Predictions for the cusp in $\eta \to 3\pi^0$ decay

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A realistic estimate of the cusp effect in the $\eta \to 3\pi^0$ decay is required for the forthcoming high precision experiments. The predictions for the size of this effect are given within the framework of nonrelativistic effective field theory.

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I. INTRODUCTION

The physical region in $M_{\pi^0\pi^0\pi^0}$ invariant mass distribution for $\eta \to 3\pi^0$ decay extends below the charged two-pion threshold. It means that a cusp structure should be visible in this distribution around $2M_{\pi\pi}$, in analogy with the pronounced cusp in $K^+ \to \pi^+\pi^0\pi^0$ decay, observed recently by NA48/2 collaboration. In this paper, in particular, it has been shown that measuring charged kaon decays in the cusp region enables one to precisely determine $S$-wave $\pi\pi$ scattering lengths $a_0$ and $a_2$, provided an accurate theoretical parameterization of the invariant mass distribution in terms of these scattering lengths is known (The strong impact of the unitarity cusp on $\pi\pi$ scattering was already mentioned in Ref. [8].). Moreover, the same logic applies to the neutral kaon decays into three pions, which have been studied in the recent experiment [7]. The theoretical framework for analysis of the neutral kaon decays is provided in Refs. [3,4,5] and the systematic inclusion of the electromagnetic effects both in charged and neutral kaon decays is considered in Ref. [6]. We further mention that the general structure of the amplitude in the neutral kaon decays is similar to the $\eta \to 3\pi^0$ decay amplitude. For this reason, e.g., the two-loop representation of the amplitude in terms of the $\pi\pi$ effective-range expansion parameters, derived in Ref. [8], can be directly used to predict the cusp in the $\eta \to 3\pi^0$ decay, which is studied in KLOE, Crystal Ball and WASA collaboration experiments [10,11,12].

It should be pointed out that the two-loop formula for the kaon decay amplitudes, which was mentioned above, have been obtained in Refs. [3,5] within the non-relativistic effective field theory framework. This framework is ideally suited for parameterizing the final-state interactions in terms of the $\pi\pi$ scattering lengths (effective-range parameters, in general), whereas the expansion of the amplitudes in Chiral Perturbation Theory (ChPT) is performed in powers of the quark masses and is less convenient for expressing the amplitude in the cusp.

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II. THEORETICAL FRAMEWORK

Below we mainly follow the notations from Ref. [8]. The tree-level amplitudes are expressed in terms of the kinetic energies $X_i$

$$X_i = E_i - M_{\pi^0},$$

(1)
where $E_i$ denote the pion energies in the eta rest frame.

Up to the quadratic terms,

$$M_{000}^{\text{tree}} = K_0 + K_1(X_1^2 + X_2^2 + X_3^2),$$

$$M_{+00}^{\text{tree}} = L_0 + L_1X_3 + L_2X_3^2 + L_3(X_1 - X_2)^2,$$  \hspace{1cm} (2)

where $L_i, K_i$ are the effective couplings in the non-relativistic Lagrangian that describe $\eta \to 3\pi$ decays at tree level. Note that we use the same notation for these couplings as in Ref. \cite{3}, where they denote the couplings describing the 3-pion decays of the neutral kaons.

Assuming $\Delta I = 1$ rule in the $\eta \to 3\pi$ vertex, the isospin symmetry relates the amplitudes for $\eta \to 3\pi^0$ and $\eta \to \pi^+\pi^-\pi^0$ (we use Condon-Shortley phase convention)

$$M_{000}(s_1, s_2, s_3) = -M_{+00}(s_1, s_2, s_3)$$

$$- M_{+00}(s_2, s_3, s_1) - M_{+00}(s_1, s_1, s_2).$$  \hspace{1cm} (3)

At tree level, this allows one to express the couplings $K_i$ through $L_i$

$$K_0 = -(3L_0 + L_1Q - L_3Q^2),$$

$$K_1 = -(L_2 + 3L_3),$$  \hspace{1cm} (4)

where $Q = M_\eta - 3M_{\pi^0}$.

In general, $\eta \to 3\pi$ decay amplitudes are given in a form of a sum of the tree, one-loop, two-loop, ... contributions $M_{000} = M_{000}^{\text{tree}} + M_{000}^{\text{1-loop}} + M_{000}^{\text{2-loops}} + \cdots$, and similarly for $M_{+00}$. The pertinent (rather lengthy) expressions are given in Ref. \cite{3}. We do not display them here. It can be checked that these amplitudes in the isospin symmetry limit explicitly obey the constraints \cite{4} at one- and two-loop level.

We wish to stress that the representations given in Refs. \cite{3,4} should be understood as a parameterization which should be fit to the data. In other words, the constants $L_i, a_0, a_2, \ldots$ are considered as free parameters to be fixed from the fit. In this paper, however, we make an attempt to predict the size of the cusp — fitting first the tree-level amplitude in order to determine $L_i$ and then using one- and two-loop representation to produce the cusp in the synthetic data. In doing this, we have fixed $a_0, a_2$ to their theoretical values \cite{24} and neglected isospin breaking in the derivative 4-pion couplings, as well as the shape parameter and the $P$-waves.

The matching of $L_i$ is done to:

1) The tree-level amplitude $\eta \to \pi^+\pi^-\pi^0$ in ChPT.

According to Eq. (4), the overall normalization of the amplitude does not play a role, only the slopes matter. The result is given by

$$L_0 = (4M_{\pi^0}^4 - 3(M_\eta - M_{\pi^0})^2)/(M_\eta^2 - M_{\pi^0}^2),$$

$$L_1 = 6M_\eta/(M_\eta^2 - M_{\pi^0}^2),$$

$$L_2 = L_3 = 0.$$  \hspace{1cm} (5)

FIG. 1: Invariant mass distribution $d\Gamma/dM_{\eta^0\pi^0}$ divided by the phase space, calculated at two loops: 1) Matching to ChPT at tree level (dashed line); 2) Matching to the KLOE parameterization \cite{25} (solid line).

2) The experimental amplitude extracted by KLOE collaboration \cite{25}.

In order to carry out the matching to the KLOE data, it is useful to introduce Dalitz variables for $\eta \to \pi^+\pi^-\pi^0$ decay

$$x = \sqrt{3}(X_1 - X_2)/Q, \quad y = 3X_3/Q - 1.$$  \hspace{1cm} (6)

For the decay $\eta \to 3\pi^0$ one defines the variable

$$z = x^2 + y^2.$$  \hspace{1cm} (7)

The phenomenological parameterization of the amplitude is given by

$$M_{+00} = A_c(1 + \alpha y + \beta y^2 + \gamma x^2),$$  \hspace{1cm} (8)

with $\alpha, \beta, \gamma$ being complex quantities. The matching of Eq. (2) to the real part of Eq. (8) yields

$$L_0 = A_c(1 - \Re \alpha + \Re \beta),$$

$$L_1 = 3A_c(\Re \alpha - 2\Re \beta)/Q,$$

$$L_2 = 9A_c \Re \beta/Q^2,$$

$$L_3 = 3A_c \Re \gamma/Q^2.$$  \hspace{1cm} (9)

The right-hand side in Eq. (9) is fixed by using Eq. (6.4) and table 1 of Ref. \cite{25}. Isospin-breaking corrections in Eqs. (8) and (9) are consistently neglected.

We would like to mention that the systematic way of fixing the parameters of the effective non-relativistic Lagrangian consists in performing a simultaneous fit of the non-relativistic representation to both charged and neutral invariant mass distributions. The results, which are contained in the present paper, should be considered only as a rough theoretical estimate of the expected size of the cusp effect in the $\eta \to 3\pi^0$ decay.
III. RESULTS

In Fig. 1 we display the calculated invariant mass distribution for $\eta \rightarrow 3\pi^0$ decay, divided by the phase space. The decay amplitude is normalized in the center of the Dalitz plot

$$|M_{000}(s_0, s_0, s_0)|^2 = 1, \quad s_0 = \frac{M_\eta^2}{3} + M_{\pi^0}^2. \quad (10)$$

We display the result for $L_i, K_i$ matched to the tree-level result of ChPT, or to the KLOE amplitude. The resulting cusp in both cases amounts roughly up to a $2\%$ effect. We would like to mention that the sign of the cusp effect is fixed by the isospin symmetry, see Eqs. (3) and (4) and is thus a robust theoretical prediction.

In order to check the convergence of the method, in Fig. 2 we show the invariant mass distribution calculated at tree level, one and two loops, with the couplings $L_i$ matched to the KLOE amplitude. It is seen that the shape of the cusp does not change much from one- to two-loop calculations, indicating a rather robust prediction for a size of this effect.

Figure 3 contains our prediction for the differential decay rate in the variable $z$ – again with $L_i, K_i$ matched either to the tree-level result of ChPT, or to the KLOE amplitude. As expected, the slope parameter in the former case has the opposite sign as compared to the experimentally observed. Apart from a small dip around $z \simeq 0.75$, corresponding to the cusp, the differential decay rate is seen to be fairly linear in $z$.

The convergence of the loop expansion for the differential decay rate in the variable $z$ and the comparison with the Crystal Ball data [10] is shown in Fig. 4. As seen, the data are described quite well.

IV. CONCLUSIONS

Using the two-loop parameterization of the $\eta \rightarrow 3\pi$ decay amplitudes [8], we have shown that the size of the cusp effect in the invariant mass distribution for $\eta \rightarrow 3\pi^0$ process amounts up to around $2\%$. Despite such tiny effect, one may expect that forthcoming high-precision experiments at Crystal Ball, KLOE and WASA-at-COSY with about $10^7$ events in the Dalitz plot will be able to observe it. It is however unlikely that one could determine $\pi\pi$ scattering lengths at a reasonable accuracy from these experiments.

Moreover, the cusp effect modifies the differential decay rate for the $\eta \rightarrow 3\pi^0$ decay in the variable $z$, producing a dip around the value $z \simeq 0.75$. We expect that, in order to carry out an accurate analysis of the Dalitz plot distributions, this effect should be taken into account.

Finally, we wish to mention that for the cusp-like structure, which has been seen recently by the Crystal Ball collaboration experiment at MAMI-C [26], the sign of the effect is claimed to be different from the theoretical
prediction. To resolve this contradiction, experimental study of the $\eta \to 3\pi^0$ decay with a better statistics would be desirable.

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