Energetics of a driven Brownian harmonic oscillator

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Abstract. We provide insight into the energetics of a Brownian oscillator in contact with a heat bath and driven by an external unbiased time-periodic force that takes the system out of thermodynamic equilibrium. Solving the corresponding Langevin equation, we compute the average kinetic and potential energies in the long-time stationary state. We also derive the energy balance equation and study the energy flow in the system. In particular, we identify the energy delivered by the external force, the energy dissipated by a thermal bath and the energy provided by thermal equilibrium fluctuations. Next, we illustrate the Jarzynski work-fluctuation relation and consider the stationary state fluctuation theorem for the total work done on the system by external force. Finally, by determining time scales in the system, we analyze the strong damping regime and discuss the problem of overdamped dynamics when inertial effects can be neglected.

Keywords: Brownian motion, driven diffusive systems, fluctuation phenomena, stochastic particle dynamics
1. Introduction

Classical thermodynamics and statistical physics usually deal with systems either isolated or at equilibrium with their environment. For example, if a system is in contact and weakly interacts with a thermostat (heat bath), its stationary state is a Gibbs canonical equilibrium state [1]. However, this ‘ideal’ situation is not always satisfied and real systems often have strong couplings with their environments or are driven far from thermodynamic equilibrium by an external agent. Recently, much theoretical development has improved our understanding of non-equilibrium thermodynamics, such as fluctuation theorems [2–6], which have been generalized both in classical and quantum statistical physics [2, 7–10]. Moreover, a number of important results have been obtained in the theory of stochastic thermodynamics [11–13]. These approaches are particularly relevant for small systems where thermal fluctuations cannot be neglected, such as colloidal particles, polymers, molecular motors, etc [14]. Systems far
from thermal equilibrium exhibit many fascinating and non-intuitive properties, such as absolute negative mobility [15], stochastic resonance [16], non-monotonic temperature dependence of a diffusion coefficient [17] or transient but long-lasting anomalous diffusion [18, 19]. Interestingly, in this recent framework, the Langevin equation, first introduced in 1908 to describe the Brownian motion [20], has become a paradigmatic study case.

The intention of this paper is to present, in a pedagogical way, an example of a non-equilibrium system. It should be useful for students in statistical physics to follow a step-by-step derivation of some formulas and, next, to perform detailed analysis of the system properties, in particular its energetics. We consider only one model: the classical harmonic oscillator in contact with a heat bath and driven by an external, unbiased time-periodic force. The system is described by a Langevin equation. It is simple enough so that the main statistical quantities characterizing its dynamics can be calculated exactly in analytical form. Moreover, it is a relevant model for many experimental applications at small scales. For example, optical tweezers used to trap and manipulate micrometric objects are very well described by a harmonic potential for small displacements [21, 22]. Nearly 40 years after their invention [23], optical tweezers have become a very common tool and are now widely used in biology and physics laboratories [24–26]. Another classical example of applications is a small torsion pendulum [27–29]. It has also been shown in a one-dimensional system of hard particles that the hard core interaction among particles can lead to a restoring force that can be modelled by a harmonic potential [30]. Moreover, a two-dimensional diffusion of a Brownian particle in a periodic asymmetric channel with soft walls can be modeled by a parabolic potential which mimics a restoring force [31, 32]. Harmonic potential has also been used in modelling confined colloidal systems [33]. More generally, any system that undergoes thermal fluctuations close enough to its stable equilibrium position can be described as a Brownian harmonic oscillator.

In this article, we have chosen to consider the case of an external sinusoidal force acting on the oscillator and we focus on the interpretation of the oscillator energetics. Namely, we analyze the long-time dynamics and compute the potential and kinetic energies, and we describe the energy balance between the system, the thermal bath and the external force. We also discuss, in detail, the regime of strong damping (frequently named overdamped) dynamics where the inertia term can be neglected. We show that the overdamped regime can be rigorously analyzed by revealing characteristic time scales in the system and then transforming the Langevin equation to its dimensionless form. A similar procedure can be applied to the energetics of the system.

The article is organized as follows: in section 2 we describe the model of a periodically-driven Brownian particle in a harmonic potential and we present the Langevin equation that governs its dynamics. We also define three kinds of averages of dynamical variables of the oscillator. In section 3, we present a solution of the Langevin equation for the position and the velocity of the particle. Next, in section 4, we compute the averaged kinetic and potential energies of the system in the long-time regime. In section 5, we derive equations for the energy balance and we discuss the energy exchanges between the system, the heat bath, and the external driving force. In section 6, we analyse the dependence of period-averaged energy on the frequency of external driving. In section 7, we study the Jarzynski-type relation for the work performed by the
external time-periodic force. In section 8, we discuss the regime of strong damping and discuss the problem of overdamped dynamics. Finally, in section 9, we conclude the paper with a discussion and a summary of the findings. The appendices contain supplementary material on Green functions (appendix A), the particle position and velocity mean square deviation (appendix B) and the first two statistical moments of the work (appendix C).

2. The forced Brownian harmonic oscillator

2.1. Langevin equation

We consider a classical Brownian particle of mass $m$ in contact with a thermal bath of temperature $T$. For simplicity we consider its motion in one dimension in the harmonic potential $U(x) = kx^2/2$, $k > 0$. It is driven by an unbiased, symmetric time-periodic force $F(t) = F_0 \cos(\omega t)$ of amplitude strength $F_0$ and angular frequency $\omega$. The dynamics of the particle is described by a Langevin equation in the form [34]:

$$m \ddot{x} + \gamma \dot{x} + kx = F(t) + \sqrt{2\gamma k_B T} \xi(t) \quad t \geq 0. \quad (1)$$

Here, $x = x(t)$ is the particle coordinate, the dot denotes differentiation with respect to time $t$ and $\gamma$ is the friction coefficient. Usually $\gamma$ is the Stokes coefficient: $\gamma = 6\pi R \eta$, with $R$ the radius of the particle and $\eta$ the viscosity of the surrounding fluid. However, in this paper we do not refer to the particle’s geometry and will simply take it to be a point-like one. The restoring potential force is $-dU(x)/dx = -kx$. Thermal fluctuations due to coupling of the particle with the thermal bath are modeled by a $\delta$-correlated Gaussian white noise $\xi(t)$ of zero mean and unit intensity:

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(u) \rangle = \delta(t-u). \quad (2)$$

The noise intensity factor $2\gamma k_B T$ (where $k_B$ is the Boltzmann constant) comes from the fluctuation–dissipation theorem [35–37]. It ensures that the system’s stationary state is the canonical Gibbs state (thermodynamic equilibrium) under the dynamics governed by equation (1) for vanishing driving, i.e. when $F_0 = 0$.

The following points are noteworthy:

(i) The system can be seen as a Brownian particle moving in a harmonic potential $U(x)$ or equivalently as a harmonic oscillator.

(ii) The stochastic process $x(t)$ determined by equation (1) is non-Markovian. However, the pair $\{x(t), \dot{x}(t)\}$ constitutes a Markovian process.

(iii) Due to the presence of the external time-periodic force $F(t)$, the system tends to a unique asymptotic non-equilibrium state characterized by a temporally periodic probability density for a long period of time [38]. In this long-time state, the average values of dynamical variables are periodic functions of time that have the same period as the acting force $F(t)$.
2.2. Notations for averages

In this problem, we define three kinds of averages:

- We denote by \( \langle Y(t) \rangle \) the ensemble averaging of a dynamical variable \( Y(t) = Y[x(t), \dot{x}(t)] \) over all realizations of the Gaussian white noise \( \xi(t) \) and over all initial conditions \( \{x(0), \dot{x}(0)\} \).

- We denote by \( \langle Y(t) \rangle_{\text{st}} \) the average value in the long-time limit state where all transient effects die out:

\[
\langle Y(t) \rangle_{\text{st}} = \lim_{\gamma \to 0} \text{ } \langle Y(t) \rangle.
\]  

(3)

In this state, the dynamics is characterized by a temporally periodic probability density with the same period \( T \) as the acting force: \( T = 2\pi/\omega \).

- We denote by \( \overline{Y} \) the time averaging of \( \langle Y(t) \rangle \) over one period \( T = 2\pi/\omega \) in the long-time regime:

\[
\overline{Y} = \lim_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \langle Y(u) \rangle \text{ du}.
\]  

(4)

By doing this, we obtain the time-independent asymptotic characteristic average of the dynamical variable \( Y(t) \).

In the following sections, we will use the three notations: \( \langle Y(t) \rangle \), \( \langle Y(t) \rangle_{\text{st}} \) and \( \overline{Y} \).

3. Solution of the Langevin equation

3.1. Position of the Brownian particle

The Langevin equation (1) is a stochastic non-homogeneous linear differential equation and its solution \( x(t) \) is in the form \( x(t) = x_c(t) + x_p(t) \), where the complementary part \( x_c(t) \) satisfies the homogeneous differential equation (i.e. when \( F_0 = 0 \) and \( \xi(t) = 0 \)) with the initial condition \( x_c(0) = x_0 \) and \( \dot{x}_c(0) = v_0 \). The particular solution \( x_p(t) \) satisfies the inhomogeneous equation with the initial conditions \( x_p(0) = 0 \) and \( \dot{x}_p(0) = 0 \). It is convenient to decompose the particular solution \( x_p(t) \) into two additive parts: \( x_p(t) = x_d(t) + x_\xi(t) \), where \( x_d(t) \) is a particular solution corresponding to the deterministic force \( F(t) \), and \( x_\xi(t) \) is a solution related to the noise term \( \xi(t) \). We therefore write

\[
x(t) = x_c(t) + x_d(t) + x_\xi(t).
\]  

(5)

The term \( x_c(t) \) is a linear combination of two independent solutions of the homogeneous equation (in the case when \( F_0 = 0 \) and \( \xi(t) = 0 \):
\[ x_c(t) = A \ e^{-\gamma t} \sin(\Omega t) + B \ e^{-\gamma t} \cos(\Omega t), \] (6)

where \( A \) and \( B \) are determined by initial conditions for the position \( x_c(0) \) and velocity \( \dot{x}_c(0) \) of the particle which can be deterministic or random. The remaining parameters are defined as

\[ \Omega^2 = \omega_0^2 - \gamma_0^2, \quad \omega_0^2 = \frac{k}{m}, \quad \gamma_0 = \frac{\gamma}{2m}. \] (7)

The second part \( x_d(t) \), related to the deterministic force \( F(t) \), can be expressed by the Green function \( G(t, u) \) of equation (1) and takes the form [39]

\[ x_d(t) = \int_0^\infty G(t, u) F(u) du, \] (8)

where the Green function reads [39, 40]

\[ G(t, u) = \Theta(t - u) \frac{1}{m\Omega} \ e^{-\gamma_0(t-u)} \sin[\Omega(t-u)] \] (9)

and \( \Theta(t) \) stands for the Heaviside step function: \( \Theta(t) = 1 \) for \( t > 0 \) and \( \Theta(t) = 0 \) for \( t < 0 \). The method of obtaining the Green’s function is explained in detail in appendix A. The form of the Green function (9) is convenient when \( \omega_0 > \gamma_0 \), i.e. when the system is weakly damped. In the opposite strongly damped regime, when \( \gamma_0 > \omega_0 \), one can use the equivalent form with the hyperbolic sine function [40],

\[ G(t, u) = \Theta(t - u) \frac{1}{m\Omega_0} \ e^{-\gamma_0(t-u)}\sinh[\Omega_0(t-u)], \quad \Omega_0^2 = \gamma_0^2 - \omega_0^2 = -\Omega^2. \] (10)

Below, in all intermediate calculations, we will use only one form, namely equation (9).

The third term \( x_\xi(t) \) is non-deterministic and is related to thermal equilibrium noise \( \xi(t) \). It can also be expressed by the Green’s function \( G(t, s) \), namely,

\[ x_\xi(t) = \sqrt{2\gamma k_B T} \int_0^\infty G(t, u) \xi(u) du. \] (11)

Because \( \xi(t) \) is a Gaussian process, \( x_\xi(t) \) as a linear functional of \( \xi(t) \) is also Gaussian, and in consequence \( x(t) \) is a Gaussian stochastic process. It means that two statistical moments \( \langle x(t) \rangle \) and \( \langle x^2(t) \rangle \) are sufficient to determine the probability density \( P(x, t) \) of the particle coordinate.

From equation (8) we get

\[ x_d(t) = \frac{F_0}{m\Omega} \int_0^t e^{-\gamma_0(t-u)} \sin[\Omega(t-u)] \cos(\omega u) \ du \]

\[ = \frac{F_0}{m\sqrt{A(\omega)}} \cos(\omega t - \delta) - \frac{F_0\gamma_0}{m\Omega\sqrt{A(\omega)}} \cos(\omega t - \zeta)e^{-\gamma_0 t}, \] (12)

where

\[ \delta = \tan^{-1} \frac{2\omega_0 \gamma_0}{\omega_0^2 - \omega^2}; \quad \zeta = \tan^{-1} \frac{\gamma_0(\omega_0^2 + \omega^2)}{\Omega(\omega_0^2 - \omega^2)}; \]

\[ A(\omega) = (\omega^2 - \omega_0^2)^2 + 4\gamma_0^2\omega^2. \] (13)

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Now, we can calculate the average value of the particle position $\langle x(t) \rangle$ at time $t$. Because $\langle \xi(t) \rangle = 0$, from equation (11) it follows that $\langle x_{\xi}(t) \rangle = 0$. Therefore $\langle x(t) \rangle = \langle x_c(t) \rangle + x_d(t)$. For any initial conditions, $\langle x_c(t) \rangle \to 0$ when $t \to \infty$ and in the stationary state, when all decaying terms vanish, we obtain [41]

$$\langle x(t) \rangle_{st} = X_d(t) = \frac{F_0}{m\sqrt{A(\omega)}} \cos(\omega t - \delta)$$

$$= \frac{F_0}{mA(\omega)} \left[ 2\gamma_0 \omega \sin(\omega t) + (\omega_0^2 - \omega^2) \cos(\omega t) \right],$$

(14)

where $X_d(t)$ is the long-time limit of the deterministic solution (12), i.e. a purely periodic, non-decaying part of (12). Let us note that according to the recipe (4), the period-averaged position in the stationary state is zero, $\bar{x} = 0$.

For long-time periods, the second moment of the position takes the form

$$\langle x^2(t) \rangle = \langle (x_c(t) + x_d(t) + x_{\xi}(t))^2 \rangle$$

$$= \int_0^t \int_0^t G(t, u)G(t, u') [F(u)F(u') + 2\gamma k_B T \langle \xi(u)\xi(u') \rangle] \, du \, du'$$

$$= X_d^2(t) + 2\gamma k_B T \int_0^t G^2(t, u) \, du,$$

(15)

where we exploited equation (2) for statistical moments of thermal noise $\xi(t)$ and neglected all exponentially decaying terms. The integral in equation (15) can be calculated, and, in the long-time limit, we again can neglect the exponentially decaying terms to obtain the relation in the stationary state,

$$\langle x^2(t) \rangle_{st} = \frac{k_B T}{m\omega_0^2} + X_d^2(t).$$

(16)

It can be verified that the mean squared deviation (MSD) of the particle position is $\langle (\Delta x)^2 \rangle = \langle x_{\xi}^2 \rangle$. In appendix B we present details of this calculation. The result turns out to be

$$\langle (\Delta x)^2(t) \rangle = \frac{k_B T}{k} + \frac{k_B T}{k\Omega^2} \left[ \gamma_0^2 \cos 2\Omega t - \gamma_0 \Omega \sin 2\Omega t - \omega_0^2 \right] e^{-2\gamma_0 t}.$$

(17)

Neglecting the exponentially decaying terms, we get the following relation in the stationary state,

$$\langle (\Delta x)^2(t) \rangle_{st} = \frac{k_B T}{k},$$

(18)

which is notably mass independent! In fact the particle’s mass only determines the relaxation to the steady state value $k_B T/k$. In figure 1, we depict the temporal evolution of MSD for several values of the particle mass. As one can see, the larger the particle’s mass, the slower the relaxation towards steady state. The role of the mass will also be discussed later when we consider the overdamped regime.
3.2. Velocity of the Brownian particle

We now come to the particles’ velocity analysis. From equation (5) it follows that the particle velocity $v(t) = \dot{x}(t)$ can be decomposed as

$$v(t) = v_c(t) + v_d(t) + v_\xi(t).$$  \hspace{1cm} (19)

The first term can be calculated from equation (6) and for any initial conditions its average value $\langle v_c(t) \rangle = \langle \dot{x}_c(t) \rangle \to 0$ if $t \to \infty$. The second term $v_d(t)$ can be calculated from equation (12) and in the long-time regime it takes the form

$$V_d(t) = \lim_{\gamma_0 \to 1} v_d(t) = - \frac{F_0 \omega}{m \sqrt{A(\omega)}} \sin(\omega t - \delta).$$  \hspace{1cm} (20)

The last term is given by:

$$v_\xi(t) = \dot{x}_\xi(t) = \sqrt{2\gamma k_B T} \int_0^\infty \frac{\partial G(t, u)}{\partial t} \xi(u) du.$$  \hspace{1cm} (21)

It can be calculated from equations (11) and (9):

$$v_\xi(t) = \dot{x}_\xi(t) = -\gamma_0 x_\xi(t)$$

$$+ (1/m) \sqrt{2\gamma k_B T} \int_0^t e^{-\gamma_0 (t-u)} \cos[\Omega(t-u)] \xi(u) du.$$  \hspace{1cm} (22)

Its mean value is zero, $\langle v_\xi(t) \rangle = 0$, because both $\langle x(t) \rangle = 0$ and $\langle x_\xi(t) \rangle = 0$. As a result, the mean velocity in the steady-state is

$$\langle v(t) \rangle_{st} = V_d(t) = \dot{X}_d(t).$$  \hspace{1cm} (23)

One can note that $\langle v(t) \rangle_{st}$ is simply the time derivative of $\langle x(t) \rangle_{st}$ given by equation (14).

The second moment of the velocity can be obtained by several methods. One of them is presented in appendix B. In the steady-state it reads

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{msd.png}
\caption{The rescaled mean squared deviation (MSD) $\langle \Delta x^2(t) \rangle / (k_B T/k)$ of the coordinate degree of freedom $x(t)$ versus rescaled time $(k/\gamma)t$ for selected values of the particle rescaled mass $M = (k/\gamma)^2 m$. The particle’s mass only affects the relaxation to the mass-independent steady state value.}
\end{figure}
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\[ \langle v^2(t) \rangle_{st} = \frac{k_B T}{m} + V^2_d(t). \]  
(24)

Finally, the mean square deviation of the velocity in the steady-state is

\[ \langle \Delta v^2(t) \rangle_{st} = \frac{k_B T}{m}, \]  
(25)

which is remarkably independent of the spring stiffness \( k \). Further, this value is the one that would be expected in the equilibrium situation (i.e. when \( F_0 = 0 \), due to the energy equipartition theorem [42].

From equations (16) and (24) it follows that in the stationary state the second moment for both position and velocity separates into two additive parts:

(i) the first part \( X^2_d(t) \) for the position (or \( V^2_d(t) \) for the velocity) is generated by the deterministic driving force \( F(t) \);
(ii) and the second part \( k_B T/k \) for the position (or \( k_B T/m \) for the velocity) is generated by the thermal noise (and is equal to the thermal equilibrium value).

This separation takes place because the system is linear and noise \( \xi(t) \) is additive.

4. Average energy in a stationary state

In this section, we compute the kinetic and potential energies of the Brownian harmonic oscillator. The kinetic energy of the oscillator:

\[ E_k(t) = \frac{1}{2} m v^2(t) \]  
(26)

is a stochastic quantity. Its potential energy:

\[ E_p(t) = U(x(t)) = \frac{1}{2} k x^2(t) \]  
(27)

is also a stochastic quantity. Naturally, the total energy \( E(t) \) of the oscillator is the sum \( E(t) = E_k(t) + E_p(t) \). Its average value is determined by the second moments of the coordinate and velocity of the oscillator. Using the relations (24) and (20), the steady-state average kinetic energy can be expressed by the relation

\[ \langle E_k(t) \rangle_{st} = \frac{k_B T}{2} + \frac{F_0^2 \omega^2}{2m A(\omega)} \sin^2(\omega t - \delta) \]

\[ = \frac{k_B T}{2} + \frac{\omega^2 F_0^2}{4mA(\omega)} - \frac{\omega^2 F_0^2}{4mA^2(\omega)} \left[ B(\omega) \cos(2\omega t) - C(\omega) \sin(2\omega t) \right], \]  
(28)

where \( A(\omega) \) is defined in equation (13) and

\[ B(\omega) = (\omega^2 - \omega_0^2)^2 - 4\gamma_0^2 \omega^2, \quad C(\omega) = 4\omega_0^2(\omega^2 - \omega_0^2). \]  
(29)
From equations (16) and (14) it follows that the steady-state average potential energy has the form

\[ \langle E_p(t) \rangle_{st} = \frac{k_B T}{2} + \frac{F_0^2 \omega_0^2}{2m A(\omega)} \cos^2(\omega t - \delta) \]

\[ = \frac{k_B T}{2} + \frac{\omega_1^2 F_0^2}{4m A(\omega)} + \frac{\omega_0^2 F_0^2}{4m A^2(\omega)} \left[ B(\omega) \cos(2\omega t) - C(\omega) \sin(2\omega t) \right]. \]  \hspace{1em} (30)

The total averaged energy of the oscillator in the stationary state is

\[ \langle E(t) \rangle_{st} = k_B T + \frac{(\omega^2 + \omega_0^2) F_0^2}{4m A(\omega)} + \frac{(\omega_0^2 - \omega^2) F_0^2}{4m A^2(\omega)} \left[ B(\omega) \cos(2\omega t) - C(\omega) \sin(2\omega t) \right]. \]  \hspace{1em} (31)

Both kinetic energy and potential energy consist of two terms: thermal equilibrium energy \( k_B T/2 \) and the driving-generated part proportional to its amplitude squared \( F_0^2 \). In turn, the driving-generated part consists of two terms: the part constant in time and the time-periodic part. For the latter, its mean value over the driving \( T \)-period is zero. In contrast to the position \( \langle x(t) \rangle_{st} \) and velocity \( \langle v(t) \rangle_{st} \), which are \( T \)-periodic functions of time, the energy is periodic with respect to time but now the period is \( T/2 = \pi/\omega \).

Time changes of energy in the stationary state are depicted in figure 2. In all panels of this figure, the rescaled kinetic energy is \( \varepsilon_k(t) = [\langle E_k(t) \rangle_{st} - k_B T/2]/E_0 \), the rescaled potential energy is \( \varepsilon_p(t) = [\langle E_p(t) \rangle_{st} - k_B T/2]/E_0 \), and the rescaled total energy is \( \varepsilon(t) = \varepsilon_k(t) + \varepsilon_p(t) \), where the characteristic energy \( E_0 = F_0^2/4m \omega_0^2 = F_0^2/4k \). One can note that the minimal values of the kinetic and potential energies are the equilibrium values \( k_B T/2 \). However, their maximal values depend on the ratio \( \omega/\omega_0 \) of two characteristic frequencies, namely, the frequency \( \omega \) of the time-periodic force \( F(t) \) and the eigenfrequency \( \omega_0 \) of the oscillator. If \( \omega < \omega_0 \) (panel (a) in figure 2) then the amplitude of the potential energy is greater than the amplitude of the kinetic energy. In turn, if \( \omega > \omega_0 \) (panel (b) in figure 2) then the amplitude of the kinetic energy is greater than the amplitude of the potential energy. In both cases, the total energy oscillates in time. Finally, if \( \omega = \omega_0 \) (panel (c) in figure 2), the total energy does not depend on time, it is constant! Remember that the system is open, not isolated. There is a dissipation of energy due to contact with the thermostat of temperature \( T \) and there is a pumping of energy by the time-periodic force \( F(t) \). Under some specific conditions (here \( \omega = \omega_0 \)), these two processes can lead to the conservation of total energy in the noisy system.

5. Energy balance

We now derive equations for the energy balance. To do this, we first rewrite equation (1) in the Ito-type form of stochastic differential equations:

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\[ \text{dx} = \text{vdt}, \]  
\[ m \text{dv} = [-\gamma \text{v} - kx + F(t)] dt + \sqrt{2\gamma k_B T} dW(t), \]

where formally \( dW(t) = \xi(t) dt \) and \( W(t) \) is the Wiener process of Gaussian statistics with zero mean \( \langle dW(t) \rangle = 0 \) and the second moment \( \langle |dW(t)|^2 \rangle = dt \). Next, we employ the Ito-differential calculus to the functions \( E_p(x) \) and \( E_k(v) \), namely,

\[ \text{d}E_p = \frac{\text{d}E_p}{\text{d}x} \text{dx} + \ldots = kxvd\text{t}, \]

\[ \text{d}E_k = \frac{\text{d}E_k}{\text{d}v} \text{dv} + \frac{1}{2} \frac{\text{d}^2E_k}{\text{d}v^2} \text{dvdv} + \ldots \]

\[ = (-\gamma v^2 - kxv + vF(t)) dt + \sqrt{2\gamma k_B T} v dW(t) + \frac{\gamma k_B T}{m} [dW(t)]^2, \]

where we have only kept the terms that, at most, are first order in \( dt \). We then perform the ensemble averaging \( \langle \ldots \rangle \) over all realizations of the Wiener process \( W(t) \) and over all
initial conditions \{x(0), v(0)\}. As a result we obtain the rate of change of the potential and kinetic energies in the form:

$$\frac{d}{dt}\langle E_p(t) \rangle = k\langle x(t)v(t) \rangle,$$

(36)

$$\frac{d}{dt}\langle E_k(t) \rangle = -\gamma\langle v^2(t) \rangle - k\langle x(t)v(t) \rangle + \langle v(t)F(t) \rangle + \frac{\gamma k_B T}{m},$$

(37)

where for the part containing the Wiener process in equation (35), we used the Itô-martingale property, i.e. \langle v(t)dW(t) \rangle = 0. By adding these two equations together, we obtain an equation for the energy balance. It reads:

$$\frac{d}{dt}\langle E(t) \rangle = -\gamma\langle v^2(t) \rangle + F(t)\langle v(t) \rangle + \frac{\gamma k_B T}{m}.$$  

(38)

We conclude that there are three processes responsible for the energy change: (i) the first term in the rhs of equation (38) is always negative and represents the rate of energy loss due to dissipation (friction of the fluid); (ii) the second term describes the pumping of energy to the system by external force \(F(t)\); and (iii) the third term is always positive and characterizes the energy supplied to the system by thermal equilibrium fluctuations. We recall that in thermal equilibrium (i.e. \(F_0 = 0\)), the theorem on the equipartition of energy states that \(\langle E_k \rangle_{eq} = k_B T/2\), from which it follows that \(\langle v^2 \rangle_{eq} = k_B T/m\). Therefore equation (38) can be rewritten in the form:

$$\frac{d}{dt}\langle E(t) \rangle = -\gamma[\langle v^2(t) \rangle - \langle v^2 \rangle_{eq}] + F(t)\langle v(t) \rangle.$$  

(39)

Let us note that when \(F(t) = 0\), in the stationary state (which is thermal equilibrium), the lhs tends to zero and we retrieve \(\langle v^2(t) \rangle \rightarrow \langle v^2 \rangle_{eq} = k_B T/m\) for any initial conditions.

Now, we perform the time averaging of equation (39) according to the prescription (4). Let us remember that in the stationary state, the average values of any function of the particle coordinate \(x(t)\) and its velocity \(v(t)\) are periodic functions of time and therefore the lhs of equation (39) vanishes after time averaging. Indeed, by definition:

$$\int_t^{t+2\pi/\omega} \frac{d}{du}\langle E(u) \rangle_{st} \, du = \frac{m}{2}[\langle v^2(t + 2\pi/\omega) \rangle_{st} - \langle v^2(t) \rangle_{st}]$$

$$+ [(U(x(t + 2\pi/\omega)))_{st} - (U(x(t)))_{st}] = 0.$$  

(40)

As a consequence, in the stationary regime the mean power \(\overline{P_{in}}\) delivered to the system by the external force \(F(t)\) over the period \(T = 2\pi/\omega\) is expressed by the relation

$$\overline{P_{in}} = \langle F(t)v(t) \rangle = \gamma \left[\langle v^2(t) \rangle - \langle v^2 \rangle_{eq} \right] = \frac{2}{\tau_1} \left[\langle E_k \rangle - \langle E_k \rangle_{eq} \right],$$  

(41)

where the characteristic time \(\tau_1 = m/\gamma = 1/(2\gamma_0)\) is the relaxation time of the particle velocity for the system when \(F(t) = 0\). This relation has an appealing physical interpretation: the input power from the external force \(F(t)\) to the system determines the difference between the averaged non-equilibrium \(\langle E_k \rangle\) and equilibrium \(\langle E_k \rangle_{eq}\) values of the kinetic energy of the Brownian particle. Moreover, the input power depends not only on the force itself (i.e. on the amplitude \(F_0\) and frequency \(\omega\)) but also on the
properties and parameters of the system: the friction coefficient $\gamma$ and the mass $m$. In contrast, the energy supplied by thermal fluctuations does not depend on the external force but only on $T, \gamma$ and $m$.

6. Period-averaged energy

In the non-equilibrium stationary state, all averaged forms of energy are periodic functions of time. Time-independent characteristics can be obtained from the formulas (28)--(31) by additional averaging over the period $T = 2\pi/\omega$ of the time-periodic force $F_0 \cos(\omega t)$. The averaging scheme is in accordance with the rule (4). In this way we obtain the period-averaged kinetic energy in the form

$$E_k = \frac{k_B T}{2} + \frac{F_0^2}{4m} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + (\gamma \omega/m)^2}.$$  

The period-averaged potential energy reads

$$E_p = \frac{k_B T}{2} + \frac{F_0^2}{4m} \frac{\omega_0^2}{(\omega^2 - \omega_0^2)^2 + (\gamma \omega/m)^2}$$

and the total period-averaged energy is given by the expression

$$E = k_B T + \frac{F_0^2}{4m} \frac{\omega^2 + \omega_0^2}{(\omega^2 - \omega_0^2)^2 + (\gamma \omega/m)^2}.$$  

The period-averaged input power can be obtained from equation (41) and the result is

$$\bar{P}_{\text{in}} = \frac{\gamma F_0^2}{2m^2} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + (\gamma \omega/m)^2}.$$  

We recall that the time-averaged input power $\bar{P}_{\text{in}}$ is independent of temperature $T$ of the system. It is not the case for non-linear systems [43] for which the input power is temperature dependent.

Now, we analyse the properties of these time-independent mean energies. The dependence of energy on the rescaled frequency $\Lambda = \omega/\omega_0$ of the external force is shown in figure 3. We observe that:

A. The kinetic energy is zero at zero driving frequency $\Lambda = 0$ and approaches zero when $\Lambda \to \infty$. It means that there must be an optimal value of the driving force frequency for which the kinetic energy is maximal. Calculation of the derivative shows that the maximal value is for the resonance frequency $\Lambda = 1$.

B. The shape of the potential energy depends on the rescaled friction coefficient $\Gamma = \gamma/\sqrt{mk}$: If $\Gamma^2 \geq 2$, the potential energy takes a bell-shape form of the maximal value at $\Lambda = 0$. However, if $\Gamma^2 < 2$, there is one minimum at $\Lambda = 0$ and maximum at $\Lambda^2 = 1 - \Gamma^2/2$.

C. The shape of the total energy also depends on the rescaled friction coefficient: If $\Gamma^2 \geq 3$, the bell-shaped total energy is maximal at $\Lambda = 0$ and tends to zero when
The frequency increases. In turn, if $\Gamma^2 < 3$, it exhibits one minimum at $\Lambda = 0$ and a maximal value at $\Lambda^2 = \sqrt{4 - \Gamma^2} - 1$.

The dependence of the input power on the rescaled friction coefficient and the rescaled mass is shown in figure 4. In panel (a), the scaled frequency is $\Lambda = \omega / \omega_0$ and the rescaled input power is given by the formula:

$$\frac{P_{\text{in}}}{P_0} = \frac{\Gamma \Lambda^2}{(\Lambda^2 - 1)^2 + \Gamma^2 \Lambda^2}$$

where the characteristic power is $P_0 = F_0^2 / (2\sqrt{mk})$. The minimum input power is always at $\Lambda = 0$, and the maximum is always at the resonance $\Lambda = 1$ and does not depend on $\Gamma$. In panel (b), the scaled frequency is $\Lambda = (\gamma / k) \omega$ and the rescaled input power now reads:

$$\frac{P_{\text{in}}}{P_1} = \frac{\Lambda^2}{(M\Lambda^2 - 1)^2 + \Lambda^2}$$

where the characteristic power is $P_1 = F_0^2 / (2\gamma)$ and the rescaled mass is $M = mk / \gamma^2$. The minimum input power is always at $\Lambda = 0$. The maximal input power is $P_1$ which is located at $\Lambda = \pm 1 / \sqrt{M}$.
Many physical observables undergo random fluctuations. The coordinate, velocity, kinetic and potential energies of a Brownian particle are generic examples of such observables. Work and fluctuation relations constitute an additional basic characterization of the statistical properties of the system. In this section, we present the counterpart of the Jarzynski work relation for the considered system. We have to stress that what we present is not an exact Jarzynski relation because conditions for the Jarzynski theorem to hold are not fulfilled and the interpretation is not in terms of a free energy—a notion which is defined for systems in thermal equilibrium but not for non-equilibrium ones.

Although the conception of work seems to be well known, there is no consensus on its precise and general definition (see the discussion in [9, 44, 45]). Here we consider the inclusive work defined as

$$ W_{\tau_0}(\tau) = -\int_{\tau_0}^{\tau_0+\tau} \dot{F}(t)x(t)dt = \omega F_0 \int_{\tau_0}^{\tau_0+\tau} \sin(\omega t)x(t)dt, $$

(48)

where $x(t)$ is given by equations (5)–(13). It is the work performed by the external force $F(t)$ on the system in the time interval $(\tau_0, \tau_0 + \tau)$. We are interested in the limit of long time periods when the asymptotic state is reached. Therefore the regime $\tau_0 \gg 1$ should be considered.

The coordinate $x(t)$, via the relation (11), is a linear functional of the Gaussian process $\xi(t)$. Therefore, for each fixed time $\tau_0$ the work (48) as a function of time $\tau$ is a stochastic Gaussian process. However, because of the time-periodic driving $F(t)$, it is a non-stationary random process and therefore the statistical properties of the work $W_{\tau_0}(\tau)$ can depend on the reference time $\tau_0$! It is a radically different case than for stationary processes.

The probability distribution of the work $W_{\tau_0}(\tau)$ is a Gaussian function in the form [46]
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\[ P_\tau(w, \tau) = \frac{1}{\sqrt{2\pi \sigma^2_\tau(\tau)}} \exp[-(w - \mu_\tau(\tau))^2/2\sigma^2_\tau(\tau)], \quad (49) \]

where

\[ \mu_\tau(\tau) = \langle W_\tau(\tau) \rangle, \quad \sigma^2_\tau(\tau) = \langle W^2(\tau) \rangle - \langle W_\tau(\tau) \rangle^2 \]

are the mean value and variance of the work, respectively.

The characteristic function \( C_\tau(z, \tau) \) of the Gaussian stochastic process \( W_\tau(\tau) \) is well known, namely [46],

\[ C_\tau(z, \tau) = \langle e^{izW_\tau(\tau)} \rangle = \exp \left[ i\mu_\tau(\tau)z - \frac{1}{2}\sigma^2_\tau(\tau)z^2 \right]. \quad (51) \]

Now, let us assume that \( z = i\beta \), where \( \beta = 1/k_B T \). Then it takes the form

\[ \langle e^{-\beta W_\tau(\tau)} \rangle = e^{-\beta \Delta F(\tau_0, \tau)}, \quad \Delta F(\tau_0, \tau) = \mu_\tau(\tau) - \beta \sigma^2_\tau(\tau)/2. \quad (52) \]

This characteristic function of the work \( W_\tau(\tau) \) for the imaginary argument \( z = i\beta \) is just a counterpart of the Jarzynski relation:

\[ \langle e^{-\beta W} \rangle = e^{-\beta \Delta F}, \quad (53) \]

where \( \Delta F \) is the free energy difference between the final and initial equilibrium states. The argument \( \Delta F(\tau_0, \tau) \) of the exponent in the rhs of (52) formally corresponds to \( \Delta F \) in (53) but cannot be interpreted as a free energy of the system.

In appendix C, we calculate both \( \mu_\tau(\tau) \) and \( \sigma^2_\tau(\tau) \) which allow us to calculate \( \Delta F(\tau_0, \tau) \). As a result we get

\[ \Delta F(\tau_0, \tau) = \frac{1}{2m} \frac{\omega^2 - \omega_0^2}{A(\omega)} \left[ F^2(\tau_0 + \tau) - F^2(\tau_0) \right] + \frac{1}{2\omega} P_{in} \{ \sin[\omega(\tau_0 + \tau)] - \sin[2\omega(\tau_0 + \tau)] - \sin(\omega\tau_0) + \sin(2\omega\tau_0) \} - \frac{1}{2} (P_{in})^2_1 \{ \cos[\omega(\tau_0 + \tau)] - \cos(\omega\tau_0) \} - \frac{1}{2} P_{in} \tau \left[ e^{-\gamma_0} I(\tau_0, \tau) - I(\tau_0, 0) \right], \quad (54) \]

where \( P_{in} \) is defined by equation (45) and the remaining notation is explained in appendix C.

In the literature [47–49], the relation (52) is called a stationary state fluctuation relation for the total work performed during the time \( \tau \). However, in [47, 49], a special case of the stationary state is considered, namely, when both statistical moments \( \mu_\tau(\tau) \) and \( \sigma^2_\tau(\tau) \) do not depend on the reference time \( \tau_0 \). In our case, both \( \mu_\tau(\tau) \) and \( \sigma^2_\tau(\tau) \) do depend on \( \tau_0 \) (see equations (C.1) and (41) in appendix C). Moreover, the well-known equality \( \beta \sigma^2_\tau(\tau) = 2\mu_\tau(\tau) \) does not hold, even in the limiting case \( \tau \to \infty \). It is only one term \( P_{in} \tau \) in \( \mu_\tau(\tau) \) and one term \( 2P_{in} \tau \) in \( \beta \sigma^2_\tau(\tau) \) which connotes this equality.

8. Strong damping regime

The regime of strong damping has played an important role in the study of deterministic and noise-assisted dynamical systems and is usually relevant for experimental micro-systems. The common way to treat the overdamped dynamics is to put mass

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zero, \( m = 0 \), in equation (1). However, this approach is not fully rigorous for several reasons. First, the mass of the particle is never exactly zero; it can be ‘small’, but then the question remains, ‘small’ in comparison to what? Second, the energy balance cannot be fully evaluated since some terms do not exist separately but only in specific combinations. It can give rise to squares of the Dirac \( \delta \)-functions which do not exist \[38\]. In other methods, the order of magnitude of the dimensionless quantity \( m\dot{x}/\gamma \dot{x} \) is considered. The problem with this criterion is that the fraction is not constant in time and it is even worse because it can diverge to infinity. A more correct procedure is to identify regimes where ‘something’ is small or ‘something’ is large. Moreover, one can use a technique of mathematical sequences and try to obtain an exact limit of a considered sequence. We present the following scheme: We transform equation (1) to its dimensionless form and work in dimensionless variables, because, from a physical point of view, only relations between scales of time, length and energy play a role and not their absolute values. For the system defined by equation (1), one can identify four characteristic times:

1. The relaxation time of the velocity \( \tau_1 = m/\gamma \), which can be extracted from the special case of equation (1), namely, \( m\dot{v} = -\gamma v \).
2. The relaxation time of the position \( \tau_2 = \gamma/k \) deduced from the equation \( \gamma \dot{x} = -kx \).
3. The reciprocal of the frequency of small oscillations \( \tau_3 = \sqrt{m/k} \) obtained from the equation \( m\ddot{x} = -kx \).
4. The periodicity time related to the frequency of the external driving \( \tau_4 = 1/\omega \).

To analyze the strong damping regime, one has to choose the characteristic time \( \tau_2 \) as a scaling time since it does not contain the particle mass \( m \) and therefore the dimensionless time is defined as \( s = t/\tau_2 \). We also introduce the dimensionless position \( y = x/l_0 \), where \( l_0 \) is some characteristic distance. In the case of the harmonic potential, the only reasonable length scale is defined by the equipartition theorem \( \langle E_p \rangle = \langle kx^2/2 \rangle = k_B T/2 = kl_0^2/2 \). In consequence, a characteristic distance is \( l_0 = \sqrt{k_B T/k} \). Then the dimensionless Langevin equation corresponding to equation (1) takes the form:

\[
M \frac{d^2 y}{ds^2} + \frac{dy}{ds} + y = \tilde{F}(s) + \sqrt{2D_0} \xi(s),
\]

where the rescaled quantities read

\[
M = \tau_1/\tau_2 = mk/\gamma^2, \quad \tilde{F}(s) = F(\tau_2 s)/k l_0, \quad D_0 = k_B T/k l_0^2 = 1.
\]

The rescaled Gaussian noise \( \tilde{\xi}(s) \) has the same statistics as \( \xi(t) \) in equation (2). Note that (i) in the lhs of equation (55) there is only one parameter \( M \) which is the same as in equation (47), and (ii) the intensity \( D_0 \) of the rescaled thermal noise is 1. The parameter \( M \) is a ratio of two characteristic times of the system and if \( \tau_1 \ll \tau_2 \) then \( M \ll 1 \), and we can neglect the inertial term. Observe that the inequality \( M \ll 1 \) means that \( mk/\gamma^2 \ll 1 \) and it defines a real physical regime of a strong damping regime. It depends
not only on mass \( m \) and the friction coefficient \( \gamma \) but also on the stiffness \( k \) of the potential trap! In the literature, this fact is often ignored.

Now, we can rewrite equations (42)–(43) in the dimensionless form. The rescaled kinetic energy is (here we consider only the contribution of the force on the energies, so we subtract the \( k_B T/2 \) contribution due to thermal agitation):

\[
\varepsilon_k = \frac{[E_k - k_B T/2]/E_0}{(M \Lambda^2 - 1)^2 + \Lambda^2},
\]

where the rescaled frequency \( \Lambda = (\gamma/k) \omega \) and \( E_0 = F_0^2/4k \). For the potential energy we get

\[
\varepsilon_p = \frac{[E_p - k_B T/2]/E_0}{(M \Lambda^2 - 1)^2 + \Lambda^2}.
\]

The rescaled input power is given by equation (47). We find that if \( M \) decreases to zero, the rescaled kinetic energy \( \varepsilon_k \) decreases to zero. In this limit, the potential energy \( \varepsilon_p \) takes a similar form to the Cauchy probability distribution, namely,

\[
\varepsilon_p = \frac{1}{1 + \Lambda^2}.
\]

Similar behavior is exhibited by the total energy because the kinetic energy is zero. As follows from equation (47), if \( M \to 0 \) the input power assumes the form

\[
\frac{\overline{P}_{\text{in}}}{P_1} = \frac{\Lambda^2}{1 + \Lambda^2}.
\]

It is zero for zero frequency (i.e. when the force \( F(t) \) is constant) and \( \overline{P}_{\text{in}} \) saturates to the value \( P_1 \) when the frequency \( \Lambda \sim \omega \to \infty \). However, it is not a correct physical result. We should stress that from the exact formulas (45)–(47) it follows that for any non-zero but arbitrary small values of parameters, \( \overline{P}_{\text{in}} \to 0 \) when \( \omega \to \infty \). In particular, in the strong damping regime, the mass parameter \( M \) is small but non-zero. So, the reader should take care with the order of limits.

Similar considerations and methods can be applied to statistical moments for the particle coordinate and velocity as well as to the stationary, time-dependent averaged values of various forms of energies, such as (28)–(31).

9. Summary

The Brownian harmonic oscillator is an extremely simple and at the same time paradigmatic model with many applications in both classical and quantum physics. In the paper, we considered a less popular problem of energetics of the system, which is permanently moved out of thermal equilibrium, but reaches a stationary state in the long-time regime. Because the model is exactly solved, all interesting characteristics can be analytically evaluated. The first conclusion is that in the stationary state, various forms of averaged energy are additive: one part is related to thermal-equilibrium energy according to the equipartition theorem, and the second part is generated by the

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externally unbiased time-periodic force of frequency $\omega$. Additivity is due to the linearity of the system (i.e. the potential force is a linear function of the particle coordinate) and for non-linear systems it does not hold true. In the long-time limit, when a non-equilibrium stationary state is reached, energy is a periodic function of time of the same period as the external driving. If $\omega > \omega_0$ (where $\omega_0$ is the eigenfrequency of the undamped oscillator), the maximal value of the kinetic energy is greater than the maximal value of the potential energy. In turn, if $\omega < \omega_0$, the maximal value of the kinetic energy is smaller than the maximal value of the potential energy. For the resonance frequency $\omega = \omega_0$, the maximal value of the potential energy is equal to the maximal value of the kinetic energy and the total average energy is conserved, i.e. it is a constant function of time. It is the second important result.

We analysed time-independent characteristics of energy which are obtained by averaging over the period of the time-periodic force. We observe that in some regimes, determined by the dissipation strength (quantified by the friction coefficient $\gamma$), the frequency dependence of energy exhibits non-monotonic behaviour. New results are obtained for the Jarzynski-type work-fluctuation relation. We analysed this problem in the regime of long-time periods, when the non-equilibrium stationary but time-dependent state is reached. Both the mean value and variance of the work performed on the system in the interval $(\tau_0, \tau_0 + \tau)$ grow linearly with time $\tau$ (as it should be) but also depend periodically on the reference time $\tau_0$. Finally, we critically discussed the so-called overdamped dynamics. We are of the opinion that the correct framework for studying a regime of strong damping has to be based on the analysis of the time scales of the system. The overdamped dynamics cannot simply be obtained by putting the particle mass $m = 0$ (which can bring physical nonsense), but rather requires the identification of the correct parameters to compute a dimensionless equation, and to calculate the limits for each quantity of interest. It is what we presented in section 7. This procedure is always correct and we hope that we were able to clarify a physically acceptable method of an approximation of some limiting dynamics on a simple example of the driven Brownian harmonic oscillator.

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Appendix A. Obtaining the Green’s function

In this appendix we obtain the Green’s function of the full Langevin equation. According to the theory of Green’s function [39], we first should find the solutions of the corresponding homogeneous differential equation:
\[ m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0. \]  

(A.1)

The solution takes the exponential form \( y(t) = \exp(rt) \), where \( r \) verifies:

\[ mr^2 + \gamma r + k = 0. \]  

(A.2)

Assuming, for simplicity, that \( \gamma^2 \geq 4mk \), we have two real solutions \( r_+ \) and \( r_- \):

\[ r_\pm = -\frac{\gamma}{2m} \pm \beta, \quad \beta = \frac{\gamma}{2m} \sqrt{1 - \frac{4km}{\gamma^2}} \]  

(A.3)

which give us two independent functions \( y_1(t) = \exp[(\beta - \gamma/2m)t] \) and \( y_2(t) = \exp[-(\beta + \gamma/2m)t] \), respectively.

Now, we construct the Green’s function \( g(t, t') \) that verifies:

\[ m \frac{d^2g(t, t')}{dt^2} + \gamma \frac{dg(t, t')}{dt} + kg(t, t') = \delta(t - t'), \]  

(A.4)

with the initial conditions \( g(0, t') = 0 \) and \( dg(t, t')/dt = 0 \) for \( t = 0 \) according to the general prescription of finding the Green’s function for the initial value problems.

The Green’s function can be written as a linear combination of solutions of the homogeneous equation as follows:

\[ g(t, t') = c_1 y_1 + c_2 y_2, \quad t < t' \]  

(A.5)

and

\[ g(t, t') = d_1 y_1 + d_2 y_2, \quad t > t'. \]  

(A.6)

We need a set of four equations in order to determine four constants \( c_1, c_2, d_1 \) and \( d_2 \). The chosen boundary conditions simply provide that \( c_1 = c_2 = 0 \). The continuity of \( g(t, t') \) at \( t = t' \) implies:

\[ c_1 y_1(t') + c_2 y_2(t') = d_1 y_1(t') + d_2 y_2(t') \]  

(A.7)

which gives

\[ d_1 y_1(t') = -d_2 y_2(t'). \]  

(A.8)

The last equation is obtained by integrating equation (A.4) from \( t'^+ \) to \( t'^- \):

\[ \int_{t'^-}^{t'^+} [m \frac{d^2g(t, t')}{dt^2} + \gamma \frac{dg(t, t')}{dt} + kg(t, t')] dt = \int_{t'^-}^{t'^+} \delta(t - t') dt. \]  

(A.9)

Because \( g(t, t') \) is continuous, therefore \( dg(t, t')/dt \) can only have a jump discontinuity which implies:

\[ \frac{dg(t, t')}{dt} \bigg|_{t = t'^+} - \frac{dg(t, t')}{dt} \bigg|_{t = t'^-} = \frac{1}{m} \]  

(A.10)

and leads to the relation

\[ \left( -\frac{\gamma}{2m} + \beta \right) d_1 y_1(t') - \left( \frac{\gamma}{2m} + \beta \right) d_2 y_2(t') = \frac{1}{m}. \]  

(A.11)

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We then use equations (A.8) and (A.11) to find:
\[
d_2(t') = -\frac{1}{2m\beta y_2(t')}, \quad d_1(t') = \frac{1}{2m\beta y_1(t')).
\]
(A.12)

Finally, the Green’s function turns out to be:
\[
g(t, t') = \Theta(t - t') \frac{1}{2m\beta} e^{-\frac{\gamma}{m}(t-t')} [e^\beta(t-t') - e^{-\beta(t-t')}].
\]
(A.13)

The expression can then be written with the help of the Heaviside step function \(\Theta(u)\) to retrieve formula (10), (and formula (9) is obtained in the same way if \(\gamma^2 \leq 4mk\)).

**Appendix B. Position and velocity mean square deviation**

In this section, we calculate the mean-squared deviation \(\langle (\Delta x^2(t)) \rangle\) and \(\langle (\Delta v^2(t)) \rangle\) of the position and velocity of the Brownian particle, respectively. From (5) it can simply be verified that \(\langle \Delta x^2(t) \rangle = \langle x^2(t) \rangle - \langle x(t) \rangle^2 = \langle x^2(t) \rangle\) and therefore we only need to evaluate the mean-squared of \(x_\xi(t)\). From (11) we have:
\[
x_\xi^2(t) = \frac{2\gamma k_B T}{m^2\Omega^2} \int_0^t \int_0^t dt' dt'' e^{-\gamma_0(2t'-t'')} \sin \Omega(t - t') \sin \Omega(t - t'') \xi(t') \xi(t'').
\]
(B.1)

Taking an average over the ensemble of noise realizations and using its statistical properties gives:
\[
\langle \Delta x^2(t) \rangle = \langle x^2_\xi(t) \rangle = \frac{2\gamma k_B T}{m^2\Omega^2} \int_0^t e^{-2\gamma_0(t-t')} \sin^2 \Omega(t - t') \, dt'.
\]
(B.2)

The evaluation of the integral leads to the following expression for the MSD:
\[
\langle \Delta x^2(t) \rangle = \frac{k_B T}{k} + \frac{k_B T}{k\Omega^2} \left[ \gamma_0^2 \cos 2\Omega t - \gamma_0 \Omega \sin 2\Omega t - \omega_0^2 \right] e^{-2\gamma_0 t}.
\]
(B.3)

Now, we evaluate the mean-squared velocity deviation:
\[
\langle \Delta v^2(t) \rangle = \langle v^2(t) \rangle - \langle v(t) \rangle^2 = \langle x^2_\xi(t) \rangle = \langle v^2_\xi(t) \rangle,
\]
where
\[
v_\xi(t) = \sqrt{2\gamma k_B T} \int_0^\infty \frac{\partial G(t, u)}{\partial t} \xi(u) du.
\]
(B.5)

The differentiation of \(G(t, u)\) gives
\[
\frac{\partial G(t, u)}{\partial t} = \frac{1}{m\Omega} e^{-\gamma_0(t-u)} \left[ \sin \Omega(t - u) [\delta(t - u) - \gamma_0 \Theta(t - u)] ight.
\]
\[+ \Omega \cos \Omega(t - u) \Theta(t - u) \right].
\]
(B.6)
Replacing (B.6) into equation (B.5) leads to the relation
\[ v_\xi(t) = \frac{\sqrt{2\gamma k_B T}}{m\Omega} \int_0^t e^{-\gamma_0(t-u)} [\Omega \cos(\Omega (t-u) - \gamma_0 \sin(\Omega (t-u))] \xi(u) du, \tag{B.7} \]
and it follows:
\[ \langle \Delta v^2(t) \rangle = \langle v_\xi^2(t) \rangle = \frac{2\gamma k_B T}{m^2\Omega^2} \int_0^t du e^{-2\gamma_0(t-u)} [\Omega \cos(\Omega (t-u) - \gamma_0 \sin(\Omega (t-u))]^2. \tag{B.8} \]
The integration yields:
\[ \langle \Delta v^2(t) \rangle = \frac{k_B T}{m} + \frac{k_B T}{m\Omega^2} (\gamma_0 \Omega \sin 2\Omega t - 2\gamma_0^2 \sin^2 \Omega t - \Omega^2)e^{-2\gamma_0 t}. \tag{B.9} \]

Appendix C. Statistical moments of the work

In the long-time regime, when \( \gamma_0 \tau_0 \gg 1 \), the mean value of the work is
\[ \mu_{\tau_0}(\tau) = \omega F_0 \int_{\tau_0}^{\tau_0+\tau} \sin(\omega t) \langle x(t) \rangle dt = \omega F_0 \int_{\tau_0}^{\tau_0+\tau} \sin(\omega t) X_d(t) dt \]
\[ = \overline{P}_m \tau - \frac{1}{2\omega} \overline{P}_m \{ \sin[2\omega (\tau_0 + \tau)] - \sin(2\omega \tau_0) \} \]
\[ + \frac{\omega^2 - \omega_0^2}{2m A(\omega)} [F^2(\tau_0 + \tau) - F^2(\tau_0)], \tag{C.1} \]
where \( \langle x(t) \rangle = X_d(t) \) is given by equation (14), \( F(t) = F_0 \cos(\omega t) \) is the external time-periodic force, \( A(\omega) \) is defined in equation (13) and the period-averaged input power \( \overline{P}_m \) is determined by equation (45). We have to emphasize that in the long-time stationary state, the mean work still depends on the reference time \( \tau_0 \): if \( \tau \) is fixed and if one translates the reference time \( \tau_0 \rightarrow \tau_0 + \epsilon \) then in general the mean work will be different. However, the mean work performed over one period \( \tau = T = 2\pi/\omega \) of the driving \( F(t) \) does not depend on \( \tau_0 \) and its value is \( \mu_{\tau_0}(T) = \overline{P}_m T \). It is in agreement with the results discussed in section 6.

The second statistical moment of the work is
\[ \langle W_{\tau_0}^2(\tau) \rangle = \omega^2 F_0^2 \int_{\tau_0}^{\tau_0+\tau} dt \sin(\omega t) \int_{\tau_0}^{\tau_0+\tau} du \sin(\omega u) \langle x(t)x(u) \rangle \tag{C.2} \]
where \( x(t) \) is given by equation (5). The correlation function of the particle position is
\[ \langle x(t)x(u) \rangle = x_d(t) x_d(u) + 2\gamma k_B T \int_0^t \int_0^u ds \int_0^u dr G(t, s) G(u, r) \langle \xi(s) \xi(r) \rangle, \tag{C.3} \]
where the Green function \( G(t, s) \) is given by equation (9). We have neglected the exponentially decaying term \( x_c(t) \) which includes the initial conditions (see equation (6)). The correlation function of thermal noise is \( \langle \xi(s) \xi(r) \rangle = \delta(s-r) \). Because the variable \( t \) and \( u \) take values in the same interval, the cases \( t > u \) and \( t < u \) should be considered:
\[ \langle x(t)x(u) \rangle = x_d(t)x_d(u) + 2\gamma k_B T \theta(t - u) \int_0^u ds \ G(t, s)G(u, s) \]
\[ + 2\gamma k_B T \theta(u - t) \int_t^u ds \ G(t, s)G(u, s). \] \hspace{1cm} (C.4)

We insert it into equation (C.2) and get
\[ \langle W^2_{\tau_0}(\tau) \rangle = \langle W^2_{\tau_0}(\tau) \rangle^2 \]
\[ + 4\gamma k_B T \omega^2 F_0^2 \int_{\tau_0}^{\tau_0 + \tau} dt \sin(\omega t) \int_{\tau_0}^t du \sin(\omega u) \int_0^u ds \ G(t, s)G(u, s). \] \hspace{1cm} (C.5)

We have used the symmetry of the correlation function \( \langle x(t)x(u) \rangle \) under the interchange of time arguments. In consequence, two terms in the r.h.s of equation (C.3) give the same contribution to the second moment of the work. From the above equation it follows that the variance of the work is
\[ \sigma^2_{\tau_0}(\tau) = \frac{4\gamma k_B T \omega^2 F_0^2}{m^2 \Omega^2} \]
\[ \times \int_{\tau_0}^{\tau_0 + \tau} dt e^{-\gamma_0 t} \sin(\omega t) \int_{\tau_0}^t du e^{-\gamma_0 u} \sin(\omega u) \int_0^u ds e^{2\gamma_0 s} \sin[\Omega(t - s)] \]
\[ \times \sin[\Omega(u - s)]. \] \hspace{1cm} (C.6)

It is convenient to express all trigonometric functions by their exponential forms. Then integrals of the exponential functions can be easily calculated. The only cumbersome task is to simplify coefficients in the final expressions. One could use computer algebra to ease this part. The final result reads
\[ \beta \sigma^2_{\tau_0}(\tau) = 2 \mathcal{F} \tau \]
\[ - \mathcal{F} \left\{ \frac{1}{\omega} \left[ \sin(\omega(\tau_0 + \tau)) - \sin(\omega \tau_0) \right] - \tau_1 C_1 \left[ \cos(\omega(\tau_0 + \tau)) - \cos(\omega \tau_0) \right] \right\} \]
\[ + \mathcal{F} \tau_1 \left[ e^{-\gamma_0 \tau} I(\tau_0, \tau) - I(\tau_0, 0) \right]. \] \hspace{1cm} (C.7)

where the characteristic (relaxation) time \( \tau_1 = m/\gamma = 1/2\gamma_0 \) and the periodic function \( I(\tau_0, \tau) \) has the following structure
\[ I(\tau_0, \tau) = \alpha_0 \cos[2\omega \tau_0 + \omega \tau] \left[ \alpha_1 \cos(\Omega \tau) + \alpha_2 \sin(\Omega \tau) \right] \]
\[ + \cos(\Omega \tau) \left[ G_1 \cos(\omega \tau) + G_2 \sin(\omega \tau) \right] + \frac{\sin(\Omega \tau)}{\Omega} \left[ G_3 \cos(\omega \tau) + G_4 \sin(\omega \tau) \right]. \] \hspace{1cm} (C.8)

All dimensionless parameters \( C_1, \alpha_k(k = 0, 1, 2) \) and \( G_n(n = 1, 2, 3, 4) \) are expressed by three system parameters \( \{\gamma_0, \omega_0, \omega\} \). We do not present their explicit form because we want to discuss a fundamental issue and not the details.

From equation (C.7) we infer that fluctuations of the work, quantified by its variance, depend on the reference time \( \tau_0 \) and consist of several characteristic contributions. The part \( 2 \mathcal{F} \tau \) grows linearly with the time-interval \( \tau \). It would correspond to normal diffusion, not in position space but in the work space. There is a part periodic
in $\tau$ and a contribution which exponentially decays as $\tau$ increases. There is also a part independent of $\tau$ (in the term $I(\tau_0, 0)$).

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