A classification point-of-view about conditional Kendall’s tau

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Abstract

We show how the problem of estimating conditional Kendall’s tau can be rewritten as a classification task. Conditional Kendall’s tau is a conditional dependence parameter that is a characteristic of a given pair of random variables. The goal is to predict whether the pair is concordant (value of 1) or discordant (value of -1) conditionally on some covariates. We prove the consistency and the asymptotic normality of a family of penalized approximate maximum likelihood estimators, including the equivalent of the logit and probit regressions in our framework. Then, we detail specific algorithms adapting usual machine learning techniques, including nearest neighbors, decision trees, random forests and neural networks, to the setting of the estimation of conditional Kendall’s tau. A small simulation study compares their finite sample properties. Finally, we apply all these estimators to a dataset of European stock indices.

Keywords: conditional Kendall’s tau, conditional dependence measure, machine learning, classification task, stock indices.

1. Introduction

Beside linear correlations, most dependence measures between two random variables are functions of the underlying copula only: Spearman’s rho, Kendall’s tau, Blomqvist’s beta, Gini’s measure of association, etc. As a consequence, they are independent of the corresponding margins, that is seen as a positive point. See Joe \cite{Joe1997} and Nelsen \cite{Nelsen2006}, for instance. Such measures are well-known and widely used by practitioners. When some covariates are available, natural extensions of these tools can be defined, providing so-called “conditional” measures of dependence. In theory, it is sufficient to replace copulas by conditional copulas to obtain the “conditional version” of any dependence measure. This simple and fruitful idea has not yet been widely used in the literature. This is surely due to the relative novelty of the conditional copula concept, introduced by Patton \cite{Patton2006, Patton2009} a few years ago. Nonetheless, in a series of papers, Gijbels et al. \cite{Gijbels2009a, Gijbels2009b, Gijbels2010, Gijbels2011} have popularized this approach, with a focus on conditional Kendall’s tau and Spearman’s rho. Note that conditional dependence measures have been invoked in different frameworks, often without any explicit link with conditional copulas: truncated data (Tsai \cite{Tsai2003}, e.g.), multivariate dynamic models (Jondeau and Rockinger \cite{Jondeau2003}, Almeida and Czado \cite{Almeida2009}, among others), vine structures (So and Yeung \cite{So2010}), etc.

Now, let us introduce our key dependence measure: for each \( z \in \mathbb{R}^p \), the conditional Kendall’s tau of a bivariate random vector \( X := (X_1, X_2) \) given some covariates \( Z = z \) may be defined as

\[
\tau_{1,2|Z=z} = \mathbb{P}((X_{2,1} - X_{1,1})(X_{2,2} - X_{1,2}) > 0|Z_1 = Z_2 = z) - \mathbb{P}((X_{2,1} - X_{1,1})(X_{2,2} - X_{1,2}) < 0|Z_1 = Z_2 = z),
\]

where \( X_1 = (X_{1,1}, X_{1,2}) \) and \( X_2 = (X_{2,1}, X_{2,2}) \) are two independent versions of \( X \). To simplify, we will assume that the law of \( X \) given \( Z = z \) is continuous w.r.t. the Lebesgue measure, for every \( z \). This implies

\[
\tau_{1,2|Z=z} = 2 \mathbb{P}((X_{2,1} - X_{1,1})(X_{2,2} - X_{1,2}) > 0|Z_1 = Z_2 = z) - 1.
\]
A conditional Kendall’s tau belongs to the interval \([-1, 1]\) and reflects a positive \((\tau_{1.2}|Z=z > 0)\) or negative \((\tau_{1.2}|Z=z < 0)\) dependence between \(X_1\) and \(X_2\), given \(Z = z\). Contrary to correlations, it has the advantage of being always defined, even if some \(X_k, k = 1, 2\), has no finite second moments (when it follows a Cauchy distribution, for example).

Some estimators of conditional Kendall’s tau have already been proposed in the literature, either as a by-product of the estimation of conditional copulas - see Gijbels et al. [6] and Fermanian and Lopez [13] - or directly, as in Derumigny and Fermanian [14]. Nonetheless, to the best of our knowledge, nobody has yet noticed the relationship between conditional Kendall’s tau and classification methods.

Let us explain this simple idea. Denote \(W := 2 \times I\{ (X_{2,1} - X_{1,1})(X_{2,2} - X_{1,2}) > 0 \} - 1\) and

\[
\mathbb{P}((X_{2,1} - X_{1,1})(X_{2,2} - X_{1,2}) > 0|Z_1 = Z_2 = z) = \mathbb{P}(W = 1|Z_1 = Z_2 = z) := p(z).
\]

Actually, the prediction of concordance/discordance among pairs of observations \((X_1, X_2)\) given \(Z\) can be seen as a classification task of such pairs. If a model is able to evaluate the conditional probability of observing concordant pairs of observations, then it is able to evaluate conditional Kendall’s tau, and the former quantity is one of the outputs of most classification techniques. Therefore, most classifiers can potentially be invoked (for example linear classifiers, decision trees, random forests, neural networks and so on [14]), but applied here to pairs of observations.

Indeed, for every \(1 \leq i, j \leq n, i \neq j\), define \(W_{(i,j)}\) as

\[
W_{(i,j)} := 2 \times I\{ (X_{j,1} - X_{i,1})(X_{j,2} - X_{i,2}) > 0 \} - 1 = \begin{cases} 
1 & \text{if } (i, j) \text{ is a concordant pair}, \\
-1 & \text{if } (i, j) \text{ is a discordant pair}.
\end{cases}
\]

A classification technique will allocate a given couple \((i, j)\) into one of the two categories \(\{1, -1\}\) (or “concordant versus discordant”, equivalently), with a certain probability, given the value of the common covariate \(Z\).

Section 2 introduces a general regression-type approach for the estimation of conditional Kendall’s tau. Some asymptotic results of consistency and asymptotic normality are stated. In Section 3, we explain how some machine learning techniques can be adapted to deal with our particular framework, and we detail the corresponding algorithms. A small simulation study compares the small-sample properties of all these algorithms in Section 4. In Section 5, these techniques are applied to European stock market data. We evaluate to what extent the dependence between pairs of European stock indices may change with respect to different covariates. All proofs have been postponed into appendices.

2. Regression-type approach

Typically, a regression-type model based on conditional Kendall’s tau may be written as

\[
\tau_{1.2}|Z=z = g_0(z, \beta^*), \quad \forall z \in Z \subset \mathbb{R}^p,
\]

for some finite dimensional parameter \(\beta^* \in \mathbb{R}^p\) and some function \(g_0\). As a particular case, a single-index approach would be

\[
\tau_{1.2}|Z=z = g(\psi(z)^T \beta^*), \quad \forall z \in Z,
\]

where \(\psi : \mathbb{R}^p \to \mathbb{R}^p\) is known, and \(g\) may be known (parametric model) or not (semi-parametric model), as in [14]. In this section, we propose an inference procedure of \(\beta^*\) under (3) when the link function \(g\) is analytically known. This procedure will be based on the signs of pairs only, and not on the specific values of the vectors \(X_i\). Then, since inference will be based on the observations of \(W \in \{1, -1\}\), our model belongs to the family of limited-dependent variable methods. One difficulty will arise from the pointwise conditioning events \(Z_i = Z_j = z\), that will necessitate localization techniques. Actually, we will consider couples of observations \(X_i\) and \(X_j\) for which the associate covariates are close to a given value \(z\). Indeed, the relationship (3) does not define the dependence levels between every couple \((X_i, Z_i)\) and \((X_j, Z_j)\), \(i \neq j\),
but only between those that share the same covariate. If the variables $Z$ were discrete, we would consider a subset of couples such that $Z_i = Z_j$. In our case of continuous variables $Z$ (see below), the latter event does not occur almost surely, and some smoothing/localization techniques have to be invoked.

Let $K$ be a $p$-dimensional kernel and $(h_n)$ be a bandwidth sequence. The bandwidth will simply be denoted by $h$ and we set $K_h(z) = K(z/h)/h^p$. The log-likelihood associated to the observation $(W_{(i,j)}, Z_i, Z_j)$ given $Z_i = Z_j = z$ is

$$\ell_\beta(W_{(i,j)}, z) := \left(\frac{1+W_{(i,j)}}{2}\right) \log P_\beta(W_{(i,j)} = 1|Z_i = Z_j = z) + \left(\frac{1-W_{(i,j)}}{2}\right) \log P_\beta(W_{(i,j)} = -1|Z_i = Z_j = z).$$

In practice, when the underlying law of $Z$ is continuous, there is virtually no couple for which $Z_i = Z_j$. Therefore, we will consider a localized “approximated” log-likelihood, based on $(W_{(i,j)}, Z_i, Z_j)$ for all pairs $(i, j)$, $i \neq j$. It will be defined as the double sum

$$L_n(\beta) := \frac{1}{n(n-1)} \sum_{i,j; i \neq j} K_h(Z_i - Z_j) \ell_\beta(W_{(i,j)}, \tilde{Z}_{i,j}),$$

for any choice of $\tilde{Z}_{i,j}$ that belongs to a neighborhood of $Z_i$ or $Z_j$. We will assume that $K$ is a compactly supported $p$-dimensional kernel of order $m \geq 2$.

The most obvious choices would be to select $\tilde{Z}_{i,j}$ among $\{Z_i, Z_j, (Z_i + Z_j)\}$. Here, we propose

$$L_n(\beta) := \frac{1}{n(n-1)} \sum_{i,j; i \neq j} K_h(Z_i - Z_j) \ell_\beta(W_{(i,j)}, Z_i)$$

$$= \frac{1}{n(n-1)} \sum_{i,j; i \neq j} K_h(Z_i - Z_j) \left\{ \left(\frac{1+W_{(i,j)}}{2}\right) \log \left(\frac{1}{2} + \frac{1}{2} g(\psi(Z_i)^T \beta)\right) \right. \right.$$

$$+ \left. \left. \left(\frac{1-W_{(i,j)}}{2}\right) \log \left(\frac{1}{2} - \frac{1}{2} g(\psi(Z_i)^T \beta)\right) \right\},$$

under (5). We can therefore derive an estimator of $\beta^*$ based on the maximization of the latter function, with a $\ell_1$ penalty (Lasso-type estimator), as

$$\hat{\beta} := \arg \max_{\beta \in \mathbb{R}^p} L_n(\beta) - \lambda_n |\beta|_1,$$

(4)

where $\lambda_n$ is a tuning parameter to be chosen. Note that $L_n(\beta)$ is not really a likelihood function since the observations $(W_{(i,j)}, Z_i, Z_j)$ for every couple $(i, j)$, $i \neq j$, are not mutually independent.

If $K \geq 0$, the objective function is a concave function of $\beta$ if it satisfies

$$\delta g''(t)(1 + \delta g)(t) \leq (g'(t))^2, \forall t,$$

(5)

for $\delta \in \{1, -1\}$. When $\beta \mapsto L_n(\beta)$ is concave, the penalized criterion above is concave too and the calculation of $\hat{\beta}$ can be led in practical terms through convex optimization routines, even with a large number of regressors ($p' > 1$). Since this will be our framework, we will show that (5) holds for some usual classification techniques. When it is not the case, we have to rely on other optimization schemes and to avoid considering too many regressors.

Moreover, note that, when $g$ is odd (i.e. $g(-t) = -g(t)$), the estimator simply becomes

$$\hat{\beta} := \arg \max_{\beta \in \mathbb{R}^p} \frac{1}{n(n-1)} \sum_{i,j; i \neq j} K_h(Z_i - Z_j) \log \left(\frac{1}{2} + \frac{1}{2} g(W_{(i,j)}\psi(Z_i)^T \beta)\right) - \lambda |\beta|_1.$$

(6)

\[3\]
Example 1 (Logit). If we choose the Fisher transform \( g(t) = (e^t - 1)/(e^t + 1) \), then \( g \) is odd and the optimization program becomes:

\[
\hat{\beta} := \arg \max_{\beta \in \mathbb{R}^p} \frac{1}{n(n-1)} \sum_{i,j \neq j} K_k(Z_i - Z_j) \log (g(W_{ij}\psi(Z_i)^T \beta) - \lambda|\beta|_1),
\]

where the so-called logit link function is defined by \( g(x) = e^x/(1 + e^x) \). Therefore \( \hat{\beta} \) can be seen as the maximizer of the log-likelihood associated to \( g(\epsilon|z)^T \beta) \), given some explanatory variables \( Z_i \). On a practical side, when \( K \geq 0 \), the \( \beta \)-criterion is concave. This allows to use the existing software and optimization routines of logistic regression without many changes.

Example 2 (Probit). Similarly, choosing \( g(t) = 2\Phi(t) - 1 \), where \( \Phi \) denotes the cdf of the standard normal distribution, yields the equivalent of a (weighted) probit regression. Indeed, this function \( g \) is odd, \( \{\} \) applies in this case and our criterion in \( [3] \) is concave wrt \( \beta \).

Let us assume that a family of models or some statistical procedure allow the calculation of the functional link \( g(\epsilon|z)^T \beta) \) and then \( p(z) \), for any \( z, \epsilon \in \{-1, 1\} \) and any given value \( \beta \). Logit, Probit, regression trees, neural networks, etc. Then, we can estimate the “true” parameter \( \beta^* \) by \( \hat{\beta} \), as given by \( [3] \), in practical terms.

Now, we state the asymptotic properties of \( \hat{\beta} \), under the assumption that \( \beta \to L_{\infty}(\beta) \) is concave. To this goal, we introduce some notations.

For any \( x \) and \( y \in \mathbb{R}^p \), denote

\[
p(x, y) := P_{\beta^*}((X_{1,1} - X_{1,1})(X_{2,2} - X_{1,2}) > 0|Z_1 = x, Z_2 = y).
\]

The latter expectations are calculated when the underlying parameter is assumed to be the true value \( \beta^* \). Note that \( p(x) := p(x, x) \) and \( 2p(z) - 1 = \gamma_{1,2}|z = z \). Moreover, for any \( x, y \) and \( z \in \mathbb{R}^p \), set

\[
p(x, y, z) := P_{\beta^*}((X_{1,1} - X_{1,1})(X_{2,2} - X_{1,2}) > 0, (X_{3,1} - X_{1,1})(X_{3,2} - X_{1,2}) > 0|Z_1 = x, Z_2 = y, Z_3 = z).
\]

This is the conditional probability that \( X_1, X_2 \) and \( X_3 \) are concordant, given their respective covariates.

Denote, for any \( \beta \in \mathbb{R}^p \),

\[
\phi(x, y, \beta) := p(x, y) \log(q(x, \beta)) + (1 - p(x, y)) \log(1 - q(x, \beta)), q(x, \beta) := 1/2 + g(\psi(x)^T \beta)/2.
\]

Note that \( q(z, \beta^*) = p(z) \). Finally, for any real function \( f \) and \( \epsilon > 0 \), denote by \( f_\epsilon \) the function \( x \to \sup_{|x - t| < \epsilon} |f(t)| \).

Regularity assumption R0: The density \( f_Z \) of \( Z \) is assumed to be \( m \)-times continuously differentiable. Moreover, the functions \( \phi(x, \cdot, \beta) \) and \( q(\cdot, \beta) \) are continuous for any \( x \in \mathbb{Z} \) and any \( \beta \in \mathbb{R}^p \). To simplify, \( (x, y, z) \to p(x, y, z) \) will be continuous on \( \mathbb{Z}^3 \).

Theorem 3. Under R0, \( \{A.2\} \) and \( \{A.3\} \) in Appendix A, if \( \lambda_n \to \lambda_\infty \) and \( n^2 h^p \to \infty \) when \( n \to \infty \), if the true model is given by \( [3] \) and \( \beta \to L_{\infty}(\beta) \) is concave, then the solution \( \hat{\beta} \) of \( [3] \) tends in probability towards \( \beta^* := \arg \max_{\beta} L_{\infty}(\beta) - \lambda_\infty|\beta|_1 \), where

\[
L_{\infty}(\beta) := \int \phi(z, z, \beta)f_Z^2(z) \, dz.
\]

In particular, when \( \lambda_\infty = 0 \), the estimator \( \hat{\beta} \) tends to \( \arg \max_{\beta} L_{\infty}(\beta) = \beta^* \), because \( \phi(z, z, \beta) \) is the expected log-likelihood associated to \( W_{(1,2)} \) given \( Z_1 = Z_2 = z \). Thus, for every z, the latter quantity is maximal when \( \beta = \beta^* \).
we can explicitly write (and code) the link function $g$ cause it is far from obvious to guess the right functional form of $\algorithm$ will "automatically" evaluate $\tau$.

Therefore, instead of estimating an assumed parametric model by penalization, as in (4), a classification of our conditional Kendall’s tau model, we recall the estimation of $\tau$-categories (concordant or discordant), given a common value $t$.

$\hat{\tau}$ simply be $\hat{\tau}$

3. Classification algorithms and conditional Kendall’s tau

In the latter section, we have studied a localized likelihood procedure to estimate $\beta^*$ under (3), when we can explicitly write (and code) the link function $g$. This may be seen as a restrictive approach, because it is far from obvious to guess the right functional form of $g$. To improve the level of flexibility of our conditional Kendall’s tau model, we recall the estimation of $\tau_{1,2|i}$ is similar to the evaluation of $\mathbb{P}(d_{1,2} = 1|Z_1 = Z_2 = z)$, i.e. the probability $p(z)$ of classifying the couple $(1, 2)$ into one of two categories (concordant or discordant), given a common value $z$ of their covariates. Formally, the answer of such a question can be directly yielded by some classification algorithms. This is the topic of this section. Therefore, instead of estimating an assumed parametric model by penalization, as in (3), a classification algorithm will “automatically” evaluate $p(z)$ by $\hat{p}(z)$. An estimator of the conditional Kendall’s tau will simply be $\hat{\tau}_{1,2|Z=z} := 2\hat{p}(z) - 1$.

Now, we show how different classification algorithms can be used and adapted to the estimation of $\tau_{1,2|Z=z}$ in practice. The first step is to transform the dataset $\mathcal{D} = (X_{1,1}, X_{1,2}, Z_{1}, \ldots, X_{n,1}, X_{n,2}, Z_{n}) \in (\mathbb{R}^{2+p})$, called the initial dataset, into an object $\hat{\mathcal{D}}$, that will be called the dataset of pairs (see Algorithm 1). Each element of this dataset of pairs is indexed by an integer $k \in \{1, \ldots, n(n-1)/2\}$, which corresponds to any (unordered) pair $(i, j)$, $i \neq j$, of observations in the initial dataset.

For any pair of observations, we compute the associated covariate $\hat{Z}_k$ which is just the average of the two covariates $Z_i$ and $Z_j$ (contrary to Section 2 where we have chosen $Z_i$). Note that we want $Z_i$ and $Z_j$ to be close to each other, so that the pair $(i, j)$ is relevant. This means that a weight variable $V_k$ is defined for any pair. It is related to the proximity between $Z_i$ and $Z_j$. Obviously, if $V_k = 0$ then the corresponding pair is not kept, finally. This selection induces also a computational benefit, by reducing the size of the dataset and the computation time. For example, suppose that $n = 4000$. Then, up to around $8 \times 10^6$ possible pairs can be constructed but only a small group of them (around $10^4$ or $10^5$ pairs, typically) will be relevant. The others are pairs for which the covariates are considered too far apart. Note that, in order to increase the proportion of $k$ such that the weight $V_k$ is zero, it is sufficient to replace any kernel $K(\cdot)$ by $\tilde{K}(\cdot) := K(\cdot) \mathbb{I}\{|\cdot| \leq \eta\}$ for some level $\eta > 0$, and to normalize these new weights so that their sum is equal to one.

3.1. The case of probit and logit classifiers

With the new dataset $\hat{\mathcal{D}}$, we can virtually apply any classification method to predict the concordance value $W_k$ of the pair $k$, given the covariate $\hat{Z}_k$ and the weight $V_k$. The logit and probit models yield some of the oldest and easiest methods in classification. They have straightforward adapted versions in our case: see Algorithm 2. These weighted penalized GLM procedures are estimated using the R package ordinalNet [16]. Note that we are still estimating $\tau_{1,2|Z=z}$ under the parametric model given by (3).
3.2. Decision trees and random forests

Now, let us discuss how partition-based methods can be used for the estimation of the conditional Kendall’s tau. Strictly speaking, such techniques are parametric: the relationship (2) implicitly applies, but for some complex untractable function $g$. And the parameter $\beta^*$ is related to some covariate thresholds, typically. Nonetheless, a classical decision tree can be directly trained on the weighted dataset $\tilde{D}$. We use the R package \texttt{tree} by Ripley \[17\], following Breiman et al. [18]. Therefore, the application of decision trees to our framework is straightforward, and do not require any special adaptation, contrary to random forests. And the tree procedure allows the calculation of the probability of observing a concordant pair, given any common value of $Z$.

In a classical classification setting, random forests are techniques of aggregation of decision trees that are built on a subset of samples and subsets of variables. More precisely, a typical random forest algorithm is the following: sample 80% of the rows of the dataset (without replacement), and 80% of the explanatory variables; estimate a tree on this, and repeat this procedure a certain number of times, with different subsamples every time. In our framework, it is not clear at which level subsampling should take place. The easiest solution would be to directly plug-in the dataset of pairs $\tilde{D}$ into a classical random forest algorithm, but it does not obviously lead to the best solution. Indeed, noting that aggregation of trees is useless if all trees are identical, it seems that the more variability in the input of the trees, the better. Following this idea, we have noticed that the observations in the dataset of pairs are not independent. For example, the pair $(1,2)$ is usually not independent of the pair $(1,3)$, because they both share the first observation $(X_{1,1},X_{1,2},Z_{1})$. Therefore, to increase the diversity of inputs in the different trees, we suggest to lead a first sampling $S_j$ on the initial dataset, and then to build a dataset of pairs on the sampled observations $D_j := (X_{i,1},X_{i,2},Z_i)_{i \in S_j}$ (see Algorithm 3). As a matter of fact, if for example the first observation does not belong to the sample $S_j$, then the dataset $D_j$ and the estimated tree $T_j$ become both independent.

\[\text{Algorithm 1: Algorithm for creating the dataset of pairs from the initial dataset.}\]

\begin{verbatim}
Input: Initial dataset $D = (X_{i,1},X_{i,2},Z_i)_{i=1,\ldots,n} \in (\mathbb{R}^{2+p})^n$;

$k \leftarrow 0$;

for $i \leftarrow 1$ to $(n-1)$ do

for $j \leftarrow (i+1)$ to $n$ do

$\tilde{Z}_k \leftarrow (Z_j + Z_i)/2$;

$W_k \leftarrow W_{(i,j)}$ as defined in Equation (11);

$V_k \leftarrow K_k(Z_i - Z_j)$;

$k \leftarrow k + 1$;

end

end

Define $K := \{k : V_k > 0\}$;

Output: A dataset of pairs $\tilde{D} := (W_k,\tilde{Z}_k,V_k)_{k \in K} \in \{-1,1\} \times \mathbb{R}^p \times \mathbb{R}_+^{n(n-1)/2}$.
\end{verbatim}

\[\text{Algorithm 2: Estimation of the conditional Kendall’s tau $\tau_{1,2|Z=z}$ using a logit (resp. probit) regression.}\]

\begin{verbatim}
Input: A dataset of pairs $\tilde{D} := (W_k,\tilde{Z}_k,V_k)_{k \in K}$
Input: A point $z \in Z$, a function $\psi$ and a penalty level $\lambda$;

Compute the usual weighted penalized logit (resp. probit) estimator $\hat{\beta}$ on the dataset $(W_k;\psi(\tilde{Z}_k),V_k)_{k \in K}$ with a tuning parameter $\lambda$;

Output: An estimator $\hat{\tau}_{1,2|Z=z} := (e^{\psi(z)^T\hat{\beta} - 1}/(e^{\psi(z)^T\hat{\beta}} + 1))$
(resp. $\hat{\tau}_{1,2|Z=z} := 2\Phi(\psi(z)^T\hat{\beta} - 1)$).
\end{verbatim}

\[\text{Algorithm 3: Estimation of the conditional Kendall’s tau $\tau_{1,2|Z=z}$ using a weighted penalized logit (resp. probit) regression.}\]

\begin{verbatim}
Input: A dataset of pairs $\tilde{D} := (W_k,\tilde{Z}_k,V_k)_{k \in K}$
Input: A point $z \in Z$, a function $\psi$ and a penalty level $\lambda$;

Define $K := \{k : V_k > 0\}$;

Output: A dataset of pairs $\tilde{D} := (W_k,\tilde{Z}_k,V_k)_{k \in K} \in \{-1,1\} \times \mathbb{R}^p \times \mathbb{R}_+^{n(n-1)/2}$.
\end{verbatim}
of this first observation \((X_{1,1}, X_{1,2}, Z_1)\). This independence property makes the trees less dependent, and significantly improves the performance in our results.

### Algorithm 3: Random forests adapted for the estimation of the conditional Kendall’s tau

**Input:** Initial dataset \(D = (X_{i,1}, X_{i,2}, Z_i)_{i=1,...,n} \in (\mathbb{R}^{2+p})^n\);

for \(j \leftarrow 1\) to \(N_{\text{tree}}\) do
- Sample a set \(S_j \subset \{1,\ldots,n\}\) without replacement;
- \(D_j \leftarrow (X_{i,1}, X_{i,2}, Z_i)_{i \in S_j}\);
- Compute the dataset of pairs \(\tilde{D}_j = (W_k, \tilde{Z}_k, V_k)_{k \in K_j}\) using Algorithm 2 on \(D_j\), providing \(K_j\);
- Sample a set \(S'_j \subset \{1,\ldots,p'\}\) without replacement;
- Estimate a classification tree \(T_j\) on the dataset \((W_k, (\psi_l(\tilde{Z}_k))_{l \in S'_j}, V_k)_{k \in K_j}\).
end

**Output:** An estimator \(\hat{\tau}_{1,2|Z=\cdot} := N_{\text{tree}}^{-1} \sum_{i=1}^{N_{\text{tree}}} T_j(\cdot)\).

### 3.3. Nearest neighbors

The nearest neighbors provide also a very popular classification algorithm and it can be used directly on the dataset \(\tilde{D}\) (see Algorithm 4). Here, we no longer assume \((\cdot)\) or even \((\cdot)\), and we live in a nonparametric framework. A pretty difficult problem is to choose a convenient number of nearest neighbors. As usual in nonparametric statistics, we must find a compromise between variance (tendency to undersmooth, i.e. to choose a too small \(N\)) and bias (tendency to oversmooth, i.e. to choose a too big \(N\)). Moreover, in our case, with \(n(n-1)/2\) possible pairs, choosing a right value for \(N\) can be challenging. Indeed, in the usual (iid) nearest neighbor framework, the asymptotically optimal \(N\) is a power of the sample size. Here, this is different because there are three potential sample sizes: \(n\), if we consider there are fundamentally \(n\) sources of randomness, \(n(n-1)/2\) by considering that the new sample has a cardinality equal to the number of pairs, or even \(|K|\) that is random and depends on \(h\). Thus, our problem is to choose a “relevant formula” for \(N\) based on the “convenient” sample size.

### Algorithm 4: Estimation of the conditional Kendall’s tau \(\tau_{1,2|Z=\cdot}\) using nearest neighbors.

**Input:** A dataset of pairs \(\tilde{D} := (W_k, \tilde{Z}_k, V_k)_{k \in K}\)

**Input:** A point \(z \in Z\), a number \(N\) of nearest neighbors and a distance \(d\) on \(\mathbb{R}^{p'}\);

\[ K_z \leftarrow \arg \min_{E \subseteq K, |E|=N} \left( \sum_{k \in E} d(\psi(z), \psi(\tilde{Z}_k)) \right) \];

**Output:** An estimator \(\hat{\tau}^{(N)}_{1,2|Z=\cdot} := (\sum_{k \in K_z} V_k W_k) / \sum_{k \in K_z} V_k \).

In applications, one might not be interested in the value of the conditional Kendall’s tau at only one point, but also in the whole function \(z \mapsto \tau_{1,2|Z=\cdot}\). The goodness of this estimation is linked to the underlying density \(f_Z\) of \(Z\): the estimation can be made more precise in regions where \(f_Z\) is high, allowing to use a higher number of neighbors with close covariates. At the opposite, in regions where \(f_Z\) is low, a smaller \(N\) should be used. Note that, in general, \(f_Z\) is unknown and its estimation may be difficult as well, due to the curse of dimensionality. Therefore, it is highly desirable to build a local number of neighbors \(N(z)\). Such a local choice \(N(z)\) will help to avoid both under- and over-smoothing in all parts of the space \(Z\).

Cross-validation techniques are widely use for the choice of tuning parameters, but might not the best solution here as one would like to find a local choice of \(N\). This problem has similarities with the classical non-parametric regression, and we propose to use a procedure inspired by Lepski’s method for choosing the bandwidth, once adapted to our setting. Lepski’s method is built on a simple principle: when two non-parametric estimators are close, the best is the smoothest. When two non-parametric estimators are
far apart, the best is the least smooth. Let \((Z_i)_{i \in \mathcal{I}}\) be a partition of \(Z\). The goal will be to choose the best estimator on each \(Z_i\), which corresponds to the choice of a local number of nearest neighbors \(N_i\). This procedure is called “local” since the diameters of the \(Z_i\) will be small. For example, if \(p = 1\) and \(Z\) is a bounded interval then the \(Z_i\) can be chosen as small intervals. We denote by \(\mathcal{N} \subset \mathbb{N}\) the finite set of possible numbers of neighbors. Following Lepski’s approach, we choose \(\mathcal{N}\) as a geometric progression, i.e. \(\mathcal{N} = \{[C_1 \times C_2^{i}], i = 1, \ldots, i_{\text{max}}\}\) for some constants \(C_1, C_2 > 0\), where \([x]\) denotes the integer part of a real \(x\).

To measure how far the estimators are from each other, we introduce a distance \(d_i\), \(i \in \mathcal{I}\). As our estimators of conditional Kendall’s tau are bounded (between \(-1\) and \(1\)) and measurable, several choices are possible. In applications, we will use

\[
d_i(f, g) = \left( \frac{1}{M_{\text{max}}} \sum_{j=1}^{M_{\text{max}}} \left( \left( f(z_{i,j}) - g(z_{i,j}) \right)/M \right)^2 \right)^{1/2}, z_{i,1}, \ldots, z_{i,M_{\text{max}}} \in Z, \tag{7}
\]

where \(M\) is a normalization factor independent of \(i\) and the subsets \(z_{i,1}, \ldots, z_{i,M_{\text{max}}}\) are arbitrarily chosen in \(Z_i\), \(i = 1, \ldots, i_{\text{max}}\). We will use \(M = (\max - \min)\tau_i^{(N)}\|z_{1,2|z=x}, N \in \mathcal{N}, z \in Z\). Indeed, in the classical nonparametric regression \(Y = f(X) + \varepsilon\), with unknown \(f\) to estimate, \(M\) should be replaced by the standard deviation of the noise \(\varepsilon\). In our case, we can define a (pseudo-)noise \(\xi_{z,N} := \tau_i^{(N)} - \tau_i^{(N)}|z=x,\) but it is unknown in practice and its distribution is complicated. Therefore \(M\) serves as a proxy of the amplitude of the variations in the estimated conditional Kendall’s tau. This normalization by \(M\) ensures a kind of adaptivity of the estimation.

**Algorithm 5**: Lepski’s method for a local choice of the number of nearest neighbors, and the corresponding estimator of the conditional Kendall’s tau.

**Input**: A set \(\mathcal{N} \subset \mathbb{N}\) of possible numbers of nearest neighbors and the corresponding estimates \(\tau_i^{(N)}\), given by Algorithm 4 for all \(N \in \mathcal{N}\);

**Input**: A partition \((Z_i)_{i \in \mathcal{I}}\) of \(Z\) and a distance \(d_i\) on a space of bounded measurable real functions defined on \(Z_i\), for every \(i \in \mathcal{I}\);

**foreach** \(i \in \mathcal{I}\) **do**

\[
S_i \leftarrow \left\{ N \in \mathcal{N} : d_i \left( \tau_i^{(N)}|z=x, \tau_i^{(N')}|z=x \right) \leq A \sqrt{(1/N') \log(\max(\mathcal{N})/N'), \forall N' \in \mathcal{N} \cap [1, N]} \right\}, \tag{8}
\]

\(N_i \leftarrow \max(S_i);\)

**end**

**Output**: An estimator \(z \mapsto \tau_i^{(N)}|z=x := \sum_{i \in \mathcal{I}} \mathbb{1}(z \in Z_i) \tau_i^{(N)}|z=x\).

### 3.4. Neural networks

Nowadays, neural networks have become very popular with a wide range of applications. In classification problems, a neural network can be seen as an estimator depending on some parameters, but in a very flexible and complex way. For every input \(z\), it yields the probability of belonging to any class. In our framework, we will train a network on the dataset of pairs \(\mathcal{D}\). It is well-known that most neural networks do not induce convex programs, and the outputs therefore depend on some initial parameter values. One strategy is to independently train networks with different starting parameter values, that may be randomly chosen, for example.

This method of using independent estimators (conditionally on the initial sample \(\mathcal{D}\)) and then aggregating them is related to the random forest approach of the previous section and the discussion therein. Therefore, the same techniques are relevant and we have noticed an improvement in performance by using an adapted
version of Algorithm 3. More precisely, we fix a number of neural networks. For each neural network, we sample without replacement a part of the initial dataset from which the corresponding dataset of pairs is constructed and used as a training set. In order to improve stability, we aggregate the predictions of the different neural networks by using the median as the final output. The precise choice of the best architecture of the network is a complicated task, which is left for future research. As we are looking for functions \( z \mapsto \tau_{1,2|Z=z} \) which are smooth almost everywhere and easy to interpret in applications, we choose a simple architecture with 10 neural networks, each having one hidden layer of 3 neurons.

### Algorithm 6: Neural networks with median bagging, adapted for the estimation of the conditional Kendall’s tau

| Input: Initial dataset \( D = (X_{i,1}, X_{i,2}, Z_i)_{i=1,...,n} \in (\mathbb{R}^{2+p})^n \); |
| for \( j \leftarrow 1 \) to \( N_{\text{net}} \) do |
| Sample a set \( S_j \subset \{1, \ldots, n\} \) without replacement; |
| \( D_j = (X_{i,1}, X_{i,2}, Z_i)_{i \in S_j} \); |
| Compute the dataset of pairs \( \tilde{D}_j = (W_k, \tilde{Z}_k, V_k)_{k \in \mathcal{K}_j} \) using Algorithm 4 on \( D_j \), providing \( \mathcal{K}_j \); |
| Estimate a neural net \( \mathfrak{R}_j \) on the dataset \( (W_k, \psi(\tilde{Z}_k), V_k)_{k \in \mathcal{K}_j} \); |
| \( \text{end} \) |
| Output: An estimator \( \hat{\tau}_{1,2|Z=} := \text{Median}\{\mathfrak{R}_j(), j = 1, \ldots, N_{\text{net}}\} \). |

### 4. Simulation study

In this section, we have studied the relative performances of our estimators by simulation. For a given model and a given method of estimation, we sample 100 different experiments, and estimate the model for each sample. We fix the sample size as \( n = 3000 \). We remark that for a given dimension \( p > 0 \) of \( Z \) and a given support \( \mathcal{Z} \) of \( Z \), we have different “blocks” of the model which can be chosen in an independent way:

1. the law \( \mathbb{P}_{Z} \) of \( Z \);  
2. the function \( z \in \mathcal{Z} \mapsto \tau_{1,2|Z=z} \); 
3. the (conditional) copula family \( (C_{\tau})_{\tau \in (0,1)} \) or \( (C_{\tau})_{\tau \in (-1,1)} \) of \( (X_1, X_2)|Z=z \), indexed by its conditional Kendall’s tau. For example the Gaussian, Student, Clayton, Gumbel, etc, copula families. Such a family can also depend on \( Z \); for example, think of a Student copula with varying degrees of freedom; 
4. the conditional margins \( X_i|Z \) and \( X_j|Z \);  
5. the choice of the functions \( \psi_i \), for \( i = 1, \ldots, p' \); 
6. the choice of the estimator \( \hat{\tau}_{1,2|Z=\cdot} \).

Our so-called “reference setting” will be defined as \( p = 1 \), \( \mathcal{Z} = [0,1] \) and (i) \( \mathbb{P}_{Z} = \mathcal{U}_{[0,1]} \); (ii) \( \tau_{1,2|Z=z} = 3z(1-z) \); (iii) \( (C_{\tau})_{\tau \in (0,1)} = \text{GaussianCopula} \); (iv) \( \mathbb{P}_{X_i|Z=z} = \mathbb{P}_{X_j|Z=z} = \mathcal{N}(z,1) \). For each tested model, the performance of the estimator will be evaluated by the mean integrated \( \ell_2 \) error. With obvious notations, it will be estimated as

\[
\text{Err} := \mathbb{E} \left[ \int_{\mathcal{Z}} (\hat{\tau}_{1,2|Z=z} - \tau_{1,2|Z=z})^2 dz \right] \approx \frac{1}{N_{\text{simu}} N_{\text{points}}} \sum_{i=1}^{N_{\text{simu}}} \sum_{j=1}^{N_{\text{points}}} (\hat{\tau}_{1,2|Z=z(i)} - \tau_{1,2|Z=z(i)})^2, \tag{9}
\]

where \( N_{\text{simu}}, N_{\text{points}} \) are positive integers, \( z^{(1)}, \ldots, z^{(N_{\text{points}})} \) are fixed points in \( \mathcal{Z} \), and \( \hat{\tau}_{1,2|Z=z(i)} \) is the estimated conditional Kendall’s tau at point \( z^{(i)} \) trained on data from the \( i \)-th simulation. We choose
$N_{simu} := 100$ experiments, and in this reference setting, the integral is discretized with $N_{points} := 100$ points equispaced on the segment $[0.01, 0.99]$ to avoid boundary numerical problems.

In the following simulations, “Logit” and “Probit” refer to Algorithm 2. “Tree” refers to the application of the method `tree()` of package `tree` by Ripley [17] on the dataset $\tilde{D}$ produced by Algorithm 1. “Random forests” refers to Algorithm 3. “Nearest neighbors” refers to the adapted version using Algorithm 4 once aggregated using Algorithm 5. Finally “Neural networks” refers to Algorithm 6. Such specifications are now part of our “reference setting”.

4.1. Choice of $\{\psi_i\}, i = 1, \ldots, p'$.

We consider six different choices of $\psi_i$, that are

1. No transformation, i.e. $\psi^{(1)}(z) = z$ ;
2. Polynomials of degree lower than 4: $\psi^{(2)}_i(z) = 2^{-i+1}(z - 0.5)^{i-1}$ for $i = 1, \ldots, 5$ ;
3. Polynomials of degree lower than 10: $\psi^{(3)}_i(z) = 2^{-i+1}(z - 0.5)^{i-1}$ for $i = 1, \ldots, 11$ ;
4. Fourier basis of order 2 with an intercept: $\psi^{(4)}_1(z) = 1$, $\psi^{(4)}_2(z) = \cos(2\pi i z)$ and $\psi^{(4)}_{2i+1}(z) = \sin(2\pi i z)$ for $i = 1, 2$ ;
5. Fourier basis of order 5 with an intercept: $\psi^{(5)}_1(z) = 1$, $\psi^{(5)}_2(z) = \cos(2\pi i z)$ and $\psi^{(5)}_{2i+1}(z) = \sin(2\pi i z)$ for $i = 1, \ldots, 5$ ;
6. Concatenation of $\psi^{(2)}$ and $\psi^{(4)}$, which will be denoted by $\psi^{(6)}$.

For each of the choices of $\psi$ above, and each estimator, we compute the criteria (9). The results are displayed in the following Table 1.

| Chosen $\psi$ | Logit | Probit | Tree | Random forests | Nearest neighbors | Neural network |
|---------------|-------|--------|------|----------------|------------------|---------------|
| $\psi^{(1)}$  | 48.1  | 48.1   | 7.5  | 4.89           | 2.26             | 0.561         |
| $\psi^{(2)}$  | 0.721 | 0.554  | 4.28 | 3.28           | 2.26             | 1.32          |
| $\psi^{(3)}$  | 0.663 | 0.528  | 4.13 | 3.41           | 2.23             | 1.73          |
| $\psi^{(4)}$  | 1.41  | 1.45   | 4.73 | 14.2           | 2.72             | 1.74          |
| $\psi^{(5)}$  | 1.05  | 1.06   | 4.76 | 10.3           | 2.79             | 2.67          |
| $\psi^{(6)}$  | 0.456 | 0.434  | 4.57 | 3.15           | 2.64             | 3.87          |

Table 1: Error criteria (9) for each choice of $\psi$ and each method, multiplied by 1000.

Neural networks and nearest neighbors globally provide the best results in most circumstances. When the true model is close to the Logit/Probit specifications, the latter methods are obviously the best ones, but they quickly deteriorate under stronger misspecifications, as when there is a single function $\psi^{(1)}$. Beside, trees and random forests yield a fairly good job.

From now on, we will choose $\psi^{(6)}$ for the methods `logit`, `probit`, `tree` and `random forests`. Indeed, for these methods, the choice of $\psi$ yield nearly the lowest error criteria and present the advantage of being diverse, which will help to combine the performances of both polynomials and oscillating functions. On the contrary, for the methods `nearest neighbors` and `neural networks`, we choose $\psi^{(1)}$ as adding new functions does not seem to increase the performance of both of these methods. Figure 4 displays a comparison of the different methods on a typical simulated sample.
Figure 1: An example of the estimation of the conditional Kendall’s tau using different estimation methods (see Table 2). The black dash-dotted curve is the true conditional Kendall’s tau that has been used in the simulation experiment.

| Method          | Logit | Probit | Tree | Random forests | Nearest neighbors | Neural network |
|-----------------|-------|--------|------|----------------|-------------------|----------------|
| Algorithm       | Algorithm 2 | Algorithm 2 | tree() of [17] | Algorithm 3 | Algorithms 4-5 | Algorithm 6 |
| Color           | orange | red    | blue | green         | purple            | black          |

Table 2: Correspondence between the colors of the curves on Figures 1 to 13 and the associated classification methods.
4.2. Comparing different copula families

Here, we keep the reference setting and we change only part (iii), i.e. the functional form of the conditional copula. The results are displayed in Table 3. We observe that such choice of a parametric copula families has nearly no effect on the performance of the estimators. Nonetheless, with the Student copula - either with fixed or variable degrees of freedom - most estimators have slightly worse performances than with other copulas.

| Copula family | Logit | Probit | Tree | Random forests | Nearest neighbors | Neural network |
|---------------|-------|--------|------|----------------|------------------|----------------|
| Gaussian      | 0.456 | 0.434  | 4.57 | 3.15           | 2.26             | 0.561          |
| Student 4 df  | 0.549 | 0.515  | 4.54 | 3.28           | 2.87             | 0.753          |
| Student (2 + 1/z) df | 0.531 | 0.518  | 4.66 | 3.23           | 2.82             | 0.805          |
| Clayton       | 0.498 | 0.472  | 4.52 | 3.36           | 2.67             | 0.742          |
| Gumbel        | 0.45  | 0.431  | 4.56 | 3.23           | 2.66             | 0.775          |
| Frank         | 0.448 | 0.42   | 4.5  | 3.28           | 2.13             | 0.615          |

Table 3: Error criteria (9) for each copula family and each method, multiplied by 1000.

4.3. Comparing different conditional margins

In this subsection, we still start from the reference setting and we change only part (iv), i.e. the functional form of the conditional margins \((X_1|Z)\) and \((X_2|Z)\). We consider the following alternatives:

1. \(P_{X_1|Z=z} = P_{X_2|Z=z} = \mathcal{N}(z, 1)\) (as in the reference case).
2. \(P_{X_1|Z=z} = \mathcal{N}(\cos(10\pi z), 1)\); \(P_{X_2|Z=z} = \mathcal{N}(z, 1)\). The idea is to make \(X_1\) oscillate fast so that the algorithms will have difficulties to localize concordant and discordant pairs;
3. \(P_{X_1|Z=z} = Exp(|z|)\); \(P_{X_2|Z=z} = \mathcal{U}[z, z+1]\). This choice allows to see how estimation is affected by changes in the conditional support of \((X_1, X_2)\) given \(Z = z\);
4. \(P_{X_1|Z=z} = \mathcal{N}(0, z^2)\); \(P_{X_2|Z=z} = \mathcal{U}[0, |z|]\). Then, we will see how estimation is affected by changes in the conditional variance of \((X_1, X_2)\) given \(Z = z\).

| Setting | Logit | Probit | Tree | Random forests | Nearest neighbors | Neural network |
|---------|-------|--------|------|----------------|------------------|----------------|
| 1.      | 0.456 | 0.434  | 4.57 | 3.15           | 2.26             | 0.561          |
| 2.      | 0.809 | 0.818  | 4.65 | 3.72           | 2.65             | 0.838          |
| 3.      | 1.15  | 1.12   | 5.29 | 4.21           | 3.57             | 1.32           |
| 4.      | 0.493 | 0.471  | 4.43 | 3.44           | 2.54             | 0.662          |

Table 4: Error criteria (9) for each choice of conditional margins and each method, multiplied by 1000.

In a similar way as in the previous section, the results of these experiments, as displayed in Table 4 show that changes in terms of conditional marginal distributions generally have a mild impact on the overall performance of the estimators. Moreover, such changes have no effect on the ranking between estimators: \textit{logit} and \textit{probit} methods are always the best, followed by the \textit{neural networks}. \textit{Nearest neighbors}, and \textit{random forests} are behind in this order. The estimator \textit{Tree} shows the lowest performance, but note that it also has the lowest computation time.
4.4. Comparing different forms for the conditional Kendall’s tau

In this part, we keep the reference setting, but we change only part (ii), i.e. the functional form of the conditional Kendall’s tau itself. We consider the following choices:

1. \( f_1(z) := 0.9 - 0.8 \mathbb{1}\{z \geq 0.5\} \),
2. \( f_2(z) := 3z(1 - z) \),
3. \( f_3(z) := 0.5 + 0.4 \sin(4\pi z) \),
4. \( f_4(z) := 0.1 + 1.6z \mathbb{1}\{z < 0.5\} + 1.6(z - 0.5) \mathbb{1}\{u \geq 0.5\} \).

The results are presented in Table 5. If the estimated model is close to be well-specified, the best methods are parametric, i.e. the logit and probit regressions. In all the other cases, neural networks seem to perform very well. There is a compromise between minimization of the error, and minimization of the computation time: in the reference setting, the neural network estimator is on average 3 times slower than nearest neighbors, 4 times slower than random forests and 98 times slower than the estimator Tree.

| Setting | Logit | Probit | Tree | Random forests | Nearest neighbors | Neural network |
|---------|-------|--------|------|----------------|------------------|---------------|
| \( f_1 \) | 11.2 | 11.6 | 4.12 | 4.03 | 3.89 | 1.48 |
| \( f_2 \) | 0.456 | 0.434 | 4.57 | 3.15 | 2.26 | 0.561 |
| \( f_3 \) | 3.77 | 3.22 | 5.95 | 4.76 | 2.35 | 2.17 |
| \( f_4 \) | 12.8 | 12.8 | 16.8 | 10 | 3.71 | 1.97 |

Table 5: Error criteria (9) for different Kendall’s tau models and each estimation method, multiplied by 1000.

4.5. Higher dimensional settings

In the previous sections, we had chosen a univariate vector \( Z \), i.e. \( p = 1 \). Since this may sound a bit restrictive, we would like to obtain some finite-sample results in dimension \( p = 2 \). Note that the latter dimension cannot be too high because of the curse of dimensionality linked with the necessary kernel smoothing (done in Algorithm 1 when creating the dataset of pairs). We also choose a simple dictionary \( \psi \) of functions, which will consists in the two projections on the coordinates of \( Z \). The performance of the estimators is still be assessed by the approximate error criteria (9). The corresponding \( z^{(j)} \) are chosen as a grid of 400 points equispaced on the square \([0.01, 0.99]^2\).

In this framework, we first choose block (iii) of the model: the conditional copula of \( X_1 \) and \( X_2 \) given \( Z \) will be Gaussian, and block (iv): \( \mathbb{P}_{X_1|Z=z} = \mathbb{P}_{X_2|Z=z} = \mathcal{N}(z_1, 1) \). We will try different combinations for the remaining blocks (i) and (ii), described as follows:

1. \( Z_1 \sim \mathcal{N}(0, 1) \), \( Z_2 \sim \mathcal{U}_{[-1,1]} \), and the copula of \((Z_1, Z_2)\) is Gaussian with a Kendall’s tau equal to 0.5. Moreover, \( \tau_{1,2|Z=z} = z_2 \tanh(z_1) \). This model is interesting because the function \( \tau_{1,2|Z=z} \) will be far away from a linear function of \( \psi(z) \), and the machine learning techniques should work better than logistic/probit regressions.
2. we keep the same model as previously, but by setting \( g(\tau_{1,2|Z=z}) = z_1 + z_2 \), using the \( g \) of Example 1 so that we recover the parametric setting of Section 2.
3. \( Z_1 \sim \text{Exp}(1) \), \( Z_2 \sim \mathcal{N}(0, 1) \) and both variables are independent. Set \( \tau_{1,2|Z=z} = \exp(-z_1|z_2|) \). Again, we have a misspecified nonlinear model that is far away from logit/probit models.

The results are given in Table 6. With the exception of the well-specified setting (2), the logit model performs worse than non-parametric methods. In all these settings, neural networks show better performances than all other methods, followed by nearest neighbors and tree-based methods. Finally, parametric methods are the worst, especially under misspecification of the model.

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To summarize, trees and random forests methods are the fastest ones, but exhibit the lowest performances. Parametric methods such as the logit and probit may be very performing under some “simple” functional forms of $g$ and $\psi$, but they deteriorate quickly when the true underlying model departs from their parametric specification. Note that they also show the longest computation time. Nearest neighbors methods are average in terms of computation time as well as performance. Finally, neural networks are the slowest of all our nonparametric methods, but they behave nearly uniformly the best ones in term of prediction.

5. Applications to financial data

In this section, we study the changes in the conditional dependence between the daily returns of MSCI stock indices during two periods: the European debt crisis (from 18 March 2009 to 26 August 2012) and the after-crisis period (26 August 2012 to 2 March 2018). We will consider the following couples: (Germany, France), (Germany, Denmark), (Germany, Greece), respectively denoted by $(X_{1},X_{2})$, $(X_{1},X_{3})$, $(X_{1},X_{4})$. We will separately consider two choices of conditioning variables $Z$:

- a proxy variable for the intraday volatility $\sigma := (High – Low)/Close$, where $High$ denotes the maximum daily value of the Eurostoxx index, $Low$ denotes its minimum and $Close$ is the index value at the end of the corresponding trading day.

- a proxy of so-called “implied volatility moves” $\Delta\sigma_{I}$. It will record the daily variations of the EuroStoxx 50 Volatility Index, whose quotes are available at [https://www.stoxx.com/index-details?symbol=V2TX](https://www.stoxx.com/index-details?symbol=V2TX): $\Delta\sigma_{I} := V2TX(i) – V2TX(i – 1)$ for each trading day $i$. The EuroStoxx 50 Volatility Index $V2TX$ measures the levels of future volatility, as anticipated by the market through option prices.

Note that, for a given couple, the levels of the estimated conditional Kendall’s tau are different (in general) for different conditioning variables. Indeed, the unconditional Kendall’s tau $\tau_{1,2}$, the average conditional Kendall’s tau with respect to $\sigma$, which is $E_{\sigma} [\tau_{1,2} | \sigma]$ and the average conditional Kendall’s tau with respect to $\Delta\sigma_{I}$, which is $E_{\Delta\sigma_{I}} [\tau_{1,2} | \Delta\sigma_{I}]$ have no reason to be equal.

Both conditioning variables $\sigma$ and $\Delta\sigma_{I}$ are of dimension 1. For each method and each conditioning variable, we will use the “best” choice of $\psi$ as determined from the simulations in Section 4.1, that is $\psi^{(6)}$ for the methods logit, probit, tree and random forests and $\psi^{(1)}$ for the methods Nearest neighbors and neural networks. On the following figures, the matching between colors and corresponding estimators still follows Table 2.

5.1. Conditional dependence with respect to the Eurostoxx’s volatility proxy $\sigma$

We will first consider the conditioning events given by $\sigma$, the proxy variable for the market intraday volatility. The results are displayed on Figures 2 to 7. Intuitively, dependence should tend to increase with market volatility: when “bad news” are announced, they are sources of stress for most dealers, especially inside the Eurozone that brings together economically connected countries. This phenomenon should be particularly sensitive during the European debt crisis, because a lot of such “bad news” were related to the Eurozone itself (economic/financial news of public debts of several European countries). Let us see whether this is the case.

On most figures, the estimated conditional Kendall’s tau seems to exhibit some kind of concavity. The behavior of these functions can be roughly broken down into two main regimes:

| Setting | Logit  | Probit  | Tree  | Random forests | Nearest neighbors | Neural network |
|---------|--------|---------|-------|----------------|------------------|--------------|
| (1)     | 35.5   | 35.5    | 9.63  | 11.7           | 6.72             | 2.21         |
| (2)     | 0.433  | 0.681   | 10.9  | 5.85           | 4.33             | 0.848        |
| (3)     | 17.8   | 17.2    | 5.72  | 9.79           | 1.84             | 1.36         |

Table 6: Error criteria (9) for each setting with 2-dimensional $Z$ random vectors and each method, multiplied by 1000.
Figure 2: Conditional Kendall’s tau between \((X_1, X_2)\) given \(\sigma\) during the European debt crisis

Figure 3: Conditional Kendall’s tau between \((X_1, X_2)\) given \(\sigma\) during the After-crisis period

Figure 4: Conditional Kendall’s tau between \((X_1, X_3)\) given \(\sigma\) during the European debt crisis

Figure 5: Conditional Kendall’s tau between \((X_1, X_3)\) given \(\sigma\) during the After-crisis period

Figure 6: Conditional Kendall’s tau between \((X_1, X_4)\) given \(\sigma\) during the European debt crisis

Figure 7: Conditional Kendall’s tau between \((X_1, X_4)\) given \(\sigma\) during the After-crisis period
1. The “moderate” volatility regime (also called the “normal regime”) in the sense that the volatility stay mild, say in the lower half of its range. In this normal regime, the conditional Kendall’s tau is an increasing function of the volatility. This is coherent with most empirical research where it is shown that dependence increases with volatility.

2. The high volatility regime: this is a “stressed regime” where $\sigma$ lies in the upper half of its range. In this less frequent regime, the influence of the European volatility $\sigma$ on the conditional Kendall’s tau appears to be less clear: the estimators become more “fluctuating” and more different from each other, as a consequence of the small number of observations in most stressed regimes.

During the European debt crisis (see Figures 2, 4 and 6), the three couples seem to exhibit the same shape of conditional dependence with respect to $\sigma$, even if their average levels are different. These similarities can be a little bit surprising considering that the economic situations of the corresponding countries are different. It can be conjectured that the heterogeneity in the “mean” levels of conditional dependence is sufficient to reflect this diversity of situations. In this perspective, the increasing pattern of conditional dependence w.r.t. the “volatility” would be a pure characteristic of that period, regardless of the chosen pair of European countries. Indeed, we have observed this pattern for most couples of European countries in the Eurozone. An explanation might be that investors were focusing on the same international news, for example, about the future of the Eurozone, and they were therefore reacting in a similar way, irrespective of the country.

For each couple of countries, the conditional Kendall’s tau is nearly always lower during the After-crisis period than during the European debt crisis. Apparently, in the after-crisis period, factors and events that are specific to each country attract more attention from investors than during the crisis, which results in lower dependence. In this context, the shapes of conditional dependence are not the same anymore for different couples. In particular, the conditional Kendall’s tau between German and French returns show a significant increase during the low volatility regime and a decrease during the high volatility regime: see Figure 3. Between German and Danish returns, the conditional dependence is also increasing during the low volatility regime, but in the high volatility, the conditional Kendall’s tau seems to be rather constant, even increasing according to the nearest neighbors and the neural networks estimators. Concerning Figure 7, we do not seem any clear tendency. It is likely that $\sigma$ has almost no impact on the conditional dependence between German and Greek stock index returns.

5.2. Conditional dependence with respect to the variations $\Delta \sigma^I$ of the Eurostoxx’s implied volatility index

The implied volatility is computed using option prices. In this sense, this financial quantity reflects investors’ anticipation of future uncertainty. When important events happen, investors most often update their anticipations, which results in a change of implied volatilities. This change, denoted by $\Delta \sigma^I$ may be linked to variations of the conditional dependence between stock returns of different countries. Figures 8 to 13 illustrate the variations of the conditional Kendall’s tau between couples of stock returns with respect to the conditioning variable $\Delta \sigma^I$ during the two periods we study.

For each couple, the levels of the conditional Kendall’s tau are higher during the European debt crisis than during the after-crisis period. This is coherent with our conclusions in the previous subsection. But here, conditional Kendall’s taus look like concave functions of $\Delta \sigma^I$ during the crisis, while they exhibit “double bumps” features after the crisis. During the crisis, when $\Delta \sigma^I$ is small in absolute value, implied volatilities do not change much and the dependence is in general higher than during big changes of the market implied volatility, i.e. when $|\Delta \sigma^I|$ is high (see Figures 10 and 12).

One exception is the couple (France, Germany), for which the conditional Kendall’s tau is roughly a decreasing function of $\Delta \sigma^I$ during the crisis. France and Germany are close countries and have strong economic relationships, but Germany is seen as a country in the “center of Europe” while France share a lot of similarities with countries of the periphery (in the South of Europe). Indeed, during the crisis, when implied volatility decreases (corresponding to a negative value of $\Delta \sigma^I$), the dependence is higher, which can be interpreted as investors seeing the two countries as close. On the contrary, when the market implied volatility increases, there are concerns in the market about the robustness of Eurozone and investors raise
Figure 8: Conditional Kendall’s tau between \((X_1, X_2)\) given \(\Delta \sigma^I\) during the European debt crisis

Figure 9: Conditional Kendall’s tau between \((X_1, X_2)\) given \(\Delta \sigma^I\) during the After-crisis period

Figure 10: Conditional Kendall’s tau between \((X_1, X_3)\) given \(\Delta \sigma^I\) during the European debt crisis

Figure 11: Conditional Kendall’s tau between \((X_1, X_3)\) given \(\Delta \sigma^I\) during the After-crisis period

Figure 12: Conditional Kendall’s tau between \((X_1, X_4)\) given \(\Delta \sigma^I\) during the European debt crisis

Figure 13: Conditional Kendall’s tau between \((X_1, X_4)\) given \(\Delta \sigma^I\) during the After-crisis period
doubts about southern European countries - including France - which tends to decrease the conditional Kendall’s tau between French and German returns.

After the crisis, the couples (Germany, France), and (Germany, Denmark) revert to a more usual shape of conditional dependence: when volatility does not change much, conditional Kendall’s tau is low; when volatility changes much, conditional Kendall’s tau is higher, reflecting more stressed situations. In this period, an exception is the couple (Germany, Greece), whose conditional Kendall’s tau has a particular shape, that looks like the one of the couple (Germany, France) during the crisis. This is coherent with the fact that, in stressed situations, when volatility increases, investors sometimes remembers that Greece still has a fragile economy, which results in a lower conditional Kendall’s tau. But three estimators suggest that, when volatility increases very much, conditional Kendall’s tau between Germany and Greece increases again, following the classical tendencies that we had already observed.

6. Conclusion

In a parametric setting, we have proposed a localized log-likelihood method to estimate conditional Kendall’s tau. When the link function is analytically tractable and explicit, it is then possible to code and optimize the full penalized criterion. The consistency and the asymptotic normality of such estimators have been stated. In particular, this is the case for Logit or Probit-type link functions. We noticed that evaluating a Kendall’s tau is equivalent to evaluating a probability of being classified as a concordant pair. Therefore, most classification procedures can be adapted to directly estimate conditional Kendall’s tau. Classification trees, random forests, nearest neighbors and neural networks have been discussed. They generally provide more flexible parametric models than previously. Finally, we have evaluated these different methods on several empirical illustrations.

Appendix A. Proof of Theorem

Simple calculations provide: if \( i \neq j \) and under (3),

\[
\begin{align*}
\mathbb{E}[L_n(\beta)] &= \mathbb{E} \left[ K_h(Z_i - Z_j) \ell_\beta(W_{(i,j)}, Z_i) \right] = \mathbb{E} \left[ K_h(Z_i - Z_j) \mathbb{E}[\ell_\beta(W_{(i,j)}, Z_i)|Z_i, Z_j] \right] \\
&= \mathbb{E} \left[ K_h(Z_i - Z_j) \left( p(Z_i, Z_j) \log \left( \frac{1}{2} + \frac{1}{2} g(\psi(Z_i)^T \beta) \right) + (1 - p(Z_i, Z_j)) \log \left( \frac{1}{2} - \frac{1}{2} g(\psi(Z_i)^T \beta) \right) \right) \right] \\
&= \mathbb{E} \left[ K_h(Z_i - Z_j) \phi(Z_i, Z_j, \beta) \right] \\
&= \int K(t) \phi(Z_i, Z_i - ht, \beta) f_Z(Z_i - ht) \, dt ,
\end{align*}
\]

that tends to \( \mathbb{E} [\phi(Z_i, Z_i, \beta) f_Z(Z_i)] = L_\infty(\beta) \) when \( n \to \infty \), if

\[
\int (\phi(z, \cdot, \beta) f_Z(\cdot)) (\epsilon(z) f_\alpha(z)) \, dz < \infty, \tag{A.1}
\]

for some \( \epsilon > 0 \) (invoke the dominated convergence Theorem and the compact support of \( K \)).

Now, let us prove that, for any \( \beta \), \( L_n(\beta) \) tends towards \( L_\infty(\beta) \) in probability, when \( n \to \infty \). It is sufficient to prove that the variance of \( L_n(\beta) \) tends to zero.

\[
\mathbb{E} \left[ \left( L_n(\beta) - \mathbb{E}[L_n(\beta)] \right)^2 \right] = \frac{1}{n^2(n - 1)^2} \sum_{i_1, j_1, i_2 \neq j_1, i_2, j_2} \sum_{i_1, j_1, i_2 \neq j_1, i_2, j_2} \psi_{i_1, j_1, i_2, j_2} \mathbb{E} \left[ K_h(Z_{i_1} - Z_{j_1}) K_h(Z_{i_2} - Z_{j_2}) \ell_\beta(W_{(i_1,j_1)}, Z_{i_1}) \ell_\beta(W_{(i_2,j_2)}, Z_{i_2}) \right] - \mathbb{E}[L_n(\beta)]^2}
\]

\[= \frac{1}{n^2(n - 1)^2} \sum_{i_1, j_1, i_2 \neq j_1, i_2, j_2} \sum_{i_1, j_1, i_2 \neq j_1, i_2, j_2} \psi_{i_1, j_1, i_2, j_2}, \]

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with obvious notations. Obviously, \(v_{i_1,j_1,i_2,j_2}\) is zero when \(i_1\) and \(j_1\) are not equal to \(i_2\) nor \(j_2\). At the opposite, in the case of equalities between some of these four indices, we get non-zero terms.

To be specific, when \(i_1 = i_2 = i\) and \(j_1 \neq j_2\), we have

\[
v_{i,j_1,i,j_2} = E \left[ K_h(Z_i - Z_{j_1})K_h(Z_i - Z_{j_2})\ell_\beta(W_{(i,j_1)}, Z_i)\ell_\beta(W_{(i,j_2)}, Z_i) - E[L_n(\beta)]^2 \right]
\]

by setting

\[
A(x, y, z) := E \left[ \ell_\beta(W_{(i,j_1)}, Z_i)\ell_\beta(W_{(i,j_2)}, Z_i) | Z_i = x, Z_{j_1} = y, Z_{j_2} = z \right]
\]

\[
= p(x, y, z) \log^2 q(x, \beta) + (p(x, y) - p(x, y, z)) \log q(x, \beta) \log(1 - q(x, \beta))
\]

If \(A(z, \cdot)_{\epsilon}(z, x) f_{\mathcal{Z}_\epsilon}(z) f_{\mathcal{Z}}(z) dz < \infty\) for some \(\epsilon > 0\), then \(v_{i,j_1,i,j_2}\) tends to a constant when \(n \to \infty\) (independently of the choice of such indices).

A similar analysis can be led for the other terms. When \(i_1 = j_2\) and \(j_1 \neq i_2\), we get

\[
v_{i_1,j_1,i_2,i_2} = E \left[ K_h(Z_{i_1} - Z_{j_1})K_h(Z_{i_2} - Z_{j_2})\ell_\beta(W_{(i_1,j_1)}, Z_{i_1})\ell_\beta(W_{(i_2,j_1)}, Z_{i_2}) - E[L_n(\beta)]^2 \right]
\]

by setting

\[
B(x, y, z) := E \left[ \ell_\beta(W_{(i_1,j_1)}, Z_{i_1})\ell_\beta(W_{(i_2,j_1)}, Z_{i_2}) | Z_{i_1} = x, Z_{j_1} = y, Z_{i_2} = z \right]
\]

\[
= p(x, y, z) \log q(x, \beta) \log q(z, \beta) + (p(x, y) - p(x, y, z)) \log q(x, \beta) \log(1 - q(z, \beta))
\]

If \(B(z, \cdot)_{\epsilon}(z, x) f_{\mathcal{Z}_\epsilon}(z) f_{\mathcal{Z}}(z) dz < \infty\), then \(v_{i_1,j_1,i_2,i_2}\) tends to a constant when \(n \to \infty\).

When \(j_1 = j_2 = j\) and \(i_1 \neq i_2\), we obtain

\[
v_{i_1,j_1,i_2,j_2} = E \left[ K_h(Z_{i_1} - Z_j)K_h(Z_{i_2} - Z_j)\ell_\beta(W_{(i_1,j)}, Z_{i_1})\ell_\beta(W_{(i_2,j)}, Z_{i_2}) - E[L_n(\beta)]^2 \right]
\]

by setting

\[
C(x, y, z) := E \left[ \ell_\beta(W_{(i_1,j)}, Z_{i_1})\ell_\beta(W_{(i_2,j)}, Z_{i_2}) | Z_{i_1} = x, Z_j = y, Z_{i_2} = z \right]
\]

\[
= p(x, y, z) \log q(x, \beta) \log q(z, \beta) + (p(x, y) - p(x, y, z)) \log q(x, \beta) \log(1 - q(z, \beta))
\]

If \(C(z, \cdot)_{\epsilon}(z, x) f_{\mathcal{Z}_\epsilon}(z) f_{\mathcal{Z}}(z) dz < \infty\), then \(v_{i_1,j_1,i_2,j_2}\) tends to a constant when \(n \to \infty\).

When \(j_1 = i_2 = i\) and \(j_2 \neq j\),

\[
v_{i_1,j_1,i_2,j_2} = E \left[ K_h(Z_{i_1} - Z_{j_1})K_h(Z_{j_1} - Z_{i_2})\ell_\beta(W_{(i_1,j_1)}, Z_{i_1})\ell_\beta(W_{(j_2,j_1)}, Z_{j_1}) - E[L_n(\beta)]^2 \right]
\]

by setting

\[
D(x, y, z) := E \left[ \ell_\beta(W_{(i_1,j_1)}, Z_{i_1})\ell_\beta(W_{(j_2,j_1)}, Z_{j_1}) | Z_{i_1} = x, Z_{j_1} = y, Z_{i_2} = z \right]
\]

\[
= p(x, y, z) \log q(x, \beta) \log q(z, \beta) + (p(x, y) - p(x, y, z)) \log q(x, \beta) \log(1 - q(z, \beta))
\]

If \(D(z, \cdot)_{\epsilon}(z, x) f_{\mathcal{Z}_\epsilon}(z) f_{\mathcal{Z}}(z) dz < \infty\), then \(v_{i_1,j_1,i_2,j_2}\) tends to a constant when \(n \to \infty\).
by setting
\[
D(x, y, z) := \mathbb{E} \left[ \ell_{\beta}(W_{(i, j), 1}) \ell_{\beta}(W_{(j, j), 2}) | Z_{i_1} = x, Z_{j_1} = y, Z_{j_2} = z \right] \\
= p(x, y) \log q(x, \beta) \log q(y, \beta) + (p(x, y) - p(x, y, z)) \log q(x, \beta) \log(1 - q(y, \beta)) \\
+ (p(y, z) - p(x, y, z)) \log q(y, \beta) \log(1 - q(x, \beta)) \\
+ (1 - p(x, y) - p(y, z) + p(x, y, z)) \log(1 - q(x, \beta)) \log(1 - q(y, \beta)).
\]

If \( \int D(\cdot, z, \cdot) e(z, z) f_{\hat{Z}, e}(z) f_Z(z) \, dz < \infty \), then \( v_{i_1, j_1, j_2} \) tends to a constant when \( n \to \infty \).

There are two cases of two equalities. If \( i_1 = i_2 = i \) and \( j_1 = j_2 = j \), this yields
\[
v_{i, j, i, j} = \mathbb{E} \left[ K_h(Z_i - Z_j)^2 \ell_{\beta}(W_{(i, j), 1}) \ell_{\beta}(W_{(j, j), 2}) | Z_{i_1} = x, Z_{j_1} = y \right] \\
= h^{-p} \mathbb{E} \left[ \int K(x)^2 E(Z_i, Z_i - h x) f_Z(Z_i - h x) \, dx \right] - \mathbb{E}[L_n(\beta)]^2,
\]
by setting
\[
E(x, y) := \mathbb{E} \left[ \ell_{\beta}(W_{(i, j), 1}) | Z_{i_1} = x, Z_{j_1} = y \right] \\
= p(x, y) \log q(x, \beta) + (1 - p(x, y)) \log(1 - q(x, \beta)).
\]

If \( \int E(z, \cdot, e(z) f_{\hat{Z}, e}(z) f_Z(z) \, dz < \infty \), then \( h^p v_{i, j, i, j} \) tends to a constant when \( n \to \infty \).

Finally, if \( i_1 = j_1 \) and \( j_1 = i_2 \), we get
\[
v_{i_1, j_1, i_1, j_1} = \mathbb{E} \left[ K_h(Z_{i_1} - Z_{j_1})^2 \ell_{\beta}(W_{(i_1, j_1), 1}) \ell_{\beta}(W_{(j_1, i_1), 2}) | Z_{i_1} = x, Z_{j_1} = y \right] \\
= h^{-p} \mathbb{E} \left[ \int K(x)^2 F(Z_i, Z_i - h x) f_Z(Z_i - h x) \, dx \right] - \mathbb{E}[L_n(\beta)]^2,
\]
by setting
\[
F(x, y) := \mathbb{E} \left[ \ell_{\beta}(W_{(i_1, j_1), 1}) \ell_{\beta}(W_{(j_1, i_1), 2}) | Z_{i_1} = x, Z_{j_1} = y \right] \\
= p(x, y) \log q(x, \beta) \log q(y, \beta) + (1 - p(x, y)) \log(1 - q(x, \beta)) \log(1 - q(y, \beta)).
\]

If \( \int F(z, \cdot, e(z) f_{\hat{Z}, e}(z) f_Z(z) \, dz < \infty \), then \( h^p v_{i_1, j_1, i_1, j_1} \) tends to a constant when \( n \to \infty \).

Summarizing the previous terms, we have obtained that \( \text{Var}(L_n(\beta)) = O(n^{-1} + n^{-3} h^{-p}) \), that tends to zero pointwise, when \( n^2 h^p \to \infty \). We deduce \( L_n(\beta) - L_{\infty}(\beta) = L_n(\beta) - \mathbb{E}[L_n(\beta)] + \mathbb{E}[L_n(\beta)] - L_{\infty}(\beta) = o_p(1) \). Since \( L_n(\cdot) \) and \( L_{\infty}(\cdot) \) are concave, invoking the convexity lemma of Geyer (1996) (see Knight and Fu, 2000, alternatively), the maximizer \( \hat{\beta} \) of \( L_n \) tends in probability towards the maximizer of \( L_{\infty} \). □

We summarize the latter technical assumptions that are sufficient to obtain the consistency of \( \hat{\beta} \): for some \( \varepsilon > 0 \),
\[
\int \left( A(z, \cdot, e(z, z) + B(\cdot, \cdot, e(z, z) + C(\cdot, z, \cdot) e(z, z) + D(\cdot, \cdot, e(z, z), Z) \right) f_{\hat{Z}, e}(z) f_Z(z) \, dz < \infty,
\]
\[
\int \left( \phi(z, \cdot, \beta) e(z) + E(z, \cdot, e(z) + F(\cdot, \cdot, e(z)) \right) f_{\hat{Z}, e}(z) f_Z(z) \, dz < \infty.
\]

Appendix B. Proof of Theorem 4

Set \( u := \sqrt{n}(\beta - \beta^*) \) and \( \bar{u} := \sqrt{n}(\hat{\beta} - \beta^*) \). Obviously,
\[
\bar{u} = \arg \max_{u \in \mathbb{R}^p} L_n(\beta^* + n^{-1/2} u) - \lambda_n |\beta^* + n^{-1/2} u|_1,
\]
\[
= \arg \max_{u \in \mathbb{R}^p} n L_n(\beta^* + n^{-1/2} u) - n L_n(\beta^*) - n \lambda_n \left\{ |\beta^* + n^{-1/2} u|_1 - |\beta^*|_1 \right\}.
\]

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Note that

\[ u \lambda_n |\beta^* + n^{-1/2} u |1 - n|\beta^* |1 = n^{1/2} \lambda_n \sum_{k, \beta_k^* = 0} |u_k| + n^{1/2} \lambda_n \sum_{k, \beta_k^* \neq 0} \text{sign}(\beta_k^*) u_k \]

\[ \longrightarrow \mu \sum_{k, \beta_k^* = 0} |u_k| + \mu \sum_{k, \beta_k^* \neq 0} \text{sign}(\beta_k^*) u_k, \]

when \( n \to \infty \). Moreover,

\[ nL_n(\beta^* + n^{-1/2} u) - n^{-1/2} L_n(\beta^*) = n^{1/2} \hat{L}_n(\beta^*) + \frac{1}{2} u^T \hat{L}_n(\beta) u + \frac{1}{6} n^{1/2} \hat{L}_n(\beta) u(\beta), \]

for some (random) \( \beta \) s.t. \(|\beta^* - \beta| < |\beta^* - \beta| \). We will successively prove that

(i) \( n^{1/2} \hat{L}_n(\beta^*) \) weakly tends to a Gaussian random vector \( \mathbb{W}, \mathbb{W} \sim \mathcal{N}(0_p, \Sigma_{\beta^*}) \);

(ii) \( \hat{L}_n(\beta^*) \) tends in probability towards a constant matrix \( \mathbb{H}(\beta^*) \);

(iii) \( \hat{L}_n(\beta) \) is \( O_P(1) \).

Then, \( \hat{u} = \arg \max_u L_n(u) \), where \( L_n(u) \) weakly tends to

\[ \mathcal{L}_\infty(u) := \mathbb{W}, u + \frac{1}{2} u^T \mathbb{H}(\beta^*) u - \mu \sum_{k, \beta_k^* = 0} |u_k| - \mu \sum_{k, \beta_k^* \neq 0} \text{sign}(\beta_k^*) u_k, \]

that is concave. The result will follow, applying Theorem 1 in Kato (2009).

First, let us prove (i), i.e. the asymptotic normality of \( n^{1/2} \hat{L}_n(\beta) \) for any given parameter \( \beta \). Consider the centered criterion

\[ M_n(\beta) := \hat{L}_n(\beta) - \mathbb{E}[\hat{L}_n(\beta)] = \frac{1}{n(n-1)} \sum_{i,j,i \neq j} \ell_{ij}(\beta), \]

where \( \ell_{ij} := K_h(Z_i - Z_j) \partial_{\beta} \ell(\psi_{i,j}, Z_i) - \mathbb{E}[\hat{L}_n(\beta)] \). We symmetrize the localized likelihood:

\[ M_n(\beta) = \frac{1}{2n(n-1)} \sum_{i,j,i \neq j} M_{ij}(\beta), \]

where \( M_{ij}(\beta) \) (or simply \( M_{ij} \)) is \( \ell_{ij}(\beta) + \ell_{ji}(\beta) \). Note that \( M_{ij} = M_{ji} \) and that \( \mathbb{E}[M_{ij}] = 0 \).

By the same reasoning as in the proof of Theorem 3 we check that \( \mathbb{E}[\partial_{\beta} \phi(Z_i, Z_i, \beta) f_Z(Z_i)] = \partial_{\beta} L_{\infty}(\beta) + o(1) \) if, for some \( \varepsilon > 0 \), we have

\[ \int (\partial_{\beta} \phi(z_{\cdot}, \beta) f_Z(\cdot)) \epsilon(z) f_Z(z) \, dz < \infty. \] (B.1)

Moreover, by simple calculations, we get, if \( i \neq j \),

\[ \mathbb{E}[M_n | Z_i] = \frac{1}{2n} \mathbb{E}[M_{ij} + M_{ji} | Z_i] = \frac{1}{n} \mathbb{E}[M_{ij} | Z_i] \]

\[ = \frac{1}{n} \int \{ K_h(z_i - z) \partial_{\beta} \phi(Z_i, z, \beta) + K_h(z - z_i) \partial_{\beta} \phi(z, Z_i, \beta) \} f_Z(z) \, dz - \frac{2}{n} \mathbb{E}[\hat{L}_n(\beta)] \]

\[ = \frac{1}{n} \int K(t) \{ \partial_{\beta} \phi(Z_i, z_i - ht, \beta) f_Z(z_i - ht) + \partial_{\beta} \phi(Z_i + ht, z_i, \beta) f_Z(z_i + ht) \} \, dt - \frac{2}{n} \mathbb{E}[\hat{L}_n(\beta)] \]

\[ = \frac{2}{n} \partial_{\beta} \phi(Z_i, Z_i, \beta) f_Z(Z_i) - \frac{2}{n} \hat{L}_{\infty}(\beta) + o(n^{-1}) + r_{n,i}, \]
where, by a $m$-order limited expansion, we obtain
\[
\|r_{n,i}\| \leq \frac{\text{Cst} \cdot h^m}{n^m!} \int |K| \|(f_{Z}(\cdot)\partial_{\beta}\phi(Z_{i},\cdot,\beta))^{(m)} + (f_{Z}(\cdot)\partial_{\beta}\phi(\cdot, \cdot, \beta))^{(m)}\|_{2}(Z_{i}),
\]
for any norm $\| \cdot \|$ on $\mathbb{R}^p$ and a positive constant Cst. We deduce that $n^{1/2} \sum_{i=1}^{n} \mathbb{E}[M_{n}|Z_{i}]$ is asymptotically normal by invoking the usual CLT, if
\[
\mathbb{E}\left[\|(f_{Z}(\cdot)\partial_{\beta}\phi(Z_{i},\cdot,\beta))^{(m)} + (f_{Z}(\cdot)\partial_{\beta}\phi(\cdot, Z_{i}, \beta))^{(m)}\|_{2}(Z_{i})\right] < \infty. \quad (B.2)
\]
To be specific, the Hájek projection of $M_{n}$ is
\[
\frac{\sqrt{n}}{2} \sum_{i=1}^{n} \mathbb{E}[M_{n}|Z_{i}] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{\partial_{\beta}\phi(Z_{i}, Z_{i}, \beta) f_{Z}(Z_{i}) - \hat{L}_{\infty}(\beta)\} + o_{p}(1) \rightarrow N(0, \Sigma_{\beta}), \quad \text{with}
\]
\[
\Sigma_{\beta} := \int \partial_{\beta}\phi(z, z, \beta) \partial_{\beta}\phi(z, z, \beta)^{T} f_{Z}^{2}(z) \, dz - \hat{L}_{\infty}(\beta) \hat{L}_{\infty}(\beta)^{T}.
\]
Note that that $\hat{L}_{\infty}(\beta^{*}) = 0$.

Now consider the “remainder term” $\Delta_{n} := M_{n}(\beta) - \sum_{i=1}^{n} \mathbb{E}[M_{n}|Z_{i}]/2$. Since $\mathbb{E}[M_{n}|Z_{i}] = n^{-1}\mathbb{E}[M_{ij}|Z_{i}]$, we deduce
\[
\Delta_{n} = M_{n}(\beta) - \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}[M_{n}|Z_{i}] = \frac{1}{2n(n-1)} \sum_{i,j:i \neq j} \{M_{ij} - \mathbb{E}[M_{ij}|Z_{i}]\}.
\]
It is relatively easy to prove that $\Delta_{n}$ is negligible, i.e. $\Delta_{n} = o_{p}(n^{-1/2})$. Indeed, let us prove that the variance of $n^{1/2}\Delta_{n}$ tends to zero with $n$:
\[
\text{Var}(n^{1/2}\Delta_{n}) = n \mathbb{E}[\Delta_{n}\Delta_{n}^{T}] = \frac{1}{4n(n-1)^{2}} \sum_{i_{1},j_{1} \neq j_{1}, i_{2},j_{2} \neq j_{2}} \delta_{i_{1},i_{2},j_{1},j_{2}} = \mathbb{E}\left[(M_{ij_{1}} - \mathbb{E}[M_{ij_{1}}|Z_{i_{1}}]) \times (M_{ij_{2}} - \mathbb{E}[M_{ij_{2}}|Z_{i_{2}}])^{T}\right].
\]
If there is no identity among the indices $(i_{1}, j_{1}, i_{2}, j_{2})$, with $i_{1} \neq i_{2}$ and $i_{2} \neq j_{2}$, then $\delta_{i_{1},i_{2},j_{1},j_{2}} = 0$. Moreover, this is the case when there is only a single identity. For instance, assume $i_{1} = i_{2} = i$ and $j_{1} \neq j_{2}$. Then,
\[
\delta_{i_{1},i_{2},j_{1},j_{2}} = \mathbb{E}\left[(M_{ij_{1}} - \mathbb{E}[M_{ij_{1}}|Z_{i_{1}}]) \times (M_{ij_{2}} - \mathbb{E}[M_{ij_{2}}|Z_{i_{2}}])^{T}\right] = \mathbb{E}\left[(M_{ij_{1}} - \mathbb{E}[M_{ij_{1}}|Z_{i_{1}}]) \times (M_{ij_{2}} - \mathbb{E}[M_{ij_{2}}|Z_{i_{2}}])^{T}\right] = \mathbb{E}\left[(M_{ij_{1}} - \mathbb{E}[M_{ij_{1}}|Z_{i_{1}}]) \times 0\right] = 0.
\]
The other terms for which a single identity between the indices can be managed similarly.

At the opposite, non-zero terms appear when $i_{1} = i_{2} = i$ and $j_{1} = j_{2} = j$. In this case, we obtain
\[
\delta_{i_{1},i_{2},j_{1},j_{2}} = \mathbb{E}\left[(M_{ij_{1}} - \mathbb{E}[M_{ij_{1}}|Z_{i_{1}}]) \times (M_{ij_{2}} - \mathbb{E}[M_{ij_{2}}|Z_{i_{2}}])^{T}\right] = \mathbb{E}[M_{ij}M_{ij}^{T}] = \mathbb{E}[M_{ij}]M_{ij}^{T} - \mathbb{E}[\mathbb{E}[M_{ij}|Z_{i}]\mathbb{E}[M_{ij}|Z_{i}^{T}].
\]
By a usual change of variable and by symmetry, we get
\[
\mathbb{E}[M_{ij}M_{ij}^{T}] = 2 \mathbb{E}\left[ (\ell_{ij}(\beta)\ell_{ij}(\beta)^{T} + \ell_{ij}(\beta)\ell_{ij}(\beta)^{T}) \right] = 2 \mathbb{E}\left[ K_{h}^{2}(Z_{i} - Z_{j})\partial_{\beta}\ell_{ij}(W_{(i,j)}, Z_{i})\partial_{\beta}\ell_{ij}(W_{(i,j)}, Z_{j})^{T} \right] = K_{h}(Z_{i} - Z_{j})K_{h}(Z_{j} - Z_{i})\partial_{\beta}\ell_{ij}(W_{(i,j)}, Z_{i})\partial_{\beta}\ell_{ij}(W_{(i,j)}, Z_{j})^{T} + O(1)
\]
\[
= \frac{2h^{-2p}}{n} \mathbb{E}\left[ \int \left( K^{2}(x)H_{1}(Z_{i}, Z_{i} - h\cdot x) + K(x)K(-x)H_{2}(Z_{i}, Z_{i} - h\cdot x) \right) f_{Z}(Z_{i} - h\cdot x) \, dx \right] + O(1)
\]
\[
= \frac{2h^{-2p}}{n} \int \left( K^{2}(x)H_{1}(z, z - h\cdot x) + K(x)K(-x)H_{2}(z, z - h\cdot x) \right) f_{Z}(z - h\cdot x) f_{Z}(x) \, dx \, dz + O(1),
\]
We deduce that \( \delta \) and \( Cst \) due to the symmetry of \( M \) in a small neighborhood of \( \beta \). By derivation, we get, for every indices \( a,b,c \),

denotes a real constant that depend on \( n \),

Thus, we have proved that \( Var(n^{1/2}\Delta_n) = O(n^{-1/2}) = o(1) \), which implies

We deduce that \( n^{1/2}M_n(\beta) \) weakly tends towards the Gaussian random vector \( N(0,p,\Sigma_{\beta}) \) for any \( \beta \). When \( \beta = \beta^* \), this yields (i).

Second, let us deal with (ii) above. It is easy to prove that \( \tilde{L}_n(\beta^*) \) tends to \( H(\beta^*) \) in probability, when \( n \) tends to the infinity. Indeed, the arguments are exactly the same as in Appendix A, where we have proved that \( L_n(\beta^*) \) is convergent in probability. We only have to replace \( \ell_{\beta}(\cdot,\cdot) \) by its second derivatives w.r.t. \( \beta \). To save space, the specific derivations of such conditions of regularity are left to the reader: simply replace the functions \( A, B,...,F \) of Appendix A by their second derivatives w.r.t. \( \beta \), taken at \( \beta = \beta^* \), and rewrite (A.2) and (A.3).

Third, to prove (iii), it is sufficient to state that \( \mathbb{E}[||\tilde{L}_n(\beta)||] \) is bounded from above, uniformly w.r.t. \( \beta \) in a small neighborhood of \( \beta^* \). By derivation, we get, for every indices \( a,b,c \) in \( \{1,...,p'\} \),

where, for every \( \delta \in \{1,-1\} \),

and \( Cst \) denotes a real constant that depend on \( g \) and \( (a,b,c) \) only. Therefore, it is sufficient to assume that

\[
\int |K| (t) \left( p(z - ht) H(1,z,\beta,a,b,c) + (1 - p(z - ht)) H(-1,z,\beta,a,b,c) \right) f(z) f(z - ht) \, dt \, dz < \infty.
\]
This is guaranteed if, for every \((a, b, c)\),

\[
\sup_{\beta \in \mathcal{V}(\beta^*)} \int \left( (p(z, \cdot) f(z(\cdot))c(z) H(1, z, \beta, a, b, c) + ((1 - p(z, \cdot)) f(z(\cdot))c(z) H(-1, z, \beta, a, b, c)) \right) f(z) \, dz < \infty,
\]

for \(\mathcal{V}(\beta^*)\) some (small) neighborhood around \(\beta^*\). Then, under the latter assumption, (iii) is stated and this finishes the proof. \(\Box\)

**Remark 5.** Note that \(\|p(\cdot, \cdot)\|_\infty \leq 1\). If \(g\) and its derivatives are bounded, Condition \((B.4)\) is satisfied if

\[
\sup_{\beta \in \mathcal{V}(\beta^*)} \sup_{\delta \in \{-1, 1\}} \int \|\psi(z)\|^3 |1 + \delta g(\psi(z)^T \beta)|^{-3} f_{Z, \varepsilon}(z) f_Z(z) \, dz < \infty.
\]

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