Matryoshka Skyrmions

Muneto Nitta\textsuperscript{1}

\textsuperscript{1} Department of Physics, and Research and Education Center for Natural Sciences, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan

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Abstract

We construct a stable Skyrmion in 3+1 dimensions as a sine-Gordon kink inside a domain wall within a domain wall in an $O(4)$ sigma model with hierarchical mass terms without the Skyrme term. We also find that higher dimensional Skyrmions can stably exist with a help of non-Abelian domain walls in an $O(N)$ model with hierarchical mass terms without a Skyrme term, which leads to a matryoshka structure of Skyrmions.

a picture from Wikipedia
I. INTRODUCTION

A half century has been passing after Skyrme proposed that Skyrmions characterized by the topological charge $\pi_3(S^3) \simeq \mathbb{Z}$ describe nucleons in the pion effective field theory or the chiral Lagrangian [1], where the Skyrme term quartic with respect to derivatives is needed to stabilize Skyrmions against shrinkage. After Skyrmions were proposed, lower dimensional analogs of Skyrmions have also been studied in both field theory and condensed matter physics. In two spatial dimensions, a lump solution characterized by $\pi_2(S^2) \simeq \mathbb{Z}$ in an $O(3)$ sigma model [2] can be regarded as a two spatial dimensional (2D) Skyrmion, whose size is arbitrary. The size of lumps is fixed in the presence of a quartic derivative term analogous to the Skyrme term and a potential term. Such 2D Skyrmions were proposed as a toy model of Skyrmions [3] and are called baby Skyrmions. A sine-Gordon kink characterized by $\pi_1(S^1) \simeq \mathbb{Z}$ is considered as the lowest dimensional Skyrmion, as was first studied by Skyrme [4].

In condensed matter physics, sine-Gordon kinks (1D Skyrmions) exist for instance in Josephson junctions of superconductors [5]. 2D Skyrmions appear in various condensed matter systems such as magnetism [6], quantum Hall liquids [7], superfluid helium 3 [8], and spinor or multi-component Bose-Einstein condensates [9]. More recently, there have been considerable efforts to realize 3D Skyrmions stably in two-component Bose-Einstein condensates [10], which are elusive because of the lack of the Skyrme term. Finally, it has been theoretically proposed that a stable 3D Skyrmion can exist as a ground state by introducing an “artificial” non-Abelian gauge fields [11].

As described above, Skyrmions exist in diverse dimensions [12] in both field theory and condensed matter physics: an $O(N + 1)$ nonlinear sigma model admits $N$ dimensional Skyrmions characterized by $\pi_N(S^N) \simeq \mathbb{Z}$, at least for $N = 1, 2, 3$. However, it has been unclear thus far whether higher dimensional Skyrmions with higher $N$ can exist stably. One of the purposes of this paper is to construct stable higher dimensional Skyrmions without a Skyrme term as composite states.

Relations between Skyrmions in diverse dimensions have been also studied. A lump (2D Skyrmion) becomes a sine-Gordon kink (1D Skyrmion) [13–15], once it is absorbed into a $CP^1$ domain wall [16]. This explains [15] a construction of a sine-Gordon kink from a holonomy of a $CP^1$ lump [17]. A 3D Skyrmion becomes a 2D Skyrmion [18, 19] once it is
absorbed into a non-Abelian domain wall with $S^2$ moduli [20, 21]. The other purpose of this paper is to extend these relations between Skyrmions in diverse dimensions to arbitrary dimensions.

In this paper, we propose relations between Skyrmions in diverse dimensions through domain walls; when a $D$ dimensional Skyrmion is absorbed into a non-Abelian domain wall, it becomes a $D - 1$ dimensional Skyrmion. By using this relation, we construct higher dimensional Skyrmions with a help of domain walls. These Skyrmions stably exist without the Skyrme term. More precisely, we consider an $O(N + 1)$ nonlinear sigma model with the target space $S^N$ and a potential term with hierarchical masses. With the largest masses, there exists a non-Abelian domain wall having normalizable moduli $\mathbb{R} \times S^{N-1}$, which connects two discrete vacua. The domain wall effective action is thus an $O(N)$ model with the target space $S^{N-1}$ with a potential term with hierarchical masses. This effective Lagrangian admits again a non-Abelian domain wall with internal moduli $\mathbb{R} \times S^{N-2}$. With repeating this procedure, we finally construct an $N$ dimensional Skyrmion as a sine-Gordon kink inside $N - 1$ non-Abelian domain walls. In Sec. II, we consider an $O(4)$ model in $d = 3 + 1$ dimensions and construct a 3D Skyrmion as a sine-Gordon kink inside a domain wall within a domain wall. In Sec. III, we generalize this to higher dimensions and find a matryoshka structure of Skyrmions. Section IV is devoted to a brief summary and discussion.

II. $O(4)$ MODEL AND 3D SKYRMIONS

A. A domain wall inside a domain wall

Let $\mathbf{n}^{(0)} = (n_1^{(0)}(x), n_2^{(0)}(x), n_3^{(0)}(x), n_4^{(0)}(x))$ be an $O(4)$ vector of scalar fields satisfying the constraint $(\mathbf{n}^{(0)})^2 = 1$. By using these fields, the Lagrangian of an $O(4)$ nonlinear sigma model in $d = 3 + 1$ dimensions can be given as ($\mu = 0, 1, 2, 3$)

$$\mathcal{L}^{(0)} = \frac{1}{2} (\partial_\mu \mathbf{n}^{(0)})^2 - V(\mathbf{n}^{(0)}).$$  (1)

Without the potential term, the Lagrangian is invariant under the $O(4)$ symmetry, which is spontaneously broken down to an $O(3)$ subgroup. Then, the target space is $O(4)/O(3) \simeq S^3 \simeq SU(2)$ [Fig. I(a)], and this model is equivalent to the $SU(2)$ principle chiral model (or the chiral Lagrangian at the leading order). We do not consider the Skyrme term or other
higher derivative terms. Here, we consider the following potential term
\[ V(n^{(0)}) = m_4^2 \left( 1 - (n_4^{(0)})^2 \right) + m_3^2 \left( 1 - (n_3^{(0)})^2 \right), \quad m_3 \ll m_4, \] (2)
which hierarchically breaks the \( O(4) \) symmetry explicitly to an \( O(3) \) subgroup by \( m_4 \) and further to an \( O(2) \) subgroup by \( m_3 \):
\[ O(4) \xrightarrow{m_4} O(3) \xrightarrow{m_3} O(2). \] (3)

FIG. 1: (a) The \( S^3 \) target space with a potential admitting two discrete vacua. (b) A wall within a wall. The widths of the outer and inner domain walls are \( \Delta x^3 = m_4^{-1} \) and \( \Delta x^2 = m_3^{-1} \), respectively.

When \( m_3 \) is negligible, the symmetry of vacua \( n_4^{(0)} = \pm 1 \) is \( O(3) \), and there exists a non-Abelian domain wall which interpolates these vacua by the field \( n_4^{(0)} \).\cite{19,21}. We place the domain wall perpendicular to the \( x^3 \)-axis. There exists a continuous family of the domain wall solutions, given by \cite{19}\[ \theta^{(0)}(x^3) = \arctan \exp(\pm \sqrt{2}m_4(x^3 - X^3)), \quad 0 \leq \theta^{(0)} \leq \pi, \] (4)
\[ n_i^{(0)} = n_i^{(1)} \sin \theta^{(0)}(x^3), \quad (n_i^{(1)})^2 = \sum_{i=1}^{3} (n_i^{(1)})^2 = 1, \] (5)
\[ n_4^{(0)} = \cos \theta^{(0)}(x^3). \] (6)
The width of the domain wall is \( \Delta x^3 = m_4^{-1} \). This domain wall solution has moduli \( \mathbf{R} \times S^2 \) parametrized by the position \( X^3 \) in the \( x^3 \)-coordinate and the internal orientation \( \mathbf{n}^{(1)} \),
respectively. This wall is called non-Abelian since it carries non-Abelian moduli $S^2$. The $O(3)$ symmetry of the vacua is spontaneously broken down to its subgroup $O(2)$ in the presence of the domain wall. Consequently, there appear Nambu-Goldstone modes $S^2 \simeq O(3)/O(2)$ which are localized in the vicinity of the wall.

Let us construct the low-energy effective theory on the domain wall using the moduli approximation:

$$\mathcal{L}^{(1)} = \frac{\sqrt{2}}{2m_4} (\partial_\mu n^{(1)})^2 + \frac{T_{\text{wall}}}{2} (\partial_\mu X^3)^2 - T_{\text{wall}}, \quad (n^{(1)})^2 = 1, \quad (7)$$

with the domain wall tension $T_{\text{wall}} = 2\sqrt{2}m_4$. Here $\mu$ runs 0, 1, 2. Hereafter, we use the same letter $\mu$ for labeling indices of various dimensions unless a confusion exists. We have seen that the above $S^2$ zero modes are in fact normalizable modes.

By turning on the small mass $m_3 (\ll m_4)$ perturbatively, we can calculate the effect on the domain wall theory as the following effective potential:

$$V^{(1)} = -\frac{\sqrt{2}}{m_4} m_3^2 (n^{(1)}_3)^2 + \text{const}. \quad (8)$$

With ignoring the position modulus $X^3$, the effective Lagrangian is summarized as

$$\mathcal{L}^{(1)} = \frac{\sqrt{2}}{m_4} \left[ \frac{1}{2} (\partial_\mu n^{(1)})^2 - m_3^2 \left( 1 - (n^{(1)}_3)^2 \right) \right] + \text{const}. \quad (9)$$

We thus have obtained an $O(3)$ sigma model with the potential term admitting two discrete vacua, given by $n^{(1)}_3 = \pm 1$. This model is equivalent to the massive $\mathbb{C}P^1$ model.

There exists a domain wall solution interpolating between the two discrete vacua $n^{(1)}_3 = \pm 1$ in the wall effective theory obtained in Eq. (9). The domain wall solution perpendicular to the $x^2$-coordinate is obtained as

$$\theta^{(1)}(x^2) = \arctan \exp(\pm \sqrt{2}m_3(x^2 - X^2)), \quad 0 \leq \theta^{(1)} \leq \pi, \quad (10)$$

$$n^{(1)}_i = n^{(2)}_i \sin \theta^{(1)}(x^2), \quad (i = 1, 2), \quad (n^{(2)})^2 = \sum_{i=1}^2 (n^{(2)}_i)^2 = 1, \quad (11)$$

$$n^{(1)}_3 = \cos \theta^{(1)}(x^2). \quad (12)$$

The width of the domain wall is $\Delta x^2 = m_3^{-1}$. Here, the index $i$ runs 1, 2; we also use the same letter $i$ for labeling different dimensional vectors unless a confusion exists. This domain wall solution has moduli $\mathbf{R} \times S^1$ parametrized by the position $X^2$ in the $x^2$-coordinate and the internal orientation $n^{(2)}$, respectively. Then, the total configuration is a domain wall
inside a domain wall as schematically shown in Fig. 1(b). This configuration was studied before in Ref. [21]. The second domain wall does not have topological meaning in the bulk outside the first domain wall. A wall within a wall has been also studied in different models [14, 15, 25].

Before going to the next subsection, let us make a comment. If one makes a loop of the second domain wall (inside the first domain wall), it is of course unstable to decay. One can twist the $U(1)$ modulus of the domain wall when making a domain wall loop. This twisted domain wall loop carries a lump charge in $d = 2 + 1$, which implies 3D Skyrme charge as discussed in [19]. Such the twisted domain wall loop may be still unstable against shrinking. In order to make it stable, one can give a linear time dependence on the $U(1)$ modulus of the domain wall. Then, such a time-dependent, twisted domain wall loop is nothing but a Q-lump in $d = 2 + 1$ dimensions [26] inside a domain wall, which was studied in [19]. This corresponds to a spinning 3D Skyrmion in the bulk.

**B. 3D Skyrmion from domain walls**

Here, we construct a 3D Skyrmion in $d = 3 + 1$ dimensions as a composite state of domain walls. To this end, let us consider the following potential term instead of the potential given in Eq. (2):

$$V(n) = m_4^2 \left( 1 - (n_4^{(0)})^2 \right) + m_3^2 \left( 1 - (n_3^{(0)})^2 \right) + m_2^2 \left( 1 - n_2^{(0)} \right), \quad m_2 \ll m_3 \ll m_4, \quad (13)$$

with a hierarchical breaking

$$O(4) \rightarrow O(3) \rightarrow O(2) \rightarrow \{0\}. \quad (14)$$

The difference with the previous potential in Eq. (2) is the last term with $m_2$. Note that we take this term linear with respect to the field $n_2^{(0)}$ unlike the other two terms, which are quadratic with respect to the fields. The reason to consider this term is clarified below.

First, we place a domain wall perpendicular to the $x^3$-axis, which interpolates the vacua $n_4^{(0)} = \pm 1$ as before. Except for the position modulus, a potential term due to $m_2$ is induced on the domain wall world-volume effective theory given in Eq. (9):

$$\mathcal{L}^{(1)} = \frac{\sqrt{2}}{m_4} \left[ \frac{1}{2} (\partial_\mu n^{(1)})^2 + (1 - m_3^2 (n_3^{(1)})^2) \right] - \frac{\sqrt{2\pi}}{2m_4} \left( 1 - m_2^2 n_2^{(1)} \right) + \text{const.}, \quad (15)$$
with $\mu = 0, 1, 2$.

Second, we consider the second domain wall perpendicular to the $x^2$-axis inside the first domain wall, interpolating between the vacua $n_3^{(1)} = \pm 1$ in the regime of $m_2 \ll m_3$. We can repeat the same procedure to obtain the effective theory on the wall inside the wall:

$$L^{(2)} = \sqrt{2} \sqrt{2} \frac{1}{m_3 m_4} (\partial_\mu n^{(2)})^2 - \sqrt{2} \sqrt{2} \frac{\pi}{2m_3} \left( 1 - m_2^2 n^{(2)}_2 \right) + \text{const.},$$

with $\mu = 0, 1$ and $n^{(2)} = (n_1^{(2)}, n_2^{(2)})$ with the constraint $(n_1^{(2)})^2 + (n_2^{(2)})^2 = 1$. This is precisely the sine-Gordon model:

$$L^{(2)} = \frac{2}{m_3 m_4} \left[ \frac{1}{2} (\partial_\mu \theta^{(2)})^2 + \hat{m}_2^2 \sin^2 \theta^{(2)} \right] + \text{const.},$$

with $(n_1^{(2)}, n_2^{(2)}) = (\sin \theta^{(2)}, \cos \theta^{(2)})$ ($0 \leq \theta^{(2)} < 2\pi$) and $\hat{m}_2 \equiv \frac{\pi}{2} m_2$.

Finally, we construct a sine-Gordon kink in the $x^1$-coordinate as the third domain wall:

$$\theta^{(2)}(x^1) = \arctan \exp(\pm \sqrt{2} \hat{m}_2 (x^1 - X^1)), \quad (18)$$

with the position modulus $X^1$ in the $x^1$-coordinate. The total configuration is the sine-Gordon kink inside the domain wall within the domain wall, as drawn in Fig. 2.

![Fig. 2](image-url)
it is identified with a lump in \( d = 2 + 1 \), one can conclude from the result in [19] that the sine-Gordon kink precisely corresponds to the 3D Skyrmion with the unit charge of the 3D Skyrmion charge \( \pi_3(S^3) \simeq \mathbb{Z} \) in \( d = 3 + 1 \) dimensional bulk. We thus have constructed a 3D Skyrmion as a sine-Gordon kink inside a \( CP^1 \) domain wall with a modulus \( U(1) \) inside a non-Abelian domain wall with moduli \( S^2 \).

If we consider the quadratic mass term \( m_2^2 \left( 1 - (n_2^{(0)})^2 \right) \) instead of the linear term in Eq. (13), we have a Skyrmion with a half charge in \( \pi_3(S^3) \) as the minimum soliton instead of a Skyrmion with the unit charge. The unit charge Skyrmion is split into two half charge Skyrmions. Therefore, we have considered the linear mass term for \( n_2^{(0)} \) although the quadratic mass term for the rest fields.

Note that we did not need a Skyrme term. Therefore this type of a 3D Skyrmion can exist in principle in condensed matter physics.

If we introduce the Skyrme term on the other hand, a 3D Skyrmion can stably exist in the bulk. However it will be absorbed into the first domain wall, subsequently it will be absorbed into the second domain wall, and becomes a sine-Gordon kink as discussed in this section.

### III. HIGHER DIMENSIONAL SKYRMIONS

We consider an \( O(N+1) \) model in \( N \) dimensional space \( (d = N + 1 \) dimensional space-time) with the target space \( S^N \simeq O(N+1)/O(N) \). Here, we show that an \( N \)-dimensional Skyrmion can be constructed as a composite state of one sine-Gordon kink and \( N-1 \) non-Abelian domain walls.

In terms of an \( O(N+1) \) vector of scalar fields \( n^{(0)} = (n_1^{(0)}(x), \cdots, n_{N+1}^{(0)}(x)) \) with the constraint \( (n^{(0)})^2 = 1 \), we consider an \( O(N+1) \) model with Lagrangian \( (\mu = 0, 1, \cdots, N) \)

\[
\mathcal{L}^{(0)} = \frac{1}{2} (\partial_\mu n^{(0)})^2 - V(n^{(0)}),
\]

\[
V^{(0)}(n^{(0)}) = \sum_{i=3}^{N+1} m_i^2 \left( 1 - (n_i^{(0)})^2 \right) + m_2^2 \left( 1 - n_2^{(0)} \right),
\]

in \( d = N + 1 \) space-time dimensions, with the hierarchical masses \( m_2 \ll m_3 \ll \cdots \ll m_{N+1} \) and the hierarchical breaking

\[
O(N+1) \xrightarrow{m_{N+1}} O(N) \xrightarrow{m_N} \cdots \xrightarrow{m_{N-k+1}} O(N-k) \xrightarrow{m_{N-k}} \cdots \xrightarrow{m_3} O(2) \xrightarrow{m_2} \{0\}. \tag{20}
\]
The target space is $O(N+1)/O(N) \simeq S^N$, see Fig. 3.

FIG. 3: The $S^N$ target space of an $O(N+1)$ model with a potential admitting two discrete vacua.

First, we consider only the largest mass $m_{N+1}$ ignoring the rests. A domain wall solution perpendicular to the $x^N$-coordinate can be obtained as before:

\begin{align}
\theta^{(0)}(x^N) &= \arctan(\pm \sqrt{2} m_{N+1} (x^N - X^N)), \quad 0 \leq \theta^{(0)} \leq \pi, \\
n_i^{(0)} &= n_i^{(1)} \sin \theta^{(0)}(x^N), \quad (i = 1, \ldots, N), \quad (n^{(1)})^2 = \sum_{i=1}^N (n_i^{(1)})^2 = 1, \\
n_4^{(0)} &= \cos \theta^{(0)}(x^N).
\end{align}

This domain wall has the moduli space $\mathbb{R} \times S^{N-1}$ parametrized by the position modulus $X^N$ and the internal orientational moduli $n^{(1)}$, respectively. We obtain an $O(N)$ model with the target space $S^{N-1}$ as the effective theory on the domain wall.

We then turn on the second largest mass $m_N$ perturbatively, which induces a potential term on the wall effective action. We obtain the second domain wall as before. We repeat the same procedures. The effective Lagrangian on the $k$-th domain wall is an $O(N-k+1)$ model in $(N-k)$ space dimensions [$d = N-k+1$ space-time dimensions] with the target space $S^{N-k} \simeq O(N-k+1)/O(N-k)$, described by fields $n^{(k)}(x) = (n_i^{(k)}(x), \ldots, n_{N-k+1}^{(k)}(x))$ with the constraint $(n^{(k)})^2 = 1$:

\begin{align}
\mathcal{L}^{(k)} &= \frac{(\sqrt{2})^k}{\prod_{a=N-k+1}^{N+1} m_a} \left[ \frac{1}{2} (\partial_\mu n^{(k)})^2 - \sum_{i=3}^{N-k} m_i^2 \left( 1 - (n_i^{(k)})^2 \right) \right] \\
&\quad - \frac{(\sqrt{2\pi/2})^k}{\prod_{a=N-k+1}^{N+1} m_a} (1 - m_2^2 n_2^{(k)}) + \text{const.}
\end{align}
The \((k+1)\)-th domain wall solution perpendicular to the \(x^{N-k+1}\)-coordinate can be obtained as

\[
\theta^{(k)}(x^{N-k+1}) = \arctan\exp(\pm\sqrt{2}m_{N-k+2}(x^{N-k+1} - X^{N-k+1})), \quad 0 \leq \theta^{(k)} \leq \pi, \tag{25}
\]

\[
n_i^{(k)} = n_i^{(k+1)} \sin \theta^{(k)}(x^{N-k+1}), (i = 1, \cdots, N - k + 1),
\]

\[
(n^{(k+1)})^2 = \sum_{i=1}^{N-k} (n_i^{(k+1)})^2 = 1, \tag{26}
\]

\[
n_4^{(k)} = \cos \theta^{(k)}(x^{N-k+1}). \tag{27}
\]

The transverse size of the \((k+1)\)-th domain wall is \(\Delta x^{N-k+1} = m_{N-k+2}^{-1}\). The \((k+1)\)-th domain wall has moduli \(\mathbb{R} \times S^{N-k-1} \simeq \mathbb{R} \times O(N-k)/O(N-k-1)\).

After repeating the same procedures \(N - 1\) times in total, we construct a sine-Gordon kink given in Eq. (18) with \(\hat{m}_2 \equiv (\pi/4)^{N-1}m_2\) by using the lightest mass \(m_2\). We thus have realized an \(N\)-dimensional Skyrmion characterized by the topological charge \(\pi_N(S^N) \simeq \mathbb{Z}\) as a sine-Gordon kink (inside a non-Abelian domain wall)\(^{N-1}\). The moduli space of this Skyrmion is \(\mathbb{R}^N \times S^{N-1}\). We did not need the Skyrme term for stability while it is unstable against shrinkage outside the walls.

**IV. SUMMARY AND DISCUSSION**

We have constructed an \(N\) dimensional Skyrmion characterized by the topological charge \(\pi_N(S^N) \simeq \mathbb{Z}\) as a composite state of one sine-Gordon kink and \(N - 1\) non-Abelian domain walls in the \(O(N + 1)\) nonlinear sigma model with the \(N\) hierarchical masses in \(d = N + 1\) space-time dimensions, where the \(k\)-th domain wall is placed orthogonal to the \(x^{N-k+1}\)-coordinate and has the width \(\Delta x^{N-k+1} = m_{N-k+2}^{-1}\) and the moduli \(\mathbb{R} \times S^{N-k-1}\). We have found a matryoshka structure of Skyrmions in diverse dimension, as summarized in Table I.

In addition to the sequence of \(N\) dimensional Skyrmions in \(O(N + 1)\) models found in this paper, there exist two exceptional sequences from solitons in gauge theories known thus far, that is, a Yang-Mills instanton to a 3D Skyrmion \[27\] and a vortex to a sine-Gordon kink \[15\]. In the former, Yang-Mills instanton becomes a 3D Skyrmion \[27\] once it is placed inside a non-Abelian domain wall with \(U(2)\) moduli \[28\], which explains a construction of 3D Skyrmion from instanton holonomy \[29\]. In the latter, a vortex becomes a sine-Gordon kink \[15\] once it is absorbed into a \(\mathbb{C}P^1\) domain wall with a \(U(1)\) modulus \[16, 24\], which
| model \ dim | 1+1  | 2+1  | 3+1  | 4+1  |
|------------|------|------|------|------|
| $O(N)$ model | 1D Skyrmion (SG kink) | 2D Skyrmion (lump) | 3D Skyrmion | 4D Skyrmion |
| Gauge theory | Vortex | | YM instanton | |

TABLE I: Hierarchical or matryoshka structure of Skyrmions in diverse dimensions.

An $N+1$ dimensional Skyrmion becomes an $N$ dimensional Skyrmion inside a non-Abelian domain wall, as indicated by arrows. In addition to the sequence of Skyrmions in the $O(N)$ models in different dimensions, two exceptional sequences from solitons in gauge theories are known, indicated by dotted arrows: a Yang-Mills instanton to a 3D Skyrmion \[27\] and a vortex to a sine-Gordon kink \[15\].

explains a construction of a sine-Gordon kink from a vortex holonomy \[17\]. In $d = 2 + 1$, lumps can be obtained in the strong gauge coupling limit of semi-local vortices. In $d = 4 + 1$, we do not know whether there exists a certain relation between Yang-Mills instantons and 4D Skyrmions.

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