Equivalence of Dimensional Reduction and Dimensional Regularisation

I. Jack, D. R. T. Jones and K. L. Roberts

DAMTP, University of Liverpool, Liverpool L69 3BX, U.K.

For some years there has been uncertainty over whether regularisation by dimensional reduction (DRED) is viable for non-supersymmetric theories. We resolve this issue by showing that DRED is entirely equivalent to standard dimensional regularisation (DREG), to all orders in perturbation theory and for a general renormalisable theory. The two regularisation schemes are related by an analytic redefinition of the couplings, under which the β-functions calculated using DRED transform into those computed in DREG. The S-matrix calculated using DRED is numerically equal to the DREG version, ensuring that both schemes give the same physics.

January 1994
1. Introduction

Dimensional reduction (DRED) was introduced some time ago\cite{1} as a means of regulating supersymmetric gauge theories which maintains manifest supersymmetry whilst retaining the elegant features of dimensional regularisation (DREG). However, its use has often been attended by controversy. Firstly, there are potential ambiguities in DRED associated with the treatment of the Levi-Civita symbol, $\epsilon^{\mu\nu\rho\sigma}$ and, relatedly, with the correct definition of $\gamma_5$\cite{2}. We shall not concern ourselves with these problems here. Secondly, questions have been raised regarding the unitarity of DRED. It was argued by van Damme and ’t Hooft\cite{3} that whilst DRED preserves unitarity in the supersymmetric case, unitarity is broken when DRED is applied to non-supersymmetric theories. However, it is our contention that when DRED is applied in the fashion envisaged in Ref.\cite{4} (which differs from that employed by van Damme and ’t Hooft in its treatment of counterterms) then unitarity is preserved for any theory. In Ref.\cite{5}, we presented evidence for our claim by computing the two-loop $\beta$-functions for a toy model using both DRED and DREG, and showing that the DRED $\beta$-functions could be transformed into those for DREG by a coupling constant redefinition. We noted that a regularisation scheme mentioned in Ref.\cite{3} (their “System 3”), which differs from the version of DRED which they use, led to the same $\beta$-functions as our version of DRED. However it was by no means clear a priori that this scheme was equivalent to the version of DRED which we advocate. In this paper we shall show that System 3 is indeed equivalent to our version of DRED, in perfect generality and to all orders. Since it will also be clear that System 3 is equivalent to DREG up to coupling constant redefinition, also to all orders and for a general theory, we shall have established the exact equivalence of DRED and DREG and hence we shall have shown that DRED preserves unitarity. Moreover, we shall also see that the S-matrix computed using DRED is equivalent to that computed using DREG.

2. Dimensional regularisation and minimal subtraction

Let us start by describing the standard dimensional regularisation procedure for a general theory. Consider a general renormalisable gauge theory coupled to scalar fields (the extension to fermions is straightforward, but introduces notational complications) with bare Lagrangian

\[ L(\lambda_B, g_B, \alpha_B) = L_G(g_B, \alpha_B) + \partial_\mu \phi_B^i \partial_\mu \phi_B^i + \lambda_B^{ij} \phi_B^i \phi_B^j + \lambda_B^{ijk} \phi_B^i \phi_B^j \phi_B^k + \lambda_B^{ijkl} \phi_B^i \phi_B^j \phi_B^k \phi_B^l \]  

(2.1)
in 4 dimensions. In Eq. (2.1), \( \phi^i_B \) represents the bare scalar fields, and \( L_G \) subsumes all the gauge-field dependent terms in the Lagrangian. The bare gauge coupling is denoted \( g_B \) (assuming a simple group so that there is only one gauge coupling), and \( \alpha_B \) represents the bare gauge-fixing parameter. The \( S \)-matrix for the theory is constructed in standard fashion from the one-particle-irreducible (1PI) \( n \)-point functions. However, it will be sufficient for our purposes to focus our attention on the 1PI \( n \)-point functions for \( n \leq 4 \), since for a renormalisable theory in 4 dimensions these are the only ones which contain potential primitive divergences. Again, since the theory is renormalisable, each potentially divergent 1PI \( n \)-point function with \( n \leq 4 \) is associated with some \( n \)-point coupling. For definiteness and notational convenience we shall restrict our discussion to the 1PI functions with external scalar lines only; however, the extension to external gauge fields is trivial. The 1PI 4-point function corresponding to a 4-point coupling \( \lambda^{ijkl}_B \) will be denoted \( \Gamma^{ijkl} \), with a similar notation for 3- and 2-point functions. The theory is regulated by continuation to \( d = 4 - \epsilon \) dimensions. We can write, for example,

\[
\Gamma^{ijkl} = \lambda^{ijkl}_B + D^{ijkl}(\lambda_B, g_B, \alpha_B) \tag{2.2}
\]

where

\[
D^{ijkl}(\lambda_B, g_B, \alpha_B) = \sum_{L=1} D^{ijkl}_L(\lambda_B, g_B, \alpha_B) \tag{2.3}
\]

with \( D^{ijkl}_L(\lambda_B, g_B, \alpha_B) \) representing the sum of \( L \)-loop diagrams contributing to \( \Gamma^{ijkl} \). (Of course \( D^{ijkl} \) and \( D^{ijkl}_L \) also depend on \( \epsilon \) and on the external momenta, but we shall not write this dependence explicitly.) In terms of \( \lambda_B \) the expression in Eq. (2.2) has divergences represented by poles in \( \epsilon \). Within standard DREG, we write \( \lambda_B \) as a Laurent series

\[
\lambda^I_B = \mu^{k_I \epsilon} (\lambda^I + \sum_{n=1} A^I_n(\lambda) \epsilon^n), \tag{2.4}
\]

(\( \lambda^I \) represents any of \( \lambda^{ijkl}_B \), \( \lambda^{ijk}_B \), \( \lambda^{ij}_B \), \( g \) or \( \alpha \), and where \( k_I \) takes, for example, the value 1 when \( I \) denotes \( ijk \), \( 1/2 \) when \( I \) denotes \( ik \) and 0 when \( I \) denotes \( ij \)) so that the Green function \( G^{ijkl} \), defined by

\[
G^{ijkl} = \sqrt{Z^{i'j'}} \sqrt{Z^{j'k'}} \sqrt{Z^{kk'}} \sqrt{Z^{ll'}} \Gamma^{i'j'k'l'}, \tag{2.5}
\]

where \( Z_\phi \) is the wave-function renormalisation for the fields \( \phi \), becomes finite as a function of the renormalised couplings \( \lambda^I \). Any momentum-dependent terms in the 1PI 4-point and
3-point functions with purely scalar external lines must be finite by renormalisability, and we have suppressed them; however this is not so for the 2-point case, for which we require the Green function

\[ G^{ij} = \sqrt{Z_\phi}^{ii'} \sqrt{Z_\phi}^{jj'} \Gamma^{i'j'}, \]  

(2.6)

to be finite, where

\[ \Gamma^{ij} = p^2 [\delta^{ij} + F^{ij}(\lambda_B)] + [\lambda_B^{ij} + D^{ij}(\lambda_B)], \]  

(2.7)

with \( p \) an external momentum. In fact, Eq. (2.2) yields a very succinct formulation of the process of constructing counterterm diagrams, which will be useful later on in comparing DRED with van Damme and ‘t Hooft’s System 3. First of all, in view of Eq. (2.5) it is convenient to define “dressed” versions of quantities by

\[ \bar{\lambda}^{ijkl} = \sqrt{Z_\phi}^{ii'} \sqrt{Z_\phi}^{jj'} \lambda^{i'j'}, \]  

(2.8)

and so on. Since \( G^{ijkl} \) as given by Eqs. (2.3) is finite as a function of the renormalised couplings \( \lambda^I \), we have

\[ \text{P.P.} [\bar{\lambda}^{ijkl}] = \bar{\lambda}^{ijkl} - \lambda^{ijkl} = -\text{P.P.} \{ D^{ijkl} \} \]  

(2.9)

where “P.P.” stands for “Pole Part” and we assume that we are working with minimal subtraction. (The pole part is taken only after substituting the expressions Eq. (2.4) for \( \lambda^I_B \) in terms of the renormalised coupling.) Next, for convenience, we define \( \lambda^I_P \equiv \sum_{n=1}^{A^I(\lambda)} \frac{A^I}{e^n} \), so that \( \lambda^I_B = \lambda^I + \lambda^I_P \). (Here we set \( \mu = 1 \) for convenience; we know that \( \lambda^I_P \) contains no explicit \( \mu \)-dependence.) Expanding \( D_L(\lambda_B) \) around \( \lambda, g \) and \( \alpha \), we obtain

\[ \bar{\lambda}^{ijkl} - \lambda^{ijkl} = -\text{P.P.} \{ \sum_{L=1}^{\infty} \sum_{l=1}^{\infty} \sqrt{Z_\phi}^{ii'} \sqrt{Z_\phi}^{jj'} \sqrt{Z_\phi}^{kk'} \sqrt{Z_\phi}^{ll'} \lambda^I_P \frac{d}{d\lambda_i} \ldots \frac{d}{d\lambda_i} D_L^{i'j'k'l'}(\lambda) \}, \]  

(2.10)

where

\[ \lambda^I \frac{\partial}{\partial \lambda^I} \equiv \lambda^{ijkl} \frac{\partial}{\partial \lambda^{ijkl}} + \lambda^{ijk} \frac{\partial}{\partial \lambda^{ijk}} + \lambda^{ij} \frac{\partial}{\partial \lambda^{ij}} + g \frac{\partial}{\partial g} + \alpha \frac{\partial}{\partial \alpha}. \]  

(2.11)

Diagrammatically, each term in the sum on the right-hand side of Eq. (2.10) represents a set of diagrams obtained by inserting a counterterm \( \lambda^I_P \) in turn at each vertex corresponding to the coupling \( \lambda^I \) in each \( L \)-loop diagram, together with an insertion of \( \sqrt{Z_\phi} \) on every external line. Hence the truncation of Eq. (2.10) at the order in \( \lambda^I \) corresponding to a given
loop order $L$, corresponds precisely to the process of computing counterterms; the right-hand side of Eq. (2.10) automatically produces the set of L-loop diagrams together with the correct associated subtraction diagrams with the correct symmetry factors. These are fairly obvious remarks but we have been unable to find this formulation in the literature. In a similar fashion, we have from Eqs. (2.11) and (2.7) that

$$Z_{ij}^{\phi} = \delta_{ij} - \text{P.P.}\{\bar{F}^{ij}\}. \quad (2.12)$$

Eqs. (2.11) (with similar expressions for the 3- and 2-point couplings involving $D^{ijk}$ and $D^{ij}$ respectively) and (2.12) clearly determine $\lambda^I_B$ as Laurent series in terms of the renormalised couplings as in Eq. (2.4). They also determine $\sqrt{Z_{\phi}}$ as a Laurent series

$$\sqrt{Z_{\phi}}^i_j = \delta_{ij} + \sum_{n=1}^{\infty} \frac{z_{ij}^n(\lambda)}{\epsilon^n}. \quad (2.13)$$

3. Dimensional reduction

We now turn to discussing two modifications of DREG (which in fact will turn out to be equivalent); namely DRED, and also van Damme and 't Hooft's System 3. The crucial distinction between DRED and DREG is that in DRED the continuation from 4 to $d = 4 - \epsilon$ dimensions is made by compactification. Therefore the number of field components remains unchanged, though they only depend on $d$ co-ordinates. Consequently $\epsilon$ of the components of a vector multiplet become scalars, termed "$\epsilon$-scalars", and which we shall denote $\phi^i$. Moreover, terms in the Lagrangian involving $\epsilon$-scalars cannot be expected to renormalise in the same way as terms involving the corresponding vector fields, since they are not related by $d$-dimensional Lorentz invariance. Hence we must introduce new couplings for the terms in the Lagrangian involving $\epsilon$-scalars. We call these “evanescent couplings”. We shall term the original fields and couplings in the Lagrangian “real”. In a natural extension of the notation for purely real couplings in Eq. (2.4), we denote them as $\lambda^i_{\bar{E}}$, $\lambda^{ijl}_{\bar{E}}$, $\lambda^{ijk}_{\bar{E}}$, etc and moreover write them generically as $\lambda^I_{\bar{E}}$. It was demonstrated in Ref. [3] that it is vital to maintain the distinction between evanescent and real couplings in order to show the equivalence between the DRED and DREG $\beta$-functions. The difference between our implementation of DRED and that adopted by van Damme and 't Hooft in Ref. [3] (their “System 4”) resides in the counterterms assigned to evanescent couplings. The renormalisation of evanescent couplings involves computing graphs with
external $\epsilon$-scalars. Our philosophy is that we should calculate the counterterms required to make graphs with external $\epsilon$-scalars finite and use those counterterms to construct subtraction diagrams. This is equivalent to the usual diagram-by-diagram subtraction procedure where we subtract from each diagram subtraction diagrams obtained by replacing divergent subdiagrams by counterterm insertions with the same pole structure. System 4 of van Damme and ’t Hooft, on the other hand, involves replacing the genuine counterterms for evanescent couplings by the counterterms for the corresponding real couplings, and this is their interpretation of DRED. Henceforth, when we use the term DRED we will refer to the former version of the scheme. (We should stress that the two versions of DRED are equivalent for supersymmetric theories.[6][3]) In either case, the calculations are done using minimal subtraction, so that the counterterms contain only poles in $\epsilon$. We must take into account factors of $\epsilon$ from the multiplicity of the $\epsilon$ scalars so that, for instance, a one-loop subdiagram with a divergent momentum integral giving an $\epsilon^{-1}$ pole but with a factor of $\epsilon$ due to the presence of $\epsilon$-scalars in the loop would be finite and would not require a subtraction.

Now let us turn to System 3. We consider first the theory obtained by performing the reduction to $4 - \epsilon$ dimensions; this will have a Lagrangian $L = L(\lambda_B) + L_E(\lambda_B, (\lambda_E)_B)$ where $L_E$ represents the extra terms involving the additional scalars (which we shall call $\epsilon$-scalars, although at present their multiplicity is $\epsilon$; shortly we shall set $\epsilon = \epsilon$). These terms also involve bare evanescent couplings denoted by $(\lambda_E)_B$. We now write the bare couplings in terms of the renormalised quantities by continuing to $d = 4 - \epsilon$ dimensions and computing the counterterms required to make the theory finite as a function of the renormalised couplings. We obtain

$$\lambda^I_B = \mu^{k_I\epsilon}(\lambda^I + \sum_{n=1}^{\infty} \sum_{m=0} A^I_{nm}(\lambda, \lambda_E) E^m \epsilon^n),$$

$$(\lambda^I_E)_B = \mu^{l_I\epsilon}(\lambda^I_E + \sum_{n=1}^{\infty} \sum_{m=0} A^I_{nm}(\lambda, \lambda_E) E^m \epsilon^n),$$

where we assume that the counterterms are defined by minimal subtraction, so that the $A^I_{nm}$ and $\tilde{A}^I_{nm}$ are independent of $\epsilon$. $k_I$ is as defined following Eq. (2.4), and $l_I$ in similar fashion. We also have expressions analogous to Eq. (2.13) for the wave-function renormalisation matrix for the real scalar fields and for the $\epsilon$-scalars:

$$\sqrt{Z_{\phi}}^{ij} = \delta^{ij} + \sum_{n=1}^{\infty} \sum_{m=0} z^{ij}_{nm}(\lambda, \lambda_E) E^m \epsilon^n,$$

$$\sqrt{Z_{\epsilon}}^{ij} = \delta^{ij} + \sum_{n=1}^{\infty} \sum_{m=0} \tilde{z}^{ij}_{nm}(\lambda, \lambda_E) E^m \epsilon^n.$$

5
(There is no wave-function renormalisation matrix mixing real scalars and $\epsilon$-scalars.) We now set $E = \epsilon$, so that we have

$$\lambda_B^I = \mu^{k_I}\epsilon (\lambda^I + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{A_{nm}^I(\lambda, \lambda_E)\epsilon^m}{\epsilon^n}),$$

$$(\lambda_E^I)_B = \mu^{l_I}\epsilon (\lambda_E^I + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\tilde{A}_{nm}^I(\lambda, \lambda_E)\epsilon^m}{\epsilon^n}),$$

$$(3.3)$$

$$\sqrt{Z_{\phi}^{ij}} = \delta^{ij} + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{z_{nm}^{ij}(\lambda, \lambda_E)\epsilon^m}{\epsilon^n},$$

$$\sqrt{Z_{\epsilon}^{ij}} = \delta^{ij} + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\tilde{z}_{nm}^{ij}(\lambda, \lambda_E)\epsilon^m}{\epsilon^n}. \tag{3.4}$$

Next we define new couplings $\lambda'^I$, $\lambda'_{E}^I$ by writing

$$\lambda_B^I = \mu^{k_I}\epsilon (\lambda'^I + \sum_{n=1}^{\infty} \frac{B_{n}^I(\lambda', \lambda'_E)}{\epsilon^n}),$$

$$(\lambda_E^I)_B = \mu^{l_I}\epsilon (\lambda'_{E}^I + \sum_{n=1}^{\infty} \frac{\tilde{B}_{n}^I(\lambda', \lambda'_E)}{\epsilon^n}) \tag{3.4}$$

and requiring that there is no explicit $\epsilon$-dependence in $B_{n}^I$ and $\tilde{B}_{n}^I$. The couplings $\lambda'^I$, $\lambda'_{E}^I$ are those of van Damme and 't Hooft’s scheme System 3. The relation between $\lambda'$, $\lambda'_E$ and $\lambda$, $\lambda_E$ rapidly becomes very complex as one looks beyond one loop; we shall shortly give an explicit expression up to two loops. It is important to note however that the relation between $\lambda$, $\lambda_E$ and $\lambda'$, $\lambda'_E$ is analytic. Next we define $Z_{\phi}^{n^{ij}}(\lambda', \lambda'_E, \epsilon)$ and $Z_{\phi}^{n^{k^{j}}}(\lambda', \lambda'_E, \epsilon)$ by

$$\sqrt{Z_{\phi}^{n^{ik}}(\lambda', \lambda'_E, \epsilon)} \sqrt{Z_{\phi}^{n^{k^{j}}}(\lambda', \lambda'_E, \epsilon)} = \sqrt{Z_{\phi}^{n^{ij}}(\lambda, \lambda_E, \epsilon)} \tag{3.5}$$

where we require

$$\sqrt{Z_{\phi}^{n^{ij}}(\lambda', \lambda'_E, \epsilon)} = \delta^{ij} + \sum_{n=1}^{\infty} \frac{z_{n}^{ij}(\lambda', \lambda'_E)}{\epsilon^n} \tag{3.6}$$

with $z^{n^{ij}}$ independent of $\epsilon$, and where $Z_{\epsilon}^{n'}$ is finite as $\epsilon \to 0$. These requirements specify $Z_{\phi}'$ and $Z_{\phi}''$ uniquely. Similarly, we also define

$$\sqrt{Z_{\epsilon}^{n^{ij}}(\lambda', \lambda'_E, \epsilon)} = \delta^{ij} + \sum_{n=1}^{\infty} \frac{\tilde{z}_{n}^{ij}(\lambda', \lambda'_E)}{\epsilon^n}, \tag{3.7}$$

together with $Z_{\epsilon}^{n^{ij}}(\lambda', \lambda'_E, \epsilon)$, by $\sqrt{Z_{\epsilon}^{n'}} \sqrt{Z_{\epsilon}^{*}} = \sqrt{Z_{\epsilon}}$, with $Z_{\epsilon}''$ analytic in $\epsilon$ and $\tilde{z}_{n}^{ij}$ independent of $\epsilon$. We will later show that the Laurent expansions of the bare couplings in Eq. (3.4)
are exactly those obtained using DRED, and that correspondingly $Z'_\phi$ and $Z'_\epsilon$ are the real scalar and $\epsilon$-scalar wave-function renormalisations obtained using DRED.

The $\beta$-functions for $\lambda$, $\lambda_E$, $\lambda'$, and $\lambda'_E$ are defined by

$$
\beta^I(\lambda, \lambda_E, \epsilon) = \mu \frac{d}{d\mu} \chi^I, \quad \beta^I_E(\lambda, \lambda_E, \epsilon) = \mu \frac{d}{d\mu} \chi^I_E
$$

(3.8)

and are given explicitly by

$$
\beta^I(\lambda, \lambda_E, \epsilon) = -k_I \epsilon \chi^I + D_I \sum_{m=0} A^I_{1m} \epsilon^m, \quad \beta^I_E(\lambda, \lambda_E, \epsilon) = -l_I \epsilon \chi^I_E + D_I \sum_{m=0} \tilde{A}^I_{1m} \epsilon^m,
$$

$$
\beta'^I(\lambda', \lambda'_E, \epsilon) = -k'_I \epsilon \chi'^I + D'_I B^I_1, \quad \beta'^I_E(\lambda', \lambda'_E, \epsilon) = -l'_I \epsilon \chi'^I_E + D'_I \tilde{B}^I_1
$$

(3.9)

where

$$
D_I = \sum_J k_J \lambda^J \frac{\partial}{\partial \lambda^J} + \sum_J l_J \lambda'_E \frac{\partial}{\partial \lambda'_E} - k_I.
$$

(3.10)

$D_I$, $D'_I$ and $D'_I$ are defined in similar fashion. (In fact the effect of the $D$ operators is to multiply an $L$-loop contribution to $A_{1m}$, $\tilde{A}_{1m}$, $B_1$ or $\tilde{B}_1$ by $L$.) By definition, the $\beta$-functions $\beta$ and $\beta'$ for $\lambda$ and $\lambda'$ are related by coupling constant redefinition, and we have from Eq. (3.8)

$$
\beta'^I(\lambda', \lambda'_E, \epsilon) = (\beta' \frac{\partial}{\partial \lambda} + \beta'_E \frac{\partial}{\partial \lambda'_E}) \chi'^I(\lambda, \lambda_E, \epsilon),
$$

(3.11)

or equivalently

$$
\beta^I(\lambda, \lambda_E, \epsilon) = (\beta' \frac{\partial}{\partial \lambda'} + \beta'_E \frac{\partial}{\partial \lambda'_E}) \chi^I(\lambda', \lambda'_E, \epsilon).
$$

(3.12)

Now the standard DREG $\beta$-function for $\lambda$, in the absence of the $\epsilon$-scalars, is clearly given by

$$
\beta^I_{\text{DREG}}(\lambda) = \lim_{\epsilon \to 0} \beta^I(\lambda, \lambda_E, \epsilon) = D_I A^I_0.
$$

(3.13)

So if we take the limit as $\epsilon \to 0$ of Eq. (3.12), we obtain, using Eqs. (3.13) that:

$$
\beta^I_{\text{DREG}}(\lambda) = (\beta'(\lambda', \lambda'_E, 0) \frac{\partial}{\partial \lambda'} + \beta'_E(\lambda', \lambda'_E, 0) \frac{\partial}{\partial \lambda'_E}) \chi^I(\lambda', \lambda'_E, 0).
$$

(3.14)

Our next step is to show that our prescription for DRED is equivalent to System 3. In order to do this, we need to consider the way in which the bare couplings are generated
diagrammatically, since our version of DRED depends crucially on the way in which counterterms are constructed for divergent subdiagrams. For the theory reduced from 4 to 4 − E dimensions, the 1PI function corresponding to a real (but not 2-point) coupling $\lambda^I$ is given by an expression analogous to Eq. (2.2),

$$\Gamma^I(\lambda_B, (\lambda_E)_B) = \lambda^I_B + \sum_{M=0} E^M D^I_M(\lambda_B, (\lambda_E)_B)$$

(3.15)

where $D^I_M$ represents the sum of all graphs which produce a multiplicity factor of $E^M$. The 1PI function corresponding to an evanescent (but not 2-point) coupling $\lambda^I_E$ is similarly given by

$$\Gamma^I(\lambda_B, (\lambda_E)_B) = (\lambda^I_E)_B + \sum_{M=0} E^M D^I_E(M)(\lambda_B, (\lambda_E)_B)$$

(3.16)

The 1PI 2-point function for real fields is given by an expression analogous to Eq. (2.7),

$$\Gamma^{ij} = p^2[\delta^{ij} + \sum_{M=0} E^M F^ij_M(\lambda_B, (\lambda_E)_B)] + [\lambda_B^i + \sum_{M=0} E^M D^{ij}_M(\lambda_B, (\lambda_E)_B)]$$

(3.17)

and there is a similar expression for the 1PI 2-point function for $\epsilon$-scalars. The definitions of $\lambda^I_B$ , $(\lambda^I_E)_B$ in terms of $\lambda^I$ , $\lambda^I_E$ in Eq. (3.1), together with the expressions for $Z_{\phi}^{ij}$ and $Z_{\epsilon}^{ij}$ in Eq. (3.3), precisely ensure that the Green functions given by the “dressed” versions of the 1PI functions, $G^I \equiv \Gamma^I$ and $G^I \equiv \Gamma^I$, are finite as a function of $\lambda^I$, $\lambda^I_E$. (Dressed versions of Green’s functions with external $\epsilon$-scalars are defined in a similar way to Eq. (2.3), but with factors of $\sqrt{Z_{\epsilon}^{ij}}$ instead of $\sqrt{Z_{\phi}^{ij}}$ where appropriate.) To take the example of the real 2-point coupling, $\sqrt{Z_{\phi}^{ij}} \sqrt{Z_{\phi}^{ij}} \Gamma^{ij}(\lambda_B, (\lambda_E)_B)$ is guaranteed to be finite. This implies the separate finiteness of

$$\sqrt{Z_{\phi}^{ij}}(\lambda, \lambda_E, \epsilon)\sqrt{Z_{\phi}^{jj}}(\lambda, \lambda_E, \epsilon)[\delta^{ij}]$$

$$+ \sum_{M=0} E^M F^{ij}_M(\lambda) + \sum_{n=1} \sum_{m=0} \frac{A^I_{nm}(\lambda, \lambda_E)E^n}{\epsilon^n}, \lambda_E + \sum_{n=1} \sum_{m=0} \frac{A^I_{nm}(\lambda, \lambda_E)E^n}{\epsilon^n}]$$

(3.18)

and

$$\sqrt{Z_{\phi}^{ij}}(\lambda, \lambda_E, \epsilon)\sqrt{Z_{\phi}^{jj}}(\lambda, \lambda_E, \epsilon)[\lambda^I_B]$$

$$+ \sum_{M=0} E^M D^{ij}_M(\lambda) + \sum_{n=1} \sum_{m=0} \frac{A^I_{nm}(\lambda, \lambda_E)E^n}{\epsilon^n}, \lambda_E + \sum_{n=1} \sum_{m=0} \frac{A^I_{nm}(\lambda, \lambda_E)E^n}{\epsilon^n}]$$

(3.19)

A fortiori, when we set $E = \epsilon$ in all the Green functions, we still have finite expressions. We then rewrite $\lambda^I_B = \lambda^I + \sum_{n=1} \sum_{m=0} \frac{A^I_{nm}(\lambda, \lambda_E)E^n}{\epsilon^n}, (\lambda^I_E)_B = \lambda^I_E + \sum_{n=1} \sum_{m=0} \frac{A^I_{nm}(\lambda, \lambda_E)E^n}{\epsilon^n}$,
\[
\lambda^I_E + \sum_{n=1} \sum_{m=0} \frac{\tilde{A}^I_m(\lambda, \lambda_E)E^m}{\epsilon^n}, \text{ in terms of } \lambda^I, \lambda^I_E \text{ according to Eq. (3.4)}, \text{ and at the same time we rewrite } Z_\phi(\lambda, \lambda_E, \epsilon) \text{ and } Z_\epsilon(\lambda, \lambda_E, \epsilon) \text{ as } Z_\phi''(\lambda', \lambda'_E, \epsilon)Z_\phi'(\lambda', \lambda'_E, \epsilon) \text{ and } Z_\epsilon''(\lambda', \lambda'_E, \epsilon)Z_\epsilon'(\lambda', \lambda'_E, \epsilon) \text{ according to Eqs. (3.5)–(3.7), obtaining expressions which are finite as functions of } \lambda', \lambda'_E. \text{ For instance, in the 2-point example above, we now have that}
\]

\[
\sqrt{Z_\phi^{ii'}(\lambda', \lambda'_E, \epsilon)}\sqrt{Z_\phi^{ii''}(\lambda', \lambda'_E, \epsilon)}\sqrt{Z_\phi^{jj'}(\lambda', \lambda'_E, \epsilon)}\sqrt{Z_\phi^{jj''}(\lambda', \lambda'_E, \epsilon)}\times
\]

\[
[\delta^{ij''} + \sum_{M=0} e^M F_{M}^{ii''}(\lambda') + \sum_{M=0} B_n^J(\lambda', \lambda'_E) + \sum_{M=0} B_n^J(\lambda', \lambda'_E)]
\]  

(3.20)

and

\[
\sqrt{Z_\phi^{ii'}(\lambda', \lambda'_E, \epsilon)}\sqrt{Z_\phi^{ii''}(\lambda', \lambda'_E, \epsilon)}\sqrt{Z_\phi^{jj'}(\lambda', \lambda'_E, \epsilon)}\sqrt{Z_\phi^{jj''}(\lambda', \lambda'_E, \epsilon)}\times
\]

\[
[\lambda^{ij''} + \sum_{M=0} e^M D_{M}^{ii''}(\lambda') + \sum_{M=0} B_n^J(\lambda', \lambda'_E) + \sum_{M=0} B_n^J(\lambda', \lambda'_E)]
\]

(3.21)

are finite as functions of \( \lambda', \lambda'_E \). We can then remove the factors of \( \sqrt{Z_\phi''} \) and \( \sqrt{Z_\epsilon''} \) in Eqs. (3.20), (3.21) and the other Green functions, still leaving finite expressions, since \( Z_\phi'' \) and \( Z_\epsilon'' \) are analytic in \( \epsilon \). These resulting finite quantities can be regarded as deriving from 1PI functions “dressed” by \( Z_\phi'(\lambda', \lambda'_E, \epsilon) \) and \( Z_\epsilon'(\lambda', \lambda'_E, \epsilon) \). Hence we now have that

\[
\hat{\lambda}^I + \sum_{M=0} e^M \hat{D}_M(\lambda') + \sum_{M=0} \frac{B_n^J(\lambda', \lambda'_E)}{\epsilon^n}, \lambda^I_E + \sum_{M=0} \frac{\hat{B}_n^J(\lambda', \lambda'_E)}{\epsilon^n}
\]

\[
\hat{\lambda}^I_E + \sum_{M=0} e^M \hat{D}_M(\lambda') + \sum_{M=0} \frac{B_n^J(\lambda', \lambda'_E)}{\epsilon^n}, \lambda^I_E + \sum_{M=0} \frac{\hat{B}_n^J(\lambda', \lambda'_E)}{\epsilon^n},
\]

(3.22)

\[
Z_\phi^{ij}(\lambda', \lambda'_E) + \sum_{M=0} e^M \hat{F}_M^{ij}(\lambda') + \sum_{M=0} \frac{B_n^J(\lambda', \lambda'_E)}{\epsilon^n}, \lambda^I_E + \sum_{M=0} \frac{\hat{B}_n^J(\lambda', \lambda'_E)}{\epsilon^n},
\]

\[
Z_\epsilon^{ij}(\lambda', \lambda'_E) + \sum_{M=0} e^M \hat{F}_M^{ij}(\lambda') + \sum_{M=0} \frac{B_n^J(\lambda', \lambda'_E)}{\epsilon^n}, \lambda^I_E + \sum_{M=0} \frac{\hat{B}_n^J(\lambda', \lambda'_E)}{\epsilon^n}
\]

(where for example \( \hat{\lambda}^{ij} \equiv \lambda^{kl} \hat{G}_\phi^{ik} \sqrt{Z_\phi''} \sqrt{Z_\phi''} \)) are finite as a function of \( \lambda^I, \lambda^I_E \). The “hatted” quantities will later turn out to be identical to quantities evaluated using DRED.
and “dressed” using the DRED wave-function renormalisations. We therefore have

\[ \lambda'^I + \sum_{n=0} \frac{\hat{B}^I_n(\lambda', \lambda'_E)}{\epsilon^n} = \lambda'^I \]

\[- \text{P.P.}\left\{ \sum_{M=0} \epsilon^M \hat{D}^I_M(\lambda'^I, \lambda'_E) + \lambda'_E \right\} \] \[ \sum_{n=0} \frac{\hat{B}^I_n(\lambda', \lambda'_E)}{\epsilon^n} = \lambda'^E \]

\[- \text{P.P.}\left\{ \sum_{M=0} \epsilon^M \hat{F}^{ij}_M(\lambda'^I, \lambda'_E) + \lambda'_E \right\} \]

\[ \sum_{n=0} \frac{z^{ij}_n(\lambda', \lambda'_E)}{\epsilon^n} = \text{P.P.}\left\{ \sum_{M=0} \epsilon^M \hat{F}^{ij}_M(\lambda'^I, \lambda'_E) + \lambda'_E \right\} \]

Thus Eq. (3.23) completely determines \( B^I_n \) and \( \hat{B}^I_n \). However, it is clear that the counterterms in DRED are obtained recursively by the same process; the diagrams appearing in DRED are precisely the same as those appearing in Eq. (3.23), the starting point at one loop is the same, and successive pole terms are determined uniquely by minimal subtraction, corresponding precisely to the fact that \( B^I_n, \hat{B}^I_n, z^{ij}_n \) and \( \hat{z}^{ij}_n \) in Eq. (3.23) have no dependence on \( \epsilon \). Hence we conclude that the expansion of the bare couplings in terms of renormalised quantities obtained using DRED is exactly that given in Eq. (3.4), and the wave-function renormalisations for DRED are those given in Eqs. (3.6) and (3.7). In other words, DRED is identical to System 3. This completes our proof that DRED is equivalent to DREG up to coupling constant redefinition, since, by virtue of Eq. (3.14), we already know that System 3 is coupling-constant-redefinition equivalent to DREG. Although our discussion has been restricted to gauge theories coupled only to scalar fields, it is straightforward to include fermions. In fact all our equations remain true if the meaning of \( \lambda^I \) and \( \lambda'^E \) is extended to include fermion couplings, and if the “dressing” by wave-function renormalisations is accomplished by fermion wave-function renormalisations where appropriate. The fermion wave-function renormalisations are transformed to System 3 in precisely the same fashion as the scalar ones.
Our final general result is that the \( S \)-matrix for DRED is equal to that for DREG. Let us first discuss processes corresponding to the various interactions that appear in the Lagrangian. The contribution to the \( S \)-matrix element corresponding to a given process consists of the appropriate \( G^I \) divided by a \( \sqrt{G^{ij}} \) for each external leg. For the theory reduced to \( 4 - E \) dimensions, this can obviously be written \( \sum_{M=0} E^M S_M \), where \( S_M \) contain no \( E \)-dependence. Clearly \( S_0 \) represents the corresponding contributions to the \( S \)-matrix for DREG. Each term \( S_M \) is individually finite as \( \epsilon \to 0 \). Hence, when we set \( E = \epsilon \) and let \( \epsilon \to 0 \), we simply obtain the DREG contribution. On the other hand, when we set \( E = \epsilon \) and make the redefinitions in Eq. (3.4) and Eq. (3.5) (and of \( Z_\epsilon \)), we obtain Eq. (3.22) and when we then let \( \epsilon \to 0 \) we obtain the DRED contribution to the \( S \)-matrix (apart from a finite wave-function renormalisation on each external line due to \( Z'' \)). So the contributions to the \( S \)-matrix from diagrams corresponding to \( \lambda^I \) within DRED are precisely equivalent to those within DREG.

So far we have concentrated on Green's functions corresponding to interactions present in the Lagrangian. The contributions from diagrams not corresponding to couplings in the Lagrangian, but with real external particles (i.e. not \( \epsilon \)-scalars) are also guaranteed to be the same in the two schemes. In fact it is easy to see that the formalism deployed in this section demonstrates this also. The process by which the DREG Green's function is transformed into the DRED one goes through in the same way, the only difference being that there is no coupling \( \lambda^I \) corresponding to the Green's function \( G^I \). Therefore the complete \( S \)-matrix is numerically the same in DRED as in DREG.

4. Discussion

By this point the reader might be forgiven for wondering why it is that anyone should wish to use DRED in the non-supersymmetric case. The most compelling motivation, as far as we are aware, arises in problems where use of Fierz identities are required, such as, for example, in the calculation of the anomalous dimensions of four-Fermi operators. Fierz identities which are straightforward in four dimensions become problematic for non-integer \( d \). When DRED is used, it is possible to factorise out the Dirac matrix algebra in a calculation so that it can be performed entirely in four dimensions\(^7\).

On the other hand, however, the importance of evanescent couplings, which we have emphasised, would appear at first sight to tend to nullify this advantage. Most actual
applications of DRED have not addressed the evanescent couplings at all, and have pro-
ceeded by implicitly setting them equal to their “natural” values. It is easy now to see
that this is in fact perfectly valid. In Section 3 we established that the DRED and DREG
S-matrices for the real particles are identical, i.e. that (suppressing all indices)

\[ S(\lambda) = S(\lambda', \lambda'_E) \]  (4.1)

where \( \lambda' = \lambda'(\lambda, \lambda_E) \) and \( \lambda'_E = \lambda'_E(\lambda, \lambda_E) \).

Evidently varying \( \lambda_E \) defines a trajectory in \((\lambda', \lambda'_E)\)-space without changing the S-
matrix. It follows that we are free to choose a point on this trajectory such that the
\( \lambda'_E \) are indeed equal to their natural values, for example \( h = g \), in the class of theories
considered towards the end of the appendix. If this is done, however, it should be clear
from our analysis that it would not be possible to relate predictions made at different
values of the renormalisation scale \( \mu \) by evolving only the \( \beta \)-functions corresponding to the
real interactions.

As its advantages become more widely appreciated, we may therefore expect to see
more widespread adoption of DRED in higher order QCD calculations. In the case of
electroweak processes, the treatment of \( \epsilon^{\mu \nu \rho \sigma} \) presents special difficulties, as we mentioned
in the introduction; these we will return to elsewhere.

Acknowledgements

I.J. and K.L.R. thank the S.E.R.C. for financial support.

Appendix A.

It seems to us worthwhile to show explicitly how \( \lambda' \) and \( \lambda'_E \) are constructed at low
orders in perturbation theory. At one loop, for instance, we have, after setting \( E = \epsilon \) in
Eq. (3.1),

\[ \lambda^I_B = \mu^I \epsilon \left( \lambda^I_A + A^{(1)I}_{11} + \frac{A^{(1)I}_{10}}{\epsilon} \right), \]

(\( \lambda_B \))^I_E = \mu^I \epsilon \left( \lambda^I_E + \tilde{A}^{(1)I}_{11} + \frac{\tilde{A}^{(1)I}_{10}}{\epsilon} \right), \]  (A.1)
where the superscript in brackets denotes the loop order. (It is of course obvious that
\[ A_{1m}^{(1)} = \tilde{A}_{1m}^{(1)} = 0 \text{ for } m > 1. \] Hence it is sufficient to take
\[
\begin{align*}
\lambda' \lambda(\lambda, \lambda, \lambda) &= \lambda + A_{11}^{(1)}(\lambda, \lambda, \lambda), \\
B_1^{(1)}(\lambda, \lambda) &= A_{11}^{(1)}(\lambda, \lambda), \\
\tilde{\lambda}' \lambda(\lambda) &= \tilde{\lambda} + \tilde{A}_{11}^{(1)}(\lambda, \lambda), \\
\tilde{B}_1^{(1)}(\lambda, \lambda) &= \tilde{A}_{11}^{(1)}(\lambda, \lambda)
\end{align*}
\tag{A.2}
\]
to obtain a Laurent expansion as in Eq. (3.4) with the correct properties. Substituting the expressions for \( \lambda' \) and \( \lambda \) in Eq. (A.2) into Eq. (3.11), we readily find that the one-loop change in \( \lambda \) induces a change in the \( \beta \)-functions
\[
\beta'(\lambda + A_{11}^{(1)}(\lambda, \lambda), \lambda, \lambda, \epsilon) = \left( \beta, \frac{\partial}{\partial \lambda} + \beta_E, \frac{\partial}{\partial \lambda_E} \right) \lambda' A_{11}^{(1)}(\lambda, \lambda), \\
\beta'(\tilde{\lambda} + \tilde{A}_{11}^{(1)}(\lambda, \lambda), \lambda, \lambda, \epsilon) = \left( \tilde{\beta}, \frac{\partial}{\partial \tilde{\lambda}} + \tilde{\beta}_E, \frac{\partial}{\partial \tilde{\lambda}_E} \right) \tilde{\lambda}' \tilde{A}_{11}^{(1)}(\lambda, \lambda)
\tag{A.3}
\]
which, using \( \beta^{(1)}(\lambda) = -k \epsilon \lambda + A_{10}^{(1)} + \epsilon A_{11}^{(1)} \) and \( \beta^{(1)}(\lambda) = -l \epsilon \tilde{\lambda} + \tilde{A}_{10}^{(1)} + \epsilon \tilde{A}_{11}^{(1)} \), yields a change in the \( \beta \)-function at the two-loop level given by
\[
\beta'^{(2)}(\lambda, \lambda, \epsilon) - \beta^{(2)}(\lambda, \lambda, \epsilon) = [A_{10}^{(1)}(\lambda, \lambda), \frac{\partial}{\partial \lambda} + \tilde{A}_{10}^{(1)}(\lambda, \lambda), \frac{\partial}{\partial \lambda_E}] A_{11}^{(1)}(\lambda, \lambda) \\
- [A_{11}^{(1)}(\lambda, \lambda), \frac{\partial}{\partial \lambda} + \tilde{A}_{11}^{(1)}(\lambda, \lambda), \frac{\partial}{\partial \lambda_E}] A_{10}^{(1)}(\lambda, \lambda)
\tag{A.4}
\]
This result has the familiar form of a Lie derivative. In the light of our general results, this should correspond to the difference between the DREG and DRED \( \beta \)-functions. The supersymmetric case was considered some time ago. The calculation here was relatively straightforward because in a supersymmetric theory, if one equates evanescent couplings to the corresponding real couplings, the \( \beta \)-functions for the evanescent couplings also become identical to those for the corresponding real couplings. This means that one can consistently equate the evanescent couplings to the real couplings throughout Eq. (A.4), obtaining
\[
\beta'^{(2)}(\lambda, \lambda, \epsilon) - \beta^{(2)}(\lambda, \lambda, \epsilon) = A_{10}^{(1)}(\lambda, \lambda), \frac{\partial}{\partial \lambda} A_{11}^{(1)}(\lambda, \lambda) - A_{11}^{(1)}(\lambda, \lambda), \frac{\partial}{\partial \lambda} A_{10}^{(1)}(\lambda, \lambda)
\tag{A.5}
\]
It was shown in Ref. [8] that this difference indeed corresponds to the difference between the 2-loop DREG and DRED \( \beta \)-functions for a supersymmetric theory.

However, in the non-supersymmetric case, the \( \beta \)-functions for evanescent couplings are different from those for the corresponding real couplings even if one equates the evanescent couplings to the corresponding real ones in the expressions for the \( \beta \)-functions. It becomes vital to distinguish evanescent couplings from real ones throughout, and consequently
Explicit calculations become somewhat involved. In Ref. [5] we explicitly demonstrated that Eq. (A.4) indeed gives the correct difference between the DREG and DRED $\beta$-functions for a toy model.

At the next order we have

$$
\lambda_B^I = \mu k^I \epsilon (\lambda^I + A_{11}^{(1)I} + A_{11}^{(2)I} + A_{22}^{(2)I} + \epsilon A_{12}^{(2)I})
+ \frac{1}{\epsilon} [A_{10}^{(1)I} + A_{10}^{(2)I} + A_{21}^{(2)I}] + \frac{A_{20}^{(2)I}}{\epsilon^2},
$$

$$(\lambda_B^I)_E = \mu k^I \epsilon (\lambda_E^I + A_{11}^{(1)I} + A_{11}^{(2)I} + A_{22}^{(2)I} + \epsilon A_{12}^{(2)I})
+ \frac{1}{\epsilon} [A_{10}^{(1)I} + A_{10}^{(2)I} + A_{21}^{(2)I}] + \frac{A_{20}^{(2)I}}{\epsilon^2}).$$

We must now take

$$
\lambda^I = \lambda^I + A_{11}^{(1)I} + A_{11}^{(2)I} + \epsilon A_{12}^{(2)I},
$$

and

$$
[B_1^{(1)I}(\lambda', \lambda'_E) + B_1^{(2)I}(\lambda', \lambda'_E)]_{2-loop} = A_{10}^{(2)I}(\lambda, \lambda_E) + A_{21}^{(2)I}(\lambda, \lambda_E),
$$

$$
[B_2^{(2)I}(\lambda', \lambda'_E)]_{2-loop} = A_{20}^{(2)I}(\lambda, \lambda_E),
$$

which leads to

$$
B_1^{(2)I} = A_{10}^{(2)I} + A_{21}^{(2)I} - (A_{11}^{(1)I} \frac{\partial}{\partial \lambda} + \tilde{A}_{11}^{(1)I} \frac{\partial}{\partial \lambda_E}) A_{10}^{(1)I},
B_2^{(2)I} = A_{20}^{(2)I};
$$

there are also entirely analogous results for the evanescent couplings. Using the consistency condition that relates the double and simple poles in $\epsilon$, we can simplify $B_1^{(2)I}$ as follows:

$$
B_1^{(2)I} = A_{10}^{(2)I} + \frac{1}{2} [A_{10}^{(1)I} \frac{\partial}{\partial \lambda} + \tilde{A}_{10}^{(1)I} \frac{\partial}{\partial \lambda_E}] A_{11}^{(1)I}
- \frac{1}{2} [A_{11}^{(1)I} \frac{\partial}{\partial \lambda} + \tilde{A}_{11}^{(1)I} \frac{\partial}{\partial \lambda_E}] A_{10}^{(1)I}.
$$

This result is consistent with Eq. (A.4). We shall not give the full results for the next order; however it is important to note that terms in $\lambda'$ proportional to $\epsilon$, as in Eq. (A.7), play an important role at this and higher orders. At third order, we write

$$
\lambda_B^I = \mu k^I \epsilon (\lambda^I + \frac{B_1^I(\lambda', \lambda'_E)}{\epsilon} + \frac{B_2^I(\lambda', \lambda'_E)}{\epsilon^2} + \frac{B_3^I(\lambda', \lambda'_E)}{\epsilon^3}).
$$

In particular, with $\lambda'$ as given by Eq. (A.7), we find $[\frac{B_1^I(\lambda', \lambda'_E)}{\epsilon}]_{3-loop}$ contains a finite term $A_{12}^{(2)I} \frac{\partial}{\partial \lambda} A_{10}^{(1)I}$, which must be cancelled by a similar term in $\lambda'$ at third order.
It is similarly quite easy to derive the forms of the wave-function renormalisations in Scheme 3 and to compare them with the results obtained using DRED in particular instances. For example, at one-loop order we have, after setting \( E = \epsilon \) in Eq. (3.2),

\[
\sqrt{Z_{\phi}}^{ij} = \delta^{ij} + z_{11}^{(1)ij} + \frac{\epsilon z_{10}^{(1)ij}}{\epsilon}.
\]

We find that Eqs. (3.5) and (3.6) are satisfied up to one loop by taking

\[
\sqrt{Z''_{\phi}}^{ij} = \delta^{ij} + z_{11}^{(1)ij}, \quad z_1' = z_{10}^{(1)ij}.
\]

At two loops we have

\[
\sqrt{Z_{\phi}}^{ij} = \delta^{ij} + z_{11}^{(1)ij} + z_{22}^{(2)ij} + \epsilon z_{12}^{(2)ij} + \frac{1}{\epsilon} \left( z_{10}^{(1)ij} + z_{10}^{(2)ij} + z_{21}^{(2)ij} \right) + \frac{z_{20}^{(2)ij}}{\epsilon^2}.
\]

We find, using Eq. (A.2), that Eqs. (3.5) and (3.6) are satisfied by taking

\[
\sqrt{Z''_{\phi}}^{(2)ij} = z_{11}^{(2)ij} + z_{22}^{(2)ij} + \epsilon z_{12}^{(2)ij} - (A_{11}^{(1)} \frac{\partial}{\partial \lambda} + \tilde{A}_{11}^{(1)} \frac{\partial}{\partial \lambda E}) z_{11}^{(1)ij},
\]

\[
z_1'' = z_{10}^{(2)ij} + z_{22}^{(2)ij} - (A_{11}^{(1)} \frac{\partial}{\partial \lambda} + \tilde{A}_{11}^{(1)} \frac{\partial}{\partial \lambda E}) z_{10}^{(1)ij} - z_{10}^{(1)ik} z_{11}^{(1)kj},
\]

\[
z_2'' = z_{20}^{(2)ij}.
\]

According to our general results, the primed quantities should yield the wave-function renormalisation matrix for real scalar fields in DRED. The most interesting quantity is \( z_1'' \), which is the only one which changes to this order. We may simplify the result for \( z_1'' \) by using the identity

\[
z_{21}^{(2)ij} = \frac{1}{2} \left[ 2z_{10}^{(1)ik} z_{11}^{(1)kj} + (A_{11}^{(1)} \frac{\partial}{\partial \lambda} + \tilde{A}_{11}^{(1)} \frac{\partial}{\partial \lambda E}) z_{10}^{(1)ij} \right] + (A_{10}^{(1)} \frac{\partial}{\partial \lambda} + \tilde{A}_{10}^{(1)} \frac{\partial}{\partial \lambda E}) z_{11}^{(1)ij},
\]

which follows from standard renormalisation group consistency conditions which determine higher-order poles in terms of simple poles. We then obtain from Eqs. (A.15) and (A.16)

\[
z_1'' = z_{10}^{(2)ij} + \frac{1}{2} \left[ (A_{10}^{(1)} \frac{\partial}{\partial \lambda} + \tilde{A}_{10}^{(1)} \frac{\partial}{\partial \lambda E}) z_{11}^{(1)ij} - (A_{11}^{(1)} \frac{\partial}{\partial \lambda} + \tilde{A}_{11}^{(1)} \frac{\partial}{\partial \lambda E}) z_{10}^{(1)ij} \right].
\]
$z_1^{(2)ij}$ in Eq. (A.17) should agree with corresponding quantities calculated in DRED. (Again, there are similar results for the wave-function renormalisation matrix for the $\epsilon$-scalars.)

We can easily check Eq. (A.17) for the case of a gauge theory with fermions but no (real) scalars, as discussed in Section (2) of Ref. [5], the notation of which we adopt below.

The relevant one-loop renormalisation constants are as follows:

\begin{align}
Z^{WW} &= Z_\alpha = 1 + \frac{1}{16\pi^2\epsilon} g^2 \left[ (\frac{13}{3} - \alpha - \frac{1}{3} E) C_2(G) - \frac{8}{3} T(R) \right] \\
Z^{\overline{\psi}\psi} &= 1 - \frac{1}{16\pi^2\epsilon} \left[ 2\alpha g^2 + Eh^2 \right] C_2(R) \\
Z_g &= 1 + \frac{1}{16\pi^2\epsilon} g^2 \left[ \frac{4}{3} T(R) + \left( \frac{1}{6} E - \frac{11}{3} \right) C_2(G) \right] \\
Z_h &= 1 + \frac{1}{16\pi^2\epsilon} \left[ (4h^2 - 6g^2) C_2(R) + 2h^2 T(R) - 2h^2 C_2(G) \right].
\end{align}

(A.18)

Here $Z^{WW}, Z^{\overline{\psi}\psi}, Z_\alpha, Z_g$ and $Z_h$ are the renormalisation constants associated with the gauge field, the fermion multiplet, the gauge parameter, the gauge coupling and the evanescent Yukawa coupling respectively.

From Eq. (A.18) it is easy to show that $\delta Z^{WW} = Z^{(2)WW} - Z^{(2)WW}$ is given by

\begin{align}
\delta Z^{WW} &= - \frac{1}{2} \frac{g^4}{(16\pi^2)^2 \epsilon} \left[ \frac{1}{6} C_2(G).2 \left( \frac{13}{3} - \alpha \right) C_2(G) - \frac{8}{3} T(R) \right] \\
&\quad - \frac{\alpha}{3} C_2(G).(-C_2(G)) - \left[ \frac{4}{3} T(R) - \frac{11}{3} C_2(G) \right] \left( -\frac{2}{3} C_2(G) \right) \\
&= \frac{g^4}{(16\pi^2)^2 \epsilon} \frac{1}{2} C_2(G)^2.
\end{align}

(A.19)

This result agrees with the original DRED calculation in Ref. [4] (see Eq. (3.12) of that reference).

By a similar calculation it is straightforward to show that

\begin{align}
\delta Z^{\overline{\psi}\psi} &= Z^{(2)\overline{\psi}\psi} - Z^{(2)\overline{\psi}\psi} \\
&= \frac{1}{(16\pi^2)^2 \epsilon} \left[ 6g^2 h^2 C_2(R)^2 - h^4 (4C_2(R)^2 - 2C_2(R)C_2(G) + 2C_2(R)T(R)) \right].
\end{align}

(A.20)

This agrees with an explicit calculation of $Z^{(2)\overline{\psi}\psi}$ in the two schemes [4].
References

[1] W. Siegel, Phys. Lett. B84 (1979) 193.
[2] H. Nicolai and P. K. Townsend, Phys. Lett. B93 (1980) 111;
   W. Siegel, Phys. Lett. B94 (1980) 37;
   D. R. T. Jones and J. P. Leveille, Nucl. Phys. B206 (1982) 473;
   V. Elias, G. McKeon and R. B. Mann, Nucl. Phys. B229 (1983) 487;
   F. B. Little, R. B. Mann, V. Elias and G. McKeon, Phys. Rev. D32 (1985) 2707;
   P. Ensign and K. T. Mahanthappa, Phys. Lett. B194 (1987) 523;
   J. G. Körner, D. Kreimer and K. Schilcher, Z. Phys. C54 (1992) 503.
[3] R. van Damme and G. ’t Hooft, Phys. Lett. B150 (1985) 133.
[4] D. M. Capper, D. R. T. Jones and P. van Nieuwenhuizen, Nucl. Phys. B167 (1980) 479.
[5] I. Jack, D. R. T. Jones and K. L. Roberts, “Dimensional Reduction in Non-
supersymmetric Theories”, Liverpool preprint LTH 320 (Z. Phys. C, to be published).
[6] I. Jack and H. Osborn, Nucl. Phys. B249 (1985) 472.
[7] J. G. Körner, G. A. Schüler and S. Sakakibara, Phys. Lett. B194 (1987) 125;
   J. G. Körner and P. Sieben, Nucl. Phys. B363 (1991) 65;
   P. J. O’Donnell and H. K. K. Tung, Phys. Rev. D45 (1992) 4342;
   P. J. O’Donnell and H. K. K. Tung, Toronto preprint UTPT-91-32;
   J. G. Körner and M. M. Tung, Mainz preprint MZ-TH/92-41;
   M. Misiak, preprint TUM-T31-46/93.
[8] I. Jack, Phys. Lett. B147 (1984) 405.
[9] D. R. T. Jones (unpublished) 1980.