{y})]_{i,j=1,...,\nu}, \quad \nu \in \mathbb{N}. \]  

(4.34)

This corresponds to equation (4.19) using the map (4.14). Note that it is highly nontrivial to derive equation (4.34) from equation (4.33) using equation (4.5).

4.5. The equivalence proof and new results for \( \beta = 1 \)

We start again from the formula in equation (4.9) expressing the scaled smallest eigenvalue distribution (with or without a fixed-trace constraint) in terms of HFMA:

\[ P_{\nu \equiv m}^{(\beta=1)}(y) = A_{m,1} y^m e^{-y/8} 0_F^{(1/2)} (-; 2m+2; \frac{y}{4}I_m), \quad m \in \mathbb{N}. \]  

(4.35)

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Next, we use again the integral representation in equation (4.27) and \( x = y/4 \) to write
\[
\int_0^{(1/2)} (-; 2 + 2m; x \mathbb{1}_m) = \hat{B}_m(4, 4) \left( \frac{1}{x} \right)^{3m/2} \left( \frac{1}{2\pi} \right)^m \times \int_{[-\pi, \pi]^m} d\theta_j e^{2\sqrt{x} \cos \theta_j} e^{3i\theta_j} |\Delta(e^{i\theta})|^4. \tag{4.36}
\]
Comparison between (4.27) and (4.35) does indeed yield \( c = 4 \) and \( \lambda = 4 \). Next, we use section 11.5 of [5], combining equations (11.5.2) and (11.5.4) there to write
\[
\prod_{j<k} |e^{i\theta_k} - e^{i\theta_j}|^4 = \det \left[ e^{i\theta_{jk}} \right], \tag{4.37}
\]
where the determinant on the right-hand side is of size \( 2m \times 2m \) and indices run as follows:
\[ 1 \leq j \leq m \quad \text{and} \quad p = -(m - 1/2), -(m - 3/2), \ldots, (m - 3/2), m - 1/2. \]
Now we can define the two sets of functions \( \{ \phi_k(\theta) \} = e^{ik\theta} \) and \( \{ \psi_k(\theta) \} = ke^{ik\theta} \) and thus use the de Bruijn identity [47]:
\[
\int \prod_{j=1}^m dx_j \omega(x_j) \det [\phi_i(x_j)\psi_i(x_j)]_{1 \leq i, j \leq 2m} = m! \text{Pf} \left[ \int dx \omega(x) (\phi_i(x)\psi_j(x) - \phi_j(x)\psi_i(x)) \right]_{1 \leq i, j \leq 2m}, \tag{4.38}
\]
to evaluate the \( m \)-fold integral in the second line of equation (4.36):
\[
m! \text{Pf} \left[ \int_{-\pi}^{\pi} d\theta e^{2\sqrt{x} \cos \theta} e^{3i\theta} (e^{ik\theta} e^{i\theta} - e^{i\theta} ke^{ik\theta}) \right] = m!(2\pi)^m \text{Pf} \left[ (j - k)I_{3+k+j}(2\sqrt{x}) \right] \tag{4.39}
\]
where the indices \( (k, j) \) all run over \( -(m - 1/2), -(m - 3/2), \ldots, (m - 3/2), m - 1/2, \ldots, (m - 1/2) \), and we have used again the integral representation for the Bessel function (4.32).

Setting \( x = y/4 \) in (4.39) and simplifying all prefactors, we eventually get for equation (4.35)
\[
P_{(\nu=2m+1)}^{(j=1)}(y) = C_m y^{-m/2} e^{-y/8} \text{Pf} \left[ (j - k)I_{3+k+j}(\sqrt{y}) \right] \tag{4.40}
\]
where \( C_m \) is the normalization constant:
\[
C_m = \frac{\sqrt{\pi}(2m + 1)!!}{16\Gamma(m + 3/2)}. \tag{4.41}
\]
Applying now equation (4.13), we get the distribution in the Dirac picture (4.16) including its explicit normalization constant for any \( \nu \), which was previously unavailable.

The proof for the cumulative distribution easily follows from (4.8), and we only quote the result, which is new, after having identified \( c = 2 \) and \( \lambda = 4 \) in equation (4.36) instead:
\[
Q_{(\nu=2m+1)}^{(j=1)}(y) = e^{-y/8} \int_0^{(1/2)} (-; 2m; (y/4) \mathbb{1}_m) = 2^m y^{-m/2} e^{-y/8} \text{Pf} \left[ (j - k)I_{1+k+j}(\sqrt{y}) \right]. \tag{4.42}
\]
The range of indices is the same as in equation (4.39). Once again, deriving equation (4.42) directly from equation (4.40) using equation (4.5) is highly nontrivial.
4.6. Moments for large $N$

We point out a universal expression for the large $N$ decay of the $\ell$th moment of the smallest eigenvalue, valid for both $\beta = 1$ and $2$:

$$
\langle \lambda_{\text{min}}^{\ell}(FT) \rangle_{N,\nu,\beta} \sim \frac{\kappa_{\ell,\nu,\beta}}{(4N^3)^{\ell}}, \quad \langle \lambda_{\text{min}}^{\ell}(WL) \rangle_{N,\nu,\beta} \sim \frac{\kappa_{\ell,\nu,\beta}}{(4N)^{\ell}} \quad \text{for } N \to \infty
$$

(4.43)

where we have defined the universal coefficient

$$
\kappa_{\ell,\nu,\beta} = \int_0^\infty ds \, s^{2\ell} P_\nu^{(\beta)}(s).
$$

(4.44)

The scaling with different powers of $N$ trivially follows from the different spacings in the definition of the microscopic limit, equations (4.1) and (4.2), as we will show now. Starting from the definition of the average of the smallest eigenvalue for FTWL (2.18) we have

$$
\langle \lambda_{\text{min}}^{\ell}(FT) \rangle_{N,\nu,\beta} = \int_0^{1/N} dx \, x^{\ell} P_{N,\nu}^{(FT)}(x) = \frac{1}{(4N^3)^{\ell}} \int_0^{4N^2} dy \, y^{\ell} P_{N,\nu}^{(FT)} \left( \frac{y}{4N^3} \right)
$$

(4.45)

after making a change of variable $x = y/4N^3$. This implies, using the limit (4.1),

$$
\lim_{N \to \infty} \left[ (4N^3)^{\ell} \langle \lambda_{\text{min}}^{\ell}(FT) \rangle_{N,\nu,\beta} \right] = \int_0^\infty dy \, y^{\ell} P_\nu^{(\beta)}(y).
$$

(4.46)

The same argument for WL with a different change of variables, $x = y/4N$, leads to

$$
\lim_{N \to \infty} \left[ (4N)^{\ell} \langle \lambda_{\text{min}}^{\ell}(WL) \rangle_{N,\nu,\beta} \right] = \int_0^\infty dy \, y^{\ell} P_\nu^{(\beta)}(y).
$$

(4.47)

Alternatively the universal right-hand side can be expressed in the Dirac picture, using the map (4.13), as given in equation (4.44).

Let us give a few examples, by simply inserting equations (4.17) and (4.15) into the integral:

$$
\kappa_{\ell,\nu,0,\beta=1} = \frac{1}{3} \int_0^\infty ds \, 2^{\ell}(2 + s)e^{-s^2/8-s/2} = 2^{3\ell}[\sqrt{2}\ell\Gamma(1/2 + \ell)]_1 F_1(1/2 + \ell; 3/2; 1/2)
$$

$$
+ \Gamma(1 + \ell)(1 + \ell; 1/2; 1/2) - F_1(1 + \ell; 3/2; 1/2))
$$

(4.48)

$$
\kappa_{\ell,\nu,0,\beta=2} = \frac{1}{3} \int_0^\infty ds \, 2^{\ell+1} e^{-s^2/4} = 4^{\ell}\Gamma(1 + \ell).
$$

(4.49)

Specializing these results to the case of the first moment $\ell = 1$ (average value), we obtain

$$
\frac{1}{4}\kappa_{1,\nu,0,\beta=1} = 2 - 2\sqrt{2}e\pi \text{erfc} \left( 1/\sqrt{2} \right) \approx 0.688641 \ldots
$$

(4.50)

$$
\frac{1}{4}\kappa_{1,\nu,0,\beta=2} = 1,
$$

(4.51)

in complete agreement with [11,18] (after switching to their conventions). However, the general asymptotic relation (4.43) provided here allows us to derive analogous, new results for any $\nu$ for which the smallest eigenvalue in the Dirac picture is known.

We have numerically verified the scaling behavior above for the FTWL ensemble for the two cases $\nu = 1, 3$ in figure 4. We plot the combination $N^3 \langle \lambda_{\text{min}} \rangle_{N,\nu,1}$ as a function of $N$ for $\nu = 1, 3$ (left and right panels respectively), which nicely converge for $N \to \infty$. 

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Figure 4. The behavior of $N^3 \langle \lambda_{\text{min}} \rangle_{N, \nu, 1}^{(\text{FT})}$ for increasing $N$ and for $\nu = 1, 3$ (left and right panels respectively). The curves converge for $N \to \infty$ to the limiting values $(1/4)\kappa_{1,1,1} = 2$ (left) and $(1/4)\kappa_{1,3,1} = e^2 - 1 \approx 6.38906$ (right).

to the predicted values $(1/4)\kappa_{1,1,1} = 2$ and $(1/4)\kappa_{1,3,1} = e^2 - 1 \approx 6.38906$, where we have used (4.44) along with (4.16).

5. Conclusions

In this paper we addressed the smallest eigenvalue distribution in fixed-trace Wishart–Laguerre (FTWL) ensembles of random matrices, as well as its cumulative distribution and moments, and the corresponding quantity for unconstrained Wishart–Laguerre (WL) ensembles. Our motivation comes from the statistical description of entangled random pure states in bipartite systems of size $MN$, where the distribution of the smallest Schmidt eigenvalue provides useful information about the degree of entanglement of these states, and as a result has been subject to intense scrutiny in recent years.

In the first part, we derived new results for the FTWL ensemble with orthogonal symmetry. Here the constraint plays an important role, and we computed explicit expressions for the smallest eigenvalue by Laplace inverting a recursion relation given by Edelman for the unconstrained WL ensemble. In particular our results extend very recent expressions to even values of $\nu = M - N$. Examples were given for several even and odd values of $\nu$ and checked via numerical simulations.

In the second part, we studied the microscopic large $N$ limit at fixed $\nu$, for both orthogonal and unitary ensembles. Here the constraint (after a proper rescaling) is immaterial due to universality. We proved the equivalence of two different sets of results in the literature, given in terms of either hypergeometric functions of matrix argument (HFMA), or Pfaffians and determinants of Bessel functions. Our technical method was translating HFMA forms into their multiple-integral representations and using the classical Andréief–de Bruijn identities. Apart from this equivalence proof, our new results include the cumulative distributions, the universality of the smallest eigenvalue for $\nu = 0$ and an explicit expression for $\nu = 2$, all in the orthogonal symmetry class. In addition we have determined the precise asymptotic behavior of all moments for large $N$, in both symmetry classes.

Several open problems persist, in particular how to find a closed universal formula for all even values of $\nu$ in the orthogonal symmetry class in the microscopic limit. Furthermore, it would be interesting to extend these results to the symplectic symmetry.
class. Here a compact expression is only known for the special integer \( \nu = 0 \). The problem is the apparent lack of a suitable integral representation for HFMA required in order to proceed.

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Appendix A. Hypergeometric functions of matrix argument (HFMA)

The hypergeometric function matrix argument takes a complex symmetric matrix \((m \times m) \mathcal{X}\) as input and provides a real number as output. More precisely, let \( p \geq 0 \) and \( q \geq 0 \) be integers, and \( \{x_1, \ldots, x_m\} \) be the real eigenvalues of \( \mathcal{X} \). We have

\[
p F_{q}^{(\beta)}(a_1, \ldots, a_p; b_1, \ldots, b_q; \mathcal{X}) := \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} (a_1)^{\beta}_{\kappa} \cdots (a_p)^{\beta}_{\kappa} k!(b_1)^{\beta}_{\kappa} \cdots (b_q)^{\beta}_{\kappa} C^{(\beta)}_{\kappa}(\mathcal{X}), \tag{A.1}
\]

where \( \beta > 0 \) is a parameter, the symbol \( \kappa \vdash k \) means that \( \kappa = (\kappa_1, \kappa_2, \ldots) \) is a partition of \( k \) (i.e. \( \kappa_1 \geq \kappa_2 \geq \cdots \geq 0 \) are integers such that \( |\kappa| = \kappa_1 + \kappa_2 + \cdots = k \)), and the following symbol \( (a)^{\beta}_{\kappa} = \prod_{(i,j) \in \kappa} (a - (i - 1)/\beta + j - 1) \) is a generalized Pochhammer symbol.

In (A.1), \( C^{(\beta)}_{\kappa}(\mathcal{X}) \) is a Jack polynomial, i.e. a symmetric, homogeneous polynomial of degree \( |\kappa| \) in the eigenvalues \( \{x_i\} \) of \( \mathcal{X} \) [48,49]. Jack polynomials generalize the Schur function, the zonal polynomial and the quaternion zonal polynomial to which they reduce for \( \beta = 1, 2, 4 \) respectively. The HFMA (A.1) generalizes the ordinary hypergeometric function to which it reduces when \( m = 1 \).

The series (A.1) converges for any \( \mathcal{X} \) if \( p \leq q \); it converges if max \( |x_i| < 1 \) and \( p = q + 1 \); and it diverges if \( p > q + 1 \) unless it terminates. An efficient evaluation of HFMA is now made possible through an algorithm devised by Koev and Edelman [50], which we used extensively to check numerically all the results given in terms of HFMA in this paper.

Appendix B. The microscopic limit of \( p_{N,\nu=0}^{(FT)}(x) \) for \( \beta = 1 \)

In this appendix we take the microscopic limit in equation (4.1) of the smallest eigenvalue for \( \beta = 1 \) and \( \nu = 0 \) in the FTWL of equation (3.20):

\[
\lim_{N \to \infty} \frac{1}{4N^3} p_{N,\nu=0}^{(FT)} \left( \frac{y}{4N^3} \right) = \lim_{N \to \infty} \frac{N \Gamma(N) \Gamma(N^2/2)}{N^{3/2} \Gamma(N/2) \Gamma((N^2 + N - 2)/2)} \left( \frac{y}{4N^3} \right)^{-N/2} \\
\times \left( 1 - \frac{Ny}{4N^3} \right)^{(N^2 + N - 4)/2} \, _2F_1 \left( \frac{N + 2}{2}, \frac{N - 1}{2}; \frac{N^2 + N - 2}{2}; \frac{N - 4N^3}{y} \right). \tag{B.1}
\]

While the first factor in the second line yields an exponential prefactor,

\[
\lim_{N \to \infty} \left( 1 - \frac{y}{4N^2} \right)^{(N^2 + N - 4)/2} = e^{-y/8}, \tag{B.2}
\]

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the asymptotic limit of the hypergeometric function requires more care. Using

\[ 2F_1 \left( \frac{N+2}{2}, \frac{N-1}{2}, \frac{N^2 + N - 2}{2}; N - \frac{4N^3}{y} \right) \]

\[ = X^{(N+2)/2} \Gamma \left( \frac{N^2/2 + N/2 - 1}{2} \right) \Gamma \left( -\frac{3/2}{2} \right) \Gamma \left( \frac{N^2/2 - 2}{2} \right) 2F_1 \left( \frac{N}{2} + 1, \frac{N^2 - 1}{2}; \frac{5}{2}; X \right) \]

\[ + X^{(N+1)/2} \Gamma \left( \frac{N^2/2 + N/2 - 1}{2} \right) \Gamma \left( \frac{3/2}{2} \right) \Gamma \left( \frac{N^2/2 - 1/2}{2} \right) 2F_1 \left( \frac{N-1}{2}, \frac{N^2 - 2}{2}; -\frac{1}{2}; X \right) \]

(B.3)

with

\[ X = \left( \frac{4N^3}{y} - N + 1 \right)^{-1}. \]  

(B.4)

In order to apply the standard asymptotic limits, equations (9.121.9) and (9.121.10) in [51],

\[ \lim_{j, k \to \infty} 2F_1 \left( j, k; \frac{1}{2}; \frac{z^2}{4jk} \right) = \cosh(z) \]  

(B.5)

\[ \lim_{j, k \to \infty} 2F_1 \left( j, k; \frac{3}{2}; \frac{z^2}{4jk} \right) = \frac{\sinh(z)}{z}, \]  

(B.6)

where we identify \( j = N/2 \) and \( k = N^2/2 \), we still have to shift the third index of the two

hypergeometric functions in equation (B.3). Using equation (9.137.1) in [51] we obtain for the first

\[ 2F_1 \left( \frac{N}{2} + 1, \frac{N^2 - 1}{2}; \frac{5}{2}; X \right) \]

\[ = \frac{-1}{(N/2 - 1/2)(N^2/2 - 2)} X \left[ \frac{3}{4}(X - 1) 2F_1 \left( \frac{N}{2} + 1, \frac{N^2 - 1}{2}; \frac{1}{2}; X \right) \right. \]

\[ \left. + \frac{3}{2} \left( \frac{1}{2} + \left( \frac{N^2}{2} + \frac{N - 3}{2} \right) X \right) 2F_1 \left( \frac{N}{2} + 1, \frac{N^2 - 1}{2}; \frac{3}{2}; X \right) \right], \]  

(B.7)

and for the second

\[ 2F_1 \left( \frac{N-1}{2}, \frac{N^2 - 2}{2}; -\frac{1}{2}; X \right) = \frac{-1}{(1/2)(-1/2)(X - 1)} \]

\[ \times \left[ \left( \frac{N}{2} - 1 \right) \left( \frac{N^2 - 5}{2} \right) X 2F_1 \left( \frac{N-1}{2}, \frac{N^2 - 2}{2}; \frac{3}{2}; X \right) \right. \]

\[ \left. + \frac{1}{2} \left( \frac{1}{2} + \left( \frac{N^2}{2} + \frac{N - 5}{2} \right) X \right) 2F_1 \left( \frac{N-1}{2}, \frac{N^2 - 2}{2}; \frac{1}{2}; X \right) \right]. \]  

(B.8)

Collecting all the results from above and all factors we obtain as a final result

\[ \lim_{N \to \infty} \frac{1}{4N^3} p_{N,y=0}^{\text{FT}} \left( \frac{y}{4N^3} \right) = \frac{1}{4} e^{-y/8} \left[ \frac{1}{2} \left( \cosh \left( \frac{\sqrt{y}}{2} \right) - \frac{2}{\sqrt{y}} \sinh \left( \frac{\sqrt{y}}{2} \right) \right) \right. \]

\[ + \frac{1}{\sqrt{y}} \left( \cosh \left( \frac{\sqrt{y}}{2} \right) - \frac{\sqrt{y}}{2} \sinh \left( \frac{\sqrt{y}}{2} \right) \right) \right] = \frac{1}{8} \left( 1 + \frac{2}{\sqrt{y}} \right) e^{-y/8 - \sqrt{y}/2} \]  

(B.9)

which agrees with equation (4.22) as we have claimed.

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