An Evolution-Based Uncertainty Analysis Method for Bending Fatigue Failure of Involute Spur Gears

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Abstract. An evolution-based uncertainty analysis model of bending fatigue failure of involute spur gears is presented. The fluctuation of transmit torque and rotational velocity of gears are introduced to the mild wear model of gear flanks. Then a novel simulation model of tooth profile evolution is developed to describe the change of meshing parameters, which include the tooth load, load angle and bending moment arm. The critical tooth-root bending stress is influenced by these time-varying parameters. Based on the evolution-based uncertainty analysis method, the load is approximately obeying logarithmic normal distribution, the load angle and bending moment arm are approximately obey geometric Brownian motion. Then the critical tooth-root bending stress is calculated as a logarithmic normal distribution quantity over time. Finally, the prediction model of time dependent reliability of the bending load capacity of spur gears is proposed.

1. Introduction

Gear transmissions are widely used in the mechanical engineering, and they efficiently transmit power. While the bending fatigue break of gear tooth is one of the most important failure modes. The degeneration of bending fatigue strength of gears is due to the initiation and extension of fatigue cracks on the root. In addition, the mild wear of gear flanks of interacting gear teeth causes unfavourable changes of the surface topography, giving non-uniform gear rate, increasing dynamic effects and perhaps more severe forms of tooth failures [1].

In previous research, Pedrero et al. presented a model of load distribution along the line of contact for involute external gears [2]. This model was used to evaluate the fatigue tooth-root stress, combined with the equations of the linear elasticity [3]. With the applied of finite element calculations, another method for the calculation of the root stresses of spur gears was presented [4].

In this paper, the reliability of involute spur gears after several working cycles, which considered the time-varying bending stress and fatigue strength, was predicted based on the evolution-based uncertainty analysis method. The assumption of random distribution was introduced to describe the dispersity of parameters which influence the critical stress for bending break of tooth root. The mild wear of gear flanks was considered to analyse the change of critical tooth-root bending stress with time.

2. The Evolution-Based Uncertainty Analysis Method

In the previous paper, an evolution-based uncertainty analysis method was developed to predict the reliability of mechanical structure [5]. For a state parameter x whose value changes over time, it can be described as a geometric Brownian motion:
\[ dx(t) = \lambda x(t)dt + \delta x(t)dw_t \]  

(1)

where, \( w_t \) is a Wiener process; \( \lambda, \delta \) is the drift rate and diffusion rate of variate \( x \), respectively. Actually, \( \lambda \) and \( \delta \) respectively represent the trend and randomness of variation.

According to the derivative chain rule of random variables proposed by Ito:

\[ d\ln x(t) = \left( \lambda - \frac{1}{2} \delta^2 \right) dt + \delta dw_t \]  

(2)

The drift rate and diffusion rate can be counted as

\[
\begin{align*}
\hat{\lambda} &= k + \frac{\delta^2}{2} \\
\hat{\delta} &= \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (q_i - \bar{q})} \\
q_i &= \frac{\ln x_i - \ln \hat{x}_i}{\sqrt{\Delta}}, \quad i = 1, 2, \ldots, m \\
\hat{x}_i &= k(i\Delta) + b, \quad i = 1, 2, \ldots, m
\end{align*}
\]  

(3)

When the Ito Lemma is extended to the multidimensional stochastic process followed by \( n \)-variate function \( G(x, t) \), the random function follows the Ito process:

\[ dG(t) = \left[ \sum_{i=1}^{n} \frac{\partial G}{\partial x_i} \lambda_i x_i(t) + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 G}{\partial x_i \partial x_k} \delta_{ij} \delta_{ik} x_j(t) x_k(t) \right] dt + \sum_{i=1}^{n} \frac{\partial G}{\partial x_i} \delta_i x_i(t) dw_t \]  

(4)

where, \( \mu_G(t), \sigma_G(t) \) is drift function and diffusion function, respectively.

The mean value and variance of \( G \) can be calculated as

\[
\begin{align*}
\hat{\mu}_G(t) &= \mu_G(0) + \int_0^t \mu_G(s) ds \\
\hat{\sigma}_G^2(t) &= \sigma_G^2(0) + \sum_{i=1}^{n} \int_0^t \sigma_G(x_i(s)) ds
\end{align*}
\]  

(5)

For the performance output \( F(t) \) and allowable performance output \([F(t)]\) of the system following the Ito process, according to equation (2), \( \ln F(t) \) and \( \ln [F(t)] \) will follow normal distribution. In order to simplify calculation, the system reliability in the future time will be written as

\[
R(t) = P\left\{ \ln F(t) \leq \ln [F(t)] \right\} = \Phi \left( \frac{\hat{\mu}_{[F]}(t) - \hat{\mu}_{F}(t)}{\left[ \hat{\sigma}_{[F]}^2(t) + \hat{\sigma}_{F}^2(t) \right]^{1/2}} \right)
\]  

(6)

3. The Evolution of Tooth Profile

3.1. Wear Model of Pinions

In the study of mild wear of spur gears, Anders Flodin and Sören Andersson developed a numerical model based on a generalized Archard’s wear equation [6]. The calculation model of wear depth at point \( K \) on the pinion and gear is expressed as
\begin{equation}
h_{kp}^{(a)} = Nkp \frac{2a_h}{v_{kp}} \left| v_{kp} - v_{kg} \right|
\end{equation}

where \( N \) is the rotation number of pinions, \( k \) is the dimensional wear coefficient, \( p_k \) is the equivalent contract pressure at a point \( K \), \( a_h \) is the semi Herzian contact width, \( v_{kp} \) is the linear velocity of pinion at point \( K \), \( v_{kg} \) is the linear velocity of gear at point \( K \).

According to the Hertz contract theory, the semi Herzian width is calculated as

\begin{equation}
a_h = \left[ \frac{4F}{\pi b} \frac{\rho_{kp}\rho_{kg}}{\rho_{kp} + \rho_{kg}} \left( \frac{1 - \mu_1^2}{E_1} + \frac{1 - \mu_2^2}{E_2} \right) \right]^{1/2}
\end{equation}

where \( F \) is the applied load at point \( K \), \( b \) is the face width, \( \rho_{kp} \) is the curvature radius of pinion at point \( K \), \( \rho_{kg} \) is the curvature radius of gear at point \( K \), \( \mu_1, \mu_2 \) are the Poisson’s ratio of pinion and gear, \( E_1, E_2 \) are the material elastic modulus of pinion and gear.

\( p_k, v_{kp} \) and \( v_{kg} \) can be calculated as

\[
\begin{aligned}
p_k &= \frac{F}{2ba_h} \\
F &= \frac{T_p}{r_{kp} \cos \alpha_n} \\
v_{kp} &= \rho_{kp} \omega_p, \quad v_{kg} = \rho_{kg} \omega_g
\end{aligned}
\]

where \( \omega_p, \omega_g \) is the angular velocity of pinion and gear at point \( K \), \( T_p \) is the transmitted torque of gear pairs, \( r_{kp} \) is the base radius of pinion, \( \alpha_n \) is the standard normal pressure angle.

3.2. Simulation of Evolution for Tooth Profile

When the accumulated wear of tooth profile up to some value, the change of involute curve which changes the rate of wear should not be ignored, and the meshing parameters need to be recalculated. The new tooth profile curve equation will be solved by using the cubic spline interpolation function.

According to the engineering experience, it’s assumed that \( T_p, \omega_p \) and \( \omega_g \) are normally distributed stochastic variables. With the reconstruction of tooth profile curve equation, \( p_k, \rho_{kp}, \rho_{kg} \) need be recounted. Thus, \( \rho_{kp} \) and \( \rho_{kg} \) are time-dependent quantities. Furthermore, \( F \) is distributed to one or two teeth. From initial point of engagement to lower boundary of single teeth-meshing area, the load on teeth increases from 0.4 to 0.6; from upper boundary of single teeth-meshing area to ends of engagement, the load on teeth decreases from 0.6 to 0.4. The transformation trend of load is approximately linear.

Based on the assumptions above, the tooth profile evolution model of involute spur gears is established. The change of parameters which influence the tooth-root bending stress will be recorded during the simulation process.

4. The Reliability Analysis of Tooth Bending Failure

4.1. The Calculation of Critical Tooth-Root Bending Stress

According to ISO 6336-3, the local tooth-root bending stress at the pinion tooth root can be calculated as

\begin{equation}
\sigma_y = \frac{6F}{s_f b} \cos \alpha_p(\xi) h_F(\xi) Y_5(\xi)
\end{equation}

where \( s_f \) is the tooth thickness at the critical section; \( \alpha_p \) is the load angle; \( h_F \) is the bending moment arm; \( Y_5 \) is the tooth correction factor; \( \xi \) is the involute profile parameter.
The tooth root suffers the maximum bending moment when the contact point at the upper boundary of single teeth-meshing area. Hence, the critical tooth-root bending stress will be calculated when this maximum arises.

Because of the wear of tooth profile and the fluctuation of transmitted torque and rotational speed, the parameters include $F$, $\alpha_F$ and $h_F$ change over time. According to equation (9), $F$ obeys a normal distribution due to the distribution feature hypothesis of $T_p$. On the basis of evolution-based uncertainty analysis method, $\alpha_F$ and $h_F$ will be known as random variables which obey geometric Brownian motion. Particularly. As is described in equation (2), they can be expressed as

$$
\left\{
\begin{array}{l}
\mathrm{dln} \alpha_F(t) = \left( \lambda_{\alpha_F} - \frac{1}{2} \delta_{\alpha_F}^2 \right) dt + \delta_{\alpha_F} dw_{\alpha_F}
\end{array}
\right.
$$

To simplify the calculation, equation (10) will be converted to:

$$
\ln \sigma_F(t) = \ln \frac{6Y_s}{s_F^2 b} + \ln F(t) + \ln \cos \alpha_F(t) + \ln h_F(t)
$$

According to equation (4):

$$
\mathrm{dln} \sigma_F(t) = \left[ -\lambda_{\alpha_F} \cdot \alpha_F(t) \tan \alpha_F(t) + \lambda_{h_F} - \frac{1}{2} \delta_{\alpha_F}^2 \cdot \frac{\alpha_F(t)}{\cos^2 \alpha_F(t)} \right] dt
$$

$$
-\delta_{\alpha_F} \alpha_F(t) \tan \alpha_F(t) dw_{\alpha_F} + \delta_{h_F} dw_{h_F}
$$

According to equation (5), the mean value and variance are calculated as

$$
\left\{
\begin{array}{l}
\hat{\mu}_{\ln \sigma_F}(t) = \ln \sigma_F(0) + \lambda_{\alpha_F} t - \int_0^t \lambda_{\alpha_F} \cdot \alpha_F(s) \tan \alpha_F(s) + \frac{1}{2} \delta_{\alpha_F}^2 \cdot \frac{\alpha_F(s)}{\cos^2 \alpha_F(s)} \, ds
\end{array}
\right.
$$

$$
\hat{\sigma}_{\ln \sigma_F}^2(t) = \hat{\sigma}_{\ln F}^2(t) + \delta_{\alpha_F}^2 t + \delta_{h_F}^2 \int_0^t \left[ \alpha_F(s) \tan \alpha_F(s) \right]^2 \, ds
$$

The mean value and variance of logarithmic normal distribution can be calculated as

$$
\left\{
\begin{array}{l}
\hat{\mu}_{\sigma_F}(t) = \exp \left[ \hat{\mu}_{\ln \sigma_F}(t) + \frac{1}{2} \hat{\sigma}_{\ln \sigma_F}^2(t) \right]
\end{array}
\right.
$$

$$
\hat{\sigma}_{\sigma_F}^2(t) = \exp \left[ \hat{\sigma}_{\ln \sigma_F}^2(t) \right] \cdot \exp \left[ 2 \hat{\mu}_{\ln \sigma_F}(t) + \hat{\sigma}_{\ln \sigma_F}^2(t) \right]
$$

4.2. The Reliability Prediction of Tooth Bending Failure

The allowable tooth-root bending stress $\sigma_{FP}$ is regarded as a parameter that degrades over time. Hence, it can be regarded as a geometric Brownian motion whose diffusion rate equals to zero. According to equation (6), the reliability of tooth bending failure can be predicted as

$$
R(t) = \Phi \left[ \frac{\hat{\mu}_{\ln \sigma_F}(t) - \hat{\mu}_{\ln \sigma_F}(t)}{\hat{\sigma}_{\ln \sigma_F}(t)} \right]
$$

4.3. Example

A pair of involute spur gears drive was designed to illustrate the application of the model. Table 1 shows the design parameters of gear pairs. The wear coefficient set to $5 \times 10^{-10}$ m$^3$/N$^{-1}$, the rotational speed of pinion set to $(150 \pm 4)$ r·min$^{-1}$, the transmitted torque of pinion set to $(1.65 \pm 0.04) \times 10^5$ N·mm. Figure 1
shows the change of $F$, $\alpha_F$ and $h_F$ over time. Figure 2 shows the change of probability density distribution of critical tooth-root bending stress over time.

Table 1. The design parameters of gear pairs.

| Module (mm) | Tooth number of the pinion | Tooth number of the gear | Tooth width (mm) | Pressure angle | Poisson ratio | Elasticity modulus (GPa) |
|-------------|---------------------------|--------------------------|------------------|----------------|---------------|--------------------------|
| 4           | 21                        | 36                       | 15               | 20°            | 0.269         | 210                      |

Figure 1. The change of meshing parameters: (a) tooth load $F$; (b) load angle $\alpha_F$; (c) bending moment arm $h_F$.

Figure 2. The probability density distribution of critical tooth-root bending stress.

5. Conclusion
A novel reliability prediction model for bending fatigue failure of involute spur gears was established on account of the evolution-based uncertainty analysis method and mild wear model. The simulation model of tooth profile evolution due to the wear of gear flanks was developed in MATLAB. The random fluctuation of transmit torque and rotational velocity of gears, which caused the uncertainly change of wear rate, were considered. According to the tooth profile evolution model, the tooth-root bending stress was time-varying because the load angle and bending moment arm were changed over time. The evolution-based uncertainty analysis method was introduced to describe the distribution characteristics of tooth-root bending stress. After that, the reliability of tooth bending failure was predicted.
Acknowledgments
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