LIPSCHITZ STABILITY FOR SOME COUPLED DEGENERATE PARABOLIC SYSTEMS WITH LOCALLY DISTRIBUTED OBSERVATIONS OF ONE COMPONENT

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Abstract. This article presents an inverse source problem for a cascade system of \( n \) coupled degenerate parabolic equations. In particular, we prove stability and uniqueness results for the inverse problem of determining the source terms by observations in an arbitrary subdomain over a time interval of only one component and data of the \( n \) components at a fixed positive time \( T' \) over the whole spatial domain. The proof is based on the application of a Carleman estimate with a single observation acting on a subdomain.

1. Introduction and main results. This article is devoted to the question of the reconstruction of all the source terms for a degenerate parabolic system of \( n \) coupled equations, with the main particularity that we observe only one component of the system. More precisely, we consider the following parabolic linear system of \( n \)-coupled degenerate equations, with \( n \) forces:

\[
\begin{align*}
\partial_t y_1 - d_1 (a(x)y_1)_x &+ \sum_{j=1}^{2} b_{1j} y_j = f_1, & (t,x) \in Q, \\
\partial_t y_2 - d_2 (a(x)y_2)_x &+ \sum_{j=1}^{3} b_{2j} y_j = f_2, & (t,x) \in Q, \\
&\vdots \\
\partial_t y_n - d_n (a(x)y_n)_x &+ \sum_{j=1}^{n} b_{nj} y_j = f_n, & (t,x) \in Q, \\
\end{align*}
\]

\[y_k(t,1) = 0, \quad y_k(t,0) = 0, \quad \text{in the weakly degenerate case (WD)},\]

\[\begin{align*}
(ay_{kx})(t,0) = 0, & \quad \text{in the strongly degenerate case (SD)}, \\
t \in (0,T), & \quad 1 \leq k \leq n, \\
y_1(0,x) = y_1^0(x), & \quad y_n(0,x) = y_n^0(x), \\
\end{align*}\]

(1)

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where \((g_{kj})_{1 \leq k \leq n} \in L^2(0,1)^n\), \(T > 0\) fixed, \(Q := (0,T) \times (0,1)\), the coupling terms \(b_{kj} = b_{kj}(t,x) \in L^\infty(Q)\) \((1 \leq k,j \leq n)\) and the function \(a\) is a diffusion coefficient which degenerates at 0 (i.e., \(a(0) = 0\)) and which can be either weakly degenerate (WD), i.e.,

\[
\left\{ \begin{array}{ll}
a \in C([0,1]) \cap C^1((0,1]), a(0) = 0, a > 0 & \text{in } (0,1], \\
\exists \alpha \in [0,1), \text{ such that } xa'(x) \leq \alpha a(x), & \forall x \in [0,1], \end{array} \right. \tag{2}
\]

or strongly degenerate (SD), i.e.,

\[
\left\{ \begin{array}{ll}
a \in C^1((0,1]), a(0) = 0, a > 0 & \text{in } (0,1], \\
\exists \alpha \in [1,2), \text{ such that } xa'(x) \leq \alpha a(x), & \forall x \in [0,1], \\
\exists \beta \in (1,\alpha], x \mapsto \frac{a(x)}{x^\beta} & \text{is nondecreasing near 0, if } \alpha > 1, \\
\exists \beta \in (0,1), x \mapsto \frac{a(x)}{x^\beta} & \text{is nondecreasing near 0, if } \alpha = 1. \tag{3}
\end{array} \right.
\]

Equivalently, the previous system can be written as

\[
\left\{ \begin{array}{l}
Y_t - D A Y + B Y = F \quad (t,x) \in Q, \\
C Y = 0, \quad (t,x) \in \Sigma, \\
Y(0,x) = Y^0(x), \quad x \in (0,1), \end{array} \right. \tag{4}
\]

where \(\Sigma := (0,T) \times \{0,1\}\), \(D\) is a \(n \times n\) matrix, \(B\) is a \(n \times n\) matrix, \(Y = (y_k)_{1 \leq k \leq n}\) is the state and \(F = (f_1, f_2, ..., f_n)^*\). The operator \(D A\) is defined by \(D A Y = (d_1(a y_1)_x, ..., d_n(a y_n)_x)\) for \(Y \in D(D A) \subset L^2(0,1)^n\).

The boundary condition \(C Y = 0\) is either \(Y(0) = Y(1) = 0\) in the weakly degenerate case (WD) or \(Y(1) = (a Y_2)(0) = 0\) in the strongly degenerate case (SD).

We are interested in answering the following inverse problem: can we retrieve the source terms \(f_1, ..., f_n\) in system (1) from incomplete data, that is to say, from a reduced number of measurements of the solution?

For this purpose, we define the zone of measurements \(\omega\) to be a nonempty open subset of \((0,1)\). For \(t_0 \in (0,T)\), we shall use the following notations \(Q_{t_0} = (t_0,T) \times (0,1), \omega_{t_0} = (t_0,T) \times \omega\) and \(T' := T + t_0\). Let us recall that in inverse source problems, the source term has to satisfy some condition otherwise uniqueness may be false, see [27]. Let \(C_0 > 0\) be given. In [23, 15], the authors make the assumption that source terms \(f\) satisfy the condition

\[
|f_t(t,x)| \leq C_0 |f(T',x)|, \text{ for almost all } (t,x) \in Q \tag{5}
\]

Therefore they define the set \(S(C_0)\) of admissible source terms as

\[
S(C_0) := \{ f \in H^1(0,T; L^2(0,1)) : f \text{ satisfies (5)} \}.
\]

The main goal of the present work is to recover all the source terms \(f_k\) \((1 \leq k \leq n)\) using the following observation data:

\[
(a y_k)_x(T',\cdot), \quad \forall k : 1 \leq k \leq n, \quad y_1|_{\omega_{t_0}} \quad \text{and} \quad y_{1t}|_{\omega_{t_0}}.
\]

Our contributions are:

- identification of all external forces term for \(n\)-coupled degenerate parabolic system (1) from a few interior measurements,
- the reduction of the number of observations, and
- a global stability estimate (of Lipschitz type).
Inverse problems for non degenerate \((a > 0)\) parabolic systems are well studied over the last decades and there have been a great number of results. The first Lipschitz stability result of an inverse source problem for a parabolic equation \((n = 1)\) was obtained by Imanuvilov and Yamamoto [23]. The essence of their methodology comes from the Bukhgeim-Klibanov method using the Carleman estimate to derive the global uniqueness in inverse problems, see [12]. Inverse problems for parabolic systems of two equations have also been studied, see for example [1, 7, 16, 17, 18, 19, 24, 28]. There are, of course, other kinds of models which involve coupled systems of partial differential equations (wave-wave, hyperbolic-parabolic coupled systems). For results related to these subjects we refer to e.g. [4] for hyperbolic systems in cascade and [6] in the case of thermoelastic systems. We refer to [5, 29] for a more detailed survey concerning the applicability of Carleman estimates to stability of inverse problems.

In the last recent years, an increasing interest has been devoted to the study of degenerate parabolic equations with degeneracy occurring at the boundary or in the interior of the space domain. For such systems, inverse problems of one equation was studied in [9, 14, 15, 25, 26]. The main result in these works is the development of adequate Carleman estimates, which are crucial tool to obtain Lipschitz stability for term sources, initial data, potentials and diffusion coefficients. The case of two coupled systems \((n = 2)\), is considered in [11], and in [10].

Recently, in [20] the authors studied the null controllability of \(n\)-coupled degenerate parabolic system when \(m\) distributed controls are exerted on the system. In this framework, up to now, inverse source problems for coupled systems of \(n\) equation with \(n > 2\) were never considered even in the case of non degenerate parabolic coupled systems.

Motivated by this reason, the present paper is devoted to the study of an inverse source problem for such coupled degenerate systems. More precisely, we will follow the approach introduced by Imanuvilov and Yamamoto in [23] for the treatment of uniformly parabolic problems which is based on the use of global Carleman estimates. For this purpose, we use and extend some recent Carleman estimates given in [20]. As a consequence, we prove a stability estimate of Lipschitz type in determining all the source terms from the knowledge of some measurements of only one component of the solution. To our knowledge, this paper is the first one concerning Lipschitz stability results in inverse problems for degenerate coupled systems such as (1).

For fixed \(T > T' > 0\), our first main result is the stability for the inverse source problem.

**Theorem 1.1.** Let \(C_0 > 0\) and assume that for some open subset \(\omega_0 \subset \omega\), we have

\[
b_{k-1k} \geq b_0 > 0, \quad \text{in} \quad (0, T) \times \omega_0, \quad \forall k : 2 \leq k \leq n. \tag{6}
\]

Then, there exists \(C = C(T, t_0, C_0) > 0\) such that, for all \(f_k \in \mathcal{S}(C_0)\) and \(y_k^0 \in L^2(0, 1)\) \((1 \leq k \leq n)\), there holds

\[
\sum_{k=1}^{n} \|f_k\|^2_{L^2(Q)} \leq C \left( \sum_{k=1}^{n} \|(ay_kx)x(T', \cdot)\|^2_{L^2_{1}(0, 1)} + \|y_1\|^2_{L^2_{1}(\omega_0)} + \|y_1t\|^2_{L^2_{1}(\omega_0)} \right). \tag{7}
\]

A brief idea of our strategy is as follows. First, we establish a Carleman estimate with a boundary observation for a single degenerate equation. Then, using a localization argument we deduce a Carleman estimate with a distributed observation
for one degenerate equation. Summing up these inequalities we obtain a Carleman estimate for the coupled system with distributed observations of each equation which could be used to show Lipschitz stability estimate in the determination of the source terms from interior measurements of all components of the system. In a second step, by using the equations we try to reduce the number of measurements obtaining a Carleman estimate with a single observation acting on a subdomain. Finally, this estimate is successfully used along with certain energy estimates to obtain the stability result for the inverse source problem of \( n \)-coupled degenerate parabolic equations by measurements of one component.

**Remark 1.** • Theorem 1.1 provides a global Lipschitz stability estimate that extend the one obtained for a single degenerate heat equation by Cannarsa, Tort and Yamamoto [15] to the case of more general cascade coupled systems. The main difference between our work and [15] is that we consider a coupled system of degenerate parabolic equations, and the additional data are given only for one component of this system.

• Although theorem 1.1 provides a useful stability result, we note that it does not ensure that the inverse problem has a unique solution because the class \( S(C_0) \) is not a vector space.

An important case is when the unknown source terms of (1) take the form

\[
f_k(t, x) = g_k(x) r_k(t, x), \quad \forall k : 1 \leq k \leq n, \tag{8}\]

where \( g_k \) are the unknown functions of \( L^2(0, 1) \) while \( r_k \in C^1([0, T] \times [0, 1]) \) are the given functions such that

\[
\forall x \in [0, 1] \quad |r_k(T', x)| > d_k, \tag{9}\]

for some given constant \( d_k > 0 \), \( 1 \leq k \leq n \). We denote by \( \mathcal{E} \) the space

\[
\mathcal{E} := \{ g(x) r(t, x) \text{ for some } g \in L^2(0, 1) \}.
\]

As an application of Theorem 1.1, we can determine \( g_k(x) \) (1 \( \leq k \leq n \)) in

\[
\begin{cases}
\partial_t y_1 - d_1(a(x)y_{1x})_x + \sum_{j=1}^2 b_{1j}y_j = g_1 r_1, & (t, x) \in Q, \\
\partial_t y_2 - d_2(a(x)y_{2x})_x + \sum_{j=1}^2 b_{2j}y_j = g_2 r_2, & (t, x) \in Q, \\
\vdots \\
\partial_t y_n - d_n(a(x)y_{nx})_x + \sum_{j=1}^n b_{nj}y_j = g_n r_n, & (t, x) \in Q, \\
y_k(t, 1) = 0, & t \in (0, T), \quad 1 \leq k \leq n, \\
y_k(t, 0) = 0, & (a y_{kx})(t, 0) = 0, \quad \text{in the weakly degenerate case (WD)}, \\
y_1(0, x) = y_1^0(x), \ldots, y_n(0, x) = y_n^0(x), & x \in (0, 1). 
\end{cases} \tag{10}\]

Here, we prove that we can uniquely recover the spatial components \( g_k \) of the source terms from the measurement of the solution over the whole spatial domain \( (0, 1) \) at any fixed moment \( T' \) plus additional local observations in space and time of one component of the solution. This is our second main result:

**Theorem 1.2.** Let \( r_k \in C^1([0, T] \times [0, 1]) \) be a function satisfying (9). Then, there exists \( C = C(T, t_0, r_k) > 0 \) such that, for all \( f_k = \hat{g}_k r_k \in \mathcal{E} \) and \( \hat{f}_k = \hat{g}_k r_k \in \mathcal{E} \) the
associated solutions \((\tilde{y}_1, \ldots, \tilde{y}_n)\) and \((\hat{y}_1, \ldots, \hat{y}_n)\) of (10) satisfy
\[
\sum_{k=1}^{n} \|\hat{y}_k - \tilde{y}_k\|_{L^2(0,1)}^2 \leq C \left( \sum_{k=1}^{n} \|a(\hat{y}_k - \tilde{y}_k)x\|_{L^2(0,1)}^2 \right)
+ \|\tilde{y}_1 - \hat{y}_1\|_{L^2(\omega_{io})}^2 + \|\hat{y}_{1t} - \tilde{y}_{1t}\|_{L^2(\omega_{io})}^2.
\] (11)

In particular, Theorem 1.2 provides the following uniqueness result: if the solutions \((\hat{y}_1, \ldots, \hat{y}_n)\) and \((\tilde{y}_1, \ldots, \tilde{y}_n)\) of (10) associated to \(\hat{y}_k\) and \(\tilde{y}_k\) satisfy
\[
(a\hat{y}_{kx})x(T', \cdot) = (a\tilde{y}_{kx})x(T', \cdot), \quad \forall k: 1 \leq k \leq n \quad \text{in} \quad (0,1),
\]
then
\[
\hat{y}_k = \tilde{y}_k \quad \text{in} \quad (0,1), \quad \forall k: 1 \leq k \leq n.
\]

For our further results, it is important to remind the following fundamental Hardy-Poincaré inequality [3, Proposition 2.1]:

**Proposition 1.** For all \(y \in H^2_0(0,1)\), the following inequality holds
\[
\int_0^1 a(x) y^2(x) \, dx \leq \frac{1}{H_P} \int_0^1 a(x) |y'(x)|^2 \, dx,
\] (12)
where \(H^2_0(0,1)\) denotes a weighted Hilbert space defined in the next section.

The article is organized as follows. In Section 2, we discuss the well-posedness of the problem (1). Then, in Section 3, we establish different Carleman estimates for parabolic equations and parabolic systems. Finally, in Section 4, we apply the Carleman estimates to prove the Lipschitz stability and uniqueness results.

2. Well-posedness and regularity results. In order to study the well-posedness of system (1), we introduce the following weighted spaces. In the (WD) case:

\[
H^1_a := \left\{ y \in L^2(0,1) : y \text{ absolutely continuous in } [0,1], \right\}
\]

\[
\sqrt{a}y_x \in L^2(0,1) \text{ and } y(1) = y(0) = 0
\]

and

\[
H^2_a := \left\{ y \in H^1_a(0,1) : ay_x \in H^1(0,1) \right\}.
\]

In the (SD) case:

\[
H^1_a := \left\{ y \in L^2(0,1) : y \text{ locally absolutely continuous in } (0,1), \right\}
\]

\[
\sqrt{a}y_x \in L^2(0,1) \text{ and } y(1) = 0
\]

and

\[
H^2_a := \left\{ y \in H^1_a(0,1) : ay_x \in H^1(0,1) \right\}
\]

\[
= \left\{ y \in L^2(0,1) : ay \text{ locally absolutely continuous in } (0,1), ay \in H^1_0(0,1), \right\}
\]

\[
\text{and } ay_x \in H^1(0,1) \text{ and } (ay_x)(0) = 0.
\]

In both cases, the norms are defined as follow
\[
\|y\|_{H^1_a}^2 := \|y\|_{L^2(0,1)}^2 + \|\sqrt{a}y_x\|_{L^2(0,1)}^2, \quad \|y\|_{H^2_a}^2 := \|y\|_{H^1_a}^2 + \|ay_x\|_{L^2(0,1)}^2.
\]
We recall from [13, 3] that the operator \( (A, D(A)) \) defined by \( Ay := (ay_x)_x \), \( y \in D(A) = H^2(0, 1) \) is closed negative self-adjoint with dense domain in \( L^2(0, 1) \). Hence, it is infinitesimal generator of an analytic semi-group of contractions in the pivot space \( L^2(0, 1) \).

At this point, as the operator \( DA \) with domain \( D(DA) = H^2(0, 1)^n \) is diagonal and since \( B \) is a bounded perturbation, the following well-posedness and regularity results hold.

**Proposition 2.** (i) For all \( Y^0 \in D(DA) \) and \( F \in H^1(0, T; L^2(0, 1)^n) \), the problem (4) has a unique solution

\[
Y \in C([0, T], D(DA)) \cap C^1(0, T; L^2(0, 1)^n).
\]

(ii) For all \( F \in L^2(Q)^n \), \( Y^0 \in L^2(0, 1)^n \), and \( \varepsilon \in (0, T) \), there exists a unique mild solution

\[
Y \in H^1([\varepsilon, T], L^2(0, 1)^n) \cap L^2(\varepsilon, T; D(DA)).
\]

If moreover, \( F \in H^1(0, T; L^2(0, 1)^n) \) and \( \varepsilon \in (0, T) \), then

\[
Y \in C([\varepsilon, T], D(DA)) \cap C^1([\varepsilon, T]; L^2(0, 1)^n).
\]

**Proof.** The proof of statements (i) and (ii) mainly follows from the fact that \( (DA, D(DA)) \) generates an analytic semi-group in the pivot space \( L^2(0, 1)^n \). Then it suffices to apply standard semi-groups theory: for example [8, Proposition 3.3] in the case \( F \in H^1(0, T; L^2(0, 1)^n) \) and [8, Proposition 3.8] in the case \( F \in L^2(Q)^n \). \( \square \)

3. Global Carleman estimates. In this section we give a new global Carleman estimate for the system (1). To this end, as in [15], we introduce the following time and space weight functions

\[
\varphi(t, x) := \theta(t)\psi(x), \quad \theta(t) := \frac{1}{(t - t_0)^4(T - t)^4},
\]

\[
\psi(x) := \gamma \left( \int_0^x \frac{y}{a(y)} \, dy - d \right), \quad \text{and} \quad \eta(t) := T + t_0 - 2t,
\]

where \( t_0 > 0 \) is a fixed initial time, \( T > 0 \) is a final time and where the parameters \( \gamma \) and \( d > d^* := \sup_{[0, 1]} \int_0^1 \frac{y}{a(y)} \, dy \) are positive constants that will be chosen later.

In order to state our fundamental result, we need to show first some Carleman estimates in the case of a single parabolic degenerate equation.

3.1. **Carleman estimate for one degenerate equation.** In this subsection we shall establish a new Carleman estimate for the solution of the following parabolic equation

\[
\begin{align*}
\begin{cases}
y_t - d(a(x)y_x)_x = f, & (t, x) \in Q, \\
y(t, 1) = 0, & t \in (0, T), \\
y(t, 0) = 0, & \text{for (WD)}, \\
(a)y_x(t, 0) = 0, & t \in (0, T), \\
y(0, x) = y^0(x), & x \in (0, 1),
\end{cases}
\end{align*}
\]

where \( d > 0 \) is a positive constant, \( f \in L^2(Q) \) and \( y^0 \in L^2(0, 1) \).

The following Carleman estimate will be crucial for the aim of this section. Note that the Carleman estimate needed in this work is different from the one showed in [3] where, however, the Carleman inequality was derived to bound just the integrals of \( s\theta(a(y)g_x^2) \) and \( s\theta^3 \frac{x^2}{a_0^2}y^2 \) (that were sufficient for control purposes). For inverse
Therefore, using the fact that \( 1 \), we have
\[
\int_{Q_{t_0}} \left( \frac{1}{s\theta} y^2 + s\theta^\frac{2}{3} |\eta\psi|y^2 + s\theta da(x) y_x^2 + s^3\theta^3 \frac{x^2}{da(x)} y^2 \right) e^{2s\varphi} \, dx \, dt 
\leq C \left( \int_{Q_{t_0}} f^2 e^{2s\varphi} \, dx \, dt + s\gamma a(1) \int_{t_0}^T \theta(t) y_x^2(t, 1) e^{2s\varphi(t, 1)} \, dt \right). \tag{15}
\]

**Theorem 3.1.** There exist two positive constants \( C \) and \( s_0 \), such that every solution \( y \) of (14) satisfies, for all \( s \geq s_0 \),
\[
\int_{Q_{t_0}} \left( \frac{1}{s\theta} y^2 + s\theta^\frac{2}{3} |\eta\psi|y^2 + s\theta da(x) y_x^2 + s^3\theta^3 \frac{x^2}{da(x)} y^2 \right) e^{2s\varphi} \, dx \, dt 
\leq C \left( \int_{Q_{t_0}} f^2 e^{2s\varphi} \, dx \, dt + s\gamma a(1) \int_{t_0}^T \theta(t) y_x^2(t, 1) e^{2s\varphi(t, 1)} \, dt \right). \tag{15}
\]

**Proof.** The proof is based on the methods developed in [15]. Given a solution \( y \in L^2(t_0, T; D(A)) \cap H^1(t_0, T; L^2(0, 1)) \) of (14) and a positive number \( s > 0 \), define \( w = ye^{\varphi} \) for a.e. \( (t, x) \in Q_{t_0} \). We first prove a Carleman-type estimate for \( w \) and then we deduce the expected estimate on \( y \). First of all, observe that \( w \) satisfies
\[
P_s^+ w + P_s^- w = f e^{\varphi},
\]
where
\[
P_s^+ w = -d(a\varphi_x)_x - s_\varphi w - s^2 da\varphi^2_x w, \\
P_s^- w = w_t + 2sda\varphi_x w_x + sd(a\varphi_x)_x w. \tag{16}
\]
Moreover, \( w(t_0, x) = w(T, x) = 0 \). This property allows us to apply the Carleman estimates established in [3] to \( w \) with \( Q_{t_0} \) in place of \( (0, T) \times (0, 1) \) and \( d\partial_x(a\partial_x) \) instead of \( \partial_x(a\partial_x) \), obtaining
\[
\|P_s^+ w\|_{L^2(Q_{t_0})}^2 + \|P_s^- w\|_{L^2(Q_{t_0})}^2 + \int_{Q_{t_0}} \left( s\theta da(x) w_x^2 + s^3\theta^3 \frac{x^2}{da(x)} w^2 \right) \, dx \, dt 
\leq C \left( \int_{Q_{t_0}} f^2 e^{2s\varphi} \, dx \, dt + s\gamma a(1) \int_{t_0}^T \theta(t) w_x^2(t, 1) \, dt \right). \tag{17}
\]
The operators \( P_s^+ \) and \( P_s^- \) are not exactly the ones of [3]. However, one can prove that the Carleman estimates do not change.

Using the previous estimate, we will bound the integral \( \int_{Q_{t_0}} \frac{1}{s\theta} w_x^2 \, dx \, dt \). In fact, we have
\[
\frac{1}{\sqrt{s\theta}} w_t = \frac{1}{\sqrt{s\theta}} \left( P_s^- w - 2sda\varphi_x w_x - sd(a\varphi_x)_x w \right) \\
= \frac{1}{\sqrt{s\theta}} P_s^- w - 2\gamma d\sqrt{s\theta} w_x - \gamma d\sqrt{s\theta} w.
\]
Therefore, using the fact that \( \frac{1}{s\theta} \) is bounded, it results
\[
\int_{Q_{t_0}} \frac{1}{s\theta} w_x^2 \, dx \, dt 
\leq C \left( \frac{1}{s\theta} \|P_s^- w\|_{L^2(Q_{t_0})}^2 + \int_{Q_{t_0}} s\theta x^2 w_x^2 \, dx \, dt + \int_{Q_{t_0}} s\theta w^2 \, dx \, dt \right) 
\leq C \left( \|P_s^- w\|_{L^2(Q_{t_0})}^2 + \int_{Q_{t_0}} s\theta \frac{x^2}{a} w_x^2 \, dx \, dt + \int_{Q_{t_0}} s\theta w^2 \, dx \, dt \right). \tag{18}
\]
Since the function \( x \mapsto \frac{x^2}{a} \) is nondecreasing, then one has
\[
\iint_{Q_t} s \theta \frac{x^2}{a} w_x^2 \, dx \, dt \leq \frac{1}{a(1)} \iint_{Q_t} s \theta a w_x^2 \, dx \, dt. \tag{19}
\]
Moreover, in what follows we will also need to estimate \( \iint_{Q_t} s \theta a w^2 \, dx \, dt \). In particular, using Young inequality, we have
\[
\iint_{Q_t} s \theta w^2 \, dx \, dt = \iint_{Q_t} \left( \theta \frac{a^{1/3}}{x^{2/3}} w^2 \right)^{\frac{3}{2}} \left( \theta \frac{x^2}{a} w^2 \right)^{\frac{1}{2}} \, dx \, dt \\
\leq \frac{3}{4} \iint_{Q_t} \theta \frac{a^{1/3}}{x^{2/3}} w^2 \, dx \, dt + \frac{s}{4} \iint_{Q_t} \theta \frac{x^2}{a} w^2 \, dx \, dt.
\]

Let \( p(x) = x^{4/3} a^{1/3} \), then since the function \( x \mapsto \frac{x^2}{a} \) is nondecreasing on \((0, 1)\) one has,
\[
p(x) = a \left( \frac{x^2}{a} \right)^{\frac{1}{3}} \leq C a(x).
\]
Thanks to the Hardy-Poincaré inequality (12), we obtain
\[
\int_0^1 \frac{a^{1/3}}{x^{2/3}} w^2 \, dx = \int_0^1 \frac{p(x)}{x^2} w^2 \, dx \leq C \int_0^1 \frac{a(x)}{x^2} w^2 \, dx \leq C \int_0^1 a(x) w_x^2 \, dx. \tag{20}
\]
This gives,
\[
\iint_{Q_t} s \theta w^2 \, dx \, dt \leq C \iint_{Q_t} \left( s \theta a w_x^2 + s^3 \theta^3 \frac{x^2}{a} w^2 \right) \, dx \, dt \\
\leq C \iint_{Q_t} \left( s \theta a w_x^2 + s^3 \theta^3 \frac{x^2}{a} w^2 \right) \, dx \, dt, \tag{21}
\]
since \( d > 0 \).

From (18)-(21), we get
\[
\iint_{Q_t} \frac{1}{s} \theta^2 w^2 \, dx \, dt \\
\leq C \left( \| P_s w \|_{L^2(Q_{t_0})}^2 + \iint_{Q_{t_0}} s \theta a w_x^2 \, dx \, dt + \iint_{Q_{t_0}} s^3 \theta^3 \frac{x^2}{a} w^2 \, dx \, dt \right). \tag{22}
\]
In a similar way, to bound the integral \( \iint_{Q_{t_0}} s \theta^2 |\eta \psi| w^2 \, dx \, dt \), we have
\[
\iint_{Q_{t_0}} s \theta^2 |\eta \psi| w^2 \, dx \, dt \leq C \iint_{Q_{t_0}} s \theta^2 w^2 \, dx \, dt, \tag{23}
\]
since \( |\eta| \leq T' + t \) and \( |\psi| \leq \gamma d \). By using inequality (20), we infer
\[
\iint_{Q_{t_0}} s \theta^2 w^2 \, dx \, dt = s \iint_{Q_{t_0}} \left( \theta \frac{a^{1/3}}{x^{2/3}} w^2 \right)^{\frac{3}{2}} \left( \theta \frac{x^2}{a} w^2 \right)^{\frac{1}{2}} \, dx \, dt \\
\leq \frac{3}{4} \iint_{Q_{t_0}} \theta \frac{a^{1/3}}{x^{2/3}} w^2 \, dx \, dt + \frac{s}{4} \iint_{Q_{t_0}} \theta \frac{x^2}{a} w^2 \, dx \, dt \\
\leq C \frac{3}{4} \iint_{Q_{t_0}} s \theta a w_x^2 \, dx \, dt + \frac{s}{4} \iint_{Q_{t_0}} \theta \frac{x^2}{a} w^2 \, dx \, dt.
\]
Therefore, since \( d > 0 \), we have for \( s \) large enough
\[
\int_{Q_{t_0}} s \theta^2 |\eta\psi| w^2 \, dx \, dt \leq C \left( \int_{Q_{t_0}} s \theta d a \, w_x^2 \, dx \, dt + \int_{Q_{t_0}} s^3 \theta^3 \frac{x^2}{da(x)} w_2^2 \, dx \, dt \right).
\] (24)
From inequalities (17), (22) and (24), one obtains
\[
\int_{Q_{t_0}} \left( \frac{1}{s \theta} w_t^2 + s \theta^2 |\eta\psi| w^2 + s \theta d a(x) w_x^2 + s^3 \theta^3 \frac{x^2}{da(x)} w_2^2 \right) \, dx \, dt
\leq C \left( \int_{Q_{t_0}} f^2 e^{2s\varphi} \, dx \, dt + s \gamma a(1) \int_{t_0}^T \theta(t) w_2^2(t, 1) \, dt \right).
\]
Consequently, we obtain the estimate (15) which completes the proof. \( \square \)

From the boundary Carleman estimate (15), we will deduce a Carleman estimate for equation (14) on a subregion
\[
\omega' := (\alpha', \beta') \subset \subset \omega_0 \subset \omega.
\] (25)
To this aim, let us set \( \omega'' = (\alpha'', \beta'') \subset \subset \omega' \) and consider a smooth cut-off function \( \xi \in C^\infty([0, 1]) \) such that \( 0 \leq \xi(x) \leq 1 \) for \( x \in (0, 1) \), \( \xi(x) = 1 \) for \( x \in [0, \alpha''[ \) and \( \xi(x) = 0 \) for \( x \in [\beta'', 1] \).

Our first intermediate Carleman estimate is thus the following.

**Proposition 3.** Let \( T > 0 \). Then, there exist two positive constants \( C \) and \( s_0 \) such that, for every \( y^0 \in L^2(0, 1) \), the solution \( y \) of equation (14) satisfies, for all \( s \geq s_0 \)
\[
\int_{Q_{t_0}} \left( \frac{1}{s \theta} y_t^2 + s \theta^2 |\eta\psi| y^2 + s \theta d a(x) y_x^2 + s^3 \theta^3 \frac{x^2}{da(x)} y_2^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt
\leq C \left( \int_{Q_{t_0}} \xi^{-2} f^2 e^{2s\varphi} \, dx \, dt + \int_{\omega'_{t_0}} \left( f^2 + s^2 \theta^2 y^2 \right) e^{2s\varphi} \, dx \, dt \right),
\] (26)
where \( \omega'_{t_0} := (t_0, T) \times \omega' \).

**Proof.** Define \( w := \xi y \) where \( y \) is the solution of (14). Then, the function \( w \) satisfies the following equation
\[
\begin{cases}
    w_t - d(a(x)w_x)_x = \xi f - d(a(x)\xi_x y)_x - da(x)\xi_x y_x, & (t, x) \in Q, \\
    w(t, 1) = 0 \quad \text{and} \quad \begin{cases}
        w(t, 0) = 0, & \text{in case (WD)} \\
        (aw_x)(t, 0) = 0, & \text{in case (SD)}
    \end{cases} & t \in (0, T), \\
    w(0, x) = \xi(x)y^0(x), & x \in (0, 1),
\end{cases}
\] (27)
Therefore, applying the Carleman estimate (15) to equation (27) and using the definition of \( w \), we have
\[
\int_{Q_{t_0}} \left( \frac{1}{s \theta} w_t^2 + s \theta^2 |\eta\psi| w^2 + s \theta d a(x) w_x^2 + s^3 \theta^3 \frac{x^2}{da(x)} w_2^2 \right) e^{2s\varphi} \, dx \, dt
\leq C \int_{Q_{t_0}} \left( \xi^2 f^2 + (d(a(x)\xi_x y)_x + da(x)\xi_x y_x)^2 \right) e^{2s\varphi} \, dx \, dt.
\] (28)
From the definition of $\xi$ and the Caccioppoli inequality (72), we obtain
\[
\int_{Q_{t_0}} \left( d(a(x)\xi y_x) + da(x)\xi y_x \right)^2 e^{2s\varphi} \, dx dt \\
\leq C \int_{\omega^\prime_{t_0}} (y^2 + y_x^2) e^{2s\varphi} \, dx dt \\
\leq C \int_{\omega^\prime_{t_0}} (f^2 + s^2\theta^2 y^2) e^{2s\varphi} \, dx dt.
\]
(29)

Moreover, since $\xi y_x = w_x - \xi y$, then we get
\[
\int_{Q_{t_0}} s\theta da(x)y_x^2 \xi e^{2s\varphi} \, dx dt \\
\leq C \left( \int_{Q_{t_0}} s\theta da(x)w^2 e^{2s\varphi} \, dx dt + \int_{\omega^\prime_{t_0}} s^2\theta^2 y^2 e^{2s\varphi} \, dx dt \right),
\]
(30)
for a positive constant $C$.

Finally, combining (28)-(30) we deduce the desired estimate. \hfill \Box

Proposition 3 gave a Carleman estimate in $(0, \alpha')$. Now, using the non degenerate Carleman estimate of [21, Lemma 1.2] which remains true when we replace $\theta(t) = \frac{1}{t(T-t)}$ by $\theta(t) = \frac{1}{(t-t_0)(T-t)^4}$, we are able to show a Carleman estimate to equation (14) on the interval $(\beta', 1)$. For more details about this modification of the time weight function, we refer to [2, Remark 1].

**Proposition 4.** Let $T > 0$. Then, there exist two positive constants $C$ and $s_0$ such that, for every $y^0 \in L^2(0, 1)$, the solution $y$ of equation (14) satisfies, for all $s \geq s_0$
\[
\int_{Q_{t_0}} \frac{1}{s^2} y_t^2 + s^2\varphi(y^2 + s\theta da(x)y_x^2 + s^3\theta^3 \frac{x^2}{da(x)} y^2) \zeta^2 e^{2s\varphi} \, dx dt \\
\leq C \left( \int_{Q_{t_0}} \zeta^2 f^2 e^{2s\varphi} \, dx dt + \int_{\omega^\prime_{t_0}} (f^2 + s^3\theta^3 y^2) e^{2s\varphi} \, dx dt \right),
\]
(31)
where $\zeta := 1 - \xi$ and $\Phi(t, x) := \theta(t)\Psi(x)$, $\Psi(x) = e^\rho\varphi - e^\rho\|\sigma\|_\infty$, with $\rho > 0$ is a positive constant to be chosen later, $\sigma$ is a $C^2([0, 1])$ function such that $\sigma(x) > 0$ in $(0, 1)$, $\sigma(0) = \sigma(1) = 0$, and $\sigma_x(x) \neq 0$ in $[0, 1] \setminus \tilde{\omega}$, $\tilde{\omega}$ is an arbitrary open subset of $\omega$.

**Proof.** The function $z := \zeta y$ is a solution of the uniformly parabolic equation
\[
\begin{aligned}
z_t - d(a(x)z_x)_x &= \zeta f - d(a(x)\zeta y)_x - da(x)\zeta y_x, & (t, x) \in Q, \\
z(t, 1) &= z(t, 0) = 0, & t \in (0, T), \\
z(0, x) &= \zeta(x)y^0(x), & x \in (0, 1),
\end{aligned}
\]
(32)
since $z$ has its support in $[0, T] \times [\alpha'', 1]$.

Hence, by [21, Lemma 1.2] we have
\[
\int_{Q_{t_0}} \frac{1}{s^2} z_t^2 + s\theta z_x^2 + s^3\theta^3 z^2 \zeta^2 e^{2s\varphi} \, dx dt \\
\leq C \left( \int_{Q_{t_0}} \left( \zeta^2 f^2 + (d(a(x)\zeta y)_x + da(x)\zeta y_x)^2 \right) e^{2s\varphi} \, dx dt + \int_{\omega^\prime_{t_0}} s^3\theta^3 z^2 e^{2s\varphi} \, dx dt \right) \\
\leq C \left( \int_{Q_{t_0}} \zeta^2 f^2 e^{2s\varphi} \, dx dt + \int_{\omega^\prime_{t_0}} (y^2 + y_x^2) e^{2s\varphi} \, dx dt + \int_{\omega^\prime_{t_0}} s^3\theta^3 z^2 e^{2s\varphi} \, dx dt \right).
\]
Therefore, using the Caccioppoli inequality (72) and the definitions of \( z \) and \( \zeta \) we deduce
\[
\iint_{Q_{t_0}} \left( \frac{1}{s^2} \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} + s \frac{\partial z}{\partial x_i} + s^3 \frac{\partial^3 z}{\partial x_i^2} \right) e^{2s \Phi} \, dx \, dt \leq C \left( \iint_{Q_{t_0}} \zeta^2 f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}} f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}} s^3 \theta^2 e^{2s \Phi} \, dx \, dt \right),
\]
(33)

From \( \zeta y_x = z_x - \zeta x_y \) and supp \( \zeta_x \subset \omega'' \), we obtain
\[
\iint_{Q_{t_0}} s \theta^2 \zeta^2 e^{2s \Phi} \, dx \, dt \leq C \left( \iint_{Q_{t_0}} s \theta^2 z_x^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} s \theta^2 y^2 e^{2s \Phi} \, dx \, dt \right)
\leq C \left( \iint_{Q_{t_0}} s \theta^2 z_x^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} s^3 \theta^2 y^2 e^{2s \Phi} \, dx \, dt \right),
\]
(34)

for \( s \) large enough.

Furthermore, using the fact that \( s \theta^2 |\eta\psi|z^2 \leq C s^3 \theta^3 z_2 \), by (33) one has
\[
\iint_{Q_{t_0}} s \theta^2 |\eta\psi|y^2 \zeta^2 e^{2s \Phi} \, dx \, dt = \iint_{Q_{t_0}} s \theta^2 |\eta\psi|z^2 e^{2s \Phi} \, dx \, dt
\leq C \left( \iint_{Q_{t_0}} \zeta^2 f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} s^3 \theta^2 y^2 e^{2s \Phi} \, dx \, dt \right),
\]
(35)

The estimates (33)-(35) lead to
\[
\iint_{Q_{t_0}} \left( \frac{1}{s^2} \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} + s \frac{\partial y}{\partial x_i} + s^3 \frac{\partial^3 y}{\partial x_i^2} \right) \zeta^2 e^{2s \Phi} \, dx \, dt
\leq C \left( \iint_{Q_{t_0}} \zeta^2 f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} s^3 \theta^2 y^2 e^{2s \Phi} \, dx \, dt \right),
\]
(36)

Taking into account the fact that \( da(x) > 0 \) and \( \frac{x^2}{da(x)} > 0 \) in \((a', 1)\), we deduce
\[
\iint_{Q_{t_0}} \left( \frac{1}{s^2} \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} + s \frac{\partial y}{\partial x_i} + s^3 \frac{\partial^3 y}{\partial x_i^2} \right) \zeta^2 e^{2s \Phi} \, dx \, dt
\leq C \left( \iint_{Q_{t_0}} \zeta^2 f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} f^2 e^{2s \Phi} \, dx \, dt + \iint_{\omega_{t_0}'} s^3 \theta^2 y^2 e^{2s \Phi} \, dx \, dt \right),
\]
(37)

This proves Proposition 4.

**3.2. Carleman estimate for \( n \)-coupled degenerate equations.** Now, we show the main result of this section, which is the \( \omega \)-Carleman estimate for the coupled system (1). For this purpose, the parameters \( \gamma, \rho \) and \( d \) will be chosen such that
\[
d > 4^n d^*, \quad \rho > \frac{1}{\| \sigma \|_{\infty}} \ln \left( \frac{4^n(d - d^*)}{d - 4^n d^*} \right),
\]
(38)

\[
e^{2\rho \| \sigma \|_{\infty}} < \gamma < \frac{4^n}{(4^n - 1)d} \left( e^{2\rho \| \sigma \|_{\infty}} - e^\rho \| \sigma \|_{\infty} \right),
\]
(39)

where \( n \) is the size of the system (1).
Remark 2. By (38) and proceeding as in [20, Lemma 3.1], it can be shown that the interval \( \left( e_{2n}^{\sigma} \xi_{\infty}, \frac{4^n}{(1-n)!} \left( 2^{2n} \sigma_{\| \sigma \|^2} - e_{2n}^{\sigma} \xi_{\infty} \right) \right) \) is nonempty. We can then choose \( \gamma \) in this interval.

From (38)-(39), we have the following result whose proof can be found in [20, Lemma 3.3].

**Lemma 3.2.** Let the sequence \( \Phi_k \) defined by
\[
\Phi_k = 4^n - k(\Phi - \varphi) + \varphi, \quad k = 1, \ldots, n.
\]

Then, we have
- \( \varphi < \Phi_k < 0, \quad k = 1, \ldots, n. \)
- \( \Phi_n = \Phi < \Phi_{n-1} < \cdots < \Phi_1. \)

Using Propositions 3 and 4 and the Hardy-Poincaré inequality we deduce an intermediate important result which could be used to show a Lipschitz stability estimate for parabolic systems of determining \( n \) term sources from measurements of all components of the solution.

**Theorem 3.3.** There exist two positive constants \( C > 0 \) and \( s_0 > 0 \) such that for all \( (y_1^0, \ldots, y_n^0) \in (L^2(0,1))^n \), the solution \( (y_1, \ldots, y_n) \) of (1) satisfies, for all \( s \geq s_0 \)
\[
\sum_{k=1}^{n} J(y_k) \leq C \sum_{k=1}^{n} \left( \int_{Q_1} f_k^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_1} s^3 \theta^3 y_k^2 e^{2s\varphi} \, dx \, dt \right),
\]
where
\[
J(y) := \int_{Q_1} \left( \frac{1}{s \theta^2} y_k^2 + s \theta^2 |\eta\psi| y^2 + s \theta a(x) y_k^2 + s^3 \theta^3 \frac{x^2}{a(x)} y^2 \right) e^{2s\varphi} \, dx \, dt.
\]

**Proof.** Since \( y_k \) is the solution of the system
\[
\begin{aligned}
\begin{cases}
\partial_t y_k - d_k(a(x)y_{kx})_x = f_k - \sum_{j=1}^{k+1} b_{kj} y_j, & (t, x) \in Q, \\
y_k(t, 0) = 0, & t \in (0, T), \\
y_k(t, 0) = 0, & \text{for (WD)}, \\
(ay_k)(t, 0) = 0, & t \in (0, T), \\
y_k(0, x) = y_k^0(x), & x \in (0, 1),
\end{cases}
\end{aligned}
\]
applying Proposition 3, for \( s \) big enough, we have
\[
\int_{Q_1} \left( \frac{1}{s \theta^2} y_k^2 + s \theta^2 |\eta\psi| y_k^2 + s \theta d_k a(x) y_k^2 + s^3 \theta^3 \frac{x^2}{d_k a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt \
\leq C \sum_{j=1}^{k+1} \int_{Q_1} \xi^2 \left( f_k - \sum_{j=1}^{k+1} b_{kj} y_j \right)^2 e^{2s\varphi} \, dx \, dt \
+ \int_{\omega_1} \left( (f_k - \sum_{j=1}^{k+1} b_{kj} y_j)^2 + s^2 \theta^2 y_k^2 \right) e^{2s\varphi} \, dx \, dt 
\]
\[
\leq C \sum_{j=1}^{k+1} \left( \int_{Q_1} \xi^2 y_j^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_1} y_j^2 e^{2s\varphi} \, dx \, dt \right) 
\]
\[
+ C \left( \int_{Q_1} f_k^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_1} \theta^2 y_k^2 e^{2s\varphi} \, dx \, dt \right).
\]
On the other hand, since $d_k > 0$, we have
\[
\min(1, d_k, \frac{1}{d_k}) \int_{Q_{t_0}} \left( \frac{1}{s\theta} y_k^2 + s\theta^2 |\eta\psi| y_k^2 + s\theta a(x) y_k^2 + s\theta d_k a(x) y_k^2 + s^3 \theta \frac{x^2}{a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt \\
\leq \int_{Q_{t_0}} \left( \frac{1}{s\theta} y_k^2 + s\theta^2 |\eta\psi| y_k^2 + s\theta d_k a(x) y_k^2 + s^3 \theta \frac{x^2}{d_k a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt.
\]

Hence,
\[
\int_{Q_{t_0}} \left( \frac{1}{s\theta} y_k^2 + s\theta^2 |\eta\psi| y_k^2 + s\theta d_k a(x) y_k^2 + s^3 \theta \frac{x^2}{a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt \\
\leq C \sum_{j=1}^{k+1} \left( \int_{Q_{t_0}} \xi^2 \eta_j^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_{t_0}} \eta_j^2 e^{2s\varphi} \, dx \, dt \right) + C \left( \int_{Q_{t_0}} \xi^2 \eta_j^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_{t_0}} s^2 \theta^2 \eta_j^2 e^{2s\varphi} \, dx \, dt \right).
\]

Moreover, since $x \mapsto \frac{x^2}{\sqrt{2\pi}}$ is nondecreasing, we have
\[
\int_{Q_{t_0}} \xi^2 \eta_j^2 e^{2s\varphi} \, dx \, dt \leq \frac{1}{a(1)} \int_{Q_{t_0}} \frac{a(x)}{x^2} \xi^2 \eta_j^2 e^{2s\varphi} \, dx \, dt.
\]

Applying Hardy-Poincaré inequality to $v := \xi y_j e^{s\varphi}$, one has
\[
\int_{Q_{t_0}} \xi^2 \eta_j^2 e^{2s\varphi} \, dx \, dt \leq C \int_{Q_{t_0}} a(x) v_j^2 \, dx \, dt,
\]
and by $v_x = \xi y_j e^{s\varphi} + \xi s \theta \frac{x}{a(x)} y_j e^{s\varphi} + \xi x y_j e^{s\varphi}$, we obtain
\[
\int_{Q_{t_0}} \xi^2 \eta_j^2 e^{2s\varphi} \, dx \, dt \leq C \left( \int_{Q_{t_0}} a(x) \left( \xi y_j e^{s\varphi} + \xi s \theta \frac{x}{a(x)} y_j e^{s\varphi} + \xi x y_j e^{s\varphi} \right)^2 \, dx \, dt \right)
\]
\[
\leq C \left( \int_{Q_{t_0}} \xi^2 a(x) y_j^2 e^{2s\varphi} \, dx \, dt + \int_{Q_{t_0}} \xi^2 s^2 \theta^2 \frac{x^2}{a(x)} y_j^2 e^{2s\varphi} \, dx \, dt + \int_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi} \, dx \, dt \right).
\]

Now, using the fact that $\theta$ is bounded from below and since $\text{supp}(\xi_x) \subset \omega'$, one has
\[
\int_{Q_{t_0}} \xi^2 \eta_j^2 e^{2s\varphi} \, dx \, dt \\
\leq C \left( \int_{Q_{t_0}} \xi^2 \theta a(x) y_j e^{2s\varphi} \, dx \, dt + \int_{Q_{t_0}} \xi^2 \theta^2 \frac{x^2}{a(x)} y_j^2 e^{2s\varphi} \, dx \, dt \right) + C \int_{\omega_{t_0}} s\theta y_j^2 e^{2s\varphi} \, dx \, dt.
\]
Therefore, by taking the Carleman parameter \( s \) large enough, we obtain

\[
\sum_{k=1}^{n} \int_{Q_{t_k}} \left( \frac{1}{s^2} y_k^2 + s \left| \psi \right| y_k^2 + s \theta a(x) y_k^2 + s^3 \frac{x^2}{a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt \\
\leq \frac{1}{2^n} \sum_{k=1}^{n} \sum_{j=1}^{k+1} \left( \int_{Q_{t_k}} s \theta a(x) y_k^2 + s^3 \frac{x^2}{a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt \\
+ C \sum_{k=1}^{n} \left( \int_{Q_{t_k}} f_k^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_{t_k}} s^2 \theta^2 y_k^2 e^{2s\varphi} \, dx \, dt \right) \\
\leq \frac{1}{2^n} \sum_{k=1}^{n} \left( \int_{Q_{t_k}} s \theta a(x) y_k^2 + s^3 \frac{x^2}{a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt \\
+ C \sum_{k=1}^{n} \left( \int_{Q_{t_k}} f_k^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_{t_k}} s^2 \theta^2 y_k^2 e^{2s\varphi} \, dx \, dt \right),
\]

and by this, it results that

\[
\sum_{k=1}^{n} \int_{Q_{t_k}} \left( \frac{1}{s^2} y_k^2 + s \left| \psi \right| y_k^2 + s \theta a(x) y_k^2 + s^3 \frac{x^2}{a(x)} y_k^2 \right) \xi^2 e^{2s\varphi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \left( \int_{Q_{t_k}} f_k^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_{t_k}} s^2 \theta^2 y_k^2 e^{2s\varphi} \, dx \, dt \right). \tag{45}
\]

Similarly, applying Proposition 4 to \( y_k \), the solution of (43), and using the Hardy-Poincaré inequality, we obtain the estimate

\[
\sum_{k=1}^{n} \int_{Q_{t_k}} \left( \frac{1}{s^2} y_k^2 + s \left| \psi \right| y_k^2 + s \theta a(x) y_k^2 + s^3 \frac{x^2}{a(x)} y_k^2 \right) \zeta^2 e^{2s\varphi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \left( \int_{Q_{t_k}} f_k^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_{t_k}} s^3 \theta^3 y_k^2 e^{2s\varphi} \, dx \, dt \right). \tag{46}
\]

Since \( e^{2s\varphi} \leq e^{2s\varphi} \), \( \frac{1}{2} \leq \xi^2 + \zeta^2 \leq 1 \) and \( s \) is bounded from below, then by adding (46) and (47) we obtain

\[
\sum_{k=1}^{n} \int_{Q_{t_k}} \left( \frac{1}{s^2} y_k^2 + s \left| \psi \right| y_k^2 + s \theta a(x) y_k^2 + s^3 \frac{x^2}{a(x)} y_k^2 \right) e^{2s\varphi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \left( \int_{Q_{t_k}} f_k^2 e^{2s\varphi} \, dx \, dt + \int_{\omega_{t_k}} s^3 \theta^3 y_k^2 e^{2s\varphi} \, dx \, dt \right),
\]

for \( s \) large enough. This ends the proof. \( \square \)
Let us recall that our goal is to determine the term sources \( f_k, \ k \in \{1, \ldots, n\} \) from measurements of one component of the solution using data on a prescribed subregion \( \omega \) of \((0, 1)\). To this aim, the key point is given by the next lemma which play a crucial role to absorb the observation on the components \( y_k, \ k \in \{2, \ldots, n\} \).

**Lemma 3.4.** Assume that Hypothesis (6) is satisfied. Let \( \varepsilon > 0, \ k \in \{2, \ldots, n\} \) and two open sets \( \mathcal{O}_0 \) and \( \mathcal{O}_1 \) such that \( \omega' \subset \mathcal{O}_1 \subset \mathcal{O}_0 \subset \omega_0 \), where we recall that \( \omega' \) is given in (25). Then, for all \( l \in \mathbb{N} \), there exist positive constants \( C_k, \mathcal{I} \) and \( J \) such that every solution \( (y_1, \ldots, y_n) \) to (1) satisfies

\[
\mathcal{L}_{\mathcal{O}_1}(l, \Phi_k, y_k) \leq \varepsilon \sum_{j=k}^{k+1} \mathcal{J}(y_j) + C_k \left( \sum_{j=1}^{k-2} \mathcal{L}_{\mathcal{O}_0}(\mathcal{I}, \Phi_{k-1}, y_j) + (1 + \frac{1}{\varepsilon}) \mathcal{L}_{\mathcal{O}_0}(J, \Phi_{k-1}, y_{k-1}) \right) + \int_{Q_{t_0}} s^{2l-2} \theta^{2l-2} f_{k-1} e^{2s\phi_{k-1}} \, dx \, dt,
\]

with \( \mathcal{L}_B(l, \phi, y) = \int_{B \times (t_0, T)} s^l \theta^l y^2 e^{2s\phi} \, dx \, dt \), \( \mathcal{I} = \max(3, 2l-3) \) and \( J = \max(3, 2l+1, 4l-5) \). In this inequality, we take \( y_{k+1} \equiv 0 \) when \( k = n \).

**Proof.** The proof follows the one of [20, Lemma 3.7], but it is different due to the fact that we have to deal here with a non-homogeneous system. For this reason we only point out the difference that appear when we consider the nonhomogeneous system (1) in place of homogeneous one. Let us consider a nonnegative smooth cut-off function \( \chi \in C^\infty(0, 1) \), such that

\[
0 \leq \chi(x) \leq 1, \quad \chi(x) = \begin{cases} 1, & x \in \mathcal{O}_1, \\ 0, & x \in (0, 1) \setminus \mathcal{O}_0, \end{cases}
\]

and \( \frac{\chi_x}{\sqrt{\chi}}, \frac{\chi_{xx}}{\sqrt{\chi}} \in L^\infty(0, 1) \).

Observe that, the \( k-1 \)th equation of the system (1) can be written as

\[
b_{k-1}y_k = -y_{k-1,t} + d_k(a(x)y_{k-1,x})_x - \sum_{j=1}^{k-1} b_{k-1,j}y_j + f_{k-1}.
\]

Multiplying the above equation by \( s^l \theta^l \chi e^{2s\phi} y_k \) and integrating on \( Q_{t_0} \), since \( \mathcal{O}_1 \subset \omega_0 \) by (6), it follows that

\[
b_0 \mathcal{L}_{\mathcal{O}_1}(l, \Phi_k, y_k) \leq \int_{Q_{t_0}} b_{k-1} s^l \theta^l \chi e^{2s\phi} y_k^2 \, dx \, dt
\]

\[
= - \sum_{j=1}^{k-1} \int_{Q_{t_0}} b_{k-1,j} s^l \theta^l \chi e^{2s\phi} y_j \, dx \, dt + \int_{Q_{t_0}} d_k(a(x)y_{k-1,x})_x s^l \theta^l \chi e^{2s\phi} y_k \, dx \, dt
\]

\[
- \sum_{j=1}^{k-1} \int_{Q_{t_0}} b_{k-1,j} s^l \theta^l \chi e^{2s\phi} y_k \, dx \, dt + \int_{Q_{t_0}} f_{k-1} s^l \theta^l \chi e^{2s\phi} y_k \, dx \, dt. \tag{50}
\]
Moreover, proceeding as in [20, Lemma 3.7], one can get

\[ K_1 + K_2 + K_3 \]

\[ \leq \frac{\varepsilon}{2} \sum_{j=k}^{k+1} \int_{Q_{t_0}} \left( s\theta a(x)y_j^2 + s^3 \theta^3 \frac{x^2}{a(x)} y_j^2 \right) e^{2s\varphi} \, dx \, dt \]

\[ + C_k \left( \sum_{j=1}^{k-2} \mathcal{L}_{O_o}(\hat{l}, \Phi_{k-1}, y_j) + (1 + \frac{1}{\varepsilon}) \mathcal{L}_{O_o}(J, \Phi_{k-1}, y_{k-1}) \right), \]  

(51)

where \( \hat{l} = \max(3, 2l - 3) \) and \( J = \max(3, 2l + 1, 4l - 5) \). On the other hand, for \( K_4 \) we have

\[ K_4 = \int_{Q_{t_0}} f_{k-1} s^l \theta^l \chi e^{2s\Phi_k} y_k \, dx \, dt \]

\[ \leq \int_{Q_{t_0}} s^{2l-1} \theta^{2l-1} f_{k-1} e^{2s\Phi_k} \, dx \, dt + \frac{\varepsilon}{4} \int_{Q_{t_0}} s^2 \theta^2 \chi^2 y_k^2 e^{2s\varphi} \, dx \, dt. \]

But by Lemma 3.2 we know that

\[ e^{2s(\Phi_k - \varphi)} \leq e^{2s\Phi_k - 1}. \]

(52)

Consequently, by (52) we have

\[ K_4 \leq \int_{Q_{t_0}} s^{2l-2} \theta^{2l-2} f_{k-1}^2 e^{2s\Phi_k - 1} \, dx \, dt + \frac{\varepsilon}{4} \int_{Q_{t_0}} s^2 \theta^2 \chi^2 y_k^2 e^{2s\varphi} \, dx \, dt. \]

Now, at this level, using the fact that \( \text{supp} \chi \subset O_0 \) and thus \( \frac{a(x)}{x^2} \) is bounded in \( \overline{O_0} \), then for \( s \) large enough,

\[ K_4 \leq \int_{Q_{t_0}} s^{2l-2} \theta^{2l-2} f_{k-1}^2 e^{2s\Phi_k - 1} \, dx \, dt + \frac{\varepsilon}{2} \int_{Q_{t_0}} s^3 \theta^3 \frac{x^2}{a(x)} y_k^2 e^{2s\varphi} \, dx \, dt. \]  

(53)

Putting together inequalities (50), (51) and (53), we finally obtain

\[ \mathcal{L}_{O_o}(l, \Phi_k, y_k) \]

\[ \leq \varepsilon \sum_{j=k}^{k+1} \mathcal{J}(y_j) + C_k \left( \sum_{j=1}^{k-2} \mathcal{L}_{O_o}(\hat{l}, \Phi_{k-1}, y_j) + (1 + \frac{1}{\varepsilon}) \mathcal{L}_{O_o}(J, \Phi_{k-1}, y_{k-1}) \right) \]

\[ + \int_{Q_{t_0}} s^{2l-2} \theta^{2l-2} f_{k-1}^2 e^{2s\Phi_k - 1} \, dx \, dt. \]

As a consequence of (41) and Lemma 3.4, we deduce the following fundamental Carleman estimate for the system (1) with one observed component.

**Theorem 3.5.** Assume Hypothesis (6). Then, there exist \( R \geq 3 \) (only depending on \( n \)), two positive constants \( C \) and \( s_0 \) such that every solution \((y_1, ..., y_n)\) of (1) satisfies, for all \( s \geq s_0 \),

\[ \sum_{k=1}^{n} \mathcal{J}(y_k) \leq C \left( \sum_{k=1}^{n} \int_{Q_{t_0}} s^R \theta^R f_{k}^2 e^{2s\Phi_k} \, dx \, dt + \int_{\omega_{t_0}} y_k^2 \, dx \, dt \right). \]  

(54)
Proof. To prove Theorem 3.5 we will follow the same argument given in [22] and which is used to obtain the null controllability property for nondegenerate cascade parabolic systems with one control force. Given \( \omega_0 \subset \omega \), we choose \( \omega' \subset \subset \omega_0 \) and let \( Y = (y_1, ..., y_n)^* \) be the solution to (1) associated to \( Y^0 \in L^2(0,1)^n \). From the definition of \( L_B(l, \phi, y) \) and recalling that \( \Phi_n = \Phi \), by (41) we have

\[
\sum_{k=1}^{n} \mathcal{J}(y_k) \leq C \sum_{k=1}^{n} \left( \mathcal{L}_{\omega'}(3, \Phi_n, y_k) + \iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n} \, dx \, dt \right).
\]

For \( k = 2, ..., n \), let us introduce the following sequence \( (\tilde{O}_k)_{2 \leq k \leq n} \) of open sets, such that \( \omega' \subset \subset \tilde{O}_n \subset \subset \tilde{O}_{n-1} \subset \subset ... \subset \subset \tilde{O}_2 \subset \subset \omega_0 \). We begin by applying formula (48), for \( k = n \), \( l = 3 \), \( O_1 = \omega' \), \( O_0 = \tilde{O}_n \) and \( \epsilon = \frac{1}{2C} \) (with \( C \) is the positive constant appearing in (55)). Thus, from (55), we obtain

\[
\sum_{k=1}^{n} \mathcal{J}(y_k) \leq C \sum_{k=1}^{n} \left[ \sum_{n=1}^{n-2} \mathcal{L}_{\omega'}(3, \Phi_{n-1}, y_k) + \frac{1}{2C} \mathcal{J}(y_n) \right.
\]
\[
+C_n \left( \sum_{k=1}^{n} \mathcal{L}_{\tilde{O}_n}(l_1, \Phi_{n-1}, y_k) + (1 + 2C) \mathcal{L}_{\tilde{O}_n}(J, \Phi_{n-1}, y_{n-1}) \right)
\]
\[
+ \iint_{Q_{t_0}} s^{2l-2} e^{2s-2} f_{n-1}^2 e^{2s\Phi_{n-1}} \, dx \, dt + \sum_{k=1}^{n} \iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n} \, dx \, dt \right].
\]

For \( l_1 := \max(3, l, J) \), using the fact that \( \Phi_n \leq \Phi_{n-1} \) and \( \mathcal{L}_{\omega'}(l_1, \Phi_{n-1}, y_k) \leq \mathcal{L}_{\tilde{O}_n}(l_1, \Phi_{n-1}, y_k) \), we deduce that

\[
\sum_{k=1}^{n} \mathcal{J}(y_k) \leq \tilde{C}_n \sum_{k=1}^{n-1} \mathcal{L}_{\tilde{O}_n}(l_1, \Phi_{n-1}, y_k) + C \iint_{Q_{t_0}} s^{R_1} e^{R_1} f_{n-1}^2 e^{2s\Phi_{n-1}} \, dx \, dt
\]
\[
+ C \sum_{k=1}^{n} \iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n} \, dx \, dt,
\]

where \( \tilde{C}_n \) is a new positive constant and \( R_1 = 2l - 2 \). Observe that in (55) we have eliminated from the right hand side the local term involving \( y_n \). We can go on applying (48) for \( k = n-1 \), \( l = l_1 \), \( O_1 = \tilde{O}_n \), \( O_0 = \tilde{O}_{n-1} \) and \( \epsilon = \frac{1}{2C_n} \) and eliminate in (56) the local term \( \mathcal{L}_{\tilde{O}_n}(J_1, \Phi_{n-1}, y_{n-1}) \). By (finite) iteration of this argument we obtain

\[
\sum_{k=1}^{n} \mathcal{J}(y_k) \leq \tilde{C}_2 \mathcal{L}_{\tilde{O}_2}(l_{n-1}, \Phi_1, y_1) + C \sum_{k=1}^{n-1} \iint_{Q_{t_0}} s^{R_{n-1}} e^{R_{n-1}} f_k^2 e^{2s\Phi_k} \, dx \, dt
\]
\[
+ C \sum_{k=1}^{n} \iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n} \, dx \, dt,
\]

for some positive constants \( l_{n-1} \) and \( R_{n-1} \).

Now, since \( \Phi_n = \Phi \leq \Phi_k \) and \( \sup_{(t,x) \in Q} s^{l_{n-1}} e^{2s\Phi_1} < +\infty \), choosing \( s \) large enough we readily deduce

\[
\sum_{k=1}^{n} \mathcal{J}(y_k) \leq C \left( \sum_{k=1}^{n} \iint_{Q_{t_0}} s^{R_{n-1}} e^{R_{n-1}} f_k^2 e^{2s\Phi_k} \, dx \, dt + \iint_{Q_{t_0}} y_1^2 \, dx \, dt \right).
\]

Finally, by setting \( R = R_{n-1} \) in the previous estimate, we end the proof. \( \square \)
4. Stability estimate and uniqueness for the inverse source problem. In this section, we establish a stability and a uniqueness result using certain ideas from [23] and [15]. More precisely, we obtain an inequality which estimates the term sources \( f_k \), \( k \in \{1, \ldots, n\} \) over the entire domain \((0, 1)\) with an upper bound given by some Sobolev norm of the solution \( Y \) at some fixed time \( T' \in (0, T) \) and the partial knowledge of \( y_1 \) and \( y_t \) over the subdomain \( \omega \subset (0, 1) \). In proving these kinds of stability estimates, the global Carleman estimate obtained in Theorem 3.5 will play a crucial part along with certain energy estimates.

**Proof of theorem 1.1.** Let us introduce \( Z := Y_t \) where \( Y = (y_k)_{1 \leq k \leq n} \) is the solution of (4). Then, thanks to Proposition 2, \( Z = (z_k)_{1 \leq k \leq n} \in L^2(t_0, T; D(A)) \cap H^1(t_0, T; L^2(0, 1)^n) \) and satisfies

\[
\begin{cases}
Z_t - DAZ + BZ = F_t, & (t, x) \in Q_{t_0}, \\
CZ = 0, & (t, x) \in \Sigma_{t_0}, \\
Z(0, x) = Y_t(0, x), & x \in (0, 1),
\end{cases}
\tag{57}
\]

where \( \Sigma_{t_0} := (t_0, T) \times \{0, 1\} \) and \( F_t = (f_{1t}, f_{2t}, \ldots, f_{nt})^* \).

Applying the Carleman estimate of Theorem 3.5 to problem (57), we get:

\[
\sum_{k=1}^n \mathcal{J}(z_k) \leq C \left( \sum_{k=1}^n \int_{Q_{t_0}} s^R y^2 \int_{t_0}^{T} e^{2s \phi_k} \, dx \, dt + \int_{\omega_{t_0}} z^2 \, dx \, dt \right). \tag{58}
\]

Let us note that, if we replace \( \int_{Q_{t_0}} s^R y^2 \int_{t_0}^{T} e^{2s \phi_k} \, dx \, dt \) by \( \int_{Q_{t_0}} f_k^2 \int_{t_0}^{T} e^{2s \phi_k} \, dx \, dt \), the inequality (58) would be the kind of estimate that one would obtain when dealing with the more standard inverse problem that consists in retrieving the source term \( f \) in the scalar equation \( y_t - (ay_x)_x = f \). Hence the next step mainly consists in adapting the reasoning of [15] to the present case, taking into account this extra term and the coupling of the equations. We shall first prove the following lemma.

**Lemma 4.1.** There exists a constant \( C = C(t_0, T) > 0 \) such that for every \( k \in \{1, \ldots, n\} \)

\[
\int_0^1 \left( z_k(T') + \sum_{j=1}^{k+1} b_{kj} y_j(T') \right)^2 e^{2s \varphi(T', x)} \, dx \leq C \left( \mathcal{J}(z_k) + \sum_{j=1}^{k+1} \mathcal{J}(y_j) \right). \tag{59}
\]

**Proof of Lemma 4.1.** Since

\[
\lim_{t \to t_0} \left( z_k(t) + \sum_{j=1}^{k+1} b_{kj} y_j(t) \right)^2 e^{2s \varphi(t, x)} \, dx = 0, \text{ for a.a. } x \in (0, 1),
\]

we can write

\[
\int_0^1 \left( z_k(T') + \sum_{j=1}^{k+1} b_{kj} y_j(T') \right)^2 e^{2s \varphi(T', x)} \, dx
\]

\[
= \int_0^1 \int_{t_0}^{T'} \frac{\partial}{\partial t} \left( z_k(t) + \sum_{j=1}^{k+1} b_{kj} y_j(t) \right)^2 e^{2s \varphi} \, dt \, dx
\]

\[
= \int_0^1 \int_{t_0}^{T'} \left( z_k(t) + \sum_{j=1}^{k+1} b_{kj} y_j(t) \right)^2 e^{2s \varphi} \, dt \, dx
\]
Thus, (60)-(64) yield the estimate (59).

On the other hand, since 
\[ |z_k + \sum_{j=1}^{k+1} b_{kj}y_j| \leq \frac{1}{s\theta} \left( z_{kt} + \sum_{j=1}^{k+1} b_{kj}y_{jt} \right) e^{2s\varphi} \] 

Moreover, applying once more Young’s inequality, one has

\[ I_2 = \int_0^1 \int_{t_0}^{t'} 2s\varphi_t \left( z_k + \sum_{j=1}^{k+1} b_{kj}y_j \right)^2 e^{2s\varphi} \, dx \, dt. \] (60)

Using Young’s inequality and taking into account the fact that \( b_{kj} \in L^\infty(Q) \), we estimate

\[ I_1 = 2 \int_0^1 \int_{t_0}^{t'} \sqrt{s\theta} \left( z_k + \sum_{j=1}^{k+1} b_{kj}y_j \right) \frac{1}{\sqrt{s\theta}} \left( z_{kt} + \sum_{j=1}^{k+1} b_{kj}y_{jt} \right) e^{2s\varphi} \, dx \, dt \]

\[ \leq C \int_{Q_{t_0}} \left( s\theta z_k^2 + \sum_{j=1}^{k+1} s\theta y_j^2 + \sum_{j=1}^{k+1} \frac{y_{jt}^2}{s\theta} \right) e^{2s\varphi} \, dx \, dt. \] (61)

Moreover, applying once more Young’s inequality, one has

\[ \int_{Q_{t_0}} s\theta y_j^2 e^{2s\varphi} \, dx \, dt = \int_{Q_{t_0}} s\theta \left( \frac{a(x)}{x^2} \right) \left( \frac{y_j^2 e^{2s\varphi}}{\theta} \right) \, dx \, dt \]

\[ \leq \frac{3}{4} \int_{Q_{t_0}} s\theta \left( \frac{a(x)}{x^2} \right) y_j^2 e^{2s\varphi} \, dx \, dt + \frac{1}{4} \int_{Q_{t_0}} s\theta \frac{x^2}{a(x)} y_j^2 e^{2s\varphi} \, dx \, dt. \]

Arguing as in (20), by the Hardy-Poincaré inequality applied to \( y_j e^{s\varphi} \), we obtain

\[ \int_{Q_{t_0}} s\theta y_j^2 e^{2s\varphi} \, dx \, dt \leq C \int_{Q_{t_0}} \left( s\theta a(x) |y_j^2 + s\theta \psi y_j|^2 + s^3 \theta x^2 a(x) y_j^2 \right) e^{2s\varphi} \, dx \, dt \]

\[ \leq C \int_{Q_{t_0}} \left( s\theta a(x) y_j^2 + s^3 \theta x^2 a(x) y_j^2 \right) e^{2s\varphi} \, dx \, dt. \] (62)

Similarly, we have

\[ \int_{Q_{t_0}} s\theta z_k^2 e^{2s\varphi} \, dx \, dt \leq C \int_{Q_{t_0}} \left( s\theta a(x) z_k^2 + s^3 \theta x^2 a(x) z_k^2 \right) e^{2s\varphi} \, dx \, dt. \] (63)

On the other hand, since \( |\varphi_t| \leq C \theta^2 |\eta \psi| \), it follows that

\[ I_2 = \int_0^1 \int_{t_0}^{t'} 2s\varphi_t \left( z_k + \sum_{j=1}^{k+1} b_{kj}y_j \right)^2 e^{2s\varphi} \, dx \, dt \]

\[ \leq C \left( \int_{Q_{t_0}} s\theta^2 |\eta \psi| z_k^2 e^{2s\varphi} \, dx \, dt + \sum_{j=1}^{k+1} \int_{Q_{t_0}} s\theta^2 |\eta \psi| y_j^2 e^{2s\varphi} \, dx \, dt \right). \] (64)

Thus, (60)-(64) yield the estimate (59). \( \square \)

Going back to the proof of Theorem 1.1, we note that the \( k \) equation of the system (4) can be written as

\[ z_k(T', .) - (a(x)y_{kx})(T', .) + \sum_{j=1}^{k+1} b_{kj}y_j(T', .) = f_k(T', .) \text{ in } (0, 1). \]
Therefore,
\[
\int_{Q_{t_0}} s^{R+1} q^{R+1} f_k^2(T', x) e^{2s} \, dx \, dt \\
\leq C \left( \int_{Q_{t_0}} (a(x)y_{xk})^2(T', x) s^{R+1} q^{R+1} e^{2s} \, dx \, dt + \int_{Q_{t_0}} \left( z_k(T') + \sum_{j=1}^{k+1} b_{kj} y_j(T') \right)^2 s^{R+1} q^{R+1} e^{2s} \, dx \, dt \right).
\]

In particular, since
\[
\sup_{(t,x) \in Q} s^{R+1} q^{R+1}(t)e^{2s} < \infty,
\]
the previous estimate yields
\[
\int_{Q_{t_0}} s^{R+1} q^{R+1} f_k^2(T', x) e^{2s} \, dx \, dt \\
\leq C \left( \left\| (a y_{xk}x)_{x}(T', \cdot) \right\|_{L^2(0,1)}^2 + \int_0^1 \left( z_k(T') + \sum_{j=1}^{k+1} b_{kj} y_j(T') \right)^2 e^{2s} \, dx \right).
\]

Using the boundedness of \( e^{-2s} \), we can prove that there exists a positive constant \( C \) such that
\[
\int_{Q_{t_0}} s^{R+1} q^{R+1} f_k^2(T', x) e^{2s} \, dx \, dt \\
\leq C \left( \left\| (a y_{xk}x)_{x}(T', \cdot) \right\|_{L^2(0,1)}^2 + \int_0^1 \left( z_k(T') + \sum_{j=1}^{k+1} b_{kj} y_j(T') \right)^2 e^{2s} \, dx \right). \quad (65)
\]

Finally, putting (54) and (58) into (65), we get
\[
\sum_{k=1}^n \int_{Q_{t_0}} s^{R+1} q^{R+1} f_k^2(T', x) e^{2s} \, dx \, dt \\
\leq C \left( \sum_{k=1}^n \left\| (a y_{xk}x)_{x}(T', \cdot) \right\|_{L^2(0,1)}^2 + \sum_{k=1}^n \left( J(z_k) + \sum_{j=1}^{k+1} J(y_j) \right) \right) \\
\leq C \left( \sum_{k=1}^n \left\| (a y_{xk}x)_{x}(T', \cdot) \right\|_{L^2(0,1)}^2 + \sum_{k=1}^n \left( J(z_k) + J(y_k) \right) \right) \\
\leq C \left( \sum_{k=1}^n \left\| (a y_{xk}x)_{x}(T', \cdot) \right\|_{L^2(0,1)}^2 + \sum_{k=1}^n \int_{Q_{t_0}} s^{R} q^{R} (f_k^2 + f_k^2) e^{2s} \, dx \, dt \\
+ \left\| y_1^2 \right\|_{L^2(0,1)} + \left\| y_{1t} \right\|_{L^2(0,1)} \right) \quad (66)
\]

Next, using the assumption that \( f_1, \ldots, f_n \in S(C_0) \), one has
\[
\left| f_k(t, x) \right| \leq C_0 \left| f_k(T', x) \right|,
\]
and
\[
\left| f_k(t, x) \right| \leq \left| f_k(T', x) \right| + \int_{T'}^t \left| f_k(t, x) \right| ds \leq C \left| f_k(T', x) \right|. \quad (67)
\]
Substituting this into (66), we obtain
\[
\sum_{k=1}^{n} \int_{Q_{t_0}} s^{R+1} \theta^{R+1} f_k^2(T', x) e^{2s \Phi_k} \, dx \, dt \\
\leq C \left( \sum_{k=1}^{n} \left\| (ay_{kx})_x(T', \cdot) \right\|_{L^2(0,1)}^2 + \sum_{k=1}^{n} \int_{Q_{t_0}} s^{R} \theta f_k^2(T', x) e^{2s \Phi_k} \, dx \, dt \\
+ \| y_1 \|_{L^2(\omega_{t_0})}^2 + \| y_1t \|_{L^2(\omega_{t_0})}^2 \right) .
\]

By choosing \( s \) large enough we can absorb the second term on the right-hand side and obtain
\[
\sum_{k=1}^{n} \int_{Q_{t_0}} s^{R+1} \theta^{R+1} f_k^2(T', x) e^{2s \Phi_k} \, dx \, dt \\
\leq C \left( \sum_{k=1}^{n} \left\| (ay_{kx})_x(T', \cdot) \right\|_{L^2(0,1)}^2 + \| y_1 \|_{L^2(\omega_{t_0})}^2 + \| y_1t \|_{L^2(\omega_{t_0})}^2 \right) .
\]

Then we observe that, for a fixed \( \varepsilon > 0 \), such that
\[
t_0 < T' - \varepsilon < T' < T' + \varepsilon < T,
\]
we may write
\[
\int_{T' - \varepsilon}^{T' + \varepsilon} \int_{0}^{1} s^{R+1} \theta^{R+1} f_k^2(T', x) e^{2s \Phi_k} \, dx \, dt \leq \int_{Q_{t_0}} s^{R+1} \theta^{R+1} f_k^2(T', x) e^{2s \Phi_k} \, dx \, dt.
\]

Consequently,
\[
2\varepsilon \kappa \sum_{k=1}^{n} \int_{0}^{1} f_k^2(T', x) \, dx \leq C \left( \sum_{k=1}^{n} \left\| (ay_{kx})_x(T', \cdot) \right\|_{L^2(0,1)}^2 \\
+ \| y_1 \|_{L^2(\omega_{t_0})}^2 + \| y_1t \|_{L^2(\omega_{t_0})}^2 \right) ,
\]
where
\[
\kappa = \min_{(t,x) \in [T' - \varepsilon, T' + \varepsilon] \times [0,1]} s^{R+1} \theta^{R+1} e^{2s \Phi_k} > 0.
\]

Thus, in view of (67), we conclude that
\[
\sum_{k=1}^{n} \int_{Q} f_k^2(t, x) \, dx \, dt \leq C \sum_{k=1}^{n} \int_{Q} f_k^2(T', x) \, dx \, dt \\
= CT \sum_{k=1}^{n} \int_{0}^{1} f_k^2(T', x) \, dx \\
\leq C \left( \sum_{k=1}^{n} \left\| (ay_{kx})_x(T', \cdot) \right\|_{L^2(0,1)}^2 + \| y_1 \|_{L^2(\omega_{t_0})}^2 + \| y_1t \|_{L^2(\omega_{t_0})}^2 \right) ,
\]
for some constant \( C = C(T, t_0, C_0) > 0 \). This gives (7) and completes the proof of Theorem 1.1. \( \square \)
Proof of Theorem 1.2. Theorem 1.2 follows directly from Theorem 1.1: if we consider two source terms $\hat{F} = (\hat{f}_k)_{1 \leq k \leq n}$ with $\hat{f}_k = \hat{g}_k r_k \in \mathcal{E}$ and $\hat{F} = (\hat{f}_k)_{1 \leq k \leq n}$ with $\hat{f}_k = \hat{g}_k r_k \in \mathcal{E}$, and if we denote by $\hat{Y} = (\hat{y}_1, ..., \hat{y}_n)^*$ and $\hat{Y} = (\hat{y}_1, ..., \hat{y}_n)^*$ the associated solutions of (4), then $Z = (z_1, ..., z_n)^* := (\hat{y}_1 - \hat{y}_1, ..., \hat{y}_n - \hat{y}_n)^*$ is the solution of the problem

$$
\begin{cases}
\partial_t z_1 - d_1(a(x)z_{1x})_x + \sum_{j=1}^2 b_{1j}z_j = (\hat{g}_1 - \hat{g}_1)r_1, & (t, x) \in Q, \\
\partial_t z_2 - d_2(a(x)z_{2x})_x + \sum_{j=1}^3 b_{2j}z_j = (\hat{g}_2 - \hat{g}_2)r_2, & (t, x) \in Q, \\
\vdots \\
\partial_t z_n - d_n(a(x)z_{nx})_x + \sum_{j=1}^n b_{nj}z_j = (\hat{g}_n - \hat{g}_n)r_n, & (t, x) \in Q, \\
z_k(t, 0) = 0, & 1 \leq k \leq n, \\
(az_{kx})(t, 0) = 0, & \text{in the weakly degenerate case (WD)}, \\
t \in (0, T), \\
z_1(0, x) = 0, ..., z_n(0, x) = 0, & x \in (0, 1).
\end{cases}
$$

(71)

One can easily check that $\hat{f}_k - \hat{f}_k \in \mathcal{S}(C_0)$. Indeed, for $\hat{f}_k = \hat{g}_k r_k \in \mathcal{E}$ and $\hat{f}_k = \hat{g}_k r_k \in \mathcal{E}$ we have, $\hat{f}_k - \hat{f}_k \in H^1(0, 1, L^2(0, 1))$. Then for all $t \in [0, T]$ and for a.e. $x \in (0, 1)$,

$$
\left| \frac{\partial (\hat{f}_k - \hat{f}_k)}{\partial t}(t, x) \right| = \left| (\hat{g}_k(x) - \hat{g}_k(x)) \frac{\partial r_k}{\partial t}(t, x) \right| \\
\leq \left| (\hat{g}_k(x) - \hat{g}_k(x)) \right| \sup_{(t, x) \in Q} \left| \frac{\partial r_k}{\partial t}(t, x) \right| \\
\leq \left| (\hat{g}_k(x) - \hat{g}_k(x)) \right| \frac{\sup_{(t, x) \in Q} \left| \frac{\partial r_k}{\partial t}(t, x) \right|}{d_k} |r_k(T', x)| \\
= C_0 |(\hat{f}_k - \hat{f}_k)(T', x)|,
$$

where, owing to (9),

$$
C_0 = \sup_{(t, x) \in Q} \frac{\left| \frac{\partial r_k}{\partial t}(t, x) \right|}{d_k}.
$$

Hence, we can apply Theorem 1.1 to obtain (11). \hfill \Box

5. Conclusion. In this work, we studied an inverse problem of reconstructing all of the source terms in a system of $n$ inhomogeneous degenerate parabolic equations coupled in cascade from interior measurements of the first component of the state vector. The Lipschitz stability for the inverse source problem is derived using a Carleman estimate with only one observed component. Besides the kind of inverse problem that has been studied in the present work, some other kind of inverse problems could be studied. For example, inverse problems concerning the stability results in determining some or all of the coefficients or the simultaneous stability result for one coefficient and for the initial conditions, which seems completely open for degenerate coupled systems. Our future research will focus on this subject.
6. Appendix. In this appendix, we show a Caccioppoli’s inequality for inhomogeneous degenerate parabolic equations that plays a crucial role to show \( \omega \)-Carleman estimates. This inequality is different from the one shown in [3] for the homogenous case.

**Lemma 6.1** (Caccioppoli’s inequality). Let \( \omega' \) and \( \omega'' \) be two nonempty open subsets of \((0,1)\) such that \( \overline{\omega''} \subset \omega' \) and \( \phi(t,x) = \theta(t)\varrho(x) \), where \( \varrho \in C^2(\overline{\omega'}, \mathbb{R}) \). Then, there exists a constant \( C > 0 \) such that any solution \( y \) of the equation (14) satisfies

\[
\int_{\omega''_0} y_x^2 e^{2s\phi} \, dx \, dt \leq C \int_{\omega'_0} (f^2 + s^2 \theta^2 y^2) e^{2s\phi} \, dx \, dt. \tag{72}
\]

**Proof of Lemma 6.1.** Define a smooth cut-off function \( \tau \in C^\infty(0,1) \) such that \( 0 \leq \tau \leq 1 \) in \((0,1)\), \( \text{supp}(\tau) \subset \omega' \) and \( \tau \equiv 1 \) on \( \omega'' \). Since \( y \) solves (14), we have

\[
0 = \int_{t_0}^T \frac{d}{dt} \left( \int_0^1 \tau^2 y_x^2 e^{2s\phi} \, dx \right) \, dt
= 2s \int_{Q_{t_0}} \tau^2 \phi_1 y_x^2 e^{2s\phi} \, dx \, dt + 2 \int_{Q_{t_0}} \left( d(a(x)y_x) + f \right) \tau^2 ye^{2s\phi} \, dx \, dt.
\]

Then, integrating by parts and using the fact that \( d > 0 \), we find

\[
\int_{Q_{t_0}} a \tau^2 y_x^2 e^{2s\phi} \, dx \, dt = s \int_{Q_{t_0}} \frac{\tau^2}{d} \phi_1 y_x^2 e^{2s\phi} \, dx \, dt + \frac{1}{2} \int_{Q_{t_0}} \left( (e^{2s\phi} \tau^2) a \right)_x y^2 \, dx \, dt
+ \int_{Q_{t_0}} \frac{\tau^2}{d} \left( e^{2s\phi} \varrho \right)_x \, dx \, dt.
\]

Since \( \tau \) is supported in \( \omega' \), \( \tau \equiv 1 \) in \( \omega'' \), \( |\hat{\theta}| \leq C\theta^2 \), \( a \in C^1(\overline{\omega'}) \), \( \min_{x \in \omega'} a(x) > 0 \) and \( \varrho \in C^2(\overline{\omega'}) \) then, using Young inequality one obtains

\[
\min_{x \in \omega'} a(x) \int_{\omega''_0} y_x^2 e^{2s\phi} \, dx \, dt \leq \int_{Q_{t_0}} a \tau^2 y_x^2 e^{2s\phi} \, dx \, dt
\leq C \left( \int_{\omega''_0} s|\hat{\theta}| + s^2 \theta^2 \right) y^2 e^{2s\phi} \, dx \, dt
+ \int_{\omega''_0} f^2 e^{2s\phi} \, dx \, dt + \int_{\omega''_0} y^2 e^{2s\phi} \, dx \, dt \right)
\leq C \int_{\omega''_0} \left( f^2 + s^2 \theta^2 y^2 \right) e^{2s\phi} \, dx \, dt,
\]

and the proof is complete. \( \square \)

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