Wheeler-DeWitt Equation in Five Dimensions and Modified QED

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Abstract

We consider the ADM splitting of the Einstein-Hilbert action in five dimensions in the presence of matter that can be either a “point particle”, or a set of scalar fields. The Hamiltonian, being a linear superposition of constraints, is equal to zero. Upon quantization, we obtain the Schrödinger equation for a wave functional, Ψ, that depends on the matter degrees of freedom, and on the 5D gravity degrees of freedom. After the Kaluza-Klein splitting, the functional Schrödinger equation decomposes so that it contains a part due to 4D gravity, a part due to electrodynamics, and a part due to matter. Depending on choice of the matter term, we obtain two different versions of a modified quantum electrodynamics. In one version, time automatically appears, and there is no problem with infinite vacuum energy density of matter fields, whereas in the other version such problems exist.

1 Introduction

Quantization of gravity is still enigmatic. A straightforward approach is to start from the Einstein-Hilbert action in the presence of matter. Because of diffeomorphism invariance, such system has constraints, called the Hamilton and momentum constraints. In the quantized theory, the constraints become operators that annihilate state vector. The Hamilton constraint gives the Wheeler-DeWitt equations [1]. The Hamiltonian, $H$, which is a linear superposition of constraints (this also involves the integration over space), is identically zero. After quantization, the equation $H = 0$ becomes $H |\Psi\rangle = 0$, in which there is no explicit time derivative term. How to obtain such a term is subject of intensive research [2].

Another enigmatic subject is the unification of gravity with other fundamental interactions. An approach that has been much investigated is to consider gravity in a higher dimensional spacetime, $M_D$. The 4-dimensional gravity and Yang-Mills interactions, including the electromagnetic U(1) interaction, are all incorporated in the metric of $M_D$, if $M_D$ is equipped with appropriate isometries [3].

As a first step, to see how the theory works, it is instructive to consider gravity in five dimensions. Beciu [4], Lacquantini and Montani [5] considered the canonical gravity in 5D, by performing the ADM [6] and Kaluza-Klein splitting of spacetime. In this Letter we will extend their work to include a matter term, $I_m$, in the action. Usually, a matter
term consists of scalar, $\varphi^\alpha$, or spinor fields, $\psi^\alpha$, minimally coupled to gravity. Upon quantization, those fields and the conjugated momenta become operators that create or annihilate particles. In the Schrödinger representation, in which the field operators are diagonal, the fields occur as arguments in the wave functional $\Psi[\varphi^\alpha, ...]$. 

In a previous work [7], we investigated an alternative approach. The idea was based on the fact that, classically, objects are described by spacetime coordinate functions $X^\mu$, $\mu = 0, 1, 2, 3$. The simplest object is a point particle, described by $X^\mu(\tau)$. However, a point particle is an idealization. In reality, there are no point particles. According to Dirac [8], even the electron can be envisaged as a charged spherical membrane, its center of mass being described by $X^\mu(\tau)$ (see [9]). Neglecting the internal degrees of freedom, we can describe a particle by an action functional $I_m[X^\mu(\tau)]$, bearing in mind that such description is only valid outside the (extended) particle. Because the particle is not a black hole, its radius is greater than the Schwarzschild radius. Since the particle is coupled to gravity, the total action contains the kinetic term for gravity, $I_g[g_{\mu\nu}]$, as well. At the classical level, the degrees of freedom are thus $X^\mu(\tau)$ and $g_{\mu\nu}(x^\rho)$. Extending the theory to five dimensions, the classical degrees of freedom are $X^M(\tau)$ and $G_{MN}(X^J)$, $M, N, J = 0, 1, 2, 3, 5$. Such theory, besides the constraints of the canonical gravity—now in 5D—has an additional constraint, due to the representation invariance of the “point particle” action $I_m[X^M, G_{MN}]$. Upon quantization, the latter constraint becomes the Klein-Gordon equation for a wave functional $\Psi[X^M, q_{ab}]$, $a, b = 1, 2, 3, 5$, where instead of $G_{MN}$ we now consider the reduced number of the metric components. We show how the Hamilton and momentum constraints, if integrated over $dx^1 dx^2 dx^3 dx^5$ and split à la Kaluza-Klein, contain quantum electrodynamics, apart from a difference that comes from our usage of $I_m[X^M, G_{MN}]$, which leads to the terms $-i\partial \Psi / \partial T$ and $-i\partial \Psi / \partial X^a$. The term $-i\partial \Psi / \partial T$ does not necessarily give infinite vacuum energy.

We then also investigate the case in which the matter term is $I_m[\varphi^\alpha, G_{MN}]$, $\alpha = 1, 2$. Upon quantization we have constraints, acting on a state vector, and no time derivative term. But otherwise, the constraints, integrated over $dx^1 dx^2 dx^3 dx^5$, closely match the Schrödinger representation of QED [10], apart from the term $H_g$ due to 4D gravity. We point out that, according to the literature [11], the term $-i\partial \Psi / \partial T$ could come from $H_g$ as an approximation. So we obtain the Schrödinger equation for the evolution of a wave functional that depends on the electromagnetic field potentials and scalar fields, $\varphi^\alpha$. This is what we also have in the usual Schrödinger (functional) representation [10] of QED. Alternatively, we might be interested in how evolves in time a wave functional that depends on the 4D gravitational field and on the electromagnetic field. We show how the time derivative term $-i\partial \Psi / \partial T$, i.e., the same term that we obtain from $I[X^M, G_{MN}]$, results as an approximation to the scalar field matter part, $H_m$, of the total Hamiltonian, $H$. Regardless of which way we generate an approximative evolution term in the quantum constraint equation, if matter consists of scalar (or spinors) fields, then it gives infinite vacuum energy density coupled to gravity.
2 ADM and Kaluza-Klein splitting of the Einstein-Hilbert action in the presence of matter

Let us consider the Einstein-Hilbert action in five dimensions in the presence of a source, whose center of mass is described by $X^M(\tau)$, $M = 0, 1, 2, 3, 5$:

$$I[X^A, G_{MN}] = M \int d\tau (\dot{X}^M \dot{X}^N G_{MN})^{1/2} + \frac{1}{16\pi G} \int d^5x \sqrt{-G} R^{(5)}.$$  \hspace{1cm} (1)

Here $G_{MN}$ is the 5D metric tensor, $G$ its determinant, and $\mathcal{G}$ the gravitational constant in five dimension. The source is not a point particle, it is an extended, ball-like or spherical membrane-like object. We are not interested in the detailed dynamics of the coupling of the ball or the membrane with the gravitational field, we will only consider the center of mass. Therefore, our description will be valid outside the object, whose radius may be small, but greater than the Schwarzschild radius.

The metric tensor $G_{MN}$ can be split according to ADM \cite{6} as:

$$G_{MN} = \left( \begin{array}{cc} N^2 - N^a N_a & -N_a \\ -N_b & -q_{ab} \end{array} \right), \hspace{1cm} (2)$$

where $N = \sqrt{1/G^{00}}$ and $N_a = q_{ab} N^b = -G_{0a}$, $a = 1, 2, 3, 5$, are the laps and shift functions in five dimensions.

Alternatively, $G_{MN}$ can be split according to Kaluza-Klein:

$$G_{MN} = \left( \begin{array}{cc} g_{\mu\nu} - \phi^2 A_\mu A_\nu & k^2 \phi^2 A_\mu \\ k^2 \phi^2 A_\nu & -\phi^2 \end{array} \right), \hspace{1cm} (3)$$

where $g_{\mu\nu}$ is the metric tensor, and $A_\mu$ the electromagnetic field in 4D, whereas $k \equiv 2\sqrt{G^{(4)}}$ is a constant to be defined later.

From Eqs. (2), (3) we obtain the following relations:

$$G_{00} = g_{00} - k^2 \phi^2 (A_0)^2 = N^2 - N^a N_a \hspace{1cm} (4)$$
$$G_{0i} = g_{0i} - k^2 \phi^2 A_0 A_i = -N_i \hspace{1cm} (5)$$
$$G_{05} = k \phi^2 A_0 = -N_5 \hspace{1cm} (6)$$
$$G_{55} = -\phi^2 = -q_{55} \hspace{1cm} (7)$$
$$G_{i5} = k \phi^2 A_i = -q_{i5} \hspace{1cm} (8)$$
$$G_{ij} = g_{ij} - k^2 \phi^2 A_i A_j = -q_{ij}, \hspace{0.5cm} i, j = 1, 2, 3. \hspace{1cm} (9)$$

For the inverse metric tensors,

$$G^{MN} = \left( \begin{array}{cc} 1/N^2, & -N^a/N^2 \\ -N^b/N^2, & -N^a N^b/N^2 - q^{ab} \end{array} \right) = \left( \begin{array}{cc} g^{\mu\nu}, & k A_\mu \\ k A_\nu, & k^2 A_\mu A_\nu - 1/\phi^2 \end{array} \right), \hspace{1cm} (10)$$

For the inverse metric tensors,
we obtain
\[ G^{00} = g^{00} = \frac{1}{N^2} \]
\[ G^{0i} = g^{0i} = -\frac{N^i}{N^2} \]
\[ G^{05} = kA^0 = -\frac{N^5}{N^2} \]
\[ G^{55} = k^2 A_\mu A^\mu - \frac{1}{\phi^2} = \frac{(N^5)^2}{N^2} - q^{55} \]
\[ G^{i5} = kA^i = -q^{i5} \]
\[ G^{ij} = g^{ij} - \frac{N^i N^j}{N^2} = q^{ij} \]  

The 4D metric \( g_{\mu\nu} \) can also be split according to ADM. This gives the 3D metric, \( \gamma_{ij} \), and its inverse, \( \gamma^{ij} \).

The matter part of the action (1) can be cast into the phase space form,
\[ I_m = \int d\tau \left[ p_M \dot{X}^M - \frac{\alpha}{2} (G^{MN} p_M p_N - M^2) \right] , \]
and split according to (2),(10). We obtain
\[ I_m = \int d\tau \left[ p_M \dot{X}^M - \frac{\alpha}{2} \left( \frac{1}{N^2} (p_0 - N^a p_a)^2 - q^{ab} p_a p_b - M^2 \right) \right] . \]

Using the ADM splitting, the gravitational part of the action can be written as
\[ I_G = \int d^5x (p^{ab} \dot{q}_{ab} - N \mathcal{H}_G - N^a \mathcal{H}_{Ga}) . \]

Here
\[ \mathcal{H}_G = -\frac{1}{\kappa} Q_{abcd} p^{cd} + \kappa \sqrt{q} \tilde{P}^{(4)} \]
\[ \mathcal{H}'_G = -2 D_b p^{ab} , \]
where \( \kappa = 1/(16\pi G) \), and \( Q_{abcd} = (1/\sqrt{q}) (-q_{ab} q_{cd}/(D - 1) + q_{ac} q_{bd} + q_{ad} q_{bc}) \) is the Wheeler-DeWitt metric in \( D \)-dimensions. In our case it is \( D = q_{ab} q^{ab} = 4 \).

Varying \( I_G \) with respect to \( p^{ab} \), we have the relation
\[ p^{ab} = \kappa \sqrt{q} (K^{ab} - K q^{ab}) , \]

where
\[ K_{ab} = \frac{1}{2N} (-\dot{q}_{ab} + D_a^{(4)} N_b - D_b^{(4)} N_a) . \]
Here, $\bar{R}^{(4)}$ and $D^{(4)}$ are, respectively, the Ricci scalar and the covariant derivative in the 4D space with the metric $g_{ab}$.

Our total phase space action

$$I = I_m + I_G \quad (24)$$

is a functional of the particle center of mass coordinates, $X^M$, of the momenta, $p_M$, of the metric $q_{ab}$ on a 4D slice, of the momenta $p^{ab}$, and of the set of the Lagrange multipliers, $\alpha, N, N^a$. Variation of the total action with respect to $\alpha$, $N$ and $N^a$ gives the following constraints:

$$\frac{1}{N^2}(p_0 - N^a p_a)^2 - q^{ab} p_a p_b - M^2 = G^{MN} p_M p_N - M^2 = 0, \quad (25)$$

$$-H_G + \delta^3(x - X) \delta(x^5 - X^5) \frac{1}{N}(p_0 - N^a p_a) = 0, \quad (26)$$

$$-H_{Ga} + \delta^3(x - X) \delta(x^5 - X^5) p_a = 0. \quad (27)$$

In deriving the last two equation we have taken into account that $(1/N^2)(p_0 - N^a p_a) = G^a_M p_M = \dot{X}^0/\alpha$, and have integrated the expressions

$$\int d\tau \frac{\alpha}{N^2}(p_0 - N^b p_b)^2 \delta^5(x - X(\tau)), \quad (28)$$

and

$$\int d\tau \frac{\alpha}{N^2} p_a (p_0 - N^b p_b) \delta^5(x - X(\tau)). \quad (29)$$

The integration $\int d^5x \delta^5(x - X(\tau)) = 1$ was inserted into $I_m$ in order to cast $I_m$ into a form, comparable to that of $I_G$. Let me repeat that $X^M(\tau)$ are the center of mass coordinates of an extended source, not of a point particle. The matter action (17) is thus an approximation to an action in which all other degrees of freedom of the extended object have been neglected.$^1$

Eqs. (26), (27) are an infinite set of constraints, one at each point $x^a = (x, x^5) \equiv \bar{x}$. If we multiply Eqs. (26), (27) by $e^{i k_a x^a}$, $a = 1, 2, 3, 5$, integrate over $d^4\bar{x} = d^3\bar{x} d\bar{x}^5$, we obtain the Fourier transformed constraints, one for each $k_a$:

$$-\int d^4\bar{x} e^{i k_a (\bar{x}^a - \bar{x}^b)} \mathcal{H}_G + \frac{1}{N}(p_0 - N^b p_b) \big|_{\bar{x} = 0} = 0, \quad (28)$$

$$-\int d^4\bar{x} e^{i k_5 (\bar{x}^a - \bar{x}^5)} \mathcal{H}_{Ga} + p_a \big|_{\bar{x} = 0} = 0. \quad (29)$$

For $k_a = 0$ (zero mode), and after fixing a gauge $N = 1, N_a = 0$, Eqs. (28), (29) become

$$\int d^4\bar{x} \mathcal{H}_G = p_0, \quad (30)$$

$$\int d^4\bar{x} \mathcal{H}_{Ga} = p_a. \quad (31)$$

$^1$See footnotes 1 and 2 of ref.[7]
Using (20), (21), we have

\[-\frac{1}{\kappa} \int d^4\bar{x} \left( Q_{abcd} \bar{p}^{ab} \bar{p}^{cd} + \kappa \sqrt{q} R^{(4)} \right) = p_a, \tag{32} \]

\[-2 \int d^4\bar{x} D_b p_a^b = -2 \oint d\Sigma_b p_a^b = p_a. \tag{33} \]

Splitting the above equations à la Kaluza-Klein by using Eqs. (4)–(16), it turns out that they contain the parts of the 4D gravity and the Maxwell theory. Eq. (32) can be written as

\[H_G = \int \! d^3x \left( H_g + H_{EM} + H_{\phi} \right) = p_0 \tag{34} \]

where according to Ref. [5]

\[H_g = -\frac{1}{\kappa^{(4)}} T_{ijkl} \pi^{ij} \pi^{k\ell} + \kappa^{(4)} \sqrt{\gamma} R^{(3)}, \tag{35} \]

\[H_{EM} = -\frac{2}{\kappa^{(4)} \sqrt{\gamma k^2 \phi^3}} \pi^i \dot{\pi}^j \gamma_{ij} - \frac{\kappa^{(4)}}{4} \sqrt{\gamma k^2 \phi^3} F_{ij} F^{ij} \tag{36} \]

\[H_{\phi} = -2\kappa^{(4)} \sqrt{\gamma} D^j D_i \phi - \frac{1}{6\kappa^{(4)}} \sqrt{\gamma} \pi^2_\phi + \frac{1}{3\kappa^{(4)} \sqrt{\gamma}} \pi^i \pi^{ij} \pi_{ij}, \tag{37} \]

with \( T_{ijkl} = (\gamma_{ik} \gamma_{j\ell} + \gamma_{il} \gamma_{jk} - \frac{2}{3} \gamma_{ij} \gamma_{k\ell}), i, j, k, \ell = 1, 2, 3 \), whereas \( \pi^{ij}, \pi^i, \pi_\phi \) are the canonical momenta conjugated to the spatial metric \( \gamma_{ij} \), the electromagnetic potential \( A_i \), and the scalar field \( \phi \), respectively.

Eq. (33) can be split according to

\[-2 \int \! d^4\bar{x} (D_i p_a^i + D_5 p_a^5) = p_a \tag{38} \]

Let us assume that \( D_5 p_a^5 = 0 \), because of the isometry along the 5th dimension (cylindricity condition). Then, for \( a = j \), we have

\[-2 \int \! d^4\bar{x} D_i p_j^i = -2 \oint d\Sigma_i p_j^i = p_j \tag{39} \]

where \( p_j^i \) can be split into the part due to the spatial metric \( \gamma^{ij} \), the part due to the electromagnetic field \( A_i \), and the part due to the scalar field \( \phi \) (see Ref. [5]).

For \( a = 5 \), using (22), (23), we find:

\[-2 \oint d\Sigma_i p_5^i = \oint d\Sigma_i \kappa \sqrt{q} \left[ -\gamma^{ij} \frac{d}{dt} (k \phi^2 A_j) + k A^i \frac{d}{dt} (\phi^2) \right] \]

\[= -\oint \kappa^{(4)} k \phi^3 \sqrt{\gamma} dS_i \dot{A}^i = p_5. \tag{40} \]
Here the hypersurface element in 4-space has been factorized according to $d\Sigma = dS_i dx^5$, and the determinant according to $\sqrt{q} = \phi \sqrt{\gamma}$. The integration over $dx^5$ then led to $\int k dx^5 \equiv \kappa^{(4)} \equiv \int d x^5 / (16\pi G) \equiv 1 / (16\pi G^{(4)})$.

Bear in mind that we have chosen the gauge $N = 1$, $N^a = 0$, which also implies $N_a = q_{ab} N^b = 0$. Then, from Eq. (3) it follows $A_0 = 0$ This is the temporal gauge for the electromagnetic potential. Therefore, the electromagnetic field, $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, has the components $F_{0i} = \partial_0 A_i - \partial_i A_0 = \partial_0 A_i \equiv \dot{A}_i$. Eq. (40) then reads

$$- \oint \kappa^{(4)} k \phi^3 \sqrt{\gamma} dS_i E_i = p_5.$$ \hspace{1cm} (41)

Because in the Kaluza-Klein theory the 5th component of a particle’s momentum is the electric charge, Eq. (41) is the Gauss law of electrodynamics.

### 3 Quantization

After quantization, the classical constraints (25)–(27) become the conditions on a state $|\Psi\rangle$:

$$(-G^{MN} p_M p_N - M^2) |\Psi\rangle = 0, \hspace{1cm} (42)$$

$$\frac{1}{\kappa} (Q_{abcd} p^{ab} p^{cd} - \kappa q^{R^{(4)}} |\Psi\rangle = -\delta^4 (\bar{x} - \bar{X}) p_0 |\Psi\rangle, \hspace{1cm} (43)$$

$$-2q_{ac} D_b p^{cb} |\Psi\rangle = \delta^4 (\bar{x} - \bar{X}) p_a |\Psi\rangle, \hspace{1cm} (44)$$

where $p_M$, $p^{ab}$ are now momentum operators, and $\delta^4 (\bar{x} - \bar{X}) \equiv \delta^3 (\bar{x} - \bar{x}) \delta (x^5 - X^5)$, $\bar{x} \equiv x^a$, $\bar{X} \equiv X^a$, $a = 1, 2, 3, 5$. The state $|\Psi\rangle$ can be represented as a wave function(al) $\langle T, X^a, q_{ab} |\Psi\rangle \equiv \Psi[T, X^a, q_{ab}]$, and the momentum operators as $p_M = -i \partial / \partial X^M$, $p^{ab} = -i \delta / \delta q_{ab}$. Integrating (43) and (44) over $d^4 \bar{x} \equiv d^3 x dx^5$ gives$^3$

$$\frac{1}{\kappa} \int d^4 \bar{x} \left( -Q_{abcd} \frac{\delta^2}{\delta q_{ab} \delta q_{cd}} - \kappa \sqrt{q} \bar{R}^{(4)} \right) |\Psi\rangle = i \frac{\partial |\Psi\rangle}{\partial T}, \hspace{1cm} (45)$$

$$-2 \int d^4 \bar{x} q_{cb} D_b \left( -i \frac{\delta |\Psi\rangle}{\delta q_{cb}} \right) = -i \frac{\partial |\Psi\rangle}{\partial X^a}. \hspace{1cm} (46)$$

Every solution to the quantum constraints (12)–(14) satisfies the Schrödinger equation (45) with the time $T \equiv X^0$. The opposite is not true: not every solution of the Schrödinger equation (45) does satisfy the full set of constraints (12)–(14). There is no term that could give infinite energy coupled to the 5D gravity. Instead of such annoying term, we have the term $i \partial |\Psi\rangle / \partial T$.

$^3$Here we neglect the ordering ambiguity issues.
We can envisage that there exists a particular, wave packet-like solution, \( \Psi[T, X^a, q_{ab}] \), that describes a 5D spacetime, split à la Kaluza-Klein. Then Eqs. (42)–(46) contain the pieces that correspond to the 4D gravity, to the electromagnetic field, and to the scalar field \( \phi \equiv -G_{55} \). For instance, Eq. (45) can then be written in the form

\[
H \left( -i \delta \gamma_{ij}, -i \delta A_i - i \delta \phi \right) \Psi[T, X^i, \gamma_{ij}, A_i, \phi] = i \partial \frac{\partial \Psi}{\partial T} \Psi[T, X^i, \gamma_{ij}, A_i, \phi]. 
\] (47)

The fifth component of Eq. (46) then becomes

\[
- \int d^3x \phi^3 \sqrt{\gamma} \partial_i \left( -i \delta \Psi \delta A_i \right) = -i \frac{\partial \Psi}{\partial X^5} = e\Psi. 
\] (48)

The above equations are the quantum versions of the classical equations (34)–(41).

In addition, the state \( |\Psi\rangle \) also satisfies Eq. (42), i.e., the 5D Klein-Gordon equation

\[
(-G^{MN} D_M D_N - M^2) \Psi = 0, 
\] (49)

that, after the Kaluza-Klein splitting becomes

\[
\left[ g^{\mu\nu}(-iD_\mu + eA_\mu)\left( -iD_\nu + eA_\nu \right) - m^2 \right] \Psi = 0, 
\] (50)

where \( m^2 = M^2 + e^2/\phi^2 \), and \( D_\mu^{(4)} \) the covariant derivative with respect to the 4D metric \( g_{\mu\nu} \).

Eq. (47) generalizes the functional Schrödinger equation for the electromagnetic field [10], whereas Eq. (48) generalizes the Gauss law constraint.

## 4 Arbitrary matter term in the action

In general, the matter term, \( I_m \), of the action is a functional of a set of fields \( \varphi^\alpha \). So we have the following total action:

\[
I = I_G[G^{MN}] + I_m[\varphi^\alpha, G^{MN}] 
\] (51)

For instance, if \( \alpha = 1, 2 \), then \( \varphi^\alpha \) can be the real an imaginary component of the charged scalar field. The matter action is then

\[
I_m = \frac{1}{2} \int d^5x \sqrt{-G} (G^{MN} \partial_M \varphi^\alpha \partial_N \varphi_\alpha - M^2 \varphi^\alpha \varphi_\alpha). 
\] (52)

After the ADM splitting, we have

\[
I_m = \frac{1}{2} \int dt d^4x N \sqrt{q} \left[ \frac{1}{2} \left( \varphi^\alpha - N^a \partial_a \varphi^\alpha \right) \left( \dot{\varphi}_\alpha - N^b \partial_b \varphi_\alpha \right) - q_{ab} \partial_a \varphi^\alpha \partial_b \varphi_\alpha - M^2 \varphi^\alpha \varphi_\alpha \right]. 
\] (53)
The Hamiltonian, corresponding to the action (51) is

\[ H = -\int d^4\bar{x} \left( N \frac{\delta I}{\delta N} + N^a \frac{\delta I}{\delta N^a} \right), \quad (54) \]

where \(-\delta I/\delta N = \mathcal{H} = \mathcal{H}_G + \mathcal{H}_m\), and \(-\delta I/\delta N^a = \mathcal{H}_a = \mathcal{H}_{G_a} + \mathcal{H}_{m_a}\) are the constraints.

Here \(H_m = \int d^4\bar{x} \mathcal{H}_m\) is the Hamiltonian for the matter fields. In the case in which \(I_m\) is given by Eq. (53), it is

\[ H_m = -\int d^4\bar{x} \frac{\delta I_m}{\delta N} = \frac{1}{2} \int d^4\bar{x} \frac{1}{\sqrt{q}} \left( \Pi^a \Pi_a + q^{ab} \partial_a \varphi^a \partial_b \varphi_a + M^2 \varphi^a \varphi_a \right), \quad (55) \]

where

\[ \Pi_a = \frac{\partial L_m}{\partial \dot{\varphi}^a} = \sqrt{q} \frac{\dot{\varphi}_a}{N} (\dot{\varphi}_a - N^a \partial_a \varphi_a). \quad (56) \]

Upon quantization, we have

\[ (H_G + H_m)\Psi = 0. \quad (57) \]

In the usual approaches to quantum field theories, where gravity is not taken into account, one does not assume the validity of the constraint equation Eq. (57), but of the Schrödinger equation

\[ H_m\Psi = i\frac{\partial \Psi}{\partial t}. \quad (58) \]

But we see, that within the more general setup with gravity, the validity of the Schrödinger equation (58) cannot be taken for granted. Eq. (58) is presumably incorporated in the constraint equation (57), and this has to be derived. Various authors have worked on such problem [11] of how to derive \(i\partial \Psi/\partial T\) from \(H_G\).

The opposite, namely how to derive \(i\partial \Psi/\partial T\) from \(H_m\) in order to obtain from (57) the equation \(H_G = i\partial \Psi/\partial T\), is also an interesting problem. There is a lot of discussion in the literature on such problem [12]. Let me show here a possible procedure. Despite that our procedure refers to the 5D gravity, it holds also for the usual, 4D, gravity.

From the stress-energy tensor

\[ T^{MN} = a \left[ \partial^M \varphi^* \partial^N \varphi - \frac{1}{2} G^{MN} (G^{JK} \partial_J \partial_K - M^2 \varphi^* \varphi) \right], \quad (59) \]

after taking the Ansatz

\[ \varphi = A e^{iS}, \quad (60) \]

we obtain the following expression for the field momentum:

\[ P^M = \int \sqrt{-G} d\Sigma_N T^{MN} = a \int \sqrt{-G} d\Sigma_N A^2 \partial^M S \partial^N S. \quad (61) \]
Here we have taken into account that \( \varphi \) satisfies the Klein-Gordon equation, which in the limit \( \hbar \to 0 \) gives
\[
\partial_M S \partial^M S - M^2 = 0,
\]
implying that the second term in Eq. (59) vanishes.

Let us now assume that \(|\varphi| = A^2\) is picked around the classical particle worldline. As a convenient approximation let us take
\[
A^2 = \int d\tau \frac{\delta^5(x - X(\tau))}{\sqrt{G}}. 
\]

Since \( p_N = \partial_N S \), we obtain
\[
P^M = a \int d\Sigma d\tau \delta^5(x - X(\tau))p^M p^N . \tag{63}
\]

Assuming that \( d\Sigma_N = p_N/\sqrt{p^2} d\Sigma \), where \( d\Sigma = d^4\bar{x} \), taking a gauge \( X^0 = \tau \), i.e., \( \dot{X}^0 = 1 \), and integrating over \( \tau \), we find
\[
P^M = a \int d\Sigma d\tau M p^M \delta^4(\bar{x} - \bar{X}) = aMp^M = p^M . \tag{64}
\]

We see that the field momentum is equal to the particle’s momentum, if the normalization constant is \( a = 1/M \).

Alternatively, if we do not integrate over \( \tau \) in Eq. (63), we have
\[
P^M = a \int d\Sigma d^5x \delta^5(x - X(\tau)). \tag{65}
\]

In the gauge in which \( \tau = x^0 \), it is \( d\Sigma d\tau = d^4\bar{x} dx^0 = d^5x \). Integrating over \( d^5x \), we obtain the same result as in Eq. (64).

This was a classical theory. Upon quantization, the momentum becomes the operator \( p_M = -i\partial/\partial X^M \), in particular, \( p_0 = -i\partial/\partial X^0 \equiv -i\partial/\partial T \). Then Eq. (57) becomes
\[
(H_G - i\frac{\partial}{\partial T})|\Psi\rangle = 0, \tag{66}
\]
which corresponds to our equation (45), derived from the total action (24) with the point particle matter term.

Since we consider a five or higher dimensional spacetime, we can perform the Kaluza-Klein splitting. Then Eq. (57) contains the terms due to the 4D gravity and the terms due to the electromagnetic or Yang-Mill fields:
\[
(H_G + H_{EM} + H_m + \ldots)|\Psi\rangle = 0. \tag{67}
\]
All those terms together form a constraint on a state vector. There is no explicit time derivative term. We have two basically different possibilities:

(a) A time derivative term comes from $H_g$ as an approximation. Then the system (67) becomes the Schrödinger equation for the electromagnetic field in the presence of "matter":

$$\left(-i \frac{\partial}{\partial T} + H_{EM} + H_m\right) |\Psi\rangle = 0. \tag{68}$$

We have considered the case in which matter consists of a charged scalar field. We could as well consider a spinor field.

(b) A time derivative term comes from $H_m$ as an approximation. Then Eq. (67) describes the evolution of the electromagnetic and the gravitational field:

$$\left(H_g + H_{EM} - i \frac{\partial}{\partial T}\right) |\Psi\rangle = 0. \tag{69}$$

In general, both equations, (68) and (69) are approximations to the constraint (67). In particular, if for the matter term in the classical action (51), instead of $I_m[\varphi^\alpha, G^{MN}]$, we take the “point particle” action $I_m[X^M, G^{MN}]$, then—as shown in Secs. 2 and 3—we also arrive at Eq. (69). This is then an “exact” equation, because the term $-i\partial/\partial T$ comes directly from $p_0$ of the “point particle”.

If in Eq. (67) we do not ticker with the term $H_m$, but leave it as it is, then it gives infinite vacuum energy.

5 Discussion

We have considered five dimensional gravity in the presence of a source whose center of mass was described by a point particle action. After performing the ADM splitting and varying the action with respect to the lapse and shift functions, we obtained the Hamiltonian constraint and four momentum constraints. In addition, we also obtained the constraint coming from the reparametrization invariance of the point particle term in the total action. In the quantized version of the theory, all those constraints act on a state that can be represented as $\Psi[T, X^a, q_{ab}]$, a function(al) of the particle’s coordinates $X^M = (T, X^a)$, and of of the 4D metric, $q_{ab}$, $a, b = 1, 2, 3, 5$. The $\Psi$ satisfies the Wheeler-DeWitt equation in which the term due to the presence of the particle is $-i\partial\Psi/\partial T$. It also satisfies quantum momentum constraints with a term $-i\partial/\partial X^a$. Besides that, the $\Psi[X^M, q_{ab}]$ satisfies the Klein-Gordon equation in curved space. Also in the usual theories the Klein-Gordon field in a curved space is a functional of the (background) metric. In our approach the metric is not a background metric. It is a dynamical metric, therefore the wave functional $\Psi[X^M, q_{ab}]$ satisfies the Wheeler-DeWitt equation as well.
If we split the 5D metric à la Kaluza-Klein, then the equations split into the terms describing the 4D gravity and electrodynamics. In the quantized theory we obtain the functional representation of quantum electrodynamics in the presence of gravity. But there are some subtleties here, because according to the usual theory [10], also a term due to the stress-energy of a charged scalar field or a spinor field should be present in Eq. (47). There is no such term in Eq. (47), because we have started from the classical action (1) with a “point particle” matter term. The corresponding stress-energy tensor has—amongst others—the five components $T_{00}$, $T_{0a}$, $a = 1, 2, 3, 5$, as given in Eqs. (26), (27). Integrating over $d^4\bar{x}$, we obtain the particle’s 5-momentum $(p_0, p_a)$ that, after quantization becomes $(-i\partial/\partial T, -i\partial/\partial X^a)$. The term $i\partial/\partial T$ in the Schrödinger equation (43) thus comes from the stress-energy of a “point particle”.

In the usual approaches, one does not start from the action (1) with a “point particle” matter term, but from an action with a charged scalar field, $\varphi$, or a spinor field, $\psi$. In Sec. 4 we explored how this works in five dimensions. The Kaluza-Klein splitting of the 5D gravity in the presence of a charged scalar or spinor field gives, after quantization, a wave functional equation (67) without the time derivative term. In such approach, the notorious “problem of time” remains. On the other hand, in the textbook formulation [10] of the Schrödinger representation of quantum electrodynamics that is not derived from a 5D or a higher dimensional gravity, one has the term $i\partial\Psi/\partial T$, besides the energy term due to $\varphi$ or $\psi$. According to the existing literature [11], such time derivative term can occur from the gravitational part of the total Hamiltonian. So we obtained Eq. (68). We have also shown how the matter part of the Lagrangian with the scalar fields can give the time derivative term. Thus we obtained Eq. (69). So we have a relation between the approach that starts from the classical action $I[X^M, G_{MN}]$, and the usual approach that start form $I[\varphi^\alpha, G_{MN}]$. But there is a crucial difference, because in the former approach, after quantization, a wave functional $\Psi[X^M, q_{ab}]$ satisfies the Klein-Gordon equation and the Wheeler-DeWitt equation, whereas in the latter approach we have a wave functional $\Psi[\varphi^\alpha, q_{ab}]$ that satisfies the Wheeler-DeWitt equation only.

Having in mind that we usually consider a classical theory and its quantization, it seems natural to start from classical objects, e.g., particles, described by $X^M$, coupled to gravity, described by $G_{MN}$, so that after quantization we obtain a wave functional $\Psi[X^M, q_{ab}]$. Having a wave functional $\Psi[X^M, q_{ab}]$, we can envisage its second quantization, so that $\Psi$ and its Hermitian conjugate are related to the operators that create at $X^M$ a particle with a surrounding gravitational field $q_{ab}$. This brings new directions for further development of quantum field theories, including gravitational, electromagnetic, and Yang-Mills fields that arise in higher dimensional spacetimes.

\footnote{Moreover, because of the infinite vacuum energy density of the charged scalar or spinor field coupled to gravity, there is the problem of the cosmological constant.}
6 Conclusion

From the Wheeler-DeWitt equation in five dimensions we have obtained, depending on choice of a matter term, two different versions of modified quantum electrodynamics in the Schrödinger representation. The five dimensional gravity with matter was only a toy model. A more realistic theory, describing all fundamental interactions, should be formulated in higher dimensions [3]. Since QED is a theory that in many respects works very well, this indicates that also the higher dimensional Wheeler-DeWitt equation, into which QED is embedded, could be—to a certain extent—a valid description of Nature. On the other hand, for many reasons gravity—regardless of the space-time dimensionality—cannot be considered as a complete, but rather as an effective theory arising from a more fundamental theory. The underlying more fundamental theory could have roots in any of the currently investigated fields of research such as strings [13], branes [14], brane worlds [15], loop quantum gravity [16], gravity as entropic force [17], etc. There could also be some new, not yet explored landscape of theoretical physics [18]–[20].

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