MEDIAN PRETREES AND FUNCTIONS OF BOUNDED VARIATION

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Abstract. We introduce functions of bounded variation on median algebras and study some properties for median pretrees. We show that if $X$ is a compact median pretree in its shadow topology then every function $f: X \to \mathbb{R}$ of bounded variation has the point of continuity property (Baire 1, if $X$, in addition, is metrizable). We prove a generalized version of Helly’s selection theorem for a sequence of functions with total bounded variation defined on a compact metrizable median pretree $X$.

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1. Introduction

Our aim is to introduce and study functions of bounded variation on median algebras and pretrees. This was motivated by recent papers [16, 10] and especially by [10, Remark 4.11], where we deal with some applications of median pretrees in topological dynamics.

In the present work we prove the following two theorems (3.12 and 3.14 below).

Theorem A. Let $X$ be a median pretree (e.g., dendron or a linearly ordered space) such that its natural shadow topology is compact or Polish. Then every function $f: X \to \mathbb{R}$ with bounded variation has the point of continuity property (Baire 1 class function, if $X$ is Polish).

Theorem B. (Generalized Helly’s selection theorem) Let $X$ be a Polish (e.g., compact metrizable) median pretree. Then every sequence $\{f_n : X \to [c, d]\}_{n \in \mathbb{N}}$ of functions with total bounded variation $\leq r$ has a pointwise converging subsequence.

Recall that a topological space $X$ is said to be Polish if it is homeomorphic to a separable complete metric space. A continuum is a compact Hausdorff connected space. A continuum $D$ is said to be a dendron [17] if every pair of distinct points $u, v$ can be separated in $D$ by a third point $w$. A metrizable dendron is called a dendrite.
The class of dendrons is an important class of 1-dimensional treelike compact spaces, \[17, 4\]. Group actions on dendrites is an attractive direction in dynamical systems theory (see \[6, 10\] and references therein).

Functions of bounded variation on median algebras we define in Section 3 (Definition 3.3). In Section 2 we recall definition and auxiliary properties of median pretrees. As to the point of continuity property and fragmented functions, see Subsection 2.3. Note that such functions play a major role in Bourgain-Fremlin-Talagrand theory, \[2\] which in turn is strongly related to the classical work of Rosenthal \[20\]. One of the results from \[2\] allows us to derive Theorem B from Theorem A.

Weaker versions of these theorems for linearly ordered spaces and BV functions proved in \[16\], for median pretrees and monotone functions in \[10\].

2. Related structures

Pretree (in terms of B.H. Bowditch) is a useful treelike structure which naturally generalizes several important structures including linear orders and the betweenness relation on dendrons.

2.1. Pretrees.

**Definition 2.1.** By a pretree (see for example \[3, 14\]) we mean a pair \((X, R)\) where \(X\) is a set and \(R\) is a ternary relation on \(X\) (we write \(\langle a, b, c \rangle \) to denote \((a, b, c) \in R)\) satisfying the following three axioms:

1. **(B1)** \(\langle a, b, c \rangle \Rightarrow \langle c, b, a \rangle\).
2. **(B2)** \(\langle a, b, c \rangle \land \langle a, c, b \rangle \iff b = c\).
3. **(B3)** \(\langle a, b, c \rangle \Rightarrow \langle a, b, d \rangle \lor \langle d, b, c \rangle\).

In \[11\] such a ternary relation is called a B-relation.

It is convenient to use also an interval approach. For every \(u, v \in X\) define

\([u, v]_X := \{x \in X : \langle u, x, v \rangle\}\).

Sometimes we write simply \([u, v]\), where \(X\) is understood.

**Remark 2.2.** The conditions (A0),(A1),(A2),(A3), as a system of axioms, is equivalent to the above definition via (B1), (B2), (B3) (see \[14\]). In every pretree \((X, R)\) we have

1. **(A0)** \([a, b] \supseteq \{a, b\}\).
2. **(A1)** \([a, b] = [b, a]\).
3. **(A2)** If \(c \in [a, b]\) and \(b \in [a, c]\) then \(b = c\).
4. **(A3)** \([a, b] \subseteq [a, c] \cup [c, b]\) for every \(a, b, c \in X\).

Every subset \(Y\) of \(X\) carries the naturally defined betweenness relation. In this case the corresponding intervals are \([a, b]_Y = [a, b] \cap Y\).

For every linear order \(\leq\) on a set \(X\) we have the induced pretree \((X, R_\leq)\) defined by

\(\langle a, b, c \rangle \iff (a \leq b \leq c) \lor (c \leq b \leq a)\).

Note that the opposite linear order defines the same betweenness relation.

A subset \(A\) of \(X\) is said to be convex if \([a, b] \subseteq A\) for every \(a, b \in A\). Intersection of convex subsets is convex (possibly empty). For a subset \(A \subseteq X\) the convex hull \(\text{co}(A)\) is the intersection of all convex subsets of \(X\) which contain \(A\).
Let us say that \( a, b, c \in X \) are collinear if
\[
a \in [b, c] \lor b \in [a, c] \lor c \in [a, b].
\]
A subset \( Y \) of \( X \) is linear (see [14, Section 3]) if all \( a, b, c \in Y \) are collinear.

By a direction on a linear subset \( Y \) in a pretree \( X \), we mean a linear order \( \leq \) on \( Y \) such that, \( R_\leq \) is just the given betweenness relation on \( Y \). Each nontrivial linear subset \( Y \) in a pretree \( X \) admits precisely two directions.

Following A.V. Malyutin [14] (which in turn follows to the terminology of P. de la Harpe and J.-P. Preaux) we define the so-called shadow topology. Alternative names in related structures are: Lawson’s topology and observer’s topology. See a discussion in [14].

Given an ordered pair \((u, v) \in X^2, u \neq v\), let
\[S_v^u := \{x \in X : u \in [x, v]\}\]
be the shadow in \( X \) defined by the ordered pair \((u, v)\). Pictorially, the shadow \( S_v^u \) is cast by a point \( u \) when the light source is located at the point \( v \). The family \( S = \{S_v^u : u, v \in X, u \neq v\} \) is a subbase for the closed sets of the topology \( \tau_s \). The complement of \( S_v^u \) is said to be a branch
\[v \in \zeta_u^v := X \setminus S_v^u = \{x \in X : u \notin [x, v]\}.
\]
The set of all branches \( \{\zeta_u^v : u, v \in X, u \neq v\} \) is a subbase neighborhood of \( v \).

In the case of a linearly ordered set we get the interval topology. In general, for an abstract pretree, the shadow topology is often (but not always) Hausdorff. Furthermore, by [14, Theorem 7.3] a pretree equipped with its shadow topology is Hausdorff if and only if, as a topological space, it can be embedded into a dendron.

**Lemma 2.3.** Let \( X \) be a pretree.

1. [14, Lemma 1.16 (A6)] \( \forall c \in [a, b] \quad [a, c] \cap [c, b] = \{c\} \).
2. [14, Lemma 2.8] For every subset \( A \subset X \) its convex hull is
\[\text{co}(A) = \bigcup\{[a, b] : a, b \in A\}\].
3. [14, Lemma 3.3.4] \( [a, b] \) is a convex linear subset for every \( a, b \in X \).
4. [14, Lemma 5.10.2] Every branch is convex. Hence, every pretree is locally convex.
5. [14, Prop. 6.5] Let \( S \) be a subset in a pretree \( X \). Then the shadow topology on \( S \) (regarded as a pretree with the structure induced by that of \( X \)) is contained in the relativization of the shadow topology on \( X \) to \( S \). If \( S \) is convex in \( X \), then the two topologies above coincide.

### 2.2. Median algebras and pretrees.

A median algebra (see, for example, [21, 3]) is a pair \((X, m)\), where the function \( m : X^3 \to X \) satisfies the following three axioms:

1. (M1) \( m(x, x, y) = x \).
2. (M2) \( m(x, y, z) = m(x, y, z) = m(y, z, x) \).
3. (M3) \( m(m(x, y, z), u, v) = m(x, m(y, u, v), m(z, u, v)) \).

This concept has been studied for a long time (Birkhoff-Kiss, Grau, Isbell) and has applications in abstract convex structures, [21].
Every distributive lattice \((L, \land, \lor)\) (e.g., any power set \(P(S) := \{A : A \subseteq S\}\)) is a median algebra with the median operation
\[
m(a, b, c) := (a \land b) \lor (b \land c) \lor (c \land a).
\]

Very particular case of this is a linearly ordered set.

Let \((X, m)\) be a median algebra. A subset \(Y \subseteq X\) is a subalgebra if it is median-closed in \(X\). In a median pretree \((X, m)\) for every subset \(A\) there exists the subalgebra generated by \(A\) which is defined as \(sp(A) := \{x \in X : x = m(a, b, c), a, b, c \in A\}\).

In every median algebra \((X, m)\) we have the naturally defined intervals \([a, b] := \{m(a, x, b) : x \in X\}\).

This leads to the natural ternary relation \(R_m\) defined by \(\langle a, c, b \rangle \iff c = m(a, c, b)\), equivalently \(\langle a, c, b \rangle \iff c \in [a, b]\). Note that not every median algebra is a pretree under the relation \(R_m\). A subset \(C\) is convex if \([a, b] \subseteq C\) for every \(a, b \in C\). Every convex subset is a subalgebra.

For every triple \(a, b, c\) in a pretree \(X\) the median \(m(a, b, c)\) is the intersection
\[
m(a, b, c) := [a, b] \cap [a, c] \cap [b, c].
\]

When it is nonempty the median is a singleton. \([3, 14]\). A pretree \((X, R)\) for which this intersection is always nonempty is called a median pretree.

**Remarks 2.4.**

1. Every median pretree \((X, R)\) is a median algebra. The corresponding ternary relation \(R_m\) induced by the median function coincides with \(R\).
2. A map \(f : X_1 \to X_2\) between two median algebras is monotone (i.e., \(f[a, b] \subseteq [f(a), f(b)]\)) if and only if \(f\) is median-preserving ([21 page 120]) if and only if \(f\) is convex ([21 page 123]) (convexity of \(f\) means that the preimage of a convex subset is convex).
3. Every median pretree is Hausdorff (and normal) in its shadow topology ([14 Theorem 7.3]).
4. ([14] Prop. 6.7) In a median pretree, the convex hull of a closed set is closed. In particular, the intervals \([a, b]\) are closed subsets.
5. It is a well known (nontrivial) fact that for every finite subset \(F \subseteq X\) in a median algebra the induced subalgebra \(sp(F)\) is finite, [21].

A compact (median) pretree is a (median) pretree \((X, R)\) for which the shadow topology \(\tau_s\) is compact. Similarly can be defined Polish pretrees.

**Examples 2.5.**

1. Every dendron \(D\) is a compact median pretree with respect to the standard betweenness relation \(R_B\) (\(w\) is between \(u\) and \(v\) in \(X\) if \(w\) separates \(u\) and \(v\) or if \(w \in \{u, v\}\)). Its shadow topology is just the given compact Hausdorff topology on \(D\) (see [17, 14]).
2. Every linearly ordered set is a median pretree with respect to the median \(m(a, b, c) = b\) iff \(a \leq b \leq c\) or \(c \leq b \leq a\). Its shadow topology is just the usual interval topology of the order.
3. Let \(X\) be a \(\mathbb{Z}\)-tree (a median pretree with finite intervals \([u, v]\)). Denote by \(Ends(X)\) the set of all its ends. According to [14 Section 12] the set \(X \cup Ends(X)\) carries a natural \(\tau_s\)-compact median pretree structure.
2.3. **Fragmented functions.** Recall the definition of fragmentability which comes from Banach space theory \([13, 18, 12]\) and effectively used also in dynamical systems theory \([15, 8, 9]\).

**Definition 2.6.** Let \(f : (X, \tau) \to (M, d)\) be a function from a topological space into a metric space. We say that \(f\) is fragmented if for every nonempty subset \(A \subset X\) and every \(\varepsilon > 0\) there exists a \(\tau\)-open subset \(O \subset X\) such that \(O \cap A\) is nonempty and \(\text{diam}(f(O \cap A)) < \varepsilon\). If \(M = \mathbb{R}\) then we use the notation \(f \in \mathcal{F}(X)\).

**Lemma 2.7.**

1. \([3]\) When \(X\) is compact or, Polish then \(f : X \to \mathbb{R}\) is fragmented iff \(f\) has the point of continuity property (i.e., for every closed nonempty \(A \subset X\) the restriction \(f|_A : A \to \mathbb{R}\) has a continuity point).

2. \([5\ p. 137]\) For every Polish space \(X\) we have \(\mathcal{F}(X) = B_1(X)\), where \(B_1(X)\) is the set of all Baire 1 functions \(X \to \mathbb{R}\).

3. \([5]\) Lemma 3.7] Let \(X\) be a compact or a Polish space. Then the following conditions are equivalent for a function \(f : X \to \mathbb{R}\).
   (a) \(f \in \mathcal{F}(X)\);
   (b) there exists a closed subspace \(Y \subset X\) and real numbers \(\alpha < \beta\) such that the subsets \(f^{-1}(-\infty, \alpha) \cap Y\) and \(f^{-1}(\beta, \infty) \cap Y\) are dense in \(Y\).

4. \([2\ Section 3]\) For every Polish space \(X\) every pointwise compact subset of \(B_1(X)\) is sequentially compact (see also \([5\ Thm. 3.13]\))

**Lemma 2.8.** Let \(f : (X, \tau) \to (M, d)\) be a function from a topological space into a metric space. Suppose that \(X = \bigcup_{i=1}^{n}Y_i\) is a finite covering of \(X\) such that every \(Y_i\) is closed in \(X\) and every restriction function \(f|_{Y_i} : (Y_i, \tau_{Y_i}) \to (M, d)\) is fragmented. Then \(f : (X, \tau) \to (M, d)\) is also fragmented.

**Proof.** Since finite union of closed subsets is closed one may reduce the proof to the case of two subsets. So, assume that \(X = Y_1 \cup Y_2\) and \(f|_{Y_1} : Y_1 \to M, f|_{Y_2} : Y_2 \to M\) are fragmented. Let \(\varepsilon > 0\) and \(A \subset X\) be a nonempty subset. We have to show that

\[
\exists O \in \tau \quad O \cap A \neq \emptyset \quad \text{and} \quad \text{diam}(f(O \cap A)) < \varepsilon.
\]

There are two cases: (a) \(A \subset Y_1 \cap Y_2\) and (b) \(A \not\subset Y_1 \cap Y_2\). In the first case using the fragmentability of \(f|_{Y_1}\) choose \(O \in \tau\) such that \((O \cap Y_1) \cap A \neq \emptyset\) and \(\text{diam}(f((O \cap Y_1) \cap A)) < \varepsilon\). Since in case (a) we have \(A \subset Y_1\) then \((O \cap Y_1) \cap A = O \cap A\). Hence, the condition (2.1) is satisfied.

Now consider (b) \(A \not\subset Y_1 \cap Y_2\). Then \((A \cap Y_1) \setminus Y_2 \neq \emptyset\) or, \((A \cap Y_2) \setminus Y_1 \neq \emptyset\). We will check only the first possibility (the second, is similar). Using the fragmentability of \(f|_{Y_1}\) choose for the subset \((A \cap Y_1) \setminus Y_2\) an open subset \(U \in \tau\) in \(X\) such that

\[
(U \cap Y_1) \cap (A \cap Y_1) \setminus Y_2 \neq \emptyset
\]

and \(\text{diam}(f((U \cap Y_1) \cap (A \cap Y_1) \setminus Y_2))) < \varepsilon\). Now observe that

\[
(U \cap Y_1) \cap (A \cap Y_1) \setminus Y_2 = (U \cap Y_1) \cap (A \cap Y_2^c) = (U \cap Y_2^c) \cap A.
\]

Then \(O := U \cap Y_2^c\) is the desired open subset in \(X\). \(\square\)
3. Functions of bounded variation

3.1. Functions on linearly ordered sets.

Definition 3.1. Let \((X, \leq)\) be a linearly ordered set. We say that a bounded function \(f: (X, <) \to \mathbb{R}\) has variation not greater than \(r\) (notation: \(f \in BV_r\)) if
\[
\sum_{i=1}^{n-1} |f(x_i) - f(x_{i+1})| \leq r
\]
for every choice of \(x_1 \leq x_2 \leq \cdots \leq x_n\) in \(X\).

The following was proved in \([16]\) using the particular case of order-preserving maps and Jordan type decomposition for functions with BV.

Theorem 3.2. Let \((K, \leq)\) be a compact linearly ordered topological space (with its interval topology). Every function \(f: K \to \mathbb{R}\) with bounded variation is fragmented.

3.2. Functions on median algebras. Let \(X\) be a set with a ternary relation \(R\) and \(Y \subseteq X\) be a subset. A two-element subset \((\text{doublet})\) \(\{a, b\} \subset Y\) is said to be \(Y\)-adjacent (or, \(Y\)-gap) if \(\langle a, c, b \rangle \Rightarrow c = a\) or \(c = b\). In terms of intervals: \([a, b]_X \cap Y = \{a, b\}\).

By \(\text{adj}(Y)\) we denote the set of all \(Y\)-adjacent doublets. For median algebras we use the standard ternary relation \(R_m\) as in Remark 2.4.1. In particular, for dendrons it is exactly the standard betweenness relation.

Definition 3.3. Let \(f: X \to \mathbb{R}\) be a bounded real valued function on a median algebra \((X, m)\) and \(\sigma \subset X\) is a finite subalgebra. By a variation \(\Upsilon(f, \sigma)\) of \(\sigma\) we mean
\[
\Upsilon(f, \sigma) := \sum_{\{a, b\} \in \text{adj}(\sigma)} |f(a) - f(b)|.
\]

The least upper bound
\[
\sup\{\Upsilon(f, \sigma) : \sigma \text{ is a finite subalgebra in } X\}
\]
is the variation of \(f\). Notation: \(\Upsilon(f)\). If it is bounded, say if \(\Upsilon(f) \leq r\) for a given positive \(r \in \mathbb{R}\), then we write \(f \in BV_r(X)\). If \(f(X) \subset [c, d]\) for some \(c \leq d\) then we write also \(f \in BV_r(X, [c, d])\). One more notation: \(BV(X) := \cup_{r>0}BV_r(X)\).

Note that \(BV(X)\) is closed under linear operations.

Remark 3.4. I am not aware of any other definition of the variation for functions on median algebras or on median pretrees. In \([7]\) the authors study a treelike system — "rooted nonmetric tree". In paragraph 7.4 they define functions of bounded variation on such objects. This definition essentially differs from our definition. In this article we examine Definition 3.3 mainly in the case when \(X\) is a median pretree. Probably Theorems A and B remain true for some classes of topological median algebras under reasonable assumptions.

Sometimes, we use the following relative version of 3.3.
Definition 3.5. Let $S \subset X$ be a subset of a median algebra $X$. By an $S$-variation $\Upsilon(f, \sigma)$ of $\sigma$ on $S$ we mean

$$\Upsilon(f, \sigma)|_S := \sum_{\{a, b\} \in \text{adj} \cap P(S)} |f(a) - f(b)|.$$  

Similarly can be defined the variation of $f$ on $S \subset X$ which we denote by $\Upsilon(f)|_S$.

Clearly, $\Upsilon(f, \sigma)|_S \leq \Upsilon(f, \sigma)$ and $\Upsilon(f)|_S \leq \Upsilon(f)$ for every $S \subset X$.

Lemma 3.6. Let $\sigma$ be a finite subalgebra in a median algebra $X$.

1. For every disjoint subsets $S_1, S_2$ in $X$ we have
   $$\Upsilon(f, \sigma) \geq \Upsilon(f, \sigma)|_{S_1} + \Upsilon(f, \sigma)|_{S_2}.$$ 

2. $\Upsilon(f, \sigma)|_S \leq \Upsilon(f, \sigma \cap S)$ for every subalgebra $S \subset X$.

3. $\Upsilon(f, \sigma)|_C = \Upsilon(f, \sigma \cap C)$ for every convex subset $C \subset X$.

4. $\Upsilon(f, \sigma) \geq \Upsilon(f, \sigma \cap C_1) + \Upsilon(f, \sigma \cap C_1)$ for every disjoint convex subsets $C_1, C_2$ of $X$.

Proof. (1) Trivial.

(2) $\sigma \cap S$ is a finite subalgebra of $X$. Hence, $\Upsilon(f, \sigma \cap S)$ is well defined. If $\{a, b\} \in \text{adj} \cap P(S)$ then $\{a, b\} \in \text{adj} \cap \sigma$.

(3) By (2) it is enough to show the inequality $\Upsilon(f, \sigma)|_C \geq \Upsilon(f, \sigma \cap C)$. It suffices to prove that if $\{a, b\} \in \text{adj} \cap C$ then $\{a, b\} \in \text{adj} \cap \sigma$. Assuming the contrary let $\langle x, a, b \rangle$ for some $x \in \sigma$ with $x \notin \{a, b\}$ and $x \in [a, b] \setminus \{a, b\} \subset C$ by the convexity of $C$ and we get $\{a, b\} \notin \text{adj} \cap C$, a contradiction.

(4) Combine (1) and (3). 

Remarks 3.7.

1. If $(X, \leq)$ is a linearly ordered set consider the induced median algebra
   $$m(x, y, z) = y \Leftrightarrow x \leq y \leq z \lor z \leq y \leq x.$$ 

   Then $\Upsilon_{R\leq}(f, \sigma) = \Upsilon(f, \sigma)$, where $\Upsilon(f, \sigma)$ computed as in Definition 3.3 and $\Upsilon_{R\leq}(f, \sigma)$ as in 3.1.

2. Let $X = \{a, b, c, m\}$ be the "4-element triod", where $m = m(a, b, c)$ is the only "nontrivial median". Then for every $f : X \to \mathbb{R}$ we have
   $$\Upsilon(f) = |f(a) - f(m)| + |f(b) - f(m)| + |f(c) - f(m)|$$

Proposition 3.8. Let $X$ and $Y$ be median pretrees, $f : Y \to \mathbb{R}$ be a bounded function and $h : X \to Y$ be a monotone map.

1. Suppose that $\sigma_1$ is a finite subalgebra in $X$ and $\sigma_2$ is a finite subalgebra in $Y$ such that $h(\sigma_1) \subset \sigma_2$. Then we have
   $$\Upsilon(f \circ h, \sigma_1) \leq \Upsilon(f, \sigma_2).$$

2. $\Upsilon(f \circ h) \leq \Upsilon(f)$.

Proof. It is enough to show (1) because (2) is a direct consequence of (1).

Let $\{s, t\} \in \text{adj} \sigma_1$. Consider the interval $[h(s), h(t)]_{\sigma_2}$. It is finite (because $\sigma_2$ is finite). By Lemma 2.3 it is a linear subset. Let
   $$[h(s), h(t)]_{\sigma_2} = \{h(s) = y_1, y_2, \ldots, y_n = h(t)\}$$
be its list of elements linearly ordered (according to the direction where \( h(s) \) is the smallest element). It is possible that \( \{h(s), h(t)\} \notin \text{adj}(\sigma_2) \) (i.e., \( n > 2 \)).

For every \( i < j < k \) we have \( \langle y_i, y_j, y_k \rangle \). Using Lemma 2.3.2 every doublet \( \{y_i, y_{i+1}\} \) (where \( 1 \leq i \leq n - 1 \)) is \( \sigma_2 \)-adjacent. Clearly,

\[
| (f \circ h)(s) - (f \circ h)(t) | = | f(h(s)) - f(h(t)) | \leq \sum_{i=1}^{n} | f(y_i) - f(y_{i+1}) | .
\]

Now, in order to check \( \Upsilon(f \circ h, \sigma_1) \leq \Upsilon(f, \sigma_2) \), it is enough to verify the following

CLAIM: If \( \{s_1, t_1\} \in \text{adj}(\sigma_1) \) and \( \{s_2, t_2\} \in \text{adj}(\sigma_1) \) then \( [h(s_1), h(t_1)]_Y \cap [h(s_2), h(t_2)]_Y \) is at most a singleton.

Proof. First of all note that the subset \( S := \{s_1, t_1, s_2, t_2\} \subset X \) is linear. Indeed, \( m(s_1, t_1, s_2) \in \{s_1, t_1\} \). Otherwise, \( \{s_1, t_1\} \) is not adjacent in the subalgebra \( \sigma_1 \). This implies that \( s_1 \in \{s_2, t_1\} \cap t_1 \in \{s_1, s_2\} \). Therefore, \( s_1, t_1, s_2 \) are collinear in \( X \). Similarly, for any other triple from \( S \). In particular we get that \( S \) is a subalgebra of \( \sigma_1 \).

Since \( h \) is monotone, \( h(S) \) is also a linear subpretree in \( Y \). Choose one of the two possible compatible directions (linear orders) \( \leq \) on \( S \). Without restriction of generality we can suppose that \( s_1 < t_1 < s_2 < t_2 \). Then \( h s_1 \leq h t_1 \leq h s_2 \leq h t_2 \) or \( h t_2 \leq h s_2 \leq h t_1 \leq h s_1 \). In both cases \( [h(s_1), h(t_1)]_h(S) \cap [h(s_2), h(t_2)]_h(S) \) is empty or the singleton \( h(t_1) = h s_2 \); otherwise, \( h \) is not monotone.

Since \( t_1, s_2 \in \{s_1, t_2\} \) and \( h \) is monotone we have \( h(t_1) = h(s_2) \in [h(s_1), h(t_2)]_Y \). Now, by Lemma 2.3.1 we get

\[
[h(s_1), h(t_1)]_Y \cap [h(s_2), h(t_2)]_Y = \{h(t_1)\} = \{h(s_2)\} .
\]

\( \square \)

\( \square \)

Corollary 3.9. Let \( X \) be a median pretree.

(1) For every pair of subalgebras \( \sigma_1, \sigma_2 \) of \( X \) with \( \sigma_1 \subseteq \sigma_2 \) and every bounded function \( f : X \rightarrow \mathbb{R} \) we have \( \Upsilon(f, \sigma_1) \leq \Upsilon(f, \sigma_2) \).

(2) Let \( h : X \rightarrow [c, d] \subset \mathbb{R} \) be a monotone bounded map on \( X \). Then \( h \in B_{\text{tr}}(X) \), where \( r = d - c \).

Proof. Apply Proposition 3.8 for:

(1) the inclusion map \( h : \sigma_1 \rightarrow \sigma_2 \) and \( f : X \rightarrow \mathbb{R} \).

(2) the map \( h : X \rightarrow [c, d] \) and the inclusion \( f = id : [c, d] \rightarrow \mathbb{R} \).

\( \square \)

Example 3.10. If we allow in Definition 3.3 that the subset \( \sigma_1 \) of \( X \) is not necessarily a subalgebra then the "monotonicity law" \( \Upsilon(f, \sigma_1) \leq \Upsilon(f, \sigma_2) \) is not true in general. For example, take the 4-element triod \( X = \{a, b, c, m\} \) (Remark 3.7.2) and define the function

\[
f : X \rightarrow [-1, 1], \quad f(a) = f(c) = f(m) = 1, f(b) = 0
\]

Then for the subset \( \sigma_1 = \{a, b, c\} \) (which is not a subalgebra) and \( \sigma_2 = X \) we have \( \Upsilon(f, \sigma_1) = 2 \) but \( \Upsilon(f, \sigma_2) = 1 \).
Proposition 3.11. Let $C_1, C_2$ be convex disjoint subsets in a pretree $X$. For every bounded function $f : X \to \mathbb{R}$ denote by $f|_{C_1} : C_1 \to \mathbb{R}$ and $f|_{C_2} : C_2 \to \mathbb{R}$ the restrictions. Then we have

$$
\Upsilon(f) \geq \Upsilon(f|_{C_1}) + \Upsilon(f|_{C_2}).
$$

Proof. Let $\sigma_1, \sigma_2$ be finite subalgebras in $X$ such that $\sigma_1 \subset C_1, \sigma_2 \subset C_2$. It is enough to show that there exists a finite subalgebra $\sigma^*$ in $X$ such that

$$
\Upsilon(f, \sigma^*) \geq \Upsilon(f, \sigma_1) + \Upsilon(f, \sigma_2).
$$

Consider the subalgebra $\sigma^* := sp(\sigma_1 \cup \sigma_2)$ of $X$ which is finite By Remark 2.4.4. Then $\sigma_1^* := \sigma^* \cap C_1$ and $\sigma_2^* := \sigma^* \cap C_2$ are finite subalgebras in $C_1$ and $C_2$ respectively. Clearly, $\sigma_1 \subset \sigma_1^*, \sigma_2 \subset \sigma_2^*$. By Lemma 3.6 we have

$$
\Upsilon(f, \sigma^*) \geq \Upsilon(f, \sigma^* \cap C_1) + \Upsilon(f, \sigma^* \cap C_2) = \Upsilon(f, \sigma_1^*) + \Upsilon(f, \sigma_2^*).
$$

Proposition 3.3 guarantees that $\Upsilon(f, \sigma_1^*) \geq \Upsilon(f, \sigma_1), \Upsilon(f, \sigma_2^*) \geq \Upsilon(f, \sigma_2)$. So we get

$$
\Upsilon(f, \sigma^*) \geq \Upsilon(f, \sigma_1) + \Upsilon(f, \sigma_2),
$$

as desired. \( \square \)

Theorem 3.12. Let $X$ be a median pretree (e.g., dendron or a linearly ordered space) such that its shadow topology is compact or Polish. Then every function $f : X \to \mathbb{R}$ with bounded variation has the point of continuity property (Baire 1 class function, if $X$ is Polish).

Proof. Let $f : X \to \mathbb{R}$ does not satisfy the point of continuity property. That is, $f$ is not fragmented (Lemma 2.7.1). Then by Lemma 2.7.3 there exists a closed (necessarily infinite) subspace $Y \subset X$ and real numbers $\alpha < \beta$ such that

$$
cl(f^{-1}(\alpha, \beta)) \cap Y = cl(f^{-1}(\alpha, \beta) \cap Y = Y.
$$

Assuming the contrary let $f : X \to \mathbb{R}$ has BV. By Definition 3.3 there exists $r \in \mathbb{R}$ such that

$$
\Upsilon(f) = \sup\{\Upsilon(f, \sigma) : \sigma \text{ is a finite subalgebra in } X\} = r.
$$

Choose a finite subalgebra $\sigma_1 \subset X$ such that

$$
r - \Upsilon(f, \sigma_1) < \beta - \alpha,
$$

where

$$
\Upsilon(f, \sigma_1) = \sum_{\{a,b\} \in ady(\sigma_1)} |f(a) - f(b)|.
$$

By Lemma 2.3.2, $co(\sigma_1) = \{[c_i, c_j] : c_i, c_j \in \sigma_1\}$. Since $\sigma_1$ is finite, by Remarks 2.4.4, its convex hull $co(\sigma_1)$ is closed (hence also compact (or, respectively, Polish) in the subspace topology) in $X$. We have to check two cases.

Case 1: $Y \subset co(\sigma_1)$.

In this case by Lemma 2.7.3 (for the compact (or, Polish) space $co(\sigma_1)$) we obtain that the restriction map $f|_{co(\sigma_1)} : co(\sigma_1) \to \mathbb{R}$ is not fragmented.

By Corollary 3.3 the variation of the restricted map $\Upsilon(f|_{co(\sigma_1)}) \leq \Upsilon(f) \leq r$ is also bounded. On the convex subset $co(\sigma_1) \subset X$ the (median) pretree structure induces exactly the subspace topology by Lemma 2.3.5.

Every interval $[c_i, c_j]$ has a linear order by Lemma 2.3.3 such that two variations defined above are the same (Remarks 3.7.1). By Theorem 3.2 every restriction $f|_{[c_i, c_j]}$ has BV. Each of the intervals $[c_i, c_j]$ is closed in the shadow topology (Remark 2.4.4).
Therefore, by Lemma 2.3 we obtain that $f \mid_{\co(\sigma_1)} : \co(\sigma_1) \to \mathbb{R}$ is fragmented. This contradiction shows that Case 1 is impossible.

Case 2: $Y \not\subset \co(\sigma_1)$.

Choose a point $y_0 \in Y$ such that $y_0 \notin \co(\sigma_1)$. Recall that $\co(\sigma_1)$ is closed in $X$. Every pretree is locally convex by Lemma 2.3.4. Therefore there exists an open nbd $O$ of $y_0$ in $X$ such that $O$ is convex (one may choose it as a finite intersection of branches) in $X$ and $O \cap \co(\sigma_1) = \emptyset$.

Choose $u, v \in O$ such that $u \in f^{-1}(-\infty, \alpha) \cap Y$ and $v \in f^{-1}(\beta, \infty) \cap Y$. Since $O$ is convex we have $[u, v] \subset O$. Then $[u, v] \cap \co(\sigma_1) = \emptyset$. Since $[u, v]$ and $\co(\sigma_1)$ are disjoint convex subsets in $X$ we can apply Proposition 3.11 which yields

$$\Upsilon(f) \geq \Upsilon(f, \sigma_1) + |f(u) - f(v)|.$$  

By our choice of $\sigma_1$ and $r$ it follows that $r < \Upsilon(f) = r$. This contradiction completes the proof. 

**Proposition 3.13.** Let $X$ be a median pretree. Then $BV_r(X, [c, d])$ is a pointwise closed and hence compact subset in $[c, d]^X$.

**Proof.** Let $\{f_i\}_{i \in I}$ be a net of functions in $BV_r(X, [c, d])$ such that $f : X \to [c, d]$ is its pointwise limit. For every finite subalgebra $\sigma$ of $X$ and every $i \in I$ we have

$$\Upsilon(f_i, \sigma) := \sum_{\{a, b\} \in \adj(\sigma)} |f_i(a) - f_i(b)| \leq r.$$  

Since $f$ is the pointwise limit of $\{f_i\}_{i \in I}$ we get $\lim |f_i(a) - f_i(b)| = \lim |f(a) - f(b)|$ for every given $\{a, b\} \in \adj(\sigma)$. This implies that $\Upsilon(f, \sigma) \leq r$ for every finite subalgebra $\sigma$. Hence, $\Upsilon(f) \leq r$. 

**3.3. Generalized Helly’s selection principle.** Note that there exists a sequence of functions $\{f_n : [0, 1] \to [0, 1]\}_{n \in \mathbb{N}}$ without any pointwise convergence subsequence. Indeed, the compact space $[0, 1]^{[0,1]}$ is not sequentially compact.

Recall the following classical result of Helly, [11, 12].

**Helly’s Selection Theorem:** For every sequence of functions $\{f_n : [a, b] \to [c, d]\}_{n \in \mathbb{N}}$ with total variation $\leq r$ there exists a pointwise convergent subsequence.

This result remains true replacing $[a, b]$ by any abstract linearly ordered set as it was proved in [10]. Our Theorem 3.12 allows us to prove the following generalization.

**Theorem 3.14.** (Generalized Helly’s selection theorem) Let $X$ be a Polish median pretree (e.g., dendrite) and $\{f_n : X \to [c, d]\}_{n \in \mathbb{N}}$ be a sequence of real functions which has total bounded variation $\leq r$. Then there exists a pointwise converging subsequence.

That is, $BV_r(X, [c, d])$ is sequentially compact.

**Proof.** By Theorem 3.12 the set $BV_r(X, [c, d])$ is a subset of $\mathcal{F}(X)$. Since $X$ is Polish we have $\mathcal{F}(X) = B_1(X)$ (Lemma 2.7.2). At the same time, $BV_r(X, [c, d])$ is compact (by Proposition 3.13). It is well known that by BFT theorem (Lemma 2.7.4) for every Polish $X$ every pointwise compact subset of $B_1(X)$ is sequentially compact. Hence, $BV_r(X, [c, d])$ is sequentially compact.  

Remark 3.15. There are many natural BV functions on dendrites which are not monotone. For example, consider the real triod $X = [u, v] \cup [v, w] \cup [u, w] \subset \mathbb{R}^2$, where $[u, v] \cap [v, w] \cap [u, w] = \{m\}$. Every “coloring” $f : X \to \{1, 2, 3\}$, provided that every "open arc" $(x, y)$ is monochromatic, is a function with BV. Much more generally, $f$ is with BV if and only if every of three restrictions on the corresponding intervals are BV functions. However, many such functions are not monotone. For example, if we use all three colors and if $f(m) \neq 2$, then $f$ is not monotone.

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